RIGIDITY OF CLOSED METRIC MEASURE SPACES WITH NONNEGATIVE CURVATURE

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Abstract. We show that one-dimensional circle is the only case for closed smooth metric measure spaces with nonnegative Bakry-Émery Ricci curvature whose spectrum of the weighted Laplacian has an optimal positive upper bound. This result extends the work of Hang-Wang in the manifold case (Int. Math. Res. Not. 18 (2007), Art. ID rnm064, 9pp).

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional closed Riemannian manifold and \(f \in C^2(M)\). We define a weighted Laplacian on \(M\)
\[
\Delta_f := \Delta - \nabla f \cdot \nabla,
\]
which is a self-adjoint operator with respect to the weighted measure \(e^{-f} dv\) (for short \(d\mu\)), where \(dv\) is the volume element induced by the metric \(g\). The weighted Laplacian is very much related to the Laplacian of a suitable conformal change of the background Riemannian metric. It also naturally arises in potential theory, probability theory and harmonic analysis on complete Riemannian manifolds. Here, the triple \((M, g, e^{-f} dv)\) is customarily called a smooth metric measure space.

On the smooth metric measure space \((M, g, e^{-f} dv)\), Bakry-Émery \cite{Bakry-Émery} (see also \cite{21, 23}) introduced the Bakry-Émery Ricci curvature
\[
Ric_f := Ric + Hess(f),
\]
where \(Ric\) denotes the Ricci curvature of the manifold and \(Hess\) denotes the Hessian with respect to the Riemannian metric. A remarkable feature of \(Ric_f\) is that this tensor relates to the weighted Laplacian via the following Bochner formula
\[
\Delta_f |\nabla u|^2 = 2|Hess(u)|^2 + 2\langle \nabla u, \nabla \Delta_f u \rangle + 2Ric_f(\nabla u, \nabla u).
\]
Moreover, Bakry-Émery Ricci curvature is related to the gradient Ricci soliton:
\[
Ric_f = \lambda g,
\]
where \(\lambda\) is some real constant. The gradient Ricci soliton is called expanding, steady and shrinking, accordingly when \(\lambda < 0\), \(\lambda = 0\) and \(\lambda > 0\). As we all know, The Ricci soliton plays an important role in the theory of the Ricci flow \cite{Chow}. It is a
special solution of the Ricci flow and often arises from the blow up analysis of the singularities of the Ricci flow \[14\].

By the variational characterization, the first nontrivial eigenvalue of the weighted Laplacian on closed metric measure space \((M, g, e^{-f} dv)\) with respect to the weighted measure \(d\mu\) is defined by

\[
\lambda_1 := \inf_{\phi \neq 0} \left\{ \int_M (\nabla \phi, \nabla \phi) d\mu, : \int_M |\phi|^2 d\mu = 1, \int_M \phi d\mu = 1, \phi \in C^\infty(M) \right\}.
\]

The above infimum can be achieved by some smooth eigenfunction \(\phi\). Meanwhile the eigenfunction \(\phi\) satisfies the Euler-Lagrange equation

\[
\Delta_f \phi = -\lambda_1 \phi.
\]

We easily see that if potential function \(f\) is constant, then \(Ric_f\) recovers the ordinary Ricci curvature and the above formulas all reduce to the classical case.

Many interesting rigid results involving Bakry-Émery Ricci curvature have been studied in large part due to their similar properties between Bakry-Émery curvature and Ricci curvature. We refer the readers to \[5\], \[8\], \[11\], \[16\], \[17\], \[18\], \[22\], \[23\], \[29\], \[30\], \[31\] and reference therein. In particular, Munteanu and Wang \[24, 25\], Su and Zhang \[28\], and the author \[32\] proved many interesting splitting results on complete noncompact metric measure spaces under some assumptions on Bakry-Émery Ricci curvature. Recently, various Liouville-type theorems on smooth metric measure spaces were obtained, see for example \[27\] and \[33\]–\[37\].

In this paper, we continue to discuss a rigid result on the closed smooth metric measure space rather than the complete noncompact case. Before introducing our result, we first recall some well-known eigenvalue estimates on closed smooth manifolds with nonnegative Ricci curvature. As we all know, Li and Yau \[20\] applied gradient estimate technique to give a lower bound of the first eigenvalue of the Laplace operator on a closed manifold with nonnegative Ricci curvature:

\[
\lambda_1 \geq \frac{\pi^2}{d^2},
\]

where \(d\) is the diameter of the manifold. Later, Zhong and Yang \[38\] improved this result to

\[
\lambda_1 \geq \frac{\pi^2}{d^2}.
\]

Recently, there exist some alternate proofs of this result in \[1, 2\] and \[26\]. We also see that the above estimate is optimal as equality holds on \(S^1\). Moreover, Hang and Wang \[15\] proved that \(S^1\) is the only case for the case \(\lambda_1 = \pi^2/d^2\). Their proof relies on a strong maximum principle and a careful geometrical analysis, which is not only to simply analyze the proof course of the Zhang-Yang’s inequality becoming the equality. On the other hand, Zhong-Yang’s result was extended by Chen and Wang \[9, 10\] via probabilistic approach, and further generalized by Bakry and Qian \[6\] to the smooth metric measure spaces. In particular, they proved that

**Theorem A.** Let \((M, g, e^{-f} dv)\) be a closed smooth metric measure space with nonnegative Bakry-Émery Ricci curvature. Then

\[
\lambda_1 \geq \frac{\pi^2}{d^2},
\]

where \(d\) is the diameter of the manifold \(M\).
We remark that Theorem A has been generalized by B. Andrews and L. Ni [3], and A. Futaki and Y. Sano [13], and further improved by A. Futaki, H.-Z. Li and X.-D. Li [12] based on the arguments of Chen and Wang [9, 10].

Motivated by the Hang-Wang’s result [15], we may ask if there exists a Hang-Wang type rigid result in closed smooth metric measure spaces. That is to say whether or not $S^1$ is the only example for the case $\lambda_1 = \frac{\pi^2}{d^2}$ in Theorem A? The purpose of this short note is to give an affirmative answer. Our main result is

**Theorem 1.1.** Let $(M, g, e^{-f}dv)$ be a closed smooth metric measure space with nonnegative Bakry-Émery Ricci curvature. Assume that the first nontrivial eigenvalue of the weighted Laplacian satisfies

$$\lambda_1 = \frac{\pi^2}{d^2},$$

where $d$ is the diameter of the manifold $M$. Then $M$ is isometric to the circle of radius $\frac{d}{\pi}$ and $f$ is constant.

The main arguments to prove Theorem 1.1 comes from Hang-Wang [15], where the gradient estimate, the maximum principle and some analysis trick are explored. In our case, the proof not only depends on Hang-Wang’s arguments [15], but also relies on the weighted gradient estimate and the weighted Bochner formula. If $f$ is constant, then Theorem 1.1 returns to Hang-Wang’s result.

**Remark 1.2.** Recently, S. Lakzian [19] extended Hang-Wang’s rigidity result to a general setting of metric measure spaces satisfying $RCD(0,N)$ curvature-dimension conditions. If manifold $M$ is complete noncompact, Munteanu and Wang [24] established a sharp upper bound of the first nonzero eigenvalue of the weighted Laplacian in terms of the linear growth rate of $f$. They also proved that if equality holds on the eigenvalue upper estimate and $M$ is not connected at infinity, then $M$ must be a cylinder.

By modifying the proof of Theorem 1.1, we also have a similar result for the first nonzero eigenvalue of the weighted Laplacian with respect to the Neumann boundary condition of a smooth metric measure space.

**Theorem 1.3.** Let $(M, g, e^{-f}dv)$ be a compact smooth metric measure space with nonnegative Bakry-Émery Ricci curvature and nonempty convex boundary. Then the first nontrivial eigenvalue of the weighted Laplacian with respect to the Neumann boundary condition satisfies

$$\mu_1 \geq \frac{\pi^2}{d^2},$$

where $d$ is the diameter of the manifold $M$. Moreover if the above inequality becomes equality, then $M$ is isometric to a line segment and $f$ is constant.

The rest of this paper is organized as follows. In Section 2 we first recall the proof of Theorem A and then give an important lemma (see Lemma 2.2). In Section 3 we apply Lemma 2.2 and the strong maximum principle to prove Theorem 1.1.

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2. A KEY LEMMA

In this section, we first recall the proof of Theorem A, which nearly follows the proofs of Li-Yau \[20\] and Zhong-Yang \[38\]. Here we sketch the proof for the reader’s convenience. Let \((M, g, e^{-f}dv)\) be a closed smooth metric measure space. Let \(\phi\) be the first eigenfunction of the weighted Laplacian. By multiplying with a constant it is possible to arrange that

\[
a - 1 = \min_M \phi, \quad a + 1 = \max_M \phi,
\]

where \(0 \leq a(\phi) < 1\) is the median of \(\phi\). Letting \(u = \phi - a\), then

\[
\Delta f u = -\lambda_1 (u + a).
\]

Following the arguments of \[20\] and \[38\], we can establish the following gradient estimate of the function \(u\).

**Proposition 2.1.** Let \((M, g, e^{-f}dv)\) be a closed smooth metric measure space with nonnegative Bakry-Émery Ricci curvature. Then

\[
|\nabla u|^2 \leq \lambda_1 (1 - u^2) + 2a\lambda_1 z(u),
\]

where \(u = \phi - a\) and \(z(u) = 2\pi (\arcsin u + u\sqrt{1 - u^2}) - u\), \(u \in [-1, 1]\).

It is clear that \(z(u)\) is continuous on \([-1, 1]\) and \(z(-u) = -z(u)\). From Proposition 2.1, we can deduce \(\lambda_1 \geq \pi^2 / d^2\) as follows. Let \(x_1, x_2 \in M\), such that \(u(x_1)\) is the maximizing point and \(u(x_2)\) is the minimizing point. Take a minimal geodesic \(\gamma\) from \(x_2\) to \(x_1\) with length at most \(d\). Integrating the estimate (2) along this segment with respect to arclength and using oddness,

\[
d\lambda_1^{1/2} \geq \lambda_1^{1/2} \int_{\gamma} |\nabla u| ds \geq \int_{\gamma} \frac{|\nabla u| ds}{\sqrt{1 - u^2 + 2az(u)}}
\]

\[
\geq \int_0^1 \left( \frac{1}{\sqrt{1 - u^2 + 2az}} + \frac{1}{\sqrt{1 - u^2 - 2az}} \right) du
\]

\[
\geq \int_0^1 \frac{1}{\sqrt{1 - u^2}} \left( 2 + \frac{3a^2 z^2}{1 - u^2} \right) du
\]

\[
\geq \pi + 3a^2 \left( \int_0^1 \frac{zdu}{\sqrt{1 - u^2}} \right)^2
\]

\[
= \pi + \frac{3a^2}{\pi^2} \left( \frac{\pi}{2} - 1 \right)^4.
\]

Hence \(\lambda_1 \geq \pi^2 / d^2\) and the inequality is strict unless \(a = 0\) (i.e. \(\min_M u = -1\)).

From the above proof, we easily see that on \(S^1\), the above inequalities all become equality. Naturally, we ask if \(S^1\) is the only case for the equality case. The answer is YES! In the rest of this note, we will explain this fact.

At first, we derive a differential inequality on the dense open set which consists of all regular points of the eigenfunction.

**Lemma 2.2.** Let \((M^n, g, e^{-f}dv)\) be a closed smooth metric measure space. Let \(u\) be a nonzero smooth function on this measure space such that

\[
\Delta f u = -\lambda u.
\]
Then on $\Omega = \{ \nabla u \neq 0 \}$,
\begin{equation}
\Delta f \psi - \frac{\nabla \psi \cdot \nabla (\psi - 2\lambda u^2)}{2|\nabla u|^2} \geq 2\text{Ric}_f(\nabla u, \nabla u),
\end{equation}
where $\psi := |\nabla u|^2 + \lambda u^2$.

**Proof of Lemma 2.2** The proof of this result follows from that of Lemma 1 in [15] with little modification, but is included for completeness. Following the computation method of [20], let $e_1, e_2, \ldots, e_n$ be a local orthonormal frame field on $M^n$. We adopt the notation that subscripts in $i, j, k$, with $1 \leq i, j, k \leq n$, mean covariant differentiations in the $e_i, e_j$ and $e_k$, directions respectively.

Differentiating $\psi$ in the direction of $e_i$, we have
\[ \psi_i = 2 \sum_j u_j u_{ij} + 2\lambda uu_i, \]
and so
\[ \left| \frac{1}{2} \nabla \psi - \lambda u \nabla u \right|^2 = \sum_i \left( \sum_j u_j u_{ij} \right)^2 \leq |\nabla^2 u|^2 \cdot |\nabla u|^2, \]
where the summation convention is adopted on repeated indices. This implies
\[ \frac{1}{4}|\nabla \psi|^2 - \lambda u \nabla u \cdot \nabla \psi \leq |\nabla u|^2(|\nabla^2 u|^2 - \lambda^2 u^2). \]
Therefore on $\Omega = \{ \nabla u \neq 0 \}$, we have
\begin{equation}
|\nabla^2 u|^2 - \lambda^2 u^2 \geq \frac{|\nabla \psi|^2 - 4\lambda uu \cdot \nabla \psi}{4|\nabla u|^2} = \frac{\nabla (\psi - 2\lambda u^2) \cdot \nabla \psi}{4|\nabla u|^2}. 
\end{equation}
On the other hand, using the Bochner formula (11), we conclude that
\[ \frac{1}{2} \Delta f \psi = |\nabla^2 u|^2 + \nabla u \cdot \nabla \Delta f u + \text{Ric}_f(\nabla u, \nabla u) + \lambda |\nabla u|^2 + \lambda u \Delta f u \]
\[ = |\nabla^2 u|^2 - \lambda^2 u^2 + \text{Ric}_f(\nabla u, \nabla u), \]
where we used $\Delta f u = -\lambda u$. Combining this with (11) yields (3).

3. **Proof of Theorem 1.1**

In this section we will prove Theorem 1.1. Since the idea of proof comes from Hang-Wang [15], we only provide main steps and omit tedious discussions.

**Proof of Theorem 1.1** Assume that $\lambda_1 = \pi^2/d^2$. From the proof of Proposition 2.1 in Section 2, we easily get $a = 0$, and hence
\[ \min_M u = -1 \quad \text{and} \quad \max_M u = 1, \]
where $u = \phi$ is a first eigenfunction. By scaling the metric, we can assume $d = \pi$. So $\lambda_1 = 1$. Let $\psi = |\nabla u|^2 + u^2$. By Lemma 2.2 on $\Omega = \{ \nabla u \neq 0 \}$, we have
\begin{equation}
\Delta f \psi - \frac{\nabla \psi \cdot \nabla (\psi - 2\lambda u^2)}{2|\nabla u|^2} \geq 0.
\end{equation}
Using the maximum principle, we conclude that
\[ \psi := |\nabla u|^2 + u^2 \leq \max_{\{\nabla u = 0\}} (|\nabla u|^2 + u^2) = 1, \]
since $\psi$ can not attain the maximum value at the point of $\Omega = \{ \nabla u \neq 0 \}$. 

Take two points \( p_0 \) and \( p_1 \) such that 
\[
u(p_0) = -1 \quad \text{and} \quad u(p_1) = 1,\]
and let \( \gamma : [0, l] \to M \) be a unit speed minimizing geodesic from \( p_0 \) to \( p_1 \). We define a function \( y(t) = u(\gamma(t)) \). Then
\[
|y'(t)| = |\nabla u(\gamma(t)) \cdot \gamma'(t)| \leq |\nabla u(\gamma(t))| \leq \sqrt{1 - y^2(t)}.
\]
Hence
\[
\pi \geq l \geq \int_{\{0 \leq t \leq l, y'(t) > 0\}} dt \geq \int_0^l \frac{y'(t) dt}{\sqrt{1 - y^2(t)}} = \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}} = \pi.
\]
Therefore
\[
l = \pi \quad \text{and} \quad y'(t) > 0
\]
for almost every \( t \in (0, \pi) \). Hence \( y(t) \) is strictly increasing on \([0, \pi]\). Moreover, we also have
\[
\int_0^\pi \frac{y'(t) dt}{\sqrt{1 - y^2(t)}} = \pi,
\]
which implies that \( y'(t) = \sqrt{1 - y^2(t)} \) for all \( t \in [0, \pi] \). Since \( y(0) = -1 \) and \( y'(0) = 0 \), then 
\[
y(t) = u(\gamma(t)) = -\cos t
\]
for \( t \in [0, \pi] \). It follows that
\[
(\nabla^2 u)(\gamma'(0), \gamma'(0)) = 1.
\]
Since \( \Delta f (p_0) = -\lambda_1 u(\gamma(0)) = 1 \), \( (\nabla f \cdot \nabla u)(p_0) = 0 \) and \( (\nabla^2 u)_{p_0} \geq 0 \), we conclude that \( \Delta u(p_0) = 1 \) and hence we must have
\[
(\nabla^2 u)_{p_0} = \lambda_{\gamma'(0)} \otimes \lambda_{\gamma'(0)},
\]
where for any tangent vector \( X \), \( \lambda_X \) is the dual cotangent vector given by \( \langle X, Y \rangle = \langle X, \lambda_Y \rangle \) for any tangent vector \( Y \).

Next, similar to the Hang-Wang’s argument \([15]\), we get

**Proposition 3.1.** The set \( \{u = \pm 1\} \) has at most four points.

**Proof.** We only discuss the case \( \{u = 1\} \) since the case \( \{u = -1\} \) is similar. For any point \( p \) with \( u(p) = 1 \), we choose a minimizing geodesic \( \gamma_p : [0, l_p] \to M \) from \( p_0 \) to \( p \). Then the same argument as before shows that
\[
l_p = \pi \quad \text{and} \quad (\nabla^2 u)_{p_0} = \lambda_{\gamma'(0)} \otimes \lambda_{\gamma'(0)},
\]
which implies \( \gamma'_p(0) = \pm \gamma'(0) \). Hence \( p = \exp(\pi \gamma'_p(0)) \) has at most two choices. \( \square \)

In the next step, to finish the proof of Theorem \([11]\) we only need to claim that the dimension of \( M \) must be one. Argue by contradiction. If the claim is not true, then we assume that \( \dim M = 2 \). If we let
\[
M^* = M \setminus \{u = \pm 1\},
\]
then \( M^* \) is still connected. In the following we want to show \( |\nabla u|^2 + u^2 = 1 \) on \( M^* \). In fact, we consider
\[
E = \{p \in M^* : |\nabla u(p)|^2 + u^2(p) = 1\}.
\]
Clearly, \( E \) is closed. On the other hand, if \( p \in E \cap \Omega \), by (5) and the strong maximum principle, we have
\[
|\nabla u|^2 + u^2 \equiv 1
\]
near \( p \). Hence \( E \) must be either an empty set or \( M^* \). Since for any \( t \in (0, \pi) \),
\[
|\nabla u(\gamma(t))|^2 + u^2(\gamma(t)) \geq \cos^2 t + \sin^2 t = 1,
\]
we see \( E \) is nonempty and therefore \( E = M^* \). Now we define \( X = \frac{\nabla u}{|\nabla u|} \) on \( M^* \).
Since \( |\nabla u|^2 + u^2 \equiv 1 \), differentiating it yields
\[
\nabla^2 u(X, X) = -u.
\]
We also notice that the proof of Lemma 2.2 easily implies that
\[
|\nabla^2 u|^2 = u^2
\]
on \( M^* \), since \( \psi = |\nabla u|^2 + u^2 \equiv 1 \). Combining the above two equalities, we have
\[
\nabla^2 u = -u\alpha \otimes \lambda X.
\]
Direct calculation shows that \( \nabla X X = 0 \), and hence all integral curves of \( X \) are geodesics. Let \( \Sigma = \{ u = 0 \} \). Since \( |\nabla u| = 1 \) on \( \Sigma \), we see that \( \Sigma \) is a hypersurface, which may have more than one components. For any \( p \in \Sigma \), let \( \alpha_p \) be the maximal integral curve of \( -X \) with \( \alpha_P(0) = p \). Then \( \alpha_p \) is a unit speed geodesic. Letting \( y_p(t) = u(\alpha_p(t)) \), we know that
\[
y_p(0) = 0 \quad \text{and} \quad y_p'(t) = -\sqrt{1 - y_p^2(t)}.
\]
It gives that
\[
y_p(t) = -\sin t \quad \text{for} \quad t \in [0, \pi/2).
\]
On the other hand, \( \alpha_p \) is a geodesic on \( M \), defined on \( [0, \infty) \). We have
\[
u(\alpha_p(t)) = -\sin t
\]
for \( t \in [0, \pi/2] \). In particular, \( u(\alpha_p(\pi/2)) = -1 \). The same argument as before shows
\[
(\nabla^2 u)_{\alpha_p}(\tilde{z}) = \lambda_{\alpha_p}(\tilde{z}) \otimes \lambda_{\alpha_p}(\tilde{z})
\]
Here \( p = \exp_{\alpha_p(0)}(-\frac{\pi}{2} \alpha_p'(\tilde{z})) \). Since there are at most two points in the set \( \{ u = -1 \} \), we may find point \( q \) satisfying \( u(q) = -1 \) and infinitely many \( p \in \Sigma \) such that \( \alpha_p(\tilde{z}) = q \). This clearly leads to a contradiction since \( \alpha_p'(\tilde{z}) \) has at most two choices. Therefore the dimension of \( M \) must be one. At this time, we easily see that \( Ric(M) = 0 \) and \( Hess(f) \geq 0 \) on \( S^1 \). Hence \( f''(t) = 0 \) on \( S^1 \), and \( f \) is constant.

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