Gravitational anomaly and fundamental forces

J. J. van der Bij
Insitut für Physik, Albert-Ludwigs Universität Freiburg
H. Herderstr. 3, 79104 Freiburg i.B., Deutschland

Abstract

I present an argument, based on the topology of the universe, why there are three generations of fermions. The argument implies a preferred unified gauge group of $SU(5)$, but with $SO(10)$ representations of the fermions. The breaking pattern $SU(5) \to SU(3) \times SU(2) \times U(1)$ is preferred over the pattern $SU(5) \to SU(4) \times U(1)$. On the basis of the argument one expects an asymmetry in the early universe microwave data, which might have been detected already.

1 Introduction

As the title of this contribution indicates I will be concerned with the fundamental forces of nature. As fundamental forces I consider the gauge forces of nature. Included are also the fermions that form representations of the gauge groups. I will in first instance not be concerned with the Higgs mechanism and the breaking of the gauge symmetries. The world described therefore consists in this picture of massless fields only. One can legitimately ask why one would study such a system and why it should be a subject for a school on Quantum Gravity and Quantum Geometry. There is a twofold reason why such a study could be interesting, one coming from particle physics phenomenology, the other being related to quantum gravity.

When one considers the status of particle physics phenomenology it is generally stated that the standard model gives an excellent description of the data, but that the model is unsatisfactory with respect to the Higgs sector. Emphasis is put on the so-called naturalness or hierarchy problem, which consists of the statement that the Higgs mass receives quadratically divergent contributions, so that the Higgs mass should be of the order of the cut-off scale for which there is no evidence in the data. Great efforts are subsequently made to solve this hierarchy problem. One introduces various extensions like supersymmetry, technicolor, higher dimensions and so forth. All these extensions have problems with the data and one has to finely tune many parameters or invoke dynamical accidents in order not to get into conflict with experiment. I think it is
fair to say that this approach has lead to an impasse. One should therefore question the basic assumption that the most important subject to study is the symmetry-breaking of the theory. At the moment we have no clue why the symmetry of nature is described by the gauge groups that are seen in experiment, in particular also why there are exactly three generations. Maybe one should first understand this question why we have the symmetry that we see in experiment, before one should try to understand its breaking. The structure of the symmetry might involve deeper principles, that ultimately will be reflected in the symmetry breaking too. At our present state of knowledge worrying about the hierarchy problem is then simply premature.

A second reason to study this system can be found in the status of quantum gravity. At present there are two leading proposals, or rather classes of proposals, that might lead to a fundamental theory of gravity, string theory and canonical gravity. Most of the effort in the last decades has been spent on string theory. The reason is apparently, that string theory promises to be not only a theory of gravity alone, but would also include matter fields. Thereby one would arrive at a fully unified theory of all forces and in the best of all worlds a rather unique one. With the arrival of the landscape the uniqueness has disappeared and thereby string theory cannot explain why the gauge forces are the way they are. Unification within the context of canonical quantum gravity has received little interest, because at first sight there is no connection. However this is where the term anomaly in the title comes in. There may be conditions on representations of gauge fields coming from anomalies, that could play a role. If these restrictions are strong enough one could argue that canonical quantum gravity is the way to a unification of forces.

So basically we therefore want to see whether there is a uniqueness to the fundamental forces and particles that exist in nature. Is there a reason for the gauge group and representations of nature? In particular we would like to understand why there are three generations of fermions. These are rather old questions. The generation question was asked by I. Rabi after the discovery of the muon: "Who ordered that?" The question about the uniqueness of nature’s laws was paraphrased by A. Einstein: "Did God have a choice when he created the world?" The argument that I present in the following will give tentative answers to these questions.

2 Ancient history

As is clear from the above we are trying to derive the fundamental forces and particles of nature from first principles. This is a rather ambitious project and one might be accused of hubris and look silly in the end. In order to try to prevent this from happening we will look at similar attempts in the past and see if we might learn from these. We know of three attempts. The first takes us to the ancient Greeks. The earliest Greek philosophers had a problem regarding nature, that is hard to understand for modern people. The problem had to do with the question of motion and the unity of nature. It was related to the
concepts of emptiness and fullness. If the world is empty there is nothing to move. If the world is full there is no place to move. Still motion exists. The solution was constructed by Leukippos and Demokritos. They split reality in two separate pieces, on the one hand space which is empty, on the other hand the atoms that are full. Movement was then explained as atoms moving through empty space. Such atoms we would nowadays probably call molecules. A second line of research was developed by Plato, Empedokles and Aristoteles, probably under the influence of Pythagorean thinking, which introduced mathematics in the description of nature. It was concerned with the elements, which we would consider the basic building blocks of the atoms. One considered four elements and associated regular bodies to them: fire (tetrahedron), air (octahedron), water (icosahedron) and earth (cube). There was some obvious phenomenology associated to this, for instance fire has to be a tetrahedron, because it has sharp points, so one burns oneself. This theory has a nice mathematical structure based on symmetry and actually made a prediction that there should be a fifth element corresponding to the dodekahedron, known as quintessence. It was variously identified as the ether, the soul and so forth. Present theories of quintessence are roughly at the same level. With hindsight we might consider this all to be nonsensical, but that is wrong. This was one of the most succesful theories for physics and chemistry, being the leading theory for about a thousand years. It did not disappear, because it was manifest nonsense, which it is not, but because of alchemical experiments and experiences in medicine, which made the theory untenable.

3 Modern history

We now skip a few thousand years and ask: where do we stand? Of course we understand better what physical laws are, basically differential equations, but at the philosophical level regarding the basic structure of reality things are still rather similar. We now have spacetime instead of space, which is not such a great leap philosophically speaking. More important however is that spacetime is not anymore static, it is a dynamical quantity, that is related to matter through the Einstein equations. Thereby the split of reality into two parts, space and atoms, has been partly closed.

Regarding the structure of matter things are very similar. Symmetry still determines the elements. Instead of Platonic solids, determined by symmetry, we now have gauge groups and representations that are determined by symmetry. In a sense we have made a step backwards, since there appear to be an infinite number of possibilities for the groups and representations. The only constraints are the ones coming from the known (chiral) anomalies.

Given the fact that the Einstein equations connect matter and spacetime it is a natural attempt in unification to connect the two via a deeper principle. One takes the largest, most symmetric (beautiful) theory one can find and postulates that nature should be described by this theory. In order to do this one normally has to assume a unification of the gauge forces, the simplest being the unification
An early attempt to derive \( SU(5) \) was made in the context of \( N = 8 \) supergravity \cite{2}. In order to derive a model that approximates nature, a number of dynamical assumptions regarding composite states had to be made. With the realization that \( N = 8 \) supergravity is not a finite theory and therefore not suitable as a fundamental theory for all forces, attempts along this direction have largely stopped.

A somewhat later attempt was to use string theory in the form of the heterotic string, which implies a gauge group \( E(8) \times E(8) \) \cite{3}. An argument similar to the one we present later was used. The group is selected by the absence of an anomaly, in this case the conformal anomaly on the world sheet. Subsequently reducing the group to something closer to the standard model appeared possible, but rather complicated. The number of fermion generations is determined by topological considerations. With the realization, that string theory allows for many vacua with different gauge groups, the idea of a unique group has been abandoned in this approach. These two attempts are similar in that they are very ambitious. The assumption is that one determines the unique form of fundamental dynamics from a given mathematical structure, which should subsequently contain the observed forces of nature.

What do we learn from these earlier attempts? The first thing we should learn is some modesty. Assuming that one knows the fundamental laws of physics and only has to construct the standard model out of these, is not a very promising approach. Mankind is not smart enough for that. Therefore we have to use experimental information. Furthermore we found that anomalies are important. As a related point also topology appears to play a role. Fortunately there are new results since 1985, in phenomenology, cosmology and mathematical physics, that under a suitable interpretation might help us to move forward. Of course this involves a certain guesswork, regarding which features are important.

### 4 Phenomenology

If one wants to construct a fundamental theory of nature, the idea of unification appears necessary. The standard model based on the gauge group \( SU(3) \times SU(2) \times U(1) \) with its complicated set of fermions, constrained by the anomaly is too peculiar to be fundamental. The situation is simply asking for a unification into a larger group. It is well known that such a unification is possible and quite natural. When one considers just the gauge bosons of theory the simplest form of unification is within the gauge group \( SU(5) \). Important in this respect is that the rank of the group is four. Even though the symmetry has to be broken, one would in first instance expect the rank of the broken group to be the same as of the unbroken group. The question is therefore if there is any evidence in the data for a higher rank of the gauge group.

This question has been studied at the LEP experiments. Various analyses are possible. The upshot is that there is no convincing evidence for the existence of extra \( Z' \)'s or \( W' \)'s. Therefore the known vector bosons point towards a unification...
within a group of rank four, namely \( SU(5) \) [4].

In the fermion sector the situation is somewhat different. At the latest with the discovery of neutrino masses, it has become clear that the natural unification for the fermions is within the group \( SO(10) \) [5], since each generation forms an irreducible spinor representation of \( SO(10) \). Moreover the spinor representation of \( SO(10) \) is somewhat special, since it is the smallest triangle anomaly free complex representation in all of group theory [6].

So naively speaking the vector bosons and the fermions point toward a different form of unification. Of course the situation can be described through the breaking of the symmetry with a number of Higgs fields, but one would hope for a more fundamental explanation for this feature. As \( SU(5) \) is a subgroup of \( SO(10) \) there ought to be an obstruction to gauging the full group \( SO(10) \).

Another fact of phenomenology is the existence of precisely three generations of fermions. This is established from the invisible decay width of the \( Z \) boson and by the precision measurements at LEP. It is natural to wonder whether the group question \( SU(5) \) versus \( SO(10) \) is related to the question of the number of generations. So we have to explain: why \( SU(5) \) for vectorbosons, why \( SO(10) \) for fermions, why 3 generations? We are therefore looking for an argument to constrain the representation content and the gauge group of the theory. The only type of argument known that can give such constraints is based on some form of an anomaly. As anomalies are intimately related to topology, one is led to the question: what is the topology of space?

5 Topology in cosmology

The question of the topology of the universe has many aspects. It has been studied in great detail in Kaluza-Klein theories, where a typical assumption is of the form \( M_4 \times S^1 \), a torus shape. The higher dimensions form circles, of which one hopes that the radii shrink to zero as the universe develops. It has appeared rather difficult to realize this picture in an attractive way in practice. We therefore take the opposite point of view and assume that the universe is lower dimensional, specifically the spatial part is supposed to be two-dimensional instead of three-dimensional. At first sight this statement appears to be obvious nonsense and would earn the author a prize from the flat earth society, if not further qualified. Of course what is meant here is that the universe is two-dimensional at the origin of the universe and the third dimension becomes large only at late times. The idea is that the universe has the topology \( M_3 \times S_1 \), where the radius of the circle shrinks to zero when one goes back in time. The question is whether such a behaviour is possible. This is where new information from cosmology comes in. The information that we need is that the universe is flat and that it has a positive cosmological constant.

Is there also direct information on cosmic topology? In principle yes, since in a topologically nontrivial universe there should be multiple images of objects in the sky. In practice this is difficult and one tries to use the cosmic microwave...
background (circles in the sky). There is no convincing evidence that topology is present. So should we therefore ignore this possibility? This is where the flatness of the universe comes in. In a curved space there is a maximal scale where topology should appear, in a flat space there is none. It could in principle be beyond the range that we can measure. Flat and possibly anisotropic spaces are called Bianchi-I type universes. Given the fact that non-trivial topology is possible in a flat space, the next question is whether the desirable dynamical behaviour is possible or likely. Namely the early universe should be highly anisotropic, one dimension being much smaller than the others, but at late times the universe should become isotropic as seen today. This is where the positive cosmological constant comes in. Late time isotropisation of the universe is actually a generic behaviour for universes with a positive cosmological constant \[7\]. The first example of such a universe is the Kasner solution \[8\]. At small times the solution looks like

\[ ds^2 \approx -dt^2 + dx^2 + dy^2 + t^2 dz^2 \]

Therefore the third dimension gets compactified to zero at early times. If this were the full solution at all times, it would be Minkowski space. One can embed this solution in a full cosmology, where then at late times one finds isotropy.

Therefore we suggest that the topology of the universe is \( M_3 \times S_1 \). The radius of \( S_1 \) may be too large to see the topology at the present time. However a remnant of the topology is the existence of a preferred direction in the universe, that might be visible. Indeed there appears to be an alignment of low multipoles along a preferred axis in the CMB data \[9\]. Apparently this can be explained in an inflationary Bianchi-I universe \[10, 11\].

6 The argument

With these arguments I hope I have convinced you that three-dimensional dynamics may be important for the four-dimensional world. The question is what does three dimensional physics look like. As an example one can take Yang-Mills theory. Beyond the ordinary Yang-Mills term in the Lagrangian, with a gauge coupling \( g \) there is a second parity violating Chern Simons term with a mass \( m \) \[12–16\]. The theory describes massive vector bosons with only one polarization. The theory is invariant under small gauge transformations, but the action gets shifted by a constant for large (topologically non-trivial) transformations. Since the Lagrangian action appears with an imaginary part in the path-integral, one will only have a full gauge invariance when there is a quantization condition. One finds, that the Yang-Mills charge \( q_{YM} = \frac{4\pi m}{g^2} \) must be an integer. What makes the quantization condition particularly interesting is that the Yang-Mills charge gets renormalized \[17\]. There are corrections to the Yang-Mills charge through fermion and vectorboson loops. As a consequence one can derive the condition, that there must be an even number of fermions in the fundamental representation \[18–21\]. For the connection with four-dimensional physics see \[22, 23\].
For the case of Yang-Mills and Maxwell fields the theory is well known in the literature. Somewhat less well known is that also for gravitational fields a similar argument exists. Only a few papers dealing with induced Chern-Simons terms exist in the literature. In three-dimensional gravity there exists the Einstein Lagrangian and a Chern-Simons term. Also here there is a quantization condition for the gravitational Chern-Simons charge.

The gravitational action in three dimensions contains two terms. One is the ordinary Einstein Lagrangian:

\[ L = -\frac{1}{\kappa^2} \sqrt{g} R \]  

where as usual, R is the curvature scalar, \( g_{\mu\nu} \) is the metric tensor, \( g \) the determinant of the metric and \( \kappa^2 \) is Newton’s constant. To this action a Chern-Simons term can be added:

\[ L_{CS} = -\frac{i}{4\kappa^2g} \epsilon^{\mu\nu\lambda}(R_{\mu\nu\lambda\beta} + \frac{2}{3} \omega^{\mu}_{\alpha\beta} \omega^{\nu}_{\beta\alpha} - \frac{2}{3} \omega^{\mu}_{\beta\alpha} \omega^{\nu}_{\alpha\beta}). \]

where

\[ R_{\mu\nu\lambda\beta} = \partial_{\mu}\omega_{\nu\lambda\beta} + \omega_{\mu\rho\lambda} \omega^{\rho}_{\nu\lambda\alpha} - (\mu \leftrightarrow \nu) \]  

is the curvature tensor and \( \omega_{\mu\nu\lambda} \) is the spin connection. The gravitational Chern-Simons charge

\[ q_{gr} = \frac{6\pi}{\mu\kappa^2} \]  

is quantized and has to be an integer. The presence of matter fields however, fermions and vector bosons with a Chern-Simons term, gives rise to an extra effective contribution to the Chern-Simons charge \( q_{gr} \).

\[ q_{gr}^{ren} = q_{gr} + \frac{1}{8} N_g \text{ sign}(m_g) - \frac{1}{16} N_f \text{ sign}(m_f) \]

where \( N_g \) is the number of vector bosons with topological mass \( m_g \) and \( N_f \) is the number of fermions of mass \( m_f \).

It is important that the corrections are only dependent on the sign of the mass and not its absolute value. This means that also at zero mass an effect is present. Within the purely three dimensional case one speaks therefore of a parity anomaly, since the basic tree level Lagrangian does not violate parity. Embedding the theory in four dimensions with a preferred direction it is easy to understand that the sign is important, since the sign of the mass in the Chern-Simons like term is fixed when one chooses an orientation for the coordinate basis vectors. We now assume that the fundamental gravitational laws have no preferred direction, implying \( q_{gr} = 0 \). The meaning of this condition is that the gravitational field equations are given by Einsteins equations, so that any early anisotropy comes from the initial conditions and not from the equations. The assumption is technically natural, since imposing it enhances the symmetry of the Lagrangian. The complete effective Chern-Simons term is then induced
by the matter fields. In this case the quantization condition gives rise to the following identity

\[ N_f \mp 2N_g = 0 \mod(16) \]  

(6)

whereby the minus sign is to be taken when the fermions and the bosons have the same sign of the mass. It is assumed that the fermions separately and the bosons separately have the same sign for the mass, which is a reasonable assumption when they are part of the same multiplets in a unified theory, since otherwise one would break the gauge symmetry. We see that the condition (6) is fulfilled for the vector bosons by themselves if the gauge group is \( SU(5) \), giving \( N_g = 24 \) and also for the fermions by themselves, when they are in the 16-dimensional spinor representation of \( SO(10) \). Moreover it is desirable that the effective renormalized gravitational Chern-Simons charge \( q_{\text{ren}}^{\text{gr}} = 0 \), since otherwise it is difficult to understand that the late universe is even approximately isotropic, because the gravitational field equations themselves would have a preferred direction. This condition is fulfilled if there are three generations of fermions \( 3 \times 16 - 2 \times 24 = 0 \).

Though at first sight these conditions look rather insignificant, they imply strong constraints, when one makes the assumptions that the fermions should be in an automatically triangle anomaly free group and that they should be in a fundamental representation. The solution appears to be essentially unique.

7 Discussion

It is to be remarked that the argument contains little speculation. On the cosmology part the Ansatz is more conservative than the standard Robertson-Walker Ansatz that implies isotropy always. The unification with the given \( SU(5) \) and \( SO(10) \) multiplets is the absolute minimum that is possible within grand unified theories. The gravitational anomaly argument is established mathematical physics.

On the speculative side one can wonder about symmetry breaking, other compactifications and extra conditions. Taking the anomaly conditions into account a breaking pattern \( SU(5) \to SU(3) \times SU(2) \times U(1) \) appears to be preferred. For this school probably the most relevant is the question into the relation with quantum geometry. From the argument it appears that a unique structure of gauge particles is needed in order to have gravity in the theory. The argument is basically topological and rather robust regarding the precise nature of quantum gravity. The argument works quite natural within the context of a quantization following the Einstein equations. There appears to be no need for string theory. Therefore maybe the canonical way towards quantum gravity is the right way. Naturally the results cry out for an underlying structure. I have not been able to come to a conclusion here. I made some attempts using octonions because of the factors of eight in the formulas, but found no convincing connection sofar. Finally we are in a position to give tentative answers to the following questions.
Rabi’s question: Who ordered that?
Answer: the early universe.

Einstein’s question: did God have a choice?
Answer: No, because He has to use perfect symmetry.

However the devil may have had something to do with the Higgs sector. Maybe one should therefore, when one mentions the Higgs sector, not talk about the God particle (L. Lederman), but about the devil’s field. More probably one should do neither.

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