Numerical investigation of stress analysis of composite materials with various elasticity

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Abstract. Composite materials have a widespread use in engineering applications and modern structures because of the qualified physical and mechanical properties. Therefore, study the structures of composite material becomes attractive and researchers to obtain novel materials that qualified environment updates. In this study, the displacements and stress analysis of two-dimensional composite materials have different modulus of elasticity and mechanical properties have discussed. Finite difference method was used in this study where the model solved using Fortran 90. The results show that, the change in elasticity in one of the layers has significant effects on the displacements and stresses distributions of the layers of the composite. In addition, the influence of passion’s ratio with elasticity at the same time is discussed. They have important effect on the stress analysis. The results were in the acceptable limit which confirms the reliability of the finite difference method.

1. Introduction
Composite materials becomes a new face of technology and more attractive for researches since they have a high quality mechanical properties. Therefore, knowing the behavior of composite materials under the action of mechanical loading is important for designing the structures. The numerical methods had become the ultimate choice by the researchers in the last few decades. Invention and improvement of numerical simulations such as finite element and finite difference methods can solve the variation in composite mechanical properties. Stress analysis of bilayer composite requires the solution of partial differential equations. There are various numerical methods available for the solution of partial differential equations of stress analysis in materials. The critical stresses occur most frequently at the boundary of the structures [1,2]. Finite difference method was used to analyse the stress distributions in composite materials. Stress analysis of composite materials that containing defects were analyzed numerically. For low deformation the von-Mises stresses increased due to increase the number of layers of composites. Fibers are the main materials that afford a high deformation for different bearing load capacity [3]. properties of Carbon fiber composite that used in
cutting force and torque processes have been studied using finite element method [4]. Carbon fiber composite has important significance to reduce surface defects, machining tool developments and effectiveness. The analyzing of using composite materials in connecting rod design that used in I.C. engines is clarified to avoid the high force, wear resistance and failure [5]. Using the composite materials for high tensile strengths is essential because they have good characteristics that reduce the failure possibilities. Propagation of stresses within the composite laminates in automobile industry is discussed [6] to improve the work efficiency. The behavior of composite material is more compatible with automobile structure than isotropic materials.

Deflection analysis of rectangular composite plates was satisfied for different loads and plate’s thickness [7]. The deflection behavior depends on the direction of the applied load and layer’s arrangement. The deflection becomes an acceptable when the length of thickness ratios decrease respectively. Successful application of the stress function in conjunction with the finite difference method was reported [8]. The main shortcoming of the stress function formulation is that it accepts boundary conditions in terms of boundary loadings only. Therefore, problems containing boundary conditions in terms of restraints only or in terms of both loading and restraints (mixed boundary value problems) could not be solved by this stress function formulation. With a view to solving the problems of mixed boundary conditions, a formulation was proposed for the solution of two-dimensional such mixed boundary value problems using the displacement potential function formulation. This formulation was applied successfully for the solution of many two dimensional elastic mixed boundary value problems [10-19]. Not only that, the formulation extended [20] to study the displacement potential function formulation for three dimensional elastic problems and obtained reliable solution for some classical problems of solid mechanics [21]. Solution of the two dimensional elastic problem with holes is successfully carried out [22]. However, these works are limited to single isotropic material only. Beside the finite difference method, another numerical method namely finite element method was first successfully applied for the two dimensional elastic problem [23-24]. Then, it became very popular and reliable with the rapid development of the digital computers and used by many researchers in both two dimension and three dimension [25,26].

In this work, the influence of elasticity for different composite materials and their development for various boundary conditions at the interface by two different isotropic materials has discussed. The investigation of displacement and stress distribution in the layers as well as at the interface were analyzed.

2. Mathmatecal Model
In the work, it will be assumed that the composite materials undergoing the effect of mechanical loadings, perfectly elastic and the deformations are very small. In general, composite materials depend on the geometry, material’s properties, external forces and supports. The deformation at any point is related to the displacement of the neighborhood of that point. The neighborhood of a point is defined as a set of points in the close vicinity of that point. The state of strain at any point could be completely defined by six components of strain: \(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \text{and } \gamma_{zx}\). By definition the normal and shear strain can be given by [14]:

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u_x}{\partial x}, \\
\varepsilon_y &= \frac{\partial u_y}{\partial y}, \\
\varepsilon_z &= \frac{\partial u_z}{\partial z}, \\
\gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \\
\gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \\
\gamma_{zx} &= \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}
\end{align*}
\]  

where, \(\varepsilon_x, \varepsilon_y, \varepsilon_z\) are the strain components parallel to the coordinate axes called normal strain and \(\gamma_{xy}, \gamma_{yz}, \gamma_{zx}\) are strain components acting on the planes \(xy, yz\) and \(zx\) planes respectively called shear strain. The stresses are related to the strains by the Hooke’s law and Poisson’s law as follows [14]:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left[ \sigma_x - \mu (\sigma_y + \sigma_z) \right], \\
\varepsilon_y &= \frac{1}{E} \left[ \sigma_y - \mu (\sigma_x + \sigma_z) \right], \\
\varepsilon_z &= \frac{1}{E} \left[ \sigma_z - \mu (\sigma_x + \sigma_y) \right]
\end{align*}
\]  

Differential equations of equilibrium and boundary conditions. The following equations can be obtained [14] for static equilibrium
These equations in equation (4) are known as the equations of equilibrium, where \( X, Y, \) and \( Z \) are the components of body force per unit volume of the element in \( x, y, \) and \( z \)-directions respectively. The body forces can be eliminated due to their negligible effect as compared to that of surface forces. For plane stress condition the cubic element reduces to a thin rectangular block and no body forces acting on that block, hence the equilibrium equations yields to

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + X = 0 \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y = 0 \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0
\]  

(4)

At the boundary they must be in equilibrium with external forces on the boundary and the external forces may be considered as the continuation of the internal stress distribution. So, the conditions of equilibrium at the boundary can be written as [14]:

\[
\sigma_n = \sigma_{xx}.l^2 + \sigma_{yy}.m^2 + 2\sigma_{xy}.lm \\
\sigma_t = \sigma_{xy}.(l^2 - m^2) + (\sigma_{yy} - \sigma_{xx}).lm
\]

(5)

where, \( \sigma_n \) and \( \sigma_t \) are the normal and tangential components of the surface forces acting on the boundary per unit area and \( l, m \) are the direction cosines of the normal to the surface. Similarly, normal component of displacement \( u_n \) and the tangential component \( u_t \) acting on the boundary surface can be expressed by

\[
u_n = u_x. l + u_y. m \\
u_t = u_y. m - u_x. l
\]

(6)

For two dimensional body three strain components can be expressed in terms of the displacement components as

\[
\varepsilon_x = \frac{\partial u_x}{\partial x}; \quad \varepsilon_y = \frac{\partial u_y}{\partial y}; \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}
\]

(7)

Since these three strain components are expressed by two functions only, they cannot be related arbitrarily among themselves. There exists a certain relationship among the strain components, which is expressed as,

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}
\]

(8)

This differential relation is called the condition of compatibility. It must be satisfied by the strain components to ensure the existence of functions \( \sigma_x \) and \( u_y \) connected with the strain components by equation 8. Elimination of strains in terms of stresses, equation 9 yields to

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0
\]

(9)

The method of solving these equations is through the introduction of a function \( \varphi(x,y) \), known as Airy stress function, defined as
which satisfies equations (equation 6) and transforms the equation (equation 10) into

\[ \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \]  

(12)

The mathematical formulation in terms of displacement potential function is given below. In absence of body forces, the equilibrium equations for two dimensional elastic problems in terms of displacements components are as follows

\[ \frac{\partial^2 u}{\partial x^2} + \left( \frac{1 - \mu}{2} \right) \frac{\partial^2 u}{\partial y^2} + \left( \frac{1 + \mu}{2} \right) \frac{\partial^2 v}{\partial x \partial y} = 0 \]
\[ \frac{\partial^2 v}{\partial y^2} + \left( \frac{1 - \mu}{2} \right) \frac{\partial^2 v}{\partial x^2} + \left( \frac{1 + \mu}{2} \right) \frac{\partial^2 u}{\partial x \partial y} = 0 \]  

(13)

A new potential function approach involves investigation of the existence of a function defined in terms of the displacement components. In this approach attempt had been made to reduce the problem to the determination of a single variable. A function \( \psi(x,y) \) is thus defined in terms of displacement components as, [14]

\[ u = \frac{\partial^2 \psi}{\partial x \partial y}, \quad v = -\left[ \left( \frac{1 - \mu}{1 + \mu} \right) \frac{\partial^2 \psi}{\partial y^2} + \left( \frac{2}{1 + \mu} \right) \frac{\partial^2 \psi}{\partial x^2} \right] \]  

(14)

with this definition of \( \psi(x,y) \), the first of the two equations (13) is automatically satisfied. Therefore, \( \psi \) has only to satisfy the second equation. Thus, the condition that \( \psi \) has to satisfy is

\[ \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \]  

(15)

In order to solve the problem by solving for the function \( \psi \) of the bi-harmonic equation (equation 15), the boundary conditions should be expressed in terms of \( \psi \). The boundary conditions are known restraints and loadings, that is, known values of components of stresses and displacements at the boundary.

2.1 Validation

A composite material under axial loading as shown in Figure 1 has been solved to validate the numerical results. The displacement and stress distributions were analysed and compared with other solutions. The problem is considered as plane stress problem. The left side of the composite material is fixed and the right side is under tensile stress. The other sides of the material are free surfaces. The boundary conditions at the left side \( u_x = 0, \ u_y = 0 \) and \( \sigma_x = \sigma_y = \sigma_y = \sigma_i/E_1 = \sigma_i/E_2 = 1.5 \times 10^{-4}, \ \sigma_i = 0 \) at the right side. The boundary conditions at the top and bottom surfaces \( \sigma_x = 0, \ \sigma_y = 0 \). The \( \sigma_y \) is the dimensionless stress. The values of \( E_1, E_2 \) are the modulus of elasticity and \( \mu_1, \mu_2 \) are the poisson’s ratios of the upper and lower material respectively. The geometry of the problem is square having \( a/b=1 \) and \( a = b = 25 \) unit. This problem is solved for stress and displacement distribution by using finite difference method and finite element method taking \( \mu_1=0.32, \ \mu_2=0.28 \). The displacement components are continuous over the bilayer composite and there is a single value for each parameter at the interface point. However, there are two values of each stress component one is for upper material and one is for lower material.
Figure 1. Bilayer composite under axial loading

The solution of the cases was obtained using finite element method and finite difference method. The distribution of $\sigma_{xy}$ that shown in Figure 2a is compatible by finite difference method and finite element method in different sections of the bilayer. The distribution of $\sigma_y$ at various sections of the bilayer composite by finite difference method is shown in Figure 2b. It indicates that for this particular problem the stress at section $y/b = 0$ is very significant as compared to the other sections of the material. At other sections of the bilayer composite, the variation of the stress $\sigma_y$ is very small and could not be discerned. Figure 2c shows that, the distribution of stress $\sigma_y$ by finite element method and finite difference method methods at $y/b = 0.1$ and $2.4$. At $y/b = 0.0$. The stress distributions matches up with each other except at the upper and lower boundary points and at interface points as shown in Figure 2d. The finite element method result shows smaller value of stress $\sigma_y$ at the boundary corner points (most critical point as it correspond the highest stress) than finite difference method result. Finite difference method results for distribution of $\sigma_y$ exactly matches up with finite element method results.

Figure 2. (a) Comparison of normalized shear stress distribution at different sections of the bilayer composite (b) Comparison of normalized normal stress ($\sigma_y$) distribution at $y/b = 0.0$. (c) Comparison of normalized normal stress ($\sigma_y/E$) distribution at $y/b =1.0$. (d) Comparison of normalized normal stress ($\sigma_y$) distribution at $y/b =0.24$. 
3. Results and Discussions

3.1 Stress analysis of composite material having different modulus of elasticity.

In this study, the distributions of stress and displacement are obtained for various combinations of material properties as well as for various types of boundary conditions. The stress and displacement distributions at the interface are also analyzed. Two dissimilar materials having modulus of elasticity of $E_1 = 200$ GPa and $E_2 = 110$ GPa (ratio of the modulus of elasticity, $\varepsilon = E_1 / E_2 = 1.8182$) are considered. The Poisson’s ratios are maintained constant ($\mu_1 = \mu_2 = 0.30$). The bilayer composite is subjected to uniform normal stress, $\sigma_n = 40$ MPa at the right boundary as shown in Figure 3(b).

The problem is solved for displacement and stress distributions by finite difference method. It is observed that, the material which has higher the modulus of elasticity will have higher stiffness and lower deformation. The distributions of displacement components are shown in Figure 4 and Figure 5. The displacement components increase, as the load increase at the right boundary approaches to the optimum value. Figure 6 shows the normalized normal stress distribution at various $y/b$ locations where the nodal stresses are normalized by each modulus of elasticity of the material corresponding to that node. It shows that the distribution of stresses is anti-symmetrical in nature. The normal and shear stress distributions as shown in Figure 7 and Figure 8. It is obvious that, the material with higher modulus of elasticity experiences higher stresses.
Figure 3. (a) Deformed shape of the bilayer composite under uniform tensile stress (deformations are 100 times magnified) (b) Uniform applied tensile stress at the right boundary.

Figure 4. Normalized displacement component (u/a) distribution at various y/b locations.
Figure 5. Normalized displacement component ($v/b$) distribution at various $y/b$ locations.

Figure 6. Distribution of normalized normal stress ($\sigma_x/\sigma_{yo}$) at various $y/b$ positions.
3.2 Stress analysis of composite have different modulus of elasticity and Poisson’s ratio
The effect of composite materials the have different poisson’s ratio and modulus of elasticity is studied. To analyze the complete combined effect of modulus of elasticity and Poissons’s ratio, a bilayer composite made of steel (E₁=200 GPa, μ₁=0.3) and copper (E₂=110 GPa, μ₂=0.33) is considered. The bilayer composite is subjected to uniform normal stress, σₙ=40 MPa at the right boundary as shown in Figure 9. The normalized stresses differ in upper and lower material. The displacement and stress distributions of the steel-copper bilayer composite are shown in Figure 10 to Figure 14. There is a variation in stresses and displacements due to the different elasticity and Poisson’s ratios in upper and lower material. Although the pattern of the distribution curves remain
unchanged but there are variations in the magnitudes of the displacements and stresses. It could be observed that, due to the change in Poisson’s ratio the bilayer composite experiences higher magnitude of displacements and stresses.

**Figure 9.** (a) Deformed shape of the bilayer composite under uniform tensile stress (deformations are 100 times magnified) (b) Uniform applied tensile stress at the right boundary.

![Figure 9 Diagram](image)

**Figure 10.** Normalized displacement component (u/a) distribution at various y/b locations of the bilayer composite.

![Figure 10 Diagram](image)
Figure 11. Normalized displacement component (v/b) distribution at various y/b locations of the bilayer composite.

Figure 12. Variation of normalized normal stress (σx/σyo) at various y/b positions of the bilayer composite.
Figure 13. Distribution of normalized normal stress ($\sigma_y/\sigma_{yo}$) at various y/b positions.

Figure 14. Distribution of normalized normal stress ($\sigma_{xy}/\sigma_{yo}$) at various y/b positions.

4. Conclusion

The stress and displacement distributions are presented in the present analysis for various combinations of mechanical properties and loadings. It is revealed that, when a bilayer composite is subjected to mechanical loading the corner zone as well as the interfacial zone is the most critical zone. The variation of modulus of elasticity in the lower material has a significant effect on the distributions of stresses and displacements in lower material as well as in the upper material. The higher the modulus of elasticity causes higher normal and shear stresses in the material.
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