An interesting property of the Friedman universes

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Abstract

We show in the paper that Friedman universes can be created from empty, flat Minkowskian spacetime by using suitable conformal rescaling of the spacetime metric.

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1 Friedman universes

Einstein equations and Cosmological Principle lead us together to Friedman universes. These universes give standard mathematical models of the real Universe.

Einstein equations

\[ G_{ik} := R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik} =: \beta T_{ik} \quad (1) \]

form system of the ten, 2-nd order quasilinear partial differential equations on ten unknown functions. Solving these equations under given initial and boundary conditions one obtains local geometry of the spacetime, i.e., \( g_{ik}(x) \rightarrow \Gamma^i_{kl}(x) \rightarrow R^i_{klm}(x) \) and local distribution and motion of matter, i.e., \( T_{ik}(x) \).

Here \( G_{ik} \) is the so-called Einstein tensor, \( T_{ik} \) is the matter energy-momentum tensor (the source of the gravitational field which is represented by tensor \( G_{ik} \) ), \( c \) is the velocity of light in vacuum, and \( G \) means Newtonian gravitational constant; \( g_{ik}(x) \) denote components of the metric tensor, and \( \Gamma^i_{kl}(x) \), \( R^i_{klm}(x) \) are the Levi-Civita connection and Riemannian curvature components respectively. \( R_{ik} \) mean components Ricci tensor and \( R \) is the so-called curvature scalar (See, eg., [1]). All Latin indices take values 0, 1, 2, 3.

The matter tensor \( T_{ik}(x) \) consists of \( g_{ik} \), \( u^i \), \( p \), \( \rho \) where \( u^i \), \( p \), \( \rho \) denote 4-velocity, pressure and density of matter respectively.

Cosmological Principle says that in the largest scale the real Universe is homogeneous and isotropic.

In the following we will use geometrized units in which \( G = c = 1 \). Friedman universes are cosmological solutions to the Einstein equations constrained by Cosmological Principle and they are foundation of the relativistic cosmology [1 2].

The line element \( ds^2 = g_{ik}(x)dx^i dx^k \) for these universes, called Friedman-Lemaître'-Robertson-Walker line element, in the comoving coordinates \( x^0 = t \), \( x^1 = \chi \), \( x^2 = \vartheta \), \( x^3 = \varphi \), reads

\[ ds^2 = dt^2 - R^2(t)[d\chi^2 + S^2(\chi)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)] \quad (2) \]

\(^2\text{We are modelling cosmological substrat by using an ideal (or perfect) fluid.}\)
where
\begin{align*}
S(\chi) &= \sin \chi, \quad \text{if } k = 1 \\
S \chi &= \chi, \quad \text{if } k = 0 \\
S(\chi) &= \sinh \chi, \quad \text{if } k = (-1).
\end{align*} \tag{3}

$t$ is the \textit{cosmic time}, i.e., the proper time for \textit{isotropic observers}, which are at rest in the coordinates $(t, \chi, \vartheta, \varphi)$.

An isotropic observer $O$ represents center of mass of a cluster of galaxies in real Universe. $R(t)$ is the so-called \textit{scale factor} (it scales spatial distances) and $k = 0, \pm 1$ means the \textit{normalized curvature} (curvature index) of the spatial sections $x^0 = t = \text{const}$.

If $k = 1$, then we have closed (spherical or elliptical) spatial sections, if $k = 0$ the geometry of the spatial section is flat, and if $k = (-1)$, then the geometry of spatial sections is hyperbolic.

Usually one chooses the moment $t = 0$ of the cosmic time $t$ when $R = 0$, i.e., usually one has $R(0) = 0$.

Einstein equations with perfect fluid (incompressible fluid, without any viscosity and not conducting heat) as source reduce, for the FLRW line element (2)-(3) to the \textit{Friedman equations}

\begin{align*}
\frac{3R^2}{R^2} + 3k &= \frac{\rho}{2\beta}, \tag{4} \\
\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} + \frac{k}{R^2} &= (-) \frac{p}{2\beta}. \tag{5}
\end{align*}

Here $\beta = 8\pi$ (We use geometrized units), $\rho = \rho(t)$ means the rest density of the fluid, and $p = p(\rho) = p(t)$ — its pressure. $\dot{R} := \frac{dR}{dt}$, and $\ddot{R} := \frac{d^2R}{dt^2}$.

Caloric equation $p = p(\rho)$ must be added to Friedman equations (4)-(5) in order to get a determined system on the three unknown functions: $R = R(t)$, $\rho = \rho(t)$, $p = p(t)$.

Usually one considers solutions to the Friedman equations (4)-(5) in the two extreme cases: $p = 0$ (dust universes or matter dominant universes, in short \textbf{MDU}), and $p = \frac{\rho}{3}$ (radiation dominant universes, in short \textbf{RDU}).

We will confine to solutions in these two extreme cases.

\textbf{Dust universes (MDU)} with $p = 0$:

\footnotetext{\[\text{A particle of this fluid represents a cluster of galaxies in real Universe.}\]}
1. k =1 (closed universe). In this case we have parametric solution

\[ R = M(1 - \cos \eta), \]
\[ t = M(\eta - \sin \eta). \]  

\[ 0 < \eta < 2\pi. \]  

2. k=0 (flat universe). In this case

\[ R = \left( \frac{9M}{2} t^2 \right)^{1/3}, \]  
\[ 0 < t < \infty. \]  

3. k =(-)1 (open universe). In this case we also have parametric solution

\[ R = M(\cosh \eta - 1), \]
\[ t = M(\sinh \eta - \eta), \]  
\[ 0 < \eta < \infty. \]  

Here \( \eta \) denotes a parameter and \( M = (4/3)\pi R^3 \rho \) is the first integral of the Friedman equations. Physically \( M \) is the mass contained inside of a “sphere” having volume \((4/3)\pi R^3\).

Radiation universes (\(\text{RDU}\)) with \( p = \frac{\rho}{3} \)

1. k = 1 (closed universe)

\[ R = \sqrt{(2bt - t^2)}, \]  
\[ b := \sqrt{\frac{8\pi C}{3}}, \]  
\[ 0 < t < 2b, \]  

where \( C = \rho R^4 = \text{const} > 0 \) is the first integral of the Friedman equations in this case.

2. k =0 (flat universe)

\[ R = \sqrt{2bt}, \]  
\[ 0 < t < \infty. \]  

3. k =(-)1 (open universe)

\[ R = \sqrt{(2bt + t^2)}, \]  
\[ 0 < t < \infty. \]
Having $R = R(t)$ one can find $\rho(t)$ from the first integrals and then $p = p(t)$ from caloric equations.

It is believed that one of the MDU correctly describes present stage of the Universe, and that one of the RDU correctly describes early Universe\footnote{The recent large–scale astronomical observations seem favorize an accelerated flat model.}. It is seen from (6)-(11) that the Friedman universes are singular at least in one moment of the cosmic time $t$ (In this moment $R = 0$). These singularities are inevitable in classical general relativity (Theorems by Hawking and Penrose, and Senovilla \footnote{The recent large–scale astronomical observations seem favorize an accelerated flat model.}); but “quantized general relativity” (loops quantum gravity) seems remove these singularities (Ashtekar, Bojowald and Lewandowski) \footnote{The recent large–scale astronomical observations seem favorize an accelerated flat model.}.

2 Conformal rescaling of metric and conformally flat spacetimes

By conformal rescaling of the metric $g$ we mean the following transformation (in established coordinates)

$$\hat{g}_{ab}(x) = \Omega^2(x)g_{ab}(x),$$

where the conformal factor $\Omega(x)$ is dimensionless, smooth and positive.

One can immediately get from (12) that

$$\hat{g}^{ab}(x) = \Omega^{-2}(x)g^{ab}(x),$$

and, after some tedious calculations one can obtain other useful transformational formulas \footnote{The recent large–scale astronomical observations seem favorize an accelerated flat model.}. For our future aims the following formulas will be needed

$$\hat{R}^b_d = \Omega^{-2}R^b_d + 2\Omega^{-1}(\Omega^{-1})_{,ac}g^{bc}$$
$$- \frac{1}{2}\Omega^{-4}(\Omega^2)_{,ac}g^{ac}\delta^b_d,$$

$$\hat{R} = \Omega^{-2}R - 6\Omega^{-3}\Omega_{,cd}g^{cd},$$

and

$$\hat{T}^k_i = \Omega^{-4}T^k_i.$$
Here \( a \) is covariant derivative with respect Levi-Civita connection of the metric in the initial gauge \( g_{ab}(x) \).

A spacetime is \textit{conformally flat} if there exist holonomic coordinates \((x^0 = t, \ x^1 = x, \ x^2 = y, \ x^3 = z)\) in which its line element \( ds^2 \) has the form

\[
\begin{align*}
    ds^2 &= \Omega^2(x^0, x^1, x^2, x^3)(dx^0^2 - dx^1^2 - dx^2^2 - dx^3^2) \\
    &\equiv \Omega^2(x^0, x^1, x^2, x^3)\eta_{ik}dx^i dx^k. \quad (16)
\end{align*}
\]

The

\[
\eta_{ik}dx^i dx^k = dx^0^2 - dx^1^2 - dx^2^2 - dx^3^2 \quad (17)
\]

means the line element of the empty, flat Minkowski spacetime in inertial coordinates.

The Theorem is true:

Necessary and sufficient condition of the conformal flatness of the \( 4 \)-dimensional (or higher, \( n > 4 \) dimensional) spacetime is vanishing its \textit{Weyl tensor} \( C_{abcd} \), where

\[
C_{abcd} := R_{abcd} + \frac{R(g_{ac}g_{bd} - g_{ad}g_{bc})}{(n - 1)(n - 2)} - \frac{(g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc})}{(n - 2)}. \quad (18)
\]

In the above formula \( R_{abcd} \) are components of the Riemann tensor, \( R_{ab} \) denote Ricci tensor components and \( R \) means Riemannian curvature scalar.

In the framework of general relativity Weyl’s tensor \( C_{abcd} \) describes free gravitational field (tidal forces).

An example of the conformally flat spacetimes give Friedman universes.

\section{Conformal transformation as Creator of the Friedman universes}

We have under conformal rescaling of the metric (12) if we use the formulas (14)-(15)

\[
\begin{align*}
    \hat{G}^d_b &= \hat{R}^d_b - \frac{1}{2}\delta^d_b\hat{R} = \Omega^{-2}G^d_b \\
    &= \frac{2}{\Omega}(\Omega^{-1})_{dce}g^{dc} + \frac{3}{\Omega^3}\delta^d_b\Omega_{ce}g^{ce} - \frac{1}{2\Omega^4}(\Omega^2)_{dce}\delta^d_c. \quad (19)
\end{align*}
\]
\[ \hat{T}^d_b = \Omega^{(-4)}T^d_b. \]  

(20)

By using Einstein equations in \textit{old gauge} \( g_{ik}(x) \)

\[ G^d_b = \beta T^d_b \]  

(21)

one can combine (19)-(20) to the form

\[ \hat{G}^d_b = \beta \Omega^2 \hat{T}^d_b + \beta \tilde{T}^d_b, \]  

(22)

where

\[ \tilde{T}^d_b := \frac{1}{\beta} \left[ \frac{2}{\Omega} (\Omega(-1))_{bc} g^{dc} \right. \]

\[ + \left. \frac{\delta^d_b}{\Omega^3} (3\Omega_{ce} g^{ce} - \frac{\Omega^2}{2\Omega} g^{ac}) \right]. \]  

(23)

(22) gives Einstein equations in \textit{new gauge} \( \hat{g}_{ik}(x) \).

The tensor \( \hat{T}^d_b(x) \) is the energy-momentum tensor of this matter \textit{which was created by conformal rescaling of the initial metric} \( g_{ik}(x) \) while the tensor \( \hat{T}^d_b(x) \) is transformed , following (20), the matter tensor \( T^d_b(x) \) which have already existed in the \textit{old gauge} \( g_{ik}(x) \).

One can rewrite (22) to the form

\[ \hat{G}^d_b = \beta \hat{T}^d_b, \]  

(24)

where

\[ \hat{T}^d_b := \Omega^2 \hat{T}^d_b + \tilde{T}^d_b. \]  

(25)

Of course, the \textit{total matter} tensor (25) is covariantly conserved.

Friedman universes are conformally flat. So, we can take in the case as “initial conditions”

\[ g_{ik}(x) = \eta_{ik}, \quad G^d_b = 0, \quad T^d_b = 0 \rightarrow \hat{T}^d_b(x) = 0, \]  

(26)

i.e., we can take empty Minkowskian spacetime as initial spacetime. Doing so, one can get the metric tensor of a Friedman universe in the form

\[ \hat{g}_{ik}(x) = \Omega^2(x) \eta_{ik}. \]  

(27)

where conformal factor \( \Omega(x) \) depends on Friedman universe.
Thus, metric $\hat{g}_{ik}(x)$ of a Friedman universe, i.e., whole geometry of a Friedman universe can be obtained from empty Minkowskian spacetime by a suitable conformal rescaling of the Minkowskian metric. Material content of this universe can be easily obtained from Einstein equations

$$\bar{T}^a_b := \frac{1}{\beta} \hat{G}^a_b, \quad (28)$$

where $\hat{G}^a_b(x)$ is Einstein tensor calculated from $\hat{g}_{ik}(x)$ or , immediately, from equations (23).

As an example we will consider a flat Friedman universe.

In this case

$$\hat{g}_{ik}(x) = \Omega^2(\tau)\eta_{ik} = \Omega^2(\tau)(d\tau^2 - dx^2 - dy^2 - dz^2) \quad (29)$$

with $\Omega(\tau) \equiv R(\tau)$. $\tau$ is here the so-called conformal time [6].

After a simple but tedious calculation one gets from (28) [or from (23)] that

$$\bar{T}^0_0 = \frac{3R'}{\beta R^4} (= \rho)$$

$$\bar{T}^1_1 = \bar{T}^2_1 = \bar{T}^3_1 = \frac{1}{\beta R^3} (2R'' - \frac{R'^2}{R}) (=-p). \quad (30)$$

Here prime denotes derivation with respect conformal time $\tau$.

Other components of the energy-momentum tensor $\bar{T}^a_b$ of the matter created by conformal rescaling (29) of the Minkowskian metric are vanishing.

For the flat dust Friedman universe we obtain

$$ds^2 = R^2(\tau)(d\tau^2 - dx^2 - dy^2 - dz^2), \quad (31)$$

where

$$R(\tau) = \frac{A^3}{9}\tau^2, \quad A = (6\pi\rho R^3)^{1/3} = const. \quad (32)$$

From that one gets

$$R' = \frac{2A^3}{9}\tau, \quad R'' = \frac{2A^3}{9}, \quad R''' = 0, \quad (33)$$

and higher derivatives also vanish.
In consequence, the material content of the universe following (30) reads
\[ \tilde{T}_0^0 = \rho = \frac{972}{\beta A_0 \tau^6}. \] (34)

The other components of the tensor $\tilde{T}_a^b$ are vanishing, i.e., $p = 0$ and stresses vanish (as it should be in the case).

Thus, we have correctly created flat, dust Friedman universe from empty Minkowskian spacetime by using the conformal transformation (31)-(32).

4 Conclusion

As we could see, Friedman universes can be created by a suitable conformal rescaling of the flat Minkowskian metric, i.e., these universes can be created from empty, flat Minkowskian spacetime by conformal transformations.

Therefore, we needn’t any “quantum gravity” in order to explain origin of the Friedman universes: classical conformal transformations are sufficient.

The analogical statement is, of course, correct for any other conformally flat spacetime.

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Interesująca własność modeli kosmologicznych Friedmana

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Streszczenie

W tej pracy pokazano, że modele kosmologiczne Friedmana, które są podstawą współczesnej kosmologii, można wykreować z pustej czasoprzestrzeni Minkowskiego przy pomocy odpowiedniej transformacji konforemnej.