Gravitational Scattering in the $c = 1$ Matrix Model

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ABSTRACT

The $c = 1$ matrix model is equivalent to $1 + 1$ dimensional string theory. However, the tachyon self-interaction in the former is local, while in the latter it is nonlocal due to the gravitational, dilaton and higher string fields. By studying scattering of classical pulses we show that the appropriate nonlocal field redefinition converts the local matrix model interaction into the expected string form. In particular, we see how the asymptotic behavior of the gravitational field appears in the scattering.
0 A Digression

The exact solution of low dimensional string theory by means of matrix models was a remarkable discovery[1, 2]. In spite of the substantial effort in this area, one must feel that the physical content of the solution has not been fully developed. In this paper we report on a further step in this direction, after a brief discussion of some general issues.

It is sometimes said that little has been learned from the matrix models. This is not true. Matrix models have taught us a vital lesson: that the “Theory of Everything” is not string theory. Let us elaborate, focussing first on the closely related string theory of two-dimensional U(N) gauge theory[3]. Canonically quantized on a circle, a typical invariant state is

\[ \text{tr}(U)^{n_1}\text{tr}(U^2)^{n_2}\ldots\text{tr}(U^m)^{n_m}, \]

where \( U \) is the holonomy around the circle. For convenience we focus on one chiral sector—that is, positive powers of \( U \). The trace \( \text{tr}(U^k) \) can be associated with a string that winds \( k \) times around the circle[4, 5]. In particular, one can introduce creation and annihilation operators for the \( k \)-times wound string,

\[ [a_k, a_l^\dagger] = k\delta_{k,l}. \]

The state (1) is then proportional to

\[ a_1^{n_1}a_2^{n_2}\ldots a_m^{n_m}|0\rangle \]

where \( |0\rangle \) is the constant wavefunction. The Hamiltonian can be written as a string tension plus a splitting-joining interaction,

\[ H = \frac{g^2L}{2} \sum_{k=1}^{\infty} a_k^\dagger a_k + \frac{g^2L}{2N} \sum_{k,k'=1}^{\infty} (a_{k+k'}^\dagger a_k a_{k'} + h.c.), \]

as well as a contact (zero-size handle) term depending on the U(1) factor.

As long as the total number of string windings \( \sum_{k=1}^{\infty} kn_k \) is less than \( N \), the states (1) are independent and in fact orthogonal under the group integration, as implied by the representation (3) and the algebra (2). But for \( N \)
or more windings this fails: for example \( \text{tr}(U^N) \) can be expanded in terms of lower traces. So while the stringy Hamiltonian correctly reproduces the perturbation series in \( 1/N \), it fails non-perturbatively. The point is not merely that there are non-analytic terms in the \( 1/N \) expansion, but the stringy description itself, the enumeration of states, is breaking down.\(^1\) In this case a better description is known—the theory can be put in fermionic form and this description is exact\(^4\)\(^5\). The breakdown of the string picture has a simple interpretation in the fermionic language. The number of windings corresponds to the total number of levels by which the fermions are promoted from the ground state. The bosonic description does not know that the Fermi sea has both an upper and a lower edge (with a total of \( N \) filled levels) and for \( N \) or more windings it includes states where a fermion is promoted from below the lower edge to above the upper—but the former state is actually empty to start with.

Exactly the same issue arises in the \( c = 1 \) matrix model. The fermionic description is well-defined. The bosonic (string) description is valid near one edge of the Fermi surface but breaks down when both the upper and lower edge become involved. This is nonperturbative in the string coupling, occurring when the density of string is of order \( 1/g_s \). Again, this is not like field theory, where the perturbation series is asymptotic but the theory is in principle exact—here the string theory itself is only an asymptotic description and new variables are needed.

It could be that this is special to the matrix model and does not apply to higher-dimensional strings, but there are several signs that it is general. One is the non-field theoretic \( e^{-O(1/g_s)} \) nonperturbative behavior\(^6\). Another is the unwieldiness of string field theory—the need to correct the covariant closed string theory at each order of perturbation theory\(^7\), and the fact that the related space of all two-dimensional field theories does not seem to have a natural definition.

So we are proposing that not only is string perturbation theory merely

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\(^1\)This point arose in discussions with M. Douglas, A. Strominger and M. Stone at the ITP Workshop on Nonperturbative String Theory. The need to supplement the string description with a projection has also been discussed recently by Taylor (seminar at UCSB).
asymptotic, but that string theory itself only generates the asymptotics of the “Theory of Everything.” Finding the correct description is vital, both because the $e^{-O(1/g_s)}$ effects are apt to be numerically as or more important than the familiar $e^{-O(1/g_s^2)}$ effects, and because we might hope that it will bring in new concepts that are essential to understanding such issues as the physics of the vacuum.

1 Introduction

We now return to a narrower issue—finding spacetime gravitational dynamics in the matrix model. The dilaton-graviton sector of two-dimensional string theory should have interesting dynamics, including black holes [8, 9]. Further, this string theory is exactly solvable through the $c = 1$ matrix model [10]. One would like to make use of this solution to address basic questions, including the effect of string theory on spacetime singularities and the full quantum evolution of the black hole. But while various proposals have been made, it is not clear how the black hole background is described in the matrix model. It should be possible to study the same processes as in dilaton gravity [11] and its generalizations, where pulses containing energy and information are sent toward the strong coupling region and a black hole forms and then evaporates. Or, if this process does not occur in the string theory, one would like a clear understanding of why this is the case.

There is an argument which would appear to indicate that gravitational effects are for some reason absent in the matrix model. Imagine sending two matter (tachyon) pulses toward the strong coupling region, one after the other, as shown in figure 1. The first pulse carries energy and so will produce a gravitational field; the second pulse will then have some amplitude to back-scatter off this field. But in the matrix model, these pulses are packets of non-interacting fermions which travel freely in the inverted harmonic oscillator potential [12]. The first packet thus does not affect the motion of the second.

The resolution of this paradox is in principle known, though it has not been developed in this time-dependent context. The back-scattering process is “bulk” scattering, which is indeed absent in the matrix model [13]. How-
ever, the string S-matrix differs by a certain wavefunction renormalization and has nonzero bulk scattering\cite{14, 15, 16}. The renormalization, although linear in the fields and merely a phase for real momenta, is able to convert an interacting theory into a non-interacting one because the kinematics restricts the scattering to particular points in the complex momentum plane where the renormalization factor vanishes.

It is sometimes stated that this wavefunction renormalization, being a phase, does not affect probabilities and so can be ignored. But an energy-dependent phase produces a time delay, and we are specifically interested in time-dependent processes.\footnote{Put differently, the renormalization is a pure phase only in a particular basis, and the states of interest to us will necessarily be superpositions of these basis elements.} Indeed we will see that the renormalization plays an essential role.

In this paper we consider only the classical scattering of pulses that are not too large, in that the Fermi surface remains single valued and does not pass over the potential barrier. In a sense all of our results are then obvious a priori, from refs. \cite{14, 15, 16}. But given the confusion in this subject, it is worth working out in detail this point of contact between the matrix model and the continuum string theory. The calculation is slightly convoluted, and is a necessary preliminary to studying the more interesting dynamics of large pulses.

## 2 Review of Matrix Model Scattering

We first review the classical solutions to the matrix model, following refs. \cite{12, 17}. The Hamiltonian is

$$ H = \frac{1}{2} \int_{-\infty}^{\infty} dx \left\{ \partial_x \psi^\dagger \partial_x \psi - x^2 \psi^\dagger \psi \right\} $$

$$ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left\{ \frac{1}{6} (p_+^3 - p_-^3) - \frac{x^2}{2} (p_+ - p_-) \right\} \quad (5) $$

where $p_{\pm}$ are the upper and lower surfaces of the Fermi sea. Our conventions are as in ref. \cite{12} with two changes. We now set $\alpha' = 1$ so that the matrix
model embedding time coincides with that of the continuum theory; this also simplifies most expressions. \(^3\) And, we now omit the factors of \(g_s\) from the definitions (11) of that paper in order that the Hamiltonian be independent of \(g_s\). \(^4\) Then \(g_s\) enters only as a parameter in the static solution,

\[
p_\pm = \pm \sqrt{x^2 - g_s^{-1}}.
\]

(6)

Focusing on one side of the barrier, say \(x < 0\), the theory can be written in terms of a canonically normalized massless scalar \(\overline{S}(q,t)\), where \(x = -e^{-q}\):

\[
p_\pm(x, t) = \mp x \pm \frac{1}{x} \epsilon_\pm(q, t)
\]

\[
\epsilon_\pm(q, t)/\sqrt{\pi} = \pm \Pi(q, t) - \partial_q \overline{S}(q, t).
\]

(7)

Here we introduce a bar to distinguish the matrix model objects here from the related string theory objects to be discussed in the next section. The Hamiltonian takes the form

\[
H = \frac{1}{2} \int_{-\infty}^{\infty} dq \left\{ \Pi^2 + (\partial_q \overline{S})^2 + e^{2q}O(\overline{S}^3) \right\}.
\]

(8)

The trilinear coupling vanishes as \(e^{2q}\) in the asymptotic region \(q \to -\infty\) and \(\overline{S}\) can be expanded asymptotically as the static solution plus a massless free field,

\[
\overline{S}(q, t) \sim -\frac{q}{2\sqrt{\pi} g_s} + \overline{S}_+(t - q) + \overline{S}_-(t + q)
\]

\[
\overline{S}_\pm(t \mp q) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\sqrt{2i\omega}} \alpha_\pm(\omega) e^{i\omega(t \mp q)}.
\]

(9)

Asymptotically,

\[
\epsilon_\pm(t \mp q) \sim \frac{1}{2g_s} + \delta_\pm(t \mp q \pm \ln \sqrt{4g_s})
\]

\[
\delta_\pm(t \mp q) = \pm \sqrt{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha_\pm(\omega)(4g_s)^{\mp i\omega/2} e^{i\omega(t \mp q)}.
\]

(10)

\(^3\)In ref. [12], only eq. (44) is affected by this.

\(^4\)Let us also here note two misprints in ref. [12]. Eq. (25) should read \(x = -e^{-q}\). In eq. (34) the last \(\pm\) should be \(\mp\)—the corrected form is given in eq. (10) below, now with \(g_s\)-dependence.
Classical solutions are described by the Fermi surface moving freely in the inverted potential. The outgoing Fermi surface is related to the incoming surface in a nonlinear way through the time delay:

$$\epsilon_-(u) = \epsilon_+(u'), \quad u' = u + \ln(\epsilon_-(u)/2).$$

By changing variables $u \rightarrow u'$ and using (11), one finds[17]

$$\int_{-\infty}^{\infty} du \, (\epsilon_-(u))^r e^{-i\omega u} = 2^{-i\omega} \frac{r}{r + i\omega} \int_{-\infty}^{\infty} du' \, (\epsilon_+(u'))^{r+i\omega} e^{-i\omega u'}$$

for arbitrary parameters $\omega$ and $r$. Expanding around the static background as in (10) gives

$$\delta_-(u) = \sum_{n=1}^{\infty} \frac{(2gs)^{n-1}}{n!} \frac{\Gamma(1 + \partial_u)}{\Gamma(2 - n + \partial_u)} (\delta_+(u))^n$$

or

$$\alpha_-(\omega) = -\sum_{n=1}^{\infty} \frac{(g_s\sqrt{8\pi})^{n-1}}{n!} \frac{\Gamma(1 + i\omega)}{\Gamma(2 - n + i\omega)} (4gs)^{-i\omega}$$

$$\left\{ \prod_{i=1}^{n} \int \frac{d\omega_i}{2\pi} \alpha_+(\omega_i) \right\} 2\pi \delta(\omega - \sum_{i=1}^{n} \omega_i).$$

This classical result becomes a tree-level operator statement, giving the tree-level S-matrix. For example, the $n \rightarrow 1$ amplitude is

$$S_{\omega_1, \ldots, \omega_n \rightarrow \omega_{n+1}} = \langle 0 | \alpha_-(-\omega_{n+1}) \alpha_+(\omega_1) \ldots \alpha_+(\omega_n) | 0 \rangle$$

$$= 2\pi i\delta(\omega_{n+1} - \sum_{i=1}^{n} \omega_i) \left\{ \prod_{i=1}^{n} \omega_i \right\} \left( \frac{\pi}{2} \right)^{-i\omega_{n+1}/2} \frac{\partial^{n-2}}{\partial \mu^{n-2}} \mu^{-i\omega_{n+1}-1}$$

where $\mu^{-1} = gs\sqrt{8\pi}$ and $[\alpha_-(\omega), \alpha_+(\omega')] = 2\pi \delta(\omega + \omega').$

### 3 Wavefunction Renormalization

At tree level, the S-matrix of two-dimensional string theory has also been obtained directly with continuum methods[14, 15], and differs from the matrix model result above only by the multiplicative factor

$$S_{\omega_1, \ldots, \omega_n \rightarrow \omega_{n+1}} = \tilde{S}_{\omega_1, \ldots, \omega_n \rightarrow \omega_{n+1}} \left( \frac{\pi}{2} \right)^{i\omega_{n+1}/2} \prod_{i=1}^{n+1} \frac{\Gamma(i\omega_i)}{\Gamma(-i\omega_i)}.$$

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In other words, these are equivalent under the redefinition

\[
\alpha_+(\omega) = \left(\frac{\pi}{2}\right)^{i\omega/4} \frac{\Gamma(i\omega)}{\Gamma(-i\omega)} \alpha_+(\omega)
\]

\[
\alpha_-(\omega) = \left(\frac{\pi}{2}\right)^{-i\omega/4} \frac{\Gamma(-i\omega)}{\Gamma(i\omega)} \alpha_-(\omega).
\]  

(17)

For real \(\omega\) this is indeed just a phase.

The string theory tachyon is

\[
S(t,\phi) \sim S_+(x^-) + S_-(x^+) \quad \text{where} \quad \phi \text{ is the Liouville field}, \quad x^\pm = t \pm \phi, \quad \text{and}
\]

\[
S_\pm(x^\mp) = \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi \sqrt{2\omega}} \alpha_\pm(\omega) e^{i\omega x^\mp}.
\]  

(18)

The relation of this to the matrix model scalar is thus

\[
S_\pm(x^\mp) = \left(\frac{\pi}{2}\right)^{\pm\partial^4/4} \frac{\Gamma(\pm\partial)}{\Gamma(\mp\partial)} S_\pm(x^\mp) = \left(\frac{\pi}{2}\right)^{-\partial^4/4} \frac{\Gamma(-\partial)}{\Gamma(\partial)} S_\pm(x^\mp).
\]  

(19)

To describe the scattering of an incoming (+) string tachyon pulse, one must (I) transform to the matrix model tachyon field via (19), (II) evolve the pulse as described in the previous section, and (III) transform back. The first and third steps can be written

(I) : \[ S_+(x^-) = \int_{-\infty}^{\infty} d\tau K(\tau) S_+(x^- - \tau) \]

(III) : \[ S_-(x^+) = \int_{-\infty}^{\infty} d\tau K(\tau) S_-(x^+ - \tau). \]  

(20)

The same kernel appears in both transformations,

\[
K(\tau) = \int_{-\infty}^{\infty} d\omega e^{i\omega \tau} \left(\frac{\pi}{2}\right)^{-i\omega/4} \frac{\Gamma(-i\omega)}{\Gamma(i\omega)}
\]

\[
= -\frac{z}{2} J_1(z), \quad z = 2(2/\pi)^{1/8} e^{\tau/2}.
\]  

(21)

This has asymptotic behaviors

\[
K(\tau) \sim -\left(\frac{\pi}{2}\right)^{-1/4} e^{\tau}, \quad \tau \to -\infty
\]

\[
\sim \left(\frac{\pi}{2}\right)^{-1/16} e^{\tau/4} \cos(z + \pi/4), \quad \tau \to \infty.
\]  

(22)
Thus, if we start with a delta-function pulse at $x^- = 0$ we get a pulse spread out in time, with an exponential tail at negative times and a tail which grows and oscillates more and more rapidly at late times. The late oscillations drop out when we have smooth wave-packets. To see this, take $S_+$ to be a gaussian wave packet of width $\ell$ in time, so that
\[ \alpha_+(\omega) \propto \omega e^{-(\omega - \omega_0)^2 \ell^2/2}. \] (23)

Then
\[ S_+(x^-) \propto \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega x^- - (\omega - \omega_0)^2 \ell^2/2} \left( \frac{\pi}{2} \right)^{-i\omega/4} \Gamma(-i\omega) \Gamma(i\omega). \] (24)
For large negative $x^-$, one can shift the integral into the lower half-plane, keeping it parallel to the real axis, and the integral is dominated by the nearest feature (pole or saddle point) in this half-plane. This is the pole at $\omega = -i$, so
\[ S_+(x^-) \propto e^{x^-}, \quad x^- \to -\infty \] (25)
the same as found for the delta-function. For large positive $x^-$, the integral is dominated by the nearest feature in the upper half-plane. There are no poles, but there is a saddle near $\omega = ix^-/\ell^2$ (note that the gaussian dominates the gamma functions at large imaginary $\omega$). The late-time behavior is then a nearly gaussian falloff. That is, a pulse with gaussian falloff is transformed to one which is still localized but with an exponential spread at early times. This early-time exponential will play an important role. The late oscillations of the kernel will play no role in the present work, though they may in more complicated situations.

Although the individual steps are simple, the net result is more complicated and less intuitive than the familiar matrix model evolution without the convolutions. In order to develop some familiarity with this, our goal in the present paper is to see how it gives rise to the gravitational effects discussed in the introduction.
4 String Scattering

The distinction between relatively slowly falling exponential wavepackets and more rapidly falling Gaussian wavepackets will be essential. This is because the gravitational field that we wish to detect itself falls off exponentially, with $G_{tt} - 1 \propto M e^{4\phi}$. Thus we need much narrower wavepackets in order to distinguish the 'long-ranged' gravitational interaction from the tachyon self-interaction, which we would expect to be local or at most smeared in a Gaussian way. One can then see how the convolution, which as we have seen turns a Gaussian into an exponential, can transmute the local matrix model interaction into a long-ranged gravitational one.

We will expand in powers of the incoming tachyon $S_+$, as in the solution $(14)$. The gravitational effect we seek appears at third order. To first order, the result of convolution-evolution-convolution is

$$S^{(1)}_-(x^-) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\sqrt{2i\omega}} e^{i\omega x^-} \mu \frac{\Gamma^2(-i\omega)}{\Gamma^2(i\omega)} \alpha_+(\omega)$$

(26)

where again $\mu^{-1} = g_s \sqrt{8\pi}$. Here $\alpha_+(\omega)$ are the modes of the incoming classical pulse. We take the incoming pulse to have a Gaussian falloff and to be centered near $x^- = 0$, perhaps a finite sum of terms of the form $(23)$. The center of the outgoing pulse is then near $x^+ = 0$. More precisely, its parametric dependence on $g_s$ is $x^+ \sim \ln g_s$, because as $g_s$ is increased the Fermi level approaches the top of the potential and the time delay increases.

We now wish to pull out the leading behavior of the outgoing wave at early times, $x^+ \to -\infty$. This is obtained by the same method as the asymptotic behavior $(23)$, being dominated by the pole at $\omega = -i$. Then,

$$S^{(1)}_-(x^+) \sim \mu \int_{-\infty}^{\infty} du^- (x^+ - u^- - c) e^{x^+ - u^-} S_+(u^-)$$

(27)

with $c = 2 + 4\Gamma'(1) - \ln \mu$. This has a simple interpretation, as we will verify by an effective Lagrangian calculation in the next section. Note that it is first order in the background $\mu$. The linear term in the integrand, from the double pole in $(23)$, comes from the linear behavior of the tachyon background.
To second order, convolution-evolution-convolution gives

\[ S^{(2)}(x^+) = -\frac{1}{2\sqrt{2}} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(2\pi)^2} \left\{ e^{i\omega x^+} \frac{\mu^{i\omega - 1} e^{i\omega x^+}}{\Gamma(-i\omega) \Gamma(-i\omega_1) \Gamma(-i\omega_2) \Gamma(i\omega_1) \Gamma(i\omega_2)} \alpha_+(\omega_1)\alpha_+(\omega_2) \right\} \]  

(28)

where \( \omega_1 + \omega_2 = \omega \). The leading behavior as \( x^+ \to -\infty \) is again governed by the first pole encountered as the \( \omega \) contour is shifted, parallel to the real axis, into the lower half-plane. Of course, as the \( \omega \) contour is shifted, \( \omega_1 \) and/or \( \omega_2 \) must also become complex. It is most efficient, in the sense of avoiding spurious leading terms which actually cancel, to keep the poles in \( \omega_1 \) and \( \omega_2 \) as far from the axis as possible by dividing the imaginary part equally between \( \omega_1 \) and \( \omega_2 \). The first pole is then at \( \omega = -i \). Evaluating the residue gives

\[ S^{(2)}(x^+) \sim -\frac{1}{2\sqrt{2}} \int_{-\infty}^{\infty} e^{x^+} \frac{d\omega_1}{2\pi} \frac{\alpha_+(\omega_1)}{\omega} \frac{\alpha_+(-\omega_1 - i)}{-\omega_1 - i}, \quad \text{Im}(\omega_1) = -\frac{1}{2} \]

\[ = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} du^- e^{x^+-u^-} S^2_+(u^-). \]  

(29)

Note that because of the gaussian falloff of \( S_+ \), its Fourier transform \( \alpha_+(\omega)/\omega \) is well-defined and analytic for all complex \( \omega \); in particular the position of the \( \omega_1 \) contour doesn’t matter in the final step. This is bulk scattering of two incoming tachyons into one outgoing. We will verify that this can be obtained from an effective Lagrangian in the next section, but the main features are easily understood. The spacetime dependence follows from the position dependence of the coupling—the outgoing ray of fixed \( t + \phi = x^+ \) meets the incoming ray of fixed \( t - \phi = u^- \) at \( 2\phi_0 = x^+ - u^- \), at which point the coupling constant is \( e^{2\phi_0} \). Also, the amplitude is zeroth order in the background \( \mu \); scatterings involving the background would involve more interactions and so are subleading as \( x^+ \to -\infty \).

To third order in the incoming field,

\[ S^{(3)}_-(x^+) = \frac{1}{6\sqrt{2}} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3} \left\{ e^{i\omega x^+} (1 - i\omega) \mu^{i\omega - 2} \right\} \]  

(30)

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with $\omega_1 + \omega_2 + \omega_3 = \omega$. Again the leading behavior as $x^+ \to -\infty$ is given by the first pole encountered in the lower $\omega$ plane, and again it is efficient to divide the imaginary part equally among $\omega_1$, $\omega_2$ and $\omega_3$. The first pole is then at $\omega = -2i$, giving

$$S^{(3)}_-(x^+) \sim -\frac{1}{12\sqrt{2}} e^{2x^+} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(2\pi)^2} \left\{ \frac{\Gamma(-i\omega_1) \Gamma(-i\omega_2)}{\Gamma(i\omega_1) \Gamma(i\omega_2)} \right\} \frac{\Gamma(i\omega_1 + i\omega_2 - 2)}{\Gamma(-i\omega_1 - i\omega_2 + 2) \alpha_+(-i\omega_1) \alpha_+(-i\omega_2) \alpha_+(i\omega_1 + i\omega_2 - 2)},$$

$$\text{Im}(\omega_1) = \text{Im}(\omega_2) = -\frac{2}{3}.$$ 

This represents bulk scattering of three incoming tachyons into one outgoing.

To identify the long-ranged gravitational interaction we now take the incoming field to be a sum of two gaussian pulses, the first centered at $x^- = 0$ and the second at $x^- = T$. That is,

$$\alpha_+(\omega) = f_{1+}(\omega) + e^{-i\omega T} f_{2+}(\omega),$$

where $\omega f_{1+}$ and $\omega f_{2+}$ are both real gaussians as in (23). The derivation of eq. (31) still goes through, and now we can extract the leading $T$-dependence as we did for $x^+$ before, thus distinguishing the exponential gravitational interaction from the gaussian local interactions. The gravitational field of the first pulse is second order in $f_{1+}$, and we wish to identify the linear scattering of the second pulse in this field, so the relevant terms from the third-order solution (31) are of the form

$$3e^{-i\omega T} f_{2+}(-i\omega_1) f_{1+}(-i\omega_2) f_{1+}(i\omega_1 + i\omega_2 - 2).$$

The first two terms at large $T$ are from $\omega_1 = -i, -2i$, giving

$$S^{(3)}_-(x^+) \sim -\frac{1}{4\sqrt{2}} e^{2x^+} f_{2+}(-i) \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{f_{1+}(\omega_2) f_{1+}(-\omega_2 - i)}{\omega_2 - \omega_2 - i} -\frac{1}{8\sqrt{2}} e^{2x^+} e^{-2T} f_{2+}(-2i) \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} f_{1+}(\omega_2) f_{1+}(-\omega_2).$$
In the final expression, the first term is the scattering of pulse 2 from the second-order tachyon background (29) produced by pulse 1, while the second is the gravitational scattering we seek. That is, these represent respectively the exchange of a tachyon and a graviton between the two pulses. Notice in particular that the gravitational term depends on pulse 1 precisely through its integrated energy flux. Also, the scattering occurs where the incoming and outgoing rays meet, which is again $2\phi_0 = x^+ - u^-$, and the gravitational term is then proportional to $e^{4\phi_0}$. Thus we reconstruct the leading correction to the metric, $\delta G_{tt} \propto M e^{4\phi}$.

5 Effective Field Theory

Let us now verify in detail that the results we have found are equivalent to those from a tachyon-graviton-dilaton effective field theory. The spacetime action is

\[
S = \frac{1}{2} e^{2\phi_0} \int_{-\infty}^{\infty} du^- e^{-u^-} S_{2+}(u^-) \int_{-\infty}^{\infty} dv^- e^{-v^-} S_{1+}^2(v^-) - \frac{1}{2} e^{2\phi_0} \int_{-\infty}^{\infty} du^- e^{-2u^-} S_{2+}(u^-) \int_{-\infty}^{\infty} dv^- \dot{S}_{1+}^2(v^-). \tag{34}
\]

The absolute normalization, which does not enter into the classical solution, is set by a shift of the dilaton $\Phi$. The constant $a_1$ setting the relative normalization of the graviton-dilaton and tachyon actions will be determined implicitly by the definition of the tachyon field below. The relevant part of the tachyon self-interaction is

\[
V(T) = a_2 T^3 / 3. \tag{36}
\]

A local quartic interaction will not contribute to the processes we consider because of our use of wavepackets to resolve the interactions in time. The cubic interaction could have been a function of the tachyon momenta, but is known from the vertex operator calculation of the three-point amplitude to be
constant up to field redefinition; we verify this below. Other higher-dimension operators are expected not to affect the leading \( x^+ \to -\infty \) behavior that we consider.

The field equations are
\[
R_{\mu \nu} + 2 \nabla_{\mu} \nabla_{\nu} \Phi - a_1^{-1} \partial_{\mu} T \partial_{\nu} T = 0
\]
\[
R + 4 \nabla^2 \Phi - 4 (\nabla \Phi)^2 + 16 - a_1^{-1} (\nabla T)^2 + 4 a_1^{-1} T^2 - \frac{2}{3} a_2 a_1^{-1} T^3 = 0
\]
\[
\nabla^2 T - 2 \nabla \Phi \nabla T + 4 T - a_2 T^2 = 0.
\] (37)

To zeroth order in the tachyon, the dilaton and metric backgrounds are
\[
\Phi_0 = 2 \phi, \quad G_{0\mu\nu} = \eta_{\mu\nu}.
\] (38)

The tachyon \( T \) is related to the massless scalar \( S \) of previous sections by
\[
T = e^{2\phi} S.
\] (39)

The \( \phi \to -\infty \) behavior of the tachyon background is given by the linearized solution,
\[
T_0 \sim (b_1 \phi + b_2) e^{2\phi}.
\] (40)

The constant \( b_2 \) is determined in terms of \( b_1 \) by the full nonlinear tachyon interaction [18], as we will see below.

Henceforth we work in conformal gauge, \( ds^2 = -e^{2\rho} dx^+ dx^- \). We again expand in powers of the incoming tachyon, \( T = T_0 + T^{(1)} + T^{(2)} + T^{(3)} + \ldots \). To the order we will be working, only the first order correction to the gravitational and dilaton backgrounds enters. Taking \( \Phi = \Phi_0 + \delta \) and linearizing in \( \delta \) and \( \rho \), the graviton-dilaton field equations to \( O(T^2) \) can be written
\[
a_1 (\partial_+ - 2) \Omega = -(\partial_+ T)^2 + T^2
\]
\[
a_1 (\partial_- + 2) \Omega = (\partial_- T)^2 - T^2
\]
\[
2 a_1 \partial_+ \partial_- \delta = 2 a_1 \Omega + T^2,
\] (41)

where \( \Omega = 2(\partial_- - \partial_+) \delta + 4 \rho \). The tachyon equation is
\[
\partial_+ \partial_- S^{(1)} = -\frac{a_2}{2} T_0 S^{(1)}
\]
\[ \partial_+ \partial_- S^{(2)} = -\frac{a_2}{4} e^{x^+-x^-} (S^{(1)})^2 - \frac{a_2}{2} T_0 S^{(2)} \]

\[ \partial_+ \partial_- S^{(3)} = \frac{1}{2} \Omega S^{(1)} + \partial_+ \delta \partial_- S^{(1)} + \partial_- \delta \partial_+ S^{(1)} \]

\[ -\frac{a_2}{2} e^{x^+-x^-} S^{(1)} S^{(2)} - \frac{a_2}{2} T_0 S^{(3)}. \]  

These are now solved using the retarded Green function \( G(x^+, x^-) = \theta(x^+)\theta(x^-) \), which satisfies \( \partial_+ \partial_- G(x^+, x^-) = \delta(x^+)\delta(x^-) \). The initial condition is

\[ S^{(1)}(t, \phi) \to S_+(x^-), \quad S^{(2,3,...)}(t, \phi) \to 0 \]  

for \( t \to -\infty \). The leading behavior of the outgoing \( S^{(1)} \) as \( x^+, x^- \to -\infty \) comes from the leading behavior of the background tachyon. Integrating the first-order equation gives

\[ S^{(1)}_-(x^+) \sim -\frac{a_2}{2} \int_{-\infty}^{\infty} du^- \left\{ b_1 (x^+ - u^-) + (b_2 - b_1) \right\} e^{x^+ - u^-} S_+(u^-). \]  

This is the same as the matrix model result (27), with \( b_2 = b_1 (-1 - 4\Gamma'(1) + \ln \mu) \) now determined, and \( b_1 = -2\mu/a_2 \). The \( \phi \) and \( \mu \)-dependence of the tachyon background is as argued in ref. [18].

In the higher order equations (42), the effect of the background tachyon is subleading as \( x^+ \to -\infty \) (both the explicit terms, and the implicit dependence through the graviton-dilaton back-reaction) and so we ignore it. The leading outgoing wave at second order is then

\[ S^{(2)}_-(x^+) \sim -\frac{a_2}{4} \int_{-\infty}^{\infty} du^- e^{x^+ - u^-} S^2_+(u^-), \]  

agreeing with the matrix model result (29) and determining \( a_2 = -2\sqrt{2} \) and \( b_1 = \mu/\sqrt{2} \). To third order we integrate the graviton-dilaton equations (41) and then the tachyon equation to get

\[ S^{(3)}_-(x^+) \sim \frac{1}{2} e^{2x^+} \int_{-\infty}^{\infty} du^- e^{-u^-} S_{2+}(u^-) \int_{-\infty}^{\infty} dv^- e^{-v^-} S^2_{1+}(v^-) \]

\[ -\frac{1}{4a_1} e^{2x^+} \int_{-\infty}^{\infty} du^- e^{-2u^-} S_{2+}(u^-) \int_{-\infty}^{\infty} dv^- S^2_{1+}(v^-). \]  

Again this agrees, and fixes the final constant \( a_1 = \frac{1}{2} \).
6 Conclusions

In a sense we have only worked out in coordinate space what is already known in momentum space, that the difference between the trivial bulk S-matrix of the matrix model and the nontrivial one of two-dimensional string theory is the normalization of the vertex operators. It is in coordinate space, however, that the significance of the difference becomes clear: it is a non-local field redefinition, which because of the simple kinematics in two dimensions can convert the local matrix model interaction into the nonlocal interaction from the gravitational and other higher fields of string theory.

In particular one learns that the simplicity of the matrix model is rather deceptive. Consider the schematic representation in figure 2 of the gravitational scattering (steps I, II, and III are as defined below eq. (19)). In step I, the exponential pre-tail produced by the convolution of pulse 2 has an overlap with pulse 1. In step II the combined pulse reflects off the end of the eigenvalue distribution, the “wall.” In step III the final convolution produces an exponential pre-tail on the outgoing pulse, which is the bulk scattering of interest. On the other hand, one believes that the actual physical picture is that pulse 2 scatters off the gravitational field of pulse 1 before it ever reaches the wall. So the matrix model does not reflect the qualitative physics of the scattering process.

One could extend our exercise to higher orders and so to higher string fields, but it seems more efficient to try to work directly at the Lagrangian level. The key seems to be to combine steps I through III so as to write the exact solution in a way which correctly represents the locality properties of the interaction. The first step would be the field redefinition (19), at least in the asymptotic free-field region. The result will be a non-local action, which presumably can be restored to a local form by additional non-linear field redefinitions as well as the introduction of additional non-dynamical fields (the string dilaton and metric, and higher).

We should emphasize that there is no local relation between the spacetime metric and the matrix model field. Such a relation has occasionally been proposed, but it is clear that it cannot exist because the gravitational field
at a given point must depend on the total energy interior to the point, as found in the scattering (34).

It is not immediately obvious how to produce a black hole from incoming tachyons, or to represent an eternal black hole in the matrix model. The former question in particular requires that we understand better the strongly nonlinear solutions to the matrix model.

It has been proposed to identify the critical string tachyon with the matrix model loop operator[19, 20],

\[ S'(t, \phi) = \int_{-\infty}^{\infty} dq \partial_q S(q, t)e^{\phi-q}. \] (47)

Like the relation (19) this is multiplicative in momentum space and a convolution in position space, but it is not of the same form and does not coincide with the tachyon field that appears in the low energy Lagrangian. We are not sure of the relation, if any, between our work and the studies of the macroscopic loop operators. We note in passing that our \( S \) satisfies a linearized equation with the tachyon background (40) having a linear term, whereas the loop operator \( S' \) satisfies a linearized equation with no linear term in the background.

Other nonlocal transformations of the tachyon field have played a role in the matrix model black hole proposals of refs. [20, 21, 22]. We again are not sure of any relation between this work and ours, but we should note that we are puzzled by the proposal [22] that processes with odd numbers of tachyons should vanish in the black hole background. There is no sign of any \( Z_2 \) symmetry in the effective spacetime action (35), and the \( 2 \rightarrow 1 \) process that we have discussed should still occur in the region exterior to the horizon. We should also note ref. [23], which discusses dynamical processes in the matrix model. This work does not include a nonlocal transformation of the tachyon, and so proposes a local relation between the metric and the matrix model fields.

In summary, the existence of the exact matrix model solutions to low-dimensional string theories ought to be a useful tool for understanding string physics in spacetime. The relation between the matrix models and the string has been a subject of some confusion. We hope that our work helps to clarify
Acknowledgements

We would like to thank S. Chaudhuri, M. Douglas, M. Stone, and A. Strominger for discussions. This work was supported in part by National Science Foundation grants PHY89-04035 and PHY91-16964.
Figure Captions

1. Successive pulses moving in the $\phi$-$t$ plane. Gravitational field of pulse 1 (dotted) should cause part of pulse 2 to backscatter, producing an outgoing wave (dashed) which precedes the main reflection from the ‘wall.’

2. How the matrix model represents the process of figure 1. The initial wavefunction renormalization (I) produces a tail on pulse 2 which overlaps pulse 1; the combined pulse reflects from the wall (II); and the final renormalization (III) produces the outgoing wave.
References

[1] D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127;
M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635;
E. Brezin and V. Kazakov, Phys. Lett. 236 (1990) 144.

[2] P. Ginsparg and G. Moore, in Recent Directions in Particle Theory,
Proceedings of the 1992 TASI, ed. J. Harvey and J. Polchinski (World
Scientific, Singapore, 1993) hep-th/9304011.

[3] D. J. Gross, Nucl. Phys. B400 (1993) 161;
D. J. Gross and W. Taylor, Nucl. Phys. B400 (1993) 181.

[4] J. A. Minahan and A. P. Polychronakos, Phys. Lett. B312 (1993) 155.

[5] M. R. Douglas, “Conformal Field Theory Techniques for Large-N Group
Theory,” Rutgers preprint (1993) hep-th/9303159.

[6] S. H. Shenker, Proceedings of the Cargese Workshop on Random Sur-
faces, Quantum Gravity, and Strings (1990).

[7] For a review see B. Zwiebach, Nucl. Phys. B390 (1993) 33.

[8] E. Witten, Phys. Rev. D44 (1991) 314.

[9] G. Mandal, A Sengupta, and S. Wadia, Mod. Phys. Lett. A6 (1991)
1685.

[10] D. J. Gross and N. Miljković, Phys. Lett. B238 (1990) 217;
E. Brézin, V. A. Kazakov, and A. B. Zamolodchikov, Nucl. Phys. B333
(1990) 673;
P. Ginsparg and J. Zinn-Justin, Phys. Lett. B240 (1990) 333.

[11] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger,
Phys. Rev. D45 (1992) R1005.

[12] J. Polchinski, Nucl. Phys. B362 (1991) 125.
[13] D. J. Gross and I. R. Klebanov, Nucl. Phys. B359 (1991) 3.

[14] A. M. Polyakov, Mod. Phys. Lett. A6 (1991) 635.

[15] P. Di Francesco and D. Kutasov, Phys. Lett. B261 (1991) 385; Nucl. Phys. B375 (1992) 119.

[16] N. Sakai and Y. Tanii, Prog. Theor. Phys. Suppl. 110 (1992) 117; Phys. Lett. B276 (1992) 41;
G. Minic and Z. Yang, Phys. Lett. B274 (1992) 27;
D. Lowe, Mod. Phys. Lett. A7 (1992) 2647.

[17] G. Moore and R. Plessar, Yale preprint YCTP-P7-92 (1992) hep-th/9203060.

[18] J. Polchinski, Nucl. Phys. B346 (1990) 253.

[19] G. Moore and N. Seiberg, Int. Jour. Mod. Phys. A7 (1992) 187.

[20] S. Das, Mod. Phys. Lett. A8 (1993) 69.

[21] A. Dhar, G. Mandal, and S. Wadia, Mod. Phys. Lett. A7 (1992) 3703.

[22] A. Jevicki and T. Yoneya, preprint NSFITP-93-67, Brown-HEP-904, UT-Komaba/93-10 (1993).

[23] J. G. Russo, Phys. Lett. B300 (1993) 336.
Figure 1
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