The algebraic Bethe Ansatz and combinatorial trees

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We present in this paper a comprehensive introduction to the algebraic Bethe Ansatz, taking as examples the six-vertex model with periodic and non-periodic boundary conditions. We propose a diagrammatic representation of the commutation relations used in the algebraic Bethe Ansatz, so that the action of the transfer matrix in the nth excited state gives place to labeled combinatorial trees. The analysis of these combinatorial trees provides in a straightforward way the eigenvalues and eigenstates of the transfer matrix, as well as the respective Bethe Ansatz equations. Several identities between the R-matrix elements can also be derived from the symmetry of these diagrams regarding the permutation of their labels. This combinatorial approach gives some insights about how the algebraic Bethe Ansatz works, which can be valuable for non-experts readers.

Keywords: Algebraic Bethe Ansatz, six-vertex model, combinatorial trees.

I. INTRODUCTION

The algebraic Bethe Ansatz is a powerful technique for solving analytically the eigenvalue problem in many-body quantum field theory and statistical mechanics. This method was created by the Leningrad group in the context of quantum field theory, as a quantum generalization of the inverse scattering method [1]. Soon after, the same mathematical structure led to an algebraic formulation of the (coordinate) Bethe Ansatz [2], this time in the field of statistical mechanics. Several models of quantum field theory and statistical mechanics were successfully solved by the algebraic Bethe Ansatz [3]: the non-linear Schrödinger equation [4] and the Sine-Gordon model [5] are typical examples in quantum field theory; in statistical mechanics we can cite the one-dimensional xxx, xxz and xyz Heisenberg spin chains, the six-vertex model and the eight-vertex model [6].

Although the algebraic Bethe Ansatz is by now a very well understood method, it is nevertheless very technical and usually a difficult matter for students or non-experts researches. In this paper we present a comprehensive introduction to the algebraic Bethe Ansatz, taking as example the six-vertex model with both periodic and non-periodic boundary conditions. We also propose – which is the main novelty of the paper – a diagrammatic representation for the commutation relations used in the algebraic Bethe Ansatz that provides the eigenvalues and eigenstates of the transfer matrix, and also the Bethe Ansatz equations, in a straightforward way. In fact, in the usual algebraic Bethe Ansatz, one needs to use the aforementioned commutation relations repeatedly in order to compute the action of the transfer matrix on the excited states, which is generally very cumbersome. In our diagrammatic approach, however, this action is represented by simple combinatorial trees and the results follow through a combinatorial analysis only.

This paper is organized as follows: in section II we consider the six-vertex model with periodic boundary conditions. We first introduce some statistical concepts as the monodromy and transfer matrices in section II A and, then, we explain step-by-step in section II B how the algebraic Bethe Ansatz works. We present our combinatorial approach in section II C, where we show how the eigenvalues and eigenstates of the transfer matrix – and also the Bethe Ansatz equations – can be easily derived from the analysis of simple diagrams. In section III, we study the six-vertex model with non-periodic boundary conditions. We discuss the role of the boundaries on the monodromy and transfer matrices in section III A and, in section III B, we also give a step-by-step presentation of the boundary algebraic Bethe Ansatz. The combinatorial approach for the non-periodic case is presented in III C and we close the paper with a brief conclusion in section IV.

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II. THE SIX-VERTEX MODEL WITH PERIODIC BOUNDARY CONDITIONS

A. The monodromy and transfer matrices of the six-vertex model with periodic boundary conditions

The first model to be solved through the algebraic Bethe Ansatz were the so-called six-vertex model. This was a statistical model introduced by Linus Pauling in an attempt to explain the statistical properties of the water ice – more specifically, to get account for the residual entropy of the ice. Remember that at low temperatures, the molecules of water arrange into an almost crystalline lattice. Pauling considered a two-dimensional approximation for the water ice lattice in which each oxygen atom is disposed in a vertex of a square lattice and that there is a hydrogen atom on each edge of this vertex. Each hydrogen atom is supposed to be either near or far from the oxygen atom, so that we have in total sixteen vertex configurations – which lead us to a sixteen vertex model – and for each vertex configuration, a respective energy and a Boltzmann weight is associated. However, the fact that each water molecule has indeed two, and only two, bound hydrogens implies that some configurations should be despised. This restriction was already noticed by Pauling in and, since then, it is known as Pauling ice rule. Taking into account the Pauling ice rule, the number of possible configurations of a given vertex representing a water molecule reduces to only six, and we get a six-vertex model. These six allowed configurations are illustrated in Fig. 1.

![Figure 1. The six-vertex configurations of the square ice model, according to the Pauling ice rule. The oxygen atoms are situated in the center of the vertices, while the hydrogen atoms are on the edges. In each edge there is exactly one hydrogen atom, which can be near or far from the oxygen atom. The Pauling ice rule states that the only physical configurations are those in which there is two, and only two, hydrogen atoms close to any oxygen atom. This rule agrees with the usual molecular formula for the water: \( \text{H}_2\text{O} \).](image)

Now, let us see how we can construct the partition function for the six-vertex model. To this end, consider a square lattice with \( L \) columns and \( N \) lines. We can impose periodic boundary conditions in both the horizontal and vertical directions, which means that any vertex at the position \( (i+N, j+L) \) is to be identified with the vertex at the position \( (i, j) \) of the lattice. To each vertex at the position \( (i, j) \) of the lattice we associate a horizontal local Hilbert space \( H_i \) (\( 1 \leq i \leq N \)) and a vertical local Hilbert space \( V_j \) (\( 1 \leq j \leq L \)), both isomorphic to \( \mathbb{C}^2 \), so that the local Hilbert space of the vertex can be written as \( H_i \otimes V_j \), which is isomorphic to \( \mathbb{C}^2 \otimes \mathbb{C}^2 \). Therefore, the Hilbert space associated with the whole lattice can be written as \( \mathcal{H} = H \otimes V \), where \( H = H_1 \otimes \cdots \otimes H_N \) stands for the horizontal spaces and \( V = V_1 \otimes \cdots \otimes V_N \) for the vertical ones. Notice that \( \mathcal{H} \) is isomorphic to \( \mathbb{C}^{2N} \otimes \mathbb{C}^{2L} \).

For a given vertex at the position \( (i, j) \) of the lattice, we introduce an \( R \)-matrix whose elements are related to the Boltzmann weights associated with the possible configurations of the vertices. The \( R \)-matrix corresponding to a vertex at the point \( (i, j) \) of the lattice has values in \( \text{End}(H_i \otimes V_j) \), so that it is a four-by-four matrix. Since there are only six possible vertex configurations, the \( R \)-matrix must have only six non-null entries. Besides, if we take into account some physical symmetries (e.g. time-inversion, parity etc.), then some vertex configurations should be equivalent, which means that some elements of the \( R \)-matrix should be the same. It turns out then that the \( R \)-matrix for the six-vertex model can be written as,

\[
R(u) = \begin{pmatrix}
    r_1(u) & 0 & 0 & 0 \\
    0 & r_2(u) & r_3(u) & 0 \\
    0 & r_3(u) & r_2(u) & 0 \\
    0 & 0 & 0 & r_1(u)
\end{pmatrix}, \tag{1}
\]
where the amplitudes $r_1(u)$, $r_2(u)$ and $r_3(u)$ are related to the Boltzmann weights of the vertex configurations (the exact expressions for them will be presented below) and $u$ is the so-called spectral parameter.

Now, let us take a given row of the lattice, say the row $i = a$. Taking the product of all $R$-matrices in this row, we get the so-called monodromy matrix,$^1$

$$M_a(u) = R_{a1}(u)R_{a2}(u)\cdots R_{aL}(u),$$

whose elements are related to the Boltzmann weights associated with all possible configurations of the row – see Fig. 2. The trace of the monodromy matrix on the space $H_a$ provides a sum over all possible row configurations, that is, it gives a reduced partition function for this row, which is called transfer matrix:

$$T_a(u) = \text{tr}_a [M_a(u)] = \text{tr}_a [R_{a1}(u)R_{a2}(u)\cdots R_{aL}(u)].$$

From this, we can easily find the total partition function of the system. In fact, this follows after we multiply all the monodromy matrices of each lattice row and we then take the trace:

$$Z = \text{tr} [M_1(u)M_2(u)\cdots M_N(u)].$$

Notice, moreover, that when the rows of the lattice are all equivalent, the eigenvalues of the transfer matrices associated with any row of the lattice will be the same. The, if $\tau_1, \ldots, \tau_L$ denote the eigenvalues of the transfer matrix, the corresponding eigenvalues of the partition function will be,

$$Z = \tau_1^N + \cdots + \tau_L^N.$$

This means that we can look for the diagonalization of the transfer matrix only. This is exactly what the algebraic Bethe Ansatz concerns with, as we shall see in the following sections.

**B. The algebraic Bethe Ansatz**

The algebraic Bethe Ansatz provides an analytical solution for the diagonalization of the transfer matrix for integrable models as the six-vertex model with periodic boundary conditions. It is implemented, however, in several steps which we may enumerate as follows:

1. **A solution of the Yang-Baxter equation providing the $R$-matrix.** The starting point is a given $R$-matrix, solution of the Yang-Baxter equation, that describes the model. The Yang-Baxter equation ensures the integrability of the model, which means that it can be solved in an exact way.

2. **The Lax representation of the monodromy and transfer matrices.** This is a representation in which the monodromy explicitly exhibit the annihilator and creator operators used in the construction of the excited states. Besides, the transfer matrix become given by sum of the diagonal monodromy elements in this representation;

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$^1$ The monodromy matrix $M_a(u)$ acts in $\text{End}(H_a \otimes V_1 \otimes \cdots \otimes V_L)$. By this reason, the $R_{aq}(u)$ matrices that appears in the definition of $M_a(u)$ should be regarded as matrices with values $\text{End}(H_a \otimes V_1 \otimes \cdots \otimes V_L)$ that act non-trivially only in $\text{End}(H_a \otimes V_q)$ and as the identity matrix in the other remaining vector spaces. The space $H_a$ is usually called auxiliary space, while the spaces $V_q$ ($1 \leq q \leq L$) are called quantum spaces.
3. The reference state. It is just a simple enough eigenstate of the transfer matrix in which the corresponding eigenvalue can be evaluated directly;

4. The construction of the excited states. They are built through the action of the aforementioned creator operators on the reference state;

5. The derivation of the commutation relations. The commutation relations are necessary to one compute the action of the transfer matrix on the excited states;

6. The computation of the eigenvalues of the transfer matrix. This is the most difficult step. To compute the eigenvalues we need to use the commutation relations mentioned above repeatedly;

7. The solution of the Bethe Ansatz equations. These are a system of coupled non-linear equations that appears as consistency conditions of the method. The Bethe Ansatz equations need to be solved in order to one obtain a completely analytical solution for the spectral problem.

Next we shall explain in details how the steps above are developed.

1. The R-matrix, solution of the Yang-Baxter equation

The start point of the periodic algebraic Bethe Ansatz is the existence a given R-matrix, a solution of the Yang-Baxter equation [8, 9],

\[ R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \] (6)

This is a matrix equation with values in \( \text{End}(V_a \otimes V_b \otimes V_c) \), where \( V_a, V_b \) and \( V_c \) are complex vector spaces isomorphic to \( \mathbb{C}^2 \) (in the case of the six-vertex model). The operators \( R_{ab} \) act as an R-matrix in \( \text{End}(V_a \otimes V_b) \) and as the identity in the other vector space \( V_c \).

![Figure 3. A graphical representation of the Yang-Baxter Equation (10). The S-matrix “twist” two adjacent rows of the lattice. The Yang-Baxter equation states that if we twist the rows 1 and 2 and then we take the product of the R matrix at the vertex (1, 3) and (2, 3), respectively, then we shall get the same result if we multiply these R matrices first and then we twist the rows.](image)

Introducing the permutator matrix,

\[ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (7)

whose action on any element \( A \otimes B \in \text{End}(V_a \otimes V_b) \) is \( P (A \otimes B) P = B \otimes A \), we can define a “twisted R-matrix”, also called S-matrix, as follows:

\[ S(u) = PR(u) = \begin{pmatrix} r_1(u) & 0 & 0 & 0 \\ 0 & r_3(u) & r_2(u) & 0 \\ 0 & r_2(u) & r_3(u) & 0 \\ 0 & 0 & 0 & r_1(u) \end{pmatrix}. \] (8)
This matrix allows us to rewrite the Yang-Baxter equation in a more symmetric form, namely, as
\[
S_{12}(u - v)S_{23}(u)S_{12}(v) = S_{23}(v)S_{12}(u)S_{23}(u - v).
\]
(9)

There is also a mixed representation that is useful:
\[
S_{12}(u - v)R_{13}(u)R_{23}(v) = R_{13}(v)R_{23}(u)S_{12}(u - v),
\]
which follows from the properties of the permutator matrix (e.g., the identity \(P_{ab}R_{bc}R_{ac}P_{ab} = R_{ac}R_{bc}\)). A graphical representation of the Yang-Baxter equation is given in Fig. 3.

Now, let us see how we can find the solutions of the Yang-Baxter equation for the six-vertex model \(R\)-matrix. From what we have seen on the previous section, the \(R\)-matrix we are looking for should have the following form\(^2\):

\[
R(u) = \begin{pmatrix}
  r_1(u) & 0 & 0 & 0 \\
  0 & r_2(u) & r_3(u) & 0 \\
  0 & r_3(u) & r_2(u) & 0 \\
  0 & 0 & 0 & r_1(u)
\end{pmatrix}.
\]
(11)

Regardless what form of the Yang-Baxter equation we use, they represent a system of coupled functional equations for the \(R\)-matrix elements. Several equations, however are dependent on each other or are null, so that we have actually only three independent equations, namely,
\[
\begin{align*}
  r_1(u - v)r_1(u)r_3(v) &= r_2(u - v)r_2(u)r_3(v) + r_3(u - v)r_3(u)r_2(v), \\
  r_3(u - v)r_1(u)r_2(v) &= r_3(u - v)r_2(u)r_1(v) + r_2(u - v)r_3(u)r_3(v), \\
  r_1(u - v)r_3(u)r_2(v) &= r_2(u - v)r_3(u)r_1(v) + r_3(u - v)r_2(u)r_3(v).
\end{align*}
\]
(12-14)

This system of functional equations can be solved through a kind of separation of variables method [11]. Eliminating the variables with the dependence on \(u - v\), we shall obtain the separated equation,
\[
\frac{r_1^2(u) + r_2^2(u) + r_3^2(u)}{r_1(u)r_2(u)} = \frac{r_1^2(v) + r_2^2(v) + r_3^2(v)}{r_1(v)r_2(v)} = \Delta,
\]
where \(\Delta\) is a constant independent of \(u\) and \(v\). The solutions of (6) follow after we find a parameterization of \(r_1(u)\), \(r_2(u)\) and \(r_3(u)\) satisfying (15). It follows that there are several possible solutions of this functional equation, each solution referring to a specific six-vertex model. The main important ones are the solution
\[
\begin{align*}
  r_1(u) &= \sinh (u + \xi), & r_2(u) &= \sinh u, & r_3(u) &= \sinh \xi,
\end{align*}
\]
(16)

which is related to the xxz Heisenberg chain, and the solution
\[
\begin{align*}
  r_1(u) &= u + \xi, & r_2(u) &= u, & r_3(u) &= \xi,
\end{align*}
\]
(17)

which is related to the xxx Heisenberg chain (in these expressions, \(\xi\) is an arbitrary parameter).

\section*{2. The Lax representation of the monodromy and transfer matrices}

Once we have an \(R\)-matrix, solution of the Yang-Baxter equation, we pass to construct the monodromy and transfer matrices. We conveniently use the a representation – the Lax representation – in which the \(R\)-matrix (11) is written as a two-by-two operator-valued matrix:
\[
R(u) = \begin{pmatrix}
  L_1^1(u) & L_1^2(u) \\
  L_2^1(u) & L_2^2(u)
\end{pmatrix},
\]
(18)

where,
\[
L_1^1(u) = \begin{pmatrix}
  r_1(u) & 0 \\
  0 & r_2(u)
\end{pmatrix}, \quad L_1^2(u) = \begin{pmatrix}
  0 & r_3(u) \\
  r_3(u) & 0
\end{pmatrix}, \quad L_2^1(u) = \begin{pmatrix}
  0 & r_2(u) \\
  r_2(u) & 0
\end{pmatrix}, \quad L_2^2(u) = \begin{pmatrix}
  r_1(u) & 0 \\
  0 & r_1(u)
\end{pmatrix}.
\]
(19)

\(^2\) We had assumed the most symmetrical case, which corresponds to the full symmetric six-vertex model. There are solutions of the Yang-Baxter equations for the non-symmetric cases as well. We indicate the reference [10] for the derivation of these most general solutions.
As we have seen in the previous section, the monodromy matrix is given by the product of the $R$-matrices running through all the sites of a given row of the lattice. That is, for a given $i = a$ of the lattice, we have, according to (2),

$$M_a(u) = R_{a1}(u) \cdots R_{aL}(u).$$

Here, $R_{aq}$ $(1 \leq q \leq L)$ means an $R$-matrix that act non-trivially only in $\text{End}(H_a \otimes V_q)$. In the Lax representation, however, the monodromy matrix becomes a two-by-two matrix with values in $\text{End}(H_a)$ — its elements are operators acting in $\text{End}(V_1 \otimes \cdots \otimes V_L)$:

$$M_a(u) = \begin{pmatrix} M^1_a(u) & M^2_a(u) \\ M^2_a(u) & M^1_a(u) \end{pmatrix} = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{21}$$

In terms of the operators (19), the monodromy elements $M^j_a(u)$ can be found explicitly by the formula:

$$M^j_a(u) = \sum_{k_1, \ldots, k_{L-1} = 1}^2 L_k^{j_1}(u) \otimes L_k^{j_2}(u) \otimes \cdots \otimes L_k^{j_{L-1}}(u) \otimes L_k^{j_L}(u). \tag{22}$$

Finally, the transfer matrix is defined as the trace (in the $H_a$ space) of the monodromy matrix, that is,

$$T(u) = \text{tr}_a[M_a(u)] = A(u) + D(u). \tag{23}$$

3. The reference state

The next step in the execution of the algebraic Bethe Ansatz consists of finding an appropriate reference state. This corresponds to a (simple enough) eigenstate of the transfer matrix so that its eigenvalue can be directly computed.

Fortunately, we can verify that the reference state for the six-vertex model is the most simple possible state, namely,

$$\Psi_0 = \frac{1}{1} \otimes \cdots \otimes \frac{1}{L}. \tag{24}$$

In fact, it follows from (19) and (22) that the action of the monodromy elements on the reference state is given by

$$A(u)\Psi_0 = \alpha(u)\Psi_0, \quad B(u)\Psi_0 \neq z\Psi_0, \quad C(u)\Psi_0 = 0\Psi_0, \quad D(u)\Psi_0 = \delta(u)\Psi_0, \tag{25}$$

where $z$ can be any complex number and

$$\alpha(u) = r^L_1(u), \quad \delta(u) = r^L_2(u). \tag{26}$$

From this is not difficult to show that the action of the transfer matrix on $\Psi_0$ is given by,

$$T(u)\Psi_0 = \tau_0(u)\Psi_0, \quad \text{where}, \quad \tau_0(u) = \alpha(u) + \delta(u), \tag{27}$$

so that $\Psi_0$ is indeed an eigenstate of the transfer matrix whose eigenvalue is $\tau_0(u)$.

Notice moreover that the operator $C(u)$ annihilates $\Psi_0$, while the operator $B(u)$ gives something not proportional to $\Psi_0$ — that is it creates another state. By this reason we say that the $C$ operators are annihilator operators, while the $B$ operators are creator operators. We shall call the $A$ and $D$ operators as diagonal operators.

4. The construction of the excited states

Once the reference state is determined, we can construct the excited states of the transfer matrix by acting with the creator operator $B$ repeatedly on the reference state $\Psi_0$. In this way, we define $n$th excited state of the transfer matrix as

$$\Psi_n(u_1, \ldots, u_n) = B(u_1) \cdots B(u_n)\Psi_0. \tag{28}$$

Notice that each creator operator in (28) depends on a different parameter $u_k$ $(1 \leq k \leq n)$; these parameters are called rapidities and they are until now undetermined. We hope to fix the values of the rapidities so that $\Psi_n$ becomes indeed an eigenstate of the transfer matrix. We shall see in a while that the rapidities can be fixed (at least implicitly) by a system of non-linear equations called the Bethe Ansatz equations.
of the algebraic Bethe Ansatz, relation

where we introduced the following amplitudes:

which shows us that the transfer matrix commutes with itself for different values of the spectral parameter. This provides a sufficient condition for the transfer matrix to commute with itself for different values of the spectral parameter.

5. The commutation relations

Our goal now is to evaluate the action of the transfer matrix on these excited states. From (23) and (28), we get that,

$$ T(u)\Psi_n(u_1, \ldots, u_n) = A(u)B(u_1) \cdots B(u_n)\Psi_0 + D(u)B(u_1) \cdots B(u_n)\Psi_0. $$

(29)

To evaluate this, the commutation relations between the diagonal operators $A(u)$ and $D(u)$ with the creator operators $B(u_k)$ ($1 \leq k \leq n$) must be known. All these commutation relations are provided by the so-called fundamental relation of the algebraic Bethe Ansatz,

$$ S_{ab}(u - v)M_a(u)M_b(v) = M_a(v)M_b(u)S_{ab}(u - v). $$

(30)

which can be thought as a representation of the Yang-Baxter equation (10) applied to an entire row of the lattice – see Fig. 4.

The fundamental relation (30) provides the commutation relation between all the elements of the monodromy matrix (which must be written in the Lax representation in order to the equation (30) make sense). For the execution of the algebraic Bethe Ansatz, however, only the following three relations are necessary:

$$ A(u)B(v) = a_1(u, v)B(v)A(u) + a_2(u, v)B(u)A(v), $$

$$ D(u)B(v) = d_1(u, v)B(v)A(u) + d_2(u, v)B(u)D(v), $$

$$ B(u)B(v) = B(v)B(u), $$

(31)-(33)

where we introduced the following amplitudes:

$$ a_1(u, v) = \frac{r_1(v - u)}{r_2(u - v)}, \quad a_2(u, v) = -\frac{r_3(v - u)}{r_2(u - v)}, \quad d_1(u, v) = \frac{r_1(u - v)}{r_2(u - v)}, \quad d_2(u, v) = -\frac{r_3(u - v)}{r_2(u - v)}. $$

(34)

The fundamental relation of the algebraic Bethe Ansatz (30) also reveals a profound property of models that have an $R$-matrix satisfying the Yang-Baxter equation. To see that, take the trace of (30) in both the $H_a$ and $V_q$ spaces. After we use the properties of the trace and the definition of the transfer matrix given by (23), we shall get directly the relation

$$ T(u)T(v) - T(v)T(u) = 0, $$

(35)

which shows us that the transfer matrix commutes with itself for different values of the spectral parameter. This means that the transfer matrix can be thought as the generator of infinitely many conserved quantities in evolution – the Hamiltonian being one of them – or, in other words, that the model is integrable.
Now we have all that is needed to compute the action of the transfer matrix on the $n$th excited state. This task, however, is not easy at all. As a matter of a fact it is the most difficult step in the execution of the algebraic Bethe Ansatz. Indeed, to compute this action we need to use repeatedly the commutation relations (31) and (32), so that we can pass the diagonal operators $A$ and $D$ over all the creator operators $B$, after which we can act with the diagonal operators on the reference state $\Psi_0$. It is necessary, therefore, to use each commutation relation $n$ times, which will generate $2^n$ terms each.

Fortunately, after we analyze the first cases (e.g., for $n = 1$ and $n = 2$), we might see that some patterns arises. In fact, we might see that some terms can be simplified and others can be grouped together, so that the we can write a direct formula for the repeated use of the commutation relations. These formulas are:

\[
A(u) \prod_{k=1}^{n} B(u_k) = \prod_{k=1}^{n} a_1(u, u_k)B(u_k)A(u) + B(u) \sum_{j=1}^{n} a_2(u, u_k) \prod_{k=1, k \neq j}^{n} a_1(u_j, u_k)B(u_k)A(u_j),
\]

\[
D(u) \prod_{k=1}^{n} B(u_k) = \prod_{k=1}^{n} d_1(u, u_k)B(u_k)D(u) + B(u) \sum_{j=1}^{n} d_2(u, u_k) \prod_{k=1, k \neq j}^{n} d_1(u_j, u_k)B(u_k)D(u_j).
\]

A rigorous proof of (36) and (37) is, nevertheless, required. This proof follows from mathematical induction: the assumption is clearly true for $n = 1$. Assume that it also holds for general $n$. Then, for $n + 1$ we have that,

\[
A(u) \prod_{k=1}^{n+1} B(u_k) = a_1(u, u_{n+1})B(u_{n+1})A(u) \prod_{k=1}^{n} B(u_k) - a_2(u, u_{n+1})B(u)A(u_{n+1}) \prod_{k=1}^{n} B(u_k),
\]

\[
D(u) \prod_{k=1}^{n+1} B(u_k) = d_1(u, u_{n+1})B(u_{n+1})D(u) \prod_{k=1}^{n} B(u_k) - d_2(u, u_{n+1})B(u)D(u_{n+1}) \prod_{k=1}^{n} B(u_k),
\]

where we made use of the fact the $B$ operators commute with themselves in order to put $B(u_{n+1})$ on the left. Now, using (36) we get that

\[
A(u) \prod_{k=1}^{n+1} B(u_k) = \prod_{k=1}^{n+1} a_1(u, u_k)B(u_k)A(u)
\]

\[
+ a_1(u, u_{n+1})B(u_{n+1}) \sum_{j=1}^{n} a_2(u, u_j)B(u) \prod_{k=1, k \neq j}^{n} a_1(u_j, u_k)B(u_k)A(u_j)
\]

\[
+ a_2(u, u_{n+1})B(u) \prod_{k=1}^{n} a_1(u_{n+1}, u_k)B(u_k)A(u_{n+1})
\]

\[
+ a_2(u, u_{n+1})B(u) \sum_{j=1}^{n} a_2(u_{n+1}, u_j)B(u_{n+1}) \prod_{k=1, k \neq j}^{n} a_1(u_j, u_k)B(u_k)A(u_j),
\]

and, similarly, using (37), we get,

\[
D(u) \prod_{k=1}^{n+1} B(u_k) = \prod_{k=1}^{n+1} d_1(u, u_k)B(u_k)D(u)
\]

\[
+ d_1(u, u_{n+1})B(u_{n+1}) \sum_{j=1}^{n} d_2(u, u_j)B(u) \prod_{k=1, k \neq j}^{n} d_1(u_j, u_k)B(u_k)D(u_j)
\]

\[
+ d_2(u, u_{n+1})B(u) \prod_{k=1}^{n} d_1(u_{n+1}, u_k)B(u_k)D(u_{n+1})
\]

\[
+ d_2(u, u_{n+1})B(u) \sum_{j=1}^{n} d_2(u_{n+1}, u_j)B(u_{n+1}) \prod_{k=1, k \neq j}^{n} d_1(u_j, u_k)B(u_k)D(u_j).
\]
Then, we can see that the second and the fourth terms in each expressions above can be grouped together and, after we use the identities,
\[
\begin{align*}
  a_1(u, u_{n+1})a_2(u, u_j) + a_2(u, u_{n+1})a_2(u_{n+1}, u_j) = a_2(u, u_j)a_1(u_j, u_{n+1}), & \quad 1 \leq j \leq n, \\
  d_1(u, u_{n+1})d_2(u, u_j) + d_2(u, u_{n+1})d_2(u_{n+1}, u_j) = d_2(u, u_j)d_1(u, u_{n+1}), & \quad 1 \leq j \leq n,
\end{align*}
\]

(42) those terms can also be grouped with the third ones in (36) and (37), after we extend the summation to \( n + 1 \). This lead us to same expressions (36) and (37) with \( n + 1 \) in the place of \( n \), which proves the assumption.

Now, from (36) and (37) it is an easy matter to compute the action of the transfer matrix on the \( n \)th excited state. It happens that the action of \( T(u) \) on \( \Psi_n(u_1, \ldots, u_n) \) can be written as,
\[
T(u)\Psi_n(u_1, \ldots, u_n) = \tau_n(u|u_1, \ldots, u_n)\Psi_n(u_1, \ldots, u_n) + \sum_{k=1}^{n} \beta_n^k(u|u_1, \ldots, u_n)\Psi_n(u_k^\alpha),
\]
(44) where,
\[
\tau_n(u|u_1, \ldots, u_n) = \alpha(u) \prod_{k=1}^{n} a_1(u, u_k) + \delta(u) \prod_{k=1}^{n} d_1(u, u_k),
\]
(45)
\[
\beta_n^k(u|u_1, \ldots, u_n) = \alpha(u_k) a_2(u, u_k) \prod_{i=1, i \neq k}^{n} a_1(u_k, u_i) + \delta(u_k) d_2(u, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i), \quad 1 \leq k \leq n,
\]
(46)

and we introduced the notation
\[
\Psi_n(u_k^\alpha) = B(u) \prod_{j=1, j \neq k}^{n} B(u_j)\Psi_0.
\]
(47)

The main point here is that the first term in (44) is proportional to \( \Psi_n(u_1, \ldots, u_n) \), while the other terms are not. Therefore, the requirement that the transfer matrix satisfies an eigenvalue equation means that all the unwanted terms – i.e., the terms which are not proportional to \( \Psi_n(u_1, \ldots, u_n) \) – need to vanish. This can be ensured by fixing appropriated values for the rapidities \( u_1, \ldots, u_n \), which until now were arbitrary. In fact, as we impose that the following system of non-linear equations,
\[
\beta_n^k(u|u_1, \ldots, u_n) = \alpha(u_k) a_2(u, u_k) \prod_{i=1, i \neq k}^{n} a_1(u_k, u_i) + \delta(u_k) d_2(u, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i) = 0, \quad 1 \leq k \leq n,
\]
(48)
is satisfied, all the unwanted terms will vanish. The system of equations are called the Bethe Ansatz equations of the six-vertex model with periodic boundary conditions [12]. Their solutions provide the correct values for the rapidities \( u_1, \ldots, u_n \) necessary to vanish the unwanted terms.

7. The solutions of the Bethe Ansatz equations

We remark that if were possible to solve the Bethe Ansatz equations analytically, then we would obtain a complete analytical answer for the spectral problem considered by the Bethe Ansatz technique but, unfortunately, they are too complex for such an ambitious endeavor be accomplished by now – it is only up to the second excited state that a complete analytical solution of the Bethe Ansatz equations were obtained so far [13, 14] –, reason by which the Bethe Ansatz equations are usually solved through numerical methods [15, 16].

C. The combinatorial approach

In the previous section we discussed how the periodic algebraic Bethe Ansatz is usually implemented. We have seen that it depends on several steps, which might make the method seem difficult, especially for introductory audiences. When some subject has several technical and complicated details, it is always desirable, when possible, to set up symbolic approach for it, so that its main features can be qualitatively understood and the final results obtained in
an easier way – a classical example of this is the use of Feynman’s diagrams in quantum field theory. In this section we shall describe such a symbolic method for the algebraic Bethe Ansatz in terms of combinatorial diagrams (for an analysis of the Bethe Ansatz through the tools of tensor-networks, see [17]). Remember that the most laborious step in the execution of the algebraic Bethe Ansatz is the computation of the action of the transfer matrix on the \( n \)th excited state. This computation relies on the repeated use of the commutation relations (31) and (32). Or method consists in representing these commutation relations by simple combinatorial diagrams, so that the repeated use of them gives place to simple combinatorial trees. As we shall see in the sequel, the analysis of these combinatorial trees provides in a straightforward way the eigenvalues and eigenstates of the transfer matrix as well as the respective Bethe Ansatz equations.

Before, however, we discuss how this combinatorial approach works, we should present some definitions and nomenclatures that we shall make extensive use in the sequel. These nomenclatures appear in the theory of graphs, more specifically in the study of combinatorial rooted trees. Following [18] and [19], a rooted tree is defined as a directed graph in which any two nodes (the vertices of the graph) are connected by exactly one path. A labeled tree is a tree whose nodes are specified by means of labels. The first node of the tree (which connects all other nodes) is called its root and the last node of a given path of the tree is called a leaf node. We say that a given node is at the level \( k \) of the tree if the path connecting this node to the root contains exactly \( k + 1 \) nodes (counting both the root and the referred node, so that the root is always at the level zero). The length of a path is defined as the number of nodes it contains, from the root to the leaves, so that the length of the tree is also the length of its longest path. If all paths of the tree have the same length (as is the case for the trees considered here), we shall refer to it as a pruned tree. Moreover, we establish a parental relationship between the nodes of the tree: given a node at the level \( k \) of the tree and another node at the level \( k + 1 \), we say that the first is the parent of the second – and, accordingly, that the second is a child of the first – if they belong to the same path of the tree. Finally, a disjoint union of trees is usually called a forest.

The combinatorial approach we are going to present is actually very simple: it consists in representing the commutation relations

\[
A(u)B(v) = a_1(u,v)B(v)A(u) + a_2(u,v)B(u)A(v),
\]

\[
D(u)B(v) = d_1(u,v)B(v)A(u) + d_2(u,v)B(u)D(v),
\]

by simple diagrams with two outputs, as below:

\[
\begin{array}{c}
\text{and} \\
\text{Notice that in these diagrams the root indicates what commutation relation we are talking about. In each diagram,} \\
\text{the nodes (hollow or filled) represent one of the two terms in the respective commutation relations. For instance, we} \\
\text{can assume that the hollow node represents the first term in the commutation relations (49) or (50), while the filled} \\
\text{node represents the second term in (49) or (50). Thus, we can write something like this:}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \quad \equiv \quad a_1(u,v)B(v)A(u), \\
A \quad \equiv \quad a_2(u,v)B(u)A(v),
\end{array} \\
\begin{array}{c}
D \quad \equiv \quad d_1(u,v)B(v)D(u), \\
D \quad \equiv \quad d_2(u,v)B(u)D(v).
\end{array}
\end{array}
\]

Finally, the labels \( u \) and \( v \) attached at the side of each node in the diagrams (51) indicate the argument of the diagonal operators, \( A \) or \( D \), in the respective term of the commutation relation (we shall see that they are very useful in the general case).

Now, the repeated use of the commutation relations can be represented by a labeled combinatorial tree. For example, let us consider the computation of \( A(u_0)B(u_1)B(u_2)\cdots B(u_n) \), which appears in the action of the operator \( A(u_0) \) on the \( n \)th excited state \( \Psi_n(u_1,\ldots,u_n) \). This can be represented\(^3\) by the following pruned binary tree of length \( n \):

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A \quad \equiv \quad a_1(u,v)B(v)A(u), \\
A \quad \equiv \quad a_2(u,v)B(u)A(v),
\end{array} \\
\begin{array}{c}
D \quad \equiv \quad d_1(u,v)B(v)D(u), \\
D \quad \equiv \quad d_2(u,v)B(u)D(v).
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[
(53)
\]

\(^3\) In this section and in section III C, we shall usually write the spectral parameter \( u \) as \( u_0 \).
and an identical diagram holds for the computation of $D(u_0)B(u_1)B(u_2) \cdots B(u_n)$.

As before, the hollow nodes in the diagram (53) mean that the first term of the commutation relations (49) or (50) is gathered at that point, while the filled nodes mean that the second term is kept there. The diagram will contain, therefore, $2^n$ paths, each one of them representing a term that would be obtained from the algebraic Bethe Ansatz. Any path $P$ of the diagram can be specified by the set $\eta = (\eta_1, \ldots, \eta_n)$, where the parameters $\eta_k$ ($1 \leq k \leq n$) can assume only the values 1 or 2, depending on whether the node in the level $k$ of the path $P$ is hollow ($\eta_k = 1$) or filled ($\eta_k = 2$).

The label $\lambda$ at the side of every node represents the argument of the diagonal operators, $A$ or $D$, on that point of the respective diagram. These labels can be easily obtained following the simple rule:

- Hollow nodes always inherit the label of his parent, while the label $\lambda_k^P$ of any filled node on the level $k$ of the path $P$ is always $u_k$ (the label of the root being defined as $u_0$).

This rule comes from a direct analysis of the commutation relations (49) and (50). In fact, starting, for instance, with $A(u_0)B(u_1)B(u_2) \cdots B(u_n)$, we may notice that the use of the commutation relation (49) has always the effect of permuting the operators $A$ and $B$. If we gather, in the one hand, the first term of the commutation relation (49), then we shall get a quantity proportional to $B(u_1)A(u_0)B(u_2) \cdots B(u_n)$, where the argument of the diagonal operator $A$ is still the same as before. If we gather, on the other hand, the second term in (49), then we would get a quantity proportional to $B(u_0)A(u_1)B(u_2) \cdots B(u_n)$, so that the argument of the diagonal operator $A$ is permuted with that of the operator $B$. We can say, therefore, that a hollow node does not change the argument of the diagonal operators $A$ or $D$, so that their labels must be the same as the labels of their parent, while a filled node permutes the argument of the diagonal operators $A$ or $D$ with that of the creator operator $B$ in that point, that is, the label of any filled node at the level $k$ of the diagram must be $u_k$. This means that the label $\lambda_k^P$ of a node at the level $k$ of a given path $P$ in any diagram ($A$ or $D$) can be determined recursively by the formula

$$\lambda_k^P = \begin{cases} \lambda_{k-1}^P, & \eta_k = 1, \\ u_k, & \eta_k = 2, \end{cases}$$ (54)

provided we define the label of the root as $\lambda_0^P = u_0$.

The key point of this approach is that there is a one-to-one correspondence between each path of the diagrams and each one of the final terms obtained from the usual algebraic Bethe Ansatz$^4$. In fact, after we use one of the commutation relations (49) or (50) $n$ times, we shall obtain $2^n$ terms, or values $V$, each of which is of the form

$$V = W |S\rangle,$$ (55)

where $W$ (the weight of the term $T$, or the path $P$ it represents) corresponds to a specific product of the coefficients appearing on the commutation relations, times the action of the diagonal operator on the reference state $\Psi_0$, and $|S\rangle$ (the state associated with the term $T$ or the corresponding path $P$) consists in a given product of the $B$ operators.

The weight of a path can be found defining the contribution of each node it contains, plus a contribution of the leaf node. In this way, we define the weights of a given node and the contribution of the leaf node as follows:

- The weight of a hollow node in the level $k$ of a given path $P_A \ [P_D]$ equals $a_1(\lambda_{k-1}^P, u_k) \ [d_1(\lambda_{k-1}^P, u_k)]$, while the weight of a given filled node in the level $k$ of a given path $P_A \ [P_D]$ equals $a_2(\lambda_{k-1}^P, u_k) \ [d_2(\lambda_{k-1}^P, u_k)]$.

- The leaf node of the path $P_A \ [P_D]$ contributes to the weight of the respective path with the factor $\alpha(\lambda_n^P) \ [\delta(\lambda_n^P)]$, which arises from the action of the operator $A(\lambda_n) \ [D(\lambda_n)]$ on $\Psi_0$.

In a more condensed way, the weight associated with each $P_A(\eta_1, \ldots, \eta_n)$ of the $A$ diagram, and the weight of each path $P_D(\eta_1, \ldots, \eta_n)$ of the $D$ diagram, are given by

$$W(P_A(\eta_1, \ldots, \eta_n)) = \alpha(\lambda_n^P) \prod_{k=1}^n a_{\eta_k}(\lambda_{k-1}^P, u_k), \quad W(P_D(\eta_1, \ldots, \eta_n)) = \delta(\lambda_n^P) \prod_{k=1}^n d_{\eta_k}(\lambda_{k-1}^P, u_k).$$ (56)

Similarly, the state associated with each path can be determined by a single rule:

$^4$ As an example, we present below the paths associated with the $A$ diagram for the second excited state ($n = 2$) and the corresponding mathematical expressions:

- $\Delta \Theta \equiv a_1(u_0, u_1) a_1(u_0, u_2) a_1(u_0) B(u_1) B(u_2) \Psi_0$,
- $\Delta \Theta \equiv a_1(u_0, u_1) a_2(u_0, u_2) a_2(u_0) B(u_0) B(u_1) \Psi_0$,
- $\Delta \Theta \equiv a_2(u_0, u_1) a_2(u_1, u_2) a_2(u_1) B(u_0) B(u_2) \Psi_0$,
- $\Delta \Theta \equiv a_2(u_0, u_1) a_1(u_1, u_2) a_1(u_2) B(u_0) B(u_1) \Psi_0$.

The paths associated with the $D$ diagram are the same as the above ones, except that the coefficients $a_1$ and $a_2$ should be replaced by $d_1$ and $d_2$, respectively.
The state $|P(u_k)\rangle$ associated with any path $P$ whose leaf node has the label $\lambda_n^P = u_k$ is given by the product of all operators $B(u_j)$, $0 \leq j \neq k \leq n$, times $\Psi_0$. That is,

$$|P(u_k)\rangle = \prod_{j=0, j \neq k}^{n} B(u_j)\Psi_0.$$  (57)

In fact, the condition for a given path to end with the label $u_k$ is that it contains a filled node in the level $k$ and no other filled node in the higher levels – which is a consequence of the rule determining the labels of the diagrams. Therefore, in both the $A$ and $D$ diagrams there will be only one path ending with the label $u_0$ and exactly $2^{k-1}$ paths that end with the label $u_k$ for $k \geq 1$.

From what was said above, it is an easy matter to determine the whole action of the transfer matrix on the $n$th excited state. In fact, the action of the $T(u_0)$ on $\Psi_n(u_1, \ldots, u_n)$ will be given by the sum of the values $V(P) = W(P) |P\rangle$ associated with all paths $P$ of the $A$ and $D$ diagrams (i.e., by a sum over the forest, in the jargon of graph theory). That is,

$$T(u_0)\Psi_n(u_1, \ldots, u_n) = \sum_{\eta_1, \ldots, \eta_n=1}^{2} [V_A(\eta_1, \ldots, \eta_n) + V_D(\eta_1, \ldots, \eta_n)] |P(\eta_1, \ldots, \eta_n)\rangle.$$  (58)

Gathering all the paths with the same state (that is, collecting all the paths terminating with the same label), we can also write this as

$$T(u_0)\Psi_n(u_1, \ldots, u_n) = \left[ W \left( P_A^{(u_0)} \right) + W \left( P_D^{(u_0)} \right) \right] |P^{(u_0)}\rangle + \sum_{k=1}^{n} \left[ W \left( P_A^{(u_k)} \right) + W \left( P_D^{(u_k)} \right) \right] |P^{(u_k)}\rangle.$$  (59)

This corresponds to partitioning the $2^n$ terms in expression (58) into the $n + 1$ terms in expression (59). Notice that this partition is only possible thanks to the identity $1 + (1 + 2 + \ldots + 2^{n-1}) = 2^n$. Identifying

$$\tau_n(u_0|u_1, \ldots, u_n) \Psi_n(u_1, \ldots, u_n) \equiv \left[ W \left( P_A^{(u_0)} \right) + W \left( P_D^{(u_0)} \right) \right] |P^{(u_0)}\rangle,$$  (60)

and

$$\beta_n^k(u_0|u_1, \ldots, u_n) \Psi_n(u_k^c) \equiv \left[ W \left( P_A^{(u_k)} \right) + W \left( P_D^{(u_k)} \right) \right] |P^{(u_k)}\rangle,$$  (61)

we can rewrite (59) as

$$T(u_0)\Psi_n(u_1, \ldots, u_n) = \tau_n(u_0|u_1, \ldots, u_n) \Psi_n(u_1, \ldots, u_n) + \sum_{k=1}^{n} \beta_n^k(u_0|u_1, \ldots, u_n) \Psi_n(u_k^c),$$  (63)

which provides a complete agreement with (44) for the action of the transfer matrix on the $n$th excited state.

The analysis above gives a complete description for the action of the transfer matrix $T(u_0)$ on the $n$th excited state $\Psi_n$. In practice, however, we are usually interested only in the eigenvalues of the transfer matrix and in the Bethe Ansatz equations. Let us show now how the eigenvalues can be obtained in a straightforward way through the analysis of the diagrams.

The eigenvalues of the transfer matrix can be determined through the analysis of the diagrams by following only one simple rule:

- The eigenvalues of the transfer matrix are determined by the sum of the weights over all paths of the diagrams $A$ and $D$ that end with the label $u_0$.

In fact, only in this case the state of the path will be proportional to $\Psi_n(u_1, \ldots, u_n)$. Notice that a path will end with the label $u_0$ if, and only if, it contains no filled nodes – i.e., if it contains only hollow nodes. Since, however, there is
only one path in each diagram satisfying this requirement – namely, the paths \( P_A(1, \ldots, 1) \) and \( P_D(1, \ldots, 1) \) –, we conclude that the eigenvalues are determined by

\[
\tau_n (u_0|u_1, \ldots, u_n) \Psi_n (u_1, \ldots, u_n) = A \quad + \quad D
\]

which means, according to the weights of these paths, that

\[
\tau_n (u_0|u_1, \ldots, u_n) = \alpha(u_0) \prod_{k=1}^{n} a_1 (u_0, u_k) + \delta (u_0) \prod_{k=1}^{n} d_1 (u_0, u_k). \tag{65}
\]

Similarly, the Bethe Ansatz equations can be determined by the rule:

\[\diamond\] The Bethe Ansatz equation fixing the rapidity \( u_k \) is determined by a sum over all paths of the \( A \) and \( D \) diagrams whose leaf node has the label \( \lambda^0_n = u_k \).

In fact, we have seen that the state associated with a given path ending with the label \( u_k \) is just \( |\Psi(u_k)\rangle = \Psi(u_k) \). The condition for a given path to end with the label \( u_k \) is that it contains a filled node at the level \( k \) and no other filled node at higher levels. In the lower levels, however, the nodes can be of any type, which means that we should sum over all the possible types of nodes in the lower levels of the diagrams. We can graphically express this as follows:

\[
\beta_n^k (u_0|u_1, \ldots, u_n) \Psi_n (u_1, \ldots, u_n|u_k) = A \quad + \quad D
\]

where a sum is to be understood on any gray node, according to the two possible types of nodes, hollow or filled. Computing the weights of such paths, we shall get the expressions

\[
\beta_n^k (u_0|u_1, \ldots, u_n) = \alpha(u_k) \sum_{\eta_1, \ldots, \eta_{k-1} = 1}^{2} \prod_{i=1}^{k-1} a_{\eta_i} \left( \lambda^0_{i-1}, u_i \right) a_2 (\lambda^0_{k-1}, u_k) \prod_{j=k+1}^{n} a_1 (u_k, u_j)
+ \delta(u_k) \sum_{\eta_1, \ldots, \eta_{k-1} = 1}^{2} \prod_{i=1}^{k-1} d_{\eta_i} \left( \lambda^0_{i-1}, u_i \right) d_2 (\lambda^0_{k-1}, u_k) \prod_{j=k+1}^{n} d_1 (u_k, u_j) = 0, \quad 1 \leq k \leq n. \tag{67}
\]

The most simple Bethe Ansatz equation is that one fixing the rapidity \( u_1 \). In this case, we have simply:

\[
\beta_n^1 (u_0|u_1, \ldots, u_n) \Psi_n (u_1, \ldots, u_n|u_1) = A \quad + \quad D
\]

that is,

\[
\beta_n^1 (u_0|u_1, \ldots, u_n) = \alpha(u_1) a_2 (u_0, u_1) \prod_{i=2}^{n} a_1 (u_1, u_i) + \delta (u_1) d_2 (u_0, u_1) \prod_{i=2}^{n} d_1 (u_1, u_i) = 0, \tag{69}
\]

which agrees with (48) for \( k = 1 \). The other Bethe Ansatz equations, however, are not yet in the same form as (48). They can, nevertheless, be simplified and then be put into the same form as (48), after we make use of the symmetry of the excited states regarding the permutation of the rapidities. To see that, notice that the commutation relation (33) implies that the \( n \)th excited state does not change if any pair of rapidities are permuted. In particular, we have that \( \Psi_n (u_1, u_2, \ldots, u_k, \ldots, u_n) = \Psi_n (u_k, u_2, \ldots, u_1, \ldots, u_n) \), where the rapidities \( u_1 \) and \( u_k \) have been permuted. If we consider the respective diagrams for the action of the diagonal operators \( A(u_0) \) and \( D(u_0) \) on this “not well-ordered” state \( \Psi_n (u_k, u_2, \ldots, u_1, \ldots, u_n) \), then we would find that these diagrams are identical to the original ones, except that the labels \( u_1 \) and \( u_k \) would be exchanged. Since the result in both cases must be the same, we conclude that the Bethe Ansatz equation fixing \( u_k \) can also be written in the same form as the Bethe Ansatz equation fixing \( u_1 \), provided that \( u_k \) and \( u_1 \) are permuted.

This leads us to a simpler rule determining the Bethe Ansatz equation for the rapidity \( u_k \) (\( 2 \leq k \leq n \)):

\[\diamond\] The Bethe Ansatz equation fixing the rapidity \( u_k \) (\( 2 \leq k \leq n \)) can be obtained by the same paths that determine the Bethe Ansatz equation fixing \( u_1 \), provided that the labels \( u_1 \) and \( u_k \) are permuted.
We conclude therefore that the final form of the Bethe Ansatz equations is,
\[ \beta_n^k (u_0|u_1, \ldots, u_n) = \alpha (u_k) a_2 (u_0, u_k) \prod_{i=1, i \neq k}^n a_1 (u_k, u_i) + \delta (u_k) d_2 (u_0, u_k) \prod_{i=1, i \neq k}^n d_1 (u_k, u_i) = 0, \quad 1 \leq k \leq n, \tag{70} \]
which is indeed equal to (48) for any value of \( k \).

We highlight that the proofs of (65) and (70) follow from combinatorial arguments only – there is no need of using mathematical induction here. It also not necessary to analyze the first cases (e.g., \( n = 1 \) or \( n = 2 \)) first: the results follow once and for all for general \( n \).

Finally, we remark that the equality between (67) and (70) also provides several intricate identities among the elements of the \( R \)-matrix, namely, the following ones:
\[ \sum_{\eta_1, \ldots, \eta_{k-1}}^2 \prod_{i=1}^{k-1} a_{\eta_i} (\lambda_{i-1}^P, u_i) a_2 (u_0, u_k) \prod_{i=1}^{k-1} a_1 (u_k, u_i), \quad 1 \leq k \leq n, \tag{71} \]
and
\[ \sum_{\eta_1, \ldots, \eta_{k-1}}^2 \prod_{i=1}^{k-1} d_{\eta_i} (\lambda_{i-1}^P, u_i) d_2 (u_0, u_k) \prod_{i=1}^{k-1} d_1 (u_k, u_i), \quad 1 \leq k \leq n. \tag{72} \]
The identities (42) and (43) are just special cases of the identities above.

### III. THE SIX-VERTEX MODEL WITH NON-PERIODIC BOUNDARY CONDITIONS

#### A. The monodromy and transfer matrices of the six-vertex model with non-periodic boundary conditions

The integrability of systems described by non-periodic boundary conditions is studied through the **boundary algebraic Bethe Ansatz**. In this case, we consider a chain with \( N \) lines and \( L \) columns as before, however, we do not identify the column \( L + 1 \) and the row \( N + 1 \) with the first ones, respectively. We assume, on the contrary, that the vertices at the boundaries are different from the remaining ones. It is convenient, therefore, to consider these boundary vertices are disposed at the columns \( j = 0 \) and \( j = L + 1 \) and at the rows \( i = 0 \) and \( i = N + 1 \) of the lattice. These boundary vertices can also have your own configurations, so that the respective Boltzmann weights can be represented by two **boundary matrices** – also known as **reflection matrices** or simply **\( K \)-matrices**. Thus, there are two \( K \)-matrices for each row of the lattice: we write \( K^+(u) \) for the left reflection matrix and \( K^-(u) \) for the right one. Since there is nothing beyond the boundaries, these \( K \)-matrices should act only in a Hilbert space that is isomorphic to \( \mathbb{C}^2 \) – this should be contrasted with the \( R \)-matrix, which is defined on a Hilbert space isomorphic to \( \mathbb{C}^2 \otimes \mathbb{C}^2 \).

![Figure 5](image)

Figure 5. A graphical representation of the double monodromy. This is defined by the product of all \( R \)-matrices in a given row \( (i = a) \) of the lattice, taken in the usual order, times the \( K \)-matrix at column \( j = L + 1 \), times the product of the inverse of these \( R \)-matrices with the opposite spectral parameter and in the reversed order. All these products follow the anti clock-wise direction of the diagram above. If we further multiply this quantity by the other \( K \)-matrix in column \( j = 0 \) of the lattice and then we take the trace over the vector space \( H_a \), we shall obtain the transfer matrix.

In the non-periodic case, we cannot define a single monodromy matrix for a given row of the lattice because we need to take account for contributions at the boundaries. One way of working this around is to consider a **double**
monodromy, which was introduced by Sklyanin [20]. This double monodromy matrix is constructed as follows: we first multiply all $R$-matrices in a given row ($i = a$) of the lattice – that is, we consider the usual periodic monodromy, $M_a(u)$, then we multiply it by $K$-matrix at the right boundary and, finally, we return back by multiplying it by the inverse of all those $R$-matrices in the reversed order and with the opposite of the spectral parameter – that is, we multiply by the inverse of the periodic monodromy, $M_a^{-1}(-u)$, which we might call the reflected monodromy. That is, the double monodromy $U_a(u)$ is therefore defined as follows:

$$U_a(u) = M_a(u)K^-(u)M_a^{-1}(-u),$$  \hspace{1cm} (73)

where,

$$M_a(u) = R_{a1}(u)\cdots R_{aL}(u), \quad \text{and} \quad M_a^{-1}(-u) = R_{aL}^{-1}(-u)\cdots R_{a1}^{-1}(-u) = \frac{R_{a1}(u)\cdots R_{aL}(u)}{r_1^+(u)r_2^+(-u)}.$$  \hspace{1cm} (74)

The last identity is obtained from the relation $R_{ij}^{-1}(u) = R_{ij}(u)/r_1(u)r_1(-u)$, which holds for the symmetric six-vertex $R$-matrix (11). See Fig. 5 for a graphical representation of the double monodromy.

After the double monodromy matrix (73) is defined, the transfer matrix for the non-periodic case can be constructed. This is given by the trace (on the vector space $H_a$) of the boundary matrix $K^+(u)$ multiplied by the double monodromy $U_a(u)$:

$$T(u) = tr_a \left[ K^+(u)U_a(u) \right].$$ \hspace{1cm} (75)

The partition function of the whole lattice can then be found by repeating the above procedure for all rows of the lattice, as we take into account the boundary matrices at the beginning and the ending of each column. When all the rows in the bulk of the lattice are equivalent, the eigenvalue for the transfer matrix of any row will be the same. Thus, as in the periodic case, the eigenvalues of the partition function will depend only on the eigenvalues of the transfer matrix, so that we can concern ourselves only with the diagonalization of the transfer matrix.

B. The algebraic Bethe Ansatz

The algebraic Bethe Ansatz can be generalized to cover the case where non-periodic boundary conditions take place. In this case, we call it the boundary algebraic Bethe Ansatz. The steps necessary to execute the algebraic Bethe Ansatz, that is, to find the eigenvalues and eigenstates of the transfer matrix, are similar to that of the periodic case, although it is more complex due to the existence of the boundary matrices. We list below the main steps:

1. The solutions of the boundary Yang-Baxter equations providing the reflection $K$-matrices. Assuming that there is known an $R$-matrix, solution of the Yang-Baxter equation, we look for the reflection matrices $K^\pm(u)$. These matrices are the solutions of the boundary Yang-Baxter equations that ensure the integrability at the boundaries;

2. The Lax representation for the double monodromy and transfer matrices. As in the periodic case, we implement a representation in which the double monodromy explicitly exhibit the annihilator and creator operators used in the construction of the excited states. The transfer matrix also become given by the diagonal operators of the monodromy matrix, times the elements of the left boundary matrix;

3. The reference state. It has the same meaning as in the periodic case;

4. The construction of the excited states. Built in the same way as in the periodic case, that is, through the action of the creator operators on the reference state;

5. The derivation of the commutation relations. They are obtained from the fundamental relation of the boundary algebraic Bethe Ansatz;

6. The computation of the eigenvalues of the transfer matrix. This is again the hardest step. To compute the eigenvalues we need to use the commutation relations repeatedly;

7. The solution of the Bethe Ansatz equations. The consistency conditions for the transfer matrix satisfy the eigenvalue equation.

We shall explain these steps in details in the following.
The integrability of the system at the boundaries is guaranteed by the boundary Yang-Baxter equations – also known as the reflection equations – which are [20],

\[ R(u-v)K_1^- (u) R(u+v)K_2^- (v) = K_2^- (v) R(u+v)K_1^- (u) R(u-v), \]

and

\[ R(v-u) [K_1^+ (u)]^{t_1} R(-u-v-2\rho) [K_2^+ (v)]^{t_2} = [K_2^+ (v)]^{t_2} R(-u-v-2\rho) [K_1^+ (u)]^{t_1} R(v-u). \]

In terms of the \( S \) matrix (8), the reflection equations above can be written, respectively, as,

\[ S(u-v) K_1^- (u) S(u+v) K_1^- (v) = K_1^- (v) S(u+v) K_1^- (u) S(u-v), \]

and

\[ S(v-u) [K_1^+ (u)]^{t_1} S(-u-v-2\rho) [K_2^+ (v)]^{t_2} = [K_1^+ (v)]^{t_2} S(-u-v-2\rho) [K_1^+ (u)]^{t_1} S(v-u), \]

which are more useful to be explained graphically – see Fig. 6.

The reflection equations for \( K^- \) ensure the integrability at the right side of the lattice, while the reflection equations for \( K^+ \) guarantee the integrability at the left. The integrability in the bulk is still provided by the periodic Yang-Baxter equation (6). The reflection equations are defined in \( \text{End} (V \otimes V) \), where \( V \) is isomorphic to \( \mathbb{C}^2 \). The reflection \( K \)-matrices, by their turn, are defined in \( \text{End} (V) \) so that \( K_1^\pm = K_1 \otimes I \) and \( K_2^\pm = I \otimes K_2^\pm \), with \( I \) denoting the identity matrix belonging to \( \text{End} (V) \). Besides, \( t_1 \) and \( t_2 \) mean the partial transposition in the first and second vector spaces, respectively, and \( \rho \) is the so-called crossing parameter – a parameter specific to the model that provides, for instance, the isomorphism \( K^+ (u) = [K^- (-u-\rho)]^t \) between the solutions of (76) and (77) – see [20] for more details.

![Figure 6](image_url)

Figure 6. A graphical representation of the boundary Yang-Baxter Equation (78). The \( K \)-matrices behave as a reflection in the horizontal direction. The arrows indicate the order in which the left and right diagrams should be read, starting from the center in the first diagram and from the upper left on the second.

For the six-vertex model, the \( K \)-matrices are two-by-two matrices. We are actually only interested in the diagonal reflection \( K \)-matrices of the six-vertex-model, which are of the form,

\[ K^\pm = \begin{pmatrix} k_{1,1}^\pm(u) & 0 \\ 0 & k_{2,2}^\pm(u) \end{pmatrix}, \]

The methods of solving (76) and (77) are somewhat similar to that employed in the periodic case, so that we shall despise it. Notwithstanding, there are several solutions for the six-vertex model with diagonal boundary conditions that rely for different integrable models – as in the periodic case again. The most important ones are the solution associated with the XXZ Heisenberg chain [21],

\[ k_{1,1}^- (u) = \sinh(u + \zeta^-), \quad k_{1,1}^+ (u) = \sinh(-u - \xi + \zeta^+), \]

\[ k_{2,2}^- (u) = -\sinh(u - \zeta^-), \quad k_{2,2}^+ (u) = -\sinh(-u - \xi - \zeta^+), \]

and that for the XXX Heisenberg chain,

\[ k_{1,1}^- (u) = u + \zeta^-, \quad k_{1,1}^+ (u) = -u - \xi + \zeta^+, \]

\[ k_{2,2}^- (u) = u + \zeta^-, \quad k_{2,2}^+ (u) = u + \xi + \zeta^+, \]

where \( \xi^\pm \) and \( \zeta^\pm \) are arbitrary parameters.
2. The Lax representation of the monodromy and transfer matrices

We continue by setting up the Lax representation, that is, a representation in which the monodromy matrix is written as a two-by-two operator-valued matrix:

\[ U_a(u) = M_a(u)K^-(u)M_a^{-1}(-u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad (83) \]

An explicit formula for the elements of \( U_a(u) \) can be found through (74) and (22).

The transfer matrix, by its turn, is given by the trace of the monodromy matrix (83), previously multiplied by \( K^+(u) \), that is,

\[ T(u) = \text{tr}_a \left[ K^+(u)U_a(u) \right] = k^+_{1,1}(u)A(u) + k^+_{2,2}(u)D(u), \quad (84) \]

where we considered only the case of diagonal \( K \)-matrices.

3. The reference state

Here as well, to diagonalize the transfer matrix (84) through the boundary algebraic Bethe Ansatz, we shall need a reference state. Fortunately, the same reference state found in the periodic case holds in this case as well, namely,

\[ \Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (85) \]

In fact, it can be verified directly from the application of (74) that the action of the boundary monodromy elements on \( \Psi_0 \) reads,

\[ A(u)\Psi_0 = \alpha(u)\Psi_0, \quad B(u)\Psi_0 \neq z\Psi_0, \quad C(u)\Psi_0 = 0\Psi_0, \quad D(u)\Psi_0 = \delta(u)\Psi_0, \quad (86) \]

for any \( z \in \mathbb{C} \). In the non-periodic case, however, \( \alpha(u) \) and \( \delta(u) \) are given by more complicated expressions:

\[ \alpha(u) = k_{1,1}^-(u) \frac{r^L_2(u)}{r^L_1(u)r^L_1(-u)}, \quad \delta(u) = k_{2,2}^-(u) \frac{r^L_2(u)}{r^L_1(u)r^L_1(-u)} + k_{1,1}^-(u)h_{L-1}(r^L_1(u), r^L_2(u)) \frac{r^L_3(u)}{r^L_1(u)r^L_1(-u)}, \quad (87) \]

where \( h_{L}(u,v) \) denotes the complete homogeneous symmetric polynomials of degree \( L \) in two variables:

\[ h_L(u,v) = \sum_{k=0}^{L} u^k v^{L-k} \frac{u^{L+1} - v^{L+1}}{u - v} \quad (88) \]

The action of the diagonal operators \( A \) and \( D \) on the reference state \( \Psi_0 \) can also be found in another way. In fact, we could had started with the fundamental relation of the periodic algebraic Bethe Ansatz (30) and evaluated it at \( v = -u \), in order get the equation,

\[ M^{-1}_a(-u)R_{ab}(2u)M_a(u) = M_a(u)R_{ab}(2u)M^{-1}_a(-u). \quad (89) \]

From this, the commutation relations between the operators \( M(u) \) and \( M^{-1}(-u) \) could be obtained and, after the action of the monodromy elements on the reference state \( \Psi_0 \) is computed, we would find the expressions,

\[ \alpha(u) = k_{1,1}^-(u) \frac{r^L_1(u)}{r^L_1(u)r^L_1(-u)}, \quad \delta(u) = \frac{f(u)\alpha(u) + [k_{2,2}^-(u) - f(u)k_{1,1}^-(u)]}{r^L_1(u)} \frac{r^L_3(u)}{r^L_1(u)r^L_1(-u)}, \quad (90) \]

where

\[ f(u) = \frac{r_3(2u)}{r_1(2u)} = \frac{r^L_3(u)}{r^L_1(u) - r^L_2(u)} \quad (91) \]

The equivalence between this result and that given by (87) is found after we use (88).

Therefore, the action of the transfer matrix on the reference state reads,

\[ T(u)\Psi_0 = \tau_0(u)\Psi_0, \quad \tau_0(u) = k_{1,1}^+(u)\alpha(u) + k_{2,2}^+(u)\delta(u) \quad (92) \]

where \( \alpha(u) \) and \( \delta(u) \) can be either written as in (87) or in (90). This proves that \( \Psi_0 \) given at (85) is indeed an eigenstate of the transfer matrix (84).
4. The construction of the excited states

The excited states can also be constructed in the same way as in the periodic case, namely, through the repeated action of the creator operator $B$ on the reference state $\Psi_0$:

$$\Psi_n (u_1, \ldots, u_n) = B(u_1) \cdots B(u_n) \Psi_0. \quad (93)$$

Hence, the action of the transfer matrix on the $n$th excited state reads:

$$T(u)\Psi_n (u_1, \ldots, u_n) = k_{1,1}^+(u) A(u) B(u_1) \cdots B(u_n) \Psi_0 + k_{2,2}^+(u) D(u) B(u_1) \cdots B(u_n) \Psi_0. \quad (94)$$

5. The commutation relations

In order to compute (94), the commutation relations between the diagonal operators $A(u)$ and $D(u)$ with the creator operators $B(u_k)$ ($1 \leq k \leq n$) are necessary. These commutation relations are provided by the fundamental relation of the boundary algebraic Bethe Ansatz:

$$R(u-v) U_a(u) R(u+v) U_b(v) = U_b(v) R(u+v) U_a(u) R(u-v), \quad (95)$$

or, in terms of the $S$ matrix (8),

$$S(u-v) U_a(u) S(u+v) U_a(v) = U_a(v) S(u+v) U_a(u) S(u-v). \quad (96)$$

A graphical interpretation of the fundamental relation of the boundary algebraic Bethe Ansatz is given in Fig. 7.

For the six-vertex model, the needed commutation relations are the following:

$$A(u) B(v) = a_1(u,v) B(v) A(u) + a_2(u,v) B(u) A(v) + a_3(u,v) B(u) D(v),$$
$$D(u) B(v) = d_1(u,v) B(v) D(u) + d_2(u,v) B(u) D(v) + d_3(u,v) B(u) A(v) + d_4(u,v) B(v) A(u),$$
$$B(u) B(v) = B(v) B(u), \quad \text{where}$$

$$a_1(u,v) = \frac{r_1(u-v)r_2(u+v)}{r_1(u+v)r_2(v-u)}, \quad a_2(u,v) = \frac{r_2(u+v)r_3(v-u)}{r_1(u+v)r_2(v-u)}, \quad a_3(u,v) = \frac{r_3(u+v)}{r_1(u+v)}, \quad (100)$$

and

$$d_1(u,v) = \frac{r_1(u-v)r_1(u+v)}{r_2(u-v)r_2(u+v)} - \frac{r_1(u-v)^2}{r_1(u+v)r_2(u-v)r_2(u+v)},$$
$$d_2(u,v) = \frac{r_3(u-v)r_3(u+v)}{r_1(u+v)r_2(u-v)r_2(u+v)} - \frac{r_1(u+v)r_3(u-v)}{r_2(u-v)r_2(u+v)},$$
$$d_3(u,v) = \frac{r_3(u-v)r_3(u+v)}{r_1(u+v)r_2(u-v)r_2(u+v)} + \frac{r_3(u-v)r_3(u+v)}{r_1(u+v)r_2(u-v)r_2(u+v)},$$
$$d_4(u,v) = \frac{r_1(u-v)r_3(u-v)r_3(u+v)}{r_1(u+v)r_2(u-v)r_2(u+v)} - \frac{r_1(u+v)r_3(u-v)r_3(u+v)}{r_1(u+v)r_2(u-v)r_2(u+v)}. \quad (104)$$

Notice that these expressions are more complicated than the periodic ones. Moreover, the commutation relation between the operators $D(u)$ and $B(v)$ has one term with no counterpart on the commutation relation between $A(u)$ and $B(v)$ – to be more specific, the term $d_4(u,v) B(v) A(u)$. This asymmetry leads to some complication in the execution of the boundary algebraic Bethe Ansatz and, by this reason, we usually introduce the following shifted diagonal operators,

$$\mathcal{A}(u) = A(u), \quad \mathcal{D}(u) = D(u) - f(u) A(u), \quad f(u) = \frac{r_3(2u)}{r_1(2u)} = \frac{r_3(u)}{r_2(u) - r_2(u)}. \quad (105)$$

order to vanish that undesirable term in (98). Notice that $f(u)$ is the same function as given by (91).
Figure 7. A graphical representation of the fundamental relation of the boundary algebraic Bethe Ansatz (96). The double monodromy is represented by rounded rectangles. The arrows indicate the order in which the left and right diagrams should be read, starting from the center in the first diagram and from the upper left on the second. The fundamental relation ensures the commutativity of the transfer matrix in the non-periodic case, and it also provides the commutation relations between the elements of the double monodromy, which are used in the computation of the transfer matrix eigenvalues.

It follows thus that the commutation relations (97), (98) and (99) can be brought into a more symmetric form, namely,

\[
A(u)B(v) = a_1(u,v)B(v)A(u) + a_2(u,v)B(u)A(v) + a_3(u,v)B(u)D(v),
\]

\[
D(u)B(v) = d_1(u,v)B(v)D(u) + d_2(u,v)B(u)D(v) + d_3(u,v)B(u)A(v),
\]

\[
B(u)B(v) = B(v)B(u),
\]

where now,

\[
a_1(u,v) = \frac{r_1(v-u)r_2(u+v)}{r_1(u+v)r_2(v-u)} - \frac{r_1(u+v)r_2(u+v)}{r_1(u+v)r_2(v-u)},
a_2(u,v) = -\frac{r_3(u+v)}{r_1(u+v)}f(v) - \frac{r_2(u+v)r_3(v-u)}{r_1(v-u)r_2(v-u)},
a_3(u,v) = -\frac{r_3(u+v)}{r_1(u+v)},
\]

and,

\[
d_1(u,v) = \frac{r_1(u-v)r_1(u+v)}{r_2(u-v)r_2(u+v)} - \frac{r_1(u-v)r_2(u+v)}{r_1(u+v)r_2(u-v)},
d_2(u,v) = \frac{r_3(u+v)}{r_1(u+v)}f(u) + \frac{r_3(u-v)r_2(u+v)}{r_1(u+v)r_2(u-v)} - \frac{r_1(u+v)r_2(u-v)}{r_1(u-v)r_2(u+v)},
d_3(u,v) = \frac{r_3(u+v)}{r_1(u+v)r_2(u+v)}r_2(u-v)r_1(u+v) - \frac{r_1(u+v)r_2(u-v)}{r_1(u+v)r_2(u+v)}f(u).
\]

The action of the shifted diagonal operators in the reference state then becomes,

\[
a(u) = \alpha(u) = k_{1,1}(u)\frac{r_1^{2L}(u)}{r_1^L(u)r_1^L(-u)},
da(u) = \delta(u) - f(u)\alpha(u)
\]

\[
= k_{2,1}(u)\frac{r_1^{2L}(u)}{r_1^L(u)r_1^L(-u)} + k_{1,1}(u)h_{L-1}(r_1^2(u),r_2^2(u))\frac{r_1^L(u)}{r_1^L(u)r_1^L(-u)} - f(u)k_{1,1}(u)\frac{r_1^{2L}(u)}{r_1^L(u)r_1^L(-u)}
\]

\[
= [k_{2,1}(u) - f(u)k_{1,1}(u)]\frac{r_1^{2L}(u)}{r_1^L(u)r_1^L(-u)}.
\]

Finally, introducing as well the shifted \(K\)-matrices elements,

\[
\kappa_{1,1}^+(u) = k_{1,1}^+(u) + f(u)k_{2,1}^+(u),
\]

\[
\kappa_{2,2}^+(u) = k_{2,2}^+(u),
\]

the transfer matrix can be rewritten as,

\[
T(u) = \kappa_{1,1}^+(u)A(u) + \kappa_{2,2}^+(u)D(u)\Psi_0,
\]

so that we have as well,

\[
T(u)\Psi_0 = \tau_0(u)\Psi_0, \quad \tau_0(u) = \kappa_{1,1}^+(u)a(u) + \kappa_{2,2}^+(u)d(u).
\]
6. The computation of the eigenvalues of the transfer matrix

At this point we have all the ingredients to execute the boundary algebraic Bethe Ansatz. The execution is, however, very laborious, since we have to use the commutation relations (106) and (107) several times in order to find the action of the transfer matrix on the excited states. In fact, we usually analyzes the first cases first – as in the periodic case –, from which the general expressions for the eigenvalues and the Bethe Ansatz equations can be guessed. This, eventually, provides a formula for repeated use of the commutation relations (106) and (107), which need to be proved further, of course. In fact, we have that,

\[ A(u) \prod_{k=1}^{n} B(u_k) = \prod_{k=1}^{n} a_1(u, u_k)B(u_k)A(u) \]

\[ + \sum_{k=1}^{n} a_2(u, u_k) \prod_{i=1, i \neq k}^{n} a_1(u, u_i)B(u)A(u_k) + \sum_{k=1}^{n} a_3(u, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i)B(u)D(u_k), \]  

(118)

\[ D(u) \prod_{k=1}^{n} B(u_k) = \prod_{k=1}^{n} d_1(u, u_k)B(u_k)D(u) \]

\[ + \sum_{k=1}^{n} d_2(u, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i)B(u)D(u_k) + \sum_{k=1}^{n} d_3(u, u_k) \prod_{i=1, i \neq k}^{n} a_1(u_k, u_i)B(u)A(u_k). \]  

(119)

These formulas can be proved through mathematical induction over \( n \), in a similar fashion as the periodic case (although the calculations are somewhat more cumbersome). In fact, for \( n = 1 \) this is trivially satisfied. Suppose then that the propositions (118) and (119) hold for general \( n \). Then, for \( n + 1 \) we have (after we use the commutative property of the creator operators in order to pass \( B(u_{n+1}) \) for the left side),

\[ A(u) \prod_{k=1}^{n+1} B(u_k) = a_1(u, u_{n+1})B(u_{n+1})A(u) \prod_{k=1}^{n} B(u_k) \]

\[ + a_2(u, u_{n+1})B(u)A(u_{n+1}) \prod_{k=1}^{n} B(u_k) + a_3(u, u_{n+1})B(u)D(u_{n+1}) \prod_{k=1}^{n} B(u_k), \]  

(120)

\[ D(u) \prod_{k=1}^{n+1} B(u_k) = d_1(u, u_{n+1})B(u_{n+1})D(u) \prod_{k=1}^{n} B(u_k) \]

\[ + d_2(u, u_{n+1})B(u)D(u_{n+1}) \prod_{k=1}^{n} B(u_k) + d_3(u, u_{n+1})B(u)A(u_{n+1}) \prod_{k=1}^{n} B(u_k), \]  

(121)

that is,

\[ A(u) \prod_{k=1}^{n+1} B(u_k) = a_1(u, u_{n+1})B(u_{n+1}) \prod_{k=1}^{n} a_1(u, u_k)B(u_k)A(u) \]

\[ + a_1(u, u_{n+1})B(u_{n+1}) \sum_{k=1}^{n} a_2(u, u_k) \prod_{i=1, i \neq k}^{n} a_1(u, u_i)B(u)A(u_k) \]

\[ + a_1(u, u_{n+1})B(u_{n+1}) \sum_{k=1}^{n} a_3(u, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i)B(u)D(u_k) \]

\[ + a_2(u, u_{n+1})B(u) \prod_{k=1}^{n} a_1(u, u_{n+1}, u_k)B(u_k)A(u_{n+1}) \]

\[ + a_2(u, u_{n+1})B(u) \sum_{k=1}^{n} a_2(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} a_1(u, u_{n+1}, u_i)B(u)A(u_{n+1}) \]

\[ + a_2(u, u_{n+1})B(u) \sum_{k=1}^{n} a_3(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} d_1(u, u_{n+1}, u_i)B(u_{n+1})D(u_k) \]
+ a_3(u, u_{n+1}) B(u) \prod_{k=1}^{n} d_1(u_{n+1}, u_k) B(u_k) D(u) \\
+ a_3(u, u_{n+1}) B(u) \sum_{k=1}^{n} d_2(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i) B(u_{n+1}) D(u_k) \\
+ a_3(u, u_{n+1}) B(u) \sum_{k=1}^{n} d_3(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} a_1(u_k, u_i) B(u_{n+1}) A(u_k) \quad (122)

and

\[ D(u) \prod_{k=1}^{n+1} B(u_k) = d_1(u, u_{n+1}) B(u_{n+1}) \prod_{k=1}^{n} d_1(u, u_k) B(u_k) D(u) \\
+ d_1(u, u_{n+1}) B(u_{n+1}) \sum_{k=1}^{n} d_2(u, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i) B(u) D(u_k) \\
+ d_1(u, u_{n+1}) B(u_{n+1}) \sum_{k=1}^{n} d_3(u, u_k) \prod_{i=1, i \neq k}^{n} a_1(u_k, u_i) B(u_{n+1}) A(u_k) \\
+ d_2(u, u_{n+1}) B(u) \prod_{k=1}^{n} d_1(u_{n+1}, u_k) B(u_k) D(u_{n+1}) \\
+ d_2(u, u_{n+1}) B(u) \sum_{k=1}^{n} d_2(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i) B(u_{n+1}) D(u_k) \\
+ d_2(u, u_{n+1}) B(u) \sum_{k=1}^{n} d_3(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} a_1(u_k, u_i) B(u_{n+1}) A(u_k) \\
+ d_3(u, u_{n+1}) B(u) \prod_{k=1}^{n} a_1(u_{n+1}, u_k) B(u_k) A(u) \\
+ d_3(u, u_{n+1}) B(u) \sum_{k=1}^{n} a_2(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} a_1(u_k, u_i) B(u_{n+1}) A(u_k) \\
+ d_3(u, u_{n+1}) B(u) \sum_{k=1}^{n} a_3(u_{n+1}, u_k) \prod_{i=1, i \neq k}^{n} d_1(u_k, u_i) B(u_{n+1}) D(u_k). \quad (123)

Now, we should realize that these expressions can be simplified if we make use of the following identities provided by the boundary Yang-Baxter equation:

\[ a_2(u, u_{n+1}) a_2(u_{n+1}, u_k) + a_3(u, u_{n+1}) d_3(u_{n+1}, u_k) = a_2(u, u_k) a_1(u_k, u_{n+1}), \quad (124) \]
\[ a_2(u, u_{n+1}) a_3(u_{n+1}, u_k) + a_3(u, u_{n+1}) d_2(u_{n+1}, u_k) = a_3(u, u_k) d_1(u_k, u_{n+1}), \quad (125) \]
\[ d_2(u, u_{n+1}) d_2(u_{n+1}, u_k) + d_3(u, u_{n+1}) a_1(u_k, u_{n+1}) = d_2(u, u_k) d_1(u_k, u_{n+1}), \quad (126) \]
\[ d_2(u, u_{n+1}) d_3(u_{n+1}, u_k) + d_3(u, u_{n+1}) a_2(u_{n+1}, u_k) = d_3(u, u_k) a_1(u_k, u_{n+1}), \quad (127) \]

which hold for any \( k \) from 1 to \( n \). Then, after we extend the summations to \( n+1 \), we will be able to group the fifth and the ninth terms in (122) and (123) with the second one and also the sixth and the eighth terms with the third. With this, we shall obtain the same expressions (118) and (119) but with \( n+1 \) replacing \( n \), so that we are done.

Finally, the action of the transfer matrix on the \( n \)th excited state follows after we multiply (118) by \( \kappa_{11}^{+1}(u) \), (119) by \( \kappa_{22}^{+1}(u) \) and take the sum of these two terms. This provides us with an expression quite analogous to that of the periodic case, namely,

\[ T(u) \Psi_n (u_1, \ldots, u_n) = \tau_n (u|u_1, \ldots, u_n) \Psi_n (u_1, \ldots, u_n) + \sum_{k=1}^{n} \beta_n^k (u|u_1, \ldots, u_n) \Psi_n (u_k^X). \quad (128) \]

Therefore, the requirement that the transfer matrix satisfies an eigenvalue equation means that all the unwanted
terms in (128) should vanish. This leads us to the eigenvalues,

\[
\tau_n(u|u_1, \ldots, u_n) = \kappa_{1,1}^+(u) a(u) \prod_{k=1}^n a_1(u, u_k) + \kappa_{2,2}^+(u) d(u) \prod_{k=1}^n d_1(u, u_k),
\]

and to the boundary Bethe Ansatz equations,

\[
\beta_n^k(u|u_1, \ldots, u_n) = [a_2(u, u_k) \kappa_{1,1}^+(u) + d_3(u, u_k) \kappa_{2,2}^+(u)] a(u_k) \prod_{i=1, i \neq k}^n a_1(u_k, u_i) \\
+ [d_2(u, u_k) \kappa_{2,2}^+(u) + a_3(u, u_k) \kappa_{1,1}^+(u)] d(u_k) \prod_{i=1, i \neq k}^n d_1(u_k, u_i) = 0, \quad 1 \leq k \leq n.
\]

7. The solution of the Bethe Ansatz equations

A completely exact solution for the spectral problem would require an analytical solution of the Bethe Ansatz equations (130). Unfortunately, this is not possible up to date due to the high complexity of these equations. Therefore, it only remains to resign ourselves with the numerical approximations available.

C. The combinatorial approach

Now let us see how the combinatorial approach can be applied as well for the case where non-periodic boundary conditions take place. Following the same approach as in the periodic case, we can represent the commutation relations (106) and (107) by the following combinatorial diagrams⁵:

\[
\begin{array}{c}
\text{A} \\
\begin{array}{c}
\circ u \\
\circ v \\
\square v
\end{array}
\end{array}
\quad \text{and} \quad \\
\begin{array}{c}
\text{D} \\
\begin{array}{c}
\circ u \\
\circ v \\
\square v
\end{array}
\end{array}
\]

Notice, however, that in the non-periodic case the A diagram has two circle nodes and one additional square node, as well as the D diagram has two square nodes and one circle node. This distinction is necessary because the third term in the commutation relations (106) and (107) will lead to an interchanging between the diagonal operators A and D when those commutation relations are applied repeatedly. In fact, the combinatorial tree representing the action of the A operator on the nth excited state \(\Psi_n\) will have the form:

\[
\begin{array}{c}
\text{A} \\
\begin{array}{c}
\circ u_0 \\
\circ u_1 \\
\circ u_2 \\
\circ u_3 \\
\circ u_4 \\
\circ \ldots \\
\circ u_{n-1} \\
\circ u_n
\end{array}
\end{array}
\]

and a similar diagram can be associated with the D operator – the only difference being that the circle nodes should be replaced by square nodes and vice-versa.

In the diagram (132) we adopted the convention that a circle node means that the commutation relation between the operators A and B is to be used at that point further, while a square node means that the commutation relation between D and B is to be used at that point further. Since each term resulting from the action of the diagonal operators on the nth excited state \(\Psi_n\) requires the use of the commutation relations \(n\) times, this will give place to a pruned ternary combinatorial trees of length \(n\). The labels on the side of every node in the diagram (132) refer to the

---

⁵ The diagrams regarding the non-shifted commutation relations (97) and (98) can also be drawn and the same analysis presented in this section can be performed with them as well. We remark, however, that in this case the diagrams representing the \(A\) and \(D\) operators will no longer be symmetrical, which makes the analysis somewhat more complicated.
argument that the $A$ operator would have if we had used the commutation relations algebraically (the same can be said about the $D$ diagram regarding the argument of the $D$ operator, of course). These labels can be obtained from the same rule as in the periodic case, namely,

- Hollow nodes inherit the label of his parent, while the label of any filled node on the level $k$ of a given path $P$ is just $\lambda_k^P = u_k$ (the label of the root being $u_0$).

Notice that only the color (hollow or filled) of the nodes, not the type (circle or square), determines the labels. Hence these labels can be obtained recursively as follows:

$$
\lambda_0^P = u_0, \quad \text{and} \quad \lambda_k^P = \begin{cases} 
\lambda_{k-1}^P, & \eta_k = 1, \\
\, u_k, & \text{otherwise},
\end{cases} \quad \text{for } 1 \leq k \leq n. 
$$

Any path $P$ of the diagram (132) can be uniquely specified by the set $\eta = (\eta_1, \ldots, \eta_n)$, where now the parameters $\eta_k$ ($1 \leq k \leq n$) can assume the values 1, 2 and 3. To any path $P$ we associate a weight $W(P)$ and a state $|P\rangle$, so that its value $V(P)$ — that represents the respective term obtained from the boundary algebraic Bethe Ansatz — can be written as $V(P) = W(P) |P\rangle$. The weight of a given path can be defined as the product of the weights of its nodes through the rules:

1. If the parent of a node in the level $k$ of a given path $P$ is a circle, then its weight equals $a_{\eta_k}(\lambda_{k-1}^P, u_k)$;
2. If the parent of a node in the level $k$ of a given path $P$ is a square, then its weight equals $d_{\eta_k}(\lambda_{k-1}^P, u_k)$;
3. Every leaf node in the path $P_A [P_D]$ of the $A$ [$D$] diagram also contributes with a factor $\kappa_{1,1}^+(u_0) a(\lambda_n^P) \Psi_0$ [$\kappa_{2,2}^+(u_0) a(\lambda_n^P) \Psi_0$] or $\kappa_{1,1}^+ (u_0) d(\lambda_n^P) \Psi_0$ [$\kappa_{2,2}^+ (u_0) d(\lambda_n^P) \Psi_0$] depending on whether this leaf node is a circle or a square, respectively, with the root defined as a circle for the $A$ diagram and as a square for the $D$ diagram. Therefore, the weight of any path belonging to the $A$ and $D$ diagrams can be written, respectively, as

$$
W [P_A (\eta_1, \ldots, \eta_n)] = \kappa_{1,1}^+ (u_0) a(\lambda_n^P) \prod_{k=1}^n b_{\eta_k}^{\eta_k-1} (\lambda_{k-1}^P, u_k),
$$

$$
W [P_D (\eta_1, \ldots, \eta_n)] = \kappa_{2,2}^+ (u_0) d(\lambda_n^P) \prod_{k=1}^n c_{\eta_k}^{\eta_k-1} (\lambda_{k-1}^P, u_k),
$$

where,

$$
b_{\eta_k}^{\eta_k-1} = \begin{cases} 
a_{\eta_k}, & \eta_{k-1} = 1, 2, \\
d_{\eta_k}, & \text{otherwise},
\end{cases} \quad c_{\eta_k}^{\eta_k-1} = \begin{cases} 
d_{\eta_k}, & \eta_{k-1} = 1, 2, \\
a_{\eta_k}, & \text{otherwise},
\end{cases}
$$

Moreover, the state associated with the paths are also given by the same rule as in the periodic case, namely,

- The state $| P^{(u_k)} \rangle$ associated with any path $P$ whose the label of its leaf node is $\lambda_n^P = u_k$ is given by,

$$
| P^{(u_k)} \rangle = \prod_{j=0, j \neq k}^n B(u_j) \Psi_0.
$$

As an example, we also present the paths associated with the $A$ diagram and the corresponding mathematical expressions for the second excited state ($n = 2$):

$$
\begin{array}{l}
\begin{aligned}
\includegraphics[width=0.3\textwidth]{path_a1} & \equiv \kappa_{1,1}^+ (u_0) a_1 (u_0, u_1) a_1 (u_0, u_2) \alpha (u_0) B(u_1) B(u_2) \Psi_0, \\
\includegraphics[width=0.3\textwidth]{path_a2} & \equiv \kappa_{1,1}^+ (u_0) a_2 (u_0, u_1) a_2 (u_0, u_2) \alpha (u_0) B(u_1) B(u_2) \Psi_0,
\end{aligned}
\end{array}
$$

The paths associated with the $D$ diagram are similar to the above ones, but we need to replace $\kappa_{1,1}^+$ by $\kappa_{2,2}^+$, $\alpha$ by $\delta$ (and vice-versa) and the coefficients $a_1, a_2$ should be replaced by $d_1, d_2$, respectively (and vice-versa).
Therefore, from the values associated with these paths, we get that
\[ \tau_n (u_0 | u_1, \ldots, u_n) \Psi_n (u_1, \ldots, u_n) = A - \circ + D - \circ \]
that is,
\[ \tau_n (u_0 | u_1, \ldots, u_n) = \kappa_{1,1}^+ (u_0) a(u_0) \prod_{k=1}^{n} a_1 (u_0, u_k) + \kappa_{2,2}^+ (u_0) d(u_0) \prod_{k=1}^{n} d_1 (u_0, u_k), \]
which agrees with (129).

For the boundary Bethe Ansatz equations, we have,

1. The eigenvalues are determined by the sum of all paths of the diagrams \( A \) and \( D \) ending with the label \( u_0 \).
2. The simplest Bethe Ansatz equation is that one fixing the rapidity \( u_k \) is determined by the sum of all paths of the \( A \) and \( D \) diagrams that end with the label \( u_k \).

Therefore, from the values associated with these paths, we get that,
\[
\beta_n^k (u_0 | u_1, \ldots, u_n) = a(u_k) \prod_{j=k+1}^{n} a_1 (u_k, u_j) \times \\
\sum_{\eta_1, \ldots, \eta_{k-1} = 1}^{3} \prod_{i=1}^{k-1} b_{\eta_i}^{n-i} (\lambda_{i-1}^P, u_i) a_2 (\lambda_{k-1}^P, u_k) \kappa_{1,1}^+ (u_0) + \prod_{i=1}^{k-1} c_{\eta_i}^{n-i} (\lambda_{i-1}^P, u_i) d_3 (\lambda_{k-1}^P, u_k) \kappa_{2,2}^+ (u_0) \\
+ d(u_k) \prod_{j=k+1}^{n} d_1 (u_k, u_j) \times \\
\sum_{\eta_1, \ldots, \eta_{k-1} = 1}^{3} \prod_{i=1}^{k-1} c_{\eta_i}^{n-i} (\lambda_{i-1}^P, u_i) d_2 (\lambda_{k-1}^P, u_k) \kappa_{2,2}^+ (u_0) + \prod_{i=1}^{k-1} b_{\eta_i}^{n-i} (\lambda_{i-1}^P, u_i) a_3 (\lambda_{k-1}^P, u_k) \kappa_{1,1}^+ (u_0) ,
\]

1 \leq k \leq n.

The simplest Bethe Ansatz equation is that one fixing the rapidity \( u_1 \). Differently from the periodic case, we have here two paths ending with the label \( u_1 \) in each diagram – namely, the paths \( P (2, 1, \ldots, 1) \) and \( P (3, 1, \ldots, 1) \). Notice
moreover that the leaf node of the path \( P_A (2, 1, \ldots, 1) \) is a circle, while the leaf node of \( P_A (3, 1, \ldots, 1) \) is a square (similarly, the leaf nodes in the paths \( P_D (2, 1, \ldots, 1) \) and \( P_D (3, 1, \ldots, 1) \) are respectively a square and a circle). From this we get that

\[
\beta^1_n (u_0 | u_1, \ldots, u_n) \Psi_n \left( u_j^\pm \right) = [A] + [D]
\]

that is,

\[
\beta^1_n (u_0 | u_1, \ldots, u_n) = a_2 (u_0, u_1) \prod_{i=2}^n a_1 (u_1, u_i) \kappa^+_{1,1} (u_0) a (u_1) + a_3 (u_0, u_1) \prod_{i=2}^n d_1 (u_1, u_i) \kappa^+_{1,1} (u_0) \vartheta (u_1) + d_2 (u_0, u_1) \prod_{i=2}^n d_1 (u_1, u_i) \kappa^+_{2,2} (u_0) \vartheta (u_1) + d_3 (u_0, u_1) \prod_{i=2}^n a_1 (u_1, u_i) \kappa^+_{2,2} (u_0) a (u_1),
\]

or, grouping the common terms,

\[
\beta^1_n (u_0 | u_1, \ldots, u_n) = [a_2 (u_0, u_1) \kappa^+_{1,1} (u_0) + d_3 (u_0, u_1) \kappa^+_{2,2} (u_0)] a (u_1) \prod_{i=2}^n a_1 (u_1, u_i) + [d_2 (u_0, u_1) \kappa^+_{2,2} (u_0) + a_3 (u_0, u_1) \kappa^+_{1,1} (u_0)] \vartheta (u_1) \prod_{i=2}^n d_1 (u_1, u_i) = 0.
\]

Finally, thanks to the symmetry of the excited states regarding the permutation of the rapidities, the other Bethe Ansatz equations given at (141) can be simplified, as we did in the periodic case. In fact, (108) implies that both the \( \mathcal{A} \) as well as the \( \mathcal{D} \) diagrams must be symmetric with respect to the permutation of their labels. From this we conclude that the Bethe Ansatz equation fixing the rapidity \( u_k \) can actually be written in the same form as that fixing \( u_1 \), except that the rapidities \( u_k \) and \( u_1 \) must be permuted. That is, likewise in the periodic case, we have the rule,

\[
\diamond \text{ The Bethe Ansatz equation fixing the rapidity } u_k \text{ (} 2 \leq k \leq n \text{) can be obtained by the same paths that determine the Bethe Ansatz equation fixing } u_1, \text{ provided that the labels } u_1 \text{ and } u_k \text{ are permuted.}
\]

This leads us to the final form of the Bethe Ansatz equations for the six-vertex model with boundaries:

\[
\beta^k_n (u_0 | u_1, \ldots, u_n) = [a_2 (u_0, u_k) \kappa^+_{1,1} (u_0) + d_3 (u_0, u_k) \kappa^+_{2,2} (u_0)] a (u_k) \prod_{i=1, i \neq k}^n a_1 (u_k, u_i) + [d_2 (u_0, u_k) \kappa^+_{2,2} (u_0) + a_3 (u_0, u_k) \kappa^+_{1,1} (u_0)] \vartheta (u_k) \prod_{i=1, i \neq k}^n d_1 (u_k, u_i) = 0,
\]

which is in agreement with (130).

Finally, the comparison between (141) and (145) provides us as well with other more intricate identities:

\[
\sum_{\eta_1, \ldots, \eta_{k-1}=1}^3 \prod_{i=1}^{k-1} b^{\eta_{i-1}}_{\eta_i} (\lambda^P_{i-1}, u_i) a_2 (\lambda^P_{i-1}, u_k) = a_2 (u_0, u_k) \prod_{i=1}^{k-1} a_1 (u_k, u_i),
\]

\[
\sum_{\eta_1, \ldots, \eta_{k-1}=1}^3 \prod_{i=1}^{k-1} b^{\eta_{i-1}}_{\eta_i} (\lambda^P_{i-1}, u_i) a_3 (\lambda^P_{i-1}, u_k) = a_3 (u_0, u_k) \prod_{i=1}^{k-1} d_1 (u_k, u_i),
\]

\[
\sum_{\eta_1, \ldots, \eta_{k-1}=1}^3 \prod_{i=1}^{k-1} c^{\eta_{i-1}}_{\eta_i} (\lambda^P_{i-1}, u_i) d_2 (\lambda^P_{i-1}, u_k) = d_2 (u_0, u_k) \prod_{i=1}^{k-1} d_1 (u_k, u_i),
\]

\[
\sum_{\eta_1, \ldots, \eta_{k-1}=1}^3 \prod_{i=1}^{k-1} c^{\eta_{i-1}}_{\eta_i} (\lambda^P_{i-1}, u_i) d_3 (\lambda^P_{i-1}, u_k) = d_3 (u_0, u_k) \prod_{i=1}^{k-1} a_1 (u_k, u_i).
\]
IV. CONCLUSION

In this paper we presented a combinatorial approach for the (periodic and non-periodic) algebraic Bethe Ansatz in which the commutation relations are represented by combinatorial diagrams, so that the action the diagonal operators on the nth excited states becomes described by labeled combinatorial trees. From the analysis of these combinatorial diagrams, every term resulting from the algebraic Bethe Ansatz can be easily recovered following simple rules. In special, the eigenvalues are obtained in a straightforward way, namely, they are provided by those paths containing only hollow nodes. The Bethe Ansatz equations also can be found directly from this analysis: we can first obtain the Bethe Ansatz Ansatz equation fixing the rapidity \( u_1 \), which is provided by the paths containing a filled node in the first level and no other filled node beyond that, and then, thanks to the symmetry of the diagrams regarding the permutation of their labels, we get that the Bethe Ansatz equations fixing the rapidity \( u_k \) will have the same form of the previous one, except that the rapidities \( u_1 \) and \( u_k \) should be permuted. This symmetry also provides several mathematical intricate identities.

As a first generalization of this combinatorial approach could be the analysis of the fifteen and nineteen vertex models. These cases are interesting because the creator operators do not commute among themselves anymore, which leads to a more elaborate form for the excited states. We expect nevertheless that these excited states can also be determined through the analysis of some combinatorial trees, for instance, imposing the symmetry requirements regarding the permutation of their labels. In this way, the present analysis could provide the eigenvalues and the eigenstates of the transfer matrix, and also the corresponding Bethe Ansatz equations, in a more direct way. Besides, perhaps this combinatorial method can be useful as well in the analysis of scalar products and correlation functions of integrable models.

Finally, since we also presented a comprehensive introduction to the algebraic Bethe Ansatz of the six-vertex model – for both periodic and non-periodic boundary conditions –, we believe that this work can also be useful for teaching the algebraic Bethe Ansatz for introductory audiences.

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