Harmonic Analysis

Spectral analysis on Damek-Ricci space

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Abstract.— We define and study the spectral projection operator for compactly supported distributions on Damek-Ricci space $N_A$. The Paley-Wiener-Schwartz theorem and the range of $S^p(N_A)(0 < p \leq 2)$ via spectral projection operator are established. The $L^2$-estimation for this operator is also given. In order to do the Paley-Wiener theorem for the non necessary radial function, the spectral projection operator can be uniquely characterized by analyticity and growth condition in $\lambda$ of Paley-Wiener theorem type on the unit disk of the complex plane as an example of Damek-Ricci space.

1 Introduction

Given a group $N$ of Heisenberg type, let $S = N_A$ be the one-solvable extension of $N$ obtained by letting $A = IR^+$ acts on $N$ by homogeneous dilatation. We equip $S$ with a natural left-invariant Riemannian structure. The

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group $S$ is in generally nonsymmetric harmonic spaces (the geodesic symmetry around the identity is not isometry (see [13] and [15])). The geodesic distance of $x \in NA$ from the identity $e$ is

$$\rho(x) = d(x, e) = \log\left(\frac{1 + r(x)}{1 - r(x)}\right), \quad 0 \leq r(x) \leq 1$$

with,

$$r(V, Z, a)^2 = 1 - \frac{4a}{(1 + a + \frac{|W|^2}{4})^2 + |Z|^2}.$$  

On such a group $S$, we consider the Laplacian $\mathcal{L}$ on $NA$ whose radial part is given by the rule

$$\mathcal{L}_r = \frac{\partial^2}{\partial \rho^2} + \left(\frac{m}{2} \coth\left(\frac{\rho}{2}\right) + k \coth(\rho)\right) \frac{\partial}{\partial \rho},$$

with $\rho$ is the geodesic distance of $x \in NA$ from the the identity. The Fourier-Helgason transform of the function $f$ in $\mathcal{D}(NA)$ (see [7]) is the function $\hat{f}$ on $\mathcal{C} \times N$ defined by

$$(1)\quad \hat{f}(\lambda, n) = \int_{NA} f(x) P_\lambda(x, n) \, dx,$$

where the kernel $P_\lambda : NA \times N \rightarrow \mathcal{C}$ is an appropriate complex power of the poisson kernel on $NA$, namely

$$P_\lambda(x, n) = [P(x, n)]^{\frac{1}{2} - i\frac{\lambda}{Q}}.$$  

For a distribution, the Fourier-Helgason transform of $T \in \mathcal{E}'(NA)$ (see [1] and [2]) is defined by

$$(2)\quad \hat{T}(\lambda, n) = \langle T(x), P_\lambda(x, n) \rangle \quad (\lambda, n) \in \mathcal{C} \times N,$$

the above formula have a sens, because the function $x \rightarrow P_\lambda(x, n)$ is an eigenfunction of Laplace-Beltrami operator with eigenvalue $-(\lambda^2 + \frac{Q^2}{4})$ (see [6] and [7]).

This transform, for $f$ in $\mathcal{D}(NA)$, can be also written as follows (see [2])

$$(3)\quad \hat{f}(\lambda, n) = \mathcal{F}_1 \circ R_n f(\lambda),$$
where $F_1$ is the Fourier transform of one variable and $R_n f(\lambda)$ is the horocyclic Radon transform (see [2]) defined as follows,

$$
R_n f(\lambda) = e^{-\lambda \frac{Q^2}{4}} \int_N f(n \sigma(n_1 \exp(\lambda H))) \, dn_1,
$$

with $\sigma$ is the geodesic inversion (see [8]) defined by the following formula

$$
\sigma(V, Z, t) = \frac{1}{(t + \frac{|V|^2}{4})^2 + |Z|^2} [((-t + \frac{|V|^2}{4}) + J_Z)V, -Z, t],
$$

for all $(V, Z, t) \in NA$, and write the Fourier-Helgason inversion formula (see [7])

$$
f(x) = \int_{-\infty}^{+\infty} \frac{c_{m,k} 4\pi}{|c(\lambda)|^2} \int_N P_{-\lambda}(x, n) \hat{f}(\lambda, n) \, dn) \, d\lambda,
$$

where $c_{m,k} = 2^{k-1} \Gamma(\frac{2m+k+1}{2}) \frac{1}{\pi^{\frac{2m+k+1}{2}}}$ and $c(\lambda)$ is the generalized Harish-chandra function.

The expression in Parentheses in the formula (1.5) is an eigenfunctions of Laplace-Beltrami operator with eigenvalue $-\left(\lambda^2 + \frac{Q^2}{4}\right)$.

For $f \in D(NA)$, We define the spectral projector as follows

$$
\mathbb{P}_\lambda f(x) = \frac{c_{m,k} 4\pi}{|c(\lambda)|^2} \int_N P_{-\lambda}(x, n) \hat{f}(\lambda, n) \, dn)
$$

so that the spectral representation is

$$
f(x) = \int_{-\infty}^{+\infty} \mathbb{P}_\lambda f(x) \, d\lambda.
$$

For a distribution, the spectral projection of $T \in \mathcal{E}'(NA)$ is defined as follows

$$
\mathbb{P} T(x) = \frac{c_{m,k} 4\pi}{|c(\lambda)|^2} (T \ast \phi_\lambda(x)),
$$

The above formula have a sens, because $\phi_\lambda$ is the spherical function, which is $\mathcal{C}^\infty(NA)$.

**Remark.** On the unit open disk $D = \{z \in \mathbb{C}; |z| = 1\}$ of the complex plane, the Laplace-Beltrami operator $\Delta_D$ on $D$ (see [20]) can be written in terms of the Euclidean Laplacian $\Delta_{\mathbb{R}^2}$ as

$$
\Delta_D = (1 - |z|^2)^2 \Delta_{\mathbb{R}^2} = 4(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}.
$$
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For $z \in D$, let $r, \theta \in \mathbb{R}$, with $r \geq 0$ be such that $z = \tanh(r)e^{i\theta}$. Since $d(0, z) = r$, then $(r, \theta)$ are called geodesic polar coordinates of $z$. In such coordinates of $z$, the Laplace-Beltrami is

$$\Delta_D = \frac{\partial^2}{\partial r^2} + 2\coth(2r)\frac{\partial}{\partial r} + 4\sinh^{-2}(2r)\frac{\partial^2}{\partial \theta^2}$$

For the Laplacian $\Delta_D$ on $D$ (see [20]), we have

$$\Delta_D(e^{(i\lambda+1)<z,w>}) = -(\lambda^2 + 1)e^{(i\lambda+1)<z,w>}, \lambda \in \mathcal{C}.$$  

With

$$e^{<z,w>} = \left(\frac{1 - |z|^2}{1 - z \cdot w^*}\right)^{\frac{i\lambda+1}{2}}.$$ 

For $\lambda \in \mathcal{C}$, let $P_\lambda$ denote the complex power of the Poisson kernel (cf. [20] p: 3) given by

$$P_\lambda(z, w) = e^{(i\lambda+1)<z,w>}$$

$$= \left(\frac{1 - |z|^2}{1 - z \cdot w^*}\right)^{\frac{i\lambda+1}{2}},$$

One can define the Fourier-Helgason transform by

$$(1.8)' \quad \hat{f}(\lambda, w) = \int_D P_{-\lambda}(z, w)f(z)\,d\mu(z),$$

for all $\lambda \in \mathcal{C}$, $w \in S^1$ for which this integral exists, where $d\mu(z) = (1 - |z|^2)^{-2}dz$, and the Fourier-Helgason inversion formula is (cf. [20] p: 33)

$$f(z) = \frac{1}{4\pi} \int_{\mathbb{R}} \left(\int_{S^1} \hat{f}(\lambda, w)P_\lambda(z, w)\,d\sigma(w)\right)\lambda \tanh\left(\frac{\pi \lambda}{2}\right)\,d\lambda.$$ 

The expression in parentheses is an eigenfunction of the Laplacian on $D$ with eigenvalue $-(\lambda^2 + 1)$, we define the spectral projection operator on the $D$ (see [20]) as follows

$$(1.8)'' \quad I P_\lambda f(z) = \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_{S^1} \hat{f}(\lambda, w)P_\lambda(z, w)\,d\sigma(w)$$

$$= \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) Q_\lambda f(z).$$
Remark. We note that $P_{\lambda} f(z)$ is not defined at $\lambda = \frac{1}{2} i (2k+1)$ for $k \in \mathbb{Z}^*$, and has a simple zero at points $\lambda_h = \frac{1}{2} i 2h$ for $h \in \mathbb{Z}^*$ and a double zeros at $\lambda = 0$. Also the function $\lambda \rightarrow Q_{\lambda} f(z)$ is even.

Remark. For $k \in \mathbb{Z}^+$, we have

$$Q_{-i(2k+1)} f(z) = (1 - |z|^2)^{-k} \int_{S^1} \hat{f}(-i(2k + 1), w) (1 - z \cdot \overline{w})^{2k} d\sigma(w).$$

Using the Libnitz formula to the function $(1 - z \cdot \overline{w})^{2k}$, we note that the integrale in the second part of the above formula is a polynomial of $z$ and $\overline{z}$ of degree $2k$.

The aim of this work is to characterize the range of $\mathcal{E}'(NA)^\#$ (respectively of $\mathcal{S}^p(NA)^\#$ for $0 < p \leq 2$) by the spectral projection operator (see Theorem 3.2) (respectively Theorem 4.1), we find the analogous of this theorem in the case of the noncompact symmetric space of rank one (see [26] and [27]), and mainly we give an estimation of this operator in $L^2(NA)$, also, we discuss in the sense of R. Strichartz (cf. [26]), the spectral Paley-Wiener theorem on the unit open disk of the complex plane.

Now we give a full description of the organization of this paper. In section 2, we recall the main definition and the know results of spherical analysis on NA groups. In section 3, we introduce the Poisson kernel and spectral projection, we state the main results, we have obtained the characterization of the $\mathcal{E}'(NA)$-range of the spectral projection which is a generalization of theorem 3.6 in [26] (see also [27]). In section 4, we give a range of $\mathcal{S}^p(NA)^\#$ (for $0 < p \leq 2$) via spectral projection. In section 5, we discut the $L^2$-estimate for this projection (see theorem 5.1 and 5.2). In the next, we discuss in the sense of R. Strichartz (cf. [26]), the spectral Paley-Wiener theorem on the unit open disk $D$ of the complex plane as an example of the hyperbolic spaces (even case). The results and ideas will be illustrated by developing a range theorem for spectral projection operator on $D$. And mainly to characterize the $C^\infty_{com}(\overline{B_R(z_0)})$-range (where $B_R(z_0)$ is the unit ball of $C$ centered at $z_0$) of spectral projection operator $P_{\lambda}$ associated to the Laplacian $\Delta_D$ on $D$.

2 Notations and Preliminaries

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra of finite dimensional (i.e., $[\mathfrak{n}, \mathfrak{n}] \neq 0$ and $[\mathfrak{g}, [\mathfrak{n}, \mathfrak{n}]] = 0$) equipped with an inner product $\langle, \rangle$, $\mathfrak{n}$ has a center.
We have then \([V, V'] \in \mathfrak{z}\) and \([V, Z] = 0\) \(\forall V, V' \in \eta\) and \(\forall Z \in \mathfrak{z}\). We write \(\eta\) as an orthogonal sum of two spaces \(p\) and \(\mathfrak{z}\) \((\eta = p \oplus \mathfrak{z})\), we have \([p, p] \subset \mathfrak{z}\), \([p, \mathfrak{z}] = 0\) and \([\mathfrak{z}, \mathfrak{z}] = 0\). According to Kaplan \([21]\), \(\eta\) is said to be an H-type Lie algebra if for every unitary \(Z \in \mathfrak{z}\) the map \(J_Z\) of \(p\) into \(p\), defined by equality \(< J_Z V, V' >_\eta = < [V, V'], Z >_\eta\), satisfy the equality \(J_Z^2 V = -|Z|^2 V\), for all \(V \in p\). A fundamental example is the Heisenberg algebra (see \([24]\)), given by the matrix.

\[
\begin{pmatrix}
0 & v_1 & \ldots & v_k & z \\
& w_1 & \ldots & w_k \\
&(0) & \ldots & \\
& & \ldots & \\
& & & 0 \\
\end{pmatrix} = (v, w, z), \quad v, w \in \mathbb{R}^k, z \in \mathbb{R}
\]

such that \(J_{(0,0,z)}(v, w, 0) = z(-w, v, 0)\).

Note that for every unit \(Z \in \mathfrak{z}\), \(J_Z\) is a complex structure on \(p\), so that \(p\) has even dimension \(m = 2m'\), we denote by \(k\) the dimension of \(\mathfrak{z}\). Let \(N\) be the connected and simply connected group of Lie algebra \(\eta\). Since \(\eta\) is nilpotent, the exponential map is surjective, we may therefore parametrize \(N\) by \(p \oplus \mathfrak{z}\) and write \((V, Z)\) for \(\exp(V + Z)\) where \(V \in p\) and \(Z \in \mathfrak{z}\). By the Baker-Campbell-Hausdorff formula, the product law in \(N\) is given by the formula

\[(V, Z).(V', Z') = (V + V', Z + Z' + \frac{1}{2}[V, V']),\]

for all \(V, V' \in p\) and for all \(Z, Z' \in \mathfrak{z}\). Let \(dV\) and \(dZ\) the lebesgue measures on \(p\) and \(\mathfrak{z}\) respectively, the measure \(dV dZ\) is the Haar measure on \(N\) whose we denote by \(dn\). Let \(A\) be a multiplicatif group isomorphe to \(\mathbb{R}^*_+\) and \(NA\) the semi-direct product of \(N\) and \(A\) relatively to the action \((V, Z) \in \eta \mapsto (a \frac{1}{2} V, a Z)\). So the Lie group \(S = NA\) (connected and simply connected) is called a Damek-Ricci space. We denote by \((V, Z, a)\) the element \(n a = \exp(V + Z)a\), the inner law on the group \(NA\) is given by the formula

\[(V, Z, a).(V', Z', a') = (V + a \frac{1}{2} V', Z + a Z' + \frac{1}{2} a \frac{1}{2}[V, V'], aa').\]

Denote by \(Q = \frac{1}{2} m + k\), with \(Q = 2q\) the homogeneous dimension of \(N\), the left Haar measure on \(NA\) is given by \(dx = a^{-Q-1} dV dZ da = a^{-Q-1} dnda\). Note that the right Haar measure on \(NA\) is \(a^{-1} dV dZ da\), then the group \(NA\) is
nonunimodulaire, so that the modular function $\delta$ is given by $\delta(V, Z, a) = a^{-Q}$.

As Riemannian manifold, $NA$ is (see [15]) a harmonic space, the noncompact symmetric space of rank one is contained in these class of $NA$ groups, $NA \approx G/K = NAK/K$ (here $NA$ is the Iwasawa group). Also the group $S$ provide an examples of nonsymmetric hamonic spaces (see [15]). The eigenfunction may be expressed as Jacobi of parameters $\alpha$ and $\beta$ via the following formula (see [22] p. 152)

$$\Phi_s(x) = \Phi_s(\rho) = \varphi^{(\alpha, \beta)}_{2\lambda}(\rho)$$

$$= 2F_1\left(\frac{1}{2}(Q-2s), \frac{1}{2}(Q+2s); \frac{m+k+1}{2}; -\sinh^2(\rho)\right).$$

where recall that $2F_1$ is the Gauss hypergeometric function with $\alpha = \frac{m+k-1}{2}$, $\beta = \frac{k-1}{2}$ and $\lambda = -iRe(s) + Im(s)$, then $Im(\lambda) = -Re(s)$.

### 3 Poisson kernel and Spectral Projection on the Damek-Ricci space

For $n_1$ fixed in $N$, we define (see [7] p. 409), the Poisson kernel on $NA$ for $n_1$ by the formula

$$\mathcal{P}(., n_1) : NA \rightarrow \mathbb{R}$$

$$na \mapsto \mathcal{P}(na, n_1) = P_a(n_1^{-1}n),$$

where, for $a > 0$, $P_a(n)$ is a function on $N$ defined by

$$P_a(n) = P_a(V, Z) = a^Q((a + \frac{|V|^2}{4})^2 + |Z|^2)^{-Q}.$$  

We have the following properties

- $\mathcal{L}\mathcal{P}(., n_1) = 0$, $\forall n_1 \in N$
- $P_a(n) = a^{-Q}P_1(a^{-1}na)$, $\forall a \in A, \forall n \in N$.  


With these properties, one may defined the kernel $P_\lambda$, ($\lambda \in \mathbb{C}$) on $NA \times N$ as follows

$$P_\lambda : NA \times N \rightarrow \mathbb{C}$$

$$(na, \pi) \rightarrow P_\lambda(na, \pi) = P(na, \pi)^{\frac{1}{2}}$$

we define the spectral projection operator on the Damek-Ricci space and we study these properties

**Definition 3.1** Let $T$ be an element of $E'(NA)$, we define the spectral projection operator on $NA$ as follows

$$\mathcal{P}_\lambda T(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \langle T * \Phi_\lambda(x), \Phi_\lambda(y^{-1}x) \rangle$$

**Proposition 3.1** Let $T$ be an element of $E'(NA)$, then for all $\lambda \in \mathbb{C}$ we have

$$\mathcal{P}_\lambda T(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \int_N P_{-\lambda}(x, n) \hat{T}(\lambda, n) dn.$$  

**Proof.** Let $T \in E'(NA)$, from the formula 3.2, we obtain

$$\mathcal{P}_\lambda T(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \langle T(y), \Phi_\lambda(d(x, y)) \rangle$$

The spherical function $\Phi_\lambda$ satisfies to the following formula (see [24] p. 42 and [7] p. 413)

$$\Phi_\lambda(x^{-1}y) = \int_N P_{-\lambda}(x, n) P_\lambda(y, n) dn.$$  

Using the Fubini-theorem, the formula (3.4) becomes

$$\mathcal{P}_\lambda T(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \langle T(y), \int_N P_{-\lambda}(x, n) P_\lambda(y, n) dn \rangle$$

$$= \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \int_N P_{-\lambda}(x, n) \langle T, P_\lambda(y, n) \rangle dn$$

$$= \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \int_N P_{-\lambda}(x, n) \hat{T}(\lambda, n) dn.$$
Let $M : D(NA) \to D(NA)^\#$ be the averaging projector on NA (see [7], [14] and [24]) defined as follows

$$(Mf)(x) = \frac{1}{|S|} \int_{S_\rho} f(y) d\sigma_\rho(y),$$

where $d\sigma_\rho$ is the surface measure induced by the left-invariant Riemannian metric on the geodesic sphere $S_\rho = \{ y \in NA : d(y, e) = \rho \}$, normalised by $\int_{S_\rho} d\sigma_\rho(y) = 1$ and $\rho(x) = d(x, e)$. Denote by $f_x$ the function $M(\tau_x f)$ where $x, y \in NA$ and $\tau_x g(y) = g(x^{-1}y)$ is the translated function.

**Proposition 3.2** Let $x \in NA$ and $f$ be in $D(NA)$, then

$$\mathcal{P}_\lambda f(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \tilde{f}_x(\lambda)$$

where $\tilde{f}$ design the spherical Fourier transform of $f \in D(NA)^\#$.

**Proof.** Let $x$ be an element of $NA$ and $f \in D(NA)$, since $f_x$ is a radial function on NA, the spherical Fourier transform of $f_x$ is given by

$$\tilde{f}_x(\lambda) = \int_{NA} f_x(y) \Phi_\lambda(y) dy$$

$$= \int_{NA} f(x^{-1}y) \Phi_\lambda(y) dy,$$

putting $x^{-1}y = z$ to obtain

$$\tilde{f}_x(\lambda) = f \ast \Phi_\lambda(x),$$

and this prove the proposition.

Remark that for $x = e$ and $f \in D(NA)^\#$, the equality (3.7) becomes

$$\mathcal{P}_\lambda f(e) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \tilde{f}(\lambda).$$

In order to do the Paley-Wiener theorem for the spectral projection operator, we will need the following lemma.
Lemma 3.1  (Koornwinder (see [22] p. 150)) For each \( \alpha, \beta \in \mathbb{C} \) and for each non-negative integer \( n \) there exists a positive constant \( C \) such that for all \( t \geq 0 \) and all \( \lambda \in \mathbb{C} \):

\[
|\left(\Gamma(\alpha + 1)\right)^{-1}d_n^{-\alpha,\beta}(t)| \leq C(1 + |\lambda|)^n(1 + t)e^{-(\Re \lambda - \Re \beta)t},
\]

where \( \mu = \alpha + \beta + 1, k = 0 \) if \( \Re \alpha > -\frac{1}{2} \) and \( k = \frac{1}{2} - \Re \alpha \) if \( \Re \alpha \leq -\frac{1}{2} \), where the function \( t \to \varphi_{\alpha,\beta}(t) \) is the Jacobi function.

Let \( B_a(z) \) be the ball of center \( z \in NA \) and of radius \( a \) for the distance \( d \). We denote by \( C^\infty_c(B_a(z)) \) the set of function \( f \in D(NA) \) which \( \text{supp} f \) is included in \( B_a(z) \).

Lemma 3.2  If \( f \in C^\infty_c(B_a(z)) \cap D(NA)^\#, \) then \( \mathcal{P}_\lambda f(x) \) satisfies to following conditions:

1) For all \( \lambda \in \mathbb{C} \), the function \( x \to \mathcal{P}_\lambda f(x) \) is a radial function

2) \( \mathcal{L}_r \mathcal{P}_\lambda f(x) = -\lambda^2 + \varrho^2 \mathcal{P}_\lambda f(x) \) (where \( \mathcal{L}_r \) is the radial part of the Laplace-Beltrami operator)

3) for all \( \lambda \in \mathbb{C} \), we have \( \mathcal{L}_r \mathcal{P}_\lambda f(x) \) is a \( C^\infty \) function on \( \mathbb{C} \times NA \)

4) for each fixed \( x \), the function \( \mathcal{P}_\lambda f(x) \) is an entire function divisible by \( |c(\lambda)|^{-2} \) and the quotient is an analytic function

5) for every \( N_0 \) there exists \( C_{N_0} \) such that

\[
|\mathcal{P}_\lambda f(x)| \leq C_{N_0} |c(\lambda)|^{-2}(1 + |\lambda|^2)^{-N_0}e^{\Re \lambda |d(x,z)+a|}.
\]

Remark.  the above theorem hold for all dimension of \( NA \)

Proof.  The spherical function \( \Phi_\lambda(x) \) is given by the formula (see [6]).

\[
\Phi_\lambda(x) = \int_{\mathbb{C}} \mathcal{P}_{-\lambda}(x, n)\mathcal{P}_\lambda(e, n)dn = \int_{\mathbb{C}} e^{(\xi+i\lambda)(t\xi)(n^{-1}x)+(\rho-i\lambda)(t\rho)(n^{-1})}dn
\]

and the equality (6.6) shows that \( (\lambda, x) \to \mathcal{P}_\lambda f(x) \) is a \( C^\infty \) function on \( \mathbb{C} \times NA \). The formula (6.6) implies, also, that \( \mathcal{L} \mathcal{P}_\lambda f(x) = -\lambda^2 + \varrho^2 \mathcal{P}_\lambda f(x) \) because \( \Phi_\lambda \) is an eigenfunction for \( \mathcal{L}_r \) and the operator \( \mathcal{L} \) has for eigenvalue \( -(\lambda^2 + \varrho^2) \). It follows from (6.6) that \( \mathcal{P}_\lambda f(x) \) is even \( (x \text{ fixed}) \) since \( \Phi_\lambda = \Phi_{-\lambda} \), the equality 6.6 shows that \( \mathcal{P}_\lambda f(x) \) is divisible by \( |c(\lambda)|^{-2} \). Showing, now, the condition 5). Assume that \( \text{supp} f \) is included in the ball \( B_a(z) \) \((z \text{ fixed})\)
and let $\mathcal{L}_0$ be the operator defined by $\mathcal{L}_0 = -\mathcal{L} + \rho^2$ with $\rho = 2Q$, where $\mathcal{L}_r$ is the radial part of the Laplace Beltrami operator (see the introduction).

Let $r$ be the integers, then

$$I\mathcal{P}_{\lambda}^r(L_0^r f)(x) = (-1)^r \lambda^{2r} I\mathcal{P}_{\lambda} f(x).$$

But, from (6.6) we have

$$I\mathcal{P}_{\lambda}^r(L_0^r f)(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} (L_0^r f \ast \Phi_{\lambda})(x)$$

$$= \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \int_{NA} \Phi_{\lambda}(xy^{-1})(L_0^r f(y)) dy.$$ 

By (6.18), the above equality becomes

$$(-1)^r \lambda^{2r} I\mathcal{P}_{\lambda} f(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \int_{NA} \Phi_{\lambda}(xy^{-1})(L_0^r f(y)) dy.$$ 

This equality implies

$$|\lambda|^{2r} |I\mathcal{P}_{\lambda} f(x)| \leq \frac{c_{m,k}}{4\pi} (|c(\lambda)|^{-2} \sup_{y \in NA} |L_0^r f(y)|)$$

$$\times (\int_{y \in B_a(z)} |\Phi_{\lambda}(xy^{-1})| dy)$$

It follows from the Koornwinder lemma (see lemma 6.1) that

$$|\lambda|^{2r} |I\mathcal{P}_{\lambda} f(x)| \leq \frac{c_{m,k}}{4\pi} (|c(\lambda)|^{-2} (\sup_{y \in NA} |L_0^r f(y)|))$$

$$\times (|B_a(z)|) \sup_{y \in B_a(z)} e^{\left|\Im \lambda \rho(xy^{-1})\right|} dy.$$ 

Where $|B_a(z)|$ design the measure of the ball $B_a(z)$. Since $\rho(xy^{-1}) = d(x, y) \leq d(x, z) + d(z, y) \leq d(x, z) + a$, the inequality (6.19) can be transformed as follows

$$|I\mathcal{P}_{\lambda} f(x)| \leq c'_r |c(\lambda)|^{-2} (1 + |\lambda|^2)^{-r} e^{\left|\Im \lambda \rho(d(x, z) + a)}.$$

Where $c'_r$ is an other absolute constant. The third condition of the above lemma is lawful because the condition (6.13) implies that $\frac{4\pi}{c_{m,k}} (I\mathcal{P}_{\lambda} f(c(\lambda))^2) = \tilde{f}(\lambda)$ for a radial function $f$. According to the theorem 3.14 in [16], we known that the function $\lambda \rightarrow \tilde{f}(\lambda)$ is analytic.
Theorem 3.1 (Abouelaz see [1]). If \( f \in C^\infty_c(B_a(z)) \cap D(NA)^\# \), then \( IP_\lambda f(x) \) satisfies to following conditions:

1) For all \( \lambda \in \mathbb{C} \), the function \( x \rightarrow IP_\lambda f(x) \) is a radial function

2) For all \( (\lambda, x) \rightarrow IP_\lambda f(x) \) is a \( C^\infty \) function on \( \mathbb{C} \times NA \)

3) for all \( \lambda \in \mathbb{C} \), we have \( \mathcal{L}IP_\lambda f(x) = -(\lambda^2 + \rho^2)IP_\lambda f(x) \) (where \( \mathcal{L} \) is the radial part of the Laplace-Beltrami operator)

4) for each fixed \( x \), the function \( IP_\lambda f(x) \) is an entire function divisible by \( |c(\lambda)|^{-2} \) and the quotient is an analytic function

5) for every \( N_0 \) there exists \( C_{N_0} \) such that

\[
|IP_\lambda f(x)| \leq C_{N_0} |c(\lambda)|^{-2}(1 + |\lambda|^2)^{-N_0}e^{\Im \lambda(d(x,z)+a)}
\]

Conversely, if \( x \rightarrow F(\lambda, x) \) (for all \( \lambda \in \mathbb{C} \)) is a radial function and \( F(\lambda, x) \) satisfies to 1),2),3),4) and 5) then there exist \( f \in C^\infty_c(B_a(z)) \cap D(NA)^\# \) such that \( IP_\lambda f(x) = F(\lambda, x) \) for all \( (\lambda, x) \in \mathbb{C} \times NA \). Where \( \mathcal{L} \) is the radial part of the Laplace-Beltrami operator (see (2.2))

Proof. The necessary condition is proved in the above lemma. Conversely, let \( F(\lambda, x) \) be a function which satisfy to condition (1)\,...,(4), the condition (5) of the above theorem shows that

\[
(a_1) \quad |F(\lambda, x)| \leq C_{N_0} |c(\lambda)|^{-2}(1 + |\lambda|^2)^{-N_0}e^{\Im \lambda(d(x,z)+a)}.
\]

Without loss of generality we take \( z=e \) (as in the proof of theorem of R. Strichartz in [26]). The function \( x \rightarrow F(\lambda, x) \) (for \( \lambda fixed \)) is radial verifying the equality

\[
(a_2) \quad \mathcal{L}F(\lambda, x) = -(\lambda^2 + \rho^2)F(\lambda, x).
\]

The function

\[
\Psi(\lambda, x) = \frac{F(\lambda, x)}{F(\lambda, e)}.
\]

verify the equality \( (a_2) \), then

\[
F(\lambda, x) = F(\lambda, e)\Phi(\lambda, x).
\]

(where \( \Phi(\lambda, x) \) is the spherical function). Replace \( F(\lambda, x) \) by its expression in the equality \( (a_1) \), we obtain

\[
(a_3) \quad |F(\lambda, e)||\Phi(\lambda, x)| \leq C_{N_0} |c(\lambda)|^{-2}(1 + |\lambda|^2)^{-N_0}e^{\Im \lambda(r+a)}.
\]
(where \( r = \rho(x) \)). The inequality \((a_3)\) implies that
\[
(a_4) \quad |F(\lambda, e)| e^{-|\text{Im}\lambda| r} |\Phi_\lambda(x)| \leq C_{N_0} |c(\lambda)|^{-2} (1 + |\lambda|^2)^{-N_0} e^{\text{Im}\lambda |c(\lambda)|}.
\]
Integrate the inequality \((a_4)\) between 0 and \( t \) (with respect \( r \)) we have
\[
(a_5) \quad |F(\lambda, e)| \left( \int_0^t e^{-|\text{Im}\lambda| r} |\Phi_\lambda(r)| \, dr \right) \leq C_{N_0} t |c(\lambda)|^{-2} (1 + |\lambda|^2)^{-N_0} e^{\text{Im}\lambda |c(\lambda)|}.
\]
Consequently
\[
(a_6) \quad |F(\lambda, e)| \left( \frac{1}{t} \int_0^t e^{-|\text{Im}\lambda| r} |\Phi_\lambda(r)| \, dr \right) \leq C_{N_0} |c(\lambda)|^{-2} (1 + |\lambda|^2)^{-N_0} e^{\text{Im}\lambda |c(\lambda)|}
\]
for all \( \lambda \in \mathcal{C} \). But
\[
1 = \lim_{t \to 0} \left( \frac{1}{t} \int_0^t e^{-|\text{Im}\lambda| r} |\Phi_\lambda(r)| \, dr \right) \quad \text{forall} \lambda \in \mathcal{C},
\]
because
\[
\lim_{t \to 0} \left( \frac{1}{t} \int_0^t e^{-|\text{Im}\lambda| r} |\Phi_\lambda(r)| \, dr \right) = \lim_{t \to 0} \left( \frac{\Psi(t)}{t} \right)
\]
\[
= \Psi'(t) |_{t=0}
\]
\[
= |\Phi_\lambda(0)|
\]
\[
= 1
\]
with
\[
\Psi(t) = \int_0^t e^{-|\text{Im}\lambda| r} |\Phi_\lambda(r)| \, dr.
\]
Then \((a_6)\) becomes
\[
|F(\lambda, e)| \leq C_{N_0} |c(\lambda)|^{-2} (1 + |\lambda|^2)^{-N_0} e^{\text{Im}\lambda |c(\lambda)|}.
\]
By Di Blasio theorem (see [16]), there exist \( f \in \mathcal{D}(NA)^# \) such that \( \text{supp} f \subset B(e, a) \). In addition
\[
\frac{F(\lambda, e)}{|c(\lambda)|^{-2}} = \tilde{f}(\lambda) \quad \text{forall} \quad \lambda \in \mathcal{C}.
\]
Whence
\[
\tilde{f}(\lambda) = \frac{I_{PA} f(e) \ 4\pi}{|c(\lambda)|^{-2} c_{m,k}} = \frac{F(\lambda, e)}{|c(\lambda)|^{-2}}.
\]
Then

$$F(\lambda, e) = \mathbb{P}_\lambda f(e) \frac{4\pi}{c_{m,k}} = \mathbb{P}_\lambda f_1(e).$$

Since

$$F(\lambda, x) = \mathbb{P}_\lambda f_1(x) \frac{4\pi}{\mathbb{P}_\lambda f_1(e)}.$$ 

We have by the above equality

$$F(\lambda, x) = \mathbb{P}_\lambda f_1(x),$$

and $f_1 \in \mathcal{D}(NA)^\#$ with $\text{supp} f_1 \subset B(e, a).$

**Conjecture 1.** It will be very interesting to generalize the theorem 3.1 for the function $f \in \mathcal{C}_c^\infty(B_0(z)) \cap \mathcal{D}(NA).$ (see [26] and [27] for the symmetric spaces of non compact of rank one.)

**Definition 3.2** Let $T \in \mathcal{E}(NA)$, we define $\mathbb{P}_\lambda T(x)$ as function on $NA$ given by the formula

$$(21) \quad \mathbb{P}_\lambda T(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} (T * \Phi_\lambda)(x)$$

$$(22) \quad = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} < T, \Phi_\lambda (d(x, \cdot)) >,$$

for all $x \in NA$ with $d(x, y)$ denote the distance from $x$ to $y$.

**Remark 3.1** If $T \in \mathcal{E}(NA)^\#$ the above equality becomes for $x = e$

$$(23) \quad \mathbb{P}_\lambda T(e) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \tilde{T}(\lambda)$$

Where $\tilde{T}(\lambda)$ is the spherical Fourier transform (see [1]).

**Theorem 3.2** Let $T \in \mathcal{E}(NA)^\#$ such that $\text{supp} T \subset B_0(z)$, then $\mathbb{P}_\lambda T(x)$ satisfies

1) For all $\lambda \in \mathcal{C}$, the function $x \rightarrow \mathbb{P}_\lambda T(x)$ is a radial function

2) $(\lambda, x) \rightarrow \mathbb{P}_\lambda T(x)$ is a $\mathcal{C}^\infty$ function on $\mathcal{C} \times NA$.
3) for all $\lambda \in C$, we have $\mathcal{L}\mathcal{P}_\lambda T(x) = -\left(\lambda^2 + \rho^2\right)\mathcal{P}_\lambda T(x)$ (where $\mathcal{L}$ is the radial part of the Laplace-Beltrami operator)

4) for each fixed $x$, the function $\mathcal{P}_\lambda T(x)$ is an entire function divisible by $|c(\lambda)|^{-2}$ and the quotient is an analytic function

5) There exists $N_0$ and $C_{N_0}$ such that

$$|\mathcal{P}_\lambda T(x)| \leq C_{N_0} |c(\lambda)|^{-2} (1 + |\lambda|^2)^{N_0} e^{[Im\lambda](d(x,z)+a)}$$

Conversely, if $x \to F(\lambda, x)$ (for all $\lambda \in C$) is a radial function and $F(\lambda, x)$ satisfies to 1), 2), 3), 4) and 5) then there exist $T \in \mathcal{E}(NA)^\#$ with $\text{supp} T \subset B_a(z)$ such that $\mathcal{P}_\lambda T(x) = F(\lambda, x)$ for all $(\lambda, x) \in C \times NA$.

**Proof.** The proof of the conditions 1)– 4) is the same as the proof of those of theorem 6.2.

Now, showing the fifth condition of theorem, recall that for all $\varphi \in C^\infty(NA)^\#$, we have for all $\Delta \in \cup_{\lambda \in C} (\text{supp} T \subset B_a(z)$) such that $\mathcal{P}_\lambda T(x) = F(\lambda, x)$ for all $(\lambda, x) \in C \times NA$.

$$\text{sup} \lambda \in C \text{sup} \varphi \in \text{sup} T$$

with $C$ is a constant, then

$$|\mathcal{P}_\lambda T(x)| \leq c \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \sup_{y \in x^{-1}\text{supp} T} \left| \frac{d^{m_0}}{d\rho^{m_0}} \Phi_\lambda(y) \right|$$
with $c'$ is an absolute constante.

since $\{ y \in NA/|d(y, x^{-1}z) \leq a \} \subset \{ y \in NA/|\rho(y) - d(x, z) \leq a \}$, we have

$$|\mathcal{P}_\lambda T(x)| \leq c' |C(\lambda)|^{-2} \sup_{\rho(y) \leq a + d(x, z)} \left| \frac{d^{m_0}}{d\rho^{m_0}} \Phi_\lambda(y) \right|.$$ 

Then by Koornwinder lemma (see lemma 3.1), we obtain

$$|\mathcal{P}_\lambda T(x)| \leq c'_m |C(\lambda)|^{-2} (1 + |\lambda|^2 m_0 e^{|Im\lambda|/(a+d(x,z))})$$

with $c'_m$ is an absolute constante.

conversely, assume $F(\lambda, x)$ satisfies to (1),(2),(3),(4) of the above theorem, where (4) is verified for some $N_0$. We construct the distribution $T$ by the rule

$$< T, \Psi > = \int_{\mathbb{R}} \left( \int_{NA} F(\lambda, x) \Psi(x) \, dx \right) d\lambda,$$

for any test function $\Psi$. This is not an absolttely convergent integral, but we can show that $\int_{NA} F(\lambda, x) \Psi(x) \, dx \in L^1(\mathbb{R}_\lambda)$.

This does not follow directly from the condition (4) of theorem, but it is easy deduced from it if we substitute $F(\lambda, x) = (-\lambda^2)^{-m}(\mathcal{L} + \rho^2)^m F(\lambda, x)$ for all $m \in \mathbb{N}$, since $(\mathcal{L} + \rho^2)^m F(\lambda, x) = (-\lambda^2)^m F(\lambda, x)$ and $F(\lambda, x)$ verifie the condition (2). Putting $\mathcal{L}_0 = \mathcal{L} + \rho^2$, then for all $\Psi \in \mathcal{D}(NA)$ we have

$$\int_{NA} F(\lambda, x) \Psi(x) \, dx = (-\lambda^2)^{-m} \int_{NA} (\mathcal{L}_0)^m F(\lambda, x) \Psi(x) \, dx,$$

an integrating by part means to:

$$\int_{NA} F(\lambda, x) \Psi(x) \, dx = (-\lambda^2)^{-m} \int_{NA} F(\lambda, x)(\mathcal{L}_0)^m \Psi(x) \, dx$$

By the fourth condition of theorem, we have the estimate

$$|\int_{NA} F(\lambda, x) \Psi(x) \, dx| \leq C_{N_0}(1 + |\lambda|^2)^{-m} |\mathcal{P}_{2r,2l}(\lambda)|(1 + |\lambda|^2)^{N_0} e^{|Im\lambda|} \times \int_{\text{supp}\Psi} e^{|Im\lambda|d(x,z)} |(\mathcal{L}_0)^m \Psi(x)| \, dx$$
Using the Hölder inequality, the above estimate becomes
\[
| \int_{NA} F(\lambda, x) \Psi(x) \, dx | \leq C'_{N_0,r,l} \left( 1 + |\lambda|^2 \right)^{N_0 + r_0 - m} e^{a|Im\lambda|} \left( \int_{\text{supp}\Psi} e^{2|Im\lambda|d(x,z)} \, dx \right)^{1/2} \times \left( \int_{NA} |(L_0^m\Psi(x)|^2 \, dx \right)^{1/2}.
\]
where \( C'_{N_0,r,l} \) is a constant which depend of \( N_0, r \) and \( l \).

Consequently
\[
\int_{\mathbb{R}} d\lambda \int_{NA} F(\lambda, x) \Psi(x) \, dx \leq C''_{N_0,r,l} \left( \int_{\mathbb{R}} \left( 1 + |\lambda|^2 \right)^{N_0 - r_0} \right) \left( \int_{\text{supp}\Psi} e^{2|Im\lambda|d(x,z)} \, dx \right)^{1/2} \times \left( \int_{NA} |(L_0^m\Psi(x)|^2 \, dx \right)^{1/2}.
\]

Since \( \sigma \) is an arbitrary integer, then
\[
\int_{NA} F(\lambda, x) \Psi(x) \, dx \in L^1(\mathbb{R}_\lambda).
\]

Next we apply a regularization argument. we choose a function \( \tilde{\theta}_\epsilon(\lambda) \) (where \( \theta_\epsilon(x) \) is the regularised function (see [1]), and \( \theta_\epsilon(\lambda) \) is the spherical Fourier transform. From [17] theorem 3.5, we have
\[
|\tilde{\theta}_\epsilon(\lambda)| \leq C_\epsilon (1 + |\lambda|)^{-n_0} e^{a|Im\lambda|\epsilon}.
\]
The function \( \tilde{\theta}_\epsilon(\lambda)F(\lambda, x) \) verifie the conditions of theorem 3.1 in [1]. Then, there exists a function \( F_\epsilon \in D(NA)^\# \) such that: \( \text{supp}F_\epsilon \subset B_{a+\epsilon}(z) \) and \( \mathcal{P}_\lambda F_\epsilon = \tilde{\theta}_\epsilon(\lambda)F(\lambda, x) \) and \( \int_{\mathbb{R}} \tilde{\theta}_\epsilon(\lambda)F(\lambda, x) \Psi(x) \, d\lambda = F_\epsilon(x) \) as \( \theta_\epsilon \to 1 \) as \( \epsilon \to 0 \) (see [1]), we have then
\[
\int_{NA} F_\epsilon(x) \Psi(x) \, dx = \int_{NA} \left( \int_{\mathbb{R}} \tilde{\theta}_\epsilon(\lambda)F(\lambda, x) \Psi(x) \, d\lambda \right) \, dx d\lambda,
\]
when \( \epsilon \to 0 \) the above equality becomes
\[
\lim_{\epsilon \to 0} \int_{NA} F_\epsilon(x) \Psi(x) \, dx = \int_{NA} \left( \int_{\mathbb{R}} F(\lambda, x) \Psi(x) \, d\lambda \right) \, dx = \langle T, \Psi \rangle.
\]
Whence , when \( \epsilon \to 0 \), we have also
\[
\langle F_\epsilon, \Psi \rangle \to \langle T, \Psi \rangle \quad \forall \Psi \in D(NA).
\]
As \( \text{supp}F_\epsilon \subset B_{a+\epsilon}(z) \) and \( \mathcal{P}_\lambda F_\epsilon = \tilde{\theta}_\epsilon(\lambda)F(\lambda, x) \), we obtain
\[
\mathcal{P}_\lambda T(x) = F(\lambda, x) \text{ and } \text{supp}T \subset B_{a}(z), \text{ and this completes the proof.} \quad \Box
**Remark 3.2** 1): If $T = \delta_{n_0}$, with $\delta_{n_0}$ the derivation of the Dirac measure $\delta_e$, the spectral projection operator of $T = \delta_{n_0}$ becomes

$$
P_\lambda T(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \delta_{n_0} \ast \Phi_\lambda(x)$$

$$
= \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \frac{d^{n_0}}{d\rho^{n_0}} \Phi_\lambda(x).
$$

According to the Koornwinder lemma (see lemma 3.1), the above equality becomes

$$
|P_\lambda T(x)| \leq \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2}(1 + |\lambda|^2)^{n_0} e^{\text{Im} \lambda |\rho(x)|},
$$

we find then the result of [1].

2): For $x = z = e$ and $T$ a radial compactly supported distribution in $\{e\}$, the spectral projection becomes also (when $\dim NA$ is odd) (see [1])

$$
P_\lambda T(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} T(\lambda)$$

$$
\leq c_{n_0} |c(\lambda)|^{-2}(1 + |\lambda|)^{n_0}.
$$

**Conjecture 2.** do we have a generalisation of theorem 3.2 for the distributions which is not necessary radials?

### 4 Characterization of the range of $S^p(NA)^\#$ by spectral projection operator

Let $\Omega$ be a left invariant differential operator on $NA$ of order $l$, defined as follows

$$
\Omega f(x) = \sum_{j=1}^{j=l} \mu_j(x) \frac{d^j}{d\rho^j} f_0(\rho(x)) \quad \forall f \in C^\infty(NA).
$$

where $\mu_j$ (j=1,2,...,l) the $C^\infty$ functions on $NA$. From lemma 2.3 in [17] p. 28, there exists a constant $c$ depending only on $\Omega$ such that

$$
\sup_{\rho(x) \geq 0} |\mu_j(x)| \leq c.
$$
For $0 < p \leq 2$ denote by, (see [17]), $S^p(NA)^#$ the space of radial and $C^\infty$ functions $f$ on $NA$ such that

$$v_p(f, \Omega, h) = \sup_{x \in NA} e^{\frac{p}{2} \rho(x)} (1 + \rho(x))^h |\Omega f(x)| < \infty$$

for all positive integers $h$ and all left invariant differential operators $\Omega$ on $NA$.

We can define the space $S^p(NA)^#$ in a different way (see [3]) instead of (4.1) we use the condition

$$\sup_{\rho \geq 0} e^{\frac{p}{2} \rho(x)} (1 + \rho(x))^h \frac{d^l}{d\rho^l} f_0(\rho) < \infty \quad \forall f \in C^\infty$$

For $\epsilon > 0$ define $\Omega_\epsilon = \{ s \in \mathcal{C} : |\text{Res}| < \epsilon^2 \}$. Denote also by $\mathcal{H}(\Omega_\epsilon)$ the space of $C^\infty$ function $\phi$ on $\Omega_\epsilon$ such that $\phi(s) = \phi(-s)$ for all $s \in \Omega_\epsilon$ and such that

$$\mathcal{V}_\epsilon(\phi, l, h) = \sup_{|\text{Res}| < \epsilon^2} (1 + |s|)^h \frac{d^l}{ds^l} \phi(s, x)| \lesssim \infty$$

for all positive integers $h$ and $l$. consider on $\mathcal{H}(\Omega_\epsilon)$ the topology defined by the semi-normes $\mathcal{V}_\epsilon(\phi, l, h)$ (see [17] p. 34).

**Lemma 4.1** (see [17], p. 34) Let $0 < p \leq 2$ and $\epsilon = \frac{2}{p} - 1$. Then the spherical transform $f \rightarrow \tilde{f}$ is a topological isomorphims from $S^p(NA)^#$ onto $\mathcal{H}(\Omega_\epsilon)$

**Proposition 4.1** The function $f$ is radial if and only if $\mathcal{P}_\lambda f$ is radial

**Proof.** If $f$ is radial, from the inversion formula of the spherical transform the spectral projection is

$$\mathcal{P}_\lambda f(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \tilde{f}(\lambda) \Phi_\lambda(x),$$

as $\Phi_\lambda$ is radial we have then that $\mathcal{P}_\lambda f$ is radial, and the reverse follow from the rule

$$f(x) = \int_{-\infty}^{+\infty} \mathcal{P}_\lambda f(x) d\lambda.$$
It follows that from (4.2) that
\[
\mathbb{P} f(e) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \tilde{f}(\lambda).
\]

One may characterize the \( S^p(NA)^\# \)- range via the spectral projection \( \mathbb{P} f(x) \) for any \( x \in NA \) and \( \lambda \in \Omega_\epsilon \). Let now define \( \mathcal{H}(\Omega_\epsilon \times NA) \) the space of \( \mathcal{C}^\infty \) function \( F \) on \( \Omega_\epsilon \times NA \) such that \( F(-\lambda, x) = F(\lambda, x) \) and \( F(\lambda, x^{-1}) = F(\lambda, x) \) for all \( \lambda \in \Omega_\epsilon \) and \( x \in NA \) and \( \mathcal{L}_r F(\lambda, \cdot) = -\left( \frac{Q^2}{4} + \lambda^2 \right) F(\lambda, \cdot) \), and for any left invariant differential operator \( D \) on \( NA \) of order \( l \), such that
\[
p_{(\epsilon, N,D)}(F) = \sup_{|\text{Re}\lambda| < \epsilon Q^{2/4}, x \in NA} (1 + |\lambda|)^N e^{-|\text{Im}\lambda|d(x,e)} |DF(\lambda, x)|
\]
for all positive integers \( N \).

**Theorem 4.1** Let \( 0 < p \leq 2 \) and \( \epsilon = \frac{2}{p} - 1 \). Then the spectral projection transform \( f \to \mathbb{P} f \) is a topological isomorphims from \( S^p(NA)^\# \) onto \( \mathcal{H}(\Omega_\epsilon \times NA) \).

The theorem 4.1 deduce from the following Theorem after having using the closed graph theorem.

**Theorem 4.2** Let \( 0 < p \leq 2 \) and \( \epsilon = \frac{2}{p} - 1 \). There exists \( f \in S^p(NA)^\# \) such that \( \mathbb{P}\lambda f(x) = IF(\lambda, x) \) if and only if
1) a) For each \( x \in NA \) we have \( F(-\lambda, x) = F(\lambda, x) \) for all \( \lambda \in \mathcal{C}' \)

b) For each fixed \( \lambda \), we have \( F(\lambda, x) = F(\lambda, x^{-1}) \) for all \( x \in NA \)

c) \( (\lambda, x) \to F(\lambda, x) \) is radial and \( \mathcal{C}^\infty \) function on \( \Omega_\epsilon \times NA \)

2) for each \( \lambda \), we have \( \mathcal{L}_r F(\lambda, x) = -(\lambda^2 + \frac{Q^2}{4}) F(\lambda, x) \)

3) for each \( N \) and each left differential operator \( D \) of order \( l \) on \( NA \), there exists \( c_{N,D} \) such that
\[
|DF(\lambda, x)| \leq c_{\epsilon,N,D}|c(\lambda)|^{-2}(1 + |\lambda|)^{-N+l} e^{\text{Im}\lambda|d(e,x)} \quad \text{if} \quad |\text{Re}\lambda| < \epsilon Q^{2/4}.
\]
Proof. Assume that \( f \in \mathcal{S}^p(NA)^\# \) (0 < \( p \leq 2 \)), since \( f \) is radial, the spectral projection becomes

\[
P_\lambda f(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \tilde{f}(\lambda) \Phi_\lambda(x).
\]

From the above lemma, we have that \( \tilde{f}(\lambda) \) is \( C^\infty \) on \( \Omega_\epsilon \) and since \( \Phi_\lambda(x) \) is \( C^\infty \) function on \( \Omega_\epsilon \times NA \), then 1) and 2) of the theorem follows immediatly. Now showing the third condition. From the lemma 4.1, there exists a constant \( C \) such that

\[
|\frac{d^k}{d\lambda^k} \tilde{f}(\lambda)| \leq C(1 + |\lambda|)^{-N} \quad if \quad |Re\lambda| < \epsilon \frac{Q}{2}
\]

for all positive integers \( N \) and \( k \), where \( C \) depend of \( N, \epsilon, k \) but not of \( \lambda \). According to the Koornwinder lemma, there exists a constant \( C'' \) such that

\[
|\frac{d^n}{dr^n} \Phi_\lambda(r)| \leq C''(1 + |\lambda|)^n e^{r|Im\lambda|}(1 + r)e^{-r \frac{m+2k}{2}},
\]

we have \((1 + r)e^{-r \frac{m+2k}{2}} \to 0 \) as \( r \to \infty \). Then the inequality 4.5 becomes

\[
|\frac{d^n}{dr^n} \Phi_\lambda(r)| \leq C''(1 + |\lambda|)^n e^{r|Im\lambda|},
\]

with \( C'' \) is an other constant. Using the formulas 4.5 and 4.6 to obtain that

\[
|\Omega P_\lambda f(x)| = \left| \sum_{j=1}^l \mu_j(x) \frac{d^j}{d\rho^j} P_\lambda f(\rho) \right|
\]

\[
= \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \left| \sum_{j=1}^l \mu_j(x) \frac{d^j}{d\rho^j} \Phi_\lambda(x) \right| |\tilde{f}(\lambda)|.
\]

Then from the formulas (4.4) and (4.6), there exists a constant \( c'_{\epsilon,N} \) such that

\[
|\Omega P_\lambda f(x)| = \left| \Omega P_\lambda f(\rho) \right|
\]

\[
\leq c'_{\epsilon,N} \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2}(1 + |\lambda|)^{-N+1} e^{r|Im\lambda|\rho} \quad if \quad |Re\lambda| < \epsilon \frac{Q}{2}.
\]

for all positive integers \( N \), with \( l \) is the order of \( \Omega \).

Conversely, assume that there exists a radial function \( IF(\lambda, x) \) that satisfies to 1), 2) and 3) of the theorem. putting

\[
\Psi(\lambda) = \frac{4\pi}{c_{m,k}} \frac{IF(\lambda, x)}{\Phi_\lambda(x)} |c(\lambda)|^{-2},
\]

with \( c_{m,k} \) is a constant depend of \( m, k \).
Spectral analysis on Damek-Ricci space

since

\[ \mathcal{L}_r \left( \frac{\mathcal{F}(\lambda, x)}{\mathcal{F}(\lambda, e)} \right) = -(\lambda^2 + \frac{Q^2}{4}) \frac{\mathcal{F}(\lambda, x)}{\mathcal{F}(\lambda, e)}. \]

and \( \mathcal{F}(\lambda, x) \) is radial, we have

\[ (31) \quad \mathcal{F}(\lambda, x) = \mathcal{F}(\lambda, e) \phi_\lambda(x). \]

From the fourth condition of theorem, we have for all \( N \in \mathbb{N} \)

\[ |\mathcal{F}(\lambda, x)| \leq c'_{\epsilon,N} (1 + |\lambda|)^{-N} e^{\nu |\lambda|d(e,x)}, \]

according to the formula (4.7), the above formula becomes

\[ |c(\lambda)|^2 |\phi_\lambda(x)||\Psi(\lambda)| \leq c'_{\epsilon,N} (1 + |\lambda|)^{-N+l} e^{\nu |\lambda|d(e,x)}. \]

Since \( \sup_{\rho \geq 0} \Phi_\lambda(\rho) \leq c' e^{\nu |\lambda|} \) (see formula (4.6)) and that there exists a constant \( C_1 \) and a constant \( b \) (see formula 7.4 in [6], p. 25 and [17] p. 37) such that

\[ |c(\lambda)|^{-2} \leq C_1 (1 + |\lambda|)^b \quad |Re\lambda| \leq \frac{\epsilon Q}{2} \]

we obtain then , for any \( N \), that

\[ |\Psi(\lambda)| \leq C''_{\epsilon,N} (1 + |\lambda|)^{-N+l+b}. \]

Where \( C''_{\epsilon,N} \) is constant which depend only of \( N, \Omega \) and \( \epsilon \). Since \( N \) is an arbitrary positive integer and according to the lemma 4.1, there exists a function \( f \in S^p(NA)^\# \) with \( 0 < p \leq 2 \) such that \( \tilde{f}(\lambda) = \Psi(\lambda) \), then

\[ \tilde{f}(\lambda) = \frac{4\pi}{c_{m,k}} \frac{\mathcal{F}(\lambda, x)}{\phi_\lambda(x)} |c(\lambda)|^{-2}, \]

consequently,

\[ \mathcal{F}(\lambda, x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \phi_\lambda(x) \tilde{f}(\lambda), \]

which is equal to \( \mathcal{F}_\lambda f(x) \) since \( \mathcal{F} \) is radial.
5 \textbf{L}^2\text{-Estimation for spectral projection operator}

The aim of this section is to do the \( L^2 \)-estimation for spectral projection. Recall that the Plancherel’s formula (see [17]) for Fourier spherical transform of \( f \in L^2(NA)^\# \) is

\begin{equation}
||f||_2^2 = \frac{c_{m,k}}{2\pi} \int_0^\infty |\tilde{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda.
\end{equation}

(32)

\textbf{Theorem 5.1} For \( x \in NA \) and \( f \in L^2(NA) \), the following inequality holds

\begin{equation}
\int_0^\infty |\mathcal{P}_\lambda f(x)|^2 |c(\lambda)|^2 d\lambda \leq \frac{c_{m,k}}{8\pi} ||f||^2_{L^2(NA)}
\end{equation}

(33)

\textbf{Proof.} Let \( f \in L^2(NA) \) and \( f_x(y) = M(\tau_{x^{-1}}f)(y) \) the averaging function of the translated function \( \tau_{x^{-1}}f \) (see [24]), we remark that if \( f \in L^2(NA) \) then \( f_x \in L^2(NA)^\# \) for any \( x \in NA \) (since \( NA \) is endowed with a left Haar measure and \( ||Mf||_2 \leq ||f||_2 \) (see [14])). From this, the formula 5.1 becomes

\begin{equation}
||f_x||_2^2 = \frac{c_{m,k}}{2\pi} \int_0^\infty |\tilde{f}_x(\lambda)|^2 |c(\lambda)|^{-2} d\lambda,
\end{equation}

and since

\[ \mathcal{P}_\lambda f(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} (f \ast \Phi_\lambda)(x) \]

\[ = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \tilde{f}_x(\lambda). \]

Using the above equality in the formula (5.3) to obtain that, for every \( x \in NA \)

\begin{align*}
||f_x||_2^2 &= ||M(\tau_{x^{-1}}f)||_2^2 \\
&= \frac{8\pi}{c_{m,k}} \int_0^\infty |\mathcal{P}_\lambda f(x)|^2 |c(\lambda)|^2 d\lambda.
\end{align*}

From the Proposition 1.3 in [14], the following properties hold for every \( f \in L^p(NA) \), with \( 1 \leq p \leq \infty \)

\[ ||Mf||_p \leq ||f||_p. \]
Then, we will have the following formula for any $x \in NA$

\begin{equation}
\int_0^\infty |\mathcal{P}_\lambda f(x)|^2 |c(\lambda)|^2 d\lambda \leq \frac{c_{m,k}}{8\pi} ||\tau_{x^{-1}} f||_2^2
\end{equation}

\begin{equation}
= \frac{c_{m,k}}{8\pi} ||f||_2^2,
\end{equation}

\[\square\]

**Theorem 5.2** Let $K$ be a compact set of $NA$ and $x$ an element of a compact $K$, we assume that $f \in L^2(NA)$, then we have the following estimate

\[\int_{-\infty}^{\infty} |c(\lambda)|^2 |\mathcal{P}_\lambda f(x)|^2 d\lambda \leq \frac{2^m \pi^{m+k}}{\Gamma(\frac{m+k}{2})} \frac{c_{m,k}}{4\pi} c(K) ||f||_2^2,\]

with $c(K)$ a constant which depend only of $K$.

**Proof.** Let $f$ be an element of $L^2(NA)$ and $x$ an element of a compact $K$ in $NA$, if we use the spectral projector as (see formula (1.6))

\begin{equation}
\mathcal{P}_\lambda f(x) = \frac{c_{m,k}}{4\pi} |c(\lambda)|^{-2} \int_N \mathcal{P}_{-\lambda}(x, n) \hat{f}(\lambda, n) dn.
\end{equation}

we will have the same result as the proposition 5.1.

Using the Hölder inequality in the formula (5.5) to obtain that

\[
|\mathcal{P}_\lambda f(x)|^2 = \left(\frac{c_{m,k}}{4\pi}\right)^2 |c(\lambda)|^{-4} \left(\int_N |\mathcal{P}_{-\lambda}(x, n)\hat{f}(\lambda, n) dn\right)^2
\]

\[
\leq \left(\frac{c_{m,k}}{4\pi}\right)^2 |c(\lambda)|^{-4} \left(\int_N |\mathcal{P}_{-\lambda}(x, n)|^2 dn\right) \times
\]

\[
(\int_N |\hat{f}(\lambda, n)|^2 dn)
\]

\[
\leq \left(\frac{c_{m,k}}{4\pi}\right) |c(\lambda)|^{-2} \int_N |\mathcal{P}_{-\lambda}(x, n)|^2 dn \times
\]

\[
\left(\frac{c_{m,k}}{4\pi}\right) \int_N |\hat{f}(\lambda, n)|^2 |c(\lambda)|^{-2} dn
\]

then

\begin{equation}
|c(\lambda)|^2 |\mathcal{P}_\lambda f(x)|^2 \leq \left(\frac{c_{m,k}}{4\pi}\right) \int_N |\mathcal{P}_{-\lambda}(x, n)|^2 dn \times
\]

\[
\left(\frac{c_{m,k}}{4\pi}\right) \int_N |\hat{f}(\lambda, n)|^2 |c(\lambda)|^{-2} dn.
\end{equation}
According to the Plancherel formula for Fourier-Helgason transform (see [7]) and since $P_{-\lambda}(x, n) \in L^2(N)$ (see [24], p. 44), we obtain that, for all compact $K$ of $NA$, there exists a constant $c(K)$ such that
\[|P_{-\lambda}(x, n)| \leq c(K)e^{|(t_o)\sigma|n} \in L^2(N),\]
the formula (5.6) becomes
\[\int_{-\infty}^{\infty} |c(\lambda)|^2 |IP_{\lambda}f(x)|^2 d\lambda \leq \frac{c_{m,k}}{4\pi} c(K)||f||_2(\int Ne^{2(\rho(t_o)n)} dn),\]
since $\int Ne^{2(\rho(t_o)n)} dn = 2^{-k} |S^{m+k-1}| = 2^{-k} 2n^{-1} \pi^{n-1} \Gamma\left(\frac{n-1}{2}\right)$ (with $n = m + k + 1$)(see [24], p. 44), then
\[\int_{-\infty}^{\infty} |c(\lambda)|^2 |IP_{\lambda}f(x)|^2 d\lambda \leq 2^{-k} 2n^{-1} \pi^{n-1} \Gamma\left(\frac{n-1}{2}\right) \frac{c_{m,k}}{4\pi} c(K)||f||_{L^2(NA)}.\]

6 Description of the eigenspace of the invariant Laplacian $\Delta$ on $D$

For every complex number $\lambda \in \mathbb{C}$, let $E_\lambda(D)$ be the space of all eigenfunctions of $\Delta_D$ in $D$ with eigenvalue $-(\lambda^2 + 1)$. Since the operator $\Delta_D$ is elliptic in $D$, the elements of $E_\lambda(D)$ are $C^\infty$-functions on $D$ i.e.,
\[(39) \quad E_\lambda(D) = \{ F \in C^\infty(D) ; \Delta_D F = -(\lambda^2 + 1)F \} .\]

Now, let $H_k$ denotes the space of restrictions to $S^1 = \partial D$ of harmonic polynomials $z^k$ and $\overline{z}^k$ which are homogeneous of degree $k$ in $z$. Then, it is well known that $H_k$ is $SO(2)$-irreductible and we have $L^2(S^1) = \bigoplus_{k \in \mathbb{Z}} H_k$.

**Proposition 6.1** (see [9]) A function $F$ is in the eigenspace $E_\lambda(D)$, if and only if $F$ can be expanded in $C^\infty(D)$ as
\[(40) \quad F(z) = \sum_{k \in \mathbb{Z}} e^{ik\theta} a_k(\lambda)(\tanh r)^{|k|} 2F_1\left(\frac{1 + i\lambda}{2}, \frac{1 - i\lambda}{2} ; 1 + |k| ; -\sin^2(r)\right) ,\]
where $a_k(\lambda)$ is a constant which depend only of $k$ and $\lambda$. 

Proof. See [9] for the proof of this proposition.

The generalized spherical function is given by

\begin{equation}
\Phi_{\lambda,k}(\tanh r) = \int_{S^1} \mathcal{P}_{\lambda}(z, e^{i\theta}) e^{ik\theta} d\sigma(\theta) \tag{41}
\end{equation}

\begin{align*}
&= (1 - (\tanh r)^2) \frac{\Gamma(|k| + \frac{1+i\lambda}{2})}{\Gamma(\frac{1+i\lambda}{2})} \frac{\Gamma(1+|k|)}{|k|!} 2F_1\left(\frac{1 + i\lambda}{2}, \frac{1 - i\lambda}{2}; 1 + |k|; -\sinh^2(r)\right) \\
&\quad + \frac{1 + i\lambda}{2}; 1 + |k|; (\tanh(r))^2 \\
&= |\tanh r|^{|k|} \frac{\Gamma(|k| + \frac{1+i\lambda}{2})}{\Gamma(\frac{1+i\lambda}{2})} \frac{\Gamma(1+|k|)}{|k|!} 2F_1\left(\frac{1 + i\lambda}{2}, \frac{1 - i\lambda}{2}; 1 + |k|; -\sinh^2(r)\right).
\end{align*}

Now, let \( X_{k,\lambda} \) denote the one-dimensional space spanned by the function

\begin{equation}
\Phi_{\lambda,k}(\tanh r)e^{ik\theta}. \tag{42}
\end{equation}

We note that

\begin{equation}
E_{\lambda} = \bigoplus_{k \in \mathbb{Z}} X_k. \tag{43}
\end{equation}

Let \( n \) be chosen fixed in \( \mathbb{Z} \) so that we have

\begin{equation}
E_{\lambda} = \bigoplus_{k \geq |n|} X_k \oplus \bigoplus_{k \leq |n|} X_k, \tag{44}
\end{equation}

Let

\begin{equation}
E'_n = \bigoplus_{k \geq |n|} X_k \quad \text{and} \quad E''_n = \bigoplus_{k \leq |n|} X_k, \tag{45}
\end{equation}

We denote by

\begin{equation}
\bar{E}_{\lambda}(D) = \{ F \in \mathcal{C}^\infty(D); (\Delta_D + \lambda^2 + 1)^2 F = 0 \}. \tag{46}
\end{equation}

It is easy to see that \( E_{\lambda}(D) \subset \bar{E}_{\lambda}(D) \).

We need to describe the functions in \( \bar{E}_{\lambda}(D) \) that are not in \( E_{\lambda}(D) \). We observe that: If \( g_{\lambda} \in E_{\lambda}(D) \) then \( \frac{d}{d\lambda} g_{\lambda} \in \bar{E}_{\lambda}(D) \). In fact, it follows from

\begin{equation}
0 = \frac{d}{d\lambda} (\Delta_D + \lambda^2 + 1) g_{\lambda} = (\Delta_D + \lambda^2 + 1) \frac{d}{d\lambda} g_{\lambda} + 2\lambda g_{\lambda}. \tag{47}
\end{equation}
We say that $f$ is $SO(2)$-finite if $f(tanh(re^{i\theta}))$ can be expanded into a finite spherical series expansion with respect to $e^{i\theta}$ to get

$$f(tanh(re^{i\theta})) = \sum_{|k| \leq m_0} f_{\lambda,k}(tanh(r))e^{ik\theta}$$

is $C^\infty(\mathbb{R}^+ \times S^1)$, for certain $m_0 \in \mathbb{Z}^+$

**Lemma 6.1** The following assertion is equivalent

i) $f(z)$ is $SO(2)$-finite

ii) $\hat{f}(\lambda, w)$ is $SO(2)$-finite

iii) $I_P f(z)$ is $SO(2)$-finite.

**Proof.** Let $f(z)$ be $SO(2)$-finite, then

$$\hat{f}(\lambda, w) = \int_D \mathcal{P}_{-\lambda}(z, w)f(z)\,d\mu(z)$$

$$= \sum_{|k| \leq m_0} \int_D \mathcal{P}_{-\lambda}(z, w)f_{\lambda,k}(tanh(r))e^{ik\theta}\,d\mu(z) \quad \text{with} \quad z = tanh(re^{i\theta})$$

$$= \sum_{|k| \leq m_0} \int_D \left(\frac{1 - |z|^2}{1 - z \cdot w}\right)^{\frac{1+i\lambda}{2}} f_{\lambda,k}(tanh(r))e^{ik\theta}\,d\mu(z)$$

$$= \sum_{|k| \leq m_0} \left(\int_0^1 (1 - r^2)^{\frac{1+i\lambda}{2}-2} f_{\lambda,k}(tanh(r)) \, dr\right) \left(\int_{S^1} \frac{1 - r^2}{|1 - r \cdot w|^2} \frac{1+i\lambda}{2} e^{ik\theta} \, d\sigma(\theta)\right)$$

$$= \sum_{|k| \leq m_0} a_{\lambda,k} w^k$$

where $w^k = e^{ik\varphi}$ and $a_{\lambda,k} = \frac{\Gamma(|k|+\frac{1+i\lambda}{2})}{\Gamma\left|\frac{1+i\lambda}{2}\right|} f_0(1-r^2)^{\frac{1-i\lambda}{2}-2} f_{\lambda,k}(tanh(r))_2 F_1(\frac{1+i\lambda}{2}, |k|+\frac{1+i\lambda}{2}; 1+|k|; (tanh(r))^2) \, dr$, then the assertion i) implique ii).

The assertion ii) $\rightarrow$ iii) is obtained from the formula of $I_P f(z)$ and the same proof as the above. The assertion iii) $\rightarrow$ i) is deduced from the inversion formula in the formula $I_P f(z)$.

Let $B_R(z_0)$ design the ball of radius $R$ and center $z_0$.

# 7 Spectral projection operator on $C^\infty_{com}(D)$ associated to the Laplacian $\Delta$

In this section, we define the spectral projection operator $I_P f(z)$ on $C^\infty_{com}(D)$ associated to the Laplacian $\Delta_D$, we give an expression exhibits $I_P f(z)$ as a
meromorphic function of $\lambda$, making appear the poles and zeros of $I_P f(z)$. We begin by giving the necessary condition.

**Theorem 7.1** Suppose $f$ is $C^\infty$ with support in $B_R(z_0)$, $f$ is $SO(2)$-finite then

1) $I_P f(z)$ is $C^\infty$ function on $\mathbb{C} - i\mathbb{Z}$ \times $D$

2) for each fixed $\lambda \in \mathbb{C} - i\mathbb{Z}$, we have $\Delta_D I_P f(z) = -(\lambda^2 + 1)I_P f(z)$

3) for each fixed $z$, $I_P f(z)$ is an even function meromorphic function of $\lambda$ with at worst simple poles at $\lambda_k = \frac{\pm i(2k+1)}{1}$ for $k \geq |n|$ where $n$ is choosed fixed in $\mathbb{Z}$, and

$$\sum_{k \in \mathbb{Z}} \text{Res}_{\lambda_k} I_P f(z) = 0$$

4) for every $N$ there exists $c_N$ such that

$$|I_P f(z)| \leq c_N (1 + |\lambda|)^{-N} e^{(R + d(z,z_0))|Im\lambda|}$$

5) $I_P f(z)$ has a simple zeros at points $\lambda_l = \frac{\pm i2l}{1}$ ($l \in \mathbb{Z}^*$) and a double zero at $\lambda = 0$ and satisfies

- $z \to \text{Res}_{\lambda = \lambda_k} I_P f(z) \in E'_n$
- $(I_P f(z) - (\lambda - \lambda_k)^{-1}\text{Res}_{\lambda = \lambda_k})|_{\lambda = \lambda_k} \in \tilde{E}'_n$
- $I_P f(z)|_{\lambda = 0} = 0$ and $\frac{I_P f(z)}{\Gamma(|k| + \frac{1+|\lambda|}{2})\Gamma(|k| + \frac{1-|\lambda|}{2})}$ has even entire expansion.

In order to do the proof of this theorem, we need some preparatory results

**Proposition 7.1** Let $f$ be an element of $C^\infty_\text{com}(D)$, then

(50) $I_P f(z) = (f \ast \varphi)(z) = \int_D \varphi(d(z,z')) f(z') \, dz'$,

where

(51) $\varphi_{\lambda}(\tanh r) = \frac{(2\pi)^{(\frac{3}{4})}}{4\pi^2} \lambda \tanh(\frac{\pi \lambda}{2}) \Phi_{\lambda}^{(0,0)}(r)$

$$= \frac{(2\pi)^{(\frac{3}{4})}}{4\pi^2} \lambda \tanh(\frac{\pi \lambda}{2}) P_{\frac{1}{4}(1+\lambda)}(\cosh(2r)),$$

and $P_{\nu}$ denote the Legendre function of the first kind with parameter $\nu$. 

Proof. Let \( f \) be an element of \( C^\infty_{com}(D) \), the equality (1.8)” in combination with (1.8)” gives

\[
I_P^\lambda f(z) = \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_{S^1} \mathcal{P}_\lambda(z, w) \left[ \int_D f(z') \mathcal{P}_{-\lambda}(z', w) \, dz' \right] d\sigma(w).
\]

According to the Fubini theorem, the above equality becomes

\[
I_P^\lambda f(z) = \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_D f(z') \left[ \int_{S^1} \mathcal{P}_\lambda(z, w) \mathcal{P}_{-\lambda}(z', w) \, d\sigma(w) \right] dz'
\]

\[
= \int_D \varphi_\lambda(d(z, z')) f(z') \, dz',
\]

where \( d(z, z') \) denotes the distance from \( z \) to \( z' \), and \( \varphi_\lambda \) is a multiple of the usual spherical function, because \( \varphi_\lambda(0) = \frac{1}{2} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \). The basic formula for \( \varphi_\lambda \) is

\[
\varphi_\lambda(d(z, z')) = \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_{S^1} \mathcal{P}_\lambda(z, w) \mathcal{P}_{-\lambda}(z', w) \, d\sigma(w),
\]

by taking \( z' = (0, 0) \) and \( z = \tanh re^{i\theta} \) to get

\[
\varphi_\lambda(\text{tanh}(r)) = \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_{S^1} \mathcal{P}_\lambda(\text{tanh} re^{i\phi}, w) \, d\sigma(w),
\]

if we substitute \( w = e^{i\theta} \), we obtain ( cf. [20] p: 38)

\[
\varphi_\lambda(\text{tanh}(r)) = \frac{1}{4\pi^2} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_0^\pi (\cosh(2r) - \sinh(2r) \cos \theta)^{-\frac{1}{2} + \frac{i}{2} \lambda} \, d\theta.
\]

In his article ( cf. [26] p: 80 formula 4.5) R. Strichartz shows that

\[
\int_0^\pi (\cosh(2r) - \sinh(2r) \cos \theta)^{-\frac{1}{2} + \frac{1}{2} i \lambda} \, d\theta = (2\pi)^{\frac{1}{2}} \mathcal{P}^{0}_{\frac{1}{2} + \frac{1}{2} i \lambda}(\cosh 2r)
\]

\[
= (2\pi)^{\frac{1}{2}} \mathcal{P}^{0}_{\frac{1}{2} + \frac{1}{2} i \lambda}(2 \cosh r)
\]

where \( \mathcal{P}^{\mu}_{\nu} \) denotes the Legendre functions.

\[
\mathcal{P}^{\mu}_{\nu}(\cosh r) = \frac{2^\nu}{\Gamma(1 - \nu)} (\sinh r)^{-\nu} \, _2F_1(1 - \nu + \mu, -\nu - \mu; 1 - \nu; 1 - \cosh(2r))).
\]
Whence

\[
P_{-\frac{1}{2}(\lambda+1)}(\cosh r) = 2F_1\left(\frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; \frac{1}{2}(1-\cosh(2r))\right)
\]

\[
= 2F_1\left(\frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; 1; \sinh^2 r\right)
\]

\[
= 2F_1\left(\frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; 1; \frac{\tanh^2 r}{\tanh^2 - 1}\right)
\]

\[
= (1-\tanh^2 r)^{\frac{1+i\lambda}{2}} 2F_1\left(\frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; 1; \tanh^2 r\right),
\]

since \(2F_1(a,b;c;z) = (1-z)^{-a} 2F_1(a,c-b;c;z)\).

and \(\phi^{(a,b)}\) denote the Jacobi function

(59)

\[\phi^{(a,b)}_\lambda = F\left(\frac{a+b+1+i\lambda}{2}, \frac{a+b+1-i\lambda}{2}; a+1; -\sinh^2(2r)\right).\]

Now we give an expression exhibits \(\mathcal{P}_\lambda f(z)\) as a meromorphic function of \(\lambda\), making appear zeros and poles.

**Proposition 7.2** If \(f\) is an element of \(\mathcal{C}_\com^\infty(\overline{B_R(z_0)})\), \((f\) be assumed of the form \(f_n(\tanh(r))e^{in\theta}, (n \in \mathbb{Z})\)) then

(60)

\[
\mathcal{P}_\lambda f(z) = \gamma(\lambda,n)(\tanh r)^{|n|}\phi^{(|n|-|n|)}_\lambda(r) e^{in\theta} \cdot \int_0^R f_n(\tanh s)\phi^{(|n|-|n|)}_\lambda(s) \times (\tanh s)^{|n|}\sinh(2s) ds
\]

with

\[
\gamma(\lambda,n) = \frac{1}{(|n|!)^2} \frac{1}{8\pi^2} \lambda \sinh\left(\frac{\pi\lambda}{2}\right) \Gamma(|n|) \Gamma\left(\frac{1+i\lambda}{2}\right) \Gamma\left(|n| + \frac{1-i\lambda}{2}\right),
\]

and \(f_n(\tanh r)\) design the Fourier's coefficient in the the finite series of Fourier of \(f\).

**Proof of proposition 7.2** Without loss of generality we can assume that \(z_0 = e = (0,0)\). Let \(f\) be an element of \(\mathcal{C}_\com^\infty(D)\) with \(\text{supp} f\) in \((\overline{B_R(e)})\).

By combining the formulas of \(\mathcal{P}_\lambda f\) (see Remark), we see that

(61)

\[\mathcal{P}_\lambda f(z) = \int_{\mathcal{S}} P_\lambda(z,w)\Psi_\lambda(w) d\sigma(w),\]
where

\begin{equation}
\Psi_\lambda(w) = \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_D f(z') \mathcal{P}_{-\lambda}(z', w) \, dz'.
\end{equation}

Recall that the hyperbolic area measure on $D$ in geodesic polar coordinate is given by

$$
\frac{1}{2} \sinh(2r) \, dr \, d\sigma(\theta) = \frac{1}{4\pi} \sinh(2r) \, dr \, d\theta.
$$

In such coordinates, the formula (7.15) becomes

\begin{align*}
\Psi_\lambda(e^{i\varphi}) &= \frac{1}{4\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \int_0^R \int_{S^1} f_n(tanh r') \mathcal{P}_{-\lambda}(tanh r' e^{i\varphi}, e^{i\varphi}) \\
&\quad \times e^{i\varphi} \frac{1}{2} \sinh(2r') \, dr' \, d\sigma(\theta') \\
&= \frac{1}{8\pi} \int_0^R f_n(tanh r') \left[ \left( \int_{S^1} \mathcal{P}_{-\lambda}(tanh r' e^{i\varphi}, e^{i\varphi}) \\
&\quad \times e^{i\varphi} d\sigma(\theta') \right) \sinh(2r') \, dr' \right] \\
&= \frac{1}{8\pi} \int_0^R f_n(tanh r') \phi_{-\lambda,n}(r') e^{i\varphi} \sinh(2r') \, dr' \\
&= \left( \frac{1}{8\pi} \int_0^R f_n(tanh r') \phi_{-\lambda,n}(r') \sinh(2r') \, dr' \right) e^{i\varphi} \\
&= K_{\lambda,n}(R) e^{i\varphi}.
\end{align*}

because $f$ is of the form $f_n(tanh(r)) e^{i\varphi}$, ($n \in \mathbb{Z}$), with $f_n(tanh(r))$ a $C^\infty$-function of support in $[-R, R]$. If we substitute the above formula in formula (7.14) we obtain

\begin{align*}
\mathcal{I}_\lambda f(z) &= K_{\lambda,n}(R) \int_{S^1} \mathcal{P}_{\lambda}(tanh re^{i\varphi}, e^{i\varphi}) e^{i\varphi} \, d\sigma(\varphi) \\
&= K_{\lambda,n}(R) \phi_{\lambda,n}(r) e^{i\varphi}
\end{align*}

According to the previous proposition, we deduce easily

\begin{equation}
\mathcal{I}_\lambda f(z) = \gamma(\lambda, n) (tanh r)^{|n|} \phi_{\lambda}^{(|n|,-|n|)}(r) e^{i\varphi} \int_0^R f_n(tanh(s)) \phi_{\lambda}^{(|n|,-|n|)}(s) \\
\times (tanh s)^{|n|} \sinh(2s) \, ds
\end{equation}

\textit{Spectral analysis on Damek-Ricci space}
Spectral analysis on Damek-Ricci space

\begin{equation}
\gamma(\lambda, n) = \frac{1}{(|n|!)^2} \frac{1}{8\pi} \lambda \tanh\left(\frac{\pi \lambda}{2}\right) \frac{\Gamma(|n| + \frac{1+i\lambda}{2}) \Gamma(|n| + \frac{1-i\lambda}{2})}{\Gamma(\frac{1+i\lambda}{2}) \Gamma(\frac{1-i\lambda}{2})}.
\end{equation}

It is clear that the expressions (7.17) and (7.18) exhibits $IP_{\lambda}f(z)$ as a meromorphic function of $\lambda$, with poles and zeros at exactly the points where $\gamma(\lambda, n)$ has poles and zeros, since $\phi_{\lambda}^{(|n|,-|n|)}$ is an entire function of $\lambda$. Recall that

\begin{equation}
\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)},
\end{equation}

then

\begin{equation}
\cosh\left(\frac{\pi \lambda}{2}\right) = \frac{\sinh\left(\pi \frac{1+i\lambda}{2}\right)}{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}
\end{equation}

From this, the formula (7.18) becomes

\begin{equation}
\gamma(\lambda, n) = \frac{1}{(|n|!)^2} \frac{1}{8\pi^2} \lambda \sinh\left(\frac{\pi \lambda}{2}\right) \frac{\Gamma(|n| + \frac{1+i\lambda}{2}) \Gamma(|n| + \frac{1-i\lambda}{2})}{\Gamma(\frac{1+i\lambda}{2}) \Gamma(\frac{1-i\lambda}{2})},
\end{equation}

Remark. We note that $IP_{\lambda}f(z)$ has a simple poles at $\lambda_k = \pm i(2k+1)$ for $k \geq |n|$, and a simple zero at a points $\lambda_h = \pm i2h$ for $h \in \mathbb{Z}^*$ and a double zero at $\lambda = 0$. From the formula (7.19), we deduce that $\frac{IP_{\lambda}f(z)}{\Gamma(|n| + \frac{1+i\lambda}{2})\Gamma(|n| + \frac{1-i\lambda}{2})}$ has an even entire expansion, since the collection of function $\{\phi_{\lambda}^{(|n|,-|n|)}\}$ has no zeros and no poles.

**Proposition 7.3** Let $f$ be an element of $C_{com}^\infty(\overline{B_R(z_0)})$, then there exist an invariant subspace $E'_n$ of $E_{\lambda}$ in which we give the condition $k \geq |n|$, such that

\begin{equation}
z \rightarrow Res_{\lambda=\lambda_k} IP_{\lambda}f(z) \in \oplus_{k \geq |n|} X_k = E'_n.
\end{equation}

**Proof.** For any function $f \in C_{com}^\infty(\overline{B_R(z_0)})$, the “spectral projection” function $IP_{\lambda}f(z)$ has the property that

\[(\Delta + \lambda^2 + 1)IP_{\lambda}f(z) = 0 \text{ for } \lambda \neq \lambda_k,
\]
then
\[
\Delta \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z) = \Delta \lim_{\lambda \to \lambda_k} (\lambda - \lambda_k) \mathcal{P}_\lambda f(z) = \lim_{\lambda \to \lambda_k} (\lambda - \lambda_k) \Delta \mathcal{P}_\lambda f(z) = -\lim_{\lambda \to \lambda_k} (\lambda - \lambda_k)(\lambda^2 + 1) \mathcal{P}_\lambda f(z) = -\lambda_k^2(\lambda_k + 1) \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z).
\]

Then, from the above remark, we must have
\[
(67) \quad \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z) \in E'_n \text{ for } k \geq |n|
\]

**Proposition 7.4** Let \( f \) be of \( C^\infty_\text{com}(\overline{B_R(z_0)}) \), then the regular part of \( \mathcal{P}_\lambda f(z) \) at \( \lambda_k \) satisfy
\[
(68) \quad (\mathcal{P}_\lambda f(z) - (\lambda - \lambda_k)^{-1} \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z))|_{\lambda = \lambda_k} \in \tilde{E}_\lambda.
\]

**Proof.** According to the Proposition (3.3), we deduce
\[
(69) \quad (\Delta + \lambda_k^2 + 1)(\mathcal{P}_\lambda f(z) - (\lambda - \lambda_k)^{-1} \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z)) = (\lambda_k^2 - \lambda^2) \mathcal{P}_\lambda f(z) \text{ for } \lambda \neq \lambda_k,
\]

and since
\[
\lim_{\lambda \to \lambda_k} (\lambda_k^2 - \lambda^2) \mathcal{P}_\lambda f(z) = -2\lambda_k \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z) \in E'_n.
\]

Then
\[
(\Delta + \lambda_k^2 + 1)(\mathcal{P}_\lambda f(z) - (\lambda - \lambda_k)^{-1} \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z))|_{\lambda = \lambda_k} \in E'_n.
\]

but
\[
(\mathcal{P}_\lambda f(z) - (\lambda - \lambda_k)^{-1} \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z))|_{\lambda = \lambda_k} \notin E_{\lambda_k}.
\]

Whence
\[
(\Delta + \lambda_k^2 + 1)^2(\mathcal{P}_\lambda f(z) - (\lambda - \lambda_k)^{-1} \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z))|_{\lambda = \lambda_k} = 0,
\]

More precisely
\[
(\mathcal{P}_\lambda f(z) - (\lambda - \lambda_k)^{-1} \text{Res}_{\lambda = \lambda_k} \mathcal{P}_\lambda f(z))|_{\lambda = \lambda_k} \in \tilde{E}_\lambda.
\]
Proof of the theorem 7.1

Let $f$ be an element of $C_{\text{com}}^\infty(B_R(z_0))$, assumed $SO(2)$-finite, then $f$ is written as a finite sum of the form $f_m(tanh(r)e^{im\theta})$ (with $m \in \mathbb{Z}$). It suffices to verify the conditions 1),..., 5) of theorem, for a functions of the form $f_m(tanh(r)e^{im\theta})$.

The condition 1) and 2) are easy, while 3) and 5) has been already established (see propositions 7.3 and 7.4 and formula (7.18)).

Now showing the estimate in 4). From the proposition 7.1, we have

\begin{equation}
\mathcal{I}_\lambda f(z) = \int_D \varphi_\lambda(d(z, z')) f(z') dz',
\end{equation}

where

\begin{equation}
\varphi_\lambda(tanh r) = \frac{(2\pi)^{\frac{d}{4}}}{4\pi^2} \lambda \tanh \left( \frac{\pi \lambda}{2} \right) P_{-\frac{1}{2}(1+i\lambda)}(\cosh(2r)),
\end{equation}

For simplicity of notation we take $z = e = (0,0)$, and write

\begin{equation}
F(r) = \int_0^{2\pi} f(tanhre^{i\theta}) d\theta.
\end{equation}

Note that $F \in C_{\text{com}}^\infty([-R - d(z, z_0), R + d(z, z_0)])$.

The formula (7.23) becomes

\begin{equation}
\mathcal{I}_\lambda f(z) = \frac{1}{4\pi} \int_0^{R+d(z, z_0)} \varphi_\lambda(tanh r) F(r) \sinh(2r) dr
\end{equation}

\begin{equation}
= \frac{(2\pi)^{\frac{d}{4}}}{2^{d/2}\pi^3} \lambda \tanh \left( \frac{\pi \lambda}{2} \right) \int_0^{R+d(z, z_0)} P_{-\frac{1}{2}(1+i\lambda)}(\cosh(2r)) F(r) \sinh(2r) dr.
\end{equation}

Now, we use the well-known identity (cf. [26] p: 87) for Legendre function

\begin{equation}
P_{-\frac{1}{2}(1+i\lambda)}(\cosh(r)) = \frac{\sqrt{2}}{\pi} \int_0^r \frac{\cos(\lambda t)}{(\cosh r - \cosh t)^{\frac{1}{2}}} dt.
\end{equation}

We interchange the order of integration in the formula (7.26) to obtain that

\begin{equation}
\int_0^{R+d(z, z_0)} P_{-\frac{1}{2}(1+i\lambda)}(\cosh(2r)) F(r) \sinh(2r) dr = \frac{\sqrt{2}}{\pi} \int_0^r \cos(\lambda t) G(t) dt,
\end{equation}
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where

\[ G(t) = \int_{t}^{R+d(z,z_0)} \frac{F(r) \sinh(2r)}{(\cosh 2r - \cosh 2t)^{\frac{1}{2}}} dr. \]

To see that \( G \) is \( C^\infty \), we note that \( \left(\frac{1}{\sinh r} \frac{d}{dr}\right)^k F(r) \) is \( C^\infty \) for any \( k \), so an integration by parts yields

\[ G(t) = \frac{(-1)^k}{(\frac{1}{2})_k} \int_{0}^{R+d(z,z_0)} (\cosh(2r) - \sinh(2r))^{k-\frac{1}{2}} \sinh(2r)(\frac{d}{d \cosh(2r)})^k F(r) \, dr, \]

is \( C^{k-1} \) by inspection.

It remains to verify that

\[ \sum_{k \in \mathbb{Z}} \text{Res}_{\lambda=\lambda_k} \mathcal{P}_\lambda f = 0. \]

from the inversion formula, we have (see the above proposition )

\[ f(z) = \int_{\mathbb{R}} \mathcal{P}_\lambda f(z), \]

with

\[ \mathcal{P}_\lambda f(z) = \gamma(\lambda, n)(\tanh r)^{|n|} \phi_\lambda^{(|n|,-|n|)}(r) e^{i \theta}. \int_{0}^{R} f_n(tanh s) \phi_\lambda^{(|n|,-|n|)}(s) \times (\tanh s)^{|n|} \sinh(2s) \, ds \]

with

\[ \gamma(\lambda, n) = \frac{1}{(|n|)!} \frac{1}{8 \pi^2} \lambda \sinh(\frac{\pi \lambda}{2}) \Gamma(|n| + \frac{1 + i \lambda}{2}) \Gamma(|n| + \frac{1 - i \lambda}{2}). \]

As \( \mathcal{P}_\lambda f(z) \) is a meromorphic function, with poles at \( \lambda_k = \pm i(2k + 1) \), thus, we can use the residue theorem to change the path of integration to obtain

\[ f(z) = \int_{\mathbb{R}+i \alpha_k} \mathcal{P}_\lambda f(z) \, d\lambda + 2i \pi \sum_{j \leq k} \text{Res}_{\lambda=\lambda_j} \mathcal{P}_\lambda f(z), \]

where \( \alpha_k = i(2k + 1 + \frac{1}{2}) \), for any integer \( k \). Since \( f(z) = 0 \) for \( z \notin B_R(z_0) \), so \( r \geq R \), then, it follows from the estimate in 4) that

\[ \lim_{\alpha_k \to \infty} \int_{\mathbb{R}+i \alpha_k} \mathcal{P}_\lambda f(z) \, d\lambda = 0. \]
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$$\sum_{j \leq k} \text{Res}_{\lambda_j} \mathbb{P}_\lambda f(z) = 0 \text{ for } z \notin B_R(z_0)$$

since the above equality is a finite sum of a fonction propre de $\Delta$, then it is a real-analytic function, so it vanishes for all $z \in \mathcal{C}'$.

**Theorem 7.2** Let $F(\lambda, z)$ a function given satisfying the following assertions

1) $F(\lambda, z)$ is $C^\infty$ function on $(\mathcal{C} - i\mathbb{Z}) \times D$

2) for each fixed $\lambda \in \mathcal{C} - i\mathbb{Z}$, we have $\Delta_D F(\lambda, z) = -(\lambda^2 + 1) F(\lambda, z)$

3) for each fixed $z$, $F(\lambda, z)$ is an even function meromorphic function of $\lambda$ with at worst simple poles at $\lambda_k = \pm i(2k + 1)$, and

$$\sum_{k \in \mathbb{Z}} \text{Res}_{\lambda=\lambda_k} F(\lambda, z) = 0 \quad (74)$$

4) for every $N$ there exists $c_N$ such that

$$|F(\lambda, z)| \leq c_N (1 + |\lambda|)^{-N} e^{(R+d(z, z_0))|\text{Im}\lambda|} \quad (75)$$

5) $F(\lambda, z)$ has a simple zeros at points $\lambda_l = \pm i2l$ ($l \in \mathbb{Z}^*$) and a double zero at $\lambda = 0$ and satisfies

- $z \to \text{Res}_{\lambda=\lambda_k} F(\lambda, z) \in E'_n$
- $(F(\lambda, z) - (\lambda - \lambda_k)^{-1} \text{Res}_{\lambda=\lambda_k})|_{\lambda=\lambda_k} \in \tilde{E}_\lambda$
- $F(\lambda, z)|_{\lambda=0} = 0$ and $\frac{F(\lambda, z)}{\Gamma(|k|+\frac{1}{2})\Gamma(|k|+1-\frac{1}{2})}$ has even entire expansion. then there exists $f$, $C^\infty$ with support in $B_R(z_0)$ given by $f(z) = \int_{\mathbb{R}} F(\lambda, z) d\lambda$ such that $F(\lambda, z) = \mathbb{P}_\lambda f(z)$

**Proof of theorem 7.2.**

Let $F(\lambda, z)$ given satisfying 1), ..., 5), and define

$$f(z) = \int_{\mathbb{R}} F(\lambda, z) d\lambda, \quad (76)$$
there is no difficulty with convergence in view of the following inequality

\[ \int_{\mathbb{R}} F(\lambda, z) \, d\lambda \leq c_N \int_{\mathbb{R}} (1 + |\lambda|)^{-N} \, d\lambda < \infty. \]

First, we want to show that \( f \) vanishes outside \( B_R(z_0) \) and \( f \) is \( C^\infty \). For this, we assume all the functions are of the form \( f_n(\tanh(r)e^{i\theta}) \) (since the condition 1), ..., 5) are preserved), then, we can assume

\[ F(\lambda, \tanh(r)e^{i\theta}) = \Psi(\lambda)e^{i\theta}\phi(\lambda, r) \]

for some meromorphic function, from the estimate in 4) we have

\[ \Psi(\lambda)e^{i\theta}\phi(\lambda, r) \leq c_N(1 + |\lambda|)^{-N}e^{(R+r)|Im\lambda|}. \]

Using the koornwinder lemma to obtain

\[ \Psi(\lambda) \leq c_N(1 + |\lambda|)^{-N}e^{r|Im\lambda|}. \]

We note that \( \Psi \) is given by the condition 5).

According to the condition 3), we use the residue theorem in the formula (7.26) to obtain

\[ f(z) = \int_{\mathbb{R}} F(\lambda, z) \, d\lambda \]
\[ = \int_{\mathbb{R}+i\alpha_k} F(\lambda, z) \, d\lambda + 2\pi \sum_{j \leq k} \text{Res}_{\lambda = \lambda_j} F(\lambda, z) \]
\[ = \int_{\mathbb{R}+i\alpha_k} F(\lambda, z) \, d\lambda \]

where \( \alpha_k = i(2k + 1 + \frac{1}{2}) \), for any integer \( k \). From the formula (7.28), we conclude that

\[ \int_{\mathbb{R}+i\alpha_k} F(\lambda, z) \, d\lambda = e^{i\theta}(\tanh r)^{|n|}\int_{\mathbb{R}+i\alpha_k} \Psi(\lambda)\phi(\lambda, r) \, d\lambda. \]

Since \( \Psi(\lambda) \) satisfy the inequality (42), then, we let \( k \to \infty \), we obtain zero in the limit in the formula (43) if \( r \geq R \). Then \( f(\tanh re^{i\theta}) = 0 \) if \( r \geq R \).

Show now that \( f \) is \( C^\infty \).
Since
\[ \Delta^N F(\lambda, z) = (-1)^N (1 + \lambda^2)^N F(\lambda, z) \text{ for any integer } N. \]

Then
\[ \Delta^N f(z) = (-1)^N \int_{\mathbb{R}} (1 + \lambda^2)^N F(\lambda, z) \, d\lambda. \]

From the estimate in 4) it follows that \( \Delta^N f \) is \( L^2(D, dm(z)) \), the usual Sobolev inequality imply that \( \Delta^N f \) is \( C^\infty \) on \( D \).

To understand the behavior of \( F(\lambda, z) \) as \( \lambda \to i(2l) \) \((l \in \mathbb{Z})\), we note that
\[ g_z(\lambda) = F(\lambda, z) \Gamma(|n| + \frac{1+i\lambda}{2}) \Gamma(|n| + \frac{1-i\lambda}{2}) \]

has an even entire expansion, satisfying the estimate in 4), then there is a constant \( c'_N \) such that
\[ g_z(\lambda) \leq c'_N (1 + |\lambda|)^{-N} e^{(R+d(z_0,z))|Im\lambda|}. \]

From the above results the function \( \lambda \to \Gamma(|n| + \frac{1+i\lambda}{2}) \Gamma(|n| + \frac{1-i\lambda}{2}) f_{\lambda,n}(\tanh r) \)

is entire and satisfying the estimate in 4). Using
\[ f_{\lambda,n}(\tanh r) = c'_{N,n} g_{\tanh r}(\lambda) \Gamma(|n| + \frac{1+i\lambda}{2}) \Gamma(|n| + \frac{1-i\lambda}{2}) \tanh r |n| |\phi^{|n|-|n|}(r), \]

which show that \( g_{\tanh r}(\lambda) \) is an even and entire function since the collection of function \( \{\phi^{|n|-|n|} \} \) has no zeros and no poles.

To complete the proof we need to show that \( F(\lambda, z) = \mathcal{P}_\lambda f(z) \) which is equivalent to showing that \( \int_{\mathbb{R}} F(\lambda, z) = 0 \) imply \( F(\lambda, z) = 0 \), and it suffices to show that \( \int_{\mathbb{R}} f_{\lambda,n} = 0 \) imply \( f_{\lambda,n} = 0 \). From the formula (7.27) we have
\[ 0 = \int_{\mathbb{R}} f_{\lambda,n}(z) \, d\lambda = |\tanh r| |n| \int_{\mathbb{R}} \Psi(\lambda) \phi^{|n|-|n|}(r) \, d\lambda. \]

Since \( \Psi(\lambda) \) has compact support, by the uniqueness of the jacobi transform (cf. [22]), we deduce that \( \Psi(\lambda) = 0 \) for \( \lambda \neq \pm i(2k+1) \). Then \( F(\lambda, z) = 0 \) for \( \lambda \neq \pm i(2k+1) \).

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