Well-Posedness for the Linearized Free Boundary Problem of Incompressible Ideal Magnetohydrodynamics Equations

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Abstract

We study the well-posedness theory for the linearized free boundary problem of incompressible ideal magnetohydrodynamics equations in a bounded domain. We express the magnetic field in terms of the velocity field and the deformation tensors in the Lagrangian coordinates, and substitute the magnetic field into the momentum equation to get an equation of the velocity in which the initial magnetic field serves only as a parameter. Then, we linearize this equation with respect to the position vector field whose time derivative is the velocity, and obtain the local-in-time well-posedness of the solution by using energy estimates of the tangential derivatives and the curl with the help of Lie derivatives and the smooth-out approximation.

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1 Introduction

This paper is concerned with the well-posedness of the linearized motion of the following incompressible ideal magnetohydrodynamics (MHD) with free boundary

(1.1) \[ v_t + v \cdot \nabla v + \nabla p = \mu H \cdot \nabla H, \quad \text{in } D, \]
\begin{equation}
H_t + v \cdot \nabla H = H \cdot \nabla v, \quad \text{in } \mathcal{D},
\end{equation}
\begin{equation}
\text{div } v = 0, \quad \text{div } H = 0, \quad \text{in } \mathcal{D},
\end{equation}
where $v$ is the velocity field, $H$ is the magnetic field, $p$ is the total pressure including the fluid pressure and the magnetic pressure, and $\mu > 0$ is the vacuum permeability. $\mathcal{D} = \bigcup_{0 \leq t \leq T} \{t\} \times \Omega_t$, $\Omega_t \subset \mathbb{R}^n$ is the domain occupied by the fluid at time $t$.

We also require boundary conditions on the free boundary $\partial \mathcal{D}$:
\begin{equation}
H \cdot \mathbf{N} = 0, \quad p = 0, \quad \text{on } \partial \mathcal{D},
\end{equation}
\begin{equation}
(\partial_t + v^i \partial_i)|_{\partial \mathcal{D}} \in T(\partial \mathcal{D}),
\end{equation}
where $\mathbf{N}$ is the exterior unit normal to $\Gamma_t := \partial \Omega_t$. The condition $p = 0$ indicates that the total pressure vanishes outside the domain. Here the fluid considered is an incompressible ideal case. The incompressibility condition prevents the body from expanding, and the fact that the pressure is positive prevents the body from breaking up in the interior. From a physical point of view, the total pressure can be thought of alternatively as being a small positive constant on the boundary instead of vanishing. The challenge of the problem is that the regularity of the boundary enters to the highest order. Roughly speaking, the velocity determines the motion of the boundary, and the boundary is the level set of the total pressure that determines the acceleration together with the magnetic tension. The condition $H \cdot \mathbf{N} = 0$ comes from the assumption that the boundary $\Gamma_t$ is a perfect conductor, and should be understood as the constraints on the initial data since it will hold true for all $t \in [0, T]$ if it holds initially as showed in [15]. The condition (1.5) means that the boundary moves with the velocity $v$ of the fluid particles at the boundary.

Given a domain $\Omega \subset \mathbb{R}^n$ that is homeomorphic to the unit ball, and initial data $(v_0, H_0)$ satisfying the constrain (1.3), we expect to find a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ and vector fields $(v, H)$ solving (1.1)-(1.5) with initial conditions
\begin{equation}
\{x : (0, x) \in \mathcal{D}\} = \Omega; \quad v = v_0, \quad H = H_0, \quad \text{on } \{0\} \times \Omega.
\end{equation}
Then, let $\Omega_t = \{x : (t, x) \in \mathcal{D}\}$. Motivated by the Taylor sign condition on the fluid pressure for the Euler equations, we raised an analogous condition based on the total pressure for ideal MHD in [15]:
\begin{equation}
\nabla_{\mathbf{N}} p \leq -c_0 < 0 \text{ on } \partial \mathcal{D},
\end{equation}
where $\nabla_{\mathbf{N}} = \mathbf{N}^i \partial_x^i$. Here we have used the summation convection over repeated upper and lower indices. In [15], we have proved a priori estimates in standard Sobolev spaces for the free boundary problem of incompressible ideal MHD system (1.1)-(1.6) under the condition (1.7). We also showed in [16] that the above free boundary problem (1.1)-(1.6) under consideration would be ill-posed at least for the case $n = 2$ if the condition (1.7) was violated. Thus, it will be much reasonable and necessary to require this condition (1.7) in the studies of well-posedness of the considering free boundary problem of incompressible ideal MHD equations.

However, the a priori estimates in [15] used all the symmetries of the nonlinear equations and so only holds for perturbations of the equations that preserved all the symmetries. Thus, we can not use those a priori estimates for solutions of linearized equations that do not preserve the symmetries. Of course, the results in [15] are important for the raise of the meaningful and reasonable condition (1.7) for the well-posedness.
In this paper, we prove the existence of solutions in Sobolev spaces for linearized equations using a new type of estimate by using some ideas in [20]. Existence for the linearized equations or some modification are crucially important to any existence proof for the nonlinear problem by putting the nonlinear problem in some iteration schemes. In the most general way, it is to linearize the equations with respect to both the velocity field and the magnetic field. However, it is almost impossible to get the well-posedness for the linearized system of this type, where many operators can not be controlled and the relations between the velocity field and the magnetic field is also destroyed. In order to preserve these important relations, we seek another way to linearize the equations. Since the magnetic field can be expressed in terms of the velocity field and the deformation tensors in the Lagrangian coordinates, we can first solve the equation (1.2) and substitute the magnetic field into the equation (1.1) to get an equation of the velocity in which the initial magnetic field serves only as a parameter. Then, we linearize this equation with respect to the position vector field whose time derivative is the velocity in the Lagrangian coordinates. As in [20], we project the linearized equation onto an equation in the interior using the orthogonal projection onto divergence-free vector fields in the $L^2$ inner product, which removes a difficult term, the differential of linearization of the pressure, and reduces a higher-order term, the linearization of the free boundary, to an unbounded symmetric operator on divergence-free vector fields. Thus, the linearized equation turns to an evolution equation in the interior for this so-called normal operator that is positive due to the condition (1.7) and leads to energy bounds. Because the operator is time dependent and nonelliptic, we can not obtain the existence of regular solutions by standard energy methods or semigroup methods. As practiced in [20], to use Lie derivatives with respect to divergence-free vector fields tangential at the boundary is an effective way. The estimates of all derivatives can be got from those of tangential derivatives, the divergence and the curl. We replace the normal operator by a sequence of bounded operators converging to it that are still symmetric and positive and have uniformly estimates in order to get the existence of solutions.

Fluids free boundary problems arising from physical, engineering and medical models are both important in applications and challenging in PDE theory. Examples include water waves, evolution of boundaries of stars, vortex sheets, multi-phase flow, reacting flow, shock waves, biomedical modeling such as tumor growth, cell deformation and etc. The most fundamental and simplest setting is for incompressible fluids for which the local well-posedness in Sobolev spaces for inviscid irrotational flow was obtained first in [36, 37] for 2D and 3D, respectively. Substantial progresses for the cases without the irrotational assumption, finite depth water waves, lower regularity, uniform estimates with respect to surface tension and etc have been made in [1-4, 7-9, 12, 17, 22, 27, 30, 31, 38] and etc. For more references, one may refer to the excellent survey [17]. For compressible inviscid flow, the local-in-time well-posedness of smooth solutions was established for liquids in [21, 33] (see also [10] for the zero surface tension limits), the study of the effects of heat-conductivity to fluid free surface can be found in [24].

However, only few results are available for free boundary problems of MHD equations. Indeed, magnetic fields are essential in many important physical situations ([11, 23, 39]), for example, solar flares in astrophysics due to the coupling between magnetic and thermomechanical degrees of freedom for which magnetic reconnection is thought to be the mechanism responsible for the conversion of magnetic energy into heat and fluid motion ([11]). Moreover, interface problems in MHD are crucial
to the theoretical and practical study of producing energy by fusion. In the study of the ideal MHD free boundary problems, a priori estimates were derived in [15] with a bounded initial domain homeomorphic to a ball, provided that the size of the magnetic field to be invariant on the free boundary. A priori estimates for the low regularity solution of this problem were given in [23] for the bounded domain with small volume. Ill-posedness was showed in [16] for the two-dimensional problem if the condition (1.7) was violated. A local existence result was established in [13] for which the detailed proof is given for an initial flat domain of the form $\mathbb{T}^2 \times (0,1)$, where $\mathbb{T}^2$ is a two-dimensional period box in $x_1$ and $x_2$. The aim of the present paper is to study the ideal MHD free surface problem with a free surface being a closed curved surface with large curvature by the geometric approach motivated by [8], [20] and [21]. For the special case where the magnetic field is zero on the free boundary and in vacuum, the local existence and uniqueness of the free boundary problem of incompressible viscous-diffusive MHD flow in three-dimensional space with infinite and finite depth setting was proved in [18] and [19] where also a local unique solution was obtained for the free boundary MHD without kinetic viscosity and magnetic diffusivity via zero kinetic viscosity-magnetic diffusivity limit. The convergence rates of inviscid limits for the free boundary problems of the three-dimensional incompressible MHD with or without surface tension was studied in [6], where the magnetic field is constant on the surface and outside of the unbounded domain. For the incompressible viscous MHD equations, a free boundary problem in a simply connected domain of $\mathbb{R}^3$ was studied by a linearization technique and the construction of a sequence of successive approximations in [26] with an irrotational condition for magnetic fields in a part of the domain. The plasma-vacuum system was investigated in [14] where the a priori estimates were derived in a bounded domain. The well-posedness of the linearized plasma-vacuum interface problem in incompressible ideal MHD was studied in [25] in an unbounded plasma domain. For other related results of MHD equations with free boundaries or interfaces, one may refer to [5, 28, 32, 34, 35].

The rest of the paper is organized as follows. In Section 2 we introduce the Lagrangian coordinates and reformulate the free boundary problem to a fixed boundary problem, and then linearize the equation. We project the linearized equation onto the divergence-free vector fields in Section 3 and derive the lowest-order energy estimates in Section 4. In Section 5 we change the linearized problem into the case of homogeneous initial data and an inhomogeneous term that vanishes to any order as time tends to zero. Next, we derive the a priori estimates of the linearized equation with homogeneous initial data in Section 6 including those of tangential derivatives and the curl. Then, we study a smoothed-out equation according to the normal operator and prove the existence of weak solution of it in Section 7 and the existence of smooth solutions for the linearized equation in Section 8. We turn to the energy estimates of the original linearized equation with inhomogeneous initial data and an inhomogeneous term in Section 9 and give the main result and its proof in Section 10. Finally, some preliminaries about the Lie derivative are given in the appendix.
2 Lagrangian coordinates and the linearization of equations

2.1 Lagrangian reformulation

In this section, we introduce the Lagrangian coordinates and reformulate the free boundary problem to a fixed boundary problem. Lagrangian coordinates \( x = x(t, y) = f_t(y) \) are given by

\[
\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega.
\]

Then \( f_t : \Omega \to \Omega_t \) is a volume-preserving diffeomorphism because of \( \text{div} \ v = 0 \), and the free boundary becomes fixed in the new \( y \)-coordinates. For simplicity, we take \( f_0 \) is the identity operator, that is, \( x(0, y) = y \) and \( \Omega \) is just the unit ball. For convenience, the letters \( a, b, c, d, e \), and \( f \) will refer to quantities in the Lagrangian frame, whereas the letters \( i, j, k, l, m \), and \( n \) will refer to ones in the Eulerian frame, e.g., \( \partial_a = \partial/\partial y^a \) and \( \partial_i = \partial/\partial x^i \).

Denote \( D_t = \partial_t + v^k \partial_k \), \( \partial_k = \frac{\partial}{\partial x^k} = \frac{\partial y^a}{\partial x^k} \frac{\partial}{\partial y^a} \).

Then, we get

\[
D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial D_t x^i}{\partial y^a} = \frac{\partial v^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a} \frac{\partial v^i}{\partial x^k}.
\]

and

\[
D_t \frac{\partial y^a}{\partial x^i} = D_t (\delta_b^a \frac{\partial y^b}{\partial x^i}) = \frac{\partial x^j}{\partial y^a} \frac{\partial y^a}{\partial x^j} D_t \frac{\partial y^b}{\partial x^i}
= \frac{\partial y^a}{\partial x^j} \frac{\partial x^j}{\partial y^b} \frac{\partial y^b}{\partial x^i} D_t \frac{\partial x^j}{\partial y^b}
= - \frac{\partial y^a}{\partial x^j} \frac{\partial x^j}{\partial y^b} \frac{\partial y^b}{\partial x^i}
= - \frac{\partial y^a}{\partial y^b} \frac{\partial x^j}{\partial y^b} \frac{\partial v^j}{\partial x^i}.
\]

From (1.2) and (2.4), we have

\[
D_t \left( H^i \frac{\partial y^a}{\partial x^i} \right) = D_t H^i \frac{\partial y^a}{\partial x^i} + H^i D_t \frac{\partial y^a}{\partial x^i} = H^i \partial_i v^j \frac{\partial y^a}{\partial x^i} - H^i \partial_i v^k \frac{\partial y^a}{\partial x^k} = 0,
\]

which yields

\[
H^i(t, x(t, y)) \frac{\partial y^a}{\partial x^i} = H^i(0, x(0, y)) \frac{\partial y^a}{\partial x^i} \bigg|_{t=0} = \tilde{H}_0^a(y) \delta_i^a = \tilde{H}_0^a(y),
\]

and

\[
H^j(t, x(t, y)) = \tilde{H}_0^a(y) \frac{\partial x^j(t, y)}{\partial y^a},
\]

(2.5)
where $\bar{H}_0^a(y) = H_0^a(x(0, y))$. Then,

$$H^k \partial_k H^i = \bar{H}_0^a \frac{\partial x^k}{\partial y^a} \frac{\partial y^c}{\partial x^k} \partial_c (\bar{H}_0^a \frac{\partial x^i}{\partial y^b}) = \bar{H}_0^a \partial_a(\bar{H}_0^b \partial_b x^i),$$

For convenience, denote the differential operator

$$B := B^a(y) \frac{\partial}{\partial y^a}, \quad \text{with} \quad B^a(y) := \sqrt{\mu \bar{H}_0^a(y)},$$

then (1.1)-(1.5) can be written as

$$\begin{cases}
D_t^2 x^i + \partial_i P = B^2 x^i, & \text{in } [0, T] \times \Omega, \\
\kappa := \det \left( \frac{\partial x}{\partial y} \right) = 1, & \text{in } [0, T] \times \Omega, \\
P = 0, & \text{on } \Gamma,
\end{cases}$$

(2.6)

where $P = P(t, y) = p(t, x(t, y))$, $\partial_i$ is thought of as the differential operator in $y$ given in (2.2) and $D_t$ is the time derivative. The initial conditions read

$$x|_{t=0} = y, \quad D_t x|_{t=0} = v_0,$$

(2.7)

satisfying the constraint $\text{div } v_0 = 0$. Taking the divergence of (2.6) gives the Laplacian of $P$:

$$\Delta P = - (\partial_i D_t x^k)(\partial_k D_t x^i) + \partial_i (B^2 x^i).$$

(2.8)

The condition (1.7) turns to be

$$\nabla_N P \leq -c_0 < 0, \quad \text{on } \Gamma,$$

(2.9)

where $N$ is the exterior unit normal to $\Gamma_t$ parametrized by $x(t, y)$.

### 2.2 Linearization

Now, we derive the linearized equations of (2.6). We assume that $(x(t, y), P(t, y))$ is a given smooth solution of (2.6) satisfying (2.8) for $t \in [0, T]$.

Let $\delta$ be a variation with respect to some parameter $r$ in the Lagrangian coordinates:

$$\delta = \frac{\partial}{\partial r} \bigg|_{(t, y) = \text{const}}.$$

(2.10)

We think of $x(t, y)$ and $P(t, y)$ as depending on $r$ and differentiating with respect to $r$, say, $\bar{x}(t, y, r)$ and $\bar{P}(t, y, r)$ respectively. Namely, $(\bar{x}, \bar{P})|_{r=0} = (x, P)$. Differentiating (2.2) and using the formula for the derivative of the inverse of a matrix, $\delta M^{-1} = -M^{-1}(\delta M)M^{-1}$, we get the commutator

$$[\delta, \partial_i] = -(\partial_i \delta x^k) \partial_k.$$

(2.11)

Let

$$\langle \delta x, \delta P \rangle = \left( \frac{\partial \bar{x}}{\partial r}, \frac{\partial \bar{P}}{\partial r} \right) \bigg|_{r=0},$$

(2.12)
which satisfies \( \text{div} \, \delta x = 0 \) and \( \delta P |_r = 0 \).

From (2.6) and (2.11), we get by noticing \([D_t, \delta] = 0\) and \([\delta, B] = 0\)

\[
D_t^2 \delta x^i = - \delta \partial_i P + B^2 \delta x^i
= (\partial_i \delta x^k) \partial_k P - \partial_i \delta P + B^2 \delta x^i.
\]  

From (2.6) again, we have

\[
\partial_i P = - D_t^2 x^i + B^2 x^i = - \partial_i v^i + B^2 x^i,
\]

and then

\[
(\partial_i \delta x^k) \partial_k P = \partial_i (\delta x^k \partial_k P) - \delta x^k \partial_i \partial_k P
= \partial_i (\delta x^k \partial_k P) - \delta x^k \partial_k (-D_t v^i + B^2 x^i)
= \partial_i (\delta x^k \partial_k P) + \delta x^k (\partial_k D_t v^i - \partial_k (B^2 x^i)).
\]

It follows from (2.13) and (2.15) that

\[
D_t^2 \delta x^i + \partial_i \delta P - \partial_i (\delta x^k \partial_k P) - \delta x^k (\partial_k D_t v^i - \partial_k (B^2 x^i)) - B^2 \delta x^i = 0.
\]

Now, we introduce new variables. Let

\[
W^a = \delta x^i \frac{\partial y^a}{\partial x^i}, \quad \delta x^i = W^b \frac{\partial x^i}{\partial y^b}, \quad q = \delta P.
\]

And recall

\[
\partial_i = \frac{\partial y^a}{\partial x^i} \partial_a, \quad \partial_a = \frac{\partial x^i}{\partial y^a} \partial_i.
\]

Let

\[
g_{ab} = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}
\]

be the metric \( \delta_{ij} \) expressed in the Lagrangian coordinates. Let \( g^{ab} \) be the inverse of \( g_{ab} \),

\[
\dot{g}_{ab} = D_t g_{ab} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^k}{\partial y^b} (\partial_k v_i + \partial_i v_k), \quad \text{and} \quad \omega_{ab} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^k}{\partial y^b} (\partial_k v_i - \partial_i v_k)
\]

be the time derivatives of the metric and the vorticity in the Lagrangian coordinates, respectively. It follows that

\[
\frac{\partial x^i}{\partial y^a} \frac{\partial x^k}{\partial y^b} \partial_k v_i = \frac{1}{2} (\dot{g}_{ab} - \omega_{ab}).
\]

Multiplying (2.16) by \( \frac{\partial x^i}{\partial y^a} \) and summing over \( i \), we obtain

\[
\delta_{il} \frac{\partial x^l}{\partial y^a} D_t^2 \delta x^i - \delta_{il} (W^c \partial_c P) - W^b \delta_{il} \frac{\partial x^l}{\partial y^a} \partial_b D_t v^i + \partial_a q
+ \delta_{il} \partial_a x^i W^d \partial_d (B^2 x^i) - \delta_{il} \partial_a x^i B^2 (W^c \partial_c x^i) = 0.
\]
The first term in (2.21) can be written as
\[
\delta_{il} \frac{\partial x^l}{\partial y^a} D_t^2 \delta x^i = \delta_{il} \frac{\partial x^l}{\partial y^a} D_t^2 (W^b \frac{\partial x^i}{\partial y^b})
\]
\[
= \delta_{il} \frac{\partial x^l}{\partial y^a} D_t (D_t W^b \frac{\partial x^i}{\partial y^b} + W^b D_t \frac{\partial x^i}{\partial y^b})
\]
\[
= g_{ab} D_t^2 W^b + 2 \delta_{il} \frac{\partial x^l}{\partial y^a} D_t W^b \frac{\partial x^i}{\partial y^b} + \delta_{il} \frac{\partial x^l}{\partial y^a} W^b \frac{\partial x^i}{\partial y^b}
\]
\[
= g_{ab} D_t^2 W^b + 2 \left( \frac{\partial x^l}{\partial y^a} \frac{\partial x^k}{\partial y^b} \partial_{kl} v_i D_t W^b + \delta_{il} \frac{\partial x^l}{\partial y^a} W^b \partial_b D_i v^i \right)
\]
\[
= g_{ab} D_t^2 W^b + (\dot{g}_{ab} - \omega_{ab}) D_t W^b + \frac{\partial x^i}{\partial y^a} W^b \partial_b D_i v^i.
\]
It follows from (2.21) that
\[
g_{ab} D_t^2 W^b + (\dot{g}_{ab} - \omega_{ab}) D_t W^b - \partial_a (W^c \partial_c P) + \partial_a g
\]
(2.22)
\[
+ \delta_{il} \partial_a x^l W^d \partial_d (B^2 x^i) - \delta_{il} \partial_a x^l B^2 (W^c \partial_c x^i) = 0,
\]
which yields by acting \(g^{da}\)
\[
D_t^2 W^d + \dot{g}_{da} (\dot{g}_{ab} - \omega_{ab}) D_t W^b - g^{da} \partial_a (W^c \partial_c P) + g^{da} \partial_a q
\]
(2.23)
\[
+ g^{da} \delta_{il} \partial_a x^l [W^c \partial_c (B^2 x^i) - B^2 (W^c \partial_c x^i)] = 0.
\]
From (2.21), we see that the energies will include \(\|BW\|^2\). But it is very complicated due to \(\text{div} (BW) \neq 0\). Indeed, we can regard \(B\) as a tangential derivative since \(B = B^a \partial_a\) is independent of time and \(\partial_a B^a = 0\). Thus, we can use the Lie derivative corresponding to \(B\) given by
\[
\mathcal{L}_B W^a = BW^a - \partial_b B^a W^b,
\]
which is divergence-free due to \(\text{div} \mathcal{L}_B W = \partial_a (B^b \partial_b W^a - \partial_b B^a W^b) = 0\) if \(\text{div} W = 0\).
We also have
\[
\mathcal{L}_B \partial_c x^i = B \partial_c x^i + \partial_c B^d \partial_d x^i.
\]
For more details and properties of Lie derivatives, one can see Appendix A.
From (2.24), it follows that
\[
\mathcal{L}_B^2 W^a = \mathcal{L}_B (BW^a - (\partial_c B^a) W^c)
\]
\[
= B (BW^a - (\partial_c B^a) W^c) - (\partial_c B^a) \mathcal{L}_B W^c
\]
\[
= B^2 W^a - B^d \partial_d B^a W^c - (\partial_c B^a) BW^c - (\partial_c B^a) \mathcal{L}_B W^c
\]
\[
= B^2 W^a - B^d \partial_c \partial_d B^a W^d - (\partial_c B^a) (\partial_d B^a W^d) - 2 (\partial_c B^a) \mathcal{L}_B W^c
\]
\[
= B^2 W^a - 2 (\partial_c B^a) \mathcal{L}_B W^c - W^d \partial_d (BB^a),
\]
and then
\[
B^2 (\partial_c x^i W^c) = \partial_c x^i B^2 W^c + 2 B \partial_c x^i BW^c + B^2 \partial_c x^i W^c
\]
\[
= \partial_c x^i (\mathcal{L}_B^2 W^c + 2 (\partial_d B^c) \mathcal{L}_B W^d + W^d \partial_d (BB^c))
\]
Hence, we get by (2.25)
\[ g^{da} \delta_{il} \partial_a x^i [W^c \partial_c (B^2 x^i) - B^2 (W^c \partial_c x^i)] = - g^{da} \delta_{il} \partial_a x^i [\partial_c x^i \mathcal{L}_B^2 W^c + 2(\partial_c B^b \partial_b x^i + B \partial_c x^i) \mathcal{L}_B W^c] = - \mathcal{L}_B^2 W^d - 2 \partial_c B^d \mathcal{L}_B W^c - 2g^{da} \delta_{il} \partial_a x^i \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c = - \mathcal{L}_B^2 W^d - 2g^{da} \delta_{il} \partial_a x^i \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c. \]

We introduce some new notations. Denote
\[
\begin{align*}
\tilde{W}^a(t, y) &:= D_t W^a(t, y), \quad \bar{W}^a := D_t^2 W^a.
\end{align*}
\]

Since \( q = \delta P \), one has \( q|\Gamma = 0 \). Thus, from (2.28) and (1.3), we have the following system
\[
\begin{align*}
\begin{cases}
\tilde{W}^d - \mathcal{L}_B^2 W^d + g^{da} \partial_a q - g^{da} \partial_a (W^c \partial_c P) + g^{da} (\dot{g}_{ab} - \omega_{ab}) \tilde{W}^b \\ - 2g^{da} \delta_{il} \partial_a x^i \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c = 0,
\end{cases}
\end{align*}
\]
(2.27)
\[
\begin{align*}
\text{div } W &= \kappa^{-1} \partial_a (\kappa W^a) = 0, \\
q|\Gamma &= 0, \\
W|_{t=0} &= W_0, \quad \bar{W}|_{t=0} = W_1,
\end{align*}
\]
where \( \text{div } W_0 = \text{div } W_1 = 0 \).

We can express (2.27) in one equation since \( q = \delta P \) is determined as a functional of \( W \) and \( \bar{W} \). Thus, we derive an elliptic equation for \( q \).

### 2.3 The equation of \( \Delta q \)

In order to get the equation of \( \Delta q \), we have to derive \( \text{div } \bar{W} \) first. Denote
\[
u^a := \frac{\partial y^a}{\partial x^i} v^i, \quad \text{and } u_a = g_{ab} u^b.
\]

From \( \text{div } W = 0 \), it follows that \( \text{div } \bar{W} = 0 \). Thus, taking the divergence of (2.27), we have,
\[
\begin{align*}
\Delta q &= \partial_d (g^{da} \partial_a (W^c \partial_c P) - g^{da} (\dot{g}_{ab} - \omega_{ab}) W^b + 2g^{da} \delta_{il} \partial_a x^i \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c), \\
q|\Gamma &= 0,
\end{align*}
\]
(2.28)

since \( \text{div } \mathcal{L}_B^2 W = 0 \).

We separate \( q \) into four parts:
\[
q = \sum_{i=1}^{4} q_i,
\]
where \(q_i\)'s satisfy the following Dirichlet problems of Poisson equations:

\[
\begin{align*}
\Delta q_1 &= \Delta (W^c \partial_c P), & q_1|\Gamma &= 0, \\
\Delta q_2 &= -\partial_d (g^{da} \dot{g}_{ab} \dot{W}^b), & q_2|\Gamma &= 0, \\
\Delta q_3 &= \partial_d (g^{da} \omega_{ab} \dot{W}^b), & q_3|\Gamma &= 0, \\
\Delta q_4 &= 2\partial_d (g^{da} \delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c), & q_4|\Gamma &= 0.
\end{align*}
\]

Then, we can write (2.27) as

\[
L_1 W := \ddot{W} - \mathcal{L}_B^2 W + \mathcal{A} W + \ddot{\mathcal{G}} W - \mathcal{C} W + \mathcal{X} \mathcal{L}_B W = 0,
\]

where

\[
\begin{align*}
\mathcal{A} W^d &= -g^{da} \partial_a (\partial_c PW^c - q_1), \\
\ddot{\mathcal{G}} W^d &= g^{da} (\dot{g}_{ab} \dot{W}^b + \partial_a q_2), \\
\mathcal{C} W^d &= g^{da} (\omega_{ab} \dot{W}^b - \partial_a q_3), \\
\mathcal{X} \mathcal{L}_B W^d &= -2g^{da} \delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c + g^{da} \partial_a q_4.
\end{align*}
\]

### 3 The projection onto divergence-free vector field

In this section, we recall some definitions on the projection onto divergence-free vector field, one can see [20] for details.

Let \(\mathbb{P}\) be the orthogonal projection onto divergence-free vector fields in the inner product

\[
\langle W, U \rangle = \int_\Omega g_{ab} W^a U^b dy.
\]

Then,

\[
\mathbb{P} U^a = U^a - g^{ab} \partial_b q,
\]

\[
\Delta q = \partial_a (g^{ab} \partial_b q) = \text{div} U = \partial_a U^a, \quad q|\Gamma = 0,
\]

because of \(g_{ab} g^{bc} = \delta^c_a\) and

\[
\langle W, (\mathbb{I} - \mathbb{P}) U \rangle = \int_\Omega g_{ab} W^a g^{bc} \partial_c q dy
\]

\[
= \int_\Gamma W^a N_a q dS - \int_\Omega q \text{div} W dy = 0, \text{ if } \text{div} W = 0,
\]

where \(N_a\) is the exterior unit conormal and \(dS\) is the surface measure. In addition, for the function vanishing on the boundary, the projection of its gradient vanishes:

\[
\mathbb{P} (g^{ab} \partial_b f) = 0, \text{ if } f|\Gamma = 0.
\]

Denote \(\|W\| := \langle W, W \rangle^{1/2}\). It is clear that

\[
\|\mathbb{P} U\| \leq \|U\|, \quad \| (\mathbb{I} - \mathbb{P}) U \| \leq \|U\|.
\]
The projection is continuous on the Sobolev spaces $H^r(\Omega)$ if the metric is sufficiently regular:

$$\|\mathbb{P}U\|_{H^r(\Omega)} \leq C_r \|U\|_{H^r(\Omega)}.$$  

Furthermore, if the metric also depends smoothly on time $t$, then

$$\sum_{j=0}^{k} \| D_t^j \mathbb{P}U \|_{H^r(\Omega)} \leq C_{r,k} \sum_{j=0}^{k} \| D_t^j U \|_{H^r(\Omega)}.$$  

(3.1)

For functions $f$ vanishing on the boundary, we define operators on divergence-free vector fields ($\partial_a W^a = 0$)

$$\mathcal{A}_f W^a = \mathbb{P} \left( - g^{ab} \partial_b (W^c \partial_c f) \right).$$  

(3.2)

$\mathcal{A}_f$ is symmetric, i.e., $\langle U, \mathcal{A}_f W \rangle = \langle \mathcal{A}_f U, W \rangle$.

Since $P$ is the total pressure, the normal operator $\mathcal{A}$ in (2.30) is

$$\mathcal{A} = \mathcal{A}_P \geq 0, \quad \langle W, \mathcal{A}W \rangle \geq 0 \text{ if } \nabla_N P \mid_\Gamma \leq 0,$$

which is true in view of the condition (1.7). In fact,

$$\langle W, \mathcal{A}W \rangle = \int_{\Omega} g_{ab} W^b A W^a dy = -\int_{\Gamma} \left( N \cdot W \right)^2 \nabla_N P dS \geq 0,$$

due to $P = 0$ and $q_1 = 0$ on the boundary $\Gamma$.

By the definition in (3.2), we have

$$\mathcal{A}_f P W^a = -g^{ab} \partial_b (W^c \partial_c (f P)) + g^{ab} \partial_b q,$$

$$\Delta q = \Delta (W^c \partial_c (f P)), \quad q \mid_\Gamma = 0.$$

Then, for the divergence-free vector field $U$, it follows that

$$\langle U, \mathcal{A}_f P W \rangle = \int_{\Omega} g_{ab} U^a A_f P W^b dy = -\int_{\Omega} (U^d \partial_d (W^c \partial_c (f P)) + U^d \partial_d q) dy = -\int_{\Gamma} U_N W^c \partial_c (f P) dS.$$
\[= - \int_{\Gamma} U_N W_N \nabla_N (fP) dS \]
\[= - \int_{\Gamma} U_N W_N f \nabla_N P dS, \]

since \(\nabla_N (fP) = f \nabla_N P\) and \(fP = 0\) on the boundary, where \(U_N = N_a U^a = N \cdot U\).

From the Cauchy-Schwarz inequality and the identity (3.3), it follows that

\[|\langle U, A fP W \rangle| \leq \|f\|_{L^\infty(\Gamma)} \left(\int_{\Gamma} |U N|^2 (-\nabla_N P) dS\right)^{1/2} \left(\int_{\Gamma} |W N|^2 (-\nabla_N P) dS\right)^{1/2} = \|f\|_{L^\infty(\Gamma)} \langle U, A \rangle \langle W, AW \rangle^{1/2}. \]

In addition, since \(P\) vanishes on the boundary, so does \(\dot{P} = D_t P\), and then we can define

\[\dot{A} = A \dot{P}, \quad \dot{A} W^a = -g^{ab} \partial_b (W^c \partial_c \dot{P} - q), \quad \Delta q = \Delta (W^c \partial_c \dot{P}), \quad q|_{\Gamma} = 0, \]

which satisfies by (3.4)

\[|\langle W, \dot{A} W \rangle| = \left| - \int_{\Gamma} |W_N|^2 \nabla_N \dot{P} dS \right| \leq \left\| \nabla_N \dot{P} \right\|_{L^\infty(\Gamma)} \langle W, AW \rangle. \]

In fact, \(\dot{A}\) is the time derivatives of the operator \(A\), considered as an operator with values in the 1-forms.

For 2-forms \(\alpha\), we define bounded projected multiplication operators, as in [20], given by

\[M_\alpha W^a = P(g^{ab} \alpha_b W^c), \quad \|M_\alpha W\| \leq \|\alpha\|_{L^\infty(\Omega)} \|W\|. \]

In particular, the operators in (2.31) and (2.32) are bounded, projected multiplication operators:

\[G = M_g, \quad C = M_\omega, \quad \dot{G} = M_{\dot{g}}, \]

for the metric \(g\), the vorticity \(\omega\), and the time derivative of the metric \(\dot{g}\).

### 4 The lowest-order energy estimates

Now, we derive the energy estimates for the linearized equations

\[L_1 W = \ddot{W} - \mathcal{L}_B^2 W + AW + \dot{G} \dot{W} - C \dot{W} + \lambda \mathcal{L}_B W = F, \]

where \(F\) is divergence-free.

We first compute the inner product of (4.1) with \(\dot{W}\) and \(W\). Since

\[D_t (g_{ab} \dot{W}^a \dot{W}^b) = \dot{g}_{ab} \dot{W}^a \dot{W}^b + 2g_{ab} \ddot{W}^a \dot{W}^b, \]

we get

\[\langle \dot{W}, W \rangle = \frac{1}{2} \frac{d}{dt} \langle \dot{W}, W \rangle - \frac{1}{2} \langle \dot{W}, \dot{G} \dot{W} \rangle, \]

\[\langle \ddot{W}, W \rangle = \frac{1}{2} \frac{d}{dt} \langle \ddot{W}, W \rangle - \frac{1}{2} \langle \ddot{W}, C \dot{W} \rangle. \]
where \( \dot{G} \) is given by (3.8). From the symmetry of \( A \), it follows
\[
\langle AW, \dot{W} \rangle = \frac{1}{2} \frac{d}{dt} \langle AW, W \rangle - \frac{1}{2} \langle \dot{A}W, W \rangle,
\]
where \( \dot{A}W = A \rho W^a \) is defined by (3.2) with \( f = \dot{P} = D_t P \). In addition,
\[
\frac{1}{2} \frac{d}{dt} \langle W, W \rangle = \langle W, \dot{W} \rangle.
\]
Thus,
\[
\frac{1}{2} \frac{d}{dt} \langle (A + I)W, W \rangle = \langle (A + I)W, \dot{W} \rangle + \frac{1}{2} \langle \dot{A}W, W \rangle.
\]
We also have
\[
-\langle \mathcal{L}_B^2 W, \dot{W} \rangle = - \int_\Omega g_{ab} \mathcal{L}_B^2 W^a \dot{W}^b dy
= - \int_\Omega \mathcal{L}_B(g_{ab} \mathcal{L}_B W^a \dot{W}^b) dy + \int_\Omega g_{ab} \mathcal{L}_B W^a \mathcal{L}_B \dot{W}^b dy
+ \int_\Omega (\mathcal{L}_B g_{ab}) \mathcal{L}_B W^a \dot{W}^b dy
= \frac{1}{2} \frac{d}{dt} \int_\Omega |\mathcal{L}_B W|^2 dy - \frac{1}{2} \int_\Omega g_{ab} \mathcal{L}_B W^a \mathcal{L}_B \dot{W}^b dy
+ \int_\Omega \delta_{il} (\partial_b x^l \mathcal{L}_B \partial_a x^i + \mathcal{L}_B \partial_b x^l \partial_a x^i) \mathcal{L}_B W^a \dot{W}^b dy.
\]
Hence, we can define the energy as
\[
\langle \mathcal{X} \mathcal{L}_B W, \dot{W} \rangle = - 2 \int_\Omega \delta_{il} \partial_b x^l \mathcal{L}_B \partial_a x^i \mathcal{L}_B W^a \dot{W}^b dy.
\]
Then, we have the following energy estimates.

**Proposition 4.1.** Let
\[
n_0(t) = \frac{1}{2} \left( 1 + \left\| \nabla_N \frac{\dot{P}}{\nabla N} \right\|_{L^\infty(\Gamma)} + 2 \left\| \partial x \right\|_{L^\infty(\Omega)} \right) \left( 2 \left\| g \right\|_{L^\infty(\Omega)} + 2 \left\| \partial x \right\|_{L^\infty(\Omega)} \right) \left( \left\| \mathcal{L}_B \partial x \right\|_{L^\infty(\Omega)} \right).
\]
It holds
\[
E_0(t) \leq e^{\int_0^t n_0(\tau) d\tau} \left( E_0(0) + \int_0^t \| F(s) \|_0 e^{-\int_0^s n_0(\tau) d\tau} ds \right).
\]

**Proof.** Due to the antisymmetry of \( \omega \), we have \( \langle \mathcal{C} \dot{W}, \dot{W} \rangle = 0 \). Then, we get
\[
\frac{1}{2} \dot{E}(t) = \langle -\frac{1}{2} \mathcal{G} \dot{W} + F, \dot{W} \rangle + \langle W, \dot{W} \rangle + \frac{1}{2} \langle \dot{A}W, W \rangle + \frac{1}{2} \langle \dot{G} \mathcal{L}_B W, \mathcal{L}_B W \rangle
+ \int_\Omega \delta_{il} (\mathcal{L}_B \partial_b x^l \partial_a x^i - \partial_b x^l \mathcal{L}_B \partial_a x^i) \mathcal{L}_B W^a \dot{W}^b dy.
\]
Thus, we obtain
\[
\frac{1}{2} \dot{E}_0 \leq n_0(t) E_0 + \| F \|,
\]
which yields the desired estimates. \( \square \)
5 Turning initial data into an inhomogeneous divergence-free term

In this section, we want to change the initial value problem (4.1) and (2.27), i.e.,
\[ L_1 W = \ddot{W} - \mathcal{L}_B^2 W + A W + \mathcal{G} \dot{W} - C \mathcal{L}_B W = F, \]
\[ W|_{t=0} = W_0, \quad \dot{W}|_{t=0} = W_1, \]
into the case of homogeneous initial data and an inhomogeneous term \( F \) that vanishes to any order as \( t \to 0 \). As in [20], we can achieve it by subtracting off a power series solution in \( t \) to (5.1):
\[ W_{0r}(t, y) = \sum_{s=0}^{r+2} \frac{t^s}{s!} W_s(y). \]
It is clear that \( W_{0r} \) is divergence-free if so does \( W_s \). Here \( W_0 \) and \( W_1 \) are the initial data given in (5.1b), \( W_2 \) is obtained from (5.1) at \( t = 0 \):
\[ W_2 = \ddot{W}(0) = F(0) + \mathcal{L}_B^2 W_0 - \mathcal{A}(0) W_0 - \mathcal{G}(0) W_1 + \mathcal{C}(0) W_1 - \mathcal{X}(0) \mathcal{L}_B W_0. \]
The higher-order terms can be obtained by differentiating the equation with respect to time first and then taking the value at \( t = 0 \). Indeed, by doing so, we can obtain an expression
\[ D_t^{k+2} W = M_k(W_0, W_1, \cdots, D_t^{k+1} W) + D_t^k F, \]
from which we inductively define
\[ W_{k+2} = M_k(W_0, W_1, \cdots, W_{k+1})|_{t=0} + D_t^k F|_{t=0}, \]
where \( M_k \) is some linear operator of order at most 1 and that is all we need to derive. Next, we calculate the explicit form of \( M_k \) as a simple model case, since we will do similar derivations later on for other operators.

It is convenient to differentiate the corresponding operator with values in 1-forms, so we denote
\[ L^a_1 W_a := g_{ab} L_1 W^b \]
\[ = g_{ab} \ddot{W}^b - g_{ab} \mathcal{L}_B^2 W^b + \partial_a q - \partial_a (\partial_c P W^c) + (\dot{g}_{ab} - \omega_{ab}) \dot{W}^b - 2 \delta_{il} \partial_a x^i \mathcal{B}_B \partial_c x^i \mathcal{L}_B W^c = g_{ab} F^b, \]
where \( q \) is chosen such that the last terms are divergence-free, and afterwards project the result to the divergence-free vector fields. Denote
\[ q^s = D_t^s q, \quad P^s = D_t^s P, \quad g_{ab}^s = D_t^s g_{ab}, \quad \omega_{ab}^s = D_t^s \omega_{ab}, \quad F_s = D_t^s F. \]
Applying the differential operator \( D_t^r \) to (5.3) and restricting \( t \) to 0, we get
\[ \sum_{s=0}^{r} c_r^s (g_{ab}^{r-s} W^b_{s+2} - g_{ab}^{r-s} \mathcal{L}_B^2 W^b_s - \partial_a (\partial_c P^{r-s} W^c_s)) + \partial_a q^r \]
\[ + \sum_{s=0}^{r} c_r^s (g_{ab}^{r-s+1} - \omega_{ab}^{r-s}) W^b_{s+1} \]
\[ + \sum_{s=0}^{r} c_r^s (g_{ab}^{r-s+2} - \omega_{ab}^{r-s+1}) W^b_{s+2} \]
\[ + \sum_{s=0}^{r} c_r^s (g_{ab}^{r-s+3} - \omega_{ab}^{r-s+2}) W^b_{s+3} = g_{ab} F^b. \]
\[-2 \sum_{s=0}^{r} C^s_r \sum_{s_1=0}^{r-s} C^s_{r-s} \delta_{il} \partial_a (D^i_t x^i_s x^i_s) \mathcal{L}_B \partial_c (D^s_{t} x^i_s) \mathcal{L}_B W^c_s = \sum_{s=0}^{r} C^s_r g^{r-s}_{ab} F^b_s.\]

Then, we need to project all terms onto divergence-free vector fields. Let \( A_i W^d := \mathcal{P}(-g^{da} \partial_a (\partial_i P^s W^c)), \)
\( G_s W^d := \mathcal{P}(g^{da} g^{sb} W^b), \)
\( C_s W^d := \mathcal{P}(g^{da} \omega^{ab}_s W^b), \)
\( \chi_s \mathcal{L}_B W^d := -2 \mathcal{P} \left( g^{da} \sum_{s_1=0}^{s} C^s_{s_1} \delta_{il} \partial_a (D^i_t x^i_{s_1}) \mathcal{L}_B \partial_c (D^s_{t} x^i_s) \mathcal{L}_B W^c \right). \]

We have
\[
W_{r+2} = - \sum_{s=0}^{r-1} C^s_r G_{r-s} W_{s+2} + \sum_{s=0}^{r} C^s_r (G_{r-s} \mathcal{L}^2_B W_s - A_{r-s} W_s - G_{r-s+1} W_{s+1})
+ \sum_{s=0}^{r} C^s_r (C_{r-s} W_{s+1} - \chi_{r-s} \mathcal{L}_B W_s + G_{r-s} F_s),
\]
which inductively defines \( W_{r+2} \) from \( W_0, W_1, \ldots, W_{r+1}. \)

By the definition of \( W_{0r} \) in \( \ref{5.2} \), it is obvious that
\[
D^i_t (L_1 W_{0r} - F)|_{t=0} = 0 \quad \text{for } s \leq r, \quad W_{0r}|_{t=0} = W_0, \quad W_{0r}|_{t=0} = W_1.
\]
Thus, we reduce \( \ref{5.1} \) to the desired case of vanishing initial data and an inhomogeneous term that vanishes to any order as \( t \to 0 \) by replacing \( W \) by \( W - W_{0r} \) and \( F \) by \( F - L_1 W_{0r}. \)

If the initial data are smooth, as similar as in \( \ref{20} \), we can also construct a smooth approximate solution \( \tilde{W} \) that satisfies the equation to all orders as \( t \to 0 \). We can realize it by multiplying the \( k \)-th term in \( \ref{5.2} \) by a smooth cutoff function \( \chi(t/\varepsilon_k) \) and summing up the infinite series where \( \chi(s) = 1 \) for \( |s| \leq \frac{1}{2} \) and \( \chi(s) = 0 \) for \( |s| \geq 1 \). If we take \( (\|\tilde{W}\|_k + 1)\varepsilon_k \leq \frac{1}{2} \), then the sequence \( \varepsilon_k \) can be chosen so small that the series converges in \( C^m([0, T], H^m) \) for any \( m. \)

## 6 A priori estimates of the linearized equation with homogeneous initial data

### 6.1 The estimates of the one more order derivatives for the linearized equation

We take the time derivative to \( \ref{5.3} \) to get
\[
g^{ab} \dddot{W}^b - g^{ab} \mathcal{L}^2_B \ddot{W}^b - \partial_a (\dot{W}^c \partial_c P) - \omega^{ab} \dot{W}^b + \partial_a \dot{q} = -2 \ddot{g}^{ab} \dot{W}^b + \dot{g}^{ab} \mathcal{L}^2_B \dot{W}^b + \partial_a (W^c \partial_c P) - (\ddot{g}^{ab} - \dot{\omega}^{ab}) \dot{W}^b + 2D^i_t (\delta_{il} \partial_a (D^i_t x^i_s x^i_s) \mathcal{L}_B \partial_c (D^s_{t} x^i_s) \mathcal{L}_B W^c + 2 \delta_{il} \partial_a (D^i_t x^i_s) \mathcal{L}_B \partial_c x^i_s \mathcal{L}_B W^c + \dot{g}^{ab} F^b + g^{ab} \dddot{W}^b.
\]

Similar to \( \ref{4.2} \) and \( \ref{4.3} \) it holds
\[
\langle \dddot{W}, \dddot{W} \rangle = \frac{1}{2} \frac{d}{dt} \langle \dddot{W}, \dddot{W} \rangle - \frac{1}{2} \langle \dddot{W}, \dddot{W} \rangle,
\]
and

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{A} \dot{W}, \dot{W} \rangle = \langle \mathcal{A} \dot{W}, \dot{W} \rangle + \frac{1}{2} \langle \mathcal{A} \dot{W}, \dot{W} \rangle.$$ 

In view of (4.4), we have

$$-\langle \mathcal{L}_B^2 \dot{W}, \dot{W} \rangle = \frac{1}{2} \frac{d}{dt} \int_\Omega |\mathcal{L}_B \dot{W}|^2 dy - \frac{1}{2} \int_\Omega \dot{g}_{ab} \mathcal{L}_B \dot{W}^a \mathcal{L}_B \dot{W}^b dy + \int_\Omega (\mathcal{L}_B g_{ab}) \mathcal{L}_B \dot{W}^a \dot{W}^b dy.$$ 

Let

$$E_{D_t} = E(D_t W) = \langle \dot{W}, \ddot{W} \rangle + \langle \dot{W}, \mathcal{A} \dot{W} \rangle + \langle \mathcal{L}_B \dot{W}, \mathcal{L}_B \dot{W} \rangle.$$ 

Then, similarly, from the antisymmetry of \(\dot{\omega}\), we have

\begin{align*}
(6.2a) \quad & E_{D_t} = 2\langle \dot{F} + \dot{\dot{g}} F, \dot{W} \rangle - 2\langle \dot{g} \dot{W}, \dot{W} \rangle + \langle \mathcal{L}_B \dot{W}, \mathcal{L}_B \dot{W} \rangle - 3\langle \dot{g} \dot{W}, \dot{W} \rangle + \langle \dot{W}, \mathcal{A} \dot{W} \rangle \\
(6.2b) \quad & + \langle \dot{g} \mathcal{L}_B \dot{W}, \mathcal{L}_B \dot{W} \rangle + 4\langle D_t (\delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i) \mathcal{L}_B \dot{W}^c, \dot{W}^a \rangle \\
(6.2c) \quad & + 2\langle \delta_{il} (\partial_a x^l \mathcal{L}_B \partial_c x^i - \mathcal{L}_B \partial_a x^l \partial_c x^i) \mathcal{L}_B \dot{W}^c, \dot{W}^a \rangle \\
(6.2d) \quad & - 2\langle \mathcal{A} \dot{W}, \dot{W} \rangle.
\end{align*}

Thus, we get

$$| (6.2a) | \leq 2\|\partial x\|_{L^\infty(\Omega)} \|\mathcal{L}_B \partial x\|_{L^\infty(\Omega)} E_{D_t}.$$ 

By Cauchy-Schwartz’ inequalities, we get

$$| (6.2a) | \leq 2(\|\dot{F}\| + \|\dot{g}\|_{L^\infty(\Omega)} \|F\|) E_{D_t}^{1/2} + 2(\|\dot{g}\|_{L^\infty(\Omega)} + \|\dot{\omega}\|_{L^\infty(\Omega)}) E_{D_t}^{1/2} E_0$$

$$+ \left( 3\|\dot{g}\|_{L^\infty(\Omega)} + \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Omega)} \right) E_{D_t},$$

and

$$| (6.2b) | \leq \|\dot{g}\|_{L^\infty(\Omega)} E_{D_t} + 4\|D_t (\delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i)\|_{L^\infty(\Omega)} E_0 E_{D_t}^{1/2}.$$ 

Now, it remains to deal with the term \(\langle \dot{A}, \dot{W} \rangle\) in (6.2d). From (3.5), it follows that

$$| \langle \dot{A}, \dot{W} \rangle | \leq \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Gamma)} \langle \dot{W}, \mathcal{A} \dot{W} \rangle^{1/2} \langle \dot{W}, \mathcal{A} \dot{W} \rangle^{1/2}.$$ 

But this does not imply that the norm of \(\dot{A}\) is bounded by the norm of \(\mathcal{A}\) because \(\langle \dot{W}, \mathcal{A} \dot{W} \rangle\) is one more order derivative than the considering energies. However, we have

$$\langle \dot{A}, \dot{W} \rangle = \frac{d}{dt} \langle \dot{A}, \dot{W} \rangle - \langle \dot{A}, \dot{W} \rangle - \langle \dot{A}, \dot{W} \rangle.$$
in which the last two terms can be bounded by \( E_{D_t} \) and \( E_0 \). Thus, we have to deal with this term in an indirect way, by including them in the energies and using (6.3).

Let

\[
D_{D_t} = 2 \langle \dot{A}W, \dot{W} \rangle,
\]

then we get

\[
\dot{D}_{D_t} = 2 \langle \dot{A}W, \dot{W} \rangle + 2 \langle \dot{A} \dot{W}, \dot{W} \rangle + 2 \langle \dot{A}W, \dot{W} \rangle,
\]

and

\[
|\dot{D}_{D_t} - 2 \langle \dot{A}W, \dot{W} \rangle| \leq 2 \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Omega)} E_0 E_{D_t}^{1/2} + 2 \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Omega)} E_{D_t},
\]

\[
|D_{D_t}| \leq 2 \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Omega)} E_0 E_{D_t}^{1/2}.
\]

Denote

\[
\bar{n}_1(t) = \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Omega)},
\]

\[
n_1(t) = 5 \frac{1}{2} \left\| \dot{g} \right\|_{L^\infty(\Omega)} + 5 \frac{1}{2} \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Omega)} + \left\| \partial x \right\|_{L^\infty(\Omega)} \left\| \mathcal{L}_B \partial x \right\|_{L^\infty(\Omega)},
\]

\[
\bar{n}_1(t) = \left\| \dot{g} \right\|_{L^\infty(\Omega)} + \left\| \dot{\omega} \right\|_{L^\infty(\Omega)} + 2 \left\| D_t (\delta_{it} \partial x^i \mathcal{L}_B \partial x^i) \right\|_{L^\infty(\Omega)} + \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\|_{L^\infty(\Omega)},
\]

and

\[
f_1(t) = \left\| \dot{F} \right\| + \left\| \dot{g} \right\|_{L^\infty(\Omega)} \left\| F \right\|.
\]

Then, we have the following energy estimates.

**Proposition 6.1.** Let \( E_1^2(t) = E_{D_t} \), and

\[
M_1(t) = 4 \bar{n}_1^2(t) E_0^2(t) + 4 e^{f_1^2(t) \int_0^t n_1(\tau) d\tau} \int_0^t (\bar{n}_1(\tau) E_0(\tau) + f_1(\tau))^2 d\tau,
\]

it holds for (5.3) with zero initial data

\[
E_1^2(t) \leq M_1(t) + \int_0^t M_1(s) e^{t-s} ds.
\]

**Proof.** From the above argument, we have obtained

\[
\left| \frac{d}{dt} E_{D_t} + D_{D_t} \right| = \left| \frac{d}{dt} (E_{D_t} + D_{D_t}) \right| \leq 2 n_1(t) |E_{D_t} + D_{D_t}| + 2 (\bar{n}_1(t) E_0(t) + f_1(t)) E_1(t),
\]

which yields

\[
|E_{D_t} + D_{D_t}| \leq 2 e^{\int_0^t n_1(\tau) d\tau} \int_0^t (\bar{n}_1(s) E_0(s) + f_1(s)) E_1(s) ds.
\]
Thus,

\[ E_1^2(t) \leq 2e^2 \int_0^t n_1(\tau) d\tau \int_0^t \left( \bar{n}_1(s) E_0(s) + f_1(s) \right) E_1(s) ds + 2\bar{n}_1(t) E_0 E_1 \]

\[ \leq e^2 \int_0^t n_1(\tau) d\tau \left( \frac{1}{2} e^{-2 \int_0^t n_1(\tau) d\tau} \int_0^t E_1^2(s) ds \right) + 2e^2 \int_0^t \left( \bar{n}_1(s) E_0(s) + f_1(s) \right)^2 ds + 2\bar{n}_1(t) E_0^2 + \frac{1}{2} E_1^2, \]

and then

\[ E_2^2(t) \leq \int_0^t E_1^2(s) ds + M_1(t), \]

which implies the desired result by the Gronwall inequality.

\[ \square \]

### 6.2 The more one order energy estimates with respect to \( L_B \)

We now analyze the higher order energy functional. Let

\[ A_B = A_B P, \quad G_B = M_B g_b, \quad g_B = L_B g_b, \]

\[ \hat{G}_B = M_B g_b, \quad C_B = M_B \omega, \quad \omega_B = L_B \omega. \]

From (6.14), it follows that

\[ L_1 L_B W^d = L_B \hat{W}^d - L_B^3 W^d + A L_B W^d + \hat{G} L_B \hat{W}^d - C L_B \hat{W}^d + \chi L_B^2 W^d \]

\[ = L_B F^d - (A_B W^d + \hat{G}_B W^d - C_B W^d + \hat{G}_B \hat{W}^d - \hat{G}_B F^d) \]

\[ + 2g^{ad}(\delta_{i\ell} \partial_a x^i \hat{L}_B \partial_c x^i) B L_B W^c. \]

As similar as for the lowest-order energies, we define

\[ E_B = E(L_B W) = \langle L_B \hat{W}, L_B \hat{W} \rangle + \langle L_B W, (A + I) L_B W \rangle + \langle L_B^2 W, L_B^2 W \rangle. \]

From (4.14), (8.13) and \( B \cdot N|_{\Sigma} = 0 \), we get

\[ - \langle L_B^3 W, L_B W \rangle = - \int_{\Omega} g_{ab} L_B^3 W^a L_B \hat{W}^b dy \]

\[ = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |L_B^2 W|^2 dy - \frac{1}{2} \int_{\Omega} \hat{g}_{ab} L_B^2 W^a L_B \hat{W}^b dy + \int_{\Omega} (L_B g_{ab}) L_B^2 W^a L_B \hat{W}^b dy. \]

One has

\[ \langle \chi L_B^2 W, \hat{W} \rangle = -2 \int_{\Omega} \delta_{i\ell} \partial_a x^i \hat{L}_B \partial_c x^i L_B^2 W^c \hat{W}_T dy. \]

Thus, by (8.13), we get

\[ - \langle L_B^3 W, L_B W \rangle + \langle \chi L_B^2 W, L_B \hat{W} \rangle \]

\[ = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |L_B W|^2 dy - \frac{1}{2} \int_{\Omega} \hat{g}_{ab} L_B^2 W^a L_B \hat{W}^b dy \]

\[ + \int_{\Omega} \delta_{i\ell} (L_B \partial_a x^i \partial_c x^i - \partial_a x^i \hat{L}_B \partial_c x^i) L_B^2 W^a L_B \hat{W}^b dy. \]
From the antisymmetry of $\dot{\omega}$, one has
\[
\dot{E}_B = 2\langle L_B \dot{W} + A \mathcal{L}_B W + \dot{\mathcal{G}} \mathcal{L}_B W, \mathcal{L}_B \dot{W} \rangle + 2\langle \mathcal{L}_B W, \mathcal{L}_B \dot{W} \rangle + D_t(\mathcal{L}_B^2 W, \mathcal{L}_B^2 W) \\
- \langle \dot{\mathcal{G}} \mathcal{L}_B \dot{W}, \mathcal{L}_B \dot{W} \rangle + \langle \mathcal{G} \mathcal{L}_B W, \mathcal{L}_B W \rangle + \langle \mathcal{L}_B W, A \mathcal{L}_B W \rangle + \langle \dot{\mathcal{G}} \mathcal{L}_B W, A \mathcal{L}_B W \rangle \\
= 2\langle L_1 L_B W, L_B \dot{W} \rangle + 2\langle \dot{\mathcal{G}} L_B \dot{W}, L_B \dot{W} \rangle + 2\langle L_3^2 W, L_B \dot{W} \rangle - 2\langle \mathcal{X} \mathcal{L}_B^2 W, L_B \dot{W} \rangle \\
+ 2\langle \mathcal{L}_B W, L_B \dot{W} \rangle + D_t(\mathcal{L}_B^2 W, \mathcal{L}_B^2 W) - \langle \dot{\mathcal{G}} \mathcal{L}_B \dot{W}, \mathcal{L}_B \dot{W} \rangle + \langle \dot{\mathcal{G}} \mathcal{L}_B W, A \mathcal{L}_B W \rangle \\
+ \langle \mathcal{L}_B W, A \mathcal{L}_B W \rangle + \langle \dot{\mathcal{G}} \mathcal{L}_B W, A \mathcal{L}_B W \rangle
\]
(6.4a)
\[
= 2\langle L_B F + \mathcal{G} B F, L_B W \rangle - 2\langle \dot{\mathcal{G}} B \dot{W}, L_B W \rangle + 2\langle \mathcal{L}_B W, L_B \dot{W} \rangle - 2\langle \mathcal{G} B \dot{W}, L_B W \rangle
\]
(6.4b)
\[
- 4\langle \mathcal{L}_B (\delta_{ij}\partial_{x^i} L_B \partial_{x^j}) \mathcal{L}_B W^c, L_B W^a \rangle
\]
(6.4c)
\[
- \langle \dot{\mathcal{G}} L_B \dot{W}, L_B \dot{W} \rangle + \langle \dot{\mathcal{G}} L_B W, L_B \dot{W} \rangle + \langle \mathcal{L}_B W, A \mathcal{L}_B W \rangle + \langle \dot{\mathcal{G}} L_B W, A \mathcal{L}_B W \rangle + \int_{\Omega} g_{ab} \mathcal{L}_B^2 W^a \mathcal{L}_B^2 W^b dy + 2\langle \mathcal{L}_B W, L_B \dot{W} \rangle
\]
(6.4d)
\[
- 2\langle A_B W, L_B \dot{W} \rangle
\]
(6.4e)

Now, we control the term $\langle A_T W, \dot{W}_T \rangle$. As the same argument as in the estimates of $E_1(t)$, we have to deal with it in an indirect way, by including it in the energies. Let
\[
D_B = 2\langle A_B W, \mathcal{L}_B W \rangle,
\]
then
\[
\dot{D}_B = 2\langle \dot{A}_B W, \mathcal{L}_B W \rangle + 2\langle A_B \dot{W}, \mathcal{L}_B W \rangle + 2\langle A_B W, L_B \dot{W} \rangle.
\]

Therefore, we obtain
\[
\dot{E}_B + \dot{D}_B = \langle 6.26a \rangle + \langle 6.26b \rangle + \langle 6.26f \rangle + \langle 6.26g \rangle \\
+ 2\langle \dot{A}_B W, \mathcal{L}_B W \rangle + 2\langle A_B \dot{W}, \mathcal{L}_B W \rangle.
\]

From (3.7), (3.8) and (3.6), it yields
\[
\langle 6.4a \rangle \lesssim 2\left( \|L_B F\| + \|L_B g\|_{L^{\infty}(\Omega)} \|F\| + \|L_B \dot{g}\|_{L^{\infty}(\Omega)} E_0 \right. \\
+ \|L_B \dot{\omega}\|_{L^{\infty}(\Omega)} E_0 + \|L_B g\|_{L^{\infty}(\Omega)} E_1 \right) E_B^{1/2},
\]
\[
\langle 6.4b \rangle \lesssim 4\|L_B (\delta_{ij} \partial_{x^i} L_B \partial_{x^j})\|_{L^{\infty}(\Omega)} E_0 E_B^{1/2},
\]
\[
\langle 6.4c \rangle + \langle 6.4d \rangle \lesssim \left( 1 + \|\dot{g}\|_{L^{\infty}(\Omega)} + \left\| \frac{\nabla N \dot{P}}{\nabla N \dot{P}} \right\| \right) E_B,
\]
and
\[
\langle 6.5 \rangle \lesssim 2 \left( \left\| \frac{\nabla N (B \dot{P})}{\nabla N P} \right\| E_0 + \left\| \frac{\nabla N (BP)}{\nabla N P} \right\| E_1 \right) E_B^{1/2}.
\]

Let
\[
E_1^B := E_B^{1/2},
\]
and
\[
\tilde{n}_1^B(t) = \left\| \frac{\nabla N(BP)}{\nabla NP} \right\|_{L^\infty(\Omega)} E_0, \quad n_1^B(t) = \frac{1}{2} \left( 1 + \|\dot{g}\|_{L^\infty(\Omega)} + \left\| \frac{\nabla N\dot{P}}{\nabla NP} \right\|_{L^\infty(\Omega)} \right),
\]
\[
\tilde{n}_1^B(t) = \|L_B\dot{g}\|_{L^\infty(\Omega)} E_0 + \|L_B\omega\|_{L^\infty(\Omega)} E_0 + \|L_B\dot{g}\|_{L^\infty(\Omega)} E_1 + 2 \left( 1 + \|\dot{g}\|_{L^\infty(\Omega)} + \left\| \frac{\nabla N\dot{P}}{\nabla NP} \right\|_{L^\infty(\Omega)} \right) f_1^B(t),
\]
\[
f_1^B(t) = \|L_B\dot{g}\|_{L^\infty(\Omega)} + \|L_B\dot{g}\|_{L^\infty(\Omega)} ||F||.
\]

Then, we have the following estimates.

**Proposition 6.2.** Let
\[
M_1^B(t) = 2\tilde{n}_1^B(t) + 2 \int_0^t (\tilde{n}_1^B(\tau) + f_1^B(\tau))d\tau,
\]

it holds
\[
E_1^B(t) \leq M_1^B(t) + 2 \int_0^t n_1^B(s)M_1^B(s) \exp \left( 2 \int_s^t n_1^B(\tau)d\tau \right) ds.
\]

**Proof.** From the above argument, we have obtained
\[
\dot{E}_B + \dot{D}_B \leq 2E_1^B(f_1^B + \tilde{n}_1^B + n_1^B E_1^B).
\]

Since \(E_B(0) = D_B(0) = 0\), the integration over \([0, t]\) in time gives
\[
E_B \leq 2E_1^B\tilde{n}_1^B + 2 \int_0^t E_1^B(n_1^B E_1^B + \tilde{n}_1^B + f_1^B)d\tau.
\]

Taking the supremum on \([0, t]\) in time and dividing by \(\sup_{[0, t]} E_1^B\), we get
\[
E_1^B(t) \leq M_1^B(t) + 2 \int_0^t n_1^B(\tau)E_1^B(\tau)d\tau.
\]

By the Gronwall inequality, we can obtain the desired estimates. 

**6.3 Construction of tangential vector fields and the div-curl decomposition**

A basic estimate in the Euclidean coordinates is that derivatives of vector fields can be estimated by derivatives of the curl, the divergence and the tangential derivatives, as proved in [20, Lemma 11.1]. But that estimate is not invariant under changes of coordinates, so we also expect to replace it by an inequality that also holds in the Lagrangian coordinates. After that we need to derive its higher-order versions as well. Both the curl and the divergence are invariant, but the other terms are not. There are two ways to make these terms to be invariant. One is to replace the differentiation by covariant differentiation as used in [8, 13], and the other is to replace it by Lie derivatives with respect to tangential vector fields introduced below, as the same as used in [20]. Both ways result in a lower-order term involving only the norm of the 1-form itself multiplied by a constant relative to the coordinates.
Definition 6.3. Let \( c_1 \) be a constant satisfying
\[
\sum_{a,b} (|g_{ab}| + |g^{ab}|) \leq c_1^2, \quad \left| \frac{\partial x}{\partial y} \right|^2 + \left| \frac{\partial y}{\partial x} \right|^2 \leq c_1^2,
\]
and let \( K_1 \) denote a continuous function of \( c_1 \).

Indeed, the bound for the Jacobian of the coordinate and its inverse follows from the bound for the metric and its inverse, and the bound for the former implies an equivalent bound for the latter with \( c_1^2 \) multiplied by \( n \).

Following [20], we now construct the tangential divergence-free vector fields which are independent of time and expressed of the form \( T^a(x, y) \frac{\partial}{\partial y^a} \) in the Lagrangian coordinates. Due to \( \det(\frac{\partial x}{\partial y}) = 1 \), the divergence-free condition reduces to
\[
\partial_a T^a = 0.
\]

Because \( \Omega \) is just the unit ball in \( \mathbb{R}^n \), the vector fields can be explicitly expressed. The rotation vector fields \( y^a \partial_b - y^b \partial_a \) span the tangent space of the boundary and are divergence-free in the interior. It is clear that \( B = B^a \partial_a \) belongs to this space. Moreover, they also span the tangent space of the level sets of the distance function from the boundary in the Lagrangian coordinates \( d(y) = \text{dist}(y, \Gamma) = 1 - |y| \) for \( y \neq 0 \) away from the origin. We denote this set of vector fields by \( S_0 \). Thus, \( B \in S_0 \).

We can also construct a finite set of vector fields, as the same as in [20, 21], which span the tangential space when \( d \geq d_0 \) and are compactly supported in the set where \( d \geq d_0/2 \). We denote this set of vector fields by \( S_1 \). Let \( S = S_0 \cup S_1 \) denote the family of space tangential vector fields, and let \( T = S \cup \{ D_i \} \) denote the family of space-time tangential vector fields.

Let the radial vector field be \( R = y^a \partial_a \). Then, \( \partial_a R^a = n \) is not 0 but it suffices for our purposes that it is constant. Let \( R = S \cup \{ R \} \), which spans the full tangent space of the space everywhere. Let \( U = S \cup \{ R \} \cup \{ D_i \} \) denote the family of all vector fields. Note that the radial vector field commutes with the rotations, i.e.,
\[
[R, S] = 0, \quad S \in S_0.
\]

Furthermore, the commutators of two vector fields in \( S_0 \) is just another vector field in \( S_0 \). For \( i = 0, 1 \), let \( R_i = S_i \cup \{ R \} \), \( T_i = S_i \cup \{ D_i \} \) and \( U_i = T_i \cup \{ R \} \).

Now, we recall some estimates as follows.

**Lemma 6.4** ([20, Lemma 11.3]). In the Lagrangian frame, with \( W_a = g_{ab} W^b \), we have
\[
|\mathcal{L}_U W| \leq K_1 \left( |\text{curl } W| + |\text{div } W| + \sum_{S \in S} |\mathcal{L}_S W| + |[g]_1 W| \right), \quad U \in R, \tag{6.6}
\]
\[
|\mathcal{L}_U W| \leq K_1 \left( |\text{curl } W| + |\text{div } W| + \sum_{T \in T} |\mathcal{L}_T W| + |[g]_1 W| \right), \quad U \in U, \tag{6.7}
\]
where \([g]_1 = 1 + |\partial g|\). Furthermore,
\[
|\partial W| \leq K_1 \left( |\mathcal{L}_R W| + \sum_{S \in S} |\mathcal{L}_S W| + |W| \right). \tag{6.8}
\]

When \( d(y) \leq d_0 \), we may replace the sums over \( S \) by the sums over \( S_0 \) and the sum over \( T \) by the sum over \( T_0 \).
Next, we recall the higher-order versions of the inequality in last lemma. We need to apply the lemma to \( W \) replaced by \( \mathcal{L}_T W \), and the divergence term will vanish in our applications. We will be able to control the curl of \( (\mathcal{L}_T W)_a = \mathcal{L}_T (g_{ab} W^b) \), which is different from the curl of \( (\mathcal{L}_T W)_a = g_{ab} \mathcal{L}_T W^b \), but the difference is lower order and can be easily controlled. We first introduce some notation.

**Definition 6.5.** Let \( \beta \) be a function, a 1- or 2-form, or vector field, and let \( \mathcal{V} \) be any of our families of vector fields. Set

\[
|\beta|_s^V = \sum_{|J| \leq s, J \in \mathcal{V}} |\mathcal{L}_J^s \beta|,
\]

\[
[\beta]_s^V = \sum_{s_1 + \cdots + s_k \leq s, s_i \geq 1} |\beta|_{s_1}^V \cdots |\beta|_{s_k}^V,
\]

and \( [\beta]_0^V = 1 \).

In particular, \( |\beta|_1^R \) and \( |\beta|_1^U \) are equivalent to \( \sum_{|\alpha| \leq r} |\partial_y^\alpha \beta| \) and \( \sum_{|\alpha| + k \leq r} |D_k^{\alpha} \partial_y^\beta| \), respectively.

**Lemma 6.6** ([20, Lemma 11.5]). With the convention that \( |\text{curl} \ W|_1^V = |\text{div} \ W|_1^V = 0 \), we have

\[
|W|^r_R \leq K_1 \left( |\text{curl} \ W|^r_R - 1 + |\text{div} \ W|^r_R - 1 + |W|^S + \sum_{s=1}^r |g|^r_s |W|^{r-1}_s \right),
\]

\[
|W|^r_U \leq K_1 \sum_{s=1}^r |g|^r_s \left( |\text{curl} \ W|^r_{r-1-s} + |\text{div} \ W|^r_{r-1-s} + |W|^S_{r-1-s} \right).
\]

The same inequalities also hold with \( R \) replaced by \( U \) everywhere and \( S \) replaced by \( T \):

\[
|W|^s_U \leq K_1 \left( |\text{curl} \ W|^s_U - 1 + |\text{div} \ W|^s_U - 1 + |W|^T_s + \sum_{s=1}^r |g|^s_s |W|^s_{r-1-s} \right),
\]

\[
|W|^s_U \leq K_1 \sum_{s=1}^r |g|^s_s \left( |\text{curl} \ W|^s_{r-1-s} + |\text{div} \ W|^s_{r-1-s} + |W|^T_{r-1-s} \right).
\]

### 6.4 Commutators between the linearized equation and Lie derivatives with respect to \( B \)

In order to get the higher-order energy estimates of tangential derivatives, we first commute tangential vector fields through the linearized equation.

Let \( T \in \mathcal{T} \) be a tangential vector field, and recall that \([\mathcal{L}_T, D_t] = 0\) and that if \( W \) is divergence-free, then so does \( \mathcal{L}_T W \). Now, we apply Lie derivatives \( \mathcal{L}_T = \mathcal{L}_{T_{i_1}} \cdots \mathcal{L}_{T_{i_r}} \) with the multi-index \( I = (i_1, \ldots, i_r) \) to \( \ref{eq:5.3} \).

From \( \ref{eq:A.3} \), we have for \( r = |I| \),

\[
\mathcal{L}_T^I (g_{ab} W^b) = \sum_{I_1 + I_2 = I} c_{I_1, I_2} \mathcal{L}_T^{I_1} g_{ab} \mathcal{L}_T^{I_2} W^b =: c_{I_1, I_2} \mathcal{L}_T^{I_1} g_{ab} \mathcal{L}_T^{I_2} W^b,
\]

where we sum over all \( I_1 + I_2 = I \) and \( c_{I_1, I_2} = 1 \) (only for the simplicity of summing over the repeated indices) in last expression.

From \( \ref{eq:A.2} \) and the identity

\[
T(\partial_e PW^c) = T^d \partial_d (\partial_e PW^c) = T^d \partial_d \partial_e PW^c + T^d \partial_d P \partial_d W^c,
\]
some new notation for the operators \( H \).

Hence, we obtain
\[
\mathcal{L}_T(\partial_a(\partial_c PW^c)) = \partial_a T(\partial_c PW^c) = \partial_a(\partial_c(TP)W^c + \partial_c PTW^c),
\]

one has
\[
\mathcal{L}_T(\partial_a(\partial_c PW^c)) = \partial_a T(\partial_c PW^c) = \partial_a(\partial_c(TP)W^c + \partial_c PTW^c).
\]

Then we have inductively
\[
(6.9) \quad \mathcal{L}_T^I(\partial_a(\partial_c PW^c)) = \partial_a T^I(\partial_c PW^c) = c_{I_1 I_2} T^I\partial_a(\partial_c(T^{I_1}P)\mathcal{L}_T^{I_2} W^c).
\]

Hence, we obtain
\[
(6.10) \quad c_{I_1 I_2}(\mathcal{L}_T^{I_1} g_{a b}) \mathcal{L}_T^{I_2} \check{W}^b - c_{I_1 I_2}(\mathcal{L}_B^{I_1} g_{a b}) \mathcal{L}_T^{I_2} \check{W}^b - c_{I_1 I_2} \partial_a(\partial_c(T^{I_1}P)\mathcal{L}_T^{I_2} W^c)
\]
\[
= - \partial_a T^I q - c_{I_1 I_2}(\mathcal{L}_T^{I_1}(\check{g}_{a b} - \omega_{a b})) \mathcal{L}_T^{I_2} \check{W}^b + c_{I_1 I_2}(\mathcal{L}_T^{I_1} g_{a b}) \mathcal{L}_T^{I_2} F^b
\]
\[
+ 2c_{I_1 I_2}(\mathcal{L}_T^{I_1}(\delta_{i b} \partial_a x^l \mathcal{L}_B \partial_c x^i)) \mathcal{L}_T^{I_2} \mathcal{L}_B W^c.
\]

Denote
\[
W_I = \mathcal{L}_T^{I} W, \quad F_I = \mathcal{L}_T^{I} F, \quad P_I = T^I P, \quad q_I = T^I q, \\
(\cdot)_I = \mathcal{L}_T^{I}(\cdot), \quad g_{I a} = \mathcal{L}_T^{I} g_{a b}, \quad \omega_{I a} = \mathcal{L}_T^{I} \omega_{a b},
\]
and \( \hat{g}_{I a} = D_t \mathcal{L}_T^{I} g_{a b} = \mathcal{L}_T^{I} \hat{g}_{a b}, \check{W}_I = D_t W_I = \mathcal{L}_T^{I} \check{W}, \) etc. Then, (6.11) can be written as
\[
c_{I_1 I_2} g_{I a} \check{W}_{I_2}^b - c_{I_1 I_2} g_{I a} \mathcal{L}_B^{I} W_{I_2}^b - c_{I_1 I_2} \partial_a(\partial_c P_I W_{I_2}^c)
\]
\[
= - \partial_a q_I - c_{I_1 I_2}(\hat{g}_{I a} - \omega_{I a}) \check{W}_{I_2}^b
\]
\[
+ 2c_{I_1 I_2}(\delta_{i b} \partial_a x^l \mathcal{L}_B \partial_c x^i)_{I_1} (\mathcal{L}_B W)_{I_2}^c + c_{I_1 I_2} g_{I a} F_{I_2}^b.
\]

Next, we project each term onto the divergence-free vector fields and introduce some new notation for the operators
\[
(6.12) \quad A_I W^a = A_{P_I} W^a, \quad G_I W^a = P(g_{a c} g_{I c} W^b),
\]
\[
(\cdot)_I W^a = P(g_{a c} \hat{g}_{I c} W^b), \quad C_I W^a = P(g_{a c} \omega_{I c} W^b),
\]
and \( \check{c}_{I_1 I_2} = c_{I_1 I_2} \) if \( I_2 \neq I \) while \( \check{c}_{I_1 I_2} = 0 \) if \( I_2 = I \). Then, we can write (6.9) as
\[
(6.13) \quad P(g_{I a} \mathcal{L}_T^{I}(g_{a c} AW^c)) = AW_{I}^b + \check{c}_{I_1 I_2} A_{I_1} W_{I_2}^b.
\]

Thus, we are able to rewrite (6.11) as
\[
(6.14) \quad L_I W_I^d = \check{W}_I^d - (\mathcal{L}_B W)_{I_1}^d + AW_{I}^d + \hat{G}_I W_I^d - C_I W_I^d + X(\mathcal{L}_B W)_I^d
\]
\[
= F_{I}^d - \check{c}_{I_1 I_2} (A_{I_1} W_{I_2}^d + \check{G}_{I_1} W_{I_2}^d - C_{I_1} W_{I_2}^d + \check{G}_{I_1} \check{W}_{I_2}^d - \check{G}_{I_1} F_{I_2}^d)
\]
\[
+ 2\check{c}_{I_1 I_2} g_{I a} (\delta_{i b} \partial_a x^l \mathcal{L}_B \partial_c x^i)_{I_1} (\mathcal{L}_B W)_{I_2}^c.
\]

Now, we define higher-order energies. For \( I \in \mathcal{V} \) with \( |I| = r \geq 2 \), let
\[
E_I = E(W_I) = \langle \check{W}_I, \check{W}_I \rangle + \langle W_I, (A + I) W_I \rangle + \langle \mathcal{L}_B W_I, \mathcal{L}_B W_I \rangle.
\]
For $V \in \{D_l, \{B\}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \mathcal{R}, \mathcal{U}\}$ where $\mathcal{B} = \{\mathcal{B}, D_l\}$, let

$$
(6.15) \quad |W|^V_s = \sum_{|I| \leq s, T \in \mathcal{V}} |L^I_W|, \quad |W|^V_{s,B} = \sum_{|I| \leq s, T \in \mathcal{V}} |\mathcal{L}_B L^I_W|,
$$

$$
(6.16) \quad \|\partial_q\|^V_{s,\infty, P-1} = \sum_{|I| = s, T \in \mathcal{V}} \left\| \nabla_N L^I_q \right\|_{L^\infty(\Omega)} , \quad \|\partial_q\|^V_{s,\infty, P-1} = \sum_{0 \leq l \leq s} \|\partial_q\|^V_{l,\infty, P-1},
$$

$$
(6.17) \quad \|f\|^V_{s,\infty} = \sum_{|I| \leq s, T \in \mathcal{V}} \|L^I_T f\|_{L^\infty(\Omega)} , \quad F^V_s = \sum_{|I| \leq s, T \in \mathcal{V}} \|F_I\|,
$$

$$
(6.18) \quad \tilde{E}^V_s = \sum_{|I| = s, T \in \mathcal{V}} \sqrt{E_I}, \quad E^V_s = \sum_{0 \leq l \leq s} \tilde{E}^V_l.
$$

### 6.5 The higher-order energy estimates for time and $\mathcal{L}_B$ derivatives

From (6.14), we have for $I \in \mathcal{B}$,

$$
(6.19) \quad \tilde{E}_I = 2\langle W_I, \dot{W}_I + AW_I \rangle + \langle \dot{W}_I, \dot{G} \dot{W}_I \rangle + 2\langle \ddot{W}_I, W_I \rangle + \langle \dot{G} W_I, (A + I)W_I \rangle
$$

$$
+ \langle W_I, \dot{A} W_I \rangle + \langle \dot{G} \mathcal{L}_B W_I, \mathcal{L}_B W_I \rangle + 2\langle \mathcal{L}_B \dot{W}_I, \mathcal{L}_B W_I \rangle
$$

$$
= 2\langle W_I, F_I \rangle - \langle \dot{G} \dot{W}_I, \dot{W}_I \rangle + 2\langle \ddot{W}_I, W_I \rangle + \langle \dot{G} W_I, (A + I)W_I \rangle
$$

$$
+ \langle W_I, \dot{A} W_I \rangle + \langle \dot{G} \mathcal{L}_B W_I, \mathcal{L}_B W_I \rangle
$$

$$
- 2\tilde{c}_{I_1 I_2} \langle \ddot{W}_I, A_I W_{I_2} \rangle + \langle \dot{W}_I, \dot{G}_{I_1} \dot{W}_{I_2} \rangle - \langle W_I, \mathcal{C}_{I_1} \dot{W}_{I_2} \rangle
$$

$$
(6.20) \quad + \langle \dot{W}_I, \mathcal{G}_{I_1} \dot{W}_{I_2} \rangle - \langle \ddot{W}_I, \mathcal{G}_{I_1} F_{I_2} \rangle
$$

$$
+ 4\tilde{c}_{I_1 I_2} \langle \ddot{W}_I, (\delta_{I_1} \partial_a x^i \mathcal{L}_B \partial_c x^i)_{I_1} (\mathcal{L}_B W)_{I_2} \rangle
$$

$$
(6.21) \quad + 2\langle \mathcal{L}_B \ddot{W}_I, \mathcal{L}_B W_I \rangle + 2\langle \ddot{W}_I, (\mathcal{L}_B^2 W_I) \rangle - 2\langle W_I, \mathcal{X}(\mathcal{L}_B W_I) \rangle.
$$

It is clear that

$$
(6.19) \quad | \tilde{E}_I | \leq 2E^{1/2}_I \| F_I \| + \left( 1 + \| \dot{g} \|_{L^\infty(\Omega)} + \left\| \nabla_N \hat{P} \right\|_{L^\infty(\Omega)} \right) E_I,
$$

and

$$
| \tilde{E}_I | \leq 4\tilde{c}_{I_1 I_2} \| (\delta_{I_1} \partial_a x^i \mathcal{L}_B \partial_c x^i)_{I_1} \|_{L^\infty(\Omega)} E^{1/2}_{I_2} E^{1/2}_I.
$$

To deal with the term $\langle \ddot{W}_I, A_I W_{I_2} \rangle$, we introduce

$$
D_I = 2\tilde{c}_{I_1 I_2} \langle W_I, A_I W_{I_2} \rangle,
$$

then

$$
\dot{D}_I = 2\tilde{c}_{I_1 I_2} (\langle \ddot{W}_I, A_I W_{I_2} \rangle + \langle W_I, \dot{A}_I W_{I_2} \rangle + \langle W_I, \dot{A}_I W_{I_2} \rangle).
$$

Thus,

$$
| \dot{D}_I | + (6.20) |.
$$
Therefore, 

\[
\|\tilde{W}_I\| \leq 2\|\bar{\tilde{W}}_I\| + \|F_I\|, 
\]

where the term \(\|\tilde{W}_I\|\) can be controlled by the energy norm taking one \(T = D_I\).

Since \(B \cdot N = 0\) on \(\Gamma\), we get by (A.3),

\[
6.22 = 2 \int_{\Omega} g_{ab} B \mathcal{L}(\tilde{W}_I^a) \mathcal{L}(W^b_I) \, dy + 4 \int_{\Omega} \delta_{ij} \delta_{ij} B \mathcal{L}(\tilde{W}_I^a) \mathcal{L}(W^b_I) \, dy
\]

\[
= -2(\mathcal{L}_B g_{ab}) \tilde{W}_I^a, B \mathcal{L}(W^b_I) + 4(\delta_{ij} \delta_{ij} B \mathcal{L}(\tilde{W}_I^a) \mathcal{L}(W^b_I)
\]

\[
= 2(\delta_{ij} \delta_{ij} B \mathcal{L}(\tilde{W}_I^a) \mathcal{L}(W^b_I) - \mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij}) W^a_I, B \mathcal{L}(W^b_I).
\]

Then,

\[
6.22 \leq 4 \|\partial x\| \|\mathcal{L}_B \partial x\| \|\mathcal{L}_B \partial x\| \|\mathcal{L}_B \partial x\| \|\mathcal{L}_B \partial x\|.
\]

Therefore,

\[
6.23 \quad \tilde{E}_I + \tilde{D}_I
\]

\[
\leq 2 E_I^{1/2} \|F_I\| \|\tilde{W}_I\| \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij}
\]

\[
= 4 \|\tilde{W}_I\| \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \| I \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \| \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \|
\]

\[
\cdot E_I^{1/2} E_I^{1/2} + 4 E_I^{1/2} E_I^{1/2} \|\delta_{ij} \delta_{ij} \mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \| I \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \| E_I^{1/2} E_I^{1/2}.
\]

Noticing that \(E_I(0) = D_I(0) = 0\), the integration over \([0, t]\) in time implies

\[
6.24 \quad E_I \leq |D_I| + \int_0^t (6.23) \, d\tau
\]

\[
\leq 2 E_I \sum_{s=0}^{r-1} \mathcal{C}^s \|\partial P\|_{r-s, \infty, p-1} \hat{E}_s^B + \int_0^t (6.23) \, d\tau.
\]

Let

\[
n_t^B = 1 + (2r - 1) \|\bar{\tilde{W}}_I\| + \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \|,
\]

\[
\bar{n}_t^B = \int_0^t (\|\partial P\|_{r, \infty, p-1} + \|\bar{\tilde{W}}_I\|_{r, \infty, p-1} + \|\bar{\tilde{W}}_I\|_{r, \infty} + \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \| + \|\bar{\tilde{W}}_I\|_{r, \infty} + \|\mathcal{L}_B \mathcal{L}_B \delta_{ij} \delta_{ij} \|) \, d\tau,
\]

\[
f_t^B = \int_0^t (1 + \|\bar{\tilde{W}}_I\|_{r, \infty}) F_t^B \, d\tau.
\]
Taking the supremum on \([0,t]\) in time of (6.24), then summing up the order from 0 to \(r\), and dividing by \(E_r^B = \sup_{[0,t]} E_r^B\), we get

\[
E_r^B \leq C (\sup_{[0,t]} \|\partial P\|_{r-1,\infty,P-1} + n_r^B) E_{r-1}^B + C f_r^B + \int_0^t n_r^B E_r^B d\tau.
\]

By the Gronwall inequality, we can obtain the following estimates.

**Proposition 6.7.** Let

\[
M_r^B = C \left( (\sup_{[0,t]} \|\partial P\|_{r-1,\infty,P-1} + n_r^B) E_{r-1}^B + f_r^B \right),
\]

it holds

\[
E_r^B(t) \leq M_r^B(t) + \int_0^t M_r^B(s) n_r^B(s) \exp \left( \int_s^t n_r^B(\tau)d\tau \right) ds.
\]

This is a recursion formula between \(E_r^B\) and \(E_{r-1}^B\), thus we can obtain inductively the estimates of \(E_r^B\) and \(E_{r}^B\) since we have proved the estimates of \(E_1^B = \sup_{[0,t]} (E_0 + E_1 + E_1^B)\) in Propositions 4.1, 6.1 and 6.9. Indeed, we have the following:

**Proposition 6.8.** Assume that \(x, P \in C_t^{r+2}([0,T] \times \Omega), B \in C_t^{r+2}(\Omega), P|_{\Gamma} = 0, \nabla N P|_{\Gamma} \leq -c_0 < 0, B^a N_a|_{\Gamma} = 0\) and \(\text{div} V = 0\), where \(V = D_1 x\). Suppose that \(W\) is a solution of (6.3) where \(F\) is divergence-free and vanishing to order \(r\) as \(t \to 0\). Then, there is a constant \(C = C(x, P, B)\) depending only on the norm of \((x, P, B)\), a lower bound for \(c_0\), and an upper bound for \(T\) such that if \(E_s^B(0) = 0\) for \(s \leq r\), then

\[
E_r^B(t) \leq C \int_0^t \|F\|_{r}^B d\tau, \quad \text{for } t \in [0,T].
\]

### 6.6 Estimates for the tangential derivatives

We have got the higher-order time and \(L_B\) derivatives that are some kinds of tangential derivatives due to \(\text{div} B = 0\) and \(B \cdot N|_{\Gamma} = 0\), but they do not give the estimates for all tangential derivatives. Thus, we need to derive the estimates for tangential derivatives \(\sum_{T \in T} |L_T W|\) of \(W\).

Let \(T \in T, W_T = L_T W, F_T = L_T F\), and similar notation as in (6.12):

\[
A_T = A_T P, \quad G_T = M_{g_T}, \quad g_{ab}^T = L_T g_{ab},
\]

\[
\hat{G}_T = M_{g_T^T}, \quad C_T = M_{\hat{g}_T}, \quad \omega_{ab}^T = L_T \omega_{ab}.
\]

Then, from (6.14), it follows that

\[
L^d W_T = \bar{W}_T^d - L_T \bar{L}_B W^d + A W_T^d + \hat{G} \bar{W}_T^d - C \bar{W}_T^d + \bar{X}_T \mathcal{L}_T \mathcal{L}_B W^d
\]

\[
= F_T^d - (A_T W_T^d + \hat{G}_T \bar{W}_T^d - C_T \bar{W}_T^d + G_T \bar{W}_T^d - G_T F^d)
\]

\[
+ 2 g^{ad}(\delta_{ai} \partial_a x^i \mathcal{L}_B \partial_c x^c) T \mathcal{L}_B W^c.
\]

As the arguments in the lowest-order energies, we define

\[
E_T = E(W_T) = \langle \bar{W}_T, W_T \rangle + \langle W_T, (A + I) W_T \rangle + \langle \mathcal{L}_B W_T, \mathcal{L}_B W_T \rangle.
\]
From (1.4), (A.8) and $B \cdot N = 0$, we get

$$-\langle L_T L_B^2 W, \dot{W}_T \rangle = - \int_{\Omega} g_{ad} L_T L_B^2 W^d \dot{W}_T^d dy $$

$$= - \int_{\Omega} g_{ad} [L_T, L_B] L_B W^d \dot{W}_T^d dy - \int_{\Omega} g_{ad} L_B [L_T, L_B] W^d \dot{W}_T^d dy $$

$$- \int_{\Omega} g_{ad} L_B^2 W^d \dot{W}_T^d dy $$

$$= - \int_{\Omega} g_{ad} L_{[T,B]} L_B W^d \dot{W}_T^d dy - \int_{\Omega} g_{ad} L_B L_{[T,B]} W^d \dot{W}_T^d dy $$

$$+ \frac{1}{2} \int_{\Omega} |L_B W_T|^2 dy - \frac{1}{2} \int_{\Omega} \dot{g}_{ab} L_B W_T^a L_B W_T^b dy + \int_{\Omega} \langle L_B g_{ab} \rangle L_B W_T^a \dot{W}_T^b dy.$$ 

One has

$$\langle X L_T L_B W, \dot{W}_T \rangle = -2 \int_{\Omega} \delta_{il} \partial_a x^i L_B \partial_c x^i L_T L_B W^c \dot{W}_T^a dy$$

$$= -2 \int_{\Omega} \delta_{il} \partial_a x^i L_B \partial_c x^i L_{[T,B]} W^c \dot{W}_T^a dy$$

$$- 2 \int_{\Omega} \delta_{il} \partial_a x^i L_B \partial_c x^i L_B W_T^c \dot{W}_T^a dy.$$ 

Thus, by (A.3), we get

$$-\langle L_T L_B^2 W, \dot{W}_T \rangle + \langle X L_T L_B W, \dot{W}_T \rangle$$

$$= -2 \int_{\Omega} g_{ad} L_{[T,B]} L_B W^d \dot{W}_T^d dy + \int_{\Omega} g_{ad} L_{[T,B]} L_B W^d \dot{W}_T^d dy + \frac{1}{2} \int_{\Omega} |L_B W_T|^2 dy $$

$$- \frac{1}{2} \int_{\Omega} \dot{g}_{ab} L_B W_T^a L_B W_T^b dy - 2 \int_{\Omega} \delta_{il} \partial_a x^i L_B \partial_c x^i L_{[T,B]} W^c \dot{W}_T^a dy$$

$$+ \int_{\Omega} \delta_{il} (L_B \partial_a x^i \partial_c x^i - \partial_a x^i L_B \partial_c x^i) L_B W_T^c \dot{W}_T^a dy.$$ 

From the antisymmetry of $\dot{\omega}$, one has

$$\dot{E}_T = 2 \langle \dot{W}_T + AW_T + \dot{G} W_T, \dot{W}_T \rangle + 2 \langle W_T, \dot{W}_T \rangle + D_t \langle L_B W_T, L_B W_T \rangle$$

$$- \langle \dot{G} W_T, W_T \rangle + \langle \dot{G} W_T, W_T \rangle + \langle W_T, AW_T \rangle + \langle G W_T, AW_T \rangle $$

$$= 2 \langle L_1 W_T, \dot{W}_T \rangle + 2 \langle \dot{G} W_T, \dot{W}_T \rangle + 2 \langle L_T L_B^2 W, \dot{W}_T \rangle - 2 \langle X L_T L_B W, \dot{W}_T \rangle$$

$$+ 2 \langle W_T, \dot{W}_T \rangle + D_t \langle L_B W_T, L_B W_T \rangle$$

$$- \langle \dot{G} W_T, W_T \rangle + \langle \dot{G} W_T, W_T \rangle + \langle W_T, AW_T \rangle + \langle G W_T, AW_T \rangle$$

(6.26a) $$= 2 \langle T + \dot{G} T + \dot{W}_T, \dot{W}_T \rangle - 2 \langle \dot{G} T W_T, \dot{W}_T \rangle + 2 \langle L_T \dot{W}_T, \dot{W}_T \rangle - 2 \langle G T \dot{W}_T, \dot{W}_T \rangle$$

$$- 4 \langle \delta_{il} \partial_a x^i L_B \partial_c x^i, L_B W_T^c \dot{W}_T^a \rangle$$

(6.26b) $$- 2 \int_{\Omega} \delta_{il} (L_B \partial_a x^i \partial_c x^i - \partial_a x^i L_B \partial_c x^i) L_B W_T^c \dot{W}_T^a dy$$

(6.26c) $$+ 4 \int_{\Omega} \delta_{il} \partial_a x^i L_B \partial_c x^i L_{[T,B]} W^c \dot{W}_T^a dy$$

(6.26d) $$+ 4 \int_{\Omega} g_{ad} L_{[T,B]} L_B W^d \dot{W}_T^a dy.$$
\[
\begin{align*}
\text{(6.26e)} & \quad - 2 \int_{\Omega} g_{aa} L_{[T,B]} W^a \dot{W}^a_1 dy \\
\text{(6.26f)} & \quad - \langle \dot{G} \dot{W}_T, \dot{W}_T \rangle + \langle \dot{G} W_T, W_T \rangle + \langle W_T, \dot{A} W_T \rangle + \langle \dot{G} W_T, \dot{A} W_T \rangle \\
\text{(6.26g)} & \quad + \int_{\Omega} \dot{g}_{ab} L_B W^a_1 L_B W^b_1 dy + 2 \langle W_T, \dot{W}_T \rangle \\
\text{(6.26h)} & \quad - 2 \langle A_T W, \dot{W}_T \rangle.
\end{align*}
\]

Now, we control the term \(\langle A_T W, \dot{W}_T \rangle\). As the same argument as in the estimates of \(E_1(t)\), we have to deal with it in an indirect way, by including it in the energies. Let

\[ D_T = 2 \langle A_T W, W_T \rangle, \]

then

\[ \dot{D}_T = 2 \langle \dot{A}_T W, W_T \rangle + 2 \langle A_T \dot{W}, W_T \rangle + 2 \langle A_T W, \dot{W}_T \rangle. \]

Therefore, we obtain

\[ \dot{E}_T + \dot{D}_T = \text{(6.26a)} + \text{(6.26b)} + \text{(6.26c)} + \text{(6.26d)} + \text{(6.26e)} + \text{(6.26f)} + \text{(6.26g)} + 2 \langle \dot{A}_T W, W_T \rangle + 2 \langle A_T \dot{W}, W_T \rangle + 2 \langle A_T W, \dot{W}_T \rangle. \]

(6.27)

From (3.7), (3.8) and (3.6), it yields

\[
\begin{align*}
|\text{(6.26a)}| & \leq 2 \left( \| F_T \| + \| g^T \|_{L^\infty(\Omega)} \| F \| + \| g^T \|_{L^\infty(\Omega)} E_0 \right. \\
& \quad + \| \omega^T \|_{L^\infty(\Omega)} E_0 + \| g^T \|_{L^\infty(\Omega)} E_1 \right) E_T^{1/2}, \\
|\text{(6.26b)}| & \leq 4 \| \delta_{ij} \partial x^j L_B \partial x^i \|_{L^\infty(\Omega)} E_0 E_T^{1/2} + 2 \| \delta_{ij} \partial x^j L_B \partial x^i \|_{L^\infty(\Omega)} E_T, \\
|\text{(6.26c)} + \text{(6.26d)}| & \leq \left( 1 + \| \dot{g} \|_{L^\infty(\Omega)} + \left\| \frac{\nabla N \dot{P}}{\nabla N P} \right\| \right) E_T, \\
|\text{(6.27)}| & \leq 2 \left( \left\| \frac{\nabla N T \dot{P}}{\nabla N P} \right\| E_0 + \left\| \frac{\nabla N T P}{\nabla N P} \right\| E_1 \right) E_T^{1/2}, \\
|\text{(6.26e)}| & \leq 4 \| \delta_{ij} \partial x^j L_B \partial x^i \|_{L^\infty(\Omega)} \| L_{[T,B]} W \| E_T^{1/2}, \\
|\text{(6.26f)}| & \leq 2 \| g \|_{L^\infty(\Omega)} \| L_{[T,B]} W \| E_T^{1/2}, \\
\text{and} \quad |\text{(6.26g)}| & \leq 4 \| g \|_{L^\infty(\Omega)} \| L_{[T,B]} L_B W \| E_T^{1/2}.
\end{align*}
\]

Since \(B \in S\) and for \(T \in S\),

\[ \text{div} [T, B] = \partial_b (T^a \partial_a B^b - B^a \partial_a T^b) = \partial_b T^a \partial_a B^b - \partial_b B^a \partial_a T^b = 0, \]
we get $[T, B] \in S$. Similarly, $[[T, B], B] \in S$. Thus, from the above estimates and observation, we see that the energies should include $E_T$ for any $T \in \mathcal{T}$ in order to deal with the commutators. Thus, we define the energy as

$$E_1^T := E_T^{1/2} \text{ for } T \in \mathcal{T}, \quad E_1^T = \sum_{T \in \mathcal{T}} E_1^T.$$ 

Let

$$\tilde{n}_1^T(t) = \left\| \frac{\nabla_T P}{\nabla_T N} \right\|_{L^\infty(\Omega)} E_0,$$

$$n_1(t) = \frac{1}{2} \left( 1 + \left\| \hat{g} \right\|_{L^\infty(\Omega)} + 3 \left\| g \right\|_{L^\infty(\Omega)} \right) + 2 \left\| \hat{\delta}_t \partial x^i \mathcal{L}_B \partial x^i \right\|_{L^\infty(\Omega)},$$

$$\dot{n}_1^T(t) = \left\| \hat{g}^T \right\|_{L^\infty(\Omega)} E_0 + \left\| \omega^T \right\|_{L^\infty(\Omega)} E_0 + \left\| g^T \right\|_{L^\infty(\Omega)} E_0 + 2 \left\| \left( \hat{\delta}_t \partial x^i \mathcal{L}_B \partial x^i \right)_T \right\|_{L^\infty(\Omega)} E_0 + \left\| \frac{\nabla_T \hat{P}}{\nabla_T N} \right\|_{L^\infty(\Omega)} E_0 + \left\| \frac{\nabla_T P}{\nabla_T N} \right\|_{L^\infty(\Omega)} E_1,$$

$$f_1^T(t) = \left\| F_T \right\| + \left\| g^T \right\|_{L^\infty(\Omega)} \left\| F \right\|,$$

$$\tilde{n}_1^T(t) = 2 \sum_{T \in \mathcal{T}} \sup_{[0, t]} \tilde{n}_1^T + 2 \int_0^t \sum_{T \in \mathcal{T}} \tilde{n}_1^T d\tau,$$

$$f_1^T(t) = 2 \int_0^t \sum_{T \in \mathcal{T}} f_1^T d\tau. \quad \text{Then, we have the following estimates.}$$

**Proposition 6.9.** It holds

$$E_1^T \leq \tilde{n}_1^T(t) + f_1^T(t) + \int_0^t (\tilde{n}_1^T(s) + f_1^T(s)) n_1(s) \exp \left( \int_s^t n_1(\tau) d\tau \right) ds.$$ 

**Proof.** From the above argument, we have obtained

$$\dot{E}_T + D_T \leq 2 E_1^T \left\{ \left\| F_T \right\| + \left\| g^T \right\|_{L^\infty(\Omega)} \left\| F \right\| + \left\| \hat{g}^T \right\|_{L^\infty(\Omega)} E_0 + \left\| \omega^T \right\|_{L^\infty(\Omega)} E_0 + \left\| g^T \right\|_{L^\infty(\Omega)} E_0 + 2 \left\| \left( \hat{\delta}_t \partial x^i \mathcal{L}_B \partial x^i \right)_T \right\|_{L^\infty(\Omega)} E_0 + \left\| \frac{\nabla_T \hat{P}}{\nabla_T N} \right\|_{L^\infty(\Omega)} E_0 + \left\| \frac{\nabla_T P}{\nabla_T N} \right\|_{L^\infty(\Omega)} E_1 \right\}.$$ 

Since $E_T(0) = D_T(0) = 0$, the integration over $[0, t]$ in time gives

$$E_T \leq 2 E_1^T \tilde{n}_1^T + 2 \int_0^t E_1^T \left[ n_1 E_1^T + \tilde{n}_1^T + f_1^T \right] d\tau.$$

Taking the supremum on $[0, t]$ in time and dividing by $\sup_{[0, t]} E_1^T$, we sum over $T \in \mathcal{T}$ to get

$$E_1^T \leq 2 \sum_{T \in \mathcal{T}} \sup_{[0, t]} \tilde{n}_1^T + 2 \int_0^t \left[ n_1 E_1^T + \sum_{T \in \mathcal{T}} \tilde{n}_1^T \right] d\tau + f_1^T$$

(6.28)
By the Gronwall inequality, we can obtain the desired estimates.

6.7 Estimates for the curl and the full derivatives of the first order

Now, we will derive the estimates of normal derivatives close to the boundary by using the estimates of the curl and the estimates of the tangential derivatives in view of Lemma 6.6. Thus, we have to derive the estimates of the curl and the time derivatives of the curl. For this reason, we need to use the 1-form of $W$ and $\bar{W}$, denoted by $w$ and $\dot{w}$ respectively, i.e.,

$$w_a = g_{ab}W^b, \quad \dot{w}_a = g_{ab}\dot{W}^b,$$

in which the latter notation is slightly confusing and $\dot{w}$ is not equal to $D_tw$, but we only try to indicate that $\dot{w}$ is the corresponding 1-form obtained by lowering the indices of the vector field $\bar{W}$.

Let

$$\text{curl} w_{ab} = \partial_a w_b - \partial_b w_a, \quad F^a = g_{ab}F^b.$$

Since $D_tw_a = D_t(g_{ab}W^b) = \dot{g}_{ab}W^b + g_{ab}\dot{W}^b$, we have

$$D_t\text{curl } w_{ab} = D_t\left(\partial_a w_b - \partial_b w_a\right)$$

$$= \partial_a(\dot{g}_{bc}W^c + g_{bc}\dot{W}^c) - \partial_b(\dot{g}_{ac}W^c + g_{ac}\dot{W}^c)$$

$$= (\partial_a\dot{g}_{bc} - \partial_b\dot{g}_{ac})W^c + \dot{g}_{bc}\partial_aW^c - \dot{g}_{ac}\partial_bW^c + \partial_a\dot{w}_b - \partial_b\dot{w}_a$$

$$= \text{curl } \dot{w}_{ab} + \partial_c\omega_{ab}W^c + \dot{g}_{bc}\partial_aW^c - \dot{g}_{ac}\partial_bW^c$$

(6.29)

$$+ \left( (\dot{g}_{bc} - \omega_{cb})\partial_a\partial_c x^k - (\dot{g}_{ca} - \omega_{ca})\partial_b\partial_c x^k \right) \frac{\partial y^c}{\partial x^k} W^c,$$

since from (2.20) we have

$$2\partial_i v_i = (\dot{g}_{ic} - \omega_{ic})\frac{\partial y^c}{\partial x^k}$$

and

$$\partial_a\dot{g}_{ab} - \partial_d\dot{g}_{ad} = \partial_a[\partial_d x^i \partial_b x^k (\partial_k v_i + \partial_i v_k)] - \partial_a[\partial_d x^i \partial_b x^k (\partial_k v_i + \partial_i v_k)]$$

$$= \partial_d x^i \partial_b x^k \partial_a x^l \partial_k v_i - \partial_d x^i \partial_b x^k \partial_d x^l \partial_i v_k$$

$$+ (\partial_d x^i \partial_a x^l \partial_b x^k - \partial_a x^l \partial_d x^i \partial_b x^k)(\partial_k v_i + \partial_i v_k)$$

$$= \partial_d x^i \partial_a x^k \partial_b (\partial_k v_i - \partial_i v_k) + (\partial_d x^i \partial_a x^k \partial_b x^l + \partial_a x^k \partial_d x^l \partial_b x^i)(\partial_k v_i - \partial_i v_k)$$

$$+ 2\partial_d x^i \partial_a x^k \partial_b x^l \partial_i v_k - 2\partial_a x^k \partial_d x^l \partial_b x^i \partial_i v_k$$

$$= \partial_d \omega_{ad} + 2(\partial_d v_k \partial_a x^k - \partial_a v_k \partial_d x^k)$$

$$= \partial_d \omega_{ad} + [(\dot{g}_{cd} - \omega_{cd})\partial_a x^k - (\dot{g}_{ca} - \omega_{ca})\partial_d x^k] \frac{\partial y^c}{\partial x^k}.$$

Due to $\text{div } W = 0$, we can get from Lemma 6.1 and (6.29) that

$$|D_t\text{curl } w| \leq |\text{curl } \dot{w}| + |\partial \omega| |W| + 2|\dot{g}| |\partial W| + (|\dot{g}| + |\omega|) |\partial^2 x| \left| \frac{\partial y}{\partial x} \right| |W|$$

$$\leq |\text{curl } \dot{w}| + K_1 |\dot{g}| \left( |\text{curl } w| + \sum_{S \in \mathcal{S}} |L_S W| + |g_1| W \right)$$
Thus, we have to derive the estimates of \( \text{curl} \hat{w} \). From (5.3), we get

\[
D_t \hat{w}_a = D_t (g_{ab} \hat{W}^b) = \hat{g}_{ab} \hat{W}_b + g_{ab} \hat{W}_b
\]

Note that the above equation can be also formulated as

\[ (6.30) \quad D_t \hat{w}_a - g_{ab} \hat{L}^2_B W^b + g_{ab} (AW^b - CW^b + X_L B W^b) = E_a. \]

Then, we have

\[
(6.31) \quad D_t \text{curl} \hat{w}_{ad} = D_t (\partial_a \hat{w}_d - \partial_d \hat{w}_a) = \partial_a D_t \hat{w}_d - \partial_d D_t \hat{w}_a
\]

where we have used the identity \( \partial_a \hat{w}_{db} - \partial_d \hat{w}_{ab} = \partial_b \hat{w}_{da} \) which can be verified by (2.19). In fact,

\[
\partial_a \hat{w}_{db} - \partial_d \hat{w}_{ab} = \partial_a \left[ \partial_d x^i \partial_b x^k (\partial_i v_k - \partial_k v_i) \right] - \partial_d \left[ \partial_a x^i \partial_b x^k (\partial_i v_k - \partial_k v_i) \right]
\]

\[
= \partial_a x^i \partial_b x^k (\partial_i v_k - \partial_k v_i) - \partial_d x^i \partial_a x^k (\partial_i v_k - \partial_k v_i)
\]

\[
+ \partial_a x^i \partial_b x^k \partial_d (\partial_i v_k - \partial_k v_i) - \partial_d x^i \partial_a x^k \partial_d (\partial_i v_k - \partial_k v_i)
\]

From (A.5), we get

\[
\hat{L}^2_B W_a = g_{ea} \hat{L}^2_B W^e = \hat{L}^2_B w_a - 2 \hat{L}^2_B g_{ea} W^e - \hat{L}^2_B g_{ea} \hat{L} B W^e,
\]

and then

\[
\text{curl} \hat{L}^2_B W_{ad} = \text{curl} \hat{L}^2_B w_{ad} - 2 \left[ \partial_a (\hat{L}^2_B g_{ed} W^e) - \partial_d (\hat{L}^2_B g_{ea} W^e) \right]
\]

\[
- \left[ \partial_a (\hat{L}^2_B g_{ea} \hat{L} B W^e) - \partial_d (\hat{L}^2_B g_{ed} \hat{L} B W^e) \right]
\]

\[ (6.32) \quad - (\text{curl} \hat{L}^2_B g_{ed})_{ad} B W^e - [\hat{L}^2_B g_{ed} \partial_a L B W^e - \hat{L}^2_B g_{ea} \partial_d L B W^e]. \]

From (6.31) and (6.32), it follows that

\[
D_t \text{curl} \hat{w}_{ad} = \text{curl} \hat{L}^2_B w_{ad} - 2 (\text{curl} \hat{L}^2_B g_{ed})_{ad} W^e - (\text{curl} \hat{L}^2_B g_{ed})_{ad} L B W^e + \partial_b \hat{w}_{da} \hat{W}^b
\]
\[-2[\mathcal{L}_B^2 g e d \partial_a W^e - \mathcal{L}_B^2 g e a \partial_d W^e] + [\omega_{db} \partial_d \hat{W}^b - \omega_{ab} \partial_d \hat{W}^b] \\
+ 20 \partial_c B^b [\partial_a g_{db} - \partial_d g_{ab}] \mathcal{L}_B W^c \\
+ 2[\delta_d \partial_d x^d \partial_a B \partial_c x^i - \delta_d \partial_a x^d \partial_d B \partial_c x^i] \mathcal{L}_B W^c \\
- [\mathcal{L}_B g e a \partial_a \mathcal{L}_B W^c - \mathcal{L}_B g e a \partial_d \mathcal{L}_B W^c] \\
+ 2 \partial_c B^b [g_{ab} \partial_a B \partial_d B^c - g_{ab} \partial_d B \partial_B W^c] \tag{6.33} \]

With the help of (A.11) and (A.7), we have

\[
\langle \text{curl} \mathcal{L}_B^2 W, \text{curl} \dot{w} \rangle \\
= \int \Omega g^{ab} g^{cd} \text{curl} \mathcal{L}_B^2 W_{ad} \text{curl} \dot{w}_{bc} dy \\
= -\frac{1}{2} D_t \langle \text{curl} \mathcal{L}_B w, \text{curl} \mathcal{L}_B w \rangle + \int \Omega g^{ab} g^{cd} \text{curl} \mathcal{L}_B w_{ad} \text{curl} \mathcal{L}_B w_{bc} dy \\
- 2 \int \Omega (\mathcal{L}_B g^{ab}) g^{cd} \text{curl} \mathcal{L}_B w_{ad} \text{curl} \dot{w}_{bc} dy \\
- \int \Omega g^{ab} g^{cd} \text{curl} \mathcal{L}_B w_{ad} \mathcal{L}_B \{ \partial_c \omega_{cb} W^e + \dot{g}_{be} \partial_c W^e - \dot{g}_{ee} \partial_b W^e \\
+ [(\dot{g}_{eb} - \omega_{eb}) \partial_e \partial_c x^k - (\dot{g}_{ec} - \omega_{ec}) \partial_b \partial_f x^k] \frac{\partial y^e}{\partial x^k} W^f \} dy \\
- \int \Omega g^{ab} g^{cd} \{ 2(\text{curl} \mathcal{L}_B^2 g e d) \dot{w}_{ad} W^e + 2[\mathcal{L}_B^2 g e a \dot{w}_{ad} W^e - \mathcal{L}_B^2 g e a \partial_d W^e] \\
+ \langle \text{curl} \mathcal{L}_B g e a \rangle \partial_a \mathcal{L}_B W^e - \mathcal{L}_B g e a \partial_d \mathcal{L}_B W^e \} \rangle \text{curl} \dot{w}_{bc} dy.
\]

Let

\[E_{\text{curl}} (t) = \langle \text{curl} w, \text{curl} w \rangle + \langle \text{curl} \dot{w}, \text{curl} \dot{w} \rangle + \langle \text{curl} \mathcal{L}_B w, \text{curl} \mathcal{L}_B w \rangle.\]

Taking the inner product of (6.33) with \text{curl} \dot{w}, we obtain, with the help of (A.9), that

\[
\frac{1}{2} \frac{d}{dt} E_{\text{curl}} (t) \leq (2\|\dot{g}\|_{L^\infty (\Omega)} + \|\mathcal{L}_B g\|_{L^\infty (\Omega)}) E_{\text{curl}} \\
+ E_{\text{curl}}^{1/2} \left[ \|\mathcal{L}_B \partial \omega\|_{L^\infty (\Omega)} + \|\partial \omega\|_{L^\infty (\Omega)} \right] E_0 \\
+ E_{\text{curl}}^{1/2} \left[ \|\mathcal{L}_B \dot{g}\|_{L^\infty (\Omega)} + \|\dot{\omega}\|_{L^\infty (\Omega)} \right] \\
\cdot \left[ \|\partial W\| + \|\partial \mathcal{L}_B W\| + \|\partial^2 B\|_{L^\infty (\Omega)} \right] E_0 \\
+ E_{\text{curl}}^{1/2} \left[ \|\mathcal{L}_B \dot{g}\|_{L^\infty (\Omega)} + \|\mathcal{L}_B \omega\|_{L^\infty (\Omega)} \right] \left[ \|\mathcal{L}_B \partial^2 x\|_{L^\infty (\Omega)} \right] \frac{\partial y}{\partial x} \left\|_{L^\infty (\Omega)} \right] E_0 \\
+ E_{\text{curl}}^{1/2} \langle \|\dot{g}\|_{L^\infty (\Omega)} + \|\omega\|_{L^\infty (\Omega)} \rangle \left[ \|\mathcal{L}_B \partial^2 x\|_{L^\infty (\Omega)} \right] \frac{\partial y}{\partial x} \left\|_{L^\infty (\Omega)} \right] E_0 \\
+ \frac{1}{2} \|\partial^2 x\|_{L^\infty (\Omega)} \left[ \mathcal{L}_B \partial \frac{\partial y}{\partial x} \right\|_{L^\infty (\Omega)} + 2 \|\partial^2 x\|_{L^\infty (\Omega)} \right] \frac{\partial y}{\partial x} \left\|_{L^\infty (\Omega)} \right] E_0 \\
+ E_{\text{curl}}^{1/2} \left[ 2 \|\text{curl} \mathcal{L}_B^2 g\|_{L^\infty (\Omega)} + \|\text{curl} \mathcal{L}_B g\|_{L^\infty (\Omega)} \right] E_0 \\
+ E_{\text{curl}}^{1/2} \left[ 4 \|\mathcal{L}_B^2 g\|_{L^\infty (\Omega)} \|\partial W\| + 2 \|\mathcal{L}_B g\|_{L^\infty (\Omega)} \|\partial \mathcal{L}_B W\| \right] \\
+ 4E_{\text{curl}}^{1/2} \left[ \|\delta_d \partial_d x^d \partial_a (B \partial_c x^i) + \partial_a (g_{db} \partial_c B^b)\|_{L^\infty (\Omega)} \right] E_0.
\]
\[
\partial W \leq K_1 \left( \| \text{curl } w \| + \sum_{\mathcal{S} \in \mathcal{S}} \| \mathcal{L}_S W \| + \|[g]_1\|_{L^\infty(\Omega)} \| W \| \right)
\]
\[(6.35)\]
\[
\partial \dot{W} \leq K_1 \left( \| \text{curl } \dot{w} \| + \sum_{\mathcal{S} \in \mathcal{S}} \| \mathcal{L}_S \dot{W} \| + \|[g]_1\|_{L^\infty(\Omega)} \| \dot{W} \| \right)
\]
\[(6.36)\]

Then, from
\[
\mathcal{L}_B W_a = g_{ab} \mathcal{L}_B W^b = \mathcal{L}_B w_a - (\mathcal{L}_B g_{ab}) W^b,
\]
we have
\[
\| \partial \mathcal{L}_B W \| \leq K_1 \left( \| \text{curl } \mathcal{L}_B W \| + \sum_{\mathcal{S} \in \mathcal{S}} \|[\mathcal{L}_S, \mathcal{L}_B] W \| + E_1^S + \|[g]_1\|_{L^\infty(\Omega)} E_0 \right)
\]
\[
\leq K_1 \left( \| \text{curl } \mathcal{L}_B W \| + (\|[B]_1^S \| + \| \mathcal{L}_B g \|_{L^\infty(\Omega)}) \| \text{curl } w \| + E_1^S \right.
\]
\[
\left. + (\|[g]_1\|_{L^\infty(\Omega)} + \| \partial \mathcal{L}_B g \|_{L^\infty(\Omega)}) E_0 \right).
\]
\[(6.37)\]

Combining (6.34), (6.35), (6.36) and (6.37), we obtain
\[
\frac{d}{dt}(E_{\text{curl}}^{1/2}(t)) \leq n_{1,\text{curl}}(E_{\text{curl}}^{1/2} + E_1^S) + \tilde{n}_{1,\text{curl}} E_0 + \| \text{curl } \mathcal{F} \|,
\]
where
\[
n_{1,\text{curl}} = K_1 \left( 1 + \|[B]_1^S \| + \| \mathcal{L}_B g \|_{L^\infty(\Omega)} \right) \left( \| \mathcal{L}_B \dot{g} \|_{L^\infty(\Omega)} + \| \dot{g} \|_{L^\infty(\Omega)} \right)
\]
\[
+ \| \mathcal{L}_B^2 g \|_{L^\infty(\Omega)} + \| \mathcal{L}_B g \|_{L^\infty(\Omega)} + \| g_{ab} \partial_d x^d B \partial_c x^c \|
\]
\[
+ \| g_{ab} \partial_c B^b \|_{L^\infty(\Omega)} + \| \omega \|_{L^\infty(\Omega)} \right),
\]
\[
\tilde{n}_{1,\text{curl}} = n_{1,\text{curl}} \left( \|[g]_1\|_{L^\infty(\Omega)} + \| \partial \mathcal{L}_B g \|_{L^\infty(\Omega)} \right)
\]
\[
+ K_1 \left( \| \mathcal{L}_B \partial \omega \|_{L^\infty(\Omega)} + \| \partial \omega \|_{L^\infty(\Omega)} + \| \mathcal{L}_B \dot{g} \|_{L^\infty(\Omega)} \right)
\]
\[
+ \| \dot{g} \|_{L^\infty(\Omega)} \| \mathcal{L}_B \|_{L^\infty(\Omega)} \left. \right) \left( \| \mathcal{L}_B \partial^2 x \|_{L^\infty(\Omega)} \left\| \frac{\partial y}{\partial x} \right\|_{L^\infty(\Omega)} \right)
\]
\[
+ \| \mathcal{L}_B \dot{g} \|_{L^\infty(\Omega)} \left\| \frac{\partial y}{\partial x} \right\|_{L^\infty(\Omega)} + 2 \| \partial^2 x \|_{L^\infty(\Omega)} \left( \frac{\partial y}{\partial x} \right)_{L^\infty(\Omega)} \right)
\]
\[
+ \| \text{curl } \mathcal{L}_B^2 g \|_{L^\infty(\Omega)} + \| \text{curl } \mathcal{L}_B g \|_{L^\infty(\Omega)}
\]
Due to $E_{\text{curl}}(0) = 0$, the integration over $[0, t]$ in time gives

$$E_{\text{curl}}^{1/2}(t) \leq \int_0^t \left[ n_{1, \text{curl}}(E_{\text{curl}}^{1/2} + E_1^S) + \tilde{n}_{1, \text{curl}} E_0 + \|\text{curl} F\|\right] d\tau.$$

From (6.28) and (6.38), we have

$$E_1^S + E_{\text{curl}}^{1/2} \leq \int_0^t (n_{1, \text{curl}} + n_1^S + \tilde{n}_{1, \text{curl}}) (E_{\text{curl}}^{1/2} + E_1^S) d\tau + \tilde{f}_1,$$

where

$$\tilde{n}_{1, \text{curl}} = K_1 \|g\|_{L^\infty(\Omega)} \|B\|_1^S (1 + \|B\|_1^S + \|B\|_{W^{1, \infty}(\Omega)} + \|\delta_t \partial x^i B \partial x^i\|_{L^\infty(\Omega)}),$$

$$\tilde{f}_1 = \tilde{n}_1^S + \int_0^t (\tilde{n}_{1, \text{curl}} + \tilde{n}_{1, \text{curl}}) \|\text{curl} \mathcal{L}_B g\|_{L^\infty(\Omega)} E_0 d\tau + 2 \int_0^t \left[ \sum_{T \in S} f_T^T + \|\text{curl} F\|\right] d\tau.$$

Therefore, by the Gronwall inequality, we have obtained the following estimates for both the first order tangential derivatives and the curl.

**Proposition 6.10.** It holds

$$E_1^S(t) + E_{\text{curl}}^{1/2}(t) \leq \tilde{f}_1(t) + \int_0^t \tilde{f}_1(s) \left[ n_{1, \text{curl}}(s) + n_1^S(s) + \tilde{n}_{1, \text{curl}}(s) \right]$$

$$\cdot \exp \left( \int_s^t (n_{1, \text{curl}}(\tau) + n_1^S(\tau) + \tilde{n}_{1, \text{curl}}(\tau)) d\tau \right) ds.$$

**Remark 6.11.** By Lemma 6.4, we have the estimates for the first-order derivative of $W$.

### 6.8 The higher-order estimates for the curl and the normal derivatives

Now, we need to get the equations for the curl of higher order derivatives. Since the Lie derivative commutes with $D_t$ and the curl, applying $\mathcal{L}_U^j$ to (6.33) and (6.29) gives

$$D_t \text{curl} \mathcal{L}_U^j \hat{w}_{ad} = \text{curl} \mathcal{L}_U^j \mathcal{L}_B^2 \hat{w}_{ad} + \text{curl} \mathcal{L}_U^j \mathcal{E}_{ad} - 2c_{J_1 J_2} (\text{curl} \mathcal{L}_U^j \mathcal{L}_B^2 g_{e_\alpha})_{ad} \mathcal{L}_U^j W^e$$

$$- c_{J_1 J_2} (\text{curl} \mathcal{L}_U^j \mathcal{L}_B g_{e_\alpha})_{ad} \mathcal{L}_U^j \mathcal{L}_B W^e + c_{J_1 J_2} \mathcal{L}_U^j \partial_a \omega_{da} \mathcal{L}_U^j \tilde{W}^b$$

$$- 2c_{J_1 J_2} \mathcal{L}_U^j \mathcal{L}_B \omega_{ed} \mathcal{L}_U^j \tilde{W}^e - \mathcal{L}_U^j \mathcal{L}_B \omega_{ed} \mathcal{L}_U^j \tilde{W}^e - \mathcal{L}_U^j \mathcal{L}_B \omega_{ed} \mathcal{L}_U^j \tilde{W}^e$$

$$+ c_{J_1 J_2} \mathcal{L}_U^j \omega_{db} \mathcal{L}_U^j \tilde{W}^b - \mathcal{L}_U^j \omega_{ad} \mathcal{L}_U^j \tilde{W}^b + \mathcal{L}_U^j \omega_{ad} \mathcal{L}_U^j \tilde{W}^b$$

$$+ 2c_{J_1 J_2} \mathcal{L}_U^j \partial_a B^b (\partial_a g_{db} - \partial_d g_{ab}) \mathcal{L}_U^j \mathcal{L}_B W^c$$

$$+ 2c_{J_1 J_2} \mathcal{L}_U^j (g_{db} \partial_a B^b - g_{ab} \partial_d B^b) \mathcal{L}_U^j \mathcal{L}_B W^c.$$
At this point, we have to derive the commutator $[\mathcal{L}_U^J, \partial]$. If $|J| = 1$, it is just the identity (A.9). For $|J| \geq 2$, we have the following identity.

Lemma 6.12. For $|J| = r \geq 1$ and $|J_r| = 1$, it holds

$$[\mathcal{L}_U^J, \partial_a] W^b = \mathcal{L}_U^J \partial_a W^b + \sum_{I_1 + I_2 + I_3 = |J|} \text{sgn}(I_1) \mathcal{L}_U^{I_1} \mathcal{L}_U^{I_2} \partial_a U^{I_3}_b.$$  

Proof. For $r = 1$, it follows from (A.4)

$$[\mathcal{L}_U, \partial_a] W^b = \partial_a U^b W^c.$$  

For $r \geq 2$, we prove it by induction argument. For $r = 2$, we have

$$[\mathcal{L}_U, \mathcal{L}_U^J, \partial_a] W^b = \mathcal{L}_U [\mathcal{L}_U^J, \partial_a] W^b + [\mathcal{L}_U, \partial_a] \mathcal{L}_U^J W^b = \mathcal{L}_U (W^d \partial_d U^b_2) + \mathcal{L}_U W^d \partial_a U^b_1 = (\mathcal{L}_U W^d) \partial_d U^b_2 + (\mathcal{L}_U W^d) \partial_d U^b_1 + W^d \mathcal{L}_U \partial_d U^b_2,$$

which satisfies (6.41).

Now, we assume that (6.41) holds for $r = s$. Then, we derive the case $r = s + 1$. For $|J| = s + 1$ and $|J_r| = 1$, one gets by using (A.9)

$$[\mathcal{L}_U^{J-s+1}, \partial_a] W^b = \mathcal{L}_U^{J-s+1} [\mathcal{L}_U^{J-s+1}, \partial_a] W^b + [\mathcal{L}_U^{J-s+1}, \partial_a] \mathcal{L}_U^{J-s+1} W^b$$
\[
\begin{align*}
\mathcal{L}_U^{J-J+1} & = \mathcal{L}_U^{J-J+1} (W^d \partial_d \partial_a U_{J+1}) + \mathcal{L}_U^{J-J+1} W^d \mathcal{L}_U^{J-J+1} \partial_d \partial_a U_{J+1} \\
& + \sum_{J-J+1 = J+1, J_3 = 1} \mathcal{L}_U^{J} \mathcal{L}_U^{J+1} W^d \mathcal{L}_U^{J} \partial_d \partial_a U_{J+1} \\
& = \sum_{J-J+1 = J+1, J_3 = 1} \mathcal{L}_U^{J} W^d \mathcal{L}_U^{J} \partial_d \partial_a U_{J+1} + \mathcal{L}_U^{J+1} W^d \mathcal{L}_U^{J+1} \partial_d \partial_a U_{J+1} \\
& + \sum_{J-J+1 = J+1, J_3 = 1} \mathcal{L}_U^{J} \mathcal{L}_U^{J+1} W^d \mathcal{L}_U^{J} \partial_d \partial_a U_{J+1} \\
& = W^d \mathcal{L}_U^{J-J+1} \partial_d \partial_a U_{J+1} + \sum_{J-J+1 = J+1, J_3 = 1} \mathcal{L}_U^{J} W^d \mathcal{L}_U^{J} \partial_d \partial_a U_{J+1},
\end{align*}
\]

which is of the form in (6.41) with \( r = s + 1 \). Thus, we proved the identity by induction.

For \( U \in \mathcal{U} \) and \( |J| = r - 1 \), let
\[
E_{r-1, \text{curl}} (t) = \langle \text{curl} \mathcal{L}_U^J w, \text{curl} \mathcal{L}_U^J w \rangle + \langle \text{curl} \mathcal{L}_U^J \hat{w}, \text{curl} \mathcal{L}_U^J \hat{w} \rangle
\]
\[
+ \langle \text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w, \text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w \rangle.
\]

Then, from (6.40) it follows

\[
\frac{d}{dt} E_{r, \text{curl}} (t) = \int_\Omega \hat{g}^{ab} g^{cd} (\text{curl} \mathcal{L}_U^J w) (\text{curl} \mathcal{L}_U^J w) + \int_\Omega \text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w (\text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w) dy
\]
\[
- \int_\Omega \mathcal{L}_B (g^{ab} g^{cd}) \text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w d\text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w dy
\]
\[
+ \langle \text{curl} \mathcal{L}_U^J \hat{w}, \text{curl} \mathcal{L}_U^J \hat{w} \rangle + \langle \text{curl} \mathcal{L}_U^J \hat{w}, D_\epsilon \text{curl} \mathcal{L}_U^J \hat{w} - \epsilon \mathcal{L}_B^2 \mathcal{L}_U^J \hat{w} \rangle
\]
\[
+ \langle \text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w, c_{J_1, J_2} \mathcal{L}_B^J \mathcal{L}_U^J \partial_a \omega_{ab} \mathcal{L}_U^J W^c + (\mathcal{L}_U^J \hat{g}_{bc} [\mathcal{L}_U^J , \partial_a] W^c - \mathcal{L}_U^J \hat{g}_{ac} [\mathcal{L}_U^J , \partial_b] W^c)
\]
\[
+ (\mathcal{L}_U^J \hat{g}_{bc} \partial_a \mathcal{L}_U^J W^c - \mathcal{L}_U^J \hat{g}_{ac} \partial_b \mathcal{L}_U^J W^c)
\]
\[
+ \mathcal{L}_B^J \{ [(\hat{g}_{eb} - \hat{g}_{eb}) \partial_a \hat{e}_c x^k - (\hat{g}_{ea} - \omega_{ea}) \partial_b \hat{e}_c x^k] \partial y^e \partial x^c \} \mathcal{L}_U^J W^c \}
\]
\[
+ \langle \text{curl} \mathcal{L}_B^J \mathcal{L}_U^J w, c_{J_1, J_2} \mathcal{L}_B^J \mathcal{L}_U^J \partial_a \omega_{ab} \mathcal{L}_U^J \mathcal{L}_U^J W^c + (\mathcal{L}_U^J \hat{g}_{bc} [\mathcal{L}_U^J , \partial_a] W^c
\]
\[
- \mathcal{L}_U^J \hat{g}_{ac} [\mathcal{L}_U^J , \partial_b] W^c) + (\mathcal{L}_U^J \hat{g}_{bc} \partial_a \mathcal{L}_U^J \mathcal{L}_U^J W^c - \mathcal{L}_U^J \hat{g}_{ac} \partial_b \mathcal{L}_U^J \mathcal{L}_U^J W^c)
\]
\[
+ \mathcal{L}_B^J \{ [(\hat{g}_{eb} - \hat{g}_{eb}) \partial_a \hat{e}_c x^k - (\hat{g}_{ea} - \omega_{ea}) \partial_b \hat{e}_c x^k] \partial y^e \partial x^c \} \mathcal{L}_U^J W^c \}
\].

Thus, by (6.39) and Lemma 6.1, we get

\[
\frac{d}{dt} E_{r-1, \text{curl}} (t) \leq (1 + \| \hat{g} \|_{L^\infty (\Omega)} + \| \mathcal{L}_B g \|_{L^\infty (\Omega)}) E_{r, \text{curl}}^{1/2} + \| \text{curl} \mathcal{L}_U^J \mathcal{L}_U^J \mathcal{L}_B^J \|_{L^\infty (\Omega)} \| \mathcal{L}_U^J \mathcal{L}_U^J \mathcal{L}_B^J \|
\]
\[
(6.43) + \| \text{curl} \mathcal{L}_U^J \mathcal{L}_U^J \mathcal{L}_B^J \|_{L^\infty (\Omega)} + \sum_{c_{J_1, J_2}} \| \text{curl} \mathcal{L}_U^J \mathcal{L}_U^J \mathcal{L}_B^J \|_{L^\infty (\Omega)} \| \mathcal{L}_U^J \mathcal{L}_U^J \mathcal{L}_B^J \|
\]
\[
+ \| \mathcal{L}_U^J \partial \omega \|_{L^\infty(\Omega)} + \| \mathcal{L}_U^J \cdot \mathbf{W} \| + \| \text{curl} \mathcal{L}_U^J \mathcal{L}_B \|_{L^\infty(\Omega)} (\| \mathcal{L}_B \mathcal{L}_U^J \mathbf{W} \|)
\]

\[ (6.44) \]
\[
+ K_1 e^{J_1, J_2} \| B_{J_1} \|_{L^\infty(\Omega)} (\| \text{curl} \mathcal{W}_{J_2} \| + \sum_{S \in S} \| \mathcal{L}_S \mathbf{W}_{J_2} \|) + \| W_{J_2} \|
\]

\[
\cdot \left( \| \text{curl} \mathcal{B}_{J_2} \|_{L^\infty(\Omega)} + \sum_{S \in S} \| \mathcal{L}_S \mathcal{B}_{J_2} \|_{L^\infty(\Omega)} + \| [g]_1 \|_{L^\infty(\Omega)} \| B_{J_1} \|_{L^\infty(\Omega)} \| W_{J_2} \|) \right)
\]

\[ (6.45) \]
\[
+ (\| \mathcal{L}_U^J \mathcal{L}_B \|_{L^\infty(\Omega)} + \left( \| \mathcal{L}_U^J \omega \|_{L^\infty(\Omega)} (\| \mathcal{L}_U^J \cdot \mathbf{W} \| + \| \text{curl} \mathcal{L}_U^J \mathbf{W} \|)
\]

\[ (6.46) \]
\[
+ \left( \| \text{curl} \mathcal{L}_U^J \mathcal{L}_B \|_{L^\infty(\Omega)} + \left( \| \text{curl} \mathcal{L}_U^J \mathcal{L}_B \|_{L^\infty(\Omega)} + \| \mathcal{L}_U^J \partial \mathbf{W} \| + \sum_{S \in S} \| \mathcal{L}_S \mathcal{L}_U^J \mathcal{L}_B \mathbf{W} \| + \| [g]_1 \|_{L^\infty(\Omega)} \| B_{J_1} \|_{L^\infty(\Omega)} \| W_{J_2} \|) \right)
\]

\[ (6.47) \]
\[
\cdot \left( \| \mathcal{L}_U^J \partial \mathbf{W} \| + \| \mathcal{L}_B \mathcal{L}_U^J \mathbf{W} \| + \| \mathcal{L}_B \mathcal{L}_U^J \mathbf{W} \| \right)
\]

We first consider the term \( |\text{curl} [\mathcal{L}_U^J, \mathcal{L}_B^2] w| \) in the line labeled (6.43). It holds

\[
[\mathcal{L}_U^J, \mathcal{L}_B^2] w = [\mathcal{L}_U^J, \mathcal{L}_B] \mathcal{L}_D w_a + L_B [\mathcal{L}_U^J, \mathcal{L}_B] w_a
\]

\[
= \mathcal{L}_U^J [L^J_1, L^J_2] \mathcal{L}_B \mathcal{L}_D w_a + \mathcal{L}_B [\mathcal{L}_U^J, \mathcal{L}_B] \mathcal{L}_D w_a
\]

\[
= \mathcal{L}_U^J [L^J_1, L^J_2] \mathcal{L}_B \mathcal{L}_D w_a + \mathcal{L}_B [\mathcal{L}_U^J, \mathcal{L}_B] \mathcal{L}_D w_a
\]

\[
(\text{curl} [\mathcal{L}_U^J, \mathcal{L}_B^2] w)_{ad}
\]
\[
\begin{align*}
&= c_I^{l_1l_2} \partial_a B_{l_1}^a \partial_b L_{U}^{l_2} L_B w_d - \partial_d B_{l_1}^d \partial_b L_{U}^{l_2} L_B w_a + B_{l_1}^d \partial_b (\text{curl } L_{U}^{l_2} L_B w)_{ad} \\
&+ (\partial_d B_{l_1}^d) \partial_b (L_{U}^{l_2} L_B w_c + L_B L_{U}^{l_2} w_c) - (\partial_d B_{l_1}^d) \partial_d (L_{U}^{l_2} L_B w_c + L_B L_{U}^{l_2} w_c) \\
&+ \partial_a L_B B_{l_1}^a \partial_b L_{U}^{l_2} w_d - \partial_d L_B B_{l_1}^d \partial_b L_{U}^{l_2} w_a + \partial_a L_B B_{l_1}^a \partial_b L_{U}^{l_2} w_d - \partial_d L_B B_{l_1}^d \partial_b L_{U}^{l_2} w_a \\
&+ L_B B_{l_1}^d \partial_b (\text{curl } L_{U}^{l_2} w)_{ad} + B_{l_1}^d \partial_a L_B \partial_b L_{U}^{l_2} w_d - B_{l_1}^d \partial_d L_B \partial_b L_{U}^{l_2} w_a \\
&+ (\partial_a L_B \partial_d B_{l_1}^d - \partial_d L_B \partial_a B_{l_1}^d) L_{U}^{l_2} w_c + (\partial_B \partial_d B_{l_1}^d \partial_a L_{U}^{l_2} w_c - (\partial_B \partial_d B_{l_1}^d) \partial_d L_{U}^{l_2} w_c).
\end{align*}
\]

Since

\[
[\mathcal{L}_B, \partial_a] w_b = -\partial_a \partial_b B^c w_c,
\]

one gets

\[
|\text{curl } [\mathcal{L}_U^{l_1}, \mathcal{L}_B^{l_2}] w| \\
\leq K_1 c_I^{l_1l_2} |\partial B_{l_1}| (|\partial L_{U}^{l_2} L_B w| + |\partial B_{l_1} L_{U}^{l_2} w| + |\partial^2 B| |L_{U}^{l_2} w|) \\
+ |B_{l_1}| |\partial (\text{curl } L_{U}^{l_2} L_B w)| + |L_B B_{l_1}| |\partial (\text{curl } L_{U}^{l_2} w)| + |B_{l_1}| (|\partial^2 L_B L_{U}^{l_2} w| \\
+ |\partial^3 B| |L_{U}^{l_2} w| + |\partial^2 B| |\partial L_{U}^{l_2} w|) + (|\partial B_B B_{l_1}| + |\partial^2 B| |B_{l_1}|) |\partial L_{U}^{l_2} w| \\
+ (|\partial^2 L_B B_{l_1}| + |\partial^3 B| |B_{l_1}| + |\partial^2 B| |\partial B_{l_1}|) |\partial L_{U}^{l_2} w|).
\]

Due to

\[
L_{U}^{l_2} w_a = L_{U}^{l_2} (g_{ab} W^b) = g_{ab} L_{U}^{l_2} W^b + c_{j_1j_2}^l g_{ab} L_{U}^{l_2} W^b,
\]

where the sum is over all \( J_1 + J_2 = J \), and \( c_{j_1j_2}^l = 1 \) for \( |J_2| < |J| \), and \( c_{j_1j_2}^l = 0 \) for \( |J_2| = |J| \), from Lemma 6.3 it follows that

\[
|L_{U}^{l_2} w| \leq |g| |L_{U}^{l_2} W| + c_{j_1j_2}^l |g_{j_1j_2} L_{U}^{l_2} W|, \\
|\partial L_{U}^{l_2} w| \leq |g| |L_{U}^{l_2} W| + |\partial g| |L_{U}^{l_2} W| + c_{j_1j_2}^l |g_{j_1j_2} L_{U}^{l_2} W| + c_{j_1j_2}^l |\partial g |L_{U}^{l_2} W| \\
\leq K_1 |g| + |\partial g| \left( |\text{curl } L_{U}^{l_2} W| + \sum_{S \in S} |L_S L_{U}^{l_2} W| + |g_1| L_{U}^{l_2} W \right) \\
+ K_1 c_{j_1j_2}^l |\partial g |L_{U}^{l_2} W| + |\partial g |L_{U}^{l_2} W|,
\]

and

\[
|\partial L_{U}^{l_2} L_B w_a| = |\partial L_{U}^{l_2} (L_B g_{ab} W^b + g_{ab} L_B W^b)| \\
\leq c_{j_1j_2}^l (|\partial L_{U}^{l_2} L_B g||L_{U}^{l_2} W| + |L_{U}^{l_2} L_B g||\partial L_{U}^{l_2} W| \\
+ |\partial L_{U}^{l_2} g||L_{U}^{l_2} L_B W| + |L_{U}^{l_2} g||\partial L_{U}^{l_2} L_B W|) \\
\leq K_1 c_{j_1j_2}^l \left( |\text{curl } L_{U}^{l_2} W| + \sum_{S \in S} |L_S L_{U}^{l_2} W| + |g_1| L_{U}^{l_2} W \right) \\
+ K_1 c_{j_1j_2}^l (|\partial L_{U}^{l_2} g| + |\partial g |L_{U}^{l_2} W|) \left( |\text{curl } L_{U}^{l_2} W| + \sum_{S \in S} |L_S L_{U}^{l_2} W| + |g_1| L_{U}^{l_2} W \right) \\
+ |g_1| c_{j_1j_2}^l \left( |L_{U}^{l_2} B| \left( |\text{curl } L_{U}^{l_2} W| + \sum_{S \in S} |L_S L_{U}^{l_2} W| + |g_1| L_{U}^{l_2} W \right) \\
+ \left( |\text{curl } L_{U}^{l_2} B| + \sum_{S \in S} |L_S L_{U}^{l_2} B| \right) |L_{U}^{l_2} W| \right),
\]
since
\begin{equation}
(6.48) \quad \| [L^I_T, L^B_T] W \| \leq c_1^{I_1 I_2} [ |B_{I_1}| |\partial W_{I_2}| + |\partial B_{I_1}| |W_{I_2}| ] \\
\leq c_1^{I_1 I_2} K_1 \left[ |B_{I_1}| \left( \sum_{S \in R} |L^S_{W I_2}| + |W_{I_2}| \right) + |\partial B_{I_1}| |W_{I_2}| \right].
\end{equation}

Now, we have to express the term, like \(|\text{curl} W_I| = |\text{curl} L^I_U W|\) in the above inequality and in the line labeled (6.44) and other lines, in term of \(w\). By Lemma 6.4, we have
\[
|\text{curl} L^I_U W_a | = |\text{curl} g_{ab} L^I_U W^b | \\
\leq |\text{curl} L^I_U w_a | + c_1^{I_1 I_2} |\text{curl} (g_{ab} L^I_U W^b) | \\
\leq |\text{curl} L^I_U w_a | + c_1^{I_1 I_2} |\partial_d (g_{ab} L^I_U W^b) | - |\partial_a (g_{ab} L^I_U W^b) | \\
\leq |\text{curl} L^I_U w_a | + 2 c_1^{I_1 I_2} (|\partial g^{I_1} | L^I_U W | + |g^{I_1} | |\partial L^I_U W |) \\
\leq |\text{curl} L^I_U w_a | + K_1 c_1^{I_1 I_2} (|\partial g^{I_1} | + |g^{I_1} |) \left( |\text{curl} L^I_U W | + \sum_{S \in S} |L^S L^I_{W} W | + |g_{I_1} | L^I_{W} W | \right).
\]

The term \(|\text{curl} L^I_U L^B_B W|\) in (6.47) can be estimated as a similar argument as above by regarding \(B\) as a tangential vector field of form \(U\). By (6.41), for the term \(|L^I_U, \partial W|\) in (6.45), we get
\[
||L^I_U, \partial W|| \leq |W| ||L^I_{U, J_2 - J_2, |J_2|, |J_2|} \partial^2 U_{J_2, |J_2|} | + \sum_{J_2 = J_1 + J_2 + J_3} \text{sgn}(|I_1|) ||L^I_{U, \partial^2 U_{I_3}} ||L^I_{U, W} |,
\]
and a similar estimate holds for the term \(|L^I_U, \partial W|\) in (6.46). Similarly, for the term \(|L^I_U, \partial L^B_B W|\) in (6.47), we have with the help of (6.48)
\[
||L^I_U, \partial L^B_B W|| \leq |L^B_B W| ||L^I_{U, J_2 - J_2, |J_2|, |J_2|} \partial^2 U_{J_2, |J_2|} | + \sum_{J_2 = J_1 + J_2 + J_3} \text{sgn}(|I_1|) ||L^I_{U, \partial^2 U_{I_3}} ||L^I_{U, W} | \\
\cdot \left[ |L^B_B L^I_U W | + K_1 c_1^{I_2 I_2} |B_{I_1}| \left( |\text{curl} W_{I_2} | + \sum_{S \in S} |L^S W_{I_2} | + |g_{I_2} | W_{I_2} \right) \\
+ \left( |\text{curl} B_{I_1} | + \sum_{S \in S} |L^S B_{I_1} | \right) |W_{I_2} | \right].
\]

For the term \(|L^S L^I_{W} L^I_{B} W|\), we can use (6.48) to get estimates.

For convenience, we introduce some new norms and notation.

**Definition 6.13.** For any family \(\mathcal{V}\) of our families of vector fields, let
\[
\| W \|_{\gamma \mathcal{V}} = \| W(t) \|_{\gamma \mathcal{V}, \Omega, t} = \sum_{|I| \leq r, T \in \mathcal{V} \cap \mathcal{V}} \left( \int_{\Omega} |L^I_{U} W(t, y)|^2 dy \right)^{1/2},
\]
\[
\| W \|_{\gamma \mathcal{V}, B} = \| W(t) \|_{\gamma \mathcal{V}, \Omega, t} = \sum_{|I| \leq r, T \in \mathcal{V \cap \gamma \mathcal{V}}} \left( \int_{\Omega} |L^I_{B} L^I_{U} W(t, y)|^2 dy \right)^{1/2},
\]
and

\[
C_r^V = \sum_{|J| \leq r-1, J \in \mathcal{V}} \left( \int_\Omega (|\text{curl} \mathcal{L}_B^J \omega|^2 + |\text{curl} \mathcal{L}_B^J \omega|^2 + |\text{curl} \mathcal{L}_B^J \omega|^2) dy \right)^{1/2},
\]

\[
C_0^V = 0.
\]

Note that the norm \( ||W(t)||_{\mathcal{R}^r(\Omega)} \) is equivalent to the usual Sobolev norm in the Lagrangian coordinates.

**Definition 6.14.** For \( \mathcal{V} \) any of our families of vector fields and \( \beta \) a function, a 1-form, a 2-form, or a vector field, let \( ||\beta||^V_s \) be as in Definition 6.5 and set

\[
||\beta||^V_{s, \infty} = ||\beta||^V_s \|L^\infty(\Omega),
\]

\[
[\beta]^V_s = \sum_{s_1 + \cdots + s_k \leq s, s_i \geq 1} ||\beta||^V_{s_1, \infty} \cdots ||\beta||^V_{s_k, \infty}, \quad [\beta]^V_{0, \infty} = 1,
\]

where the sum is over all combinations with \( s_i \geq 1 \). Furthermore, let

\[
m^V_r = [\beta]^V_{r, \infty},
\]

\[
\bar{m}^V_r = \sum_{s + u \leq r} \sum_{s_1 + \cdots + s_k \leq s, s_i \geq 1} ||\beta||^V_{s_1, \infty} \cdots ||\beta||^V_{s_k, \infty},
\]

\[
\bar{m}^V_r = \sum_{s \leq r} \left( \|B\|_{s+2, \infty} + \|\partial x\|_{s, \infty} + \|\partial^2 x\|_{s, \infty} + \|\partial y\|_{s, \infty} \right).
\]

Let \( F_{r, \text{curl}}^\mathcal{U} = ||\text{curl} \mathcal{U}||_{\mathcal{U}^r(\Omega)} \). Then, it follows from the above arguments in this subsection that

\[
\left| \frac{d}{dt} C_r^{\mathcal{U}} \right| \leq K_1 \sum_{s=0}^{r} \left( \bar{m}^{\mathcal{U}}_{r-s} + \bar{m}^{\mathcal{U}}_{r-s} \right) (C_s^{\mathcal{U}} + E_s^T) + F_{r, \text{curl}}^\mathcal{U},
\]

where \( E_s^T \) is the energy of the tangential derivatives defined in (6.17). Here, we note that the same inequalities hold with \( \mathcal{U} \) and \( \mathcal{T} \) replaced by \( \mathcal{R} \) and \( \mathcal{S} \), respectively. Thus, by the Gronwall inequality, we obtain for \( r \geq 1 \)

\[
C_r^{\mathcal{U}} \leq K_1 e^{K_1 (\bar{m}^{\mathcal{U}}_0 + \bar{m}^{\mathcal{U}}_0) dx} \int_0^t \left( \text{sgn}(r-1) \sum_{s=1}^{r-1} (\bar{m}^{\mathcal{U}}_{r-s} + \bar{m}^{\mathcal{U}}_{r-s}) C_s^{\mathcal{U}} + \sum_{s=0}^{r} (\bar{m}^{\mathcal{U}}_{r-s} + \bar{m}^{\mathcal{U}}_{r-s}) E_s^T + F_{r, \text{curl}}^\mathcal{U} \right) d\tau.
\]

Since we have already proved a bound for \( E_s^T \) in Proposition 6.8, it inductively follows that \( C_r^{\mathcal{U}} \) is bounded. By Lemma 6.6, we obtain

\[
\|W(t)\|_{\mathcal{U}^r(\Omega)} + \|\dot{W}(t)\|_{\mathcal{U}^r(\Omega)} + \|\mathcal{L}_B W(t)\|_{\mathcal{U}^r(\Omega)} \leq K_1 \sum_{s=0}^{r} m^{\mathcal{U}}_{r-s} (C_s^{\mathcal{U}} + E_s^T).
\]

Therefore, we have the following estimates.
Proposition 6.15. Suppose that $x, P \in C^{r+2}(0, T] \times \Omega$, $B \in C^{r+2}(\Omega)$, $P|_{\Gamma} = 0$, $\nabla_{N} P|_{\Gamma} \leq -c_{0} < 0$, $B^{a} N_{a}|_{\Gamma} = 0$ and $\text{div} V = 0$, where $V = D_{t} x$. Then, there is a constant $C = C(x, P, B)$ depending only on the norm of $(x, P, B)$, a lower bound for $c_{0}$, and an upper bound for $T$ such that if $E_{T}^{T}(0) = C_{U}^{T}(0) = 0$ for $s \leq r$, then

$$\|W\|_{U}^{T} + \|\dot{W}\|_{U}^{T} + \|\mathcal{L}_{B} W\|_{U}^{T} + E_{T}^{T} \leq C \int_{0}^{T} \|F\|_{U}^{T} d\tau,$$

for $t \in [0, T]$.

7 The smoothed-out equation and existence of weak solutions

7.1 The smoothed-out normal operator

In order to prove the existence of solutions, we need to replace the normal operator $\mathcal{A}$ by a sequence $\mathcal{A}_{\varepsilon}$ of bounded symmetric and positive operators that converge to $\mathcal{A}$ as $\varepsilon \to 0$.

Let $\rho = \rho(d)$ be a smooth function of the distance $d = d(y) = \text{dist}(y, \Gamma)$ such that $\rho' \geq 0$, $\rho(d) = d$ for $d \leq \frac{1}{4}$, and $\rho(d) = \frac{1}{2}$ for $d \geq \frac{3}{4}$.

Let $\chi(\rho)$ be a smooth function such that $\chi'(\rho) \geq 0$, $\chi(\rho) = 0$ for $\rho \leq \frac{1}{4}$, and $\chi(\rho) = 1$ for $\rho \geq \frac{3}{4}$.

For a function $f$ vanishing on the boundary, we define

$$\mathcal{A}_{\varepsilon}^{f} W^{a} = \mathbb{P}(-g^{ab} \chi_{\varepsilon}(\rho) \partial_{b} (f \rho^{-1}(\partial_{c} \rho) W^{c})), $$

where $\chi_{\varepsilon}(\rho) = \chi(\rho/\varepsilon)$. The integration by parts gives

$$\langle U, \mathcal{A}_{\varepsilon}^{f} W \rangle = \int_{\Omega} f \rho^{-1} \chi'_{\varepsilon}(\rho) (U^{a} \partial_{a} \rho) (W^{b} \partial_{b} \rho) dy,$$

which yields the symmetry of $\mathcal{A}_{\varepsilon}^{f}$. In particular, $\mathcal{A}^{f} = \mathcal{A}^{f}_{\rho}$ is positive if $P \geq 0$, at least close to the boundary, i.e.,

$$\langle W, \mathcal{A}^{f} W \rangle \geq 0.$$

We have another expression for $\mathcal{A}_{\varepsilon}^{f}$:

$$\mathcal{A}_{\varepsilon}^{f} W^{a} = \mathbb{P}(g^{ab} \chi'_{\varepsilon}(\rho) (\partial_{b} \rho) f \rho^{-1}(\partial_{c} \rho) W^{c}).$$

Since the projection is continuous on $H^{r}(\Omega)$, if the metric and pressure are sufficiently regular, one gets, as in [20, 21], that

$$\sum_{j=0}^{k} \|D_{t}^{j} \mathcal{A}^{f} W\|_{H^{r}(\Omega)} \leq C_{\varepsilon, r, k} \sum_{j=0}^{k} \|D_{t}^{j} W\|_{H^{r}(\Omega)}.$$

Moreover, we have

$$\|\mathcal{A}^{f} W - AW\|^{2} = \langle \mathcal{A}^{f} W - AW, \mathcal{A}^{f} W - AW \rangle.$$
\[ = \langle \mathcal{A}^\varepsilon W - \mathcal{A} W, \mathcal{A} W \rangle - \langle \mathcal{A}^\varepsilon W - \mathcal{A} W, A W \rangle \]
\[
= - \int_\Omega (\mathcal{A}^\varepsilon W^a - \mathcal{A} W^a) \chi_\varepsilon(\rho) \partial_a (P^{-1}(\partial_\varepsilon \rho) W^c) dy + \int_\Omega (\mathcal{A}^\varepsilon W^a - \mathcal{A} W^a) \partial_a (\partial_\varepsilon PW^c) dy \\
= - \int_\Omega (\mathcal{A}^\varepsilon W^a - \mathcal{A} W^a) \chi_\varepsilon(\rho) \partial_a [(P^{-1}(\partial_\varepsilon \rho - \partial_\varepsilon P) W^c)] dy + \int_\Omega (\mathcal{A}^\varepsilon W^a - \mathcal{A} W^a) (1 - \chi_\varepsilon(\rho)) \partial_a (\partial_\varepsilon PW^c) dy \\
= \int_\Omega (\mathcal{A}^\varepsilon W^a - \mathcal{A} W^a) \chi_\varepsilon'(\rho) \partial_a \rho [(P^{-1}(\partial_\varepsilon \rho - \partial_\varepsilon P) W^c)] dy + \int_\Omega (\mathcal{A}^\varepsilon W^a - \mathcal{A} W^a) (1 - \chi_\varepsilon(\rho)) \partial_a (\partial_\varepsilon PW^c) dy \\
\]

due to \( P^{-1} \partial_\varepsilon \rho = \partial_\varepsilon P \) on the boundary, which yields

\[
\| \mathcal{A}^\varepsilon W - AW \| \leq \| \chi_\varepsilon'(\rho) \|_{L^\infty(\Omega)} \| \partial_a \rho (P^{-1} \partial_\varepsilon \rho - \partial_\varepsilon P) W^c \| \\
+ \| (1 - \chi_\varepsilon(\rho)) \|_{L^\infty(\Omega)} \| \partial_a (\partial_\varepsilon PW^c) \| \to 0, \text{ as } \varepsilon \to 0,
\]

since \( \chi_\varepsilon'(\rho) \to 0 \) and \( \chi_\varepsilon(\rho) \to 1 \) in \( L^\infty(\Omega) \) as \( \varepsilon \to 0 \). Thus, we obtain

\[
(7.3) \quad \mathcal{A}^\varepsilon U \to AU \text{ in } L^2(\Omega), \quad \text{if } U \in H^1(\Omega).
\]

As in (3.5), it holds

\[
\left| \langle U, \mathcal{A}_{\hat{f}P} W \rangle \right| \leq \langle U, \mathcal{A}_{\hat{f}P} U \rangle^{1/2} \langle W, \mathcal{A}_{\hat{f}P} W \rangle^{1/2} \\
\leq \| f \|_{L^\infty(\Omega \setminus \Omega^\varepsilon)} \langle U, \mathcal{A}^\varepsilon U \rangle^{1/2} \langle W, \mathcal{A}^\varepsilon W \rangle^{1/2},
\]

where

\[
(7.5) \quad \Omega^\varepsilon = \{ y \in \Omega : \text{dist} (y, \Gamma) > \varepsilon \}.
\]

In fact, it suffices to take the supremum over the set where \( d(y) \leq \varepsilon \) since \( \chi_\varepsilon' = 0 \) when \( d(y) \geq \varepsilon \). The only difference with (3.5) is that the supremum is over a small neighborhood of the boundary instead of on the boundary. Since \( P \) vanishes on the boundary, \( P > 0 \) in the interior, and \( \nabla_N P \leq -c_0 < 0 \) on the boundary, it follows that \( \hat{P} = D_t P \) vanishes on the boundary and \( \hat{P}/P \) is a smooth function. Let \( \mathcal{A}^\varepsilon = \mathcal{A}_{\hat{P}}^\varepsilon \) be the time derivative of the operator \( \mathcal{A}^\varepsilon \), which satisfies

\[
(7.6) \quad \left| \langle W, \mathcal{A}^\varepsilon W \rangle \right| \leq \| \hat{P}/P \|_{L^\infty(\Omega \setminus \Omega^\varepsilon)} \langle W, \mathcal{A}^\varepsilon W \rangle.
\]

The commutators between \( \mathcal{A}^\varepsilon_f \) and the Lie derivatives \( \mathcal{L}_T \) with respect to tangential vector fields \( T \) are basically the same as for \( \mathcal{A} \). Since \( Td = 0 \) for \( T \in T_0 = S_0 \cup \{ D_1 \} \), we have

\[
(7.7) \quad \mathcal{P}(g^{ca} \mathcal{L}_T (g_{ab} \mathcal{A}^\varepsilon_f W^b)) = \mathcal{A}^\varepsilon_f \mathcal{L}_T W^c + \mathcal{A}^\varepsilon_f T f W^c.
\]

In order to get additional regularity in the interior, we include the vector fields \( S_1 \) that span the tangent space in the interior. The vector fields in \( S_1 \) satisfy \( S_\rho = L_\rho S_\rho = 0 \) when \( d \leq d_0/2 \). Due to \( \chi_\varepsilon'(\rho) = 0 \) when \( d \geq \varepsilon \), the above relation (7.7) holds for these as well if we assume that \( \varepsilon \leq d_0/2 \).
Now, it remains to estimate the curl of $A^\varepsilon$. Although the curl of $A$ vanishes, it is not the case for the curl of $A^\varepsilon$. However, it will vanish away from the boundary. Let $(\mathcal{A}^\varepsilon W)_a = g_{ab}A^\varepsilon W^b$, we have
\[
(\mathcal{A}^\varepsilon W)_a = g_{ab}P(-g^{bd}\chi_\varepsilon(\rho)\partial_d(P\rho^{-1}(\partial_c\rho)W^c)) \\
= -\chi_\varepsilon(\rho)\partial_a(P\rho^{-1}(\partial_c\rho)W^c) - \partial_a q_1,
\]
for some function $q_1$ vanishing on the boundary and determined so that the divergence vanishes. Then, when $d(y) \geq \varepsilon$, we get $\chi_\varepsilon'(\rho) = 0$ and
\[
(7.8) \quad (\text{curl } \mathcal{A}^\varepsilon W)_{ab} = \partial_a(\mathcal{A}^\varepsilon W)_b - \partial_b(\mathcal{A}^\varepsilon W)_a \\
= -\partial_a(\chi_\varepsilon(\rho)\partial_b(P\rho^{-1}(\partial_c\rho)W^c)) + \partial_b(\chi_\varepsilon(\rho)\partial_a(P\rho^{-1}(\partial_c\rho)W^c)) \\
= -\chi_\varepsilon'(\rho)[\partial_a\rho\partial_b(P\rho^{-1}(\partial_c\rho)W^c) - \partial_b\rho\partial_a(P\rho^{-1}(\partial_c\rho)W^c)] \\
= 0.
\]

### 7.2 The smoothed-out equation and existence of weak solutions

We introduce the following $\varepsilon$ smoothed-out linear equation
\[
(7.9a) \quad \dot{W}_\varepsilon^a - \mathcal{L}_B^2 W_\varepsilon^a + \mathcal{A}^\varepsilon W_\varepsilon^a + \mathcal{G} \dot{W}_\varepsilon^a - \mathcal{C} \mathcal{L}_B W_\varepsilon^a = F^a, \\
(7.9b) \quad W_\varepsilon|_{t=0} = 0, \quad \dot{W}_\varepsilon|_{t=0} = 0.
\]
It is a wave equation with variable coefficients, one can get the existence of weak solutions in $H^1(\Omega)$ by standard methods and noticing that $B^a N_a = 0$ on the boundary, or in $H^r(\Omega)$ by (7.2), since all operators are bounded and $\mathcal{L}_B$ can be regarded as the first-order derivative with respect to spatial variables.

In order to obtain the additional regularity in time as well, we need to apply more time derivatives using (7.2) and (3.1), the initial data for these vanish as well since we constructed $F$ in (7.9) vanishing to any given order. If the initial data, encoded in $F$, are smooth, we hence have a smooth solution of the $\varepsilon$ approximate linear equation.

We will show that $W_\varepsilon \to W$ weakly in $L^2$, where $W \in H^r(\Omega)$ for some large $r$. From the weak convergence, it will follow that $W$ is a weak solution, and then from the additional regularity of $W$, we can obtain that it is indeed a classical solution; hence the a priori bounds in the earlier section hold.

The norm of $\mathcal{A}^\varepsilon$ tends to infinity as $\varepsilon \to 0$, but we can include it in the energy because it is a positive operator. The energy will be the same as before with $A$ replaced by $A^\varepsilon$, so (4.5) becomes
\[
E^\varepsilon(t) = \langle \dot{W}_\varepsilon, W_\varepsilon \rangle + \langle (\mathcal{A}^\varepsilon + I)W_\varepsilon, W_\varepsilon \rangle + \langle \mathcal{L}_B W_\varepsilon, \mathcal{L}_B W_\varepsilon \rangle.
\]
Since $D_t d = 0$, it follows from taking the time derivative of (7.11), with $f = P$, that
\[
\frac{d}{dt} \langle \mathcal{A}^\varepsilon W_\varepsilon, W_\varepsilon \rangle = 2 \langle \mathcal{A}^\varepsilon W_\varepsilon, \dot{W}_\varepsilon \rangle + \langle \mathcal{A}^\varepsilon W_\varepsilon, W_\varepsilon \rangle,
\]
where the last term is bounded by (7.6). Thus, by (4.7), one has
\[
|\dot{E}^\varepsilon| \leq \left(1 + \frac{\dot{P}}{P}\right) \|g\|_{L^\infty(\Omega)} + 2\|g\|_{L^\infty(\Omega)}\|\partial B\|_{L^\infty(\Omega)} + |E^\varepsilon| + 2\sqrt{E^\varepsilon}\|F\|,
\]
from which we obtain a uniform bound for $t \in [0, T]$ independent of $\varepsilon$, i.e., $E_\varepsilon(t) \leq C$.

Since $\|W_\varepsilon\| \leq C$, we can choose a subsequence $W_{\varepsilon_n} \rightharpoonup W$ weakly in the inner product. Now, we show that the limit $W$ is a weak solution of the equation. Multiplying (7.9a) by a smooth divergence-free vector field $U$ that vanished for $t \geq T$ and integrating by parts, we have

$$
\int_0^T \int_\Omega g_{ab} U^b F^a dy dt
$$

$$
= \int_0^T \int_\Omega g_{ab} (\tilde{W}_\varepsilon^a - L_B^2 W_\varepsilon^a + A^a W_\varepsilon^a + \dot{G} W_\varepsilon^a - \mathcal{C} W_\varepsilon^a + \mathcal{X} \mathcal{L}_B W_\varepsilon^a) U^b dy dt
$$

$$
= \int_0^T \int_\Omega (D_t g_{ab} \dot{W}_\varepsilon^a) U^b - g_{ab} L_B^2 W_\varepsilon^a U^b + \chi'_\varepsilon (\partial_b \rho) P \rho^{-1} (\partial_c \rho) W^c U^b - \omega_{bc} \dot{W}_\varepsilon^c U^b
$$

$$
- 2\delta_{il} \partial_h x^l B \partial_c x^i L_B W_\varepsilon^c U^b - 2g_{ab} \partial_c B^a \mathcal{L}_B W_\varepsilon^c U^b dy dt
$$

$$
= - \int_0^T \int_\Omega g_{ab} \dot{W}_\varepsilon^a U^b dy dt + \int_0^T \int_\Omega ((L_B g_{ab}) U^b + g_{ab} \mathcal{L}_B U^b) \mathcal{L}_B W_\varepsilon^a dy dt
$$

$$
+ \int_0^T \int_\Omega g_{ab} A^c U^b W_\varepsilon^a dy dt + \int_0^T \int_\Omega (\dot{\omega}_{bc} U^b + \omega_{bc} \dot{U}^b) W_\varepsilon^a dy dt
$$

$$
+ 2\int_0^T \int_\Omega \delta_{il} \partial_h x^l B \partial_c x^i L_B U^b dy dt + 2\int_0^T \int_\Omega \mathcal{L}_B (\delta_{il} \partial_h x^l B \partial_c x^i) W_\varepsilon^c U^b dy dt
$$

$$
+ 2\int_0^T \int_\Omega \mathcal{L}_B (g_{ab} \partial_c B^a) U^b W_\varepsilon^c dy dt + 2\int_0^T \int_\Omega g_{ab} \partial_c B^a \mathcal{L}_B U^b W_\varepsilon^c dy dt
$$

$$
= \int_0^T \int_\Omega g_{ab} (\ddot{U}^b - L_B^2 U^b + A^c U^b + \dot{G} U^b - \mathcal{C} U^b + \mathcal{X} \mathcal{L}_B U^b) W_\varepsilon^a dy dt
$$

$$
+ \int_0^T \int_\Omega \dot{\omega}_{bc} U^b W_\varepsilon^c dy dt + \int_0^T \int_\Omega \mathcal{L}_B (\delta_{il} \partial_h x^l B \partial_c x^i - \delta_{il} B \partial_h x^l \partial_c x^i) U^b W_\varepsilon^c dy dt
$$

$$
+ \int_0^T \int_\Omega \mathcal{L}_B (g_{ab} \partial_c B^a - g_{ac} \partial_b B^a) U^b W_\varepsilon^c dy dt.
$$

From (7.3), we know that $A^c U$ converges to $AU$ strongly in the norm if $U \in H^1$. Because $W_{\varepsilon_n} \rightharpoonup W$ weakly, this proves that we have a weak solution $W$ of the equation

$$
\int_0^T \int_\Omega g_{ab} (\ddot{U}^b - L_B^2 U^b + AU^b + \dot{G} U^b - \mathcal{C} U^b + \mathcal{X} \mathcal{L}_B U^b) dy dt
$$

$$
+ \int_0^T \int_\Omega \dot{\omega}_{bc} U^b W^c dy dt + \int_0^T \int_\Omega \mathcal{L}_B (\delta_{il} \partial_h x^l B \partial_c x^i - \delta_{il} B \partial_h x^l \partial_c x^i) U^b W^c dy dt
$$

$$
+ \int_0^T \int_\Omega \mathcal{L}_B (g_{ab} \partial_c B^a - g_{ac} \partial_b B^a) U^b W^c dy dt = \int_0^T \int_\Omega g_{ab} U^b F^a dy dt
$$

for any smooth divergence-free vector field $U$ that vanishes for $t \geq T$. Moreover, due to $\text{div} W_\varepsilon = 0$, we get

$$
\int_0^T \int_\Omega (\partial_a q) W_\varepsilon^a dy dt = 0
$$

for any smooth $q$ that vanishes on the boundary and thus

(7.11)$$
\int_0^T \int_\Omega (\partial_a q^a) W^a dy dt = 0.
$$
Therefore, $W$ is weakly divergence-free.

## 8 Existence of smooth solutions for the linearized equation

In order to show that $W$ is divergence-free classical solution, we need to prove the additional regularity, i.e., $W, \dot{W} \in H^r(\Omega)$ for any $r \geq 0$. Then, the integration by parts for (7.11) yields

$$
\int_0^T \int_\Omega q \partial_a W^a dy dt = 0
$$

for any smooth function $q$ that vanishes on the boundary. Thus, $W$ is divergence-free.

Moreover,

$$
\int_0^T \int_\Omega g_{ab} U^b (\ddot{W}^a - \mathcal{L}_B^2 W^a + \mathcal{A} W^a + \dot{\mathcal{G}} W^a - \mathcal{C} W^a + \mathcal{X} \mathcal{L}_B W^a) dy dt
$$

for any smooth, divergence-free vector field $U$ that vanished for $t \geq T$. Since $W$ is divergence-free, it follows that $\dot{W}^a - \mathcal{L}_B^2 W^a + \mathcal{A} W^a + \dot{\mathcal{G}} W^a - \mathcal{C} W^a + \mathcal{X} \mathcal{L}_B W^a$ is divergence-free. By construction, $F$ is also divergence-free, it follows that (8.1) holds for any smooth vector field $U$ that vanishes for $t \geq T$. Thus, we conclude that

$$
\ddot{W}^a - \mathcal{L}_B^2 W^a + \mathcal{A} W^a + \dot{\mathcal{G}} W^a - \mathcal{C} W^a + \mathcal{X} \mathcal{L}_B W^a = F^a, \quad \text{div } W = 0.
$$

Therefore, it only remains to prove that $W \in H^r(\Omega)$. We must show that we have uniform bounds for the $\varepsilon$ smoothed-out equation similar to the a priori bounds for the linearized equation. The uniform tangential bounds for the $\varepsilon$ smoothed-out equation follow the proof of the a priori tangential bounds in Section 6.5, which is just a change of notation. Let

$$
E^\varepsilon_I = \langle \dot{W}_{\varepsilon I}, \dot{W}_{\varepsilon I} \rangle + \langle W_{\varepsilon I}, (A + I) W_{\varepsilon I} \rangle + \langle \mathcal{L}_B W_{\varepsilon I}, \mathcal{L}_B W_{\varepsilon I} \rangle, \quad W_{\varepsilon I} = \mathcal{L}^I_T W_{\varepsilon}.
$$

If $\varepsilon < d_0$, then the commutator relation for $A^\varepsilon$, (7.7), is exactly the same as for $A$, (6.13). Moreover, the positivity property for $A^\varepsilon_I$ only differs from the one for $A_I$ in which the supremum over the boundary in (3.5) is replaced by the supremum over a neighborhood of the boundary where $d(y) < \varepsilon$ in (7.4). Thus, all the calculations and inequalities in Sections 6.5, 6.7 and 6.8 hold with $A$ replaced by $A^\varepsilon$ if we replace the supremum of $\nabla_N q / \nabla_N P$ over the boundary in (6.16) by the supremum of $q / P$ over the domain $\Omega \setminus \Omega^\varepsilon$, where $\Omega^\varepsilon$ is given by (7.5). Hence, we will reach the energy bound (6.25) for $E^T_I$ replaced by

$$
E^{T,\varepsilon}_I = \sum_{|I| \leq r, I \in T} \sqrt{E^\varepsilon_I},
$$

namely, Proposition 6.8 holds for $E^T_I$ replaced by $E^{T,\varepsilon}_I$ with a constant independent of $\varepsilon$. It is where we need to have vanishing initial data and an inhomogeneous term that vanishes to higher order when $t = 0$ so that the higher-order time derivatives of
the solution of (7.9a) also vanish when \( t = 0 \). If the initial data for higher-order time derivatives were obtained from the \( \varepsilon \) smoothed-out equation, then they would depend on \( \varepsilon \), and so we would not have been able to get a uniform bound for the energy \( E^T,\varepsilon \).

The estimate for the curl is simple since the curl of \( A_\varepsilon \) vanishes in \( \Omega^\varepsilon \) by (7.8), it follows that all the formula in Sections 6.7 and 6.8 hold when \( d(y) \geq \varepsilon \). This follows from replacing \( A \) in (6.30) by \( A_\varepsilon \) and vanishing of its curl for \( d(y) \geq \varepsilon \). Let

\begin{equation}
C^T,\varepsilon_r = \sum_{|J| \leq r-1, J \in \mathcal{U}} \left( \int_{\Omega^\varepsilon} |\text{curl} \mathcal{L}_U^{\varepsilon} w_\varepsilon|^2 + |\text{curl} \mathcal{L}_U^{\varepsilon} w_\varepsilon|^2 dy \right)^{1/2},
\end{equation}

\begin{equation}
\|W(t)\|_{U^r(\Omega^\varepsilon)} = \sum_{|J| \leq r, J \in \mathcal{U}} \left( \int_{\Omega^\varepsilon} |\mathcal{L}_U^{\varepsilon} W(t, y)|^2 dy \right)^{1/2}.
\end{equation}

Because all the used estimates from Section 6.3 are pointwise estimates, we conclude that the inequality in Proposition 6.15 holds with a constant \( C \) independent of \( \varepsilon \) if we replace \( C^T,\varepsilon \) by \( C^T,\varepsilon \) and the norms by \( (8.3) \), as follows.

**Proposition 8.1.** Suppose that \( x, P \in C^{r+2}([0, T] \times \Omega), B \in C^{r+2}(\Omega), |P| = 0 \), \( \nabla \mathcal{N} \mathcal{P} \leq -c_0 < 0, B^a N_a r |r| = 0 \), where \( V = D_t x \). Suppose that \( W_\varepsilon \) is a solution of (7.9a) where \( F \) is divergence-free and vanishing to order \( r \) as \( t \to 0 \). Let \( E^T,\varepsilon \) be defined by (8.2). Then, there is a constant \( C = C(x, P, B) \) depending only on the norm of \( (x, P, B) \), a lower bound for \( c_0 \), and an upper bound for \( T \), but independent of \( \varepsilon \) such that if \( E^T,\varepsilon (0) = C^T,\varepsilon (0) = 0 \) for \( s \leq r \), then for \( t \in [0, T] \)

\[ \|W_\varepsilon\|_{U^r(\Omega^\varepsilon)} + \|\dot{W}_\varepsilon\|_{U^r(\Omega^\varepsilon)} + \|\mathcal{L}_B W_\varepsilon\|_{U^r(\Omega^\varepsilon)} + E^T,\varepsilon (t) \leq C \int_0^t \|F\|^T,\varepsilon d\tau. \]

Therefore, it implies that the limit \( W \) satisfies the same bound with \( \Omega^\varepsilon \) replaced by \( \Omega \), and so the weak solution in Section 7.2 is indeed a smooth solution.

### 9 The energy estimate with inhomogeneous initial data

In this section, we consider the original equations with inhomogeneous initial data and an inhomogeneous term:

\begin{equation}
\dot{W}^a - \mathcal{L}_B^2 W^a + AW^a + \hat{g} \dot{W}^a - CW^a + \mathcal{X} \mathcal{L}_B W^a = F^a.
\end{equation}

We need some estimates of the commutators with the operator \( \mathcal{A}, \hat{g}, \mathcal{C} \) and tangential vector fields. We recall them from [20],

\[ [\mathcal{L}_S, \mathcal{A}]W = (A_S - G_S \mathcal{A})W, \quad [\mathcal{L}_S, \hat{g}S]W = (G_{TS} - G_T \hat{G}_S)W, \]

\[ [\mathcal{L}_S, \mathcal{C}]W = (C_S - G_S \mathcal{C})W, \quad [\mathcal{L}_T, \hat{g}S]W = \delta_I^J I^k \hat{g}_{I_1} \cdots \hat{g}_{I_{k-1}} A_{I_{k-1}} W_{I_k} - \mathcal{L}_S g_{bc}, \]

\[ [\mathcal{L}_S, \hat{g}]W = \delta_I^J I^k \hat{g}_{I_1} \cdots \hat{g}_{I_{k-1}} A_{I_{k-1}} W_{I_k}, \quad [\mathcal{L}_S, \mathcal{C}]W = \delta_I^J I^k \hat{g}_{I_1} \cdots \hat{g}_{I_{k-1}} C_{I_{k-1}} W_{I_k}, \]

where \( A_S = A_{SP}, G_S = M_{g^S} \) defined by \( G_S W^a = \mathbb{P}(g^{ac} g^S_{bc} W^b), g^T_{bc} = \mathcal{L}_T \mathcal{L}_S g_{bc}, \)

\( G_{TS} W^a = \mathbb{P}(g^{ab} g_{bc} T_{wc}), L_T W^a = \mathbb{P}(g^{ab} \omega_{bc} T_{wc}), \omega_{bc} = \mathcal{L}_T \omega_{bc}, \hat{g}_J W^a = M_{g^J} W^a = \mathbb{P}(g^{ac} g^J_{bc} W^b), g^J_{ac} = \mathcal{L}_S g_{ac}, A_J = A_{SP}, \) and \( W_J = \mathcal{L}_J W, \) the sum is over all combinations with \( I_1 + I_2 + \cdots + I_k = I \) in last three identities, with \( k \geq 2 \) and \( |I_k| < |I|, \)
\[ \partial_t^I \Phi \text{ and } \hat{\partial}_t^I \Phi \text{ are some constants. For the operator } \mathcal{X}, \text{ we have similar equality for its commutator with tangential vector fields. Denote } \beta_{bc} = \delta_{il} \partial_b x^l \mathcal{L}_B \partial_c x^l, \text{ we have} \]

\[ \mathcal{L}_T \mathcal{X} W^a = \mathcal{L}_T (g^{ab} (-2 \beta_{bc} W^c + \partial_b q)) \]

\[ = (\mathcal{L}_T g^{ab}) (-2 \beta_{bc} W^c + \partial_b q) - 2 g^{ab} (\mathcal{L}_T \beta_{bc}) W^c - 2 g^{ab} \beta_{bc} \mathcal{L}_T W^c + g^{ab} \partial_b T q, \]

where

\[ \Delta q = 2 \partial_a (g^{ab} \beta_{bc} W^c), \quad q|_\Gamma = 0. \]

Projecting each term onto the divergence-free vector fields, we get

\[ [\mathcal{L}_T, \mathcal{X}] W = (\mathcal{X}_T - \mathcal{G}_T \mathcal{X}) W, \]

where \( \mathcal{X}_T W^a = \mathbb{P} (-2 g^{ab} \mathcal{L}_T (\delta_{il} \partial_b x^l \mathcal{L}_B \partial_c x^l) W^c) \). Similarly, we have

\[ [\mathcal{L}_S^I, \mathcal{X}] W = \hat{e}_I^I \mathcal{G}_I \cdots \mathcal{G}_{I_{k-2}} \mathcal{X}_{I_{k-1}} W_{I_k}. \]

These commutators are bounded operator and lower order since \( |I_k| < |I| \). In addition, \([\mathcal{L}_T^I, \mathcal{L}_B^2]\) is also a bounded operator since \( B \) is a tangential vector field. Thus, we obtain

\[ L_I W = \hat{W}_I - \mathcal{L}_B^2 W_I + AW_I + \hat{G} \hat{W}_I - C \hat{W}_I + \mathcal{X} \mathcal{L}_B W_I = H_I, \]

with

\[ H_I = F_I + \hat{e}_I^I \mathcal{G}_I \cdots \mathcal{G}_{I_{k-2}} A_{I_{k-1}} W_{I_k} \]

\[ + [\mathcal{L}_T^I, \mathcal{L}_B^2] W + \hat{e}_I^I \mathcal{G}_I \cdots \mathcal{G}_{I_{k-2}} X_{I_{k-1}} [\mathcal{L}_T^I, \mathcal{L}_B^2] W \]

\[ + \hat{e}_I^I \mathcal{G}_I \cdots \mathcal{G}_{I_{k-2}} \hat{G}_{I_{k-1}} \hat{W}_{I_k} + \hat{e}_I^I \mathcal{G}_I \cdots \mathcal{G}_{I_{k-2}} C_{I_{k-1}} \hat{W}_{I_k}, \]

where \( |I_k| < |I| \) and \( F_I = \mathcal{L}_T^I F \). We consider only \( W_I = \mathcal{L}_S^I W \) with \( S \in S \), as before, let

\[ E_I = \langle \hat{W}_I, \hat{W}_I \rangle + \langle W_I, (A + I) W_I \rangle + \langle \mathcal{L}_B W_I, \mathcal{L}_B W_I \rangle. \]

The energy estimate is as similar as before, and we only need to estimate the \( L^2 \)-norm of the \( H_I \). It is obvious that (9.3) and (9.4) are bounded by \( E_J \) for some \( |J| \leq |I| \). Since \( A_{I_k} \) is of order 1, it contains derivatives in any direction, thus the term has to be estimated by \( \| \partial W_{I_k} \|_{L^2(\Omega)} \), and then it does not directly get an estimate for \( \| \mathcal{L} S W_{I_k} \|_{L^2(\Omega)} \) for all tangential derivatives \( S \). However, we can combine the estimates for the curl to get the desired estimate.

Let \( G_r^R \) and \( E_r^S \) be defined as in (6.49) and (6.17), respectively. Let \( m_r^V, \hat{m}_r^V \) and \( \hat{m}_r^V \) be defined as in (6.50). Then, we have by (6.52)

\[ \| W \|_r + \| \hat{W} \|_r + \| \mathcal{L}_B W \|_r \leq K_1 \sum_{s=0}^r m_{r-s}^R (C_r^R + E_r^S), \]

where \( \| W \|_r = \| W(t) \|_{R^r(\Omega)} \). Since the projection \( \mathbb{P} \) has norm 1, and \( \| G_J W \| \leq \| g_J \|_{L^\infty(\Omega)} \| W \| \), it follows that

\[ \| \mathcal{G}_I \cdots \mathcal{G}_{I_{k-2}} \hat{G}_{I_{k-1}} \hat{W}_{I_k} \| \leq \| g^I \|_{L^\infty(\Omega)} \cdots \| g^{I_{k-2}} \|_{L^\infty(\Omega)} \| \hat{g}^{I_{k-1}} \|_{L^\infty(\Omega)} \| \hat{W} \|_s \]
\[ \| \mathcal{G}_{I_k} \cdots \mathcal{G}_{I_k} \mathcal{L}_{I_k} W_{I_k} \| \leq \| g_{I_1} \|_{L^\infty(\Omega)} \cdots \| g_{I_k} \|_{L^\infty(\Omega)} \| \omega_{I_k} \|_{L^\infty(\Omega)} \| \hat{W} \|_s \]

and

\[ \| \mathcal{G}_{I_k} \cdots \mathcal{G}_{I_k} \mathcal{L}_{I_k} W \| \leq \| g_{I_1} \|_{L^\infty(\Omega)} \cdots \| g_{I_k} \|_{L^\infty(\Omega)} \| \mathcal{L}_{I_k}^{-1} (\delta_{I_i} \partial_0 x_j \mathcal{L}_B \partial_0 x_i) \|_{L^\infty(\Omega)} \| \mathcal{L}_{I_k} \| \mathcal{L}_B \| W \|
\]

where \( s = |I_k| < r = |I| \). Denote

\[ P^R_r = \sum_{s=0}^{r} \sum_{|J| = s+1, J \in S} \| \partial^J P \|_{L^\infty(\partial \Omega)}. \]

Then, we have

\[ \| \hat{d}^I_{I_k} \mathcal{G}_{I_k} \cdots \mathcal{G}_{I_k} \mathcal{A}_{I_k} W_{I_k} \| \leq \| g_{I_1} \|_{L^\infty(\Omega)} \cdots \| g_{I_k} \|_{L^\infty(\Omega)} \| \mathcal{A}_{I_k} W_{I_k} \|_{L^\infty(\Omega)} \]

Similar to (4.7), we can get

\[ \| \mathcal{H} \| \leq \left( 1 + 2 \| \hat{g} \|_{L^\infty(\Omega)} + \frac{\| \partial \hat{P} \|_{L^\infty(\Gamma)}}{c_0} + 2 \| \partial x \|_{L^\infty(\Omega)} \| \mathcal{L}_B \partial x \|_{L^\infty(\Omega)} \right) E_I \]

\[ + 2 \sqrt{E_I \| H_I \|}, \]

where \( c_0 \) is the constant in the condition (1.7). From (9.8) and (9.5), we get

\[ \| H_I \| \leq C \sum_{s=0}^{r-1} (m^R_{I_{r-s}} \| \hat{W} \|_s + (m^R_{I_{r-s}} + \bar{m}^R_{I_{r-s}}) \| W \|_{s, B} + P^R_{r-s} \| W \|_s) \]

\[ + P^R_0 \| W \|_r + \| F \|_r \]

\[ \leq K_1 \sum_{s=0}^{r-1} (m^R_{I_{r-s}} + \bar{m}^R_{I_{r-s}} + P^R_{r-s}) (C^R_s + E^R_s) \]

\[ + K_1 P^R_0 (C^R_r + E^R_r) + \| F \|_r. \]

Summing (9.9) over all \( I \in S \) with \(|I| = r \) and using (9.10), we have

\[ \left| \frac{dE^S_r}{dt} \right| \leq K_1 \left( 1 + 2 \| \hat{g} \|_{L^\infty(\Omega)} + \frac{\| \partial \hat{P} \|_{L^\infty(\Gamma)}}{c_0} + 2 \| \partial x \|_{L^\infty(\Omega)} \| \mathcal{L}_B \partial x \|_{L^\infty(\Omega)} \right) (C^R_r + E^R_r) \]

\[ + K_1 \sum_{s=0}^{r-1} (m^R_{I_{r-s}} + \bar{m}^R_{I_{r-s}} + P^R_{r-s}) (C^R_s + E^R_s) + \| F \|_r. \]

Since (6.51) holds with \( \mathcal{U} \) and \( \mathcal{T} \) replaced by \( \mathcal{R} \) and \( S \), respectively, we get

\[ \left| \frac{dC^R_r}{dt} \right| \leq K_1 (m^R_0 + \bar{m}^R_0) (C^R_r + E^R_r) \]
Thus, (9.11) and (9.12) yield a bound for $C^r + E^s_r$ in terms of $C^r_s + E^s_s$ for $s < r$, namely

$$C^r_r(t) + E^s_r(t) \leq K_1 \epsilon^{K_1} \int_0^t \left( C^r_r(0) + E^s_r(0) + \int_0^t \left( \sum_{s=0}^{r-1} (\tilde{m}^r_{r-s} + \tilde{m}^r_{r-s})(C^r_s + E^s_s) + \|F\|_r \right) dt \right),$$

where

$$n = 1 + 2 \|\dot{g}\|_{L^\infty(\Omega)} + \|\dot{P}\|_{L_{\infty}(\Gamma)} / c_0 + 2 \|\dot{x}\|_{L^\infty(\Omega)} \|L_{\partial x}\|_{L^\infty(\Omega)} + \sum_{S \in \mathcal{S}} \|\partial S P\|_{L^\infty(\Gamma)} + \|\omega\|_{L^\infty(\Omega)} + \|L_B g\|_{L^\infty(\Omega)} + \|L_B B\|_{L^\infty(\Omega)} + \|B\|_{L^2, \infty}^R + \|\partial x\|_{L^\infty(\Omega)} + \|\partial^2 x\|_{L^\infty(\Omega)} + \|\partial y / \partial x\|_{L^\infty(\Omega)}.$$

Because the bound for $E^s_0 = E_0$ have been already proven, we can get the bound for $C^r + E^s$ inductively. Therefore, from (9.3), we obtain the following estimates.

**Proposition 9.1.** Suppose that $x, P \in C^{r+2}([0, T] \times \Omega)$, $B \in C^{r+2}(\Omega)$, $P|_{\Gamma} = 0$, $\nabla_N P|_{\Gamma} \leq -c_0 < 0$, $B^a N_a|_{\Gamma} = 0$ and $\text{div} V = 0$, where $V = D_t x$. Let $W$ be the solution of (9.1) where $F$ is divergence-free. Then, there is a constant $C$ depending only on the norm of $(x, P, B)$, a lower bound for the constant $c_0$, and an upper bound for $T$, such that, for $s \leq r$, we have

$$\|W(t)\|_r + \|\dot{W}(t)\|_r + \|L_B W(t)\|_r + \langle W(t)\rangle_{A, r} + \|\dot{x}\|_{L^\infty(\Omega)} + \|\dot{y}\|_{L^\infty(\Omega)},$$

where

$$\|W(t)\|_r = \sum_{|I| \leq r, I \in \mathcal{R}} \|L^I_B W(t)\|_{L^2(\Omega)},$$

$$\langle W(t)\rangle_{A, r} = \sum_{|I| \leq r, I \in \mathcal{S}} \langle L^I_S W(t) , A L^I_S W(t) \rangle^{1/2}.$$

**10 The main result**

As the same as in [20], $\|W(t)\|_r$ is equivalent to the usual time-independent Sobolev norm; $\langle W(t)\rangle_{A, r}$ is only a seminorm on divergence-free vector fields, which is not only equivalent to a time-independent seminorm given by (3.2) with $f$ the distance function $d(y)$ due to $0 < c_0 \leq -\nabla_N P \leq C$, but also equivalent to the normal component of the vector field $W_N = N_a W^a$ being in $H^r(\Gamma)$ in view of (3.2), up to lower-order terms that can be controlled by $\|W(t)\|_r$, since we only apply tangential vector fields.

We define $H^r(\Omega)$ to be the completion of $C^\infty(\Omega)$ in the norm $\|W(t)\|_r$ and define $N^r(\Omega)$ to be the completion of the $C^\infty(\Omega)$ divergence-free vector fields in the norm
∥W∥_{N^r} = ∥W(t)∥_r + (W(t))_{A,r}. Because the projection \mathbb{P} is continuous in the \(H^r\) norm, it implies that \(H^r\) is also the completion of the \(C^\infty(\Omega)\) divergence-free vector fields in the \(H^r\) norm. We state the main result as follows.

**Theorem 10.1.** Suppose that \(x, P \in C^{r+2}([0, T] \times \Omega), B \in C^{r+2}(\Omega), P|_{\Gamma} = 0, \nabla N P|_{\Gamma} \leq -c_0 < 0, B^a N_a|_{\Gamma} = 0\) and \(\text{div} D_t x = 0\). Then, if initial data and the inhomogeneous term in (5.1b) are divergence-free and satisfy

\[
(W_0, W_1, \mathcal{L}_B W_0) \in N'(\Omega) \times H'(\Omega) \times H'(\Omega), \quad F \in L^1([0, T], H^r(\Omega)),
\]

the linearized equations (5.1) have a solution

\[
(W, \dot{W}, \mathcal{L}_B W) \in C([0, T], N'(\Omega) \times H'(\Omega) \times H'(\Omega)).
\]

**Proof.** If \(W_0, W_1\) and \(F\) are divergence-free and \(C^\infty\), and \(F\) is supported in \(t > 0\), then there exists a solution by the arguments in Section 8. It follows, by approximating \(W_0, W_1\) and \(F\) with \(C^\infty(\Omega)\) divergence-free vector fields and applying the estimate (9.13) to the differences, that we obtain a convergent sequence in (10.1), thus the limit must be in the same space. \(\square\)

### A Lie derivatives

Let us review the Lie derivative of the vector field \(W\) with respect to the vector field \(T\) constructed in the previous section,

\[
\mathcal{L}_T W^a = TW^a - (\partial_c T^a)W^c.
\]

For those vector fields, it holds \(\text{div} T = 0\), so \(\text{div} W = 0\) implies that

\[
\text{div} \mathcal{L}_T W = T\text{div} W - W\text{div} T = 0.
\]

The Lie derivative of a 1-form is defined by

\[
\mathcal{L}_T \alpha_a = T\alpha_a + (\partial_a T^c)\alpha_c.
\]

The Lie derivatives also commute with the exterior differentiation, \([\mathcal{L}_T, d] = 0\), so if \(q\) is a function, then

\[
\mathcal{L}_T \partial_a q = \partial_a T q.
\]

The Lie derivative of a 2-form is given by

\[
\mathcal{L}_T \beta_{ab} = T\beta_{ab} + (\partial_a T^c)\beta_{cb} + (\partial_b T^c)\beta_{ac}.
\]

In general, in local coordinate notation, for a type \((r, s)\) tensor field \(\beta\), the Lie derivative along \(T\) is given by

\[
\mathcal{L}_T \beta^{a_1...a_r}_{b_1...b_s} = T\beta^{a_1...a_r}_{b_1...b_s} - (\partial_a T^{a_1})\beta^{c_2...a_r}_{b_1...b_s} - \ldots - (\partial_a T^{a_r})\beta^{a_1...a_{r-1} c_1}_{b_1...b_s} + (\partial_b T^c)\beta^{a_1...a_r}_{c b_2...b_s} + \ldots + (\partial_s T^c)\beta^{a_1...a_r}_{b_1...b_{s-1} c}.
\]
It follows that the Lie derivative satisfies the Leibnitz rule, e.g.

\[ \mathcal{L}_T(\alpha_c W^c) = (\mathcal{L}_T \alpha_c) W^c + \alpha_c \mathcal{L}_T W^c, \]
\[ \mathcal{L}_T(\beta_{ac} W^c) = (\mathcal{L}_T \beta_{ac}) W^c + \beta_{ac} \mathcal{L}_T W^c, \]
\[ \mathcal{L}_T(g^{ab} \alpha_b) = \mathcal{L}_T g^{ab} \alpha_b + g^{ab} \mathcal{L}_T \alpha_b, \]

and

\[ \mathcal{L}_T(W^c \partial_c \beta_{ab}) = \mathcal{L}_T W^c \partial_c \beta_{ab} + W^c \mathcal{L}_T \partial_c \beta_{ab}. \]

If \( w \) is a 1-form and \( \text{curl} \, w_{ab} = dw_{ab} = \partial_a w_b - \partial_b w_a \), then

\[ \mathcal{L}_T \text{curl} \, w_{ab} = \text{curl} \, \mathcal{L}_T w_{ab}, \]

since the Lie derivative commutes with exterior differentiation.

From \((A.1)\), we have the following relation on the commutator of two Lie derivatives

\[ [\mathcal{L}_T, \mathcal{L}_B]W^a = \mathcal{L}_{[T,B]}W^a. \]

From \((A.4)\), we get the commutator of Lie derivative and \( \partial_a \)

\[ [\mathcal{L}_T, \partial_a]W^b = W^d \partial_d \partial_a T^b. \]

Furthermore, we also treat \( D_t \) as if it were a Lie derivative and we set

\[ \mathcal{L}_{D_t} = D_t. \]

Of course, this is not a space Lie derivative but rather could be interpreted as a space-time Lie derivative in the domain \([0, T] \times \Omega\). It is important that it satisfies all the properties of the other Lie derivatives considered, such as \( \text{div} \, W = 0 \) implies that \( \text{div} \, D_t W = 0 \) and \( D_t \text{curl} \, w = \text{curl} \, D_t w \), because it commutes with partial differentiation with respect to the \( y \) coordinates. It is more efficient with the same notation, since we will apply products of Lie derivatives and \((A.10)\). Moreover,

\[ [\mathcal{L}_{D_t}, \mathcal{L}_T] = 0, \]

because this quantity is \( \mathcal{L}_{[D_t, T]} \) and \([D_t, T] = 0\) for the vector fields we considered, or it follows from \((A.1)\) and that \( T^a = T^a(y) \) is independent of \( t \).

**Acknowledgments.** Hao’s research was supported by National Natural Science Foundation of China (Grant No. 11671384). Luo’s research was supported by a grant from the Research Grants Council of Hong Kong (Project No. 11305818).

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