PROPERTIES OF A TERNARY INFINITE WORD

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Abstract. We study the properties of the ternary infinite word

\[ p = 0121021010210102101021012 \cdots, \]

that is, the fixed point of the map \( h : 0 \to 01, 1 \to 21, 2 \to 0 \). We determine its factor complexity, critical exponent, and prove that it is 2-balanced. We compute its abelian complexity and determine the lengths of its bispecial factors. Finally, we give a characterization of \( p \) in terms of avoided factors.

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1. Introduction

One of the themes of combinatorics on words is the study of particular infinite words with interesting properties. For example, in one of the very earliest results in this area, Thue proved that the Thue-Morse word

\[ t = 0110100110010110 \cdots \]

avoids overlaps: factors of the form \( axaxa \) with \( a \) a single letter and \( x \) a possibly empty word \([1, 22]\). He also proved that the word

\[ vtm = 21020121012021012021 \cdots, \]

avoids squares: factors of the form \( yy \) with \( y \) nonempty.

More generally, one can study other kinds of repetitions. We say that a finite word \( w = w[1..n] \) has period \( p \geq 1 \) if \( w[i] = w[i + p] \) for \( 1 \leq i \leq n - p \). The smallest period of a word \( w \) is called the period, and we write it as \( \text{per}(w) \). The exponent of a finite word \( w \), written \( \text{exp}(w) \) is defined to be \( |w|/\text{per}(w) \). We say a word (finite or infinite) is \( \alpha \)-free if the exponent of all its nonempty factors is \( > \alpha \). We say a word is \( \alpha^+ \)-free if the exponent
of all its nonempty factors is $\geq \alpha$. The critical exponent of a finite or infinite word $x$ is the supremum, over all nonempty finite factors $w$ of $x$, of $\exp(w)$; it is written $\text{ce}(x)$. The critical exponent of a word can be either rational or irrational. If it is rational, then it can either be attained by a particular finite factor, or not attained. For example, the critical exponent of both $t$ and $vtm$ is 2, but it is attained in the former case and not attained in the latter. If the critical exponent $\alpha$ is attained, we typically write it as $\alpha^+$. An overlap is a $2^+$ power, so the Thue-Morse word is $2^+$-free.

The Fibonacci word $f = 0100100101001001010 \cdots$ is the fixed point of the morphism $0 \rightarrow 01$, $1 \rightarrow 0$. Karhumäki proved [11] that $f$ has no fourth powers (i.e., blocks of the form $xxxx$, with $x$ nonempty), and Mignosi and Pirillo [12] proved that the critical exponent of $f$ is $(5 + \sqrt{5})/2$.

Another aspect of infinite words that has been studied is balance. We say that a finite or infinite word $x$ is $t$-balanced if for all equal-length factors $y, z$ of $x$, and all letters $a$, the inequality $||y|_a - |z|_a| \leq t$ is satisfied. As is well-known, the Fibonacci word $f$ (and more generally, every Sturmian word) is 1-balanced [5, 13].

A third aspect is factor complexity, also called subword complexity. For an infinite word $x$, the factor complexity function $\rho_x(n)$ counts the number of distinct factors of length $n$ in $x$. Morse and Hedlund [13] proved that $\rho_f(n) = n + 1$ for all $n \geq 0$. There is also the abelian analogue of factor complexity, where we count two factors as the same if they are permutations of each other [17].

In this note we study various aspects of the infinite ternary word $p = 012102101021012101021012 \cdots$, fixed point of the map $h$ sending $0 \rightarrow 01$, $1 \rightarrow 21$, $2 \rightarrow 0$. This word is not automatic (because, as we will see, letters occur with irrational densities). It is not Sturmian (because it is over a 3-letter alphabet). Neither is it episturmian [10], because its set of subwords is not closed under reversal: $p$ contains 02, but avoids 20.

We determine its factor complexity, its critical exponent, and prove that it is 2-balanced. A novel aspect of our work is that much of it is carried out using the Walnut theorem-prover [14, 20]. This software tool can prove or disprove assertions phrased in first-order logic about automatic sequences and their generalizations. These ideas were used previously to study the Tribonacci word [16], but the word $p$ provides some new complications. All the files required to carry out the computations are available at the last author’s website:

https://cs.uwaterloo.ca/~shallit/papers.html.

The word $p$ has been studied previously. For example, it is sequence A287072 in the OEIS. It also appears implicitly in [3], where it is (up to renaming of the letters) the fixed point of the morphism $c_2c_1$. A very similar word is the word $G$ studied in [21, Sect. 3.2]. Indeed, the factors of $p$ can be obtained from the factors of $G$ by reversal and applying the permutation (012). So $G$ and $p$ have the same critical exponent.

The results in this paper are applied to an avoidability problem in the companion paper [6].

2. A Pisot numeration system

The properties of $p$ are intimately related to a particular numeration system $P4$, which we discuss now. Consider the following linear recurrence:

$$X_1 = 1, X_2 = 2, X_3 = 4, X_4 = 7, \text{ and } X_n = X_{n-1} + X_{n-2} + X_{n-4} \text{ for } n \geq 0.$$ 

Table 1 gives the first few terms of this recurrence: This is sequence A005251 in the OEIS. It is easy to verify it also satisfies the simpler recurrence $X_n = 2X_{n-1} - X_{n-2} + X_{n-3}$ for $n \geq 4$. 


Table 1. The recurrence $X_n$.

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $X_n$ | 1  | 2  | 4  | 7  | 12 | 21 | 37 | 65 | 114| 200| 351| 616| 1081| 1897|

Table 2. Representations in the $P_4$ numeration system.

| $n$ | $(n)_P$ | $n$ | $(n)_P$ |
|-----|---------|-----|---------|
| 1   | 1       | 13  | 10001   |
| 2   | 10      | 14  | 10010   |
| 3   | 11      | 15  | 10011   |
| 4   | 100     | 16  | 10100   |
| 5   | 101     | 17  | 10101   |
| 6   | 110     | 18  | 10110   |
| 7   | 1000    | 19  | 11000   |
| 8   | 1001    | 20  | 11001   |
| 9   | 1010    | 21  | 100000  |
| 10  | 1011    | 22  | 100001  |
| 11  | 1100    | 23  | 100010  |
| 12  | 10000   | 24  | 100011  |

Figure 1. Automaton recognizing the valid representations.

We can consider representing natural numbers as a sum of the $X_i$, as follows: $N = \sum_{1 \leq i \leq t} e_i X_i$, where $e_i \in \{0, 1\}$. If we impose the following two rules on such a representation, namely

(a) $(e_i, e_{i+1}, e_{i+2}, e_{i+3}) \neq (1, 0, 1, 1)$; and
(b) $(e_i, e_{i+1}, e_{i+2}) \neq (1, 1, 1)$.

then this representation is unique, and can be written as a binary string $(N)_P := e_t e_{t-1} \cdots e_1$. For example, Table 2 gives the first few representations of numbers in this numeration system:

These representations are, in fact, the ones resulting by applying the greedy algorithm, and the conditions (a) and (b) follow from a theorem of Fraenkel [8].

A representation in this system $P_4$ is valid if and only if it contains no occurrence of 111 or 1101. It follows that the language of all valid representations is recognized by the automaton in Figure 1.

The characteristic polynomial of the recurrence is $X^4 - X^3 - X^2 - 1 = (X + 1)(X^3 - 2X^2 + X - 1)$. The second term has one real zero, namely

$$\beta_1 = \left(\frac{100 + 12\sqrt{69}}{6}\right)^{1/3} + \frac{2}{3(100 + 12\sqrt{69})^{1/3}} + 2/3 \approx 1.7548776662466927600495,$$

and two imaginary zeros that lie inside the unit circle. Therefore $\beta_1$ is a Pisot number, and so, by the results in [2, 9] we know that there is a finite automaton $A$ recognizing the addition relation $x + y = z$, where $x, y, z$
are represented in the numeration system described above, with inputs in parallel and the shorter padded with leading zeros. Furthermore, there is an algorithm to compute $A$ for any Pisot number.

Instead of applying this algorithm to compute $A$, we took a different approach. Namely, we “guessed” the adder using the Myhill-Nerode theorem, and then verified it using an incrementer constructed and verified by hand. This incrementer computes the relation $y = x + 1$, and is illustrated in Figure 2. Here the inputs are $x$ and $y$ in parallel, both represented in the numeration system $P4$.

Once we have the incrementer, the correctness of the adder can be verified as done in [15]. The adder has 64 states.

It turns out that the word $p$ is automatic in this numeration system; this means it is computed by a deterministic finite automaton with output (DFAO) taking the $P4$ representation of $n$ as input and outputting (in the last state reached) the value of $p[n]$. The DFAO computing it is depicted in Figure 3.

This can be verified as follows. Let $h: 0 \rightarrow 01, 1 \rightarrow 21, 2 \rightarrow 0$, and let the automaton be represented by a morphism $\varphi : \Sigma \rightarrow \Sigma^*$ encoding the transitions of the automaton, and a coding $\tau : \Sigma \rightarrow \Delta$ giving the outputs for each state. Here the elements of $\Sigma$ correspond to the state numbers of the automaton in Figure 3, except that we code the states $10, 11, \ldots, 19$ by the capital letters $A, B, \ldots, J$. Then we need to verify the following identities:

\[
\begin{align*}
\tau(\varphi^n(0)) &= h^n(0) \\
\tau(\varphi^n(1)) &= h^n(1) \\
\tau(\varphi^n(23)) &= h^n(21) \\
\tau(\varphi^n(45)) &= h^n(21)
\end{align*}
\]
\[\tau(\varphi^n(78)) = h^n(21)\]
\[\tau(\varphi^n(BC)) = h^n(10)\]
\[\tau(\varphi^n(4FG8)) = h^n(2101)\]
\[\tau(\varphi^n(JA)) = h^n(12)\]

\[\tau(\varphi^n(9A)) = h^n(02)\]
\[\tau(\varphi^n(2DE3)) = h^n(2101)\]
\[\tau(\varphi^n(HIA)) = h^n(012)\]
\[\tau(\varphi^n(6)) = h^n(2)\]

where \(A = 10, B = 11\), etc. This can be done by a tedious induction on \(n\), which we omit. Just to demonstrate one needed identity:

\[\tau(\varphi^n(9A)) = \tau(\varphi^{n-1}(0BC))\]
\[= \tau(\varphi^{n-1}(0))\tau(\varphi^{n-1}(BC))\]
\[= h^{n-1}(0)h^{n-1}(10)\]
\[= h^{n-1}(010)\]
\[= h^n(02),\]

as desired.

3. Factor Complexity of \(p\)

In this section we prove that the factor complexity of \(p\) is \(2n + 1\). This is also a consequence of more general results of [3]. Also see [4].

There is a well-established computational method for determining factor complexity, as discussed in [18]. The first step is to create an automaton that, given integers \(i, j, n\) as input, decides if \(p[i..i+n-1] = p[j..j+n-1]\). Normally we would do this with the following \texttt{Walnut} command, where PI is a file containing the automaton in Figure 3:

\texttt{def pisotfaceq "\texttt{?msd_pisot4 At (t<n) => PI[i+t]=PI[j+t]}":}

However, in this case, the attempt fails. \texttt{Walnut} tries to determinize an automaton with 37351 states, and fails even with 5 Terabytes of storage and many hours of computation, so we need a different approach.

Instead, we use the approach discussed in [20, Sect. 6.3]. We “guess” an automaton for “pisotfaceq” using the Myhill-Nerode theorem. States are labeled with prefixes of Pisot representations. Two states are guessed to be the same if all suffixes of length \(\leq 5\) (representing \(1 + 8^2 + \cdots + 8^5 = 37449\) words) give the same result. This gives us an automaton with 1080 states that we can represent as a file called \texttt{pisi.txt}.

Next, we can use induction and \texttt{Walnut} together to prove that our guess is correct. We do this as follows:

\texttt{eval zeros "\texttt{?msd_pisot4 Ai,j $pisi(i,j,0)}":}
\texttt{eval induc "\texttt{?msd_pisot4 Ai,j,n ($pisi(i,j,n) & PI[i+n]=PI[j+n])}

=> $pisi(i,j,n+1)"":}

Both of these return \texttt{TRUE}, so our guess is correct.

Next, we create a linear representation \((v, \zeta, w)\) for the subword complexity function, using the following \texttt{Walnut} command:

\texttt{eval pisotsc n "\texttt{?msd_pisot4 Aj j<i => ~$pisi(i,j,n)}":}

This gives us a linear representation of rank 131. When we minimize it using a Maple program, we get this linear representation of rank 16:

\[v = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]\]
Theorem 3.1. The factor complexity of \( p \) is \( 2n + 1 \).

Proof. It remains to show that the linear representation \((v, \zeta, w)\) computes the function \( 2n + 1 \). To do this, we compute a linear representation for \( 2n + 1 \) using the following \texttt{Walnut} command:

\[
\text{eval c2n1 n "?msd_pisot4 i<2*n+1":}
\]

This gives us a linear representation \((v', \zeta', w')\) of rank 108. When we minimize it, we get exactly the same linear representation \((v, \zeta, w)\). This shows that the subword complexity is indeed \( 2n + 1 \).

4. Critical exponent of \( p \)

In this section we determine the critical exponent of \( p \).

Theorem 4.1. The critical exponent of \( p \) is

\[
\gamma + 1 = (8 - \beta_1 + 2\beta_2^2)/5 \approx 2.480862716147236962394265321.
\]

Here \( \gamma + 1 \) is the real zero of \( 5X^3 - 26X^2 + 43X - 23 \).

Proof. We use the strategy previously employed for the Tribonacci word [16].

Now that we have a DFAO for \( p \), we can find all the periods corresponding to overlaps. We do this with the following \texttt{Walnut} command.

\[
\text{def pisotlargepow "?msd_pisot4 Ei (n>=1) & At (t>i & t<=i+n) => PI[t]=PI[t+n]:"}
\]

This gives the automaton depicted in Figure 4. Constructing this automaton involved determinizing an NFA with 10,859 states and minimizing a DFA with 13,114,119 states; the calculation needed 90 Gigs of storage and 1100 seconds of CPU time.

As you can see from inspecting it, the periods are given by the representations 1010* and 10000*. We now compute the maximal repetitions in \( p \) by looking only at the periods given above:
This gives the automaton in Figure 5. It recognizes all pairs \((n,p)\) such that \(n/p+1\) is a maximal power in \(p\) with period \(p\), for the \(p\) given above. By inspection we see these are as follows:

\[
\begin{align*}
[1, 1][1, 0][0, 1] \\
[1, 1][0, 0][1, 0][0, 0][1, 0]^*[0, 0] \\
[1, 1][0, 0][1, 0][0, 0][1, 0]^*[0, 0][0, 0] \\
[1, 0][0, 1][0, 0][0, 1][(1, 0)[0, 0][0, 0][0, 0]]^*[0, 0] \\
[1, 0][0, 1][0, 0][0, 1][(1, 0)[0, 0][0, 0][0, 0]]^*[0, 0][1, 0] \\
[1, 0][0, 1][0, 0][0, 1][(1, 0)[0, 0][0, 0][0, 0]]^*[0, 0][1, 0][1, 0] \\
[1, 0][0, 1][0, 0][0, 1][(1, 0)[0, 0][0, 0][0, 0]]^*[0, 0][1, 0][1, 0][0, 0]
\end{align*}
\]

These correspond to, respectively, exponents of

\[
\frac{6/5}{(\sum_{1 \leq i \leq n+2} X_{2i})/X_{2n+4}, \quad n \geq 0} \]

\[
(\sum_{1 \leq i \leq n+2} X_{2i+1})/X_{2n+5}, \quad n \geq 0
\]

\[
(\sum_{1 \leq i \leq n+1} X_{4i+1})/(X_{4n+4} + X_{4n+2}), \quad n \geq 0
\]

\[
(1 + \sum_{1 \leq i \leq n+1} X_{4i+2})/(X_{4n+5} + X_{4n+3}), \quad n \geq 0
\]

\[
(3 + \sum_{1 \leq i \leq n+1} X_{4i+3})/(X_{4n+6} + X_{4n+4}), \quad n \geq 0
\]

\[
(6 + \sum_{1 \leq i \leq n+1} X_{4i+4})/(X_{4n+7} + X_{4n+5}), \quad n \geq 0.
\]
It now remains to check that these expressions are all less than $\gamma$ and the last six tend to $\gamma$ from below. We explain how to do this for the second expression; the others are similar.

First, by the standard theory of linear recurrences (see, e.g., [7]), we know that

$$X_n = \alpha_1 \beta_1^n + \alpha_2 \beta_2^n + \alpha_3 \beta_3^n,$$

where $\beta_1, \beta_2, \beta_3$ are the zeros of $X^3 - 2X^2 + X - 1$, and $\beta_1 = 1.7548776662466927600495$ is the unique real zero. By solving the appropriate linear system, we find that $\alpha_1 = (\beta_2^2 + 6\beta_1 + 3)/23$ and $\alpha_2 + \alpha_3 = (20 - \beta_1^2 - 6\beta_1)/23 = 0.2778558169688758856 \cdots < 1$. Furthermore, $|\alpha_2| + |\alpha_3| = ((8\beta_1^2 - 20\beta_1 + 16)/23)^{1/2} < 1/2$. Since $|\beta_2| = |\beta_3|$ and the product of the zeros is equal to the constant term of the defining polynomial, which is 1, we get $|\beta_2|^2 \beta_1 = 1$, which gives $|\beta_2| = \sqrt{1/\beta_1} = \beta_1 - 1$ by the defining equation. It follows that

$$|X_n - \alpha_1 \beta_1^n| = |\alpha_2 \beta_2^n + \alpha_3 \beta_3^n| \leq (|\alpha_2| + |\alpha_3|)(\beta_1 - 1)^n < (\beta_1 - 1)^n.$$

Next, one can prove by induction (or using Walnut!) that

$$\sum_{1 \leq i \leq n} X_{2i} = \frac{3X_{2n} - X_{2n+1} + 2X_{2n+2} - 6}{5}.$$

It follows that

$$\sum_{1 \leq i \leq n} X_{2i} \leq \frac{3X_{2n} - X_{2n+1} + 2X_{2n+2} - 6}{5} \leq \frac{3 - \beta_1 + 2\beta_2}{5}(\alpha_1 \beta_1^{2n}) + 6(\beta_1 - 1)^{2n} - 6/5.$$

Dividing by $X_{2n}$, we see that the quotient tends to $\gamma$ from below, where $\gamma = \frac{3 - \beta_1 + 2\beta_2}{5}$. Then $\gamma + 1$ is the desired critical exponent. \qed

### 5. Synchronization and Balance

We now show that the functions $c_i : n \rightarrow |p[i..i + n - 1]|$ are synchronized in this numeration system, for $i \in \{0, 1, 2\}$. This means that there exists an automaton $A_i$ taking $n$ and $x$ as inputs (in $P4$ representation) and accepting if $x = c_i(n)$, for $i = 0, 1, 2$. For more information about synchronization, see [19].

To do so, we “guess” the automata for the $c_i$ using the Myhill-Nerode theorem and then verify our guesses using Walnut. Here is the code for verification, where $\text{psynch0}$, $\text{psynch1}$, $\text{psynch2}$ are the guessed automata:

```wsl
eval tmp0 "$\text{msd_pisot4 An,x ((}$\text{psynch0(n,x) \& PI[n]=00}$) => $\text{psynch0(n+1,x+1)}$)
 & ((}$\text{psynch0(n,x) \& PI[n]!=00}$) => $\text{psynch0(n+1,x+1)}$)"
& (}$\text{psynch1(n,x) \& PI[n]=01}$) => $\text{psynch1(n+1,x+1)}$)
 & ((}$\text{psynch1(n,x) \& PI[n]!=01}$) => $\text{psynch1(n+1,x+1)}$)"
& (}$\text{psynch2(n,x) \& PI[n]=02}$) => $\text{psynch2(n+1,x+1)}$)
 & ((}$\text{psynch2(n,x) \& PI[n]!=02}$) => $\text{psynch2(n+1,x+1)}$)"
and all of these return TRUE.

Next, we create synchronized automata for $|p[i..i + n - 1]|$:

def pcount0 "$\text{msd_pisot4 Ex,y}$ $\text{psynch0(i,x) \& psynch0(i+n,y) \& y=x+z}$":
 # 1687 states
def pcount1 "$\text{msd_pisot4 Ex,y}$ $\text{psynch1(i,x) \& psynch1(i+n,y) \& y=x+z}$":
 # 2626 states
def pcount2 "?msd_pisot4 Ex,y $psynch2(i,x) & $psynch2(i+n,y) & y=x+z":  
# 2773 states

We can now prove the following result.

**Theorem 5.1.** The word $p$ is 2-balanced.

*Proof.* We use the following Walnut commands.

def twobalanced0 "?msd_pisot4 Ai,j,n,x,y ($pcount0(i,n,x) & $pcount0(j,n,y)) => (y<=x+2 & x<=y+2)";

def twobalanced1 "?msd_pisot4 Ai,j,n,x,y ($pcount1(i,n,x) & $pcount1(j,n,y)) => (y<=x+2 & x<=y+2)";

def twobalanced2 "?msd_pisot4 Ai,j,n,x,y ($pcount2(i,n,x) & $pcount2(j,n,y)) => (y<=x+2 & x<=y+2)";

Walnut returns TRUE for all of these.

### 6. Abelian complexity

We can compute the abelian complexity of $p$ with Walnut in much the same way that it was done for the Tribonacci word in [16], with some minor modifications. For each $n$ and $i \in \{0, 1, 2\}$, we compute the vector $u_i(n) = \min_x |x|_i$, where the minimum is over all the length-$n$ factors of $p$.

def min0 "?msd_pisot4 Ei $pcount0(i,n,x) & Aj,y $pcount0(j,n,y) => y>=x":  
# 169 states

def min1 "?msd_pisot4 Ei $pcount1(i,n,x) & Aj,y $pcount1(j,n,y) => y>=x":  
# 169 states

def min2 "?msd_pisot4 Ei $pcount2(i,n,x) & Aj,y $pcount2(j,n,y) => y>=x":  
# 223 states

Once we have this, we can show that $\psi(p[i..i+n-1]) - (u_0(n), u_1(n), u_2(n)) \in \{(0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (1, 0, 0), (1, 0, 1), (1, 0, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (2, 0, 0), (2, 0, 1), (2, 1, 0)\}$

for $i \geq 0$ and $n \geq 1$, as follows:

def validtriples "?msd_pisot4 Ei,n,a,b,c $pcount0(i,n,a+x) & $min0(n,a) & $pcount1(i,n,b+y) & $min1(n,b) & $pcount2(i,n,c+z) & $min2(n,c)":

Next, we can show that

$\{\psi(p[i..i+n-1]) - (u_0(n), u_1(n), u_2(n)) : i \geq 0\}$

is one of the following 18 possible sets:

$S_1 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$
$S_2 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$
$S_3 = \{(0, 1, 1), (1, 0, 1), (2, 0, 0)\}$
$S_4 = \{(0, 0, 2), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$
$S_5 = \{ (0,1,1),(0,2,0),(1,0,1),(1,1,0) \}$
$S_6 = \{ (0,0,2),(0,1,1),(1,0,1),(1,1,0),(2,0,0) \}$
$S_7 = \{ (0,1,2),(1,0,2),(1,1,1),(2,0,1),(2,1,0) \}$
$S_8 = \{ (0,2,1),(1,1,1),(1,2,0),(2,0,1),(2,1,0) \}$
$S_9 = \{ (0,1,1),(0,2,0),(1,0,1),(1,1,0),(2,0,0) \}$
$S_{10} = \{ (0,0,2),(0,1,1),(0,2,0),(1,0,1),(1,1,0) \}$
$S_{11} = \{ (0,1,2),(0,2,1),(1,0,2),(1,1,1),(1,2,0) \}$
$S_{12} = \{ (0,1,2),(0,2,1),(1,1,1),(1,2,0),(2,0,1),(2,1,0) \}$
$S_{13} = \{ (0,1,2),(0,2,1),(1,0,2),(1,1,1),(2,0,1),(2,1,0) \}$
$S_{14} = \{ (0,1,2),(0,2,1),(1,0,2),(1,1,1),(1,2,0),(2,0,1),(2,1,0) \}$
$S_{15} = \{ (0,1,2),(1,0,2),(1,1,1),(1,2,0),(2,0,1),(2,1,0) \}$
$S_{16} = \{ (0,2,1),(1,0,2),(1,1,1),(1,2,0),(2,0,1),(2,1,0) \}$
$S_{17} = \{ (0,1,2),(0,2,1),(1,0,2),(1,1,1),(1,2,0),(2,0,1) \}$
$S_{18} = \{ (0,1,2),(0,2,1),(1,0,2),(1,1,1),(1,2,0),(2,0,1),(2,1,0) \}$

**Theorem 6.1.** There is a P4-automaton of 144 states that, on input a P4 representation of $n$, computes the number of distinct length-$n$ factors of $p$, up to abelian equivalence.

**Proof.** To create the automaton, use the following Walnut code:

```walnut
def a001 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x) & $min0(n,x) & $pcount1(i,n,y) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)":
  #6 states

def a002 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x) & $min0(n,x) & $pcount1(i,n,y) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)":
  #125 states

def a010 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x) & $min0(n,x) & $pcount1(i,n,y+1) & $min1(n,y) & $pcount2(i,n,z) & $min2(n,z)":
  #6 states

def a011 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x) & $min0(n,x) & $pcount1(i,n,y+1) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)":
  #132 states

def a012 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x) & $min0(n,x) & $pcount1(i,n,y+1) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)":
  #129 states

def a020 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x) & $min0(n,x) & $pcount1(i,n,y+2) & $min1(n,y) & $pcount2(i,n,z) & $min2(n,z)":
  #126 states

def a021 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x) & $min0(n,x) & $pcount1(i,n,y+2) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)":
  #131 states
```


def a100 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+1) & $min0(n,x) &
$pcount1(i,n,y) & $min1(n,y) & $pcount2(i,n,z) & $min2(n,z)";
# 6 states

def a101 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+1) & $min0(n,x) &
$pcount1(i,n,y) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)";
# 132 states

def a102 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+1) & $min0(n,x) &
$pcount1(i,n,y) & $min1(n,y) & $pcount2(i,n,z+2) & $min2(n,z)";
# 127 states

def a110 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+1) & $min0(n,x) &
$pcount1(i,n,y+1) & $min1(n,y) & $pcount2(i,n,z) & $min2(n,z)";
# 132 states

def a111 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+1) & $min0(n,x) &
$pcount1(i,n,y+1) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)";
# 131 states

def a120 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+1) & $min0(n,x) &
$pcount1(i,n,y+2) & $min1(n,y) & $pcount2(i,n,z) & $min2(n,z)";
# 134 states

def a200 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+2) & $min0(n,x) &
$pcount1(i,n,y) & $min1(n,y) & $pcount2(i,n,z) & $min2(n,z)";
# 110 states

def a201 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+2) & $min0(n,x) &
$pcount1(i,n,y) & $min1(n,y) & $pcount2(i,n,z+1) & $min2(n,z)";
# 131 states

def a210 "?msd_pisot4 Ei,x,y,z $pcount0(i,n,x+2) & $min0(n,x) &
$pcount1(i,n,y+1) & $min1(n,y) & $pcount2(i,n,z) & $min2(n,z)";
# 127 states

def num1 "?msd_pisot4 $a001(n) & ~$a002(n) & $a010(n) & ~$a011(n) & ~$a012(n) &
~$a020(n) & ~$a021(n) & $a100(n) & ~$a101(n) & ~$a102(n) & ~$a110(n) &
~$a111(n) & ~$a120(n) & ~$a200(n) & ~$a201(n) & ~$a210(n)";

def num2 "?msd_pisot4 ~$a001(n) & ~$a002(n) & ~$a010(n) & $a011(n) & ~$a012(n) &
~$a020(n) & ~$a021(n) & ~$a100(n) & $a101(n) & ~$a102(n) & $a110(n) &
~$a111(n) & ~$a120(n) & ~$a200(n) & ~$a201(n) & ~$a210(n)";

def num3 "?msd_pisot4 ~$a001(n) & ~$a002(n) & ~$a010(n) & $a011(n) & ~$a012(n) &
~$a020(n) & ~$a021(n) & ~$a100(n) & ~$a101(n) & ~$a102(n) & $a110(n) &
~$a111(n) & ~$a120(n) & $a200(n) & ~$a201(n) & ~$a210(n)";

def num4 "?msd_pisot4 ~$a001(n) & ~$a002(n) & ~$a010(n) & $a011(n) & ~$a012(n) &
```python
def num5 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land a020(n) \land \sim a021(n) \land \sim a100(n) \land a101(n) \land \sim a102(n) \land a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num6 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land a101(n) \land \sim a102(n) \land \sim a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num7 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land a101(n) \land \sim a102(n) \land \sim a110(n) \land a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num8 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land a101(n) \land \sim a102(n) \land a110(n) \land \sim a111(n) \land \sim a120(n) \land a200(n) \land \sim a201(n) \land \sim a210(n)";

def num9 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land \sim a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land \sim a101(n) \land \sim a102(n) \land a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num10 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land a101(n) \land \sim a102(n) \land \sim a110(n) \land \sim a111(n) \land \sim a120(n) \land a200(n) \land \sim a201(n) \land \sim a210(n)";

def num11 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land \sim a101(n) \land \sim a102(n) \land \sim a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num12 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land \sim a101(n) \land \sim a102(n) \land a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num13 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land \sim a101(n) \land \sim a102(n) \land \sim a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num14 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land \sim a101(n) \land \sim a102(n) \land \sim a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num15 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land \sim a101(n) \land \sim a102(n) \land \sim a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";

def num16 "\text{\texttt{msd\_pisot4}} \sim a001(n) \land \sim a002(n) \land \sim a010(n) \land \sim a011(n) \land \sim a012(n) \land \sim a020(n) \land \sim a021(n) \land \sim a100(n) \land \sim a101(n) \land \sim a102(n) \land \sim a110(n) \land \sim a111(n) \land \sim a120(n) \land \sim a200(n) \land \sim a201(n) \land \sim a210(n)";
```
Corollary 6.2. The abelian complexity of \( p \), for \( n \geq 1 \), lies in \( \{3, 4, 5, 6, 7\} \). Each possibility occurs infinitely often.

7. Palindromes

Theorem 7.1. The only palindromes occurring in \( p \) are

\[ \{0, 1, 2, 121, 101, 0101, 01210, 21012, 1012101\} \]

Proof. It suffices to list all the factors of length \( \leq 9 \), since any longer palindrome would have these as factors. We can be sure we have examined all of them, by Theorem 3.1.

8. Bispecial factors

We say a factor \( w \) of an infinite word \( x \) is right-special (resp., left-special) if there exist two distinct letters \( a, b \) such that \( wa \) and \( wb \) (resp., \( aw \) and \( bw \)) are both factors of \( x \). A word \( w \) is bispecial if it is both right- and left-special.

We can determine the lengths of bispecial factors occurring in \( p \).

The result is displayed in Figure 6.

Theorem 8.1. The word \( p \) has a bispecial factor of length \( n \) iff

\[ (n)_P \in \{1, 11, 110\} \cup (10)^+\{e, 0\} \cup (1000)^+\{0, 01, 011, 0110\} \]
9. LETTER DENSITY

In this section we obtain the letter densities of $p$. We know from [3] that the densities exist, and therefore it suffices to determine them on prefixes of the form $h^n(0)$.

**Theorem 9.1.**

- The density of 0 is $1/\beta_1^2 \approx 0.324717957244746$;
- The density of 1 is $1/\beta_1^2 + 1/\beta_1^4 \approx 0.430159709001946734$;
- The density of 2 is $1/\beta_1^3 + 1/\beta_1^5 \approx 0.2451223337533$.

**Proof.** An easy induction gives $\psi(|h^n(0)|) = (X_{n-1}, X_{n-1} + X_{n-2}, X_{n-2} + X_{n-4})$, from which the desired result follows immediately. \qed

**Remark 9.2.** These densities were also given in [4] for a slightly different morphism with the same incidence matrix.

10. RECURRENCE AND APPEARANCE

Let $x$ be an infinite word. The recurrence function $R(n)$ is defined to be the smallest positive integer $m$ such that every occurrence of a length-$n$ factor $x$ is followed by another occurrence of the same word at distance at most $R(n)$. We now compute it for the word $p$.

**Theorem 10.1.** Define the sequences $(B_i)$ and $(C_i)$ as follows:

\[
B_{2i+3} = [1(01)^i10]_p
\]
\[
B_{2i+4} = [1(01)^i100]_p
\]
\[
C_{4i+2} = [10(0010)^i]_p
\]
\[
C_{4i+3} = [10(0010)^i0]_p
\]
\[
C_{4i+4} = [10(0010)^i00]_p
\]
\[
C_{4i+5} = [10(0010)^i0001]_p
\]


for \( i \geq 0 \). Then

\[
R(n) = \begin{cases} 
5, & \text{if } n = 1; \\
12, & \text{if } n = 2; \\
16, & \text{if } n = 3; \\
21, & \text{if } n = 4; \\
28, & \text{if } n = 5; \\
X_{i+4}, & \text{if } B_i \leq n \leq C_{i+1} \text{ for } i \geq 3; \\
X_{i+4} + X_{i+2}, & \text{if } C_{i+1} < n < B_{i+1} \text{ for } i \geq 3. 
\end{cases}
\]

**Proof.** We use the following Walnut code to compute \( R \).

```walnut
def pocc  "?msd_pisot4 Ai Ej j>i & j<=i+x & $pisi(i,j,n)";
def prec  "?msd_pisot4 $poccur(n,x) & ~$poccur(n,x-1)";
reg pix msd_pisot4 "0*10*";
def precB "?msd_pisot4 Ex $pix(x) & $prec(n,x) & ~$prec(n-1,x)";
def precC "?msd_pisot4 Ex $pix(x) & $prec(n,x) & ~$prec(n+1,x)"
```

**Corollary 10.2.** We have \( R(n) \leq \xi n \) for \( \xi = (2\beta_1^2 - \beta_1 + 2) \doteq 6.40431358 \).

The appearance function \( A(n) \) is defined to be the smallest positive integer \( m \) such that every occurrence of a length-\( n \) factor \( x \) begins at a position \( \leq m \). We now compute it for the word \( p \).

**Theorem 10.3.** Define the sequences \( (B_n) \) and \( (C_n) \) as in Theorem 10.1. We have

\[
A(n) = \begin{cases} 
2, & \text{if } n = 1; \\
4, & \text{if } n = 2; \\
7, & \text{if } n = 3; \\
11, & \text{if } n = 4; \\
13, & \text{if } n = 5; \\
X_{2i+4} - 1, & \text{if } B_{2i+1} < n \leq C_{2i+2} \text{ for } i \geq 1; \\
C_{2i+4}, & \text{if } C_{2i+2} < n < B_{2i+2} \text{ for } i \geq 1; \\
X_{2i+5} - 1, & \text{if } B_{2i+2} \leq n \leq C_{2i+3} \text{ for } i \geq 1; \\
C_{2i+5}, & \text{if } C_{2i+3} < n \leq B_{2i+3} \text{ for } i \geq 1. 
\end{cases}
\]

**Corollary 10.4.** We have \( A(n) \leq \zeta n \) for \( \zeta = (2\beta_1^2 - 2\beta_1 + 1) \doteq 3.6494359 \).

11. **Characterization of factors**

In this section, we give a useful definition of \( p \) by forbidden factors. We thank Lucas Mol for his work on this result.

**Theorem 11.1.** Every bi-infinite ternary cube-free word avoiding

\[
F = \{00, 11, 22, 20, 212, 0101, 02102, 121012, 01021010, 21021012102\}
\]

has the same set of factors as \( p \).
Proof. First, we check that \( p \) is cube-free by Theorem 4.1 and contains no factor in \( F \).

Let \( w \) be a bi-infinite ternary cube-free word avoiding \( F \). Since \( w \) avoids \( \{00, 11, 22, 20, 212\} \), the only factors of \( w \) of the form \( 0z0 \) with \( z \in \{1, 2\} \) are 010, 01210, 0210. So \( w \in \{01, 0121, 021\}^{\omega} \) and thus \( w \in \{01, 21, 0\}^{\omega} \).

So we write \( w = h(v) \) where \( h \) is the morphism \( 0 \rightarrow 01, 1 \rightarrow 21, 2 \rightarrow 0 \).

Now it suffices to show that \( L \) contains \( v \) too. Since \( w \) is cube-free, its pre-image \( v \) is also cube-free.

To show that \( v \) avoids \( F \), we consider every \( f \in F \) and we show by contradiction that \( f \) is not a factor of \( v \).

(a) if \( v \) contains 00, then \( h(00) = 0101 \in F \).
(b) if \( v \) contains 11, then \( h(11) = 2121 \in F \).
(c) if \( v \) contains 22, then \( h(22) = 00 \in F \).
(d) if \( v \) contains 20, then \( h(20) = 001 \in F \).
(e) if \( v \) contains 212, then \( v \) contains 212 by (c) and (d).
\[ h(212) = 021021 \in F. \]
(f) if \( v \) contains 0101, then \( h(0101) = 01210121 \in F \).
(g) if \( v \) contains 02102, then \( h(02102) = 01021010 \in F \).
(h) if \( v \) contains 121012, then \( v \) contains 121012 by (c) and (d).
\[ h(121012) = 21021012102 \in F. \]
(i) if \( v \) contains 01021010, then \( v \) contains 010210102 by (a) and (f).
\[ v \text{ contains 010210102 by (a) and (f).} \]
\[ v \text{ contains 010210102 by (a) and (f).} \]
\[ v \text{ contains 1010210102101 by (a) and (f).} \]
\[ v \text{ contains 21010210102101 by (f) and (b).} \]
\[ v \text{ contains 21010210102101 by (f) and (b).} \]
\[ v \text{ contains 1210102101021012 by (c) and to avoid (21010)^3.} \]
\[ v \text{ contains 1210102101021012 by (c) and to avoid (02101)^3.} \]
\[ h(1210102101021012) = 21021012101210121012101210 = 2(102101210)^3. \]
(j) if \( v \) contains 21021012102, then \( v \) contains 121021012102 by (g) and (c).
\[ v \text{ contains 121021012102 by (g) and (c).} \]
\[ v \text{ contains 0121021012102 by (a) and (e).} \]
\[ v \text{ contains 102101210121012 by (a) and (f).} \]
\[ v \text{ contains 210102101021012 by (f) and (b).} \]
\[ v \text{ contains 210102101021012 by (f) and (b).} \]
\[ v \text{ contains 10210121012101210 by (h) and (c).} \]
\[ v \text{ contains 210102101021012 by (a) and (e).} \]
\[ v \text{ contains 1021012101021012 by (a) and (e).} \]
\[ v \text{ contains 01210210121021010 by (h) and (c).} \]
\[ v \text{ contains 210102101021012 by (a) and (e).} \]
\[ v \text{ contains 01210210121021010 by (h) and (c).} \]
\[ v \text{ contains 210102101021012 by (a) and (e).} \]
\[ v \text{ contains 01210210121021010 by (h) and to avoid (02101)^3.} \]
\[ h(01210210121021010) = 2101021012101210121012101210 = (210102101210)^3.1. \]

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