On the homotopy type of intersections of two real Bruhat cells

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January 19, 2022

Abstract

Real Bruhat cells give an important and well studied stratification of such spaces as $\text{GL}_{n+1}$, $\text{Flag}_{n+1} = \text{SL}_{n+1}/B$, $\text{SO}_{n+1}$ and $\text{Spin}_{n+1}$. We study the intersections of a top dimensional cell with another cell (for another basis). Such an intersection is naturally identified with a subset of the lower nilpotent group $L^1_{n+1}$. We are particularly interested in the homotopy type of such intersections. In this paper we define a stratification of such intersections. As a consequence, we obtain a finite CW complex which is homotopically equivalent to the intersection.

We compute the homotopy type for several examples. It turns out that for $n \leq 4$ all connected components of such subsets of $L^1_{n+1}$ are contractible: we prove this by explicitly constructing the corresponding CW complexes. Conversely, for $n \geq 5$ and the top permutation, there is always a connected component with even Euler characteristic, and therefore not contractible. This follows from formulas for the number of cells per dimension of the corresponding CW complex. For instance, for the top permutation $S_6$, there exists a connected component with Euler characteristic equal to 2. We also give an example of a permutation in $S_6$ for which there exists a connected component which is homotopically equivalent to the circle $S^1$.

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1 Introduction

For a permutation \( \sigma \in S_{n+1} \), let \( P_\sigma \) be the corresponding permutation matrix, with entries \((P_\sigma)_{i,\sigma(i)} = 1 \) and 0 otherwise. The top permutation is denoted by \( \eta \), so that \( \eta(i) = n + 2 - i \) (for all \( i \in [n+1] = \{1, 2, \ldots, n, n+1\} \)). Let \( L_{o_{n+1}}^1 \) be the nilpotent group of real lower triangular matrices with diagonal entries equal to 1. Following the Bruhat decomposition, partition \( L_{o_{n+1}}^1 \) into subsets \( BL_\sigma \) for \( \sigma \in S_{n+1} \):

\[
BL_\sigma = \{ L \in L_{o_{n+1}}^1 \mid \exists U_0, U_1 \in U_{p_{n+1}}, L = U_0 P_\sigma U_1 \}. \tag{1.1}
\]

Here \( U_{p_{n+1}} \) is the group of real upper triangular matrices with non zero diagonal entries, a Borel subgroup of \( GL_{n+1} \). Notice that \( BL_\eta \) is open and dense. The set \( BL_\sigma \) is naturally homeomorphic to the intersection of two Bruhat cells with different basis, one of the of top dimension: see Section 2. The aim of this paper is to study the homotopy type of the sets \( BL_\sigma \). The following theorem sums up some of our main results.

**Theorem 1.** Consider \( \sigma \in S_{n+1} \) and \( BL_\sigma \subset L_{o_{n+1}}^1 \) as in Equation (1.1).

1. For \( n \leq 4 \), every connected component of every set \( BL_\sigma \) is contractible.

2. For \( n = 5 \) and \( \sigma = 563412 \in S_6 \), there exist connected components of \( BL_\sigma \) which are homotopically equivalent to the circle \( S^1 \).

3. For \( n \geq 5 \), there exist connected components of \( BL_\eta \) which have even Euler characteristic.

We also list the number of connected components per permutation. Here 563412 denotes the permutation \( \sigma \) with \( 1^\sigma = 5 \), \( 2^\sigma = 6 \), \( 3^\sigma = 3 \) and so on until \( 6^\sigma = 2 \).

In order to compute the homotopy type of \( BL_\sigma \), we first choose a reduced word (also called reduced decomposition) for \( \sigma \in S_{n+1} \):

\[
\sigma = a_{i_1} \cdots a_{i_\ell}, \quad \ell = \text{inv}(\sigma). \tag{1.2}
\]

We extensively use the standard Coxeter-Weyl generators \( a_1, \ldots, a_n \) of the symmetric group \( S_{n+1} \); \( a_i \) is the simple transposition \((i, i+1)\). Also, \( \text{inv}(\sigma) \) is the number of inversions of \( \sigma \), so that a reduced word is a word of shortest length. A reduced word for a permutation is represented by a wiring diagram, as in Figure [ ] (Notice that there are different conventions in the literature; in our system, each crossing is a generator, read left-to-right.)

There are usually several reduced words for a given permutation \( \sigma \) but we shall keep our word fixed in the construction of the stratification of \( BL_\sigma \). A preancestry
Figure 1: The words $a_1a_3a_2a_1a_4a_3a_2$ and $a_3a_2a_1a_2a_4a_3a_2$ are both reduced and represent the same permutation $\sigma = 43512 \in S_5$, $\text{inv}(\sigma) = 7$.

for a reduced word is obtained by marking certain crossings with black and white squares, as in Figure 2. The first example is the empty preancestry. We always mark the same number $d$ of black and white squares: $d$ is called the dimension of the preancestry. There are other conditions in the definition of a preancestry, to be made precise in Section 4 (see Equations (4.1) and (4.2)).

Figure 2: Preancestries for the word $a_3a_2a_1a_2a_4a_3a_2$ with dimensions 0, 1, 2.

An ancestry is obtained from a preancestry by marking the remaining crossings with black and white disks, as in Figure 3 (see also Section 7). Given a preancestry of dimension $d$, there are $2^{\ell-2d}$ corresponding ancestries. Figure 3 shows examples of ancestries. More formally, an ancestry is a sequence $\varepsilon$ of length $\ell$ assuming values in $\{\pm1, \pm2\}$: square stands for $\pm2$, disk stands for $\pm1$, black stands for negative and white for positive.

Figure 3: Ancestries corresponding to the preancestries in Figure 2. These examples are $(-1, +1, -1, -1, +1, -1, +1)$, $(+1, -2, -1, +2, -1, +1, -1)$ and $(-2, -2, +1, +2, -1, +2, +1)$.

For each ancestry $\varepsilon$, we define a smooth contractible submanifold $\text{BLS}_\varepsilon \subset \text{BL}_\sigma$ of codimension $d = \dim(\varepsilon)$. For a fixed reduced word, such submanifolds are disjoint and their union is $\text{BL}_\sigma$. This decomposition is not as nice as might be
desired. In particular, except in the simplest cases, it does not satisfy Whitney’s
c Condition: Example 13.4 exhibits ancestries \( \varepsilon_0 \) and \( \varepsilon_1 \) such that
\[
\text{BLS}_{\varepsilon_0} \cap \text{BLS}_{\varepsilon_1} \neq \emptyset, \quad \text{BLS}_{\varepsilon_0} \not\subseteq \overline{\text{BLS}_{\varepsilon_1}}.
\]
Nevertheless, a dual construction is possible.

**Theorem 2.** For \( \sigma \in S_{n+1} \), there exist a finite CW complex \( \text{BLC}_\sigma \) and a continuous map \( i_\sigma : \text{BLC}_\sigma \rightarrow \text{BL}_\sigma \) with the following properties:

1. The map \( i_\sigma \) is a homotopy equivalence.

2. The cells \( \text{BLC}_{\varepsilon} \) of \( \text{BLC}_\sigma \) are labeled by ancestries \( \varepsilon \). For each ancestry \( \varepsilon \) of dimension \( d \), the cell \( \text{BLC}_{\varepsilon} \) has dimension \( d \).

The desired CW complex \( \text{BLC}_\sigma \) and the homotopy equivalence are constructed
in Section 14. The recursive step is provided by Lemma 14.1, a topological result
similar to many well known results but for which we could not locate a published
proof. The CW complex \( \text{BLC}_\sigma \) is, roughly speaking, a dual cell structure to
the stratification (see [8], Section 3.3, particularly the figure in page 232). The
construction is reasonably explicit, but the description of the glueing maps is not
as direct as might be desired. For a small value of \( n \) and an explicit \( \sigma \), the CW
complex \( \text{BLC}_\sigma \) can be constructed by hand. We shall construct \( \text{BLC}_\sigma \) explicitly
for several examples in the text and this is how we prove the first two items of
Theorem 1.

![Figure 4: A connected component of BLC\( _\sigma \) for \( \sigma = 563412 \); see also Figure 5.](image)

Figures 4 and 5, for instance, show a connected component of \( \text{BLC}_\sigma \), \( \sigma = 563412 = a_2a_1a_3a_2a_4a_3a_5a_4a_2a_1a_3a_2 \in S_6 \). In Figure 5, cells of dimension 0
are indicated by boxed ancestries and cells of dimension 1 by edges. Cells of
dimension 2 are shown in Figure 4, there are two octagons and two hexagons.
There are no cells of dimension 3 or higher. It should be clear from Figure 4
that this is homotopically equivalent to \( S^1 \), as claimed in the second item of
Theorem 1.
Figure 5: A connected component of $\text{BLC}_\sigma$ for $\sigma = 563412$. 
In order to prove the third item of Theorem 1 we need a better understanding of the connected components of $\mathcal{B}_\sigma$: we need to know which subsets $\mathcal{B}_\sigma$ are contained in which connected component. We shall discuss the connected components further in this paper. For the moment let us introduce a partition of $\mathcal{B}_\sigma$ into $2^{n+1}$ subsets which are both open and closed (but which may be empty or disconnected):

$$\mathcal{B}_\sigma = \bigsqcup_{z \in \hat{\operatorname{quat}}_{n+1}} \mathcal{B}_z. \quad (1.3)$$

The definition of the sets $\mathcal{B}_z$ is given in Equation (10.1), Section 9. Following the notation of [5], we have a finite group $\tilde{B}^+_{n+1} \subset \operatorname{Spin}_{n+1}$ and a surjective homomorphism $\Pi : \tilde{B}^+_{n+1} \to S_{n+1}$ with kernel $\operatorname{quat}_{n+1}$, a normal subgroup of order $2^{n+1}$. For $\sigma \in S_{n+1}$, we have $\tilde{\sigma} \in \tilde{B}^+_{n+1}$ with $\Pi(\tilde{\sigma}) = \sigma$ so that $\tilde{\sigma} \operatorname{quat}_{n+1} = \Pi^{-1}[\{\sigma\}] \subset \tilde{B}^+_{n+1}$ is a coset (both left and right). We shall go over this in Sections 6 and 9.

**Example 1.1.** Take $n = 2$ and $\sigma = \eta = a_1a_2a_1$. The subset $\mathcal{B}_\eta \subset \mathcal{L}_{0,3}$ is open with 6 connected components, all contractible:

$$\mathcal{B}_\eta = \{L \mid z \neq 0, z \neq xy\}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}. \quad (1.4)$$

The connected component $\mathcal{B}_{z_0} = \{L \mid z > \max\{0, xy\}\}$ is decomposed into three strata, two of them open (corresponding to $y > 0$ and $y < 0$). The third stratum is the half plane $y = 0, z > 0$. Figure 6 shows $\mathcal{BLC}_{z_0} = i^{-1}_\eta[\mathcal{B}_{z_0}]$, the connected component of $\mathcal{BLC}_\eta$ corresponding to $\mathcal{B}_{z_0}$. This is the simplest non-trivial example and shall be discussed in Examples 10.1, 10.2, 10.3, 11.1 and 12.3.

![Figure 6: The CW complex $\mathcal{BLC}_{z_0} \subset \mathcal{BLC}_\sigma$ is homotopically equivalent to $\mathcal{B}_{z_0}$; both are contractible. The box on the left and right represent the vertices $\mathcal{BLC}_{(-1,+1,+1)}$ and $\mathcal{BLC}_{(+1,-1,-1)}$, respectively. The only edge is $\mathcal{BLC}_{(-2,+1,+2)}$.](image)

In order to prove the first item of Theorem 1 we construct the CW complexes $\mathcal{BLC}_z$ for several $z \in \tilde{B}^+_{n+1}$, as in the previous example, and verify that they are, indeed, contractible. It turns out that we have a somewhat stronger result.
Theorem 3. For $n \leq 4$ and $z \in \tilde{B}_{n+1}^+$, each connected component $X \subseteq BLC_z$ collapses to a point: $X \searrow \{\bullet\}$.

Theorem 3 implies the first item of Theorem 1.

Given a preancestry $\varepsilon_0$ and $z \in \sigma \text{Quat}_{n+1}$, our next main result gives a formula for the number $N_{\varepsilon_0}(z)$ of ancestries $\varepsilon$ for which $BLS_\varepsilon \subseteq BL_z$. Such a formula yields information about the connected components of $BL_\sigma$, implying in particular the last item of Theorem 1. The formula is also very helpful as a check when working out concrete examples, such as in Figure 5.

Before stating our next main result, we need a few remarks about notation. We interpret $\text{Spin}_{n+1}$ to be contained in the Clifford algebra $\text{Cl}^0_{n+1}$. For $z \in \text{Cl}^0_{n+1}$, let $\Re(z) = \langle 1, z \rangle \in \mathbb{R}$ be its real part, so that $2^n\Re: \text{Spin}_{n+1} \rightarrow \mathbb{R}$ is a character. We shall see that if $z \in \sigma \text{Quat}_{n+1} \subset \tilde{B}_{n+1}^+$ then $\Re(z)$ equals either 0 or $\pm 2^{-(n+1-c)/2}$, where $c = nc(\sigma)$ is the number of cycles of $\sigma$. There is always an element $z_0 \in \sigma \text{Quat}_{n+1}$ with $\Re(z_0) = 2^{-(n+1-c)/2} > 0$. Also, given a preancestry $\varepsilon_0$ we define a subgroup $H_{\varepsilon_0} \leq \text{Quat}_{n+1}$ in Equation (7.3), Section 7.

Theorem 4. Consider a permutation $\sigma \in S_{n+1}$, a reduced word and a preancestry $\varepsilon_0$. Let $z_0 \in \sigma \text{Quat}_{n+1}$ be such that $\Re(z_0) > 0$.

For any $z = qz_0 \in \sigma \text{Quat}_{n+1}$, we have:

$$N_{\varepsilon_0}(z) - N_{\varepsilon_0}(-z) = 2^{\ell-2d}\Re(z);$$

$$N_{\varepsilon_0}(z) + N_{\varepsilon_0}(-z) = \begin{cases} 2^{\ell-2d+1}/|H_{\varepsilon_0}|, & q \in H_{\varepsilon_0}, \\ 0, & q \notin H_{\varepsilon_0}. \end{cases}$$

Theorem 4 above can be used (together with previous results by several authors, detailed in Section 2) to give an effective enumeration of the $3 \cdot 2^n$ connected components of $BL_\eta$ for $n \geq 5$ (see Proposition 11.3). The theorem also implies that, if $z \in \eta \text{Quat}_{n+1}$ and $\Re(z) < 0$ then the Euler characteristic $\chi(BL_z)$ is even. For $n \leq 4$, $BL_z$ has an even number of contractible connected components. For $n \geq 5$, however, the number of connected components is odd and all except one are contractible: this last connected component has therefore even Euler characteristic.

Section 2 outlines the context and history of this problem. In Section 3 we review some notation and useful facts about permutations; we are particularly interested in the Bruhat order and in reduced words. In Section 4 we define preancestries, one of the main combinatorial tools in our work. In Section 5 we present a quick construction of Clifford algebras and prove a few results. In Section 6 we construct the finite group $\tilde{B}_{n+1}^+ \subset \text{Spin}_{n+1}$. Ancestries are defined based on preancestries and on the group $\tilde{B}_{n+1}^+$ in Section 7. The concept of a
thin ancestry is discussed in Section 8: this will tie in with the concept of total
positivity. Section 9 is largely a review of [5]: we discuss the Bruhat decomposition
of Spin_{n+1} and define the lifted Bruhat order in \( \tilde{B}_{n+1}^+ \). This induces a partial
order in the set of ancestries (for a fixed reduced word). We already mentioned in
Equation (1.3) that the set \( \text{BL}_\sigma \) is the disjoint union of closed and open disjoint
subsets \( \text{BL}_\varepsilon \): this is explained in Section 10.

The second part of the paper begins with Section 11 where we present first
examples of the stratification of \( \text{BL}_\sigma \) (or of \( \text{BL}_\varepsilon \)) into subsets \( \text{BLS}_\varepsilon \) where \( \varepsilon \) is an
ancestry. The stratification is presented in full generality in Section 12: this is
where we complete the proof of Theorem 1. Basic properties of the strata \( \text{BLS}_\varepsilon \)
are proved in Section 13. These are smooth contractible submanifolds. Whitney’s
property does not hold (Example 13.4). Nevertheless, the partial order among
ancestries defined in Section 9 (based on the lifted Bruhat order) allows us to
construct a dual CW complex and prove Theorem 2 is Section 14. Lemma 14.1
gives us the recursive step: this is a topological result similar to many well known
results, such as Poincaré duality, but we did not find a reference with the required
generality. In Section 15 we apply Theorem 2 to deduce information about the
Euler characteristic of \( \text{BL}_\varepsilon \) and of its connected components: we complete the
proof of the third item of Theorem 1 in Corollary 15.3 and Remark 15.4. As with
any CW complex, the glueing maps for \( \text{BLC}_\sigma \) are extremely important: they are
studied in Section 16. We define the concept of a tame ancestry: ancestries in
low dimensional examples are tame and for tame ancestries, the glueing map is
easy. In Section 17 we apply our results to compute the homotopy type of \( \text{BL}_\varepsilon \)
in several examples; we also complete the proof of Theorems 1 and 3. The proof of
Theorem 3 is by then a long computation. Some examples are given in Examples
17.1, 17.2, 17.4 and 17.5. Other examples are discussed in greater detail in [1]. In
particular, the case \( n = 4 \) (corresponding to \( S_5 \)) and \( \sigma = \eta \), the top permutation,
is a larger example than the ones discussed here, with cells of dimension up to 4.
This example is carefully described in [1].

Acknowledgements. The authors warmly thank Dan Burghelea, Victor Goulart,
Giovanna C. Leal, Boris Shapiro, Michael Shapiro and Cong Zhou for helpful
conversations and the referee for a very careful report. The first author wants to
acknowledge the hospitality of the Department of Mathematics, Stockholm Uni-
versity in February 2019 when Boris Shapiro proposed the problem which would
eventually lead to this paper. The first author was working at the Mathematics
Department of PUC-Rio during part of the time when this work was produced.
The second author gratefully acknowledges the support of CNPq, CAPES and
Faperj (Brazil).
2 Context and history of the problem

Bruhat/Schubert cell decompositions of Grassmannians and various spaces of flags have been used in mathematics for more than a century and are standard objects/tools in e.g. topology, enumerative geometry and representation theory. Intersections of pairs and more general collections of Bruhat cells appear naturally in several areas such as singularity theory, Kazhdan-Lusztig theory, matroid theory, to mention a few. In spite of their importance, to the best of our knowledge, there is hardly any topological information available about such intersections, see e.g. [16] and references therein.

One exception from this general situation is the problem of counting connected components in pairwise intersections of big (i.e., top-dimensional) Bruhat cells over the reals. Substantial progress was obtained in the late 90’s, see [14, 15, 17, 11, 12, 4, 18]. In short, this problem can be reduced to counting the orbits of a certain finite group of symplectic transvections acting on a finite-dimensional vector space over the finite field \( \mathbb{F}_2 \) (also denoted by \( \mathbb{Z}/(2) \)). Both the group and the vector space are uniquely determined by the pair of Bruhat cells under consideration, see [17]. Further information about counting such orbits can be found in [13].

Consider the real flag space \( \text{Flag}_{n+1} = \text{SL}_{n+1} / \text{Up}_{n+1} = \text{Spin}_{n+1} / \text{Quat}_{n+1} \): notice that \( \text{Up}_{n+1} \) is the Borel subgroup (usually called \( B \)). The intersection of two opposite big Bruhat cells in \( \text{Flag}_{n+1} \) is homeomorphic to \( BL_\eta \) (as in Equation (1.1)). The number of connected components of \( BL_\eta \) equals 2, 6, 20, 52 for \( n = 1, 2, 3, 4 \) respectively. Starting from \( n = 5 \), the number of connected components stabilizes and is given by \( 3 \cdot 2^n \). This is explained by the possibility to embed, for \( n \geq 5 \), the lattice \( E_6 \) in a certain lattice arising in this problem, see [15].

Observe that the relative positions of two big Bruhat cells in \( \text{Flag}_{n+1} \) are in bijective correspondence with \( S_{n+1} \). In particular, opposite big Bruhat cells correspond to the longest permutation \( \eta \). The study of the number of connected components in the intersection of two big cells in a given relative position \( \sigma \) was initiated in § 7 of [15]. For each concrete \( \sigma \), the number can, in principle, be deduced from the results of [13] obtained about two decades ago. However, to the best of our knowledge, there is no closed formula.

The main goal of the present paper is to introduce the appropriate tools which allow us to study the homotopy type the latter intersections. As a corollary, we obtain in Proposition 11.3 an effective labeling of the connected components of \( BL_\eta \). We make extensive use of notation and results from [5], where the Bruhat cell decomposition of \( \text{Spin}_{n+1} \) is studied: we review the needed facts in Section 9. We use Clifford algebras. In order to keep this paper more self-contained we review the basic constructions in Section 5, see [2, 10] for far more complete
discussions. We also need some basic facts from the theory of totally positive matrices; [3] is an excellent reference and contains much more than we need.

The stratification we construct appears to be different but closely related to the one constructed by V. V. Deodhar. As of this writing, the precise relationship between the two stratifications is unfortunately not clear to us.

3 Permutations

Recall that if \( \sigma \in S_{n+1} \) is a permutation, we write \( i^\sigma \) (and not \( \sigma(i) \)) and we compose left to right so that \( i^\sigma \circ \sigma_1 = (i^\sigma)_{\sigma_1} \); we use several notations for permutations \( \sigma \in S_{n+1} \). The complete notation just lists the values of \( 1^\sigma, 2^\sigma, \ldots, (n + 1)^\sigma \); we usually enclose the list in square brackets but sometimes omit them for brevity. Another common notation is to write \( \sigma \) as a product of disjoint cycles: Figure 1 shows \( \sigma = [34512] = (14)(235) \). Perhaps the most important notation for us is to write \( \sigma \) as a reduced word, a word of minimal length \( \ell = \text{inv}(\sigma) \) in the standard Coxeter-Weyl generators \( a_1, \ldots, a_n \) (see Equation (1.2)). Here \( a_i \) is the transposition \((i, i + 1)\). Figure 1 shows a wiring diagram, a visual representation of a reduced word. In that example, \( \sigma = a_1a_3a_2a_1a_4a_3a_2 \); in our system, each crossing is a generator, read left-to-right, and the row indicates the value of \( i \). The identity permutation is denoted by \( 1 \in S_{n+1} \).

Given \( \sigma \in S_{n+1} \), an inversion is a pair \((i, j) \in [n+1]^2 \) with \( i < j \) and \( i^\sigma > j^\sigma \). Let \( \text{Inv}(\sigma) \) be the set of inversions of \( \sigma \); let \( \text{inv}(\sigma) = |\text{Inv}(\sigma)| \). Crossings in a wiring diagram for a reduced word for \( \sigma \) naturally correspond to inversions of \( \sigma \). Indeed, for \( i \in [n+1] \), the wire \( i \) is the curve joining the point \( i \) on the left to the point \( i^\sigma \) on the right. Given an inversion \((i, j)\), the wires \( i \) and \( j \) cross exactly once: that is the crossing \((i, j)\); sometimes we omit parenthesis and comma for brevity. For the first reduced word in Figure 1, the inversions are, in order, 12, 34, 14, 12, 35, 15, 25. For the second reduced word the inversions are the same but the order is different: 34, 24, 14, 12, 35, 15, 25. More algebraically, if \( \ell = \text{inv}(\sigma) \) and \( k \) is the crossing \((i, j)\) then

\[
\sigma = a_{i_1} \cdots a_{i_k} \cdots a_{i_\ell} = (ij) a_{i_1} \cdots a_{i_{k-1}} a_{i_{k+1}} \cdots a_{i_\ell} = a_{i_1} \cdots a_{i_{k-1}} a_{i_{k+1}} \cdots a_{i_\ell} (j^\sigma i^\sigma). \tag{3.1}
\]

Notice that the subword \( a_{i_1} \cdots a_{i_{k-1}} a_{i_{k+1}} \cdots a_{i_\ell} \) is usually not reduced.

We use the strong Bruhat order throughout the paper, and write \( \sigma_0 < \sigma_1 \) if \( \sigma_0 < \sigma_1 \) (in the strong Bruhat order) and \( \text{inv}(\sigma_1) = 1 + \text{inv}(\sigma_0) \). Equivalently, \( \sigma_0 < \sigma_1 \) if there exists a reduced word \( \sigma_1 = a_{i_1} \cdots a_{i_\ell} \) such that \( \sigma_0 = a_{i_1} \cdots a_{i_{k-1}} a_{i_{k+1}} \cdots a_{i_\ell} \) is also a reduced word.

A permutation \( \sigma \in S_{n+1} \) blocks at \( j \) (where \( 1 \leq j \leq n \)) if and only if \( i \leq j \) implies \( i^\sigma \leq j \). Equivalently, \( \sigma \) blocks at \( j \) if and only if \( a_j \) does not appear in
a reduced word for $\sigma$. Let $\text{Block}(\sigma)$ be the set of $j$ such that $\sigma$ blocks at $j$ and $b = \text{block}(\sigma) = |\text{Block}(\sigma)|$. A permutation $\sigma$ does not block if $\text{block}(\sigma) = 0$.

Given $\sigma_0 \in S_j$ and $\sigma_1 \in S_k$ define

$$\sigma = \sigma_0 \oplus \sigma_1 \in S_{j+k}, \quad \sigma^i = \begin{cases} \sigma_0^i, & i \leq j, \\ \sigma_1^{i-j} + j, & i > j. \end{cases}$$

If $\sigma \in S_{n+1}$ blocks at $j$ then there exist permutations $\sigma_0 \in S_j$ and $\sigma_1 \in S_{n+1-j}$ such that $\sigma = \sigma_0 \oplus \sigma_1$. In this case, the permutation matrices satisfy $P_\sigma = P_{\sigma_0} \oplus P_{\sigma_1}$, i.e., the matrix $P_\sigma$ has two diagonal blocks $P_{\sigma_0}$ and $P_{\sigma_1}$ (and is zero elsewhere).

**Remark 3.1.** If $\sigma = \sigma_0 \oplus \sigma_1$ then $BL_\sigma = BL_{\sigma_0} \oplus BL_{\sigma_1}$, meaning that $L \in BL_\sigma$ if and only if there exist $L_0 \in BL_{\sigma_0}$ and $L_1 \in BL_{\sigma_1}$ with $L = L_0 \oplus L_1$. As a manifold, we have that $BL_\sigma$ is diffeomorphic to $BL_{\sigma_0} \times BL_{\sigma_1}$. Since our aim is to study the homotopy type of the sets $BL_\sigma$, we may focus on permutations $\sigma$ which do not block.

### 4 Preancestries

A **preancestry** for a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$ is a sequence $(\rho_k)_{0 \leq k \leq \ell}$ of permutations with the following properties:

1. $\rho_0 = \rho_\ell = \eta$;
2. for all $k \in \llbracket \ell \rrbracket$, we either have $\rho_k = \rho_{k-1}$ or $\rho_k = \rho_{k-1}a_{i_k}$;
3. for all $k \in \llbracket \ell \rrbracket$, if $\rho_{k-1}a_{i_k} > \rho_{k-1}$ then $\rho_k = \rho_{k-1}a_{i_k}$.

It is usually more convenient to represent a preancestry $(\rho_k)$ by the sequence $\varepsilon_0 : \llbracket \ell \rrbracket \to \{0, \pm 2\}$ with

$$\varepsilon_0(k) = \begin{cases} 0, & \rho_k = \rho_{k-1}, \\
-2, & \rho_k = \rho_{k-1}a_{i_k} < \rho_{k-1}, \\
+2, & \rho_k = \rho_{k-1}a_{i_k} > \rho_{k-1}. \end{cases} \quad (4.1)$$

Thus, a sequence $\varepsilon_0$ is a **preancestry** if the sequence of permutations $(\rho_k)_{0 \leq k \leq \ell}$ defined by the recursion below is a preancestry:

$$\rho_0 = \eta, \quad \rho_k = \begin{cases} \rho_{k-1}a_{i_k}, & \varepsilon_0(k) \neq 0, \\
\rho_{k-1}, & \varepsilon_0(k) = 0. \end{cases} \quad (4.2)$$
Figure 7: Two preancestries for $\sigma = a_2a_3a_2a_1a_2a_4a_3a_2 \in S_5$. The shaded region corresponds to the preancestry of dimension 1. The second preancestry has dimension 2.

The preancestry $\varepsilon_0$ is represented in the wiring diagram for $\sigma$: a black (resp. white) square indicates $-2$ (resp. $+2$).

Clearly, in any preancestry $\varepsilon_0$, the number of $k \in [\ell]$ for which $\varepsilon_0(k) = -2$ equals the number of $k \in [\ell]$ for which $\varepsilon_0(k) = +2$: this number is denoted by $d = \dim(\varepsilon_0)$, the dimension of the preancestry. Figure 7 shows two preancestries for $\sigma = a_2a_3a_2a_1a_2a_4a_3a_2 \in S_5$, of dimensions 1 and 2. Figure 8 shows all preancestries for $\eta = a_1a_2a_1a_3a_2a_1a_4a_3a_2a_1 \in S_5$.

Figure 8: The reduced word $\eta = a_1a_2a_1a_3a_2a_1a_4a_3a_2a_1 \in S_5$ admits respectively 1, 6, 10, 5, 1 preancestries of dimensions 0, 1, 2, 3, 4.

**Lemma 4.1.** Consider a permutation $\sigma \in S_{n+1}$. The number of preancestries per dimension does not depend on the choice of the reduced word.

**Proof.** We know that two reduced words for $\sigma$ can be joined by a finite number of steps of the two following moves:

\[
\begin{align*}
    a_{i_1} \cdots a_{i_k} a_{i_{k+1}} \cdots a_{i_\ell} &= a_{i_1} \cdots a_{i_{k+1}} a_{i_k} \cdots a_{i_\ell}, & |i_k - i_{k+1}| \neq 1; \\
    a_{i_1} \cdots a_{i_k} a_{i_{k+1}} \cdots a_{i_\ell} &= a_{i_1} \cdots a_{i_{k+1}} a_{i_k} a_{i_{k+1}} \cdots a_{i_\ell}, & |i_k - i_{k+1}| = 1.
\end{align*}
\]
The effect of the first move is trivial. In fact, in Figures 5 and 8 diagrams have been drawn with several disjoint crossings on the same vertical line, thus blurring the distinction between reduced words differing by the first move.

We must therefore prove that if two reduced words differ by the second move then the number of preancentries of each dimension is the same. This can be verified by a finite computation, considering which of the three intersections involved in the move are marked by a preancestry. We omit the rather long case-by-case verification. Notes that there are three wires involved in the second move: they do not cross to the left or right of the move.

Remark 4.2. There is always a unique preancestry of dimension 0, obtained by marking no vertices. Preancestries of dimension 1 are also easy to classify: we must mark two consecutive intersections on the same row. In other words, a preancestry of dimension 1 corresponds to a bounded component of the complement of the wiring diagram, and the two marked intersections are the left and right extremes. The component is shown shaded in Figure 7. The number of preancestries of dimension 1 is $\ell - n + b$ where $\ell = \text{inv}(\sigma)$ and $b = \text{block}(\sigma)$.

For dimension 2 the situation is slightly more complicated. For a preancestry $\varepsilon_0$ of dimension 2, let $k_1 < k_2 < k_3 < k_4$ be such that $|\varepsilon_0(k_i)| = 2$. We always have $\varepsilon_0(k_1) = -2$ and $\varepsilon_0(k_4) = +2$. If $\varepsilon_0(k_2) = +2$, we have $\varepsilon_0(k_3) = -2$, $i_{k_1} = i_{k_2}$, $i_{k_3} = i_{k_4}$. In this case, intersections $k_1$ and $k_2$ are consecutive on row $i_{k_1}$ and intersections $k_3$ and $k_4$ are consecutive on row $i_{k_4}$. The first three preances on the third row of Figure 8 are examples. If $\varepsilon_0(k_2) = -2$ and $|i_{k_1} - i_{k_2}| > 1$ we also have two pairs of consecutive intersections on two rows. The preancestry on the fourth row, third column of Figure 8 is an example. In both cases, the preancestry $\varepsilon_0$ is said to be of type I.

If $\varepsilon_0(k_2) = -2$ (which implies $\varepsilon_0(k_3) = +2$) and $|i_{k_1} - i_{k_2}| = 1$ then $\varepsilon_0$ is said to be of type II. In this case, $i_{k_1} = i_{k_2}$, $i_{k_2} = i_{k_3}$ and intersections $k_2$ and $k_3$ are consecutive on row $i_{k_2}$. Intersection $k_1$ is the last on row $i_{k_1}$ before $k_2$ and intersection $k_4$ is the first on row $i_{k_4}$ after $k_3$. There is no bound, however, on the number of intersections on row $i_{k_1}$ between $k_2$ and $k_3$. Figure 7 shows a preancestry of type II for which there are two such intersections.

The subword formed by all marked letters has value 1. The subword formed by unmarked letters is also informative, as the following lemma shows. It also gives an estimate for the possible dimensions of preances.

Lemma 4.3. Consider $\sigma \in S_{n+1}$ and a fixed reduced word of length $\ell = \text{inv}(\sigma)$. Consider a preancestry $\varepsilon_0$ of dimension $d = \text{dim}(\varepsilon_0)$. There are $\delta = \ell - 2d$ unmarked crossings $k_1, \ldots, k_\delta$. Assume that the unmarked crossing $k_j$ is $(i_{j,0}, i_{j,1}) \in \text{Inv}(\sigma)$. We then have

$$\sigma = (i_{\delta,0}i_{\delta,1}) \cdots (i_{1,0}i_{1,1}).$$

If $c = \text{nc}(\sigma)$ is the number of cycles then $2d \leq l + c - n - 1$. 

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Proof. Join marked crossings into subwords $\sigma_j$ to write

$$\sigma = \sigma_0 a_{i_{k_1}} \sigma_1 \cdots \sigma_{\delta - 1} a_{i_{k_\delta}} \sigma_\delta, \quad 1 = \sigma_0 \sigma_1 \cdots \sigma_{\delta - 1} \sigma_\delta.$$ 

Apply Equation (3.1) to the crossing $k_\delta$ to obtain

$$\sigma = (\iota_{\delta,0}\iota_{\delta,1}) \sigma_0 a_{i_{k_1}} \sigma_1 \cdots \sigma_{\delta - 1} \sigma_\delta.$$ 

Repeat to obtain

$$\sigma = (\iota_{\delta,0}\iota_{\delta,1}) \cdots (\iota_{1,0}\iota_{1,1}) \sigma_0 \sigma_1 \cdots \sigma_{\delta - 1} \sigma_\delta = (\iota_{\delta,0}\iota_{\delta,1}) \cdots (\iota_{1,0}\iota_{1,1}),$$

proving the first claim. The second claim follows from observing that any cycle of length $k$ can be written as a product of $k - 1$ transpositions, but not fewer. \qed

![Preancestries of top dimension $d_{\text{max}}$ for $\eta \in S_{n+1}$, $n = 5$ and $n = 6$.](image)

**Lemma 4.4.** For $n \geq 2$, let $\eta \in S_{n+1}$ be the top permutation. The largest possible dimension among all preancestries is

$$d_{\text{max}} = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Furthermore, there exists a unique preancestry of dimension $d_{\text{max}}$.

**Proof.** From Lemma 4.1 we may use the reduced word

$$\eta = a_1 a_2 a_1 a_3 a_2 a_1 \cdots a_n a_{n-1} \cdots a_2 a_1.$$ 

The case $n = 4$ is addressed by Figure 8. Figure 9 shows a preancestry of dimension $d_{\text{max}}$ for cases $n = 5$ and $n = 6$. The pattern should be clear. The value of $d_{\text{max}}$ follows from these examples and from Lemma 4.3. Since $\eta$ can be written as product of $\left\lceil \frac{n}{2} \right\rceil$ transpositions in a unique way, the result follows. \qed
5 Clifford algebras

For each $k \in \mathbb{N}^*$, let $J_k$ be the $2^k \times 2^k$ real matrix defined by:

$$J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad J_k = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}, \quad I \in \mathbb{R}^{2^k-2 \times 2^{k-2}}, \quad k > 1.$$

For $k \in \mathbb{Z} = \{1, 2, \ldots, n\}$, let $\hat{a}_k$ be the real $2^n \times 2^n$ matrix with $2^{n-k}$ blocks $J_k$ along the diagonal:

$$\hat{a}_k = \begin{pmatrix} J_k \\ \vdots \\ J_k \end{pmatrix}.$$

The following relations are easily verified:

$$(\hat{a}_i)^2 = -I, \quad \hat{a}_j \hat{a}_i = (-1)^{|i-j|} [\hat{a}_i \hat{a}_j].$$

We use Iverson’s bracket notation: thus, for instance, $[|i-j| = 1] = 1$ if $|i-j| = 1$ and $[|i-j| = 1] = 0$ otherwise. Let $\mathbb{C}_{n+1}^0$ be the associative algebra generated by $\hat{a}_1, \ldots, \hat{a}_n$. In the notation of [2, 10], this is the subalgebra $\mathbb{C}_{n+1}^0 \subset \mathbb{C}_{n+1}$ of even elements of the Clifford algebra $\mathbb{C}_{n+1}$; we shall abuse notation and refer to $\mathbb{C}_{n+1}^0$ as the Clifford algebra. The set $\mathrm{HQuat}_{n+1}$ below is a real basis of $\mathbb{C}_{n+1}^0$:

$$\mathrm{HQuat}_{n+1} = \{[\hat{a}_1] [\hat{a}_2] \cdots [\hat{a}_n]\} = \{1, \hat{a}_1, \hat{a}_2, \hat{a}_1 \hat{a}_2, \hat{a}_3, \hat{a}_1 \hat{a}_2 \hat{a}_3, \ldots, \hat{a}_1 \hat{a}_2 \cdots \hat{a}_n\};$$

here the square brackets denote that the presence of the corresponding element in the product is optional. Notice that from now on we write 1 for $I \in \mathbb{C}_{n+1}^0$.

The set $\mathrm{Quat}_{n+1} = \mathrm{HQuat}_{n+1} \cup (- \mathrm{HQuat}_{n+1}) \subset \mathbb{C}_{n+1}^0$ is a group of cardinality $2^{n+1}$ generated by $\hat{a}_i$, $i \in \mathbb{Z}$. The subgroup $\{\pm 1\} \subset \mathrm{Quat}_{n+1}$ is normal and contained in the center: we have the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Quat}_{n+1} \to \mathrm{Diag}_{n+1}^+ \to 1; \quad (5.1)$$

the quotient $\mathrm{Diag}_{n+1}^+$ is naturally identified with the subgroup of diagonal matrices in $\mathrm{SO}_{n+1}$. Group-theoretic commutators in $\mathrm{Quat}_{n+1}$ will be occasionally important: set

$$[q_0, q_1] = q_0^{-1} q_1^{-1} q_0 q_1 \in \{\pm 1\} = [\mathrm{Quat}_{n+1}, \mathrm{Quat}_{n+1}] \subset \mathrm{Quat}_{n+1}. \quad (5.2)$$

For instance, if $q = \pm \hat{a}_1^\varepsilon(1) \cdots \hat{a}_n^\varepsilon(n)$ (with $\varepsilon(k) \in \mathbb{Z}$) then $[\hat{a}_i, q] = (-1)^{\varepsilon(i-1)+\varepsilon(i+1)}$ (we follow the convention that $\varepsilon(0) = \varepsilon(n+1) = 0$).

Define in $\mathbb{C}_{n+1}^0$ a positive definite inner product so that $\mathrm{HQuat}_{n+1}$ is an orthonormal basis: $\langle z_1, z_2 \rangle = 2^{-n} \mathrm{Trace}(z_1 z_2^\top)$. Define the real part of $z \in \mathbb{C}_{n+1}^0$ as...
by \( \mathcal{R}(z) = 2^{-n} \text{Trace}(z) = \langle z, 1 \rangle \): equivalently, \( \mathcal{R}(z) = c_1 \) if \( z = \sum_{q \in \text{Quat}} c_q q \). From now on we almost never consider the elements of \( \text{Cl}^0_{n+1} \) as matrices, rather as elements of an explicitly given associative algebra.

Define \( a_i^{\text{Spin}} = \frac{1}{2} \hat{a}_i \in \text{Cl}^0_{n+1} \) and \( a_i^{\text{SO}} \) to be the \( (n+1) \times (n+1) \) real skew symmetric matrix whose only non zero entries are \( (a_i^{\text{SO}})_{i+1,i} = 1 \) and \( (a_i^{\text{SO}})_{i,i+1} = -1 \). Let \( \text{spin}_{n+1} \subset \text{Cl}^0_{n+1} \) be the Lie algebra generated by \( a_i^{\text{Spin}} \), \( i \in [n] \). It can easily be verified that there exists a unique isomorphism of Lie algebras \( \Pi : \text{spin}_{n+1} \to \text{so}_{n+1} \) satisfying \( \Pi(a_i^{\text{Spin}}) = a_i^{\text{SO}} \). From now on we often identify the two isomorphic Lie algebras \( \text{spin}_{n+1} \) and \( \text{so}_{n+1} \). We write \( \alpha_i(\theta) = \exp(\theta a_i) \), defining one parameter subgroups \( \alpha_i^{\text{Spin}} : \mathbb{R} \to \text{Spin}_{n+1} \) and \( \alpha_i^{\text{SO}} : \mathbb{R} \to \text{SO}_{n+1} \); we often omit the superscripts. More explicitly, we have

\[
\alpha_i^{\text{SO}}(\theta) = \begin{pmatrix} I & \cos(\theta) - \sin(\theta) \\ \sin(\theta) & \cos(\theta) & I \end{pmatrix}, \quad \alpha_i^{\text{Spin}}(\theta) = \cos \left( \frac{\theta}{2} \right) + \sin \left( \frac{\theta}{2} \right) \hat{a}_i;
\]

in the first formula, the central block occupies rows and columns \( i \) and \( i+1 \). Let \( \text{Spin}_{n+1} \subset \text{Cl}^0_{n+1} \) be the group generated by \( \alpha_i^{\text{Spin}}(\theta) \) for \( i \in [n] \) and \( \theta \in \mathbb{R} \). Taking exponentials, we have the familiar double cover \( \Pi : \text{Spin}_{n+1} \to \text{SO}_{n+1} \). The function \( \mathcal{R} : \text{Spin}_{n+1} \to \mathbb{R} \) is \( 2^{-n} \) times a character, the Clifford algebra being the corresponding representation.

Let \( \text{Diag}_{n+1} \) be the group of diagonal matrices with diagonal entries in \( \{\pm 1\} \); this group acts by conjugations on \( \text{SO}_{n+1} \). The quotient \( \mathcal{E}_n = \text{Diag}_{n+1} / \{\pm 1\} \) is naturally isomorphic to \( \{\pm 1\}^{[n]} \); the matrix \( D \in \text{Diag}_{n+1} \) is taken to \( E \in \mathcal{E}_n = \{\pm 1\}^{[n]} \), \( E_i = D_{i,i}D_{i+1,i+1} \). The group \( \mathcal{E}_n \) also acts by automorphisms on \( \text{SO}_{n+1} \).

This action can be lifted to \( \text{Spin}_{n+1} \) and then extended to \( \text{Cl}^0_{n+1} \). Indeed, each \( E \in \mathcal{E}_n \) defines automorphisms of \( \text{Spin}_{n+1} \) and \( \text{Cl}^0_{n+1} \) given by

\[
(\hat{a}_i)^E = E_i \hat{a}_i, \quad (\alpha_i(\theta))^E = \alpha_i(E_i \theta).
\]

**Remark 5.1.** If \( n \) is even, conjugation by an element of \( \text{Quat}_{n+1} \) defines the same set of automorphisms as the action above. If \( n \) is odd, however, the element \( \hat{a}_1 \hat{a}_3 \cdots \hat{a}_{n-2} \hat{a}_n \) is in the center of \( \text{Cl}^0_{n+1} \). In this case, conjugation accounts for only half of the automorphisms above.

The following result gives a formula for \( \mathcal{R}(z) \), \( z \in \text{Spin}_{n+1} \), in terms of the eigenvalues of \( Q = \Pi(z) \in \text{SO}_{n+1} \).

**Lemma 5.2.** For \( z \in \text{Spin}_{n+1} \subset \text{Cl}^0_{n+1} \), let \( Q = \Pi(z) \in \text{SO}_{n+1} \) have eigenvalues \( \exp(\pm \theta_1 i), \ldots, \exp(\pm \theta_k i), 1, \ldots, 1 \). Then

\[
\mathcal{R}(z) = \pm \cos \left( \frac{\theta_1}{2} \right) \cdots \cos \left( \frac{\theta_k}{2} \right).
\]
Notice, in particular, that $\Re(z) = 0$ if and only if $-1$ is an eigenvalue of $Q$.

Proof. The function $\Re : \Spin_{n+1} \to \mathbb{R}$ is invariant under conjugation. We may therefore assume that $Q = \Pi(z) \in \SO_{n+1}$ is of the form:

$$Q = \text{diag} \left( \left( \begin{array}{cc} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{array} \right), \ldots, \left( \begin{array}{cc} \cos \theta_k & -\sin \theta_k \\ \sin \theta_k & \cos \theta_k \end{array} \right) \right), 1, \ldots, 1,$$

$$z = \alpha_1 (\theta_1) \alpha_3 (\theta_2) \cdots \alpha_{2k-1} (\theta_k)$$

$$= \left( \cos \left( \frac{\theta_1}{2} \right) + \sin \left( \frac{\theta_1}{2} \right) \hat{a}_1 \right) \cdots \left( \cos \left( \frac{\theta_k}{2} \right) + \sin \left( \frac{\theta_k}{2} \right) \hat{a}_{2k-1} \right).$$

The result now follows from an explicit computation. \qed

6 The group $\tilde{B}_{n+1}^+$

Let $B_{n+1}$ be the Coxeter-Weyl group of signed permutation matrices; let $B_{n+1}^+ = B_{n+1} \cap \SO_{n+1}$. For the double cover $\Pi : \Spin_{n+1} \to \SO_{n+1}$, let $\tilde{B}_{n+1}^+ = \Pi^{-1}[B_{n+1}^+]$. The finite group $\tilde{B}_{n+1}^+ \subset \Spin_{n+1}$ is generated by

$$\hat{a}_i = \alpha_i \left( \frac{\pi}{2} \right) = \frac{1}{\sqrt{2}} \hat{a}_i, \quad \hat{a}_i = (\hat{a}_i)^{-1} = \alpha_i \left( -\frac{\pi}{2} \right) = \frac{1}{\sqrt{2}} \hat{a}_i, \quad i \in [n]$$

(of course, $\alpha_i = \alpha_i^{\Spin}$). Clearly, we have $\hat{a}_i = (\hat{a}_i)^2 = -(\hat{a}_i)^2$. There is a natural short exact sequence

$$1 \to \text{Quat}_{n+1} \to \tilde{B}_{n+1}^+ \to \SO_{n+1} \to 1. \quad (6.1)$$

If $\sigma = a_{i_1} \cdots a_{i_r}$ is a reduced word, define $\hat{\sigma} = \hat{a}_{i_1} \cdots \hat{a}_{i_r} \in \tilde{B}_{n+1}^+$: it turns out that different reduced words yield the same answer ([5], Lemma 3.2). If $\Pi = \Pi_{\tilde{B}_{n+1}^+, \SO_{n+1}} : \tilde{B}_{n+1}^+ \to \SO_{n+1}$ is as in Equation (6.1), the set $\Pi^{-1}[\{\sigma\}] = \hat{\sigma} \text{Quat}_{n+1} \subset \tilde{B}_{n+1}^+$ is a coset (both left and right). Recall that the group $\mathcal{E}_n = \{\pm 1\}^{[n]}$ acts by automorphisms on $\Spin_{n+1}$ (see Equation (5.3)): these are also automorphisms of $\tilde{B}_{n+1}^+$. Furthermore, $\Re(z^E) = \Re(z)$ for all $z \in \tilde{B}_{n+1}^+$ and $E \in \mathcal{E}_n$.

We now compute $\Re(z)$ for $z \in \tilde{B}_{n+1}^+$ using information about $\sigma = \Pi(z) \in \SO_{n+1}$.

Remark 6.1. The Chebyshev polynomials $T_k$ are recursively defined by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$$

and satisfy $\cos(kt) = T_k(\cos(t))$. Thus, for $k > 0$, the leading term of $T_k(x)$ is $2^{k-1}x^k$ and the independent term is 0 if $k$ is odd and $(-1)^{k/2}$ if $k$ is even. \hfill ♦
Example 6.2. Consider \( z_0 \in \tilde{B}_{n+1}^+ \) such that \( Q_0 \in B_{n+1}^+ \) is a cycle of odd length \( k \). More precisely, there exist \( i_1, \ldots, i_k \) with \( (e_{i_j})^T Q_0 = (e_{i_{j+1}})^T \) for \( 1 \leq j < k \), \( (e_{i_k})^T Q_0 = (e_{i_1})^T \) and \( (e_i)^T Q_0 = (e_i)^T \) otherwise. The eigenvalues of \( Q_0 \) are 1 (with some large multiplicity) and simple eigenvalues

\[
\exp \left( \pm \frac{2\pi i}{k} \right), \exp \left( \pm \frac{4\pi i}{k} \right), \ldots, \exp \left( \pm \frac{(k-1)\pi i}{k} \right).
\]

It follows from Lemma 5.2 that \( \Re(z) = \pm P \) where \( P \) is the product

\[
P = \cos \left( \frac{\pi}{k} \right) \cos \left( \frac{2\pi}{k} \right) \cdots \cos \left( \frac{(k-3)\pi}{2k} \right) \cos \left( \frac{(k-1)\pi}{2k} \right).
\]

But \( -P^4 \) is the product of the roots of \( T_{2k}(x) - 1 = 0 \): since the leading coefficient of \( T_{2k} \) is \( 2^{2k-1} \) and the independent coefficient is \( -1 \) we have \( P^4 = 2^{-2k+2} \). We therefore have \( \Re(z_0) = \pm 2^{(-k+1)/2} \).

Example 6.3. Consider \( z_0 \in \tilde{B}_{n+1}^+ \) such that \( Q_0 \in B_{n+1}^+ \) is a cycle of even length \( k \). More precisely, there exist \( i_1, \ldots, i_k \) with \( (e_{i_j})^T Q_0 = (e_{i_{j+1}})^T \) for \( 1 \leq j < k \), \( (e_{i_k})^T Q_0 = -(e_{i_1})^T \) and \( (e_i)^T Q_0 = (e_i)^T \) otherwise. By adapting the computations in Example 6.2 we have \( \Re(z_0) = \pm 2^{(-k+1)/2} \).

Let \( \text{Diag}_{n+1}^+ = \text{Diag}_{n+1} \cap \text{SO}_{n+1} \). Given a partition \( X \) of the set \( [n + 1] = \{1, \ldots, n + 1\} \) we define a subgroup \( H_{\text{Diag}, X} \leq \text{Diag}_{n+1}^+ \) of index \( 2^{|X|-1} \). Given \( X, H_{\text{Diag}, X} \) is the set of matrices \( E \in \text{Diag}_{n+1} \subset \text{SO}_{n+1} \) such that, if \( A = \{i_1, i_2, \ldots, i_k\} \in X \) then the product \( E_{i_1} E_{i_2} \cdots E_{i_k} \) equals 1. Let \( H_X = \Pi^{-1}[H_{\text{Diag}, X}] \leq \text{Quat}_{n+1} \), where \( \Pi : \text{Quat}_{n+1} \to \text{Diag}_{n+1}^+ \) is the restriction of \( \Pi : \text{Spin}_{n+1} \to \text{SO}_{n+1} \).

For a permutation \( \sigma \), consider the partition \( X_{\sigma} \) of \( [n + 1] \) into cycles. Let \( H_{\sigma} = H_{X_{\sigma}} \leq \text{Quat}_{n+1} \). We have \( |H_{\sigma}| = 2^{n+2-c} \).

Example 6.4. Take \( n = 4 \), \( \sigma = 53421 = (15)(234) = a_1 a_2 a_3 a_2 a_1 a_4 a_3 a_2 a_1 \). The subgroup \( H_{\text{Diag}, \sigma} \) is spanned by

\[
\text{diag}(-1, 1, 1, 1, 1), \text{diag}(1, -1, -1, 1, 1), \text{diag}(1, 1, -1, -1, 1) \in \text{Diag}_{5}^+:
\]

these diagonal matrices change an even number of signs in a single cycle. Lifting to \( H_{\sigma} \) we have the generators

\[
\hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4, \hat{a}_2, \hat{a}_3 \in \text{Quat}_5,
\]

\[
H_{\sigma} = \{ \pm 1, \pm \hat{a}_2, \pm \hat{a}_3, \pm \hat{a}_2 \hat{a}_3, \pm \hat{a}_1 \hat{a}_4, \pm \hat{a}_1 \hat{a}_2 \hat{a}_4, \pm \hat{a}_1 \hat{a}_3 \hat{a}_4, \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \}.
\]

Notice that have \( c = 2 \) and therefore \( |H_{\sigma}| = 2^4 = 16 \).
Lemma 6.5. Consider $\sigma \in S_{n+1}$: assume that $\sigma$ has $c = nc(\sigma)$ disjoint cycles. Choose $z_0 \in \sigma \text{Quat}_{n+1}$ such that $\Re(z_0) > 0$. For $q \in \text{Quat}_{n+1}$, we have

$$|\Re(qz_0)| = |\Re(z_0q)| = \begin{cases} 2^{-(n+1-c)/2}, & q \in H_\sigma, \\ 0, & q \notin H_\sigma. \end{cases}$$

There are $2^{n+1-c}$ values of $q \in \text{Quat}_{n+1}$ such that $\Re(qz_0) > 0$ (and similarly for $\Re(z_0q)$). Also, if $z_0$ is expanded in the canonical basis as $z_0 = \sum_{q \in H\text{Quat}_{n+1}} c_q q$ then $c_q \neq 0$ if and only if $q \in H_\sigma$.

Proof. For $z \in \sigma \text{Quat}_{n+1}$, take $Q = \Pi(z)$ and consider its cycles and submatrices $Q_1, \ldots, Q_c$ of dimensions $k_1, \ldots, k_c$ with $k_1 + \cdots + k_c = n + 1$. If $\det(Q_i) = -1$ for some $i$ then $-1$ is an eigenvalue of $Q_i$ and therefore of $Q$ which implies that $\Re(z) = 0$. Otherwise, apply Lemma 5.2 to compute $\Re(z)$ as product, imitating Examples 6.2 and 6.3 to obtain $\Re(z) = \pm 2^{-(n+1-c)/2}$. It is easy to relate these properties with the definition of the subgroup $H_\sigma$. □

Example 6.6. Take $n = 4$ and $\sigma = a_1 a_2 a_3 a_2 a_1 a_4 a_3 a_2 a_1$ as in Example 6.4. A computation gives

$$\dot{\sigma} = \frac{-\dot{a}_1 - \dot{a}_2 + \dot{a}_1 \dot{a}_2 + \dot{a}_1 \dot{a}_3 - \dot{a}_1 \dot{a}_2 \dot{a}_3 - \dot{a}_4 + \dot{a}_2 \dot{a}_4 - \dot{a}_3 \dot{a}_4 - \dot{a}_2 \dot{a}_3 \dot{a}_4}{2\sqrt{2}};$$

notice that $1/\sqrt{8} = 2^{-(n+1-c)/2}$. We may choose

$$z_0 = \dot{a}_1 \dot{\sigma} = \frac{1 + \dot{a}_2 - \dot{a}_3 + \dot{a}_2 \dot{a}_3 - \dot{a}_1 \dot{a}_4 + \dot{a}_1 \dot{a}_2 \dot{a}_4 - \dot{a}_1 \dot{a}_3 \dot{a}_4 - \dot{a}_1 \dot{a}_2 \dot{a}_3 \dot{a}_4}{2\sqrt{2}};$$

notice that the nonzero coefficients match the elements of $H_\sigma$. □

7 Ancestries

An ancestry is a sequence $(\varrho_k)_{0 \leq k \leq \ell}$ of elements of $\tilde{B}_{n+1}^+$ such that:

1. $\varrho_0 = \dot{\eta}$, $\varrho_\ell \in \dot{\eta} \text{Quat}_{n+1}$;

2. for all $k$, we have $\varrho_k = \varrho_{k-1}$ or $\varrho_k = \varrho_{k-1} \dot{a}_{i_k}$ or $\varrho_k = \varrho_{k-1} \dot{a}_{i_k}$;

3. the sequence $(\rho_k)$ defined by $\rho_k = \Pi_{\tilde{B}_{n+1}^+, S_{n+1}}(\varrho_k)$ is a preancestry.

In this case, we say that the ancestry $(\varrho_k)$ corresponds to the preancestry $(\rho_k)$. The third condition is equivalent to saying that $\Pi(\varrho_{k-1}) < \Pi(\varrho_{k-1}) a_{i_k}$ implies $\varrho_k = \varrho_{k-1} a_{i_k}$ (for all $k$).
There are two sequences of integers and a sequence of elements of \( \text{Quat}_{n+1} \) which conveniently describe an ancestry. The sequence \( \xi : [\ell] \to \{0,1,2\} \) is recursively defined by \( \varrho_k = \varrho_{k-1}(\hat{a}_{ik})^{\xi(k)} \). Equivalently, we have
\[
\varrho_k = (\hat{a}_{i1})^{\xi(1)} \cdots (\hat{a}_{ik})^{\xi(k)}.
\]
The sequence \( (q_k)_{0 \leq k \leq \ell} \), \( q_k \in \text{Quat}_{n+1} \), is defined by \( \hat{\rho}_k q_k = \varrho_k \), so that in particular \( q_\ell = \eta \varrho_\ell \).

The third sequence may seem at first artificial, but we shall later see that it appears naturally in our problem. Given an ancestry (and therefore \( \xi \), \( (\rho_k) \) and \( (q_k) \)), define a sequence \( \varepsilon : [\ell] \to \{\pm 1, \pm 2\} \)
\[
\varepsilon(k) = \begin{cases} 
-2, & \xi(k) = 1, \rho_k < \rho_{k-1}, \\
+2, & \xi(k) = 1, \rho_k > \rho_{k-1}, \\
(1 - \xi(k))\hat{a}_{ik}, q_{k-1} & \xi(k) \neq 1.
\end{cases}
\] (7.1)

Conversely, \( \xi \) and \( (q_k) \) can be recovered from \( \varepsilon \) by:
\[
\xi(k) = \begin{cases} 
0, & \varepsilon(k) = [\hat{a}_{ik}, q_{k-1}], \\
2, & \varepsilon(k) = -[\hat{a}_{ik}, q_{k-1}], \\
1, & |\varepsilon(k)| = 2; \\
\end{cases}
\] (7.2)

Here \( [\hat{a}_{ik}, q_{k-1}] = (\hat{a}_{ik})^{-1}q_{k-1}^{-1}\hat{a}_{ik}q_{k-1} \in \{\pm 1\} \) is the commutator, as in Equation (5.2). We also have \( q_k = (\hat{\rho}_k)^{-1}q_k = (\hat{a}_{ik})^{-\text{sign}(\varepsilon(k))}q_{k-1}^{-1}\hat{a}_{ik} \). Thus, given the reduced word, each of the sequences \( (q_k) \), \( \xi \) and \( \varepsilon \) allows us to obtain \( (q_k) \) and the other two sequences; given the preancestry and \( (q_k) \), we can also obtain the three sequences above. We thus consider these three sequences to be alternate descriptions of an ancestry. An ancestry is represented in a wiring diagram by the sequence \( \varepsilon : -2 \) and \( +2 \) are represented by black and white squares, respectively, as for pre ancestries; \( -1 \) and \( +1 \) are represented by black and white disks. For an ancestry \( \varepsilon \), set \( P(\varepsilon) = \hat{\sigma} q_\ell^{-1} \in \hat{\sigma} \text{Quat}_{n+1}^+ \); we therefore have \( q_\ell = \eta(P(\varepsilon))^{-1}\hat{\sigma} \).

**Lemma 7.1.** For any ancestry \( \varepsilon \), we have
\[
P(\varepsilon) = (\hat{a}_{i1})^{\text{sign}(\varepsilon(1))} \cdots (\hat{a}_{i\ell})^{\text{sign}(\varepsilon(\ell))}.
\]

**Proof.** Let \( \sigma_k = a_{i1} \cdots a_{ik} \); we prove by induction that
\[
\hat{\sigma} q_k^{-1} = (\hat{a}_{i1})^{\text{sign}(\varepsilon(1))} \cdots (\hat{a}_{ik})^{\text{sign}(\varepsilon(k))}.
\]
The result follows by taking \( k = \ell \).

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Lemma 7.4. For any \( \varepsilon \) corresponding to \( \varepsilon \), we have \( P(\varepsilon) \) consistently with Lemma 7.1. ⋄

Remark 7.3. Recall that \( \varepsilon \) is defined as follows. If \( \varepsilon \) is the first summation is understood to be over all ancestries \( \varepsilon \). We claim that \( \xi = (2, 2, 0, 0, 2, 2, 0) \), \( \gamma = -\eta \hat{a}_2 \hat{a}_4 \), \( \eta = -\eta \hat{a}_2 \hat{a}_4 \) and therefore \( P(\varepsilon_1) = \delta(-\hat{a}_2 \hat{a}_4) = \hat{a}_3 \hat{a}_2 \hat{a}_1 \hat{a}_4 \hat{a}_3 \hat{a}_2 \), consistently with Lemma 7.1. ⋄

Example 7.2. Let \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) be the three ancestries shown in Figure 3.

We have \( \varepsilon_1 = (-1, +1, -1, -1, +1, -1, +1) \) for this ancestry we have \( \rho_k = \eta \) for all \( k \). We apply the recursion in Equation 7.2. We have \( q_0 = 1 \) and therefore \([\hat{a}_3, q_0] = 1\): thus, \( \xi(1) = 2 \), \( \gamma_1 = \eta \hat{a}_3 \) and \( q_1 = \hat{a}_3 \). We have \([\hat{a}_2, q_1] = -1 \) and therefore \( \xi(2) = 2 \), \( \gamma_2 = \eta \hat{a}_3 \hat{a}_2 \) and \( q_2 = -\hat{a}_2 \hat{a}_3 \). We have \([\hat{a}_1, q_2] = -1 \) and therefore \( \xi(3) = 0 \), \( \gamma_3 = -\eta \hat{a}_2 \hat{a}_4 \) and \( \gamma_3 = -\hat{a}_2 \hat{a}_3 \). Proceed in this manner to obtain \( \xi = (2, 2, 0, 0, 2, 2, 0) \), \( \gamma = -\eta \hat{a}_2 \hat{a}_4 \), \( \eta = -\hat{a}_2 \hat{a}_4 \) and therefore

\[
P(\varepsilon_1) = \delta(-\hat{a}_2 \hat{a}_4) = \hat{a}_3 \hat{a}_2 \hat{a}_1 \hat{a}_4 \hat{a}_3 \hat{a}_2 ,
\]

\[
\text{Remark 7.3. Recall that } N_{\varepsilon_0}(z) \text{ appears in the statement of Theorem 4. We shall see in Section 12 that } N_{\varepsilon_0}(z) = NL_{\varepsilon_0}(z) \text{. The following result is therefore one important step towards the proof of Theorem 4.} \quad \diamondsuit
\]

Lemma 7.4. For any \( z \in \sigma \text{ Quat}_{n+1} \), we have \( NL_{\varepsilon_0}(z) - NL_{\varepsilon_0}(-z) = 2^{\ell - 2d} \Re(z) \).

Proof. Set \( \delta = \ell - 2d \). We define the element of the Clifford algebra

\[
S = \sum_{\varepsilon} P(\varepsilon) = \sum_{z} NL_{\varepsilon_0}(z) z \in Cl_{n+1}^{0} ;
\]

the first summation is understood to be over all ancestries \( \varepsilon \) associated with \( \varepsilon_0 \). We claim that \( S = 2^{\frac{\ell}{2}} \cdot 1 \) (where \( 1 \in Cl_{n+1}^{0} \) is the unit of the Clifford algebra). Indeed, \( S \) can be written as a product \( S = 2^{\frac{\ell}{2}} w_1 w_2 \cdots w_\ell \) where \( w_k \in Cl_{n+1}^{0} \) is defined as follows. If \( \varepsilon_0(k) = 2 \) set \( w_k = \hat{a}_{i_k} \); if \( \varepsilon_0(k) = -2 \) set \( w_k = \hat{a}_{i_k} \); if \( \varepsilon_0(k) = 0 \) set \( w_k = (\hat{a}_{i_k} + \hat{a}_{i_k})/\sqrt{2} = 1 \in Cl_{n+1}^{0} \). Set \( s_k = w_1 w_2 \cdots w_k \); we have \( S = 2^{\frac{\ell}{2}} s_\ell \). From the recursive definition of \( \rho_k \) in Equation (4.2), we have \( s_k = \eta \hat{\rho}_k \).

It follows that \( s_\ell = \eta \hat{\eta} = 1 \), completing the proof of the claim.

On the other hand, since two distinct elements of \( \sigma \text{ Quat}_{n+1} \) are either antipodal or orthogonal, we have \( 2^{\frac{\ell}{2}} \Re(z) = \langle z, S \rangle = NL_{\varepsilon_0}(z) - NL_{\varepsilon_0}(-z) \), completing the proof of the lemma. \( \square \)

Given a preancestry \( \varepsilon_0 \), we define a partition \( X_{\varepsilon_0} \) which is coarser than \( X_\sigma \). Recall that \( X_\sigma \) is the partition of \([n + 1]\) into cycles of \( \sigma \). If \( k \in [\ell] \) is such that \( \varepsilon_0(k) = 0 \) and the \( k \)-th crossing is \((i_0, i_1)\) then \( \{i_0, i_1\} \) is contained in some \( A \in X_{\varepsilon_0} \). The partition \( X_{\varepsilon_0} \) is the finest partition respecting the conditions above. Let

\[
H_{\varepsilon_0} = H_{X_{\varepsilon_0}} \leq \text{Quat}_{n+1} \quad \quad (7.3)
\]

we clearly have \( H_{\sigma} \leq H_{\varepsilon_0} \).
Example 7.5. Let $n = 4$ and $\sigma = \eta = a_1a_2a_1a_3a_2a_1a_4a_3a_2a_1 = (15)(24)(3)$. We have $X_{\sigma} = \{\{1, 5\}, \{2, 4\}, \{3\}\}$ and $H_{\sigma} = \{\pm 1, \pm a_2a_3, \pm a_1a_4, \pm a_1a_2a_3a_4\}$.

As seen in Figure 8, there is only one preancestry of top codimension $d_{\text{max}} = 4$, which is $(-2, -2, -2, -2, 0, 2, 0, 2, 2, 2)$: then we have $X = X_{\sigma}$ and therefore $H = H_{\sigma}$.

There are 5 preancestries of codimension 3. For $(-2, -2, 0, -2, 0, 0, 0, 2, 2, 2)$, we have $X = \{\{1, 5\}, \{2, 3, 4\}\}$ and $H$ is generated by $H_{\sigma}$ and $a_2$ so that

$$H = \{\pm 1, \pm a_2, \pm a_3, \pm a_2a_3, \pm a_1a_4, \pm a_1a_2a_3a_4\}.$$  

For $(-2, 0, 2, -2, 0, 0, 2, 0, 2, 2)$, we have $X = \{\{1, 3, 5\}, \{2, 4\}\}$ and $H$ is generated by $H_{\sigma}$ and $a_1a_2$. For the other three, we have $X = \{\{1, 2, 4, 5\}, \{3\}\}$ and $H$ is generated by $H_{\sigma}$ and $\hat{a}_1$.

There are 10 preancestries of codimension 2. We have $X = \{\{1, 2, 3, 4, 5\}\}$ and therefore $H = \text{Quat}_5$ for 9 of them. The exception is $(0, -2, -2, 0, 0, 2, 0, 2, 0)$ for which $X = \{\{1, 2, 4, 5\}, \{3\}\}$. For all 6 preancestries of codimension 1 and for the preancestry of codimension 0, we have $X = \{\{1, 2, 3, 4, 5\}\}$ and $H = \text{Quat}_5$.

The following result is another important step towards Theorem 4.

Lemma 7.6. Consider a preancestry $\varepsilon_0$ and the subgroup $H_{\varepsilon_0} \leq \text{Quat}_{n+1}$. Choose $z_0 \in \mathfrak{d} \text{Quat}_{n+1}$ with $R(z) > 0$. For $z = qz_0$, we have

$$\text{NL}_{\varepsilon_0}(z) + \text{NL}_{\varepsilon_0}(-z) = \begin{cases} 2^{\ell - 2d + 1} / |H_{\varepsilon_0}|, & q \in H_{\varepsilon_0}, \\ 0, & q \notin H_{\varepsilon_0}. \end{cases}$$

Proof. For this proof it is convenient to introduce the (associative but not commutative) group ring $\mathbb{R}[B_{n+1}^+]$. Elements of $\mathbb{R}[B_{n+1}^+]$ are formal linear combinations $\sum_{Q \in B_{n+1}^+} c_Q Q$, $c_Q \in \mathbb{R}$. Thus, the set $B_{n+1}^+$ is a real basis for $\mathbb{R}[B_{n+1}^+]$; multiplication is inherited from $B_{n+1}^+$. We use names of elements of $B_{n+1}^+$, such as $\hat{a}_i$, as abbreviations of $1 \cdot Q \in \mathbb{R}[B_{n+1}^+]$, $Q = \Pi(\hat{a}_i)$. We thus have (in $\mathbb{R}[B_{n+1}^+]$) $(\hat{a}_i)^2 = 1$, $\hat{a}_i + \hat{a}_i = \hat{a}_i(1 + \hat{a}_i) = \hat{a}_i(1 + \hat{a}_i)$ and $(1 + \hat{a}_i)^2 = 2(1 + \hat{a}_i)$.

The commutative subring $\mathbb{R}[\text{Diag}_{n+1}^+] \subset \mathbb{R}[B_{n+1}^+]$ will also interest us. Given a subgroup $H \leq \text{Diag}_{n+1}^+$, define

$$p_H = \frac{1}{|H|} \sum_{E \in H} 1 \cdot E \in \mathbb{R}[\text{Diag}_{n+1}^+] .$$

We have $p_H^2 = p_H$ so that multiplication by $p_H$ on $\mathbb{R}[\text{Diag}_{n+1}^+]$ is the projection onto the subring $R_H$:

$$\sum_{E \in \text{Diag}_{n+1}^+} c_E \cdot E \in R_H \iff \forall E_0, E_1 \in \text{Diag}_{n+1}^+, (E_0E_1^{-1} \in H \rightarrow c_{E_0} = c_{E_1}).$$

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If $H_0, H_1 \leq \text{Diag}^+_{n+1}$ we have $p_{H_0} p_{H_1} = p_{H_0 H_1}$.

For a transposition $\tau = (ij) \in S_{n+1}$ (with $i < j$), write

$$E_{\tau} = \Pi(\hat{a}_i \hat{a}_{i+1} \cdots \hat{a}_{j-1}) \in \text{Diag}^+_{n+1},$$

the diagonal matrix with entries $(E_{\tau})_{ii} = (E_{\tau})_{jj} = -1$ and $(E_{\tau})_{kk} = +1$ for $k \notin \{i, j\}$. Write $H_{\tau} = \{I, E_{\tau}\}$ and $p_{\tau} = p_{H_{\tau}}$. A simple computation verifies the following identities in $\mathbb{R}[B^+_{n+1}]$:

$$\hat{a}_i p_{\tau} = p_{\tau a} \hat{a}_i, \quad \hat{a}_i p_{\tau} = p_{\tau a} \hat{a}_i. \tag{7.4}$$

Notice that $\tau^{a_i} = a_i \tau a_i$ is also a transposition.

We need to compute $N_z = \text{NL}_{z_0}(z) + \text{NL}_{z_0}(-z)$, which is the coefficient $c_Q$ of $Q = \Pi(z) \in B^+_{n+1}$ in $S = s_\ell \in \mathbb{R}[B^+_{n+1}]$, defined recursively by

$$s_0 = 1, \quad s_k = \begin{cases} s_{k-1} \cdot (\hat{a}_{ik} + \hat{a}_{ik}'), & \varepsilon_0(k) = 0, \\ s_{k-1} \cdot (\hat{a}_{ik}) \text{sign}(\varepsilon_0(k)), & \varepsilon_0(k) \neq 0. \end{cases}$$

Let $\delta = \ell - 2d$. Choose $\varepsilon$ with $P(\varepsilon) = z_0$ and rewrite $S$ as $S = 2^d \tilde{S} = 2^d \tilde{s}_\ell$,

$$\tilde{s}_0 = 1, \quad \tilde{s}_k = \begin{cases} \tilde{s}_{k-1} \cdot (\hat{a}_{ik}) \text{sign}(\varepsilon(k)) \cdot p_{\alpha_k}, & \varepsilon_0(k) = 0, \\ \tilde{s}_{k-1} \cdot (\hat{a}_{ik}) \text{sign}(\varepsilon_0(k)), & \varepsilon_0(k) \neq 0. \end{cases}$$

As in the proof of Lemma 4.3, let $k_1, \ldots, k_\delta$ be the unmarked crossings. Again assume that the unmarked crossing $k_j$ is $(i_{j,0}, i_{j,1}) \in \text{Inv}(\sigma)$. Let $\tau_j = (i_{j,0} i_{j,1})$. Applying Equation (7.4) to bring the terms $p_\sigma$ to the left, we have

$$\tilde{S} = p_{\tau_1} \cdots p_{\tau_\delta} \Pi(z_0) \in \mathbb{R}[B^+_{n+1}].$$

On the other hand, we have $H_{\tau_1} \cdots H_{\tau_\ell} = H_{\text{Diag}, X_{z_0}}$ and therefore

$$p_{\tau_1} \cdots p_{\tau_\delta} = p_{H_{\text{Diag}, X_{z_0}}} = \frac{2}{|H_{z_0}|} \sum_{\prod(E) \in H_{z_0}} 1 \cdot E \in \mathbb{R}[\text{Diag}^+_{n+1}]$$

and $S = 2^\delta p_{H_{\text{Diag}, X_{z_0}}} \Pi(z_0) \in \mathbb{R}[B^+_{n+1}]$, completing the proof. \hfill $\square$

## 8 Thin ancestries

Assume given a permutation $\sigma \in S_{n+1}$ and a reduced word for $\sigma$. An ancestry $\varepsilon$ of dimension 0 is thin if $i_{k_0} = i_{k_1}$ implies $\varepsilon(k_0) = \varepsilon(k_1)$. Otherwise, an ancestry is thick. There are therefore $2^{n-b}$ thin ancestries, where $b = \text{block}(\sigma)$. Consistently with Remark 3.1, we assume from now on that $\sigma$ does not block, i.e., that $b = 0$. 

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We shall see in Sections 11 and later that thin ancestries correspond to certain contractible connected components of \( BL_\sigma \).

For now, we are interested in counting thin solutions for equations of the form \( P(\varepsilon) = z \). In other words: let \( \varepsilon_0 \) be the empty preancestry and consider a fixed element \( z \in \dot{\mathrm{Quat}}_{n+1} \). By definition, there are \( NL_{\varepsilon_0}(z) \) ancestries \( \varepsilon \) corresponding to \( \varepsilon_0 \) and satisfying \( P(\varepsilon) = z \). Lemmas 7.4 and 7.6 allow us to compute \( NL_{\varepsilon_0}(z) \). Among these, let \( NL_{\text{thin}}(z) \) be the number of such ancestries \( \varepsilon \) which are thin. We want to compute \( NL_{\text{thin}}(z) \).

Recall that the multiplicative abelian group \( E_n \) acts as an automorphism group of \( \text{Cl}_{n+1} \) (see Equation (5.3) and Remark 5.1). For \( z \in \text{Spin}_{n+1} \), define \( E^z \subseteq E_n \) as the isotropy group of \( z \), i.e.,

\[
E^z = \{ E \in E_n \mid z^E = z \}.
\]

We want to study the group \( \mathcal{E}_n \). Let \( \tilde{\sigma}_n \) be the orbit of \( \tilde{\sigma} \) under \( E_n \):

\[
\tilde{\sigma} \mathcal{E}_n = \{ \tilde{\sigma}E, E \in \mathcal{E}_n \}.
\]

The following result is now easy.

**Lemma 8.3.** Let \( \tilde{c} \) be as in Lemma 8.1. For \( z \in \sigma \mathcal{Qut}_{n+1} \), we have

\[
NL_{\text{thin}}(z) = \begin{cases} 2^{n-\tilde{c}}, & z \in \sigma \mathcal{E}_n, \\ 0, & z \notin \sigma \mathcal{E}_n. \end{cases}
\]

Furthermore, \( |\sigma \mathcal{E}_n| = 2^{\tilde{c}} \).
Proof. Each $E \in \mathcal{E}_n$ gives us a thin ancestry $\varepsilon$ with $P(\varepsilon) = \sigma^E$. The result follows by the usual combinatorics of actions over finite sets.

Corollary 8.4. Consider $\sigma \in S_n$ which does not block and consider the empty preancestry $\varepsilon_0$. If $\ell = \text{inv}(\sigma) > 2n + 2$ then for all $z \in \sigma \text{ Quat}_{n+1}$ we have $NL_{\varepsilon_0}(z) > N\text{L}_{\text{thin}}(z)$.

Remark 8.5. In particular, for $\sigma = \eta$ and $n > 4$ the above condition holds.

Proof of Corollary 8.4. From Lemma 7.6, $NL_{\varepsilon_0}(z) + NL_{\varepsilon_0}(-z) \geq 2^{\ell - n}$. From Lemma 7.4, $NL_{\varepsilon_0}(z) - NL_{\varepsilon_0}(-z) \geq -2^{\ell/2}$. From Lemma 8.3, $NL_{\text{thin}} \leq 2^n$. Thus $NL_{\varepsilon_0}(z) - NL_{\text{thin}} \geq 2^{\ell - n - 1} - 2^{\ell/2 - 1} - 2^n \geq 2^{\ell - n - 1} - 2^{\ell/2} > 0$.

9 Bruhat cells in the spin group

This section contains further review of notation and results from [5]. We define the Bruhat cells in the orthogonal group. For $\sigma \in S_{n+1}$,

$$\text{Bru}_\sigma^{SO} = \{Q \in \text{SO}_{n+1} \mid \exists U_0, U_1 \in \text{Up}_{n+1}, Q = U_0P_\sigma U_1\};$$

compare with Equation (1.1). Let $\text{Bru}_\sigma = \Pi^{-1}[\text{Bru}_\sigma^{SO}] \subset \text{Spin}_{n+1}$ (where $\Pi : \text{Spin}_{n+1} \rightarrow \text{SO}_{n+1}$ is the double cover). The set $\text{Bru}_\sigma$ has $2^{n+1}$ connected components, each one containing an element $z$ of $\sigma \text{ Quat}_{n+1}$. For $z \in \tilde{B}_{n+1}^+$, let $\text{Bru}_z$ be the connected component of $\text{Bru}_\Pi(z)$ containing $z$. The set $\text{Bru}_z$ is a smooth contractible submanifold of $\text{Spin}_{n+1}$ of dimension $\ell = \text{inv}(\sigma)$, $\sigma = \Pi(z)$. We have

$$\text{Bru}_\sigma = \bigsqcup_{z \in \sigma \text{ Quat}_{n+1}} \text{Bru}_z, \quad \text{Spin}_{n+1} = \bigsqcup_{z \in \tilde{B}_{n+1}^+} \text{Bru}_z.$$ 

For a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$, an explicit parametrization of $\text{Bru}_\sigma$ is given by $\Phi : (0, \pi)^\ell \rightarrow \text{Bru}_\sigma$ (see Theorem 1, [5]):

$$\Phi(\theta_1, \ldots, \theta_\ell) = \alpha_{i_1}(\theta_1) \cdots \alpha_{i_\ell}(\theta_\ell).$$

The (strong) Bruhat order in $S_{n+1}$ can be defined by

$$\sigma_0 \leq \sigma_1 \quad \iff \quad \text{Bru}_{\sigma_0} \subseteq \text{Bru}_{\sigma_1}.$$ 

If $z \in \text{Bru}_{\sigma_0}$, $\sigma_0 = \Pi(z_0) < \sigma_1 = \sigma_0a_i$ then $z\alpha_i(\theta) \in \text{Bru}_{\sigma_0a_i}$ for $\theta \in (0, \pi)$. The order here is the strong Bruhat order: if $\text{inv}(\sigma_0) = \ell$ then $\text{inv}(\sigma_1) = \ell + 1$ and if $\sigma_0 = a_{i_1} \cdots a_{i_\ell}$ is a reduced word then so is $\sigma_1 = a_{i_1} \cdots a_{i_\ell}a_i$. On the other hand, if $z \in \text{Bru}_{\sigma_0}$, $\sigma_1 < \sigma_0 = \Pi(z_0) = \sigma_1a_i$ then there exists $\theta = \Theta_i(z) \in (0, \pi)$ such that $z\alpha_i(-\theta) \in \text{Bru}_{\sigma_1}$, $z_0 = z_1\alpha_i$. We also have $z\alpha_i(\theta) \in \text{Bru}_{\sigma_1}$ for all $\theta \in (-\theta, \pi - \theta)$.  

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We define a partial order in $\tilde{B}_{n+1}^+$, the \textit{lifted Bruhat order}:

$$z_0 \leq z_1 \iff \text{Bru}_{z_0} \subseteq \text{Bru}_{z_1}. \quad (9.1)$$

Clearly, $z_0 \leq z_1$ implies $\Pi(z_0) \leq \Pi(z_1)$ but the converse does not hold: for instance, $\Pi(z_0) = \Pi(z_1)$ and $z_0 \leq z_1$ together imply $z_0 = z_1$. We also define a partial order on the set of ancestries (for a given permutation $\sigma$). Given two ancestries $\varepsilon$ and $\tilde{\varepsilon}$, let $(\varrho_k)$ and $(\tilde{\varrho}_k)$ be as in Section 7. We define

$$\varepsilon \preceq \tilde{\varepsilon} \iff (\forall k, \varrho_k \leq \tilde{\varrho}_k). \quad (9.2)$$

Notice that $\varepsilon \preceq \tilde{\varepsilon}$ implies $P(\varepsilon) = P(\tilde{\varepsilon})$. The fact that this is a partial order is straightforward. A set $U$ of ancestries is an \textit{upper set} if $\varepsilon \in U$ and $\varepsilon \preceq \tilde{\varepsilon}$ imply $\tilde{\varepsilon} \in U$. The upper set generated by $\varepsilon$ is $U_\varepsilon = \{\tilde{\varepsilon} \mid \varepsilon \preceq \tilde{\varepsilon}\}$.

**Remark 9.1.** If $\varepsilon$ is an ancestry and $\dim(\varepsilon) = 0$ then $\varepsilon$ is $\preceq$-maximal. Conversely, consider $\varepsilon$ with $\dim(\varepsilon) > 0$. Define $\tilde{\varepsilon}$ by $\tilde{\varepsilon}(k) = \text{sign}(\varepsilon(k))$: we have $\varepsilon \prec \tilde{\varepsilon}$. In a wiring diagram, the ancestry $\tilde{\varepsilon}$ is obtained by replacing each square by a disk of the same color.

**Example 9.2.** If $\dim(\varepsilon) = 1$ then the upper set $U_\varepsilon$ generated by $\varepsilon$ consists of $\varepsilon$ itself and two ancestries of dimension 0. One of them is $\tilde{\varepsilon} = \text{sign} \circ \varepsilon$, obtained by replacing the two squares by disks of the same color, as in Remark 9.1. The second one is obtained from $\tilde{\varepsilon}$ by changing the colors of all disks on the boundary of the region corresponding to $\varepsilon$. Figure 10 shows an example.

![Figure 10: An ancestry of dimension one and the upper set generated by it.](image)

If $\dim(\varepsilon) > 1$ then the description of the upper set $U$ generated by $\varepsilon$ is not so direct. Recall from Remark 4.2 that preancestries (and therefore ancestries) of dimension 2 are classified as type I or type II.

**Example 9.3.** Let $\varepsilon$ be an ancestry of dimension 2, type I. The set $U_\varepsilon$ contains 4 elements of dimension 0, 4 elements of dimension 1 and one element of dimension 2 (which is $\varepsilon$). An example is shown in Figure 11.

![Figure 11: An example of a type I ancestry and its upper set.](image)

If $\varepsilon$ is an ancestry of dimension 2, type II, the structure of $U_\varepsilon$ is somewhat more complicated. Indeed, let $\varepsilon$ and $\tilde{\varepsilon}$ be the first and second ancestries in Figure 12, respectively. We have $\varepsilon \preceq \tilde{\varepsilon}$, $\tilde{\varepsilon} \not\preceq \varepsilon$ and $\dim(\varepsilon) = \dim(\tilde{\varepsilon}) = 2$.

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10 The sets $BL_z$

For a matrix $L \in Lo^{1}_{n+1}$, perform the usual $QR$ factorization:

$$L = QR, \quad Q \in SO_{n+1}, R \in Up^{+}_{n+1};$$

here $Up^{+}_{n+1} \subset Up_{n+1}$ is the group of upper triangular matrices with positive diagonal entries. This defines a smooth map:

$$Q_{SO} : Lo^{1}_{n+1} \rightarrow SO_{n+1}, \quad Q_{SO}(L) = Q.$$ 

Lift this map to define $Q : Lo^{1}_{n+1} \rightarrow \text{Spin}_{n+1}$ with $Q(I) = 1$. The set $U_1 = Q[Lo^{1}_{n+1}] \subset \text{Spin}_{n+1}$ is an open contractible neighborhood of $1 \in \text{Spin}_{n+1}$. We have $U_1 = \hat{\eta} \text{Bru}_{\eta}$. In other words, $U_1$ is a top-dimensional Bruhat cell for the basis described by $\hat{\eta}$, which is, up to signs, $e_{n+1}, e_n, \ldots, e_2, e_1$. The inverse map $L = Q^{-1} : U_1 \rightarrow Lo^{1}_{n+1}$ is also a smooth diffeomorphism: it corresponds to the $LU$ factorization.

We are now ready to define the sets $BL_z$, already mentioned in Equation (1.3) in the Introduction: for $z \in B^{+}_{n+1}$, take

$$BL_z = Q^{-1}[\text{Bru}_z] = Q^{-1}[\text{Bru}_z \cap \hat{\eta} \text{Bru}_{\eta}] \subseteq Lo^{1}_{n+1}. \quad (10.1)$$
The set $\text{BL}_z$ is diffeomorphic to $\text{Bru}_z \cap \hat{\eta}\text{Bru}_q$, the intersection of two Bruhat cells for different bases in $\text{Spin}_{n+1}$. Indeed, the diffeomorphisms are given by the restrictions of $L$ and $Q$.

**Example 10.1.** As in Example 1.1, take $n = 2$ and $\eta = a_1a_2a_1 = 321 = (13) \in S_3$. We have

$$
\hat{\eta} = \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}, \quad \hat{\eta}\text{Quat}_3 = \left\{ \pm \frac{1 \pm \hat{a}_1 \hat{a}_2}{\sqrt{2}}, \pm \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}} \right\};
$$

sign are chosen freely, so that $|\hat{\eta}\text{Quat}_3| = 8$. Notice that the value of $\Re(z)$ matches Lemma 6.5. We have

$$
\text{Lo}_3^1 = \left\{ L = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}; x, y, z \in \mathbb{R} \right\}, \quad \text{BL}_n = \{ L \mid z \neq 0, z \neq xy \} \subset \text{Lo}_3^1.
$$

A computation yields

$$
\text{BL}_{1-\hat{a}_1\hat{a}_2} = \{ L \mid z > \max \{0, xy\} \}, \quad \text{BL}_{1+\hat{a}_1\hat{a}_2} = \{ L \mid z < \min \{0, xy\} \},
$$

$$
\text{BL}_{\hat{a}_1+\hat{a}_2} = \{ L \mid x > 0, 0 < z < xy \}, \quad \text{BL}_{\hat{a}_1-\hat{a}_2} = \{ L \mid x > 0, xy < z < 0 \},
$$

$$
\text{BL}_{\hat{a}_1-\hat{a}_2} = \{ L \mid x < 0, 0 < z < xy \}, \quad \text{BL}_{1+\hat{a}_1\hat{a}_2} = \{ L \mid x < 0, xy < z < 0 \},
$$

$$
\text{BL}_{1+\hat{a}_1\hat{a}_2} = \text{BL}_{1-\hat{a}_1\hat{a}_2} = \emptyset.
$$

Notice that the six non empty sets are contractible.

Let $\mathfrak{lo}_{n+1}^1$ be the Lie algebra of $\text{Lo}_{n+1}^1$, i.e., the set of lower strictly triangular matrices. For $i \in \{1, \ldots, n\}$, let $t_i \in \mathfrak{lo}_{n+1}^1$ be the matrix whose only non zero entry is $(t_i)_{(i+1,i)} = 1$. Let $\lambda_i : \mathbb{R} \to \text{Lo}_{n+1}^1$ be the corresponding one-parameter subgroup:

$$
\lambda_i(t) = \exp(t_i) = I + t_i \in \text{Lo}_{n+1}^1.
$$

We saw in Equation 5.3 how the group $\mathcal{E}_n = \{ \pm 1 \}^{[n]}$ acts by automorphisms on $\text{Spin}_{n+1}$. The same group acts by automorphisms on $\text{Lo}_{n+1}^1$:

$$
(\lambda_i(t))^E = \hat{\lambda}_i(E, t).
$$

We therefore have $Q(L^E) = (Q(L))^E$ for all $L \in \text{Lo}_{n+1}^1$ and $E \in \mathcal{E}_n$. Thus, the set $\mathcal{U}_I$ is invariant: $\mathcal{U}_I^E = \mathcal{U}_I$ for all $E \in \mathcal{E}_n$. We also have $L(z^E) = (L(z))^E$ for all $z \in \mathcal{U}_I$ and $E \in \mathcal{E}_n$. For $z \in \hat{B}_{n+1}^+$ and $E \in \mathcal{E}_n$, we have $(\text{BL}_z)^E = \text{BL}_{z^E}$: in particular, the sets $\text{BL}_z$ and $\text{BL}_{z^E}$ are diffeomorphic through $L \mapsto L^E$. Thus, in order to determine the homotopy type of $\text{BL}_z$ we decompose $\hat{\sigma}\text{Quat}_{n+1}$ into $\mathcal{E}_n$-orbits. For each orbit, we take a representative $z$ and determine the homotopy type of $\text{BL}_z$. 

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Example 10.2. For $\eta \in S_3$, the set $\eta \text{Quat}_3$ partitions into three $E_\eta$-orbits defined by $\Re(z)$. From Example 10.1 it is clear that, indeed, $\text{BL}_{(1+\hat{a}_1\hat{a}_2)/\sqrt{2}}$ and $\text{BL}_{(1-\hat{a}_1\hat{a}_2)/\sqrt{2}}$ are diffeomorphic by an action of $E$, i.e., by changes of signs. Similarly, the four sets $\text{BL}_{(\pm\hat{a}_1\pm\hat{a}_2)/\sqrt{2}}$ are diffeomorphic. Finally, the two sets $\text{BL}_{(-1\pm\hat{a}_1\hat{a}_2)/\sqrt{2}}$ are empty.

Let $\eta = a_{i_1} \cdots a_{i_m}$ be a reduced word for the top permutation $\eta$. A matrix $L \in \text{Lo}_{n+1}$ is totally positive if and only if there exist positive numbers $t_1, \ldots, t_m$ such that

$$L = \lambda_{i_1}(t_1) \cdots \lambda_{i_m}(t_m).$$

The set $\text{Pos}_\eta$ of totally positive matrices is an open semigroup and a contractible connected component of $\text{BL}_\eta$. Furthermore, the set does not depend on the choice of the reduced word. For $\sigma \in S_{n+1}$, set $\text{Pos}_\sigma = \text{BL}_\sigma \cap \text{Pos}_\eta$. This set can be equivalently defined as follows. Let $\sigma = a_{i_1} \cdots a_{i_\ell}$ be a reduced word for $\sigma$: we have $L \in \text{Pos}_\sigma$ if and only if there exist positive numbers $t_1, \ldots, t_\ell$ such that

$$L = \lambda_{i_1}(t_1) \cdots \lambda_{i_\ell}(t_\ell).$$

The set $\text{Pos}_\sigma$ is also a contractible connected component of $\text{BL}_\sigma$.

Example 10.3. We return to Example 1.1. For $n = 2$, it follows easily from Example 10.1 that

$$\text{Pos}_\eta = \text{BL}_{\eta'}, \quad \eta' = \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}.$$

As we shall see, for all $n$ and for all $\sigma \in S_{n+1}$, we have $\text{Pos}_\sigma \subseteq \text{BL}_{\hat{a}}$. However, for larger values of $n$ and for most permutations $\sigma$, $\text{Pos}_\sigma$ is a small connected component of a much larger set $\text{BL}_{\hat{a}}$.  

11 First examples of stratata $\text{BLS}_\varepsilon$

Consider a permutation $\sigma$ and a fixed reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$. We define the open subset $\text{BLS}_\varepsilon \subset \text{BL}_\sigma$ if $\varepsilon$ is an ancestry of dimension 0: the general definition will be given later.

There is exactly one preancestry of dimension 0, the empty preancestry, as the first example in Figure 2. An ancestry $\varepsilon$ of dimension 0 is a function (or sequence) $\varepsilon : \left[\ell\right] \rightarrow \{\pm 1\}$. As a diagram, we mark intersections with black and white disks (no squares), as in the first example in Figure 3 or in the boxed ancestries shown in Figure 5. Recall that a black ball denotes $-1$ and a white ball denotes $+1$.

Given an ancestry $\varepsilon$ of dimension 0, we have

$$\text{BLS}_\varepsilon = \{\lambda_{i_1}(t_1) \cdots \lambda_{i_\ell}(t_\ell) \mid t_k \in \mathbb{R} \setminus \{0\}, \text{sign}(t_k) = \varepsilon_k\} \subset B_\sigma. \quad (11.1)$$
These subsets have been extensively studied (see Section 2 for references). We quote the relevant results without proof: not only are they in the literature but also they will follow from our results for arbitrary ancestries. It follows from Corollary 6.5 in [5] that:

$$BLS_{\varepsilon} \subseteq BL_{\sigma}, \quad z = P(\varepsilon) = (\dot{a}_{i_1})^{\varepsilon(1)} \cdots (\dot{a}_{i_\ell})^{\varepsilon(\ell)} \in \dot{\sigma} \text{Quat}_{n+1}.$$  \hspace{1cm} (11.2)

The subsets \(BLS_{\varepsilon} \subset BL_{\sigma}\) are open, and the union over all ancestries of dimension 0 is open and dense.

We shall define and study the sets \(BLS_{\varepsilon}\) for any ancestry in Sections 12 and 13. The sets \(BLS_{\varepsilon}\) will be disjoint, and \(BLS_{\varepsilon}\) will be a contractible smooth submanifold \(BLS_{\varepsilon} \subseteq BL_{\sigma(P(\varepsilon))}\) of codimension \(d = \dim(\varepsilon)\). Before presenting the general definition, we discuss some examples and special cases.

Recall that an ancestry \(\varepsilon\) (of dimension 0) is thin if \(i_{k_0} = i_{k_1}\) implies \(\varepsilon_{k_0} = \varepsilon_{k_1}\). If \(\varepsilon\) is a thin ancestry, the subset \(BLS_{\varepsilon} \subset BL_{\sigma}\) is also called thin. Notice that \(\varepsilon = (+, +, \ldots, +, +)\) is thin, with \(P(\varepsilon) = \dot{\sigma}\) and \(BLS_{\varepsilon} = \text{Pos}_{\sigma} \subseteq BL_{\dot{\sigma}}\), a contractible connected component. More generally, if \(\varepsilon\) is a thin ancestry then there exists \(E \in \mathcal{E}_n\) such that \(\varepsilon(k) = (\dot{a}_{i_1})^E\) (for all \(k\)). We therefore have \(P(\varepsilon) = \dot{\sigma}^E\) and \(BLS_{\varepsilon} = (\text{Pos}_{\sigma})^E\). It follows that \(BLS_{\varepsilon} \subseteq BL_{\dot{\sigma}^E}\) is a contractible connected component. The set

$$BL_{\varepsilon, \text{thick}} = BL_{\varepsilon} \setminus \bigcup_{\varepsilon \text{ thin}} BLS_{\varepsilon}$$

(11.3)

is called the thick part of \(BL_{\varepsilon}\). As we shall see, in general \(BL_{\varepsilon, \text{thick}}\) can be empty or disconnected.

**Example 11.1.** As in Examples 1.1 and 10.1 take \(n = 2\) and \(\eta \in S_3\). There are two reduced words: \(\eta = a_1a_2a_1 = a_2a_1a_2\), shown as diagrams in Figure 13. For each reduced word, there are two preancestries, with dimensions 0 and 1, also shows in Figure 13.

![Figure 13](image-url)

Figure 13: Two reduced words for \(\eta = a_1a_2a_1 = a_2a_1a_2 \in S_3\). For each word, there are two preancestries.

We first consider the reduced word \(\eta = a_1a_2a_1\). Write \(L\) as in Equation (10.2). The eight ancestries of dimension 0 are \((\pm 1, \pm 1, \pm 1)\); the two ancestries of dimension 1 are \((-2, \pm 1, +2)\). A simple computation gives

$$\lambda_1(t_1)\lambda_2(t_2)\lambda_1(t_3) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2t_3 & t_2 & 1 \end{pmatrix}$$
and therefore
\[ BL_{(-1,+1,+1)} = \{ L \mid z > \max\{0, xy\}, y > 0 \}, \]
\[ BL_{(+1,-1,-1)} = \{ L \mid z > \max\{0, xy\}, y < 0 \} \]

Take \( z_0 = (1 - \hat{a}_1 \hat{a}_2) / \sqrt{2} \). Notice that \( P(-1,+1,+1) = P(+1,-1,-1) = P(-2,+1,+2) = z_0 \). These are the only ancestries \( \varepsilon \) with \( P(\varepsilon) = z_0 \), as can be verified either by computing \( P \) for the other ancestries or from Lemmas 7.4 and 7.6. As we shall see later,
\[ BL_{(-2,+1,+2)} = \{ L \mid y = 0, z > 0 \}\]
\[ BL_{z_0} = BL_{(-1,+1,+1)} \sqcup BL_{(-2,+1,+2)} \sqcup BL_{(+1,-1,-1)},\]
consistently with Equation (11.2). The subset \( BL_{(-2,+1,+2)} \subset BL_{z_0} \) is a contractible submanifold of codimension \( d = 1 \). The CW complex in Figure 6 is homotopically equivalent to \( BL_{z_0} \).

A similar decomposition holds for \( BL_{(1+\hat{a}_1 \hat{a}_2) / \sqrt{2}} \). The four connected components \( BL_{(\pm \hat{a}_1 \pm \hat{a}_2) / \sqrt{2}} \) are thin, consistently with Example 10.3.

Recall that the permutation \( \eta \) also admits the reduced word \( \eta = a_2 a_1 a_2 \). A similar decomposition exists for the other reduced word, but the strata are different. For this other word, the set \( BL_{(1-\hat{a}_1 \hat{a}_2) / \sqrt{2}} \) contains two open strata:
\[ BL_{(+1,+1,-1)} = \{ L \mid z > \max\{0, xy\}, x > 0 \}, \]
\[ BL_{(-1,-1,+1)} = \{ L \mid z > \max\{0, xy\}, x < 0 \} \]
and a third stratum of codimension 1:
\[ BL_{(-2,-1,+2)} = \{ L \mid x = 0, z > 0 \}. \]

Clearly, the meaning of the ancestry depends on the reduced word being used. ⋄

**Example 11.2.** Take \( n = 3 \) and \( \sigma = a_2 a_1 a_3 a_2 = (3, 4, 1, 2) \in S_4 \). Up to transposing adjacent commuting generators, as in \( a_2 a_1 a_3 a_2 = a_2 a_3 a_1 a_2 \), the permutation \( \sigma \) admits only one reduced word, shown in Figure 14.

![Figure 14: The permutation \( \sigma = a_2 a_1 a_3 a_2 \in S_4 \) and its two preancestries.](image)

There are two preancestries and 20 ancestries:
\[(\pm 1, \pm 1, \pm 1, \pm 1), \quad d = 0; \quad (-2, \pm 1, \pm 1, \pm 2), \quad d = 1.\]
For the preancestry of dimension $d = 1$, $(\rho_k)$ (as in Equations (4.1) and (4.2)) satisfies $\rho_1 = \rho_2 = \rho_3 = a_1a_2a_3a_1$. Take $z_0 = -\hat{\sigma}a_3 = (1 - \hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_3 - \hat{a}_2a_3)/2$. The three ancestries with $P(\varepsilon) = z_0$ are $(+1, +1, -1, -1)$, $(-2, -1, +1, +2)$ and $(-1, -1, +1, +1)$. At this point, the reader may guess that $\text{BL}_{z_0}$ is homotopically equivalent to the CW complex in Figure 15, which is indeed correct.

Figure 15: A CW complex homotopically equivalent to $\text{BL}_{z_0}$.

Write $L \in \text{Lo}_4^1$ as

$$\text{Lo}_4^1 = \left\{ L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ u & y & 1 & 0 \\ w & v & z & 1 \end{pmatrix}, u, v, w, x, y, z \in \mathbb{R} \right\}.$$

By applying the definition, the set $\text{BL}_{\sigma}$ is

$$\text{BL}_{\sigma} = \{ L \mid w = 0, u \neq 0, v \neq 0, xyz = xv + zu \}.$$

If $L = \lambda_2(t_1)\lambda_1(t_2)\lambda_3(t_3)\lambda_2(t_4)$ then

$$u = t_1t_2, \quad v = t_3t_4, \quad w = 0, \quad x = t_2, \quad y = t_1 + t_4, \quad z = t_3.$$

Let $U \subset (0, +\infty)^2 \times \mathbb{R}^2$ (with coordinates $(u, v, x, y)$) be the contractible open set defined by $xy < u$. Consider the map $\Phi: U \to \text{BL}_{\sigma}$ taking $(u, v, x, y)$ to the matrix $L \in \text{BL}_{\sigma}$ with the prescribed values of $u, v, x, y$ and $z = xv/(xy - u)$.

If $x > 0$ we have $L = \Phi(u, v, x, y) \in \text{BL}_{(+1,+1,-1,-1)}$; if $x < 0$ we have $L = \Phi(u, v, x, y) \in \text{BL}_{(-1,-1,+1,+1)}$. We will see that

$$\text{BL}_{(-2,-1,+1,+2)} = \Phi[(0, +\infty)^2 \times \{0\} \times \mathbb{R}],$$

$$\text{BL}_{z_0} = \text{BL}_{(-1,-1,+1,+1)} \sqcup \text{BL}_{(-2,-1,+1,+2)} \sqcup \text{BL}_{(-1,-1,+1,+1)}.$$

It follows that $\text{BL}_{z_0}$ is contractible. Similar computations verify that $\text{BL}_{\sigma}$ has 12 connected components, 8 thin and 4 thick, all contractible.

We conclude this section with an effective enumeration of the connected components of $\text{BL}_{\eta}$ for $n \geq 5$. Recall that it is known that there are $3 \cdot 2^n$ connected components (see Section 2 and 15).
Proposition 11.3. For $n \geq 5$, the $3 \cdot 2^n$ connected components of $\text{BL}_n$ are

$$\text{Pos}^E_n, E \in \mathcal{E}_n, \quad \text{BL}_{z, \text{thick}}, \quad z \in \hat{\eta} \text{Quat}_{n+1}.$$ 

The first list are the $2^n$ thin connected components; the second are the $2^{n+1}$ thick connected components.

Proof. We must prove that each $\text{BL}_{z, \text{thick}}$ is connected and non empty. It follows from Remark 3.5 that $\text{BL}_{z, \text{thick}} \neq \emptyset$ for each $z \in \hat{\eta} \text{Quat}_{n+1}$. If one of them were to be disconnected, we would have more than $3 \cdot 2^n$ connected components, contradicting the number of connected components. \hfill \quad \Box

12 The stratification $\text{BLS}_\varepsilon$

Consider as usual a fixed reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$. For $L \in \text{BL}_\sigma$, we show how to determine the ancestry of $L$.

Let $\hat{z}_\ell = Q(L)$. Take $q_\ell \in \text{Quat}^1_{n+1}$ such that $z_\ell = \hat{z}_\ell q_\ell \in \text{Bru}_\ell$. Define recursively $\sigma_0 = 1$, $\sigma_1 = a_{i_1}$, $\sigma_k = \sigma_{k-1} a_{i_k} = a_{i_1} \cdots a_{i_k}$ so that $\sigma = \sigma_\ell$. From Theorem 1 from [5], we have well-defined sequences $(\theta_k)_{0 < k \leq \ell}$ and $(z_k)_{0 \leq k \leq \ell}$ such that $z_0 = 1 \in \text{Spin}_{n+1}$ and

$$z_k = z_{k-1} \alpha_{i_k}(\theta_k) \in \text{Bru}_{\sigma_k}, \quad \theta_k \in (0, \pi). \quad (12.1)$$

Take $\varrho_k \in \hat{B}^+_{n+1}$ such that $z_k \in \hat{\eta} \text{Bru}_{\varrho_k}$; the sequence $(\varrho_k)$ is the desired ancestry. The corresponding preancestry is $(\rho_k)$, $\rho_k = \prod_{k=1}^{\ell} \alpha_{i_k}(\theta)$ so that $z_k \in \hat{\eta} \text{Bru}_{\rho_k}$. Given an ancestry $\varepsilon$, let $\text{BLS}_\varepsilon \subset \text{BL}_\sigma$ be the set of matrices $L$ with ancestry $\varepsilon$.

We verify that the sequences $(\rho_k)$ and $(\varrho_k)$ are indeed a preancestry and an ancestry in the sense defined in Sections 4 and 7. Indeed, we have $\rho_0 = \eta$ and $\varrho_0 = \hat{\eta}$ since $z_0 = 1 \in \hat{\eta} \text{Bru}_\eta \subset \hat{\eta} \text{Bru}_\eta$. If $z_{k-1} \in \hat{\eta} \text{Bru}_{\varrho_{k-1}}$ and $\rho_{k-1} \alpha_{i_k}(\theta) \notin \hat{\eta} \text{Bru}_{\varrho_{k-1} a_{i_k}}$ for all $\theta \in (0, \pi)$: this implies $\varrho_k = \varrho_{k-1} \alpha_{i_k}(\theta)$ and $\rho_k = \rho_{k-1} a_{i_k} \alpha_{i_k}(\theta)$ for $\theta \in (0, \pi)$ belongs to one of the following three sets: $\hat{\eta} \text{Bru}_{\varrho_{k-1}}$, $\hat{\eta} \text{Bru}_{\varrho_{k-1} \alpha_{i_k}}$, $\hat{\eta} \text{Bru}_{\varrho_{k-1} \alpha_{i_k}}$. This implies $\varrho_k$ to be one among $\varrho_{k-1}$, $\varrho_{k-1} \alpha_{i_k}$, $\varrho_{k-1} \alpha_{i_k}$. Finally, $\hat{z}_k \in \mathcal{U}_1$ implies $z_k \in \text{Bru}_\eta$ and $q_\ell \in \hat{\eta} \text{Quat}_{n+1}$. We thus have

$$\text{BL}_\sigma = \bigsqcup_{\varepsilon} \text{BLS}_\varepsilon, \quad \text{BL}_z = \bigsqcup_{P(\varepsilon)=z} \text{BLS}_\varepsilon, \quad (12.2)$$

where $\varepsilon$ varies over ancestries (for a fixed reduced word for $\sigma$). We are ready to complete the proof of Theorem 4.
Proof of Theorem 4. It follows from the definition of $\operatorname{BLS}_c$ above and from Equation (12.2) that $\operatorname{BLS}_c \subseteq \operatorname{BL}_{P(\varepsilon)}$. For any prebranching $\varepsilon(z)$ and any $z \in \delta \operatorname{Quat}_{n+1}$, we have $\operatorname{NL}_{\varepsilon(z)}(z) = h_{\varepsilon(z)}(z)$; see Remark 7.3. Theorem 4 now follows directly from Lemmas 7.4 and 7.6. \hfill \Box

We present the interpretation of $\xi(k)$. Consider $z_{k-1} \in \eta \operatorname{Bru}_{\rho_{k-1}}$. If $\rho_{k-1} < \rho_{k-1}a_i$ then $z_{k-1}a_i(\theta) \in \eta \operatorname{Bru}_{\rho_{k-1}a_i}$ for all $\theta \in (0, \pi)$: in this case we take $\xi(k) = 1$. If $\rho_{k-1} > \rho_{k-1}a_i$ then there exists a unique $\theta \in (0, \pi)$ such that $z_{k-1}a_i(\theta) \in \eta \operatorname{Bru}_{\rho_{k-1}a_i}$. If $\theta < \theta_\ast$ we have $z_k \in \eta \operatorname{Bru}_{\rho_{k-1}}, \rho_k = \rho_{k-1}$ and $\xi(k) = 0$. If $\theta > \theta_\ast$ we have $z_k \in \eta \operatorname{Bru}_{\rho_{k-1}a_i}, \rho_k = \rho_{k-1}a_i$ and $\xi(k) = 2$. Finally, if $\theta = \theta_\ast$ we have $z_k \in \eta \operatorname{Bru}_{\rho_{k-1}a_i}, \rho_k = \rho_{k-1}a_i$ and $\xi(k) = 1$. Thus, $\xi(k)$ tells us about the size of $\theta_k$: $\xi(k) = 0$ means that $\theta_k$ is small, $\xi(k) = 2$ means that $\theta_k$ is large and $\xi(k) = 1$ means that $\theta_k$ is just right.

Let us introduce some more notation. Let

$$
\mathcal{U}_i^\circ = \bigcup_{\sigma \in S_{n+1}} \eta \operatorname{Bru}_{\sigma}, \quad \mathcal{U}_1 \subseteq \mathcal{U}_i^\circ \subseteq \mathcal{U}_i \subseteq \operatorname{Spin}_{n+1}.
$$

The set $\mathcal{U}_i^\circ$ is a fundamental domain for the action of $\operatorname{Quat}_{n+1}$ on $\operatorname{Spin}_{n+1}$; given $z \in \operatorname{Spin}_{n+1}$ there exists a unique $q \in \operatorname{Quat}_{n+1}$ such that $zq \in \mathcal{U}_i^\circ$ (see [5]). For each $k$, write $z_k = \hat{z}_kq_k$, $\hat{z}_k \in \mathcal{U}_i^\circ$, $q_k \in \operatorname{Quat}_{n+1}$. We clearly have $\hat{z}_k \in \eta \operatorname{Bru}_{\rho_k}$.

Lemma 12.1. There exist unique $\hat{\theta}_k \in (-\pi, 0) \cup (0, \pi)$ such that $\hat{z}_k = \hat{z}_{k-1}a_i(\hat{\theta}_k)$. Furthermore, for $s = [a_i, q_{k-1}] \in \{\pm 1\}$ we have either $\hat{\theta}_k = s\theta_k$ or $\hat{\theta}_k = s(\theta_k - \pi)$. In the first case we have $q_k = q_{k-1}$; in the second case, $q_k = q_{k-1}a_i$.

Proof. If $z \in \mathcal{U}_i^\circ$ and $\theta \in (0, \pi)$ then either $z\alpha_i(\theta) \in \mathcal{U}_i^\circ$ or $z\alpha_i(\theta - \pi) \in \mathcal{U}_i^\circ$ (but not both). Similarly, either $z\alpha_i(-\theta) \in \mathcal{U}_i^\circ$ or $z\alpha_i(-\theta + \pi) \in \mathcal{U}_i^\circ$ (but not both).

We have $z_k = z_{k-1}a_i(\theta_k)$ and $z_{k-1} = \hat{z}_{k-1}q_{k-1}$. We therefore have $z_k = \hat{z}_{k-1}q_{k-1}a_i(\theta_k)$ and therefore $z_k = \hat{z}_{k-1}a_i(s\theta_k)q_{k-1}$. By definition, $q_k$ is such that $z_k = \hat{z}_kq_k$, $\hat{z}_k \in \mathcal{U}_i^\circ$. If $\hat{z}_{k-1}a_i(s\theta_k) \not\in \mathcal{U}_i^\circ$ we take $\hat{\theta}_k = s\theta_k$, $\hat{z}_k = \hat{z}_{k-1}a_i(s\theta_k)$ and $q_k = q_{k-1}$. If $s = +1$ and $\hat{z}_{k-1}a_i(s\theta_k) \not\in \mathcal{U}_i^\circ$ we take $\hat{\theta}_k = \theta_k - \pi$, $\hat{z}_k = \hat{z}_{k-1}a_i(\theta_k - \pi)$ and $q_k = q_{k-1}a_i = \hat{a}_i(q_{k-1})$. Finally, if $s = -1$ and $\hat{z}_{k-1}a_i(\theta_k - \pi) \not\in \mathcal{U}_i^\circ$ we take $\hat{\theta}_k = -\theta_k + \pi$, $\hat{z}_k = \hat{z}_{k-1}a_i(-\theta_k + \pi) \in \mathcal{U}_i^\circ$ and $q_k = q_{k-1}a_i = -a_iq_{k-1}$.

We are finally ready to give the interpretation of $\varepsilon(k)$.

Lemma 12.2. We have $\operatorname{sign}(\varepsilon(k)) = \operatorname{sign}(\hat{\theta}_k)$. Also, $\varepsilon(k) = -2$ if and only if $\rho_k < \rho_{k-1}$; $\varepsilon(k) = +2$ if and only if $\rho_k > \rho_{k-1}$.

Proof. Compare the formula for $\hat{\theta}_k$ in Lemma 12.1 with Equation (7.1). \hfill \Box

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The above construction can be thought of as a way to extend the construction in Section 11, based on the functions $\lambda_i$, to infinity in the cases where it is impossible in $\text{Lo}_{n+1}^1$. The group $\text{Lo}_{n+1}^1$ is unsuitable: we work instead in the compact group $\text{Spin}_{n+1}$ (or $\text{SO}_{n+1}$), using the functions $\alpha_i$ instead of $\lambda_i$ and performing the required adaptations. The examples should clarify these remarks.

**Example 12.3.** As in Examples 1.1 and 10.1 set $n = 2$ and $\sigma = \eta = a_1a_2a_1$. Consider

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \tilde{z}_3 = \mathbf{Q}(L_0) = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}.$$  

(More correctly, the matrix shown is $\Pi(\tilde{z}_3) \in \text{SO}_3$. Here and in other occasions it is easier to do computations in $\text{SO}_{n+1}$ instead of $\text{Spin}_{n+1}$.)

We have $\tilde{z}_3 = \alpha_1(-\frac{\pi}{2})\alpha_2(\frac{\pi}{2})\alpha_1(\frac{\pi}{2})$. We have $\rho_1 = \rho_2 = a_1a_2$ and therefore (denoting ancestries by $\varepsilon$) $L_0 \in \text{BL}_{(-2,1,2)}$. More generally, it is not hard to verify that, for $L \in \text{Lo}_3^1$, we have $L \in \text{BL}_{(-2,1,2)}$ if and only if $y = 0$ and $z > 0$. This justifies the claims made in Example 11.1, particularly Figure 6.

**Example 12.4.** As in Example 11.2 set $n = 3$ and $\sigma = a_2a_1a_3a_2$. Consider

$$L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \tilde{z}_4 = \mathbf{Q}(L_0) = \frac{\sqrt{3}}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$  

We have $\tilde{z}_4 = \alpha_2(-\frac{\pi}{2})\alpha_1(-\frac{\pi}{2})\alpha_3(\frac{\pi}{4})\alpha_2(\frac{\pi}{2})$, $\rho_1 = \rho_2 = \rho_3 = a_1a_2a_3a_2a_1$ and therefore (denoting ancestries by $\varepsilon$) $L_0 \in \text{BL}_{(-2,-1,1,2)}$. The claims in Example 11.2 are now easy to verify.

13 The strata $\text{BLS}_\varepsilon$

We now prove that the strata $\text{BLS}_\varepsilon$ are reasonably well-behaved.

**Lemma 13.1.** Consider a permutation and reduced word $\sigma = a_{i_1} \cdots a_{i_\ell} \in S_{n+1}$ and an ancestry $\varepsilon$. The subset $\text{BLS}_\varepsilon \subseteq \text{BL}_\sigma$ is a smooth submanifold of codimension $d = \dim(\varepsilon)$.

**Proof.** Let $(\varrho_k)_{0 \leq k \leq \ell}$ be the ancestry encoded by $\varepsilon$. Consider $L \in \text{BLS}_\varepsilon$ and $\tilde{z}_\ell = \mathbf{Q}(L)$. Construct $(z_k)_{0 \leq k \leq \ell}$ and $(\theta_k)_{0 \leq k \leq \ell}$ as above, so that $z_\ell = \tilde{z}_\ell q_\ell$, $z_k = z_{k-1}a_{i_k}(\theta_k) \in \hat{\eta} \text{Bru}_\varrho_k$. Let $K_0 = \{ k \in \mathbb{Z}, 1 \leq k \leq \ell, g_k < g_{k-1} \}$ so that $|K_0| = d$. For $0 \leq j \leq \ell$, let $V_j = \{ x \in \mathbb{R}^j \mid k \in K_0 \rightarrow x_k = 0 \} \subset \mathbb{R}^j$, a linear subspace; $V_\ell$ has codimension $d$. We construct a compact neighborhood $U \subset \mathbb{R}^\ell$ of $0$
and a smooth local diffeomorphism $\Phi : U \to \text{Bru}_d$ such that $\Phi(0) = z_\ell$ and $\Phi(x) \in \mathbb{Q}[\text{BLS}_\varepsilon]q_\ell$ if and only if $x \in V_\varepsilon$.

The set $U$ has the form $U = U_\ell$ where

$$U_k = [-\varepsilon_1, \varepsilon_1] \times \cdots \times [-\varepsilon_k, \varepsilon_k].$$

We recursively define $\varepsilon_k > 0$ and maps $\Phi_k : U_k \to \text{Bru}_{\delta_k}$ with $\Phi_k(0) = z_k$. In every case we shall have $\Phi_k(x_{k-1}, \theta) = \Phi_{k-1}(x_{k-1})\alpha_{i_k}(*)$, where $*$ stands for a smooth function of $x_{k-1} \in U_{k-1}$ and $\theta \in [-\varepsilon_k, \varepsilon_k]$. The case $k = 0$ is trivial.

If $k \notin K_0$, take $\varepsilon_k > 0$ sufficiently small such that the following two conditions hold. For all $x_{k-1} \in U_{k-1}$ and $\theta \in [-\varepsilon_k, \varepsilon_k]$, we have $\Phi_k(x_{k-1})\alpha_{i_k}(\theta_k + \theta) \in \text{Bru}_{\delta_k}$. For all $x_{k-1} \in U_{k-1} \cap V_{k-1}$ and $\theta \in [-\varepsilon_k, \varepsilon_k]$, we have $\Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \theta) \in \tilde{\text{Bru}}_{\delta_k}$. The existence of such $\varepsilon_k > 0$ follows from Theorem 1 of \cite{5}. We then define $\Phi_k(x_{k-1}, \theta) = \Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \theta)$.

If $k \in K_0$, there exists a smooth function $\vartheta : U_{k-1} \cap V_{k-1} \to (0, \pi)$ such that, for all $x_{k-1} \in U_{k-1} \cap V_{k-1}$, we have $\vartheta \Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \theta) \in \tilde{\text{Bru}}_{\delta_k}$ (here we again use Theorem 1 of \cite{5}). Notice that $\vartheta(0) = 0$. Let $\Pi : U_{k-1} \to U_{k-1} \cap V_{k-1}$ be the orthogonal projection. Extend $\vartheta$ to $U_{k-1}$ by defining $\vartheta(x_{k-1}) = \vartheta(\Pi(x_{k-1}))$; notice that this is a smooth function. Define

$$\Phi_k(x_{k-1}, \theta) = \Phi_{k-1}(x_{k-1})\alpha_{i_k}(\theta_k + \vartheta(x_{k-1}) + \theta).$$

Notice that, for $x_{k-1} \in U_{k-1} \cap V_{k-1}$, we have $\Phi_k(x_{k-1}, \theta) \in \tilde{\text{Bru}}_{\delta_k}(\alpha_{i_k} \text{sign}(\theta))$ (with $\text{sign}(0) = 0$). Choose sufficiently small $\varepsilon_k > 0$ and we are done. $\square$

**Lemma 13.2.** Consider a permutation $\sigma$, a reduced word $\sigma = a_{i_1} \cdots a_{i_\ell}$ and an ancestry $\varepsilon$ with $d = \dim(\varepsilon)$. The smooth submanifold $\text{BLS}_\varepsilon \subset \text{BL}_\sigma$ is diffeomorphic to $\mathbb{R}^{\ell-d}$.

**Proof.** Let $(\theta_k)$ be the sequence of elements of $\tilde{B}_{n+1}^+$ defining the ancestry $\varepsilon$; let $\xi$ be as usual. Let $\Psi_k : (0, \pi)^k \to \text{Bru}_{\delta_k}$ be the diffeomorphism

$$\Psi_k(\theta_1, \ldots, \theta_k) = \alpha_{i_1}(\theta_1) \cdots \alpha_{i_k}(\theta_k).$$

For $0 \leq k \leq \ell$, let $X_k \subseteq (0, \pi)^k$ be defined by:

$$(\theta_1, \ldots, \theta_k) \in X_k \iff \forall j, (0 \leq j \leq k) \to (\Psi_j(\theta_1, \ldots, \theta_j)) \in \tilde{\text{Bru}}_{\theta_j}.$$  

As in the proof of Lemma \textbf{[13.1]} let $K_0 = \{k \in \mathbb{Z}, 1 \leq k \leq \ell, \theta_k < \theta_{k-1}\}$. We prove by induction on $k$ that each set $X_k$ is diffeomorphic to $\mathbb{R}^{k-[K_0\cap[1,k]]}$. By definition, the restriction $\Psi_{\ell} : X_\ell \to \text{BLS}_\varepsilon$ is a bijection; it follows from Lemma \textbf{[13.1]} that it is a diffeomorphism.

The set $X_0$ has a single element, the empty sequence. We have $X_k \subseteq X_{k-1} \times (0, \pi)$. For $(\theta_1, \ldots, \theta_k) \in X_{k-1} \times (0, \pi)$, we have $(\theta_1, \ldots, \theta_k) \in X_k$ if and only if
\[ \Psi_k(\theta_1, \ldots, \theta_k) \in \hat{\text{h}} \text{Br}_\varrho_k. \] We again divide our discussion into cases. If \( \xi(k) = 1 \) and \( \varrho_{k-1} < \varrho_k = \varrho_{k-1} \hat{a}_k \) then \( X_k = X_{k-1} \times (0, \pi) \) and we are done.

Otherwise, we have \( \rho_{k-1} a_i < \rho_{k-1}. \) There exists a smooth function \( \vartheta : X_{k-1} \to (0, \pi) \) such that, for all \( \tilde{\theta} = (\theta_1, \ldots, \theta_{k-1}) \in X_{k-1}, \) we have

\[ \Psi_{k-1}(\tilde{\theta}) \alpha_{i_k}(\vartheta(\tilde{\theta})) \in \text{Br}_{\varrho_{k-1} \hat{a}_k}. \]

Indeed, in the notation of Remark 6.6 of [5], we have \( \vartheta(\tilde{\theta}) = \pi - \Theta_{\varrho_k}(\Psi_{k-1}(\tilde{\theta})). \) If \( \xi(0) = 0 \) we have

\[ X_k = \left\{ (\tilde{\theta}, \theta_k) \in X_{k-1} \times (0, \pi) \mid \theta_k < \vartheta(\tilde{\theta}) \right\}, \]

which is diffeomorphic to \( X_{k-1} \times (0, 1) \) and therefore to an open ball. If \( \xi(0) = 2 \) we have

\[ X_k = \left\{ (\tilde{\theta}, \theta_k) \in X_{k-1} \times (0, \pi) \mid \theta_k > \vartheta(\tilde{\theta}) \right\}, \]

and we are done. Finally, if \( \xi(0) = 1 \) we have

\[ X_k = \left\{ (\tilde{\theta}, \theta_k) \in X_{k-1} \times (0, \pi) \mid \theta_k = \vartheta(\tilde{\theta}) \right\}, \]

which is diffeomorphic to \( X_{k-1}. \) This completes the proof. \( \square \)

Recall that in Equation 9.2 we define a partial order on the set of ancestries (for a fixed reduced word). The following lemma is the reason why we defined a partial order among ancestries.

**Lemma 13.3.** Let \( \varepsilon, \bar{\varepsilon} \) be ancestries. If \( \text{BLS}_\varepsilon \cap \overline{\text{BLS}_\bar{\varepsilon}} \neq \emptyset \) then \( \varepsilon \preceq \bar{\varepsilon}. \)

**Proof.** Assume that \( \text{BLS}_\varepsilon \cap \overline{\text{BLS}_\bar{\varepsilon}} \neq \emptyset. \) Thus, there exists a sequence \( (L_j) \) of elements of \( \text{BLS}_\varepsilon \) converging to an element \( L_\infty \) of \( \text{BLS}_\bar{\varepsilon}. \) This implies that \( P(\bar{\varepsilon}) = P(\varepsilon). \) If \( \bar{z}_{j,\ell} = Q(L_j) \) and \( \bar{z}_{\infty,\ell} = Q(L_\infty), \) we have \( \lim_j \bar{z}_{j,\ell} = \bar{z}_{\infty,\ell}. \) This implies that \( P(\varepsilon) = P(\bar{\varepsilon}). \) Thus, for \( z_{j,\ell}, z_{\infty,\ell} \in \text{Br}_\varrho \) defined as usual we have \( \lim_j z_{j,\ell} = z_{\infty,\ell}. \) Since \( z_k \) is a smooth function of \( z_\ell, \) we have \( \lim_j z_{j,k} = z_{\infty,k}. \) We have \( z_{j,k} \in \hat{\text{h}} \text{Br}_{\varrho_k} \) and \( z_{\infty,k} \in \hat{\text{h}} \text{Br}_{\varrho_k} \) and therefore \( \text{Br}_\varepsilon \cap \overline{\text{Br}_\bar{\varepsilon}} \neq \emptyset. \) By definition of the lifted Bruhat order in \( \text{B}_{n+1}^+ \) we have \( \varrho_k \leq \tilde{\varrho}_k. \) Since this holds for all \( k, \) we have \( \varepsilon \preceq \bar{\varepsilon}, \) as desired. \( \square \)

Notice that in Lemma 13.3 we neither claim equivalence, nor that either of the above conditions imply \( \text{BLS}_\varepsilon \subseteq \overline{\text{BLS}_\bar{\varepsilon}}. \) The next example shows that Whitney’s condition does not always hold.

**Example 13.4.** Consider \( \sigma = a_2a_3a_2a_1a_2a_4a_3a_2 \) and let \( \varepsilon, \bar{\varepsilon} \) be the two ancestries shown in Figure 12 in order:

\[ \varepsilon = (-2, -2, +1, -1, -1, +1, +2, +2), \quad \bar{\varepsilon} = (-2, -1, +2, -1, -2, +1, +1, +2), \]
\[ \xi = (1, 1, 0, 0, 2, 0, 1, 1), \quad \tilde{\xi} = (1, 0, 1, 0, 1, 0, 0, 1). \]
We claim that
\[ \varepsilon \leq \tilde{\varepsilon}, \quad \text{BL}_\varepsilon \cap \text{BL}_{\tilde{\varepsilon}} \neq \emptyset, \quad \text{BL}_\varepsilon \not\subseteq \text{BL}_{\tilde{\varepsilon}}. \]

The fact that inclusion does not hold follows from the fact that \( \text{BL}_\varepsilon \) and \( \text{BL}_{\tilde{\varepsilon}} \) are disjoint smooth submanifolds of the same dimension. Consider

\[ z_8 = \alpha_2(\theta_1)\alpha_3(\theta_2)\alpha_2(\theta_3)\alpha_1(\theta_4)\alpha_2(\theta_5)\alpha_4(\theta_6)\alpha_3(\theta_7)\alpha_2(\theta_8) \in \text{Bru}_\delta \]

and corresponding \( L \). Take \( \theta_1 = \pi/2, \theta_3 = \theta_4 = \theta_6 = \theta_7 = \theta_8 = \pi/4 \). A computation shows that for \( \theta_2 = \pi/2 \) and \( \theta_5 = \pi - \arctan(\sqrt{2}) \), we have \( L \in \text{BL}_\varepsilon \).

For \( \theta_2 \in (\pi/3, \pi/2) \) and \( \theta_5 = \pi - \arctan(\sqrt{2}) \), we have \( L \in \text{BL}_{\tilde{\varepsilon}} \). \( \Box \)

**Corollary 13.5.** If \( U \) is an upper set of ancestries then

\[ \bigcup_{\varepsilon \in U} \text{BL}_\varepsilon \subseteq \text{BL}_\sigma \]

is an open subset.

**Proof.** This follows from Lemma 13.3. \( \square \)

### 14 Proof of Theorem 2

In this section we construct the CW complex \( \text{BLC}_\sigma \), the continuous map \( i_\sigma : \text{BLC}_\sigma \to \text{BL}_\sigma \) and prove Theorem 2. The construction is recursive and for the induction step we need a topological result, Lemma 14.1 below.

The idea is that the CW complex \( \text{BLC}_\sigma \) is a dual cell structure to the stratification. Examples of this construction under nicer conditions should be familiar from Poincaré duality (see, for instance, [8], Section 3.3, particularly the figure in page 232). As we shall see, we have sufficient conditions to apply a similar construction. We did not find a reference in the required generality, hence the need to state and prove Lemma 14.1. Another example of such a construction, but involving Hilbert manifolds, can be found in [7] (see Theorems 1 and 4).

Here \( S^{k-1}_r \) denotes the sphere of radius \( r \), \( B^k_r \) denotes the open ball, \( D^k_r \) denotes the compact disk and \( K^{k}_{r_0,r_1} \) denotes the corona:

\[
S^{k-1}_r = \{ v \in \mathbb{R}^k \mid |v| = r \}, \quad B^k_r = \{ v \in \mathbb{R}^k \mid |v| < r \},
\]
\[
D^k_r = \{ v \in \mathbb{R}^k \mid |v| \leq r \}, \quad K^k_{r_0,r_1} = \{ v \in \mathbb{R}^k \mid r_0 \leq |v| \leq r_1 \};
\]

also, \( D^k = D^k_1 \). For a CW complex \( X \), let \( X^{[j]} \subseteq X \) denote the skeleton of dimension \( j \), that is, the union of cells of dimension at most \( j \).
Lemma 14.1. Let $M_0 \subset M_1$ be smooth manifolds of dimension $\ell$. Assume that $N_1 = M_1 \setminus M_0 \subset M_1$ is a smooth submanifold of codimension $k$, $0 < k \leq \ell$, and that $N_1$ is diffeomorphic to $\mathbb{R}^{\ell-k}$. Assume that $X_0$ is a finite CW complex and that $i_0 : X_0 \to M_0$ is a homotopy equivalence.

There exists a map $\beta : S^{k-1} \to X_0^{[k-1]}$ with the following properties. Let $X_1$ be obtained from $X_0$ by attaching a cell $C_1$ of dimension $k$ with glueing map $\beta$. There exists a map $i_1 : X_1 \to M_1$ with $i_1|_{X_0} = i_0$ such that $i_1 : X_1 \to M_1$ is a homotopy equivalence.

Remark 14.2. Notice that since $M_0 \subset M_1$ is a submanifold of codimension $0$ it follows that $M_0$ is an open subset of $M_1$. The subset $N_1 \subset M_1$ is therefore closed. If $k < \ell$ it follows that $M_1$ is not compact.

The maps $i_0$ and $i_1$ can be taken to be inclusions in many examples but are not required to be so. The first paragraph of the proof provides us with a construction of the map $\beta$ and therefore of the CW complex $X_1$. The map $i_1 : X_1 \to M_1$ is constructed slightly later.

Proof of Lemma 14.1. Consider a small transversal section to $N_1$, the map $\alpha_1 : \mathbb{D}^k_0 \to M_1$ with $\alpha_1(0) = z_1 \in N_1$. Consider the restriction $\beta_1 = \alpha_1|_{S^{k-1}} : S^{k-1} \to M_0$. Take $\beta : S^{k-1} \to X_0^{[k-1]}$ such that $\beta_1$ and $i_0 \circ \beta$ are homotopic in $M_0$. This is the desired glueing map $\beta : S^{k-1} \to X_0^{[k-1]}$, completing the construction of the CW complex $X_1$. We must now construct $i_1 : X_1 \to M_1$ and prove that $i_1$ is a homotopy equivalence.

We first go again through the construction of the map $\beta$, but more carefully, collecting more notation and information. By our assumptions, the map $i_0 : X_0 \to M_0$ is a homotopy equivalence. Thus, there exist a continuous map $p_0 : M_0 \to X_0$ and two homotopies $H_0 : [0, 1] \times M_0 \to M_0$ and $\bar{H}_0 : [0, 1] \times X_0 \to X_0$ with

$$H_0(0, z) = z, \quad H_0(1, z) = i_0(p_0(z)), \quad \bar{H}_0(0, x) = x, \quad \bar{H}_0(1, x) = p_0(i_0(x))$$

for all $z \in M_0$ and $x \in X_0$. Consider a tubular neighborhood of $N_1$ disjoint from the compact set $i_0[X_0]$. Since $N_1$ is diffeomorphic to $\mathbb{R}^{\ell-k}$, the tubular neighborhood may be assumed to be a smooth injective map $\Phi : \mathbb{D}^k_0 \times \mathbb{R}^{\ell-k} \to M_1$ with $\Phi([0] \times \mathbb{R}^{\ell-k}) = N_1$. Let $\alpha_1 : \mathbb{D}^k_0 \to M_1$, $\alpha_1(x) = \Phi(x, 0)$, $z_1 = \alpha_1(0)$ and consider the restriction $\beta_1$, as above.

Define $\beta_2 = p_0 \circ \beta_1 : S^{k-1} \to X_0$: notice that $\beta_1$ and $i_0 \circ \beta_2$ are homotopic in $M_0$, with homotopy $H_0(\cdot, \beta_1(\cdot))$. Also, there exists $\beta : S^{k-1} \to X_0^{[k-1]}$ such that $\beta_2$ and $\beta$ are homotopic in $X_0$; let $H_X : [0, 1] \times S^{k-1} \to X_0$ be such a homotopy. Thus, $\beta_1$ and $i_0 \circ \beta$ are homotopic in $M_0$. 

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We proceed to construct $i_1 : X_1 \to M_1$. Let $\alpha_2 : \mathbb{R}^k_{\frac{1}{2}, 1} \to M_0$ satisfy $\alpha_2|_{\mathbb{R}^k_{0, 1}} = \beta_1$, $\alpha_2|_{\mathbb{R}^k_{1/2, 1}} = i_0 \circ \beta_2$ and $\alpha_2|_{\mathbb{R}^k_{1, 1}} = i_0 \circ \beta$. More precisely, for $r \in [\frac{1}{2}, 1]$ and $u \in S^{k-1}$, set

$$\alpha_2(ru) = \begin{cases} H_0(4r - 2, \beta_1(u)), & r \in [\frac{1}{2}, \frac{3}{4}], \\ i_0(H_X(4r - 3, u)), & r \in [\frac{3}{4}, 1]. \end{cases}$$

Define $\alpha : \mathbb{D}^k \to M_1$ by $\alpha|_{\mathbb{D}^k_{\frac{1}{2}}} = \alpha_1$ and $\alpha|_{\mathbb{D}^k_{\frac{1}{2}, 1}} = \alpha_2$. Define $i_1$ by $i_1|_{X_0} = i_0$ and by $i_1(x) = \alpha(x)$ for $x \in C_1 = \mathbb{D}^k$.

We need to prove that $i_1$ is a homotopy equivalence. We could at this point construct $p_1 : M_1 \to X_1$ and homotopies $H_1$ and $\tilde{H}_1$. Since that construction is rather cumbersome, we prefer to proceed in a slightly different way: we first prove that for any $j \geq 0$ the map $\pi_j(i_1) : \pi_j(X_1) \to \pi_j(M_1)$ is a bijection.

We first prove the surjectivity of $\pi_j(i_1)$. Let $\gamma_M : S^j \to M_1$: we want to prove that there exists $\gamma_X : S^j \to X_1$ such that $\gamma_M$ and $i_1 \circ \gamma_X$ are homotopic. We may assume that $\gamma_M$ is smooth and transversal to $N_1$: let $N_S = \gamma_M^{-1}[N_1] \subset S^j$, a smooth submanifold of codimension $k$. By transversality, there exist $\epsilon \in (0, \frac{1}{2})$ and a tubular neighborhood $\Psi : \mathbb{D}^k_\epsilon \times N_S \to S^j$ with $\Psi(0, s) = s$ (for all $s \in N_S$). We may furthermore assume that there exists a smooth function $f_0 : \mathbb{D}^k_\epsilon \times N_S \to \mathbb{R}_k$ such that $\gamma_M(\Psi(v, s)) = \Phi(v_{\epsilon/2}, f_0(v, s))$ (here $\Phi$ is the tubular neighborhood of $N_1$ described above).

Multiplication of $f_0$ by a bump function takes us from $\gamma_M$ to a homotopic function $\gamma_{M, 1}$ such that $\gamma_M$ and $\gamma_{M, 1}$ coincide in $S^j \setminus \Psi[\mathbb{D}^k_\epsilon \times N_S]$ and $\gamma_{M, 1}(\Psi(v, s)) = \alpha_1(v)$ for all $s \in N_S$ and $v \in \mathbb{D}^k_\epsilon$. Another homotopy takes us to $\gamma_{M, 2}$ such that $\gamma_{M, 1}$ and $\gamma_{M, 2}$ coincide in $S^j \setminus \Psi[\mathbb{D}^k_\epsilon/2 \times N_S]$ and, if $s \in N_S$ and $v \in \mathbb{D}^k_\epsilon$ then

$$\gamma_{M, 2}(\Psi(v, s)) = \begin{cases} \alpha_1(2v/\epsilon), & |v| \leq \epsilon/4; \\ \alpha_1(\epsilon/2\epsilon), & \epsilon/4 \leq |v| \leq 3\epsilon/8; \\ \alpha_1((2 - (3\epsilon/2) - (4(1 - \epsilon)/\epsilon)|v|)v), & 3\epsilon/8 \leq |v| \leq \epsilon/2. \end{cases}$$

We now compose with $H_0$ to obtain $\gamma_{M, 3}$ as follows:

$$\gamma_{M, 3}(s) = \begin{cases} (i_0 \circ p_0 \circ \gamma_{M, 2})(s), & s \notin \Psi[\mathbb{D}^k_{3\epsilon/8} \times N_S], \\ H_0(8|v| - (\epsilon/4))/\epsilon, \gamma_{M, 2}(s), & s = \Psi(v, s_0), |v| \in [\epsilon/4, 3\epsilon/8], \\ \gamma_{M, 2}(s), & s \in \Psi[\mathbb{D}^k_{\epsilon/4} \times N_S]. \end{cases}$$

Notice that for $s \in N_S$ and $|v| \leq 3\epsilon/8$ we have $\gamma_{M, 3}(\Psi(v, s)) = \alpha(2v/\epsilon)$. For $s \in S^j \setminus \Psi[\mathbb{D}^k_{3\epsilon/8} \times N_S]$, we have $\gamma_{M, 3}(s) = (i_0 \circ p_0 \circ \gamma_{M, 2})(s)$. A small adjustment using $H_X$ takes us to $\gamma_{M, 4} = i_1 \circ \gamma_{X, 4}$, completing the proof of surjectivity.

The proof of injectivity of $\pi_j(i_1)$ is similar, and will be presented in less detail. Consider $\gamma_X : S^j \to X_1$ and assume that $\pi_j(i_1)(\gamma_X) = 0 \in \pi_j(M_1)$. We may
assume that $\gamma_X$ is smooth in the interior of the new cell $C_1$ and that the center $0_{C_1} \in C_1$ is a regular value. By hypothesis, there exists $\Gamma_M : \mathbb{D}^{j+1} \to M_1$ such that $\Gamma_M|S_j = i_1 \circ \gamma_X$. Again, we may assume that $\Gamma_M$ is smooth in a neighborhood of $\Gamma_M^{-1}[0_{C_1}]$ and that $0_{C_1}$ is a regular value. Let $N_D = \Gamma_M^{-1}[0_{C_1}] \subset \mathbb{D}^{j+1}$: this is a smooth submanifold of codimension $k$ with boundary $\partial N_D \subset S_j$, also a smooth submanifold of codimension $k$. As above, pulling back $\Phi$ gives us a tubular neighborhood $\Psi$. Again as above, we construct $\Gamma_M^*$ of the form $\Gamma_M^* = i_1 \circ \Gamma_X^*$, where $\Gamma_X^* : \mathbb{D}^{j+1} \to X_1$ satisfies $\Gamma_X^*|S_j = \gamma_X$. We thus have $[\gamma_X] = 0 \in \pi_j(X_1)$, completing the proof of injectivity.

At this point we know that $i_1 : X_1 \to M_1$ is such that $\pi_j(i_1)$ is bijective for all $j$. In other words, $i_1$ is a weak homotopy equivalence. The set $X_1$ is a CW complex and $M_1$ is a manifold and therefore homeomorphic to a CW complex. By Whitehead’s Theorem, the map $i_1$ is a homotopy equivalence, as desired. □

**Proof of Theorem 2** Ancestries of dimension 0 are maximal elements under the partial order $\succeq$. Let $BL_{\sigma,0} \subseteq BL_{\sigma}$ be the union of the open, disjoint, contractible sets $BL_{\varepsilon,i}$ for $\varepsilon$ an ancestry of dimension 0. The set $BL_{\sigma,0}$ is homotopically equivalent to a finite set with one vertex per ancestry, which is of course a CW complex of dimension 0. This is the basis of a recursive construction.

We can list the set of ancestries of positive dimension as $(\varepsilon_i)_{1 \leq i \leq N_\varepsilon}$ in such a way that $\varepsilon_i \succeq \varepsilon_j$ implies $i \leq j$. Define recursively the subsets $BL_{\sigma,i} = BL_{\sigma,i-1} \cup BL_{\varepsilon,i} \subseteq BL_{\sigma}$. The family of sets $BL_{\sigma,i}$ defines a filtration:

$$BL_{\sigma,0} \subset BL_{\sigma,1} \subset \cdots \subset BL_{\sigma,N_\varepsilon-1} \subset BL_{\sigma,N_\varepsilon} = BL_{\sigma}.$$  

The partial order $\succeq$ and Lemma 13.3 guarantee that $BL_{\sigma,i-1} \subset BL_{\sigma,i}$ is an open subset. Lemma 13.1 tells us that $BL_{\varepsilon,i} = BL_{\sigma,i} \setminus BL_{\sigma,i-1} \subset BL_{\sigma,i}$ is a smooth submanifold of codimension $d = \dim(\varepsilon_i)$ and Lemma 13.2 tells us that $BL_{\varepsilon,i}$ is diffeomorphic to $\mathbb{R}^{\ell-d}$. Notice that $BL_{\varepsilon,i} \subset BL_{\sigma,i}$ is a closed subset (see Remark 14.2). We may therefore apply Lemma 14.1 to the pair $M_0 = BL_{\sigma,i-1} \subset BL_{\sigma,i} = M_1$, completing the recursive construction and the proof. □

The proofs of Lemma 14.1 and of Theorem 2 give us instructions for the actual construction of the CW complex $BLC_{\sigma}$ and of the map $i_{\sigma}$. This construction of the CW complex and of the glueing maps are not as direct as might perhaps be desired. This is the subject of Section 16.

### 15 Euler characteristic

We begin with an easy formula for the Euler characteristic of $BL_{\varepsilon}$.  

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Lemma 15.1. For $\sigma \in S_{n+1}$ and $z \in \sigma \text{Quat}_{n+1}$, we have

$$\chi(\text{BL}_z) = \sum_{\epsilon_0} (-1)^{\dim \epsilon_0} N_{\epsilon_0}(z).$$

The summation is taken over all preancestries $\epsilon_0$.

**Proof.** This follows directly from Theorem 2 and the definition of $N_{\epsilon_0}(z)$. $\square$

Lemma 15.2. Let $z_0 \in \eta \text{Quat}_{n+1}$ be such that $\Re(z_0) > 0$. We have that $\chi(\text{BL}_{z_0})$ is odd and $\chi(\text{BL}_{-z_0})$ is even.

**Proof.** We have $\ell = \text{inv}(\eta) = n(n + 1)/2$ and $c = \text{nc}(\eta) = 1 + [n/2]$. From Lemma 6.5, $\Re(z_0) = 2^{-(n+1)/2} = 2^{-(n/2)/2}$.

From Lemma 4.4, the highest possible dimension $d_{\text{max}} = [n^2/4]$ is achieved by a unique preancestry $\epsilon_{\text{max}}$ (see Figure 9). This preancestry shall be addressed separately. We have $\ell - 2d_{\text{max}} = [n/2]$. Thus, from the first equation in Theorem 4, we have $N_{\epsilon_{\text{max}}}(z_0) - N_{\epsilon_{\text{max}}}(z_0) = 1$. We have $H_{\epsilon_{\text{max}}} = H_\eta$ and $|H_\eta| = 2^{n+2-c} = 2^{1+[n/2]}$. The second equation in Theorem 4 thus gives us $N_{\epsilon_{\text{max}}}(z_0) + N_{\epsilon_{\text{max}}}(z_0) = 1$. We thus have $N_{\epsilon_{\text{max}}}(z_0) = 1$ and $N_{\epsilon_{\text{max}}}(z_0) = 0$.

Consider a preancestry $\epsilon_0$ with $d = \dim(\epsilon_0) = d_{\text{max}} - 1$. The first equation in Theorem 4 tells us that $N_{\epsilon_0}(z_0) - N_{\epsilon_0}(z_0) = 2$. The second equation tells us that $N_{\epsilon_0}(z_0) + N_{\epsilon_0}(z_0) = 4|H_\sigma|/|H_{\epsilon_0}|$, a power of 2. By definition of $H_{\epsilon_0}$, however, we have $|H_{\epsilon_0}| > |H_\sigma|$ and therefore $N_{\epsilon_0}(z_0) + N_{\epsilon_0}(z_0) \leq 2$. We thus have $N_{\epsilon_0}(z_0) = 2$ and $N_{\epsilon_0}(z_0) = 0$.

Consider finally a preancestry $\epsilon_0$ with $d = \dim(\epsilon_0) < d_{\text{max}} - 1$. The first equation in Theorem 4 tells us that $N_{\epsilon_0}(z_0) - N_{\epsilon_0}(z_0)$ is a power of 2 and a multiple of 4. The second equation tells us that $N_{\epsilon_0}(z_0) + N_{\epsilon_0}(z_0)$ is also a power of 2 and therefore also a multiple of 4. Thus, $N_{\epsilon_0}(z_0)$ and $N_{\epsilon_0}(z_0)$ are both even.

Summing up, $N_{\epsilon_0}(z_0)$ is even for every preancestries. Similarly, $N_{\epsilon_0}(z_0)$ is even for every preancestries except $\epsilon_{\text{max}}$. The result follows from Lemma 15.1 $\square$

Corollary 15.3. Consider $n \geq 5$ and $z_0 \in \eta \text{Quat}_{n+1}$ with $\Re(z_0) > 0$. Then $\text{BL}_{-z_0,\text{thick}}$ is non empty, connected and its Euler characteristic $\chi(\text{BL}_{-z_0,\text{thick}})$ is even.

**Proof.** From Proposition 11.3, $\text{BL}_{-z_0,\text{thick}}$ is non empty and connected. From Lemma 15.2, $\chi(\text{BL}_{z_0})$ is even. From Lemma 8.3, $N_{\text{thin}}(z_0)$ is even. We have $\chi(\text{BL}_{-z_0,\text{thick}}) = \chi(\text{BL}_{-z_0}) - N_{\text{thin}}(z_0)$: the result follows. $\square$
Remark 15.4. Corollary 15.3 implies the third item in Theorem 1. For \( n = 5 \), \( BLC_{-z_0} \) is connected and has: 480 vertices, 1120 cells of dimension 1, 864 cells of dimension 2, 228 cells of dimension 3, 6 cells of dimension 4 and no cells of higher codimension. It follows that \( \chi(BL_{-z_0}) = 480 - 1120 + 864 - 228 + 6 = 2 \): in particular, \( BL_{-z_0} \) is not contractible. It would be very interesting to compute its homotopy type.

\[ \diamond \]

16 The glueing maps

The following results gives us a little more information about the CW complex \( BLC_\sigma \), particularly about the glueing maps.

Recall that \( U_\varepsilon \) is the upper set of ancestries \( \bar{\varepsilon} \) with \( \varepsilon \preceq \bar{\varepsilon} \). Let \( U_\varepsilon^* = U_\varepsilon \setminus \{ \varepsilon \} \), also an upper set. In general, for an upper set \( U \) of ancestries, define

\[
BLS_U = \bigcup_{\varepsilon \in U} BLS_\varepsilon \subseteq BL_\sigma, \quad BLC_U = \bigcup_{\varepsilon \in U} BLC_\varepsilon \subseteq BLC_\sigma.
\]

We know from Corollary 13.5 that \( BLS_U \subseteq BL_\sigma \) is an open subset.

Lemma 16.1. Let \( U \) be an upper set of ancestries. The subset \( BLC_U \subseteq BLC_\sigma \) is closed and a CW complex. The restriction \( i_\sigma|_{BLC_U} : BLC_U \to BLS_U \) is a homotopy equivalence.

Proof. We use induction on \(|U|\), the cases \(|U| \leq 1\) being trivial. Let \( \varepsilon \in U \) be a minimal element, so that \( U^* = U \setminus \{ \varepsilon \} \) is also an upper set. By induction hypothesis, the result holds for \( U^* \). It follows from Lemma 14.1 and the proof of Theorem 2 that the image of the glueing map for the cell \( BLC_\varepsilon \) is contained in \( BLC_{U^*} \), proving the first claim for \( U \). The second claim follows from Lemma 14.1. \( \square \)

Corollary 16.2. The image of the glueing map for \( BLC_\varepsilon \) in contained in \( BLC_{U^*} \).

Proof. This follows directly from Lemma 16.1. \( \square \)

Consider an ancestry \( \varepsilon \) of positive dimension \( d = \text{dim}(\varepsilon) > 0 \). We define two non empty subsets \( U_\varepsilon^\pm \subseteq U_\varepsilon^* \). Let \( k_\bullet \) be the largest index \( k \) for which \( \varepsilon(k) = -2 \). We have \( \varrho_{k_\bullet} = \varrho_{k_\bullet - 1} \hat{a}_{ik} \): define \( \bar{\varrho}_{k_\bullet} = \varrho_{k_\bullet - 1} \) and \( \bar{\varrho}_{k_\bullet}^+ = \varrho_{k_\bullet - 1} \hat{a}_{ik} \). For \( \bar{\varepsilon} \in U_\varepsilon \), let \( (\bar{\varrho}_{k})_{0 \leq k \leq \ell} \) be defined as usual. We have

\[
\bar{\varepsilon} \in U_\varepsilon^\pm \iff ((\bar{\varrho}_{k_\bullet} = \bar{\varrho}_{k_\bullet}^\pm) \land (\forall k, (0 \leq k < k_\bullet) \rightarrow (\bar{\varrho}_k = \varrho_k))). \quad (16.1)
\]

The sets \( U_\varepsilon^\pm \) are clearly disjoint.
Example 16.3. We already discussed $U_\varepsilon$ for $\dim(\varepsilon) = 1$ in Example 9.2. In this case, the sets $U_\varepsilon^\pm$ have one element each.

If $\varepsilon$ is an ancestry of dimension two, type I (see Remark 4.2 and Example 9.3) then the sets $U_\varepsilon^\pm$ also have one element each. In Figure 11 these are the edges at the top and bottom of the figure.

If $\varepsilon$ is an ancestry of dimension two, type II then the sets $U_\varepsilon^\pm$ can be more complicated. In particular, if the ancestries in Figure 12 are $\varepsilon_0$ and $\varepsilon_1$, we have $\varepsilon_1 \in U_{\varepsilon_0}^-$ (see also Example 13.4).

Define
\[
BLS_\varepsilon^\pm = \bigcup_{\tilde{\varepsilon} \in U_\varepsilon^\pm} BLS_{\tilde{\varepsilon}}.
\] (16.2)

The following result describes these sets near $BLS_\varepsilon$.

Lemma 16.4. Let $\varepsilon$ be an ancestry of dimension $d = \dim(\varepsilon) > 0$. If $W$ is a sufficiently thin open tubular neighborhood of $BLS_\varepsilon$ then $(BLS_\varepsilon \cup BLS_\varepsilon^\pm) \cap W \subset W$ are smooth submanifolds with boundary. Both manifolds have codimension $d - 1$ and boundary equal to $BLS_\varepsilon$.

Let $W^* = W \setminus BLS_\varepsilon$. There exists a diffeomorphism $\Phi : S^{d-1} \times (0, r) \times \mathbb{R}^{d-d} \to W^*$ such that
\[
\Phi^{-1}[BLS_\varepsilon^+] = \{N\} \times (0, r) \times \mathbb{R}^{d-d}, \quad \Phi^{-1}[BLS_\varepsilon^-] = \{S\} \times (0, r) \times \mathbb{R}^{d-d},
\]
where $N,S \in S^{d-1}$ are the north and south poles.

Proof. Let $(z_k)_{0 \leq k \leq \ell}$ be sequences as in Equation (12.1). For $W \supset BLS_\varepsilon$ as above and $L \in W$, we identify $L \in W$ with such a sequence with $z_\varepsilon \in \hat{\eta}\text{Bru}_{\varepsilon_k}$. For $L \in W$, we have $L \in BLS_\varepsilon^\pm$ if and only if:
\[
(z_{k_*} \in \hat{\eta}\text{Bru}_{\varepsilon_{k_*}}) \land (\forall k, (0 \leq k < k_*) \to (z_k \in \hat{\eta}\text{Bru}_{\varepsilon_k})).
\]
The condition above includes $d-1$ transversal equations, corresponding to $k < k_*$, $\varepsilon(k) = -2$: the remaining are open conditions. This completes the proof of the first paragraph (the first three sentences in the statement). The other claims are then easy. \(\Box\)

Let $M$ be a smooth manifold and $N \subset M$ be a transversally oriented submanifold of codimension $k$ which is also a closed set. Intersection with $N$ defines an element of $H^k(M; \mathbb{Z})$. In $W^*$ (as defined in Lemma 16.4), intersection with either $BLS_\varepsilon^\pm$ defines a generator of $H^{d-1}(W^*; \mathbb{Z}) \approx \mathbb{Z}$.

An ancestry $\varepsilon$ with $d = \dim(\varepsilon) > 0$ is tame if the following conditions hold:

1. The manifold $BLS_{U_\varepsilon^*}$ is homotopically equivalent to $S^{d-1}$. 44
2. Intersection with BLS\(_\varepsilon^\pm\) defines generators of \(H^{d-1}(\text{BLS}_{U^\varepsilon}; \mathbb{Z}) \approx \mathbb{Z}\).

Otherwise, \(\varepsilon\) is wild.

Recall that BLS\(_{U^\varepsilon}\) is homotopically equivalent to the CW complex BLC\(_{U^\varepsilon}\). Let us translate the definition above in terms of BLC\(_{U^\varepsilon}\). The first condition says of course that BLC\(_{U^\varepsilon}\) is homotopically equivalent to \(S^{d-1}\). The second condition says that we can build cocycles \(\omega_{\text{BLC}}^\pm \in Z^{d-1}(\text{BLC}_{U^\varepsilon}; \mathbb{Z})\) by taking the elements of \(U^\varepsilon\) of dimension \(d-1\) and interpreting them as cells of BLC\(_{U^\varepsilon}\): the elements \(\omega_{\text{BLC}}^\pm\) are then generators of \(H^{d-1} \approx \mathbb{Z}\).

Example 16.5. It follows from Example 9.2 that \(\dim(\varepsilon) = 1\) implies that \(\varepsilon\) is tame. Similarly, Example 9.3 that if \(\varepsilon\) is of dimension two type I then \(\varepsilon\) is tame.

The following lemma tells us how to obtain the glueing map in tame cases.

**Lemma 16.6.** If \(\varepsilon\) is tame then the glueing map \(\beta : S^{d-1} \to \text{BLC}_{U^\varepsilon}\) is a homotopy equivalence.

**Proof.** Let \(W^*\) be as in Lemma 16.4. Let \(\omega_{W^*}^\pm \in H^{d-1}(W^*; \mathbb{Z})\) be defined by intersection with BLS\(_\varepsilon^\pm\), by definition of tameness, either one is a generator. Let \(\beta_1 : S^{d-1} \to W^*\) be as in the first lines of the proof of Lemma 14.1 (with \(M_0 = \text{BLS}_{U^\varepsilon}\), \(M_1 = \text{BLS}_{U^\varepsilon}\) and \(k = d\)). We have a pairing \(H^{d-1}(W^*; \mathbb{Z}) \times \pi_{d-1}(W^*) \to \mathbb{Z}\). By Lemma 16.4, \(|\omega_{W^*}^\pm, \beta_1| = 1\).

Let \(i : W^* \to \text{BLS}_{U^\varepsilon}\) be the inclusion. Let \(\omega_{\text{BLS}}^\pm \in H^{d-1}(\text{BLS}_{U^\varepsilon}; \mathbb{Z})\) be defined by intersection with BLS\(_\varepsilon^\pm\), as in the definition of tameness. Let

\[
i^* = H^{d-1}(i) : H^{d-1}(\text{BLS}_{U^\varepsilon}; \mathbb{Z}) \to H^{d-1}(W^*; \mathbb{Z})
\]

we have \(i^*(\omega_{\text{BLS}}^\pm) = \omega_{W^*}^\pm\). We thus have \(\omega_{\text{BLS}}^\pm(i \circ \beta_1) = \omega_{W^*}^\pm \beta_1\) and \(i \circ \beta_1\) is therefore a generator of \(\pi_{d-1}(\text{BLS}_{U^\varepsilon})\). From the proof of Lemma 14.1, so is the glueing map \(\beta\). The result follows.

Example 16.7. It follows from Lemma 16.6 together with Examples 9.2 and 16.5 that cells BLC\(_\varepsilon\) of dimension 1 in BLC\(_\sigma\) are edges joining the two vertices corresponding to the elements of dimension 0 in \(U^\varepsilon\). This is illustrated in Figure 10 (among many others).

Similarly, Examples 9.3 and 16.5 imply that if \(\varepsilon\) has dimension two type I then BLC\(_\varepsilon\) fills in a square hole, as in Figure 11.

In the next sections we shall see several other examples of tame ancestries. So far we have not seen any example of a wild ancestry, which raises the following doubt:

**Question 16.8.** Are there any wild ancestries?

We do not know the answer: it seems entirely plausible that in higher dimension wild ancestries exist.
Examples and proof of Theorems 1 and 3

In this section we apply the previous results, particularly Theorem 4, Lemmas 16.6 and 15.1 to compute the homotopy type of \( BL_\sigma \) and of its subsets \( BL_z \) for several examples. Let \( N(z) = N_{z_0}(z) \) (where \( \varepsilon_0 \) is the empty preancestry) be the number of vertices in the CW complex \( BLC_z \).

Example 17.1. Consider \( n = 3 \) and \( \sigma = 4231 = a_1a_2a_3a_2a_1 \). A simple computation gives \( \dot{\sigma} = (\hat{a}_2 + \hat{a}_1\hat{a}_3)/\sqrt{2} \) and

\[
\dot{\sigma} \text{ Quat}_4 = \left\{ \frac{\pm 1 \pm \hat{a}_1\hat{a}_2\hat{a}_3}{\sqrt{2}}, \frac{\pm \hat{a}_1 \pm \hat{a}_2\hat{a}_3}{\sqrt{2}}, \frac{\pm \hat{a}_2 \pm \hat{a}_1\hat{a}_3}{\sqrt{2}}, \frac{\pm \hat{a}_3 \pm \hat{a}_1\hat{a}_2}{\sqrt{2}} \right\}.
\]

The action of \( E_3 \) on \( \dot{\sigma} \text{ Quat}_4 \) has 5 orbits. The orbit \( O_{\dot{\sigma}} = \{(\pm \hat{a}_2 \pm \hat{a}_1\hat{a}_3)/\sqrt{2}\} \) has size 4, and for each \( z \) we have \( N(z) = N_{\text{thin}}(z) = 2 \) so that \( BL_z \) has two thin (and therefore contractible) connected components. The orbits \( O_{\hat{a}_1\dot{\sigma}} = \{(\pm \hat{a}_3 \pm \hat{a}_1\hat{a}_2)/\sqrt{2}\} \) and \( O_{\hat{a}_3\dot{\sigma}} = \{(\pm \hat{a}_1 \pm \hat{a}_2\hat{a}_3)/\sqrt{2}\} \) both have size 4. For each \( z \) in one of these two orbits, we have \( N(z) = 2 \) and \( N_{\text{thin}}(z) = 0 \). Figure 16 shows the stratification of \( BL_z \) for one representative of each orbit: it follows that such sets \( BL_z \) are contractible. The orbit \( O_{\hat{a}_2\dot{\sigma}} = \{(\pm 1 \pm \hat{a}_1\hat{a}_2\hat{a}_3)/\sqrt{2}\} \) has size 2. For \( z \) in this orbit, we have \( N(z) = 0 \), so that the corresponding sets \( BL_z \) are empty.

Finally, the orbit \( O_{-\hat{a}_2\dot{\sigma}} = \{(1 \pm \hat{a}_1\hat{a}_2\hat{a}_3)/\sqrt{2}\} \) also has size 2: consider \( z = -\hat{a}_2\dot{\sigma} = \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_2\hat{a}_1 = (1 + \hat{a}_1\hat{a}_2\hat{a}_3)/\sqrt{2} \). Figure 16 also shows the stratification of \( BL_z \). A straightforward computation verifies that the set \( BL_z \) consists of matrices of the form:

\[
L = \begin{pmatrix}
1 & l_{21} & l_{31} & l_{41} \\
l_{21} & 1 & l_{32} & l_{42} \\
l_{31} & l_{32} & 1 & l_{43} \\
l_{41} & l_{42} & l_{43} & 1
\end{pmatrix}, \quad l_{41} > \max\{0, l_{21}l_{42}, l_{31}l_{43}\}, \quad l_{32} = \frac{l_{31}l_{42}}{l_{41}}.
\]

The above description makes it clear that \( BL_z \) is contractible, but we want to explore its decomposition into strata. The set \( BL_z \) contains 4 open strata, 4
strata of codimension 1 and one stratum of codimension 2, with ancestry \( \epsilon = (-2, -2, +1, +2, +2) \). A computation shows that \( \epsilon \) is tame. The set \( \text{BL}_\epsilon \) is the set of matrices of the above form with \( l_{32} = l_{42} = l_{43} = 0 \). The open strata are characterized by the signs of \( l_{42} \) and \( l_{43} \). In our CW complex, the ancestry \( \epsilon = (-2, -2, +1, +2, +2) \) corresponds to a square, glued along the four edges (corresponding to ancestries of codimension 1) in the obvious way. Thus, \( \text{BL}_\epsilon \) is also contractible. Summing up, \( \text{BL}_\sigma \) has 18 connected components. Each connected component of \( \text{BL}_\sigma \) collapses to a point. Each connected component of \( \text{BL}_\sigma \) is therefore contractible.

\[ \square \]

**Example 17.2.** We take \( n = 3 \) and \( \sigma = \eta = a_1a_2a_1a_3a_2a_1 \), the top permutation (with \( \ell = 6 \)); see Figure 17. We shall verify that the set \( \text{BL}_\eta \) has 20 connected components, all contractible. The total number of connected components of \( \text{BL}_\eta \) was first calculated by Vl. Kostov, B. Shapiro and M. Shapiro using ad hoc methods back in 1987 (unpublished); see also [4, 14].

![Figure 17: The CW complexes BLC_{-\hat{a}_1\hat{\eta}} and BLC_{-\hat{a}_2\hat{\eta}}.](image)

We have

\[ \hat{\eta} = \frac{-1 + \hat{a}_2 + \hat{a}_1\hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \quad \hat{\hat{\eta}} = \frac{-1 - \hat{a}_2 + \hat{a}_1\hat{a}_3 + \hat{a}_1\hat{a}_2\hat{a}_3}{2}, \]

\[ \hat{\eta}\text{ Quat}_4 = \left\{ \pm \hat{a}_2 \pm \hat{a}_1\hat{a}_3 \pm \hat{a}_1\hat{a}_2\hat{a}_3, \pm \hat{a}_1 \pm \hat{a}_1\hat{a}_2 \pm \hat{a}_3 \pm \hat{a}_2\hat{a}_3 \right\} \]

where we must take an even number of ‘−’ signs (so that the above set has 16 elements). There are three \( \mathcal{E}_3 \)-orbits, determined by real part (two of size 4, one of size 8). If \( \Re(z) = -\frac{1}{2} \), the set \( \text{BL}_z \) has two thin connected components (and no thick ones).

In order to study the orbit \( \Re(z) = 0 \), take

\[ z_1 = \hat{a}_1\hat{a}_2\hat{a}_1\hat{a}_3\hat{a}_2\hat{a}_1 = (-\hat{a}_1)\hat{\eta} = \hat{\eta}(\hat{a}_3) = \frac{\hat{a}_1 - \hat{a}_1\hat{a}_2 + \hat{a}_3 - \hat{a}_2\hat{a}_3}{2}. \]

A case-by-case verification shows that \( \text{BL}_{z_1} \) has 4 ancestries of dimension 0, 3 ancestries of dimension 1 and no ancestries of dimension higher than 1. These numbers can also be obtained from Theorem 4. In any case, \( \text{BLC}_{z_1} \) is the first graph in Figure 17: \( \text{BLC}_{z_1} \) therefore collapses to a point and \( \text{BL}_{z_1} \) is contractible.
Figure [17] also shows BLC where

\[ z_0 = \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5 = \hat{a}_2 \hat{a}_1 = \frac{1 + \hat{a}_2 + \hat{a}_1 \hat{a}_3 + \hat{a}_1 \hat{a}_2 \hat{a}_3}{2}. \]

In BLC there are 6 ancestries of dimension 0, 6 ancestries of dimension 1 and exactly one ancestry of dimension 2: \( \varepsilon = (-2, -2, +1, +1, +2, +2) \). A computation shows that \( \varepsilon \) is tame. Thus, in the CW complex shown in Figure [17] the cell of dimension 2 glues in the obvious way, so that BLC collapses to a point and is contractible. This completes this example.

\[ \diamond \]

**Example 17.3.** Set \( n = 5 \) and \( \sigma = 563412 = a_2 a_1 a_3 a_2 a_4 a_3 a_5 a_4 a_2 a_1 a_3 a_2 \). We have \( \ell = 12, b = |\text{Block}(\sigma)| = 0, c = \text{nc}(\sigma) = 4 \) and

\[ \sigma = -\sigma = \frac{-\hat{a}_1 - \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5}{2}. \]

Consider \( z_0 = \hat{a}_1 \sigma \): we have \( \Re(z_0) = \frac{1}{2} \) and \( N(-z_0) = 48 \). A computation shows that BLC has 56 cells of dimension 1, 8 cells of dimension 2 and no cells of higher dimension. This implies that the Euler characteristic is \( \chi(\text{BL}_{-z_0}) = 0 \). It turns out that BLC has two connected components. We construct the CW complex corresponding to one of the components in Figure [5]; the other connected component is similar, changing all signs for strata of codimension 0. The ancestries of dimension 2 can be verified to be tame. Thus, the 2-cells in the CW complex fill in the hexagons and octagons in the figure, including the hexagon which joins to top and bottom of the figure, as in Figure [4]. Thus, each connected component of BLC is homotopically equivalent to the circle \( S^1 \). This completes the proof of the second item in Theorem [1].

\[ \diamond \]

**Example 17.4.** Set \( \sigma = \eta = a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1 \). We have \( \ell = 10, b = |\text{Block}(\eta)| = 0, c = \text{nc}(\eta) = 3 \) and

\[ \eta = -\eta = \frac{-\hat{a}_1 - \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4}{2}. \]

It follows from Theorem [3] that \( N(z) = 32 + 16 \Re(z) \). In this example, it turns out that \( \eta \text{Quat}_5 \) contains 4 elements with \( \Re(z) = \frac{1}{2} \), 4 elements with \( \Re(z) = -\frac{1}{2} \) and 24 elements with \( \Re(z) = 0 \). The set \( \eta \text{Quat}_5 \) has 5 orbits under \( E_4 \) of sizes
8, 4, 4, 8, 8, shown below.

\[
\mathcal{O}_\eta = \left\{ \frac{\pm \hat{a}_1 \pm \hat{a}_2 \hat{a}_3 \pm \hat{a}_4 \pm \hat{a}_2 \hat{a}_3 \hat{a}_4}{2} \right\}, \quad N(z) = 32, \quad N_{\text{thin}}(z) = 2,
\]

\[
\mathcal{O}_{\hat{a}_1 \hat{a}_2} = \left\{ \frac{1 \pm \hat{a}_2 \hat{a}_3 \pm \hat{a}_1 \hat{a}_4 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4}{2} \right\}, \quad N(z) = 40, \quad N_{\text{thin}}(z) = 0,
\]

\[
\mathcal{O}_{-\hat{a}_1 \hat{a}_2} = \left\{ \frac{-1 \pm \hat{a}_2 \hat{a}_3 \pm \hat{a}_1 \hat{a}_4 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4}{2} \right\}, \quad N(z) = 24, \quad N_{\text{thin}}(z) = 0,
\]

\[
\mathcal{O}_{\hat{a}_2 \hat{a}_3} = \left\{ \frac{\pm \hat{a}_1 \hat{a}_2 \pm \hat{a}_1 \hat{a}_3 \pm \hat{a}_2 \hat{a}_4 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4}{2} \right\}, \quad N(z) = 32, \quad N_{\text{thin}}(z) = 0,
\]

\[
\mathcal{O}_{\hat{a}_1 \hat{a}_2 \hat{a}_3} = \left\{ \frac{\pm \hat{a}_2 \pm \hat{a}_3 \pm \hat{a}_1 \hat{a}_2 \hat{a}_4 \pm \hat{a}_1 \hat{a}_3 \hat{a}_4}{2} \right\}, \quad N(z) = 32, \quad N_{\text{thin}}(z) = 0,
\]

In the expressions in the Clifford algebra notation, we must always have an even number of ‘−’ signs.

In order to count connected components and obtain further information about the topology of the sets BL\(_z\), \(z \in \hat{\eta} \text{Quat}_{n+1}\), we can pick one representative from each orbit and draw the CW complex BLC\(_z\). As a sample, we do this in Figure 18 for \(z = -\hat{a}_1 \hat{\eta} = -\hat{\eta} \hat{a}_4\). In this case, there are exactly two ancestries of dimension 2, both tame:

\[
(+1, -2, -2, +1, +1, +2, +1, +1, +2, +1),
\]

\[
(-1, -2, -2, +1, -1, +2, -1, +1, +2, -1);
\]

there are no ancestries of higher dimension. It follows that BL\(_{-\hat{a}_1 \hat{\eta}}\) is homotopically equivalent to the disjoint union of two points. In other words, each of the two connected components of BLC\(_{-\hat{a}_1 \hat{\eta}}\) = BLC\(_{-\hat{a}_1 \hat{\eta}, \text{thick}}\) collapses to a point.

The other cases are more laborious but the CW complexes can be drawn. In all other cases, BL\(_z, \text{thick}\) is non empty, connected and contractible. This confirms that BL\(_\eta\) has 52 connected components and that they are all contractible. ⊗

**Example 17.5.** Consider now \(\sigma = 54231 = a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1\); Figure 19 shows this reduced word as a diagram. In the cycle notation, \(\sigma = (15)(243)\); we therefore have \(n = 4, \ell = 9, c = 2\) and \(b = 0\). Theorem 4 tells us that, for \(z \in \hat{\sigma} \text{Quat}_5\), we have \(N(z) = 16 + 8\sqrt{2} \Re(z)\). We have

\[
\hat{\sigma} = \frac{-\hat{a}_1 + \hat{a}_1 \hat{a}_2 + \hat{a}_1 \hat{a}_3 - \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_2 \hat{a}_4 + \hat{a}_3 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4}{2\sqrt{2}}
\]

and \(\Re(\pm \hat{a}_1 \hat{\sigma}) = \pm \sqrt{2}/4\).
Figure 18: The CW complex $\text{BLC}_{-\hat{a}_1\hat{\sigma}}$; see Example 17.4.

It turns out that $\Re(z) = 0$ implies $c_{\text{anti}} = 1$: the set $\hat{\sigma} \text{Quat}_5$ thus has 3 orbits under $\mathcal{E}$, of sizes 16, 8 and 8:

- $O_{\hat{\sigma}}, \quad \Re(z) = 0, \quad N(z) = 16, \quad N_{\text{thin}}(z) = 1$,
- $O_{\hat{a}_1\hat{\sigma}}, \quad \Re(z) = \frac{\sqrt{2}}{4}, \quad N(z) = 20, \quad N_{\text{thin}}(z) = 0$,
- $O_{-\hat{a}_1\hat{\sigma}}, \quad \Re(z) = -\frac{\sqrt{2}}{4}, \quad N(z) = 12, \quad N_{\text{thin}}(z) = 0$.

The thick part of $\text{BL}_{\hat{\sigma}}$ is connected, so that $\text{BL}_{\hat{\sigma}}$ has two connected components. The set $\text{BL}_{\hat{a}_1\hat{\sigma}}$ is also connected, but $\text{BL}_{-\hat{a}_1\hat{\sigma}}$ has two connected components. In Figure 20 we show the two connected components of $\text{BL}_z$ for $z = -\hat{\sigma}\hat{a}_1$. There are no ancestries of codimension 2 or higher and therefore these connected components are contractible. Notice that an involution takes one component of
Figure 19: The permutation $\sigma \in S_5$; see Example 17.5.

BL$_z$ to the other. The total number of connected components of BL$_{\sigma}$ is therefore equal to 56.

Figure 20: The CW complex BLC$^{-\delta_{\hat{a_1}}}$; see Example 17.5.

Proof of Theorem 3. This follows from a long computation: samples are given in Examples 17.1, 17.2, 17.4 and 17.5. More examples are presented in greater detail in [1]. In particular, for $n = 4$ and $\sigma = \eta$, we have a larger example than anything presented in this paper, including cells of dimension up to 4. In [1] we present an explicit sequence of collapses taking to a CW complex homeomorphic to a 2-disk. Such computations rely of course on such results as Lemma 16.6.

Proof of Theorem 1. The first item follows from Theorems 2 and 3. The second item follows from Example 17.3. The third item follows from Corollary 15.3 and Remark 15.4.

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