Doubling Rational Normal Curves

Roberto Notari, Ignacio Ojeda and Maria Luisa Spreafico

Abstract. In this paper, we study double structures supported on rational normal curves. After recalling the general construction of double structures supported on a smooth curve described in [11], we specialize it to double structures on rational normal curves. To every double structure we associate a triple of integers \((2r, g, n)\) where \(r\) is the degree of the support, \(n \geq r\) is the dimension of the projective space containing the double curve, and \(g\) is the arithmetic genus of the double curve. We compute also some numerical invariants of the constructed curves, and we show that the family of double structures with a given triple \((2r, g, n)\) is irreducible. Furthermore, we prove that the general double curve in the families associated to \((2r, r + 1, r)\) and \((2r, 1, 2r - 1)\) is arithmetically Gorenstein. Finally, we prove that the closure of the locus containing double conics of genus \(g \leq -2\) form an irreducible component of the corresponding Hilbert scheme, and that the general double conic is a smooth point of that component. Moreover, we prove that the general double conic in \(\mathbb{P}^3\) of arbitrary genus is a smooth point of the corresponding Hilbert scheme.

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1. Introduction

Non-reduced projective curves arise naturally when one tries to classify smooth curves, where a projective curve is a dimension 1 projective scheme without embedded or isolated 0-dimensional components. In fact, two of the main tools to classify projective curves are liaison theory and deformation theory.

Given two curves \(C\) and \(D\) embedded in the projective space \(\mathbb{P}^n\), we say that they are geometrically linked if they have no common component and their union is an arithmetically Gorenstein curve. More than the geometric links, a modern
treatment of the theory takes as its base the algebraic link where two curves are algebraically linked via the arithmetically Gorenstein curve $X$ if $I_X : I_C = I_D$ and $I_X : I_D = I_C$, where $I_C, I_D, I_X$ are the saturated ideals that define the curves $C, D, X$, respectively, in the projective space $\mathbb{P}^n$. If $C$ and $D$ have no common irreducible component, the two definitions agree. Liaison theory and even liaison theory are the study of the equivalence classes of the equivalence relation generated by the direct link, and by an even number of direct links, respectively. In $\mathbb{P}^3$, a curve is arithmetically Gorenstein if, and only if, it is the complete intersection of two algebraic surfaces. A pioneer in the study of this theory for curves in $\mathbb{P}^3$ was F. Gaeta (see [13]). In the quoted paper, he proved that every arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ is in the equivalence class of a line. More in general, every curve sits in an equivalence class, and, for curves in $\mathbb{P}^3$, it is known that every curve in a bilaision class can be obtained from the curves of minimal degree in the class via a rather explicit algorithm. This property is known as Lazarsfeld-Rao property ([23], Definition 5.4.2), and it was proved in ([22], Ch. IV, Theorem 5.1). The existence of minimal curves and their construction is proved in ([22], Ch. IV, Proposition 4.1, and Theorem 4.3). In [1], the authors proved the Lazarsfeld-Rao property for the curves in the same bilaision class, without the explicit construction of the minimal curves. The minimal arithmetically Cohen-Macaulay curves are the lines. Hence, the Lazarsfeld-Rao property can be seen as a generalization of Gaeta’s work. Also if one wants to study smooth curves, the minimal curves in the bilaision class can have quite bad properties, e.g., they can be non-reduced, or they can have a large number of irreducible components. Moreover, the minimal curves in a bilaision class form an irreducible family of curves with fixed degree and arithmetic genus. Today, it is not known if the equivalence classes of curves in $\mathbb{P}^n$ have the same properties as those in $\mathbb{P}^3$ (see [25], [28], [8], [16] for evidence both ways).

To study the properties of smooth curves, one can also try to deform the smooth curve to a limit curve and investigate the properties one is interested in on the limit curve. If those properties are shared by the limit curve and the deformation behaves well with respect to the considered properties, then the general curve shares the same properties of the limit curve. Often, the limit curves are non-reduced curves. In the papers [12], [3], [10], the authors study Green’s conjecture concerning the free resolution of a canonical curve by reducing it to the study of a similar conjecture for double structures on $\mathbb{P}^1$ called ribbons.

Both described approaches lead to the study of families of curves. The universal family of curves of fixed degree $d$ and arithmetic genus $g$ is the Hilbert scheme $\mathcal{Hilb}_{d+1-g}(\mathbb{P}^n)$, where, for us, $\mathcal{Hilb}_{d+1-g}(\mathbb{P}^n)$ is the open locus of the full Hilbert scheme corresponding to locally Cohen-Macaulay 1-dimensional schemes, i.e., corresponding to curves. Since A. Grothendieck proved its existence in [14], the study of the properties of the Hilbert scheme attracted many researchers. In spite of their efforts, only a few properties are known, such as the connectedness of the full Hilbert scheme proved by R. Hartshorne in [15]. A current trend of