Fluctuations of Quantum Fields in a Classical Background and Reheating

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We consider the particle creation process associated with a quantum field χ in a time-dependent, homogeneous and isotropic, classical background. It is shown that the field square χ2, the energy density and the pressure of the created particles have large fluctuations comparable to their vacuum expectation values. Possible effects of these fluctuations on the reheating process after inflation are discussed. After determining the correlation length of the fluctuations in two different models, corresponding to the decay in the parametric resonance regime and in the perturbation theory, it is found that these fluctuations should be taken into account in the final thermalization process, in the back-reaction effects and when the formation of primordial black holes is considered. In both models, by comparing quantum and thermal fluctuations with each other it is observed that very quick thermalization after the complete inflaton decay is not always possible even when the interaction rates are large. On the other hand, when the back-reaction effects are included during the preheating stage, the coherence of the inflaton oscillations is shown to be lost because of the fluctuations in χ2. Finally, we note that a large fluctuation in the energy density may cause a black hole to form and we determine the fraction of total energy density that goes into such primordial black holes in the model of preheating we consider.

I. INTRODUCTION

The theory of quantum fields in curved spacetime is by now a well established subject (see e.g. [1–3]). The issues like the non-uniqueness of the vacuum and the particle creation process, the renormalization of the stress-energy-momentum tensor are all thoroughly understood, and the current research is mainly focused on the incorporation of interactions and alternative rigorous definitions of the theory (see e.g. [4–6]). There are also some unsolved problems, the primary example being the black hole information paradox, but it seems that these can be explained in the full quantum theory of gravity and not in the semi-classical approximation which treats the background classically.

Recently, we have pointed out in [7, 8] an alternative description of the cosmological particle creation process, which tries to overcome two potential issues which may arise in the standard treatments based on the Fourier decomposition. The first one is a (possible) conflict between causality, which requires the particle creation process to take place independently in each Hubble volume, and the introduction of the globally defined Fourier modes as the main physical observable. The second issue is related to a subtlety in the interpretation of the vacuum expectation value of the number operator of a momentum mode, since one can show that the number of created particles with a fixed momentum largely fluctuates about its mean value. To resolve these two issues, in [7, 8] we have analyzed the particle creation process using wave packets localized in a Hubble volume. As a result, the issue of causality is naturally settled by treating the creation of the modes in different Hubble volumes independently and the mean values are interpreted as statistical averages over distinct Hubble volumes. By applying this formalism to reheating process after inflation, in [7, 8] it is shown that there exists small density perturbations on Hubble length scales at the end of reheating. Although the same result can also be obtained in the standard formulation by introducing a suitable window function to probe the horizon scale, the construction presented in [7, 8] is physically more direct and transparent.

Since field theory respects locality by construction, the first issue mentioned above should not be a problem as long as the modes are physically interpreted properly. Indeed, the fact that field variables must commute at space-like separations, i.e. [χ(t′, x′), χ(t, x)] = 0 when g_{µν}∂_{x'μ}∂_{x′ν} > 0, guarantees causality. The Fourier mode expansion of the field variable χ(t, x) is just a convenient way of describing physics. Any other complete set can be used for expansion and physics should not depend on the basis chosen. However, the second matter related to the interpretation of expectation values can be tricky, especially if there are large fluctuations about the mean values.

In this paper, we consider the particle creation effects in a time-dependent, homogeneous and isotropic, classical background and try to determine the magnitude of the fluctuations of the energy-momentum tensor T_{µν} and the field strength χ2 about their mean values. Namely, by defining the fluctuation operators δT_{µν} = T_{µν} − ⟨T_{µν}⟩ and δχ2 = χ2 − ⟨χ2⟩, whose expectation values vanish ⟨δT_{µν}⟩ = ⟨δχ2⟩ = 0, we calculate the variances ⟨(δT_{µν})2⟩ ≠ 0 and ⟨(δχ2)2⟩ ≠ 0. As we will see in section II, for most of the components of the energy-momentum tensor including the energy density and the pressure, and for χ2 the deviations have the same order of magnitude as the corresponding mean values, which

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shows the existence of large fluctuations. We emphasize that the spatial scale of these fluctuations is given by the correlation length of the quantum field \( \chi \).

In the second part of this paper, in section III, we discuss possible implications of these results for the reheating process in single scalar field inflationary models. We consider the decay of the inflaton field \( \phi \) into bosonic \( \chi \) particles in two different prototype models corresponding to perturbative and parametric resonance regimes, and in both models we determine the correlation length of the field excited by the inflaton oscillations. We find that the existence of these fluctuations may affect various events during reheating. It is pointed out in section III that thermalization process cannot be completed unless quantum fluctuations become smaller than the thermal fluctuations in equilibrium and this may delay the final moment of reaching thermal equilibrium. Secondly, in the model with preheating, when the field starts affecting the frequency of the background inflaton oscillations, the fluctuations in \( \chi^2 \) are found to destroy the coherence of the oscillations. We show that because of this loss of coherence it is no longer possible to neglect the spatial derivatives of the inflaton when the back-reaction effects become important. Finally in section III, the possibility of black hole formation because of the fluctuations in the energy density of created particles is considered. For a large fluctuation, which of course happens rarely, the corresponding Schwarzschild radius may become larger than the correlation length which would produce a black hole. We show that the observational constraint on the fractional energy density that can go into primordial black holes puts some new restrictions, especially in models of preheating. We finally conclude by reviewing our findings and discussing future directions in section IV.

II. THE FLUCTUATIONS

We consider a real scalar field \( \chi \) propagating in a cosmological Robertson-Walker metric

\[
 ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2),
\]

which has the following action

\[
 S = -\frac{1}{2} \int \sqrt{-g} (|\nabla \chi|^2 + M^2 \chi^2). 
\]  (1)

We assume that in addition to the scale factor \( a \), the mass parameter \( M \) may also depend on time: \( M = M(t) \). Therefore, particle creation can be induced both by the evolution of the metric and by the externally varying time dependent mass parameter \( M \). The energy-momentum tensor can be obtained from (2) as

\[
 T_{\mu \nu} = \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu \nu} [\nabla_\alpha \chi \nabla^\alpha \chi + M^2 \chi^2].
\]  (3)

More explicitly, one can read the energy density \( \rho \), the pressure \( P \), the "momentum" \( U_i \) and (trace-free) stress components \( \tau_{ij} \) as

\[
 \rho = \frac{1}{2} \chi^2 + \frac{1}{2} g^{ij} (\partial_i \chi)(\partial_j \chi) + \frac{1}{2} M^2 \chi^2,
\]

\[
 P = \frac{1}{2} \chi^2 - \frac{1}{6} g^{ij} (\partial_i \chi)(\partial_j \chi) - \frac{1}{2} M^2 \chi^2,
\]

\[
 U_i = T_{ti} = \frac{1}{2} \left[ (\partial_i \chi) + (\partial_i \chi) \right],
\]  (4)

\[
 \tau_{ij} = \partial_i \chi \partial_j \chi - \frac{1}{3} g_{ij} \left[ g^{kl} (\partial_k \chi)(\partial_l \chi) \right],
\]

where dot denotes time derivative and all indices refer to the obvious coordinate basis of (1). Note that the spatial components of the energy-momentum tensor are decomposed as \( T_{ij} = \tau_{ij} + P g_{ij} \), where \( g^{ij} \tau_{ij} = 0 \). In quantum theory, there is an ordering ambiguity in \( U_i \) and in (4) we perform a symmetric ordering.

For quantization it is convenient to define a new field \( X \) by

\[
 X = a^{3/2} \chi.
\]  (5)

Then, the action (2) up to surface terms becomes

\[
 S = \frac{1}{2} \int \left[ X^2 - g^{ij} (\partial_i X)(\partial_j X) - (M^2 - \frac{9}{4} H^2 - \frac{3}{2} \dot{H}) X^2 \right],
\]

where \( H = \dot{a}/a \) is the Hubble parameter. The momentum variable \( \Pi \) conjugate to \( X \) is given by \( \Pi = \dot{X} \) and one can easily apply the standard canonical quantization procedure. Introducing the mode function \( X_k \) and time independent ladder operator corresponding to a fixed time \( t_0 \), \( a_k \), as

\[
 X = \frac{1}{(2\pi)^{3/2}} \int d^3 k \left[ a_k X_k e^{-i k \cdot x} + a_k^\dagger X_k^* e^{i k \cdot x} \right],
\]  (6)

the canonical commutation relation \([X(t,x), \Pi(t,x')] = i\hbar (x - x')\) can be satisfied by imposing

\[
 [a_k, a_{k'}^\dagger] = \delta(k - k'),
\]  (7)

and

\[
 X_k X_k^* - X_k^* X_k = i,
\]  (8)

where boldface letters \( k \) and \( x \) refer to the spatial 3-vectors \( k_i \) and \( x_i \), \( k \cdot x = k_i x_i \) and \( k^2 = \delta^{ij} k_i k_j \). On the other hand, the field equations imply

\[
 \ddot{X}_k + \omega_k^2 X_k = 0,
\]  (9)

where

\[
 \omega_k^2 = M^2 + \frac{k^2}{a^2} - \frac{9}{4} H^2 - \frac{3}{2} \dot{H}.
\]  (10)

Because of the homogeneity and isotropy of the background, the mode function \( X_k \) depends only on the magnitude \( k \) of \( k \).
The ground state of the system at time \( t_0 \) can be defined as
\[
a_k|0\rangle = 0, \quad (11)
\]
and in terms of the mode functions this corresponds to the initial conditions
\[
X_k(t_0) = \frac{1}{\sqrt{2\omega_k}}, \quad \dot{X}_k(t_0) = -i\sqrt{\frac{\omega_k}{2}}. \quad (12)
\]
Using (12), the Hamiltonian at time \( t_0 \) can be found as
\[
H(t_0) = \int d^3k \left[ a_k^\dagger a_k + \frac{1}{2} \right] \omega_k, \quad (13)
\]
which justifies the identification of \( |0\rangle \) defined in (11) as the ground state. Let us note that the above formulation is identical to the standard formulation of particle creation in terms of Bogoliubov transformations (see [9]).

The correspondence can be achieved by defining the time dependent \( \alpha_k \) and \( \beta_k \) coefficients as
\[
X_k = \frac{1}{\sqrt{2\omega_k}} \left[ \alpha_k e^{-i \int_{t_0}^t \omega_k dt'} + \beta_k e^{i \int_{t_0}^t \omega_k dt'} \right]. \quad (14)
\]
By imposing the following equations
\[
\dot{\alpha}_k = \frac{\omega_k}{2\omega_k} e^{-2i \int_{t_0}^t \omega_k dt'} \beta_k, \\
\dot{\beta}_k = \frac{\omega_k}{2\omega_k} e^{2i \int_{t_0}^t \omega_k dt'} \alpha_k,
\]
and the initial conditions \( \alpha_k(t_0) = 1, \beta_k(t_0) = 0 \), the equivalence of both formulations can easily be established. For future use let us also calculate \( X_k \) as
\[
\dot{X}_k = \sqrt{\frac{\omega_k}{2}} \left[ -i \alpha_k e^{-i \int_{t_0}^t \omega_k dt'} + i \beta_k e^{i \int_{t_0}^t \omega_k dt'} \right]. \quad (15)
\]
Note that because of the equations obeyed by \( \alpha_k \) and \( \beta_k \), in taking time derivative of \( X_k \) from (14) one can treat \( \alpha_k \) and \( \beta_k \) as if they are constant parameters.

The particle creation process is usually described by giving the vacuum expectation value of the number operator for a mode of momentum \( k \). Alternatively, one can also calculate the vacuum expectation value of the energy-momentum tensor \( < T_{\mu\nu} > = \langle 0 | T_{\mu\nu} | 0 \rangle \). One benefit of working with the energy-momentum tensor is that the potential problem mentioned in the introduction, namely the creation of a global Fourier mode in an expanding universe, is naturally evaded. Using (6) in (4), one can straightforwardly calculate
\[
< \rho > = T + V + G, \\
< P > = T - V - \frac{1}{3} G, \quad (16)
\]
where being the real functions of time \( T, V \) and \( G \) are given by
\[
T = \frac{1}{2(2\pi a)^3} \int d^3k |X_k - \frac{3}{2} H X_k|^2, \\
V = \frac{1}{2(2\pi a)^3} \int d^3k M^2 |X_k|^2, \quad (17)
\]
\[
G = \frac{1}{2(2\pi a)^3} \int d^3k \frac{k^2}{a^2} |X_k|^2.
\]
Here, \( T \) and \( V \) play the roles of the kinetic and potential energies, respectively, and \( G \) is like the energy stored in the gradient of the field. It is an easy exercise to show that when \( M \) is constant the expectation value of the energy-momentum tensor is conserved
\[
M(t) = M_0 \Rightarrow \nabla_\mu < T^{\mu\nu} > = 0, \quad (18)
\]
which actually reduces to a single nontrivial equation \( < \rho > + 3H(< \rho > + < P >) = 0 \). Thus, in that case \( < T^{\mu\nu} > \) can be fed into the right hand side of the Einstein's equations to include the back-reaction effects.

The functions \( T, V \) and \( G \) defined in (17) diverge in general. Therefore, they should be regularized to be interpreted physically. Adiabatic (or WKB) regularization is a suitable and physically appealing way of getting finite expectation values for \( < \chi^2 >, < \rho > > > P > \) [10, 11]. One can see that the function \( V \) is related to \( < \chi^2 > \) and thus it can be made finite by adiabatic regularization. Similarly, the functions \( T \) and \( G \) can also be expressed in terms of \( < \rho >, < P > < \chi^2 > \), thus they can also be regularized. When we mention about the magnitudes of the variables \( T, V \) and \( G \) below, we implicitly refer to their regularized finite values.

The fact that \( < U_i > = 0 \) and \( < \tau_{ij} > = 0 \) does not imply \( U_i \) and \( \tau_{ij} \) identically. Similarly, \( \rho \) and \( P \) do not exactly equal their mean values. All these physical quantities fluctuate about their expectation values and we would like to find out the size of these fluctuations. For that we define the fluctuation operator
\[
\delta T_{\mu\nu} = T_{\mu\nu} - < T_{\mu\nu} >, \quad (19)
\]
and try to compute the variance \( < (\delta T_{\mu\nu})^2 > \neq 0 \).

Let us start with the energy density \( \delta \rho = \rho - < \rho > \). Using (6) in the definition of \( \delta \rho \) given in (4) and reading \( < \rho > \) from (16), a relatively long but straightforward calculation gives
\[
< \delta \rho^2 > = 2T^2 + 2V^2 + \frac{2}{3} G^2 \quad (20)
\]
\[
+ \frac{M^2}{(2\pi a)^6} \left[ \int d^3k \cos(\phi_k) |\dot{X}_k - \frac{3}{2} H X_k|^2 \right]^2, 
\]
\[
- \frac{M^2}{(2\pi a)^6} \left[ \int d^3k \sin(\phi_k) |\dot{X}_k - \frac{3}{2} H X_k|^2 \right]^2,
\]
where \( \phi_k \) is the phase difference between \( \dot{X}_k - \frac{3}{2} H X_k \) and \( X_k \). The only "nontrivial" step in this calculation
is to deal with a double integral which has the following form

$$
\int \int d^3k_1 \, d^3k_2 \, (k_1 \cdot k_2)^2 |X_{k_1}|^2 |X_{k_2}|^2.
$$

(21)

This can be converted by suitable angular integrations into

$$
\frac{1}{3} \left[ \int d^3k_1(\vec{k}_1^2) |X_{\vec{k}_1}|^2 \right] \left[ \int d^3k_2(\vec{k}_2^2) |X_{\vec{k}_2}|^2 \right],
$$

(22)

which can then be expressed in terms of the function $G$. As a check on this computation one can see from (17) and (20) that

$$
< \delta \rho^2 > \geq \frac{2}{27} G^2,
$$

(24)

\[ - \frac{M^2}{(2\pi)^6} \left[ \int d^3k \cos(\phi_k) |\vec{X}_k - \frac{3}{2} H \vec{X}_k||\vec{X}_k| \right]^2 \]

\[ + \frac{M^2}{(2\pi)^6} \left[ \int d^3k \sin(\phi_k) |\vec{X}_k - \frac{3}{2} H \vec{X}_k||\vec{X}_k| \right]^2. \]

Here also one has $< \delta P^2 > \geq 0$, as expected.

Both in (20) and in (24), there are integrals involving the phase $\phi_k$, which cannot be expressed in terms of the functions $T$, $V$ and $G$. Interestingly, however, these terms appear with opposite signs in $< \delta \rho^2 >$ and $< \delta P^2 >$, and from (20) and (24) one sees that

$$
< \delta \rho^2 > + < \delta P^2 > = 4T^2 + 4V^2 + \frac{20}{27} G^2.
$$

(25)

Comparing with the average $< \rho >$ given in (16), eq. (25) shows that $< \delta \rho^2 > + < \delta P^2 >$ has the same order of magnitude as the mean energy density square

$$
< \delta \rho^2 > + < \delta P^2 > = c(< \rho >)^2,
$$

(26)

where $c$ is a number close to unity, which varies depending on the hierarchy between the functions $T$, $V$ and $G$. For instance, $c \sim \frac{32}{27} / 27$ if $T \sim G \gg V$, or $c \sim 0.97$ if $T \sim V \sim G$.

Before proceeding with the calculations of other fluctuations, let us make a few comments. It is easy to see from (23) that the integrals involving $\cos(\phi_k)$ and $\sin(\phi_k)$ in (20) and (24) are bounded by $2T^2 + 2V^2$. Therefore, one would expect them to be regularized in a suitable scheme. On the other hand, the sum of the fluctuations $< \delta \rho^2 > + < \delta P^2 >$ can already be expressed in terms of finite functions $T$, $V$ and $G$ after adiabatic regularization. In general, one would also expect the oscillating $\cos(\phi_k)$ and $\sin(\phi_k)$ integrals in (20) and (24) to be smaller than the previous terms. Therefore, both $< \delta \rho^2 >$ and $< \delta P^2 >$ should have the same order of magnitude as $(< \rho >)^2$. In any case, (25) shows that the fluctuations $< \delta \rho^2 >$ and $< \delta P^2 >$ cannot be simultaneously small compared to the average energy density. This can be viewed as an uncertainty relation between the energy density and pressure fluctuations.

Because of these large fluctuations, it may not always be possible to use a simple equation of state $P = w \rho$ to characterize the energy-momentum tensor in quantum particle creation. One can calculate the correlation of the energy and pressure fluctuations as

$$
< \{ \delta \rho, \delta P \} > = 4T^2 - 4V^2 - \frac{4}{9} G^2,
$$

(27)

where the symmetric ordering $< \{ \delta \rho, \delta P \} > = \delta \rho \delta P + \delta P \delta \rho$ is chosen inside the brackets. Thus, depending on the case the correlation can be negligible, which would forbid the use of an effective equation of state parameter $w$. One can also determine the expectation value of the commutator, which should give information about simultaneous measurability of fluctuations. We find

$$
< [\delta P, \delta \rho] > = \frac{4M^2}{(2\pi)^6} \left[ \int d^3k \cos(\phi_k) |\vec{X}_k - \frac{3}{2} H \vec{X}_k||\vec{X}_k| \right]

\times \left[ \int d^3k \sin(\phi_k) |\vec{X}_k - \frac{3}{2} H \vec{X}_k||\vec{X}_k| \right],
$$

(28)

therefore the integrals involving the phase $\phi_k$ is related to the expectation value of the commutator. Note that the expectation value is equal to zero for $M = 0$.

Let us now continue with the other components of the energy-momentum tensor, namely $U_i$ and $\tau_{ij}$. As noted in (16), their expectation values vanish. To determine fluctuations, one can calculate

$$
< U_i U_j > = \frac{4}{3} T G g_{ij},
$$

(29)

and

$$
< \tau_{ij} \tau_{ij} > = \begin{cases} \frac{16}{27} G^2 & \text{if } i \neq j, \\ \frac{16}{27} G^2 & \text{if } i = j, \end{cases}
$$

where in the last equation the summation convention is not used, i.e. $i$ and $j$ are treated as free indices. The correlation of field variables carrying different set of indices can be seen to vanish because of the homogeneity and isotropy of the background. However, when $G$ has the same order of magnitude as $T$ and $V$, the fluctuations of the stress components $\tau_{ij}$ cannot simply be ignored. Remember that $G$ is the energy stored in the gradient.
of the field, therefore it is not surprising to discover that
the fluctuations of the stress components depend on \( G \). Similarly, for large \( G \) and \( t \), the "momentum" compo-
nents \( \bar{U}_i \) has large fluctuations, of the order of average
energy density. Again it is natural to see both \( T \) and
\( G \) in the fluctuations of momentum, since \( T \) measures
kinetic energy and \( G \) measures spatial variations.

When back-reaction or symmetry breaking effects are
considered, \( \chi^2 \) becomes an important parameter and thus it is crucial to determine the fluctuations about the mean \( \langle \chi^2 \rangle \). Defining as before the variation operator
\( \delta \chi = \chi^2 - \langle \chi^2 \rangle \) and using (5) and (6), one can easily calculate
\[
\langle (\delta \chi)^2 \rangle = 2 \langle \chi^2 \rangle^2.
\]
Therefore, viewing \( \chi^2 \) as a random variable one sees
that the corresponding standard deviation is equal to \( \sqrt{2} \)
times the average, which again indicates the existence of
large fluctuations about the mean. This result can indeed
be anticipated without doing any computation since the
free field \( \chi(t, x) \) can be viewed as a collection Gaussian
random variables defined at each point in space and (29)
is true for any Gaussian distribution.

Let us remind that the expectation values of physical
quantities determined above should be interpreted as sta-
tistical averages over different points in space at a given
time. However, the field variables at nearby points are
not independent, i.e. they are correlated with each other.
The size of a region containing such correlated field vari-
ables is given by the correlation length \( \xi_c \), which can be
determined from the two point function
\[
\langle \chi(t, x) \chi(t, 0) \rangle = \frac{1}{(2\pi a)^3} \int d^3k |X_k|^2 e^{ik.x} = \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 |X_k|^2 \left[ \frac{\sin(kr)}{kr} \right],
\]
where \( r^2 = \delta_{ij} x^i x^j \). The comoving correlation length \( \xi_c \)
the minimum value of \( r \) that makes (30) to vanish.
Physically, it gives the spatial comoving size of a region
in which the \( \chi \) field is homogenous. Similarly, it is also
possible to define the correlation lengths of the energy
density, the pressure etc., however these are expected to
have the same order of magnitude as \( \xi_c \). Therefore, one
can imagine the space to be divided into comoving re-
gions of typical size \( \xi_c \), and in each region the physical
quantities like the \( \chi \) field, the energy density \( \rho \) and the
pressure \( P \) are uniformly distributed. The fluctuations,
on the other hand, give variations from one region to
another (see figure 1).

As \( r \rightarrow 0 \), (30) should give the (regularized) mean
value \( \langle \chi^2 \rangle \). Therefore, in calculating the correlation
length from (30), a suitable regularization should be per-
formed before the momentum integral is taken (note that
(30) is finite for \( r \neq 0 \) without any need of regularization).
To employ WKB regularization [11] one uses (14)

\[
\text{FIG. 1: The space divided into regions of typical (comoving)
size } \xi_c. \text{ At a given instant, the } \chi \text{ field can be assumed to
vary appreciably from one region to another, while it is nearly
uniform in each region.}
\]

in (30) and sets \( |\alpha_k|^2 = |\beta_k|^2 + 1 \), which yields
\[
\langle \chi(t, x) \chi(t, 0) \rangle = \frac{1}{2\pi^2 a^3} \int_0^\infty dk \frac{k^2}{\omega_k} \times (31)
\left[ |\beta_k|^2 + \text{Re} \left( \alpha_k \beta_k^* e^{-2i \int_0^t \omega_k d\tau} \right) \right] \left[ \frac{\sin(kr)}{kr} \right],
\]
where an additive factor of \( 1/2 \) is ignored in the first
square brackets for regularization. Since initially one
imposes \( \beta_k(t_0) = 0 \), the momentum integral in (31) equals
zero at \( t = t_0 \) and it is expected to converge also at later
times. We will use this regularization in determining the
correlation length in the next section.

Let us note that in our calculations we ignore the
back-reaction effects and assume that the evolution of
the background does not change as the particles are
created. This is an approximation and, for instance, as the
energy of the created particles increases the energy of
the background should decrease. Naively, one may then
tend to view these fluctuations as "isocurvature" pertur-
bations, however, this depends on how the energy density
of the background and the created particles redshift. On
the other hand, keeping in mind that the back-reaction
effects are ignored may help to solve the following con-
cern. As noted in (18), when \( M \) is constant the average of
the energy-momentum tensor is conserved and one may
worry that this can be spoiled if fluctuations are added
on top of the mean values. However, all that is needed
is to satisfy the energy-momentum conservation for the
whole system, and this should be guaranteed when the
back-reaction effects are properly taken into account.

It is interesting to compare the direct evaluation of the
dispersion used in this paper with the standard formula-
tion which employs a smearing of the dispersion (or power
spectrum) in momentum space with a suitable window
function to probe a given scale. In general these two
computations should agree, at least on large scales, how-
ever there are some important differences. While using a window function can be a convenient way of regularizing some expressions, it is not possible to identify the correlation length, since the power spectrum does not contain any information about the correlation of variables at different points. Moreover, some results may be sensitive on the choice of a window function whereas the adiabatic regularization offers both technically and physically unique way of getting finite and consistent values from singular expressions. To emphasize the difference, let us point out that recently in [30, 31] adiabatic regularization is applied to cosmological perturbation theory and some standard results, such as the form of the power spectrum, are shown to be modified significantly depending on the regularization.

III. IMPLICATIONS FOR REHEATING

In this section, we analyze the particle creation process in the reheating period after inflation and consider the single scalar field inflationary models. In general, the evolution of the metric and the inflaton $\phi$ is governed by the Friedmann and the scalar field equations

$$H^2 = \frac{8\pi}{3M_p^2} \left[ \frac{1}{2} \phi^2 + V(\phi) \right],$$

$$\ddot{\phi} + 3H \dot{\phi} + \frac{\partial V}{\partial \phi} = 0,$$  \hspace{1cm} (32)

where $V(\phi)$ is the scalar potential. In reheating, the scalar oscillates about the minimum of the potential and we focus on models in which the potential at that stage can be taken as

$$V = \frac{1}{2} m^2 \phi^2,$$  \hspace{1cm} (33)

where $m$ is the inflaton mass. Because of the expansion of the universe, the amplitude of the oscillations slowly decreases in time so one can assume a solution of the form

$$\phi = \Phi(t) \sin(mt).$$  \hspace{1cm} (34)

When $\dot{\Phi} \ll m\Phi$, the field equations (32) become

$$H^2 = \frac{4\pi m^2}{3M_p^2} \Phi^2, \hspace{1cm} \ddot{\Phi} + \frac{3}{2} H \dot{\Phi} = 0,$$  \hspace{1cm} (35)

and these can be solved as

$$a = a_0 \left( \frac{t}{t_0} \right)^{2/3}, \hspace{1cm} \Phi = \frac{M_p}{\sqrt{3\pi mt}}.$$  \hspace{1cm} (36)

Thus, the evolution is equivalent to the dust dominated universe and in that case the combination $9H^2/4 + 3H/2$, which appears in (10), is equal to zero.

We assume that reheating occurs due to the coupling of the inflaton to a bosonic field, $\chi$, and consider two different types of interactions suitable for preheating and perturbative decay, respectively. We first determine the correlation length $\xi$ of the excited $\chi$ field at the end of decay and evaluate the functions $T$, $V$ and $G$ introduced in the previous section in (17). Later we study possible implications of our findings on the three important processes, which are the final thermalization of the decay products, the back-reaction effects and the formation of primordial black holes.

Preheating

Let us first start with preheating. The decay of the inflaton field in the parametric resonance regime has been analyzed in detail in [12–15]. Specifically, it is shown in [13, 15] that the following interaction

$$\mathcal{L}_{int} = -\frac{1}{2} g^2 \phi^2 \chi^2,$$  \hspace{1cm} (37)

can give a decay due to broad parametric resonance if the parameter $q = g^2 \Phi^2/m^2 \gg 1$. This condition can naturally be satisfied in chaotic inflationary scenario in which $\Phi_0 \sim M_p$, where $\Phi_0$ is the initial value of the inflaton amplitude and $M_p$ is the Planck mass. In this model the frequency (10) becomes

$$\omega_k^2 = \frac{k^2}{a^2} + g^2 \Phi^2 \sin^2(mt),$$  \hspace{1cm} (38)

and the particle creation mainly occurs due to the oscillating term. As shown in [13, 15], for momenta in certain instability bands, the solution of (9) exponentially grows, which corresponds to $|\alpha_k| \approx |\beta_k| \gg 1$. Actually in the expanding universe the real process is much more complicated where a given momentum, which is initially in the first band, jumps over many different instability bands in time. Fortunately, it can still be treated analytically by approximating the process in terms of successive scattering on parabolic potentials [15]. The final result for the Bogoliubov coefficients can effectively be described by an index $\mu_k$ such that

$$|\alpha_k| \approx |\beta_k| = e^{\mu_k mt}.$$  \hspace{1cm} (39)

Moreover, the first and the most important resonance band is initially peaked around the physical momentum $k_\ast = \sqrt{gm \Phi_0}$, with a width of the order of $k_\ast$ (here we only consider the first stage of preheating where back-reaction and re-scattering effects are ignored). The effective index $\mu_k$ can be approximated as

$$\mu_k \simeq \mu - \frac{1}{2} \mu_k''(k_\ast)(k - k_\ast)^2,$$  \hspace{1cm} (40)

where $\mu_k''(k_\ast) \simeq 2\mu/(k_\ast)^2$, $\mu$ is the average index depending on the coupling constant $g$ and it typically varies in-between 0.1 and 0.2 [15].

To calculate the correlation length we use (39) in (31). As discussed in [15], the integral of the second term in
the square bracket in (31) (the one containing the real part) gives a time dependent oscillating correction which is less than 1 compared to $|β_k|^2$ integral (see (89) in [15]). Therefore, (31) can be written as

$$<χ(t,x)χ(t,0)> ≥ \frac{(1+C)}{2\pi^2a^2} \int_0^\infty dk \frac{k^2|β_k|^2}{ω_k} \left[ \sin(kr) kr \right],$$

(41)

where $C$ is the correction factor mentioned above. Using (39) and (40), the momentum integral in the last equation can now be performed using the steepest-decent approximation, which gives

$$<χ(t,x)χ(t,0)> ≥ \frac{(1+C)}{2\pi^2a^2} \int_0^\infty dk \frac{k^2|β_k|^2}{ω_k} \left[ \sin(kr) kr \right].$$

(42)

Therefore, the comoving correlation length can be determined as

$$ξ_c ≥ \frac{1}{k_s} = \frac{1}{\sqrt{gmΦ_0}}.$$  

(43)

In this computation, the effects of the modes in other instability bands and the time evolution of the first band caused both by the redshift of momenta and the decrease of the inflaton amplitude $Φ$, which would lessen the width of the band, are ignored (note that $k_s$ given below (39) is the physical momentum). However, as shown in [15], the modes that have been amplified from the very beginning become exponentially larger than the others. Thus, in practice one can treat $k_s$ as a comoving momentum scale such that the first instability band does not change in time and (43) should give a good estimate for the comoving correlation length of fluctuations.

It is interesting to compare the size of the correlation length to the Hubble radius at the end of the preheating. The broad parametric resonance ends when $q = g^2Φ^2/m^2 ≃ 1$ and this can be used to estimate the value of the amplitude at that time as $Φ ≃ m/g$. The Hubble parameter can be determined from (35) as

$$H ≥ \frac{m^2}{gM_p}.$$  

(44)

By comparing the initial and the final values of $q$, the amplitude $Φ$ can be seen to decrease $\sqrt{q_0}$ times, where $q_0$ is the initial value of $q$ at the beginning of preheating. Then, from (36), one sees that the universe expands $q_0^{1/3}$ times and thus the physical correlation length at the end of broad parametric resonance is given by

$$ξ_c^{phys} ≥ q_0^{1/3}ξ_c = \frac{q_0^{1/3}}{\sqrt{gmΦ_0}} = q_0^{1/2}m^{-1}.$$  

(45)

As a result, (43) and (44) gives the ratio of the physical correlation length to the Hubble radius as

$$\frac{ξ_c^{phys}}{H} ≥ q_0^{5/12} \frac{Φ_0}{M_p}.$$  

(46)

One can have $Φ_0 ≃ M_p$, but $q_0 ≫ 1$ in the broad parametric resonance regime, thus the ratio is in general less than unity.

In this model, the variables $T$, $V$ and $G$ can be determined in terms of $<χ^2>$ as follows. Firstly, from (17), one sees that

$$V = \frac{1}{2}M^2 <χ^2> = \frac{1}{2}g^2Φ^2 sin^2(mt) <χ^2>.$$  

(47)

Therefore, $V$ oscillates between zero and the maximum value $V_{max} = g^2Φ^2 <χ^2>/2$. To determine $G$, one can again use (39) and (40) in (17), and apply the steepest-decent approximation. The steps are identical to the derivation of (42) and one finds

$$G ≃ \frac{k_s^2}{a^2} <χ^2>.$$  

(48)

To find $T$, one can first ignore the expansion of the universe to a very good approximation. Using (15) in (17) one then obtains

$$T = \frac{1}{4π^2a^2} \int_0^\infty k^2ω_k \left[ |β_k|^2 - Re \left( α_kβ^*_k e^{-2i∫_0^\infty ω_kdt} \right) \right] dk$$

where we regularize this expression by ignoring an additive factor of $1/2$ in the square brackets. Applying again the steepest-decent approximation, we find

$$T ≃ \frac{1-C}{1+C} <χ^2>.$$  

(49)

where $C$ is the factor defined in (41). Since from (38) $ω_k$ is an oscillating function of time, $T$ is also oscillating. However, it is easy to see that $V + G ≃ T$.

Finally, it is also possible to determine the phase $ϕ_k$, which first appeared in (20). Ignoring the expansion of the universe, $ϕ_k$ equals the difference between the arguments of $X_k$ and $X_k$. Using $|α_k| ∼ |β_k|$, one can see that $Arg(X_k) ≃ ϕ_k/2$ and $Arg(X_k) ≃ ϕ_k/2 + π$, where $ϕ_k$ is the phase difference between $α_k$ and $β_k$. Therefore

$$ϕ_k ≃ π.$$  

(50)

From (20) and (24), the value of $ϕ_k$ can be seen to favor the energy fluctuations compared to pressure fluctuations.

**Perturbative decay**

Let us now consider the reheating process in perturbation theory. For that we assume the following trilinear coupling

$$L_{int} = \frac{1}{2}σΦ^2,$$  

(51)

which may arise after spontaneous symmetry breaking. In this model, the frequency (10) becomes

$$ω_k^2 = \frac{k^2}{a^2} + σΦ sin(mt).$$  

(52)
As discussed in [17], the perturbation theory is applicable when $\sigma \Phi / m^2 \ll 1$, which can in general be satisfied if the amplitude $\Phi$ is small. It is known that the preheating picture completely changes when both interactions (37) and (51) present in the Lagrangian, see [16]. Therefore, the perturbative decay due to (51) should be considered on its own as a different model, i.e. it is not to be preceded by the preheating considered above. Let us remind that the evolution of the background fields is still given by (36).

In perturbation theory i.e. for $|\beta_k| \ll 1$, Bogoliubov coefficients can iteratively be solved and to first order they can be determined as

$$\alpha_k \simeq 1,$$

$$\beta_k \simeq \frac{1}{2} \int_0^t dt' \frac{\dot{\omega}_k(t')}{\omega_k(t')} \exp \left(-2i \int_{t'}^t \omega_k(t'') dt'' \right)$$

(53)

The time integral in (53) can be evaluated using the stationary phase method [18] (note that in almost all models of inflation $H \ll m$, which is important for the applicability of the stationary phase approximation). There are two oscillatory terms in (53), one is the explicit pure phase exponential and the other is coming from $\dot{\omega}_k / \omega_k$. It can be shown that for a given $k$, the main contribution to the integral comes from an interval near $t_*$ fixed by $\omega_k(t_*) = m/2$, which, in the perturbative regime $\sigma \Phi / m^2 \ll 1$, implies

$$\frac{k}{a_*} = \frac{m}{2}.$$  

(54)

This can be interpreted as the decay of the inflaton at time $t_*$ to two $\chi$ particles with comoving momentum $k$ (see e.g. [17]). Thus, only the modes in the following interval

$$\frac{a_0 m}{2} < k < \frac{a_1 m}{2}$$

(55)

significantly produced, where $a_0$ and $a_1$ are the scale factors at the beginning and at the end of the decay, respectively. We quote from [7] the result of the relatively straightforward calculation:

$$|\beta_k|^2 = \frac{\pi \sigma^2 M_p \Phi_0}{2 m^{5/2} k} \left( \frac{a_0}{a_k} \right)^{3/2}.$$  

(56)

where $\Phi_0$ is the initial value of the amplitude.

To determine the correlation length, we use (53) in (31) and find

$$< \chi(t, x) \chi(t, 0) > = \frac{1}{2 \pi^2 a^3} \int_0^\infty dk \frac{k^2}{\omega_k} \times$$

$$\left[ |\beta_k|^2 + \int_0^t dt' \frac{\dot{\omega}_k(t')}{\omega_k(t')} \cos \left( 2 \int_{t'}^t \omega_k(t'') dt'' \right) \right] \frac{\sin(kr)}{kr}.$$  

(57)

The time integral in the square brackets in (57) can also be performed using stationary phase method. However, the phase integral now produces an extra cosine term,

$$\cos \left( \int_{t_0}^{t_1} [2 \omega_k(t') + m] dt' \right).$$  

(58)

where $t_*$ is the time corresponding (54). This term can be seen to oscillate very rapidly in the decay range (55) and thus the second term in the square brackets in (57) can be neglected since its contribution will be very small after performing the momentum integral. Using (56) in (57) then gives

$$< \chi(t, x) \chi(t, 0) > \sim B \int_{a_0 m/2}^{a_1 m/2} dk \frac{\sin(kr)}{k^{3/2}},$$

(59)

where $B$ is a $k$-independent constant. The indefinite integral can be explicitly evaluated in terms of the Fresnel cosine integral $C(x)$ as

$$2 \sqrt{2 \pi r} C \left( \sqrt{\frac{2\pi}{\pi}} \frac{2\sin(kr)}{kr} \right).$$

(60)

One can now see that to a very good accuracy the comoving correlation length is independent of the upper limit of the integral $a_1 m/2$, instead it is fixed by the lower limit as

$$\xi_c \simeq \frac{1}{a_0 m}.$$ 

(61)

Thus, the physical correlation length at the end of the decay is given by

$$\xi_c^{\text{phys}} \sim \frac{a_1}{a_0 m},$$

(62)

which is larger than $1/m$ by the expansion factor of the universe during reheating.

By noting from (56) that the spectrum is given by $|\beta_k| \sim k^{-3/4}$, it is easy to understand (62) physically as follows. The dependence of $|\beta_k|$ on the comoving momentum $k$ indicates that, as far as the correlation length is concerned, the particle creation effects decrease with increasing momentum. Since the process can be thought as the decay of the inflaton particle with mass $m$ into two $\chi$ particles with physical momentum $m/2$, the particles created in the beginning of the process have the smallest comoving momentum $a_0 m/2$ and thus are the most important ones. The corresponding wavelength, which redshifts in time, gives the correlation length (62). All the particles created in due time have larger momenta and smaller wavelengths, and they produce sub-leading corrections to the correlation length.

To compare the correlation length to the Hubble radius, let us determine the Hubble parameter at the end of the decay. As discussed, e.g. in [17], the decay process described by the interaction (51) is equivalent to decay with a constant decay rate $\Gamma_\phi \sim \sigma^2/m$, which gives a reheating temperature $T_R \simeq \sqrt{\Gamma_\phi M_p} \sim \sqrt{\sigma^2 M_p/m}$. The Hubble parameter corresponding to this temperature is given by

$$H \sim \frac{\sigma^2}{m}.$$ 

(63)
Therefore,
\[ \frac{\delta U_{\mathrm{phys}}}{U_{\mathrm{phys}}} \sim \frac{a_1 \sigma^2}{a_0 m^2}. \]  
(64)

In these models one usually imposes \( \sigma \ll m \) for perturbation theory to be applicable. On the other hand, the expansion factor during reheating depends on the initial value of the inflaton amplitude, but it is not expected to be a very large number. Thus, one again finds that the correlation length is smaller than the Hubble radius.

In perturbation theory, the functions (17) can also be approximately determined in terms of \( \langle \chi^2 \rangle \). By definition the potential energy \( V \) is given by
\[ V = \frac{1}{2} \sigma \Phi \sin(mt) \langle \chi^2 \rangle. \]  
(65)

The calculations of the variables \( T \) and \( G \) proceed as follows. Compared to the momentum integral in \( \langle \chi^2 \rangle \), the integrand in \( T \) and \( G \) contains two more powers of \( k \) (note that in the decay range \( \omega_k \simeq k/a \)). Evaluating the integrals, the dominant contribution comes from the Ultra-Violet (UV) end of the limit, which is equal to \( m \), and this gives
\[ T \simeq G \simeq m^2 \langle \chi^2 \rangle. \]  
(66)

Since in perturbation theory \( \sigma \Phi/m^2 \ll 1 \), one finds that \( T \sim G \gg V_{\text{max}} \), where \( V_{\text{max}} \) is the maximum value of the potential energy \( V \).

From (14) and (15) the phase \( \phi_k \) can also be determined easily when the expansion of the universe is neglected. Using \( |\alpha_k| \gg |\beta_k| \), one finds that \( \phi_k \simeq 3\pi/2 \). As oppose to the preheating, the integrals involving \( \phi_k \) in (20) and (24) now tend to decrease \( \delta \rho \) and increase \( \delta P \). Finally, by using (56) in (31) (and taking the \( r \to 0 \) limit) one can obtain
\[ \langle \chi^2 \rangle \sim \frac{\sigma^2 \Phi M_P}{m^2}. \]  
(67)

Not surprisingly, one sees from (67) that in perturbation theory \( \langle \chi^2 \rangle \) is smaller by several orders of magnitude compared to \( M_P^2 \), and it is also much smaller than \( m^2 \).

**Thermalization**

After determining the correlation length of quantum fluctuations, let us now discuss possible effects that they might produce during reheating. We first consider the thermalization process of the decay products. This is a difficult process to study and unfortunately there is not much work done in the literature (see e.g. [19–26]). In general, the distribution of particles produced during the decay of the inflaton can be seen to be far from thermal equilibrium [19]. Thus, depending on the interaction rates the full thermalization can be delayed resulting a low reheat temperature. Here, we would like to point out that a very quick or instant thermalization of the decay products may also be prohibited by the existence of quantum fluctuations in the energy density.

It is known that in thermal equilibrium at temperature \( T \) the energy density is given by
\[ \rho = \gamma T^4, \]  
(68)

where \( \gamma \) is a constant depending on the number of bosonic and fermionic species in equilibrium. Then, \( U_V = \rho V \), where \( U_V \) is the average energy inside a physical volume \( V \). In the canonical ensemble, the dispersion of the energy about its mean value is given by
\[ \overline{U_V^2} - U_V^2 = -\frac{\delta U_V}{d\beta}, \]  
(69)

which, using (68), implies (see e.g. [27])
\[ \frac{\delta U_V}{U_V} = \frac{2}{(\gamma V)^{1/2}} \frac{1}{T^{3/2}}. \]  
(70)

In (70), the dependence of the relative deviation on \( V \) is a characteristic feature of thermal fluctuations.

Thermal equilibrium can only be justified if thermal fluctuations at a given scale is larger than the quantum fluctuations at the same scale. Otherwise, one should wait for some time for the energy to be transferred in space for uniformization. (see the discussion following (76) below). In a volume \( V \), there are on the average \( V/(\xi_{\text{phys}}^3) \) number of uncorrelated regions as far as quantum fluctuations are concerned. Since in each region the relative deviation of the energy density is of the order of unity, the relative order of quantum fluctuations in the volume \( V \) can be found as
\[ \frac{\delta U_V}{U_V} \sim \frac{\xi_{\text{phys}}^3/2}{V^{1/2}}. \]  
(71)

Comparing with (70), one sees that instant thermal equilibrium after the decay can only be justified if
\[ T < \frac{1}{\xi_{\text{phys}}^3/\gamma^{1/3}}, \]  
(72)

i.e. when the thermal correlation length is larger than \( \xi_{\text{phys}}^3 \). Eq. (72) places an upper bound on the reheating temperature in terms of the correlation length of quantum fluctuations. Note that since thermal fluctuations can occur locally, the volume \( V \) about which the fluctuations are compared should be taken inside the horizon and thus it is important to have \( \xi_{\text{phys}}^3 \) to be less than the Hubble radius.

Since the Hubble parameter should not change abruptly, the final equilibrium temperature \( T_R \) can be determined from the Friedmann equation
\[ H^2 \sim \frac{\rho}{M_P^2} \sim \frac{T^4}{M_P^2}, \]  
(73)
which gives \( T_R \approx \sqrt{H M_p} \). The full thermalization can be achieved when \( \Gamma \chi \sim H \), where \( \Gamma \) is the total interaction rate of the decay products and should not be confused with \( \Gamma_\phi \), i.e. the decay rate of the inflaton.

In the preheating model reviewed above, the Hubble parameter at the end of the broad parametric resonance is determined in (44). If thermal equilibrium sets in a very short time following the end of the broad parametric resonance, the reheating temperature must be fixed as

\[
T_R \sim \sqrt{\frac{m^2}{g}}. \tag{74}
\]

Then, (45) and (72) give

\[
g > \gamma^{2/3} q_0^{1/6}. \tag{75}
\]

Since \( q_0 \gg 1 \), the coupling constant becomes large \( g > 1 \), which shows that the quantum fields \( \phi \) and \( \chi \) are strongly interacting. This is a very difficult regime to study and the results about the preheating period reviewed above can no longer be trusted. For \( g < 1 \), (72) cannot be satisfied, which shows that even when the total interaction rate is large to yield quick thermal equilibrium in principle, this cannot be achieved since would be thermal fluctuations are (much) smaller than quantum fluctuations.

On the other hand, in the perturbative decay described by the interaction (51), the reheating temperature is given by \( T_R \sim \sqrt{\sigma^2 M_p/m} \) (here one assumes \( \Gamma_\phi \sim \Gamma_\chi \sim \sigma^2/m \)). Then, (62) and (72) imply

\[
\left( \frac{a_0}{a_1} \right)^2 \frac{m^3}{\sigma^2} > M_p. \tag{76}
\]

It is not very difficult to satisfy this condition since one usually assumes \( \sigma \ll m \ll M_p \).

If (72) is not satisfied, one should wait for the energy to be redistributed for thermal equilibrium. This process is different than the redistribution of \textit{mean} energy or \textit{mean} number density by scattering or particle decays, as analyzed, for instance, in [22]. Namely, fluctuations should propagate in space-time to achieve uniformization and this cannot happen instantaneously because of causality.

To illustrate what would happen when (72) is violated, i.e. when quantum fluctuations are larger than thermal fluctuations, assume that \( \Gamma_\chi \) is large enough to convert \( \chi \) particles into radiation in local thermal equilibrium in a very short time. Consider now two different regions that have the size of the correlation length and assume that in one region all the energy stored in the inflaton oscillations is converted into \( \chi \) particles and in the other region only half of the energy is converted into \( \chi \) particles. This is a typical situation since the deviation in the energy density of the created \( \chi \) particles is comparable to its mean value. Let us now try to imagine what would happen as the universe expands twice. In the first region the energy density, being composed of radiation in equilibrium, will be redshifted by 16 times, while in the second region half of the energy density, i.e. the radiation, will be redshifted by 16 times and the other half corresponding to inflaton oscillations will be redshifted 8 times. Therefore, there appears a genuine density contrast \( \delta \rho \) and one can easily see that because of the different redshifts of the constituent energy densities, \( \delta \rho/\rho \) grows with the expansion like \( \delta \rho/\rho \sim a \).

On the other hand, the free propagating of radiation is expected to work for the uniformization of energy, but this process is slowed down by causality. Namely, by looking at the evolution of radial null geodesics, a comoving spherical region of volume \( V \) can be seen to expand (in the comoving grid) as \( a^{3/2} \). Therefore, \( \delta \rho/\rho \) is expected to decrease like \( \delta \rho/\rho \sim 1/\sqrt{V} \sim a^{-3/4} \) which is smaller than the increase \( \delta \rho/\rho \sim a \) noted above. As a result, one sees that the fluctuations in the \( \chi \) field induce changes in the total energy density during the decay process. Later on, when everything is converted into radiation, the fluctuations tend to be smoothed out, but depending both on the magnitude and the scale, one should wait for some time for full thermal equilibrium to set in.

This naive scenario should be sharpened by considering the evolution of all fields, especially by taking into account the back-reaction of the created particles on the inflaton oscillations and geometry. In any case, one sees that quantum fluctuations can be larger than (would be) thermal fluctuations in equilibrium and thus they should be taken into account when the thermalization of the decay products is studied.

**Back-reaction**

Another interesting process that can potentially be affected by quantum fluctuations is the back-reaction of created particles on the evolution of the background fields. It is important to understand back-reaction effects to get a complete picture of the reheating process. Here, we consider the preheating model with interaction (37) reviewed above. When the created \( \chi \) particles are taken into account, the background scalar field equation in (32) should be modified as

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{m^2 \Phi}{g} - g^2 (\phi_0 \phi) + g^2 \chi^2 \Phi = 0. \tag{77}
\]

One usually invokes Hartree approximation and uses the expectation value \( \langle \chi^2 \rangle \) for \( \chi^2 \) in (77). Since \( \langle \chi^2 \rangle \) does not depend on spatial coordinates and since initially (i.e. just after the end of inflation) the \( \phi \) field is homogeneous, only the zero mode continues to exist and one can ignore the spatial derivatives in (77). In that case, when the mean value of \( \chi^2 \) grows to satisfy

\[
g^2 < \chi^2 \sim m^2, \tag{78}
\]

the back-reaction effects become important. According to [15] this happens when

\[
\pi_\chi \sim \frac{m^2 \Phi}{g}. \tag{79}
\]
where $n_\chi$ is the average total number density of $\chi$ particles.

However, as shown in the previous section, $\chi^2$ has fluctuations comparable to its vacuum expectation value, therefore as $<\chi^2>$ grows and (78) is satisfied, the "actual value" of $\chi^2$ depends highly on the position, especially when it is compared at scales larger than the correlation length $\xi_c$. To determine the subsequent evolution of the inflaton zero mode, we imagine the space to be divided into regions of volume $\xi_c^3$ and analyze (77) in each region independently, trying to predict the full motion by gluing the results. Note that if only the zero mode survives even when the fluctuations are taken into account, then this should be a good approximation.

From (77) and by ignoring the expansion of the universe, the frequency of the oscillations in the $i$th region is given by

$$\omega_{(i)}^2 = m^2 + g^2\chi_i^2,$$

where $\chi^2_i$ denotes the value of $\chi^2$ in that region. Since $\omega_i$ is (nearly) uniform in the $i$th region, one can approximately write

$$\phi_i \simeq \Phi \sin(\omega_i t),$$

where $\phi_i$ denotes the restriction of $\phi$ in that region.

Let us now try to see if these local solutions, which are valid in different regions, can smoothly be glued. From (80), one sees that when $q^2 < \chi^2 > \sim m^2$, the frequencies start to differ from one region to another as

$$\frac{\delta \omega}{\omega} \sim \frac{\delta \chi_i^2}{\chi_i^2} \sim 1.$$

Therefore, in time $t \sim 1/\omega \sim 1/m$, nearly corresponding to a single average oscillation, the oscillations of the inflaton field in different regions become completely out of phase.

Assume now that $\phi = \sum_i \phi_i$, where each $\phi_i$ has its support in the $i$th region. In that case, one can estimate the spatial derivatives in (77) as

$$\partial_i \phi \sim \frac{\Phi}{\xi_c}, \quad g^{ij}(\partial_i \partial_j \phi) \sim \frac{\Phi}{(\xi_c^{\text{phys}})^2}.$$  

By comparing with the mass term, the spatial derivatives can only be ignored if

$$m^2 |\phi| \gg \frac{\Phi}{(\xi_c^{\text{phys}})^2}.$$  

It is not possible to satisfy (84), when the inflaton passes through its minimum $\phi = 0$. When it reaches its maximum $\phi = \Phi$, spatial derivatives can be neglected if

$$m \gg \frac{1}{\xi_c^{\text{phys}}}.$$  

Using (45) as an estimate for $\xi_c^{\text{phys}}$, (85) requires $q_0^{1/12} \gg 1$, which can only be satisfied if $q_0 \gg 10^6$, or so. As a result, one sees that when back-reaction effects become important the coherence of the inflaton oscillations is lost and the spatial derivatives must not be neglected.

If preheating continues after back-reaction, our findings indicate a major change in the whole process. First of all, with the inclusion of spatial derivatives it becomes much more difficult, if not impossible, to determine the subsequent evolution of the inflaton field. Moreover, one should also revise the particle creation process since the evolution of the background drastically changes. Presumably, it will no longer be possible to talk about resonance bands in momentum space, but instead it would be more appropriate to analyze the process region by region.

**Formation of primordial black holes**

Finally, in this subsection we point out a new mechanism for the formation of primordial black holes, which can be efficient especially in models of preheating. As we will see, this process is different than the one considered, e.g. in [28], where a sufficiently large density contrast, whose amplitude is greater than a critical value as it enters the horizon, dynamically evolves in time to form a black hole.

We show in the previous section that the energy density $\rho$ has fluctuations about its mean value. This is not surprising, since in quantum theory (4) implies that $\rho$ becomes a random variable as it is a function of a Gaussian random variable $\chi$. In general one can try to determine (at least an approximate) probability distribution for $\rho$, which must be close to a chi-squared distribution, but this is in general a difficult task since it contains two "non-commuting" random variables $\chi$ and $\dot{\chi}$. Instead of considering the full energy density, in the following we focus on the potential energy part

$$\rho_P = \frac{1}{2} M^2 \chi^2,$$

which has a simple probability distribution determined in terms of the Gaussian distribution of $\chi$. It is clear that $\rho > \rho_P$ and if $\rho_P$ exceeds the critical value for a black hole to form then so does $\rho$.

Consider now the potential energy $E_P$ stored in a region of size $\xi_c^{\text{phys}}$, which is given by

$$E_P = \rho_P (\xi_c^{\text{phys}})^3 \simeq M^2 \chi^2 (\xi_c^{\text{phys}})^3.$$  

If the Schwarzschild radius corresponding to this energy becomes greater than $\xi_c^{\text{phys}}$, then a black hole must form in that region (as discussed in the previous subsections, $\xi_c^{\text{phys}}$ is in general smaller than the Hubble radius and thus it is legitimate to ignore the expansion of the universe). Recalling that the Schwarzschild radius for the energy $E$ is given by $R_S \simeq E/M^2$, the condition $R_S > \xi_c^{\text{phys}}$ for the energy (87) becomes

$$\chi^2 > \chi_0^2 \equiv \frac{M^2}{M^2 (\xi_c^{\text{phys}})^2}.$$  


Since \( \chi \) is a Gaussian random variable with zero mean, the normalized probability distribution can be written as
\[
P(\chi) = \frac{1}{\sqrt{2\pi} \chi^2} \exp\left( -\frac{\chi^2}{2} \right). \tag{89}
\]
Therefore, the probability of getting \( \chi^2 = \chi^2_0 \), which is also the probability for a black hole to form, is given by
\[
P = 2 \int_{\chi_0}^{\infty} P(\chi) \, d\chi. \tag{90}
\]
Note that, we underestimate this probability, since only the contribution of the potential energy is considered. Nevertheless, \((90)\) should give a good estimate since one would not expect a large hierarchy between the potential and total energies.

On the other hand, because of the time dependence of the variance \( \langle \chi^2 \rangle \) (and in some cases \( \chi_0 \)), the probability distribution \((89)\) and the probability \((90)\) changes in time. In that case, the black hole formation process can be visualized in time as follows. Assume at a fixed time \( t \), some \( P(t) \) fraction of regions collapse to form black holes. Then one should wait for some time for the field \( \chi \) to evolve according to the new probability distribution and then regions containing large \( \chi^2 \) fluctuations collapse again. Therefore, the process is actually cumulative. Moreover, the energy density in a collapsing region is larger than the average energy density in the universe, thus the actual fraction of energy that goes into primordial black holes must be larger than \( P \). All these arguments support the use of \((90)\) as a conservative estimate for black hole formation probability.

If \( \beta \) denotes the observational upper limit of the fraction of energy that can go into the primordial black holes, then one should impose
\[
\beta > P. \tag{91}
\]
It is known that \( \beta < 10^{-20} \) (see, e.g., [28]). The magnitude of \( P \) depends very sensitively on the ratio \( \langle \chi^2 \rangle / \chi^2_0 \) and numerically one can find that to satisfy \((91)\) with a (small) margin the following condition must be obeyed:
\[
\frac{\chi^2_0}{\chi^2} > 22. \tag{92}
\]
Using the definition of \( \chi^2_0 \) from \((88)\), this implies
\[
\langle \chi^2 \rangle < \frac{M_p^2}{22M_p^2(\xi^{physy})^2}. \tag{93}
\]
Therefore, in a model giving a large vacuum expectation value for \( \chi^2 \) and a small correlation length \( \xi^{physy} \), formation of primordial black holes can be a problem. It is clear that this can be a dangerous issue especially for preheating.

Let us check, for example, if the model \((37)\) reviewed above passes the condition \((93)\). Assume that the broad parametric resonance ends just when the back reaction effects become important, i.e. \( q = 1 \) when \( q^2 < \chi^2 > \sim m^2 \).

In this paper, we consider the well known particle creation effects in a time-dependent, homogenous and isotropic, classical background and point out a feature that has not been elaborated in detail previously. Namely, we examine the fluctuations of important physical quantities characterizing the particle creation process about their vacuum expectation values. We specifically consider the energy density, the pressure and the other components of the energy-momentum tensor, and find that all these quantities have in general large fluctuations comparable to average energy density. We also note that the deviation corresponding to the field square \( \chi^2 \) also equals \( \sqrt{2} \) times the mean value of \( \chi^2 \). It is possible to make sense of all these fluctuations by using adiabatic or WKB regularization, therefore they have direct and unambiguous physical meaning.

The spatial scale of these fluctuations is given by the correlation length of the quantum field \( \chi \) excited by the classical background. This is an important observation, which allows one to think of these fluctuations being defined in regions that have the size of the correlation length, instead of imagining them as independent random variables defined at different points. If one still maintains the view that the field variables are defined point-wise, then one should always keep in mind that the variables inside the same region are correlated with each other and behave in the same way.

In the second part of this paper, we focus on the reheating process in single scalar field driven inflationary models and investigate possible implications of our findings. We consider two well known models, which are typical examples of the decay in broad parametric resonance regime and in perturbation theory, and determine the correlation length of quantum fluctuations. In both models, the physical correlation length becomes smaller than the Hubble radius at the end of the decay, where the ratio tends to be smaller in perturbation theory.
We investigate three possible effects of these fluctuations during reheating period. The first one is related to the final thermalization process of the decay products. One usually assumes that the decay products reach thermal equilibrium in a very short time, which is possible if the interaction rates are comparable to the expansion rate at the end of the decay. Although, this assumption is difficult to satisfy in realistic scenarios (see, e.g. [22]), it is important to estimate the maximum reheating temperature in a given model. We show that even when the interaction rates are presumed to be large enough and thermal equilibrium is expected to be set in a very short time, the existence of quantum fluctuations may delay this process. Namely, we observe that if on a given (subhorizon) scale quantum fluctuations are larger than would be thermal fluctuations in equilibrium then one should wait for energy to be redistributed in space to obtain the real equilibrium. Especially for larger temperatures, the thermal correlation length can be many orders of magnitude smaller than the quantum correlation length, which requires more time for energy to spread. On the other hand, the whole process can actually be much more difficult to study since back-reaction effects must be considered to obtain the complete picture.

As a second event which might be affected by quantum fluctuations, we consider the back-reaction of the created particles on the inflaton oscillations in the model of preheating. Back-reaction becomes important when the mean value of $\chi^2$ sufficiently grows to modify the frequency of the inflaton oscillations given by the inflaton mass. When this happens, because of the fluctuations in $\chi^2$ the oscillation frequency starts to appreciably vary on scales larger than the correlation length. We show that in a very short time the coherence of the oscillations is lost and spatial derivatives of the inflaton can no longer be neglected in determining its dynamical evolution. From that moment on, the whole back-reaction process changes and the problem becomes much more difficult to study since it involves non-homogenous fields in space.

Finally, we notice that a large fluctuation in the energy density, which occurs in a region that has the size of the correlation length, causes a black hole to form if the corresponding Schwarzschild radius becomes greater than the correlation length. To our knowledge, this process is considered for the first time in the context of reheating as a plausible mechanism for the formation of primordial black holes. In previous studies (see e.g. [28]), the collapse of a sufficiently large, horizon size density contrast, which is known to produce a black hole as a result of its dynamical evolution under the influence of classical gravity, is considered as the main mechanism. We estimate the fraction of energy that can go into primordial black holes because of large energy fluctuations and show that observational constraints impose some new restrictions for the models.

It is clear from our findings that, as far as the reheating after inflation is concerned, the correlation length is one of the most important parameters characterizing the quantum fluctuations in the particle creation process. While sometimes a larger correlation length amplifies the impacts of quantum fluctuations as in the final thermalization process, in some instances a smaller correlation length can increase the effects; formation of primordial black holes being an example as discussed below (93).

It would be interesting to extend the present work in different directions. For example, it is of interest to study the back-reaction effects in more detail in the preheating model. Particularly, new physics may arise if the broad parametric resonance continuous to exists when the back-reaction effects become important. It would also be interesting to extend the fluctuation analysis to include perturbations of the other fields, i.e. the metric and the inflaton. The fluctuations in pressure and other components of the energy-momentum tensor are expected to play a role in this study. Finally, it is clear that the existence of these fluctuations should be considered when symmetry breaking effects or formation of topological defects are studied. For example, in [29] formation of cosmic strings in a preheating model is studied by solving classical field equations numerically. Although in a linear theory such as considered in [29], the classical field equations determine the evolution of the mean values, it would not be surprising to see the destruction of cosmic strings longer than the correlation length because of the existence of large, order one fluctuations. It would be interesting to check this expectation by an explicit computation.

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