TRACE MAPPINGS ON QUASI-BANACH MODULATION SPACES AND APPLICATIONS TO PSEUDO-DIFFERENTIAL OPERATORS OF AMPLITUDE TYPE

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Abstract. We deduce trace properties for modulation spaces (including certain Wiener-amalgam spaces) of Gelfand-Shilov distributions. We use these results to show thatΨdos with amplitudes in suitable modulation spaces, agree with normal type Ψdos whose symbols belong to (other) modulation spaces. In particular we extend and improve the style of trace results for modulation spaces in [9, 45] to include quasi-Banach modulation spaces. We also apply our results to obtain Schatten-von Neumann and nuclearity properties for Ψdos with amplitudes in modulation spaces, extending earlier work in [12, 41, 45, 51].

0. Introduction

In the paper we deduce continuity properties of trace mappings when acting on extended classes of modulation spaces. Thereafter we apply these properties to get continuity and identification properties for amplitude type pseudo-differential operators with amplitudes in modulation spaces. In contrast to most of earlier approaches, e.g. [21, 45] we allow the Lebesgue exponents of the involved modulation spaces to stay in the full interval (0, ∞]. In particular, the modulation spaces in our investigations are quasi-Banach spaces, which might not be Banach spaces. Note that these Lebesgue exponents in [45] should belong to the smaller interval [1, ∞], while in [21] they should belong to (0, ∞) and thereby not allowed to attain ∞. In particular, our investigations include trace mapping properties of the modulation space $M^{\infty,q}$, $q \in (0,1)$, while analogous investigations in [21, 45] do not host these spaces.

A trace map is an operator which reduce the dimension of the domain for functions or distributions, by fixing some coordinates. For example, let $f(x_1, x_2)$ be a function which depends on $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$, let $z \in \Omega_2$ be fixed. Then the map $T_z$ which takes $(x_1, x_2) \mapsto f(x_1, x_2)$ into $x_1 \mapsto f(x_1, z)$, i.e.,

$$(T_z f)(x_1) = f(x_1, z),$$

can be considered as the archetype of trace mappings, provided $x_1 \mapsto f(x_1, z)$ makes sense as a function. (See [34] and Section 1 for notations.)

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Trace mappings appear in natural ways in different kinds of problems, e.g.
boundary value problems of partial differential equations. If \( P(t, x, D_t, D_x) \) is a partial differential operator, and \( f \) and \( u_0 \) are fixed functions or distributions, then
\[
\begin{align*}
P(t, x, D_t, D_x)u(t, x) &= f(x, t), \quad t > 0, \\
u(0, x) &= u_0(x),
\end{align*}
\]
is an example of an initial value problem. It is expected that the solution \( u(t, x) \) should possess suitable continuity and differentiability properties as well as the trace map which takes \( u(t, x) \) into \( u_0(x) = u(0, x) \) is well-defined. See e.g. [10] and the references therein for more facts on this.

Another example where trace mappings appear naturally concerns pseudo-differential operators of amplitude types. For any amplitude \( a \in C^0(\mathbb{R}^{2d}) \), the pseudo-differential operator \( \text{Op}(a) \) is the linear and continuous map from \( \mathcal{S}(\mathbb{R}^{d}) \) to \( \mathcal{S}(\mathbb{R}^{d}) \), given by
\[
(\text{Op}(a)f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(x, y, \zeta) e^{i(x-y, \zeta)} f(y) \, dyd\zeta.
\]
Any such operator may in a unique way be expressed as a pseudo-differential operator of standard or Kohn-Nirenberg type with symbol in \( \mathcal{S}(\mathbb{R}^{2d}) \). That is, there is a unique \( a_0 \in \mathcal{S}(\mathbb{R}^{2d}) \) such that \( \text{Op}(a) \) above is equal to \( a_0(x, D) = \text{Op}_0(a_0) \), where
\[
(\text{Op}_0(a_0)f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a_0(x, \zeta) e^{i(x-y, \zeta)} f(y) \, dyd\zeta.
\]
The drop of variable \( y \) when passing from \( a(x, y, \zeta) \) to \( a_0(x, \zeta) \) implies that somewhere in the process, a trace map appears. Indeed, \( a_0 \) is obtained by the formula
\[
a_0(x, \zeta) = (e^{i(D_{\zeta}, D_y)}a)(x, y, \zeta) \big|_{y=x} = (e^{i(D_{\zeta}, D_y)}a)(x, x+y, \zeta) \big|_{y=0}
\]
(see e.g. [34], Section 18.2).

Trace mappings often possess convenient continuity properties when acting on spaces of continuous functions. For example, the mappings
\[
\begin{align*}
\text{Tr}_z : C^N(\mathbb{R}^{d_1+d_2}) &\to C^N(\mathbb{R}^{d_1}), \\
\text{Tr}_z : \mathcal{S}(\mathbb{R}^{d_1+d_2}) &\to \mathcal{S}(\mathbb{R}^{d_1}),
\end{align*}
\]
and
\[
\text{Tr}_z : \Sigma_1(\mathbb{R}^{d_1+d_2}) \to \Sigma_1(\mathbb{R}^{d_1}),
\]
are continuous (and surjective). More sensitive situations appear when the intended domains of \( \text{Tr}_z \) contain elements with lack of continuity, or, more dreadful, host elements with heavy singularities.

A classical example on such situations concerns trace mappings when acting on Sobolev spaces \( H^2_s(\mathbb{R}^{d_1+d_2}) \). For \( s > \frac{d_1}{2} \) one has that the mappings (0.2) and (0.3) are uniquely extendable to a continuous map
\[
\text{Tr}_z : H^2_s(\mathbb{R}^{d_1+d_2}) \to H^2_{s_0}(\mathbb{R}^{d_1}),
\]
provided \( s \geq s_0 + \frac{d_1}{2} \) (see e.g. [2]).

The latter trace property was extended in [11] to modulation spaces which include the Sobolev spaces above as special cases. We recall that the modulation spaces \( M_p^q(\mathbb{R}^d) \)
and $W_{p,q}^{\omega}(\mathbb{R}^d)$, introduced by Feichtinger in [14,15] and further developed in [17,20,26] are the sets of (Gelfand-Shilov) distributions whose short-time Fourier transforms belong to the weighted and mixed Lebesgue spaces $L_{p,q}^{\omega}(\mathbb{R}^{2d})$ respectively. Here $\omega$ is a weight function on phase (or time-frequency shift) space and $p,q \in (0,\infty]$. Note that $W_{p,q}^{\omega}(\mathbb{R}^d)$ is also an example on Wiener-amalgam spaces (cf. [17]).

Since the introduction, modulation spaces have entered several fields within mathematics and science, e.g. the theory of pseudo-differential operators (see e.g. [3,9,29,41,43,48] and the references therein for more recent progresses).

It follows from Theorem 3.2 in [45] that if $p,q \in [1,\infty]$ and $\omega_0(x,\xi) = (\eta)^{t}$, $t \geq \frac{d_2}{q'}$ for some constant $C > 0$, with the latter inequality strict when $q > 1$, then (0.3) is uniquely extendable to a continuous map

$$\text{Tr}_z : M_{p,q}^{\omega}(\mathbb{R}^{d_1+d_2}) \to M_{p,q}^{\omega_0}(\mathbb{R}^{d_1}),$$

(0.6)

that is,

$$\|\text{Tr}_z f\|_{M_{p,q}^{\omega_0}} \lesssim \|f\|_{M_{p,q}^{\omega}}.$$

(0.7)

Here $\langle x \rangle \equiv (1 + |x|^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^d$, as usual. If

$$p = q = 2, \quad \omega(x,y,\xi,\eta) = (\langle \xi,\eta \rangle)^{s}, \quad \omega_0(x,\xi) = \langle \xi \rangle^{s_0}, \quad s \geq s_0 + \frac{d_2}{2},$$

then (0.6) agrees with (0.4).

There are other extensions of (0.4). In [21, Theorem 3.3], Feichtinger, Huang and Wang use uniform-frequency decomposition techniques to establish trace properties on $\alpha$-modulation spaces, and thereby achieve trace properties on modulation and Besov spaces as special cases (cf. [21, Theorem 3.1]). In [39], Schneider deduce trace mapping results for Besov and Triebel-Lizorkin spaces, allowing the involved Lebesgue exponents to belong to the full interval $(0,\infty]$.

In Section 2 we extend the trace mapping result (0.6) in the sense of relaxing the conditions of the involved weight functions, allowing the Lebesgue exponents to belong to the full interval $(0,\infty]$, and complete (0.6) and (0.7) with trace maps for $W_{p,q}^{\omega}(\mathbb{R}^d)$ spaces. Especially we prove

$$\text{Tr}_z : W_{p,q}^{\omega}(\mathbb{R}^{d_1+d_2}) \to W_{p,q}^{\omega_0}(\mathbb{R}^{d_1}),$$

is continuous when $\omega$ and $\omega_0$ are the same as for (0.6). (See Theorems 2.2 and 2.3.) The involved weight functions should satisfy conditions of the form

$$\omega(x+y) \lesssim \omega(x)e^{r|y|}$$

(0.8)

for some $r > 0$, i.e., we permit general moderate weights (cf. [28]). For example, for any $r \in \mathbb{R}$ and $\theta \in (0,1]$, we allow the weights $(1+|x|)^r$ and $e^{r|x|^\theta}$. Note that in [21,45] the condition (0.8) is replaced by

$$\omega(x+y) \lesssim \omega(x)(y)^r,$$

(0.9)

for some $r \geq 0$, which is more restricted. In particular, weights like $e^{r|x|^\theta}$ are not allowed in [21,45].
The conditions on the weights have strong impact on the shape of modulation spaces. For example, conditions of the form \((0.9)\) imply that the modulation space \(M^{p,q}_{\omega}(\mathbb{R}^d)\) stay between \(\mathcal{S}(\mathbb{R}^d)\) and \(\mathcal{S}'(\mathbb{R}^d)\), while conditions of the form \((0.8)\) imply that \(M^{p,q}_{\omega}(\mathbb{R}^d)\) may contain the whole \(\mathcal{S}'(\mathbb{R}^d)\), or might be contained in \(\mathcal{S}(\mathbb{R}^d)\). In this respect, in contrast to [21, 45], our trace mapping results for modulation spaces in Section 2 also include ultra-distributions which are outside \(\mathcal{S}(\mathbb{R}^d)\).

In particular we show that \((0.6)\) holds for such weights and \(p,q \in (0,\infty)\) after \((0.5)\) is relaxed into

\[
\omega_0(x,\xi)(\eta)^t \leq C\omega(x,y,\xi,\eta)e^{r|b|}, \quad t \geq d_2 \left( \max \left( 1, \frac{1}{p}, \frac{1}{q} \right) - \frac{1}{q} \right),
\]

\((0.5)'\)

for some \(r \geq 0\), where the latter inequality in \((0.5)'\) should be strict when \(q > \max(p,1)\). (See Theorem 2.2)

To reach such general results for modulation spaces, we first use Gabor expansions to deduce the quasi-norm estimate \((0.7)\) for elements in the Gelfand-Shilov space \(\Sigma_1(\mathbb{R}^d)\) (which is dense in \(\mathcal{S}(\mathbb{R}^d)\)). Here convolution properties for weighted discrete Lebesgue spaces (see 2.14 and 2.15 for details) play key roles to achieved the desired estimates. Thereafter we deduce extensions of \((0.10)\) to the whole \(M^{p,q}_{\omega}(\mathbb{R}^d)\) in the Banach space case, \(p,q \geq 1\) by applying Hahn-Banach’s theorem. In similar way as in [45], some critical cases when \(p = \infty\) are obtained by using the narrow convergence, a weaker form of convergence compared to norm convergence, but sufficiently strong to guarantee needed uniqueness properties (see 2.14).

Extensions to the general case, \(p,q \in (0,\infty)\) are then obtained by applying suitable embedding results for modulation spaces, exploiting the fact that \(M^{p,q}_{\omega}(\mathbb{R}^d)\) increases with \(p\) and \(q\). Finally the uniqueness of \((0.6)\) follows in the case \(p,q < \infty\) by using the fact that \(\Sigma_1(\mathbb{R}^d)\) is dense in \(M^{p,q}_{\omega}(\mathbb{R}^d)\). For general \(p\) and \(q\), the uniqueness of \((0.6)\) is then reached by embedding \(M^{p,q}_{\omega}(\mathbb{R}^d)\) into other modulation spaces, where uniqueness assertions hold. More precisely, we find suitable weights \(\omega_1\) and \(\omega_{0,1}\) such that the diagram

\[
\begin{array}{ccc}
M^{p,q}_{\omega}(\mathbb{R}^d) & \xrightarrow{\text{Tr}_x} & M^{p,q}_{\omega_1}(\mathbb{R}^d) \\
I & & I \\
M^{1,1}_{\omega_{0,1}}(\mathbb{R}^d) & \xrightarrow{\text{Tr}_x} & M^{1,1}_{\omega_{0,1}}(\mathbb{R}^d)
\end{array}
\]

\((0.10)\)

commutes. Here \(I\) denotes continuous inclusions. The uniqueness of the first map \(\text{Tr}_x\) in \((0.10)\) then follows from the uniqueness of the second one in \((0.10)\). Note that these arguments remind on those in [39, Remark 3.1], where Schneider explain how trace mappings on Besov and Triebel-Lizorkin spaces can be extended to avoid the involved Lebesgue exponents to stay in the full interval \((0,\infty)\).

In contrast to Theorems 3.1 and 3.3 in [21], our results do not include any trace results for Besov or, more generally, those \(\alpha\)-modulation spaces which are not modulation spaces. It is not obvious whether the methods in Section 2 are well-designed for such investigations. On the other hand, the restrictions that the Lebesgue exponents in Theorems 3.1 and 3.3 in [21] are not allowed to attain \(\infty\) is removed in Theorem 2.2 in Section 2. This restriction might also be removed by using the ideas in [39, Remark 3.1] or behind \((0.10)\) in combination with embedding results in [24,33,53]. (See Remark 2.3)
In Section 3 we apply our trace mapping results in Section 2 to show that pseudo-differential operators with amplitudes in suitable modulation spaces can be formulated as pseudo-differential operators of Kohn-Nirenberg type with symbols in other modulation spaces. For example, as a consequence of our investigations it follows that if \( p, q \in (0, \infty] \), \( \omega \) and \( \omega_0 \) are moderate weights such that

\[
\omega(x, x + z, \zeta + \eta, \xi - \eta, \eta, z) \approx \omega_0(x, \zeta, \xi, z)(\eta)^t, \quad t \geq \frac{d}{q} \]

with strict inequality when \( q > 1 \), then

\[
\text{Op}(M^{p,q}(R^d)) = \text{Op}_0(M^{p,q}(R^d)). \tag{0.11}
\]

(See Theorem 3.3.) In particular, for any \( a \in M^{p,q}(R^d) \), there is a unique \( a_0 \in M^{p,q}(R^d) \) such that \( \text{Op}(a) = \text{Op}_0(a_0) \). On the other hand, for any \( a_0 \in M^{p,q}(R^d) \), there exists an \( a \in M^{p,q}(R^d) \) such that \( \text{Op}(a) = \text{Op}_0(a_0) \). (See Proposition 3.4.) Hence, the map which takes \( a \in M^{p,q}(R^d) \) into \( a_0 \in M^{p,q}(R^d) \) (which is uniquely defined) is surjective.

In order to deduce the surjectivity, a natural idea might be to choose \( a(x, y, \zeta) = a_0(x, \zeta) \) as candidate for \( a \) above. On the other hand, if \( p < \infty \) and \( a_0 \in M^{p,q}(R^d) \), then \( (x, y, \zeta) \mapsto a_0(x, \zeta) \) fails to belong to \( M^{p,q}(R^d) \). Hence the choice \( a(x, y, \zeta) = a_0(x, \zeta) \) does not work in this case. Instead the choice

\[
a(x, y, \zeta) = e^{-i(x, y, \zeta)}(a_0(x, \zeta) \varphi(y - x))
\]

works when \( \varphi \) is a suitable function such that \( \varphi(0) = 1 \). (Cf. Remark 2.10 and Theorem 3.3 and its proof.)

The identity \(0.11\) (supplied by Theorem 3.3) leads to that any continuity or compactness property for \( \text{Op}_0(M^{p,q}(R^d)) \) carry over to the class \( \text{Op}(M^{p,q}(R^d)) \). Here \( M^{p,q}(R^d) \) is a modification of \( M^{p,q}(R^d) \), obtained by suitable linear pullbacks of the involved elements (see Section 2 for strict definition). In Theorem 0.1 in Section 3 we use \( \omega \) Theorem 3.1] to achieve such continuity and compactness properties on modulation spaces, which are more general compared to existing results in the literature (e.g. in [45]). For example, the following result is an immediate consequence of Theorem 0.1 in Section 3 and is obtained by choosing \( p = \infty \) and \( q \in (0, 1] \) in that result. Here

\[
\frac{\omega_2(x, \zeta)}{\omega_1(z, \zeta)} \lesssim \omega(x, z, \zeta + \eta, \xi - \zeta - \eta, \eta, z - x), \quad x, z, \xi, \eta, \zeta \in R^d. \tag{0.12}
\]

**Theorem 0.1.** Let \( p \in (0, \infty], q \in (0, 1], \omega \in \mathcal{P}_E(R^{6d}) \) and \( \omega_1, \omega_2 \in \mathcal{P}_E(R^{2d}) \) be such that \(0.12\) holds. If \( a \in M^{p,q}(R^{3d}) \), then \( \text{Op}(a) \) is continuous map from \( M^{p,q}(R^{3d}) \) to \( M^{p,q}(R^d) \).

In the case \( q = 1 \), \( \omega = 1 \) and \( \omega_j = 1, j = 1, 2 \), Theorem 0.1 is proved already in [41]. For weights of polynomial type and Lebesgue exponents are obeyed to stay in the restricted subinterval \([1, \infty]\) of \((0, \infty]\), Theorem 0.1 is essentially proved in [45].

Section 3 also includes some investigations on pseudo-differential operators of amplitude types with symbols in modulation spaces of Wiener amalgam types. For such operators we deduce continuity between suitable modulation spaces and Wiener amalgam spaces (see Theorem 3.7).
In Section 3 we also combine Theorem 3.3 with suitable results in [50, 51] to obtain detailed compactness results for pseudo-differential operators of amplitude types. Especially the following Schatten-von Neumann and nuclearity results are special cases of Theorem 0.2 and Theorem 0.3 in Section 3 (See [50] Theorem 3.4 and [51] Theorem 4.2 for related results.)

**Theorem 0.2.** Let $p, q \in (0, \infty]$ be such that $q \leq \min (p, p')$, $\omega \in \mathcal{P}_E (\mathbb{R}^{d \alpha})$ and $\omega_1, \omega_2 \in \mathcal{P}_E (\mathbb{R}^{d \beta})$ be such that (0.12) holds. If $a \in \mathcal{M}^p_q (\omega_1 \omega_2)$, then $\text{Op} (a) \in \mathcal{P}_p (\mathcal{M}^p_1 (\omega_1 \omega_2), \mathcal{M}^p_2 (\omega_1 \omega_2))$, and

\[ \| \text{Op} (a) \|_{\mathcal{P}_p (\mathcal{M}^p_1 (\omega_1 \omega_2), \mathcal{M}^p_2 (\omega_1 \omega_2))} \lesssim \| a \|_{\mathcal{M}^p_q (\omega_1 \omega_2)}, \quad a \in \mathcal{M}^p_q (\omega_1 \omega_2). \]

**Theorem 0.3.** Let $p \in (0, 1]$, $\omega \in \mathcal{P}_E (\mathbb{R}^{d \alpha})$ and $\omega_1, \omega_2 \in \mathcal{P}_E (\mathbb{R}^{d \beta})$ be such that (0.12) holds. If $a \in \mathcal{M}^p_q (\omega_1 \omega_2)$, then $\text{Op} (a) \in \mathcal{N}_p (\mathcal{M}^p_1 (\omega_1 \omega_2), \mathcal{M}^p_2 (\omega_1 \omega_2))$, and

\[ \| \text{Op} (a) \|_{\mathcal{N}_p (\mathcal{M}^p_1 (\omega_1 \omega_2), \mathcal{M}^p_2 (\omega_1 \omega_2))} \lesssim \| a \|_{\mathcal{M}^p_q (\omega_1 \omega_2)}, \quad a \in \mathcal{M}^p_q (\omega_1 \omega_2). \]

The paper is organized as follows. In Section 1 we recall some basic facts for Gelfand-Shilov, modulation spaces and pseudo-differential operators. Thereafter we deduce trace results for modulation spaces in Section 2. Finally, we apply these trace results in Section 3 to transform common continuity and compactness properties for pseudo-differential operators of standard or Kohn-Nirenberg types into related properties for pseudo-differential operators of amplitude types.

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**1. Preliminaries**

In this section we recall some facts on Gelfand-Shilov spaces, modulation spaces and pseudo-differential operators. After introducing classes of weight functions and mixed norm spaces of Lebesgue types, we recall some properties of the Gelfand-Shilov space $\Sigma_1 (\mathbb{R}^d)$ and its distribution space $\Sigma'_1 (\mathbb{R}^d)$. Thereafter we consider a class of modulation spaces which contains the classical modulation spaces, introduced by Feichtinger in [15], but are not that general as in the more general approach, given by Feichtinger in [17]. In the last part we recall the definition of pseudo-differential operators and present some basic facts.

**1.1. Weight functions.** A weight on $\mathbb{R}^d$ is a positive function $\omega \in L^\infty_{loc} (\mathbb{R}^d)$ such that $1 / \omega \in L^\infty_{loc} (\mathbb{R}^d)$. The weight $\omega$ on $\mathbb{R}^d$ is called moderate if there is a positive locally bounded function $\nu$ on $\mathbb{R}^d$ such that

\[ \omega (x + y) \leq C \omega (x) \nu (y), \quad x, y \in \mathbb{R}^d, \tag{1.1} \]

for some constant $C \geq 1$. If $\omega$ and $\nu$ are weights on $\mathbb{R}^d$ such that (1.1) holds, then $\omega$ is also called $\nu$-moderate. The set of all moderate weights on $\mathbb{R}^d$ is denoted by $\mathcal{P}_E (\mathbb{R}^d)$.

We let $\mathcal{P} (\mathbb{R}^d)$ be the set of all weights of polynomial type. That is, (1.1) holds true for some (positive) polynomial $\nu$ on $\mathbb{R}^d$. For $s \geq 1$ we also let $\mathcal{P}_E, s (\mathbb{R}^d)$ ($\mathcal{P}_{E, s} (\mathbb{R}^d)$) be the set of all weights on $\mathbb{R}^d$ such that for some $r > 0$ (every $r > 0$) there is a constant $C > 0$ such that

\[ \omega (x + y) \leq C \omega (x) e^{r \| y \|^2}, \quad x, y \in \mathbb{R}^d, \tag{1.1'} \]
The mixed quasi-norm space $L^p$ and let $\sigma \in \mu$. Let $\mu$ be positive Borel measures on $\mathbb{R}^d$. Here and in what follows we write $A(\theta) \lesssim B(\theta)$, $\theta \in \Omega$, if there is a constant $c > 0$ such that $A(\theta) \leq c B(\theta)$ for all $\theta \in \Omega$. We also set $A(\theta) \asymp B(\theta)$ when $A(\theta) \lesssim B(\theta) \lesssim A(\theta)$.

If $\omega$ is a moderate weight on $\mathbb{R}^d$, then by [15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50] and above, there is a submultiplicative weight $v$ on $\mathbb{R}^d$ such that \((1.1)\) and \((1.2)\) hold. Moreover if $v$ is submultiplicative on $\mathbb{R}^d$, then

$$1 \lesssim v(x) \lesssim e^{r|x|}$$

(1.3)

for some constant $r > 0$ (cf. [28]). In particular, if $\omega$ is moderate, then

$$\omega(x + y) \lesssim \omega(x)e^{r|y|} \quad \text{and} \quad e^{-r|x|} \leq \omega(x) \lesssim e^{r|x|}, \quad x,y \in \mathbb{R}^d$$

(1.4)

for some $r > 0$.

1.2. Mixed quasi-normed spaces of Lebesgue types. Let $p, q \in (0, \infty]$, and let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$. Then $L^{p,0}(\mathbb{R}^{2d})$ and $L^{0,q}(\mathbb{R}^{2d})$ consist of all measurable functions $F$ on $\mathbb{R}^{2d}$ such that

$$\|g_1\|_{L^p(\mathbb{R}^{2d})} < \infty, \quad \text{where} \quad g_1(\xi) \equiv \|F(\cdot, \xi)\omega(\cdot, \xi)\|_{L^p(\mathbb{R}^{2d})}$$

and

$$\|g_2\|_{L^q(\mathbb{R}^{2d})} < \infty, \quad \text{where} \quad g_2(x) \equiv \|F(x, \cdot)\omega(x, \cdot)\|_{L^q(\mathbb{R}^{2d})},$$

respectively.

More generally, as in [50] we consider general classes of mixed quasi-normed spaces of Lebesgue types, parameterized by

$$p = (p_1, \ldots, p_d) \in (0, \infty]^d, \quad q = (q_1, \ldots, q_d) \in (0, \infty]^d,$$

$\sigma \in S_d$ and $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$. Here $S_d$ is the set of permutations on $\{1, \ldots, d\}$. In fact, let $p \in (0, \infty]^d$, $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, and let $\sigma \in S_d$. Moreover, let $\Omega_j \subseteq \mathbb{R}$ be Borel-sets, $\mu_j$ be positive Borel measures on $\Omega_j$, $j = 1, \ldots, d$, and let $\Omega = \Omega_1 \times \cdots \times \Omega_d$ and $\mu = \mu_1 \otimes \cdots \otimes \mu_d$. For every measurable and complex-valued function $f$ on $\Omega$, let $g_j, j = 1, \ldots, d$ be defined inductively by

$$g_0(\omega, \mu)(x_1, \ldots, x_d) \equiv |f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(d)})\omega(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(d)})|,$$

$$g_k(\omega, \mu)(x_{k+1}, \ldots, x_d) \equiv \|g_k & \omega, \mu(\cdot, x_k, \ldots, x_d)\|_{L^{p_k}(\mu_k)}, \quad k = 1, \ldots, d - 1,$$

and let

$$\|f\|_{L^p(\omega, \mu)} \equiv \|g_{d-1, \omega, \mu}\|_{L^{p_d}(\mu_d)}.$$

The mixed quasi-norm space $L^{p,\omega}(\mu)$ of Lebesgue type is defined as the set of all $\mu$-measurable functions $f$ such that $\|f\|_{L^p(\omega, \mu)} < \infty$. 

In the sequel we have $\Omega = \mathbb{R}^d$ and $d\mu = dx$, or $\Omega = \Lambda$ and $\mu(j) = 1$ when $j \in \Lambda$, where

$$\Lambda = \Lambda(\theta) = T_\theta \mathbb{Z}^d \equiv \{ (\theta_1 j_1, \ldots, \theta_d j_d) : (j_1, \ldots, j_d) \in \mathbb{Z}^d \},$$

(1.5)

and $T_\theta$ denotes the diagonal matrix with diagonal elements $\theta_1, \ldots, \theta_d$. In the former case we set $L^p_{\theta} = L^p_{\theta} = L^p_{\theta}(\mathbb{R}^d)$, and in the latter case we set $L^p_{\theta} = L^p_{\theta}(\mathbb{R}^d)$. For convenience we set

$$L^p_{\omega} = L^p_{\omega} = L^p_{\omega}$$

when $p, q \in (0, \infty]^d$, $\omega \in \mathcal{P}_F(\mathbb{R}^d)$, $\sigma_1 \in S_{2d}$ is the identity map and $\sigma_2 \in S_{2d}$ is given by

$$\sigma_2(j) = j + d \quad \text{and} \quad \sigma_2(j + d) = j, \quad j = 1, \ldots, d. \quad (1.6)$$

Later on it is common that $p$ above is split in some ways. If $d_j \in \mathbb{N}$, $j = 1, \ldots, n$, then

$$p_j = (p_{j,1}, \ldots, p_{j,d_j}) \in (0, \infty)^{d_j}, \quad j \in \{1, \ldots, n\},$$

and

$$p = (p_1, \ldots, p_n) = (p_{1,1}, \ldots, p_{d_1}, \ldots, p_{n,1}, \ldots, p_{n,d_n})$$

then set

$$L^p_{\omega} = L^p_{\omega} = L^p_{\omega}$$

and

$$\|f\|_{L^p_{\omega}} = \|f\|_{L^p_{\omega}} = \|f\|_{L^p_{\omega}}, \quad (1.8)$$

when $f$ is complex-valued and measurable on $\mathbb{R}^{d_1 + \cdots + d_n}$. If in addition $d_1 = \cdots = d_n = d$ for some $d \geq 1$, then the space in (1.7) becomes

$$L^p_{\omega}(\mathbb{R}^{nd}) = L^p_{\omega}(\mathbb{R}^{nd})$$

(1.9)

with $p_j \in (0, \infty]^d$ for every $j = 1, \ldots, n$.

1.3. **Gelfand-Shilov spaces.** Let $0 < h, s \in \mathbb{R}$ be fixed. Then $S_{h,s}(\mathbb{R}^d)$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{S_{h,s}} = \sup \frac{|x^\beta \partial^n f(x)|}{h^{2|\alpha|} + \beta!}$$

is finite. Here the supremum is taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

Obviously $S_{h,s} \to F$ is a Banach space which increases with $h$ and $s$. Here and in what follows we use the notation $A \to B$ when the topological spaces $A$ and $B$ satisfy $A \subseteq B$ with continuous inclusion.

The **Gelfand-Shilov spaces** $S_h(\mathbb{R}^d)$ and $\Sigma_h(\mathbb{R}^d)$ are the inductive and projective limits respectively of $S_{h,s}(\mathbb{R}^d)$ with respect to $h > 0$. This implies that

$$S_s(\mathbb{R}^d) = \bigcup_{h>0} S_{h,s}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h>0} S_{h,s}(\mathbb{R}^d),$$

(1.9)

and that the topology for $S_s(\mathbb{R}^d)$ is the strongest possible one such that the inclusion map from $S_{h,s}(\mathbb{R}^d)$ to $S_s(\mathbb{R}^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma_s(\mathbb{R}^d)$
is a Fréchet space with semi norms $\| \cdot \|_{S_{s,h}}$, $h > 0$. Moreover, $\Sigma_s(\mathbb{R}^d) \neq \{0\}$, if and only if $s > 1/2$, and $\Sigma_s(\mathbb{R}^d) \neq \{0\}$ if and only if $s \geq 1/2$ (cf. [23,30]).

Let $S'_{s,h}(\mathbb{R}^d)$ be the dual of $S_{s,h}(\mathbb{R}^d)$. Then the Gelfand-Shilov distribution spaces $S'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the projective and inductive limit respectively of $S'_{s,h}(\mathbb{R}^d)$ with respect to $h > 0$. This means that

$$S'_s(\mathbb{R}^d) = \bigcap_{h>0} S'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h>0} S'_{s,h}(\mathbb{R}^d).$$

We remark that $S'_s(\mathbb{R}^d)$ is the (strong) dual of $S_s(\mathbb{R}^d)$ when $s \geq \frac{1}{2}$, and $\Sigma'_s(\mathbb{R}^d)$ is the (strong) dual of $\Sigma_s(\mathbb{R}^d)$ when $s > \frac{1}{2}$ (cf. [30]). We also remark that the form $(\cdot, \cdot)_{L^2} = (\cdot, \cdot)_{L^2(\mathbb{R}^d)}$ restricted to $S_s(\mathbb{R}^d) \times S_s(\mathbb{R}^d)$ ($\Sigma_s(\mathbb{R}^d) \times \Sigma_s(\mathbb{R}^d)$) is uniquely extendable to a continuous map from $S'_s(\mathbb{R}^d) \times S'_s(\mathbb{R}^d)$ ($\Sigma'_s(\mathbb{R}^d) \times \Sigma'_s(\mathbb{R}^d)$) to $C$.

We have

$$S_{1/2}(\mathbb{R}^d) \hookrightarrow \Sigma_{s_1}(\mathbb{R}^d) \hookrightarrow S_{s_1}(\mathbb{R}^d) \hookrightarrow \Sigma_{s_2}(\mathbb{R}^d) \hookrightarrow S_{s_2}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)$$

$$\hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \Sigma'_{s_2}(\mathbb{R}^d) \hookrightarrow S'_{s_1}(\mathbb{R}^d) \hookrightarrow \Sigma'_{s_1}(\mathbb{R}^d) \hookrightarrow S'_{1/2}(\mathbb{R}^d), \quad \frac{1}{2} < s_1 < s_2, \quad (1.10)$$

with dense embeddings.

The Gelfand-Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations and under (partial) Fourier transformations. We also note that the map $(f_1, f_2) \mapsto f_1 \otimes f_2$ is continuous from $S_s(\mathbb{R}^{d_1}) \times S_s(\mathbb{R}^{d_2})$ to $S_s(\mathbb{R}^{d_1+d_2})$, and similarly when each $S_s$ are replaced by $\Sigma_s$, $S'_s$ or by $\Sigma'_s$. (See also [52].)

We let $\mathcal{F}$ be the Fourier transform which takes the form

$$(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} \, dx$$

when $f \in L^1(\mathbb{R}^d)$. Here $(\cdot, \cdot)$ denotes the usual scalar product on $\mathbb{R}^d$. The map $\mathcal{F}$ extends uniquely to homeomorphisms on $\mathcal{S}'(\mathbb{R}^d)$, $\Sigma'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$, and restricts to homeomorphisms on $\mathcal{S}(\mathbb{R}^d)$, $\Sigma_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$, and to a unitary operator on $L^2(\mathbb{R}^d)$.

There are several characterizations of Gelfand-Shilov spaces and their distribution spaces (cf. [6,13,49] and the references therein). For example, it follows from [6,13] that $f \in S_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$), if and only if

$$|f(x)| \lesssim e^{-r|x|^{1/2}} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim e^{-r|\xi|^{1/2}} \quad (1.11)$$

is true for some $r > 0$ (for every $r > 0$).

Gelfand-Shilov spaces and their distribution spaces can also be characterized by estimates on their short-time Fourier transforms. Let $\phi \in S_s(\mathbb{R}^d)$ ($\phi \in \Sigma_s(\mathbb{R}^d)$) be fixed. Then the short-time Fourier transform of $f \in S'_s(\mathbb{R}^d)$ (of $f \in \Sigma'_s(\mathbb{R}^d)$) with respect to $\phi$ is defined by

$$(V_\phi f)(x,\xi) \equiv (2\pi)^{-d/2} \langle f, \phi(\cdot - x) e^{i(\cdot,\xi)} \rangle_{L^2} \quad (1.12)$$

We observe that

$$(V_\phi f)(x,\xi) = \mathcal{F} \left( f \cdot \overline{\phi(\cdot - x)} \right)(\xi) \quad (1.12')$$
Remark 1.2. If in addition \( f \in L^p(\mathbb{R}^d) \) for some \( p \in [1, \infty] \), then
\[
(V_\phi f)(x, \xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(y)\phi(y - x)e^{-i(y, \xi)} dy. \tag{1.12}
\]

In the next lemma we present characterizations of Gelfand-Shilov spaces and their distribution spaces in terms of estimates on the short-time Fourier transforms of the involved elements. The proof is omitted, since the first part follows from \([31]\), and the second part from \([46, 49] \).

Lemma 1.1. Let \( p \in [1, \infty], f \in S'_s(\mathbb{R}^d), s \geq \frac{1}{2}, \phi \in S_s(\mathbb{R}^d) \setminus \{0\} \) and
\[
v_r(x, \xi) = e^{r(|x|^\frac{1}{2} + |\xi|^\frac{1}{2})}, \quad r \geq 0.
\]
Then the following is true:
(1) \( f \in S_s(\mathbb{R}^d) \) (\( f \in S_s(\mathbb{R}^d) \)), if and only if
\[
\|V_\phi f \cdot v_r\|_{L^p} < \infty \tag{1.13}
\]
for some \( r > 0 \) (for every \( r > 0 \));
(2) \( f \in S'_s(\mathbb{R}^d) \) (\( f \in S'_s(\mathbb{R}^d) \)), if and only if
\[
\|V_\phi f / v_r\|_{L^p} < \infty \tag{1.14}
\]
for every \( r > 0 \) (for some \( r > 0 \)).

Remark 1.2. Evidently, if \( d_j \in \mathbb{N} \) are such that \( d = d_1 + d_2 + d_3 \geq 1, j = 1, 2, 3, x_j \in \mathbb{R}^{d_j}, x = (x_1, x_2, x_3) \in \mathbb{R}^d \) and \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^d \), then we may replace \( v_r \) in \((1.13)\) and \((1.14)\) by
\[
v_r(x, \xi) = e^{r(|x|^\frac{1}{2} + |\xi_1|^\frac{1}{2} + |\xi_2|^\frac{1}{2} + |\xi_3|^\frac{1}{2})},
\]
in order for Lemma 1.1 should hold true. In particular it follows that if \( \phi \in \Sigma_1(\mathbb{R}^d) \) and \( f \in S'_s(\mathbb{R}^d) \), then \( f \in \Sigma'_s(\mathbb{R}^d) \), if and only if
\[
(x, \xi) \mapsto V_\phi f(x, \xi)e^{-r(|x| + |\xi_1| + |\xi_2| + |\xi_3|)} \quad \text{\((1.14)\)' for } r > 0.
\]

1.4. Modulation spaces. As in \([50]\) we need a broader family of modulation spaces than what is presented in \([15, 35]\). See also \([17]\) for even more general modulation spaces.

Definition 1.3. Let \( p, q \in (0, \infty]^d, \sigma_1 \in S_{2d}, \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( \phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\} \). Then the modulation space \( M_{\sigma_1(\omega)}^p(\mathbb{R}^d) \) consists of all \( f \in \Sigma'_1(\mathbb{R}^d) \) such that
\[
\|f\|_{M_{\sigma_1(\omega)}^p} = \|V_\phi f\|_{L^p_{\sigma_1(\omega)}}
\]
is finite.

Remark 1.4. Let \( p, q \in (0, \infty]^d, \sigma_1 \in S_{2d} \) be the identity map, \( \sigma_2 \in S_{2d} \) be given by \((1.6)\) and \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \). Then set
\[
M_{\sigma_1(\omega)}^{p, q}(\mathbb{R}^d) = M_{\sigma_1(\omega)}^{(p, q)} \quad \text{and} \quad W_{\sigma_2(\omega)}^{p, q}(\mathbb{R}^d) = M_{\sigma_2(\omega)}^{(q, p)}.
\]
We observe that \( M_{\sigma_1(\omega)}^{p, q}(\mathbb{R}^d) \) is a slight generalization of the standard modulation spaces, introduced in \([13]\), and \( W_{\sigma_2(\omega)}^{p, q}(\mathbb{R}^d) \) is a Wiener amalgam space, considered in \([14]\).
In some situations later on one has that \( p \) and \( q \) are given by
\[
p = (p_1, \ldots, p_n) = (p_{1,1}, \ldots, p_{1,d_1}, \ldots, p_{n,1}, \ldots, p_{n,d_n}),
\]
\[
q = (q_1, \ldots, q_n) = (q_{1,1}, \ldots, q_{1,d_1}, \ldots, q_{n,1}, \ldots, q_{n,d_n}),
\]
for some \( d_j \in \mathbb{N} \) \( j = 1, \ldots, n \), where
\[
p_j = (p_{j,1}, \ldots, p_{j,d_j}) \in (0, \infty)^{d_j}, \quad q_j = (q_{j,1}, \ldots, q_{j,d_j}) \in (0, \infty)^{d_j}.
\]
It follows that
\[
M^{p,q}_{(\omega)}(\mathbb{R}^d) = M^{p_1,\ldots,p_n,q_1,\ldots,q_n}_{(\omega)}(\mathbb{R}^d) = M^{p_1,\ldots,p_n,q_1,\ldots,q_n}_{(\omega)}(\mathbb{R}^{d_1+\cdots+d_n})
\]
and
\[
W^{p,q}_{(\omega)}(\mathbb{R}^d) = W^{p_1,\ldots,p_n,q_1,\ldots,q_n}_{(\omega)}(\mathbb{R}^d) = W^{p_1,\ldots,p_n,q_1,\ldots,q_n}_{(\omega)}(\mathbb{R}^{d_1+\cdots+d_n})
\]
consist of all \( f \in \Sigma_d'(\mathbb{R}^d) \) such that
\[
\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}} \quad \text{respectively} \quad \|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}}
\]
are finite. Here \( d = d_1 + \cdots + d_n \).

In our situations it is common that \( n = 3 \) in Remark \([1,4]\) i.e.,
\[
M^{p,q}_{(\omega)}(\mathbb{R}^{d_1+d_2+d_3}) = M^{p_1,p_2,p_3,q_1,q_2,q_3}_{(\omega)}(\mathbb{R}^{d_1+d_2+d_3}).
\]
It is also common that \( d_1 = d_2 = d_3 = d \geq 1 \). In this situation we write
\[
M^{p,q}_{(\omega)}(\mathbb{R}^{d+d+d}) = M^{p,q}_{(\omega)}(\mathbb{R}^{3d}) = M^{p_1,p_2,p_3,q_1,q_2,q_3}_{(\omega)}(\mathbb{R}^{3d})
\]
and
\[
W^{p,q}_{(\omega)}(\mathbb{R}^{d+d+d}) = W^{p,q}_{(\omega)}(\mathbb{R}^{3d}) = W^{p_1,p_2,p_3,q_1,q_2,q_3}_{(\omega)}(\mathbb{R}^{3d}).
\]
In what follows, the conjugate exponent \( p' \in (0, \infty] \) of \( p \in (0, \infty] \) is defined by
\[
p' = \begin{cases} 1 & \text{when } p = \infty \\ \frac{p}{p-1} & \text{when } 1 < p < \infty \\ \infty & \text{when } p \leq 1. \end{cases}
\]
For any \( p = (p_1, \ldots, p_n) \in (0, \infty]^n \) and \( q = (q_1, \ldots, q_n) \in (0, \infty]^n \) we set
\[
p^{-1} = \frac{1}{p} = (p_1^{-1}, \ldots, p_n^{-1}), \quad p' = (p_1', \ldots, p_n'), \quad p + q = (p_1 + q_1, \ldots, p_n + q_n)
\]
and
\[
p \leq q \quad (p < q) \quad \text{when} \quad p_j \leq q_j \quad (p_j < q_j), \quad j = 1, \ldots, n.
\]
For \( r \in (0, \infty] \) we also set
\[
p = r, \quad p \leq r \quad \text{respectively} \quad p < r
\]
when \( p_j = r, p_j \leq r \) respectively \( p_j < r \) for every \( j = 1, \ldots, n \).

In the following proposition we list some basic properties of modulation spaces. We omit the proof since the result follows by straightforward generalizations of the analysis in \([22,26,47]\) (see also \([18,20,37,38]\)).

**Proposition 1.5.** Let \( p, q \in (0, \infty]^n, r \in (0, 1] \) be such that \( r \leq \min(p, q), d_j \in \mathbb{N}, j = 1, \ldots, n \), \( d = d_1 + \cdots + d_n, \omega, v \in \mathcal{D}_E'(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, and let \( \phi \in \Sigma_d'(\mathbb{R}^d) \setminus \{0\} \).
(1) $\sum_1(\mathbb{R}^d) \subseteq M_{(\omega)}^{p,q}(\mathbb{R}^d) \subseteq \sum_1(\mathbb{R}^d)$. If in addition $\max(p,q) < \infty$, then $\sum_1(\mathbb{R}^d)$ is dense in $M_{(\omega)}^{p,q}(\mathbb{R}^d)$;

(2) the definitions of $M_{(\omega)}^{p,q}(\mathbb{R}^d)$ is independent of the choices of $\phi \in M_{(1)}^{1}(\mathbb{R}^d) \setminus 0$, and different choices of $\phi$ give rise to equivalent quasi-norms;

(3) $M_{(\omega)}^{p,q}(\mathbb{R}^d)$ is a quasi-Banach space which increase with $p$ and $q$, and decrease with $\omega$. If in addition $\min(p,q) \geq 1$, then $M_{(\omega)}^{p,q}(\mathbb{R}^d)$ is a Banach space;

(4) The $L^2(\mathbb{R}^d)$ scalar product, $(\cdot,\cdot)_{L^2(\mathbb{R}^d)}$, on $\sum_1(\mathbb{R}^d) \times \sum_1(\mathbb{R}^d)$ is uniquely extendable to a duality between $M_{(\omega)}^{p,q}(\mathbb{R}^d)$ and $M_{(1/\omega)}^{p,q'}(\mathbb{R}^d)$.

If in addition $p,q < (\infty,\ldots,\infty)$, then the dual of $M_{(\omega)}^{p,q}(\mathbb{R}^d)$ can be identified with $M_{(1/\omega)}^{p',q'}(\mathbb{R}^d)$, through $(\cdot,\cdot)_{L^2(\mathbb{R}^d)}$;

(5) Let $\omega_0(x,\xi) = \omega(-\xi,x)$. Then $\mathcal{F}$ on $\sum_1(\mathbb{R}^d)$ restricts to a homeomorphism from $M_{(\omega)}^{p,q}(\mathbb{R}^d)$ to $W_{(\omega)}^{p,q}(\mathbb{R}^d)$.

Similar facts hold true with $W_{(\omega)}^{p,q}$ spaces in place of corresponding $M_{(\omega)}^{p,q}$ at each occurrence.

We set $M^{p,q} = M_{(\omega)}^{p,q}$ and $W^{p,q} = W_{(\omega)}^{p,q}$ when $\omega = 1$ and $p,q \in (0,\infty]^n$. We observe that

$$M^{p_0,q_0} = M^{p,q}, \quad W^{p_0,q_0} = W^{p,q} \quad \text{and} \quad M^{p_0} = M^{p_0,p_0}$$

when

$$p = (p_0,\ldots,p_0) \in (0,\infty]^n \quad \text{and} \quad q = (q_0,\ldots,q_0) \in (0,\infty]^n.$$

An important property for modulation spaces is that it is possible to discretize them in terms of Gabor expansions. The following result is based on [25] Theorem 1 and the Gabor analysis in [22]. (See also Proposition 3.6 and Theorem 3.7 in [47]).

**Proposition 1.6.** Let $\omega, \psi \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $\nu$-moderate, $p,q \in (0,\infty]^d$ and let $r \in (0,1]$ be such that $r \leq \min(p,q)$. Then there is an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, there are $\phi \in M_{(\omega)}^{r}(\mathbb{R}^d)$ and $\psi \in \sum_1(\mathbb{R}^d)$ such that

$$f = \sum_{j,i \in \mathbb{Z}^d} (V_\phi f)(j,i)e^{i\langle \cdot,\cdot \rangle} \psi(\cdot - j)$$

$$= \sum_{j,i \in \mathbb{Z}^d} (V_\psi f)(j,i)e^{i\langle \cdot,\cdot \rangle} \phi(\cdot - j), \quad f \in M_{(\omega)}^{\infty}(\mathbb{R}^d), \quad (1.15)$$

with convergence of the series with respect to the weak$^*$ topology. Furthermore,

$$\|f\|_{M_{(\omega)}^{p,q}} \approx \|V_\phi f\|_{\ell_{(\omega)}^{p,q}(\mathbb{Z}^{2d})} \approx \|V_\psi f\|_{\ell_{(\omega)}^{p,q}(\mathbb{Z}^{2d})}.$$ 

If in addition $\max(p,q) < \infty$, then the series in (1.15) converge unconditionally to $f$ with respect to the $M_{(\omega)}^{p,q}$ norm.

The same holds true with $W_{(\omega)}^{p,q}$ and $\ell_{(\omega)}^{p,q}$ in place of $M_{(\omega)}^{p,q}$ and $\ell_{(\omega)}^{p,q}$, respectively, at each occurrence.

1.5. Pseudo-differential operators. Next we discuss some issues in pseudo-differential calculus. Let $M(d,\Omega)$ be the set of all $d \times d$-matrices with entries in the set $\Omega$, and let
\( a \in \Sigma_1(\mathbb{R}^{2d}) \) and \( A \in \mathbf{M}(d, \mathbb{R}) \) be fixed. Then the pseudo-differential operator \( \text{Op}_A(a) \) with symbol \( a \) is the linear and continuous operator from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma_1(\mathbb{R}^d) \), defined by

\[
(\text{Op}_A(a)f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(x-A(x-y),\xi)f(y)e^{i(x-y,\xi)}\,dyd\xi, \tag{1.16}
\]

when \( f \in \Sigma_1(\mathbb{R}^d) \). For general \( a \in \Sigma_1(\mathbb{R}^{2d}) \), the pseudo-differential operator \( \text{Op}_A(a) \) is defined as the linear and continuous operator from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma'_1(\mathbb{R}^d) \) with distribution kernel given by

\[
K_{a,A}(x,y) = (2\pi)^{-d} \hat{a}(\mathcal{F}_2^{-1}a)(x-A(x-y),x-y). \tag{1.17}
\]

Here \( \mathcal{F}_2F \) is the partial Fourier transform of \( F(x,y) \in \Sigma'_1(\mathbb{R}^{2d}) \) with respect to the \( y \) variable. This definition makes sense, since the mappings

\[
\mathcal{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x-A(x-y),x-y)
\]

are homeomorphisms on \( \Sigma'_1(\mathbb{R}^{2d}) \). In particular, the map \( a \mapsto K_{a,A} \) is a homeomorphism on \( \Sigma'_1(\mathbb{R}^{2d}) \).

We set \( \text{Op}_t(a) = \text{Op}_{tI}(a) \), when \( t \in \mathbb{R} \) and \( I = I_d \in \mathbf{M}(d, \mathbb{R}) \) is the \( d \times d \) identity matrix. The normal or Kohn-Nirenberg representation, \( a(x,D) \), is obtained when \( t = 0 \), and the Weyl quantization, \( \text{Op}^w(a) \), is obtained when \( t = \frac{1}{2} \). That is,

\[
a(x,D) = \text{Op}_0(a) \quad \text{and} \quad \text{Op}^w(a) = \text{Op}_{1/2}(a).
\]

The following result explains the relationship between \( a_1 \) and \( a_2 \) in the identity \( \text{Op}_A(a_1) = \text{Op}_A(a_2) \). We refer to Propositions 1.1 and 2.8 [15] for the proof (see also [34] for background ideas).

**Proposition 1.7.** Let \( p,q \in (0,\infty] \), \( A, A_1, A_2 \in \mathbf{M}(d, \mathbb{R}) \), \( \omega \in \mathcal{P}_E(\mathbb{R}^{4d}) \) and let

\[
\omega_A(x,\xi,\eta,y) = \omega(x+Ay,\xi+\xi^*,\eta,y). \tag{1.19}
\]

Then the following is true:

1. \( e^{i(AD_\xi,D_x)} \) is homeomorphic on \( \Sigma_1(\mathbb{R}^{2d}) \) and uniquely extendable to a homeomorphism on \( \Sigma'_1(\mathbb{R}^{2d}) \);
2. if \( a_1, a_2 \in \Sigma'_1(\mathbb{R}^{2d}) \), then
   \[
   \text{Op}_{A_1}(a_1) = \text{Op}_{A_2}(a_2) \iff e^{i(A_1D_\xi,D_x)}a_1(x,\xi) = e^{i(A_2D_\xi,D_x)}a_2(x,\xi); \tag{1.20}
   \]
3. \( e^{i(AD_\xi,D_x)} \) on \( \Sigma'_1(\mathbb{R}^{2d}) \) restricts to a homoeomorphism from \( M^{p,q}_p(\omega)(\mathbb{R}^{2d}) \) to \( M^{p,q}_p(\omega)(\mathbb{R}^{2d}) \).

Note here that the latter equality in (1.20) makes sense since it is equivalent to

\[
e^{i(A_2x,\xi)}a_2(\xi,x) = e^{i(A_1x,\xi)}a_1(\xi,x),
\]

and that the map \( a \mapsto e^{i(AX,\xi)}a \) is continuous on \( \Sigma'_1(\mathbb{R}^{2d}) \) (cf. e.g. [50],[51]).

The next result is a slight extension of [50, Theorem 3.1], and follows from [50, Theorem 3.1] and Proposition [17]. The details are left for the reader.

**Theorem 1.8.** Let \( A \in \mathbf{M}(d, \mathbb{R}) \), \( \sigma \in S_{2d}, \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^{4d}) \) be such that

\[
\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \lesssim \omega_0(x-A(x-y),\eta-A^*(\eta-\xi),\xi-\eta,y-x). \tag{1.21}
\]

Also let \( p_1, p_2 \in [0,\infty]^{2d} \), \( p, q \in (0,\infty] \) be such that

\[
\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{p} + \min \left( \frac{1}{2}, \frac{1}{q} - 1 \right), \quad q \leq \min(p_2) \leq \max(p_2) \leq p, \tag{1.22}
\]
hold, and let \( a \in M^{p,q}_{\omega_0}(\mathbb{R}^d) \). Then \( \text{Op}_A(a) \) from \( S_{1/2}(\mathbb{R}^d) \) to \( S'_{1/2}(\mathbb{R}^d) \) extends uniquely to a continuous map from \( M^{p,q}_{\omega_1}(\mathbb{R}^d) \) to \( M^{p,q}_{\omega_2}(\mathbb{R}^d) \), and

\[
\| \text{Op}_A(a)f \|_{M^{p,q}_{\omega_2}} \lesssim \|a\|_{M^{p,q}_{\omega_0}} \|f\|_{M^{p,q}_{\omega_1}}, \quad a \in M^{p,q}_{\omega_0}(\mathbb{R}^d), \quad f \in M^{p,q}_{\omega_1}(\mathbb{R}^d). \tag{1.23}
\]

By choosing \( \sigma \) as the identity map, Theorem 1.8 gives continuity properties for pseudo-differential operators acting on modulation spaces of standard type, i.e. acting between spaces of the form \( M^{p,q}_{\omega}(\mathbb{R}^d) \), where \( p, q \in (0, \infty] \) or \( p, q \in (0, \infty]^d \) (see e.g. [11, 13, 45, 48]). If instead \( \sigma \) equals \( \sigma_2 \) in (1.6), Theorem 1.8 gives continuity properties for pseudo-differential operators acting on Wiener amalgam type spaces of the form \( W^{p,q}_{\omega}(\mathbb{R}^d) \). We remark that such continuity properties were also established in [7].

Since \( W^{p,q}_{\omega}(\mathbb{R}^d) \) spaces are Fourier images of \( M^{p,q}_{\omega}(\mathbb{R}^d) \) spaces, any continuity result valid for \( M^{p,q}_{\omega}(\mathbb{R}^d) \) spaces can be transformed into a continuity result for \( W^{p,q}_{\omega}(\mathbb{R}^d) \) spaces (see e.g. [4] for ideas on such transitions).

For \( A = 0 \), i.e. the standard or Kohn-Nirenberg case, (1.24) becomes

\[
\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \lesssim \omega_0(x, \eta, \xi - \eta, y - x), \tag{1.24}
\]

and Theorem 1.8 implies

\[
\text{Op}_0(a) : M^{q', p'}_{\omega_1}(\mathbb{R}^d) \to M^{p,q}_{\omega_0}(\mathbb{R}^d), \quad a \in M^{p,q}_{\omega_0}(\mathbb{R}^d), \quad p, q \in [1, \infty], \quad q \leq p. \tag{1.25}
\]

For slightly relaxed conditions on \( p \) and \( q \) we also have

\[
\text{Op}_0(a) : M^{q', p'}_{\omega_1}(\mathbb{R}^d) \to W^{p,q}_{\omega_0}(\mathbb{R}^d), \quad a \in W^{p,q}_{\omega_0}(\mathbb{R}^d), \quad p, q \in [1, \infty], \tag{1.26}
\]

correcting mapping properties for pseudo-differential operators with symbols in Wiener amalgam type spaces \( W^{p,q}_{\omega_0}(\mathbb{R}^d) \). (See e.g. [13, Theorem 3.9].) We observe that the condition \( q \leq p \) in (1.26) is removed in (1.26).

1.6. Pseudo-differential operators of amplitude types. Let \( a \in \Sigma_1(\mathbb{R}^d) \). Then the pseudo-differential operator \( \text{Op}(a) \) of amplitude type with amplitude \( a \) is the linear and continuous operator from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma_1(\mathbb{R}^d) \), defined by

\[
(\text{Op}(a)f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(x, y, \xi) f(y) e^{i(x-y, \zeta)} dy d\zeta. \tag{1.27}
\]

The definition of \( \text{Op}(a) \) extends to more general \( a \). For example, in Section 3 we observe that \( \text{Op}(a) \) makes sense when \( a \) belongs to certain modulation spaces.

Let \( A \in M(d, \mathbb{R}) \) be fixed. It is evident that the definition of \( \text{Op}_A(a_0) \) is a special case of \( \text{Op}(a) \), since we may choose

\[
a(x, y, \zeta) = a_0(x - A(x - y), \zeta). \tag{1.28}
\]

On the other hand, it follows by Fourier inversion formula in combination with kernel theorems for functions and distributions, it follows that any continuous and linear operator from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma_1'(\mathbb{R}^d) \) is given by \( \text{Op}_A(a_0) \) for a unique \( a_0 \in \Sigma_1(\mathbb{R}^{2d}) \). Consequently, the set of linear and continuous operators from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma_1'(\mathbb{R}^d) \) is not enlarged when passing from operators of the form \( \text{Op}_A(a_0) \) into the form \( \text{Op}(a) \).

In particular, if \( a \in \Sigma_1(\mathbb{R}^{2d}) \), then there is a unique \( a_0(\mathbb{R}^{2d}) \in \Sigma_1'(\mathbb{R}^{2d}) \) such that \( \text{Op}(a) = \text{Op}_0(a_0) \). By straightforward computations it follows that

\[
\text{Op}_0(a_0) = \text{Op}(a). \tag{1.28}
\]
when
\[ a_0(x, \zeta) = \left. (e^{i(D\cdot D_y) a})(x, y, \zeta) \right|_{y=x} = \left. (e^{i(D\cdot D_y) a})(x, x + y, \zeta) \right|_{y=0} \] (1.29)
(see e.g. [34]).

2. Trace properties of modulation spaces

In this section we deduce continuity properties of trace mappings on modulation spaces. Especially we extend such mapping properties to modulation spaces with general moderate weights.

More precisely, for any fixed \( z \in \mathbb{R}^d \), we consider continuity of the trace function map which takes a suitable function or distribution \( f(x_1, x_2, x_3) \) into
\[ (\text{Tr}_z f)(x_1, x_3) \equiv f(x_1, z, x_3), \quad x_1 \in \mathbb{R}^d, \ x_3 \in \mathbb{R}^d. \] (2.1)
Here \( z \in \mathbb{R}^d \) is fixed and \( x_j \in \mathbb{R}^d \) are variables, \( j = 1, 2, 3 \). By straight-forward computations it follows that Tr\(_z\) is a linear and continuous map from \( C^\infty(\mathbb{R}^{d_1+d_2+3}) \) to \( C^\infty(\mathbb{R}^{d_1+d_2}) \), and that similar fact holds with \( \Sigma_s, \mathcal{S}_s \) or \( \mathcal{S} \) in place of \( C^\infty \) at each occurrence.

**Remark 2.1.** Let \( d = d_1 + d_2 + d_3 \), \( z \in \mathbb{R}^d \) be fixed, \( x_j \in \mathbb{R}^d \) for \( j = 1, 2, 3 \),
\[ f \in \Sigma_1(\mathbb{R}^d) \quad \text{and} \quad g_z(x_0) = f(x_1, z, x_3), \quad x_0 = (x_1, x_3). \]
Also let \( d_0 = d_1 + d_3 \),
\[ \phi_0 \in \Sigma_1(\mathbb{R}^{d_0}) \setminus 0, \quad \phi_2 \in \Sigma_1(\mathbb{R}^{d_2}) \setminus 0 \quad \text{and} \quad \phi(x_1, x_2, x_3) = \phi_0(x_0)\phi_2(x_2). \]
If \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^d \), \( \xi_j \in \mathbb{R}^d \) and \( \xi_0 = (\xi_1, \xi_3) \), then
\[ (V_{\phi_0} g_z)(x_0, \xi_0) = (2\pi)^{-\frac{d_0}{2}} \| \phi_2 \|^{-2}_{L^2} \int_{\mathbb{R}^{2d_2}} V_{\phi_0} f_Y(x_0, \xi_0) \phi_2(z - y) e^{-i(z, \eta)} \, dY, \] (2.2)
where
\[ f_Y(x_0) = (V_{\phi_2}(f(x_1, \cdot, x_3)))(y, \eta), \quad Y = (y, \eta) \in \mathbb{R}^{2d_2}. \] (2.3)

In fact, (2.2) follows by first evaluating the integral with respect to \( \eta \) and using Parseval’s formula, and thereafter integrate with respect to \( y \). For future references we notice that (2.2) is the same as
\[ (V_{\phi_0} g_z)(x_0, \xi_0) = (2\pi)^{-\frac{d_0}{2}} \| \phi_2 \|^{-2}_{L^2} \int_{\mathbb{R}^{2d_2}} V_{\phi} f(x_1, y, x_3, \xi_1, \eta, \xi_3) \phi_2(z - y) e^{-i(z, \eta)} \, dy \, d\eta. \] (2.2’)

For modulation spaces we have the following trace result. Here the involved Lebesgue exponents and weight functions should satisfy conditions of the form
\[ p_0 = (p_1, p_3), \quad p = (p_1, p_2, p_3), \quad q_0 = (q_1, q_3), \quad q = (q_1, q_2, q_3), \] (2.4)
\[ \max \left( \frac{1}{p_0}, \frac{1}{q_1}, \frac{1}{q_2}, 1 \right) - 1 \quad \frac{1}{q_2} \leq \frac{1}{r}, \] (2.5)
\[ \sup_{\xi_1 \in \mathbb{R}^d} \left( \frac{\sup_{(x, \xi_1) \in \mathbb{R}^{d+1}} \left( \frac{\omega_0(x_0, \xi_0) e^{-r_0|x_1|}}{\omega(x, \xi_1, \cdot, \xi_3)} \right) \right) \left( L^r(\mathbb{R}^d) \right) < \infty \] (2.6)
and
\[ \omega(x, \xi) \lesssim \omega_0(x_0, \xi_0) e^{-r_0|\xi_2|}. \tag{2.7} \]
Here
\[ x = (x_1, x_2, x_3) \in \mathbb{R}^d_1 \times \mathbb{R}^d_2 \times \mathbb{R}^d_3, \quad x_0 = (x_1, x_3) \]
\[ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^d_1 \times \mathbb{R}^d_2 \times \mathbb{R}^d_3 \quad \text{and} \quad \xi_0 = (\xi_1, \xi_3). \tag{2.8} \]

We observe that (2.6) is the same as
\[ \omega_0(x_0, \xi_0) e^{-r_0|x_2|} \lesssim \omega(x, \xi) \vartheta(\xi_2, \xi_3), \quad C_0 \equiv \sup_{\xi_3 \in \mathbb{R}^d_3} \| \vartheta(\cdot, \xi_3) \|_{L^r} < \infty, \tag{2.9} \]
where \( \vartheta \in \mathcal{P}_E(\mathbb{R}^{d_2+d_3}) \) is given by
\[
\vartheta(\xi_2, \xi_3) \equiv \sup_{x \in \mathbb{R}^d} \left( \sup_{\xi_1 \in \mathbb{R}^d_1} \left( \frac{\omega_0(x_0, \xi_0) e^{-r_0|x_2|}}{\omega(x, \xi)} \right) \right), \tag{2.10} \]
which indicates similarities between the conditions (2.6) and (2.7). We observe

**Theorem 2.2.** Let \( d_j \geq 0 \) be integers, \( z \in \mathbb{R}^{d_2} \) be fixed, \( p_j, q_j \in (0, \infty)^{d_j}, \) \( r \in (0, \infty)^{d_2}, \) \( j = 1, 2, 3, \) be such that (2.12) holds and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2(d_2+d_3)}), \) \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^{2(d_2+d_3)}), \) \( p, p_0, q \) and \( q_0 \) be such that (2.1) and (2.2) hold true for some \( r_0 \geq 0. \) Then the following is true:

1. The map \( \text{Tr}_z \) from \( \Sigma_1(\mathbb{R}^{d_2+d_3}) \) to \( \Sigma_1(\mathbb{R}^{d_1+d_3}) \) extends uniquely to a continuous map from \( M^{p_0,q_0}(\mathbb{R}^{d_2+d_3}) \) to \( M^{p_0,q_0}(\mathbb{R}^{d_1+d_3}), \) and
2. if in addition (2.7) holds true for some \( r_0 \geq 0, \) then the map in (1) is surjective.

For modulation spaces of Wiener-amalgam types we also have the following. Here the conditions (2.5) and (2.6) are relaxed into
\[
\left\{ \frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{r} \right\} - \frac{1}{q_2} \leq 1, \tag{2.12} \]
and
\[ \sup_{(x, \xi_3) \in \mathbb{R}^{d_2+d_3}} \left( \sup_{(\xi_1, \xi_3) \in \mathbb{R}^{d_3}} \left( \frac{\omega_0(x_1, x_3, \xi_1, \xi_3) e^{-r_0|x_2|}}{\omega(x_1, x_2, x_3, \xi_1, \xi_3, \xi_3)} \right) \right) < \infty. \tag{2.6} \]

**Theorem 2.3.** Let \( d_j \geq 0 \) be integers, \( z \in \mathbb{R}^{d_2} \) be fixed, \( p_j, q_j \in (0, \infty)^{d_j}, \) \( r \in (0, \infty)^{d_2}, \) \( j = 1, 2, 3, \) be such that (2.12) holds and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2(d_2+d_3)}), \) \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^{2(d_2+d_3)}), \) \( p, p_0, q \) and \( q_0 \) be such that (2.1) and (2.6) hold true for some \( r_0 \geq 0. \) Then the following is true:

1. The map \( \text{Tr}_z \) from \( \Sigma_1(\mathbb{R}^{d_2+d_3}) \) to \( \Sigma_1(\mathbb{R}^{d_1+d_3}) \) extends uniquely to a continuous map from \( W^{p_0,q_0}(\mathbb{R}^{d_2+d_3}) \) to \( W^{p_0,q_0}(\mathbb{R}^{d_1+d_3}), \) and
2. if in addition (2.7) holds true for some \( r_0 \geq 0, \) then the map in (1) is surjective.
For the proof of Theorems 2.2 and 2.3 we recall the Young type inequality
\[ \|f_1 * f_2\|_{\ell^p_\omega} \leq \|f_1\|_{\ell^p_\omega} \|f_2\|_{\ell^{p,1}_{\omega}} \quad f_1 \in \ell^p_\omega(\mathbb{Z}^d), \ f_2 \in \ell^{min[p,1]}_{\omega}(\mathbb{Z}^d) \]  \hspace{1cm} (2.14)
for discrete Lebesgue spaces, when \(\omega, v \in \mathcal{E}(\mathbb{R}^d)\) satisfy \(\omega(x + y) \leq \omega(x)v(y)\). This gives
\[ \|f \ast e^{-r \cdot 1} \|_{\ell^p_\omega(\mathbb{Z}^d)} \lesssim \|f\|_{\ell^p_\omega(\mathbb{Z}^d)}, \quad f \in \ell^p_\omega(\mathbb{Z}^d), \]  \hspace{1cm} (2.15)
provided \(r > 0\) is chosen large enough.

**Remark 2.4** Let \(p \in (0, 1], \mathcal{B}\) be a vector space and let \(\| \cdot \|_s\) be a \(p\)-norm, i.e. \(\| \cdot \|_s\) is a quasi-norm on \(\mathcal{B}\) which fulfills
\[ \|f + g\|_s^p \leq \|f\|_s^p + \|g\|_s^p, \quad f, g \in \mathcal{B}. \]
Then it follows that
\[ \|f + g\|_s^q \leq \|f\|_s^q + \|g\|_s^q, \quad f, g \in \mathcal{B}, \]
for every \(q \in (0, p]\).

**Proof of Theorem 2.2** First we prove (1). Let \(d = d_1 + d_2 + d_3, d_0 = d_1 + d_3, \rho = \min(p_0, q_1, q_2, 1), \rho_0 = \min(p, q_1) = \min(p, q_3)\) and \(g_2 = \text{Tr}_z f\) and \(v \in \mathcal{E}(\mathbb{R}^d)\) be such that \(\omega\) is \(v\)-moderate. Also let \(x, \xi, x_0, \xi_0\) and \(\vartheta\) be given by (2.8) and (2.10). Then \(\vartheta \in \mathcal{E}(\mathbb{R}^{d_2 + d_3})\) fulfills (2.9).

First suppose that \(f \in \Sigma_1(\mathbb{R}^d)\). By Proposition 1.6 there are \(\phi_m \in M_{(\omega)}^0(\mathbb{R}^d), \psi_m \in \Sigma_1(\mathbb{R}^{dm}), m = 1, 2, 3\) such that
\[ f(x) = c_\varepsilon \sum_{j, i \in \mathbb{Z}^d} V_\phi f(j, \iota)e^{i(x, \iota)}\psi(x - j), \]
where
\[ \phi = \phi_1 \otimes \phi_2 \otimes \phi_3 \quad \text{and} \quad \psi = \psi_1 \otimes \psi_2 \otimes \psi_3. \]

We have
\[ g_2(x_1, x_3) = c_\varepsilon \sum_{j, i \in \mathbb{Z}^d} V_\phi f(j, \iota)e^{i((x_1, \iota_1)+(x_2, \iota_2)+(x_3, \iota_3))}\psi(x_1 - j_1, z - j_2, x_3 - j_3), \]  \hspace{1cm} (2.16)
when
\[ g_2(x_1, x_3) = f(x_1, z, x_3), \]  \hspace{1cm} (2.17)
and observe that the series possess strong convergence properties, because
\[ |V_\phi f(j, \iota)| \lesssim e^{-r_1(|j| + |\iota|)} \quad \text{and} \quad |\psi(x_1 - j_1, z - j_2, x_3 - j_3)| \lesssim e^{-r_1(|x_1 - j_1| + |j_2| + |x_3 - j_3|)} \]
for every \(r_1 > 0\).

An application of the short-time Fourier transform on (2.10) gives
\[ V_{\psi_0} g_2(x_0, \xi_0) \]
\[ = c_\varepsilon \sum_{j, i \in \mathbb{Z}^d} V_\phi f(j, \iota)e^{i((\xi, \iota_2)+(j_0, \xi_0 - \xi_0))}(V_{\psi_0} \psi_0)(x_0 - j_0, \xi_0 - \xi_0)\psi_2(z - j_2), \]  \hspace{1cm} (2.18)
where
\[ j_0 = (j_1, j_3), \quad \iota_0 = (\iota_1, \iota_3) \quad \text{and} \quad \psi_0 = \psi_1 \otimes \psi_3. \]
By letting
\[ F_\omega = |V_0 f \cdot \omega| \quad \text{and} \quad F_{0,\omega_0} = |V_0 g_2 \cdot \omega_0|, \]
and using that
\[ \omega_0(x_0, \xi_0) \lesssim \omega(x_1, j_2, x_3, \xi_1, t_2, \xi_3)e^{r_0|j_2|}\vartheta(t_2, \lambda_3) \lesssim \omega(j, i)e^{r_0(|x_0-j_0|+|j_2|+|\xi_0-\xi_1|)}\vartheta(t_2, \lambda_3). \]
and
\[ |(V_\psi V_\theta)(x_0 - j_0, \xi_0 - t_0)\psi_2(z - j_2)e^{r_0(|x_0-j_0|+|j_2|+|\xi_0-\xi_1|)}| \leq e^{-r_1(|x_0-j_0|+|j_2|+|\xi_0-\xi_1|)} \]
for every \( r_1 > 0 \), it follows from (2.18) that
\[ F_{0,\omega_0}(l_0, \lambda_0) \lesssim \sum_{j, i \in \mathcal{E} \mathbb{Z}^d} F_\omega(j, i)e^{-r_1(|l_0-j_0|+|j_2|+|\lambda_0-\lambda_1|)}\vartheta(t_2, \lambda_3), \quad r_1 > 0, \tag{2.19} \]
where
\[ l_0 = (l_1, l_3) \in \mathcal{E} \mathbb{Z}^{d_0} \quad \text{and} \quad \lambda_0 = (\lambda_1, \lambda_3) \in \mathcal{E} \mathbb{Z}^{d_0}. \]
Since \( \rho \in (0, 1] \), (2.19) gives
\[ F_{0,\omega_0}(l_0, \lambda_0)\rho \lesssim \sum_{j, i \in \mathcal{E} \mathbb{Z}^d} F_\omega(j, i)\rho e^{-r_1(|l_0-j_0|+|j_2|+|\lambda_0-\lambda_1|)}\vartheta(t_2, \lambda_3)\rho, \quad r_1 > 0. \tag{2.19}' \]
Now let
\[ \Lambda_1 = \mathcal{E} \mathbb{Z}^{d_2+d_3+d}, \quad \Lambda_2 = \mathcal{E} \mathbb{Z}^{d_3+d}, \quad \Lambda_3 = \mathcal{E} \mathbb{Z}^{d_2+d_3}, \]
\[ F_{p_1,\omega}(x_2, x_3, \xi) = \|F_\omega(\cdot, x_2, x_3, \xi)\|_{p_1(\mathcal{E} \mathbb{Z}^{d_1})}, \quad F_{p_1,p_2,\omega}(x_3, \xi) = \|F_\omega(\cdot, x_3, \xi)\|_{p_1,p_2(\mathcal{E} \mathbb{Z}^{d_1+d_2})}, \]
\[ F_{p,\omega}(\xi) = \|F_\omega(\cdot, \xi)\|_{p(\mathcal{E} \mathbb{Z}^d)}, \quad F_{p,q_1,\omega}(\xi_2, \xi_3) = \|F_\omega(\cdot, \xi_2, \xi_3)\|_{p,q_1(\mathcal{E} \mathbb{Z}^{d_1+d_2})}, \]
\[ F_{p,q_1,q_2,\omega}(\xi_3) = \|F_\omega(\cdot, \xi_3)\|_{p,q_1,q_2(\mathcal{E} \mathbb{Z}^{d_1+d_2})} \]
and
\[ G_{r_1,\rho,\omega}(l_1, j_2, j_3, t) = (F_\omega(\cdot, j_2, j_3, t)\rho * e^{-r_1| \cdot |})(l_1), \]
where the convolution is the discrete convolution with respect to \( \mathcal{E} \mathbb{Z}^{d_1} \). By (2.19)' it follows that
\[ F_{0,\omega_0}(l_0, \lambda_0)\rho \lesssim \sum_{(j_2, j_3, t) \in \Lambda_1} (F_\omega(\cdot, j_2, j_3, t)\rho * e^{-r_1| \cdot |})(l_1)e^{-r_1(|j_2|+|j_3-t_2|+|\lambda_0-\lambda_1|)}\vartheta(t_2, \lambda_3)\rho, \]
for every \( r_1 > 0 \). If we apply the \( \ell^{p_1/\rho} \) norm with respect to \( l_1 \) variable, and use Minkowski’s, Young’s and Hölder’s inequalities, we obtain

\[
\| F_{0,\omega_0}(\cdot, \lambda_0) \|_{\ell^{p_1/\rho}} \leq \| F_{0,\omega_0}(\cdot, l_3, \lambda_0) \|_{\ell^{p_1}}
\]

\[
\lesssim \sum_{(j_2,j_3,t) \in A_1} \| G_{r_1,\rho,\omega,\lambda_0}(j_2,j_3,t) e^{-r_1 (|j_2| + |j_3| + |\lambda_0|)} \theta(t_2, \lambda_3) \|_{\ell^{p_1/\rho}}
\]

\[
\lesssim \sum_{(j_2,j_3,t) \in A_1} \| F_{r_1,\omega_0}(\cdot, j_2, j_3, t) e^{-r_1 (|j_2| + |j_3| + |\lambda_0|)} \theta(t_2, \lambda_3) \|_{\ell^{p_1/\rho}}
\]

\[
\lesssim \sum_{(j_2,j_3,t) \in A_1} \| G_{r_1,\rho,\omega,\lambda_0}(j_2,j_3,t) \|_{\ell^{p_1/\rho}} \| e^{-r_1 (|j_2| + |j_3| + |\lambda_0|)} \theta(t_2, \lambda_3) \|_{\ell^{p_1/\rho}}
\]

\[
= \sum_{(j_2,j_3,t) \in A_1} F_{r_1,\omega_0}(j_2,j_3,t) e^{-r_1 (|j_2| + |j_3| + |\lambda_0|)} \theta(t_2, \lambda_3)
\]

\[
\lesssim \sum_{(j_2,j_3,t) \in A_1} \| F_{r_1,\omega_0}(j_2,j_3,t) \|_{\ell^{p_1/\rho}} e^{-r_1 (|j_2| + |j_3| + |\lambda_0|)} \theta(t_2, \lambda_3) \rho
\]

for every \( r_1 > 0 \). In the last inequality we have used the fact that \( \| f \|_{\ell^p(\mathbb{Z}^{d_2})} \leq \| f \|_{\ell^p(\mathbb{Z}^{d_2})} \) for every sequence \( f \) on \( \mathbb{Z}^{d_2} \) and \( p \in (0, \infty) \). Hence, if

\[
G_{r_1,\omega_0}(l_3,\lambda) = \left( F_{r_1,\omega_0}(\cdot, \cdot) \right)^{\rho} e^{-r_1 |\cdot|}(l_3)
\]

where the convolution is the discrete convolution with respect to \( \varepsilon \mathbb{Z}^{d_0} \), we get

\[
\| F_{0,\omega_0}(\cdot, \lambda_0) \|_{\ell^{p_1}} \lesssim \sum_{l \in \mathbb{Z}^d} G_{r_1,\omega_0}(l_3,\lambda) e^{-r_1 |\lambda_0|} \theta(t_2, \lambda_3)^\rho, \quad r_1 > 0.
\]

An application of the \( \ell^{p_1/\rho} \) norm with respect to \( l_3 \) variable, and using Minkowski’s, Young’s and Hölder’s inequalities now give

\[
\| F_{0,\omega_0}(\cdot, \lambda_0) \|_{\ell^{p_0}} \lesssim \sum_{l \in \mathbb{Z}^d} G_{r_1,\omega_0}(l_3,\lambda) e^{-r_1 |\lambda_0|} \theta(t_2, \lambda_3)^\rho
\]

\[
\lesssim \sum_{l \in \mathbb{Z}^d} \| F_{r_1,\omega_0}(\cdot, \cdot) \|_{\ell^{p_1/\rho}} e^{-r_1 |\lambda_0|} \theta(t_2, \lambda_3)^\rho
\]

\[
\lesssim \sum_{l \in \mathbb{Z}^d} \| F_{r_1,\omega_0}(\cdot, \cdot) \|_{\ell^{p_1/\rho}} e^{-r_1 |\lambda_0|} \theta(t_2, \lambda_3)^\rho
\]

\[
= \sum_{l \in \mathbb{Z}^d} F_{r,\omega_0}(l) e^{-r_1 |\lambda_0|} \theta(t_2, \lambda_3)^\rho
\]

for every \( r_1 > 0 \). That is

\[
\| F_{0,\omega_0}(\cdot, \lambda_0) \|_{\ell^{p_0}} \lesssim \sum_{(l_2,l_3) \in A_3} G_{r,\omega_0}(\lambda_1, \lambda_2, \lambda_3) e^{-r_1 |\lambda_2-\lambda_3|} \theta(t_2, \lambda_3)^\rho, \quad r_1 > 0.
\]
where
\[ G_{p,r_1,\rho,\omega}(t) = (F_{p,\omega}(\cdot, t_2, t_3))^\rho * e^{-r_1|\cdot|}(t_1) \]
and the convolution is the discrete convolution with respect to $\varepsilon Z^d_1$. If we apply the $\ell_{\rho,1}$ norm with respect to $\lambda_1$ variable, and use Minkowski’s, Young’s and Hölder’s inequalities, we obtain

\[
\|F_{0,\omega_0}(\cdot, \lambda_3)\|_{\ell_{\rho,1}} \lesssim \left\| \sum_{(t_2, t_3) \in A_3} G_{p,r_1,\rho,\omega}(\lambda_1, t_2, t_3)e^{-r_1|\lambda_3-t_3|}\vartheta(t_2, \lambda_3)^\rho \right\|_{\ell_{\rho,1}}
\leq \sum_{(t_2, t_3) \in A_3} \|F_{p,\omega}(\cdot, t_2, t_3)^\rho * e^{-r_1|\cdot|}\|_{\ell_{\rho,1}}e^{-r_1|\lambda_3-t_3|}\vartheta(t_2, \lambda_3)^\rho
\leq \sum_{(t_2, t_3) \in A_3} \|F_{p,\omega}(\cdot, t_2, t_3)^\rho\|_{\ell_{\rho,1}}\|\vartheta(\cdot, \lambda_3)^\rho\|_{\ell_{\rho,1}} e^{-r_1|\lambda_3-t_3|}
= \sum_{(t_2, t_3) \in A_3} F_{p,q_1,\omega}(t_2, t_3)^\rho e^{-r_1|\lambda_3-t_3|}\vartheta(t_2, \lambda_3)^\rho
\leq C_{q_1} \sum_{t_3 \in Z^d_3} F_{p,q_1,\omega}(t_3)^\rho e^{-r_1|\lambda_3-t_3|} \lesssim (F_{p,q_1,\omega}^\rho * e^{-r_1|\cdot|})(\lambda_3)
\]
for every $r_1 > 0$. Here $C_{q_1}$ is given by (2.22).

By applying the $\ell_{\rho,1}$ quasi-norm on the latter inequality we obtain

\[
\|F_{0,\omega_0}\|_{\ell_{\rho,1}(\varepsilon Z^{d_3})} \lesssim \left\| F_{p,q_1,\omega}^\rho \right\|_{\ell_{\rho,1}(\varepsilon Z^{d_3})} \leq \left\| F_{p,q_1,\omega}^\rho \right\|_{\ell_{\rho,1}(\varepsilon Z^{d_3})} \|e^{-r_1|\cdot|}\|_{\ell_{\rho,1}(\varepsilon Z^{d_3})}
\]
for every $r_1 > 0$, which gives

\[
\|F_{0,\omega_0}\|_{\ell_{\rho,1}(\varepsilon Z^{d_3})} \lesssim \|F_{p,q_1,\omega}^\rho \|_{\ell_{\rho,1}(\varepsilon Z^{d_3})} = \|F_{\omega}^\rho\|_{\ell_{\rho,1}(\varepsilon Z^{d_3})}.
\]
This gives (2.11) for $f \in \Sigma(R^{d_1+d_2+d_3})$, i.e.

\[
\| Tr_2 f \|_{M^{\rho,\omega}_{\varphi_0}} \lesssim \| f \|_{M^{\rho,\omega}_{\varphi_0}}, \quad f \in \Sigma_1(R^{d_1+d_2+d_3}) \quad (2.11)'
\]
The assertion now follows in the case $\max(p, q) < \infty$ from (2.11)' and the fact that $\Sigma_1(R^{d_1+d_2+d_3})$ is dense in $M^{\rho,\omega}_{\varphi_0}(R^{d_1+d_2+d_3})$ in view of (1) in Proposition 2.13.

For the case $\max(p, q) = \infty$, we first assume that $\min(p, q) \geq 1$. Then all involved modulation spaces are Banach spaces, and it follows by (2.11)' and Hahn-Banach’s theorem that $Tr_2$ extends to a continuous map from $M^{\rho,\omega}_{\varphi_0}(R^{d_1+d_2+d_3})$ to $M^{\rho,\omega}_{\varphi_0}(R^{d_1+d_3})$.

In order to show that this extension is unique, let

\[
\omega_{0,1}(x_0, \xi_0) = \omega_0(x_0, \xi_0)e^{-r_0(|x_0|+|\xi_0|)}
\]
and 
\[ \omega_1(x, \xi) = \omega(x, \xi) e^{-r_0(|x|+|\xi_0|)} \vartheta(\xi_2, \xi_3). \]

Then
\[ M^{p,q}(\mathbb{R}^d) \hookrightarrow M^{1,1}_{(\omega_1)}(\mathbb{R}^d), \quad M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d) \hookrightarrow M^{1,1}_{(\omega_0, 1)}(\mathbb{R}^d), \] (2.20)
and (2.6) implies
\[ \sup_{\xi_3} \left( \sup_{x, \xi_1} \left( \frac{\omega_{0,1}(x_0, \xi_0)e^{-r_0|\xi_2|}}{\omega(x, \xi_1, \cdot, \xi_3)} \right) \right) \| \|_{L^\infty(\mathbb{R}^d)} < \infty \]
(see also (0.10)). Since the assertion holds in the case \( \max(p, q) < \infty \), it now follows that \( \text{Tr}_z \) is uniquely defined and continuous from \( M^{1,1}_{(\omega_1)}(\mathbb{R}^d) \) to \( M^{1,1}_{(\omega_0, 1)}(\mathbb{R}^d) \). In particular, if \( f \in M^{p,q}_{(\omega)}(\mathbb{R}^d) \), then \( \text{Tr}_z f \) is uniquely defined as an element in \( M^{1,1}_{(\omega_0, 1)}(\mathbb{R}^d) \) due to the inclusions above. The asserted uniqueness now follows from the fact that if \( g_1, g_2 \in M^{p,q}_{(\omega_0)}(\mathbb{R}^d) \) and \( g_1 = g_2 \) as elements in \( M^{1,1}_{(\omega_0, 1)}(\mathbb{R}^d) \), then \( g_1 = g_2 \) as elements in \( M^{p,q}_{(\omega_0)}(\mathbb{R}^d) \).

Finally suppose that \( p \) and \( q \) are general, and let \( \omega_1 \) and \( \omega_{0, 1} \) be as above. Since (2.20) holds when \( \min(p, q) \geq 1 \) and that \( M^{p,q}_{(\omega)}(\mathbb{R}^d) \) increases with \( p \) and \( q \), it follows that (2.20) still holds without the restriction \( \min(p, q) \geq 1 \). Hence, if \( f \in M^{p,q}_{(\omega)}(\mathbb{R}^d) \), then \( \text{Tr}_z f \) is uniquely defined as an element in \( M^{1,1}_{(\omega_0, 1)}(\mathbb{R}^d) \). By the computations which lead to (2.11'), it follows that \( \text{Tr}_z f \in M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d) \), and that (2.11) holds. This gives (1).

It remains to prove (2). Let \( f_0 \in M^{p,q}_{(\omega_0)}(\mathbb{R}^d) \) be fixed, \( \phi_j \) and \( \phi \) be as above, \( \varphi \in \Sigma_1(\mathbb{R}^2) \) be such that \( \varphi(0) = 1 \), and let
\[ f(x_1, x_2, x_3) = f_0(x_1, x_3) \varphi(x_2 - z). \]
We have
\[ V_{\varphi_2} \varphi(x_2, \xi_2) \lesssim e^{-2r_1(|x_2|+|\xi_2|)} \]
for every \( r_1 > r_0 \), in view of Lemma 1.1. Hence, by letting
\[ F = |V_{\varphi_0} f| \cdot \omega, \quad F_0 = |V_{\varphi_0} f_0| \cdot \omega_0 \]
and using (2.7), we obtain
\[ F(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) = |V_{\varphi_0} f_0(x_1, x_3, \xi_1, \xi_2, \xi_3) V_{\varphi_2} \varphi(x_2 - z, \xi_2) \omega(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)| \]
\[ \lesssim F_0(x_1, x_3, \xi_1, \xi_3) e^{-r_0|x_2|} e^{-r_1(|x_2|+|\xi_2|)} \lesssim F_0(x_1, x_3, \xi_1, \xi_3) e^{-r_1(|x_2|+|\xi_2|)}. \]
An application of the \( L^{p,q} \) norm on the latter inequality now gives
\[ \| f \|_{M^{p,q}_{(\omega)}} \lesssim \| g \|_{L_{p_2,q_2}} \| f_0 \|_{M^{p_0,q_0}_{(\omega_0)}} \approx \| f_0 \|_{M^{p_0,q_0}_{(\omega_0)}}, \quad g(x_2, \xi_2) = e^{-r_1(|x_2|+|\xi_2|)}, \]
which in turn implies that \( f \in M^{p,q}_{(\omega)}(\mathbb{R}^d) \), and the surjectivity follows. \( \square \)

**Proof of Theorem 2.2** The result follows by similar arguments as in the proof of Theorem 2.2. In order for clarifying some details, we present the first part of the proof.

...
We use the same notation as in the proof of Theorem 2.2 except that we let $\rho = \min(q_1, q_2, 1)$,

$$
\vartheta(x, \xi_2, \xi_3) \equiv \sup_{\xi_1 \in \mathbb{R}^d} \left( \frac{\omega_0(x_0, \xi_0)e^{-r_0|\xi_2|}}{\omega(x, \xi)} \right) \in \mathcal{P}_E(\mathbb{R}^{d_2+d_3}),
$$

$$
\Lambda_1 = \varepsilon \mathbb{Z}^{d_2+d_3}, \quad \Lambda_2 = \varepsilon \mathbb{Z}^{d_3}, \quad \Lambda_3 = \varepsilon \mathbb{Z}^{d_2+d_3},
$$

$$
F_{q_1, \omega}(x, \xi_2, \xi_3) = \|F_\omega(x, \cdot, \xi_2, \xi_3)\|_{p_1(\varepsilon \mathbb{Z}^{d_1})},
$$

and

$$
G_{r_1, \rho, \omega}(j, \lambda_1, \tau_2, \tau_3) = (F_\omega(j, \cdot, \tau_2, \tau_3)^\rho * e^{-r_1|\cdot|})(\lambda_1),
$$

where the convolution is the discrete convolution with respect to $\varepsilon \mathbb{Z}^{d_1}$. Then

$$
\omega_0(x_0, \xi_0)e^{-r_0|\xi_2|} \leq \vartheta(x, \xi_2, \xi_3)\omega(x, \xi),
$$

and (2.21) implies

$$
C_\rho = \sup_{x, \xi_3} \|\vartheta(x, \cdot, \xi_3)\|_{L^\rho} < \infty.
$$

Suppose that $f \in \Sigma_4(\mathbb{R}^d)$. By (2.19)' and (2.21) it follows that

$$
F_{0, \omega_0}(l_0, \lambda_0)^\rho
\lesssim \sum_{(j, \tau_2, \tau_3) \in \Lambda_1} (F_\omega(j, \cdot, \tau_2, \tau_3)^\rho * e^{-r_1|\cdot|})(\lambda_1)e^{-r_1(|l_0-j_0|+|\tau_2|+|\lambda_3-\varepsilon_3|)}\vartheta(j, \tau_2, \lambda_3)^\rho, \quad r_1 > 0.
$$

If we apply the $\ell^{q_1/\rho}$ norm with respect to the $\lambda_1$ variable, and use Minkowski’s and Young’s inequalities, we obtain

$$
\|F_{0, \omega_0}(l_0, \cdot, \lambda_3)^\rho\|_{\ell^{q_1/\rho}} = \|F_{0, \omega_0}(l_0, \cdot, \lambda_3)^\rho\|_{\ell^{q_1}}
\lesssim \left\| \sum_{(j, \tau_2, \tau_3) \in \Lambda_1} G_{r_1, \rho, \omega}(j, \cdot, \tau_2, \tau_3)e^{-r_1(|l_0-j_0|+|\tau_2|+|\lambda_3-\varepsilon_3|)}\vartheta(j, \tau_2, \lambda_3)^\rho \right\|_{\ell^{q_1/\rho}}
\lesssim \sum_{(j, \tau_2, \tau_3) \in \Lambda_1} \left( \left\| F_\omega(j, \cdot, \tau_2, \tau_3)^\rho * e^{-r_1|\cdot|} \right\|_{\ell^{q_1/\rho}} \right) e^{-r_1(|l_0-j_0|+|\tau_2|+|\lambda_3-\varepsilon_3|)}\vartheta(j, \tau_2, \lambda_3)^\rho
\lesssim \sum_{(j, \tau_2, \tau_3) \in \Lambda_1} F_{q_1, \omega}(j, \tau_2, \tau_3)^\rho e^{-r_1(|l_0-j_0|+|\tau_2|+|\lambda_3-\varepsilon_3|)}\vartheta(j, \tau_2, \lambda_3)^\rho,
$$

for every $r_1 > 0$. By using Hölder’s inequality with respect to the $\tau_2$ variable in the sum we obtain

$$
\|F_{0, \omega_0}(l_0, \cdot, \lambda_3)^\rho\|_{\ell^{q_1/\rho}}
\lesssim \sum_{(j, \tau_3) \in \Lambda_1} \|F_{q_1, \omega}(j, \cdot, \tau_3)^\rho\|_{\ell^{q_2/\rho}} e^{-r_1(|l_0-j_0|+|\tau_3|+|\lambda_3-\varepsilon_3|)}\vartheta(j, \cdot, \lambda_3)^\rho\|_{\ell^{q_2/\rho}}^\rho.
\leq C_\rho \sum_{(j, \tau_3) \in \Lambda_1} \|F_{q_1, \omega}(j, \cdot, \tau_3)^\rho\|_{\ell^{q_2/\rho}} e^{-r_1(|l_0-j_0|+|\tau_3|+|\lambda_3-\varepsilon_3|)}.
for every $r_1 > 0$. The remaining part of the proof is performed in similar ways as in the proof of Theorem 2.2 by applying Minkowski’s and Young’s inequalities in suitable ways. The details are left for the reader.

By letting $\omega$ and $\omega_0$ be related as
\[
\omega_0(x_1, x_3, \xi_1, \xi_3) \leq \omega(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \vartheta(\xi_2, \xi_3),
\]
we get the following special case of Theorem 2.2. The details are left for the reader. □

Proposition 2.5. Let $d_j$ be integers, $z \in \mathbb{R}^{d_2}$ be fixed, $p, p_0, q, q_0$ and $r$ be the same as in Theorem 2.2 and suppose that $\omega \in \mathcal{P}_E(\mathbb{R}^{2(d_1+d_2+d_3)})$, $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2(d_1+d_3)})$ and $\vartheta \in \mathcal{P}_E(\mathbb{R}^{d_2+d_3})$ satisfy (2.22) and (2.22). Then $\mathcal{T}_{r, z}$ is continuous from $M_{p_0, q_0}(\mathbb{R}^{d_1+d_2+d_3})$ to $M_{p, q}(\mathbb{R}^{d_1+d_3})$, and from $W_{p_0, q_0}(\mathbb{R}^{d_1+d_2+d_3})$ to $W_{p, q}(\mathbb{R}^{d_1+d_3})$.

Example 2.6. An important case of Proposition 2.5 appears by letting $\vartheta$ in (2.22) be given by
\[
\vartheta(\xi_2, \xi_3) = \langle (\xi_2, \xi_3) \rangle^{-s} \langle \xi_3 \rangle^{s_0}.
\]
Here
\[
s \geq \frac{d_2}{r}, \quad s_0 \leq s - \frac{d_2}{r},
\]
with the first inequality strict when $r < \infty$. Note that (2.22) is fulfilled for such $\vartheta$. In fact, for $r < \infty$ we have
\[
\|\vartheta(\cdot, \xi_3)\|_{L^r}^r = \int_{\mathbb{R}^{d_2}} \langle (\xi_2, \xi_3) \rangle^{-sr} \langle \xi_3 \rangle^{s_0 r} d\xi_2
\]
\[
= \langle \xi_3 \rangle^{(s_0 - s)r} \int_{\mathbb{R}^{d_2}} \langle \xi_2/\langle \xi_3 \rangle \rangle^{-sr} d\xi_2
\]
\[
= \langle \xi_3 \rangle^{(s_0 - s)r + d_2} \int_{\mathbb{R}^{d_2}} \langle \xi_2 \rangle^{-sr} d\xi_2 \lesssim 1,
\]
where the inequality follows from the assumptions on $s$ and $s_0$. By similar arguments one gets (2.23) in the case when $r = \infty$. The details are left for the reader.

By choosing $\omega_0(x_0, \xi_0) = \langle \xi_3 \rangle^{s_0}, \omega(x, \xi) = \langle (\xi_2, \xi_3) \rangle^s$ and $p = q = 2$, we regain Sobolev’s embedding theorem (see (1.1) in the introduction).

By letting
\[
d_j = d, \quad p_{j,k} = p_j \in (0, \infty] \quad \text{and} \quad q_{j,k} = q_j \in (0, \infty], \quad j \in \{1, 2, 3\}, \quad k \in \{1, \ldots, d\},
\]
and replacing the assumption (2.22) by
\[
\omega_0(x_1, x_3, \xi_1, \xi_3) \leq \omega(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \vartheta(\xi_2, \xi_3),
\]
Theorems 2.2 and 2.3 give the following.

Proposition 2.7. Let $z \in \mathbb{R}^d$ be fixed, $p = (p_1, p_2, p_3) \in (0, \infty]^3$, $q = (q_1, q_2, q_3) \in (0, \infty]^3$ and $r \in (0, \infty)$ be such that (2.23) holds, and suppose that $\omega \in \mathcal{P}_E(\mathbb{R}^{6d})$ and $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{d})$ satisfy (2.23), for some
\[
\theta \geq d \left( \max \left( \frac{1}{p_1}, \frac{1}{p_3}, \frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3} \right) - \frac{1}{q_2} \right)
\]
with strict inequality when
\[
\min(p_1, p_3, q_1, q_3, 1) < q_2.
\]
Then $\text{Tr}_z$ is continuous and surjective $M^{p,q}_\omega(R^{3d})$ to $M^{p_0,q_0}_\omega(R^{2d})$, and from $W^{p,q}_\omega(R^{3d})$ to $W^{p_0,q_0}_\omega(R^{2d})$.

By choosing $p_1 = p_2 = p_3$ and $q_1 = q_2 = q_3$ in the previous proposition, we get the following.

**Corollary 2.8.** Let $z \in R^d$ be fixed, $p, q, r \in (0, \infty]$ be such that (2.23) holds, and suppose that $\omega \in \mathcal{P}_E(R^{3d})$ and $\omega_0 \in \mathcal{P}_E(R^{2d})$ satisfy (2.28), for some

$$\theta \geq d \left( \max \left( \frac{1}{p}, \frac{1}{q} \right) - \frac{1}{q} \right)$$

with strict inequality when $\min(p, 1) < q$. Then $\text{Tr}_z$ is continuous and surjective $M^{p,q}_\omega(R^{3d})$ to $M^{p_0,q_0}_\omega(R^{2d})$, and from $W^{p,q}_\omega(R^{3d})$ to $W^{p_0,q_0}_\omega(R^{2d})$.

**Remark 2.9.** Let $p, q, r \in (0, \infty)$ be such that $0 < p < q$, $r = \min(1, p, q)$,

$$p_1 = (p, \ldots, p) \in (0, \infty]^d, \quad p_2 = (p, \ldots, p) \in (0, \infty)^{d-1},$$

$$q_1 = (q, \ldots, q) \in (0, \infty]^d \quad \text{and} \quad q_2 = (q, \ldots, q) \in (0, \infty)^{d-1}.$$  

Also let $s \in R$,

$$\omega_1(x_1, \xi_1) = \langle \xi_1 \rangle^s \quad \text{and} \quad \omega_2(x_2, \xi_2) = \langle \xi_2 \rangle^s, \quad x_1, \xi_1 \in R^d, \ x_2, \xi_2 \in R^{d-1}.$$  

Then it is proved in [21] Theorem 3.1] that the trace map which takes

$$(x_1, \ldots, x_d) \mapsto f(x_1, \ldots, x_d) \quad \text{into} \quad (x_1, \ldots, x_{d-1}) \mapsto f(x_1, \ldots, x_{d-1}, 0)$$

is continuous from $M^{p_1,q_1}_{\omega_1}(R^d)$ to $M^{p_2,q_2}_{\omega_2}(R^{d-1})$.

See also [10] Theorem 2 for similar results of trace operators on Wiener amalgam type spaces $W^{p,q}_\omega(R^d)$.

We note that Theorem 2.2 is more general compared to the above result in the sense of relaxing the conditions of the involved weight functions, as well as including more exponents in the definition of modulation spaces. For example, we notice that the trace result above only deals with polynomial type weights of special kinds, and the range of the Lebesgue exponents $p$ and $q$ is $(0, \infty)$ instead of the full interval $(0, \infty]$, which includes $\infty$.

On the other hand, [21] Theorem 3.1] is a special case of [21] Theorem 3.3], which concern trace results for broader classes of $\alpha$-modulation spaces. Evidently, Theorem 2.2, Propositions 2.5, 2.7 and Corollary 2.8 do not show any properties for $\alpha$-modulation spaces which are not modulation spaces. We note however that some arguments in [39] or in the proof of Theorem 2.2 might be suitable when extending Theorems 3.1 and 3.3 in [21] to allow the Lebesgue exponents to belong to the full interval $(0, \infty]$. (Cf. [39] Remark 3.1] and [10].)

**Remark 2.10.** Let $p, q, \theta, \omega_0, \omega, z$ be the same as in Proposition 2.7. It is clear that Proposition 2.7 is more general compared to Corollary 2.8. On the other hand, the involved modulation space $M^{p,\infty,p,q}_{\omega}(R^{3d})$ in the domain of $\text{Tr}_z$ in Proposition 2.7 is in some sense more complicated compared to the corresponding space $M^{p,q}_{\omega}(R^{3d})$ in Corollary 2.8. An interesting question concerns whether Proposition 2.7 involve important situations which are not included in Corollary 2.8.
In this respect, on one hand we have
\[ f \in M^{p,\infty,p,q,q,q}_{(\omega)}(\mathbb{R}^{3d}) \setminus M^{p,q}_{(\omega)}(\mathbb{R}^{3d}) \] when \( f(x, y, \zeta) = g(x, \zeta), \) \( x, y, \zeta \in \mathbb{R}^d, \)
\[ g \in M^{p,q}_{(\omega)}(\mathbb{R}^{3d}) \setminus 0, \]
if in addition \( p < \infty. \) Here \( \omega \) and \( \omega_0 \) should satisfy (2.23) and (2.24). Hence we may identify \( M^{p,q}_{(\omega)}(\mathbb{R}^{3d}) \) with a subset of \( M^{p,\infty,p,q,q,q}_{(\omega)}(\mathbb{R}^{3d}); \) but not as a subset of \( M^{p,q}_{(\omega)}(\mathbb{R}^{3d}). \)
By straightforward computations it also follows that \( f \) and \( g \) in (2.26) fulfill \( \text{Tr}_z f = g \) for every \( z \in \mathbb{R}^d. \)

On the other hand, if \( p, p_0 \in (0, \infty], \) then Proposition 2.7 shows that
\[ \text{Tr}_z : M^{p,p_0,p,q,q,q}_{(\omega)}(\mathbb{R}^{3d}) \to M^{p,q}_{(\omega)}(\mathbb{R}^{3d}) \] (2.27)
is surjective.

3. Continuity for pseudo-differential operators of amplitude type on modulation spaces

In this section we shall apply Proposition 2.3 to deduce identification properties between pseudo-differential operators of amplitude types and standard types, when the symbol and amplitude classes are modulation spaces. Then we combine this with suitable continuity and compactness results in [50, 51] to deduce continuity, Schatten-von Neumann and nuclearity properties for pseudo-differential operators of amplitude types.

3.1. Pseudo-differential operators with amplitudes in modulation spaces. It is suitable to consider a modified version of the modulation space \( M^{p}_{(\omega)}(\mathbb{R}^d) \) in order to formulate the main result. Let
\( p_j, q_j \in (0, \infty)^d, \) \( j \in \{1, 2, 3\}, \) \( p = (p_1, p_2, p_3) \in (0, \infty)^{3d}, \) \( q = (q_1, q_2, q_3) \in (0, \infty)^{3d}, \)
\( r = \min(1, p, q), \) \( \omega \in \mathcal{S}_E(\mathbb{R}^d) \) and \( \phi \in \Sigma_1(\mathbb{R}^d). \) Then let
\[ \|F\|_{L^{p,q}(\mathbb{R}^{6d})} \equiv \|G_1\|_{L^{p,q}(\mathbb{R}^{6d})}, \quad G_1(x, y, \zeta, \xi, \eta, z) = F(x, x + y, \zeta, \xi, \eta, z), \]
and
\[ \|F\|_{L^{p,q}_r(\mathbb{R}^{6d})} \equiv \|G_2\|_{L^{p,q}_r(\mathbb{R}^{6d})}, \quad G_2(\xi, \eta, z, x, y, \zeta) = F(x, x + y, \zeta, \xi, \eta, z), \]
when \( F \in L^{p,q}_{loc}(\mathbb{R}^{6d}). \) The modulation spaces
\[ \mathcal{M}^{p,q}_{(\omega)}(\mathbb{R}^{3d}) = \mathcal{M}^{p_1,p_2,p_3,q_1,q_2,q_3}_{(\omega)}(\mathbb{R}^{3d}) \quad \text{and} \quad \mathcal{W}^{p,q}_{(\omega)}(\mathbb{R}^{3d}) = \mathcal{W}^{p_1,p_2,p_3,q_1,q_2,q_3}_{(\omega)}(\mathbb{R}^{3d}) \]
are the sets of all \( a \in \Sigma_1(\mathbb{R}^{3d}) \) such that
\[ \|a\|_{\mathcal{M}^{p,q}_{(\omega)}} = \|a\|_{\mathcal{M}^{p_1,p_2,p_3,q_1,q_2,q_3}_{(\omega)}} \equiv \|V\phi a \cdot \omega\|_{L^{p,q}} \]
respectively
\[ \|a\|_{\mathcal{W}^{p,q}_{(\omega)}} = \|a\|_{\mathcal{W}^{p_1,p_2,p_3,q_1,q_2,q_3}_{(\omega)}} \equiv \|V\phi a \cdot \omega\|_{L^{p,q}} \]
is finite. The spaces \( \mathcal{M}^{p,q}_{(\omega)}(\mathbb{R}^{3d}) \) and \( \mathcal{M}^{p,q}_{(\omega)}(\mathbb{R}^{3d}) \) are equipped with the topologies supplied by the quasi-norms \( \| \cdot \|_{\mathcal{M}^{p,q}_{(\omega)}} \) and \( \| \cdot \|_{\mathcal{W}^{p,q}_{(\omega)}} \), respectively.

As for the classical modulation spaces \( M^{p,q}_{(\omega)}(\mathbb{R}^{3d}), \) we put
\[ M^{p_0,1,p_0,2,p_0,3,q_0,1,q_0,2,q_0,3}_{(\omega)} = M^{p_0,1,p_0,2,p_0,3,q_0,1,q_0,2,q_0,3}_{(\omega)} \]
Lemma 3.1. Let $p_j = (p_0,\ldots,p_{0,j}) \in (0,\infty]^d$ and $q_j = (p_0,\ldots,q_{0,j}) \in (0,\infty]^d$, $j = 1, 2, 3$.

The following lemma justifies the introduction of $M_{(\omega)}^{p,q}(\mathbb{R}^{d})$.

Lemma 3.2. Let $p, q, \omega$ be the same as in Lemma 3.1, let
$$\omega_0(x, y, \zeta, \xi, \eta, z) = \omega(x, y + z, \xi + \eta, \eta, z),$$
and let $T_2$ be the map given by
$$(T_2a)(x, y, \zeta, \xi, \eta, z) = (e^{i\langle D_i, D_j \rangle} a)(x, y, \zeta, \xi, \eta, z), \quad a \in \Sigma_1'(\mathbb{R}^{d}).$$
Then $T_2$ from $\Sigma_1'\left(\mathbb{R}^{d}\right)$ to $\Sigma_1'(\mathbb{R}^{d})$ restricts to a homeomorphism from $M_{(\omega)}^{p,q}(\mathbb{R}^{d})$ to $M_{(\omega)}^{p,q}(\mathbb{R}^{d})$.

Proof. By straight-forward computations we get
$$|V_{T_2}(Ta))(x, y, \zeta, \xi, \eta, z)| = |V_\varphi a(x, y + z, \zeta + \eta, \eta, z)|$$
when $a \in \Sigma_1'(\mathbb{R}^{d})$ and $\varphi \in \Sigma_1(\mathbb{R}^{d})$. The result now follows by multiplying the equality with $\omega_0$ and then apply the $L^{p,q}$ quasi-norm.

We also need the following lemma. We omit the proof since the result follows from [48 Proposition 2.8] and its proof.

Lemma 3.3. Let $p, q, \rho, \omega$ be the same as in Lemma 3.1, let
$$\omega_0(x, y, \zeta, \xi, \eta, z) = \omega(x, y + z, \zeta + \eta, \zeta, \eta, z),$$
and let $T_2$ be the map given by
$$(T_2a)(x, y, \zeta, \xi, \eta, z) = (e^{i\langle D_i, D_j \rangle} a)(x, y, \zeta, \xi, \eta, z), \quad a \in \Sigma_1'(\mathbb{R}^{d}).$$
Then $T_2$ from $\Sigma_1'(\mathbb{R}^{d})$ to $\Sigma_1'(\mathbb{R}^{d})$ restricts to a homeomorphism from $M_{(\omega)}^{p,q}(\mathbb{R}^{d})$ to $M_{(\omega)}^{p,q}(\mathbb{R}^{d})$.

Let $a \in \Sigma_1(\mathbb{R}^{d})$, $T_1$ and $T_2$ be as in Lemmas 3.1 and 3.2. Then (3.1) shows that
$$a_0 = (T_1 \circ T_2 \circ T_1)a.$$
Proposition 3.4. Let 

\[ \omega(x, \xi, \zeta, z) \leq \omega(x, x + y + z, \zeta + \eta, \xi - \eta, \eta, z) e^{r_0 |y| \vartheta(\eta, z)} \]  

should hold for some \( r_0 > 0 \), due to the moderateness of \( \omega \). The details are left for the reader.

The assertion (2) follows by letting

\[ a(x, y, \zeta) = e^{- \langle D_x, D_y \rangle}(a_0(x, \zeta) \varphi(y - x)), \]

where \( \varphi \in \Sigma_1(\mathbb{R}^d) \) fulfills \( \varphi(0) = 1 \), and applying Theorem 2.2 (2), Lemma 3.1 and Lemma 3.2. The details are left for the reader.

In a similar way as when passing from Theorem 2.2 into Proposition 3.3, it follows that the following result is a special case of Theorem 3.3. The details are left for the reader. Here the involved weights are related as

\[ \omega_0(x, \xi, \zeta, z) \sim \omega(x, x + z, \zeta + \eta, \xi - \eta, \eta, z) \vartheta(\eta, z) \]  

(3.3)

Proposition 3.4. Let \( p, p_0, q, q_0, r \) and \( \vartheta \) be the same as in Theorem 3.3 and suppose that \( \omega \in \mathcal{P}_E(\mathbb{R}^{4d}) \) and \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^{4d}) \) satisfy (3.3). Then

\[ \{ \text{Op}(a) : a \in M^{p,q}_{\omega}(\mathbb{R}^{3d}) \} = \{ \text{Op}_0(a_0) : a_0 \in M^{p,q}_{\omega}(\mathbb{R}^{2d}) \}. \]

In the same way it follows that the next result is a special case of the previous one. The details are left for the reader. Here the relationship between the weights are more specified into

\[ \omega_0(x, \xi, \zeta, z) \sim \omega(x, x + z, \zeta + \eta, \xi - \eta, \eta, z) \vartheta(\eta, z)^{-\theta} \]  

(3.3')

Proposition 3.5. Let \( p, q \in (0, \infty) \), and suppose that \( \omega \in \mathcal{P}_E(\mathbb{R}^{4d}) \) and \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^{4d}) \) satisfy (3.3'), for some \( \theta \) such that (2.25) holds with strict inequality when \( q > \min(p, 1) \). Then

\[ \{ \text{Op}(a) : a \in M^{p,q}_{\omega}(\mathbb{R}^{3d}) \} = \{ \text{Op}_0(a_0) : a_0 \in M^{p,q}_{\omega}(\mathbb{R}^{2d}) \}. \]

We observe that \( M^{p,q}_{\omega}(\mathbb{R}^{3d}) = M^{p,q}_{\omega_0}(\mathbb{R}^{2d}) \) in the last proposition.

We may combine the previous results with other established results for pseudo-differential operators when acting on modulation spaces. For example, the following result is a straight-forward consequence of Theorem 3.3 or Proposition 3.3 with Theorem 1.8. The details are left for the reader. Here the condition (0.12) from the introduction needs to be modified into

\[ \frac{\omega_2(x, \xi)}{\omega_1(z, \zeta)} \leq \omega(x, z, \xi + \eta, \xi - \zeta - \eta, \eta, z - x) \vartheta(\eta, z), \quad x, z, \xi, \eta, \zeta \in \mathbb{R}^d. \]  

(0.12')

Recall also Subsection 1.4 for ordering relations between elements in \((0, \infty]\) and elements in \((0, \infty]^d\).

Theorem 3.6. Let \( \sigma \in S_{2d} \), \( p, q, r \in (0, \infty] \) and \( p_1, p_2 \in (0, \infty)^{2d} \) be such that

\[ \frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{p} + \min \left( \frac{0}{q} - \frac{1}{p} \right), \quad q \leq \min(p_2) \leq \max(p_2) \leq p, \]

\[ \frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{p} + \min \left( \frac{0}{q} - \frac{1}{p} \right), \quad q \leq \min(p_2) \leq \max(p_2) \leq p, \]
It is now evident that Theorem 0.1′ then holds with latter inequality strict when \( q > \min(1, p) \). Also let \( \omega \in \mathcal{P}_E(\mathbb{R}^{6d}) \), \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( \vartheta \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that (1.12)′ and (2.9) hold. If \( a \in \mathcal{M}_{\sigma,(\omega)}^{p,\infty,p,q,q,q}(\mathbb{R}^{3d}) \), then \( \text{Op}(a) \) from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma_1'(\mathbb{R}^d) \) is uniquely extendable to a continuous map from \( M_{\sigma,(\omega)}^{p_1}(\mathbb{R}^d) \) to \( M_{\sigma,(\omega_2)}^{p_2}(\mathbb{R}^d) \), and

\[
\| \text{Op}(a)f \|_{M_{\sigma,(\omega_2)}^{p_2}} \lesssim \| a \|_{\mathcal{M}_{\sigma,(\omega)}^{p,\infty,p,q,q,q}} \| f \|_{M_{\sigma,(\omega_1)}^{p_1}},
\]

\( a \in \mathcal{M}_{\sigma,(\omega)}^{p,\infty,p,q,q,q}(\mathbb{R}^{3d}), \ f \in M_{\sigma,(\omega_1)}^{p_1}(\mathbb{R}^d) \). (3.5)

As a special case of the previous result we have the following extension of Theorem 0.1 in the introduction.

Theorem 0.1. Let \( p, q, p_j, q_j, r \in (0, \infty], \ j = 1, 2 \), be such that

\[
\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{q_2} - \frac{1}{q_1} = \frac{1}{p} + \min\left(0, \frac{1}{q} - 1\right), \ \text{q} \leq p_2, q_2 \leq p,
\]

and (3.3) holds with strict inequality when \( q > \min(1, p) \). Also let \( \omega \in \mathcal{P}_E(\mathbb{R}^{6d}) \), \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( \vartheta \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that (1.12)′ and (2.9) hold. If \( a \in \mathcal{M}_{\omega}^{p,\infty,p,q,q,q}(\mathbb{R}^{3d}) \), then \( \text{Op}(a) \) is continuous from \( M_{\omega}^{p_1}(\mathbb{R}^d) \) to \( M_{\omega_2}^{p_2}(\mathbb{R}^d) \), and from \( W_{\omega}^{p_1,q_1}(\mathbb{R}^d) \) to \( W_{\omega_2}^{p_2,q_2}(\mathbb{R}^d) \).

We observe that if \( p = \infty \) and \( q \in (0, 1] \) in Theorem 0.1, then \( p_1 = p_2 \in [q, \infty] \), \( q_1 = q_2 \in [q, \infty] \) and we may choose \( r = \infty \) and thereby choose \( \vartheta(\eta, z) = 1 \) everywhere. It is now evident that Theorem 0.1 takes the form of Theorem 0.1 in the introduction for such choices of \( p \) and \( q \).

By combining Theorem 3.3 or Proposition 3.3 with (1.26) instead of Theorem 1.8 we get the following. The details are left for the reader.

Theorem 3.7. Let \( p, q, r \in [1, \infty] \) be such that and

\[
\frac{1}{q} \leq \frac{1}{r},
\]

hold. Also let \( \omega \in \mathcal{P}_E(\mathbb{R}^{6d}) \), \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( \vartheta \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that (1.12)′ and (2.9) hold. If \( a \in \mathcal{W}_{(\omega)}^{p,\infty,p,q,q,q}(\mathbb{R}^{3d}) \), then \( \text{Op}(a) \) from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma_1'(\mathbb{R}^d) \) is uniquely extendable to a continuous map from \( M_{(\omega)}^{p',q'}(\mathbb{R}^d) \) to \( W_{(\omega_2)}^{p,q}(\mathbb{R}^d) \), and

\[
\| \text{Op}(a)f \|_{W_{(\omega_2)}^{p,q}} \lesssim \| a \|_{\mathcal{W}_{(\omega)}^{p,\infty,p,q,q,q}} \| f \|_{M_{(\omega)}^{p',q'}},
\]

\( a \in \mathcal{W}_{(\omega)}^{p,\infty,p,q,q,q}(\mathbb{R}^{3d}), \ f \in M_{(\omega)}^{p',q'}(\mathbb{R}^d) \). (3.7)

3.2. Pseudo-differential operators with amplitudes in weighted Gevrey analogies of the Hörmander class \( S_{0,0}^0 \). We shall next apply Theorem 0.1 to deduce some continuity properties for amplitude type pseudo-differential operators with certain smooth symbols when acting on modulation spaces. For any \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \) and
Lemma 3.8. Definition of involved weight classes. (see also [8]). The details are left for the reader. Here recall Subsection 1.1 for the

Then the following is true:

Proposition 3.9. Proposition 2.7], Proposition 1.5 (2) and the fact that

For \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \) and \( s \geq 1 \), set

\[ \omega_r(x, \xi) = \omega(x)\xi^{-r} \quad \text{and} \quad \omega_{r,s}(x, \xi) = \omega(x)e^{-r\|\xi\|^2}. \]  

The following lemma is a straight-forward consequence of [1, Proposition 2.5], [30, Proposition 2.7], Proposition 1.5 (2) and the fact that

\( M_{(\omega_1)}^{p,q}(\mathbb{R}^d) \subseteq M_{(\omega_2)}^{p,q}(\mathbb{R}^d) \) when \( q \in (0, \infty], \quad N > \frac{d}{q}, \quad \omega_2(x, \xi) = \omega_1(x, \xi)\xi^N \)

(see also [8]). The details are left for the reader. Here recall Subsection 1.1 for the definition of involved weight classes.

Lemma 3.8. Let \( q \in (0, \infty] \), \( s \geq 1 \), \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), and let \( \omega_r \) and \( \omega_{r,s} \) be as in (3.12).

Then the following is true:

1. If in addition \( \omega \in \mathcal{P}(\mathbb{R}^d) \), then

\[ S^{(\omega)}(\mathbb{R}^d) = \bigcap_{r>0} M_{(1/\omega_r)}^{\infty,q}(\mathbb{R}^d); \]

2. If in addition \( \omega \in \mathcal{P}_{E,s}(\mathbb{R}^d) \), then

\[ S^{(\omega)}_{s}(\mathbb{R}^d) = \bigcup_{r>0} M_{(1/\omega_{r,s})}^{\infty,q}(\mathbb{R}^d); \]

3. If in addition \( \omega \in \mathcal{P}_{E,s}(\mathbb{R}^d) \), then

\[ S^{(\omega)}_{0,s}(\mathbb{R}^d) = \bigcap_{r>0} M_{(1/\omega_{r,s})}^{\infty,q}(\mathbb{R}^d). \]

We have now the following. Here the involved weights should satisfy

\[ \frac{\omega_1(y, \xi)}{\omega_2(x, \xi)} \gtrsim \omega_0(x, y, \xi). \]  

(3.13)

Proposition 3.9. Let \( p, q \in (0, \infty] \) and \( s \geq 1 \). Then the following is true:

1. If \( \omega_0 \in \mathcal{P}(\mathbb{R}^{3d}), \omega_j \in \mathcal{P}(\mathbb{R}^{2d}), j = 1, 2 \), satisfy (3.13) and \( a \in S^{(\omega)}(\mathbb{R}^{3d}) \), then \( \text{Op}(a) \) is continuous from \( M_{(\omega_1)}^{p,q}(\mathbb{R}^d) \) to \( M_{(\omega_2)}^{p,q}(\mathbb{R}^d) \);

2. If \( \omega_0 \in \mathcal{P}_{E,s}(\mathbb{R}^{3d}), \omega_j \in \mathcal{P}_{E,s}(\mathbb{R}^{2d}), j = 1, 2 \), satisfy (3.13) and \( a \in S^{(\omega)}_{s}(\mathbb{R}^{3d}) \), then \( \text{Op}(a) \) is continuous from \( M_{(\omega_1)}^{p,q}(\mathbb{R}^d) \) to \( M_{(\omega_2)}^{p,q}(\mathbb{R}^d) \);

3. If \( \omega_0 \in \mathcal{P}_{E,s}^0(\mathbb{R}^{3d}), \omega_j \in \mathcal{P}_{E,s}^0(\mathbb{R}^{2d}), j = 1, 2 \), satisfy (3.13) and \( a \in S^{(\omega)}_{0,s}(\mathbb{R}^{3d}) \), then \( \text{Op}(a) \) is continuous from \( M_{(\omega_1)}^{p,q}(\mathbb{R}^d) \) to \( M_{(\omega_2)}^{p,q}(\mathbb{R}^d) \).
We also set for the reader. \( \{P' \} \) is said to be in the class \( B \) call that if in addition \( B \) is a Banach space with dual \( B' \). Theorem 0.2 and Theorem 3.3. The details are left for the reader.

Proof. We only prove (3). The other assertions follow by similar argument and are left for the reader.

Since all weights are moderate, (3.13) gives

\[
\frac{\omega_2(x, \xi)}{\omega_1(z, \zeta)} \lesssim \omega_0(x, z, \zeta)^{-1} e^{r|\xi - \zeta|^\frac{1}{q}}
\]

for every \( r > 0 \). Hence, using that

\[
e^{r|x + y|^\frac{1}{q}} \leq e^{r|x|^\frac{1}{q}} e^{r|y|^\frac{1}{q}}
\]

we obtain

\[
\frac{\omega_2(x, \xi)}{\omega_1(z, \zeta)} \lesssim \omega_0(x, z, \zeta + \eta)^{-1} e^{r(|\xi - \zeta|^\frac{1}{q} + |\xi - \eta|^\frac{1}{q} + |z - x|^\frac{1}{q})}
\]

for every \( r > 0 \). The result now follows by combining Theorem 0.1 with Lemma 3.8. □

3.3. Schatten and Nuclearity properties for pseudo-differential operators of amplitude type. Next we shall combine Theorem 3.3 with results in [51] to find Schatten-von Neumann and nuclear properties for pseudo-differential operators of amplitude type with symbols in suitable modulation spaces.

Let \( p \in (0, \infty] \), \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be quasi-Banach spaces, and let \( T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \). Then the singular number of \( T \) of order \( j \geq 1 \) is defined by

\[
\sigma_j(T) = \inf \|T - T_j\|_{\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)},
\]

where the infimum is taken over all \( T_j \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \) with rank at most \( j - 1 \). Then \( \mathcal{I}_p(\mathcal{B}_1, \mathcal{B}_2) \) consists of all \( T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \) such that

\[
\|T\|_{\mathcal{I}_p(\mathcal{B}_1, \mathcal{B}_2)} = \|\{\sigma_j(T)\}_{j=1}^{\infty}\|_{\ell^{p}(\mathbb{N})}
\]

is finite. We observe that \( \mathcal{I}_\infty(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \) with the same quasi-norms. We recall that if in addition \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are Hilbert spaces, then \( \mathcal{I}_2(\mathcal{B}_1, \mathcal{B}_2) \) and \( \mathcal{I}_1(\mathcal{B}_1, \mathcal{B}_2) \) are the sets of Hilbert-Schmidt and trace class operators from \( \mathcal{B}_1 \) to \( \mathcal{B}_2 \), respectively.

The following result is now a straight-forward consequence of [50] Theorem 3.4] and Theorem 3.3. The details are left for the reader.

Theorem 0.2. Let \( p, q, r \in (0, \infty] \) be such that \( q \leq \min(p, p') \) and (3.4) hold, \( \omega \in \mathcal{P}_{E}(\mathbb{R}^d) \) and \( \omega_1, \omega_2 \in \mathcal{P}_{E}(\mathbb{R}^{2d}) \) be such that (0.12) holds with \( \vartheta \) being the same as in Theorem 0.1. If \( a \in M^{p,\infty,p,q,q}(\mathbb{R}^d) \), then \( \text{Op}(a) \in \mathcal{I}_p(M^2_{(\omega_1)}(\mathbb{R}^d), M^2_{(\omega_2)}(\mathbb{R}^d)) \), and

\[
\|\text{Op}(a)\|_{\mathcal{I}_p(M^2_{(\omega_1)}(\mathbb{R}^d), M^2_{(\omega_2)}(\mathbb{R}^d))} \lesssim \|a\|_{M^{p,\infty,p,q,q}_{(\omega_1)}}, \quad a \in M^{p,\infty,p,q,q}_{(\omega_1)}(\mathbb{R}^d).
\]

Next we perform similar discussions for nuclear operators. Let \( p \in (0, 1] \), \( \mathcal{B}_1 \) be a Banach space with dual \( \mathcal{B}_1' \), \( \mathcal{B}_2 \) be a quasi-Banach space, and let \( T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \). Then \( T \) is said to be in the class \( \mathcal{N}_p(\mathcal{B}_1, \mathcal{B}_2) \) of nuclear operators of order \( p \), if there are sequences \( \{\varepsilon_j\}_{j=1}^{\infty} \subseteq \mathcal{B}_1' \) and \( \{\varepsilon_j\}_{j=1}^{\infty} \subseteq \mathcal{B}_2 \) such that

\[
Tf = \sum_{j=1}^{\infty} \langle \varepsilon_j, f \rangle e_j \quad \text{and} \quad \sum_{j=1}^{\infty} (\|\varepsilon_j\|_{\mathcal{B}_1'} \|e_j\|_{\mathcal{B}_2})^p < \infty, \quad f \in \mathcal{B}_1.
\]

We also set

\[
\|T\|_{\mathcal{N}_p(\mathcal{B}_1, \mathcal{B}_2)} = \inf \left( \sum_{j=1}^{\infty} (\|\varepsilon_j\|_{\mathcal{B}_1'} \|e_j\|_{\mathcal{B}_2})^p \right)^{\frac{1}{p}}.
\]
where the infimum is taken over all \( \{ \varepsilon_j \}_{j=1}^{\infty} \subseteq \mathcal{B}_1 \) and \( \{ \varepsilon_j \}_{j=1}^{\infty} \subseteq \mathcal{B}_2 \) such that (3.14) holds true. (See e.g. [11, 12].)

We observe that \( \mathcal{N}_p(\mathcal{B}_1, \mathcal{B}_2) \) decreases with \( \mathcal{B}_1 \), and increases with \( p \) and \( \mathcal{B}_2 \). If in addition \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are Hilbert spaces, then the spectral theorem implies that

\[
\mathcal{N}_p(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{I}_p(\mathcal{B}_1, \mathcal{B}_2)
\]

with the same quasi-norms. (See e.g. [32].) As a consequence we have

\[
\mathcal{N}_p(M^\infty_\omega(R^d), M^p_\omega(R^d)) \subseteq \mathcal{N}_p(M^2_\omega(R^d), M^2_\omega(R^d)) = \mathcal{I}_p(M^2_\omega(R^d), M^2_\omega(R^d)).
\]

The following result is now a consequence of [51, Theorem 4.2] and Theorem 3.3. The details are left for the reader.

**Theorem 3.3.** Let \( p \in (0, 1) \), \( \omega \in \mathcal{P}_E(R^{2d}) \) and \( \omega_1, \omega_2 \in \mathcal{P}_E(R^{2d}) \) be such that (1.12) holds. If \( a \in \mathcal{M}^{p, \infty-p, p-p, p}(R^{2d}) \), then \( Op(a) \in \mathcal{N}_p(M^\infty_\omega(R^d), M^p_\omega(R^d)) \), and

\[
\| Op(a) \|_{\mathcal{N}_p(M^\infty_\omega(R^d), M^p_\omega(R^d))} \lesssim \| a \|_{\mathcal{M}^{p, \infty-p, p-p, p}(R^{2d})}, \quad a \in \mathcal{M}^{p, \infty-p, p-p, p}(R^{2d}).
\]

Due to (3.15) it is evident that Theorem 3.3 improves Theorem 0.2 when \( p \in (0, 1) \).

**Remark 3.10.** There are other nuclearity properties for pseudo-differential operators of the types \( Op(a) \) with symbols in modulation spaces, available in the literature and which are comparable with those in [51] (see e.g. [11, 12]). In this context we have chosen to use the nuclearity results in [51] because they seem to be sharp.

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