Analysis and Modeling of Process of Residual Deformations Accumulation in Soils and Granular Materials

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Abstract. It is established that under the influence of repeated loads the process of plastic deformation in soils and discrete materials is hereditary. To perform the mathematical modeling of plastic deformation, the authors applied the integral equation by solution of which they manage to obtain the power and logarithmic dependencies connecting plastic deformation with the number of repeated loads, the parameters of the material and components of the stress tensor in the principal axes. It is shown that these dependences generalize a number of models proposed earlier in Russia and abroad. Based on the analysis of the experimental data obtained during material testing in the dynamic devices of triaxial compression at different values of the stress deviator, the coefficients in the proposed models of deformation are determined. The authors determined the application domain for logarithmic and degree dependences.

1. Introduction
Consumer properties of the road depend on the smoothness of the coatings, which causes the urgency of the work aimed at predicting changes in the smoothness of the coatings. The depth of the unevenness at the considered point is determined by the difference in the plastic displacements of the surface of the coating at this point and at the point with the smallest plastic displacement [1].

For example, the depth of a gauge at a point located within the track line, is determined by the difference in the residual displacements of the coating at this point and at the point, located outside the track line, in which the amount of residual movement of the surface of the coating is minimal. Since the maximum value of the residual deformation accumulates in the center of the track line, the criterion for calculating the pavement by limiting the gauge depth can be written in the form [1]:

\[ S_{\text{max}} - S_{\text{min}} \leq h_{\text{lim}} \]

where \( S_{\text{max}} \) – maximum value of the irreversible movement of the surface of the coating in the section located in the center of the track line, In which the largest number of transport traffic passes, mm; \( S_{\text{min}} \) – minimum value of the irreversible displacement of the surface of the coating in a section located beyond the track line, in which there is the smallest number of passages traffic loads, mm; \( h_{\text{lim}} \) – limit gauge depth limited by regulatory documents.

The plastic displacement of the surface of the road construct element in the section by the symmetry axis of the load is determined by integrating the plastic deformation function over the depth of its propagation zone. The sum of the plastic displacements of the surfaces of the pavement layers and the ground of the subgrade determines the plastic displacement of the surface of the coating in the
section under consideration. The outlined scheme for solving the problem is written down by the integral equation:

\[ S_n = \sum_{i=1}^{0} \int_{z_i}^{\infty} \varepsilon [\sigma_1(z); \sigma_2(z); \sigma_3(z); a(z), b(z), c(z)...m(z); N; t] dz \]  

(2)

where \( i \) and \( n \) – number and amount of layers of the road structure, including the subgrade; \( z_i \) – ordinate of the point, which limits the zone of propagation of plastic deformations in the section along the symmetry axis of the load; \( \varepsilon \) – plastic deformation, which is a function of a number of material parameters; \( \sigma_1(z), \sigma_2(z), \sigma_3(z) \) – the main stresses, which are a function of depth \( z \), Pa; \( a(z), b(z), c(z)...m(z) \) – parameters of material, which are a function of depth and physical properties (density, moisture content, temperature, porosity, etc.); \( N \) – number of loads applied; \( t \) – time of action of one load, s.

Thus, the determination of the integrand of the equation (2) is an important task of predicting the change in the smoothness of the road structure.

2. Review of the literature and statement of the research task

Mathematical modeling of plastic deformation under conditions of triaxial compression of discrete materials and the impact of repeated (cyclic) loads is the goal of works performed worldwide [2-8]. A feature of such models is the representation, accumulated plastic deformation \( \varepsilon_N \), multiplication of the number of loads \( f(N) \) and deformation \( \varepsilon_n \), accumulated from a relatively small number of repeated loads \( n (n<<N) \). That is

\[ \varepsilon_N = \varepsilon_n \cdot f(N) \]  

(3)

When choosing empirical formulas (3) number of loads is taken within \( n=1...10^3 \), a \( N=10^5...10^6 \) and more. Testing of samples from soils and crushed stone materials is carried out using dynamic triaxial compression devices [2,9].

At present time received, the logarithmic, degree, and exponential functions of the number of loads, used in models of the type (3). A proposal to use logarithmic dependencies to predict the value of plastic deformation is due to R.D. Barksdale [9]. G.T.H. Sweere in his thesis [10] suggested the use degree functions, which were developed in the works [11 – 13]. Exponential dependencies are used in the works [14, 15].

The accuracy of these dependencies is determined by the number of loads \( N \), from the effect of which the prediction of accumulated deformation is performed \( \varepsilon_N \). With a relatively small number of applied loads \( N\leq10^3 \) best approximation is given by exponential functions. With the number of loads varying in the range \( 10^3>N\leq10^5 \), greatest accuracy have the logarithmic dependence. Degrees dependencies make it possible with the greatest accuracy to predict the plastic with the number of loads \( 10^5>N\leq3\cdot10^6 \), as a result of which they are most suitable for calculating the deformations materials of road constructs. In addition, degrees functions make it possible to obtain results with an acceptable accuracy and for a smaller number of loads.

Due to the dependence of the accuracy of calculation of plastic deformations from the number of repeated loads for each type of function it's possible to specify the application area. For example, logarithmic functions can be used to calculate the plastic deformations of materials and soils used in road structure on roads IV or V of technical category. Degree functions should be used when predicting deformations of materials and soils for roads of I-III technical categories. Exponential functions for our purposes are not applicable.

The review shows that mathematical modeling of plastic deformation of materials and soils is performed by selecting empirical formulas. The abundance of empirical formulas and the lack of theoretical justification leads to difficulties in choosing a mathematical model. By virtue of the fact that such a formula is an integrand of equation (2), the solution of the tasks of the plastic displacement of the surface of the coating, and hence the depth of the irregularities is also hampered.
Therefore, the authors will set the task of finding a theoretical solution that will allow us to obtain a generalizing mathematical model for a number of well-known empirical formulas.

3. Main part
Analysis data of laboratory triaxial dynamic test allows us to conclude, that the magnitude of the observed plastic deformation acquired by the sample during the realization N-th load, depends both on this and on all previous loads. Consequently, the plastic deformation of materials and soils when applying repeated loads has a hereditary nature. Therefore, for mathematical modeling of such plastic deformations, we can use the integral equations of the creep theory, in which the time function must be replaced by a function of the number of loads.

Analyzing the deformation of materials and soils, we note that during each application of the load, the plastic deformation increases in time, that is, it continuously increases from the beginning to the end of observe load. Thus, the process of accumulation of plastic deformations can be considered continuous, and for their calculation it is possible to perform integration over the number of loads. Integrands take as a power function, determining an increment of plastic deformation from the load with a sequence number \( n \). Then we give the kernels of the integral equations in the form:

\[
\Delta \varepsilon_{\text{ins}} = a \cdot n^{-1} ; \quad \Delta \varepsilon_{\text{vis}} = a \cdot n^{-1} \tag{4}
\]

\[
\Delta \varepsilon_{\text{ins}} = b \cdot n' ; \quad \Delta \varepsilon_{\text{vis}} = b \cdot n' \tag{5}
\]

where \( \Delta \varepsilon_{\text{ins}} \) and \( \Delta \varepsilon_{\text{vis}} \) – respectively, instantaneous and viscoplastic deformations arising from the load with the serial number \( n \); \( a, b \) and \( c \) – model parameters that take into account the type of material and the magnitude of the stresses.

Then we give the integral equations in the form:

\[
\varepsilon_N = (\varepsilon_{\text{ins}} + \varepsilon_{\text{vis}}) \cdot \left[ 1 + a \cdot \int_1^N n^{-1} dn \right] \tag{6}
\]

\[
\varepsilon_N = (\varepsilon_{\text{ins}} + \varepsilon_{\text{vis}}) \cdot \left[ 1 + b \cdot \int_1^N n' dn \right] \tag{7}
\]

Integrating (6), we obtain

\[
\varepsilon_N = (\varepsilon_{\text{ins}} + \varepsilon_{\text{vis}}) \cdot \left[ 1 + a \cdot \left( \ln N \right) \right] \tag{8}
\]

where \( \varepsilon_{\text{ins}} \) and \( \varepsilon_{\text{vis}} \) – instantaneous and viscous components of plastic deformation accumulated upon exposure to the n-th number of loads \( (n<<N) \); \( a \) – parameter of the logarithmic model, accounting the magnitude of the stresses, the type of material and the indices of its physical properties (density, humidity, etc.).

Taking the integral of (7), we have

\[
\varepsilon_N = (\varepsilon_{\text{ins}} + \varepsilon_{\text{vis}}) \cdot \left[ 1 + b \cdot \frac{N^{c+1} - n^{c+1}}{c+1} \right] \tag{9}
\]

where \( b \) and \( c \) – Parameters of the degree model, accounting the same factors as the parameter \( a \) in the model (8).

If the number of loads \( n \) is relatively small and is \( n=10; n=100 \) or \( n=200 \), then the connection of constituents \( \varepsilon_{\text{ins}} \) and \( \varepsilon_{\text{vis}} \) plastic deformation with this number of loads and residual deformations from the first action of load it is expedient to look for in the form
\[ \varepsilon_\text{str} + \varepsilon_\text{vis} = (\varepsilon_\text{str1} + \varepsilon_\text{vis1}) \cdot \left[ 1 + d \cdot \frac{n}{n^*} \right] \]

Taking the integral (10), we get
\[ \varepsilon_\text{str} + \varepsilon_\text{vis} = (\varepsilon_\text{str1} + \varepsilon_\text{vis1}) \cdot \left[ 1 + d \cdot \ln(n) \right] \]

Taking into account expression (11) in the formulas (8) and (9), these models are reduced to the form:
\[ \varepsilon_N = (\varepsilon_\text{str1} + \varepsilon_\text{vis1}) \cdot \left[ 1 + a \cdot \ln\left( \frac{N}{n^*} \right) \right] \]
\[ \varepsilon_N = (\varepsilon_\text{str1} + \varepsilon_\text{vis1}) \cdot \left[ 1 + b \cdot \frac{n^*}{c+1} \right] \]

Application area of models (12) and (13), as well as their parameters \(a\), \(b\), \(c\) and \(d\) it is necessary to install on the basis of the analysis of experimental data at triaxial compression of materials and soils by cyclic loading. Moreover, from the whole variety of experimental data for the problem being suitable only those that are obtained with a sufficiently large number applications of load \(N>10^7\). Such experiments are performed abroad using dynamic triaxial compression devices. The constructions of triaxial compression devices and methods of experimental tests differ in variety. Therefore, the analysis of such devices and experimental methods deserves consideration in a separate article. In this publication, the authors confine themselves to data Barksdale R.D. [9], Werkmeister S. [16], presented in figure 1 and figure 2.

Analyzing the data in figure 1 and figure 2, as well as other dependencies proposed in the paper [16], authors determined the coefficients of the models (12) and (13). Here we stipulate that the experimental data about the plastic deformation of materials under the action of cyclic loads were obtained by other specialists, and the values of the coefficients \(a\), \(b\), \(c\) and \(d\) were established by the authors according to the data of these tests. Therefore, the table shown below with coefficients \(a\), \(b\), \(c\) and \(d\) are new, but calculated from known experimental data.

**Figure 1.** Dependence of vertical plastic deformation of granodiorite crushed stone from the number of loads and stresses at \(\sigma_3=40\) kPa [16]
1 - 6 - at \((\sigma_1-\sigma_3)\) 40; 80; 120; 160; 200 and 360 kPa; at \(\sigma_3=40\) kPa.

**Figure 2.** Dependence of vertical plastic deformation sandy-gravel mixture from the number of loads and stresses at \(\sigma_3=40\) kPa [16]
1 - 4 - at \((\sigma_1-\sigma_3)\) 40; 80; 120; and 160 kPa; at \(\sigma_3=40\) kPa;
In table 1 shows the values of the coefficients $n$, $a$ and $d$ of model (12) to calculate the deformation accumulated by crushed stone from granite or gneiss. Presented in this table material parameters in the model (12) are determined from the analysis of the work data [9].

**Table 1. Parameters of the model (11) for calculating the residual deformation accumulated by a sample of granite or gneiss crushed stone.**

| Characteristic $(\sigma_1-\sigma_3)/\sigma_3$ | Model parameters (11) |
|---------------------------------------------|------------------------|
|                                            | $n$  | $d$  | $a$  |
| 1.5                                        | 100  | 1,0134 | 0,3722 |
| 1.94                                       | 100  | 4,3009 | 0,2895 |
| 2.83                                       | 100  | 3,1082 | 0,2141 |
| 4.6                                        | 100  | 0,8723 | 0,415 |

In tables 2 and 3 show the values of the coefficients $n$, $c$, $b$ and $d$ of the model (13) for calculating the residual deformation accumulated by the sand-gravel mixture. The parameters of material presented in this table in the model (13) are determined from the analysis of the work data [16].

**Table 2. Parameters $b$ and $c$ of the model (13) for calculating the residual deformation accumulated by the sand-gravel mixture.**

| Characteristic $(\sigma_1-\sigma_3)/\sigma_3$ | Parameter $n$ | $\sigma_3 \leq 40$ kPa | $\sigma_3 = 70$ kPa | $\sigma_3 \geq 210$ kPa |
|---------------------------------------------|---------------|-------------------------|---------------------|-------------------------|
|                                            | $b$ | $c$ | $b$ | $c$ | $b$ | $c$ |
| $\leq 0.5$                                  | 100 | –   | –   | –   | 0,073 | -0,84 |
| 1                                           | 100 | 0,072 | -0,832 | 0,085 | -0,818 | 0,071 | -0,865 |
| 1.5                                         | 100 | –   | –   | 0,071 | -0,809 | 0,074 | -0,742 |
| 2                                           | 100 | 0,065 | -0,871 | 0,060 | -0,915 | 0,056 | -0,741 |
| 3                                           | 100 | 0,079 | -0,699 | 0,071 | -0,861 | – | – |
| 4                                           | 100 | 0,072 | -0,656 | 0,079 | -0,805 | – | – |
| $\geq 5$                                    | 100 | –   | –   | 0,046 | -0,505 | – | – |

**Table 3. Parameter $d$ of the model (13) for calculating the residual deformation accumulated by the sand-gravel mixture.**

| Characteristic $(\sigma_1-\sigma_3)/\sigma_3$ | $\sigma_3 \leq 40$ kPa | $\sigma_3 = 70$ kPa | $\sigma_3 \geq 210$ kPa |
|---------------------------------------------|-------------------------|---------------------|-------------------------|
|                                            | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ |
| $\leq 0.5$                                  | 1,734 | 0,895 | – | – | 0,378 |
| 1                                           | 1,566 | 0,883 | – | – | 0,611 |
| 1.5                                         | 1,397 | 0,871 | – | – | 0,770 |
| 2                                           | 1,229 | 0,920 | – | – | 0,644 |
| 3                                           | 0,682 | 0,887 | – | – | 0,611 |
| 4                                           | 2,457 | 2,295 | – | – | 2,019 |
| $\geq 5$                                    | 2,213 | 2,050 | – | – | 1,774 |

Analogous coefficients are determined by the authors for the calculation of deformations which accumulated by soils and crushed stone from various rocks (diabase, granodiorite, limestone, dolomite, etc.). The range of materials to which are applicable models (12) and (13) and for which
model parameters are determined is large. Therefore, to illuminate the parameters of models (12) and (13) for different materials and types of soils, it is advisable to devote a separate publication, which is a continuation of this work. In this article we confine ourselves to the data presented in table 1 - table 3, as well as comparing the results of calculating the deformation accumulated by the sand-gravel mixture with the experimental data. Such a comparison is shown in figure 3. The experimental data and the results of the calculation were obtained with a minimum principal stress of 40 kPa.

![Figure 3. Dependence of vertical plastic deformation sandy-gravel mixture from the number of loads and stresses at $\sigma_3=40$ kPa](image)

1 - 4 - experiment at $(\sigma_1-\sigma_3)/\sigma_3$ 1; 2; 3; and 4; at $\sigma_3=40$ kPa;
5 - 8 - calculation at $(\sigma_1-\sigma_3)/\sigma_3$ 1; 2; 3; and 4; at $\sigma_3=40$ kPa;

From the analysis of the data in figure 3 it follows that proposed by us generalized power-law model (13) allows us to predict the process of accumulation of plastic deformation with acceptable accuracy. The accuracy of the model (12) can also be considered satisfactory, as demonstrated in the works [17, 18].

4. Conclusion
Based on the materials of the work, it is possible to formulate the conclusions and tasks of future studies and publications:

1. It is established that under the action of cyclic loading in soils and discrete materials, the process of accumulation of plastic deformation has a hereditary character. To determine the increment of plastic deformation from the $n$-th load application, the degrees functions are proposed that are considered as kernels of integral equations of the hereditary theory.

2. Integration of the equations gives the logarithmic and power-law models (12) and (13), which generalize a number of known empirical formulas and have greater accuracy in calculating the plastic deformation.

3. From the analysis of data of the triaxial test, established the coefficients of the proposed models for a wide range of crushed stone and soils.

4. The tasks of further publications are:
   - the development of the methodology and its application in determining the parameters of models (12) and (13) for various materials;
   - the development of a method for calculating the viscous component of plastic deformation $\varepsilon_{\text{visc}}$, which will allow to take into account the influence of the duration of the impact of the load, and hence the speed of movement;
• the development of a method for calculating the plastic displacement of the coating surface and the depth of irregularities formed in the longitudinal and transverse directions.

References
[1] Gercog V N, Dolgih G V and Kuzin V N 2015 Calculation of pavements for roughness criteria Part 1 Justification standards evenness of asphalt-concrete coatings Magazine of Civil Engineering 5(57) pp 45–57
[2] Mirsayaynov I T, Brechman A I, Koroleva I V and Ivanova O A 2012 Strength and deformation of sandy soils under triaxial cyclic loading Izvestiya KGASU 3(21) pp 58–63
[3] C Chen, L Ge and J Zhang 2010 Modeling Permanent Deformation of Unbound Granular Materials under Repeated Loads International journal of geomechanics vol 10 pp 236–241
[4] Aleksandrov A S and Kiseleva N Yu 2012 Plastic deformation of the gneiss and diabazmaterial when exposed to repetitive loads News of higher educational institutions. Construction 6 pp 49–59
[5] Perez I, Medina L and Gallego J 2010 Plastic deformation behaviour of pavement granular materials under low traffic loading Granular Matter 1 pp 57–68
[6] Rondon H A 2009 Deformacion permanente de materiales granulares en pavimentos flexibles: estado del conocimiento Revista Ingenierias Universidad de Medellin vol 8 14 pp 71–94
[7] Gidel G, Hornych P, Chauvin J, Breysse D and Denis A 2001 A new approach for investigating the permanent deformation behaviour of unbound granular material using the repeated load triaxial apparatus Bulletin des Laboratoires des Ponts et Chaussées 14(233) pp 5–21
[8] Aleksandrov A S 2016 A Generalizing model of plastic deformation of discrete materials of road structures under impact of cyclic loads Construction Materials pp 27–30
[9] Barksdale R D 1972 Laboratory Evaluation of Rutting in Base course Materials. Proceedings of the 3-rd International Conference on Asphalt Pavements (London) pp 161–174
[10] Sweere G T H 1990 Unbound granular bases of roads PhD thesis (The Netherlands: Delft University of Technology)
[11] Erlingsson S and Ahmeda A 2014 Performance prediction modelling of flexible pavement structures Transport Research Arena (Paris) pp 1–10
[12] Siripun K, Nikraz H and Jitsangiam P 2011 Mechanical Behavior of Unbound Granular Road Base Materials under Repeated Cyclic Loads International Journal of Pavement Research and Technology vol 4 1 pp 56–66
[13] Hornych P, Corte J F and Paute J L 1993 Étude des déformations permanentes sous chargements répétés de trois graves non traitées Bulletin de Liaison des Laboratoires des Ponts et Chaussées 184 pp 77–84
[14] Theyse H L 2000 The development of mechanistic-empirical permanent deformation design models for unbound pavement materials from laboratory accelerated pavement Proceedings of the 5-th International symposium on unbound aggregates in road (Nottingham) pp 285–293
[15] Wolff H and Visser A 1994 Incorporating elasto-plasticity in granular layer pavement design Proceedings of Institution of Civil Engineers Transport 105 pp 259–272
[16] Werkmeister S 2003 Permanent deformation behavior of unbound granular materials in pavement constructions PhD thesis (The Germany University of Technology Dresden)
[17] Aleksandrov A S 2013 Plastic deformation granodiorite gravel and sand and gravel when exposed to cyclic loading triaxial Magazine of Civil Engineering 4 pp 22–34
[18] Aleksandrov A S, Semenova T V and Aleksandrova N P 2016 Analysis of permanent deformations in granular materials of road structures Road and Bridges vol 15 pp 263–276