All-optical Imprinting of Geometric Phases onto Matter Waves

Qi Zhang,1 Jiangbin Gong,2,3 and C.H. Oh1

1Centre of Quantum Technologies and Department of Physics, National University of Singapore, 117543, Singapore
2Department of Physics and Centre for Computational Science and Engineering, National University of Singapore, 117542, Singapore
3NUS Graduate School for Integrative Sciences and Engineering, Singapore 117597, Republic of Singapore

(Dated: April 2, 2009)

PACS numbers: 03.65.Vf, 03.75.-b, 32.80.Qk

When the Hamiltonian of a quantum system adiabatically changes along a path in its parameter space, a non-degenerate (degenerate) eigenstate will acquire, on top of an obvious dynamical phase, an Abelian (non-Abelian) geometric phase [1, 2]. The discovery of geometric phases has motivated many proposals on built-in fault-tolerant quantum logical operations [3]. Geometric phases also provide an elegant framework to understand the coupling between translational motion and internal degrees of freedom. In particular, a slowly moving system subject to an external field naturally causes adiabatic changes in its internal Hamiltonian, giving rise to an Abelian or non-Abelian vector potential for the translational motion [4, 5, 6]. Interestingly, one can also engineer laser-matter interaction to synthesize effective vector potentials from geometric phases [7, 8, 9, 10].

One implicit assumption in geometric phase-based quantum gate studies is that the qubit systems are frozen in space. By contrast, here we examine the geometric phases acquired by a system coherently delocalized in space. In particular, we propose to optically imprint geometric phases at each local space point onto delocalized matter waves of ultracold systems. We show that the geometric phases acquired by a system coherently delocalized in space. In particular, we propose to optically imprint geometric phases at each local space point onto delocalized matter waves of ultracold systems. We show that the geometric phases may re-direct atom wavepackets, split a wavepacket, or create interesting matter wave patterns. The all-optical geometric phase imprinting is insensitive to the dynamical details. Hence it is superior to conventional optical phase imprinting approach to matter wave engineering [12, 13], where the imprinted phase is proportional to the duration of optical imprinting and to the magnitude of an optical potential.

Consider first a two-level atom in a plane-wave laser field. In the rotating wave approximation (RWA), the internal Hamiltonian is given by

$$H_{\text{RWA}} = \frac{1}{2} \omega_0 | 1 \rangle \langle 1 | - | 2 \rangle \langle 2 | + \frac{\Omega_0}{2} (| 1 \rangle \langle 2 | + | 2 \rangle \langle 1 |)$$

where | 1 | and | 2 | are the two internal states of the atom, Ω0 is assumed to be real, k is the wavevector of the laser field, and Ω0 is the detuning from one-photon resonance, and R the coordinate of the atom. In the | 1 \rangle \langle 1 | representation, H_{\text{RWA}} has the eigenstate

$$| \phi^+ \rangle = \frac{1}{\chi(\Delta)} \left( \sqrt{\frac{\Omega_0^2 + \Delta^2}{\Omega_0 e^{i k \cdot R}}} \right),$$

and an analogous eigenstate | φ^- \rangle orthogonal to | φ^+ \rangle, where

$$\chi(\Delta) = \sqrt{\Omega_0^2 + (\Omega_0^2 + \Delta^2 + \Delta)^2}$$

is the normalization factor. Note that these internal eigenstates vary with the coordinate R. Within the adiabatic approximation and in the absence of a trapping potential, the overall eigenstate (internal plus translational) for the system, denoted | ψ^± \rangle, is given by

$$| \Psi^\pm \rangle = | \phi^\pm \rangle e^{i k \cdot R},$$

where k is a wavevector associated with the translational motion of the atom.

We begin with a test case that is well-known in laser chirping control [14] (which realizes a Landau-Zener process). Here the detuning parameter Δ is adiabatically varied from Δ1 > 0 to Δ2 < 0, with |Δ1/Ω0| ≫ 1 and |Δ2/Ω0| ≫ 1. Accordingly, k changes from k1 to k2, but with a fixed direction. The initial state is assumed to be

$$| \Psi^+(\Delta = \Delta_1) \rangle \approx (1, 0)^T e^{i k \cdot R}.$$
one finds the following geometric phase for an arbitrary $R$,

$$\beta(R) = i \int_{\Delta_1}^{\Delta_2} \langle \phi^+ | \frac{\partial}{\partial \Delta} | \phi^+ \rangle d\Delta$$

$$\approx -\frac{R \cdot \mathbf{k}_l^i}{\hbar c} \int_{\Delta_1}^{\Delta_2} \frac{\Omega_0^2}{\chi^2(\Delta)} d\Delta$$

$$\approx -R \cdot [k^i_l - k^i_r],$$

(2)

where $\mathbf{k}_l^i$ denotes the laser wavevector when $\Delta = 0$, and throughout $\mathbf{k}$ represents a unit vector along the vector $\mathbf{k}$. Given that the evolving state remains in the system eigenstate, the dynamical phase is evidently $R$-independent. Imprinting the geometric phase $\beta(R)$ onto the adiabatic state $|\phi^+\rangle$ in Eq. (1) and neglecting the overall dynamical phase, one obtains the final state

$$|\Psi^+_f(\Delta_2)\rangle = \frac{e^{i[\beta(R)+|\mathbf{k}|]|\mathbf{R}|}}{\chi(\Delta_2)} \left( \sqrt{\Omega_0^2 + \Delta_2^2} + \Delta_2 \right) |0, e^{ik^i_r \cdot \mathbf{R}}\rangle = \left( \begin{array}{c} 0 \\ e^{ik^i_r \cdot \mathbf{R}} \end{array} \right) e^{i(k^i_l + k^i_r) \cdot R}. \quad (3)$$

Equation (3) shows that the population between the two levels is adiabatically inverted and the atom momentum increases by $\hbar k^i_l$. This result is expected, because (i) the chirping scenario is long known to induce the absorption of one on-resonance photon and (ii) momentum conservation requires the atom momentum to increase by $\hbar k^i_l$. But quite noteworthy, here the obvious consequence of momentum conservation is manifested as an effect of the spatially-dependent geometric phase $\beta(R)$. Note also that if we further change $\Delta$ along a reversed path until it reaches its initial value, then Eq. (2) gives a zero Berry phase and hence zero momentum change. This is again correct because the atom releases a photon during the reversed process. One can also carry out similar analysis if the chirped laser field possesses a nonzero orbital angular momentum. That is, the transfer of orbital angular momentum from photon to the matter wave (hence vortex generation) can also be interpreted as geometric phase imprinting.

As a second Abelian example, we examine the system of $H_{\text{RWA},2}$ for $\Delta = 0$ and with $\mathbf{k}_r^i$ being rotated adiabatically. We assume below both $\mathbf{k}_r^i$ and $\mathbf{k}$ are in the $x$-$z$ plane. Let $\gamma$ be the angle by which $\mathbf{k}_r^i$ is rotated clockwise with respect to the $x$-axis. Then $\mathbf{k}_r^i(\gamma) = k^i_l [\cos(\gamma) \hat{e}_x - \sin(\gamma) \hat{e}_z]$, where $k^i_l = |k^i_l|$, $\hat{e}_x$ and $\hat{e}_z$ are the unit vectors along the $x$-axis and $z$-axis. For convenience we also define the polar coordinates $(R, \theta)$ for $\mathbf{R}$ confined on the $x$-$z$ plane, i.e., $x = R \cos(\theta)$, and $z = R \sin(\theta)$. Then, the eigenstate in Eq. (1) for $\Delta = 0$ can be rewritten as

$$|\phi^+(\gamma)\rangle = \frac{1}{\sqrt{2}} \left( e^{i k^i_r \cdot R \cos(\theta + \gamma)} \right). \quad (4)$$

Consider an initial state $|\Psi^+\rangle = |\phi^+(\gamma = 0)\rangle e^{ik^i_r \cdot R}$. As the adiabatic parameter $\gamma$ varies from 0 to $\alpha$, the acquired geometric phase at an arbitrary local point $(R, \theta)$ is found to be

$$\beta(R, \theta) = i \int_0^\alpha \langle \phi^+ | \frac{\partial}{\partial \gamma} | \phi^+ \rangle d\gamma$$

$$= \frac{k^i_l}{2} \left[ R \cos(\theta) - R \cos(\theta + \alpha) \right]. \quad (5)$$

Neglecting the translational motion during the adiabatic process, the final state $|\Psi^+_f(\alpha)\rangle$ can be obtained by imprinting $\beta(R, \theta)$ onto the adiabatic state $|\phi^+(\gamma = \alpha)\rangle e^{ik^i_r \cdot R}$, yielding

$$|\Psi^+_f(\alpha)\rangle = e^{i[\beta(R,\theta)+k^i_r \cdot R]} |\phi^+(\gamma = \alpha)\rangle$$

$$= \frac{1}{\sqrt{2}} \left( e^{i k^i_r \cdot R \sin(\theta + \frac{\alpha}{2}) \sin(\frac{\alpha}{2})} \right) e^{ik^i_r \cdot R \cos(\theta + \frac{\alpha}{2}) \cos(\frac{\alpha}{2})} \right). \quad (6)$$

Unlike in the above test case that apparently involves the absorption of one on-resonance photon, here neither the population on states $|1\rangle$ and $|2\rangle$ nor the atom mechanical momentum changes. However, the geometric phases $\beta(R, \theta)$ imprinted onto the matter wave will induce differences in time evolution if we suddenly switch off the laser field, after which the matter waves associated with each of the two components of $|\Psi^+_f(\alpha)\rangle$ will evolve independently and freely. The momentum in free space for each component of $|\Psi^+_f(\alpha)\rangle$ can be found by taking the spatial derivative of that component. Without loss of generality we set $k = 0$. One then finds that upon a sudden laser field switch-off, the state $|\Psi^+_f(\alpha)\rangle$ will split into two components: atoms with the internal state $|2\rangle$ will move in the direction of $\cos(\alpha/2) \hat{e}_x - \sin(\alpha/2) \hat{e}_z$, with a momentum $\hbar k^i_l \cos(\alpha/2)$, but the atoms on the state $|1\rangle$ will move in the direction of $\sin(\alpha/2) \hat{e}_x + \cos(\alpha/2) \hat{e}_z$ with a momentum $\hbar k^i_l \sin(\alpha/2)$. These motion characteristics are clearly $\alpha$-dependent and intriguing. In particular, if there is no geometric phase imprinting, $\alpha = 0$, the $|1\rangle$ component is static and the $|2\rangle$ component moves along $\hat{e}_x$; but if $\alpha = \pi$, then the $|2\rangle$ component is static and the $|1\rangle$ component moves along $\hat{e}_x$. This could be useful for filtering the internal states.

We have also carried out numerical wavepacket simulations based on the Hamiltonian $H_{\text{RWA},2}$ (where being the mass of the atom) that undergoes slow changes in its parameter $\gamma$. In our simulations we replace the plane-wave factor $\exp(i k^i_r \cdot R)$ of $|\Psi^+\rangle$ by a Gaussian wavepacket. Numerical results do confirm our predictions.

We now discuss the imprinting of non-Abelian geometric phases. As in Refs. [10, 13, 17], we adopt the so-called “tripod scheme” of four-level atoms interacting with three on-resonance laser fields. The internal state is denoted as $|n\rangle$, $n = 0 \sim 3$. Each of the three transitions $|0\rangle \leftrightarrow |1\rangle$, $|0\rangle \leftrightarrow |2\rangle$, and $|0\rangle \leftrightarrow |3\rangle$ is coupled by one laser field. This coupling scheme can be...
realized if states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are degenerate magnetic sub-levels and the three coupling fields have different polarizations. Alternatively, states $|1\rangle$, $|2\rangle$, $|3\rangle$ are non-degenerate and the three coupling fields then carry different frequencies (same polarization is then allowed). For convenience we adopt the same configuration as in Ref. \cite{17}, where two laser beams are counter-propagating along the $x$-axis and the third laser beam is along the $z$-axis. The associated internal Hamiltonian under RWA is given by $H_{\text{RWA,4}} = \sum_{n=1}^{3} \Omega_{n} |n\rangle \langle n| + \text{h.c.}$, with $\Omega_{1} = \Omega_{0} \sin(\xi)/\sqrt{2} e^{-ikz\xi}$, $\Omega_{2} = \Omega_{0} \sin(\xi)/\sqrt{2} e^{ikz\xi}$, and $\Omega_{3} = \Omega_{0} \cos(\xi) e^{ikz\xi}$, where the parameter $\xi$ is set to satisfy $\cos(\xi) = \sqrt{2} - 1$. Note that even if the three lasers carry different frequencies, their effective $k_{r}$ in the $x$-$z$ plane can still be the same by titling the laser beams out of the $x$-$z$ plane.

The Hamiltonian $H_{\text{RWA,4}}$ has two degenerate and spatially-dependent dark (null-eigenvalue) states $|D_{1(2)}\rangle$, which are given by \cite{17},

\[
|D_{1}\rangle = (|\tilde{1}\rangle - |\tilde{2}\rangle) e^{-i\kappa z/\sqrt{2}}
\]
\[
|D_{2}\rangle = \left[\cos(\xi) (|\tilde{1}\rangle + |\tilde{2}\rangle) / \sqrt{2} - \sin(\xi)|3\rangle\right] e^{-i\kappa z},\tag{7}
\]

where $\kappa' \equiv k_{r} (1 - \cos(\xi))$, $|\tilde{1}\rangle \equiv |1\rangle e^{ik_{r}(x+z)}$, and $|\tilde{2}\rangle \equiv |2\rangle e^{-ik_{r}(x-z)}$. Clearly, within the dark-state subspace the dynamical phase is always zero. So we focus on the time evolution in this dark subspace. Any state therein can be expanded as $c_{1}|D_{1}\rangle + c_{2}|D_{2}\rangle$. This expansion hence defines a dark state representation. In this representation, the mechanical momentum operator becomes \cite{17,18}

\[
P^{D} = -i \hbar \vec{\nabla} + \hbar \kappa (\sigma_{z} \hat{e}_{x} + \sigma_{x} \hat{e}_{z}),\tag{8}
\]

where $\sigma_{x,z}$ are Pauli matrices, $\kappa = \cos(\xi) k_{r}$, and $\nabla$ represents the gradient in the dark-state representation. As in Ref. \cite{17}, we also introduce an additional constant shift to state $|3\rangle$, denoted by a matrix operator $V_{s}$, such that the total effective Hamiltonian becomes $H^{\text{eff}}_{D} = \frac{P^{D \cdot 2}}{2m_{s}}$. The eigenstates of $H^{\text{eff}}_{D}$ are

\[
|\Psi^{D,\pm}\rangle = \frac{1}{2} \left( 1 \mp ie^{i\varphi_{k}} \right) e^{i\kappa \mathbf{R}},\tag{9}
\]

where $\varphi_{k}$ is the angle between the $x$-axis and the atom wavevector $\mathbf{k}$ in Eq. \cite{9}. According to Eq. \cite{5}, it is straightforward to show that the eigenstates $|\Psi^{D,\pm}\rangle$ have the momentum $(k \pm \kappa \mathbf{k}) \hbar$.

Consider then two simple scenarios of laser field manipulation. In the first scenario, we adiabatically move all the laser beams along the negative $z$ axis. The non-Abelian geometric phases thus induced are determined by

\[
\frac{d}{dz} \begin{pmatrix} c_{1} \\ c_{2} \end{pmatrix} = - \begin{pmatrix} i[D_{1} |\sigma_{z} |D_{1}\rangle & i[D_{1} |\sigma_{z} |D_{2}\rangle \\ i[D_{2} |\sigma_{z} |D_{1}\rangle & i[D_{2} |\sigma_{z} |D_{2}\rangle \end{pmatrix} \begin{pmatrix} c_{1} \\ c_{2} \end{pmatrix} = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix} \begin{pmatrix} c_{1} \\ c_{2} \end{pmatrix}.\tag{10}
\]

Because the matrix on the right side of Eq. \cite{10} turns out to be independent of $\mathbf{R}$, one may naively conclude that the non-Abelian geometric phases induced here will not affect matter wave propagation. But this is incorrect. To see this let us assume the initial state to be the $|\Psi^{D,-}\rangle$ state in Eq. \cite{9} with $\varphi_{k} = 0$. This initial state has a mechanical momentum $\hbar(k-\kappa)$ along the $x$-axis, where $k \equiv |\mathbf{k}|$. Let the final state, a function of the displacement $d_{z}$ along $-z$, be $|\Psi^{D}_{f}(d_{z})\rangle$. It is enlightening to consider a specific case, e.g., $d_{z} = \frac{\pi}{2\kappa}$. Neglecting matter wave propagation during the period of geometric phase imprinting, Eq. \cite{10} then gives

\[
|\Psi^{D}_{f}(\pi/4\kappa)\rangle = \begin{pmatrix} 1 + i & 1 + i \\ 1 + i & 1 - i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{ik\kappa}.\tag{11}
\]

The two components on the right side of Eq. \cite{11} represent a superposition of the two eigenstates $|\Psi^{D,-}\rangle$ and $|\Psi^{D,-}\rangle$ in Eq. \cite{9}. Because these two eigenstates possess different mechanical momenta $(k+\kappa)\hbar$ and $(k-\kappa)\hbar$, we thus have the result that the non-Abelian geometric phases generated here can split matter waves.

This result is also verified by our numerical simulations based on the full Hamiltonian $-\frac{\hbar^{2} \nabla^{2}}{2m_{s}} + H_{\text{RWA,4}} + V_{s}$. In particular, in our simulations the initial state is chosen as a Gaussian wavepacket instead of a plane wave considered in Eq. \cite{9}. As an example we choose $\Omega_{0}/\hbar = 1200\hbar\kappa^{2}/m$. The duration of the imprinting process is chosen to be $0.063\mu_{s}/(\hbar\kappa^{2})$. (In real units, for $m = 10^{-25}$ Kg, $\kappa \sim 10^{6}$ m$^{-1}$, the duration is about $\sim 60$ $\mu$s and $\Omega_{0}/\hbar \sim 10^{6}$ Hz. The cold-atom version of the Zitterbewegung oscillation \cite{10} is negligible for these parameters). Figure 1 depicts the simulation results for $\varphi_{k} = 0$, $k = \kappa$. At $t = 0$ we have initiated the first non-Abelian geometric phase imprinting for $d_{z} = \pi/(4\kappa)$. As seen in Figs.
1(b)-(c), an initial wavepacket of zero group velocity indeed splits into two parts: one part holds a zero group velocity and the other part moves along the $x$ axis at an average mechanical momentum of $2\hbar k$. Then, at $t = 20$, we move all the lasers up by $d_z = \pi/(4\kappa)$ to induce a second imprinting process. Figure 3(d) shows that this further splits the wavepacket into four copies of equal weights, a result also derivable from Eq. (10). Choosing a different $d_z$ at $t = 20$ for the second imprinting process, we may even exchange the momenta between the two separated sub-wavepackets. Extending the strategy here, in principle one can split a wavepacket into $2^n$ copies with an arbitrary integer $n$. This should be useful for atom optics applications such as atomic interferometry. Other simulation results (not shown) also indicate that so long as $\Omega_0$ is sufficiently large, then on one hand the matter wave propagation during the adiabatic processes can be made negligible and on the other hand the adiabaticity in the internal motion is well maintained.

In the second scenario, we slowly rotate all the three laser beams clockwise in the $x$-$z$ plane. The non-Abelian geometric phase at the local point $(R, \theta)$ is determined by

$$i\frac{d}{dt} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \begin{pmatrix} i\langle D_1 | \frac{\partial}{\partial R} | D_1 \rangle & i\langle D_1 | \frac{\partial}{\partial \theta} | D_2 \rangle \\ i\langle D_2 | \frac{\partial}{\partial R} | D_1 \rangle & i\langle D_2 | \frac{\partial}{\partial \theta} | D_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \kappa R \begin{pmatrix} \cos(\theta + \gamma) & -\sin(\theta + \gamma) \\ -\sin(\theta + \gamma) & -\cos(\theta + \gamma) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where $0 \leq \gamma \leq \alpha$ is the angle of rotation that serves as the adiabatic parameter. An analytical solution to Eq. (12) is not found and hence we solve it computationally. After obtaining the numerical non-Abelian geometric phases for each point $(R, \theta)$ from Eq. (12), we imprint the spatially dependent geometric phases onto the system at $t = 0$ and then let it evolve under $H_D^{\text{eff}} = \frac{P_D^2}{2m}$ (also with the rotated laser fields). For the example shown in Fig. 2, the initial state is the state $|\Psi^{D,\gamma}\rangle$ in Eq. (9) multiplied by a Gaussian profile, with $k = \kappa$ and $\varphi_k = -\pi/2$. This initial state hence has a zero group velocity according to Eqs. (5) and (9). As shown in Fig. 2 for $\alpha = 3\pi/2$, the phase imprinting substantially impacts the ensuing evolution. The evolving wavepacket first displays interesting interference patterns, followed by a certain degree of self-focusing, and later it starts to display some moon-like patterns. As expected, the patterns in real space at later times begin to resemble the $t = 0$ wavepacket pattern plotted as a function of $P_D^x$ and $P_D^z$, namely, the mechanical momenta along the $x$-axis and $z$-axis. Remarkably, even if we let $\alpha = 2\pi$, i.e., with all laser fields returning to their original configuration, similar results can still be obtained. We have checked that essentially the same results can be obtained using the full Hamiltonian $\frac{\hbar^2 \nabla^2}{2m} + H_{\text{RWA},4} + V_A$. However, we find that in order to fully neglect matter wave propagation during the imprinting, the duration of the adiabatic process here should be about two orders of magnitude shorter than in the first non-Abelian example. Hence $\Omega_0$ must be about two orders of larger.

In summary, we have demonstrated the concept of all-optical imprinting of both Abelian and non-Abelian geometric phases onto matter waves. The adiabatic paths considered here are among the simplest. Exploring more complicated adiabatic paths might offer extensive laser control of matter wave propagation. Extending this work to the imprinting of non-adiabatic geometric phases is also of considerable interest.

This work was supported by WBS grant No. R-710-000-008-271 (ZQ and CH) under the project “Topological Quantum Computation”, and by the NUS “YIA” fund (WBS grant No.: R-144-000-195-123) (JG).

[1] M.V. Berry, Proc. R. Soc. London A 392, 45 (1984).
[2] F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).
[3] For example, P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94 (1999); L.-M. Duan, J.I. Cirac and P. Zoller, Science 292, 1695 (2001); S.L. Zhu and Z.D. Wang, Phys. Rev. Lett. 91, 187902 (2003); R.G. Unanyan and M. Fleischhauer, Phys. Rev. A 69, 050302 (R) (2004); E. Sjoqvist, Physics 1, 35 (2008).
[4] M.V. Berry, Proc. R. Soc. Lond. A 442, 641 (1993); M.V. Berry, Proc. R. Soc. Lond. A 442, 659 (1993).
[5] R.G. Littlejohn and S. Weigert, Phys. Rev. A 48, 924 (1993).
[6] G. Sundaram and Q. Niu, Phys. Rev. B 59, 14195 (1999).
[7] R. Dum and M. Olshanii, Phys. Rev. Lett. 76, 1788 (1996).
[8] G. Juzeliunas and P. Ohberg, Phys. Rev. Lett. 93, 033602 (2004).
[9] G. Juzeliunas et al., Phys. Rev. A 71, 053614 (2005).
[10] J. Ruseckas et al., Phys. Rev. Lett. 95, 010404 (2005).
[11] M. Möttönen, V. Pietilä, and S. M. M. Virtanen, Phys. Rev. Lett. 99, 250406 (2007).
[12] L. Dobrak et al., Phys. Rev. A 60, 3381(R) (1999).
[13] J. Denschlag et al., Science 287, 97 (2000).
[14] For example, T. Lu, X. Miao, and H. Metcalf, Phys. Rev. A, 71, 061405(R) (2005).
[15] The last equality in Eq. (2) will become mathematically exact if $-\Delta_{\min} = \Delta_{\max}$ or if $|\Delta_{\min}/\Omega_0| \to \infty$ and $|\Delta_{\max}/|\Omega_0| \to \infty$.
[16] R. G. Unanyan, B. W. Shore, and K. Bergmann, Phys. Rev. A 59, 2910 (1999).
[17] G. Juzeliunas et al., Phys. Rev. Lett. 100, 200405 (2008).
[18] This is clear, because the actual mechanical momentum is shifted from $-\hbar \hat{\nabla}$ by an effective vector potential derived in Ref. [17].
[19] J.Y. Vaishnav and C.W. Clark, Phys. Rev. Lett. 100, 153002 (2008).