Cross correlation of low-lying levels of even-even nuclei

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Abstract

We consider the cross correlations of low-lying energy levels of nuclei belonging to small intervals of the excitation-energy ratio $R_{4/2}$ of the first $4^+$ to $2^+$ states. The mean value of the cross-correlation coefficients, plotted as a function of $R_{4/2}$ has deep minima at values of $R_{4/2} = 2.2$ and 2.9, which correspond to the critical dynamical symmetries of the interacting Boson model. The distribution of the calculated coefficients for nuclei belonging to different $R_{4/2}$ classes has different pattern.
I. INTRODUCTION

Geometrical collective models provide a suitable framework for studying low lying levels of several even-even nuclei [1]. In particular, the interacting boson model (IBM) has been very successful in describing the collective behavior algebraically [2]. In the simplest version of this model, known as IBM-1, an even-even nucleus with \( n \) valence nucleons is treated as a system of \( n/2 \) bosons interacting via two-body force. The model has three dynamical symmetries, obtained by constructing the chains of subgroups of the \( U(6) \) group that end with the angular momentum group \( SO(3) \). The symmetries are labeled by the first subgroup appearing in the chain which are \( U(5) \), \( SU(3) \), and \( O(6) \) corresponding, respectively, to vibrational, rotational and \( \gamma \)-unstable nuclei. There are few nuclei close to these symmetry limits. The majority are in an earlier or later stage of the of a phase (or shape) transition between the dynamical symmetries. Recently, Iachello [3, 4] introduced new dynamical symmetries, labeled as \( E(5) \) and \( X(5) \) at the critical point of the \( U(5)-O(6) \) and \( U(5)-SU(3) \) phase transitions, respectively.

It is not \textit{a priori} clear to what extent can one distinguish the location of prevalence of the dynamical symmetries in the chart of nuclei. Most of the efforts are focused on comparing the level schemes of selected nuclei with the predictions of IBM. A variety of signatures have been employed in the identification of the symmetry properties of numerous individual nuclei that have large numbers of low-lying levels with definite spin-parity assignment (see, e.g. [5]). However, individual nuclei still remain individual. The need to associate the spectral properties of one nucleus with another calls for the definition of a model-independent parameter that enables one to understand the evolution of the collective nuclear structure. Among the proposed parameters are the product of the number of valence protons relative to the nearest closed shell and the valence neutron number, \( N_pN_n \), and the ratio of the excitation energies of the first 4\(^+\) and the first 2\(^+\) level in each nucleus, \( R_{4/2} \) [6].

During the past decades, a vast amount of nuclear spectroscopic data has been accumulated. Level schemes involving tens and sometimes hundreds of levels with reliably known values of spin and parity are now available for hundreds of nuclei (see Ref. [7]). The wealth of published spectroscopic data allows for an extensive study of the level statistics of nuclei at low excitation energies. Recent statistical analyses of level spacings of low–lying states with spin and parity 2\(^+\) [8] lead to more definitive and precise statements about regular-
ity versus chaos in this domain than has been possible so far. These nuclei are grouped into classes that have common collective behavior. The classes are defined in terms of the excitation-energy ratio $R_{4/2}$. The parameter that measures the degree of chaoticity has deep minima at $R_{4/2} = 2.0, 2.5, \text{ and } 3.3$. These minima correspond, respectively, to the $U(5)$, $SO(6)$, and $SU(3)$ dynamical symmetries of the IBM.

In the present paper, we add a new criterion to the existing set to identify the collective motion in nuclei. The low lying levels of collective nuclei are expressed as functions of the parameters of some simple model, e.g. IBM. If a group of nuclei belong to the same symmetry group, then levels of the same rank (e.g., the $n$th level of given spin-parity $J^\pi$) in these nuclei are strongly correlated, since their excitation energies can be evaluated by the same formula but with different values of the model parameters. In the case of IBM, where the level energies linearly depend on the parameters, one even expects a linear relationship between several levels of nuclei belonging to the same symmetry group. Cross correlation is a standard method for estimating the degree to which two series are linearly correlated. In the present case the series consists of the energies of a given excited state of the nuclei belonging to the class under investigation. The the cross-correlation coefficient (CCC) determines the extent to which the energies of the two levels are linearly related. In this work, we calculate CCC for each pair of the low lying levels of nuclei belonging to the same class of the $R_{4/2}$ ratio. If the nuclei of the $R_{4/2}$-class under investigation belongs to the same symmetry group, most of the level pairs will have their CCC’s close to 1 (or -1). On the other hand, if the nuclei fall in the domain of phase transition between the dynamical symmetries, one expects large fluctuations of the CCC’s around a small mean value. We describe the data in Section 2 and give a brief account of the $R_{4/2}$ classification in Section 3. Section 4 is devoted to the calculation of CCC. The conclusion of this work is outlined in Section 5.

II. DATA SET

The data on low–lying levels of even–even nuclei with spins ranging from 0 to 6 are taken from the compilation by Tilley et al. for mass numbers $16 \leq A \leq 20$, from that of Endt for $20 \leq A \leq 44$, and from the Nuclear Data Sheets for heavier nuclei. We considered within each nucleus those levels of spin–parity $J^\pi$, for which the spin–parity assignments of at least five consecutive levels are unambiguous. In cases, where
the spin-parity assignments were uncertain and where the most probable value appeared in brackets, we accepted this value. We terminated the sequence when we arrived at a level with unassigned $J^\pi$, or when an ambiguous assignment involved $J^\pi$ among several possibilities, as e.g. $(J^\pi = (2^+, 4^+))$. We made an exception when only one such level occurred and was followed by several unambiguously assigned levels containing at least two $J^\pi$ levels, provided that the ambiguous $J^\pi$ level is found in a similar position in the spectrum of a neighboring nucleus. However, this situation occurred for less than 5% of the levels considered. In this way, we obtained enough levels to obtain statistically significant results. In this way, we obtained an ensemble of 4177 levels of spin–parity $0^+, 1^+, 1^-, 2^+, 2^-, 3^+, 3^-, 4^+, 4^-, 5^+, 5^-$ and $6^+$ belonging to 478 nuclei. The composition of this ensemble is shown in Table 1 where the vacancies correspond to cases with number of levels less than three. The number of states involved to the set is large enough to justify the statistical analysis with few exceptions consisting mainly of states that have no counterpart in the standard collective models.

III. $R_{4/2}$ CLASSIFICATION

We grouped nuclei, involved in the data set described in Section 2, into classes. Within each class the ratio

$$R_{4/2} = \frac{E(4^+_1)}{E(2^+_1)}$$

(1)

of excitation energies of the first $4^+$ and the first $2^+$ excited states, must lie in a fixed interval. The width of the intervals was taken to be 0.1 when the total number of nuclei falling into the corresponding class was 20 or more. Otherwise, the width of the interval was increased (see Fig. 1). The use of the parameter (1) as an indicator of collective dynamics is justified both empirically and by theoretical arguments. We recall some of these arguments in turn.

(i) Casten et al. plotted $E(4^+_1)$ versus $E(2^+_1)$ for all nuclei with $38 \leq Z \leq 82$ and with $2.05 \leq R_{4/2} \leq 3.15$. The authors found that the data fall on a straight line. This suggests that nuclei in this wide range of $Z$–values behave like anharmonic vibrators with nearly constant anharmonicity. In a subsequent paper it was found that a linear relation between $E(4^+_1)$ and $E(2^+_1)$ holds for pre–collective nuclei with $R_{4/2} < 2$. Thus, from an empirical perspective, the dynamical structure of medium–weight and heavy nuclei can be quantified in terms of $R_{4/2}$. 
(ii) Theoretical calculations based on the IBM-1 model support the conclusion that $R_{4/2}$ is an appropriate measure for collectivity in nuclei. The IBM calculation of energy levels yields values of $R_{4/2} = 2.00, 3.33,$ and $2.50$ for the dynamical symmetries $U(5), SU(3),$ and $O(6)$, respectively. In this respect, a recent systematic analysis of the NNS distributions for $2^+$ levels of even–even nuclei aimed to determine the chaoticity parameter $f$ for nuclei at low excitation showed that this parameter has deep minima at these values of $R_{4/2}$. Thus, we may expect increased regularity that leads to an enhanced correlation between the energy levels of nuclei having any of the IBM dynamical symmetries.

(iii) Recently, Iachello has introduced a new class of symmetries that applies to nuclei undergoing a phase transition between the limiting symmetries of IBM. In particular, the "critical symmetry" $E(5)$ has been suggested to describe critical points in the phase transition from spherical to $\gamma$-unstable shapes while $X(5)$ describes systems lying at the critical point in the transition from spherical to axially deformed systems. Iachello described these symmetries using a Bohr-type geometric Hamiltonian with a flat-bottomed potential that allows for analytic solutions. The calculation of energy levels for this Hamiltonian yields $R_{4/2} = 2.20$ and $2.91$ for the critical symmetries $E(5)$ and $X(5)$, respectively. We expect increased fluctuation that leads to a reduce cross correlation between the energy levels of nuclei having $R_{4/2}$ near one of these values.

IV. CALCULATION OF CCC

The cross correlation coefficient is a measure of degree to which linear model may describe the relationship between two variables. It may take any value between -1 and +1. A positive CCC means that as the value of one variable increases, the value of the other variable increases; as one decreases the other decreases. A negative CCC indicates that as one variable increases, the other decreases, and vice-versa. Taking the absolute value of the correlation coefficient measures the strength of the relationship between the corresponding. Thus a CCC coefficient of zero ($C_{ij} = 0$, for variable labeled $i$ and $j$) indicates the absence of a linear relationship and CCC’s of $C_{ij} = \pm 1$ indicate a perfect linear relationship. A convenient way of summarizing a large number of correlation coefficients is to put them in a single table, called a correlation matrix. A correlation matrix is a table of all possible correlation coefficients between a set of variables. One would not need to calculate all
possible correlation coefficients, however, because the correlation of any variable with itself is necessarily 1.00. Thus the diagonals of the matrix need not be computed. In addition, CCC is non-directional. For this reason the correlation matrix is symmetrical around the diagonal.

In this section we analyze CCC’s for energy levels of the nuclei that belong to each of the seventeen \( R_{4/2} \) classes, which are described in the previous section and shown in Table 1. Collective models express level energies of low excited states in terms of formulae, which linearly depend on one or more the parameters. For example, the rotational model divides the levels into bands, within which the energy of a level with spin \( J \) equals \( E_J = (1/2I)\hbar^2J(J + 1) \) where \( I \) is the effective moment of inertia. One thus expects a linear relationship between several levels of collective nuclei belonging the same symmetry group. As a consequence, the mean value of the matrix elements of their correlation matrix to be close to one.

Consider an \( R_{4/2} \) class in which the spin-parity assignment of \( N_i \) nuclei allow the identification of the level with label \( i \) \((= J_i^\pi)\), which denotes the spin \( J \) and parity \( \pi \) of the level and its rank \( r \) of excitation (i.e., 1st, 2nd, etc) within the spin-parity group. Let \( E_i(\nu) \) be the excitation energy of the level \( i \) in a nucleus, labeled by \( \nu \), belonging to this class. Then, CCC is defined as

\[
C_{ij} = \frac{\sum_{\nu=1}^{N_i} [E_i(\nu) - \overline{E_i}] [E_j(\nu) - \overline{E_j}]}{\sqrt{\sum_{\nu=1}^{N_i} [E_i(\nu) - \overline{E_i}]^2} \sqrt{\sum_{\nu=1}^{N_j} [E_j(\nu) - \overline{E_j}]^2}},
\tag{2}
\]

where \( \overline{E_i} \) and \( \overline{E_j} \) are the mean values of energies of levels \( i \) and \( j \) of nuclei belonging to this class, e.g. \( \overline{E_i} = (1/N_i) \sum_{\nu=1}^{N_i} E_i(\nu) \). The sums run over the nuclei within the class and \( N = \min(N_i, N_j) \). We substitute for \( E_i(\nu) \) in Eq. (2) the values of the excitation energies of the levels listed in Table 1. Thus, for each \( R_{4/2} \)-class, we construct an \( 18 \times 18 \) correlation matrix having unit diagonal elements and off-diagonal elements calculated by using Eq. (2). We then calculate the mean value \( \langle C \rangle \) and standard deviation \( \sigma \) of the elements of each of these matrices. The result of calculation are shown in Fig. 1. Most of the mean values of the CCC’s are close to 1, which suggests that nuclei belonging to the same \( R_{4/2} \)-class indeed have similar structure. A particularly enhanced values of \( \langle C \rangle \) are observed in the classes of nuclei belonging to the regions of \( R_{4/2} = 1.95 - 2.05, R_{4/2} = 2.25 - 2.85 \) with a summit
at $R_{4/2} \approx 2.5$, and $R_{4/2} = 3.25 - 3.33$ that correspond to the $U(5)$, $SO(6)$, and $SU(3)$ dynamical symmetries. Another maximum is observed in the class of $R_{4/2} = 1.45 - 1.65$ which consists mainly of nuclei with a single closed shell. Two deep minima of $\langle C \rangle$ are observed in the intervals of $R_{4/2} = 2.05 - 2.15$ and $R_{4/2} = 2.85 - 2.95$. The histogram for $\sigma$ has maxima at these positions. These regions are nearly at the values $R_{4/2} = 2.20$ and 2.91 corresponding the critical symmetries $E(5)$ and $X(5)$, respectively. Another broad minimum occurs at $R_{4/2} \approx 1.8$, corresponding to a transition from non-collective (with single or double closed shells) to collective dynamics. The standard deviations are large for values of $R_{4/2}$, which correspond to these minima of $\langle C \rangle$. It is well known from thermodynamics that systems undergoing phase transitions suffer from large-scale fluctuations. Therefore, there is no wonder that the CCC has minima and its standard deviation has maxima at locations expected for the critical points of the phase transitions between dynamical symmetries.

Figures 2 and 3 show the probability density function $P(C_{ij})$ of CCC’s for the classes that respectively correspond to the maxima and minima of the histogram in Fig. 1. The four distributions shown in Fig. 2 are essentially different from those of Fig. 3. About 80% of the coefficients of the four classes in Fig. 2 are greater that 0.95 and practically none is less that 0.75. In contrast, the coefficients in Fig. 3 have wide distributions that extends deeply into the domain negative values. However, most of the coefficients still have large positive values, implying that positively-correlated behavior is considerably more prevalent that negatively-correlated (anti-correlated) behavior even in the transition between dynamical symmetries.

V. SUMMARY AND CONCLUSION

Since the proposal of IBM, several papers have been devoted to the identification of nuclei belonging to the $U(5)$, $SO(6)$, and $SU(3)$ dynamical symmetries. Several stringent criteria has been proposed for this purpose, which are mainly based on the comparison between the level schemes of selected nuclei and the predictions of IBM. A question has been raised concerning the definition of an empirical characteristic that allows to select candidate nuclei for testing dynamical symmetries and serve as a ”control parameter” for the related structural phase transitions. Among the proposed characteristics is the ratio $R_{4/2}$ of the excitation energies of the lowest $4^+$ and $2^+$ levels. Some of the argument in favor of classifying nuclei according to the $R_{4/2}$ are given in Section 3. We have shown in a previous
paper that the statistical analysis of energy levels of nuclei belonging to fixed intervals of $R_{4/2}$ may contribute to the selection. This analysis suggests that nuclei belonging to the $R_{4/2}$ domains expected for the three dynamical symmetries have less chaotic dynamics than other nuclei.

In the present paper, we have carried out a systematic analysis of the collective behavior of the low-lying states of even–even nuclei by calculating the CCC’s for pairs of levels taken from similar nuclei. As the measure of similarity we have taken the ratio $R_{4/2}$. As seen in Figure 1, the mean value $\langle C \rangle$ of the CCC’s is indeed dependent on $R_{4/2}$. It is largest at $R_{4/2} = 2.0, 2.5,$ and $3.3$. These maxima correspond, respectively, to the $U(5)$, $SO(6)$, and $SU(3)$ dynamical symmetries of the IBM as well as a plateau at CCC $\sim 0.95$ around $R_{4/2} = 1.5$. It has deep minima at $R_{4/2} = 2.10 \pm 0.05$ and $2.85 \pm 1.0$. These minima correspond, respectively, to the $E(5)$ and $X(5)$ critical symmetries that have been recently introduce to describe phase transitional behavior. The standard deviations of the CCC’s are much larger at these minima that at other values of $R_{4/2}$, adding further evidence that the minima are spotting phase transitions. Figures 2 and 3 show the distribution CCC’s for individual classes of $R_{4/2}$, indicating that the behavior is different at the maxima and minima of $\langle C \rangle$.

In conclusion, we have established a correspondence between CCC of low-lying levels of nuclei in narrow ranges of $R_{4/2}$ and the collective behavior of nuclei as expressed by IBM. This finding may be added to the diverse attempt to identify nuclei that could be located at the critical points of shape/structural transitional regions, e.g. in [14]. We note that the analysis of the statistics of energy levels alone is not sufficient for testing the collective behavior. Statistical analysis of other quantities such as transition probabilities are required to do so. For example, recent IBM calculation [15] show that the E0-transition strengths are particularly large in spherical-deformed transition regions. The analysis of CCC’s for transition strengths of nuclei belonging to classes of $R_{4/2}$ is in progress.

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Table 1. Number of levels considered for nuclei belonging to different classes of the energy ratio $R_{4/2}$. The notation $J^\pi_r$ is used to label a level with spin $J$, parity $\pi$ and rank $r$.

| $R_{4/2}$ \ Level | $0^+_1$ | $0^+_2$ | $1^+_1$ | $1^-_1$ | $2^+_1$ | $2^+_2$ | $2^-_3$ | $3^+_1$ | $3^-_2$ | $3^+_2$ | $4^+_1$ | $4^-_2$ | $4^-_1$ | $5^+_1$ | $5^-_1$ | $6^+_1$ |
|-------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1.058 – 1.449     | 15     | 10     | 6      | 4      | 27     | 17     | 14     | 4      | 9      | 4      | 20     | 7      | 13     | 11     | 7      | 12     | 22     |
| 1.45 – 1.649      | 17     | 11     | 3      | 6      | 23     | 16     | 13     | 3      | 3      | –      | 10     | 7      | 23     | 11     | 5      | –      | 12     | 15     |
| 1.65 – 1.749      | 10     | 4      | 3      | 3      | 19     | 14     | 8      | –      | 6      | –      | 9      | –      | 19     | 8      | 3      | 3      | 5      | 11     |
| 1.75 – 1.849      | 12     | 8      | –      | 5      | 24     | 18     | 10     | –      | 7      | –      | 14     | 5      | 24     | 13     | 4      | 3      | 10     | 20     |
| 1.85 – 1.949      | 15     | 8      | –      | –      | 20     | 16     | 12     | –      | 7      | –      | 15     | 5      | 20     | 15     | 3      | –      | 14     | 14     |
| 1.95 – 2.049      | 13     | 4      | –      | –      | 20     | 16     | 12     | –      | 12     | 3      | 12     | 7      | 20     | 16     | 6      | 3      | 10     | 13     |
| 2.05 – 2.149      | 18     | 13     | 6      | 5      | 28     | 20     | 19     | 3      | 13     | 4      | 15     | 4      | 28     | 19     | 4      | 4      | 14     | 25     |
| 2.15 – 2.249      | 14     | 9      | 3      | 6      | 26     | 16     | 13     | 5      | 8      | 5      | 17     | 7      | 26     | 13     | 6      | 6      | 13     | 18     |
| 2.25 – 2.349      | 30     | 23     | 5      | 11     | 47     | 35     | 26     | 5      | 21     | 4      | 30     | 11     | 47     | 28     | 12     | 12     | 25     | 38     |
| 2.35 – 2.449      | 20     | 14     | 4      | 5      | 30     | 23     | 19     | 4      | 18     | 4      | 21     | 4      | 30     | 18     | 7      | 13     | 15     | 21     |
| 2.45 – 2.549      | 30     | 18     | 10     | 8      | 36     | 30     | 26     | 3      | 28     | 9      | 20     | 8      | 36     | 25     | –      | 14     | 18     | 26     |
| 2.55 – 2.649      | 16     | 10     | 6      | 4      | 25     | 20     | 14     | –      | 17     | 5      | 12     | 7      | 25     | 17     | 6      | 6      | 9      | 18     |
| 2.65 – 2.749      | 13     | 4      | –      | 3      | 21     | 17     | 14     | –      | 11     | –      | 9      | –      | 21     | 14     | 3      | 9      | 9      | 15     |
| 2.75 – 2.949      | 14     | 5      | –      | 5      | 28     | 21     | 13     | –      | 12     | –      | 13     | –      | 18     | 17     | 7      | 9      | 12     | 23     |
| 2.95 – 3.149      | 15     | 7      | –      | 6      | 21     | 16     | 13     | 5      | 11     | 3      | 10     | 4      | 21     | 9      | 7      | 7      | 8      | 19     |
| 3.15 – 3.249      | 14     | 10     | 3      | 8      | 25     | 16     | 14     | 8      | 14     | 4      | 15     | 4      | 25     | 14     | 13     | 10     | 12     | 22     |
| 3.25 – 3.333      | 42     | 26     | 9      | 33     | 58     | 46     | 37     | 34     | 36     | 21     | 43     | 28     | 58     | 39     | 35     | 25     | 36     | 53     |
Figure Captions

Figure 1. Mean value and standard deviation of the correlation coefficients for all levels considered for different classes of the $R_{4/2}$ ratio.

Figure 2. Distribution of level CCC for nuclei belonging to classes of $R_{4/2}$ that involve pre-collective, vibrational, $\gamma$-unstable and rotational nuclei.

Figure 3. Distribution of level CCC for nuclei belonging to classes of $R_{4/2}$ that involve nuclei undergoing shape/structural phase transitions.
