Non-trivial 1-classes in the homology of the real moduli spaces $\overline{M}_{0,n}$ and related structures

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March 29, 2022

Abstract

We give lower bounds for the rank of the first homology group of the real points of the Deligne-Mumford-Knudsen compactification of stable $n$-pointed curves of genus 0, which coincides with the Chow quotient $(\mathbb{R}P^1)^n//\text{PGL}(2, \mathbb{R})$. This study is a diagrammatic version of Loday’s Theorems. Also, it has connections with spectral geometry and combinatorics.

A.M.S. Classification: 14P25 (14F35, 14J10, 57M15, 20F99)

Keywords: compactification, moduli spaces, homology

1 Discrete Spectral warm-up

We built a sequence of 2 dimensional complexes. Its study, inspired by spectral graph theory, can guide the study of the real moduli spaces $\overline{M}_{0,n}$. Also, these complexes give a counterexample to a tempting conjecture.

1. Let $G(n)$ be the Schreier coset graph of $S(n)$, the symmetric group, modulo $D(n)$, the dihedral group, with respect to the following set $T$ of generators. $T$ is a set of $n(n-3)/2$ involutions from $S_n$. For every pair $(k, l)$, $0 < k < l < n + 1$, where $(k,l)$ is different from $(1, n)$, we define $d:=d(k, l)(x)=l+k-x$, if $x$ is between $k$ and $l$. Otherwise, $x$ is a fixed point.

The vertices of $G(n)$ are the right coset classes $S(n)/D(n)$. There is an edge between $D(n)g$ and $D(n)h$, if there is $t \in T$ such that $D(n)g=D(n)ht$.

Any regular graph (a graph where any vertex has the same degree) is a Schreier
coset graph of a group $G$, modulo a subgroup $H$, with respect to a set of generators $T$. [14]

Let $B(n)$ the following 2 dimensional cellular complex. Its 1-skeleton is given by $G(n)$. For any $u$ and $v$ in $T$, such that $u$ and $v$ commutes, there is a face around $D(n)a$, $D(n)av$, $D(n)avu$, and $D(n)au$.

As we will prove, rank $H_1(B(n)) = \text{rank } H_1(M_{0,n}(R))$. So, the study of $B(n)$ is important. The first barycentric subdivision transforms any cellular complex into a simplicial one.

2. Definition 1. For any 2 dimensional simplicial complex, the link of a vertex $v$ is the graph $L(v)$ whose vertices are the edges incident to $v$, and whose edges are the faces between 2 incident edges to $v$. So, the graph is the intersection between the cone from $v$ and a plane.

The Laplacian of a simple finite connected graph is the matrix of entries $a(i,j)/\sqrt{d(i)d(j)}$, where the rows and columns are indexed over vertices of the graph. $d(v)$ is the degree of a vertex $v$, i.e the number of vertices joined with $v$. $a(i,j)$ is 1 if and only if there is an edge between $i$ and $j$. [13].

Why do we need this graph? The cohomology with real coefficients of a compact simplicial complex is zero IF:

a. [Ballmann]: for every vertex $v$, the first positive eigenvalue of the Laplacian on $L(v)$ is greater than 1/2. See also [12].

or IF:
b. [Zuk]: for any 2 vertices $a$ and $b$ of the complex, joined by an edge, the average of their first positive eigenvalue of the Laplacians on $L(a)$ and $L(b)$ is greater than 1/2. [16].

Let us formulate the tempting conjecture: The first cohomology with real coefficients is zero, if the average of first positive eigenvalue of any 3 vertices of the same triangle is greater than 1/2. This conjecture is not true. A counterexample is given by the spaces $B(n)$. As we will see, their first cohomology is always
different from zero, even if the conditions from the statement of this conjecture are satisfied for n greater than 70!

The first barycentric subdivision of B(n) has 3 types of vertices. The first one is a center of a face. Its link is an 8-gon. The first positive eigenvalue is $1 - \sin(\pi/4)$. The second one is a center of an edge. Its link is a bipartite graph and its first positive eigenvalue is 1. A wonderful treatment of this theoretical and computational subject is given in \[10\]. Every vertex from G(n) has the same isomorphic link, because of the regularity of the Schreier coset graphs. The last type of vertex is a vertex from the old G(n). The first positive eigenvalue of its link, $\lambda(n)$ was computed by Matlab 6.1 using a C++ program, available at the author’s web page. $\lambda(n)$ is an increasing function of n.

$$\lambda(6) = 0.1888 \quad \lambda(7) = 0.1891 \quad \lambda(50) = 0.2070 \quad \lambda(70) = 0.2084$$

There is the following explanation of the increasing behavior of $\lambda(L(n))$, ($L(n)$ is the link of a vertex from G(n)).

$L(n)$ is the graph whose vertices are the diagonals of an n-gon. 2 vertices are joined by an edge if the interiors of the diagonals do not intersect. L(n) has a D(n) symmetry. Also, it is included in L(n+1). If K(n) is the complete graph of n vertices, there is a covering map from the cartesian product L(n) X K(n+1) to L(n+1). So, any eigenvalue of the latter graph is an eigenvalue of the former graph, too (see Chung). Also, the eigenvalues of a cartesian product are sums of separate eigenvalues, because the Laplacian of a cartesian product is a tensor product of separate matrices. Zero is always an eigenvalue, so the smallest eigenvalue of the cartesian product is the minimum between the smallest eigenvalue of L(n) and K(n). In this case, it is the smallest positive eigenvalue of L(n). We’ve got the desired inequality between 2 consecutive $\lambda(n)$’s.

Note: applying Zuk’s Theorem to hypergraphs. \[10\], we get that any closed path in G(n) is a sum of closed paths of length 4. We did not find an equivalent statement in algebraic geometry.
2 The first homology group of $\overline{M_{0,n}}(R)$

In their Quantum Invariants and Knot Theory books and articles, Turaev, Kassel and Dror bar-Nathan use ”graphical proofs” and diagrammatic calculus, for algebraic or categorical data. There is an intricate relation between formal algebraic structures and concrete geometrical objects. Even if it is elementary, it is useful to describe the path towards $H_1(\overline{M_{0,n}}(R))$ because this is a way to understand the proofs and the formalism of Loday and the appearance of these spaces as Operads.

To keep track of the first homology group, we will use the cellular decomposition of $M=\overline{M_{0,n}}(R)$, after Devadoss. We apply the dual block complex, [18] pp.380, and restrict to level two of its filtration.

Let $K_{n-1}$ be the $n$-convex polytope whose partial order set of its faces is isomorphic with the partial order set of an $n$-gon with a couple of non-intersecting diagonals. The partial orders are given by inclusions. $K_{n-1}$ is called the associahedron. There is such a convex polytope [Ziegler-Lectures on Polytope p.310]! $K_{n-1}$ has the acyclic carrier property, so we can apply the classical theorems from [Munkres pp 225]. The codimension $k$ faces of the associahedron are indexed by $n$-gons with $k$ non-intersecting diagonals.

DIAG is the following set of $n(n-3)/2$ involutions from $S_n$. For every pair $(k, l)$, $0 < k < l < n + 1$, where $(k,l)$ is different from $(1, n)$, we define $d:=d(k, l)(x)=l+k-x$, if $x$ is between $k$ and $l$. Otherwise, $x$ is a fixed point. Let $P$ be a fixed $n$-gon, with edges labeled $1,2,3...n$. For every diagonal of $P$ we can associate an element of DIAG in the following way: any diagonal determines a partition of $1,2,3...n$. Take the one which doesn’t contain $n$: it’s between 2 numbers, $k$ and $l$. Then the associated $d$ will be $d(k,l)$, and we say that $d(k,l)$ is supported by the diagonal of the $n$-gon $P$. Throughout the paper, the word ”diagonal” means a diagonal of the $n$-gon, or the involution carried by the diagonal.
Take \( n! \) copies of \( K_{n-1} \). For every permutation of \( S_n \), label the edges of the \( n \)-gon with \( \sigma(1), \sigma(2), \ldots, \sigma(k), \ldots, \sigma(n) \). So the codimension \( k \) faces are labeled by decorated \( n \)-gons with \( k \) non-intersecting diagonals. Now we build our space \( \overline{M}_{0,n}(R) \). Two codimension \( k \) faces of different \( K_{n-1} \)'s are identified (glued) if the permutations \( \sigma_1 \) and \( \sigma_2 \) which color the edges of the \( n \)-gons satisfy the following condition “flip” or gluing condition: there are \( d_1, d_2, \ldots, d_i \) couple of elements of \( \text{DIAG} \), supported by the diagonals of the second face, such that \( \sigma_1 = \sigma_2 \circ d_1 \circ d_2 \circ \ldots \circ d_i \). ( \( \circ \) means composition of functions ).

The top dimensional faces (without diagonals) are identified by the action of \( D_n \), the dihedral group. So we can begin with \((n-1)!/2 \) copies of \( K_{n-1} \), indexed over \( S_n/D_n \). Two codimension \( k \) faces are identified if their classes modulo the dihedral group \( D_n \) contain 2 permutations which satisfy the flip condition from the previous paragraph.

**The Homology** is encoded in the gluing process above. \( M \)(our moduli space) is a smooth compact \((n-3)\)-manifold, non-orientable if the dimension is higher than 1.

Recall the construction of the dual block complex. Let \( X \) be a compact homology \( n \)-manifold. Let \( sdX \) be the first barycentric subdivision. The simplices of \( sdX \) are \([\bar{a}, \bar{b}, \ldots, \bar{z}]\) where \( a \supset b, \ldots, \supset z \) and \( \bar{\sigma} \) is the barycenter of \( \sigma \), a simplex of \( X \).

Given a simplex \( \sigma \), \( D(\sigma) \), the block of \( \sigma \), is the union of the open simplices of \( sdX \), where \( \sigma \) is the final vertex; i.e. ”\( \sigma = z \)” in the notation above. We have \( \dim \sigma + \dim D(\sigma) = \dim \text{manifold} \).

2.0.Let \( X_p \) be the dual \( p \)-skeleton of \( X \), the union of all \( D(\sigma) \) such that \( \dim D(\sigma) \) is smaller or equal to \( p \). \( D_p = H_p(X_p , X_{p-1}) \). The boundary operator is the boundary operator in the exact sequence of the triple \((X_p , X_{p-1}, X_{p-2})\). \( D_p = H_p(X_p , X_{p-1}) \) is the free abelian group generated by the blocks of dimension \( p \) of \( X \), arbitrarily oriented, i.e. the blocks from the previous statement label the
elements of a basis. How can we deal with the boundary operator of the dual block complex? Fortunately it has a nice geometric meaning. The dual blocks form a CW decomposition of $X$. Let $a$ and $b$ be 2 cells, $\text{dim}(a)=p$ and $\text{dim}(b)=p-1$. $\partial a$ is a formal sum of $p-1$ cells, with integer coefficients. The coefficient of $b$ is given by "the incidence coefficient", which is the degree of a map that sends the boundary of a (i.e. $S^{p-1}$) to a bouquet of $p-1$ spheres = $X_{p-1}/X_{p-2}$, and then projected to a $p-1$ sphere. So the boundary map shows how the boundary of the cell is patched by $p-1$ spheres.

2.1 We would like to apply the previous settings to $X = \overline{M}_{0,n}(R)$. The associahedra give a cellular, not a simplicial decomposition of $X$. We have to take the first barycentric subdivision of $X$. Fortunately, the barycenters of the faces of the associahedra are already labelled by n-gons with a couple of diagonals, where the edges of the n-gons bear a permutation.

A maximal simplex in $X$ is a sequence of n-3 barycenters, i.e a chain of length n-3, which joins 2 vertices at the distance n-2, in the graph $G(n)$ from the previous section. Its dual block, of dimension 0, is its barycenter, which can be labelled by the simplex itself. In the notation from 2.0 section, $D_0$ is the free abelian group generated by these simplices.

A codimension 1 simplex of a simplex above is a face of the simplex above. Its dual block is a segment between 2 maximal simplices which share the same face ($X$ is a manifold !). $D_1$ is the free abelian group generated by these segments, arbitrarily oriented.

There are 2 types of segments: the segments inside the same associahedron, and the segments between 2 different associahedra.

Similarly, $D_2$ is the free abelian group generated by 4 segments which form the boundary of a 2-cell, arbitrarily oriented. Any dim 2 block is shared by exactly 4 dim 1 blocks and exactly 4 dim 0 blocks, thereby building a structure similar with the structure of 4 cubes in 3 dimensions.
The boundary morphisms between D’s are ”normal”: a segment goes to the difference of its vertices, and a 2-face goes to a sum of edges, correlated by signs.

\[ H_1(\overline{M}_{0,n}(R)) = \text{Ker}(\partial_1)/\text{Im}(\partial_2) \]

The barycenters of the maximal simplices above and the edges among them form a new graph, called GG(n). There is a following pictorial transformation between G(n) and GG(n): -the vertices of G(n) become circles. Between these circles, instead of one edge e, there are m(e) edges. m(e) is the following number: any edge e is decorated by a diagonal of the n-gon. m(e) is the number of diagonals which do not intersect e. The tips and the tails of these edges are inside the circles and there are connected by a system of pipes given by the barycentric subdivision of the associahedra.

The barycentric subdivision gives a simplicial decomposition of the real moduli.
The barycenters of the maximal simplices form the vertices of GG(n)
The arrows among them are the edges. G(n) shows the incidence relations between associahedra. GG(n) shows the incidence relations among the simplices of the first barycentric subdivision.

An element of \( D_1 \) is a formal sum of edges, decorated with reals. We can
decorate the edges of with any algebraic structure V. Between 2 vertices of G(n) there is a vector from the tensor product of m(e) copies of V.

Theorem 1. There is an isomorphism between the first homology groups with real coefficients of B(n) and $\mathcal{M}_{0,n}$. It is given by the following function F: If the m(e) edges (between circles) are decorated by a couple of real numbers in G(n), we associate their sum to the edge e in G(n). Using the fact that the associahedron is homeomorphic with a closed ball, it is easy to prove the statement above. It is true only for real coefficients! There are 3 questions to be answered in this proof:

1. If the definition is independent of the numbers (it depends only on the homology class), the image of a boundary is a boundary. By a boundary, we mean a formal sum of 4 edges, colored by + or -1, according to their orientation, which are geometric boundaries of 2-blocks etc.

2. If we can define an inverse of F. We can define a function G from the homology of $B(n)$ to $H_1(M)$, defined on generators in the following way: the edge e colored by number x goes to the sum of the m(e) edges from G(n), colored by the same number x/m(e) - a kind of trace-diagonal process. We do not have any obstructions: the associahedron is aspherical, so it is possible to assign numbers to the edges inside them, such that the result is a flux in G(n). (a flux is an assignment of numbers to edges such that the sum for every vertex is zero, see section 4). A physics of this process is given by the concept of ”pressure”: the pressure of a gas inside the associahedron is zero. It is just a distribution of pressures given by the numbers from the external faces.

3. If they are inverse to each other. It is easy, to see that this is the case.

3 Koszulness and Koszul Duality for Operads.

Combinatorics. The beginning of a project.

If, instead of real numbers, we decorate the graphs above with vector spaces, and ”arrows” means morphisms (we already fixed an arbitrarily orientation for edges),
our result is exactly the Theorem 4.3 of Loday [10]: the simplex and the Stasheff quadratic operads are Koszul.

The 2 operads above are Koszul dual to each other. A natural question is: what is the dual diagrammatic calculus?

An associahedron can be divided in standard simplices using the barycentric subdivision. Is it possible to divide a standard simplex in a couple of associahedra? For example, a 2-simplex can be divided in 9 pentagons.

Maybe an inductive study can show that it is true for any n. Is it the right structure for the dual picture?

In a very recent paper by Stanley [22] the ”right” picture showed up: A simplex can be divided in chambers, such that these chambers are products of lower dimensional simplices, and the partial order set, under inclusion, of the faces of the chambers are isomorphic with the associahedron. We call it ”the associahedron internal to an n-simplex”. Instead of ”associahedra” we can use standard simplices, glued together as the corresponding associahedra are glued together. This is just the beginning of the picture. An axiomatic treatment is needed to complete the duality. Also, it is worth to study the universality or the freeness of the above structures, as the role played by tangles for braided categories.

The subject above opened the following questions:
1. Is it possible to divide an n-simplex in a couple of associahedra. Is it possible
to apply induction?
2. Does the Stanley construction give rise to a simplicial complex equivalent with the real moduli spaces? A study of the automorphism group of Stanley’s construction can show how can we glue the faces of the simplices. This simplicial decomposition can be useful in homological computations.
3. Is it possible to build an operad based on simplices, following the ideas above? Lower dimensional faces of associahedra are products of lower dimensional associahedra. The chambers of the decomposition of an n-simplex are products of lower simplices. The natural boundary or co-boundary maps among chambers can give this structure.

4. Non-trivial 1-classes in the homology of the real moduli spaces $\overline{M}_{0,n}$

Because of the homology isomorphism above, it is enough to study $B(n)$, to get the desired results of this section. A partial study of these spaces is given in [5].

The notes below is a study of real vector spaces or subspaces generated by formal sums of oriented edges in $G(n)$. We found 4 special types of cycles in $B(n)$:

1. Cycles given by $k$ non-intersecting diagonals in the $n$-gon, and by $k$ numbers of zero sum. (picture 1 for $k=3$).
the vertices are the classes modulo D(n) of the following permutations: p, pa, pb, pc, pab, pac, pbc, pabc, where p is an arbitrary permutation and a,b,c are 3 non-intersecting diagonals (i.e. 3 commuting involutions from T, represented by 3 diagonals of an n-gon).

the orientation of the edges is not shown. The tips of the edges are the vertices with the smallest number of permutations in the product. i.e there is an arrow from pc to pbc; and from pab to pabc and so on.

for every vertex, the sum of the numbers attached to the edges is zero, so the picture shows a cycle in homology of B(n). This cycle is zero because it can be written as a sum of boundaries, represented by its faces. The boundaries have the coefficients +/- x,y or z.

In a similar way, k numbers with zero sum, and k commuting involutions (k non-intersecting diagonals) give a zero cycle in homology.

These cycles are boundaries, they are zero in homology. A cycle of this type is the 1-skeleton of a k-dimensional cube. For every k numbers $a_i$ with zero sum, we can decorate the vertices of the cycle with + or - $a_i$ in such a way we get a cycle. We can write the 1-skeleton like a sum of cycles of length 4, so it is zero in homology.

2. Cycles given by regular k-gons. (picture 2 for k=4)
3. Cycles given by Reiner’s duality [20]: Let $a$ and $b$ two vertices of $G(n)$ at the distance $n-2$. The vertices which are on the paths of length $n-2$ form a self-dual lattice. So, it is enough to color the edges of $G(n)$ with numbers, only for the first half of the lattice. The remaining part is the anti-symmetric.

We did not find a way to decide if the previous 2 types of cycles are zero or not in homology.

4. Cycles give by the graph $G(n, k)$, the subgraph of $G(n)$ induced by the vertices of distance at most $k$ from the identity vertex.
These are cycles of "finite type". For \( k=2 \), the computation of the dimension \( M \) of the space of cycles was made in \(^3\). These cycles survive in the whole homology of \( B(n) \), they cannot be cancelled by external boundaries.

Let us mention the formulas for \( M \), and a sketch of the proof.

\[
M = \begin{cases}
1 & \text{if } n = 4 \\
4 & \text{if } n = 5 \\
n(n - 3)/2 + P & \text{if } n > 5, n \neq 3k \\
n(n - 3)/2 + P + n/3[n+3/6] & \text{if } n > 5, n = 3k
\end{cases}
\]

\( P \) is the number of unordered pairs \((a,b)\), where \( a,b \) are 2 parallel diagonals in a regular \( n \)-gon. It is not hard to find a precise formula for \( P \): for every direction \( w \), we have to count how many diagonals are on this direction. If there are \( z = z(w) \) diagonals, then \( \binom{z}{2} \) is the contribution of \( w \). Any direction is given by the diagonals which pass through a fixed vertex \( A \). We still have 3 more directions, given by \( AB, AC \) and \( BC \), where \( AB \) and \( AC \) are edges of the \( n \)-gon.

Let \( A \) be a maximal set of vertices of \( G(n) \), such that the distance between any 2 vertices is greater than 4. In this case the small graphs of distance at most 2 from every vertex of \( A \) are disjointed. They give independent non-zero cycles in \( B(n) \) (see the proof of Theorem 2).

So, the rank of the first homology is at least \( M \) times the cardinal of \( A \).

Theorem: The cardinal of \( A \) is at least \((n-9)!\), if \( n \) is big enough.

Proof: \( A \) is maximal. so for any vertex \( x \) from \( G(n)-A \), there is a vertex \( b(x) \) from \( A \) such that the distance from \( x \) to \( b(x) \) is at most 4. Otherwise, we could add \( x \) to \( A \), in contradiction with the maximality of \( A \).

Let’s say that \( A \) is smaller than \((n-9)!\). Then, \( G(n)-A \) is greater than \( n! - (n-9)! = (n-9)!*P(n) \), where \( P \) is a polynomial of degree 9. \( b \) is a function from a set with at least \((n-9)!*P(n)\) elements, to a set \( A \) of at most \((n-9)!\) elements. So, there is an element \( y \) from \( A \), covered by at least \( P(n) \) elements from pre-image. So there are at least \( P(n) \) elements at the distance at most 4 from \( y \). The degree of every vertex is \( n(n-3)/2 \), so at most \( n^2 \). There are at most \( n^4 \) elements at the distance
2 from y. There are at most \( n^6 \) elements at the distance 3 from y. There are at most \( n^8 \) elements at the distance 2 from y. Totally, there are at most \( Q(n) = n^2 + n^4 + n^6 + n^8 \) vertices at distance at most 4 from x. For \( n \) big enough, \( P > Q \). Contradiction.

The sketch of the proof for the lower bound \( M \).

Definition. Let \( G(n, k) \) and \( B(n, k) \) be the graph and the sub-complex of \( B(n) \) generated by the vertices of distance at most \( k \) from the vertex represented by the identity.

Theorem 2. The inclusion \( i: B(n, k) \to B(n) \) gives a monomorphism in the 1-homology with real coefficients, if \( k \) is smaller than \( n-3 \).

Note: the condition ”\( k \) is smaller than \( n-3 \)” is essential. \( G(n) \) is a graph divided in levels but it natural metric. there are \( n-2 \) levels. All vertices from the first \( n-3 \) levels can be represented by ”permutations of type \( k \)” . A permutation of type \( k \) is a product of \( k \) non-intersecting involutions (”diagonals”) from \( T \).

Proof. Let \( a \) be a formal sum of edges which satisfy \( i_*(a) = 0 \). So, \( a \) is a sum of boundaries= a sum of boundaries from \( B(n, k) \) + a sum of boundaries outside \( B(n, k) \). The support of \( a \) is in \( B(n, k) \), so the last sum , let’s call it \( A \), is zero. \( A \) is a sum of partial sums, given by a couple of non-intersecting diagonals. The sum of the corresponding edges can be recovered by boundaries inside \( B(n, k) \), so \( a \) is zero in the homology of \( B(n, k) \). So, the kernel of \( i_* \) is zero.
The proof of the imbedding theorem.
i(a) is a sum of boundaries of 3 types:
(1)-boundaries inside $G(n,2)$
(2)-boundaries outside $G(n,2)$
(3)-boundaries with vertices on the levels 1, 2 and 3, i.e. level = the distance from the identity vertex. Also, there are boundaries of type (A), obtained by adding 2 non-intersecting diagonals and of type (B), when the picture is symmetrized about an extra diagonal.

We want to prove that $a$ is a sum of boundaries of type 1. There is no boundaries of type B outside $G(n,2)$: the support of $a$ (the set of edges with non-zero coefficients) is in $G(n,2)$. And the dotted edge below, from a boundary of type B, cannot be cancelled by other boundaries:

![Diagram 1](image1)

Boundaries of type 2 and 3 satisfy only the A condition. They can be canceled only by structures of the following type:

![Diagram 2](image2)

Otherwise, we cannot cancel an edge of a boundary, using other boundaries with vertices on a lower level. (we could go on lower levels indefinitely). Much more, this structure shows that sums of boundaries of type (3) equals sums of boundaries of type (1). QED.

Note: the left hand side picture shows only the diagonals 1,2,3,4. Other non-intersecting diagonals do not interfere.
Theorem 3. The rank of $H_1(B(n,2))$ is $M$.

We need "the theory of fluxes" from graph theory. For every oriented graph $G$, let $A$ be the real vector space of basis the oriented edges. The vectors are formal sums of these edges, with real coefficients. A flux is a formal sum which satisfy the following property: for every vertex $v$, the sum of the coefficients of the edges which enter in $v$ is equal to the sum of the coefficients of the edges which leave $v$. The vector space of all fluxes has the dimension $m-n+1$. [Bollobas pp.51-54], where $m$ is the number of edges and $n$ is the number of vertices of $G$. This formula gives us the opportunity to compute the rank as the difference between 2 dimensions of vectors spaces (the kernel and the imagine), the dimensions being separately computed.

The kernel is the space of fluxes. The image is studied using the properties of the graph $G(n,2)$. Using a similar proof as above, we can restrict to the complex $A(n,2)$, induced in $G(n,2)$ by all vertices except the vertex represented by identity.

Notation: $(M - N + 1)_T$ = the dimension of the space of fluxes in the graph $T$, of $M$ edges and $N$ vertices.

The rank of $H_1(A(n,2)) = (M - N + 1)_{A(n,2)} - \dim(V \cup W) = (M - N + 1)_{A(n,2)} - \dim(V) - \dim(W) + \dim(V \cap W)$, where

$V =$ the space generated by boundaries which are inside $A(n,2)$ (called ”relations”).

$W=$ the space of fluxes generated by boundaries outside $A(n,2)$, which contain the identity vertex.

$M$=the number of edges of $A(n,2) = n(n-3) / 2 * [n(n-3) / 2 - 1]$

$N$=the number of vertices of $A(n,2) = n(n-3) / 2 + S$, where $S = 1 / 3 \binom{n-3}{2} \binom{n+1}{2}$ (the number of pairs of non-intersecting diagonals).

$\dim(W) = m - n + 1_W = S - n(n-3) / 2 + 1$

$\dim(V \cap W)$= the dimension of the space $E$ generated by some special cycles of length 3 (supported by lines which form an equilateral triangle).

$\dim(V) = S$ - the number of pairs of parallel diagonals - $A$. $A$ is the number of independent relations between cycles of length 3, supported by lines which form
an equilateral triangle. So it is equal to the number of these cycles minus the dimension of the space generated by them in W. Fortunately, we do not need to compute this last number. \( A + \dim( V \cap W) \) = the number of equilateral triangles supported by 3 diagonals of an n-gon. We apply the formulas above and we compute the announced result on M.

There are 2 types of boundaries: in the first case, 2 vertices are represented by symmetric pictures of diagonals, with respect to a diameter d. the last 2 vertices are obtained from the first ones by adding a non-intersecting diagonal which is perpendicular to d.

In the second type of boundary, 2 non-intersecting diagonals are added the vertices are represented by : p,pa,pb,pab, where a and b are 2 commuting involutions from T, and p is an arbitrary permutation.

In A(n,2), the first type of boundary can be cancelled only by the structure above ,where 3 diagonals support an equilateral triangle.
a couple of diagonals a,b,c,d... such that 2 consecutive ones do not intersect
is a closed path (a cycle) in L(n)- the graph from the first section.
This structure gives a sum of boundaries with support in A(n,2). i.e it will be a
relation in A(n,2).

The result is a consequence of a combinatorial analysis on the graph $G(n, 2)$. It is
fine enough to be carefully performed, but it is elementary: it is a study of formal
sums of n-gons with 1 or 2 diagonals, with real coefficients.

**Notes about the graphs G and $G(n, 2)$:** $\text{diam}(G)=n-2$ (proof by induction).
Every vertex has degree $n(n-3)/2$. For any vertex $v$, take a permutation $s$ from
$v$, which is a class modulo $D_n$. Then $s\mathcal{d}$, where $\mathcal{d}$ runs in DIAG, represents the
neighbors of $v$.

2 vertices are joined by an edge in only 2 situations: ”a non-intersecting di-
agonal is added.” Or the picture is symmetrized about a diameter perpendicular
to a diagonal $d$, and $d$ disappears. More precisely: in these vertices, there are 2
permutations represented by a product of a couple of permutations from DIAG,
and there are only 2 situations when the vertices are joined.

There are 2 types of good cycles of length 4: when the opposite vertices are
on different levels, or when the opposite vertices are on the same level. The
first situation is possible when we add 2 non-intersecting diagonals.In the second
situation, the 4 vertices are like: $(a, b, ac, bc)$, where $b$ is the symmetric of $a$ with
respect to the diameter perpendicular on $c$ (called the symmetry diagonal). $a$ and
c are non-intersecting diagonals.

So the good cycles from $V$ are linearly independent (because the edges are
met only one time, in these structures), except for one situation: if there are
a,b,c whose lines form an equilateral triangle, the 3 good cycles whose symmetry
diagonals are a,b,c are not independent: their sum is a vector of W. But we can have a couple of these triads, and the sum can be zero (so we have a relation). This explains the computations of the dimensions of vector spaces above.

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