Abstract:

The Hawking temperature for Schwarzschild black hole $T_H = 1/8\pi M$ is singular in the limit of vanishing mass $M \to 0$. However, the Schwarzschild metric itself is regular when the black hole mass $M$ tends to zero, it is reduced to the Minkowski metric and there are no reasons to believe that the temperature becomes infinite. It is pointed out that this discrepancy may be due to the fact that the Kruskal coordinates are singular in the limit of the vanishing mass of the black hole. To improve the situation, new coordinates for the Schwarzschild metric are introduced, called thermal coordinates, which depend on the black hole mass $M$ and the parameter $b$. The thermal coordinates are regular in the limit of vanishing black hole mass $M$. In this limit, the Schwarzschild metric is reduced to the Minkowski metric, written in coordinates dual to the Rindler coordinates.

Using the thermal coordinates the Schwarzschild black hole radiation is reconsidered and it is found that the Hawking formula for temperature is valid only for large black holes while for small black holes the temperature is $T = 1/2\pi(4M + b)$. The thermal observer in Minkowski space sees radiation with temperature $T = 1/2\pi b$, similar to the Unruh effect with non-constant acceleration. Now, during evaporation, in the thermal coordinates the black hole mass is decreasing inverse proportional to time and the black hole life time is infinite. The thermal coordinates for more general spherically symmetric metrics, including the Reissner-Nordstrom, de Sitter and anti-de Sitter are also considered. In these coordinates one sees a Planck distribution with constant temperature. One obtains that the property to have a temperature distribution for quantum fields in classical gravitational background is not restricted to the cases of black holes or constant acceleration, but is valid for any spherically symmetric metric written in thermal coordinates. Implications for primordial black holes and for the information loss problem are mentioned.
1 Introduction

Hawking showed that black holes emit radiation like black bodies with a temperature of $T_H = 1/8\pi M$, where $M$ is the mass of the black hole [1, 2]. Considering the scenario where a black hole that emits its energy ends up completely evaporating, a problem arises. From Hawking’s formula it follows that the energy density of radiation emitted by a black hole according to the Stefan-Boltzmann formula behaves at small $M$ as $M^{-4}$. If during evaporation the mass of the black hole disappears, then the black hole releases an infinite amount of energy, which is clearly not physical. This can be called the problem of the big bang of black holes. The information loss problem [3–5] is closely related to this big bang problem, since the radiation entropy diverges for small $M$ as $M^{-3}$. Considerations in more complicated cases, such as de Sitter Schwarzschild black hole evaporation, do not improve the situation in an essential way.

Standard transformation from the Schwarzschild coordinates to the Kruskal ones includes first transformation from the Schwarzschild coordinates to Eddington-Finkelstein coordinates [6–9]. The Schwarzschild metric in the Schwarzschild coordinates

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2, \quad r > 2M > 0,$$

(1.1)

obviously admits the $M \to 0$ limit, that defines the Minkowski space. The Kruskal coordinates $(U, V)$ are defined as

$$U = -e^{-u/4M}, \quad V = e^{v/4M}$$

(1.2)

here $u$ and $v$ are the Eddington-Finkelstein coordinates. The Kruskal coordinates are used to get a maximal analytic extension, but we note, that in the limit of vanished mass $M$ black hole, even outside of horizon, $r > 2M$, the Kruskal coordinates and the metric get a singularity, instead to become the Minkowski one. This leads also to the singular behaviour of the Hawking temperature $T_H = 1/8\pi M$ in the limit $M \to 0$. To improve the situation, E(xponential)-coordinates $\mathcal{U}$ and $\mathcal{V}$ for the Schwarzschild metric are introduced,

$$\mathcal{U} = -e^{-\mathcal{u}/4\mathcal{M}}, \quad \mathcal{V} = e^{\mathcal{v}/4\mathcal{M}}$$

(1.3)

which depend on the black hole mass $M$ and a parameter $b > 0$. The E-coordinates are regular in the limit of the vanishing black hole mass $M$. Obviously in this limit the Schwarzschild metric is reduced to the Minkowski one written in the E-coordinates.

Black hole radiation is considered and it is found that the Hawking formula for temperature is approximately valid only for large black holes while for small black
holes for the temperature of black hole the following formula is obtained:

\[ T = \frac{1}{2\pi(4M + b)}. \]  

(1.4)

As a result, black holes could completely evaporate in terms of classical geometry, but it is shown that this requires the infinite time because the mass is decreasing in the inverse proportional to time,

\[ M(t) = \frac{C}{t}, \quad t \to \infty \]  

(1.5)

It is noted that E-observer in Minkowski space will see radiation with the temperature

\[ T = \frac{1}{2\pi b}. \]  

(1.6)

This effect is similar (dual) to the Unruh effect [10, 11] for the Rindler metric [12] but in our case the acceleration is not a constant.

We define the E(xponential)-coordinates \((U, V)\) and L(ogarithmic)-coordinates \((\nu, \vartheta)\) for the arbitrary static metric of the form

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2 = -f(r)dudv + r^2d\Omega^2 \]  

(1.7)

as

\[ U = -e^{-\frac{u}{B}}, \quad V = e^{\frac{v}{B}}, \quad B > 0, \]  

(1.8)

\[ \vartheta = -\frac{1}{a}\log(av), \quad \nu = -\frac{1}{a}\log(-au) \quad a > 0. \]  

(1.9)

It will be shown that one has the Planck distribution with temperature \(T = 1/2\pi B\) and \(T = a/2\pi\) for quantum fields in the gravitational background (1.7) with an arbitrary function \(f(r)\) in E- and L-coordinates, respectively. We have the following general scheme (duality)

\[ \begin{pmatrix} \text{E-coord.} \, (U, V) \rightarrow \nu = e^{\frac{u}{B}} \\ \vartheta = -e^{-\frac{v}{B}} \end{pmatrix} \begin{pmatrix} M \\ (u, v) \end{pmatrix} \rightarrow \begin{pmatrix} \nu = -\frac{1}{a}\log(-au) \\ \vartheta = -\frac{1}{a}\log(av) \end{pmatrix} \begin{pmatrix} \text{L-coord.} \, (\nu, \vartheta) \end{pmatrix} \]  

(1.10)

The physical meaning of above formulae for temperature is that they give temperature of radiation for different observers moving along different trajectories in the same background. The simplest examples of such special trajectories are ones in the Minkowski space. The standard Rindler observer moves with constant acceleration and sees radiation from Minkowski vacuum as it has temperature defined by its acceleration. The E-observer moves along a hyperbola and feels the temperature. E- and
L-coordinates can be called the thermal coordinates since in these coordinates one sees a Planck distribution with constant temperature. Actually, the property to have a temperature is not connected with black holes and horizons but with the usage of the thermal coordinates and the temperature can be obtained for any metric. Implications for the information loss problem and primordial black holes are mentioned.

The paper is organized as follows. We start Sect.2 reminding the standard definition of the Kruskal coordinates and also we discuss problem that arises with them when one considers the limit $M \to 0$. Then in Sect.2.2 we introduce in E-coordinates for Schwarzschild metric, and in Sect.2.3 temperature of Schwarzschild black holes in E-coordinates. In the next Sect.3 we discuss E-coordinates in Minkowski space. We show in Sect.3.1 that two-dimensional Minkowski space can be represented as a union of 4 disconnected regions, right ($\mathcal{R}$), future ($\mathcal{F}$), left ($\mathcal{L}$) and past ($\mathcal{P}$), and each of them is isometric to two-dimensional Minkowski space. In 3.2 we study geodesics in E-coordinates. In Sect.3.3 we calculate acceleration of a E-observer. Sect.3.4 is devoted to comparison of E- and Rindler coordinates in Minkowski space. In Sect.4 we introduce general E-coordinates. In Sect.4.2 acceleration along special trajectories $\mathcal{E} = \mathcal{E}_0$ in black hole backgrounds are calculated. Then in next sections we consider some examples, In next Sect.5 general L-coordinates are introduced. In Sect.5.4 temperature in L-coordinates is calculated and it has been found that it is given by an universal formula that does not depend on characteristic of the black hole under consideration. The origin of this phenomena is that the choice of the coordinate system depends in essentially way on the metric itself. In Sect.6 we present an estimation of evaporation time as it can been seen by observers in different coordinate systems. In Sect.7 we summarise the obtained results and discuss their physical applications.

2 Exponential Coordinates

2.1 Kruskal coordinates

Standard transformation from the Schwarzschild coordinates to the Kruskal ones includes first transformation from the Schwarzschild coordinates to Eddington-Finkelstein coordinates. The Schwarzschild metric in the Schwarzschild coordinates is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. It is obvious that the exterior Schwarzschild spacetime ($r > r_h = 2M$) admits the $M \to 0$ limit, that defines the Minkowski spacetime\footnote{The Kretschmann invariant $K = \frac{48M^2}{r^6} \to 0$ as $M \to 0$ for any fixed $r > 2M$.}.

One can introduce the tortoise coordinate $r_*$

$$r_* = r + 2M \log \left( \frac{r}{2M} - 1 \right).$$

$$\frac{48M^2}{r^6} \to 0$$ as $M \to 0$ for any fixed $r > 2M$.}
which solves the equation \( dr_\ast = (1 - 2M/r)^{-1}dr \). To keep the reality condition one has to assume \( r > 2M \). Then one defines

\[
u = t - r_\ast, \quad v = t + r_\ast.
\] (2.3)

(coordinates \( u, v \) cover whole \( \mathbb{R}^2 \)), and one has

\[
ds_2^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 = -(1 - \frac{2M}{r})dudv
\] (2.4)

The Kruskal coordinates are

\[
U = -e^{-\frac{r}{2M}}, \quad V = e^{v/4M}
\] (2.5)

and the Schwarzschild metric becomes

\[
ds^2 = -\frac{32M^3}{r}e^{-r/2M}dUdV + r^2d\Omega^2,
\] (2.6)

where \( r \) is defined from equation

\[
\left( \frac{r}{2M} - 1 \right)e^{\frac{r}{2M}} = -UV
\] (2.7)

Note that the Kruskal coordinates (2.5) and metric (2.6) are singular in the limit \( M \to 0 \).

2.2 E-coordinates for the Schwarzschild metric

To be able to send the mass \( M \) of black hole to zero we define coordinates (we call them E-coordinates) as follows

\[
\mathcal{U} = -e^{-\frac{r}{2M+b}}, \quad \mathcal{V} = e^{v/4M+b},
\] (2.8)

here \( b \) is a positive constant. The coordinates run over the region \( \mathcal{U} < 0, \mathcal{V} > 0 \). The question of the existence of an extension of the Schwarzschild metric that is analytic not only with respect to space and time variables, but also with respect to the mass parameter, requires a separate consideration.

The Schwarzschild metric in the E-coordinates is \( r > 2M \)

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega^2
\]

\[
= (4M + b)^2(1 - \frac{2M}{r})\frac{d\mathcal{U}d\mathcal{V}}{\mathcal{U}\mathcal{V}} + r^2d\Omega^2,
\] (2.9)

here \( r \) is derived from the relation

\[
e^{2r_\ast/(4M+b)} = -\mathcal{U}\mathcal{V}
\] (2.10)
It is clear that in the limit \( b \to 0 \) the metric (2.9) rewritten as

\[
\begin{align*}
ds^2 &= -16(M + b/4)^2 (2M)^{M/4(1+1/4)} \frac{r - 2M}{r} \frac{r}{\pi M} e^{-\frac{r}{\pi M (M+b/4)}} \cdot d\mathcal{U} d\mathcal{V} + r^2 d\Omega^2 
\end{align*}
\]

becomes the Schwarzschild-Kruskal metric (2.6).

At the limit \( M \to 0 \) the metric (2.9) becomes

\[
\begin{align*}
ds^2 &= -b^2 e^{-\frac{2\pi}{r}} d\mathcal{U} d\mathcal{V} + r^2 d\Omega^2, 
\end{align*}
\]

here \( r \) is defined by

\[
\begin{align*}
- \mathcal{U} \mathcal{V} &= e^{\frac{2\pi}{r}}. 
\end{align*}
\]

Equation (2.13) is nothing but the formula (2.10) rewritten as

\[
\begin{align*}
(r - 2M)^{M/4(1+1/4)} = (2M)^{M/4(1+1/4)} (-\mathcal{U} \mathcal{V}) e^{-\frac{r}{\pi M (M+b/4)}} 
\end{align*}
\]

in the limit \( M \to 0 \). Explicit check shows that metric (2.12) is the Minkowski metric.

### 2.3 Temperature of Schwarzschild black holes in E-coordinates

We consider the scalar field on the Schwarzschild background in two systems of coordinates, the Eddington-Finkelstein \((u,v)\) and E-coordinates \((\mathcal{U}, \mathcal{V})\) related as

\[
\begin{align*}
\mathcal{U} &= -\exp\left\{-\frac{u}{4M+b}\right\}, \\
\mathcal{V} &= \exp\left\{\frac{v}{4M+b}\right\}, \\
u, v &\in \mathbb{R}, \\
M > 0, \ b > 0. 
\end{align*}
\]

The two-dimensional parts of the Schwarzschild metric in these coordinate systems read

\[
\begin{align*}
ds^2_2 &= -(1 - \frac{2M}{r}) du dv = (4M + b)^2 \frac{r}{\pi r (M+b/4)} d\mathcal{U} d\mathcal{V}, \quad r > 2M. 
\end{align*}
\]

The wave equations for the scalar field \( \phi(u,v) = \Phi(\mathcal{U}, \mathcal{V}) \) in these coordinate systems are

\[
\begin{align*}
\partial_u \partial_v \phi &= 0, \quad u, v \in \mathbb{R} \\
\partial_\mathcal{U} \partial_\mathcal{V} \Phi &= 0, \quad \mathcal{U} < 0, \ \mathcal{V} > 0. 
\end{align*}
\]

They can be represented as combinations of the left and right modes, \( \phi(u,v) = \phi_R(u) + \phi_L(v) \). For the real right mode (for the left mode all consideration is similar and will be omitted) one has

\[
\begin{align*}
\phi_R(u) &= \int_0^\infty d\omega (f_\omega b_\omega \mathbf{e} + f_\omega^* b_\omega^* \mathbf{e}), \\
f_\omega(u) &= \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega u}, 
\end{align*}
\]

where

\[
[b_\omega, b_\omega^*] = \delta(\omega - \omega'). 
\]
One also has representation for Φ-field  
\[ \Phi_R(\mathcal{U}) = \int_0^\infty d\mu \left( b_\mu f_\mu(\mathcal{U}) + b_\mu^* f_\mu^*(\mathcal{U}) \right), \quad f_\mu(\mathcal{U}) = \frac{1}{\sqrt{4\pi \mu}} e^{-i\mu \mathcal{U}} \]  
(2.21)

where

\[ [b_\mu, b_{\mu'}] = \delta(\mu - \mu') \]  
(2.22)

Right (and left) modes in different coordinate system are related \( \phi_R(u) = \Phi_R(U(u)) \) and therefore,

\[ \int_0^\infty d\omega (f_{\omega} b_{\omega} + f_{\omega}^* b_{\omega}^*) = \int_0^\infty d\mu (f_{\mu} b_{\mu} + f_{\mu}^* b_{\mu}^*), \quad u \in \mathbb{R} \]  
(2.23)

Multiplying (2.23) on \( f_{\omega'}(u) \) and integrate the first equation over \( \mathbb{R} \) one gets \(^2\)

\[ b_{\omega} = \int d\mu \left( \beta_{\omega}^* b_{\mu}^* + \alpha_{\omega} b_{\mu} \right), \quad b_{\omega}^* = \int d\mu \left( \beta_{\omega} b_{\mu} + \alpha_{\omega}^* b_{\mu}^* \right), \]  
(2.24)

where

\[ \beta_{\omega} = \int_\mathbb{R} \frac{du}{2\pi} \sqrt{\frac{\omega}{\mu}} \frac{e^{-iuu}}{e^{-i\mu \mathcal{U}}}, \]  
(2.25)

\[ \alpha_{\omega} = \int_\mathbb{R} \frac{du}{2\pi} \sqrt{\frac{\omega}{\mu}} \frac{e^{-iuu}}{e^{i\mu \mathcal{U}}}. \]  
(2.26)

The E-observer has the E-vacuum

\[ b_{\omega}|0_E\rangle = 0, \]  
(2.27)

i.e. the state \( |0_E\rangle \) does not contain \( b \) particles. But it contains the Minkowski \( b \) particles:

\[ \langle 0_E|N_{\omega}(b)|0_E\rangle \equiv \langle 0_E|b_{\omega}^* b_{\omega}|0_E\rangle = \int_0^\infty d\mu |\beta_{\omega\mu}|^2. \]  
(2.28)

The Bogoliubov coefficient \( \beta_{\omega\nu} \) is given by (2.25) with \( \mathcal{U} \) as in (2.15) and we have

\[ \beta_{\omega\mu} = \frac{B}{2\pi} \frac{\omega}{\mu} e^{-\frac{\mu}{B}} \left( \mu \right)^{-iB\omega} \Gamma(iB\omega), \quad B = 4M + b \]  
(2.29)

Using the formula \( |\Gamma(ix)|^2 = \pi/(x \sinh(\pi x)) \) we get he Planck distribution

\[ |\beta_{\omega\mu}|^2 = \frac{B}{2\pi \mu e^{2\pi B\omega} - 1}, \]  
(2.30)

with the temperature

\[ T = \frac{1}{2\pi B} = \frac{1}{2\pi (4M + b)} \]  
(2.31)

\(^2\)For \( b = 0 \) we get the standard formula for the Schwarzschild metric in the Kruskal coordinates.
3 E-Coordinates in Minkowski Space

3.1 Minkowski space in terms of new coordinates $U, V$ and $T, X$

Starting from Minkowski coordinates

$$ds^2 = ds_2^2 + r^2 d\Omega^2, \quad ds_2^2 = -dt^2 + dr^2, \quad r \in \mathbb{R}_+, \quad t \in \mathbb{R}. \quad (3.1)$$

we introduce the E-ones

$$U = U^{(R)}(t, r) = -\exp\left\{\frac{r - t}{b}\right\} = -\exp\left\{-\frac{u}{b}\right\}, \quad u = t - r \quad (3.2)$$

$$V = V^{(R)}(t, r) = \exp\left\{\frac{t + r}{b}\right\} = \exp\left\{\frac{v}{b}\right\}, \quad v = t + r, \quad b > 0, \quad (3.3)$$

$r > 0$ corresponds to $UV < -1$. One has an expression of $r$ in terms of coordinates $U, V$

$$r = b/2 \log(-UV) \quad (3.4)$$

The 2-dimensional Minkowski metric in (3.1) after this change becomes

$$ds_2^2 = b^2 \frac{dU dV}{UV} \quad (3.5)$$

One can see that the metric (2.12) admits an extension to the region $-UV < 0$. In this case by using formula (3.4) the coordinate $r$ is extending to the region $r \leq 0$, see Fig.1.

We also define the coordinates

$$T = \frac{U + V}{2}, \quad \mathcal{X} = \frac{V - U}{2} \quad (3.6)$$

We cover the $\mathbb{R}$-region

$$\mathbb{R} = \{(T, \mathcal{X}) \in \mathbb{R}^2 | \mathcal{X}^2 - T^2 > 0, \mathcal{X} > 0\}$$

by the map

$$R : \quad T = e^{r/b} \sinh \frac{t}{b}, \quad \mathcal{X} = e^{r/b} \cosh \frac{t}{b}, \quad (t, x) \in \mathcal{M}^{(1,1)} \quad (3.7)$$

The inverse transformation is

$$t = b \arctanh \frac{T}{\mathcal{X}} \quad r = b/2 \log(\mathcal{X}^2 - T^2), \quad (T, \mathcal{X}) \in \mathbb{R} \quad (3.8)$$

The metric here is

$$ds_2^2 = -dt^2 + dr^2 = \frac{b^2}{\mathcal{X}^2 - T^2}(-dT^2 + d\mathcal{X}^2) \quad (3.9)$$
Let us introduce the future (F) E-coordinates

\[
\begin{align*}
\mathcal{U} &= \mathcal{U}(F)(r, t) = \exp\left\{ \frac{r - t}{b} \right\} = \exp\left\{ \frac{-u}{b} \right\}, \\
\mathcal{V} &= \mathcal{V}(F)(r, t) = \exp\left\{ \frac{t + r}{b} \right\} = \exp\left\{ \frac{v}{b} \right\},
\end{align*}
\]

(3.10)

and they cover the F-part of (\(\mathcal{U}, \mathcal{V}\))-plane, see the top part of Fig.1,

\[\mathcal{U}\mathcal{V} > 0 \quad (3.11)\]

The Minkowski metric (3.1) after (3.10) becomes

\[ds^2 = -b^2 \frac{d\mathcal{U} d\mathcal{V}}{\mathcal{U}\mathcal{V}} \quad (3.12)\]

One also introduces the left (L) E-coordinates

\[
\begin{align*}
\mathcal{U} &= \mathcal{U}(L)(t, r) = \exp\left\{ \frac{r - t}{b} \right\} = \exp\left\{ \frac{-u}{b} \right\}, \\
\mathcal{V} &= \mathcal{V}(L)(t, r) = -\exp\left\{ \frac{t + r}{b} \right\} = -\exp\left\{ \frac{v}{b} \right\},
\end{align*}
\]

(3.13)

with the inequality

\[\mathcal{U}\mathcal{V} < 0 \quad (3.14)\]

and past (P) E-coordinates

\[
\begin{align*}
\mathcal{U} &= \mathcal{U}(P)(t, r) = -\exp\left\{ \frac{r - t}{b} \right\} = \exp\left\{ \frac{-u}{b} \right\}, \\
\mathcal{V} &= \mathcal{V}(P)(t, r) = -\exp\left\{ \frac{t + r}{b} \right\} = -\exp\left\{ \frac{v}{b} \right\},
\end{align*}
\]

(3.15)

with the inequality

\[\mathcal{U}\mathcal{V} > 0 \quad (3.16)\]

The metric can be written in the universal way

\[ds^2 = -dudv = -b^2 \frac{d\mathcal{U} d\mathcal{V}}{|\mathcal{U}\mathcal{V}|}, \quad \mathcal{U}\mathcal{V} \neq 0 \quad (3.17)\]

These maps are shown in Fig.1.

To summarize, we have obtained the 2-dimensional plane divided on 4 disconnected regions with E-coordinates (\(\mathcal{U}, \mathcal{V}\)) and the metric given by (3.17). Or in other words, we have obtained that two-dimensional space is represented as a union of 4 disconnected regions R,F,P and L, each of which of them is isometric to two-dimensional Minkowski space.
3.2 Geodesics in E-coordinates

The geodesics in E-coordinates \((U, V)\) can be obtained simply by changing of variables from geodesics in Minkowski space. However it seems instructive and useful for quantization to investigate geodesics directly in E-coordinates. Geodesics equations for the metric (2.12) are

\[
\frac{V'(s)^2 - V(s)V''(s)}{U(s)V(s)^2} = 0, \quad (3.18)
\]

\[
\frac{U'(s)^2 - U(s)U''(s)}{U(s)^2V(s)} = 0. \quad (3.19)
\]
These geodesics equations (3.18) and (3.19) can be solved to get
\[ V(s) = c_2 e^{c_1 s}, \quad U(s) = c_4 e^{c_3 s}; \]  
(3.20)
here \(-\infty < s < \infty\).

To guaranty that the geodesics run in one of regions R, F, L, P we have to take
\[ R: \quad c_2 > 0, \quad c_4 < 0, \quad F: \quad c_2 > 0, \quad c_4 > 0 \]  
(3.21)
\[ L: \quad c_2 < 0, \quad c_4 > 0, \quad P: \quad c_2 < 0, \quad c_4 < 0 \]  
(3.22)
here \(-\infty < s < \infty\), \(c_1\) and \(c_3\) can have any signs.

The geodesics (3.20) in \((\mathcal{T}, \mathcal{X})\) coordinates are
\[ \mathcal{X}(s) = \frac{1}{2} (c_4 e^{c_3 s} - c_2 e^{c_1 s}), \quad \mathcal{T}(s) = \frac{1}{2} (c_2 e^{c_1 s} + c_4 e^{c_3 s}); \]  
(3.23)

One can check that these geodesics after mapping to \(t, x\) (Minkowski space-coordinates) are (as should be) the straight lines
\[ t = t(s) = b \arctanh \frac{T(s)}{\mathcal{X}(s)} = \frac{1}{2} \left( s(-c_1 - c_3) - \log \left( \frac{-c_2}{c_4} \right) \right) \]  
(3.24)
\[ r = r(s) = 2b \log(\mathcal{X}(s)^2 - \mathcal{T}(s)^2) = 2b \left( (c_1 + c_3)s + \log(-c_2 c_4) \right) \]  
(3.25)

We have 3 types of geodesics:

- \(c_1 c_3 < 0\), in this case both ”ends” of geodesics are in infinities, see for example plot Fig. 3.A
- \(c_1 c_3 > 0\), in this case one ”end” of geodesics is at infinity and the second one is at zero, see for example plot Fig. 3.B
- \(c_1 = 0, c_3 \neq 0\) or \(c_3 = 0, c_1 \neq 0\), in this case geodesics are bounded by characteristics \(U = 0\) or \(V = 0\), see Fig. 3.C

From (3.20) follows that
\[ \left( \frac{V}{c_2} \right)^{c_3} = \left( \frac{U}{c_4} \right)^{c_1} \]  
(3.26)
3.3 Acceleration in E-coordinates

Let us consider the time-like trajectory with located at $X = X_0$. The proper time $\tau$

$$\tau = b \arctan \left( \frac{T}{\sqrt{X_0^2 - T^2}} \right) + \tau_0 \quad (3.27)$$

From (3.27) one has

$$T = \pm X_0 \sin \left( \frac{\tau - \tau_0}{b} \right) \quad (3.28)$$

We see that it takes the finite time $T$, or $\tau$, to reach the characteristic $X = T$.

In Fig.4.A and Fig.4.B, we plot trajectories with $X_0 = 1$ (red) and $X_0 = 4$ (blue) in $(X, T)$ and $(t, x)$ planes, respectively. We can parameterize these trajectories in
Figure 4. Trajectories with $X_0 = 1$ (red) and $X_0 = 4$ (blue) in $(X, T)$ plane (A) and $(x, t)$ plane (B). We can take $\tau_0 = 0$.

the Minkowski by $\tau$:

$$t = b \arctanh \left( \frac{\tau - \tau_0}{b} \right)$$  \hspace{1cm} (3.29)

$$r = \frac{b}{2} \log \left( X_0^2 \cos^2 \left( \frac{\tau - \tau_0}{b} \right) \right)$$  \hspace{1cm} (3.30)

We get

$$V^0 = \frac{dt}{d\tau} = \frac{X_0}{\sqrt{X_0^2 - T^2}}, \quad V^1 = \frac{dx}{d\tau} = \frac{T}{\sqrt{X_0^2 - T^2}}.$$  \hspace{1cm} (3.31)

$$W^0 = \frac{dV^0}{d\tau} = \frac{X_0 T}{b(X_0^2 - T^2)}, \quad W^1 = \frac{dV^1}{d\tau} = \frac{X_0^2}{b(X_0^2 - T^2)}.$$  \hspace{1cm} (3.32)

We have

$$W^2 = -(W^0)^2 + (W^1)^2 = \frac{X_0^2}{b^2(X_0^2 - T^2)}.$$  \hspace{1cm} (3.33)

One can also rewrite (3.33) in term of $t$

$$W^2 = \frac{1}{b^2} \cosh^2 \frac{t}{b}.$$  \hspace{1cm} (3.34)

or in term of $\tau$

$$W^2 = \frac{1}{b^2 \cos^2 \left( \frac{\tau - \tau_0}{b} \right)}.$$  \hspace{1cm} (3.35)

These calculations show that the acceleration $W$ of the E-trajectory with ”$X = X_0$” in the inertial coordinate system $(t, x)$ increases with increasing of the inertial time $t > 0$ as $W \sim \frac{1}{b} e^{t/b}$. If $b \to 0$ this acceleration increases to infinity, and decreases to 0 when $b \to \infty$.

Acceleration of the E-observer located at $X_0$ depends on its E-time $T$ (see Fig.5) as well as it own proper time. The proper time does not depend on the location. In all cases, the acceleration is inversely proportional to the $b$ parameter.
Figure 5. Accelerations of E-observers located at $X_0 = 1$ (red) and $X_0 = 2$ (blue) against their E-time $T$, $b = 1$ (A) and proper time $\tau$, $b = 0.5$ (gray) and $b = 1$ (green) (B).

3.4 Comparison of Exponential and Rindler coordinates

The accelerated observer traveling in Minkowski with constant acceleration $a$ is described by the Rindler coordinates $(\nu, \vartheta)$ related with the inertial coordinates $(u, v)$ [11, 12]

\[

u = -\frac{1}{a}e^{-a\nu}, \quad v = \frac{1}{a}e^{a\vartheta}, \quad a > 0. \quad \left[\frac{-\log(-au)}{a} = \nu\right]
\]

These transformations define the Rindler observer as an observer that is ”at rest” in Rindler coordinates, i.e., maintaining constant $\xi$ and only varying $\eta$; $a$ represents the proper acceleration (along the hyperbola $\xi = 0$) of the Rindler observer, whose proper time is defined to be equal to Rindler coordinate time). The Rindler observer in the rest in $(\eta, \xi)$ Rindler coordinate travels along the hyperbola

\[

uv = e^{\xi_0}
\]

in the inertial coordinates $(t, x)$, see Fig.6.

The transformation formulas between the inertial coordinates $(t, x)$ and the Rindler coordinates $(\xi, \eta)$ (for simplicity we consider 2-dimensional case) are different in four parts of the inertial coordinates plane $\mathbb{M}^{1,1}$. Four different maps of $\mathbb{M}^{1,1}$ to four different parts of inertial plane [11, 12]. Below we present them in light-cone coordinates $u, v$ and $\nu, \vartheta$:

\[

u = t - x, \quad v = t + x, \quad \nu = \eta - \xi, \quad \vartheta = \eta + \xi,
\]

\[

(3.38)
\]

\[

(3.39)
\]
Figure 6. Rindler observers "in the rest" (located at $\xi = 1, 2$) move along hyperbolae in the inertial coordinates with constant acceleration.

and

\[
\begin{align*}
R : & \quad u = \frac{1}{a} e^{-a\nu}, \quad v = \frac{1}{a} e^{a\vartheta}, \\
F : & \quad u = \frac{1}{a} e^{-a\nu}, \quad v = \frac{1}{a} e^{a\vartheta}, \\
L : & \quad u = -\frac{1}{a} e^{-a\nu}, \quad v = -\frac{1}{a} e^{a\vartheta}, \\
P : & \quad u = -\frac{1}{a} e^{-a\nu}, \quad v = -\frac{1}{a} e^{a\vartheta}.
\end{align*}
\] (3.40) \quad (3.41) \quad (3.42) \quad (3.43)

In all cases $\vartheta \in (-\infty, +\infty)$, $\nu \in (-\infty, +\infty)$, see Fig.7.

As it is well known, the velocity and acceleration along the trajectory $\xi = \xi_0$ in the inertial coordinates

\[
t = \frac{e^{a\xi_0}}{a} \sinh(a\eta), \quad x = \frac{e^{a\xi_0}}{a} (\cosh(a\eta) - 1)
\] (3.44)

are

\[
\begin{align*}
u^0 &= \frac{dt}{ds} = \cosh(a\eta), \quad u^1 = \frac{dx}{ds} = \sinh(a\tau), \quad \text{since} \quad ds = e^{a\xi_0} d\eta \\
\end{align*}
\] (3.45)

and we get, that the velocity squared is equal to 1 and the acceleration squared is equal to $a^2 e^{-2a\xi_0}$.

We see that the formulas for maps (3.40)-(3.43) are the same as in the E-case, but there is the essential difference in the forms of the corresponding metrics. In Rindler case the metrics in all 4 spaces with $(\xi, \eta)$ coordinates are non trivial:

\[
ds_{Rindler}^2 = \pm a^2 e^{a(\vartheta - \nu)} d\vartheta d\nu
\] (3.46)
with “−” for \((u, v) \in \mathbb{R}\) or \(L\), and “+” for \((u, v) \in F\) or \(P\).

In the case of E-thermal, the metrics in all 4 spaces with coordinates \((u, v)\) are trivial, as the first formula in (3.9), but the metric in 2-dimensional space with coordinates \((\mathcal{T}, \mathcal{X})\) is nontrivial and given by (3.17), or in details by (2.12) and (3.12). Therefore, the maps between the pairs of coordinates \((\mathcal{U}, \mathcal{V}) \leftrightarrow (u, v)\) and \((\mathcal{W}, \mathcal{Y}) \leftrightarrow (\nu, \vartheta)\) are given by the same formulas, but the metrics are different.

![Diagram](image)

Figure 7. Map of 4 copies of \(\mathbb{M}^{1,1}\) to one \(\mathbb{M}^{1,1}\). Here \(a = 1\)
4 General E-coordinates

4.1 E-coordinates

We define here an extension of E-coordinates to the arbitrary static metric of the form

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2 \]  

(4.1)

Define also

\[ r_* = r_*(r) = \int \frac{dr}{f(r)} \]  

(4.2)

and general Eddington-Finkelstein coordinates

\[ u = t - r_*, \ v = t + r_* \]  

(4.3)

Then the 2-dim part of the metric becomes

\[ ds^2_2 = -f(r)dudv. \]  

(4.4)

E-coordinates \( \mathcal{U}, \mathcal{V} \) are defined by the relations

\[ \mathcal{U} = -e^{-u/B}, \ \mathcal{V} = e^{v/B}, \ B > 0. \]  

(4.5)

Now the the 2-dim part of the metric (4.4) reads

\[ ds^2_2 = -f(r)dudv = f(r)B^2d\mathcal{U}/\mathcal{V}, \]  

(4.6)

where \( r \) is implicitly defined by the relation

\[ e^{2r_*B} = -\mathcal{U}\mathcal{V}. \]  

(4.7)

By introducing the coordinates \( \mathcal{X} = (\mathcal{V} + \mathcal{U})/2 \) and \( \mathcal{T} = (\mathcal{V} - \mathcal{U})/2 \) the metric can be written in the form

\[ ds^2 = f(r)B^2\frac{-d\mathcal{T}^2 + d\mathcal{X}^2}{-\mathcal{T}^2 + \mathcal{X}^2}, \]  

(4.8)

where \( \mathcal{X}^2 - \mathcal{T}^2 > 0 \).

Note that if \( r_* \) and \( r \) are constants, then from (4.7) rewritten as

\[ \mathcal{X}^2 - \mathcal{T}^2 = e^{2r_*B} = \text{const}, \]  

(4.9)

It follows that we deal with the motion on a hyperbola (4.9).

In the same way as in Sect.2.3, one can show that in the E-coordinates introduced above for a rather general function \( f(r) \) the E-observer will see the Planck distribution of quanta with temperature

\[ T = \frac{1}{2\pi B}. \]  

(4.10)
It is interesting that one can get the temperature distribution for the function \( f(r) \) that has no zeros, i.e for a metric (1.7) that does not describe a black hole. For instance, take
\[
f(r) = e^{-\alpha r}, \quad r > 0, \quad \alpha > 0
\] (4.11)
in this case
\[
r_* = \frac{1}{\alpha}(e^{-\alpha r} - 1), \quad u = t - r_*, \quad v = t + r_*.
\] (4.12)
In E-coordinates (4.5) one gets the temperature (4.10).

Note that if there is a black hole with horizon at \( r_h \), \( f(r_h) = 0, \) \( f'(r_h) > 0, \) the metric for arbitrary \( B \) has a coordinate singularity. One can fix \( B = B_0 \) to avoid this singularity,
\[
B_0 = \frac{2}{f'(r_h)}, \quad \kappa = \frac{1}{2}f'(r_h) \Rightarrow B_0 = \frac{1}{\kappa},
\] (4.13)
where \( \kappa \) is the surface gravity for metric (4.1). Indeed, taking into account, that from (4.2) it follows that as \( r \to r_h \)
\[
r_* = \frac{\ln(r - r_h)}{f'(r_h)} + ... = \frac{\ln(r - r_h)}{2\kappa} + ...
\] (4.14)
we get that
\[
\mathcal{U} \sim e^{2r_*B} \sim e^{\ln(r - r_h)} \sim (r - r_h)^{\frac{4}{\kappa}}
\] (4.15)
To compensate the first order zero coming from \( f(r) \) near \( r_h \) we take \( B \) as in (4.13).

We denote
\[
\mathcal{U} \big|_{B = B_0} = U, \quad \mathcal{V} \big|_{B = B_0} = V.
\] (4.16)

Note that these formula reproduce usual formula, in particular, for the Schwarzschild metric (2.1).

The coordinates with
\[
B = B_0 + b, \quad B_0 = \frac{1}{\kappa},
\] (4.17)
we call the general E-coordinates.

4.2 Acceleration of the E-observer in black holes

Now we can fix \( \mathcal{S} = \mathcal{S}_0 \). This trajectory can be parametrized by \( \mathcal{T} \), i.e. in the Schwarzschild coordinates
\[
t = t(\mathcal{T}), \quad r = r(\mathcal{T}),
\] (4.18)
\[
t = t(\mathcal{T}) = \frac{B}{2} \log\left(\frac{\mathcal{T} + \mathcal{S}_0}{\mathcal{S}_0 - \mathcal{T}}\right),
\] (4.19)
\[
r_* = r_*(\mathcal{T}) = \frac{B}{2} \log(\mathcal{S}_0^2 - \mathcal{T}^2).
\] (4.20)
Due to (4.8) along the trajectory we have

\[
\frac{d\mathcal{T}}{ds} = \frac{\mathcal{T}_0^2 - \mathcal{T}^2}{B \sqrt{f(r)}} \tag{4.21}
\]

Therefore the velocity components for the observer moving along this trajectory are

\[
u^0 = \frac{dt}{ds} = \frac{\mathcal{T}_0}{\sqrt{f(r) \mathcal{T}_0^2 - \mathcal{T}^2}}, \tag{4.22}
\]

\[
u^1 = \frac{dr}{ds} = -\frac{\sqrt{f(r) \mathcal{T}}}{\sqrt{\mathcal{T}_0^2 - \mathcal{T}^2}}. \tag{4.23}
\]

We have

\[-fu^0u^0 + f^{-\frac{1}{2}}u^1u^1 = -1 \tag{4.24}\]

The components of the moving observer acceleration are defined as

\[
w^0 = \frac{du^0}{ds} + \Gamma^0_{\mu\nu}u^\mu u^\nu = \frac{du^0}{dr} \frac{dr}{ds} + \frac{du^0}{d\mathcal{T}} \frac{d\mathcal{T}}{ds} + \frac{f'}{f}u^0u^1 \tag{4.25}
\]

\[
= \left( -\frac{1}{2} f' + \frac{1}{B} \right) \frac{1}{f} \frac{\mathcal{T}}{\mathcal{T}_0^2 - \mathcal{T}^2} \tag{4.26}
\]

\[
w^1 = \frac{du^1}{ds} + \Gamma^1_{\mu\nu}u^\mu u^\nu = \frac{du^1}{dr} \frac{dr}{ds} + \frac{du^1}{d\mathcal{T}} \frac{d\mathcal{T}}{ds} + \frac{f'}{2f} \left( f^2u^0u^0 - u^1u^1 \right) \tag{4.27}
\]

\[
= \left( \frac{f'}{2} - \frac{1}{B} \right) \frac{\mathcal{T}_0^2}{(\mathcal{T}_0^2 - \mathcal{T}^2)} \tag{4.28}
\]

The square of the acceleration is

\[
w^2 = -f(w^0)^2 + f^{-\frac{1}{2}}(w^1)^2 = \frac{1}{f} \left( \frac{f'}{2} - \frac{1}{B} \right)^2 \frac{\mathcal{T}_0^2}{\mathcal{T}_0^2 - \mathcal{T}^2} \tag{4.29}
\]

One can check that \(-fu^0 \cdot w^0 + f^{-\frac{1}{2}}u^1 \cdot w^1 = 0.\)

### 4.3 Examples. Black holes in E-coordinates

#### 4.3.1 E-observer in the Schwarzschild black hole

For the Schwarzschild solution

\[
f(r) = 1 - \frac{2M}{r} \tag{4.30}
\]

and

\[
r_* = r + 2M \log \left( \frac{r}{2M} - 1 \right), \tag{4.31}
\]
and due to (4.20)

\[ r_* = r_*(\mathcal{I}) = \frac{B}{2} \log(\mathcal{I}_0^2 - \mathcal{I}^2) \]  

(4.32)

and

\[ \mathcal{I}^2 = \mathcal{I}_0^2 - e^{2r/B} \left( \frac{r}{2M} - 1 \right)^{4M/B} \]  

(4.33)

For the acceleration we have

\[ w^2 = \frac{\mathcal{I}_0^2 \left( \frac{1}{r} - \frac{M}{r^2} \right)^2}{1 - \frac{2M}{r}} e^{-2r/B} \left( \frac{r}{2M} - 1 \right)^{-4M/B} \]  

(4.34)

The dependence of acceleration (4.34) on \( r \) for fixed \( M = 1 \) and along the trajectory (4.19), (4.20) with \( \mathcal{I}_0 = 1.5 \) is presented in Fig. 8. We see that for \( B < 4M \) the acceleration is infinite at the horizon \( z_h = 2M \). For \( B = 4M \) the acceleration at the horizon is related with the surface gravity \( \kappa = 1/4M \),

\[ w^2 |_{r=r_h} = \frac{4\mathcal{I}_0^2}{e} \kappa^2 \]  

(4.35)

where \( \kappa = 1/4 \) and \( w^2 = 0.20693 \). For \( B > 4M \) the acceleration near horizon is infinite and tends to zero at \( r = r_0 = \sqrt{BM} \); for \( B = 3.5 \) \( r_0 = \sqrt{4.1} = 2.02485 \). The locations of these zeros are shown at the contourplot in Fig. 9.B by the magenta line.

**Figure 8.** A) The trajectories of stationary observer at \((\mathcal{I}, \mathcal{J})\) coordinates with \( \mathcal{J} = \mathcal{J}_0 \) in the the Schwarzschild coordinates for different \( B \) and \( M = 1, X_0 = 1.5 \). B) The acceleration \( w^2 \) vs \( r \) (blue lines) and \( T^2 \) vs \( r \) (green lines) for trajectories in shown on A). The red dashed line shows \( r_* = r_*(r) \) for the same \( M \) and \( X_0 \). The dashed-dotted lines show the restrictions on \( r \) following from the requirement \( T^2 > 0 \).
Figure 9. The contourplot versions of the plot presented in Fig.8. Here on horizontal axes we show $r$ and on vertical $B$, $M = 1$, $X_0 = 1.5$. The acceptable region with $T^2 > 0$ is bounded by solid green line. Behaviour of $w^2$ mainly for $B < 4M$ (A) and for $B > 4M$ only (B). Here the magenta line shows locations of points with $w^2 = 0$. These points exist only for $B > 4M$.

4.3.2 E-observer in the Reissner-Nordstrom black hole

For the RN solution one has

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

$$r_\pm = r + \frac{r_+ - r_-}{r_+ - r_-} \ln(r - r_+) - \frac{r_-}{r_+ - r_-} \ln(r - r_-),$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$, $M \geq Q$. $u, v$ are defined as $u = t - r_+, v = t + r_+$, [6].

Using the general formula (4.29) we find the dependence of the acceleration on $r$ along the trajectory $\mathcal{Z} = \mathcal{Z}_0$ and the results are presented in Fig.10. We see that the qualitative behaviour of acceleration dependence on $r$ is similar to the previous case considered in Sect.4.3.1. For $B < 1/\kappa(M, Q)$, here $1/\kappa(1, 0.2) = 4.00042$, the acceleration near horizon is infinite and monotonically decreases to zero for $r \to \infty$. For $B = 1/\kappa(M, Q)$ the acceleration at the horizon $r = r_+$ is finite, here $r_+ = 1.9798$ and $w^2|_{r_+} = 0.10448$, and for $r \to \infty$ decreases to 0. For $B > 1/\kappa(M, Q)$ the acceleration becomes infinite at the horizon and decreases up to zero at $r = r_0(M, Q)$.
For the Schwarzschild-AdS solution one has

\[
f(r) = 1 - \frac{2M}{r} + k^2 r^2 \quad (4.38)
\]

and

\[
r^*(r) = \int_{r_0}^{r} \frac{dr'}{f(r')} = \frac{r_h}{3k^2r_h^2 + 1} \left[ \log \left| 1 - \frac{r}{r_h} \right| - \frac{1}{2} \log \left( 1 + \frac{k^2 r (r_h + r)}{k^2 r_h^2 + 1} \right) \right]
\]

\[
+ \frac{(3k^2r_h^2 + 2)}{k \sqrt{3k^2r_h^2 + 4}} \arctan \left( \frac{kr \sqrt{3k^2r_h^2 + 4}}{2 (k^2r_h^2 + 1) + k^2 rr_h} \right) \quad (4.39)
\]

where

\[
r_h = \sqrt[3]{\frac{M}{k^2}} \left( 1 - \sqrt{1 - \frac{1}{27k^2M^2} + 1} + \sqrt{\frac{1}{27k^2M^2} + 1 + 1} \right) \quad (4.40)
\]

Fig.11 shows the dependence of the acceleration along trajectories with \( \mathcal{X}_0 = 1.5 \) and different \( B \) in the AdS-Sch metric with \( M = 1 \) and varying \( B \). We see that the acceleration becomes infinite near the horizon, decreases monotonically for \( B < B_c \) (the thick line in Fig.A) and has minimum equal to zero for \( B > B_c \) (the line of moderate thickness in Fig.A). For \( B > B_c \) this minimal value is reached out of the acceptable region (the thin line in Fig.A).

**4.3.3 E-observer in Schwarzschild-AdS**

Figure 10. A) The trajectories with \( \mathcal{X} = \mathcal{X}_0 \) in the RN spacetime in the Schwarzschild coordinates for different \( B \) and \( M = 1 \), \( X_0 = 1.5, Q^2 = 0.2 \). B) \( w^2 \) vs \( r \) (blue) and \( T^2 \) vs \( r \) (green) for trajectories shown in A). The red dashed line shows \( r^* = r^*(r) \) for the same \( M, X_0 \). The dashed-dotted lines show the restrictions on \( r \) following from the requirement \( T^2 > 0 \).
Figure 11. A) The trajectories with $X_0 = 1.5$ in the AdS-Schwarzschild spacetime in the Schwarzschild coordinates for different $B$ and $M = 1$ are shown by darker cyan lines. The acceleration $w^2$ along these trajectories are shown by blue lines. $T^2$ for these trajectories for the same $M = 1$ and $X_0$ are shown by green lines. The red dashed line shows $r_\ast = r_\ast(r)$ for the same $M, X_0$. The dashed-dotted lines show the restrictions on $r$ following from the requirement $T^2 > 0$. The in-plot shows the zoom of the original plot for $B = 2.615$. B) The contour plot version of A) with $r$ in the horizontal direction and $B$ on the vertical one. The darker magenta line shows the acceleration $w^2$ zeroes locations.

4.3.4 E-observer in Schwarzschild-dS

For the Schwarzschild-dS solution one has

$$f(r) = 1 - \frac{2M}{r} - k^2 r^2$$  \hspace{1cm} (4.41)

and for $0 < 27k^2M^2 < 1$ there exist two positive roots $r_1$ and $r_2$ of $f(r)$ such that $0 < 2M < r_1 < 3M < r_2$,

$$r_1 = \frac{2}{k\sqrt{3}} \cos(\alpha/3 + 4\pi/3), \hspace{0.5cm} r_2 = \frac{2}{k\sqrt{3}} \cos(\alpha/3) \hspace{0.5cm} \text{with} \hspace{0.5cm} \cos \alpha = -3Mk\sqrt{3} \hspace{1cm} (4.42)$$

There also is a negative root

$$r_3 = \frac{2}{k\sqrt{3}} \cos(\alpha/3 + 2\pi/3). \hspace{1cm} (4.43)$$

Here $r_1$ and $r_2$ describe the black-hole event horizon and the cosmological event horizon, respectively. Now we can write $r_\ast$ in the form

$$r_\ast = \int \frac{dr}{f(r)} = -\frac{1}{k^2} (A \ln(r_1 - r) + B \ln(r - r_2) + C \ln(r - r_3)) + D, \hspace{1cm} (4.44)$$

where

$$A = \frac{r_1}{(r_1 - r_-)(r_1 - r_2)}, \hspace{0.5cm} B = \frac{-r_2}{(r_2 - r_-)(r_1 - r_2)}, \hspace{0.5cm} C = \frac{r_-}{(r_2 - r_-)(r_1 - r_-)} \hspace{1cm} (4.45)$$
We adjust $D$ to remove the imaginary part from expression for $r_\ast$.

Fig. 12 shows the dependence of the acceleration along trajectories with $X_0 = 1.5$ and different $B$ in the dS-Sch metric with $M = 1$ and varying $k$. We see that the acceleration becomes infinite near the horizon and decreases monotonically for $B < B_0$ (the thick line in Fig.A). For $B = B_0$ the acceleration is finite at the horizon and for $B > B_0$ it is infinite at the horizon and has minimum equal to zero. For $B$ large enough this minimal value is reached out of the acceptable region (the thin line in Fig.A).

**Figure 12.** A) The trajectories with $X_0 = 1.5$ in the Schwarzschild coordinates for dS-Schwarzschild metric. B) $w^2$ vs $r$ (blue) and $T^2$ vs $r$ (green) for different $B$ and $M = 1$. The red dashed line shows $r_\ast = r_\ast(r)/25$ for the same $M, X_0$. 

\begin{figure} [h]
\includegraphics[width=\textwidth]{figure12.png}
\caption{A) The trajectories with $X_0 = 1.5$ in the Schwarzschild coordinates for dS-Schwarzschild metric. B) $w^2$ vs $r$ (blue) and $T^2$ vs $r$ (green) for different $B$ and $M = 1$. The red dashed line shows $r_\ast = r_\ast(r)/25$ for the same $M, X_0$.}
\end{figure}
5 General L-coordinates

5.1 L-coordinates

Having in mind formulas (1.7), (4.2), (4.3) and (4.4) we define L-coordinates $\nu, \vartheta$ by the relations

$$
\nu = \frac{1}{a} \log(-au), \quad \vartheta = \frac{1}{a} \log(av), \quad a > 0 \tag{5.2}
$$

In term of coordinates $(\vartheta, \nu)$ the metric (4.4) reads

$$
ds^2 = -f(r)du dv = -f(r)e^{a(\vartheta-\nu)}d\vartheta d\vartheta, \quad \nu, \vartheta \in (-\infty, \infty) \tag{5.3}
$$

where $r = r(\vartheta, \nu)$ is defined in two steps: first $r$ is defined as $r = r(r_*)$ by the relation (4.2) and then $r_*$ as a function of $\vartheta, \nu$ using

$$
r_* = \frac{1}{2a}(e^{a\vartheta} + e^{-a\nu}) \tag{5.4}
$$

By introducing the coordinates

$$
\eta = (\vartheta + \nu)/2, \quad \xi = (\vartheta - \nu)/2 \tag{5.5}
$$

the metric (5.3) can be rewritten as

$$
ds^2 = -f(r)e^{a(\vartheta-\nu)}d\vartheta d\vartheta = f(r)e^{2a\xi}(-d\eta^2 + d\xi^2), \tag{5.6}
$$

that is up to the conformal factor $f(r)$ the Rindler metric on the $(\eta, \xi)$-plane,

$$
ds^2_{\text{Rindler}} = e^{2a\xi}(-d\eta^2 + d\xi^2). \tag{5.7}
$$

From (5.4) and (5.5) follow relations

$$
r_* = \frac{e^{a\xi} \cosh(a\eta)}{a}, \quad t = \frac{e^{a\xi} \sinh(a\eta)}{a}. \tag{5.8}
$$

Hence in the case of (5.4) one gets

$$
r_*^2 - t^2 = \frac{e^{2a\xi}}{a^2}, \tag{5.9}
$$

in the case of (5.1) one gets

$$
(r_* + \frac{1}{a})^2 - t^2 = \frac{e^{2a\xi}}{a^2}. \tag{5.10}
$$

3To have a possibility to send $a \to 0$ one can use a modified definition

$$
u = \frac{1}{a}(e^{-av} - 1), \quad \vartheta = \frac{1}{a}(e^{a\vartheta} - 1), \quad a > 0, \tag{5.1}
$$

4In the case of (5.1) $r_* = \frac{1}{2a}(e^{a\vartheta} + e^{-a\nu} - 2)$. 

- 25 -
and in both cases if $\xi$ is a constant one has a motion along hyperbola. In particular, if one takes $\xi = 0$ then the parameter $1/a$ is a semi-axis of the hyperbola. So, for the two-dimensional part of general spherically symmetric metric (5.6) the parameter $1/a$ is a semi-axis of the hyperbola.

It will be shown below, in Sect.5.4, that for a rather general metric in the form (5.6) the temperature is

$$T = \frac{a}{2\pi}.$$ (5.11)

The temperature (5.22) does not depend of the form of the function $f(r)$ in the metric (1.7). But the trajectory of the L-observer does depend on $f(r)$ through the form of $r_*$.

5.2 Acceleration along trajectories $\xi = \xi_0$ in black hole backgrounds

Let us consider an observer moving along this hyperbola, i.e. along the trajectory (5.8) with $\xi = \xi_0$ in $(t, r_*)$-plane. One can parametrize this trajectory as

$$r_* = r_*(\eta) = \frac{e^{a\xi_0} \cosh(a\eta)}{a},$$ (5.12)

$$t = t(\eta) = \frac{e^{a\xi_0} \sinh(a\eta)}{a}. $$ (5.13)

This parametrization means that $r_* > 0$, or, more precisely, that $r_* > \frac{e^{a\xi_0}}{a}$.

One can find velocity and acceleration along this trajectory. Indeed, along this trajectory the interval is

$$ds = \sqrt{f(r)e^{a\xi_0} d\eta}, \quad \text{here} \quad ds = |ds|,$$ (5.14)

and components of the velocity along this trajectory are

$$u^0 = \frac{dt}{ds} = \frac{1}{\sqrt{f(r)}} \cosh(a\eta),$$

$$u^1 = \frac{dr}{ds} = \sqrt{f(r)} \sinh(a\eta). $$ (5.15)

We see that the square of velocity is equal to -1, $-f(u^0)^2 + f^{-1}(u^1)^2 = -1$.

The components of acceleration are

$$w^0 = \frac{du^0}{ds} + \Gamma^0_{\mu\nu} u^\mu u^\nu = \frac{\sinh(a\eta)}{f} \left( ae^{-a\xi_0} + \frac{f'}{2} \cosh a\eta \right),$$ (5.16)

and

$$w^1 = \frac{du^1}{ds} + \Gamma^1_{\mu\nu} u^\mu u^\nu = \cosh(a\eta) \left( \frac{f'}{2} \cosh(a\eta) + ae^{-a\xi_0} \right),$$ (5.17)

$$w^\theta = \Gamma^\theta_{\mu\nu} u^\mu u^\nu = 0,$$ (5.18)

$$w^\phi = \Gamma^\phi_{\mu\nu} u^\mu u^\nu = 0,$$ (5.19)
and
\[ w^2 = -f(w^0)^2 + \frac{1}{f(w^1)^2} = \frac{a^2 e^{-2a \xi_0}}{f} \left( \frac{f'}{2} r_*(r) + 1 \right)^2. \] (5.20)

One can also check the orthogonality condition \( -fu^0 w^0 + f^{-1}u^1 w^1 = 0. \)

5.3 Examples.

5.3.1 Schwarzschild in L-coordinates

According (5.20) the acceleration along the trajectory (5.12),(5.13) is given by
\[ w^2 = \frac{a^2 e^{-2a \xi_0}}{1 - \frac{2M}{r}} \left( 1 + \frac{M}{r} \left( 1 + \frac{2M}{r} \log \left( \frac{r}{2M} - 1 \right) \right) \right)^2. \] (5.21)

Acceleration along these trajectories as function of \( r \) is presented in Fig.14.A and as function of \( \eta \) in Fig.14.B.

5.3.2 Reissner-Nordstrom in L-coordinates

For the RN solution with \( f(r) \) given by (4.36) the acceleration dependence on \( r \) can be calculated using general formula (5.20). The results are presented in Fig.15.A.
\[ a = 1 \Rightarrow M = 1, \xi_0 = 1, a = 1 \]

\[ a = 1 \Rightarrow M = 1, \xi_0 = 0.8 \]

\[ a = 0.5 \Rightarrow M = 1, \xi_0 = 0.8 \]

\[ a = 1 \Rightarrow M = 1.5, \xi_0 = 0.8 \]

\[ a = 1 \Rightarrow M = 1.5, \xi_0 = 1 \]

\[ a = 0.5 \Rightarrow M = 1.5, \xi_0 = 0.8 \]

\[ a = 0.5 \Rightarrow M = 1.5, \xi_0 = 1 \]

\[ \omega^2 (r, 1) \]

\[ \omega^2 (r, 1.5) \]

\[ A \]

\[ B \]

**Figure 14.** The acceleration along the trajectories shown in Fig.13 as function of \( r \) B) Contourplot for varying \( r \) (horizontal) and \( M \) (vertical). We see that in the admissible area the acceleration \( w^2 \) decreases when \( r \) increases.

\[ M = 1, \xi_0 = 1, a = 1 \]

\[ w^2 \]

\[ Q = 0 \]

\[ Q = 0.5 \]

\[ Q = 0.9 \]

\[ r^* (r, 1) \]

\[ r^* (r, 1.5) \]

\[ A \]

\[ B \]

**Figure 15.** A) \( w^2 \) as function on \( r \) along the trajectories with \( \xi = \xi_0 \) for different \( Q \). The physical acceptable regions are on the right of dashed lines shown \( r^* \), here \( M = 1 \). B) Contour plot of A) for varying \( r \) and \( Q \), \( M = 1 \). The physically acceptable domain is bounded by the green line. We see that the character of the acceleration dependence on \( r \) is the same as for the Schwarzschild case shown in Fig.14.

### 5.3.3 Schwarzschild-AdS in L-coordinates

Here we consider the Schwarzschild-AdS metric in L-coordinates. For the Schwarzschild-AdS solution \( f(r) \) and \( r_\ast = r_\ast (r) \) are given by (4.38) and (4.39), respectively. Using general formula (5.20) we calculate the acceleration. The result is presented in Fig.??.
Figure 16. A) Trajectories in the AdS-Schwarzschild spacetime in the Schwarzschild coordinates corresponding to the observer with the fixed space coordinate $\xi_0$ in the $(\eta, \xi)$ coordinates; acceleration $w^2$ vs $r$ for trajectories in shown on A) for different values of $k$; C) zoom of B).

5.3.4 Schwarzschild-dS in L-coordinates

Here we consider Schwarzschild-dS metric in L-coordinates. The geometry is characterized by formula (4.44) and (4.41) The dependences of acceleration $w^2$ on $r$ are shown in Fig.18.

[Graphs and diagrams shown here]
Figure 17. Trajectories in the dS-Schwarzschild spacetime in the Schwarzschild coordinates corresponding $\xi = \xi_0 = 1, M = 1, a = 1$ for $k = 0.1$ (A) and $k = 0.19$ (B)

Math file: Geodeics-Rindler-short.nb+Rindler-Gen.nb.

Figure 18. A) The accelerations along the trajectories with $\xi = \xi_0 = 1$ for different $k = 0.1$ (darker reds line), 0.15 (blue lines), 0.19 (green lines). B), C) and D) are zooms of A) as well as the same plots for $\xi_0 = 0.8$. 
5.4 Temperature in L-coordinates

Here we show that the accelerated observer moving along special trajectories defined by requirement $\xi = \xi_0$ [[specified this condition in term of the original coordinates, the form of trajectory essentially depends on the blackening function $f(r)$]] see Hawking radiation with temperature, defined by only by parameter $a$.

$$T = \frac{a}{2\pi}.$$  
(5.22)

Indeed, comparing the solution of the wave equation in $(u, v)$ and $(\nu, \vartheta)$ coordinates related as in (5.1)

$$\partial_{\nu} \partial_{\vartheta} \phi = 0, \ \nu, \vartheta \in \mathbb{R}$$  
(5.23)

$$\partial_{u} \partial_{v} \Phi = 0, \ u < 0, v > 0$$  
(5.24)

we get relation between corresponding creation and annihilation operators. For this purpose we write as usual, the representation for solutions of 2-dimensional wave equations as combinations of the left and right modes,

$$\phi(\nu, \vartheta) = \phi_R(\nu) + \phi_L(\vartheta), \ \Phi(u, v) = \Phi_R(u) + \Phi_L(v).$$

For the real right mode (for the left mode all consideration is similar) one has

$$\phi_R(\nu) = \int_0^\infty d\omega (f_\omega B_\omega + f^*_\omega B^*_\omega), \ f_\omega(\nu) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\nu},$$  
(5.25)

where $[B_\omega, B^*_\omega'] = \delta(\omega - \omega')$ and

$$\Phi_R(u) = \int_0^\infty d\mu (\mathcal{B}_\mu f_\mu(u) + \mathcal{B}^*_\mu f_{\mu}^*(u)), \ f_\mu(u) = \frac{1}{\sqrt{4\pi\mu}} e^{-i\mu u},$$  
(5.26)

where $[\mathcal{B}_\mu, \mathcal{B}^*_\mu'] = \delta(\mu - \mu')$.

Right (and left) modes in different coordinate system are related as $\phi_R(\nu) = \Phi_R(u)$ and therefore,

$$\int_0^\infty d\omega (f_\omega B_\omega + f^*_\omega B^*_\omega) = \int_0^\infty d\mu (f_\mu \mathcal{B}_\mu + f^*_\mu \mathcal{B}^*_\mu).$$  
(5.27)

Multiplying (2.23) on $f_\omega(\nu)$ and integrate the first equation over $\mathbb{R}$ one gets

$$B_\omega = \int d\mu (\beta^{\omega*}_{\omega\mu} \mathcal{B}_\mu + \alpha^{\omega}_{\omega\mu} \mathcal{B}^*_\mu), \ B^*_\omega = \int d\mu (\beta_{\omega\mu} \mathcal{B}_\mu + \alpha_{\omega\mu} \mathcal{B}^*_\mu),$$  
(5.28)

(5.29)

where (compare with calculations in Sect.2.3)

$$\beta^{\omega*}_{\omega\mu} = \int d\nu \sqrt{\omega \mu} e^{-i\omega \nu} e^{-i\mu u}, \ \alpha^{\omega}_{\omega\mu} = \int d\nu \sqrt{\omega \mu} e^{-i\omega \nu} e^{i\mu u}.$$  
(5.30)
The EF (Eddington-Finkelstein) observer has the EF vacuum
\[ \mathcal{B}_\omega |0_{EF} \rangle = 0, \quad (5.31) \]
i.e. the state \(|0_{EF}\rangle\) does not contain \(\mathcal{B}\) particles. But it contains \(B\) particles:
\[ \langle 0_{EF} | N_\omega(B) | 0_{EF} \rangle \equiv \langle 0_{EF} | B^*_\omega B_\omega | 0_{EF} \rangle = \int_0^\infty d\mu |\beta_{\omega\mu}|^2. \quad (5.32) \]
The Bogoliubov coefficient \(\beta_{\omega\mu}\) is given by (2.25) with \(u\) as in (5.1), so we have
\[ \beta_{\omega\mu} = \frac{1}{2\pi a} \sqrt{\frac{\omega}{\mu}} e^{-\frac{\pi}{2a}} \left( \frac{\mu}{i} \right)^{-\frac{1}{2}} \Gamma(i \frac{\omega}{a}), \quad (5.33) \]
and as in (2.30) we get the Planck distribution
\[ |\beta_{\omega\mu}|^2 = \frac{1}{2\pi a \mu} \frac{1}{e^{\frac{\pi}{a}} - 1} \quad (5.34) \]
with the temperature (5.22). So we have obtained that the temperature depends only on acceleration \(a\), but the equation of the trajectory along which the observer is moving depends on the metric through the coordinate \(r^*\).

6 Characteristic times in the black holes space times

6.1 Time of black hole evaporation
The change of the black hole mass due to evaporation is described by equation
\[ \frac{dM(t)}{dt} = -L \quad (6.1) \]
where \(L\) is the luminosity, \(L = CT^4 \cdot \text{Area}\) and \(C\) is a constant. In our case \(T = 1/2\pi(b + 4M)\), and \(\text{Area} = 16\pi M^2\) so
\[ L = \frac{CM^2}{\pi^3(b + 4M)^4} \quad (6.2) \]
and we have the equation
\[ \frac{dM}{dt} = -\frac{CM^2}{\pi^3(b + 4M)^4}, \quad M(0) = M_0 \quad (6.3) \]
Therefore,
\[ \int_{M_0}^M \frac{(b + 4M')^4}{M'^2} \, dM' = -\frac{C}{\pi^3} t, \quad (6.4) \]
Taking the integral we get
\[- \frac{b^4}{M} + 8b^3 \log(M) + 96b^2 M + 128b M^2 + \frac{256 M^3}{3} = \frac{C}{\pi^3} (t_0 - t),\] (6.6)
where $t_0$ is
\[t_0 = \frac{\pi^3}{C} \left( - \frac{b^4}{M_0} + 8b^3 \log(M_0) + 96b^2 M_0 + 128b M_0^2 + \frac{256 M_0^3}{3} \right).\] (6.7)
The evaporation time $t = t_{\text{evap.time}}$ is the time, when $M(t_{\text{evap.time}}) = \epsilon \to 0$. In our case the leading terms are the first two terms in the LHS of (6.6) and
\[- \frac{b^4}{\epsilon} + 8b^3 \log(\epsilon) = \frac{C}{\pi^3} (t_0 - t_{\text{evap.time}}),\] (6.8)
so we obtain
\[t_{\text{evap.time}} \sim \frac{\pi^3}{C} \left( - \frac{b^4}{\epsilon} - 8b^3 \log(\epsilon) \right) + t_0 \to \infty.\] (6.9)
Or, in other words
\[M(t) = \frac{\pi^3 b^4}{C t} \to 0 \quad \text{when} \quad t \to \infty \] (6.10)
Note that the leading term is independent on the initial mass of the black hole. Note also that if we set $b = 0$ then the evaporation time is
\[t_{\text{evap.time}} = \frac{256 \pi^3}{3C} M_0^3.\] (6.11)

### 6.2 Small black holes and free falling observer

Light falling on black hole in the Schwarzschild spacetime is described by equation
\[t = \int_{r_h}^{r} \frac{dr}{(1 - \frac{r}{2M})},\] (6.12)
where $r_h = 2M$. Here the integral is divergent at $r = r_h$ as $r_h \ln(r - r_h)$.
Usually one concludes an asymptotics of approaching to the horizon is
\[r - r_h = \text{const} \ e^{-\frac{r}{r_h}}.\] (6.13)
We consider the question about the limit $r_h \to 0$ in more detailed. The solution of the equation
\[- dt = \frac{dr}{(1 - \frac{r}{2M})}, \quad r(0) = r_0 > r_h\] (6.14)
is
\[- t + r_0 - r = 2M \log \left( \frac{r - 2M}{r_0 - 2M} \right).\] (6.15)
When $M \to 0$ one gets $r = r_0 - t$, as it should be.
From the other site if we rewrite equation (6.15) in the form

\[ r = 2M + (r_0 - 2M)e^{\frac{r_0-x}{2M}} \]  \hspace{1cm} (6.16)

then it is not obvious how to take the limit \( M \to 0 \).

Let us discuss the leading term when the regularization parameters \( B_1 \) and \( B_2 \) are introduced

\[ t = \int_{r_h+B_1}^{r} \frac{dr}{(1 - \frac{r_h+B_2}{r})}, \quad B_1 > B_2 > 0. \]  \hspace{1cm} (6.17)

We have

\[ r = r_h + B_2 + (B_1 - B_2)e^{-\frac{t+r-r_h-B_1}{r_h+B_2}} \]  \hspace{1cm} (6.18)

and for \( r \) near \( r_h \) and large \( t \) one obtains

\[ r = r_h + B_2 + (B_1 - B_2)e^{-\frac{t}{r_h+B_2}} \]  \hspace{1cm} (6.19)

If \( r_h \) is large we can set \( B_2 = 0 \) and we get the asymptotic formula (6.13). However for small black holes we can take the limit \( r_h \to 0 \) in (6.18) to get

\[ r = B_2 + (B_1 - B_2)e^{-\frac{t}{r_h+B_2}} \]  \hspace{1cm} (6.20)

This consideration was a motivation to introduce the E-coordinates.

7 Discussions and Conclusions

It is shown that the property to have a temperature distribution for quantum fields in classical gravitational background is not restricted to the cases of black holes or constant acceleration, but is valid for any spherically symmetric metric written in thermal coordinates.

The Hawking temperature for Schwarzschild black hole \( T_H = 1/8\pi M \) is singular in the limit of vanishing mass \( M \to 0 \). So, as the result of evaporation one gets an explosion of black hole. This is clearly unphysical since the Schwarzschild metric in the original coordinates is regular when the black hole mass \( M \) tends to zero. It is reduced to the Minkowski metric and there are no reasons to believe that the temperature becomes infinite.

To improve the situation, new coordinates, called thermal coordinates, which depend on the black hole mass \( M \) and the parameter \( b \) that defines the semi-axis of a hyperbola along which an observer is moving are used. Using the thermal coordinates the Schwarzschild black hole radiation is reconsidered and it is found that the Hawking formula for temperature is valid only for large black holes while for small black holes the temperature is \( T = 1/2\pi(4M + b) \). The thermal coordinates are
regular in the limit of vanishing black hole mass $M$. In this limit, the Schwarzschild metric is reduced to the Minkowski metric, written in coordinates dual to the Rindler coordinates. The thermal observer in Minkowski space sees radiation with temperature $T = 1/2\pi b$, similar to the Unruh effect but in our case the acceleration is not a constant.

During evaporation, in the thermal coordinates with $b \neq 0$ the black hole mass is decreasing inverse proportional to time and the black hole life time is infinite. This is in contrast with the case $b = 0$ when the time life of black hole is finite. Sometimes the information paradox is formulated as follows. A collapsing black hole, described by a wave function, completely evaporates and leaves only radiation, described by a dense matrix. So one gets a transformation of pure state to a mixed state, that contradicts to the unitary evolution in quantum mechanics. Thus, a transformation of a pure state into a mixed one is occurred, which contradicts the unitary evolution in quantum mechanics, and means the loss of information. This formulation of the information lost paradox has a meaning only if the black hole time life is finite. In the case of using the E-coordinates the black hole life time is infinite. Therefore, formally speaking, the information loss paradox disappears.

It would be interesting to estimate the entropy balance during the evaporation in the thermal coordinates. It would also be interesting to study the fate of primordial black holes in the thermal coordinates and to use the thermal coordinates for investigation of massive fields of various spins in different dimensions in gravitational backgrounds.
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