LOCAL ARONSON-BÉNILAN GRADIENT ESTIMATES AND HARNACK INEQUALITY FOR THE POROUS MEDIUM EQUATION ALONG RICCI FLOW

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Abstract. In this paper, we prove some new local Aronson-Bénilan type gradient estimates for positive solutions of the porous medium equation

$$u_t = \Delta u^m, \quad m > 1$$

coupled with Ricci flow, assuming that the Ricci curvature is bounded. As an application, the related Harnack inequality is derived. Our results generalize known results. These results may be regarded as the generalizations of the gradient estimates of Lu-Ni-Vázquez-Villani and Huang-Huang-Li to the Ricci flow.

1. Introduction and main results. In this paper, we mainly derive the parabolic version of gradient estimates and Harnack inequality for positive solutions to the porous medium equation (PME for short)

$$u_t = \Delta u^m, \quad m > 1$$

along Ricci flow.

Let $\textbf{(M}^n, g\textbf{)}$ be a complete Riemannian manifold. Li and Yau [10] established a famous gradient estimate for positive solutions to the heat equation. In 1991, Li in [11] deduced gradient estimates and Harnack inequalities for positive solutions to some nonlinear parabolic equation on $\textbf{M} \times [0, \infty)$. In 1993, Hamilton in [6] generalized the constant $\alpha$ in Li and Yau’s result to the function $\alpha(t) = e^{2Kt}$ (see (5) for details). In 2006, Sun [22] also proved gradient estimates with different $\alpha$. In 2011, Li and Xu in [12] further generalized Li and Yau’s result, and found two new functions $\alpha(t)$. Recently, the first author and Zhang in [23] further generalized Li and Xu’s results to the nonlinear parabolic equation. Related results can be found...
In 2016, Li in [14] proved an improved version of the dimension free Hamilton Harnack inequality for the heat equation for the Witten Laplacian on complete weighted Riemannian manifolds. Recently, in [16] the Hamilton differential Harnack inequality for Witten Laplacian on Riemannian manifolds was proved.

In 1979, Aronson and Bénilan [1] obtained a famous second order differential inequality

\[
\sum_i \frac{\partial}{\partial x_i} \left( m u^{m-2} \frac{\partial u}{\partial x_i} \right) \geq - \frac{\kappa}{t}, \quad \kappa = \frac{n}{n(m-1) + 2},
\]

for all positive solutions of (1) on the Euclidean space \( \mathbb{R}^n \) with \( m > 1 - \frac{2}{n} \).

In 2009, Lu, Ni, Vázquez and Villani in [18] studied the PME on manifolds, and obtained the result below.

**Theorem A (Lu, Ni, Vázquez and Villani).** Let \((M^n, g)\) be an \( n \)-dimensional complete Riemannian manifold with \( \text{Ric}(B_p(2R)) \geq -K, K > 0 \). Assume that \( u \) is a positive solution to (1). Let \( v = \frac{m}{m-1} u^{m-1} \) and \( M = \max_{B_p(2R) \times [0,T]} v \). Then for any \( \alpha > 1 \), we have

\[
\frac{|\nabla v|^2}{v} - \frac{\alpha v_t}{v} \leq \frac{C M a \alpha^2}{R^2} \left( \frac{\alpha^2}{\alpha - 1} a(m-1) M K + \frac{aa^2}{t} \right) + \frac{\alpha^2}{\alpha - 1} a(m-1) M K + \frac{aa^2}{t}
\]

on the ball \( B_p(R) \), where \( a = \frac{n(m-1)}{n(m-1)+2} \) and the constant \( C \) depends only on \( n \).

Moreover, when \( R \to \infty \), the following gradient estimate on complete noncompact Riemannian manifold \((M^n, g)\) can be deduced:

\[
\frac{|\nabla v|^2}{v} - \frac{\alpha v_t}{v} \leq \frac{\alpha^2}{\alpha - 1} a(m-1) M K + \frac{aa^2}{t}
\]

Huang, Huang and Li in [8] generalized the results of Lu, Ni, Vázquez and Villani, and obtained Li-Yau type, Hamilton type and Li-Xu type gradient estimates. Wang and Chen [24, 25, 26] proved Perelman type W-entropy monotonicity formula and various differential Harnack inequalities for porous medium equation on compact Riemannian manifolds. Recently, above some results had been generalized to the Ricci flow.

The Ricci flow

\[
\partial_t g(x,t) = -2 \text{Ric}(x,t)
\]

was first introduced by Hamilton [7], and was an important tool of analyzing the structure of manifolds. In 2008, Kuang and Zhang [9] proved a gradient estimate for positive solutions to the conjugate heat equation under Ricci flow on a closed manifold. Soon afterwards, gradient estimates for positive solutions to the heat equation under Ricci flow were further studied, one can see [2, 13, 19, 21]. Recently, Song and Li [15, 17] proved some Li-Yau and Hamilton’s Harnack inequalities for the heat equation of Witten Laplacian on supper Ricci flow.

Recently, Cao and Zhu [4] investigated PME (1) with a linear forcing term,

\[
u_t = \Delta u^m + Ru, \quad m > 1
\]

along the Ricci flow on a complete manifold coupled with bounded curvature and nonnegative curvature operator, and derived Aronson-Bénilan type estimates for any bounded positive solution.
We first introduce three $C^1$ functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t) : (0, +\infty) \to (0, +\infty)$. Suppose that $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ satisfy the following conditions:

(C1) $\alpha(t) > 1$.

(C2) $\alpha(t)$ and $\varphi(t)$ satisfy the following system

\[
\begin{cases}
\frac{2\varphi}{n(n-1)} - 2(m-1)MK \geq (\frac{2\varphi}{n(m-1)} - \alpha') \frac{1}{\alpha}, \\
\frac{2\varphi}{n(m-1)} - \alpha' > 0, \\
\frac{\varphi^2}{n(m-1)} + \alpha\varphi' \geq 0.
\end{cases}
\]

(C3) $\gamma(t)$ satisfies

\[
\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n(m-1)} - \alpha') \frac{1}{\alpha} \leq 0.
\]

(C4) $\alpha(t)$ and $\gamma(t)$ are non-decreasing. Where $\alpha' = \frac{d\alpha}{dt}$, $\varphi' = \frac{d\varphi}{dt}$ and $\gamma' = \frac{d\gamma}{dt}$.

Our results state as follows.

**Theorem 1.1.** Let $(M^n, g(x,t))_{t \in [0,T]}$ be a complete solution to the Ricci flow (2). Suppose that there exist three functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ satisfy conditions (C1), (C2), (C3) and (C4).

Given $x_0 \in M^n$ and $R > 0$, let $u$ be a positive solution of the nonlinear parabolic equation (1) in the cube $B_{2R,T} := \{(x,t) | d(x,x_0,t) \leq 2R, 0 \leq t \leq T\}$. Assume that $|\text{Ric}(x,t)| \leq K$ for some $K > 0$ on $B_{2R,T}$. Let $v = \frac{m}{m-1}u^{m-1}$. Assume that $v \leq M$ on $B_{2R,T}$.

If $\frac{\gamma}{n-1} \leq C_1$ for some constant $C_1$, then for any $(x,t) \in B_{R,T}$,

\[
\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq C_1 a \alpha^2(t) \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{K} R \right) + K \right]
\]

\[+ \frac{CC_1 a M m^2(m-1)}{R^2 \gamma} + \alpha^2(t) K \sqrt{a(m-1)(n+1)} + \alpha \varphi \quad (3)
\]

holds.

If $\frac{\gamma}{n-1} \leq C_2$ for some constant $C_2$, then for any $(x,t) \in B_{R,T}$,

\[
\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq C_2 a \alpha^2(t) \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{K} R \right) + K \right]
\]

\[+ \frac{CC_2 a M m^2(m-1) \alpha^4(t)}{R^2 \gamma} + \alpha^2(t) K \sqrt{a(m-1)(n+1)} + \alpha \varphi, \quad (4)
\]

holds, where $a = \frac{n(m-1)}{n(m-1)+1}$, and the constant $C$ depends only on $n$.

**Remark 1.** If $\alpha(t)$ is bounded, then there exist a constant $C_3$ such that $\frac{\gamma}{n-1} \leq C_3$ and $\frac{\gamma^2}{n-1} \leq C_3$. In this case, the two inequalities (3) and (4) are the same.

Let us give some special functions to illustrate Theorem 1.1 holds for different circumstances and we leave the detailed calculation to the appendix in section 4.
Remark 2. 1. Li-Yau type:

\[ \alpha(t) = \text{constant}, \quad \varphi(t) = \frac{an(m-1)}{t} + \frac{n(m-1)^2MK}{\alpha - 1}, \]
\[ \gamma(t) = t^\theta \quad \text{with} \quad 0 < \theta \leq 2. \]

Then
\[ \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq Ca^2 \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] \]
\[ + \frac{Cam^2(m-1)\alpha^4M}{R^2\gamma} + \alpha^2K \sqrt{a(m-1)(n+1) + \alpha\varphi}. \]

2. Hamilton type:

\[ \alpha(t) = e^{2(m-1)MKt}, \quad \varphi(t) = \frac{n(m-1)}{t}e^{4(m-1)MKt}, \quad \gamma(t) = te^{2(m-1)MKt}. \]

Then
\[ \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq Ca^2 \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] \]
\[ + \frac{Cam^2(m-1)\alpha^4M}{R^2\gamma} + \alpha^2K \sqrt{a(m-1)(n+1) + \alpha\varphi}. \]

3. Li-Xu type:

\[ \alpha(t) = 1 + \frac{\sinh((m-1)MKt) \cosh((m-1)MKt) - (m-1)MKt}{\sinh^2((m-1)MKt)} , \quad \varphi(t) = 2n(m-1)^2MK[1 + \coth((m-1)MKt)], \quad \gamma(t) = \tanh((m-1)MKt). \]

Then
\[ \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq Ca \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] \]
\[ + \frac{Cam^2(m-1)M}{R^2\gamma} + \alpha^2K \sqrt{a(m-1)(n+1) + \alpha\varphi}, \]

where \( \alpha(t) \) is bounded uniformly.

4. Linear Li-Xu type:

\[ \alpha(t) = 1 + (m-1)MKt, \quad \varphi(t) = \frac{n(m-1)}{t} + n(m-1)^2MK, \]
\[ \gamma(t) = (m-1)MKt. \]

Then
\[ \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq Ca^2 \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] \]
\[ + \frac{Cam^2(m-1)\alpha^4M}{R^2\gamma} + \alpha^2K \sqrt{a(m-1)(n+1) + \alpha\varphi}. \]

The local estimates above imply global estimates.

Corollary 1. Let \((M^n, g(0))\) be a complete noncompact Riemannian manifold without boundary, and assume \(g(t)\) evolves by Ricci flow in such a way that \(|\text{Ric}| \leq K\) for \(t \in [0, T]\). Suppose that there exist three functions \(\alpha(t), \varphi(t)\) and \(\gamma(t)\) which satisfy conditions (C1), (C2), (C3) and (C4). Let \(u(x, t)\) be a positive solution to
the equation (1), and let $v = \frac{m}{m-1} u^{m-1}$ and $v \leq M$. Then for $(x, t) \in M^n \times (0, T)$, we have

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \alpha^2(t) K \left[ Ca + \sqrt{a(m-1)(n+1)} \right] + \alpha \varphi.$$ 

The following Harnack inequality can be derived from Corollary 1.

**Theorem 1.2 (Harnack Inequality).** Let $(M^n, g(0))$ be a complete noncompact Riemannian manifold without boundary, and assume $g(t)$ evolves by Ricci flow in such a way that $|\text{Ric}| \leq K$ for $t \in [0, T]$. Suppose that there exist three functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ which satisfy conditions (C1), (C2), (C3) and (C4). Let $u(x, t)$ be a positive solution to the equation (1), and let $v = \frac{m}{m-1} u^{m-1}$ and $M = \max_{M^n \times [0, T]} v$. Then for all $(x_1, t_1) \in M^n \times (0, T)$ and $(x_2, t_2) \in M^n \times (0, T)$ with $t_1 < t_2$, we have

$$u(x_1, t_1) \leq u(x_2, t_2) \exp \left\{ \int_{t_1}^{t_2} \left[ \frac{|\gamma'(s)|^2}{2(t_2 - t_1)^2} ds + \int_{t_1}^{t_2} \frac{\alpha^2(t)}{32M^2} dt \right] \right\},$$

where $\gamma(s)$ is a smooth curve connecting $x_1$ and $x_2$ with $\gamma(1) = x_1$ and $\gamma(0) = x_2$, $C$ is a constant depending only on $n$.

**2. Preliminary.** Let $v = \frac{m}{m-1} u^{m-1}$ and put into equation (1), we get

$$v_t = (m-1) v \Delta v + |\nabla v|^2,$$

which is equivalent to the following form:

$$\frac{v_t}{v} = (m-1) \Delta v + \frac{|\nabla v|^2}{v}.$$ (6)

**Lemma 2.1.** Assume that $(M^n, g(x, t))$ satisfies the hypotheses of Theorem 1.1. We introduce the differential operator

$$\mathcal{L} = \partial_t - (m-1)v \Delta.$$

Let $F = |\nabla v|^2 - \alpha \frac{v_t}{v} - \alpha \varphi$, where $\alpha = \alpha(t) > 1$. Then we have

$$\mathcal{L}(F) \leq -(m-1)v^2_{ij} + (m-1)\alpha^2 K^2 + 2(m-1)K|\nabla v|^2 + 2m\nabla v \nabla F - \left[ (m-1)\Delta v \right]^2 + 2(\alpha - 1)K \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} - \alpha \varphi - \alpha \varphi'.$$ (7)

**Proof.** Simple calculation shows

$$\partial_t \left( \frac{v_t}{v} \right) = \partial_t \left[ (m-1) \Delta v + \frac{|\nabla v|^2}{v} \right]$$

$$= (m-1)(\Delta v)_t + \frac{2 \nabla v \nabla v_t}{v} + \frac{2}{v} \text{Ric}(\nabla v, \nabla v) - \frac{|\nabla v|^2 v_t}{v^2}$$

$$= (m-1)(\Delta v)_t + \frac{2 \nabla v}{v} \nabla \left[ (m-1)v \Delta v + |\nabla v|^2 \right]$$

$$- \frac{|\nabla v|^2}{v^2} \left[ (m-1)v \Delta v + |\nabla v|^2 \right] + \frac{2}{v} \text{Ric}(\nabla v, \nabla v)$$

$$= (m-1)(\Delta v)_t + (m-1) \frac{|\nabla v|^2 \Delta v}{v} + 2(m-1)(\nabla v)(\Delta v)$$

$$+ \frac{2 \nabla v \nabla |\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} + \frac{2}{v} \text{Ric}(\nabla v, \nabla v),$$ (8)
and

\[ \Delta \left( \frac{v_t}{v} \right) = \frac{(\Delta v)_t - 2R_{ij}v_{ij}}{v} - 2\nabla v\nabla v_t - \frac{v_t \Delta v}{v^2} + \frac{2|\nabla v|^2 v_t}{v^3}, \quad (9) \]

where we use the fact that

\[ (\Delta v)_t = \Delta(v_t) + 2R_{ij}v_{ij}, \quad (\nabla v)^2)_t = 2\nabla v\nabla (v_t) + 2\text{Ric}(\nabla v, \nabla v). \quad (10) \]

Combining (8) and (9), we have

\[ L \left( \frac{v_t}{v} \right) = (m - 1)\frac{|\nabla v|^2 \Delta v}{v} + 2(m - 1)\nabla v\nabla (\Delta v) + \frac{2\nabla v \nabla |\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} \]

\[ - \frac{|\nabla v|^2}{v^2} + 2(m - 1)\frac{\nabla v \nabla v_t}{v} + (m - 1)\frac{v_t \Delta v}{v} \]

\[ - 2(m - 1)\frac{|\nabla v|^2 v_t}{v^2} + 2(m - 1)R_{ij}v_{ij} + \frac{2}{v} \text{Ric}(\nabla v, \nabla v). \quad (12) \]

On the other hand, similar calculations show

\[ \partial_t \left( \frac{|\nabla v|^2}{v} \right) = \frac{2\nabla v \nabla \left( (m - 1)v \Delta v + |\nabla v|^2 \right)}{v} + \frac{2\text{Ric}(\nabla v, \nabla v)}{v} \]

\[ - \frac{|\nabla v|^2}{v} \left[ (m - 1)\Delta v + \frac{|\nabla v|^2}{v} \right] \]

\[ = 2(m - 1)\frac{\nabla v |\nabla v|^2}{v} + 2(m - 1)\nabla v \nabla (\Delta v) + \frac{2}{v} \nabla v \nabla |\nabla v|^2 \]

\[ - (m - 1)\Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} + \frac{2\text{Ric}(\nabla v, \nabla v)}{v}, \quad (14) \]

where we utilize the formula (11) above (14).
By utilizing Bochner’s formula, we have

\[
\Delta \left( \frac{|\nabla v|^2}{v} \right) = \frac{2v^2}{v} + \frac{2\nabla v \Delta (\nabla v)}{v} - 2 \frac{\nabla v \nabla |\nabla v|^2}{v^2} - \Delta v \frac{|\nabla v|^2}{v^2} + 2 \frac{|\nabla v|^4}{v^3}
\]

\[
= \frac{2v^2}{v} + 2 \frac{Ric(\nabla v, \nabla v)}{v} + \frac{2}{v} \nabla v \nabla (\Delta v) - \Delta v \frac{|\nabla v|^2}{v^2}
\]

\[- 2\nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla (\log v). \quad (15)
\]

From (14) and (15), we obtain

\[
\mathcal{L} \left( \frac{|\nabla v|^2}{v} \right) = 2(m-1)\Delta v \frac{|\nabla v|^2}{v} + \frac{2}{v} \nabla v \nabla |\nabla v|^2 - 2(m-1)R_{ij} |\nabla v|^2
\]

\[- 2(m-1)v^2_{ij} - \frac{|\nabla v|^4}{v^2} + 2(m-1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla (\log v)
\]

\[+ 2 \frac{Ric(\nabla v, \nabla v)}{v}. \quad (16)
\]

By utilizing (13) and (16), we have

\[
\mathcal{L}(F) = \mathcal{L} \left( \frac{|\nabla v|^2}{v} \right) - \alpha \mathcal{L} \left( \frac{v_t}{v} \right) - \alpha v_t \frac{\nabla v}{v} - \alpha v_t \frac{v_t}{v} - \alpha v_t \frac{v_t}{v} - \alpha \mathcal{L} \left( \frac{v_t}{v} \right) - \alpha v_t \frac{\nabla v}{v}
\]

\[- \alpha(2\alpha - 1) \frac{v_t}{v} \left[ \frac{|\nabla v|^2}{v} - \alpha v_t \right] \nabla (\log v)
\]

\[- \alpha(m-1)\Delta v \frac{v_t}{v} - \frac{2(\alpha-1)}{v} \frac{v_t}{v} \nabla (\log v)
\]

\[- \alpha(2\alpha - 1) \frac{v_t}{v} \mathcal{L}(\nabla v, \nabla v) - \alpha v_t \frac{v_t}{v} - \alpha v_t \frac{v_t}{v} - \alpha \mathcal{L} \left( \frac{v_t}{v} \right). \quad (17)
\]

It is not difficult to calculate that

\[
2(m-1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla (\log v) - 2\alpha(m-1)v \nabla \left( \frac{v_t}{v} \right) \nabla (\log v)
\]

\[= 2(m-1)\nabla v \nabla \left[ \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \right] \nabla (\log v)
\]

\[= 2(m-1)\nabla v \nabla F, \quad (18)
\]

and

\[
\frac{2}{v} \nabla v \nabla |\nabla v|^2 - \frac{2}{v} \nabla v \nabla v_t = \frac{2}{v} \nabla v \nabla (|\nabla v|^2 - \alpha v_t)
\]

\[= \frac{2}{v} \nabla v \nabla (F v + \alpha v v)
\]

\[= 2 \nabla v \nabla F + 2F \frac{|\nabla v|^2}{v} + 2\alpha v \frac{|\nabla v|^2}{v}. \quad (19)
\]
We deduce from (18) and (19) that
\[
2(m - 1)v \nabla \left( \frac{|
abla v|^2}{v} \right) \nabla (\log v) - 2\alpha (m - 1) \nabla \left( \frac{v_t}{v} \right) \nabla (\log v) + \frac{2}{v} \nabla v \nabla |\nabla v|^2
\]
\[
= 2m \nabla v \nabla F + 2F \frac{|\nabla v|^2}{v} + 2\alpha \frac{|\nabla v|^2}{v}
\]
\[
= 2m \nabla v \nabla F + 2 \left( \frac{|\nabla v|^2}{v} - \frac{v_t}{v} \right) |\nabla v|^2.
\]

Besides,
\[
2(m - 1)\Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} - \alpha (m - 1) \Delta v \frac{v_t}{v} + \alpha v_t \frac{|\nabla v|^2}{v}
\]
\[
= 2 \frac{|\nabla v|^2}{v} \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v^2} \right) - \frac{|\nabla v|^4}{v^2} - \alpha v_t \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) + \alpha v_t \frac{|\nabla v|^2}{v
\]
\[
= (2\alpha + 2) \frac{v_t}{v} \frac{|\nabla v|^2}{v^2} - 3 \frac{|\nabla v|^4}{v^2} - \alpha \left( \frac{v_t}{v} \right)^2.
\]

From (20) and (21), we have
\[
2(m - 1)\Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} - \alpha (m - 1) \Delta v \frac{v_t}{v} + \alpha v_t \frac{|\nabla v|^2}{v}
\]
\[
= 2m \nabla v \nabla F - \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right)^2 + (1 - \alpha) \left( \frac{v_t}{v} \right)^2
\]
\[
\leq 2m \nabla v \nabla F - [(m - 1)\Delta v]^2 \quad \text{for} \quad \alpha > 1.
\]

Substituting (22) into (17), we arrive at
\[
\mathcal{L}(F) \leq -2(m - 1)(v_{ij}^2 + \alpha R_{ij}v_{ij}) - 2(m - 1)R_{ij}|\nabla v|^2 + 2m \nabla v \nabla F
\]
\[
- [(m - 1)\Delta v]^2 - 2(\alpha - 1) \frac{\text{Ric}(\nabla v, \nabla v)}{v} - \alpha' \frac{v_t}{v} - \alpha' \varphi - \alpha \varphi'.
\]

Further, applying Young’s inequality
\[
|R_{ij}v_{ij}| \leq \frac{\alpha}{2} R_{ij}^2 + \frac{1}{2\alpha} v_{ij}^2
\]
to (23), we conclude the result. We complete the proof of Lemma 2.1. \(\square\)

**Lemma 2.2.** Suppose that \((M^n, g(t))_{t \in [0, T]}\) satisfies the hypotheses of Theorem 1.1. We also assume that \(\alpha(t) > 1\) and \(\varphi(t) > 0\) satisfy the following system
\[
\begin{align*}
\frac{2\varphi}{n(m - 1)} - 2(m - 1)MK & \geq \left( \frac{2\varphi}{n(m - 1)} - \alpha' \right) \frac{1}{\alpha}, \\
\frac{2\varphi}{n(m - 1)} & > \alpha' > 0, \\
\frac{\varphi^2}{n(m - 1)} + \alpha \varphi' & \geq 0,
\end{align*}
\]
and $\alpha(t)$ is non-decreasing. Then
\[
\mathcal{L}F \leq -(m - 1) \left[ v_{ij} + \frac{\varphi}{n(m - 1)} \delta_{ij} \right]^2 - \left[ \frac{2\varphi}{n(m - 1)} - \alpha' \right] \frac{1}{\alpha} F + (m - 1)\alpha^2 K^2 + 2(\alpha - 1)K \frac{\|\nabla v\|^2}{v} + 2m\nabla v \nabla F - [(m - 1)\Delta v]^2.
\]

Proof. By utilizing the unit matrix $(\delta_{ij})_{n \times n}$ and (7), we obtain
\[
\mathcal{L}(F) \leq -(m - 1) \left[ v_{ij} + \frac{\varphi}{n(m - 1)} \delta_{ij} \right]^2 + \frac{\varphi^2}{n(m - 1)} + \frac{2\varphi}{n} \Delta v
+ (m - 1)\alpha^2 K^2 + 2(m - 1)KM \frac{\|\nabla v\|^2}{v} + 2m\nabla v \nabla F
- [(m - 1)\Delta v]^2 + 2(\alpha - 1)K \frac{\|\nabla v\|^2}{v} - \alpha' \frac{\|v\|}{v} - \alpha' \gamma - \alpha' \varphi - \alpha' \varphi'.
\]
Applying (6) to above inequality, we have
\[
\mathcal{L}(F) \leq -(m - 1) \left[ v_{ij} + \frac{\varphi}{n(m - 1)} \delta_{ij} \right]^2 - \left[ \frac{2\varphi}{n(m - 1)} - 2(m - 1)MK \right] \frac{\|\nabla v\|^2}{v}
+ \left[ \frac{2\varphi}{n(m - 1)} - \alpha' \right] \frac{\|v\|}{v} + \left[ \frac{2\varphi}{n(m - 1)} - \alpha' \right] \frac{\|v\|^2}{v} + \frac{2\varphi}{n(m - 1)} \frac{\|v\|^2}{v} - \alpha' \gamma - \alpha' \varphi - \alpha' \varphi'.
\]
Again using (24), we follow (25). \hfill \square

Lemma 2.3. Let $G = \gamma(t) F$ and $F > 0$. Then
\[
\mathcal{L}G \leq -\frac{1}{aa^2\gamma} G^2 + \left[ \frac{\gamma}{\gamma} - \left( \frac{2\varphi}{n(m - 1)} - \alpha' \right) \frac{1}{\alpha} \right] G
- \frac{2(\alpha - 1)\gamma}{n^2(m - 1)} \frac{\|\nabla v\|^2}{v} - \frac{\gamma(\alpha - 1)^2}{n^2(m - 1)} \frac{\|v\|^4}{v^2}
+ (m - 1)\alpha^2 \gamma K^2 + 2\gamma(\alpha - 1)K \frac{\|\nabla v\|^2}{v} + 2m\nabla v \nabla G,
\]
where $a = \frac{n(m - 1)}{n(m - 1) + 1}$.

Proof. Simple calculation gives
\[
\mathcal{L}G = \mathcal{L}F + \gamma' F
\leq -(m - 1) \left[ v_{ij} + \frac{\varphi}{n(m - 1)} \delta_{ij} \right]^2 + \left[ - \left( \frac{2\varphi}{n(m - 1)} - \alpha' \right) \frac{1}{\alpha} + \frac{\gamma'}{\gamma} \right] G
+ (m - 1)\alpha^2 \gamma K^2 + 2\gamma(\alpha - 1)K \frac{\|\nabla v\|^2}{v} + 2m\nabla v \nabla G
- \gamma[(m - 1)\Delta v]^2.
\]
Since
\[
\left[ v_{ij} + \frac{\varphi}{n(m-1)} \delta_{ij} \right]^2 \geq \frac{1}{n} (\Delta v + \frac{1}{m-1} \varphi)^2
= \frac{1}{n\alpha^2 (m-1)^2} \left[ F + (\alpha - 1) |\nabla v|^2 \right]^2,
\]
and
\[
(m-1)\Delta v = -\frac{F}{\alpha} - \frac{\alpha - 1}{\alpha} \frac{|\nabla v|^2}{v} - \varphi \leq -\frac{F}{\alpha}.
\]
Therefore, we follow that from (27), (28) and (29)
\[
\mathcal{L}G \leq -\frac{\gamma}{n\alpha^2 (m-1)} \left[ F + (\alpha - 1) \frac{|\nabla v|^2}{v} \right]^2
+ \left[ -\left( \frac{2\varphi}{n(m-1)} - \frac{\gamma'}{\alpha} \right) \frac{1}{\alpha} + \frac{\gamma'}{\gamma} \right] G + (m-1)\alpha^2 \gamma K^2
+ 2\gamma(\alpha - 1)K \frac{|
abla v|^2}{v} + 2m\nabla v \nabla G - \frac{G^2}{\alpha^2 \gamma}.
\]
From (30), we infer (26). The proof is be completed. \[\Box\]

3. Proof of main results. In this section, we will prove our main results.

Proof of Theorem 1.1. Now let \( \varphi(r) \) be a \( C^2 \) function on \([0, \infty)\) such that
\[
\varphi(r) = \begin{cases} 
1, & \text{if } r \in [0,1], \\
0, & \text{if } r \in [2, \infty),
\end{cases}
\]
and
\[
0 \leq \varphi(r) \leq 1, \quad \varphi'(r) \leq 0, \quad \varphi''(r) \leq 0, \quad \frac{\varphi'(r)}{\varphi(r)} \leq C,
\]
where \( C \) is an absolute constant. Let define by
\[
\phi(x,t) = \varphi(d(x,x_0,t)) = \varphi \left( \frac{d(x,x_0,t)}{R} \right) = \varphi \left( \frac{\rho(x,t)}{R} \right),
\]
where \( \rho(x,t) = d(x,x_0,t) \). By using the maximum principle, the argument of Calabi [3] allows us to suppose that the function \( \phi(x,t) \) with support in \( B_{2R,T} \), is \( C^2 \) at the maximum point. By utilizing the Laplacian comparison theorem, we deduce that
\[
\frac{|
abla \phi|^2}{\phi} \leq \frac{C}{R^2},
\]
\[
-\Delta \phi \leq \frac{C}{R^2} (1 + \sqrt{K} R),
\]
where \( C \) is a constant depending on \( n \).

For any \( 0 \leq T_1 \leq T \), let \( H = \phi G \) and \((x_1, t_1)\) be the point in \( B_{2R,T_1} \) at which \( G \) attain its maximum value. We can suppose that the value is positive, because otherwise the proof is trivial. Then at the point \((x_1, t_1)\), we infer
\[
\mathcal{L}(H) \geq 0, \quad \nabla G = -\frac{C}{\phi} \nabla \phi.
\]
By the evolution formula of the geodesic length under the Ricci flow [4], we calculate
\[
\phi_t G = -G\phi' \left( \frac{\rho}{R} \right) \frac{1}{R} \frac{d\rho}{dt} = G\phi' \left( \frac{\rho}{R} \right) \int_{\gamma_t} \text{Ric}(S, S) ds
\]
\[
\leq G\phi' \left( \frac{\rho}{R} \right) \frac{1}{R} K\rho \leq G\phi' \left( \frac{\rho}{R} \right) K \leq G\sqrt{C}K,
\]
where \( \gamma_t \) is the geodesic connecting \( x \) and \( x_0 \) under the metric \( g(t) \), \( S \) is the unite tangent vector to \( \gamma_t \), and \( ds \) is the element of the arc length. Hence, by applying (33) and (34), we have
\[
0 \leq \mathcal{L}(H) \leq \phi\mathcal{L}G - (m-1)vG \left( \Delta\phi - 2\frac{|
abla\phi|^2}{\phi} \right) + \phi\phi G
\]
\[
\leq -\frac{1}{aa^2\gamma} \phi^2 G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2\phi}{n(m-1) - \alpha'} \right) \frac{1}{\alpha} \right] \phi G
\]
\[
- \frac{2(a-1)}{na^2(m-1)} \frac{|
abla v|^2}{v} \phi G - \frac{\gamma(a-1)^2}{na^2(m-1)} \frac{|
abla v|^4}{v^2} \phi^2
\]
\[
+ (m-1)a^2\gamma K^2 + 2\gamma \phi(a-1)K \frac{|
abla v|^2}{v} + 2m\phi \nabla v \nabla G
\]
\[- (m-1)vG \left( \Delta\phi - 2\frac{|
abla\phi|^2}{\phi} \right) + \sqrt{C}K\phi G.
\]
Multiply \( \phi \), we have
\[
0 \leq -\frac{1}{aa^2\gamma} \phi^2 G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2\phi}{n(m-1) - \alpha'} \right) \frac{1}{\alpha} \right] \phi G
\]
\[
- \frac{2(a-1)}{na^2(m-1)} \frac{|
abla v|^2}{v} \phi^2 G - \frac{\gamma(a-1)^2}{na^2(m-1)} \frac{|
abla v|^4}{v^2} \phi^2
\]
\[
+ (m-1)a^2\gamma K^2 + 2\gamma \phi(a-1)K \frac{|
abla v|^2}{v} - 2m\phi \frac{\nabla v}{\phi} G \nabla v - (m-1)v\phi G \left( \Delta\phi - 2\frac{|
abla\phi|^2}{\phi} \right) + \sqrt{C}K\phi G.
\]
Further using the inequality \( Ax^2 + Bx \geq -\frac{B^2}{4A} \) with \( A > 0 \), we have
\[
- \frac{2(a-1)}{na^2(m-1)} \frac{|
abla v|^2}{v} \phi^2 G - 2m\phi \frac{\nabla v}{\phi} G \nabla v \leq \frac{nm^2(m-1)a^2}{2(a-1)} \frac{|
abla\phi|^2}{\phi} - \phi G,
\]
\[
- \frac{\gamma(a-1)^2}{na^2(m-1)} \frac{|
abla v|^4}{v^2} \phi^2 + 2\gamma \phi(a-1)K \frac{|
abla v|^2}{v} \leq na^2K^2(m-1)\phi^2\gamma.
\]
Hence, we deduce that
\[
0 \leq -\frac{1}{aa^2\gamma} \phi^2 G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2\phi}{n(m-1) - \alpha'} \right) \frac{1}{\alpha} \right] \phi G
\]
\[
- (m-1)v \left( \Delta\phi - 2\frac{|
abla\phi|^2}{\phi} \right) + \sqrt{C}K \phi G
\]
\[+ (m-1)(n+1)a^2K^2\phi^2\gamma.\]
Combine (31), (32) and (35), we have

\[ 0 \leq -\frac{1}{aa^2\gamma} \phi^2G^2 + \left[ \frac{\gamma'}{\gamma} \phi - \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \frac{\phi}{\alpha} + \frac{Cm^2(m-1)\alpha^2}{R^2(\alpha - 1)} \right. \]
\[ + \frac{C(m-1)M}{R^2} (1 + \sqrt{K}R) + \sqrt{CK} \phi \]
\[ \left. + (m-1)(n+1)\alpha^2K^2\phi^2\gamma. \right] \]

This inequality becomes

\[ \frac{1}{aa^2\gamma} \phi^2G^2 - \left[ \frac{\gamma'}{\gamma} \phi - \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \frac{\phi}{\alpha} + \frac{Cm^2(m-1)\alpha^2}{R^2(\alpha - 1)} \right. \]
\[ + \frac{C(m-1)M}{R^2} (1 + \sqrt{K}R) + \sqrt{CK} \phi \]
\[ \leq (m-1)(n+1)\alpha^2K^2\phi^2\gamma. \]

For the inequality $Ax^2 - 2Bx \leq C$, one has $x \leq \frac{2B}{A} + (\frac{C}{A})^{\frac{1}{2}}$, where $A, B, C > 0$. 

$$\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \leq \left\{ Ca\alpha^2\gamma \left[ \frac{m^2(m-1)\alpha^2}{R^2(\alpha - 1)} + \frac{(m-1)M}{R^2} (1 + \sqrt{K}R) + K \right] \right.$$
\[ + a\alpha^2\gamma \phi \left[ \frac{\gamma'}{\gamma} - \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \frac{1}{\alpha} \right] \]
\[ + \alpha^2K\gamma\phi \sqrt{a(m-1)(n+1)} \} (x_1, t_1). \]

If $\gamma$ is nondecreasing which satisfies the system

\[ \left\{ \begin{array}{l} \frac{\gamma'}{\gamma} - \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} \leq 0, \\ \frac{\alpha^4}{\alpha - 1} \leq C_1. \end{array} \right. \]

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have

$$\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \leq Ca\alpha^2(T_1)\gamma(T_1) \left[ \frac{(m-1)M}{R^2} (1 + \sqrt{K}R) + K \right]$$
\[ + \alpha^2(T_1)K\gamma(T_1)\phi \sqrt{a(m-1)(n+1)}. \]

Hence, we have for $\phi \equiv 1$ on $B_{R,T}$,

$$F(x, T_1) \leq Ca\alpha^2(T_1) \left[ \frac{1}{R^2} (1 + \sqrt{K}R) + K \right] + \frac{Cam^2}{R^2\gamma(T_1)}$$
\[ + \alpha^2(T_1)K\sqrt{a(m-1)(n+1)}. \]

If $\gamma$ is nondecreasing which satisfies the system

\[ \left\{ \begin{array}{l} \frac{\gamma'}{\gamma} - \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} \leq 0, \\ \frac{\gamma}{\alpha - 1} \leq C_2. \end{array} \right. \]
Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have

$$
\phi G(x, T_1) \leq (\phi G)(x_1, t_1)
\leq C a \alpha^2(T_1) \gamma(T_1) \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{K} R \right) + K \right]
+ \frac{C C a m^2 (m-1) \alpha^4(T_1)}{R^2} + \alpha^2(T_1) K \gamma(T_1) \phi \sqrt{a(m-1)(n+1)}.
$$

Hence, we have for $\phi \equiv 1$ on $B_{R,T}$,

$$
F(x, T_1) \leq C a \alpha^2(T_1) \left[ \frac{(m-1)M}{R^2} \left( 1 + \sqrt{K} R \right) + K \right]
+ \frac{C C a m^2 (m-1) \alpha^4(T_1)}{R^2} K \sqrt{a(m-1)(n+1)}.
$$

Because $T_1$ is arbitrary, so the conclusion is valid. \qed

**The proof of Theorem 1.2.** By Corollary 1, we have

$$
-\frac{v_t}{v} \leq -\frac{1}{\alpha(t)} \frac{|\nabla v|^2}{v} + \left[ K \alpha(t) \left( Ca + \sqrt{a(m-1)(n+1)} \right) + \varphi \right]
\leq -\frac{1}{\alpha(t)} M \frac{|\nabla v|^2}{v^2} + \left[ K \alpha(t) \left( Ca + \sqrt{a(m-1)(n+1)} \right) + \varphi \right].
$$

Let $l(s) = \log v(\gamma(s), (1-s)t_2 + st_1)$. Obviously, we infer that $l(0) = \log v(x_2, t_2)$ and $l(1) = \log v(x_1, t_1)$. Therefore, we have

$$
\frac{\partial l(s)}{\partial s} = (t_2 - t_1) \left( \frac{\nabla v}{v} \frac{\gamma'(s)}{t_2 - t_1} - \frac{v_t}{v} \right)
\leq (t_2 - t_1) \left[ \frac{\nabla v}{v} \frac{\gamma'(s)}{t_2 - t_1} - M \frac{|\nabla v|^2}{\alpha(t) v^2}
+ K \alpha(t) \left( Ca + \sqrt{a(m-1)(n+1)} \right) + \varphi \right]
\leq \frac{\alpha(t) |\gamma'(s)|^2}{4M (t_2 - t_1)} + (t_2 - t_1) \left[ \varphi + K \alpha(t) (Ca + \sqrt{a(m-1)(n+1)}) \right].
$$

Integrating above inequality over $\gamma(s)$, we can derive that

$$
\begin{align*}
\log \frac{v(x_1, t_1)}{v(x_2, t_2)} &= \int_0^1 \frac{\partial l(s)}{\partial s} ds \\
&\leq \int_0^1 \frac{|\gamma'(s)|^4}{2(t_2 - t_1)^2} ds + \int_{t_1}^{t_2} \frac{\alpha^2(t)}{32M^2} dt \\
&+ \int_{t_1}^{t_2} \left[ \varphi + K \alpha(t) (Ca + \sqrt{a(m-1)(n+1)}) \right] dt.
\end{align*}
$$

The proof is be completed. \qed
4. **Appendix.** We will check some functions $\alpha(t) > 1$, $\varphi(t) > 0$ and $\gamma(t) > 0$ in Remark satisfy the following two systems

$$
\begin{align*}
\begin{cases}
\frac{2\varphi}{n(m-1)} - 2(m-1)MK & \geq \frac{2\varphi}{n(m-1)} - \alpha' \frac{1}{\alpha}, \\
\frac{2\varphi}{n(m-1)} - \alpha' > 0, \\
\frac{\varphi^2}{n(m-1)} + \alpha \varphi' & \geq 0.
\end{cases}
\end{align*}
$$

(36)

and

$$
\begin{align*}
\begin{cases}
\gamma' - \left( -\frac{2\varphi}{n(m-1)} - \alpha' \frac{1}{\alpha} \right) & \leq 0, \\
\gamma(\alpha - 1) \leq C, \text{ or } \frac{\gamma}{\alpha - 1} & \leq C.
\end{cases}
\end{align*}
$$

(37)

Besides, $\alpha(t)$ and $\gamma(t)$ are non-decreasing.

1. Let $\alpha(t) = 1 + (m-1)MKt$, $\varphi(t) = \frac{\alpha(m-1)}{t} + n(m-1)^2MK$ and $\gamma(t) = (m-1)MKt$.

Direct calculation gives

\( (i) \)

$$
\frac{2\varphi}{n(m-1)} - \alpha' = \frac{2}{t} + 2(m-1)MK - (m-1)MK > 0,
$$

\( (ii) \)

$$
\frac{\varphi^2}{n(m-1)} + \alpha \varphi' = \frac{n(m-1)}{t^2} + n(m-1)^3M^2K^2 + \frac{2}{t}n(m-1)^2MK
+ \left[ 1 + (m-1)MKt \right] \left[ - \frac{n(m-1)}{t^2} \right] > 0
$$

\( (iii) \)

$$
\begin{align*}
& \alpha \left( \frac{2\varphi}{n(m-1)} - 2(m-1)MK \right) - \left( -\frac{2\varphi}{n(m-1)} - \alpha' \right) \\
& = (\alpha - 1) \frac{2\varphi}{n(m-1)} - 2(m-1)MK \alpha + \alpha' \\
& = 2(m-1)MK + 2(m-1)^2M^2K^2t - 2(m-1)MK \\
& - 2(m-1)^2M^2K^2t + \alpha' > 0.
\end{align*}
$$

Hence, $\alpha(t) = 1 + \frac{1}{3}(m-1)MKt$, $\varphi(t) = \frac{n(m-1)}{t} + \frac{1}{3}n(m-1)^2MK$ satisfy the system (36).

On the other hand, one has

$$
\begin{align*}
\frac{\gamma'}{\gamma} - \left( -\frac{2\varphi}{n(m-1)} - \alpha' \frac{1}{\alpha} \right) \frac{1}{\alpha} & = \frac{1}{t} - \left[ \frac{2}{t} + \frac{2}{3}(m-1)MK - \frac{1}{3}(m-1)MK \right] \frac{1}{1 + \frac{1}{3}(m-1)MKt} \\
& = \frac{1}{t(1 + \frac{2}{3}(m-1)MKt)} \\
& \leq 0, \text{ for } t \geq 0.
\end{align*}
$$

and $\frac{\gamma}{\alpha - 1} = 1$. So, (37) is also satisfied.
(2) \( \alpha(t) = e^{2(m-1)MKt} \), \( \varphi(t) = \frac{n(m-1)}{t} e^{4(m-1)MKt} \) and \( \gamma(t) = te^{2(m-1)MKt} \).

Direct calculation shows

(i) \( \frac{2\varphi}{n(m-1)} - \alpha' = 2 \frac{2}{t} e^{2(m-1)MKt}(e^{2(m-1)MKt} - (m-1)MKt) > 0, \)

(ii) \( \frac{\varphi^2}{n(m-1)} + \alpha \varphi' = \frac{n(m-1)}{t^2} e^{6(m-1)MKt}(e^{2(m-1)MKt} - 1 + 4(m-1)MKt) > 0, \)

(iii) \( \frac{2\varphi}{n(m-1)} - 2(m-1)MK - (\frac{2\varphi}{n(m-1)} - \alpha') \frac{1}{\alpha} \)
\[= \frac{2}{t} e^{4(m-1)MKt} - 2(m-1)MK - \frac{2}{t} e^{2(m-1)MKt} + 2(m-1)MK \]
\[= \frac{2}{t} e^{2(m-1)MKt}(e^{2(m-1)MKt} - 1) \geq 0. \]

Hence, \( \alpha(t) = e^{2(m-1)MKt} \) and \( \varphi(t) = \frac{n(m-1)}{t} e^{4(m-1)MKt} \) satisfy the system (36).

Besides, we have
\[\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n(m-1)} - \alpha') \frac{1}{\alpha} \]
\[= \frac{1}{t} + 2(m-1)MKt - \frac{2}{t} e^{2(m-1)MKt} - 2(m-1)MK \]
\[\leq 0, \quad \text{for} \quad t \geq 0. \]

and as \( t \to 0^+, \frac{\gamma}{\alpha-1} = \frac{te^{2MKt}}{e^{\alpha t} - 1} \to \frac{1}{\alpha - 1}. \) This implies \( \frac{\gamma}{\alpha-1} \leq C. \) So, (37) is also satisfied.

(3) \( \alpha(t) = 1 + \sinh((m-1)MKt) \cosh((m-1)MKt) - (m-1)MKt, \varphi(t) = 2n(m-1)^2MK[1 + \coth((m-1)MKt)] \) and \( \gamma(t) = \tanh((m-1)MKt). \) Direct calculation shows

\( \alpha'(t) = 2(m-1)MK - 2(\alpha - 1)(m-1)MK \coth((m-1)MKt). \)

Then

(i) \( \frac{2\varphi}{n(m-1)} - \alpha' = 2(m-1)MK[1 + \coth((m-1)MKt)] - 2(m-1)MK \]
\[+ 2(\alpha - 1)(m-1)MK \coth((m-1)MKt) > 0, \]

(ii) \( \alpha \left( \frac{2\varphi}{n(m-1)} - 2(m-1)MK \right) - \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \]
\[= 2(m-1)MK[1 + \coth((m-1)MKt)] - 2(m-1)MK \alpha \]
\[+ 2(\alpha - 1)(m-1)MK[1 + \coth((m-1)MKt)] + \alpha' \]
\[= 2(m-1)MK(\alpha - 1)[1 + \coth((m-1)MKt)] \]
\[+ 2(m-1)MK(\alpha - 1) \coth((m-1)MKt) - 2(m-1)MK + \alpha' = 0 \]

(iii) \( \frac{\varphi^2}{n(m-1)} + \alpha \varphi' \)
\[= \frac{n(m-1)^3 M^2 K^2}{\sinh^2(m-1)MKt} \left[ [1 + \coth((m-1)MKt)]^2 \sinh^2((m-1)MKt) - \alpha \right]. \]
Let $x = (m - 1)MKt$, then
\[
[1 + \coth((m - 1)MKt)]^2 \sinh^2((m - 1)MKt) - \alpha
= \frac{e^{2x}}{(e^{2x} - 1)^2} \left[ e^{4x} - 2e^{2x} + 3 + 4x \right].
\]

Let $f(x) = e^{4x} - 2e^{2x} + 3 + 4x$ with $x \leq 0$. Obviously, $f(0) > 0$ and
\[f'(x) = 2(e^{4x} - e^{2x} + 2) > 0.
\]

Then we get $f(x) > 0$ for $x > 0$. Hence, we have
\[
\frac{\varphi^2}{n(m-1)} + \alpha \varphi' > 0.
\]

Hence, $\alpha(t) = 1 + \frac{\sin((m-1)MKt) \cosh((m-1)MKt) - (m-1)MKt}{\sinh((m-1)MKt)}$ and $\varphi(t) = 2n(m-1)^2MK[1 + \coth((m-1)MKt)]$ satisfy the system (36).

On the other hand, as $t \to 0$, we have $\frac{2\alpha'}{\alpha - 1} \to 2$; $\frac{2\alpha^2}{\alpha - 1} \to 1$ for $t \to \infty$. These imply $\frac{2\alpha^2}{\alpha - 1} \leq C$, here $C$ is a universal constant.

Besides, we have
\[
\frac{\gamma'}{\gamma} - \left(\frac{2\varphi}{n(m-1)} - \alpha'\right) \frac{1}{\alpha}
= \frac{1}{\alpha} \left[ \frac{x}{\sinh(xt) \cosh(xt)} - 2x - 2x(1 + \alpha) \coth(xt) \right]
= \frac{1}{\alpha} \left[ \frac{x}{\sinh(xt) \cosh(xt)} [\alpha - 2(1 + \alpha) \cosh^2(xt)] - 2x \right]
= \frac{1}{\alpha} \left[ \frac{x}{\sinh(xt)} [\alpha(1 - 2 \cosh(xt)) - 2 \cosh(xt)] - 2K \right]
\leq 0, \quad \text{for} \quad t \geq 0,
\]

where $x = (m - 1)MK$. So, (37) is also satisfied.

(4) $\alpha(t) = \text{constant}$, $\varphi(t) = \frac{2n(m-1)}{t} + \frac{n(m-1)^2MK}{\alpha - 1}$ and $\gamma(t) = t^\theta$ with $0 < \theta \leq 2$.

Direct calculation gives
\[
(i) \quad \frac{2\varphi}{n(m-1)} - \alpha' = \frac{2\alpha}{t} + \frac{(m - 1)MK}{\alpha - 1} > 0,
\]
\[
(ii) \quad \frac{\varphi^2}{n(m-1)} + \alpha \varphi' = \frac{n(m-1)\alpha^2}{t^2} - \frac{n(m-1)\alpha^2}{t^2}
+ \frac{n^2(m-1)^4MK^2}{(\alpha - 1)^2} + \frac{2n^2(m-1)^3MK}{(\alpha - 1)t} > 0,
\]
\[
(iii) \quad \alpha \left(\frac{2\varphi}{n(m-1)} - 2(m - 1)MK \right) - \left(\frac{2\varphi}{n(m-1)} - \alpha'\right)
= (\alpha - 1) - \frac{2\varphi}{n(m-1)} - 2(m - 1)MK > 0.
\]

Hence, $\alpha(t) = \text{constant}$, and $\varphi(t) = \frac{2n(m-1)}{t} + \frac{n(m-1)^2MK}{\alpha - 1}$ satisfy the system (36).
On the other hand, we have
\[
\frac{\gamma'}{\gamma} - \left(\frac{2\varphi}{n(m-1)} - \alpha'\right) \frac{1}{\alpha} \frac{\theta}{t} - \frac{2}{t} - \frac{(m-1)MK}{(\alpha-1)\alpha} \leq 0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad 0 < \theta \leq 2.
\]
So, (37) is also satisfied.

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