Attempts at a determination of the fine-structure constant from first principles: A brief historical overview

U. D. Jentschura$^{1,2}$ and I. Nándori$^2$

$^1$Department of Physics, Missouri University of Science and Technology, Rolla, Missouri 65409-0640, USA
$^2$MTA–DE Particle Physics Research Group, P.O.Box 51, H–4001 Debrecen, Hungary

It has been a notably elusive task to find a remotely sensical ansatz for a calculation of Sommerfeld’s electrodynamic fine-structure constant $\alpha_{QED} \approx 1/137.036$ based on first principles. However, this has not prevented a number of researchers to invest considerable effort into the problem, despite the formidable challenges, and a number of attempts have been recorded in the literature. Here, we review a possible approach based on the quantum electrodynamic (QED) $\beta$ function, and on algebraic identities relating $\alpha_{QED}$ to invariant properties of “internal” symmetry groups, as well as attempts to relate the strength of the electromagnetic interaction to the natural cutoff scale for other gauge theories. Conjectures based on both classical as well as quantum-field theoretical considerations are discussed. We point out apparent strengths and weaknesses of the most prominent attempts that were recorded in the literature. This includes possible connections to scaling properties of the Einstein–Maxwell Lagrangian which describes gravitational and electromagnetic interactions on curved space-times. Alternative approaches inspired by string theory are also discussed. A conceivable variation of the fine-structure constant with time would suggest a connection of $\alpha_{QED}$ to global structures of the Universe, which in turn are largely determined by gravitational interactions.

PACS:
12.20.Ds (Quantum electrodynamics — specific calculations) ;
11.25.Tq (Gauge field theories) ;
11.15.Bt (General properties of perturbation theory) ;
04.60.Cf (Gravitational aspects of string theory) ;
06.20.Jr (Determination of fundamental constants).

I. INTRODUCTION

Today, the determination of a viable analytic formula for the fine-structure constant remains an extremely elusive problem. The fine-structure constant $\alpha_{QED} \approx 1/137.036$ is a dimensionless physical constant, and any conceivable variation of it with time $\frac{\alpha}{\alpha_0}$ would necessarily imply a connection of electromagnetic interactions to other global properties of the Universe, such as its age. Alternatively, one may point out that expressions for the fine-structure constant in terms of well-defined mathematical invariants of an underlying symmetry group $\{3,4\}$ are incompatible with a variation of the fine-structure constant with time (unless the symmetry group changes with time also). Indeed, the problem of finding a conceivable analytic formula for the fine-structure constant is of such fundamental interest that considerable field-theoretical insight and effort has been invested into the task, despite the formidable challenges. It thus appears useful to review the historical development and status of these attempts, and to indicate possible future directions of research, while noting that considerable scrutiny and scepticism are appropriate with regard to the elusive and formidable challenge.

Let us start by recalling that the quantum electrodynamic (QED) $\beta$ function $\frac{\beta}{\beta_0}$ describes the evolution of the QED running coupling $\alpha_{QED}$ over different momentum scales; one may naturally ask the question if the physical value of $\alpha_{QED}$ is related to a specific momentum scale. Indeed, QED is first and foremost defined at a high-energy scale (cutoff scale), and the renormalization-group (RG) evolution of $\alpha_{QED}$ can thus be used to evolve the coupling into the low-energy domain. If one postulates certain constraining properties of $\alpha_{QED}$ either in the high-energy, or the low-energy, limit, then one might hope $\frac{\beta}{\beta_0}$ to obtain a constraint equation which determines a physically reasonable approximation to $\alpha_{QED}$. However, the so-called triviality of QED $\frac{\alpha}{\alpha_0}$ poses a very interesting question for physicists, namely, to explain the numerical value of $\alpha_{QED}$ without having, as an “anchor point”, a “critical value” for the coupling: From the point of view of renormalization-group (RG) theory $\frac{\alpha}{\alpha_0}$, QED does not have a phase transition. Consequences of these observations are discussed in Sec. II A

Another intuitive ansatz for the determination of $\alpha_{QED}$ would a priori involve algebraic considerations which relate $\alpha_{QED}$ to certain invariants of internal symmetry groups, e.g., those which describe the intrinsic spin of a Dirac particles. Other attempts are based on possible connections of the QED coupling to invariants of higher-dimensional internal symmetry groups, whose projection onto four dimensions yields a value for $\alpha_{QED}$ close to the observed parameter. Related attempts are discussed in Sec. II B

However, one may point out that “stand-alone” approaches to the calculation of $\alpha_{QED}$, discussed in Sec. II would
necessarily imply that the value of $\alpha_{\text{QED}}$ is determined by invariants of an underlying symmetry group. (In the case of the QED $\beta$ function, the $U(1)$ gauge group determines the coefficients of the $\beta$ function.) Under these assumptions, the symmetry group fixes the value of the fine-structure constant, and $\alpha_{\text{QED}}$ will remain constant with time. (This of course does not hold if the symmetry group changes with time, i.e., if QED suddenly evolves from a $U(1)$ gauge theory to a different internal symmetry group.) Dirac’s large number hypothesis \[1\] conjectures that $\alpha_{\text{QED}}$ changes with the age of the Universe, and a number of recent papers \[\text{[11-12]}\] claim to have established a variation of $\alpha_{\text{QED}}$ with time (see Sec. \[\text{[III]}\]).

The large-scale structure of the Universe is determined first and foremost by gravitational interactions. This has inspired some researchers to look for connections of quantum fluctuations of the quantum fields on the curved space-time, and possible connections of the effective Lagrangian obtained after the integration of the “heavy” degrees of freedom, to gravity (see Sec. \[\text{[IV]}\]). Indeed, if there is a variation of $\alpha_{\text{QED}}$ connected to changes in the global properties of the Universe (such as its age which trivially evolves with time), then a change in the strength of the electromagnetic interaction suggests a connection of the electrodynamic and gravitational interactions, because the latter in turn determine the global properties of the Universe (see Sec. \[\text{[V]}\]). One of the first attempts at a unification of two fundamental forces, namely, gravitation and electromagnetism, involves a five-dimensional generalization of space-time (Kaluza–Klein theories, see Refs. \[\text{[16-18]}\]). This ansatz involves a compactification of the fifth dimension with which in turn leads to a natural charge quantization condition, and results in a formula connecting the fine-structure constant with a background field (see Sec. \[\text{[V B]}\]). It is also instructive to recall (see Sec. \[\text{[VA]}\]) that the known, manifestly nonvanishing photon-graviton conversion amplitudes establish a connection of gravity and electromagnetism within a fully quantized formalism.

An alternative attempt at analyzing a connection of gravity and electromagnetism is inspired by string theory \[19,20\]. Roughly speaking, the scattering amplitudes derived from open as opposed to closed strings strongly suggest that a connection of $\alpha_{\text{QED}}$ and $\alpha_G$ (the fine-structure constant of gravitational interactions) might exist. Furthermore, the functional relationship suggests that $\alpha_{\text{QED}}$ (related to an open-string amplitude) should be proportional to the square root of $\alpha_G$ of the gravitational fine-structure constant (closed-string amplitude), modulo a proportionality factor that might depend on other fundamental constants. Possible consequences of this observation are discussed in Sec. \[\text{[V C]}\].

We frequently use the Newtonian gravitational constant $G$, Planck’s unit of action $\hbar$, the vacuum permittivity $\varepsilon_0$, the electron and proton masses $m_e$ and $m_p$, and the speed of light $c$. SI mksA units are employed throughout the paper.

\[\text{II. ALGEBRAIC AND ANALYTIC APPROACH TO THE FINE–STRUCTURE CONSTANT}\]

\[\text{A. Renormalization Group and the Fine–Structure Constant}\]

It is very interesting to explore possible explanations of the observed relation $\alpha_{\text{QED}} \approx 1/137.036$ to the RG evolution of the coupling constant of quantum electrodynamics (QED) with the momentum scale. We recall that the RG evolution of $\alpha_{\text{QED}}$ is described by the equation \[\text{[5-7,21-23]}\]

\[\int_{G_{\text{as}}(\alpha,x_1)}^{G_{\text{as}}(\alpha,x_2)} \frac{dz}{\beta(z)} = \ln \left( \frac{x_2}{x_1} \right),\]  

(1)

where $G_{\text{as}}(\alpha, x)$ is the running QED coupling in the asymptotic, high-energy region, and $\alpha$ is its reference value at the momentum scale $x$ where the RG evolution starts (i.e., where the coupling is “matched” against the physical value of $\alpha$). Of course, the assumption is that an analytic continuation of the QED $\beta$ function to the non-asymptotic, low-energy region exists which allows us to use the functional form \[\text{[11]}\] in the low-energy region. Indeed, the RG equation of Callen and Symanzik \[\text{[24,25]}\] clarifies that, in the low-energy region, the “running” of the mass parameters cannot be ignored, but we ignore this possible complication in the following discussion. Pertinent remarks on the “running” of the mass parameter can be found in Chap. 13 of Ref. \[\text{[22]}\], Refs. \[\text{[10,26]}\]. For a discussion of the mass running within QED, we refer \[\text{[27]}\].

Let us now assume that we are evolving the QED coupling downward with respect to the momentum scale. We assume that $x_1 \to \infty$ defines a large momentum scale, e.g., the momentum scale of the bare theory, while we also assume that $x_2$ defines the scale where the coupling $G_{\text{as}}(\alpha, x_2) = \alpha_{\text{QED}}$. For $x_1 \to \infty$, the QED coupling describes the “bare” charge. If we plainly enter with $x_1 \to \infty$ into the right-hand side of Eq. \[\text{[1]}\], then it diverges logarithmically. So, if QED is assumed to be valid across all momentum scales, then the only way to make the left-hand side of Eq. \[\text{[1]}\] also diverge logarithmically is to assume that

\[\beta(G_{\text{as}}(\alpha, x_2)) = \beta(\alpha^*) = \beta(\alpha_{\text{QED}}) = 0,\]  

(2)
because the integrand $1/\beta(z)$ in Eq. (1) would in this case also diverge near $z \approx \alpha^* = G^{\alpha}(x, z) = \alpha_{\text{QED}}$. This consideration motivates the conjecture [8, 9] that $\alpha_{\text{QED}}$ constitutes a zero of the QED $\beta$ function, where we recall that the asterisk in the superscript $\alpha^*$ is used to denote a generic critical point of the RG evolution [10]. It has been pointed out in Ref. [23] that this argument also holds if the QED $\beta$ function is restricted to subclasses of diagrams with only one closed fermion loop. Indeed, the replacement $\beta \rightarrow F[1]$ advocated in Ref. [23] singles out the one-fermion-loop diagrams (where $F[1]$ singles out the one-loop diagrams) but does not change the overall physical picture discussed below.

We should also be careful, because we have in fact used the RG evolution equation in a region which is manifestly non-asymptotic with respect to the momentum scale, namely, in a region where the running of the coupling cannot be separated from the running of the mass [22]. A more serious objection comes from the fact that the conjecture is in obvious disagreement with the concrete numerical values of the first two perturbative coefficients of the QED $\beta$ function (in the momentum scheme, see Ref. [23]),

$$
\beta(\alpha) = \frac{\alpha^2}{3\pi} + \frac{\alpha^3}{4\pi^2} + \frac{\alpha^4}{\pi^3} \left( \frac{1}{3} \zeta(3) - \frac{101}{288} + \frac{\alpha^5}{\pi^4} \left( -\frac{5}{3} \zeta(5) + \frac{1}{3} \zeta(3) + \frac{93}{128} \right) + \mathcal{O}(\alpha^6) \right).
$$

While the coefficients of order $\alpha^2$ and $\alpha^3$ are both positive and thus exclude a nontrivial positive of $\beta(\alpha)$ for small and positive $\alpha$, one could argue that this says nothing about the large-order (or strong-coupling) asymptotics of the $\beta$ function. The apparent absence of a zero of the QED $\beta$ function, based on a consideration of the first perturbative coefficient, is also known as the “triviality” of QED (see Ref. [29]). Recently, using lattice calculations and exact RG approaches, the triviality of QED (from the RG point of view, i.e., the absence of a phase transition) has been confirmed in Refs. [30–32].

One might still speculate about possible zeroes of the QED $\beta$ function in the high-momentum region. However, analytic arguments [33, 34] based on a rather sophisticated extrapolation of the perturbative approach of the QED $\beta$ function to the nonperturbative (strong-coupling) domain suggest that the value of the QED $\beta$ function increases with the momentum scale, with $\beta(g) \propto g$ in the asymptotic region. While the linear increase with the momentum scale, if confirmed, would constitute a rather surprising functional dependence, one may otherwise remark that the result of Ref. [34] is otherwise consistent with the absence of zeros of the QED beta function. The linear increase $\beta(g) \propto g$ of the QED coupling with the momentum scale, proposed in Ref. [34], appears to be at variance with the logarithmic dependence suggested by the dominant one-loop evolution [22], and also, at variance with the results of a sophisticated effective-charge approach [35]. However, the precise functional form of the monotonous increase of the QED coupling does not really matter: The QED $\beta$ function, as it follows from the RG program applied at face value to QED, is universally assumed to remain positive over all momentum scales, thus excluding a nontrivial zero.

One may also explore the possibility of an ultraviolet (UV) fixed point of QED, with $\beta(\alpha = \alpha^*) = 0$ in the UV, where the QED $\beta$ function in the on-mass-shell scheme runs into this fixed point in the ultraviolet, i.e.,

$$
\alpha_\infty = G^{\alpha}(\alpha^*, x \rightarrow \infty), \quad \beta(\alpha^*) = 0.
$$

In this case, the effective QED coupling should attain a well-defined, finite value at infinite momentum transfer, and $\alpha^*$ would have to be interpreted as the finite bare charge. This assumption is at variance with predictions of the Landau pole theory [22, 23, 33, 34], which would predict the QED coupling to diverge at an intermediate momentum scale $q_0^2$ [interestingly, this pole persists even at finite temperature, see Refs. [36–38]], and also, with more general arguments [33, 34] that suggest a monotonous increase of the QED coupling at high momenta (without intermediate poles of the Landau type).

Let us conclude this discussion by observing that the QED $\beta$ function has an (almost trivial) zero at the Gaussian fixed point,

$$
\beta(\alpha^*) = \beta(\alpha^* = 0) = 0.
$$

(The notation $\alpha^*$ is normally reserved for the first nontrivial zero of the QED $\beta$ function, i.e., for a nonvanishing value of $\alpha^*$, but we extend the notation here to the value $\alpha^* = 0$, which is the trivial Gaussian fixed point.) The famous argument of Dyson [39] identifies the point $\alpha^* = 0$ (transition from positive to negative QED coupling) as the point where the vacuum becomes unstable against the creation of electron-positron pairs, due to quantum tunneling. In that sense, the value $\alpha^* = 0$ constitutes a critical point of the RG evolution of the QED coupling, but it does not match the physical value of $\alpha_{\text{QED}}$ which is manifestly nonvanishing.

The dependence on the order of perturbation theory of the first coefficients of the QED $\beta$ function in Eq. [9] does not exhibit the asymptotic (factorially divergent) structure which is otherwise useful in determining critical exponents as a function of the critical coupling, e.g., in the $N$-vector model [40]. We have carried out, independently, numerical attempts to determine an estimate for the first positive or negative zero $\alpha^*$ of the QED $\beta$ function, based
on the coefficients given in Eq. (3). These attempts have been unsuccessful (here, the “first zero” is to be interpreted as the one with minimum modulus, for either positive or negative $\alpha_{\text{QED}}$). This conclusion remains valid even if Padé approximants are used in order to improve the convergence of the perturbative expansion of the $\beta$ function. Furthermore, the zero of the RG evolution, determined by the first two coefficients of Eq. (3),

$$\beta(\alpha^*) \approx \frac{(\alpha^*)^2}{3\pi} + \frac{(\alpha^*)^3}{4\pi^2} = 0, \quad \alpha^* = -\frac{4\pi}{3} \approx -4.189,$$

lies in the “unphysical”, “unstable” region where the QED vacuum becomes unstable against the creation of electron-positron pairs (which would repel each other, so that the vacuum energy can be lowered by separating the charges into distant regions of space, still conserving electric neutrality of the Universe, see Ref. [39]). Also, since the modulus of $\alpha^* = -4.189$ is of order unity (or, larger than unity), one may expect that the first perturbative terms from Eq. (3) cannot give a reliable estimate for its numerical value. However, one may speculate about the physical significance of $\alpha^* = -4.189$ (or, of an improved determination thereof based on higher-order perturbative terms) as follows: Namely, for negative $\alpha$ of unit modulus, the binding energy of two (mutually attracting, in this case) electrons, or two (mutually attracting) positrons becomes commensurate with their rest energy. The negative critical value $\alpha^* = -4.189$ might thus describe a phase transition where the spontaneous pair creation from the vacuum, which sets in immediately at $\alpha = 0$ (branch point of the QED perturbation series), in fact turns into spontaneous creation of electron-positron pairs in bound as opposed to free states (i.e., pair creation into bound “Cooper pairs” consisting of either two mutually attracting electrons, or two mutually attracting positrons). In these modified “Cooper pairs”, the dominant attractive binding force would come from the strong, mutually attracting, electrostatic Coulomb interaction (which is reversed in sign because we consider a hypothetical situation with $\alpha < 0$).

All considerations reported in the current section are consistent with the absence of a phase transition of QED for any positive value of $\alpha$, and with the lack of an explanation for the physically observed value of $\alpha_{\text{QED}}$, based on the RG evolution of the QED coupling parameter.

B. Algebraic Relations and the Fine–Structure Constant

The scientific literature is not free from attempts to determine $\alpha_{\text{QED}}$ based on algebraically simple combinations of transcendental numbers like $\pi$, or logarithms of characteristic dimensionless physical quantities, which approximate the numerical value of the QED coupling $\alpha_{\text{QED}} \approx 1/137.036$. Other attempts are inspired by characteristic ratios in classical “spin” orbits of the electron [41]. The latter approach is perhaps not totally unreasonable because the QED fine-structure constant describes perturbation series for phenomena which are related to circular motion. E.g., in the semi-classical approximation, one may consider the Larmor precession frequency of an electron in a magnetic field, which is proportional to the electron (anomalous) magnetic moment, and yet, in its classical analogue, is equivalent to the numerical value of $\alpha_{\text{QED}}$.

One may also counter-argue that it would be rather unusual if the precise numerical value of a constant which is so intimately related to quantum physics, such as $\alpha_{\text{QED}}$, could be explained by invariant characteristic numbers occurring in the classical analogue of the quantum phenomenon. The emergence of a dimensionless fundamental constant such as $\alpha_{\text{QED}}$ would naturally be assumed to be an inherent property of the field theory which describes the underlying phenomena (namely, QED), rather than a characteristic invariant of some quantum “motion” which is always smeared out because of quantum fluctuations. The approach discussed in Refs. [41] therefore appears doubtful. Other numerical coincidences fulfilled by the fine-structure constant have recently been compiled in Ref. [42].

In principle, it would appear to be more promising to determine the fine-structure constant based on invariants of symmetry groups. Wyler [3, 4] considers an a priori massless particle propagating in five-dimensional space, where the mass, in the fifth dimension, according to the replacement $m \rightarrow i\partial/\partial x_5$, parameterizes an internal time (or “aging speed”) of the particle. The invariance group $O(5, 2)$ of the five-dimensional Klein–Gordon equation is investigated in Refs. [3, 4]. The space $\text{SO}(n, 2)/[\text{SO}(n) \otimes \text{SO}(2)]$ relates the complex hypersphere $D^n$ and its characteristic boundary $Q^n$ to the spherical surface $S^{n-1}$. Putting $n = 5$, one obtains [43]

$$V(D^n) = \frac{\pi^n}{2^{n-1} n!} = \frac{\pi^5}{2^4 5!}, \quad V(Q^n) = \frac{2\pi^{n+1}}{\Gamma(n/2)} = \frac{2^3 \pi^3}{3}, \quad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \frac{2^3 \pi^2}{3}.$$

(7)
From these quantities, Wyler \[3, 4, 43\] assembles the following approximation to the fine-structure constant,

\[
\alpha_{\text{QED}} \approx 2 \times 4\pi \frac{1}{V(S^4)} \frac{1}{V(Q^5)} \left[ V(D^5) \right]^{1/4} = 0.0072973 \ldots = \frac{1}{137.0360824}.
\]

(8)

One may argue that Wyler’s considerations follow from the internal symmetry group of an equation describing a noninteracting particle (the free Klein–Gordon equation, generalized to five dimensions), while QED necessitates the covariant coupling of the particle to the electromagnetic field operator (and thus involves an interaction Hamiltonian). Mathematically, Pease \[44\] argues that Wyler’s calculation is based on an incorrect value of the coefficient of the Poisson kernel of certain five-dimensional domains considered in his formalism, and so the value of \(\alpha_{\text{QED}}\) may not be derivable from his assumptions. Robertson \[45\] argues that the introduction of an additional scaling factor in Wyler’s formula destroys the (approximate) agreement with experiment. Additionally, the numerical value derived from \(\alpha_{\text{QED}}\) in Eq. (8) is in disagreement with the latest CODATA adjustment of the fundamental constants \[46\].

A different approach is taken by Rosen \[47\] who assumes that the electromagnetic field operators might be sums over \(N = 42\) “hidden” field operators, where \(N = 42\) is the order of the transitive subgroup \(G\) of the of the symmetric group of degree 7, namely, \(S_7\). The physical value of \(\alpha\) is determined by matching the vacuum expectation value of the sum over the “hidden” field operators, invariant under the internal symmetry group, against the “effective”, “physical” QED field operator. Rosen \[47\] obtains the approximation

\[
\alpha_{\text{QED}} \approx \frac{4\pi}{N(N - 1)} \approx \frac{1}{137.032406} \ldots, \quad N = 42.
\]

(9)

However, both the choice of the order of the symmetry group (\(S_6\) as well as \(S_8\) are a priori not excluded by physical considerations but yield markedly different value for \(\alpha_{\text{QED}}\)). Another main criticism is that the group-theoretical considerations leave almost no room for an “adjustment” of the fine-structure constant with regard to the observed deviations of the physical value of \(\alpha_{\text{QED}}\) from the values given in Eqs. (8) and (9). Either the values are exact, or group theoretical explanations fail. At present, one should reemphasize that the value predicted for \(\alpha_{\text{QED}}\) by Eq. (9), just like the value given in Eq. (8), is incompatible with the CODATA value of the fine-structure constant \[46\].

III. CONJECTURES BASED ON CLASSICAL PHYSICS

A. Weyl’s Hypothesis

In the year 1919, Weyl \[48\] formulated a conjecture relating the radius of the Universe \(R_U\) and the classical electron radius \(r_e\),

\[
r_e = \frac{\alpha \hbar}{m_e c}.
\]

(10)

He speculated that \(r_e\) and \(R_U\) might be related to a hypothetical “radius” of a particle whose rest mass \(m_H\) is equal to the naive expression for the gravitational “self-energy” of the electron (with dimension equal to its classical electron radius),

\[
r_H = \frac{\alpha \hbar}{m_H c}, \quad m_H c^2 = \frac{Gm_e^2}{r_e}.
\]

(11)

The ratio of the two large quantities is observed to be of the same order-of-magnitude, namely

\[
\frac{r_H}{r_e} \sim 10^{42} \sim \frac{R_U}{r_e}.
\]

(12)

However, this observation is scrutinized by the lack of a consistent interpretation of the classical electron “radius” from a modern point of view.
B. Dirac Large Number Hypothesis

Dirac’s famous Large Number Hypothesis dates back to the year 1938 (see Ref. [49]). It is based on the observation that the age $T$ of the Universe and the time it takes light to travel a distance equal to the classical electron radius, which is equal to $r_e/c$, are approximately proportional to each other,

$$\frac{T}{(r_e/c)} \sim 10^{40} \sim \frac{e^2}{4\pi\epsilon_0 G m_e m_p}. \quad (13)$$

Dirac conjectured that the equality holds exactly, and that the gravitational interaction constant $G$ might be inversely proportional to the age of the Universe $T$.

Alternatively, one might identify the expression $e^2/(4\pi\epsilon_0)$ on the right-hand side of Eq. (13) as $\alpha_{\text{QED}} \hbar c$, and conjecture that the fine-structure constant might be varying with the age of the Universe. Indeed, such variations have been investigated recently by Flambaum and others [11–15]. While observational data may indicate slightly lower values of $\alpha$ in the distant past (see Fig. 1 of Ref. [13]), a direct proportionality of the fine-structure constant to the age of the Universe has not been established conclusively. In particular, in its most basic form, Dirac’s hypothesis is incompatible with recent claims [15] regarding a spatial variation (in addition to the temporal variation) of the fine-structure constant.

C. Eddington Conjecture

Eddington [50] was the first to suggest a connection of the gravitational and electromagnetic interactions, probably inspired by his seminal work [51] on the theory of general relativity where, as well shall see in the following, a certain connection of gravitational and electromagnetic interactions is suggested by the structure of the Lagrangian. Furthermore, as shown in Appendix A, there exist certain analogies of the gravitational and electromagnetic bound-state problems. The gravitational fine-structure constant, defined in complete analogy with the electromagnetic bound-state problem, depends on the masses of the two involved particles which form the bound state and is typically much smaller than $\alpha_{\text{QED}}$ for the known elementary particles. Eddington conjectured (see Ref. [50]) that the electromagnetic fine-structure constant $\alpha_{\text{QED}} \approx 1/137.036$ and the gravitational fine-structure constant $\alpha^{(ee)}_G$ for two gravitationally interacting electrons should be proportional to each other,

$$\alpha_{\text{QED}} = \frac{e^2}{4\pi\epsilon_0 \hbar c}, \quad \alpha^{(ee)}_G = \frac{G m_e^2}{\hbar c}, \quad \alpha^{(ee)}_G = \frac{e^2}{4\pi\epsilon_0 G m_e^2} \approx 4.2 \times 10^{42} \approx \sqrt{N_C}, \quad (14)$$

where $N_C$ is the number of charged particles in the Universe. Eddington then went on to conjecture that $N_C$ should be given explicitly in terms of an integer number with 42 decimals (the “Eddington number”), invariant over time, giving the number of positrons and electrons in the globally neutral Universe. Invariably, the Eddington conjecture was formulated before the advent of modern particle accelerators and lasers where particle creation processes may be studied: e.g., a now rather famous experiment at the Stanford Linear Accelerator Center (SLAC) described in Refs. [52, 53] has shown that electrons may be created in strong laser fields, after the injection of a highly energetic γ ray emitted by Compton backscattering from an energetic oncoming electron.

IV. QUANTUM AND CLASSICAL FLUCTUATIONS AND THE FINE-STRUCTURE CONSTANT

A. Casimir Effect and the “Mousetrap” Model

The fine-structure constant is proportional to the ratio $e^2/(\hbar c)$. This particular combination of physical quantities gives rise to a conjecture formulated by Casimir [54], as follows. The charge distribution of a truly elementary charged particle (like the electron, not the proton), in its rest frame, is approximated as a conducting spherical shell carrying a homogeneous surface charge of total magnitude $e$. The electrostatic self-energy of such an object is given as

$$E_S = \frac{e^2}{8\pi\epsilon_0 a}, \quad (16)$$
where $a$ is the radius of the shell. This energy needs to be compared to the Casimir energy of the spherical shell configuration, because vacuum fluctuations of the electromagnetic field are influenced by the boundary conditions set forth by the spherical shell configuration.

E.g., according to the formalism of QED (see Chap. 3.2.4 of Ref. [22]), parallel plates are known to attract each other due to the vacuum fluctuations; the latter are suppressed in the region in between the plates, while they are not suppressed outside the plates, resulting in a net attractive force. Analogously, vacuum fluctuations are suppressed inside the spherical shell. Intuitively, one would assume that the vacuum fluctuations outside the spherical shell, which are only marginally influenced by the boundary conditions set forth by the (small) spherical shell of radius $a$, would tend to press the spherical shell together, “trapping” the charge. Hence, the name “mousetrap model” has been used in the literature [55].

Dimensional analysis shows that the Casimir energy of the spherical shell contribution has to be of the functional form

$$E_C = -C \frac{\hbar c}{2a},$$

where $C$ is a constant to be determined. Equating the surface tensions (or the restoring forces) derived from the self-energy $E_S$ and the Casimir energy $E_C$, one arrives at the equilibrium condition

$$C = \frac{e^2}{\epsilon_0 \hbar c}.$$  

If we now assume that the model holds universally for all charged elementary particles, then $C$ is promoted to the role of a dimensionless fundamental constant of nature, exclusively determined by the Casimir configurations (boundary conditions), and proportional to the same combination of quantities $e$, $\hbar$, $\epsilon_0$ and $c$ as the fine-structure constant. Furthermore, the formalism remains valid in the limit $a \to 0$, thus avoiding discussion regarding the role of $a$ as a yet-to-be-specified parameter of the “mousetrap” model (which thus would work for arbitrarily small “mice”).

Unfortunately, closer inspection [55] reveals that the most immediate ansatz for the shape of the fundamental charge distribution, namely, the spherical shell just discussed, leads to a repulsive rather than attractive Casimir energy [55], thus invalidating (at least the most immediate version of) the model. However, one may point out that it would be very interesting to consider alternative shapes, and the last word on “mousetrap models” is yet to be spoken. Another aspect which would need to be taken into consideration is as follows. While $C$ is related to the fine-structure constant, it is not necessarily equal to it, i.e., the model is derived from the self-interaction of a single elementary particle, not from the interaction of different elementary particles, which characterizes the strength of the interaction. Still, it might be worthwhile to explore variants of the “mousetrap model” based on alternative shapes of the charge distribution in the future.

### B. Electromagnetic Fluctuations and Gravitational Interactions

Electrostatic interactions can either be attractive or repulsive, while gravity always is attractive. However, fluctuations of the electromagnetic field in neutral objects typically lead to attractive interactions. Examples are the charge fluctuations of induced dipoles that lead to the van-der-Waals and Casimir interactions among atoms [56]. Hence, it is tempting to explore a possible ansatz that would identify gravitational interactions as possible residual manifestations generated by fluctuations electromagnetic (or even electroweak) interactions. While these approaches typically do not lead to any concrete formula for the fine-structure constant, we still include a brief discussion of related works in part because they naturally lead to the field-theoretical conjectures discussed in Sec. V.

In Refs. [57, 58], one such possibility is explored. It is shown that if every charged (or neutral) object is endowed with a fundamental set of spherically symmetric intrinsic electromagnetic oscillations, then these oscillations of the charge distributions will induce a force law at large distances which resembles Newton’s gravitational force law, and which is always attractive. Gravity and electromagnetic interactions can then be unified provided a relation of the following type [see Eq. (38) of Ref. [57]] holds universally for macroscopic bodies of masses $M_1$ and $M_2$,

$$G M_1 M_2 \propto \frac{q_1 A_1^2 - \omega_1^2 q_2 A_2^2 - \omega_2^2}{4\pi \epsilon_0},$$

the proportionality factor being of order unity. Here, $q_1$ and $q_2$ are characteristic charges of the microscopic entities in the two macroscopic bodies, while $A_1$ and $A_2$ are conjectured to be of the order of an Angstrom ($10^{-10}$ m), while the angular frequencies $\omega_1$ and $\omega_2$ are conjectured to be of order $10^8$ Hz. One may object, though, that the precise details of the material properties should otherwise enter the formalism, and that it would be surprising if
a general relationship of the type given in Eq. \((19)\) could be established universally for all macroscopically relevant physical samples, relating the microscopic and macroscopic properties. Nevertheless, the interpretation of gravitational interactions are residual interactions stemming from fluctuations of fundamentally different physical origin is intriguing and in fact, has been explored further.

One of the most prominent approaches in this direction has been given by Sakharov \([59]\), with a modern interpretation being supplemented by Visser \([60]\). The basic idea is to study the quantum (as opposed to classical) fluctuations of fundamental quantum fields (of the “quantum vacuum”), and to try to relate these, on a macroscopic scale, to the gravitational interactions, i.e., to space-time curvature. The idea can roughly be formulated as follows: One first assumes that the quantum fields should exist on a (possibly curved) Lorentzian manifold. One does not attempt to quantize gravity itself, but rather, one interprets gravity as being created as a residual interaction, due to the quantum fluctuations of the fields which “live” on the Lorentzian manifold. Given a (possibly nonminimal) coupling of the quantum fields to the space-time curvature, it can be shown that the one-loop quantum fluctuations, evaluated according to the Schwinger proper-time method \([60]\), give rise to terms of the form

\[
S = \int d^4x \sqrt{-\text{det} g} \left( c_0 + c_1 R(g) + \ldots \right),
\]

where \(c_0\) and \(c_1\) are coefficients which are determined by the model, while the space-time curvature is \(R(g)\), and the ellipsis “(…)” denotes higher-order correction terms proportional to \(R^2\). This action needs to be matched against the Einstein–Hilbert action (with a cosmological constant term),

\[
S = \int d^4x \sqrt{-\text{det} g} \left( \Lambda + \frac{R(g)}{16\pi G} + \ldots \right)
\]

where again we ignore higher-order corrections proportional to \(R^2\).

One possibility to generate the terms in Eq. \((20)\) is to consider the quantum fluctuations described by the one-loop effective action for a non-minimally coupled scalar field [see Eq. (3) of Ref. \([60]\)],

\[
S_g = -\frac{1}{2} \ln \text{det} (\Box_g + m^2 + \xi R(g))
\]

where \(\Box_g\) is the “quabla” operator on the curved space-time manifold, while \(m\) is the mass of the scalar field and \(\xi\) is an effective coupling constant. With a convenient reference metric \(g_0\), one can write in the Schwinger proper-time representation [see Chap. 4 of \([22]\) and Eq. (3) of \([60]\)],

\[
S_g = S_{g_0} + \frac{1}{2} \text{Tr} \int_{\kappa^{-2}}^{\infty} \frac{ds}{s} \left[ \exp(-s[\Box_g + m^2 + \xi R(g)]) - \exp(-s[\Box_{g_0} + m^2 + \xi R(g_0)]) \right]
\]

where \(\kappa\) is a cutoff parameter in the proper-time. According to Eq. (21) of Ref. \([60]\), and in agreement with arguments originally put forward by Sakharov \([59]\), the matching of 1/\(G\) in Eq. \((21)\) against the quantum fluctuations leads to an expression which is quadratically divergent in \(\kappa\). Thus, without recourse to further cancellations, which could potentially be mediated by supersymmetry, the physical value of \(G\) would crucially depend on the precise value of the cutoff. This might not seem appealing. Also, the approach outlined in Refs. \([53, 60]\) does not automatically lead to a formula expressing \(G\) in terms of \(\alpha_{\text{QED}}\), or vice versa. If the quantum fluctuation ansatz formulated in Refs. \([53, 60]\) is to yield a connection of \(G\) and \(\alpha_{\text{QED}}\), then it will be necessary to study the quantum fluctuations of all quantized fields in the Universe, including the electrically charged fermion fields (see also Sec. \([\text{V}\text{C}]\) below). The quantum fluctuation-inspired approaches are important because, if containing elements of truth, they will most likely imply at least a partial dismissal of the quantum gravity program, as gravitation will most likely no longer be seen as a fundamental interaction.

Finally, let us briefly mention that recently \([61, 62]\), a model has been put forward which modifies gravity, at short distance scales, into a theory which incorporates quantum fluctuations of the weak gauge bosons. The result is a modified gravitational force law, of the form

\[
V_{\text{eff}}(\vec{r}) = -\frac{G_N m_1 m_2}{r} \left[ 1 + \left( \frac{\bar{G}_N}{G_N} - 1 \right) \exp \left( -\frac{r}{\Lambda_p} \right) \right], \quad \bar{G}_N = \frac{1}{\sqrt{2}} \left( \frac{c}{\hbar} \right)^2 G_F, \quad \Lambda_p = \sqrt{\frac{2\hbar G_N}{c^3}}.
\]

Here, \(\bar{G}_N\) is a modified gravitational constant into which \(G_N\), the Newtonian gravitational constant, is assumed to “morph” at short distances (\(G_F\) is the Fermi coupling constant), while \(\Lambda_p\) is the “Planck length” corresponding to the “short-distance-version” of gravity. The model formulated in Refs. \([61, 62]\) might explain the proton radius
puzzle \cite{63,64} connected with the Lamb shift in muonic hydrogen, while its generalization to time-like momentum transfer might lead to additional corrections to the electron and muon $g$ factors \cite{65}, which remain to be explored. Also, a unification of the gravito-weak model formulated in Refs. \cite{61,62} with electromagnetism is likely to yield correction terms to Eq. (23). Still, the unification ansatz formulated in Refs. \cite{61,62} is interesting because it highlights the need to search for clues toward unifications of gravity with other fundamental interactions.

V. CONJECTURES BASED ON QUANTUM FIELD THEORY

A. Graviton–Photon Conversion and Gravito–Electromagnetic RG

Inspired by the works of Gertsenshtein \cite{66}, and by Zel’dovich and Novikov \cite{67}, let us consider the possibility of a connection of the gravitational and electromagnetic interactions on the quantum level. In the quantized gravitational interaction, with a metric $g_{\mu\nu}$ of the curved space-time, the deviation $h_{\mu\nu}$ from the flat space-time metric $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$ is quantized. The quantized $h_{\mu\nu}$ field operator (graviton) [see Eq. (5) of Ref. \cite{68}] is proportional to $\sqrt{G}$ where $G$ is Newton’s gravitational constant. The investigation of quantized gravity and relevant Ward identities \cite{69,72} as well as the influence of gravitational interactions on the RG evolution of other gauge couplings \cite{73} is a matter of ongoing investigations. These combined gravito-electromagnetic models significantly enhance the scope of nonlinear field-theoretical extensions of Maxwell theory alone \cite{74,75}.

In particular, we consider the tree-level amplitude contained in the coupling $h_{\mu\nu} T^{\mu\nu}$ of the graviton $h_{\mu\nu}$ to the energy-momentum tensor $T^{\mu\nu}$ of the electromagnetic field,

$$T^{\mu\nu} = F^{\mu\alpha} F^\nu_{\alpha} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu}. \tag{25}$$

Taking $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + f^{\mu\nu}$, where $F^{\mu\nu}_{\text{ext}}$ is the external field-strength tensor and $f^{\mu\nu}$ the photon field, the term $h_{\mu\nu} T^{\mu\nu}$ yields the trilinear form \cite{66,67,71}

$$h_{\mu\nu} (F^{\mu\alpha}_{\text{ext}} f^\nu_{\alpha} + f^{\mu\alpha} F^{\nu\alpha}_{\text{ext}}) - \frac{1}{2} g^{\mu\nu} F^{\alpha\beta}_{\text{ext}} f_{\alpha\beta}, \tag{26}$$

which mixes the graviton $h_{\mu\nu}$ and the photon $f_{\mu\nu}$. Indeed, photon-graviton mixing near a pulsar has been considered in Ref. \cite{76}. The possibility of converting photons into gravitons may hint at a deeper connection of the two interactions, even in the low-energy domain. In any case, the existence of graviton-photon mixing implies that the running of the QED and gravitational coupling constants cannot be treated independently of each other \cite{73}.

Indeed, the RG evolution of the fine-structure constant naturally leads to a conjecture involving the Planck and electron mass scales. Generally, it is assumed that the coupling constants of physics unify at a length scale commensurate with the Planck scale,

$$\ell_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.616 \times 10^{-25} \text{ m}, \tag{27}$$

where we restore the factors of $\hbar$ and $c$. The basic paradigm \cite{35} is that the coupling strengths of gravity and the electroweak model increase with the momentum scale, whereas the coupling constant of quantum chromodynamics decreases. At $\ell_P$, they are generally assumed to unify at a value of order unity (see, e.g., Ref. \cite{35}). If we assume $\alpha_{QED}$ to be of order unity at the Planck scale, and consider that the natural length scale of quantum electrodynamics is the mass scale of the lightest fermion, the electron, and furthermore conjecture that the logarithmic one-loop running of the QED coupling approximates the full RG evolution sufficiently well, then it is natural to conjecture that

$$\alpha_{QED} \sim \left[\ln \left(\frac{\Lambda}{m_e}\right)\right]^{-1} \sim \left[\ln \left(\frac{\Lambda}{\ell_P}\right)\right]^{-1}, \tag{28}$$

where $\lambda_e$ is the electron’s Compton wavelength, and $\Lambda$ is the cutoff (this formula also illustrates that the QED coupling constant goes to zero as the cutoff goes to infinity \cite{26}. Indeed, in Refs. \cite{77,78}, different approaches have led to relations similar to Eq. (17) of Ref. \cite{23}, which we quote here as

$$\alpha_{QED} = \frac{3\pi}{(\sum Q^2)} \left[\ln \left(\frac{4\pi}{\kappa_0 N_0 G m_e^2}\right)\right]^{-1}, \tag{29}$$
where $\kappa_0 = 5/9$ or $2/3$, depending on whether a smooth, Lorentz invariant or straight cutoff is used in the RG \cite{79}. In Eq. (29), the number $N_0$ is related to the number of active fermions in a Weinberg–Salam multiplet. In particular, according to Ref. \cite{79}, we have $N_0 = \frac{15}{2} N$ and $\sum Q^2 = \frac{5}{3} N$ for $N$ multiplets (generations). If the numerical prefactor in Eq. (29) is exact, then a brief numerical calculation suggests there should exist five or six additional heavy charged leptons, neutrinos, and quark multiplets, as noted by the authors of Ref. \cite{79}. The extra generations have not been experimentally confirmed up to now. One should also note that the precise evaluation of the prefactor in Eq. (29) may have to be revisited and may depend on the details of the unification model used.

B. Kaluza–Klein Theories and the Fine–Structure Constant

It is well known that Kaluza–Klein theories \cite{16,17} represent one of the first attempts to unify gravity and electromagnetism. It is less well appreciated that these theories also predict a direct proportionality of the gravitational and electromagnetic coupling constants. In order to illustrate this point, let us start by recalling that the compactification of the fifth dimension in Kaluza–Klein theories leads to a natural explanation of charge quantization, while relating the fine-structure constant to a quantity which can either be interpreted as the vacuum expectation value of a scalar field, or, to a physical constant which relates the fifth component of the metric to the $(4 \times 4)$-space-time components. The covariant coupling in QED is of course given by the covariant derivative, $\partial_\mu$, differently, by the replacement $\partial_\mu \to \nabla_\mu = \partial_\mu + i e A_\mu$. Let $x = (x^0, x^1, x^2, x^3)$ denote a four-vector, while $y$ adds the fifth dimension. Let $A, B$, denote indices in the extended space, i.e., including the extra dimensions. We start from Eq. (22) of Ref. \cite{18},

$$S_\psi = - \int d^4 x \, dy \sqrt{-\det(\hat{g})} \, \partial^A \hat{\psi}(x, y) \, \partial_A \hat{\psi}(x, y),$$

with the metric given in Eq. (16) of Ref. \cite{18},

$$(g_{AB}) = \phi^{-1/3} \left( g_{\alpha\beta} + \kappa^2 \phi \hat{A}_\alpha \hat{A}_\beta \right. \begin{array}{c} \kappa \phi \hat{A}_\alpha \kappa \phi \end{array} \right).

(31)$$

Here, $\kappa = \sqrt{16 \pi G}$. The expansion into Fourier modes in the compactified dimension leads us to the formula

$$\hat{\psi}(x, y) = \sum_{n=-\infty}^{n=\infty} \psi^{(n)}(x) e^{in y/r},

(32)$$

where the $\psi^{(n)}(x)$ denote massive scalar fields coupled to the electromagnetic field, while $r$ is the scale of the compactification of the fifth dimension. (We reserve the hat over a symbol for the five-dimensional generalization of a quantity otherwise defined on four-dimensional space-time.) The action is given as follows,

$$S_\psi = - \left( \int dy \right) \sum_{n=-\infty}^{n=\infty} \int d^4 x \, dy \sqrt{-\det g}

\times \left\{ \left( (i \partial^\mu - \frac{n}{r} \kappa \hat{A}^\mu) \psi^{(n)}(x) \right) \left[ (i \partial_\mu - \frac{n}{r} \kappa \hat{A}_\mu) \psi^{(n)}(x) \right] - \frac{n^2}{\phi r^2} \left( \dot{\psi}^{(n)}(x) \right)^2 \right\},

(33)$$

where $g$ is the four-dimensional restriction of $\hat{g}$. We now renormalize the four-vector potential and define the four-dimensional gravitational coupling constant as follows,

$$\hat{A}_\alpha = \frac{A_\alpha}{(\phi \int dy)^{1/2}}, \quad G = \frac{\hat{G}}{\int dy}.

(34)$$

The scaling of the four-vector potential is necessary in order to eliminate an otherwise disturbing factor $\phi$ in front of the gauge boson term in the action functional (which we do not explicitly write out, see Ref. \cite{18} for details). Otherwise, this term blows up when we choose $\phi$ to be large. The sum over spinless fields, coupled to the electromagnetic field,
is given as follows,

\[ S_{\psi} = - \left( \int dy \right) \sum_{n=-\infty}^{\infty} \int d^4x \, dy \sqrt{-\det g} \left\{ \left( (i\partial^{\mu} - \frac{n \kappa}{r(\phi)dy^{1/2}} A^\mu) \psi^{(n)}(x) \right) \right. \\
\left. \times \left[ (i\partial_{\mu} - \frac{n \kappa}{r(\phi)dy^{1/2}} A_{\mu}) \psi^{(n)}(x) \right] - \frac{n^2}{\phi r^2} \left( \psi^{(n)}(x) \right)^2 \right\} . \tag{35} \]

Charge is found to be naturally quantized according to the formula

\[ q_n = \frac{n \kappa}{r(\phi)dy^{1/2}} = \frac{n \sqrt{16 \pi}}{r(\phi)dy^{1/2}} \sqrt{G} = \frac{n \sqrt{16 \pi}}{r(\phi)dy^{1/2}} \sqrt{G} . \tag{36} \]

Setting \( n = 1 \), one obtains

\[ \alpha_{\text{QED}} = \frac{e^2}{4\pi} = \frac{q_1^2}{4\pi r(\phi)dy^{1/2}} = \frac{16\pi}{4\pi(r(\phi)dy^{1/2})} G = \frac{4}{(r(\phi)dy^{1/2})} G , \tag{37} \]

suggesting a proportionally of the electromagnetic and gravitational fine-structure constants. If the field \( \phi \) varies in space and/or time, this could otherwise explain a time variation of the fundamental constants. The main problem of the Kaluza–Klein formalism is due to the very large generated mass parameters, \( m^2 = n^2/(\phi r^2) \) which are obtained when a realistic physical value for the quantity \( r(\sqrt{\phi}) \), determined from Eq. \( (37) \), is inserted into the formula \( m^2 = n^2/(\phi r^2) \). Another problem is related to the instability of the radius \( r \) of the compact dimension against small perturbations. None of these problems have been satisfactorily addressed in the literature up to this point, while the general idea of Kaluza and Klein has inspired many technical developments in field theory over the last decades.

C. String Theory and Connections of Gravity and Electromagnetism

String theories, and superstring theories contain both gravitational and gauge interactions \[ \text{[19, 20]}. \] In principle, they thus offer a possible way to unify the gauge field and gravity. Gravitons and gauge particles are assumed to correspond to massless states of closed and open strings. Relations between closed and open strings then imply the relations between gravitational interactions and gauge fields. In the consideration of strings on a curved space-time background \[ \text{[19]} \], it is almost universally assumed [see Eq. \( (3.2.7) \) of Ref. \[ \text{[19]} \)] that a relationship of the form

\[ g_o \sim g_c \sim \exp(\lambda) \tag{38} \]

holds, where \( g_o \) is the gauge coupling, \( g_c \) is the gravitational coupling, and \( \lambda \) parameterizes the string interaction. According to the text following Eq. \( (3.7.17) \) of Ref. \[ \text{[19]} \], the value of \( \lambda \) determines the coupling strength between strings, but this does not necessarily imply that string theories with different values of this parameter describe different physics; namely, \( \lambda \) defines a basic unit of length, and can be absorbed in a redefinition of the coordinates \( X^\mu \) of the string world sheet. Specialized to quantum chromodynamics, analogous relations are also known as the Kawai–Lewellen–Tye relations \[ \text{[80]} \].

Now, combining Eqs. \( (3.7.26) \) and Eq. \( (6.6.18) \) of Ref. \[ \text{[19]} \], we have

\[ \kappa = (8\pi G)^{1/2} = 2\pi g_c , \quad g_c^2 = \frac{\kappa^2}{4\pi^2} = \frac{2}{\pi} G . \tag{39} \]

According the Eq. \( (3.6.26) \) of Ref. \[ \text{[19]} \], the photon vertex operator is given as

\[ -i \frac{g_o}{(2\alpha')^{1/2}} e_\mu \int_{\partial M} ds \left[ X^\mu \exp(i k \cdot X) \right] \tag{40} \]

where \( \alpha' \) is a numerical parameter of order unity \( [\alpha' = 2 \text{ for the closed string, while } \alpha' = \frac{1}{2} \text{ for the open string}] \), \( k \) is the exchange four-momentum, \( e_\mu \) the polarization vector, and the interaction is evaluated on the string world-sheet \( (\Gamma, \partial M) \). Furthermore, according to Eq. \( (6.6.23) \) of Ref. \[ \text{[19]} \], the four-point open (gauge coupling) \( A_o \) and closed (gravitational) string amplitudes \( A_c \) are related as follows,

\[ g_c^4 A_c(s, t, u, \alpha', g_c) = g_o^2 \pi i \alpha' \sin[\pi \alpha_o(t)] A_o(s, t, \frac{1}{2} \alpha', g_o) A_o(t, u, \frac{1}{2} \alpha', g_o)^* . \tag{41} \]
where \( \alpha_c(t) = 1 + \alpha' x \), and the \( s, t \) and \( u \) are the Mandelstam variables. Both the basic equation (38) and the more detailed formulation given in Eq. (41) suggest a proportionality of the form, inspired by string theory,

\[
\alpha_{\text{QED}} \propto \sqrt{G},
\]

when expressed in terms of the physical couplings of QED and gravity. We recall Eq. (A7), which implies that the gravitational fine-structure constant of electron and proton reads as

\[
\alpha \equiv \frac{G m_e m_p}{\hbar c}.
\]

Based on the physical values of \( \alpha_{\text{QED}} \) and \( \alpha_G \), and the inspiration from string theory [19, 20], one may attempt to express the proportionality factor in Eq. (12) in a simple form. Based on the coincidence

\[
\frac{\alpha_{\text{QED}}}{\sqrt{\alpha_G}} \approx 4.07 \times 10^{18},
\]

one may investigate the following relationship, which is numerically fulfilled to relatively good accuracy,

\[
\frac{1}{\sqrt{\alpha_G}} \exp \left( -\frac{m_p}{m_e} \right) = \left( \frac{G m_e m_p}{\hbar c} \right)^{-1/2} \exp \left( -\frac{m_p}{m_e} \right) = 136.976(8) \approx \frac{1}{\alpha_{\text{QED}}}.
\]

This relation is consistent with the “string-inspired conjecture” \( \alpha_{\text{QED}} \propto \sqrt{G} \) and identifies the proportionality factor as being approximately given by the expression \([\hbar c/(m_e m_p)]^{1/2} \exp[(m_p/m_e)^{1/2}]\). In Eq. (45), the latest CODATA value [46] for the gravitational constant has been used, \( G = 6.67384(80) \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \). Its uncertainty dominates the uncertainty of the numerical value of the expression on the left-hand side of the relation [45]. One may observe that recent determinations of Newton’s gravitational constant are not all in mutual agreement [81]; the scatter of the experimental data has not been resolved [82]. One should add that observations related to Eq. (45) had previously been made in Ref. [83] and interpreted as a means of determining \( G \), not \( \alpha_{\text{QED}} \).

The square root of the string-inspired on the right-hand side of Eq. (12) otherwise finds a physical interpretation as follows. Namely, according to Ref. [84], there exists a relativistic correction term in the gravitational bound-state problem which can naturally be identified as the gravitational quiver motion (zitterbewegung) term and which is proportional to

\[
H_{Z} \propto \frac{\hbar^2 G m_p}{c^2 m_e} \delta^{(3)}(\vec{r}) \equiv \hbar c r_{Z}^2 \delta^{(3)}(\vec{r}),
\]

where \( r_Z \) is the radius of the quiver motion (zitterbewegung) of the gravitationally bound electron. The identification of the Darwin term with the quiver motion is discussed on p. 71 in Sec. 2.2.4 of Ref. [22]. Solving for \( r_{Z}^2 \), we obtain

\[
r_{Z}^2 = \frac{\hbar G m_p}{c^2 m_e} = \frac{\hbar G m_p}{c^2 m_e} = \ell_{P}^2 \frac{m_p}{m_e}.
\]

where \( \ell_{P} = (\hbar G/c^3)^{1/2} \) is the Planck length. The argument of the exponential factor in Eq. (45) is thus identified as the ratio of the gravitational zitterbewegung term to the Planck length,

\[
\left( \frac{m_p}{m_e} \right)^{1/2} = \frac{r_Z}{\ell_{P}}.
\]

It appears that the square root of the mass ratio of the proton and electron naturally occurs in the gravitational bound-state problem.

Let us also consider the following scaling transformation (a “global dilaton”) of the fields,

\[
A^\mu \rightarrow \lambda A^\mu, \quad A_\mu \rightarrow \lambda A_\mu, \quad \psi \rightarrow \lambda \psi,
\]

combined with the following transformation of the coordinates,

\[
x^\mu \rightarrow \lambda^{-1/2} x^\mu, \quad x_\mu \rightarrow \lambda^{1/2} x_\mu, \quad g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \lambda^{-1} g^{\mu\nu},
\]

combined with the following transformation of the coordinates,
where the transformation of the metric entails the following transformation of the curved-space Dirac matrix, \( \bar{\gamma}^\mu \to \lambda^{-1/2} \bar{\gamma}^\mu \). This transformation modifies the Einstein–Maxwell Lagrangian, originally given as

\[
S = \int d^4x \sqrt{-\det g} \left\{ \frac{R}{16 \pi G} - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}(x) \left[ i \bar{\gamma}^\mu \left( \nabla_\mu - e A_\mu \right) - m \right] \psi(x) \right\},
\]

into

\[
S' = \int \frac{d^4x}{\lambda^2} \sqrt{-\lambda^4 \det g} \left\{ \frac{R}{16 \pi G} - \lambda^2 \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \lambda^2 \bar{\psi}(x) \left[ i \lambda^{-1/2} \bar{\gamma}^{\mu} \left( \lambda^{1/2} \nabla_\mu - e \lambda A_\mu \right) - m \right] \psi(x) \right\},
\]

which can be rearranged into

\[
S' = \lambda^2 \int \frac{d^4x}{\lambda^2} \sqrt{-\lambda^4 \det g} \left\{ \frac{R}{16 \pi G} - \lambda^2 \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \lambda^2 \bar{\psi}(x) \left[ \lambda^1/2 \nabla_\mu - e \lambda A_\mu \right] - m \right\} \psi(x) \right\}.
\]

The coupling constants have transformed as follows,

\[
G \to \lambda^2 G, \quad e^2 \to \lambda e^2.
\]

This scaling is consistent with the proportionality \( \alpha_{\text{QED}}^2 \propto e^2 \propto \lambda^2 \propto G \propto \alpha_G \), and thus with the string-inspired conjecture given in Eq. (45). The conformal transformation (49) leaves the following action integrals

\[
\int d^4x |\psi|^2 \to \left( \frac{1}{\lambda^{1/2}} \right)^4 \int d^4x (\lambda |\psi|^2) = \int d^4x |\psi|^2,
\]

\[
\int d^4x F_{\mu\nu} F_{\mu\nu} \to \left( \frac{1}{\lambda^{1/2}} \right)^4 \int d^4x \lambda^2 F_{\mu\nu} F_{\mu\nu},
\]

invariant. In order to arrive at a variational principle, we again isolate from Eq. (52) the terms in the Lagrangian which depend on the scale parameter \( \lambda \),

\[
S'' = \frac{S'}{\lambda^2} = \int \frac{d^4x}{\lambda^2} \sqrt{-\det g} \mathcal{L}' = \mathcal{L}' \to \frac{R}{16 \pi G \lambda^2} - e \lambda^{1/2} \bar{\psi}(x) A_\mu \psi(x).
\]

Most particles in the Universe are in bound states, both gravitationally as well as electromagnetically. Integrating the Lagrangian density over all space, one obtains the Lagrangian of the Universe. As matter is found in the bound states of atoms, we have as an order-of-magnitude estimate

\[
-\lambda^{1/2} \int d^3x \left\langle e \bar{\psi}(x) A_\mu \psi(x) \right\rangle \sim \lambda^{1/2} E_0 \sim \lambda^{1/2} 27.2 \text{ eV},
\]

where \( E_0 \) is a typical binding energy in an atom, which is distributed over the electrons and protons, measured in terms of the Hartree energy of 27.2 eV, and \( \psi \) can be the wave function of a proton or an electron.

The gravitational contribution per proton can be estimated as the contribution to the curvature integral of the Schwarzschild metric for an average star,

\[
\frac{1}{\lambda^2} \int d^3x \frac{R}{16 \pi G} \sim \frac{1}{\lambda^2} \left( \frac{G m_p M_{\text{Sun}}}{r_{\text{Sun}}} \right) = \frac{1}{\lambda^2} E_G \sim \frac{1}{\lambda^2} 1991.3 \text{ eV}.
\]

We here use the radius \( r_{\text{Sun}} \) and its mass \( m_{\text{Sun}} \) in our solar system as a measure of an average particle bound to a typical star in the Universe. If \( \lambda \) acts as a variational parameter, then the variational condition reads as

\[
\frac{\partial}{\partial \lambda} \left( \frac{1}{\lambda^2} E_G + \lambda^{1/2} E_0 \right)^{1/4} = 0, \quad \lambda = \left( \frac{E_G}{E_0} \right)^{1/3}.
\]

With our order-of-magnitude estimates (56) and (57), we obtain \( \lambda \approx 9.70 \) for the current Universe, which differs from unity by less than one order of magnitude. A more precise variational calculation remains elusive to this date and requires a better understanding of the mass distribution in the Universe ("dark matter").
VI. CONCLUSIONS

We have discussed attempts to find approximate formulas for the fine-structure constant, based on purely algebraic relations (see Sec. III). The most promising ansatz in this direction has been a hypothetical zero of the QED $\beta$ function. However, it appears that nature does not do us the favor of providing a fixed point for the RG evolution of the QED coupling in the low-energy regime which could form a suitable candidate equation for the determination of $\alpha_{\text{QED}} \approx 1/137.036$ (see Sec. III A). Group-theoretical approaches discussed in Sec. III B are very rigid in predicting fixed numerical values for $\alpha$ which therefore cannot receive any further quantum field-theoretical loop corrections, but so far, no compelling predictions have been obtained for manifestly interacting particles.

Arguments based on classical physics (see Sec. III) meet considerable difficulty when confronted with the additional wisdom obtained from modern particle physics experiments, which challenge some of the concepts on which the original conjectures were based. Theories relating classical and quantum fluctuations (e.g., of electromagnetic origin) to gravitational interactions (see Sec. IV) have yet to produce a viable conjecture for a formula that would relate the fine-structure constant to other fundamental constants.

In turn, conjectures inspired by quantum field-theory (see Sec. IV) have been strongly inspired by the graviton-photon conversion amplitude analyzed in Refs. 66, 67 discussed in Sec. V A. The renormalization-group inspired formula (29), studied in Refs. 77, 78, relates the QED coupling at the Planck scale to the observed coupling parameter $\alpha_{\text{QED}}$, which is relevant to the low-energy (electron mass) scale. We note that the RG running of the gravitational and electromagnetic coupling constants are intertwined [73] so that Eq. (29), even if it holds approximately, will receive higher-order loop corrections from graviton-photon conversion. However, in terms of promising directions for future research, we believe that it may be useful to put Eq. (29) into the context of other conjectures which have been discussed in the literature.

In Sec. V B we discuss Kaluza–Klein theories. Often, it is not really appreciated that Kaluza–Klein theories with a compactified fifth dimension predict rather unambiguously that $\alpha_{\text{QED}}$ and $G$ should be directly proportional to each other [see Eq. (37)]. However, the mass hierarchy problems connected with such models have never been resolved in the literature, so that related conjectures should probably be taken with a grain of salt.

String theory [19, 20] predicts a different functional relationship which implies that $\alpha_{\text{QED}}$ should be proportional to $\sqrt{G}$ [see Eq. (3.2.7) of Ref. 19]. In turn, a dimensionless variant of the gravitational constant is given by the electron-proton coupling strength given in Eq. (A7), which we refer to as $\alpha_G$. One may thus attempt to search for simple analytic formulas which might describe the constant of proportionality in the predicted relationship $\alpha_{\text{QED}} \propto \sqrt{G}$. Surprisingly, numerical experimentation yields quite good agreement if one simply sets the proportionality constant equal to $\exp(\sqrt{m_p/m_e})$ [see Eq. (42)]. Most probably, this observation is a numerical coincidence and does not allow us to gain any deeper insight; however, it may be permissible to record the observation (14) as it may inspire models which attempt to match a few of the parameters of string-theoretical, and supersymmetric string-theory models with low-energy properties of the Standard Model, notably, the electron and proton masses.

The future will tell if any of the conjectures survive the tests of scrutiny that will confront these with other developments from the renormalization-group analysis of the intertwined gravitational and electromagnetic interactions, or, more sophisticated string-theoretical models, and experiments which may fix parameters of the models based on independent relations.

Acknowledgements

Helpful discussions with P. J. Mohr, M. M. Bush and E. Lötstedt are gratefully acknowledged. Work on this project has been supported by the National Science Foundation (Grants PHY–1068547 and PHY–1403973).

Appendix A: Analogy of the Electrostatic and Gravitational Bound–State Problem

Recently, the gravitational bound-state problem has been investigated in a relativistic quantum framework [34, 38]. Gravity and QED are the only two long range interactions mediated by a massless gauge boson (photon and graviton), while gluons have never been observed as free particles due to their confinement into hadrons. It is reasonable to search for a connection of the value of $\alpha_{\text{QED}}$ to other physical constants, derived from the low-energy regime alone, and the gravitational quantum bound-state problem provides for an interesting starting point.

The gravitational coupling constant for the interaction of electron and proton can be derived based on the Bohr–Sommerfeld quantization condition alone. The centripetal force $F$ exerted on the electron in its circular orbit is as follows, $F = m_e v^2/R$, where $R$ is the radius of the orbit. On the circular orbit, the centripetal force is equal to the
gravitational attractive force, and we have

\[ F = \frac{m_e v^2}{R} = \frac{G m_e m_p}{R^2}, \quad \frac{1}{2} m_e v^2 = \frac{G m_e m_p}{2R}, \]  

(A1)

where the latter equation is obtained from the former via multiplication by \( R/2 \) (the modulus of the electron velocity in the circular orbit is \( v \)). The virial theorem states that

\[ E = \frac{1}{2} m_e v^2 - \frac{G m_e m_p}{R} = -\frac{G m_e m_p}{2R}, \]  

(A2)

i.e. the potential energy is twice as large as the kinetic energy on the circular orbit, and has the opposite sign. Let us now implement the Bohr–quantization condition in its most basic form,

\[ \int p \, dq = n \hbar, \quad m_e v (2\pi R) = n \hbar, \quad m_e v R = n \hbar. \]  

(A3)

Let us square the latter equation,

\[ \frac{(n \hbar)^2}{2m_e} = \frac{(m_e v R)^2}{2m_e} = \frac{m_e v^2}{2} R^2 = \frac{G m_e m_p}{2R} R^2, \]  

(A4)

so that, on the quantized orbits, we have

\[ (n \hbar)^2 = G m_e^2 m_p R. \]  

(A5)

We plug this into the virial theorem and obtain the result

\[ E_n = -\frac{1}{2} \left( \frac{G m_e m_p}{\hbar c} \right)^2 \frac{m_e c^2}{n^2} = -\frac{1}{2} \alpha_G^2 \frac{m_e c^2}{n^2}, \]  

(A6)

where we use the “gravitational fine-structure constant” as \[87\]

\[ \alpha_G = \frac{G m_e m_p}{\hbar c}. \]  

(A7)

The numerical value is about \( 3.21 \times 10^{-42} \). The gravitational Schrödinger formula \[87\] is based on the quantization of the orbit according to Eq. \( \text{A3} \). By way of comparison, the Schrödinger formula for the electromagnetically bound atomic hydrogen energy levels is

\[ E_n = -\frac{1}{2} \alpha_{\text{QED}}^2 \frac{m_e c^2}{n^2}. \]  

(A8)

Equation \[84\] is obtained from Eq. \( \text{A6} \) by the simple substitution \( \alpha_G \rightarrow \alpha_{\text{QED}} \). Based on an analysis of the Dirac–Schwarzschild central-field problem in general relativity \[84\] \[85\] \[87\], it has recently been verified that \( \alpha_G \) is the analogue of the QED coupling constant for gravitational phenomena, on the basis of a calculation of the relativistic corrections for the Dirac-Schwarzschild central-field problem \[84\] \[87\].

[1] P. A. M. Dirac, Proc. Roy. Soc. London, Ser. A 167, 148 (1938).
[2] M. Fischer, N. Kolachevsky, M. Zimmermann, R. Holzwarth, T. Udem, T. W. Hänsch, M. Abgrall, J. Grünert, I. Mak-simovic, S. Bize, H. Marion, F. Pereira Dos Santos, P. Lemonde, G. Santarelli, P. Laurent, A. Clairon, C. Salomon, M. Haas, U. D. Jentschura, and C. H. Keitel, Phys. Rev. Lett. 92, 230802 (2004).
[3] A. Wyler, C. R. Acad. Sci. Paris Ser. A–B 269, A743 (1969).
[4] A. Wyler, C. R. Acad. Sci. Paris Ser. A 271, 186 (1971).
[5] M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).
[6] M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969).
[7] A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics (Nauka, Moscow, 1969).
[61] R. Onofrio, Mod. Phys. Lett. A 28, 1350022 (2013).
[62] R. Onofrio, Europhys. Lett. 104, 20002 (2013).
[63] R. Pohl, A. Antognini, F. Nez, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, L. M. P. Fernandes, A. Giesen, T. Graf, T. W. Hänsch, P. Indelicato, L. Julien, C.-Y. Kao, P. Knowles, E.-O. Le Bigot, Y. W. Liu, J. A. M. Lopes, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. A. Schaller, K. Schuhmann, C. Schwob, D. Taquini, J. F. C. A. Veloso, and F. Kottmann, Nature (London) 466, 213 (2010).
[64] A. Antognini, F. Nez, K. Schuhmann, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, M. Diepold, L. M. P. Fernandes, A. Giesen, A. L. Gouvea, T. Graf, T. W. Hänsch, P. Indelicato, L. Julien, C.-Y. Kao, P. Knowles, F. Kottmann, E.-O. Le Bigot, Y.-W. Liu, J. A. M. Lopes, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. A. Schaller, C. Schwob, D. Taquini, J. F. C. A. Veloso, J. Vogelsang, and R. Pohl, Science 339, 417 (2013).
[65] U. D. Jentschura, Ann. Phys. (N.Y.) 326, 516 (2011).
[66] M. E. Gertsenshtein, Sov. Phys. JETP 14, 84 (1962).
[67] Y. B. Zeldovich and I. D. Novikov, The structure and evolution of the universe, Rel. Astrophys. Vol. 2, Chicago University Press, 1983.
[68] L. F. Abbott and D. D. Harari, Nucl. Phys. B 264, 487 (1986).
[69] F. Bastianelli and C. Schubert, J. High Energy Phys. 0502, 069 (2005).
[70] F. Bastianelli, U. Nucamendi, C. Schubert, and V. M. Villanueva, J. High Energy Phys. 0711, 099 (2007).
[71] F. Bastianelli and C. Schubert, J. High Energy Phys. 0903, 086 (2009).
[72] F. Bastianelli, O. Corradini, J. M. Davila, and C. Schubert, Phys. Lett. B 716, 345 (2012).
[73] S. P. Robinson and F. Wilczek, Phys. Rev. Lett. 96, 231601 (2006).
[74] M. Born and L. Infeld, Proc. Roy. Soc. London, Ser. A 144, 425 (1934).
[75] L. Infeld, Nature (London) 137, 658 (1936).
[76] G. Raffelt and L. Stodolsky, Phys. Rev. D 37, 1237 (1988).
[77] L. D. Landau, Niels Bohr and the Development of Physics, edited by W. Pauli (McGraw–Hill, New York), p. 52., 1955.
[78] H. Terazawa, K. Akama, and Y. Chikashige, Prog. Theor. Phys. (Kyoto) 56, 1935 (1976).
[79] H. Terazawa, Y. Chikashige, K. Akama, and T. Matsuki, Phys. Rev. D 15, 1181 (1977).
[80] H. Kawai, D. C. Lewellen, and S.-H. H. Tye, Nucl. Phys. B 289, 1 (1986).
[81] G. T. Gillies, Rep. Prog. Phys. 60, 151 (1997).
[82] P. J. Mohr, private communication, 2014.
[83] J. E. Brandenburg, IEEE Trans. Plasma Science 20, 944 (1992).
[84] U. D. Jentschura and J. H. Noble, Phys. Rev. A 88, 022121 (2013).
[85] U. D. Jentschura, Phys. Rev. A 87, 032101 (2013), [Erratum Phys. Rev. A 87, 069903(E) (2013)].
[86] U. D. Jentschura and J. H. Noble, J. Phys. A 47, 045402 (2014).
[87] U. D. Jentschura, Ann. Phys. (Berlin) 526, A47 (2014).
[88] U. D. Jentschura, Phys. Rev. A 90, 022112 (2014).