Swimming of Microorganism
and the String- and Membrane-like Algebra

Masako KAWAMURA, Akio SUGAMOTO

Department of Physics, Faculty of Science
Ochanomizu University
1-1 Otsuka 2, Bunkyo-ku, Tokyo 112, Japan

and

Shin’ichi NOJIRI

Department of Mathematics and Physics
National Defence Academy
Yokosuka, 239, Japan

Abstract

Swimming of microorganisms is further developed from a viewpoint of strings and membranes swimming in the incompressible fluid of low Reynolds number. In our previous paper the flagellated motion was analyzed in two dimensional fluid, by using the method developed in the ciliated motion with the Joukowsky transformation. This method is further refined by incorporating the inertia term of fluid as the perturbation. Understanding of the algebra controlling the deformation of microorganisms in the fluid is further developed, obtaining the central extension of the algebra with the help of the recent progress on the $W_{1+\infty}$ algebra. Our previous suggestion on the usefulness of the $N$-point string- and membrane-like amplitudes for studying the collective swimming motion of $N-1$ microorganisms is also examined.
Introduction

In our previous paper[1] we have studied swimming of microorganisms viewed from string and membrane theories. Our target is still the understanding of the following ways of microorganisms’ swimming[2]: There exist only three different universality classes of the swimming ways of microorganisms; (1) Swimming with cilia is adopted by the spherical organisms with the length scale of $20 \sim 2 \times 10^4 \mu m$, an example of which is paramecium; (2) the smaller microorganisms with the size of $1 \sim 50 \mu m$ swim with flagella, an example of which is the sperm; (3) the bacteria with the size of $0.2 \sim 5 \mu m$ swim with bacterial flagella the motion of which resembles the screwing of the wine-opener. Why is it possible that such simple classification be realized in the swimming problem of microorganisms?

Recapitulating our previous work[1], we will comment on a few newly obtained results. Starting from the work by Shapere and Wilczek[3], we have developed the ciliated and flagellated motions and studied the microorganisms’ swimming from the viewpoint of algebraic structure existing in the deformation operation of the microorganisms in the low Reynolds liquid, as well as from the swimming dynamics of a group of microorganisms, which may be connected to the $N$-point correlation function of string and membrane theories.

The microorganisms with the length scale $L \ll 1$, the Reynolds number $R$ satisfies $R \ll 1$, so that the hydrodynamics in this case leads to the following equations of motion for the incompressible fluid:

$$\nabla \cdot \mathbf{v} = 0,$$

and

$$\Delta \mathbf{v} = \frac{1}{\mu} \nabla p,$$

or equivalently

$$\Delta (\nabla \times \mathbf{v}) = 0,$$

where $p$ is the pressure and $\mathbf{v}(x)$ is the velocity field of the fluid. The surface of a microorganism swimming in the $D = 3$ dimensional fluid of the real world forms a closed membrane at a fixed times $t$, the position of which can be parametrized by introducing $(D - 1)$ parameters $\xi^i (i = 1, \cdots, D - 1)$ as $X^\mu = X^\mu(t; \xi^1, \cdots, \xi^{D-1})$. It is sometimes instructive to consider the $D = 2$ dimensional fluid. Then, the surface of a ciliate (flagellate) becomes a closed (open) string and its position can be described by a complex number

$$Z = x^1 + ix^2 = Z(t; \theta),$$

with $-\pi \leq \theta \leq \pi$. In the sticky fluid of $\mu \neq 0$, there is no slipping between the surface of a microorganism and the fluid, namely, we have the matching condition

$$\mathbf{v}(x = X(t; \xi)) = \dot{X}(t; \xi),$$

or in the general coordinate system with the metric tensor $g^{\mu\nu}(x)$

$$g^{\mu\nu}(x)v_\mu(x)|_{x=X(t;\xi)} = \dot{X}^\nu(t;\xi).$$

A The ciliated motion in the two dimension fluid

The ciliated motion can be viewed as a small but time-dependent deformation of a unit circle in a properly chosen scale,

$$Z(t, \theta) = s + \alpha(t, s),$$

where $s = e^{i\theta}$ and $\alpha(t, s)$ is arbitrary temporally periodic function with period $T$ satisfying $|\alpha(t, s)| \ll 1$ with $-\pi \leq \theta \leq \pi$. The complex representation of the velocity vector $v_\mu$ can be denoted as

$$2v_\xi(z, \bar{z}) = (v_1 + iv_2)(z, \bar{z})$$

and

$$2v_\xi(z, \bar{z}) = (v_1 - iv_2)(z, \bar{z}).$$
By estimating the translational and rotational flows at spatial infinity caused by the deformation of the cilia, we have obtained $O(\alpha^2)$ expression of the net translationally swimming velocity $v_T^{(cilia)}$ of the ciliated microorganism as follows:

$$
2v_T^{(cilia)} = -\dot{\alpha}_0(t) + \sum_{n \leq 1} n(\dot{\alpha}_n \alpha_{n+1} - \dot{\alpha}_n \alpha_{n-1} - \dot{\alpha}_n \alpha_{n+2}) - \sum_{n > 1} n\alpha_n \alpha_{n-1},
$$

(9)

where $\alpha_n(t)$ is defined by $\alpha(t, s) = \sum_{n=-\infty}^{+\infty} \alpha_n(t) s^n$. On the other hand, the net angular momentum $v_R^{(cilia)}$ gained by the microorganism from the fluid becomes

$$
2v_R^{(cilia)} = -\text{Im}\{\dot{\alpha}_1(t) \}
$$

$$
- \sum_{n \leq 1} n(\dot{\alpha}_n \alpha_{n+2} - \dot{\alpha}_n \alpha_{n} - \dot{\alpha}_n \alpha_{n+2}) + \sum_{n > 1} n\alpha_n \alpha_n\}
$$

(10)

The net translation and rotation resulted after the period $T$ come from $O(\alpha^2)$ terms since the $O(\alpha)$ terms cancel after the time integration over the period.

B  The flagellated motion in two dimensional fluid

Microorganisms swimming using a single flagellum can be viewed as an open string with two endpoints, $H$ and $T$, where $H$ and $T$ represent the head and the tail-end of a flagellum, respectively. Our discussion will be given by assuming that the distance between $H$ and $T$ is time-independent and is chosen to be 4 in a proper length scale. This assumption can be shown to be valid for the flagellated motion by small deformations in the incompressible fluid. Then, at any time $t$, we can take a complex plane of $z$, where $H$ and $T$ are fixed on $z = 2$ and $-2$, respectively. This coordinate system $z$ can be viewed as that of the space of standard shapes of Shapere and Wilczek. Time dependent, but small deformation of the flagellate can be parametrized as

$$
Z(t, \theta) = 2(\cos \theta + i \sin \theta \alpha(t, \theta)),
$$

(11)

where the small deformation $\alpha(t, \theta)$ can be taken to be a real number$^1$ satisfying

$$
\alpha(t, \theta) = -\alpha(t, -\theta).
$$

(12)

Here, we parametrize the position of the flagellum twice, starting from the endpoint $T$ at $\theta = -\pi$, coming to the head $H$ at $\theta = 0$, and returning to $T$ again at $\theta = \pi$. Motion of the two branches corresponding to $-\pi \leq \theta < 0$ and $\pi \geq \theta \geq 0$ should move coincidentally, which requires the condition (12). The Joukowski transformation $z = z(w) = w + w^{-1}$, separates the two coincident branches in the $z$ plane to form lower and upper parts of a unit circle in the $w$ plane, outside domain of which we are able to study the swimming problem of the flagellate in a quite similar fashion to that of the ciliate. The parametrization of our microorganism in the $w$ plane corresponding to Eq.(11) is now

$$
W(t, \theta) = e^{i\theta}(1 + \alpha(t, \theta)) + O(\alpha^2).
$$

(13)

Using the mode expansion satisfying Eq.(12),

$$
\alpha(t, \theta) = \sum_{n=1}^{\infty} \alpha_n(t) \sin n\theta,
$$

(14)

$^1$When $\alpha$ is taken to be a complex number, the length of the flagellum is locally changeable at $O(\alpha)$. For such an elastic flagellum, we have similar results to that of the ciliated motion. In case of real $\alpha$, its length is locally preserved at $O(\alpha)$, giving a non-elastic flagellum, which is the more realistic one.
we are able to determine the net swimming velocity $v_T^{(flagella)}$ gained by the flagellate motion of microorganisms:

$$2v_T^{(flagella)} = -i\dot{\alpha}_1 - \sum_{m \geq 1} m\alpha_m \dot{\alpha}_{m+1} + \sum_{m \geq 2} m\alpha_m \dot{\alpha}_{m-1}. \tag{15}$$

On the other hand, the angular momentum $v_R^{(flagella)}$ is given by

$$2v_R^{(flagella)} = -\frac{1}{2}\dot{\alpha}_2. \tag{16}$$

After the time integration over the period $T$, $v_R^{(flagella)}$ vanishes since in our first order approximation, the length of the flagellum is fixed in the incompressible fluid. Therefore the second order approximation is necessary for the non-vanishing $v_R^{(flagella)}$.

C The selection rules and the symmetry of microorganisms’ swimming

Even though the results in Eqs. (15), (16), (18) and (19) are obtained perturbatively, we are able to read from them the characteristics of the microorganisms’ swimming: In order for the ciliates to swim or rotate, they need the coexistence of the two different Fourier modes of $n_1$, $n_2$. The selection rules for the allowed $(n_1, n_2)$ combinations are

1) $|n_1 + n_2| = 3$ or $|n_1 \pm n_2| = 1$ for the ciliated translation

2) $n_1 = n_2$ or $|n_1 + n_2| = 2$ for the ciliated rotation,

The corresponding selection rules for the flagellate motion are

3) $|n_1 - n_2| = 1$ for the flagellate translation,

where the Fourier modes are $\sin n\theta$ in this case.

Viewing these selection rules, we are tempted to elucidate the algebraic structure possibly existing in the background of the swimming mechanism. It is similar to the Virasoro algebra, but is different from it. For such a purpose, introduction of the “action” will be convenient. The “action” $S$ reproducing the classical equations of motion of the swimming of $N$ microorganisms in the incompressible liquid with low Reynolds number may be given by

$$S_N = \sum_{i=1}^{N} \int dt \int d^{D-1}x P_{\mu}^{(i)}(t; \xi(i)) \left[ \dot{X}^\mu_{(i)}(t; \xi(i)) - v^\mu(X_{(i)}(t; \xi(i))) \right]$$

$$+ \frac{1}{2\pi\alpha'} \int d^{D}x \sqrt{g(x)} \left[ -\frac{1}{\mu} p(x) \partial_{\nu} v^\nu(x) + \frac{1}{4} \omega_{\mu\nu}(x) \omega^{\mu\nu}(x) \right] \tag{20}$$

where the velocity field $\omega_{\mu\nu}(x)$ is given by

$$\omega_{\mu\nu}(x) = \partial_\mu v_\nu - \partial_\nu v_\mu, \tag{21}$$

whose $D = 2$ expression is $\omega_{z\bar{z}}(z, \bar{z}) = \partial_z v_{\bar{z}} - \partial_{\bar{z}} v_z$.

We have introduced the parameter $\alpha'$ so as to make $S$ dimensionless, where $\alpha'$ has the dimension of $(\text{Length})^D(\text{Time})^{-2}$. The reason why we have used the notation $\alpha'$, familiar in the string theories to describe the Regge slope, will be understood later. The Lagrange multiplier fields of $P_{\mu}^{(i)}(t; \xi(i))$ ($i = 1, \cdots, N, \mu = 1, \cdots, D$) guarantee the matching condition of (18) or (19) for $i$-th microorganism at any time, and the pressure $p(x)$ is also such multiplier giving the incompressibility given in Eq. (1). The field equation (20) can be easily reproduced. In the action (20), time $t$ appears only in the first term of representing the matching conditions, that is, the time evolution is triggered only by the self-motion of the
microorganisms, of which influence spreads instantaneously over the whole space and causes the change of the fluid velocity there. Because of the lacking of the kinetic term, we may call $S$ as the “action”. The additional metric contribution such as $\sqrt{g(x)}$ is only relevant for the curved space, an example of which has appeared in the flagellate swimming on the $w$ plane. The later discussion is given for the flat metric.

Now, we will define the following local transformation at a fixed time $t$:

$$
\delta \dot{X}^\mu_{(i)} = \lambda^\mu(X_i(t); \xi_i)) \tag{22}
$$

$$
\delta P^\mu_{(i)} = 0 \tag{23}
$$

$$
\delta v^\mu(x) = \lambda^\mu(x) \tag{24}
$$

$$
\delta p(x) = \kappa(x), \tag{25}
$$

where we have assumed that the transformation parameters $\lambda^\mu(x)$ and $\kappa(x)$ are restricted by the equations of motion,

$$
\partial_\mu \lambda^\mu(x) = 0, \quad \text{and} \quad \partial_\mu \lambda^\mu_{(i)}(x) = \frac{1}{\mu} \partial_\nu \kappa(x), \tag{26}
$$

where $\lambda_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$ is the vorticity for $\lambda^\mu$. Meaning of the transformations (22)~(25) are quite simple; the deformation of the microorganisms (22) triggers the increase of the velocity field (24) and of pressure (25) so that they can be consistent with the incompressible fluid dynamics of the low Reynolds number. It is also important to note that the succession of these time-independent transformations result in the time evolution of our problem. Therefore, the transformations (22)~(25) resemble the ordinary canonical transformation generated by the Hamiltonian.

The transformation also generates the deformation of the shapes of microorganism (22). The generator of this deformation per unit time can be written as

$$
\hat{L}_{[\lambda]} \equiv \int d^{D-1} \xi \lambda^\mu (X(t; \xi)) \frac{\delta}{\delta X^\nu (t; \xi)}, \tag{27}
$$

which gives the volume (area for $D = 2$) preserving diffeomorphisms owing solely to the incompressibility condition in Eq.(26). The second condition in Eq.(26) adds the further restriction on $\hat{L}_{[\lambda]}$: By the help of the stream function $\sigma_\lambda(x)$, the incompressibility condition is automatically satisfied through $\lambda^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu \sigma_\lambda$ for $D = 3$, so that the Eq.(26) becomes the constraint on the stream function

$$
\Delta (\Delta g_{\mu\nu} - \partial_\mu \partial_\nu) \sigma^\nu(x) = 0, \tag{28}
$$

or in $D = 2$, in terms of the only non-vanishing component $\sigma = \sigma_3$

$$
\Delta^2 \sigma(x) = 0. \tag{29}
$$

This constraint has been already solved generally in $D = 3$ and $D = 2$ fluid.

The deformation operator of the fluid corresponding to (27)

$$
L_{[\lambda]} \equiv \lambda^\mu (x) \frac{\partial}{\partial x^\mu}, \tag{30}
$$

becomes in $D = 2$ as

$$
L_\sigma = 2(\lambda_3 \partial_z + \lambda_3 \partial_{\bar{z}})
= 2(\partial_{\bar{z}} \sigma \partial_z - \partial_z \sigma \partial_{\bar{z}}), \tag{31}
$$

where the stream function $\sigma(z, \bar{z})$ contains $z \bar{z}^k$, $z^k \bar{z}$, $\ln z$, $\ln \bar{z}$, and $\bar{z} \ln z \bar{z}$ terms ($k$: integer). Then, $L_\sigma$ can be considered as a Liouville operator of a dynamical system moving in the phase space of $(z, \bar{z})$, having $-\sigma(z, \bar{z})$ as its Hamiltonian. Invariance of the phase volume during the temporal evolution of the dynamical system shows that $L_\sigma$ is really the area preserving diffeomorphism. The commutation relation is simple, namely

$$
[L_{\sigma_1}, L_{\sigma_2}] = -L_{\{\sigma_1, \sigma_2\}}, \tag{32}
$$
where the \( \{\sigma_1, \sigma_2\} \) is the Poisson bracket defined by

\[
\{\sigma_1, \sigma_2\} = \partial_z \sigma_1 \partial_{\bar{z}} \sigma_2 - \partial_{\bar{z}} \sigma_1 \partial_z \sigma_2.
\]  

(33)

Recently we have tried to obtain a closed algebra including \( \ln z \) and \( \ln \bar{z} \) in \( \sigma(z, \bar{z}) \). The algebra so obtained consists of \( L_{(l,m,n)} \equiv L(z, \bar{z})^{m+1} \), and \( M \equiv L_{\ln z - \ln \bar{z}} \); They satisfy

\[
[L_{(l,m,n)}, L_{(p,q,r)}] = -(lp - mp) L_{(l+p-1,m+q-1,n+r)} \\
+ (mr + nr - np) L_{(l+p-1,m+q-1,n+r-1)} \\
+ c(m - l) \delta_{l+p,0} \delta_{m+q,0} \delta_{n+r,0} ,
\]

(34)

\[
[M, M] = 0 ,
\]

(35)

and

\[
[M, L_{(l,m,n)}] = -(l + m) L_{(l-1,m-1,n)} - 2n L_{(l-1,m-1,n-1)} \\
+ \frac{1}{2} c \delta_{l,0} \delta_{m,0} \delta_{n,0} .
\]

(36)

In the above expression we add the central charge \( c \), corresponding to the possible central extension of the algebra in which the Jacobi identities are kept to hold and the generators are understood to be properly redefined.

The reason why the \( \ln z \) or \( \ln \bar{z} \) is permitted in the stream function \( \sigma(z, \bar{z}) \) is that the existing singularities at \( z = 0 \) can be hidden inside the body of the microorganism itself. Therefore, if we are not interested in the circulation flow of the fluid(topological flow) around the microorganism we can ignore the logarithmic contribution in \( \sigma \). In that case, the algebra becomes \( w_{1+\infty} \): With the notation \( T_{n,m} = L_{(n,m,0)} \), we have\(^2\)

\[
[T_{n,m}, T_{k,l}] = -(nl - mk) T_{n+k-1,m+l-1} \\
+ c(m - n) \delta_{m+k,0} \delta_{n+l,0} .
\]

(37)

There is a discrepancy between the definition of the \( W_{1+\infty} \) and the above expression\(^3\). Corresponding to the classical generator \( z^k D^n \) of \( W_{1+\infty} \) algebra, we define the quantum version of the generator as \( V_k^n = W(z^k D^n) \left[ D = \frac{\partial}{\partial z} \right] \). Then, the operator \( \hat{T}_{n,m} = \alpha^{n+m-2}W(z^{n-m} D^m) \) satisfy the algebra \(^4\) in the limit of \( \alpha \to 0 \) for \( c = 0 \).

Recently the representation theory of this \( W_{1+\infty} \) algebra is progressing considerably\(^5\) so that we are wishing to apply it to our problem. Successive application of the infinitesimal deformation on the microorganism forms a swimming motion. On the other hand a representation of the \( W_{1+\infty} \) algebra is obtained by a successive operation of raising operators \( V_{n,r}^k \) \((n \geq 1, r \geq 1)\) on a highest weight state \(|\lambda>\) which is characterized by a set of eigenvalues for \( V_0^n \) \((n = 1, 2, 3, \ldots)\). This conceptual correspondence between the swimming motion of microorganisms and the representation theory of the algebra, both of which are controlled by the area-preserving diffeomorphisms, may be useful to understand algebraically the typical pattern of the microorganisms’ swimming.

## D Small but Non-Vanishing Reynolds number

Next, we will study the case in which the Reynolds number \( R \) is small but non-vanishing. This has not been studied in our previous work\(^1\). In fact the length size \( L \) of the microorganisms ranges from \( \mu \) to

\(^2\) For the ordinary \( W_{1+\infty} \) algebra, \( m \geq 0 \) but in our algebra of microorganisms’ swimming, \( m \) can be negative.

\(^3\)
to mm. Therefore, for the larger microorganisms of $L = O(1 \text{mm})$, $R$ of $O(1)$ can not be eliminated from the beginning. The incompressible fluid with $R \neq 0$ has the following equation of motion:

$$
\frac{\rho}{\mu} \{ \vec{\dot{v}} + (\vec{v} \cdot \nabla) \vec{v} \} = -\frac{1}{\mu} \nabla p + \Delta \vec{v},
$$

or

$$
\epsilon \{ \nabla \times \vec{\dot{v}} + \nabla \times ((\vec{v} \cdot \nabla) \vec{v}) \} = \Delta (\nabla \times \vec{v}),
$$

with $\epsilon = \rho/\mu$. This equation reduces to eq. (B) or (F) for $R = 0$. We will use $\epsilon$ as an expansion parameter of incorporating small perturbation from the non-vanishing $R$. Expansion of $\vec{v}$ in terms of $\epsilon$, $\vec{v} = \vec{v}^{(0)} + \vec{v}^{(1)} + \cdots$, leads to

$$
\Delta (\nabla \times \vec{v}^{(0)}) = 0
$$

(40)

$$
\Delta (\nabla \times \vec{v}^{(1)}) = \epsilon \{ \nabla \times \vec{\dot{v}}^{(0)} + \nabla \times ((\vec{v}^{(0)} \cdot \nabla) \vec{v}^{(0)}) \}
$$

(41)

for $D = 3$, but for $D = 2$ they can be written as

$$
4 \partial_z \partial_z (\partial_z v^{(0)}_z - \partial_z v^{(1)}_z) = 0
$$

(42)

$$
4 \partial_z \partial_z (\partial_z v^{(0)}_z - \partial_z v^{(1)}_z) \\
= \epsilon \left\{ \partial_z \dot{v}^{(0)}_z - \partial_z \dot{v}^{(1)}_z \\
+ 2 \partial_z ((v^{(0)}_z \partial_z + v^{(0)}_z \partial_z) v^{(0)}_z) - 2 \partial_z ((v^{(0)}_z \partial_z + v^{(0)}_z \partial_z) v^{(0)}_z) \right\}.
$$

(43)

The matching condition (41) can be understood as

$$
\vec{v}^{(0)}(\vec{x} = \vec{X}(t, \xi)) = \dot{\vec{X}}(t, \xi)
$$

(44)

and

$$
\vec{v}^{(1)}(\vec{x} = \vec{X}(t, \xi)) = 0
$$

(45)

As an example we will take up the ciliated motion in the $D = 2$ fluid. Then $\vec{v}^{(0)}$ satisfying (H) and (44) reads as before

$$
v^{(0)}_z(z, \bar{z}) = v^{(-)}(z) + v^{(+))(\bar{z})} + (\bar{z}^{-1} - z) v^{(-)}(z)
$$

(46)

where

$$
v^{(-)}(z) = \frac{1}{2} \left\{ \sum_{n<1} \alpha_n z^n - \sum_{n<1, m+n<2} n \alpha_m \alpha_n z^{m+n-1} + \sum_{n<1, m<n} n \alpha_m \alpha_n z^{m-n+1} \\
+ \sum_{n \geq 1, m > n} n \alpha_m \alpha_n z^{-m+n+1} + \sum_{n < 1, m+n > 2} n \alpha_m \alpha_n z^{m-n-3} \right\},
$$

(47)

$$
v^{(+)}(\bar{z}) = \frac{1}{2} \left\{ \sum_{n \geq 1} \alpha_n \bar{z}^{-n} - \sum_{n < 1, m+n \geq 2} n \alpha_m \alpha_n \bar{z}^{-m-n+1} + \sum_{n < 1, m<n} n \alpha_m \alpha_n \bar{z}^{m-n-1} \\
+ \sum_{n \geq 1, m \leq n} n \alpha_m \alpha_n \bar{z}^{m-n-1} + \sum_{n < 1, m+n \leq 2} n \alpha_m \alpha_n \bar{z}^{m+n+3} \right\}.
$$

(48)

Substitution of $\vec{v}^{(0)}$ into eq. (41), we can find $\vec{v}^{(1)}$ satisfying eq. (45) as well as the boundary condition at spacial infinity where $\vec{v}^{(1)}$ is at least finite. This is carried out by adding properly the arbitrary solution $\vec{v}^{(1) \prime}$ satisfying the homogeneous equation $\Delta (\nabla \times \vec{v}^{(1) \prime}) = 0$. We have found, however, unwanted terms
behaving $z^2\zeta^{-2}$, $\zeta^2z^{-2}$, $\ln z$, and $\ln\zeta$ in the solution $v^{(1)}$. To eliminate these unwanted terms, we impose the following restriction on $\alpha(s, t)$:

$$\alpha_k + \sum_{n<0} n\left(\alpha_{-n+k+1}\alpha_n + \alpha_{-n+k+1}\alpha_n\right)
- (\alpha_{n+k-1}\alpha_n + \alpha_{n+k-1}\alpha_n) - (\alpha_{n-k+3}\alpha_n + \alpha_{n-k+3}\alpha_n)
+ \sum_{n>1} n(\alpha_{n+k+1}\alpha_n + \alpha_{n-k+1}\alpha_n)
+ \frac{2}{\kappa} \sum_{n<1} (n-k)(n-k)\alpha_n\alpha_{n+k+1} = 0, \quad (k = -1, -2)$$

$$\sum_{n<1} n(n+1)\alpha_n\alpha_{n-1} = 0, \quad (k < 0)$$

$$\frac{1}{(k+1)(\alpha_k\alpha_{k+1} + \alpha_{-k}\alpha_{k})} = 0, \quad (k < 0)$$

Now, we have the final solution of $v^{(1)}$. This gives the net swimming velocity $v_T^{(\text{cilia})}$ and the angular momentum gained $\psi_R^{(\text{cilia})}$ when the microorganism is swimming in the fluid with small but non-vanishing Reynolds number:

$$v_T^{(\text{cilia})} = -\dot{\alpha}_0(t)
+ \sum_{k \leq 1} k(\dot{\alpha}_k\alpha_{-k+1} - \alpha_k\dot{\alpha}_{k-1} - \alpha_k\dot{\alpha}_{k+1}) - \sum_{k>1} k\dot{\alpha}_k\alpha_{k-1}
+ \epsilon \left\{ \left( \sum_{k>1} \frac{2(k-3)}{(k-1)(k-2)} \alpha_k\alpha_{-k+3} - \sum_{k>1} \frac{2}{k-1} \dot{\alpha}_k\dot{\alpha}_{k+1}
- \sum_{k<1} \frac{2(3k+1)}{k(k-1)(k+1)} \dot{\alpha}_k\alpha_{k+1} \right)
+ \sum_{k<1} \frac{1}{k+1} (\alpha_{-k+1}\dot{\alpha}_k - \alpha_{k-1}\dot{\alpha}_k + \dot{\alpha}_k\alpha_{k-1} - \alpha_{k+1}\alpha_{k+1})
- (\alpha_{-2}\dot{\alpha}_1 + \alpha_2\dot{\alpha}_1) \right\},$$

$$\psi_R^{(\text{cilia})} = -\text{Im}\{\dot{\alpha}_1(t)
- \sum_{k \leq 1} k(\dot{\alpha}_k\alpha_{-k+2} - \alpha_k\dot{\alpha}_{k-2}) + \sum_{k>1} k\dot{\alpha}_k\alpha_{k} + \text{Im} \left\{ \sum_{k>1} \frac{2}{k-1} (\alpha_{-k+2}\dot{\alpha}_k - \dot{\alpha}_{k+2}\alpha_k) - \alpha_{-1}\dot{\alpha}_{-1} + \alpha_{-1}\dot{\alpha}_{-1} \right) + \sum_{k<1} \frac{1}{k+1} (\alpha_{-k+2}\dot{\alpha}_k - \alpha_{k+2}\dot{\alpha}_k + \alpha_{k+2}\dot{\alpha}_k - \alpha_{k+2}\dot{\alpha}_k) \right\}$$

It is interesting to note that the selection rules (17) and (18) are also valid in this case of small but non-vanishing Reynolds number.
E Collective motion of microorganisms and $N$-point correlation function

Our previous study in this problem is summarized in the following. Consider the situation where the vortices are created and annihilated, so that the probability of having the vortex distribution (field) $\omega^{\mu\nu}(x)$ is given by

$$P[\omega^{\mu\nu}(x)] \sim \exp \left\{ -\frac{1}{2\pi i\alpha'} \int d^Dx \sqrt{g(x)} \frac{1}{4} \omega^{\mu\nu}(x) \omega^{\mu\nu}(x) \right\},$$

(55)

where $i\alpha'$ is the external parameter controlling the fluctuation of the vortex distribution. [The $\alpha' \to 0$ limit corresponds to the classical limit without the fluctuation.] In this situation we should sum over all the possible configurations of the velocity fields with Eq. (57) as their probability. The probability of having $N$ microorganisms with their surfaces located at $X(1), X(2), \cdots, X(N)$, and with their time derivatives $\dot{X}(1), \dot{X}(2), \cdots, \dot{X}(N)$, is given by the following $N$-point correlation function (amplitude);

$$G_N(X(1), \dot{X}(1); \cdots; X(N), \dot{X}(N))$$

$$= \int \mathcal{D}P^{(i)} \exp \left\{ i \sum_{i=1}^{N} \int dt \int d^{D-1}\xi(i) \ P^{(i)}(\dot{X}) \right\}$$

$$\times \tilde{G}_N(X(1), P^{(1)}; \cdots; X(N), P^{(N)}),$$

(56)

where

$$\tilde{G}_N(X(1), P^{(1)}; \cdots; X(N), P^{(N)})$$

$$= \exp \left[ \frac{2\pi i\alpha'}{2} \sum_{i,j} \int dt(i) \int d^{D-1}\xi(i) \int dt(j) \int d^{D-1}\xi(j) \ P^{(i)}(t(i); \xi(i)) G_\perp^{\mu\nu} (X(i)(t(i)) - X(j)(t(j))) \ P^{(j)}(t(j); \xi(j)) \right],$$

(57)

with the Green’s function of the transverse waves. Then, we have obtained $N$-point correlation function for the collective swimming of $N$ microorganisms. It is quite similar to the $N$-point function of strings for $D = 2$ case and membranes for $D = 3$ case. It is also related to the string field theory, since the incoming and outgoing strings are not point-like, but the Reggeons.

Here we will discuss the use of $N$-point correlation function (56) in the collective swimming motion of microorganisms. If $G_N$ represents the probability of having $N$ microorganisms whose surfaces are located at $X(i)$ with velocity $\dot{X}(i)$ ($i = 1, \cdots, N$), then it can be viewed as the probability distribution of $\dot{X}_N$ of the imaginary microorganism $N$ located at spacial infinity under the given data of $X_i$ and $X_i(i = 1, \cdots, N - 1)$. Following the usual strategy, the counterflow $-\dot{X}_N$ can be indentified to the collective swimming motion of $N - 1$ microorganisms. Therefore, the averaged collective swimming motion over the fluctuation distribution is given by

$$-\langle \dot{X}_N \rangle = -\sum_{X_N} \dot{X}_N G_N(X_1, \dot{X}_1; \cdots; X_N, \dot{X}_N).$$

(58)

It is also an interesting problem to include the kinetic terms of the microorganisms themselves, or to consider the stochastic behavior in the swimming motion of the microorganisms.

Summary

In this paper, recapitulating our previous work on the swimming of microorganisms, we have given a few newly obtained results: we have obtained the closed algebra and its central extension, which is a
generalization of the $W_{1+\infty}$ algebra, controlling the swimming motion of microorganism. For the ciliated motion, we have extended our previous result to the larger microorganisms for which the Reynolds number is small but non-vanishing. We have developed the perturbation theory with respect to the Reynolds number.

Acknowledgements

This work is partly supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.06221229).

References

[1] M. Kawamura, A. Sugamoto and S. Nojiri, “Swimming of Microorganisms Viewed from String and Membrane Theories”, to appear in Modern Physics Letters A.

[2] T. Motokawa, Elephant Time and Mouse Time (in Japanese), Chuô-Kôron(1992).

[3] A. Shapere and F. Wilczek, “Geometry of self-propulsion at low Reynolds number”, J. Fluid Mech. 198, 557 (1989).

[4] V. Kac, A. Radul, Comm. Math. Phys. 157 (1993)429-457.

[5] H. Awata, M. Fukuma, S. Odake, Y.-H. Quano, “Eigensystem and Full Character Formula of the $W_{1+\infty}$ Algebra with $c = 1$”, YITP/K-1049, SULDP-1993-1, RIMS-959.

[6] Y. Matsuo, Phys. Lett. B326 (1994)95.

[7] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, “Determinant Formulae of Quasi-Finite Represetation of $W_{1+\infty}$ Algebra at Lower Levels”, YITP/K-1054, UT-669, SULDP-1994-1.

[8] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, “Character and Determinant Formulae of Quasifinite Representation of the $W_{1+\infty}$ Algebra”, YITP/K-1060, YITP/U-94-17, SULDP-1994-3.

[9] E. Frenkel, V. Kac, A. Radul, W. Wang, “$W_{1+\infty}$ and $W(gl_N)$ with central charge $N$”, hep-th-9405121.