Exact entanglement renormalization for string-net models

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We construct an explicit renormalization group (RG) transformation for Levin and Wen’s string-net models on a hexagonal lattice. The transformation leaves invariant the ground-state “fixed-point” wave function of the string-net condensed phase. Our construction also produces an exact representation of the wave function in terms of the multi-scale entanglement renormalization ansatz (MERA). This sets the stage for efficient numerical simulations of string-net models using MERA algorithms. It also provides an explicit quantum circuit to prepare the string-net ground-state wave function using a quantum computer.

The fractional quantum Hall effect provides the first experimental evidence [1] for the existence of topological phases of quantum matter. It has motivated the study of topological order and its characterization [2], and has spurred considerable theoretical efforts to find condensed matter systems exhibiting the relevant features, that is, topological ground space degeneracy and anyonic excitations. The interest in concrete Hamiltonian models is manifold: They provide an important testbed for theoretical concepts such as the topological entanglement entropy [3, 4], and may serve as a guide to the experimental search for evidence of their existence. Moreover, systems supporting anyons with computationally universal braiding are a promising avenue for the realization of a quantum computer [5, 6].

A fruitful approach to the realization of topological phases is the study of model systems whose degrees of freedom are geometric objects such as loops or so-called string-nets (labeled trivalent graphs) embedded in a surface [7, 8]. To respect the topology, one attempts to find Hamiltonians whose ground states are topologically invariant, i.e., assign equal amplitudes to configurations that can be smoothly deformed into each other. This invariance property is not sufficient to uniquely fix a topological phase, however. To constrain the system further, it is assumed that the ground states are—as representatives of a particular phase—fixed under a renormalization group (RG) flow, and thus scale-invariant. A corresponding Hamiltonian consisting of local terms can then be constructed by expressing topological invariance and the “fixed-point” property in terms of local constraints [10]. Following this program, Levin and Wen [8] have constructed an exactly soluble “fixed-point” Hamiltonian that realizes, starting from an (essentially arbitrary) modular tensor category, a spin Hamiltonian corresponding to the associated doubled (PT-symmetric) topological phase.

The postulated fix-point property of the ground space under RG is a key ingredient of Levin and Wen’s construction as it motivates the choice of the local constraints. However, the outlined procedure does not provide an RG transformation; in fact, the mere existence of an RG that fixes the ground states is a priori unclear. Here we construct an explicit RG transformation for (2+1)-dimensional string-net models with this property. This establishes that the “fixed-point” wave functions and Hamiltonians of Ref. [8] are indeed the infrared limit of string-net condensed phases, and thus confirms the validity of the heuristic reasoning underlying [8].

The proposed RG transformation can be seen as an instance of entanglement renormalization [9], that is, it proceeds by locally eliminating part of the ground-state entanglement before each coarse-graining step [11]. Due to its structure based on local transformation rules, our RG transformation conserves topological degrees of freedom. In fact, it maps the ground space exactly into the ground-space of the coarse-grained system. These features are analogous to results obtained in [10] for Kitaev’s toric code [5] and its generalizations [18]. In particular, they give rise to an efficient representation of the ground-states as tensor networks (i.e., in terms of the multi-scale entanglement renormalization ansatz (MERA) [9]). They also imply that our RG transformation is a reasonable choice of initial point for numerical (variational) algorithms [11] when studying, e.g., the stability of topological phases under perturbations. Finally, our RG transformation gives an explicit prescription for efficiently preparing “fixed-point” wave functions or reading out topological information using a quantum computer.

Following [8], let $G$ be a trivalent graph embedded in a surface $S$ so that the components of $S \setminus G$ are simply connected (“plaquettes”). The Hilbert space $\mathcal{H}_G$ of a string-net model is spanned by the different networks of labeled, oriented strings living on $G$’s edges. A standard basis for this space is obtained by orienting $G$ and associating to each edge $e$ a Hilbert space $V_e \cong \mathbb{C}^{N+1}$.
with orthonormal basis \( \{ |i\rangle \}_{i=0}^{N} \). Here, \( i \) determines the type and direction of string, with \( i = 0 \) corresponding the absence of a string across edge \( e \). For each \( i \), label \( i^* \) corresponds to a string of the same type but with the opposite direction; \( 0^* = 0 \). Then \( \mathcal{H}_G = \bigotimes_{v} \mathcal{V}_v \). The model is further characterized by branching rules, the set of triples \( \{ i, j, k \} \) of string types that are allowed to come together at a vertex, e.g., \( \{ i, i^*, 0 \} \) is always allowed. We define the physical subspace \( \mathcal{H}^{\text{phys}}_G \subset \mathcal{H}_G \) as the span of all string-net configurations that have an allowed triple at every vertex.

Define a Hamiltonian \( H_G \) acting on \( \mathcal{H}_G \) by

\[
H_G = - \sum_{v} Q_v - \sum_{\text{plaquettes } p} B_p .
\] (1)

Here, for each vertex \( v \), \( Q_v \) is the projection onto the set of allowed net edge triples at \( v \). Thus the first term projects onto \( \mathcal{H}^{\text{phys}}_G \). The second term has a more complicated definition. Let \( F_{ijm}^{kln} \) be an order-six tensor, indexed by string types; satisfying certain conditions roughly described as self-consistency, unitarity and compatibility with the branching rules; see appendix for full details.

For each plaquette \( p \) the plaquette operator \( B_p \) is a projection on the edges bordering \( p \) controlled by the edges with one endpoint on \( p \). More precisely, \( B_p = \sum_i d_i B_i^p / \sum_i d_i^2 \) where \( d_i = 1 / F_{ii^*0}^{ii^*0} \) and \( B_i^p \) acts on a simple plaquette \( p \) with \( r \) boundary edges as

\[
B_i^p |m \rangle_{ijm} = \sum_{k_1,...,k_r} \left( \prod_{\nu=1}^r F_{v^*k_{\nu-1}k_{\nu}}^{v_k k_{\nu-1}k_{\nu}} \right) |m \rangle_{ijm} \to |m \rangle_{ijm} .
\] (2)

identifying \( j_0 = j_e \) and \( k_0 = k_r \). The plaquette and vertex operators commute, and thus the ground space of \( H_G \) is the space simultaneously fixed by all these projections.

In the appendix, we give a natural definition of \( B_i^p \) for more general plaquettes; roughly, \( B_i^p \) adds a loop of type \( i \) around a puncture in the center of \( p \) followed by reduction to the basis of \( \mathcal{H}_G \). Eq. (2) is a special case.

We now focus on the case where \( G \) is the honeycomb lattice \( \mathcal{L} \). Our RG transformation is a map \( \mathbf{R} : \mathcal{H}_\mathcal{L} \to \mathcal{H}_\mathcal{L}' \), where \( \mathcal{L}' \) is a coarser hexagonal lattice, that satisfies:

(i) The physical subspace \( \mathcal{H}^{\text{phys}}_{\mathcal{L}} \) is mapped into \( \mathcal{H}^{\text{phys}}_{\mathcal{L}'} \).

(ii) Local operators on \( \mathcal{H}_{\mathcal{L}} \) are mapped under conjugation by \( \mathbf{R} \) to local operators on \( \mathcal{H}_{\mathcal{L}'} \).

Each plaquette \( p \) of \( \mathcal{L} \) is either retained or eliminated by renormalization. We can show that the form of the plaquette part of the Hamiltonian is preserved under the map \( \mathbf{R} \), in the following sense:

(iii) If \( q \) is a retained plaquette of \( \mathcal{L} \) and \( \tilde{q} \) the corresponding plaquette of \( \mathcal{L}' \), then \( B_q|\mathcal{H}_{\mathcal{L}}' = \mathbf{R}^\dagger B_q \mathbf{R} |\mathcal{H}_{\mathcal{L}}' \), where \( \mathcal{H}_{\mathcal{L}}' \subset \mathcal{H}^{\text{phys}}_{\mathcal{L}} \) is the subspace simultaneously fixed by all \( B_p \) operators for eliminated plaquettes \( p \).

Furthermore,

(iv) The ground space of \( H_\mathcal{L} \) is mapped bijectively to the ground space of \( H_{\mathcal{L}'} \).

The map \( \mathbf{R} \) is defined by a sequence of \( F \)-moves, elementary trivalent graph transformations. As shown in \textbf{FIG. 1}, \( F_e(G) \) is a graph \( G' \) that is the same as \( G \) except with an edge \( e \) reconnecting in a way that corresponds to flipping an edge in the dual graph. Using the tensor \( F_{ijm}^{kln} \), \( F_e \) also defines a linear transformation \( \mathcal{H}_G \to \mathcal{H}_{G'} \), controlled by the labels \( |i,j,k\rangle \) of the edges adjacent to \( e \):

\[
F_e |i\rangle_j |k\rangle = \sum_n F_{ijm}^{kln} |i\rangle_j |n\rangle_k ,
\] (3)

in the standard string-net bases defined above. For each edge \( e \), \( F_e \) maps \( \mathcal{H}^{\text{phys}}_G \) isomorphically to \( \mathcal{H}^{\text{phys}}_{G'} \), and \( F_e |\mathcal{H}^{\text{phys}}_{G} \) can be extended to a unitary on \( \mathcal{H}_{\mathcal{L}'} \).

A second ingredient of \( \mathbf{R} \) are transformations that reduce the number of degrees of freedom by eliminating edges. Suppose that after some \( F \)-moves, the resulting graph \( G \) contains a "tadpole," i.e., a subgraph of the form shown in \textbf{FIG. 2} consisting of a self-loop around plaquette \( p \), and three other edges. We associate with this tadpole the local operator \( Z_p : \mathcal{H}_G \to \mathcal{H}_G \), where \( G' \) is obtained from \( G \) by deleting the tadpole subgraph and replacing edges \( e_3, e_4 \) by a single edge \( e' \):

\[
Z_p = \langle \Phi | e_3 \otimes (0)_{e_2} \sum_i |ii\rangle_{e_3 e_4} \otimes \mathbf{1}_{G\setminus \{e_1,...,e_4\}} ,
\] (4)

where \( |\Phi\rangle = \frac{1}{\sqrt{\sum_i d_i}} \sum_i |i\rangle \). Observe \( Z_p^\dagger \) is an isometry.

The map \( \mathbf{R} \) from the lattice \( \mathcal{L} \) into the coarser lattice \( \mathcal{L}' \) is now given by the sequence of \( F \)-moves indicated in...
followed by eliminating the tadpoles using the $Z_p$ maps.

The properties of the map $R$ rely on two basic claims about the behavior of plaquette operators under $F$-moves and the removal of tadpoles. We show that

**Lemma 1.** For every edge $e$ and plaquette $p$,

$$F_e B_p = B_p F_e,$$

where $p'$ corresponds to the plaquette $p$ in the graph $G' = F_e(G)$. Roughly speaking, $F$-moves “commute” with plaquette operators.

**Lemma 1** implies that the plaquette part $H_G$ is mapped to the plaquette part of $H_{G'}$ under conjugation by $F_e$. A similar statement applies to the removal of a tadpole with head $p$ inside a plaquette $q$: this operation “commutes” with $B_q$ provided we restrict to the subspace fixed by $B_q$.

**Lemma 2.** Consider a tadpole around plaquette $p$ inside a plaquette $q$ as shown in Fig. 3, and let $q'$ be the modified plaquette after removal of the tadpole. Then $B_p$ is a rank-one projection,

$$B_p = |\Phi\rangle\langle\Phi| e_1 \otimes |0\rangle e_2 \otimes \text{id}_{G\setminus\{e_1,e_2\}}$$

with $|\Phi\rangle$ defined as in Eq. (4) and

$$B_q Q_v B_p = Z_p^i B_q Z_p^j.$$

Every ground state $|\Psi\rangle_G$ of $H_G$ is a product state,

$$|\Psi\rangle_G = Z_p^i |\Psi\rangle_{G'}$$

$$= |\Phi\rangle e_1 \otimes |0\rangle e_2 \otimes \left(\sum_i |ii\rangle e_3 e_4 |i\rangle e'_c\right) |\Psi\rangle_{G'}$$

where $|\Psi\rangle_{G'}$ is a ground state of $H_{G'}$.

**Lemma 1** and **Lemma 2** can in principle be verified directly from the explicit expression [2] for the plaquette operators in terms of standard basis vectors. A simpler proof is based on the interpretation of $B_q'$ as adding a “virtual loop” to the surface as explained in [8, Appendix C]. The consistency of this interpretation is guaranteed by Mac Lane’s coherence theorem [12], which shows the required reductions yield the same result independently of the sequence of local rules applied. In terms of this interpretation, **Lemma 1** is immediate since the virtual loops are added in a region that is not affected by $F$-moves. Similarly, **Lemma 2** follows since the operator $B_q$ effectively removes a puncture in the surface located at the center of $p$. We present these details and the proofs in the appendix.

Let us now justify properties (i) and (iv) of $R$. It is easy to check that both $F$-moves as well as the operators $Z_p$ preserve the branching rule at every vertex; this proves (i). Similarly, (ii) immediately follows from the fact that $R$ is made of local operations. Statement (iii) is a direct consequence of **Lemma 1** and **Lemma 2** since Eq. (5) implies $B_q |\Psi\rangle_{H_G} = Z_p^i B_q Z_p^j |\Psi\rangle_{H_{G'}}$. For property (iv), note that the three rounds of $F$-moves in $R$ are unitaries. Therefore we only need to check that $Z_p$, removing a tadpole around $p$ from a graph $G$, is a bijection from the ground space of $H_G$ to the ground space of $H_{G'}$. Again, this directly follows from **Lemma 2**.

Let us remark that **Lemma 1** and **Lemma 2** generalize considerably. In particular, Property (iii) holds even if $B_q$ is replaced by the more general Wilson loop operators discussed in [8] that can act nontrivially on the ground space. The operator $Z_p$ is a special case of surgery between two surfaces, one of which is the sphere in this case. A version of **Lemma 2** holds for general surgery.

Every iteration of the RG transformation $R$ reduces the number of sites of the lattice $L$ by one-third. In the case that $L$ is embedded in the infinite plane, the unique ground state $|\Psi\rangle_L$ is a fixed point of $R$ (by property (iv)). More interesting are cases with a topological ground space degeneracy, e.g., a finite system on a torus $L$. A ground state $|\Psi\rangle_L$ of $H_L$ is eventually reduced to a ground state $|\Psi\rangle_{\text{top}}$ of an effective Hamiltonian on a small number of edges; both the state and the Hamiltonian encode the topological features of the original state/model.

In the terminology of entanglement renormalization [4], we can think of $R$ as being made of disentanglers $U : V^S \rightarrow V^S$ (e.g., the first round of $F$-moves) and isometries $W : V^O \rightarrow V^O$ (the remaining $F$- and $Z$-moves). $W$ replaces a triangle by a single vertex. This pattern of operations has also been applied in the context of an RG transformation for classical partition functions [13]. By reversing $R$, we obtain an explicit, logarithmic-depth quantum circuit $C$ to prepare $|\Psi\rangle_L$ from $|\Psi\rangle_{\text{top}}$ using local gates [21]. This is a consequence of the recursive character of the RG transformation. It should be contrasted with [14], where it is shown that...
that the creation of a topologically ordered state takes a time linear in the system size if it is based on local Hamiltonian evolution.

In summary, the RG transformation presented here provides both a theoretical foundation and a concrete tool for the study of string-net condensation as a model for topologically ordered phases. Its simple description in terms of the underlying tensor category translates into an efficient representation of the ground-states. This gives a theoretical indication of the suitability of appropriate numerical RG procedures in the study of topologically ordered systems, thereby adding to the evidence for their remarkable precision \[11\].

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APPENDIX A: BASIC DEFINITIONS FOR GENERAL STRING-NET MODELS

We first review the properties that the tensor \( F_{ijkl}^{ijlm} \) needs to satisfy in order to define a string-net model. Start by encoding the branching rules into a tensor \( \delta_{ijk} \), with \( \delta_{ijk} = 1 \) if string types \( i, j, k \) are allowed to come together at a vertex, and \( \delta_{ijk} = 0 \) otherwise. The branching rules are assumed to satisfy the Kronecker delta. Assume that the \( F \) tensor satisfies

\[ F_{ijkl}^{ijlm} \text{ needs to satisfy } \delta_{ijk} = \delta_{ijl}, \]

\[ F_{ijkl}^{ijlm} = F_{jikl}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{ikjl}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{klji}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{ljik}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{kjil}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{klij}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{lkji}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{iljk}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{jikl}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{ikjl}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{lji}^{ijlm}, \]

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\[ F_{ijkl}^{ijlm} = F_{lji}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{jikl}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{ikjl}^{ijlm}, \]

\[ F_{ijkl}^{ijlm} = F_{lkji}^{ijlm}, \]
for all $i, j, \ldots, s$:

\[
F_{kln}^{ijm} \delta_{ijm} \delta_{kln} = F_{kln}^{ijm} \delta_{sln} \delta_{jkn} \quad (A1)
\]

physicality: \quad pentagon identity: \quad unitarity: \quad tetrahedral symmetry:

\[
\sum_{n=0}^{N} F_{kpm}^{iql} F_{ijm}^{jkl} F_{jnm}^{klm} = F_{kpm}^{iql} F_{ijm}^{jkl} F_{jnm}^{klm} \quad (A2)
\]

\[
(F_{kln}^{ijm})^* = F_{i*nl}^{*j*m} \quad (A3)
\]

\[
F_{kln}^{ijm} = F_{kln}^{kjm} = F_{kln}^{kim} \sqrt{\prod_{d=1}^{d=3} d_d} \quad (A4)
\]

normalization:

\[
F_{i*j*k}^{i*} = \sqrt{\prod_{d=1}^{d=3} d_d} \delta_{ijk} \quad (A5)
\]

where $d^{-1} = F_{ii*0}^{*0} \neq 0$. Then via Eqs. (1) (2) and

\[
Q_v = \sum_{i,j,k} \delta_{ijk} |i\rangle \langle j| |j\rangle \langle k| , \quad (A6)
\]

the tensor $F_{kln}^{ijm}$ gives rise to a Hamiltonian $H_L$ of a string-net model on the honeycomb lattice $\mathcal{L}$. To define $H_G$ for more general trivalent graphs $G$, though, we need to extend the definition [2] of the operators $B_p^i$ to arbitrary plaquettes.

Recall that $G$ is embedded in a surface $S$. Put a puncture in the interior of each plaquette of $G$, and let $S^*$ be the resulting punctured surface. A smooth string net is an equivalence class of directed trivalent graphs embedded in $S^*$, where the edges carry string labels [cf. 8, Appendix C] for the case of the honeycomb lattice). The equivalences consist of isotopy, i.e., smooth deformations of the embedding in $S^*$ (for example, crossing punctures is not allowed), and of reversing the direction of an edge labeled $i$ while changing the label to $i^*$. Any smooth string net representative embedded in $G \subset S^*$ can be associated with one of the basis vectors of $\mathcal{H}_G = \bigotimes \mathcal{V}_e$ in the natural way, assigning $|0\rangle$ for any edge not crossed by the smooth string net. More generally, every smooth string net on $S^*$ uniquely determines an element of $\mathcal{H}_G$ by applying some sequence of the following local substitution rules to obtain a linear combination of smooth string nets in $G$:

\[
\text{Example 1.} \quad \text{A smooth string-net “bubble” with three incoming edges and no interior punctures can be simplified to a trivalent vertex by, e.g., applying an F-move to the edge labeled } l, \text{ using Eqs. (A10) and (A9) followed by applying an F-move to the edge labeled } m \text{ and simplifying with Eqs. (A9) (A8) and (A5)}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example1}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Example 2.} \quad \text{The operator } B_p^i \text{ adds a loop of type } i, \text{ followed by expanding the resulting smooth string net into a sum of standard basis vectors. For example,}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example2}
\end{array}
\end{array}
\]

\[
\text{Here in the first step we have applied three F-moves, and in the second step we have applied Eq. (A11) three times and simplified. The puncture in plaquette } p \text{ is marked by } x. \text{ Thus we have derived Eq. (2) for the case that } p \text{ has } r = 3 \text{ sides.}
\]

\[
\text{Remark. The derivation in Example 2 suggests a convenient shorthand rule for determining the action of } B_p^i \text{. First, draw a loop with label } i \text{ going counterclockwise along the boundary inside plaquette } p. \text{ Then, formally replace each T-junction as shown:}
\]

\[
\text{Finally, identify primed variables at adjacent junctions, and sum over the remaining primed variables. It is easy to check that this rule computes } B_p^i, \text{ although special care must be taken to apply the rule to a plaquette with degenerate boundary.}
\]
APPENDIX B: PROOFS OF LEMMAS 1 AND 2

Proof of Lemma 1. We claim that \( F_e B_p^i = B_p^i F_e \). Since \( B_p^i \) is defined as adding a loop of type \( i \) followed by reduction to the standard basis of the graph, this claim is equivalent to the following diagram commuting:

\[
\begin{align*}
\begin{array}{c}
\Large{\star_p} \\
\end{array}
\end{align*}
\]

To simplify the diagram, we have drawn only \( G \) and \( G' = F_e(G) \), instead of writing superpositions of basis states.

Now the left half of this diagram commutes since \( e \) is separated away from the puncture. The right half of the diagram commutes by Mac Lane’s coherence theorem, since the two ways around it are different ways of reducing to \( \mathcal{H}_G' \).

Thus Lemma 1 is a nearly immediate corollary of Mac Lane’s coherence theorem. This simple proof shows the usefulness of defining \( B_p^i \) using smooth string nets. A similar argument shows that \( [B_p^i, B_q^j] = 0 \) for all plaquettes \( p, q \) and all string-net types \( i, j \), as we asserted below Eq. (2).

For the proof of Lemma 2, we first show the following rule that applies to smooth string nets:

Lemma 3.

\[
B_p^i \star_p = B_p^i \star_p \quad (B1)
\]

Intuitively, Lemma 3 says that applying \( B_p \) effectively removes from \( S^* \) the puncture \( p \) by allowing strings to be carried over it isotopically. The proof is by applying two \( F \)-moves. Let \( D = \sqrt{\sum_k d_k^2} \), the “total quantum dimension.”

Proof. By definition of \( B_p \),

\[
D^2 B_p^i \star_p = \sum_j d_j B_p^i \star_p
\]

\[
= \sum_j d_j \star_j
\]

\[
= \sum_j d_j F_p^{\star_j} \star_j
\]

\[
= \sum_{j,k} \sqrt{\sum d_i d_k} \delta_{i,j} \star_k
\]

We have made an \( F \)-move and used Eq. (A5). Every smooth string net depicted above represents the corresponding element of \( \mathcal{H}_G \); the use of Mac Lane’s theorem is implicit. Now by symmetry,

\[
\sum_{j,k} \sqrt{\sum d_i d_k} \delta_{i,j} \star_k = D^2 B_p^i \star_p = D^2 B_p^i \star_p
\]

Proof of Lemma 2. First, note that

\[
B_p^j \star_p = B_p^j \star_p
\]

\[
= \delta_{j0} B_p^j \star_p
\]

\[
= \delta_{j0} d_k B_p^j \star_p
\]

\[
= \delta_{j0} d_k \sum_i H_e^i \star_p = D B_p^i \Delta \star_p
\]

where we have applied Lemma 3 and Eqs. (A8) and (A9). Eq. (6) follows since \( B_p \) is a projection.

Now we can argue that \( B_q^i Q_v B_p^i = Z_q^i B_q^i Z_p^i \), from which Eq. (7) follows. On the left-hand side we know from (6) and (B2)

\[
Q_v B_p^i = |\Phi\rangle_{e_1} \otimes |0\rangle_{e_2} \otimes \sum_i |ii\rangle_{e_3 e_4}
\]

\[
= DB_p^i |0\rangle_{e_1} \otimes |0\rangle_{e_2} \otimes \Delta \Delta
\]

where \( \Delta = \sum_j |j\rangle_{e'} \langle j|_{e_3 e_4} \). Similarly, we have

\[
Z_q^i B_q^i Z_p^i = DB_p^i \Delta \Delta \otimes |0\rangle_{e_1} \otimes |0\rangle_{e_2} \otimes \Delta \Delta
\]

Thus we need only verify that \( B_p^i B_q^j \Delta \Delta \otimes |0\rangle_{e_1 e_2} = B_p^i |00\rangle_{e_1 e_2} \otimes \Delta \Delta \). Indeed, letting \( \text{red}_G \) (resp. \( \text{red}_G' \)) mean reducing the smooth string net to \( \mathcal{H}_G \) (resp. \( \mathcal{H}_G' \)),

\[
B_p^i B_q^j (\text{red}_G \otimes \Delta \Delta |00\rangle_{e_1 e_2}) = B_p^i (|00\rangle_{e_1 e_2} \otimes \Delta \Delta)
\]

\[
= B_p^i \text{red}_G
\]

\[
= B_p (|00\rangle_{e_1 e_2} \otimes \Delta \Delta)
\]
where the first and last equalities are by definition of $B_q^i$ and $B_q'^i$, the second equality is by Lemma 2, and the third equality is because the exact same sequence of steps can be used to reduce the pictured smooth string net to $\mathcal{H}_G$ as can be used to reduce it to $\mathcal{H}_G'$. Eq. (8) now follows immediately.
