From a non-Markovian system to the Landau equation

Juan J. L. Velázquez †, Raphael Winter ‡

†University of Bonn, Institute for Applied Mathematics
Endenicher Allee 60, D-53115 Bonn, Germany

July 25, 2017

Abstract

In this paper, we prove that in macroscopic times of order one, the solutions to the truncated BBGKY hierarchy (to second order) converge in the weak coupling limit to the solution of the nonlinear spatially homogeneous Landau equation. The truncated problem describes the formal leading order behavior of the underlying particle dynamics and can be reformulated as a non-Markovian hyperbolic equation, which converges to the Markovian evolution described by the parabolic Landau equation. The analysis in this paper is motivated by Bogolyubov’s derivation of the kinetic equation by means of a multiple time scale analysis of the BBGKY hierarchy.

Contents

1 Introduction

2 Main results, notation and auxiliary lemmas
   2.1 Formulation of the main results
   2.2 Strategy of the proofs of Theorems 2.6 and 2.8
   2.3 A well-posedness result for the regularized problem (2.29)

3 The linear equation

4 A priori estimate for the nonlinear problem
   4.1 Continuity of the fixed point mapping Ψ
   4.2 Invariance of the set Ω under the mapping Ψ
      4.2.1 Recovering the quadratic decay in Laplace variables
      4.2.2 Boundary Layer Estimate

5 Existence of solutions and Markovian Limit
   5.1 Existence of a solution to the non-Markovian equation
   5.2 Non-Markovian to Markovian limit
## 1 Introduction

A central objective in kinetic theory is the derivation of effective equations for macroscopic densities of particles in a plasma or gas. Two of the main equations in this context are the Boltzmann equation and the Landau equation, and a large portion of the mathematical research in this area is devoted to the study of these equations. For an extensive overview over mathematical kinetic theory we refer to [25,32]. For the Boltzmann equation, rigorous results have been proved, both on the level of the equation itself, as well as on that of its derivation from particle systems. Results on well-posedness, entropic properties of solutions and rate of convergence to equilibrium can be found in [13,14,29,33]. For the derivation of the equation from interacting particle systems we refer to [15,20,24], and to [7,9,16,25] for the derivation of the linear equation from Lorentz models.

Many of these problems, including the derivation starting from particle systems, are still open for the Landau equation. The equation was introduced by Landau in [19] (see also [22]) to describe the dynamics: scaling limits, Landau ([19]) formally derived the following equation for the number density $f(t,v)$ of particles in the velocity space $\mathbb{R}^3$:

$$
\partial_t f(t,v) = \sum_{j=1}^{3} \partial_{v_j} \left( \int_{\mathbb{R}^3} a_{i,j}(v-v')(\partial_{v_j} - \partial_{v'_j})(f(t,v)f(t,v')) \, dv' \right)
$$

(1.2)

where the matrix valued function $a$ is determined by the pair interaction potential $\phi$:

$$
a_{i,j}(w) = \int_{\mathbb{R}^3} \delta(k \cdot w) |\phi(k)|^2 \, dk = \frac{\Lambda}{|w|} \left( \delta_{i,j} - \frac{w_i w_j}{|w|^2} \right) \text{ for some } \Lambda > 0.
$$

(1.3)

In the physically most relevant case of Coulomb interaction, i.e. $\phi(x) = \frac{e}{|x|}$, considered in [19], the constant $\Lambda$ is logarithmically divergent.

A rather general approach to deriving kinetic equations from (1.1) was later developed by Bogolyubov ([6]). We will briefly summarize this method here. Assume that the random initial distribution of particles $(X_i(0), V_i(0))_{i \in I}$ is uncorrelated and translation invariant in space. Furthermore assume the velocities $V_i$ are always of order one and let $Z$ be the average number of particles per unit of volume. Then we can consider the $n$-particle number densities $F_n(x_1,v_1,\ldots,x_n,v_n)$. In order to work with functions of order one, we define the functions $f_n$ by:

$$
F_n(x_1,v_1,\ldots,x_n,v_n) = Z^n f_n(x_1,v_1,\ldots,x_n,v_n).
$$
Then the densities $f_n$ satisfy the so called BBGKY hierarchy (see e.g. [3]):

$$
\partial_t f_n + \sum_{i=1}^n v_i \nabla_x f_n - \sum_{i=1}^n \int Z \theta^2 \nabla \phi(x_i - x_{n+1}) \nabla v_i f_{n+1} \, dx_{n+1} \, dv_{n+1} = \theta^2 \sum_{i \neq j} \nabla \phi(X_i - X_j) \nabla v_i f_n.
$$

(1.4)

In the forthcoming analysis we will assume that $Z \theta^2 \rightarrow 0$, which is usually referred to as weak coupling limit. Here Bogolyubov’s argument identifies the Landau equation (1.2) as the limiting equation for $f_1$. Bogolyubov’s technique can also be applied in the case $Z \theta^2 \approx 1$, yielding the Balescu-Lenard equation (see [3][2][21]). In this case the system has to be considered as some kind of effective medium, in which the interaction of pairs of particles is modified due to collective effects. In the physics literature this is characterized by means of the so-called dielectric function, that gives a nontrivial correction to the limit kinetic equation, but we will not consider this situation here.

Our assumption $Z \theta^2 \rightarrow 0$ has a clear interpretation in terms of dimensionless quantities. Observe that $Z \theta^2$ describes the ratio of the average potential and kinetic energy of a particle:

$$
\frac{\langle \theta^2 \sum_{j \in \mathbb{N} : j \neq i} \phi(X_i - X_j) \rangle}{\langle V_i^2 \rangle} \sim Z \theta^2.
$$

When $Z \theta^2 \rightarrow 0$, the kinetic energy of particles is dominant, hence the absence of collective effects. Our objective is to study the evolution of the one particle density function $f_1$. We will denote the timescale on which this evolution takes place as macroscopic time. For shorter notation let $(x_i, v_i) = \xi_i$ and introduce the correlation functions $g_2, g_3, \ldots$ as:

$$
g_2(\xi_1, \xi_2) = f_2(\xi_1, \xi_2) - f_1(\xi_1) f_1(\xi_2),
$$

$$
g_3(\xi_1, \xi_2, \xi_3) = f_3(\xi_1, \xi_2, \xi_3) - f_1(\xi_1) g_2(\xi_2, \xi_3) - f_1(\xi_2) g_2(\xi_1, \xi_3) - f_1(\xi_3) g_2(\xi_1, \xi_2) - f_1(\xi_1) f_1(\xi_2) f_1(\xi_3) + g_4(\xi_1, \xi_2, \xi_3, \xi_4) = \ldots.
$$

From (1.4) we can derive equations for $g_2, g_3$ and higher order correlations. A crucial observation is that we can expect to have a separation of orders of magnitude $f_1 \gg g_2 \gg g_3$ as $\theta^2 \rightarrow 0$. To see this, consider now the exact equations satisfied by $g_2$ and $f_1$. For shorter notation we introduce the function $\sigma$ with $\sigma(1) = 2, \sigma(2) = 1$. By a straightforward algebraic computation, the BBGKY hierarchy (1.4) implies:

$$
\partial_t f_1 = \nabla \cdot \left( \nabla \phi(x_1 - x') g_2(x_1, v_1, x', v') \, dx' \, dv' \right)
$$

$$
\partial_t g_2 + v \nabla_x g_2 - Z \theta^2 \sum_{k=1}^2 \int \nabla \phi(x_k - x_3) \nabla_{v_k} \left( f_1(\xi_k) g_2(\xi_2, \xi_3) + g_3(\xi_1, \xi_2, \xi_3) \right) \, d\xi_3
$$

$$
= \theta^2 \sum_{k=1}^2 \nabla_{v_k} \left( f_1(\xi_k) f_1(\xi_2) + g_2(\xi_1, \xi_2) \right) \nabla \phi(x_k - x_{\sigma(k)}).
$$

(1.5)

Indeed, the sources on the right-hand side of the equation are of order $\theta^2 \ll 1$, hence we expect $f_1 \gg g_2$, a similar argument suggests $g_2 \gg g_3$. A key point in the argument by Bogolyubov is that
this separation of orders of magnitudes implies a separation of timescales. The correlations have size \( g_2 \approx \theta^2 \), so by (1.5) we can expect \( f_1 \) to evolve on a macroscopic timescale \( t = Z \theta^4 \tau \), and \( g_2 \) to evolve on the faster timescale \( \tau \). Therefore, on the macroscopic timescale, the correlation \( g_3(t) \) can be expected to be a functional \( g_3(t) = A_2[f_1(t)] \) of \( f_1 \). More generally, Bogolyubov argues that on the timescale \( t \) all correlations evolve "adiabatically" as functionals:

\[
\begin{align*}
g_2(t) &= A_2[f_1(t)] \\
g_3(t) &= A_3[f_1(t)] \\
&\ldots
\end{align*}
\]

This argument allows us to derive the limit kinetic equation of scaling limits in a straightforward fashion. Since we consider the case of weak interaction, i.e. \( Z \theta^2 \to 0 \), the integral term in (1.5) is of lower order and (1.5) can be replaced by:

\[
\begin{align*}
\partial_t f_1 &= Z \theta^2 \nabla \cdot \left( \int \nabla \phi(x_1 - x_3)g_2(\xi_1, \xi_3) \, d\xi_3 \right) \\
\partial_t g_2 + \sum_{k=1}^{2} v_k \nabla \cdot g_2 &= \theta^2 (\nabla_{\xi_1} - \nabla_{\xi_2}) (f_1(\xi_1) f_1(\xi_2)) \nabla \phi(x_1 - x_2).
\end{align*}
\]

(1.6)

Now the functional \( A_2[f_1] \) can be computed explicitly by solving the steady state equation for \( g_2 \) in (1.6). We substitute \( g_2 = A_2[f_1] \) in the equation for \( f_1 \) and identify the Landau equation (1.2) as the limit equation. In the case \( Z \theta^2 \approx 1 \) the functional \( A_2[f_1] \) was computed explicitly in [21], formally yielding the Balescu-Lenard equation. It is possible to go from (1.6) to the Landau equation, reformulating the problem as a non-Markovian evolution. To this end, we rewrite (1.6) as a single equation, involving only terms depending on \( f_1 \). We can integrate the equation for \( g_2 \) along characteristics (by assumption the initial correlations vanish):

\[
g_2(\tau, \xi_1, \xi_2) = \int_0^\tau \theta^2 (\nabla_{\xi_1} - \nabla_{\xi_2})(f_1(s, \xi_1) f_1(s, \xi_2)) \nabla \phi(x_1 - x_2 - (t - s)(v_1 - v_2)) \, ds.
\]

Inserting this back into (1.6), and changing to the macroscopic timescale \( Z \theta^4 \tau = t \), we obtain a closed equation for the one particle distribution function. Write \( \epsilon = Z \theta^4 \) and let \( f_\epsilon(t, v) \) be the one particle density function on the macroscopic timescale, then \( f_\epsilon \) satisfies the equation

\[
\begin{align*}
\partial_t f_\epsilon &= \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t K[f_\epsilon(s)] \left( \frac{t - s}{\epsilon}, v \right) \nabla f_\epsilon(s, v) - \nabla \cdot K[f_\epsilon(s)] \left( \frac{t - s}{\epsilon}, v \right) f_\epsilon(s, v) \, ds \right) \\
f_\epsilon(0, v) &= f_0(v),
\end{align*}
\]

(1.7)

where \( K \) is given by the formula

\[
K[f](t, v) := \int \int \nabla \phi(x) \otimes \nabla \phi(x - t(v - v')) f(v') \, dv' \, dx.
\]

By Bogolyubov’s argument, (1.7) should display the leading order nonlinear behavior of the one particle density \( f_\epsilon \) in the scaling limit procedure and converge to a solution \( f \) of the Landau equation (1.2).

There are multiple gaps to bridge in order to make Bogolyubov’s argument rigorous. First one has to prove the well-posedness of the infinite system of ODEs (1.1). Sufficient conditions on the
potential and initial data for this can be found for example in [27]. Proving the separation of orders of magnitude \( f_1 \gg g_2 \gg \ldots \) and the validity of the truncation of the BBGKY hierarchy is a key problem and still open. Indeed, this assumption seems to be wrong in general, at least when the relative velocity of particles becomes very small. The two particle correlation function \( g_2 \) measures the effect of deflections due to particle interaction. Now let us consider the mutual deflection due to the interaction of two particles at initial positions \( \xi_1, \xi_2 \) for \( \theta^2 \) very small. When the relative velocity \( v_1 - v_2 \approx 0 \) is sufficiently small, the total deflection can be of order one, since the collision time diverges like \( |v_1 - v_2|^{-1} \). Hence in this region \( g_2 \approx f_1 \) is not necessarily small. Due to the integrability of the singularity, one could expect the approach to derive the equation is still valid. The singular region \( v_1 \approx v_2 \) is also an important issue in the analysis of the Landau equation. The function \( a \) (cf. (1.3)) diverges for small relative velocity, i.e. when \( w = v - v' \approx 0 \) is very small. Notice that the singularity \( |v - v'|^{-1} \) appears independently of the choice of interaction potential \( \phi \).

Due to the mathematical problems arising from the singularity in relative velocity, a number of variants of (1.2) have been studied. An important class of Landau type equations are obtained by replacing the singularity \( |v - v'|^{-1} \) by \( |v - v'|^{\gamma+2} \). For \( \gamma \in (0, 1] \), existence, uniqueness and regularity have been proved in [11], results on entropic properties in [12]. The case \( \gamma = 0 \) is covered in [31], including existence and uniqueness of classical solutions, as well as characterization of the qualitative behavior of solutions. For so-called "soft" kernels, meaning \(-3 < \gamma < 0\), the equation on a periodic domain is studied in [28] and it is proved that the Maxwellian is exponentially stable under small perturbations. For \( \gamma \geq -3 \), the global existence of solutions in a periodic box is proved in [17] close to equilibrium. In [30], a concept of \( H\)-solution is introduced for exponents \(-3 \leq \gamma \leq 2\), as well as sufficient conditions under which the equation can be obtained as a grazing collision limit from Boltzmann equation, which has also been proved in the spatially inhomogeneous case ([11]). Lower bounds on the entropy dissipation in the physically most relevant case \( \gamma = -3 \) can be found in [8].

As mentioned above, the derivation of the Landau-type equations from particle systems is still largely open. The linear Landau equation has been derived in [4] as a scaling limit of systems with a single particle traveling through a random but fixed configuration of scatterers. Furthermore it is shown in [5] that the Landau equation (1.2) is consistent with a scaling limit of interacting particle systems. More precisely it is shown that the time derivative of the macroscopic density of particles in the weak coupling limit at \( t = 0 \) is correctly predicted by the Landau equation. The technique follows a similar line of reasoning like Bogolyubov, proving the validity of a truncation of the BBGKY hierarchy to a system like (1.7), and proving convergence to the Landau equation on a shorter timescale than the macroscopic.

In this paper we will prove that Bogolyubov’s adiabatic approach to deriving the Landau equation (1.2) from the system (1.7) is indeed correct, when the singularity \( v \approx v' \) is cut. To be precise, we consider the modified Landau equation

\[
\partial_t f = \sum_{i,j=1}^{3} \partial_{v_i} \left( \int_{\mathbb{R}^3} a_{i,j}(v - v')(\partial_{v_j} - \partial_{v'_j}) (f(t, v)f(t, v')) \eta(|v - v'|^2) \, dv' \right)
\]

(1.8)

\[
f(0, v) = f_0(v),
\]

where \( \eta(r) \) vanishes for \( r \) small. We will derive the equation (1.8) from the system (1.7), where \( K \) is now given by:

\[
K[f](t, v) := \int \int \nabla \phi(x) \otimes \nabla \phi(x - t(v - v')) f(v') \eta(|v - v'|^2) \, dv' \, dx.
\]

(1.9)
The main results of the paper are the existence of strong solutions \( f_\varepsilon \) to (1.7) with \( K \) as in (1.9), and the convergence of these solutions to a strong solution \( f \) of the Landau equation (1.8) for macroscopic times of order one. We assume that \( f_0 \) is close to the Maxwellian steady state of the limit equation and choose a particular short range potential \( \phi \). In contrast to the diffusive, parabolic Landau equation, equation (1.7) is hyperbolic. We show that regularity and decay of the initial datum \( f_0 \) are conserved. Furthermore, the evolution given by (1.7) is clearly non-Markovian, since the time derivative depends on the whole history of the function \( f_\varepsilon \) until time \( t \). In the limit \( \varepsilon \to 0 \), this memory effect disappears and we recover the Markovian dynamics of the Landau equation.

The techniques used in this paper are reminiscent of the theory of symmetric hyperbolic systems (18, 23). The main difference is that the evolution at hand is non-Markovian, therefore we have to introduce a more general notion of coercivity that holds in a time averaged sense.

The paper is structured as follows: In Section 2, we give a precise formulation of the main results Theorem 2.6 and 2.8 as well as the proofs of some auxiliary results. In Section 3 we prove the result in the linear case. Section 4 proves that the a priori estimates are stable under certain small perturbations, and that these smallness assumptions are conserved by the equation. In Section 5 we give the proofs of the two main theorems.

2 Main results, notation and auxiliary lemmas

2.1 Formulation of the main results

Our goal is to prove the existence of a strong solution to the equation

\[
\begin{align*}
\partial_t u_\varepsilon &= \frac{1}{\varepsilon} \nabla_v \cdot \left( \int_0^t K[u_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s,v) \, ds \right) \\
&\quad - \frac{1}{\varepsilon} \nabla_v \cdot \left( \int_0^t P[u_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s,v) \, ds \right) \\
&= -\frac{1}{\varepsilon} \left( \int_0^t P[u_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s,v) \, ds \right) + \frac{1}{\varepsilon} \left( \int_0^t K[u_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s,v) \, ds \right),
\end{align*}
\]

(2.1)

where \( K \) and \( P \) denote the following operators:

\[
\begin{align*}
K[u](t,v) &= \int \nabla \phi(x) \otimes \nabla \phi(x-t(v-v')) u(v') \eta(|v-v'|^2) \, dv' \, dx \\
P[u](t,v) &= \nabla_v \cdot K(t,v) = \int \nabla \phi(x) \otimes \nabla \phi(x-t(v-v')) \nabla u(v') \eta(|v-v'|^2) \, dv' \, dx.
\end{align*}
\]

(2.2)

We will specify the potential \( \phi \) and the cutoff function \( \eta \in C^\infty(\mathbb{R}) \) below. Formally, as \( \varepsilon \to 0 \), the functions \( u_\varepsilon \) converge to a strong solution \( u \) of:

\[
\begin{align*}
\partial_t u &= \nabla \cdot (K[u] \nabla u) - \nabla \cdot (P[u] u) \\
u(0,v) &= u_0(v) \\
K[u](v) &= \int (k \otimes k) \hat{\phi}(k)^2 \delta(k \cdot (v-v')) \eta(|v-v'|^2) u(v') \, dk \, dv' \\
P[u](v) &= \int (k \otimes k) \hat{\phi}(k)^2 \delta(k \cdot (v-v')) \eta(|v-v'|^2) \nabla u(v') \, dk \, dv'.
\end{align*}
\]

(2.3)
We will prove this result close to the Maxwellian distribution \( m \), which is the steady state of the limit equation (2.3). Furthermore we choose the potential \( \phi \) to have a particular form, making the computations considerably easier.

**Notation 2.1** Let \( \eta \in C^\infty(\mathbb{R}) \) be a fixed cutoff function with \( 0 \leq \eta \leq 1 \), \( \eta(r) = 1 \) for \( |r| \geq \kappa \) and \( \eta(r) = 0 \) for \( |r| \leq \frac{\kappa}{2} \) for some \( \frac{1}{2} > \kappa > 0 \) that we will not further specify in the following analysis. We choose the potential \( \phi(x) \) to be given by

\[
\phi(x) = \sqrt{\frac{2}{\kappa}} K_0(|x|),
\]

where \( K_0 \) is the modified Bessel function of second type.

**Remark 2.2** The potential \( \phi \) is monotone decreasing, decays exponentially at infinity and diverges logarithmically at the origin. Our approach also seems to work for other potentials with analogous properties, but becomes significantly less technical with this particular choice. The Fourier transform of the potential is given by:

\[
\hat{\phi}(k) = \frac{1}{(1 + |k|^2)^{\frac{1}{2}}}.
\]

The function spaces we are going to work with in the forthcoming analysis are the following ones.

**Definition 2.3** Let \( \lambda(v) \), \( \tilde{\lambda}(v) \) be the weight functions given by \( \lambda(v) := e^{\nu v} \), \( \tilde{\lambda}(v) := \frac{e^{\nu v}}{1 + |v|^2} \). For \( n \in \mathbb{N} \) and \( v = \lambda, \tilde{\lambda} \), we define the weighted Sobolev space \( H^v_\lambda \) as the closure of \( C_c^\infty(\mathbb{R}^3) \) with respect to the norm:

\[
|u|^2_{H^v_\lambda} := \sum_{a \in \mathbb{N}^3, |a| \leq n} \| \nabla^a \cdot u(\cdot) \|_{L^2}^2.
\]

In the case \( n = 0 \) we also write \( H^v_\lambda = L^2_\lambda \). For functions \( f(t, v) \) with an additional time dependence, we define the spaces \( V^n_{\nu, \lambda} \) as the closure of \( C_c^\infty([0, \infty) \times \mathbb{R}^3; \mathbb{R}^d) \) with respect to:

\[
\|f(\cdot, \cdot)\|_{{V^n_{\nu, \lambda}}} := \int_0^\infty e^{-\nu t} \sum_{j=1}^d \|f_j(t, \cdot)\|_{H^n_\lambda}^2 \, dt, \quad \text{where } \nu \geq 1.
\]

Let \( X_\lambda^n \) be the function space given by:

\[
X_\lambda^n := \{ (f, g) \in V_\lambda^n \times V_{\nu, \lambda}^{n-1} : f = \nabla \cdot g, \supp f, g \subset [0, 1] \times \mathbb{R}^3 \},
\]

with norm \( \| (f, g) \|_{X_\lambda^n} := \|f\|_{V^n_{\nu, \lambda}} + \|g\|_{V^{n-1}_{\nu, \lambda}} \).

For \( u = (f, g) \in X_\lambda^n \), we write \( \partial_t u = (\partial_t f, \partial_t g) \) whenever the right-hand side is well-defined.

**Remark 2.4** The validity of our analysis is not subject to the choice of the particular exponent in the weight function, weights of the form \( \lambda(v) = e^{\nu v} \) or fast power law decay would work equally well.

The choice of the weight functions \( \lambda, \tilde{\lambda} \) is motivated by the following compactness property, that we will later use to prove the existence of fixed points.
Lemma 2.5 Let \((u_i)_{i\in\mathbb{N}} = ((f_i, g_i))_{i\in\mathbb{N}} \subset X^{n+1}_{A,\lambda}\) be a bounded sequence, such that the sequence 
\((\partial_t f_i, \partial_t g_i) \in X^{n+1}_{A,\lambda}\) is bounded as well. Then the sequence \((u_i)\) is precompact in \(X^n_{A,\lambda}\).

Proof: For some \(C > 0\) there holds \(\|\hspace{2pt}(f_i, g_i)\|_{X^{n+1}_{A,\lambda}} + \|\hspace{2pt}(\partial_t f_i, \partial_t g_i)\|_{X^{n+1}_{A,\lambda}} \leq C\). Denote by \((\varphi_R)_{R>0} \in C^\infty_c\) a standard sequence of cutoff functions that is one on \(B_R\) and vanishes outside of \(B_{R+1}\). We construct a convergent subsequence \(u_{\ell(k)}\) inductively. The region \([0, 1] \times B_{R+1}\) is compact, so by Rellich’s theorem the sequences \((f_i \varphi_1), (g_i \varphi_1)\) have convergent subsequences \(f_{\ell(i)} \varphi_1 \rightarrow F_1, g_{\ell(i)} \varphi_1 \rightarrow G_1\) in \(V^n_{A,\lambda}\) and \(V^{n-1}_{A,\lambda}\) respectively. Since \(V^n_{A,\lambda} \subset V^{n,d}_{A,\lambda}\) embed continuously (actually Lipschitz with constant \(L \leq 1\)), the sequences are also convergent in the latter spaces. Now we inductively extract further convergent subsequences \(f_{\ell(i)} \varphi_k \rightarrow F_k\) and \(g_{\ell(i)} \varphi_k \rightarrow G_k\). By construction we have \(F_m = F_k, G_m = G_k\) on \(B_k\) for \(m \geq k\). We pick a sequence \(u_{\ell(k)}\) such that:

\[
\| f_{\ell(k)} \varphi_k - F_k \|_{V^n_{A,\lambda}} + \| g_{\ell(k)} \varphi_k - G_k \|_{V^{n-1}_{A,\lambda}} \leq \frac{1}{k}.
\]

The sequences \(f_{\ell(k)}, g_{\ell(k)}\) are Cauchy sequences in \(V^n_{A,\lambda}\) and \(V^{n-1}_{A,\lambda}\) respectively. To see this, take \(i, j \geq k\) and bound:

\[
\| f_{\ell(i)} - f_{\ell(j)} \|_{V^n_{A,\lambda}} \leq \| f_{\ell(i)} - f_{\ell(j)} \|_{V^n_{A,\lambda}} + \| f_{\ell(j)} - f_{\ell(j)} \|_{1-\varphi_k} \|_{V^n_{A,\lambda}}.
\]

\[
\leq \frac{2}{k} + \frac{1}{k} \| f_{\ell(j)} - f_{\ell(j)} \|_{1-\varphi_k} \|_{V^n_{A,\lambda}} \rightarrow 0,
\]

where we have used that \(\lambda'(v) \leq \frac{1}{|v|^2}\lambda(v)\) for \(|v| \geq k\). Hence \(f_{\ell(k)}\) is a Cauchy sequence. The proof for \(g_{\ell(k)}\) is similar. Therefore \(u_{\ell(k)}\) is precompact in \(X^n_{A,\lambda}\). \(\Box\)

We can now formulate the precise statement for the existence of solutions \(u_\varepsilon\) of (2.1) and convergence to a solution of the nonlinear Landau equation (2.3).

Theorem 2.6 Let \(m_0, \sigma > 0\) and \(m(\sigma^2, m_0)\) be the Maxwellian with mass \(m_0\) and standard deviation \(\sigma\):

\[
m(\sigma^2, m_0)(v) := m_0 e^{-\frac{|v|^2}{2\sigma^2}}.
\]

Let \(n \geq 6\) and \(v_0 \in H^n_A\) satisfy:

\[0 \leq v_0(v) \leq C e^{-\frac{1}{2}|v|}.
\]

There exist \(A, C(A) > 0, \delta_1, \varepsilon_0 \in (0, \frac{1}{2})\) such that for all \(\varepsilon, \delta_2 \in (0, \varepsilon_0) > 0\) the equation

\[
\begin{align*}
\partial_t u_\varepsilon = & \frac{1}{\varepsilon} \nabla \cdot \left( \int_0^1 K[u_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s, v) \, ds \right) \\
- & \frac{1}{\varepsilon} \nabla \cdot \left( \int_0^1 P[u_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s, v) \, ds \right)
\end{align*}
\]

(2.10)

has a strong solution \(u_\varepsilon \in V^n_{A,\lambda} \cap C^1([0, \delta_1]; H^{n-2}_A)\) up to time \(\delta_1\) with uniform bound:

\[
\| u_\varepsilon \|_{V^n_{A,\lambda}} + \| \partial_t u_\varepsilon \|_{V^{n-1}_{A,\lambda}} \leq C(A).
\]

(2.11)
Remark 2.7 Our result is valid for small initial perturbations \( u_0 + \delta \varepsilon u_0 \) of the Maxwellian and small times \( 0 \leq t \leq \delta_1 \). Notice that the functions \( u_t \) are solutions to (2.10) up to time \( \delta_1 \), but are defined also for later times. In the following, we will write \( C, c > 0 \) for generic large/small constants that are not dependent on other parameters.

Theorem 2.8 For \( n \geq 6 \) pick \( A \geq 1 \), \( \delta_1 \in (0, \frac{1}{2}] \) and \( \varepsilon, \delta_2 \) small enough such that Theorem 2.6 ensures the existence of solutions \( u_t \in V^n_{A, \lambda} \cap C^1([0, \delta_1]; H^{n-2}_A) \) of (2.10). Along a sequence \( \varepsilon_j \to 0 \) the \( u_{\varepsilon_j} \) converge \( u_{\varepsilon_j} \to u \) in \( V^{n-3}_{A, \lambda} \), \( u_{\varepsilon_j} \to u \) in \( V^n_{A, \lambda} \). \( \partial u_{\varepsilon_j} \to \partial u \) in \( V^{n-2}_{A, \lambda} \). The function \( u \in V^n_{A, \lambda} \cap C^1([0, \delta_1]; H^{n-4}_A) \) solves the limit equation up to times \( 0 \leq t \leq \delta_1 \):

\[
\begin{align*}
\partial_t u &= \nabla \cdot (K[u] \nabla u) - \nabla \cdot (P[u] u) \\
u(0, v) &= m(v) + \delta_2 v_0(v) \\
K[u](v) &= \int (k \otimes k) |\hat{\phi}(k)|^2 \delta(k \cdot (v - v')) \eta(|v - v'|^2) u(v') \, dk \, dv' \\
P[u](v) &= \int (k \otimes k) |\hat{\phi}(k)|^2 \delta(k \cdot (v - v')) \eta(|v - v'|^2) \nabla u(v') \, dk \, dv'.
\end{align*}
\]  

(2.12)

In order to show the existence of a strong solution to (2.10), we will consider mollifications of the equations first, and derive priori estimates that are independent of the mollification. We introduce the following notation.

Notation 2.9 Let \( \varphi_\gamma \) be a standard mollifier on \( \mathbb{R}^3 \). For \( 0 < \gamma \leq 1 \), define the regularized gradient \( r \nabla f(v) \) as \( r \nabla f(v) := \nabla (\varphi_\gamma \ast f) \). We define \( r \nabla \) to be the standard gradient for \( \gamma = 0 \). We will use the following conventions for Laplace transform and Fourier transform:

\[ \mathcal{L}(u)(z) = \int_0^\infty u(t) e^{-zt} \, dt \]

(2.13)

\[ \hat{u}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} u(v) e^{-ik \cdot v} \, dv. \]

(2.14)

Now we observe that if \( u_\varepsilon = u_0 + f_\varepsilon \) is a solution of (2.10), an equivalent way of stating this is

\[
\begin{align*}
\partial_t u_\varepsilon &= \frac{r}{\varepsilon} \nabla \cdot \left( \int_0^t K[u_\varepsilon + f_\varepsilon(s, \cdot)] \left( \frac{t-s}{\varepsilon} \right) r \nabla u_\varepsilon(s, v) \, ds \right) \\
&\quad - \frac{r}{\varepsilon} \nabla \cdot \left( \int_0^t P_\gamma[u_\varepsilon + f_\varepsilon(s, \cdot)] \left( \frac{t-s}{\varepsilon} \right) u_\varepsilon(s, v) \, ds \right) \\
u_\varepsilon(0, \cdot) &= u_0(\cdot), \quad P_\gamma = r \nabla \cdot K, \quad K \text{ as defined in } (2.2)
\end{align*}
\]  

(2.15)

holds for \( \gamma = 0 \). We will show a priori estimates for the above equation for \( 0 < \gamma \leq 1 \) and later recover the case \( \gamma = 0 \) as a limit. We start our analysis by writing \( K \) and \( P \) in a more convenient form.

Lemma 2.10 The operator \( K \) defined in (2.2) and \( P_\gamma = r \nabla \cdot K \) can be expressed by the formulas:

\[ K[u](t, v) = \int (k \otimes k) |\hat{\phi}(k)|^2 \cos(t(v - v') \cdot k) \eta(|v - v'|^2) u(v') \, dk \, dv' \]

(2.16)

\[ P_\gamma[u](t, v) = \int (k \otimes k) |\hat{\phi}(k)|^2 \cos(t(v - v') \cdot k) \eta(|v - v'|^2) r \nabla u(v') \, dk \, dv'. \]

(2.17)
Proof: The formula for $P_t$ follows from the one for $K$, so we only prove this one. Plancherel’s theorem allows to rewrite:

$$K[u](t,v) = \int \nabla \phi(x) \otimes \nabla \phi(x - t(v-v'))u(v')\eta(|v-v'|^2)\, dv'\, dx$$

$$= \int (k \otimes k)[\hat{\phi}(k)]^2 e^{-ik\cdot(v-v')}u(v')\eta(|v-v'|^2)\, dv'\, dk.$$

Since $K$ only takes real values, we can symmetrize the exponential and obtain

$$\int (k \otimes k)[\hat{\phi}(k)]^2 e^{-ik\cdot(v-v')}u(v')\eta(|v-v'|^2)\, dv'\, dk$$

$$= \int (k \otimes k)[\hat{\phi}(k)]^2 \cos (k \cdot (v-v')) u(v')\eta(|v-v'|^2)\, dv'\, dk,$$

proving the claim. \qed

We will omit the index $\gamma \geq 0$ in notation, when there is no risk of confusion. Controlling the non-linearity inside $K$ and $L$ strongly relies on being able to bound spatial derivatives of $u_e$. Therefore we consider differentiations of the equation. Let $\alpha \in \mathbb{N}^3$ be a multi-index. With the convention $(\vec{a}) = \prod_{j=1}^{3} (\alpha_j)$, the function $D^\alpha u_e = \frac{\partial^\alpha u_e}{\partial u_{x_1} \partial u_{x_2} \partial u_{x_3}}$ (formally) satisfies the equation:

$$\partial_t D^\alpha u_e = \sum_{\beta_1 + \beta_2 = \alpha} \left( \alpha \frac{1}{\beta_1} \left( \nabla \cdot \left( \int_0^t D^\beta K D^{\beta_2} u_e \, ds \right) - \nabla \cdot \left( \int_0^t D^\beta P D^{\beta_2} u_e \, ds \right) \right) \right).$$

In order to have a short notation for the terms appearing on the right-hand side of the equation above, we introduce the following notation.

Notation 2.11 Let $n \in \mathbb{N}$ and $\alpha, \beta$ be multi-indices with $\beta \leq \alpha$, $|\alpha| \leq n - 1$ and $v, u_e \in V_{A,\lambda}^n$. For $\gamma \in (0,1]$ we define:

$$A_{\gamma}^{\alpha,\beta}[v](u_e) = \frac{1}{\epsilon} \left( \int_0^t D^\gamma K[v](s) \left( \frac{t-s}{\epsilon}, v \right) \nabla D^{\alpha-\beta} u_e(s,v) \, ds \right)$$

$$- \frac{1}{\epsilon} \left( \int_0^t D^\gamma P_t[v](s) \left( \frac{t-s}{\epsilon}, v \right) D^{\alpha-\beta} u_e(s,v) \, ds \right).$$

Furthermore, for $m \in \mathbb{N}$, $u \in V_{A,\lambda}^m$, we set:

$$|u|_{Fm}(z,v) := \sum_{|\beta| \leq m} |L(D^\beta u)(z,v)|.$$  \hspace{1cm} (2.19)

The equation (2.15) has an averaged in time coercivity property, which we will prove by showing nonnegativity for certain quadratic functionals $Q$. This allows to show that $u_e$ inherits decay and regularity properties from the initial datum. We have the following basic a priori estimate for solutions $u_e$ of (2.15):

Lemma 2.12 Let $n \in \mathbb{N}$, $A, \epsilon, \gamma > 0$ and $u_e \in C^1([0,T];H_{-1}^n)$ be a solution to (2.15) for $T > 0$ arbitrary. Then for $|\alpha| \leq n$ we can bound:

$$A \int_0^T \int_0^v \lambda(v)|D^\alpha u_e(t,v)|^2 e^{-\lambda t} \, dt \, dv \leq -2Q_{\gamma,n}^{\alpha}[u_0 + f_{\epsilon}(u_e 1_{[0,T]})] + \| \lambda^2 D^\alpha u_0 \|_{L^2}^2.$$  \hspace{1cm} (2.18)
Here $Q^u_{t,A}[v](u)$ is given by (we drop the index $\gamma$ if there is no risk of confusion):

$$Q^u_{t,A}[v](u) = \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) Q^{u,\beta}_{t,A}[v](u)$$  \hspace{1cm} (2.20)

$$Q^{u,\beta}_{t,A}[v](u) = \int_0^T \int_{\mathbb{R}^3} \frac{e^{-\beta t}}{\varepsilon} \nabla(D^u u(t)) \lambda \int_0^t D^{u-\beta} K[v(s)] \frac{I-s}{\varepsilon} \nabla D^\beta u(s) \, ds \, dt$$

$$- \int_0^T \int_{\mathbb{R}^3} \frac{e^{-\beta t}}{\varepsilon} \nabla(D^u u(t)) \int_0^t D^{u-\beta} P_r[v(s)] \frac{I-s}{\varepsilon} D^\beta u(s) \, ds \, dt. \hspace{1cm} (2.21)$$

$$Q^{u,\beta}_{t,A}[v](u) = \int_0^T \int_{\mathbb{R}^3} \frac{e^{-\beta t}}{\varepsilon} \nabla(D^u u(t)) \lambda \int_0^t D^{u-\beta} K[v(s)] \frac{I-s}{\varepsilon} \nabla D^\beta u(s) \, ds \, dt$$

$$- \int_0^T \int_{\mathbb{R}^3} \frac{e^{-\beta t}}{\varepsilon} \nabla(D^u u(t)) \int_0^t D^{u-\beta} P_r[v(s)] \frac{I-s}{\varepsilon} D^\beta u(s) \, ds \, dt. \hspace{1cm} (2.22)$$

**Proof:** Follows by a simple computation:

$$\mathcal{A} \int_0^T \int_{\mathbb{R}^3} \lambda(t) \left\{ D^u u_e(t) \right\}^2 e^{-\beta t} \, dt \, dv$$

$$= - \int_0^T \int_{\mathbb{R}^3} \lambda(t) \int_0^t \frac{e^{-\alpha t}}{\varepsilon} \nabla(D^u u_e(t)) \frac{I-s}{\varepsilon} \nabla D^\alpha u_e(s) \, ds \, dv$$

$$\leq 2 \int_0^T \int_{\mathbb{R}^3} \lambda(t) \left\{ D^u u_e(t) \right\}^2 \, dt \, dv + \int \lambda(t) \left\{ D^u u_0 \right\}^2 \, dv$$

$$- 2Q^{u,\alpha}_{t,A}[u_0 + f_e \mathfrak{L} \{ u \cdot 1_{[0,T]} \} + \lambda \frac{1}{2} D^u u_0 \right\}_L^2,$$

where in the last line the equation is used. $\square$

The following analogue of Plancherel’s theorem for Laplace transforms will be useful throughout the paper.

**Lemma 2.13** Let $\mu_A(dt) := e^{-\alpha t} \, dt$. Then for $u, v \in L^2(\mu_A)$ we have:

$$(2\pi)^\frac{1}{2} \int_0^\infty e^{-\alpha t} \mathfrak{m}(t) v(t) \mu_A(dt) = \int_\mathbb{R} \mathcal{L}(u) \left( \frac{A}{2} + i\omega \right) \mathcal{L}(v) \left( \frac{A}{2} + i\omega \right) \, d\omega.$$

Our proof strongly relies on the geometry of both complex and real vectors. To avoid confusion we introduce the following notation.

**Definition 2.14** For $v, w \in \mathbb{R}^3$ we will use the notation $v \cdot w = \sum_i v_i w_i$ for the Euclidean scalar product. The inner product of complex vectors $V, W \in \mathbb{C}^3$ we denote by $\langle V, W \rangle = \sum_i \overline{V_i} W_i$. We will use the notation $| \cdot |$ for the vector norms induced by each of the inner products, as well as the matrix norm induced by this norm. Moreover for $0 \neq V \in \mathbb{C}^3$ and $W \in \mathbb{C}^3$ we define the orthogonal projections $P_v W$ and $P_v^\perp W$ as:

$$P_v W := \left( \frac{\langle V, W \rangle}{|V|^2} \right) \frac{V}{|V|}, \hspace{1cm} P_v^\perp W := W - P_v W.$$  \hspace{1cm} (2.23)

For future reference, we compute the Laplace transform of $K[u](t, v)$ in $t$. With our particular choice of potential, some of the integrals are explicitly computable, as is stated in the following auxiliary Lemma.

**Lemma 2.15** For $\Re(z) \geq 0$, $v \in \mathbb{R}^3$ let $M_1(z, v)$, $M_2(z, v)$ be the matrix-valued functions defined by

$$M_1(z, v) := \pi^2 \frac{1}{4|z|} \frac{1}{1 + \frac{z}{|v|}} P_v^\perp, \hspace{1cm} M_2(z, v) := \pi^2 \frac{1}{4|z|} \left( \frac{z}{|v|} \right)^2 P_v.$$

$$M_1(z, v) := \pi^2 \frac{1}{4|z|} \frac{1}{1 + \frac{z}{|v|}} P_v^\perp, \hspace{1cm} M_2(z, v) := \pi^2 \frac{1}{4|z|} \left( \frac{z}{|v|} \right)^2 P_v.$$  \hspace{1cm} (2.24)
Then we have the following identity:

\[
\int (k \otimes k) |\hat{\phi}(k)|^2 \frac{z}{z^2 + (k \cdot v)^2} \, dk = M_1(z, v) + M_2(z, v).
\] (2.25)

**Proof:** We decompose \( k \in \mathbb{R}^3 \) into \( k = uw + u\perp, \) where \( w = \frac{v}{|v|} \). We insert the explicit form of the Fourier transform of \( \phi \) (cf. (2.5)) to rewrite the integral as (here \( a \otimes a = a \otimes a \)):

\[
\int (k \otimes k) |\hat{\phi}(k)|^2 \frac{z}{z^2 + (k \cdot v)^2} \, dk = \int_{\mathbb{R}} \int_{\text{span}(w)^\perp} \frac{(uw + u\perp)^\otimes}{(1 + u^2 + |u\perp|^2)^3} \, du \frac{z}{z^2 + (u|v|)^2} \, du
\]

\[
= \frac{1}{|v|} \int_{\mathbb{R}} \int_{\text{span}(w)^\perp} \frac{(uw + u\perp)^\otimes}{(1 + u^2 + |u\perp|^2)^3} \, du \frac{z}{(\frac{z}{|v|})^2 + u^2} \, du
\]

\[
= \frac{1}{|v|} \int_{\mathbb{R}} \int_{\text{span}(w)^\perp} \frac{(uw)^\otimes + (u\perp)^\otimes}{(1 + u^2 + |u\perp|^2)^3} \, du \frac{z}{(\frac{z}{|v|})^2 + u^2} \, du.
\]

where we used that the mixed terms \( uw \otimes u\perp \) do not contribute to the integral due to the symmetry of the integrand. Now the inner integral is explicit:

\[
\int_{\text{span}(w)^\perp} \frac{(uw)^\otimes + (u\perp)^\otimes}{(1 + u^2 + |u\perp|^2)^3} \, du = u^2 \int_0^\infty \frac{2\pi r P_w}{(1 + u^2 + r^2)^3} \, dr + \int_0^\infty \frac{\pi r^3 P_{u\perp}}{(1 + u^2 + r^2)^3} \, dr
\]

\[
= \frac{\pi u^2}{2(1 + u^2)^2} P_w + \frac{\pi}{4(1 + u^2)} P_{u\perp}^\perp.
\]

Inserting this back into the full integral gives two explicit integrals:

\[
\frac{1}{|v|} \int_{\mathbb{R}} \int_{\text{span}(w)^\perp} \frac{(uw)^\otimes + (u\perp)^\otimes}{(1 + u^2 + |u\perp|^2)^3} \, du \frac{z}{(\frac{z}{|v|})^2 + u^2} \, du
\]

\[
= \frac{1}{|v|} \int_{\mathbb{R}} \left( \frac{\pi u^2}{2(1 + u^2)^2} P_w + \frac{\pi}{4(1 + u^2)} P_{u\perp}^\perp \right) \frac{z}{(\frac{z}{|v|})^2 + u^2} \, du
\]

\[
= \frac{\pi^2}{4|v|} \left( \frac{z}{(1 + \frac{z}{|v|})^2} P_w + \frac{1}{1 + \frac{z}{|v|}} P_{u\perp}^\perp \right)
\]

\[
= M_1(z, v) + M_2(z, v),
\]

which implies the statement of the lemma. \( \square \)

Now the Laplace transform \( \mathcal{L}(K[u]) \) can be rewritten in a more explicit form.

**Lemma 2.16** Let \( u \in \mathcal{H}_q^n, n \geq 2 \) and \( \mathcal{L}(K[u])(z, v) \) be the Laplace transform of \( K[u] \), i.e.

\[
\mathcal{L}(K[u])(z, v) = \int_0^\infty K[u](t, v)e^{-zt} \, dt.
\]

Then \( \mathcal{L}(K[u]) \) is given by the formula:

\[
\mathcal{L}(K[u])(z, v) = \int (M_1 + M_2)(z, v - v')u(v')\eta(|v - v'|^2) \, dv'.
\] (2.26)
In particular, the matrix $\mathcal{L}(K[u])$ is symmetric. For the operator $P_\gamma$ introduced in (2.15) we have the formula:

$$\mathcal{L}(P_\gamma[u])(z, v) = \int (M_1 + M_2)(z, v - v')\nabla u(v')\eta(|v - v'|^2) \, dv'.$$  (2.27)

Proof: Follows from $\mathcal{L}(\cos(\alpha t))(z) = \frac{1}{z^2 + \alpha^2}$, Lemma 2.10 and Lemma 2.15.

2.2 Strategy of the proofs of Theorems 2.6 and 2.8

We can now outline the structure of this paper, and introduce the key steps in the proofs of the Theorems 2.6 and 2.8.

(i) In Section 3 we prove that the linear equation

$$\partial_t u_\epsilon = \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t K[u_\epsilon] \left( \frac{t - s}{\epsilon}, v \right) \nabla u_\epsilon(s, v) \, ds \right)$$

$$- \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t P_\gamma[u_\epsilon] \left( \frac{t - s}{\epsilon}, v \right) u_\epsilon(s, v) \, ds \right)$$  (2.28)

has a solution $u_\epsilon \in V_{A, \lambda}^n \cap C^1(\mathbb{R}^+; H_{\lambda}^{n-2})$. The proof is based on the fact that the equation is dissipative in a time averaged sense, and strongly relies on the convolution structure of the equation in Laplace variables. Symbolically the equation in Laplace variables looks similar to:

$$z\mathcal{L}(u)(z, v) = \nabla \cdot (K(z, v)\nabla \mathcal{L}(u)(z, v)) + u_0(v).$$

We show that for $\Re(z) > 0$, the real part of the matrix $K(z, v)$ is nonnegative. This is quantified in Lemma 3.7 in terms of the quadratic operators $Q_{\epsilon, A}^n[u_0]$ (cf. (2.20)).

(ii) In order to solve the nonlinear problem, we have to allow for time dependent functions inside the operator $K$. We therefore consider equation (2.15) for a fixed function $f_\epsilon$ and mollified derivatives $\nabla$:

$$\partial_t u_\epsilon = \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t K[u_\epsilon + f_\epsilon(s, \cdot)] \left( \frac{t - s}{\epsilon}, v \right) \nabla u_\epsilon(s, v) \, ds \right)$$

$$- \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t P_\gamma[u_\epsilon + f_\epsilon(s, \cdot)] \left( \frac{t - s}{\epsilon}, v \right) u_\epsilon(s, v) \, ds \right)$$  (2.29)

In Subsection 4.1 we identify a closed, nonempty, convex subset $\Omega$ of $X_{A, \lambda}^n$ (defined in (4.6)) such that the local in time solution operator $\Psi_{\delta_1}$ to (2.29):

$$\Psi_{\delta_1} : \Omega \rightarrow X_{A, \lambda}^n$$

$$(f, F) \mapsto (u - u_0)\kappa_{\delta_1}, A_{\gamma, 0}^n[f](u)\kappa_{\delta_1}) \quad \text{where } u \text{ solves (2.29)}$$  (2.30)

is well-defined. Here $\kappa_{\delta_1}$ is a cutoff function that localizes to small times. Notice that the solution operator maps from $X_{A, \lambda}^n$ to $X_{A, \lambda}^n$, thus we gain decay. The proof is based on proving that
replacing the constant kernel $K[u_0]$ by $K[u_0 + f]$ amounts to a small perturbation. The main assumption for this, and the defining property of the set $\Omega$ is that for some $A, R > 0$ and small $\delta > 0$, we can bound $\mathcal{L}(f)$ on the line $\Re(z) = \frac{A}{2}$ by:

$$|\mathcal{L}(f)(z, v)| \leq \left( \frac{\delta}{1 + |z|^2} + \frac{Re[z]}{(1 + \varepsilon |z|)(1 + |z|^2)} \right) e^{-\frac{1}{2}|v|}. \quad (2.31)$$

Under assumption (2.31) we obtain an a priori estimate on the solutions and their time derivatives:

$$\|\Psi_{\delta_1}(f, F)\|_{X^{n+1}_{A, d}} + \|\partial_t \Psi_{\delta_1}(f, F)\|_{X^{n+2}_{A, d}} \leq C$$

$$\|\Psi_{\delta_1}(f, F)\|_{X^{n+1}_{A, d}} + \|\partial_t \Psi_{\delta_1}(f, F)\|_{X^{n+2}_{A, d}} \leq C(\gamma). \quad (2.32)$$

It is crucial that the first estimate is uniform in the mollifying parameter $\gamma > 0$. In Section 4.2 we prove that the operator $\Psi_{\delta_1}$ introduced in (2.30) leaves the set $\Omega$ invariant, for $\delta_1 > 0$ small, close to the Maxwellian and $\varepsilon > 0$ small.

Now, for $\gamma > 0$, we infer the existence of a fixed point of $\Psi_{\delta_1}$ from (2.32) and Schauder’s theorem. Here we use bounded sequences in $X^{n+1}_{A, d}$ with bounded time derivative are precompact in $X^{n+1}_{A, d}$, as proved in Lemma 2.5. This compactness property allows to take the limit $\gamma \to 0$ and thus to prove Theorem 2.6. Here we make use of the uniform estimate in (2.32). The proof of Theorem 2.8 follows by passing $\varepsilon \to 0$ using Lemma 2.5 yet again.

A key point of the analysis is the invariance of the set $\Omega$ under $\Psi_{\delta_1}$, which is proved in Section 4.2. The proof relies on recovering the decay assumption (2.31). We can think of functions $f$ satisfying (2.31) as a sum $f = f_1 + f_2$. Here $f_1$ satisfies $|\mathcal{L}(f_1)(z)| \leq \frac{\delta}{1+|z|^2}$, which can be thought of as an estimate of the form $\|\partial_t^2 f_1\|_{L^1} \leq \delta$, and $f_2$ satisfies $|\mathcal{L}(f_2)(z)| \leq \frac{Re[z]}{(1 + \varepsilon |z|)(1 + |z|^2)}$, which can be understood as $\|\partial_t f_2\|_{L^1} \leq Re$ and $\|\partial_t^2 f_2\|_{L^1} \leq R$. This is only a heuristic consideration, since $L^\infty/L^1$ duality does not hold for Laplace transforms. A typical function of this form is $f^t\gamma(t) = e^{2t}\Phi(t/\varepsilon)$. The behavior of $f_1$ close to $t = 0$ is more complicated, since it involves a boundary layer. Indeed, there is necessarily a boundary layer in $\partial_t u_\varepsilon$ in equation (2.29). To see this, let $u$ be the solution of the limit (Landau-) equation (2.3), and $u_\varepsilon$ the solution to (2.29). Then, starting away from equilibrium, we have:

$$\partial_t u_\varepsilon(0, v) = 0, \quad \partial_t u(0, v) \neq 0.$$

So in the limit $\varepsilon \to 0$, the second derivative necessarily grows infinitely large close to the origin.

The quadratic decay of the Laplace transforms can be obtained by a bootstrap argument. To fix ideas, we observe that (2.29) in Laplace variables is similar to:

$$z\mathcal{L}(u - u_0) = \nabla \cdot \left( \tilde{K}(\varepsilon z)(\nabla \mathcal{L}(u) + \nabla \mathcal{L}(u) * \mathcal{L}(f)) \right). \quad (2.33)$$

In Subsection 4.1 we prove that $\nabla^m \mathcal{L}(u)$ are bounded in a weighted $L^2$ space in time and velocities. This can be bootstrapped to pointwise estimates: First we remark that localizing supp $u \subset [0, 1] \times \mathbb{R}^3$ gives an $L^\infty$ estimate for $\nabla^m \mathcal{L}(u)$. Assuming $|\tilde{K}(z)| \leq \frac{C}{1 + |z|^2}$, equation (2.33) gives an estimate like:

$$|\nabla^m \mathcal{L}(u - u_0)(z, v)| \leq \frac{C}{(1 + \varepsilon |z|)|z|} e^{-\frac{1}{2}|v|}.$$

Plugging this estimate back into (2.33) proves quadratic decay of the Laplace transforms:

$$|\nabla^m \mathcal{L}(u - u_0)(z, v)| \leq \frac{C}{(1 + \varepsilon |z|)|z|^2} e^{-\frac{1}{2}|v|}.$$
In order to show invariance of the set $\Omega$ we need the same estimate with a small prefactor, as in estimate (2.31). We split the solution into a well-behaved part and the boundary layer mentioned before. For the first part, we use smallness of the cutoff time $\delta_1 > 0$ to get a small prefactor additional to the quadratic decay. The estimate of the boundary layer, close to the Maxwellian, is obtained by isolating and estimating it explicitly. This is the content of Subsection 2.2.2 and the most delicate part of the analysis.

### 2.3 A well-posedness result for the regularized problem (2.29)

Before we start with the analysis of the equation in more detail, we first prove that the equation (2.29) with frozen nonlinearity indeed has a solution. This standard Picard-iteration argument is given in the following Lemma.

**Lemma 2.17** Let $n \in \mathbb{N}$, $\gamma, \epsilon > 0$ and $u_0 \in H^n$. Further assume there is a constant $C > 0$ such that $|f_\epsilon(t, v)| \leq Ce^{-\frac{1}{\gamma}|v|}$ and $\text{supp } f_\epsilon \subset [0, 1]$. Then there exists a (unique) global in time solution $u_\epsilon \in C^1([0, \infty); H^n)$ to:

\[
\partial_t u_\epsilon = \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t K[u_0 + f_\epsilon(s)] \left( \frac{t-s}{\epsilon}, v \right) \nabla u_\epsilon(s, v) \, ds \right) - \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t P_\gamma[u_0 + f_\epsilon(s)] \left( \frac{t-s}{\epsilon}, v \right) u_\epsilon(s, v) \, ds \right) \quad (2.34)
\]

\[u_\epsilon(0, \cdot) = u_0(\cdot).\]

**Proof:** For better notation, we introduce a shorthand for the right-hand side of the equation:

\[B(u)(t, t', v) := \frac{1}{\epsilon} \nabla \cdot \left( \int_{t'}^t K[u_0 + f_\epsilon(s)] \left( \frac{t-s}{\epsilon}, v \right) \nabla u(s, v) \, ds \right) - \frac{1}{\epsilon} \nabla \cdot \left( \int_{t'}^t P_\gamma[u_0 + f_\epsilon(s)] \left( \frac{t-s}{\epsilon}, v \right) u(s, v) \, ds \right).\]

The claim follows from a standard Picard-type argument. Let $T > 0$ to be chosen later. Consider the mapping

\[D : C^1([0, T]; H^n) \rightarrow C^1([0, T]; H^n) \]

\[u \mapsto D(u),\]

where $D(u)$ is given by:

\[D(u)(t, v) := u_0(v) + \int_0^t B(u)(s, v) \, ds. \quad (2.35)\]

The mapping is $D$ contractive for small times. More precisely we have:

\[\|B(u)(t, t', \cdot)\|_{H^n} \leq C|t - t'| \sup_{s \leq s' \leq t} \|u(s, \cdot)\|_{L^2}. \quad (2.36)\]

Hence, there exists a $T_1 > 0$ such that $D$ is contractive and we obtain a unique solution for $T \leq T_1$. Assume we already have constructed the solution $u$ up to time $mT_1$ for $m \in \mathbb{N}$. Consider the mapping:

\[D_m : C^1([mT_1, (m + 1)T_1]; H^n) \rightarrow C^1([mT_1, (m + 1)T_1]; H^n) \]

\[w \mapsto D_m(w) = u(mT_1, v) + \int_{mT_1}^T B(w)(s, v) \, ds.\]
By (2.36) this mapping is contractive and we can pick the same small time $T_1$ in each step of the induction.

3 The linear equation (2.28)

The linear equation (2.28) has an average in time coercivity property. We will prove this using geometric arguments that resemble the ones used for the Landau equation, see for instance [11]. For shortness we introduce the following notation.

\textbf{Notation 3.1} For $z \in \mathbb{C}$ and $v \in \mathbb{R}^3$ define:

$$\alpha(z, v) := \frac{|\Im(z)|}{1 + |v|}, \quad \beta(z, v) := \frac{|\Re(z)|}{1 + |v|}. \quad (3.1)$$

Further we define the following positive functions $C_1$, $C_2$ and $C_3$:

$$C_1(z, v) = \frac{1}{(1 + |v|)(1 + \alpha(z, v))^2} \quad (3.2)$$

$$C_2(z, v) = \frac{\beta(z, v) + \alpha(z, v)^2}{(1 + |v|)(1 + \alpha(z, v))^4} \quad (3.3)$$

$$C_3(z, v) = \frac{\beta(z, v) + \alpha(z, v) + \alpha(z, v)^2}{(1 + |v|)(1 + \alpha(z, v))^4} \quad (3.4)$$

Let $0 \neq v \in \mathbb{R}^3$, $V, W \in \mathbb{C}^3$. We define the anisotropic norm:

$$|W|_v := |P^+_v W| + \frac{|P_v W|}{1 + |v|}. \quad (3.5)$$

and the weight functionals $B_1(z, v)(V, W)$, $B_2(z, v)(V, W)$ given by:

$$B_1(V, W) = C_1(z, v)|V|_v|W|_v + C_2(z, v)|P_v V||P_v W| \quad (3.6)$$

$$B_2(V, W) = C_1(z, v)|V|_v|W|_v + C_3(z, v)|P_v V||P_v W|. \quad (3.7)$$

The following straightforward analysis lemma we will use to bound real and imaginary part of the matrices $M_t$ defined in (2.24) from above and below.

\textbf{Lemma 3.2} Let $z \in \mathbb{C}$ with $0 \leq \Re(z) \leq 1$. The following bounds hold:

$$\Re\left(\frac{z}{1 + z}\right) \geq c \frac{\Re(z) + |\Im(z)|^2}{(1 + |\Im(z)|)^4} \quad (3.8)$$

$$|\Im\left(\frac{z}{1 + z}\right)| \leq C \frac{\Re(z) + |\Im(z)| + |\Im(z)|^2}{(1 + |\Im(z)|)^3} \quad (3.9)$$

$$\Re\left(\frac{1}{1 + z}\right) \geq c \frac{1}{(1 + |\Im(z)|)^2} \quad (3.10)$$

$$|\Im\left(\frac{1}{1 + z}\right)| \leq C \frac{|\Im(z)|}{(1 + |\Im(z)|)^2}. \quad (3.11)$$
Proof: To prove (3.8)-(3.9), we rewrite the fraction as:

\[
\frac{z}{(1+z)^2} = \frac{z+2|z|^2 + \overline{z}|z|^2}{|1+z|^4}.
\]

Since the real part of \(z\) is bounded and nonnegative by assumption, (3.8) follows immediately. For the proof of (3.9) we include the computation:

\[
|\Re\left(\frac{z}{(1+z)^2}\right)| \leq C \frac{|\Re(z)| + (\Re(z)^2 + |\Im(z)|^2)(1 + |\Im(z)|)}{|1+z|^4} \leq C \frac{\Re(z) + |\Im(z)| + |\Im(z)|^2}{|1+z|^3},
\]

proving also the second claim. The inequalities (3.10) and (3.11) are immediate. \(\square\)

The following simple lemma provides an estimate for the derivatives of the matrices \(M_i\) defined in (2.24).

**Lemma 3.3** For a multi-index \(\beta \in \mathbb{N}^3\), \(\Re(z) \geq 0\), \(i = 1, 2\) and \(v \in \mathbb{R}^3\), \(V, W \in \mathbb{C}^3\), we can estimate:

\[
|\langle V, D^{\beta}(M_i(z,v)\eta(|v|^2))W \rangle| \leq \frac{C_{|\beta|}|V||W|}{(1 + |v|^{\beta_1+1})(1 + a(z,v))} \eta(16|v|^2).
\]

(3.12)

Here \(\eta\) is the cutoff function introduced in Notation 2.1.

**Proof:** With Leibniz’s rule, we can split the derivative into:

\[
D^{\beta}(\langle M_1 + M_2\rangle(z,v)\eta(|v|^2)) = \sum_{\beta_1 \leq \beta} \binom{\beta}{\beta_2} D^{\beta_1}(\langle M_1 + M_2\rangle(z,v))D^{\beta_2}(\eta(|v|^2)).
\]

By construction of the fixed cutoff function \(\eta\) we can estimate:

\[
|\nabla^m \eta(r)| \leq \frac{C}{1 + |r|^m} \eta(16r).
\]

(3.13)

We write \(M_1, M_2\) defined in (2.24) as:

\[
M_1(z,v) = \frac{\pi^2}{4(z+|v|)}P_v^\perp, \quad M_2(z,v) = \frac{\pi^2 z}{4(z + |v|)^2}P_v.
\]

The operators \(P_v, P_v^\perp\) are zero-homogeneous in \(v\). So for every \(c > 0\) we can estimate:

\[
|\nabla^m v M_i(z,v)| \leq \frac{C|M_i(z,v)|}{1 + |v|^n} \leq \frac{C}{(1 + |v|)^{n+1}(1 + a(z,v))} \quad \text{for } i = 1, 2, |v| \geq c > 0.
\]

(3.14)

Combining (3.13) and (3.14) gives the claim. \(\square\)

The following Lemmas prove coercivity of the matrix \(L(K)|u|(v)\), which becomes anisotropic as \(|v| \to \infty\). The crucial geometric argument is contained in the following Lemma, that in our setting needs to be valid for complex vectors (since we apply it to Laplace transforms).
Lemma 3.4  For $0 \neq V \in \mathbb{C}^3$ and $0 \leq r \leq 1$, let $D_V(r)$ be given by:

$$D_V(r) = \{ v' \in \mathbb{R}^3 : \frac{1}{2} \leq |v'| \leq 1, \frac{|\langle v', V \rangle|}{|v'||V|} \geq r \}.$$ 

There exists a constant $c > 0$ such that for all $v \in \mathbb{R}^3, |v| \geq 2$ the following statements hold:

for $0 \neq V \in \mathbb{C}^3$:

$$\text{Vol}(D_V(1/8)) \geq c, \quad (3.15)$$

for $V \in \mathbb{C}^3 \exists \ 0 \neq W \in \mathbb{C}^3 \forall v' \in D_W(1/8) : |P_{v' \cdot V}^1 V| + |P_{v' \cdot (-v)}^1 V| \geq c|V|_v, \quad (3.16)$$

where the anisotropic norm $| \cdot |_v$ was introduced in (3.15). Furthermore for $v \in \mathbb{R}^3, V \in \mathbb{C}^3$, define

$$E(v, V) = \{ v' \in B_1(0) \subseteq \mathbb{R}^3 : |\langle v' + v, V \rangle| \geq |\langle v, V \rangle| \}.$$ 

There exists $c > 0$ such that for all $v \in \mathbb{R}^3, |v| \geq 2$:

$$|P_{v' \cdot V}^1 V| \geq c|P_v V| \quad \text{for } v' \in E(v, V) \quad (3.17)$$

$$\text{Vol}(E(v, V)) \geq c > 0. \quad (3.18)$$

Proof: The inequality (3.15) is clear if $0 \neq V \in \mathbb{R}^3$ is real. Moreover, there is a constant $c > 0$ such that $\text{Vol}(D_V(r)) \geq c > 0$ for $0 \leq r \leq \frac{3}{4}$ and $V \in \mathbb{R}^3$. Let now $V = V_R + iV_I \in \mathbb{C}^3$, where at least one of the vectors $V_R, V_I \in \mathbb{R}^3$ is nonzero, and let $W$ be the longer vector of $V_R, V_I$. We define $\tilde{D}_V = D_W(\frac{1}{2})$. Then we have $\frac{|\langle v', V \rangle|}{|v'||V|_v} \geq \frac{|W|}{|V||V|} \geq \frac{1}{8}$ for $v' \in \tilde{D}_V$. Since $W \in \mathbb{R}^3$ we have $\text{Vol}(D_W(\frac{1}{2})) \geq c > 0$, so in particular

$$U(v, V) := \{ v' \in \mathbb{R}^3 : \frac{|\langle v', V \rangle|}{|v'||V|} \geq \frac{1}{8} \}$$

satisfies $\text{Vol}(U(v, V)) \geq c > 0$. Since $U(v, V)$ is homogeneous, the set

$$U(v, V) \cap \{ v' \in \mathbb{R}^3 : \frac{1}{2} \leq |v'| \leq 1 \} \subseteq D_V(\frac{1}{8})$$

also has volume uniformly bounded below, which implies the claim (3.15). For the proof of (3.16), let $v \in \mathbb{R}^3, |v| \geq 2$ and $V \in \mathbb{C}^3$ be a unit vector such that $V = V_1 + V_2, V_1 = P_v V, V_2 = P_{-v} V$. Let us first assume that $V_2 \neq 0$. We claim that (3.16) holds with $W = V_2$. To this end, let $|v| \geq 2$ and $v' \in D_{V_2}(1/8)$, so in particular $|v'| \leq 1$. Then the angle $\psi$ between $v$ and $v - v'$ is bounded by $|\psi| \leq \frac{\pi}{6}$, hence:

$$|P_{v' \cdot V_2}^1 V_2| = |P_{v' \cdot v'}^1 P_v V| \leq \frac{1}{2}|V_2|,$$ 

therefore:

$$|P_{v' \cdot V}^1 V_2| = |V_1 - P_{v' \cdot V}^1 V_1 + V_2 - P_{v' \cdot V}^1 V_2| \geq |V_1 - P_{v' \cdot V}^1 V_1 + V_2| - \frac{1}{2}|V_2|$$

$$\geq |P_{V_2}(V_1 - P_{v' \cdot V}^1 V_1 + V_2)| - \frac{1}{2}|V_2| = |V_2 - P_{V_2} P_{v' \cdot V}^1 V_1| - \frac{1}{2}|V_2|. \quad (3.19)$$

We rewrite the first term on the right-hand side as:

$$|V_2 - P_{V_2} P_{v' \cdot V}^1 V_1| = |V_2| - \langle \frac{V_2}{|V_2|}, P_{v' \cdot V}^1 V_2 \rangle. \quad (3.20)$$
Let $\zeta(v') = \langle \frac{V_2}{|V_2|}, P_{v-v'} V_1 \rangle$. We observe that $V_2 = P_v^+ V$ and $V_1 = p v$ for some $p \in \mathbb{C}$, so:

$$
\zeta(v') = \langle \frac{V_2}{|V_2|}, \frac{v - v'}{|v - v'|} \rangle \langle \frac{v - v'}{|v - v'|}, V_1 \rangle = \frac{p}{|v - v'|} \langle V_2, -v' \rangle \langle v - v', v \rangle.
$$

(3.21)

Since $|v'| \leq \frac{1}{2} |v|$, we have $\langle \frac{v - v'}{|v - v'|}, v \rangle \geq \frac{1}{2} |v|$. This implies the lower bound:

$$
|\zeta(v')| \geq \frac{1}{4} \left( \frac{|pv|}{1 + |v|} \right) \left( \frac{V_2}{|V_2|}, -v' \right) \geq \frac{c|V_1|}{1 + |v|} \quad \text{for } v' \in D_{V_2}(1/8).
$$

(3.22)

Now we claim that the real part of $\zeta(v')$ is nonpositive, after possibly changing the sign of $v'$:

$$
\Re(\zeta(v')) \leq 0, \quad \text{or } \Re(\zeta(-v')) \leq 0.
$$

(3.23)

To see this, we use (3.21) and $\langle \frac{v - v'}{|v - v'|}, v \rangle \geq 0$. Inserting the estimates (3.22), (3.23) and the lower bound $|z| \geq \frac{1}{\sqrt{2}} \left( |\Re(z)| + |\Im(z)| \right)$ into (3.20) we obtain:

$$
|V_2 - P_{v} P_{v-v'} V_1| + |V_2 - P_{-v} P_{v-(-v')} V_1| \geq \frac{1}{\sqrt{2}} \left( |V_2| + \frac{|z| |V_1|}{1 + |v|} \right).
$$

We plug this back into (3.19) and add the corresponding term for $-v'$ to prove (3.16) in the case $V_2 \neq 0$. In order to prove (3.16) for $V_2 = 0$, we remark that the estimate is homogeneous in $V$, so it suffices to prove it for $|V| = 1$, when it follows by continuity from the case $V_2 = 0$.

The estimate (3.17) follows from the observation that for $v' \in E(v, V)$ we have

$$
|P_{v-v'} V| = \left| \langle \frac{v - v'}{|v - v'|}, V \rangle \right| \geq \frac{1}{2} |P_{v} V|.
$$

Finally (3.18) is a consequence of $E(v, V)$ containing either $v'$ or $-v'$ for every $v' \in B_1(0)$.

Lemma 3.4 proves lower bounds for the projections $|P_{v-v'} V|$ respectively $|P_{v-(-v')} V|$ on a set (of $v'$) with uniformly positive Lebesgue measure. We now show that this implies a lower bound for the integrals (2.26), (2.27) representing $L(K)$, $L(P)$.

**Lemma 3.5** Let $z \in C$ with $0 \leq \Re(z) \leq 1$ and $\beta$ be a multi-index. Let $V, W \in C^3$ be complex vectors. Further let $n \geq 1$ and $u_0 \in H^\alpha$ satisfy the pointwise estimates:

$$
c \mathbb{1}_{|v| \leq 4}(v) \leq u_0(v) \leq C e^{-\frac{1}{4} |v|}, \quad \text{for } c > 0.
$$

Recall $B_1, B_2$ as defined in (3.6)-(3.7) and $C_1$ defined in (3.2). Then there holds:

$$
\int_{\mathbb{R}^3} \langle V, \Re(M_1 + M_2)(z, v - v') V \rangle u_0(v') \eta \, dv' \geq c B_1(z, v) \langle V, V \rangle
$$

(3.24)

$$
\int_{\mathbb{R}^3} |\langle V, (M_1 + M_2)(z, v - v') W \rangle | u_0(v') \eta \, dv' \leq C (1 + \alpha(z, v)) B_2(z, v) \langle V, W \rangle
$$

(3.25)

$$
\int_{\mathbb{R}^3} |\langle V, D^\beta ((M_1 + M_2)(z, v - v') \eta) W \rangle | u_0(v') \, dv' \leq C \frac{(1 + \alpha(z, v))}{(1 + |v|)^{|eta|}} C_1(z, v) |V||W|.
$$

(3.26)
Proof: First we prove (3.24). We remark that the integrand is nonnegative:

\[ \langle V, \mathcal{R}(M_1)V \rangle = \langle V, \mathcal{R} \left( \frac{\pi^2}{4|v|} \frac{1}{1 + \frac{v^2}{|v|^2}} \right) P^\perp_{v'} V \rangle \]

\[ = \mathcal{R} \left( \frac{\pi^2}{4|v|} \frac{1}{1 + \frac{v^2}{|v|^2}} \right) |P^\perp_{v'} V|^2 \geq 0. \]

by (3.10). By a similar computation the same is true for \( M_2 \). We use (3.8) to bound the real part of \( M_2 \) (cf. (2.24)) below. Using nonnegativity of the integrand, the lower bound on \( u_0(v') \) and \( \eta(|r|) = 1 \) for \( |r| \geq 1 \) we can estimate from below by (C_2 as in (3.3)):

\[ \int_{\mathbb{R}^3} \langle V, \mathcal{R}(M_2)(z, v - v')V \rangle u_0(v') \eta(|v - v'|^2) \, dv' \geq c \int_{B_4(0) \setminus B_1(v)} C_2(z, v - v') |P^\perp_{v'} V|^2 \, dv'. \]

Now there are \( c_1, c_2 > 0 \) s.t. for \( |v| \leq 2 \) we have \( |P_{v-v'} V| \geq c_1 |V|_v \) for all \( v' \) in a set \( G(v, V) \subset B_4(0) \setminus B_1(v) \) with \( |G(v, V)| \geq c_2 \). To see this we remark that the inequality is homogeneous in \( V \), so we can restrict to \( |V| = 1 \) and \( v \) bounded, when the claim follows by contradiction. For \( |v| \geq 2 \) we use (3.17) to obtain a set of positive measure on which we have \( |P_{v-v'} V| \geq c |P_v V| \). We find the lower bound:

\[ \int_{\mathbb{R}^3} \langle V, \mathcal{R}(M_2)(z, v - v')V \rangle u_0(v') \eta(|v - v'|^2) \, dv' \geq c C_2(z, v) |P_v V|^2. \]  (3.27)

We apply the same strategy for the term containing \( M_1 \) (cf. (2.24)):

\[ \int_{\mathbb{R}^3} \langle V, \mathcal{R}(M_1)(z, v - v')V \rangle u_0(v') \eta(|v - v'|^2) \, dv' \geq c C_1(z, v) \int_{B_4(0) \setminus B_1(v)} |P^\perp_{v'} V|^2 \, dv'. \]

For \( |v| \geq 2 \) we use (3.15) to obtain:

\[ \int_{\mathbb{R}^3} \langle V, \mathcal{R}(M_1)(z, v - v')V \rangle u_0(v') \eta(|v - v'|^2) \, dv' \geq c C_1(z, v) |V|_v^2. \]  (3.28)

for \( |v| \leq 2 \) the same follows again by rescaling \( |V| = 1 \) and contradiction. Combining (3.27) and (3.28) we obtain (3.24). We now show the upper bound (3.25). The estimates (3.8)-(3.9) allow to estimate the contribution of \( M_2 \) (cf. (2.24)) by \( C_3 \) as defined in (3.4):

\[ \int_{\mathbb{R}^3} |\langle V, M_2(z, v - v')W \rangle| u_0(v') \eta(|v - v'|^2) \, dv' \leq C \int_{\mathbb{R}^3} |M_2(z, v - v')||\langle P_{v-v'} V, P_{v-v'} W \rangle| u_0(v') \eta(|v - v'|^2) \, dv' \]

\[ \leq C \int_{\mathbb{R}^3} |M_2(z, v - v')||P_{v-v'} V||P_{v-v'} W| |u_0(v')| \eta(|v - v'|^2) \, dv' \]

\[ \leq C(1 + a(z, v)) C_3(z, v) \left( |P_v V||P_{v-v'} W| + |V|_v |W|_v \right). \]

Since \( C_3(z, v) \leq C C_1(z, v) \) for \( 0 \leq \mathcal{R}(z) \leq 1 \), this shows the contribution of \( M_2 \) can be estimated by the right-hand side of (3.25). For bounding the contribution of \( M_1 \) we proceed similarly, using (3.11):

\[ \int_{\mathbb{R}^3} |\langle V, M_1(z, v - v')W \rangle| u_0(v') \eta(|v - v'|^2) \, dv' \leq C \int_{\mathbb{R}^3} |(1 + a(z, v - v')) C_1(z, v - v') |P^\perp_{v-v'} V||P^\perp_{v-v'} W| e^{-\frac{1}{2}|v'|^2} \eta(|v - v'|^2) \, dv'. \]
Write $V = P_2V + P_1^\perp V = V_1 + V_2$ and $W = W_1 + W_2$ respectively. Then we have

$$|P_1^\perp V| \leq C \left( \frac{|V_1||v'|}{1 + |v|} + |V_2| \right).$$

This implies that we can bound:

$$\int_{\mathbb{R}} |(V, M_1(z, v - v'))W_0(v')\eta| dv' \leq C \int_{\mathbb{R}} (1 + \alpha(z, v - v'))C_1(z, v - v')(\frac{|V_1||v'|}{1 + |v|} + |V_2|)(\frac{|W_1||v'|}{1 + |v|} + |W_2|)\eta dv' \leq C(1 + \alpha(z, v))C_1(z, v)|V_1||W_1|, $$

which concludes the proof of (3.25). Estimate (3.26) follows from a similar computation, using Lemma 3.3.  

The following Lemma uses the symmetry of the highest order term in the functionals $Q$ to show it can be expressed by the real part of $F(K)$, $F(P)$ only, which surprisingly has a sign.

**Lemma 3.6** Let $n \geq 1$ and $u_0 \in H^n_A$ satisfy the pointwise estimates

$$c 1_{|x| \leq 4}(v) \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad \text{for } c > 0. \quad (3.29)$$

Furthermore let $\varepsilon > 0$, $A > 0$ such that $\varepsilon A \leq 1$ and write $z = a + i\omega = \frac{A}{2} + i\omega$. Let $u \in V^n_{A,\omega}$ for some $n \in \mathbb{N}$ and $\gamma \in (0, 1]$. The term in $Q_{\varepsilon, A}^{\alpha,\beta}$ (as defined in (2.20), (2.22)), where $|\alpha| \leq n$, depends on the real part of $F(K)$ only. Writing $V = \nabla D^\alpha F(u)(z, v)$ we have:

$$(2\pi)^{-2}Q_{\varepsilon, A}^{\alpha,\beta}u_0(u) = \int_{\mathbb{R}} \int \langle \nabla V(z, v)\dot{\lambda}(v), F(K)[u_0](\varepsilon z, v)\nabla V(z, v) \rangle dv d\omega$$

$$= \int_{\mathbb{R}} \int \langle \nabla V(z, v)\dot{\lambda}(v), R(F(K))[u_0](\varepsilon z, v)\nabla V(z, v) \rangle dv d\omega. \quad (3.30)$$

**Proof:** Follows from the observation that the left-hand side is real by Plancherel’s Lemma and that $K$ is a symmetric matrix. 

The following lemma amounts to a coercivity result, and shows that for a function $u \in V^n_{A,\omega}$, the functional $Q_{\varepsilon, A}^{\alpha,\beta}[u_0](u)$ can be controlled by the first $n$ derivatives of $u$ only. Here we use that to leading order, the functional is actually dissipative. The exact form of the dissipation $D$ is of particular importance, since we use it later to show that the nonlinearity can be handled as a perturbation.

**Lemma 3.7** Let $n \geq 1$ and $u_0 \in H^n_A$ satisfy the pointwise estimates

$$c 1_{|x| \leq 4}(v) \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad \text{for } c > 0. \quad (3.31)$$

For $A > 0$, let $a = \frac{A}{2}$ and assume $\varepsilon \in (0, 1], \gamma \in (0, 1]$ arbitrary and $|\alpha| \leq n$ for an $\alpha \in \mathbb{N}_0^d$. Define the dissipation $D_{\varepsilon, A}^{\alpha}$ as $(z = a + i\omega)$:

$$D_{\varepsilon, A}^{\alpha}(u) := \int B_1(\varepsilon z, v)[\nabla D^\alpha F(u)(z, v), \nabla D^\alpha F(u)(z, v)]\dot{\lambda}(v) dv d\omega. \quad (3.32)$$

$$21 $$
Then the leading order quadratic form satisfies the lower bound:
\[ Q_{\varepsilon,A}^{\alpha}[u_0](u) \geq c D_{\varepsilon,A}^{\alpha}(u) - C \| u \|^2_{V_{\varepsilon,A}}. \] (3.33)

We will denote by \( D_{\varepsilon,A}^{\alpha} \) the dissipation of the equation. The lower order terms can be estimated by the dissipation:
\[ \sum_{\beta < \alpha} \left( \alpha \right) |Q_{\varepsilon,A}^{\alpha}[u_0](u)| \leq \frac{c}{2} D_{\varepsilon,A}^{\alpha}(u) + C \| u \|^2_{V_{\varepsilon,A}}. \] (3.34)

The constants can depend on \( u_0 \) and \( n \), but not on \( A \geq 1, \varepsilon > 0 \).

**Proof:** In the proof, we drop the dependence on \( \gamma \) for shortness. We start with proving the lower bound (3.33). As a first step we rewrite \( Q_{\varepsilon,A}^{\alpha}[u_0](u) \) in terms of Laplace transforms (write \( z = a + i \omega \) for shortness):
\[
Q_{\varepsilon,A}^{\alpha}[u_0](u) = \frac{1}{\varepsilon} \int_0^\infty e^{-\varepsilon t} \int K[u_0]\left( \frac{t-s}{\varepsilon}, v \right) \nabla D_{\varepsilon,A}^{\alpha}(u) \, ds \, dv \, dt
- \frac{1}{\varepsilon} \int_0^\infty e^{-\varepsilon t} \int P[u_0]\left( \frac{t-s}{\varepsilon}, v \right) D_{\varepsilon,A}^{\alpha}(u) \, ds \, dv \, dt
= (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \left( \langle (D_{\varepsilon,A}^{\alpha}(u(z,v)) \lambda), \mathcal{L}(u) \rangle \right) \mathcal{L}(K)[u_0](\varepsilon z, v) \mathcal{L}(u(z,v)) \, dv \, d\omega
- \frac{1}{2} \int_\mathbb{R} \left( \langle (D_{\varepsilon,A}^{\alpha}(u(z,v)) \lambda), \mathcal{L}(u) \rangle \right) \mathcal{L}(P)[u_0](\varepsilon z, v) \mathcal{L}(u(z,v)) \, dv \, d\omega
= J_1 + J_2. \] (3.35)

We recall the representation of \( \mathcal{L}(K) \) given in Lemma 2.16:
\[
\mathcal{L}(K)[u](z, v) = \int \langle M_1 + M_2 \rangle(z, v - v') \eta(|v - v'|^2) \, dv'. \] (3.36)

We start by estimating \( J_1 \). For shortness, we write \( V = \nabla D_{\varepsilon,A}^{\alpha}(u) \). Then use (3.36), Lemma 3.6 and the pointwise estimates proven in Lemma 3.3:
\[
J_1 = (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \left( \langle V(z,v) \lambda(v), \mathcal{L}(K)[u_0](\varepsilon z, v) V(z,v) \rangle \right) \, dv \, d\omega
+ (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \left( \langle D_{\varepsilon,A}^{\alpha}(u(z,v)) \nabla(\lambda(v)), \mathcal{L}(K)[u_0](\varepsilon z, v) V(z,v) \rangle \right) \, dv \, d\omega
\geq c D_{\varepsilon,A}^{\alpha}(u) + (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \left( \langle D_{\varepsilon,A}^{\alpha}(u(z,v)) \nabla(\lambda(v)), \mathcal{L}(K)[u_0](\varepsilon z, v) V(z,v) \rangle \right) \, dv \, d\omega
= c D_{\varepsilon,A}^{\alpha}(u) + I_3. \] (3.37)

It remains to estimate \( J_2 \) given by (3.35) and \( I_3 \) given by (3.37). To this end, we recall the definition of \( \| \cdot \|_{V_{\varepsilon,A}} \) in (2.7) and use the Plancherel identity in Lemma 2.13 to estimate:
\[
\int_\mathbb{R} \int_\mathbb{R}^3 \| D_{\varepsilon,A}^{\alpha}(u)(z,v) \|^2 \lambda(v) \, dv \, d\omega \leq C \| u \|^2_{V_{\varepsilon,A}}. \] (3.38)
In order to estimate $I_3$, we observe that $\nabla \lambda = P_u \nabla \lambda$. Then we combine (3.36) with (3.25) in Lemma 3.5 to obtain the estimate (recall $B_2$, cf. (5.17)):

$$|I_3| \leq C \int_{\mathbb{R}} \int |D^a \mathcal{L}(u)| \lambda C(1 + \alpha(|\epsilon z, v|))B_2(\epsilon z, v)[P_u \nabla \lambda(v), V] \, dv \, dw \leq C \int_{\mathbb{R}} \int \left(|D^a \mathcal{L}(u)| \lambda \right) \left(\frac{|V(z, v)|}{(1 + \alpha(|\epsilon z, v|))(1 + |v|^2)} + \frac{(\beta + a + \alpha^2)|P_u V(z, v)|}{(1 + a)^2(1 + |v|)}\right) \, dv \, dw.$$  

We apply Young’s inequality and (3.38) to get the bound ($D^a_{\epsilon,A}$ defined in (3.32)):

$$|I_3| \leq \frac{C}{4} D^a_{\epsilon,A} + C \|D^a u\|_{V_{\epsilon,A}}^2. \quad (3.39)$$

It remains to estimate $J_2$ to finish the proof of (3.33). We recall that $P[u_0] = \nabla \cdot K[u_0]$. We apply (3.26) with $|\beta| = 1$ and recall the definition of $C_1$ (cf. (3.2)) to obtain an upper estimate for $J_2$:

$$|J_2| \leq C \int_{\mathbb{R}} \int \left(\lambda^\frac{1}{2}(v) \frac{1 + \alpha(|\epsilon z, v|)}{1 + |v|} C_1(\epsilon z, v)|V|\right) \left(\lambda^\frac{1}{2}(v)|D^a \mathcal{L}(u)(z, v)|\right) \, dv \, dw.$$

Notice that (3.26) provides $\frac{1}{|v|}$ more decay than naively expected, which is essential here. Young’s inequality in combination with (3.38) implies:

$$|J_2| \leq C \frac{D^a_{\epsilon,A}}{4} + C \|D^a u\|_{V_{\epsilon,A}}^2. \quad (3.40)$$

Combining the estimates (3.35), (3.39) and (3.40) proves (3.33). In the case $\beta < a$ we use (3.26) in Lemma 3.5 and Young’s inequality to prove (3.34).

The linear result follows as a corollary. The statement can be generalized significantly, the assumptions in our a priori estimates are designed for the nonlinear case and therefore more restrictive than needed for the linear equation.

**Theorem 3.8** Let $n \geq 6$ and $u_0 \in H^n_\lambda$ satisfy the pointwise estimate

$$c \|v\|_{L^4} \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad c, C > 0. \quad (3.41)$$

There exists $A > 0$ s.t. for $\epsilon > 0$ small, there is a solution $u_\epsilon \in V_{A,\lambda}^n \cap C^1(\mathbb{R}^+; H^{n-2}_\lambda)$ to:

$$\partial_t u_\epsilon = \frac{1}{\epsilon} \nabla \cdot \left(\int_0^t K[u_\epsilon](\frac{t-s}{\epsilon}, v) \nabla u_\epsilon(s, v) \, ds\right) - \frac{1}{\epsilon} \nabla \cdot \left(\int_0^t P[u_\epsilon](\frac{t-s}{\epsilon}, v) u_\epsilon(s, v) \, ds\right) \quad (3.42)$$

$$u_\epsilon(0, \cdot) = u_0(\cdot).$$

There is a function $u \in V_{A,\lambda}^n \cap C^1(\mathbb{R}^+; H^{n-4}_\lambda)$ s.t. $u_\epsilon \to u$ in $V_{A,\lambda}^n$ along a sequence $\epsilon_j \to 0$. The function $u$ solves the limit equation ($\mathcal{K}$, $P$ defined in (2.3)):

$$\partial_t u = \nabla \cdot (\mathcal{K}[u_0]\nabla u) - \nabla \cdot (P[u_0]u) \quad (3.43)$$

$$u(0, v) = u_0(v).$$
Proof: For $0 < \gamma \leq 1$, the existence of solutions $u_{\epsilon, \gamma}$ to \eqref{2.34} follows from Lemma 2.17. In order to prove well-posedness for \eqref{3.42}, i.e. $\gamma = 0$, we derive a priori estimates that are uniform in $\gamma$. Combining Lemma 2.12 and Lemma 3.7 shows that for $A^0 > 0$ large enough
\[ \|u_{\epsilon, \gamma}\|_{V^4_{A, \lambda}} \leq C \] (3.44)
are uniformly bounded in $0 < \gamma, \epsilon \leq \frac{1}{A}$. Now we use the Laplace representation in Lemma 2.16 to infer the uniform boundedness:
\[ |\nabla^m L(K[u_0])(z, v)| + |\nabla^m L(P_{\gamma}[u_0])(z, v)| \leq C(m) \quad \text{for } m \in \mathbb{N}. \] (3.45)

We rewrite \eqref{2.34} in Laplace variables and obtain:
\[ z L(u_{\epsilon, \gamma}) = i \nabla \cdot \left( L(K[u_0])(\epsilon z) i \nabla L(u_{\epsilon, \gamma}) - L(P_{\gamma}[u_0])(\epsilon z) L(u_{\epsilon, \gamma}) \right) + u_0(v). \] (3.46)
The right-hand side of \eqref{3.46} is bounded in $V^{-2}_{A, \lambda}$ due to \eqref{3.45} and \eqref{3.44}, so we get a bound of:
\[ \|u_{\epsilon, \gamma}\|_{V^{-2}_{A, \lambda}} + \|\partial_t u_{\epsilon, \gamma}\|_{V^{-3}_{A, \lambda}} \leq C. \] (3.47)

By the Rellich type Lemma 2.5 and the fact that $V^m_{A, \lambda}$ is a separable Hilbert space, there is a $u_\epsilon \in V^m_{A, \lambda}$ and a sequence $\gamma_j \to 0$ s.t. $u_{\epsilon, \gamma_j} \to u_\epsilon$ in $V^m_{A, \lambda}$ and $u_{\epsilon, \gamma_j} \to u_\epsilon$ in $V^{-3}_{A, \lambda}$. We need to show that the weak limit $u_\epsilon$ indeed solves the equation \eqref{3.42}. Both sides of \eqref{3.46} converge pointwise a.e. to the respective sides with $\gamma = 0$ along a subsequence of $\gamma_j \to 0$. Since the Laplace transform defines the function uniquely, $u_\epsilon$ is indeed a solution. Finally, the solutions $u_\epsilon$ are in $C^1(\mathbb{R}^+; H^{n-2}_\lambda)$ since they are bounded in $V^m_{A, \lambda}$ and the equation \eqref{3.42} in combination with $|\nabla^m K[u_0]| + |\nabla^m P[u_0]| \leq C(m)$ allows to control the time derivative in $C^0(\mathbb{R}^+; H^{n-2}_\lambda)$.

The convergence of $u_{\epsilon, \gamma_j}$ to a solution $u$ of \eqref{3.43} follows similarly. We use the uniform bound \eqref{3.47} to find a subsequence $\gamma_j \to 0$ and $u \in V^m_{A, \lambda}$ such that $u_{\epsilon, \gamma_j} \to u$ in $V^m_{A, \lambda}$ and $u_{\epsilon, \gamma_j} \to u$ in $V^{-3}_{A, \lambda}$. Now the claim follows from the observation that for $\gamma = 0$ we can take the limits on both sides of \eqref{3.46} and pointwise a.e. along a subsequence there holds:
\[ L(u_{\epsilon, \gamma_j}) \to L(u), \quad L(K)[u_0](\epsilon_j z, v) \to K[u_0](v), \quad L(P)[u_0](\epsilon_j z, v) \to P[u_0](v). \]

Repeating the argument above, we find that the weak limit $u_{\epsilon, \gamma_j} \to u \in V^m_{A, \lambda}$ is actually $u \in V^m_{A, \lambda} \cap C^1(\mathbb{R}^+; H^{n-4}_\lambda)$ and is indeed a solution of the equation \eqref{3.43}. \hfill \Box

4 A priori estimate for the nonlinear problem

4.1 Continuity of the fixed point mapping $\Psi$

In this subsection we prove that solutions of equation \eqref{2.15} satisfy an a priori estimate, for small perturbations $f_{\epsilon}$. Here smallness is measured in terms of the size and decay of the Laplace transform, i.e. the smoothness of the perturbation $f_{\epsilon}$. The necessary framework is provided by the definition below. Notice that we always assume that $f_{\epsilon} = \nabla \cdot g_{\epsilon}$ is a divergence, so it has zero average. This is the key point to obtain an additional decay $\frac{1}{|t|}$ in Lemma 4.7. Furthermore it is essential that the highest order term $Q^{\sigma, \alpha}_{\epsilon, \lambda}[f_{\epsilon}][u]$ introduced in \eqref{2.20} is a symmetric integral, which induces a cancellation for large Laplace frequencies. In the subsequent subsection we will prove that our smallness assumption is consistent, i.e. if the condition is satisfied by $f_{\epsilon}$, then it is also satisfied by $u_{\epsilon} - u_0$ when $u_{\epsilon}$ solves \eqref{2.15}.
Definition 4.1 We define a sequence of cutoff functions $\kappa_{\delta_i} \in C_c^\infty(\mathbb{R})$ by
\[
\kappa_{\delta_i}(s) := \kappa\left(\frac{s}{\delta_i}\right),
\]
where $\kappa \in C_c^\infty(\mathbb{R})$, $0 \leq \kappa \leq 1$, $\kappa(s) = 1$ for $|s| \leq 1$ and $\kappa(s) = 0$ for $|s| \geq 2$. Let $R, \varepsilon, \delta > 0$ and $z \in \mathbb{C}$. We define $Y_{R,\varepsilon,\delta}(z)$ by
\[
Y_{R,\varepsilon,\delta}(z) := \frac{\delta}{1 + |z|^2} + \frac{\Re|z|}{(1 + \varepsilon|z|)(1 + |z|^2)}.
\]
We will consider $u = (f, g) \in X_{A,\lambda}^n$ (defined in (2.8)), s.t. a.e. on the line $\Re(z) = \frac{A}{2} = a > 0$:
\[
|\ell(f)(z, v)| \leq Y_{R,\varepsilon,\delta}(z)e^{-\frac{1}{2}|v|},
|\ell(g)(z, v)| \leq Y_{R,\varepsilon,\delta}(z)e^{-\frac{1}{2}|v|}
\]
(4.3)
\[
|\ell(f)(z, v)| \leq \frac{\Re^{-\frac{1}{2}|v|}}{|1 + \varepsilon z|(1 + |z|^2)},
|\ell(g)(z, v)| \leq \frac{\Re^{-\frac{1}{2}|v|}}{|1 + \varepsilon z|(1 + |z|^2)}
\]
(4.4)
\[
|\partial_{\alpha} f(t, v)| \leq \Re^{-\frac{1}{2}|v|}.
\]
(4.5)

For $R, \varepsilon, \delta > 0$, $A \geq 1$, $a = \frac{A}{2}$ and $n \in \mathbb{N}$, let $\Omega_{A,\varepsilon,\delta, \lambda}^n \subset X_{A,\lambda}^n$ be the set of functions given by:
\[
\Omega_{A,\varepsilon,\delta, \lambda}^n = \{u = (f, g) \in X_{A,\lambda}^n : \|u\|_{X_{A,\lambda}^n} \leq R, (4.3) \text{ and } (4.4) \text{ for } \Re(z) = a\}.
\]
(4.6)

Since the estimates (4.3)-(4.4) are stable under convex combinations of functions, we have:

Lemma 4.2 For all $R, \varepsilon, \delta > 0$, $A \geq 1$ and $n \in \mathbb{N}$, the set $\Omega_{A,\varepsilon,\delta, \lambda}^n$ is a nonempty, bounded, closed and convex subset of $X_{A,\lambda}^n$.

The following theorem is the main result of this subsection, giving an a priori estimate for the solution operator to (2.15) under the smallness assumption $(f, g) \in \Omega_{A,\varepsilon,\delta, \lambda}^n$ for small $\varepsilon, \delta$. We prove the error term can be controlled by the dissipation $D_{A, \varepsilon, \lambda}^n$ (cf. (3.32)) provided by the linear equation. Observe that existence of (unique) global solutions of (2.15) has been proved in Lemma 2.17. Here we will prove a priori estimates that are uniform in the mollifying parameter $\gamma > 0$ and $\varepsilon > 0$.

Theorem 4.3 Let $n \in \mathbb{N}$, $n \geq 2$. Assume $u_0 \in H_A^n$ satisfies:
\[
c \|v\|_A \leq \|v_0\|_A \leq Ce^{-\frac{1}{2}|v|}.
\]
Then there exist $A, \delta > 0$ such that for all $R > 0$ there is an $\varepsilon_0 > 0$ with the property that the operator $\psi_{\delta_i}$ given by:
\[
\psi_{\delta_i} : \Omega_{A,\varepsilon,\delta, \lambda}^n \rightarrow X_{A,\lambda}^n,
(f, g) \mapsto \left((u - u_0)k_{\delta_i}, A^{0.0}_\gamma[f](u)k_{\delta_i}\right), A^{0.0}_\gamma
\]
as in (2.11) and $u$ solution to:
\[
\begin{align*}
\partial_t u &= \frac{1}{\varepsilon} \Delta \cdot \left(\int_0^s K[u_0 + f(s)] \left(\frac{t-s}{\varepsilon}, v\right) \nabla u(s, v) \, ds\right) \quad (4.7) \\
- \frac{1}{\varepsilon} \Delta \cdot \left(\int_0^s P_s[u_0 + f(s)] \left(\frac{t-s}{\varepsilon}, v\right) u(s, v) \, ds\right)
\end{align*}
\]
u(0, \cdot) = u_0(\cdot),
\]
25
We split the kernel in Laplace variables. Furthermore, the solutions satisfy the following estimate:

$$\|\Psi_{\delta_1}(f, g)\|_{X_{A^2}} + \|\partial_i \Psi_{\delta_1}(f, g)\|_{X_{A^2}} \leq C(A, \delta_1).$$

Notice that the operator $\phi_{\delta_1}$ maps functions in $X_{A^2}$ to functions in $X_{A^2}$, thus yields better decay. As can be seen from Lemma 2.12, this follows from the fast decay of the initial datum, provided we can control the quadratic terms $Q$. In Section 3, we have shown that the quadratic functionals $Q(u_0)$ defined in (2.20) satisfy a coercivity estimate. In this subsection we will prove smallness for the perturbation $Q(u_1)$, so the sum $Q[u_0 + f]$ still has a sign. To this end we first include an auxiliary Lemma to represent those functionals in Laplace variables.

**Lemma 4.4** The quadratic functionals $Q_{\epsilon, A}[v](u)$ defined in (2.20) can be represented by means of the Laplace transform of $u$ as:

$$\begin{align*}
(2\pi)^{1/2}Q_{\epsilon, A}[v](u) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\nabla(D^\alpha \Lambda(\omega)\zeta(z), D^{\alpha-\beta} \Lambda[v](\epsilon z, \omega - \theta)\mathcal{L}(\nabla(D^\beta u)(p))\right) dv \ d\omega \\
&- \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\nabla(D^\alpha \Lambda(\omega)\zeta(z), \nabla(D^{\alpha-\beta} \Lambda[v](\epsilon z, \omega - \theta)\mathcal{L}(D^\beta u)(p))\right) dv \ d\omega.
\end{align*}$$

We use the short notation $z = a + i\omega, p = a + i\theta$ and $\Lambda$ is given by $M_1, M_2$ (cf. (2.24)) as:

$$\Lambda[v](z, \tau, \nu) = \int_{\mathbb{R}} \int_{\mathbb{R}} (M_1 + M_2)(z, \nu - \nu') e^{i\tau \nu} \eta(|\nu - \nu'|^2) v(s, \nu') ds \ d\nu'.$$

**Proof:** Follows directly from the elementary properties of the Laplace Transform.

Exploiting the symmetry properties of the functional $Q_{\epsilon, A}[f_1]$ is essential to proving that this term is small compared to the dissipation $D_{\epsilon, A}^\alpha$ (cf. (3.32)). For better notation we first include some definitions.

**Definition 4.5** For $\epsilon > 0, v \in \mathbb{R}^3, z = a + i\omega, p = a + i\theta \in \mathbb{C}$, define the matrices $L_1, L_2$:

$$\begin{align*}
L_1(\epsilon, z, p, v) &:= \frac{1}{2}(M_1(\epsilon z, v) + M_1(\epsilon \overline{p}, v)) \\
L_2(\epsilon, z, p, v) &:= \frac{1}{2}(M_2(\epsilon z, v) + M_2(\epsilon \overline{p}, v))
\end{align*}$$

and the associated symmetrized kernel $\Lambda_\beta$ by:

$$\begin{align*}
\Lambda_1[v](\epsilon, z, p, v) &:= \Lambda_1[v](\epsilon, z, p, v) + \Lambda_2[v](\epsilon, z, p, v) \\
\Lambda_1[v](\epsilon, z, p, v) &:= \int_{\mathbb{R}^3} L_1(\epsilon, z, p, v - \nu') \left(\int_0^\infty e^{-i(\nu' - \theta) \nu} v(s, \nu') ds\right) \eta(|\nu - \nu'|^2) d\nu' \\
\Lambda_2[v](\epsilon, z, p, v) &:= \int_{\mathbb{R}^3} L_2(\epsilon, z, p, v - \nu') \left(\int_0^\infty e^{-i(\nu' - \theta) \nu} v(s, \nu') ds\right) \eta(|\nu - \nu'|^2) d\nu'.
\end{align*}$$

We split the kernel $L_2$ further into:

$$\begin{align*}
N_2(\epsilon, z, p, v) &= L_2(\epsilon, z, p, v) - N_1(\epsilon, z, p, v), \text{where} \\
N_1(\epsilon, z, p, v) &= \frac{1}{|v|^2} \frac{\epsilon(\epsilon + i(\theta - \omega))}{(1 + \frac{\epsilon^2}{|v|^2})(1 + \frac{\epsilon^2}{|v|^2})} P_v.
\end{align*}$$
Lemma 4.6 Let \( a > 0 \) and \( z = a + i\omega, p = a + i\theta \). Further let \( \varepsilon \leq \frac{1}{a} \). For \( V, W \in \mathbb{C}^3 \) and \( N_1 \), as in the definition above, and \(|v| \geq c > 0\), we have the estimates:

\[
|\langle V, L_1(\varepsilon, z, p, v)W \rangle| \leq C \left| \frac{P^1_v V}{1 + |v|} \right| \frac{1 + \varepsilon |\theta - \omega|}{(1 + a(\varepsilon z, v))(1 + a(\varepsilon p, v))} \tag{4.15}
\]

\[
|\langle V, N_2(\varepsilon, z, p, v)W \rangle| \leq C \left| \frac{W|\varepsilon|^2 |p||z| + \varepsilon^2 |p||z|(1 + \varepsilon |\theta - \omega|)}{1 + |v|^3 (1 + a(\varepsilon z, v))^2(1 + a(\varepsilon p, v))^2} \right| \tag{4.16}
\]

Proof: We start by proving (4.15). Using \( \varepsilon \leq \frac{1}{a}, |v| \geq c > 0 \) and the definition of \( L_1 \) (cf. (4.10)) and \( M_1 \) (cf. (2.23)) we can bound:

\[
|\langle V, L_1(\varepsilon, z, p, v)W \rangle| \leq C \left| \frac{P^1_v V}{1 + |v|} \right| \frac{1}{1 + \varepsilon |\theta - \omega|} \left| \frac{|v|}{1 + a(\varepsilon z, v)}(1 + a(\varepsilon p, v)) \right|
\]

The decomposition of \( L_2 \) (defined in (4.11)) follows from the identity:

\[
\frac{b}{(1 + b)^2} + \frac{\bar{c}}{(1 + \bar{c})^2} = \frac{b + \bar{c}}{(1 + b)^2(1 + \bar{c})^2} + \left( \frac{4bc}{(1 + b)^2(1 + \bar{c})^2} + \frac{b\bar{c}(b + \bar{c})}{(1 + b)^2(1 + \bar{c})^2} \right) \tag{4.17}
\]

We insert \( b = \frac{\varepsilon z}{|v|}, \bar{c} = \frac{\varepsilon p}{|v|} \) and multiply (4.17) with \( \frac{\varepsilon^2 p}{4|v|} \). Then the first term on the right gives \( N_1 \), so the second gives \( N_2 \) as defined in (4.13). The latter is bounded by:

\[
|N_2| \leq \frac{\varepsilon^2 p}{4|v|} \left( \frac{4bc}{(1 + b)^2(1 + \bar{c})^2} + \frac{b\bar{c}(b + \bar{c})}{(1 + b)^2(1 + \bar{c})^2} \right)
\]

This proves estimate (4.16).

Our goal is to prove estimates for the functional \( Q^{\varepsilon,\omega}_{a,\theta}[f_j] \). This will be done estimating \( \Lambda_j \), as defined in (4.12), which is given by \( L_1, L_2 \) (cf. (4.10), (4.11)). We have decomposed \( L_1 + L_2 = L_1 + N_1 + N_2 \), and Lemma 4.6 gives estimates for \( L_1 \) and \( N_2 \). It remains to prove an estimate for \( N_1 \). Here we rely on the additional decay provided by the divergence property \( f = \nabla \cdot g \) of functions in \( \Omega \). Under the divergence assumption we get the following Lemma.

Lemma 4.7 Let \( N_1 \) be given by (4.14). Let \( h = \nabla \cdot G \), where \( G \in H^1_{\varepsilon} |G(v)| \leq R_1 e^{-\frac{1}{2}|v|} \). For \( a > 0 \), \( \varepsilon \in (0, \frac{1}{a}) \), \( z = a + i\omega, p = a + i\theta \in \mathbb{C} \) we have:

\[
\left| \int \langle V, N_1(\varepsilon, z, p, v - v')W \rangle h(v') \eta(|v - v'|^2) \, dv' \right| \leq \frac{CR_1 |V||W|(1 + \varepsilon |\omega - \theta|)}{(1 + |v|^3)(1 + a(\varepsilon z, v))^2(1 + a(\varepsilon p, v))^2}. \tag{4.18}
\]

Proof: We simply use that \( h = \nabla \cdot G \) is a divergence and write:

\[
\int_{\mathbb{R}^3} N_1(\varepsilon, z, p, v - v') \eta h(v') \, dv' = -\int_{\mathbb{R}^3} \nabla_{v'} \left( N_1(\varepsilon, z, p, v - v') \eta(|v - v'|^2) \right) G(v') \, dv'. \tag{4.19}
\]
Explicitly computing the derivative of $N_1$ as defined in (4.14) gives:
\[
|\nabla_{\nu} \left( N_1(\epsilon, z, p, v)\eta(|v|^2) \right)| \leq C \frac{1 + \epsilon|\theta - \omega|}{(1 + |v|^3)(1 + a(\epsilon z, v))^2(1 + a(\epsilon p, v))^2}.
\]

Now plugging the assumption $|G(v)| \leq R_1 e^{-\frac{1}{2}|v|}$ into (4.19) gives the claim. \hfill \Box

Lemma 4.8 For $A > 0, n \in \mathbb{N}, n \geq 2, R, \delta, \varepsilon > 0$ and all $(f, g) \in \Omega_{A,R,\delta,\varepsilon}^n$ we have:
\[
\left| \int_0^\infty e^{-ist} f(s, v) \, ds \right| \leq C(A) \min\{Y_{R,\delta,\varepsilon}(\tau), \frac{R}{(1 + \varepsilon|\tau|)(1 + |\tau|^2)}\} e^{-\frac{1}{2}|\tau|},
\]
\[
\left| \int_0^\infty e^{-ist} g(s, v) \, ds \right| \leq C(A) \min\{Y_{R,\delta,\varepsilon}(\tau), \frac{R}{(1 + \varepsilon|\tau|)(1 + |\tau|^2)}\} e^{-\frac{1}{2}|\tau|},
\]
for $\tau \in \mathbb{R}$. Here $Y_{R,\delta,\varepsilon}(\tau)$ is the function defined in (4.2).

Proof: By definition of $\Omega_{A,R,\delta,\varepsilon}^n$ (see (4.6)) for $\Re(z) = a$ there holds:
\[
|\mathcal{L}(f)(z, v)| \leq \min\{Y_{R,\delta,\varepsilon}(z), \frac{R}{(1 + \varepsilon|z|)(1 + |z|^2)}\} e^{-\frac{1}{2}|z|},
\]
\[
|\mathcal{L}(g)(z, v)| \leq \min\{Y_{R,\delta,\varepsilon}(z), \frac{R}{(1 + \varepsilon|z|)(1 + |z|^2)}\} e^{-\frac{1}{2}|z|}.
\]
Notice that the estimate is the same for $f$ and $g$. We rewrite the left-hand side of (4.20) as:
\[
\left| \int_0^\infty e^{-ist} f(s, v) \, ds \right| = \left| \int_0^\infty e^{-ist} e^{-as} f(s, v) \kappa_2(s) e^{as} \, ds \right| = \frac{1}{2\pi} \left| \int_\mathbb{R} \mathcal{L}(f)(a + i(\tau - \omega)) F(\tau - \omega) \, d\omega \right|,
\]
where $F(\omega) = \int_\mathbb{R} e^{-isl} \kappa_2(|s|) e^{as} \, ds$. The function $F$ is the Fourier transformation of a fixed Schwartz function, hence decays faster than any polynomial. For the rational function $Y_{R,\delta,\varepsilon}$ defined in (4.2), a straightforward computation shows $|Y_{R,\delta,\varepsilon} \ast F| \leq C|Y_{R,\delta,\varepsilon}|$ with $C > 0$ independent of $\varepsilon \in (0, \frac{1}{n}]$ and $R, \delta > 0$.

Lemma 4.9 Let $A \geq 1, n \geq 2, a$ a multi-index with $|a| \leq n$ and $c > 0$ arbitrary be given. There exists $\delta_0(c, A, n) > 0$ such that for all $\delta \in (0, \delta_0)$ and $R > 0$, we can estimate:
\[
|Q_{A,\delta}^\epsilon f| \leq \sum_{\beta \leq a} \left( \frac{\alpha}{\beta} \right) |Q_{A,\delta}^\epsilon f| \leq c D_{\epsilon,\delta}^a(u) + \|u\|_{V_{\epsilon,\delta}}^2,
\]
for all $(f, g) \in \Omega_{A,R,\delta,\varepsilon}^n$, when $0 < \epsilon \leq \epsilon_0(\delta, R, A, c, n)$ is small.

Proof: Fix $A \geq 1$ and $n \in \mathbb{N}, n \geq 2$ and $c > 0$ as in the assumption. We first estimate the highest order term $\beta = \alpha$ in the quadratic form $Q$. We start our estimate from the representation in Lemma 4.3 (we write $\nabla = \frac{\partial}{\partial z} \nabla$ for shortwrite):
\[
(2\pi)^{\frac{1}{2}} Q_{A,\delta}^\epsilon(v) = \int_\mathbb{R} \int_\mathbb{R} \int \langle \lambda \mathcal{L}(V D^2 u)\nu(z), \Lambda(\epsilon z, \omega - \theta) \mathcal{L}(V D^2 u)(p) \rangle \, dv \, d\theta \, d\omega \tag{4.23}
\]
\[
+ \int_\mathbb{R} \int_\mathbb{R} \int \langle \mathcal{L}(D^2 u)(z)\nu(\lambda), \Lambda(\epsilon z, \omega - \theta) \mathcal{L}(D^2 u)(p) \rangle \, dv \, d\theta \, d\omega \tag{4.24}
\]
\[
- \int_\mathbb{R} \int_\mathbb{R} \int \langle \nabla \mathcal{L}(D^2 u)\nu(z), \nabla \Lambda(\epsilon z, \omega - \theta) \mathcal{L}(D^2 u)(p) \rangle \, dv \, d\theta \, d\omega \tag{4.25}
\]
\[= J_1 + J_3 + J_2.\]
We start with estimating the critical term \( J_1 \). We can symmetrize in \( p, z \), and replace \( \Lambda \) by \( \Lambda_x \) as introduced in Definition 4.5. The symmetrization gives (for shortness write \( V = L(\nabla D^4 u) \)):

\[
J_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \int \lambda(V(z, v), \Lambda(\epsilon z, \omega - \theta, v)V(p, v)) \, dv \, d\theta \, d\omega = \int_{\mathbb{R}} \int_{\mathbb{R}} \int \lambda(V(z, v), (\Lambda_1 + \Lambda_2)V(p, v)) \, dv \, d\theta \, d\omega = I_1 + I_2.
\] (4.26)

We estimate \( I_1 \) using the estimate on \( L_1 \) in (4.15) and use Lemma 4.8 to bound \( L(f) \):

\[
|I_1| \leq C \int_{\mathbb{R}} \left( \int \int \frac{\lambda P_{(v^2 - v^3)}(z, v) |P_{(v^2 - v^3)}(p, v)|}{|v - v'| \lambda |V(z, v)|_\nu |V(p, v)|_\nu} (1 + |v - v'|)^{(1 + \alpha(\epsilon z, v - v'))(1 + \alpha(\epsilon p, v - v'))} \right) \, dv \leq C(A) \int_{\mathbb{R}} \left( \int \int \frac{\lambda |V(z)|_\nu |V(p)|_\nu}{(1 + |v|)(1 + \alpha(\epsilon z, v))} (1 + \alpha(\epsilon p, v)) (1 + |\theta - \omega|)^2 \right) \, dv \leq C(A) \int_{\mathbb{R}} \left( \int \int \frac{\lambda |V(z)|_\nu |V(p)|_\nu}{(1 + |v|)(1 + \alpha(\epsilon p, v))} Y_{R, \delta}(\theta - \omega). \right) \leq C(A) \int_{\mathbb{R}} \left( \int \int \frac{\lambda |V(z)|_\nu |V(p)|_\nu}{(1 + |v|)(1 + \alpha(\epsilon p, v))} Y_{R, \delta}(\theta - \omega). \right)
\] (4.27)

Observe the following straightforward integral estimates hold:

\[
\int_{\mathbb{R}} \frac{R\lambda |\tau|}{(1 + |v|)(1 + |\tau|)^2} \, d\tau \leq C R\lambda^\frac{1}{2}, \quad \int_{\mathbb{R}} Y_{R, \delta}(\tau) \, d\tau \leq C(\delta + \epsilon^\frac{1}{2} R). \] (4.28)

We apply Young’s inequality to (4.27) and use (4.28) to obtain a total bound of:

\[
|I_1| \leq \int_{\mathbb{R}} \left( \int \int \frac{C(A)\lambda |V(z)|_\nu |V(p)|_\nu}{(1 + |v|)(1 + \alpha(\epsilon z, v))} \, dv \frac{\lambda |V(z)|_\nu |V(p)|_\nu}{(1 + |v|)(1 + \alpha(\epsilon p, v))} \right) (1 + \alpha(\epsilon p, v - v')) \right) Y_{R, \delta}(\theta - \omega) \, d\tau \right) \, dv \leq \frac{c}{6} D_{\epsilon, A}(u),
\]

for \( 0 < \delta < \delta_0(n, A), 0 < \epsilon \leq \epsilon_0(\delta, R, A, c, n) \) small and \( D_{\epsilon, A}(u) \) as defined in (3.32). The term \( I_2 \) (cf. (4.26)) can be controlled similarly. We split \( I_2 \) further into:

\[
|I_2| = \left| \int_{\mathbb{R}} \left( \int \int \lambda(V(z, v, z, v) N_1(\epsilon, z, p, v) V(p, v)) L(f)(i(\theta - \omega), v') \eta \, dv' \, d\theta \, d\omega \right) \right| + \left| \int_{\mathbb{R}} \left( \int \int \lambda(V(z, v, z, v) N_2(\epsilon, z, p, v) V(p, v)) L(f)(i(\theta - \omega), v') \eta \, dv' \, d\theta \, d\omega \right) \right| = I_{2,1} + I_{2,2}.
\]

The integral \( I_{2,2} \) can be bounded using (4.16) and (4.28) (adapting \( 0 < \delta_0, \epsilon_0 \) if needed):

\[
|I_{2,2}| \leq \int_{\mathbb{R}} \left( \int \int \lambda |V(z)|_\nu |V(p)|_\nu \frac{\epsilon^2 |p| |z| + \epsilon^2 |p| |z|(1 + \epsilon |\theta - \omega|)}{(1 + \alpha(\epsilon z, v - v'))^2(1 + \alpha(\epsilon p, v - v'))^2} \right) L(f) \eta \, dv' \, d\theta \, d\omega \leq C(A) \left( \delta + \epsilon^\frac{1}{2} \right) \int_{\mathbb{R}} \left( \int \int \frac{\lambda |V(z, v)|_\nu^2 (\epsilon z, |v|) + \alpha^2(\epsilon z, |v|)}{(1 + |v|)(1 + \alpha(\epsilon z, v - v'))^4} \right) \, dv \, d\omega \leq \frac{c}{4} D_{\epsilon, A}(u),
\]

where we use that \( C_2 \leq C C_1 \). It remains to control \( I_{2,1} \), which we estimate by means of (4.18). We obtain:

\[
|I_{2,1}| \leq C(A) \left( \delta + \epsilon^\frac{1}{2} \right) \int_{\mathbb{R}} \left( \int \int \frac{\lambda |V(z, v)|_\nu^2}{(1 + |v|)(1 + \alpha(\epsilon z, v)))^2} \right) \, dv \, d\omega \leq \frac{c}{4} D_{\epsilon, A}(u).
\]
Here we use the notation $\langle A, B \rangle = \sum_{i,j} A_{i,j} B_{i,j}$ for matrices $A, B$. The first two lines are bounded by $(4.30)$ and the Plancherel Lemma 2.12. The third line can be estimated like the corresponding term $I_3$ in Lemma 3.7. The lower order terms $\beta < \alpha$ are estimated in the same way using Lemma 3.5, so we indeed obtain (4.29). Combining all the estimates, we obtain the upper estimate $|Q_{\varepsilon, A}^a[f](u)| \leq c D_{\varepsilon, A}^a(u) + \|u\|^2_{V_{\varepsilon, A}}$, as claimed.

We obtain the main result of this subsection, Theorem 4.3, as a Corollary.

Proof of Theorem 4.3. We have proved the existence of solutions $u$ to $(4.7)$ in Lemma 2.17. We need to show continuity of the mapping $\Psi_{\delta_1}$ and the a priori estimate (4.8). First we use Lemma 2.12 to bound the norm of the solution by:

$$A \|u\|^2_{V_{\varepsilon, A}} \leq C \|u_0\|^2_{W^n} - 2 \sum_{|a| \leq n} Q_{\varepsilon, A}^a[u_0](u) + Q_{\varepsilon, A}^a[f](u).$$

(4.31)

Applying Lemma 3.7 to $Q_{\varepsilon, A}^a[u_0](u)$ and Lemma 4.9 to $Q_{\varepsilon, A}^a[f](u)$ we find that for $A > 0$ and $\delta > 0$ sufficiently small, $R > 0$ and $\varepsilon > 0$ small enough we have:

$$Q_{\varepsilon, A}^a[u_0](u) + Q_{\varepsilon, A}^a[f](u) \geq \frac{c}{2} D_{\varepsilon, A}^a - C \|u\|^2_{V_{\varepsilon, A}}.$$
Plugging this back into (4.31) we find $A, \delta > 0$ such that for all $R > 0$ and $\varepsilon > 0$ small we have, independently of $0 < \gamma \leq 1$:
\[
\|u\|_{\nu_\gamma^+} \leq \|u_0\|_{\nu_\gamma^*}.
\] (4.32)

Now define $U := \int_0^1 A^{0,1}[u_0 + f](u)(A \text{ as in Notation 2.1})$. Then by equation (4.7) we have $(u-u_0) = \nabla \cdot U$. Using Lemma 2.13 we write:
\[
L(\partial_t U)(z,v) = \int (M_1 + M_2)(\varepsilon z, v - v')\eta L((u_0(v') + f(\cdot, v')) \nabla u(\cdot, v))(z) \, dv' - \int \nabla \cdot (M_1 + M_2)(\varepsilon z, v - v')\eta L((u_0(v') + f(\cdot, v'))u(\cdot, v))(z) \, dv'.
\] (4.33)

Now $M_1, M_2$ as well as their derivatives are bounded. Further Lemma 4.8 and (4.28) imply:
\[
\|L(f)(z,v)\|_{L_0^1} \leq C(A)(\delta + \text{Re} \gamma) e^{-\frac{1}{2} \gamma^2}.
\] (4.34)

Hence for $\delta > 0$ and $\varepsilon(A, R) > 0$ sufficiently small, combining (4.33), (4.34), and the Plancherel Lemma 2.13 gives the desired estimate for $U$ in (4.8). Plugging this back into (4.7) gives (4.8):
\[
|((u-u_0)\kappa_{\delta_1}, U \kappa_{\delta_1})|_{X_{A,d}^n} + \|\partial_t((u-u_0)\kappa_{\delta_1}, U \kappa_{\delta_1})\|_{X_{A,d}^n} \leq C.
\]

It remains to show continuity of the operator $\Psi_{\delta_1}$ for positive $\gamma, \varepsilon$. Let $(f_i, g_i) \in \Omega_{A,R,\delta,\varepsilon}^n, i = 1, 2$ and $u_1, u_2$ the corresponding solutions to (4.7). For shortness write
\[
K_i = \frac{1}{\varepsilon} K[u_0 + f_i(s)] \left( \frac{t-s}{\varepsilon}, v \right), \quad P_i = \frac{1}{\varepsilon} P[u_0 + f_i(s)] \left( \frac{t-s}{\varepsilon}, v \right).
\]

Then the difference $u_1 - u_2$ satisfies $(u_1(0) - u_2(0)) = 0$ and:
\[
\partial_t(u_1 - u_2) = \frac{\gamma}{\varepsilon} \nabla \cdot \left( \int_0^t K_i \nabla u_1(s, v) - P_i u_1 - K_2 \nabla u_2(s, v) + P_2 u_2 \, ds \right).
\] (4.35)

For $m \in \mathbb{N}$ arbitrary, $\|K[f]\|_{L^1([0,1]; C^m(\mathbb{R}^+; \mathbb{R}))} + \|P[f]\|_{L^1([0,1]; C^m(\mathbb{R}^+; \mathbb{R}))} \leq C\|f, g\|_{X_{A,d}^n}$ are continuous. Recalling that $\nabla$ are mollifying operators, the continuity of $\Psi_{\delta_1}$ now follows from (4.35) by Gronwall’s Lemma.

\[\square\]

### 4.2 Invariance of the set $\Omega$ under the mapping $\Psi$

#### 4.2.1 Recovering the quadratic decay in Laplace variables

In the last subsection we have shown that for $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ as defined in (4.6), the equation
\[
\partial_t u_\varepsilon = \frac{1}{\varepsilon} \nabla \cdot \left( \int_0^t K[u_0 + f_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s, v) \, ds \right) - \frac{1}{\varepsilon} \nabla \cdot \left( \int_0^t P[u_0 + f_\varepsilon(s)] \left( \frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s, v) \, ds \right)
\] (4.36)
\[u_\varepsilon(0, \cdot) = u_0(\cdot),\]

has solutions in $X_{A,d}^n$. The goal of this section is to show that the associated solution operator $\Psi_{\delta_1}$ defined in (4.7) leaves the set $\Omega_{A,R,\delta,\varepsilon}^n$ (cf. (4.6)) invariant. More precisely, we will prove the following theorem.
Lemma 4.10 Let \( n \geq 6 \) and assume \( v_0 \in H^n_\Delta \) satisfies the bounds:

\[
0 \leq v_0(v) \leq Ce^{-\frac{1}{2}|v|}.
\]

Let \( A, \delta > 0 \) be as in Theorem 4.3 and \( \Psi_{\delta_1} \) the solution operator to (4.36):

\[
\Psi_{\delta_1} : \Omega^n_{A,R,\delta,R} \longrightarrow X^n_{A,\delta}
\]

\[
(f, g) \mapsto \left( (u_\varepsilon - u_0)k_{\delta_1}, A^{0,0}_\gamma |u_0 + f(v)| k_{\delta_1} \right), \quad u_\varepsilon \text{ solves (4.36) with } u_0 = m + \delta_2 v_0.
\]

There exist \( \delta_1, \epsilon_0, R > 0 \) such that for \( \delta_2, \epsilon \in (0, \epsilon_0] \) and for all \( \gamma \in (0, 1] \), the set \( \Omega^n_{A,\delta,R} \) is invariant under the mapping \( \Psi_{\delta_1} \).

As a first step, we will prove estimate (4.4). Differentiating equation (4.36) yields, where \( \Lambda^{\alpha}_{\gamma} \) is defined in (2.18):

\[
\partial_t D^\alpha u_\varepsilon = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \nabla \cdot \left( \Lambda^{\alpha}_{\gamma} |u_0 + f(v)| \right).
\]

Therefore in order to characterize the properties of \( D^\alpha u_\varepsilon \) in Laplace variables, we first need to understand the right-hand side of the above equation in this framework.

Lemma 4.11 Let \( n \geq 0 \) and \((f, g) \in \Omega^n_{A,R,\delta,R} \). Further let \( u_0 \in C(\mathbb{R}^3) \) satisfy

\[
0 \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}.
\]

Let \( a = \frac{A}{2} \geq \frac{1}{2}, \gamma \in (0, 1] \) and \( \beta \leq \alpha \) be multi-indexes with \( |\alpha| = m < n \). Then for almost every \( z \in \mathbb{C} \)

\[
|\mathcal{L}(\Lambda^{\alpha}_{\gamma} |u_0|)(z,v)| \leq C(A)\frac{|u|_F^{m+1}}{|1 + \varepsilon z|}, \quad |\mathcal{L}(\Lambda^{\alpha}_{\gamma} |f|(v))(z,v)| \leq C(A)\frac{Y_{\varepsilon,\delta} * u |u|_F^{m+2}}{|1 + \varepsilon z|}.
\]

Here the convolution \( *_{a} \) is to be understood as \((z = a + i\omega)\):

\[
(f *_{a} g)(a + i\omega) = \int_{\mathbb{R}} f(i\theta)g(a + i(\omega - \theta)) \, d\theta.
\]

**Proof:** Is a direct consequence of elementary properties of the Laplace transform, Lemma 4.3 and the defining formula (2.18) of \( \Lambda^{\alpha}_{\gamma} \).

Lemma 4.12 Let \( u \in V^n_{A,\Delta} \) for \( n \geq 2 \). For \( a \in (0, 1] \) and \( \delta_1 \in (0, 1] \) we have:

\[
|\mathcal{L}(u \kappa_{\delta_1})(\cdot, v)|_{L^2_{\mathbb{R}(v)} R^{v}_{(a)}} \leq C(a, \delta_1)\|\mathcal{L}(u)(\cdot, v)\|_{L^2_{\mathbb{R}(v)} R^{v}_{(a)}} \quad (4.39)
\]

\[
|\mathcal{L}(u \kappa_{\delta_1})(\cdot, v)|_{L^2_{\mathbb{R}(v)} R^{v}_{(a)}} \leq C(a, \delta_1)\|\mathcal{L}(u)(\cdot, v)\|_{L^2_{\mathbb{R}(v)} R^{v}_{(a)}} \quad (4.40)
\]

**Proof:** We start by proving (4.39). Consider the two-sided Laplace transform \( \tilde{\mathcal{L}} \):

\[
\tilde{\mathcal{L}}(f)(z) = \int_{-\infty}^{\infty} e^{-zt} f(t) \, dt.
\]
Extending \( u(t) = 0 \) for negative \( t \), we find that for \( \Re(z) = a \geq \frac{1}{2} \):

\[
\tilde{L}(\kappa_{\delta_{1}}) = \tilde{L}(\kappa_{\delta_{1}}) *_{a} \tilde{L}(u).
\]

Since \( \tilde{L}(\kappa_{\delta}) \) is a Schwartz function, the claim follows from Young’s inequality and the assumption \( n \geq 2 \) (so both sides of (4.39), (4.40) are continuous). The proof of (4.40) follows similarly.

Now that we can characterize the properties of the operators \( A_{\gamma}^{a,b} \) in Laplace variables, we are able to prove bounds for the Laplace transforms of the solution \( u_{\epsilon} \).

**Lemma 4.13** Let \( n \geq 2 \) and \( A = 2a \geq \frac{1}{2} \), \( \delta > 0 \) be as in Theorem 4.3. For \( R > 0, \gamma \in (0,1], \) \((f_{x}, g_{x}) \in \Omega_{A,R,\delta}^{n}\) let \( u_{x} \in V_{A,R}^{n} \) be the solution to (4.36), and let \( |\alpha| = m \leq n - 2 \). Recall the family of cutoff functions \( \kappa_{\delta} \) defined in (4.1). For \( \delta_{1}, \epsilon \in (0,1] \), we have:

\[
|\mathcal{L}(\kappa_{\delta_{1}}D^{a}(u_{\epsilon} - u_{0}))| \leq \frac{C(A, \delta_{1})}{1 + \epsilon z}|(u_{\epsilon} \kappa_{2\delta_{1}}|_{F^{\epsilon} + 2} + \gamma_{\epsilon, \delta_{1}} *_{a} |u_{\epsilon} \kappa_{2\delta_{1}}|_{F^{\epsilon} + 2})| (4.41)
\]

\[
|\mathcal{L} \left( \kappa_{\delta_{1}}A^{0,0}_{\gamma}[u_{\epsilon} + f_{\epsilon}](u_{\epsilon}) \right) | \leq \frac{C(A, \delta_{1})}{1 + \epsilon z} |(u_{\epsilon} \kappa_{2\delta_{1}}|_{F^{\epsilon} + 2} + \gamma_{\epsilon, \delta_{1}} *_{a} |u_{\epsilon} \kappa_{2\delta_{1}}|_{F^{\epsilon} + 2})|, (4.42)
\]

\( \Re \) a.e. on the line \( \Re(z) = a \). Again we use the shorthand \( *_{a} \) as introduced in (4.38).

**Proof:** Integrating the equation (4.36) we find:

\[
(u_{\epsilon} - u_{0})(T) = \int_{0}^{T} \frac{1}{\epsilon} \nabla \cdot \left( \int_{0}^{t} K[u_{0} + f_{\epsilon}(s)] \left( \frac{t - s}{\epsilon}, v \right) \nabla u_{\epsilon}(s, v) \, ds \right)
\]

\[
- \frac{1}{\epsilon} \nabla \cdot \left( \int_{0}^{t} P[u_{0} + f_{\epsilon}(s)] \left( \frac{t - s}{\epsilon}, v \right) u_{\epsilon}(s, v) \, ds \right) \, dt.
\]

Since \( \kappa_{2\delta_{1}} = 1 \) on the support of \( \kappa_{\delta_{1}} \), the Volterra structure of the equation allows to rewrite:

\[
\kappa_{\delta_{1}}(u_{\epsilon} - u) = \kappa_{\delta_{1}} \int_{0}^{T} \frac{1}{\epsilon} \nabla \cdot \left( \int_{0}^{t} K[u_{0} + f_{\epsilon}(s)] \left( \frac{t - s}{\epsilon}, v \right) \nabla (\kappa_{2\delta_{1}} u_{\epsilon})(s, v) \, ds \right)
\]

\[
- \frac{1}{\epsilon} \nabla \cdot \left( \int_{0}^{t} P[u_{0} + f_{\epsilon}(s)] \left( \frac{t - s}{\epsilon}, v \right) (\kappa_{2\delta_{1}} u_{\epsilon})(s, v) \, ds \right) \, dt.
\]

Hence in Laplace variables we have:

\[
z \mathcal{L}(D^{a}(u_{\epsilon} - u_{0})\kappa_{\delta_{1}}) = \mathcal{L} \left( \kappa_{\delta_{1}} \sum_{\beta \leq a} \frac{\alpha}{\beta} \nabla \cdot \left( A_{\gamma}^{a,b}[u_{\epsilon} + f_{\epsilon}](u_{\epsilon} \kappa_{2\delta_{1}}) \right) \right)
\]

Estimate (4.41) now follows from Lemma 4.11 and Lemma 4.12. Estimate (4.42) is proved in the same way.

**Lemma 4.14** (L∞ estimate in Laplace variables) Let \( n \geq 2 \) and \( A = 2a \geq \frac{1}{2} \), \( \delta > 0 \) be as in Theorem 4.3. For \( R > 0, \gamma \in (0,1], \) \((f_{x}, g_{x}) \in \Omega_{A,R,\delta}^{n}\) let \( u_{x} \in V_{A,R}^{n} \) be the solution to (4.36) with \( u_{0} = m(v) + \delta_{1}v_{0}(v) \), where \( v_{0} \in H_{A}^{1} \) satisfies:

\[
0 \leq v_{0}(v) \leq Ce^{-\frac{1}{2}|v|}.
\]

Then for \( m \in \mathbb{N}, m \leq n - 2, \epsilon > 0 \) small enough and \( \delta_{1} \in (0,1] \), there holds:

\[
\| \mathcal{L}(\nabla^{m} u_{\epsilon} \kappa_{\delta_{1}}) \|_{L^{\infty}(\Re(z) = a)} \leq C(A, \delta_{1})e^{-\frac{1}{2}|v|}.
\]
Proof: We solve equation (4.36) with \((f, g) \in \Omega^n_{\text{A}, \text{R}, \delta, \varepsilon}\). Theorem 4.9 shows there are \(A, \delta, C(A) > 0\) such that for all \(R > 0\) a solution \(u_\varepsilon\) to (4.36) satisfies:

\[
\|u_\varepsilon\|_{V^{\alpha}_{A, R}} \leq C(A),
\]

provided \(\varepsilon > 0\) is small enough. By Plancherel Lemma 2.13 this implies in particular

\[
\|\mathcal{L}(D^\alpha u_\varepsilon)\|_{L^2_{\text{R}^{n+1}}} \leq C(A) \quad \text{for } |\alpha| \leq n.
\]

With Sobolev inequality we can infer the existence of a constant \(C(A) > 0\) such that for every multiindex \(\alpha\) with \(|\alpha| \leq n - 2\) we have:

\[
\|\mathcal{L}(D^\alpha u_\varepsilon(\cdot, v))\|_{L^2_{\text{R}^{n+1}}} \leq C(A)e^{-\frac{1}{2}|v|}.
\]

Now with Lemma 4.12 we can estimate:

\[
\|\mathcal{L}(\nabla^m u_\varepsilon(\cdot, v))\|_{L^\infty_{\text{R}^{n+1}}} \leq C(A, \delta_1)e^{-\frac{1}{2}|v|},
\]

as claimed. □

We can plug the \(L^\infty\) estimate for the Laplace transform back into (4.13) and bootstrap it to a pointwise estimate.

Lemma 4.15 (Linear decay in Laplace variables) Let \(n \geq 4\) and \(A = 2a \geq \frac{1}{2}\), \(\delta > 0\) be as in Theorem 4.3. For \(R > 0\), \(\gamma, \delta_2 \in (0, 1]\), \((f, g) \in \Omega^n_{\text{A}, \text{R}, \delta, \varepsilon}\) let \(u_\varepsilon \in V^n_{\text{A}, \delta, \varepsilon}\) be the solution to (4.36) with \(u_0 = m(v) + \delta_2 v_0(v)\), where \(v_0 \in H^\delta_2\) satisfies:

\[
0 \leq v_0(v) \leq C e^{-\frac{1}{2}|v|}.
\]

Then for \(m \in \mathbb{N}, m \leq n - 4, \varepsilon > 0\) small enough and \(\delta_1 \in (0, 1]\) there holds:

\[
|\mathcal{L}(\nabla^m((u - u_\varepsilon)k_\delta_1)(z, v))| \leq \frac{C(A, \delta_1)e^{-\frac{1}{2}|v|}}{1 + |z|},
\]

\[
|\mathcal{L}(\nabla^m(A^0_{\gamma} u_0 + f_\varepsilon)(u_\varepsilon k_\delta_1)(z, v))| \leq \frac{C(A, \delta_1)e^{-\frac{1}{2}|v|}}{1 + |z|}.
\]

Proof: Follows by combining Lemma 4.13 with Lemma 4.14. □

Bootstrapping the estimate in Lemma 4.15 gives an additional quadratic decay, which is the content of the following Lemma.

Lemma 4.16 (Quadratic decay of Laplace Transforms) Let \(n \geq 4\) and \(A = 2a \geq \frac{1}{2}\), \(\delta > 0\) be as in Theorem 4.3. For \(R > 0\), \(\gamma, \delta_2 \in (0, 1]\), \((f, g) \in \Omega^n_{\text{A}, \text{R}, \delta, \varepsilon}\) let \(u_\varepsilon \in V^n_{\text{A}, \delta, \varepsilon}\) be the solution to (4.36) with \(u_0 = m(v) + \delta_2 v_0(v)\), where \(v_0 \in H^\delta_2\) satisfies:

\[
0 \leq v_0(v) \leq C e^{-\frac{1}{2}|v|}.
\]

Then for \(m \in \mathbb{N}, m \leq n - 4, \varepsilon > 0\) small enough and \(\delta_1 \in (0, 1]\) there holds:

\[
|\mathcal{L}(\nabla^m(u_\varepsilon - u_0)k_\delta_1)(z, v)| \leq \frac{C(A, \delta_1)e^{-\frac{1}{2}|v|}}{1 + \varepsilon|z|(1 + |z|^2)},
\]

\[
|\mathcal{L}(\nabla^m(A^0_{\gamma} u_0 + f_\varepsilon)(u_\varepsilon k_\delta_1)(z, v))| \leq \frac{C(A, \delta_1)e^{-\frac{1}{2}|v|}}{1 + \varepsilon|z|(1 + |z|^2)}.
\]
Proof: Follows by iterating Lemma 4.13 further with the estimate Lemma 4.15. For completeness we remark that the linear decay of $|u_t \kappa t^j|_{m+2}$ is stable under convolution with $Y_{t, \delta}$. To see this we estimate the convolution explicitly ($z = a + i \omega$, $y = a + i \theta$, $a \geq \frac{1}{2}$):

$$Y_{t, \delta} * u_t \kappa t^j|_{m+2} \leq \int_R \frac{C(A, \delta_1)}{1 + |z - y|} \left( \frac{\delta}{1 + |z|} + \frac{\text{Re}|z|}{(1 + \epsilon|y|(1 + |z|)^2)} \right) \, d\theta \leq \frac{C(A, \delta_1)}{1 + |z|} + \int_R \frac{C(A, \delta_1)}{1 + |z - y|} \frac{\text{Re}|z|}{(1 + \epsilon|y|(1 + |z|)^2)} \, d\theta. $$

It remains to show that the last integral decays linearly with a prefactor independent of $R > 0$. This can be seen by splitting the integral into the regions

$$D_d(x) := \{ y : \Re(y) = a, |y| \geq 2|x| \text{ or } |y| \leq \frac{1}{2}|x| \},$$

$$D_c(x) := \{ y : \Re(y) = a, \frac{1}{2}|x| \leq |y| \leq 2|x| \},$$

when the integral can be estimated as $(C(A, \delta_1)$ might change from line to line):

$$\int_{D_d(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{\text{Re}|y|}{(1 + \epsilon|y|(1 + |y|^2)} \, d\theta \leq \frac{C(A, \delta_1)}{1 + |z|} \int_{D_d(x)} \frac{\text{Re}}{1 + |z - y|} \frac{\text{Re}|y|}{(1 + \epsilon|y|(1 + |y|^2)} \, d\theta \leq \frac{C(A, \delta_1)}{1 + |z|} \int_{D_d(x)} \frac{\text{Re}}{1 + |z - y|} \frac{\text{Re}}{(1 + \epsilon|y|(1 + |y|^2)} \, d\theta,$$

with $C(A, \delta_1)$ is independent of $R > 0$, provided $\epsilon(R) > 0$ is small enough. We can bound the second integral by:

$$\int_{D_d(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{\text{Re}}{(1 + \epsilon|y|(1 + |y|^2)} \, d\theta \leq \frac{\text{Re}}{(1 + \epsilon|z|(1 + |z|)}} \int_{D_d(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{\text{Re}}{(1 + \epsilon|y|(1 + |y|)^2)} \, d\theta \leq \frac{C(A, \delta_1) \text{Re} \log(1 + |z|)}{(1 + \epsilon|z|(1 + |z|)^2)},$$

for $\epsilon > 0$ small enough.

As a corollary we obtain the uniform boundedness of the sequence $u_t$.

Lemma 4.17 (Uniform boundedness) Let $n \geq 4$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 4.3. For $R > 0$, $\gamma, \delta_2 \in (0, 1)$, $(f_t, g_t) \in \Omega^n_{A, R, \delta, t}$ let $u_t \in V^n_{A, \lambda}$ be the solution to (4.36) with $u_0 = m(v) + \delta_2 v_0(v)$, where $v_0 \in H^1_{\lambda}$ satisfies:

$$0 \leq v_0(v) \leq Ce^{-\frac{1}{2}|v|}.$$

Then for $m \in \mathbb{N}$, $m \leq n-4$, $\epsilon > 0$ small enough there holds:

$$|\nabla^m(u_t - u_0(t, v))| \leq C(A)e^{-\frac{1}{2}|v|}, \quad \text{for } 0 \leq t \leq 1. \quad (4.44)$$
4.2.2 Boundary Layer Estimate

To obtain smallness for the Laplace transforms, we separate the contributions of $M_1$ and $M_2$ to $u_\varepsilon$.

**Lemma 4.18 (Decomposition)** Let $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta, \varepsilon}^n$ and $u_\varepsilon \in V_{A, \lambda}^n$ a solution to (4.36). Then $u_\varepsilon - u_0 = p_\varepsilon + q_\varepsilon$. Here $p_\varepsilon = \nabla \cdot P_\varepsilon$ is a divergence and $P_\varepsilon$ is given by:

$$
\partial_1 P_\varepsilon = \left( \int_0^t \int_\mathbb{R}^3 \frac{\pi^2 \varepsilon^{-\frac{10 n}{11}}}{4} \rho(t-s, v-v') \eta(|v'|^2) \nabla u_\varepsilon(t-s, v) \, dv' \, ds \right) \left( u_0 + f_\varepsilon(t) \right) (t-s, v-v') \eta(|v'|^2) \nabla u_\varepsilon(t-s, v) \, dv' \, ds
$$

$$
\left( \int_0^t \int_\mathbb{R}^3 \frac{\pi^2 \varepsilon^{-\frac{10 n}{11}}}{4} \rho(t-s, v-v') \eta(|v'|^2) \nabla u_\varepsilon(t-s, v) \, dv' \, ds \right) \left( u_0 + f_\varepsilon(t) \right) (t-s, v-v') \eta(|v'|^2) \nabla u_\varepsilon(t-s, v) \, dv' \, ds
$$

$$
P_\varepsilon(0) = 0.
$$

Similarly, $q_\varepsilon = \nabla \cdot Q_\varepsilon$, where $Q_\varepsilon$ is given by:

$$
z\mathcal{L}(Q_\varepsilon) = \left( \int_\mathbb{R}^3 \frac{\pi^2 \varepsilon^{-\frac{10 n}{11}}}{4} \rho(t-s, v-v') \eta(|v'|^2) \mathcal{L} \left( (u_0 + f_\varepsilon(t-s, v-v') \nabla u_\varepsilon(s, v) \right) \, dv' \right)
$$

$$
- \left( \int_\mathbb{R}^3 \nabla \cdot (u_0 + f_\varepsilon(s, v-v') \nabla u_\varepsilon(s, v) \right) \mathcal{L} \left( (u_0 + f_\varepsilon(t-s, v-v') \nabla u_\varepsilon(t-s, v) \right) \, dv' \right).
$$

**Proof:** We take the Laplace transform of equation (4.36) and use Lemma 2.15 to obtain:

$$
z\mathcal{L}(u_\varepsilon)(z, v) - u_0(v)
$$

$$
= \nabla \cdot \left( \int_\mathbb{R}^3 (M_1 + M_2)(\varepsilon z, v') \eta(|v'|^2) \mathcal{L} \left( (u_0 + f_\varepsilon)(s, v-v') \nabla u_\varepsilon(s, v) \right) \, dv' \right)
$$

$$
- \nabla \cdot \left( \int_\mathbb{R}^3 \nabla \cdot (M_1 + M_2)(\varepsilon z, v') \eta(|v'|^2) \mathcal{L} \left( (u_0 + f_\varepsilon)(s, v-v') \nabla u_\varepsilon(s, v) \right) \, dv' \right).
$$

Now introduce the functions $p_\varepsilon, q_\varepsilon$ given by the splitting:

$$
z\mathcal{L}(q_\varepsilon)(z, v) = \nabla \cdot \left( \int_\mathbb{R}^3 M_2(\varepsilon z, v') \eta(|v'|^2) \mathcal{L} \left( (u_0 + f_\varepsilon)(s, v-v') \nabla u_\varepsilon(s, v) \right) \, dv' \right)
$$

$$
- \nabla \cdot \left( \int_\mathbb{R}^3 \nabla \cdot M_2(\varepsilon z, v') \eta(|v'|^2) \mathcal{L} \left( (u_0 + f_\varepsilon)(s, v-v') \nabla u_\varepsilon(s, v) \right) \, dv' \right)
$$

$$
z\mathcal{L}(p_\varepsilon)(z, v) = \nabla \cdot \left( \int_\mathbb{R}^3 M_1(\varepsilon z, v') \eta \mathcal{L} \left( (u_0 + f_\varepsilon)(s, v-v') \nabla u_\varepsilon(s, v) \right) \, dv' \right)
$$

$$
- \nabla \cdot \left( \int_\mathbb{R}^3 \nabla \cdot M_1(\varepsilon z, v') \eta \mathcal{L} \left( (u_0 + f_\varepsilon)(s, v-v') \nabla u_\varepsilon(s, v) \right) \, dv' \right).
$$

Therefore $q_\varepsilon = \nabla \cdot Q_\varepsilon$, with $Q_\varepsilon$ as in (4.36). To show $p_\varepsilon = \nabla \cdot P_\varepsilon$ we transform the equation for $p_\varepsilon$ back to the variables $(t, v)$. To do so we remark that $M_1$ is the Laplace transform of:

$$
\frac{\pi^2}{4} \mathcal{L} \left( \frac{e^{-\frac{d}{\varepsilon}}}{\varepsilon} \right) (z) \rho = M_1(\varepsilon z, v).
$$

Therefore $p_\varepsilon = \nabla \cdot Q_\varepsilon$ and $u_\varepsilon - u_0 = q_\varepsilon + p_\varepsilon$ as claimed. 

\[\square\]
Splitting the function \( u_e \) into \( u_e = p_e + q_e \) allows to estimate the contributions of \( M_1 \) and \( M_2 \) (as in (2.15)) separately. The function \( q_e \) can be estimated in a straightforward fashion.

**Lemma 4.19 (Estimate for \( q_e \))** Let \( n \geq 4 \) and \( A = 2a \geq \frac{\gamma}{2} \), \( \delta > 0 \) be as in Theorem 4.3. For \( R > 0 \), \( \gamma, \delta_2 \in (0, 1) \), \((f_e, g_e) \in \Omega^n_{A,R,\delta,\varepsilon} \) let \( u_e \in V^n_{A,\varepsilon} \) be the solution to (4.36) with \( u_0 = m(v) + \delta_2 \nu_0(v) \), where \( \nu_0 \in H^n_{A,\varepsilon} \) satisfies:

\[
0 \leq \nu_0(v) \leq Ce^{-\frac{1}{2}[\varepsilon]}.
\]

Let \( \nabla \cdot Q_e = q_e \in V^n_{A,\varepsilon} \) be given by (4.46). Then for \( m \in \mathbb{N} \), \( n \leq m - 4 \), \( \varepsilon > 0 \) small enough there holds:

\[
|\mathcal{L}(\nabla^m q, \kappa_{\delta_1})| \leq \frac{C(A, \delta_1)\varepsilon|z|}{(1 + \varepsilon|z|)^2(1 + |z|^2)} e^{-\frac{1}{2}|\varepsilon|}
\]

\[
|\mathcal{L}(\nabla^m Q_e, \kappa_{\delta_1})| \leq \frac{C(A, \delta_1)\varepsilon|z|}{(1 + \varepsilon|z|)^2(1 + |z|^2)} e^{-\frac{1}{2}|\varepsilon|}.
\]

In particular, for \( 0 \leq t \leq 1 \), \( m \leq n - 4 \) we have

\[
|\partial_t \nabla^m q_e| \leq C(A)e^{-\frac{1}{2}|\varepsilon|} \quad |\partial_t \nabla^m Q_e| \leq C(A)e^{-\frac{1}{2}|\varepsilon|}.
\]

**Lemma 4.20 (\( L^\infty \) estimate for time derivative)** Let \( n \geq 4 \) and \( A = 2a \geq \frac{\gamma}{2} \), \( \delta > 0 \) be as in Theorem 4.3. For \( R > 0 \), \( \gamma, \delta_2 \in (0, 1) \), \((f_e, g_e) \in \Omega^n_{A,R,\delta,\varepsilon} \) let \( u_e \in V^n_{A,\varepsilon} \) be the solution to (4.36) with \( u_0 = m(v) + \delta_2 \nu_0(v) \), where \( \nu_0 \in H^n_{A,\varepsilon} \) satisfies:

\[
0 \leq \nu_0(v) \leq Ce^{-\frac{1}{2}|\varepsilon|}.
\]

Then for \( m \in \mathbb{N} \), \( n \leq m - 4 \), \( \varepsilon > 0 \) small enough there holds:

\[
|\partial_t \nabla^m u_e(t, v)| \leq C(A)e^{-\frac{1}{2}|\varepsilon|} \quad \text{for } 0 \leq t \leq 1.
\]

**Proof:** We use the decomposition \( u_e = p_e + q_e \) introduced in Lemma 4.18. By the previous Lemma 4.19 we know

\[
|\partial_t \nabla^m q_e(t, v)| \leq C(A)e^{-\frac{1}{2}|\varepsilon|} \quad \text{for } 0 \leq t \leq 1.
\]

It remains to estimate \( p_e \). The sequence \( e^{-t/\varepsilon}/|\varepsilon| \) is bounded in \( L^1 \). Therefore the claim follows by inserting the estimate (4.44) into the definition (4.43) of \( p_e = \nabla \cdot P_e \).

**Notation 4.21** Let \( b \) be the function given by:

\[
b(t, r) := \frac{e^{-t/r} + t - 1}{r^2}.
\]

For \( u_0 \in H^n_{A,\varepsilon} \), define the boundary layer \( B(t, v; u_0) = \nabla \cdot B_F(t, v; u_0) \) by:

\[
B_F(t, v; u_0) := \int_0^\pi \frac{1}{4} \frac{b(t, \frac{\nu_0'}{\varepsilon}) P_{\nu'}^*}{\varepsilon} \eta \left( u_0(v - v') \nabla u_0(v) - \nabla u_0(v - v') u_0(v) \right) dv'.
\]
Lemma 4.22 (Boundary Layer property) The function $B = \nabla \cdot B_F$, as defined in (4.53) satisfies:

$$\partial_t B(t, v) = \nabla \left( \int \frac{\pi^2 e^{-\frac{|v'|^2}{4}} P^\perp_{\nu'}}{\epsilon} \eta(|v'|^2) (u_0(v - v') \nabla u_0(v) - \nabla u_0(v - v') u_0(v)) \, dv' \right),$$

$$B(0, v) = 0 \quad \partial_t B(0, v) = 0.$$

**Proof:** Differentiating $b$ gives:

$$\partial_t b(t, r) = \frac{1 - e^{-rt}}{r}, \quad \partial_r b(t, r) = e^{-rt}.$$

Therefore the second time derivative of $B$ is:

$$\partial_{tt} B(t, v) = \nabla \left( \int \frac{\pi^2 e^{-\frac{|v'|^2}{4}} P^\perp_{\nu'}}{\epsilon} \eta(|v'|^2) (u_0(v - v') \nabla u_0(v) - \nabla u_0(v - v') u_0(v)) \, dv' \right).$$

The initial data $B(0, v) = 0$, $\partial_t B(0, v) = 0$ follow by simply putting $t = 0$. \hfill \Box

**Lemma 4.23 (Remainder estimate)** Let $n \geq 4$ and $p_\epsilon$ solve (4.45) and $\|u_\epsilon\|_{V_{n, \delta}} \leq C$. There exists a $C_0 > 0$ such that for all $m \leq n - 2$ there exists $\epsilon$ small enough such that:

$$|\partial_t (p_\epsilon - B)(t, v)| \leq C_0 e^{-\frac{1}{2}|v|}, \quad \text{for } t \in [0, 1].$$

**Proof:** Take the time derivative of (4.45). We can split using Lemma 4.22:

$$\partial_{tt} p_\epsilon = \nabla \left( \int_0^t \frac{\pi^2 e^{-\frac{|v'|^2}{4}} P^\perp_{\nu'}}{\epsilon} \partial_t ((u_0 + f_\epsilon)(t - s, v, v') \eta \nabla u_\epsilon(t - s, v)) \, dv' \, ds \right)$$

$$- \nabla \left( \int_0^t \frac{\pi^2 e^{-\frac{|v'|^2}{4}} P^\perp_{\nu'}}{\epsilon} \partial_t \nabla ((u_0 + f_\epsilon)(t - s, v, v') \eta u_\epsilon(t - s, v)) \, dv' \, ds \right)$$

$$+ \nabla \left( \int_0^t \frac{\pi^2 e^{-\frac{|v'|^2}{4}} P^\perp_{\nu'}}{\epsilon} \eta(|v'|^2) (u_0(v - v') \nabla u_\epsilon(v) - \nabla u_\epsilon(v - v') u_\epsilon(v)) \, dv' \right)$$

$$= R_1 + R_2 + \partial_{tt} B.$$

Since $|\partial_t f_\epsilon| \leq C e^{-\frac{1}{2}|v|}$ by assumption, we obtain:

$$|\partial_t (p_\epsilon - B)(t, v)| = |R_1(t, v) + R_2(t, v)| \leq C_0 e^{-\frac{1}{2}|v|}, \quad \text{for } t \in [0, 1],$$

as claimed. \hfill \Box

**Lemma 4.24 (Smallness of $\mathcal{L}(p_\epsilon - B)$)** Let $p_\epsilon$ solve (4.45) and $\|u_\epsilon\|_{V_{n, \delta}} \leq C$ for some $A = 2a > 0$. We have $p_\epsilon - B = \nabla \cdot (p_\epsilon - B_F)$, and there is a $C_0 > 0$ such that for all $m \leq n - 2$, $\delta_1 > 0$ and $\epsilon > 0$ small enough:

$$|\mathcal{L}(p_\epsilon - B)\kappa_{\delta_1})(z, v)| + |\mathcal{L}(p_\epsilon - B_F)\kappa_{\delta_1})(z, v)| \leq \frac{\delta_1 C_0 e^{-\frac{1}{2}|v|}}{1 + |z|^2}.$$
Proof: By definition of $B$ the difference $p_e - B$ vanishes initially, as well as the time derivative:

$$(p_e - B)(0,v) = \partial_t(p_e - B)(0,v) = 0.$$ Combining with the lemma above this shows:

$$|\partial_t((p_e - B)\kappa_{\delta_1})| \leq C_0 e^{-\frac{1}{2}|v|}(1 + \frac{t}{\delta_1} + \frac{t^2}{\delta_1^2})\kappa_{\delta_1}, \quad \text{for } t \in [0,1].$$ After integrating by parts twice this allows to bound the Laplace transform by:

$$|\mathcal{L}((p_e - B)\kappa_{\delta_1})(z,v)| \leq \frac{C_0 e^{-\frac{1}{2}|v|}}{|z|^2} \int_0^\infty e^{-\frac{1}{2}|v|}(1 + \frac{t}{\delta_1} + \frac{t^2}{\delta_1^2})\kappa_{\delta_1} \, dt \leq \frac{C_0 e^{-\frac{1}{2}|v|}}{|z|^2}\delta_1.$$ The estimate for $p_e - B_F$ is proved similarly. \hfill \Box

Lemma 4.25 (Stationarity of $m$) Let $\sigma^2, m_0 > 0$, $m(\sigma^2, M_0)(v)$ be the Maxwellian defined in (2.9). Then for all $t \geq 0$, $v \in \mathbb{R}^3$ we have:

$$B(t,v;m) = 0.$$ \hfill (4.54)

Proof: The argument is identical to the one proving that $m$ is a stationary point of the Landau equation: First we observe that

$$\nabla m(v) = -\frac{v}{\sigma^2}m(v).$$ This however implies that:

$$P(v') (m(v - v')\nabla m(v) - \nabla m(v - v')m(v)) = -P(v') \frac{v'}{\sigma^2}m(v - v')m(v) = 0.$$ Inserting this into the definition of $B(t,v;m)$ in (4.53) gives the claim. \hfill \Box

We use the stationarity of the Maxwellian $m$ to obtain smallness of the boundary layer, provided the evolution starts sufficiently close to $m$.

Lemma 4.26 (Boundary layer estimate) Let $u_0 = m(v) + \delta_2v_0$, for $v_0$ some fixed smooth function satisfying

$$0 \leq v_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad |\nabla^i v_0| \leq Ce^{-\frac{1}{2}|v|} \quad \text{for } i = 0, 1, 2.$$ Let $B$ be the associated Boundary Layer defined by (4.53). Then the Laplace transforms of $B$ and $B_F$ satisfy:

$$|\mathcal{L}(B\kappa_{\delta_1})(z,v)| + |\mathcal{L}(B_F\kappa_{\delta_1})(z,v)| \leq C(\delta_1)\frac{\delta_2e^{-\frac{1}{2}|v|}}{1 + |z|^2}. \hfill (4.55)$$
Proof: Using Lemma \ref{Lemma4.25} we can simplify $B$ to:

$$B(t, v) = \nabla \cdot \left( \int \frac{b(t, \frac{|v|}{\varepsilon}) P_1}{\varepsilon} (m + \delta_2 v_0)(v - v') \eta \nabla (m + \delta_2 v_0)(v) \, dv' \right)$$

$$- \nabla \cdot \left( \int \frac{b(t, \frac{|v|}{\varepsilon}) P_1}{\varepsilon} (m + \delta_2 v_0)(v - v') \eta (m + \delta_2 v_0)(v) \, dv' \right)$$

$$= \nabla \cdot \left( \int \frac{b(t, \frac{|v|}{\varepsilon}) P_1}{\varepsilon} \eta \left[ \delta_2 v_0(v') \nabla m(v) + (\delta_2 v_0 + m)(v - v') \delta_2 \nabla v_0(v) \right] \, dv' \right)$$

$$- \nabla \cdot \left( \int \frac{b(t, \frac{|v|}{\varepsilon}) P_1}{\varepsilon} \eta \left[ \delta_2 \nabla v_0(v') m(v) + \nabla (\delta_2 v_0 + m)(v - v') \delta_2 v_0(v) \right] \, dv' \right).$$

The Laplace transform of $b$ can be computed explicitly:

$$\mathcal{L}(b(\cdot, r))(z) = \frac{1}{rz^2} - \frac{1}{r(z + r)z}.$$

Inserting this above we obtain the estimate:

$$|\mathcal{L}(B_{\kappa_{\delta_1}})(z, v)| + |\mathcal{L}(B_F \kappa_{\delta_1})(z, v)| \leq C(\delta_1) \frac{\delta_2 e^{-\frac{1}{2}|v|}}{1 + |z|^2}.$$

which is the claim of the Lemma.

We are in the position to now prove Theorem \ref{Theorem4.10}.

**Proof of Theorem \ref{Theorem4.10}** Let $A, \delta > 0$ as in Theorem \ref{Theorem4.3}. Then the theorem ensures that for $R > 0$, $\delta_2 \in (0, 1]$ arbitrary, and $(f, g) \in \Omega^n_{A, R, \delta}$ the solution $u_\varepsilon$ to \eqref{4.56} with $u_\varepsilon - u_0 = \nabla \cdot U_\varepsilon$ can be bounded by:

$$\|u_\varepsilon \kappa_{\delta_1}\|_{V_n^A} + \|U_\varepsilon \kappa_{\delta_1}\|_{V_n^A} \leq C.$$  \hfill (4.56)

We use that $\psi_{\delta_1}(f, g) = (\kappa_{\delta_1}(u_\varepsilon - u_0), \kappa_{\delta_1} U_\varepsilon)$ and decompose $u_\varepsilon$ into three pieces:

$$(u_\varepsilon - u_0) \kappa_{\delta_1} = (p_\varepsilon - B) \kappa_{\delta_1} + B F \kappa_{\delta_1} + Q \kappa_{\delta_1}$$

$$U_\varepsilon \kappa_{\delta_1} = (P_\varepsilon - B_F) \kappa_{\delta_1} + B \kappa_{\delta_1} + Q \kappa_{\delta_1}.$$  \hfill (4.57)

Using estimate \eqref{4.56} and Lemmas \ref{Lemma4.19}, \ref{Lemma4.24}, \ref{Lemma4.26} we can find $\delta_1, \varepsilon_0 > 0$ small enough and $R > 0$ large enough, such that for $\delta_2, \varepsilon \in (0, \varepsilon_0]$ the Laplace transforms of the summands in \eqref{4.57} can be estimated by:

$$|\mathcal{L}(u_\varepsilon \kappa_{\delta_1})| + |\mathcal{L}(U_\varepsilon \kappa_{\delta_1})| \leq \frac{\delta_2 e^{-\frac{1}{2}|v|}}{1 + |z|^2} + \frac{R \varepsilon |z| e^{-\frac{1}{2}|v|}}{(1 + \varepsilon |z|)^2(1 + |z|)^2}.$$

So we recover \eqref{4.3}, one of the defining estimates of $\Omega^n_{A, R, \delta, \varepsilon}$. The upper bound \eqref{4.4} is the content of Lemma \ref{Lemma4.16}. The remaining estimate \eqref{4.5} is proved in Lemma \ref{Lemma4.20}. \hfill \Box
5 Existence of solutions and Markovian Limit

5.1 Existence of a solution to the non-Markovian equation

With the a priori estimates proved in the last section, we can now prove Theorem 2.6.

**Proof of Theorem 2.6.** Without loss of generality, let \( m \) be the standard Gaussian, i.e. \( \sigma = m_0 = 1 \). First let \( \gamma > 0 \). We invoke Theorems 4.3 and 10 to find \( A, \delta, R, \delta_1 > 0 \) and \( \epsilon_0 > 0 \) such that for all \( \epsilon, \delta_2 \in (0, \epsilon_0] \) the mapping \( \Psi_{\delta_1} : \Omega^n_{A, R, \delta, \epsilon} \to \Omega^n_{A, R, \delta, \epsilon} \) is continuous with respect to the topologies of \( X^n_{A, \lambda} \), \( X^n_{A, \lambda} \) hence also as a map from \( X^n_{A, \lambda} \) to itself. By Lemma 4.7 we know that \( \Omega^n_{A, R, \delta, \epsilon} \) is a closed, convex, bounded and nonempty subset of \( X^n_{A, \lambda} \). Therefore, existence of a fixed point of \( \Psi_{\delta_1} \) follows from Schauder’s theorem, provided we can show that the mapping is compact. To see this, we use that \( \Psi_{\delta_1} \) is smoothing, the defining equation (4.36) of \( \Psi_{\delta_1} \) implies:

\[
\| \Psi_{\delta_1}(f, g) \|_{X^n_{A, \lambda}} + \| \partial_{t, \gamma} \Psi_{\delta_1}(f, g) \|_{X^n_{A, \lambda}} \leq C(A).
\]  

(5.1)

Since \( \nabla \) is smoothing, the defining equation (4.36) of \( \Psi_{\delta_1} \) implies:

\[
\| \Psi_{\delta_1}(f, g) \|_{X^n_{A, \lambda}} + \| \partial_{t, \gamma} \Psi_{\delta_1}(f, g) \|_{X^n_{A, \lambda}} \leq C(A, \gamma).
\]

This implies compactness of the mapping \( \Psi_{\delta_1} \) by the Rellich type Lemma 2.5. Hence for \( \gamma \in (0, 1] \), we have proved the existence of solutions \( u_{t, \gamma} \) to:

\[
\partial_{t, \gamma} u_{t, \gamma} = \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t K[u_{t, \gamma}] \left( \frac{t - s}{\epsilon}, v \right) \nabla u_{t, \gamma}(s, v) \, ds \right) - \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t P[u_{t, \gamma}] \left( \frac{t - s}{\epsilon}, v \right) u_{t, \gamma}(s, v) \, ds \right)
\]

(5.2)

\[ u_{t, \gamma}(0, \cdot) = u_0(\cdot), \]

for times \( 0 \leq t \leq \delta_1 \). It remains to pass \( \gamma \to 0 \) to obtain a solution of the non-mollified equation. The uniform estimate (5.1) shows that for \( \epsilon > 0 \) there is a sequence \( \gamma_j \to 0 \) such that \( u_{t, \gamma_j} \to u_\epsilon \) in \( V^{n-3}_{A, \lambda} \), \( u_{t, \gamma_j} \to u_\epsilon \) in \( V^n_{A, \lambda} \) and \( \partial_{t, \gamma_j} \to \partial_{t, \gamma} \) in \( V^{n-2}_{A, \lambda} \). Hence both sides of (5.2) converge weakly in \( V^{n-2}_{A, \lambda} \), and it suffices to identify the limit of the right-hand side. Indeed, from the convergence in \( V^{n-3}_{A, \lambda} \) we conclude that pointwise a.e. along a subsequence:

\[
\frac{1}{\epsilon} \nabla \cdot \left( \int_0^t K[u_{t, \gamma_j}] \left( \frac{t - s}{\epsilon}, v \right) \nabla u_{t, \gamma_j}(s, v) \, ds \right) - \frac{1}{\epsilon} \nabla \cdot \left( \int_0^t P[u_{t, \gamma_j}] \left( \frac{t - s}{\epsilon}, v \right) u_{t, \gamma_j}(s, v) \, ds \right).
\]

(5.3)

Estimate (2.11) follows from (5.1), and inserting the estimate back into equation (2.10) proves that \( u_\epsilon \in C^1([0, \delta_1]; H^{n-2}_{\lambda}) \).

\( \square \)

5.2 Non-Markovian to Markovian limit

In this section we prove the transition from non-Markovian to Markovian dynamics on the macroscopic timescale. As \( \epsilon \to 0 \), the solutions \( u_\epsilon \) to the non-Markovian equations (2.10) converge to solutions of the Landau equation.
Proof of Theorem 2.8. For the solutions $u_\varepsilon$ of (2.10) constructed in Theorem 2.6 we have the a priori bound:

$$\|((u_\varepsilon - u_0)\kappa_{\delta_1}, U_\varepsilon \kappa_{\delta_1})\|_{X_{A,d}^n} + \|\partial_i((u_\varepsilon - u_0)\kappa_{\delta_1}, U_\varepsilon \kappa_{\delta_1})\|_{X_{A,d}^{n-2}} \leq C(A).$$

Using the compactness Lemma 2.5 and the fact that $V^n_{A,d}$ is a separable Hilbert space, we can find $u \in V^n_{A,d}$, s.t. along a sequence $\varepsilon_j \to 0$ we have $u_{\varepsilon_j} \to u$ in $V^{n-3}_{A,d}$ and $\partial_i u_{\varepsilon_j} \to \partial_i u \in V^{n-2}_{A,d}$. We need to show that $u$ solves the equation (2.12). Since both sides of the equation are well-defined and have a well-defined Laplace transform, it is sufficient to show that $u$ solves the equation in Laplace variables. To this end, we take the Laplace transform of (2.10):

$$\mathcal{L}(\partial_i u_{\varepsilon_j})(z, v) = \nabla \cdot \left( \int_{\mathbb{R}^3} (M_1 + M_2)(\varepsilon_j z, v') \mathcal{L}(u_{\varepsilon_j}(s, v - v')\nabla u_{\varepsilon_j}(s, v))(z)\eta \, dv' \right)$$

$$- \nabla \cdot \left( \int_{\mathbb{R}^3} \mathcal{V}(M_1 + M_2)(\varepsilon_j z, v') \mathcal{L}(u_{\varepsilon_j}(s, v - v')u_{\varepsilon_j}(s, v))(z)\eta \, dv' \right).$$

The left-hand side converges pointwise to $\mathcal{L}(\partial_i u) = z\mathcal{L}(u) + u_0$, up to choosing a further subsequence. The right-hand side of (5.4) converges pointwise along a subsequence to:

$$\nabla \cdot \left( \int_{\mathbb{R}^3} \frac{\pi^2}{4|v'|^2} \frac{1}{1 + \frac{z}{|v'|}} P_{v'} \mathcal{L}(u(s, v - v')\nabla u(s, v))(z)\eta \, dv' \right)$$

$$- \nabla \cdot \left( \int_{\mathbb{R}^3} \frac{\pi^2}{4|v'|^2} \frac{1}{1 + \frac{z}{|v'|}} P_{v'} \mathcal{L}(\nabla u(s, v - v')u(s, v))(z)\eta \, dv' \right)$$

$$= \mathcal{L}(\nabla \cdot (\mathcal{K}[u]\nabla u) - \nabla \cdot (P[u]u)).$$

Therefore $u \in V^n_{A,d} \cap C^1([0, \delta_1]; H^{n-4}_d)$ solves equation (2.12) as claimed.

Acknowledgement. The authors acknowledge support through the CRC 1060 The mathematics of emergent effects at the University of Bonn that is funded through the German Science Foundation (DFG).
References

[1] R. Alexandre and C. Villani. On the Landau approximation in plasma physics. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(1):61–95, 2004.

[2] R. Balescu. *Statistical mechanics of charged particles*. Monographs in Statistical Physics and Thermodynamics, Vol. 4. Interscience Publishers John Wiley & Sons, Ltd., London-New York-Sydney, 1963.

[3] R. Balescu. *Equilibrium and nonequilibrium statistical mechanics*. Interscience Publishers John Wiley & Sons, Ltd., London-New York-Sydney, 1975.

[4] G. Basile, A. Nota, and M. Pulvirenti. A diffusion limit for a test particle in a random distribution of scatterers. *J. Stat. Phys.*, 155(6):1087–1111, 2014.

[5] A. Bobylev, M. Pulvirenti, and C. Saffirio. From particle systems to the Landau equation: a consistency result. *Comm. Math. Phys.*, 319(3):683–702, 2013.

[6] N. Bogoliubov. Problems of a dynamical theory in statistical physics. In *Studies in Statistical Mechanics, Vol. I*, pages 1–118. North-Holland, Amsterdam; Interscience, New York, 1962.

[7] C. Boldrighini, L. A. Bunimovich, and Y. Sinai. On the Boltzmann equation for the Lorentz gas. *J. Statist. Phys.*, 32(3):477–501, 1983.

[8] L. Desvillettes. Entropy dissipation estimates for the Landau equation in the Coulomb case and applications. *J. Funct. Anal.*, 269(5):1359–1403, 2015.

[9] L. Desvillettes and M. Pulvirenti. The linear Boltzmann equation for long-range forces: a derivation from particle systems. *Math. Models Methods Appl. Sci.*, 9(8):1123–1145, 1999.

[10] L. Desvillettes and V. Ricci. A rigorous derivation of a linear kinetic equation of Fokker-Planck type in the limit of grazing collisions. *J. Statist. Phys.*, 104(5-6):1173–1189, 2001.

[11] L. Desvillettes and C. Villani. On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness. *Comm. Partial Differential Equations*, 25(1-2):179–259, 2000.

[12] L. Desvillettes and C. Villani. On the spatially homogeneous Landau equation for hard potentials. II. $H$-theorem and applications. *Comm. Partial Differential Equations*, 25(1-2):261–298, 2000.

[13] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.*, 159(2):245–316, 2005.

[14] R. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math.* (2), 130(2):321–366, 1989.

[15] I. Gallagher, L. Saint-Raymond, and B. Texier. *From Newton to Boltzmann: hard spheres and short-range potentials*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[16] G. Gallavotti. Grad-Boltzmann limit and Lorentz’s Gas. *Statistical Mechanics. A short treatise*, 1999.
[17] Y. Guo. The Landau equation in a periodic box. *Comm. Math. Phys.*, 231(3):391–434, 2002.

[18] F. John. *Partial differential equations*, volume 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, fourth edition, 1991.

[19] L. Landau. Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung. *Phys. Zs. Sow. Union*, 10(154), 1936.

[20] O. Lanford. Time evolution of large classical systems. *Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974)*, pages 1–111. Lecture Notes in Phys., Vol. 38, 1975.

[21] A. Lenard. On Bogoliubov’s kinetic equation for a spatially homogeneous plasma. *Ann. Physics*, 10:390–400, 1960.

[22] E. Lifshitz and L. Pitaevskii. *Course of Theoretical Physics*. Pergamon Press, Oxford, 1981.

[23] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[24] M. Pulvirenti, C. Saffirio, and S. Simonella. On the validity of the Boltzmann equation for short range potentials. *Rev. Math. Phys.*, 26(2):64, 2014.

[25] H. Spohn. The Lorentz process converges to a random flight process. *Comm. Math. Phys.*, 60(3):277–290, 1978.

[26] H. Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.*, 52(3):569–615, 1980.

[27] H. Spohn. *Large Scale Dynamics of Interacting Particles*. Springer Science & Business Media, December 2012.

[28] R. Strain and Y. Guo. Exponential decay for soft potentials near Maxwellian. *Arch. Ration. Mech. Anal.*, 187(2):287–339, 2008.

[29] G. Toscani and C. Villani. Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. *Comm. Math. Phys.*, 203(3):667–706, 1999.

[30] C. Villani. On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal.*, 143(3):273–307, 1998.

[31] C. Villani. On the spatially homogeneous Landau equation for Maxwellian molecules. *Math. Models Methods Appl. Sci.*, 8(6):957–983, 1998.

[32] C. Villani. *A review of mathematical topics in collisional kinetic theory*, volume 1 of *Handbook of mathematical fluid dynamics*. North-Holland, Amsterdam, 2002.

[33] B. Wennberg. Stability and exponential convergence in $L^p$ for the spatially homogeneous Boltzmann equation. *Nonlinear Anal.*, 20(8):935–964, 1993.