Analytic approximation for the modified Bessel function $I_{-2/3}(x)$

Pablo Martin$^1$, Jorge Olivares$^2$, and Fernando Maass$^1$

$^1$Department of Physics, Universidad de Antofagasta, Antofagasta, Chile
$^2$Department of Mathematics, Universidad de Antofagasta, Antofagasta, Chile

E-mail: pablo.martin@uantof.cl, jorge.olivares@uantof.cl, fernando.maass@uantof.cl

Abstract. In the present work an analytic approximation to modified Bessel function of negative fractional order $I_{-2/3}(x)$ is presented. The validity of the approximation is for every positive value of the independent variable. The accuracy is high in spite of the small number (4) of parameters used. The approximation is a combination of elementary functions with rational ones. Power series and assymptotic expansions are simultaneously used to obtain the approximation.

1. Introduction

The modified Bessel functions appear in Electrodynamics and other areas of Physics [1-4]. In particular those of fractional order como $I_{-2/3}$ are also important, because they are connected to Airy functions. This kind of functions of negative and fractional order have a negative branch point at zero. To obtain a good approximation for small values of the independent variable, this singularity should appear in the approximation, which leads to some peculiar problems, which will be solved in the paper. As in a previous work [5], elementary functions will be combined with rational ones, to obtain analytic approximations valid for $x \geq 0$, and with high accuracy. The key of the present method will be the simultaneous used of power series and asymptotic expansion. The present method is an improvement of previous technique applied to 2D-Quadratic Zeemann Effect, Instantons and others [6-9].

In Section 2, the theoretical basis of the present technique will be shown, as well as, the description of the structure of the analytic approximation. The determination of the parameters and numerical results will be also treated in this Section, and the Conclusion will be the last Section.

2. Theoretical Treatment

The starting point will be the well known power series

$$I_{-2/3}(x) = \left(\frac{x}{2}\right)^{-2/3} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma\left(\frac{1}{3} + k\right)} \left(\frac{x}{2}\right)^{2k}$$

The asymptotic expansion will be also

$$I_{-2/3}(x) \sim \left(\frac{e^x}{\sqrt{2\pi x}}\right) \left(1 - \frac{7}{72x} + \ldots\right)$$
In order to obtain analytic approximations with good accuracy for the function \( I_{-2/3}(x) \), the approximate function, should have a factor \( x^{-2/3} \). However this is a branch point, which are coming by pairs, thus this factor will introduce a second branch point at the infinite, which is different of the characteristic branch point in Eq.2. In order to take care of this problem the factor to be introduce must be \((1 + \chi^2 x^2)^{1/3}/x^{2/3}\), but this is not convenient, since the adjoin power series in Eq.1, has only even powers. In order to keep this condition, the selection of the factor must be \((1 + \lambda^2 x^2)^{1/3}/x^{2/3}\), where \(\lambda\) is a free parameter, which will be very useful later, as it will be shown. This last factor introduce also a new branch point at \(x^2 = -1/\lambda^2\), however this new singularity is out of the region of interest, which is the right half complex plane, where the real part of the variable \(x\) is positive.

Now, it is necessary to consider the branch point at the infinite which is \(x^{-1/2}\). Similarly as in the precedent case, this factor introduces a branch point at \(x = 0\), which is not in \(I_{-2/3}\). The procedure to follow is to use \((1 + \lambda^2 x^2)^{-1/4}\), where the second branch point is out of the region of interest, and furthermore looking for the power series expansion of this function, only even powers are obtained. The same \(\lambda\) is used to avoid additional parameters. Finally both factors, the one from the power series, and that from the asymptotic expansion, can be combined in a unique factor \((1 + \chi^2 x^2)^{1/12}/x^{2/3}\).

The additional factor \(e^x\) at the asymptotic expansion is taken care using the auxiliary function \(\cosh(x)\), since in this way the exponential factor is obtained at the infinite, and only even powers are coming in the power series expansion. Considering now, the rational function to be combined with all the previous elementary functions, and taken care of all the restrictions previously described, the structure of the analytic approximation \(\tilde{I}_{-2/3}(x)\) will be considered as

\[
\tilde{I}_{-2/3}(x) = \frac{\sqrt[3]{3}(1 + \lambda^2 x^2)^{1/12}}{\Gamma(1/3) x^{2/3}} \cosh(x) \sum_{j=1}^{n} p_j x^{2j} \frac{1}{1 + \sum_{k=1}^{n} q_k x^{2k}}
\]

where the value of \(n\), will be chosen, according with the number of parameters to use. This will lead to higher accuracy with increasing number of parameters.

The simplest approximation will be

\[
\tilde{I}_{-2/3}(x) = \frac{\sqrt[3]{3}(1 + \lambda^2 x^2)^{1/12}}{\Gamma(1/3) x^{2/3}} \frac{p_0 + p_1 x^2}{1 + q_1 x^2} \cosh(x)
\]

The condition with the leading term of \(\tilde{I}_{-2/3}(x)\) in Eq.(2) leads to

\[
\frac{p_1}{q_1} = \frac{\Gamma(1/3)}{\sqrt{\pi} \sqrt{2\lambda}}
\]

Two more equations for \(p_0\), \(p_1\) and \(q_1\) are coming from the power series Eq.(1). However before to equalize coefficients between series, it is convenient to rationalize both terms of the equation, so

\[
\frac{\Gamma(1/3)}{\sqrt[4]{3}} x^{2/3} (1 + \lambda^2 x^2)^{-1/12} (1 + q_1 x^2 I_{-2/3}(x) \approx \cosh(x)(p_0 + p_1 x^2)
\]

\[
(1 - \frac{1}{12} \lambda^2 x^2)(1 + q_1 x^2)(1 + \frac{3}{4} x^2)^{12} \approx (1 + \frac{x^2}{2})(p_0 + p_1 x^2) + ...
\]
From Eq.(7), it is obtained
\[-\frac{1}{12} \lambda^2 + q_1 + \frac{3}{4} = \frac{p_0}{4} + p_1 = \frac{1}{2} + \frac{\sqrt{2}}{\sqrt{\pi}\Gamma(1/3)}\lambda^{1/12} q_1 \quad ; \quad p_0 = 1 \quad (8)\]

From Eqs. (8) and (5), $q_1$ and $p_1$ are obtained giving

\[q_1 = \frac{\lambda^2 - 3}{12(1 - \Gamma(1/3)/\sqrt{2}\lambda\sqrt{\pi})} \quad (9)\]

and

\[p_1 = \frac{\Gamma(1/3)}{\sqrt{\pi}\sqrt{2}\lambda} \frac{\lambda^2 - 3}{12(1 - \Gamma(1/3)/\sqrt{2}\lambda\sqrt{\pi})} \quad (10)\]

A plot of $q_1$ as a function of $\lambda$ is shown in Figure(1).

![Figure 1. Value of $q_1$ as a function of $\lambda$](image)

Looking the Figure(1), it is clear that $q_1$ is positive only for $\lambda$ in the interval $(0, \sqrt{3})$ and $\lambda \geq 5.96$. The best value for $\lambda$ is considered as that where there is a minimum of maximum relative error $\varepsilon$ defined as

\[
\varepsilon(x, \lambda) = \frac{|\tilde{I}_{-2/3}(x, \lambda) - I_{-2/3}(x)|}{I_{-2/3}(x)}
\]

(11)

In this way the best value of $\lambda$ is $\lambda=0.6$.

The relative error $\varepsilon(x,\lambda=0.6)$ is shown in Fig (2) as a function of the independent variable $x$. The function $\tilde{I}_{-2/3}(x)$ is given in this case as

\[
\tilde{I}_{-2/3}(x) = \frac{(1 + 0.36x^2)^{1/12}}{1.6876x^{2/3}} \frac{1 + 0.6919x^2}{1 + 0.4719x^2} \cosh(x)
\]

(12)
Figure 2. Relative error $\varepsilon(x, \lambda = 0.6)$ of the approximation $I_{-2/3}(x)$ as a function of the variable $x$

The maximum relative error $\varepsilon_{max}$ is about 0.0065 for $x \sim 6$, but taken out the region around $x \sim 6$, the relative error becomes smaller than 0.002.

3. Conclusion

Here it is presented a method to obtain approximations to the modified Bessel functions of negative and fractional order. The case of $I_{-2/3}(x)$ is considered in detail. A general formula is shown, and the most simple approximation is considered. The form of the approximations are determined using the power series and asymptotic expansion, and they are rational functions combined with fractional power and hyperbolic functions. In the case here treated the number of parameters to be determined is four, and the maximum relative error is 0.0065, however the error is smaller than 0.002 for most of the values of $x$, except an small region around the values of $x$, where there is maximum relative error. The approximation here found is valid for any positive value of the variable $x$, and therefore can be integrated and differentiated. Furthermore it is easy to calculate, and this can be done with a simple pocket calculator.

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