NON-ABELIAN HODGE THEORY FOR ALGEBRAIC CURVES IN CHARACTERISTIC \( p \)

TSAO-HSIEN CHEN, XINWEN ZHU

Abstract. Let \( G \) be a reductive group over an algebraically closed field of positive characteristic. Let \( C \) be a smooth projective curve over \( k \). We give a description of the moduli space of flat \( G \)-bundles in terms of the moduli space of \( G \)-Higgs bundles over the Frobenius twist \( C' \) of \( C \). This description can be regarded as the non-abelian Hodge theory for curves in positive characteristic.

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1. INTRODUCTION

Let \( C \) be a Riemann surface, and \( G_c \) be a compact Lie group with \( G \) its complexification. Let \((E, \nabla, \phi)\), where \( E \) is a \( C^\infty \)-principal \( G_c \)-bundle, \( \nabla \) is a \( C^\infty \)-connection on \( C \), and \( \phi \in \text{ad}(E) \otimes A^{1,0} \) is a \((1,0)\) form on \( C \) with valued in the adjoint bundle \( \text{ad}(E) \). Then the famous Hitchin’s equations are

\[
\begin{align*}
F(\nabla) + [\phi, \phi^*] &= 0, \\
\nabla''(\phi) &= 0,
\end{align*}
\]

where \( F(\nabla) \) is the curvature of \( \nabla \), \( \phi^* \) is the complex conjugation of \( \phi \) (with respect to \( G_c \)), and \( \nabla'' \) is the \((0,1)\)-component of the connection \( \nabla \).

Hitchin obtained his equations by a dimension reduction from \( 4D \) to \( 2D \) of the self-dual Yang-Mills equations, and found that the space of solutions of (1.1) has spectacular geometric properties. Namely, let \((E, \nabla, \phi)\) be a solution. Then \( \nabla'' \) endows \( E \otimes \mathbb{C} \) with a structure as a holomorphic \( G \)-bundle, and \( \phi \) as a holomorphic \( \text{ad}(E) \)-valued 1-form, i.e. \( \phi \in \Gamma(C, \text{ad}(E) \otimes \omega_C) \). Then the pair \((E, \phi)\) form a Higgs bundle. On the other hand, the connection \( \nabla = \nabla + (\phi + \phi^*) \) is flat by (1.1). It is a well-known fact that a flat \( C^\infty \)-bundle on \( C \) is the same as a holomorphic \( G \)-bundle with a flat connection, i.e., the pair \((E, \nabla)\) form a de Rham \( G \)-local system. Hitchin showed that if a Higgs bundle \((E, \phi)\) on \( C \) arises as a solution of (1.1) if and only it is (poly)stable with vanishing Chern class. Furthermore, the solution corresponding to \((E, \phi)\) is unique up to gauge transforms. In the other direction, Donaldson proved that every (semi)simple de Rham \( G \)-local system arises as a (unique up to gauge transforms) of the solution of (1.1). In this way, one established a bijection between the stable \( G \)-Higgs bundles with vanishing Chern classes and the irreducible \( G \)-local systems on \( C \), which is furthermore a homeomorphism of the corresponding moduli spaces. This is
what Simpson[4] called the non-abelian Hodge theory for $C^1$. Note that the $(0,1)$-form $\phi^*$ plays a fundamental role as its presence changes the holomorphic structure on $E$.

It has been for a while to search a version of this correspondence in characteristic $p$ case ([O] [OV]). However, compared with the story over $C$, the picture is not very complete. In [OV], the Simpson correspondence is only established for connections with nilpotent $p$-curvatures.

In this note, we will establish a full version of the non-abelian Hodge theory for curves. For this purpose, let us try to write down the analogue of the Hitchin’s equation in characteristic $p$. Assume that $G$ and $C$ are defined over an algebraically closed field $k$ of characteristic $p > 0$. We assume that $p$ does not divide the order of the Weyl group of $G$. Let $F_C : C \rightarrow C'$ be the relative Frobenius map, where $C'$ is the pullback of $C$ along the absolute Frobenius of $k$. Let $(E, \nabla)$ be a de Rham $G$-local system on $X$. In characteristic $p$, one can associate $(E, \nabla)$ its $p$-curvature, which is an $F$-Higgs field $\Psi(\nabla) \in \text{ad}(E) \otimes F_C^*\omega_C$. Let $\theta \in \text{ad}(E) \otimes \omega_C$. The analogue of Hitchin’s equations (1.1) in characteristic $p$ we have in mind are

\begin{equation}
\begin{cases}
\Psi(\nabla - \theta) = 0, \\
(\nabla - \theta)(\Psi(\nabla)) = 0.
\end{cases}
\end{equation}

Let us explain the meaning of these equations. Let $(E, \nabla, \theta)$ be a solution. Then we can endow $E$ with a new flat connection $\nabla = \nabla - \theta$, whose $p$-curvature vanishes by the first equation of (1.2). Therefore, by the Cartier decent, $E$ with this new connection is a pullback of a $G$-torsor $E'$ on $C'$. The second equation in (1.2) implies that the $F$-Higgs field $\Psi(\nabla)$ is horizontal with respect to $\nabla$ and therefore is a pull back of $\phi^* \in \text{ad}(E') \otimes \omega_{C'}$, again by the Cartier decent. In other words, a solution of (1.2) gives rises to a Higgs bundle $(E', \phi')$ on $C'$. Note that similar to (1.1), our first equation is about the $(p)$-curvature and the second equation is about the horizontality of some Lie algebra valued one-from. Note that the role $\theta$ is similar to the role of $\phi^*$ in (1.1).

As is well-known, the $p$-curvature $\Psi(\nabla)$ is horizontal with respect to the connection $\nabla$. Therefore, the second equation in (1.2) reduces to a matrix equation

\begin{equation}
[\Psi(\nabla), \theta] = 0.
\end{equation}

On the other hand, the first equation (1.2) is an ODE of order $p - 1$. Indeed, it is well-known if we choose a coordinate $z$ on $C$, and let $A \in M_n(k[[z]])$ be a $k[[z]]$-valued $n \times n$ matrix, then the $p$-curvature of the connection $\partial_z + A$ can be represented as the matrix $(\partial_z + A)^p - A$. Therefore, (1.2) in principle should be much simpler than (1.1) as the latter are non-linear PDEs.

Unfortunately, despite of its simple nature, we do not know how to solve (1.2) directly. The work of [OV] [4.1] essentially shows that if $G = \text{GL}_n$, (1.2) has solutions étale locally on $C$. The proof is not elementary and is based on the Azumaya property of the sheaf of crystalline operators. It seems that this approach has some difficulties to generalize to groups other than $\text{GL}_n$. The main problem, as we shall see, is that there is no canonical solution of (1.2) unless $\Psi(\nabla)$ is nilpotent. So it is not clear (for us) how to apply the Tannakian formalism here. In addition, the methods of loc. cit. seems not to give the information of the global solutions. In fact, Equation (1.2) themselves are just local avatar of some global equations we now explain.

Note that (1.3) says that $\theta \in \text{LieAut}(E, \Psi(\nabla))$. The group scheme $\text{Aut}(E, \Psi(\nabla))$ on $C$ is not well-behaved as the dimension of its fibers might jump. By the work of [DG] [N1], the $F$-Higgs bundle $(E, \Psi(\nabla))$ defines a smooth commutative group scheme $J_{bp}$ over $C$. 

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1And what other people called the Simpson correspondence.
2But presumably this is the most interesting case since flat connections of geometric origin have nilpotent $p$-curvatures.
where $b^p$ is the characteristic polynomial of $\Psi(\nabla)$. This is the so-called regular centralizer of the Higgs field $(E, \Psi(\nabla))$, which canonically maps to Aut$(E, \Psi(\nabla))$. Our first observation is that instead of imposing $\theta \in \text{LieAut}(E, \Psi(\nabla))$, it is better to require $\theta \in \text{Lie}J_{\theta}$. Our second observation is that the group scheme $J_{\theta}$ is the Frobenius pullback of a smooth group scheme $J_{\theta}^0$ on $C'$ and by [14] it makes sense to talk about $J_{\theta}$-torsors with flat connections. In addition, given a $J_{\theta}$-torsor with a connection $(P, \nabla_P)$, there is a “product” connection $\nabla_{P \otimes E}$ on the $G$-torsor $P \otimes E := P \times_{J_{\theta}} E$. Now the true global analogue of $\theta$ is a $J_{\theta}$-torsor with a connection $(P, \nabla_P)$ and the global equation is given by

\begin{equation}
\Psi(\nabla_{P \otimes E}) = 0.
\end{equation}

Note that as $\Psi(\nabla)$ is preserved by $J_{\theta}$, $(P \otimes E, \Psi(\nabla))$ is an $F$-Higgs bundle and (1.4) implies that it is a pullback of a Higgs bundle $(E', \phi')$ on $C'$. One can show that after a trivialization of $P$ and write $\nabla_P = d + \theta$, where $d$ is the canonical connection on the trivial $J_{\theta}$-bundle, (1.4) becomes (1.2). We will call a solution $(P, \nabla_P)$ of (1.4) harmonic bundle, a name introduced by Simpson [S]. See Remark 3.18 for an explanation of the terminology.

Let us denote by $\mathcal{H}$ the moduli space of harmonic bundles. One can show that this is an algebraic stack over the Hitchin base $B'$ for the curve $C'$. In fact, in the main context, we will give another definition of $\mathcal{H}$, which makes it clear an algebraic stack over $B'$. Let $J'$ denote the regular centralizer group scheme on $C' \times B'$ and $\mathcal{P}'$ denote the Picard stack over $B'$ of $J'$-torsors. Then there is an action of $\mathcal{P}'$ on $\mathcal{H}$ by tensoring a harmonic bundle with the Frobenius pullback of a $J'$-torsor. Our first result is

**Theorem 1.1.** Under this action, the stack $\mathcal{H}$ form a $\mathcal{P}'$-torsor.

It is not difficult to prove that the action of $\mathcal{P}'$ on $\mathcal{H}$ is simply-transitive. So the essentially point is to show that the map $\mathcal{H} \to B'$ is surjective.

Now we state the main theorem of the note. Let Higgs$_G$ denote the moduli space of Higgs bundles on $C'$. Recall that according to [N1, N2], there is a canonical action of $\mathcal{P}'$ on Higgs$_G$.

**Theorem 1.2.** Over $B'$, there is a canonical isomorphism

$$\mathcal{C} : \mathcal{H} \times \mathcal{P}' \text{ Higgs}_G \to \text{LocSys}_G.$$

**Corollary 1.3.** étale locally on $B'$, LocSys$_G$ is isomorphic to Higgs$_G$.

**Proof.** The Picard stack $\mathcal{P}'$ is smooth over $B'$, and therefore $\mathcal{H}$ can be trivialized étale locally on $B'$.

\[\Box\]

**Remark 1.4.** In the case $G = \text{GL}_n$, this corollary is one of the main theorems of [G], which extends a result of [BB] that establishes the above isomorphism away from the discriminant locus of $B'$. In fact, for $G = \text{GL}_n$, the $\mathcal{P}'$-torsor $\mathcal{H}$ can be identified with the stack $\mathcal{S}$ introduced in [G §3.1]. However, the isomorphism $\mathcal{C}$ was not obtained in loc. cit..

Let us discuss some new features of non-abelian Hodge theory in characteristic $p$. First, our theorem is stated for all $G$-Higgs fields and all flat $G$-bundles, while over $\mathbb{C}$ such a correspondence could only be possible for (poly)stable Higgs bundles with vanishing Chern classes and (semi)simple flat $G$-local systems. Second, in characteristic $p$, what exists is the isomorphism $\mathcal{C}$ rather than a direct isomorphism between Higgs$_G$ and LocSys$_G$. This is the reminiscence in characteristic $p$ of the transcendental nature of the Simpson correspondence over $\mathbb{C}$. Third, as the isomorphism $\mathcal{C}$ is algebraic, one can compare directly between the tangent spaces of Higgs$_G$ and LocSys$_G$. In particular, one can obtain the comparison between Higgs cohomology and de Rham cohomology as a consequence of the global isomorphism $\mathcal{C}$.

The relation between our work and the construction of [OV] (in the curve case) is as follows. The construction of [OV] amounts to the restriction of $\mathcal{C}$ to $0 \in B'$. Namely, upon

\[\text{But } F\text{-Higgs bundle } (E, \Psi(\nabla)) \text{ is of course not the Frobenius pullback!}\]
Let us briefly summarize the following sections. We will review the theory of Hitchin fibrations in §2 mostly following [N1, N2]. Readers familiar with this theory can skip most of it and go to §2 directly. Our main theorem is proved in §3. We construct the $p$-Hitchin map in §3.1 and then prove Theorem 1.1 in §3.2 by some cohomological argument. A briefly discuss the trivialization of the $\mathcal{P}_0'$-torsor $\mathcal{H}$ is in §3.3. We establish our main theorem in §3.4. The proof is rather formal, after we develop some general theories in Appendix A which might be of independent interests. More precisely, we first develop the notion of the scheme of horizontal sections of a $\mathcal{D}_X$-scheme, which is the right adjoint of the pullback of $X'$-schemes along the relative Frobenius. This is analogous to the same named notion in characteristic zero case developed in ([BD, §2.6]). Next, we develop the theory of de Rham $\mathfrak{g}$-local systems for a non-constant group scheme $\mathfrak{g}$ over $X$. The main observation is that the only additional input is a flat connection on $\mathfrak{g}$ itself compatible with the group structure. We finish the note with a construction for every smooth commutative group scheme $\mathfrak{g}'$ over $X'$, a 4-term exact sequence of étale sheaves on $X'$, which for $\mathfrak{g}' = \mathbb{G}_m$ specializes to famous exact sequence as in [II, §2.1].

We introduce the notations used throughout the note. We fix a smooth projective curve $C$ of genus at least 2 over an algebraically closed field $k$ of characteristic $p > 0$. Let $\omega = \omega_C$ be the canonical line bundle of $C$.

Let $S$ be a Noetherian scheme and $\mathcal{X} \to S$ be an algebraic stack over $S$. If $pO_S = 0$, we denote by $\mathcal{F}_{\mathcal{X}} : S \to S$ be the absolute Frobenius map of $S$. We have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\mathcal{F}_{\mathcal{X}}/S} & \mathcal{X}^{(S)} \\
\downarrow & & \downarrow \mathcal{F}_{\mathcal{X}/S} \\
S & \to & S
\end{array}
\]

where the square is Cartesian. We call $\mathcal{X}^{(S)}$ the Frobenius twist of $\mathcal{X}$ along $S$, and $\mathcal{F}_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X}^{(S)}$ the relative Frobenius morphism. If the base scheme $S$ is clear, $\mathcal{X}^{(S)}$ is also denoted by $\mathcal{X}'$ for simplicity and $\mathcal{F}_{\mathcal{X}/S}$ is denoted by $F_{\mathcal{X}}$ or $F$.

Let $\mathfrak{g}$ be a smooth affine group scheme over $X$, and $E$ be a $\mathfrak{g}$-torsor on $X$. We denote by $\text{Aut}(E) = E \times^\mathfrak{g} \mathfrak{g}$ the adjoint torsor and $\text{ad}(E)$ the adjoint bundle.

Let $G$ be a reductive algebraic group over $k$ of rank $l$. We denote by $\mathfrak{g}$ the Lie algebras of $G$. We assume that $p$ does not divide the order of the Weyl group $W$ of $G$.

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2. Recollections of Hitchin fibrations

In this section, we review some basic geometric facts of Hitchin fibrations, mostly following \[1\] \[2\]. Readers familiar with this theory can directly jump to the next section, probably except \[2\].

2.1. The Hitchin fibration. Let \(k[\mathfrak{g}]\) and \(k[t]\) be the algebra of polynomial function on \(\mathfrak{g}\) and \(t\). By Chevalley’s theorem, we have an isomorphism \(k[\mathfrak{g}]^G \simeq k[t]^W\). Moreover, \(k[t]^W\) is isomorphic to a polynomial ring of \(t\) variables \(u_1, \ldots, u_l\) and each \(u_i\) is homogeneous in degree \(e_i\). Let \(\mathfrak{c} = \text{Spec}(k[t]^W)\). Let

\[
\chi : \mathfrak{g} \to \mathfrak{c}
\]

be the map induced by \(k[\mathfrak{c}] \simeq k[\mathfrak{g}]^G \to k[\mathfrak{g}]\). It is \(G \times \mathbb{G}_m\)-equivariant map where \(G\) acts trivially on \(\mathfrak{c}\), and \(\mathbb{G}_m\) acts on \(\mathfrak{c}\) through the gradings on \(k[t]^W\). Let \(\mathcal{L}\) be an invertible sheaf on \(C\) and \(\mathcal{L}^x\) be the corresponding \(\mathbb{G}_m\)-torsor. Let \(\mathfrak{g}_{\mathcal{L}} = \mathfrak{g} \times \mathbb{G}_m \mathcal{L}^x\) and \(\mathfrak{c}_\mathcal{L} = \mathfrak{c} \times \mathbb{G}_m \mathcal{L}^x\) be the \(\mathbb{G}_m\)-twist of \(\mathfrak{g}\) and \(\mathfrak{c}\) with respect to the natural \(\mathbb{G}_m\)-action.

Let \(\text{Higgs}_{G,\mathcal{L}} = \text{Sect}(C, [\mathfrak{g}_{\mathcal{L}}/G])\) be the stack of section of \([\mathfrak{g}_{\mathcal{L}}/G]\) over \(C\), i.e., for each \(k\)-scheme \(S\) the groupoid \(\text{Higgs}_{G,\mathcal{L}}(S)\) consists of maps:

\[
h_{\mathcal{L},\phi} : C \times S \to [\mathfrak{g}_{\mathcal{L}}/G].
\]

Equivalently, \(\text{Higgs}_{G,\mathcal{L}}(S)\) consists of a pair \((E, \phi)\) (called a Higgs bundle), where \(E\) is a \(G\)-torsor over \(C \times S\) and \(\phi\) is an element in \(\Gamma(C \times S, \text{Ad}(E) \otimes \mathcal{L})\). If the group \(G\) is clear from the content, we simply write \(\text{Higgs}_{\mathcal{L}}\) for \(\text{Higgs}_{G,\mathcal{L}}\).

Let \(B_{\mathcal{L}} = \text{Sect}(C, \mathfrak{c}_{\mathcal{L}})\) be the scheme of section of \(\mathfrak{c}_{\mathcal{L}}\) over \(C\), i.e., for each \(k\)-scheme \(S\), \(B_{\mathcal{L}}(S)\) is the set of section

\[
b : C \times S \to \mathfrak{c}_{\mathcal{L}}.
\]

This is called the Hitchin base of \(G\).

The natural \(G\)-invariant projection \(\chi : \mathfrak{g} \to \mathfrak{c}\) induces a map

\[
[\chi_{\mathcal{L}}] : [\mathfrak{g}_{\mathcal{L}}/G] \to \mathfrak{c}_{\mathcal{L}},
\]

which in turn induces a natural map

\[
h_{\mathcal{L}} : \text{Higgs}_{\mathcal{L}} = \text{Sect}(C, [\mathfrak{g}_{\mathcal{L}}/G]) \to \text{Sect}(C, \mathfrak{c}_{\mathcal{L}}) = B_{\mathcal{L}}.
\]

We call \(h_{\mathcal{L}} : \text{Higgs}_{\mathcal{L}} \to B_{\mathcal{L}}\) the Hitchin map associated to \(\mathcal{L}\). For any \(b \in B_{\mathcal{L}}(S)\) we denote by \(\text{Higgs}_{\mathcal{L},b}\) the fiber product \(S \times_{B_{\mathcal{L}}} \text{Higgs}_{\mathcal{L}}\).

We are mostly interested in the case \(\mathcal{L} = \omega\). For simplicity, from now on we denote \(B = B_{\omega}\), \(\text{Higgs} = \text{Higgs}_{\omega}\), \(h = h_{\omega}\), and \(\text{Higgs}_{\mathcal{L}} = \text{Higgs}_{\omega\mathcal{L},b}\), etc. We sometimes also write \(\text{Higgs}_G\) for \(\text{Higgs}\) to emphasize the group \(G\).

We fix a square root \(\kappa = \omega^{1/2}\) (called a theta characteristic) of \(\omega\). Recall that in this case, there is a section \(\epsilon_{\mathcal{L}} : b \to \text{Higgs}\) of \(h : \text{Higgs} \to B\), induced by the Kostant section \(k_{\mathcal{L}} : \mathfrak{c} \to \mathfrak{g}\). Sometimes, we also call \(\epsilon_{\mathcal{L}}\) the Kostant section of the Hitchin fibration, and denote it by \(\kappa\) for simplicity.

2.2. Symmetries of Hitchin fibration. Consider the group scheme \(I\) over \(\mathfrak{g}\) consisting of pair

\[
I = \{(g, x) \in G \times \mathfrak{g} \mid \text{Ad}_g(x) = x\}.
\]

The group scheme \(I\) is not flat, but when restricted to the open subset of regular elements \(\mathfrak{g}^{reg}\), it is smooth. We define \(J = k^\times I\), This is called the universal centralizer group scheme of \(\mathfrak{g}\), which is a smooth commutative group scheme over \(\mathfrak{c}\). The following proposition is proved in \[3\], \[4\].

**Proposition 2.1.** There is a canonical isomorphism of group schemes \(\chi^* J|_{\mathfrak{g}^{reg}} \simeq I|_{\mathfrak{g}^{reg}},\) which extends to a morphism of group schemes \(a : \chi^* J \to I \subset G \times \mathfrak{g}\).
All the above constructions can be twisted. Namely, there are $\mathbb{G}_m$-actions on $I$ and $J$, so that the natural morphisms $J \to c$ and $I \to g$ are $\mathbb{G}_m$-equivariant. Therefore by twisting we have $J_L \to c_L$, $I_L \to g_L$, where $J_L = J \times \mathbb{G}_m \times L^\times$ and $I_L = I \times \mathbb{G}_m \times L^\times$. Moreover, there is a $G$-action on $I$ given by $h \cdot (x, g) = (x, hgh^{-1})$. The group scheme $I_L \to g_L$ is $G$-equivariant, and hence descends to a group scheme $[I_L]$ over $[g_L/G]$. As above, we denote $J = J_L$ and $I = I_L$ if no confusion will arise.

Let $b : S \to B_L$ be $S$-point of $B_L$. It corresponds to a map $b : C \times S \to c_L$. Pulling back $J \to c_L$ using $b$ we get a smooth groups scheme $J_b = b^* J$ over $C \times S$. Let $(E, \phi) \in \text{Higgs}_{L,b}$ and let $h_{E,\phi} : C \times S \to [g_L/G]$ be the corresponding map. Observe that the morphism $\chi^* J \to I$ in Proposition 2.1 induces $[\chi_L]^* J \to [I]$ of group schemes over $[g_L/G]$. Pulling back to $C \times S$ by $h_{E,\phi}$, we get a map

$$a_{E,\phi} : J_b \to h_{E,\phi}^*[I] = \text{Aut}(E, \phi) \subset \text{Aut}(E).$$

Therefore, we can twist $(E, \phi) \in \text{Higgs}_{L,b}$ by a $J_b$-torsor. This defines an action of $\mathcal{P}_L$ on $\text{Higgs}_L$ over $B_L$, where $\mathcal{P}_L$ is the Picard stack over $B_L$, whose fiber $\mathcal{P}_b$ over $b$ is the Picard category of $J_b$-torsors over $C \times S$.

2.3. The tautological section $\tau : c \to \text{Lie}J$. Recall that by Proposition 2.1 there is a canonical isomorphism $\chi^* J_{|g^{reg}} \simeq I_{|g^{reg}}$. The sheaf of Lie algebras $\text{Lie}(I_{|g^{reg}}) \subset g^{reg} \times g$ admits a tautological section given by $x \mapsto x \in I_x$ for $x \in g^{reg}$. Clearly, this section descends to give a tautological section $\tau : c \to \text{Lie}J$. We have the following property of $\tau$.

Lemma 2.2. Let $x \in g$, and $a_x : J_{\chi(x)} \to I_x \subset G$ be the homomorphism as in Proposition 2.1. Then $da_x(\tau(x)) = x$.

Proof. Consider the universal situation $x = \text{id} : g \to g$. Then we need to show that $da \circ \tau : g \to \chi^* \text{Lie}J \to g \times g$ is the diagonal map. But by definition, this is true when restricted to $g^{reg} \subset g$. Therefore it holds over $g$.

Remark 2.3. In particular, if we take $x = 0$, this shows that $da_0 : \text{Lie}J_{\chi(0)} \to g$ is not injective. I.e. the map $a : \chi^* J \to I$ is either injective nor surjective over a general $x \in g$.

Observe that $\mathbb{G}_m$ acts on $g^{reg} \times g$ via natural homotheties on both factors, and therefore on $\chi^* \text{Lie}(I_{|g^{reg}}) \subset g^{reg} \times g$. This $\mathbb{G}_m$-action on $\chi^* \text{Lie}(I_{|g^{reg}})$ descends to a $\mathbb{G}_m$-action on $\text{Lie}J$ and for any line bundle $\mathcal{L}$ on $C$, the $L^\times$-twist $(\text{Lie}J) \times_{\mathbb{G}_m} L^\times$ under this $\mathbb{G}_m$-action is $\text{Lie}(J_L) \otimes \mathcal{L}$, where $J_L$ is introduced in 2.2. In addition, $\tau$ is $\mathbb{G}_m$-equivariant with respect to this $\mathbb{G}_m$ action on $\text{Lie}J$ and the natural $\mathbb{G}_m$ action on $c$. Therefore, if we define a vector bundle $B_{J,L}$ over $B_L$, whose fiber over $b \in B_L$ is $\Gamma(C, \text{Lie}J_b \otimes \mathcal{L})$, then by twisting $\tau$ by $\mathcal{L}$, we obtain

$$\tau_{\mathcal{L}} : B_L \to B_{J,L},$$

which is a canonical section of the projection $\text{pr}_J : B_{J,L} \to B_L$. As before, we omit the subscript $\mathcal{L}$ if $\mathcal{L} = \omega$ for brevity.

3. The non-abelian Hodge theory

Let LocSys$_G$ (or LocSys for brevity) be the moduli space of $G$-local systems on the curve $C$. The natural map LocSys$_G \to \text{Bun}_G$ can be regarded as a deformation of the map $\pi : T^* \text{Bun}_G \to \text{Bun}_G$. On the other hand, $T^* \text{Bun}_G$ is just the moduli of Higgs field $\text{Higgs}_G$, and there is the Hitchin fibration $h : \text{Higgs}_G \to B$. In this section, we show that in the case char $k = p > 0$, there is also a deformation of the Hitchin fibration $h$. More precisely, we will construct the $p$-Hitchin map $h_p : \text{LocSys}_G \to B'$, which is a reminiscence of the classical Hitchin map in many aspects. In particular, we prove the non-abelian Hodge theory for $C$, which among other things, implies that étale locally on $B'$, $h_p$ is the “same” as $h' : \text{Higgs}_G' \to B'$. For some general discussions of local systems on a smooth variety, we refer to Appendix A.
3.1. The $p$-Hitchin map for $G$-local systems. Let $\text{LocSys}_{G}$ be the stack of $G$-local system on $C$, i.e., for every scheme $S$ over $k$, $\text{LocSys}_{G}(S)$ is the groupoid of all $G$-torsors $E$ on $C \times S$ together with a connection $\nabla : \mathcal{T}_{C \times S/S} \to \mathcal{T}_{E}$. For any $(E, \nabla) \in \text{LocSys}_{G}$ the $p$-curvature of $E$ is an element $\Psi(\nabla) \in \Gamma(C, \text{ad}(E) \otimes \omega^p)$. We call such a pair an $F$-Higgs field. The assignment $(E, \nabla) \mapsto (E, \Psi(\nabla))$ defines a map

$$\Psi_{G} : \text{LocSys}_{G} \to \text{Higgs}_{G, \omega^p}.$$

Observe that the pullback along $F_{C} : C \to C'$ induces a natural map

$$F^p : B' \to B_{\omega^p}.$$

In what follows, for $b' \in B'$, we denote by $b^p$ the image of $b'$ under the map $F^p : B' \to B_{\omega^p}$.

**Theorem 3.1.** There is a unique morphism $h_p : \text{LocSys}_{G} \to B'$, making the following diagram commute: Consider the following diagram

$$\begin{array}{ccc}
\text{LocSys}_{G} & \xrightarrow{h_p} & B' \\
\Psi_{G} \downarrow & & \downarrow F^p \\
\text{Higgs}_{G, \omega^p} & \xrightarrow{h_{\omega^p}} & B_{\omega^p}.
\end{array}$$

**Proof.** The argument is a variation of the proof of [LP] Proposition 3.2]. Let $(E, \nabla) \in \text{LocSys}_{G}$ and let $\Psi \in \Gamma(C, \text{ad}(E) \otimes \omega^p)$ be the $p$-curvature of $\nabla$. As $c_{\omega^p}$ is the pullback along $F_{C} : C \to C'$ of $c_{\omega}$, it has a canonical connection $\nabla^{\text{can}}$ and the scheme of horizontal sections $c_{\omega^p}^{\text{can}}$ of $c_{\omega^p}$ is $c_{\omega}^{\text{can}}$ (see [A, 1]), it is enough to show that for any $G$-invariant polynomial $f$ of degree $i$ on $\mathfrak{g}$, the section $f(\Psi) \in \Gamma(C, \omega^p)$ is flat with respect to $\nabla^{\text{can}}$ on $\omega^p$. This is a local question, therefore, we can assume that $C = \text{Spec}(k[[z]])$ and that $\omega$, $E$ are trivial.

Then $\nabla^{\text{can}} = \partial_2$ and $\nabla = \partial_2 - A$, where $A \in \mathfrak{g}[[z]]$. The $p$-curvature $\Psi$ can be written as

$$\Psi = (\partial_2 - A)^p \in \mathfrak{g}[[z]].$$

We have $[A, \Psi] = [\partial_2, \Psi]$ and $\partial_2 \Psi = [\partial_2, \Psi]$. Hence

$$\partial_2 \Psi = [A, \Psi],$$

where we regard $\partial_2$, $\Psi$ and $A$ as elements in $\mathcal{T}_{E}$, the Lie algebroid of infinitesimal symmetry of $E$ and $[\cdot, \cdot]$ is the natural bracket on $\mathcal{T}_{E}$ (cf. Appendix A).

Now we claim that (3.1) implies for any polynomial function $f$ on $\mathfrak{g}$ we have

$$\partial_2 f(\Psi) = (A \cdot f)(\Psi),$$

where we denote by $A \cdot f$ the action of $A \in \mathfrak{g}$ on $f \in k[\mathfrak{g}]$ induced by the adjoint action of $G$ on $\mathfrak{g}$. The claim implies the theorem because for an invariant polynomial $f$ we have $A \cdot f = 0$. To prove this, note that

$$A \cdot f(\Psi) = \frac{d}{dt}(f(\Psi + tf[A, \Psi]))|_{t=0} = \frac{d}{dt}(f(\Psi + t\partial_2 \Psi))|_{t=0}.$$

Now by a direct calculation (considering $f$ as a polynomial function on the vector space $\mathfrak{g}$), one can show that it is equal to $\partial_2 f(\Psi)$. The theorem follows. \qed

We call $h_p$ the $p$-Hitchin fibration. Its fiber over $b'$ is denoted by $\text{LocSys}_{G, b'}$.

Similarly to the usual Hitchin fibration, there is a large symmetry of the $p$-Hitchin fibration. We need a lemma. Let $(E, \nabla)$ be a $G$-local system on $C$ and $\Psi = \Psi(\nabla) \in \text{ad}(E) \otimes \omega^p$ be the $p$-curvature. It induces a morphism $a_{E, \Psi} : J_{b^p} \to \text{Aut}(E)$ as in (2.1). Observe that $J_{b^p} = F^p J_{b'}$ is a $\mathcal{D}_C$ group scheme so that $(J_{b^p})^\nabla = J_{b'}^\nabla$. We refer to [A] for the detailed discussion of $\mathcal{D}_C$-group schemes.

**Lemma 3.2.** The morphism $a_{E, \Psi}$ is horizontal.
We apply the above construction to the following diagram

\[ G \times \mathfrak{g} \xrightarrow{a} \chi^* J \xrightarrow{\chi} J \]

where \( a : \chi^* J \to G \times \mathfrak{g} \) is as in Proposition 3.1 (1). We thus obtain a horizontal morphism

\[ a_{E \times \mathcal{L}^\times} : \chi^*_{E \times \mathcal{L}^\times} J_{\mathcal{L}} \to \text{Aut}(E) \]

of group schemes over the total space of \( \text{ad}(E) \otimes \mathcal{L} \). If \( \phi : C \to \text{ad}(E) \otimes \mathcal{L} \) is a horizontal section, then \( b = h_{\mathcal{L}}(E, \phi) \in B_{\mathcal{L}} \) is a horizontal section of \( \mathcal{L} \) and the pullback of \( a_{E \times \mathcal{L}^\times} \) along \( \phi \) is \( a_{E, \phi} : J_b \to \text{Aut}(E, \phi) \) as in (2.1), which is horizontal.

Finally, the lemma follows by applying the construction to \( \mathcal{L} = \omega^p \) with the canonical connection, and \( \phi = \Psi = \Psi(\nabla) \).

Now applying the formulation §A.7 we obtain a morphism \( \text{LocSys}_{J_b} \times \text{LocSys}_G \to \text{LocSys}_G \).

**Lemma 3.3.** Assume that \( h_p(E, \nabla) = b' \). Let \( P' \) be the \( J'_b \) local system. Let \( P = F^* P' \) be the \( J'_b \)-bundle with the canonical connection. Then \( h_p(P \otimes E, \nabla_{P \otimes E}) = b' \).

The question is local so one can assume that bundles are trivial. Then it is a direct calculation. We refer to Lemma 3.10 for more details, where a similar statement is proved. We identify \( \mathcal{P}' \) with the substack of \( \text{LocSys}_{J_b} \) with vanishing \( p \)-curvature. Then we obtain,

**Proposition 3.4.** There is a natural action \( \mathcal{P}' \times B' \to \text{LocSys}_G \to \text{LocSys}_G \).

**Remark 3.5.** Alternatively, one can see the action of \( \mathcal{P}' \) on \( \text{LocSys}_G \) as follows. One can define a sheaf \( \text{Aut}(E, \nabla) \) on \( X'_b \) by assigning every étale map \( f : U' \to X' \),

\[ \text{Aut}(E, \nabla)(U') = \text{Aut}(f^* E, f^* \nabla). \]

One can show that there \( \text{Aut}(E)^{\nabla} = \text{Aut}(E, \nabla) \) as étale sheaves on \( X' \). Then it follows from Lemma 3.2 that an action of \( J'_b \) on \( (E, \nabla) \), and therefore we can twist \( (E, \nabla) \) by a \( J'_b \)-torsor.

### 3.2. The stack \( \mathcal{H} \)

We will establish a way to study the \( p \)-Hitchin map \( h_p : \text{LocSys} \to B' \) in terms of the classical Hitchin map \( h' : \text{Higgs}' \to B' \). They are related by a \( \mathcal{P}' \)-torsor \( \mathcal{H} \).

Let \( b' \in B' \), and \( J'_{b'} \) be the group scheme over \( C' \). Then \( F_{C_*} J'_{b'} = J_{b'} \) is a \( \mathcal{D}_C \)-group scheme (A.2). Therefore, it makes sense to introduce the stack \( \text{LocSys}_{J_{b'}} \) of \( J_{b'} \)-local systems on \( C' \), where \( b' \in B' \). This is representable because the map \( \text{LocSys}_{J_{b'}} \to \text{Picard stack} \mathcal{P}_{b'} \) of \( J_{b'} \) bundles on \( C \) is schematic. As Theorem 3.1 (or A.9), there is the \( p \)-Hitchin map

\[ h_{J_{b'}} : \text{LocSys}_{J_{b'}} \to B'_{J'_{b'}} : = \Gamma(C', \text{Lie} J'_{b'} \otimes \omega_{C'}). \]

Recall that \( B'_{J'_{b'}} \) is just the fiber of the vector bundle \( B'_{J'} \) over \( B' \) introduced in (2.3). It is convenient to work with the universal situation, i.e. \( b' : B' \to B' \) is the identity map. Then the \( p \)-Hitchin map can be organized into

\[ h_{J'} : \text{LocSys}_{J_{b'}} \to B'_{J'}. \]

The result in (A.10) implies

**Lemma 3.6.** This morphism is smooth.
There exists a character \( \rho \) in the “second spectral sequence” that calculating the hypercohomology of the de Rham complex \( \text{Lie}J^p \to \text{Lie}J^p \otimes \omega_C \). But the latter is surjective in this case. \( \square \)

Recall the section \( \tau' : B' \to B'_J \), defined in \( \text{(2.2)} \). Let us define
\[
\mathcal{H} = \text{LocSys}_{JP}(\tau') := (\tau')^* \text{LocSys}_{JP}.
\]

By Corollary \( \text{(A.7)} \) this is a pseudo \( \mathcal{P}' \)-torsor. An object in \( \mathcal{H} \) is a \( JP \)-local system \( (P, \nabla) \) with a specific \( p \)-curvature. We will call such a pair \( (P, \nabla) \) a harmonic bundle, a name borrowed from Simpson (\( [S] \)). See Remark \( \text{B.18} \) for an explanation.

**Theorem 3.7.** The stack \( \mathcal{H} \) is a \( \mathcal{P}' \)-torsor.

The rest of this subsection is devoted to the proof of this theorem.

By Lemma \( \text{3.9} \) \( \mathcal{H} \) is smooth over \( B' \). So it is enough to show that \( \mathcal{H} \to B' \) is surjective. We will fix \( b' \in B'(k) \). Let \( \mathcal{G}_{\tau'} \) be the \( J' \)-gerbe corresponding to the section \( \tau'(b') \in \Gamma(C', \text{Lie}J' \otimes \omega_{C'}) \) via the four term short exact sequence in Proposition \( \text{A.8} \). Then \( \mathcal{H}_b \) is just the pseudo-Bun\( J' \)-torsor of splittings of \( \mathcal{G}_{\tau'} \). We denote by \( [b'] \in H^2(C', J'_{b'}) \) the class corresponding to the gerbe \( \mathcal{G}_{\tau'_{b'}} \). We need to show that \( [b'] \) is trivial. We begin with the following lemma.

**Lemma 3.8.** The class \( [b'] \) lies in the image of the map \( H^2(C', (J'_{b'})^0) \to H^2(C', J'_{b'}) \). Here \( (J'_{b'})^0 \) is the neutral component of \( J'_{b'} \).

**Proof.** By definition, the class \( [b'] \) is obtained from \( \tau'(b') \) via the exact sequence in Proposition \( \text{A.8} \). But we can also apply this sequence to \( (J'_{b'})^0 \) to produce a class in \( H^2(C', (J'_{b'})^0) \). Clearly, this new class will map to \( [b'] \). \( \square \)

**Proposition 3.9.** For any \( b \in B(k) \) we have \( H^2(C, J^0_b) = 0 \)

Clearly this proposition will imply \( [b'] \) is zero, thus finished the proof of surjectivity of \( \mathcal{H} \to B' \). So it is enough to prove this proposition.

**Proof.** Let \( K \) be the function field of \( C \) and let \( j : \eta = \text{Spec} K \to C \) be the inclusion. Fixed \( b \in B(k) \) and we write \( J^0_\eta := J^0_{b|\eta} \). The group \( J^0_\eta \) is smooth connected and commutative.

Moreover, for a choice of a trivialization of \( \omega_C^* \) at \( \eta \) we have \( J^0_\eta \simeq (G^x_\eta)^0 \), where \( x \in g^{reg}(K) \) is the image of \( b \in c(K) \) via \( \text{kos} \). and \( (G^x_\eta)^0 \) the neutral component of the centralizer of \( x \) in \( G_\eta = G \times_k K \). There exists a largest affine subgroup \( J^0_{\eta,s} \subset J^0_\eta \) of multiplicative type such that the quotient \( U := J^0_\eta/J^0_{\eta,s} \) is unipotent. Recall that an unipotent group over a field \( K \) is called \( K \)-split if it admits a composition series with successive quotients \( K \)-isomorphic to \( \mathbb{G}_a \).

**Lemma 3.10.** The unipotent group \( U \) is \( K \)-split.

**Proof.** Let \( F \) be the separable closure of \( K \). By [CGP, Theorem B.3.4], it is enough to show that \( U_F := U \otimes K F \) is \( F \)-split. We first construct a \( \mathbb{G}_m \)-action on \( U_F \) defined over \( F \).

Let \( x = x_s + x_u \) be the Jordan decomposition and \( L^x_{\eta} := G^x_{\eta} \) be the centralizer of \( x_s \) in \( G_{\eta} = G \times_k \tilde{F} \).

It is known that \( x_u \in \text{Lie} L^x_{\eta} \) is regular nilpotent and we have
\[
J^0_\eta := J^0_F \otimes F \tilde{F} \simeq (L^x_{\eta})^0 = (L^x_u)^0.
\]

There exists a co-character \( 2\rho_L : \mathbb{G}_m \to L^x_{\eta} \) such that \( \text{Ad}(2\rho_L(t))x_u = t^2x_u \). The co-character \( 2\rho_L \) defines a \( \mathbb{G}_m \)-action \( 2\rho_L(\mathbb{G}_m) : \mathbb{G}_m \times J^0_\eta \to J^0_\eta \) on \( J^0_\eta \) by conjugation action.

\[\text{(Notice that } x_s, x_u \in g(\tilde{F}) \text{ are not necessary in } g(F) \text{ since } F \text{ is not perfect.}\]
We claim that the action map $2\rho_L(G_m)$ is defined over $F$. Since the closed embedding $i : J^0_\eta \hookrightarrow G_\eta$ is defined over $F$, it is enough to show that the composition

$$a : G_m \times J^0_\eta \to J^0_\eta \hookrightarrow G_\eta$$

is defined over $F$. But in fact, $a$ factors as

$$G_m \times J^0_\eta \xrightarrow{2\rho_L,G \times i} G_\eta \times G_\eta \xrightarrow{\text{Ad}} G_\eta$$

where $2\rho_L,G$ is the composition $2\rho_L,G : G_m \to L_\eta \hookrightarrow G_\eta$, which is defined over $F$ since it factors through a maximal torus of $G_\eta$ and any maximal torus of $G_\eta$ and homomorphism between torus are defined over $F$. Since the the adjoint action $\text{Ad}$ of $G_\eta$ is also defined over $F$, so is $a$. This finishes the construction of a $G_m$-action on $J^0_F$ over $F$. Clearly, this $G_m$-action will preserve $J^0_F$, hence induced a $G_m$-action on $U_F \simeq J^0_F/J^0_{F,s}$. By abuse of notation we still denote the action by $2\rho_L(G_m)$.

We next show that the action $2\rho_L(G_m)$ on $U_F$ has only nontrivial weights on Lie$U_F$. It is enough to prove the same statement over $\overline{F}$. Let $B_L$ be the Borel subgroup of $L_\eta$ such that $x_a \in \text{Lie}U_\eta$, here $U_\eta$ is the unipotent radical of $B_L$. Without loss of generality, we can assume $x_a = \sum_{\alpha \in \Delta_L} x_\alpha$ and $2\rho_L = \sum_{\alpha \in \Delta_L} \alpha^\vee$, where $\Delta_L$ is the set of simple roots of $L_\eta$ determined by $B_L$ and $x_\alpha$ is a non-zero element in the corresponding root space. We have $U_\overline{F} = U_\overline{F}^\alpha \subset U_L$ and the statement follows from the fact that the action $2\rho_L(G_m)$ on $U_\overline{F}$ is the restriction of the conjugate action of $2\rho_L$ on $U_L$ and the later action has only nontrivial weights on Lie$U_\overline{F}$.

We have constructed an action of $G_m$ on $U_F$ over $F$ with only nontrivial weights on Lie$U_F$. Then the lemma will follow from the general result. \qed

**Lemma 3.11.** Let $U$ be a smooth connected commutative affine unipotent group over a separably closed field $F$ of $\text{char} F = p > 0$. If we have an action $\chi(G_m)$ of $G_m$ on $U$ over $F$ with only nontrivial weights on Lie$U$, then $U$ is $F$-split.

**Proof.** Now by [CGP] Theorem B.3.4], there is a unique smooth connected $F$-split $F$-subgroup $U_s$ of $U$ such that $U_w := U/U_s$ is $F$-wound (see loc. cit. for the definition of $F$-wound). We need to show $U_w$ is trivial. By the uniqueness of $U_s$, the action $\chi(G_m)$ will perverse $U_s$ and it induces a $G_m$-action on $U_w$. On the other hand, by [CGP] Theorem B.4.3], any $G_m$-action over $F$ on an $F$-wound smooth connected unipotent group is trivial, thus $G_m$ acts trivially on $U_w$ and its Lie algebra Lie$U_w$ is in the zero weight space of LieU which is zero by assumption. It implies $U_w$ is trivial since $U_w$ is smooth and connected. \qed

**Lemma 3.12.** We have $H^i(\text{Spec}K, J^0_n) = 0$ for $i \geq 1$.

**Proof.** We have an exact sequence $H^i(\text{Spec}K, J^0_\eta,s) \to H^i(\text{Spec}K, J^0_\eta) \to H^i(\text{Spec}K,U)$. Thus it suffices to show $H^i(\text{Spec}K, J^0_\eta,s) = H^i(\text{Spec}K,U) = 0$ for $i \geq 1$. Since dim $K \leq 1$ and $J^0_\eta,s$ is a connected torus, we have $H^i(\text{Spec}K, J^0_\eta,s) = 0$ for $i \geq 1$. On the other hand, $U$ is $K$-split we have $H^i(\text{Spec}K,U) = 0$ for $i \geq 1$. We are done. \qed

Finally, we prove Proposition 3.13. Since $C$ is a curve we have $H^2(C, J^0_\eta) \simeq H^2(\text{Spec}K, j_* J^0_\eta)$. In addition, $R^i j_* J^0_\eta = 0$ for $i \geq 1$. Indeed, the $i$-th direct image is the sheaf associated with the functor $S \to H^i(\text{Spec}K \times_C S, J^0_\eta)$. Now Spec$K \times_C S$ is an etale extension over Spec$K$, and thus it made up by finitely many separable extension of $K$. Their cohomology vanishes by Lemma 3.12. By Leray spectral sequence, this gives us

$$H^2(C, J^0_\eta) \simeq H^2(\text{Spec}K, j_* J^0_\eta) \simeq H^2(\text{Spec}K, J^0_\eta),$$

which vanishes by Lemma 3.12. This finished the proof. \qed
**Example 3.13.** Let us look at the most singular case $b = 0 \in B(k)$. We assume that $G$ is semisimple for simplicity. Then $J_0 = G^e \times \widetilde{G}^\omega$ where $e \in g^{reg}$ is regular nilpotent. So the group $J_0^e$ is isomorphic to $J_0^e \simeq (G^e)^0 \otimes_k K$. As the group $(G^e)^0$ is smooth connected and unipotent over an algebraically closed field $k$, $J_0^e$ is $K$-split.

### 3.3. The main theorem

**Theorem-Definition 3.14.** There is a canonical morphism of stacks over $B'$

$$\mathcal{C} : \mathcal{H} \times \mathcal{P}' \xrightarrow{\rho} \text{Higgs}_G \to \text{LocSys}_G.$$  

**Proof.** The construction of the morphism $\mathcal{C}$ is given as follows. For any $(E', \phi') \in \text{Higgs}_G'$, there is a canonical morphism $\rho_{E', \phi'} : J_{\phi'} \to \text{Aut}(E', \phi') \subset \text{Aut}(E')$ (see (2.1)), and therefore via pullback there is a horizontal morphism of $\mathcal{D}_C$-groups

$$F^* a_{E', \phi'} = a_{F^* E', F^* \phi'} : J_{\phi'} \to \text{Aut}(F^* E'),$$

where $F^* E'$ is equipped with the canonical connection. Now, let $\mathcal{G}_1 = J_{\phi'}$, $\mathcal{G}_2 = \text{Aut}(F^* E')$ and $\mathcal{G}_3 = G$ as in (3.7), and apply the construction there. Then given $(P, \nabla) \in \text{LocSys}_{J_{\phi'}}$, one can therefore form a $G$-local system

$$((P, \nabla), (E', \phi')) \mapsto (F^* a_{E', \phi'})(P \otimes F^* E').$$

It is easy to see that this induces a morphism $\mathcal{C} : \text{LocSys}_{J_{\phi'}} \times \mathcal{P}' \to \text{Higgs}_G \to \text{LocSys}_G$. Indeed, let $P'$ be a $J_{\phi'}$-torsor on $C'$. By (2.1), one can twist the Higgs field $(E', \phi')$ by $P'$ and let $P' \otimes (E', \phi')$ denote the new Higgs field. Then the claim amounts to

$$(F^* a_{E', \phi'})(P \otimes F^* P') \otimes F^* E' = (F^* a_{P' \otimes (E', \phi')})(P \otimes F^* E'),$$

which can be checked directly by definitions.

To show that $\mathcal{C}$ is a morphism over $B'$, it is enough to show the following lemma.  

**Lemma 3.15.** If $(E', \phi') \in \text{Higgs}_G'$, and $(P, \nabla) \in \text{LocSys}_{J_{\phi'}}$ whose $p$-curvature is $\tau'(b') \in B'_{J_{\phi'}}$, then $h_p(F^* a_{E', \phi'})_\ast P \otimes F^* E' = b'$.

**Proof.** Regard $b' : C' \to \mathcal{E}'_\phi$ as a section. Then the question is local on $C$. Therefore, we can assume that $E'$ and $P$ are trivialized, and can denote the map $F^* a_{E', \phi'} : J_{\phi'} \to \text{Aut}(F^* E')$ by $F^* a_{\phi'} : J_{\phi'} \to G \times C$. This is exactly the map $J_{\phi'} \simeq J_{\phi'} \to G \times C$.

By definition, the $p$-curvature of the $G$-local system $F^* E' = G \times C$ is zero, and the $p$-curvature of $(P, \nabla_P)$ is $\Psi(\nabla_P) = F^* \tau(b') \in \text{Lie}J_{\phi'} \otimes F^* \omega_C$. By the diagram (A.6), it is then easy to see that the $p$-curvature of $(F^* a_{E', \phi'})(P \otimes F^* E')$ is given by the image of $\tau(b') = F^* \tau(b')$ under $dF^* a_{\phi'} : \text{Lie}J_{\phi'} \otimes F^* \omega_C \to \mathfrak{g} \otimes F^* \omega_C$. But by (2.3), $dF^* a_{\phi'}(\tau(b')) = F^* \phi'$, and therefore its image under $a_{\phi'} = \epsilon_{\phi'}$ is $b'$. The lemma follows.

**Remark 3.16.** It is clear from the construction that $\mathcal{C}$ is $\mathcal{P}'$-equivariant.

Our main theorem is

**Theorem 3.17.** The morphism $\mathcal{C}$ is an isomorphism.

**Proof.** To prove that $\mathcal{C}$ is an isomorphism, we will construct the inverse morphism. This is essentially explained in the introduction: Given $(\tilde{E}, \nabla)$ is $G$-local system, a solution of the equation (1.4) defines a Higgs bundle $(E', \psi')$. Here we make it precise. Namely, by Corollary (A.7) the pseudo $\mathcal{P}'$-torsor $\text{LocSys}_{J_{\phi'}}(\tau')$ of $J_{\phi'}$-local systems with $p$-curvature $\tau'$ is the inverse of $\mathcal{H} = \text{LocSys}_{J_{\phi'}}(\tau')$. In particular, it is a $\mathcal{P}'$-torsor. Therefore, it is enough to construct a morphism

$$\mathcal{C}^{-1} : \mathcal{H} \to \text{LocSys}_G \xrightarrow{\rho} \text{Higgs}_G.$$  

We can apply the construction of Appendix (A.7) to $\mathcal{G}_1 = J_{\phi'}$, $\mathcal{G}_2 = \text{Aut}(E)$ and $\mathcal{G}_3 = G$. Let $(P, \nabla_P)$ be a $J_{\phi'}$-local system with the $p$-curvature $-\tau(b')$ so that $\tilde{E} = (a_{E, \psi})_\ast P \otimes E$ is a $G$-local system. In addition, there is an F-Higgs field $\Psi$ on $\tilde{E}$. Indeed, $(\tilde{E}, \Psi)$ is the...
twist of \((E, \Psi)\) by the underlying \(J_{bp}\)-torsor \(P\). In particular, under the classical Hitchin map \(h_{\omega_p}: \text{Higgs}_{G, \omega_p} \to B_{\omega_p}, h_{\omega_p}(\tilde{E}, \tilde{\Psi}) = b_p\).

We show that the connection on the \(G\)-local system \(\tilde{E}\) has vanishing \(p\)-curvature and \(\tilde{\Psi}\) is horizontal. Then \((\tilde{E}, \tilde{\Psi}) = F^*(E', \Psi')\), and \(h'(E', \Psi') = b'\). This construction provides the inverse map of \(\mathcal{C}\).

The question is local on \(C\). We can trivialize \(P\) and \(E\) and \(\omega_C\). Then \(\tilde{\Psi} \in \mathfrak{g} \otimes \mathcal{O}_C, \Psi_P \in \text{Lie}_{J_{bp}}\), and by Lemma 2.2 \(\text{da}_{E, \Psi}: \text{Lie}_{J_{bp}} \to \mathfrak{g} \otimes \mathcal{O}_C\) will send \(\Psi_P\) to \(-\Psi\).

Note that the \(p\)-curvature of \(E\) with respect to its connection \(\nabla\) (do not confuse with \(\tilde{\Psi}\)) is given by \(\Psi_P \otimes 1 + 1 \otimes \Psi \in \text{Der}_{G_C}((0_p \otimes \mathcal{O}_E)^{\text{triv}})\). Here \(\Psi_P \otimes 1 + 1 \otimes \Psi\), which is a priori an element in \(\text{Der}_{G_C}((0_p \otimes \mathcal{O}_E)^{\text{triv}})\) preserves \(\mathfrak{g}\). It is clear that under the isomorphism \((0_p \otimes \mathcal{O}_E)^{\text{triv}} \simeq \mathcal{O}_E\) induced by the trivialization, \(\Psi_P \otimes 1\) maps to \(\text{da}_{E, \Psi}(\Psi_P)\). Therefore, \(\Psi_P \otimes 1 + 1 \otimes \Psi = 0\).

Finally, we show that \(\tilde{\Psi}\) is horizontal. Again, the question is local and we pick up the trivialization of \(P\). Then under the isomorphism \(\tilde{E} \simeq E\) of \(\mathcal{O}\)-bundles, \(\nabla_{\tilde{z}} = \nabla_{z} + A\) for some \(A \in \text{Im}(a_{E, \Psi}: \text{Lie}_{J_{bp}} \to \mathfrak{g} \otimes \mathcal{O}_C)\). Then the claim follows from \([A, \Psi] = 0\) and \(\nabla_{\tilde{z}}(\Psi) = 0\). \(\square\)

Let \(\text{Higgs}^{	ext{reg}}_{G, \mathcal{L}}\) denote the open substack of \(\text{Higgs}_{G, \mathcal{L}}\) consisting of \((E, \phi): C \to [\mathfrak{g}_{\mathcal{L}}/G]\) that factors through \(C \to [(\mathfrak{g}^{\text{reg}}_{\mathcal{L}})/G]\). It is known from [DG, N1] that \(\text{Higgs}^{	ext{reg}}_{G, \mathcal{L}}\) is a \(\mathcal{P}_{\mathcal{L}}\)-torsor, which is trivialized by a choice of Kostant section \(\epsilon_{\mathcal{L}, 1/2}\).

Let us define an open substack \(\text{LocSys}^{	ext{reg}}_{G, \mathcal{L}}\) of \(\text{LocSys}_{G, \mathcal{L}}\) consisting of those \((E, \nabla)\) such that the \(F\)-Higgs field \(\Psi(\nabla) \in \text{Higgs}^{	ext{reg}}_{G, \omega_p}\). As it is the image of \(\mathcal{H} \times 1^p\) \(\text{Higgs}^{	ext{reg}}_{G, \mathcal{L}}\) under \(\mathcal{C}\), the map \(\text{LocSys}^{	ext{reg}}_{G, \mathcal{L}} \to B^p\) is surjective, which is not obvious from its definition.

**Remark 3.18.** Upon a choice of the Kostant section, we have \(\mathcal{H} \simeq \text{LocSys}^{	ext{reg}}_{G, \mathcal{L}}\). Therefore objects in \(\mathcal{H}\) can be regarded as a \(\mathcal{G}\)-bundle \(E\) with a connection and a Higgs field that induces the reduction of the structure group of \(E\) to \(J_P\). This is an analogue of a harmonic bundle introduced in [S], which is a \(C^\infty\)-bundle equipped with a Higgs field and a flat connection that are related by a harmonic metric. For this reason, we call \(\mathcal{H}\) the stack of harmonic bundles.

**Remark 3.19.** It is instructive to look at the isomorphism \(\mathcal{C}\) in the case when \(G = \text{GL}_n\). For \(b' \in B^p\), let \(S_{b'} \subset T^*C\) be the corresponding spectral curve. Recall that a Higgs field \((E', \phi')\) with the characteristic polynomial \(b'\) can be regarded as a coherent sheaf on \(S_{b'}\). Let

\[S_{b'} \subset T^*C \times_C C\]

be the pullback of \(S_{b'}\), i.e. the corresponding spectral curve for \(b' = F^*b'\). Let \(\pi_{b'}: S_{b'} \to C\) be the projection and \(W: S_{b'} \to S_{b'}\) be the finite morphism of degree \(p\). Then

\[J_{b'} = \pi_{b', *} \mathbb{G}_m,\]

and \(\mathcal{H}\) can be regarded as the stack of rank \(n\) vector bundles with connections \((E, \nabla)\) on \(C\), such that the \(F\)-Higgs field \((E, \phi(\nabla))\) realizes \(E\) as an invertible sheaf \(\mathcal{L}\) on \(S_{b'}\). Therefore, for such a \((E, \nabla)\), the direct image of \(\mathcal{L}\) along \(S_{b'} \to S_{b'}\) is locally free of rank \(p\) on \(S_{b'}\) and therefore is a splitting module of the Azumaya algebra of the ring of crystalline differential operators \(\mathcal{D}_C\) of \(C\) restriction \(S_{b'}\). As a result, the map \(\mathcal{C}\) is just tensoring a Higgs field \((E', \phi')\), regarded as a coherent sheaf on \(S_{b'}\), with a splitting module \(W_* \mathcal{L}\) of \(\mathcal{D}_{C|S_{b'}}\) and the inverse map \(\mathcal{C}^{-1}\) is given by \(\text{Hom}_{\mathcal{O}_C}(W_* \mathcal{L}, -)\) from \(\text{LocSys}_{G, \mathcal{L}}\) to \(\text{Higgs}_{G, \mathcal{L}}\). Over \(0 \in B'\), the spectral curve \(S_0\) is just the \((n - 1)\)th infinitesimal neighborhood of \(C\) in \(T^*C\). As we assume that \(p > n\), the \(\mathcal{D}_C\)-module \(\mathcal{B}_{X/S}\) (restricted to \(S_0\)) as in [OV] Theorem 2.8 gives a splitting of \(\mathcal{D}_C\) over \(S_0\) and therefore a trivialization of \(\mathcal{H}|_0\).
3.4. Trivialization of \( \mathcal{H}_0 \). We briefly study the trivialization of the \( \mathcal{P}' \)-torsor \( \mathcal{H} \) over \( 0 \in B' \). We first deal the case \( G = \text{PGL}_2 \), which is closely related to the geometry of the curve itself.

**Lemma 3.20.** If \( G = \text{PGL}_2 \), then \( J'_0 \simeq \omega_{C'}^{-1} \), where \( \omega_{C'}^{-1} \) is regarded as an affine vector group over \( C' \).

As a result, \( \mathcal{P}'_0 \simeq H^1(C', \omega_{C'}^{-1}) \), and \( \mathcal{H}_0 \) is an \( H^1(C', \omega_{C'}^{-1}) \)-torsor.

**Lemma 3.21.** Under the above identification, the \( \mathcal{P}'_0 \)-torsor \( \mathcal{H}_0 \) is canonically isomorphic to the \( H^1(C', \omega_{C'}^{-1}) \)-torsor of liftings of \( C' \) to \( W_2(k) \).

**Proof.** This is essentially a reformulation of [OV, Theorem 4.5] in the curve case. Given a lifting \( \tilde{C}' \) of \( C' \) to \( W_2(k) \), we construct an object in \( \mathcal{H}_0 \) as follows. As explained in [OV §1], a lifting of \( C' \) to \( W_2(k) \) defines an extension of \( \mathcal{O}_C \) by \( F^*\omega_{C'}^{-1} \) as \( \mathcal{D}_C \)-modules

\[
0 \to F^*\omega_{C'}^{-1} \to \mathcal{E} \xrightarrow{\pi} \mathcal{O}_C \to 0.
\]

such that the \( p \)-curvature of \( \mathcal{E} \) is given by

\[
\mathcal{E} \to \mathcal{O}_C \simeq F^*\omega_{C'}^{-1} \otimes F^*\omega_{C'} \subset \mathcal{E} \otimes F^*\omega_{C'}.
\]

Then \( \pi^{-1}(1) \) is an \( F^*\omega_{C'}^{-1} \) torsor on \( C \), equipped with a connection\(^5\). This defines an object of \( \mathcal{H}_0 \).

This construction induces a morphism from the \( H^1(C', \omega_{C'}^{-1}) \)-torsor of liftings of \( C' \) to \( W_2(k) \) to the \( \mathcal{P}'_0 \)-torsor \( \mathcal{H}_0 \), which clearly intertwines the action \( H^1(C', \omega_{C'}^{-1}) \simeq \mathcal{P}'_0 \).

As a corollary, in the case \( G = \text{PGL}_2 \) a choice of the lifting of \( C' \) to \( W_2(k) \) gives rise to a trivialization of \( \mathcal{H}_0 \). The same is true for \( G = \text{SL}_2 \), as \( J'_0 \simeq \omega_{C'}^{-1} \times \mu_2 \). Now for a general \( G \), we fix an principal \( \varphi : \text{SL}_2 \to G \). Via pushout, we see the same is true for general \( G \).

**Appendix A. The Stack of \( \mathfrak{S} \)-local systems.**

In this appendix we discuss the notion of (de Rham) \( \mathfrak{S} \)-local systems and their \( p \)-curvatures. We will fix a smooth morphism \( X \to S \) of noetherian schemes (however all discussions carry through without change if \( X \) is a smooth Deligne-Mumford stack), and a smooth affine group scheme \( \mathfrak{S} \) over \( X \). We do not assume that \( \mathfrak{S} \) is constant or is fiberwise connected. The main example is the regular centralizer group scheme \( J_{br} \) as in the note.

For any scheme \( X \to S \) smooth over \( S \), we denote by \( T_{X/S} \) (resp. \( \Omega_{X/S} \)) its tangent (resp. cotangent) sheaf relative to \( S \), or sometimes by \( T_X \) (resp. \( \Omega_X \)) if no confusion will likely arise. We denote by \( \mathcal{D}_{X/S} \) (or by \( \mathcal{D}_X \) for simplicity) the sheaf of crystalline differential operators on \( X \) as in [BB, OV].

**A.1. The scheme of horizontal sections.** We begin to develop the theory of scheme of horizontal sections of a \( \mathcal{D}_X \)-scheme in characteristic \( p \), analogous to [BB Proposition 2.6.2].

**Lemma A.1** (Cartier descent).

1. For any quasi-coherent sheaf \( \mathcal{L} \) on \( X' \), there is a canonical \( \mathcal{D}_X \)-module structure on \( F^*(\mathcal{L}) \) and the assignment \( \mathcal{L} \to F^*(\mathcal{L}) \) defines an equivalence between the category of quasi-coherent sheaves on \( X' \) and the category of \( \mathcal{D}_X \)-modules on \( X \) with zero \( p \)-curvature, with an inverse functor given by taking flat sections \( \mathfrak{S} \to F^N \).

2. The above equivalence is a tensor equivalence, i.e., for any \( \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{Q}_{\text{Coh}}(X') \) the natural isomorphism of \( \mathcal{O}_{X'} \)-modules

\[
m : F^*(\mathcal{L}_1) \otimes F^*(\mathcal{L}_2) \simeq F^*(\mathcal{L}_1 \otimes \mathcal{L}_2)
\]

is compatible with their \( \mathcal{D}_X \)-modules structures coming from part (1).

\(^5\)This is in fact the torsor of lifting of the Frobenius \( F : C \to C' \) to \( W_2(k) \).
(3) Let \( B \) be a \( O_X \)-algebra on \( X' \). Then there is a canonical \( \mathcal{D}_X \)-algebra structure on \( F^*(B) \) and the assignment \( L \to F^*(L) \) defines an equivalence between the category of \( O_X \)-algebras on \( X' \) and the category of \( \mathcal{D}_X \)-algebra on \( X \) with zero p-curvature.

**Proof.** Part (1) is the standard Cartier descent. Part (3) follows from Part (2), which can be proved by a direct computation. □

**Lemma A.2.**

1. The functor \( M' \to F^*M' \) admits a left adjoint functor, i.e. for a \( \mathcal{D}_X \)-algebra \( N \) there is a \( O_X \)-algebra \( H_\nabla(N) \) such that
   \[
   \text{Hom}_{\mathcal{D}_X \text{-alg}}(N, F^*M') = \text{Hom}_{O_X \text{-alg}}(H_\nabla(N), M')
   \]
   for any \( O_X \)-algebra \( M' \).

2. The canonical map \( N \to F^*(H_\nabla(N)) \) is surjective.

**Proof.** Let \( \nabla \) be the connection of \( N \). We denote by \( \Psi \) the p-curvature of \( \nabla \). We can think of \( \Psi \) as a map \( \Psi : F^*T_{X'} \otimes_{O_X} N \to N \). Let \( \Psi(N) \) be the ideal of \( N \) generated by the image of \( \Psi \). We define \( N_\Psi = N/\Psi(N) \). Since \( \nabla \) commutes with \( \Psi \), the \( O_X \)-algebra \( N_\Psi \) carries a connection and we define the following \( O_X \)-algebra
   \[
   H_\nabla(N) = (N_\Psi)^\nabla.
   \]
Let us show that \( H_\nabla(N) \) satisfies our requirement. For any \( O_X \)-algebra \( M' \), the p-curvature of \( F^*M' \) is zero, and therefore we have
   \[
   \text{Hom}_{\mathcal{D}_X \text{-alg}}(N, F^*M') = \text{Hom}_{\mathcal{D}_X \text{-alg}}(N_\Psi, F^*M')
   \]
On the other hand, since the p-curvature of \( N_\Psi \) is zero, by Lemma A.1 we have
   \[
   \text{Hom}_{\mathcal{D}_X \text{-alg}}(N_\Psi, F^*M') = \text{Hom}_{O_X \text{-alg}}((N_\Psi)^\nabla, M') = \text{Hom}_{O_X \text{-alg}}(H_\nabla(N), M').
   \]
Therefore
   \[
   \text{Hom}_{\mathcal{D}_X \text{-alg}}(N, F^*M') = \text{Hom}_{O_X \text{-alg}}(H_\nabla(N), M').
   \]
This proved part 1).

By Lemma A.1 again, we have \( N_\Psi = F^*(H_\nabla(N)) \) and it implies the canonical map \( N \to N_\Psi = F^*(H_\nabla(N)) \) is surjective. This proved part 2). □

Let \( N \) be a commutative \( \mathcal{D}_X \)-algebra. Then the \( O_X \)-algebra \( H_\nabla(N) \) is commutative by (2). We called the \( X' \)-scheme
   \[
   N^\nabla := \text{Spec}(H_\nabla(N))
   \]
the scheme of horizontal sections. Let \( N \) and \( N_\Psi \) be the \( \mathcal{D}_X \)-scheme associated to the commutative \( \mathcal{D}_X \)-algebras \( N \) and \( N_\Psi \). We have
   \[
   (A.1) \quad N_\Psi = X \times_{X', N^\nabla} N^\nabla, \quad (N_\Psi)^\nabla = N^\nabla,
   \]
and \( N_\Psi \subset N \) is the maximal closed subscheme of \( N \) that is constant with respect to the connection.

**A.2. Connection on \( \mathcal{G} \)-torsor.** In order to talk about a connection on a \( \mathcal{G} \)-torsor, we need to assume that \( \mathcal{G} \) itself is a \( \mathcal{D}_{X/S} \)-group scheme. Then it carries a flat connection which is compatible with the group structure, i.e. there is a connection
   \[
   \nabla_\mathcal{G} : \mathcal{G} \to \mathcal{G} \otimes_{O_X} \Omega_{X/S}
   \]
which is compatible with the unit, the multiplication and the co-multiplication on \( \mathcal{G} \).

Given a \( \mathcal{G} \)-torsor \( E \), we define a flat connection on \( E \) to be the following data
1) A connection \( \nabla : O_E \to O_E \otimes_{O_X} \Omega_{X/S} \) which is compatible with the multiplication of
\(O_E\), i.e. \(\nabla\) makes \(O_E\) into a \(D_X\)-algebra.

2) We have the following commutative diagram

\[
\begin{array}{ccc}
O_E & \xrightarrow{\sigma} & O_E \otimes O_S \\
\downarrow \phi & & \downarrow \phi \otimes 1 + 1 \otimes \nabla_S \\
O_E \otimes \Omega_{X/S} & \xrightarrow{\sigma} & (O_E \otimes O_S) \otimes \Omega_{X/S}
\end{array}
\]

where \(a : O_E \rightarrow O_E \otimes O_S\) is the co-action map.

We denote by \(\text{LocSys}_G\) the stack of \(G\)-torsors with flat connections (or \(G\)-local systems). Note that the following discussions do not require the representability of this stack.

**Example A.3.** In the case of constant group scheme \(G = S \times S X\), there is a canonical connection on \(O_S = O_G \otimes O_S O_X\) coming from \(O_X\). The above definition reduces to the standard one.

**Example A.4.** Assume that \(pO_S = 0\). Let \(X'\) be the Frobenius twist of \(X\) along \(S\) and \(F : X \rightarrow X'\) be the relative Frobenius map. Let \(G'\) be a smooth affine group scheme over \(X'\). Then the group scheme \(F'G'\) has a canonical connection, and for any \(G'-\text{torsor} E'\), there is a canonical connection on the \(F'G'\)-torsor \(F'E'\). Indeed, we have \(O_{F'G'} = O_X \otimes O_X, O_{G'},\) and therefore the connection on \(O_X\) will induce a connection on \(O_{F'G'}\) which is compatible with group structure. For the same reason, \(F'E'\) also carries a connection and one can check that this connection satisfies above requirements.

### A.3. Lie algebroid definition

We give a Lie algebroid definition of connection. Let \(\mathfrak{g}\) be an affine group scheme with connection. Let \(E\) be an \(\mathfrak{g}\)-torsor. We denote by \(\overline{T_E}\) the Lie algebroid of infinitesimal symmetry of \(E\): a section of \(\overline{T_E}\) is a pair \((v, \tilde{v})\), where \(v \in T_{X/S}\) and \(\tilde{v} \in T_E\) is a vector field on \(E\) such that:

1. The restriction on \(\tilde{v}\) to \(O_X \subset O_E\) is equal to \(v\) (i.e. \(\tilde{v}\) is a lifting of \(v\)).
2. \(\tilde{v}\) is \(\mathfrak{g}\)-invariant, i.e. the following diagram commutes

\[
\begin{array}{ccc}
O_E & \xrightarrow{\sigma} & O_E \otimes O_S \\
\downarrow \phi & & \downarrow \phi \otimes 1 + 1 \otimes \nabla_S(v) \\
O_E & \xrightarrow{\sigma} & O_E \otimes O_S
\end{array}
\]

Let \(\sigma : \overline{T_E} \rightarrow T_{X/S}\) be the projection map \((v, \tilde{v}) \rightarrow v\). We have the following exact sequence

(A.2) \[0 \rightarrow \text{ad}(E) \rightarrow \overline{T_E} \xrightarrow{\sigma} T_{X/S}\].

A connection \(\nabla\) on \(E\) is a splitting of this exact sequence, i.e. \(\nabla\) is a map \(\nabla : T_{X/S} \rightarrow \overline{T_E}\) such that \(\sigma \circ \nabla = \text{id}\). If in addition \(\nabla\) is a Lie algebroid homomorphism, we say that \(\nabla\) is a flat connection.

### A.4. Grothendieck’s approach

Let \(\Delta : X \rightarrow X \times_S X\) be the diagonal embedding and let \(X^{(1)}\) be the first infinitesimal neighborhood of \(\Delta\). We denote by \(p_1, p_2 : X^{(1)} \rightarrow X\) to be the first and second projections. Let \(G\) be a smooth affine group scheme over \(X\). A flat connection on \(\mathfrak{g}\) is an isomorphism of group schemes \(\nabla\mathfrak{g} : p_1^* (\mathfrak{g}) \simeq p_2^* (\mathfrak{g})\) such that:

1. The restriction of \(\nabla\mathfrak{g}\) to \(\Delta\) is the identity map.
2. Let \(\Delta_3 : X \rightarrow X \times X \times X\) be the diagonal embedding. Let \(X^{(1)}_3\) be the first infinitesimal neighborhood of \(\Delta_3(X)\). We denote by \(\pi_1, \pi_2, \pi_3\) three projections from \(X \times X \times X\) to \(X\). The isomorphism \(\nabla\mathfrak{g}\) induces isomorphisms \(f_{ij} : p_i^*(\mathfrak{g})|_{X^{(1)}_3} \simeq p_j^*(\mathfrak{g})|_{X^{(1)}_3}\). We require \(f_{13} = f_{23} \circ f_{12}\).
Let $E$ be a $G$-torsor over $X$. A flat connection on $E$ is an isomorphism $\nabla : p_1^*(E) \simeq p_2^*(E)$ of $G$-torsors, i.e. the following diagram commutes

$$\begin{array}{ccc}
p_1^*\mathcal{G} \times p_1^*E & \longrightarrow & p_1^*E \\
\nabla \times \nabla & \downarrow & \nabla \\
p_2^*\mathcal{G} \times p_2^*E & \longrightarrow & p_2^*E
\end{array}$$

satisfying:

1) The restriction of $\nabla$ to $\Delta(X)$ is the identity map.
2) Let $g_0 : \pi_1^*(E)|_{X_1^{(1)}} \simeq \pi_2^*(E)|_{X_2^{(1)}}$ be isomorphisms induced by $\nabla$. Then $g_{13} = g_{23} \circ g_{12}$.

A.5. Connections on the trivial $G$-torsor. Let $E = E^0$ be the trivial $G$-torsor. Then it is equipped with a canonical flat connection coming from $\nabla_G$, denoted by $\nabla^0$. Then we denote the subsheaf of flat connections on $E^0$ by $(\text{Lie}\mathcal{G} \otimes \Omega_{X/S})^\text{cl}$. If $G = G \times X$ is constant, then $(\text{Lie}\mathcal{G} \otimes \Omega_{X/S})^\text{cl} = \text{LieG} \otimes \mathcal{Z}_{X/S}$, where $\mathcal{Z}_{X/S} \subset \Omega_{X/S}$ is the sheaf of closed one-forms.

There is always the following map of sheaves on $X$

$$d\log : \mathcal{G} \to (\text{Lie}\mathcal{G} \otimes \Omega_{X/S})^\text{cl},$$

defined as follows. We regard $g \in \mathcal{G}$ as an element in $\text{Hom}_X(\mathcal{O}_G, \mathcal{O}_X)$, which carries on a natural connection, still denoted by $\nabla_g$. Let $J = \ker g \subset \mathcal{O}_G$. Then it is easy to see $\nabla_g(g) \in \text{Hom}(\mathcal{O}_G, 0) \otimes \Omega_{X/S}$ annihilates $\mathcal{J}_g^2$, and therefore induces

$$\nabla_g(g) : \mathcal{J}_g/\mathcal{J}_g^2 \to \Omega_{X/S}.$$

Then $g^{-1}\nabla_g(g)$ can be regarded as an element in $\text{Lie}\mathcal{G} \otimes \Omega_{X/S}$. Then $\nabla^0 + d\log(g)$ is a connection on $E^0$.

Note that $\mathcal{G}$ acts on $E^0$, and therefore on $\tilde{T}_{E^0}$. It is easy to see that $g : (E^0, \nabla^0 + d\log(g)) \simeq (E^0, \nabla^0)$ is an isomorphism. In particular, $\nabla^0 + d\log(g)$ is a flat connection, i.e. $d\log(g) \in (\text{Lie}\mathcal{G} \otimes \Omega_{X/S})^\text{cl}$.

A.6. Bitorsors and connection on bitorsors. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two smooth affine group schemes with flat connections. We called a scheme $E$ over $X$ a $(\mathcal{G}_1 \times \mathcal{G}_2)$-bitorsor if it has an $(\mathcal{G}_1 \times \mathcal{G}_2)$-action and this action makes $E$ into a left $\mathcal{G}_1$-torsor and a right $\mathcal{G}_2$-torsor. We can define a similar notion of a flat connection of a $(\mathcal{G}_1 \times \mathcal{G}_2)$-bitorsor $E$, i.e. a flat connection on $E$ is an isomorphism $\nabla : p_1^*(E) \simeq p_2^*(E)$ together with compatibility conditions similar to §A.4.

We denote by $\text{LocSys}_{\mathcal{G}_1 \times \mathcal{G}_2}$ the stack of $(\mathcal{G}_1 \times \mathcal{G}_2)$-bitorsors with flat connections.

Example A.5. Let $\mathcal{G}$ be a group scheme over $X$ with connection and let $E \in \text{LocSys}_G$. Let $\text{Aut}(E)$ be the group scheme of automorphism of $E$ (as a $G$-torsor). Then $E$ has a natural structure of $(\text{Aut}(E) \times \mathcal{G})$-bitorsor. In addition, we claim that the group scheme $\text{Aut}(E)$ has a canonical flat connection and $E \in \text{LocSys}_{\text{Aut}(E) \times \mathcal{G}}$. To see this, observe that $\text{Aut}(E) \simeq E \times^G \mathcal{G}$ and for any $Y \to X$ we have $\text{Aut}(E)_Y \simeq \text{Aut}(E)_Y \times^G \mathcal{G}_Y$. Thus, using the definition of connection in §A.4, we see that connections on $E$ and $\mathcal{G}$ induces an isomorphism $p_1^*\text{Aut}(E) \simeq p_2^*\text{Aut}(E)$, where $p_1, p_2 : X^{(1)} \to X$. In addition, one can easily check that this isomorphism satisfies the compatibility conditions and it defines a connection on $\text{Aut}(E)$.

From above construction, it is clear that $E \in \text{LocSys}_{\text{Aut}(E) \times \mathcal{G}}$.

A.7. Some functors. Let $f : Y \to X$ be morphism between smooth schemes and let $\mathcal{G}_X$ be a smooth group scheme on $X$ with flat connection. Let $\mathcal{G}_Y := f^*\mathcal{G}_X$. Then it is easy to see that $\mathcal{G}_Y$ also carries a flat connection and the pullback of $f$ defines a functor

$$(A.3) \quad f^* : \text{LocSys}_{\mathcal{G}_X} \to \text{LocSys}_{\mathcal{G}_Y},$$

sometimes called the pullback functor.
Given two smooth affine group schemes $\mathcal{G}_1$ and $\mathcal{G}_2$ with connections and a groups scheme homomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ which is compatible with the connection (we call $h$ horizontal). For any $(E, \nabla) \in \text{LocSys}_{\mathcal{G}_1}$, we can form the fiber product $h_*E = E \times^h \mathcal{G}_2$, which is a $\mathcal{G}_2$-torsor. Moreover, one can show (for example by using the formulation in Example A.4) that the connection on $E$ will induce a connection $h_*\nabla$ on $h_*E$. Thus the assignment $(E, \nabla) \to (h_*E, h_*\nabla)$ defines a functor

$$h_* : \text{LocSys}_{\mathcal{G}_1} \to \text{LocSys}_{\mathcal{G}_2},$$

sometimes called the induction functor.

Let $\mathcal{G}_2$ and $\mathcal{G}_3$ be two $\mathcal{D}_{X/S}$-groups. Let $P \in \text{LocSys}_{\mathcal{G}_2}$ and $E \in \text{LocSys}_{\mathcal{G}_1 \times \mathcal{G}_3}$. We can form the fiber product

$$P \otimes E := P \times_{\mathcal{G}_1} E,$$

which is a $\mathcal{G}_3$-torsor. Moreover, connections on $P$ and $E$ will induce a connection $\nabla_{P \otimes E}$ on $P \otimes E$ and the assignment $(P, \nabla_P) \times (E, \nabla_E) \to (P \otimes E, \nabla_{P \otimes E})$ defines a functor

$$\otimes : \text{LocSys}_{\mathcal{G}_2} \times \text{LocSys}_{\mathcal{G}_1 \times \mathcal{G}_3} \to \text{LocSys}_{\mathcal{G}_3},$$

sometimes called the tensor functor.

A.8. The $p$-curvature. Let us now assume that $pO_S = 0$. The $p$-curvature of the $\mathcal{D}_{X/S}$-group $\mathcal{G}$ is a mapping

$$\Psi(\nabla_\mathcal{G}) : F^* T_X' \to \text{Der}_{O_X}(O_\mathcal{G}),$$

such that for any $v \in F^* T_X'$, the following diagram is commutative

$$\begin{array}{ccc}
O_\mathcal{G} & \xrightarrow{m} & O_\mathcal{G} \otimes_{O_X} O_\mathcal{G} \\
\Psi(\nabla_\mathcal{G})(v) \downarrow & & \downarrow \Psi(\nabla_\mathcal{G})(v) \otimes \text{id} + \text{id} \otimes \Psi(\nabla_\mathcal{G})(v) \\
O_\mathcal{G} & \xrightarrow{m} & O_\mathcal{G} \otimes_{O_X} O_\mathcal{G}.
\end{array}$$

By Example A.1, $\Psi(\nabla_\mathcal{G}) = 0$ if and only if the canonical map $F^* \mathcal{G} \nabla \to \mathcal{G}$ is an isomorphism.

Likewise, let $(E, \nabla) \in \text{LocSys}_{\mathcal{G}}$. Then the $p$-curvature of $(E, \nabla)$ is a mapping

$$\Psi(\nabla) : F^* T_X' \to \text{Der}_{O_X}(O_E, O_E),$$

such that for any $v \in F^* T_X'$, the following diagram is commutative

$$\begin{array}{ccc}
O_E & \xrightarrow{m} & O_E \otimes_{O_X} O_\mathcal{G} \\
\Psi(\nabla)(v) \downarrow & & \downarrow \Psi(\nabla)(v) \otimes \text{id} + \text{id} \otimes \Psi(\nabla_\mathcal{G})(v) \\
O_E & \xrightarrow{m} & O_E \otimes_{O_X} O_\mathcal{G}.
\end{array}$$

If $\Psi(\nabla_\mathcal{G}) = 0$, then the image of $\Psi(\nabla)$ lands in $\text{ad}(E)$, and therefore the $p$-curvature mapping is regarded as a section

$$\Psi(\nabla) \in \text{ad}(E) \otimes F^* \Omega_X'.$$

Example A.6. Consider the case of Example A.4. The $F^* \mathcal{G}'$-torsor $F^* E'$ carries a connection and one can easily check that the $p$-curvature of this connection is zero. Therefore, we have a functor

$$F^* : \text{Bun}_{\mathcal{G}'} \to \text{LocSys}_{\mathcal{G}}, \quad E' \mapsto F^* E'.$$

The functor $F^*$ induces an equivalence between the category of $\mathcal{G}'$-torsors on $X'$ and the category of $\mathcal{G}$-local systems with zero $p$-curvature. In fact, the inverse functor is given by $E \mapsto E^\nabla$. 

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A.9. The commutative case. Assume that \( \mathcal{G}' \) is commutative on \( X' \) and \( \mathcal{G} = F^*\mathcal{G}' \). For any \( \mathcal{G} \)-torsor \( E \), \( \text{ad}(E) \simeq \text{Lie} \mathcal{G} \) canonically. If \( (E, \nabla) \in \text{LocSys}_\mathcal{G} \), then by the Cartier descent, \( \Psi(\nabla) \in \Gamma(X, \text{Lie} \mathcal{G} \otimes F^*\Omega_{X'/S}) \) is the pullback of a unique element in \( \Gamma(X', \text{Lie} \mathcal{G}' \otimes \Omega_{X'/S}) \). Therefore, taking p-curvature can be regarded as a map
\[
h_p : \text{LocSys}_\mathcal{G} \to \Gamma(X', \text{Lie} \mathcal{G}' \otimes \Omega_{X'/S}),
\]
where \( T = \Gamma(X', \text{Lie} \mathcal{G}' \otimes \Omega_{X'/S}) \) is regarded as a space over \( S \), whose fiber over \( s \in S \) is \( \Gamma(X'_s, \text{Lie} \mathcal{G}'|_{X'_s} \otimes \Omega_{X'_s}) \). This is called the p-Hitchin map (see Theorem A.6 for the non-commutative analogue).

As \( \mathcal{G} \) is commutative, given two \( \mathcal{G} \)-local systems \( (E_1, \nabla_1), (E_2, \nabla_2) \), the induction \( E_1 \times^\mathcal{G} E_2 \) is a natural \( \mathcal{G} \)-local system. Then we have
\[
(\text{A.7}) \quad h_p(E_1 \times^\mathcal{G} E_2) = h_p(E_1) + h_p(E_2).
\]
Combining with Example A.6, we have

Corollary A.7. Assume that \( \mathcal{G}' \) is commutative. Then the stack of \( \mathcal{G} \)-local systems with a fixed p-curvature \( \psi \in T \) is a (pseudo) torsor under the Picard stack \( \text{Bun}_{\mathcal{G}'} \).

We will give an interpretation of this (pseudo) \( \text{Bun}_{\mathcal{G}'} \)-torsor, generalizing [OV, Proposition 4.13] for \( \mathcal{G} = \mathbb{G}_m \).

Let \( \mathcal{Z}_{X/S} \) be the sheaf of closed one forms on \( X \). Then \( F_* \mathcal{Z}_{X/S} \) is an \( \mathcal{O}_{X'} \)-module. Recall the sheaf of flat connections \( (\text{Lie} \mathcal{G} \otimes \Omega_{X'/S})^{\text{ct}} \) on the trivial \( \mathcal{G} \)-torsor \( E^0 \) as in A.5. Under our assumptions of \( \mathcal{G} \), \( F_* (\text{Lie} \mathcal{G} \otimes \Omega_{X'/S})^{\text{ct}} = \text{Lie} \mathcal{G}' \otimes \mathcal{O}_{X'} \), \( F_* \mathcal{Z}_{X/S} \), where \( \mathcal{Z}_{X/S} \) is the sheaf of closed one-forms on \( X \). Then the p-Hitchin map \( h_p \) induces an additive map of étale sheaves on \( X' \),
\[
h_p : \text{Lie} \mathcal{G}' \otimes F_* \mathcal{Z}_{X/S} \to \text{Lie} \mathcal{G}' \otimes \Omega_{X'/S}.
\]
As in the case \( \mathcal{G} = \mathbb{G}_m \), this maps fits into the following four term exact sequence of sheaves on \( X'_et \).

Proposition A.8. Over \( X'_et \), there is the exact sequence of étale sheaves
\[
1 \to \mathcal{G}' \to F_*\mathcal{G} \xrightarrow{F_*d \log} \text{Lie} \mathcal{G}' \otimes F_* F_* \mathcal{Z}_{X/S} \xrightarrow{h_p} \text{Lie} \mathcal{G}' \otimes \Omega_{X'/S} \to 1,
\]
where \( d \log \) is defined as in A.6.

Proof. It is clear from the definition of \( d \log \) that the kernel of \( F_*d \log \) is \( \mathcal{G}' = \mathcal{G}' \). Next, we show that the sequence is exact at \( \text{Lie} \mathcal{G}' \otimes F_* \mathcal{Z}_{X/S} \). As explained in A.5 \( (E^0, \nabla^0 + d \log(g)) \simeq (E^0, \nabla^0) \). As the latter has zero p-curvature, we see that \( h_p \circ F_*d \log = 0 \). On the other hand, if \( \nabla = \nabla^0 + \omega \) is a flat connection on \( E^0 \) with zero p-curvature, then by Example A.6 \( E = (E^0)^{\nabla} \) is a \( \mathcal{G}' \)-torsor. and there is a canonical isomorphism \( \alpha : F^* E \simeq E^0 \). Étale locally, we can trivialize \( E' \) so we can choose \( \beta : (E^0)^{\nabla} \simeq E' \). Then we obtain a section \( g = \alpha \circ F^* \beta \in \mathcal{G} \), and it is not hard to check that \( \omega = d \log(g) \). Note that if \( \mathcal{G} = \mathbb{G}_m \), the argument shows that this sequence is exact at \( \text{Lie} \mathcal{G}' \otimes F_* \mathcal{Z}_{X/S} \) even Zariski locally on \( X' \), which is well-known.

Finally, we show that \( h_p \) is surjective. Unlike \( \mathcal{G} = \mathbb{G}_m \), there is no explicit formula for the p-linear map of \( \text{Lie} \mathcal{G} \), and the usual argument of the explicit calculation does not apply here.

Recall that if \( \mathcal{V} \) is a locally free \( \mathcal{O}_{X'} \)-module of finite rank, \( \text{Spec Sym}_{\mathcal{O}_{X'}} \mathcal{V}^\vee \) is a vector bundle on \( X' \) whose sheaf of sections are \( \mathcal{V} \). Let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) be the vector bundle on \( X' \) corresponding to \( \text{Lie} \mathcal{G}' \otimes F_* \mathcal{Z}_{X/S} \) and \( \text{Lie} \mathcal{G}' \otimes \Omega_{X'/S} \) respectively. Although \( h_p \) is not \( \mathcal{O}_{X'} \)-linear, we have

Lemma A.9. The map \( h_p \) is induced by a smooth surjective homomorphism of commutative group schemes \( h_p : \mathcal{V}_1 \to \mathcal{V}_2 \).
Assuming the lemma, the surjectivity of \( h_p \) then is clearly. Namely, let \( s : X' \to \mathcal{V}_2 \) be a section, then étale locally on \( X' \), \( s \) can be lifted to a section of \( \mathcal{V}_1 \) as \( h_p \) is smooth surjective.

It remains to prove the lemma. We use an argument similar to [OV] Proposition 2.5 (1). We first give another description of the map \( h_p \). Let \( \mathfrak{C}_{X/S} : F_*Z_{X/S} \to \Omega^0_{X/S} \) be the Cartier operator. Let \( \pi_{X/S} : X' \to X \) be the map over the absolute Frobenius \( F_{R_S} \) of \( S \). Let \((-)^p : \text{Lie}_{\mathfrak{G}'} \to \text{Lie} \mathfrak{G}' \) be the map given by the \( p \)-Lie algebra structure of \( \text{Lie} \mathfrak{G}' \). We claim that for \( z \otimes \omega \in \text{Lie} \mathfrak{G}' \otimes F_*Z_{X/S}, \)
\[
H_p(z \otimes \omega) = z^p \otimes \pi_{X/S}^*(\omega) - z \otimes \mathfrak{C}_{X/S}(\omega).
\]
Indeed, let \( \nabla^0 \) be the canonical connection on the trivial \( \mathfrak{G}' \)-torsor. Then for a vector field \( \xi \in T_{X/S} \), and \( z \otimes \omega \in \text{Lie} \mathfrak{G}' \otimes F_*Z_{X/S} \), one has (as sections of \( T_{\mathcal{V}^0} \))
\[
\langle \nabla^0_\xi, (\omega, \xi) \rangle^p - \langle (\nabla^0_\xi)^p, (\omega, \xi^p) \rangle = \xi^p - (\omega, \xi^p)z,
\]
by the Jacobson identity. By the definition of the Cartier operator, \( (\omega, \xi^p) - \xi^p - (\omega, \xi^p)z = (\mathfrak{C}_{X/S}(\omega), \pi_{X/S}^*(\xi)) \). The claim follows.

We apply \((A.8)\) as follows: The Cartier operator \( \mathcal{C}_{X/S} \) is \( \mathcal{O}_{X/S} \)-linear and surjective, and therefore is induced by a smooth homomorphism \( C_{X/S} : \mathcal{V}_1 \to \mathcal{V}_2 \) of the underlying commutative group schemes. The map \( z \otimes \omega \to z^p \otimes \pi_{X/S}^*(\omega) \) is not \( \mathcal{O}_{X/S} \)-linear, but is still induced from an inseparable homomorphism \( \Phi : \mathcal{V}_1 \to \mathcal{V}_2 \) of the commutative group schemes. In fact, let \( F_{R_X} : X' \to X' \) be the absolute Frobenius of \( X' \). We have \( \mathcal{O}_{X/S} \)-linear maps \( F_{R_X}^*, \text{Lie} \mathfrak{G}' \to \text{Lie} \mathfrak{G}' \) given by the \( p \)-Lie algebra structure and \( F_{R_X}^*, F_*Z_{X/S} \to \Omega^1_{X/S} \) given by the composition \( F_{R_X}^*, F_*Z_{X/S} \to F_{R_X}^*F_* \Omega_{X/S} \to \Omega^1_{X/S} \). Therefore, we have a commutative group scheme homomorphism \( \mathcal{V}_1^{(X')} \to \mathcal{V}_2 \). It is then readily to see that \( \Phi \) is the composition of the relative Frobenius \( \mathcal{V}_1 \to \mathcal{V}_1^{(X')} \) (which is a group homomorphism) with the above homomorphism.

Now \( h_p = \Phi - C_{X/S} \) is a smooth morphism, as \( \Phi \) is inseparable and the differential of \( C_{X/S} \) is surjective. It remains to show that \( \Phi - C_{X/S} \) is surjective at the level of points. We can base change \( h_p \) to a geometric point of \( x \in X' \). As \( h_p \) is smooth, the image contains an open subgroup of \( (\mathcal{V}_2)_x \), as \( (\mathcal{V}_2)_x \) is connected, this subgroup must be the entire \( (\mathcal{V}_2)_x \). The lemma follows.

Now, we will give an interpretation of the pseudo-\( \mathfrak{G}' \)-torsor in Corollary \((A.7)\). Namely, the \( p \)-curvature \( \psi \) gives rise to a section of \( \text{Lie} \mathfrak{G}' \otimes \Omega^1_{X/S} \), and the 4-term exact sequence \((A.8)\) induces a \( \mathfrak{G}' \)-gerbe \( \mathcal{G}_\psi \) on \( X' \).

**Proposition A.10.** Then the \( \text{Bun}_{\mathfrak{G}'} \)-torsor as in Corollary \((A.7)\) is the stack of splittings of the gerbe \( \mathcal{G}_\psi \) over \( S \).

The proof is similar to [OV] §4, Proposition 4.2. One only needs to replace the invertible sheaf \( L \) associated to the \( F_{Y/S} \mathcal{O}_Y \)-torsor \( \mathcal{L} \) in loc. cit. by the structure sheaf \( \mathcal{O}_E \) of the \( F_* \)-torsor \( E \) in the current setting.

A.10. In this subsection we assume that \( S = \text{Spec} \mathbf{k} \), where \( k \) algebraically closed of characteristic \( p > 0 \). We describe the tangent map of \( h_p : \text{LocSys}_S \to \Gamma(X', \text{Lie} \mathfrak{G}' \otimes \Omega^1_{X/k}) \) at the trivial \( \mathfrak{G} \)-torsor with the canonical connection \( x = (E^0, \nabla^0) \). First, the tangent space of \( \text{LocSys}_S \) at \( x \) is given by hypercohomology \( H^1 \) of the following deRham complex
\[
\Omega^0_{X/k}(\text{Lie} \mathfrak{G}) := \{ 0 \to \text{Lie} \mathfrak{G} \to \text{Lie} \mathfrak{G} \otimes \Omega^1_{X/k} \to \text{Lie} \mathfrak{G} \otimes \Omega^2_{X/k} \to \cdots \}.
\]
Recall that there is the “second spectral sequence” with \( E^1 = H^1(X, \Omega^0_{X/k}(\text{Lie} \mathfrak{G})) \) and \( E^2_2 = H^1(X, \mathfrak{H}^1(\Omega^0_{X/k}(\text{Lie} \mathfrak{G}))) \). When \( \mathfrak{G} = \mathbb{G}_m \), this is also known as the conjugate spectral sequence. In particular
\[
E^0_{-1} = \Gamma(X, \mathfrak{H}^1(\Omega^0_{X/k}(\text{Lie} \mathfrak{G}))) \simeq \text{Lie} \mathfrak{G}' \otimes \Omega^1_{X/k}.
\]
Here the last isomorphism is obtained by the Lie algebra version of the exact sequence as in Proposition A.8. The edge morphism of the spectral sequence induces
\[ c : T_s \text{LocSys}_S \cong H^1(X, \Omega^1_{X/k}(\text{Lie} G)) \to H^0(X', \text{Lie} G' \otimes \Omega^1_{X'/k}). \]

**Lemma A.11.** The map \( c \) is equal to \(-dh_p.\)

**Proof.** We follow the argument in [OV, Lemma 4.12]. Recall that we have a natural inclusion \( i : H^0(X', \text{Lie} G' \otimes F_* \mathcal{Z}_{X/k}) \to \text{LocSys}_G \) of the substack of flat connections on the trivial torsor \( E^0. \) It follows by definition that
\[ -d(h_p \circ i) = c \circ di : H^0(X, \text{Lie} G' \otimes F_* \mathcal{Z}_{X/k}) \to H^0(X', \text{Lie} G' \otimes \Omega^1_{X'/k}). \]
More explicitly, they are equal to the Cartier map \( \mathcal{C}_{X/S} \) as introduced in the proof of Proposition A.8. Now let \( v \in H^1(X, \Omega^1_{X/k}(\text{Lie} G)) \) and let \((E, \nabla)\) be the corresponding \( S\)-torsor with connection on \( X[\epsilon]. \) The section \( dh_p(v) \in H^0(X', \text{Lie} G' \otimes \Omega^1_{X'/k}) \) is determined by its pull back to any étale cover of \( X'. \) We can choose an étale cover on which \( E \) is trivial, hence reduced to the case when \( E \) is trivial and \( v \in H^0(X, \text{Lie} G' \otimes F_* \mathcal{Z}_{X/k}). \) \( \square \)

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**Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, 60208, USA**

*E-mail address: chenth@math.northwestern.edu*

**Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, 60208, USA**

*E-mail address: xinwenz@math.northwestern.edu*