The Structure of the Closure of the Rational Functions in $L^q(\mu)$

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Abstract

Let $K$ be a compact subset in the complex plane and let $A(K)$ be the uniform closure of the functions continuous on $K$ and analytic on $K^\circ$. Let $\mu$ be a positive finite measure with its support contained in $K$. For $1 \leq q < \infty$, let $A^q(K, \mu)$ denote the closure of $A(K)$ in $L^q(\mu)$. The aim of this work is to study the structure of the space $A^q(K, \mu)$. We seek a necessary and sufficient condition on $K$ so that a Thomson-type structure theorem for $A^q(K, \mu)$ can be established. Our theorem deduces J. Thomson’s structure theorem for $P^q(\mu)$, the closure of polynomials in $L^q(\mu)$, as the special case when $K$ is a closed disk containing the support of $\mu$.

Keywords.

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Introduction

Let $1 \leq q < \infty$ and let $\mu$ be a positive finite (regular Borel) measure with compact support in the complex plane $\mathbb{C}$. Let $K$ be a compact subset that contains the support of $\mu$. The purpose of this paper is to investigate the following problem:

What is the closure of $A(K)$ in $L^q(\mu)$?

This is a very difficult question to get a complete answer. For a given measure $\mu$, the answer depends on $K$. Let $K$ be a closed disk that contains the support of $\mu$. Then every $f$ in $A(K)$ can be uniformly approximated by polynomials, and hence $A^q(K, \mu) = P^q(\mu)$, which is the closure of polynomials in $L^q(\mu)$. In this case, J. Thomson proved a structure theorem for $P^q(\mu)$ in [28, 1991].

Roughly speaking, Thomson’s theorem says that there exists a Borel partition $\{\Delta_n\}_{n=0}^\infty$ of the support of $\mu$ such that

$$P^q(\mu) = L^q(\mu|\Delta_0) \oplus P^q(\mu|\Delta_1) \oplus \ldots \oplus P^q(\mu|\Delta_n) \oplus \ldots,$$

where each $P^q(\mu|\Delta_n)$ is identified with a space consisting of analytic functions on a simply connected domain $U_n$ via a so-called evaluation map. The union, $\cup U_n$,

\footnote{This paper basically gives ultimate solution to the most important research problem raised in [5, 28] and it is also very satisfactory generation of Thomson’s theorem for polynomials. It is the close-up work of this kind (in an once quite active research area) even it received no attentions since its publication in 2007.}
is known as the set of analytic bounded point evaluations \((abpes)\) for \(P^q(\mu)\). In this special case, since \(A(K)\) is the uniform closure of polynomials, Thomson’s theorem shows that \(A(K)\) is dense in \(L^q(\mu)\) if and only if \(P^q(\mu)\) (= \(A^q(K, \mu)\)) has no \(abpes\) (this special result answered an open question raised by D. Sarason in 1972 [27]). For an arbitrary compact subset \(K\), the author shows in [20] that it is still true: \(A(K)\) is dense in \(L^q(\mu)\) if and only if \(A^q(K, \mu)\) has no \(abpes\). However, the corresponding result for \(A^q(K, \mu)\) is not always possible even the set of \(abpes\) for \(A^q(K, \mu)\) is not empty. J. Conway and N. Elias give an example in [2] that shows the set of \(abpes\) for \(A^q(K, \mu)\) is a simply connected domain \(U\), but \(A^q(K, \mu) \cap L^\infty(\mu)\) cannot be identified with \(H^\infty(U)\) via the evaluation map.

For \(R^q(K, \mu)\), the closure in \(L^q(\mu)\) of \(R(K)\) (which is the uniform closure of the rational functions with poles off \(K\)), the situation is worse since the set of \(abpes\) may be empty even \(R(K)\) is not dense in \(L^q(\mu)\) (see [1]).

Assuming the existence of \(abpes\) for \(R^q(K, \mu)\), Conway and Elias proved a structure-type theorem for \(R^q(K, \mu)\) under additional conditions. But their result does not imply Thomson’s theorem. Then, can we have a structure theorem for \(A^q(K, \mu)\) or \(R^q(K, \mu)\) that is beyond the polynomial case and that also covers Thomson’s theorem? Prior our work, it was unknown whether such a structure theorem for \(A^q(K, \mu)\) is possible. Thomson was unable to offer any result for \(R^q(K, \mu)\) that is beyond \(P^q(\mu)\) (that is, the disk case in our setting) (see [28, p. 505]).

To tackle the problem, we first need to restrict our effort on those \(K\) such that the components of \(K^o\) are finitely connected. In fact, the author shows in [21] Theorem 2 even when \(K\) is a simplest kind of infinitely connected domains, such as a ”road-runner”, our main theorem (Theorem 2) for \(A^q(K, \mu)\) could fail.

In this paper, we seek a necessary and sufficient condition on \(K\) so that a Thomson type of structure theorem holds for \(A^q(K, \mu)\).

A domain is called a circular domain if its boundary consists of finitely many disjoint circles. We call a domain \(U\) multi-nicely connected if there is a circular domain \(W\) and a conformal map \(\alpha\) from \(W\) onto \(U\) such that \(\alpha\) is almost 1-1 on \(\partial W\) with respect to the arclength measure.

Our main theorem, Theorem 2 extends Thomson’s theorem to \(A^q(K, \mu)\) in the case when the components of \(K^o\) are multi-nicely connected and the harmonic measures of the components of \(K^o\) are mutually singular. We also show that the condition of Theorem 2 is necessary.

If every \(f\) in \(H^\infty(K^o)\) a pointwise limit of a bounded sequence in \(A(K)\), then \(K\) satisfies the condition of Theorem 2. In particular, when \(K\) is such that \(R(K)\) is a hypo-Dirichlet algebra [18], \(K\) satisfies the condition of Theorem 2(in this case, \(R^q(K, \mu) = A^q(K, \mu)\)). If the complement of \(K\) has only finitely many components (note, \(K^o\) may still have infinitely many components in this case), then \(R(K)\) is a hypo-Dirichlet algebra and hence \(K\) satisfies the hypothesis of Theorem 2. Since a quite large of class of \(R^q(K, \mu)\) satisfies our conditions, Theorem 2 is also a theorem for the rational \(R^q(K, \mu)\).

Since the polynomial case is just the disk case in our setting and since a general compact subset \(K\) is much more complicated than a disk in nature, one can expect the extension needs much more work. To get a structure theorem for \(A^q(K, \mu)\) (here \(K\) is an arbitrary compact subset), we need more than Thomson’s technique and method. In fact, we need a new Thomson-type approximation scheme as developed in [20] that takes all of what is used in Thomson’s paper [28]. In addition, we need what was not involved in the case of \(P^q(\mu)\): we make extensive use of results and techniques related to uniform algebra or rational approximation:
such as peak points, harmonic measures, hypo-Dirichlet algebra, multi-nicely connected domains, representing measures, pointwise bounded approximation, etc (these concepts are not needed for $P^q(\mu)$). So, besides Thomson’s technic and method, we need a significant part of the theories from the uniform algebras and the rational approximation to get the work done. Combination for Thomson’s technic and uniform rational approximation theory is the key to prove Theorem 2. However, not everything we need in uniform algebra theory is ready for us. We have to prove some results in that theory by ourselves. In doing so, we first introduce the concept of multi-nicely connected domains, then we prove Proposition 3.1 and an interesting result in uniform algebra, Lemma 5, which is crucial for us to prove Lemma 7. That is one of our key lemmas and it is needed to prove another key lemma, Lemma 10. The rest of the paper is to use these two lemmas and results in [20] and [21] to prove the main theorem and extend those lemmas and results that were proved for the polynomial case in [28]. So far, we are unable to offer any other proof that is less involved with the theory of uniform algebra. Actually, due to the nature of this problem, we believe the rational approximation theory is the right tool in studying this type of problems.

Now we would like to point out the relation between this paper and some other related papers. This paper is the sequel of the author’s work [20, 21]. Thomson’s paper consist of two parts of important results. One is to give a sufficient and necessary condition on when $\nabla P^q(\mu)$ is not empty and another one is to have a structure theorem for $P^q(\mu)$. In [20], we only study the problem of when $\nabla A^q(K, \mu)$ ($\nabla R^q(K, \mu)$) is not empty and we show it is empty if and only if $A(K)$ is dense in $L^q(\mu)$. In [21], our effort was primarily to establish the result that is a part of 4) and 5) of Theorem 2 in this paper and to solve a problem in [6]. In this paper, our effort is to establish a full version of Thomson’s theorem for $A^q(K, \mu)$ ($R^q(K, \mu)$) and this paper is based on [20, 21]. The readers may notice that our theorem (Theorem 2) not only completely covers Thomson’s theorem, it also has more important consequences, such as 4) and 5) (which are not in [28] and are important facts. One needs to have them when applying it to operator theory. For example, see [17] and [24]). In [6], Conway and Elias studied the same problem as that in this paper. However, since their theorem is based on the assumption that $K$ is the closure of $\nabla R^q(K, \mu)$ and $R^q(K, \mu)$ is pure, so their result does not cover Thomson’s theorem. For a given measure $\mu$, one can not tell when their conditions are satisfied. In contrast, our paper deals with arbitrary measures just as [28] does.

In 1972, D. Sarason [27] established a structure theorem for $P^\infty(\mu)$, the weak star closure of polynomials in $L^\infty(\mu)$, which has a similar form to that of our theorem.

1 Preliminaries

For a compact subset $K$ in the complex plane $\mathbb{C}$. Let $C(K)$ denote the algebra of continuous functions on $K$. For an open subset $G$ in the sphere $\mathbb{C}^\infty$ whose boundary does not contain $\infty$, let $A(G)$ be the closed subalgebra of $C(G)$ that consists of functions continuous on $G$ and analytic on $G$. Notice that $A(\overline{G}) \neq A(\overline{G})$ in general.

A point $w$ in $\mathbb{C}$ is called an analytic bounded point evaluation (abpe) for $A^q(K, \mu)$ if there is a neighborhood $G$ of $w$ and $c > 0$ such that for all $\lambda \in G$

$$|f(\lambda)| \leq c \|f\|_{L^q(\mu)} \text{ for all } f \in A(K).$$
So the map, \( f \rightarrow f(\lambda) \), extends to a functional in \( A^q(K, \mu)^* \). Thus, there is a (kernel) function \( k_\lambda \) in \( A^q(K, \mu)^* \) such that \( f(\lambda) = \int f k_\lambda \, d\mu, \ f \in A(K) \). Clearly, the set of abpes is open. For each \( f \in A^q(K, \mu) \), let \( \tilde{f}(\lambda) = \int f k_\lambda \, d\mu \). Then \( \tilde{f}(\lambda) \) is analytic on the set of abpes. The abpes for \( R^q(K, \mu) \) are defined similarly.

We shall use \( \nabla A^q(K, \mu) \) to denote the set of abpes for \( A^q(K, \mu) \). The following is one of the main results in [20] which our main theorem relies on:

**Theorem 1** Let \( K \) be a compact subset of \( \mathbb{C} \) and let \( \mu \) be a positive finite measure supported on \( K \). Then \( A(K) \) is dense in \( L^q(\mu) \) if and only if \( \nabla A^q(K, \mu) = \emptyset \).

For the case when \( K \) is the polynomially convex hull of the support of \( \mu \), the above theorem is a consequence of Thomson’s theorem. However, since it was known long before Thomson’s paper [28] that there is a compact \( K \) and a measure \( \mu \) on \( K \) such that \( R(K) \) is not dense in \( L^q(\mu) \) but \( R^q(K, \mu) \) has no abpe, and since

\[
P^q(\mu) \subset R^q(K, \mu) \subset A^q(K, \mu)
\]

always holds, Theorem 1.1 was unexpected before [20]. Somehow, it was a surprise that the theorem is true for the spaces on both sides of the inequality above, but fails for the spaces between.

**Nicely connected domains.** Following Glicksburg [10], we call a domain \( \Omega \) nicely connected if it is multi-nicely connected and if it is simply connected.

**Harmonic measures.** Let \( \Omega \) be a domain in the extended plane \( \mathbb{C}_\infty \) such that it is solvable for the Dirichlet problem and \( \infty \) is not in \( \partial \Omega \). For \( u \in C(\partial \Omega) \), let \( \hat{u} = \sup \{ f : f \text{ is subharmonic on } \Omega \text{ and } \limsup_{z \to a} f(z) \leq u(a), a \in \partial \Omega \} \). The function \( \hat{u} \) turns out to be harmonic on \( \Omega \) and continuous on \( \overline{\Omega} \), and the map \( u \to \hat{u}(z) \) defines a positive linear functional on \( C(\partial \Omega) \) with norm one, so the Riesz representing theorem implies that there is a probability measure \( \omega_z \) on \( \partial \Omega \) such that

\[
\hat{u}(z) = \int_{\partial \Omega} u \omega_z, \quad u \in C(\partial \Omega).
\]

The measure \( \omega_z \) is called the harmonic measure of \( \Omega \) evaluated at \( z \). The harmonic measures evaluated at two different points are boundedly equivalent. We shall use \( \omega_{\partial \Omega} \) to denote a harmonic measure of \( \Omega \).

**HypoDirichlet algebras.** A closed subalgebra \( \mathcal{B} \) of \( C(K) \) is said to be a hypo-Dirichlet algebra, if the uniform closure of \( Re(\mathcal{B}) = \{ R(f) : f \in \mathcal{B} \} \) has finite codimension in \( C_R(K) = \{ f : f \in C(K) \text{ and } f \text{ is real} \} \) and the linear span of \( \log|B^{-\infty}| \) is uniformly dense in \( C_R(K) \), where \( B^{-\infty} \) is the subset in \( \mathcal{B} \) consisting of invertible elements. A function algebra \( \mathcal{B} \) is called a Dirichlet algebra if \( Re(\mathcal{B}) \) is uniformly dense in \( C_R(K) \). Clearly, a Dirichlet algebra is also a hypo-Dirichlet algebra. An good example is that \( R(K) \) is hypo-Dirichlet if \( \mathbb{C} \setminus K \) has only finitely many components. This covers a large class of domains that have been studied.

If \( R(K) \) is a hypo-Dirichlet algebra, then \( A(K) = R(K) \) [8] p. 116].

**Peak points.** A point \( a \in K \) is a peak point for a function algebra \( \mathcal{B} \subset C(K) \) if there is a function in \( \mathcal{B} \) such that \( f(a) = 1 \) at \( z = a \) and \( |f(z)| < 1 \) for \( z \neq a \).

**Pure and irreducible spaces.** The space \( A^q(K, \mu) \) is called pure if there is no Borel subset \( \Delta \) of supp(\( \mu \)) such that the restriction of \( A(K) \) on \( \Delta \) is dense in \( L^q(\mu|\Delta) \).

An observation is that for any \( A^q(K, \mu) \), there is a Berel partition \( \{ \Delta_0, \Delta_1 \} \) of the support of \( \mu \) such that \( A^q(K, \mu|\Delta_1) \) is pure and

\[
A^q(K, \mu) = L^q(\mu|\Delta_0) \oplus A^q(K, \mu|\Delta_1).
\]
The space $A^q(K, \mu)$ is said to be irreducible if it contains no nontrivial characteristic functions. So an irreducible space must be pure.

**Nontangential limits.** Let $G$ be a bounded domain that is conformally equivalent to a circular domain $W$ in the plane and let $u$ be a conformal map from $W$ onto $\Omega$. Then $u$ has well-defined boundary values on $\partial W$, which are equal to the nontangential limits of $u$ for almost every point on $\partial W$ with respect to $\omega_W$ (the harmonic measure of $W$). We still use $u$ to denote the boundary value function.

Now, if $E$ is a Borel subset of $\partial W$ such that $u$ is 1-1 on $E$ a.e. $[\omega_W]$, then each $f \in H^\infty(G)$ has nontangential limits almost everywhere on $u(E)$ with respect to $\omega_G$. That is,

$$f(a) = \lim_{z \to u^{-1}(a)} f \circ u(z) \text{ a.e. on } u(E) \text{ with respect to } \omega_G.$$

So, if $\mu$ is such that $\mu \ll \omega_G$ on $u(E)$, then each $f \in H^\infty(G)$ has nontangential limits on $u(E)$ almost everywhere with respect to $\mu$.

## 2 The Main Result

In this section, we introduce our main result, Theorem 2. Recall that the connectivity of a finitely connected domain is defined to be the number of the components of its complement.

**Theorem 2** Let $K$ be a compact subset and let $\mu$ be a finite positive measure supported on $K$. If each of the components of $K^\circ$ is multi-nicely connected and the harmonic measures of the components of $K^\circ$ are mutually singular, then there exists a Borel partition $\{\Delta_n\}_{n=0}^\infty$ of $\text{supp}(\mu)$ such that

$$A^q(K, \mu) = L^q(\mu|\Delta_0) \oplus A^q(K, \mu|\Delta_1) \oplus ... \oplus A^q(K, \mu|\Delta_n) \oplus ...$$

and for each $n \geq 1$, if $U_n$ denotes $\nabla A^q(K, \mu|\Delta_n)$, then

1) $\overline{U_n} \supset \Delta_n$ and $A^q(K, \mu|\Delta_n) = A^q(U_n, \mu|\Delta_n)$;

2) each $U_n$ is a finitely connected domain that is conformally equivalent to a circular domain $W_n$; the connectivity of $U_n$ does not exceed the connectivity of the component of $K^\circ$ that contains $U_n$;

3) the map $e$, defined by $e(f) = \hat{f}$, is an isometrical isomorphism and a weak-star homeomorphism from $A^q(K, \mu|\Delta_n) \cap L^\infty(\mu|\Delta_n)$ onto $H^\infty(U_n)$;

4) $\mu\partial U_n \ll \omega_{U_n}$; and if $u_n$ is a conformal map from $W_n$ onto $U_n$, then for each $f \in H^\infty(U_n)$ has nontangential limits on $\partial U_n$ a.e. $[\mu]$ and

$$e^{-1}(f)(a) = \lim_{z \to u_n^{-1}(a)} f \circ u_n(z) \text{ a.e. on } \partial U_n \text{ with respect to } \mu|\partial U_n;$$

5) for each $f \in H^\infty(U_n)$, if let $f^*$ be equal to its nontangential limit values on $\partial U_n$ and let $f^* = \hat{f}$ on $U_n$, then the map $m$, defined by $m(f) = f^*|\Delta_n$, is the inverse of the map $e$.

**Remark 1** Thomson proved 1), 2) and 3) of Theorem 2 in the case when $A^q(K, \mu) = P^q(\mu)$. For the polynomial case, 4) is the main result in [10]. The author proved 4) for $A^q(K, \mu)$ with a different method in [27].

**Remark 2** 3) clearly implies that each $A^q(K, \mu_n)$ is irreducible.
Remark 3 The condition on $K$ is the best possible one. What we mean here is that in order to have Theorem 2 holds for any positive finite measure supported on $K$, it is necessary and sufficient that each component of $K^o$ is multi-nicely connected and the harmonic measures of the components of $K^o$ are mutually singular.

Now we outline a proof for this fact. Let $\Omega$ be a component of $K^o$ and let $\mu$ be a harmonic measure for $\Omega$. By Theorem 3 of \cite{21}, the map $f \to \tilde{f}$, from $A^q(\Omega, \mu) \cap L^\infty(\mu)$ onto $H^\infty(\Omega)$ is surjective if and only if $\Omega$ is a multi-nicely connected domain. Hence we know that $\Omega$ must be multi-nicely connected.

Now let $\Omega_1$ and $\Omega_2$ be two components of $K^o$. We want to show that $\omega_{\Omega_1}$ and $\omega_{\Omega_2}$ are mutually singular. Set $\mu = \omega_{\Omega_1} + \omega_{\Omega_2}$. Then it is easy to see that $A^q(K, \mu)$ is pure. Since the harmonic measure at a given point is a representing measure, it follows by the definition of abpe and the Harnack’s inequality that $\nabla A^q(K, \mu) \supset \Omega_1 \cup \Omega_2$. Since each $\Omega_i$ has no boundary slit, it follows clearly that $\nabla A^q(K, \mu) = \Omega_1 \cup \Omega_2$. If Theorem 2 holds for $A^q(K, \mu)$, then

$$A^q(K, \mu) = A^q(\Omega_1, \mu_1) \oplus A^q(\Omega_2, \mu_2),$$

where $\mu_i, i = 1, 2$, are as in Theorem 2. Let $v$ be a conformal map of $\Omega_1$ onto a circular domain $W$. Theorem 2 implies that there exists $v_v \in A^q(\Omega_1, \mu_1)$ such that $v_v = v$. Set $\eta = \mu_1 \circ v_v^{-1}$. According to Lemma 2 in \cite{21}, $\eta$ is a measure on $\partial W$ such that $A^q(W, \eta)$ is irreducible and $\nabla A^q(W, \eta) = W$. Moreover, $\eta \ll \omega_W$ by Lemma 3 of \cite{21}. On the other hand, since $W$ is circular and since $\nabla A^q(W, \eta) = W$, it is easy to see that $A^\infty(W, \eta)$, the weak-star closure of $A(W)$ in $L^\infty(\eta)$, is equal to $\tilde{H}^\infty(W)$, which is the image of the map $f \to \tilde{f}$ from $H^\infty(W)$ into $L^q(\omega_W)$ (where $\tilde{f}$ is the boundary value function of $f$ on $\partial W$). Since the support of $\mu_1 \subset \partial \Omega_1$, it follows by a classical result that $[\omega_W] = [\eta]$. Now, applying Lemma 3 in \cite{21}, we conclude that $[\omega_{\Omega_1}] = [\mu_1]$. Similarly, we have $[\mu_2] = [\omega_{\Omega_2}]$. But $\mu_1$ and $\mu_2$ are mutually singular, therefore $\omega_{\Omega_1}$ and $\omega_{\Omega_2}$ must be mutually singular.

3 The proof of the main result

Lemma 1 For each $f \in A^q(K, \mu)$, $\tilde{f} = f$ on $\nabla A^q(K, \mu)$ a.e. $[\mu]$.

Proof. Let $a$ be an abpe and choose a sequence $\{f_n\}$ in $A(K)$ such that $f_n \to f$ in $L^q(\mu)$. Since $f_n \to \tilde{f}$ uniformly in a neighborhood of $a$, it follows (by passing a sequence if necessary) that $f_n \to f$ a.e. $[\mu]$ and consequently

$$\tilde{f}(a) = \lim_{n \to \infty} \tilde{f}_n = \lim_{n \to \infty} f_n = f(a) \text{ a.e. } [\mu].$$

The following lemma is elementary too.

Lemma 2 If $f \in L^\infty(\mu) \cap A^q(K, \mu)$ and $g \in A^q(K, \mu)$, then $fg \in A^q(K, \mu)$ and $\tilde{fg} = \tilde{fg}$.

The next lemma is proved in \cite{21}.

Lemma 3 Let $\Omega = \nabla A^q(K, \mu)$. If $\Omega$ is finitely connected, then every component of $(\mathbb{C} \setminus \Omega)$ has nonempty interior.
Representing measures. Let $B$ be a closed subalgebra of $C(K)$. A complex representing measure of $B$ for $a \in K$ is a finite measure $\nu$ on $K$ such that

$$f(a) = \int f d\nu, \ f \in B.$$ 

A representing measure for $a$ is a probability measure that satisfies the above condition. Note, if $a$ is a peak point, then the only representing measure for $a$ is the point mass $\delta_a$.

The sweep of a measure. Let $G$ be a domain that is regular for the Dirichlet problem and let $\mu$ be a measure on $\partial G$. The sweep of $\mu$ is the unique positive measure $\tilde{\mu}$ on $\partial G$ that satisfies $\int_{\partial G} \tilde{\mu} d\mu = \int_{\partial G} u d\tilde{\mu}, \ u \in C(\partial G)$, where $\tilde{u}$ is the solution of the Dirichlet problem for $u$. A simple fact is that if $\mu$ is a measure on $\partial G$, then $\tilde{\mu} = \mu|\partial G + \mu(G)$.

Lemma 4 Let $K$ be a compact subset such that the components of $K^\circ$ are multi-nicely connected and the harmonic measures of the components of $K^\circ$ are mutually singular. Let $\Omega$ be a component of $K^\circ$. If $A^\circ(K, \mu)$ is pure and if $K^\circ$ is dense in $K$, then $\mu|\partial \Omega \ll \omega_\Omega$.

Proof. Let $\{\Omega_j\}_{j=1}^\infty$ be the collection of the components of $K^\circ$. Fix an integer $j \geq 0$ and let $E$ be a component of $\partial \Omega_j$. Let $G_j$ be the unique simply connected domain in the sphere $C_\infty$ that has $E$ as its boundary and contains $\Omega_j$. Since $\Omega_j$ is multi-nicely connected, $G_j$ must be nicely connected. For $i \neq j$, let $G_i$ be the bounded simply connected domain that contains $\Omega_i$ and whose boundary is a component of $\partial \Omega_i$. Clearly, $G_i$ is also nicely connected. Now let $\Omega$ be the union of those $G_i$’s for which $G_i \cap G_j = \emptyset$ (different $G_i$’s are either disjoint or one contains other). Set $G = \Omega \cup G_j$. Then each component of $G$ is equal to some $G_i$. Let $\{G_{ik}\}$ be the collection of all the components of $G$. Then our hypothesis on $K$ implies that the harmonic measures of the components of $G$ are mutually singular. It follows from [7] that $A(G)$ is a Dirichlet algebra on $\partial G$. Hence, every point in $\partial G$ is a peak point for $A(G)$ and every trivial Gleason part of $A(G)$ consists of a single point. Therefore, $\{G_{ik}\}$ is the collection of all the nontrivial Gleason parts of $A(G)$.

Let $\eta \perp A(G)$. By the Abstract F. and M. Riesz Theorem [8], $\eta = \sum_{m \geq 0} \eta_m$, where each $\eta_m \perp A(G), \ \eta_m \ll \nu_m$ for a representing measure $\nu_m$ at some point $a_m$ in $\partial G$, $\nu_m$’s are mutually singular. Let $a \in \partial G$. Then $a$ is a peak point. Let $f \in A(G)$ be a peak function for $a$. Then $f^n(z) \to \chi(a)$ pointwise, and thus $0 = \int \lim_{n \to \infty} f^n d\eta_m = \eta_m(\{a\})$. Hence $a_m \in G$ (otherwise, $\nu_m$ is the point mass at $a_m$ and hence $\nu_m(G - \{a_m\}) = 0$. So we conclude that $\eta_m(\partial G - \{a_m\}) = 0$. Then for each $\eta_m(\{a_m\}) + \eta_m(\{a_m\}) = 0 + 0 = 0$, a contradiction).

Let $G_{ik}$ be the component that contains $a_m$ and let $\tilde{v}_m$ be the sweep of $\nu_m$ on $\partial G_{ik}$. Then for each $g \in A(G)$

$$\int_{\partial G} g(z) d\tilde{v}_m = \int_{\partial G_{ik}} g(z) d\tilde{v}_m = \int_{\partial G_{ik}} g(z) d\nu_m = \int_{\partial G_{ik}} g(z) d\omega_{am},$$

where $\omega_{am}$ is the harmonic measure of $G_{ik}$ evaluated at $a_m$. It follows by the uniqueness that $\tilde{v}_m = \omega_{am}$. Hence, we have

$$\eta|\partial G \ll \sum |v_{am}|\partial G \ll \sum \tilde{v}_m|\partial G = \sum \omega_{am}|\partial G$$
In particular, we have that \( \eta |E | \ll \omega_{\Omega_j} |E | \).

Finally, suppose that \( g \in L^p(\mu) \) such that \( \int f g d\mu = 0 \), for \( f \in A(K) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). Then \( g \perp A(G) \) as well. Hence, \( g|E | \ll \omega_{\Omega_j} |E | \). This implies that \( (g_\mu)|E | = 0 \), where \( (g_\mu)_x \) is the singular part of the Lebesgue decomposition of \( g_\mu \) with respect to \( \omega_{\Omega_j} |E | \). Consequently, we have \( g \perp \chi_{\Delta \cap E} \), where \( \Delta \) is the carrier of \( \mu_x |\partial \Omega_j \) and \( \mu_x \) is the singular part of the Lebesgue decomposition of \( \mu \) with respect to \( \omega_{\Omega_j} \).

Now an application of Hahn-Banach theorem yields \( \chi_{\Delta \cap E} \in A^q(K, \mu) \). By purity, \( \chi_{\Delta \cap E} = 0 \) a.e. \( [\mu] \) and therefore \( \mu |E | \ll \omega_{\Omega_j} |E | \). Since \( E \) is an arbitrary component of \( \partial \Omega_j \), it follows that \( \mu |\partial \Omega_j | \ll \omega_{\Omega_j} \).

\[
\text{Proposition 3.1} \quad \text{Let } K \text{ be a compact subset such that the components of } K^0 \text{ are multi-nicely connected and the harmonic measures of } K^0 \text{ are mutually singular. Let } \{\Omega_j\}_{j=0}^\infty \text{ denote the collection of the components of } K^0 \text{. Set } \omega = \sum_{j=0}^\infty \frac{1}{2^j} \omega_{\Omega_j} \text{. If } A^q(K, \mu) \text{ is pure, then } \mu |\partial K | \ll \omega .
\]

\[\text{Proof.} \quad \text{By Lemma 17.10 in [5 p. 246], there exists a function } g \in A^q(K, \mu)^\perp \text{ such that } |f| \mu \ll |g| \mu \text{ for each } f \in A^q(K, \mu)^\perp \text{. Since } A^q(K, \mu) \text{ is pure, we see that } g \neq 0 \text{ on } \text{supp}(\mu) \text{ a.e. } [\mu] \text{. Thus } |g| \mu = |\mu| \text{ a.e. } [\mu] \text{. Set } \nu = |g| \mu .
\]

Let \( a \in K - K^0 \) be such that \( \int \frac{d\nu}{|z - a|} < \infty \) and let \( f \) be a peak function for \( a \).

For each integer \( n \geq 1, \frac{1 - f^n(z)}{|z - a|} \in A(K) \), so we have

\[
\hat{\nu}(a) = \lim_{n \to \infty} \int \frac{1 - f^n(z)}{z - a} d\nu = 0.
\]

Since the set \( \{a : \int \frac{d\nu}{|z - a|} < \infty\} \) has full area measure in the plane, it follows by a well-known fact (see Theorem [3] or the comments after it) that \( \nu \) is the zero measure off \( K^0 \).

Finally, since \( \mu |\partial \Omega_j | \ll \omega_{\Omega_j} \) for each \( j \) (by Lemma [4]), the conclusion follows.

The proof of the next lemma can be found in [26].

\[
\text{Lemma 5} \quad \text{Let } \Omega \text{ be an open subset whose boundary does not contain } \infty \text{. Suppose the components of } \Omega \text{ are multi-nicely connected and the harmonic measures of the components of } \Omega \text{ are mutually singular. Let } \{\Omega_j\}_{j=0}^\infty \text{ be the collection of the components of } \Omega \text{. If all but finitely many components of } \Omega \text{ are simply connected, then } A(\Omega) \text{ is boundedly pointwise dense in } H^\infty(\Omega) .
\]

For a finite positive measure \( \mu \), let \( A^\infty(K, \mu) \) denote the weak-star closure of \( A(K) \) in \( L^\infty(\mu) \).

\[
\text{Lemma 6} \quad \text{Let } \Omega \text{ be a multi-nicely connected domain. Let } W \text{ be a circular domain that is conformally equivalent to } \Omega \text{ and let } \phi \text{ be a conformal map of } W \text{ onto } \Omega . \text{ Then the boundary value function } \hat{\phi} \text{ has a well-defined inverse, } \phi^{-1} \text{, on } \partial \Omega \text{. Moreover, } \phi^{-1} \in A^\infty(\Omega, \omega_{\Omega}) .
\]

\[\text{Proof.} \quad \text{Since } \phi \text{ is almost 1-1 on } \partial W \text{ with respect to } \omega_W , \text{ it is apparent that } \hat{\phi} \text{ has a well-defined inverse function on a set of full } \omega_{\Omega} \text{ measure. By Lemma [5] } A(\Omega) \text{ is boundedly pointwise dense in } H^\infty(\Omega) . \text{ Choose a bounded sequence } \{f_n\} \text{ in } A(\Omega) \text{ such that } f_n \to \phi^{-1} \text{ on } \Omega . \text{ Then one can show that } f_n \to \phi^{-1} \text{ in the weak-star topology of } L^\infty(\omega_{\Omega}) . \text{ Hence } \phi^{-1} \in A^\infty(\overline{W}, \omega_{\Omega}) .
\]

\[\text{The following is one of our key lemmas.}\]
Lemma 7 Suppose that each component of $K^\circ$ is multi-nicely connected and the harmonic measures of the components of $K^\circ$ are mutually singular. Let $U$ be a component of $\nabla A^q(K, \mu)$ and let $\Omega$ be the component of $K^\circ$ that contains $U$. Set $\tau = \mu(\Omega)$. Then $A^q(\Omega, \tau) \subset A^q(K, \mu)$ and $U \subset \nabla A^q(\Omega, \tau)$.

Proof. We first assume that $A^q(K, \mu)$ is pure. Let $\{\Omega_i\}_{i=0}^\infty$ be the collection of all the components of $K^\circ$. Without loss of generality, let $\Omega_0 = \Omega$. Suppose $h \in A^q(\Omega, \tau)$ and choose a sequence $\{r_n\}$ in $A(\Omega)$ such that $r_n \rightarrow h$ in $L^q(\tau)$. Let $\omega = \sum_i \frac{1}{\mu(\Omega_i)}$. Fix a function $r_n$. Extend both $h$ and $r_n$ to be functions on the whole plane by defining their values to be zero off $\Omega$. We claim that there exists a sequence $\{q_n\}$ in $A(\tilde{K}^\circ)$ such that it weak-star converges to $r_n$ in $L^\infty(\tilde{\omega})$.

For each $i \geq 1$, let $G_i$ be the bounded simply connected domain that contains $\Omega_i$ and whose boundary is a component of $\partial \Omega_i$. Let $G$ be the union of $\tilde{\Omega}$ with those domains $G_i$ that do not intersect $\Omega$. Then all but finitely many components of $G$ are simply connected domains and each component of $G$ is multi-nicely connected.

By Lemma 5, $A(G)$ is boundedly pointwise dense in $H^\infty(G)$. Thus, there exists a bounded sequence $\{q_m\}$ in $A(\tilde{G})$ so that it pointwise converges to $r_n$ on $G$. Now for given $\epsilon > 0$, let $f \in L^1(\tilde{\omega})$. Then

$$\left| \int_{\partial K} f(r_n - q_m) d\omega \right| \leq \left| \int_{\bigcup_i \partial \Omega_i} f(r_n - q_m) d\omega \right| + \frac{\epsilon}{2}$$

for all $m$ whenever $k$ is sufficiently large. Observe that $\{q_m\}$ weak-star converges to $r_n$ in $L^\infty(\tilde{\omega} \Omega_i)$ for each $i$. Thus,

$$\left| \int_{\partial K} f(r_n - q_m) d\omega \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

when $m$ is sufficiently large.

Hence, for each $f \in L^1(\tilde{\omega})$ we have

$$\lim_{n \rightarrow \infty} \int_{\partial K} f(r_n - q_m) d\omega = 0.$$  

That is, $\{q_m\}$ weak-star converges to $r_n$ in $L^\infty(\tilde{\omega})$. This proves the claim.

Next we show that $r_n$ belongs to the weak-star closure of $A(\tilde{K}^\circ)$ in $L^\infty(\mu)$. By Proposition 7.1, $\mu \partial K \ll \omega$. Thus

$$\lim_{m \rightarrow \infty} \int_{\partial K} f q_m \ d\mu = \int_{\partial K} f r_n d\mu, \ f \in L^1(\mu).$$

Since $\{q_m\}$ is bounded and pointwise converges to $r_n$ on $K^\circ$, it follows by the bounded convergence theorem that

$$\lim_{m \rightarrow \infty} \int q_m \ d\mu = \int r_n \ d\mu, \ f \in L^1(\mu).$$

Therefore, $r_n$ belongs the weak-star closure of $A(\tilde{K}^\circ)$ in $L^\infty(\mu)$. But this closure is contained in the weak closure of $A(\tilde{K}^\circ)$ in $L^q(\mu)$. Because a convex set is norm-closed if and only if it is weakly closed in $L^q(\mu)$, we have $r_n$ is in $A^q(\tilde{K}^\circ, \mu)$.

Now for each $n \geq 1$, choose $x_n$ in $A(\tilde{K}^\circ)$ such that

$$\|r_n - x_n\|_{A^q(\tilde{K}^\circ, \mu)} \leq \frac{1}{n}.$$
Then
\[ \|x_n - h\|_{L^q(\mu)} = \|x_n - r_n\|_{L^q(\mu)} + \|r_n - h\|_{L^q(\mu)} \leq \frac{1}{n} + \|r_n - h\|_{L^q(\tau)} \to 0, \text{ as } n \to \infty. \]

Therefore we have that \( h \in A^q(K^\circ, \mu) \). Because \( h \) is an arbitrary element in \( A(\Omega, \tau) \), we conclude that \( A^q(\Omega, \tau) \subset A^q(K^\circ, \mu) \). Since \( A^q(K, \mu) \) is pure and since \( \mu \) is supported on \( K^\circ \) (for \( \mu|\partial K \ll \nu \) and \( \nu \) is supported on \( \partial K^\circ \)), we have that
\[ A^q(K, \mu) = A^q(K^\circ, \mu). \]

Hence \( A^q(\Omega, \tau) \subset A^q(K, \mu) \).

Now let \( b \in U \). Then \( b \) is an abpe for \( A^q(K, \mu) \) and thus there exists \( d > 0 \) and a small open disk \( D_b \subset U \) such that for all \( r \in A(K) \)
\[ |r(a)| \leq d \int |r| \, d\mu, \quad a \in D_b. \]

Let \( y \in A(\Omega) \) and extend \( y \) to be zero off \( \Omega \). Then \( y \in A^q(K, \mu) \) and so there is a sequence \( \{y_n\} \) in \( A(K) \) so that it converges to \( y \) in \( L^q(\mu) \). Then \( \{y_n\} \) converges to \( y \) uniformly on \( D_b \). Hence, it follows by the expression above that for all \( y \in A(\Omega) \)
\[ |y(a)| \leq d \int |y| \, d\mu, \quad a \in D_b. \]

Thus \( a \in \nabla A^q(\Omega, \tau) \). By the definition of abpe, \( U \subset \nabla A^q(\Omega, \tau) \).

If \( A^q(K, \mu) \) is not pure, let \( \mu = \mu_0 + \mu_1 \) be the decomposition so that \( A^q(K, \mu) = L^q(\mu_0) \oplus A^q(K, \mu_1) \) and \( A^q(K, \mu_1) \) is pure. Then
\[ \nabla A^q(K, \mu) \supset \nabla A^q(K, \mu_1) \supset U. \]

So the conclusion of the lemma follows.

A function \( f \) analytic at \( \infty \) can be written as a power series of the local coordinate \( \frac{1}{z - z_0} \) at \( \infty \):
\[ f(z) = f(\infty) + \frac{a_1}{z - z_0} + \frac{a_2}{(z - z_0)^2} + \ldots. \]
The coefficient \( a_1 \) is called the derivative of \( f \) at \( \infty \) and is denoted by \( f'(\infty) \). It is easy to see that \( f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)) \). Define \( \beta(f, z_0) = a_2 \).

The next lemma is elementary.

**Lemma 8** Let \( \delta > 0 \) and let \( a \) in \( \mathbb{C} \). Let \( B(a, \delta) = \{z : |z - a| \leq \delta\} \). If \( f \) is a bounded analytic function on \( \mathbb{C}_\infty \setminus B(a, \delta) \), then \( |f'(\infty)| \leq \delta \|f\|_\infty \) and \( |\beta(f, a)| \leq \delta^2 \|f\|_\infty \).

**Thomson’s Scheme.** Now, we introduce an approximation scheme originally developed by J. Thomson in [28].

For an integer \( k \geq 1 \), let \( \{S_{k,p}\}_{p=1}^\infty \) be the collection of all open squares with sides \( 2^{-k} \), parallel to the coordinate axes and corners at the points whose coordinates are both integral multiples of \( 2^{-k} \). A finite sequence \( \{S_i\}_{i=1}^\infty \) of squares is

\[ \text{In this paper, we don’t directly use this scheme. But we need the concept of light and heavy points and results related to them (Lemma 9 and Theorem 3).} \]
called a path of squares if the interior of $\bigcup S_1$ is connected. In this case we say $S_1$ and $S_n$ are joined by a path of squares. The collection of $\{S_{kp}\}_{p=1}^{\infty}$ is called the $k$-th generation of squares.

Let $\phi$ be a nonnegative function in $L^1(Area)$. An open square $S$ is said to be light with respect to $\phi$ if $\int_S \phi \, d\text{Area} \leq [\text{Area}(S)]^2$.

Now we begin with the scheme. Let $a \in \mathbb{C}$ and let $S$ be a square in $\{S_{kp}\}_{p=1}^{\infty}$ such that $a \in \overline{S}$. Color $S$ yellow and let $\Gamma_k = \partial S$. We then move to the squares in the next generation. First, color green every light square in $\{S_{(k+1)p}\}_{p=1}^{\infty}$ that lies outside $\Gamma_k$ and has a side on $\Gamma_k$. Second, color green every light square that can be joined to a green square in the first step by a path of light squares in $\{S_{(k+1)p}\}_{p=1}^{\infty}$. Now if there is an unbounded green path (that is made up of infinitely many squares), then this coloring process ends. Otherwise, let $\gamma_{k+1}$ be the boundary of the polynomially convex hull of the union of $\Gamma_k$ and the closure of the set of the light points. We then color red every square $S$ in $\{S_{(k+1)p}\}_{p=1}^{\infty}$ if $S$ is outside $\gamma_{k+1}$ and $S$ has a side on $\gamma_{k+1}$. After that, color a square $T$ yellow if $T$ is outside $\gamma_{k+1}$ and $T$ has no side lying on $\gamma_{k+1}$ and the distance from $T$ to some red square in $\{S_{(k+1)p}\}_{p=1}^{\infty}$ is less or equal to $(k+1)2^{-(k+1)}$. Now let $\Gamma_{k+1}$ be the boundary of the polynomially convex hull of the union and the closure of the colored squares in the $(k+1)$-th generation. To this step, the coloring process in $(k+1)$-th generation is completed.

Next we continue this process to the $(k+2)$-th generation of squares and keep this process to all higher generations unless there is an unbounded green path in the coloring scheme in some $(m+l)$-th generation ($l \geq 1$). We use $(\phi, a, k)$ to denote this colored scheme.

**Light and heavy points.** For a nonnegative function $\phi \in L^1(Area)$, a point $\lambda$ in $\mathbb{C}$ is called light (with respect to $\phi$) if there exists $\delta > 0$ such that for each $\delta_0 \leq \delta$

\[ \{z : |z - a| = \delta_0\} \cap \text{all colored squares in } (\phi, k, a) \neq \emptyset, \]

whenever $k$ is a sufficiently large integer. If a point is not light, then it is called a heavy point.

**Remark 4** The construction of our colored scheme is exactly the same as that in [23]. But the light and heavy points improved ‘light route to $\infty$’ and ‘heavy barrier’ in Thomson’s original work. Let us explain the difference: For a given $\phi$, if there is an unbounded green path in the colored scheme $(\phi, k, a)$ for every $k$, it is said that there is a sequence of light routes from $a$ to $\infty$. This is essentially the definition of ‘light’ points in Thomson’s paper. Because most of the light points for a given $\phi$ in our definition don’t have a sequence of light routes from $a$ to $\infty$, the set of the light points is much larger than the set of points that have a sequence of light routes from $\infty$.

The Cauchy transform of a measure (with compact support) $\mu$ is defined as $\hat{\mu}(z) = \int \frac{d\mu(w)}{w - z}$. Because $\frac{1}{z}$ is local integrable with respect to the area measure, it follows that $\hat{\mu}(z)$ is defined everywhere except a subset of zero area.

The following is a practically useful result coming out of our light point concept [20] Theorem 2.4.

**Theorem 3** Let $\mu$ be a finite measure with compact support. Let $V$ be an open subset in $\mathbb{C}$. If every point in $V$ is light with respect to $|\mu|$, then $|\mu|(V) = 0$.

\[ \text{The polynomially convex hull of a compact subset } K \text{ in the plane is defined as the union of } K \text{ and all the bounded components of } \mathbb{C} \setminus K. \]
The above theorem generalizes a well-known result in the theory of uniform approximation: if \( \mu = 0 \) a.e. on an open subset with respect to the area measure, then the restriction of \( \mu \) on the open subset is zero. The following is the key lemma in [20, Lemma 2.3].

**Lemma 9** Let \( \nu_1, ..., \nu_k \) be finite measures. Let \( \phi(z) = \max\{|\hat{\nu}_j(z)| : 1 \leq j \leq k\} \). If \( a \) is a light point with respect to \( \phi \), then there is an arbitrarily small positive number \( \delta \) such that for any \( \epsilon > 0 \) and \( \alpha, \beta \in \{z : |z - 1| \leq 1\} \), there is a function in \( C(C_{\infty}) \) that has the following properties: 1) \( ||f||_{\infty} \leq C \) (a universal constant), 2) \( f \) is analytic on \( \{z : |z - a| > \delta\} \), 3) \( f(\infty) = 0 \), 4) \( f'(\infty) = \alpha \delta \), 5) \( \beta(f, a) = \beta \delta^2 \), 6) \( |\int f d\nu_j| \leq \epsilon \) for all \( 1 \leq j \leq k \).

**Vitushkin covering.** For a natural number \( k \), let \( \{S_{kl}\}_{l=1}^{\infty} \) is the \( k \)-th generation of squares with sides of length \( 2^{-k} \). For each \( S_{kl} \), let \( F_{kl} \) be the square obtained by enlarging \( S_{kl} \) \( \frac{1}{2} \) times. The collection \( \{F_{kl}\} \) is called a regular Vitushkin covering of the plane. We suppress \( k \) and let \( z_j \) be the center of \( F_i \). Then there exists a \( C^1 \) partition of unity \( \{\phi_i\} \) subordinate to \( \{F_i\} \) with \( ||\text{grad} \phi_i|| \leq 100 \cdot 2^k \) such that

\[
\sum_{l=1}^{\infty} \min(1, \frac{2^{-3k}}{|z - z_l|^3}) \leq C \min\{1, \frac{2^{-k}}{\text{dist}(z, \cup_i F_i)}\}, \quad z \in \mathbb{C}.
\]

One may consult [5] for a proof of the inequality.

**Lemma 10** Suppose that each component of \( K^c \) is multi-nicely connected and the harmonic measures of the components are mutually singular. Let \( U \) be a component of \( \nabla A^\theta(K, \mu) \) and let \( f \in H^\infty(U) \). Then there exists a function \( h \in A^\theta(K, \mu) \cap L^\infty(\mu) \) such that \( \hat{h}(z) = f(z) \) on \( U \) and \( h = 0 \) off \( U \).

**Proof.** First, we assume that \( K \) is finitely connected, \( K = \overline{K}^o \) and \( K^c \) is connected. Let \( \{x_j\} \) be a countable dense subset of \( \partial \nabla A^\theta(K, \mu)^c \). For an integer \( k \geq 1 \), let \( \Phi(z) = \max\{|(x_j, \mu)(z)| : j \leq k\} \). Let \( \{F_i\} \) be the regular Vitushkin covering of squares with sides of length \( \frac{1}{2^k} \) and center \( z_l \). There is then a \( C^1 \) partition \( \{\phi_i\} \) subordinate to the covering \( \{F_i\} \). For each \( l \), let \( f_l = T_{\phi_l} f = \frac{1}{2} \int f(z) \cdot (w, \phi_l) d \text{Area} \). Then \( f_l \) is analytic off \( F_l \), \( f_l(\infty) = 0 \), and \( ||f_l||_{\infty} \leq 2 ||\text{grad} \phi_l|| ||\text{dist}(f(z) - f(w)) : z, w \in \text{supp} \phi_l|| \leq C_0 \), where \( C_0 \) is a positive universal constant.

Let \( l \) be such that \( F_l \cap \partial U \neq \emptyset \) and let \( a \in \partial U \cap F_l \). We claim that \( a \) is a light point with respect to \( \Phi \). In fact, first let \( a \in K^c \). Set \( V = K^c \setminus \partial U \), it follows by Lemma 3.3 in [20] that \( |(x_j, \mu)(z)| = 0 \) on \( V \) for each \( j \) and thus \( \Phi = 0 \) on \( V \). By Lemma 3.7 in [20] we see \( a \) is light. Now suppose that \( a \) is on \( \partial K \). Since \( \nu_j \perp A(K) \supset R(K) \), we have that \( \nu_j = 0 \) off \( K \). Thus, again it follows from Lemma 3.7 in [20] that \( a \) is also a light point. This proves the claim.

Next let \( d_l = \frac{1}{2} d_l^{2^{-k}} \) and let \( B(a, d_l) \) be open disk having radius \( d_l \) and the center at \( a \). Applying Lemma 3.3 to \( B(a, d_l) \), it follows that \( |f_l(\infty)| \leq C_0 d_l \) and \( \beta(f_l, z_l) \leq C_0 d_l^2 \). Let \( \alpha = \frac{f_l(\infty)}{C_0 d_l} \) and \( \beta \leq \frac{2(f_l(z_l), C_0 d_l^2)}{C_0 d_l} \). Then \( |\alpha| \leq 1 \) and \( |\beta| \leq 1 \). Let \( n \) be the number of those \( F_i \)’s for which \( F_i \cap \partial U \neq \emptyset \). Then \( n \) is a positive integer. Because \( a \) is a light point, applying Lemma 3 with \( \alpha, \beta, \frac{1}{\text{dist}(a, K)} \), there exists a function \( g_l \) in \( C(C_{\infty}) \) that is analytic off \( B(a, d_l) \) and satisfies: 1) \( ||g_l|| \leq C_1 \) (\( C_1 \) is a

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4 By combining the proof Lemma 3.3 in [20] and that of Theorem 4.8 in [28], we can extend Lemma 3.3 in [20] so that it has the conclusion that Theorem 4.8 in [28] has (so it can be applied to \( \sum_{j \leq k} |x_j, \mu| \)). From this, we also see any point in \( K^c \) that is not an \( abse \) must be light.
positive universal constant), 2) $g_l(\infty) = 0$, 3) $g_l(\infty) = \alpha d_l$, 4) $\beta(g_l, z_l) = \beta d_l^2$, 5) $\int g_l x_j \, d\mu_l \leq \frac{1}{\eta_k}$ for all $1 \leq j \leq k$. Then $h_l$ has the following properties: 1) $\|h_l\| \leq C_0 C_l 1$, 2) $h_l$ is analytic off $B(a, d_l)$, 3) $\int h_l x_j \, d\mu_l \leq \frac{1}{\eta_k}$ for $j \leq k$, 4) $h_l - f_l$ has a triple zero at $\infty$, that is, $(h_l - f_l)(\infty) = 0$, $(h_l - f_l)(\infty) = 0$ and $\beta(h_l - f_l, z_l) = 0$.

Let $h_l$ be the length of a side of $F_l$. Since $a \in F_l$, it is evident that $B(a, d_l)$ is contained in the square with center $z_l$ and sides of length $2h_l$. $h_l$ is clearly analytic off $\{ z : |z - z_l| \leq 2h_l \}$. Since $f_l - h_l$ has a triple zero at $\infty$, $(z - z_l)^3(f_l - h_l)$ is also analytic off $\{ z : |z - z_l| \leq 2h_l \}$. So the maximum principle implies that $|(z - z_l)^3(f_l - h_l)| \leq 2^3 \delta_l^3 \|f_l - h_l\|_{\infty} \leq C_0 (C_1 + 1) 2^3 \delta_l^3$ whenever $|z - z_l| \geq 2h_l$.

Let $C_2 = 8 C_0 (C_1 + 1)$. Then, for each $z \in \{ f_l(z) - h_l(z) \} \leq \min(C_2, \frac{C_2 \delta_l^2}{|z - z_l|^3})$. So it follows that for every $z$ (the first sum is taken over those $U$ for which $F_l \cap \partial U \neq \emptyset$),

$$\sum |f_l - h_l| \leq \sum \min(C_2, \frac{C_2 \delta_l^3}{|z - z_l|^3}) = \sum \min(C_2, \frac{C_2 (z - z_l)^{-3}}{|z - z_l|^3})$$

$$\leq C_2 \frac{5}{4} \delta_l^3 \min\left(1, \frac{2 - k}{\text{dist}(z, \cup F_l)} \right).$$

Notice that $f$ is analytic on those $F_l$’s for which $F_l \cap \partial U = \emptyset$. It follows that

$$f_l = T_{\phi_l} f = \frac{1}{\pi} \int_{\partial U} \frac{f(z) - f(w) \partial \phi_l(z)}{z - w} \, d\text{Area} = -\frac{1}{\pi} \int_{\partial U} \frac{\partial f(z)}{\partial \nu} \phi_l(z) \, d\text{Area} = 0$$

for those $l$’s. So there are only finitely many $f_l$’s that are not zero.

Now define $h_l = 0$ if $l$ is such that $f_l = 0$ and set $y_k = f + \sum (h_l - f_l)$. Then $y_k = \sum h_l$. For any $z$ off $\partial U$, it is clear that $\text{dist}(z, \cup F_l) \rightarrow \text{dist}(z, \partial U)$ as $k \rightarrow \infty$, and hence it follows from the above inequalities that $y_k \rightarrow f(z)$ for each $z$ in $C \setminus \partial U$. According to 3), we have $\| \int y_k x_j \, d\mu_l \| \leq \frac{1}{k},$ for $1 \leq j \leq k$. Notice that

$$|y_k| \leq |f| + \sum (h_l - f_l) \leq \|f\|_{\infty} + C_2 \frac{5}{4} \delta_l^3,$$

so $\{y_k\}$ is a bounded sequence. Since the weak-star topology on the unit ball of the dual space of a separable Banach space is metrizable, it follows by Alaoglu’s theorem that there exists a subsequence $\{y_k\}$ that weak-star converges to some $h \in L^\infty(\mu)$. According to the last inequality above, we have that $\int h \, x_j \, d\mu_l = 0$ for all $j \geq 1$. Consequently, we have that $h \in A^q(K, \mu)$. Because $y_k \rightarrow f$ pointwise off $\partial U$ and $f = 0$ off $\partial \mathcal{U}$, we have $h = 0$ off $\partial \mathcal{U}$.

Finally, we show that $\hat{h} = f$ on $U$. If supp$(\mu)$ contains an open subset of $G$, then this is easy to see this is true (since $y_k \rightarrow f$ pointwise on $U$ and hence $f = h = \hat{h}$ a.e. $[\mu]$ on $G$. Because both $f$ and $\hat{h}$ are analytic on $U$, hence $f = \hat{h}$ on $U$). Otherwise, let $G$ be open so that $\mathcal{U} \subset U$ and let $\rho = \mu + \text{Area} [\mathcal{U}]$. Then $\| \cdot \|_\rho$ and $\| \cdot \|_\mu$ are equivalent norms. By the definition of $abpe$, we see that $U \subset \nabla A^q(K, \mu) \subset \nabla A^q(K, \rho)$. Therefore, there exists $f_1 \in A^q(K, \rho) \cap L^\infty(\rho)$ such that $f_1 = f$ and $f_1 = 0$ off $\partial U$. Now set $h = f_1 |\text{supp}(\mu)$. We show $h$ is the desired function. Let $\{f_n\} \subset A(K)$ such that $f_n \rightarrow f_1$ in $L^q(\rho)$. Then $f_n \rightarrow h$ in $L^q(\mu)$. Thus, $f_n \rightarrow h$ uniformly on $G$. Since $f_n \rightarrow f$ uniformly on $G$ as well, it follows that $h = f$ on $G$. Because $U$ is the union of such open subsets $G$, we conclude that $\hat{h} = f$ on $U$. 13
Now we consider a general $K$ that satisfies the hypothesis of this lemma. Let $\Omega$ be the component of $K^\circ$ that contains $U$. Then the multi-nicely connectivity of $\Omega$ insures that there is circular domain $W$ and a conformal map $v$ from $\Omega$ onto $W$ such that $v$ is almost 1-1 on $\partial \Omega$ with respect the harmonic measure of $\Omega$. Let $\mu = \mu_0 + \tau$ be the decomposition such that $A^q(K, \tau)$ is pure and $A^q(K, \mu) = L^q(\mu_0) \oplus A^q(K, \tau)$. By Proposition 3.1, $\tau|\partial \Omega$ is absolutely continuous with respect to the harmonic measure. Extend $v$ to $\overline{\Omega}$ by defining its boundary values as its nontangential limits and set $\nu = \tau \circ v^{-1}$. It is easy to check that $\nu(U)$ is a component of $\nabla A^q(\overline{W}, \nu)$ and there is $h_1 \in A^q(\overline{W}, \nu)$ such that $h_1 = f \circ v^{-1}$. Set $h = h_1 \circ v^{-1}$. Then, $h \in A^q(\overline{\Omega}, \tau)$ and it is straightforward to verify that $\hat{h} = f$. Extend $h$ to be a function on $K$ by defining $h = 0$ off $\overline{\Omega}$. By Lemma 7, $h \in A^q(\overline{\Omega}, \mu|\overline{\Omega}) \subset A^q(K, \mu)$. Clearly, $h$ does the job.

Lemma 11 If $a \in \nabla A^q(K, \mu)$, then $\frac{f(z) - f(a)}{z - a} \in A^q(K, \mu)$ for each $f \in A^q(K, \mu)$.

Proof. Let $W = \nabla A^q(K, \mu)$. Then there exists $\{f_n\} \subset A(K)$ such that $f_n \to f$ in $L^q(\mu)$ and so $f_n \to f$ uniformly on compact subset of $W$. Thus $f(z) - f(a) = \int f_n(z) - f(a)\frac{1}{z - a}d\mu \to f(z) - f(a)\frac{1}{z - a}$ uniform on a small closed disk $B(a, \delta) \subset W$. Note,

$$\int_K |f_n(z) - f(a)|^q d\mu \leq M \int_K |f_n(z) - f(z) - (f_n(a) - f(a))|^q d\mu + \int_{B(a, \delta)} |f_n(z) - f_n(a)|^q d\mu \to |f(z) - f(a)|^q d\mu,$$

where $M = \sup_{z \in K \setminus B(a, \delta)} |\frac{1}{z - a}|^q$. Thus, $\frac{f_n(z) - f_n(a)}{z - a} \to \frac{f(z) - f(a)}{z - a}$ in $L^q(\mu)$. Since $\frac{f_n(z) - f_n(a)}{z - a} \in A(K)$ for each $n$, the conclusion of the lemma follows.

Lemma 12 Suppose that $\nu \perp A(K)$ and $\text{supp}(\nu) \subset K$. Let $U$ be a component of $K^\circ \setminus \text{supp}(\nu)$. If $\tilde{\nu}(a) \neq 0$ at some $a \in U$, then $U \subset \nabla A^1(K, |\nu|)$.

Proof. Clearly $\tilde{\nu}(z) = \int \frac{1}{z - w}d\nu(w)$ is analytic on $U$. Observe that for $f \in A(K)$, $\frac{f(z) - f(a)}{z - a} \in A(K)$ for every $a \in K^\circ$. Suppose that $\hat{\nu}(a) \neq 0$ for some $a \in U$. Then there exists a small closed disk $\overline{B(a, \delta)} \subset U$ so that $\hat{\nu}(z) \neq 0$ on $\overline{B(a, \delta)}$. For each $\lambda \in B(a, \delta)$, $\int \frac{f(z) - f(\lambda)}{z - \lambda}d\nu = 0$ and hence

$$f(\lambda) = \int \frac{f(z)}{z - \lambda}d\nu, \text{ for every } f \in A(K).$$

Since $\overline{B(a, \delta)}$ does not interest $\text{supp}(\nu)$, we see that

$$|f(\lambda)| \leq c\|f\|_{L^1(\nu)} \text{ for some } c > 0 \text{ on } B(a, \delta).$$

Hence, $a \in \nabla A^1(K, |\nu|)$. Since the zeros of $\hat{\nu}$ is isolated on $U$, it follows by Lemma 8 that $U \subset \nabla A^1(K, |\nu|)$.

Lemma 13 Let $h \in A^q(K, \mu) \perp$ and set $\nu = h\mu$. Then $\nabla A^1(K, |\nu|) \subset \nabla A^q(K, \mu)$. 

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Proof. For $f \in A(K)$, by Hölder’s inequality $\|f\|_{L^1(\mu)} \leq \|h\|_{L^p(\nu)} \|f\|_{L^q(\mu)}$, where $\frac{1}{q} + \frac{1}{p} = 1$. The theorem of the lemma clearly follows.

**Proposition 3.2** Let $\mu$ be a positive finite measure with $\text{supp}(\mu) \subset K$. Let $U$ be a component of $K^c \setminus \text{supp}(\mu)$. If $U \cap \nabla A^q(K, \mu) \neq \emptyset$, then $U \subset \nabla A^q(K, \mu)$.

**Proof.** Let $a \in U \cap \nabla A^q(K, \mu)$. Then there is $g \in A^q(K, \mu)^\perp$ such that $\int \frac{gd\mu}{z-a} \neq 0$. Clearly,

$$g\mu \perp A(K) \quad \text{and} \quad U \subset K^c \setminus \text{supp}(g\mu).$$

So it follows from the previous lemmas that

$$U \subset \nabla A^1(K, |g\mu|) \subset \nabla A^q(K, \mu).$$

**Lemma 14** Let $\Omega = \nabla A^q(K, \mu)$ and let $U$ be a component of $\nabla A^q(K, \mu)$. Suppose that $A^q(K, \mu)$ is pure. Let $f \in A^q(K, \mu)$. If $\hat{f} = 0$ and $f = 0$ off $\partial U$ a.e. $[\mu]$, then $f = 0$.

**Proof.** Since $A^q(K, \mu)$ is pure, as we argued in the proof of Proposition 3.1, there exists $h \in A^q(K, \mu)^\perp$ such that $h \neq 0$ a.e. $[\mu]$ and $h = 0$ off $K^c$. So $h \perp A(K)$ as well. Let $\nu = hf\mu$. Then $\nu$ is a measure such that it is perpendicular to $A(K)$ and $\text{supp}(\nu) \subset \partial U$. We show that $\hat{\nu}(a) = 0$ off $\partial U$.

Let $W$ be a component of $K^c \setminus \text{supp}(\nu)$. We claim that $\hat{\nu}(z) = 0$ on $W$. Suppose $\hat{\nu}(a) \neq 0$ for some $a \in W$. According to Lemma 14,

$$W \subset \nabla A^1(K, |\nu|) \subset \nabla A^q(K, \mu).$$

By Lemma 14 and the hypothesis, we have that $\frac{\nu}{z-a} \in A^q(K, \mu)$. But $h \perp A(K)$. So we conclude that $\hat{\nu}(a) = 0$, which contradicting our assumption above. Therefore, $\hat{\nu} = 0$ on $W$. In particular, $\hat{\nu} = 0$ on $K^c \setminus \partial U$. It is easy to see that $K^c \setminus \partial U \supset R(K^c)$. So, by the continuity we have that $\hat{\nu} = 0$ on $K^c - \partial U$. Because $h \perp A(K^c) \supset R(K^c)$, $\hat{\nu} = 0$ off $\partial K^c$. Hence, we conclude that $\hat{\nu} = 0$ off $\partial U$.

Now, according to our definition, it is apparent that every point off $\partial U$ is a light point with respect to $|\nu|$. So it follows from Lemma 3.7 in [20] that every point in $\partial U$ is light as well. Consequently, every point in the plane $C$ is a light point. Applying Theorem 3 we conclude that $\nu = hf\mu = 0$. Since $h \neq 0$ a.e. on $K$, $f$ must be the zero function in $L^q(\mu)$. So we are done.

**Lemma 15** Let $K$ be a compact subset in $C$ such that each component of $K^c$ is finitely connected. Let $\mu$ be a positive finite measure supported on $K$. Then each component $U$ of $\nabla A^q(K, \mu)$ is a finitely connected domains conformally equivalent to a circular domain in the plane. Moreover, the connectivity of $U$ does not exceed the connectivity of the component of $K^c$ that contains $U$.

**Proof.** Suppose $\nabla A^q(K, \mu) \neq \emptyset$. Let $U$ be a component of $\nabla A^q(K, \mu)$ and let $\Omega$ be the component of $K^c$ that contains $U$. Let $M$ be the connectivity of $\Omega$.

Now suppose $F$ is a component of $C \setminus U$. We claim that $F \cap (C \setminus \Omega) \neq \emptyset$. First, $F$ is unbounded, this is obvious. So we assume $F$ is a bounded subset in the plane and assume that $F \cap (C \setminus \Omega) = \emptyset$. Then $F \subset \Omega$. Since $U$ is a connected
domain, $F$ is polynomially convex (this means the complement of $F$ is connected). Since $U$ is finitely connected, there exists a Jordan curve $\gamma$ in $U$ such that $F$ is contained in $V$, the bounded Jordan domain enclosed by $\gamma$. Let $f \in A^q(K, \mu)$ and choose a sequence of functions $\{r_n\}$ in $A(K)$ such that $r_n \to f$ in $L^q(\mu)$. Since $\gamma$ is contained in $U \subset \nabla A^q(K, \mu)$, it follows that $r_n \to f$ uniformly on $\gamma$. Also, it is clear that we can choose $\gamma$ such that $\text{dist}(F, \gamma)$ small enough that the closure of $V$ is contained in $\Omega \subset K^\circ$. Then each $r_n$ is analytic on $\overline{V}$ and thus the maximum principle implies that $\{r_n\}$ uniformly converges to a function $h$ near $F$. By the definition of abpes, we have that $F \subset \nabla A^q(K, \mu)$. But $F \cap \partial U \neq \emptyset$, and hence we conclude that $\partial U \cap \nabla A^q(K, \mu) \neq \emptyset$, which is a contradiction. Hence $F \cap (C \setminus \Omega) = \emptyset$.

Now let $\{E_i\}$ be the collection of all the components of $(C \setminus \Omega)$ that intersect $F$. Then $F \cup (\cup E_i)$ is connected compact subset and 

$$[F \cup (\cup E_i)] \cap U = \emptyset.$$ 

So $F \cup (\cup E_i)$ is a component of $C \setminus U$ that contains $F$. Hence $F \cup (\cup E_i) = F$ and therefore $E_i \subset F$ for each $i$. Consequently, each component of $C \setminus U$ contains at least a component of $(C \setminus \Omega)$. Since the number of the components of $C \setminus \Omega$ is $M$, we see that the number of the components of $C \setminus U$ is less or equal to $M$.

Finally, since $U$ is finitely connected and since $\partial U$ contains no single-point component (by Lemma 3), it follows by a classical result [30, Tsuji, p. 424] that $U$ is conformally equivalent to a circular domain.

The next two propositions are Theorem 1 and Theorem 3 in [21], respectively. We include them for readers convenience and self contained.

**Proposition 3.3** Let $A^q(K, \mu)$ be irreducible. Let $U = \nabla A^q(K, \mu)$ be a finitely connected domain. If the map $e$, defined by $e(f) = f$, from $A^q(K, \mu) \cap L^\infty(\mu)$ to $H^\infty(U)$ is surjective, then $\mu|\partial U \preccurlyeq \omega_U$, the harmonic measure of $U$.

**Proposition 3.4** Let $A^q(K, \mu)$ be irreducible. Let $U = \nabla A^q(K, \mu)$ be a finitely connected domain and let $u$ be a conformal map from a circular domain $W$ onto $U$. If the map $e$, defined by $e(f) = f$, from $A^q(K, \mu) \cap L^\infty(\mu)$ to $H^\infty(U)$ is surjective, then for each $f \in H^\infty(U)$

$$e^{-1}(f)(a) = \lim_{z \to u^{-1}(a)} f \circ u(z) \text{ a.e. on } \partial U \text{ with respect to } \mu|\partial U.$$ 

Moreover, $A(U) \subset A^q(K, \mu)$.

The proof of Theorem 2.

Let $\mu = \mu_0 + \tau$ be the decomposition such that $A^q(K, \tau)$ is pure and

$$A^q(K, \mu) = L^q(\mu_0) \oplus A^q(K, \tau).$$

Suppose $A(K)$ is not dense in $L^q(\mu)$. Then $\tau \neq 0$ in the decomposition. According to Theorem 1 $\nabla A^q(K, \tau) \neq \emptyset$.

Let $\{U_n\}_{n=1}^\infty$ be the components of $\nabla A^q(K, \tau)$. For each $n \geq 1$, by Lemma 10 there exists $f_n$ in $A^q(K, \tau) \cap L^\infty(\tau)$ such that $f_n = \chi_{U_n}$ and $f_n = 0$ off $U_n$. Since $U_n$’s are pairwise disjoint, we have $\hat{f}_n \hat{f}_m = \hat{f}_n \hat{f}_m = 0$. It follows by Lemma 14 that $f_n f_m = 0$. Similarly, since $\hat{f}_n^2 = \hat{f}_n$, we get that $f_n^2 = f_n$. Therefore, we
Conclude that \( f_n = \chi_{\Delta_n} \) for some Borel subset \( \Delta_n \). But \( \tau(\Delta_n \cap \Delta_m) = 0 \) (because \( f_n f_m = 0 \)). Thus, \( \Delta_n \)’s can be chosen to be pairwise disjoint. Moreover, since \( f_n = 0 \) off \( U_n \), we can also require that \( \Delta_n \subset U_n \).

For each \( n \geq 1 \), let \( K_n = \overline{U}_n \) and let \( \mu_n = \tau|\Delta_n \). We claim that \( U_n = \nabla A^q(K_n, \mu_n) \). Let \( \Omega \) be the component of \( K^o \) that contains \( U_n \). By Lemma 7, \( A^q(\Omega, \tau|\Omega) \subset A^q(K, \tau) \). Note, every function \( f \) in \( A^q(\Omega, \tau|\Omega) \) has zero values off \( \Omega \). Clearly, \( f_n(\chi_{\Delta_n}) \) belongs to \( A^q(\Omega, \tau|\Omega) \) also. Set \( F_n = U_n \cup (\Delta_n \setminus U_n) \). Because \( \Delta_n \subset \overline{U}_n \), we see that \( \chi_{F_n} = f_n \in A^q(\Omega, \tau|\Omega) \). So by Lemma 2, \( \chi_{F_n} \subset A^q(\Omega, \tau|\Omega) \) for each \( f \in A(\Omega) \). This implies that \( A^q(\Omega, \mu_n) = A^q(\Omega, \tau|F_n) \subset A^q(\Omega, \tau) \). Let \( a \in U_n \). Then there exists \( c > 0 \) and an open open disk \( D_a \subset U_n \) such that for all \( f \in A(\Omega) \)

\[
|f(a)| \leq c \left( \int |f|^q d\tau \right)^{1/q}, \quad a \in D_a.
\]

Let \( g \in A(\Omega) \). Then there is a sequence \( \{q_i\}_{i=1}^\infty \in A(\Omega) \) so that \( q_i \to g \chi_{F_n} \) in \( L^q(\tau) \). Then \( q_i \to g \) uniformly on \( D_a \). Hence, it follows that for \( a \in D_a \)

\[
|g(a)| = \lim |q_i(a)| \leq c \lim \left( \int |q_i|^q d\tau \right)^{1/q} = c \left( \int g \chi_{F_n} \, d\tau \right)^{1/q} = c \left( \int |g|^q \, d\mu_n \right)^{1/q}.
\]

Thus \( a \in \nabla A^q(\Omega, \mu_n) \). Therefore, \( U_n \subset \nabla A^q(\Omega, \mu_n) \). By Lemma 7, \( \nabla A^q(\Omega, \mu_n) \subset \nabla A^q(K, \tau) \). So we see (notice that \( U_n \) is a component of \( \nabla A^q(K, \tau) \)) that \( U_n = \nabla A^q(\Omega, \mu_n) \).

The hypothesis and Lemma 15 together imply that \( U_n \) is a finitely connected domain. It is also easy to see that \( A^q(\Omega, \mu_n) \) is irreducible. Applying Proposition 3.3, we have \( A(U_n) \subset A^q(\Omega, \mu_n) \). Consequently, \( A^q(\overline{U}_n, \mu_n) \subset A^q(\Omega, \mu_n) \).

Therefore, we conclude that \( A^q(\overline{U}_n, \mu_n) \subset A^q(K, \mu_n) \). Hence, \( U_n = \nabla A^q(K, \mu_n) \). This proves the claim.

Since each \( A^q(K_n, \mu_n) \) is contained in \( A^q(K, \mu) \) and since \( \{A^q(K_n, \mu_n)\} \) are pairwise orthogonal, we have

\[
A^q(K, \mu) \supset L^q(\mu) \oplus A^q(K_1, \mu|\Delta_1) \oplus \cdots \oplus A^q(K_n, \mu|\Delta_n) \oplus \cdots
\]

For the other direction of the equality, let \( f \) be the pointwise limit of \( \{\chi_{f_n}\}_{n=1}^\infty \). Then the bounded convergence theorem implies that \( f \in A^q(K, \tau) \). We show that \( 1 - f = 0 \) a.e. \( \tau \). Otherwise, there exists a Borel subset \( E \) of the support of \( \tau \) such that \( 0 \neq \chi_E = 1 - f \). Since both 1 and \( f \) are in \( A^q(K, \tau) \), we have that \( \chi_E \in A^q(K, \tau) \). By the purity, we have \( L^q(\tau|E) \neq A^q(K, \tau|E) \). So it follows by Theorem 3 that \( \nabla A^q(K, \tau|E) \neq 0 \). But \( \chi_E f_n = 0 \) for each \( n \geq 1 \), thus we have \( \nabla A^q(K, \tau|E) \cap (\cup U_n) = \emptyset \). But by the definition of \( abpes \), \( \nabla A^q(K, \tau|E) \subset \nabla A^q(K, \mu) = \cup U_n \). This is a contradiction, and hence \( f - 1 = 0 \). Therefore, \( \{\Delta_n\} \) is a Borel partition. Let \( g \in A^q(K, \tau) \). By the Lebesgue dominated convergence theorem, we conclude that \( g = \lim_{n \to \infty} \sum_{i=1}^n f_i g \) in \( L^q(\tau) \). Therefore,

\[
A^q(K, \mu) \subset L^q(\mu) \oplus A^q(K_1, \mu|\Delta_1) \oplus \cdots \oplus A^q(K_n, \mu|\Delta_n) \oplus \cdots
\]

Consequently,

\[
A^q(K, \mu) = L^q(\mu) \oplus A^q(K_1, \mu|\Delta_1) \oplus \cdots \oplus A^q(K_n, \mu|\Delta_n) \oplus \cdots
\]

Now we prove the rest of Theorem 2. For 1), since we have already proved \( \overline{U}_n \subset \Delta_n \) above, we only need to show that \( A^q(K, \mu_n) \subset A^q(\overline{U}_n, \mu_n) \). Because \( A(K) \subset A(\overline{U}_n) \), it follows by the definition of \( abpe \) that

\[
U_n \subset \nabla A^q(\overline{U}_n, \mu_n) \subset \nabla A^q(K, \mu_n).
\]

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Notice that $A^q(K, \mu_n) \subset A^q(U_n, \mu_n)$ and the latter is irreducible. So we see that $A^q(K, \mu_n)$ is irreducible also. This implies that $\nabla A^q(K, \mu_n)$ have only one component. Also, it is clear that $\nabla A^q(K, \mu_n) \subset \nabla A^q(K, \tau)$. So we conclude that $\nabla A^q(K, \mu_n) = U_n$. Thus, there is $h_n \in A^q(K, \mu_n)$ such that $h_n = \chi_{U_n}$ and $h_n = 0$ off $U_n$. By the uniqueness, we get that $h_n = f_n = \chi_{\Delta_n}$. As we proved above, $f = \sum_{n=1}^{\infty} f_n$ for each $f \in A^q(K, \tau)$. Hence, we conclude that

$$A^q(K, \tau) \subset A^q(K, \mu_1) \oplus ... \oplus A^q(K, \mu_n) \oplus ...$$

$$\subset A^q(U_1, \mu_1) \oplus ... \oplus A^q(U_n, \mu_n) \oplus ...$$

$$= A^q(K, \tau).$$

Consequently, $A^q(K, \mu_n) = A^q(U_n, \mu_n)$ for each $n \geq 1$.

2) follows from Lemma 11

For 3), let $e$ be the map, $f \rightarrow \hat{f}$, from $L^\infty(\mu_n) \cap A^q(K_n, \mu_n)$ into $H^\infty(U_n)$. Then $e$ is surjective by Lemma 10 and is injective by Lemma 13. Since $e(fg) = e(f)e(g)$, $e$ is an algebraic isomorphism between two commutative Banach algebras and thus $e$ is an isometry.

Next we need to show that $e$ is a weak-star homeomorphism. To do this, we will argue as in [6]. Using Krein-Smulian theorem it suffices to show that $e$ is weak-star sequentially continuous.

Recall that a sequence of functions in $H^\infty(U_n)$ is weak-star Cauchy sequence if and only if it is uniformly bounded on $U_n$ and it is a Cauchy sequence in the topology of pointwise convergence. Let $\{h_i\}$ be a sequence in $A^q(K_n, \mu_n) \cap L^\infty(\mu_n)$ that converges to zero in the weak star topology. By the uniform boundedness, $\{h_i\}$ is bounded and hence $\{e(h_i)\}$ is also bounded. Let $a \in U_n$ and let $k_a$ be kernel function. Then

$$\lim_{i \rightarrow \infty} e(h_i) = \lim_{i \rightarrow \infty} \hat{h}_i(a) = \lim_{i \rightarrow \infty} \int h_i k_a d\mu_n = 0.$$

So $e(h_i)$ weak-star converges to zero. Therefore, $e$ is a weak-star homeomorphism.

4) follows from Proposition 4.3 and Proposition 5.4

For 5), by Proposition 5.4, each $f \in H^\infty(U_n)$ has nontangential limits almost everywhere on $\partial U_n$ with respect to $\mu|\partial U_n$ and the nontangential limits are equal to $e^{-1}(f) a.e. \ [\mu]|\partial U_n].$ Since $f_e = f = f^* a.e. \ [\mu]$ on $U_n$, we see that $f^* = e^{-1}(f)|\Delta_n a.e. \ [\mu].$ Evidently

$$m(e(f)) = m(\hat{f}) = \begin{cases} \hat{f} & \text{on } U_n \\ e^{-1}(\hat{f}) & \text{on } \partial U_n \end{cases}$$

$$= \begin{cases} f & \text{on } U_n \\ f & \text{on } \partial U_n \end{cases}$$

$$= f, \text{ for each } f \in A^q(K_n, \mu_n).$$

Therefore, $m$ is the inverse map of $e$. So the proof of Theorem 2 is complete.

Remark: This paper contains the best and close-up result in an once quite active research area. However, this paper has not been cited by any other authors but me, while [28] has been cited more than 80 times. I re-published this paper on Arxiv to hope to bring more attentions from future generation of mathematicians to the results in this paper, which I believe is great and can stay in history of mathematics.
Acknowledgement. During 1994, I gave several talks on the result of this work in turn in the SouthEast Analysis Meeting at Virginia Tech, UC-Berkeley, AMS Summer Research Conference at Mt. HolyOak, Brown University and Wabash Conferences at IUPUI. When $R(K)$ is a hypodirichlet algebra, A slightly simper version of Theorem 2 was stated as the last theorem in my work [20].

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