Symmetries and Observables
in Topological Gravity

Clisthenis P. Constantinidis*, Aldo Deandrea**, François Gieres**
Matthieu Lefrançois** and Olivier Piguet*

* Universidade Federal do Espírito Santo (UFES), CCE, Departamento de Física,
Campus Universitário de Goiabeiras, BR-29060-900 - Vitória - ES (Brasil).
** Institut de Physique Nucléaire, Bat. Paul Dirac, Université C. Bernard Lyon 1,
4, rue Enrico Fermi, F - 69622 - Villeurbanne Cedex (France).

E-mails: clisthen@cce.ufes.br, deandrea@ipnl.in2p3.fr,
gieres@ipnl.in2p3.fr, lefrancois@ipnl.in2p3.fr, opiguet@yahoo.com

Abstract: After a brief review of topological gravity, we present a superspace approach to this theory. This formulation allows us to recover in a natural manner various known results and to gain some insight into the precise relationship between different approaches to topological gravity. Though the main focus of our work is on the vielbein formalism, we also discuss the metric approach and its relationship with the former formalism.
# Contents

1 Introduction .............................................. 1

2 Topological gravity .................................... 3
   2.1 Geometric setting .................................. 3
   2.2 First order formalism .............................. 4
      2.2.1 Horizontality conditions ...................... 5
      2.2.2 Observables .................................. 9
   2.3 Second order formalism ............................ 12

3 Superspace approach .................................. 13
   3.1 Supersymmetry and superspace ..................... 13
   3.2 Fields and symmetries .............................. 15
   3.3 BRST transformations in superspace ................ 15
   3.4 Projection to component fields ..................... 16
      3.4.1 General gauge ................................ 16
      3.4.2 Wess-Zumino gauge ............................ 18
   3.5 Alternative approach ............................... 19
   3.6 Observables .................................... 21

4 Remarks on the gauge fixing .......................... 21

5 Concluding comments .................................. 22

A Appendix: Metric approach ............................ 23
   A.1 Horizontality conditions .......................... 25
   A.2 Comparison with the vielbein approach ............. 26
   A.3 Observables .................................... 27
1 Introduction

Topological field theories of Witten-type have been introduced some fifteen years ago [1] and have been widely studied ever since. In recent years, they have gained particular attention in relation with non-topological field theories, most notably with non-perturbative quantum gravity, e.g. see ref. [2]. While topological Yang-Mills theories are pretty well understood by now [3], this is not true to the same extend for topological gravities due to the presence of diffeomorphisms. The complexity of symmetry algebras and Lagrangians, as well as the variety of possible approaches for topological gravity also makes it difficult to compare the results obtained using different approaches or formalisms. Let us shortly expand on these points.

A topological gravity theory can be constructed by gauge fixing an action that is a topological invariant. Alternatively, it can be introduced by twisting an extended supergravity theory\(^1\). The latter theories involve diffeomorphisms and local supersymmetry transformations and thereby have a sensibly more complex structure than super Yang-Mills theories.

We now review briefly the different formulations of topological gravity which have been considered in the past so as to situate the present work. The first papers on topological gravity were devoted to the construction of the model [4, 5], while many of the subsequent and recent papers [6]-[18] were rather concerned with the determination of non-trivial observables. Some of the early papers view topological gravity as a topological version of Weyl (conformal) gravity [4, 19], but these theories do not allow for non-trivial observables. The remaining work is related to ordinary Einstein gravity. The construction of topological gravity by a twist of extended supergravity [20, 21, 22] has led to the study of topological Einstein-Maxwell theory since extended supergravity theories involve a Maxwell field (the so-called graviphoton) in addition to the vielbein fields. Topological gravity can also be viewed as a BF-type model and has been studied from this point of view in a series of papers [23].

Though most works on topological gravity concern space-time manifolds of dimension two or four, generalizations to higher dimensions have recently been introduced [24].

Just as for ordinary gravity, different geometric formulations can be – and indeed have been – developed for topological gravity. The most common one is the metric approach: it relies only on the metric tensor field and general coordinate transformations as symmetries. If the metric is decomposed with respect to vielbein fields, local Lorentz transformations also appear as symmetries (second order formalism). In addition to the vielbein fields, one can consider an independent Lorentz

\(^1\)In the present work, we have in mind Lagrangian models as concrete realizations of topological theories. We do not touch upon the issue of defining cohomological theories in the most general way nor do we address the question of whether or not the Lagrangian versions of these theories can always be constructed by twisting some extended supersymmetric model.
connection as basic variable (first order formalism) [8]. For ordinary gravity, the latter formalism is equivalent to the standard metric approach once the connection has been eliminated in terms of the vielbein by requiring the torsion to vanish. The equivalence also holds for topological gravity, but the comparison is more subtle due to the presence of extra symmetries\(^2\).

Topological field theories of Witten-type involve one or several shift symmetries. This kind of invariance can be viewed as a relic of supersymmetry transformations characterizing the extended supersymmetric theories from which topological models may be constructed by performing a twist. Thus, the shift invariance is also referred to as *supersymmetry transformation* and it can be described conveniently in a superspace with an odd coordinate [25, 26] (or several odd coordinates for more complex models [27]). This formulation, which has been explored previously for topological Yang-Mills theories, allows to obtain the symmetries, Lagrangian, etc., in a compact form and to apply the standard methods of supersymmetry to topological models. In particular, standard results on the ordinary BRST cohomology can be used [28] to determine the *equivariant cohomology* describing the observables of topological field theories [26, 29]. For the case of topological gravity, some partial results exist concerning the symmetries and the Lagrangian in two dimensions [30].

The present paper has two parts. The first part (and the appendix) deals with previous work on the symmetries and observables of topological Einstein-Maxwell theory. In our presentation, we have tried to be geometric and concise, and to clarify the relationship between different formulations considered in the literature. Apart from the known observables related to the topological invariants involving curvature, we construct new observables related to a topological invariant which involves torsion and which is not widely known. In the sequel, we develop a superspace approach which leads to a complete off-shell formulation for the symmetries. Simple field redefinitions allow us to recover the results discussed in the first part. Since our superspace approach explicitly involves local supersymmetry transformations (parametrized by a single odd variable), it also allows us to compare directly with the on-shell results which have previously been obtained by twisting extended supergravity transformations [20, 22]. In an appendix, we discuss the metric approach and compare with the results obtained for the symmetries and observables within the vielbein formalism. Though the metric approach has the advantage of introducing a minimal number of fields, it is harder to tackle due to the shift transformations which act on the metric tensor field and thereby on covariant indices. We hope that our study will contribute to a better understanding of the general structure of a certain number of results and of the precise relationship between different approaches to topological gravity.

\(^2\)We note that, although topological gravity can be formulated without vielbein fields, the latter necessarily appear – due to the presence of spinor fields – in the extended supergravity theory from which the topological model arises by virtue of a twist.
2 Topological gravity

After specifying the geometric framework, we will discuss the symmetries and observables for topological gravity within the first order formalism. The reinterpretations to be made in the second order formalism will be commented upon thereafter.

2.1 Geometric setting

The geometric set-up and the symmetry algebras presented in the sequel are well defined in any space-time dimension $d$. We will only specify the dimension for the discussion of observables where we focus on the dimensions two and four (subsection 2.2.2). Thus, the geometric arena is a real $d$-dimensional pseudo-Riemannian manifold $M_d$, the local coordinates being denoted by $x = (x^\mu)_{\mu=0,...,d-1}$. Let us briefly recall some geometric notions and results that we will use in the sequel [31, 32, 33].

BRST formalism  Within the BRST formalism, the parameters of infinitesimal symmetry transformations are turned into ghost fields. The latter have ghost-number $g=1$ while the basic fields appearing in the invariant action have a vanishing ghost-number. The Grassmann parity of an object is given by the parity of its total degree defined as the sum $p+g$ of its form degree $p$ and ghost-number $g$. All commutators and brackets are assumed to be graded according to this grading.

The BRST operator, which is denoted by $S$, acts on the algebra of fields as an antiderivation which increases the ghost-number (and thus the total degree) by one unit. It is assumed to anticommute with the exterior derivative $d$.

Vector fields, inner product and Lie derivative  For a vector field $w = w^\mu \partial_\mu$ on $M_d$, the total degree is given by its ghost-number which we denote by $[w]$. It is said to be even (odd) if $[w]$ is even (odd).

The Lie bracket $[u,v]$ of two vector fields $u$ and $v$ is again a vector field: this bracket is assumed to be graded so that its components are given by

$$ [u,v]^\mu = u^\nu \partial_\nu v^\mu - (-1)^{[u][v]} v^\nu \partial_\nu u^\mu .$$

(2.1)

The interior product $i_w$ with respect to the vector field $w = w^\mu \partial_\mu$ is defined in local coordinates by $i_w \varphi = 0$ for 0-forms and $i_w (dx^\mu) = w^\mu$. If $w$ is even, the operator $i_w$ acts as an antiderivation (odd operator), otherwise it acts as a derivation (even operator).

The Lie derivative $\mathcal{L}_w$ with respect to $w$ acts on differential forms according to

$$ \mathcal{L}_w \equiv [i_w, d] = i_w d + (-1)^{[w]} d i_w$$

(2.2)
and we have the graded commutation relations

\[ [\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u,v]} \ , \quad [\mathcal{L}_u, i_v] = i_{[u,v]} . \quad (2.3) \]

In the following, the quantity \( \xi = \xi^\mu \partial_\mu \) always denotes a vector field of ghost-number 1 (representing the ghost for diffeomorphisms). We then have the following identities involving the vector fields \( \xi \) and \( \xi^2 \equiv \frac{1}{2} [\xi, \xi] \) as well as the previously introduced operators (in particular the exponential \( e^{i\xi} \) of the linear operator \( i\xi \)):

\[
e^{i\xi}(XY) = (e^{i\xi}X)(e^{i\xi}Y)
\]

\[
e^{-i\xi}de^{i\xi} = d - \mathcal{L}_\xi - i\xi^2
\]

\[
[S, e^{i\xi}] = iS_\xi e^{i\xi} \quad , \quad [S, e^{-i\xi}] = -iS_\xi e^{-i\xi} .
\]

2.2 First order formalism

In the first order formalism of the theory, the basic variables are the vielbein 1-forms \( e = (e^a)_{a=0,...,d-1} \) and the Lorentz connection 1-form \( \omega = (\omega^a_b) \). The tangent space indices \( a, b, \ldots \) are raised or lowered using the constant tangent space metric \( (\eta_{ab}) \) which can be of Minkowskian or of Euclidean signature. In the following, we will use the matrix notation \( e, \omega, \ldots \) so as to avoid spelling out the tangent space indices \( a, b, \ldots \).

Since topological gravity is expected to originate from a twisted extended supergravity theory, we also introduce an Abelian (Maxwell or \( U(1) \)) gauge connection 1-form \( a \), the so-called graviphoton field that generally appears in extended supergravity theories.

The respective field strengths of \( e, \omega \) and \( a \) are the torsion 2-form \( T = De \equiv de + \omega e \), the curvature 2-form \( R = d\omega + \frac{1}{2} [\omega, \omega] \) and the Abelian curvature 2-form \( F_a = da \). They satisfy the Bianchi identities

\[
DR = 0 , \quad \text{where} \quad DR \equiv dR + [\omega, R]
\]

\[
DT = Re , \quad \text{where} \quad DT \equiv dT + \omega T
\]

\[
dF_a = 0 .
\]

The basic symmetries are diffeomorphisms and local Lorentz transformations parametrized in a BRST setting by ghosts \( \xi = \xi^\mu \partial_\mu \) and \( c = (c^a_b) \), as well as local \( U(1) \) transformations parametrized by a ghost \( u \). Note that both \( \omega \) and \( c \) take their values in the Lie algebra of the Lorentz group, i.e. \( \omega_{ab} = -\omega_{ba} \) and \( c_{ab} = -c_{ba} \).
2.2.1 Horizontality conditions

We introduce the *generalized differential* \( \hat{d} = d + S \) and the *generalized fields* \([34, 32]\)

\[
\hat{\omega} \equiv e^{i\xi}(\omega + c) = \omega + c + i\xi \omega, \quad \hat{e} \equiv e^{i\xi}e = e + i\xi e
\]

\[
\hat{a} \equiv e^{i\xi}(a + u) = a + u + i\xi a,
\]

\[
\hat{\hat{R}} \equiv \hat{d}\hat{\omega} + \frac{1}{2}[\hat{\omega}, \hat{\omega}], \quad \hat{\hat{T}} \equiv \hat{D}\hat{e} = \hat{d}\hat{e} + \hat{\omega}\hat{e}
\]

which imply Bianchi identities for the generalized curvature and torsion forms:

\[
\hat{D}\hat{R} = 0, \quad \hat{D}\hat{T} = \hat{\hat{R}}
\]

By expanding the generalized 2-forms \( \hat{R}, \hat{T} \) and \( \hat{F}_a \) with respect to the ghost-number we find

\[
\hat{R} = R_0^0 + R_1^1 + R_0^2, \quad \text{with} \quad \begin{cases} 
R_0^0 = R \\
R_1^1 = S\omega + Dc_\xi \\
R_0^2 = Sc_\xi + c_\xi^2,
\end{cases} \quad (c_\xi \equiv c + i\xi \omega)
\]

(2.7)

as well as similar expressions for \( \hat{T} \) and \( \hat{F}_a \).

The BRST transformations of all space-time fields then follow from relations (2.6) by imposing a horizontality condition, i.e. by specifying \( R_1^1 \) and \( R_0^2 \) (\( R_0^0 \) being necessarily equal to the curvature 2-form \( R \)) and by specifying the corresponding components of \( \hat{T} \) and \( \hat{F}_a \). For topological gravity, one imposes the following *horizontality conditions* \([21]\) which generalize those of topological Yang-Mills theories\(^3\):

\[
\hat{R} = e^{i\xi}(R + \tilde{\psi} + \tilde{\phi}), \quad \hat{T} = e^{i\xi}(T + \psi + \phi)
\]

(2.8)

Here, \( \tilde{\psi}^a_b, \psi^a \) and \( \eta \) are 1-forms with ghost-number 1, while \( \tilde{\phi}^a_b, \phi^a \) and \( t \) are 0-forms with ghost-number 2. The fields \( \tilde{\psi} \) and \( \tilde{\phi} \) are Lorentz algebra-valued, i.e. \( \tilde{\psi}_{ab} = -\tilde{\psi}_{ba} \) and \( \tilde{\phi}_{ab} = -\tilde{\phi}_{ba} \).

By substituting the expansion (2.7) and the analogous expansions for \( \hat{T} \) and \( \hat{F}_a \) into (2.8), we see that the ghost-number 2 fields \( \tilde{\phi}^a_b, \phi^a \) and \( t \) appear in the \( S \)-variations of the ghost fields so that they represent *ghosts for ghosts*. Their appearance expresses the reducibility of the resulting symmetry algebra, see remark (i) below.

\(^3\)The geometrical interpretation of horizontality conditions for ordinary and topological Yang-Mills theories are discussed in references \([32]\) and \([35]\), respectively.
The action of the operator $e^{i\xi}$ can be factorized in all terms of equations (2.6) and (2.8) by virtue of the following operatorial relation [31, 33] which results from equations (2.4):

$$(d + \mathcal{S})e^{i\xi} = e^{i\xi}(d + \mathcal{S} - \mathcal{L}_\xi + i\mathcal{S}_\xi - \mathcal{S}_\xi^2).$$

(2.9)

Let $\varphi \equiv \varphi^\mu \partial_\mu$ denote the vector field $\mathcal{S}\xi - \xi^2$ which appears on the right-hand-side and which is of ghost-number 2. The requirement of nilpotency for the variation $\mathcal{S}\xi$ then implies that the $\mathcal{S}$-variation of the vector field $\varphi$ is given by its Lie derivative:

$$\mathcal{S}\xi = \xi^2 + \varphi, \quad \mathcal{S}\varphi = [\xi, \varphi].$$

(2.10)

Since $\varphi$ describes a local shift of the diffeomorphism ghost, it parametrizes *vector supersymmetry* transformations [36]. By expanding equations (2.8) and (2.6) with respect to the ghost-number, we get the $\mathcal{S}$-variations of all fields as well as the relation

$$\varphi \equiv \varphi^\mu \partial_\mu - \varphi^{\mu a} e^\mu_a$$

which is missing in [21] and [18], respectively, and which ensure the nilpotency of $\mathcal{S}$.

After the change of variables

$$\tilde{\phi} \to \tilde{\varphi} := \tilde{\phi} - i\varphi \omega$$

$$t \to \tau := t - i\varphi a,$$

(2.11)

the $\mathcal{S}$-variations of the basic fields take the simple form

$$S e = \mathcal{L}_\xi e - ce + \psi,$$

$$S \psi = \mathcal{L}_\xi \psi - c\psi - \mathcal{L}_\varphi e + \varphi e,$$

$$S \omega = \mathcal{L}_\xi \omega - \mathcal{L}_\varphi \omega - D\varphi,$$

$$S \tilde{\psi} = \mathcal{L}_\xi \tilde{\psi} - [c, \tilde{\psi}] - \mathcal{L}_\varphi \omega - D\varphi,$$

$$S \phi = \mathcal{L}_\xi \phi - c\phi + \psi e - D\psi,$$

$$S c = \mathcal{L}_\xi c - c^2 + \varphi,$$

$$S \varphi = \mathcal{L}_\xi \varphi - [c, \varphi] - \mathcal{L}_\varphi c,$$

$$S \tilde{\varphi} = \mathcal{L}_\xi \tilde{\varphi} - [c, \tilde{\varphi}] - \mathcal{L}_\varphi \omega,$$

(2.12)

and those of the field strengths read as

$$S R = \mathcal{L}_\xi R - [c, R] - D\tilde{\psi},$$

$$S T = \mathcal{L}_\xi T - cT + \psi e - D\psi,$$

$$S F_a = \mathcal{L}_\xi F_a - d\eta.$$

(2.13)

By construction, the so-defined $\mathcal{S}$-operator is nilpotent and the results coincide with those given in references [21, 18], except for some terms in the $\mathcal{S}$-variations of $\tilde{\varphi}$ and $T$ which are missing in [21] and [18], respectively, and which ensure the nilpotency of $\mathcal{S}$. 

6
Remarks: (i) It is easy to understand the origin of all terms appearing in the transformation laws (2.12). We only consider the gravitational sector since the argumentation for the Maxwell sector proceeds along the same lines. The $S$-variations of the basic fields $e$ and $\omega$ describe diffeomorphisms (parametrized by $\xi$), local Lorentz transformations (parametrized by $c$) and the topological (or shift) symmetry (parametrized by $\psi$ and $\tilde{\psi}$), which is characteristic for topological field theories of Witten-type. Obviously, the $S$-variation of $e$ is reducible: $Se$ is invariant under a shift $\delta \xi = \varphi$ which comes together with the transformation $\delta \psi = -\mathcal{L}_\varphi e$. Furthermore, it is invariant under the shift $\delta c = \tilde{\varphi}$ that goes together with $\delta \tilde{\psi} = D\tilde{\varphi}$, respectively. The BRST algebra can then be completed by assuming all fields to change linearly under Lorentz transformations and to transform with the Lie derivative under diffeomorphisms: this leads to the reducibility of $Sc$ under the shift $\delta \xi = \varphi$ accompanied by the transformation $\delta \tilde{\varphi} = -\mathcal{L}_\varphi c$. Thus, all terms have a natural interpretation and the signs are simply a matter of nilpotency.

(ii) The supersymmetry or shift operator is given by

$$\tilde{Q} \equiv S - \mathcal{L}_{\xi} - \delta^{(L)}_e - \delta^{(M)}_u$$

where $\delta^{(L)}_e$ and $\delta^{(M)}_u$ denote, respectively, infinitesimal Lorentz and Maxwell transformations. When applied to the basic fields and ghosts, it satisfies

$$\tilde{Q}^2 = -\mathcal{L}_\varphi - \delta^{(L)}_\tilde{\varphi} - \delta^{(M)}_\tau,$$

i.e. $\tilde{Q}$ is nilpotent up to infinitesimal diffeomorphism, Lorentz and Maxwell transformations with parameters $\varphi$, $\tilde{\varphi}$ and $\tau$, respectively.

(iii) An arbitrary shift $\psi^a_\mu$ of the vielbein $e^a_\mu$ does not preserve the positivity of the determinant of the metric [6]. This “problem” can be solved [6] by assuming that the shift of the vielbein is described by a local infinitesimal gauge transformation associated to the general linear group $GL(n, \mathbb{R})$, i.e. by assuming $\psi^a_\mu$ to be of the form $G^a_b e^b_\mu$ with $(G^a_b) \in GL(n, \mathbb{R})$. The BRST algebra then takes a form which is quite similar to (2.12).

However, topological invariants are inert under arbitrary shifts of the metric (or vielbein) and therefore the positivity of the determinant of the metric does not necessarily have to be imposed at this point.

(iv) By changing generators according to reparametrizations of the form $c' = c + i\xi \omega$, $\psi' = \psi + (i\xi \omega)e$, ..., the BRST algebra (2.12) can be cast into equivalent forms. One such reformulation can be obtained from a different reading of the generalized fields and horizontality conditions. This parametrization naturally appears in the group manifold approach and the associated rheonomic parametrization
of curvatures [20]. In fact, let us read the generalized fields (2.5) as
\[ \hat{\omega} = \omega + c_\xi, \quad \text{with} \quad c_\xi \equiv c + i_\xi \omega \]
\[ \hat{e} = e + \varepsilon_\xi, \quad \text{with} \quad \varepsilon_\xi \equiv \xi^a = \xi^m e^a_m \]
\[ \hat{a} = a + u_\xi, \quad \text{with} \quad u_\xi \equiv u + i_\xi a, \quad (2.14) \]
and the horizontality conditions (2.8) as
\[ \hat{R} = R + \tilde{\psi}_\xi + \tilde{\phi}_\xi, \quad \hat{T} = T + \psi_\xi + \phi_\xi \]
\[ \hat{F}_a = F_a + \eta_\xi + t_\xi, \quad (2.15) \]
with \( \tilde{\psi}_\xi = \tilde{\psi} + i_\xi R, \tilde{\phi}_\xi = \tilde{\phi} + i_\xi \tilde{\psi} + \frac{1}{2} i_\xi i_\xi R, \) etc. Expansion of relations (2.15) with respect to the ghost-number immediately yields the \( S \)-variations in their “Lorentz- and Maxwell-covariantized form”:
\[ S e = -D \varepsilon_\xi - c_\xi e + \psi_\xi, \quad S \varepsilon_\xi = -c_\xi \varepsilon_\xi + \tilde{\phi}_\xi \]
\[ S \psi_\xi = -D \tilde{\phi}_\xi - c_\xi \tilde{\psi}_\xi + \tilde{\psi} \varepsilon_\xi + \tilde{\phi} e, \quad S \tilde{\phi}_\xi = -c_\xi \tilde{\phi}_\xi + \phi_\xi \varepsilon_\xi \]
\[ S \omega = -Dc_\xi + \tilde{\psi}_\xi, \quad S c_\xi = -c_\xi^2 + \tilde{\phi}_\xi \]
\[ S \tilde{\psi}_\xi = -D \tilde{\phi}_\xi - [c_\xi, \tilde{\psi}_\xi], \quad S \tilde{\phi}_\xi = -[c_\xi, \tilde{\phi}_\xi] \quad (2.16) \]
\[ S a = -du_\xi + \eta_\xi, \quad S u_\xi = t_\xi \]
\[ S \eta_\xi = -dt_\xi, \quad S t_\xi = 0 \]
and
\[ S R = -D \tilde{\psi}_\xi - [c_\xi, R], \quad S T = -D \psi_\xi - c_\xi T + R \varepsilon_\xi + \tilde{\psi}_\xi e \]
\[ S F_a = -d\eta_\xi. \quad (2.17) \]
These expressions coincide with those of reference [20]. An advantage of this parametrization consists of the fact that the field \( \tilde{\phi}_\xi \) simply transforms like a commutator, exactly as the ghost for ghost in topological Yang-Mills theories. Henceforth, BRST invariant polynomials in this variable are generated by \( \text{Tr}(\tilde{\phi}_\xi)^n \) with \( n = 1, 2, \ldots \).
We will come back to this point in the next subsection.

In conclusion, we mention one more change of generators which allows to cast the BRST algebra into another equivalent form which appears more or less explicitly in the early works on the subject, e.g. see references [13, 14, 8]. In fact, by virtue of the reparametrization \( c \rightarrow c_\xi = c + i_\xi \omega \) and \( \tilde{\phi} \rightarrow \tilde{\phi} = \tilde{\phi} + i_\xi \omega, \) the \( S \)-variations of the gravitational sector, as given by equations (2.12), take the following form involving the Lorentz-covariant Lie derivative \( L_\xi \equiv i_\xi D_\xi - Di_\xi \):
\[ S e = L_\xi e - c_\xi e + \psi, \quad S \xi = \xi^2 + \varphi \]
\[ S \psi = L_\xi \psi - c_\xi \psi - L_\varphi e + \tilde{\phi} e, \quad S \varphi = [\xi, \varphi] \]
\[ S \omega = i_\xi R - D c_\xi + \tilde{\psi}, \quad S c_\xi = i_\xi \tilde{\psi} + \frac{1}{2} i_\xi i_\xi R - c_\xi^2 + \phi \]
\[ S \tilde{\psi}_\xi = L_\xi \tilde{\psi} - [c_\xi, \tilde{\psi}] - i_\varphi R - D \tilde{\phi}, \quad S \phi = L_\xi \tilde{\phi} - [c_\xi, \tilde{\phi}] - i_\varphi \tilde{\psi}. \quad (2.18) \]
2.2.2 Observables

The construction of observables for topological gravity is based on gauge invariant polynomials of the curvature form (e.g. $\text{Tr}\{R^k\}$ with $k = 1, 2, \ldots$) and of the torsion form. A topological invariant involving torsion has first been introduced in four dimensions by Nieh and Yan [37] and has been further discussed by the authors of reference [38] who also constructed higher dimensional generalizations. The following discussion of the cases $k = 2$ and $k = 1$ applies to manifolds which are at least of dimension four and two, respectively.

Case $k = 2$: We first consider the gravitational sector and comment on the Maxwell sector thereafter.

The Pontryagin density, i.e. the 4-form $W^0_4 \equiv -\frac{1}{2} \text{Tr}\{RR\}$ is closed by virtue of the Bianchi identity $DR = 0$:

$$dW^0_4 = -\text{Tr}\{(DR)R\} = 0.$$ 

Accordingly, the generalized 4-form $\hat{W} \equiv -\frac{1}{2} \text{Tr}\{R\hat{R}\}$ is annihilated by the generalized differential $\hat{d} \equiv d + s$, i.e.

$$s\hat{W} = -d\hat{W}.$$ 

By substituting (2.8) into $\hat{W}$ and expanding with respect to the ghost-number, we obtain

$$\hat{W} = -\frac{1}{2} e^{i\xi} \text{Tr}\{(R + \tilde{\psi} + \tilde{\phi})(R + \tilde{\psi} + \tilde{\phi})\} = \sum_{k=0}^{4} W^{k}_{4-k}(\xi).$$

Here, the $\xi$-dependence is explicitly given by

$$W^{k}_{4-k}(\xi) = \sum_{n=0}^{k} \frac{1}{n!} (i\xi)^n W^{k-n}_{4-k+n},$$

where the polynomials $W^{k}_{4-k}$ appearing on the right-hand-side have the same form as the Donaldson polynomials in topological Yang-Mills theory:

$$W^{0}_{4} = -\frac{1}{2} \text{Tr}\{RR\}, \quad W^{1}_{3} = -\text{Tr}\{\tilde{\psi}R\}, \quad W^{2}_{2} = -\text{Tr}\{\tilde{\phi}R + \frac{1}{2}\tilde{\psi}\tilde{\psi}\},$$

$$W^{3}_{1} = -\text{Tr}\{\tilde{\phi}\tilde{\psi}\}, \quad W^{4}_{0} = -\frac{1}{2} \text{Tr}\{\tilde{\phi}\tilde{\phi}\}.$$ 

In particular, $W^{0}_{4}(\xi) = W^{0}_{4} = -\frac{1}{2} \text{Tr}\{RR\}$. If one uses the relation $\tilde{\phi} = \tilde{\varphi} + i\varphi\omega$ to express $\tilde{\phi}$ in terms of the variable $\tilde{\varphi}$ which appears in the BRST transformations (2.12), then the polynomials $W^{2}_{2}(\xi)$, $W^{3}_{1}(\xi)$ and $W^{4}_{0}(\xi)$ also depend on the shift $\varphi$ of $\xi$. 

9
By virtue of the relation (2.19), the polynomials (2.21) satisfy the descent equations$^4$

\[
\begin{align*}
    dW^0_4(\xi) &= 0 \\
    SW^k_{4-k}(\xi) + dW^{k+1}_{3-k}(\xi) &= 0 \quad \text{with} \quad 0 \leq k \leq 3 \\
    SW^0_4(\xi) &= 0.
\end{align*}
\]

The polynomials $W^k_{4-k}(\xi)$ represent elements of the so-called *equivariant cohomology* $[26, 29]$ of topological gravity. By contrast to the case of topological Yang-Mills theories, a ghost associated to gauge transformations, namely the diffeomorphism ghost $\xi$, appears in the cohomology classes $[7]$. However, this ghost does not play the same rôle as the ghosts $c$ and $u$ associated to Lorentz and Maxwell gauge transformations (since its action amounts to moving points on the space-time manifold) and its presence is actually necessary [15].

In four dimensions, another set of observables can be obtained from the *Euler class* $V^0_4 \equiv -\frac{1}{2} \Tr \{\varepsilon_{abcd} R^{ab} R^{cd}\}$ whose integration yields the Euler characteristic. One follows the same procedure, i.e. one expands $\hat{V} \equiv -\frac{1}{2} \Tr \{\varepsilon_{abcd} \hat{R}^{ab} \hat{R}^{cd}\}$ with respect to the ghost-number as in equation (2.20): $\hat{V} = \sum_{k=0}^{4} V^k_{4-k}(\xi)$.

Since $d\hat{W} = 0 = d\hat{V}$, we also have $d(\hat{W}^m \hat{V}^n) = 0$ for $m, n \in \{0, 1, \ldots\}$. By expanding $\hat{W}^m \hat{V}^n$ with respect to the ghost-number, one obtains further representatives of the cohomology algebra $[17]$:

\[
\hat{W}^m \hat{V}^n = w_0^{4(m+n)}(\xi) + w_1^{4(m+n)-1}(\xi) + \cdots + w_4^{4(m+n)-4}(\xi),
\]

with

\[
\begin{align*}
    w_0^{4(m+n)}(\xi) &= [W^4_0(\xi)]^m [V^4_0(\xi)]^n, \\
    w_1^{4(m+n)-1}(\xi) &= n [W^4_0(\xi)]^m [V^4_0(\xi)]^{n-1} V^3(\xi) + m [W^4_0(\xi)]^{m-1} W^3(\xi) [V^4_0(\xi)]^n, \quad \text{etc.}
\end{align*}
\]

An obvious question is whether or not there exist further elements in the gravitational sector of the equivariant cohomology which are related to the curvature form. To address this problem, it is useful to recall the variables $\tilde{\psi}_\xi$ and $\tilde{\phi}_\xi$ introduced in equation (2.15) and to invoke a simple argument put forward in reference [20]. Due to the very definition of $\tilde{\psi}_\xi$ and $\tilde{\phi}_\xi$, we can read expression (2.20) as

\[
\hat{W} = -\frac{1}{2} \Tr \{(R + \tilde{\psi}_\xi + \tilde{\phi}_\xi)(R + \tilde{\psi}_\xi + \tilde{\phi}_\xi)\},
\]

so that the polynomials $W^k_{4-k}(\xi)$ are the Donaldson polynomials (2.22) with $\tilde{\psi}$ and $\tilde{\phi}$ replaced by $\tilde{\psi}_\xi$ and $\tilde{\phi}_\xi$, respectively. In particular, we have $W^4_0(\xi) = -\frac{1}{2} \Tr \{\tilde{\phi}_\xi \tilde{\phi}_\xi\}$. This result is consistent with the comment (made after equation (2.17)) that the

$^4$We note that the $\mathcal{S}$-variation of $\tilde{\phi}$ is given by $\mathcal{S} \tilde{\phi} = \mathcal{L}_\xi \tilde{\phi} - [\tilde{\phi}, \tilde{\phi}] - i_\phi \tilde{\psi}$. 

10
\( S \)-invariant polynomials in \( \tilde{\phi}_\xi \) are generated by the Lorentz invariant polynomials \( \text{Tr} (\tilde{\phi}_\xi)^n \) with \( n = 1, 2, \ldots \). More specifically, for the four dimensional case that we consider here, the \( 4 \times 4 \) matrix \( \tilde{\phi}_\xi \) has the four invariants \( \text{Tr} \tilde{\phi}_\xi \), \( \text{Tr} (\tilde{\phi}_\xi)^2 \), \( \text{Tr} (\tilde{\phi}_\xi)^3 \), \( \text{Tr} (\tilde{\phi}_\xi)^4 \), but \( \text{Tr} \tilde{\phi}_\xi \) and \( \text{Tr} (\tilde{\phi}_\xi)^3 \) vanish due to the antisymmetry of the matrix \( \tilde{\phi}_\xi \). Thus, the only non-trivial invariants are \( \text{Tr} (\tilde{\phi}_\xi)^3 \), \( \text{Tr} (\tilde{\phi}_\xi)^4 \), but \( \text{Tr} \tilde{\phi}_\xi \) and \( \text{Tr} (\tilde{\phi}_\xi)^3 \) vanish due to the antisymmetry of the matrix \( \tilde{\phi}_\xi \). Thus, the torsion is the integral over the Nieh-Yan form \([37, 38]\), i.e. the 4-form over the space-time manifold. In four dimensions, this topological invariant related to fields whose integral only depends on the topology, i.e. on the global properties of the manifold. Accordingly, we can proceed as for the Pontryagin or Euler density, i.e. exploit the fact that the part of the BRST algebra which involves the Lorentz connection can be cast into a form that is isomorphic to the BRST algebra of topological Yang-Mills theory allows us to invoke the known results concerning the equivariant cohomology of the latter, e.g. see ref. [28].

Let us now turn to the vielbein part of the gravitational BRST algebra. Quite remarkably, there exists a local expression given by the vielbein and connection fields whose integral only depends on the topology, i.e. on the global properties of the space-time manifold. In four dimensions, this topological invariant related to the torsion is the integral over the Nieh-Yan form \([37, 38]\), i.e. the 4-form

\[
Z^0_4 \equiv -\frac{1}{2} (TT - eRe) = -\frac{1}{2} (T^aT_a - e^aR_{ab}e^b).
\]

(2.26)

The latter vanishes if the torsion vanishes (due to the Bianchi identity \( Re = DT \)). Furthermore, this form is closed and locally exact: \( d(eT) = D(eT) = -2Z^0_4 \). Accordingly, we can proceed as for the Pontryagin or Euler density, i.e. exploit the fact that the generalized 4-form \( \tilde{Z} \equiv -\frac{1}{2}(T\tilde{T} - \tilde{e}\tilde{R}\tilde{e}) \) is annihilated by the generalized differential \( \tilde{d} = d + S \). Thus, we expand \( \tilde{Z} \) with respect to the ghost-number analogously to the expansion of \( \tilde{W} \) in equations (2.20) and (2.21). By construction, the polynomials \( Z^k_{4-k}(\xi) \) appearing in this expansion satisfy the descent equations (2.23). Explicit expressions can readily be obtained from those of \( \tilde{e}, \tilde{T}, \tilde{R} \) given in equations (2.5) and (2.8). The results take a concise form when written in terms of the reparametrized ghosts \( \varepsilon_\xi, \tilde{\psi}_\xi, \tilde{\phi}_\xi, \psi_\xi, \phi_\xi \) introduced in (2.14) and (2.15):

\[
Z^0_4(\xi) = -\frac{1}{2} (TT - eRe) \\
Z^1_4(\xi) = -(T\psi_\xi - \varepsilon_\xi Re - \frac{1}{2} e\tilde{\psi}_\xi e) \\
Z^2_4(\xi) = -(T\phi_\xi + \frac{1}{2} \psi_\xi \psi_\xi - \varepsilon_\xi \tilde{\psi}_\xi e - \frac{1}{2} e\tilde{\phi}_\xi e - \frac{1}{2} \varepsilon_\xi R\varepsilon_\xi) \\
Z^3_4(\xi) = - (\psi_\xi \phi_\xi - \varepsilon_\xi \tilde{\phi}_\xi e - \frac{1}{2} \varepsilon_\xi \tilde{\psi}_\xi \varepsilon_\xi) \\
Z^4_4(\xi) = - \frac{1}{2} (\phi_\xi \phi_\xi - \varepsilon_\xi \tilde{\phi}_\xi \varepsilon_\xi). \\
\]

(2.27)
Last, we consider the Maxwell sector. It represents a topological Yang-Mills theory with Abelian gauge group which entails that the observables are generated by the 4-form \( F_a F_a \).

By combining all of these results, one concludes that the most general elements of the equivariant cohomology for topological gravity are obtained by considering appropriate products of the expressions constructed in the gravitational and Maxwell sectors.

**Case \( k = 1 \):** This case can be treated along the same lines while starting from the closed 2-form \( W_2^0 \equiv \varepsilon^{ab} R_{ab} \): the generalized 2-form

\[
\bar{W} \equiv \varepsilon^{ab} \bar{R}_{ab} = W_2^0 + W_1^1(\xi) + W_0^2(\xi)
\]

then satisfies \( \hat{d} \bar{W} = 0 \) (i.e. \( S \bar{W} = -d \bar{W} \)) and involves the polynomials

\[
W_1^1(\xi) = \varepsilon^{ab} (\bar{\psi}_{ab} + i \xi R_{ab}) \quad \text{(2.29)}
\]

\[
W_2^2(\xi) = \varepsilon^{ab} (\bar{\phi}_{ab} + i \xi \bar{\psi}_{ab} + \frac{1}{2} i \xi R_{ab}) .
\]

From \( \hat{d}(\bar{W})^n = 0 \) for \( n = 1, 2, \ldots \) and from the expansion \( (\bar{W})^n = w_0^{2n} + w_1^{2n-1} + w_2^{2n-2} \), one obtains more general representatives of the equivariant cohomology algebra in two dimensions [10, 12]:

\[
w_0^{2n} = [W_0^2(\xi)]^n, \quad w_1^{2n-1} = n[W_1^1(\xi)][W_0^2(\xi)]^{n-1}, \quad w_2^{2n-2} = n[W_2^0(\xi)][W_0^2(\xi)]^{n-1} + \frac{1}{2} n(n - 1)[W_0^2(\xi)]^{n-2}[W_1^1(\xi)]^2 .
\]

There is no topological invariant related to the torsion in two dimensions [38]. In the Maxwell sector, the basic invariant is given by the 2-form \( F_a \).

### 2.3 Second order formalism

We only discuss the gravitational sector since the Maxwell sector is not modified.

If we require the torsion form to vanish, the connection becomes a function of the vielbein (and its inverse) which is now the only independent field in the gravitational sector:

\[
\omega_{abc} = \frac{1}{2} (P_{abc} + P_{bca} - P_{cab}) , \quad \text{with} \quad P^a_{bc} \equiv (\partial_{\mu} e^a_{\nu} - \partial_{\nu} e^a_{\mu}) e^b_{\nu} e^c_{\epsilon} .
\]

For consistency, one also has to require the \( S \)-variation of the torsion to vanish. By virtue of equation (2.13), this implies that \( \bar{\psi} \) is no longer an independent ghost, rather it is a function of the ghost \( \psi \) and the vielbein:

\[
\bar{\psi}_{\delta} e^b = D \psi^a .
\]
Note that relation (2.31), as well as the solution of (2.32) with respect to \(\tilde{\psi}\), again rely on the assumption that the vielbein is invertible.

In summary, in the gravitational sector, the basic variable is the vielbein \(e\) and we have ghosts \(\xi, c, \psi\) as well as ghosts for ghosts \(\varphi\) and \(\tilde{\varphi}\) with the nilpotent \(S\)-variations

\[
\begin{align*}
Se &= \mathcal{L}_\xi e - ce + \psi, \\
Sc &= \mathcal{L}_\xi c - c^2 + \tilde{\varphi}, \\
S\xi &= \xi^2 + \varphi,
\end{align*}
\]

(2.33)

Henceforth, the only difference with the first order formalism consists of the fact that \(\omega, \tilde{\psi}\) and their \(S\)-variations

\[
\begin{align*}
S\omega &= \mathcal{L}_\xi \omega - Dc + \tilde{\psi}, \\
S\tilde{\psi} &= \mathcal{L}_\xi \tilde{\psi} - [c, \tilde{\varphi}] - \mathcal{L}_{\varphi} \omega - D\tilde{\varphi}
\end{align*}
\]

(2.34)

are no longer independent expressions, but merely consequences of \(T = 0, ST = 0\) and of the variations (2.33). A fortiori, the Lorentz part of the equivariant cohomology is not modified up to the fact that \(\omega\) and \(\tilde{\psi}\) have the explicit form (2.31) and (2.32) in terms of \(e\) and \(\psi\). Some of the results of the metric approach simply follow by combining the vielbein fields into a metric tensor \(g_{\mu\nu} \equiv \eta_{ab}e^a_{\mu}e^b_{\nu}\), see appendix A.

## 3 Superspace approach

After introducing superspace and the geometric objects that it supports, we derive the symmetry algebra of topological gravity from a few simple transformation laws in superspace. As before, we do not need to specify the space-time dimension for the discussion of symmetries.

### 3.1 Supersymmetry and superspace

*Rigid supersymmetry* is defined by an odd generator \(Q\) satisfying the Abelian superalgebra \(Q^2 = 0\). Field theoretic representations are given by doublets and singlets, and they are readily obtained from a superspace construction: we extend the \(d\)-dimensional space-time manifold by a single Grassmann variable \(\theta\) so as to obtain a superspace parametrized by local coordinates \((x, \theta)\). Then, a *superfield* is, by definition, a function on superspace,

\[
F(x, \theta) = f(x) + \theta f_\theta(x),
\]

(3.1)

transforming under an infinitesimal supersymmetry transformation as

\[
QF = \partial_\theta F,
\]

(3.2)
which yields the following action of $Q$ on the component fields:

$$Qf = f'_\theta, \quad Qf'_\theta = 0.$$  \hfill (3.3)

In expression (3.1), the component field $f$ has the same Grassmann grading as $F$ while its superpartner $f'_\theta$ has the opposite one. We assign a “supersymmetry ghost-number” (SUSY-number or SUSY-charge for short) to all fields and variables: this charge is defined by assigning the value $-1$ to the variable $\theta$ and, quite generally, an upper or lower $\theta$-index on a field corresponds to a charge $-1$ or $+1$, respectively. The generator $Q$ raises the SUSY-number by one unit.

A $p$-superform admits the expansion

$$\hat{\Omega}_p(x, \theta) = \sum_{k=0}^{p} \Omega_{p-k}(x, \theta) (d\theta)^k,$$  \hfill (3.4)

where $\Omega_{p-k}$ has $k$ $\theta$-indices that we did not spell out. The components $\Omega_q(x, \theta)$ of the $p$-superform are $q$-forms whose coefficients are superfields:

$$\Omega_q(x, \theta) = \frac{1}{q!} \Omega_{\mu_1...\mu_q}(x, \theta) dx^{\mu_1} \cdots dx^{\mu_q} = \omega_q(x) + \theta \omega'_q(x).$$

In the previous expression and in the following, the wedge product symbol is always omitted. Moreover, we will adhere to the notational conventions considered in the previous expressions: functions or forms on ordinary space-time are denoted by small case letters, superfields (or space-time forms having superfields as coefficients) by upper case letters and super $p$-forms with $p \geq 1$ (i.e. $p$-forms in superspace with superfields as coefficients) by upper case letters with a “hat”.

A supervector field has the form $\hat{\Xi}(x, \theta) = 2^\mu (x, \theta) \partial_\mu + 2^\theta (x, \theta) \partial_\theta \equiv 2^M \partial_M$ where $M$ denotes a supercoordinate index (i.e. $M = \mu$ or $M = \theta$). The graded Lie bracket of two vector fields $\hat{\Xi}_1$ and $\hat{\Xi}_2$ is again a vector field whose components are given by

$$[\hat{\Xi}_1, \hat{\Xi}_2]^M = 2^N \partial_N 2^M \pm 2^N \partial_N 2^M,$$

with a plus sign if both $\hat{\Xi}_1$ and $\hat{\Xi}_2$ have odd ghost-number, and a minus sign otherwise.

We now proceed to introduce the standard differential operators in superspace. The exterior derivative is given by $\hat{d} = d + d\theta \partial_\theta$ with $d = dx^\mu \partial_\mu$. We have the relations $0 = \hat{d}^2 = d^2 = (d\theta \partial_\theta)^2 = [d, d\theta \partial_\theta]$ where the bracket $[\cdot, \cdot]$ denotes the graded commutator. The Lie derivative $\mathcal{L}_{\hat{\Xi}}$ with respect to the supervector field $\hat{\Xi}$ acts on a superform according to $\mathcal{L}_{\hat{\Xi}} = [i_{\hat{\Xi}}, \hat{d}]$ (where $i_{\hat{\Xi}}$ denotes the inner product operation) and we have the graded commutation relation $[\mathcal{L}_{\hat{\Xi}_1}, \mathcal{L}_{\hat{\Xi}_2}] = \mathcal{L}_{[\hat{\Xi}_1, \hat{\Xi}_2]}$.

A local, infinitesimal supersymmetry transformation is given by a $x$-dependent translation of the $\theta$-variable, i.e. $\theta \rightarrow \theta + \varepsilon^\theta(x)$. Thus, it is a supercoordinate
transformation generated by the vector field \( \varepsilon^\theta(x) \partial_\theta \). The latter acts in superspace by virtue of the Lie derivative, e.g. on a superfield:

\[
\delta_\varepsilon F(x, \theta) = \varepsilon^\theta(x) \partial_\theta F(x, \theta). \tag{3.5}
\]

The induced variations of the component fields read as

\[
\delta_\varepsilon f(x) = \varepsilon^\theta(x) f'_\theta(x), \quad \delta_\varepsilon f'_\theta(x) = 0. \tag{3.6}
\]

Obviously, the rigid supersymmetry transformations \( \delta F = \varepsilon Q F \) with \( \varepsilon \) constant represent a special case.

### 3.2 Fields and symmetries

The basic variables in the gravitational sector of the theory are the connection super 1-forms \( \hat{\Omega}^a_{\theta}(x, \theta) \) associated to local Lorentz transformations and the vielbein super 1-form \( \hat{E}^a(x, \theta) \). We do not introduce superforms with \( \theta \)-indices, \( \hat{\Omega}^\theta_{\theta}(x, \theta) \) or \( \hat{E}^\theta(x, \theta) \). In fact, \( \hat{\Omega}^\theta_{\theta} = 0 \) since the action of the Lorentz algebra on scalars is trivial and \( \hat{E}^\theta \) only transforms linearly and solely under supercoordinate transformations, henceforth there is no obstruction for its vanishing. We shall however come back to this point in subsection 3.5. In the Maxwell sector, the basic variable is the connection super 1-form \( \hat{A} \) associated to local \( U(1) \) transformations.

Within the BRST formalism, the parameters of infinitesimal symmetry transformations are turned into ghost fields (having a ghost-number 1): thus, we have the Lorentz and Maxwell ghosts \( C^a_{\theta}(x, \theta) \) and \( U(x, \theta) \) which are superfields and the superdiffeomorphism ghost \( \hat{\Xi} \) which is a supervector field. The connection \( \hat{\Omega} \) and ghost \( C \) both take their values in the Lorentz algebra, i.e. \( \hat{\Omega}_{ab} = -\hat{\Omega}_{ba} \) and \( C_{ab} = -C_{ba} \). For the ghost vector field \( \hat{\Xi} \), it is convenient to introduce the notation

\[
\hat{\Xi}^2 \equiv \frac{1}{2} [\hat{\Xi}, \hat{\Xi}], \quad \text{i.e.} \quad (\hat{\Xi}^2)^M = \Xi^\mu \partial_\mu \Xi^M + \Xi^\theta \partial_\theta \Xi^M,
\]

in terms of which we can write \( (\mathcal{L}_{\hat{\Xi}})^2 = \mathcal{L}_{\hat{\Xi}^2} \).

### 3.3 BRST transformations in superspace

The Grassmann parity of an object is given by the parity of its total degree which is now defined as the sum \( p + g + s \) of its form degree \( p \), ghost-number \( g \) and SUSY-number \( s \). All commutators and brackets are assumed to be graded according to this grading.

We collect all symmetry transformations in the superspace BRST transformations which can be written in the following way, using obvious matrix notation like

15
\( \hat{E} \) for \( \hat{E}^a \) and \( \hat{\Omega} \) for \( \hat{\Omega}_{ab} \):

\[
\begin{align*}
\hat{S} \hat{E} &= \mathcal{L}_{\hat{\Xi}} \hat{E} - C \hat{E}, \\
\hat{S} \hat{\Xi} &= \hat{\Xi}^2,
\end{align*}
\]

\[
\begin{align*}
\hat{S} \hat{\Omega} &= \mathcal{L}_{\hat{\Xi}} \hat{\Omega} - \hat{D} C, \\
\hat{S} \hat{C} &= \mathcal{L}_{\hat{\Xi}} C - C^2, \\
\hat{S} \hat{A} &= \mathcal{L}_{\hat{\Xi}} \hat{A} - \hat{d} U, \\
\hat{S} U &= \mathcal{L}_{\hat{\Xi}} U.
\end{align*}
\]

(3.7)

Here, \( \hat{D} C \equiv \hat{d} C + [\hat{\Omega}, C] \) and the given \( \hat{S} \)-operator is nilpotent.

We note [39] that the transformations laws (3.7) may be deduced from horizontality conditions involving the torsion and curvature superforms

\[
\hat{T} \equiv \hat{d} \hat{E} + \hat{\Omega} \hat{E}, \quad \hat{R} \equiv \hat{d} \hat{\Omega} + \hat{\Omega}^2, \quad \hat{F} \equiv \hat{d} \hat{A}.
\]

(3.8)

Indeed, let us introduce the extended superforms

\[
\hat{E} \equiv \hat{E}, \quad \hat{\Omega} \equiv \hat{\Omega} + C, \quad \hat{A} \equiv \hat{A} + U
\]

and the extended differential \( \Delta \equiv \hat{d} + \mathcal{S} - \mathcal{L}_{\hat{\Xi}} \). The nilpotency requirement for \( \Delta \) is equivalent to the transformation law \( \hat{S} \hat{\Xi} = \hat{\Xi}^2 \).

The extended torsion and curvature superforms

\[
\hat{T} \equiv \Delta \hat{E}, \quad \hat{R} \equiv \Delta \hat{\Omega} + \hat{\Omega}^2, \quad \hat{F} \equiv \Delta \hat{A}
\]

(3.9)

then satisfy the extended Bianchi identities

\[
\begin{align*}
\Delta \hat{T} + \hat{\mathcal{O}} \hat{T} - \hat{R} \hat{E} &= 0, \\
\Delta \hat{R} + [\hat{\mathcal{O}}, \hat{R}] &= 0, \\
\Delta \hat{F} &= 0.
\end{align*}
\]

(3.10)

The BRST transformations (3.7) now result from the horizontality conditions

\[
\hat{T} = \hat{T}, \quad \hat{R} = \hat{R}, \quad \hat{F} = \hat{F}
\]

and substitution of these conditions into the extended Bianchi identities (3.10) directly yields the transformation laws of the torsion and curvature superforms:

\[
\begin{align*}
\hat{S} \hat{T} &= \mathcal{L}_{\hat{\Xi}} \hat{T} - C \hat{T}, \\
\hat{S} \hat{R} &= \mathcal{L}_{\hat{\Xi}} \hat{R} - [C, \hat{R}], \\
\hat{S} \hat{F} &= \mathcal{L}_{\hat{\Xi}} \hat{F}.
\end{align*}
\]

(3.11)

### 3.4 Projection to component fields

#### 3.4.1 General gauge

In order to obtain the space-time BRST transformations, we introduce the superfield components of superforms,

\[
\begin{align*}
\hat{E}^a &= E^a(x, \theta) + d\theta E^a_\theta(x, \theta) \quad \text{with} \quad E^a = dx^\mu E^a_\mu, \\
\hat{\Omega}^a_{\dot{b}} &= \Omega^a_{\dot{b}}(x, \theta) + d\theta \Omega^a_{\dot{b}}(x, \theta) \quad \text{with} \quad \Omega^a_{\dot{b}} = dx^\mu \Omega^a_{\dot{b}}^\mu, \\
\hat{A} &= A(x, \theta) + d\theta A_\theta(x, \theta) \quad \text{with} \quad A = dx^\mu A_\mu,
\end{align*}
\]

(3.12)

as well as the space-time components of the latter:

\[
\begin{align*}
E^a(x, \theta) &= e^a(x) + \theta \psi^a_\theta(x), \quad E^a_\theta(x, \theta) = \chi^a_\theta(x) + \theta \phi^a_\theta(x), \\
\Omega^a_{\dot{b}}(x, \theta) &= \omega^a_{\dot{b}}(x) + \theta \tilde{\omega}^a_{\dot{b}}(x), \quad \Omega^a_{\dot{b}}(x, \theta) = \tilde{x}^a_{\dot{b}}(x) + \theta \tilde{\phi}^a_{\dot{b}}(x), \\
A(x, \theta) &= a(x) + \theta \eta_\theta(x), \quad A_\theta(x, \theta) = \sigma_\theta(x) + \theta t_\theta(x).
\end{align*}
\]

(3.13)
Similarly, we define the component fields of the ghost superfields:

\[
\begin{align*}
\Xi^\mu(x, \theta) &= \xi^\mu(x) + \theta \xi'^\mu(x), \\
\Xi^\theta(x, \theta) &= \varepsilon^\theta(x) + \theta \varepsilon'(x), \\
C^a_b(x, \theta) &= c^a_b(x) + \theta c'^a_b(x), \\
U(x, \theta) &= u(x) + \theta u'(x).
\end{align*}
\] (3.14)

In the sequel, we will omit the indices labeling space-time fields in order to simplify the notation and we will use the short-hand notation \(\xi(x) \equiv \xi^\mu(x)\partial_\mu\) and \(\xi'(x) \equiv \xi'^\mu(x)\partial_\mu\).

From equations (3.7) it follows that the BRST transformations of space-time fields take the following form (where \(Dc \equiv dc + [\omega, c]\) denotes the Lorentz covariant derivative):

\[
\begin{align*}
Se &= L_\xi e - ce + \varepsilon \psi - d\varepsilon \chi \\
S\psi &= L_\xi \psi - L_{\xi'} e - c\psi + c' e - \varepsilon' \psi - d\varepsilon' \chi - d\phi \\
S\chi &= L_\xi \chi - i_{\xi'} e - c\chi + \varepsilon \phi - \varepsilon' \chi \\
S\phi &= L_\xi \phi - L_{\xi'} \chi - i_{\xi'} \psi - c\phi - c' \chi - 2\varepsilon' \phi \\
S\omega &= L_\xi \omega - Dc + \varepsilon \tilde{\psi} - d\varepsilon \tilde{\chi} \\
S\tilde{\psi} &= L_\xi \tilde{\psi} - L_{\xi'} \omega - [c, \tilde{\psi}] - D\tilde{c} - \varepsilon' \tilde{\psi} - d\varepsilon' \tilde{\chi} - d\varepsilon \tilde{\phi} \\
S\tilde{\chi} &= L_\xi \tilde{\chi} - i_{\xi'} \omega - [c, \tilde{\chi}] - c' + \varepsilon \tilde{\phi} - \varepsilon' \tilde{\chi} \\
S\tilde{\phi} &= L_\xi \tilde{\phi} - L_{\xi'} \tilde{\chi} - i_{\xi'} \tilde{\psi} - [c, \tilde{\phi}] + [c', \tilde{\chi}] - 2\varepsilon' \tilde{\phi} \\
Sc &= L_\xi c - c^2 + \varepsilon c' \\
Sc' &= L_\xi c' - L_{\xi'} c - [c, c'] - \varepsilon' c' \\
S\xi &= \xi^2 + \varepsilon \xi', \\
S\xi' &= [\xi, \xi'] - \varepsilon' \xi' \\
S\varepsilon &= L_\xi \varepsilon + \varepsilon \varepsilon', \\
S\varepsilon' &= L_{\xi'} \varepsilon + L_{\xi'} \varepsilon' .
\end{align*}
\] (3.15)

\[
\begin{align*}
Sa &= L_\xi a - du + \varepsilon \eta - d\varepsilon \sigma \\
S\eta &= L_\xi \eta - L_{\xi'} a - du' - \varepsilon' \eta - d\varepsilon' \sigma - d\varepsilon t \\
S\sigma &= L_\xi \sigma - i_{\xi'} a - u' + \varepsilon t - \varepsilon' \sigma \\
St &= L_\xi t - L_{\xi'} \sigma - i_{\xi'} \eta - 2\varepsilon' t \\
Su &= L_\xi u + \varepsilon u', \\
Su' &= L_{\xi'} u - L_\xi u - \varepsilon' u'.
\end{align*}
\] (3.16)

We note that these \(S\)-variations describe eight local symmetries, parametrized by the ghosts \(\xi, \varepsilon, c, u\) and \(\xi', \varepsilon', c', u'\). The first four ones are the diffeomorphism, local supersymmetry, local Lorentz and local Maxwell transformations whereas the last four ones may be called vector supersymmetry, \(R\)- (or Fayet) transformations and supergauge transformations. As we shall see in the next subsection, one may, if one wishes, gauge fix the three local invariances parametrized by \(\xi', c', u'\) in an algebraic way. In addition, the positive SUSY-numbers can be traded for positive ghost-numbers by rescaling fields with appropriate powers of \(\varepsilon\), the consequence being that the parameters \(\varepsilon\) and \(\varepsilon'\) disappear from the BRST transformations.
3.4.2 Wess-Zumino gauge

Besides the physically relevant fields and symmetries, the superfield formalism generally introduces some additional fields and symmetries which can be eliminated in an algebraic way by imposing supergauge conditions of Wess-Zumino (WZ) type. In the present case, the WZ gauge is defined by the choices

$$\chi = 0 \quad \tilde{\chi} = 0 \quad \sigma = 0.$$  \hspace{1cm} (3.19)

and it corresponds to the gauge-fixing of the local invariances parametrized by the ghosts $\xi'$, $c'$ and $u'$. In fact, by virtue of equations (3.15), (3.16) and (3.18), the $S$-invariance of the choices (3.19) requires the conditions

$$\varepsilon \phi - i \xi' e = 0 \quad \varepsilon \tilde{\phi} - i \xi' \omega - c' = 0 \quad \varepsilon t - i \xi' a - u' = 0.$$  \hspace{1cm} (3.20)

The latter allow us to eliminate the ghosts $\xi'$, $c'$ and $u'$,

$$\xi'' = \varepsilon \varphi^\mu \quad c' = \varepsilon \tilde{\varphi} \quad u' = \varepsilon \tau,$$  \hspace{1cm} (3.21)

where $\varphi^\mu$, $\tilde{\varphi}$ and $\tau$ are defined by

$$\varphi^\mu = \phi^a e^\mu_a \quad \tilde{\varphi} = \tilde{\phi} - i \varphi \omega \quad \tau = t - i \varphi a.$$  \hspace{1cm} (3.22)

Here, $(\varepsilon^\mu)$ denotes the inverse vielbein, i.e. the vielbein $(e^a_\mu)$ is assumed to be invertible at this point. If we consider $\xi'^a = \xi''^a e^a_\mu$, the first equations in (3.21) and (3.22) can be rewritten as $\xi'^a = \varepsilon \phi^a$ and $0 = \phi^a - i \varphi e^a$, respectively. Thus, each of the three expressions appearing in equations (3.19),(3.21) and (3.22) have the same form.

By substituting the WZ gauge choices (3.19) and their stability conditions (3.21) into the $S$-variations (3.15)-(3.18), we find the BRST transformations in the WZ gauge. Since the WZ gauge choices do not affect diffeomorphisms, Lorentz and Maxwell transformations, we will only display the other contributions to the BRST transformations, i.e. the parts parametrized by $\varepsilon$ and $\varepsilon'$. For any space-time field $f$ with SUSY-number $\alpha_f$, the $\varepsilon'$-variation reads as

$$\delta_{\varepsilon'} f = \alpha_f \varepsilon' f,$$  \hspace{1cm} (3.23)

very much like Fayet’s $R$-transformation in ordinary (i.e. Poincaré) supersymmetric field theory. The local supersymmetry transformations read as

$$\delta \varepsilon = \varepsilon \psi, \quad \delta \varepsilon' = -\varepsilon (L_\varphi e - \tilde{\varphi} e) - 2(d\varepsilon)(i \varphi e), \quad \delta \xi = \varepsilon^2 \varphi,$$
$$\delta \varepsilon = \varepsilon \psi, \quad \delta \varepsilon' = -\varepsilon (L_\varphi \omega + D \tilde{\varphi}) - 2(d\varepsilon)(\tilde{\varphi} + i \varphi \omega), \quad \delta \xi = \varepsilon^2 \tilde{\varphi},$$
$$\delta \varepsilon = \varepsilon \eta, \quad \delta \varepsilon' = -\varepsilon (L_\varphi a + d \tau) - 2(d\varepsilon)(\tau + i \varphi a), \quad \delta \xi = \varepsilon^2 \tau$$  \hspace{1cm} (3.24)

and $\delta_{\varepsilon} \varepsilon' = -\varepsilon L_\varphi \varepsilon$. These results coincide in parts with the on-shell expressions obtained in references [22, 20] by twisting the on-shell version of $N = 2$ euclidean supergravity\(^5\).

\(^5\)A comparison of both the field content and symmetries before and after the twisting of an extended (rigid or local) supersymmetric field theory shows that the operations of twisting and of reduction to the mass-shell (i.e. elimination of auxiliary fields) commute with each other. We wish to thank B. Spence for kindly illustrating this point to us.
In order to obtain the *shift symmetries* of topological gravity, one has to absorb the parameter \( \varepsilon(x) \) of local supersymmetry into the fields, just as one does in topological Yang-Mills theory for the constant parameter \( \varepsilon \) of rigid supersymmetry, e.g. see reference [28]. More precisely, we absorb all \( \theta \)-indices of the fields (which have been explicitly displayed in expressions (3.13)) by rescaling these fields with appropriate powers of \( \varepsilon \equiv \varepsilon^\theta \). Since \( \varepsilon \) has SUSY-number \(-1\) and ghost-number \(1\), this rescaling amounts to assigning positive ghost-numbers to these fields rather than positive SUSY-numbers. Thus, let us redefine the variables according to

\[
\psi_0 = \varepsilon \psi, \quad \tilde{\psi}_0 = \varepsilon \tilde{\psi}, \quad \eta_0 = \varepsilon \eta, \quad \tau_0 = \varepsilon^2 \tau,
\]

without modifying the basic fields \( e, \omega, a \) and the ghosts \( \xi, c, u \). Then, the BRST transformations in the WZ gauge take the form

\[
S e = \mathcal{L}_\xi e - ce + \psi_0, \quad S \xi = \xi^2 + \varphi_0, \quad S \psi_0 = \mathcal{L}_\xi \psi_0 - c\psi_0 - \mathcal{L}_{\varphi_0} e + \tilde{\varphi}_0 e, \quad S \varphi_0 = [\xi, \varphi_0]
\]

\[
S \omega = \mathcal{L}_\xi \omega - Dc + \tilde{\psi}_0, \quad S c = \mathcal{L}_\xi c - c^2 + \varphi_0, \quad S \psi_0 = \mathcal{L}_\xi \psi_0 - [c, \psi_0] - \mathcal{L}_{\varphi_0} \omega - D\tilde{\varphi}_0, \quad S \varphi_0 = \mathcal{L}_\xi \varphi_0 - [c, \varphi_0] - \mathcal{L}_{\varphi_0} c
\]

and

\[
S a = \mathcal{L}_\xi a - du + \eta_0, \quad S u = \mathcal{L}_\xi u + \tau_0, \quad S \eta_0 = \mathcal{L}_\xi \eta_0 - \mathcal{L}_{\varphi_0} a - d\tau_0, \quad S \tau_0 = \mathcal{L}_\xi \tau_0 - \mathcal{L}_{\varphi_0} u.
\]

Supersymmetry is now realized as a rigid symmetry, as usual for the shift supersymmetry of topological field theories. Remarkably enough, the transformation laws (3.26)(3.27) coincide with those obtained in equations (2.12), i.e. those of reference [21]. The parameter \( \varepsilon \) has disappeared from these \( S \)-variations, because it has been absorbed into the fields so as to define new fields with vanishing SUSY-number. Consequently, the parameter \( \varepsilon' \) parametrizing the SUSY-number symmetry according to eq.(3.23) does not occur either in these transformation laws. To be more precise, \( \varepsilon \) and \( \varepsilon' \) only appear in \( S \varepsilon \) and \( S \varepsilon' \) which variations can simply be omitted since they decouple from the others.

### 3.5 Alternative approach

Interestingly enough, the BRST algebra (3.26)(3.27) can also be obtained by starting from different fields and symmetries in superspace. More precisely, let us discard the \( U(1) \) superconnection \( \hat{A} \) as well as the associated ghost \( \hat{U} \), and let us supplement the super 1-form \( \hat{E}^a \) with a \( \theta \)-component \( \hat{E}^\theta \) transforming as \( S \hat{E}^\theta = \mathcal{L}_\xi \hat{E}^\theta \). In other words, we are now considering the complete superspace vielbein matrix:

\[
\begin{pmatrix}
E^a_{\mu} & E^a_{\theta} \\
E^\theta_{\mu} & E^\theta_{\theta}
\end{pmatrix}
\]

(3.28)
The expansion of $\hat{E}^\theta$ reads as

$$
\hat{E}^\theta = E^\theta(x, \theta) + d\theta E_\theta^\theta(x, \theta)
$$

with $E^\theta = dx^\mu E_\mu^\theta$

and

$$
E^\theta(x, \theta) = a^\theta(x) + \theta \eta(x), \quad E_\theta^\theta(x, \theta) = \sigma^\theta_0(x) + \theta t^\theta(x).
$$

Here, the space-time components of $E^\theta$ and $E_\theta^\theta$ have been denoted by the same letters as the components of $A$ and $A_\theta$ in expressions (3.13), except for the fact that these components now carry an extra upper index $\theta$. By projecting the superspace BRST transformations to space-time components, one gets the $S$-variations (3.15)-(3.18) with $u = 0 = u'$ in the last set of equations.

The WZ gauge choices are again given by $\chi = 0 = \tilde{\chi}$ (see eqs.(3.19)), but the condition $\sigma \equiv \sigma_\theta = 0$ is now replaced by $\sigma^\theta_\theta = 1$. As before, the stability of the gauge choices $\chi = 0 = \tilde{\chi}$ under $S$-variations implies $\xi^\mu = \varepsilon \varphi^\mu$ and $c' = \varepsilon \tilde{\varphi}$. The stability of the condition $\sigma^\theta_\theta = 1$ now leads to

$$
\varepsilon t_\theta - i\varepsilon a^\theta - \varepsilon' = 0,
$$

and thus allows us to eliminate the ghost $\varepsilon'$ by virtue of the relation $\varepsilon' = \varepsilon \tau$ with $\tau \equiv t_\theta - i\varphi a^\theta$. If we substitute all of these expressions into the transformation rules (3.15)-(3.18) and subsequently perform the field redefinitions (3.25), we again obtain the BRST transformations (3.26), as well as (3.27) with $u$ replaced by $\varepsilon$. Henceforth, the $\varepsilon$-transformations (which only concern the field $a$ and its partners $\eta_0$, $\tau_0$) should no longer be interpreted as local supersymmetry transformations, but rather as $U(1)$ gauge transformations. Accordingly, the space-time field $a$ is to be viewed as Maxwell potential, i.e. as graviphoton field. Of course, this reinterpretation of the variables $\varepsilon$, $a$, $\eta_0$, $\tau_0$ requires a change of statistics for each of them. Since all of these fields carry an (upper) index $\theta$, our reinterpretation is tantamount to dropping this index, i.e. shifting their SUSY-number from one to zero. There is no obstruction to this shift, because the ghost $\varepsilon'$ parametrizing the SUSY-number symmetry has been eliminated by virtue of the WZ gauge choices.

**Remark:** The stability of a non-vanishing value for $\sigma^\theta_\theta$ is ensured by condition (3.29) which determines the ghost $\varepsilon'$ in terms of the ghost $\varepsilon$. Since the variable $\sigma^\theta_\theta$ represents the lowest component of the superfield $E_\theta^\theta$, this condition (together with the invertibility of the vielbein matrix $(e_\mu^a)$ which is related to the stability of the gauge choice $\chi = 0$) ensures that the supervielbein matrix (3.28) is invertible. Let us stress that this invertibility has only to be imposed in the Wess-Zumino gauge. Degenerate supervielbeins may well appear in a general gauge. This situation is somewhat reminiscent of the fact that the invertibility problem does not manifest itself in 3-dimensional quantum gravity if the latter is expressed as a topological theory of Chern-Simons or of $BF$ type [40, 33].
It is puzzling that two approaches involving different fields, symmetries and gauge choices lead to space-time results of the same form. To elucidate this point, we consider the $S$-variations (3.18) which have been obtained in a general gauge, in the case where Maxwell transformations were included at the superspace level:

\[
S_a = -du - (d\varepsilon)\sigma + \ldots \\
S_\eta = -du' - (d\varepsilon')\sigma + \ldots \\
(S - L_\xi)\sigma = -u' - \varepsilon'\sigma + \varepsilon t - i\phi a.
\]

Thus, the parts of the $S$-variations in superspace that are parametrized by $U \equiv u + \theta u'$ and $\Xi^\theta \equiv \varepsilon + \theta \varepsilon'$ yield the same space-time results for $S_a$, $S_\eta$ and $S\sigma$ up to a factor $\sigma$: for every term in $u$ or $u'$, respectively, there is analogous term in $\varepsilon$ or $\varepsilon'$, multiplied by $\sigma$. In the superspace approach based on $\hat{A}$ and $\hat{U}$, the gauge choice $\sigma = 0$ implies

\[
S_a = -du + \ldots \\
S_\eta = -du' + \ldots , \quad \text{with } u' = \varepsilon(t - i\phi a).
\]

By contrast, in the approach based on $\hat{E}^\theta$ and $U = 0$ (i.e. $u = 0 = u'$), the gauge choice $\sigma = 1$ implies

\[
S_a = -d\varepsilon + \ldots \\
S_\eta = -d\varepsilon' + \ldots , \quad \text{with } \varepsilon' = \varepsilon(t - i\phi a).
\]

### 3.6 Observables

Superspace expressions for the observables related to the curvature may be obtained by viewing the theory as a topological gauge theory associated to the Lorentz group and to a $U(1)$ group: the methods developed for topological Yang-Mills theories in reference [28] can then be applied. They also allow us to obtain space-time expressions for the observables in a general gauge (and not just in the WZ gauge).

### 4 Remarks on the gauge fixing

The complete Lagrangian for topological gravity can be constructed by gauge fixing the shift symmetry (characterizing a topological invariant) by virtue of a condition which localizes the path integral so as to describe a moduli space of interest, e.g. see ref. [21]. Examples of such gauge choices which can be imposed onto the Lorentz connection are the flatness condition $R_{\mu\nu} = 0$ (which is admissible in any space-time dimension), the half-flatness or self-duality condition $R_{\mu\nu}^- = 0$ (in four dimensions) or the condition of constant scalar curvature (in two dimensions) [41, 14, 16].

On a four-dimensional Riemannian manifold with $SU(2)$ holonomy, one has the following remarkable result concerning self-duality [42, 21]. The curvature two-form
$R^a_b$ of a torsionless connection $\omega^a_b(e)$ satisfies the self-duality condition $R_{ab}^- = 0$ (where $X_{ab}^- \equiv \frac{1}{2}(X_{ab} - \frac{1}{2} \varepsilon_{abcd}X_{cd})$) if and only if $\omega^a_b(e)$ is self-dual, i.e. $\omega^-_{ab}(e) = 0$. In this respect, we note that the gauge group $SO(4)$ is locally given by $SU(2)_+ \otimes SU(2)_-$ and that the condition $R_{ab}^- = 0$ is $SO(4)$-invariant, whereas the condition $\omega^-_{ab}(e) = 0$ is only $SU(2)_-$-invariant. The self-dual part $\omega^+_{ab}(e)$ transforms like a connection and the $SU(2)$ holonomy corresponds to a reduction of the $SO(4)$-frame bundle to a $SU(2)$-bundle. Accordingly, one expects a restricted action of local orthonormal transformations on fields like the one encountered in reference [22].

Alternatively, the complete Lagrangian may be obtained by twisting $N = 2$ euclidean supergravity [20, 22] on a Riemannian four-manifold with $SU(2)$ holonomy. Indeed such a manifold admits two covariantly constant chiral spinors that can be used to perform the twist of the gravitinos and of the parameters of supergravity transformations so as to give rise to shift transformations parametrized by a variable $\varepsilon(x)$. (This was recently done in detail using an on-shell formulation [22]). Our discussion in subsection 3.4.2 shows that the variable $\varepsilon(x)$ has to be absorbed in an appropriate way into the fields if one wants to cast the shift transformations into a standard form and to compare with the models constructed by gauge fixing a topological invariant.

The twist of supergravity transformations not only gives rise to local supersymmetry transformations parametrized by the scalar $\varepsilon(x)$, but also to a vector and a tensor supersymmetry for which on-shell expressions have been given in reference [22]. By contrast to the global vector supersymmetry transformations encountered in topological Yang-Mills theory [43], the local vector supersymmetry of topological gravity does not leave invariant the fundamental fields (i.e. the vielbein and graviphoton) and therefore appears to act non-trivially on the topological invariant from which the complete Lagrangian originates by gauge fixing. Thus, this symmetry may be more restrictive for the perturbative renormalization of topological gravity than it is for topological Yang-Mills theory.

5 Concluding comments

We have shown that the graviphoton field $a_\mu$ can be implemented in the superspace approach in two different ways, namely using an independent Abelian superconnection as in subsection 3.4 or, maybe more geometrically, using a complete superspace vielbein as in subsection 3.5. Although both implementations involve different local symmetries, they turn out to be equivalent in the sense that they lead to the same space-time BRST algebra, once the supergauge is fixed according to suitable Wess-Zumino type conditions.

It is worth mentioning once more that the vielbein matrix does not need to be invertible in superspace, although the formulation in the Wess-Zumino gauge (that
corresponds to the various formulations considered in the literature), necessitates
an invertible vielbein, i.e. a metric which is nonsingular at every space-time point.
Thus, the theory written in superspace in a general supergauge might have further
significance than the one defined in the Wess-Zumino gauge.

Acknowledgments

It is a great pleasure to thank M. Blau, F. Delduc, B. Spence, R. Stora and
G. Thompson for valuable discussions. Olivier Piguet thanks the members of the
Institut de Physique Nucléaire of the University of Lyon for a very kind invitation
as Visiting Professor for two periods during which a large part of this work has been
done.

A Appendix: Metric approach

We will only discuss the gravitational sector since the Maxwell sector can be treated
as in the vielbein formalism.

Notation: The metric formulation involves tensor fields, e.g. the metric tensor
field \( g = g_{\mu\nu} dx^\mu \otimes dx^\nu \). Their variation under an infinitesimal change of coordinates
generated by the vector field \( w = w^\mu \partial_\mu \) is given by the Lie derivative \( \mathcal{L}_w \) as acting
on generic tensor fields, e.g.

\[
\mathcal{L}_w g_{\mu\nu} \equiv (\mathcal{L}_w g)_{\mu\nu} = w^\rho \partial_\rho g_{\mu\nu} + (\partial_\mu w^\rho) g_{\rho\nu} + (\partial_\nu w^\rho) g_{\rho\mu}.
\]  

(A.1)

In particular, the action of \( \mathcal{L}_w \) on a vector field \( \varphi = \varphi^\mu \partial_\mu \) is the Lie bracket (2.1):
\( \mathcal{L}_w \varphi^\mu = [w, \varphi]^\mu \).

The metric tensor field can also be viewed as a 0-form with values in the covariant
rank-two tensors. Along the same vein, the collection \( (\Gamma^\nu_{\mu\sigma}) \) of Christoffel symbols
may be regarded as a matrix-valued 1-form, \( \Gamma^\nu_{\sigma} = \Gamma^\nu_{\mu\sigma} dx^\mu \). Thus, these geometric
quantities can be acted upon by the linear operator \( l_w \equiv [i_w, d] = i_w d + (-1)^{|w|} di_w \),
e.g.

\[
l_w g_{\mu\nu} = w^\rho \partial_\rho g_{\mu\nu},
\]

\[
(l_w \Gamma)_\mu = w^\rho \partial_\rho \Gamma_{\mu\nu} + (\partial_\mu w^\rho) \Gamma_{\rho\nu}.
\]  

(A.2)

Note that the operator \( l_w \) and the Lie derivative \( \mathcal{L}_w \) act in the same way on forms
which do not carry extra curved space indices \( \mu, \nu, \ldots \), like the forms appearing
in the vielbein formalism (which carry extra tangent space indices \( a, b, \ldots \)). This
should be kept in mind when comparing the results below with those presented in
the main body of the text.
Fields: As in the second order formalism, we eliminate the Lorentz connection $\omega$ in terms of the vielbein $e$ by requiring the torsion to vanish. The metric tensor given by $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ can then be considered as the only independent variable.

Symmetries: The BRST algebra reduces to the following well known form [6]:

$$
\begin{align*}
Sg &= \mathcal{L}_\xi g + \Psi, & S\Psi &= \mathcal{L}_\xi \Psi - \mathcal{L}_\varphi g \\
S\xi &= \xi^2 + \varphi, & S\varphi &= [\xi, \varphi].
\end{align*}
$$

(A.3)

Here, the symmetric tensor field $\Psi$ with components

$$
\Psi_{\mu\nu} \equiv \eta_{ab} (e^a_\mu \psi^b_\nu + e^a_\nu \psi^b_\mu) = \psi_{\mu\nu} + \psi_{\nu\mu}
$$

is to be viewed as a new variable that is usually referred to as gravitino field.

We note that the diffeomorphisms can be completely decoupled by introducing the operator $\tilde{S} \equiv S - \mathcal{L}_\xi$ which satisfies $\tilde{S}^2 = -\mathcal{L}_\varphi$. The variations of $g$, $\Psi$ and $\varphi$, as given by (A.3), then read as

$$
\tilde{S}g = \Psi , \quad \tilde{S}\Psi = -\mathcal{L}_\varphi g , \quad \tilde{S}\varphi = 0.
$$

(A.5)

Thus, one has a close analogy with topological Yang-Mills theory where $\tilde{S}$ corresponds to the SUSY-generator $\tilde{Q}$ that is nilpotent up to an infinitesimal gauge transformation.

From equations (A.3), it follows that the Christoffel symbols describing the Levi-Civita connection transform as

$$
S\Gamma^\rho_{\mu\sigma} = (l_\xi \Gamma)^\rho_{\mu\sigma} + \nabla_\mu (\partial_\sigma \xi^\rho) + \tilde{\Psi}^\rho_{\mu\sigma}.
$$

(A.6)

Here, $l_\xi \Gamma$ is given by (A.2) and $\nabla_\mu$ denotes the covariant derivative with respect to the Levi-Civita connection, i.e. $\nabla_\mu v^\rho_\sigma \equiv \partial_\mu v^\rho_\sigma + \Gamma^\rho_{\mu\lambda} v^\lambda_\sigma - \Gamma^\lambda_{\mu\sigma} v^\rho_\lambda$, while the components of the third rank tensor $\tilde{\Psi}$ are defined by

$$
\tilde{\Psi}^\rho_{\mu\sigma} \equiv \frac{1}{2} (\nabla_\mu \Psi^\rho_\sigma + \nabla_\sigma \Psi^\rho_\mu - \nabla^\rho \Psi_{\mu\sigma})
$$

(A.7)

Since the variable $\partial_\sigma \xi^\rho$ appearing in equation (A.6) does not define a tensor field, the expression $\nabla_\mu (\partial_\sigma \xi^\rho)$ only represents a convenient notation. As a matter of fact, the right-hand-side of equation (A.6) can also be written in terms of the Lie derivative acting on tensor fields [32]:

$$
S\Gamma^\rho_{\mu\sigma} = \mathcal{L}_\xi \Gamma^\rho_{\mu\sigma} + \partial_\mu (\partial_\sigma \xi^\rho) + \tilde{\Psi}^\rho_{\mu\sigma}.
$$

(A.8)

However, just as for expression (A.6), this only represents a convenient notation since the Christoffel symbols do not define a tensor field.
A.1 Horizontality conditions

By using matrix notation, we can derive the BRST algebra (A.3)(A.6) from an horizontality condition. To do so, let us introduce the matrix-valued forms

\[ \Gamma = \Gamma_\mu dx^\mu, \quad \text{with} \quad \Gamma_\mu = (\Gamma^\rho_{\sigma \mu}) \]

\[ \mathbf{R} = \frac{1}{2} R_{\mu \nu} dx^\mu dx^\nu, \quad \text{with} \quad R_{\mu \nu} = (R^\rho_{\sigma \mu \nu}), \]

where \( \mathbf{R} \) denotes the curvature 2-form associated to the Levi-Civita connection. The connection forms \( \omega \) and \( \Gamma \) are related by a formal gauge transformation involving the matrix of vielbein fields \( E \equiv (e^a_\mu) \):

\[ \omega_\mu = E \Gamma_\mu E^{-1} + E \partial_\mu E^{-1}. \]  

(A.10)

Accordingly, the curvature 2-forms \( \mathbf{R} \) and \( \mathbf{R} \) associated to \( \omega \) and \( \Gamma \), respectively, are related by a similarity transformation:

\[ \mathbf{R} = E \mathbf{R} E^{-1}. \]

Let us now consider the generalized fields [44]

\[ \hat{\Gamma} \equiv e^{i \xi} (\Gamma + \nu) = \Gamma + \nu + i \xi \Gamma, \quad \text{with} \quad \nu = (\nu^\rho_\sigma) \equiv (\partial_\sigma \xi^\rho) \]

\[ \hat{\mathbf{R}} \equiv \hat{d} \hat{\Gamma} + \hat{\Gamma}^2, \quad \text{with} \quad \hat{d} = d + S. \]

(A.11)

By construction, \( \hat{\mathbf{R}} \) satisfies the generalized Bianchi identity \( 0 = \hat{\nabla} \hat{\mathbf{R}} \equiv \hat{d} \hat{\mathbf{R}} + [\hat{\Gamma}, \hat{\mathbf{R}}] \).

Since \( (\nu + i \xi \Gamma)_\rho^\sigma = \nabla_\sigma \xi^\rho \), we also consider the covariant derivative \( \nabla_\sigma \varphi^\rho \) as well as the combination of covariant derivatives (A.7) which describes the shift of \( \hat{\Gamma} \):

\[ \tilde{\Phi} = (\tilde{\Phi}^\rho_\sigma), \quad \text{with} \quad \tilde{\Phi}^\rho_\sigma = \nabla_\sigma \varphi^\rho \]

\[ \tilde{\Psi} = \tilde{\Psi}_\mu dx^\mu, \quad \text{with} \quad \tilde{\Psi}_\mu = (\tilde{\Psi}^\rho_{\sigma \mu}). \]

(A.12)

The horizontality condition then reads as

\[ \hat{\mathbf{R}} = e^{i \xi} (\hat{\mathbf{R}} + \tilde{\Psi} + \tilde{\Phi}) \]  

(A.13)

and we can proceed as in subsection 2.2.1 to derive the BRST transformations. (Instead of assuming the fields \( \tilde{\Psi}_1^1 \) and \( \tilde{\Phi}_2^2 \) appearing in (A.13) to be explicitly given by (A.12), we could also assume them to be undetermined. The consistency of the resulting BRST transformations with the known \( S \)-variations of \( \xi \) and \( g \) and with the expression for the Christoffel symbols in terms of the metric, then implies the relations (A.12).)

Thus, we use the operatorial relation (2.9), the definitions (2.10) which are part of the basic algebra (A.3), as well as a change of variables that is analogous to (2.11):

\[ \tilde{\Phi} \to \tilde{\hat{\phi}} \equiv \tilde{\Phi} - i \varphi \Gamma, \quad \text{i.e.} \quad \hat{\varphi}^\rho_\sigma = \partial_\sigma \varphi^\rho. \]
Thereby, the $S$-variations following from the expansion of (A.13) and of the Bianchi identity $\tilde{\nabla} R = 0$ take the form

$$
\begin{align*}
S \Gamma &= l_\xi \Gamma - \nabla v + \Psi, \\
S \tilde{\Psi} &= l_\xi \tilde{\Psi} - [v, \tilde{\Psi}] - l_\varphi \Gamma - \nabla \tilde{\varphi}, \\
S \varphi &= [\xi, \varphi],
\end{align*}
$$

(A.14)

and

$$
S R = l_\xi R - [v, R] - \nabla \tilde{\Psi},
$$

(A.15)

where $\nabla v \equiv dv + [\Gamma, v]$. Due to relation (A.10), the BRST algebra (A.14) of the metric approach has exactly the same form as the BRST algebra (2.34)(2.33) of the vielbein approach (which entails that the BRST variations (A.14) are nilpotent). In fact, when passing from the vielbein formalism to the metric approach, a tangent space index that is acted upon by the Lorentz parameter $c_{ab}$ (with $S c_{ab} = \tilde{\varphi}_{ab} + \ldots$) becomes a curved space index that is acted upon by diffeomorphisms in the disguise of the parameter $v^\mu_\nu = \partial_\nu \xi^\mu$ (with $S v^\mu_\nu = \tilde{\varphi}^\mu_\nu + \ldots = \partial_\nu \varphi^\mu + \ldots$).

We note that the symmetry algebras of the prepotential $g$, as given by equations (A.3), and of the potential $\Gamma$, as given by (A.14), are consistent with each other and that these symmetry algebras have the same structure:

$$
\begin{align*}
S g &= \delta \xi g + \Psi, \\
S \tilde{\Gamma} &= \delta \xi \Gamma + \tilde{\Psi}, \\
S \varphi &= \delta \xi \varphi + \xi^2 + \varphi,
\end{align*}
$$

(A.16)

### A.2 Comparison with the vielbein approach

Let us compare the variables appearing, respectively, in the metric approach and in the second order formalism. The shift transformations of fields are symbolized by a vertical arrow:

| Basic fields: | $g_{\mu\nu}$ | $e^a_\mu$ |
|---------------|--------------|------------|
| Ghosts:       | $\psi_{\mu\nu}$, $\xi^\mu$ | $\psi^a_\mu$, $\xi^\mu$, $e^{ab}$ |
| Ghosts for ghosts: | $\varphi^\mu$, $\varphi^{ab}$ | $\varphi^\mu$, $\tilde{\varphi}^{ab}$ |

The basic field $e^a_\mu$ of the second order formalism involves a Lorentz index, which implies that a Lorentz ghost and the corresponding ghost for ghost appear as independent variables, in addition to those that are present in the metric approach. In particular, the ghost for ghost $\tilde{\varphi}^{ab}$ appears in the variation $S \psi^a_\mu$ and thereby in the observables of the vielbein formalism, though it can only appear in those of the metric approach under the disguise of the dependent variable $\tilde{\varphi}^\rho_\sigma = \partial_\sigma \varphi^\rho$, see next subsection. One expects that the observables in these different approaches are
cohomologically equivalent, i.e. that they only differ by \( S \)- and \( d \)-exact terms, just as the gravitational anomaly in Einstein gravity can manifest itself under different disguises (Lorentz anomaly or diffeomorphism anomaly, as well as Weyl or chirally split anomaly in two dimensions) [32, 45].

**A.3 Observables**

In view of the formal gauge transformation (A.10), one would expect that the expressions for the observables in the second order formalism have exactly the same form as those in the metric approach. Yet, this is not quite true as we will see in the following.

Let us denote the observables in the metric approach by \( M_0^d, M_1^d, \ldots, M_d^d \) so as to distinguish them from those of the vielbein formalism denoted by \( W_0^d, W_1^d, \ldots, W_d^d \). The polynomials \( M_{d-k}^k \) satisfy descent equations that are analogous to equations (2.23) which correspond to the special case \( d = 4 \). Of course, the topological invariant \( M_0^d(g_{\mu\nu}) \) coincides with the topological invariant \( W_0^d(e^a_\mu) \) since the metric \( g_{\mu\nu} \) can be expressed in terms of the vielbein fields \( e^a_\mu \). Furthermore, the polynomial \( M_1^d(g_{\mu\nu}, \Psi_{\mu\nu}, \xi^\mu) \) coincides with \( W_1^d(e^a_\mu, \psi^a_\mu, \xi^\mu) \) by virtue of relations (A.4) and (2.32). However, the polynomials of ghost-number \( k \geq 2 \), i.e. \( M_{d-k}^k(g_{\mu\nu}, \Psi_{\mu\nu}, \xi^\mu, \varphi^\mu) \), do not depend on the same set of independent variables as \( W_{d-k}^k(e^a_\mu, \psi^a_\mu, \xi^\mu, \phi^\mu) \). And even if \( \varphi^{ab} \) (or \( \tilde{\phi}^{ab} \)) is viewed as the Lorentz analogue of \( \tilde{\phi}^{\mu\nu} \equiv \partial_{\nu}\varphi^\mu \) (or \( \tilde{\phi}^{\mu\nu} \equiv \nabla_{\nu}\varphi^\mu \)), the polynomials of ghost-number \( k \geq 2 \) do not quite have the same form: the expressions \( M_{d-k}^k, \ldots \) involve extra contributions which are not present in \( W_{d-k}^k, \ldots \). We will see that the appearance of these terms can be drawn back to the shift transformations affecting the metric tensor which raises or lowers covariant indices.

**Two-dimensional case:** One starts from the 2-form \( M_2^0 = \text{As} \, R \) where \( R \) is the matrix-valued 2-form defined in equations (A.9) and ‘As’ the antisymmetric part of this matrix:

\[
M_2^0 = \text{As} \, R \equiv \sqrt{g} \varepsilon_{\rho\sigma} R^{\rho\sigma} = \frac{1}{2} \sqrt{g} \varepsilon_{\rho\sigma} R^{\rho\sigma} dx^\mu dx^\nu = \frac{1}{2} \sqrt{g} \varepsilon_{\mu\nu} \mathcal{R} \, dx^\mu dx^\nu .
\]

Here, \( g \) denotes the determinant of the metric tensor, \( \varepsilon_{\rho\sigma} \) the antisymmetric tensor in flat space (which is just a numerical tensor normalized by \( \varepsilon_{12} = 1 \)) and \( \mathcal{R} \) the curvature scalar.

By laboriously solving the descent equations

\[
S M_2^0 = -d M_1^1(\xi), \quad S M_1^1(\xi) = -d M_0^2(\xi), \quad S M_0^2(\xi) = 0 ,
\]

27
one finds the following expressions [12], which correspond to the Mumford classes [11]:

\[
\mathcal{M}_1^1(\xi) = \sqrt{g} \left[ \varepsilon_{\mu\nu} \nabla^\nu \Psi_\mu^\rho + \varepsilon_{\rho\sigma} \xi^\mu \mathcal{R} \right] dx^\rho \tag{A.17}
\]

\[
\mathcal{M}_0^2(\xi) = \sqrt{g} \varepsilon_{\mu\nu} \left[ \nabla^\mu \varphi^\nu - \frac{1}{4} \Psi^\mu_\rho \Psi_\rho^\nu - \xi^\rho \nabla^\nu \Psi_\rho^\mu + \frac{1}{2} \xi^\mu \xi^\nu \mathcal{R} \right].
\]

Using the two-dimensional identity \( "0 = \varepsilon_{\mu\nu} V_\rho + \text{cyclic permutations of indices}" \), the term involving a derivative of \( \Psi_\mu^\rho \) may also be expressed in terms of the traceless part of the symmetric tensor \( \Psi \):

\[
\varepsilon_{\mu\nu} \nabla^\nu \Psi_\rho^\mu = -\varepsilon_{\mu\rho} \nabla_\nu (\Psi_\nu^\sigma - g_\nu^\sigma \Psi_\sigma^\rho).
\]

In reference [16], the results (A.17) have been obtained by applying the mathematical techniques of equivariant cohomology, thereby justifying earlier calculations and discussions along these lines [7, 15].

Alternatively, one could try to proceed as in subsection 2.2.2 (see equations (2.28) and (2.29)), i.e. consider the expansion

\[
\hat{\mathcal{M}} \equiv \text{As } \hat{R} = e^{i\xi} \text{As } (R + \Psi + \Phi)
\]

\[
= \text{As } R + \text{As } (\Psi + i\xi R) + \text{As } (\Phi + i\xi \Psi + \frac{1}{2} i\xi \xi R).
\]

The latter expressions have exactly the same form as the polynomials \( W_0^0, W_1^1(\xi) \) and \( W_2^2(\xi) \) appearing in eqs.(2.28)(2.29) of the vielbein approach. By spelling them out, one immediately finds the results (A.17) up to the quadratic term \( \Psi^\mu_\rho \Psi_\rho^\nu \) that is present in \( \mathcal{M}_0^2(\xi) \). Such a term is generated from the variation

\[
S \Psi_\nu^\mu = -\Psi^\mu_\rho \Psi_\rho^\nu + \mathcal{L}_\xi \Psi^\mu_\nu - g_\mu^\rho (\mathcal{L}_\varphi g_\rho^\nu),
\]

i.e., it is due to the fact that the metric tensor, which raises or lowers indices, is subject to shift transformations. This shows that the purely algebraic passage from ordinary to generalized fields and from the ordinary differential \( d \) to the generalized differential \( \hat{d} = d + S \) is a subtle business for topological models in the metric approach. A proper geometric treatment requires to extend the action of symmetries from the space-time manifold to the infinite-dimensional space of all metrics, whence the use of global differential geometric machinery, see ref. [16].

In conclusion, we note that the two-dimensional metric tensor (and thus the two-dimensional observables) can equally well be parametrized in terms of Beltrami differentials, see references [11, 16].

**Four-dimensional case:** One starts from the 4-form

\[
\mathcal{M}_4^0 = E_\lambda^\mu R_\mu^\lambda R_\rho^\chi,
\]

(A.19)
where

\[ E^\mu_\nu_{\lambda\chi} = \begin{cases} 
\delta^\mu_\lambda \delta^\rho_\chi - \delta^\mu_\chi \delta^\rho_\lambda & \text{for the Pontryagin density} \\
\frac{1}{\sqrt{g}} \varepsilon^{\mu\nu\rho\sigma} g_{\nu\lambda} g_{\sigma\chi} & \text{for the Euler density},
\end{cases} \]

i.e.

\[ \mathcal{M}_4^0 = R^\mu_\nu R^\nu_\mu \quad \text{or} \quad \mathcal{M}_4^0 = \frac{1}{\sqrt{g}} \varepsilon^{\mu\nu\rho\sigma} R^\mu_\nu R^\rho_\sigma. \]

The first descendant can readily be obtained by expanding the generalized 4-form \( \hat{\mathcal{M}} \equiv E^\mu_\nu \hat{R}^\lambda_\mu \hat{R}^\chi_\rho \) with respect to the ghost-number:

\[ \mathcal{M}_3^1(\xi) = 2 E^{\nu\sigma}_\mu R^\mu_\nu \left[ \frac{1}{2} \left( \nabla_\sigma \Psi^\rho_\beta - \nabla_\rho \Psi^\sigma_\beta \right) + \xi^\alpha R^\rho_\sigma \right] dx^\beta. \]

For a determination and explicit expression of the other polynomials, we refer to the work [17].

Finally, we note that the Nieh-Yan 4-form (2.26), which yields the observables related to torsion, takes the form [37]

\[ Z_4^0 = \frac{1}{4} \sqrt{g} \varepsilon^{\mu\nu\rho\sigma} \left( R_{\mu\nu\rho\sigma} - \frac{1}{2} T^\lambda_\mu T^\lambda_{\rho\sigma} \right) dx^1 \ldots dx^4. \]

References

[1] E. Witten, “Topological quantum field theory”, Commun.Math.Phys. 117 (1988) 353;
E. Witten, “Introduction to cohomological theories”, Int.J.Mod.Phys. A6 (1991) 2775.

[2] L. Smolin and A. Starodubtsev, “General relativity with a topological phase: an action principle”, hep-th/0311163;
A. Perez, “Spin foam models for quantum gravity”, Class.Quant.Grav. 20 (2003) R43, gr-qc/0301113;
T. Thiemann, “Lectures on loop quantum gravity”, in Quantum Gravity: From Theory to Experimental Search, D.J.W. Giulini, C. Kiefer and C. Lämmerzahl, eds., Lecture Notes in Physics Vol.631 (Springer Verlag, 2003), gr-qc/0210094;
I. Oda, “A relation between topological quantum field theory and the Kodama state”, hep-th/0311149;
G. Bonelli and A.M. Boyarski, “Six dimensional topological gravity and the cosmological constant problem”, Phys.Lett. B490 (2000) 147, hep-th/0004058.

[3] E. Witten, in “Quantum Fields and Strings: A Course for Mathematicians, Vol. 2”, P. Deligne et al. (eds.), (American Mathematical Society, 1999).
[4] E. Witten, “Topological gravity,” *Phys.Lett.* **B206** (1988) 601;
J. M. F. Labastida and M. Pernici, “A Lagrangian for topological gravity and its BRST quantization,” *Phys.Lett.* **B213** (1988) 319;
R. Brooks, D. Montano and J. Sonnenschein, “Gauge fixing and renormalization in topological quantum field theory,” *Phys.Lett.* **B214** (1988) 91;
D. Montano and J. Sonnenschein, “Topological strings,” *Nucl.Phys.* **B313** (1989) 258.

[5] D. Montano and J. Sonnenschein, “The topology of moduli space and quantum field theory,” *Nucl.Phys.* **B324** (1989) 348;
J. M. F. Labastida, M. Pernici and E. Witten, “Topological gravity in two dimensions,” *Nucl.Phys.* **B310** (1988) 611.

[6] R. Myers and V. Periwal, “Topological gravity and moduli space,” *Nucl.Phys.* **B333** (1990) 536.

[7] R. Myers, “New observables for topological gravity,” *Nucl.Phys.* **B343** (1990) 705.

[8] R. Myers, “On alternate formulations of topological gravity,” *Phys.Lett.* **B252** (1990) 365.

[9] R. Myers and V. Periwal, “Invariants of smooth 4-manifolds from topological gravity,” *Nucl.Phys.* **B361** (1991) 290.

[10] E. Verlinde and H. Verlinde, “A Solution Of Two-Dimensional Topological Quantum Gravity”, *Nucl.Phys.* **B348** (1991) 457.

[11] L. Baulieu and I.M. Singer, “Conformally invariant gauge fixed actions for 2-d topological gravity”, *Commun.Math.Phys.* **135** (1991) 253.

[12] C. M. Becchi, R. Collina and C. Imbimbo, “On the semi-relative condition for closed (topological) strings”, *Phys.Lett.* **B322** (1994) 79, hep-th/9311097;
C. M. Becchi, R. Collina and C. Imbimbo, “A functional and Lagrangian formulation of two dimensional topological gravity”, in *Symmetry and Simplicity in Physics: a symposium on the occasion of Sergio Fubini’s 65th Birthday*, W.M. Alberico and S. Sciuto, eds. (World Scientific, 1994), hep-th/9406096;
C. M. Becchi and C. Imbimbo, “Gribov horizon, contact terms and Čech-De Rham cohomology in 2D topological gravity”, *Nucl.Phys.* **B462** (1996) 571, hep-th/9510003;
C. M. Becchi and C. Imbimbo, “A Lagrangian formulation of 2-dimensional topological gravity and Čech-De Rham cohomology,” hep-th/9511156.

[13] A. Nakamichi, A. Sugamoto and I. Oda, “Topological four-dimensional self-dual gravity,” *Phys.Rev.* **D44** (1991) 3835.
[14] I. Oda, “Topological four-dimensional gravity,” *Progr.Theor.Phys.Suppl.* **110** (1992) 41.

[15] S. Wu, “Appearance of universal bundle structure in four-dimensional topological gravity”, *J.Geom.Phys.* **12** (1993) 205.

[16] R. Stora, F. Thuillier and J. C. Wallet, “Algebraic structure of cohomological field theory models and equivariant cohomology”, in *Infinite dimensional geometry, non commutative geometry, operator algebras and fundamental interactions*, Proceedings of the Caribbean Spring School of Mathematics and Theoretical Physics (Guadeloupe 1993), R. Coquereaux, M. Dubois-Violette, P. Flad, eds. (World Scientific, 1995).

[17] F. Thuillier, “Some remarks on topological 4d-gravity”, *J.Geom.Phys.* **27** (1998) 221, *hep-th/9707084*.

[18] M. Menaa and M. Tahiri, “BRST-anti-BRST symmetry and observables for topological gravity,” *Phys.Rev.* **D57** (1998) 7312.

[19] M. J. Perry and E. Teo, “Topological conformal gravity in four dimensions,” *Nucl.Phys.* **B401** (1993) 206, *hep-th/9211063*.

[20] D. Anselmi and P. Fré, “Twisted N=2 supergravity as topological gravity in four dimensions”, *Nucl.Phys.* **B392** (1993) 401, *hep-th/9208029*.

[21] L. Baulieu and A. Tanzini, “Topological gravity versus supergravity on manifolds with special holonomy”, *JHEP* **0203** (2002) 015, *hep-th/0201109*.

[22] P. de Medeiros and B. Spence, “Four-dimensional topological Einstein-Maxwell gravity”, *Class.Quant.Grav.* **20** (2003) 2075, *hep-th/0209115*.

[23] H.Y. Lee, A. Nakamichi and T. Ueno, “Topological two form gravity in four dimensions”, *Phys.Rev.* **D47** (1993) 1563, *hep-th/9205066*;

M. Abe, A. Nakamichi and T. Ueno, “Gravitational instantons and moduli spaces in topological two form gravity”, *Phys.Rev.* **D50** (1994) 7323.

[24] L. Baulieu, M. Bellon and A. Tanzini, “Eight-dimensional topological gravity and its correspondence with supergravity”, *Phys.Lett.* **B543** (2002) 291, *hep-th/0207020*;

L. Baulieu, M. Bellon and A. Tanzini, “Supergravity and the knitting of the Kalb-Ramond two-form in eight-dimensional topological gravity”, *Phys.Lett.* **B565** (2003) 211, *hep-th/0303165*;

L. Baulieu, “Gravitational topological quantum field theory versus N = 2, D = 8 supergravity”, *hep-th/0304221*.
[25] J. H. Horne, “Superspace versions of topological theories”, *Nucl. Phys.* **B318** (1989) 22;

C. Aragao de Carvalho and L. Baulieu, “Local BRST symmetry and superfield formulation of the Donaldson-Witten theory”, *Phys. Lett.* **B275** (1992) 323.

[26] S. Ouvry, R. Stora and P. van Baal, “On the algebraic characterization of Witten’s topological Yang-Mills theory”, *Phys. Lett.* **B220** (1989) 159.

[27] M. Blau and G. Thompson, “Aspects of $N_T \geq 2$ topological gauge theories and D-branes”, *Nucl. Phys.* **B492** (1997) 545, hep-th/9612143;

B. Geyer and D. Mülsch, “$N_T = 4$ equivariant extension of the 3D topological model of Blau and Thompson”, *Nucl. Phys.* **B616** (2001) 476, hep-th/0108042;

B. Geyer and D. Mülsch, “Higher dimensional analogue of the Blau-Thompson model and $N_T = 8, D = 2$ Hodge-type cohomological gauge theories”, *Nucl. Phys.* **B662** (2003) 531, hep-th/0211061;

C.P. Constantinidis, O. Piguet and W. Spalenza, “Superspace gauge fixing of topological Yang-Mills theories”, *European Phys. J.* (2004) to appear, hep-th/0310184.

[28] J.L. Boldo, C.P. Constantinidis, F. Gieres, M. Lefrançois and O. Piguet, “Observables in topological Yang-Mills theories”, *Int. J. Mod. Phys.* **A** (2003) to appear, hep-th/0303053;

J.L. Boldo, C.P. Constantinidis, F. Gieres, M. Lefrançois and O. Piguet, “Topological Yang-Mills theories and their observables: a superspace approach”, *Int. J. Mod. Phys.* **A18** (2003) 2119, hep-th/0303084.

[29] H. Kanno, “Weil algebra structure and geometrical meaning of BRST transformation in topological quantum field theory”, *Z. Physik* **C43** (1989) 477;

H. Kanno, “Observables in topological Yang-Mills theory and the Gribov problem”, *Lett. Math. Phys.* **19** (1990) 249.

[30] F. Yu and Y. S. Wu, “BRST superspace formulation for world sheet topological gravity with gauged Lorentz and Weyl symmetries”, *Class. Quant. Grav.* **6** (1989) L199;

M. Tahiri, “Unconstrained superspace formalism of topological 2-d gravity”, *Mod. Phys. Lett.* **A10** (1995) 1949.

[31] L. Baulieu and M. Bellon, “$p$-forms and supergravity: gauge symmetries in curved space”, *Nucl. Phys.* **B266** (1986) 75.

[32] R. A. Bertlmann, *Anomalies in quantum field theory* (Clarendon Press, Oxford 1996).
[33] C.P. Constantinides, F. Gieres, O. Piguet and M.S. Sarandy, “On the symmetries of BF models and their relation with gravity”, *JHEP 0201* (2002) 017, hep-th/0111273.

[34] F. Langouche, T. Schücker and R. Stora, “Gravitational anomalies of the Adler-Bardeen type”, *Phys.Lett. B145* (1984) 342;
L. Baulieu and J. Thierry-Mieg, “Algebraic structure of quantum gravity and the classification of the gravitational anomalies”, *Phys.Lett. B145* (1984) 53.

[35] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, “Topological field theory”, *Phys.Rept. 209* (1991) 129.

[36] O. Piguet, “Ghost equations and diffeomorphism invariant theories”, *Class.Quant.Grav. 17* (2000) 3799, hep-th/0005011.

[37] H.T. Nieh and M.L. Yan, “An identity in Riemann-Cartan geometry”, *J.Math.Phys. 23* (1982) 373;
H.T. Nieh and M.L. Yan, “Quantized Dirac field in curved Riemann-Cartan background. 1. Symmetry properties, Green’s function”, *Annals Phys. 138* (1982) 237.

[38] O. Chandía and J. Zanelli, “Torsional topological invariants (and their relevance for real life)”, in *Trends in Theoretical Physics*, AIP Conference Proceedings Vol. 419, H. Falomir, R.E. Gamboa-Saraví, F.A. Schaposnik, eds. (Amer. Inst. Phys., 1998), hep-th/9708138;
O. Chandía and J. Zanelli, “Topological invariants, instantons and chiral anomaly on spaces with torsion”, *Phys.Rev. D55* (1997) 7580, hep-th/9702025;
A. Mardones and J. Zanelli, “Lovelock-Cartan theory of gravity”, *Class.Quant.Grav. 8* (1991) 1545;
H.Y. Guo, K. Wu and W. Zhang, “On torsion and Nieh-Yan form”, *Commun.Theor.Phys. 32* (1999) 381, hep-th/9805037.

[39] L. Baulieu, M. Bellon and R. Grimm, “BRS symmetry of supergravity in superspace and its projection to component formalism”, *Nucl.Phys. B294* (1987) 279.

[40] E. Witten, “(2+1)-dimensional gravity as an exactly soluble system”, *Nucl.Phys. B311* (1988) 46;
E. Witten, “Topology changing amplitudes in (2+1)-dimensional gravity”, *Nucl.Phys. B323* (1989) 113.

[41] D. Birmingham and M. Rakowski, “Equivariance in topological gravity”, *Phys.Lett. B289* (1992) 271.
[42] T. Eguchi and A.J. Hanson, “Self-dual solutions to euclidean gravity”, *Annals Phys.* **120** (1979) 82.

[43] A. Brandhuber, O. Moritsch, M. W. de Oliveira, O. Piguet and M. Schweda, “A renormalized supersymmetry in the topological Yang-Mills field theory”, *Nucl.Phys.* **B431** (1994) 173, hep-th/9407105;

F. Gieres, J. Grimstrup, T. Pisar and M. Schweda, “Vector supersymmetry in topological field theories”, *JHEP* **0006** (2000) 018, hep-th/0002167.

[44] J. P. Ader, F. Gieres and Y. Noirot, “Gauged BRST symmetry and covariant gravitational anomalies”, *Phys.Lett.* **B256** (1991) 401.

[45] M. Knecht, S. Lazzarini and F. Thuillier, “Shifting the Weyl anomaly to the chirally split diffeomorphism anomaly in two dimensions”, *Phys.Lett.* **B251** (1990) 279.