1/2-HEAVY SEQUENCES DRIVEN BY ROTATION

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Abstract. We investigate the set of $x \in S^1$ such that for every positive integer $N$, the first $N$ points in the orbit of $x$ under rotation by irrational $\theta$ contain at least as many values in the interval $[0, 1/2]$ as in the complement. By using a renormalization procedure, we show both that the Hausdorff dimension of this set is the same constant (strictly between zero and one) for almost-every $\theta$, and that for every $d \in [0, 1]$ there is a dense set of $\theta$ for which the Hausdorff dimension of this set is $d$.

1. Introduction

Let $X = \mathbb{R}/\mathbb{Z}$ be the unit circle, with addition modulo one, and for irrational $\theta$ fixed define

$$ f(x) = \chi_{[0,1/2]}(x) - \chi_{(1/2,1)}(x), \quad T(x) = x + \theta \mod 1, \quad S_n(x) = \sum_{i=0}^{n-1} f \circ T^i(x). $$

We will explicitly present the sets

$$ H_\theta = \{ x \in [0,1) : S_n(x) \geq 0, \quad n = 1, 2, \ldots \}, $$

$$ H^*_\theta = \{ x \in [0,1) : S_n(x) > 0, \quad n = 1, 2, \ldots \}. $$

These sets are known as the heavy and strictly heavy sets, respectively. The sequence $S_n(x)$ is the 1/2-discrepancy sequence defined by a given $x$ and the rotation parameter $\theta$. It follows from [4] that $H_\theta$ is nonempty and from [3] that it is of measure zero. The special case of $x = \theta$ was studied in [1] (where $S_n(\theta) \geq 0$ was termed ‘$\{n\theta\}$ is a 1/2-heavy sequence’), and Theorem 1 in that work may be interpreted as saying that $0 \in H^*_\theta$ if and only if $\theta$ has a particular expansion as a continued fraction: all partial quotients of odd index are themselves even.

Theorem 1. For every irrational $\theta$, $H^*_\theta$ is a singleton.

Corollary 1.0.1. $H^*_\theta = \{0\}$ if and only if all partial quotients of $\theta$ of odd index are themselves even

Theorem 2. There is some constant $c \in (0,1)$ such that for almost-every $\theta$, $\dim_H(H_\theta) = c$.

Theorem 3. Given any $d \in [0,1]$, there is a dense set of $\theta$ for which $\dim_H(H_\theta) = d$.

Several examples making explicit use of the techniques developed are presented.

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2. The Renormalization Procedure

We will freely use standard continued fraction notation. For $\theta \in (0, 1) \setminus \mathbb{Q}$, let

$$\theta = [a_1, a_2, a_3, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}.$$ 

The first of the following inequalities is standard, and the second is easy to verify:

\begin{align*}
(1a) & \quad \frac{1}{(2n + 1)(m + 1)} < 1 - 2n\theta < \frac{1}{2nm} \quad (a_1 = 2n, a_2 = m) \\
(1b) & \quad \frac{1}{2n + 1} < 1 - 2n\theta < \frac{1}{n + 1} \quad (a_1 = 2n + 1 \neq 1)
\end{align*}

The Gauss map is given by

$$\gamma(\theta) = \frac{1}{\theta} \mod 1, \quad \gamma([a_1, a_2, \ldots]) = [a_2, a_3, \ldots].$$

Define the function $g$ by

$$g(\theta) = \begin{cases} 
1 - \theta = [a_2 + 1, a_3, \ldots] & (a_1(\theta) = 1) \\
\frac{1}{1 + \gamma(\theta)} = [1, a_2, a_3, \ldots] & (a_1(\theta) = 2n + 1 \neq 1) \\
\gamma^2(\theta) = [a_3, \ldots] & (a_1(\theta) = 2n)
\end{cases}$$

As rotation by any irrational $\theta$ is topologically minimal, for any proper interval $I' \subset [0, 1)$, we define for every $x$

$$n(x) = \min \{ n \in \mathbb{N} : T^n(x) \in I' \},$$

and the induced map on $I'$ is defined for $x \in I'$ by $T_I'(x) = T^n(x)$. Let $[x] = x - (x \mod 1)$ denote the integer part or floor of $x$. The following information may all be derived from [5, §3], where the specified renormalization is studied in detail:

**Proposition 2.1.** For a fixed $\theta \in (0, 1) \setminus \mathbb{Q}$, let $I'$ be the interval $[0, \delta)$, where

$$\delta = 1 - 2 \left\lfloor \frac{a_1}{2} \right\rfloor \theta.$$

Then if the endpoints of $I'$ are identified with one another, the first return map $T_{I'}$ is rotation by $g(\theta)\delta$.

For those $\theta < 1/2$ (so that $I' \neq [0, 1)$), for all $x \in [0, \delta/2]$ we have

$$1 \leq i \leq n(x) \implies S_i(x) \geq 1,$$

and $S_{n(x)}(x) = 1$.

For those $x \in (\delta/2, \delta)$, on the other hand,

$$1 \leq i \leq n(x) \implies S_i(x) \geq -1,$$

and $S_{n(x)}(x) = -1$.

This ergodic properties of this renormalization map have also been presented:

**Theorem 1, [6].** There exists a unique probability measure $\mu_g$ on the circle, mutually absolutely continuous with respect to Lebesgue measure, such that $\mu_g$ is $g$-invariant and ergodic. The system $\{S^t, \mu_g, g\}$ is in fact exponentially CF-mixing, and both Radon-Nikodym derivatives $d\mu_g/dx$ and $dx/d\mu_g$ are essentially bounded.
Denote the Hausdorff, upper Minkowski box, and lower Minkowski box dimensions of a set $S$ by $\dim_H(S)$, $\dim_B(S)$, and $\dim_b(S)$, respectively, so for any set $S$ we have

$$\dim_H(S) \leq \dim_b(S) \leq \dim_B(S).$$

Let $E_0$ be a closed, connected interval in $[0, 1)$, and define a sequence of sets

$$E_0 \supset E_1 \supset E_2 \supset \ldots$$

by requiring that each $E_i$ be a disjoint union of closed intervals and that the maximal length of an interval in $E_i$ tends to zero as $i \to \infty$. Define $F = \bigcap_{i=0}^\infty E_i$. Suppose further that each $E_i$ contains at least $m_{i+1}$ intervals of $E_{i+1}$, and that the intervals of $E_i$ are separated by at least $\epsilon_i > 0$, where $\epsilon_{i+1} < \epsilon_i$. Then by [2, Example 4.6]:

$$\dim_H(F) \geq \liminf_{k \to \infty} \frac{\sum_{i=1}^k \log m_i}{-\log m_{k+1} - \log \epsilon_{k+1}}.$$

A sequence of numbers $\{m_i\}$ will be said to be not exceptionally irregular if for every $\epsilon > 0$, there is some number $N$ such that for all $k \geq N$, we have

$$m_k < \prod_{i=1}^{k-1} m_i^\epsilon.$$

Proposition 3.1. Suppose that each interval of $E_i$ contains exactly $m_{i+1} \geq 1$ intervals of $E_{i+1}$, each of which is of length exactly $\delta_i \leq 1$ times the length of the intervals in $E_i$, and furthermore suppose that the intervals of $E_i$ are separated by gaps at least as large as the intervals themselves, and this length converges to zero as $i \to \infty$. Finally, assume that the $m_k$ are not exceptionally irregular. Then

$$\dim_H(F) = \dim_b(F) = \liminf_{k \to \infty} \frac{\sum_{i=1}^k \log m_i}{-\sum_{i=1}^k \log \delta_i}.$$

Proof. First, note that all intervals in $E_i$ are of length $\delta_1 \cdots \delta_i$ times the length of $E_0$, and this length also serves as $\epsilon_i$ in (3). From (3), the definition of lower Minkowski box dimension $\dim_b$ and the fact that $\dim_H \leq \dim_b$, we have that

$$\liminf_{k \to \infty} \frac{\sum_{i=1}^k \log m_i}{-\log m_{k+1} - \sum_{i=1}^k \log \delta_i} \leq \dim_H(F) \leq \dim_b(F) = \liminf_{k \to \infty} \frac{\sum_{i=1}^k \log m_i}{-\sum_{i=1}^k \log \delta_i},$$

and the condition (4) exactly ensures that equality is forced.

Returning our attention to the sets $H_\theta$, let $\theta$ be fixed. Note that we trivially have $H_\theta^* \subset H_\theta \subset [0, 1/2]$.

Lemma 3.1.1. If $a_1(\theta) = 1$, then (with subtraction defined pointwise)

$$H_\theta = \frac{1}{2} - H_{g(\theta)}, \quad H_\theta^* = \frac{1}{2} - H_{g(\theta)}^*.$$

Proof. $g(\theta) = 1 - \theta$ for $a_1(\theta) = 1$, and for any $\theta$, the reader may verify that

$$x + i\theta \leq \frac{1}{2} \iff (1/2 - x) + i(1 - \theta) \leq 1/2.$$
Lemma 3.1.2. Let $\theta$ be fixed with $a_1(\theta) \neq 1$, and $\delta = 1 - 2|a_1/2|\theta$. Then

$$H_\theta \cap [0, \delta) = \delta \cdot H_{g(\theta)},$$

where multiplication is defined pointwise.

Proof. Suppose $x \in H_\theta \cap [0, \delta)$. Then as all ergodic sums are nonnegative, certainly the ergodic sums along the subsequence of times $n_i(x)$, the $i$-th return to $[0, \delta)$, are also nonnegative. However, by Proposition 2.1 we have

$$S_n(x) = \sum_{j=0}^{i-1} \left( \chi_{[0,\delta/2]} - \chi_{(\delta/2,\delta]} \right) (T^j(x)),$$

as the cumulative sums through return to $I'$ are exactly $\pm 1$ on these intervals. As this function is simply $f(\delta x)$, and $T_\nu$ is rotation by $g(\theta) \cdot \delta$ (recall that the endpoints of $I'$ are identified with one another), we have

$$H_\theta \cap [0, \delta) \subset \delta \cdot H_{g(\theta)}.$$

For the other containment, we note (again using Proposition 2.1) that the cumulative sum $S_{n_i(x)}(x)$ is in fact the minimal value reached until return to $n(x)$, so if

$$\sum_{j=0}^{i-1} \left( \chi_{[0,\delta/2]} - \chi_{(\delta/2,\delta]} \right) (T^j(x)) \geq 0$$

for $k = 0, 1, \ldots, i$, then $S_n(x) \geq 0$ for all $n = 0, 1, \ldots, n_i(x)$.

Corollary 3.1.1. $H_\theta \cap (\delta/2, \delta) = \emptyset$.

Proof. $H_{g(\theta)} \subset [0, 1/2]$.

Lemma 3.1.3. Let $a_1(\theta) \neq 1$. Then $H_\theta \cap (\delta/2 + \theta, 1) = \emptyset$.

Proof. Clearly no $x > 1/2$ is in $H_\theta$. So assume that $a_1(\theta) = 2n$ or $2n + 1$, but in either case $n \neq 0$. If $n = 1$, then $\delta/2 + \theta = 1/2$, so without loss of generality assume $n \geq 2$. Let $x \in (1/2 - (n - 1)\theta, 1/2]$. As $x > 1/2 - (n - 1)\theta$, evaluating $f$ along the orbit of $x$ produces no more than $n - 1$ consecutive values of $+1$. Since $n\theta < 1/2$, however, after this initial string of positive values, the orbit includes at least $n$ consecutive values of $-1$, producing a negative sum in the orbit of $x$.

Lemma 3.1.4. Suppose $a_1(\theta) \neq 1$. Then for $x \in [\delta, \delta/2 + \theta]$,

$$x \in H_\theta \iff (x + 2n\theta) \in H_\theta.$$

Proof. Let $a_1(\theta) = 2n$ or $2n + 1$; in either case $n \neq 0$. As

$$\delta/2 = 1/2 - n\theta < x < 1/2 - (n - 1)\theta,$$

the orbit begins with exactly $n$ consecutive values within $[0, 1/2]$. As in Lemma 3.1.3 it then contains at least $n$ consecutive values within $(1/2, 1)$. So the first $2n$ values of $f$ along the orbit of $x$ consist of exactly $n$ consecutive values of $+1$ followed by $n$ consecutive values of $-1$. The ergodic sums have not been negative before time $2n$, and for $k \geq 0$ we have

$$S_{2n+k}(x) = S_{2n}(x) + S_k(x + 2n\theta) = S_k(x + 2n\theta).$$

Corollary 3.1.2. $H^*_\theta \subset [0, \delta/2]$.

Proof. We have shown that for $x > \delta/2$, there is some $n$ for which $S_n(x) \leq 0$. □
Lemma 3.1.5. If $a_1(\theta) = 2n + 1 \neq 1$, then

\[
\frac{\delta}{2} + \delta > \frac{1}{2} - (n - 1)\theta.
\]

If $a_1(\theta) = 2n$ and $a_2(\theta) = a_2$, then

\[
\frac{\delta}{2} + a_2\delta < \frac{1}{2} - (n - 1)\theta < \frac{\delta}{2} + (a_2 + 1)\delta.
\]

Furthermore, if $a_1(\theta) = 2n$, then

\[
(a_2 + 1)\delta < \frac{1}{2} - (n - 1)\theta
\]

if and only if $a_3(\theta) = 1$.

Proof. All of these facts are simple computations using elementary inequalities of continued fractions. We include only the computations behind the last statement to give an indication of the techniques necessary, where each line is equivalent to the next:

\[
(a_2 + 1)\delta < \frac{1}{2} - (n - 1)\theta
\]
\[
a_2\left(\frac{1}{\theta} - 2n\right) + \frac{1}{\theta} - 2n < \frac{1}{2}\left(\frac{1}{\theta} - 2n\right) + 1
\]
\[
\gamma(\theta)(a_2 + 1) < \frac{\gamma(\theta)}{2} + 1
\]
\[
\frac{1}{2} < \frac{1}{\gamma(\theta)} - a_2
\]
\[
\frac{1}{2} < \gamma(\theta).
\]

□

Corollary 3.1.3. For all $\theta$ such that $a_1(\theta) \neq 1$, $H_\theta^* = \delta \cdot H_{g(\theta)}^*$. If $a_1(\theta) = 2n + 1 > 1$, then (with addition and multiplication defined pointwise),

- $H_\theta \subset [0, \delta/2] \cup [\delta, 1/2 - (n - 1)\theta]$,
- $H_\theta \cap [0, \delta/2] = \delta \cdot H_{g(\theta)}$,
- $H_\theta \cap [\delta, 1/2 - (n - 1)\theta] = (H_\theta \cap [0, (n + 1)\theta - 1/2]) + \delta$.

If $a_1(\theta) = 2n$ and $a_3(\theta) \neq 1$,

- $H_\theta \subset \bigcup_{i=0}^{a_2} ([0, \delta/2] + i\delta)$, for $i = 0, 1, \ldots, a_2$,
- $H_\theta \cap ([0, \delta/2] + i\delta) = \delta \cdot H_{g(\theta)} + i\delta$.

If $a_1(\theta) = 2n$ and $a_3(\theta) = 1$,

- $H_\theta \subset [(a_2 + 1)\delta, 1/2 - (n - 1)\theta] \cup \bigcup_{i=0}^{a_2} ([0, \delta/2] + i\delta))$,
- $H_\theta \cap ([0, \delta/2] + i\delta) = \delta \cdot H_{g(\theta)} + i\delta$, for $i = 0, 1, \ldots, a_2$.
- $H_\theta \cap [(a_2 + 1)\delta, 1/2 - (n - 1)\theta] = (H_\theta \cap [0, (n + 1)\theta - 1/2]) + (a_2 + 1)\delta$.

Proof. The proof is a combination of all previous discussion and is left to the reader. See Figure 1 for a ‘proof by picture’ which illustrates all the necessary steps for the case $a_1 = 2n$. □
δ is less than one half. This shows both that the intersection of all $E \leq n$ Corollary 4.0.4.

The set $I \setminus H$ and the set $\mathcal{S}$ are accounted for by $p$ and on the interval $I$ Inductively, then, $E$ It follows from Proposition 2.1 (reversals of orientation induced by $a$ $p$ Now, inductively define $I$ where $\#$ denotes cardinality and $p$ where $I$ is the length of $g \theta$ in the interval $\{a_2 + 1\} \delta, 1/2 - (n - 1)\theta$. Placement of the cutoff point $\delta/2 + \theta$ is determined by Lemma 3.1.5.

4. Proof of Theorem 1

Let $\theta_i = g^i(\theta)$, $\delta_i = 1 - 2[a_1(\theta_i)/2] \theta_i$, and

$$p_i = \# \{j = 0, 1, \ldots, i - 1: a_1(\theta_i) = 1\} \mod 2,$$

where $\#$ denotes cardinality and $p_0 = 0$.

We trivially know that $H_\delta \subset [0, 1/2]$, so denote $E_0^\ast = [0, 1/2]$ and $I_0 = [0, 1)$. Now, inductively define $I_{i+1}$ and $E_{i+1}^\ast$, using $E_i^\ast = [a, b]$. If $p_{i+1} = 0$, let

$$I_{i+1} = [a, a + \delta_i |I_i|], \quad E_{i+1}^\ast = [a, a + \delta_i |I_i|/2],$$

where $|I_i|$ is the length of $I_i$, which is easily inductively seen to be $2(b - a)$. If $p_{i+1} = 1$, let

$$I_{i+1} = (b - \delta_i |I_i|, b], \quad E_{i+1}^\ast = [b - \delta_i |I_i|/2, b].$$

It follows from Proposition 2.3 (reversals of orientation induced by $a_1(\theta_i) = 1$ are accounted for by $p_i$) that ergodic sums with respect to the function $\chi_{E_i^\ast}(x) - \chi_{I_i \setminus E_i^\ast}(x)$ and the transformation $T_{\theta_i}$ are never less than one through the return time to $I_{i+1}$ exactly on the set $E_{i+1}^\ast$, which is the total ergodic sum at this time, and on the interval $I_{i+1} \setminus E_{i+1}^\ast$ the ergodic sums are never less than minus one through their return, and this is the value of the ergodic sums at the return time. Furthermore, on $I_i \setminus E_{i+1}^\ast$ there is some non-positive ergodic sum (Corollary 3.1.2). Inductively, then, $E_{i+1}^\ast$ is exactly the set of points in $I_0$ for which $S_n(x) \geq 1$ for all $n \leq n(x)$, the return time to $I_i$.

The length of $I_i$ is given by $\delta_\theta \delta_1 \cdots \delta_i$. For $a_3(\theta_i) \neq 1$ (i.e. $\theta_i < 1/2$), we have $\delta_i < 1/2$, and since $g : (1/2, 1) \to (0, 1/2)$, of any consecutive $\delta_i, \delta_{i+1}$, at least one is less than one half. This shows both that the intersection of all $E_i^\ast$ is a singleton and the set $H_\delta^\ast$ (the return time to $I_i$ must diverge as the length decreases to zero).

Corollary 4.0.4. The set $H_\theta$ is always infinite.
singleton by (5). We already know that $H$ is a singleton (and therefore an isolated point within $H$) is a singleton (and therefore an isolated point within $H$). For each such $N_i$ we see

$$S_k(H^*_\theta + N_i\theta) = S_{N_i + k}(H^*_\theta) - S_{N_i}(H^*_\theta) \geq 1 - 1 = 0. \quad \square$$

That $S_n(x)$ form an additive cocycle allows us to immediately conclude (for $a_1(\theta) \neq 1$, i.e. $\theta < 1/2$):

$$H_\theta \cap [\theta, 1/2] = H^*_\theta + \theta. \quad (5)$$

5. Proof of Theorem 3

Before we construct the sets $E_i$ to determine $\dim_H(H_\theta)$, we refine Corollary 3.1.3 using Theorem 4.

**Lemma 5.0.6.** If $a_1(\theta) = 2n + 1 \neq 1$, then

$$H_\theta \cap \delta, 1/2$$

is a singleton (and therefore an isolated point within $H_\theta$). If $a_1(\theta) = 2n$ and $a_3(\theta) = 1$, then

$$H_\theta \cap [(a_2(\theta) + 1)\delta, 1/2 - (n - 1)\theta]$$

is a singleton (and therefore an isolated point within $H_\theta$).

**Proof.** First let $a_1(\theta) = 2n + 1$. In this case $\delta/2 < \theta < \delta$, and $H_\theta \cap [\theta, 1/2]$ is a singleton by (5). We already know that $H_\theta \cap (\delta/2, \delta) = \emptyset$, so that singleton is in $[\delta, 1/2]$. In the event that $a_1(\theta) = 2n$ and $a_1(\theta) = 1$, we verify

$$\frac{1}{2} < \gamma^2(\theta) < 1$$

$$\frac{1}{2} + a_2(\theta) < \frac{1}{\gamma(\theta)} < 1 + a_2(\theta)$$

$$\left(\frac{1}{2} - 2n\right) \left(\frac{1}{2} + a_2\right) < 1 < \left(\frac{1}{2} - 2n\right)(1 + a_2)$$

$$\delta\left(\frac{1}{2} + a_2\right) < \theta < \delta(1 + a_2).$$

So, $H_\theta \cap [\theta, 1/2]$ is a singleton within $[(a_2 + 1)\delta, 1/2 - (n - 1)\theta]$; from Corollary 3.1.3 we have

$$H_\theta \cap \left(\delta\left(\frac{1}{2} + a_2\right), \delta(1 + a_2)\right) = \emptyset. \quad \square$$

We now let $E_0 = [0, 1/2]$, and $\theta_i$, $\delta_i$, $p_i$ are defined as in the proof of Theorem 1. We further define $S_0 = \emptyset$, $\Delta_0 = 1$, and

$$\Delta_i = \prod_{j=0}^{i-1} \delta_j.$$ 

Assume that $E_i$ is given by a collection of closed intervals, each of which are of length $(1/2)\Delta_i$ and which are separated from one another by at least $(1/2)\Delta_i$. If $a_1(\theta_i) = 1$, then if we let $E_{i+1} = E_i$ and $S_{i+1} = S_i$, this assumption will be maintained, as for $\theta_i > 1/2$ we have $\delta_i = 1$.

Inductively assume that for each interval $[a, b] \in E_i$, we have

$$H_\theta \cap [a, b] = \Delta_i \cdot H_{\theta_i} + a,$$

which is clearly true for $i = 0$. Then for each such $[a, b] \in E_i$ we add to $E_{i+1}$ a collection of intervals according to several cases, given in Table 1.
| Case | Interval(s) in $E_{i+1}$ |
|------|-------------------------|
| $a_1(\theta_i) = 1$ | $[a, b]$ |
| $a_1(\theta_i) = 2n + 1 \neq 1$ | $[a, a + (1/2)\Delta_{i+1}]$ |
| $a_1(\theta_i) = 2n, a_2(\theta) = 2$ | $[b - (1/2)\Delta_{i+1}, b]$ |

Table 1. The decomposition of $E_i$ into $E_{i+1}$ according to $\theta_i$.

So each interval in $E_{i+1}$ is of length $(1/2)\Delta_{i+1}$ and separated from other intervals in $E_{i+1}$ by the same amount, by applying Corollary 3.1.3 and Proposition 2.1, we know that each such interval contains the desired affine image of $H_{\theta_{i+1}}$. The terms $p_i$ track the parity of the number of changes of orientation caused by applying Lemma 3.1.1 as in the proof of Theorem 1.

Finally, if $a_1(\theta_i) = 2n + 1$ or $a_1(\theta_i) = 2n, a_3(\theta_i) = 1$, we define $S_{i+1}$ to be the union of $S_i$ along with the (finitely many, possible zero) isolated points which arise in those smaller intervals listed in Corollary 3.1.3 (applying Lemma 5.0.6).

Proposition 5.1. $H_\theta = E \cup S$, where

$$E = \bigcap_{i=0}^{\infty} E_i, \quad S = \bigcup_{i=0}^{\infty} S_i.$$ 

Proof. First, let $x \in S_i$, where $i$ is the minimal index such that $x \in S_i$. Then $x$ was added to $S_i$ precisely because it is the affine image of the point $H^*_\theta$ within $H_{\theta_i} \subset [a, b] \in E_{i-1}$, and our inductive hypothesis ensures that $x \in H_\theta$. Similarly, if $x \in E$, then $x \in E_i$ for all $i$. Just as in the proof of Theorem 1, if $x \in H_\theta$; nonnegative sums are maintained through return times to an interval whose length is converging to zero.

Suppose now that $x \notin (E \cup S)$. Our inductive construction of the $E_i$ and $S_i$ then guarantees that $x \notin H_\theta$: by applying Corollary 3.1.3 it must fail to be in any of the affine images of $H_{\theta_i}$ that comprise the set $H_\theta$. \hfill \Box

Note that $S$ is a countable set, as each $S_i$ is finite (alternately, from Lemma 5.0.6 we know that all points in $S$ are separated from each other). Therefore, $\dim_H(H_\theta) = \dim_H(E)$. Define the functions

$$f_1(\theta) = 1 - 2 \left\lfloor \frac{a_1(\theta)}{2} \right\rfloor \theta,$$

$$f_2(\theta) = \begin{cases} 1 & (a_1(\theta) = 1 \mod 2) \\ a_2(\theta) + 1 & (a_1(\theta) = 0 \mod 2). \end{cases}$$

Then the set $E_{i+1}$ is formed from $E_i$ by letting each interval in $E_i$ be replaced with $f_2(\theta_i)$ intervals of length scaled by $f_1(\theta_i)$, all of which are separated from one another by gaps at least as large as the intervals themselves.

Lemma 5.1.1. For almost every $\theta$, the sequence $\{f_2(\theta_i)\}$ is not exceptionally irregular (4).
Proof. Recall that \( \theta_i = g^i(\theta) \), and \( g \) is ergodic with respect to the continuous measure \( \mu_g \). By the Birkhoff ergodic theorem, then, for almost every \( \theta \) we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(f_2(\theta_i)) = \int_{S^1} \log(f_2(\theta))d\mu_g,
\]

provided that \( \log(f_2) \in L^1(S^1, \mu_g) \). Since \( \mu_g \) is continuous, however, it suffices to show that \( \log(f_2) \in L^1(S^1, dx) \). Note that \( f_2 \) is constant on intervals of \( \theta \) whose continued fraction expansion begins \([2n, m, \ldots] \), and off these intervals \( f_2(\theta) = 1 \):

\[
\int_{S^1} \log f_2(\theta) d\theta = \sum_{n,m=1}^{\infty} \int_{\frac{2nm}{m+n+1}}^{\frac{2m+n+1}{m+n+1}} \log m d\theta
\]

\[
= \sum_{n,m=1}^{\infty} \frac{\log m}{(2nm + 1)(2n(m+1) + 1)} < \infty.
\]

Finally, note that if \( \{f_2(\theta_i)\} \) is exceptionally irregular, then the ergodic average of \( \log(f_2) \) cannot converge to a positive real number, while as the integrable function \( \log(f_2) \) is positive on a set of positive measure, the ergodic theorem guarantees such convergence almost surely. \( \square \)

**Lemma 5.1.2.** \( \log(f_1) \in L^1(S^1, d\mu_g) \).

Proof. The proof is similar to the computation in Lemma 5.1.1 using both (1a) and (1b) to show that

\[
\int_0^1 |\log(f_1(x))| dx < \sum_{k=1}^{\infty} \frac{\log(2k+1)}{(2k+1)(2k+2)} + \sum_{n,m=1}^{\infty} \frac{\log(2n+1) + \log(m+1)}{(2nm + 1)(2n(m+1) + 1)} \quad \square
\]

Combining Lemma 5.1.1 and Proposition 3.1, for almost every \( \theta \) we have

\[
\dim_H H_\theta = \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \log(f_2(\theta))}{\sum_{i=0}^{n-1} \log(f_1(\theta))}.
\]

By the ergodic theorem, however, for almost every \( \theta \) the right side converges as a proper limit to the ratio

\[
c = \frac{\int_{S^1} \log(f_2(\theta))d\mu_g}{\int_{S^1} \log(f_1(\theta))d\mu_g}.
\]

That \( c \neq 0 \) is seen by noting again that the nonnegative function \( \log(f_2) \) is in fact positive on a set of positive measure. By verifying the pointwise inequality \( -\log(f_1) > \log(f_2) \) for \( \theta < 1/2 \) (using (1a) for \( a_1 \) even and \( \log(f_2) = 0 \) for \( a_1 \) odd), we see \( c \neq 1 \) (both \( f_1(\theta) \) and \( f_2(\theta) \) equal one for \( \theta > 1/2 \)).

Note that we have proved something slightly stronger than the original statement of Theorem 2. The set \( H_\theta \) is always comprised of a (possibly empty) set of isolated points \( S \) together with a set \( E \), where \( E \) is almost surely of both Hausdorff and box dimension \( c \), where

\[
0 < c = \frac{\int_{S^1} \log(f_2(\theta))d\mu_g}{\int_{S^1} \log(f_1(\theta))d\mu_g} < 1.
\]

We also see that \( E \) is a perfect set (closed with no isolated points) for all \( \theta \) such that \( f_2(\theta_i) \neq 1 \) infinitely many times, which happens almost surely (via ergodicity
of $g$ and continuity of $\mu_g$). Note that $H_\theta$ is exactly this perfect set exactly when $S = \emptyset$, which happens exactly when $a_1(\theta_i) = 0 \mod 2$ for all $i$. These are exactly the $\theta$ for which all partial quotients of odd index are themselves even, which we have remarked are exactly the $\theta$ for which $H_\theta^* = \{0\}$:

**Corollary 5.1.1.** $H_\theta$ is a perfect set if and only if $H_\theta^* = \{0\}$, if and only if $a_{2i+1}(\theta) = 0 \mod 2$ for all $i = 0, 1, \ldots$

6. **Proof of Theorem 3**

Let $d \in [0, 1]$; we begin by showing that there is some $\theta$ such that $\dim_H(H_\theta) = d$. As the general proof is simplified by the assumption $d \neq 0$, let us take care of that particular case by hand. Let 

$$
\theta = [2!, 1, 3!, 1, 4!, 1, \ldots],
$$

so $\theta_i = [(i+2)!, 1, (i+3)!, 1, \ldots]$. As the sequence $f_2(\theta_i)$ is not exceptionally irregular (it is constantly two), we have (using (1a))

$$
\dim_H(H_\theta) = \liminf_{n \to \infty} \frac{n \log 2}{-\sum_{i=0}^{n-1} \log(1 - (i + 2)! \theta_i)} \leq \liminf_{n \to \infty} \frac{n \log 2}{\sum_{i=0}^{n-1} \log((i + 2)!)},
$$

which is clearly zero. Note that $H_\theta$ is a perfect set and therefore uncountable.

So now let us proceed on the assumption that $d \neq 0$. Define $m_i = 2^i$ and

$$
n_i = 1 + \left[ \frac{(m_i + 1)^{1/2} - 1}{2} \right].
$$

By construction,

$$
\log(m_i + 1) \log(2n_i) + \log(m_i + 1) \to d.
$$

That the values $\log(m_i/(m_i+1))$ vanish exponentially in $i$, and $\log(2n_i/(2n_i+1))$ are either constant (if $d = 1$) or vanish exponentially in $i$ (otherwise) further guarantees that

$$
\liminf_{n \to \infty} \frac{\sum_{i=0}^{n} \log(m_i + 1)}{\log(2n_i) + \log(m_i + 1)} = \liminf_{n \to \infty} \frac{\sum_{i=0}^{n} \log((2n_i + 1)(m_i + 1))}{\sum_{i=0}^{n} \log(2n_i + 1)(m_i + 1)}
$$

which is seen to imply

$$
\liminf_{n \to \infty} \frac{\sum_{i=0}^{n} \log(m_i + 1)}{\sum_{i=0}^{n} \log(2n_i m_i)} = \liminf_{n \to \infty} \frac{\sum_{i=0}^{n} \log(m_i + 1)}{\sum_{i=0}^{n} \log((2n_i + 1)(m_i + 1))}
$$

Let $\theta = [2n_0, m_0, 2n_1, m_1, \ldots]$, so $\theta_i = [2n_i, m_i, 2n_{i+1}, m_{i+1}, \ldots]$. As $m_i$ are not exceptionally irregular [4], we have

$$
\dim_H(H_\theta) = \liminf_{n \to \infty} \frac{\sum_{i=0}^{n} \log(m_i + 1)}{\log(1 - 2n_i \theta_i)}.
$$

The denominator may be approximated from above by $\log((2n_i + 1)(m_i + 1))$ and from below by $\log(2n_i m_i)$ using (1a):
\[
\liminf_{n \to \infty} \frac{\sum_{i=0}^{n} \log(m_i + 1)}{\sum_{i=0}^{n} \log((2n_i + 1)(m_i + 1))} \leq \dim_H(H_\theta) \leq \liminf_{n \to \infty} \frac{\sum_{i=0}^{n} \log(m_i + 1)}{\sum_{i=0}^{n} \log(2n_i m_i)},
\]
and by applying (7), the two sides have the same limit infimum, and by (6) this common value is \(d\).

As \(\dim_H(H_\theta)\) is a \(g\)-invariant function of \(\theta\) for every irrational \(\theta\), the theorem now follows from the fact that
\[
\bigcup_{i=0}^{\infty} g^{-i}(\theta)
\]
is a dense set for every irrational \(\theta\); given any finite string of initial partial quotients \(a_i(\theta)\), a sufficiently high power of \(g\) will have truncated all of them from the continued fraction expansion.

## 7. Examples

**Example 7.0.1.** For \(\theta = \sqrt{2} \mod 1 = [2, 2, 2, \ldots]\), \(H_\theta\) is a perfect set of both Hausdorff and box dimension
\[
\frac{\log(3)}{\log(3 + 2\sqrt{2})} \approx .623,
\]
and \(H_\theta^* = \{0\}\).

*Proof.* Since \(\theta_i = [2, 2, 2, \ldots]\) for all \(i\), we always form \(E_{i+1}\) from \(E_i\) by considering \(a_2 + 1 = 3\) subintervals scaled in length by \(1 - 2\theta = 3 - 2\sqrt{2} = (3 + 2\sqrt{2})^{-1}\). So the Hausdorff and box dimensions of this simple Cantor set are both given by
\[
\dim_H(H_\theta) = \dim_B(H_\theta) = \frac{\log 3}{\log(3 + 2\sqrt{2})}.
\]
We have already remarked that if all \(a_{2i+1} (\theta) = 0 \mod 2\) then \(H_\theta^* = \{0\}\). \(\square\)

It is tempting to think that the tail of the continued fraction expansion of \(\theta\) is what controls the development of the heavy set. This is not the case, however: it is dependent on orbit of \(\theta\) under \(g\), which might be significantly different than the orbit under the Gauss map \(\gamma\).

**Example 7.0.2.** Let \(\theta: \sqrt{2}/2 = [1, 2, 2, 2, \ldots]\). Then \(H_\theta\) consists of countably many isolated points and exactly one accumulation point, given by \(H_\theta^* = \{\theta/2\}\).

*Proof.* In this case, \(a_1(\theta_i) \in \{1, 3\}\) for all \(i\), so \(E_i = E_i^*\) for all \(i\). So the set \(E\) is in fact the singleton \(H_\theta^*\); every other point in \(H_\theta\) (which must be infinite by Corollary 4.0.3) therefore belongs to the set of isolated points \(S\). Note that even though \(a_i(\theta) = 2\) for all \(i \neq 1\), \(a_1(\theta_i) \neq 2\) for all \(i\).

The accumulation point of \(H_\theta\) is given by \(\cap E_i\), but for this choice of \(\theta\) we have \(E_i = E_i^*\), so the accumulation point is \(H_\theta^*\). To compute the value of this point, note that \(a_1(\theta_i) \neq 1\) if and only if
\[
\theta_i = [3, 2, 2, \ldots] = 1/(2 + \sqrt{2}),
\]
for which \(\delta_i = 1 - 2\theta_i = \sqrt{2} - 1\). Using Table I to construct the sets \(E_i = E_i^*\), we see that as every other \(a_1(\theta_i) = 1\), we alternate placing \(E_{i+1}\) at the top or bottom of \(E_i\), each time scaled by \(\delta = \sqrt{2} - 1\):
\[
E_0 = E_1 = [0, 1/2], \quad E_2 = E_3 = [1/2 - \delta/2, 1/2], \quad E_4 = E_5 = [1/2 - \delta, 1/2 + \delta^2/2], \ldots
\]
and the intersection is given by the geometric series

$$H^*_\theta = \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i (\sqrt{2} - 1)^i = \frac{1}{2(1 + (\sqrt{2} - 1))} = \frac{1}{2\sqrt{2}} = \frac{\theta}{2}. \quad \Box$$

**Example 7.0.3.** As a final pair of examples of general interest, consider

\[
\alpha = e - 2 = [1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2k, 1, \ldots]
\]

\[
\beta = \frac{e - 1}{e + 1} = [2, 6, 10, 14, 18, \ldots, 4k - 2, \ldots].
\]

Then \(\dim_H(H_\alpha) = 0, \dim_H(H_\beta) = 1/2\).

The value \(\beta\) is easier to consider: \(\beta_1 = [8i + 2, 8i + 6, \ldots]\). The sequence \(f_2(\beta_i)\) is not grow exceptionally irregular (it is an arithmetic sequence), so

\[
\dim_H(H_\beta) = \lim\inf_{n \to \infty} \sum_{i=0}^{n-1} \frac{\log(8i + 7)}{\log(8i + 2) + \sum_{i=0}^{n-1} \log(8i + 6)},
\]

which is bounded above and below (using (1a)) by

\[
\lim\inf_{n \to \infty} \frac{\sum_{i=0}^{n-1} \log(8i + 7)}{\sum_{i=0}^{n-1} \log(8i + 2) + \sum_{i=0}^{n-1} \log(8i + 6)} \geq \dim_H(H_\beta) \geq \lim\inf_{n \to \infty} \frac{\sum_{i=0}^{n-1} \log(8i + 7)}{\sum_{i=0}^{n-1} \log(8i + 3) + \sum_{i=0}^{n-1} \log(8i + 7)}.
\]

Both sides are seen to have proper limit one half. As all \(a_1(g^i\beta) = 0 \mod 2\), there are no isolated points in \(H_\beta\).

Now considering \(\alpha\), we must examine how \(H_\alpha\) decomposes into the sets \(E_i\) based on \(\alpha_i\). Referring to Table 2 we see how the sequence \(\alpha_i\) develops with a 5-step pattern with respect to the functions \(f_1\) and \(f_2\). We directly compute the lower box dimension of the set of non-isolated points in \(H_\alpha\):

\[
\dim_H H_\alpha \leq \dim_B(E) = \lim\inf_{n \to \infty} \frac{\sum_{k=0}^{n-1} (2 \log 2)}{\sum_{k=0}^{n-1} (\log(2k + 2) + \log(2) + \log(4k + 4))} = 0.
\]

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