On projectively equivalent metrics near points of bifurcation

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Abstract

Let Riemannian metrics \( g \) and \( \bar{g} \) on a connected manifold \( M^n \) have the same geodesics (considered as unparameterized curves). Suppose the eigenvalues of one metric with respect to the other are all different at a point. Then, by the famous Levi-Civita’s Theorem, the metrics have a certain standard form near the point.

Our main result is a generalization of Levi-Civita’s Theorem for the points where the eigenvalues of one metric with respect to the other bifurcate.

1 Introduction

1.1 Metrics with the same geodesics

Definition 1. Two Riemannian metrics \( g \) and \( \bar{g} \) on \( M^n \) are called \( \text{projectively equivalent} \), if they have the same geodesics considered as unparameterized curves.

Trivial examples of projectively equivalent metrics can be obtained by considering proportional metrics \( g \) and \( C \cdot g \), where \( C \) is a positive constant.

Definition 2. The Riemannian metrics \( g \) and \( \bar{g} \) are said to be \( \text{strictly non-proportional at} \ x \in M^n \), if the eigenvalues of \( g \) with respect to \( \bar{g} \) are all different at \( x \).

In the present paper we study projectively equivalent metrics which are strictly non-proportional at least at one point of the manifold. It is a very classical material. In 1865, Beltrami [1] found the first examples and formulated a problem of finding all pairs of projectively equivalent metrics. Locally the

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problem almost has been solved by Dini [6] for dimension two and Levi-Civita [16] for an arbitrary dimension: they obtained a local description of projectively equivalent metrics near the points where the eigenvalues of one metric with respect to the other do not bifurcate:

**Theorem 1 (Levi-Civita [16]).** Consider two Riemannian metrics on an open subset $U^n \subset M^n$. Suppose the metrics are strictly-non-proportional at every point $x \in U^n$.

Then the metrics are projectively equivalent on $U^n$, if and only if for every point $x \in U^n$ there exist coordinates $x_1, x_2, ..., x_n$ in some neighborhood of $x$ such that in these coordinates the metrics have the following model form:

$$ds^2_{g_{\text{model}}} = \Pi_1 dx_1^2 + \Pi_2 dx_2^2 + \cdots + \Pi_n dx_n^2,$$

$$ds^2_{\bar{g}_{\text{model}}} = \rho_1 \Pi_1 dx_1^2 + \rho_2 \Pi_2 dx_2^2 + \cdots + \rho_n \Pi_n dx_n^2,$$

(1) (2)

where the functions $\Pi_i$ and $\rho_i$ are given by

$$\Pi_i \overset{\text{def}}{=} (\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \cdots (\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i) \cdots (\lambda_n - \lambda_i),$$

$$\rho_i \overset{\text{def}}{=} \frac{1}{\lambda_1 \lambda_2 \cdots \lambda_{n-1} \lambda_i} \frac{1}{\lambda_i},$$

(3)

where, for any $i$, the function $\lambda_i$ is a smooth function of the variable $x_i$.

However, the global behavior of projectively equivalent metrics is not understood completely.

One of the main difficulties for global description of projectively equivalent metrics was the absence of local description of projectively equivalent metrics near the points where the eigenvalues of one metric with respect to the other bifurcate. As it has been proven in [23], if the manifold is not covered by the torus, it must have such points.

Our main result is a description of (strictly non-proportional at least at one point) projectively equivalent metrics near points where the eigenvalues of one metric with respect to the other bifurcate. The description needs some preliminary work and will be formulated at the very end of the paper, see Theorem 10 in Section 2.6. Here, we would like to note that the points of bifurcation behave quite regularly: the multiplicity of an eigenvalue is at most three; the points where the multiplicity of an eigenvalue is at least two are organized in totally-geodesic submanifold $U^{n-2}$ of co-dimension two; the points where the multiplicity of the eigenvalue is three are organized in a closed totally-geodesic hypersurface (submanifold of co-dimension one) of $U^{n-2}$.

This local description is the main tool of the following

**Theorem 2 (Topalov, Matveev).** Suppose $M^n$ is a connected closed manifold. Let Riemannian metrics $g$ and $\bar{g}$ on $M^n$ be projectively equivalent. Suppose they are strictly non-proportional at least at one point. Then the manifold can be finitely covered by a product of spheres.
We will neither prove nor explain this theorem in this paper. The proof will appear elsewhere, in a joint paper with P. Topalov.

As we will show in Section 2.3, see Corollary 5, a product of spheres always admits projective equivalent metrics strictly non-proportional at least at one point.

1.2 Geodesic rigidity conjecture

Conjecture 1. Suppose $M^n$ is closed connected. Let Riemannian metrics $g$ and $\bar{g}$ on $M^n$ be projectively equivalent and be nonproportional. Then the manifold can be covered by the sphere, or it admits a local product structure.

Definition of the notion “local product structure” is in Section 2.2. Roughly speaking, local product structure is a metric and two orthogonal foliations of complimentary dimensions such that locally all three objects look as they came from the direct product of two Riemannian manifold.

Conjecture 1 is true for dimensions two [19, 20, 26] and three [25, 27]. Theorem 2 shows that conjecture is true, if we assume in addition that the metrics are strictly-non-proportional at least at one point. The following theorem shows that the conjecture is true on the level of fundamental groups:

Theorem 3 ([28]). Let $M^n$ be a closed connected manifold. Suppose two non-proportional Riemannian metrics $g$, $\bar{g}$ on $M^n$ are projectively equivalent. If the fundamental group of $M^n$ is infinite, then there exist a local product structure on $M$.

If a manifold admits a local product structure, it admits two non-proportional projectively equivalent metrics. The direct product of manifolds admitting strictly-non-proportional projectively equivalent metrics also admit strictly-non-proportional projectively equivalent metrics (see Section 2.3 and Lemma 3 there). Every sphere admits strictly-non-proportional projectively equivalent metrics (essentially [11]; see also Corollary 5 in Section 2.3).

From the other side, not every manifold covered by the sphere can carry non-proportional projectively equivalent metrics. The lowest dimension where such phenomena is possible is three:

Theorem 4 ([25, 27]). A closed connected 3-manifold admits non-proportional projectively equivalent metrics if and only if it is homeomorphic to a lens space or to a Seifert manifold with zero Euler number.

Then, the Poincare homology sphere (which is certainly covered by $S^3$ [11]) does not admit non-proportional projectively equivalent metrics.

1.3 Integrable systems in the theory of projectively equivalent metrics

There exist two classical techniques for working with projectively equivalent metrics:
One is due to Dini [6], Lie [17], Levi-Civita [16], Weyl [36] and Eisenhart [7], and was actively developed by Russian and Japan geometry schools in the 50th-70th. Projective equivalence of two metrics is equivalent to a certain system of quasi-linear differential equation on the entries of these metrics. Combining this system with additional geometric assumptions (mostly written as a tensor formula) one can deduce topological restrictions on the manifold, see the surveys [30].

The second classical technique is due to E. Cartan [5] and was very actively developed by the French geometry school. The main observation is that, on the level of connections, projective equivalence is a very easy condition. In particular, given a connection, it is easy to describe all projectively equivalent connections. So the goal is to understand which of these connections are Levi-Civita connections of a Riemannian metric.

Unfortunately, both techniques are local, and all global (when the manifold is closed) result obtained with the help of these techniques require additional strong geometric assumptions (i.e., the metric is assumed to be Kähler or semisymmetric or generally semisymmetric or T-generalized semisymmetric or Einstein or of constant curvature or Ricci-flat or recurrent or admitting a concircular vector field or admitting a torse-forming vector field).

New methods for global investigation of projectively equivalent metrics have been suggested in [18, 35]. The main observation of [18, 35] is that the existence of $\bar{g}$ projectively equivalent to $g$ allows one to construct commuting integrals for the geodesic flow of $g$, see Theorem 6 in Section 2.1.

When the metrics are strictly non-proportional at least at one point, the number of functionally independent integrals equals the dimension of the manifold so that the geodesic flow of $g$ is Liouville-integrable and we can apply the well-developed machinery of integrable geodesic flows. In particular, in dimension two, Theorem 2 and Conjecture 1 follow directly from the description of metrics on surfaces with quadratically integrable geodesic flows from [2, 3, 12, 15].

For arbitrary dimension, the following theorem has been obtained in [29] (the three-dimensional version is due to [21, 24]) by combining the ideas applied in [33] for analytically-integrable geodesic flows with technique developed in [13] for quadratically-integrable geodesic flows:

**Theorem 5 ([29])**. Let $M^n$ be a connected closed manifold and $g$, $\bar{g}$ be projectively equivalent Riemannian metrics on $M^n$. Suppose there exists a point of the manifold where the number of different eigenvalues of $g$ with respect to $\bar{g}$ equals $n$. Then the following holds:

1. The first Betti number $b_1(M^n)$ is not greater than $n$.
2. The fundamental group of the manifold is virtually Abelian.
3. If there exists a point where the number of different eigenvalues of $g$ with respect to $\bar{g}$ is less than $n$, then $b_1(M^n) < n$.
4. If there exists no such point, then $M^n$ can be covered by the torus $T^n$. 
For dimensions three, Theorem 2 follows immediately from Theorem 5 modulo the Poincaré conjecture.

More precisely, by the second statement of Theorem 5, there exists a finite cover with the fundamental group isomorphic to $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}}_{k}$. By [8], the fundamental group of a compact 3-manifold has no subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, so $k$ is at most 3.

By a result of Reidemeister (which has been formulated and proven in [21] and [34]), it follows that $Z \times Z$ can not be the fundamental group of a closed 3-manifold. Thus, $k$ is either 3 or 1 or 0. If $k = 3$, then, by the fourth statement of Theorem 5, $M^3$ is covered by the torus $S^1 \times S^1 \times S^1$. If $k = 0$, the fundamental group of $M^3$ is finite so that (modulo the Poincaré conjecture) it is covered by $S^3$. If $k = 1$, then, modulo the Poincaré conjecture and in view of results of [37], $M^3$ is covered by the product $S^2 \times S^1$, see [24] for details.

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2 Local description near bifurcations

Within this section, we assume that Riemannian metrics $g$ and $\bar{g}$ are projectively equivalent, and that the manifold is connected. Our goal is to give a local description of (strictly-non-proportional at least at one point) projectively equivalent metrics $g$, $\bar{g}$ near the points where the eigenvalues of one metric with respect to the other bifurcate (i.e. are not all different). Sections 2.1, 2.2, 2.3 shows that it is sufficient to do these for two- and three-dimensional manifolds only; we do this in Sections 2.4, 2.5 (Theorems 8, 9) and combine all results in Section 2.6, see Theorem 10.

2.1 The multiplicity of an eigenvalue is not greater than three.

Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be Riemannian metrics on a manifold $M^n$. Consider the $(1,1)$-tensor $L$ given by the formula

$$L^i_j \overset{\text{def}}{=} \left( \frac{\det(\bar{g})}{\det(g)} \right)^{\frac{1}{n+1}} \bar{g}^{\alpha \beta} g_{\alpha j}.$$  

(4)
Then, $L$ determines the family $S_t$, $t \in R$, of $(1,1)$-tensors
\[
S_t \defeq \det(L - t \operatorname{Id}) (L - t \operatorname{Id})^{-1}.
\]

**Remark 1.** Although $(L - t \operatorname{Id})^{-1}$ is not defined for $t$ lying in the spectrum of $L$, the tensor $S_t$ is well-defined for every $t$. Moreover, $S_t$ is a polynomial in $t$ of degree $n - 1$ with coefficients being $(1,1)$-tensors.

We will identify the tangent and cotangent bundles of $M^n$ by $g$. This identification allows us to transfer the natural Poisson structure from $T^*M^n$ to $TM^n$.

**Theorem 6** ([18, 35]). If $g$, $\bar{g}$ are projectively equivalent, then, for every $t_1, t_2 \in R$, the functions
\[
I_{t_i} : TM^n \to R, \quad I_{t_i}(v) \defeq g(S_{t_i}(v), v)
\]
are commuting integrals for the geodesic flow of $g$.

Since $L$ is self-adjoint with respect to $\bar{g}$, the eigenvalues of $L$ are real. At every point $x \in M^n$, let us denote by $\lambda_1(x) \leq ... \leq \lambda_n(x)$ the eigenvalues of $L$ at the point.

**Remark 2.** The notation $\lambda$ for the eigenvalues of $L$ is compatible with the notations used inside Levi-Civita’s Theorem: in Levi-Civita coordinates from Theorem 1, the tensor $L$ is given by the diagonal matrix
\[
\operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n).
\]

**Definition 3.** A Riemannian manifold is called **geodesic**, if every two points can be connected by a geodesic.

**Corollary 1** ([26, 28]). Let $(M^n, g)$ be a geodesic Riemannian manifold. Let a Riemannian metric $\bar{g}$ on $M^n$ be projectively equivalent to $g$. Then, for every $i \in \{1, ..., n - 1\}$, for every $x, y \in M^n$, the following holds:

1. $\lambda_i(x) \leq \lambda_{i+1}(y)$.

2. If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.

In order to prove Corollary 1, we need the following technical lemma. For every fixed $v = (\xi_1, \xi_2, ..., \xi_n) \in T_x M^n$, the function (5) is a polynomial in $t$. Consider the roots of this polynomial. From the proof of Lemma 1 it will be clear that they are real. We denote them by
\[
t_1(x, v) \leq t_2(x, v) \leq ... \leq t_{n-1}(x, v).
\]
Lemma 1 ([26, 28]). The following holds for every $i \in \{1, \ldots, n-1\}$:

1. For every $v \in T_xM^n$,
   \[
   \lambda_i(x) \leq t_i(x, v) \leq \lambda_{i+1}(x).
   \]
   In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$, then $t_i(x, v) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. If $\lambda_i(x) < \lambda_{i+1}(x)$, then for every $\tau \in \mathbb{R}$ the Lebesgue measure of the set
   \[
   V_\tau \subset T_xM^n, \quad V_\tau \overset{\text{def}}{=} \{ v \in T_xM^n : t_i(x, v) = \tau \},
   \]
   is zero.

Proof of Lemma. By definition, the tensor $L$ is self-adjoint with respect to $g$. Then, for every $x \in M^n$, there exists "diagonal" coordinates in $T_xM^n$ where the metric $g$ is given by the diagonal matrix $\text{diag}(1, 1, \ldots, 1)$ and the tensor $L$ is given by the diagonal matrix $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then, the tensor (5) reads:

\[
S_t = \det(L - t\text{Id})(L - t\text{Id})^{-1} = \text{diag}(\Pi_1(t), \Pi_2(t), \ldots, \Pi_n(t)),
\]

where the polynomials $\Pi_i(t)$ are given by the formula

\[
\Pi_i(t) \overset{\text{def}}{=} (\lambda_1 - t)(\lambda_2 - t)\ldots(\lambda_{i-1} - t)(\lambda_{i+1} - t)\ldots(\lambda_{n-1} - t)(\lambda_n - t).
\]

Hence, for every $v = (\xi_1, \ldots, \xi_n) \in T_xM^n$, the polynomial $I_t(x, v)$ is given by

\[
I_t = \xi_1^2\Pi_1(t) + \xi_2^2\Pi_2(t) + \ldots + \xi_n^2\Pi_n(t). \tag{8}
\]

Evidently, the coefficients of the polynomial $I_t$ depend continuously on the eigenvalues $\lambda_i$ and on the components $\xi_i$. Then, it is sufficient to prove the first statement of the lemma assuming that the eigenvalues $\lambda_i$ are all different and that $\xi_i$ are non-zero. For every $\alpha \neq i$, we evidently have $\Pi_{\alpha}(\lambda_i) \equiv 0$. Then,

\[
I_{\lambda_i} = \sum_{\alpha=1}^n \Pi_{\alpha}(\lambda_i)\xi_{\alpha}^2 = \Pi_i(\lambda_i)\xi_i^2.
\]

Hence $I_{\lambda_i}(x, v)$ and $I_{\lambda_{i+1}}(x, v)$ have different signs. Hence, the open interval $[\lambda_i, \lambda_{i+1}]$ contains a root of the polynomial $I_t(x, v)$. The degree of the polynomial $I_t$ is equal $n - 1$; we have $n - 1$ disjoint intervals; each of these intervals contains at least one root so that all roots are real and the $i$th root lies between $\lambda_i$ and $\lambda_{i+1}$. The first statement of the lemma is proved.

Let us prove the second statement of Lemma. Suppose $\lambda_i < \lambda_{i+1}$. Let first $\lambda_i < \tau < \lambda_{i+1}$. Then, the set

\[
V_\tau \overset{\text{def}}{=} \{ v \in T_xM^n : t_i(x, v) = \tau \},
\]
consists of the points $v$ where the function $I_t(x, v) \overset{\text{def}}{=} (I_t(x, v))_{|t=\tau}$ is zero; then it is a nontrivial quadric in $T_xM^n \equiv \mathbb{R}^n$ and its measure is zero.

Let $\tau$ be one of the endpoints of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can suppose $\tau = \lambda_i$. Let $k$ be the multiplicity of the eigenvalue $\lambda_i$. Then, every coefficient $\Pi_i(t)$ of the quadratic form $S$ has the factor $(\lambda_i - t)^{k-1}$. Hence,

$$\hat{I}_t \overset{\text{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is a polynomial in $t$ and $\hat{I}_t$ is a nontrivial quadratic form. Evidently, for every point $v \in V_\tau$, we have $\hat{I}_\tau(v) = 0$ so that the set $V_\tau$ is a subset of a nontrivial quadric in $T_xM^n$ and its measure is zero. Lemma 1 is proved.

**Proof of Corollary 1.** The first statement of Corollary 1 follows immediately from the first statement of Lemma 1. Let us join the points $x, y \in M^n$ by a geodesic $\gamma : R \to M^n$, $\gamma(0) = x$, $\gamma(1) = y$. Consider the one-parametric family of integrals $I_t(x, v)$ and the roots

$$t_1(x, v) \leq t_2(x, v) \leq \ldots \leq t_{n-1}(x, v).$$

By Theorem 1, each root $t_i$ is constant on every orbit $(\gamma, \dot{\gamma})$ of the geodesic flow of $g$ so that

$$t_i(\gamma(0), \dot{\gamma}(0)) = t_i(\gamma(1), \dot{\gamma}(1)).$$

Using Lemma 1, we obtain

$$\lambda_i(\gamma(0)) \leq t_i(\gamma(0), \dot{\gamma}(0)), \quad \text{and} \quad t_i(\gamma(1), \dot{\gamma}(1)) \leq \lambda_{i+1}(\gamma(1)).$$

Thus $\lambda_i(\gamma(0)) \leq \lambda_{i+1}(\gamma(1))$ and the first statement of Corollary 1 is proved.

Let us prove the second statement of Corollary 1. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for every point $y$ of some subset $V \subset M^n$. Then, the value of $\lambda_i$ is a constant (independent of $y \in V$). Indeed, by the first statement of Corollary 1

$$\lambda_i(y_0) \leq \lambda_{i+1}(y_1) \quad \text{and} \quad \lambda_i(y_1) \leq \lambda_{i+1}(y_0),$$

so that $\lambda_i(y_0) = \lambda_i(y_1) = \lambda_{i+1}(y_1) = \lambda_{i+1}(y_0)$ for every $y_0, y_1 \in V$.

We denote this constant by $\tau$. Let us join the point $x$ with every point of $V$ by all possible geodesics. Consider the set $V_\tau \subset T_xM^n$ of the initial velocity vectors (at the point $x$) of these geodesics.

By the first statement of Lemma 1 for every geodesic $\gamma$ passing through at least one point of $V$, the value $t_i(\gamma, \dot{\gamma})$ is equal to $\tau$. By the second statement of Lemma 1, the measure of the set $V_\tau$ is zero. Since the set $V$ lies in the image of the exponential mapping of the set $V_\tau$, the measure of the set $V$ is also zero. Corollary 1 is proved.

**Corollary 2.** Suppose $M^n$ connected. Let $g$ and $\tilde{g}$ be projectively equivalent on $M^n$. Suppose they are strictly non-proportionally at least at one point. Then the following holds:

1. The metrics are strictly non-proportionally at almost every point of $M^n$. 


2. At every point of \( M^n \), the multiplicity of any eigenvalue of \( L \) is at most three.

**Proof:** Suppose the metrics are simply non-proportional at \( x_1 \). Since the manifold is connected, for every point \( x \in M^n \) there exists a sequence of open convex balls \( B_i \subset M^n \) such that the first ball \( B_1 \) contains the point \( x_1 \), the last ball \( B_m \) contains the point \( X \) and the intersection \( B_i \cap B_{i+1}, i < m, \) is not empty. By the second statement of Corollary \([1]\) the metrics are strictly non-proportional at almost every points of \( B_1 \). Then there are strictly non-proportional at almost every points of \( B_1 \cap B_2 \). Then, there exists a point of \( B_2 \) where the metrics are strictly non-proportional. Hence, by the second statement of Corollary \([1]\) they are strictly non-proportional at almost every points of \( B_2 \). Iterating this argumentation \( m - 2 \) times, we obtain that the metrics are strictly non-proportional at almost every points of \( B_m \). Since the point \( x \) is arbitrary, the metrics are strictly non-proportional at almost every points of \( M^n \).

Then, the multiplicity of an eigenvalue at the point \( x \) is not greater than 3: indeed, if we suppose that \( \lambda_k(x) = \lambda_{k+3}(x) \), then, by the first statement of Corollary \([1]\) \( \lambda_{k+1} = \lambda_{k+2} \) at each point of \( B_m \), which contradicts that the metrics are strictly non-proportional at almost every points of \( B_m \). Corollary \([2]\) is proved.

### 2.2 Splitting procedure

The goal of the next two sections is to show that for a local description of projectively equivalent metrics strictly non-proportional at least at one point it is sufficient to describe them on two- and three-dimensional manifold only. We will need the following statement.

**Corollary 3 ([4]).** Suppose the Riemannian metrics \( g, \bar{g} \) on \( M^n \) are projectively equivalent. Then the Nijenhuis torsion of \( L \) vanishes.

A self-contained proof of Corollary \([3]\) is found in \([4]\); here we prove the theorem assuming that the manifold is connected and the metrics are strictly non-proportional at least at one point, which is sufficient for our goals.

**Proof:** Nijenhuis torsion is a tensor, so it is sufficient to check its vanishing at almost every point. By Corollary \([1]\) almost every point of \( M^n \) is stable. In the Levi-Civita coordinates from Theorem \([1]\) the tensor \( L \) is given by the diagonal matrix

\[
\text{diag}(\lambda_1, \ldots, \lambda_n).
\]

Since the eigenvalue \( \lambda_i \) depends on the variable \( x_i \) only, the Nijenhuis torsion of \( L \) is zero \([10]\). Corollary \([3]\) is proved.

**Definition 4.** A local-product structure on \( M^n \) is the triple \((h, B_r, B_{n-r})\), where \( h \) is a Riemannian metrics and \( B_r, B_{n-r} \) are transversal foliations of dimensions \( r \) and \( n - r \), respectively (it is assumed that \( 1 \leq r < n \)), such that every point \( p \in M^n \) has a neighborhood \( U(p) \) with coordinates

\[
(\bar{x}, \bar{y}) = ((x_1, x_2, \ldots, x_r), (y_{r+1}, y_{r+2}, \ldots, y_n))
\]
such that the $x$-coordinates are constant on every leaf of the foliation $B_{n-r} \cap U(p)$, the $y$-coordinates are constant on every leaf of the foliation $B_r \cap U(p)$, and the metric $h$ is block-diagonal such that the first $(r \times r)$ block depends on the $x$-coordinates and the last $((n-r) \times (n-r))$ block depends on the $y$-coordinates.

A model example of manifolds with local-product structure is the direct product of two Riemannian manifolds $(M^r_1, g_1)$ and $(M^{n-r}_2, g_2)$. In this case, the leaves of the foliation $B_r$ are the products of $M^r_1$ and the points of $M^{n-r}_2$, the leaves of the foliation $B_{n-r}$ are the products of the points of $M^r_1$ and $M^{n-r}_2$, and the metric $h$ is the product metric $g_1 + g_2$.

Below we assume that

(a) The metrics $g$ and $\bar{g}$ are projectively equivalent on a connected $M^n$.

(b) They are strictly-non-proportional at least at one point of $M^n$.

(c) There exists $r, 1 \leq r < n$, such that $\lambda_r < \lambda_{r+1}$ at every point of $M^n$.

We will show that (under the assumptions (a,b,c)) we can naturally define two local-product structures $(h, B_r, B_{n-r})$ and $(\bar{h}, B_r, B_{n-r})$ such that the restrictions of $h$ and $\bar{h}$ to every leaf are projectively equivalent and strictly non-proportional at least at one point.

At every point $x \in M^n$, denote by $V^r_x$ the subspaces of $T_xM^n$ spanned by the eigenvectors of $L$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_r$. Similarly, denote by $V^{n-r}_x$ the subspaces of $T_xM^n$ spanned by the eigenvectors of $L$ corresponding to the eigenvalues $\lambda_{r+1}, \ldots, \lambda_n$. By assumption, for any $i, j$ such that $i \leq r < j$, we have $\lambda_i \neq \lambda_j$ so that $V^r_x$ and $V^{n-r}_x$ are two smooth distributions on $M^n$. By Corollary 3 the distributions are integrable so that they define two transversal foliations $B_r$ and $B_{n-r}$ of dimensions $r$ and $n-r$, respectively.

By construction, the distributions $V_r$ and $V_{n-r}$ are invariant with respect to $L$. Let us denote by $L_r$, $L_{n-r}$ the restrictions of $L$ to $V_r$ and $V_{n-r}$, respectively. We will denote by $\chi_r$, $\chi_{n-r}$ the characteristic polynomials of $L_r$, $L_{n-r}$, respectively.

Consider the $(1,1)$-tensor

$$C \overset{\text{def}}{=} ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

and the metric $h$ given by the relation

$$h(u, v) \overset{\text{def}}{=} g(C^{-1}(u), v)$$

for any vectors $u, v$. (In the tensor notations, the metrics $h$ and $g$ are related by $g_{ij} = h_{\alpha\beta}C^\alpha_\beta$).

Consider the $(1,1)$-tensor

$$\bar{C} \overset{\text{def}}{=} \left( \frac{(-1)^r}{\det(L_{n-r})} \chi_r(L) + \frac{1}{\det(L_r)} \chi_{n-r}(L) \right)$$
and the metric $\bar{h}$ given by the relation

$$\bar{h}(u, v) \overset{\text{def}}{=} \bar{g}(C^{-1}(u), v)$$

for any vectors $u, v$.

**Lemma 2.** The following statements hold:

1. The triples $(h, B_r, B_{n-r})$ and $(\bar{h}, B_r, B_{n-r})$ are local-product structures on $M^n$.

2. For any leaf of $B_r$, the restrictions of $h$ and $\bar{h}$ to the leaf are projectively equivalent and strictly non-proportional at least at one point. For any leaf of $B_{n-r}$, the restrictions of $h$ and $\bar{h}$ to the leaf are projectively equivalent and strictly non-proportional at least at one point.

**Proof:** First of all, $h$ and $\bar{h}$ are well-defined Riemannian metrics. Indeed, take an arbitrary point $x \in M^n$. At the tangent space to this point, we can find a coordinate system where the tensor $L$ and the metric $g$ are diagonal. In this coordinate system, the characteristic polynomials $\chi_r, \chi_{n-r}$ are given by

$$\chi_r = (-1)^r(t - \lambda_1)(t - \lambda_2)\ldots(t - \lambda_r),$$

$$\chi_{n-r} = (\lambda_{r+1} - t)(\lambda_{r+2} - t)\ldots(\lambda_n - t).$$

Then, the $(1,1)$-tensors

$$C = ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

$$\bar{C} = ((-1)^r \det(L_{n-r})\chi_r(L) + \det(L_r)\chi_{n-r}(L))$$

are given by the diagonal matrices

$$\text{diag}\left( \prod_{j=r+1}^n (\lambda_j - \lambda_1), \ldots, \prod_{j=r+1}^n (\lambda_j - \lambda_r), \prod_{j=1}^r (\lambda_{r+1} - \lambda_j), \ldots, \prod_{j=1}^r (\lambda_n - \lambda_j) \right),$$

$$\text{diag}\left( \prod_{j=r+1}^n (\lambda_j - \lambda_1), \ldots, \prod_{j=r+1}^n (\lambda_j - \lambda_r), \prod_{j=1}^r (\lambda_{r+1} - \lambda_j), \ldots, \prod_{j=1}^r (\lambda_n - \lambda_j) \right).$$

We see that the tensors are diagonal and all its diagonal components are positive. Then, the tensors $C^{-1}, \bar{C}^{-1}$ are well-defined and $h, \bar{h}$ are Riemannian metrics.

By construction, $B_r$ and $B_{n-r}$ are well-defined transversal foliations of supplementary dimensions. In order to prove Lemma 2 we need to verify that, locally, the triples $(h, B_r, B_{n-r})$ and $(\bar{h}, B_r, B_{n-r})$ are as in Definition 4 that the restriction of $h$ to a leaf is projectively equivalent to the restriction of $\bar{h}$,

$$\text{diag}\left( \prod_{j=r+1}^n (\lambda_j - \lambda_1), \ldots, \prod_{j=r+1}^n (\lambda_j - \lambda_r), \prod_{j=1}^r (\lambda_{r+1} - \lambda_j), \ldots, \prod_{j=1}^r (\lambda_n - \lambda_j) \right).$$

We see that the tensors are diagonal and all its diagonal components are positive. Then, the tensors $C^{-1}, \bar{C}^{-1}$ are well-defined and $h, \bar{h}$ are Riemannian metrics.

By construction, $B_r$ and $B_{n-r}$ are well-defined transversal foliations of supplementary dimensions. In order to prove Lemma 2 we need to verify that, locally, the triples $(h, B_r, B_{n-r})$ and $(\bar{h}, B_r, B_{n-r})$ are as in Definition 4 that the restriction of $h$ to a leaf is projectively equivalent to the restriction of $\bar{h}$,
and that the restriction of $h$ to a leaf is strictly non-proportional (at least at a point) to the restriction of $h$.

It is sufficient to verify the first two statements at almost every point of $M^n$. More precisely, it is known that the triple $(h, B_r, B_{n-r})$ is a local-product structure if and only if the foliations $B_r$ and $B_{n-r}$ are orthogonal and totally geodesic \[31\]. Clearly, if the foliations and the metric are globally given and smooth, if the foliations are orthogonal and totally-geodesic at almost every point then they are orthogonal and totally-geodesic at every point.

Similarly, if a foliation and two metrics are globally-given and smooth, if the restriction of the metrics to the leaves of the foliation is projectively equivalent almost everywhere then it is so at every point.

By Corollary \[1\] at almost every point of $M^n$ the eigenvalues of $L$ are different. Consider the Levi-Civita coordinates $x_1, ..., x_n$ from Theorem \[1\]. In the Levi-Civita coordinates, $L$ is given by (7). Then, by constructions of the foliations $B_r$ and $B_{n-r}$, the coordinates $x_1, ..., x_r$ are constant on every leaf of the foliation $B_{n-r}$, the coordinates $x_{r+1}, ..., x_n$ are constant on every leaf of the foliation $B_r$.

Using (10,11), we see that, in the Levi-Civita coordinates, $h, \bar{h}$ are given by

\[
h(x, \dot{x}) = \bar{\Pi}_1 dx_1^2 + \cdots + \bar{\Pi}_r dx_r^2 + \bar{\Pi}_{r+1} dx_{r+1}^2 + \cdots + \bar{\Pi}_n dx_n^2,
\]

\[
\bar{h}(x, \dot{x}) = \bar{\rho}_1 \bar{\Pi}_1 dx_1^2 + \cdots + \bar{\rho}_r \bar{\Pi}_r dx_r^2 + \bar{\rho}_{r+1} \bar{\Pi}_{r+1} dx_{r+1}^2 + \cdots \bar{\rho}_n \bar{\Pi}_n dx_n^2,
\]

where the functions $\bar{\Pi}_i, \bar{\rho}_i$ are as follows: for $i \leq r$, they are given by

\[
\bar{\Pi}_i \overset{\text{def}}{=} \frac{(\lambda_i - \lambda_1)(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)(\lambda_r - \lambda_i)}{\lambda_i (\lambda_1 \lambda_2 \cdots \lambda_r)},
\]

\[
\bar{\rho}_i \overset{\text{def}}{=} \frac{1}{\lambda_i (\lambda_1 \lambda_2 \cdots \lambda_r)}.
\]

For $i > r$, the functions $\bar{\Pi}_i, \bar{\rho}_i$ are given by

\[
\bar{\Pi}_i \overset{\text{def}}{=} \frac{1}{\lambda_i (\lambda_{r+1} \lambda_2 \cdots \lambda_n)},
\]

\[
\bar{\rho}_i \overset{\text{def}}{=} \frac{1}{\lambda_i (\lambda_{r+1} \lambda_2 \cdots \lambda_n)}.
\]

We see that the restrictions of the metrics on the leaves of the foliations have the form from Levi-Civita’s Theorem and, therefore, are projectively equivalent. We see that the metrics $h, \bar{h}$ are block-diagonal with the first $r \times r$ block depending on the variables $x_1, ..., x_r$ and the second $(n-r) \times (n-r)$ block depending on the remaining variables, so that $(h, B_r, B_{n-r})$ and $(\bar{h}, B_r, B_{n-r})$ are local-product structure.

The last thing to show is that the restrictions of the metrics to every leaf are strictly non-proportional at least at one point. Suppose it is not so; that is, there exists a leaf (say, of foliation $B_r$) and $k$, $0 < k < r$ such that $\lambda_k = \lambda_{k+1}$ at each point of the leaf. Then, by the first statement of Corollary \[1\] the
eigenvalues $\lambda_k, \lambda_{k+1}$ are constant on the leaf. Since the Nijenhuis torsion of $L$ is zero, the eigenvalues $\lambda_k, \lambda_{k+1}$ are constant along the leaves of the foliation $B_{n-r}$ [10]. Then $\lambda_k = \lambda_{k+1}$ at each point of a neighborhood of the leaf, which contradicts Corollary [2]. Thus the restrictions of the metrics to every leaf are strictly non-proportional at least at one point. Lemma [2] is proved.

2.3 Gluing procedure

Let $B_r$ and $B_{n-r}$, $0 < r < n$, be transversal foliations of supplementary dimensions $r$ and $n-r$ on a connected $M^n$. Suppose there exist Riemannian metrics $h$ and $\tilde{h}$ such that

(i) The triples $(h, B_r, B_{n-r})$ and $(\tilde{h}, B_r, B_{n-r})$ are local-product structures.

(ii) For every fiber of $B_r$ the restrictions of $h, \tilde{h}$ to the fiber are projectively equivalent and strictly-non-proportional at least at one point. For every fiber of $B_{n-r}$ the restrictions of $h, \tilde{h}$ to the fiber are projectively equivalent and strictly-non-proportional at least at one point.

Let us denote by $L_r$ (by $L_{n-r}$, respectively) the tensor (4) constructed for the restrictions of $h, \tilde{h}$ to the tangent spaces of the leaves of $B_r$ (of $B_{n-r}$, respectively). Assume in addition that

(iii) for any point of $M^n$, the eigenvalues of $L_{n-r}$ are greater than that of $L_r$.

At every point of $M^n$, let us denote by $\chi_r, \chi_{n-r}$ the characteristic polynomials of $L_r, L_{n-r}$, respectively. Let us denote by $P_r : T_x M^n \to T_x B_r, P_{n-r} : T_x M^n \to T_x B_{n-r}$ the orthogonal projections to the tangent spaces of the foliations $B_r, B_{n-r}$. (Since the foliations are orthogonal with respect to both metrics, it does not matter what metric we take here.) There exists a unique $(1,1)$-tensor $L$ such that $L \circ P_r = L_r$ and $L \circ P_{n-r} = L_{n-r}$.

Consider the $(1,1)$-tensor

$$C \overset{\text{def}}{=} ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

(14)

and the metric $g$ given by the relation

$$g(u, v) \overset{\text{def}}{=} h(C(u), v)$$

(15)

for any vectors $u, v$. (In the tensor notations, the metric $g$ is given by $h_{i\alpha}C^\alpha_j$.)

Consider the $(1,1)$-tensor

$$\bar{C} \overset{\text{def}}{=} \left(\frac{(-1)^r}{\det(L_{n-r})}\chi_r(L) + \frac{1}{\det(L_r)}\chi_{n-r}(L)\right)$$

(16)

and the metric $\bar{h}$ given by the relation

$$\bar{g}(u, v) \overset{\text{def}}{=} \bar{h}(\bar{C}(u), v)$$

(17)

for any vectors $u, v$. 

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Lemma 3. The metrics $g$ and $\bar{g}$ are projectively equivalent on $M^n$ and are strictly-non-proportional at almost every point of $M^n$.

Proof: First of all, $g$ and $\bar{g}$ are well-defined Riemannian metrics. More precisely, at the tangent space to every point we can find a coordinate system where $L$ is given by the diagonal matrix $\text{diag}(\lambda_1, ..., \lambda_n)$ assuming

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_r < \lambda_{r+1} \leq ... \leq \lambda_n.$$

In this coordinate system, the characteristic polynomials $\chi_r$ and $\chi_{n-r}$ are given by (9). Hence, the tensors $C, \bar{C}$ are given by (10,11), and the condition (iii) guarantees that the metrics $g$ and $\bar{g}$ are well-defined Riemannian metrics.

Let us show that the metrics $g$ and $\bar{g}$ are projectively equivalent. It is sufficient to check it at almost every point. Clearly, at almost every point of $M^n$ the eigenvalues of the tensor $L$ are all different. Since by Levi-Civita’s Theorem, the restriction of $h, \bar{h}$ to the leaves of $B_r, B_{n-r}$ have the model form (12,13), we obtain that the metrics $h, \bar{h}$ are given by (12,13). Using that $C, \bar{C}$ are given by (10,11), we obtain that the metrics $g, \bar{g}$ have precisely the form from Levi-Civita’s Theorem, and, therefore, are projectively equivalent. Lemma 3 is proved.

Remark 3. The notation $C, \bar{C}, L, \lambda_i, g$ and $\bar{g}$ used in this section are not misleading and are compatible with the notations in Section 2.2. More precisely, if we take $g, \bar{g}$ satisfying assumptions (a,b,c), construct the metrics $h, \bar{h}$ and foliations $B_r, B_{n-r}$, then (by Lemma 2) the triples $(h, B_r, B_{n-r})$, $(\bar{h}, B_r, B_{n-r})$ satisfy conditions (i, ii, iii). Moreover, the tensor $L$ given by (4) coincide with the tensor $L$ constructed in this section. Therefore, the tensors (14,16) coincide with the tensors $C, \bar{C}$ from Section 2.2, and, therefore, the metrics constructed by (15,17) coincide with the initial metrics $g, \bar{g}$.

Remark 4. Levi-Civita’s Theorem 1 follows from Lemmas 2,3.

Proof of Remark 4. First of all, by direct calculation it is possible to verify that if the metrics are given by the Levi-Civita’s model form (12), then they are projectively equivalent.

In order to prove that strictly non-proportional projectively equivalent metrics have (locally) the form (12), we use induction in dimension of the manifold.

For one-dimensional manifold it is nothing to prove; suppose Levi-Civita’s Theorem is true for dimension $n-1$. Let us prove that, for dimension $n$, strictly non-proportional projectively equivalent metrics are locally given by (12).

If the metrics are strictly non-proportional at $p$, then $\lambda_1 < \lambda_2$ in a small neighborhood of $p$. Put $r = 1$ and construct the local-direct-product structures $(h, B_1, B_{n-1})$ and $(\bar{h}, B_1, B_{n-1})$. By definition 4, there exists a smaller neighborhood of $p$ such that the foliations $B_1$ and $B_{n-1}$ look as they came from the direct product of the interval and the $(n-1)$-dimensional disk. Let us choose on leaf of the foliation $B_1$ and one leaf of the foliation $B_{n-1}$.

Since the leaf of $B_1$ is one-dimensional, there exists a coordinate $x_1$ there such that the restriction of the metrics $h, \bar{h}$ are respectively given by $dx_1^2$ and
Since the restrictions of the metrics $h, \bar{h}$ to the leaf of $B_{n-1}$ are strictly non-proportional and projectively equivalent, by Levi-Civita’s Theorem, there exists a coordinate system $x_2, \ldots, x_n$ there such that the restrictions of the metrics $h, \bar{h}$ to the leaf of $B_{n-1}$ are respectively given by
\[
\hat{\Pi}_2 dx_2^2 + \cdots + \hat{\Pi}_n dx_n^2, \quad \text{and} \quad \frac{1}{\hat{\rho}_2} \hat{\Pi}_2 dx_2^2 + \cdots + \frac{1}{\hat{\rho}_n} \hat{\Pi}_n dx_n^2,
\]
where the functions $\hat{\rho}_i$ and $\hat{\Pi}_i$ are related to the functions $\rho_i$ and $\Pi_i$ from Levi-Civita’s Theorem by the formulae
\[
\hat{\rho}_i = \lambda_1 \rho_i, \quad \hat{\Pi}_i = \frac{\Pi_i}{\lambda_i - \lambda_1}.
\]

Because of the local-product structure, these coordinates of the leaf of $B_1$ and on the leaf of $B_{n-1}$ give us a coordinate system in the neighborhood of $p$. By direct calculations, $-\chi_1 = (t - \lambda_1), \chi_{n-1} = (\lambda_2 - t)(\lambda_3 - t)\ldots(\lambda_n - t)$. Then,
\[
-\chi_1(L) = \text{diag}(0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots, \lambda_n - \lambda_1),
\]
\[
\chi_{n-1}(L) = \text{diag}(\Pi_1, 0, 0, \ldots, 0),
\]
and the determinants $\det(L_1), \det(L_{n-1})$ are equal to $\lambda_1, \lambda_2\lambda_3\ldots\lambda_n$, respectively.

Using that the metric $h, \bar{h}$ are the products of their restrictions to the leaf of $B_1$ and the leaf of $B_{n-1}$, and in view of Remark 3, we obtain that the metrics $g, \bar{g}$ are precisely in the model form \((12)\).

### 2.4 Dimension 2

The goal of this section is to give the local description of projectively equivalent metrics (on surfaces) near the points where the eigenvalues of $L$ bifurcate. In dimension two, the inverse of Theorem 6 also takes place:

**Theorem 7** ([18, 22, 19]) Let $g, \bar{g}$ be Riemannian metrics on $M^2$. They are projectively equivalent if and only if the function
\[
F : TM^2 \to R, \quad F(\xi) = \left( \frac{\det(g)}{\det(\bar{g})} \right)^{\frac{1}{2}} \bar{g}(\xi, \xi)
\]
is an integral of the geodesic flow of $g$.

We see that the integral $F$ is quadratic in velocities. Thus the existence of an integral quadratic in velocities (for the geodesic flow of $g$) allows one to construct a metric projectively equivalent to $g$ (at least locally).

Now, in in the two-dimensional case, the local description of metrics with quadratically integrable geodesic flows has been obtained in \([24]\), see also \([3]\) (Basing on the technique developed in \([13]\)). Combining this description with Theorem 7 we obtain the following
Theorem 8. Let \( g \) and \( \bar{g} \) be projectively equivalent on a (2-dimensional) connected surface \( M^2 \). Suppose they are non-proportional at least at one point. Assume they are proportional at \( p \in M^2 \). Then, precisely one of the following possibilities takes place:

1. There exist coordinates \( u, v \) in a neighborhood of \( p \), and there exists function \( \lambda \) of one variable such that the metrics have the following model form
   \[
   ds_\bar{g}^2 = \frac{2\lambda(u + \rho) - \lambda(u - \rho)}{\rho}(du^2 + dv^2) \quad (19)
   \]
   \[
   ds_\bar{g}^2 = \left(\frac{\lambda(u + \rho) - \lambda(u - \rho)}{\rho\lambda(u - \rho)\lambda(u + \rho)}\right)^2 \left[\left(\frac{\rho\lambda(u + \rho) + \lambda(u - \rho)}{\rho\lambda(u + \rho) - \lambda(u - \rho)}\right)(du^2 + dv^2)
   \right.
   - \left. udu^2 - 2vdudv + udv^2\right] \quad (20)
   \]
   where \( \rho \overset{\text{def}}{=} \sqrt{u^2 + v^2} \).

2. There exist coordinates \( u, v \) in a neighborhood of \( p \), there exists a functions \( f \) of one variable, and there exists a positive constant \( \lambda_1 \) such that the metrics have the following model form
   \[
   ds_\bar{g}^2 = f(\rho^2)(du^2 + dv^2) \quad (21)
   \]
   \[
   ds_\bar{g}^2 = \frac{f(\rho^2)}{\lambda_1(\lambda_1 + \lambda_1\rho^2f(\rho^2))} \left(\left(1 + f(\rho^2)v^2\right)du^2 - 2f(\rho^2)uvdudv + (1 + f(\rho^2)u^2)dv^2\right), \quad (22)
   \]
   where \( \rho \overset{\text{def}}{=} \sqrt{u^2 + v^2} \).

3. There exist coordinates \( u, v \) in a neighborhood of \( p \), there exists function \( f \) of one variable, and there exists a positive constant \( \lambda_2 \) such that the metric \( g \) has the form \( (21) \) and the metric \( \bar{g} \) is given by
   \[
   ds_\bar{g}^2 = \frac{f(\rho^2)}{\lambda_2(\lambda_2 - \lambda_2f(\rho^2))^2} \left(\left(1 - f(\rho^2)v^2\right)du^2 + 2f(\rho^2)uvdudv + (1 - f(\rho^2)u^2)dv^2\right), \quad (23)
   \]
   where \( \rho \overset{\text{def}}{=} \sqrt{u^2 + v^2} \).

Theorem 8 is true also in the other direction: if Riemannian metrics are given by formulae \( (19,20) \) or \( (21,22) \) or \( (21,23) \), then they are projectively equivalent. Of course, in order the formulae to define Riemannian metrics (at least in a small neighborhood of \( (0,0) \)), the functions \( f \) and \( \lambda \) must be smooth, positive and satisfy the conditions \( \lambda'(0) > 0, f'(0) > 0 \).
Remark 5. The most natural coordinate system for projectively equivalent metrics near a point of bifurcation has singularity at the point:

1. In the elliptic coordinate system $x_1^2 = \rho - u$, $x_2^2 = \rho + u$ (which has a singularity at $(0, 0)$), the metrics (19, 20) are given (up to multiplication by 4) by

$$ (\lambda(x_1) - \lambda(x_2))(dx_1^2 + dx_2^2), $$

$$ \left( \frac{1}{\lambda(x_2)} - \frac{1}{\lambda(x_1)} \right) \left( \frac{dx_1^2}{\lambda(x_1)} + \frac{dx_2^2}{\lambda(x_2)} \right), $$

which is precisely the Levi-Civita form for dimension two. In particular, the eigenvalues $\lambda_1(u, v)$, $\lambda_2(u, v)$ of $L$ are equal to $\lambda(u - \rho)$, $\lambda(u + \rho)$, respectively.

2. In the polar coordinate system $u = e^r \cos(\phi)$, $v = e^r \sin(\phi)$ the metrics (21, 22) and (23) are given by

$$ (e^{2r} f(e^{2r})) \left( dr^2 + d\phi^2 \right), $$

$$ \frac{1}{\lambda_1} \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \lambda_1 e^{2r} f(e^{2r})} \right) \left( \frac{d\phi^2}{\lambda_1} + \frac{dr^2}{\lambda_1 + \lambda_1 e^{2r} f(e^{2r})} \right), $$

$$ \frac{1}{\lambda_2} \left( \frac{1}{\lambda_2 - \lambda_2 e^{2r} f(e^{2r})} - \frac{1}{\lambda_2} \right) \left( \frac{dr^2}{\lambda_2 - \lambda_2 e^{2r} f(e^{2r})} + \frac{d\phi^2}{\lambda_2} \right), $$

respectively. We see that they are in the Levi-Civita form (up to the factors $\lambda_1$ and $\lambda_2$), and that the eigenvalues of $L$ for the pair of metrics (21, 22) are $\lambda_1(u, v) = \lambda_1$, $\lambda_2(u, v) = \lambda_1 + \lambda_1 \rho^2 f(\rho^2)$, and the eigenvalues of $L$ for the pair of metrics (21, 23) are $\lambda_1(u, v) = \lambda_2$, $\lambda_2(u, v) = \lambda_2$.

We see that the first possibility (from Theorem 8) for projectively equivalent metrics is characterized by the condition that $\lambda_1$, $\lambda_2$ of $L$ for are non-constant, and the second possibility is characterized by the condition $\lambda_1$ is constant, $\lambda_2$ is not constant, and the third possibility is characterized by the condition $\lambda_2$ is constant, $\lambda_1$ is not constant.

2.5 Dimension 3

The goal of this section is to describe (strictly-non-proportional at least at a point) projectively equivalent metrics on a 3-manifold near the points where the metrics are proportional. This will be made in Theorem 9. In order to prove it, we need Corollary 4 and Lemma 4.

Corollary 4. Let $g$ and $\bar{g}$ be projectively equivalent on connected $M^3$ and be strictly non-proportional at least at one point of $M^3$. Let $p \in M^3$. Suppose in a neighborhood of $p \in M^3$ the eigenvalue $\lambda_2$ is constant, and suppose $\lambda_1(p) <$
\( \lambda_2 = \lambda_3(p) \). Then, there exists a neighborhood of \( p \) with coordinates \( x_1, x_2, x_3 \) where the metrics have the following model form:

\[
 ds^2_g = (\lambda_1(x_1) - 1)(\lambda_1(x_1) - 1 - \rho^2 f(\rho^2)) dx_1^2 + f(\rho^2)(1 + x_2^2 f(\rho^2) - \lambda_1(x_1)) dx_2^2 \\
 - 2f(\rho^2)x_2 x_3 dx_2^2 dx_3^2 + (1 + x_3^2 f(\rho^2) - \lambda_1(x_1)) dx_3^2, \\
= \frac{(\lambda_1(x_1) - 1)(\lambda_1(x_1) - 1 - \rho^2 f(\rho^2))}{\lambda_1^2(x_1)(1 + \rho^2 f(\rho^2))} dx_1^2 \\
+ \frac{f(\rho^2)}{\lambda_1(x_1)(1 + \rho^2 f(\rho^2))} \left( 1 - \lambda_1(x_1) \frac{1 + x_3^2 f(\rho^2)}{1 + \rho^2 f(\rho^2)} \right) dx_2^2 \\
+ \frac{2\lambda_1(x_1)x_2 x_3}{1 + \rho^2 f(\rho^2)} dx_2 dx_3 \\
+ \left( 1 - \lambda_1(x_1) \frac{1 + x_3^2 f(\rho^2)}{1 + \rho^2 f(\rho^2)} \right) dx_3^2, \\
(24)
\]

where \( \rho = \sqrt{x_2^2 + x_3^2} \); \( f \) and \( \lambda_1 \) are functions of one variable and \( \lambda_2 \) is a positive constant.

\textbf{Remark 6.} In the cylindrical coordinates \( x_1 = u_1, x_2 = u_3 \cos(u_2), v = u_3 \sin(u_2) \) the metrics \( g, \bar{g} \) almost have Levi-Civita form \([12]\).

Proof of Corollary 4 Since \( \lambda_1 < \lambda_2 \) at the point \( p \), there exists a neighborhood of \( p \) where \( \lambda_1 < \lambda_2 \). Put \( r = 1 \), apply the splitting procedure from Section 2.2 and construct the metrics \( h \) and \( \bar{h} \) and the foliations \( B_1, B_2 \). By Lemma 2, there exists a (possible, smaller) neighborhood of \( p \) isomorphic to a direct product of an interval and a 2-disc such that the metric \( h \) is the product metric \( g_1 + g_2 \) (where the metric \( g_1 \) is a metric on the interval and \( g_2 \) is a metric on the disc) and the metric \( \bar{h} \) is the product metric \( \bar{g}_1 + \bar{g}_2 \) (where the metric \( \bar{g}_1 \) is a metric on the interval and \( \bar{g}_2 \) is a metric on the disc projectively equivalent to the metric \( g_2 \) and strictly-non-proportional to \( g_2 \) at least at one point). Since \( \lambda_2 \) is constant, the smallest eigenvalue of the tensor \( L \) constructed for the metrics \( g_2, \bar{g}_2 \) is constant. Since \( \lambda_2 = \lambda_3 \) at \( p \), the metrics \( g_2, \bar{g}_2 \) are proportional at one point and, therefore, are given by \([21, 22]\) in an appropriate coordinate system \( x_2, x_3 \). There evidently exists a coordinate \( x_1 \) on the interval such that the metrics \( g_1, \bar{g}_1 \) are given by \( dx_1^2, dx_2^2 + \frac{dx_3^2}{\lambda_1^2(x_1)} \) for an appropriate function \( \lambda \). Applying the gluing procedure from Section 2.3 we obtain precisely the form \([24, 25]\). Corollary 4 is proved.

\textbf{Lemma 4.} Consider the Riemannian metrics given by the formulae \([24, 25]\) in a neighborhood of the point \((0, 0, 0)\). Then,

- the plane \( P := \{(x_1, x_2, x_3) : x_2 = 0\} \) is a totally geodesic submanifold;
- the eigenvalue \( \lambda_3 \) equals \( \lambda_2 \) precisely at the points where \( x_2 = x_3 = 0 \);
- the action of \( S^1 \) by the rotations
  \[
  (x_1, x_2, x_3) \mapsto (x_1, x_2 \cos(\phi) - x_3 \sin(\phi), x_2 \sin(\phi) + x_3 \cos(\phi))
  \]

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preserves both metrics, and at every point \( x \notin \{(x_1, x_2, x_3): x_2 = x_3 = 0 \} \), its orbits are tangent to the eigenspace of \( L \) corresponding to \( \lambda_2 \).

- at every point of the plane \( P \), the vector \( \left( \frac{\partial}{\partial x_2} \right) \) is the eigenvector of \( L \) with the eigenvalue \( \lambda_2 \).

The lemma could be proved by direct computations. Actually, the first statement follows from the fact that the symmetry \((x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)\) is evidently an isometry; the second, third and the fourth statements follow from the observation that splitting-gluing procedure using in construction of metrics [(24,25)] is invariantly given in terms of metrics and therefore inherits all symmetries of the metrics \( g_1 \) and \( g_2 \).

**Theorem 9.** Let \( g \) and \( \bar{g} \) be projectively equivalent on a (3-dimensional) connected manifold \( M^3 \). Suppose they are strictly non-proportional at least at one point. Assume they are proportional at \( p \in M^3 \). Then, there exist coordinates \( u_1, u_2, u_3 \) in a neighborhood of \( p \), a function \( \lambda \) of one variable and a positive constant \( C \), such that the metrics have the following model form

\[
ds_g^2 = 2 \frac{\lambda(\rho + u_1) - \lambda(u_1 - \rho)}{\rho} \left( du_1^2 + \left( \sqrt{u_2^2 + u_3^2} \right)^2 \right) + C(\lambda(0) - \lambda(u_1 - \rho)) \frac{(\lambda(u_1 + \rho) - \lambda(0))}{\rho \lambda(u_1 - \rho) \lambda(u_1 + \rho)} (u_3 du_2 - u_2 du_3)^2
\]

\[
ds_{\bar{g}}^2 = \left( \frac{\lambda(u_1 + \rho) - \lambda(u_1 - \rho)}{\rho \lambda(u_1 - \rho) \lambda(u_1 + \rho)} \right)^2 \left( \frac{\lambda(u_1 + \rho) + \lambda(u_1 - \rho)}{\lambda(u_1 + \rho) - \lambda(u_1 - \rho)} \right) \left( du_1^2 + \left( \sqrt{u_2^2 + u_3^2} \right)^2 \right)
- u_1 du_1^2 - 2 \sqrt{u_2^2 + u_3^2} du_1 \sqrt{u_2^2 + u_3^2} + u_1 \left( \sqrt{u_2^2 + u_3^2} \right)^2
+ C \frac{C(\lambda(0) - \lambda(u_1 - \rho))(\lambda(u_1 + \rho) - \lambda(0))}{\lambda(0)^2 \lambda(u_1 - \rho) \lambda(u_1 + \rho)} (u_3 du_2 - u_2 du_3)^2
\]

where \( \rho = \sqrt{u_1^2 + u_2^2 + u_3^2} \).

**Remark 7.** The most natural coordinate system here are cylindrical-elliptic:

\[
x_1 = \rho - u_1; \ x_2 = \sqrt{C} \arccos \left( \frac{u_2}{\sqrt{u_2^2 + u_3^2}} \right); \ x_3 = \rho + u_1,
\]

where the metrics have the Levi-Civita form [(24,25)] (with \( \lambda_1 = \lambda(\rho - u_1) \), \( \lambda_2 = \lambda(0) \), and \( \lambda_3 = \lambda(\rho + u_1) \).) In particular, if two Riemannian metrics are given in the form [(24,25)], they are projectively equivalent. If \( \lambda \) is smooth and positive with \( \lambda'(0) > 0 \), then the formulae [(24,25)] define Riemannian metrics.

Proof: Let the metrics be proportional at \( x_0 \). Take a small \( \epsilon \)-ball \( B_\epsilon \) (in the metric \( g \)) with the center in \( x_0 \). If \( \epsilon \) is small enough, \( x_0 \) is the only point of the ball where the metrics are proportional. Indeed, suppose they are proportional.
at three points $y_1, y_2, y_3$ of the ball. Then, for almost every point $x$ of the ball, there exist three geodesics $\gamma_1, \gamma_2, \gamma_3$ such that $\gamma_i(0) = y_i$, $\gamma_i(1) = x$ and the velocity vectors $\dot{\gamma}_1(1), \dot{\gamma}_2(1), \dot{\gamma}_3(1)$ are mutually transversal. Let us show that $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$ at $x$. Indeed, by Lemma [1] for every $i = 1, 2, 3$, we have that $\lambda_2$ is a double root of $I_i(\gamma_i(1), \dot{\gamma}_i(1))$. At $T_xB$, consider the coordinate system where $g$ and $L$ are given by diagonal matrices $\text{diag}(1, 1, 1)$ and $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$, respectively. In this coordinate system, the polynomial $I_i$ is given by
\[
(\lambda_2 - t)(\lambda_3 - t)\xi_i^2 + (\lambda_1 - t)(\lambda_3 - t)\xi_i^2 + (\lambda_1 - t)(\lambda_2 - t)\xi_i^2. \tag{28}
\]
Then, the components $(\xi_1, \xi_2, \xi_3)$ of the velocity vectors $\dot{\gamma}_i(1)$ satisfy
\[
\begin{cases}
(\lambda_1(x) - \lambda_2)(\lambda_3(x) - \lambda_2)\xi_i^2 = 0 \\
(\lambda_1(x) - \lambda_2)(\xi_i^2 + \xi_j^2) + (\lambda_3(x) - \lambda_2)(\xi_i^2 + \xi_j^2) = 0. \tag{29}
\end{cases}
\]
If $\lambda_1(x) < \lambda_2 < \lambda_3(x)$, the solutions of (29) are organized into two intersected straight lines so that the velocity vectors $\dot{\gamma}_i(1)$ are not mutually transversal. Then, at almost every point of $B$, we have $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$, which contradicts Corollary [1].

Consider the set
\[
U \overset{\text{def}}{=} \{ x \in B : (\lambda_1(x) - \lambda_2)(\lambda_3(x) - \lambda_2) = 0 \}.
\]

Lemma 5. The set $U$ is a totally geodesic connected submanifold of $B$ of dimension 1. (In other words, $U$ is a geodesic segment).

Proof: First of all, there exists $x_1 \in B$ where precisely two eigenvalues of $L$ coincide. Indeed, take the $\frac{\pi}{2}$-sphere $S_2$ with the center in $x_0$ and consider the exponential mapping $\exp : T_{x_0}B \to B$. Suppose there exists no point of $S_2$ where precisely two eigenvalues of $L$ coincide. Then, at every point of the sphere $S_2$ the eigenspace of $L$ corresponding to $\lambda_2$ has dimension one. Let us show that it is tangent to the sphere. It is sufficient to show that it is orthogonal to the geodesic connecting the point of the sphere with the point $x_0$. Denote the initial velocity vector of the geodesic by $\xi$. At the tangent space at this point, consider the coordinate system where $g$ and $L$ are given by diagonal matrices $\text{diag}(1, 1, 1)$ and $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$, respectively. In this coordinate system, the polynomial $I_i$ is given by (28). Since the geodesic goes through the point where $\lambda_1 = \lambda_2 = \lambda_3$, by Lemma [1] $\lambda_2$ is the double root $I_i(\xi)$. Then the components $\xi_1, \xi_2, \xi_3$ of $\xi$ satisfy the system (29). From the first equation of the system, in view of $\lambda_1 < \lambda_2 < \lambda_3$, we obtain that the component $\xi_2$ is zero so that $\xi$ is orthogonal to the eigenspace of $L$ corresponding to $\lambda_2$. Thus, the eigenspaces of $L$ corresponding to $\lambda_2$ give us a smooth one-dimensional distribution on the 2-sphere which is impossible because the Euler characteristic of the sphere.

Finally, there exists $x_1 \in S_2$ where precisely two eigenvalues of $L$ coincide.

Denote by $\gamma$ the geodesic going through $x_1$ and $x_0$. Let us show that at every point of this geodesic at least two eigenvalues of $L$ coincide.

We assume that $\gamma(1) = x_1$ and $\gamma(0) = x_0$. 

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At each tangent space, we can find coordinates such that \( g \) and \( L \) are given by the diagonal matrices \( \text{diag}(1,1,1) \) and \( \text{diag}(\lambda_1, \lambda_2, \lambda_3) \), respectively. Consider the function
\[
I_t \overset{\text{def}}{=} - \left( \frac{d}{dt} I \right).
\]
For every fixed \( t \), the function \( I'_t \) is an integral of the geodesic flow. By (28),
\[
I'_t = (\lambda_2 - t + \lambda_3(x) - t)\xi_1 + (\lambda_1(x) - t + \lambda_3(x) - t)\xi_2 + (\lambda_2 - t + \lambda_1(x) - t)\xi_3,
I'_t = (\lambda_1(x) - \lambda_2) (\xi_1^2 + \xi_2^2) + (\lambda_3(x) - \lambda_2) (\xi_1^2 + \xi_3^2).
\]
(30)
By Lemma 1, \( \lambda_2 \) is a double-root of the polynomial \( I_t(\gamma, \dot{\gamma}) \). It follows from Lemma 1 that the leading coefficient of the polynomial \( I'_t(\gamma, \dot{\gamma}) \) is not zero.
Let us prove that the differential \( dI_{\lambda_2}' \) vanishes at each point of the geodesic orbit \((\gamma, \dot{\gamma})\). Since \( I'_{\lambda_2} \) is an integral, it is sufficient to show this at the point \((\gamma(1), \dot{\gamma}(1))\) only. We have,
\[
I'_{\lambda_1}(x) = (\lambda_3(x) - \lambda_1(x))(\xi_1^2 + \xi_2^2) + (\lambda_2 - \lambda_1(x))(\xi_1^2 + \xi_3^2).
\]
We see that the function \( I'_{\lambda_1}(x) \) is non-negative. At the point \((\gamma(1), \dot{\gamma}(1))\), it coincides with \( I'_{\lambda_2} \) and, therefore, is zero. Then, it has a minimum at the point \((\gamma(1), \dot{\gamma}(1))\), and its differential vanishes.
Let us show that the differential of the function \( I'_{\lambda_1}(x) - I'_{\lambda_2} \) also vanishes at the point \((\gamma(1), \dot{\gamma}(1))\). Indeed, the function \( I'_t \) is a linear polynomial in \( t \) with non-zero leading coefficient at the point \((\gamma(1), \dot{\gamma}(1))\). Since \( \lambda_1(x) < \lambda_2 \), then the function \( I'_{\lambda_1}(x) - I'_{\lambda_2} \) is either everywhere positive or everywhere negative. Since it vanishes at \((\gamma(1), \dot{\gamma}(1))\), the differential of \( I'_{\lambda_1}(x) - I'_{\lambda_2} \) vanishes at the point \((\gamma(1), \dot{\gamma}(1))\).
Thus the differential \( dI'_{\lambda_2} \) is zero at each point of the geodesic orbit \((\gamma, \dot{\gamma})\).
At each point of \( \gamma \), the components \( \frac{\partial I'_{\lambda_2}}{\partial \xi_i} \) of \( dI'_{\lambda_2} \) are
\[
2(\lambda_3 - \lambda_2)\xi_1, \ 2(\lambda_3 - \lambda_2 - \lambda_2 + \lambda_1)\xi_2, \ 2(\lambda_1 - \lambda_2)\xi_3.
\]
Since the differential vanishes, all its components are equal to zero. Then, \( \lambda_2 = \lambda_3 \) or \( \lambda_1 = \lambda_2 \) or \( \xi_2 \neq 0 \).
On the other hand, by (28), using that \( I_{\lambda_2} = 0 \), we see that \( (\lambda_2 - \lambda_1(x))(\lambda_2 - \lambda_3(x)) = 0 \) or \( \xi_2 = 0 \).
Finally, every point of \( \gamma \) lies in \( U \).
Now let us prove that \( \gamma \) actually coincides with \( U \). Assume \( \gamma \) does not coincide with \( U \); that is, there exist a point \( x_2 \) where \( \lambda_1 = \lambda_2 \) or \( \lambda_2 = \lambda_3 \) not lying on the geodesic \( \gamma \). Then, for almost every \( x \in B_t \), there exist three geodesics \( \gamma_1, \gamma_2, \gamma_3 \) such that \( \gamma_i(1) = x \) and \( \gamma_i(0) \in U \) and the vectors \( \dot{\gamma}_i(1) = x \) are linearly independent. From other side, the components \( \xi_1, \xi_2, \xi_3 \) of each of these vectors (in the coordinate system where \( g \) and \( L \) are given by diagonal matrices \( \text{diag}(1,1,1) \) and \( \text{diag}(\lambda_1, \lambda_2, \lambda_3) \)) satisfy the first equation of the system (29). Then, \( x \in U \) which contradicts Lemma 1. Finally, \( U \) coincides with \( \gamma \).
Now let us show that there exists a smooth vector field \( v_2 \) on \( B_t \) such that:
1. $v_2$ is a Killing vector field with respect to both metrics.

2. $g(v_2, v_2) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$ (in particular, $v_2$ vanishes at $U$).

3. $Lv_2 = \lambda v_2$.

4. the integral curves of $v_2$ are homeomorphic to the circle.

Let us prove that there exists $v_2$ on $B_e \setminus U$ satisfying (2), (3). Since at every point of $B_e \setminus U$ the eigenspace of $L$ corresponding to $\lambda_2$ is one-dimensional, at every point there exist precisely two vectors satisfying conditions (2), (3).

Take a point $x_2 \in B_e \setminus U$ and choose a vector satisfying conditions (2), (3) at $x_2$. Then we can smoothly choose the vector field satisfying (2), (3) locally, and, therefore, along any curve. Thus it is sufficient to show that the result does not depend on the curve. Take two curves $c_1$, $c_2$ connecting $x_2$ and an arbitrary point $x \in B_e \setminus U$.

Take a point $x_1 \in U$, $x_1 \neq x_0$. Without loss of generality we can assume that $\lambda_1(x_1) < \lambda_2(x_1) = \lambda_3(x_1)$. By Corollary 4 in a small neighborhood $W$ of $x_1$ the metrics are given by the model form (21,22,23). Clearly, there exists curves $c_1$, $c_2$ such that $c_1$ is homotop to $c_1$, $c_2$ is homotop to $c_2$, and $c_1$ coincides with $c_2$ outside of $W$. As it follows from Corollary 4 we can choose $v_2$ satisfying (2), (3) in $W$. Since the curves coincide outside of $W$, there exists $v_2$ on $B_e \setminus U$ satisfying (2), (3).

In Levi-Civita coordinates from Theorem 1 the vector field $v_2$ equals $\pm \frac{\partial}{\partial x_2}$ and therefore is Killing with respect to the metrics $g$ and $\bar{g}$. Put the vector field $v_2$ equal to zero at every point of $U$. Since any isometry of $g$ is a diffeomorphism, the vector field is smooth everywhere. Consider the exponential mapping from the point $x_0$. As we have proven before, the $v_2$ is tangent to the images of the spheres on $T_{x_0}B_e$, and, therefore, generates a Killing vector field on every such sphere. Then its integral curves are closed [2].

Consider the point $x_1$ where $\lambda_1(x_1) < \lambda_2(x_1) = \lambda_3(x_1)$. In the neighborhood of the point there exists coordinates $x_1, x_2, x_3$ where the metrics are given by the form (24,25). Consider the exponential mapping $exp : T_{x_1}B_e \rightarrow B_e$, the 2-plane $P \subset T_{x_1}B_e$ spanned by $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial x_3}$, and the image $exp(P)$ of this plane under the exponential mapping.

Since $\epsilon$ is small, $exp(P)$ is a two-dimensional submanifold of $B_e$. Let us show that $exp(P)$ is totally geodesic, (so that the restrictions of the metrics to $exp(P)$ are projectively equivalent), that the eigenvalues of the tensor $L$ constructed for the restriction of the metrics to $exp(P)$ are $\lambda_1, \lambda_3$, and that at every point $exp(P)$ is orthogonal to the vector $v_2$.

It is sufficient to prove both facts at almost every point of $exp(P)$. Take a point $p_2 \in exp(P)$ where $\lambda_1 < \lambda_2 < \lambda_3$. Denote by $\gamma$ the geodesic connecting this point with $p_1$, $\gamma(1) = p_1$, $\gamma(2) = p_2$. In a neighborhood of $p_1$, the listed above statements are true because of Corollary 4. In a neighborhood of another points, the listed above statements are true because of Levi-Civita’s Theorem.

Finally, since $exp(P)$ is totally geodesic, the restrictions of the metrics $g$ and $\bar{g}$ to $exp(P)$ are projectively equivalent. They are proportional at $x_0$, they
are non-proportional at every other point, and the eigenvalues of $L$ constructed for the restriction are non-constant. By Theorem 5 the metrics are given by (21,22). Finally, the metrics $g$ and $\bar{g}$ are as in Theorem 9.

### 2.6 General case and realization

Let $M_1^{n_1}$, $M_2^{n_2}$ be connected manifolds. Suppose Riemannian metrics $g_1, \bar{g}_1$ on $M_1^{n_1}$ are projectively equivalent and are strictly-non-proportional at least at one point. Suppose Riemannian metrics $g_2, \bar{g}_2$ on $M_2^{n_2}$ are projectively equivalent and are strictly-non-proportional at least at one point. We denote by $L_1$ the tensor (4) corresponding to $g_1, \bar{g}_1$, and by $L_2$ the tensor (4) corresponding to $g_2, \bar{g}_2$. Assume in addition the eigenvalues of $L_1$ are less than the eigenvalues of $L_2$. Consider the direct product $M_1^{n_1} \times M_2^{n_2}$ with the canonical transversal foliations $B_{n_1}$ and $B_{n_2}$. (The leaves of the foliation $B_{n_1}$ are the products of the point of $M_1^{n_1}$ and a point of $M_2^{n_2}$, the leaves of the foliation $B_{n_2}$ are the products of the point of $M_1^{n_1}$ and $M_2^{n_2}$.) Consider the Riemannian metric $h \equiv g_1 + g_2$ and $\bar{h} \equiv \bar{g}_1 + \bar{g}_2$ on the product $M_1^{n_1} \times M_2^{n_2}$. It is easy to see that the foliations and the metrics satisfy the assumptions (i,ii,iii) of Section 2.3. Then, by Lemma 3 the metrics $g$ and $\bar{g}$ given by (15,17) are projectively equivalent and are strictly-non-proportional at least at one point of the manifold. Thus, given two triples $(M_1^{n_1}, g_1, \bar{g}_1)$, $(M_2^{n_2}, g_2, \bar{g}_2)$, we constructed the triple $(M_1^{n_1} \times M_2^{n_2}, g, \bar{g})$. We will denote this operation by “⊕”:

$$(M_1^{n_1}, g_1, \bar{g}_1) \oplus (M_2^{n_2}, g_2, \bar{g}_2) = (M_1^{n_1} \times M_2^{n_2}, g, \bar{g}).$$

It is easy to check that the operation is associative:

$$(((M_1^{n_1}, g_1, \bar{g}_1) \oplus (M_2^{n_2}, g_2, \bar{g}_2)) \oplus (M_3^{n_3}, g_3, \bar{g}_3)) = ((M_1^{n_1}, g_1, \bar{g}_1) \oplus ((M_2^{n_2}, g_2, \bar{g}_2) \oplus (M_3^{n_3}, g_3, \bar{g}_3))).$$

Consider connected manifolds $M_1^{k_1}, M_2^{k_2}, \ldots, M_m^{k_m}$ with projectively equivalent metric $g_1, \bar{g}_1; g_2, \bar{g}_2; \ldots; g_m, \bar{g}_m$, respectively. Assume the metrics are strictly-non-proportional at least at one point. We denote by $L_i$ the tensor (4) constructed for $g_i, \bar{g}_i$, $i = 1, \ldots, m$. Assume in addition that, for any $i < j$, the eigenvalues of $L_i$ are less than the eigenvalues of $L_j$. Then, we can canonically construct projectively equivalent metric on the product of the manifolds, and these metrics are strictly-non-proportional at least at one point.

**Corollary 5.** A product of spheres admits projectively equivalent metrics which are strictly-non-proportional at least at one point.

**Proof:** Basically we will show that if connected closed manifolds $M_1^{k_1}, M_2^{k_2}, \ldots, M_m^{k_m}$ admit projectively equivalent strictly-non-proportional at least at one point metric $g_1, \bar{g}_1; g_2, \bar{g}_2; \ldots; g_m, \bar{g}_m$, then the product of the $M_1^{k_1} \times M_2^{k_2} \times \ldots \times M_m^{k_m}$ also admit projectively equivalent metrics strictly-non-proportional at least at one point.
Since the manifolds $M^k_i$ are closed, the eigenvalues of the tensors $\overline{4}$ constructed for the metrics $g_i, \bar{g}_i$ are bounded. By definition, the tensor $\overline{4}$ constructed for the metrics $g_i, \bar{g}_i$ and the tensor $\overline{4}$ constructed for the metrics $C \cdot g_i, \bar{g}_i$, where $C$ is a positive constant, are related by

$$L_{new} = C L_{old}.$$

Thus without loss of generality we can assume that for $i < j$ the eigenvalues of the tensor $\overline{4}$ constructed for the metrics $g_j, \bar{g}_j$ are greater than the eigenvalues of the tensor $\overline{4}$ constructed for the metrics $g_i, \bar{g}_i$. Then, by Lemma 3, $(M_1^{k_1}, g_1, \bar{g}_1) \oplus (M_2^{k_2}, g_2, \bar{g}_2) \oplus \ldots \oplus (M_m^{k_m}, g_m, \bar{g}_m)$

gives us projectively equivalent metrics on $M_1^{k_1} \times M_2^{k_2} \times \ldots \times M_m^{k_m}$ strictly-non-proportional at least at one point.

Finally, in order to prove Corollary 5 we need to show that the sphere $S^n$ admits two projectively equivalent metrics which are strictly-non-proportional at least at one point. Essentially, it was done in [1]: the metric $g$ is the restriction of the Euclidean metrics $dx_1^2 + \ldots + dx_{n+1}^2$ to the sphere

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}: x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}.$$

The metric $\bar{g}$ is the pull-back $l^*g$, where the mapping $l : S^n \to S^n$ is given by $l : v \mapsto \frac{A(v)}{\|A(v)\|}$, where $A$ is an arbitrary linear non-degenerate transformation of $R^{n+1}$.

The metrics $g$ and $\bar{g}$ are projectively equivalent. Indeed, the geodesics of $g$ are great circles (the intersections of planes that go through the origin with the sphere). The mapping $A$ is linear and, hence, takes planes to planes. Since the normalization $w \mapsto \frac{w}{\|w\|}$ takes planes to their intersections with the sphere, the mapping $l$ takes great circles to great circles. Thus the metrics $g$ and $\bar{g}$ are projectively equivalent. It is easy to verify that for almost all linear transformations $A$, (and in particular for $A = \text{diag}(a_1, \ldots, a_{n+1})$, where $a_1 < a_2 < \ldots < a_{n+1}$), the metrics $g$ and $\bar{g}$ are strictly-non-proportional at almost every point. Corollary 5 is proved.

Definition 5. By prime standard triples we will mean:

- In the one-dimensional case, $(I, dx^2, \frac{1}{\lambda(x)}dx^2)$, where $I$ is an interval with the coordinate $x$ and $\lambda$ is a smooth positive function.
- In the two-dimensional case, $(D^2, g, \bar{g})$, where $D^2$ is a 2-disc with coordinates $u, v$ and the metrics $g, \bar{g}$ are given either by the formulae (19,20), (21,22) or (21,23), respectively.
- In the three-dimensional case, $(D^3, g, \bar{g})$, where $D^3$ is a 3-disc with coordinates $u_1, u_2, u_3$ and the metrics $g, \bar{g}$ are given by the formulae (24,25).
A disk $D^n$ with two projectively equivalent metrics $g$, $\bar{g}$ will be called standard if there exists the prime standard triples $(D_1, g_1, \bar{g}_1), i = 1, \ldots, m$, such that

$$(D_1, g_1, \bar{g}_1) \oplus (D_2, g_2, \bar{g}_2) \oplus \ldots \oplus (D_m, g_m, \bar{g}_m) = (D^n, g, \bar{g}).$$

The sign “=” here mean the existence of a diffeomorphism between $D_1 \times \ldots \times D_m$ and $D^n$ which is an isometry with respect to both metrics.

**Theorem 10.** Suppose projectively equivalent Riemannian metrics $g$ and $\bar{g}$ on a connected manifold $M^n$ and are strictly non-proportional at least at one point. Then, every point $p$ of the manifold has a neighborhood $U$ such that the triple $(U, g, \bar{g})$ is standard.

**Proof:** We use induction in dimension. For dimension one the statement is trivial; for dimension two, the statement follows from Theorem 9. Assume $n \geq 3$, and suppose the theorem is true for dimensions less than $n$. Let us prove that it is true for dimension $n$. By Corollary 1 there exists $1 \leq r \leq 3$ such that $\lambda_1(p) = \ldots = \lambda_r(p) < \lambda_{r+1}(p) \leq \ldots \leq \lambda_n(p)$. If $r = n$, then $n = 3$; then, the theorem follows from Theorem 9.

Suppose $n > r$. Then, it is so at every point of a small neighborhood of $p$, and we can apply the splitting procedure from Section 2.2. We obtain the metrics $h, \bar{h}$ and the foliations $B_r$ and $B_{n-r}$ such that the local product structures $(h, B_r, B_{n-r})$ and $(\bar{h}, B_r, B_{n-r})$ satisfy the conditions (i,ii,iii) of Section 2.3. By definition of a local product structure, a (possibly, smaller) neighborhood of $p$, is a direct product of two discs $D^r$ and $D^{n-r}$, the metrics $h$ and $\bar{h}$ are the product metrics $g_1 + g_2, \bar{g}_2 + \bar{g}_2$, where $g_i$ is projectively equivalent to $\bar{g}_i$. By the induction assumption, the triples $(D^r, g_1, \bar{g}_1)$ and $(D^{n-r}, g_2, \bar{g}_2)$ are standard; therefore, $(D^r, g_1, \bar{g}_1) \oplus (D^{n-r}, g_2, \bar{g}_2)$ is standard as well. In view of Remark 3, $(D^r, g_1, \bar{g}_1) \oplus (D^{n-r}, g_2, \bar{g}_2)$ is precisely $(D^r \times D^{n-r}, g, \bar{g})$. Theorem 10 is proved.

**References**

[1] E. Beltrami, *Resoluzione del problema: riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette*, Ann. Mat., 1(1865), no. 7, 185–204.

[2] A.V. Bolsinov, V.S. Matveev, A.T. Fomenko, *Two-dimensional Riemannian metrics with an integrable geodesic flow. Local and global geometries*, Sb. Math. 189(1998), no. 9-10, 1441–1466.

[3] A.V. Bolsinov, A.T. Fomenko, *Integrable geodesic flows on two-dimensional surfaces*, Monographs in Contemporary Mathematics. Consultants Bureau, New York, 2000.

[4] Alexei V. Bolsinov, Vladimir S. Matveev, *Geometrical interpretation of Benenti’s systems*, accepted by J. of Geometry and Physics. Preprinted at Freiburg University, Nr. 11/2001.
[5] E. Cartan, *Lecons sur la théorie des espaces à connexion projective*. Rediges par P. Vincensini, Paris: Gauthier-Villars., 1937.

[6] U. Dini, *Sopra un problema che si presenta nella teoria generale delle rappresentazioni geografiche di una superficie su un’altra*, Ann. Mat., ser.2, 3(1869), 269–293.

[7] L. P. Eisenhart, *Riemannian Geometry. 2d printing*. Princeton University Press, Princeton, N. J., 1949.

[8] D. B. A. Epstein, *Projective planes in 3-manifolds*, Proc. London Math. Soc. (3) 11(1961), 469–484.

[9] D. B. A. Epstein, *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford Ser. (2) 12(1961), 205–212.

[10] J. Haantjes, *On $X_m$-forming sets of eigenvectors*, Nederl. Akad. Wetensch. Proc. Ser. A. 58(1955) = Indag. Math. 17(1955), 158–162.

[11] R.C. Kirby, M. G. Scharlemann, *Eight faces of the Poincaré homology 3-sphere*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), 113–146, Academic Press, New York-London, 1979.

[12] K. Kiyohara, *Compact Liouville surfaces*, J. Math. Soc. Japan 43(1991), 555-591.

[13] K. Kiyohara, *Two Classes of Riemannian Manifolds Whose Geodesic Flows Are Integrable*, Memoirs of the AMS, Vol.130(1997), no. 619.

[14] S. Kobayashi, K. Nomizu, *Foundations of differential geometry. Vol I*. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[15] V. N. Kolokol’tzov, *Geodesic flows on two-dimensional manifolds with an additional first integral that is polynomial with respect to velocities*, Math. USSR-Izv. 21(1983), no. 2, 291–306.

[16] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche*, Ann. di Mat., serie 2a, 24(1896), 255–300.

[17] S. Lie, *Untersuchungen über geodätische Kurven*, Math. Ann. 20(1882). Can be found in Sophus Lie Gesammelte Abhandlungen, Band 2, erster Teil, 267–374. Teubner, Leipzig 1935.

[18] V. S. Matveev, P. J. Topalov, *Trajectory equivalence and corresponding integrals*, Regular and Chaotic Dynamics, 3(1998), no. 2, 30–45.

[19] V. S. Matveev, P. J. Topalov, *Geodesic equivalence of metrics on surfaces, and their integrability*, Dokl. Akad. Nauk 367(1999), no. 6, 736–738.
[20] V. S. Matveev and P. J. Topalov, Metric with ergodic geodesic flow is completely determined by unparameterized geodesics, ERA-AMS, 6(2000), 98–104.

[21] Vladimir S. Matveev, Geschlossene hyperbolische 3-Mannigfaltigkeiten sind geodätisch starr, Manuscripta Math. 105 (2001), no. 3, 343–352.

[22] V. S. Matveev, P. J. Topalov, Quantum integrability for the Beltrami-Laplace operator as geodesic equivalence, Math. Z. 238(2001), 833–866.

[23] Vladimir S. Matveev, Commuting operators and separation of variables for Laplacians of projectively equivalent metrics, Let. Math. Phys., 119(2001), 193–201.

[24] Vladimir S. Matveev, Low-dimensional manifolds admitting metrics with the same geodesics, Contemporary Mathematics, 308(2002), 229–243.

[25] Vladimir S. Matveev, Three-manifolds admitting metrics with the same geodesics, Math. Research Letters, 9(2002), no. 2-3, 267–276.

[26] Vladimir S. Matveev, Petar J, Topalov, Geodesic equivalence via integrability, Geometriae Dedicata 96(2003), 91–115.

[27] Vladimir S. Matveev, Three-dimensional manifolds having metrics with the same geodesics, Topology 42(2003) no. 6, 1371-1395.

[28] Vladimir S. Matveev, Hyperbolic manifolds are geodesically rigid, Invent. math. 151(2003), 579–609.

[29] Vladimir S. Matveev, Projectively equivalent metrics on the torus, Diff. Geom. Appl. 20: 251-265, 2004.

[30] J. Mikes, Geodesic mappings of affine-connected and Riemannian spaces. Geometry, 2., J. Math. Sci. 78(1996), no. 3, 311–333.

[31] A. M. Naveira, A classification of Riemannian almost-product manifolds, Rend. Mat. (7) 3(1983), no. 3, 577–592.

[32] S. Tachibana, Some theorems on locally Riemannian spaces, Tohoku Math. J. (2) 12(1960), 281–292.

[33] I. A. Taimanov, Topological obstructions to the integrability of geodesic flow on nonsimply connected manifold, Math.USSR-Izv., 30(1988), no. 2, 403–409.

[34] C. B. Thomas, On 3-manifolds with finite solvable fundamental group, Invent. Math. 52(1979), no. 2, 187–197.

[35] P. J. Topalov and V. S. Matveev, Geodesic equivalence and integrability, Preprint of Max-Planck-Institut f. Math. no. 74(1998).
[36] H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung*, Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse; “Selecta Hermann Weyl”, Birkhäuser Verlag, Basel und Stuttgart, 1956.

[37] J. H. C. Whitehead, *On finite cocycles and the sphere theorem*, Colloq. Math., 6(1958), 271–281.