Casimir energies with finite-width mirrors

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We use a functional approach to the Casimir effect in order to evaluate the exact vacuum energy for a real scalar field in $d+1$ dimensions, in the presence of backgrounds that, in a particular limit, impose Dirichlet boundary conditions on one or two parallel surfaces. Outside of that limit, the backgrounds are described by a nonlocal effective action and may be thought of as modelling finite-width mirrors with frequency-dependent transmission and reflection coefficients. We obtain formal expressions for the Casimir energy in general backgrounds, and provide new explicit results in some particular cases.

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I. INTRODUCTION

The last years have seen a renewed interest in the Casimir effect \cite{1}. The main reason for this has been the design and successful implementation of precision experiments \cite{2}. The magnitudes involved in those experiments posed an important challenge to the first theoretical calculations, usually based on simplified theoretical models. Indeed, in those models, the materials were usually regarded as perfect mirrors, imposing strict boundary conditions on the quantum fields.

In order to explain the experimental results, however, it is important to use more accurate models, including the corrections due to, for example, rugosity, and to finite temperature and conductivity. In this respect, there is a running controversy about how to model theoretically the finite conductivity of the material, and also about its influence on the Casimir forces \cite{3}. The computation of the self-energies due to quantum fluctuations is even more controversial: while the Casimir forces between two objects become finite and independent on the properties of the materials in the limit of perfect conductivity, the situation for the self-energy is, by far, not so clear. The reason may be traced back to the fact that the local energy density diverges close to an idealized boundary \cite{4}, and it is therefore necessary to introduce surface counterterms in order to get finite results for the self-energy \cite{5}. Since the self-energy is relevant, for instance, to analyze physical problems like gravitational effects produced by the vacuum energy, a careful analysis of the sources of divergences in the stress tensor as well as in the total energy is in order.

Motivated by those problems, in this paper we are concerned with the calculation of the vacuum energy distortion that results from the presence of some particular backgrounds, which are meant to represent more realistic boundaries, for a real scalar field theory in $d+1$ spacetime dimensions. Following \cite{6}, we will choose backgrounds which may be used to reproduce exact Dirichlet boundary conditions on flat surfaces, when a particular limit for the parameters that define the background is taken (see also \cite{7}). As in \cite{6}, we also introduce two properties in the descriptions of the plates: finite widths and finite strengths. The former is represented by a smooth function $\sigma$, depending on a single variable (the normal coordinate to the plate), which has a characteristic width $2\varepsilon$. Regarding the strength of the coupling between the walls and the fluctuating field, we extend the kind of background considered in \cite{6}, by allowing for a frequency and (parallel) momentum dependent form for a function $\lambda$, which determines the strength of that coupling.

Any non-trivial dependence on those variables shall amount, in coordinate space, to a nonlocal form for the action that describes the interaction between field and plates. The main new contribution of this paper is the derivation of expressions for the Casimir energy, taking into account nonlocal effective actions that model
finite-width mirrors with frequency dependent reflection and transmission coefficients, using the functional formalism as a tool. We do that for general cases, obtaining formal expressions, as well as for particular examples, where we derive more concrete results.

We will also be able to consider smooth backgrounds, in order to test the dependence of the Casimir energy with the ‘sharpness’ of the boundaries. As already stressed, these generalizations are of interest for the calculation of the Casimir forces in realistic situations, and also from a formal point of view, since many of the divergences that appear in the calculations of the zero point self-energies are related to the introduction of sharp interfaces and/or ideal boundary conditions. In this way, with finite-widths and frequency-momentum dependent strengths, one can simultaneously control the two sources of UV divergences that pop-up during the calculation of vacuum energies. This will also allow us to avoid the unnecessary introduction of ill-defined quantities at the intermediate steps, in the course of calculating physical observables.

The calculation of the Casimir energy in the presence of realistic mirrors is a subject that have been considered in a large number of previous papers. In particular, within the Lifshitz approach it is, in principle, possible to compute the zero point energies for slabs of arbitrary width and arbitrary electromagnetic properties, bounded by flat interfaces [7]. In our approach, we will be able to consider smooth surfaces, including some which approximate slabs (to an arbitrary degree). Moreover, the functional method is particularly well adapted to consider the effects on the zero point energy of the degrees of freedom inside the material. Indeed, the nonlocal effective action may be thought of as resulting precisely from the integration of those degrees of freedom. Thus our initial nonlocal effective action can be considered as a toy model for the interaction of the would-be electromagnetic field (here the quantum scalar field) and the charges confined in the material. In order to illustrate the method, in the present paper we will consider a particular class of nonlocal effective actions for a scalar field, leaving the generalization to the electromagnetic case and the derivation of the effective action from first principles for a future work. Nonlocal effective actions have also been briefly considered by other authors, see for instance [8].

From a more technical point of view, we shall use an extended version of a previously used functional approach [9] to the Casimir effect with perfect mirrors, in order to cope with the more general situations considered here.

The organization of this paper is as follows: in Section II we introduce the general method used to calculate Casimir energies. We obtain formal expressions for the total vacuum energy for one and two mirrors. We then present explicit results: in Section III for the change in the vacuum energy under the introduction of a single mirror, to consider afterwards, in Section IV the case of two mirrors. For the single mirror case, we compute the Casimir energy associated to smooth and piecewise constant backgrounds. We discuss the dependence of the energy with the width $\epsilon$, the cutoff frequency, and the sharpness of the boundary. For the case of two mirrors, we present a general expression, suitable for numerical evaluation, and derive some of its general properties. The expression becomes much more simple for the particular situation in which the width of the mirrors is much smaller than the distance between them, but not necessarily smaller than the inverse of the strength of the coupling $\lambda^{-1}$. In Section V we summarize our conclusions.

II. THE METHOD

In this Section we set up the problem, and derive the general expressions subsequently used for the calculation of vacuum energies in some specific cases.

We shall consider either one or two identical, flat, parallel and finite width ($\sim 2\epsilon$) mirrors in $d$ spatial dimensions. A coordinate system has been chosen such that their ‘centers’ correspond to $x_d = 0$ for the case of a single mirror, and to $x_d = 0$ and $x_d = a$, for the case of two parallel mirrors. In both cases, the system will be described by an Euclidean action:

$$S(\varphi) = S_0(\varphi) + S_I(\varphi),$$

(1)
where $S_0$ defines the free theory:

$$S_0(\varphi) = \frac{1}{2} \int d^{d+1}x \partial_\mu \varphi \partial_\mu \varphi ,$$

(2)

while $S_I(\varphi)$ is a term, quadratic in the field, that contains the interaction with the walls. Introducing a number $N = 1, 2$ depending on whether one has one or two walls, respectively, we have:

$$S_I = \sum_{\alpha=1}^{N} S_I^{(\alpha)} ,$$

(3)

where:

$$S_I^{(\alpha)}(\varphi) = \frac{1}{2} \int dx_0 \int dx_0' \int d^{d-1}x_\parallel \int d^{d-1}x_\parallel' \int dx_d \varphi(x_0, x_\parallel, x_d) \lambda(x_0-x_0', x_\parallel-x_\parallel') \sigma_\epsilon(x_d-a_\alpha) \varphi(x_0', x_\parallel', x_d) ,$$

(4)

with $a_1 \equiv 0$ and $a_2 \equiv a$. $x_\parallel$ denotes the $d-1$ coordinates parallel to the mirror: $x_1, x_2, \ldots, x_{d-1}$. We regard the interaction term as an effective action coming from the integration of the degrees of freedom confined to the walls that interact with the scalar field $\varphi(x)$. On general grounds, the quadratic part of that effective action should be of the form:

$$S_{\text{eff}} = \frac{1}{2} \int d^{d+1}x \int d^{d+1}x' \varphi(x) \Gamma_2(x; x') \varphi(x') .$$

(5)

Taking into account translation invariance in $x_0$ and $x_\parallel$, the dependence of the kernel $\Gamma_2$ on the coordinates must be of the form:

$$\Gamma_2(x; x') = \Gamma_2(x_d, x_d', x_0-x_0', x_\parallel-x_\parallel') .$$

(6)

If, in addition, we assume locality in the perpendicular coordinate $x_d$, we arrive at (4), where the functions $\lambda$ and $\sigma_\epsilon$ encode the structure of the walls, such as their reflection and transmission coefficients and their widths. We will assume, without any lose of generality, that $\lambda$ is an even function (any odd part would cancel away in the action). Therefore its Fourier transform, $\tilde{\lambda}$, is real. Besides, we will also assume that $\tilde{\lambda}$ is strictly positive, so that the interaction with the walls is repulsive at all frequencies. Note that the nonlocality in configuration space will produce reflection and transmission coefficients dependent on the frequency $\omega$, as well as on the wave vector along the parallel plane, $k_\parallel$.

On the other hand, $\sigma_\epsilon$ is an even, strictly positive and continuous function, approximately constant on a region of size $\sim 2\epsilon$ around 0. There is no big loose of generality by assuming strict positivity for this function. Indeed, assuming that one wanted to consider, for example, a function whose support is the interval $[-\epsilon, \epsilon]$, we could approximate it by a strictly positive one that vanished, arbitrarily fast, outside that interval (see Section III where this case is dealt with in some detail). Since $\sigma_\epsilon$ appears multiplied by $\lambda$, we may impose the condition:

$$\int_{-\infty}^{+\infty} dx_d \sigma_\epsilon (x_d) = 1 ,$$

(7)

without restricting the actual interaction at all. Rather, it is a way of disentangling the ‘shape’ (attributed to $\sigma_\epsilon$) from the strength (carried by $\lambda$) of the interaction.

In our conventions, the Euclidean coordinates shall be denoted as $x_\mu$, $\mu = 0, 1, \ldots, d$. The $x_d$ coordinate points along the normal direction to the walls, while $x_\parallel \equiv (x_1, \ldots, x_{d-1})$ are parallel to them. The $d$ spatial coordinates are collectively denoted by $x$.

We now consider the vacuum energy cost, of distorting the vacuum by the introduction of the mirrors. This quantity may be written as follows:

$$E_0 = - \lim_{T \to -\infty} \frac{1}{T} \ln \left( \frac{Z}{Z_0} \right)$$

(8)
where

\[ Z = \int D\varphi e^{-S(\varphi)}, \quad Z_0 = \int D\varphi e^{-S_0(\varphi)}, \]  

and \( T \) is the extension of the (imaginary) time interval.

In more than one spatial dimension, \( E_0 \) is proportional to the ‘area’ \( L^{d-1} \) of the mirrors (assumed to be \((d - 1)\)-dimensional squares of side \( L \)). Since \( L \to \infty \), it is convenient to introduce the energy density \( E_0 \), such that

\[ E_0 = \lim_{T,L \to \infty} \frac{1}{L^{d-1}T} \ln \left( \frac{Z}{Z_0} \right). \]  

Besides, in the case of two mirrors, one is usually interested not in \( E_0 \), but rather in a subtracted quantity, \( \tilde{E}_0 \), defined as the difference:

\[ \tilde{E}_0 \equiv E_0 - E_0(\infty) \]  

where \( E_0(\infty) \) denotes the surface energy density when the mirrors are separated by an infinite distance.

\( \tilde{E}_0 \) is finite even if ideal mirrors (i.e., imposing Dirichlet boundary conditions) were considered. The reason is that the self-energies of the mirrors, being translation invariant, are cancelled when subtracting the \( a \to \infty \) contribution. This, however, would involve a regularization in order to avoid the having to deal with the difference between two ill-defined (divergent) quantities. In our case, the self-energies are finite due to the presence of a physical regularization mechanism. Therefore, the subtraction of the self-energies is a well-defined step, without having to invoke any additional regulator.

To proceed, we note that a quite natural extension of the functional approach followed in previous works \cite{9} can be implemented here. Indeed, we can introduce an auxiliary field, \( \xi_\alpha \), in order to rewrite the exponential factor \( S^{(\alpha)}_I(\varphi) \):

\[ e^{-S^{(\alpha)}_I(\varphi)} = \frac{1}{\mathcal{N}} \int D\xi_\alpha e^{-\frac{1}{2} \int d^{d+1}x \int d^{d+1}x' \xi_\alpha(x) \lambda^{-1}(x_0-x_0';x_\parallel-x_\parallel') \sigma_{\parallel}(x_\parallel-a_\parallel) \delta(x_\parallel-x_\parallel') \xi_\alpha(x')} \times \int e^{i \int d^{d+1}x \xi_\alpha(x) \sigma_{\parallel}(x_\parallel-a_\parallel) \varphi(x)}, \]  

where the factor \( \mathcal{N} \), independent of \( \alpha \), is given by:

\[ \mathcal{N} = \int D\xi_\alpha e^{-\frac{1}{2} \int d^{d+1}x \int d^{d+1}x' \xi_\alpha(x) \lambda^{-1}(x_0-x_0';x_\parallel-x_\parallel') \sigma_{\parallel}(x_\parallel-a_\parallel) \delta(x_\parallel-x_\parallel') \xi_\alpha(x')}, \]  

and we have used \( \lambda^{-1}(x_0-x_0';x_\parallel-x_\parallel') \) as a notation for the inverse kernel associated to \( \lambda(x_0-x_0';x_\parallel-x_\parallel') \), i.e.,

\[ \int_{x_0';x_\parallel'} \lambda(x_0-x_0';x_\parallel-x_\parallel') \lambda^{-1}(x_0''-x_0';x_\parallel''-x_\parallel') = \int_{x_0'';x_\parallel''} \lambda^{-1}(x_0''-x_0';x_\parallel''-x_\parallel') \lambda(x_0''-x_0';x_\parallel''-x_\parallel') = \delta(x_0-x_0') \delta(x_\parallel-x_\parallel'). \]  

An important difference with the case of ideal, zero-width mirrors, is in that the auxiliary fields now live in \( d + 1 \) dimensions, rather than on the \( d \)-dimensional submanifold \( x_\parallel = 0 \). The present case reduces to the ideal situation if \( \sigma_{\parallel} \) is replaced by a \( \delta \) function (of which it may be regarded as an approximation when \( \epsilon \) is finite). In other words, there is a ‘dimensional reduction’ in the auxiliary fields when \( \epsilon \to 0 \). Particular profiles for the function \( \sigma_{\parallel} \) are introduced in Sections \[II\] and \[IV\] obtaining for them explicit results.

We now use (12) to rewrite \( Z \) and afterwards integrate out the scalar field \( \varphi \), to obtain:

\[ Z = Z_0 \frac{1}{(\mathcal{N})^N} \int \prod_{\alpha=1}^N D\xi_\alpha e^{-\frac{1}{2} \int d^{d+1}x \int d^{d+1}x' \sum_{\alpha,\beta=1}^N \xi_\alpha(x) \Pi_{\alpha,\beta}(x,x') \xi_\beta(x')} \]  

\[ \]
with a matrix kernel whose elements $\Omega_{\alpha\beta}$ are defined by:

$$\Omega_{\alpha\beta}(x, x') = \delta_{\alpha\beta} \lambda^{-1}(x_0 - x_0'; x_{\parallel} - x_{\parallel}') \sigma_\epsilon(x_d - a_\alpha) \delta(x_d - x_d') + \sigma_\epsilon(x_d - a_\alpha) \Delta(x_0, x_{\parallel}; x_d, x_0'; x_{\parallel}', x_d'; x_0, x_{\parallel}, x_d - a_\beta),$$

where $\Delta$ is the free scalar-field propagator:

$$\Delta(x, x') = \Delta(x - x') = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik(x-x')} \frac{1}{k^2}.$$

We then make, for each auxiliary field $\xi_\alpha$, the redefinition:

$$\xi_\alpha(x_0, x_{\parallel}, x_d) \rightarrow \int_{x_0', x_{\parallel}'} \lambda^\frac{3}{2}(x_0 - x_0'; x_{\parallel} - x_{\parallel}') \xi_\alpha(x_0', x_{\parallel}', x_d - a_\alpha),$$

(18) both in the explicit integral over $\xi$ in (15) and in the implicit one in the definition of $N$. We have assumed that the square-root kernel is well-defined, what is consistent with the assumptions about its properties. This redefinition produces a non-trivial ($\lambda$-dependent) Jacobian for each factor $D\xi_\alpha$. An identical Jacobian does, however, appear also in each denominator $N$, and therefore they cancel each other. Besides, after the redefinition, the $N$ factor changes: $N \rightarrow \tilde{N}$, which is independent of $\lambda$. Thus, after a trivial shift in the coordinates,

$$Z = Z_0 \frac{1}{(\tilde{N})^N} \prod_{\alpha}^{N} D\xi_\alpha e^{-\frac{1}{2} \int d^{d+1}x_0 \int d^{d+1}x' \sum_{\alpha, \beta} \xi_\alpha(x) \tilde{\Omega}_{\alpha\beta}(x, x') \xi_\beta(x')} ,$$

(19) where:

$$\tilde{\Omega}_{\alpha\beta}(x, x') = \delta_{\alpha\beta} \delta^{(d+1)}(x - x') \sigma_\epsilon(x_d) + \sigma_\epsilon(x_d) \int_{x_0'', x_{\parallel}''} \lambda^\frac{3}{2}(x_0 - x_0''; x_{\parallel} - x_{\parallel}'') \Delta(x_0'', x_{\parallel}'', x_d + a_\alpha; x_0'', x_{\parallel}'', x_d + a_\beta) \times \lambda^\frac{3}{2}(x_0'' - x_0'; x_{\parallel}'' - x_{\parallel}') \sigma_\epsilon(x_d') .$$

(20) We then proceed to perform yet another redefinition of the auxiliary fields, now involving a diffeomorphism in the $x_d$ coordinate. We introduce the one-to-one mapping $x_d \rightarrow z$ that results from the differential equation:

$$\frac{dz}{dx_d} = \sigma_\epsilon(x_d) ,$$

(21) where the assumed positivity of $\sigma_\epsilon$ comes in handy to ensure that the change of variables is non-singular. Imposing the condition $z(0) = 0$, the solution to the previous equation is unique. We also note that, as a consequence of the change of variables, (22)

$$\delta(x_d - x_d') = \sigma_\epsilon(x_d) \delta(z - z') , \quad dx_d \sigma_\epsilon(x_d) = dz .$$

Equation (21), together with the condition $z(0) = 0$ and the normalization condition for $\sigma_\epsilon$ imply that the range of $z$ is (regardless of the particular profile used for $\sigma_\epsilon$) the finite interval $[-\frac{1}{2}, \frac{1}{2}]$.

Keeping the same notation for the auxiliary fields when written in terms of the new variables, we then have, in a simplified notation:

$$Z = Z_0 \int \prod_{\alpha}^{N} D\xi_\alpha e^{-\frac{1}{2} \int x_0 \xi_\alpha(x_0, x_{\parallel}, z) \sum_{\alpha, \beta} \xi_\alpha(x_0, x_{\parallel}, z) \tilde{K}_{\alpha\beta}(x_0, x_{\parallel}, z) \xi_\beta(x_0, x_{\parallel}, z')} ,$$

(23)
with

\[ K_{\alpha\beta}(x_0, x_{||}; z; x_{0}', x_{||}') = \delta_{\alpha\beta} \delta(x_0 - x_{0}') \delta^{(d-1)}(x_{||} - x_{||}') \delta(z - z') \]

\[ + \int_{x_{0}'', x_{||}'', x_{||}''} \lambda^{\alpha\beta}(x_0 - x_{0}'', x_{||} - x_{||}'') \Delta(x_{0}'', x_{||}'', x_{||}'', x_{||}'') \delta(z - z') \delta(z' - z') + a_{\alpha\beta} \lambda^{\alpha\beta}(x_0, x_{||}') \]

\[ (24) \]

We note that, had any Jacobian arisen because of the last field diffeomorphism, it would have, again, been 
cancelled against an equal object coming from \( \tilde{N} \). Furthermore, since after the redefinition \( \tilde{N} \) became 
independent of \( \lambda \) and \( \sigma_\epsilon \), that factor has been dropped.

Then we Fourier transform all the spacetime coordinates for which there is translation invariance, namely, 
\( x_0, x_{||} \). The transformed operator has the following form:

\[ \tilde{K}_{\alpha\beta}(z; z') = \delta_{\alpha\beta} \delta(z - z') + \tilde{\lambda}(\omega, k_{||}) D_{\alpha\beta}(z; z'), \]

where

\[ \tilde{\lambda}(\omega, k_{||}) = \int_{-\infty}^{+\infty} dx_0 \int d^{d-1} x_{||} e^{-i(\omega x_0 + k_{||} x_{||})} \lambda(x_0, x_{||}) \]

\[ (25) \]

and

\[ D_{\alpha\beta}(z; z') = \frac{e^{-\kappa|a_\alpha(z) - a_\beta(z') + a_\alpha - a_\beta|}}{2\kappa}, \]

\[ (27) \]

where \( \kappa = \sqrt{\omega^2 + k_{||}^2} \) (to simplify the notation, we omit writing the dependence of \( D_{\alpha\beta}(z; z') \) on \( k_{||} \) and \( \omega \) explicitly).

Then we have the following expression for \( Z \):

\[ Z = Z_0 \left( \det \tilde{K} \right)^{-\frac{1}{2}}, \]

\[ (28) \]

and:

\[ \mathcal{E}_0 = \lim_{T, L \to \infty} \frac{1}{2TL^{d-1}} \text{Tr} \left( \ln \tilde{K} \right), \]

\[ (29) \]

where ‘Tr’ denotes trace over all the variables (including \( \omega \) and \( k_{||} \)). Introducing the symbol ‘\( \tilde{\text{Tr}} \)’ for the 
(reduced) trace over the Hilbert space of functions depending on \( z \in [-\frac{1}{2}, \frac{1}{2}] \), we may write a more explicit 
formula, where the trace over the variables for which the kernel is translation-invariant is explicit:

\[ \mathcal{E}_0 = \int \frac{d\omega}{2\pi} \int \frac{d^{d-1} k_{||}}{(2\pi)^{d-1}} \tilde{\text{Tr}} \left( \ln \tilde{K} \right). \]

\[ (30) \]

From the last expression one can, in principle, extract the vacuum energy relevant to each case. However, 
since there are important differences between them, we present now separate calculations of \( \mathcal{E}_0 \), corresponding 
to two different physical situations: for the case of one mirror, and for a two-mirror system.

A. One mirror

This case amounts to a single index \( \alpha = 1 \); thus we suppress the \( \alpha, \beta \) indices, and consider just the kernel:

\[ \tilde{K}(z; z') = \delta(z - z') + \tilde{\lambda}(\omega, k_{||}) D(z, z'), \]

\[ (31) \]
with
\[ D(z; z') \equiv \frac{e^{-\kappa|x_d(z)-x_d(z')|}}{2\kappa}. \] (32)

We shall calculate the trace of the logarithm of \( \widetilde{K} \), by finding the eigenvalues of \( \widetilde{K} \). The equation for \( \psi_\alpha(z) \), the eigenfunction associated to the eigenvalue \( \alpha \), may be written as follows:
\[
\int_{-\frac{1}{2}}^{+\frac{1}{2}} \! 
\frac{d\omega}{2\pi} \frac{\lambda(\omega, k_{\|})}{\alpha - 1} \psi_\alpha(z') = \alpha \psi_\alpha(z). \]
(33)

This can be converted into a differential equation, taking into account the fact that \( D(z; z') \) is obtained by performing a Fourier transformation plus a change of variables in the Green’s function for the real scalar field. Indeed, starting from:
\[
\left( -\frac{\partial^2}{\partial x_d^2} + \kappa^2 \right) \tilde{G}_0(x_d, x'_d) = \delta(x_d - x'_d), \]
(34)
where \( \tilde{G}_0(x_d, x'_d) \) is the Fourier transformed of the free propagator:
\[
\tilde{G}_0(x_d, x'_d) \equiv \frac{e^{-\kappa|x_d-x'_d|}}{2\kappa}, \]
(35)
we obtain, by changing variables:
\[
\left[ -\tilde{\sigma}_e(z) \frac{\partial}{\partial z} (\tilde{\sigma}_e(z) \frac{\partial}{\partial z}) + \kappa^2 \right] D(z; z') = \tilde{\sigma}_e(z) \delta(z - z'), \]
(36)
where
\[
\tilde{\sigma}_e(z) \equiv \sigma_e(x_d(z)). \]
(37)

We see that, by acting with the differential operator that appears on the left hand side of (36) on both sides of (33), it becomes a differential equation with the structure:
\[
L[\psi_\alpha](z) = \xi_\alpha \psi_\alpha(z), \]
(38)
where \( L \) denotes a linear differential operator of the Sturm-Liouville type:
\[
L[f](z) = -\frac{d}{dz} \left[ p(z) \frac{df(z)}{dz} \right] + q(z) f(z), \]
(39)
with
\[
p(z) \equiv \tilde{\sigma}_e(z), \quad q(z) \equiv \frac{\kappa^2}{\sigma_e}, \]
(40)
and the eigenvalue of \( L, \xi_\alpha \), determined \( \alpha \) through the relation \( \xi_\alpha = \frac{\lambda}{\alpha - 1} \). The eigensystem corresponding to the operator whose determinant we need has the coefficient functions \( p \) and \( q \), which are in turn determined by the properties of the mirror.

An important fact to note is that the space of functions to be used for the calculation of the eigenvalues of \( L \) corresponds to functions that vanish at \( z = \pm \frac{1}{2} \). Indeed, the \( \tilde{K} \) operator is self-adjoint under the \( L^2(\mathbb{R}) \) scalar product defined by:
\[
(f, g) \equiv \int_{-\infty}^{+\infty} dx_d f^*(x_d)g(x_d) \]
(41)
for any pair of square integrable functions: \( f, g \). This scalar product becomes, when written in terms of the new variable \( z \):

\[
(f, g) \equiv \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{dz}{\tilde{\sigma}(z)} f^*(z) g(z) .
\]  

(42)

In particular, this implies that normalizable functions in the interval \([-\frac{1}{2}, \frac{1}{2}]\) must vanish when \( z = \pm \frac{1}{2} \) (of course, they should also decrease sufficiently fast for the integral to converge). The reason is that, from the defining properties of \( \sigma_\epsilon \), it follows that \( \tilde{\sigma}_\epsilon \) vanishes at least linearly at those points.

It also implies that eigenfunctions of the Sturm-Liouville operator corresponding to different eigenvalues shall be orthogonal, for the scalar product (42), which includes the weight function \( 1/\tilde{\sigma}_\epsilon \).

Thus, we conclude that, to evaluate the vacuum energy in the one mirror case, this procedure has lead us to the expression

\[
\tilde{E}_0 = \frac{1}{2} \int \frac{d\omega}{2\pi} \left( \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \sum_l \ln [\alpha_l(\omega, k_\parallel, \epsilon)] \right) ,
\]

(43)

where \( l \) labels the eigenvalues \( \alpha \). They are, in general, non trivial functions of their arguments, and they depend on the particular function \( \sigma_\epsilon \) considered.

### B. Two mirrors

We shall now consider the situation of two identical mirrors, located at a distance \( a \) apart. Thus, we are concerned with the quantity:

\[
\tilde{\mathcal{E}}_0(a) \equiv E_0(a) - E_0(\infty)
\]

(44)

We have to consider now the \( 2 \times 2 \) matrix kernel:

\[
[\tilde{K}_{\alpha \beta}(z; z')] = \begin{pmatrix} \tilde{\mathcal{K}}(z; z') & \mathcal{P}(z; z') \\ \mathcal{Q}(z; z') & \tilde{\mathcal{K}}(z; z') \end{pmatrix}
\]

(45)

where the two diagonal elements in the expression above coincide with the (identically noted) kernel corresponding to the single-wall case, equation (31). There appear also two new kernels \( \mathcal{P} \) and \( \mathcal{Q} \):

\[
\mathcal{P}(z; z') = \tilde{\lambda}(\omega, k_\parallel) \tilde{G}_0(x_d(z) + 0; x_d(z') + a) = \tilde{\lambda}(\omega, k_\parallel) \frac{e^{-\kappa|x_d(z) - x_d(z') - a|}}{2\kappa} ,
\]

(46)

and \( \mathcal{Q} \equiv \mathcal{P}|_{a \to -a} \).

One can then use the expression for the determinant of a matrix in terms of a combination of its blocks,

\[
\det [K_{ij}] = (\det \tilde{\mathcal{K}})^2 \det(1 - \mathcal{O})
\]

(47)

where

\[
\mathcal{O} = \tilde{\mathcal{K}}^{-1} \mathcal{P} \tilde{\mathcal{K}}^{-1} \mathcal{Q} .
\]

(48)

Then we note that, since \( \det \tilde{\mathcal{K}} \) is independent of \( a \), the Casimir energy density, obtained by subtracting the contribution with the plates at an infinite distance, becomes:

\[
\tilde{\mathcal{E}}_0(a) = \lim_{T, L \to \infty} \frac{1}{2L^{d-1}T} \text{Tr} \ln(1 - \mathcal{O}) .
\]

(49)
It is important to note that, in the last equation, the $a$-independent contribution, corresponding to the self-energies of the plates, have been completely subtracted. Note that that contribution yields exactly twice the vacuum energy for one plate. Of course, had $\lambda$ been assumed to be a constant, those self energies would have been, generally, divergent.

Thus the UV behaviour of (11) is milder that in other approaches, where those contributions have to be dealt with in order to give sense to the Casimir energy. Here, as we shall see in the examples, it is already finite. Assuming that $\vartheta_l(\omega, k_\parallel, \epsilon, a)$ are the eigenvalues of $O$:

$$
\tilde{\mathcal{E}}_0(a) = \frac{1}{2} \int \frac{d\omega}{2\pi} \int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \sum_l \ln \left[ 1 - \vartheta_l(\omega, k_\parallel, \epsilon, a) \right],
$$

the expression that we will use as a starting point for the calculation of the Casimir energy density in concrete examples.

The finiteness of (50) will follow from the fact that, as a function of the frequency and momenta, $O$ is bounded by the exponentially decreasing factors of those variables, carried by $P$ and $Q$.

### III. RESULTS FOR THE TOTAL VACUUM ENERGY IN THE SINGLE-MIRROR CASE

#### A. Smooth $\sigma_\epsilon$

We first assume a particular form for the function $\sigma_\epsilon$. A very convenient choice, from the calculational point of view is:

$$
\sigma_\epsilon(x_d) = \frac{1}{2\epsilon} \text{sech}^2 \left( \frac{2x_d}{\epsilon} \right),
$$

where the $\frac{1}{2\epsilon}$ factor has been introduced in order to satisfy $\int_{-\infty}^{+\infty} dx_d \sigma_\epsilon(x_d) = 1$, the normalization condition corresponding to an approximant of the $\delta$-function. Of course, the wall is then localized around $x_d = 0$, with a width $\sim 2\epsilon$. Besides, for this profile we may find the explicit relation between $x_d$ and $z$:

$$
x_d(z) = \epsilon \text{arctanh}(2z),
$$

$$
z = \frac{1}{2} \tanh \left( \frac{x_d}{\epsilon} \right). \quad (52)
$$

Note that $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We immediately find:

$$
\tilde{\sigma}_\epsilon(z) = \frac{1}{2\epsilon} \left[ 1 - (2z)^2 \right]. \quad (53)
$$

Applying the differential operator on the lhs of (36) to both sides of (33), and then changing variables: $z \rightarrow u \equiv 2z$, we obtain a differential equation for the eigenfunctions:

$$
\frac{d}{du} \left[ (1 - u^2) \frac{d}{du} \psi_\alpha(u) \right] + \left[ \frac{\epsilon\tilde{\lambda}(\omega, k_\parallel)}{2(\alpha - 1)} - \frac{(\epsilon\kappa)^2}{1 - u^2} \right] \psi_\alpha(u) = 0. \quad (54)
$$

We recognize the associated Legendre equation, which has regular independent solutions when:

$$
\frac{\epsilon\tilde{\lambda}(\omega, k_\parallel)}{2(\alpha - 1)} = l(l + 1), \quad l = 0, 1, \ldots
$$

$$
\epsilon\kappa = m, \quad m = 1, \ldots, l. \quad (55)
$$
Note that the \( m = 0 \) solutions have been discarded, since the last condition does not lead to any non-trivial solution. The eigenvectors are then the polynomials \( P_m^l(2z) \) which (since \( m \neq 0 \)) vanish at \( z = \pm \frac{1}{2} \), as expected from our general analysis at the end of the previous section.

From the above, we conclude that the eigenvalues are:

\[
\alpha = \alpha(l, \omega) = 1 + \frac{e\tilde{\lambda}(\omega, k_\parallel)}{2l(l+1)},
\]

while, for each \( l \), we have the constraints:

\[
\epsilon \sqrt{\omega^2 + k_\parallel^2} = 1, \ldots, l,
\]

which restrict the allowed values of \( k_\parallel \equiv |k_\parallel| \) and \( \omega \). We may then use the explicit form of the eigenvalues to obtain:

\[
\mathcal{E}_0 = \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=1}^{l} \int \frac{d\omega}{2\pi} \int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}
\times \delta[\epsilon \sqrt{\omega^2 + k_\parallel^2} - m] \ln \left[ 1 + \frac{e\tilde{\lambda}(\omega, k_\parallel)}{2l(l+1)} \right],
\]

Note that we are writing a general expression as a function of the dimension of space, \( d \), but we are not meaning by that that a dimensional regularization is used. Indeed, as we will see, this expression will be regularized by the function \( \tilde{\lambda} \).

In the case \( d = 1 \), there is no integral over \( k_\parallel \). Furthermore, the integral over \( \omega \) becomes trivial because of the constraint, and thus we derive an expression for the vacuum energy in terms of a double sum:

\[
\mathcal{E}_0 = E_0 = \frac{1}{4\pi \epsilon} \sum_{l=1}^{\infty} \sum_{m=1}^{l} \left\{ \ln \left[ 1 + \frac{e\tilde{\lambda}(\omega, k_\parallel)}{2l(l+1)} \right] \right. \\
+ \left. \ln \left[ 1 + \frac{e\tilde{\lambda}(\omega, k_\parallel)}{2l(l+1)} \right] \right\},
\]

or,

\[
\mathcal{E}_0 = \frac{1}{4\pi \epsilon} \sum_{l=1}^{\infty} \sum_{m=1}^{l} \ln \left| 1 + \frac{e\tilde{\lambda}(\omega, k_\parallel)}{2l(l+1)} \right|^2,
\]

since \( \lambda(x_0 - x'_0) \) is assumed to be real (i.e., no dissipative coupling to the mirror).

The result (60) shows an interesting relationship between the large-\( \omega \) behaviour of \( \tilde{\lambda} \) and the size of the mirror, \( \epsilon \). Indeed, assume that \( \tilde{\lambda} \) becomes negligible small above some cutoff frequency \( \omega_c \), then we have the approximate expression:

\[
\mathcal{E}_0 \simeq \frac{1}{4\pi \epsilon} \sum_{l=1}^{\infty} \sum_{m=1}^{\min\{l, [\omega_c] \}} \ln \left| 1 + \frac{e\tilde{\lambda}(\omega, k_\parallel)}{2l(l+1)} \right|^2.
\]

Note that the energy is zero if \( \epsilon < 1/\omega_c \). This may be understood from the fact that no modes could be trapped inside the mirror, since it is transparent for wavelengths smaller than \( \epsilon \).

To proceed in the case \( d > 1 \), we will assume that \( \tilde{\lambda}(\omega, k_\parallel) = \tilde{\lambda}(\kappa) \). The motivation for this assumption is that, with this choice, the reflection and transmission coefficients will depend on the wave number \( k_d \) in the
perpendicular direction (note that when rotated back to Minkowski space, $\kappa$ becomes $k_d$). In this case, the integrals in Eq. (58) can be easily computed. We obtain

$$
E_0 = \frac{\Omega_{d-1}}{2(2\pi^d)} \sum_{l=1}^{\infty} \sum_{m=1}^{l} m^{d-1} \ln \left[ 1 + \frac{\tilde{\lambda}(\omega)}{2l(l+1)} \right],
$$

where $\Omega_{d-1} = \ldots$ is the solid angle in $d-1$ dimensions. If there is a cutoff $\omega_c$, as in the previous example, and besides we have that for $\kappa < \omega_c$ it is constant: $\tilde{\lambda} = \tilde{\lambda}_0$, then:

$$
E_0 = \frac{\Omega_{d-1}}{2(2\pi^d)} \sum_{l=1}^{\infty} \ln \left[ 1 + \frac{\tilde{\lambda}_0}{2l(l+1)} \right] \left( \sum_{m=1}^{\min(l, \lfloor \epsilon \omega_c \rfloor)} m^{d-1} \right).
$$

Of course, we also have the same relationship between $\omega_c$ and $\epsilon$ as in the previous example with $d = 1$, since the defect is essentially one-dimensional.

Let us see that the introduction of the cutoff $\omega_c$ produces a finite result, in any number of dimensions. To see this, we just consider the case of a $\tilde{\lambda}$ which is constant below the cutoff, and zero above. Then we may rewrite (63) more explicitly, as follows:

$$
E_0 = \frac{\Omega_{d-1}}{2(2\pi^d)} \sum_{l=1}^{\lfloor \epsilon \omega_c \rfloor} \ln \left[ 1 + \frac{\epsilon \tilde{\lambda}_0}{2l(l+1)} \right] \left( \sum_{m=1}^{l} m^{d-1} \right) + \frac{\Omega_{d-1}}{2(2\pi^d)} \sum_{m=1}^{\lfloor \epsilon \omega_c \rfloor} m^{d-1} \ln \left[ 1 + \frac{\epsilon \tilde{\lambda}_0}{2(l+1)} \right].
$$

Now, the first term on the rhs above is a finite sum, while the second one involves a convergent series, so the result is finite. Of course, the same holds true for the case of a (continuous) $\tilde{\lambda}$ function which is not necessarily constant below the cutoff. Indeed, one may simply bound that function by a constant, and then use the previous result.

B. Piecewise constant profile

We consider now another important profile for the function $\sigma_{\epsilon}$. We assume that this function takes the approximately constant value $1/(2\epsilon)$ inside the interval $[-\frac{1}{2}, \frac{1}{2}]$, and vanishes very fast outside. The function is then extremely smooth, except for small intervals around $x_d = \pm \epsilon$ where all the variation is concentrated. Of course, one can approximate a ‘square barrier’ function with this kind of profile, and this is the motivation for considering it.

In the limit when that profile is approached, we are left with the equation for the eigenvalues:

$$
-\frac{d^2}{dz^2} \psi_\alpha(z) + (2\epsilon\kappa)^2 \psi_\alpha(z) = \frac{2\epsilon \tilde{\lambda}(\omega, k_\parallel)}{\alpha - 1} \psi_\alpha
$$

in the interval $(-\frac{1}{2}, \frac{1}{2})$. With the boundary conditions we find the eigenvalues of $\tilde{\mathcal{K}}$, which form a discrete set:

$$
\alpha = \alpha_l = 1 + \frac{2\epsilon \tilde{\lambda}(\omega, k_\parallel)}{l^2 \pi^2 + (2\epsilon\kappa)^2}, \quad l \in \mathbb{N}.
$$

Thus, in $d$ spatial dimensions, the expression for $\mathcal{E}$ becomes:

$$
\mathcal{E}_0 = \frac{1}{2} \int \frac{d\omega}{2\pi} \int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \sum_{l=1}^{\infty} \ln \left[ 1 + \frac{2\epsilon \tilde{\lambda}(\omega, k_\parallel)}{l^2 \pi^2 + (2\epsilon \kappa)^2} \right].
$$
which differs from the one obtained for the other profile used for $\sigma$. It should be noted that the procedure above for the square barrier profile relies upon the existence of a region where that function is approximately constant. This would not make sense if we wanted to consider the $\delta$-function ($\epsilon \to 0$) limit, which we shall study (see next section) using the $(\text{sech})^2$ function instead.

We assume, as stated before, that $\tilde{\lambda}$ depends on $\omega$ and $k_\parallel$ through the particular combination $\kappa = \sqrt{\omega^2 + k_\parallel^2}$, i.e., $\tilde{\lambda} = \tilde{\lambda}(\kappa)$. Then we may write a more explicit formula for $E_0$:

$$
E_0 = \frac{1}{2d\pi^{d/2}\Gamma(d/2)} \int_0^\infty d\kappa \kappa^{d-1} \sum_{l=1}^\infty \ln \left[ 1 + \frac{2\epsilon \tilde{\lambda}(\kappa)}{l^2\pi^2 + (2\epsilon\kappa)^2} \right].
$$

The series is convergent, and one can immediately find its large-$\kappa$ behaviour:

$$
\sum_{l=1}^\infty \ln \left[ 1 + \frac{2\epsilon \tilde{\lambda}(\kappa)}{l^2\pi^2 + (2\epsilon\kappa)^2} \right] \sim \frac{\tilde{\lambda}(\kappa)}{\kappa}.
$$

From the previous expression, we see that if $\tilde{\lambda}$ vanishes above a cutoff, the energy is finite. We see that it would also be finite if it vanished following an exponential law, or even a power-law: $\tilde{\lambda} \sim 1/\kappa^\alpha$, with $\alpha > d - 1$.

### C. UV divergences

We conclude this section by presenting a brief study of the UV behaviour (divergences) of the expressions that we used to calculate the energies in the one mirror case; i.e., of the self-energies, when one approaches the constant-$\tilde{\lambda}$ case.

We will study here that limit starting from the self-energy for a finite-width mirror with a cutoff for $\kappa$, so that the energy is finite, and then take the relevant limit, which corresponds to a constant $\tilde{\lambda}$ (that eventually tends to infinity), and a vanishing $\epsilon$. The most transparent way to take these two limits is by letting first $\epsilon \to 0$, and then making $\tilde{\lambda}$ go to a constant. The advantage of using this order is that the first limit can be taken exactly. Indeed, when $\epsilon \to 0$, we have:

$$
\tilde{K}(z; z') \to \delta(z - z') + \frac{\tilde{\lambda}(\kappa)}{2\kappa},
$$

or, in operatorial form,

$$
\tilde{K} \to (1 + \frac{\tilde{\lambda}(\kappa)}{2\kappa})\rho + \eta,
$$

where $\rho$ and $\eta$ are projection operators whose kernels are:

$$
\rho(z; z') = 1, \quad \eta(z; z') = \delta(z - z') - \rho(z, z').
$$

Thus, the expression for the vacuum energy in this limit is

$$
[\mathcal{E}_0]_{\epsilon \to 0} = \frac{1}{2d\pi^{d/2}\Gamma(d/2)} \int_0^\infty d\kappa \kappa^{d-1} \ln \left[ 1 + \frac{\tilde{\lambda}(\kappa)}{2\kappa} \right].
$$

Note that the large-$\kappa$ behaviour of the integral is:

$$
\int_0^\infty d\kappa \kappa^{d-1} \ln \left[ 1 + \frac{\tilde{\lambda}(\kappa)}{2\kappa} \right] \sim \frac{1}{2} \int_0^\infty d\kappa \kappa^{d-2}\tilde{\lambda}(\kappa),
$$

(74)
as in the sharp-$\sigma$ case. Assuming a constant $\tilde{\lambda}$, and introducing an Euclidean cutoff $\Lambda$ for the $\kappa$-integration, we see that:

$$\int_0^\infty d\kappa \kappa^{d-1} \ln \left[ 1 + \frac{\tilde{\lambda}(\kappa)}{2\kappa} \right] \sim \frac{1}{2} \tilde{\lambda}_0 \int_0^\Lambda d\kappa \kappa^{d-2} ,$$

(75)

where the $\sim$ refers to the large-$\kappa$ behaviour of the integrals only. The UV divergences in the previous expression are then quite immediate to extract: in $d = 1$ (two spacetime dimensions) there is a logarithmic divergence:

$$[\mathcal{E}_0]_{d \to 0, \text{div}} = \frac{\tilde{\lambda}_0}{4\pi} \ln(\frac{\Lambda}{\mu}) ,$$

(76)

where $\mu$ is a momentum scale. It should be noted that these divergences may be interpreted in quantum field theoretic terms. Indeed, when $\tilde{\lambda}$ is a constant, the model we consider may be regarded as a free scalar field theory with $\varphi^2$ insertions; the latter coming from an expansion of the term proportional to $\tilde{\lambda}$ in the action (which is local when $\tilde{\lambda}$ is a constant). For $n$ insertions (i.e., the term of order $\tilde{\lambda}^n$) in $d + 1$ dimensions, the superficial degree of divergence $\delta$ of a term contributing to $\mathcal{E}_0$ with $n$ insertions of $\varphi^2$ is, by power counting:

$$\delta = (d + 1) + n(d - 1) - n(d + 1) .$$

(77)

For the $d = 1$ case, we see that only the $n = 1$ term (linear in $\tilde{\lambda}$) diverges and it does so logarithmically, as we have seen above. In two spatial dimensions, there is only a linear divergence, in the $n = 1$ term, while for $d = 3$ there are divergences for $n = 1$ (quadratic) and $n = 2$ (logarithmic). Therefore, we expect to have one, two and three renormalization conditions for $d = 1$, $d = 2$ and $d = 3$, respectively.

To make sense of those divergences, rather that dealing with the one particle irreducible functions corresponding to the operator insertions, we apply the renormalization program in a more straightforward way, in terms of the vacuum energy, as follows: subtracting from the integrand in the expression for $\mathcal{E}_0$ with a constant $\tilde{\lambda}$ its MacLaurin expansion in $\tilde{\lambda}$ up to an order $\delta + 1$, where $\delta$ corresponds to the higher degree of divergence at the given $d$, we have a convergent expression. The renormalized energy will then have a finite degree polynomial, whose coefficients must be fixed by imposing suitable renormalization conditions. For example, in $d = 1$:

$$\mathcal{E}_0 = c_1 \tilde{\lambda}_0 + \Delta \mathcal{E}_0 ,$$

(78)

where $c_1$ is a constant (depending logarithmically on the cutoff), and $\Delta \mathcal{E}_0$ is the finite expression:

$$\Delta \mathcal{E}_0 = \frac{1}{2\pi} \int_0^\infty d\kappa \left[ \ln \left( 1 + \frac{\tilde{\lambda}_0}{2\kappa} \right) - \frac{\tilde{\lambda}_0}{2\kappa} \right] .$$

(79)

A simple rescaling shows that $\Delta \mathcal{E}_0$ is also linear in $\tilde{\lambda}_0$ (although with a cutoff-independent, finite coefficient). Thus we conclude that the renormalized energy will have the form:

$$[\mathcal{E}_0]_{d=2} = C_1 \tilde{\lambda}_0 ,$$

(80)

where $C_1$ requires, to be fixed, to know the vacuum energy at some scale. In $d = 2$, on the other hand,

$$\mathcal{E}_0 = c_1 \tilde{\lambda}_0 + c_2 \tilde{\lambda}_0^2 + \Delta \mathcal{E}_0 ,$$

(81)

where now,

$$\Delta \mathcal{E}_0 = \frac{1}{4\pi} \int_0^\infty d\kappa \kappa \left[ \ln \left( 1 + \frac{\tilde{\lambda}_0}{2\kappa} \right) - \frac{\tilde{\lambda}_0}{2\kappa} + \frac{1}{2} \left( \frac{\tilde{\lambda}_0}{2\kappa} \right)^2 \right] ,$$

(82)
which goes like $\tilde{\lambda}^2_0$; then:

$$[\mathcal{E}_0]_{\text{ren}} = C_1 \tilde{\lambda}_0 + C_2 \tilde{\lambda}^2_0.$$  \hfill (83)

Finally, in $d = 3$, a similar procedure yields:

$$[\mathcal{E}_0]_{\text{ren}} = C_1 \tilde{\lambda}_0 + C_2 \tilde{\lambda}^2_0 + C_3 \tilde{\lambda}^3_0.$$  \hfill (84)

As already mentioned, the vacuum energy is simply related to the effective action in the presence of $\varphi^2$ operator insertions; hence the renormalization conditions on the vacuum energy may be related to conditions for the corresponding one particle irreducible functions. From a more phenomenological point of view, the meaning of the renormalization conditions is best put in a negative way: they summarize the ignorance one has about the vacuum energy, when it is divergent. Namely, one can predict the dependence of the vacuum energy with the coupling constant, except for the first few terms in a McLaurin expansion of the energy in terms of $\tilde{\lambda}_0$, the $\kappa \to \infty$ part of that coupling constant. It is perhaps worth emphasizing that in a real situation there must be a cutoff; hence $\tilde{\lambda}_0 = 0$, and the divergences above are replaced by finite, cutoff dependent terms, which one does not need to renormalize, and whose precise form depends on the details on the defect (its profile, for example).

We see that we would need to impose three of those conditions in order to completely fix the renormalization constants $C_i$. We agree with the results of [6], for the case of a sharp defect.

Finally, note that the would be $n = 0$ ($\tilde{\lambda}$-independent) divergences do not appear because we measure energies with respect to the vacuum in the absence of mirrors.

IV. RESULTS ABOUT THE CASIMIR ENERGY FOR TWO MIRRORS

We shall now consider the evaluation of the Casimir energy, as a function of the different parameters, for different profiles.

A. Thin mirrors

As a first check of the method, we consider firstly the case of thin walls ($\epsilon \to 0$) and arbitrary $\lambda$. This corresponds, physically, to $\epsilon$ much smaller than the other two parameters with the dimensions of a length, $a$ and $\tilde{\lambda}^{-1}$. Later on we shall also impose the condition that $\lambda$ tends to infinity to recover the well-known Dirichlet case.

The $\epsilon \to 0$ condition can be easily imposed on $\tilde{K}$; indeed, when $\epsilon \to 0$, one has the following expression for that kernel:

$$\tilde{K}^{-1}(z; z') \to \delta(z - z') - \frac{\tilde{\lambda}(\kappa)}{2\kappa + \lambda(\kappa)},$$  \hfill (85)

and

$$P(z; z') \to \frac{\tilde{\lambda}(\kappa)}{2\kappa} e^{-\kappa a}.$$  \hfill (86)

Then it is immediate to see that, in the same limit,

$$O(z; z') \to \left(\frac{\tilde{\lambda}(\kappa)}{2\kappa + \lambda(\kappa)}\right)^2 e^{-2\kappa a},$$  \hfill (87)
i.e., it becomes independent of \( z \) and \( z' \). Thus the Casimir energy density for \( \epsilon \to 0 \) is given by the expression:

\[
\hat{E}_0(a) = \frac{1}{2d\pi^{d/2}\Gamma(d/2)} \int_0^\infty d\kappa \kappa^{d-1} \ln \left[ 1 - \left( \frac{\lambda(\kappa)}{2\kappa + \lambda(\kappa)} \right)^2 e^{-2\alpha a} \right],
\]

(88)

which could be evaluated numerically for different interesting functional forms of \( \lambda \), depending on the material considered.

We can obtain exact results using some particular profiles. The Dirichlet case is obtained by considering a constant \( \lambda \) which tends to infinity. Note that the integral is convergent for any finite \( \lambda \), thus the limit could be taken after evaluating the integral over momenta. Nevertheless, taking the \( \lambda \to \infty \) limit before we can perform the integral exactly, in any number of dimensions, \( d \). For example,

\[
\hat{E}_0(a) = \begin{cases} 
\frac{-\frac{\pi}{2\lambda^2\zeta(3)}}{16\pi^a} & \text{for } d = 1 \\
\frac{\pi}{2\lambda^2\zeta(3)} & \text{for } d = 2 \\
-\frac{\pi}{4\lambda^2\zeta(3)} & \text{for } d = 3
\end{cases}
\]

(89)

There is another profile for \( \lambda \) which allows us to find exact results, albeit it is unphysical regarding the properties of the function \( \lambda \). However, it serves the purpose of illustrating a property of the Casimir effect in the Dirichlet case: consider the profile \( \lambda(\kappa) = 2\alpha \kappa \), where \( \alpha \) is a constant, which has the unphysical property of growing with the frequency and momentum. In this case we have:

\[
\hat{E}_0(a) = \frac{1}{2d\pi^{d/2}\Gamma(d/2)} \int_0^\infty d\kappa \kappa^{d-1} \ln \left[ 1 - \left( \frac{\alpha}{1+\alpha} \right)^2 e^{-2\alpha a} \right] = I(d, \alpha) \frac{1}{a^d},
\]

(90)

where

\[
I(d, \alpha) = \frac{1}{2d\pi^{d/2}\Gamma(d/2)} \int_0^\infty dx x^{d-1} \ln \left[ 1 - \left( \frac{\alpha}{1+\alpha} \right)^2 e^{-2\alpha x} \right],
\]

(91)

is a finite number, depending on the constant \( \alpha \) and the dimension, \( d \). Note that this kind of profile produces a dependence of the energy with the distance that is identical to the Dirichlet case, although with a smaller coefficient, in spite of the fact that the coupling constant is a function that grows with \( \kappa \). This is a reflection of the fact that, although \( \lambda \) is not infinite, its particular form introduces reflection and transmission coefficients that are independent of \( \kappa \), as in the Dirichlet case.

Of course, when \( \alpha \to \infty \), we recover the Dirichlet result, which shows that one can approach it not just from the constant \( \lambda \) case, but also starting from a rather different profile.

If, on the other hand, \( \epsilon \) is negligible with respect to \( a \), but not necessarily in comparison with \( \lambda^{-1} \), we may use the approximation:

\[
|x_d(z) - x_d(z') + a| \sim a
\]

(92)

in the expressions defining \( P \) and \( Q \). Then, from the definition of \( O \), we see that:

\[
O(z; z') \sim \left( \frac{\lambda(\kappa)}{2\kappa} \right)^2 \left[ \int dz_2 \int dz_3 \tilde{K}^{-1}(z_2; z_3) \right] \int dz_1 \tilde{K}^{-1}(z; z_1) e^{-2\alpha a},
\]

(93)

which is independent of \( z' \).

Then, \( O \) has only one non-vanishing eigenvalue, \( \vartheta \), for each \( \kappa \), as in the \( \epsilon \to 0 \) case:

\[
\vartheta = \left( \frac{\lambda(\kappa)}{2\kappa} \right)^2 \left[ \int dz \int dz' \tilde{K}^{-1}(z; z') \right]^2 e^{-2\alpha a},
\]

(94)
and

$$\hat{E}_0(a) = \frac{1}{2^{d/2} \pi^{d/2} (d/2)} \int_0^{\infty} d\kappa \kappa^{d-1} \ln \left\{ 1 - \left( \frac{\lambda(\kappa)}{2\kappa} \right)^2 \left[ \int dz \int dz' \hat{K}^{-1}(z; z') \right]^2 e^{-2\kappa a} \right\}. \quad (95)$$

Of course, the result depends on $\epsilon$ because of the object $\int dz \int dz' \hat{K}^{-1}(z; z')$. For the case of the piecewise constant defect, for example, we find:

$$\int dz \int dz' \hat{K}^{-1}(z; z') = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{(2k+1)^2 \pi^2 + (2\kappa)^2}{(2k+1)^2 \pi^2 + 2\epsilon \lambda(\kappa)} \quad (96)$$

We conclude the study of the thin wall case by mentioning that it is possible to study analytically the net-to-leading term in an expansion in powers of $\epsilon$ for $\hat{E}_0(a)$, by extending the method used in thin-wall approximation. A rather lengthy calculation shows that the first-order term exactly vanishes. Thus, the first non-trivial correction to $\hat{E}_0(a)$ (if present), can be of the form $C\epsilon^2/a^{d+2}$, in $d$ spatial dimensions.

We also wish to pinpoint to a phenomenon, which appears when considering the finite size limit in expressions obtained by using the eigenvalues corresponding to a particular profile. The $\epsilon \to 0$ and $\lambda \to \infty$ limits, which we have studied exactly at the beginning of IV A, have been found by using the explicit form of $\hat{K}^{-1}$. It corresponds, in fact, to the exact solution when the mirrors are $\delta$-like potentials. That limit, however, cannot be taken from the finite-$\epsilon$ expressions we have derived for particular profiles, since their derivation assumes that, for all $x_d$, $\sigma_\epsilon(x_d) > 0$, in order to map $\mathbb{R}$ to a finite interval, finding the eigenfunctions in that finite region. The use of that mapping to find those eigenfunctions collapses when $\epsilon \to 0$, however, since then $\sigma_\epsilon(x_d)$ becomes a $\delta$-function.

### B. Finite width mirrors

We now study some general properties of $\hat{E}_0(a)$, which are independent of any assumption regarding the form of the defect.

We first note that, denoting by $O_{ll'} = \langle \psi_l | O | \psi_{l'} \rangle$ the matrix elements of the operator $O$ in the basis of eigenfunctions of $\hat{K}$, and a similar convention for the $P$ and $Q$ kernels, we have:

$$O_{ll'} = \frac{1}{\alpha_l} \sum_m P_{lm} \frac{1}{\alpha_m} Q_{ml'} . \quad (97)$$

To study the properties of $\hat{E}_0$ in more detail, we consider the explicit form of $P_{lm}$ and $Q_{lm}$:

$$P_{lm} = \frac{\hat{\lambda}(\kappa)}{2\kappa} \int dz \int dz' \psi_l(z) e^{-\kappa|x_d(z) - x_d(z') - a|} \psi_m(z')$$

$$Q_{lm} = \frac{\hat{\lambda}(\kappa)}{2\kappa} \int dz \int dz' \psi_l(z) e^{-\kappa|x_d(z) - x_d(z') + a|} \psi_m(z') = P_{ml} . \quad (98)$$

Then,

$$O_{ll'} = \frac{1}{\alpha_l} \sum_m P_{lm} P_{lm} \frac{1}{\alpha_m} . \quad (99)$$

An eigenvalue $\vartheta$ corresponding to $O$ (required in order to evaluate the Casimir energy by applying [55]) is then determined by $\sum_{l'} O_{ll'} \vartheta_{ll'} = \vartheta \vartheta_l$. Using [55], and taking into account the fact that $\alpha_l > 0$, this may be equivalently written in matrix form as follows:

$$A v = \vartheta g v \quad (100)$$
where \( A \) is a symmetric and positive definite matrix whose elements are given by

\[
A_{kl} = \sum_m \frac{P_{km} P_{lm}}{\alpha_m}
\]

and \( g \) is a diagonal matrix: \( g = \text{diag}\{\alpha_1, \alpha_2, \ldots\} \).

Note that (100) has the form of a generalized eigenvalue problem for the symmetric matrix \( A \) and the diagonal positive matrix \( g \). Taking advantage of the fact that \( g \) is diagonal, one can show that the eigenvalues of \( A \) can be found as the ones of a matrix \( B \) such that \( B_{kl} \equiv \alpha_{-\frac{1}{2}} A_{kl} \alpha_{\frac{1}{2}} \). This matrix \( B \) may be given an even more convenient form:

\[
B = C C^\dagger,
\]

where

\[
C_{kl} \equiv \frac{P_{kl}}{\sqrt{\alpha_k \alpha_l}}.
\]

Being symmetric and definite positive, there are many available numerical algorithms to compute the eigenvalues of the matrix \( B \).

V. CONCLUSIONS

In this paper we have computed the Casimir energy for a real scalar field in different backgrounds that describe finite width, semitransparent mirrors. The properties of the mirrors are described by \( \lambda(x_0 - x'_0; x_\parallel - x'_\parallel) \) and \( \sigma_e(x_d) \). The former is a two-point function that depends on time and on the parallel coordinates, and represents frequency-dependent transmission and reflection coefficients of the mirrors, while the latter depends only on the coordinate normal to the mirrors, and describes the spatial dependence of their electromagnetic properties. Our starting point, a nonlocal effective action for the quantum scalar field, can be thought as arising from the interaction with the degrees of freedom of the mirrors.

The functions \( \lambda \) and \( \sigma_e \) act as physical regulators for the divergences of the zero point energy, and the usual ‘perfect conductor’ limit can be obtained when \( \sigma_e \) becomes a \( \delta \)-function and \( \lambda \) tends to infinity. In this case, the effect of the mirrors is to impose Dirichlet boundary conditions on the scalar field.

We have computed explicitly the self-energy of a mirror using smooth and piecewise constant profiles \( \sigma_e \). In both cases, the self-energy is finite under the assumption that the mirror becomes transparent at high frequencies. We have discussed in detail the dependence of the results with the cutoff frequency \( \omega_c \) at which the mirror becomes transparent, and with the width \( \epsilon \) of the mirror, showing that the zero point energy vanishes for \( \epsilon \omega_c < 1 \). Finally, we have also analyzed the UV divergences that arise as \( \epsilon \to 0 \), recovering previous results in \[8\].

We hope that an analogous computation of local quantities like \( < T_{\mu\nu} > \) will, in this context, be useful to discuss the gravitational effects of the zero point fluctuations, without having to deal neither with bulk nor surface divergences.

For the case of two mirrors, we have discussed some general properties of the interaction energy, for arbitrary functions \( \lambda \) and \( \sigma_e \). In the thin wall limit, the expression for the energy is considerably simpler, and we have evaluated it explicitly for some particular cases, reproducing also the well known results for perfect mirrors.

Regarding future research, as already mentioned, it would certainly be of interest to compute also local quantities, like the energy and pressure densities. Moreover, a realistic model for the interaction between the quantum field and the degrees of freedom in the mirror would allow us to derive a nonlocal effective action suitable for a detailed analysis of the dissipation effects. This realistic model would necessarily involve the electromagnetic field. Due to gauge invariance, on general grounds we expect the nonlocal effective action
to be of the form:

\[ S_{\text{eff}} = \int d^{d+1}x \int d^{d+1}x' \ F_{\mu\nu}(x)K^{\mu\nu\rho\sigma}(x; x')F_{\rho\sigma}(x') , \]  

(104)

where the kernel \( K^{\mu\nu\rho\sigma}(x; x') \) encodes the electromagnetic properties of the mirrors. Here gauge invariance is inherited from the (assumed) gauge invariant coupling between the material and the gauge field. On the other hand, this is consistent with the limiting case of ideal plates, where the boundary conditions are given in terms of \( F_{\mu\nu} \) rather than \( A_\mu \). In spite of the fact that the effective action shall involve derivatives of the gauge field, we expect a conveniently adapted version of the method will make it possible to consider also this case.

The extension of the results of this paper to non planar and/or non static mirrors would also be of high interest.

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