Behaviour of torsional Alfvén waves and field line resonance on rotating magnetars

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ABSTRACT

Torsional Alfvén waves are likely excited with bursts in rotating magnetars. These waves are probably propagated through corotating atmospheres toward a vacuum exterior. We have studied the physical effects of the azimuthal wavenumber and the characteristic height of the plasma medium on wave transmission. In this work, explicit calculations were carried out based on the three-layered cylindrical model. We found that the coupling strength between the internal shear and the external Alfvén modes is drastically enhanced, when resonance occurs in the corotating plasma cavity. The spatial structure of the electromagnetic fields in the resonance cavity is also investigated when Alfvén waves exhibit resonance.

Key words: stars: magnetic fields – stars: neutron – gamma-rays: bursts.

1 INTRODUCTION

Soft gamma-ray repeaters (SGRs) and anomalous X-ray pulsars (AXPs) are known as strongly magnetized neutron stars, ‘magnetars’. The hallmark of these objects is to repeat X-ray or gamma-ray emissions irregularly. So far, four or five known active objects which show frequent X-ray emission of tremendously energetic and shorter initial bursts (typically $E \lesssim 10^{41}$ erg and $\Delta t \sim 0.5$ s) have been identified with SGRs (Thompson & Duncan 2001). Some of these occasionally exhibit more energetic events. The giant flare 1900+14, now associated with SGR 0525−66, was observed in 1979 (Mazets, Golenetskii & Gur'yan 1979b) for the first time and became active again in 1992 (Kouveliotou et al. 1993) and also in 1998 (Kouveliotou et al. 1998; Hurley et al. 1999a, 1999b). Recently, the intense gamma-ray flare from SGR 1806−20, which occurred on 2004 December 27, was reported (Hurley et al. 2005; Mereghetti et al. 2005; Palmer et al. 2005). Rea et al. (2005) have investigated the pulse profile and flare spectrum of SGR 1806−20. Their study may potentially give information about the global field structure in the magnetosphere and may further promote theoretical magnetar models. Their extreme peak luminosity extends up to $10^6 \dot{L}_{\text{Edd}}$ estimated at a distance of 10 kpc (Mazets et al. 1999; Feroci, Duncan & Thompson 2001). Typical X-ray luminosities of SGRs have been measured to be $L_x = 10^{34}–10^{36}$ erg s$^{-1}$ (Hurley et al. 2000; Thompson et al. 2000), except with giant bursts SGRs 1900+14 and 1806−20. Shorter durations of initial flares of SGRs are comparable to the Alfvén crossing time of the core. The energy distribution shows good agreement with the Gutenberg–Lichter law, which may indicate statistical similarity to earthquakes or solar flares (Cheng et al. 1996; Gögus et al. 1999, 2000). The SGRs have spin periods in a small range of $P = 5–8$ s with rapid spin-down rate $\dot{P} \simeq 10^{-10}$ ss$^{-1}$, which therefore give characteristic ages $P/\dot{P} \sim 10^5$ yr (Mazets et al. 1999a; Kouveliotou et al. 1998; Hurley et al. 1999a).

On the other hand, the physical nature of AXPs is still uncertain due to poor observations, but there seem to be some similarities and differences between SGRs and AXPs. AXPs are energetic sources of pulsed X-ray emission, whose periods lie in a narrow range $P = 6–12$ s, characteristic ages $P/\dot{P} = 3 \times 10^5 \sim 4 \times 10^5$ yr and X-ray luminosities $L_x = 5 \times 10^{32}–10^{36}$ erg s$^{-1}$ (Mereghetti 2002; Thompson & Duncan 2001). AXP sources likely have somewhat larger active ages and some of them have softer X-ray spectra compared with SGRs. One of the primary differences between them will be that AXPs so far have shown only quiescent X-ray emission with no bright active bursts such as giant flares. The active ages of SGRs and AXPs are consistent with the observed evidence that these compact objects often give their location close to the edge of shell-type supernova remnants. Neither SGRs nor AXPs show the presence of conspicuous counterparts at other wavelengths (Mereghetti & Stella 1995; Mereghetti et al. 2002). The origin of these enormous energies is likely to be their strong magnetic fields $B = 10^{14}–10^{15}$ G estimated by their spin periods and spin-down rates. Only the magnetar model has been able to account for the enigmatic properties of a rare class of SGRs or AXPs.

Shear and Alfvénic waves play an essential role on the energy transfer to the exterior in the burst-like phenomena observed in SGRs or giant flares. The Alfvén wave propagation in such a strongly magnetized star should be clarified theoretically. In our previous work, the propagation and transmission of torsional Alfvén waves have been studied, focused only on the fundamental mode of azimuthal wavenumber $m = 1$ as a first step. In that work, the exterior of the star is examined qualitatively only, assuming two extreme cases: (i) corotating together with the star and (ii) static state independent
of the stellar rotation. The former will come true when the plasma gas is trapped by closed magnetic field lines, and the latter will be realized without any other force. In any case, it might be inadequate at least in the following two points to place such strong constraints on the wave mode and on ambient circumstances of the star in the previous paper. First, under more realistic situations various modes with azimuthal wavenumber \( m \gg 1 \) are probably triggered by starquake. Secondly, it is natural to assume that a scaleheight of the plasma gas is neither \( L = 0 \) nor \( L \to \infty \), but has a finite value of \( L \). Huang, Dai & Lu (1998) pointed out that a relativistic fireball like those in classical GRBs may exist in SGRs. Recently, Thompson & Duncan (2001) have also suggested that after the initial hard spike emission, some lumps of hot plasma gas involving electron–positron pairs and high-energy photons, that is, a fireball, would be created on closed magnetic field lines. In fact, the fast decline and complete evaporation on X-ray light curve observed in the August 27 burst provides clear evidence of the trapped fireball. Motivated by this, we thus extend our study to include some plasma gas spreading over the stellar surface.

The main aim of this paper is to study the physical behaviour of the torsional Alfvén waves with azimuthal wavenumber \( m \) in the presence of corotating plasma confined within a certain finite distance \( L \), and then to investigate the effects of these quantities \( m \) and \( L \) on the wave propagation and transmission. In Section 2, our model is self-consistently constructed and the relevant basic equations are formulated. These equations are almost the same as those derived in Kojima & Okita (2004), but are summarized here for the paper to be self-contained. In Section 3, we derive the dispersion relation in a Wentzel–Kramers–Brillouin (WKB) way to show that the rotating plasma atmosphere plays a crucial role as a resonant cavity of the wave when a certain condition holds. In Section 4, transmission rates of the Alfvén waves are numerically calculated. In Section 5, their electromagnetic field structure is also discussed with the numerical results. In Section 6, we give a brief summary of our findings and their implications for the torsional Alfvén waves on rotating magnetars.

2 ELECTROMAGNETICS ON ROTATING MAGNETARS

2.1 Model

Both the magnetic field and the rotation of a star lead to a complicated geometrical configuration. In this paper we assume some simplified conditions to understand the physical processes of the wave propagation. We here consider a three-layered cylindrical model with radius \( \sigma_{pc} \). It is composed of the neutron star crust between \( z = -q \) and \( z = 0 \) (region 1), corotating plasma above the stellar surface between \( z = 0 \) and \( z = L \) (region 2) and static pure vacuum at \( z > L \) (region 3) as shown in Fig. 1. The plasma gas filled in the atmosphere corotates with the crust at the same angular velocity \( \Omega \). Local magnetic fields permeate uniformly in each layer and point along the \( z \)-direction, \( B = B_{r} \hat{e}_{r} \), which represents open magnetic fields extending to infinity.

Alfvén waves excited at the bottom of the crust \( q \approx 10^{5} \) cm, owing to some mechanisms, travel upward along the local magnetic field lines. They are partially reflected and transmitted at the boundaries \( z = 0 \) and \( z = L \), where the physical property of Alfvén waves significantly changes because of the effect of the background rotation, as shown below. This model, by setting \( L \to \infty \), reduces to the case in which the plasma gas extends infinitely to the exterior of the star, while setting \( L \to 0 \) reduces to the simple case in which the exterior is filled with pure vacuum only. Up to the present time, we have no observational information of the plasma size \( L \). In this work, \( L \) is regarded as a free parameter in order to investigate its effect on the wave propagation and transmission.

We now give some comments on the validity of this model by comparing the physical sizes \( \sigma_{pc}, q \) and \( L \) with the stellar radius \( R \).

In this model we have explicitly assumed that the local magnetic field has a \( z \)-component only. Our model can be applied to the polar cap region, whose cylindrical radius \( \sigma_{pc} \) is given by \( \sigma_{pc} = R \sin \theta_{pc} \approx 10^{4}(T/1 s)^{-1/2} \text{cm} \ll R = 10^{6} \text{cm} \). Curvature of the stellar surface and the magnetic field lines may be neglected within the polar cap region. In a similar way, it may be valid to assume that the local magnetic field lines are uniform if the thickness of each layer is smaller than the star radius, \( L, q \ll R \). We can also extend our model to the extreme case \( L \sim R \). Even in this case, the validity of this model nevertheless holds good near the \( z \)-axis. In the remainder of this paper we restrict our explicit calculations only to the axially symmetric small region within the polar cap radius, unless otherwise stated.

2.2 Linear perturbation

In this section, we consider the propagation of torsional shear-Alfvén waves with various azimuthal modes on rotating magnetars. Such waves are probably excited by the turbulent motion of the starquake in the deep crust, but above the neutron drip \(( z > z_{nd} \approx 10^{-5} \text{cm}) \). Many proposals for the starquake model have been put forward (e.g. Pacini & Ruderman 1974), but all of them are generally argued only for weakly magnetized neutron stars with \( 10^{11}–10^{12} \text{G} \). Therefore, their treatments are inadequate for magnetars, as they are. However, an analogous mechanism may also occur in magnetars. In this paper, we do not discuss the triggering mechanism of the starquake itself, but focus only on the process by which electromagnetic shear waves, once generated, are propagated and transmitted from the deep crust, through a magnetized plasma, toward the vacuum exterior.

We assume that the horizontal Lagrange displacement \( \xi = (\xi_{\theta}, \xi_{\varphi}, 0) \) is suddenly shaken in the deep interior at depth \( q \approx 10^{5} \text{cm} \); nevertheless, the matter remains immobile in the vertical direction.
due to strong gravity of the neutron star, \( g \sim 10^{14} \text{ cm s}^{-2} \). Therefore, the following transverse wave condition may be easily satisfied:

\[
\nabla \cdot \mathbf{\xi} = \left( \frac{1}{m} \frac{\partial}{\partial \sigma} \left( \sigma \xi_{\sigma} \right) \right) + 1 \frac{\partial \hat{e}_\phi}{\partial \phi} = 0.
\]

As one of the simplest solutions, wave Ansatz under this condition can be written formally as

\[
\xi(\sigma, \phi, z, t) = (e_{\sigma} \pm i e_{\phi}) \left( \frac{\sigma}{\sigma_{pc}} \right)^{m-1} \xi_{\pm, \alpha, \omega}(\zeta) e^{-i(\omega t + m\phi)}.
\]

\((m = 1, 2, 3 \ldots)\).

Equations (11) and (12) imply that if stellar rotation can be completely ignored, then \( \delta E, \delta B \) and the propagation vector \( k = k e_\sigma \) form a mutually orthogonal set of vectors, say, transverse electromagnetic modes (TEM). However, in the presence of rotation, such an orthogonality breaks down and thus longitudinal modes of the perturbed electric fields and currents are excited [transverse magnetic (TM) modes]. Other perturbed quantities related to the displacement are calculated by using above the results (11) and (12):

\[
\delta j = \frac{c}{4\pi} \nabla \times \delta B - \frac{1}{4\pi} \frac{\partial}{\partial \phi} \delta E
\]

\[
\approx \frac{e}{4\pi} B_0 \left( \frac{\sigma}{\sigma_{pc}} \right)^{m-1} \left\{ \left( \frac{\partial^2 \xi_{\pm}}{\partial z^2} + \frac{\omega (\omega \pm m \Omega)}{c^2} \xi_{\pm} \right) \right. 

\times \left( e_{\sigma} \pm i e_{\phi} \right) \frac{\omega \sigma}{c^2} \left[ \frac{d\xi_{\pm}}{dz} e_\sigma \right] e^{-i(\omega t + m\phi)}.
\]

\((13)\)

\[
\delta \rho_e = \frac{1}{4\pi} \nabla \cdot \delta E
\]

\[
\approx \frac{B_0 \omega}{4\pi c} \left( \frac{\sigma}{\sigma_{pc}} \right)^{m-1} \frac{\partial^2 \xi_{\pm}}{\partial z^2} e^{-i(\omega t + m\phi)}.
\]

\((14)\)

The linearized equation of motion for uniformly rotating background from a static equilibrium state governing the torsional Alfvén waves is now given by

\[
\rho \delta \delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \delta \mathbf{v} + (\delta \mathbf{v} \cdot \nabla) \mathbf{v}
\]

\[
= \delta \mathbf{S}_i + \delta F_i + (g \delta \rho) \nabla \times (\nabla \delta \rho), \quad (i = \sigma, \phi, z).
\]

\((15)\)

The profile of the mass density \( \rho \) in the crust is obtained by solving the equation of state for degenerate electrons as follows (Blaes et al. 1989)

\[
\rho = \frac{\mu \rho_n a^{8/3}}{3 \pi c^2 h^3} \left( \frac{g^2 \mu \rho_n a}{c^2} + 2 \mu g \rho_n \right)^{3/2}
\]

\[
\approx 8 \times 10^{10} \left( \frac{|z|}{\text{cm}} \right)^3 + 2.5 \times 10^{-4} \left( \frac{|z|}{\text{cm}} \right)^2 \text{ g cm}^{-3}.
\]

\((16)\)

where \( \mu \) is an atomic mass unit, \( \mu \) is the mean molecular weight per electron and the other symbols have their usual physical meanings.

The first term on the right-hand side of equation (15), \( \delta S_i \), is a perturbed elastic stress tensor associated with distorted matter in the crust and can be written in terms of \( \xi \)

\[
\delta S_i = \nabla_j \left[ \left( \frac{\xi_j - 2 \mu \xi_j}{3} \right) \delta_{ij} \nabla \cdot \mathbf{\xi} \right] + \nabla_j \left( \mu \left( \frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right) \right)
\]

\((17)\)

where \( \xi \) is a bulk modulus, \( \mu \) is a shear modulus given by

\[
\mu = 0.295 Z^2 \sigma_{pc}^{3/2}
\]

\[
\approx 4.8 \times 10^{77} \left( \frac{\rho}{10^{11} \text{ g cm}^{-3}} \right)^{4/3} \text{ erg cm}^{-3},
\]

\((18)\)

with the ion number density \( n_i = \rho / Z \mu \sigma_{pc} \), and \( \delta_{ij} \) is the Kronecker delta. In the following calculations, \( Z = 32 \) will be adopted as a typical value in the crust. If one considers the complete transverse oscillation mode, the first term of equation (17) vanishes owing to \( \nabla \cdot \mathbf{\xi} = 0 \).

The second term on the right-hand side of equation (15) can be resolved into perturbations of Lorentz and Coulomb forces

\[
\delta F = \delta \rho_e \mathbf{E} + \rho_e \delta \mathbf{E} + \frac{1}{c} \left( \delta \mathbf{j} \times \mathbf{B} + j \times \delta \mathbf{B} \right)
\]

\[
\approx \frac{1}{c} \left( \delta \mathbf{j} - \rho_e \delta \mathbf{v} \right) \times \mathbf{B}.
\]

\((19)\)
Here we used equation (9) and unperturbed current \( \mathbf{j} = \rho_e \mathbf{v} \) induced by the background rotation. We further dropped the term \( \delta \mathbf{B} \times \mathbf{E} \propto (\Omega \mathbf{\omega} / c^2) \), which is small near the \( z \)-axis or within the actual stellar radius \( R \). Eliminating \( \delta \mathbf{J} \) with the help of equation (8), the electromagnetic force reduces to

\[
\delta \mathbf{F} \simeq \frac{1}{4\pi} (\mathbf{B} \cdot \nabla)^2 \xi - \frac{B_0^2}{4\pi c^2} \partial_v \delta \mathbf{v}.
\]  
\( 20 \)

The first term in equation (20) implies the tension of the perturbed magnetic field, while the second term shows the magnetic pressure generated by the distorted matter.

The last two terms of equation (15) represent gravitational force (\( g \delta \rho \)), and pressure gradient (\( \nabla \delta p \)), respectively. However, unless one takes the p- or f-mode such as compressional waves and/or sonic waves into consideration, one can ignore them for simplicity in this model.

### 2.3 Wave equation

We now derive the wave equation in region (1) shown in Fig. 1. Substituting equations (17) and (20) into equation of motion (15), we can obtain the following differential wave equation in terms of the displacement \( \xi^{(1)}_{\pm} \):

\[
\frac{d^2 \xi^{(1)}_{\pm}}{dz^2} + \frac{1}{\tilde{\mu}} \frac{1}{\mu} \frac{d\tilde{\mu}}{dz} - \frac{\tilde{\rho}}{\rho} \frac{d\rho}{dz} \times \left[ \sigma_{\pm} \pm \left\{ m(1 - h) + 2 \right\} \Omega \right] \xi^{(1)}_{\pm} = 0,
\]  
\( 21 \)

where \( \tilde{\mu} \) and \( \tilde{\rho} \) denote the effective shear modulus and the effective mass density defined as

\[
\tilde{\mu} = \mu + \frac{B_0^2}{4\pi},
\]  
\( 22 \)

\[
\tilde{\rho} = \rho + \frac{B_0^2}{4\pi c^2}.
\]  
\( 23 \)

The ratio of these quantities gives the shear-Alfvén wave speed in the crust \( v = \sqrt{\tilde{\mu}/\tilde{\rho}} \). In the above expression, the frequency \( \sigma_{\pm} \equiv \omega \mp m \Omega \) measured in the corotating frame for each helicity state has been introduced. We can limit this frequency to the positive regime \( \sigma_{\pm} > 0 \) for the symmetry \( \xi_{\pm m \rightarrow -m} = \xi_{\pm m \rightarrow +m} \). The dimensionless function \( h \) in equation (21) is formally defined as

\[
h \equiv \frac{4\pi \rho c^2}{4\pi c^2 + B_0^2},
\]  
\( 24 \)

which has a great influence on the dispersion relation of the wave especially with large \( m \) not only in the inner surface, but also in the rotating plasma cloud, as discussed in the following section. Note that if one considers the static background (\( \Omega = 0 \)), wave equation (21) coincides with that already derived by Blaes et al. (1989).

We now look for a WKB solution. In the deep interior, the solution of wave equation (21) can be well asymptotically given by

\[
\xi^{\text{asym}}_{\pm} \approx |z|^\beta \left[ A_{\pm} \exp\left\{ -i \left[ \psi_{\pm}(z) + \omega t \right] \right\} + B_{\pm} \exp\left\{ \left[ \psi_{\pm}(z) - \omega t \right] \right\} \right],
\]  
\( 25 \)

with \( \beta = -7/4 \). Here \( \psi_{\pm}(z) \) denotes eikonals defined as

\[
\psi_{\pm}(z) \equiv \int_{-\infty}^z dz' \frac{\sqrt{\sigma_{\pm} \pm \left\{ m(1 - h) + 2 \right\} \Omega}}{v},
\]  
\( 26 \)

for each mode. The first and second terms in equation (25) represent the upward-propagating Alfvén wave with a complex incident amplitude \( A_{\pm} \) and a downward-propagating wave with a complex reflection amplitude \( B_{\pm} \) bounced at the stellar surface, respectively.

We now turn to the wave behaviour in region (2) (0 < \( z < L \)). Mass density in this region is so small that one can formally take the limit \( h \to 0 \). Thus equation (21) reduces to

\[
\frac{d^2 \xi^{(2)}_{\pm}}{dz^2} + \frac{1}{c^2} \sigma_{\pm} [\sigma_{\pm} \pm (m + 2) \Omega] \xi^{(2)}_{\pm} = 0.
\]  
\( 27 \)

The solution of this equation can be analytically written as

\[
\xi^{(2)}_{\pm} = C_{\pm} \exp\left\{ \left[ k_{\pm}^{(2)} z - \omega t \right] \right\} + D_{\pm} \exp\left\{ \left[ -i k_{\pm}^{(2)} z + \omega t \right] \right\},
\]  
\( 28 \)

where the wavenumber \( k_{\pm}^{(2)} \) with each mode in the plasma is defined as

\[
k_{\pm}^{(2)} \equiv \frac{\sqrt{\sigma_{\pm} \pm (m + 2) \Omega}}{c}.
\]  
\( 29 \)

In region (3) with pure vacuum (\( z > L \)), the wave equation becomes

\[
\frac{d^2 \xi^{(3)}_{\pm}}{dz^2} + \omega^2 \xi^{(3)}_{\pm} = 0.
\]  
\( 30 \)

Owing to the absence of rotating matter, two helical states satisfy the same equation. We here dropped the notation ‘\( \pm \)’. The solution is easily given by

\[
\xi^{(3)}_{\pm} = E \exp\left\{ \left[ k_{\pm}^{(3)} z - \omega t \right] \right\}.
\]  
\( 31 \)

The wavenumber thus takes an ordinal form as

\[
k^{(3)}_{\pm} = \frac{\omega}{c}.
\]  
\( 32 \)

In the above expressions, \( C_{\pm} \), \( D_{\pm} \) and \( E \) are complex incidence, reflection and transmission amplitudes in each region, respectively. These wave amplitudes and the wavenumbers determine the transmission rate of the wave based on some boundary conditions. A mathematical treatment is given in Section 2.4.

### 2.4 Boundary condition

The physical property of Alfvén waves changes at the bottom and at the top of the rotating plasma layer, depending on the azimuthal wavenumber and the angular velocity of the background. We now require boundary conditions in the usual way in order to connect each wave solution continuously. From equations (28) and (31), the continuity at the upper surface of the plasma, \( z = L \), gives

\[
C_{\pm} = E \frac{k_{\pm}^{(2)} + k^{(3)}_{\pm}}{2k_{\pm}^{(2)}} \exp\left\{ -i \left[ k_{\pm}^{(2)} - k^{(3)} \right] L \right\},
\]  
\( 33 \)

\[
D_{\pm} = E \frac{k_{\pm}^{(2)} - k^{(3)}_{\pm}}{2k_{\pm}^{(2)}} \exp\left\{ i \left[ k_{\pm}^{(2)} + k^{(3)} \right] L \right\}.
\]  
\( 34 \)

Substituting equations (33) and (34) into equation (28) and then taking the derivative at the stellar surface, we obtain

\[
\frac{d}{dz} \ln \left[ \frac{\xi^{(2)}_{\pm}}{\xi^{(3)}_{\pm}} \right] |_{z=0} = i k^{(2)}_{\pm} \gamma,
\]  
\( 35 \)

where \( \gamma \) is a modification quantity due to background rotation of the surrounding plasma defined as

\[
\gamma = \frac{\omega}{c} \exp\left\{ -i \left[ \omega_L - \omega_c \right] L \right\} - \frac{\omega}{c} \exp\left\{ i \left[ \omega_L - \omega_c \right] L \right\},
\]  
\( 36 \)

\[
\gamma = \frac{\omega}{c} \exp\left\{ -i \left[ \omega_L - \omega_c \right] L \right\} + \frac{\omega}{c} \exp\left\{ i \left[ \omega_L - \omega_c \right] L \right\},
\]  
\( 37 \)

\( 36 \)

\( 37 \)
A detailed treatment and the physical meaning of $\gamma$ are given in Appendix A.

In order to solve wave equation (21), we consider the complex linear combination $\zeta = \zeta_1 + i\zeta_2$ with specific solutions $\zeta_1$ and $\zeta_2$ to be satisfied with equation (35). At the ends, we can obtain the explicit boundary condition at the stellar surface $z = 0$

$$\frac{d}{dz} \ln (\zeta_1 + i\zeta_2) \bigg|_{z=0} = -\frac{k_B^2}{\zeta_1(0)} \gamma y + \zeta_2(0) \gamma y \bigg|_{z=0}
+ i k_B^2 \frac{1}{\zeta_1(0)} \gamma y - \zeta_2(0) \gamma y \bigg|_{z=0}.$$

(38)

The logarithmic derivative of the asymptotic solution has to be equal to that of numerical solution in the deep interior $z \ll 0$, so that we request

$$\frac{d}{dz} \ln \left( \xi_{\text{asym}} \right) \bigg|_{z=-q} = \frac{d}{dz} \ln \left( \zeta \right) \bigg|_{z=-q}.$$  

(39)

This yields

$$B_\pm = \frac{\left| \beta \right| z^{-1} \pm i(\psi_\pm /dz)}{\left| -\beta \right| z^{-1} \pm i(\psi_\pm /dz)} \left[ -\left( d\psi_\pm /dz \right) \right]^{-1} \exp(-2i\psi_\pm).$$  

(40)

Finally, the reflection $R_\pm$ and transmission coefficients $T_\pm$ of the waves with each helicity propagating from the crust through the plasma toward the vacuum exterior are respectively expressed as

$$R_\pm = \frac{|B_\pm|^2}{|A_\pm|^2},$$  

(41)

$$T_\pm = \frac{k_B^2}{k_B^2} \frac{|E|^2}{|A|^2} = 1 - R_\pm.$$  

(42)

3 PROPAGATION

3.1 Dispersion relation

In this section, the behaviour of the torsional Alfvén waves is discussed based on the dispersion relations. Eikonal equation (26) depends on the depth through the functions $h$ and $\gamma$. For strong magnetic fields $B_\pm \geq 10^{14}$ G considered in this paper, it can be shown that $\psi_\pm$ in equation (26) varies weakly with the depth except for the inner region close to the star surface. The local dispersion relation can be approximately written as

$$k_\pm \approx \frac{1}{v} \sqrt{\sigma_\pm \left[ \sigma_\pm \pm \left( m(1 - h) + 2 \right) / \Omega \right]}.$$  

(43)

This relation can also be obtained by taking a short wavelength limit, namely, the high-frequency limit. The dimensionless function $h$ involved in equation (43) varies from $h = 1$ in the deep interior to $h = 0$ in the surface and exterior. This function represents the ratio between the rest-mass energy density and the effective magnetic energy density. For $h = 1$, the rest-mass energy dominates over the magnetic energy. This case corresponds to the classical limit. On the other hand, strong magnetic fields $B_\pm \geq 10^{14}$ G easily overwhelms the rest-mass energy of the electrons or ions in the low-density region, where $h \approx 0$. In this case, relativistic displacement current should be included in the analysis.

In order to investigate the propagation property of the wave, it is useful to define the phase velocity $v_p$ in a straightforward manner as

$$v_p^2 = \frac{c^2}{\alpha^2 + (c k_\pm)^2},$$  

(44)

which is also related to the refractive index $N_\pm = c v_p$ for each mode. Note that $v_p^2$ is not necessarily positive. It is well known in plasma physics (e.g. Wolfgang & Rudolf 1996) that waves are in general reflected at the cut-off points where $N_\pm \to 0$, and are absorbed at the resonant absorption points where $N_\pm \to \infty$. Since the frequency $\sigma_\pm$ in the rotating frame is here confined within the positive regime $\sigma_\pm > 0$, $k_\pm$ is always real, that is, neither cut-off nor resonant absorption points appear in the positive helicity. On the other hand, there are both absorption and cut-off points for the negative mode. In the remainder of this section, our attention will thus be paid only to this mode for physical interests. The dispersion relations with negative mode are schematically shown for the classical limit in Fig. 2 and for the relativistic limit in Fig. 3, respectively. In both diagrams, the region $v_p^2 c^2 < 0$ stands for the non-propagation, i.e. the evanescent zone. The vertical dotted line denotes the cut-off frequency of the wave.

(i) Classical limit. As seen in Fig. 2, the evanescent zone appears only in the low frequency and narrow-band width $0 < \sigma_- / \Omega < 2$. In this classical treatment, the cut-off of the wave appears at $\sigma_- / \Omega = 2$, which can be physically interpreted as the fact that the Coriolis force interrupts the wave propagation in the rotating frame. In a high-frequency region $\sigma_- / \Omega > 2$, waves are almost capable of propagating except for $\sigma_- / \Omega = m$, at which waves are strongly absorbed by resonance with a rotating background. This is because refractoriness diverges in this particular frequency. It is noticed that

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure2.png}
  \caption{Dispersion relation for negative helicity waves with $m \geq 3$ in a classical limit, $h \to 1$. The region below the horizontal axis corresponds to the evanescent zone. The vertical dotted line denotes the cut-off frequency $\sigma_- / \Omega = 2$. The resonant absorption frequency is given by $\sigma_- / \Omega = 2$.}
\end{figure}

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure3.png}
  \caption{Same as Fig. 2 in a relativistic limit, $h \to 0$. The cut-off frequency is $\sigma_- / \Omega = m + 2$. The resonant absorption occurs within an evanescent regime at $\sigma_- / \Omega = m$.}
\end{figure}
such a resonant absorption frequency $\sigma_- = m\Omega$ in our model is formally analagous to the electron–cyclotron resonant frequency in the standard plasma physics (Wolfgang & Rudolf 1996).

(ii) Relativistic limit. It is important to understand the new effect that appears in the relativistic case. Fig. 3 demonstrates that the evanescent zone prevails in the frequency regime $0 < \sigma_-/\Omega < m + 2$, whose bandwidth is therefore broadened by large $m$ modes. The cut-off frequency is given by $\sigma_-/\Omega = m + 2$. It is only a high-frequency wave with $\sigma_-/\Omega > m + 2$ that can always propagate in a WKB sense. By contrast with the classical limit, the resonant absorption $\sigma_-/\Omega = m$ does not appear in this propagation regime, but in the evanescent regime.

### 3.2 Structure of the potential barrier

As is obvious from Fig. 3, evanescency cannot be neglected in large $m$ mode. In this subsection, we turn to the investigation of the spatial structure in the evanescent zone for large $m$. We now introduce a new dependent variable $\Psi_- = \bar{\mu}^{1/2}\xi_-$ for the amplitude of the negative mode with $\bar{\mu} \equiv \sqrt{\sigma_-}$. Exploiting some arithmetic algebra, wave equation (21) is now rewritten into a Sturm–Liouville-type differential equation

$$\frac{d^2 \Psi_-}{dz^2} + \left[ \frac{1}{4\bar{\mu}^2} \left( \frac{d\bar{\mu}}{dz} \right)^2 - \frac{1}{2\mu} \frac{d^2 \bar{\mu}}{dz^2} + \frac{1}{v^2(\bar{E} - \bar{V}_m)} \right] \Psi_- = 0,$$

where

$$\bar{E} = \sigma_-,$$

$$\bar{V}_m = |m(1 - h) + 2|\Omega.$$

The first two terms concerning $\bar{\mu}$ in square brackets in equation (45) become very small for strong magnetic fields $B_\Omega > 10^{14}$ G, since $\bar{\mu}$ can be almost regarded as a constant. We can compare equation (45) with the one-dimensional Schrödinger equation for the box-type potential in a stationary state. Physical quantities $\bar{E}$ and $\bar{V}_m$, respectively, correspond to ‘wave energy’ and ‘potential’ in a formal sense. Strictly, the potential $\bar{V}_m$ depends on the wave energy $\bar{E}$ through the frequency $\sigma_-$. However, $\sigma_-$ is here treated as a constant value irrespective of the position, so that the following discussion is valid without loss of generality.

A schematic profile of the effective potential $\bar{V}_m$ and the wave energy $\bar{E}$ is given in Fig. 4 as a function of distance $z$. We draw the curve in region (1) somewhat exaggerated. The potential varies with the position through the function $h$ and the background rotation $\Omega$. The potential height is determined by the coupled quantity $(m + 2)\Omega$ and its width is given by the size $L$ of the corotating zone. This result means that the large wavenumber lifts up the potential barrier, only if the background stellar medium rotates. In other words, if the surrounding matter is not dragged by the star, say, being in static state $\Omega = 0$, then the potential never rises, even though some high azimuthal waves exist.

Whether or not the wave can propagate and transmit out depends on the wave energy, that is, wave frequency $\sigma_-$. Our argument deserves to be specially emphasized in two explicit regimes: the evanescent frequency mode $2\Omega < \sigma_- < (m + 2)\Omega$ and the propagation mode $\sigma_- > (m + 2)\Omega$.

#### 3.2.1 Evanescent mode: $2\Omega < \sigma_- < (m + 2)\Omega$

As is apparent in Fig. 4, such low-frequency waves excited in the interior cannot help striking the potential barrier. Since refractivity is zero on the critical curve $\sigma_-/\Omega = m(1 - h) + 2$, outgoing waves with $m(1 - h) + 2 < \sigma_-/\Omega < m + 2$ are generally reflected, when they reach the potential wall.

#### 3.2.2 Propagation mode: $\sigma_- > (m + 2)\Omega$

In this case, perturbation of the electromagnetic fields can propagate as a wave. The present context in our model is almost concerned with the WKB frequency range. The propagation sometimes exhibits a remarkable property, if certain conditions are satisfied. As long as the wave frequency is much greater than the potential strength $\sigma_\Omega \gg (m + 2)\Omega$, the potential itself does not have much influence on the wave behaviour. However, if the frequency becomes commensurable to the potential height $\sigma_- \cong (m + 2)\Omega$, the existence of a potential barrier cannot be ignored. Especially if the wavelength is comparable to the potential width, i.e. $k_L \sim 1$, then the waves will interfere with the potential barrier and their behaviour will be strongly altered. This inherent property is expected to be more evident near the threshold frequency $\sigma_- \sim (m + 2)\Omega$.

Motivated by this general consideration, we explicitly calculate the transmission rate in the next section. The numerical parameters are adopted to satisfy the above condition, $L \sim 1/k_- \sim 10^6$ cm and $m = \omega_{\Omega}/\Omega \sim 10^6$. Using these parameters, we examine how and to what extent the propagation and transmission are affected by the potential barrier.

### 4 TRANSMISSION

#### 4.1 Numerical calculation

We numerically calculate the transmission rates (42) of the wave propagating from the deep crust, through the corotating plasma envelope, toward the vacuum exterior subject to matching conditions at each boundaries. As already mentioned above, only the negative helicity wave is intriguing for physical interests. Evanescent modes in a low-frequency regime $0 < \sigma_- < [m(1 - h) + 2]\Omega$ are excluded from this calculation. This constraint on wave frequency thereby guarantees that all waves are capable of propagating in a WKB sense. We can roughly estimate a typical wave frequency $\omega$ measured in the inertial frame by approximating as $\omega \sim v/q$. An appropriate frequency thus lies in a finite range $10^3 \lesssim \omega \lesssim 10^6$ s$^{-1}$.
The objective of this section is to explore the effects of azimuthal wavenumber \( m \) and the size \( L \) of the corotating plasma medium on the wave transmission. We here consider two kinds of specific torsional waves whose azimuthal number is (i) the fundamental, \( m = 1 \), and (ii) much larger than one, \( m = \omega_{\text{max}}/\Omega = 10^6 \). In each case, we further consider two apparently different circumstances in the exterior \( L = R \sim \lambda_{\text{A}} \sim 10^6 \) cm and in the absence of plasma \( L = 0 \) cm.

Figs 5 and 6 respectively demonstrate the transmission coefficients for the negative helicity waves with two extreme cases (i) \( m = 1 \) and (ii) \( m = 10^6 \) as a function of the wave frequency measured in the inertial frame when \( B_0 = 10^{15} \) G, typical for magnetars. In both figures, the solid and dotted lines denote the results of \( L = 10^6 \) and \( L = 0 \) cm, respectively. The surrounding plasma is here assumed to corotate with the same angular velocity as that of the star, \( \Omega = 1 \) s\(^{-1}\). In each case, we obtained the following results.

(i) Fundamental wavenumber: \( m = 1 \). As seen in Fig. 5, the transmission curve in the case of \( L = 10^6 \) cm slightly shows a wiggling behaviour. This curve intersects that of \( L = 0 \) at \( \omega \sim 10^3 \) s\(^{-1}\). However, the difference between both cases becomes indistinguishable in the high-frequency regime \( \omega \gg 10^5 \) s\(^{-1}\), since the wave frequency is much higher than the critical one \((m + 2)/\Omega = 3 \) s\(^{-1}\), which appeared as the potential barrier in Section 3. As the wave frequency becomes higher, the transmission rate approaches unity asymptotically. Since the overall property does not depend on \( L \), the potential barrier due to rotating plasma does not make any significant influence on the wave transmission for small azimuthal number \( m \sim 1 \).

(ii) Highly azimuthal wavenumber: \( m = 10^6 \). In this high \( m \) mode, we can find some surprising results. Fig. 6 shows that the transmission rate of \( L = 10^6 \) cm is drastically enhanced at some selected frequencies. More importantly, such enhancements occur periodically at specific frequencies. At the top of the first wing \((n = 1)\omega_1 = 2.2 \times 10^3 \) s\(^{-1}\), the rate reaches the maximum \( T_{\text{max}} = 0.71 \), which corresponds to approximately 70 times larger than that of the first bottom \( \omega_b = 7.0 \times 10^3 \) s\(^{-1}\). Compared with the \( L = 0 \) case at the same frequency \( \omega_b \), this maximum \( T_{\text{max}} \) amounts to 230 times larger. In this way, the plasma effect is important at each top frequency, but unimportant at any bottom frequency. As shown in Fig. 6, the transmission coefficient through the plasma layer is exactly equal to that of the \( L = 0 \) case at arbitrary wing bottom. In general, the rotating plasma has an effect in helping the escape of the waves except for the bottom frequencies. This point is different from that of case (i).

In order to understand the transmission enhancement, we have also numerically computed the integral \( \int T \omega \, \omega \) in some frequency bands for both \( L = 10^6 \) cm and \( L = 0 \) cm. We obtain some 13 times enhancement in the narrow band \( 2\Omega < \omega < \omega_b \) and 1.4 times in the broad band \( 2\Omega < \omega < \omega_{\text{max}} \) compared to the results of \( L = 0 \). Such enhancement and periodic variation slowly decrease with increasing wave frequency. This property almost vanishes when the frequency becomes comparable to the critical frequency \((m + 2)/\Omega \sim 10^6 \) s\(^{-1}\).

Periodic enhancements in our numerical calculations can be interpreted as a consequence of wave interference within the rotating plasma cavity. In the following subsection, we investigate our results more quantitatively in a simplified analytical method.

### 4.2 Analytical Calculation

Periodic enhancements on the wave transmission in our model can be satisfactorily rationalized by comparing with the wave propagation in homogeneous multimedia. Non-relativistic Alfvén resonance itself in the magnetosphere around the Earth and the Sun has been widely recognized and discussed theoretically and observationally (for recent reviews, see Leonovich & Mazur 1997; Waters 2000; Li & Wang 2001). Hollweg (1983) has theoretically studied WKB wave propagation in three homogeneous layers labelled by (1), (2) and (3) separated at \( z = 0 \) and \( z = L \) as shown in Fig. 7. In his work Alfvén waves are simply assumed to have constant wavenumbers \( k^{(1)}, k^{(2)} \) and \( k^{(3)} \) in each region. In this homogeneous model, the transmission coefficient \( T \) can be analytically calculated by taking

---

**Figure 5.** Transmission coefficients of the negative helicity wave with the fundamental mode \( m = 1 \) as a function of frequency when \( B_0 = 10^{15} \) G and \( \Omega = 1 \) s\(^{-1}\). Solid and dotted lines denote the results of \( L = 10^6 \) and \( L = 0 \) cm, respectively.

**Figure 6.** Same as in Fig. 5, but for high azimuthal number \( m = 10^6 \). In this high \( m \) mode, the transmission rate is highly enhanced at some selected frequencies due to the plasma envelope.

**Figure 7.** Three homogeneous models separated by two discontinuities at \( z = 0 \) and \( z = L \). If the periodic boundary condition \( k^{(2)}L = n\pi \) \((n = 1, 2, \ldots)\) holds, the intermediate layer acts as a resonant cavity and transmission is highly enhanced.

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boundary conditions at the discontinuities \( z = 0 \) and \( z = L \):

\[
T = \frac{4k^{(1)}k^{(3)}}{k^{(1)} + k^{(3)}} \left\{ 1 + \frac{k^{(1)}}{k^{(3)}} \right\}^2 \cos^2 \left[ k^{(2)}L \right] + \left[ k^{(3)} / k^{(2)} \right] \sin^2 \left[ k^{(2)}L \right] \right\}^{-1}.
\]  

(48)

Although the physical situation is different from our present work, similar expressions could be found also in our model. This formula (48) implies that the transmission rate has a periodic structure depending on the wavenumber \( k^{(2)} \) and the characteristic scale \( L \) of the intermediate layer unless \( k^{(2)}L \sim 0 \). Extrema of the transmission rate work out to be

\[
T_{\text{max}} = \begin{cases} \frac{4k^{(1)}k^{(3)}}{k^{(1)} + k^{(3)}} \right\{ 1 + \frac{k^{(1)}}{k^{(3)}} \right\}^2, & \text{at } k^{(2)}L = n\pi, \\ \frac{4k^{(1)}k^{(3)}}{k^{(1)} + k^{(3)}} \right\{ 1 + \frac{k^{(1)}}{k^{(3)}} \right\}^2, & \text{at } k^{(2)}L = (2n - 1)\pi/2, \\ \end{cases}
\]

(49)

with \( n = 1, 2, \ldots \). Provided that the wavenumber in each region satisfies the inequalities

\[
k^{(1)} \gg k^{(2)} \gg k^{(3)},
\]

(50)

\[
k^{(2)}L \gtrsim 1,
\]

(51)

the coefficient of \( \cos^2[k^{(2)}L] \) in equation (48) becomes dominant compared to that of \( \sin^2[k^{(2)}L] \). In this limit, equation (48) can be well approximated by

\[
T \approx \frac{4k^{(1)}k^{(3)}}{k^{(1)} + k^{(3)}} \cos^2 \left[ k^{(2)}L \right].
\]

(52)

It is noticed that if \( k^{(2)} \sim k^{(2)} \sim k^{(3)} \), the transmission shows neither periodic variations nor enhancements. Consequently, inequalities (50) and (51) give a set of resonant conditions of the wave. Sterling & Hollweg (1984) have subsequently considered a three-layer model for a solar flare, which is composed of a chromosphere, a spicule and a corona. In that work they have confidently suggested the possibility of Alfvénic resonance on solar spicules and have shown a new aspect of spicules which may account for occasionally twisting motions of magnetic field lines, even when the above resonant conditions approximately hold. Conditions (46) and (47) may hold true also in our model except close to the thin regime beneath the stellar surface, whenever large twisted waves with \( m \gg 1 \) propagate in the rotating background.

We can now quantify the particular frequencies at which the wave resonance occurs. By solving the quadratic equation \( (\sigma - (m + 2)\Omega \sigma - \Gamma^2)^2 = 0 \), together with the periodic condition \( k^{(2)}L = (2n - 1)\pi \), the eigenfrequencies of the negative mode are obtained as

\[
\omega_{m,n} = \frac{(2 - m)}{2} \Omega + \sqrt{\left\{ \frac{m + 2}{2} \right\}^2 + \left[ \frac{(2n - 1)\pi}{2L} \right]^2},
\]

(53)

with \( m, n = 1, 2, \ldots \). This formula shows that the resonant frequencies depend critically on the azimuthal number \( m \) and the plasma cavity length \( L \). They gently decrease with an increase in \( L \) or \( m \). Physically, this means that it takes a longer time for the wave to go back and forth between the stellar surface and the top of the plasma layer, and then this wave interferes with another one propagating from the crust into the cavity. In extremal cases, we find

\[
\lim_{L \to \infty} \omega_{m,n} = \lim_{m \to \infty} \omega_{m,n} = 2\Omega,
\]

(54)

which coincides with the cut-off frequency in the classical limit. At the same time, transmission peaks become blended with neighbouring resonance

\[
\lim_{L \to \infty} \Delta \omega_{m} = \lim_{m \to \infty} \Delta \omega_{m} = 0,
\]

(55)

where \( \Delta \omega_{m} \equiv \omega_{m,n} - \omega_{m,n-1} \).

For \( \Omega = 1 \) s\(^{-1} \) and \( L = 10^6 \) cm, from equation (53) we can calculate some representative resonant frequencies: the fundamental mode frequency \( \omega_{1,0} = 8.8 \times 10^5 \) s\(^{-1} \), the second mode \( \omega_{2,0} = 7.4 \times 10^5 \) s\(^{-1} \) and the third \( \omega_{3,0} = 1.9 \times 10^5 \) s\(^{-1} \). Our numerical work also gives similar results; \( \omega_{1,0} = 2.2 \times 10^5 \) s\(^{-1} \), \( \omega_{2,0} = 1.9 \times 10^5 \) s\(^{-1} \) and \( \omega_{3,0} = 5.2 \times 10^4 \) s\(^{-1} \). Our results exhibit slightly positive deviations from the analytical results. Recall that the wavenumber \( k^{(1)} \) in our magnetar model cannot be regarded as a constant, since the mass density drastically changes in the vicinity of the stellar surface. This discrepancy would probably be attributed to the inhomogeneity in the crust. If a sharp boundary is formed at the stellar surface, such a difference may probably become small. From a comparison with the analytical model, we have concluded that the transmission enhancements obtained in our model are thought to be a result of the Alfvén resonance on the rotating plasma cavity due to the potential barrier.

5 Electromagnetic Field Structure

In the preceding section we have elucidated that the transmission rates of waves are highly enhanced at some selected frequencies owing to the resonance effect in the rotating plasma cavity. Resonance in the plasma portion has another spectacular nature. In this section, we show that resonance has a great impact not only on the transmission rates, but also on the electromagnetic field structure associated with Alfvén waves. Most of our applications will once again concern the WKB frequency regime of the negative helicity \( \sigma_\gamma \geq |m(1 - h) + 2\Omega| \). Hereafter we omit the subscript ‘−’ for simplicity.

By substituting equation (28) into equations (11) and (12), we can explicitly have the expressions for electromagnetic field amplitudes in the plasma zone in terms of \( k^{(1)} \), normalized by a local magnetic field \( B_0 \)

\[
\frac{|\delta B_m|}{B_0} = \frac{|\delta B_0|}{B_0} = \frac{1}{\Omega^2 \sigma_\gamma^2} \frac{|c E_z|}{B_0}^2
\]

\[
= f_m(\sigma_\gamma) \left| \frac{dE_z}{dz} \right|^2 = f_m(\sigma_\gamma) \frac{k^{(1)}}{k^{(3)}} T_{a_m} |A|^2
\]

\[
\times \left\{ k_m^{(2)} \right\}^2 \sin^2 k_m^{(2)}(z - L) + \left[ k^{(3)} \right]^2 \cos^2 k_m^{(2)}(z - L),
\]

(56)
and

\[
\frac{|\varepsilon E_r|^2}{B_0^2} = \frac{|\varepsilon E_r|^2}{B_0^2} = f_m(\sigma)^2 \left| k_{(2)}^m \right|^2
\]

\[
= f_m(\sigma) \left( \frac{1}{2m^2} \right) \left| k_{(3)}^m \right|^2 |A|^2 T_m
\]

\[
\times \left\{ \left[ \frac{T_{m}}{2} \right] \cos^2 k_{(2)}^m (z - L) + \left[ k_{(3)}^m \right] \sin^2 k_{(2)}^m (z - L) \right\}
\]

\[
(57)
\]

where \( f_m(\sigma) \) is a radial profile given by \( f_m(\sigma) \equiv (\sigma / \sigma_p)^{2(m-1)} \) and the other notations have the same meaning as the previous ones. The longitudinal component of the perturbed magnetic fields is always zero \( B_z = 0 \), because the matter has been assumed to be vertically immobile in the present work. On the contrary, only if the background rotates, the electric fields have a longitudinal component whose structure is essentially identical to that of the magnetic fields. Thereby, we only have to investigate the transverse components \((\sigma, \phi)\) of the fields.

For the sake of exploring the dependence of the azimuthal number \( m \) on the field structure, we once more restrict our discussions to two kinds of extreme cases: (i) the fundamental mode \( m = 1 \) and (ii) the large azimuthal wavenumber \( m \rightarrow \infty \). In this limit, equations (49) and (50) can be well approximated by

\[
\frac{|\delta B_z|^2}{B_0^2} = \frac{|\delta B_z|^2}{B_0^2} \approx \frac{1}{2 \sigma^2 \sigma^2} \frac{|\varepsilon E_r|^2}{B_0^2}
\]

\[
\left\{ \begin{array}{ll}
 f_m(\sigma) k_{(1)}^{(m)} T_m |A|^2 & \text{for } m = 1, \\
 f_m(\sigma) k_{(1)}^{(m)} \left( \frac{k_{(2)}^m}{k_{(3)}^m} \right)^2 T_m |A|^2 \sin^2 k_{(2)}^m (z - L) & \text{for } m = 1,
\end{array} \right.
\]

\[
(58)
\]

and

\[
\frac{|\delta E_r|^2}{B_0^2} = \frac{|\delta E_r|^2}{B_0^2} \approx \frac{1}{2 \sigma^2 \sigma^2} \frac{|\varepsilon E_r|^2}{B_0^2}
\]

\[
\left\{ \begin{array}{ll}
 f_m(\sigma) \sigma_r^2 \frac{k_{(1)}^{(m)}}{k_{(3)}^m} T_m |A|^2 & \text{for } m = 1, \\
 f_m(\sigma) \sigma_r^2 \frac{k_{(1)}^{(m)}}{k_{(3)}^m} T_m |A|^2 \cos^2 k_{(2)}^m (z - L) & \text{for } m = 1.
\end{array} \right.
\]

\[
(59)
\]

Here we have dropped some small terms coupled to \( k_{(3)}^m \), since inequality \( k_{(1)}^m \gg k_{(2)}^m \gg k_{(3)}^m \) holds for large \( m \). Equations (58) and (59) have the implication that the fields are almost constant for small \( m \) modes, but are sinusoidally changed with \( z \) for large \( m \).

In principle, the absolute value of the incident wave amplitude \( |A| \) cannot be determined in our linearized theory. If we are allowed to assume that \( f_{m=1}(\sigma) |A|_{m=1} \sim f_m(\sigma) |A|_{m=1} \), then the ratios of electromagnetic field amplitude with \( m = l \) to those with \( m = 1 \) are approximately given by

\[
\frac{|\delta B_z|^2}{B_0^2} \approx \frac{|\delta B_z|^2}{B_0^2} \sim \left( 1 + \frac{i \Omega \omega}{2} \right)^2 \left( 1 - \frac{2 \Omega \omega}{\omega} \right) T_{m=1} \sin^2 k_{(2)}^m (z - L)
\]

\[
+ \frac{|\delta E_r|^2}{B_0^2} \approx \left( 1 + \frac{i \Omega \omega}{2} \right)^2 \left( 1 - \frac{2 \Omega \omega}{\omega} \right) T_{m=1} \cos^2 k_{(2)}^m (z - L),
\]

\[
(61)
\]

with \( i = \sigma, \phi \). These quantities are much larger than unity because of the extra factor \( (1 + i \Omega \omega) \) except for some special locations \( z = n \pi / k_{(2)}^m \) and \( z = (2n - 1) \pi / k_{(2)}^m \) \( (m, n = 1, 2, \ldots) \), which correspond to the nodes of the standing Alfvén wave.

We now examine the effect of the resonance in the rotating plasma cavity, whose thickness is given by \( L \sim R = 10^6 \text{ cm} \), on the spatial structure of the perturbed electromagnetic fields. Let us designate by \( \alpha \) the amplitude of the perturbed magnetic field within the plasma normalized by that in the absence of the plasma

\[
\alpha(z) \equiv \frac{|\delta B_z|^2}{B_0^2} \approx \left( 1 + \frac{i \Omega \omega}{2} \right)^2 \left( 1 - \frac{2 \Omega \omega}{\omega} \right) T_{m=1} \sin^2 k_{(2)}^m (z - L) + \frac{\omega^2}{c^2}.
\]

\[
(62)
\]

The first term in square brackets is related to the deviation due to high torsional modes in the rotating background from the dispersion relation in a vacuum. The ratio \( \alpha \) is plotted against the height from the stellar surface for (i) \( m = 1 \) in Fig. 8 and (ii) \( m = 10^6 \) in Fig. 9. Solid lines denote the numerical results at the top of the wing \( \omega_l = 2.2 \times 10^5 \text{ s}^{-1} \), while dotted lines denote the bottom \( \omega_b = 7.0 \times 10^4 \text{ s}^{-1} \). The transmission rates at resonant frequencies obtained in Section 3 are appropriately used in this calculation.

(i) Fundamental wavenumber: \( m = 1 \). As shown in Fig. 8, in the case of \( m = 1 \), the wave amplitude keeps one order of magnitude in the plasma. This result agrees well with the fact that the dispersion relation for small \( m \sim 1 \) is almost equal to that of the electromagnetic fields in pure vacuum, \( k_{(2)}^m)^2 \sim \omega^2 / c^2 \). The background rotation therefore does not affect this small \( m \) mode.

(ii) Highly azimuthal wavenumber: \( m = 10^6 \). Some significant differences can be found in this high mode. The perturbed magnetic field strength \( \alpha \) for \( m = 10^6 \) at the resonant frequency \( \omega_l \) is drastically amplified up to \( \alpha \sim 10^7 \) at \( z \lesssim 10^{5} \text{ cm} \). Then the field strength \( \alpha \) falls down to \( \alpha = T_{l=R} / T_{L=0} \sim 230 \) at the top of the plasma layer, in which the first node of the standing Alfvén wave is formed. Even

\[\text{Figure 8. Spatial structure of the perturbed magnetic field with } m = 1 \text{ in the rotating plasma region normalized by that in the absence of plasma at some resonant frequencies in the first wing. The solid and dotted lines denote the result at the top of the wing } \omega_l = 2.2 \times 10^5 \text{ s}^{-1} \text{ and at the bottom } \omega_b = 7.0 \times 10^4 \text{ s}^{-1} \text{, respectively.} \]
at the bottom frequency \( \omega_b \) the strength \( \alpha \) has about \( 10^2 \) at \( z \approx 5 \times 10^3 \) cm.

In the same way, the normalized amplitude of the perturbed electric field is expressed by

\[
\beta(z) = \frac{\delta E_z|_{L=R}}{\delta E_z|_{L=0}} = \frac{T_{L=R}}{T_{L=0}} \left( 1 + \frac{m\Omega}{\omega} \right)^2 \frac{1}{k_m(2)_{\Omega}} \times \left[ \left( \frac{k_m(2)_{\Omega}}{c} \right)^2 - \frac{\omega^2}{c^2} \right] \cos^2 k_m(2)_{\Omega} (z - L) + \frac{\omega^2}{c^2} \right],
\]

which is plotted in Figs 10 and 11. When \( m = 10^6 \), the field amplitude \( \beta \) of \( \omega_l \) has almost the same magnitude \( 10^5 \) as \( \alpha \) near the stellar surface \( z \lesssim 10^4 \) cm. However, near the top of the plasma \( z \sim 10^6 \) cm, in contrast to \( \alpha \), \( \beta \) has a maximum \( 5 \times 10^7 \), which corresponds to the loops of the standing wave. Approximately, \( \beta \) is much greater than \( \alpha \) because of the extra term \( (1 + m\Omega/\omega)^2 \sim 10^2 \) in equation (63).

Such a field amplification can also be explained by considering a conservation of energy flux \( F \sim \rho v \omega |\xi|^2 \). The phase velocity \( v \) of the mode \( m > 1 \) in the plasma atmosphere is much smaller than that in the vacuum, as is confirmed in equations (43) and (44). Assuming the same value \( |\xi| \) at the stellar surface, which is irrelevant to \( L \), the amplitude \( |\xi| (\propto v^{-1/2}) \) is enhanced in the rotating plasma region so as to compensate for the slowing down of the wave propagation.

### 6 SUMMARY AND DISCUSSION

We have studied the propagation and transmission of torsional Alfvén waves along the rotation axis on magnetars by using a three-layered cylindrical model. If the intermediate plasma layer rotates with angular velocity \( \Omega \), the propagation property for largely twisted waves with \( m > \omega/\Omega \) is drastically modified. This middle zone can be regarded as a kind of a potential barrier, whose height is explicitly specified by \( (m + 2)\Omega \). Waves having an angular velocity of \( 2\Omega \) originate from the Coriolis force in the classical limit, while an additional quantity \( m\Omega \) comes from the relativistic effect. This therefore means that the potential is lifted up higher in the large \( m \) modes on the rotating background. It should be emphasized that this new finding is purely attributed to having incorporated the displacement current into the model.

We have numerically computed the transmission rates of the torsional waves with a large value \( m \gg 1 \) driven in the crust, through the corotating plasma with a finite size \( L \sim R \), into the exterior. We found that the transmissions are strongly enhanced for large \( m \approx \omega/\Omega \) at selected wave frequencies satisfied with a periodic condition \( kL \approx (2n - 1)\pi/2(n = 1, 2, \ldots) \). Such a transmission enhancement arises because the rotating plasma forms a resonant cavity which traps the wave energy by virtue of the strong reflections occurring at the transition region at the bottom and at the top of the plasma layer. Efficient transmissions due to resonance are also compatible with the fact that the reflection of waves at the stellar surface can almost completely be extinguished. A magnetospheric waveguide or resonance cavity can thus actually generate a set of coherent eigenmodes for high \( m \). Resonance may be a signature of fundamental processes by which high-order torsional oscillations of waves, if these exist, can transfer much more energy out of the magnetars especially in the low-frequency regime.

It is well known in quantum mechanics that when the potential width is comparable to (or integer times) the de Broglie wavelength of electrons \( L \sim n\hbar/m_e v \), incident wave flux is bounded by finely tuning a phase relation with inversely propagating waves (e.g. Schiff 1968). Then resonance is excited in the potential zone and the wave flux spends much of its time in the resonance cavity. Similar resonance may occur also in our macroscopic model, if the characteristic scale of the evanescent zone is comparable to the wavelength of the Alfvén wave.
We have investigated the spatial structure of the perturbed electromagnetic fields propagating in the rotating plasma, when standing Alfvén waves are formed. Resonance in the plasma cavity for high \( m \) modes has another spectacular aspect. At some points magnetic fields of the Alfvén waves are drastically amplified even up to a few \( 10^3 \) times as large as those in the absence of plasma or in the static exterior. This amplification can be straightforwardly explained by both the transmission enhancement due to resonance and the conservation of WKB energy flux.

In this way, once the Alfvén waves are resonantly excited and transmissions are greatly enhanced, a substantial part of the wave energy will probably be transferred into the ambient charged particles such as electrons or positrons confined within the fireballs associated with the bursts on magnetars. The charged particles entwined around magnetic field lines will then be violently swayed. These disturbances may eventually produce high-energy gamma-ray or X-ray emissions from the magnetars. More importantly, if magnetic field lines at some altitudes are strongly distorted from the equilibrium state by resonance and plasma fluid contracts into the surface, the field lines with antiparallel components will approach each other. Such geometry of field lines will give rise to possible magnetic reconnection type events as Thompson & Duncan (2001) have suggested.

The behaviour of resonant magnetohydrodynamics waves within the corona or spicules erupted from the solar surface has been studied so far. In fact, field line resonance has been found to occur on magnetic shells in the magnetosphere of the Earth. However, their physical treatment is inevitably restricted to the very weak magnetic fields. The resonant property found in our magnetar model is very similar to that in solar astrophysics or planetary physics. We can expect that the same mechanism works also on the relativistic Alfvén waves even in a different environment, that is, in an extreme circumstance accompanied by very strong magnetic fields such as magnetars.

As a final remark, we address the angular velocity \( \Omega \) of the star and the azimuthal wavenumber \( m \). In our model, the rotating background is throughout assumed to have a slow angular velocity \( \Omega = 1 \text{ s}^{-1} \), which is actually observed in SGRs and AXPs. We concentrate our treatment on the torsional waves with a specific value \( \Omega = 10^{-1} \text{ s}^{-1} \) as it is not clear whether or not such modes with high wavenumber \( m \) realistically exist on the magnetars. However, one should keep it in mind that the local dispersion relation of the Alfvén waves is almost determined by the coupled quantity \( m\Omega \) except for the high-frequency regime. This means that the physical behaviour of the wave with high \( m = 10^3 \text{–} 10^5 \) on the slowly rotating background \( \Omega = 1 \text{ s}^{-1} \) is equivalent to that of small \( m = 1 \text{–} 10 \) and fast rotation \( \Omega = 10^3 \text{ s}^{-1} \). Neutron stars are often expected to have been a rapid rotator with \( \Omega \sim 10^3 \text{ s}^{-1} \) at a star-born period. Our results would thus become much more effective especially on young magnetars.

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### APPENDIX A: RESONANCE MECHANISM

The physical effects of \( \gamma \) can be clarified by taking some explicit limits. Specifically, if we ignore the substantial thickness of the plasma layer \( L \rightarrow 0 \), then the wavenumber of region (2) should be replaced with that of region (3), \( k_\perp^2 \rightarrow k_\perp \). This yields that \( \kappa_\perp \rightarrow 2k_\perp \), \( \kappa_\perp \rightarrow 0 \) and \( \gamma \rightarrow 1 \). In this limit, the boundary condition reduces to \( d/dz(\ln \xi_\perp^{(2)}) = ik_\perp^{(2)} \), which clearly corresponds to the simple case that the exterior of the star is filled with pure vacuum or static plasma gas. While, if we imagine the very huge plasma gas corotating in the exterior and take the limit formally \( L \rightarrow \infty \), then the wavenumber of region (3) should be equal to that of region (2), \( k_\perp^{(3)} \rightarrow k_\perp^{(2)} \). We thus have that \( \kappa_\perp \rightarrow 2k_\perp \), \( \kappa_\perp \rightarrow 0 \) and \( \gamma \rightarrow 1 \), which yield the boundary condition \( d/dz(\ln \xi_\perp^{(2)}) = ik_\perp^{(2)} \).

We can draw some important facts from the boundary condition (35). Especially when the wavenumber satisfies the periodic condition \( \kappa_\perp L = n\pi (n = 1, 2, \ldots) \), this yields \( \gamma = k_\perp^{(3)}/k_\perp^{(2)} \) and thereby the boundary condition for pure vacuum is recovered \( d/dz(\ln \xi_\perp^{(2)}) = ik_\perp^{(2)} \). The background rotation has no influence only on the Alfvén waves satisfied with this condition, whereas, when the wavenumber satisfies the another periodic condition \( \kappa_\perp L = (2n - 1)\pi/2 \).
(n = 1, 2, . . .), we obtain γ = k_n^2/k^3 and thus the boundary condition works out to be d/dz[ln k ± n] = i k^3 [k_n^2/k^3]² ≫ i k^5. The right-hand side of this expression turns out that the waves satisfied with this condition are highly transmitted, as far as k_±(1) does not change much over one wavelength, say the validity of WKB approximation (1/k_±(1)) |dk_±(1)/dz| ≪ k_±(1) holds. This peculiar result means that the internal shear modes can be coupled on to the high-frequency Alfvén modes at the stellar surface, which drastically enhances their transmission rate at certain wavenumbers or frequencies.

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