The role of $D_{(s)}^*$ and their contributions in $B_{(s)} \to D_{(s)}hh'$ decays

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Abstract

We demonstrate the roles of $D_{(s)}^*$ and their contributions in the quasi-two-body decays $B_{(s)} \to D_{(s)}hh'$ ($h, h' = \{\pi, K\}$) in the perturbative QCD approach, stemming from the quark flavour changing $\bar{b} \to \bar{c} q_2 \bar{q}_1$ and $\bar{b} \to c \bar{q}_1 \bar{q}_2$ with $q_1, q_2 = \{s/d, u\}$. The main motivation of this study is the measurements of significant derivations from the simple phase-space model in the channels $B_{(s)} \to D_{(s)}hh'$ at $B$ factories and LHC, which is now clarified as the Breit-Wigner-tail effects from the corresponding intermediate resonant states $D_{(s)}^*$. We confirm that these effect from $D^*$ is small ($\sim 5\%$) in the quasi-two-body $B_{(s)} \to D\pi\pi(K)$ decaying channels, and predict the tiny (< 1%) contributions from $D_{s}^*$ in the $B_{(s)} \to D_{s}K\pi(K)$ decaying channels, our result for the $B_{s} \to DK\pi(K)$ decaying channels contributed only from the Breit-Wigner-tail effect of $D_{s}^*$ is in agreement with the current LHCb measurement. The hierarchy of Breit-Wigner-tail effect in different channels shown in this paper can be understood intuitively by the tail of energy dependent width distributed in the invariant mass spectral, which locates above the $D_{(s)}h$ threshold.

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I. INTRODUCTION

Three-body $B$ decays have much richer phenomenology with the number of channels being about ten times larger than the number of two-body decays. With the non-trivial kinematics described by two invariant masses of three-body final states, it provides another wonderful site to study the hadron spectroscopy and the intermediate resonant structures. From the QCD side it is also interesting to investigate the factorisation theorem and also the nonfactorizable contribution in three-body decaying channels [1]. In 2013, as the first time, the LHCb collaboration observed the appreciable local $CP$ violation in the dalitz plot of $B^\pm \rightarrow K^\pm \pi^+ \pi^-$ and $B^\pm \rightarrow K^\pm K^+ K^-$ decays [2], which switched on a new era to study the mater-antimatter asymmetry. In order to understand and explain the physical observables in the full dalitz plot with abundant phase space and complicated dynamics, the QCD-based approaches, such as the perturbative QCD (pQCD) approach [3–6] and QCD factorization (QCDF) approach [7–10] did the pioneer studies on the quasi-two-body $B$ decays [11–24]. Furthermore, some phenomenological analyses are also done within the $U$-spin, isospin and flavour SU(3) symmetries for the relevant three-body $B$ decays [25–30].

In the traditional framework of QCD-based approaches, $D^*_s$ is usually treated as a stable vector meson state by embodying the heavy quark effective theory (HQET) [31, 32]. Two-body $B$ decays with one charm meson in the final state $B_s \rightarrow D^*_s h'$ have two categories, one is the Cabibbo-Kobayashi-Maskawa (CKM) favoured transition induced by $b \rightarrow c$ decay [33, 34], and the other one induced by $b \rightarrow u$ transition is CKM suppressed for the amplitudes [35, 36]. With the interplay between $b \rightarrow c$ and $b \rightarrow u$ transitions at tree level, the decaying channels $B_s \rightarrow D^*_s K^{(*)}$ give the dominant constraint to the CKM angle $\gamma$ [37]. Theoretical studies on this type of decaying channels are carried out with the factorization-assisted topological-amplitude (FAT) approach [38], the QCDF approach [39] and also the pQCD approach [40–42]. Recently, collaborations at B factories [43–47] and LHC [48–52] have performed lots of dalitz analysis of the processes $B_s \rightarrow D_s h h'$ and shown clearly the resonant structures $D^*_s$ in the $D_s h$ invariant mass spectroscopy, which without doubt enrich our knowledge of $D^*_s$ and promote us to study their contributions in the corresponding three-body $B$ decaying channels.

A new issue attracted attention recently in three-body $B$ decays is the virtual contribution arose from the Breit-Wigner-tail (BWT) effect of the resonant state whose pole mass is lower than the threshold value of the corresponding invariant mass. Within the pQCD approach, the BWT effect from $\rho(770)$ is found to be half larger than the pole mass contribution from the first excited state $\rho(1450)$ in the channel $B^\pm \rightarrow \rho_{\pi^\pm} \rightarrow K^+ K^- \pi^\pm$ [53]. Inspired by the Belle [43], the BaBar [45] and the LHCb [49, 52] collaborations measurements, the BWT effect from resonant state $D^*$ are discussed in $B \rightarrow D \pi h$ decays [54] with showing the indispensable role of $D^{*0}(2007)$ and $D^{*+}(2010)$. This process is also calculated in the pQCD approach with the invariant mass $m_{D^*}>2.1$ GeV [55], with the result
indicating ~ 5% contribution from the BWT effect to the branching ratios. In the channels with resonant states $D^*_s$, some derivations from the single phase-space model have been observed at $B$ factories in the $B \to D_s K \pi(K)$ decays [46, 47], moreover, the dalitz plot analysis from the LHCb collaboration give a rather large fit result of the virtual contribution from $D^*_s$ in the $B_s \to D^0 K^- \pi^+$ decays [48]. These measurements motivate a systemic study of the BWT effect from $D^*_s$ in the three-body $B_{(s)} \to D_{(s)}h h'$ decays.

In this paper we implement the pQCD approach to calculate the branching ratios of quasi-two-body decays $B_{(s)} \to D^*_s h' \to D_{(s)}h h'$ with totally 46 channels, aiming to explore the role of different resonant states, especially to clarify the contributions from possible BWT effect of the ground states $D^*$ and $D^*_s$. We will not discuss the CP violation here because there is no contributions from penguin operators in the single charmed $B$ decays. These type of quasi-two-body $B$ decays happen in two phases, the first one is the weak decay of $b$ quark and the second one is the subsequent strong decay from the resonant states to two stable final states. The pQCD calculation is performed in the standard formalism for two-body $B$ decays with replacing the single meson wave function by the di-meson one, in which the strong decays are represented by means of time-like form factor and parameterized by the relativistic Breit-Wigner function. We will also check the quasi-two-body decays in the narrow width approximation, with which the light-cone distribution amplitude (LCDA) of di-meson system shrinks into a Delta function at the physical pole mass, then the result from directly two-body calculation should be recovered.

The rest of the paper is organized as follows. In Sec. II, we give a brief introduction for the theoretical framework. In Sec. III, the numerical results will be showed. Discussions and conclusions will be given in Sec. IV. Decaying amplitudes and the factorization formulas in the pQCD approach are collected in appendix.

II. FRAMEWORK AND THE (DI-)MESON WAVE FUNCTIONS

Quasi-two-body $B$ decays are usually treated as a marriage problem, where the first ingredient is the weak decay described by the low energy effective hamiltonian [56]

$$\mathcal{H}_{e.f.f} = \frac{G_F}{\sqrt{2}} V_{q_b}^* V_{q'd(s)} \left[ C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu) \right], \quad (1)$$

and the cascaded second ingredient is the strong decay described by matrix element $\langle M_1 M_2| R \rangle$ from the resonant state $R$ to two stable mesons, with the energy eigenstate of $R$ writing by means of Breit-Wigner formula or others. In the case of $B_{(s)} \to D^*_s h' \to D_{(s)}h h'$ as depicted in figure 1, $R = D^*_s$ and $q, q' \in \{ c, u, d \}$, the decay amplitude
FIG. 1. Typical Feynman diagrams for the decay processes $B_{(s)} \rightarrow D_{(s)}^{*} h' \rightarrow D_{(s)} h h'$, $h = (\pi, K)$, $h' = (\pi, K)$. The symbol $\otimes$ and $\times$ denote the weak vertex and all the possible attachments of hard gluons, respectively, the green rectangle represents the vector states $D_{(s)}^{*}$.

can be intuitively understood by

$$
A (B_{(s)} \rightarrow Rh' \rightarrow D_{(s)} h h') = \langle [D_{(s)}] h | R h' | H_{\text{eff}} | B_{(s)} \rangle
$$

$$
= \langle D_{(s)} h | R \rangle \frac{1}{m_{R}^{2} - s - im_{R} \Gamma_{R}(s)} \langle Rh' | H_{\text{eff}} | B_{(s)} \rangle .
$$

We use the conventional kinematics for two-body $B$ decays in the factorizable approaches under the rest frame of $B$ meson,

$$
p_{1} = \frac{m_{B}}{\sqrt{2}} (1, 1, 0_{T}) , \quad k_{1} = \left(0, x_{1} \frac{m_{B}}{\sqrt{2}}, k_{1T} \right),
$$

$$
p_{R} = \frac{m_{B}}{\sqrt{2}} (1, \zeta, 0_{T}) , \quad k_{R} = \left(x_{R} \frac{m_{B}}{\sqrt{2}}, 0, k_{RT} \right),
$$

$$
p_{3} = \frac{m_{B}}{\sqrt{2}} (0, 1 - \zeta, 0_{T}) , \quad k_{3} = \left(0, x_{3} (1 - \zeta) \frac{m_{B}}{\sqrt{2}}, k_{3T} \right).
$$

Here $p_{1}$ and $k_{1}$ represent the momentum of $B$ meson and the light spectator quark in $B$ meson, respectively, with $x_{1}$ being the longitudinal momentum fraction. $p_{R}$ and $p_{3}$ are the momentum of resonant state $D_{(s)}^{*}$ and pseudoscale meson $h'$, with the corresponding longitudinal momentum fractions $x_{R}$ and $x_{3}$, respectively. The variable $\zeta \equiv p_{R}^{2} / m_{B}^{2}$ describes the momentum transfer from $B$ meson to resonant state $R$, in the case of full recoiled $\zeta = 1$.

For the sake of generality, we go beyond the narrow resonance approximation and take the energy dependent width [48–52],

$$
\Gamma_{R}(s) = \Gamma_{R}^{\text{tot}} \left( \frac{\beta(s)}{\beta_{R}} \right) \left( \frac{m_{R}}{\sqrt{s}} \right) \left( \frac{1 + s \left[ \beta_{R} r_{BW} \right]^{2}}{1 + s \left[ \beta(s) r_{BW} \right]^{2}} \right),
$$

in which $\beta(s) = \frac{1}{s} \sqrt{s - (m_{D_{(s)}} + m_{h})^{2}} \left[ s - (m_{D_{(s)}} - m_{h})^{2} \right]$ is the non-dimensional phase space factor of $D_{(s)} h$ system, $\beta_{R} \equiv \beta(m_{R})$, the dimensionnal parameter $r_{BW} = 4.0 \text{ GeV}^{-1}$. Nowadays the total width of charged vector $D$ meson is precisely measured $\Gamma_{D_{(s)}^{+}}^{\text{tot}} = 83.4 \pm 1.8 \text{ KeV}$, while the width of its strange partner meson is
still studied with the upper limit $\Gamma_{D_s^{(*)}}^{\text{tot}} = 1.9$ MeV [57]. There are also some theoretical attempts, the dominant partial width in radiative decay is evaluated by lattice QCD with the result $\Gamma_{D^+ \rightarrow D_s^0 \gamma} = 0.066 \pm 0.026$ KeV [58], and also by QCD sum rules with the result $\Gamma_{D^0 \rightarrow D_s^0 \gamma} = 0.59 \pm 0.15$ KeV [59], the second dominant partial width $\Gamma_{D_s^0 \rightarrow D_s^+ \pi^0} = 8.1^{+3.0}_{-2.6}$ eV is obtained from the heavy meson chiral perturbation [60]. We will use the upper limit value in the numerical evaluation to include the largest uncertainty$^1$. For the neutral vector $D$ meson, the result from the isospin analysis $\Gamma_{D_s^0}^{\text{tot}} = 55.3 \pm 1.4$ KeV [61] consists with our previous extraction value 53 KeV [55]. In order to access the virtual contributions (BWT effect) from the state $D_{(s)}^*$ whose pole mass is lower than the threshold value of invariant mass, i.e., $m_R < m_D + m_h$, the pole mass $m_R$ in $\beta_R$ shall be replaced by the effective mass $m_R^{\text{eff}}$ to avoid the kinematical singularity appeared in the phase space factor $\beta(s)$ [49, 52]

$$m_R^{\text{eff}}(m_0) = m_0^\text{min} + (m_0^\text{max} - m_0^\text{min}) \left[1 + \tanh \left(\frac{m_0 - (m_0^\text{max} + m_0^\text{min})/2}{m_0^\text{max} - m_0^\text{min}}\right)\right],$$

here $m_0^\text{max} = m_{D(s)} - m_h$ and $m_0^\text{min} = m_{B(s)} + m_h$ are the upper and lower thresholds of $\sqrt{s}$, respectively.

In the pQCD framework of a quasi-two-body $B$ decays, the physics of strong interactions subsequent to the weak decay are absorbed into the wave functions of di-meson system and single meson, the decay amplitude in Eq. (2) is exactly written as a convolution of the hard kernel $H$ with the hadron distribution amplitudes (DAs) $\phi_B, \phi_{h'}$ and $\phi_{D_{(s)}h}$

$$A(B_{(s)} \rightarrow R h' \rightarrow D_{(s)} h h') = \phi_B(x_1, b_1, \mu) \otimes H(x_1, b_1, \mu) \otimes \phi_{Dh}(x_1, b_1, \mu) \otimes \phi_{h'}(x_3, b_3, \mu),$$

here $\mu$ is the factorization scale, $b_i$ are the conjugate distances of transversal momenta.

$B$ meson DAs are defined under the HQET by dynamical twist expansion [62], at leading twists level the nonlocal matrix element associated with $B$ meson for pQCD calculation is

$$\int d^4z_1 e^{ik\cdot z_1}(0|\bar{d}_\sigma(z_1)b_\beta(0)|B^0(p_1)) = -\frac{if_B}{4N_c} \left\{ (\eta_1 + m_B)\gamma_5 \left[ \phi_B(x_1, b_1) - \frac{\eta_{h'}}{\sqrt{2}} \phi_{h'}(x_1, b_1) \right] \right\}_{\beta\sigma},$$

with the antiquark momentum in the minus direction along the light cone $x_1 = k^-/p_1^-$. We omit the transversal projection term, and the underlying integral $\varphi_{(x_1, b_1)} = \int d\eta^+ d^2k_{1T} e^{i\eta_1 b_1} \varphi(x_1, k_1)$ is implemented. The DA $\bar{\phi}_B$ is highly suppressed by $O(\ln \frac{\Lambda}{m_B})$ in contrast to $\phi_B$ with $\Lambda \simeq m_B - m_h$ [63], and it is zero in the symmetry limit, which would be employed in our calculation to match the current accuracy. The most usual expression of DA is parameterized in the exponential model

$$\phi_B(x_1, b_1) = N_B x_1^2 (1 - x_1)^2 \exp \left[ -\frac{x_1^2 m_B^2}{2 \omega_B^2} - \frac{(\omega_B b_1)^2}{2} \right],$$

$^1$ This value is also employed by LHCb collaboration in the study of virtual contribution from $D_s^*$ in the decaying channel $B_s^0 \rightarrow \bar{D}^0 K^- \pi^+$ [48].
with the normalization condition

$$\int_0^1 dx_1 \phi_B(x_1, b_1 = 0) = 1,$$

(9)

and the first inverse moment $\omega_B = 2/3\Lambda$ [64, 65].

Because the transversal polarisation of $V$ meson does not contribute in $B \to VP$ decays, we take into account only the longitudinal polarisation wave function of $D_s h$ system, whose P-wave component with possible resonance $D_{(*)}^r$ is expressed by [31, 32]

$$\Phi_{D_{(*)}^r}^P = \frac{1}{\sqrt{2N_c}} \int L \left( p_R + \sqrt{s} \right) \phi_{D_{(*)}^r}(x, b, s),$$

(10)
in which the LCDA at leading twist has the same Gegenbauer expansion as the vector meson $D_{(*)}^r$

$$\phi_{D_{(*)}^r}(x, b, s) = \frac{F_{D_{(*)}^r}(s)}{2\sqrt{2N_c}} \frac{6x(1-x)}{[1+a_{D_{(*)}^r} (1 - 2x)] \exp \left[ -\frac{\omega_{D_{(*)}^r}^2 b^2}{2} \right]}.$$

The only difference is the time-like form factor $F_{D_{(*)}^r}(s)$ which in the case of single meson is the decay constant $f_{D_{(*)}^r}$. In the $D_{(*)}^r$ dominant approximation this form factor is defined and expressed as

$$F_{D_{(*)}^r}(s) \equiv \left\{ \frac{s \bar{p}_R \langle D_{(*)}^r | \bar{c} \gamma_\mu (1 - \gamma_5) q | 0 \rangle}{s^2 - 2s \left( m_{D_{(*)}^r}^2 + m_b^2 \right) + \left( m_{D_{(*)}^r}^2 - m_b^2 \right)^2} \right\} = \frac{\sqrt{s} f_R \Gamma_{D_{(*)}^r}}{m_{D_{(*)}^r}^2 - s - im_{D_{(*)}^r}} (R),$$

(12)

where $f_R$ is the decay constant of resonant state, $\bar{p}_R$ denotes the momentum difference of $D_{(*)}$ and $h$ mesons in the $D_{(*)}h$ system, and the strong coupling is defined by means of the matrix element $g_{RD_{(*)}h} \equiv \langle D_{(*)}h | R \rangle$. With the precisely measured of $g_{D^* \pi^0 \pi^+} = 16.92 \pm 0.13 \pm 0.14$ [66, 67] and the the universal relation [68]

$$\frac{g_{D^* \pi^0}}{2\sqrt{m_{D^*} m_D}} = \frac{g_{D^* D \pi}}{2\sqrt{m_{D^*} m_D}} = 1, \quad g_{D^* D}$$(13)

we obtain $g_{D^* D} = 14.6 \pm 0.06 \pm 0.07$ and $g_{D^* D, K} = 14.6 \pm 0.10 \pm 0.13$. They are consistent with $g_{D^* D} = 14.6 \pm 1.7$ and $g_{D^* D, K} = 14.7 \pm 1.7$ [69] extracted from the CLEO collaboration [70], and also comparable to the predictions $g_{D^* D} = 15.2$, $g_{D^* D, K} = 15.2$ from quark model [71]. By the way, $a_{Dh}$ is the first gegenbauer coefficient in the polynomial expansion, $\omega_{Dh}$ denotes the first inverse momentum of P-wave $D_{(*)}h$ state, for these two parameters, we use the value of their partner vector meson $D_{(*)}^r$ in the numerical evaluation.

Wave function of the single pseudoscalar meson $h' = \pi, K$ is defined by the nonlocal matrix element [72, 73], we here take $\pi^-$ for example,

$$\int d^4 z_2 e^{i k_3 z_3} \langle \pi^- (p_3) | \bar{d}_3 (z_3) u_\alpha (0) | 0 \rangle$$

$$= \frac{-if_{\pi}}{4N_c} \left\{ \gamma_5 \left[ \bar{p}_3 \phi_\pi (x_3, b_3) + m_0^\pi \phi_\pi^P (x_3, b_3) + m_0^\pi \left( \not \! p + \not \! n_+ - 1 \right) \phi_\pi^P (x_3, b_3) \right] \right\} \alpha \delta.$$

(14)
Once again, the integral \( \phi_\pi(x, b_3) = \int dk_3^+ d^2k_3 T e^{ik_3T \cdot b_3} \phi_\pi(k_3) \) is indicated. The decay constant \( f_\pi \) reflects the local matrix element between the vacuum and the pion meson state,

\[
\langle \pi^+(p) | \bar{u}(0)(\mp \gamma_\tau \gamma_5) d(0) | 0 \rangle = \pm if_\pi p_\tau .
\] (15)

\( \phi_\pi \) is the leading twist LCDA, and \( \phi_\pi^{p,t} \) are the twist three ones, with whom the chiral mass \( m_0^\pi \equiv m_\pi^2 / (m_u + m_d) \) originates from the equation of motion. The light-cone vectors are defined as \( n_+ = (1, 0, 0) \), \( n_- = (0, 1, 0) \).

The differential branching ratios for the quasi-two-body \( B_{(s)} \rightarrow D_{(s)}^* h' \rightarrow D_{(s)} h h' \) decays is written as

\[
\frac{d\mathcal{B}}{d\zeta} = \frac{\tau_B q_{h'}^3 q_\zeta^3}{48 \pi^3 m_B^3} |A|^2 ,
\] (16)

in which \( q_{h'} \) is the magnitude of momentum for the bachelor meson \( h' \)

\[
q_{h'} = \frac{1}{2} \sqrt{\left[ (m_B^2 - m_{h'}^2)^2 - 2 (m_B^2 + m_{h'}^2) s + s^2 \right] / s} ,
\] (17)

and \( q \equiv \beta(s) \sqrt{s} \) is the magnitude of momentum for the daughter meson \( D_{(s)} \) or \( h \) in the rest frame of the \( D_{(s)}^* \). We present all the decay amplitudes \( A \) in the appendix.

III. NUMERICS AND DISCUSSIONS

The input parameters in our numerical evaluation for quasi-two-body decaying process are listed in table I, besides these, the CKM matrix elements in the effective Hamiltonian are determined by Wolfenstein parameters \( \lambda = 0.22650 \pm 0.00048 \), \( A = 0.790^{+0.017}_{-0.012} \), \( \bar{\rho} = 0.141^{+0.016}_{-0.017} \) and \( \bar{\eta} = 0.357 \pm 0.01 \) [57], the chiral masses of light

| TABLE I. The list of input parameters for the quasi-two-body decaying process in our prediction [74–77]. |
|---|---|---|---|---|---|
| meson | \( m \) (MeV) | \( f_M \) (MeV) | \( \Gamma_{D_{(s)}^*}^{\text{tot}} \) (KeV) | \( \omega \) (MeV) | Gegen. moment |
| \( \pi^+ / \pi^0 \) | 140/135 | 130 | — | — | \( a_2 = 0.25 \) |
| \( K^+ / K^0 \) | 494/498 | 156 | — | — | \( a_1 = 0.05 \), \( a_2 = 0.25 \) |
| \( D^{*+} \) | 2010 | 250 ± 11 | 83.4 ± 1.8 | 100 ± 20 | \( a_1 = 0.5 \pm 0.1 \) |
| \( D^{*0} \) | 2007 | 250 ± 11 | 55.3 ± 1.4 | 100 ± 20 | \( a_1 = 0.5 \pm 0.1 \) |
| \( D_s^{*+} \) | 2112 | 270 ± 19 | 1900 | 200 ± 40 | \( a_1 = 0.4 \pm 0.1 \) |
| \( B^{\pm} \) | 5279 | 189 | 1.638 ± 0.004 | 400 ± 40 | — |
| \( B^0 \) | 5280 | 189 | 1.520 ± 0.004 | 400 ± 40 | — |
| \( B_s^0 \) | 5367 | 231 | 1.509 ± 0.004 | 500 ± 50 | — |
TABLE II. The pQCD predictions of the branching ratios for quasi-two-body decays $B^0 \to D^*_{(s)} h^* \to Dhh'$, where the result of channels happened by the BWT effect are denoted by $B_v$. Theoretical uncertainties come from the inputs of $\omega_B$, $f_{D^*}$, $a_{Dh}$, $A$, $\omega_{Dh}$ in turn.

| Decay modes | $B/\mathcal{B}_v$ | Results | Units |
|-------------|------------------|---------|-------|
| $B^0 \to D^{*+}\pi^-$ $\to \bar{D}^0_\pi^-\pi^+$ | $B$ | $1.69^{+0.57+0.15+0.13+0.07+0.04}_{-0.52-0.15-0.11-0.05-0.02}$ | $10^{-3}$ |
| $\to D^-\pi^0\pi^+$ | $B$ | $7.79^{+3.00+0.70+0.33+0.12}_{-2.34-0.67-0.23-0.13}$ | $10^{-4}$ |
| $\to D_s^-K^0\pi^+$ | $B_v$ | $1.25^{+0.62+0.11+0.14+0.05+0.02}_{-0.38-0.11-0.09-0.04-0.01}$ | $10^{-5}$ |
| $B^0 \to D^{*-}K^-\to D^0\pi^+\pi^-$ | $B$ | $1.01^{+0.39+0.09+0.00+0.04+0.02}_{-0.25-0.08-0.00-0.03-0.01}$ | $10^{-6}$ |
| $\to D^+\pi^-\pi^-$ | $B$ | $4.64^{+1.72+0.41+0.03+0.18+0.01}_{-1.22-0.40-0.15+0.01}$ | $10^{-7}$ |
| $\to D^+_sK^0\pi^-$ | $B_v$ | $1.64^{+0.59+0.15+0.00+0.06+0.00}_{-0.42-0.14-0.00-0.04}$ | $10^{-8}$ |
| $B^0 \to D^{*-}K^0\to D^0\pi^+\pi^-$ | $B$ | $1.38^{+0.64+0.16+0.04+0.06+0.04}_{-0.42-0.09-0.10-0.04}$ | $10^{-4}$ |
| $\to D^-\pi^-K^+$ | $B$ | $6.39^{+2.78+0.57+0.15+0.04+0.01}_{-2.13-0.56-0.19-0.28}$ | $10^{-5}$ |
| $\to D^-sK^+\to D_s^0K^0$ | $B_v$ | $1.02^{+0.48+0.10+0.09+0.04+0.03}_{-0.31-0.09-0.08-0.03}$ | $10^{-6}$ |
TABLE III. The same as table II, but for the quasi-two-body $B^+ \rightarrow D^{*+}h^l \rightarrow Dhhl'$ decaying channels.

| Decay modes | $B/B_v$ | Results | Units |
|-------------|---------|---------|-------|
| $B^+ \rightarrow D^{*+}\pi^0 \rightarrow D^{0+}\pi^0$ | $B$ | 5.81$\pm$1.45$\pm$0.52$\pm$0.03$\pm$0.23$\pm$0.05 | $10^{-7}$ |
| $\rightarrow D^+\pi^0\pi^0$ | $B$ | 2.65$\pm$0.70$\pm$0.24$\pm$0.00$\pm$0.10$\pm$0.01 | $10^{-7}$ |
| $\rightarrow D_s^+K^0\pi^0$ | $B_v$ | 9.04$\pm$3.18$\pm$0.44$\pm$0.03$\pm$0.36$\pm$0.01 | $10^{-9}$ |

and annihilation typological diagrams give contributions, the channel $B^0 \rightarrow D^{*+}K^+$ is CKM suppressed $\mathcal{O}(\lambda)$ and only the emission typology contributes to the amplitude, while the channel $B^0 \rightarrow D^{*+}\pi^-$ is CKM doublely suppressed $\mathcal{O}(\lambda^2)$ and simultaneously color suppressed, resulting to the branching ratios four powers smaller in magnitudes than it of the channel $B^0 \rightarrow D^{*+}\pi^+$. For each case of two-body decay, we go a step further to show the possible different strong couplings between $D^{*+}_{(s)}$ and the $D_{(s)}h$ state, say, with $\bar{u}u$, $\bar{d}d$ and $\bar{s}s$ configurations in turn. Once again, we take the two-body decaying channel $B^0 \rightarrow D^{*-}\pi^+$ as the example to explain more. The result of the two strong decaying channels with $u$- and $d$-quark pair configurations obey the isospin relation $g_{D^{*-}\bar{D}^0}\pi^- = -\sqrt{2}g_{D^{*-}\bar{D}^0}\pi^0$, of course, this relation also works for other two similar channels, like $D^{*\mp} \rightarrow D^{0\mp}\pi^\mp$ and $D^{*\mp} \rightarrow D^{\mp}\pi^{0\mp}$ which are both happened by the pole mass dynamics ($m_{D^{*\mp}} > m_{D^{0\mp}} + m_{\pi^\mp}, m_{D^{\mp}} + m_{\pi^{0\mp}}$). The strong decay $D^{*-} \rightarrow D_s^+K^0$ happens by the BWT effect and the branching ratio is apparently much smaller with comparing to the strong decays happened by the pole mass dynamics.

In this work, we do not take into account the CKM and color suppressed pure annihilation quasi-two-body decays, whose pole mass contributions to the branching ratios are already rather small ($[10^{-12}, 10^{-8}]$), with considering the weak decay process $D \rightarrow \pi K$ used to rebuild the events of $D$ mesons has the branching ratio ($[10^{-2}, 10^{-1}]$), these
of this work is a litter bit larger than the previous predictions [55], the reason is that we here consider the invariant mass branchings comes from the first inverse momentum of channels are impossible in the near future experiment. As shown in tables (II-IV), the largest error in our prediction for branching ratios comes from the first inverse momentum of \( B_{(s)} \) meson (\( \omega_{B_{(s)}} \)), the second one comes from the decay constant of intermediate resonant states (\( f_{D_{(s)}} \)), the gegenbauer moment of resonant state (\( a_{Dh} \)) gives the third error, the Wolfenstein parameter (A) is the fourth uncertainty source, and the last uncertainty source is the inverse moment of resonant state (\( \omega_{Dh} \)). For the decay channels \( B^+ \rightarrow D^- \pi^+ \pi^+(K^+) \) happened by the BWT effect, the result in this work is a litter bit larger than the previous predictions [55], the reason is that we here consider the invariant mass of \( D\pi \) stating from their threshold value 2.01 GeV, while in Ref. [55] the evaluation is chosen to start at \( \sqrt{s_0} = 2.1 \) GeV.

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TABLE IV. The same as table II, but for the quasi-two-body \( B_s^0 \rightarrow D_{(s)}^* h' \rightarrow Dh\bar{h}' \) decaying channels.

| Decay modes                              | \( B/B_v \)          | Results                        | Units |
|------------------------------------------|-----------------------|--------------------------------|-------|
| \( B_s^0 \rightarrow D_s^- \pi^+ \rightarrow \bar{D}^0 K^- \pi^+ \)  | \( B_v \)            | \( 1.90 \pm 0.94 \pm 0.28 \pm 0.16 \pm 0.08 \pm 0.14 \) | \( 10^{-5} \) |
| \( \rightarrow D^- \bar{K}^0 \pi^+ \)  | \( B_v \)            | \( 1.83 \pm 0.94 \pm 0.27 \pm 0.15 \pm 0.08 \pm 0.14 \) | \( 10^{-5} \) |
| \( \rightarrow D^- \bar{K}^0 \pi^+ \)  | \( B_v \)            | \( 1.23 \pm 0.66 \pm 0.18 \pm 0.09 \pm 0.05 \pm 0.09 \) | \( 10^{-6} \) |
| \( B_s^0 \rightarrow D_s^- K^+ \rightarrow \bar{D}^0 K^- K^+ \)  | \( B_v \)            | \( 1.28 \pm 0.66 \pm 0.19 \pm 0.10 \pm 0.06 \pm 0.10 \) | \( 10^{-6} \) |
| \( \rightarrow D^- \bar{K}^0 K^+ \)  | \( B_v \)            | \( 1.67 \pm 0.68 \pm 0.15 \pm 0.00 \pm 0.06 \pm 0.04 \) | \( 10^{-8} \) |
| \( B_s^0 \rightarrow D^*^+ K^- \rightarrow D^0 \pi^+ K^- \)  | \( B_v \)            | \( 1.13 \pm 0.46 \pm 0.10 \pm 0.00 \pm 0.04 \pm 0.00 \) | \( 10^{-6} \) |
| \( \rightarrow D^+ \pi^0 K^- \)  | \( B_v \)            | \( 1.54 \pm 2.10 \pm 0.46 \pm 0.01 \pm 0.20 \pm 0.00 \) | \( 10^{-7} \) |
| \( \rightarrow D^+_s \bar{K}^0 K^- \)  | \( B_v \)            | \( 1.67 \pm 0.68 \pm 0.15 \pm 0.00 \pm 0.06 \pm 0.04 \) | \( 10^{-8} \) |
| \( B_s^0 \rightarrow D^+_s \pi^0 \rightarrow \bar{D}^0 \pi^0 \pi^0 \)  | \( B_v \)            | \( 1.46 \pm 0.24 \pm 0.37 \pm 0.50 \pm 0.18 \pm 0.07 \) | \( 10^{-7} \) |
| \( \rightarrow D^- \pi^+ \pi^0 \)  | \( B_v \)            | \( 2.30 \pm 0.25 \pm 0.21 \pm 0.28 \pm 0.10 \pm 0.05 \) | \( 10^{-8} \) |
| \( \rightarrow D^+_s \bar{K}^+ \pi^0 \)  | \( B_v \)            | \( 2.75 \pm 0.23 \pm 0.25 \pm 0.41 \pm 0.12 \pm 0.06 \) | \( 10^{-9} \) |
FIG. 2. The differential branching ratios for the quasi-two-body $B^0 \to D^{*-} \pi^+ \to Dh\pi^+$ decays (left panel) and $B^+ \to \bar{D}^{*0} \pi^+ \to Dh\pi^+$ decays (right panel). The embedded graphs denote the ratios $R_{D^{*-}\to D\pi}$.

In figure 2, we depict the differential branching ratios of channels $B^0 \to D^{*-} \pi^+ \to Dh\pi^+$ and $B^+ \to \bar{D}^{*0} \pi^+ \to Dh\pi^+$ to reveal the relative strength in different strong couplings following a same two-body weak decay. The processes with pole mass dynamical strong decays $D^{*-} \to D^0\pi^-$ (in red), $D^-\pi^0$ (in blue) and the process happened by BWT effect $D^{*-} \to D^- K^0$ (in magenta) following the same $B^+ \to \bar{D}^{*0}\pi^+$ weak decay are shown explicitly on the left panel. In parallel, the processes with pole mass dynamical strong decay $\bar{D}^{*0} \to \bar{D}^0\pi^0$ (in red) and the processes happened by BWT effect $\bar{D}^{*0} \to D^-\pi^+$ (in blue), $D^- K^+$ (in magenta) following the same $B^+ \to \bar{D}^{*0}\pi^+$ weak decay are shown on the right panel. Within the embedded graphs, we display the evolution of ratios

$$R_{D^{*-}\to D\pi} \equiv \frac{d\mathcal{B}(B^0 \to D^{*-}\pi^+ \to D^-\pi^0\pi^+)}{d\mathcal{B}(B^0 \to D^{*-}\pi^+ \to D^0\pi^-\pi^+)}$$

$$R_{\bar{D}^{*0}\to D\pi} \equiv \frac{d\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^+ \to \bar{D}^0\pi^0\pi^+)}{d\mathcal{B}(B^+ \to D^0\pi^+ \to D^-\pi^+\pi^+)}$$

(18)

on the invariant mass to check the isospin relation. It can be seen that the ratio $R_{D^{*-}}$ goes to 0.5 suddenly around the $D^{*-}$ pole, which is natural due to the pole mass dynamics for both the strong decays $D^{*-} \to D^- \pi^0$ and $D^{*-} \to D^0\pi^-$. Nevertheless, the ratio $R_{\bar{D}^{*0}}$ trends to 0.5 from infinity smoothly, the underlying reason is that the $\bar{D}^{*0} \to D^-\pi^+$ process happens by the BWT effect with the threshold value of $D^-\pi^+$ state being a litter bit larger than the $\bar{D}^{*0}$ pole mass, hence the peak of its $d\mathcal{B}/d\sqrt{s}$ curve emerges at the invariant mass a bit bitter larger than the pole mass. Concerning on the BWT effect in the channels $B^0 \to D^{*-} \pi^+ \to D^- K^0\pi^+$ and $B^+ \to D^{*0} \pi^+ \to D^- K^+\pi^+$ (in magenta) induced by the $s$-quark pair configuration, we multiply their result by ten to show clearly for the evolution behaviour, their curves are smooth and the locations of the largest distribution ($2.8 - 2.9$ GeV) are far away from the resonance pole masses by $0.8 - 0.9$ GeV because the threshold values of $D^- K$ states are far away from their resonance pole masses by 0.455 GeV.

In figure 3, we plot the differential branching ratio of $B_s^0 \to D_s^-\pi^+ \to \bar{D}^0 K^-\pi^+$ decay with the invariant mass
FIG. 3. The differential branching ratios for the quasi-two-body decay $B_s^0 \to D_s^{*-} \pi^+ \to \bar{D}^0 K^- \pi^+$ with the invariant mass $\sqrt{s} \in [2.3, 4.0]$ GeV. The embedded graph indicates the evolution on $m_0$ from the $\bar{D}^0 K^-$ threshold value to $m_0^{eff}$.

of $\bar{D}^0 K^-$ state varying in [2.3, 4.0] GeV, we also embed the evolution of total branching ratio on the effective mass. We find that the BWT effect in this channel is at the same order as in the channels $B^0 \to D^{*-} \pi^+ \to D_s^- K^0 \pi^+$ and $B^+ \to D^{*0} \pi^+ \to D_s^- K^+ \pi^+$ (magenta curves in figure 2), while the location of the largest distribution here is more closer to the $\bar{D}^0 K^-$ threshold, because the threshold value 2.359 GeV here is more closer to the resonance pole mass $m_{D_s^{*-}} = 2.112$ GeV. Exceeding the threshold value, the total branching ratio does not displace a dependence on the effective mass, in another word, the width effect in Eq. (4) of $D_s^{*-}$ is negligible here. This can be understood by the Breit-Winger formula in Eq. (2), where the real part is much larger than the imaginary part in the denominator, say, $|m_{D_s^{*-}}^2 - s| \gg |m_{D_s^{*-}} \Gamma_{D_s^{*-}}(s)|$, when $\sqrt{s} > 2.359$ GeV and the total width $\Gamma_{D_s^{*-}} < 1.9$ MeV.

We compare our predictions, in table V, with the available measurements for some channels induced by the BWT effect, where the theoretical and experimental errors are both added in quadrature. The result listed in the second column is obtained with the integral of invariant mass starting from the threshold value. Because the data of the first two channels is obtained by taking the integral by a cut $\sqrt{s} \geq 2.1$ GeV, which is a litter bit larger than the threshold, we list in the third column the pQCD predictions with the same cut [55], denoting by $B^0_{v^{cut}}$. For the channel $B^+ \to \bar{D}^{*0} \pi^+ \to D^+ \pi^- \pi^+$, the prediction is more inclined to the Belle data, while the measurements from BABAR and LHCb do not consist with Belle. For the channel $B^+ \to \bar{D}^{*0} K^+ \to D^+ \pi^- K^+$, the central value of pQCD prediction is about two times larger in magnitude than the LHCb measurement, even though a large uncertainty is associated with experiment data. With considering the well consistence between the measurements and the pQCD predictions for the relevant two-body decays $B^+ \to \bar{D}^{*0} \pi^+$ and $B^+ \to \bar{D}^{*0} K^+$ as shown in table VI, and the fact
that they process the same strong decay, the power behaviour displayed in table V is expected for these two channels.

We hope that the Belle-II and the LHCb will restudy these two channels to reveal the important information of \( D^{*0} \) and the strong decay \( D^{*0} \to D^+\pi^- \). For the channel \( B_s^0 \to D_s^*\pi^+ \to \bar{D}^0K^+ \), the pQCD prediction is in the same power as the data, more data will explain more.

With the calculations for these quasi-two-body decays, we can extract the branching ratios of single charmed two-body \( B \) decays by using the narrow width approximation

\[
B(B \to D_{(s)}^o h' \to D_{(s)} h h') \approx B(B \to D_{(s)}^o h') \cdot B(D_{(s)}^o \to D_{(s)} h)
\]  

(19)

and the measurements \( B(D^{*+} \to D^0 \pi^+) = 67.7\% \), \( B(D^{*+} \to D^+\pi^0) = 30.7\% \) and \( B(D^{*0} \to D^0\pi^0) = 64.7\% \) [57]. The result of the CKM favoured channels is shown in table VI, which is consistent with the direct two-body calculations and agree with the data. For the CKM suppressed decays, only the channel \( B^+ \to D^{*0}K^+ \) has been measured with the branching ratio \( B(B^+ \to D^{*0}K^+) = (7.8 \pm 2.2) \times 10^{-6} \) [57], our extraction here gives \((1.54^{+0.45}_{-0.49}) \times 10^{-6}\), consisting with the result \((0.71^{+0.76}_{-0.53}) \times 10^{-6}\) by direct two-body calculation from pQCD approach [42], however, deviating from the data by \( 4\sigma \). We mark that the result from FAT approach \((11.8^{+3.3}_{-2.5}) \times 10^{-6}\) is in agreement with the data, because the nonfactorizable annihilation-type contribution at there is fit from data and much larger than it calculated from the pQCD approach. LHCb will accumulate much more data to clarify this problem.

### IV. CONCLUSION

In this paper we studied systematically the role of \( D_{(s)}^* \) and their contributions in \( B_{(s)} \to D_{(s)} h h' (h' = \pi, K) \) decays by taking the di-meson LCDAs of \( D_{(s)} h \) system in the framework of the pQCD approach. With the weak
certify the smallness (\(<\ 5\%\)) of the BWT effect in the three-body \(B_{(s)}\) decays with the intermediate resonant states \(D^*\),

decays \(B_{(s)} \rightarrow D^*\ h'\) originating only from the tree level current-current operator, there are no source of weak phase differences to generate \(CP\) violations, so we predicted only the branching ratios. In the local kinematic region where the invariant mass of final \(D(s)\ h\) system locates in/around the interval of \(D^*\) and simultaneously the other invariant mass approaching zero, three-body decay can be treated as a quasi-two-body decaying process and divided into two ingredients. The first one is the \(b\) quark weak decays which have the same formula as in the two-body \(B\) decays, and the strong decays happened subsequently are absorbed into the di-meson wave functions of \(D(s)\ h\) system by means of the time-like form factor.

We calculated in total 46 channels for possible intermediate \(D^*\) contributions in \(B^0, B^+\) and \(B_s\) decays, and clarified the strong decays \(D^*_3 \rightarrow D(s)\ h'\) by \(u, d\)- and \(s\)-quark pair configurations for each resonant structure. Concerning on the charged resonance \(D^*\), the strong decays with the \(u\)- and \(d\)-quark configurations happen by the pole mass dynamics, while the decay with \(s\)-quark configuration happens by the BWT effect. For the neutral resonances \(\bar{D}^*\) and \(D^*\), the pole mass dynamical strong decay happens for the \(u\)-quark configuration, and the BWT effect induced strong decays work for the \(d\)- and \(s\)-quark configurations. The strong decays of \(D^*_3\) following from the \(B_s\) weak decay can only happen by the BWT effect for both the \(u\)- and \(d\)-quark configurations. Our predictions certify the smallness (\(<\ 5\%\)) of the BWT effect in the three-body \(B_{(s)}\) decays with the intermediate resonant states \(D^*\),

TABLE VI. Branching ratios of \(B_{(s)} \rightarrow D^*\ h'\) decays obtained from quasi-two-body processes under the narrow width approximation. The previous two-body pQCD calculation [41] and the experimental measurements are also listed for comparison.

| Decay modes   | pQCD \((10^{-4})\) | This work \((10^{-4})\) | Data \((10^{-4})\) [57] |
|---------------|---------------------|------------------------|------------------------|
| \(B^0 \rightarrow D^{*-}\pi^+\) | \(26.1^{+8.90}_{-9.30}\) | \(25.0^{+8.90}_{-8.10}\) | \(25.4^{+12.4}_{-8.20}\) |
| \(B^0 \rightarrow D^{*-}K^+\)   | \(2.21^{+0.82}_{-0.83}\) | \(2.04^{+0.98}_{-0.65}\) | \(2.12 \pm 0.15\) |
| \(B^0 \rightarrow \bar{D}^{*0}\pi^0\) | \(2.30^{+0.87}_{-0.83}\) | \(1.62^{+0.71}_{-0.44}\) | \(2.20 \pm 0.60\) |
| \(B^0 \rightarrow D^{*0}K^0\)    | \(0.25^{+0.10}_{-0.09}\) | \(0.26^{+0.14}_{-0.07}\) | \(0.36 \pm 0.12\) |
| \(B^+ \rightarrow \bar{D}^{*0}\pi^+\) | \(51.1^{+14.7}_{-14.2}\) | \(49.8^{+20.7}_{-15.6}\) | \(49.0 \pm 1.70\) |
| \(B^+ \rightarrow D^{*0}K^+\)   | \(3.94^{+1.24}_{-1.32}\) | \(3.80^{+1.86}_{-1.16}\) | \(3.97^{+0.31}_{-0.28}\) |
| \(B^0_s \rightarrow D^{*0}\ K^0\) | \(4.14^{+2.01}_{-1.52}\) | \(2.64^{+1.15}_{-0.90}\) | \(2.80 \pm 1.10\) |
the litter tension between our predictions and the current data requires the future measurements with high accuracy. For the quasi-two-body decay in channel $B_s \to D_s^*\pi^+ \to \bar{D}^0 K^-\pi^+$ happened by the BWT effect, the pQCD prediction is consistent with the current LHCb measurement with in the large error. We also checked the narrow width approximation of these resonant states by extracting the branching ratios of relative two-body decays from these quasi-two-body processes in the pQCD approach, and found it works well for the CKM-favoured channels.

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Appendix A: Decay amplitudes

The amplitudes of two-body decays $B_{(s)} \to D_{(s)}^*\pi, \ D_{(s)}^*K$ in the factorization approaches are [41, 42].

\[
A(B^0 \to D^*\pi^+) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} \left[ (\frac{c_1}{3} + c_2) F_{TD^*} + c_1 M_{TD^*} + (c_1 + \frac{c_2}{3}) F_{A\pi} + c_2 M_{A\pi} \right], \quad \text{(A1)}
\]

\[
A(B^0 \to D^{*+}\pi^-) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} \left[ (c_1 + \frac{c_2}{3}) F_{AD^*} + c_2 M_{AD^*} + (\frac{c_1}{3} + c_2) F_{T\pi} + c_1 M_{T\pi} \right], \quad \text{(A2)}
\]

\[
A(B^0 \to D^{*-}\pi^+) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} \left[ (\frac{c_1}{3} + c_2) F_{TD^*} + c_1 M_{TD^*} \right], \quad \text{(A3)}
\]

\[
A(B^0 \to \bar{D}^{*0}\pi^0) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{us} \left[ (c_1 + \frac{c_2}{3}) F_{T\pi} - c_2 M_{T\pi} + (c_1 + \frac{c_2}{3}) F_{A\pi} + c_2 M_{A\pi} \right], \quad \text{(A4)}
\]

\[
A(B^0 \to D^{*0}K^0) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{us} \left[ (c_1 + \frac{c_2}{3}) F_{T\pi} + c_2 M_{T\pi} \right], \quad \text{(A5)}
\]

\[
A(B^0 \to \bar{D}^{*0}K^0) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{us} \left[ (c_1 + \frac{c_2}{3}) F_{T\pi} + c_2 M_{T\pi} \right]; \quad \text{(A6)}
\]

\[
A(B^+ \to D^{*+}\pi^0) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} \left[ (-\frac{c_1}{3} + c_2) F_{AD^*} - c_1 M_{AD^*} + (\frac{c_1}{3} + c_2) F_{T\pi} + c_1 M_{T\pi} \right], \quad \text{(A7)}
\]

\[
A(B^+ \to \bar{D}^{*0}\pi^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} \left[ (\frac{c_1}{3} + c_2) F_{TD^*} + c_1 M_{TD^*} + (\frac{c_1}{3} + c_2) F_{T\pi} + c_1 M_{T\pi} \right], \quad \text{(A8)}
\]

\[
A(B^+ \to \bar{D}^{*0}K^+) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} \left[ (\frac{c_1}{3} + c_2) F_{TD^*} + c_1 M_{TD^*} + (c_1 + \frac{c_2}{3}) F_{T\pi} + c_2 M_{T\pi} \right], \quad \text{(A9)}
\]

\[
A(B^+ \to D^{*0}K^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs} \left[ (c_1 + \frac{c_2}{3}) F_{AD^*} + c_1 M_{AD^*} + (c_1 + \frac{c_2}{3}) F_{T\pi} + c_2 M_{T\pi} \right]; \quad \text{(A10)}
\]
\[ A(B_s^0 \rightarrow D_s^+ \pi^-) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} [(\frac{c_1}{3} + c_2) F_{TD_s^+} + c_1 M_{TD_s^+}], \tag{A11} \]
\[ A(B_s^0 \rightarrow D_s^- K^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{us} [(\frac{c_1}{3} + c_2) F_{TD_s^-} + c_1 M_{TD_s^-} + (c_1 + \frac{c_2}{3}) F_{AK} + c_2 M_{AK}], \tag{A12} \]
\[ A(B_s^0 \rightarrow D^+ \pi^-) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} [(\frac{c_1}{3} + \frac{c_2}{3}) F_{AP} + c_2 M_{AP}], \tag{A13} \]
\[ A(B_s^0 \rightarrow D^+ K^-) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [(\frac{c_1}{3} + c_2) F_{TK} + c_1 M_{TK}], \tag{A14} \]
\[ A(B_s^0 \rightarrow \bar{D}^0 \pi^+) = \frac{G_F}{2} V_{cb}^* V_{us} [(c_1 + \frac{c_2}{3}) F_{AP} + c_2 M_{AP}], \tag{A15} \]
\[ A(B_s^0 \rightarrow \bar{D}^+ K^-) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [(c_1 + \frac{c_2}{3}) F_{TK} + c_2 M_{TK}], \tag{A16} \]

where \( c_{i=1,2}(\mu) \) is the tree-level Wilson coefficients carrying the physics extending in the energy scale regions from \( m_W \) to \( m_B \). The factorizable and non-factorizable scattering amplitudes \( F \) and \( M \) carry the physics below the \( m_B \) energy scale, they are represented by means of the hadron matrix elements with certified four fermion effective operators, as expressed in Eq. (1). The pQCD calculation result in

\[
F_{TD_s'} = 8\pi C_F m_B^4 f_{\pi(K)} \int dx_1 dx_R \int b_1 db_1 b_R db_R \phi_B(x_1, b_1) \phi_{D\pi}(x_R, b_R, s) \\
\times \left\{ \left[ \sqrt{\zeta} (2x_R - 1) - x_R - 1 \right] E_1 ^{(1)} (t_a) h_{a}(x_1, x_R, b_1, b_R) - (\zeta + r_c) E_1 ^{(2)} (t_b) h_{b}(x_1, x_R, b_1, b_R) \right\}, \tag{A17} \]
\[
M_{TD_s'} = 32\pi C_F m_B^4 / \sqrt{2N_c} \int dx_1 dx_R dx_3 \int b_1 db_1 b_3 db_3 \phi_B(x_1, b_1) \phi_{D\pi}(x_R, b_R, s) \phi^A \\
\times \left\{ \left[ (x_R - x_3 + 1) - x_R \sqrt{\zeta} + x_1 + x_3 - 1 \right] E_0 (t_c) h_{c}(x_1, x_R, x_3, b_1, b_3) \\
+ \left[ x_3 (1 - \zeta) + x_R \left( 1 - \sqrt{\zeta} \right) - x_1 \right] E_0 (t_d) h_{d}(x_1, x_R, x_3, b_1, b_3) \right\}, \tag{A18} \]
\[
F_{AD_s'} = 8\pi C_F m_B^4 f_{\pi(K)} \int dx_R dx_3 \int b_R db_R b_3 db_3 \phi_D (x_R, b_R, s) \\
\times \left\{ \left[ (1 - x_R) \phi^A + 2r_0 x_R \zeta \sqrt{\zeta} \phi^P \right] E_1 ^{(1)} (t_c) h_{c}(x_R, x_3, b_R, b_3) \\
+ \left[ x_3 (\zeta - 1) - \zeta \phi^A - r_0 \zeta \phi^P + (\zeta + 1) \phi^T \right] E_1 ^{(2)} (t_f) h_{f}(x_R, x_3, b_R, b_3) \right\}, \tag{A19} \]
\[
M_{AD_s'} = 32\pi C_F m_B^4 / \sqrt{2N_c} \int dx_1 dx_R dx_3 \int b_1 db_1 b_R db_R \phi_B(x_1, b_1) \phi_{D\pi}(x_R, b_R, s) \\
\times \left\{ \left[ (\zeta (x_R + 1) + x_1 - x_3 (\zeta - 1)) \phi^A + r_0 \zeta \sqrt{\zeta} ((1 - x_3) (1 - \zeta) - x_1) (\phi^P + \phi^T) \\
+ x_R (\phi^T - \phi^P) \right] E_0 (t_g) h_{g}(x_1, x_R, x_3, b_1, b_R) + \left[ (1 - x_R) (\zeta - 1) \phi^A - r_0 \zeta \sqrt{\zeta} ((x_R - 1) (\phi^P + \phi^T) \\
+ x_1 + x_3 \zeta - \zeta - x_3) (\phi^T - \phi^P) \right] E_0 (t_h) h_{h}(x_1, x_R, x_3, b_1, b_R) \right\}; \tag{A20} \]
\[
F_{T\pi(K)} = 8\pi C_F m_B^4 F_{D\pi}(s) \int dx_1 dx_3 \int b_1 db_1 b_3 db_3 \phi_B(x_1, b_1) \\
\times \left\{ \left[ \phi^A (x_3 (1 - \zeta) + 1) - r_0 [\phi^P (2x_3 - 1) + \phi^T \zeta (2x_3 (\zeta - 1) + \zeta + 1)] \right] E_1 ^{(1)} (t_m) \\
\times h_m(x_1, x_3, b_1, b_3) + \left[ 2r_0 \zeta \phi^P (1 - x_1 - 1) - \zeta x_1 \phi^A \right] E_1 ^{(2)} (t_n) \\
\times h_n(x_1, x_3, b_1, b_3) \right\}, \tag{A21} \]
\[ M_{\pi(K)} = 32\pi C_F m_B^2 / \sqrt{2N_c} \int dx_1 dx_R dx_3 \int b_1 db_1 b_R db_R \phi_B(x_1, b_1) \phi_{D\pi}(x_R, b_R, s) \times \{ [\zeta - 1] (1 - x_1 - x_R) \phi - r_0 \zeta \sqrt{\zeta} (x_R + x_1) (\phi^T + \phi^T) + x_3 (1 - \zeta) (\phi^T - 2\zeta \phi^T) \} \times E_f(t_0) h_o(x_1, x_R, x_3, b_1, b_R) - \left[ r_c \sqrt{\zeta} + x_3 (\zeta - 1) - x_R + x_1 \right] \phi^T + r_0 \zeta (x_R - x_1) (\phi^T - \phi^T) + r_0 [x_3 (1 - \zeta) \phi^T - (4r_c \sqrt{\zeta} + x_3 - x_3 \phi^T)] E_f(t_R) h_p(x_1, x_R, x_3, b_1, b_R) \} , \]  
(A22)

\[ M'_{\pi(K)} = 32\pi C_F m_B^2 / \sqrt{2N_c} \int dx_1 dx_R dx_3 \int b_1 db_1 b_R db_R \phi_B(x_1, b_1) \phi_{D\pi}(x_R, b_R, s) \times \{ [\phi^T + x_R - 1 - \sqrt{\zeta} r_c - \zeta (x_R + x_1 - 1)] - r_0 \zeta \sqrt{\zeta} (\phi^T + \phi^T) + x_3 (1 + x_R - x_3 - 2) \} + r_0 x_3 \zeta (\phi^T - \phi^T) E_f(t_0) h_o(x_1, x_R, x_3, b_1, b_R) - \left[ (\zeta - 1) x_3 + x_1 - x_R \right] \phi^T + r_0 x_3 (\phi^T + \phi^T) \] 
\[ + r_0 \zeta (x_R - x_1) (\phi^T - \phi^T) E_f(t_R) h_p(x_1, x_R, x_3, b_1, b_R) \} , \]  
(A23)

\[ F_{AK} = 8\pi C_F m_B^4 f_B \int dx_R dx_3 \int b_R db_R lb_3 \phi_{D\pi(x_R, b_R, s)} \times \left[ \left[ (\zeta - 1) x_3 + x_1 \right] \phi^T - r_0 \zeta \sqrt{\zeta} (x_R + x_1 - 1) \phi^T + (1 - \zeta) (x_R - 1) \phi^T \right] E_g^{(1)}(t_0) h_s(x_R, x_3, b_3) \] 
\[ - \left[ -x_R \phi^T + 2r_0 \sqrt{\zeta} (\zeta + x_R - 1) \phi^T \right] E_g^{(2)}(t_1) h_t(x_R, x_3, b_3) \} , \]  
(A24)

\[ M_{AK} = 32\pi C_F m_B^2 / \sqrt{2N_c} \int dx_1 dx_R dx_3 \int b_1 db_1 b_R db_R \phi_B(x_1, b_1) \phi_{D\pi}(x_R, b_R, s) \times \left[ [1 - \zeta (x_R + x_1 + \zeta) \phi^T - r_0 \zeta \sqrt{\zeta} (x_R + x_1 + 1 - x_3 \zeta) \phi^T + (1 - \zeta) (x_R - 3 - 1) \phi^T \right] + r_0 \zeta \sqrt{\zeta} (x_R + x_1) \phi^T + (1 - \zeta) (x_R - 3 - 1) \phi^T \right] E_A(t_0) h_a(x_1, x_R, x_3, b_1, b_R) + \left[ (\zeta + x_R - x_1 - x_3) + x_3 - 1 \right] \phi^T \] 
\[ + r_0 \sqrt{\zeta} (x_R + x_1 - 1) (\phi^T - \phi^T) + \zeta (x_1 - x_R) (\phi^T + \phi^T) \right] E_A(t_0) h_v(x_1, x_R, x_3, b_1, b_R) \} , \]  
(A25)

with \( \hat{\zeta} = 1 / (\zeta - 1) \).

The hard scale \( t_1 \) in the pQCD approach to deal with hard scatterings is chosen as the largest virtuality of the internal momentum transition,

\[ t_a = \text{Max}\{ m_B \sqrt{|a_1|}, m_B \sqrt{|a_2|}, 1/b_R, 1/b_1 \}, \quad t_b = \text{Max}\{ m_B \sqrt{|b_1|}, m_B \sqrt{|b_2|}, 1/b_1, 1/b_R \}; \]
\[ t_c = \text{Max}\{ m_B \sqrt{|c_1|}, m_B \sqrt{|c_2|}, 1/b_1, 1/b_3 \}, \quad t_d = \text{Max}\{ m_B \sqrt{|d_1|}, m_B \sqrt{|d_2|}, 1/b_1, 1/b_3 \}; \]
\[ t_e = \text{Max}\{ m_B \sqrt{|e_1|}, m_B \sqrt{|e_2|}, 1/b_1, 1/b_3 \}, \quad t_f = \text{Max}\{ m_B \sqrt{|f_1|}, m_B \sqrt{|f_2|}, 1/b_3, 1/b_R \}; \]
\[ t_g = \text{Max}\{ m_B \sqrt{|g_1|}, m_B \sqrt{|g_2|}, 1/b_R, 1/b_1 \}, \quad t_h = \text{Max}\{ m_B \sqrt{|h_1|}, m_B \sqrt{|h_2|}, 1/b_R, 1/b_1 \}; \]
\[ t_m = \text{Max}\{ m_B \sqrt{|m_1|}, m_B \sqrt{|m_2|}, 1/b_3, 1/b_1 \}, \quad t_n = \text{Max}\{ m_B \sqrt{|n_1|}, m_B \sqrt{|n_2|}, 1/b_1, 1/b_3 \}; \]
\[ t_o = \text{Max}\{ m_B \sqrt{|o_1|}, m_B \sqrt{|o_2|}, 1/b_1, 1/b_R \}, \quad t_p = \text{Max}\{ m_B \sqrt{|p_1|}, m_B \sqrt{|p_2|}, 1/b_1, 1/b_R \}; \]
\[ t_{o'} = \text{Max}\{ m_B \sqrt{|o'_1|}, m_B \sqrt{|o'_2|}, 1/b_1, 1/b_R \}, \quad t_{p'} = \text{Max}\{ m_B \sqrt{|p'_1|}, m_B \sqrt{|p'_2|}, 1/b_1, 1/b_R \}; \]
\[ t_s = \text{Max}\{ m_B \sqrt{|s_1|}, m_B \sqrt{|s_2|}, 1/b_3, 1/b_R \}, \quad t_t = \text{Max}\{ m_B \sqrt{|t_1|}, m_B \sqrt{|t_2|}, 1/b_R, 1/b_3 \}; \]
\[ t_u = \text{Max}\{ m_B \sqrt{|u_1|}, m_B \sqrt{|u_2|}, 1/b_R, 1/b_1 \}, \quad t_v = \text{Max}\{ m_B \sqrt{|v_1|}, m_B \sqrt{|v_2|}, 1/b_R, 1/b_1 \}. \]  
(A26)
In the above expressions, the non-dimensional kinematical factors are

\[ a_1 = x_R, \quad a_2 = x_R x_1; \]
\[ b_1 = r_c^2 + x_1 - \xi, \quad b_2 = a_2; \]
\[ c_1 = a_2, \quad c_2 = x_R [x_1 - (1 - \xi)(1 - x_3)]; \]
\[ d_1 = a_2, \quad d_2 = x_R [x_1 - (1 - \xi)x_3]; \]
\[ e_1 = x_R - 1, \quad e_2 = (1 - x_R) [x_3(\xi - 1) - \zeta]; \]
\[ f_1 = r_c^2 + x_3(\xi - 1) - \zeta, \quad f_2 = e_2; \]
\[ g_1 = e_2, \quad g_2 = x_R [x_1 + (x_3 - 1)(1 - \xi)] + 1; \]
\[ h_1 = e_2, \quad h_2 = (1 - x_R)(x_1 + x_3\xi - x_3 - \zeta); \]
\[ m_1 = (1 - \xi)x_3, \quad m_2 = (1 - \xi)x_3 x_1; \]
\[ n_1 = (1 - \xi)x_1, \quad n_2 = m_2; \]
\[ o_1 = m_2, \quad o_2 = (1 - x_1 - x_R)(x_3\xi - x_3 - \zeta); \]
\[ p_1 = m_2, \quad p_2 = r_c^2 + x_3(\xi - 1)(x_R - x_1); \]
\[ q'_1 = (1 - \xi)x_3 x_1, \quad q'_2 = r_c^2 - (x_R + x_1 - 1)(x_3(\xi - 1) - \zeta); \]
\[ p'_1 = q'_1, \quad p'_2 = x_3(\xi - 1)(x_R - x_1); \]
\[ s_1 = r_c^2 - x_3\xi + x_3 - 1, \quad s_2 = x_R(x_3 - 1)(1 - \xi); \]
\[ t_1 = x_R(\xi - 1), \quad t_2 = s_2; \]
\[ u_1 = s_2, \quad u_2 = (x_R + x_1 - 1)(\xi + x_3 - x_3\xi) + 1; \]
\[ v_1 = s_2, \quad v_2 = (x_3 - 1)(1 - \xi)(x_R - x_1). \]  \tag{A27} 

The hard functions \( h_i \) \((i \in \{a, b, c, d, e, f, g, h, m, n, o, p, q', p', s, t, u, v\})\) in scattering amplitudes is expressed in terms of transversal distances \( b_i \) conjugated to the transversal momentum by fourier transform.

\[
\begin{align*}
    h_{ii}(x_1, x_2, (x_3), b_1, b_2) &= h_{ii1}(\beta, b_2) \times h_{ii2}(\alpha, b_1, b_2), \\
    h_{ii1}(\beta, b_2) &= \begin{cases} 
        K_0(\sqrt{3}b_2), & \beta > 0 \\
        \frac{i\pi}{2} H_0^{(1)}(\sqrt{3}b_2), & \beta < 0
    \end{cases} \\
    h_{ii2}(\alpha, b_1, b_2) &= \begin{cases} 
        \theta(b_2 - b_1)I_0(\sqrt{3}b_1)K_0(\sqrt{3}b_2) + (b_1 \leftrightarrow b_2), & \alpha > 0 \\
        \frac{i\pi}{2} \theta(b_2 - b_1)J_0(\sqrt{3}b_1)H_0^{(1)}(\sqrt{3}b_2) + (b_1 \leftrightarrow b_2), & \alpha < 0
    \end{cases} 
\end{align*}
\]  \tag{A28} 

where \( J_0 \) is the Bessel function, \( K_0 \) and \( I_0 \) are modified Bessel functions, \( N_0 \) is the Neumann function, and \( H_0 \) is the
Hankel function of the first kind with relation \( H_0^{(1)}(x) = J_0(x) + iN_0(x) \). The kinematic factors \( \alpha \) and \( \beta \) are exactly the certain case of \( i_1 \) and \( i_2 \) defined in Eq. (A27), respectively.

The evolution functions in the scattering amplitudes take into account the strong coupling constant and also the Sudakov suppressed factors from the resummations of end-point singularity [4, 5].

\[
E_1^{(1)}(t) = \alpha_s(t) \exp[-S_B(t) - S_C(t)] S_t(x_R), \\
E_1^{(2)}(t) = \alpha_s(t) \exp[-S_B(t) - S_C(t)] S_t(x_1), \\
E_b(t) = \alpha_s(t) \exp[-S_B(t) - S_C(t) - S_P(t)]|_{b_R=b_1}, \\
E_2^{(1)}(t) = \alpha_s(t) \exp[-S_C(t) - S_P(t)] S_t(x_R), \\
E_2^{(2)}(t) = \alpha_s(t) \exp[-S_C(t) - S_P(t)] S_t(x_3), \\
E_d(t) = \alpha_s(t) \exp[-S_B(t) - S_C(t) - S_P(t)]|_{b_3=b_R}, \\
E_3^{(1)}(t) = \alpha_s(t) \exp[-S_B(t) - S_P(t)] S_t(x_3), \\
E_3^{(2)}(t) = \alpha_s(t) \exp[-S_B(t) - S_P(t)] S_t(x_1), \\
E_f(t) = \alpha_s(t) \exp[-S_B(t) - S_C(t) - S_P(t)]|_{b_3=b_1}, \\
E_4^{(1)}(t) = \alpha_s(t) \exp[-S_C(t) - S_P(t)] S_t(x_3), \\
E_4^{(2)}(t) = \alpha_s(t) \exp[-S_C(t) - S_P(t)] S_t(x_R), \\
E_h(t) = \alpha_s(t) \exp[-S_B(t) - S_C(t) - S_P(t)]|_{b_3=b_R}.
\]

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