Transmuted complementary exponential power distribution: properties and applications

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Abstract
In this study, we introduce a new lifetime distribution by using quadratic rank transmutation map. The some properties of this new distribution is provided. Furthermore, the parameters of this new distribution are estimated by the maximum likelihood method. The performances of the estimates are examined according to bias and mean squared errors (MSEs) criteria through a Monte Carlo simulation study. Finally, two applications with real data are presented to evaluate the fits of introduced distribution.

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1. Introduction
In reliability analysis, every statistical tools are presented based on assumption of the distribution of lifetimes. Therefore, lifetime distributions are hearts of survival and reliability theory. Nowadays, introducing the new lifetime distributions is gaining much more attention. There are a lot of distributions introduced in last two decades. One of the them is complementary exponential power (CEP) distribution suggested by [1] using exponential power (EP) distribution introduced in [2]. The probability density function (pdf) and cumulative distribution function (cdf) of CEP distribution are given, respectively, by

\[ g(x;\gamma) = \frac{\beta x^{\beta-1}}{\alpha^\beta} \exp\left(1 + \left(\frac{x}{\alpha}\right)^\beta - \exp\left(\frac{x}{\alpha}\right)^\beta\right) \]

\[ \times \left[1 - \exp\left(1 - \exp\left(\frac{x}{\alpha}\right)^\beta\right)\right]^{\beta-1} I_A(x) \] (1)

and

\[ G(x;\gamma) = \left[1 - \exp\left(1 - \exp\left(\frac{x}{\alpha}\right)^\beta\right)\right]^\alpha, \] (2)

where \( I_A(x) \) is indicator function on set \( A \), \( \gamma = (\alpha, \beta, \theta) \) is parameter vector, \( \alpha > 0 \) is a scale parameter, \( \beta > 0 \) and \( \theta > 0 \) are shape parameters.

In this study, we aim to introduce a new distribution named transmuted complementary exponential power (TCEP) using Quadratic Rank Transmutation Map (QRTM) proposed by [3]. In the literature, there are many lifetime distributions generated by QRTM such as [4], [5], [6] and [7]. The pdf and cdf of QRTM family are given by

\[ f(x;\delta) = g(x;\gamma)[1 + \lambda - 2\lambda G(x;\gamma)] \] (3)

and
\[ F(x; \delta) = (1 + \lambda)G(x; \gamma) - \lambda G(x; \gamma)^2, \quad (4) \]

where \( G \) and \( g \) are cdf and corresponding pdf of any lifetime, \( \gamma \) is parameter vector of distribution with cdf \( G \) and \( \lambda \in [-1, 1] \) is extra parameter beside \( \delta = (\gamma, \lambda) \). Hence, new distribution includes a parameter to baseline distribution \( G \). For more information on QRTM, see [3].

In this paper, a new lifetime distribution is introduced by QRTM family. In Section 2, the pdf and cdf of distribution are described. The raw moments are derived under a condition, quantile, survival and hazard functions are also given. In Section 3, the point and interval estimations are discussed by maximum likelihood (ML) methodology. A simulation study is conducted to observe the behaviours of ML estimates (MLEs) in Section 4. In Section 5 and Section 6, two numerical examples are also provided to close the paper.

2. TCEP Distribution

In this study, we introduce a new lifetime distribution obtained by using Eqs. (1-2) in Eqs. (3-4). Then pdf and cdf of introduced distribution are given, respectively, by

\[
f(x; \delta) = \frac{\beta \theta x^{\beta - 1}}{\alpha^\theta} \exp \left\{ 1 + \left( \frac{x}{\alpha} \right)^\theta - \exp \left( \left( \frac{x}{\alpha} \right)^\theta \right) \right\} \times \left[ 1 - \exp \left( 1 - \exp \left( \left( \frac{x}{\alpha} \right)^\theta \right) \right) \right]^{\gamma - 1} \left[ 1 + \lambda - 2\lambda \left[ 1 - \exp \left( 1 - \exp \left( \left( \frac{x}{\alpha} \right)^\theta \right) \right) \right] \right] I_s, (x) \quad (5)\]

and

\[
F(x; \delta) = (1 + \lambda) \left[ 1 - \exp \left( 1 - \exp \left( \left( \frac{x}{\alpha} \right)^\theta \right) \right) \right]^{\gamma - 1} - \lambda \left[ 1 - \exp \left( 1 - \exp \left( \left( \frac{x}{\alpha} \right)^\theta \right) \right) \right]^{2\theta}, \quad (6)\]

where \( \delta = (\alpha, \beta, \theta, \lambda) \) is parameter vector, \( \lambda \in [-1, 1], \alpha, \beta, \theta \in \mathbb{R}_+ \) are parameters. The distribution with pdf (5) and cdf (6) is called Transmuted Complementary Exponential Power (TCEP) \( (\delta) \) distribution. When \( \lambda = 0 \), TCEP \( (\delta) \) reduces to CEP\( (\gamma) \). In Fig. 1, the pdf of TCEP \( (\delta) \) are plotted for some selected parameter values.
2.1. Moments

In this subsection, the raw moments of the TCEP(\(\delta\)) distribution are derived, explicitly. We obtain the raw moments using following lemma under the condition that \(r/\beta\) is an integer.

**Lemma 1** For \(\nu, \mu > 0\) and \(m \in \mathbb{N}\)

\[
\int x^{m-1} \left( \log x \right)^m \exp(-\mu x) dx = \frac{\partial^m \mu^\nu \Gamma(\nu, \mu)}{\partial \nu^m}, \quad m = 0, 1, 2, \ldots
\]

where \(\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} \exp(-t) dt\) is the incomplete gamma function [8]. Using Lemma 1, following theorem gives the raw moments of TCEP(\(\delta\)) distribution.

**Theorem 1**

If \(r/\beta\) is an positive integer, the \(r\) th moments of TCEP(\(\delta\)) distribution are given by

\[
E(X^r) = (1 + \lambda)\alpha^r \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\theta + 1) \exp(j + 1)}{\Gamma(\theta - j) j!} \left( \frac{\partial^m \mu^\nu \Gamma(\nu, \mu)}{\partial \nu^m} \right)_{\nu = j + 1}
\]

\[
-\lambda\alpha^r \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(2\theta + 1) \exp(j + 1)}{\Gamma(2\theta - j) j!} \left( \frac{\partial^m \mu^\nu \Gamma(\nu, \mu)}{\partial \nu^m} \right)_{\nu = j + 1},
\]

where \(r = 1, 2, \ldots\) and \(\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t) dt\) is the well-known gamma function.

**Proof.** Using pdf in Eq. (5), the raw moments can be written by
\[ E(X') = \left(1 + \lambda\right) \frac{\beta \theta^\beta}{\alpha^\beta} \int_0^\infty \exp\left[1 + \left(\frac{x}{\alpha}\right)^\beta - \exp\left(\frac{x}{\alpha}\right)^\beta\right] \left(1 - \exp\left[1 - \exp\left(\frac{x}{\alpha}\right)^\beta\right]\right)^{\theta-1} dx \]

\[ - \frac{2\lambda \beta \theta^\beta}{\alpha^\beta} \int_0^\infty \exp\left[1 + \left(\frac{x}{\alpha}\right)^\beta - \exp\left(\frac{x}{\alpha}\right)^\beta\right] \left(1 - \exp\left[1 - \exp\left(\frac{x}{\alpha}\right)^\beta\right]\right)^{2\theta-1} dx \]

for \( \theta > 0 \). Let us consider the identity

\[ (1 - z)^{\theta-1} = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\theta) z^j}{\Gamma(\theta - j) j!} \]

By using expansion (10) in (9), we can write

\[ \mu' = \frac{(1 + \lambda) \Gamma(\theta + 1) \beta}{\alpha^\beta} \sum_{j=0}^{\infty} \frac{(-1)^j e^{j/j^\beta}}{\Gamma(\theta - j) j!} \int_0^\infty \exp\left[\left(\frac{x}{\alpha}\right)^\beta\right] \exp\left[-(j+1)\left(\frac{x}{\alpha}\right)^\beta\right] dx \]

\[ - \frac{\lambda \Gamma(2\theta + 1) \beta}{\alpha^\beta} \sum_{j=0}^{\infty} \frac{(-1)^j e^{j/j^\beta}}{\Gamma(2\theta - j) j!} \int_0^\infty \exp\left[\left(\frac{x}{\alpha}\right)^\beta\right] \exp\left[-(j+1)\left(\frac{x}{\alpha}\right)^\beta\right] dx. \]

Using transformation of \( y = \exp\left[\left(\frac{x}{\alpha}\right)^\beta\right] \) in integrals in (11), we get

\[ \mu' = (1 + \lambda) \alpha' \Gamma(\theta + 1) \sum_{j=0}^{\infty} \frac{(-1)^j e^{j/j^\beta}}{\Gamma(\theta - j) j!} y^{j+1} \exp\left[-(j+1)y\right] dy \]

\[ - \lambda \alpha' \Gamma(2\theta + 1) \sum_{j=0}^{\infty} \frac{(-1)^j e^{j/j^\beta}}{\Gamma(2\theta - j) j!} y^{j+1} \exp\left[-(j+1)y\right] dy \]

By using Lemma 1 in (12), the proof is completed.

2.2. Quantile function and random number generation

The quantile function of TCEP (\( \delta \)) distribution is obtained by solving \( F(x; \delta) = p \) for \( p \in (0,1) \) and it is given by

\[ x_p = \alpha \left[ \log \left( 1 - \log \left( 1 - \left[ \frac{1 + \lambda - \sqrt{(\lambda + 1)^2 - 4\lambda p}}{2\lambda} \right]^\theta \right) \right) \right]^{1/\beta}, \]

where \( F(x; \delta) \) is given in (6).

2.3. Reliability and hazard functions

The reliability function and hazard function of TCEP (\( \delta \)) distribution are given, respectively, by

\[ R(t) = 1 - \left[ (1 + \lambda) \left(1 - \exp\left[1 - \exp\left(\frac{t}{\alpha}\right)^\beta\right]\right) \right]^{\theta} - \lambda \left(1 - \exp\left[1 - \exp\left(\frac{t}{\alpha}\right)^\beta\right]\right)^{\theta-1} \]

and
\[
g(t; \alpha, \beta) = \frac{1 - \exp\left(1 - \exp\left(\frac{t}{\alpha}\right)\right)^{\theta-1} \left[1 + \lambda - 2\lambda \left[1 - \exp\left(1 - \exp\left(\frac{t}{\lambda}\right)\right)^{\theta}\right]\right]}{1 - \left(1 + \lambda\right) \left[1 - \exp\left(1 - \exp\left(\frac{t}{\lambda}\right)\right)^{\theta}\right] - \lambda \left[1 - \exp\left(1 - \exp\left(\frac{t}{\lambda}\right)\right)^{\theta}\right]^2}.
\]

where
\[
g(t; \alpha, \beta) = \frac{\beta \theta t^{\theta-1}}{\alpha^\theta} \exp\left(1 + \left(\frac{t}{\alpha}\right)^\theta\right) - \exp\left(\left(\frac{t}{\alpha}\right)^\theta\right).
\]

Fig. 2 presents the shapes of the hazard function of TCEP(\(\delta\)) distribution for selected parameter values.

![Hazard function plots](image)

**Figure 2.** TCEP hazard functions

### 3. Maximum Likelihood Estimation and Asymtotic Confidence Intervals

Let \(X_1, X_2, ..., X_n\) be the independent random variables having TCEP(\(\delta\)) distribution. The log-likelihood function based on this sample is given by
\[\ell(\delta | x) = n(1 + \log(\beta) + \log(\theta) - \beta \log(\alpha)) \]

\[+ (\beta - 1) \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \left( \frac{x_i}{\alpha} \right)^\theta - \sum_{i=1}^{n} \exp \left( \frac{x_i}{\alpha} \right) \]

\[+ \sum_{i=1}^{n} \log \left[ \left( 1 - \exp \left( 1 - \exp \left( \frac{x_i}{\alpha} \right)^\theta \right) \right)^{(\theta - 1)} \left( 1 + \lambda - 2 \lambda \left( 1 - \exp \left( \frac{x_i}{\alpha} \right)^\theta \right) \right) \right] \]

and associated gradients found to be

\[\frac{\partial \ell(\delta | x)}{\partial \alpha} = \frac{2 \lambda \theta (\theta - 1) \left[ \sum_{i=1}^{n} \log(k(x, \alpha, \beta))(k(x, \alpha, \beta))^\theta \right]}{k(x, \alpha, \beta)} \]

\[+ \frac{(\theta - 1) \left( \frac{1}{2} + \frac{\lambda}{2} - \lambda \left( k(x, \alpha, \beta) \right)^\theta \right) \beta \left( \frac{x_i}{\alpha} \right)^\theta \exp \left( \frac{x_i}{\alpha} \right) (1-k(x, \alpha, \beta))}{k(x, \alpha, \beta)} \]

\[+ \frac{\beta}{\alpha} \left[ \sum_{i=1}^{n} \left( \frac{x_i}{\alpha} \right)^\theta \exp \left( \frac{x_i}{\alpha} \right) + \left( n + \sum_{i=1}^{n} \frac{x_i}{\alpha} \right)^\theta \right] \]

\[\frac{\partial \ell(\delta | x)}{\partial \beta} = \frac{n}{\beta} - n \log(\alpha) + \sum_{i=1}^{n} \log(x_i) + \left( \sum_{i=1}^{n} \frac{x_i}{\alpha} \right)^\theta \]

\[+ \log \left( \sum_{i=1}^{n} \frac{x_i}{\alpha} \right) - \sum_{i=1}^{n} \left( \frac{x_i}{\alpha} \right)^\theta \log \left( \frac{x_i}{\alpha} \right) \exp \left( \frac{x_i}{\alpha} \right) \]

\[-2 \lambda \theta \sum_{i=1}^{n} \frac{x_i}{\alpha} \log(k(x, \alpha, \beta))(k(x, \alpha, \beta))^\theta \]

\[-2 \lambda \theta \sum_{i=1}^{n} \frac{x_i}{\alpha} \log(k(x, \alpha, \beta))(k(x, \alpha, \beta))^\theta \]

\[\frac{\partial \ell(\delta | x)}{\partial \theta} = n \left( 1 + \frac{1}{\theta} + \lambda \right) - 2 \lambda \sum_{i=1}^{n} \left( k(x, \alpha, \beta) \right)^\theta + 2 \lambda (\theta + 1) \log(k(x, \alpha, \beta))(k(x, \alpha, \beta))^\theta \]

\[\ell(\delta | x) = \sum_{i=1}^{n} \log \left( k(x, \alpha, \beta) \right)^{(\theta - 1)} \left( 1 - 2 \left( k(x, \alpha, \beta) \right)^\theta \right) \]

where

\[k(x, \alpha, \beta) = 1 - \exp \left( \left( \frac{x_i}{\alpha} \right)^\theta \right) \]

The log-likelihood function \( \ell(\delta | x) \) can be maximized by using numerical methods such as Nelder-Mead. Let \( \hat{\delta} \) denote the MLEs of \( \delta \). Under some mild regularity conditions, one can write

\[\sqrt{n} \left( \hat{\delta} - \delta \right) \to N \left( 0, \tau^{-1}(\hat{\delta}) \right) \]
where

$$I(\hat{\delta}) = \begin{pmatrix}
  -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \alpha^2} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \alpha \beta} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \alpha \theta} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \alpha \lambda} \right] \\
  -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \beta \alpha} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \beta^2} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \beta \theta} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \beta \lambda} \right] \\
  -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \theta \alpha} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \theta^2} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \theta \lambda} \right] \\
  -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \lambda \alpha} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \lambda \beta} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \lambda \theta} \right] & -E \left[ \frac{\ell^2(\hat{\delta} | x)}{\partial \lambda^2} \right]
\end{pmatrix}$$

is expected Fisher information matrix. $I(\hat{\delta})$ can be approximated by $I(\hat{\hat{\delta}})$ which is observed Fisher Information Matrix. Using asymptotic normality of MLEs, we can write the approximate confidence intervals (CIs)

$$P\left(\tilde{\delta} - z_\frac{\eta}{2} \sqrt{\text{Var}(\hat{\delta})} < \delta_i < \tilde{\delta} + z_\frac{\eta}{2} \sqrt{\text{Var}(\hat{\delta})}\right) = 1 - \eta, \ i=1,2,3,4,$$

where $\text{Var}(\hat{\delta})$ is \((i,i)\) (diagonal) elements of $I^{-1}(\hat{\delta})$, $\tilde{\delta}=(\delta_1,\delta_2,\delta_3,\delta_4)=(\alpha,\beta,\theta,\lambda)$ and $\hat{\delta}=(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3,\hat{\delta}_4)=(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\lambda})$.

### 4. Simulation Study

In this section, Monte Carlo simulation study is performed in order to compare the performances of the MLEs of $\tilde{\delta}$ according to MSE and bias. In the simulation study, the biases and MSEs of the MLEs are empirically estimated by 1000 trials. The sample sizes are fixed as 50, 100, 250, 500, 750, 1000, 5000 and four different parameter settings are considered. The bias and MSEs of MLEs are given in Table 1 while the average lengths (AL) and coverage probabilities (CPs) of MLEs for TCEP$(\tilde{\delta})$ are presented in Table 2.

According to Table 1, when the sample size increases, the MSEs and bias of MLEs decrease for all selected parameters settings. On the other hand, it is observed that the CPs of confidence intervals approach to nominal level 0.95 and AL of intervals decrease when the sample size increases for all the parameters.
Table 1. Biases and MSEs of MLEs for TCEP ($\hat{\alpha}$) parameters

| $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | $n$ | $\hat{\alpha}$ bias | $\hat{\alpha}$ MSE | $\hat{\beta}$ bias | $\hat{\beta}$ MSE | $\hat{\theta}$ bias | $\hat{\theta}$ MSE | $\hat{\lambda}$ bias | $\hat{\lambda}$ MSE |
|----------|---------|----------|-----------|-----|----------------------|---------------------|---------------------|---------------------|----------------------|---------------------|----------------------|---------------------|
| 0.3      | 0.5     | -0.2     |           | 50  | 0.1157               | 0.0596              | 0.5440              | 0.8602              | -0.0728              | 0.3321              | 0.1192              | 0.2737              |
| 0.4      | 0.6     | 0.3      |           | 100 | 0.1124               | 0.0469              | 0.3407              | 0.3864              | -0.1340              | 0.0768              | 0.0939              | 0.2586              |
| 0.2      | 0.3     | 0.4      |           | 50  | 0.0781               | 0.0384              | 0.0620              | 0.0198              | -0.0361              | 0.0137              | 0.0100              | 0.0784              |
| 0.5      | 0.7     | 0.8      | -0.5     | 100 | 0.1124               | 0.0790              | 0.1472              | 0.0741              | -0.0629              | 0.0434              | -0.0715             | 0.2099              |
| 0.2      | 0.3     | 0.4      |           | 50  | 0.1157               | 0.0909              | 0.2334              | 0.1456              | -0.0536              | 0.1047              | -0.0963             | 0.2066              |
| 0.5      | 0.7     | 0.8      |           | 100 | 0.1124               | 0.0651              | 0.0952              | 0.0368              | -0.0554              | 0.0214              | -0.0324             | 0.1345              |
| 0.5      | 0.7     | 0.8      |           | 50  | 0.0470               | 0.0133              | 0.1037              | 0.0778              | -0.0561              | 0.0380              | 0.0778              | 0.1330              |
| 0.2      | 0.3     | 0.4      |           | 100 | 0.1124               | 0.0041              | 0.0485              | 0.0164              | -0.0329              | 0.0211              | 0.0472              | 0.0707              |
| 0.5      | 0.7     | 0.8      |           | 50  | 0.0505               | 0.0511              | 0.0134              | 0.0029              | -0.01               | 0.0028              | 0.0103              | 0.0135              |
5. Real Data Analysis I

In this section, an application with real data is provided to compare the fitting ability of TCEP(\(\theta\)) distribution with some lifetime distributions such as Complementary Exponential Power (CEP) [1], Log-Kumaraswamy (LKw) [9], Weibull and Exponentiated Exponential (EE) [10]. The pdfs of these distributions are given by

\[ LKw: \quad f(x) = \alpha \beta e^{-\beta x} \left(1 - e^{-\beta x}\right)^{\alpha-1} \left[1 - \left(1 - e^{-\beta x}\right)^\alpha\right]^\beta - 1 \nu s(x) \]

\[ Weibull: \quad f(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \nu s(x) \]

\[ EE: \quad f(x) = \alpha \beta (1 - e^{-\beta x})^{\alpha-1} e^{-\beta x} \nu s(x) \]

where \(\alpha, \beta > 0\) are parameters. We have considered the comparison criteria as the -2*log-likelihood value, Akaike’s Information Criterion (AIC), Kolmogorov-Smirnov test statistics (KS) and its (p-value) as comparison criteria. The data related to the failure stresses of single carbon fibers (length 1mm) is considered in the analysis. Note that this data firstly analyzed by [11]. The MLEs and the selection criteria statistics are given in Table 3. Furthermore, Fig. 3 presents the fitted cdfs to real data.
Table 3. Selection criteria statistics and MLEs for carbon fibres data

|                | Weibull | TCEP   | LKw     | EE       | CEP       |
|----------------|---------|--------|---------|----------|-----------|
| LogL           | -71.0240| -69.9704| -72.0352| -73.7699| -70.0187  |
| -2LogL         | 142.0479| 139.9408| 144.0705| 147.5399| 140.0375  |
| AIC            | 146.0479| 147.9408| 148.0705| 151.5399| 146.0375  |
| BIC            | 150.1340| 156.1130| 152.1566| 155.6260| 152.1666  |
| CAIC           | 146.2701| 148.7100| 148.2927| 151.7621| 146.4903  |
| HQIC           | 147.6359| 151.1168| 149.6585| 153.1279| 148.4195  |
| K-S            | 0.0859  | 0.1521 | 0.4354  | 0.6913  | 0.1595    |
| W*             | 0.0591  | 0.0212 | 0.0729  | 0.1067  | 0.0227    |
| p-value(K-S)   | 0.7618  | 0.9859 | 0.8273  | 0.6261  | 0.9777    |
| p-value (A*)   | 0.8598  | 0.9985 | 0.8124  | 0.5655  | 0.9978    |
| p-value (W*)   | 0.8219  | 0.9961 | 0.7359  | 0.5545  | 0.9941    |
| \(\hat{\alpha}\) | 4.5752  | 4.0176 | 68.7284 | 114.5288| 3.5979    |
| \(\hat{\beta}\) | 5.5930  | 1.5294 | 1.7687  | 1.2421  | 1.3105    |
| \(\hat{\theta}\) | -       | 5.7577 | -       | -       | 7.5679    |
| \(\hat{\lambda}\) | -       | 0.3693 | -       | -       | -         |
| LB for \(\alpha\) | 4.3507  | 1.9357 | 47.7744 | 15.5453 | 2.0178    |
| LB for \(\beta\) | 4.4972  | 0.0795 | 1.0788  | 1.0039  | 0.2921    |
| LB for \(\theta\) | -       | -4.7440 | -       | -       | -5.2705   |
| LB for \(\lambda\) | -       | -1.1059 | -       | -       | -         |
| UB for \(\alpha\) | 4.7998  | 6.0995 | 89.6825 | 213.5122| 5.1779    |
| UB for \(\beta\) | 6.6887  | 2.9793 | 2.4587  | 1.4804  | 2.3290    |
| UB for \(\theta\) | -       | 16.2593 | -       | -       | 20.4064   |
| UB for \(\lambda\) | -       | 1.8444 | -       | -       | -         |
| SE of \(\hat{\alpha}\) | 0.1146  | 1.0622 | 10.6910 | 50.5027 | 0.8062    |
| SE of \(\hat{\beta}\) | 0.5591  | 0.7397 | 0.3520  | 0.1215  | 0.5196    |
| SE of \(\hat{\theta}\) | -       | 5.3581 | -       | -       | 6.5503    |
| SE of \(\hat{\lambda}\) | -       | 0.7526 | -       | -       | -         |
Figure 3. Fitted cdfs and empirical cdf for carbon fibres data

From Table 3 and Fig. 3, it can be said that the TCEP(6) distribution is candidate to fitting the real data and it is competitor to the other existing models according to all criteria discussed here.

6. Real Data Analysis II

Let us consider lifetime regression analysis and let \( Y = \log(X) \). Then cdf and pdf of \( Y \) is given by

\[
F_Y(y; \kappa) = (1 + \lambda) \left[ 1 - \exp \left( \frac{y - \mu}{\sigma} \right) \right]^{\nu} - \lambda \left[ 1 - \exp \left( \frac{y - \mu}{\sigma} \right) \right]^{\nu+1},
\]

and

\[
f_Y(y; \tau) = \frac{\theta}{\sigma} \exp \left( \frac{y - \mu}{\sigma} \right) \exp \left( 1 + \exp \left( \frac{y - \mu}{\sigma} \right) - \exp \left( \frac{y - \mu}{\sigma} \right) \right) \times \left[ 1 - \exp \left( \frac{y - \mu}{\sigma} \right) \right]^{\nu-1} \times \left[ 1 + \lambda - 2\lambda \left[ 1 - \exp \left( \frac{y - \mu}{\sigma} \right) \right] \right]^{\nu} I_y(y)
\]

where \( \kappa = (\mu, \sigma, \theta, \lambda) \) is parameter vector. Let \( \mu = 0 \) and \( \sigma = 1 \) in Eq. (22). Then, Eq. (22) is reduce to

\[
F_Y(z; \kappa) = (1 + \lambda) \left[ 1 - \exp \left( 1 - \exp \left( \frac{z}{\sigma} \right) \right) \right]^{\nu} - \lambda \left[ 1 - \exp \left( 1 - \exp \left( \frac{z}{\sigma} \right) \right) \right]^{\nu+1}.
\]
Let us consider regression model

$$Y = \mu + \sigma Z,$$

where $Y = (Y_1, Y_2, \ldots, Y_n)^T$ is a random vector and $Y_1, Y_2, \ldots, Y_n$ are iid random variables (they are also called dependent variables) with cdf (22). $Z = (Z_1, Z_2, \ldots, Z_n)^T$ is a random error vector and $Z_1, Z_2, \ldots, Z_n$ are iid random variables with cdf (24) and $\sigma > 0$. Assume that location is linked to covariates by $\mu = X \beta$, where $X$ is $n \times (p + 1)$ matrix consist of covariates (First column is 1) and $\beta = (\beta_0, \beta_1, \ldots, \beta_p)^T$. Let $T_i = \min(Y_i, c_i)$ and $c_i$ is censoring time for $i$th individual or any component. Then the log-likelihood function is written by

$$\ell(k) = \sum_{i=1}^{n} \left\{ \omega_i \log \left( f_i(t_i; k) \right) + (1 - \omega_i) \log \left( 1 - F_i(t_i; k) \right) \right\},$$

where $\omega_i$ is the indicator function given by

$$\omega_i = \begin{cases} 1, & t_i \leq c_i \\ 0, & t_i > c_i \end{cases}$$

Let us consider the data given in page 335 in [12], [13] carried out an experiment and obtained a data on the lifetime of specimens of solid epoxy electrical-insulation in an accelerated voltage life test. 20 specimens were put on a life test at each of three voltage levels: 52.5, 55.0, and 57.5 kV. Failure times were measured in minutes. Six lifetimes of specimens are censored at a random. Based on the data, the log-likelihood (25) is maximized and the MLEs of parameters, AIC criteria are presented in Table 4 for TCEP regression. For a comparison Weibull and TLGBXII (see [14]) regression results are also given in Table 4. From the Table 4 and Fig. 4, it can be conclude that TCEP regression can be alternative lifetime regression analysis to Weibull and TLGBXII regression.

Figure 4. Fitted survival functions and the empirical survivals
distribution: A generalization of the Weibull probability distribution. In Table 5, Method “1” indicates Nelder-Mead whereas Method “2” indicates BFGS.

Table 5. First 10 best solutions with initial values for TCEP regression

| Trial | Method | -\ell | \hat{\theta}_0 | \hat{\beta}_0 | \hat{\sigma} | \hat{\lambda} | se(\hat{\lambda}) | se(\hat{\sigma}) | se(\hat{\theta}_0) | se(\hat{\beta}_0) | se(\hat{\beta}_0) | se(\hat{\sigma}) | se(\hat{\lambda}) | \hat{\beta}_1(t) | \hat{\beta}_1(s) | \hat{\sigma}(s) | \lambda(t) |
|-------|--------|-------|----------------|--------------|-------------|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 359   | 1      | -78.375 | 11.361     | -0.191       | 9.942       | 126.844     | 0.619           | 6.656           | 0.060           | 257.279        | 0.502           | 26.936         | -0.357          | 0.845           | 0.620           | 0.366           |                 |
| 971   | 1      | -78.455 | 12.328     | -0.192       | 8.719       | 86.939      | 0.464           | 3.994           | 0.059           | 70.926         | 0.640           | 21.215         | -0.225          | 0.840           | 0.894           | 0.685           |                 |
| 262   | 2      | -78.483 | 13.115     | -0.198       | 8.318       | 72.457      | 0.509           | 4.963           | 0.059           | 101.625        | 0.570           | 27.251         | -0.048          | 0.670           | 0.905           | 0.892           |                 |
| 309   | 2      | -78.494 | 13.225     | -0.199       | 8.221       | 69.651      | 0.510           | 4.904           | 0.059           | 96.054         | 0.565           | 16.533         | -0.096          | 1.146           | 0.541           | 0.103           |                 |
| 631   | 2      | -78.497 | 13.205     | -0.198       | 8.216       | 60.078      | 0.517           | 4.868           | 0.059           | 93.937         | 0.556           | 15.886         | -0.073          | 1.352           | 0.733           | 0.910           |                 |
| 487   | 2      | -78.513 | 12.409     | -0.186       | 8.287       | 70.057      | 0.523           | 4.481           | 0.059           | 79.607         | 0.532           | 27.940         | -0.330          | 1.276           | 0.640           | 0.161           |                 |
| 727   | 2      | -78.521 | 13.492     | -0.200       | 8.003       | 63.720      | 0.509           | 4.872           | 0.059           | 87.711         | 0.558           | 22.355         | -0.118          | 1.386           | 0.138           | 0.374           |                 |
| 218   | 1      | -78.529 | 12.926     | -0.193       | 8.255       | 67.090      | 0.587           | 4.058           | 0.059           | 60.461         | 0.473           | 26.250         | -0.356          | 0.991           | 0.727           | 0.879           |                 |
| 781   | 2      | -78.535 | 13.363     | -0.197       | 7.946       | 61.167      | 0.521           | 4.547           | 0.059           | 73.457         | 0.541           | 19.195         | 0.260           | 1.329           | 0.730           | 0.276           |                 |
| 890   | 1      | -78.537 | 12.225     | -0.196       | 8.658       | 102.473     | 0.090           | 4.362           | 0.085           | 138.953        | 3.308           | 23.208         | -0.295          | 0.839           | 0.903           | 0.297           |                 |

Conflicts of interest

There is no conflict of interest.

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