A NEW CHARACTERIZATION OF $q_{\omega}$-COMPACT ALGEBRAS

M. SHAHRYARI

Abstract. In this note, we give a new characterization for an algebra to be $q_{\omega}$-compact in terms of super-product operations on the lattice of congruences of the relative free algebra.

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1. Introduction

In this article, our notations are the same as [2], [3], [4], [5] and [6]. The reader should review these references for a complete account of the universal algebraic geometry. However, a brief review of fundamental notions will be given in the next section.

Let $\mathcal{L}$ be an algebraic language, $A$ be an algebra of type $\mathcal{L}$ and $S$ be a system of equation in the language $\mathcal{L}$. Recall that an equation $p \approx q$ is a logical consequence of $S$ with respect to $A$, if any solution of $S$ in $A$ is also a solution of $p \approx q$. The radical $\text{Rad}_A(S)$ is the set of all logical consequences of $S$ with respect to $A$. This radical is clearly a congruence of the term algebra $T_{\mathcal{L}}(X)$ and in fact it is the largest subset of the term algebra which is equivalent to $S$ with respect to $A$. Generally, this logical system of equations with respect to $A$ does not obey the ordinary compactness of the first order logic. We say that an algebra $A$ is $q_{\omega}$-compact, if for any system $S$ and any consequence $p \approx q$, there exists a finite subset $S_0 \subseteq S$ with the property that $p \approx q$ is a consequence of $S_0$ with respect to $A$. This property of being $q_{\omega}$-compact is equivalent to

$$\text{Rad}_A(S) = \bigcup_{S_0} \text{Rad}_A(S_0),$$

where $S_0$ varies in the set of all finite subsets of $S$. If we look at the map $\text{Rad}_A$ as a closure operator on the lattice of systems of equations in the language $\mathcal{L}$, then we see that $A$ is $q_{\omega}$-compact if and only if $\text{Rad}_A$...
is an algebraic. The class of \( q_\omega \)-compact algebras is very important and it contains many elements. For example, all equationally noetherian algebras belong to this class. In [4], some equivalent conditions for \( q_\omega \)-compactness are given. Another equivalent condition is obtained in [7] in terms of geometric equivalence. It is proved that (the proof is implicit in [7]) an algebra \( A \) is \( q_\omega \)-compact if and only if \( A \) is geometrically equivalent to any of its filter-powers. We will discuss geometric equivalence in the next section. We will use this fact of [7] to obtain a new characterization of \( q_\omega \)-compact algebras. Although our main result will be formulated in an arbitrary variety of algebras, in this introduction, we give a simple description of this result for the case of the variety of all algebras of type \( \mathcal{L} \).

Roughly speaking, a super-product operation is a map \( C \) which takes a set \( K \) of congruences of the term algebra and returns a new congruence \( C(K) \) such that for all \( \theta \in K \), we have \( \theta \subseteq C(K) \). For an algebra \( B \) define a map \( T_B \) which takes a system \( S \) of equations and returns

\[
T_B(S) = \{ \text{Rad}_B(S_0) : S_0 \subseteq S, |S_0| < \infty \}.
\]

Suppose for all algebra \( B \) we have \( C \circ T_B \leq \text{Rad}_B \). We prove that an algebra \( A \) is \( q_\omega \)-compact if and only if \( C \circ T_A = \text{Rad}_A \).

2. Main result

Suppose \( \mathcal{L} \) is an algebraic language. All algebras we are dealing with, are of type \( \mathcal{L} \). Let \( V \) be a variety of algebras. For any \( n \geq 1 \), we denote the relative free algebra of \( V \), generated by the finite set \( X = \{ x_1, \ldots, x_n \} \), by \( F_V(n) \). Clearly, we can assume that an arbitrary element \( (p, q) \in F_V(n)^2 \) is an equation in the variety \( V \) and we can denote it by \( p \approx q \). We introduce the following list of notations:

1- \( P(F_V(n)^2) \) is the set of all systems of equations in the variety \( V \).

2- \( \text{Con}(F_V(n)) \) is the set of all congruences of \( F_V(n) \).

3- \( \Sigma(V) = \bigcup_{n=1}^{\infty} P(F_V(n)^2) \).

4- \( \text{Con}(V) = \bigcup_{n=1}^{\infty} \text{Con}(F_V(n)) \).

5- \( \text{PCon}(V) = \bigcup_{n=1}^{\infty} P(\text{Con}(F_V(n))) \).

6- \( q_\omega(V) \) is the set of all \( q_\omega \)-compact elements of \( V \).

Note that, we have \( \text{Con}(V) \subseteq \Sigma(V) \). For any algebra \( B \in V \), the
map \( \text{Rad}_B : \Sigma(\mathbf{V}) \to \Sigma(\mathbf{V}) \) is a closure operator and \( B \) is \( q_\omega \)-compact, if and only if this operator is algebraic. Define a map

\[
T_B : \Sigma(\mathbf{V}) \to \text{PCon}(\mathbf{V})
\]

by

\[
T_B(S) = \{ \text{Rad}_B(S_0) : S_0 \subseteq S, \ |S_0| < \infty \}. 
\]

**Definition 1.** A map \( C : \text{PCon}(\mathbf{V}) \to \text{Con}(\mathbf{V}) \) is called a super-product operation, if for any \( K \in \text{PCon}(\mathbf{V}) \) and \( \theta \in K \), we have \( \theta \subseteq C(K) \).

There are many examples of such operations; the ordinary product of normal subgroups in the varieties of groups is the simplest one. For another example, we can look at the map \( C(K) = \text{Rad}_B(\bigcup_{\theta \in K} \theta) \), for a given fixed \( B \in \mathbf{V} \). We are now ready to present our main result.

**Theorem 1.** Let \( C \) be a super-product operation such that for any \( B \in \mathbf{V} \), we have \( C \circ T_B \leq \text{Rad}_B \). Then

\[
q_\omega(\mathbf{V}) = \{ A \in \mathbf{V} : C \circ T_A = \text{Rad}_A \}. 
\]

To prove the theorem, we first give a proof for the following claim. Note that it is implicitly proved in \([7]\) for the case of groups.

An algebra is \( q_\omega \)-compact if and only if it is geometrically equivalent to any of its filter-powers.

Let \( A \in \mathbf{V} \) be a \( q_\omega \)-compact algebra and \( I \) be a set of indices. Let \( F \subseteq P(I) \) be a filter and \( B = A^I / F \) be the corresponding filter-power. We know that the quasi-varieties generated by \( A \) and \( B \) are the same. So, these algebras have the same sets of quasi-identities. Now, suppose that \( S_0 \) is a finite system of equations and \( p \approx q \) is another equation. Consider the following quasi-identity

\[
\forall \mathbf{r}(S_0(\mathbf{r}) \to p(\mathbf{r}) \approx q(\mathbf{r})). 
\]

This quasi-identity is true in \( A \), if and only if it is true in \( B \). This shows that \( \text{Rad}_A(S_0) = \text{Rad}_B(S_0) \). Now, for an arbitrary system \( S \), we have

\[
\text{Rad}_A(S) = \bigcup_{S_0} \text{Rad}_A(S_0) \\
= \bigcup_{S_0} \text{Rad}_B(S_0) \\
\subseteq \text{Rad}_B(S). 
\]
Note that in the above equalities, $S_0$ ranges in the set of finite subsets of $S$. Clearly, we have $\text{Rad}_B(S) \subseteq \text{Rad}_A(S)$, since $A \leq B$. This shows that $A$ and $B$ are geometrically equivalent. To prove the converse, we need to define some notions. Let $\mathfrak{X}$ be a prevariety, i.e., a class of algebras closed under product and subalgebra. For any $n \geq 1$, let $F_{\mathfrak{X}}(n)$ be the free element of $\mathfrak{X}$ generated by $n$ elements. Note that if $V = \text{var}(\mathfrak{X})$, then $F_{\mathfrak{X}}(n) = F_V(n)$. A congruence $R$ in $F_{\mathfrak{X}}(n)$ is called an $\mathfrak{X}$-radical, if $F_{\mathfrak{X}}(n)/R \in \mathfrak{X}$. For any $S \subseteq F_{\mathfrak{X}}(n)^2$, the least $\mathfrak{X}$-radical containing $S$ is denoted by $\text{Rad}_\mathfrak{X}(S)$.

**Lemma 1.** For an algebra $A$ and any system $S$, we have
\[
\text{Rad}_A(S) = \text{Rad}_{\text{pvar}(A)}(S),
\]
where $\text{pvar}(A)$ is the prevariety generated by $A$.

**Proof.** Since $F_{\mathfrak{X}}(n)/\text{Rad}_A(S)$ is a coordinate algebra over $A$, so it embeds in a direct power of $A$ and hence it is an element of $\text{pvar}(A)$. This shows that
\[
\text{Rad}_{\text{pvar}(A)}(S) \subseteq \text{Rad}_A(S).
\]
Now, suppose $(p, q)$ does not belong to $\text{Rad}_{\text{pvar}(A)}(S)$. So, there exists $B \in \text{pvar}(A)$ and a homomorphism $\varphi : F_{\mathfrak{X}}(n) \to B$ such that $S \subseteq \ker \varphi$ and $\varphi(p) \neq \varphi(q)$. But, $B$ is separated by $A$, hence there is a homomorphism $\psi : B \to A$ such that $\psi(\varphi(p)) \neq \psi(\varphi(q))$. This shows that $(p, q)$ does not belong to $\ker(\psi \circ \varphi)$. Therefore, it is not in $\text{Rad}_A(S)$.

Note that, since $\text{pvar}(A)$ is not axiomatizable in general, so we cannot give a deductive description of elements of $\text{Rad}_A(S)$. But, for $\text{Rad}_{\text{var}(A)}(S)$ and $\text{Rad}_{\text{qvar}(A)}(S)$ this is possible, because the variety and quasi-variety generated by $A$ are axiomatizable. More precisely, we have:

1- Let $\text{Id}(A)$ be the set of all identities of $A$. Then $\text{Rad}_{\text{var}(A)}(S)$ is the set of all logical consequences of $S$ and $\text{Id}(A)$.

2- Let $\text{Q}(A)$ be the set of all identities of $A$. Then $\text{Rad}_{\text{qvar}(A)}(S)$ is the set of all logical consequences of $S$ and $\text{Q}(A)$.

We can now, prove the converse of the claim. Suppose $A$ is not $q_\omega$-compact. We show that
\[
\text{pvar}(A)_\omega \neq \text{qvar}(A)_\omega.
\]
Recall that for and arbitrary class $\mathfrak{X}$, the notation $\mathfrak{X}_\omega$ denotes the class of finitely generated elements of $\mathfrak{X}$. Suppose in contrary we have the
equality

\[ pvar(A)_{\omega} = qvar(A)_{\omega}. \]

Assume that \( S \) is an arbitrary system and \((p, q) \in \text{Rad}_A(S)\). Hence, the infinite quasi-identity

\[ \forall \overline{x}(S(\overline{x}) \rightarrow p(\overline{x}) \approx q(\overline{x})) \]

is true in \( A \). So, it is also true in \( pvar(A) \). As a result, every element from \( qvar(A)_{\omega} \) satisfies this infinite quasi-identity. Let \( F_{A}(n) = F_{\text{var}(A)}(n) \). We have \( F_A(n) \in qvar(A)_{\omega} \) and hence \( \text{Rad}_{qvar(A)}(S) \) depends only on \( qvar(A)_{\omega} \). In other words, \((p, q) \in \text{Rad}_{qvar(A)}(S)\), so \( p \approx q \) is a logical consequence of the set of \( S + Q(A) \). By the compactness theorem of the first order logic, there exists a finite subset \( S_0 \subseteq S \) such that \( p \approx q \) is a logical consequence of \( S_0 + Q(A) \). This shows that \((p, q) \in \text{Rad}_{qvar(A)}(S_0)\). But \( \text{Rad}_{qvar(A)}(S_0) \subseteq \text{Rad}_A(S_0) \). Hence \((p, q) \in \text{Rad}_A(S_0)\), violating our assumption of non-\( q_{\omega} \)-compactness of \( A \). We now showed that

\[ pvar(A)_{\omega} \neq qvar(A)_{\omega}. \]

By the algebraic characterizations of the classes \( pvar(A) \) and \( qvar(A) \), we have

\[ SP(A)_{\omega} \neq SPP_u(A)_{\omega}, \]

where \( P_u \) is the ultra-product operation. This shows that there is an ultra-power \( B \) of \( A \) such that

\[ SP(A)_{\omega} \neq SP(B)_{\omega}. \]

In other words the classes \( pvar(A)_{\omega} \) and \( pvar(B)_{\omega} \) are different. We claim that \( A \) and \( B \) are not geometrically equivalent. Suppose this is not the case. Let \( A_1 \in pvar(A)_{\omega} \). Then \( A_1 \) is a coordinate algebra over \( A \), i.e. there is a system \( S \) such that

\[ A_1 = \frac{F_V(n)}{\text{Rad}_A(S)}. \]

Since \( \text{Rad}_A(S) = \text{Rad}_B(S) \), so

\[ A_1 = \frac{F_V(n)}{\text{Rad}_B(S)}, \]

and hence \( A_1 \) is a coordinate algebra over \( B \). This argument shows that

\[ pvar(A)_{\omega} = pvar(B)_{\omega}, \]

which is a contradiction. Therefore \( A \) and \( B \) are not geometrically equivalent and this completes the proof of the claim. We can now complete the proof of the theorem. Assume that \( C \circ T_A = \text{Rad}_A \). We show that \( A \) is geometrically equivalent to any of its filter-powers. So,
let $B = A^I/F$ be a filter-power of $A$. Note that we already proved that for a finite system $S_0$, the radicals $\text{Rad}_A(S_0)$ and $\text{Rad}_B(S_0)$ are the same. Suppose that $S$ is an arbitrary system of equations. We have

$$\text{Rad}_A(S) = C(T_A(S)) = C(\{\text{Rad}_A(S_0) : S_0 \subseteq S, |S_0| < \infty\}) \subseteq \text{Rad}_B(S).$$

So we have $\text{Rad}_A(S) = \text{Rad}_B(S)$ and hence $A$ and $B$ are geometrically equivalent. This shows that $A$ is $q_\omega$-compact. Conversely, let $A$ be $q_\omega$-compact. For any system $S$, we have

$$\text{Rad}_A(S) = \bigcup_{S_0} \text{Rad}_A(S_0) = \bigvee \{\text{Rad}_A(S_0) : S_0 \subseteq S, |S_0| < \infty\} = \bigvee T_A(S),$$

where $\bigvee$ denotes the least upper bound. By our assumption, $C(T_A(S)) \subseteq \text{Rad}_A(S)$, so $C(T_A(S)) \subseteq \bigvee T_A(S)$. On the other hand, for any finite $S_0 \subseteq S$, we have $\text{Rad}_A(S_0) \subseteq C(T_A(S))$. This shows that

$$C(T_A(S)) = \bigvee T_A(S),$$

and hence $C \circ T_A = \text{Rad}_A$. The proof is now completed.

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M. Shahryari: Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

E-mail address: mshahryari@tabrizu.ac.ir