C₀-POSITIVITY AND A CLASSIFICATION OF CLOSED THREE-DIMENSIONAL CR TORSION SOLITONS

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ABSTRACT. A closed CR 3-manifold is said to have C₀-positive pseudohermitian curvature if \((W + C₀T or)(X, X) > 0\) for any \(0 \neq X \in T_{1,0}(M)\). We discover an obstruction for a closed CR 3-manifold to possess C₀-positive pseudohermitian curvature. We classify closed three-dimensional CR Yamabe solitons according to C₀-positivity and C₀-negativity whenever C₀ = 1 and the potential function lies in the kernel of Paneitz operator. Moreover, we show that any closed three-dimensional CR torsion soliton must be the standard Sasakian space form. At last, we discuss the persistence of C₀-positivity along the CR torsion flow starting from a pseudo-Einstein contact form.

1. Introduction

Self-similar solutions, also known as geometric solitons, of various geometric flows have attracted lots of attentions in recent years because their close ties with singularity formations in the flows. In particular, important progress has been made in the study of Ricci solitons, self-similar solutions to the mean curvature flow, as well as Yamabe solitons, etc.

There have been several flows proposed to investigate the geometry and topology of CR manifolds, such as CR Yamabe flow, Q-curvature flow, CR Calabi flow, and CR torsion flow. Among them, CR torsion flow behaves more like the Riemannian Ricci flow. Similar to their Riemannian counterparts, for the CR Yamabe flow it is relatively easier to prove long time existence and convergence results than the CR torsion flow, but it provides less information about the local geometry.

The CR torsion flow is to deform the CR contact form \(\theta\) and the complex structure \(J\) by Tanaka-Webster curvature \(W\) and torsion \(A\) respectively. Namely,

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2A_{J,\theta}, \\
\frac{\partial \theta}{\partial t} &= -2W \theta.
\end{align*}
\]

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Note that the second equation alone is called the CR Yamabe flow, where to a certain extent $J$ is freely changed. So the first equation is added to confine the behavior of $J$ and the CR torsion tensor $A$. In the paper [CW], it was shown that there exists a unique smooth solution to the CR torsion flow (1.1) in a small time interval with a certain CR pluriharmonic function as the initial data. In spirit, it is the CR analogue of the Cauchy-Kovalevskaya local existence and uniqueness theorem for analytic partial differential equations associated with Cauchy initial value problems.

Unlike its Riemannian analogue, the problem of asymptotic convergence of three-dimensional CR torsion flows is widely open ([CKW]). Inspired by Perelman’s work [Pe1] and [Pe2] on Hamilton’s Ricci flow and its solitons, we introduce the analogous notion of Ricci solitons - CR torsion solitons for the CR torsion flow. The structure of CR torsion solitons may be a necessary ingredient towards understanding the asymptotic convergence of solutions of the CR torsion flow (1.1). Indeed, one expects CR torsion solitons to model singularity formations of the CR torsion flow. The equations of three-dimensional CR torsion solitons are

\[
\begin{align*}
W + \frac{1}{2} f_0 &= \mu, \\
 f_{11} + iA_{11}f &= -A_{11},
\end{align*}
\]

and the ones for CR Yamabe solitons are

\[
\begin{align*}
W + \frac{1}{2} f_0 &= \mu, \\
 f_{11} + iA_{11}f &= 0,
\end{align*}
\]

for some smooth function $f$ on a pseudohermitian 3-manifold $(M^3, J, \theta)$. Note that they are the same when $M$ has vanishing torsion. In [CCC], we have derived some classification theorems for CR Yamabe solitons. Here we study CR torsion solitons and their classifications, especially under a curvature assumption called $C_0$-positive.

We say that a CR 3-manifold is $C_0$-positive if the pseudohermitian curvature is $C_0$-positive, i.e.,

\[
(W + C_0 Tor)(X, X) > 0, \text{ for any } 0 \neq X = x^1Z_1 \in T_{1,0}(M),
\]

where $W(X, X) := Wx^1 \bar{x}^1$ is the Tanaka-Webster operator, sometimes called the pseudohermitian Ricci curvature, and $Tor(X, X) := 2\text{Re}(iA_{11}x^1 \bar{x}^1)$ is the pseudohermitian torsion tensor. This condition is a key to drive the most general CR Li-Yau gradient estimate that serves as a generalization of the CR analogue of Cao-Yau gradient estimate ([CY]) in a closed pseudohermitian manifold with such a nonnegative pseudohermitian curvature condition and non-vanishing torsion as in [CKL], [CFTW] and [CCL]. We obtain the following result for $C_0$-positive CR 3-manifolds.
Theorem 1.1. Let $M$ be a closed CR 3-manifold. For any $C_0 \geq 0$, $C_0$-positive is equivalent to the curvature-torsion pinching condition $W(x) > 2C_0|A_{11}|(x)$ for all $x \in M$. Moreover, if $M$ is $C_0$-positive with $C_0 \geq \frac{1}{2}$, then $M$ admits a Riemannian metric of positive scalar curvature.

For closed three-dimensional CR Yamabe solitons $(M^3, J, \theta)$, we derive the following classification according to $C_0$-(non)positivity.

Theorem 1.2. (i) Any simply connected closed three-dimensional CR Yamabe soliton with $(W + \text{Tor})(X,X) > 0$ for $0 \neq X \in T_{1,0}(M)$ and $P_0 f = 0$ must be the standard CR three sphere.

(ii) Any closed three-dimensional CR Yamabe soliton with $(W + \text{Tor})(X,X) = 0$ for $0 \neq X \in T_{1,0}(M)$ and $P_0 f = 0$ must be the standard Heisenberg space form which is a Seifert fiber space over a euclidean 2-orbifold with nonzero Euler number.

(iii) Any closed three-dimensional CR Yamabe soliton with $(W + \text{Tor})(X,X) < 0$ for $0 \neq X \in T_{1,0}(M)$ and nonnegative CR Paneitz operator must be the standard Lorentz space form which is a Seifert fiber space over a hyperbolic orbifold with nonzero Euler number.

Remark 1.1. The condition $P_0 f = 0$ is not very restrictive in the sense that the kernel of the CR Paneitz operator $P_0$ is infinite dimensional, containing all CR-pluriharmonic functions (see [Hi]).

On the other hand, we can classify closed three-dimensional CR torsion solitons without using the pseudohermitian curvature condition. This is the CR analogue of the fact that closed three-dimensional Ricci solitons must be Einstein.

Theorem 1.3. Every closed three-dimensional CR torsion soliton has constant Tanaka-Webster curvature and zero torsion $A_{11} = 0$. Therefore, they must belong to one of the following three cases:

(i) The standard CR spherical space form in case $W = 1$.

(ii) The standard Heisenberg space form in case $W = 0$.

(iii) The standard Lorentz space form in case $W = -1$.

In the last section, we derive some basic evolution equations for curvature and torsion along the CR torsion flow and then obtain the preserving property for the uniform pinching pseudohermitian curvature condition under certain assumptions.
**Theorem 1.4.** Let \((M^3, J, \theta)\) be a closed CR manifold with pseudo-Einstein \(\theta\). If there exists a pluriharmonic function \(\gamma(0)\) such that \(W(0) = e^{2\gamma(0)}\theta\) is pluriharmonic, then the solution \(\theta(t)\) of the CR torsion flow

\[
\begin{cases}
\frac{\partial J}{\partial t} = 2A_{J,\theta}, \\
\frac{\partial \theta}{\partial t} = -2W\theta, \\
\partial_0\gamma(t) = 0, \quad \partial_0W(t) = 0
\end{cases}
\]

has the following evolution equations

\[
\begin{cases}
\frac{\partial W}{\partial t} = 5\Delta bW + 2(W^2 - |A_{11}|^2), \\
\frac{\partial |A_{11}|^2}{\partial t} = \Delta b|A_{11}|^2 - 2|\nabla|A_{11}||^2.
\end{cases}
\]

Moreover, if \(W > 2C_0\max |A_{11}|\) initially with \(C_0 \geq \frac{1}{2}\), then it holds for all \(t < T\). In particular, the solution \(\theta(t)\) is \(C_0\)-positive for all \(t < T\), i.e.,

\[(W + C_0\text{Tor})(X, X) > 0 \text{ for all non-vanishing } X \in T_{1,0}(M) \text{ and for all } t < T.\]

This theorem is motivated by the flow approach for proving Frankel’s conjecture in Kähler geometry, e.g. [CT1, CT2]. As a CR analogue of it, we conjecture that the CR torsion flow which starts from a closed CR manifold with pseudo-Einstein \(\theta(0)\), positive constant curvature \(W(0)\) and \(W + \frac{1}{2}\text{Tor}(X, X) > 0\) must converge to the Sasakian space form. We believe Theorem 1.4 will be useful for proving the conjecture.

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## 2. Preliminaries

We give a brief introduction to pseudohermitian geometry on a closed 3-manifold (see [L1, L2] for more details). Let \(M\) be a closed 3-manifold with an oriented contact structure \(\xi\). There always exists a global contact form \(\theta\), obtained by patching together local ones with a partition of unity. The characteristic vector field of \(\theta\) is the unique vector field \(T\) such that \(\theta(T) = 1\) and \(\mathcal{L}_T\theta = 0\) or \(d\theta(T, \cdot) = 0\). A CR structure compatible with \(\xi\) is a smooth endomorphism \(J : \xi \to \xi\) such that \(J^2 = -Id\). A pseudohermitian structure compatible with \(\xi\) is a CR-structure \(J\) compatible with \(\xi\) together with a global contact form \(\theta\). The CR structure \(J\) can extend to \(\mathbb{C} \otimes \xi\) and decomposes \(\mathbb{C} \otimes \xi\) into the direct sum of \(T_{1,0}\) and \(T_{0,1}\) which are eigenspaces of \(J\) with respect to \(i\) and \(-i\), respectively.
Let \( \{ T, Z_1, Z_\bar{1} \} \) be a frame of \( TM \otimes \mathbb{C} \), where \( Z_1 \) is any local frame of \( T_{1,0} \), \( Z_\bar{1} = \overline{Z_1} \in T_{0,1} \) and \( T \) is the characteristic vector field. Then \( \{ \theta, \theta^1, \theta^{\bar{1}} \} \), the coframe dual to \( \{ T, Z_1, Z_\bar{1} \} \), satisfies
\[
(2.1) \quad d\theta = ih_{1\bar{1}} \theta^1 \wedge \theta^{\bar{1}},
\]
for some positive function \( h_{1\bar{1}} \). Actually we can always choose \( Z_1 \) such that \( h_{1\bar{1}} = 1 \); hence, throughout this paper, we assume \( h_{1\bar{1}} = 1 \).

The Levi form \( \langle \ , \ \rangle_{L_\theta} \) is the Hermitian form on \( T_{1,0} \) defined by
\[
\langle Z, Y \rangle_{L_\theta} = -i \langle d\theta, Z \wedge Y \rangle.
\]
We can extend \( \langle \ , \ \rangle_{L_\theta} \) to \( T_{0,1} \) by defining \( \langle Z, Y \rangle_{L_\theta} = \overline{\langle Z, Y \rangle_{L_\theta}} \) for all \( Z, Y \in T_{1,0} \). The Levi form induces naturally a Hermitian form on the dual bundle of \( T_{1,0} \), denoted by \( \langle \ , \ \rangle_{L_\theta^*} \), and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over \( M \) with respect to the volume form \( d\mu = \theta \wedge d\bar{\theta} \), we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation \( \langle \ , \ \rangle \). For example,
\[
(2.2) \quad \langle \varphi, \psi \rangle = \int_M \varphi \overline{\psi} \, d\mu,
\]
for functions \( \varphi \) and \( \psi \).

The pseudohermitian connection of \( (J, \theta) \) is the connection \( \nabla \) on \( TM \otimes \mathbb{C} \) (and extended to tensors) given in terms of a local frame \( Z_1 \in T_{1,0} \) by
\[
\nabla Z_1 = \theta_1^1 \otimes Z_1, \quad \nabla Z_\bar{1} = \theta_\bar{1}^1 \otimes Z_\bar{1}, \quad \nabla T = 0,
\]
where \( \theta_1^1 \) is the 1-form uniquely determined by the following equations:
\[
(2.3) \quad d\theta^1 = \theta^1 \wedge \theta_1^1 + \theta \wedge \tau^1, \quad \tau^1 \equiv 0 \mod \theta^1, \quad 0 = \theta_1^1 + \theta_\bar{1}^1,
\]
where \( \tau^1 = A_{1\bar{1}} \theta^{\bar{1}} \) is the pseudohermitian torsion. Moreover, the structure equation for \( \nabla \) is
\[
(2.4) \quad d\theta_1^1 = W\theta^1 \wedge \theta^1 + 2i \text{Im}(A_{1,1} \theta^1 \wedge \theta),
\]
where \( W \) is the Tanaka-Webster scalar curvature and the index preceded by a comma denotes the covariant derivative. The indices 0, 1 and \( \bar{1} \) indicate derivatives with respect to \( T, Z_1 \) and \( Z_\bar{1} \). For derivatives of a scalar function, we will often omit the
comma, for instance, \( \varphi_1 = Z_1 \varphi \), \( \varphi_{11} = Z_1 Z_1 \varphi - \theta_1^1 (Z_1) Z_1 \varphi \), \( \varphi_0 = T \varphi \) for a (smooth) function \( \varphi \).

For a real function \( \varphi \), the subgradient \( \nabla_b \varphi \) is defined by \( \nabla_b \varphi \in \xi \) and \( \langle Z, \nabla_b \varphi \rangle_{L_0} = d \varphi(Z) \) for all vector fields \( Z \) tangent to the contact plane. Locally \( \nabla_b \varphi = \varphi_1 Z_1 + \varphi_1 Z_1 \).

We can use the connection to define the subhessian as the complex linear map

\[
(\nabla^H)^2 \varphi : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}
\]

by

\[
(\nabla^H)^2 \varphi(Z) = \nabla_Z \nabla_b \varphi.
\]

Also

\[
\Delta_b \varphi = Tr ((\nabla^H)^2 \varphi) = (\varphi_{11} + \varphi_{11}).
\]

For all \( Z = x^1 Z_1 \in T_{1,0} \), we define

\[
Ric(Z,Z) = W x^1 x^1 = W |Z|_{L_0}^2,
\]
\[
Tor(Z,Z) = 2 \text{Re } i A_{11} x^1 x^1.
\]

Next we recall the definition of CR Paneitz operator.

**Definition 2.1.** Let \((M,J,\theta)\) be a closed three-dimensional pseudohermitian manifold. We define (11)

\[
P \varphi = (\varphi_{11} + i A_{11} \varphi^1) \theta^1 = (P_1 \varphi) \theta^1,
\]

which is an operator that characterizes CR-pluriharmonic functions. Here \( P_1 \varphi = \varphi_{11} + i A_{11} \varphi^1 \) and \( P \varphi = \overline{P_1 \varphi} \theta^1 \), the conjugate of \( P \). The CR Paneitz operator \( P_0 \) is defined by

\[
P_0 \varphi = (\delta_b (P \varphi) + \overline{\delta_b (P \varphi)}) ,
\]

where \( \delta_b \) is the divergence operator that takes \((1,0)\)-forms to functions by \( \delta_b (\sigma_1 \theta^1) = \sigma_{11} \), and similarly, \( \overline{\delta_b (\sigma_1 \theta^1)} = \sigma_{11} \).

We observe that

\[
\int_M \langle P \varphi + \overline{P \varphi}, d_b \varphi \rangle_{L_0^*} \, d \mu = - \int_M P_0 \varphi \cdot \varphi \, d \mu
\]

with \( d \mu = \theta \wedge d \theta \). One can check that \( P_0 \) is self-adjoint, that is, \( \langle P_0 \varphi, \psi \rangle = \langle \varphi, P_0 \psi \rangle \) for all smooth functions \( \varphi \) and \( \psi \). For the details about these operators, the reader can make reference to [GL], [Hi], [LL], [GL] and [FH].
3. **Geometry and Topology of $C_0$-positive CR 3-manifolds**

In this section, we study the geometry and topology of CR 3-manifolds which are $C_0$-positive, namely, for all $0 \neq X = x^1 Z_1 \in T_{1,0}(M)$,

$$W + C_0 \text{Tor}(X, X) = W x^1 x^1 + 2C_0 \text{Re}[i(A_{11} x^1 x^1)]$$

$$= W x^1 x^1 + C_0 i(A_{11} x^1 x^1 - A_{11} x^1 x^1) > 0.$$  

Our aim in this section is to prove the following

**Theorem 1.1.** Let $M$ be a closed CR 3-manifold. For any $C_0 \geq 0$, $C_0$-positive is equivalent to the curvature-torsion pinching condition $W(x) > 2C_0 |A_{11}|(x)$ for all $x \in M$. Moreover, if $M$ is $C_0$-positive with $C_0 \geq \frac{1}{2}$, then $M$ admits a Riemannian metric of positive scalar curvature.

We divide the proof into two lemmas. The first lemma contains results more than the theorem states.

**Lemma 3.1.** Let $M$ be a closed CR 3-manifold. For any $C_0 \geq 0$, $C_0$-positive curvature condition is equivalent to the pinching condition $W(x) > 2C_0 |A_{11}|(x)$ for all $x \in M$. Similarly, $C_0$-negative curvature condition is equivalent to $W(x) < -2C_0 |A_{11}|(x)$ for all $x \in M$.

**Proof.** Fix a point $x \in M$ and denote $A_{11}(x) = a(x) + b(x)i$. Without loss of generality, one may consider $X = x^1 Z_1 = (1 + si)Z_1$ for some $s \in \mathbb{R}$. The positivity condition (3.1) reads as

$$W(1 + si)(1 - si) > -C_0 i[(a - bi)(1 - si)^2 - (a + bi)(1 + si)^2]$$

$$= -C_0[4as + 2b - 2bs^2],$$

i.e.,

$$W > -4C_0 \frac{as + b}{1 + s^2} + 2C_0 b.$$  

Since $X$ is arbitrary, this inequality holds for all $s \in \mathbb{R}$. Denote $f(s) = -4\frac{as + b}{1 + s^2} + 2b$. When $a \neq 0$, we have

$$f(s) \leq \max_{s \in \mathbb{R}} f(s) = f(s_0) = 2\sqrt{a^2 + b^2},$$

where $s_0 = \frac{b + \sqrt{a^2 + b^2}}{-a}$ is a critical number of $f$. On the other hand, it is easy to see that $f(s) \leq 2|b|$ when $a = 0$. All these imply $f(s) \leq 2|A_{11}|$ for all $s \in \mathbb{R}$ and thus $W(x) > 2C_0 |A_{11}|(x)$ for all $x \in M$. Therefore, $C_0$-positive is equivalent to $W(x) > 2C_0 |A_{11}|(x)$.
Similarly, at the minimum point \( s_1 = \frac{b - \sqrt{a^2 + b^2}}{-a} (a \neq 0) \), we have

\[
f(s) \geq \min_{s \in \mathbb{R}} f(s) = f(s_1) = -2\sqrt{a^2 + b^2}.
\]

Then it follows analogously that \( C_0 \)-negative curvature condition is equivalent to

\[
W(x) < -2C_0|A_{11}|(x).
\]

**Remark 3.1.** Note that \( C_0 \)-positive curvature can also be expressed as

\[
(W - C_0\text{Tor})(Y, Y) > 0
\]

for all \( 0 \neq Y = y^1Z_1 \in T_{1,0}(M) \) with \( y^1 = ix^1 \). This is to say that \( C_0 \)-positive actually implies \( W > C_0|\text{Tor}| \), although this does not imply \( W > 2C_0|A_{11}| \) literally without using the lemma above.

Next, it is interesting to know the obstruction for the \( C_0 \)-positivity when \( C_0 \geq \frac{1}{2} \).

The following lemma completes the proof of Theorem 3.1.

**Lemma 3.2.** Let \((M^3, J, \theta)\) be a closed CR 3-manifold with \( W(x) > |A_{11}(x)| \). Then \( M \) admits a Riemannian metric of positive scalar curvature.

**Proof.** We recall that the Webster (adapted) Riemannian metric on \( M \) is defined by

\[
g_\lambda = d\theta + \lambda^{-2}\theta^2
\]

for any parameter \( \lambda > 0 \). Now it follows from the paper by Chang and Chiu ([CCCh]) that the Ricci curvature \( R^\lambda_{ij} \) with respect to \( g_\lambda \) is

\[
R^\lambda_{11} = 2W - 2\lambda^{-2} - 2i\lambda^2\text{Im}A_{11}\theta_1^1(T) + 2\text{Im}A_{11} - \lambda^2\text{Re}A_{11,0}
\]

\[
R^\lambda_{22} = 2W - 2\lambda^{-2} + 2i\lambda^2\text{Im}A_{11}\theta_1^1(T) - 2\text{Im}A_{11} + \lambda^2\text{Re}A_{11,0}
\]

\[
R^\lambda_{33} = -2\lambda^2|A_{11}|^2 + 2\lambda^{-2}
\]

\[
R^\lambda_{12} = 2i\lambda^2\text{Re}A_{11}\theta_1^1(T) - 2\text{Re}A_{11} - \lambda^2\text{Im}A_{11,0}
\]

\[
R^\lambda_{13} = 2\lambda\text{Re}A_{11,1}, \quad R^\lambda_{23} = -2\lambda\text{Im}A_{11,1}.
\]

and then the scalar curvature is

\[
R^\lambda = 4W - 2\lambda^2|A_{11}|^2 - 2\lambda^{-2}.
\]

We want to find \( \lambda \) so that \( R \) is positive. This is not always solvable. For instance, when \( W = |A_{11}| = 0 \), \( R^\lambda \) is always negative. So we need to find an admissible condition on \( W \) and \( |A_{11}| \). Let \( \mu = \lambda^2 \). Then \( R^\lambda > 0 \) if and only if the quadratic polynomial \( 2|A_{11}|^2\mu^2 - 4W\mu + 2 \) attains negative values, i.e., the coefficients should satisfy

\[
0 > (4W)^2 - 4(2|A_{11}|^2)(2) = 16(W^2 - |A_{11}|^2).
\]
It follows that if $W > |A_{11}|$, then there is a positive constant $\lambda$ such that $R^\lambda > 0$. Although $\lambda$ varies pointwisely, there exists a uniform $\lambda > 0$ on the closed manifold $M$. This completes the proof. □

4. CR Harnack Quantity

For CR manifolds, we have the concept of CR Yamabe solitons (4.2) which are self-similar solutions to the CR Yamabe flow on a pseudohermitian $(2n + 1)$-manifold. Similarly, one can introduce CR torsion solitons which correspond to self-similar solutions to the CR torsion flow (1.1) on a pseudohermitian $(2n + 1)$-manifold.

**Definition 4.1.** A pseudohermitian $(2n+1)$-manifold $(M, \xi, J, \theta)$, with CR structure $J$ and compatible contact form $\theta$, is called a CR torsion soliton if there exist an infinitesimal contact diffeomorphism $X$ and a constant $\mu \in \mathbb{R}$ such that

$$\begin{cases} 
W \theta + \frac{1}{2} L_X \theta = \mu \theta, \\
L_X J = 2A_{J,\theta},
\end{cases}$$

where $W$ is the Tanaka-Webster scalar curvature of $(M, \xi, J, \theta)$ and $L_X$ denotes Lie derivative by $X$. It is called shrinking if $\mu > 0$, steady if $\mu = 0$, and expanding if $\mu < 0$. In particular, it is called a CR Yamabe soliton ([CCC]) if

$$\begin{cases} 
W \theta + \frac{1}{2} L_X \theta = \mu \theta, \\
L_X J = 0,
\end{cases}$$

where $J$ is invariant under the contact diffeomorphism.

**Lemma 4.2.** A quintuple $(M^3, J, \theta, f, \mu)$ is a three-dimensional CR torsion soliton if

$$\begin{cases} 
W + \frac{1}{2} f_0 = \mu, \\
f_{11} + iA_{11} f = -A_{11}.
\end{cases}$$

**Proof.** We first recall a result from [G].

**Lemma 4.3.** Let $(M^{2n+1}, J, \theta)$ be a pseudohermitian $(2n + 1)$-manifold. For any smooth function $\tilde{f}$ on $M$, let $X_{\tilde{f}}$ be the vector field uniquely defined by

$$X_{\tilde{f}}| d\theta = \tilde{f} d\theta \mod \theta \quad \text{and} \quad X_{\tilde{f}}|\theta = -\tilde{f}.$$

Then $X_{\tilde{f}} = i\tilde{f}_\alpha Z_\alpha - i\tilde{f}_\bar{\alpha} Z_{\bar{\alpha}} - \tilde{f} T$ and it is a smooth infinitesimal contact diffeomorphism of $(M^{2n+1}, \theta)$. Conversely, every smooth infinitesimal contact diffeomorphism is of the form $X_{\tilde{f}}$ for some smooth function $\tilde{f}$. Moreover, $L_{X_{\tilde{f}}} J$ has the following expression

$$L_{X_{\tilde{f}}} J \equiv 2(f_{aa} + iA_{aa} \tilde{f}) \theta^a \otimes Z_\alpha + 2(f_{\bar{a}\bar{a}} - iA_{\bar{a}\bar{a}} \tilde{f}) \theta^{\bar{a}} \otimes Z_\alpha \mod \theta.$$
Next, we relate CR torsion solitons and self-similar solutions to the CR torsion flow. A special class of solutions to the CR torsion flow \((1.1)\) is given by self-similar solutions, whose contact forms \(\theta_t\) deform under the CR Yamabe flow only by a scaling function depending on \(t\)

\[
\theta(t) := \rho(t)\Phi_t^*\theta(0), \quad \rho(t) > 0, \quad \rho(0) = 1
\]

and

\[
J(t) = \Phi_t^*(J(0)),
\]

where \(\Phi_t : M \to M\) is a one-parameter family of contact diffeomorphisms generated by a CR vector field \(X_{\tilde{f}}\) as above on \(M\) with \(\Phi_0 = id_M\).

As in the paper [CCC], it follows from (4.4) and \(\partial_t \theta = -2W\theta\) that we put \(f = -\tilde{f}\) and denote \(\mu = -\frac{1}{2}\rho'(0)\), then we have

\[W + \frac{1}{2}f_0 = \mu.\]

On the other hand, it follows from (4.5) that

\[
\frac{\partial}{\partial t} J(t) = \frac{\partial}{\partial t} \Phi_t^*(J(0)) = L_{X_{\tilde{f}}} J(t)
\]

and then

\[A = (\tilde{f}_{\alpha\bar{\alpha}} + iA_{\alpha\bar{\alpha}} \tilde{f})\theta^\alpha \otimes Z_\bar{\alpha} + (\tilde{f}_{\bar{\alpha}\bar{\alpha}} - iA_{\bar{\alpha}\bar{\alpha}} \tilde{f})\theta^{\bar{\alpha}} \otimes Z_\alpha.\]

Hence, when \(n = 1\), we obtain

\[f_{11} + iA_{11} f = -A_{11}\]

for \(f = -\tilde{f}\) and \(n = 1\). We refer to [CCC] for more details.

To prove Theorem 1.2, we shall need a certain differential Harnack quantity for the three-dimensional CR Yamabe or torsion flow. For the Ricci flow, Hamilton found a conserved quantity which vanishes identically for expanding Ricci solitons and showed that such a quantity is nonnegative for solutions to the Ricci flow with positive curvature operator. This quantity is called the differential Harnack quantity, or LYH quantity, because one can obtain the Harnack inequality for the evolving scalar curvature from it. Recall that the authors have derived a CR Harnack quantity for CR Yamabe soliton in [CCC]:

\[
4\Delta_b W + 2W(W - \mu) + \langle \nabla_b W, X_{\tilde{f}} \rangle + \langle W_0 T, X_{\tilde{f}} \rangle = 0.
\]

In this paper, we obtain the CR Harnack quantity for the CR torsion flow \((1.1)\).
Theorem 4.4. A three-dimensional CR torsion soliton satisfies the following CR Harnack quantity

\begin{equation}
4\Delta_b W + 2W(W - \mu) + \langle \nabla_b W, X_f \rangle + \langle W_0 T, X_f \rangle - i(A_{11,11} - A_{11,11}) - 2|A_{11}|^2 = 0.
\end{equation}

That is,

\begin{equation}
3\Delta_b W + 2W(W - \mu) + \langle \nabla_b W, X_f \rangle + \langle W_0 T, X_f \rangle - 2Q - 2|A_{11}|^2 = 0.
\end{equation}

Here $Q$ is the CR $Q$-curvature (see Section 6).

Proof. Recall that $\Delta_b W = W_{11} + W_{11}$. We first differentiate the soliton equation (1.2) and obtain

\[ W_{11} = -\frac{1}{2} f_{011} = \frac{i}{2} (f_{11} - f_{11,11}). \]

The two terms appear on the right hand side are higher derivatives of $f$ which can be reduced by using

\[ f_{11} + iA_{11}f = -A_{11}. \]

Indeed, by using commutation relations, one derives

\[
\begin{align*}
f_{1111} &= f_{1111} - if_{10} - f_1 W \\
&= -A_{11,1} - i(A_{11}f)_1 - if_{01} + iA_{11}f_1 - f_1 W \\
&= -A_{11,1} - iA_{11,1}f + 2iW_1 - f_1 W.
\end{align*}
\]

Differentiate this in the direction $Z_1$, one achieves

\[ f_{1111} = -A_{11,11} - iA_{11,11}f - iA_{11,1}f_1 + 2iW_{11} - f_{11} W - f_1 W_1. \]

On the other hand, the differentiation of the conjugation in the $Z_1$ direction gives an expression of $f_{1111}$. After changing the 3rd and 4th indices, one obtains

\[ f_{1111} = f_{1111} + if_{10} = -A_{11,11} + iA_{11,11}f + iA_{11,1}f_1 - 2iW_{11} - f_{11} W - f_1 W_1 + if_{110}. \]

Note that the bad term $f_{110}$ can be rewritten as

\[
\begin{align*}
f_{110} &= f_{011} - A_{11}f_{11} - A_{11,1}f_1 - A_{11}f_{11} - A_{11,1}f_1 \\
&= -2W_{11} - A_{11}f_{11} - A_{11,1}f_1 - A_{11}f_{11} - A_{11,1}f_1 \\
&= 2|A_{11}|^2 - 2W_{11} - A_{11,1}f_1 - A_{11,1}f_1.
\end{align*}
\]

Substituting our expressions for $f_{1111}$ and $f_{1111}$ into the equation for $W_{11}$, we get

\[
\begin{align*}
2W_{11} &= -i(A_{11,11} - A_{11,11}) + A_{11,11}f + A_{11,1}f_1 - 2W_1 - if_{11} W - if_1 W_1 \\
&+ A_{11,11}f + A_{11,1}f_1 - 2W_{11} + if_{11} W + if_1 W_1 \\
&+ 2|A_{11}|^2 - 2W_{11} - A_{11,1}f_1 - A_{11,1}f_1.
\end{align*}
\]
By the CR Bianchi identity $A_{11,11} + A_{11,11} = W_0$, we have the Harnack quantity

\begin{equation}
4\Delta_b W = -i(A_{11,11} - A_{11,11}) + 2|A_{11,11}|^2 + (A_{11,11}^2 + A_{11,11})f + (f_{11} - f_{11})W - i(f_{11}W_1 - f_{11}W_1) = -i(A_{11,11} - A_{11,11}) + 2|A_{11,11}|^2 + W_0f - i(f_{11}W_1 - f_{11}W_1)
\end{equation}

\begin{equation}
= -2W(W - \mu) + \langle \nabla_b W, J(\nabla_b f) \rangle.
\end{equation}

Notice that

\[
\langle \nabla_b W + W_0 T, X_f \rangle = \langle \nabla_b W + W_0 T, -fT - if_1Z_1 + if_1Z_1 \rangle
\]

\[
= -W_0f - \langle \nabla_b W, J(\nabla_b f) \rangle a,
\]

so the proof is completed. \hfill \Box

5. Classification of CR Solitons

In the first part of this section, we derive a classification of closed three-dimensional CR Yamabe solitons according to pseudohermitian curvature condition. In the second part, we prove the complete classification of closed three-dimensional CR torsion solitons.

In [CCC], we use Harnack quantity (4.6) to prove that every closed three-dimensional CR Yamabe soliton satisfies $\int_M (W - \mu)^2 = 0$ and thus has constant Tanaka-Webster curvature. This is also true for higher dimensional cases, as proved by P.-T. Ho in [Ho]. However, this is not enough for us to classify the underlying CR manifold due to the lack of information on CR torsion. We suggest that $C_0$-positivity, which involves both curvature and torsion, may be a suitable notion for achieving a complete classification of closed three-dimensional CR Yamabe solitons.

**Theorem 1.2.** (i) Any simply connected closed three-dimensional CR Yamabe soliton with $(W + \text{Tor})(X, X) > 0$ for $0 \neq X \in T_{1,0}(M)$ and $P_0f = 0$ must be the standard CR three sphere.

(ii) Any closed three-dimensional CR Yamabe soliton with $(W + \text{Tor})(X, X) = 0$ for $0 \neq X \in T_{1,0}(M)$ and $P_0f = 0$ must be the standard Heisenberg space form which is a Seifert fiber space over a euclidean 2-orbifold with nonzero Euler number.

(iii) Any closed three-dimensional CR Yamabe soliton with $(W + \text{Tor})(X, X) < 0$ for $0 \neq X \in T_{1,0}(M)$ and nonnegative CR Paneitz operator must be the standard Lorentz space form which is a Seifert fiber space over a hyperbolic orbifold with nonzero Euler number.
Proof. (i) Since $f_{11} + iA_{11}f = 0$, by using commutation relations, one derives

\[
\begin{align*}
  f_{111} &= f_{111} - if_{10} - f_{1}W \\
  &= -i(A_{11}f)_{11} - if_{10} + iA_{11}f_{1} - f_{1}W \\
  &= -iA_{11}f + 2iW_{1} - f_{1}W.
\end{align*}
\]

Recall that $W = \mu$, so the term $W_{1}$ vanishes (see Theorem 1.1 in [CCC]). Multiplying both sides by $f_{1}$ and integrating them, we have

\[- \int_{M} |f_{11}|^2 d\mu + i \int_{M} A_{11,1} f_{1} d\mu + \int_{M} W f_{1} f_{1} d\mu = 0.\]

Because

\[
i \int_{M} A_{11,1} f_{1} f_{1} d\mu = -i \int_{M} A_{11} f_{1} f_{1} d\mu - i \int_{M} A_{11} f_{11} f_{1} d\mu
\]

\[
= -i \int_{M} A_{11} f_{1} f_{1} d\mu + \int_{M} |A_{11}|^2 f_{2} d\mu,
\]

we have

\[0 = \int_{M} |A_{11}|^2 f_{2} d\mu - \int_{M} |f_{11}|^2 d\mu - i \int_{M} A_{11} f_{1} f_{1} d\mu + \int_{M} W f_{1} f_{1} d\mu.\]

Using the soliton equation and the pluriharmonic operator $P_{1} \varphi = \varphi_{1}^1 + iA_{11}\varphi^1$, we can break the second term into

\[
\int_{M} |f_{11}|^2 d\mu = \int_{M} |f_{11}|^2 d\mu + i \int_{M} f_{11} f_{0} d\mu = i \int_{M} A_{11} f_{1} f_{1} d\mu - \int_{M} (P_{1} f)_{1} f_{1} d\mu.
\]

All these imply

\[0 = \int_{M} |A_{11}|^2 f_{2} d\mu + \int_{M} W f_{1} f_{1} d\mu - i \int_{M} A_{11} f_{1} f_{1} d\mu + \int_{M} (P_{1} f)_{1} f_{1} d\mu.\]

By taking conjugate and summing with the original equation, we have

\[0 = \int_{M} |A_{11}|^2 f_{2} d\mu + \int_{M} W f_{1} f_{1} d\mu - i \int_{M} (A_{11} f_{1} f_{1} - A_{11} f_{1} f_{1}) d\mu - \frac{1}{2} \int_{M} (P_{0} f) f_{1} d\mu,
\]

i.e.,

\[0 = \int_{M} |A_{11}|^2 f_{2} d\mu + \int_{M} (W + Tor)((\nabla b f)_{C}, (\nabla b f)_{C}) d\mu - \frac{1}{2} \int_{M} (P_{0} f) f_{1} d\mu.\]

By our assumptions and the identity (5.1), it is easy to see $A_{11} = 0$. Since a pseudohermitian 3-manifold $(M, J, \theta)$ of constant Tanaka-Webster scalar curvature and vanishing pseudohermitian torsion must be spherical, it follows from a result of Y. Kamishima and T. Tsuboi ([KT]) that one can have a complete classification of such closed spherical torsion-free CR torsion solitons. We also refer to [T] for closed CR 3-manifolds of Sasakian space forms.

The cases (ii) and (iii) can be similarly derived as (i).
The condition \( P_0 f = 0 \) in Theorem 1.2 is not very restrictive in the sense that the kernel of the CR Paneitz operator \( P_0 \) is infinite dimensional, containing all CR-pluriharmonic functions (Cf. [Hi]). For a closed pseudohermitian 3-manifold of transverse symmetry, we have \( \ker P_1 = \ker P_0 \). Moreover, these kernels are invariant under rescaling \( \tilde{\theta} = e^{2g} \theta \), since \( \tilde{P}_1 = e^{-3g} P_1 \) and \( \tilde{P}_0 = e^{-4g} P_0 \). Note that CR-pluriharmonic function is naturally related to holomorphic functions on \( \mathbb{C}^2 \). In fact, let \( M \) be a hypersurface in \( \mathbb{C}^2 \), i.e., \( M = \partial \Omega \) for a bounded domain \( \Omega \) in \( \mathbb{C}^2 \), then for any pluriharmonic function \( u : U \rightarrow \mathbb{R} (\partial \bar{\partial} u = 0) \) with a simply connected \( U \subset \overline{\Omega} \), there exists a holomorphic function \( w \) in \( U \) such that \( u = \text{Re}(w) \). Now define \( f := u|_M \), it follows that \( f \) is a CR-pluriharmonic function (see Definition 2.1) and thus \( P_0 f = 0 \).

We have mentioned that every closed three-dimensional CR Yamabe soliton has constant Tanaka-Webster curvature. This can be proven by integrating the CR Yamabe Harnack quantity (4.6). For CR torsion solitons, we obtain similar results by integrating the CR torsion Harnack quantity (4.7). Therefore we obtain the following classification.

**Theorem 1.3.** Every closed three-dimensional CR torsion soliton (1.3) has constant Tanaka-Webster curvature and zero torsion \( A_{11} = 0 \). Therefore, they must belong to one of the following three cases:

(i) The standard CR spherical space form in case \( W = 1 \).

(ii) The standard Heisenberg space form in case \( W = 0 \).

(iii) The standard Lorentz space form in case \( W = -1 \).

**Proof.** By integrating the Harnack quantity (4.7), one derives that

\[
2 \int_M |A_{11}|^2 d\mu = \int_M [2W(W - \mu) - W_0 f + i(f_1 W_1 - f_1 W_0)]d\mu
\]

\[
= 2 \int_M W(W - \mu)d\mu + \int_M W f_0 d\mu - \int_M i(f_{11} - f_{11})Wd\mu
\]

\[
= 2 \int_M W(W - \mu)d\mu + 2 \int_M W f_0 d\mu
\]

\[
= 2 \int_M W(W - \mu)d\mu - 4 \int_M W(W - \mu)d\mu
\]

\[
= -2 \int_M W(W - \mu)d\mu.
\]

Together with the fact that \( \int_M (W - \mu)d\mu = -\frac{1}{2} \int_M f_0 d\mu = 0 \), we obtain

\[
\int_M (W - \mu)^2 d\mu = \int_M W(W - \mu)d\mu - \mu \int_M (W - \mu)d\mu
\]

\[
= \int_M W(W - \mu)d\mu
\]

\[
= -\int_M |A_{11}|^2 d\mu.
\]

Hence \( W = \mu \) and \( A_{11} = 0 \). This completes the proof. □
6. Curvature Evolution Equations and Preserving Positive Curvature

Let \((M^3, J, \theta)\) be a closed CR manifold. Recall that, as in Definition 2.1, a function \(u\) is called pluriharmonic w.r.t. \(\theta\) if
\[
Pu := (u\bar{1}\bar{1} + iA_{11}u)\theta^1 = (P_1u)\theta^1 = 0.
\]
We denote \(W^\perp = W - W^\ker\), where \(W^\ker\) is the pluriharmonic portion w.r.t. \(\theta\), i.e., \(PW^\ker = 0\) and \(PW = PW^\perp\). Note that \(\theta\) and \(\hat{\theta} := e^{-2\gamma}\theta\) characterize the same pluriharmonic functions, because \(P = e^{-4\gamma}\hat{P}\).

In Theorem 1.1, we have seen that \(C_0\)-positivity is equivalent to the pointwise pinching condition \(W > 2C_0|A_{11}|\). Here we show that the uniform pinching condition \(W > 2C_0\max|A_{11}|\) is preserved by the CR torsion flow with specific initial data. To be precise, given a background CR manifold \((M^3, J, \theta^0)\) which is pseudo-Einstein, if there exists a pluriharmonic function \(\gamma(0)\) such that \(W(0)\) of \(\theta(0) := e^{2\gamma(0)}\theta^0\) is pluriharmonic, then the CR torsion flow
\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2A_{J, \theta}, \\
\frac{\partial \theta}{\partial t} &= -2W\theta, \\
P\gamma(t) = 0, \ PW(t) = 0
\end{align*}
\]
has a short-time solution \(\theta(t) = e^{2\gamma(t)}\theta^0\) on \([0, T]\) which preserves the uniform pinching condition.

Note that the initial contact form \(\theta(0)\) of the flow differs from the background contact form \(\theta^0\) by the factor \(e^{2\gamma(0)}\). According to Lee \([L1]\) and Hirachi \([H]\), \(\theta(0) := e^{2\gamma(0)}\theta^0\) is still pseudo-Einstein because \(\gamma(0)\) is a pluriharmonic function. This “gauge changing” plays an analogue role as DeTurck’s trick of the Ricci flow in Chang and Wu’s proof of the existence of the torsion flow \(\theta(t)\). Moreover, \(P\gamma(t) = 0\) and \(PW(t) = 0\) imply that \(\gamma(t) = \frac{1}{2}\left(\ln \theta(t) - \ln \theta^0\right)\) and \(W(t)\) are still pluriharmonic for all \(t \in [0, T]\) \([CW, p.20]\). This makes the evolution equations of curvature and torsion easier to handle. Our result in this section is based on these facts. For more properties about this CR torsion flow, one can consult \([CKW, CW]\).

Before we prove the preservation of the uniform pinching condition, we clarify the definition of pseudo-Einstein mentioned above. Traditionally, the contact form \(\theta\) of a CR manifold \((M^{2n+1}, J, \theta)\) is called pseudo-Einstein if \(R_{a\bar{b}} - \frac{1}{n}Rh_{a\bar{b}} = 0\). Note that this condition holds trivially when \(n = 1\) and thus gives no information about \(\theta\). When \(n \geq 2\), pseudo-Einstein is equivalent to \(W_a - iA_{a\bar{b}, \bar{b}} = 0\). However, \(W_a - iA_{a\bar{b}, \bar{b}}\) does not always vanishes when \(n = 1\). This can be seen as an alternative definition of
pseudo-Einstein. So in this article a CR manifold \((M^3, J, \theta)\) is called \textit{pseudo-Einstein} if

\[ H_1 := W_1 - iA_{11,1} = 0. \]

Note that to be pseudo-Einstein is neither sufficient to be a self-similar solution of CR torsion flow or CR Yamabe flow, nor sufficient to conclude that \(W(0)\) is a constant.

\textbf{Remark 6.1.} 1. There always exists a pseudo-Einstein contact form on a closed hypersurface in \(\mathbb{C}^2\).

2. Let \((M, J, \theta)\) be a closed CR 3-manifold. It is still open whether there exists a pseudo-Einstein contact form \(\hat{\theta}\) of pluriharmonic scalar curvature (in particular, the constant scalar curvature)! We refer to [CKLin] for some details.

Now we demonstrate the preservation of the uniform pinching condition.

\textbf{Theorem 1.4.} Let \((M^3, J, 0 \theta)\) be a closed CR manifold with pseudo-Einstein \(0 \theta\). If there exists a pluriharmonic function \(\gamma(0)\) such that \(W(0)\) of \(\theta(0) := e^{2\gamma(0)} 0 \theta\) is pluriharmonic, then the solution \(\theta(t)\) of the CR torsion flow (6.1) has the following evolution equations

\[ \begin{aligned}
&\frac{\partial W}{\partial t} = 5\Delta_b W + 2(W^2 - |A_{11}|^2), \\
&\frac{\partial |A_{11}|^2}{\partial t} = \Delta_b |A_{11}|^2 - 2|\nabla|A_{11}||^2.
\end{aligned} \]

Moreover, if \(W > 2C_0 \max |A_{11}| \) initially with \(C_0 \geq \frac{1}{2}\), then it holds for all \(t < T\). In particular, the solution \(\theta(t)\) is \(C_0\)-positive for all \(t < T\), i.e.,

\[ (W + C_0 \text{Tor})(X, X) > 0 \] for all non-vanishing \(X \in T_{1,0}(M)\) and for all \(t < T\).

\textbf{Proof.} For generic CR torsion flow, we have ([CW pp.12-13])

\[ \frac{\partial}{\partial t} A_{11} = 2(iW_{11} + W A_{11}) - iA_{11,0} \]

and

\[ \frac{\partial}{\partial t} W = 2\text{Re} \left( iA_{11,11} - A_{11} A_{11} \right) + (4\Delta_b W + 2W^2). \]

Hence we have

\[ \begin{aligned}
&\frac{\partial W}{\partial t} = 4\Delta_b W + 2(W^2 - |A_{11}|^2) + i(A_{11,11} - A_{11,11}), \\
&\frac{\partial |A_{11}|^2}{\partial t} = -2i W_{11} + A_{11,11} - A_{11,11}.
\end{aligned} \]

Recall that CR \(Q\)-curvature is defined by

\[ Q := -\text{Re}(W_1 - iA_{11,1})_1 = -\text{Re}(W_{11} - iA_{11,11}) = -\frac{1}{2}[\Delta_b W - i(A_{11,11} - A_{11,11})]. \]

If we denote \(H_1 := W_1 - iA_{11,1}\), then \(H_{11} + H_{11} = -2Q\) and

\[ \begin{aligned}
&\frac{\partial W}{\partial t} = 5\Delta_b W + 2(W^2 - |A_{11}|^2) + 2Q, \\
&\frac{\partial |A_{11}|^2}{\partial t} = \Delta_b |A_{11}|^2 - 2|\nabla|A_{11}||^2 + 4\text{Im}(H_{11} A_{11}).
\end{aligned} \]
Since \( \theta(t) \) is pseudo-Einstein for all \( t \), we have \( H_1 = 0 \), thus \( Q = 0 \) and \( 4 \text{Im}(H_{11}A_{11}) = 0 \). Therefore, we obtain the desired evolution equation. Now applying the maximum principle to the second evolution equation, one can show that

\[ |A_{11}(t)| \leq \max |A_{11}(0)| \]

for all \( t < T \). But from the first evolution equation, we have

\[
\frac{\partial W}{\partial t} \geq 5\Delta_b W + 2(W^2 - \max |A_{11}(0)|^2) \\
= 5\Delta_b W + 2(W + \max |A_{11}(0)|)(W - \max |A_{11}(0)|).
\]

It follows from the maximal principle that the condition \( W(t) > \max |A_{11}(t)| \) is preserved under the flow and

\[ W(t) > \max |A_{11}(0)| \geq \max |A_{11}(t)|, \]

which is equivalent to say that

\[ (W + \frac{1}{2} Tor)(Z, Z) > 0, \quad Z = x^1Z_1 \in T_{1,0}(M). \]

holds for all \( t < T \).

\( \square \)

**Remark 6.2.** 1. Note that if \( M \) is an embeddable CR 3-manifold in \( \mathbb{C}^N \), then there exists a contact form \( \theta \) of vanishing CR \( Q \)-curvature \( \hat{Q} = 0 \) \([\text{CS}]\). Hence \( Q = 0 \) and the generic evolution equations (6.2) become

\[
\begin{align*}
\frac{\partial W}{\partial t} &= 5\Delta_b W + 2(W^2 - |A_{11}|^2), \\
\frac{\partial |A_{11}|^2}{\partial t} &= \Delta_b |A_{11}|^2 - 2|\nabla|A_{11}|^2 + 4\text{Im}(H_{11}A_{11}).
\end{align*}
\]

2. Under the same condition \( W > 2C_0 \max |A_{11}| \), we also obtain that \( (W - C_0 Tor)(X, X) > 0 \) for all non-vanishing \( X \in T_{1,0}(M) \) and for all \( t < T \).

The torsion flow greatly simplifies if the torsion vanishes. This only happens in very special setups. Indeed, CR 3-manifolds with vanishing torsion are \( K \)-contact, meaning that the Reeb vector field is a Killing vector field for the contact Riemannian metric \( g = \frac{1}{2}d\theta + \theta^2 \). In general, one can still hope that the torsion flow improves properties of the contact manifold underlying the CR structure. It is the case in a closed homogeneous pseudohermitian 3-manifold \([\text{CKW}]\). Long-time existence and asymptotic convergence of solutions for the (normalized) torsion flow can be achieved in this special case.
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