On Planar Algebraic Curves and Holonomic $D$-modules in Positive Characteristic

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Abstract

In this paper we study a correspondence between cyclic modules over the first Weyl algebra and planar algebraic curves in positive characteristic. In particular, we show that any such curve has a preimage under a morphism of certain ind-schemes. This property might pave the way for an indirect proof of existence of a canonical isomorphism between the group of algebra automorphisms of the first Weyl algebra over the field complex numbers and the group of polynomial symplectomorphisms of $\mathbb{C}^2$.

1 Overview

Let $R$ be an associative unitary ring. For $n \in \mathbb{N}$ the $n$-th Weyl algebra $A_{n,R}$ over $R$ is defined as the quotient

$$A_{n,R} = R\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle/I$$

with the ideal $I$ being generated by all elements of the form

$$x_ix_j - x_jx_i, \ y_jy_i - y_iy_j, \ y_ix_j - x_jy_i - \delta_{ij}$$

for $1 \leq i, j \leq n$.

The $R$-algebra $A_{n,R}$ is associative, unital and is a rank $2n$ free $R$-module with the generators given by the images of $x_i, y_i$ under the standard projection (we will from here on denote these generators simply as $x_i, y_i$).

The case when $R \equiv k$ is a field is of main interest. The Weyl algebra then coincides with the algebra $D(A^n_k)$ of polynomial differential operators on $A^n_k = \text{Spec} \ k[x_1, \ldots, x_n]$, with $x_i$ acting as multiplication by an indeterminate $x_i$ and $y_i$ as taking partial derivative $\frac{\partial}{\partial x_i}$.

Let us now recall several open conjectures in which the Weyl algebra emerges, along with a number of objects naturally associated to the free algebra, the polynomial algebra and itself. The statements outlined below have been demonstrated to possess a profound interconnection as well as relation to other problems of mathematical physics.

Let $\text{Aut}(A_{n,k})$ be the group of algebra automorphisms of $A_{n,k}$, and let $\text{End}(A_{n,k})$ be the monoid of algebra endomorphisms of $A_{n,k}$. 

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The **Dixmier Conjecture** $\text{DC}_n$, first stated in [1], asserts that every algebra endomorphism of $A_{n,k}$ is invertible, that is, $\text{Aut}(A_{n,k}) = \text{End}(A_{n,k})$, whenever $\text{char } k = 0$. By the Lefschetz principle it is sufficient to set the base field to be the field of complex numbers $\mathbb{C}$. The $\text{DC}_n$ implies $\text{DC}_m$ for all $n > m$; the conjunction $\bigwedge_{n \in \mathbb{N}} \text{DC}_n = \text{DC}_\infty$ is referred to as the stable Dixmier conjecture. The conjecture $\text{DC}_n$ is open for all $n \in \mathbb{N}$.

The **Jacobian Conjecture** $\text{JC}_n$ states that for any field $k$ of characteristic zero any polynomial endomorphism $\phi$ of $A_{n,k}$ with unital Jacobian

$$\det || \frac{\partial \phi^*(x_i)}{\partial x_j} ||_{1 \leq i,j \leq n} = 1 \tag{1.2}$$

is an automorphism. Again, by the Lefschetz principle one may set $k = \mathbb{C}$. The $\text{JC}_n$ implies $\text{JC}_m$ whenever $n > m$, and $\text{JC}_\infty$ denotes the stable Jacobian conjecture. Evidently $\text{JC}_1$ is true, as linear maps are globally invertible; $\text{JC}_n$, however, is open for all $n \geq 2$. A detailed description of the Jacobian conjecture and its equivalent formulations can be found in [2].

It is known that $\text{DC}_n \Rightarrow \text{JC}_n$, and that $\text{JC}_{2n} \Rightarrow \text{DC}_n$, which together imply that $\text{JC}$ and $\text{DC}$ are stably equivalent. The implication $\text{JC}_{2n} \Rightarrow \text{DC}_n$ is, much unlike its converse, very non-trivial, and it was proved by Belov-Kanel and Kontsevich in [3] and independently by Tsuchimoto, [13] (also cf. [5]).

Another conjecture involving natural structures over quotients of the free algebra, such as automorphism groups, was formulated by Belov-Kanel and Kontsevich, [4], and is called the **Belov-Kanel - Kontsevich Conjecture** $\text{B-KKC}_n$. Let

$$P_{n,\mathbb{C}} = \mathbb{C}[z_1, \ldots, z_{2n}] \tag{1.3}$$

be the polynomial $\mathbb{C}$-algebra over $2n$ variables equipped with the Poisson bracket:

$$\{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j} \tag{1.4}$$

Denote $\text{Aut}(P_{n,\mathbb{C}})$ the group of Poisson structure-preserving automorphisms of $P_{n,\mathbb{C}}$; this is the group of polynomial symplectomorphisms of $A_{2n,\mathbb{C}}$. The $\text{B-KKC}_n$ states that the groups $\text{Aut}(A_{n,\mathbb{C}})$ and $\text{Aut}(P_{n,\mathbb{C}})$ are canonically isomorphic.

The $\text{B-KKC}_n$ is true for $n = 1$. The proof is essentially a straightforward description of the groups involved: the structure of $\text{Aut}(P_{n,\mathbb{C}})$ was obtained by Jung and van der Kulk in mid-twentieth century, (see [6] and [7]), and is represented as the quotient of a free product of two groups as follows:

$$\text{Aut}(P_{n,\mathbb{C}}) \simeq G_1 \ast G_2/(G_1 \cap G_2) \tag{1.5}$$

where

$$G_1 \simeq SL(2, \mathbb{C}) \ltimes \mathbb{C}^2$$

is the special affine group and $G_2$ is a group of polynomial transformations of the form

$$(x_1, x_2) \mapsto (ax_1 + f(x_2), a^{-1}x_2), \quad a \in \mathbb{C}^\times, \quad f \in \mathbb{C}[z]$$

Thirty years ago Makar-Limanov, [8], [9], showed that the automorphism groups of the corresponding Weyl algebra and the free algebra of two variables admit a similar description. Subsequently, the case $n = 1$ is resolved positively. Higher-dimensional case is open to this day.
2 Correspondence between holonomic $\mathcal{D}$-modules and lagrangian subvarieties

Let $\psi \in \text{Aut}( P_n, \mathbb{C} )$ be a smooth symplectomorphism, that is

$$\psi : \mathbb{A}^{2n}_\mathbb{C} \to \mathbb{A}^{2n}_\mathbb{C}$$

(2.1)

is an isomorphism preserving the standard symplectic structure on $\mathbb{A}^{2n}_\mathbb{C}$, in local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$ given by a closed 2-form $\Omega = \sum_{i=1}^{n} dx_i \wedge dp_i$.

Let $P = (\mathbb{A}^{2n}_\mathbb{C}, \Omega)$ represent the symplectic manifold, $\bar{P} = (\mathbb{A}^{2n}_\mathbb{C}, \bar{\Omega})$, $\bar{\Omega} = -\Omega$ be its dual manifold. Consider the (tensor) product $P \times \bar{P} = (\mathbb{A}^{4n}_\mathbb{C}, \pi_1^*\Omega + \pi_2^*\bar{\Omega})$ with $\pi_i^*$ being the duals of cartesian projections

$$\pi_i : \mathbb{A}^{2n}_\mathbb{C} \times \mathbb{A}^{2n}_\mathbb{C} \to \mathbb{A}^{2n}_\mathbb{C}, \ i = 1, 2$$

(2.2)

We have the following

**Proposition 2.1.** Let $P_1 = (V_1, \Omega_1)$, $P_2 = (V_2, \Omega_2)$ be smooth manifolds equipped with symplectic structures $\Omega_1$, $\Omega_2$ respectively. For every smooth symplectomorphism $\psi : P_1 \to P_2$ its graph

$$\Gamma_\psi = \{(z, \psi(z)) \mid z \in V_1\}$$

(2.3)

is a lagrangian submanifold of $P_1 \times P_2$ (where $P_2$ is the dual manifold of $P_2$).

**Proof** Indeed, $\Gamma_\psi$ is isotropic:

$$\Gamma_\psi^\perp \equiv \{w \in \Gamma_\psi \mid \Omega(w, w') = 0, \ \forall w' \in \Gamma_\psi\}$$

$$\forall w = (z, \psi(z)) \in \Gamma_\psi, \ w' = (z', \psi(z')) \in \Gamma_\psi \Rightarrow$$

$$\Omega(w, w') = \Omega_1(z, z') - \Omega_2(\psi(z), \psi(z')) = 0$$

because $\psi$ preserves the symplectic structure.

Then, $\Gamma_\psi$ is also coisotropic. Take any $v = (z_1, z_2)$ in the orthogonal complement $\Gamma_\psi^\perp$ and any $w = (z, \psi(z))$ in $\Gamma_\psi$. By definition

$$\Omega(v, w) = 0$$

so that

$$\Omega_1(z_1, z) = \Omega_2(z_2, \psi(z))$$

As $\psi$ is an isomorphism, $\exists y \in P_1 : z_2 = \psi(y)$, but then

$$\Omega_1(y - z_1, z) = 0$$

and $y = z_1$ follows from non-degeneracy. □

The converse is easily seen to be also true.

Now, by this proposition, any polynomial symplectomorphism $\psi \in \text{Aut}( P_n, \mathbb{C} )$ corresponds to a lagrangian subvariety $L_\psi$ of the form (2.3) in $\mathbb{A}^{2n}_\mathbb{C}$ endowed with
symplectic structure formed from that of $A^{2n}_C$ as above.

One can establish along the lines of Proposition 2.1 a similar statement involving polynomial symplectomorphisms of affine space in positive characteristic.

As noted in [4], any automorphism $\phi \in \text{Aut}(A_{n,k})$ gives a bimodule, which can be viewed as a holonomic $A^{2n,k}$-module $M_\phi$. It might be possible to arrive at the B-KKC by establishing a canonical correspondence between such modules and lagrangian subvarieties of $A^{2n}_{k}$, thus constructing an inverse map by means of appropriate lifting to characteristic zero

$$\text{Aut}(P_{n,C}) \to \text{Aut}(A_{n,C})$$ (2.4)

The base case $n = 1$ seems to be penetrable in positive characteristic. Any lagrangian subvariety of $A^{2}_{k}$ has dimension 2 and therefore corresponds to a system of two polynomial equations of the form

$$f_1(x_1, x_2, y_1, y_2) = 0,$$
$$f_2(x_1, x_2, y_1, y_2) = 0;$$ (2.5)

Then, the subvariety can be projected separately onto subspaces spanned by $(x_1, x_2, y_1)$ and $(x_1, x_2, y_2)$, leading in each case to a set of planar curves parameterised by $x_2$. In the next section we establish that any such curve corresponds to at least one differential operator, so that the projections of (2.5) would naturally generate elements of $A^{2}_k$. However, it still remains to be seen whether the appropriate $A^{2,k}$-module is non-trivial. This problem is closely related to a situation which emerges when one considers the integrability of certain systems of partial differential equations.

3 Weyl algebra and planar curves in positive characteristic

Let $\text{char } k = p > 0$ and let $A_{1,k}$ be the first Weyl algebra over $k$. It is a free module of rank $p^2$ over its center $C(A_{1,k}) = k[x^p, y^p]$ together with the standard basis $\mathfrak{B}_1 = \{x^iy^j \mid 0 \leq i, j \leq p-1\}$. A key observation is that $A_{1,k}$ is an Azumaya algebra of rank $p$ over $k[x_1, x_2]$ (and, in general, that $A_{n,k}$ is Azumaya of rank $p^n$ over the polynomial algebra in $2n$ variables). In particular, we perform the following procedure.

Let us extend the center $k[x^p, y^p]$ of the first Weyl algebra by adding central variables $\bar{x}$, $\bar{y}$, such that

$$\bar{x}^p = x^p, \quad \bar{y}^p = y^p$$ (3.1)

One has now

**Lemma 3.1.** The extension of $A_{1,k}$ is isomorphic to the matrix algebra

$$\text{Mat}(p \times p, k[\bar{x}, \bar{y}])$$ (3.2)

**Proof** Indeed, for any prime $p$ the unital algebra $A$ over $k$, ($\text{char } k = p$) generated by two elements $y_1, y_2$ satisfying the relations

$$[y_1, y_2] = 1, \quad y_1^p = y_2^p = 0$$ (3.3)

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is isomorphic to Mat\((p \times p, k)\). Namely, a direct calculation shows there is an isomorphism
\[
A \rightarrow \text{End}_{k-mod}(k[x]/(x^p))
\]
given by
\[
y_1 \mapsto d/dx, \quad y_2 \mapsto x
\]
differentiation and multiplication by \(x\) operators).

Now, the extension
\[
k[x^p, y^p] \rightarrow k[\tilde{x}, \tilde{y}]
\]
is faithfully flat, as \(k[\tilde{x}, \tilde{y}]\) is a flat \(k[x^p, y^p]-\text{algebra}\) and the induced morphism
\[
\text{Spec } k[\tilde{x}, \tilde{y}] \rightarrow \text{Spec } k[x^p, y^p]
\]
is surjective.

The pullback algebra
\[
A = A_{1,k} \otimes_{k[\tilde{x}, \tilde{y}]} k[\tilde{x}, \tilde{y}]
\]
as a \(k[\tilde{x}, \tilde{y}]-\text{algebra}\) is generated by \(x, y\) with relations
\[
[y, x] = 1, \quad x^p = \tilde{x}^p, \quad y^p = \tilde{y}^p
\]
Shifted generators \(x' = x - \tilde{x}, \quad y' = y - \tilde{y}\) satisfy
\[
[y', x'] = 1, \quad (x')^p = (y')^p = 0
\]
which means that \(A\) is isomorphic to the \(k\)-tensor product of \(k[\tilde{x}, \tilde{y}]\) with a \(k\)-algebra generated by \(x', y'\) as above. The remark at the beginning of the proof yields the statement of the lemma. □

The above statement is a special case of a more general construction provided in [3]. Also, a similar result holds for rings of differential operators over smooth affine schemes \(X\) in positive characteristic, with the Weyl algebra emerging in the case \(X = \mathbb{A}^n_k\) (cf. [11]).

The structure of a generic holonomic left \(A_{1,k}\)-module is well known - any such module is cyclic of the form
\[
M = A_{1,k}/A_{1,k} \cdot L
\]
for some differential operator \(L = \sum_{i+j \leq N} a_{ij} x^i y^j \in A_{1,k}\).

Let \(X_p, Y_p\) denote the \(p \times p\) matrices representing the shifted generators (3.6), \(I_p\) be the identity matrix of size \(p \times p\). Following [11], consider for every differential operator \(L = \sum_{i+j \leq N} a_{ij} x^i y^j\) the \(p\)-determinant polynomial
\[
\text{Det}_p L := \det \left( \sum_{i+j \leq N} a_{ij} (X_p + \tilde{x} I_p)^i (Y_p + \tilde{y} I_p)^j \right)
\]

**Lemma 3.2**\(^1\) For \(L = \sum_{i+j \leq N} a_{ij} x^i y^j\) the corresponding \(p\)-determinant \(\text{Det}_p L\) is a polynomial in \(\tilde{x}^p, \tilde{y}^p\) of degree \(\leq N\).

**Proof** It suffices to show that the \(p\)-determinant vanishes after taking partial derivatives
\[
\frac{\partial}{\partial \tilde{x}} \text{Det}_p L = \frac{\partial}{\partial \tilde{y}} \text{Det}_p L = 0
\]
\(^1\)This statement was communicated to us by Maxim Kontsevich, also cf. [11]
The proof of that particular statement is elementary and reduces to an application of a well-known result of Jacobi, namely that
\[
\frac{\partial \det(A)}{\partial x} = \text{tr}(\text{adj}(A) \frac{\partial A}{\partial x}) \tag{3.9}\]
where "tr" means taking the trace of a matrix, and adj(A) is the adjugate matrix of A. Note that the adjugate can be expressed as a finite sum of powers of A: if \( P_A(\lambda) = \det(A - \lambda I) \) is the characteristic polynomial of A and \( f_A(t) = (P_A(0) - P_A(\lambda))/\lambda \), then
\[
\text{adj}(A) = f_A(A) \]
In particular, the adjugate commutes with A.

Let us consider the derivative with respect to \( \tilde{x} \). Just as in characteristic zero, we have
\[
[Y_p, X_p^{k}] = kX_p^{k-1} \tag{3.10}\]
so that taking the derivative of the matrix \( A(\tilde{x}, \tilde{y}) \) from the \( p \)-determinant expression is equivalent to taking the commutator with \( Y_p \):
\[
\frac{\partial A}{\partial \tilde{x}} = -[A, Y_p] \tag{3.11}\]
Plugging it in (3.9) yields
\[
\frac{\partial \det(A)}{\partial \tilde{x}} = -\text{tr}(\text{adj}(A)[A, Y_p]) \tag{3.12}\]
which equals zero by the cyclic property of the trace (applied after one permutes the commuting adj(A) and A). The vanishing of the derivative with respect to \( \tilde{y} \) is shown similarly. □

(This proof is somewhat different from the one suggested in [11], although the presence of the trace indicates that it revolves around the same idea).

We are also going to need the following observation

**Lemma 3.3.** If \( L \neq 0 \) then \( \text{Det}_p L \neq 0 \).

**Proof** The statement is true if \( L \) has degree zero.

Suppose \( \text{deg}(L) > 0 \). Let \((i_0, j_0)\) correspond to the leading term in \((x, y)\)-lexicographical ordering. The determinant of this term is easy to evaluate - it is the product of powers of characteristic polynomials of nilpotent matrices \( X_p, Y_p \), multiplied by \( a_{i_0j_0}^{p,i_0j_0} \tilde{x}^{p,i_0} \tilde{y}^{p,j_0} \).

The other non-zero terms in the \( p \)-determinant expansion are lexicographically smaller than the determinant of the leading term. This is justified by the following sublemma.

**Sublemma:** For an arbitrary collection \( \{A^1, ..., A^m\} \) of \( p \times p \)-matrices the determinant of its sum decomposes into a sum of determinants as follows:
\[
\det(A^1 + ... + A^m) = \sum_{\sigma} \det(A^\sigma) \tag{3.13}\]
where the sum extends over all maps \( \sigma : \{1, ..., p\} \to \{1, ..., m\} \), and the entries of \( A^\sigma \) are
\[
(A^\sigma)_{ij} = A^{\sigma(i)}_{ij} \tag{3.14}\]
Proof of Sublemma. Indeed, let $A = A^1 + ... + A^m$. Expanding the determinant of $A$ along the first row yields

$$
\det A = \sum_{i=1}^{m} \det A^{(i)}
$$

where the first row of $A^{(i)}$ is assembled out of elements of $A^i$, and the remaining rows are from $A$. Each $\det A^{(i)}$ is then expanded along the second row:

$$
\det A^{(i)} = \sum_{j=1}^{m} \det A^{(i,j)}
$$

where in $A^{(i,j)}$ the first row is the first row of $A^i$, the second one is the second row of $A^j$, so that

$$
\det A = \sum_{(i,j) \in [m] \times [m]} \det A^{(i,j)}
$$

Iterating the process, we obtain

$$
\det A = \sum_{1 \leq i_1, ..., i_p \leq m} \det A^{(i_1, ..., i_p)}
$$

Each term $(i_1, ..., i_p) \in [m]^p$ defines a map $\sigma : \{1, ..., p\} \rightarrow \{1, ..., m\}$, likewise every map $[p] \rightarrow [m]$ is present. The $k$-th row of the matrix $A^{(i_1, ..., i_p)}$ is the $k$-th row of $A^{i_k}$, which implies $A^{(i_1, ..., i_p)} = A^{\sigma}$. Sublemma is proved.

Applying the sublemma, one concludes that the terms smaller than $(i_0, j_0)$, including the mixed-row terms, cannot contain the monomial $\tilde{x}^{p_{i_0}} \tilde{y}^{p_{j_0}}$, and thus the $p$-determinant is not identically zero. □

Let $k$ be an algebraically closed field. Denote by $\mathcal{C}(\mathbb{A}_k^2)$ the set of all algebraic curves in $\mathbb{A}_k^2 = \text{Spec} \, k[z_1, z_2]$. This set possesses a natural structure of an ind-scheme. Indeed, any curve in $\mathcal{C}(\mathbb{A}_k^2)$ is the zero-locus of some polynomial in two variables. The correspondence between curves and polynomials is one-to-one modulo multiplicative constant, therefore the set of curves of degree $\leq d$ is the projective space $\mathbb{P}^{\frac{(d+1)(d+2)}{2} - 1}$ over $k$. Indexed by $d \in \mathbb{N}$, these spaces together with obvious embeddings form an inductive system with a direct limit

$$
\mathcal{C}(\mathbb{A}_k^2) = \lim_{\rightarrow} \mathbb{P}^{\frac{(d+1)(d+2)}{2} - 1}
$$

This expression can be viewed as the definition of the infinite-dimensional projective space.

The first Weyl algebra $A_{1,k}$ together with the Bernstein filtration

$$
\mathfrak{F} = \{ F_k \mid k \in \mathbb{Z}_+ \}
$$

$$
F_k = \{ L \in A_{1,k} \mid \text{deg}_z(L) + \text{deg}_y(L) \leq k \}
$$

$$
\text{deg}_z(x) = \text{deg}_y(y) = 1
$$

can be turned into an infinite-dimensional projective space by the equivalence relation $\mathfrak{F}/k^\times$, which glues together elements differing by a non-zero factor. Note that its
dimensional structure as a direct limit is the same as that of the ind-scheme $\mathcal{C}(\mathbb{A}^2_k)$.

By Lemma 3.2, the $p$-determinant map induces a morphism
\[
\Theta : A_{1,k}/k^\times \to \mathcal{C}(\mathbb{A}^2_k), \quad [L] \mapsto C_L \subset \mathbb{A}^2_k
\] (3.17)

Our main result reduces to the following statement

**Theorem 3.4.** $\Theta$ is surjective.

**Proof of the Theorem** It is a well-known fact (see, for instance, [12]) that any dominant morphism of projective varieties with finite fibers is surjective. In this case $\Theta$ can be thought of as a system of morphisms $\Theta_d$, which map (projective classes of) differential operators of degree $d$ into algebraic curves of degree $d$. The surjectivity of all such morphisms will imply that of $\Theta$ itself.

For any $d \in \mathbb{N}$ the morphism $\Theta_d$ is formed by compactifying a morphism
\[
\Theta_d^{\text{aff}} : \mathbb{A}^{(d+1)(d+2)}_k \to \mathbb{A}^{(d+1)(d+2)}_k
\] (3.18)
which is essentially the same $p$-determinant. The induced homomorphism of coordinate rings is injective by lemma 3.3, therefore $\Theta_d^{\text{aff}}$ is dominant.

It now remains to show that a generic fiber $\Theta_d^{-1}(C)$ over a planar curve $C$ is zero-dimensional. In order to do so, consider a one-parametric family of differential operators
\[
k \to A_{1,k}, \quad t \mapsto L(t)
\]
\[
L(t) = \sum_{i+j \leq N} a_{ij}(t)x^iy^j
\]
with $a_{ij}(t)$ being algebraic functions (up to a projective equivalence). The $p$-determinant will map this set onto a subset
\[
\mathcal{L}(t) \subset \mathcal{C}(\mathbb{A}^2_k)
\]

**Claim:** If cardinality $|\{L(t) \mid t\}| > 1$ then $|\mathcal{L}(t)| > 1$.

**Proof:** Suppose at first that all $a_{ij}(t)$ are polynomials. Then, if we denote $\tilde{L}$ the matrix in the $p$-determinant expression built out of $L$, the terms in it can be rearranged, so that
\[
\tilde{L}(t) = \tilde{A} + \tilde{B}t + \ldots + \tilde{U}t^m, \quad \tilde{U} \neq 0
\] (3.19)
with $\tilde{U}$ corresponding to some non-zero element $U \in A_{1,k}$. Since $\{L(t) \mid t\}$ consists of more than one point, $m > 0$. The $p$-determinant of $\tilde{L}$ is just the determinant of the $\tilde{L}$ matrix. Using the preceding sublemma, we deduce that the leading term in the determinant of (3.19) is
\[
U^m \det \tilde{L}
\]
which is non-zero by lemma 3.3.

The general case of algebraic $a_{ij}(t)$ is processed in a very much similar fashion. Viewing $t$ as a formal parameter, consider the case when there is a term $a_{ij}(t)$ such that it has a singularity at a point $t = t_0 \neq \infty$. We may expand the functions $a_{ij}(t)$ as their respective Puiseux series in the vicinity of $t_0$ and, by making appropriate substitutions (and using, if need be, the fact that $k$ is algebraically closed) turning them into polynomials in some parameter $q(t, t_0)$. We therefore may view $L(t)$ locally near $t_0$ as a polynomial curve $L_1(q)$, so that the preceding argument applies and the claim is proved.

The claim shows that the fibers of $\Theta_d$ are discrete, which together with the fact that $\Theta_d$ are dominant for all $d$ implies surjectivity of $\Theta$. The Theorem is proved. $\square$
4 Conclusion and outlook

By establishing the fact that the map (3.17) is epimorphic, we are looking at a way of associating to every polynomial symplectomorphism $\psi \in \text{Aut}(P_{1,k})$ an algebra automorphism of $A_{1,k}$ in positive characteristic. After that we are expecting to find a lift to characteristic zero. Combined with the results in the opposite direction which are described in [10], this could lead to a proof of the B-KKC$_1$ suitable for a generalization to a higher-dimensional case. However, issues beyond technical remain (see the end of section 2), which necessitates further investigation.

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