Finite dimensional semigroup quadratic algebras
with the minimal number of relations

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Abstract

A quadratic semigroup algebra is an algebra over a field given by the generators $x_1, \ldots, x_n$ and a finite set of quadratic relations each of which either has the shape $x_j x_k = 0$ or the shape $x_j x_k = x_l x_m$. We prove that a quadratic semigroup algebra given by $n$ generators and $d \leq \frac{n^2 + n}{2}$ relations is always infinite dimensional. This strengthens the Golod–Shafarevich estimate for the above class of algebras. Our main result however is that for every $n$, there is a finite dimensional quadratic semigroup algebra with $n$ generators and $\delta_n$ relations, where $\delta_n$ is the first integer greater than $\frac{n^2 + n}{2}$. That is, the above Golod–Shafarevich-type estimate for semigroup algebras is sharp.

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1 Introduction

Throughout the paper, $K$ is an arbitrary field and $\mathbb{N}$ is the set of positive integers. For a set $X$, $K\langle X \rangle$ stands for the free associative algebra over $K$ generated by $X$. For $n \in \mathbb{N}$, $\langle X \rangle_n$ denotes the set of monomials in $K\langle X \rangle$ of degree $n$: $\langle X \rangle_n = \{x_{i_1} \ldots x_{i_n} : x_{i_j} \in X\}$.

We deal with quadratic algebras, that is, algebras $R$ given as $K\langle X \rangle/I$, where $I$ is the ideal in $K\langle X \rangle$ generated by a collection of homogeneous elements (called relations) of degree 2.

Algebras of this class, their growth, their Hilbert series, rationality and nil/nilpotency properties have been extensively studied (see [11, 12, 13] and references therein). One of the most challenging questions in the area (see the ICM paper [12], or [17]) is whether there exists an infinite dimensional nil algebra in this class. This is a version of the Kurosh problem with an additional constraint formulated in terms of defining relations. A version of the Kurosh problem with a restriction on the rate of growth was recently solved in [9]. As it is well-known, the original Kurosh problem was solved with the help of the Golod–Shafarevich lower estimate for the Hilbert series of an algebra. This shows that the behavior of the Hilbert series plays a crucial role in the structural theory of algebras. There is a vast literature devoted to the better understanding of the Golod–Shafarevich inequality. For instance, in [10, 3] a number of results on asymptotic (in various senses) tightness of the Golod–Shafarevich inequality is obtained. In [3] we also prove some partial non-asymptotic results results on the Anick conjecture.

The Golod–Shafarevich-type estimates have very deep connections to other areas of algebra. First, it is necessary to mention their classical applications to $p$-groups and class field theory [1, 2, 15]. As evidence of the connection to certain deep homological properties one can consider the fact that for quadratic algebras with $n$ generators and $d$ relations, the interval $\frac{2}{3} < d < \frac{2}{3}$, where the Anick conjecture on the tightness of the Golod–Shafarevich estimate is difficult to tackle, is exactly the interval, for which the generic algebra is non-Koszul (see [11]). Furthermore the notion of noncommutative complete intersection [4] can be defined in terms of the Hilbert series and refers to a class of algebras for which the Golod–Shafarevich inequality turns into equality.
On the other hand, this concept is also characterized by the nice behavior of the noncommutative Koszul (Shafarevich) complex.

In this paper, we mostly deal with quadratic semigroup algebras, which traditionally serve as a main source of examples in this area and where an interesting combinatorics of words can applied. This class was extensively studied, one can find an account of main results and methods in [8].

The Golod–Shafarevich lower estimate [5, 11] for the Hilbert series of an algebra given by quadratic relations implies [15] that $R$ is infinite dimensional if $d \leq \frac{n^2}{4}$, where $R$ is an algebra defined by $n$ generators and $d$ quadratic relations. Anick [1, 2] conjectured that the Golod–Shafarevich estimate is attained in the case of quadratic algebras. In particular, this conjecture, if true, implies that for all $d, n \in \mathbb{N}$ with $d > \frac{n^2}{4}$, there is a finite dimensional quadratic algebra with $n$ generators defined by $d$ relations. This conjecture in its full generality still remains open.

We study the same question for a subclass of the class of quadratic algebras.

**Definition 1.1.** A quadratic semigroup algebra is an algebra given by generators and relations such that each of its defining relations is either a degree 2 monomial or a difference of two degree 2 monomials.

Many partial solutions of Anick’s conjecture were obtained by means of examples, which are quadratic semigroup algebras. An interesting combinatorial result dealing with semigroup relations was obtained by Wisliceny. We are moving in the same direction. Wisliceny [16] proved that the conjecture of Anick is asymptotically correct. Namely, for every $n \in \mathbb{N}$, he constructed a quadratic algebra $R_n$ with $n$ generators defined by $d$ relations such that $R_n$ is finite dimensional and $\lim_{n \to \infty} d_n = \frac{1}{4}$. More specifically, $d_n = \frac{n^2 + 2n}{4}$ if $n$ is even and $d_n = \frac{n^2 + 2n + 1}{4}$ if $n$ is odd. The algebras $R_n$, constructed by Wisliceny, are quadratic semigroup algebras.

It turns out that it is not enough to consider quadratic semigroup algebras to prove Anick’s conjecture. Namely, the Golod–Shafarevich estimate can be improved for this class.

**Theorem 1.2.** Let $R$ be a quadratic semigroup algebra with $n$ generators given by $d \leq \frac{n^2 + n}{4}$ relations. Then $R$ is infinite dimensional.

However, this leaves a gap between the $\frac{n^2}{4} + \frac{n}{4}$ of Theorem 1.2 and approximately $\frac{n^2}{4} + \frac{n}{2}$ of the Wisliceny example. Surprisingly, it turns out that the estimate provided by Theorem 1.2 is tight.

**Theorem 1.3.** Let $d, n \in \mathbb{N}$ be such that $d > \frac{n^2 + n}{4}$. Then there exists a finite dimensional quadratic semigroup algebra with $n$ generators given by $d$ relations.

Thus the minimal possible number of defining relations of a finite dimensional quadratic semigroup algebra with $n$ generators is precisely the first integer greater than $\frac{n^2 + n}{4}$. Theorems 1.2 and 1.3 give a solution to the natural analog of Anick’s conjecture for the class of semigroup quadratic algebras. Furthermore, it roughly halves the gap between the number of relations in the original Anick’s conjecture and the number of relations in Wisliceny’s example, which gave the best bound known before the result of this paper.

2 Proof of Theorem 1.2

**Definition 2.1.** Let $g = \sum_{a, b \in X} \lambda_{a,b}ab$ be a homogeneous degree 2 element of $\mathbb{K}\langle X \rangle$. Then the finite set $\{ab : \lambda_{a,b} \neq 0\} \subseteq \langle X \rangle_2$ is called the support of $g$ and is denoted $\text{supp}(g)$.

**Lemma 2.2.** Let $X$ be a non-empty set, $M$ be a family of homogeneous degree 2 elements of $\mathbb{K}\langle X \rangle$ and $R = \mathbb{K}\langle X \rangle/I$, where $I$ is the ideal generated by $M$. Assume also that there are $a, b \in X$ such that $ab, ba \notin \text{supp}(f)$ for every $f \in M$. Then $R$ is infinite dimensional.
Proof. Since \( ab \) and \( ba \) are not in the support of any \( f \in M \), the monomial \( (ab)^n \) does not feature in the polynomial \( pfq \) for every \( f \in M \) and \( p, q \in \mathbb{K}(X) \). Hence \( (ab)^n \not\in I \) for each \( n \in \mathbb{N} \). It immediately follows that \( R \) is infinite dimensional.

\[ \] 

**Lemma 2.3.** Let \( X \) be a non-empty set, \( M \) be a family of homogeneous degree 2 elements of \( \mathbb{K}(X) \) and \( R = \mathbb{K}(X)/I \), where \( I \) is the ideal generated by \( M \). Assume also that for each \( f \in M \), the coefficients of \( f \) sum up to 0. Then \( R \) is infinite dimensional.

**Proof.** Since the coefficients of each \( f \) sum up to 0, it follows that the coefficients of every element of \( I \) sum up to 0. Hence \( I \) contains no monomials and therefore \( R \) is infinite dimensional.

We are ready to prove Theorem 1.2. Denote by \( X \) the \( n \)-element set of generators of \( R \). If there are \( a, b \in X \) such that \( ab \) and \( ba \) are both not in the support of any relation, then \( R \) is infinite dimensional according to Lemma 2.2. Thus we can assume that for every \( a, b \in X \) either \( ab \) or \( ba \) (or both) belongs to the union \( N \) of the supports of the relations. It follows that \( |N| \geq n^2+4n \). Since each relation is either a monomial or a difference of two monomials, their supports have at most two elements. Taking into account that we have \( d \leq n^2+2n \) relations, we are left with the only possibility that \( d = \frac{n^2+2n}{2} \) and each relation is a difference of two monomials. In this case, the coefficients of each relation sum up to 0. By Lemma 2.3, \( R \) is infinite dimensional. The proof of Theorem 1.2 is complete.

### 3 Quasierhöhungssysteme

We find the algebras from Theorem 1.3 within the following class considered by Wisliceny 16.

**Definition 3.1.** Let \( X \) be a finite set of generators carrying a total ordering \( < \). A set \( M \subset \mathbb{K}(X) \) of homogeneous elements of degree 2 is called a *Quasierhöhungssysteme* (QHS) on \( X \) if

\[
\bigcup_{g \in M} \text{supp}(g) = \{ab : a, b \in X, a \geq b\}
\]

and for every \( g \in M \) one of the following three possibilities holds:

- either \( g = ab \) with \( a, b \in X, a \geq b \);
- or \( g = ab - cd \) with \( a, b, c, d \in X, a \geq b > c \geq d \);
- or \( g = ab - cd \) with \( a, b, c, d \in X, a > b = c > d \).

Each QHS \( M \) generates an ideal \( J_M \) in \( \mathbb{K}(X) \) and \( R_M = \mathbb{K}(X)/J_M \) is a quadratic semigroup algebra. It is easy to see that the cardinalities of QHSs on an \( n \)-element set \( X \) range from the first integer \( \delta_n \) greater than \( \frac{n^2+2n}{4} \) to \( \frac{n^2+2n}{2} \). In the subsequent sections we shall prove Theorem 1.3 by showing that there is a QHS \( M \) on an \( n \)-element \( X \) of cardinality \( \delta_n \) such that the corresponding quadratic semigroup algebra \( R_M \) is finite dimensional.

This class of algebras was introduced by Wisliceny 16 and his example is from the same class. He also provided an example of a QHS \( M \) on a 6-element \( X \) for which the corresponding algebra is infinite dimensional. There is a criterion in 16 for a QHS to produce a finite dimensional algebra, but although it can be handy for treating specific examples with small number of generators with an aid of a computer, it is nearly impossible to use when \( X \) has a large number of elements. Our proof does not make use of this criterion.

**Remark 3.2.** Let \( X \) be a finite totally ordered set and \( M \) be a QHS on \( X \). Let also \( a \) be the minimal element of \( X \) and \( b \) be the maximal element of \( X \). Form the definition of a QHS it easily follows that the monomial \( ba \) belongs to \( M \). We shall frequently deal with the set \( M' = M \setminus \{ba\} \) and the ideal \( I_M \) in \( \mathbb{K}(X) \) generated by \( M' \).
We will also use the following notation. If \( I \) is an ideal in \( \mathbb{K}\langle X \rangle \) and \( f, g \in \mathbb{K}\langle X \rangle \), the equality \( f = g \pmod{I} \) means that \( f - g \in I \). Furthermore, if \( X \) is totally ordered and \( m \in \mathbb{N} \), we shall always endow the set of monomials \( \langle X \rangle_m \) of degree \( m \) with the right-to-left lexicographical ordering. That is, \( u_1 \ldots u_m < v_1 \ldots v_m \) if there is \( j \in \{1, \ldots, m\} \) such that \( u_j < v_j \) and \( u_l = v_l \) whenever \( l > j \).

**Definition 3.3.** Let \( X \) be a finite totally ordered set, \( M \) be a QHS on \( X \), \( a = \min X \), \( b = \max X \), \( M' = M \setminus \{ab\} \) and \( J_M \) be the ideal in \( \mathbb{K}\langle X \rangle \) generated by \( M' \). A monomial \( u = u_1 \ldots u_m \in \langle X \rangle_m \) is called minimal (with respect to \( M' \)) if \( u \notin J_M \) and \( u \leq v \) whenever \( v \in \langle X \rangle_m \) and \( u = v \pmod{I_M} \) (in other words, \( u \) is minimal in the right-to-left lexicographical ordering among the degree \( m \) monomials belonging to the coset \( u + J_M \)).

We say that a minimal monomial \( u \in \langle X \rangle_m \) is tame if there is \( v \in \langle X \rangle_m \) such that \( u = v \pmod{I_M} \) and there is \( j \in \{1, \ldots, m\} \) such that \( v_j = b \) and \( v_l = u_l \) for every \( l > j \).

A non-tame minimal monomial will be called singular.

We say that the QHS \( M \) is regular if there is \( m \in \mathbb{N} \) such that there are no singular monomials of degree \( m \).

**Remark 3.4.** It is easy to see that under the assumptions of Definition 3.3 for any \( v \in \langle X \rangle_m \) such that \( v \notin I_M \), there is a unique minimal monomial \( u \in \langle X \rangle_m \) satisfying \( v = u \pmod{I_M} \). Furthermore, if \( v \) does not belong to the ideal \( J_M \) in \( \mathbb{K}\langle X \rangle \) generated by \( M \), then \( u \notin J_M \) as well.

The latter happens because \( I_M \subset J_M \).

As usual, a submonomial of a monomial \( u_1u_2 \ldots u_m \) is a monomial of the shape \( u_ju_{j+1} \ldots u_k \), where \( 1 \leq j \leq k \leq m \). From the above definition it immediately follows that a submonomial of a minimal monomial is minimal and that a minimal monomial which has a tame submonomial is tame itself. This, in turn, implies that a submonomial of a singular monomial is singular. These observations are summarized in the next lemma.

**Lemma 3.5.** Let \( X \) be a finite totally ordered set and \( M \) be a QHS on \( X \). Let also \( u \in \langle X \rangle_m \) and \( v \) be a submonomial of \( u \). Then minimality of \( u \) implies minimality of \( v \) and singularity of \( u \) implies singularity of \( v \).

The following lemma shows the relevance of the above definition.

**Lemma 3.6.** Let \( X \) be a finite totally ordered set and \( M \) be a regular QHS on \( X \). Then the corresponding quadratic semigroup algebra \( R_M \) is finite dimensional.

**Proof.** Let \( a = \min X \), \( b = \max X \), \( M' = M \setminus \{ab\} \), \( I_M \) and \( J_M \) be the ideals in \( \mathbb{K}\langle X \rangle \), generated by \( M' \) and \( M \) respectively. Since \( M \) is regular, there is \( m \in \mathbb{N} \) such that there are no singular monomials of degree \( m \). In order to show that \( R_M = \mathbb{K}\langle X \rangle/J_M \) is finite dimensional it suffices to show that \( R_M \) is \((m + 1)\)-step nilpotent, or equivalently, that \( \langle X \rangle_{m+1} \subset J_M \). Assume the contrary. Then (see Remark 3.4) there is a minimal monomial \( u = u_1 \ldots u_mu_{m+1} \in \langle X \rangle_{m+1} \), which does not belong to \( J_M \). By Lemma 3.5 the submonomial \( \widetilde{u} = u_1 \ldots u_{m+1} \) is also minimal. Since there are no singular monomials of degree \( m \), \( \widetilde{u} \) is tame. That is, there are \( j \in \{1, \ldots, m\} \) and \( \widetilde{v} = v_1 \ldots v_m \in \langle X \rangle_m \) such that \( \widetilde{u} = \widetilde{v} \pmod{I_M} \), \( v_j = b \) and \( v_l = u_l \) whenever \( j > l \leq m \). If \( v_ju_{j+1} = bu_{j+1} \in M \), then

\[
\mod{J_M}
\]

and therefore \( u \in J_M \), which is a contradiction. If \( bu_{j+1} \notin M \), then since \( b = \max X \), the definition of a QHS implies that there are \( c, d \in X \) such that \( bu_{j+1} - cd \in M' \) and \( u_{j+1} > d \). Hence

\[
\mod{I_M}\]

Since the last monomial in the above display is less than \( u \) (in the right-to-left lexicographical ordering) we have obtained a contradiction with the minimality of \( u \). \( \Box \)
We also need the following property of singular monomials.

**Lemma 3.7.** Let $X$ be a finite totally ordered set, $a = \min X$, $b = \max X$, $M$ be a QHS on $X$, $M' = M \setminus \{b\}$, $I_M$ be the ideal in $\mathbb{K} \langle X \rangle$ generated by $M'$ and $u = u_1 \ldots u_m \in \langle X \rangle_m$ be a singular monomial. Assume also that $1 \leq k \leq m$ and $v = v_1 \ldots v_k \in \langle X \rangle_k$ is such that $v = u_1 \ldots u_k \pmod{I_M}$.

Then there exists $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m \geq v_k$ and $u = w \pmod{I_M}$.

**Proof.** We shall use induction by $m - k$. If $m - k = 0$, then $m = k$ and $w = v$ satisfies all desired conditions. Thus we have our basis of induction. Assume now that $n \in \mathbb{N}$ and that the statement of the lemma holds whenever $m - k < n$. We have to prove it the case $m - k = n$.

First, consider the case $u_{k+1} \geq v_k$. Since $v_1 \ldots v_k u_{k+1} = u_1 \ldots u_{k+1} \pmod{I_M}$ and the difference between the degrees of $u$ and $v_1 \ldots v_k u_{k+1}$ is $n - 1$, the induction hypothesis provides us with $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m \geq u_{k+1}$ and $u = w \pmod{I_M}$. Since $u_{k+1} \geq v_k$, we have $w_m \geq v_k$, as required.

It remains to consider the case $u_{k+1} < v_k$. Observe that $v_k \neq b$. Indeed, otherwise the equality
\begin{equation}
    v_1 \ldots v_{k-1} v_k u_{k+1} \ldots w_m = u \pmod{I_M}
\end{equation}
implies that $u$ is tame. Since $u_{k+1} < v_k$, the degree 2 monomial $v_k u_{k+1} \in M'$ features in exactly one of the supports of the elements of $M$ (see the definition of a QHS). Since $v_k \neq b$, $v_k u_{k+1} \neq ba$ and therefore there is exactly one $g \in M'$ such that $v_k u_{k+1} \in \supp(g)$. There are three possibilities: either $g = v_k u_{k+1}$ or $g = v_k u_{k+1} - cd$ with $d < u_{k+1}$ or $g = cd - v_k u_{k+1}$ with $d \geq v_k$. If $g = v_k u_{k+1}$, we have $v_k u_{k+1} \in I_M$ and therefore (3.1) implies that $u \in I_M$, which contradicts the minimality of $u$. Hence $g = v_k u_{k+1} - cd$ with $d < u_{k+1}$ or $g = cd - v_k u_{k+1}$ with $d \geq v_k$. In either case $v_k u_{k+1} = cd \pmod{I_M}$ and (3.1) implies that
\begin{equation}
    u = v_1 \ldots v_{k-1} c du_{k+1} \ldots w_m \pmod{I_M}.
\end{equation}

In the case $d < u_{k+1}$ the monomial in the right-hand-side of (3.2) is less than $u$, which contradicts the minimality of $u$. It remains to consider the case $d \geq v_k$. Since the difference between the degrees of $u$ and $v_1 \ldots v_{k-1} cd$ is $n - 1$, by the induction hypothesis, there is $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m \geq d$ and $u = w \pmod{I_M}$. Since $d \geq v_k$, we have $w_m \geq v_k$, as required.

We also need a slight improvement of Lemma 3.7 in a particular case.

**Definition 3.8.** Let $X$ be a finite totally ordered set and $M$ be a QHS on $X$. We say that $d \in X$ is pure if $a > b > c > d$ whenever $ab - cd \in M$.

**Lemma 3.9.** Let $X$ be a finite totally ordered set, $a = \min X$, $b = \max X$, $M$ be a QHS on $X$, $M' = M \setminus \{ba\}$, $I_M$ be the ideal in $\mathbb{K} \langle X \rangle$ generated by $M'$ and $u = u_1 \ldots u_m \in \langle X \rangle_m$ be a singular monomial such that $u_m$ is pure. Assume also that $1 \leq k < m$ and $v = v_1 \ldots v_k \in \langle X \rangle_k$ is such that $v = u_1 \ldots u_k \pmod{I_M}$. Then there exists $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m > v_k$ and $u = w \pmod{I_M}$.

**Proof.** If $u_m \geq v_k$, we can just take $w = u$. Thus we can assume that $v_k \geq u_m$. By Lemma 3.5, $u_1 \ldots u_{m-1}$ is singular and therefore Lemma 3.7 provides $y = y_1 \ldots y_{m-1} \in \langle X \rangle_{m-1}$ such that $y = u_1 \ldots u_{m-1} \pmod{I_M}$ and $y_{m-1} \geq v_k$. Clearly,
\begin{equation}
    u = y_1 \ldots y_{m-2} y_{m-1} u_m \pmod{I_M}.
\end{equation}

Since $y_{m-1} \geq v_k \geq u_m$, the monomial $y_{m-1} u_m$ features in one of the elements of $M$. Next, $y_{m-1} \neq b$ (otherwise (3.2) implies that $u$ is tame). Hence $y_{m-1} u_m \neq ba$ and therefore $y_{m-1} u_m$ belongs to the support of some $g \in M'$. Taking into account that $u_m$ is pure, we have three possibilities: either $g = y_{m-1} u_m$ or $g = y_{m-1} u_m - cd$ with $d < u_m$ or $g = cd - y_{m-1} u_m$ with $d > y_{m-1}$.
If \( g = y_{m-1}u_m, y_{m-1}u_m \in I_M \) and therefore \((3.3)\) implies that \( u \in I_M \), which contradicts the minimality of \( u \). Hence \( g = y_{m-1}u_m - cd \) with \( d < u_m \) or \( g = cd - y_{m-1}u_m \) with \( d > y_{m-1} \). In either case \( y_{m-1}u_m = cd \mod I_M \) and \((3.3)\) gives

\[
u = y_1 \ldots y_{m-2}cd \mod I_M.
\]

In the case \( d < u_{k+1} \) the monomial in the right-hand-side of \((3.3)\) is less than \( u \), which contradicts the minimality of \( u \). It remains to consider the case \( d > y_{m-1} \geq v_k \). In this case \( w = y_1 \ldots y_{m-2}cd \) satisfies all desired conditions.

**Corollary 3.10.** Let \( X \) be an \( n \)-element totally ordered set, \( a = \min X \), \( b = \max X \), \( M \) be a QHS on \( X \), \( M' = M \setminus \{ba\} \), \( I_M \) be the ideal in \( \mathbb{K}\langle X \rangle \) generated by \( M' \) and \( u = u_1 \ldots u_m \in \langle X \rangle_m \) be a singular monomial. Then the set \( \{j : u_j \text{ is pure}\} \) has at most \( n - 1 \) elements.

**Proof.** Assume the contrary. Then we can choose \( j_1, \ldots, j_n \in \mathbb{N} \) such that \( 1 \leq j_1 < \ldots < j_n \leq n \) and each \( u_{j_k} \) is pure. Let \( a_1 = u_{j_1} \) and take \( y^{(1)} = u_1 \ldots u_{j_1-1} \). Clearly \( u_1 \ldots u_{j_1} = y^{(1)}a_1 \) and therefore \( u_1 \ldots u_{j_1} = y^{(1)}a_1 \mod I_M \). By Lemma 3.9 there are \( a_2 > a_1 \) and a monomial \( y^{(2)} \in \langle X \rangle_{j_2-1} \) such that \( u_1 \ldots u_{j_2} = y^{(2)}a_2 \mod I_M \). Applying Lemma 3.9 again, we see that there are \( a_3 > a_2 \) and a monomial \( y^{(3)} \in \langle X \rangle_{j_3-1} \) such that \( u_1 \ldots u_{j_3} = y^{(3)}a_3 \mod I_M \). Proceeding this way we obtain \( a_1, \ldots, a_n \in X \) and the monomials \( y^{(k)} \in \langle X \rangle_{j_k-1} \) such that \( u_1 \ldots u_{j_k} = y^{(k)}a_k \mod I_M \) for \( 1 \leq k \leq n \) and, most importantly, \( a_1 < \ldots < a_n \). Since \( X \) has \( n \) elements, the latter can only happen if \( a_n = b \). Then the equality \( u_1 \ldots u_{j_n} = y^{(n)}a_n \mod I_M \) implies that \( u = y^{(n)}u_{j_n+1} \ldots u_m \mod I_M \), which means that \( u \) is tame. This contradiction completes the proof.

4 The extension construction for a Quasierhöhungssysteme

Throughout this section \( n \geq 5 \) is an integer, \( X = \{x_1, \ldots, x_n\} \) is an \( n \)-element set with the total ordering \( x_1 < x_2 < \ldots < x_n \) and the \((n-4)\)-element set \( X_0 = X \setminus \{x_1, x_2, x_{n-2}, x_{n-1}, x_n\} = \{x_3, \ldots, x_{n-2}\} \) carries the total ordering inherited from \( X \). Let also \( M_0 \) be a QHS on \( X_0 \). By Remark 3.2, \( x_{n-2}x_3 \in M_0 \). As usual, \( M'_0 = M_0 \setminus \{x_{n-2}x_3\} \).

We consider \( M, M' \subset \mathbb{K}\langle X \rangle \) defined in the following way:

\[
M' = M'_0 \cup \{x_nx_j - x_jx_1 : 2 \leq j \leq n - 2\} \cup \{x_{n-1}x_{j+1} - x_jx_2 : 2 \leq j \leq n - 3\} \\
\cup \{x_nx_n - x_n-1x_1, x_nx_{n-1} - x_{n-2}x_2, x_{n-1}x_{n-1} - x_{n-2}x_3, x_{n-1}x_2 - x_1x_1\}
\]

and

\[
M = M' \cup \{x_nx_1\}.
\]

It is straightforward to verify that \( M \) is a QHS on \( X \).

Symbols \( J \) and \( I \) stand for the ideals in \( \mathbb{K}\langle X \rangle \) generated by \( M \) and \( M' \) respectively. Similarly, \( J_0 \) and \( I_0 \) are the ideals in \( \mathbb{K}\langle X_0 \rangle \) generated by \( M_0 \) and \( M'_0 \) respectively.

**Lemma 4.1.** Let \( m \in \mathbb{N} \) and \( u = u_1u_2 \ldots u_m \in \langle X \rangle_m \) be a singular (with respect to \( M \)) monomial. Then the set \( \{j : u_j \in \{x_2, x_n, x_{n-1}\}\} \) has at most \( n - 1 \) elements.

**Proof.** From the way we defined \( M \) it easily follows that \( x_n, x_{n-1} \) and \( x_2 \) are pure. It remains to apply Corollary 3.10.

**Lemma 4.2.** Let \( k \) be a non-negative integer and \( a \in \{x_2, \ldots, x_{n-2}\} \). Then \( x_n^ka = ax_1^k \mod I \).

**Proof.** From the definition of \( M \) it follows that \( x_na - ax_1 \in M' \subset I \). That is, \( x_na = ax_1 \mod I \). The required inequality now follows via an obvious inductive argument.
Lemma 4.3. Let \( q = \frac{5n-9}{2} \) if \( n \) is odd and \( q = \frac{5n-8}{2} \) if \( n \) is even. Then there exists a monomial \( u = u_1 \ldots u_q \in \langle X \rangle_q \) such that \( u_q = x_n \) and \( u = x_1^r (\mod I) \).

Proof. Since \( x_{n-1}x_2 - x_1x_1 \in M' \subset I \), we see that for every \( k \geq 2 \), \( x_1^k = x_{n-1}x_2x_1^{k-2} (\mod I) \). By Lemma 4.2, \( 2x_2x_1^{k-2} = x_2x_1^{k-2} (\mod I) \). Hence

\[
x_1^k = x_{n-1}x_2x_1^{k-2} (\mod I) \quad \text{for every } k \geq 2. \tag{4.1}
\]

Since \( x_nx_n - x_{n-1}x_1 \in M' \) and \( x_nx_{n-1} - x_{n-2}x_2 \in M' \), we get \( x_n^3 = x_nx_{n-1}x_1 = x_{n-2}x_2x_1 (\mod I) \). Hence \( x_n^k = x_n^{k-3}x_{n-2}x_2x_1 (\mod I) \). By Lemma 4.2, \( x_n^{k-3}x_{n-2} = x_{n-2}x_1^{k-3} (\mod I) \). Combining the last two equalities, we obtain

\[
x_n^k = x_{n-2}x_1^{k-3}x_2x_1 (\mod I) \quad \text{for every } k \geq 3. \tag{4.2}
\]

For \( k \geq 5 \), we can apply (4.2) to \( x_n^{k-2} \) in (4.1), which gives

\[
x_1^k = x_{n-1}x_n^{k-2}x_2x_1x_2 (\mod I) \quad \text{for every } k \geq 5. \tag{4.3}
\]

Since \( x_nx_2 - x_2x_1 \in M' \) and \( x_nx_3 - x_2x_2 \in M' \), we have

\[
x_2x_1x_2 = x_nx_2 = x_nx_{n-1}x_3 (\mod I). \tag{4.4}
\]

Next, for \( 2 \leq j \leq n - 4 \), we have \( x_{n-1}x_{j+1} - x_jx_2 \in M' \), \( x_nx_{j+1} - x_{j+1}x_1 \in M' \) and \( x_{n-1}x_{j+2} - x_{j+1}x_2 \in M' \). Hence,

\[
x_jx_2x_1x_2 = x_{n-1}x_{j+1}x_1x_2 = x_{n-1}x_nx_{j+2}x_1 (\mod I) \quad \text{for } 2 \leq j \leq n - 4. \tag{4.5}
\]

Finally, since \( x_{n-1}x_{n-2} - x_{n-3}x_2 \in M' \), \( x_nx_{n-2} - x_{n-2}x_1 \in M' \), \( x_nx_{n-1} - x_{n-2}x_2 \in M' \) and \( x_nx_n - x_{n-1}x_1 \in M' \), we get

\[
x_{n-3}x_1x_2x_1x_2 = x_{n-1}x_{n-2}x_1x_2x_1 = x_{n-1}x_nx_{n-2}x_1x_2x_1 =
\]

\[
= x_{n-1}x_nx_{n-1}x_1 = x_{n-1}x_nx_nx_n (\mod I). \tag{4.6}
\]

Case 1: \( n \) is odd. In this case \( n = 2k + 3 \) and \( q = 5k + 3 \) for some \( k \in \mathbb{N} \). Using (4.3) consecutively \( k \) times, we see that

\[
x_q^1 = (x_{n-1}x_{n-2})^k x_1x_1 (x_2x_1x_2)^k x_1 (\mod I).
\]

Since \( x_1x_1 = x_{n-1}x_2 (\mod I) \), we obtain

\[
x_q^1 = (x_{n-1}x_{n-2})^k x_1x_{n-2} (x_2x_1x_2)^k x_1 (\mod I).
\]

Applying (4.5) consecutively \( k - 1 \) times, we have

\[
x_q^i = (x_{n-1}x_{n-2})^k x_{n-1} x_{n-3}x_2x_1x_2x_1 (\mod I).
\]

According to (4.6),

\[
x_q^i = (x_{n-1}x_{n-2})^k x_{n-1} x_{n-3}x_2x_1x_2x_1 (\mod I),
\]

which completes the proof in the case of odd \( n \).

Case 2: \( n \) is even. In this case \( n = 2k + 2 \) and \( q = 5k + 1 \) for some \( k \geq 2 \). Using (4.3) consecutively \( k \) times, we see that

\[
x_q^1 = (x_{n-1}x_{n-2})^k (x_2x_1x_2)^k x_1 (\mod I).
\]
By (4.4),
\[ x_1^q = (x_{n-1}x_{n-2})^k x_n x_{n-1} x_3 (x_2 x_1 x_2)^{k-1} x_1 \pmod{I}. \]

Applying (4.5) consecutively \( k - 2 \) times, we have
\[ x_1^q = (x_{n-1}x_{n-2})^k x_n x_{n-1} (x_{n-1}x_n x_{n-1})^{k-2} x_{n-3} x_2 x_1 x_2 x_1 \pmod{I}. \]

According to (4.6),
\[ x_1^q = (x_{n-1}x_{n-2})^k x_n x_{n-1} (x_{n-1}x_n x_{n-1})^{k-2} x_{n-1} x_n x_n x_n \pmod{I}, \]

which completes the proof in the case of even \( n \).

\[ \square \]

**Lemma 4.4.** Let \( k \) be a non-negative integer. If \( ab \in M'_0 \), then \( ax_k^b \in I \). If \( ab - cd \in M'_0 \), then \( ax_k^b = c x_i^d \pmod{I} \).

**Proof.** If \( ab \in M'_0 \), we have \( ab \in I_0 \subset I \) and therefore Lemma 4.2 implies that \( ax_k^b = x_n^k ab \in I \).

If \( ab - cd \in M'_0 \), we can use Lemma 4.2 to see that \( ax_k^b = x_n^k ab = x_n^k cd = cx_i^d \pmod{I} \).

Since \( I_0 \) is the ideal in \( \mathbb{K}(X_0) \) generated by \( M'_0 \), the above lemma immediately implies the following result.

**Corollary 4.5.** Let \( u, v \in \langle X_0 \rangle_m \) and \( u = v \pmod{I_0} \). Then \( \tilde{u} = \tilde{v} \pmod{I} \) for every non-negative integers \( k_1, \ldots, k_{m-1} \), where \( u = u_1 \ldots u_m \), \( v = v_1 \ldots v_m \), \( \tilde{u} = u_1 x_{1}^{k_1} u_2 x_{1}^{k_2} \ldots x_{1}^{k_{m-1}} u_m \) and \( \tilde{v} = v_1 x_{1}^{k_1} v_2 x_{1}^{k_2} \ldots x_{1}^{k_{m-1}} v_m \).

**Lemma 4.6.** Let \( m \in \mathbb{N}, m \geq 3 \) and \( u_1, \ldots, u_m \in \langle X_0 \rangle_m \) be such that the monomial \( u_1 \cdots u_{m-2} \) is \( M_0 \)-tame. Let also \( k_1, \ldots, k_{m-1} \) be non-negative integers and \( w = u_1 x_1^{k_1} \ldots u_{m-2} x_1^{k_{m-2}} u_m \). Then \( w \) is not \( M \)-singular.

**Proof.** Since \( u_1 \cdots u_{m-2} \) is \( M_0 \)-tame, there are \( v_1 \cdots v_m \in \langle X_0 \rangle_{m-2} \) and \( j \in \{1, \ldots, m-2\} \) such that
\[ u_1 \cdots u_{m-2} = v_1 \cdots v_{m-2} \pmod{I_0}, \quad v_j = x_{n-2} \quad \text{and} \quad v_l = v_l \quad \text{for} \quad j < l \leq m-2. \quad (4.7) \]

According to (4.7) and Corollary 4.5,
\[ w = v_1 x_1^{k_1} \ldots v_{j-1} x_1^{k_{j-1}} x_{n-2} x_1^{k_j} u_{j+1} x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod{I}. \]

Applying Lemma 4.2 to \( x_{n-2} x_1^{k_j} \) and using the above display, we obtain
\[ w = y x_{n-2} u_{j+1} x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod{I}, \quad \text{where} \quad y = v_1 x_1^{k_1} \ldots v_{j-1} x_1^{k_{j-1}} x_{n-2} x_1^{k_j} \pmod{I}. \quad (4.8) \]

Since \( x_{n-2} = \max X_0 \), the monomial \( x_{n-2} u_{j+1} \) features in exactly one \( g \in M_0 \). If \( u_{j+1} > x_3 \), \( g \in M'_0 \) and there are two possibilities: either \( g = x_{n-2} u_{j+1} \) or \( g = x_{n-2} u_{j+1} - cd \) with \( d < u_{j+1} \). In the first case (4.8) implies that \( w \in I \) and therefore \( w \) is non-minimal and therefore non-singular. In the second case (4.8) gives
\[ w = y c x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod{I}. \]

The monomial in the right-hand side of the above display is less than \( w \) and again we see that \( w \) is non-minimal and therefore non-singular.

It remains to consider the case \( u_{j+1} = x_3 \). Since \( x_{n-1} x_{n-1} = x_{n-2} x_3 \in M' \), we have \( x_{n-2} x_3 = x_{n-1} x_{n-1} \pmod{I} \). Thus in this case (4.8) implies that
\[ w = y x_{n-1} x_{n-1} x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod{I}. \quad (4.9) \]
Again, we have two possibilities: either $k_{j+1} > 0$ or $k_{j+1} = 0$. First, assume that $k_{j+1} = 0$. In this case (4.9) reads
\[ w = yx_{n-1}x_{n-1}u_{j+2}x_1^{k_{j+2}} \cdots u_{m-1}x_1^{k_{m-1}}u_m \pmod{I}. \]
Since $u_{j+2} \in X_0$, from the definition of $M$ it follows that there is $g \in M'$ such that $g = x_{n-1}u_{j+2}-cd$ with $d < u_{j+1}$. Then $x_{n-1}u_{j+2} = cd \pmod{I}$ and according to the above display
\[ w = yx_{n-1}cdx_1^{k_{j+2}} \cdots u_{m-1}x_1^{k_{m-1}}u_m \pmod{I}. \]
The monomial in the right-hand side of the above display is less than $w$ and therefore $w$ is non-minimal and hence non-singular. Finally, assume that $k_{j+1} > 0$. Since $x_nx_n - x_{n-1}x_1 \in M'$, we have $x_{n-1}x_1 = x_nx_n \pmod{I}$ and (4.9) implies that
\[ w = yx_{n-1}x_nx_1^{k_{j+1}-1}u_{j+3}x_1^{k_{j+3}} \cdots u_{m-1}x_1^{k_{m-1}}u_m \pmod{I}. \]
The above display shows that $w$ is tame if it is minimal. Thus in any case, $w$ is non-singular. \[\square\]

The main result of this section is the following theorem.

**Theorem 4.7.** If $M_0$ is regular, then $M$ is also regular.

**Proof.** Assume that $M_0$ is regular. Then there is $m \in \mathbb{N}$ such that there are no $M_0$-singular monomials in $\langle X_0 \rangle_{m-2}$. Let $q = \frac{5m-9}{2}$ if $n$ is odd and $q = \frac{5m-8}{2}$ if $n$ is even and set $r = nmq$.

It suffices to show that there are no $M$-singular monomials in $\langle X \rangle_r$. Assume that there exists an $M$-singular $u \in \langle X \rangle_r$. By Lemma 4.1, the set $\{ j : u_j \in \{ x_2, x_n, x_{n-1} \} \}$ has at most $n-1$ elements. It follows that there is a submonomial $v$ of $u$ of degree $mq$ such that $v$ does not contain $x_2$, $x_{n-1}$ or $x_n$. By Lemma 3.5, $v$ is singular. If at least $m$ entries of $v$ belong to $X_0$, Lemma 4.6 implies that $v$ is non-singular and we arrive to a contradiction. Thus the degree $mq$ monomial $v$ contains at most $m-1$ entries from $X_0$. Since all other entries of $v$ are $x_1$, it follows that $x_1^q$ is a submonomial of $v$. Hence $x_1^q$ is singular, which contradicts Lemma 4.3. \[\square\]

It is straightforward to see that the cardinalities of $M$ and $M_0$ are related by the equality $|M| = |M_0| + 2n - 3$. Hence Theorem 4.7 immediately implies the following result.

**Corollary 4.8.** Suppose that $n \geq 5$ and that there is a $k$-element regular QHS on an $(n-4)$-element set. Then there is a $(k + 2n - 3)$-element regular QHS on an $n$-element set.

Applying Corollary 4.8 several times, we, after an exercise on summing up an arithmetic series, obtain the following result.

**Proposition 4.9.** Let $m, j \in \mathbb{N}$ and there is a $k$-element regular QHS on an $m$-element set. Then there is a $(k + 2jm + 4j^2 + j)$-element regular QHS on an $(m + 4j)$-element set.

## 5 Proof of Theorem 1.3

We start with the following 4 specific examples of QHS. For $1 \leq m \leq 4$, we set $X_m = \{ x_1, \ldots, x_m \}$ ordered by $x_1 < x_2 < x_3 < x_4$. Now we define a QHS $M_m$ on the $m$-element totally ordered set $X_m$:

- $M_1 = \{ x_1x_1 \}$
- $M_2 = \{ x_2x_2 - x_1x_1, x_2x_1 \}$
- $M_3 = \{ x_3x_3 - x_2x_1, x_3x_2 - x_1x_1, x_3x_1, x_2x_2 \}$
- $M_4 = \{ x_4x_4 - x_3x_1, x_4x_3 - x_2x_1, x_4x_2 - x_1x_1, x_3x_3 - x_2x_2, x_3x_2, x_4x_1 \}$

**Lemma 5.1.** For each $m \in \{ 1, 2, 3, 4 \}$, the QHS $M_m$ is regular.
Proof. It is straightforward to see that for each $M_m$, each element of $X_m$ is pure. By Corollary \[3.10\] an $M_m$-singular monomial can not have the degree more than $m - 1$. Hence there are no $M_m$-singular monomials of degree $m$. By definition, each $M_m$ is regular.

Let $j$ be an arbitrary non-negative integer. By Lemma \[5.1\] there is a regular 1-element QHS on a 1-element set. By Proposition \[4.9\] there is a $(4j^2 + 3j + 1)$-element regular QHS on a $(4j + 1)$-element set. By Lemma \[5.1\] there is a regular 2-element QHS on a 2-element set. By Proposition \[4.9\] there is a $(4j^2 + 5j + 2)$-element regular QHS on a $(4j + 2)$-element set. By Lemma \[5.1\] there is a regular 4-element QHS on a 3-element set. By Proposition \[4.9\] there is a $(4j^2 + 7j + 4)$-element regular QHS on a $(4j + 3)$-element set. Finally, by Lemma \[5.1\] there is a regular 6-element QHS on a 4-element set. By Proposition \[4.9\] there is a $(4j^2 + 9j + 6)$-element regular QHS on a $(4j + 4)$-element set.

For $n \in \mathbb{N}$, we define $\delta_n = 4j^2 + 3j + 1$ if $n = 4j + 1$, $\delta_n = 4j^2 + 5j + 2$ if $n = 4j + 2$, $\delta_n = 4j^2 + 7j + 4$ if $n = 4j + 3$ and $\delta_n = 4j^2 + 9j + 6$ if $n = 4j + 4$. The above observations mean that for every $n \in \mathbb{N}$, there is a $\delta_n$-element regular QHS on an $n$-element set. According to Lemma \[3.6\] this means that for every $n \in \mathbb{N}$, there is a quadratic semigroup algebra $R_n$ with $n$ generators given by $\delta_n$ relations such that $R_n$ is finite dimensional. On the other hand, a routine check shows that $\delta_n$ is precisely the smallest integer greater than $\frac{n^2 + n}{4}$. This completes the proof of Theorem \[1.3\].

We conclude with the following remarks.

Remark 5.2. We have just seen that for $n \leq 4$, there is a $\delta_n$-element QHS on an $n$-element set $X$ for which every $c \in X$ is pure. It turns out that $n = 4$ is the last $n$ for which this phenomenon occurs. More precisely, one can easily see that a QHS on an $n$-element set $X$, for which every $c \in X$ is pure, must contain at least $\frac{n^2 + 2n}{4}$ elements.

Remark 5.3. Vershik [14] conjectured that for every $n \geq 3$, there is a finite dimensional quadratic algebra with $n$ generators given by $\frac{n^2 - n}{2}$ relations. This conjecture is proved in [7] (see also [3] for the proof in the case $3 \leq n \leq 7$). Note that for $n \geq 4$, this statement could also be obtained as a consequence of Theorem \[1.3\].

Remark 5.4. It would be interesting to get a tight enough estimate of the order of nilpotency of quadratic semigroup algebras which we construct while proving Theorem \[1.3\]. An estimate, that follows from the proof is way higher than the actual value for small $n$.

Remark 5.5. Let $R$ be a quadratic algebra with $n$ generators given by $d \leq n^2$ relations. For $k \in \mathbb{N}$, $R_k$ stands for the degree $k$ homogeneous component of $R$. By the Golod–Shafarevich estimate $\dim R_3 \geq \min\{0, n^3 - 2dn\}$. Anick [11] [2] proved that this estimate for $\dim R_3$ is actually attained for every $n$ and $d$. An analysis of his proof shows that it is also attained if we restrict ourselves to semigroup algebras.

On the other hand (see [7]), there is an $R$ with $n = d = 3$ for which $R_5 = \{0\}$, while Theorem \[1.2\] implies that $\dim R_5 > 0$ if $R$ is a semigroup algebra with $n = d = 3$. So, for every pair $(n, d)$, the minimal dimensions of the third component for general quadratic algebras and for semigroup quadratic algebras coincide, while the same statement fails for the dimension of the fifth component. The situation with the fourth component remains unclear. Its clarification is a part of the following much more general question.

Question 5.6. What is the minimal dimension of $R_k$, where $R$ is a semigroup quadratic algebra with $n$ generators given by $d \leq n^2$ relations?

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