Self-dual gravity via Hitchin’s equations

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Abstract

In this work half-flat metrics are obtained from Hitchin’s equations. The $\text{SU}(\infty)$ Hitchin’s equations are obtained and as a consequence of them, the Husain-Park equation is found. Considering that the gauge group is $\text{SU}(2)$, some solutions associated to Liouville, sinh-Gordon and Painlevé III equations are taken and, through Moyal deformations, solutions of the $\text{SU}(\infty)$ Hitchin’s equations are obtained. From these solutions, hamiltonian vector fields are determined, which in turn are used to construct the half-flat metrics. Following an approach of Dunajski, Mason and Woodhouse, it is also possible to construct half-flat metrics on $\mathcal{M} \times \mathbb{C}P^1$.

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1 Introduction

Several years ago Ward in Ref. [1, 2] conjectured that all non-linear integrable systems in field theory in lower dimensions than four might be obtained by dimensional reduction from four-dimensional self-dual Yang-Mills (SDYM) equations using finite gauge group. This is called Ward’s conjecture and it can be generalized to include all sl(N, C) algebras as well as infinite dimensional algebras of hamiltonian vector fields. This conjecture has been a guiding principle in a great deal of work in integrable systems done since then (for some complete reviews see, [3, 4]).

In a remarkable article Ashtekar, Jacobson and Smolin (AJS) [5], reduced the self-dual gravity (SDG) equations to a set of three volume-preserving vector fields satisfying a Nahm-like equation. Later Mason and Newman [6], shown that starting from the SDYM equations with gauge group G in four dimensions, it is possible to obtain self-dual gravity equations (in the AJS version) in vacuum space. This is achieved by assuming that the vector potentials are independent of all the spacetime coordinates and the Lie algebra generators of G are replaced by the volume preserving hamiltonian vector fields X_f of the underlying four manifold M, where f is the corresponding hamiltonian function. Thus the Lie brackets of the Lie algebra of G are changed by the Lie bracket of hamiltonian vector fields. At the level of hamiltonian functions, the Lie brackets are replaced by Poisson brackets. The relationship of the obtained AJS equations to the Plebański heavenly equations [7] was later obtained in Ref. [8]. In this same direction Dunajski made a generalization of AJS equations to the hyper-Hermitian case [9].

It is also possible to obtain SDG from SDYM if a two dimensional reduction is performed with gauge group G =SDiff(Σ) [10, 11, 12]. Mason observed that SL(N, C) is a subgroup of SDiff(Σ) only for N = 2 [10]. It is know that solutions of the SDYM equations with two-dimensional reductions are singular if the solution is real and is regular if the solution is complex [13, 14].

Moreover, starting from two-dimensional conformal field theories it is also possible to obtain SDG in four dimensions by promoting the gauge group G of the two-dimensional theory to be the area-preserving diffeomorphism group, SDiff(Σ). Some of these examples were explicitly given by Park [15], who showed that SDG arises from two-dimensional SU(N) non-linear sigma model with Wess-Zumino term in the large-N limit. Later Park extended this work and shown that it is possible to obtain SDYM equations on a self-dual space if the gauge group SDiff(Σ) is extended to an arbitrary Lie algebra [16]. Later Husain [17] showed how an alternative to the Plebański heavenly equation [7] could be derived from the two-dimensional chiral model. Following this line of thought the non-linear graviton was constructed from chiral fields in Ref. [18].

Another way to find solutions to the heavenly equations is by using the Moyal deformation quantization method [19]. The Moyal deformation of the heavenly equations was considered in [20, 21, 22, 23, 24]. Using this Moyal deformation on the two-dimensional chiral model with the Lie bracket replaced by the Moyal one, it is possible to obtain solutions of the Husain-Park heavenly equation [25]. More generally, starting from the SDYM equations a six-dimensional master equation can be obtained
through the deformation quantization procedure [26]. From this master equation, by dimensional reduction, it is possible to find various integrable systems in four dimensions, one of them is the Husain-Park heavenly equation. The most general case of the SDYM fields in a SDG background and its Moyal deformation was considered by Formański and Przanowski in Refs. [27, 28]. For other applications, see [29, 30]. In other contexts, the self-dual sector of gravity has been also explored as related to the $N = 2$ strings [31, 32] and to the double copy conjecture [33, 34, 35, 36]. Furthermore, Monteiro and O’Connell in [33] were able to identify an algebra underlying the color-kinematics duality with that of area preserving diffeomorphisms.

Very recently a great deal of activity on Hitchin’s equations, and in general the theory of Higgs bundles, has been taken place and their impact in physics and mathematics is strongly discussed. Hitchin’s equations [37] were originally obtained as a two-dimensional reduction from the Euclidean SDYM equations in four dimensions. He obtained a set of equations involving two-dimensional gauge fields coupled to one-form section of a holomorphic vector bundle over a Riemann surface $\Sigma$ called the Higgs bundle. Thus these equations are characterized by the pairs $\{(E, \Phi)\}$, where $E$ is a holomorphic bundle over $\Sigma$ and $\Phi$ is a section of the bundle $\Omega^1 \otimes E$ such that $\Phi \wedge \Phi = 0$. The moduli space of Higgs bundles has very interesting properties. For instance, in the context of Riemann surfaces, it describe an integrable system [38, 39]. Further generalizations for holomorphic bundles over general Kähler manifolds was considered by Simpson in Ref. [40]. Recently, a four-dimensional version of the Hitchin’s system called the Kapustin-Witten (KW) equations was obtained also in the context of the physical approach to the geometric Langlands problem [41]. In this approach, the KW equations were obtained from the dimensional reduction of a topological Yang-Mills theory in six dimensions. Moreover a great deal of work has been done in this context, mainly in its relation to Khovanov homology (for some reviews, see for instance, [42]). Another interesting spinoff of Hitchin’s equations in mathematics is its relationship with the wall-crossing formula of Kontsevich-Soibelman (see for instance [43]).

The relationship between Hitchin’s equations and gravity has been few investigated. For instance, Ueno obtained the Hitchin’s equations starting from the SDG [44]. Later Etesi gives a AdS/CFT correspondence between classical $(2 + 1)$-dimensional vacuum gravity on $\Sigma \times \mathbb{R}$ and SO(3) Hitchin’s equations on the space-like past boundary $\Sigma$, a compact, oriented Riemann surface of genus greater than 1 [45]. Moreover, Calderbank showed that if we take $\text{Diff}(S^1)$ to be the gauge group in the Hitchin’s equations then it is obtained the hyperCR Einstein-Weyl equation. Otherwise if we take the gauge group to be $\text{SDiff}(\Sigma)$ then we obtain an Euclidean analogue of Plebański’s heavenly equations [46]. KW equations were recently considered in the context of deformation quantization [47]. In this paper the $\text{SU}(\infty)$ KW equations arise in a natural way and the question of the possible relation of this equation to gravity was raised.

In the present paper we will continued studying how self-dual gravity arises directly from the $\text{SU}(N)$ Hitchin’s equations in the large-$N$ limit and from finite group. We will find the relationship between $\text{SU}(\infty)$ Hitchin’s equations and Husain-Park equation, also the corresponding self-dual metrics of three different family of solutions to the Hitchin’s equations, namely: Liouville, sinh-Gordon and Painlevé III models, will be also found.
The structure of the paper is as follows. In Section 2 we review the Hitchin’s equations and give the Lax pair formalism. Using SU(2) as gauge group, we review three families of solutions of Hitchin’s equations which are expressed in terms of the Liouville, sinh-Gordon and Painlevé III equations. In the case of the Liouville equation there are a singular and regular solution depending if the gauge fields are complex or real. The solutions given by sinh-Gordon and Painlevé III equations are singular and regular respectively. In section 3 we give the SU(∞) version of the Hitchin’s equations in terms of the hamiltonian functions and find a relation with Husain-Park equation. Using Moyal deformation, given a solutions of the Hitchin’s equations one gets solutions of the SU(∞) version, we can construct explicit forms of the hamiltonian functions and their corresponding hamiltonian vectors fields. In section 4 we use the fact that SU(2) is a sub group of SU(∞) and we construct vector fields from hamiltonian vector fields over $\mathbb{C}P^1$. Also, in section 5 we construct half-flat metrics from the vector fields obtained previously using the correspondence between Lax pair formulation of the Hitchin’s equation and the self-dual gravity. The half-flat metrics obtained are the Husain’s metric and other six half-flat metrics that depend on the solutions of the Liouville, sinh-Gordon and Painlevé equations, thus these metrics can be singular or regular. Finally, in section 6 we give our conclusions and some open questions are raised.

2 Hitchin’s equations and some solutions

In this section we review the Hitchin’s equations coming from two-dimensional reduction of SDYM equations over $\mathbb{R}^4$, the equations obtained are conformally invariant. Later we find the Lax pair for Hitchin’s equations from a two-dimensional reduction of the Lax pair for SDYM equations. Finally in this section, we review three families of solutions of the SU(2) Hitchin’s equations coming from two ansätze. These solutions will depend on three different differential equations: Liouville, sinh-Gordon and Painlevé III equations. Later in this paper we use these solutions to build hamiltonian vectors fields and half-flat metrics.

On a coordinate system $\{x_i\}$ ($i = 1, \ldots, 4$) of Euclidean space $\mathbb{R}^4$, the self-dual Yang-Mills (SDYM) equations $\star F = F$ can be expressed as

$$F_{12} = F_{34}, \quad F_{13} = F_{42}, \quad F_{14} = F_{23},$$

where $\star$ is the Hodge star operator. The components of the curvature are given by $F_{ij} = [D_i, D_j]$, where the brackets are the Lie brackets and $D_i = \partial_i + A_i$ is the covariant derivative. Moreover $A = \sum_i A_i(x)dx^i$ is the connection one-form associated to the curvature $F$ i.e. $F = dA + A \wedge A$. If we consider that $A_i$ will be only functions of the coordinates $x_1$ and $x_2$, then the SDYM equations are explicitly written as

$$\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2] = [A_3, A_4],$$
$$\partial_1 A_3 + [A_1, A_3] = -\partial_2 A_4 + [A_4, A_2],$$
$$\partial_1 A_4 + [A_1, A_4] = \partial_2 A_3 + [A_2, A_3].$$

Now we define the following field combinations

$$A_+ = \frac{1}{2}(A_1 + iA_2), \quad A_- = \frac{1}{2}(A_1 - iA_2),$$

$$A_3 = \frac{1}{2}(A_1 + iA_2), \quad A_4 = \frac{1}{2}(A_1 - iA_2),$$

where $A_0 = iA_2 - A_1$.
\[ \Phi = A_3 - i A_4, \quad \Phi^* = -A_3 - i A_4, \]  
(3)

where we have taken the complex coordinate \( z = x + iy, \) \( x_1 = x, \) \( x_2 = y. \) In terms of these fields and coordinates, equations from (2) can be rewritten as

\[
F_{\bar{z}z} + \frac{1}{4}[\Phi, \Phi^*] = 0, \\
D_{\bar{z}}\Phi = \partial_{\bar{z}}\Phi + [A_{\bar{z}}, \Phi] = 0.
\]  
(4)

These equations are the well known Hitchin’s equations [37], \( \Phi \) and \( \Phi^* \) are the Higgs fields. They are invariant under gauge transformation of the form

\[
A_{\bar{z}} \rightarrow g^{-1}A_{\bar{z}}g + g^{-1}g_{,z}, \quad \Phi \rightarrow g^{-1}\Phi g, \\
A_\bar{z} \rightarrow g^{-1}A_\bar{z}g + g^{-1}g_{,\bar{z}}, \quad \Phi^* \rightarrow g^{-1}\Phi^* g,
\]  
(6)

where \( g = g(z, \bar{z}) \) is an element of the gauge group \( G. \)

### 2.1 Lax pair

Now, the Lax pair for the Hitchin’s system is obtained from a two-dimensional reduction from the Lax pair for SDYM and the reduced system is obtained by making a particular choose of the fields. The SDYM equations (1) have a compatibility condition given by the Lax pair

\[
[L, M] = 0,
\]  
(7)

with

\[
L = D_1 - i D_2 - \lambda(D_3 - i D_4), \\
M = -D_3 - i D_4 - \lambda(D_1 + i D_2),
\]  
(8)

where \( D_i = \partial_i + A_i. \) If now we perform the two-dimensional reduction, we assume that connections will be functions independent on the coordinates \( x_3, x_4, \) then from the definitions in (3), the Lax pair for the Hitchin’s system are given by

\[
L = D_z - \frac{\lambda}{2}\Phi, \quad M = \frac{1}{2}\Phi^* - \lambda D_{\bar{z}}.
\]  
(9)

Thus the compatibility condition reduces to

\[
\left[ D_z - \frac{\lambda}{2}\Phi, D_{\bar{z}} - \frac{\lambda^{-1}}{2}\Phi^* \right] = 0,
\]  
(10)

where we also have de equation \( D_z \Phi^* = \partial_z \Phi^* + [A_z, \Phi^*] = 0, \) which can be obtained from the SDYM reduction equations.

In the case for a particular value of spectral parameter \( \lambda = 1, \) the compatibility condition has the form

\[
\left[ D_z - \frac{\Phi}{2}, D_{\bar{z}} - \frac{\Phi^*}{2} \right] = 0.
\]  
(11)
This gives us the single equation

\[ \left( A_\tau - \frac{\Phi^*}{2} \right)_{,\tau} - \left( A_z - \frac{\Phi}{2} \right)_{,z} + \left[ A_z - \frac{\Phi}{2}, A_\tau - \frac{\Phi^*}{2} \right] = 0. \] (12)

Then there exist a gauge transformation in which we can take \( \Phi = 2A_z \) and \( \Phi^* = 2A_\tau \). Substituting these values into (11) and (12), and taking into account also \( D_z \Phi^* = \partial_z \Phi^* + [A_z, \Phi^*] = 0 \), the Hitchin’s equations reduce to

\[ A_z,\tau - [A_z, A_\tau] = 0, \quad A_\tau, z + A_z, \tau = 0. \] (13)

Later these equations will be related with Husain-Park equation when the gauge group is SU(\( \infty \)).

### 2.2 Some solutions

In this subsection we review some solutions of Hitchin’s equations, with gauge group SU(2), related to Liouville, sinh-Gordon and Painlevé III equations. In the case of the Liouville equation it is possible to obtain analytic solutions which will be singular or regular depending whether the connections are real or complex and are defined over \( S^2 \). In the case of sinh-Gordon equation we have an analytic solution but it is singular. For the Painlevé III case, the corresponding solution is defined over \( \mathbb{R}^2 \), it is regular but there is not an analytic solution.

In Ref. [48] Mosna and Jardim obtained solutions to Hitchin’s equations given the following ansatz. For the gauge group SU(2), they proposed the following form for the gauge connections

\[
A_z = \frac{1}{2}(f_1 - if_2)\tau_1, \quad A_\tau = \frac{1}{2}(f_1 + if_2)\tau_1, \\
\Phi = (i)^ng\tau_2 - (i)^{n+1}h\tau_3, \quad \Phi^* = -(i)^ng\tau_2 - (i)^{n+1}h\tau_3, \] (14)

where \( \tau_i \) is the basis for the Lie algebra su(2) and \( n = 0, 1 \). Hitchin’s equations hold if the functions \( f_1, f_2, g \) and \( h \) satisfy the following relations

\[
\partial_x g = f_2 h, \quad \partial_y g = -f_1 h, \\
\partial_x h = f_2 g, \quad \partial_y h = -f_1 g, \\
\partial_x f_2 - \partial_y f_1 = (i)^2 gh. \] (15)

We have an integration constant \( \kappa \) given by

\[ \kappa^2 = g^2 - h^2. \] (16)

There are two independent cases \( \kappa = 0 \) and \( \kappa \neq 0 \). In the first case it gives rise to the Liouville equation, which has singular and regular solutions, depending if the solutions of the SDYM equations are real or complex. In the second case, the equation is the sinh-Gordon and it determines a real (singular) solution of the SDYM equations.
2.2.1 Liouville equation

The solution of the Hitchin’s equations corresponding to the Liouville equation is defined on $S^2$ and we will consider the real and complex cases depending on the choice of $n = 0$ or $n = 1$ respectively. Thus we have $\kappa = 0$ and $g = h$, the functions $f_1$ and $f_2$ are then given as a function of $g$

\[
\begin{align*}
    f_1 &= -\partial_y \ln g, \quad f_2 = \partial_x \ln g, \\
    \partial_x f_2 - \partial_y f_1 &= (i)^{2n} g^2. 
\end{align*}
\]

(17)

Then the function $g^2 = \lambda$ is obtained from the Liouville equation, depending on the choise of $n$, i.e.

\[\nabla^2 (\ln \lambda) \mp 2\lambda = 0.\]

(18)

Here the top sign corresponds to $n = 0$, while the bottom sign corresponds to $n = 1$ and $\nabla^2 = \partial_x^2 + \partial_y^2$. Equation (18) can be rewritten in complex coordinates $z = x + iy$ as follows

\[
\partial_z \partial_{\bar{z}} (\ln \lambda) \mp \frac{\lambda}{2} = 0.
\]

(19)

Then a solution of this equation has the general form

\[
\lambda(z, \bar{z}) = 4 \frac{\psi'(z) \eta'(\bar{z})}{[1 \mp \psi(z) \eta(\bar{z})]^2},
\]

(20)

where $\psi$ and $\eta$ are arbitrary functions and $'$ stands for derivative with respect to $z$. Real solutions can be found by taking $\eta(\bar{z}) = \bar{\psi}(z)$. If in addition we have $\psi = z^\nu$, the function $g$ will be a function only depending on $r = |z|$, and it is given by

\[
g^2(r) = 4\nu^2 \frac{r^{2\nu - 2}}{(1 \mp r^{2\nu})^2}.\]

(21)

The gauge connection defined on the plane $x - y$ has the following form

\[A = A_1 dx_1 + A_2 dx_2 = r \partial_r (\ln g) \tau_1 d\theta.\]

(22)

Taking into account the form of $g$ in (21), the connection can be written as

\[A = \left( \nu - 1 \pm 2\nu \frac{r^{2\nu}}{1 \mp r^{2\nu}} \right) \tau_1 d\theta.\]

(23)

In the case $\nu = 1$ it is reduced to

\[A = \pm 2 \frac{r^2}{1 \mp r^2} \tau_1 d\theta = M \tau_1,\]

(24)

which can be related with an abelian magnetic monopole $M = \pm \frac{2\nu^2}{1 \mp r^2} d\theta$. 

7
With \( g \) given by (21), we obtain an explicit solution to the Hitchin’s equations

\[
A_z = -i \partial_z (\ln g) \tau_1 = -i e^{-i\theta} \left( \nu - 1 \pm 2\nu \frac{r^{2\nu}}{1 \mp r^{2\nu}} \right) \tau_1,
\]

\[
A_{\bar{z}} = i \partial_{\bar{z}} (\ln g) \tau_1 = i e^{i\theta} \left( \nu - 1 \pm 2\nu \frac{r^{2\nu}}{1 \mp r^{2\nu}} \right) \tau_1,
\]

\[
\Phi = g(i)^n (\tau_2 - i\tau_3) = 2(i)^n \nu \frac{r^{\nu-1}}{1 \mp r^{2\nu}} (\tau_2 - i\tau_3),
\]

\[
\Phi^* = -g(i)^n (\tau_2 + i\tau_3) = -2(i)^n \nu \frac{r^{\nu-1}}{1 \mp r^{2\nu}} (\tau_2 + i\tau_3).
\]

2.2.2 Sinh-Gordon equation

For this case we have \( \kappa \neq 0 \). We only will consider the case corresponding to \( n = 0 \), which is a real and singular solution. Now we define

\[
g = \kappa \cosh(\alpha), \quad h = \kappa \sinh(\alpha).
\]

Then functions \( f_1 \) and \( f_2 \) are given in terms of \( \alpha \), i.e.,

\[
f_1 = -\partial_y \alpha, \quad f_2 = \partial_x \alpha,
\]

\[
\partial_x f_2 - \partial_y f_1 = \frac{\kappa^2}{2} \sinh(2\alpha).
\]

The function \( \alpha \) satisfies the sinh-Gordon equation

\[
\nabla^2 \alpha - \frac{\kappa^2}{2} \sinh(2\alpha) = 0.
\]

An analytic solution of this equation is given by [49]

\[
\alpha = 2 \tanh^{-1} \left[ \exp \left( \kappa \left( \frac{(z - z_0)e^{-i\omega}}{2} + \frac{(\bar{z} - \bar{z}_0)e^{i\omega}}{2} \right) \right) \right],
\]

with \( z_0 \) complex constant and \( \omega \) real constant. In this case, the Hitchin’s connections are

\[
A_z = -i \partial_z (\alpha) \tau_1, \quad \Phi = \kappa (\cosh(\alpha) \tau_2 - i \sinh(\alpha) \tau_3),
\]

\[
A_{\bar{z}} = i \partial_{\bar{z}} (\alpha) \tau_1, \quad \Phi^* = -\kappa (\cosh(\alpha) \tau_2 + i \sinh(\alpha) \tau_3),
\]

and \( \alpha \) is given by (29).

2.2.3 Painlevé III equation

We now discuss a solution given by Ward in Ref. [50]. Ward considered smooth SU(2) solutions of the Hitchin’s equations on \( \mathbb{R}^2 \) with boundary conditions involving
an integer $n$, which is the degree of the determinant of the Higgs field $\Phi$. Then we take the following ansatz for the fields

$$A_\bar{z} = \frac{i}{2}(\partial_{\bar{z}} \psi) \tau_3 + \alpha \Phi, \quad A_z = -\frac{i}{2}(\partial_z \psi) \tau_3 - \bar{\alpha} \Phi^*,$$

$$\Phi = \begin{pmatrix} 0 & \mu_+ e^{\psi/2} \\ \mu_- e^{-\psi/2} & 0 \end{pmatrix},$$  \hspace{1cm} (31)

where the functions $\mu_+, \mu_-, \psi$ and $\alpha$ are functions of the complex variables $z$ and $\bar{z}$. The fields $A_z, A_{\bar{z}}, \Phi, \Phi^*$ satisfy Hitchin’s equations if the four functions fulfil the following equations

$$\Delta \psi = 2(1 + 4|\alpha|^2)(|\mu_+|^2 e^\psi - |\mu_-|^2 e^{-\psi}),$$  \hspace{1cm} (32)

$$0 = e^{-\psi/2} \partial_z (e^\psi \mu_+ \alpha) + e^{\psi/2} \partial_{\bar{z}} (e^{-\psi} \mu_- \bar{\alpha}),$$  \hspace{1cm} (33)

where $\Delta = 4\partial_z \partial_{\bar{z}}$. In the special case $\alpha = 0$ the previous equations are locally equivalent to the sinh-Gordon equation, at the cost of more complicated global condition on the field $\psi$.

For future reasons, it is convenient to write $\Phi$ in the basis of the Lie algebra $su(2)$, this yields

$$\Phi = \begin{pmatrix} 0 & \mu_+ e^{\psi/2} \\ \mu_- e^{-\psi/2} & 0 \end{pmatrix}$$

$$= -i(\mu_+ e^{\psi/2} + \mu_- e^{-\psi/2}) \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (\mu_+ e^{\psi/2} - \mu_- e^{-\psi/2}) \frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= f_1 \tau_1 + f_2 \tau_2,$$  \hspace{1cm} (34)

where

$$f_1 = -i(\mu_+ e^{\psi/2} + \mu_- e^{-\psi/2}), \quad f_2 = \mu_+ e^{\psi/2} - \mu_- e^{-\psi/2}. \hspace{1cm} (35)$$

The simplest case is when the determinant of $\Phi$ is $z$, i.e. $n = 1$. In this situation the $\psi$ is only function of $r$ and, by symmetries we have $\alpha = 0$. Consequently we have the following definitions

$$\mu_+ = z, \quad \mu_- = -1, \quad \psi = \psi(r), \quad r = |z|. \hspace{1cm} (36)$$

Thus equation (33) is trivial and (32) reduces to

$$\Delta \psi = 4\psi_{z\bar{z}} = 2(|z|^2 e^\psi - e^{-\psi}).$$  \hspace{1cm} (37)

Now defining $t = r^{3/2}$ and $h(t) = e^{-\psi/2} t^{-1/3}$, we have

$$h'' - \frac{(h')^2}{h} + \frac{h'}{t} + 4 \frac{h}{9h} - \frac{4h^3}{9} = 0,$$  \hspace{1cm} (38)

which is the Painlevé III equation. In this case fields take the form

$$A_\bar{z} = \frac{i}{2} \psi_{\bar{z}} \tau_3, \quad A_z = -\frac{i}{2} \psi_z \tau_3,$$

$$\Phi = f_1 \tau_1 + f_2 \tau_2, \quad \Phi^* = -\bar{f}_1 \tau_1 - \bar{f}_2 \tau_2,$$  \hspace{1cm} (39)

with the functions $f_1$ and $\bar{f}_2$ given by

$$f_1 = -i(ze_{\bar{z}} - e^{-\bar{z}}), \quad f_2 = ze_{\bar{z}} + e^{-\bar{z}}. \hspace{1cm} (40)$$
3 \textbf{SU}(\infty) Hitchin’s equations}

In this section we obtain the SU(\infty) Hitchin’s equations. Taking particular values for the hamiltonian functions, the SU(\infty) Hitchin’s equations reduce to Husain-Park equation. These particular values for the hamiltonian functions are inspired in the reduced Hitchin system \([13]\). Solutions to the SU(\infty) Hitchin’s equations are obtained from solutions of the SU(2) Hitchin’s equations using WWMG-correspondence. We obtain three different sets of hamiltonian functions and their hamiltonian vector fields corresponding with the Liouville, sinh-Gordon and Painlevé equations respectively.

Consider the case with gauge group SU(N). In the large-N limit, \((N \to \infty)\), the Lie group can be written in terms of the area-preserving diffeomorphism group of a certain surface \(\Sigma\) i.e.

\[
SU(N) \to \text{SDiff}(\Sigma),
\]

with \(A_i \in \text{sdiff}(\Sigma)\). Then \(A_i\) are hamiltonian vector fields over \(\Sigma\) and they have the following form

\[
A_i = A_{\theta_i} = \theta_{i, p} \partial_q - \theta_{i, q} \partial_p,
\]

where \(p\) and \(q\) are the local coordinates on the two-manifold \(\Sigma\), \(\theta_i\) are the hamiltonian functions and \(\theta_{i, p} = \partial_p \theta_i, \theta_{i, q} = \partial_q \theta_i\). In a more compact form, we can use the antisymmetric matrix \(\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and rewrite \(A_i\) as follows

\[
A_i = A_{\theta_i} = \epsilon^\mu\nu \theta_{i, \mu} \partial_{\nu}.
\]

Now the Lie bracket of two vector fields is

\[
[A_i, A_j] = \epsilon^\mu\nu \theta_{i, \mu} \epsilon^\sigma\rho \theta_{j, \sigma} - \epsilon^\mu\nu \theta_{j, \mu} \epsilon^\sigma\rho \theta_{i, \sigma} \partial_{\rho}
= \epsilon^\sigma\rho \{\theta_i, \theta_j\}_{, \sigma} \partial_{\rho}
= A_{\{\theta_i, \theta_j\}},
\]

where \(\{\theta_i, \theta_j\} = \theta_{i, p} \theta_{j, q} - \theta_{i, q} \theta_{j, p}\) is the Poisson bracket over \(\mathbb{R}^2\).

Hitchin’s equations \([11]\) and \([15]\), and also \(\partial_z \Phi^* + [A_z, \Phi^*] = 0\), with the gauge group SDiff(\(\Sigma\)) have the form

\[
\begin{align*}
H_{z, z} + \{H_{z, z}, H\} &= 0, \\
H_{z, z} + \{H_{z, z}, H_{z, z}\} &= 0, \\
H_{z, z} - H_{z, z} + \{H_{z, z}, H_{z, z}\} + \frac{1}{4}\{H, H_{z, z}\} &= 0,
\end{align*}
\]

where \(H_{z, z}, H_{z, z}, H_{z, z}\) are the hamiltonian functions depending on the coordinates \((z, \bar{z}, p, q)\) associated to \(A_z, A_{\bar{z}}, \Phi, \Phi^*\) respectively. Right hand side of Eqs. \([11]\) and \([15]\) are actually equal to three corresponding arbitrary functions \(f_i\) \((i = 1, 2, 3)\), depending only on the coordinates \((z, \bar{z})\). In the rest of this paper we only consider the case \(f_i = 0\).

\subsection{3.1 Husain-Park equation}

If we take the particular case when \(H = 2H_z\) and \(H_{\ast} = 2H_{\bar{z}}\), then we have the simplified system

\[
H_{z, z} - \{H_{z, z}, H\} = 0, \\
H_{z, z} + H_{z, z} = 0.
\]
Furthermore we can use the integrability condition and take $H_z = \Lambda_z$ and $H_{\bar{z}} = -\Lambda_{\bar{z}}$ with $\Lambda = \Lambda(z, \bar{z}, p, q)$. In this case we obtain a second order equation

$$\Lambda_{zz} + \{\Lambda_z, \Lambda_{\bar{z}}\} = 0, \quad (47)$$

which is a complex version of the Husain-Park equation [25]. Indeed, if we take $z = x + iy$, $\Lambda \to i\Lambda$ and a rescaling in the coordinates $p$ and $q$ we arrive to

$$\Lambda_{yy} + \Lambda_{xx} + \Lambda_{yp} \Lambda_{xq} - \Lambda_{yq} \Lambda_{xp} = 0. \quad (48)$$

Notice that equations (46) are equations (13) when the gauge group is SDiff($\Sigma$). Consequently for a particular value of the spectral parameter, the Hitchin’s equations with gauge group SDiff($\Sigma$) describe the complex version of a Husain-Park equation.

### 3.2 Weyl-Wigner-Moyal-Groenewold correspondence

In this section we survey the Weyl-Wigner-Moyal-Groenewold (WWMG) correspondence in order to use this to find solutions of the SU($\infty$) Hitchin’s equations. Following Ref. [25], we use Moyal deformation to find solutions of (45) and (44). First of all we associate to each Lie algebra-valued element $K_a$ corresponding unitary operator $\hat{K}$ acting on a Hilbert space. Then, using the WWMG correspondence, we can associate to $\hat{K}$ a function $\tilde{K}(z, \bar{z}, p, q) = \tilde{K}(z, \bar{z}, p, q)$ by

$$\tilde{K} = \int_{-\infty}^{\infty} \left< p - \frac{\zeta}{2} \right| \hat{K}|p + \frac{\zeta}{2} \right> \exp\left( \frac{i q \zeta}{\hbar} \right) d\zeta. \quad (49)$$

For a commutator of two operators, we have

$$\{\tilde{K}_1, \tilde{K}_2\}_M := \int_{-\infty}^{\infty} \left< p - \frac{\zeta}{2} \right| [\hat{K}_1, \hat{K}_2] \right| p + \frac{\zeta}{2} \right> \exp\left( \frac{i q \zeta}{\hbar} \right) d\zeta. \quad (50)$$

Right hand side defines the Moyal bracket which can be expressed in a more practical way

$$\{\tilde{K}_1, \tilde{K}_2\}_M = \frac{2}{\hbar} \tilde{K}_1 \sin\left( \frac{\hbar}{2} \right) \tilde{K}_2, \quad (51)$$

where

$$\frac{\hbar}{2} \frac{\partial}{\partial p} \frac{\partial}{\partial q} - \frac{\hbar}{2} \frac{\partial}{\partial q} \frac{\partial}{\partial p}. \quad (52)$$

Using the previous procedure we can associate functions $\tilde{H}_z$, $\tilde{H}_{\bar{z}}$, $\tilde{H}$, $\tilde{H}_*$ to fields $A_z, A_{\bar{z}}, \Phi, \Phi^*$ respectively and the Hitchin’s equations take the form

$$\tilde{H}_{z,zz} - \tilde{H}_{z,\bar{z}} + \{\tilde{H}_z, \tilde{H}_{\bar{z}}\}_M + \frac{1}{4} \{\tilde{H}, \tilde{H}_*\}_M = 0, \quad (53)$$

$$\tilde{H}_{z,\bar{z}} + \{\tilde{H}_z, \tilde{H}_{\bar{z}}\}_M = 0. \quad (54)$$

Taking the limit $\hbar \to 0$, the Moyal bracket reduces to the Poisson bracket

$$\lim_{\hbar \to 0} \{\cdot, \cdot\}_M = \{\cdot, \cdot\}. \quad (55)$$
We have assumed that the function $\tilde{K}$ is an analytic function of $\hbar$, i.e.

$$\tilde{K} = \sum_{n=0}^{\infty} \hbar^n K_n, \quad K_n = K_n(z, \bar{z}, p, q). \quad (56)$$

Then the functions obtained from the Moyal deformation provides the hamiltonian functions which are solutions of the SU(∞) Hitchin’s equations (44) and (45), in the limit $\lim_{\hbar \to 0} \tilde{H} = H_0 = H$, etc.

3.3 Solutions

Using the WWMG-correspondence, we can obtain solutions to the SU(∞) Hitchin’s equations, (44) and (45), from solutions to the SU(2) Hitchin’s equations considered previously as: the Liouville, sinh-Gordon and Painlevé III equations. In these cases we have

$$A(z, \bar{z}) = A_i(z, \bar{z}) \tau_i, \quad (57)$$

for the Yang-Mills field. In the previous equation, summation over the repeated indices should be understood. Now we consider that $\tau_i$, a basis for the $\mathfrak{su}(2)$, can be substituted by the corresponding basis of self-dual operators over the Hilbert space $\hat{\tau}_i$. The explicit form of the $\hat{\tau}_i$ are

$$\hat{\tau}_1 = i\beta \hat{q} + \frac{1}{2\hbar}(\hat{q}^2 - 1)\hat{p}, \quad \hat{\tau}_2 = -\beta \hat{q} + \frac{i}{2\hbar}(\hat{q}^2 + 1)\hat{p},$$

$$\hat{\tau}_3 = -i\beta \hat{1} - \frac{1}{\hbar} \hat{q}\hat{p}. \quad (58)$$

Using the WWMG-correspondence (49), these operators $\hat{\tau}_i$ is mapped to functions $X_i(p, q)$ in the coordinates $p, q$ of $\Sigma$. Thus, the field $A(z, \bar{z})$ is now a function of all the coordinates $A(z, \bar{z}, p, q) = i\hbar A_i(z, \bar{z})X_i(p, q)$ given by

$$\frac{i}{2} A_1 p(q^2 - 1) - \frac{1}{2} A_2 p(q^2 + 1) - iA_3 pq + \hbar \left( \beta + \frac{1}{2} \right) (-A_1 q - iA_2 q + A_3). \quad (59)$$

In a limit when $\hbar \to 0$, the previous solution leads to a solution for the SU(∞) Hitchin’s equations (44) and (45).

Now we use the solutions of the SU(2) Hitchin’s equation, expressed in terms of Liouville, sinh-Gordon and Painlevé III equations, to find three different sets of hamiltonian functions which are solutions of SU(∞) Hitchin’s equations. We also build three different sets of hamiltonian vector fields and in section 5 we use them to build half-flat metrics.

3.3.1 Liouville equation

From solutions (25) to Hitchin’s equations, the corresponding hamiltonian function can be obtained by using the WWMG-correspondence. They are given by

$$H_z = -\frac{1}{4} p(q^2 - 1)e^{-ig} \partial_r (\ln g), \quad H = \frac{(i)^n q p}{2} (q + 1)^2,$$
\[ H_\tau = \frac{1}{4} p(q^2 - 1)e^{i\theta} \partial_r (\ln g), \quad H_* = -\frac{(i^n g p}{2} (q - 1)^2, \] (60)

which are solutions of the Hitchin’s equations (45) and (44). The corresponding hamiltonian vector fields are given by

\[ A_z = H_{z,p} \partial_q - H_{z,q} \partial_p = -\frac{1}{4} (q^2 - 1)e^{-i\theta} \partial_r (\ln g) \partial_q + \frac{1}{2} p q e^{-i\theta} \partial_r (\ln g) \partial_p, \]
\[ A_\tau = H_{\tau,p} \partial_q - H_{\tau,q} \partial_p = \frac{1}{4} (q^2 + 1)h \partial_q - \frac{1}{2} p q e^{i\theta} \partial_r (\ln g) \partial_p, \]
\[ \Phi = H_{,p} \partial_q - H_{,q} \partial_p = (i^n g (q + 1)^2 \partial_q - (i^n g p (q + 1) \partial_p, \]
\[ \Phi^* = H_{*,p} \partial_q - H_{*,q} \partial_p = -\frac{(i^n g}{2} (q - 1)^2 \partial_q + (i^n g p (q - 1) \partial_p. \] (61)

### 3.3.2 Sinh-Gordon equation

In this case, the solutions are given in Eqs. (30). Consequently the hamiltonian functions obtained from the sinh-Gordon equation are

\[ H_z = -\frac{1}{2} p(q^2 - 1)\alpha_z, \quad H = -\kappa p \left[ \frac{1}{2} (q^2 + 1) \cosh(\alpha) + q \sinh(\alpha) \right], \]
\[ H_\tau = \frac{1}{2} p(q^2 - 1)\alpha_\tau, \quad H_* = -\kappa p \left[ -\frac{1}{2} (q^2 + 1) \cosh(\alpha) + q \sinh(\alpha) \right]. \] (62)

The hamiltonian vectors fields are given by

\[ A_z = -\frac{1}{2} (p^2 - 1)\alpha_z \partial_q + pq \alpha_z \partial_p, \]
\[ A_\tau = \frac{1}{2} (p^2 + 1)\alpha_\tau \partial_q - pq \alpha_\tau \partial_p, \]
\[ \Phi = -\kappa \left[ \frac{1}{2} (q^2 + 1) \cosh(\alpha) + q \sinh(\alpha) \right] \partial_q + \kappa p \left[ q \cosh(\alpha) + \sinh(\alpha) \right] \partial_p, \]
\[ \Phi^* = -\kappa \left[ -\frac{1}{2} (q^2 + 1) \cosh(\alpha) + q \sinh(\alpha) \right] \partial_q + \kappa p \left[ -q \cosh(\alpha) + \sinh(\alpha) \right] \partial_p. \] (63)

### 3.3.3 Painlevé III equation

In the case of the Painlevé III equation, the solutions are given by Eqs. (39). Consequently the hamiltonian functions are given by

\[ H_z = -\frac{1}{2} p(q^2 - 1)\psi_z, \quad H = -\frac{1}{2} p(q^2 - 1)\psi_z, \]
\[ H_\tau = \frac{1}{2} p(q^2 - 1)\psi_\tau, \quad H_* = -\frac{1}{2} p(q^2 - 1)\psi_z, \]
\[ H = \frac{i}{2} p(q^2 - 1) f_1 - \frac{1}{2} p(q^2 + 1) f_2 = -e^{-\psi} p(q^2 + e^{\psi} z), \]
\[ H_* = -\frac{i}{2} p(q^2 - 1) \bar{f}_1 + \frac{1}{2} p(q^2 + 1) \bar{f}_2 = -e^{-\psi} p(1 + e^{\psi} q^2 z), \] (64)
where the functions $f_1$ and $f_2$ are given by (40). These hamiltonian functions are solutions of Hitchin’s equations (45) and (44), where the condition (37) is satisfied. The hamiltonian vector fields are

\[
A_z = H_{z,p} \partial_q - H_{z,q} \partial_p = -\frac{1}{2} q \psi_z \partial_q + \frac{1}{2} p \psi_z \partial_p, \\
A_\bar{z} = H_{\bar{z},p} \partial_q - H_{\bar{z},q} \partial_p = \frac{1}{2} q \psi_{\bar{z}} \partial_q - \frac{1}{2} p \psi_{\bar{z}} \partial_p, \\
\Phi = H_{z,p} \partial_q - H_{z,q} \partial_p = -e^{-\frac{1}{2} \psi} (q^2 + e^\psi z) \partial_q + 2e^{-\frac{1}{2} \psi} pq \partial_p, \\
\Phi^* = H_{\bar{z},p} \partial_q - H_{\bar{z},q} \partial_p = -e^{-\frac{1}{2} \psi} (1 + e^\psi q^2 \bar{z}) \partial_q + 2e^{-\frac{1}{2} \psi} pq \bar{z} \partial_p.
\] (65)

4 Conformal compactification case of SU($\infty$) Hitchin’s equations

In this section we represent solutions of the SU(2) Hitchin’s equations as hamiltonian vector fields. To do this we need reexpress $\tau_i$, the basis of Lie algebra $\mathfrak{su}(2)$, as hamiltonian vector fields $X_{H_i}$ on $\mathbb{CP}^1$. From this we build three different sets of hamiltonian vector fields depending of three different equations: Liouville, sinh-Gordon and Painlevé III. In the following section we use this information to build half-flat metrics.

As we mentioned before, Mason conjectured that SL($N, \mathbb{C}$) is a subgroup of SL($\infty$) only for $N = 2$. In Ref. [51], it was found a way to reproduce self-dual gravity from the linear action of SU(2) on $\mathbb{CP}^1$. In this case, the covariant derivative can be expressed in terms of the hamiltonian vector fields and it is given by

\[
D_i = \partial_i + A_i = \partial_i + A^k_i X_{H_k}, \tag{66}
\]

where $X_{H_k}$ are the hamiltonian vector fields of $\mathfrak{su}(2)$ over $\mathbb{CP}^1$ and $H_i$ are hamiltonian functions. Moreover $\mathbb{CP}^1$ has a symplectic form given by

\[
\Omega_{\mathbb{CP}^1} = \Omega dp \wedge dq, \tag{67}
\]

with $\Omega = \frac{i}{(1 + pq)^2}$ and the hamiltonian functions are given by

\[
H_1 = -\frac{p + q}{1 + pq}, \quad H_2 = -i\frac{p - q}{1 + pq}, \quad H_3 = \frac{2}{1 + pq}, \tag{68}
\]

where $p$ and $q$ are local coordinates over $\mathbb{CP}^1$. The hamiltonian vector fields can be written as

\[
X_{H_i} = X_{H_i}^a \partial_a = \frac{1}{\Omega} \varepsilon^{ab} H_{i,a} \partial_b = \frac{1}{\Omega} \left( H_{i,p} \partial_q - H_{i,q} \partial_p \right). \tag{69}
\]

Explicitly we have

\[
X_{H_1} = -i(q^2 - 1) \partial_q + i(p^2 - 1) \partial_p, \quad X_{H_2} = -(q^2 + 1) \partial_q - (p^2 + 1) \partial_p, \\
X_{H_3} = 2i(q \partial_q - p \partial_p). \tag{70}
\]
Hitchin’s equations which fulfil the previous hamiltonian vector fields are given by

\[
\partial_z A_\sigma - \partial_\sigma A_z + \frac{1}{2} [A_z, A_\sigma] + \frac{1}{8} [\Phi, \Phi^*] = 0, 
\]

(71)

\[
\partial_\sigma \Phi + \frac{1}{2} [A_{\sigma}, \Phi] = 0, 
\]

(72)

where the bracket is the Lie bracket of two vector fields.

**4.1 Liouville equation**

In the case of the Liouville equation, the hamiltonian vector fields have the form

\[
A_z = -\frac{1}{2} (q^2 - 1) e^{-i\theta} \partial_r (\ln g) \partial_q + \frac{1}{2} (p^2 - 1) e^{-i\theta} \partial_r (\ln g) \partial_p, 
\]

\[A_\sigma = \frac{1}{2} (q^2 - 1) e^{i\theta} \partial_r (\ln g) \partial_q - \frac{1}{2} (p^2 - 1) e^{i\theta} \partial_r (\ln g) \partial_p, \]

\[
\Phi = -g(i)^n (q - 1)^2 \partial_q - g(i)^n (p + 1)^2 \partial_p, 
\]

\[
\Phi^* = g(i)^n (q + 1)^2 \partial_q + g(i)^n (p - 1)^2 \partial_p, 
\]

(73)

They are solutions of equations (71) and (72) if \(g\) has the form described in Eq. (21).

**4.2 Sinh-Gordon equation**

For the case of the sinh-Gordon equation, the hamiltonian vector fields are

\[
A_z = (q^2 - 1) \alpha_z \partial_q - (p^2 - 1) \alpha_z \partial_p, 
\]

\[
A_\sigma = -(q^2 - 1) \alpha_\sigma \partial_q + (p^2 - 1) \alpha_\sigma \partial_p, 
\]

\[
\Phi = \kappa [2q \sinh(\alpha) - (q^2 + 1) \cosh(\alpha)] \partial_q - \kappa [-2p \sinh(\alpha) + (p^2 + 1) \cosh(\alpha)] \partial_p, 
\]

\[
\Phi^* = \kappa [2q \sinh(\alpha) + (q^2 + 1) \cosh(\alpha)] \partial_q + \kappa [-2p \sinh(\alpha) + (p^2 + 1) \cosh(\alpha)] \partial_p, 
\]

(74)

where \(\alpha\) has the form given in Eq. (29).

**4.3 Painlevé III equation**

In the case of the Painlevé III equation, we obtain the following hamiltonian vector fields

\[
A_z = \psi_z q \partial_q - \psi_z p \partial_p, 
\]

\[
A_\sigma = -\psi_\sigma q \partial_q + \psi_\sigma p \partial_p, 
\]

\[
\Phi = [\psi f_1 (q^2 - 1) + f_2 (q^2 + 1)] \partial_q + [i f_1 (p^2 - 1) - f_2 (p^2 + 1)] \partial_p 
\]

\[
= -2 e^{-\frac{\psi}{2}} (1 + e^{\psi} q^2 z) \partial_q + 2 e^{-\frac{\psi}{2}} (p^2 + e^{\psi} z) \partial_p, 
\]

\[
\Phi^* = [i f_1 (q^2 - 1) + f_2 (q^2 + 1)] \partial_q + [-i f_1 (p^2 - 1) + f_2 (p^2 + 1)] \partial_p 
\]

\[
= 2 e^{-\frac{\psi}{2}} (q^2 + e^{\psi} z) \partial_q - 2 e^{-\frac{\psi}{2}} (1 + e^{\psi} p^2 z) \partial_q, 
\]

(75)

with \(f_1\) and \(f_2\) given by Eqs. (40) respectively, and \(\psi\) is a solution of (37).
5 Self-dual gravity

In this section we finally build half-flat metrics on $\mathcal{M}_1 \times \mathcal{M}_2$, where $\mathcal{M}_i$ are two-dimensional manifolds. We make this using the Mason and Newman formulation of SDG which connect the Lax pair formalisms of SDYM and SDG. The half-flat metrics are coming from hamiltonian vector fields which solve the SU($\infty$) Hitchin’s equations.

In the case of the infinite dimensional gauge group we first give the most general half-flat metric and then show that this metric is reduced to Husain’s metric with a specific choose of the hamiltonian functions. Later we build six different half-flat metrics using the Liouville, sinh-Gordon and Painlevé equations. In each case also the dual frame is given.

In Ref. [6], Mason and Newman shown that given four independent vector fields $V_a = (W, \tilde{W}, Z, \tilde{Z})$ over a complex four-manifold $M$, and given a non-zero four-form $\nu$ (volume form) which satisfies

$$[L, M] = 0, \quad L_L \nu = -L_M \nu = 0, \quad (76)$$

where $L$ is the Lie derivative, $L = Z - \lambda \tilde{W}$ and $M = W - \lambda \tilde{Z}$. Then $f^{-1}V_a$ determines a null-tetrad for a half-flat metric (i.e. with vanishing Ricci tensor and self-dual Weyl tensor), where $f^2 = \nu(W, \tilde{W}, Z, \tilde{Z})$. Moreover, every half-flat metric arises in this way.

Consequently Eq. (76) impose the following conditions over the vector fields

$$[W, Z] = 0, \quad [\tilde{W}, \tilde{Z}] = 0, \quad [W, \tilde{W}] + [\tilde{Z}, Z] = 0. \quad (77)$$

If we make the identifications

$$Z = D_z, \quad \tilde{Z} = D_{\overline{z}}, \quad W = \frac{1}{2} \Phi^*, \quad \tilde{W} = \frac{1}{2} \Phi, \quad (78)$$

then Eqs. (77) expresses the Hitchin’s equations (4) and (5). This shown that the vector fields $f^{-1}(D_z, D_{\overline{z}}, \frac{1}{2} \Phi, \frac{1}{2} \Phi^*)$ form a null-tetrad for a half-flat metric.

In this case the metric is expressed in the dual frame $e_{V_a}$

$$g = f^2(e_Z \odot e_{\overline{Z}} - e_W \odot e_{\overline{W}}), \quad (79)$$

where $\odot$ stands for the symmetrized tensor product and

$$e_W = f^{-2} \nu(-, \tilde{W}, Z, \tilde{Z}), \quad e_{\overline{W}} = f^{-2} \nu(W, -, Z, \tilde{Z}),$$

$$e_Z = f^{-2} \nu(W, \tilde{W}, -, \tilde{Z}), \quad e_{\overline{Z}} = f^{-2} \nu(W, \tilde{W}, Z, -). \quad (80)$$

In the case of the Hitchin’s vector fields, the dual frame (80) and the function $f$, are given by

$$e_{\Phi^*} = f^{-2} \nu\left(-, \frac{1}{2} \Phi, D_z, D_{\overline{z}}\right), \quad e_{\Phi} = f^{-2} \nu\left(\frac{1}{2} \Phi^*, -, D_z, D_{\overline{z}}\right),$$

$$e_Z = f^{-2} \nu\left(\frac{1}{2} \Phi^*, \frac{1}{2} \Phi, -, D_z\right), \quad e_{\overline{Z}} = f^{-2} \nu\left(\frac{1}{2} \Phi^*, \frac{1}{2} \Phi, D_z, -\right),$$

$$f^2 = \nu\left(\frac{1}{2} \Phi^*, \frac{1}{2} \Phi, D_z, D_{\overline{z}}\right). \quad (81)$$
5.1 Infinite gauge group

Now we construct half-flat metrics coming from solutions of the SU(∞) Hitchin’s equations obtained by Moyal deformation. The four-form is given by \( \nu = dz \wedge d\bar{z} \wedge dp \wedge dq \) and the Hamiltonian vector fields are

\[
D_z = \partial_z + \varepsilon^{\mu\nu} H_{z,\mu} \partial_\nu, \quad \Phi = \varepsilon^{\mu\nu} H_{,\mu} \partial_\nu,
\]

\[
D_{\bar{z}} = \partial_{\bar{z}} + \varepsilon^{\mu\nu} \bar{H}_{\bar{z},\mu} \partial_\nu, \quad \Phi^* = \varepsilon^{\mu\nu} \bar{H}_{,\mu} \partial_\nu,
\]

where \( H_z, H_{\bar{z}}, H, H_\ast \) are solutions of equations (45) and (44). In this case the dual frame (81) and the function \( f \) are

\[
e_{\Phi^*} = \frac{f^{-2}}{2} [H_{,q} Dq + H_{,p} Dp], \quad e_z = dz,
\]

\[
e_{\Phi} = -\frac{f^{-2}}{2} [H_{,q} Dq + H_{,p} Dp], \quad e_{\bar{z}} = d\bar{z},
\]

\[
f^2 = \frac{1}{4} \{H_\ast, H\},
\]

where \( Dp \) and \( Dq \) are defined as follows

\[
Dp := dp + H_{\mu,q} dx^\mu, \quad Dq := dq - H_{\mu,p} dx^\mu,
\]

with \( H_{\mu} dx^\mu = H_z dz + H_{\bar{z}} d\bar{z} \). Moreover, the metric is given by

\[
ds^2 = \frac{1}{\{H_\ast, H\}} \left\{ \{H, H_z\} \{H_\ast, H_z\} dz^2 + \{H, H_{\bar{z}}\} \{H_\ast, H_{\bar{z}}\} d\bar{z}^2 + H_{\mu,p} H_{\ast,p} dp^2 + H_{\mu,q} H_{\ast,q} dq^2 + \{H, H_z\} H_\ast \{H_{\mu,p} H_{\ast,p} + H_{\mu,q} H_{\ast,q}\} dp dq + \{H, H_{\bar{z}}\} H_\ast \{H_{\mu,p} H_{\ast,p} + H_{\mu,q} H_{\ast,q}\} d\bar{z} dq + \{H_{,q} \{H_\ast, H_{\bar{z}}\} + H_{,p} \{H_\ast, H_z\}\} d\bar{z} dp + \{H_{,q} \{H_\ast, H_z\} + H_{,p} \{H_\ast, H_{\bar{z}}\}\} dz dq \right\}.
\]

5.1.1 Husain-Park metric

For a particular value of the spectral parameter, \( \lambda = 1 \), we have \( H = 2H_z, H_\ast = 2H_{\bar{z}} \) and \( H_z = \Lambda z, H_{\bar{z}} = -\Lambda \bar{z} \), then the previous metric reduces to

\[
ds^2 = (\Lambda_{,zp} dp + \Lambda_{,zq} dq) dz - (\Lambda_{,zp} dp + \Lambda_{,zq} dq) d\bar{z}
\]

\[
+ \frac{1}{\{\Lambda_{,z}, \Lambda_{,\bar{z}}\}} \left( \Lambda_{,zp} \Lambda_{,zq} dp^2 + (\Lambda_{,zp} \Lambda_{,q} + \Lambda_{,zq} \Lambda_{,p}) dp dq + \Lambda_{,zq} \Lambda_{,p} dq^2 \right). \quad (86)
\]

Taking \( z = x + iy, \lambda \rightarrow i\lambda \) and rescaling the coordinates we recover the Husain metric\(^3\)

\[
ds^2 = (\Lambda_{,xp} dp + \Lambda_{,xq} dq) dx + (\Lambda_{,yp} dp + \Lambda_{,yq} dq) dy
\]

\[
+ \frac{1}{\{\Lambda_{,x}, \Lambda_{,y}\}} \left( (\Lambda_{,xp} dp + \Lambda_{,xq} dq)^2 + (\Lambda_{,yp} dp + \Lambda_{,yq} dq)^2 \right). \quad (87)
\]

\(^3\)There is an overall factor \( i \).
5.1.2 Liouville equation

If we take $H_z, H_{\bar{z}}, H, H_*$ as given in Eqs. (60), then we obtain

$$Dp = dp - irpq \partial_r (\ln g) d\theta, \quad Dq = dq + \frac{i(q^2 - 1)}{2} r \partial_r (\ln g) d\theta,$$

$$f^2 = \pm \frac{p(q^2 - 1)g^2}{4},$$

(88)

where $g$ is given by (21) and the + sign corresponds to $n = 0$, while the − sign corresponds to $n = 1$. In this case $e_{\Phi}$ and $e_{\Phi^*}$ have the form

$$e_{\Phi^*} = \frac{1}{inp(q - 1)g} [2pDq + (q + 1)Dp],$$

$$e_{\Phi} = \frac{1}{inp(q + 1)g} [2pDq + (q - 1)Dp].$$

(89)

Then metric is written as

$$ds^2 = \frac{p(q^2 - 1)}{4} \left[ \pm g^2 dr^2 + r^2 (\pm g^2 - (\partial_r (\ln g))^2) d\theta^2 \right] - pdq^2 - \frac{q^2 - 1}{4p} dp^2$$

$$+ ipr \partial_r (\ln g) dq d\theta - qdpdq.$$

(90)

If we take the top sign, $n = 0$, we obtain a singular metric corresponding to a real solution of the SDYM equations in two dimensions. On the other hand, the bottom sign, $n = 1$, gives rise to a regular metric related to complex solutions of SDYM equations in two dimensions.

As we mentioned previously, if we take the value $\nu = 1$ this is related with the abelian magnetic monopole, $M = \pm \frac{2\pi^2}{15r^2} d\theta$. Thus we have

$$ds^2 = \frac{p(q^2 - 1)}{(1 + r^2)^2} dr^2 + \frac{p(q^2 - 1)}{2} M d\theta + ipMdq - pdq^2 - \frac{q^2 - 1}{4p} dp^2 - qdpdq.$$

(91)

5.1.3 Sinh-Gordon equation

If we define $\beta(z, \bar{z})$ in the following form

$$\beta = \kappa \left( \frac{(z - z_0)e^{-i\omega}}{2} + \frac{(\bar{z} - \bar{z}_0)e^{i\omega}}{2} \right),$$

(92)

then using the hamiltonian functions (62) we obtain

$$Dp = dp + \frac{1}{2} \text{csch}(\beta) pq \kappa (e^{-i\omega} dz - e^{i\omega} d\bar{z}),$$

$$Dq = dq - \frac{1}{4} \text{csch}(\beta) (q^2 - 1) \kappa (e^{-i\omega} dz - e^{i\omega} d\bar{z}),$$

$$f^2 = \frac{\kappa^2 p(q^2 - 1)}{4} \text{csch}(\beta) \coth(\beta).$$

(93)
Gathering all that information we can write down the metric as
\[
\frac{\sinh(\beta)}{\kappa p(q^2 - 1)} \left[ 2p(q + \text{sech}(\beta))Dq + (1 + q^2 + 2q \text{sech}(\beta))Dp \right],
\]
\[
\frac{\sinh(\beta)}{\kappa p(q^2 - 1)} \left[ 2p(q - \text{sech}(\beta))Dq + (1 + q^2 - 2q \text{sech}(\beta))Dp \right].
\]

In the simplest case, we take \( \omega = 0 \) and \( z_0 = 0 \), then the function \( \beta \) is only function of \( x \), i.e. \( \beta = \kappa x \). The metric obtained in this case is given by
\[
ds^2 = \frac{p(q^2 - 1)}{4} \kappa^2 \coth(\kappa x) \text{csch}(\kappa x)dx^2 + \frac{p(q^2 + 1)}{4(q^2 - 1)} \kappa^2 \text{sech}(\kappa x)dy^2
\]
\[- \frac{iq(q^2 + 1)}{q^2 - 1} \kappa \tanh(\kappa x)d density - \frac{2ip\kappa}{q^2 - 1} \left[ -1 + q^2 \cosh(2\kappa x) \right] \text{cosh}(2\kappa x)d q d y
\]
\[+ \frac{4q^2 \text{sech}(\kappa x) - (q^2 + 1)^2 \cosh(\kappa x)}{4p(q^2 - 1)} dp^2 + \frac{2q \text{sech}(\kappa x) - q(q^2 + 1) \cosh(\kappa x)}{q^2 - 1} dp dq
\]
\[+ \frac{p[1 - q^2 \cosh^2(\kappa x)] \text{sech}(\kappa x)}{q^2 - 1} dq^2. \]

5.1.4 Painlevé III equation

Now, we take the values of the hamiltonian functions given in (64), then we have
\[
Dp = dp - \frac{i}{2} pr\psi, d \theta; \quad Dq = dq + \frac{i}{2} qr\psi, d \theta,
\]
\[f^2 = \frac{1}{2} pq \left( e^\psi r^2 - e^{-\psi} \right). \]

Then, \( e_\Phi \) and \( e_{\Phi^*} \) are given by
\[
e_{\Phi^*} = - \frac{e^{-\psi}}{pq \left( e^\psi r^2 - e^{-\psi} \right)} \left[ 2pq Dq + (q^2 + e^\psi r^2 e^{i\theta}) Dp \right],
\]
\[
e_\Phi = - \frac{e^\psi}{pq \left( e^\psi r^2 - e^{-\psi} \right)} \left[ 2pq e^{-i\theta} Dq + (e^{-\psi} + q^2 e^{-i\theta}) Dp \right].
\]

Gathering all that information we can write down the metric as
\[
ds^2 = \frac{1}{8} pq \left( e^\psi r^2 - e^{-\psi} \right) (dr^2 + r^2 d\theta^2) - \frac{2}{pq \left( e^\psi r^2 - e^{-\psi} \right)} \left[ e^{-\psi} \left( 1 + q^2 e^\psi e^{-i\theta} \right) (q^2 + e^\psi e^{-i\theta}) dp^2
\]
\[+ 4pq^2 re^{-i\theta} dq^2 + \frac{e^{-\psi} p^2 r^2}{4} \left( q^2 e^\psi - q^2 \right) (e^\psi e^{-i\theta} - e^{-\psi}) (\psi, r) dp d\theta
\[+ ip^2 qre^{-\psi} \left( 1 - 2q^2 e^\psi e^{-i\theta} + e^2\psi r^2 \right) (\psi, r) dq d\theta + 2pq \left( e^{-\psi} + 2q^2 e^{-i\theta} + e^\psi r^2 \right) dq dp \]
\[. \]
5.2 Finite gauge group

In this subsection we work out with hamiltonian vector fields whose hamiltonian functions correspond to the $su(2)$ on $\mathbb{C}P^1$ with coordinates $(p, q)$. In this case the metrics are defined on $M_1 \times \mathbb{C}P^1$, the four-form is $\nu = dz \wedge d\bar{z} \wedge \Omega_{\mathbb{C}P^1}$ and the vector fields are

\begin{equation}
W = \frac{1}{2} \Phi^* i X_H, \quad Z = \partial_z + A^i_2 X_H, \\
\tilde{W} = \frac{1}{2} \Phi^* X_H, \quad \tilde{Z} = \partial_{\bar{z}} + A^i_\bar{z} X_H.
\end{equation}

The dual frame (80) is given by

\begin{align}
e_W &= \frac{f^{-2}}{2} \left[ \{ \Phi^i H_i, A^i_2 H_1 \} dz + \{ \Phi^i H_i, A^i_\bar{z} H_1 \} d\bar{z} + d\Sigma (\Phi^i H_i) \right] \\
&= \frac{f^{-2}}{2} \Phi^i [H_{i,p} dp + H_{i,q} dq - 2 \varepsilon^k_{ij} H_k (A^i_2 dz + A^i_\bar{z} d\bar{z})] \\
&= \frac{f^{-2}}{2} \Phi^i (H_{i,p} D_p + H_{i,q} D_q), \\
e_{\tilde{W}} &= \frac{f^{-2}}{2} \left[ - \{ \Phi^* i H_i, A^i_2 H_1 \} dz - \{ \Phi^* i H_i, A^i_\bar{z} H_1 \} d\bar{z} - d\Sigma (\Phi^* i H_i) \right] \\
&= - \frac{f^{-2}}{2} \Phi^* i [H_{i,p} dp + H_{i,q} dq - 2 \varepsilon^k_{ij} H_k (A^i_2 dz + A^i_\bar{z} d\bar{z})] \\
&= - \frac{f^{-2}}{2} \Phi^* i (H_{i,p} D_p + H_{i,q} D_q), \\
e_Z &= dz, \\
e_{\bar{Z}} &= d\bar{z},
\end{align}

where $D_p$ and $D_q$ are defined as follows

\begin{align}
D_p := dp + \frac{A^i_p}{\Omega} H_i d\mu, \quad D_q := dq - \frac{A^i_q}{\Omega} H_i d\mu,
\end{align}

with $A_\mu d\mu = A_z dz + A_{\bar{z}} d\bar{z}$. Moreover, the function $f^2$ is obtained by

\begin{equation}
f^2 = 2 \Phi^* \frac{\Phi^j \varepsilon_{ji}^k h_k}{1 + pq}.
\end{equation}

5.2.1 Liouville equation

In the case of solutions (25), the hamiltonian vector fields are given by

\begin{align}
D_z = \partial_z - i \partial_z (\ln g) X_{H_1}, \quad \frac{1}{2} \Phi^* = \frac{1}{2} (- g^{i_1} X_{H_2} - g^{i_2} X_{H_3}), \\
D_{\bar{z}} = \partial_{\bar{z}} + i \partial_{\bar{z}} (\ln g) X_{H_1}, \quad \frac{1}{2} \Phi = \frac{1}{2} (g^{i_1} X_{H_2} - g^{i_2} X_{H_3}).
\end{align}

In the present case $D_p$, $D_q$ and $f^2$ take the form

\begin{align}
D_p &= dp - i (p^2 - 1) r \partial_r (\ln g) d\theta, \quad D_q = dq + i (q^2 - 1) r \partial_r (\ln g) d\theta, \\
&= \pm ig^2 \frac{p + q}{1 + pq}.
\end{align}
The dual frame is

\[ e_{\Phi^*} = \frac{1}{2i^n(p+q)(1+pq)g} [(1+p)^2 Dq - (q-1)^2 Dp], \]

\[ e_{\Phi} = \frac{1}{2i^n(p+q)(1+pq)g} [-(p-1)^2 Dq + (q+1)^2 Dp]. \] (105)

Gathering all that, the metric is a function of \( g \), which is obtained from (21), and it has the following form

\[
ds^2 = \pm \frac{i(p+q)g^2}{1+pq} dr^2 + \frac{1}{(p+q)(1+pq)} \left\{ i r^2 [\pm (p+q)^2 g^2 - (p^2 - 1)(q^2 - 1)(\partial_r (\ln g))^2] d\theta^2 \\
+ r(q^2 - 1) [\partial_r (\ln g)] dp d\theta - r(p^2 - 1) [\partial_r (\ln g)] dq d\theta + \frac{i(q^2 - 1)^2}{4(1+pq)^2} dp^2 \\
- \frac{i \left[ ((1+pq)^2 + (p+q)^2) \right]}{2(1+pq)^2} dp dq + \frac{i(p^2 - 1)^2}{4(1+pq)^2} dq^2 \right\}.
\] (106)

If we take \( \nu = 1 \) we get

\[
ds^2 = \pm \frac{4i(p+q)}{(1+pq)(1+pq)} dr^2 + \frac{1}{(p+q)(1+pq)} \left\{ \pm i \left( \frac{p+q}{r^2} - (p^2 - 1)(q^2 - 1) \right) M^2 \\
- (q^2 - 1) M dp + (p^2 - 1) M dq + \frac{i(q^2 - 1)^2}{4(1+pq)^2} dp^2 - \frac{i \left[ ((1+pq)^2 + (p+q)^2) \right]}{2(1+pq)^2} dp dq \\
+ \frac{i(p^2 - 1)^2}{4(1+pq)^2} dq^2 \right\}.
\] (107)

5.2.2 Sinh-Gordon equation

In this case we use the solutions (62) and then

\[ D_z = \partial_z - i \partial_z(\alpha) X_{H_1}, \quad \frac{1}{2} \Phi = \frac{\kappa}{2} \cosh(\alpha) X_{H_2} - \frac{i\kappa}{2} \sinh(\alpha) X_{H_3}, \]

\[ D_{\bar{z}} = \partial_{\bar{z}} + i \partial_{\bar{z}}(\alpha) X_{H_1}, \quad \frac{1}{2} \Phi^* = -\frac{\kappa}{2} \cosh(\alpha) X_{H_2} + \frac{i\kappa}{2} \sinh(\alpha) X_{H_3}. \] (108)

Thus we find

\[ Dp = dp + \frac{\kappa(p^2 - 1)}{2} \cosh(\beta) (e^{-i\omega} dz - e^{i\omega} d\bar{z}), \]

\[ Dq = dq - \frac{\kappa(q^2 - 1)}{2} \cosh(\beta) (e^{-i\omega} dz - e^{i\omega} d\bar{z}), \]

\[ f^2 = \frac{i\kappa^2 (p+q)}{1+pq} \cosh(\beta) \coth(\beta). \] (109)

The frame is written as

\[ e_{\Phi^*} = \frac{\tanh(\beta)}{2\kappa(p+q)(1+pq)} \left\{ - [2p + (1+p^2) \cosh(\beta)] Dq + [-2q + (1+q^2) \cosh(\beta)] Dp \right\}, \]

\[ e_{\Phi} = \frac{\tan(\beta)}{2\kappa(p+q)(1+pq)} \left\{ [2p + (1+p^2) \cosh(\beta)] Dq - [2q + (1+q^2) \cosh(\beta)] Dp \right\}. \] (110)
We take the simplest case, $\beta = \kappa x$, and it yields the following metric

\[
ds^2 = \frac{1}{1 + pq} \left\{ i\kappa^2 (p + q) \coth(\kappa x) \csch(\kappa x) dx^2 - \frac{i\kappa^2 [1 - 3q^2 + p^2(q^2 - 3) + (p(q - 1) - q - 1)(p + q + pq - 1)\cosh(2\kappa x)]}{2(p + q)} \csch(\kappa x) \sech(\kappa x) dy^2 \right. \\
- \frac{\kappa [1 - 1 + 3q^2 + pq(q^2 - 3) + (pq - 1)(q^2 + 1)\cosh(2\kappa x)]}{(p + q)(1 + pq)} \csch(2\kappa x) \sech(\kappa x) \frac{dydp}{(p + q)(1 + pq)} \\
+ \frac{i\kappa [p^2 + 1)(pq - 1)\coth(\kappa x) + 4p(p - q)\csch(2\kappa x)]}{4(p + q)(1 + pq)^2} \sech(\kappa x) dp^2 \\
- \frac{i[pq + (1 + p^2)(1 + q^2)\cosh^2(\kappa x)]}{2(p + q)(1 + pq)^2} \csch(\kappa x) dpdq \\
+ \left. \frac{i[-4q^2 + (p^2 + 1)^2\cosh^2(\kappa x)]}{4(p + q)(1 + pq)^2} \sech(\kappa x) dq^2 \right\}. \tag{111}
\]

### 5.2.3 Painlevé III equation

In the case of the solutions given by Ward, from (39), the hamiltonian vector fields are given by

\[
D_z = \partial_z - \frac{i}{2} \psi_z X_{H_3}, \quad \frac{1}{2} \Phi^* = -\frac{i}{2}(\zeta e^{\psi r} - e^{-\psi}) X_{H_1} - \frac{1}{2}(\zeta e^{\psi r} + e^{-\psi}) X_{H_2}, \\
D_{\bar{z}} = \partial_{\bar{z}} + \frac{i}{2} \psi_{\bar{z}} X_{H_3}, \quad \frac{1}{2} \Phi = -\frac{i}{2}(ze^{\psi r} - e^{-\psi}) X_{H_1} + \frac{1}{2}(ze^{\psi r} + e^{-\psi}) X_{H_2}. \tag{112}
\]

Now the form of $Dp, Dq, f^2$ is given as follows

\[
Dp = dp + ipr \psi_r d\theta, \quad Dq = dq - iq r \psi_r d\theta, \\
f^2 = \frac{i(e^{\psi r^2} - e^{-\psi})}{2(1 + pq)} \tag{113}
\]

While the dual frame takes the form

\[
e_{\Phi^*} = e^{-\psi} \frac{e^{-\psi}}{(1 + pq)(e^{\psi r^2} - e^{-\psi})} \left[ (p^2 + re^{i\theta + \psi})Dq - (1 + q^2 re^{i\theta + \psi})Dp \right], \\
e_{\Phi} = e^{-\psi} \frac{e^{-\psi}}{(1 + pq)(e^{\psi r^2} - e^{-\psi})} \left[ -(1 + p^2 re^{-i\theta})Dq + (q^2 + re^{-i\theta})Dp \right]. \tag{114}
\]
In this case the metric is given by

\[
\begin{align*}
\mathrm{d}s^2 &= \frac{1}{2(pq + 1)} \left( r^2e^{\psi} - e^{-\psi} \right)^2 \mathrm{d}r^2 \\
&\quad - ir^2 \left[ (e^{-\psi-i\theta}(p+q)e^{\psi+i\theta}) \left( p + q(re^{\psi+i\theta}) \right) \psi, r - (r^2e^{\psi} - e^{-\psi})^2 \right] \mathrm{d}\theta^2 \\
&\quad + ie^{-\psi-i\theta} \left( r(e^{\psi} + e^{i\theta}q^2) \left( 1 + q^2r^2 e^{\psi+i\theta} \right) \right) dp^2 + \frac{ie^{-\psi-i\theta}(p^2 + re^{\psi+i\theta}) \left( p^2re^{\psi} + e^{i\theta} \right) dq^2}{(pq + 1)^2} \\
&\quad - \frac{r \left[ 2re^{-i\theta}(p + e^{2i\theta}q^3) + q(qp + 1)e^{\psi} + qr^2(qp + 1)e^{\psi} \right] \psi, r}{pq + 1} \mathrm{d}p \mathrm{d}\theta \\
&\quad + \frac{r \left[ 2re^{-i\theta}(p^3 + e^{2i\theta}q) + p(qp + 1)e^{-\psi} + pr^2(qp + 1)e^{\psi} \right] \psi, r}{pq + 1} \mathrm{d}q \mathrm{d}\theta \\
&\quad - \frac{i \left[ e^{-\psi}(p^2q^2 + 1) + r^2(p^2q^2 + 1)e^{\psi} + 2re^{-i\theta}(p^2 + e^{2i\theta}q^2) \right]}{(pq + 1)^2} \mathrm{d}p \mathrm{d}q. 
\end{align*}
\]

(115)

6 Conclusions

In the present paper we briefly survey Hitchin’s equations and some of their solutions. Besides we focused in two types of models leading to three non-linear equations, namely: Liouville, sinh-Gordon and Painlevé III.

On the other hand, Hitchin’s equations constitutes a system of PDE’s among a set of integrable reductions to two dimensions coming from the SDYM equation in four dimensions. It is well known that some other reductions to two-dimensional models, such as the WZW model and the chiral model are strongly related to self-dual gravity in four dimensions. Unlike these cases, a straightforward relationship between Hitchin’s equations and self-dual gravity is not so evident. In the present paper we intended to fill this gap. In order to do that we first follow the strategy employed in several works [25, 26, 27, 28, 29, 30]. In this work we first promote SU(\(N\)) Hitchin’s equations to a unitary anti-self-dual operator Lie algebra and then it is taken the gauge fields valued on this algebra. We used the WWMG-formalism by writing this operator algebra in a phase-space representation, i.e. in terms of the coordinate and momentum operators (\(\hat{p}, \hat{q}\)) satisfying the Heisenberg algebra. In this way we obtained the Moyal deformation of the SU(\(N\)) Hitchin’s equations. For the well known relationship between the Moyal bracket and the Poisson bracket it is possible to find the large-\(N\) limit of the Hitchin’s system by taking the limit \(\hbar \to 0\). This is explicitly done for the three cases we mentioned before: Liouville, Sinh-Gordon and Painlevé III. This limit is characterized by the set of AJJS equations for four hamiltonian vector fields. The volume form of the underlying spacetime manifold determines the tetrad system and consequently the half-flat metric.

In section 3, the WWMG-formalism was employed and we found that SU(\(\infty\)) Hitchin’s equations (14) and (15) are related to Husain-Park equation (18). Indeed, we see that if we choose a gauge in which \(\Phi = 2A_\pi\) and \(\Phi^* = 2A_\pi\), then the Hitchin’s equations reduce to (13) and then, taking the large-N limit, these equations describe
the Husain-Park equation (48). In section 5 the half-flat metric was determined for the mentioned three cases. We found the general metric which is given by Eq. (85). For the specific value of spectral parameter $\lambda = 1$, it reduces to the well known Husain-Park metric (87). For the three mentioned models the half-flat metrics are given by Eqs. (91), (95) and (98) respectively.

In Ref. [51] it was alternatively found a description of a finite dimensional sub-algebra of the infinite dimensional one of vector fields for the gauge theory and its correspondence with self-dual gravity. For these Hitchin’s equations with Lie algebra $\mathfrak{su}(2)$ we take the underlying manifold as $\mathbb{C}P^1$ instead of $\mathbb{R}^2$. In section 4 we took this approach and construct hamiltonian vectors fields satisfying Hitchin’s equations and later in section 5 we use these vectors fields and found the corresponding self-dual metrics, which for the three models are given by Eqs. (107), (111) and (115) respectively.

There are some questions that we would like to address in the near future and which we comment briefly in what follows. In Ref. [28] it was found that the Husain-Park equation can be obtained by dimensional reduction from a master equation and then a solution, using a Cauchy-Kovalevski form and initial Cauchy data [53], was given. This description is used to provide an explicit example of a sequence of $su(N)$ chiral fields giving rise to a curved heavenly space for $N \to \infty$, see also [18]. For that reason, it would be interesting to determine a Cauchy-Kovalevski form of the $SU(\infty)$ Hitchin’s equations. This can be done by looking for explicit sequences of $su(N)$ chiral field tending to a curved heavenly space. We will examine this in a future work.

On the other hand, the Plebański heavenly equations are the most usual equations for describing SDG, it is then a natural question to ask if there exist a relation between $SU(\infty)$ Hitchin’s equations and the heavenly equations. In Ref. [54] Jakimowicz and Tafel shown that Husain-Park equation is equivalent to the first heavenly equation making a Bäcklund transformation between these equations. In this direction it would be interesting to look for a corresponding transformation between $SU(\infty)$ Hitchin’s equations and the heavenly equation.

Finally, as we mentioned in the Introduction, the KW system is a four-dimensional version of the Hitchin system. These equations are a dimensional reduction of the Haydys-Witten theory in five dimensions [42]. Moreover, Haydys-Witten equations are in turn reductions from the eight-dimensional Spin(7) instanton equations to five dimensions [55]. In Ref. [47] it was obtained the $SU(\infty)$ Kapustin-Witten equations and one possible interesting question is to study what kind of gravity emerges from them as a higher-dimensional version of the present work.

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