On the dimension of Chowla–Milnor space

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Abstract. In a recent work, Gun, Murty and Rath defined the Chowla–Milnor space and proved a non-trivial lower bound for these spaces. They also obtained a conditional improvement of this lower bound and noted that an unconditional improvement of their lower bound will lead to irrationality of $\zeta(k)/\pi^k$ for odd positive integers $k > 1$. In this paper, we give an alternate proof of their theorem about the conditional lower bound.

Keywords. Hurwitz zeta function; Chowla–Milnor spaces.

1. Introduction

For any complex number $s \in \mathbb{C}$, with $\Re(s) > 1$, one defines the Riemann zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which has an Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

The Riemann zeta function defines an analytic function in the region $\Re(s) > 1$ and can be extended meromorphically to the whole complex plane with a simple pole at $s = 1$ having residue 1. Hurwitz generalized the Riemann zeta function by $\zeta(s, x)$ which is defined as

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

where $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. He proved that $\zeta(s, x)$ can be extended meromorphically to the entire complex plane with a pole at $s = 1$. Note that for $x = 1$, $\zeta(s, 1)$ is the classical Riemann zeta function.

DEFINITION

For integers $k > 1, q \geq 2$, define the Chowla–Milnor space $V_k(q)$ by

$$V_k(q) := \mathbb{Q} - \text{span of } \{\zeta(k, a/q) : 1 \leq a < q, \ (a, q) = 1\}.$$
As described in [1], the conjecture of Chowla and Milnor is the assertion that the dimension of $V_k(q)$ is equal to $\varphi(q)$, where $\varphi$ is the Euler's phi-function. Gun et al. [1] showed that the dimension of the above spaces is at least $\varphi(q)/2$. They also derived the following theorem.

**Theorem.** Let $k > 1$ be an odd integer and $q, r > 2$ be two co-prime integers. Then either
\[
\dim_Q V_k(q) \geq \frac{\varphi(q)}{2} + 1 \quad \text{or} \quad \dim_Q V_k(r) \geq \frac{\varphi(r)}{2} + 1.
\]

The proof in [1] uses the expansion of Bernoulli polynomials. In this note, we give an alternate proof of the theorem by an explicit evaluation of co-tangent derivatives.

2. **Proof of the theorem**

The following Lemma 1 due to Okada [2] about the linear independence of co-tangent values at rational arguments plays a significant role in proving the theorem.

**Lemma 1.** Let $k$ and $q$ be positive integers with $k \geq 1$ and $q > 2$. Let $T$ be a set of $\varphi(q)/2$ representations mod $q$ such that the union $T \cup (-T)$ constitutes a complete set of co-prime residue classes mod $q$. Then the set of real numbers
\[
\frac{d^{k-1}}{dz^{k-1}} \cot(\pi z)|_{z=a/q}, \quad a \in T
\]
are linearly independent over $\mathbb{Q}$.

We first have the following lemma.

**Lemma 2.** For an integer $k \geq 1$,
\[
D^{k-1}(\pi \cot \pi z) = \pi^k \times \mathbb{Z} \text{ linear combination of } (\csc \pi z)^{2l}(\cot \pi z)^{k-2l},
\]
for some non-negative integer $l$. Here $D^{k-1} = \frac{d^{k-1}}{dz^{k-1}}$.

**Proof.** We will prove this by induction on $k$. For $k = 1$, we have $D^{k-1}(\pi \cot(\pi z)) = \pi \cot(\pi z)$. Assume that the statement is true for $k - 1$, i.e.
\[
D^{k-2}(\pi \cot(\pi z)) = \pi^{k-1} \sum a_i (\csc \pi z)^{2l_i} (\cot \pi z)^{(k-1)-2l_i},
\]
where $a_i$'s are integers.

Differentiating both sides with respect to $z$, we get
\[
D^{k-1}(\pi \cot \pi z) = \pi^k \sum [b_i (\csc \pi z)^{2l_i} (\cot \pi z)^{k-2l_i} + c_i (\csc \pi z)^{2l_i+2} (\cot \pi z)^{k-(2l_i+2)}],
\]
where $b_i, c_i$'s are integers. This completes the proof of Lemma 2. \qed
Lemma 3. For an integer \( k \geq 2 \),
\[
\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q}.
\]

Proof.

L.H.S. \( = \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \)
\[
= \sum_{n \geq 0} \frac{1}{(n + a/q)^k} + (-1)^k \sum_{n \geq 0} \frac{1}{(n + 1 - a/q)^k}
\]
\[
= \sum_{n \geq 0} \frac{1}{(n + a/q)^k} + (-1)^k \sum_{n=1}^{\infty} \frac{1}{(n - a/q)^k}
\]
\[
= \sum_{n \geq 0} \frac{1}{(n + a/q)^k} + (-1)^{2k} \sum_{n=1}^{\infty} \frac{1}{(-n + a/q)^k}
\]
\[
= \sum_{n \in \mathbb{Z}} \frac{1}{(n + a/q)^k}.
\]

Again we know that for \( z \notin \mathbb{Z} \),
\[
\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{z+n}.
\]

This implies that
\[
D^{k-1}(\pi \cot \pi z) = (-1)^{k-1}(k-1)! \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k}.
\]

So,
\[
\frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q} = \sum_{n \in \mathbb{Z}} \frac{1}{(n + a/q)^k},
\]
which completes the proof of Lemma 3. \( \square \)

Finally, we have Lemma 4, whose proof is standard.

Lemma 4. Let \( P \) be the set of primes. We have
\[
\zeta(k) \prod_{\substack{p \in P, \\
p|q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\
(a,q)=1}}^{q-1} \zeta(k, a/q).
\]

Proof of the theorem. First note that the space \( V_k(q) \) is also spanned by the following sets of real numbers:
\[
\{\zeta(k, a/q) + \zeta(k, 1 - a/q)|(a, q) = 1, \ 1 \leq a < q/2\},
\]
\[
\{\zeta(k, a/q) - \zeta(k, 1 - a/q)|(a, q) = 1, \ 1 \leq a < q/2\}.
\]
Now from Lemma 3, we have the following:

\[ \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q}. \]

Applying the above Lemma 1, we see that

\[ \dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2}. \]

Now from Lemmas 2 and 3 for an odd integer \( k \), we have

\[ \frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} = \frac{i}{2^k} \times \mathbb{Q} \text{ linear combinations of } (\csc \pi a/q)^{2l}(\cot \pi a/q)^{k-2l}. \]

We note that

\[ i \cot(\pi a/q) = \frac{1 + \zeta_a}{1 - \zeta_a} \]

belongs to \( \mathbb{Q}(\zeta_q) \) and so do the numbers \( \csc(\pi a/q)^{2l} \) and \( \cot(\pi a/q)^{2l} \). Since \( k \) is odd, we have

\[ \frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} \in \mathbb{Q}(\zeta_q) \tag{1} \]

Now we go back to the main part of the proof. Let \( q \) and \( r \) be two co-prime integers. Suppose that

\[ \dim_{\mathbb{Q}} V_k(q) = \frac{\varphi(q)}{2}. \]

Then the numbers

\[ \zeta(k, a/q) - \zeta(k, 1 - a/q), \text{ where } (a, q) = 1, \ 1 \leq a < q/2 \]

generate \( V_k(q) \). Now from Lemma 4, we get

\[ \zeta(k) \prod_{p|q} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1, \\ (a,q)=1}}^{q-1} \zeta(k, a/q) \in V_k(q). \]

and hence

\[ \zeta(k) = \sum_{(a,q)=1}^{1 \leq a < q/2} \lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)], \quad \lambda_a \in \mathbb{Q} \]

so that

\[ \frac{\zeta(k)}{(2\pi i)^k} = \sum_{\substack{(a,q)=1, \\ 1 \leq a < q/2}}^{\lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)]} \frac{(2\pi i)^k}{(2\pi i)^k}. \]
Thus by (1),
\[
\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_q).
\]
Similarly, if
\[
\dim_{\mathbb{Q}} V_k(r) = \frac{\varphi(r)}{2},
\]
then
\[
\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_r)
\]
and hence
\[
\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r).
\]
Since any non-trivial finite extension of \( \mathbb{Q} \) is ramified, if \( \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) \neq \mathbb{Q} \), there exists a prime which is ramified in \( \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) \), and also in \( \mathbb{Q}(\zeta_q) \) and \( \mathbb{Q}(\zeta_r) \). Note that a prime which ramifies in this intersection must necessarily divide both \( q \) and \( r \). This is impossible because \( (q, r) = 1 \). So \( \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q} \). Hence we arrive at a contradiction as \( \frac{\zeta(k)}{i\pi^k} \) is a real number. Thus
\[
\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1 \quad \text{or} \quad \dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.
\]
This completes the proof of the theorem. \( \square \)

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References
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