PARAMETRIC DWARF SPHEROIDAL TIDAL INTERACTION

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ABSTRACT

The time-dependent tidal interaction of the Local Group dwarf spheroidal (dSph) galaxies with the Milky Way (MW) can fundamentally affect their dynamical properties. The model developed here extends earlier numerical descriptions of dSph-MW tidal interactions. We explore the dynamical evolution of dSph systems in circular or elliptical MW orbits in the framework of a parametric oscillator. An analytic model is developed and compared with more general numerical solutions and N-body simulation experiments.

Subject headings: galaxies: dwarf — galaxies: kinematics and dynamics — Local Group — stellar dynamics

1. INTRODUCTION

The Local Group dwarf spheroidal (dSph) galaxies may be sensitive probes of the chemical and dynamical formation history of the Milky Way (MW; e.g., Harbeck et al. 2001). They have also been used to constrain dark matter models and early structure formation scenarios in the universe (e.g., Łokas 2002).

Controversy over whether the dSph galaxies are equilibrium dynamical systems (e.g., Mateo 1998) or tidally driven nonequilibrium objects (Kuhn & Miller 1989, hereafter KM; Kroupa 1997) persists. The primary observational evidence for tidally dominated dynamics comes not only from the dSph high velocity dispersions and highly elliptical shapes but also from measurements of a spatially variable velocity dispersion (Kleyna et al. 2002) and significant stellar populations beyond the dSph nominal tidal radii (Kuhn, Smith, & Hawley 1996; Martínez-Delgado et al. 2001). While there is undisputed evidence that the dSph galaxy in Sagittarius is tidally disrupted (Ibata, Gilmore, & Irwin 1994), some practitioners have devised more complex multiparametric models that could formally account for the kinematic data in some of the high-M/L dwarfs using dark matter (Kleyna et al. 2002; Wilkinson et al. 2002; Łokas 2002).

A theme of the tidal excitation argument presented here is that the gravitational interaction scale for transferring energy to the dSph stellar system from the dSph-MW orbit can be much larger than the dSph classical tidal radius. In those systems where the stellar crossing time can be comparable to the “period” of the external tide, the simple static arguments for equilibrium break down. Early calculations (KM) demonstrated this possibility using particle-mesh gravitational N-body simulations in idealized circular dSph orbits.

This paper develops an analytic formulation of the dSph-MW tidal interaction problem. Once again, by treating the response of the dSph in terms of galaxy oscillations, but now incorporating the tidal interaction through the Mathieu equation, it is possible to understand the evolution of these tidally interacting systems in noncircular orbits far from the simple resonance condition. As a parametric oscillator, we find that a dSph stellar system can be “inflated” even when its characteristic pulsational spectrum is not tuned to its MW orbital dynamics.

2. PARAMETRIC TIDAL INTERACTIONS

An oscillator driven by a periodic forcing function can sometimes be described by the Mathieu equation

$$\ddot{x} + \omega^2_0(1 - \epsilon \cos 2\omega t)x = 0. \quad (1)$$

Here $x(t)$ describes the oscillator amplitude function, $\omega_0$ is the oscillator resonant frequency, $\omega$ is the driving frequency, and $\epsilon$ is a small constant. Solutions to equation (1) are most easily developed by a perturbative multiscales analysis (Bender & Orzag 1978). One finds that, in general, growing (unstable) oscillatory solutions exist for $\omega = \omega_0/n, n \in \mathbb{N}$.

An important property of equation (1) is that even for a weakly damped (high-“Q”) oscillator a significant, nonzero frequency range for the driving force leads to growing unstable solutions. For small intrinsic damping this result is independent of the frequency width of the natural resonance. For example, near the first resonance $\omega$ between $\omega_0(1 - \epsilon/4)$ and $\omega_0(1 + \epsilon/4)$ leads to instability. The second resonance near $\omega = \omega_0/2$ is unstable for a narrower frequency range

$$\frac{1}{24}\epsilon^2\omega > \omega - \frac{\omega_0}{2} > -\frac{5}{24}\epsilon^2\omega. \quad (2)$$

We show here that the internal dynamics of stellar systems in the MW’s gravitational environment can be described as a two-dimensional coupled parametric oscillator. These equations can be solved analytically using multiscale perturbation methods to explain the dynamical and morphological properties of the Local Group dSph.

2.1. Tide Geometry

Each of the dSph galaxies orbits the MW at a distance that is long compared to the Galactic disk scale length so that it is reasonable to treat the MW as a purely spherical central force problem. We define the MW central force

$$F_{\text{MW}} = F(r) = F(r)\hat{r}. \quad (3)$$

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Letting \( r \) describe the displacement between the MW and dSph centers and \( \delta r \) be the displacement to a dSph star from the dSph center, we write the MW tidal force on the star at lowest order as

\[
\Delta F_{MW} = F(r + \delta r) - F(r) = [F(||r + \delta r||) - F(||r||)]\delta r + \sin(\Delta \varphi)F(||r||)\delta \varphi. \tag{4}
\]

Here \( \Delta \varphi \) is the central angle between a dSph star, the MW, and the dSph center and \( \delta r \) and \( \delta \varphi \) are unit vectors in the radial and central angle directions, respectively (Fig. 1). Since the spatial extent of the dSph is small compared to its distance from the Galactic center, we expand equation (4) using

\[
F(||r + \delta r||) - F(||r||) = \frac{dF}{dr}(r)\delta r \cdot \hat{r}, \tag{5a}
\]

\[
\sin(\Delta \varphi) = \frac{1}{r} \delta r \cdot \hat{\varphi}. \tag{5b}
\]

We define a Cartesian \((x, y)\)-coordinate system centered on the dSph that contains the MW and the star so that we can express \( \delta r \), \( \delta \varphi \), and \( \hat{\varphi} \) in terms of \( \hat{x} \) and \( \hat{y} \):

\[
\delta r = x\hat{x} + y\hat{y}, \tag{6a}
\]

\[
\hat{r} = \cos \varphi \hat{x} + \sin \varphi \hat{y}, \tag{6b}
\]

\[
\hat{\varphi} = -\sin \varphi \hat{x} + \cos \varphi \hat{y}. \tag{6c}
\]

The MW orbital time dependence is contained in \( r \) and \( \varphi \). To this order of approximation these equations also describe the three-dimensional stellar problem where \((x, y)\) are the projected coordinates in the dSph-MW orbital plane. The tidal force perpendicular to the orbital plane yields harmonic motion that is not coupled to the MW orbital motion and is ignorable. Thus, the tidal force becomes

\[
\Delta F_x = \frac{dF}{dr}(r) \left( x\cos^2 \varphi + \frac{y}{2} \sin 2\varphi \right) + \frac{F(r)}{r} \left( x\sin^2 \varphi - \frac{y}{2} \sin 2\varphi \right), \tag{7a}
\]

\[
\Delta F_y = \frac{dF}{dr}(r) \left( y\sin^2 \varphi + \frac{x}{2} \sin 2\varphi \right) + \frac{F(r)}{r} \left( y\cos^2 \varphi - \frac{x}{2} \sin 2\varphi \right). \tag{7b}
\]

The form of the MW potential determines \( dF/dr \) and \( F \). For a Keplerian potential it is \( dF/dr = -2F(r)/r \), and for the logarithmic case it is \( dF/dr = -F(r)/r \). Hence, we can write a simplified form of the tide for these two potential forms:

\[
\Delta F_{\text{Kepl}} = \frac{3}{2} \frac{dF}{dr} \left[ \left( x\cos 2\varphi - \frac{x}{3} + \sin 2\varphi \right) \hat{x} \right. - \left. \left( -y\cos 2\varphi - \frac{y}{3} + x \sin 2\varphi \right) \hat{y} \right], \tag{8a}
\]

\[
\Delta F_{\text{log}} = \frac{dF}{dr} \left[ \left( x\cos 2\varphi + y \sin 2\varphi \right) \hat{x} + \left( -y\cos 2\varphi + x \sin 2\varphi \right) \hat{y} \right]. \tag{8b}
\]

The numerical factors are of course different for different potential assumptions, but the form of the resonant solution remains unchanged. It seems likely that the MW potential is actually logarithmic in the domain of most dSph galaxies so that we assume the form

\[
\phi = v_c^2 \ln \left( \frac{r}{r_0} \right). \tag{9}
\]

2.2. Solution of the Coupled Mathieu Equations
at First Order

2.2.1. Equation of Motion for a Star near the Core

In the dSph reference frame a star experiences forces due to the MW tide and the dSph potential. The dSph potential generates a radial acceleration, \( a_r = -GM(r)/r^2 \), where \( M(r) \) is the mass enclosed within a spherical volume of radius \( r \) centered on the dSph. Near the center of the dSph we approximate \( M(r) \approx \rho_04\pi r^3/3 \) so that we can write \( a_r \approx -\omega_0^2r \) with \( \omega_0^2 = 4\pi G\rho_0/3 \). In this case the total dSph-frame acceleration of a star can be written as

\[
\frac{d^2r}{dt^2} = -\omega_0^2r + \frac{\Delta F}{m}. \tag{10}
\]

With additional definitions,

\[
k = \frac{1}{m} \frac{dF}{dr}(r), \tag{11a}
\]

\[
\varepsilon = \frac{k}{\omega_0^2}; \quad \omega_0^2 = \frac{\omega_c^2}{\omega_0^2}, \tag{11b}
\]

where the circular frequency of the dSph around the MW is \( \omega_c = v_c/r \). Combining these terms, we finally obtain a recognizable set of coupled equations that describe the behavior of a star in the center of a dSph as projected in the \((x, y)\)-plane:

\[
\ddot{x} + \omega_0^2(1 - \varepsilon \cos 2\varphi)x = \varepsilon \omega_c^2 \sin(2\varphi)y, \tag{12a}
\]

\[
\ddot{y} + \omega_0^2(1 + \varepsilon \cos 2\varphi)y = \varepsilon \omega_c^2 \sin(2\varphi)x. \tag{12b}
\]
Notice that the coupling between $x$ and $y$ motion results from the anisotropic form of the galactic tide where the expansive and compressive tidal contributions are both important in determining the dSph dynamical response.

2.2.2. Galaxy Oscillations

Individual stars also experience forces due to the dSph gravitational perturbations, but equations (12a) and (12b) neglect temporal variations [due to $\rho(t)$] in the dSph potential. Thus, there is an additional acceleration term that should be included here at late times in the dSph-tide evolution when the temporal variability of the dSph density is important.

The resulting oscillations in the mean stellar orbits are not included above but can be described in a manner that is analogous to polytropic stellar normal mode oscillations. This description of galaxy dynamics in terms of linear oscillations was the basis of KM’s prediction that Local Group dSph galaxies could be tidally inflated even when the stars were contained within the static tidal radius. Miller & Smith (1994) and Vandervoort (1999) have now demonstrated that weakly damped or even growing finite amplitude normal mode galaxy oscillations are a physically interesting feature of realistic galaxy dynamics simulations.

The lowest order oscillation mode should have no radial nodes, as is the case for radial oscillations in stellar polytropes (e.g., Cox 1980). Vandervoort (1999) showed that the total Lagrangian radial displacement summed over all stars in an $N$-body system exhibits a very similar oscillatory behavior. This is not entirely intuitive given that the collisionless orbital motion of stars in a galaxy occurs over the length scale of the galaxy, while a mass element in a stellar polytrope oscillates only over the small scale of the oscillation displacement amplitude. Nevertheless, Vandervoort (1999) proves that the fundamental $N$-body radial eigenmode for the mean Lagrangian radial particle displacement, $\xi$, satisfies

$$\frac{d^2 \xi}{dt^2} = -\omega_0^2 \xi, \quad (13)$$

with $\omega_0^2 = -W/I = G \int M(r)\rho(r) dr/\int \rho(r)r^4 dr$ and $\xi \propto r$. Here $W$ and $I$ are the potential energy and moment of inertia, respectively, while $\rho(r)$ and $M(r)$ are the galaxy density and enclosed mass distribution, respectively, and $G$ is the gravitational constant. Realistic density distributions lead to frequencies that are of order $\pi(G\rho_0)^{1/2}$, where $\rho_0$ is the central density.

2.2.3. Circular Orbit

If we launch the dSph on a circular orbit around the MW, then $\varphi = \omega t$ and equations (12a) and (12b) describe a set of two-dimensional Mathieu equations. We now show that these equations exhibit resonant instability around $\omega = \omega_0$.

To obtain a valid solution near the resonance, we use a multimescale analysis in terms of $t$ and $\tau = \epsilon t$ and develop the relevant variables in terms of the perturbation parameter $\epsilon$:

$$\omega_0 = \omega + \epsilon \omega_1, \quad (14a)$$

$$x(t, \tau) = x_0(t, \tau) + \epsilon x_1(t), \quad (14b)$$

$$y(t, \tau) = y_0(t, \tau) + \epsilon y_1(t). \quad (14c)$$

The equations obtained from the lowest order ($\epsilon^0$) for $x_0$ and $y_0$ are simple harmonic oscillators with frequency $\omega_0$ as we would expect in a harmonic system. From the lowest order $x_0$ and $y_0$ solutions and first-order equations we can construct a refined solution. Using complex notation, we write

$$x_0(t, \tau) = A(\tau)e^{i\omega_0 t} + c.c., \quad (15a)$$

$$y_0(t, \tau) = B(\tau)e^{i\omega_0 t} + c.c. \quad (15b)$$

Gathering the terms of order $\epsilon^1$, we obtain differential equations for $x_1$ and $y_1$:

$$\frac{\partial^2 x_1}{\partial t^2} + \omega^2 x_1 = -2\omega_1 x_0 + \omega^2 \cos(2\omega t)x_0 + \omega^2 \sin(2\omega t)y_0 - 2 \frac{\partial^2 x_0}{\partial \tau^2} \quad (16a)$$

$$\frac{\partial^2 y_1}{\partial t^2} + \omega^2 y_1 = -2\omega_1 y_0 - \omega^2 \cos(2\omega t)y_0 + \omega^2 \sin(2\omega t)x_0 - 2 \frac{\partial^2 y_0}{\partial \tau^2}. \quad (16b)$$

These are combined with the lowest order solutions for $x_0$ and $y_0$. We construct a solution for $A(\tau)$ and $B(\tau)$ using the well-known Ansatz (Fredholm alternative) of forcing all harmonic terms that lead to unstable (secular) solutions to vanish. Obtaining a solution with unbounded coefficients implies that there is no stable oscillatory solution for $x$ and $y$. Thus, we force the coefficients of $e^{i\omega t}$ and $e^{-i\omega t}$ on the right-hand side of equations (16a) and (16b) to vanish to obtain two differential equations (and their conjugates) for the functions $A(\tau)$ and $B(\tau)$:

$$-2\omega_1 A(\tau) + \omega^2 A^*(\tau) + \omega \frac{dA(\tau)}{d\tau} = 0, \quad (17a)$$

$$-2\omega_1 B(\tau) - \omega^2 B^*(\tau) - \omega \frac{dB(\tau)}{d\tau} = 0. \quad (17b)$$

We solve these four coupled differential equations in terms of the real and imaginary parts of $A$ and $B$, i.e., $(\Re(A), \Im(A), \Re(B), \Im(B))$. This yields the matrix equation

$$\frac{dX}{d\tau} = MX, \quad (18)$$

where

$$X = \begin{pmatrix} \Re(A) \\ \Im(A) \\ \Re(B) \\ \Im(B) \end{pmatrix}, \quad (19)$$

$$M = \begin{pmatrix} 0 & -(4\omega_1 + \omega) & -\omega & 0 \\ 4\omega_1 - \omega & 0 & 0 & \omega \\ -\omega & 0 & 0 & -(4\omega_1 - \omega) \\ 0 & \omega & (4\omega_1 + \omega) & 0 \end{pmatrix}. \quad (20)$$

This equation is solved most readily by finding the
eigenvectors of $M$. Taking

$$H = \sqrt{\omega + 2\omega_1 \over \omega - 2\omega_1},$$

(21)

the eigenvectors of $M$ of respective eigenvalues $(\omega^2 - 4\omega_1^2)^{1/2}/2, -(\omega^2 - 4\omega_1^2)^{1/2}/2, i\omega_1,$ and $-i\omega_1$ are

$$
\begin{pmatrix}
-H \\
1 \\
1 \\
H
\end{pmatrix},
\begin{pmatrix}
H \\
1 \\
1 \\
-H
\end{pmatrix},
\begin{pmatrix}
1 \\
-i \\
i \\
1
\end{pmatrix},
\begin{pmatrix}
1 \\
i \\
-i \\
1
\end{pmatrix}.
\end{equation}

(22)

The solution to equation (18) is obtained in terms of four complex constants of integration $C = (\alpha, \beta, \gamma, \delta)$ and the eigenvectors as

$$
\alpha \exp \left( \frac{\sqrt{\omega^2 - 4\omega_1^2}}{2} \right) \tau = -H \Re(A) + \Im(A) + \Re(B) + H\Im(B),
\end{equation}

(23a)

$$
\beta \exp \left( -\frac{\sqrt{\omega^2 - 4\omega_1^2}}{2} \right) \tau = H\Re(A) + \Im(A) + \Re(B) - H\Im(B),
\end{equation}

(23b)

$$
\gamma e^{i\omega_1 \tau} = \Re(A) - i\Im(A) + i\Re(B) + \Im(B),
\end{equation}

(23c)

$$
\delta e^{-i\omega_1 \tau} = \Re(A) + i\Im(A) - i\Re(B) + \Im(B).$
\end{equation}

(23d)

Note that in the case $\omega > |2\omega_1|$, which will interest us in the following, $\alpha$ and $\beta$ are necessarily real.

At late times the growing exponential term dominates, and these equations simplify to yield expressions for $A$ and $B$.

$$A(\tau) = \alpha \frac{4H}{\omega - 2\omega_1} \exp \left( \frac{\sqrt{\omega^2 - 4\omega_1^2}}{2} \right) \frac{\tau}{\varepsilon t} (1 \mp iH),$$

(24a)

$$B(\tau) = \alpha \frac{4H}{\omega + 2\omega_1} \exp \left( \frac{\sqrt{\omega^2 - 4\omega_1^2}}{2} \right) \frac{\tau}{\varepsilon t} \left( H + i \right).$$

(24b)

Evidently $A$ and $B$ define the mean position of a star in the dSph,

$$x(t) = \alpha' \exp \left( \frac{\sqrt{\omega^2 - 4\omega_1^2}}{2} \right) \frac{\varepsilon t}{\omega} \cos(\omega t + \theta),$$

(25a)

$$y(t) = \alpha' \exp \left( \frac{\sqrt{\omega^2 - 4\omega_1^2}}{2} \right) \frac{\varepsilon t}{\omega} \sin(\omega t + \theta).$$

(25b)

Equations (25a) and (25b) show that instability occurs if $\omega > |2\omega_1|$. Thus, if the circular frequency of the orbit satisfies

$$|\omega_0 - \omega_c| < \frac{\omega_c \varepsilon}{2},$$

(26)

the dSph will be tidally excited. The constants $C$ determine the initial position and velocity of a given star in the dSph. Notice that the growing eigenmode solution involves stars oriented along a line that makes an angle $\theta$ with respect to the $x$-axis where $\tan \theta = -H$. Thus, we find in general that stars expand away from the center of the dSph depending on the difference between the dSph orbital frequency and $\omega_0$ and on the perpendicular distance a star makes to a line in the $(x, y)$-plane of the dSph at angle $\theta$ with respect to the line of sight toward the MW.

Extending this calculation to second order in $\varepsilon$ reveals that, unlike the simple one-dimensional case, the next resonance near $\omega_0/2$ is stable with no secular amplitude variation.

### 2.2.4. Elliptical Orbit

Equations (25a) and (25b) assume a circular orbit. Their applicability is also restricted, depending on the central concentration and oscillation frequency $\omega_0$ of the dSph. For example, the perturbation parameter, $\varepsilon = \omega_0^2/\omega_c^2$, can approach unity for a circular orbit in a logarithmic potential. Under these conditions the static tide force alone is strong enough to disrupt the dSph. Since the dSph galaxies are generally not in circular orbits, it is important to extend the model to elliptical orbits. In a logarithmic MW potential we can recover the form of the Mathieu equation with the epicycle approximation. We define an “eccentricity” $\varepsilon$ from $r(t)$, where $\kappa = \sqrt{2\omega}$ is the epicycle frequency for a logarithmic potential. It follows that (Binney & Tremaine 1987)

$$r(t) = r_c [1 - \varepsilon \cos(\kappa t)],$$

(27a)

$$\varphi(t) = \omega_0 \tau + \varepsilon \sqrt{2} \sin(\kappa t).$$

(27b)

Here $k = [v_c/r(t)]^2 \approx \omega_0^2 [1 + 2\varepsilon \cos(\kappa t)],$ where $v_c = r_c \omega_c$ is the circular velocity of the potential. We note that with this convention for $r(t)$, the dSph is at perigalacticon at $t = 0$. Replacing $k(t) = \varepsilon(t) \omega_0^2$ and $\varphi(t)$ in equations (12a) and (12b) gives, at first order in $\varepsilon$,

$$\dot{x} + \omega_0^2 \left[ 1 - \varepsilon \cos(\omega t) - \sum_x b_x e^{i\delta_x} \cos(2a_x \omega_0 t) \right] x = \omega_0^2 \left[ e_x \sin(2\omega_0 t) + \sum_x b_x e^{i\delta_x} \sin(2a_x \omega_0 t) \right] y,$$  

(28a)

$$\dot{y} + \omega_0^2 \left[ 1 + \varepsilon \cos(\omega t) + \sum_x b_x e^{i\delta_x} \cos(2a_x \omega_0 t) \right] y = \omega_0^2 \left[ e_x \sin(2\omega_0 t) + \sum_x b_x e^{i\delta_x} \sin(2a_x \omega_0 t) \right] x,$$

(28b)

where $a_x = 1 \pm \sqrt{2}/2$, $b_x = 1 \pm \sqrt{2}$, and $\varepsilon_x = \omega_0^2/\omega_c^2$. Equations (28a) and (28b) have a form identical to equations (12a) and (12b) where $\omega$ is replaced by $a_x \omega_c$ and $\varepsilon$ becomes $b_x e_x$. Evidently two sets of resonant frequencies occur when $\omega_c (1 + \sqrt{2}/2)$ or $\omega_c (1 - \sqrt{2}/2)$ is near $\omega_0$. These frequencies are sufficiently different that we treat them as independent resonant solutions. It follows that the form of the growing mode late-time solution for $x(t)$ and $y(t)$ in an elliptical orbit is

$$x(t) = \alpha' \exp \left[ \frac{\sqrt{(a_x \omega_c)^2 - 4\omega_0^2}}{2} \right] b_x |\varepsilon_x| t \cos(a_x \omega_0 t + \delta_x),$$

(29a)
where $\omega_1$ has been defined by $\omega_0 = a_2 \omega_2 + b_2 e e_2 \omega_1$. Here $\tan \theta_+ = -H_+ \text{ and } \tan \theta_- = 1/H_+ \text{, where } H_\pm$ has the form of equation (21) with $\omega$ replaced by $a_2 \omega_2$. Comparing to the circular orbit solution, we see that the expansion parameter $\varepsilon$ becomes $(\omega_0^2/\omega_2)^2 b_2 \text{ so that instability occurs when } |\omega_0 - a_2 \omega_2| < \omega_2 (\omega_0^2/\omega_2^2)^2/b_2 |/2$. This expression is comparable to the circular orbit solution given by equation (26), but resonance is “easier” to reach even as the strength of the tidal force compared to the binding force [which is given by $\varepsilon = k_e/\omega_0^2 = (\omega_2/\omega_0)^2 \sim 1/a_2^2$] can be weaker by a factor of 3 or more compared to the circular orbit.

Expanding $\varphi(t)$ and $\varepsilon(t)$ to order 3 in $e$ yields additional resonant behavior. Analogous to the above expansion, we find two new sets of excitation frequencies. The additional terms in the differential equation for $x(t)$ and $y(t)$ above have the form $d_2 \varepsilon^2 e \cos(2b_2 \omega_1 t)$ and $d_3 \varepsilon^2 e \cos(2b_3 \omega_1 t)$, with corresponding sine terms as in equations (28a) and (28b). Here $c_b = 1 \pm (3/2)^{1/2}$ and $d_{b+n}$, $n \in \{2, 3\}$, are numerical factors of order unity.

Thus, the higher order terms lead to excitations at frequencies where $\omega_0$ is $b_2 \omega_1$ (order 2) and at $c_b \omega_1$ (order 3). The growth rates of these modes and instability range for $\omega_0$ are smaller by corresponding powers of $e$ so we expect to excite higher order modes only as the orbital eccentricity increases.

2.2.5. Projection on the Line of Sight

An interesting, and perhaps observable, signature from the simulations is the elliptical shape caused by the strong geometric asymmetry in the growing mode solution for the distance of a star away from the center of the dSph. Here we consider how the induced elliptical shape varies with respect to the line of sight from the MW center of force toward the dSph center. Consider, for the sake of generality, that the excitation is encountered at a frequency $a \omega_1$, where $a$ can be chosen from the set $\{1, a_2, b_2, c_b\}$, with the corresponding exponential term $E(t)$. Thanks to equations (6b) and (6c), we obtain (where $t \equiv 0$ corresponds to perigalacticon)

\begin{align*}
(x \ddot{x} + y \ddot{y}) \cdot \ddot{r} &= E(t) \cos[(a - 1) \omega_1 t + \theta], \\
(x \ddot{x} + y \ddot{y}) \cdot \dot{e} &= E(t) \sin[(a - 1) \omega_1 t + \theta].
\end{align*}

(30a)

(30b)

In the circular case ($a = 1$) the dSph is expanded along an axis rotated away from the line of sight to the MW by an angle $\theta$ with $\tan \theta = -H$. For elliptical orbits the bar appears to rotate. At order $n$ the bar formed by the tidal interaction will be turning at a frequency $n \omega_0/2$. Thus, the dSph orbits between successive perigalacticon, the bar will turn $n/2$ times.

3. NUMERICAL CALCULATIONS AND EXPERIMENTS

These analytic calculations suggest that a broad range of dSph-MW orbit dynamic conditions can lead to parametric tidal dSph excitation. Evidently this excitation can occur even when the orbital and dSph galaxy resonant frequencies are widely “mismatched.” Although our analytic calculations simplified the force equation terms from the exact spatial and temporal dependence of the dSph self-gravity, we can show by numerical methods how many of the properties are retained in more realistic self-gravitating stellar systems.

In the discussion below we assume a system of units where the gravitational constant $G = 1$. We fix one numeric length unit to a physical scale of 1 kpc, and we have taken the dSph to have a total initial mass of $10^6 M_\odot$. With the gravitational constant set to unity, this implies a numerical time unit of 415 Myr. The logarithmic external MW potential has been chosen to have the form $\phi(r) = \varepsilon \log(r/r_0)$, where we assume a circular velocity $v_c = 220 \text{ km s}^{-1}$ (or 100 numeric units).

We first consider the problem of a more realistic nonharmonic, but still static, dSph potential/density model. This is addressed by numerical integration of the modified two-dimensional Mathieu differential equation. Finally, we consider direct N-body calculations to explore the collective and time-dependent effects of the dSph potential as it responds to an external tide.

3.1. Plummer Potential Solutions

The Plummer model, $\phi(r) = -M/[r^2 + b^2]^{1/2}$ (see Binney & Tremaine 1987), is a rough but useful approximation for a self-gravitating dSph stellar distribution. We use this model both to initialize our numerical simulations and in the differential equation integrations below. For example, in this case the local Cartesian $x$ acceleration of a star can be written as $\ddot{x} = -x_0^2/[b^2/(b^2 + r^2)^{3/2}]$, where $x_0 = 4\pi\rho_0/3$ and $\rho_0$ is the central density. Notice that the parameter $b$ conveniently describes potentials that range from completely unbound ($b = 0$) to homogenous, purely harmonic ($b \rightarrow \infty$) models. Substituting for the harmonic dSph force term in equations (12a) and (12b), we obtain

\begin{align*}
\ddot{x} + x^2_0 &- x_0^2 (b^2/(b^2 + r^2)^{3/2}) = -x_0^2 \omega_0^2 \cos 2\varphi = y \varepsilon \omega_0^2 \sin 2\varphi, \\
\ddot{y} + y^2_0 &- y_0^2 (b^2/(b^2 + r^2)^{3/2}) = y \varepsilon \omega_0^2 \cos 2\varphi = x \varepsilon \omega_0^2 \sin 2\varphi.
\end{align*}

(31a)

(31b)

We were unable to construct analytic solutions to equations (31a) and (31b), except for the cases $b = 0$ or $b \rightarrow \infty$. Notice that this more general equation preserves the form of the two-dimensional parametric oscillator but with a spatially variable “resonance” frequency $\omega_0^2(r) = \omega_0^2 b^2/[b^2 + r^2]^{3/2}$ and with $\varepsilon = \varepsilon \omega_0^2/\omega_0^2$. For example, stars on circular orbits with initially constant $r$ should be tidally excited at the first Mathieu equation resonance condition when the dSph orbit satisfies $\omega = \omega_p(r)$. We can expect that, to the extent that individual stellar orbits are not circular and there is a broad range in $\omega_p(r)$, the dSph tidal response should be a broad function of its orbital frequency, $\omega$. For finite $b$ we also note that $\omega_p(r) < \omega_0$, so that resonant behavior must occur at lower frequencies than the dSph harmonic oscillation frequency.

We can explore this solution space numerically. To distinguish stable and unstable solutions as a function of orbital frequency $\omega$, we numerically integrate equations (31a) and (31b) using a fourth-order Runge-Kutta algorithm. For example, according to the analytic solution, with $b \rightarrow \infty$, $\omega_0 = 4$, and $\varepsilon = 0.1$ we should obtain solutions for $r(t) = (x^2 + y^2)^{1/2}$, which are growing only for dSph orbits with circular frequencies $\omega \approx \omega_0$. To verify this, we integrate an ensemble of solutions with $\omega$ between 0 and 8 over
\[ t = 0 - 50 \] and compute the mean square radius over time for each solution. Here \( x(0) = 1 \) and the stellar orbits were chosen with a zero initial velocity condition to yield radial orbits. Figure 2 plots this mean solution radius for each dSph orbital frequency.

Figure 2 illustrates several features of the analytic parametric solution. First, instability occurs when the orbital frequency is close to the dSph resonant frequency (\( \omega_p \)), and, unlike in the one-dimensional parametric oscillator, the next resonance where the orbital frequency is one-half of the oscillation frequency is stable. This agrees with our analytic result. It is perhaps surprising that a circular orbit with half the orbital frequency of the dSph resonance does not lead to instability, since the tide force has 180° symmetry. This is a consequence of the compressive and expansive parts of the tensor tide interaction.

The instability frequency width is determined by the tidal strength and form of the MW potential (through \( \varepsilon \) in eq. [26]). With \( \omega_0 = \omega_c = 4 \) and \( \varepsilon = 0.1 \) we expect a resonance width of 0.4, which is confirmed in Figure 2. The growth rate of \( r(t) \), as determined by equation (25), is also reproduced in the numerical calculation. For \( \omega_1 = 0 \) we expect a stellar orbit to expand like \( \exp(\omega_0 t/2) \exp(0.2t) \), which by \( t = 50 \) yields the amplitude plotted here.

The other extreme, where \( b = 0 \), corresponds to a dSph that is completely unbound, i.e., where the dSph self-gravity is unimportant. For example, at late times a dSph might be tidally distended so that its self-gravity becomes negligible. A naive interpretation of the MW effect here might be to conclude that unbound dSph member stars should be tidally accelerated to large distances from the dSph center of mass. Figure 3 shows the logarithm of the rms stellar orbit radius for the unbound \((b = 0)\) case with the same initial conditions, range of dSph-MW orbital frequencies, and integration parameters as in Figure 2. We see that for static and slowly rotating tides the stellar system does indeed rapidly blow up, saturating our numerical dynamic range before \( t = 50 \) for orbits near \( \omega = 0 \).

Analytic solutions using computer-aided symbolic manipulation imply that the leading growing mode term in the solution varies as \( \exp((\epsilon \omega_0^2 - \omega^2)^{1/2} t) \). Interestingly, as this shows, for dSph orbits with short enough periods satisfying

\[ \omega > \sqrt{\varepsilon \omega_0} = \sqrt{k}, \]

the ejected stars are localized near the dSph center of mass, despite being unbound to the dSph.

Intermediate potential models \((0 < b < \infty)\) yield unstable orbits depending on the initial conditions of the stellar orbit. The dSph orbital frequency that leads to instability of a dSph star depends on the initial radius of the star with respect to the dSph center of mass in units of \( b \) and the dSph central density that determines \( \omega_0 \). The previous two cases have shown that dSph stars that begin their orbits at normalized distances of 0 and \( \infty \) develop secular instability for dSph-MW orbits that satisfy, respectively, \( \omega \approx \omega_b \) and \( \omega \approx 0 \). Intermediate cases lead to intermediate resonant frequencies.

The top panel of Figure 4 plots the orbital expansion of a star starting from \( r/b = 1 \) using the same integration parameters \((\omega_0 = 4)\) as the previous Runge-Kutta solutions. Secular growth now occurs at lower frequencies near \( \omega \approx 2.8 \). In a Plummer potential/density model the half-mass radius occurs at \( 1.4b \). The bottom panel of Figure 4 demonstrates for this radius how the dSph resonant orbital frequency continues to decrease for stars at larger normalized dSph distances. Evidently stars near the half-mass radius are tidally excited when \( \omega = 1.1 \). The initial position of a star within the dSph potential helps to determine its fate in the time-dependent MW tide.

These solutions describe a system where the tidal amplitude \( \varepsilon \) and the orbital frequency \( \omega \) are independent. A logarithmic MW potential satisfies \( \varepsilon \omega_0^2 = \omega^2 \). Thus, for increasing orbital frequency, \( \varepsilon \) in equations (31a) and (31b) is also increasing. Qualitatively we find that this increases the orbital frequency domain over which resonance expands...
elliptical orbits with fixed perigalacticon distance. Here \( e = 1 - \omega/2 \) so that \( \omega = 2.0 \) is a circular orbit. The tidal amplitude and circular frequency are constrained as they are in a logarithmic potential so that \( k = \omega^2 \). Orbital and dSph parameters correspond approximately to the N-body simulation parameters. The broad frequency response of the dSph is a consequence of elliptical orbits.

We note that a circular orbit (\( \omega = 2 \)) can be less dispersive than an elliptical orbit (\( \omega < 2 \)) even if the tide is always stronger in the circular case.

### 3.2. Direct N-Body Calculations

In order to account for the self-gravity and collective dynamics of many stars within the dSph, we have used a direct N-body calculation. The numerical simulations were computed using the modified TREECODE version 1.4 (Barnes 1990). Typically between 1000 and 10,000 particles were introduced into simulations that included several different “external” forces designed to study various dSph internal oscillations and the tidal coupling of the stellar system to an imposed large-scale logarithmic external MW potential.

The effects of numerical viscosity and small spatial scale potential fluctuations can be important in these calculations, and we used a range of integration time step and potential “softening” parameter to understand this dependence. Most of the simulations were done with a time step of 0.008 units. We explored the effects of the interparticle potential softening parameter over values ranging from 0.025 to 0.3. Significant damping of oscillations appears to be minimal in these models with softening parameter \( a > 0.1 \).

With a crossing time of about 0.2 time units the relaxation timescale for \( N = 1000 \) simulations was at least 2 time units and much larger for softer potentials. Many of our results only depend on measuring differential changes between simulations, e.g., to determine the frequency dependence in the particle ejection rate between models with different orbital frequencies. Some results are computed from relatively long model simulation runs.

Several different initial particle configurations were generated although most of the results we present here used the tree code Plummer model realization (see Aarseth, Hénon, & Wielen 1974 as implemented by Barnes 1990). The rms particle radius was typically 0.1 kpc and the Plummer parameter was typically \( b = 0.16 \) for a softening parameter of \( a = 0.1 \).
To achieve this, an initially isotropic velocity dispersion of about 1 km s$^{-1}$ was used and allowed to relax before imposing any external time-dependent tidal forces.

We have not tried to reproduce any of the dSph galaxies in detail since their MW orbital uncertainties and the non-uniqueness of the model solutions make detailed comparisons difficult to interpret. Nevertheless, we believe that these $N$-body calculations are most representative of a dSph like Draco or Ursa Minor.

### 3.2.1. Oscillations

The parametric galaxy model described by equations (12a) and (12b) neglects the effect of the oscillating potential of the dSph. Thus, we can expect normal mode galaxy oscillations to dominate even the growing parametric tidal modes if the system’s normal mode frequencies are ever excited by the MW orbit. To explore this numerically, we need to understand the normal mode oscillation spectrum of our dSph system.

Based on previous experience with particle-mesh calculations (KM) involving $10^5$ particles or more, we expected free galaxy oscillations to be readily generated by small deviations from equilibrium in the initial conditions of the particle configuration. For example, Miller & Smith (1999) had difficulty damping and suppressing such oscillations. Even though the total energy in our simulations of isolated dSph was conserved to within 0.03%, we initially had difficulty detecting oscillations. To excite an oscillation spectrum, we applied an impulsive radial stretching acceleration at $t = 0$ and looked for the resulting ringing. From a temporal Fourier analysis of projections of the particle motion onto the fundamental eigenmode, $\xi(r) = r$, we measure the oscillation spectrum.

In detail, we compute a time series by summing over particles (labeled with $i$)

$$ c_j = \frac{1}{N} \sum_i r_i(t_j) \cdot \mathbf{v}_i(t_j). \quad (34) $$

The logarithms of the temporal Fourier transforms of $c_j$ from two different simulation runs are indicated in Figure 6. For a homogeneous sphere, the power spectrum should peak at the oscillation frequency $\omega_0$. Increasing the softening parameter changes both the peak frequency in the power spectrum and the peak amplitude response to the excitation. The virial equation (13) yields a reasonable estimate of the observed oscillation peak frequency. For example, with softening parameter of $a = 0.1$ and using the observed $N$-body density distribution, we compute a frequency of $\omega_0 = 3.54$, which is consistent with the peak in Figure 6 (bottom panel).

The shorter time step minimizes the integration errors and the intrinsic numerical damping noise (Fig. 7). Similarly, by increasing the softening parameter, we can decrease the effects of integration error from the short-range particle interactions. Figure 6 confirms that as the softening parameter increases there is an enhancement in the fundamental mode amplitude.

It is difficult to compute the true physical damping of galaxy oscillations, but comparing our results with Miller & Smith (1999) suggests that the direct $N$-body calculations have a larger numerical damping than particle-cell approximations. We expect the dSph instability growth rates from these $N$-body simulations to be underestimated compared to the analytic and Runge-Kutta solutions.

![Fig. 6.—Influence of the softening parameter. The potential softening parameter also affects the relaxation time and numerical dissipation. Here we plot the galaxy oscillation spectrum for identical initial $N$-body conditions but with softening parameter values of $a = 0.025$ (top) and 0.1 (bottom). Curves are generated from the time series defined in eq. (34).](image)

### 3.2.2. Parametric Resonance

In order to demonstrate equations (25a) and (25b) for a dSph in a circular orbit, we decouple the strength of the tidal force from its time dependence. We isolate the dependence of the tide amplitude and frequency by fixing the amplitude while independently setting the orbital frequency ($\omega$). This is analogous to the approach KM used in their particle-mesh calculations. It allows the frequency dependence of the resonance to be distinguished from the effect of a growing static tide strength as the dSph galactocentric distance is decreased in order to increase $\omega$. Evidence of this resonant behavior is illustrated in Figure 8, which shows a snapshot of the particle positions for three different models corresponding to $\omega = \omega_p/2$, $\omega_p$, and $2\omega_p$. The direction of the instantaneous tide is indicated by the line on each plot.

Figure 9 shows the effect of changing $\omega$ while keeping the tidal amplitude (determined by $\varepsilon$) fixed. Here the total number of particles ejected from the dSph is plotted versus time and frequency $\omega$. Recall that $\omega_p$ is approximately 2 numerical units and the tide amplitude corresponds to $\varepsilon = 0.5$. Successive curves displaced upward in this plot.
show how the ensemble of models have ejected more particles at later times.

Several points are illustrated by Figure 9. The most important is that, as expected from the Mathieu equation solutions, the resonant behavior (measured by particle ejection from the dSph) extends to frequencies relatively far from the effective resonant frequency \( \omega_p \). According to our solution to equations (25a) and (25b) and for our choice of \( \varepsilon \), we expect an instability frequency range \( \Delta \omega / \omega_p = \varepsilon / 2 = \frac{3}{4} \), which is consistent with the numerical results. It is also notable that there is no second-order resonance at \( \omega_p / 2 \), consistent with the analytic predictions of the coupled Mathieu equation model.

We also find in the N-body solutions that for dSph circular frequencies \( \omega \leq 1.5 \) the ejected stars migrate toward \( r \to \infty \) as the dSph evolves. For \( \omega \geq 1.6 \) the ejected stars (more distant than 1 kpc from the core) appear to remain in a bounded region of space, at least over the duration of these simulations. This result is anticipated by equation (32). In these simulations \( k = 2 \) so the cutoff frequency should be about \( \omega = 1.4 \), in fair agreement with the N-body calculation.

3.2.3. Elliptical Orbits

More realistic orbital calculations also demonstrate how the dynamical tide interaction affects the internal dynamics of a dwarf spheroidal. We adapted the numerical calculations described above to describe a self-gravitating ensemble of masses orbiting within a logarithmic external potential. We use a 1024 point dSph galaxy characterized by oscillation frequency, \( \omega_p \approx 7 \) numeric units. Setting this dSph onto a circular orbit at 50 kpc from the galactic center yields a stable system with 75% of the initial stars and with a particle-loss rate of less than 50 stars over 250 time units.

To generate a family of elliptical orbits, we launched the dSph at 50 kpc along the \( x \)-axis with variable speed \( v_0 \geq v_c \) along the \( y \)-axis in a constant galactic potential \( \phi(r) = v_c^2 \log(r) \). Thus, in our units when \( \omega_c = 2 \) the orbit is circular and in general the eccentricity, \( e \), varies like \( e = 1 - \omega_c / 2 \). Thus, the perigalacticon of all runs is exactly 50 kpc and the mean tidal force on the dSph is largest for the circular case (\( e = 0, \omega_c = 2 \)) and decreases with \( \omega_c \) or as the eccentricity \( e \) increases.

Figure 10 shows how the number of particles ejected from the dSph varies with orbital ellipticity and time. Here the orbit time is measured from the injection of the equilibrated dSph into the logarithmic potential. The broad frequency response of the dSph in an elliptical orbit is confirmed from the Runge-Kutta solutions (Fig. 5). It is interesting that elliptical orbits, with weaker tides than a circular orbit, can lose more stars. Eccentric variable tides can eject stars where a stronger constant tide cannot.

3.3. Velocity Dispersion

The velocity dispersion of our parametrically excited dSph galaxies varies by a large factor over their orbits and can be a strong function of the extent of the dSph over which the dispersion is calculated. For example, a general feature of these models is that the dispersion increases outward from the core of the dSph.

We find, as KM did, that it is also possible to inflate the dispersion by an order of magnitude or more depending on the dSph orbit. For example, a mildly resonant system with orbital eccentricity of 0.5 can exhibit a velocity dispersion that is 10 times larger than its equilibrium value. The fraction of the orbital period during which this dSph exhibits a large dispersion is relatively small in our simulations (a few percent of the orbit period in the \( e = 0.5 \) simulation), but it also retains its central core concentration for several perigalacticon passages. Figure 11 shows the projected appearance of this system when its apparent \( M/L \) was 100 times larger than its initial equilibrium value. In contrast, KM found that the velocity dispersion near dSph dissolution was high throughout the orbit. We have yet to investigate the differences between these calculations.

3.4. Interpreting Galaxy Morphology from N-Body Calculations

Our analytic parametric oscillator calculations and numerical integrations of more realistic Plummer model systems showed that resonant excitation of dSph stars should produce elliptical systems with their long axis rotated by an angle \( \theta \) away from the direction toward the force center in the plane of the orbit. For a circular or elliptical orbit this angle is calculated from \( \tan \theta = -H / \omega_c \) (§ 2.2.3). At the first parametric resonance this angle is 45°. Further from the resonance condition this angle varies from an orientation along the center-of-force direction to being perpendicular to it. The existence of a nonzero dSph bar angle is evidence of parametric resonance.

We found in § 2.2.5 that this angle also describes the dSph elongation in an elliptical orbit at perigalacticon. During the orbit the bar of a dSph should rotate with respect to the center-of-force direction with increasing frequency, depending on the order of the resonance in orbital ellipticity, \( e \).

We have computed the direction of the dSph bar from the moment of inertia tensor of the dSph stars. Figure 12 shows a series of snapshots from the simulation with \( e = 0.22 \). In this figure the center of force is always to the left and \( \theta(t) \) is the angle between the two solid lines that intersect at the
center of the dSph. Figure 13 shows how the angle varies with time for these calculations. Each spike in the graph corresponds to one rotation of the bar. We find that the rotation rate increases with increasing eccentricity (as it should) as higher order modes in powers of $e$ are excited.

The bar rotation indicated in Figure 13 is not strictly sinusoidal with the epicycle period. We also note that at perigalacticon our dSph systems tend to have the bar directed toward the force center ($\theta = 0$). Recall from our discussion of the Plummer model that not all stars in the dSph are characterized by the same resonance frequency $\omega$, so that stars near the center of the dSph have larger frequencies than stars at larger distances. Thus, for given orbital circular and epicyclic frequencies the response of many of the dSph stars is non-resonant. Thus, near perigalacticon many of the stars (especially near the core of the dSph) simply respond to the tide like any nonresonant fluid system. Our moment of inertia calculation is thus controlled by contributions from both resonant and nonresonant stellar components. This can lead to a more complex behavior of the bar pointing direction than our model predicts in detail.

The outer parts of the dSph (see Fig. 12) are curved as if $H$ depends on distance from the dSph. This is anticipated from the Plummer calculations since the effective $\omega_p$ decreases outward. The curvature in the bar at large distances from the dSph raises the possibility that snapshot observations of real tidally excited stellar systems may provide useful constraints on the form of the dSph potential.

4. COMPARING THE PARAMETRIC OSCILLATOR MODEL WITH OBSERVATIONS

Our earlier comparisons of dSph dynamics and morphology with a simpler resonant tidal excitation model depended on the commensurability of the dSph oscillation frequency (and damping width) and its MW orbital driving frequency to generate elliptical systems with large velocity dispersions. The parametric oscillator model presented here more accu-
rately describes the dSph-MW interaction and is a more precise explanation for dSph properties. In particular, their elliptical shape and sometimes large velocity dispersions are a natural consequence of this model.

Parametric dSph galaxy oscillations effectively increase the dSph gravitational interaction cross section with the MW. Properly accounting for the time-dependent tidal interaction of a dSph with the MW, even when the dynamical frequency of the dSph ($\omega_d$) is not commensurate with its orbital or epicyclic frequency around the MW, is critical to describing the dSph dynamics. We have shown how the frequency domain near the dSph fundamental frequency that describes resonance is not characterized by the galaxy oscillation damping time but by the tide amplitude ($\epsilon$). For a logarithmic MW potential in a circular orbit the fractional resonant growth frequency domain is simply $\omega_c^2/\omega_0$ (the square of the ratio of the circular orbit frequency and the internal galaxy oscillation frequency). Equation (26) implies that any dSph with oscillation frequency less than about 1.4 $\omega_c$ will be tidally disrupted.

As we noted above, elliptical orbits can be even more likely to disrupt dSph stellar systems because of their rich epicyclic harmonic structure. Eccentric orbits lead to disruption of “stiffer” dSph with internal frequencies as large as 1.7, 2.4, and 3.1 (or larger) times the circular orbital frequency, corresponding to first-, second-, and third-order terms in orbital eccentricity. Note that parametric resonance increases the mean dSph central distance of its member stars and their velocities (and velocity dispersions). The exponential growth time depends on how close the dSph is to a resonance condition, but as equations (25a) and (25b) show, the growth times of, for example, the velocity dispersion can be comparable to the orbit period. In general, we expect larger growth rates as the ratio of circular to internal dSph frequency increases toward unity.

Despite the undetermined orbital characteristics of the dSph, it is interesting to compare our best estimate of $\epsilon$ for each of the MW dSph galaxies with their kinematic properties. It is our contention that the dSph kinematics is not dominated by dark matter halos, but by their MW tidal interaction. Thus, we estimate their mass from their

![Fig. 9.—Number of particles lost vs. circular orbit frequency. These simulations were performed using a simple fixed amplitude turning tide with $k = 2$ (corresponding to $\epsilon \approx 0.5$). Particles are defined to be lost when they reach a distance greater than 1 kpc from the center-of-mass position. The multiple curves plot the number of particles lost at successive times. The curves are separated by 6.25 time units. The effective resonant frequency of the dSph is $\omega_p = 1.7$ initially. As stars are lost and the central density decreases, $\omega_p$ decreases and we also observe the resonant peak shift to lower frequencies. We also observe two regimes in the simulation results: if $\omega \leq 1.5$, lost stars are ejected to reach $r = \infty$, but when $\omega \geq 1.6$, stars remain in a bounded region of space near the dSph.](image1)

![Fig. 10.—Parametric excitation on elliptical orbits. This figure plots the variation in the number of particles lost vs. time and orbit frequency. Multiple curves in the figure show the dSph system at successively later times $\Delta t = 25$. With a fixed perigalacticon distance at 50 kpc we have $\omega_e \approx 2(1 - e)$, so that $\omega_e = 2$ corresponds to the circular case $e = 0$. The maximum loss is 1024 stars, but all simulations began with 734 stars when injected into their MW orbit. Elliptical orbit tidal interactions can expand and entirely disrupt a galaxy as a result of parametric excitation where a stronger static tide has only a small effect. These results compare favorably with the nongravitating Runge-Kutta elliptical orbit calculations above (see Fig. 5).](image2)
luminosity and not their velocity dispersion, since the dSph galaxies are not in virial equilibrium. Core (half-light) radii ($r_c$), galactocentric distances ($d$), and $M/L$ values are used from the data compilations of Mateo et al. (1993) and Mateo (1998). A pulsation frequency $\omega_0$ is computed from the Vandervoort (1999) approximation $\omega_0 = \pi (G\rho_0)^{1/2}$. We compute the central density $\rho_0$ from the observed total luminosity (assumed equal to the total dSph mass in solar units) and the expression $\rho_0 = 3M(0.64)^2/4\pi r_c^3$. This is exact when the density distribution equals a Plummer model and is a reasonable empirical approximation for the MW dSph. We also take $\omega_c = v_c/d$ (with $v_c = 220$ km s$^{-1}$) as the circular frequency around the MW. Adequate for our purposes, we note that the parametric growth rate increases with $\varepsilon$ and the ratio $A = \omega_c/\omega_0$. If the dynamics of the dSph is dominated by tidal interactions, we expect observed dSph virial $M/L$ ratios to increase with $A$. Figure 14 plots $A$ against $M/L$. The general trend of increasing $M/L$ with $A$ is in good agreement with the parametric model, especially given the uncertainty in the dSph orbital parameters.

Our "snapshot" knowledge of the Local Group dSph galaxies means that we cannot measure their orbital...
ellipticity (although see Kuhn 1993 for a discussion of this point); nevertheless, knowledge of the dSph radial velocities and galactocentric distances is sufficient to suggest that at least the nearest dSph galaxies must be tidally inflated by parametric resonance.

Another interesting and perhaps observationally verifiable consequence of parametric excitation is that the long axis of the dSph bar should, in general, be inclined with respect to the separation vector between the MW center and the dSph. Orbits that are nearly circular should yield a dSph with the leading edge closer to the MW than the trailing edge and inclined at an angle of 45°. Elliptical orbits produce a rotating bar whose phase with respect to the radius vector direction depends on how close the dSph is to resonance and the proximity of the dSph in its orbit to perigalacticon.

5. SUMMARY AND CONCLUSIONS

We have demonstrated an analytic model of time-dependent dSph-MW tidal interactions that extends the Mathieu equation for parametric oscillations to two dimensions. Several important general conclusions follow from this model:

1. Secular instability can cause dSph star orbits and velocities to grow, depending on the ellipticity and circular frequency of the MW orbit and the resonant frequency of the dSph.

2. Exponential growth can occur over a broad range of frequencies, not just near the characteristic dSph harmonic frequency. Growth times can be comparable to the MW orbital period.

3. Direct integration of more realistic, nonharmonic dSph potentials yields the same qualitative behavior predicted by the Mathieu equation.

4. Self-gravitating N-body calculations also confirm the existence of broad resonance conditions that lead to stellar ejection from the dS system, elliptical bar formation, and inflated (nonvirial) velocity dispersions.

5. Model expectations for the parametric growth rate in MW dwarf galaxies have been contrasted with the observed $M/L$ measurements for the eight MW dSph galaxies. The expected trend of increasing $M/L$ with $\omega_i/\omega_p$ is confirmed.

Considerable attention has been given to finding dark matter density distributions that will stabilize the luminous components of the dSph galaxies so they survive MW tidal encounters, but not so much that they do not allow stars to be ejected to populate "tidal tails." These tails are now clearly observed by several methods (Kuhn et al. 1996; Smith, Kuhn, & Hawley 1997; Martinez-Delgado et al. 2001) and are an observational hurdle that any dSph model must pass. While it appears that there may be dark matter models that cannot be ruled out (e.g., Mayer et al. 2002), here we have developed a consistent model of MW-dSph interactions that does not require any invisible dSph mass component and that appears to account for the morphology and dynamics of the dSph. The prediction of extratidal dSph stars in the simple KM resonance model was later observed. We can hope that the considerably refined predictions of the parametric oscillator model (for example, the inclined bar) may also be empirically established.

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