ON THE AUTOMORPHISM GROUP OF A POSSIBLE SYMMETRIC $(81, 16, 3)$ DESIGN

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Abstract. In this paper we study the automorphism group of a possible symmetric $(81, 16, 3)$ design.

1. Introduction

Let $v, k$ and $\lambda$ be non-negative integers such that $v > k > \lambda$. By a symmetric $(v, k, \lambda)$ design, we mean a pair $D = (V, B)$, where $V$ is a $v$-set and $B$ is a set of $k$-subsets of $V$ such that the following four requirements are satisfied by $D$:

1. $|B| = v$.
2. any element of $V$ belongs to precisely $k$ members of $B$.
3. any two distinct members of $B$ intersect in exactly $\lambda$ elements of $V$.
4. any two distinct elements of $V$ are in exactly $\lambda$ members of $B$.

As usual, the elements of $V$ are called points of $D$ and the members of $B$ are called blocks of the design $D$. An automorphism of a symmetric design $D = (V, B)$ is a permutation on $V$ which sends blocks to blocks. The set of all automorphisms of $D$ with the composition rule of maps forms the full automorphism group of $D$ which will be denoted by $Aut(D)$. If $\alpha$ is an automorphism

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of $D$, we denote by $F(\alpha)$ the set of all points which are fixed by $\alpha$; and $F_b(\alpha)$ denotes the set of all blocks which are fixed by $\alpha$.

Over the years, researchers have tackled problems related to symmetric designs. The question of existence still remains unsettled for many parameter sets. Indeed, if we list the parameters $(v, k, \lambda)$ in order of increasing $n = k - \lambda$, then $(81, 16, 3)$ would be the smallest unknown case [8]. On the other hand, the success of almost all the design construction methods depends heavily on a proper choice of possible automorphism groups [4].

As far as we know, the only known results on a possible $(81, 16, 3)$ design are the following:

**Theorem 1.1.** (See [2]) There is no symmetric $(81, 16, 3)$ design with an abelian regular 3-group of automorphisms.

**Theorem 1.2.** (See [7]) Let $\alpha$ be an automorphism of a possible symmetric $(81, 16, 3)$ design of order 2. Then $|F(\alpha)| = 9$.

**Theorem 1.3.** (See [5]) The alternating group $A_5$ of degree 5 cannot be isomorphic to a group of automorphisms of a possible symmetric $(81, 16, 3)$ design.

T. Spence has announced in his home page [http://www.maths.gla.ac.uk/~es/] that there is no symmetric $(81, 16, 3)$ designs having a “certain” fixed-point free automorphism of order 3.

Our main result is:

**Theorem 1.4.** If $G$ is the full automorphism group of a possible symmetric $(81, 16, 3)$ design, then $|G| = 2^\gamma 3^3 5^1 13^\sigma$, where $\gamma \leq 1$, $\sigma \leq 1$. Moreover, $G$ has no subgroup of order 65, and has no elements of orders 10 or 26; and $G$ does not contain any abelian 2-subgroup of rank greater than 3.

In Section 2, some general results on the automorphism groups of a symmetric design are given and in Section 3, we prove a series of Lemmata. Based on them we can prove Theorem 1.4.
2. Some general results on the automorphism group of a symmetric design

**Lemma 2.1.** (See [6]) Let $\alpha$ be an automorphism of a nontrivial symmetric $(v, k, \lambda)$ design. Then $|F(\alpha)| = |F_b(\alpha)|$.

**Lemma 2.2.** (See [6, Corollary 3.7, p. 82]) Let $D$ be a non trivial symmetric $(v, k, \lambda)$ design and $\alpha$ a non trivial automorphism of $D$. Then $|F(\alpha)| \leq k + \sqrt{k - \lambda}$.

**Lemma 2.3.** Let $D$ be a symmetric $(v, k, \lambda)$ design and $\alpha$ an automorphism of $D$ of prime order $p$ such that $\lambda < p$. If $B$ is a block of $D$ such that $|F(\alpha) \cap B| \geq 2$, then $B^\alpha = B$.

**Proof.** Let $x, y$ be two distinct elements of $F(\alpha) \cap B$. Then $x, y \in B = B^{\alpha_0}, B^{\alpha_1}, \ldots, B^{\alpha_\lambda}$. Since every two distinct points are in exactly $\lambda$ blocks, $B^{\alpha_i} = B^{\alpha_j}$ for some distinct $i, j \in \{0, 1, \ldots, \lambda\}$. Thus $B^{\alpha_{i-j}} = B$. Since $p$ is prime and $1 \leq |i - j| \leq \lambda < p$, $\gcd(i - j, p) = 1$. Therefore $B^\alpha = B$ as required. □

**Lemma 2.4.** Let $B_1$ and $B_2$ be two distinct fixed blocks of the automorphism $\alpha$ of prime order $p$ of a symmetric $(v, k, \lambda)$ design with $\lambda < p$. Then $B_1 \cap B_2 \subseteq F(\alpha)$.

**Proof.** Suppose, for a contradiction, that there exists a point $x \in (B_1 \cap B_2) \setminus F(\alpha)$. Then $x^{\alpha} \neq x^{\alpha_j}$, for any two distinct $i, j \in \{0, 1, \ldots, p - 1\}$; since otherwise $x^{\alpha_{i-j}} = x$ and so $x^{\alpha} = x$, as $\gcd(i - j, p) = 1$. It follows that $p = |\{x^\beta \mid \beta \in \langle \alpha \rangle\}|$. Since $B_i^\alpha = B_i$ for $i \in \{1, 2\}$, we have that $\{x^\beta \mid \beta \in \langle \alpha \rangle\} \subseteq B_1 \cap B_2$. Therefore $|B_1 \cap B_2| \geq p > \lambda$, a contradiction; since in symmetric $(v, k, \lambda)$ designs, two distinct blocks intersect in exactly $\lambda$ points. □

**Lemma 2.5.** Let $\alpha$ be an automorphism of prime order $p$ of a symmetric $(v, k, \lambda)$ design with $\lambda < p$. Then

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \setminus F(\alpha)| \leq v.$$  

**Proof.** It follows from Lemma 2.4 that for any two distinct blocks $B_1$ and $B_2$ in $F_b(\alpha)$, $(B_1 \setminus F(\alpha)) \cap (B_2 \setminus F(\alpha)) = \emptyset$. This completes the proof. □
Lemma 2.6. Let $\alpha$ be an automorphism of a symmetric $(v, k, \lambda)$ design of prime order $p$ such that $1 < \lambda < p$. Then $B \not\subset F(\alpha)$ for all blocks $B$.

Proof. Suppose, for a contradiction, that there exists a block $B$ such that $B \subset F(\alpha)$. Since every block $B_1 \neq B$ intersects $B$ in $\lambda \geq 2$ points, it follows from Lemma 2.3 that every block is fixed under $\alpha$. Thus $|F_b(\alpha)| = |F(\alpha)| = v$, by Lemma 2.1. Hence $\alpha$ is the identity automorphism; a contradiction. This completes the proof. □

The following lemma is Theorem 2.7 of Aschbacher’s paper [1].

Lemma 2.7. (Theorem 2.7 of [1]) Let $p$ be a prime divisor of the automorphism group of a symmetric $(v, k, \lambda)$ design such that $1 < \lambda < p$ and gcd($p, v$) = 1. Then $p \leq k$.

Proof. Suppose that $\alpha$ is an automorphism of the design of order $p$. Since $\alpha$ is a permutation on the point set, $F(\alpha) \equiv v \pmod{p}$ and since gcd($p, v$) = 1, we have that $|F(\alpha)| \geq 1$. Thus, by Lemma 2.1, there exists a block $B$ such that $B^\alpha = B$. Thus by Lemma 2.6, there exists an element $x \in B \setminus F(\alpha)$ and so $|\{x^\beta \mid \beta \in \langle \alpha \rangle \}| = p$. Since $B^\alpha = B$, we have that $\{x^\beta \mid \beta \in \langle \alpha \rangle \} \subseteq B$ and so $p \leq k$, as required. □

3. Automorphism Group of a Possible Symmetric $(81,16,3)$ Design

Lemma 3.1. Let $G$ be an automorphism group of a possible symmetric $(81, 16, 3)$ design which is elementary abelian 2-group. Then $|G| \leq 8$.

Proof. Let $r$ be the number of orbits of the action of $G$ on the point set of the design. Then by the Cauchy-Frobenius Lemma (see [6] Proposition A.2, p. 246),

$$r = \frac{1}{|G|} \sum_{\alpha \in G} |F(\alpha)|.$$

Since $G$ is an elementary abelian 2-group, it follows from Theorem 1.2 that $|F(\alpha)| = 9$ for all non-identity elements $\alpha$ of $G$. 
Let $|G| = 2^n$. Then, since $r = (2^n + 8) \cdot 9/2^n$ is an integer, we must have that $2^n$ divides $2^n + 8$ and so $n \leq 3$, as required. \[ \square \]

**Lemma 3.2.** Let $G$ be an automorphism group of a possible symmetric $(81, 16, 3)$ design. Then $G$ has no element of order 7 or 11.

**Proof.** Suppose, for a contradiction, that $G$ has an automorphism $\alpha$ of order $p$, where $p \in \{7, 11\}$. Since $\alpha$ is a permutation on a set with 81 elements, we have $|F(\alpha)| \equiv 81 \mod p$. Then it follows from Lemma 2.2 that

$$|F(\alpha)| \in \begin{cases} \{4, 11, 18\} & \text{if } p = 7 \\ \{4, 15\} & \text{if } p = 11 \end{cases} \quad (I)$$

Thus there are at least two distinct blocks which are fixed by $\alpha$ and so

$$|F(\alpha)| \geq 3 \quad (\ast)$$

by Lemma 2.4. Now if $B \in F_b(\alpha)$, then $\alpha$ induces a permutation on the set $B$. Therefore $|F(\alpha) \cap B| \equiv 16 \mod p$ and so by $(\ast)$ and Lemma 2.6 we have

$$|F(\alpha) \cap B| = \begin{cases} 9 & \text{if } p = 7 \\ 5 & \text{if } p = 11 \end{cases} \quad (II)$$

If $p = 11$, then it follows from $(I)$ and $(II)$ that $|F(\alpha)| = 15$ and $|B \setminus F(\alpha)| = 11$ for all blocks $B \in F_b(\alpha)$; and if $p = 7$, then $|B \setminus F(\alpha)| = 7$ for all blocks $B \in F_b(\alpha)$ and $|F(\alpha)| \in \{11, 18\}$. Both cases contradict Lemma 2.5. This completes the proof. \[ \square \]

**Lemma 3.3.** Let $\alpha$ be an automorphism of a possible symmetric $(81, 16, 3)$ design of order 5. Then $|F(\alpha)| = 1$.

**Proof.** Since $\alpha$ is a permutation on the point set, it follows from Lemma 2.2 that $|F(\alpha)| \in \{1, 6, 11, 16\}$. Suppose, for a contradiction, that $|F(\alpha)| \neq 1$. Let $B = B_1$ be an arbitrary block in $F_b(\alpha)$. Since $|F_b(\alpha)| = |F(\alpha)| \geq 2$, there exists a block $B_2 \neq B_1$ in $F_b(\alpha)$. By Lemma 2.4 $B_1 \cap B_2 \subseteq F(\alpha)$ and so there exist distinct elements $x$ and $y$ in $F(\alpha)$ which are both in $B_1$ and $B_2$. Therefore there exists a block $B_3$ distinct from $B_1$ and $B_2$ containing both $x$ and $y$. Thus $3 = |B_1 \cap B_3| \geq |B_1 \cap B_2 \cap B_3| \geq 2$ for
any two distinct \( i, j \in \{1, 2, 3\} \). Now by Lemma 2.4, \( B_i^\alpha = B_i \) for all \( i \in \{1, 2, 3\} \) and so \( \alpha \) is a permutation on \( B_i \). Therefore, it follows from Lemmas 2.4 and 2.6 that \( |F(\alpha) \cap B| \in \{6, 11\} \) for all blocks \( B \in F_i(\alpha) \). Thus \( |F(\alpha) \cap (B_1 \cup B_2 \cup B_3)| \geq 11 \) and so \( |F(\alpha)| \in \{11, 16\} \). If \( |F(\alpha)| = 16 \), then
\[
|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \backslash F(\alpha)| \geq 16 + 16 \cdot 5 = 85,
\]
which is a contradiction by Lemma 2.5. If \( |F(\alpha)| = 11 \), then there is no block \( B \in F_b(\alpha) \) such that \( |F(\alpha) \cap B| = 11 \), since otherwise \( |(B' \cup B) \cap F(\alpha)| \geq 11 + 6 - 3 = 14 \) for any block \( B' \in F(\alpha) \) distinct from \( B \). Hence, in this case,
\[
|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \backslash F(\alpha)| \geq 11 + 11 \cdot 10 = 121,
\]
which contradicts Lemma 2.5. This completes the proof. □

**Lemma 3.4.** Let \( G \) be an automorphism group of a possible symmetric \((81, 16, 3)\) design which is a 5-group. Then \( |G| \leq 5 \).

**Proof.** It is enough to show that \( G \) has no subgroup \( H \) of order \( 5^2 \). If \( \alpha \in G \) is of order 25, then by Lemma 3.3 \( |F(\alpha)| = 1 \), since \( \emptyset \neq F(\alpha) \subseteq F(\alpha^5) \). Then, by Lemma 3.3 the number of orbits of the action of \( H \) on \( G \) is equal to
\[
r = \frac{1}{5^2} \sum_{h \in H} |F(h)| = \frac{81 + 24 \cdot 1}{5^2} = \frac{21}{5}.
\]
This is a contradiction, since \( r \) should be an integer. □

**Lemma 3.5.** Let \( \alpha \) be an automorphism of a possible symmetric \((81, 16, 3)\) design of order 13. Then \( |F(\alpha)| = 3 \).

**Proof.** Since \( \alpha \) is a permutation on the point set, it follows from Lemma 2.2 that \( |F(\alpha)| \in \{3, 16\} \). Suppose, for a contradiction, that \( |F(\alpha)| = |F_b(\alpha)| = 16 \). Then, by Lemma 2.6 \( |F(\alpha) \cap B| = 3 \) for all \( B \in F_b(\alpha) \). Thus
\[
|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \backslash F(\alpha)| \geq 16 + 16 \cdot 13 = 224,
\]
contradicting Lemma 2.5. This completes the proof. □
Lemma 3.6. Let $G$ be an automorphism group of a possible symmetric $(81, 16, 3)$ design which is a 13-group. Then $|G| \leq 13$.

Proof. It is enough to show that $G$ has no subgroup $H$ of order $13^2$. Since $13^2 > 81$, $G$ has no element of order $13^2$. Thus $H$ is an elementary abelian 13-group. Then, by Lemma 3.5, the number of orbits of the action of $H$ on $G$ is equal to

$$r = \frac{1}{13^2} \sum_{h \in H} |F(h)| = \frac{81 + 12 \cdot 3}{13^2} = \frac{9}{13}.$$ 

This is a contradiction, since $r$ should be an integer. □

Lemma 3.7. Let $G$ be an automorphism group of a possible symmetric $(81, 16, 3)$ design. Then $G$ has no element with the following orders: 10, 26, 65.

Proof. (1) Suppose that $G$ has an element of order 10. Then $G$ contains two automorphisms $\alpha$ and $\beta$ of orders 5 and 2 respectively such that $\alpha \beta = \beta \alpha$. Since $\alpha$ and $\beta$ commutes, $\alpha(F(\beta)) = F(\beta)$. By Theorem 1.2 we have that $|F(\beta)| = 9$. Now by considering the cycle decomposition of $\alpha$ on $F(\beta)$, it follows that $|F(\alpha) \cap F(\beta)| \in \{4, 9\}$ which contradicts Lemma 3.3.

(2) Suppose that $G$ has an element of order 26. Then $G$ contains two automorphisms $\alpha$ and $\beta$ of orders 13 and 2 respectively such that $\alpha \beta = \beta \alpha$. Since $\alpha$ and $\beta$ commutes, $\alpha(F(\beta)) = F(\beta)$ and by Theorem 1.2 $|F(\beta)| = 9$, the cycle decomposition of $\alpha$ on $F(\beta)$ shows that $F(\beta) \subseteq F(\alpha)$ which contradicts Lemma 3.5.

(3) Suppose that $G$ has an element of order 65. Then $G$ contains two automorphisms $\alpha$ and $\beta$ of orders 13 and 5 respectively such that $\alpha \beta = \beta \alpha$. Since $\alpha$ and $\beta$ commutes, $\beta(F(\alpha)) = F(\alpha)$. But by Lemma 3.5 we have that $|F(\alpha)| = 3$ so the cycle decomposition of $\beta$ on $F(\alpha)$ implies that $F(\alpha) \subseteq F(\beta)$ which contradicts Lemma 3.3. □

Proof of Theorem 1.4. It follows from Lemmas 3.1, 3.2, 3.4, 3.6 and 3.7.
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