Tables of parameters of symmetric configurations $v_k^*$

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Abstract

Tables of the currently known parameters of symmetric configurations are given. Formulas for parameters of the known infinite families of symmetric configurations are presented as well. The results of the recent paper [18] are used. This work can be viewed as an appendix to [18], in the sense that the tables given here cover a much larger set of parameters.

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1 Introduction

Configurations as combinatorial structures were defined in 1876. For an introduction to the problems connected with configurations, see [35,37] and the references therein.

Definition 1.1. [36]

(i) A configuration $(v_r, b_k)$ is an incidence structure of $v$ points and $b$ lines such that each line contains $k$ points, each point lies on $r$ lines, and two distinct points are connected by at most one line.

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(ii) If $v = b$ and, hence, $r = k$, the configuration is symmetric, and it is referred to as a configuration $v_k$.

(iii) The deficiency $d$ of a configuration $(v, b)$ is the value

$$d = v - r(k - 1) - 1.$$ 

A symmetric configuration $v_k$ is cyclic if there exists a permutation of the set of its points mappings blocks to blocks, and acting regularly on both points and blocks. Equivalently, $v_k$ is cyclic if one of its incidence matrix is circulant.

Steiner systems are configurations with $d = 0$.

The deficiency $d$ of a symmetric configuration $(v, r, b)$ is the value

$$d = v - r(k - 1) - 1.$$ 

A symmetric configuration $v_k$ is cyclic if there exists a permutation of the set of its points mappings blocks to blocks, and acting regularly on both points and blocks.

Denote by $M(v, k)$ an incidence matrix of a symmetric configuration $v_k$. Any matrix $M(v, k)$ is a $v \times v$ 01-matrix with $k$ ones in every row and column; moreover, the $2 \times 2$ matrix consisting of all ones is not a submatrix of $M(v, k)$. Two incidence matrices of the same configuration may differ by a permutation on the rows and the columns.

A matrix $M(v, k)$ can be considered as a biadjacency matrix of the Levi graph of the configuration $v_k$ which is a $k$-regular bipartite graph without multiple edges [36, Sec. 7.2]. Clearly, the graph has girth at least six, i.e. it does not contain 4-cycles. Such graphs are useful for the construction of bipartite-graph codes that can be treated as low-density parity-check (LDPC) codes. If $M(v, k)$ is circulant, then the corresponding LDPC code is quasi-cyclic; it can be encoded with the help of shift-registers with relatively small complexity; see [3, 4, 19–21, 28, 40, 41] and the references therein.

Matrices $M(v, k)$ consisting of square circulant submatrices have a number of useful properties, e.g. they are more suitable for LDPC codes implementation. We say that a 01-matrix $A$ is block double-circulant (BDC for short) if $A$ consists of square circulant blocks whose weights give rise to a circulant matrix (see Definition 3.1). A configuration $v_k$ with a BDC incidence matrix $M(v, k)$ is called a BDC symmetric configuration. Symmetric and non-symmetric configurations with incidence matrices consisting of square circulant blocks are considered, for example, in [3, 4, 19–21, 40, 41, 52]. In [3, 4], BDC symmetric configuration are considered in connection with $Z_\mu$-schemes (see Remarks 3.2 and 3.3 in Section 3).

Cyclic configurations are considered, for instance, in [3, 4, 19–21, 26, 31, 40, 41, 47, 50]. A standard method to construct cyclic configurations (or, equivalently, circulant matrices $M(v, k)$) is based on Golomb rulers.

Definition 1.2. [26, 54]

(i) A Golomb ruler $G_k$ of order $k$ is an ordered set of $k$ integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j | 1 \leq j < i \leq k\}$ are distinct.
The length $L_G(k)$ of the ruler $G_k$ is equal to $a_k - a_1$. Let $L_{\overline{G}}(k)$ be the length of the shortest known Golomb ruler $\overline{G}_k$.

(ii) A $(v, k)$ modular Golomb ruler is an ordered set of $k$ integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j | 1 \leq i, j \leq k; i \neq j\}$ are distinct and nonzero modulo $v$.

For any $\delta \geq 0$, Golomb rulers $(a_1, a_2, \ldots, a_k)$ and $(a_1 + \delta, a_2 + \delta, \ldots, a_k + \delta)$ have the same properties. Usually, $a_1 = 0$ is assumed. We say that a 0,1-vector $u = (u_0, u_1, \ldots, u_{v-1})$ corresponds to a (modular) Golomb ruler if the increasing sequence of integers $j \in \{0, 1, \ldots, v - 1\}$ such that $u_j = 1$ form a (modular) Golomb ruler.

Recall that weight of a circulant 0,1-matrix is the number of ones in each its row.

**Theorem 1.3.** \cite[Sec. 4]{31}, \cite{48}

(i) Any Golomb ruler $G_k$ of length $L_G(k)$ is a $(v, k)$ modular Golomb ruler for all $v$ such that $v \geq 2L_G(k) + 1$.

(ii) A circulant $v \times v$ 0,1-matrix of weight $k$ is an incidence matrix $M(v, k)$ of a cyclic symmetric configuration $v_k$ if and only if the first row of the matrix corresponds to a $(v, k)$ modular Golomb ruler.

(iii) For all $v$ such that $v \geq 2L_{\overline{G}}(k) + 1$, there exists a cyclic symmetric configuration $v_k$.

We call the value $G(k) = 2L_{\overline{G}}(k) + 1$ the Golomb bound. On the other hand, we call $P(k) = k^2 - k + 1$ the projective plane bound.

Let $v_{\delta}(k)$ be the smallest possible value of $v$ for which a $(v, k)$ modular Golomb ruler (or, equivalently, a cyclic symmetric configuration) exists.

In \cite{IS}, two bounds are considered. The existence bound $E(k)$ is the least integer such that for any $v \geq E(k)$, there exists a symmetric configuration $v_k$. Similarly, the cyclic existence bound $E_c(k)$ is the least integer such that for any $v \geq E_c(k)$, there exists a cyclic $v_k$. Clearly, for a fixed $k$, we have

$$k^2 - k + 1 = P(k) \leq E(k) \leq E_c(k) \leq G(k) = 2L_{\overline{G}}(k) + 1. \quad (1.1)$$

$$k^2 - k + 1 = P(k) \leq v_{\delta}(k) \leq E_c(k) \leq G(k) = 2L_{\overline{G}}(k) + 1. \quad (1.2)$$

The aim of this work is to give tables of the currently known parameters of symmetric configurations $v_k$, including those arising from the recent work \cite{IS}. We consider the spectrum of possible parameters of $v_k$ (with special attention to parameters of cyclic symmetric configurations) in the interval

$$k^2 - k + 1 = P(k) \leq v < G(k) = 2L_{\overline{G}}(k) + 1. \quad (1.3)$$

Also, we pay attention to parameters of circulant and block double-circulant incidence matrices $M(v, k)$. Some upper bounds on $E(k)$ and $E_c(k)$ are pointed out.

From the standpoint of applications, including Coding Theory, it is sometimes useful to have different matrices $M(v, k)$ for the same $v$ and $k$. This is why we remark situations when different constructions provide configurations with the same parameters.
The Generalized Martinetti Construction (Construction GM) proposed in [25] plays a key role for the investigation of the spectrum of possible parameters of symmetric configurations as it provides, for a fixed $k$, intervals of values of $v$ for which a $v_k$ exists. Construction GM has been considered also in [3,5,6]. To be successfully applied, Construction GM needs a convenient starting incidence matrix. To this end, BDC matrices turn out to be particularly useful. In this work new starting matrices proposed in [18] are considered as well as those originally proposed in [3,6].

We remark that new cyclic configurations provide new modular Golomb rulers, i.e. new deficient cyclic difference sets.

The work is organized as follows. In Section 2, we briefly summarize some constructions and parameters of configurations $v_k$. Preliminaries on BDC matrices are given in Section 3. In Section 4, parameters of block double-circulant incidence matrices $M(v,k)$ are reported, according to some results from [18]. In Section 5, parameters of configurations $v_k$ obtained by the Construction GM from the starting matrices proposed in [3,6,18] are given. In Sections 6 and 7, results on the spectra of parameters of cyclic and non-cyclic configurations are reported. Finally, Section 8 contains the tables of parameters of symmetric configurations which are the main object of the paper. In particular, tables of parameters of BDC configurations $v_k$ based on projective planes and punctured affine planes are given, as well as tables of values $v$ for which a cyclic symmetric configuration $v_k$ exists. Finally, aggregated tables on the existence of symmetric configurations are given. In the tables, the new parameters obtained from [18] are written in bold font.

2 Some known constructions and parameters of configurations $v_k$ with $P(k) \leq v < G(k)$

The aim of this section is to provide a list of pairs $(v,k)$ for which a (cyclic) symmetric configuration $v_k$ is known to exist, see Equations (2.1)–(2.16). Infinite families of configurations $v_k$ given in this section are considered in [1–8, 10, 13, 17, 19–21, 25–27, 29, 31–37, 47, 50, 53, 54, 58]; see also the references therein.

Throughout the work, $q$ is a prime power and $p$ is a prime. Let $F_q$ be Galois field of $q$ elements. Let $F_q^* = F_q \setminus \{0\}$. Let $0_u$ be the zero $u \times u$ matrix. Denote by $P_u$ a permutation matrix of order $u$.

We recall that several pairs $(v, k - \delta)$ can be actually obtained from a given $v_k$; it is a basic result on symmetric configurations.

**Theorem 2.1.** [3, Sec. 2], [34, Sec. 5.2], [37, Sec. 2.5] [50] If a (cyclic) configuration $v_k$ exists, then for each $\delta$ with $0 \leq \delta < k$ there exists a (cyclic) configuration $v_{k-\delta}$ as well.

We note that from a cyclic configuration $v_k$, a cyclic configurations $v_{k-\delta}$ can be obtained by dismissing $\delta$ ones in the 1-st row of its incidence matrix. For the general case,
Theorem 2.1 is based on the fact that an incidence matrix $M(v, k)$ can be represented as a sum of $k$ permutations $v \times v$ matrices (in different ways). This fact follows from the results of Steinits (1894) and König (1914), see e.g. [34, Sec. 5.2] and [37, Sec. 2.5].

The value $\delta$ appearing in Equations (2.1)–(2.16) is connected with Theorem 2.1. When a reference is given, it usually refers to the case $\delta = 0$.

The families giving rise to pairs (2.1)–(2.3) below are obtained from $(v, k)$ modular Golomb rulers [23, Ch. 5], [24], [31, Sec. 5], [54, Sec. 19.3]; see Theorem 1.3(ii).

- cyclic $v_k : v = q^2 + q + 1$, $k = q + 1 - \delta$, $q + 1 > \delta \geq 0$; (2.1)
- cyclic $v_k : v = q^2 - 1$, $k = q - \delta$, $q > \delta \geq 0$; (2.2)
- cyclic $v_k : v = p^2 - p$, $k = p - 1 - \delta$, $p - 1 > \delta \geq 0$. (2.3)

The configurations giving rise to (2.1) use the incidence matrix of the cyclic projective plane $PG(2, q)$ [23, Sec. 5.5], [24, 54, Th. 19.15], [58]. The family with parameters (2.2) is obtained from the cyclic punctured affine plane $AG(2, q)$ [13], [23, Sec. 5.6], [24,26], [54, Th. 19.17]; see also [21, Ex. 5] and [27] where the configurations are called anti-flags. We recall that the punctured plane $AG(2, q)$ is the affine plane without the origin and the lines through the origin. Punctured affine planes are also called elliptic (Desarguesian) semiplanes of type L. Finally, the configurations with parameters (2.3) follow from Ruzsa’s construction [23, Sec. 5.4], [24, 53], [54, Th. 19.19].

The non-cyclic families with parameters (2.4) and (2.5) are given in [1, Constructions (i),(ii), p.126] and [29, Constructions 3.2,3.3, Rem. 3.5]; see also the references therein and [3, 6, Sec. 3], [21, Sec. 7.3], [27,51].

- $v_k : v = q^2 - qs$, $k = q - s - \delta$, $q > s \geq 0$, $q - s > \delta \geq 0$; (2.4)
- $v_k : v = q^2 - (q - 1)s - 1$, $k = q - s - \delta$, $q > s \geq 0$, $q - s > \delta \geq 0$. (2.5)

For $q$ a square, in [1, Conjec. 4.4, Rem. 4.5, Ex. 4.6], [3, Th. 6.4], and [29, Constructions 3.7, Th. 3.8], families of non-cyclic configuration $v_k$ with parameters (2.6) are provided; see also [21, Ex. 8]. Taking $c = q - \sqrt{q}$, we obtain (2.7), see also [19, Ex. 2(ii)], [27].

- $v_k : v = c(q + \sqrt{q} + 1), k = \sqrt{q} + c - \delta, c = 2, 3, \ldots, q - \sqrt{q}, \delta \geq 0$; (2.6)
- $v_k : v = q^2 - \sqrt{q}, k = q - \delta, q > \delta \geq 0$. (2.7)

In [27, Th. 1.1], a non-cyclic family with parameters

$$v_k : v = 2p^2, k = p + s - \delta, 0 < s \leq q + 1, q^2 + q + 1 \leq p, p + s > \delta \geq 0$$ (2.8)

is given. In [21, Sec. 6], a construction of non-cyclic configuration based on the cyclic punctured affine plane, is provided with parameters

$$v_k : v = c(q - 1), k = c - \delta, c = 2, 3, \ldots, b, b = q \text{ if } \delta \geq 1,$$ (2.9)

$$b = \left\lfloor \frac{q}{2} \right\rfloor \text{ if } \delta = 0, c > \delta \geq 0.$$
In [19, Sec. 2], [21, Sec. 3], the following geometrical construction which uses point orbits under the action of a collineation group is described.

**Construction A.** Take any point orbit \( P \) under the action of a collineation group in an affine or projective space of order \( q \). Choose an integer \( k \leq q + 1 \) such that the set \( \mathcal{L}(P, k) \) of lines meeting \( P \) in precisely \( k \) points is not empty. Define the following incidence structure: the points are the points of \( P \), the lines are the lines of \( \mathcal{L}(P, k) \), the incidence is that of the ambient space.

**Theorem 2.2.** In Construction A the number of lines of \( \mathcal{L}(P, k) \) through a point of \( P \) is a constant \( r_k \). The incidence structure is a configuration \((v_{rk}, b_k)\) with \( v_{rk} = |P| \), \( b_k = |\mathcal{L}(P, k)| \).

By Definition [1,1] if \( r_k = k \) then Construction A produces a symmetric configuration \( v_k \). It is noted in [19,21] that Construction A works for any 2-(\( v, k, 1 \)) design \( D \) and for any group of automorphism of \( D \). The size of any block in \( D \) plays the role of \( q + 1 \).

Families of non-cyclic configuration \( v_k \) obtained by Construction A with the following parameters are given in [21, Exs 2, 3].

\[
\begin{align*}
v_k & : v = \frac{q(q - 1)}{2}, \quad k = \frac{q + 1}{2} - \delta, \quad \frac{q + 1}{2} > \delta \geq 0, \quad q \text{ odd.} \tag{2.10} \\
v_k & : v = \frac{q(q + 1)}{2}, \quad k = \frac{q - 1}{2} - \delta, \quad \frac{q - 1}{2} > \delta \geq 0, \quad q \text{ odd.} \tag{2.11} \\
v_k & : v = q^2 + q - q\sqrt{q}, \quad k = q - \sqrt{q}, \quad q - \sqrt{q} > \delta \geq 0, \quad q \text{ square.} \tag{2.12}
\end{align*}
\]

In [7,8,10], non-cyclic families with parameters (2.13)–(2.16) are described in connection with graph theory; see also [1] for another construction of (2.14).

\[
\begin{align*}
v_k & : v = q^2 - rq - 1, \quad k = q - r - \delta, \quad q - r > \delta \geq 0, \quad q - 3 \geq r \geq 0. \tag{2.13} \\
v_k & : v = q^2 - q - 2, \quad k = q - 1 - \delta, \quad q - 1 > \delta \geq 0. \tag{2.14} \\
v_k & : v = tq - 1, \quad k = t - \delta, \quad t > \delta \geq 0, \quad q > t \geq 3. \tag{2.15} \\
v_k & : v = tq - 2, \quad k = t - \delta, \quad t > \delta \geq 0, \quad q > t \geq 3. \tag{2.16}
\end{align*}
\]

A classical construction by V. Martinetti for configurations \( v_3 \), going back to 1887 [49], is described in detail, e.g. in [3, 6, 12, 15, 25, 32, 37, Sec. 2.4, Fig. 2.4.1]. In [25] a **Generalized Martinetti Construction** (Construction GM) for configurations \( v_k, k \geq 3 \), is proposed. Use of Construction GM to obtain a wide spectrum of symmetric configurations parameters is considered in [3,5,6].

Construction GM can be presented from different points of view, see [3,25]. Here we focus on an approach based on incidence matrices, which will be used for obtaining new values of \( v, k \).

**Definition 2.3.** Let \( M(v, k) \) be an incidence matrix of a symmetric configuration \( v_k \). In \( M(v, k) \), we consider an aggregate \( A \) of \( k - 1 \) rows corresponding to pairwise disjoint lines of \( v_k \) and \( k - 1 \) columns corresponding to pairwise non-collinear points of \( v_k \). If a
$$(k - 1) \times (k - 1)$$ submatrix $C(A)$ formed by the intersection of the rows and columns of $A$ is a permutation matrix $P_{k-1}$ then $A$ is called an extending aggregate (or E-aggregate). The matrix $M(v, k)$ admits an extension if it contains at least one E-aggregate. The matrix $M(v, k)$ admits $\theta$ extensions if it contains $\theta$ E-aggregates that do not intersect each other.

**Procedure E (Extension Procedure).** Let $M(v, k) = [m_{ij}]$ be an incidence matrix of a symmetric configuration $v_k = (P, L)$. Assume that $M(v, k)$ admits an extension. A matrix $M(v + 1, k)$ can be obtained by two steps.

1. To the matrix $M(v, k)$, add a new row from below and a new column to the right. Denote the new $(v + 1) \times (v + 1)$ matrix by $B = [b_{ij}]$, and let $b_{v+1,v+1} = 1$ whereas $b_{v+1,i} = \ldots = b_{v+1,v} = 0$, $b_{1,v+1} = \ldots = b_{v,v+1} = 0$.

2. One of E-aggregates of $M(v, k)$, say $A$, is chosen. In the matrix $B$, we “clone” all $k - 1$ ones of the submatrix $C(A)$ writing their “projections” to the new row and column. Finally, the ones cloned are changed by zeroes. In other words, let the aggregate $A$ consist of rows with indexes $i_u$, $u = 1, 2, \ldots, k - 1$, and columns with indexes $j_d$, $d = 1, 2, \ldots, k - 1$. Then the ones of $C(A)$ are as follows: $m_{i_u,j_d(v)} = 1$, $u = 1, 2, \ldots, k - 1$, for some permutation $\pi$ of the indexes $1, \ldots, k - 1$. Then $B$ arising from Step 1 is changed as follows: $b_{i_u,v+1} = 1$, $b_{v+1,j_d} = 1$, $b_{i_u,j_d} = 0$, $u = 1, 2, \ldots, k - 1$, $d = 1, 2, \ldots, k - 1$.

As the Golomb bound $2L_{G}(k) + 1$ is important for studying parameters $v, k$, we note that for sufficiently large orders $k$, relatively short Golomb rulers are constructed and are available online, see [23, 55, 56] and the references therein. For $k \leq 150$, the order of magnitude of the lengths $L_{G}(k)$ of the shortest known Golomb rulers is $ck^2$ with $c \in [0.7, 0.9]$, see [23, 26, 31, 36, 54, 55]. Moreover, $L_{G}(k) < k^2$ for $k < 65000$, see [23]. Constructions of Golomb rulers for large $k$ can be found in [24]. Remind also that Sidon sets are equivalent to Golomb rulers, see [23, 51] and the references therein.

### 3 Preliminaries on block double-circulant incidence matrices

Results of this section are taken from [18], see also the references therein, in particular, discussions of [18, Rem. 1, 2].

Recall that the weight of a circulant binary is the number of 1’s in each its rows.

**Definition 3.1.** Let $v = td$. A binary $v \times v$ matrix $A$ is said to be a block double-circulant matrix (BDC matrix for short) if

$$A = \begin{bmatrix}
C_{0,0} & C_{0,1} & \cdots & C_{0,t-1} \\
C_{1,0} & C_{1,1} & \cdots & C_{1,t-1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{t-1,0} & C_{t-1,1} & \cdots & C_{t-1,t-1}
\end{bmatrix},$$

(3.1)

where $C_{i,j}$ is a circulant $d \times d$ binary matrix for all $i, j$, and submatrices $C_{i,j}$ and $C_{i,m}$
with $j - i \equiv m - l \pmod{t}$ have the same weight. The matrix

$$W(A) = \begin{bmatrix}
  w_0 & w_1 & w_2 & w_3 & \cdots & w_{t-2} & w_{t-1} \\
  w_{t-1} & w_0 & w_1 & w_2 & \cdots & w_{t-3} & w_{t-2} \\
  w_{t-2} & w_{t-1} & w_0 & w_1 & \cdots & w_{t-4} & w_{t-3} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  w_1 & w_2 & w_3 & w_4 & \cdots & w_{t-1} & w_0 \\
\end{bmatrix} \tag{3.2}$$

is a circulant $t \times t$ matrix whose entry in position $i, j$ is the weight of $C_{i,j}$. $W(A)$ is called the weight matrix of $A$. The vector $\overline{W}(A) = (w_0, w_1, \ldots, w_{t-1})$ is called the weight vector of $A$.

**Remark 3.2.** If in Definition 1 the matrices $C_{i,j}$ were assumed to be right-circulant and not left-circulant, then they would have been the sum of some right-circulant permutation matrices. A right-circulant $d \times d$ permutation matrix is always associated to a permutation of the set $\{1, 2, \ldots, d\}$ in the subgroup generated by the cycle $(1 \ 2 \ 3 \ \ldots \ d)$. Then the notion of a BDC matrix is substantially equivalent to that of a $\mathbb{Z}_d$-scheme, as defined in [3].

Let $A$ be as in Definition 3.1. In addition, assume that $A$ is the incidence matrix of a symmetric configuration $v_k$ with $k = \sum_{i=0}^{t} w_i$. From $A$ one can obtain BDC incidence $v' \times v'$ matrices $A'$ of symmetric configurations $v'_k$, by the following way.

(i) For each $h \in \{0, 1, \ldots, t - 1\}$, in each row of every submatrix $C_{i,j}$ with $j - i \equiv h \pmod{t}$ replace $\delta_h \geq 0$ values of 1 with zeros, in such a way that the obtained submatrix is still circulant. As a result, a BDC incidence matrix of a configuration $v'_k$ with

$$v' = v, \quad k' = k - \sum_{h=0}^{t-1} \delta_h, \quad 0 \leq \delta_h \leq w_h, \quad w'_h = w_h - \delta_h, \quad \overline{W}(A') = (w'_0, \ldots, w'_{t-1}) \tag{3.3}$$

is obtained.

(ii) Fix some non-negative integer $j \leq t - 1$. Let $m$ be such that $w_m \leq w_h$ for all $h \neq j$. Cyclically shift all block rows of $A$ to the left by $j$ block positions. A matrix $A^*$ with $\overline{W}(A^*) = (w_0^* = w_j, \ldots, w_m^* = w_{m+j} \pmod{t}, \ldots, w'_{t-1} = w_{j-1})$ is obtained. By applying (i), construct a matrix $A^{**}$ with $w_0^{**} = w_0^* = w_j, \quad w_h^{**} = w_m, \quad h \geq 1$. Now remove from $A^{**}$ $t - c$ block rows and columns from the bottom and the right. In this way an incidence $cd \times cd$ BDC matrix $A'$ of a configuration $v'_k$ is obtained with

$$v' = cd, \quad k' = w_j + (c-1)w_m, \quad c = 1, 2, \ldots, t, \quad \overline{W}(A') = (w_j, w_m, \ldots, w_m) \tag{3.4}$$

(iii) Let $t$ be even. Let $A^*$ be as in (ii). Let $w_O, w_E$ be weights such that $w_O \leq w_h^*$ for odd $h$ and $w_E \leq w_h^*$ for even $h$. By applying (i), construct a matrix $A^{**}$ with $w_0^{**} = w_0^* = w_j, \quad w_h^{**} = w_O$ for odd $h, \quad w_h^{**} = w_E$ for even $h \geq 2$. Let $f = 1, 2, \ldots, t/2$. From $A^{**}$ remove $t - 2f$ block rows and columns from the bottom and the right. An incidence $2fd \times 2fd$ BDC matrix $A'$ of a configuration $v'_k$ is obtained with

$$v' = 2fd, \quad k' = w_j + w_O + (f-1)(w_E + w_O), \quad \overline{W}(A') = (w_j, w_O, w_E, w_O, \ldots, w_E, w_O) \tag{3.5}$$

\[ f-1 \text{ pairs} \]

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Remark 3.3. Construction (i) in this section essentially follows from Theorem 2.1 in [3] it is referred to as 1-factor deletion. Construction (ii) is essentially a different formulation of Proposition 4.3 in [3], which is stated in terms of $\mathcal{S}_m$-schemes. Apart from terminology, the only difference is that here the case $m > 1$ is considered. Other methods for obtaining families of symmetric configurations from $A$ can be found in [21, Sec. 4].

4 Constructions and parameters of block double-circulant incidence matrices from [18]

All the results of this section are taken from [18], apart from Tables 3.1 and 3.2 which essentially present more examples than the corresponding [18, Tab. 1]. In Subsections 4.1 and 4.2 we give some results based on a general method connected with the action of the automorphism group $S$ of a configuration. The method was originally proposed in [19,21] and then developed in [18].

4.1 BDC incidence matrices from projective planes

In this subsection, the projective plane $PG(2, q)$ is considered as a cyclic symmetric configuration $(q^2 + q + 1)_{q+1}$ [21, Sec. 5], [23, Sec. 5.5], [54, Th. 19.15], [58]. The Singer group of $PG(2, q)$ is used as the automorphism group $S$.

The following BDC matrices with $d \times d$ circulant submatrices and the corresponding BDC configurations $v_k$ are given in [18, Sec. 4.1]:

\[
\text{BDC } v_k : d = \frac{q^2 + q + 1}{3}, \quad v = 2d, \quad k = \frac{2q + \sqrt{q} + 2}{3}, \quad q = p^{4m+2}, \quad p \equiv 2 \pmod{3};
\]

(4.1)

\[
\text{BDC } v_k : d = \frac{q^2 + q + 1}{3}, \quad v = 2d, \quad k = \frac{2q - \sqrt{q} + 2}{3}, \quad q = p^{4m}, \quad p \equiv 2 \pmod{3}.
\]

BDC $v_k : d = \frac{q^2 + q + 1}{t}, \quad v = cd, \quad k = \frac{q + 1 \pm (1 - t)\sqrt{q}}{t} + \frac{(c - 1)q + 1 \pm \sqrt{q}}{t}, \quad (4.2)$

\[c = 1, 2, \ldots, t, \quad q = p^{2m}, \quad t \text{ prime},\]

where $p \pmod{t}$ is a generator of the multiplicative group of $\mathbb{Z}_t$.

The needed for (4.2) hypothesis that $p \pmod{t}$ is a generator of the multiplicative group of $\mathbb{Z}_t$ holds, for example, in the following cases: $q = 3^4, t = 7; q = 2^5, t = 13; q = 5^4, t = 7; q = 2^{12}, t = 19; q = 3^8, t = 7; q = 2^{16}, t = 13; q = 17^4, t = 7; p \equiv 2 \pmod{t}, t = 3$.

In Table 4.1, parameters of configurations $v'_n$ with BDC incidence matrices are given. We use both (ii) and (iii) of Section 3. The starting weights $w'_i$ are obtained by computer by considering orbits of subgroups of a Singer group of $PG(2, q)$. For $q = 81$ we use (4.2). The values $k', v'$ are calculated by (3.4), (3.5). Only cases with $v' < G(k')$ are included in
the tables. Then the smallest value $k^\#$ for which $v' < G(k^\#)$ is found. As a result, each row of the table provides configurations $v_n'$ with $v' < G(n)$, $n = k^\#, k^\# + 1, \ldots, k'$, see (i) of Section 3 and (3.3). In column $w_i^*$, an entry $s_j$ indicates that the weight $s$ should be repeated $j$ times.

### 4.2 BDC incidence matrices from punctured affine planes

In this subsection, the cyclic punctured affine plane is considered as a cyclic symmetric configuration $(q^2 - 1)q$, see [13], [23, Sec. 5.6], [54, Th. 19.17], as well as [21, Ex. 5, Sec. 6]. The affine Singer group of $AG(2, q)$ is used as the automorphism group $S$.

The following BDC matrices with $d \times d$ circulant submatrices and the corresponding BDC configurations $v_k$ are given in [18, Sec. 4.2] on the base [18, Th. 4]:

\[
\text{BDC } v_k : \quad d = (\sqrt{q} - 1)(q + 1), \quad v = 2fd, \quad k = (2f - 1)\sqrt{q},
\]

\[
f = 1, 2, \ldots, \frac{\sqrt{q} + 1}{2}, \quad q \text{ odd square.}
\]

\[
\text{BDC } v_k : \quad d = 2(\sqrt{q} - 1)(q + 1), \quad v = cd, \quad k = (2c - 1)\sqrt{q},
\]

\[
c = 1, 2, \ldots, \frac{\sqrt{q} + 1}{2}, \quad q \text{ odd square, } \sqrt{q} \equiv 1 \pmod{4}.
\]

\[
\text{BDC } v_k : \quad d = 2(\sqrt{q} - 1)(q + 1), \quad v = 2fd, \quad k = (4f - 1)\sqrt{q},
\]

\[
f = 1, 2, \ldots, \frac{\sqrt{q} + 1}{4}, \quad q \text{ odd square, } \sqrt{q} \equiv 3 \pmod{4}.
\]

\[
\text{BDC } v_k : \quad d = 4(\sqrt{q} - 1)(q + 1), \quad v = cd, \quad k = (4c - 1)\sqrt{q},
\]

\[
c = 1, 2, \ldots, \frac{\sqrt{q} + 1}{4}, \quad q \text{ odd square, } \sqrt{q} \equiv 3 \pmod{4}, \quad \frac{\sqrt{q} + 1}{4} \text{ is odd.}
\]

\[
\text{BDC } v_k : \quad d = 4(\sqrt{q} - 1)(q + 1), \quad v = 2fd, \quad k = (8f - 1)\sqrt{q},
\]

\[
f = 1, 2, \ldots, \frac{\sqrt{q} + 1}{8}, \quad q \text{ odd square, } \sqrt{q} \equiv 3 \pmod{4}, \quad \frac{\sqrt{q} + 1}{4} \text{ is even.}
\]

In Table 4.2 parameters of configurations $v'_n$ with BDC incidence matrices are given. We use both (ii) and (iii) of Section 3. The starting weights $w_i^*$ are obtained by computer through the constructions of the orbits of subgroups of the affine Singer group. For notations $k'$ and $k^\#$ see Table 4.1.

### 5 Parameters of symmetric configurations $v_k$ admitting an extension

The following infinite family of symmetric configuration $v_k$ is obtained in [3, Th. 6.2], [6, Th. 1(i)] with the help of Construction GM:

\[
v_k : \quad v = q^2 - qs + \theta, \quad k = q - s - \Delta, \quad q > s \geq 0, \quad q - s > \Delta \geq 0, \quad \theta = 0, 1, \ldots, q - s + 1. \quad (5.1)
\]
Theorem 5.1. [18 Corollary 1] Let \( v = td, t \geq k, d \geq k - 1 \), and let \( v_k \) be a symmetric configuration. Assume that an incidence matrix \( A \) of \( v_k \) is a BDC matrix as in \((3.1)\) with weight vector \( \overline{W}(A) \). If \( \overline{W}(A) = (0, 1, \ldots, 1) \) or \( \overline{W}(A) = (1, 1, \ldots, 1) \) then one can obtain a family of symmetric configurations \( v_k \) with parameters \((5.2)\) or \((5.3)\), respectively

\[
v_k : v = cd + \theta, \quad k = c - 1 - \delta, \quad c = 2, 3, \ldots, t, \quad \theta = 0, 1, \ldots, c + 1, \quad \delta \geq 0. \quad (5.2)
\]

\[
v_k : v = cd + \theta, \quad k = c - \delta, \quad c = 2, 3, \ldots, t, \quad \theta = 0, 1, \ldots, c + 1, \quad \delta \geq 0. \quad (5.3)
\]

Configurations with parameters \((5.4)\) were first obtained in [13 Th. 6.3] from the punctured affine plane by using Construction GM; see also [20 Eqn. (8)] and [18 Ex. 6(i)].

\[
v_k : v = c(q - 1) + \theta, \quad k = c - 1 - \delta, \quad c = 2, 3, \ldots, q + 1, \quad \theta = 0, 1, \ldots, c + 1, \quad \delta \geq 0. \quad (5.4)
\]

The families of configurations \( v_k \) \((5.5)\) and \((5.6)\) are obtained in [18 Sec. 5] by Construction GM, starting from some new starting matrices proposed in [18].

\[
v_k : v = cp + \theta, \quad k = c - \delta, \quad c = 2, 3, \ldots, p - 1, \quad \theta = 0, 1, \ldots, c + 1, \quad \delta \geq 0, \quad p \text{ prime.} \quad (5.5)
\]

\[
v_k : v = c(p - 1) + \theta, \quad k = c - 1 - \delta, \quad c = 2, 3, \ldots, p, \quad \theta = 0, 1, \ldots, c + 1, \quad \delta \geq 0, \quad p \text{ prime.} \quad (5.6)
\]

The family of configurations \( v_k \) \((5.7)\) is obtained in [18].

\[
v_k : v = c(q + \sqrt{q} + 1) + \theta, \quad k = c - \delta, \quad c = 2, 3, \ldots, q - \sqrt{q} + 1, \quad \theta = 0, 1, \ldots, c + 1, \quad \delta \geq 0, \quad q \text{ square.} \quad (5.7)
\]

6 The spectrum of parameters of cyclic symmetric configurations

Current data on the existence of cyclic configurations \( v_k, k \leq 51 \), are given in Table 6.1. New parameters obtained in [18] are written in bold font.

In Table 6.1, the values of \( v \) for which cyclic symmetric configurations \( v_k \) exist (resp. do not exist) are written in normal (resp. in italic) font. Moreover, \( \overline{v} \) means that no configuration \( v_k \) exists, whereas \( \overline{v} \) indicates that no cyclic configuration \( v_k \) exists. Data from [26 [31] [33] [36] [42] [47] [50] are used in the 4-th column of the table. We take into account that an entry of the form “+” in the row “n” of [56 Tab. 1] means the existence of cyclic symmetric configurations \( v_n \) with \( v \geq t \). An absence of a value “+” in the row “n” of [56 Tab. 1] means the non-existence of a cyclic symmetric configuration \( v_n \). Also, we use the following non-existence results: \( \overline{32}^{[31]} \text{ Th. 4.8]; \overline{33}^{[42]}; \overline{34}; \overline{59}^{[47]; \overline{75}; \overline{97}; \overline{81}; \overline{85}; \overline{26}; \overline{97}; \overline{106}; \overline{121}; \overline{132}; \overline{134}; \overline{155}; \overline{157}; \overline{169}; \overline{182}; \overline{184} \); \overline{256}; \overline{260}; \overline{258}; \overline{192} \); \overline{224}; \overline{256}; \overline{260}; \overline{263} \); [56 Tab. 1]; see also Theorem 5.1.

The values of \( k \) for which the spectrum of parameters of cyclic symmetric configurations \( v_k \) is completely known are indicated by a dot “.”; the corresponding values of \( E_c(k) \) are sharp and they are noted by the dot “.” too.
An entry \(v\rightarrow w\) indicates an interval of sizes from \(v\) to \(w\) without gaps. If an already known value lies within an interval \(v\rightarrow w\) obtained in work [18], then it is written immediately before the interval.

Entries \(v_a, v_b,\) and \(v_c\) (here and in all tables) mean, respectively, that relations (2.1), (2.2), and (2.3) are applied.

The value \(v_3(k)\) is defined in Introduction. In the second column, the exact values of \(v_3(k)\) are marked by the dot “.”. For \(k \leq 16\), the exact values of \(v_3(k)\) are taken from [30 Tab.IV], [38 Tab.2], [50 Tab.1a], [61]. Also, if \(k - 1\) is a prime power then \(v_3(k) = P(k) = k^2 - k + 1\). The remaining entries in the second column are lower bounds of \(v_3(k)\). By the Bruck-Ryser Theorem, planes \(PG(2, k - 1)\) with \(k - 1 = 6, 14, 21, 22, 30, 33, 38, 42, 46, 54, 57, 62\) do not exist. It is well-known that \(P(k) \leq v_3(k)\), and that a cyclic symmetric configuration \((k^2 - k + 1)_k\) exists if and only if a cyclic projective plane of order \(k - 1\) exists. By [11], no cyclic projective planes exist with non-prime power orders \(\leq 2 \cdot 10^9\). Therefore cyclic projective planes \(PG(2, k - 1)\) with \(k - 1 = 18, 20, 24, 26, 28, 34, 35, 36, 39, 40, 44, 45, 48, 50\) do not exist. Also we use Theorem 6.1 taken from [33]. The mentioned non-existence cases of cyclic configurations are marked in Table 6.1 by subscripts br (Bruck-Ryser Theorem), s (11), and t (Theorem 6.1).

For \(k \leq 22\) the filling of the interval \(P(k) - G(k)\) is expressed as a percentage in the last column of Table 6.1.

**Theorem 6.1.** [33 Th.2.4] There is no symmetric configuration \((k^2 - k + 2)_k\) if \(5 \leq k \leq 10\) or if neither \(k\) or \(k - 2\) is a square.

In order to widen the ranges of parameter pairs \((v, k)\) for which a cyclic symmetric configuration \(v_k\) exists, we consider a number of procedures that allow to define a new modular Golomb ruler from a known one. Some methods have already been introduced in the paper, see Theorem 2.1.

Here we first recall a result from [54], which describes a method to construct different rulers with the same parameters.

**Theorem 6.2.** [54] If \((a_1, a_2, \ldots, a_k)\) is a \((v, k)\) modular Golomb ruler and \(m\) and \(b\) are integers with gcd\((m, v) = 1\) then \((ma_1 + b \mod v), ma_2 + b \mod v, \ldots, ma_k + b \mod v)\) is also a \((v, k)\) modular Golomb ruler.

It should be noted that a \((v, k)\) modular Golomb ruler can be a \((v + \Delta, k)\) modular Golomb ruler for some integer \(\Delta\) [31]. This property does not depend on parameters \(v\) and \(k\) only. This is why Theorem 6.2 can be useful for our purposes.

**Example 6.3.** We consider the \((31, 6)\) modular Golomb ruler

\[(a_1, \ldots, a_6) = (0, 1, 4, 10, 12, 17)\]

obtained from \(PG(2, 5)\), see [55]. We can apply Theorem 6.2 for \(m = 19, b = 0\). The \((31, 6)\) modular Golomb ruler \((ma_1 \mod 31, \ldots, ma_6 \mod 31)\) is

\[(a'_1, \ldots, a'_6) = (0, 4, 11, 13, 14, 19).\]
Now we take $\Delta = 4$ and calculate the set of differences $\{a_{i}^{'} - a_{j}^{'} \pmod{35} \mid 1 \leq i, j \leq 6; i \neq j\}$, that is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34\}$. As the all differences are distinct and nonzero, the starting $(31, 6)$ modular Golomb ruler is also a $(35, 6)$ modular Golomb ruler.

For $k \leq 81$, we performed a computer search starting from the $(v, k)$ modular Golomb rulers corresponding to (2.1)–(2.3). For projective and affine planes, we got a concrete description of the ruler from [55].

For Ruzsa’s construction, we use the following known relations. Let $p$ be a prime. Let $g$ be a primitive element of $F_{p}$. The following Ruzsa’s sequence [53], [23, Sec. 5.4], [54, Th. 19.19] forms a $(p^2 - p, p - 1)$ modular Golomb ruler:

$$e_{u} = pu + (p - 1)g^{u} \pmod{p^2 - p}, \ u = 1, 2, \ldots, p - 1, \ v = p^2 - p. \quad (6.1)$$

For every starting $(v, k)$ modular ruler we first considered all possible $m$ with $\gcd(m, v) = 1$, and applied Theorem 6.2 for all $b < v$ to get new rulers with the same parameters $v$ and $k$. Then, we checked whether this ruler was also a $(v + \Delta, k)$ for some $\Delta$.

In Table 6.2, for $52 \leq k \leq 83$, the upper bounds on the cyclic existence bound $E_{c}(k)$ obtained in [18] are listed in bold font. An entry, say $A(u)$, in the column $E_{c}(k)$ on the row “$u$”, means that in [18] all cyclic symmetric configurations $v_{u}$ in the region $A(u), A(u) + 1, \ldots, G(u) - 1$ are obtained. These configurations are new. We obtained also many other new cyclic symmetric configurations $v_{k}$ for $52 \leq k \leq 83$. However, we do not give here their sizes $v$ here in order to save space.

The upper bounds on $E_{c}(k)$ in Tables 6.1 and 6.2, obtained in [18], are written in bold font.

Some new cyclic symmetric configurations important for Table 7.1 of Section 7 are given in Table 6.3 where we write the first rows of their incidence matrices; these rows may be the same for distinct $v$.

7 The spectrum of parameters of symmetric (not necessarily cyclic) configurations

The known results regarding to parameters of symmetric configurations can be found in [1–10, 12–13, 15, 17–21, 25–27, 29–38, 42, 44, 47, 49, 50, 53, 54, 56]; see also the references therein.

The known families of configurations are described in Section 2, see also (5.1), (5.4). New families obtained in the work [18] are given in Sections 4 and 5. In Table 7.1, for $k \leq 51$, $P(k) \leq v < G(k)$, values of $v$ for which a symmetric configuration $v_{k}$ from one of the families of Sections 2–5 exists are given. The new parameters obtained in the paper [18] are written in bold font.
An entry of type $v_{\text{subscript}}$ indicates that one of the following is used: (2.i), (4.j), (5.k), Table 4.1, Table 6.1, the Bruck-Ryser Theorem, Theorem 6.1. More precisely $v_a$ indicates that $v$ is obtained from (2.1), and similarly $v_b \rightarrow (2.2)$, $v_c \rightarrow (2.3)$, $v_f \rightarrow (2.5)$, $v_g \rightarrow (2.6)$, $v_h \rightarrow (2.7)$, $v_j \rightarrow (2.8)$, $v_k \rightarrow (2.9)$, $v_\lambda \rightarrow (2.13) - (2.16)$, $v_m \rightarrow (5.1)$, $v_P \rightarrow (4.3)$, $v_r \rightarrow (5.4)$, $v_S \rightarrow (5.5)$, $v_T \rightarrow (5.6)$, $v_W \rightarrow$ Table 4.1, $v_y \rightarrow$ Table 6.1 with $k \leq 15$, $v_Z \rightarrow$ Table 6.1 with $k > 15$, $v_{br} \rightarrow$ the Bruck-Ryser Theorem, $v_t \rightarrow$ Theorem 6.1.

Here capital letters in subscripts remark new results and constructions of [18] while lower case letters indicate the known ones.

An entry $v_{\text{subscript}1}$ with more than one subscript means that the same value can be obtained from different constructions. An entry of type $v_{\text{subscript}1} - v_{\text{subscript}2} \ldots$ indicates that a whole interval of values from $v$ to $v'$ can be obtained from the constructions corresponding to the subscripts. We use the following known results on the existence of sporadic symmetric configurations: 457 [9]; 829 [27, Tab. 1]; 13512, see [33] with reference to Mathon’s talk at the British Combinatorial Conference 1987; 346 [44], see also [4]. The non-existence of configuration 11211 is proven in [43]. The non-existence of the plane $PG(2,10)$ implies the non-existence of configuration 11111.

The values of $k$ for which the spectrum of parameters of symmetric configurations $v_k$ is completely known are indicated by a dot ”/”; the corresponding values of $E(k)$ are exact and they are indicated by a dot as well.

To save space, in Table 7.1 for given $v, k$, we do not write all the constructions providing a configuration $v_k$, but often we describe some of them as a matter of illustration.

For the convenience of the reader we give also Table 7.2 where constructions are not indicated and for $k \leq 64$, $P(k) \leq v < G(k)$, values of $v$ for which a symmetric configuration $v_k$ exists are written.

The filling of the interval $P(k) - G(k)$ is expressed as a percentage in the last column of Tables 7.1 and 7.2. It is interesting to note that such a percentage is quite high and that most gaps occur for $v$ close to $k^2 - k + 1$.

We note that a number of parameters obtained in the work [18] are new, see bold font in Table 7.1. Note that parameters of some new families are too big to be included in Tables 7.1 and 7.2. Recall also (see Introduction) that from the stand point of applications, including Coding Theory, it is useful to have different matrices $M(v, k)$ for the same $v$ and $k$.

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8 Tables of parameters

8.1 Tables for Section 4

Table 4.1. Parameters of configurations $v'_n$ with BDC incidence matrices, $v' < G(n)$, $n = k^#, k^# + 1, \ldots, k'$, by (ii) of Section 3 from the cyclic projective plane $PG(2, q)$

| $q$ | $t$ | $d$ | $v'_n$ | $c$ | $k'$ | $v'$ | $G(k')$ | $k^#$ | $G(k^#)$ |
|-----|-----|-----|--------|-----|------|------|----------|------|----------|
| 25  | 3   | 217 | 12,7,7 | 2   | 19   | 434  | 493      | 19   | 493      |
| 25  | 7   | 93  | 8,3,3,3,3,3,3 | 4 | 17   | 372  | 399      | 17   | 399      |
| 25  | 7   | 93  | 8,3,3,3,3,3,3 | 5   | 20   | 465  | 567      | 19   | 493      |
| 25  | 7   | 93  | 8,3,3,3,3,3,3 | 6   | 23   | 558  | 745      | 20   | 567      |
| 32  | 7   | 151 | 0,5,5,6,5,6,6 | 6 | 25   | 906  | 961      | 25   | 961      |
| 37  | 3   | 469 | 16,9,13  | 2   | 25   | 938  | 961      | 25   | 961      |
| 43  | 3   | 631 | 19,13,12 | 2   | 31   | 1262 | 1495     | 30   | 1361     |
| 49  | 3   | 817 | 21,16,13 | 2   | 34   | 1634 | 1877     | 33   | 1719     |
| 61  | 3   | 1261| 25,21,16 | 2   | 41   | 2522 | 2611     | 40   | 2565     |
| 64  | 3   | 1387| 27,19,19 | 2   | 46   | 2774 | 3407     | 42   | 2795     |
| 64  | 19  | 219 | 11,3\textsuperscript{18} | 7   | 32   | 1533 | 1569     | 32   | 1569     |
| 64  | 19  | 219 | 11,3\textsuperscript{18} | 8   | 35   | 1752 | 1975     | 34   | 1877     |
| 64  | 19  | 219 | 11,3\textsuperscript{18} | 9   | 38   | 1971 | 2293     | 35   | 1975     |
| 66  | 19  | 219 | 11,3\textsuperscript{18} | 10  | 41   | 2190 | 2611     | 37   | 2199     |
| 66  | 19  | 219 | 11,3\textsuperscript{18} | 17  | 62   | 3723 | 6431     | 48   | 3775     |
| 66  | 19  | 219 | 11,3\textsuperscript{18} | 18  | 65   | 3942 | 7187     | 50   | 4189     |
| 67  | 3   | 1519| 28,19,21 | 2   | 47   | 3038 | 3609     | 44   | 3193     |
| 73  | 3   | 1801| 28,27,19 | 2   | 47   | 3602 | 3609     | 47   | 3609     |
| 79  | 3   | 2107| 31,21,28 | 2   | 52   | 4214 | 4541     | 51   | 4381     |
| 81  | 7   | 949 | 4,13,13,13,13,13,13 | 6 | 69   | 5694 | 8291     | 58   | 5703     |
| 81  | 7   | 949 | 4,13,13,13,13,13,13 | 5   | 56   | 4745 | 5451     | 54   | 4747     |
| 97  | 3   | 3169| 39,28,31 | 2   | 67   | 6338 | 7639     | 62   | 6431     |
| 103 | 3   | 3571| 39,28,37 | 2   | 67   | 7142 | 7639     | 65   | 7187     |
| 107 | 7   | 1651| 24,15,15,15,15,13,13 | 6 | 89   | 9906 | 13557    | 75   | 9965     |
| 107 | 7   | 1651| 24,15,15,15,15,13,13 | 5   | 76   | 8255 | 10179    | 69   | 8291     |
| 109 | 3   | 3997| 43,36,31 | 2   | 74   | 7994 | 9507     | 69   | 8291     |
| 109 | 7   | 1713| 8,15,15,19,15,19,19 | 6   | 83   | 10278| 12041    | 77   | 10409    |
| 121 | 37  | 399 | 14,3\textsuperscript{36} | 25  | 89   | 9975 | 13557    | 76   | 10179    |
| 127 | 3   | 5419| 49,43,36 | 2   | 85   | 10838| 12821    | 80   | 11127    |
| 128 | 7   | 2359| 24,21,21,14,21,14,14 | 6   | 94   | 14154| 15769    | 91   | 15085    |
| 137 | 7   | 2701| 24,15,15,23,15,23,23 | 6   | 99   | 16206| 17081    | 96   | 16243    |
Table 4.1 (continue). Parameters of configurations $v'_n$ with BDC incidence matrices, $v' < G(n)$, $n = k^#, k^# + 1, \ldots, k'$, by (ii) of Section 3 from the cyclic projective plane $PG(2, q)$

| $q$ | $t$ | $d$ | $w^*_i$ | $c$ | $k'$ | $v'$ | $G(k')$ | $k^#$ | $G(k^#)$ |
|-----|-----|-----|---------|-----|------|------|--------|------|--------|
| 139 | 3   | 6487| 52, 39, 49 | 2   | 91   | 12974| 15085  | 86   | 13075  |
| 149 | 7   | 3193| 12, 25, 21, 25, 21, 21 | 6   | 117  | 19158| 25035  | 104  | 19163  |
| 149 | 7   | 3193| 12, 25, 21, 25, 21, 21 | 5   | 96   | 15965| 16243  | 96   | 16243  |
| 121 | 3   | 4921| 48, 37, 37    | 2   | 85   | 9842 | 12821  | 75   | 9965   |
| 121 | 7   | 2109| 21, 20, 13, 13, 21, 13, 21 | 6   | 86   | 12654| 13075  | 85   | 12821  |
| 121 | 37  | 399 | 14, 3, 36     | 20  | 71   | 7980 | 8661   | 69   | 8291   |
| 151 | 3   | 7651| 57, 43, 52    | 2   | 100  | 15302| 17663  | 93   | 15453  |
| 151 | 7   | 3279| 32, 19, 19, 21, 19, 21, 21 | 6   | 127  | 19674| 28921  | 105  | 19769  |
| 151 | 7   | 3279| 32, 19, 19, 21, 19, 21, 21 | 5   | 108  | 16395| 20831  | 97   | 16715  |
| 151 | 7   | 3279| 32, 19, 19, 21, 19, 21, 21 | 4   | 89   | 13116| 13557  | 87   | 13417  |
| 157 | 3   | 8269| 61, 48, 49    | 2   | 109  | 16538| 21167  | 97   | 16715  |
| 163 | 3   | 8911| 63, 49, 52    | 2   | 112  | 17822| 27043  | 102  | 18437  |
| 163 | 7   | 3819| 32, 25, 25, 19, 25, 19, 19 | 6   | 127  | 22914| 28921  | 114  | 23529  |
| 163 | 7   | 3819| 32, 25, 25, 19, 25, 19, 19 | 5   | 108  | 19095| 20831  | 104  | 19163  |
| 169 | 3   | 9577| 64, 57, 49    | 2   | 113  | 19154| 22847  | 104  | 19163  |
| 179 | 7   | 4603| 24, 21, 21, 31, 21, 31, 31 | 6   | 129  | 27618| 30151  | 124  | 27897  |
| 181 | 3   | 10981| 67, 63, 52   | 2   | 119  | 21962| 25823  | 111  | 22217  |
Table 4.2. Parameters of configurations \( \nu'_n \) with BDC incidence matrices, \( \nu' < G(n) \), \( n = k^#, k^# + 1, \ldots, k' \), by (ii) and (iii) of Section 3 from the cyclic punctured affine plane \( AG(2, q) \)

| \( q \) | \( t \) | \( d \) | \( w_i^* \) | \( c \) | \( f \) | \( \nu' \) | \( G'(\nu') \) | \( k^# \) | \( G(k^#) \) |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 3 | 85 | 8, 4, 4 | 2 | 12 | 170 | 171 | 12 | 171 |
| 31 | 3 | 320 | 14, 9, 8 | 2 | 22 | 640 | 713 | 21 | 667 |
| 37 | 3 | 456 | 16, 9, 12 | 2 | 25 | 912 | 961 | 25 | 961 |
| 49 | 4 | 600 | 16, 12, 9, 12 | 3 | 34 | 1800 | 1877 | 34 | 1877 |
| 49 | 6 | 400 | 4, 9, 12, 8, 8, 8 | 5 | 36 | 2000 | 2011 | 36 | 2011 |
| 53 | 4 | 702 | 17, 12, 10, 14 | 3 | 37 | 2106 | 2199 | 37 | 2199 |
| 61 | 3 | 1240 | 25, 16, 20 | 2 | 41 | 2480 | 2611 | 39 | 2505 |
| 61 | 4 | 930 | 18, 18, 12, 13 | 3 | 42 | 2790 | 2795 | 42 | 2795 |
| 61 | 5 | 744 | 18, 9, 10, 12, 12 | 4 | 45 | 2976 | 3375 | 43 | 3015 |
| 61 | 6 | 620 | 15, 8, 8, 10, 8, 12 | 5 | 47 | 3100 | 3609 | 44 | 3193 |
| 64 | 9 | 455 | 0, 8, 8, 8, 8, 8, 8, 8 | 8 | 56 | 3640 | 5451 | 48 | 3775 |
| 64 | 9 | 455 | 0, 8, 8, 8, 8, 8, 8, 8 | 7 | 48 | 3185 | 3775 | 44 | 3193 |
| 67 | 3 | 1496 | 26, 24, 17 | 2 | 43 | 2992 | 3015 | 43 | 3015 |
| 71 | 5 | 1008 | 8, 14, 15, 16, 18 | 4 | 50 | 4032 | 4189 | 50 | 4189 |
| 73 | 3 | 1776 | 30, 21, 22 | 2 | 51 | 3552 | 4381 | 47 | 3609 |
| 79 | 3 | 2080 | 32, 25, 22 | 2 | 54 | 4160 | 4747 | 50 | 4189 |
| 79 | 6 | 1040 | 8, 14, 13, 14, 18, 12 | 5 | 56 | 5200 | 5451 | 56 | 5451 |
| 79 | 6 | 1040 | 18, 12, 8, 14, 13, 14 | 2 | 50 | 4160 | 4189 | 50 | 4189 |
| 81 | 4 | 1640 | 25, 20, 16, 20 | 3 | 57 | 4920 | 5547 | 55 | 5197 |
| 81 | 4 | 1640 | 25, 20, 16, 20 | 1 | 45 | 3280 | 3375 | 45 | 3375 |
| 81 | 8 | 820 | 16, 8, 8, 8, 9, 12, 8, 12 | 7 | 64 | 5740 | 7055 | 59 | 5823 |
| 81 | 8 | 820 | 16, 8, 8, 8, 9, 12, 8, 12 | 6 | 56 | 4920 | 5451 | 55 | 5187 |
| 83 | 8 | 861 | 6, 10, 10, 10, 15, 11, 10, 11 | 7 | 66 | 6027 | 7515 | 60 | 6039 |
| 83 | 8 | 861 | 6, 10, 10, 10, 15, 11, 10, 11 | 6 | 56 | 5166 | 5451 | 55 | 5187 |
| 89 | 4 | 1980 | 26, 25, 18, 20 | 3 | 62 | 5940 | 6431 | 60 | 6039 |
| 89 | 8 | 990 | 17, 8, 10, 12, 8, 10, 10, 14 | 7 | 65 | 6930 | 7187 | 64 | 7055 |
| 97 | 3 | 3136 | 37, 34, 26 | 2 | 63 | 6272 | 6783 | 62 | 6431 |
| 97 | 4 | 2352 | 29, 26, 20, 22 | 3 | 69 | 7056 | 8291 | 65 | 7187 |
| 97 | 6 | 1586 | 21, 20, 14, 16, 14, 12 | 5 | 69 | 7930 | 8291 | 69 | 8291 |
| 101 | 4 | 2550 | 30, 25, 20, 26 | 3 | 70 | 7650 | 8435 | 68 | 7913 |
| 101 | 4 | 2550 | 30, 25, 20, 26 | 1 | 55 | 5100 | 5197 | 55 | 5197 |
| 103 | 3 | 3536 | 41, 32, 30 | 2 | 71 | 7072 | 8661 | 65 | 7187 |
| 103 | 6 | 1768 | 24, 18, 14, 17, 14, 16 | 5 | 80 | 8840 | 11127 | 72 | 8947 |
| 103 | 6 | 1768 | 24, 18, 14, 17, 14, 16 | 4 | 66 | 7072 | 7515 | 65 | 7187 |
| 103 | 6 | 1768 | 24, 18, 14, 17, 14, 16 | 2 | 70 | 7072 | 8435 | 65 | 7187 |
| 107 | 8 | 1431 | 17, 15, 10, 13, 10, 12, 17, 13 | 7 | 77 | 10017 | 10409 | 76 | 10179 |
| 107 | 8 | 1431 | 17, 15, 10, 13, 10, 12, 17, 13 | 3 | 73 | 8586 | 9027 | 71 | 8661 |
Table 4.2 (continue). Parameters of configurations $v'_n$ with BDC incidence matrices, $v' < G(n)$, $n = k^#, k^# + 1, \ldots, k'$, by (ii) and (iii) of Section 3 from the cyclic punctured affine plane $AG(2, q)$.

| $q$ | $t$ | $d$ | $w^*_i$ | $c$ | $f$ | $k'$ | $v'$ | $G(k')$ | $k^#$ | $G(k^#)$ |
|-----|-----|-----|--------|-----|-----|------|------|---------|-------|----------|
| 109 | 3   | 3960| 42, 30, 37 | 2   | 72  | 7920 | 8947 | 69      | 8291  |
| 109 | 4   | 2970| 32, 26, 22, 29 | 3   | 76  | 8910 | 10179| 72      | 8947  |
| 109 | 6   | 1980| 12, 19, 24, 18, 18, 18 | 5   | 84  | 9900 | 12319| 75      | 9965  |
| 109 | 9   | 1320| 8, 14, 20, 10, 13, 10, 12, 10, 12 | 8   | 78  | 10560| 10599| 78      | 10599 |
| 113 | 4   | 3192| 32, 32, 24, 25 | 3   | 80  | 9576 | 11127| 75      | 9965  |
| 113 | 7   | 1824| 10, 20, 14, 17, 22, 16, 14 | 6   | 80  | 10944| 11127| 80      | 11127 |
| 121 | 4   | 3660| 36, 30, 25, 30 | 3   | 86  | 10980| 13075| 80      | 11127 |
| 121 | 4   | 3660| 36, 30, 25, 30 | 3   | 86  | 10980| 13075| 80      | 11127 |
| 121 | 5   | 2928| 16, 24, 28, 25, 28 | 4   | 88  | 11712| 13491| 83      | 12041 |
| 125 | 4   | 3906| 37, 32, 26, 30 | 3   | 89  | 11718| 13557| 83      | 12041 |
| 125 | 8   | 1953| 24, 16, 14, 15, 13, 16, 12, 15 | 7   | 96  | 13671| 16243| 90      | 13935 |
| 125 | 8   | 1953| 24, 16, 14, 15, 13, 16, 12, 15 | 3   | 93  | 11718| 15453| 83      | 12041 |
| 127 | 3   | 5376| 49, 36, 42 | 2   | 85  | 10752| 12821| 79      | 10817 |
| 127 | 6   | 2688| 28, 18, 18, 21, 18, 24 | 5   | 100 | 13440| 17663| 88      | 13491 |
| 127 | 6   | 2688| 28, 18, 18, 21, 18, 24 | 4   | 82  | 10752| 11629| 79      | 10817 |
| 131 | 5   | 3432| 19, 24, 32, 26, 30 | 4   | 91  | 13728| 15085| 90      | 13935 |
| 137 | 4   | 4692| 40, 32, 29, 36 | 3   | 98  | 14076| 16925| 91      | 15085 |
| 139 | 3   | 6440| 54, 41, 44 | 2   | 95  | 12880| 15935| 86      | 13075 |
| 149 | 4   | 5550| 42, 41, 32, 34 | 3   | 106 | 16650| 20271| 97      | 16715 |
| 151 | 3   | 7600| 58, 44, 49 | 2   | 102 | 15200| 18437| 92      | 15235 |
| 151 | 6   | 3800| 18, 24, 32, 26, 25, 26 | 2   | 93  | 15200| 15453| 93      | 15453 |
### 8.2 Tables for Section [6]

**Table 6.1.** Values of $v$ for which a cyclic symmetric configuration $v_k$ exists, $5 \leq k \leq 51$, $P(k) \leq v \leq G(k) - 1$

| $k$ | $P(k)$ | $v_k(k)$ | $v_k(k) \leq v \leq G(k) - 1$ | $E_c(k)$ | $G(k)$ | filling |
|-----|--------|----------|-------------------------------|----------|--------|---------|
| 5.  | 21     | 21.      | $21_a, 22_e$                  | 23.      | 23     | 100%    |
| 6.  | 31     | 31.      | $31_a, 32_a, 33, 34_e$        | 35.      | 35     | 100%    |
| 7.  | 43     | 48.      |                                | 48.      | 51     | 100%    |
| 8.  | 57     | 57.      | $57_a, 58_a, 59_f - 62_f, 63_a, 64 - 68$ | 63.      | 69     | 100%    |
| 9.  | 73     | 73.      | $73_a, 74_b, 75_f - 79_f, 80_b, 81_f - 84_f, 85 - 88$ | 85.      | 89     | 100%    |
| 10. | 91     | 91.      | $91_a, 92_a, 93_f - 106_f, 107 - 109, 110_c$ | 107.     | 111    | 100%    |
| 11. | 111    | 120.     | $120_b, 121_f - 132_f, 133_a, 134_f, 135 - 144$ | 135.     | 145    | 100%    |
| 12. | 133    | 133.     | $133_a, 134_a, 135 - 155_e, 156_c, 157_f, 158_e, 159_f - 160_f, 168_b, 161 - 170$ | 161.     | 171    | 100%    |
| 13. | 157    | 168.     | $168_b, 169_f - 172_f, 183_a, 174_f - 179_f, 193 - 212$ | 193.     | 213    | 100%    |
| 14. | 183    | 183.     | $183_a, 184_a, 185_f - 224_f, 225 - 254$ | 225.     | 255    | 100%    |
| 15. | 211    | 255.     | $255_b, 256_f - 260_f, 263_c, 272_c, 273_a, 288_b, 267 - 302$ | 267      | 303    | 95%     |
| 16. | 241    | 255.     | $255_b, 272_c, 273_a, 288_b, 307_a, 313, 317, 318, 320 - 354$ | 320      | 355    | 43%     |
| 17. | 273    | 273.     | $273_a, 274_a, 288_b, 307_a, 342_c, 343, 349, 353, 360_b, 381_a, 356 - 398$ | 356      | 399    | 42%     |
| 18. | 307    | 307.     | $307_a, 342_a, 360_b, 381_a, 389, 391, 395 - 398, 401, 403 - 432$ | 403      | 433    | 32%     |
| 19. | 343    | $\geq 345_{s,t}$ | $360_b, 381_a, 445, 450, 453, 455 - 458, 460 - 492$ | 460      | 493    | 29%     |
| 20. | 381    | 381.     | $381_a, 382, 482, 497, 498, 501 - 503, 506_c, 505 - 509, 528_b, 553_a, 511 - 566$ | 511      | 567    | 37%     |
| 21. | 421    | $\geq 423_{s,t}$ | $506_c, 528_b, 553_a, 586, 589, 591, 592, 594, 595, 597, 598, 624_b, 600 - 666$ | 600      | 667    | 32%     |
| 22. | 463    | $\geq 465_{b,r}$ | $506_c, 528_b, 553_a, 624_b, 633, 637, 640 - 642, 651_a, 644 - 712$ | 644      | 713    | 33%     |

Key to Table 6.1: $a \rightarrow (2.1)$, $b \rightarrow (2.2)$, $c \rightarrow (2.3)$, $br \rightarrow$ Bruck-Ryser Theorem, $s \rightarrow [11]$, $t \rightarrow$ Theorem [6.1]
Table 6.1 (continue 1) Values of $v$ for which a cyclic symmetric configuration $v_k$ exists, $5 \leq k \leq 51$, $P(k) \leq v < G(k)$

| $k$ | $P(k)$ | $v_0(k)$ | $v_0(k) \leq v \leq G(k) - 1$ | $E_c(k)$ | $G(k)$ |
|-----|--------|----------|--------------------------------|----------|--------|
| 23  | 507    | $\geq 509_{br,t}$ | $528_b, 553_a, 624_b, 651_a, 683, 686 - 688, 692, 695 - 700, 728_b, 702 - 744$ | 702      | 745    |
| 24  | 553    | 553.     | $553_a, 557_{t,1}, 624_b, 651_a, 728_b, 738, 739, 742, 747 - 749, 752, 753, 755, 757_{a,1}$, $757 - 850$ | 757      | 851    |
| 25  | 601    | $\geq 602_s$ | $624_a, 651_a, 728_b, 757_a, 812_c, 830, 840_b, 871_a, 930_c, 960_b, 837 - 960$ | 837      | 961    |
| 26  | 651    | 651.     | $651_a, 657_{t,1}, 728_b, 757_a, 812_c, 840_b, 871_a, 885, 888, 895, 900, 903, 905 - 907, 910 - 913, 915 - 917, 919 - 925, 927, 930_c, 960_b, 929 - 984$ | 929      | 985    |
| 27  | 703    | $\geq 704_s$ | $728_b, 757_a, 812_c, 840_b, 871_a, 930_c, 960_b, 970, 971, 972, 975, 977, 978, 985, 987, 988, 991, 993_a, 993 - 997, 1000, 1001, 1003 - 1015, 1023_b, 1057_a, 1017 - 1106$ | 1017     | 1107   |
| 28  | 757    | 757.     | $757_a, 758_b, 812_c, 840_b, 871_a, 930_c, 960_b, 993_a, 1006, 1023_b, 1045, 1051, 1053, 1057_a, 1063 - 1067, 1070 - 1972, 1074, 1075, 1077, 1079 - 1170$ | 1079     | 1171   |
| 29  | 813    | $\geq 815_{s,t}$ | $840_b, 871_a, 930_c, 960_b, 993_a, 1023_b, 1057_a, 1091, 1127, 1135, 1137, 1141, 1143, 1145, 1146, 1151 - 1246$ | 1151     | 1247   |
| 30  | 871    | 871.     | $871_a, 877_{t,1}, 930_c, 960_b, 993_a, 1023_b, 1057_a, 1196, 1198 - 1201, 1206, 1207, 1216, 1217, 1219 - 1224, 1332_c, 1226 - 1360$ | 1226     | 1361   |
| 31  | 931    | $\geq 933_{br,t}$ | $960_b, 993_a, 1023_b, 1057_a, 1298, 1309, 1314, 1315, 1320, 1321, 1324, 1325, 1332_c, 1330 - 1335, 1339 - 1346, 1368_b, 1348 - 1494$ | 1348     | 1495   |
| 32  | 993    | 993.     | $993_a, 997_{t,1}, 1023_b, 1057_a, 1332_c, 1366, 1368_b, 1383, 1388, 1391 - 1395, 1397, 1398, 1400, 1401, 1403, 1407_a, 1406 - 1409, 1411 - 1414, 1416, 1420, 1421, 1424 - 1434, 1436, 1436 - 1568$ | 1436     | 1569   |

Key to Table 6.1: $a \rightarrow (2,1)$, $b \rightarrow (2,2)$, $c \rightarrow (2,3)$, $br \rightarrow$ Bruck-Ryser Theorem, $s \rightarrow [11]$, $t \rightarrow$ Theorem 6.1.
Table 6.1 (continue 2) Values of $v$ for which a cyclic symmetric configuration $v_k$ exists, $5 \leq k \leq 51$, $P(k) \leq v < G(k)$

| $k$ | $P(k)$ | $v_{\delta}(k)$ | $v_{\delta}(k) \leq v \leq G(k) - 1$ | $E_{\circ}(k)$ | $G(k)$ |
|-----|--------|-----------------|--------------------------------|--------------|--------|
| 33  | 1057   | 1057.           | 1057, 1058, 1332, 1368b, 1407a, 1492, 1506, 1507, 1515, 1518, 1520, 1521, 1528, 1529, 1533, 1535, 1537, 1540, 1542, 1543, 1545, 1547 – 1553, 1555 – 1559, 1640c, 1680b, 1561 – 1718 | 1561         | 1719   |
| 34  | 1123   | $\geq 1125_{br,t}$ | 1332c, 1368b, 1407a, 1640c, 1664, 1665, 1670, 1676, 1680a, 1686, 1693, 1698, 1699, 1700, 1705, 1708 – 1712, 1714, 1717, 1721, 1723a, 1723 – 1726, 1728, 1730 – 1742, 1744 – 1752, 1806c, 1848b, 1754 – 1876 | 1754         | 1877   |
| 35  | 1191   | $\geq 1193_{s,t}$ | 1332c, 1368b, 1407a, 1640c, 1680b, 1723a, 1777, 1781, 1783, 1788, 1792, 1793, 1795, 1798, 1800 – 1803, 1806c, 1805 – 1807, 1810, 1812 – 1815, 1848b, 1893a, 1817 – 1974 | 1817         | 1975   |
| 36  | 1261   | $\geq 1262_s$  | 1332c, 1368b, 1407a, 1640c, 1680b, 1723a, 1806c, 1848b, 1853, 1855, 1860, 1867 – 1870, 1872 – 1876, 1878, 1882 – 1884, 1893a, 1886 – 2010 | 1886         | 2011   |
| 37  | 1333   | $\geq 1335_{s,t}$ | 1368b, 1407a, 1640c, 1680b, 1723a, 1806c, 1848b, 1892, 1893a, 1910, 1922, 1930, 1934, 1938, 1943, 1944, 1947 – 1953, 1957, 1959, 1960, 1962, 1963, 1965 – 1967, 1969, 2162c, 1972 – 2198 | 1972         | 2199   |
| 38  | 1407   | 1407.           | 1407a, 1640c, 1680b, 1723a, 1806c, 1848b, 1893a, 2059, 2061, 2073, 2088, 2089, 2092, 2094, 2096, 2097, 2099 – 2101, 2103, 2105 – 2108, 2110, 2111, 2114 – 2116, 2118, 2123, 2124, 2126 – 2130, 2135 – 2153, 2155 – 2157, 2162c, 2159 – 2164, 2166 – 2170, 2208b, 2257a, 2172 – 2292 | 2172         | 2293   |

Key to Table 6.1: $a \rightarrow$ (2.1), $b \rightarrow$ (2.2), $c \rightarrow$ (2.3), $br \rightarrow$ Bruck-Ryser Theorem, $s \rightarrow$ [11], $t \rightarrow$ Theorem 6.1

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Table 6.1 (continue 3) Values of $v$ for which a cyclic symmetric configuration $v_k$ exists, $5 \leq k \leq 51$, $P(k) \leq v < G(k)$

| $k$ | $P(k)$ | $v_k$ | $v_k \leq v \leq G(k) - 1$ | $E_c(k)$ | $G(k)$ |
|-----|--------|-------|----------------------------|----------|--------|
| 39  | 1483   | $\geq 1485_{br,t}$ | $1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2187, 2195, 2208_b, 2240, 2241, 2243, 2247, 2248, 2251, 2252, 2254, 2255, 2257_a, 2258 - 2260, 2263 - 2265, 2269, 2270, 2277 - 2279, 2281, 2283, 2287, 2289 - 2309, 2311, 2313 - 2328, 2400_b, 2451_a, 2330 - 2504$ | 2330 | 2505 |
| 40  | 1561   | $\geq 1563_{s,t}$ | $1640_b, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, 2326, 2338, 2345, 2349, 2353, 2355, 2357, 2360, 2361, 2363, 2364, 2366 - 2368, 2370, 2372, 2374 - 2377, 2379 - 2381, 2387 - 2389, 2393, 2395 - 2397, 2399, 2400_b, 2401, 2403, 2404, 2406, 2407, 2409, 2411 - 2418, 2451_a, 2420 - 2436, 2438 - 2564$ | 2438 | 2565 |
| 41  | 1641   | $\geq 1643_{s,t}$ | $1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, 2345, 2399, 2400_b, 2436, 2449, 2451_a, 2459, 2460, 2465, 2471, 2472, 2479, 2480, 2483, 2485, 2491, 2493, 2494, 2496 - 2500, 2502, 2503, 2505, 2507 - 2513, 2515 - 2525, 2528 - 2540, 2542, 2544 - 2610$ | 2544 | 2611 |
| 42  | 1723   | $1723_a, 1723_t$ | $1723_a, 1723_t, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, 2400_b, 2451_a, 2510, 2522, 2539, 2541, 2557 - 2559, 2562, 2564, 2566 - 2568, 2570, 2573, 2577, 2578, 2580 - 2584, 2586 - 2590, 2593, 2595, 2597 - 2601, 2603 - 2610, 2612, 2613, 2615 - 2626, 2756, 2628 - 2794$ | 2628 | 2795 |

Key to Table 6.1: $a \rightarrow (2.1), b \rightarrow (2.2), c \rightarrow (2.3), br \rightarrow$ Bruck-Ryser Theorem, $s \rightarrow \text{[11]}, t \rightarrow$ Theorem 6.1
Table 6.1 (continue 4) Values of $v$ for which a cyclic symmetric configuration $v_k$ exists, $5 \leq k \leq 51$, $P(k) \leq v < G(k)$

| $k$ | $P(k)$ | $v_{a}(k)$ | $v_{b}(k) \leq v \leq G(k) - 1$ | $E_{c}(k)$ | $G(k)$ |
|-----|--------|------------|---------------------------------|-------------|--------|
| 43  | 1807   | $\geq 1809_{br,t}$ | 1848$_b$, 1893$_a$, 2162$_c$, 2208$_b$, 2257$_a$, 2400$_b$, 2451$_a$, 2684, 2686, 2688, 2715, 2725, 2728, 2734, 2737, 2739, 2744, 2752, 2756$_c$, 2757, 2759, 2762, 2763, 2766 – 2768, 2771, 2772, 2776, 2777, 2783, 2786 – 2789, 2791, 2792, 2794 – 2798, 2800, 2801, 2808$_b$, 2803 – 2811, 2813 – 2815, 2817 – 2858, 2863$_a$, 2860 – 3014 | 2860 | 3015 |
| 44  | 1893   | 1893,      | 1893$_a$, 1894$_t$, 2162$_c$, 2208$_b$, 2257$_a$, 2400$_b$, 2451$_a$, 2756$_c$, 2808$_b$, 2811, 2821, 2826, 2834, 2836, 2844, 2848, 2849, 2861, 2862, 2863$_a$, 2865 – 2867, 2870, 2871, 2873 – 2875, 2879 – 2881, 2884, 2887, 2890 – 2895, 2898, 2899, 2901 – 2912, 2914, 2916 – 3192 | 2916 | 3193 |
| 45  | 1981   | $\geq 1983_{a,t}$ | 2162$_c$, 2208$_b$, 2257$_a$, 2400$_b$, 2451$_a$, 2756$_c$, 2808$_b$, 2863$_a$, 2994, 3013, 3014, 3019, 3038, 3052, 3054, 3056, 3066, 3069, 3082 – 3085, 3087, 3088, 3090, 3091, 3093, 3095 – 3098, 3101 – 3103, 3105 – 3112, 3114, 3116 – 3120, 3122 – 3163, 3165 – 3374 | 3165 | 3375 |
| 46  | 2071   | $\geq 2073_{a,t}$ | 2162$_c$, 2208$_b$, 2257$_a$, 2400$_b$, 2451$_a$, 2756$_c$, 2808$_b$, 2863$_a$, 3124, 3171, 3188, 3191, 3194, 3196, 3197, 3198, 3206, 3216, 3218, 3219, 3221, 3227, 3228, 3231, 3233, 3234, 3238 – 3241, 3244 – 3247, 3249, 3250, 3252 – 3265, 3267 – 3271, 3273 – 3278, 3280 – 3406 | 3280 | 3407 |

Key to Table 6.1: $a \rightarrow (2.1)$, $b \rightarrow (2.2)$, $c \rightarrow (2.3)$, $br \rightarrow$Bruck-Ryser Theorem, $s \rightarrow [11]$, $t \rightarrow$Theorem 6.1
Table 6.1 (continue 5) Values of \( v \) for which a cyclic symmetric configuration \( v_k \) exists, 
\[ 5 \leq k \leq 51, \quad P(k) \leq v < G(k) \]

| \( k \) | \( P(k) \) | \( v_k \) | \( v(k) \leq v \leq G(k) - 1 \) | \( E_c(k) \) | \( G(k) \) |
|-------|--------|---|-----------------|-----|-----|
| 47    | 2163   | 2165_{br,t} | 2208, 2257, 2400, 2451, 2756, 2808, 2863, 3255, 3261, 3271, 3280, 3285, 3292, 3301, 3312, 3327, 3331, 3342, 3343, 3346 – 3348, 3351, 3353, 3355 – 3357, 3360 – 3371, 3375 – 3379, 3381 – 3384, 3387, 3389, 3390, 3392 – 3402, 3404 – 3408, 3412, 3413, 3422, 3415 – 3427, 3480, 3514, 3429 – 3608 | 3429 | 3609 |
| 48    | 2257   | 2257, 2258 | 2257, 2258, 2400, 2451, 2756, 2808, 2863, 3418, 3422, 3431, 3445, 3459, 3480, 3487, 3491, 3492, 3495, 3499, 3509, 3512, 3515, 3518, 3519, 3522, 3523, 3526 – 3528, 3535, 3540, 3541, 3545 – 3547, 3549 – 3552, 3554 – 3563, 3565, 3567, 3568, 3570, 3572 – 3578, 3580 – 3583, 3586, 3587, 3589 – 3595, 3597 – 3605, 3607 – 3618, 3620 – 3630, 3632 – 3640, 3660, 3642 – 3774 | 3642 | 3775 |
| 49    | 2353   | \geq 2354_s | 2400, 2451, 2756, 2808, 2863, 3422, 3480, 3541, 3608, 3627, 3637, 3640, 3642, 3644, 3646, 3647, 3649, 3652, 3653, 3655, 3660, 3665, 3669, 3671, 3675, 3677, 3678, 3680 – 3683, 3685, 3688 – 3692, 3694 – 3701, 3707, 3709, 3720, 3711 – 3722, 3724 – 3730, 3732 – 3735, 3737 – 3740, 3742, 3743, 3745, 3747 – 3755, 3783, 3757 – 3825, 3827 – 3837, 3839 – 3916 | 3839 | 3917 |

Key to Table 6.1: \( a \rightarrow (2.1), \ b \rightarrow (2.2), \ c \rightarrow (2.3), \) \( br \rightarrow \) Bruck-Ryser Theorem, \( s \rightarrow [11], \) \( t \rightarrow \) Theorem 6.1
Table 6.1 (continue 6). Values of \( v \) for which a cyclic symmetric configuration \( v_k \) exists, \( 5 \leq k \leq 51 \), \( P(k) \leq v < G(k) \)

| \( k \) | \( P(k) \) | \( v_3(k) \) | \( v_3(k) \leq v \leq G(k) - 1 \) | \( E_c(k) \) | \( G(k) \) |
|---|---|---|---|---|---|
| 50 | 2451 | 2451 | 2451\(_a\), 2452\(_b\), 2756\(_c\), 2808\(_b\), 2863\(_a\), 3422\(_c\), 3480\(_b\), 3541\(_a\), 3660\(_b\), 3685, 3688, 3692, 3712, 3714, 3716, 3720\(_b\), 3726, 3743, 3745, 3749, 3750, 3752, 3753, 3758, 3762, 3766, 3767, 3769, 3770, 3772, 3775, 3779, 3780, 3783\(_a\), 3782 - 3785, 3788 - 3791, 3796, 3803, 3805, 3811, 3813, 3817 - 3823, 3825, 3828 - 3832, 3834 - 3840, 3842, 3843, 3845, 3846, 3848 - 3869, 4095\(_b\), 4161\(_a\), 3871 - 4188 | 3871 | 4189 |
| 51 | 2551 | 2552\(_a\) | 2756\(_c\), 2808\(_b\), 2863\(_a\), 3422\(_c\), 3480\(_b\), 3541\(_a\), 3660\(_c\), 3720\(_b\), 3783\(_a\), 3871, 3894, 3927, 3938, 3986, 3998, 4004, 4018, 4032, 4035, 4037, 4042, 4048 - 4050, 4053, 4060, 4064 - 4066, 4068, 4072, 4076, 4079 - 4083, 4085, 4087, 4090, 4091, 4093, 4095\(_b\), 4095 - 4100, 4102 - 4104, 4107 - 4109, 4112 - 4120, 4124, 4125, 4127 - 4131, 4133 - 4156, 4161\(_a\), 4158 - 4164, 4166 - 4196, 4199, 4200, 4202 - 4206, 4208 - 4231, 4233 - 4380 | 4233 | 4381 |

Key to Table 6.1: \( a \rightarrow (2.1) \), \( b \rightarrow (2.2) \), \( c \rightarrow (2.3) \), \( br \rightarrow \text{Bruck-Ryser Theorem} \), \( s \rightarrow (11) \), \( t \rightarrow \text{Theorem 6.1} \)

Table 6.2. Upper bounds on the cyclic existence bound \( E_c(k) \), \( 52 \leq k \leq 83 \)

| \( k \) | \( E_c(k) \) | \( G(k) \) | \( k \) | \( E_c(k) \) | \( G(k) \) | \( k \) | \( E_c(k) \) | \( G(k) \) | \( k \) | \( E_c(k) \) | \( G(k) \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 52 | 4359 | 4541 | 60 | 5687 | 6039 | 68 | 7463 | 7913 | 76 | 10023 | 10179 |
| 53 | 4463 | 4695 | 61 | 5994 | 6269 | 69 | 8111 | 8291 | 77 | 10229 | 10409 |
| 54 | 4513 | 4747 | 62 | 6150 | 6431 | 70 | 8125 | 8435 | 78 | 10395 | 10599 |
| 55 | 5195 | 5197 | 63 | 6611 | 6783 | 71 | 8288 | 8661 | 79 | 10800 | 10817 |
| 56 | 5341 | 5451 | 64 | 6796 | 7055 | 72 | 8694 | 8947 | 80 | 10977 | 11127 |
| 57 | 5501 | 5547 | 65 | 6853 | 7187 | 73 | 8813 | 9027 | 81 | 11396 | 11435 |
| 58 | 5551 | 5703 | 66 | 7279 | 7515 | 74 | 8965 | 9507 | 82 | 11443 | 11629 |
| 59 | 5612 | 5823 | 67 | 7359 | 7639 | 75 | 9883 | 9965 | 83 | 11593 | 12041 |
| $k$ | $v$ | the 1-st row of incidence matrix |
|-----|-----|----------------------------------|
| 17  | 382 | 0,25,69,81,88,89,112,123,126,128,141,174,196,202,206,223,232 |
| 17  | 383 | 0,5,15,34,35,42,73,75,86,89,98,134,151,155,177,183,201 |
| 17  | 385 | 0,68,70,84,90,107,111,120,139,151,185,186,193,196,211,244,249 |
| 17  | 386 | 0,10,27,35,50,74,94,103,105,108,146,159,165,191,195,207,228 |
| 18  | 387 | 0,1,4,15,35,42,75,85,94,111,133,139,141,157,162,194,206,219 |
| 18  | 388 | 0,71,73,81,85,88,117,118,141,167,172,183,192,210,231,244,250,272 |
| 18  | 389 | 0,79,89,106,107,114,129,153,173,182,184,187,225,238,244,270,274,286 |
| 18  | 390 | 0,36,50,56,59,74,78,85,122,139,147,149,179,180,192,226,231,247 |
| 18  | 391 | 0,18,30,32,71,84,90,93,119,127,152,169,176,192,196,197,207,243 |
| 18  | 392 | 0,71,94,104,136,160,164,176,195,217,238,243,256,263,290,292,293,301 |
| 18  | 393 | 0,2,10,14,17,46,47,70,96,101,112,121,139,160,173,179,201,236 |
| 18  | 394 | 0,2,10,22,53,56,82,83,89,98,130,148,153,167,188,192,205,216 |
| 18  | 395 | 0,49,59,62,97,99,117,141,173,180,184,192,201,206,207,237,253,278 |
| 18  | 396 | 0,14,15,27,50,53,60,81,97,115,137,139,145,156,188,208,213,217 |
| 18  | 397 | 0,91,193,195,203,215,246,249,275,276,282,291,323,341,346,360,381,385 |
| 18  | 398 | 0,36,68,103,110,111,121,124,130,161,176,200,202,225,230,247,259,263 |
| 18  | 399 | 0,93,195,197,205,217,248,251,277,278,284,293,325,343,348,362,383,387 |
### 8.3 Tables for Section 7

**Table 7.1.** The currently known parameters of symmetric configurations \( v_k \) (cyclic and non-cyclic)

| \( k \) | \( P(k) \) | \( P(k) \leq v \leq G(k) - 1 \) | \( E(k) \leq G(k) \) filling |
|---|---|---|---|
| 3 | 7 | 7 | 7, 7 100% |
| 4 | 13 | 13 | 13, 13 100% |
| 5 | 21 | \( 21_a, 22_a \) | 23, 23 100% |
| 6 | 31 | \( 31_a, 32_b, 33, 34 \) | 34, 35 100% |
| 7 | 43 | \( 73_a, 74_t, 45_b, 48_b, 49_m, 50_m \) | 48, 51 75% |
| 8 | 57 | \( 73_a, 74_t, 63_b, 64_m - 68_m \) | 63, 69 67% |
| 9 | 73 | \( 73_a, 74_t, 78_b, 80_b, 81_m - 88_m \) | 80, 89 75% |
| 10 | 91 | \( 91_a, 92_t, 98, 107_y - 109_y, 110_c, k \) | 107, 111 35% |
| 11 | 111 | \( T71, T72, 120_b, 121_m - 133_m, 135_y - 142_y, 143_m, S, 144_m, g \) | 135, 145 76% |
| 12 | 133 | \( 33_a, 33_t, 135, 154, 155, 156_m - 169_m, S, T, X \) | 154, 171 52% |
| 13 | 157 | \( 75_b, 168_b, 169_m - 183_m, 189, 193_y - 209_y, 210_m, 212_m \) | 193, 213 68% |
| 14 | 183 | \( 183_a, 174_t, 210_g, 222_l, 223_l, 224_m, 225_m, 254_m \) | 222, 255 50% |
| 15 | 211 | \( 211_b, 272_t, 231_g, 238_l, 239_l, 240_m - 266_m, 267_m - 302_m \) | 238, 303 73% |
| 16 | 241 | \( 252_b, 255_b, 256_m - 321_m, 322_l - 345_m \) | 255, 355 89% |
| 17 | 273 | \( 273_a, 274_t, 288_b, 289_m - 307_m, 321_l, 322_l, 323_m, 324_g, 381_m, 382_z - 388_z \) | 321, 399 79% |
| 18 | 307 | \( 307_a, 340_l, 341_l, 342_m, 389_z, 391_z, 395_z - 398_z, 401_z, 403_g \) | 403, 433 63% |

Key to Table 7.1: \( a \rightarrow \{21\}, b \rightarrow \{22\}, c \rightarrow \{23\}, f \rightarrow \{25\}, g \rightarrow \{26\}, h \rightarrow \{27\}, j \rightarrow \{28\}, k \rightarrow \{29\}, \lambda \rightarrow \{213\} - \{216\}, m \rightarrow \{51\}, P \rightarrow \{43\}, r \rightarrow \{54\}, S \rightarrow \{55\}, T \rightarrow \{56\}, W \rightarrow \) Table 4.1, \( X \rightarrow \) Table 4.2, \( y \rightarrow \) Table 6.1 with \( k \leq 15 \), \( Z \rightarrow \) Table 6.1 with \( k > 15 \), \( br \rightarrow \) Bruck-Ryser Theorem, \( t \rightarrow \) Theorem 6.1
Table 7.1 (continue 1). The currently known parameters of symmetric configurations $v_k$ (cyclic and non-cyclic)

| $k$ | $P(k)$ | $P(k) \leq v \leq G(k) - 1$ | $E(k)$ | filling |
|-----|--------|-----------------------------|--------|---------|
| 19  | 343    | $344_{4t}, 360_{bfr}, 361_{m-}, 381_{m-}, 434_{gW}, 435\lambda, 436\lambda, 437_{mS} - 457_{mS}, 458\lambda r T, 459\lambda r T$ | 434 | 493 | 54% |
| 20  | 381    | $381_a, 382_t, 458\lambda, 459\lambda, 469_W, 460_{m-}, 482_{r T}$ | 458 | 567 | 59% |
| 21  | 421    | $422_t, 481\lambda, 482_t, 558_W, 558_{m-} - 666_{m-}$ | 481 | 667 | 76% |
| 22  | 463    | $463_{br}, 464_t, 504\lambda, 505_{m-}, 558_{W}, 506_{m-}, 573_{m-}, 574\lambda r, 640_X, 575_{m-} - 712_{m-}$ | 504 | 713 | 84% |
| 23  | 507    | $507_{br}, 508_t, 528_{f r}, 529_{m-}, 558_{g W}, 573\lambda, 574_{m-}, 575_{m-} - 744_{m-}$ | 573 | 745 | 84% |
| 24  | 553    | $553_{a}, 554_t, 589_{g}, 589\lambda, 599\lambda, 600_{m-} - 673_{m-}, 674_{m-}, 675_{m-} - 850_{m-}$ | 598 | 851 | 85% |
| 25  | 601    | $620_{gh}, 624_{f r}, 625_{m-}, 651_{m-}, 673\lambda, 674_X, 675_{m-}, 906_W, 912_X, 938_W, 676_{m-} - 960_{m-}$ | 673 | 961 | 88% |
| 26  | 651    | $651_{a}, 652_t, 700_{\lambda}, 701_{\lambda}, 702_{m-} - 781_{m-}, 782_{r T}, 783_{m-} - 984_{m-}$ | 700 | 985 | 85% |
| 27  | 703    | $728_{br}, 729_{m-} - 757_{m-}, 781_{\lambda}, 782_{\lambda}, 783_{mS}, 784_{m-}, 1065_{m-}, 1066_{r T Z}, 1072_{r T Z}$, $1073_{mS Z}, 1103_{mS Z}, 1104_{r T Z}, 1105_{r T Z}, 1110_{r T Z}$, $1106_{r T Z}$ | 781 | 1107 | 88% |
| 28  | 757    | $757_{a}, 758_t, 810_{\lambda}, 811_{\lambda}, 812_{m-} - 1065_{m-}, 1066_{r T}, 1072_{r T}, 1073_{mS}, 1103_{mS}$, $1104_{r T Z}, 1109_{r T Z}, 1110_{mS Z}, 1114_{mS Z}, 1114_{r T Z}$, $1142_{r T Z}, 1145_{r T Z}, 1146_{r T Z}$, $1147_{mS Z}, 1170_{mS Z}$ | 810 | 1171 | 87% |

Key to Table 7.1: $a \rightarrow (2.1)$, $b \rightarrow (2.2)$, $c \rightarrow (2.3)$, $f \rightarrow (2.5)$, $g \rightarrow (2.6)$, $h \rightarrow (2.7)$, $j \rightarrow (2.8)$, $k \rightarrow (2.9)$, $\lambda \rightarrow (2.13)$, $m \rightarrow (5.1)$, $P \rightarrow (4.3)$, $r \rightarrow (5.4)$, $S \rightarrow (5.5)$, $T \rightarrow (5.6)$, $W \rightarrow$ Table 4.1, $X \rightarrow$ Table 4.2, $y \rightarrow$ Table 6.1 with $k \leq 15$, $Z \rightarrow$ Table 6.1 with $k > 15$, $br \rightarrow$ Bruck-Ryser Theorem, $t \rightarrow$ Theorem 6.1.
Table 7.1 (continue 2). The currently known parameters of symmetric configurations \( v_k \) (cyclic and non-cyclic)

| \( k \) | \( P(k) \) | \( P(k) \leq v \leq G(k) - 1 \) | \( E(k) \leq G(k) \) filling |
|---|---|---|---|
| 29 | 813 | \( 871_q, 872_q, 840_{bf{r}}, 841_{mr}, 871_{m}, 897_\lambda, 898_\lambda, 899_{ms}, 900_{mr} - 1057_{mr}, 1071_\lambda, 1072_\lambda, 1073_{ms} - 1103_{ms}, 1104_{r-T}, 1109_{r-T}, 1110_{ms} - 1141_{ms}, 1142_{r-T} - 1146_{r-T}, 1147_{ms} - 1179_{ms}, 1180_{r-T}, 1183_{r-T}, 1184_{ms} - 1219_{ms}, 1220_{r-T}, 1221_{ms} - 1246_{ms} \) | 1071 1247 85% |
| 30 | 871 | \( 871_a, 872_a, 928_\lambda, 929_\lambda, 930_{mr} - 1057_{mr}, 1108_\lambda, 1109_\lambda, 1110_{ms} - 1141_{ms}, 1142_{r-T} - 1146_{r-T}, 1147_{ms} - 1179_{ms}, 1180_{r-T} - 1183_{r-T}, 1184_{ms} - 1217_{ms}, 1218_{r-T} - 1220_{r-T}, 1262_{W}, 1221_{ms} - 1360_{ms} \) | 1108 1361 78% |
| 31 | 931 | \( 931_{br}, 932_t, 960_{bf{r}}, 961_{mr} - 1057_{mr}, 1145_\lambda, 1146_\lambda, 1147_{ms} - 1179_{ms}, 1180_{r-T} - 1183_{r-T}, 1184_{ms} - 1217_{ms}, 1218_{r-T} - 1220_{r-T}, 1221_{ms} - 1255_{ms}, 1256_{r-T}, 1257_{r-T}, 1262_{W}, 1258_{ms} - 1494_{ms} \) | 1145 1495 79% |
| 32 | 993 | \( 993_a, 994_t, 1023_{bf{r}}, 1024_{mr} - 1057_{mr}, 1182_\lambda, 1183_\lambda, 1184_{ms} - 1217_{ms}, 1218_{r-T} - 1220_{r-T}, 1221_{ms} - 1255_{ms}, 1256_{r-T}, 1257_{r-T}, 1258_{ms} - 1293_{ms}, 1294_{r-T}, 1533_{W}, 1295_{ms} - 1568_{ms} \) | 1182 1569 73% |
| 33 | 1057 | \( 1057_a, 1058_t, 1219_\lambda, 1220_\lambda, 1221_{ms} - 1255_{ms}, 1256_{r-T}, 1257_{r-T}, 1258_{ms} - 1293_{ms}, 1294_{r-T}, 1634_{W}, 1295_{mr} - 1718_{mr} \) | 1219 1719 75% |
| 34 | 1123 | \( TT2_{br}, TT2_{tl}, 1256_\lambda, 1257_\lambda, 1258_{ms} - 1293_{ms}, 1294_{r-T}, 1295_{mr} - 1429_{mr}, 1430_{r-T} - 1434_{r-T}, 1634_{W}, 1800_\lambda, 1435_{ms} - 1876_{ms} \) | 1256 1877 82% |

Key to Table 7.1: \( a \rightarrow (2.1), b \rightarrow (2.2), c \rightarrow (2.3), f \rightarrow (2.5), g \rightarrow (2.6), h \rightarrow (2.7), j \rightarrow (2.8), k \rightarrow (2.9), \lambda \rightarrow (2.13) - (2.16), m \rightarrow (5.1), P \rightarrow (4.3), r \rightarrow (5.4), S \rightarrow (5.5), T \rightarrow (5.6), W \rightarrow Table 4.1, X \rightarrow Table 4.2, y \rightarrow Table 6.1 with \( k \leq 15, Z \rightarrow Table 6.1 with k > 15, br \rightarrow Bruck-Ryser Theorem, t \rightarrow Theorem 6.1 \)
Table 7.1 (continue 3). The currently known parameters of symmetric configurations \( v_k \) (cyclic and non-cyclic)

| \( k \) | \( P(k) \) | \( P(k) \leq v \leq G(k) - 1 \) | \( E(k) \) | \( G(k) \) | filling |
|-------|---------|---------------------------------|--------|--------|--------|
| 35    | 1191    | \( T793, 1293, 1294, 1295m-S, \) & 1433 \( \leq \) 1975 \( 83\% \)
|       |         | \( 1472, m-r \leq 1724, 1476m-S \) & \( 1723 \) |
| 36    | 1261    | \( 1330, 1331, 1332m-r \leq 1474 \), \( 1475, 1476m-S \leq 1514m-T \) & 1474 \( 2011 \) \( 82\% \)
|       |         | \( 1517m-S \leq 1519m-S, 2000 \leq 1520m-r \leq 1525 \) |
| 37    | 1333    | \( T564, 1368b, 1369m-r \leq 1474 \), \( 1515, 1516 \leq 1555m-S \) & 1515 \( 2199 \) \( 83\% \)
|       |         | \( 1556m-r \leq 1557m-r \leq 1558m-r \leq 2198m-r \) |
| 38    | 1407    | \( 1407, 1556, 1557, 1558m-S, 1559m-S \) & 1556 \( 2293 \) \( 83\% \)
|       |         | \( 1560m-r \leq 1597m-r \leq 1598m-r \) |
| 39    | 1483    | \( T483, 1509 \leq 1598 \leq 1599m-S \) & 1597 \( 2505 \) \( 89\% \)
|       |         | \( 1600m-r \leq 1601m-r \leq 1762m-r \) |
| 40    | 1561    | \( T564, 1638, 1639, 1640m-r \leq 1761m-r \), \( 1762m-r \leq 1921m-r \) & 1638 \( 2565 \) \( 92\% \)
|       |         | \( 1927m-r \leq 1931m-r \leq 1932m-r \leq 2564m-r \) |
| 41    | 1641    | \( T642, 1680b, 1681m-r \leq 1723m-r \) & 1925 \( 2611 \) \( 92\% \)
|       |         | \( 1763m-r \leq 1764m-r \leq 1893m-r \leq 1925m-r \) |
| 42    | 1723    | \( 1723, 1724, 1804 \leq 1805 \leq 1806m-r \) & 1972 \( 2795 \) \( 85\% \)
|       |         | \( 1893m-r \leq 1972m-r \leq 1973m-r \) |
| 43    | 1807    | \( T808, 1848b, 1849m-r \leq 1893m-r \) & 2019 \( 3015 \) \( 86\% \)
|       |         | \( 2020m-r \leq 2065m-S \leq 2066m-r \leq 2067m-r \) |
|       |         | \( 2491m-S \leq 2593m-S \leq 2594m-r \leq 2595m-r \) |

Key to Table 7.1: \( a \rightarrow \) Table 4.1, \( b \rightarrow \) Table 4.2, \( c \rightarrow \) Table 4.3, \( f \rightarrow \) Table 6.1, \( g \rightarrow \) Table 6.2, \( h \rightarrow \) Table 6.3, \( j \rightarrow \) Table 6.4, \( k \rightarrow \) Table 6.5, \( \lambda \rightarrow \) Table 6.6, \( m \rightarrow \) Table 6.7, \( P \rightarrow \) Table 6.8, \( r \rightarrow \) Table 6.9, \( S \rightarrow \) Table 6.10, \( T \rightarrow \) Table 6.11, \( W \rightarrow \) Table 4.4, \( X \rightarrow \) Table 4.5, \( Y \rightarrow \) Table 6.11 with \( k \leq 15 \), \( Z \rightarrow \) Table 6.11 with \( k > 15 \), \( br \rightarrow \) Bruck-Ryser Theorem, \( t \rightarrow \) Theorem 6.1
Table 7.1 (continue 4). The currently known parameters of symmetric configurations $v_k$ (cyclic and non-cyclic)

| $k$ | $P(k)$ | $P(k) \leq v \leq G(k) - 1$ | $E(k) \leq G(k)$ | filling |
|-----|--------|-------------------------------|------------------|--------|
| 44  | 1893   | 1893, $1894$, 2066, 2067, 2068, 2113, 2301, 2302, 2303, 2485 | 2066 3193 | 86% |
| 45  | 1981   | 1982, 2113, 2114, 2115, 2116, 2301, 2302, 2303, 2485, 2486, 2490, 2540, 2543, 2544, 2594, 2596, 2647, 2648, 2649, 2650 | 2113 3375 | 90% |
| 46  | 2071   | 2072, 2160, 2161, 2162, 2301, 2302, 2303, 2485, 2486, 2490, 2540, 2543, 2544, 2594, 2596, 2647, 2648, 2649, 2650, 2701, 2702 | 2160 3407 | 93% |
| 47  | 2163   | 2164, 2208, 2209, 2301, 2302, 2303, 2451, 2489, 2490, 2491, 2540, 2543, 2544, 2594, 2596, 2647, 2648, 2649, 2650 | 2489 3609 | 91% |

Key to Table 7.1: $a \rightarrow [2,1], b \rightarrow [2,2], c \rightarrow [2,3], f \rightarrow [2,5], g \rightarrow [2,6], h \rightarrow [2,7], j \rightarrow [2,8], k \rightarrow [2,9], \lambda \rightarrow [1,3] [2,13], m \rightarrow [5,1], P \rightarrow [1,3], r \rightarrow [5,4], S \rightarrow [5,5], T \rightarrow [5,6], W \rightarrow $ Table 4.1, $X \rightarrow $ Table 4.2, $y \rightarrow $ Table 6.1 with $k \leq 15$, $Z \rightarrow $ Table 6.1 with $k > 15$, $br \rightarrow $ Bruck-Ryser Theorem, $t \rightarrow $ Theorem 6.1
Table 7.1 (continue 5). The currently known parameters of symmetric configurations \(v_k\) (cyclic and non-cyclic)

| \(k\) | \(P(k)\) | \(P(k) \leq v \leq G(k) - 1\) | \(E(k)\) | \(G(k)\) | filling |
|-------|---------|-------------------------------|---------|---------|---------|
| 48    | 2257    | \(2257_a, 2258_t, 2350 \lambda, 2351 \lambda, 2352_m \cdot r - 2451_m \cdot r,\) \(2542 \lambda, 2543 \lambda, 2544_m \cdot S - 2593_m \cdot S,\) \(2594_r \cdot T - 2596_r \cdot T, 2597_m \cdot S - 2647_m \cdot S,\) \(2648_r \cdot T - 2649_r \cdot T, 2650_m \cdot r \cdot S - 2701_m \cdot r \cdot S \cdot T,\) \(2702_r \cdot T, 2703_m \cdot r - 2881_m \cdot r,\) \(2882_r \cdot T - 2890_r \cdot T, 2891_m \cdot S - 3774_m \cdot S\) | 2542 | 3775 | 88% |
| 49    | 2353    | \(2400_b \cdot r, 2401_m \cdot r - 2451_m \cdot r, 2595 \lambda, 2596 \lambda,\) \(2597_m \cdot S - 2647_m \cdot S, 2648_r \cdot T - 2649_r \cdot T,\) \(2650_m \cdot r \cdot S - 2701_m \cdot r \cdot S \cdot T, 2702_r \cdot T,\) \(2703_m \cdot r - 2863_m \cdot r, 2889 \lambda, 2890 \lambda,\) \(2891_m \cdot S - 2941_m \cdot S, 2942_r \cdot T - 2949_r \cdot T,\) \(2950_m \cdot S - 3916_m \cdot S\) | 2889 | 3917 | 86% |
| 50    | 2451    | \(2451_a, 2452_t, 2648 \lambda, 2649 \lambda, 2650_m \cdot S - 2701_m \cdot S,\) \(2702_r \cdot T, 2703_m \cdot r - 2863_m \cdot r, 2948 \lambda, 2949 \lambda,\) \(2950_m \cdot S - 3001_m \cdot S, 3002_r \cdot T - 3008_r \cdot T,\) \(3009_m \cdot S - 4188_m \cdot S\) | 2948 | 4189 | 83% |
| 51    | 2551    | \(2701 \lambda, 2702 \lambda, 2703_m \cdot S, 2704_m \cdot r - 2863_m \cdot r,\) \(3007 \lambda, 3008 \lambda, 3009_m \cdot S - 3061_m \cdot S,\) \(3062_r \cdot T - 3067_r \cdot T, 3068_m \cdot S - 4380_m \cdot S\) | 3007 | 4381 | 83% |

Key to Table 7.1: \(a \rightarrow (2.1), b \rightarrow (2.2), c \rightarrow (2.3), f \rightarrow (2.4), g \rightarrow (2.6), h \rightarrow (2.7),\) \(j \rightarrow (2.8), k \rightarrow (2.9), \lambda \rightarrow (2.13), m \rightarrow (5.1), P \rightarrow (4.3), r \rightarrow (5.4), S \rightarrow (5.5),\) \(T \rightarrow (5.6), W \rightarrow \text{Table 4.1}, X \rightarrow \text{Table 4.2}, y \rightarrow \text{Table 6.1} \text{with } k \leq 15, Z \rightarrow \text{Table 6.1} \text{with } k > 15, br \rightarrow \text{Bruck-Ryser Theorem}, t \rightarrow \text{Theorem 6.1} \)
Table 7.2. The currently known parameters of symmetric configurations $v_k$ (cyclic and non-cyclic). Constructions are not remarked.

| $k$ | $P(k)$ | $P(k) \leq v \leq G(k) - 1$ | $E(k) \leq G(k)$ | filling |
|-----|--------|-----------------------------|------------------|---------|
| 3.  | 7      | 7                           | 7, 7             | 100%    |
| 4.  | 13     | 13                          | 13, 13           | 100%    |
| 5.  | 21     | 21, 22                      | 23, 23           | 100%    |
| 6.  | 31     | 31, 32, 33, 34              | 34, 35           | 100%    |
| 7   | 43     | &overline{43, 44, 45, 48, 49, 50} | 48, 51          | 75%     |
| 8   | 57     | 57, 58, 63 − 68             | 63, 69           | 67%     |
| 9   | 73     | 73, &overline{77, 78, 80 − 88} | 80, 89           | 75%     |
| 10  | 91     | 91, &overline{92, 98, 107 − 110} | 107, 111        | 35%     |
| 11  | 111    | &overline{111, 112, 120 − 133, 135 − 144} | 135, 145        | 76%     |
| 12  | 133    | 133, &overline{134, 135, 154 − 170} | 154, 171        | 52%     |
| 13  | 157    | &overline{158, 168 − 183, 189, 193 − 212} | 193, 213        | 68%     |
| 14  | 183    | 183, &overline{184, 210, 222 − 254} | 222, 255        | 50%     |
| 15  | 211    | &overline{211, 212, 231, 238 − 302} | 238, 303        | 73%     |
| 16  | 241    | 252, 255 − 354              | 255, 355         | 89%     |
| 17  | 273    | 273, &overline{274, 288 − 307, 321 − 398} | 321, 399        | 79%     |
| 18  | 307    | 307, 340 − 381, 389, 391, 395 − 398, 401, 403 − 432 | 403, 433        | 63%     |
| 19  | 343    | &overline{344, 360 − 381, 434 − 492} | 434, 493        | 54%     |
| 20  | 381    | 381, &overline{382, 458 − 566} | 458, 567        | 59%     |
| 21  | 421    | &overline{422, 481 − 666}   | 481, 667        | 76%     |
| 22  | 463    | &overline{463, 464, 504 − 712} | 504, 713        | 84%     |
| 23  | 507    | &overline{507, 508, 528 − 553, 558, 573 − 744} | 573, 745        | 84%     |
| 24  | 553    | 553, &overline{554, 589, 598 − 850} | 598, 851        | 85%     |
| 25  | 601    | 620, 624 − 651, 673 − 960   | 673, 961        | 88%     |
| 26  | 651    | 651, &overline{652, 700 − 984} | 700, 985        | 85%     |
| 27  | 703    | 728 − 757, 781 − 1106       | 781, 1107       | 88%     |
| 28  | 757    | 757, &overline{758, 810 − 1170} | 810, 1171       | 87%     |
| 29  | 813    | &overline{814, 840 − 871, 897 − 1057, 1071 − 1246} | 1071, 1247      | 85%     |
| 30  | 871    | 871, &overline{872, 928 − 1057, 1108 − 1360} | 1108, 1361      | 78%     |
| 31  | 931    | &overline{931, 932, 960 − 1057, 1145 − 1494} | 1145, 1495      | 79%     |
| 32  | 993    | 993, &overline{994, 1023 − 1057, 1182 − 1568} | 1182, 1569      | 73%     |
Table 7.2 (continue). The currently known parameters of symmetric configurations $v_k$
(cyclic and non-cyclic). Constructions are not remarked.

| $k$ | $P(k)$ | $P(k) \leq v \leq G(k) - 1$ | $E(k) \leq G(k)$ | filling |
|-----|--------|----------------------------|----------------|---------|
| 33  | 1057   | 1057, $1058$, 1219 – 1718   | 1219           | 1719    | 75%    |
| 34  | 1123   | $1123$, $T_{124}$, 1256 – 1876 | 1256           | 1877    | 82%    |
| 35  | 1191   | $1192$, 1293 – 1407, 1433 – 1974 | 1433           | 1975    | 83%    |
| 36  | 1261   | 1330 – 1407, 1474 – 2010      | 1474           | 2011    | 82%    |
| 37  | 1333   | $T_{334}$, 1368 – 1407, 1515 – 2198 | 1515           | 2199    | 83%    |
| 38  | 1407   | 1407, 1556 – 2292             | 1556           | 2293    | 83%    |
| 39  | 1483   | $T_{483}$, $T_{484}$, 1597 – 2504 | 1597           | 2505    | 89%    |
| 40  | 1561   | $T_{562}$, 1638 – 2564        | 1638           | 2565    | 92%    |
| 41  | 1641   | $T_{642}$, 1680 – 1723, 1761 – 1893, 1925 – 2610 | 1925           | 2611    | 92%    |
| 42  | 1723   | $T_{724}$, 1804 – 1893, 1972 – 2794 | 1972           | 2795    | 85%    |
| 43  | 1807   | $T_{807}$, $T_{808}$, 1848 – 1893, 2019 – 3014 | 2019           | 3015    | 86%    |
| 44  | 1893   | 1893, $T_{894}$, 2066 – 3192  | 2066           | 3193    | 86%    |
| 45  | 1981   | $T_{982}$, 2113 – 3374        | 2113           | 3375    | 90%    |
| 46  | 2071   | $2072$, 2160 – 3446           | 2160           | 3407    | 93%    |
| 47  | 2163   | $2163$, $2164$, 2208 – 2257, 2301 – 2451, 2489 – 3608 | 2489           | 3609    | 91%    |
| 48  | 2257   | 2257, $2258$, 2350 – 2451, 2542 – 3774   | 2542           | 3775    | 88%    |
| 49  | 2353   | 2400 – 2451, 2595 – 2863, 2889 – 3916  | 2889           | 3917    | 86%    |
| 50  | 2451   | 2451, $2452$, 2648 – 2863, 2948 – 4188  | 2948           | 4189    | 83%    |
| 51  | 2551   | 2701 – 2863, 3007 – 4380       | 3007           | 4381    | 83%    |
| 52  | 2653   | $2654$, 2754 – 2863, 3066 – 4540 | 3066           | 4541    | 84%    |
| 53  | 2757   | $2758$, 2808 – 2863, 3125 – 4694 | 3125           | 4695    | 83%    |
| 54  | 2863   | $2863$, $2867$, 3184 – 4746    | 3184           | 4747    | 83%    |
| 55  | 2971   | $2971$, $2972$, 3243 – 5196    | 3243           | 5197    | 88%    |
| 56  | 3081   | $3082$, 3302 – 5450           | 3302           | 5451    | 91%    |
| 57  | 3193   | $3194$, 3361 – 5546           | 3361           | 5547    | 93%    |
| 58  | 3307   | $3307$, $3308$, 3420 – 5702    | 3420           | 5703    | 95%    |
| 59  | 3423   | $3424$, 3480 – 3541, 3597 – 5822 | 3597           | 5823    | 95%    |
| 60  | 3541   | $3542$, 3658 – 3783, 3838 – 6038 | 3838           | 6039    | 93%    |
| 61  | 3661   | $3662$, 3720 – 3783, 3902 – 6268 | 3902           | 6269    | 93%    |
| 62  | 3783   | 3783, 3966 – 6430             | 3966           | 6431    | 93%    |
| 63  | 3907   | $3907$, $3908$, 4030 – 4161, 4219 – 6782  | 4219           | 6783    | 94%    |
| 64  | 4033   | 4095 – 4161, 4286 – 7054       | 4286           | 7055    | 94%    |