AN UPPER BOUND FOR THE VOLUMES OF COMPLEMENTS OF PERIODIC GEODESICS

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Abstract. Any periodic geodesic on a surface has a natural lift to the unit tangent bundle of the surface. The volume of the complement of the geodesic in the unit tangent bundle typically grows as the geodesic gets longer and we give an upper bound for this volume which is linear in the geometric length of the geodesic. For the special case of the modular surface, we also get a linear bound relative to a complexity measure of the corresponding word in the fundamental group.

1. Introduction

A closed curve on a surface $S$ can be naturally lifted to the unit tangent bundle in two ways, by associating a tangent direction to the curve at each point. Such a lift is an embedding of $S^1$ into the 3-manifold $T^1S$, and thus, when considering its ambient isotopy class rather than the curve itself, a knot.

If we equip $S$ with a hyperbolic Riemannian metric, the isotopy class of each (non-peripheral and non-trivial) closed curve contains a periodic geodesic (the representative of shortest length). The associated knot is a closed orbit of the geodesic flow on the unit tangent bundle and does not depend on the chosen metric. Such periodic geodesics have long been an object of interest, both from the topological \cite{11,15} and the dynamical \cite{2} points of view.

The complement of a lift of a periodic geodesic in the unit tangent bundle is a hyperbolic 3-manifold as soon as the curve is filling \cite{10}, in which case its volume is a topological invariant. We are grateful to Juan Souto for suggesting that we study this invariant and for offering many useful insights. Our main result is that the volume of the complement grows at most linearly with the geometric length of the geodesic (c.f. Bridgeman \cite{3}). More precisely, we prove:

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Theorem 1.1. Let $S = \mathbb{H}^2/\Gamma$ be a hyperbolic surface. There is a constant $C_S > 0$ such that, for any finite set $\gamma$ of (primitive) periodic geodesics on $S$, we have

$$\Vol(T^1S \setminus \gamma) \leq C_S |\gamma|$$

where $|\gamma|$ is the total length of the geodesics in $\gamma$.

Remark. Here, $S$ comes equipped with a fixed hyperbolic metric in which we measure the length of $\gamma$. However, the linearity of the bound is topological and a change in metric will only affect the constant.

Our proof of Theorem 1.1 begins with the case of the modular surface $M = \mathbb{H}^2/\text{PSL}_2(\mathbb{Z})$. In this special case, a precise topological description was given by Ghys [12], together with a coding of periodic geodesics by positive words in the two letters $x$ and $y$. Using this, along with previous work of Birman and Williams [6, 5], he determined many of their knot properties. Some growth functions relating topological properties to the length were then determined by Sarnak [14].

The key to our linear bound lies in the connection between a decomposition of the unit tangent bundle into basic building blocks and a complexity measure for the coding of the geodesics in $\gamma$. Explicitly, we first bound the volume in terms of the number $n_{\gamma}$ of transitions from $x$ to $y$ in the words coding the set $\gamma$ of geodesics:

Theorem 1.2. There is a constant $C > 0$ such that, for any finite collection $\gamma$ of periodic geodesics on $M$, we have

$$\Vol(T^1M \setminus \gamma) \leq C n_{\gamma}.$$ 

Remark. In Section 2, we exhibit a family of arbitrarily long geodesics on the modular surface whose associated knot complements have uniformly bounded volume. On the other hand, in a forthcoming paper [4], we give a sublinear lower bound for the growth of the volume in terms of the combinatorics of the coding for geodesics. Nevertheless, numerical evidence [7] indicates that in some situations the growth is in fact linear in the geometric length of the geodesics.

It is interesting to note that $T^1M$ embeds homeomorphically into the complement of a trefoil knot in $\mathbb{S}^3$. In this setting, $n_{\gamma}$ also plays an important rôle when computing topological invariants of these knots as done in [6]. For instance, the genus of a geodesic $\gamma$ for this embedding is bounded from below by $n_{\gamma}(n_{\gamma} - 1)/2$.

We conclude this introduction with a brief outline of the paper. In Section 2, we begin by reviewing the coding of geodesics on $M$ by positive words. Then, in Section 3, we prove Theorem 1.2 and obtain...
We start with the isomorphism $\text{PSL}_2(\mathbb{Z}) \simeq C_2 \ast C_3$. For later reference, we fix elements $\kappa_0$ and $\omega \in \text{PSL}_2(\mathbb{Z})$ of order 2 and 3, respectively, so that $\text{PSL}_2(\mathbb{Z}) = \langle \kappa_0 \rangle \ast \langle \omega \rangle$. Restricting every homomorphism to the generating set, we see that $\text{PSL}_2(\mathbb{Z})$ has an essentially unique surjection onto $C_3$, and hence a unique normal subgroup $\Gamma_3$ of index 3. We write $M_3 := \mathbb{H}^2/\Gamma_3$ for the associated hyperbolic manifold, the unique normal three-fold cover of $M$, depicted in Figure 1.

Remark. For an alternative construction of $\Gamma_3$, note that $\text{PSL}_2(\mathbb{F}_3) \simeq S_4$ (consider the action on $\mathbb{P}^1(\mathbb{F}_3)$) and that the 2-Sylow subgroup of $S_4$ is normal of index 3; this gives a surjection onto $C_3$ and shows that the kernel contains $\Gamma(3)$ so that $\Gamma_3$ is a congruence subgroup.

Let $p \in \mathbb{H}^2$ be the point fixed by $\omega$ and notice that, since $\text{Stab}_{\text{PSL}_2(\mathbb{Z})}(p)$ coincides with $\langle \omega \rangle$, $p$ is not a fixed point of an elliptic element of $\Gamma_3$. It follows that the associated Dirichlet domain $U_3$ (the set of points of $\mathbb{H}^2$ closer to $p$ than to any other point of the orbit $\Gamma_3 \cdot p$) is a fundamental domain for $M_3$. In fact, as shown in Figure 2, $U_3$ is an ideal triangle. Moreover, $\kappa_0$ fixes a point $q_0$ along one of the sides of this triangle which we label $J_0$, and its conjugates $\kappa_i = \omega^i \kappa_0 \omega^{-i}$ fix points $q_i = \omega^i q_0$
along the other two sides $J_1$ and $J_2$. For convenience (as shown in the figure) we choose our identification of $\mathbb{H}^2$ with the disc so that $p$ is the centre of the disc. In that case $\omega$ acts by rotation by the angle $2\pi/3$, cyclically permuting the vertices of the triangle, the arcs $J_i$ connecting them, and the elliptic fixed points $q_0$, $q_1$ and $q_2$.

We now use this picture to study certain geodesics on $M$. Observe first that any set $\gamma$ of periodic geodesics lifts to a set $\tilde{\gamma}$ of closed curves in $M_3$ (each periodic geodesic in $M$ lifts either to three periodic geodesics of the same length, or to a single geodesic of three times the length, but this is immaterial for our arguments). Moreover, any geodesic in $M_3$ has a lift to an (infinite) geodesic on $\mathbb{H}^2$ connecting two points on the boundary. This lift may be chosen to cross any particular fundamental domain for $\text{PSL}_2(\mathbb{Z})$, specifically $U_3$. If the geodesic is periodic, its ends cannot lie on the cusps of $\text{PSL}_2(\mathbb{Z})$ and, in particular, on the vertices of $U_3$. Hence, acting by the element of order 3, we may choose the lift to start at $I_0 \subset \partial \mathbb{H}^2$, so that it will enter the triangle $U_3$ through its side $J_0$.

![Figure 2. A fundamental domain, $U_3$, for a three-fold cover $M_3$ of the modular surface.](image)

We have seen that it is enough to encode geodesics through $U_3$ starting at $I_0$. Accordingly, let $\tilde{\gamma}$ be such an infinite geodesic. We now construct a different (disconnected) lift of $\gamma$ to $M$, consisting of a sequence of segments in $U_3$. The first part of this new lift will be the segment of $\tilde{\gamma}$ between its entry point to $U_3$ along $J_0$ and its exit point.
along either $J_1$ or $J_2$. We begin our code by $x$ or $y$ depending on the two possibilities.

Suppose our segment ends on $J_i$. Acting by $\kappa_i$ the remaining ray of $\tilde{\gamma}$ (the part after the end of the segment) now begins on $J_i$ (but usually not on the point where the segment ended). Rotating by a power of $\omega$, we may assume instead that the remaining ray again enters via $J_0$. It will then exit via one of the other sides and we again break off a segment, record a letter $x$ or $y$, apply $\kappa_1$ or $\kappa_2$ and a rotation, and continue.

If $\tilde{\gamma}$ projects to a periodic geodesic $\gamma$ on $M$, the resulting infinite word will be periodic. In that case the code $w_\gamma$ will be the the primitive part of this periodic word. We write $n_\gamma$ for the number of (cyclic) subwords of the form $xy$ in $w_\gamma$. In more generality, for $\tilde{\gamma} = \cup_{i=1}^k \tilde{\gamma}_i$ a set of geodesics projecting to a collection of periodic geodesics $\gamma$ on $M$, we define $n_\gamma$ to be the sum $n_\gamma = \sum_{i=1}^k n_{\tilde{\gamma}_i}$. It is shown in [12] that, up to cyclic permutation, the word $w_\gamma$ only depends on the projection of $\tilde{\gamma}$ to $M$ and, conversely, that any such word encodes a periodic geodesic.

This code has a natural interpretation in terms of a template embedded in $T^1 M$ whose existence is due to Birman and Williams [5]. More precisely, this template is a branched surface with boundary equipped with a semi-flow such that any finite set $\tilde{\gamma}$ of geodesics may be deformed to a collection of closed orbits on the template by an ambient isotopy (of $\tilde{\gamma}$ together with the tangent direction). Ghys [12] proved that the template can be chosen to have a single branchline lying over $J_0$ in $T^1 M$ together with two bands such that, after the isotopy, one contains the geodesic segments passing in $U_3$ from $J_0$ to $J_1$ while the other contains the segments passing from $J_0$ to $J_2$. The coding described above thus encodes the sequence in which the image of $\tilde{\gamma}$ under the isotopy travels through the two template bands. This template constructed by Ghys is depicted in Figure 3 as embedded in the complement of a trefoil knot.

**Example.** Consider the family of modular geodesics that have codes of the form $x^ny^m$ for some $n, m \in \mathbb{N}$. It follows from the Prime Geodesic Theorem that their lengths go to infinity. On the other hand, they correspond to a knot winding more and more around the trefoil in one template band, and then in the other. Thus, their volumes are all bounded by the volume of $T^1 M \setminus (\gamma \cup \alpha \cup \beta)$ where $\gamma$ is the geodesic corresponding to $xy$, and $\alpha$ and $\beta$ are both trivial knots, encircling one strand of $\gamma$ and the strand of the trefoil in the centre of the corresponding ear (c.f Adams [1]).
3. An upper bound for modular geodesics

Let $\tilde{\gamma}$ be a collection of periodic geodesics on $M_3$. Once the geodesics are isotoped as in the previous section and we have the coding for $\tilde{\gamma}$, we proceed by drilling out $\tilde{\gamma}$ from $T^1 M_3$. Since $T^1 M_3$ is a Seifert fibered manifold (and thus not hyperbolic) the drilling turns $\tilde{\gamma}$ into a collection of cusps and we obtain a hyperbolic manifold $T^1 M_3 \setminus \tilde{\gamma}$ which we now cut into blocks:

**Lemma 3.1.** If $\gamma$ is a finite collection of periodic geodesics on $M$, then we can decompose $T^1 M_3 \setminus \tilde{\gamma}$ into simple blocks of three types, each containing at most two ideal vertices.
Proof. Let $\hat{J}_i$ denote the image of $J_i$ in $M_3$ and cut $T^1M_3$ open along $T^1 \cup_{i=0}^2 \hat{J}_i$ to obtain $T^1U_3$. Next, cut $T^1U_3$ along the three sections of $T^1\mathbb{H}^2$ defined by associating to each point in $\mathbb{H}^2$ the tangent vector pointing away from one of the three cusps of $U_3$. This cuts $T^1U_3$ into three pieces, each containing the lifts of geodesic segments in $U_3$ emanating from $J_i$ for a certain $i$, depicted in Figure 4. We call these blocks type I.

As a second step, to obtain even simpler blocks, we now cut $T^1U_3$ along its fibres above the lifts of $\gamma$ to $U_3$. In other words, we cut the blocks of type I vertically along each of the $\tilde{\gamma}$ arcs they contain. In this way, we obtain three new types of blocks that are shown in Figure 5, we call them $A$, $B$ and $C$ blocks. The complement of any geodesics can be decomposed into these three block types.

We now use this block decomposition to prove our linear bound on the volume as a function of $n_\gamma$. In the next section, we will bound the volume of each of these blocks to get an estimate on the coefficients of our linear expression.

Proof of Theorem 1.2. Denote by $m_1$ the number of arcs of $\tilde{\gamma}$ appearing in a given block of type I (by symmetry, there will be the same number of $\tilde{\gamma}$ arcs through each type I block). It follows from the decomposition into sub-blocks that:

\[(1) \quad \text{Vol}(I) \leq 2 \text{Vol}(A) + m_1 \text{Vol}(B) + \text{Vol}(C).\]

We now come to the keystone of the proof. Typically, there will be many arcs of $\tilde{\gamma}$ in the blocks of type I, and thus, there will be many consecutive $B$ sub-blocks. The idea is to merge such blocks together whenever possible to reduce their count. More precisely, consider a segment $\beta$ in $U_3$ contributing an $x$ (resp. $y$) symbol to the code of a geodesic in $\tilde{\gamma}$. If the segment preceding $\beta$ also contributes an $x$ (resp.
y) to the code, i.e., if the $x$ symbol coming from $\beta$ is not preceded by a $y$ (resp. $x$) in the code, we call $\beta$ negligible.

To make sense of this terminology, consider a $B$ sub-block whose intersection with $\tilde{\gamma}$ consists of two arcs arising from negligible segments. This $B$ block then corresponds to a rectangle in the template, bounded by two $\tilde{\gamma}$ arcs within the $x$ (resp. $y$) band, such that both its boundary arcs are connected at their starting points to their preceding arcs which are also within the $x$ (resp. $y$) band. In such a case, since there is no “shuffling” within the arcs coming from the same band, the two preceding arcs are again adjacent (see Figure 3). This means both arcs continue to another $B$ sub-block, once again as adjacent arcs. In particular, the first $B$ block bounded by them has an entire face glued to a second $B$ block, bounded by the two preceding arcs, in such a way that merging them together yields a longer $B$ block. This will be the case for all but at most $2n_\gamma$ of the $B$ blocks. Since we will be bounding the volume of such blocks by using an ideal triangulation, no harm is done by counting these pairs as a single $B$ block. We may therefore replace $m_I$ by $2n_\gamma$ in Equation (1).

Since there are three type I blocks of equal volume in our decomposition of $T^1U_3 \setminus \tilde{\gamma}$ and since $\text{Vol}(T^1M \setminus \gamma) = \frac{1}{3} \text{Vol}(T^1M_3 \setminus \gamma)$, it follows that

$$\text{Vol}(T^1M \setminus \gamma) = \text{Vol}(I) \leq 2 \text{Vol}(A) + 2n_\gamma \text{Vol}(B) + \text{Vol}(C),$$

which completes the proof. $\square$

At this point, we use the bound given in Equation (2) to prove:

**Theorem 3.2.** There is a constant $A > 0$ such that, for any set $\gamma$ of periodic geodesics on $M$, we have

$$\text{Vol}(T^1M \setminus \gamma) \leq A \cdot l(\gamma)$$

where $l(\gamma)$ is the sum of the geometric lengths of the geodesics in $\gamma$.

**Proof.** The idea is very simple: any transition from $x$ to $y$ in the coding of $\gamma$ corresponds to a geodesic segment $\delta_1$ passing from $J_0$ to $J_1$, followed by a segment $\delta_2$ from $J_1$ to $\kappa_1J_0$ (see Figure [3]). The length of the segment $\delta_1$ may be very small, but the sum of the lengths $l(\delta_1) + l(\delta_2)$ is bounded from below by some constant $l_0$, the length of the minimal geodesic crossing a regular ideal quadrilateral. This ensures that the number of transitions between $x$ and $y$ in the coding of a geodesic $\gamma$ of length $l$ is bounded from above as follows:

$$n_\gamma \leq \frac{l}{l_0}.$$
3.1. A quantitative upper bound. In this section we convert Equation (2) into a quantitative statement. In order to estimate the coefficients in the equation, we decompose the $A$, $B$ and $C$ blocks into tetrahedra and bound their individual volumes. We then use the basic fact [3] that the volume of a hyperbolic tetrahedron $T \subset \mathbb{H}^3$ is bounded by $v_0 \sim 1.01494$, the volume of the ideal regular hyperbolic tetrahedron. We do not claim that the coefficients obtained in this way are optimal and this remains an open question for now.

Theorem 3.3. For any finite collection of periodic geodesics $\gamma$ on $M$, we have the following bound on the volume of $T^1 M \setminus \gamma$:

$$\text{Vol}(T^1 M \setminus \gamma) \leq (24 v_0) n_\gamma + 32 v_0.$$  

Proof. We start with $A$ blocks. Here, Figures 7 and 8 show the step by step decomposition into tetrahedra. We find that an $A$ block can be decomposed into 8 tetrahedra, and thus $\text{Vol}(A) \leq 8 v_0$.

Next, Figure 9 gives the decomposition of $B$ blocks into four tetrahedra and four pyramids with base a hyperbolic square along with one ideal vertex. Such a pyramid can be decomposed into two tetrahedra as in the last step for the $A$ block, and thus $\text{Vol}(B) \leq 12 v_0$.

We now turn to blocks of type $C$. These can be decomposed as in Figure 10 into a block of type $B$, and two pyramids, where the base is a square with one ideal vertex and the vertex of the pyramid is also ideal.
Figure 7. The decomposition of a type $A$ block into two sub-blocks, $A_1$, which is a pyramid with a square base and an ideal vertex, and $A_2$ that is a pentagonal prism.

Figure 8. The decomposition of a sub-block of a type $A_2$ into six tetrahedra. The last step also shows the decomposition of a sub-block of type $A_1$ into two tetrahedra.

These can be decomposed in the same way as a pyramid with a regular square base as above into two tetrahedra. This yields, $\text{Vol}(C) \leq 16 v_0$.

Summarizing, we have shown that

$$(3) \quad 2 \text{Vol}(A) + 2n_\gamma \text{Vol}(B) + \text{Vol}(C) \leq 16v_0 + 2n_\gamma 12v_0 + 16v_0$$

and this completes the proof of Theorem 3.3.
Figure 9. The decomposition of a $B$ block into four tetrahedra, and four pyramids with a square base and one ideal vertex.

Figure 10. The decomposition of a $C$ block.

4. AN UPPER BOUND FOR ARBITRARY HYPERBOLIC SURFACES

We now use Theorem 3.2 to prove Theorem 1.1.

Proof of Theorem 1.1. To begin, consider the case where our hyperbolic surface $S$ is a $d$-fold regular topological cover of the modular surface $M$. Using this cover, we can pull back a new hyperbolic metric from $M$ to $S$ for which the deck transformations are given by isometries. We refer to this new hyperbolic surface as $\hat{S}$ and note that the covering map from $\hat{S}$ to $M$ lifts to a covering map of the unit tangent bundles.

Next, observe that replacing $S$ with $\hat{S}$ will only change the length of a geodesic on $S$ by a multiplicative constant (c.f. Thurston [16]). To do
this precisely, we use results of Lenzhen, Rafi and Tao [13, Propositions 3.1 and 3.2] from which it follows that

\[
I_S(\gamma) \leq C_1 \sum_{\alpha \in \mu \hat{S}} i(\gamma, \alpha) I_S(\alpha) = C_2 \sum_{\alpha \in \mu \hat{S}} i(\gamma, \alpha) I_{\hat{S}}(\alpha) = C_3 I_{\hat{S}}(\gamma),
\]

where \( \mu \hat{S} \) is a short marking on \( \hat{S} \), \( \alpha \) denotes the dual curve of \( \alpha \) (also in \( \mu \hat{S} \)) intersecting \( \alpha \) minimally (once or twice) and the \( C_i \)'s are constants.

Although the results in [13] are stated for a simple closed curve \( \gamma \), we are grateful to Kasra Rafi for indicating that their proof does not depend on the fact that the curve is simple. We may therefore replace \( S \) by \( \hat{S} \) without loss of generality in our pursuit of a linear bound.

Now, if \( \gamma \) is a filling set of periodic geodesics on \( \hat{S} \), we denote its orbit under the group of deck transformations by \( \gamma' \). Letting \( \delta \) denote the projection of \( \gamma \) onto \( M \) by the covering map, we immediately obtain that

\[
\text{Vol}(T^1 \hat{S} \setminus \gamma) \leq \text{Vol}(T^1 \hat{S} \setminus \gamma') = d \cdot \text{Vol}(T^1 M \setminus \delta) = d \cdot A \cdot I(\delta) \leq d \cdot A \cdot I(\gamma)
\]

where \( A \) is the constant provided by Theorem 3.2. This establishes Theorem 1.1 for such surfaces.

\[\text{Figure 11. The cover of } M \text{ by a once punctured torus}\]

Before proving the theorem for an arbitrary hyperbolic surface, we describe some surfaces to which the preceding argument applies. To begin, consider the cover of \( M \) by a once punctured torus given in Figure 11. Notice, moreover, that a twice punctured torus (endowed with the appropriate metric) covers a once punctured torus (arrange the punctures symmetrically about the central axis which doesn’t intersect the torus and use a rotation by \( \pi \)) and, in the same manner, that we
have a $6k$-fold cover from a $k$ punctured torus to $M$. On the other hand, a rotation through the two cusps in a symmetrically arranged twice punctured surface of genus $g$ (see Figure 12) yields a cover of the twice punctured torus, and thus of $M$. In fact, we obtain a covering map from $S_{g, 2+kg}$ (the surface of genus $g$ with $2 + kg$ punctures) to $M$ in exactly the same way by permuting the punctures and mapping it to the $k + 2$ punctured torus. Lastly, recall that the thrice punctured sphere is a 6-fold cover of the modular surface. The same strategy now produces a $6(n - 3)$-fold covering map from a punctured sphere $S_{0,n}$ with $n > 3$ to $M$ by arranging an axis through two of the punctures and using a rotation acting transitively on the rest of the punctures.

Consider now a collection of periodic geodesics $\gamma$ on an arbitrary hyperbolic surface $S_{g,n}$. Since we have already dealt with the case where $n \equiv 2 \mod g$, we just need a way to increase the number of punctures by $k$ (chosen such that $n + k \equiv 2 \mod g$) to obtain a hyperbolic surface $S_{g,n+k}$ which is a topological cover of $M$. To do this precisely, we begin by marking the points on the surface at which we are going to puncture it. Then, following the procedure given by Brooks in [9], we isotope $\gamma$ off small neighbourhoods of the marked points in such a way that its length is not severely altered by puncturing (the punctures will only disturb the geometry inside these small neighbourhoods). Puncturing $S_{g,n}$ at the marked points we obtain $S_{g,n+k}$ and, according to Brooks

Figure 12. The cover of the twice punctured torus by a twice punctured genus $g$ surface.
there is a fixed $\varepsilon > 0$ for which the resulting set $\gamma'$ of geodesics on $S_{g,n+k}$ satisfies
\begin{equation}
 l(\gamma') \leq (1 + \varepsilon) \cdot l(\gamma)
\end{equation}
where $l(\gamma)$ denotes the length of $\gamma$ on $S_{g,n}$ while $l(\gamma')$ denotes the length of $\gamma'$ on $S_{g,n+k}$. We can now proceed as before: if we let $\delta$ denote the projection of $\gamma'$ onto $M$ and enlarge $\gamma'$ to its orbit $\gamma''$ under the group of deck transformations we have
\begin{equation}
 \text{Vol}(T^1 S_{g,n+k} \setminus \gamma') \leq \text{Vol}(T^1 S_{g,n+k} \setminus \gamma'') = d \cdot \text{Vol}(T^1 M \setminus \delta) \leq d \cdot A \cdot l(\delta)
\end{equation}
where the second inequality follows by Theorem 3.2. Putting this all together, we finally obtain
\begin{equation}
 \text{Vol}(T^1 S_{g,n} \setminus \gamma) \leq \text{Vol}(T^1 S_{g,n+k} \setminus \gamma') \leq d \cdot A \cdot l(\delta)
\end{equation}
\begin{equation}
 \leq d \cdot A \cdot l(\gamma') \leq d \cdot A \cdot (1 + \varepsilon) \cdot l(\gamma) = C \cdot l(\gamma)
\end{equation}
where the fourth inequality follows from Equation (6).
Note that if $\gamma$ is not filling then its complement in the unit tangent bundle is not hyperbolic. In this case, we set the volume to be zero and the bound trivially holds.

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