Regular spherical dust spacetimes

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Physical (and weak) regularity conditions are used to determine and classify all the possible types of spherically symmetric dust spacetimes in general relativity. This work unifies and completes various earlier results. The junction conditions are described for general non-comoving (and non-null) surfaces, and the limits of kinematical quantities are given on all comoving surfaces where there is Darmois matching. We show that an inhomogeneous generalisation of the Kantowski-Sachs metric may be joined to the Lemaître-Tolman-Bondi metric. All the possible spacetimes are explicitly divided into four groups according to topology, including a group in which the spatial sections have the topology of a 3-torus, appears not to have been discussed previously.

The new results include the matching between an exact solution in the generalised Kantowski-Sachs family and the LTB solutions, and junction conditions for arbitrary non-null surfaces. The possible types of centre are extended. (Note that spherically symmetric (SS) dust solutions need not possess a centre [7].) All possible composite SS dust models are found and classified into four topological groups. One such topology, in which spatial sections have the topology of a 3-torus, appears not to have been discussed previously.

We use comoving spatial coordinates, since these are best adapted to matching problems. For an analysis of astronomical observations, coordinates based on the past light cones of the observer are a better choice [8].

The paper is laid out as follows. In section II, all of the solutions of Einstein’s field equations (with a smooth SS dust source) are expressed in the forms (10) and (14). The geometrical requirements at any junction in spacetimes composed from these solutions are analysed in section III. In section IV, the reasonable physical regularity conditions are made explicit. This leads to regularity conditions within the domain of any one solution (section V), at any centre of symmetry (section VI), and at any junction between solution domains (section VII). Also in section VII, all the possible composite models are listed. In section VIII, the results are summarised and used to give a simple proof of the recollapse conjecture for these models.

I. Introduction

The convenient Lemaître-Tolman-Bondi (LTB) exact solutions have been exploited as the main inhomogeneous models in relativity and cosmology for many years. The remarkably rich structure of these solutions has many subtleties, in particular concerning the regularity of the metric [1–3]. The purpose of this paper is to clarify, unify and complete existing results on regularity. This topic has important implications, for example in the exact modelling of gravitational collapse in an expanding universe [4], or the exact modelling of cosmological voids in an expanding universe [5, 6].

The new results include the matching between an exact solution in the generalised Kantowski-Sachs family and the LTB solutions, and junction conditions for arbitrary non-null surfaces. The possible types of centre are extended. (Note that spherically symmetric (SS) dust solutions need not possess a centre [7].) All possible composite SS dust models are found and classified into four topological groups. One such topology, in which spatial sections have the topology of a 3-torus, appears not to have been discussed previously.

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II. Spherical Dust Solutions

In standard comoving coordinates $x^a = \{t, r, \theta, \phi\}$, the dust 4-velocity is $u^a = \delta_t^a$ and the metric is [1]

$$ds^2 = -dt^2 + X(r, t)^2 dr^2 + R(r, t)^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and we choose $R \geq 0$. The energy-momentum tensor is $T^b_a = \rho u_a u^b = -\rho \delta_t^a \delta_t^b$, where $\rho$ is the proper matter energy density. The Einstein tensor is [1]

$$G^t_t = -2 \frac{\dot{X} \ddot{R}}{X R} + \frac{1}{X^2} \left( 2 \frac{R''}{R} - \frac{R''}{R^2} - 2 \frac{X' R'}{X R} \right),$$

$$G^r_r = -2 \frac{\dot{R}}{R} + \frac{R'}{X^2 R^2},$$

$$G^\theta_\theta = G^\phi_\phi = -\frac{\dot{X}}{X} - \frac{\ddot{R}}{R} + \frac{X' R'}{X R} + \frac{1}{X^2} \left( \frac{R''}{R} - \frac{X' R'}{X R} \right),$$

$$G^r_r = X^2 G^t_t = 2 \left( \frac{R'}{R} - \frac{X' R'}{X R} \right).$$
where an overdot denotes \( u^a \partial_a = \partial / \partial t \) and a prime denotes \( \partial / \partial r \). The units satisfy \( G = c = 1 \).

In order for solutions of Einstein’s equations \( G^i_a \) to exist, the Einstein tensor \( G^i_a \) must be defined through (3-4). In such regions of spacetime, it is therefore required that

\[
X \neq 0 \neq R; \ X \text{ and } R \text{ are } C^2 \text{ in } t; \ X \text{ and } \dot{R} \text{ are } C^1 \text{ in } r; \ R \text{ is } C^2 \text{ in } r.
\]

(6)

However, these regions may be joined together to form a composite spacetime, in which \( G^i_a \) is not defined by (2-5) on the boundaries.

Bondi [1] integrated the system as follows. The \( G^r_r = 0 \) field equation integrates to (assuming \( R' \neq 0 \))

\[
X = \frac{R'}{\sqrt{1 + E(r)}},
\]

(7)

where \( E \) is an arbitrary function. The remaining independent equations reduce to

\[
\dot{R}^2 = \frac{2M(r)}{R} + E,
\]

(8)

with \( M \) another arbitrary function. The corresponding proper density is given by

\[
\rho = \frac{M'}{4\pi R' R^2}.
\]

(9)

There are five solutions of (8):

- (s1) for \( \{ E = M = 0 \} \):
  \( R = -T \) \( \quad \quad \quad \quad \quad \quad \quad [T \leq 0] \),

- (s2) for \( \{ E > 0, M = 0 \} \):
  \( R = \tau \sqrt{E} \), \( \tau \equiv c t - T \) \( \epsilon = \pm 1 \) \( \quad \quad \quad \quad \quad \quad \quad [\tau \geq 0] \),

- (s3) for \( \{ E = 0, M > 0 \} \):
  \( R = (9M/2)^{1/3} \tau^{2/3} \) \( \quad \quad \quad \quad \quad \quad \quad [\tau \geq 0] \),

- (s4) for \( \{ E > 0, M > 0 \} \):
  \( R = M (\cosh \eta - 1) E^{-1}, \quad \sinh \eta - \eta = \tau E^{3/2} M^{-1} \quad \quad \quad \quad \quad \quad \quad [0 < \eta < +\infty, \ \tau > 0] \),

- (s5) for \( \{ E < 0, M > 0 \} \):
  \( R = M (\cos \eta - 1) E^{-1}, \quad \eta - \sin \eta = \tau |E|^{3/2} M^{-1} \quad \quad \quad \quad \quad \quad \quad [0 < \eta < 2\pi, \ \tau > 0] \),

(10)

where \( T(r) \) is a third arbitrary function and we denote the five solutions by (s1), . . . , (s5). No physical solutions exist for \( (E < 0, M \leq 0) \) or for \( (E = 0, M < 0) \). Note that (s1) and (s2) are (locally) Minkowski spacetime. Motivated by equation (8), Bondi [1] describes \( M \) as a relativistic generalisation of Newtonian mass, and \( \frac{1}{2}E \) as a total energy. The surfaces \( \tau = 0 \) are spacelike singularities.

The above integration necessarily requires \( R' \neq 0 \). If instead \( R' = 0 \), then a different solution results, as follows. The \( G^r_r = 0 \) equation integrates to

\[
\dot{R} = \pm \sqrt{\frac{2M}{R} - 1},
\]

(11)

where \( \dot{M} > 0 \) is an arbitrary constant and \( \dot{R} \neq 0 \) has been assumed, since the converse immediately leads to an inconsistency. The second integration (assuming \( 0 < R \leq 2\dot{M} \), otherwise \( \dot{R}^2 < 0 \)) reveals that

\[
R = \dot{M} (1 - \cos \eta), \quad \eta - \sin \eta = \dot{M}^{-1} \left( c t - \dot{T} \right),
\]

(12)

where \( 0 < \eta < 2\pi \) and \( \dot{T} \) is an arbitrary constant. By (12), equation (4) implies the linear equation

\[
\frac{\partial^2 X}{\partial \eta^2} + \frac{X}{\cos \eta - 1} = 0,
\]

(13)

which transforms to a first-order Ricatti equation under \( X \to X^{-1} \partial_\eta X \). Hence the general solution of (13) may be found provided one solution is known [9]. One particular solution is \( X = \sin \eta (1 - \cos \eta)^{-1} \), and the general solution is:

\[
X = A(r) \frac{\sin \eta}{1 - \cos \eta} + B(r) \left[ 1 - \frac{\eta \sin \eta}{2 (1 - \cos \eta)} \right],
\]

(14)
where \( A \) and \( B \) are arbitrary functions, and we denote this solution by \((s6)\). Finally (by \(2\)) the density reduces to
\[
\rho = \frac{B}{8\pi M^2 (1 - \cos \eta) \left[A \sin \eta + B (1 - \cos \eta - \frac{1}{2} \eta \sin \eta)\right]}.
\]
(15)

This solution is an SS variant of an inhomogeneous generalisation of the \( k = +1 \) Kantowski-Sachs metric, as was discovered previously \(10\). The form of the solution presented in \(11\) (in which \( \rho \neq 0 \) was assumed) is
\[
ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} ds^2 + \tilde{r}^2 d\Omega^2, \quad e^{2\nu} = \frac{\tilde{t}}{a - \tilde{t}}, \quad e^{\lambda} = e^{-\nu} \left[ \int e^{\nu} \frac{2x^2}{1 + x^2} dx + C(\tilde{r}) \right].
\]
(16)

The coordinate transformation between \(11\) and \((s6)\) is given by \( \tilde{t} = R, \tilde{r}' = A/C, \tilde{r} = 0 \) (with the identifications \( C = 2A/B \) and \( a = 2\tilde{M} \)).

In summary, the six possible SS dust solutions are \((s1)-(s6)\). Matching these dust solutions to form composite space times is a focus in the remaining sections.

### III. Junction Conditions

The differentiability conditions \(10\) on the metric need not be satisfied at the interfaces between domains of separate solutions in a composite model. Instead (weaker) matching conditions (for the geometry) must be satisfied \(12\). At these junctions, it is assumed that the coordinate \( r \) remains regular, which is not the case at a shell-crossing (by definition). Such singularities are excluded in this paper. The matching of two general SS spacetimes has been considered in \(13\), in which necessary conditions for the matching were presented, which are valid for any equation of state. Here necessary and sufficient conditions are found in the special case of dust.

Consider firstly the case of a comoving boundary \( r = \) constant (which was analysed previously \(2\)). The unit outward normal is \( n_a = |X| \delta^n_a \), the metric intrinsic to the surface is
\[
\hat{h}_{ab} = g_{ab} - n_a n_b = \text{diag} \left(-1, 0, R^2, R^2 \sin^2 \theta\right),
\]
and the extrinsic curvature is
\[
\hat{K}_{ab} = \hat{h}^c_a \hat{h}^d_b \nabla_d n_c = \text{diag} \left(0, 0, \frac{RR'}{|X|}, \frac{RR'}{|X| \sin^2 \theta}\right).
\]
(18)

The Darmois matching conditions \(2, 12, 14\) state that \( \hat{h}_{ab} \) must match across the surface and any discontinuity in \( \hat{K}_{ab} \) gives rise to a surface-density layer as represented by the Lanczos equation \(14\)
\[
8\pi \left(3T_{ab}\right) = g^{cd} (\Delta K_{cd}) h_{ab} - \Delta K_{ab},
\]
(19)

where \( \Delta \) denotes the limit of a quantity on the \('r+'\) side of the interface, minus the limit on the \('r−'\) side. The surface energy-momentum tensor \(3T_{ab}\) can be expressed in perfect fluid form \(3T_{ab} = (3\rho + 3p) u_a u_b + 3p \hat{h}_{ab} \) with \(3\rho\) and \(3p\) the surface energy density and surface (isotropic) pressure. Now
\[
3\rho = 3T_{ab} u^a u^b = -\frac{1}{4\pi R} \Delta \left( \frac{R'}{|X|} \right), \quad 3p = \frac{1}{2} \left(3T_{ab} \hat{h}_{ab} + 3\rho\right) = -\frac{3\rho}{2}.
\]

Following Bonnor \(2\), this equation of state is regarded as unphysical, i.e. we require the extrinsic curvature on comoving surfaces to be continuous in \( r \). By \(17\) and \(18\), the junction conditions reduce to
\[
\begin{align*}
R & \text{ continuous in } r, \\
R' \frac{|X|}{|X|} & \text{ continuous in } r.
\end{align*}
\]
(20)
(21)

For solution \((s6)\), \( R' \equiv 0 \), hence \((s6)\) may only be matched (across a comoving surface) to one other SS dust solution, i.e. \((s5)\) \(\{ R' \to 0, \ X \neq 0 \} \) requires \( E \to -1 \). From \(11\), \(12\) and \(20\) the matching also requires \( M \to \tilde{M} \) and \( T \to \tilde{T} \) in the \((s5)\) region. This motivates a characterisation of \((s6)\): solution \((s6)\) may be characterised within the LTB family by the conditions \( \{ M' = T' = 0, \ M > 0, \ E = -1 \} \).
As we show below in Section VII, in addition to (s5) to (s6), the other possible matchings across a comoving boundary are: (s2) to (s4), (s3) to (s4) or (s5) and (s4) to (s5).

Consider now the junction conditions on the spacelike surfaces \( t = \text{constant} \). In this case, the unit normal is \( u^a \) and the intrinsic and extrinsic curvatures are

\[
h_{ab} = g_{ab} + u_a u_b = \text{diag} \left( 0, X^2, R^2, R^2 \sin^2 \theta \right), \quad K_{ab} = h_{a}^{c} h_{b}^{d} \nabla_c u_d = \text{diag} \left( 0, \dot{X} R, \dot{R} R \right).
\]  

(22)

Hence continuity of \( h_{ab} \) and \( K_{ab} \) in \( t \) merely implies that \( R, X, \dot{R}, \dot{X} \) are continuous in \( t \). (Spacelike surface layers, which imply an instanton transition, are treated as unphysical a priori.) In fact, by inspection of equations (7), (8) and (11), the metric tensors for (s1)-(s6) are infinitely differentiable \((C^\infty)\) in \( t \).

An analysis of the Darmois matching problem for solutions (s1)-(s6) across a general non-comoving (and non-null) SS surface provides insight into the nature of general dust models. From (11), the unit normal \( n^a \) and unit SS tangent \( t^a \) to such a surface satisfy

\[
n_a n^a = -t_a t^a = \lambda, \quad n_a = \left( -P, |X|\sqrt{\lambda + P^2}, 0, 0 \right), \quad t_a = \left( -\sqrt{\lambda + P^2}, P|X|, 0, 0 \right),
\]

where the surface is timelike (spacelike) for \( \lambda = 1 \) (\( \lambda = -1 \)) and \( P \) is a function determined by the equation of the surface:

\[
r = s(t), \quad \frac{ds}{dt} = \frac{P}{|X|\sqrt{\lambda + P^2}}.
\]  

(23)

A lengthy calculation leads to the extrinsic curvature of the surface:

\[
\hat{K}_{ab} = \begin{pmatrix}
- (1 + \lambda P^2) F_1 & F_1 |X| \lambda \lambda |X| \sqrt{\lambda + P^2} & 0 & 0 \\
F_1 |X| \lambda \lambda |X| \sqrt{\lambda + P^2} & -\lambda P^2 X^2 F_1 & 0 & 0 \\
0 & 0 & F_2 & 0 \\
0 & 0 & 0 & F_2 \sin^2 \theta
\end{pmatrix},
\]

(24)

and the intrinsic curvature of the surface is

\[
\hat{h}_{ab} = \begin{pmatrix}
- (1 + \lambda P^2) & \lambda P |X| \sqrt{\lambda + P^2} & 0 & 0 \\
\lambda P |X| \sqrt{\lambda + P^2} & -\lambda P^2 X^2 & 0 & 0 \\
0 & 0 & R^2 & 0 \\
0 & 0 & 0 & R^2 \sin^2 \theta
\end{pmatrix},
\]

(25)

where

\[
F_1 = \frac{P_a t^a}{\sqrt{\lambda + P^2}} + \frac{P \dot{X}}{X}, \quad F_2 = \frac{PR}{\sqrt{\lambda + P^2}} R_a t^a + \frac{\lambda RR'}{|X|\sqrt{\lambda + P^2}}.
\]

Since the coordinates continue through the surface, all four coordinates are induced on the surface, and one of \( r, t \) is redundant there. Now, the Darmois conditions with no surface layers take the form

\[
\Delta h_{ab} = 0 \Rightarrow \Delta R = \Delta |X| = 0,
\]

(26)

\[
\Delta \hat{K}_{ab} = 0 \Rightarrow \Delta \dot{R} = \Delta (\dot{X}/X) = 0,
\]

(27)

where it has been assumed that \( P \neq 0 \) and \( P \neq 1 \), as these cases are already considered above. By equations (23), (24) and (25), \( \Delta P = 0 \). We now show that:

**If there are no surface layers, then all boundaries between domains of different solutions (s1)--(s6) in a composite dust model must be comoving, i.e. \( \{ r = \text{constant} \} \).**

(See also (13) and (14).)

The proof is as follows. In matching together two different solutions (s1)-(s5), \( |X| = |R'|/\sqrt{1 + E} \). Then by (26) and (27), \( \Delta E = 0 \). From (23)

\[
\dot{R} = \frac{R_a t^a}{\sqrt{\lambda + P^2}} - \frac{PR'}{|X|\sqrt{\lambda + P^2}},
\]

(29)
so that (26) and (27) imply $\Delta \dot{R} = 0$. Then by (8), $\Delta M = 0$. Finally $\Delta R = 0$ forces $\Delta T = 0$. Note that $\Delta (R_a t^a) = (\Delta R)_a t^a = 0$, since $t^a$ is tangent to the boundary. Hence all three LTB functions $E$, $M$ and $T$ ‘carry through’ the surface. Since $E$, $M$ and $T$ depend only on $r$, the surface must be of the form $r =$ const. By (24), matching between one of (s1)-(s5) and solution (s6) requires $R' \to 0$ on the (s1)-(s5) side. This is only possible in (s5), with $E \to -1$. However [by (7)], $E(r) = -1$ only at isolated values of $r$, i.e. only on a comoving surface. The result follows.

It is of some interest however to establish the nature of non-comoving singular surfaces (surface layers) in these solutions, in which case $\Delta h_{ab} = 0 \neq \Delta K_{ab}$. Restricting to $\lambda = 1$ (timelike surface layers) one may construct $^3T_{ab}$ once again, by (19):

$$8\pi \left( ^3T_{ab} \right) = \begin{pmatrix} -2\sqrt{1 + P^2} \Delta R'/\left(|X| X \right) & 2 P \Delta R'/R & 0 & 0 \\ 2 P \Delta R'/R & -2 |X|^3 \Delta R'/\left(R \sqrt{1 + P^2} \right) & 0 & 0 \\ 0 & 0 & -F_3 & 0 \\ 0 & 0 & 0 & -F_3 \sin^2 \theta \end{pmatrix},$$  

where

$$F_3 = \frac{R^2 P^2 \Delta \left(X'/X \right)}{|X| \sqrt{1 + P^2}}.$$  

A comparison between (28) and the perfect fluid energy tensor $^3T_{ab} = \left( ^3\rho + \frac{3}{2} \right) t_a t_b + \frac{3}{2} \hat{p}_{ab}$ reveals that the surface layer is always of perfect fluid form. The surface energy density and pressure are

$$^3\rho = \frac{-\Delta R'}{4\pi R |X| \sqrt{1 + P^2}} \quad \text{and} \quad ^3p = ^3\rho \left( -\frac{1}{2} + \frac{P^2 R \Delta \left(X'/X \right)}{2 \Delta R'} \right).$$

The nature of the matching problem changes greatly in moving away from the comoving case. In the non-comoving case, conditions at the wall must be satisfied through some range of $r$. These conditions are therefore ‘dragged’ into the adjoining spacetimes, since the arbitrariness in these solutions resides purely in functions of $r$. This approach to surface layers and its application to models of voids is further discussed in [15].

From now on, singular surfaces are ruled out. All boundaries are necessarily comoving, and the metrics are fully determined then by choices of the arbitrary functions $E(r)$, $M(r)$ and $T(r)$ [and $A(r)$ and $B(r)$ in regions where $E(r) = -1$]. Throughout these SS dust models, junction condition (20) reduces to

$$M(r) \geq 0, \ E(r) \geq -1 \text{ and } T(r) \text{ are continuous.}$$  

IV. Regularity Requirements

In this section, the physical requirements (and one coordinate constraint) to be imposed on the metric are made explicit and justified. From section III, the metrics of (s1)-(s6) are $C^\infty$ in $t$, hence attention is focused on radial differentiability.

For a well-behaved radial coordinate, $g_{rt}$ must be piecewise continuous in $r$ (that is, continuous except at isolated values of $r$, where both left and right limits must be finite). It is also required that $\lim_{r \to 0} g_{rt} \neq 0$ everywhere, where $a + (–)$ denotes a right (left) limit. This extra condition purely defines a ‘good’ spatial coordinate, so that $dr$ is everywhere proportional to the differential increase in radial proper distance. This subtlety ensures that the continuity properties of physical quantities may be expressed unambiguously through their differentiabilities in $r$.

The dust is characterised (in the SS case) by the density $\rho$, expansion rate $\Theta = K_a a$ and shear $\sigma_{ab} = K_{ab} - \frac{1}{2} \Theta h_{ab}$. For physically reasonable matter, $\rho$, $\Theta$ and $\sigma_{ab}$ must each be piecewise continuous in $r$. By spherical symmetry, $\sigma_{ab} \to 0$ wherever $R \to 0$. For the metric (11)

$$\sigma^b_a = \frac{1}{3} \left[ \frac{\dot{X}}{X} \frac{\dot{R}}{R} \right] \times \text{diag} (0, 2, -1, -1), \quad \Theta = 2 \frac{\dot{R}}{R} + \frac{\dot{X}}{X},$$

where $\dot{X}/X$ and $\dot{R}/R$ are ‘radial’ and ‘azimuthal’ expansion rates.

To ensure that the spacetime itself is regular at each point [16] it is required that $R_{(i)(j)(k)(l)} = R_{abcd} e_{(i)}^a e_{(j)}^b e_{(k)}^c e_{(l)}^d$ is piecewise continuous in $r$, where $e^a_{(i)}$ is an orthonormal tetrad basis. Here a
natural choice is made: $e^{(a)}_0 = u^a$, $e^{(a)}_1 = |X|^{-1} \delta^a_r$, $e^{(a)}_2 = R^{-1} \delta^a \theta$, $e^{(a)}_3 = R^{-1} \cosec \theta \delta^a \phi$. For solutions (s1)-(s5), the nontrivial components are found to be

$$R_{(0)(1)(0)(1)} = 4\pi \rho - \frac{2M}{R^3}, \quad R_{(0)(2)(0)(2)} = \frac{M}{R^3},$$

$$R_{(1)(2)(1)(2)} = 4\pi \rho - \frac{M}{R^3}, \quad R_{(2)(3)(2)(3)} = \frac{2M}{R^3},$$

and for (s6)

$$R_{(0)(1)(0)(1)} = \frac{\dot{X} \dot{R}}{XR} - \frac{M}{R^3}, \quad R_{(0)(2)(0)(2)} = \frac{M}{R^3},$$

$$R_{(1)(2)(1)(2)} = \frac{\dot{X} \dot{R}}{XR}, \quad R_{(2)(3)(2)(3)} = \frac{2M}{R^3}.$$

Hence, for regular spacetimes: at points where $R \to 0$, $\lim(M/R^3)$ must be finite [using (29), and since $\rho$, $\dot{X}/X$ and $\dot{R}/R$ are already required to be piecewise continuous in $r$]. Solution (s6) does not admit central points. To summarise, throughout the models it is required that:

R1. The junction condition (21) is satisfied,

R2. $g_{rr}$ is piecewise continuous in $r$ and $\lim g_{rr} \neq 0$,

R3. $\rho \geq 0$ is piecewise continuous in $r$,

R4. $\dot{R}/R$ is piecewise continuous in $r$,

R5. $\dot{X}/X$ is piecewise continuous in $r$, and $\dot{X}/X \to \dot{R}/R$ wherever $R \to 0$,

R6. $M/R^3$ is finite wherever $R \to 0$,

(except, trivially, at the spacelike singularities $\tau \to 0$).

In sections V-VII the above conditions [with (29) satisfied a priori] are enforced for a general SS dust metric, to guarantee regularity.

V. Regular Solutions

Here the conditions of section IV are verified in turn, for points in the domain of a solution. This domain does not include the origin (treated in section VI) or interfaces between solutions (treated in section VII).

R1. Junction condition (21) is automatically satisfied for (s6). For any of (s1)-(s5) it implies that [using (7)]

$$R' \text{ may change sign only at values of } r \text{ satisfying } E(r) = -1,$$

as was noted in [2]. Hence $R' \geq 0$ throughout the domain of each solution (s1)-(s4), or $R' \leq 0$ throughout. In (s5), $R'$ may change sign where $E = -1$. Now for (s4), $R'$ may be written as

$$R' = \frac{M'}{E} \left[ \frac{\eta \sinh \eta}{\cosh \eta - 1} - 2 \right] - T' E^{1/2} \left[ \frac{\sinh \eta}{\cosh \eta - 1} \right] + \frac{E' M}{E^2} \left[ \frac{\sinh \eta (\sinh \eta - 3\eta)}{2 (\cosh \eta - 1)} + 2 \right],$$

and for a positive density [by equation (9)], $R'$ and $M'$ must have the same sign. At large $\eta$ (large $t$), the third term in (32) dominates, so that $R'$ and $E'$ must have the same sign. At small $\eta$, the second term dominates, so $T'$ must have the opposite sign to $R'$. These conditions are also sufficient (to ensure $R'$ has a constant sign), since in (32) each function of $\eta$ in square brackets is strictly positive for all allowed $\eta$. Hence [for (s4)] equation (31) implies that

$$\pm R' \geq 0 \Rightarrow \{\pm M' \geq 0, \quad \pm E' \geq 0, \quad \pm T' \leq 0\}.$$
For any of (s1)-(s3), similar reasoning also leads to (33). Now for regions of (s5) which satisfy $E \neq -1$ (in which $R'$ cannot change sign), it is useful to write $R'$ in the form

$$R' = \left( \frac{E' M}{E^2} - \frac{2 M'}{3 E} + \frac{T'|E|^{1/2}}{3 \pi} \right) \left[ \frac{\sin \eta (\sin \eta - 3 \eta)}{2 (1 - \cos \eta)} + 2 \right] - T'|E|^{1/2} \left[ \frac{\sin \eta}{1 - \cos \eta} + \frac{1}{3 \pi} \left\{ 2 + \frac{\sin \eta (\sin \eta - 3 \eta)}{2 (1 - \cos \eta)} \right\} \right] + \frac{M'}{3|E|} \left[ 1 - \cos \eta \right], \quad (34)$$

where each function in square brackets is always positive. At small $\eta$ the second term dominates. As $\eta \to 2\pi$ the first term dominates, and $M'$ must have the same sign as $R'$ for $\rho \geq 0$. Hence in these regions of (s5)

$$\pm R' \geq 0 \Rightarrow \left\{ \pm M' \geq 0, \quad \pm \left( E'M - \frac{2}{3} M'E + \frac{T'|E|^{5/2}}{3 \pi} \right) \geq 0, \quad \pm T' \leq 0 \right\}. \quad (35)$$

Equations (33) and (35) are the Hellaby and Lake no-shell-crossing conditions [3], derived here from the junction conditions. Violation of (33) or (35) would necessitate either a pathological choice of radial coordinate or true caustic formation.

**R2.** From (7), piecewise continuity of the radial coordinate within the domain of one of (s1)-(s4) [and in regions of (s5) where $E(r) \neq -1$] requires piecewise continuity of $R'$ in $r$, for which it is necessary and sufficient that $M'(r)$, $T'(r)$ and $E'(r)$ are piecewise continuous. Now $\lim_{\pm g_{rr}} \neq 0$ implies $\lim_{\pm} R' \neq 0$ in these regions, which [from (33) and (35)] forces one of $\lim_{\pm} M'$, $\lim_{\pm} T'$ and $\lim_{\pm} E'$ to be nonzero at each point (likewise for the left limits). At points in (s5) where $E(r) = -1$, $\lim_{\pm} g_{rr}$ is finite if and only if

$$\lim_{\pm} \frac{M'}{\sqrt{1 + E}}, \quad \lim_{\pm} \frac{T'}{\sqrt{1 + E}} \quad \text{and} \quad \lim_{\pm} \frac{E'}{\sqrt{1 + E}} \quad \text{are finite, and at least one is nonzero,} \quad (36)$$

as follows from (7), (34) and (35) (and analogously for $\lim_{\pm} g_{rr}$). One necessary consequence of (36) is that $\lim_{\pm} R' = \lim_{\pm} M' = \lim_{\pm} T' = \lim_{\pm} E' = 0$ at these points. Finally, for (s6) the radial coordinate is well-behaved if $A(r)$ and $B(r)$ are piecewise continuous and if at least one of $\lim_{\pm} A$ and $\lim_{\pm} B$ is nonzero [by (14), and likewise for $\lim_{\pm}$].

**R3.** The density is piecewise continuous in (s1)-(s5) as a consequence of the junction conditions, which is seen as follows. From (9), $\rho$ is at least continuous (in $r$) except at isolated points, since $R'$ and $M'$ have this property (see above) and $R' = 0$ only at isolated points. Trivially $\rho = 0$ for (s1) and (s2). For (s3)-(s5), if $M' = 0$ in a finite region, then $\rho = 0$ there. Otherwise $\lim_{\pm} \rho$ are finite if and only if $\lim_{\pm} (R'/M') \neq 0$, but this is automatically satisfied in (s3)-(s5) [e.g. in (s4)]

$$\lim_{\pm} \frac{R'}{M'} \geq \frac{1}{E} \left[ \frac{\eta \sinh \eta}{\cosh \eta - 1} - 2 \right] > 0,$$

by (32), (33). Hence $\lim_{\pm} \rho$ are finite, and $\rho$ is piecewise continuous in $r$. In fact, $\lim_{\pm} \rho = 0$ (with $M' \neq 0$) only if either $\lim_{\pm} (E'/M') = +\infty$ or $\lim_{\pm} (T'/M') = -\infty$. Note that none of these remarks require any modification at points in (s5) with $E = -1$. Now the density in (s6) [given by (15)] vanishes if $B = 0$ and $A \neq 0$ [and (s6) degenerates to part of the exterior Schwarzschild solution]. Otherwise $\rho$ is finite and positive at all times in (s6) if and only if

$$\lim_{\pm} \frac{B}{A} \geq \frac{1}{\pi}, \quad (37)$$

which ensures no zeroes in the denominator of (15). Piecewise continuity of $\rho$ in (s6) follows from the piecewise continuity of $A(r)$ and of $B(r)$.

**R4.** Full continuity of $\dot{R}/R$ in $r$ is guaranteed throughout the SS dust spacetimes by (8) and (29).
IV. Only comoving origins are possible, and they may join only to solutions (s1)-(s5). All the results are given in 

\[ \lim_{R' \to 0} (E'/R') = \text{finite} \quad \text{wherever} \quad R' \to 0. \]

This is trivially satisfied in (s3). In (s2), (s4) and (s5) it is also automatically satisfied since \( \lim_{R' \to E'} \frac{R'}{|E'|} > 0 \)

(e.g. in (s4))

\[ \lim_{R' \to E'} \frac{R'}{E'} \geq \frac{M}{E'^2} \left[ \frac{\sinh \eta (\sinh \eta - 3\eta)}{2(\cosh \eta - 1)} + 2 \right] > 0 \]

by [32, 33]. Finally, solution (s6) has

\[ \frac{\dot{X}}{X} = \frac{\epsilon}{M(1 - \cos \eta)} \left[ \frac{1}{2} B (\eta - \sin \eta) - A \right] \]

which is automatically piecewise continuous in \( r \) by [37] and by the piecewise continuity of \( A(r) \) and \( B(r) \).

VI. Regular Centres

In this section the possible types of origin (for which \( R \to 0 \)) are determined by imposing the conditions of section IV. Only comoving origins are possible, and they may join only to solutions (s1)-(s5). All the results are given in Table 1 in which (i) the condition \( R \to 0 \), (ii) \( \lim_{R' \to 0} g_{rr} \neq 0 \), and (iii) \( \lim \frac{g_{rr}}{E'^2} \) is finite wherever \( g_{rr} \neq 0 \). The logarithmic term \( \frac{g_{rr}}{E'^2} \) diverges by (38). Hence (39) is the only previous results.

One central limit suffices to illustrate the arguments used to obtain Table 1. At an origin of (s4), suppose \( \eta \to \infty \). Then by (10), on approaching the origin

\[ e^\eta \approx \frac{2E^{3/2}}{M} \to \infty, \quad (38) \]

and \( R \to 0 \) forces \( E \to 0 \), \( M \to 0 \). Now

\[ \frac{\dot{R}}{R} \to \frac{\epsilon}{\tau}, \quad \frac{\dot{X}}{X} \to \frac{\epsilon (\frac{1}{2} E' E^{-1/2} + T'E^{-1} M^{-1/2} + M'E^{-1} \tau^{-1})}{[M' E^{-1} \log \left( E^{3/2} M^{-1} \right) - T'E^{1/2} + \frac{1}{2} E' E^{-1/2} \tau]} \]

so that vanishing shear requires either

\[ \lim \left( \frac{T'M}{E'^{1/2} M'} \right) = \lim \left( \frac{M'}{E'^{1/2} M'} \right) = \lim \left[ - T'E + \frac{M'}{E'^{1/2} E'} \log \left( \frac{E'^{3/2}}{M} \right) \right] = 0 \quad (39) \]

or

\[ \lim \left( \frac{E'^{1/2} E'}{M'} \right) = \lim \left( \frac{M'T'}{M'} \right) = 0, \quad \lim \left[ \log \left( \frac{E'^{3/2}}{M} \right) - T'E^{1/2} \right] = 1. \quad (40) \]

However the latter case (40) is ruled out since the logarithmic term must diverge, by (38). Hence (39) is the only possibility, and this reduces to

\[ \lim \left( \frac{T'E}{E'} \right) = \lim \left[ \frac{M'}{E'^{1/2} E'} \log \left( \frac{E'^{3/2}}{M} \right) \right] = 0, \quad (41) \]

by consideration of (33) and (38). From (32)

\[ \sqrt{g_{rr}} \to R' \to \frac{M'-E \log \left( \frac{E'^{3/2}}{M} \right) - T'E^{1/2} + \frac{E' \tau}{2E^{1/2}}}{E} \]

so that \( \lim g_{rr} \neq 0 \) requires \( \lim (E'E^{-1/2}) \) finite and nonzero. Finally

\[ 4\pi \rho \to \left[ r^2 \log \left( \frac{E'^{3/2}}{M} \right) - \frac{E'^{3/2} T' r^2}{M'} + \frac{E'^{1/2} E' r^3}{2M'} \right]^{-1}, \quad (42) \]

which vanishes (again by the divergence of the logarithmic term).
In this section the conditions of section IV are considered on the comoving interfaces between domains of solutions (s1)-(s6) in a composite model. The solution domains are assumed to be regular (in the sense of section V) and this generally ensures regular interfaces.

By (s1), the sign of $R'$ cannot change across these interfaces, since $E \neq -1$ on them [except on interfaces between (s5) and (s6), but $R' \equiv 0$ in (s6)]. Solution (s1) may not be matched to any other solution, since $R' \equiv 0$ in (s1).

VII. Regular Interfaces

| Soln. | Behaviour of $E$, $M$ and $T$ | Kinematics | Example |
|-------|-------------------------------|------------|---------|
| (s1)  | (i) $T \to 0$ | $\Theta \equiv 0$ | $T = -r$ |
|       | (ii) $\lim T'$ finite, nonzero | $\rho \equiv 0$ |         |
| (s2)  | (i) $E \to 0$ | $\Theta \to 3\tau^{-1}$ | $E = r^2$ |
|       | (ii) $\lim(E^{-1/2}E')$ finite, nonzero | $\rho \equiv 0$ | $T = 0$ |
|       | (iii) $ET'/E' \to 0$ | $E \to 3\tau^{-1}$ |         |
| (s3)  | (i) $M \to 0$ | $\Theta \to 2\tau^{-1}$ | $M = r^3$ |
|       | (ii) $\lim(M^{-2/3}M')$ finite, nonzero | $\rho \equiv 0$ | $T = 0$ |
|       | (iii) $MT'/M' \to 0$ | $E \to 2\tau^{-1}$ |         |
| (s4)  | (i) $E^{3/2}/M \to 0$, $M \to 0$ | $\Theta \to 2\tau^{-1}$ | $E = r^3$ |
|       | (ii) $\lim(E^{-1/2}E')$ finite, nonzero | $\rho \equiv 0$ | $T = 0$ |
|       | (iii) $\lim(ET'/E') = \lim(M^{1/3}E'/M') = 0$ | $E \to 3\alpha (cosh \eta - 1)^2$ | $M = r^3$ |
|       | $\equiv \lim \left[ E^{-1/2}M'E^{-1}\log(E^{3/2}M^{-1}) \right] = 0$ | $M \to 0$ |         |
| (s5)  | (i) $|E|^{3/2}/M \to 0$, $M \to 0$ | $\Theta \to 3\alpha \sin \eta/(1 - \cos \eta)^2$ | $E = -r^3$ |
|       | (ii) $\lim(M^{-2/3}M')$ finite, nonzero | $\rho \equiv 0$ | $M = r^3$ |
|       | (iii) $\lim(MT'/M') = \lim(M^{1/3}E'/M') = 0$ | $E \to 2\tau^{-1}$ | $T = 0$ |

TABLE I: Central behaviour for regular centres. (i) follows from $R \to 0$, (ii) from $\lim \pm g_{\tau r} \neq 0$, and (iii) from $\sigma_{ab} \to 0$. 

(whereas the other solutions are cosmological). Solution (s2) does not match to (s3), because $M \to 0$ forces $R \to 0$ in (s3). Also, (s2) does not match to (s5), since $(E \to 0, M \to 0)$ forces $R \to 0$ in (s5). From section III, (s6) only matches to (s5). There remain just five physical types of junction, given below. At each of the five, equation (29) ensures continuity of $R$.

a. Matching (s2) to (s4)
The (s2) side of this interface is unconstrained by the matching. On approaching the interface from (s4), $M \to 0$, $E \neq 0$ and $M^\prime > 0$ throughout some finite region [by the piecewise continuity of $M^\prime$, and since $M > 0$ in (s4)]. Hence by (31) and (32), $R$ must increase in the direction (s2)→(s4). Now $\eta$ obeys (38), so that in (s4)

$$ \frac{\sqrt{g_{rr}}}{\sqrt{1 + \epsilon}} \left[ \frac{M^\prime}{M} \log \left( \frac{E^{3/2}}{M} \right) - T^\prime E^{1/2} \right] + \frac{E^\prime}{2E^{1/2}} $$

Therefore $r$ is a good coordinate if $\lim_{(s4)} T^\prime$, $\lim_{(s4)} E^\prime$ and $\lim_{(s4)} M^\prime \log M$ are finite, and at least one is nonzero.

On the (s4) side, the density reduces to (42), and vanishes by the divergence of the logarithmic term. [Note that $\rho \equiv 0$ in (s2).]

On the (s4) side, since $(M \to 0, M^\prime \log M$ finite) forces $M^\prime \to 0$, we have

$$ \frac{\dot{X}}{X} \to \left\{ \begin{array}{ll} 0 & \text{if } E^\prime \to 0, \\ \epsilon \frac{\tau - 2ET^\prime/E^\prime - 2M^\prime \log M/(E^{1/2} E^\prime)}{1} & \text{otherwise,} \end{array} \right. $$

whereas on the (s2) side

$$ \frac{\dot{X}}{X} \to \frac{\epsilon}{\tau - 2ET^\prime/E^\prime}, $$

and on both sides $\dot{R}/R \to \epsilon/\tau$. Hence the shear is necessarily finite on both sides of the interface, as required.

b. Matching (s3) to (s4)
The (s3) side is unconstrained by the matching. On approaching the interface from (s4), $E \to 0$ while $M \neq 0$, so that $\eta \approx (6\tau/M)^{1/3} E^{1/2} \to 0$, and $E^\prime > 0$ throughout some finite region [since $E > 0$ in (s4)]. Hence by (31) and (33), $R$ must increase in the direction (s3)→(s4). On the (s4) side

$$ \frac{\sqrt{g_{rr}}}{\sqrt{1 + \epsilon}} \left[ \frac{M^\prime}{M} \log \left( \frac{E^{3/2}}{M} \right) - T^\prime E^{1/2} \right] + \frac{E^\prime}{2E^{1/2}} $$

so that $r$ is a good coordinate provided

$$ \lim_{(s4)} M^\prime, \lim_{(s4)} T^\prime \text{ and } \lim_{(s4)} E^\prime \text{ are finite, and at least one is nonzero.} \quad (44) $$

On both sides of the interface, the density reduces to

$$ 4\pi \rho \to \left\{ \begin{array}{ll} 0 & \text{if } M^\prime \to 0, \\ \left[ 3\tau^2/2 - 3MT^\prime \tau/M^\prime + (6\tau)^{8/3} M^{1/3} E^\prime/(160M^4) \right]^{-1} & \text{otherwise,} \end{array} \right. $$

and on both sides,

$$ \frac{\dot{X}}{X} \to \frac{\epsilon \left[ T^\prime + M^\prime \tau/M^\prime + \left( 243\tau^5/(250M^2) \right)^{1/3} E^\prime \right]}{3M^\prime \tau^2/(2M) - 3T^\prime \tau + E^\prime \left( 9\tau^2/2 \right)^{4/3} / \left( 10M^{2/3} \right)}, \quad \frac{\dot{R}}{R} \to \frac{2\epsilon}{3\tau}. \quad (46) $$

Therefore both $\rho$ and $\dot{X}/X$ are well-behaved.
c. Matching (s3) to (s5)
This interface is similar to b. On approaching the interface in (s5), \( \eta \approx (6\pi/M)^{1/3}|E|^{1/2} \to 0 \). Then all the results in b for the kinematics and radial coordinate follow [with \('(s4)' replaced by \('(s5)'\)].

However, \( R \) must increase in the opposite sense to that in b - here \( R \) must increase in the direction \((s5)\to(s3)\), as we now show. Taking the limit of equation (35) on the \((s5)\) side gives sign\((R')E' > 0\). Since \( E < 0 \) in \((s5)\) and \( E' \) is piecewise continuous, the result follows. Note that the coordinate condition (44) is crucial to this proof. Note also that since \( \dot{R}/R \to \frac{2}{3}\sqrt{\pi} -1 > 0 \) at any interface between (s3) and (s5), the r-continuity of \( \dot{R}/R \) forces the existence of a finite region in (s5) adjoining the interface for which the azimuthal expansion rate is positive \( \dot{R}/R > 0 \) [even though all points in (s5) eventually satisfy \( \dot{R} < 0 \)].

d. Matching (s4) to (s5)
Both sides are constrained by \( E \to 0 \) with \( M > 0 \), and the resulting junction is given precisely by combining the results of b and c. In this case, \( R \) must increase in the direction \((s5)\to(s4)\), by a similar argument to that given in b.

e. Matching (s5) to (s6)
At this interface the \((s6)\) side is unconstrained by the matching. Equation (31) places no restriction, and \( R \) may increase in either direction on approaching the interface from (s5). Now \( \lim_{(s5)} E = -1 \). Therefore, as discussed in section V, \( r \) is a good coordinate provided (36) is satisfied.

The \((s5)\)-limits of \( \rho \) and \( X/X \) are just those of an ‘ordinary’ point, i.e. one in the domain of (s5). In this sense, the matching conditions at this type of interface are considerably less restrictive than those at the other four.

Combining these results with the rest of the paper, all the regular SS dust models may be classified into four topologies:

i. Open models with one origin
By noting the sense in which \( \dot{R} \) must increase at the interfaces a-e above, the only possible composite models are:

\[
\begin{align*}
\mathcal{O}(s1)^+ & , & \mathcal{O}(s2)^+(s4)^+ , \\
\mathcal{O}(s2)^+ & , & \mathcal{O}(s3)^+(s4)^+ , \\
\mathcal{O}(s3)^+ & , & \mathcal{O}(s5)^+S(s5)^+(s3)^+ , \\
\mathcal{O}(s4)^+ & , & \mathcal{O}(s5)^+S(s5)^+(s4)^+ , \\
\mathcal{O}(s5)^+S & , & \mathcal{O}(s5)^+S(s5)^+(s3)^+(s4)^+ ,
\end{align*}
\]

where \( \mathcal{O} \) denotes an origin, and a superscript + (−) implies that \( R \) increases (decreases) from left to right. Here \( S \) is any combination of \((s5)^-\), \((s5)^+\) and \((s6)\). Note that open models can be constructed from collapsing solutions [e.g. \( \mathcal{O}(s5)^+(s6) \)]. Papapetrou [17] discussed a particular example of \( \mathcal{O}(s5)^+(s3) \).

In the above construction, we have noted from (43) that on \( t = \text{const.} \), \( d\chi = |dR|/\sqrt{1+E} \), where \( \chi \) is radial proper distance. Hence by (44), if \( E > \alpha > -1 \) for all \( \chi > \beta \) (\( \alpha, \beta \) constants) then:

\[
\chi \to \infty \text{ forces } R \to \infty \quad \text{if } \frac{dR}{d\chi} > 0,
\]

there is a finite value of \( \chi > \beta \) for which \( R = 0 \), \( \text{ if } \frac{dR}{d\chi} < 0. \) (48)

However, if \( E \to -1 \) as \( \chi \to \infty \), then neither of (47), (48) are necessary. An example of \( \mathcal{O}(s5)^+(s5)^- \) of type i is

\[
\begin{align*}
\mathcal{E} = \begin{cases} 
-\sin^2 r [1 - e^{-2r_0}] /\sin^2 r_0 & \text{for } 0 < r < r_0, \\
-1 + e^{-2r} & \text{for } r > r_0 
\end{cases}, \\
\mathcal{M} = \begin{cases} 
\sin^3 r [M_\infty + e^{-r_0}] /\sin^3 r_0 & \text{for } 0 < r < r_0, \\
M_\infty + e^{-r} & \text{for } r > r_0 
\end{cases}, \\
T = 0, \quad M_\infty > 0, \quad \pi < r_0 < 2\pi,
\end{align*}
\]
and
\[
E = \begin{cases} 
-r^2 \left[1 - e^{-2r_0}\right]/r_0^3 & \text{for } 0 < r < r_0, \\
-1 + e^{-2r} & \text{for } r > r_0, 
\end{cases} \\
M = \begin{cases} 
r^3 \left[M_\infty - e^{-r}\right]/r_0^3 & \text{for } 0 < r < r_0, \\
M_\infty - e^{-r} & \text{for } r > r_0, 
\end{cases}
\]
\[T = 0, \quad 0 < M_\infty < 2/3, \quad r_0 > 0,\]
is an example of \(\mathcal{O}(s5)^+\). In each of \([49]\) and \([50]\), \(R \rightarrow \text{const} > 0\) as \(\chi \rightarrow \infty\). There are no SS dust models with \(R \rightarrow 0\) as \(\chi \rightarrow \infty\) [by \([17]\)] and since, by \([10]\), \(R \rightarrow 0\) requires \(E \rightarrow 0\).

ii. Open models with no origin
By \([28]\), to avoid a zero in \(R\), a model with no origin must either be composed entirely of \((s6)\), or it must contain a section of \((s5)\), in order to allow (at least one) minimum in \(R\). Then the possible matchings are evident:

\[
S = \begin{cases} 
(s3)^- (s5)^- \\
(s4)^- (s5)^- \\
(s4)^- (s3)^- (s5)^- 
\end{cases} \quad S = \begin{cases} 
(s5)^+ (s3)^+ \\
(s5)^+ (s4)^+ \\
(s5)^+ (s3)^+ (s4)^+ 
\end{cases}
\]

Examples and a detailed analysis of such models are provided in \([2]\). In these models, due to the presence of collapsing solutions \((s5),(s6)\), an origin does eventually form, but gravitational collapse will violate the regularity conditions in any case.

iii. Closed models with two origins
These models must contain a section of \((s5)\), since there must be (at least one) turning point in \(R\). The models cannot contain a section of \((s2)\), \((s4)\) or \((s2)(s4)\), since the section would either contain an origin and match to another solution, or would match to other solutions on both sides. Hence \(E\) would vanish on both sides, and since \(E > 0\) throughout the domains of \((s2)\) and \((s4)\), \(E'\) could not have the same sign throughout, contrary to \([33]\) [with \([31]\)]. There can be no \((s1)\) section in the closed model, since it does not match to any other solution. There can be no \((s3)\) region in the model either, since \(R\) must increase in the direction \((s5)\rightarrow(s3)\). Hence if \((s3)\) contains an origin, it cannot match to \((s5)\). Conversely, if \((s3)\) does not contain an origin, it cannot match to \((s5)\) on both sides, leaving the model open. This leaves just \((s5)\) and \((s6)\) to construct these models, and the possibilities are:

\[
\mathcal{O}(s5)^+ S(s5)^- \mathcal{O}
\]

iv. Closed models with no origin
Consider an SS dust model which has \(R > 0\) in some range \(0 \leq r \leq d\) (and at some \(t\)). This final possibility of composite models is obtained by identifying (matching) the surfaces \(r = 0\) and \(r = d\). Since \(\Delta R = 0\), the model must be everywhere \((s6)\) or else it must contain a section of \((s5)\) [otherwise \(\text{sign}(R')\) is constant in \(0 \leq r \leq d\), which forces \(R(0) \neq R(d)\)]. No sections composed from the solutions \((s1)-(s4)\) may be present, since they would be forced to match to \((s5)\) on both sides. This would force \(R'\) to change sign in the section (since \(R\) must increase away from \((s5)\) into these solutions) and this is not possible, by \([31]\). Hence the models may only be constructed from \((s5)\) and \((s6)\), with the possibilities:

\[
\mathcal{I} \mathcal{S} \mathcal{I}
\]

where \(\mathcal{I}\) denotes the surfaces which are identified (at which the standard matching conditions must be satisfied, as we have described). The spatial sections of these models have the topology of a 3-torus. An example is provided by

\[
E = \begin{cases} 
ar^2 - 1 & \text{for } 0 < r < \frac{1}{4}d, \\
a(r - \frac{1}{2}d)^2 - 1 & \text{for } \frac{1}{4}d < r < \frac{3}{4}d, \\
a(r - d)^2 - 1 & \text{for } \frac{3}{4}d < r < d, 
\end{cases} \\
M = \begin{cases} 
b + cr^2 & \text{for } 0 < r < \frac{1}{4}d, \\
b + \frac{1}{8}cd^2 - c(r - \frac{1}{2}d)^2 & \text{for } \frac{1}{4}d < r < \frac{3}{4}d, \\
b + c(r - d)^2 & \text{for } \frac{3}{4}d < r < d, 
\end{cases}
\]
\[T = 0, \quad a \left(2b + \frac{1}{4}cd^2\right) < \frac{4}{3}c,\]
\[\text{(51)}\]
where $a, \ldots, d$ are positive constants. Note that a model of type iv cannot be constructed from the homogeneous (Friedmann-Lemaître-Robertson-Walker) subclass of LTB (since the elliptic homogeneous solution has only one point with $E = -1$, at which $R$ is maximum).

There are no further possible topologies or composite models. Examples of models of types i-iii are given in previous literature (see especially [7]).

VIII. Conclusion

In this paper, full regularity conditions have been derived and discussed for SS dust spacetimes. From section III, the solutions for the metric in section II (which are all $C^\infty$ in $t$) may be joined only on comoving surfaces. Hence the models are fully determined by choices of the functions $E(r) \geq -1$, $M(r) \geq 0$, $T(r)$, $A(r)$ and $B(r)$. Existence of the Einstein tensor places a basic restriction on these functions [equation (6)]: $E$, $M$ and $T$, $A$ and $B$ must be $C^1$, except at a finite number of points. At these points, there may be discontinuities in any of $E'$, $M'$, $T'$, $A$ or $B$. However there are additional, more subtle requirements of the functions. For example, at points where $E \to -1$, equation (36) must hold. These differentiability conditions ensure the good behaviour of all the relevant physical quantities.

In the current context, the recollapse conjecture [13], i.e. ‘all closed SS dust models must recollapse everywhere’, follows directly from the results of section VII. We simply note that all the possible closed models contain only the solutions (s5) and (s6), which both recollapse in a finite time. This is a simple alternative proof to that of Burnett [19] (which involved considering the lengths of timelike curves in these spacetimes). Our proof slightly strengthens that of Bonnor [2], in that no mathematical assumptions are required (as were used in [2]) other than those explicitly demanded by the regularity.

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