Global regularity for the 3D Hall-MHD equations with low regularity axisymmetric data

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Abstract
In this paper, we consider the global well-posedness of the incompressible Hall-MHD equations in $\mathbb{R}^3$. We prove that the solution of this system is globally regular if the initial data is axisymmetric and the swirl components of the velocity and magnetic vorticity are trivial. It should be pointed out that the initial data can be arbitrarily large and satisfy low regularity assumptions.

Keywords Hall-MHD equations · Axisymmetric solutions · Global regularity

Mathematics Subject Classification 35Q35 · 76D03

1 Introduction
This paper is concerned with the following 3D incompressible Hall-MHD equations

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla P - \Delta u &= B \cdot \nabla B \\
\partial_t B + u \cdot \nabla B + \text{curl}(\text{curl} B \times B) &= \Delta B + B \cdot \nabla u \\
\text{div} u &= \text{div} B = 0
\end{aligned}
\]  

in $\mathbb{R}^3 \times \mathbb{R}_+$, \quad (1)

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with the initial data \((u(x, 0), B(x, 0)) = (u_0(x), B_0(x))\). Here vector functions \(u = (u^1, u^2, u^3)^T\) and \(B = (B^1, B^2, B^3)^T\) denote the fluid velocity field and the magnetic field respectively; the function \(P\) denotes the scalar pressure.

The Hall term \(\text{curl}(\text{curl} \, B \times B)\) is derived from Ohm’s law and describes deviation from charge neutrality between the electrons and the ions. Hence, the Hall-MHD equations can describe some physical phenomena that cannot be characterized appropriately by the classical MHD equations, for example, magnetic reconnection in plasmas, neutron stars, star formation, etc. For more physical background, see [13, 21].

Recently, there has been tremendous interest in developing the Hall-MHD system. In [2], Acheritogaray et al gave a derivation of the Hall-MHD system from either two-fluid or kinetic models in a mathematically rigorous way. Chae et al in [5] established for initial data \((u_0, B_0)\) the global existence of weak solutions and the local well-posedness of smooth solutions of the system (1). In [4], the authors showed the local existence of smooth solutions in \(H^2(\mathbb{R}^3)\). Later on, the initial data was weakened to \((u_0, B_0) \in H^1(\mathbb{R}^3) \times H^{1+1-\varepsilon}(\mathbb{R}^3)\) with \(s > \frac{1}{2}\) and any small enough \(\varepsilon > 0\) such that \(s - \varepsilon > \frac{1}{2}\) by Dai in [9]. Very recently, Danchin et al in [10] showed the global well-posedness for small initial conditions \(u_0, B_0\) and \(\nabla \times B_0\) in critical spaces \(\dot{B}^{\frac{3}{p} - 1}_{p,1}\), \(1 \leq p < \infty\). More interesting results, we recommend [8, 17, 18, 20, 22, 27].

It is well-known that the global well-posedness to the 3D incompressible Hall-MHD equations with large initial data is still unsolved. As far as we know, with respect to this problem, only some partial results are known. For example, the solutions satisfy some special structures where an important case is axisymmetric. Fan et al in [12] obtained that (1) is global well-posed for a class of special axisymmetric initial data, which extended the well-known result for MHD equations by Lei in [23]. More precisely, they showed that there exists a unique global solution if the initial data satisfy swirl components of the velocity field and magnetic vorticity field vanish, that is,

\[ u(0, x) = u_0^r(r, z)e_r + u_0^\theta(r, z)e_\theta \quad \text{and} \quad B(0, x) = B_0^\theta(r, z)e_\theta, \]

and also assume that

\[ (u_0, B_0) \in H^2(\mathbb{R}^3) \quad \text{and} \quad \frac{B_0^\theta}{r} \in L^\infty(\mathbb{R}^3). \]

Recently, the first author and Cui [19] establish the global well-posedness of the incompressible Hall-MHD system with horizontal dissipation.

Inspired by [12, 23], the aim of this paper is to establish the global well-posedness of 3D Hall-MHD equations (1) for a class of large axisymmetric data without swirl, which is stated as follows.

**Theorem 1** Suppose that \(u_0\) and \(B_0\) are two axisymmetric divergence free vector fields such that \(u_0^r = B_0^r = B_0^\theta = 0\). Assume that \((u_0, B_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)\), and \((\omega_0, \frac{\omega_0}{r}, \frac{B_0}{r}) \in L^\alpha(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^\sigma(\mathbb{R}^3)\) with \(6 < \alpha \leq \infty\) and \(\sigma \in (3, \alpha] \cap (3, \infty)\). Then the system (1) has a unique global solution \((u, B)\) satisfying

\[ \square \]
\[ u \in L^\infty([0, \infty); H^1(\mathbb{R}^3)) \cap L^2([0, \infty); H^2(\mathbb{R}^3)), \]
\[ B \in L^\infty([0, \infty); H^2(\mathbb{R}^3)) \cap L^2([0, \infty); H^3(\mathbb{R}^3)), \]
\[ \omega \in L^\infty([0, \infty); L^\sigma(\mathbb{R}^3)), \quad \frac{\omega}{r} \in L^\infty([0, \infty); L^2(\mathbb{R}^3)), \quad \frac{B}{r} \in L^\infty([0, \infty); L^\sigma(\mathbb{R}^3)), \]
where \( \omega = \text{curl } u \).

**Remark 1** It should be stressed that we improve the result of [12] in two ways. Since \( B_0(x) = B_0^\theta(r, z)e_\theta, \) we extend the condition \( \frac{B_0^\theta}{r} \in L^\infty(\mathbb{R}^3) \) to \( \frac{B_0^\theta}{r} \in L^\sigma(\mathbb{R}^3), 6 < \sigma \leq \infty. \) In addition, notice that in cylindrical coordinates the vorticity of the swirl-
free axisymmetric velocity is given by \( \omega = \text{curl } u = \omega^\theta e_\theta \) with \( \omega^\theta := \partial_z u^r - \partial_r u^z, \)
and \( |\nabla^2 u| \sim |\nabla \omega^\theta| + |\frac{\omega^\theta}{r}|. \) Moreover, applying the Sobolev embedding inequality, we
know \( \|\omega^\theta\|_{L^p(\mathbb{R}^3)} \leq C\|\omega^\theta\|_{H^1(\mathbb{R}^3)}, 2 \leq p \leq 6. \) Thus, our result weakens the
condition \( u_0 \in H^2(\mathbb{R}^3) \) when \( 3 < \sigma \leq 6. \)

**Remark 2** Due to the presence of the Hall term \( \text{curl } (\text{curl } B \times B), \) we need the estimate
\( \nabla B \in L^2(0, T; L^\infty(\mathbb{R}^3)) \) to obtain the \( L^\infty(0, T; H^2(\mathbb{R}^3)) \) estimate of \( B, \) for more
details see \( J_4 \) in Step 8 below. Thus we add the assumption condition \( (\omega_0, \frac{B_0}{r}) \in L^\sigma(\mathbb{R}^3) \times L^\sigma(\mathbb{R}^3) \) with \( 6 < \sigma \leq \infty \) and \( \sigma \in (3, \alpha] \cap (3, \infty). \) In the near future we
will further study whether or not the condition can be removed.

**Remark 3** When the Hall term \( \text{curl } (\text{curl } B \times B) \) is neglected, the system (1) reduces to
the incompressible viscous MHD system. In fact, we prove that under the conditions
\( u_0^\theta = B_0^\theta = B_0^\nu = 0, (u_0, B_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \) and \( \frac{\omega}{r} \in L^2(\mathbb{R}^3), \) there exists a
unique global solution to the incompressible viscous MHD system. Clearly, we do not
need the assumption condition \( (\omega_0, \frac{B_0}{r}) \in L^\sigma(\mathbb{R}^3) \times L^\sigma(\mathbb{R}^3) \) with \( 6 < \sigma \leq \infty \) and \( \sigma \in (3, \alpha] \cap (3, \infty). \) This is because we emphasize that for the incompressible viscous
MHD system, in order to get the estimate \( B \in L^\infty(0, T; H^2(\mathbb{R}^3)), \) we only need the
\( L^2(0, T; L^6(\mathbb{R}^3)) \) estimate of \( \nabla B, \) which can be similarly estimated as (3.23) below.
This is a new result for the incompressible MHD system and it improves the
related result of [23].

The proof of the main result is achieved by using more deeply the structure of
the Hall-MHD equations in axisymmetric case with zero swirl component of the
velocity and the magnetic vorticity. In contrast with the proof in [12], since
the absence of the condition \( u_0 \in H^2(\mathbb{R}^3), \) we use the dyadic decomposition to obtain
the \( \nabla B \in L^2((0, T); \text{Lip}(\mathbb{R}^3)), \) see Step 5 below. We also cannot directly obtain
\( \nabla B \in L^2((0, T); L^\infty(\mathbb{R}^3)), \) which plays a key role in the proof. To do this, we
assume \( (\omega_0, \frac{B_0}{r}) \in L^\sigma(\mathbb{R}^3) \times L^\sigma(\mathbb{R}^3) \) with \( 6 < \alpha \leq \infty \) and \( \sigma \in (3, \alpha] \cap (3, \infty). \) It
should be emphasized that we fully use the Gagliardo–Nirenberg inequality in order
to appropriately choose the value range of \( \sigma \) and \( \alpha, \) for more details see Step 4 and
Step 6 below. With \( u \in L^1((0, T]; \text{Lip}(\mathbb{R}^3)) \) and \( \nabla B \in L^2((0, T); L^\infty(\mathbb{R}^3)) \) in hand,
we show the \( H^2 \) estimate of \( B \) in Step 8.

**Notations:** We shall denote \( \int \cdot \text{d}x := \int_{\mathbb{R}^3} \cdot \text{d}x, \quad \| \cdot \|_{L^p} := \| \cdot \|_{L^p(\mathbb{R}^3)}, \) and
\( \| (A, B, C) \|_X := \| A \|_X + \| B \|_X + \| C \|_X, \) where \( X \) is a Banach space. We use the
letter C to denote a generic constant, which may vary from line to line. We always use \( A \lesssim B \) to denote \( A \leq CB \) and omit a generic positive constant \( C \) in \( \exp Ct^\alpha \).

## 2 Preliminaries

In this section, we will transform the system (1) into the cylindrical coordinate, recall some useful inequalities and introduce the Besov spaces.

Firstly, we derive the system (1) in the cylindrical coordinate. In the cylindrical coordinate, the solutions to (1) have the following form

\[
\begin{align*}
  u(t, x) &= u^r(r, z, t) e_r + u^\theta(r, z, t) e_\theta + u^z(r, z, t) e_z, \\
  B(t, x) &= B^r(r, z, t) e_r + B^\theta(r, z, t) e_\theta + B^z(r, z, t) e_z, \\
  P(t, x) &= P(r, z, t),
\end{align*}
\]

where

\[
\begin{align*}
  e_r &= \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right)^T, \\
  e_\theta &= \left( -\frac{x_2}{r}, \frac{x_1}{r}, 0 \right)^T, \\
  e_z &= (0, 0, 1)^T, \\
  r &= \sqrt{x_1^2 + x_2^2}.
\end{align*}
\]

Taking advantage of the local well-posedness result for the system (1) in \( \mathbb{R}^3 \), see [9], we can obtain the following lemma.

**Lemma 2** [9] Let \((u_0, B_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)\) be an axisymmetric divergence-free vector. Then there exists \( T > 0 \) and a unique axisymmetric solution \((u, B)\) on \([0, T)\) to the system (1) such that

\[
\begin{align*}
  u &\in L^\infty(0, T; H^1(\mathbb{R}^3)), \\
  B &\in L^\infty(0, T; H^2(\mathbb{R}^3)).
\end{align*}
\]

It is not difficult to deduce from the uniqueness of local solutions that \( u^\theta_0 = B^\theta_0 = B^z_0 = 0 \) implies \( u^\theta = B^r = B^z = 0 \) for all later times. In this case, by direct computations, we have

\[
\text{curl}(\text{curl} B \times B) = -\frac{2}{r} B^\theta \partial_z B^\theta e_\theta = -\partial_z (\Pi B^\theta e_\theta), \\
(B \cdot \nabla)u = \frac{u^r}{r} B^\theta e_\theta,
\]

and

\[
\omega = \text{curl } u = (\partial_z u^r - \partial_r u^z) e_\theta := \omega^\theta e_\theta.
\]
Hence the system (1) can be rewritten as

\[
\begin{align*}
\partial_t u^r + u^r \partial_r u^r + u^z \partial_z u^r + \partial_r P &= \left(\Delta - \frac{1}{r^2}\right) u^r - \frac{(B^\theta)^2}{r}, \\
\partial_t u^z + u^r \partial_r u^z + u^z \partial_z u^z + \partial_z P &= \Delta u^z, \\
\partial_t B^\theta + u^r \partial_r B^\theta + u^z \partial_z B^\theta &= \left(\Delta - \frac{1}{r^2}\right) B^\theta + \frac{u^r}{r} B^\theta + \frac{1}{r} \partial_z (B^\theta)^2, \\
\partial_r u^r + \frac{u^r}{r} + \partial_z u^z &= 0, \\
\end{align*}
\]

(2.1)

where \( \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \) is the Laplacian operator.

Define

\[ \Pi := \frac{B^\theta}{r} \quad \text{and} \quad \Omega := \frac{\omega^\theta}{r}, \]

then the system (2.1) is equivalent to

\[
\begin{align*}
\partial_t \Pi + u \cdot \nabla \Pi &= \left(\Delta + \frac{2}{r} \partial_r\right) \Pi + \partial_z \Pi^2, \\
\partial_t \Omega + u \cdot \nabla \Omega &= \left(\Delta + \frac{2}{r} \partial_r\right) \Omega - \partial_z \Pi^2, \\
\end{align*}
\]

(2.3)

Secondly, we present two known lemmas, which will play an important role in our proof.

**Lemma 3** (See [1, Proposition 4.2]) Let \( u \) be a smooth axisymmetric vector field with zero divergence and \( \omega = \omega^\theta \mathbf{e}^\theta \) be its curl. Then we have

\[
\|u\|_{L^\infty} \leq C \|\omega^\theta\|_{L^2}^{\frac{1}{2}} \|\nabla \omega^\theta\|_{L^2}^{\frac{1}{2}}
\]

and

\[
\|\frac{u^r}{r}\|_{L^\infty} \leq C \|\frac{\omega^\theta}{r}\|_{L^2}^{\frac{1}{2}} \|
abla \left(\frac{\omega^\theta}{r}\right)\|_{L^2}^{\frac{1}{2}}.
\]

**Lemma 4** (See [16, Lemma 4.1]) Let \( u \) be divergence-free and \( 1 < p < \infty \). Then there is a constant \( C > 0 \) depending only on the dimension \( n \) such that

\[
\|\nabla u\|_{L^p} \leq \frac{C p^2}{p - 1} \|\text{curl } u\|_{L^p}.
\]

Thirdly, we review the definition of Besov spaces and some useful inequalities. Let us first recall the classical dyadic decomposition in \( \mathbb{R}^3 \), for more details see [6].

For every \( u \in S'(\mathbb{R}^3) \), we set

\[
\Delta_q u := \varphi(2^{-q} \mathcal{D}) u, \quad \Delta_{-1} u := \chi(\mathcal{D}) u \quad \text{and} \quad S_q u := \sum_{-1 \leq j \leq q-1} \Delta_j u, \quad \text{for } q \in \mathbb{N}
\]
and
\[ \dot{\Delta}_q u := \varphi(2^{-q}D)u \quad \text{and} \quad \dot{\delta}_q u := \sum_{j \leq q - 1} \dot{\Delta}_j u, \quad \text{for} \quad q \in \mathbb{Z}. \]

Here \( \varphi \) and \( \chi \) are two smooth functions with compact support satisfying
\[
\begin{align*}
\text{supp} \varphi &= C := \{ \xi \in \mathbb{R}^3 \mid \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \}, \\
\text{supp} \chi &= B := \{ \xi \in \mathbb{R}^3 \mid |\xi| \leq \frac{3}{4} \}, \\
\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) &= 1 \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \text{and} \quad \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^3.
\end{align*}
\]

(2.4)

Then for every tempered distribution \( u \), the following decomposition holds
\[ u = \sum_{q \geq -1} \Delta_q u, \quad \forall \ u \in S'(\mathbb{R}^3), \]
or
\[ u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u, \quad \forall \ u \in S'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3), \]

where \( \mathcal{P}(\mathbb{R}^3) \) is the set of polynomials in \( \mathbb{R}^3 \), see [3].

Now we are in a position to recall the definition of the nonhomogeneous Besov spaces \( B^{s}_{p,r}(\mathbb{R}^3) \) and the homogeneous Besov spaces \( \dot{B}^{s}_{p,r}(\mathbb{R}^3) \), abbreviated as \( B^{s}_{p,r} \) and \( \dot{B}^{s}_{p,r} \), without confusion, respectively.

**Definition 1 ([3, 6])** For \( 1 \leq p, r \leq \infty \) and \( s \in \mathbb{R} \), the nonhomogeneous Besov space \( B^{s}_{p,r}(\mathbb{R}^3) \) is the space of the tempered distribution \( u \) such that
\[ \| u \|_{B^{s}_{p,r}} := \left( 2^{qs} \| \Delta^q u \|_{L^p} \right)_{\ell^r} < +\infty, \]

and the homogeneous Besov space \( \dot{B}^{s}_{p,r} \) is the space of the tempered distribution \( u \) such that
\[ \| u \|_{\dot{B}^{s}_{p,r}} := \left( 2^{qs} \| \dot{\Delta}^q u \|_{L^p} \right)_{\ell^r} < +\infty. \]

It is well-known (for example, see [3]) that the Besov space \( B^{s}_{2,2} \) coincides with the Sobolev space \( H^s \) and \( \dot{B}^{s}_{2,2} \) coincides with \( \dot{H}^s \).

Next, we recall the Bernstein inequalities, tame estimates, and commutator estimates.
Lemma 5 (See [6, Lemma 2.1]) Let $B$ be a ball and $C$ be an annulus in $\mathbb{R}^3$. There exists a constant $C$ such that for any positive $\delta$, non-negative integer $k$, smooth homogeneous function $\sigma$ of degree $m$, real numbers $q \geq p \geq 1$, and $u \in L^p(\mathbb{R}^3)$, we have

\[
\sup \hatsu \subset \delta B \Rightarrow \sup_{|\alpha|=k} \|\partial^{\alpha} u\|_{L^q} \leq C^{k+1} \delta^{k+3(\frac{1}{p} - \frac{1}{2})} \|u\|_{L^p},
\]

\[
\sup \hatsu \subset \delta C \Rightarrow C^{-1-k} \delta^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^{\alpha} u\|_{L^p} \leq C^{1+k} \delta^k \|u\|_{L^p},
\]

\[
\sup \hatsu \subset \delta C \Rightarrow \|\sigma(D) u\|_{L^q} \leq C_{\sigma,m} \delta^{m+3(\frac{1}{p} - \frac{1}{2})} \|u\|_{L^p},
\]

where $\hatsu$ denotes the Fourier transform of $u$.

We deduce from the Bernstein inequalities that the following continuous embedding

\[
B^{s}_{p_1,r_1} \hookrightarrow B^{s+3(\frac{1}{p_2} - \frac{1}{r_1})}_{p_2,r_2}
\]

holds for $p_1 \leq p_2$ and $r_1 \leq r_2$.

The following inequality is called tame estimates, see [6].

Lemma 6 [6] Suppose that $s > 0$ and $1 \leq p, q \leq \infty$. Then there exists a constant $C > 0$ such that

\[
\|fg\|_{B^{s}_{p,q}(\mathbb{R}^3)} \leq \frac{C^{s+1}}{s} \left( \|f\|_{L^\infty(\mathbb{R}^3)} \|g\|_{B^{s}_{p,q}(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)} \|f\|_{B^{s}_{p,q}(\mathbb{R}^3)} \right).
\]

Lemma 7 (Commutator Estimates) Let $1 < p < \infty$, $s > 0$ and $\Lambda^s := (-\Delta)^{\frac{s}{2}}$. Then for any $f, g \in C_c^\infty(\mathbb{R}^3)$,

\[
(1) \text{ (See [14, Lemma 3.1])}
\]

\[
\|\Lambda^s f g - f \Lambda^s g\|_{L^p(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^{p_1}(\mathbb{R}^3)} \|\Lambda^{s-1} g\|_{L^{q_1}(\mathbb{R}^3)} + \|\Lambda^s f\|_{L^{p_2}(\mathbb{R}^3)} \|g\|_{L^{q_2}(\mathbb{R}^3)},
\]

where $1 < p_1, p_2, q_1, q_2 < \infty$ satisfy

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.
\]

(2) (See [15, Lemma X1]).

\[
\|\Lambda^s f g - f \Lambda^s g\|_{L^p(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^\infty(\mathbb{R}^3)} \|\Lambda^{s-1} g\|_{L^p(\mathbb{R}^3)} + \|\Lambda^s f\|_{L^p(\mathbb{R}^3)} \|g\|_{L^\infty(\mathbb{R}^3)}.
\]

Lemma 8 (Gagliardo–Nirenberg inequality, [24]) Assume that $1 \leq p, q, r \leq \infty$ and $\delta \in [0, 1]$ such that $\frac{1}{p} = \delta(\frac{1}{r} - \frac{1}{n}) + \frac{1-\delta}{q}$. Suppose also, if $n \geq 2$, that $p \neq \infty$ or $r \neq n$. Then there exists a positive constant $C$ depending on $p, q, r, n$ such that

\[
\|u\|_{L^p(\mathbb{R}^n)} \leq C \|u\|^{1-\delta}_{L^q(\mathbb{R}^n)} \|\nabla u\|^{\delta}_{L^r(\mathbb{R}^n)},
\]

for all $u \in C^1_0(\mathbb{R}^n)$. 
Finally, we introduce the Chemin–Lerner spaces \( L^p_T(\mathbb{R}^3) \) and \( \dot{L}^p_T(\mathbb{R}^3) \), abbreviated as \( L^p_T(\dot{H}^s) \) and \( \dot{L}^p_T(\dot{H}^s) \), respectively, without confusion.

**Definition 2** ([7]) For \( s \in \mathbb{R}, p \in [1, \infty] \) and \( T > 0 \), the Chemin-Lerner spaces \( L^p_T(\dot{H}^s) \) is the space of the tempered distribution \( u \) such that
\[
\|u\|_{L^p_T(\dot{H}^s)} := \left( \sum_{\ell=1}^{2qs} \|\Delta_q u\|_{L^p(\mathbb{R}^3)}^2 \right)^{1/2} < +\infty.
\]

\( L^p_T(\dot{H}^s) \) is the space of the tempered distribution \( u \) such that
\[
\|u\|_{L^p_T(\dot{H}^s)} := \left( \sum_{\ell=1}^{2qs} \|\dot{\Delta}_q u\|_{L^p(\mathbb{R}^3)}^2 \right)^{1/2} < +\infty.
\]

**Remark 4** It is easy to observe that for any \( s \in \mathbb{R} \) and \( \varepsilon > 0 \), we have
\[
\|u\|_{L^1(\mathbb{R}^3); H^{s-\varepsilon}} \leq C \|u\|_{L^1_T(\dot{H}^s)}.
\]

Moreover, Minkowski’s inequality implies that
\[
\|u\|_{L^p_T(\dot{H}^s)} \leq \|u\|_{L^p_T(\dot{H}^s)} \quad \text{if} \quad p \leq 2,
\]
and
\[
\|u\|_{L^p_T(\dot{H}^s)} \leq \|u\|_{L^p_T(\dot{H}^s)} \quad \text{if} \quad p \geq 2.
\]

### 3 Proof of Theorem 1

The purpose of this section is to derive a priori estimates and then complete the proof of Theorem 1. Indeed, we will prove the following energy estimates.

**Proposition 9** Under the assumptions of Theorem 1, if \((u, B)\) is a smooth solution to the system (1), then there exists some constant \( C(t) \) such that
\[
\|u(t)\|_{H^1}^2 + \|B(t)\|_{H^2}^2 + \int_0^t \|u(s)\|_{H^2}^2 ds + \int_0^t \|B(s)\|_{H^3}^2 ds \leq C(t).
\]

**Proof** The proof is split into the following eight steps.

**Step 1.** \( L^\infty([0, t]; L^2(\mathbb{R}^3)) \) estimate of \((u, B)\).
Multiplying the first and second equations in (1) by $u$ and $B$, respectively, integrating by parts over $\mathbb{R}^3$ and adding up, we arrive at

$$
\frac{1}{2} \frac{d}{dt} \left\| (u(t), B(t)) \right\|^2_{L^2} + \left\| (\nabla u(t), \nabla B(t)) \right\|^2_{L^2} = \int (B \cdot \nabla B) \cdot u \, dx + \int (B \cdot \nabla u) \cdot B \, dx + \int \text{curl}(\text{curl} \times B) \cdot B \, dx
$$

$$
= \int (B \cdot \nabla B) \cdot u \, dx - \int (B \cdot \nabla B) \cdot u \, dx + \int (\text{curl} \times B) \cdot \text{curl} B \, dx
$$

$$
= 0.
$$

Then integrating over $[0, t]$ yields

$$
\left\| (u(t), B(t)) \right\|^2_{L^2} + 2 \int_0^t \left\| (\nabla u(s), \nabla B(s)) \right\|^2_{L^2} \, ds \leq \left\| (u_0, B_0) \right\|^2_{L^2}. \tag{3.5}
$$

**Step 2.** $L^\infty([0, t]; L^2(\mathbb{R}^3))$ estimate of $\Omega$.

For $p \in [2, \alpha] \cap [2, \infty)$ with $6 < \alpha < \infty$, multiplying the first equation in (2.3) by $|\Pi|^p \Pi$, we have

$$
\frac{1}{p} \frac{d}{dt} \left\| \Pi(t) \right\|^p_{L^p} + (p - 1) \int |\Pi|^{p-2} |\nabla \Pi|^2 \, dx
$$

$$
= \frac{2}{p} \int \partial_r |\Pi|^p \, dr + \frac{2}{p+1} \int \partial_z |\Pi|^{p+1} \, dx
$$

$$
= \frac{4\pi}{p} \int_{-\infty}^\infty \left( \int_0^\infty \partial_r |\Pi|^p \, dr \right) \, dz + \frac{4\pi}{p+1} \int_0^\infty r \left( \int_{-\infty}^\infty \partial_z |\Pi|^{p+1} \, dz \right) \, dr
$$

$$
\leq 0, \tag{3.6}
$$

where we have used the fact that $\lim_{|z| \to \infty} \Pi(r, z, t) = 0$ and $\lim_{r \to \infty} \Pi(r, z, t) = 0$.

Noted that

$$
|\nabla B|^2 = |\nabla B^0|^2 + |\Pi|^2,
$$

we observe

$$
\left\| \Pi_0 \right\|_{L^2} \leq \left\| B_0 \right\|_{H^1}.
$$

Integrating (3.6) over $[0, t]$ gives that, for any $p \in [2, \alpha] \cap [2, \infty)$,

$$
\left\| \Pi(t) \right\|_{L^p} \leq \left\| \Pi_0 \right\|_{L^p} \leq \left\| \Pi_0 \right\|_{L^2}^{\frac{2(\alpha-p)}{(\alpha-2)p}} \left\| \Pi_0 \right\|_{L^{\alpha-2}}^{\frac{\alpha(p-2)}{(\alpha-2)p}} \leq \left\| B_0 \right\|_{H^1}^{\frac{2(\alpha-p)}{(\alpha-2)p}} \left\| \Pi_0 \right\|_{L^2}^{\frac{\alpha(p-2)}{(\alpha-2)p}}. \tag{3.7}
$$

In the above process, there are two special cases:
Case 1. When $\alpha = \infty$, letting $p \to \infty$ in (3.7), we obtain
\[
\| \Pi(t) \|_{L^\infty} \leq \| \Pi_0 \|_{L^\infty},
\] (3.8)
which together with (3.7) yields that (3.7) holds for any $p \in [2, \alpha]$.

Case 2. When $p = 2$, (3.6) can be rewritten as
\[
\frac{1}{2} \frac{d}{dt} \| \Pi(t) \|_{L^2}^2 + \| \nabla \Pi(t) \|_{L^2}^2 \leq 0,
\]
which implies
\[
\| \Pi(t) \|_{L^2}^2 + 2 \int_0^t \| \nabla \Pi(t) \|_{L^2}^2 dt \leq \| \Pi_0 \|_{L^2}^2.
\] (3.9)

On the other hand, taking $L^2$-inner product of the second equation in (2.3) with $\Omega$ yields
\[
\frac{1}{2} \frac{d}{dt} \| \Omega(t) \|_{L^2}^2 + \| \nabla \Omega(t) \|_{L^2}^2 \leq 0,
\]
where we have used the fact that $\text{div} u = 0$ and $\lim_{r \to \infty} \Omega(r, z, t) = 0$.

A direct calculation gives
\[
- \int \Omega \partial_z \Pi^2 dx = -2\pi \int_0^\infty r \left( \int_{-\infty}^\infty \Omega \partial_z \Pi^2 dz \right) dr
\]
\[
= 2\pi \int_0^\infty r \left( \int_{-\infty}^\infty \Pi^2 \partial_z \Omega dz \right) dr.
\]
Therefore
\[
\frac{1}{2} \frac{d}{dt} \| \Omega(t) \|_{L^2}^2 + \| \nabla \Omega(t) \|_{L^2}^2 = \int \Pi^2 \partial_z \Omega dx \leq \| \Pi \|_{L^4}^2 \| \nabla \Omega \|_{L^2},
\]
which together with the Young inequality and (3.7) yields
\[
\frac{d}{dt} \| \Omega(t) \|_{L^2}^2 + \| \nabla \Omega(t) \|_{L^2}^2 \leq \| \Pi_0 \|_{L^4}^4 \leq \| B_0 \|_{H^1}^{2p - 8} \| \Pi_0 \|_{L^{2p}}^{2p}.
\]
Integrating over $[0, t]$, we observe
\[
\|\Omega(t)\|_{L^2}^2 + \int_0^t \|\nabla \Omega(s)\|_{L^2}^2 \, ds \leq \|\Omega_0\|_{L^2}^2 + t\|B_0\|_{L^{2^{\frac{3\alpha-8}{\alpha-2}}}} \|\Pi_0\|_{L^{2^{\frac{3\alpha}{\alpha-2}}}} \lesssim 1 + t.
\]
(3.10)

Furthermore, it is easy to deduce from the Hölder inequality, Lemma 3 and (3.10) that
\[
\int_0^t \|\frac{u^r}{r}(s)\|_{L^\infty} \, ds \leq C \sup_{0 \leq s \leq t} \|\Omega(s)\|_{L^2}^\frac{1}{2} \int_0^t \|\nabla \Omega(s)\|_{L^2}^\frac{1}{2} \, ds \lesssim t^\frac{5}{4}.
\]
(3.11)

**Step 3.** $L^\infty([0, t]; L^2(\mathbb{R}^3))$ estimate of $\nabla u$.

For any $p \in [2, \infty)$, multiplying the third equation in (2.1) by $|B^\theta|^{p-2}B^\theta$, thanks to integrating by parts and Hölder’s inequality, we get
\[
\frac{1}{p} \frac{d}{dt} \|B^\theta\|_{L^p}^p + (p - 1) \int |B^\theta|^{p-2} |\nabla B^\theta|^2 \, dx + \|\frac{|B^\theta|^p}{r}\|_{L^2}^2 = \int |B^\theta|^p |\frac{u^r}{r}| \, dx
\]
\[
\leq \|B^\theta\|_{L^p}^p \|\frac{u^r}{r}\|_{L^\infty},
\]
which together with Gronwall’s inequality and (3.11) yields
\[
\|B^\theta\|_{L^p} \lesssim \|B_0^\theta\|_{L^p} \exp \int_0^t \|\frac{u^r}{r}(s)\|_{L^\infty} \, ds \lesssim \|B_0^\theta\|_{L^p} \exp(Ct^\frac{5}{4}), \quad \forall \ 2 \leq p < \infty.
\]

Passing $p \to \infty$ in the above estimate, we know that
\[
\|B^\theta\|_{L^\infty} \lesssim \|B_0^\theta\|_{L^\infty} \exp(Ct^\frac{5}{4}).
\]
(3.12)

Recall that in cylindrical coordinates the vorticity of the swirl-free axisymmetric velocity is given by
\[
\omega = \text{curl } u = \omega^\theta e_\theta,
\]
which satisfies
\[
\partial_t \omega^\theta + u \cdot \nabla \omega^\theta - \left( \Delta - \frac{1}{r^2} \right) \omega^\theta - \frac{u^r}{r} \omega^\theta = - \partial_z \left( \frac{B^\theta}{r} \right)^2.
\]
(3.13)

Taking the $L^2$-inner product of (3.13) with $\omega^\theta$ and then using the incompressible condition $\text{div } u = 0$, we see
\[
\frac{1}{2} \frac{d}{dt} \|\omega^\theta\|_{L^2}^2 + \|\nabla \omega^\theta\|_{L^2}^2 + \|\frac{\omega^\theta}{r}\|_{L^2}^2 \leq \int \frac{u^r}{r} |\omega^\theta|^2 \, dx - \int \partial_z \left( \frac{B^\theta}{r} \right)^2 \omega^\theta \, dx
\]
\[ : = I_1 + I_2. \quad (3.14) \]

For \( I_1 \), we have
\[
|I_1| \leq \| \frac{u^r}{r} \|_{L^\infty} \| \omega^\theta \|^2_{L^2}. \]

For \( I_2 \), it follows from the integration by parts that
\[
|I_2| = \left| \int \frac{(B^\theta)^2}{r} \partial_z \omega^\theta \, dx \right| \leq \| B^\theta \|_{L^\infty} \| \frac{B^\theta}{r} \|_{L^2} \| \partial_z \omega^\theta \|_{L^2} \\
\leq \frac{1}{2} \| B^\theta \|^2_{L^\infty} \| \Pi \|^2_{L^2} + \frac{1}{2} \| \partial_z \omega^\theta \|^2_{L^2}.
\]

Substituting the above estimates into (3.14) and using (3.12), we obtain
\[
\frac{d}{dt} \| \omega^\theta \|^2_{L^2} + \| \nabla \omega^\theta \|^2_{L^2} + \| \frac{\omega^\theta}{r} \|^2_{L^2} \leq \frac{1}{2} \| \frac{u^r}{r} \|_{L^\infty} \| \omega^\theta \|^2_{L^2} + \| B^\theta \|^2_{L^2} \| \Pi \|^2_{L^2} + \frac{1}{2} \| \partial_z \omega^\theta \|^2_{L^2}.
\]

where we have used the Sobolev embedding \( H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \) for \( s > \frac{3}{2} \).

Then we deduce from the Gronwall inequality and (3.11) that
\[
\| \omega^\theta(t) \|^2_{L^2} + \int_0^t \| \nabla \omega^\theta(s) \|^2_{L^2} \, ds + \int_0^t \| \frac{\omega^\theta}{r}(s) \|^2_{L^2} \, ds \leq \left( \| \omega^\theta_0 \|^2_{L^2} + \| B^\theta \|^2_{H^2} \int_0^t \| s \frac{5}{2} \|_{L^2} \, ds \right) \exp \int_0^t \| \frac{u^r}{r}(s) \|_{L^\infty} \, ds \leq (1 + t) \exp(Ct^{\frac{5}{4}}).
\]

In view of
\[
\| \omega \|_{L^2} = \| \omega^\theta \|_{L^2} \quad \text{and} \quad \| \nabla \omega \|^2_{L^2} = \| \nabla \omega^\theta \|^2_{L^2} + \| \frac{\omega^\theta}{r} \|^2_{L^2},
\]
we arrive at
\[
\| \omega(t) \|^2_{L^2} + \int_0^t \| \nabla \omega(s) \|^2_{L^2} \, ds \lesssim (1 + t) \exp(Ct^{\frac{5}{4}}), \quad (3.15)
\]
which together with Lemma 4 yields
\[
\| \nabla u(t) \|^2_{L^2} + \int_0^t \| \nabla^2 u(s) \|^2_{L^2} \, ds \lesssim (1 + t) \exp(Ct^{\frac{5}{4}}). \quad (3.16)
\]
Moreover, Lemma 3 and (3.15) lead to

\[ \int_0^t \| u(s) \|_{L^\infty}^2 \, ds \lesssim \int_0^t \| \omega^\theta(s) \|_{L^2} \| \nabla \omega^\theta(s) \|_{L^2} \, ds \]

\[ \lesssim \sup_{0 \leq s \leq t} \| \omega^\theta(s) \|_{L^2} \left( \int_0^t \| \nabla \omega^\theta(s) \|_{L^2}^2 \, ds \right)^{1/2} \left( \int_0^t 1 \, ds \right)^{1/2} \]

\[ \lesssim (1 + t)^{3/2} \exp(Ct^{5/4}). \tag{3.17} \]

**Step 4.** \( L^\infty([0, t]; L^\infty(\mathbb{R}^3)) \) estimate for \( u \).

For any \( 3 < \sigma < \infty \), taking the \( L^2 \)-inner product of (3.13) with \( |\omega^\theta|^\sigma - 2 \omega^\theta \) yields

\[ \frac{1}{\sigma} \frac{d}{dt} \| \omega^\theta \|_{L^\sigma}^\sigma + \frac{4(\sigma - 1)}{\sigma^2} \| \nabla (|\omega^\theta|^{\sigma/2}) \|_{L^2}^2 + \| |\omega^\theta|^{\sigma/2} \|_{L^2}^2 \leq \int \frac{u^r}{r} |\omega^\theta| \sigma \, dx - \int |\omega^\theta|^\sigma \omega^\theta \partial_z \left( \frac{B^\theta}{r} \right)^2 \, dx \]

\[ \leq \| \frac{u^r}{r} \|_{L^\infty} \| \omega^\theta \|_{L^\sigma}^\sigma + \int \left( \frac{B^\theta}{r} \right)^2 \partial_z(|\omega^\theta|^{\sigma/2} - 2 \omega^\theta) \, dx. \tag{3.18} \]

Due to the fact that

\[ \partial_z(|\omega^\theta|^{\sigma/2} - 2 \omega^\theta) = |\omega^\theta|^{\sigma/2} \partial_z \omega^\theta + \omega^\theta \partial_z(|\omega^\theta|^{\sigma/2}) \]

\[ = \frac{2(\sigma - 1)}{\sigma} |\omega^\theta|^{\sigma/2 - 1} \text{sign}(\omega^\theta) \partial_z(|\omega^\theta|^{\sigma/2}), \]

we have

\[ \int \frac{(B^\theta)^2}{r} \partial_z(|\omega^\theta|^{\sigma/2} - 2 \omega^\theta) \, dx \]

\[ \leq \frac{2(\sigma - 1)}{\sigma} \int |B^\theta| \left| \frac{B^\theta}{r} \right| |\omega^\theta|^{\sigma/2 - 1} |\nabla (|\omega^\theta|^{\sigma/2})| \, dx \]

\[ \leq \frac{2(\sigma - 1)}{\sigma} \| B^\theta \|_{L^\infty} \| \Pi \|_{L^\sigma} \| \nabla (|\omega^\theta|^{\sigma/2}) \|_{L^2} \| \omega^\theta \|_{L^\sigma}^{\sigma/2 - 1} \]

\[ \leq \frac{2(\sigma - 1)}{\sigma^2} \| \nabla (|\omega^\theta|^{\sigma/2}) \|_{L^2}^2 + C_\sigma \| B^\theta \|_{L^\infty} \| \Pi \|_{L^\sigma} \| \omega^\theta \|_{L^\sigma}^{\sigma}, \]

where \( C_\sigma > 0 \) is a constant depending only on \( \sigma \).

From this and (3.18), we obtain

\[ \frac{d}{dt} \| \omega^\theta \|_{L^\sigma}^\sigma \leq \left( 1 + \| \frac{u^r}{r} \|_{L^\infty} \right) \| \omega^\theta \|_{L^\sigma}^\sigma + C_\sigma \| B^\theta \|_{L^\infty} \| \Pi \|_{L^\sigma} \| \omega^\theta \|_{L^\sigma} \sigma, \quad \forall \sigma \in (3, \alpha] \cap (3, \infty). \]
Thanks to the Gronwall inequality, we get from (3.7), (3.11) and (3.12) that
\[
\|\omega^t(t)\|_{L^\sigma} \lesssim \left( \|\omega_0\|_{L^\sigma} + \int_0^t \|B^\theta(s)\|_{L^\infty} \|\Pi(s)\|_{L^\sigma} \, ds \right) \exp \int_0^t \left( 1 + \|\frac{u^t}{r}\|_{L^\infty} \right) \, ds \\
\lesssim (1 + t) \exp(Ct^{\frac{5}{4}}), \quad \forall \sigma \in (3, \alpha) \cap (3, \infty).
\] (3.19)

Then we deduce from (3.19) and Lemma 4 that
\[
\|\nabla u\|_{L^\sigma} \lesssim \|\omega^t(t)\|_{L^\sigma} \lesssim (1 + t) \exp(Ct^{\frac{5}{4}}).
\]

Notice that \( \sigma \in (3, \alpha) \cap (3, \infty) \), by the Lemma 8, we know that
\[
\|u\|_{L^\infty([0, t]; L^\infty(\mathbb{R}^3))} \lesssim \|u\|_{L^\infty([0, t]; L^2(\mathbb{R}^3))} + \|\nabla u\|_{L^\infty([0, t]; L^\sigma(\mathbb{R}^3))} \lesssim (1 + t) \exp(Ct^{\frac{5}{4}}).
\]

**Step 5.** \( L^1([0, t]; \text{Lip}(\mathbb{R}^3)) \) estimate for \( u \).

Making use of the vector identity
\[
f \cdot \nabla f = \text{curl} f \times f + \frac{1}{2} \nabla |f|^2,
\]
we can deduce from a routine computation that
\[
\text{curl}(B \cdot \nabla B) = \text{curl}(\text{curl} B \times B) = -\partial z (\Pi B^\theta e_\theta).
\]

Thus, rewriting the equation (1)\(_1\) by the vorticity \( \omega = \text{curl} u \), we see
\[
\partial_t \omega - \Delta \omega = -\text{curl}(u \cdot \nabla u) - \partial z (\Pi B^\theta e_\theta).
\] (3.20)

Let \( q \in \mathbb{N} \) and \( \omega_q := \Delta_q \omega \). Then localizing in frequency to the vorticity equation (3.20) and applying the Duhamel formula, we find
\[
\omega_q = e^{t \Delta} \omega_q(0) - \int_0^t e^{(t-s) \Delta} \Delta_q (\text{curl}(u \cdot \nabla u)) (s) \, ds \\
- \int_0^t e^{(t-s) \Delta} \Delta_q (\partial z (\Pi B^\theta e_\theta)) (s) \, ds.
\]

By the estimate, see [6],
\[
\|e^{t \Delta} \Delta_q f\|_{L^m} \leq C e^{-ct} 2^{2q} \|\Delta_q f\|_{L^m}, \quad \forall 1 \leq m \leq \infty,
\]
and the Bernstein inequality, we get
\[
\|\omega_q\|_{L^p} \lesssim e^{-ct} 2^{2q} \|\omega_q(0)\|_{L^p} + 2^{2q} \int_0^t e^{-c(t-\tau)2q} \|\Delta_q (u \otimes u)(s)\|_{L^p} \, ds \\
+ 2^{2q} \int_0^t e^{-c(t-\tau)2q} \|\Delta_q (\Pi B^\theta)(s)\|_{L^p} \, ds.
\]

Then integrating over time and using convolution inequalities yield
\[
\int_0^t \|\omega_q(s)\|_{L^p} \, ds \lesssim 2^{-2q} \|\omega_q(0)\|_{L^p} + \int_0^t \|\Delta_q (u \otimes u)(s)\|_{L^p} \, ds \\
+ 2^{-2q} \int_0^t \|\Delta_q (\Pi B^\theta)(s)\|_{L^p} \, ds,
\]

which implies that
\[
\int_0^t \|\omega(s)\|_{B^{3p}_{p,1}} \, ds \lesssim \int_0^t \|\Delta_{-1} \omega(s)\|_{L^p} \, ds + \|\omega_0\|_{B^{3p-2}_{p,1}}^{\frac{3}{p}} \\
+ \int_0^t \|u \otimes u(s)\|_{B^{3p}_{p,1}}^{\frac{3}{p}} \, ds + \int_0^t \|\Pi B^\theta(s)\|_{B^{3p-1}_{p,1}}^{\frac{3}{p}} \, ds.
\]

Taking $3 < p \leq 6$, applying the Bernstein inequality and (3.15) to the first term of the right hand side leads to
\[
\int_0^t \|\Delta_{-1} \omega(s)\|_{L^p} \, ds \lesssim t \|\omega\|_{L^\infty([0,t];L^2(\mathbb{R}^3))} \lesssim (1 + t)^{\frac{3}{2}} \exp(C t^{\frac{5}{4}}).
\]

For the second term of the right hand side, the Besov embedding implies
\[
\|\omega_0\|_{B^{3p-2}_{p,1}} \lesssim \|u_0\|_{B^{3p-1}_{p,1}} \lesssim \|u_0\|_{B^{7}_{2,1}}^{\frac{1}{2}} \lesssim \|u_0\|_{H^1}.
\]

Applying the Besov embedding, law products and interpolation inequality, we have
\[
\|u \otimes u\|_{B^{3p}_{p,1}}^{\frac{3}{p}} \lesssim \|u \otimes u\|_{B^{3p}_{2,1}}^{\frac{3}{p}} \\
\lesssim \|u\|_{L^\infty} \|u\|_{B^{\frac{3}{2}}_{2,1}}^{\frac{3}{p}} \\
\lesssim \|u\|_{L^\infty} \|u\|_{L^2} + \|u\|_{L^\infty} \|\nabla u\|_{B^{\frac{1}{2}}_{2,1}}^{\frac{1}{2}} \\
\lesssim \|u\|_{L^\infty} \|u\|_{L^2} + \|u\|_{L^\infty} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}},
\]

which together with (3.16) implies
\[
\|u \otimes u\|_{L^1([0, t], B_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3))} + t \frac{1}{2} \|u\|_{L^2([0, t], L^\infty(\mathbb{R}^3))} \|u\|_{L^\infty([0, t], L^2(\mathbb{R}^3))} \\
\lesssim t \frac{1}{2} \|u\|_{L^2([0, t], L^\infty(\mathbb{R}^3))} \|u\|_{L^\infty([0, t], L^2(\mathbb{R}^3))} \\
+ t \frac{1}{4} \|u\|_{L^2([0, t], L^\infty(\mathbb{R}^3))} \|u\|_{L^2([0, t], L^2(\mathbb{R}^3))} \\
\lesssim (1 + t)^{\frac{3}{2}} \exp(Ct^{\frac{5}{2}}).
\]

Using the embedding \(L^p(\mathbb{R}^3) \hookrightarrow B_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3)\) for \(p > 3\), we know

\[
\|\Pi B^\theta\|_{B_{p,1}^{\frac{3}{p}-1}} \lesssim \|\Pi B^\theta\|_{L^p} \lesssim \|B^\theta\|_{L^\infty} \|\Pi\|_{L^p},
\]

which implies that, for \(3 < p \leq 6\),

\[
\int_0^t \|\Pi B^\theta(s)\|_{B_{p,1}^{\frac{3}{p}-1}} ds \lesssim \Pi_0 \|\Pi\|_{L^p} \int_0^t \|B^\theta(s)\|_{L^\infty} ds \lesssim t \exp(Ct^{\frac{5}{2}}).
\]

Hence we find

\[
\int_0^t \|\omega(s)\|_{B_{p,1}^{\frac{3}{p}}} ds \lesssim (1 + t)^{\frac{3}{2}} \exp(Ct^{\frac{5}{2}}).
\]

For \(p \in (3, 6]\), using the Besov embedding \(B_{p,1}^{\frac{3}{p}+1}(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)\) and the Sobolev embedding \(H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)\), we deduce from (3.16) that

\[
\int_0^t \|\nabla u(s)\|_{L^\infty} ds \lesssim \int_0^t \|u(s)\|_{B_{p,1}^{\frac{3}{p}+1}} ds \\
\lesssim \int_0^t \|\omega(s)\|_{B_{p,1}^{\frac{3}{p}}} ds + \int_0^t \|u(s)\|_{L^p} ds \\
\lesssim (1 + t)^{\frac{3}{2}} \exp(Ct^{\frac{5}{2}}).
\]

(3.21)

**Step 6.** \(L^2([0, t]; \text{Lip}(\mathbb{R}^3))\) estimate of \(B\).

We first rewrite the second equation in (1) as

\[
\partial_t B + u \cdot \nabla B = \Delta B + \frac{B^\theta}{2} \partial_r e_\theta + 2 \frac{B^\theta}{r} \partial_r \partial_r e_\theta.
\]

(3.22)
Taking $L^2$-inner product of $(3.22)$ with $-\Delta B$, by the Hölder inequality, we get

$$
\frac{1}{2} \frac{d}{dt} \| \nabla B \|_{L^2}^2 + \| \Delta B \|_{L^2}^2 = \int \left( u \cdot \nabla B - \frac{u^}{r} B^\theta e_\theta - 2 \frac{B^\theta}{r} \partial_\theta B^\theta e_\theta \right) \cdot \Delta B \, dx
$$

\begin{align*}
\leq & \left( \| u \|_{L^\infty} \| \nabla B \|_{L^2} + \| \frac{u^}{r} \|_{L^\infty} \| B^\theta \|_{L^2} + 2 \| \frac{B^\theta}{r} \|_{L^6} \| \partial_\theta B^\theta \|_{L^3} \right) \| \Delta B \|_{L^2} \\
\leq & \left( \| u \|_{L^\infty} \| \nabla B \|_{L^2} + \| \frac{u^}{r} \|_{L^\infty} \| B^\theta \|_{L^2} + 2 \| \frac{B^\theta}{r} \|_{L^6} \| \nabla B \|_{L^2} \right) \| \Delta B \|_{L^2}.
\end{align*}

In view of $\| \nabla B \|_{L^3} \leq \| \nabla B \|_{L^2}^{\frac{1}{2}} \| \nabla B \|_{L^6}^{\frac{1}{2}}$ and $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \| \nabla B(t) \|_{L^2}^2 + \| \Delta B(t) \|_{L^2}^2 \\
\leq \| u \|_{L^\infty} \| \nabla B \|_{L^2} \| \Delta B \|_{L^2} + \| \frac{u^}{r} \|_{L^\infty} \| B^\theta \|_{L^2} \| \Delta B \|_{L^2} + \| \frac{B^\theta}{r} \|_{L^6} \| \nabla B \|_{L^2}^2 \| \Delta B \|_{L^2}^2 \\
\leq \frac{1}{2} \| \Delta B \|_{L^2}^2 + C \| u \|_{L^\infty} \| \nabla B \|_{L^2}^2 + C \| \frac{u^}{r} \|_{L^\infty} \| B^\theta \|_{L^2}^2 + C \| \frac{B^\theta}{r} \|_{L^6} \| \nabla B \|_{L^2}^2,
$$

which implies

$$
\frac{d}{dt} \| \nabla B \|_{L^2}^2 + \| \Delta B \|_{L^2}^2 \lesssim \left( \| u \|_{L^\infty}^2 + \| \frac{B^\theta}{r} \|_{L^6}^4 \right) \| \nabla B \|_{L^2}^2 + \| \frac{u^}{r} \|_{L^\infty} \| B^\theta \|_{L^2}^2.
$$

Thanks to Gronwall’s inequality, we observe

$$
\| \nabla B(t) \|_{L^2}^2 + \int_0^t \| \Delta B(s) \|_{L^2}^2 \, ds \\
\lesssim \left( \| \nabla B_0 \|_{L^2}^2 + \int_0^t \| B^\theta(s) \|_{L^2}^2 \| \frac{u^}{r}(s) \|_{L^\infty} \, ds \right) \exp \int_0^t \left( \| u(s) \|_{L^\infty}^2 + \| \frac{B^\theta}{r}(s) \|_{L^6}^4 \right) \, ds \\
\lesssim \exp \left\{ (1 + t)^{\frac{3}{2}} \exp\left(C t^{\frac{5}{8}}\right) \right\},
$$

where we have used $(3.5)$, $(3.7)$, $(3.17)$ and

$$
\int_0^t \| \frac{u^}{r}(s) \|_{L^\infty} \, ds \leq C \sup_{0 \leq s \leq t} \| \Omega(\cdot, s) \|_{L^2} \left( \int_0^t \| \nabla \Omega(s) \|_{L^2} \, ds \right)^{\frac{1}{2}} \left( \int_0^t 1 \, ds \right)^{\frac{1}{2}} \\
\lesssim t^{\frac{5}{8}}.
$$

Therefore

$$
\| B \|_{L^\infty([0, t]; H^1(\mathbb{R}^3))} + \| B \|_{L^2([0, t]; H^2(\mathbb{R}^3))} \lesssim \exp \left( (1 + t)^{\frac{3}{2}} \exp\left(C t^{\frac{5}{8}}\right) \right).
\quad (3.23)
$$
Set
\[
\gamma = \gamma(\alpha) = \begin{cases} 
\frac{6\alpha}{6+\alpha} & \text{if } \alpha \in (6, \infty), \\
6 & \text{if } \alpha = \infty.
\end{cases}
\]

Then we can easily get \(3 < \gamma \leq 6\).

Making use of the H"older inequality and the Sobolev embedding \(H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\), we find
\[
\left\| \frac{u^r}{r} B^\theta \right\|_{L^2([0, t]; L^\gamma(\mathbb{R}^3))} \leq \left\| \frac{B^\theta}{r} \right\|_{L^\infty([0, t]; L^6(\mathbb{R}^3))} \left\| u \right\|_{L^2([0, t]; L^6(\mathbb{R}^3))} \\
\lesssim \left\| \Pi \right\|_{L^\infty([0, t]; L^6(\mathbb{R}^3))} \left\| \nabla u \right\|_{L^2([0, t]; L^6(\mathbb{R}^3))} \\
\lesssim \left\| \Pi_0 \right\|_{L^6} \left( u_0, B_0 \right)_{L^2},
\]
and
\[
\left\| u \cdot \nabla B \right\|_{L^2([0, t]; L^\gamma(\mathbb{R}^3))} \leq \left\| u \right\|_{L^\infty([0, t]; L^6(\mathbb{R}^3))} \left\| \nabla B \right\|_{L^2([0, t]; L^6(\mathbb{R}^3))} \\
\lesssim \left\| u \right\|_{L^\infty([0, t]; L^6(\mathbb{R}^3))} \left\| u \right\|_{L^\infty([0, t]; L^\infty(\mathbb{R}^3))} \| \Delta B \|_{L^2([0, t]; L^2(\mathbb{R}^3))} \\
\lesssim \exp \left\{ (1 + t)^{\frac{3}{2}} \exp(C t^{\frac{5}{2}}) \right\},
\]

Thus we know that
\[
\partial_t B - \Delta B = -u \cdot \nabla B + \frac{u^r}{r} B^\theta \theta + \frac{2 B^\theta}{r} \partial_z B^\theta \in L^2([0, t]; L^\gamma(\mathbb{R}^3)).
\]

By the regularity theory of the heat equation, we obtain that for any \(3 < \gamma \leq 6\),
\[
\Delta B \in L^2([0, t]; L^\gamma(\mathbb{R}^3)),
\]
which together with Lemma 8 ensures
\[
\left\| \nabla B \right\|_{L^2([0, t]; L^\infty(\mathbb{R}^3))} \lesssim \left\| \nabla B \right\|_{L^2([0, t]; L^2(\mathbb{R}^3))} + \left\| \nabla^2 B \right\|_{L^2([0, t]; L^\gamma(\mathbb{R}^3))} \\
\lesssim \exp \left\{ (1 + t)^{\frac{3}{2}} \exp(C t^{\frac{5}{2}}) \right\}.
\]

(3.24)

Step 7. \(L^1([0, t]; H^{2-\varepsilon}(\mathbb{R}^3))\) estimate of \(\nabla u\).

From (3.20), we can deduce from the regularity theory of the heat equation that
\[
\| \omega \|_{L^1_t(\dot{H}^2)} \lesssim \| \omega_0 \|_{L^2} + \| \text{curl}(u \cdot \nabla u) \|_{L^1([0, t]; L^2(\mathbb{R}^3))} + \| \partial_z (\Pi B^\theta) \|_{L^1([0, t]; L^2(\mathbb{R}^3))}.
\]

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Making use of Lemma 6, we see that

\[ \| \text{curl} (u \cdot \nabla u) \|_{L^2(\mathbb{R}^3)} \leq \| u \cdot \nabla u \|_{H^1} \]
\[ \lesssim \| u \|_{L^\infty} \| \nabla u \|_{H^1} + \| u \|_{H^1} \| \nabla u \|_{L^\infty}, \]

and

\[ \| \partial_z (\Pi \Phi) \|_{L^2(\mathbb{R}^3)} \leq \| \Phi \|_{L^\infty(\mathbb{R}^3)} \| \nabla \Pi \|_{L^2(\mathbb{R}^3)} + \| \Pi \|_{L^2(\mathbb{R}^3)} \| \nabla \Phi \|_{L^\infty(\mathbb{R}^3)}. \]

Collecting estimates (3.9), (3.12), (3.16), (3.17) and (3.24) together, we have

\[ \| \omega \|_{L^1_t(\dot{H}^2)} \lesssim \| \omega_0 \|_{L^2} + \| \text{curl} (u \cdot \nabla u) \|_{L^1([0,t];L^2(\mathbb{R}^3))} \]
\[ + \| \partial_z (\Pi \Phi) \|_{L^1([0,t];L^2(\mathbb{R}^3))} \]
\[ \lesssim \| \omega_0 \|_{L^2} + \| u \|_{L^2([0,t];L^\infty(\mathbb{R}^3))} \| \nabla u \|_{L^2([0,t];H^1(\mathbb{R}^3))} \]
\[ + \| u \|_{L^\infty([0,t];H^1(\mathbb{R}^3))} \| \nabla u \|_{L^1([0,t];L^\infty(\mathbb{R}^3))} \]
\[ + \| \Pi_0 \|_{L^2} \| \nabla \Phi \|_{L^2([0,t];L^\infty(\mathbb{R}^3))} \]
\[ \lesssim \exp \left\{ (1 + t)^{\frac{3}{2}} \exp \left\{ Ct^{\frac{5}{4}} \right\} \right\}, \]

which together with Remark 4 and (3.15) implies for any \( 0 < \varepsilon < 1 \),

\[ \| \omega \|_{L^1([0,t];H^{2-\varepsilon}(\mathbb{R}^3))} \lesssim \| \omega \|_{L^1_t(\dot{H}^2)} \]
\[ \lesssim \int_0^t \| \omega(s) \|_{L^2} \, ds + \| \omega \|_{L^1_t(\dot{H}^2)} \]
\[ \lesssim \exp \left\{ (1 + t)^{\frac{3}{2}} \exp \left\{ Ct^{\frac{5}{4}} \right\} \right\}. \]

Therefore

\[ \int_0^t \| \nabla u(s) \|_{H^{2-\varepsilon}} \, ds \lesssim \int_0^t \| \omega(s) \|_{H^{2-\varepsilon}} \, ds \]
\[ \lesssim \exp \left\{ (1 + t)^{\frac{3}{2}} \exp \left\{ Ct^{\frac{5}{4}} \right\} \right\}. \] (3.25)

**Step 8.** \( L^\infty([0, t]; H^2(\mathbb{R}^3)) \) estimate of \( B \).

Applying the operator \( \nabla^2 \) to the third equation in (2.1) leads to
\[
\partial_t \nabla^2 B + u \cdot \nabla \nabla^2 B - B \cdot \nabla \nabla^2 u + \nabla^2 \text{curl}(\text{curl } B \times B)
= \nabla^2 (B \cdot \nabla) u - \nabla^2 (u \cdot \nabla) B + \Delta \nabla^2 B.
\]

Taking the \(L^2\)-inner product of the above equation with \(\nabla^2 B^\theta\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^2 B \|^2_{L^2} + \| \nabla^3 B \|^2_{L^2}
= \int B \cdot \nabla \nabla^2 u : \nabla^2 B \, dx - \int [\nabla^2, B \cdot \nabla] u : \nabla^2 B \, dx
+ \int [\nabla^2, B \cdot \nabla] u : \nabla^2 B \, dx - \int \nabla^2 \text{curl}(\text{curl } B \times B) : \nabla^2 B \, dx
:= \sum_{i=1}^4 J_i.
\]

Next we estimate \(J_i\) term by term.

For \(J_1\), we deduce from Hölder’s inequality, Young’s inequality and the Sobolev embedding \(\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)\) that
\[
|J_1| = \left| \int B \cdot \nabla \nabla^2 u : \nabla^2 B \, dx \right|
= \left| \int B \cdot \nabla \left( \nabla^2 u : \nabla^2 B \right) \, dx - \int B \cdot \nabla^2 B : \nabla^2 u \, dx \right|
\leq \| \nabla^3 B \|_{L^2} \| B \|_{L^6} \| \nabla^2 u \|_{L^3}
\leq \frac{1}{4} \| \nabla^3 B \|^2_{L^2} + 2 \| \nabla^2 u \|^2_{\dot{H}^{\frac{1}{2}}} \| \nabla B \|^2_{L^2}.
\]

Thanks to the Sobolev embeddings \(\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)\) and \(\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\), we see
\[
|J_2| = \left| \int [\nabla^2, u \cdot \nabla] B : \nabla^2 B \, dx \right|
\lesssim \sum_{i, j=1}^3 \left| \int \partial_i \partial_j u \cdot \nabla B \cdot \partial_i \partial_j B \, dx \right| + \sum_{i, j=1}^3 \left| \int \partial_i u \cdot \nabla \partial_j B \cdot \partial_i \partial_j B \, dx \right|
\lesssim \| \nabla^2 u \|_{L^3} \| \nabla B \|_{L^6} \| \nabla^2 B \|_{L^2} + 2 \| \nabla u \|_{L^\infty} \| \nabla^2 B \|^2_{L^2}
\lesssim \left( \| \nabla^2 u \|^2_{\dot{H}^{\frac{1}{2}}} + \| \nabla u \|_{L^\infty} \right) \| \nabla^2 B \|^2_{L^2}.
\]

Similarly,
\[ |J_3| = \left| \int [\nabla^2, B \cdot \nabla] u : \nabla^2 B \, dx \right| \]

\[
\leq \sum_{i, j=1}^{3} \left| \int \partial_i \partial_j B \cdot \nabla u \cdot \partial_i \partial_j B \, dx \right| + \sum_{i, j=1}^{3} \left| \int \partial_i B \cdot \nabla \partial_j u \cdot \partial_i \partial_j B \, dx \right| 
\]

\[
\lesssim \left( \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{H^{1/2}} \right) \|\nabla^2 B\|_{L^2}^2.
\]

For \( J_4 \), we have

\[
J_4 = - \sum_{i, j=1}^{3} \int \partial_i \partial_j \text{curl}(\text{curl} B \times B) \cdot \partial_i \partial_j B \, dx 
\]

\[
= - \sum_{i, j=1}^{3} \int \partial_i \partial_j (\text{curl} B \times B) \cdot \partial_i \partial_j \text{curl} B \, dx 
\]

\[
= - \sum_{i, j=1}^{3} \int (\partial_i \partial_i (\text{curl} B \times B) - \partial_i \partial_i \text{curl} B \times B) \cdot \partial_i \partial_j \text{curl} B \, dx,
\]

i.e.,

\[
J_4 = - \int (\Delta (\text{curl} B \times B) - \Delta \text{curl} B \times B) : \Delta \text{curl} B \, dx 
\]

\[
= \int (\Delta (B \times \text{curl} B) - B \times (\Delta \text{curl} B)) : \Delta \text{curl} B \, dx 
\]

\[
\leq \|\Delta (B \times \text{curl} B) - B \times \Delta \text{curl} B\|_{L^2} \|\Delta \text{curl} B\|_{L^2}.
\]

Thus it follows from Lemma 7 that

\[
|J_4| \leq C (\|\nabla B\|_{L^\infty} \|\nabla \text{curl} B\|_{L^2} + \|\nabla^2 B\|_{L^2} \|\text{curl} B\|_{L^\infty}) \|\nabla^3 B\|_{L^2}
\]

\[
\leq C \|\nabla B\|_{L^\infty} \|\nabla^2 B\|_{L^2} \|\nabla^3 B\|_{L^2}
\]

\[
\leq \frac{1}{4} \|\nabla^3 B\|_{L^2}^2 + C \|\nabla B\|_{L^\infty}^2 \|\nabla^2 B\|_{L^2}^2.
\]

Collecting all the above estimates together, we find

\[
\frac{d}{dt} \|\nabla^2 B(t)\|_{L^2}^2 + \|\nabla^3 B(t)\|_{L^2}^2 
\]

\[
\lesssim \left( \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{H^{1/2}} + \|\nabla B\|_{L^\infty} \right) \|\nabla^2 B\|_{L^2}^2 
\]

\[
\lesssim \left( \|\nabla u\|_{L^\infty} + \|\nabla u\|_{H^{1/2}} + \|\nabla B\|_{L^\infty} \right) \|\nabla^2 B\|_{L^2}^2,
\]
which together with (3.12), (3.16), (3.21), (3.24), (3.25) and Gronwall’s inequality ensures

\[
\|\nabla^2 B(t)\|_{L^2}^2 + \int_0^t \|\nabla^3 B(s)\|_{L^2}^2 ds \\
\lesssim \|B_0\|_{H^2}^2 \exp \left\{ \int_0^t \left( \|\nabla u(s)\|_{L^\infty} + \|\nabla B(s)\|_{L^2}^2 + \|\nabla u\|_{H^\frac{5}{2}}^2 \right) ds \right\} \\
\lesssim \exp \left\{ \exp \left\{ (1 + t)^{3/2} \exp\{Ct^{5/4}\} \right\} \right\}.
\]

Combining Step 1 to Step 8 together, we complete the proof of Proposition 9. □

**Proof of Theorem 1** With the Proposition 9, taking advantage of the local existence and uniqueness result, that is, Lemma 2, we finish the proof of Theorem 1. □

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