Parallel Construction of Compact Planar Embeddings

Leo Ferres¹, José Fuentes-Sepúlveda², Travis Gagie³, Meng He⁴, and Gonzalo Navarro²

1 Faculty of Engineering, Universidad del Desarrollo, Chile
lferres@udd.cl
2 Department of Computer Science, University of Chile, Chile
jfuentess@dcc.uchile.cl, gnvarro@dcc.uchile.cl
3 School of Computer Science and Telecommunications, Diego Portales University, Chile
travis.gagie@mail.udp.cl
3 Faculty of Computer Science, Dalhousie University, Canada
mhe@cs.dal.ca

Abstract
The sheer sizes of modern datasets are forcing data-structure designers to consider seriously both parallel construction and compactness. To achieve those goals we need to design a parallel algorithm with good scalability and with low memory consumption. An algorithm with good scalability improves its performance when the number of available cores increases, and an algorithm with low memory consumption uses memory proportional to the space used by the dataset in uncompact form. In this work, we discuss the engineering of a parallel algorithm with linear work and logarithmic span for the construction of the compact representation of planar embeddings. We also provide an experimental study of our implementation and prove experimentally that it has good scalability and low memory consumption. Additionally, we describe and test experimentally queries supported by the compact representation.

1998 ACM Subject Classification E.4 Coding and Information Theory

Keywords and phrases planar graph, multicore algorithm, compact data structure

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

Planar embeddings are present in several applications that need an underlying representation of topological information, such as in the mesh representation in finite-element simulations, road networks and Geographical Information Systems (GIS) in general. Because of their very nature, the size of such planar embeddings is large and dynamic, meaning it is constantly growing. For example, the underlying planar embedding to represent OpenStreetMap, has

* The second and fifth authors received travel funding from EU grant H2020-MSCA-RISE-2015 BIRDS GA No. 690941. The second author received funding from Conicyt Fondecyt 3170534. The third and fifth authors received funding Basal Funds FB0001, Conicyt, Chile. The third author received funding from Academy of Finland grant 268324. Early parts of this work were done while the third author was at the University of Helsinki and while the third and fifth authors were visiting the University of A Coruña.
more than 3 billion nodes\footnote{In OpenStreetMap, a node is defined as a specific point on the earth’s surface defined by its latitude and longitude. Updated statistics are available at \url{http://www.openstreetmap.org/stats/data_stats.html} (last access: April 06, 2017).}. Manipulating those large planar embeddings, storing and updating them efficiently is of practical importance. Having a data structure with a small memory footprint to represent this class of graphs will help us manipulate huge graphs in small devices with little (on-board, fast) memory.

Since the 1990s, compact data structures have become a viable alternative to represent large data in a small space, storing data in space close to its information-theoretic lower bound while still supporting queries efficiently. Their usefulness has been practically demonstrated in several libraries, such as, LibCDS\cite{4}, SDSL\cite{11}, Sur\cite{21}, Dynamic\cite{17} and Succinct\cite{16}. However, there has been no attempt to implement and test the practical efficiency of compact data structures for planar graphs with planar embeddings.

The construction stage of compact data structures is of particular interest, since fast construction algorithm can simulate real-time updates, effectively making it behave as a dynamic version of it. In this work, we take advantage of multicore architectures to provide a fast implementation of an algorithm to construct compact data structures for planar embeddings.

Recently, Ferres et al.\cite{9} extended the compact representation of planar embeddings of Turán\cite{20}. The extension supports navigation queries, without greatly increasing the complexity introduced in the original. In the same work, Ferres et al. also introduced a work-optimal parallel algorithm with logarithmic depth for the extended compact representation. In our work, we provide the algorithm engineering of the parallel algorithm of Ferres et al. We discuss the implementation of each used parallel algorithm, such as, Euler Tour and spanning tree, and discuss some practical trade-offs. We provide a set of experiments to prove the scalabitily and good space-usage of our implementation, using a small portion of the original input. Finally, we also provide implementations of useful queries that also behave efficiently.

The layout of the rest of the paper is as follows: in Section 2 we summarize the compact representation (for more details, we refer the reader to \cite{9}); in Section 3 we describe the parallel algorithm of \cite{9} and discuss the details of its implementation; in Section 4 we describe our experiments for construction and query, and present their results; finally, in Section 5 we present our conclusions and future work.

\section{Representation}

In \cite{9}, we introduced an extension of the compact representation of Turán\cite{20} for connected planar multi-graph with $n$ vertices and $m$ edges is introduced. The extended representation works as follows: First, we chose an arbitrary spanning tree $T$ of the planar graph, rooted at a vertex on the outer face. Second, by traversing the spanning tree in DFS order, we create three binary sequences $A$, $B$ and $B^*$. The first time we traverse an edge of $T$, we write a 0 in the sequence $B$, otherwise we write a 1 in $B$. Similarly, the first time that we reach an edge that does not belong to $T$, we write a 0 in $B^*$, otherwise we write a 1 in $B^*$. Finally, each time that we traverse an edge of $T$, we write a 1 in $A$, otherwise we write a 0. The sequence $B$ has length $2n - 2$ and corresponds to the balanced-parentheses representation of $T$. The sequence $B^*$ has length $2(m - n + 1)$ and corresponds to reversed balanced-parentheses representation of the complementary spanning tree of the dual of the graph. The sequence $A$
Figure 1 Left: A planar embedding of a planar graph $G$, with a spanning tree $T$ of $G$ shown in red and the complementary spanning tree $T^*$ of the dual of $G$ shown in blue. Right: The two spanning trees, with $T$ rooted at the vertex 1 on the outer face and $T^*$ rooted at the vertex A corresponding to the outer face. We can represent this embedding of $G$ with the three bitvectors $A[1..28] = 01101101110010100010100$, $B[1..14] = 00101100110011$ and $B^*[1..14] = 01001001110101$.

has length $2m$ and indicates how the sequences $B$ and $B^*$ are interleaved during the traversal. See Figure 1 as an example of the construction of $A$, $B$ and $B^*$.

We can add sublinear number of bits to each balanced-parentheses representation, we can support fast navigation in the trees, and by storing the sequence $A$ as a bitvector, we can support fast navigation in the graph. In particular, for the balanced-parentheses representation we are interested on the constant time queries match and parent, where match($i$) returns the position of the parenthesis matching the $i$th parenthesis and parent($v$) returns the parent of $v$, given as its pre-order rank in the traversal, or 0 if $v$ is the root of its tree. For the binary sequence $A$ we are interested in the constant time queries rank and select, where rank$_b(i)$ returns the number of bits set to $b$ in the prefix of length $\ell$ of the $A$ and select$_b(j)$ returns the position of the $j$th bit set to $b$. For more details on bitvectors and balanced-parentheses representations, we refer the reader to Navarro’s text [15]. Based on the previous queries, we defined the following basic queries:

**first($v$):** return $i$ such that the first edge we process while visiting $v$ is the $i$th we process during our traversal;

**next($i$):** return $j$ such that if we are visiting $v$ when we process the $i$th edge during our traversal, then the next edge incident to $v$ in counterclockwise order is the one we process $j$th;

**mate($i$):** return $j$ such that we process the same edge $i$th and $j$th during our traversal;

**vertex($i$):** return the vertex $v$ such that we are visiting $v$ when we process the $i$th edge during our traversal.

The compact representation support these basic queries in constant time as follows:

$$\text{first}(v) = \begin{cases} \text{A.select}_1(B.\text{select}_0(v-1)) + 1 & \text{if } m \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{next}(i) = \begin{cases} i + 1 & \text{if } A[i] = 0 \text{ and } i < 2m \\ \text{mate}(i) + 1 & \text{if } A[i] = 1 \text{ and } B[A.\text{rank}_1(i)] = 0 \\ 0 & \text{otherwise} \end{cases}$$
Parallel Construction of Compact Planar Embeddings

mate(i) = \begin{cases} 
A.\text{select}_0(B^*.\text{match}(A.\text{rank}_0(i))) & \text{if } A[i] = 0 \\
A.\text{select}_1(B.\text{match}(A.\text{rank}_1(i))) & \text{otherwise}
\end{cases}

vertex(i) = \begin{cases} 
B.\text{rank}_0(A.\text{rank}_1(i)) + 1 & \text{if } A[i] = 0 \text{ and } B[A.\text{rank}_1(i)] = 0 \\
B.\text{parent}(B.\text{rank}_0(B.\text{match}(A.\text{rank}_1(i)))) + 1 & \text{if } A[i] = 0 \text{ and } B[A.\text{rank}_1(i)] = 1 \\
B.\text{parent}(B.\text{rank}_0(A.\text{rank}_1(i))) + 1 & \text{if } A[i] = 1 \text{ and } B[A.\text{rank}_1(i)] = 0 \\
B.\text{rank}_0(B.\text{match}(A.\text{rank}_1(i))) + 1 & \text{otherwise.}
\end{cases}

With those basic queries, it is possible to define more complex queries. As an example, we present three queries based on the basic ones: counting(v), the number of neighbors of vertex v; listing(v), the list of neighbors of vertex v, in counterclockwise order; face(e), the list of vertices, in clockwise order, of the face where the edge e belongs. The implementation of the three queries follows. For queries listing(v) and face(e), we report each vertex by using the function print.

| Function counting | Function listing | Function face |
|-------------------|------------------|---------------|
| **Input:** node v | **Input:** node v | **Input:** edge e |
| 1 d = 0 | 1 nxt = first(v) | 1 nxt = e, flag = true |
| 2 nxt = first(v) | 2 while nxt < 2m do | 2 while nxt \(\neq e \) or flag do |
| 3 while nxt < 2m do | 3 mt = mate(nxt) | 3 flag = false |
| 4 nxt = next(nxt) | 4 print(vertex(mt)) | 4 mt = mate(nxt) |
| 5 d = d + 1 | 5 nxt = next(nxt) | 5 print(vertex(mt)) |
| 6 | | 6 nxt = next(nxt) |

In the rest of this paper, we explain how to construct A, B and B* efficiently in parallel, and demonstrate experimentally that our representation is practical.

3 Parallel algorithm for compact planar embeddings

In this section we discuss the parallel construction of the compact representation of planar embeddings. Since the compact representation is based on spanning trees and tree traversals, we can borrow ideas of well-known parallel algorithms, such as Euler Tour traversal in parallel or parallel computation of spanning trees.

For the analysis of our algorithm, we use the Dynamic Multithreading (DyM) Model. In this model, a multithreaded computation is modelled as a directed acyclic graph (dag) where vertices are instructions and an edge (u, v) represents precedence among instruction u and v. The model is based on two parameters of the multithreaded computation: its work \(T_1\) and its span \(T_\infty\). The work is the running time on a single thread, that is, the number of nodes (i.e., instructions) in the dag, assuming each instruction takes constant time. The span is the length of the longest path in the dag; the intrinsically sequential part of the computation. The time \(T_p\) needed to execute the computation on \(p\) threads has complexity \(\Theta(T_1/p + T_\infty)\), which can be reached with a greedy scheduler. The improvement of a multithreaded computation using \(p\) threads is called speedup, \(T_1/T_p\). The upper bound on the achievable speedup, \(T_1/T_\infty\), is called parallelism. Finally, the efficiency is defined as
\( T_1/pT_p \) and can be interpreted as the percentage of improvement achieved by using \( p \) cores or how close we are to the linear speedup. In the DyM model, the workload of the threads is balanced by using the work-stealing algorithm \(^8\).

To describe parallel algorithms in the DyM model, we augment sequential pseudocode with three keywords. The \textit{spawn} keyword, followed by a procedure call, indicates that the procedure should run in its own thread and may thus be executed in parallel to the thread that spawned it. The \textit{sync} keyword indicates that the current thread must wait for the termination of all threads it has spawned. Finally, \textit{parfor} is “syntactic sugar” for spawning one thread per iteration in a for loop, thereby allowing these iterations to run in parallel, followed by a \textit{sync} operation that waits for all iterations to complete. In practice, the \textit{parfor} keyword is implemented by halving the range of loop iterations, spawning one half and using the current procedure to process the other half recursively until reaching one iteration per range. After that, the iterations are executed in parallel. Therefore, this implementation adds an overhead bounded above the logarithm of the number of loop iterations. In our algorithm, we include such overhead in our complexities.

For the rest of this section, each tree \( T \) is represented with an adjacency list representation. Such representation consists of an array of nodes of size \( n \), \( V_T \), and an array of edges of size \( m \), \( E_T \). Each node \( v \in V_T \) stores two indices in \( E_T \), \( v\.first \) and \( v\.last \), indicating the adjacency list of \( v \), sorted counterclockwise around \( v \) and starting with \( v \)’s parent edge (except the root). Notice that the number of children of \( v \) is \( (v\.last - v\.first) \). Each edge \( e \in E_T \) has three fields, \( e\.src \), which is a pointer to the source vertex, \( e\.tgt \), which is a pointer to the target vertex and \( e\.cmp \), which is the position in \( E_T \) of the complement edge of \( e \), \( e' \), where the \( e'.src = e\.tgt \) and \( e\.tgt = e\.src \). For \( x \in \{e\.src,e\.tgt\} \), we use \textit{next}(\( x \)) and \textit{first}(\( x \)) to denote the indices in \( E_G \) of \( e \)’s successor and of the first element (parent edge) in \( x \)’s adjacency list, respectively. Both are easily computed in constant time by following pointers. Notice that \( |E_T| = 2(|V_T| - 1) \). The representation of graphs is similar, with the exception that the concept of parent of a vertex is not valid in graphs, therefore the first edge in the adjacency list of a vertex \( v \) cannot be interpreted as the \( v \)’s parent edge.

### 3.1 Parallel construction of compact planar embeddings

Given a planar embedding of a connected planar graph \( G = (V_G, E_G) \), for the moment we assume a spanning tree of \( T = (V_T, E_T) \) of \( G \) and an array \( C \) that stores the number of edges of \( G \setminus T \) between two consecutive edges in \( T \), in counterclockwise order, are given as part of input. Later, in Section 3.2, we explain how to obtain the spanning tree \( T \) and the array \( C \) in parallel. With the spanning tree, we construct the bitvectors \( A, B \) and \( B^* \) by performing an Euler Tour over \( T \). During tour, by writing a 0 for each forward (parent to child) edge and a 1 for each backward (child to parent) edge, we obtain the bitvector \( B \), by counting the number of edges of \( G \setminus T \) between two consecutive edges of \( T \) (array \( C \)), representing them by 0’s and the edges of \( T \) by 1’s, we obtain the bitvector \( A \), and by using the previous Euler Tour and the array \( C \) we can obtain the bitvector \( B^* \). Algorithm 1 shows this idea in more details. The algorithm works in six steps: In the first step, the algorithm creates an auxiliar array \( LE \) (line 4) that is used to store the traversal of the tree following the Euler Tour. Each entry of \( LE \) represents one traversed edge of \( T \) and stores four fields: \textit{value} is 0 or 1 depending on whether the edge is a forward or a backward edge, respectively; \textit{suc} is the index in \( LE \) of the next edge in the Euler tour; \textit{rankA} is the rank of the edge in \( A \); and \textit{rankB} is the rank of the edge in \( B \).

In the second step, the algorithm traverses \( T \) (lines 6–22). For each edge \( e_j \in E_T \), \textit{rankA} is set as \( C[E_T[j].cmp] + 1 \) and \textit{rankB} as 1 (lines 9–10). Those ranks will be used later to...
Algorithm 1: Parallel compact planar embedding algorithm (PAR-SPE).

\textbf{Input}: A planar embedding of a planar graph $G = (V_G, E_G)$, a spanning tree $T = (V_T, E_T)$ of $G$, an array $C$ of size $|E_T|$, the starting vertex $\text{init}$ and the number of threads, $\text{threads}$.

\textbf{Output}: Bitvectors $A$, $B$ and $B^*$ induced by $G$ and $T$.

\begin{algorithmic}
1 $A$ = a bitvector of length $|E_G|$
2 $B$ = a bitvector of length $|E_T| - 2$
3 $B^*$ = a bitvector of length $|E_G| - |E_T| + 2$
4 $LE$ = an array of length $|E_T|$
5 $chk = |E_T|/\text{threads}$
6 \textbf{parfor} $t = 0$ to $\text{threads} - 1$ do
7 \hspace{1em} for $i = 0$ to $chk - 1$ do
8 \hspace{2em} $j = t * chk + i$
9 \hspace{2em} $LE[j].rankA = C[E_T[j].cmp] + 1$
10 \hspace{2em} $LE[j].rankB = 1$
11 \hspace{2em} if $E_T[j].src == \text{init}$ OR first($E_T[j].src) \neq j$ then // forward edge
12 \hspace{3em} $LE[j].value = 0$ // opening parenthesis
13 \hspace{3em} if $E_T[j].tgt$ is a leaf then
14 \hspace{4em} $LE[j].suc = E_T[j].cmp$
15 \hspace{4em} else
16 \hspace{5em} $LE[j].suc = \text{first}(E_T[j].tgt) + 1$
17 \hspace{2em} else // backward edge
18 \hspace{3em} $LE[j].value = 1$ // closing parenthesis
19 \hspace{3em} if $E_T[j].tgt$ is the last edge in the adjacency list of $E_T[j].src$ then
20 \hspace{4em} $LE[j].suc = \text{first}(E_T[j].tgt)$
21 \hspace{4em} else
22 \hspace{5em} $LE[j].suc = \text{next}(E_T[j].tgt)$
23 \parfor $t = 0$ to $\text{threads} - 1$ do
24 \hspace{1em} for $i = 0$ to $chk - 1$ do
25 \hspace{2em} $j = t * chk + i$
26 \hspace{2em} $AL[LE[j].rankA] = 1$ // By default, all elements of $A$ are 0’s
27 \hspace{2em} $BL[LE[j].rankB] = LE[j].value$
28 $D_{pos}, D_{edge}$ = two arrays of length $|E_G| - |E_T| + 2$
29 \parfor $t = 0$ to $\text{threads} - 1$ do
30 \hspace{1em} for $i = 0$ to $chk - 1$ do
31 \hspace{2em} $j = t * chk + i$
32 \hspace{2em} $pos = LE[j].rankA - LE[j].rankB$
33 \hspace{2em} $lim = \text{ref}(E_T[j].cmp) + C[E_T[j].cmp]$
34 \hspace{2em} for $k = \text{ref}(E_T[j].cmp) + 1$ to $lim$ do
35 \hspace{3em} $D_{pos}[k] = pos$
36 \hspace{3em} $D_{edge}[pos] = k$
37 \hspace{3em} $pos = pos + 1$
38 $chk = [(|E_G| - |E_T| + 2)/\text{threads}$
39 \parfor $t = 0$ to $\text{threads} - 1$ do
40 \hspace{1em} for $i = 0$ to $chk - 1$ do
41 \hspace{2em} $j = t * chk + i$
42 \hspace{2em} $cmp = E_T[D_{edge}[j].cmp]$
43 \hspace{2em} if $j > D_{pos}[\text{cmp}]$ then
44 \hspace{3em} $B^*[j + 1] = 1$ // By default, all elements of $B^*$ are 0’s
46 createRankSelect($A$), createBP($B$), createBP($B^*$)
\end{algorithmic}
compute the final position of the edges in A, B and B\*. For each forward edge, a 0 is written in the corresponding value field and the succ field is connected to the next edge in the Euler Tour. For backward edges is similar. Considering the adjacency list representation of T, all the edges in the adjacency list of a node (except the root) of T are forward edges, except the first one (parent edge). For the root, all the edges of its adjacency list are forward edges.

In the third step, the algorithm computes the final ranks in A and B using a parallel list ranking algorithm (line 23). We use the algorithm introduced in \cite{12} over the rank\! A and rank\! B fields of LE to obtain the final position of each edge in A and B, respectively.

In the fourth step, bitvectors A and B are written. If initially all the elements of A are 0’s, it is enough to set to 1 all the elements given by the fields rank\! A’s. For B, the algorithm copies the content of field value at position rank\! B, for all the elements in LE.

In the fifth step, the algorithm computes the position of each edge of G \setminus T in B\*. That information is implicit in the fields rank\! A and rank\! B of LE (line 33), after the list ranking of the third step. For each edge e_i \in E_T, the algorithm computes the positions, in B\*, of the edges in G \setminus T that follow, in counterclockwise order, the complement edge of e_i (lines 34–38). The algorithm uses two auxiliary arrays, D_{pos} and D_{edge}. The entry D_{pos}[j] stores the position of the edge e_j of G \setminus T in B\*. The array D_{edge} is the inverse of D_{pos}. It stores the position of the j-th edge of B\* in G \setminus T. Thus, D_{pos}[i] = j \iff D_{edge}[j] = i. In this step, the function ref(E_T[j]) returns the position of the edge e_j of E_T in E_G.

In the sixth step, the algorithm computes if the edges stores in D_{pos} are forward or backward edges. For each edge e in G \setminus T, it is done by comparing the position in B\* of e and its complement. If the position of e is greater than the position of its complement, then e is a backward edge and, therefore, represented by a 1. By default, we assume that all the elements of B\* are 0’s.

Finally, the structures to support operations \texttt{rank}, \texttt{select}, \texttt{match} and \texttt{parent} are constructed. For the bitvector A, Algorithm 1 uses the parallel algorithms of Labeit et al. \cite{13} (\texttt{createRankSelect}). In the case of B and B\*, the algorithm uses the parallel algorithm of Ferres et al. \cite{10} (\texttt{createBP}) for balanced parenthesis sequences.

Now we present the analysis of our algorithm. In the first step there is not computation involved, therefore, we do not include it in the complexity. In the second step, the algorithm traverse the edges of T, performing an independent computation in each edge, therefore, with the overhead of the \texttt{parfor} loop, we obtain T_1 = O(n) and T_\infty = O(\lg n) time. The third step uses the algorithm of Helman and JáJá for the parallel list ranking problem over m elements, with complexities T_1 = O(n) and T_\infty = O(\lg n) time. In the fourth step, the assignation of the values to A and B can be done independently for each entry of the bitvectors. With the overhead of the parallel loop, we have T_1 = O(n) and T_\infty = O(\lg n) time. In the fifth step, the algorithm traverses all the edges in G \setminus T. Observe that the range of the loop in line 35 can be processed in parallel using a domain decomposition technique. With that, we obtain T_1 = O(m - n) and T_\infty = O(\lg(m - n)) time. We decide to implement it as it appears in line 35, because we obtained good practical results. If we need to process the line 35 in parallel, we can use the same domain decomposition technique of lines 30–32. Similar to the fourth step, in the sixth step the algorithm sets the entries of the bitvector B\*, which can be done independently for each entry. Therefore, T_1 = O(m - n) and T_\infty = O(\lg(m - n)) time.

The rank/select structures can be constructed in T_1 = O(m) and T_\infty = O(\lg m) time by using the results of \cite{13}. A structure that supports \texttt{match} and \texttt{parent} operations over a balanced parentheses sequence can be constructed in T_1 = O(m) and T_\infty = O(\lg m) time with the results of \cite{10}.

In addition to the size of the compact data structure, the memory consumption of
our algorithm depends on the size of arrays \(LE, D_{pos}\) and \(D_{edge}\). The array \(LE\) uses \(O(n \lg n)\) bits, and arrays \(D_{pos}\) and \(D_{edge}\) uses \(O(m - n) \lg n)\) bits. Thus, the total memory consumption of our algorithm is \(O(m \lg n)\) bits plus the output data structure. Notice that the memory consumption is independent of the number of threads.

In summary, we have the following theorem.

**Theorem 1.** Given a planar embedding of a connected planar graph \(G = (V_G, E_G)\) with \(m\) edges and a spanning tree of \(G\), we can compute in parallel compact representation of \(G\), using \(4m + o(m)\) bits and supporting navigational operations, in \(O(m)\) work, \(O(\lg m)\) span, \(O(m/p + \lg n)\) time, using \(O(m \lg n)\) bits of additional memory, where \(p\) is the number of available threads.

### 3.2 Parallel computation of spanning trees

In this section we discuss the parallel computation of the spanning tree \(T = (V_T, E_T)\) and the array \(C\) used in Section [3.1]

The work of Bader and Cong [2] can be used to compute a spanning tree of a planar graph. Their algorithm works as follows: Given a starting vertex of the graph \(G\) with \(n\) vertices and \(m\) edges, the algorithm computes sequentially a spanning tree of size \(O(p)\), called stub spanning tree, where \(p\) is the number of available threads. Then, an evenly number of leaves of the stub spanning tree are assigned to the \(p\) threads as starting vertices. Each thread traverses \(G\), using its starting vertices, constructing spanning trees with a DFS traversal using a stack. For each vertex, a reference to its parent is assigned. Since a vertex can be visited by several threads, the assignment of the parent of the vertex may generate a race condition. However, since the parent assigned by any thread already belongs to a spanning tree, any assignment will generate a correct tree. Thus, the race condition is a benign race condition. Once a thread has no more vertices on its stack, the thread tries to steal vertices from the stack of other thread by using the work stealing algorithm. Since the spanning trees generated by all the threads are connected to the stub spanning tree, the union of all the spanning tree generates a spanning tree of \(G\). Thus, the algorithm gives an array of parent references for each vertex. With such array of references, we can construct the corresponding adjacency list representation of the spanning tree. To do that, we mark with a 1 each edge \(E_G\) that belongs to \(E_T\) and with 0 the rest of the edges. Using a parallel prefix sum algorithm over \(E_G\), we compute the position of all the marked edges of \(E_G\) in \(E_T\). The first and last fields of each node in the spanning tree are computed similarly. As a byproduct of the computation of \(E_T\), we can compute an array \(C\) which stores the number of edges of \(G \setminus T\) between two consecutive edges in \(T\), in counterclockwise order. It can be done by using the marks in the edges, counting the number of 0’s between two consecutive 1’s. Notice the starting vertex for the stub spanning tree must be in the outer face of \(G\), to meet the description of the compact data structure for planar embeddings.

The complexities of the spanning tree algorithm depends on both the random traversal of the threads and the diameter of \(G\): \(T_1 = O(m + n)\) work, \(T_{\infty} = O(m + n)\) span, since the stub spanning tree is computed sequentially and its size is proportional to the number of threads, and \(T_p = O((m + n)/p)\) expected time for general random graphs, for \(p \ll n\). This algorithm has a worst case when \(G\) has diameter of \(O(m)\) and low connectivity. In that case, the expected time is \(T_p = (m + n)\). Despite its span and worst case, the algorithm of Bader and Cong has a good practical behavior and its implementation is simple. The adjacency list representation of the spanning tree \(T\) and the array \(C\) are computed by using a parallel prefix sum algorithm, which is well-known to have \(T_1 = O(m)\) and \(T_{\infty} = O(\lg m)\) time.
Since the spanning tree of $G$ is part of the input of our algorithm for planar embeddings, we decide to explore different parallel models to overcome the worst case of Bader and Cong’s algorithm, and thus improve the overall complexity of our algorithm. The algorithm introduced in this manuscript is simple enough to be adapted to other parallel models, because it is based on Euler Tour and list ranking algorithm, and parallel filling of arrays, which are well-known algorithms in most of the parallel models. In particular, we can adapt our parallel algorithm to the CRCW PRAM model, since we can use the CRCW PRAM Euler Tour algorithm of [6], with $O(\log n)$ parallel time using $O(m)$ cores, and the CRCW PRAM list ranking algorithm of [5], with $O(\log n)$ parallel time using $O(n/\log n)$ cores. For spanning trees, we have the algorithms of [1] and [18] in the CRCW PRAM model. The algorithm in [1] takes $O(\log n)$ parallel time using $O(m)$ cores, and the algorithm in [18] takes $O(\log n)$ parallel time using $n + 2m$ cores.

Thus, in the CRCW PRAM model, our algorithm for compact planar embeddings reaches logarithmic parallel time, including the computation of the spanning tree. The array $C$ can be constructed using the strategy explained before, based on parallel prefix sum.

Despite its the worst case, we used the spanning tree algorithm of Bader and Cong in our implementation and experiments, since it showed good practical results.

4 Experiments

We implemented the par-spe algorithm in C and compiled it using GCC 5.4 with Cilk Plus extension, an implementation of the DyM model[2]. In our implementation of the parallel spanning tree algorithm of Bader and Cong, to reduce the worst case, we included a threshold of $O(n/p)$ elements in the stack size of each thread. Each time that a thread has more nodes that the threshold, that thread create a new parallel task with the half of its stack. Additionally, we also return for each node the reference to its parent. Returning the references to parents gives better performance than forcing the first edge of each node to be the reference to its parent. However, in the description of the PAR-SPE algorithm we assume that the first edge of each node is the reference to its parent to make it more readable. Additionally, we implemented a sequential algorithm called seq, which corresponds to the serialization of the PAR-SPE algorithm and a sequential DFS algorithm to compute the spanning tree. To serialize a parallel algorithm in the DyM model, we replaced each parfor keyword for the for keyword and deleted the spawn and sync keywords.

The experimental trials consisted in running the implementation on artificial datasets of different number of nodes and threads. The datasets are shown in Table 1. Each dataset was generated in three stages: In the first stage, we used the function $rnorm$ of R to generate random coordinates $(x, y)$. The only exception was the dataset wc, which corresponds to the coordinates of 2,243,467 unique cities in the world. In the second stage, we generated the Delaunay Triangulation of the coordinates generated in the first stage. The triangulations

---

2 The code and data needed to replicate our results are available at [http://www.dcc.uchile.cl/~jfuentess/pemb/](http://www.dcc.uchile.cl/~jfuentess/pemb/)

3 The $rnorm$ function generates random numbers for the normal distribution given a mean and a standard deviation. In our case, the $x$ and $y$ components was generated using mean 0 and standard deviation 10000. For more information about the $rnorm$ function, please visit [https://stat.ethz.ch/R-manual/R-devel/library/stats/html/Normal.html](https://stat.ethz.ch/R-manual/R-devel/library/stats/html/Normal.html)

4 The dataset containing the coordinates was created by MaxMind, available from [https://www.maxmind.com/en/free-world-cities-database](https://www.maxmind.com/en/free-world-cities-database). The original dataset contains 3,173,959 cities, but some of them have the same coordinates. We selected the 2,243,467 cities with unique coordinates to build our dataset worldcities.
Parallel Construction of Compact Planar Embeddings

| Dataset | Vertices \((n)\) | Edges \((m)\) |
|---------|-----------------|----------------|
| wc      | 2,243,467       | 6,730,395      |
| pe5M    | 5,000,000       | 14,999,983     |
| pe10M   | 10,000,000      | 29,999,979     |
| pe15M   | 15,000,000      | 44,999,983     |
| pe20M   | 20,000,000      | 59,999,975     |
| pe25M   | 25,000,000      | 74,999,979     |

Table 1 Datasets used in the experiments of the PAR-SPE algorithm.

| Number of threads | Speedup |
|-------------------|---------|
| 1                 | 4       |
| 2                 | 8       |
| 4                 | 12      |
| 8                 | 16      |
| 16                | 20      |
| 32                | 24      |
| 64                | 28      |
| 128               | 32      |
| 256               | 36      |

Figure 2 Speedup of the PAR-SPE algorithm.

Table 2 Running times of PAR-SPE algorithm in seconds.

were generated using Triangle, a piece of software dedicated to the generation of meshes and triangulations.

In the final stage, we generated planar embeddings of the Delaunay triangulations computed in the second stage. The planar embedding was generated with the The Edge Addition Planarity Suite.

The experiments were carried out on a NUMA machine with two NUMA nodes. Each NUMA node includes a 14-core Intel® Xeon® CPU (E5-2695) processor clocked at 2.3GHz. The machine runs Linux 2.6.32-642.el6.x86_64, in 64-bit mode. The machine has per-core L1 and L2 caches of sizes 64KB and 256KB, respectively and a per-processor shared L3 cache of 35MB, with a 768GB DDR3 RAM memory (384GB per NUMA node), clocked at 1867MHz. Hyperthreading was enabled, giving a total of 28 logical cores per NUMA node.

Table 2 shows the running times obtained in our experiments, and Figure 2 shows the speedups compared with the seq algorithm. On average, the seq algorithm took about 76% of the time obtained by the PAR-SPE algorithm running with 1 thread. With \(p \geq 2\), the PAR-SPE algorithm shows better times than the seq algorithm. We observe an almost linear speedup up to \(p = 24\), with an efficiency of at least 56%, considering all the datasets. With \(p = 28\) the speedup has a slowdown, due to the topology of our machine. Up to 24 cores, all the threads were running in the same NUMA node. With \(p \geq 28\), both NUMA nodes

5 The software is available at [http://www.cs.cmu.edu/~quake/triangle.html](http://www.cs.cmu.edu/~quake/triangle.html). Our triangulations were generated using the options `-cezCBVPNE`.

6 The suite is available at [https://github.com/graph-algorithms/edge-addition-planarity-suite](https://github.com/graph-algorithms/edge-addition-planarity-suite). Our embeddings were generated using the options `-s -q -p`. 
Figure 3 shows the memory consumption of our algorithm. Specifically, the figure shows the space used by each dataset in adjacency list representation (inputGraph), the peak of consumption of our implementation (peakMem) and the size of the compact representation of each dataset (compGraph). Compared with the space consumption of the adjacency list representation, our implementation uses 36% more space and the compact representation uses about 4.6% of it. The consumption per edge was 5 bits, which matches the Theorem 1.

Finally, we tested the three queries introduced in Section 2: counting, listing and face. Observe that, given the adjacency list representation described in Section 3, to answer counting and listing queries is straightforward. In our experiments, we tested counting and listing 10 times for each vertex, and face 10 times per edge. Figure 4 shows the median time per query, both for the adjacency list (al-counting, al-listing and al-face) and compact representation (comp-counting, comp-listing and comp-face). The adjacency list representation allows to answer counting and listing queries 100 and 80 times faster than the compact representation. This result was expected, since the adjacency list representation we assumed already has the list of neighbors in counterclockwise order. For the face query, the adjacency list representation is only 14 times faster.

In summary, our parallel algorithm has good scalability to construct the compact representation of planar embeddings of [9]. In particular, using only one NUMA node, our algorithm scales linearly. To answer queries, compact representations of data structures are slower than their uncompacted counterparts. However, such compact representation use less memory, allowing to fit data structures close to fast memories, such as main memory and caches, speeduping up the overall computation for large datasets. In the particular case of the face query, a query closer to what we expect when solving more realistic problems, our compact implementation is 14 slower and uses 20 times less memory than the uncompacted representation.
5 Conclusions

In this paper, we presented the algorithm engineering of the parallel algorithm for the construction of compact representations of planar embeddings introduced in [9]. We also show empirically that our proposed implementation has good scalability in shared-memory architectures. Finally, we tested three different queries supported by our implementation and show that they have good execution-time behavior, making them of practical importance.

Notice, interestingly, that the compact representation can be extended to unconnected planar graphs. To do this, we first need to find all the connected components of the graph. Then, we compute an arbitrary spanning tree for each connected component. Then, we construct the binary sequences: the sequence $B$ will represent the forest of the spanning trees, concatenating all the balanced-parentheses representations; the sequence $B^*$ will represent complementary spanning tree of the dual of the graph. Since the outer face of all the connected components is the same, $B^*$ represents a tree and not a forest. Finally, sequence $A$ indicates the interleaving of the sequences $B$ and $B^*$. For that, we arbitrarily choose a connected component to start the traversal. For the computation of the connected components, we can use the work of Shun et al. [19] which has good theoretical and practical results.

As future work, we will compare our implementation of the Bader and Cong algorithm with the algorithm for connected components of Shun et al. [19]. The algorithm of Shun et al. computes the connected components of a graph by calling the partitioning algorithm of Miller et al. [14]. During the partitioning step, the algorithm performs multiple BFS’s over the graph. All the vertices in a partition are contracted to generate a new graph with less edges. The process is recursively repeated until the contracted graph has not edges. We can modify this algorithm in order to obtain the spanning tree by returning the edges traversed in each BFS.

References

1 B. Awerbuch and Y. Shiloach. New connectivity and msf algorithms for shuffle-exchange network and pram. IEEE Transactions on Computers, C-36(10):1258–1263, Oct 1987. doi: 10.1109/TC.1987.1676869
2 David A. Bader and Guojing Cong. A fast, parallel spanning tree algorithm for symmetric multiprocessors (SMPs). Journal of Parallel and Distributed Computing, 65(9):994–1006, 2005.
3 Robert D. Blumofe and Charles E. Leiserson. Scheduling multithreaded computations by work stealing. Journal of the ACM, 46(5):720–748, 1999.
4 Francisco Claude. A compressed data structure library. https://github.com/fclaude/libcds. Last accessed: April 07, 2017.
5 Richard Cole and Uzi Vishkin. Faster optimal parallel prefix sums and list ranking. Information and Computation, 81(3):334 – 352, 1989. URL: http://www.sciencedirect.com/science/article/pii/0890540189900369 doi:http://dx.doi.org/10.1016/0890-5401(89)90036-9
6 Guojin Cong and D. A. Bader. The euler tour technique and parallel rooted spanning tree. In International Conference on Parallel Processing, 2004. ICPP 2004., pages 448–457 vol.1, Aug 2004. doi:10.1109/ICPP.2004.1327954
7 T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. Multithreaded algorithms. In Introduction to Algorithms, pages 772–812. The MIT Press, 3rd edition, 2009.
8 Ulrich Drepper. What every programmer should know about memory, 2007. URL: http://people.redhat.com/drepper/cpumemory.pdf.
Leo Ferres, José Fuentes, Travis Gagie, Meng He, and Gonzalo Navarro. Fast and compact planar embeddings. In Workshop on Algorithms and Data Structures (WADS), 2017. To appear.

Leo Ferres, José Fuentes-Sepúlveda, Meng He, and Norbert Zeh. Parallel construction of succinct trees. In Proceedings of the 14th Symposium on Experimental Algorithms (SEA), pages 3–14, 2015.

Simon Gog. Succinct data structure library 2.0. https://github.com/simongog/sdsl-lite. Last accessed: April 07, 2017.

David R. Helman and Joseph JáJá. Prefix computations on symmetric multiprocessors. Journal of Parallel and Distributed Computing, 61(2):265–278, 2001.

Julian Labeit, Julian Shun, and Guy E. Blelloch. Parallel lightweight wavelet tree, suffix array and FM-index construction. In Proceedings of the Data Compression Conference (DCC), pages 33–42, 2016.

Gary L. Miller, Richard Peng, and Shen Chen Xu. Parallel graph decompositions using random shifts. In Proceedings of the Twenty-fifth Annual ACM Symposium on Parallelism in Algorithms and Architectures, SPAA ’13, pages 196–203, New York, NY, USA, 2013. ACM. URL: http://doi.acm.org/10.1145/2486159.2486180, doi:10.1145/2486159.2486180.

Gonzalo Navarro. Compact Data Structures: A Practical Approach. Cambridge University Press, 2016.

Giuseppe Ottaviano. Succinct. https://github.com/ot/succinct. Last accessed: April 07, 2017.

Nicola Prezza. Dynamic: a succinct and compressed dynamic data structures library. https://github.com/nicolaprezza/DYNAMIC. Last accessed: April 07, 2017.

Yossi Shiloach and Uzi Vishkin. An o(logn) parallel connectivity algorithm. Journal of Algorithms, 3(1):57 – 67, 1982. URL: http://www.sciencedirect.com/science/article/pii/0196677482900086, doi:http://dx.doi.org/10.1016/0196-6774(82)90008-6

Julian Shun, Laxman Dhulipala, and Guy Blelloch. A simple and practical linear-work parallel algorithm for connectivity. In Proceedings of the 26th ACM Symposium on Parallelism in Algorithms and Architectures, SPAA ’14, pages 143–153, New York, NY, USA, 2014. ACM. URL: http://doi.acm.org/10.1145/2612669.2612692, doi:10.1145/2612669.2612692.

György Turán. On the succinct representation of graphs. Discrete Applied Mathematics, 8(3):289–294, 1984.

Sebastiano Vigna. Six: Implementing succinct data structures. http://sux.di.unimi.it/ Last accessed: April 07, 2017.