Travelling Wave Solutions in Nonlinear Diffusive and Dispersive Media

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Abstract

We investigate the presence of soliton solutions in some classes of nonlinear partial differential equations, namely generalized Korteweg-de Vries-Burgers, Korteweg-de Vries-Huxley, and Korteweg-de Vries-Burgers-Huxley equations, which combine effects of diffusion, dispersion, and nonlinearity. We emphasize the chiral behavior of the travelling solutions, whose velocities are determined by the parameters that define the equation. For some appropriate choices, we show that these equations can be mapped onto equations of motion of relativistic 1 + 1 dimensional $\phi^4$ and $\phi^6$ field theories of real scalar fields. We also study systems of two coupled nonlinear equations of the types mentioned.

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1 Introduction

Nonlinear partial differential equations are ubiquitous in physics. Their applications range from magnetofluid dynamics to water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas, among several others. In a number of them, it is well known that localized travelling solutions, or solitons, appear. Since the seminal work of Korteweg and de Vries (KdV) [1], in which a third order nonlinear equation was studied to explain the shallow-water solitary wave experiments by Rusell [2], high order equations have been used to describe physical phenomena occurring in a broad range of fields [3, 4], from liquid crystals to dynamics of growing interfaces and domain walls, also including many applications in electromagnetism, nonlinear optics, acoustics, and elasticity. In addition, another interesting possibility arises in some special cases in which the soliton solutions present chirality, i.e., non-arbitrary velocities of propagation determined by the parameters of the differential equation, and with definite sign. In this context, recent works [5, 6] have shown that the nonlinear Schrödinger equation presents chiral solitons when nonlinearity enters the game via derivative coupling. The nonlinear Schrödinger equation that appears in these works is obtained from a very interesting dimensional reduction to one space dimension of a planar model [7] describing non-relativistic matter coupled to a Chern-Simons gauge field. This equation can be cast to the form

\[ iu_t + \lambda \, j \, u + \mu \, u_{xx} = \frac{dV}{d\rho} \, u, \]  

where \( j = -i\nu (u^* u_x - u u_x^*) \) is the current density, \( \lambda, \mu, \nu \) are real parameters and \( V = V(u^* u) = V(\rho) \) is the potential, expressed in terms of the charge density. This equation should be contrasted with

\[ iu_t + \mu \, u_{xx} = \frac{dV}{d\rho} \, u, \]  

which is the standard nonlinear Schrödinger equation – recall that one usually considers \( V(\rho) \) as quadratic or cubic in \( \rho \). For travelling waves in the form \( u(x, t) = \rho(x - ct) \exp[i\theta(x, t)] \), one finds solutions to the above nonlinear derivative Schrödinger equation that present velocity restricted to just one sense, and the system is then chiral. This is nicely illustrated in a more recent work on the same subject [8], where the soliton structure for vanishing and non-vanishing boundary conditions are investigated. See also Ref. [9] for the case of coupled equations. Evidently, the chiral solitons found in these works may play important role within the context of the fractional quantum Hall effect, where chiral excitations are known to appear [8].

On the other hand, some very recent works have shown that chiral solitons also appear in generalized KdV equations [10]

\[ u_t + f_x - \delta \, u_{xxx} = 0, \]  

and in generalized Burgers [3] equations [11]

\[ u_t + f_x - \nu \, u_{xx} = 0. \]
Here \( f = f(u) \) is some smooth function of \( u \) and \( f_x = (df/du)u_x \) is the term that introduces nonlinearity. The KdV equation combines dispersion, controlled by the real parameter \( \delta \), and nonlinearity, described by \( f(u) \). The Burgers equation combines nonlinearity with diffusion related to the real parameter \( \nu \). In Ref.\[11\] we have also studied the generalized Burgers-Huxley equation, in which extra nonlinear terms are introduced:

\[
 u_t + f_x - \nu u_{xx} = B \frac{df}{du} .
\]  

(5)

Within this context, it seems interesting to extend the presence of chiral solitons to the new scenario where the KdV and Burgers-Huxley equations are added to play the game. The equation that we consider is given by

\[
 u_t + f_x + g_{xx} - \delta u_{xxx} = h(u) ,
\]  

(6)

where \( f, g, \) and \( h \) are smooth functions in \( u \). We name this equation the generalized KdV-Burgers-Huxley or gKdVBH equation. It contains several interesting particular cases. For \( h = 0 \), it corresponds to the generalized KdV-Burgers or gKdVB equation \[11\]. For \( f(u) = (1/2)\lambda u^2 \), \( g(u) = -\nu u \), and \( h(u) = 0 \), we get to the standard KdVB equation. The KdV and Burgers equations were first added in Ref.\[12\] to describe propagation of waves in liquid-filled elastic tubes. For \( f(u) = \lambda u^3 \) and \( g \) and \( h \) as above, it represents the modified KdVB equation \[3\]. For \( g \) trivial, i.e. \( g = 0 \), we name the equation the generalized KdV-Huxley or gKdVH equation since it is similar to the generalized Burgers-Huxley equation considered in Ref.\[11\], but with the diffusion term present in the Burgers-Huxley case changed by the dispersion term present in the KdV case.

In this work we investigate the above-mentioned nonlinear equations, with focus on the search for soliton solutions that present chirality. For some appropriate choices of the functions \( f(u), g(h), \) and \( h(u) \), we show that it is possible to map each one of the equations onto the equation of motion for localized travelling configurations in relativistic field theoretical systems of real scalar fields \( \phi(x, t) \), with general Lagrangian density in bidimensional 1 + 1 spacetime given by \[10, 11, 13, 14, 15\]

\[
 \mathcal{L} = \frac{1}{2} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x_\alpha} - U(\phi) ,
\]  

(7)

where \( x^\alpha = (x^0 = t, x^1 = x) \) and \( x_\alpha = (x^0 = t, x^1 = x) \), and \( U(\phi) \) is the potential. In this context, soliton field configurations propagating with velocity \( c \) as \( \phi(x, t) = \phi(x - ct) = \phi(y) \) arise \[10, 11\] as solutions of the equation of motion associated with the above Lagrangian density,

\[
 \frac{d^2 \phi}{dy^2} = \frac{\partial U}{\partial \phi} .
\]  

(8)

In particular, we present suitable choices of functions in Eq.(6) that provide mappings of the mentioned nonlinear equations onto the \( \phi^4 \) and \( \phi^6 \) field theories, whose solutions present interesting topological features \[10, 11\].

3
The article is organized as follows: in the next section we study the gKdVH equation. In Sec. 3 we investigate the gKdVB equation, and in Sec. 4 we consider the combination of the previous ones, the gKdVBH equation. In each Section, we analyze both cases of a single equation and a system of coupled nonlinear equations. Finally, conclusions are presented in Sec. 5.

2 The gKdVH Equation

We start by considering the gKdVH equation, i.e., Eq.(6) without the term with second order derivative in space related to diffusion, \( g(u) = 0 \), but with nonlinear and dispersion effects present:

\[
\frac{du}{dt} + \frac{df}{du} u_x - \delta u_{xxx} = h(u) .
\] (9)

For propagating waves of the type \( u(x,t) = u(x-ct) = u(y) \), Eq.(9) can be solved by the first order equation

\[
\frac{du}{dy} = Bh(u) ,
\] (10)

if \( f(u) = h(u) (dh/du) \), \( B^2 = 1/\delta \), and the velocity is set by \( c = -1/B \), a property that confers chirality to the travelling solutions. We now present an explicit example in which the solutions are solitons. By considering the case in which

\[
h(u) = \lambda(a^2 - u^2) ,
\] (11)

which implies,

\[
f(u) = -2\lambda^2 u(a^2 - u^2) ,
\] (12)

with \( \lambda \) and \( a \) real parameters, we obtain the chiral solitons

\[
u(x,t) = a \tanh[\lambda a B(x - ct - \bar{x})] ,
\] (13)

where \( \bar{x} \) is the center of the solution, which is an arbitrary point in space. We observe that the gKdVH equation, Eq.(9), defined by the choices of Eqs.(11) and (12), represents the equation of motion for localized travelling field configurations of a relativistic 1 + 1 dimensional \( \phi^4 \) field theory of a real scalar field \( \phi \) [14]. Indeed, one can attest this fact by differentiating Eq.(10) with respect to \( y \), and then making use of Eqs.(11) and (13). The resulting equation, when compared with Eq.(8), allows the identification of the associated \( \phi^4 \) potential,

\[
U(\phi) = -\frac{\lambda^2 B^2}{2} \phi^2(2a^2 - \phi^2) .
\] (14)

In this context, studies [15] performed on field theories of two coupled scalar fields motivate us to generalize the above analysis to the case of a pair of coupled gKdVH
equations,
\[ u_t + \frac{\partial f}{\partial u} u_x + \frac{\partial f}{\partial v} v_x - \delta u_{xxx} = h(u, v), \]  
(15)  
\[ v_t + \frac{\partial \bar{f}}{\partial u} u_x + \frac{\partial \bar{f}}{\partial v} v_x - \bar{\delta} v_{xxx} = \bar{h}(u, v). \]  
(16)

A similar procedure applied to the travelling waves \( u(x, t) = u(y) \) and \( v(x, t) = v(y) \) leads to the pair of first order coupled equations,
\[ \frac{du}{dy} = B h(u, v), \]  
(17)  
\[ \frac{dv}{dy} = \bar{B} \bar{h}(u, v), \]  
(18)

provided that \( B^2 = 1/\delta, B = \bar{B}, c = -1/B, \) and
\[ f(u, v) = h \frac{\partial h}{\partial u} + \bar{h} \frac{\partial \bar{h}}{\partial v}, \]  
(19)  
\[ \bar{f}(u, v) = \bar{h} \frac{\partial \bar{h}}{\partial v} + h \frac{\partial h}{\partial u}. \]  
(20)

As an example also related to the field theory of a pair of coupled scalar fields interacting through a potential up to the forth power in the fields [15], we consider
\[ h(u, v) = \frac{1}{B} (\lambda - \lambda u^2 - \mu v^2), \]  
(21)  
and
\[ \bar{h}(u, v) = -\frac{2\mu uv}{B}, \]  
(22)

which lead to,
\[ f(u, v) = -2\lambda^2 c^2 u(1 - u^2) + 2\mu c^2 uv(\lambda + 2\mu), \]  
(23)  
\[ \bar{f}(u, v) = -2\mu c^2 v(1 - u^2) + 2\mu^2 c^2 v^2(2 + 2u^2), \]  
(24)

with coupled chiral solutions for \( \lambda/\mu > 2 \) given by
\[ u(x, t) = \tanh[2\mu(x - ct - \bar{x})], \]  
(25)  
\[ v(x, t) = \sqrt{\frac{\lambda}{\mu}} - 2 \sec h[2\mu(x - ct - \bar{x})]. \]  
(26)
3 The gKdVB Equation

Let us now focus on the gKdVB equation in which nonlinearity, diffusion, and dispersion are present, Eq.(6) with $h(u) = 0$:

$$u_t + \frac{df}{du} u_x + \frac{dg}{du} u_x + \frac{d^2 g}{du^2} u_{xx} + \frac{dg}{du} u_{xx} - \delta u_{xxx} = 0 .$$

(27)

Travelling solutions $u(y)$ are obtained from

$$\left(-c + \frac{df}{du}\right) u_y + \frac{d^2 g}{du^2} u_y + \frac{dg}{du} u_{yy} - \delta u_{yyy} = 0 .$$

(28)

In the case of $f(u) = -f(-u)$ and $g(u) = -g(-u)$, the gKdVB equation is invariant under the discrete symmetry $u \rightarrow -u$, so that a trivial integration leads to

$$\frac{dg}{du} u_y - \delta u_{yy} = c u - f(u) ,$$

(29)

if we impose that the resulting equation mantain the original symmetry. Now, for $f(u) = A u$, with $c = A$, Eq.(29) is solved by

$$\frac{du}{dy} = \frac{1}{\delta} g(u) ,$$

(30)

such as found for the gKdVH equation. To introduce a distinct example, let us consider now that

$$g(u) = \lambda u (a^2 - u^2) ,$$

(31)

which leads to the chiral solution

$$u(x,t) = \sqrt{(a^2/2)} \{1 + \tanh[(\lambda/\delta)a^2(x - ct - \bar{x})]\} ,$$

(32)

since its velocity is just $c = A$. In the context of field theoretical representations, these choices for $f(u)$ and $g(u)$ correspond to a $\phi^6$ theory of real scalar fields [14], with potential given by,

$$U(\phi) = \frac{\lambda^2}{2\delta^2} \phi^2(\phi^2 - a^2)^2 .$$

(33)

Furthermore, as in the previous case we can also extend the analysis to a system of coupled gKdVB equations in the form,

$$u_t + f_x + g_{xx} - \delta u_{xxx} = 0 ,$$

(34)

$$v_t + \bar{f}_x + \bar{g}_{xx} - \delta v_{xxx} = 0 .$$

(35)

Here $f = f(u,v)$ and $g = g(u,v)$ are odd in $u$ and even in $v$, and $\bar{f} = \bar{f}(u,v)$ and $\bar{g} = \bar{g}(u,v)$ are even in $u$ and odd in $v$, in order to preserve the symmetries in the $(u,v)$
space of the original equations. These smooth functions allow us to write the above equations in the form

\[
\begin{align*}
  &u_t + \frac{\partial f}{\partial u} u_x + \frac{\partial f}{\partial v} v_x + \frac{\partial g}{\partial u} u_{xx} + \frac{\partial g}{\partial v} v_{xx} + \frac{\partial^2 g}{\partial u^2} u_x^2 + \frac{2}{\partial u\partial v} u_x v_x + \frac{\partial^2 g}{\partial v^2} v_x^2 - \delta u_{xxx} = 0, \\
  &v_t + \frac{\partial \bar{f}}{\partial u} u_x + \frac{\partial \bar{f}}{\partial v} v_x + \frac{\partial \bar{g}}{\partial u} u_{xx} + \frac{\partial \bar{g}}{\partial v} v_{xx} + \frac{\partial^2 \bar{g}}{\partial u^2} u_x^2 + \frac{2}{\partial u\partial v} u_x v_x + \frac{\partial^2 \bar{g}}{\partial v^2} v_x^2 - \bar{\delta} v_{xxx} = 0.
\end{align*}
\]

(36)  
(37)

For travelling waves \( u(y) \) and \( v(y) \) we obtain, after integrating them once,

\[
\begin{align*}
  &\frac{\partial g}{\partial u} \frac{du}{dy} + \frac{\partial g}{\partial v} \frac{dv}{dy} - \delta \frac{d^2 u}{dy^2} = cu - f(u, v), \\
  &\frac{\partial \bar{g}}{\partial u} \frac{du}{dy} + \frac{\partial \bar{g}}{\partial v} \frac{dv}{dy} - \bar{\delta} \frac{d^2 v}{dy^2} = cv - \bar{f}(u, v).
\end{align*}
\]

(38)  
(39)

In order to present an example of coupled chiral soliton solutions of Eqs.(38) and (39), we consider for instance,

\[
\begin{align*}
  &\frac{dg}{dy} = au + bv^2, \\
  &\frac{d\bar{g}}{dy} = \bar{a}u^2 + \bar{b}v,
\end{align*}
\]

(40)  
(41)

along with,

\[
\begin{align*}
  &f(u, v) = -cu + cu^3 + \frac{\delta}{\bar{\delta}}(1 + 2\frac{\delta}{\bar{\delta}})uv^2 - bv^2, \\
  &\bar{f}(u, v) = -cv + \frac{\delta}{\bar{\delta}}v^3 + c(1 + 2\frac{\delta}{\bar{\delta}})u^2v - \bar{a}u^2.
\end{align*}
\]

(42)  
(43)

By substituting Eqs.(40)-(43) in Eqs.(38) and (39), we obtain after setting \( c = a = \bar{b} \), the following system of coupled differential equations:

\[
\begin{align*}
  &\frac{\delta}{c} \frac{d^2 u}{dy^2} = -(1 - u^2)u + \frac{\delta}{\bar{\delta}}(1 + 2\frac{\delta}{\bar{\delta}})uv^2, \\
  &\frac{\bar{\delta}}{c} \frac{d^2 v}{dy^2} = -(1 - \frac{\delta}{\bar{\delta}})v^2 + (1 + 2\frac{\delta}{\bar{\delta}})u^2v.
\end{align*}
\]

(44)  
(45)
which can also be seen as the equations of motion for localized travelling configurations of a relativistic field theory of two coupled scalar fields with fourth order potential. Eqs. (44) and (45) present solutions given by

\[ u(x, t) = \tanh\left[2\left(\frac{c\delta}{2\bar{\delta}^2}\right)^{1/2}(x - ct - \bar{x})\right], \tag{46} \]

\[ v(x, t) = \left(\frac{\bar{\delta}}{\delta} - 2\right)^{1/2} \sec h\left[2\left(\frac{c\delta}{2\bar{\delta}^2}\right)^{1/2}(x - ct - \bar{x})\right], \tag{47} \]

with \( \bar{\delta} > 2\delta \), and chiral behavior related to the identification of the velocity \( c \) with the parameters of the functions \( f \) and \( g \).

4 The gKdVBH Equation

At last, we generalize the previous results to the gKdVBH equation, Eq. (6), in which extra nonlinear terms are included with respect to the gKdVB equation, Eq. (27), through the function \( h(u) \). By considering travelling solutions of Eq. (6), we obtain

\[ \left(-c + \frac{df}{du}\right)u_y + \frac{d^2g}{du^2}u_y^2 + \frac{dg}{du}u_{yy} - \delta u_{yyy} = h(u). \tag{48} \]

At this point, by assuming in Eq. (48) the presence of symmetry under \( u \to -u \), then the function \( h(u) \) must also have odd parity, as discussed in Sec. 3 for the functions \( f(u) \) and \( g(u) \). After a trivial integration, the gKdVBH equation can be mapped onto the gKdVB equation, Eq. (28), if

\[ f(u) = \bar{f}(u) - \int_y^\infty h[u(y')]dy'. \tag{49} \]

Therefore, from the analysis presented in Sec. 3 we conclude that chiral solitons can also be found as solutions of the gKdVBH equation, as well as of a pair of coupled gKdVBH equations. In addition, mappings of Eq. (6) onto field theories of real scalar coupled fields are also possible.

5 Conclusion

In conclusion, we have investigated and introduced explicit examples of chiral soliton solutions in generalized equations such as the gKdVH, the gKdVB and the gKdVBH equations. These nonlinear partial differential equations combine diffusion, dispersion, and nonlinearity in distinct ways. We have also shown that they can be mapped onto equations of motion that appear in relativistic systems of real scalar fields, provided appropriate choices of the functions that define them are made. In particular, we illustrate this point by presenting examples related to \( \phi^4 \) and \( \phi^6 \) field theories. In each case considered, the
velocity of the solutions is determined in terms of the parameters of the nonlinear equation, therefore characterizing their chiral feature. We have also found chirality in systems of two coupled equations, which were also shown to be mapped onto field theories of pairs of real scalar coupled fields.

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