The Uniform Convergence of Eigenfunction Expansions of Schrödinger Operator in the Nikolskii Classes $H^a_p(\bar{\Omega})$

N. A Jamaludin$^1$, A Ahmedov$^2$

$^1$ Department of Mathematic, Centre of Foundation Studies, National Defence University of Malaysia, 57000, Kem Sg Besi, Kuala Lumpur, Malaysia

$^2$ Faculty of Industrial Sciences & Technology, Universiti Malaysia Pahang, Gambang, 26300 Kuantan, Pahang, Malaysia

E-mail: amalinajamaludin@upnm.edu.my, anvarjon@ump.edu.my

Abstract. Many boundary value problems in the theory of partial differential equations can be solved by separation methods of partial differential equations. When Schrödinger operator is considered then the influence of the singularity of potential on the solution of the partial differential equation is interest of researchers. In this paper the problems of the uniform convergence of the eigenfunction expansions of the functions from corresponding to the Schrödinger operator with the potential from classes of Sobolev are investigated. The spectral function corresponding to the Schrödinger operator is estimated in closed domain. The isomorphism of the Nikolskii classes is applied to prove uniform convergence of eigenfunction expansions of Schrödinger operator in closed domain.

1. Introduction

Let $\Omega$ be a three-dimensional bounded domain with smooth boundary $\partial \Omega$. We denote the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ by the symbol $\alpha$ made up of 3 integral nonnegative numbers $\alpha_j$ and set $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. The symbol $\partial^\alpha f(x)$ is used to denote the generalized partial derivative

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

We say that a function $f(x)$, defined in domain $\Omega$, belongs to the Sobolev class $W^l_p(\Omega), p \geq 1$, where $l$ is an integral nonnegative number, if $\partial^\alpha f \in L_p(\Omega)$ for all $\alpha : |\alpha| \leq l$. The norm of the function $f \in W^l_p(\Omega)$ is defined by

$$\|f\|_{W^l_p} = \|f\|_{L_p} + \sum_{|\alpha| = l} \|\partial^\alpha f\|_{L_p}.$$

Let's proceed with the definition of the Nikolskii class $H^a_p(\Omega), a = l + \sigma, l = 0, 1, 2, \ldots$ and $0 < \sigma \leq 1$. For any positive $h$ we denote by $\Omega_h$ a subset of $\Omega$ defined as follows: $\Omega_h = \{x \in \Omega : \text{dist}(x, \partial \Omega) < h\}$. We say that a function $f(x)$ belongs to the Nikolskii class $H^a_p(\Omega)$, if $f \in L_p(\Omega), p \geq 1$, and for any $y \in \Omega$ and multi index $\alpha$ satisfying the conditions $|\alpha| = l$, the following inequality holds:
\[ ||\partial^\alpha f(x + y) - 2\partial^\alpha f(x) + \partial^\alpha f(x - y)||_{L_p(\Omega_\sigma)} = O(|y|^\sigma). \]

Using the notation \( \Delta_2^2 f(x) = f(x + y) - 2f(x) + f(x - y) \) the norm in \( H_2^\sigma(\Omega) \) can be defined as follows:

\[ ||f||_{H_2^\sigma} = ||f||_{L_p(\Omega)} + \sum_{|\alpha|=2} \sup_{y} |y|^{-\sigma} ||\Delta_2^2 \partial^\alpha f(x)||_{L_p(\Omega_\sigma)}. \]

The Schrödinger operator

\[ H\psi(x) = [-\Delta + q(x)]\psi(x) \]

with domain of definition \( C_0^\infty(\Omega) \) is symmetric and semibounded, where \( q(x) \) is a spherically symmetrical potential function. By Friedrich’s Theorem [1] it has at least one non-negative self-adjoint extension. We denote by \( \hat{H} \) a self adjoint extension of the Schrödinger operator with discrete spectrum. Let \( 0 < \lambda_1 \leq \lambda_2 \leq ... \) be a sequence of the eigenvalues and \( \{ \psi_n(x) \} \) be a complete system of eigenfunctions in \( L_2(\Omega) \) Schrödinger operator. The spectral expansions of the function \( f(x) \in L_2(\Omega) \) can be defined as follows:

\[ E_\lambda f(x) = \sum_{\lambda_n < \lambda} f_n \psi_n(x), \]

where

\[ f_n = \int_\Omega f(x) \psi_n(x) dx, \quad n = 1, 2, 3, \ldots. \]

In order to reconstruct the function from its eigenfunction expansions will be convenient to introduce the Riesz means of the spectral expansion:

\[ E_s^\alpha f(x) = \sum_{\lambda_n < \lambda} \left( 1 - \frac{\lambda_n}{\lambda} \right)^s f_n \psi_n(x), \quad s \geq 0. \]

The current research is devoted to the problems of the uniformly convergence of eigenfunction expansions of the Schrödinger operator in closed domain. There were many works on the convergence of the eigenfunction expansions related to the general elliptic differential operators (we refer the readers to the survey [2]). Methodology of an estimation of eigenfunctions in compact subsets of the domain is well developed and known (see [3]). But in estimation of eigenfunctions in closed domain some difficulties occur in connection with the behaviour of the eigenfunctions near the boundary. These difficulties can be avoided by considering boundary conditions that help to estimate eigenfunctions near the boundary with more accurate values. There are only few results when the problems concern the convergence of the eigenfunction expansions up to the boundary. The estimations for the eigenfunctions of the Laplace operator in closed domain, which were obtained by E.I. Moiseev in [4], made possible to prove the uniform convergence of the eigenfunction expansions in closed domain. Some latest results on the uniformly convergent of the spectral expansions of the distributions are obtained by Rakhimov [5]. Here we are combining the technique suggested in the paper [6] and ideas of the [5], [7] with modification for the differential operator with singular potential. We would like to note here that the spectral function of the biharmonic operator on closed domain are investigated in the works [8]. Some new results related to the spectral expansions of the pseudodifferential operator can be found in [9]. Note that the localization problems of the spectral expansions of the distribution are investigated in [10]. For more references we refer the reader to [11]-[17].

We proceed with the formulation of the main result of the paper.
Theorem 1.1 Let \( f \in H^p_\mu(\Omega) \) be continuous on closed domain \( \overline{\Omega} \) and the numbers \( \alpha > 0, p \geq 1, 0 \leq s \leq 1 \) be related by the following inequalities \( \alpha + s > 1, \alpha p \geq 3 \), then the Riesz means \( E^\alpha_p f(x) \) uniformly converges to \( f(x) \) on \( \overline{\Omega} \).

The case of convergence in any compactum subset of \( \Omega \), the statement of Theorem 1.1 is established by Alimov and Joo [6] for eigenfunction expansions from Liouville classes \( L^1_\mu(\Omega) \). Meanwhile the case of closed domain for \( q = 0 \) is established by Rakhimov [5].

2. Estimate of a solution of the Schrödinger equation in closed domain

In this section we obtain an estimation for eigenfunctions of Schrödinger operator by modifying the method of Moiseev [4] and Rakhimov [5] for the case \( N = 3 \). Moiseev deals with estimation of eigenfunction expansions for Laplace equation in closed domain. Meanwhile Rakhimov developed new technique for estimation of sum squares eigenfunction expansions associated with Schrödinger equation \( (N = 2) \) in a closed domain.

Corollary 2.1 Let \( q \in W^1_2(\Omega) \) and \( \mu = \mu_0 + i \alpha, \mu_0 > 0, \alpha \neq 0 \). Then for any \( f \in L^2(\Omega) \), the solution \( \psi(x) \) of first boundary value problem

\[
\begin{align*}
(-\Delta + q(x))\psi(x) + \mu^2 \psi(x) &= f(x), \\
\psi|_{\partial \Omega} &= 0,
\end{align*}
\]

satisfies the following inequality

\[
\|\psi(x)\|_{C(\overline{\Omega})} \leq C\left(\sqrt{\ln \mu_0}\right)\|f\|_{L^2(\Omega)}, \quad \mu_0 \to +\infty.
\]

uniformly for all \( x \in \overline{\Omega} \).

Proof. Let \( \psi(x) \) be a solution of first boundary value problem. The application of second Green’s formula to the functions \( \psi(y) \) and \( |x - y|^{-\frac{3}{2}} H^{\frac{1}{2}}(\delta |x - y|/\mu) \) by \( y \) gives

\[
\begin{align*}
\psi(x) &= C\sqrt{\mu} \iint_{\Omega} \left( r^{-\frac{3}{2}} H^{\frac{1}{2}}(\delta \mu) \right) q \psi d\tau - C\sqrt{\mu} \iint_{\Omega} \left( r^{-\frac{3}{2}} H^{\frac{1}{2}}(\delta \mu) \right) f d\tau \\
&+ C\sqrt{\mu} \iint_{\partial \Omega} \left( r^{-\frac{3}{2}} H^{\frac{1}{2}}(\delta \mu) \right) \frac{\partial \psi}{\partial n} d\sigma,
\end{align*}
\]

where \( H_\nu(t) \) denotes Hankel function of order \( \nu \). Using Cauchy-Schwart inequality we obtain

\[
\begin{align*}
|\psi(x)| &\leq C\sqrt{\mu} \left( \iint_{\Omega} r^{-1} |H^{\frac{1}{2}}(\delta \mu)|^2 d\tau \right)^\frac{1}{2} \|q\|_{L^2(\Omega)} \|\psi\|_{C(\overline{\Omega})} \\
&+ C\sqrt{\mu} \left( \iint_{\Omega} r^{-1} |H^{\frac{1}{2}}(\delta \mu)|^2 d\tau \right)^\frac{1}{2} \|f\|_{L^2(\Omega)} \\
&+ C\sqrt{\mu} \left( \iint_{\partial \Omega} r^{-1} |H^{\frac{1}{2}}(\delta \mu)|^2 d\sigma \right)^\frac{1}{2} \left\| \frac{\partial \psi}{\partial n} \right\|_{L^2(\partial \Omega)},
\end{align*}
\]

Using the asymptotic estimations for the Hankel functions one has

\[
\iint_{\Omega} r^{-1} |H^{\frac{1}{2}}(\delta \mu)|^2 d\tau = O \left( \frac{1}{\mu} \right) + O \left( \frac{\ln \mu}{\mu^2} \right),
\]

and
\[ \iint_{\partial \Omega} r^{-1}|H_2^1(r \mu)|^2 \, ds = O \left( \frac{1}{\mu} \right). \]

Which allows us to have following estimation

\[ \frac{1}{\sqrt{\mu}}|\psi(x)| = \left[ O \left( \frac{1}{\sqrt{\mu}} \right) + O \left( \frac{\sqrt{\ln \mu}}{\mu} \right) \right] \left[ ||q||_{L_2(\Omega)} ||\psi||_{L_2(\Omega)} + ||f||_{L_2(\Omega)} \right] \]

+ \( O \left( \frac{1}{\sqrt{\mu}} \right) ||\frac{\partial \psi}{\partial n}||_{L_2(\partial \Omega)} \)

The following estimation can be obtained using the technique from Moiseev and with modification made by Rakhimov

\[ \left\| \frac{\partial \psi}{\partial n} \right\|_{L_2(\partial \Omega)}^2 \leq C ||f||_{L_2(\Omega)}^2 + C \frac{C}{\mu_0} ||q||_{L_2} ||\psi||_{C(\Omega)} ||f||_{L_2}. \]

Hence,

\[ |\psi(x)| = O \left( \sqrt{\ln \mu_0} \right) ||f||_{L_2(\Omega)}, \]

which completes the proof of Lemma 2.1.

We denote eigenfunctions and eigenvalues of the first boundary value problem for Schrödinger operator by \( \{ \psi_n(x) \} \) , \( \{ \lambda_n \} \) respectively.

\[ [-\Delta + q(x)] \psi_n(x) = \lambda_n \psi_n(x), x \in \Omega, \]

\[ \psi_n(x) = 0, x \in \partial \Omega. \]

where \( q \in W^{1,2}_0(\Omega) \).

**Theorem 2.2** For all \( x \in \overline{\Omega} \) one has

\[ \sum_{|\sqrt{\lambda_n - \mu_0}| \leq 1} \psi_n^2(x) = O(\mu_0^2 \ln \mu_0), \]

where \( \mu_0 \) is sufficiently large.

**Proof.** Define \( R(x, y, \mu) \) be the resolvent operator of the boundary value problem, thus any solution \( \psi(x) \) of the problem can be represented as

\[ \psi(x) = \int_{\Omega} R(x, y, \mu) f(y) \, dy. \]

From (1) using duality properties of Hilbert spaces it follows that

\[ ||R(x, y, \mu)||_{L_2(\Omega)} = O \left( \sqrt{\ln \mu_0} \right), \]

uniformly on \( x \in \overline{\Omega}. \)

Since the fact \( \psi_n(x) = (\mu^2 - \lambda_n) \int_{\Omega} \psi_n(y) R(x, y, \mu) \, dy \) shows that
For fixed \( I \) and \( \mu \), uniformly for all \( x \in \Omega \), we obtain:

\[
\sum_{n \geq 1} \left| \psi_n(x) \right|^2 \leq C |\mu_0^2| \int_{\Omega} |R(x, y, \mu)|^2 dy \int_{\Omega} |\psi_n(y)|^2 dy
\]

\[
= C \mu_0^2 \int_{\Omega} |R(x, y, \mu)|^2 dy
\]

\[
= O(\mu_0^2 \ln \mu_0),
\]

and we conclude the sum square of eigenfunctions give

\[
\sum_{n = 1}^{\infty} \psi_n^2(x) \leq O(\mu_0^2 \ln \mu_0), \quad x \in \overline{\Omega}.
\]

Which completes a proof of the Theorem 2.2 is proved. After the statement of the Theorem 2.2 is established the following lemma can be proven by the methods of the paper [1].

**Corollary 2.3** For any positive \( \epsilon \) we have

\[
\sum_{\lambda_n < \lambda} \psi_n^2(x) \cdot \lambda_n^{-\frac{\epsilon}{2}} = O(\lambda' \cdot \ln^2 \lambda), \forall x \in \overline{\Omega}.
\]

\[
\sum_{\lambda_n > \lambda} \psi_n^2(x) \cdot \lambda_n^{-\frac{\epsilon}{2}} = O(\lambda' \cdot \ln^2 \lambda), \forall x \in \overline{\Omega}.
\]

We applied Corollary 2.3 into the following estimation for Theorem 1.1

**Proof of Theorem 1.1**

Let \( f(x) \in H_0^\alpha(\Omega) \cap C(\overline{\Omega}) \), for \( \alpha \geq 1 - s \), \( \alpha p = 3, p \geq 1 \). Using the estimations obtained in Theorem 2.2 and Corollary 2.3 the Riesz means of the eigenfunction expansions of the function \( f \) we estimate as follows (for the details of obtaining this inequality we refer the reader to the thesis of first author [14] )

\[
|E_\lambda^s f(y)| = ||f||_{H^s_0(\overline{\Omega})} + ||f||_\infty + \frac{c_3}{\sqrt{\lambda^2}} \ln \lambda||f||_{H^s_0}.
\]

We fix an arbitrary \( \epsilon > 0 \) and let denote by \( f_1(x) \in C_0^\infty(\overline{\Omega}) \), such that

\[
||f - f_1||_{H^s_0(\Omega)} + \max_{x \in \Omega} |f(x) - f_1(x)| < \frac{\epsilon}{3C}.
\]

As we know Riesz means \( E_\lambda^s f_1 \) of \( f_1 \in C_0^\infty(\overline{\Omega}) \) converges uniformly on \( \overline{\Omega} \). For fixed \( \epsilon > 0 \). There exist \( \lambda_0(\epsilon) > 0 \) such that for all \( x \in \overline{\Omega} \) and for all \( \lambda > \lambda_0(\epsilon) \) we have

\[
|E_\lambda^s f_1(x) - f_1(x)| < \frac{\epsilon}{3}, \quad s > 0.
\]

Finally, we note that

\[
|E_\lambda^s(f - f_1)(x)| < \frac{\epsilon}{3}, \quad \forall x \in \overline{\Omega}.
\]

From these estimates we obtain:

\[
|E_\lambda^s f(x) - f(x)| = |E_\lambda^s(f - f_1) + [E_\lambda^s f_1 - f_1] + f_1(x) - f(x)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leq \epsilon,
\]

uniformly for all \( x \in \overline{\Omega} \). [The proof of Theorem 1.1 is complete].
Acknowledgments
The current research is supported by Universiti Malaysia Pahang under Universiti Research Grant Scheme RDU170364.

References
[1] Sh.A. Alimov, V.A. Il'in, and E.M. Nikishin, 1976 Convergence, Uspekhi Mat. Nauk, 31: 6 (192) 28.
[2] Sh. A. Alimov, R.R. Ashurov, A.K. Pulatov, 1992 Multiple Fourier Series and Fourier Integrals, Commutative Harmonic Analysis- IV, Springer-Verlag, 1-97.
[3] V.A. Il'in, 1991 Spectral theory of differential operators, Nauka , Moscow, 367.
[4] E. I. Moiseev, 1977 Dokl. Akad. Nauk, 233 6, 1042-1045.
[5] A.A. Rakhamov and K. Zakaria, 2010 Mathdigest, UPM, Malaysia, 1-4.
[6] Sh.A. Alimov and I. Joo, 1983 Acta. Sci. Math., 45 5-18.
[7] Sh.A. Alimov and I. Joo, 1983 Acta. Math. Hung., 42(1-2) 121-129.
[8] A. Anvarjon and Z. Hishamuddin, 2009 Bulletin of Universiti Putra Malaysia, 2(1) 6-11.
[9] A. Anvarjon and R. R. Ashurov, 2010 Journal of Pseudo-Differential Operators and Applications, 1-16.
[10] A. Anvarjon, A. A. Rakhimov and Z. Hishamuddin, 2011 Australian Journal of Basic and Applied Sciences, 5(5) 1-4.
[11] R.R. Ashurov, A. A. Anvarjon and M. Rodzi, 2010 Journal of Mathematical Analysis and Applications, 371 832-841.
[12] A. Anvarjon and R. Ashurov, 2011 Malaysian Journal of Mathematical Sciences, 5(2) 185-196.
[13] A. Anvarjon, 2009 Journal of Mathematical Analysis and Applications, 356 310-321.
[14] Nur Amalina bt Jamaludin, 2017 PhD Thesis, Universiti Putra Malaysia, 6(5).
[15] S. M. Nikolskii, 1969 Approximation of functions of several variables and embedding theorems, Nauka, Moscow.
[16] A.A. Rakhimov, 2003 Modern problems of math., 1 167-172