Isometric immersions and self-similar buckling in Non-Euclidean elastic sheets.

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The edge of torn elastic sheets and growing leaves often form a hierarchical buckling pattern. Within non-Euclidean plate theory this complex morphology can be understood as low bending energy isometric immersions of hyperbolic Riemannian metrics. With this motivation we study the isometric immersion problem in strip and disk geometries. By finding explicit piecewise smooth solutions of hyperbolic Monge-Ampere equations on a strip, we show there exist periodic isometric immersions of hyperbolic surfaces in the small slope regime. We extend these solutions to exact isometric immersions through resummation of a formal asymptotic expansion. In the disc geometry we construct self-similar fractal-like isometric immersions for disks with constant negative curvature. The solutions in both the strip and disc geometry qualitatively resemble the patterns observed experimentally and numerically in torn elastic sheets, leaves and swelling hydrogels. For hyperbolic non-Euclidean sheets, complex wrinkling patterns are thus possible within the class of finite bending energy isometric immersions. Further, our results identify the key role of the degree of differentiability (regularity) of the isometric immersion in determining the global structure of a non-Euclidean elastic sheet in 3-space.

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Within the last ten years there has been a significant effort to extend the theory of finite elasticity to model growth in thin elastic sheets. One approach is through the non-Euclidean formalism which was developed to understand the complex rippling morphologies observed in torn plastic sheets [1–3], leaves [4,7] and swelling hydrogels [8,10]. This model posits that growth permanently deforms the intrinsic distance between material points. Material points on the center surface are labelled by material coordinates \((x_1, x_2) \in \Omega\) a subset of \(\mathbb{R}^2\), and the distances between points in the center surface are encoded in the arc length element:

\[
ds^2 = g_{11}(x_1, x_2)dx_1^2 + 2g_{12}(x_1, x_2)dx_1dy + g_{22}(x_1, x_2)dx_2^2, \quad (1)
\]

with \(g_{ij}\) the components of a Riemannian metric \(g\). By making the connection between “growth” due to localized swelling and a Riemannian metric, one can use swelling as a method for designing actively deforming soft matter devices with specific geometries. Indeed, hydrogels capable of reversibly deforming between flat and curved surfaces with Gaussian curvature that is positive, negative or some combination of the two have been realized experimentally [2,13]. By controlling the local swelling factor at each point, exotic shapes such as Enneper’s surface have also been created [10].

Given the swelling pattern, or equivalently given \(g\), can one deduce the equilibrium shape of the sheet? By the Kirchhoff hypothesis [14], the conformation of the sheet as a 3-dimensional object in \(\mathbb{R}^3\) is determined by an immersion \(F : \Omega \rightarrow \mathbb{R}^3\) of the center surface. The elastic energy is then modeled as a sum of stretching and bending contributions:

\[
E[F] = S[\gamma] + t^2B[D^2 F] = \int_{\Omega} Q_1(\gamma) dxdy + t^2 \int_{\Omega} Q_2(D^2 F) dxdy, \quad (2)
\]

where \(\gamma = (\nabla F)^T \cdot \nabla F - g\) denotes the in-plane strains in the center surface, \(t\) is the thickness of the sheet, \(D^2 F\) denotes the Hessian of \(F\) and \(Q_1, Q_2\) are quadratic forms [11,15]. The equilibrium configuration minimizes the energy \(E[F]\). Due to the relative strength of stretching to bending rigidity it is natural to expect that thin sheets deform into low bending energy configurations that remove all in-plane strain. This corresponds to the “restricted” problem of minimizing the bending energy over all isometric immersions of the Riemannian 2-manifold \((\Omega, g)\), i.e. deformations satisfying \((\nabla F)^T \cdot \nabla F = g\). Indeed, provided they exist, finite bending energy isometric immersions of \((\Omega, g)\) are the \(t \rightarrow 0\) limits of minimizers of the “full” elastic energy [2,16]. Therefore, studying the immersions of a given manifold can shed light on (3D) shape selection for thin non-Euclidean sheets.

In this work, we will only be concerned with sheets whose center surface is hyperbolic, i.e has negative Gaussian curvature \((K < 0)\) [17]. We first recall various results on isometric immersions of hyperbolic surfaces – Hilbert showed there are no analytic isometric immersions of the entire hyperbolic plane \(\mathbb{H}^2\) (i.e. \(K = -1\)) into \(\mathbb{R}^3\) [18]. This result was improved by Holmgren who proved that smooth isometric immersions of bounded subsets of \(\mathbb{H}^2\) can only be extended a finite distance before forming singularities [19]. Amsler showed that, for sufficiently smooth \(K = -1\) surfaces these singularities form curves – “singular edges” – along which the principal
curvatures diverge \[20\]. This phenomenon is readily observed as the singular rim bounding surfaces of revolution with constant negative curvature, e.g. the tractricoid (a.k.a. “the” pseudosphere). Efimov generalized Hilbert’s results to (geodesically) complete surfaces with uniformly negative Gaussian curvature \( K \leq - \epsilon < 0 \), by showing that any isometric immersion will necessarily have singularities where the immersed surface is not \( C^2 \), i.e. fails to be twice differentiable \[21\].

In this letter we investigate the relation between singularities for isometric immersions of hyperbolic metrics and the observed morphologies of swelling elastic sheets. For a large class of metrics we show there exist exact isometric immersions with mild singularities where the surface fails to be twice differentiable in the classical sense (is not \( C^2 \)), but is nonetheless weakly twice differentiable with finite bending energy (is \( W^{2,2} \)). The construction of these surfaces consists of gluing together local solutions of the isometric immersion problem in a manner that preserves continuity of the tangent plane and is a generalization of the techniques in \[22\].

The main motivation of this work is to explain the fractal like periodic patterns observed both experimentally \[1\] \[3\] and numerically \[2\] \[6\] \[23\] in torn elastic sheets and swelling hydrogels \[9\]. Such complex, self-similar patterns are usually associated with “strongly frustrated” systems, e.g. elastic sheets with boundary conditions that preclude the possibility of finite energy isometric immersions \[24\] \[26\], or at alloy-alloy interfaces between distinct crystal structures \[27\] \[28\]. Why then do we observe self-similar buckling patterns despite the existence of smooth isometrics in the form of surfaces of revolution \[7\] and helices \[26\]? Our construction of finite energy isometric immersions corresponding to periodic, and self similar wrinkling profiles addresses this puzzle.

We first consider a strip \( \Omega \) defined by \(-\infty < x_1 < \infty, 0 \leq x_2 < \infty \) with free boundary conditions and

\[
g = (1 + \epsilon^2 f(x_2)) \, dx_1^2 + dx_2^2,
\]

where \( f \) is a function satisfying \( f(x_2) > 0, f(x_2) \rightarrow 0 \) as \( x_2 \rightarrow \infty, f''(x_2) > 0 \) and \( \epsilon > 0 \). This metric corresponds to \( x_2 \) dependent growth in the \( x_1 \) direction and is a generalization of the metrics considered in \[2\] \[3\] \[6\] \[26\] \[29\]. To construct approximate isometries, we consider a Föppl - von Kármán ansatz in out-plane deformations \( \epsilon^2 u_1(x_1, x_2), \epsilon^2 u_2(x_1, x_2) \) and out-of-plane deformation \( \epsilon w(x_1, x_2) \). With this ansatz, the in-plane strain is given by

\[
\gamma_{ij} = \epsilon^2 \left( u_{i,j} + u_{j,i} + w_{i,j} - \delta_{i,j} \epsilon f \right) + \epsilon^4 \left( u_{i,1,1} + u_{j,1,1} \right),
\]

where \( \delta \) denotes the derivative of any quantity \( z \) with respect to \( x_i \) and \( \delta_{i,j} \) is the Kronecker delta. Formal asymptotic expansions \( u_i = \sum_{\alpha=0} a^{2\alpha} u_i^{(\alpha)}, w = \sum_{\alpha=0} a^{2\alpha} w^{(\alpha)} \) yield the following equations for an isometry at \( \mathcal{O}(1) \):

\[
0 = u_{i,1}^{(1)} + u_{j,1}^{(1)} + w_{i,1}^{(1)} - \delta_{i,j} \epsilon f \]

Eliminating \( u_1^{(0)} \) and \( u_2^{(0)} \) yields the solvability condition

\[
\left[ w^{(0)}, w^{(0)} \right] = -\frac{f'' \epsilon^4}{2},
\]

where \( [a, b] = \frac{1}{2} (h_{1,1} g_{2,2} - 2 h_{1,2} g_{1,2} + g_{1,1} h_{2,2}) \). The hyperbolic Monge-Ampère equation \( \epsilon^4 = 0 \) is a small-slopes version of Gauss’s Theorema Egregrium \[30\].

We seek solutions of \( \psi \) in the product form \( u^{\phi_1}(x_1, x_2) = 2^{-\frac{1}{2}} \psi(x_1) \phi(x_2) \) with \( \psi(x_1) \) periodic in \( x_1 \). The variables separate if \( \delta \phi''(x_2) \phi(x_2) = (\phi'(x_2))^2 = f''(x_2) > 0 \) for some constant \( \delta \). Requiring \( f(x_2) \rightarrow 0 \) as \( x_2 \rightarrow \infty \) yields \( 0 < \delta \leq 1 \). For \( \delta = 1 \) we obtain \( \phi(x_2) = e^{-x_2/l}, f(x_2) = \frac{1}{2} e^{-2z_2/l}. \)

For \( 0 < \delta < 1 \) we obtain \( \phi(x_2) = (1 + x_2/l)^{-\delta/(1-\delta)}, f(x_2) = \delta/(2(1 + \delta))(1 + x_2/l)^{-2\delta/(1-\delta)} \), where \( l > 0 \) is the length scale of swelling. \( \psi \) is determined by

\[
\psi'(x_1)^2 - \delta^{-1} \psi''(x_1) \psi(x_1) = 1.
\]

Observing that \( \psi(x_1) \psi'' \psi(x_1) = \frac{1}{2} \frac{d\psi^2}{dx_1} \) and letting \( k \) denote a constant of integration, we have

\[
\psi^2 \pm k^2 |\psi|^{2\delta} = 1,
\]

identifying solutions \( \psi(x) \) with the motion of a unit mass particle in the potential \( V(\psi) = \pm \frac{1}{2} (k|\psi|^{2\delta} \right). With V(\psi) = +\frac{1}{2} (k|\psi|^{2\delta} \right) we get periodic solutions with amplitude \( 1/k \) and half wavelength \( \lambda_k \sim k^{-1} \).

Note it follows from Eq. \( 8 \) that solutions to Eq. \( 7 \) satisfy \( \psi'' = 2\delta \psi^{2\delta-1} \psi \) and hence for \( \delta \neq 1 \) solutions to Eq. \( 7 \) are not smooth across the lines \( \psi = 0 \). To construct periodic profiles we take odd periodic extensions \( \bar{\psi} \) of \( \psi \) across these lines. For example, by solving Eq. \( 7 \) with boundary conditions \( \psi(-\lambda_k/2) = \psi(\lambda_k/2) \) and extending periodically we obtain a wavy profile \( \bar{\psi} \) that (weakly) solves Eq. \( 7 \); see Fig. \[1\]. By construction, \( \bar{\psi}, \bar{\psi} \) are continuous across the lines \( \bar{\psi} = 0 \) while \( \bar{\psi}' \) has a jump discontinuity or diverges. Moreover, for \( 1/4 < \delta \leq 1 \) it follows that \( \bar{\psi}' \) is square integrable and thus these deformations have finite bending energy. Since the concavity of \( \bar{\psi} \) changes across the lines \( \bar{\psi} = 0 \) we refer to these as lines of inflection.

FIG. 1. Piecwise smooth small slope isometric immersion with out-of-plane displacement \( w^{\phi_1}(x_1, x_2) = \phi(x_1) \phi(x_2) \) for \( \delta = 1/2 \). The out-of-plane displacement vanishes along the green lines i.e. \( \phi(x_1) = 0 \), and the periodic surface is created by odd periodically reflecting the smooth surface across these lines.

We now explore the possibility of “lifting” the small slope isometries to full isometric immersions using the formal asymptotic expansion in \( \epsilon \). For \( n \geq 1 \) the equations for an isometry at \( \mathcal{O}(\epsilon^{2n}) \), and the corresponding solvability condi-
tion for \( w^n \) are:

\[
0 = u_{i,j}^n + u_{j,i}^n + \sum_{a=0}^{n-1} \left( u_{i+a,j}^a u_{i,a}^{n-a-1} + u_{j+a,i}^a u_{j,a}^{n-a-1} \right) + \sum_{a=0}^{n} w_j^a w_{i,j}^{n-a}.
\]

0 = \sum_{a=0}^{n-1} \left( [u_{i,j}^a, u_{i,j}^{n-a}] + [u_{j,i}^a, u_{j,i}^{n-a}] \right) + \sum_{a=0}^{n} [w_j^a, w_{i,j}^{n-a}]. \tag{9}

For \( \delta = 1 \) with boundary conditions \( \psi(-\lambda_k/2) = \psi(\lambda_k/2) = 0 \) we obtain \( w^0(x_1, x_2) = k^{-1} e^{-\epsilon x_1^2/\lambda} \cos(kx_1) \). By induction it follows that solutions to Eq. (9) are of the form:

\[
w^n(x_1, x_2) = k^{-1} e^{-2(\pi + 1) x_2^2/\lambda} \sum_{a=0}^{\alpha} a_{a,n} (kl)^{-1} \cos((2n + 1) x_1),
\]

where \( a_{a,n} \) are polynomials.

The full series representation for \( w \) is only asymptotic in \( \epsilon \), i.e., it could diverge. Indeed, for increasing values of \( \epsilon \) – equivalently Gaussian curvature – we expect the surface to develop a singular edge, with the first singularity appearing at \((0, 0)\) where \([w_{x_2} w_{x_2}']\) attains its maximum value on \( \Omega \). The order-by-order calculation computes the partial sums

\[
w_{x_2}(0, 0) \approx S_N = -\sum_{a=0}^{\alpha} \sum_{n=0}^{N} \epsilon^{2n} (2a + 1) (kl)^{-1} a_{a,n} ((kl)^{-1}),
\]

\[
w_{x_2}(0, 0) \approx S'_N = \frac{1}{1} \sum_{a=0}^{\alpha} \sum_{n=0}^{N} \epsilon^{2n} (2a + 1)^2 (kl)^{-1} a_{a,n} ((kl)^{-1}).
\]

If \( w_{x_2}(x_1, x_2) \) has a singularity of the form \( w_{x_2}(x_1, x_2) \sim A(x_2 + \epsilon - 6^2) - (2(\pi + 1)) \) near \((0, 0)\), we can approximate \( \epsilon_0 \) and \( \beta \) by the poles and residues of the Padé approximants to the logarithmic derivative \( w_{x_2}(x_1, x_2) \). In Fig. 2 we plot the Dlog Padé approximants for the specific value \( \epsilon_0 \). The inset figure is a plot of the approximants to \( w_{x_2}(x_1, x_2) \) evaluated at \((0, 0)\). The dashed curve indicates values at which the singular curve first appears at \((0, 0)\).

\[\text{FIG. 2. } [N, N+1] \text{ Padé approximants to the logarithmic derivative } w_{x_2}/w_{x_2} \text{ evaluated at } (0, 0). \text{ The inset figure is a plot of the approximate value of } \epsilon_0 \text{ determined from the } [N, N+1] \text{ Padé approximants.}\]

The principal curvature \( \kappa_{x_2} \) diverges with increasing \( \epsilon \) for all values of \( k \) and \( l \). This has consequences for modeling non-Euclidean elastic sheets. In particular minimizers of the small slope approximation to the elastic energy do not adequately approximate the full elastic energy. E.g., with \( \delta = 1 \) and \( kl = 1 \) the solution \( w^0(x_1, x_2) = k^{-1} e^{-\epsilon x_2^2/\lambda} \cos(kx_1) \) is harmonic and hence a global minimizer of the bending energy over small slope isometries. However, in Fig. 3(a) and Fig. 3(b) we present contour plots of the \([9, 9]\) Padé approximants to \( \Delta w \) evaluated at \((0, 0)\) and the following proxy for the bending energy averaged over one half wavelength:

\[B(\Delta w) = \chi_{R}^{-1} \int_{x_2=0}^{x_2=\alpha} (\Delta w)^2 dx_2 dx_1.\]

\([9, 10]\) indicates that for moderate values of \( \epsilon \), i.e., \( \epsilon \gtrsim 2 \), the full elastic energy selects shorter wavelengths than what is predicted by the small slope theory, i.e., \( \epsilon \ll 1 \). In particular, due to the existence of the singular edge, it follows that for large values of \( \epsilon \) the wavelength must satisfy \( kl > 1 \) in order for the isometric immersion

As a second model system, we consider solutions to the Monge-Ampère equation \([w, w] = -1\), the small slopes approximation to \( K = -1 \), on the disk \( \Omega_R \) defined by \( x_1^2 + x_2^2 \leq R^2 \). In reference [22], solutions to this equation with \( n\)-fold rotational symmetry were constructed by taking odd periodic extensions of the one parameter family of solutions \( w^0_n(x_1, x_2) = x_2(x_1 - \cot(\pi/n)x_2) \) about the line of inflection \( x_2 = \tan(\pi/n)x_1 \); see Fig. 4(a–b). The key to this construction is that any quadratic surface is ruled by asymptotic lines whose projection onto the \( x_1, x_2 \) plane form two sets of parallel lines; see Fig. 4(c). The surfaces \( w^0_n(x_1, x_2) \) are constructed so that they are locally ruled by asymptotic lines that are parallel to lines of inflection; see Fig. 4(d). We adopt the terminology in [20] and refer to points where lines of inflection intersect – in this case \( x_1 = x_2 = 0 \) as branch points. They correspond to branch points of the map \( (x_1, x_2) \mapsto \nabla w \), or equivalently to branch points for the Gauss normal map [17];
see Fig. 4(e-f). Note that, if a hyperbolic surface is $C^2$, every point is locally a (regular) saddle (as in Fig. 4(a)) and therefore cannot contain branch points. Non-$C^2$ immersions are therefore qualitatively different from $C^2$ immersions in that they admit 3-saddles (“monkey saddles”) and higher order saddles, which can mediate a local refinement of the buckling wavelength (See Fig. 5).

**FIG. 4.** (a-b) Small slope isometric immersions $w_0^n(x_1, x_2)$ and $\overline{w}_0^n(x_1, x_2)$ for constant Gaussian curvature $K = -1$. $\overline{w}_0^n(x_1, x_2)$ is constructed by taking odd periodic reflections of the piece of $w_0^n(x_1, x_2)$ bounded between the green lines. The mesh on both of these surfaces correspond to their asymptotic lines. (c-d) Projection of the asymptotic lines of $w_0^n(x_1, x_2)$ and $\overline{w}_0^n(x_1, x_2)$ onto the $x_1, x_2$ plane. (e-f) Direction of the gradient $\nabla w$ along circles centered at the origin. The regular saddle in (a) corresponds to a gradient field with winding number $-1$, so the gradient map is $1:1$. The 4-saddle in (b) has winding number $-3$, so the gradient map is a 3-sheeted covering near the origin.

Multiple branch points can be introduced on the surface by replicating the above process at any point, not just the origin. For example, consider the surface $w_0^n(x_1, x_2) = x_1 x_2$ which is ruled by the asymptotic lines $x_1, x_2 = \text{const}$. A branch point can be added at $(x_1, x_2) = (1/\sqrt{2}, 1/\sqrt{2})$ by removing the sector $x_1, x_2 \geq 1/\sqrt{2}$ and in this region fitting three rotated and translated copies of $w_0^n(x_1, x_2) = x_2(x_1 - \sqrt{3}x_2)$ so that the resulting surface has continuous partial derivatives across the cut; see Fig 5(a). Three more branch points $b_{2,1}, b_{2,2}, b_{2,3}$ at a radial distance of $1/4$ from $b_{1,1}$ can be added along rays emanating from $b_{1,1}$ that bisect the lines of inflection; see Fig 5(b). This construction can be continued so that at the $n$-th iteration $3^n$ new branch points are added at a radial distance of $(1/2)^n$ from the previous branch points. The surface $\overline{w}(x_1, x_2)$ formed in the limit $n \rightarrow \infty$ is a fractal with an infinite number of subwrinkles in the region $x_1 \geq 0, x_2 \geq 0, x_1^2 + x_2^2 \geq 1$, and it satisfies $|\overline{w}, \overline{w}| = -1$. The solution can be extended by odd periodic reflections to give a small-slopes isometric immersion of the unit disk with $K = -1$. To illustrate the wrinkling behavior near the edge we map $\overline{w}$ to a strip geometry through a conformal map $h[x + iy] = \overline{w}[e^{x+iy}]$; see Figs. 5(c-d).

**FIG. 5.** Finite bending energy solutions to the Monge-Ampere equation $|w^0, w^0| = -1$. (a) Three subwrinkle solution created by inserting three rotated and translated copies of the solution $w_0^n(x_1, x_2) = x_2(x_1 - \sqrt{3}x_2)$ onto the solution $w_0^n(x_1, x_2) = x_1 x_2$ at a branch point. (b) Nine subwrinkle solution created by inserting nine copies of $w_0^n(x_1, x_2) = x_1 x_2 - (2 + \sqrt{3})x_2$ at three branch points added onto the three subwrinkle solution. (c) Extension of the nine subwrinkle solution to the full circular domain. (d) The nine subwrinkle solution mapped to the strip geometry by a conformal map.

The existence of self-similar isometric immersions has implications to the modeling of non-Euclidean elastic sheets. As for the strip with $\delta = 1$, the solution $w_0^n(x_1, x_2)$ is harmonic yet the extension of $w_0^n(x_1, x_2)$ to an exact isometric immersion has divergent bending energy for $R \geq 1.25$ with the bending content concentrated near the singular point $x_1 = x_2 \approx 1.25/\sqrt{2}$ [22]. We can isometrically immerse disks with larger $R$ by a global refinement of the wavelength i.e. taking $n > 2$. These solutions increase the bending energy globally. An energetically favorable alternative might be to introduce a branch point in the $n = 2$ solution near the singular point, and locally refining the wavelength instead. Indeed, numerics for $\delta = 1/3$ in the strip geometry indicate that, even within the small slopes approximation, localized self similar wrinkling profiles may be energetically preferred over global refinement of the wavelength [22][23].

Crumpled sheets have an energy scale $t^{5/3}$ which is intermediate between the stretching and bending energies [12][33]. In contrast, the existence of $W^{2,2}$ isometric immersions for
hyperbolic free sheets ensures that the energy scales like $t^2$. Our results show that there are multiple low-energy states as $W^{2,2}$ isometric immersions can be constructed by appropriately gluing together low energy building blocks using lines of inflection and branch points in a variety of ways. The energy barriers between distinct low energy states are small, $\sim t^2$, and these sheets are thus “floppy”. Consequently, thin elastic sheets are easily deformed by weak stresses, and the pattern selected may be sensitive to the dynamics of the swelling process, experimental imperfections, or other external forces. A key first step for better understand the buckling patterns of hyperbolic free sheets would be to analyze how the optimal elastic energy of the two types of singularities – lines of inflection and branch points – scale with the various length scales in the problem: $k^{-1}, l$ and $t$.

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