A vertex operator approach for correlation functions of Belavin’s \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model

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Abstract

Belavin’s \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model is considered on the basis of bosonization of vertex operators in the \(A^{(1)}_{n-1}\) model and vertex–face transformation. The corner transfer matrix (CTM) Hamiltonian of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model and tail operators are expressed in terms of bosonized vertex operators in the \(A^{(1)}_{n-1}\) model. Correlation functions of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model can be obtained by using these objects, in principle. In particular, we calculate spontaneous polarization, which reproduces the result we obtained in 1993.

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Dedicated to Professor Tetsuji Miwa on the occasion of his 60th birthday

1. Introduction

In this paper we consider Belavin’s \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model [1] on the basis of bosonization of vertex operators in the \(A^{(1)}_{n-1}\) model [2] and vertex–face transformation. Belavin’s \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model is a higher rank generalization of Baxter’s eight-vertex model [3] in the sense that the former model is an \(n\)-state model. The \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model is a vertex model on a two-dimensional lattice such that the state variables take on values of \((\mathbb{Z}/n\mathbb{Z})\)-spin. A local weight \(R_{jl}^{ik}\) is assigned to spin configuration \(j, l, i, k\) around a vertex. The model is \((\mathbb{Z}/n\mathbb{Z})\)-symmetric in a sense that \(R_{jl}^{ik}\) satisfies the two conditions: (i) \(R_{jl}^{ik} = 0\) unless \(j + l = i + k\) (mod \(n\)) and (ii) \(R_{j+pl+p}^{i+k+p} = R_{jl}^{ik}\) for any \(p \in (\mathbb{Z}/n\mathbb{Z})\). Since there are \(n^3\) non-zero weights among \(R_{jl}^{ik}\)s, we may call the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model by the \(n^3\)-vertex model. (When \(n = 2\), it becomes the eight-vertex model.)

In [4], Lashkevich and Pugai presented the integral formulae for correlation functions of the eight-vertex model [3] using bosonization of vertex operators in the eight-vertex SOS model [5] and vertex–face transformation. The present paper aims to give an \(sl(n)\)-generalization of Lashkevich–Pugai’s construction. For our purpose, we use the vertex–face correspondence...
between the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model and unrestricted \(A_{n-1}^{(1)}\) model. First, we note that the \(A_{n-1}^{(1)}\) model \([6]\) is a restricted model, while we should relate the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model to the unrestricted \(A_{n-1}^{(1)}\) model. Second, we note that the original vertex–face correspondence \([6]\) maps the \(A_{n-1}^{(1)}\) model in regime III to the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model in the disordered phase. We should relate the former to the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model in the principal regime.

In this paper, we present integral formulae for correlation functions of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model on the basis of the free field representation formalism. As the simplest example, we perform the calculation of the integral formulae for a one-point function, in order to obtain the spontaneous polarization of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model.

There is another approach to find the expression for correlation functions. It was shown in \([7]\) that the correlation functions of the eight-vertex model satisfy a set of difference equations, the quantum Knizhnik–Zamolodchikov equation of level \(-4\). On the basis of the difference equation approach, we obtained the expression of the spontaneous polarization of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model \([8]\). In this paper, we show that the expressions for the spontaneous polarization of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model obtained on the basis of the free field representation formalism reproduce the known result in \([8]\). This coincidence indicates the relevance of the free field representation formalism.

The present paper is organized as follows. In section 2, we review the basic definitions of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model \([1]\), the corresponding dual face model \([6]\) and the vertex–face correspondence. In section 3, we introduce the corner transfer matrix (CTM) Hamiltonians and the vertex operators of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model and \(A_{n-1}^{(1)}\) model, and also introduce the tail operators which relates those two CTM Hamiltonians. In section 4 we construct the free field formalism of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model. In section 5 we present trace formulae for correlation functions of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model. Furthermore, we calculate the spontaneous polarization of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model in this formalism. Sections 4 and 5 are main original parts of the present paper. In section 6 we give some concluding remarks.

2. Basic definitions

The present section aims to formulate the problem, thereby fixing the notation.

2.1. Theta functions

The Jacobi theta function with two pseudo-periods 1 and \(\tau\) (\(\text{Im} \tau > 0\)) are defined as follows:

\[
\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] (v; \tau) := \sum_{m \in \mathbb{Z}} \exp [\pi \sqrt{-1} (m + a)(m + a) \tau + 2(v + b)],
\]

for \(a, b \in \mathbb{R}\). Let \(n \in \mathbb{Z}_{\geq 2}\) and \(r \in \mathbb{R}\) such that \(r > n - 1\), and also fix the parameter \(x\) such that \(0 < x < 1\). We will use the abbreviations,

\[
[v] = x^{2^{-v} \Theta_2 (x^{2v})}, \quad [v'] = x^{2^{-v} \Theta_2 (x^{2v})},
\]

where

\[
\Theta_q(z) = (z; q)_{\infty} (q z^{-1}; q)_{\infty} (q; q)_{\infty} = \sum_{m \in \mathbb{Z}} q^{m(m-1)/2} (-z)^m,
\]

\[
(z; q_1, \ldots, q_m) = \prod_{i_1, \ldots, i_m \geq 0} (1 - z q_1^{i_1} \cdots q_m^{i_m}).
\]
Note that
\[ R(v) = \frac{1}{2 \Gamma} \left( \frac{\pi (\frac{1}{x} - \frac{1}{r})}{r} \right) = \sqrt{\frac{\pi}{\Gamma}} \exp \left( -\frac{x}{4} \right) [v]. \]

where \( x = e^{-\epsilon} (\epsilon > 0). \)

For later conveniences we also introduce the following symbols:
\[ R_{li}(v) = z^{i-l-1} g_{li}(z) \]
where \( z = x^{2r}, 1 \leq l \leq n \) and
\[ |z| = (z; x^{2r}, x^{2n})_{\infty}. \]

These factors will appear in the commutation relations among the type I vertex operators.

The integral kernel for the type I vertex operators will be given as the products of the following elliptic functions:
\[ f(v, w) = \frac{v + \frac{1}{2} - w}{v - \frac{1}{2}}, \quad g(v) = \frac{v - 1}{v + 1}. \]

2.2. Belavin’s vertex model

Let \( V = \mathbb{C}^n \) and \( \{ e_{\mu} \}_{0 \leq \mu \leq n-1} \) be the standard orthonormal basis with the inner product \( \langle e_{\mu}, e_{\nu} \rangle = \delta_{\mu\nu}. \) Belavin’s \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model is a vertex model on a two-dimensional square lattice \( L \) such that the state variables take on values of \((\mathbb{Z}/n\mathbb{Z})\)-spin. In the original papers [1, 9], the \( R \)-matrix in the disordered phase is given. For the present purpose, we need the following \( R \)-matrix:

\[ R(v) = \frac{1}{1 - v} r_{1}(v) \bar{R}(v), \quad \bar{R}(v) = \frac{1}{n} \sum_{\alpha \in G_{n}} \theta \left[ \frac{1}{4} + \frac{n_{\alpha}}{n} \right] \left( \frac{1}{nr} - \frac{v}{r}, \frac{\pi \sqrt{-1}}{\epsilon r} \right) \left( \frac{1}{nr} + \frac{v}{r}, \frac{\pi \sqrt{-1}}{\epsilon r} \right) I_{\alpha} \otimes I_{\alpha}^{-1}. \]

Here \( G_{n} = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}), \) and \( I_{\alpha} = g^{\alpha} h^{\alpha} \) for \( \alpha = (\alpha_{1}, \alpha_{2}), \)
\[ g_{vi} = \omega^{|i|} v_{i}, \quad h_{vi} = v_{i-1}, \]
with \( \omega = \exp(2\pi \sqrt{-1}/n). \) We assume that the parameters \( v, \epsilon \) and \( r \) lie in the so-called principal regime:
\[ \epsilon > 0, \quad r > 1, \quad 0 < v < 1. \]

When \( n = 2, \) the principal regime (2.8) lies in one of the antiferroelectric phases of the eight-vertex model [3]. We describe \( n \) kinds of ground states of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model in the principal regime in section 3.1.

The \( R \)-matrix satisfies the Yang–Baxter equation (YBE),
\[ R_{12}(v_{1} - v_{2}) R_{13}(v_{1} - v_{3}) R_{23}(v_{2} - v_{3}) = R_{23}(v_{2} - v_{3}) R_{13}(v_{1} - v_{3}) R_{12}(v_{1} - v_{2}), \]

where \( R_{ij}(v) \) denotes the matrix on \( V_{i,j} \otimes V_{i,j} \), which acts as \( R(v) \) on the \( i \)th and \( j \)th components and as identity on the other one.
If \( i + k = j + l \mod n \), the elements of the \( R \)-matrix \( R(v)_{ik}^{jl} \) is given as follows:

\[
R(v)_{ik}^{jl} = \frac{h(v)\vartheta \left[ \frac{1}{2} + \frac{i-k}{n} \right] (1-v; \frac{-1}{n\epsilon r}) \vartheta \left[ \frac{1}{2} + \frac{j-l}{n} \right] (1-v; \frac{-1}{n\epsilon r})}{\vartheta \left[ \frac{1}{2} + \frac{i-k}{n} \right] (v; \frac{-1}{n\epsilon r}) \vartheta \left[ \frac{1}{2} + \frac{j-l}{n} \right] (0; \frac{-1}{n\epsilon r})},
\]

(2.10)

where

\[
h(v) = \prod_{j=0}^{n-1} \vartheta \left[ \frac{1}{2} + \frac{j}{n} \right] (v; \frac{\pi \sqrt{-1}}{n\epsilon r}) \prod_{j=1}^{n-1} \vartheta \left[ \frac{1}{2} + \frac{j}{n} \right] (0; \frac{\pi \sqrt{-1}}{n\epsilon r}),
\]

and otherwise \( R(v)_{ik}^{jl} = 0 \).

Note that the weights (2.10) reproduce those of the eight-vertex model in the principal regime when \( n = 2 \) [3].

2.3. The weight lattice and the root lattice of \( A_{n-1}^{(1)} \)

Let \( V = \mathbb{C}^n \) and \( \{\varepsilon_\mu\}_{0 \leq \mu \leq n-1} \) be the standard orthonormal basis as before. The weight lattice of \( A_{n-1}^{(1)} \) is defined as follows:

\[
P = \bigoplus_{\mu=0}^{n-1} \mathbb{Z}\varepsilon_\mu,
\]

(2.11)

where

\[
\bar{\varepsilon}_\mu = \varepsilon_\mu - \varepsilon, \quad \varepsilon = \frac{1}{n} \sum_{\mu=0}^{n-1} \varepsilon_\mu.
\]

We denote the fundamental weights by \( \omega_\mu (1 \leq \mu \leq n-1) \),

\[
\omega_\mu = \sum_{\nu=0}^{\mu-1} \bar{\varepsilon}_\nu,
\]

and also denote the simple roots by \( \alpha_\mu (1 \leq \mu \leq n-1) \),

\[
\alpha_\mu = \varepsilon_{\mu-1} - \varepsilon_\mu = \bar{\varepsilon}_{\mu-1} - \bar{\varepsilon}_\mu.
\]

The root lattice of \( A_{n-1}^{(1)} \) is defined as follows:

\[
Q = \bigoplus_{\mu=1}^{n-1} \mathbb{Z}\alpha_\mu,
\]

(2.12)

For \( a \in P \) we set

\[
a_{\mu\nu} = a_{\mu} - a_{\nu}, \quad a_{\mu} = \langle a + \rho, \varepsilon_\mu \rangle = \langle a + \rho, \bar{\varepsilon}_\mu \rangle, \quad \rho = \sum_{\mu=1}^{n-1} \omega_\mu.
\]

(2.13)

Useful formulae are

\[
\langle \varepsilon_\mu, \varepsilon_\nu \rangle = \langle \bar{\varepsilon}_\mu, \bar{\varepsilon}_\nu \rangle = \delta_{\mu\nu} - \frac{1}{n}, \quad \langle \alpha_\mu, \omega_\nu \rangle = \delta_{\mu\nu},
\]

\[
\langle \varepsilon_\mu, \omega_\nu \rangle = \theta(\mu < \nu) - \frac{\nu}{n}, \quad \langle \omega_\mu, \omega_\nu \rangle = \min(\mu, \nu) - \frac{\mu\nu}{n}.
\]
When \( a + \rho = \sum_{\mu=0}^{n-1} k^\mu \omega_\mu \), we have \( a_{\mu \nu} = k^{\mu+1} + \cdots + k^\nu \) when \( \mu < v \), and
\[
\langle a + \rho, a + \rho \rangle = \frac{1}{n} \sum_{\mu < v} a_{\mu \nu}^2, \quad \langle a + \rho, \rho \rangle = \frac{1}{2} \sum_{\mu < v} a_{\mu \mu}.
\]

Let \( \sum_{\mu=0}^{n-1} k^\mu = r \), where \( a + \rho = \sum_{\mu=0}^{n-1} k^\mu \omega_\mu \), then we denote \( a \in P_{r-n} \).

2.4. The \( A_{n-1} \) face model

An ordered pair \((a, b) \in P_{2n}^2\) is called admissible if \( b = a + \bar{\epsilon}_\mu \), for a certain \( 0 \leq \mu \leq n-1 \).

For \((a, b, c, d) \in P_{4n}^4\), let \( W[\begin{array}{c} c \\ b \\ d \\ a \end{array} | v] \) be the Boltzmann weight of the \( A_{n-1}^{(1)} \) model for the state configuration \([cd | ab | v]\) round a face. Here the four states \( a, b, c \) and \( d \) are ordered clockwise from the SE corner. In this model, \( W[\begin{array}{c} c \\ b \\ d \\ a \end{array} | v] = 0 \) unless the four pairs \((a, b), (a, d), (b, c)\) and \((d, c)\) are admissible. Non-zero Boltzmann weights are parametrized in terms of the elliptic theta function of the spectral parameter \( v \) as follows:
\[
W[\begin{array}{c} a + 2\bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\mu \\ a \end{array} | v] = r_1(v),
\]
\[
W[\begin{array}{c} a + \bar{\epsilon}_\mu + \bar{\epsilon}_v \\ a + \bar{\epsilon}_v \\ a \end{array} | v] = -r_1(v) \frac{[v][v, a_{\mu \mu}+1]}{[v][v, a_{\mu \mu}]} (\mu \neq v),
\]
\[
W[\begin{array}{c} a + \bar{\epsilon}_\mu + \bar{\epsilon}_v \\ a + \bar{\epsilon}_\mu \\ a \end{array} | v] = r_1(v) \frac{[v][v, a_{\mu \mu}+1]}{[v][v, a_{\mu \mu}]} (\mu \neq v).
\]

We consider the so-called regime III in the model, i.e., \( 0 < v < 1 \).

The Boltzmann weights (2.14) solve the YBE for the face model [6]:
\[
\sum_g W[\begin{array}{c} d \\ c \\ g \\ v_1 \end{array}] W[\begin{array}{c} c \\ g \\ v_2 \end{array}] W[\begin{array}{c} e \\ f \\ a \end{array} | v_1 - v_2] = \sum_b W[\begin{array}{c} g \\ b \\ a \end{array} | v_1] W[\begin{array}{c} c \\ g \\ v_2 \end{array}] W[\begin{array}{c} d \\ g \\ a \end{array} | v_1 - v_2].
\]

2.5. Vertex–face correspondence

In this paper, we use the \( R \)-matrix of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model in the principal regime while Belavin’s original paper used that in the disordered phase. Thus, we need different intertwining vectors from that by Jimbo–Miwa–Okado [6].

Let
\[
t(v)a_{\mu}^{\alpha} = \sum_{\nu=0}^{n-1} \epsilon_{\nu} v^{\alpha} \left[ 0, \frac{v}{2} \right] \left( \frac{v}{n^r + \frac{\tilde{a}_\mu}{r}} ; \frac{\pi \sqrt{-1}}{n \epsilon r} \right).
\]

Then we have (cf. figure 1)
\[
R(v_1 - v_2)R(v_1)^{\alpha} \otimes R(v_2)^{b} = \sum_b t(v_1)^{\alpha} \otimes t(v_2)^{b} W[\begin{array}{c} c \\ b \\ a \end{array} | v_1 - v_2].
\]
3. Vertex–face transformation

The basic objects in the vertex operator approach are the CTMs and the vertex operators [10]. In sections 3.1 and 3.2 we recall the CTM Hamiltonians, the type I vertex operators and the space of states of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model and the \(A_{n-1}^{(1)}\) model, respectively.

In [4], Lashkevich and Pugai introduced the nonlocal operator called the tail operator, in order to express the correlation functions of the eight-vertex model in terms of those of the SOS model. In section 3.3, we introduce the tail operator for the present purpose; i.e., in order to express the correlation functions of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model in terms of those of the \(A_{n-1}^{(1)}\) model. The commutation relations among the tail operators and the type I vertex operators are given in section 3.4.

3.1. The CTM Hamiltonian for the vertex model

Let us consider the ‘low-temperature’ limit \(x \to 0\). Then the elements of the \(R\)-matrix behave

\[
R_{\mu \nu}^{\mu' \nu'}(v) \simeq \xi H_v(\mu, \nu) \delta_{\mu \mu'} \delta_{\nu \nu'},
\]

(3.1)

where \(z = x^2 = \xi^n\) and

\[
H_v(\mu, \nu) = \begin{cases} 
\mu - \nu - 1 & \text{if } 0 \leq \nu < \mu \leq n - 1 \\
\mu - \nu - 1 + \nu & \text{if } 0 \leq \mu < \nu \leq n - 1.
\end{cases}
\]

(3.2)

Thus the CTM Hamiltonian of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model in the principal regime is given as follows:

\[
H_{\text{CTM}}(\mu_1, \mu_2, \mu_3, \ldots) = \sum_{j=1}^{\infty} j H_v(\mu_j, \mu_{j+1}).
\]

(3.3)

The CTM Hamiltonian diverges unless \(\mu_j = i + 1 - j \ (\text{mod } n)\) for \(j \gg 0\) and a certain \(0 \leq i \leq n - 1\).

Let \(\mathcal{H}^{(i)}\) be the \(\mathbb{C}\)-vector space spanned by the half-infinite pure tensor vectors of the forms

\[
\varepsilon_{\mu_1} \otimes \varepsilon_{\mu_2} \otimes \varepsilon_{\mu_3} \otimes \cdots \quad \text{with} \quad \mu_j \in \mathbb{Z}/n\mathbb{Z}, \mu_j = i + 1 - j \ (\text{mod } n) \quad \text{for} \quad j \gg 0.
\]

(3.4)

Let \(\mathcal{H}^{(i)\ast}\) be the dual of \(\mathcal{H}^{(i)}\) spanned by the half-infinite pure tensor vectors of the forms

\[
\cdots \otimes \varepsilon_{\mu_{-2}} \otimes \varepsilon_{\mu_{-1}} \otimes \varepsilon_{\mu_0} \quad \text{with} \quad \mu_j \in \mathbb{Z}/n\mathbb{Z}, \mu_j = i + 1 - j \ (\text{mod } n) \quad \text{for} \quad j \ll 0.
\]

(3.5)

1 We fix \(\mathcal{H}^{(i)}\) by (3.4) such that it coincides with \(V_0(\omega_i)\), the level 1 highest weight irreducible \(U_q(\hat{s}_n)\)-module, in the trigonometric limit \(r \to \infty\). For example, see [11], keeping in mind that our \(i\) should be read as \(-i\) in [11].
Introduce the type I vertex operator by the following half-infinite transfer matrix

\[
\Phi^\mu(v_1 - v_2) = v_1^{\mu_1} v_2^{\mu_2} v_2^{\mu_3} v_2^{\mu_4} \cdots
\]

(3.6)

Then the operator (3.6) is an intertwiner from \(\mathcal{H}^{(i)}\) to \(\mathcal{H}^{(i+1)}\). The type I vertex operators satisfy the following commutation relation:

\[
\Phi^\mu(v_1) \Phi^\nu(v_2) = \sum_{\mu',\nu'} R(v_1 - v_2)_{\mu',\nu'} \Phi^{\mu'}(v_2) \Phi^{\nu'}(v_1).
\]

(3.7)

Introduce the CTM in the south–east (SE) corner.

The diagonal form of \(A^{(i)}_{\text{SE}}(v)\) can be determined from the 'low-temperature' limit of the \(R\)-matrix (3.1)–(3.2):

\[
A^{(i)}_{\text{SE}}(v) \sim \zeta^{HCTM} = z^{\frac{1}{2}HCTM} : \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)},
\]

(3.8)

where \(\sim\) refers to an equality modulo a divergent scalar in the infinite lattice limit. Likewise, other three types of the CTMs are given as follows:

\[
\begin{align*}
A^{(i)}_{\text{NE}}(v) : & \quad \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)}, \\
A^{(i)}_{\text{NW}}(v) : & \quad \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)}, \\
A^{(i)}_{\text{SW}}(v) : & \quad \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)},
\end{align*}
\]

(3.9)

where NE, NW and SW stand for the corners north–east, north–west and south–west, respectively. It seems to be rather general [3] that the product of four CTMs in the infinite lattice limit is independent of \(v\):

\[
\rho^{(i)} = A^{(i)}_{\text{SE}}(v) A^{(i)}_{\text{SW}}(v) A^{(i)}_{\text{NW}}(v) A^{(i)}_{\text{NE}}(v) = x^{2HCTM}.
\]

(3.10)

Since \(H(\mu_j, \mu_{j+1})\) takes on the value of \(\{0, 1, \ldots, n-1\}\), the eigenvalues of \(HCTM\) are of the form

\[
N = \sum_{j=1}^{\infty} jm_j, \quad 0 \leq m_j \leq n - 1.
\]

This stands for the partition of \(N\) such that the multiplicity of each \(j\) is at most \(n - 1\). Thus, the character is given by

\[
\chi^{(i)} = \text{tr}_{H^{(i)}}(\rho^{(i)}) = \frac{(x^{2\mu}; x^{2\mu})_\infty}{(x^2; x^2)_\infty}.
\]

(3.11)
3.2. CTM for the $A_{n-1}^{(1)}$ model

After gauge transformation [6], the CTM Hamiltonian of the $A_{n-1}^{(1)}$ model in the regime III is given as follows:

$$H_{\text{CTM}}(a_0, a_1, a_2, \ldots) = \sum_{j=1}^{\infty} j H_f(a_{j-1}, a_j, a_{j+1}),$$

where $H_f(v, \mu)$ is defined by (3.2). The CTM Hamiltonian diverges unless $a_j = \xi + \alpha_{j+1-j}$ for $j \gg 0$ and a certain $\xi \in P_{r-\omega-1}$ and $0 \leq i \leq n - 1$.

For $k = a + \rho, l = \xi + \rho$ and $0 \leq i \leq n - 1$, let $\mathcal{H}_{l,k}^{(i)}$ be the space of admissible paths $(a_0, a_1, a_2, \ldots)$ such that

$$a_0 = a, \quad a_j - a_{j+1} \in \{\bar{\xi}_0, \bar{\xi}_1, \ldots, \bar{\xi}_{n-1}\},$$

for $j = 1, 2, 3, \ldots$, $a_j = \xi + \alpha_{j+1-j}$ for $j \gg 0$. (3.13)

Also, let $\mathcal{H}_{l,k}^{(a)}$ be the space of admissible paths $(\ldots, a_{-2}, a_{-1}, a_0)$ such that

$$a_0 = a, \quad a_j - a_{j+1} \in \{\bar{\xi}_0, \bar{\xi}_1, \ldots, \bar{\xi}_{n-1}\},$$

for $j = 1, 2, 3, \ldots$, $a_j = \xi + \alpha_{j+1-j}$ for $j \ll 0$. (3.14)

Introduce the type I vertex operator by the following half-infinite transfer matrix

$$\Phi(v_1 - v_2)^{a+\bar{\xi} \rho} = \Phi(v_1)_{la} \Phi(v_2)_{lb}$$

Then the operator (3.15) is an intertwiner from $\mathcal{H}_{l,k}^{(i)}$ to $\mathcal{H}_{l,k+\bar{\xi}}^{(i+1)}$. The type I vertex operators satisfy the following commutation relation:

$$\Phi(v_1)^c_{cd} \Phi(v_2)^d_{ab} = \sum_d W_{cd}^{ab} \Phi(v_1)^c_{cd} \Phi(v_2)^d_{ab}. \quad (3.16)$$

Introduce the CTM of the $A_{n-1}^{(1)}$ model in the SE corner

$$A_{\text{SE}}^{(l,k)}(v_1 - v_2)^{a_0 a_1 a_2 a_3 \cdots} =$$
The diagonal form of $A^{(l,k)}_{SE}(v)$ can be determined from the ‘low-temperature’ limit (3.12):

$$A^{(l,k)}_{SE}(v) \sim \xi^{H_{CTM}} \rightarrow H^{(l)}_{l,k}.$$  
(3.17)

where $\sim$ refers to an equality modulo a divergent scalar in the infinite lattice limit. Likewise other three types of the CTMs are given as follows:

$$A^{(l,k)}_{NE}(v) : H^{(l)}_{l,k} \rightarrow H^*(l,k),$$  
$$A^{(l,k)}_{NW}(v) : H^*(l,k) \rightarrow H^*(l,k),$$  
$$A^{(l,k)}_{SW}(v) : H^*(l,k) \rightarrow H^{(l)}_{l,k}.$$  
(3.18)

The product of four CTMs for the $A^{(1)}_{n-1}$ model in the infinite lattice limit is also independent of $v$ [6]:

$$\rho^{(i)}_{l,k} = G_a x^{2nH^{(i)}_{l,k}}.$$  
(3.19)

where

$$G_a = \prod_{\mu < \nu} [a_{\mu \nu}].$$

The character of the $A^{(1)}_{n-1}$ model was obtained in [6]:

$$\chi^{(i)}_{l,k} = \text{tr}_{H^{(i)}_{l,k}}(\rho^{(i)}_{l,k}) = x^{n[\beta_1 + \beta_2]} \left(\frac{1}{y^{2n}; x^{2n}}\right)_\infty G_a,$$  
(3.20)

where

$$t^2 - \beta_0 t - 1 = (t - \beta_1)(t - \beta_2), \quad \beta_0 = \frac{1}{\sqrt{t(\bar{t} - 1)}}, \quad \beta_1 < \beta_2.$$  
(3.21)

We note the following sum formula:

$$\sum_{k \equiv l + n_0 \, (\text{mod} \, Q)} \chi^{(i)}_{l,k} = \frac{\left(\frac{x^{2n}; x^{2n}}{x^{2}; x^{2}}\right)_\infty \left(\frac{(x^{2r}; x^{2r})_\infty}{(x^{2r-2}; x^{2r-2})_\infty}\right)}{(n-1)(n-2)/2} G'_\xi,$$  
(3.22)

where

$$G'_\xi = \prod_{\mu < \nu} [\xi_{\mu \nu}]^{-1}.$$  

Equations (3.22) and (3.11) imply that

$$\chi^{(i)} = \frac{1}{b_l} \sum_{k \equiv l + n_0 \, (\text{mod} \, Q)} \chi^{(i)}_{l,k},$$  
(3.23)

where

$$b_l = \left(\frac{(x^{2r}; x^{2r})_\infty}{(x^{2r-2}; x^{2r-2})_\infty}\right)^{(n-1)(n-2)/2} G'_\xi.$$  
(3.24)
\[
\sum_{\mu=0}^{n-1} a^{\mu}_a v^{\mu} = \delta^{\mu}_{\text{fac}}, \quad \sum_{a'} a a' = \delta^{\mu}_{\text{lit}}.
\]

Figure 2. Picture representation of the dual intertwining vectors.

\[
\sum_{d} v_1 \rightarrow c \quad \sum_{d} v_1 \rightarrow c
\]

Figure 3. Vertex-face correspondence by dual intertwining vectors.

3.3. Tail operator

Let us introduce the dual intertwining vectors (see figure 2) satisfying

\[
\sum_{\mu=0}^{n-1} t^\mu (v \alpha_{\mu})^{a'} = \delta^{a'}_{\text{fac}}, \quad \sum_{\mu=0}^{n-1} t\mu (v \alpha_{-\mu}) t^\mu (v \alpha_{-\mu}) = \delta^{\mu}_{\text{lit}}.
\]

(3.25)

From (2.17) and (3.25), we have (cf. figure 3)

\[
t^*(v_1)^b \otimes t^*(v_2)^d R(v_1 - v_2) = \sum_{d} W \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] t^*(v_1)^a \otimes t^*(v_2)^{d}. \]

(3.26)

Now introduce the intertwining operators between \( H^{(i)} \) and \( H^{(i)}_{l,k} (k = l + \omega_i \text{ (mod Q)}) \):

\[
T(u) \xi_a = \prod_{j=0}^{\infty} t^\mu (-u)^{a_{\mu}} : H^{(i)} \rightarrow H^{(i)}_{l,k},
\]

(3.27)

where \( k = a_0 + \rho \) and \( l = \xi + \rho \), and \( 0 < \Re(u) < \frac{n}{2} + 1 \). The tail operator \( \Lambda \) (see figure 4) is defined by

\[
\Lambda(u)^a_{\alpha} = T(u)\xi^a_{\alpha} T(u)\xi_a.
\]

(3.28)

Let

\[
L \left[ \begin{array}{cc} d_0' & a_{\alpha} \\ d_0 & a_{\alpha} \end{array} \right] u := \sum_{\mu=0}^{n-1} t^\mu (-u)^{a_{\mu}} t^\mu (-u)^{a_{\alpha}}.
\]

(3.29)

Then we have

\[
\Lambda(u)^{a_{\alpha}}_{0} = \prod_{j=0}^{\infty} L \left[ \begin{array}{cc} d_j' & a_{\alpha} \\ d_j & a_{\alpha} \end{array} \right] u.
\]

(3.30)

Here we note that in the ‘low-temperature’ limit, \( t^\mu (-u)^{a_{\mu}} t^\mu (-u)^{a_{\alpha}} \) is much greater than other, \( t^\mu (-u)^{a_{\mu}} t^\mu (-u)^{a_{\alpha}} (\mu \neq j) \).
\[ \Lambda(u)_{a_0}^{\alpha_0} = a_0 \begin{array}{cccc} a_0' & a_1' & a_2' & a_3' \\ -u & a_1 & a_2 & a_3 \\ \end{array} \xi \cdots \xi + \omega_1 \xi + \omega_2 \xi \cdots \xi + \omega_2 \xi + \omega_1 \xi \]

Figure 4. Tail operator \( \Lambda(u)_{a_0}^{\alpha_0} \). The upper (resp. lower) half stands for \( T(u)\xi a_{a_0} \) (resp. \( T(u)\xi a_{a_0} \)).

Note that

\[ L \left[ \begin{array}{cc} a' & \bar{a}' - \bar{\xi}_v \\ a & \bar{a} - \bar{\xi}_u \end{array} \right] = \frac{[u + \bar{a}_{\mu} - \bar{a}'_{\nu}]}{[\bar{a}]} \prod_{j \neq \mu} \frac{[\bar{a}'_v - \bar{a}_j]}{[a_{aj}]} \right]. \tag{3.31} \]

It is obvious from (3.25) that we have

\[ L \left[ \begin{array}{cc} a & a' \\ a & a'' \end{array} \right] = \delta_{a'^{\prime}}^{a^\prime}. \tag{3.32} \]

We therefore have

\[ \Lambda(u)^{\alpha_0}_a = 1. \tag{3.33} \]

From (3.23) and (3.33), we may assume that

\[ \rho^{(i)} = \frac{1}{b_j} \sum_{k \equiv l + \omega_i (\text{mod } Q)} T(u)\xi a \rho_{i.a}^{(j)} T(u)\xi a. \tag{3.34} \]

3.4. Commutation relations between \( \Lambda \) and \( \phi \)

By using the vertex–face correspondence (see figure 5), we obtain

\[ T(u)^{\xi_b} \Phi^b(v) = \sum_a i^\mu (v - u)^b_a \Phi^b(v)^{\xi_a} T(u)^{\xi_a}, \tag{3.35} \]

\[ T(u)^{\xi_b} \Phi^b(v)^{\xi_a} = \sum_{\mu} i^\mu (v - u)^b_{\mu} \Phi^\mu(v) T(u)^{\xi_a}. \tag{3.36} \]

From these commutation relations and the definition of the tail operator (3.28), we have

\[ \Lambda(u)^{\xi_b}_a \Phi^b(v)^{\xi_a} = \sum_d L \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] u - v \right] \Phi(v)^{\xi_d}_a \Lambda(u)^{\xi_d}_a. \tag{3.37} \]

4. The vertex operator approach

One of the most standard ways to calculate correlation functions is the vertex operator approach [10] on the basis of free field representation. In section 4.2, we recall the free field representation for the \( A(1)_{n-1} \) model [2]. The type I vertex operators of the \( A(1)_{n-1} \) model can be constructed in terms of basic bosons introduced in [12, 13]. The \( A(1)_{n-1} \) model has the so-called \( \sigma \)-invariance. The free field representation of type I vertex operator given in section 4.2 is not invariant under \( \sigma \)-transformation. Thus, we give other free field
Let us define the basic operators for

4.2. Type I vertex operators

We also need the bosonized CTM Hamiltonian of the $A_{n-1}^{(1)}$ model [14] in order to obtain correlation functions of the $A_{n-1}^{(1)}$ model. In section 4.4 we discuss the space of states of the unrestricted $A_{n-1}^{(1)}$ model. The free field representation of the tail operator is presented in section 4.5.

4.1. Bosons

Let us consider the bosons, $B^j_m(1 \leq j \leq n-1, m \in \mathbb{Z} \setminus \{0\}$, with the commutation relations

$$[B^j_m, B^k_n] = \begin{cases} \frac{[n-1]m_x}{[nm]} \frac{[r-1]m_y}{[rm]} \delta_{m+m',0}, & (j = k) \\ \frac{m_x^{\mu(n-j)} m_y}{[nm]} \frac{m_x}{[rm]} (r-1)^{m+y} \delta_{m+m',0}, & (j \neq k). \end{cases} \tag{4.1}$$

where the symbol $[a]_x$ stands for $(x^n - x^{-n})/(x - x^{-1})$. Define $B^0_m$ by

$$\sum_{j=1}^{n} x^{-2jm} B^0_m = 0.$$ 

Then the commutation relations (4.1) holds for all $1 \leq j, k \leq n$. These oscillators were introduced in [12, 13].

For $\alpha, \beta \in h^* := \mathbb{C} \omega_0 \oplus \mathbb{C} \omega_1 \oplus \cdots \mathbb{C} \omega_{n-1}$, let us define the zero-mode operators $P_\alpha, Q_\beta$ with the commutation relations

$$[P_\alpha, \sqrt{-1} Q_\beta] = \langle \alpha, \beta \rangle, \quad [P_\alpha, B^k_m] = [Q_\beta, B^k_m] = 0. \tag{4.2}$$

We will deal with the bosonic Fock spaces $\mathcal{F}_{l,k}$, $(l, k \in h^*)$ generated by $B^l_m(m > 0)$ over the vacuum vectors $|l, k\rangle$:

$$\mathcal{F}_{l,k} = \mathbb{C}[\{B^l_m, B^j_{-m}, \ldots\}_{1 \leq j \leq n}]|l, k\rangle,$$

where $B^l_m|l, k\rangle = 0 (m > 0)$,

$$P_\alpha |l, k\rangle = \langle \alpha, \beta | l + \beta_2 | l, k \rangle |l, k\rangle,$$

$$|l, k\rangle = \exp(\sqrt{-1} (\beta_1 Q_\delta + \beta_2 Q_\delta))|0, 0\rangle,$$

where $\beta_1$ and $\beta_2$ are defined by (3.21).

4.2. Type I vertex operators

Let us define the basic operators for $j = 1, \ldots, n-1$:

$$U_{-\omega_j}(v) = z^{\frac{\omega_j}{2}} : \exp \left( -\beta_1 (\sqrt{-1} Q_{\omega_j} + P_{\omega_j} \log z) + \sum_{m \neq 0} \frac{1}{m} (B^l_m - B^l_{m+1})(x^j z)^{-m} \right) : , \tag{4.2}$$
These type I vertex operators satisfy the following commutation relations on $U_{\omega_j}(v)$:

$$U_{\alpha_j}(v)U_{\omega_j}(v') = r_j(v-v')U_{\omega_j}(v')U_{\alpha_j}(v),$$  

$$U_{-\alpha_j}(v)U_{\alpha_j}(v') = -f(v-v',0)U_{\omega_j}(v')U_{-\alpha_j}(v),$$  

$$U_{-\alpha_j}(v)U_{-\alpha_{j+1}}(v') = -f(v-v',0)U_{-\alpha_{j+1}}(v')U_{-\alpha_j}(v),$$  

$$U_{-\alpha_j}(v)U_{-\alpha_j}(v') = g(v-v')U_{-\alpha_j}(v')U_{-\alpha_j}(v).$$

(4.3)

where $\beta_1 = -\sqrt{\frac{r-1}{r}}$ and $z = x^{2\nu}$ as usual. Following commutation relations are useful:

$$U_{\omega_j}(v)U_{\omega_j}(v') = r_j(v-v')U_{\omega_j}(v')U_{\omega_j}(v),$$

$$U_{-\alpha_j}(v)U_{\omega_j}(v') = -f(v-v',0)U_{\omega_j}(v')U_{-\alpha_j}(v),$$

$$U_{-\alpha_j}(v)U_{-\alpha_{j+1}}(v') = -f(v-v',0)U_{-\alpha_{j+1}}(v')U_{-\alpha_j}(v),$$

$$U_{-\alpha_j}(v)U_{-\alpha_j}(v') = g(v-v')U_{-\alpha_j}(v')U_{-\alpha_j}(v).$$

(4.4)

In the sequel we set

$$\pi_{\mu} = \sqrt{r(r-1)}P_{\mu}, \quad \pi_{\mu\nu} = \pi_{\mu} - \pi_{\nu}.$$  

The $\pi_{\mu\nu}$ acts on $\mathcal{F}_{l,k}$ as a scalar $(\varepsilon_{\mu - \nu, (r - 1)l}).$  

For $0 \leq \mu \leq n - 1$ define the type I vertex operator [2] by

$$\phi_{\mu}(v) = \int \prod_{j=1}^{\mu} \frac{dz_j}{2\pi \sqrt{-1z_j}} U_{\omega_n}(v)U_{-\alpha_n}(v_1) \cdots U_{-\alpha_n}(v_\mu) \prod_{j=0}^{\mu-1} f(v_{j+1} - v_j, \pi_{j\mu}) \prod_{j=0}^{\mu-1} \sum_{j \neq \mu} \frac{\pi_{j\mu}}{[\pi_{j\mu}]^{-1}},$$

(4.5)

where $v_0 = v$ and $z_j = x^{2\nu_j}$. The integral contour for $z_j$-integration encircles the poles at $z_j = x^{1 + 2kr}z_{j-1}(k \in \mathbb{Z}_{\geq 0})$, but not the poles at $z_j = x^{-1 - 2kr}z_{j-1}(k \in \mathbb{Z}_{\geq 0})$, for $1 \leq j \leq \mu$. Note that

$$\phi_{\mu}(v) : \mathcal{F}_{l,k} \rightarrow \mathcal{F}_{l,k + \mu}.$$  

(4.6)

These type I vertex operators satisfy the following commutation relations on $\mathcal{F}_{l,k}$:

$$\phi_{\mu_1}(v_1)\phi_{\mu_2}(v_2) = \sum_{v_1 + v_2 = v_1} W \left[ a + \bar{\varepsilon}_{\mu_1}, \bar{\varepsilon}_{\mu_2}, a + \bar{\varepsilon}_{\mu_2} \mid a \right] \phi_{\mu_1}(v_1)\phi_{\mu_2}(v_2).$$

(4.7)

We thus denote the operator $\phi_{\mu}(v)$ by $\Phi(v)_{\mu}$ on the bosonic Fock space $\mathcal{F}_{l,a+p}$. We notice that our vertex operator (4.8) has different normalization from that originally constructed in [2] because of the difference of the Boltzmann weight $W$. Furthermore, the range of $\mu$ is shifted from that of [2] by 1 so that our $\phi_{\mu}(v)$ corresponds to $\phi_{\mu+1}(v)$ in [2], up to normalization.

Dual vertex operators are likewise defined as follows:

$$\phi_{\mu}^*(v) = c_{\mu}^{-1} \int \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi \sqrt{-1z_j}} U_{\omega_n}(v - \frac{n}{2}) U_{-\alpha_{n-1}}(v_{n-1}) \cdots U_{-\alpha_{n-1}}(v_{\mu+1})$$

(4.9)
\[
\times \prod_{j=\mu+1}^{n-1} f(v_j - v_{j+1}, \pi_{\mu}) \\
= c_n^{-1} (-1)^{n-1-\mu} \oint_{\gamma} \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi \sqrt{-1}z_j} U_{-\alpha_{\mu}}(v_{\mu+1}) \cdots U_{-\alpha_{n-1}}(v_{n-1}) U_{\alpha_{n}} \left( v - \frac{n}{2} \right) \\
\times \prod_{j=\mu+1}^{n-1} f(v_{j+1} - v_j, 1 - \pi_{\mu}) \tag{4.11}
\]

where \( v_0 = v - \frac{n}{2} \), and
\[
c_n = x^{\frac{1}{2} - \frac{1}{m}} \frac{g_{n-1}(x^n)}{(x^2, x^{-2\nu}y^\omega, x^{2\nu}, x^{-2\nu}y^{-\omega})}\frac{2}{(x^2, y^\omega, x^{-2\nu}, x^{2\nu})}.
\]

The integral contour for \( z_j \)-integration encircles the poles at \( z_j = x^{1+2k}z_{j+1} (k \in \mathbb{Z}_{\geq 0}) \), but not the poles at \( z_j = x^{1-2k}z_{j+1} (k \in \mathbb{Z}_{\geq 0}) \), for \( \mu + 1 \leq j \leq n - 1 \). Note that
\[
\phi_\mu^* (v) : \mathcal{F}_{\ell,k} \longrightarrow \mathcal{F}_{\ell,k-\epsilon_\mu}.
\tag{4.12}
\]

The operators \( \phi_\mu(v) \) and \( \phi^*_\mu(v) \) are dual in the following sense:
\[
\sum_{\mu=0}^{n-1} \phi_\mu^* (v) \phi_\mu (v) = 1. \tag{4.13}
\]

We notice that our dual vertex operator \( \phi_\mu^* (v) \) coincides with \( \phi_\mu^{*(n-1)} (v - \frac{n}{2}) \) in [2].

### 4.3. Other representations

The present face model has the so-called \( \sigma \)-invariance:
\[
W \left[ \begin{array}{cc} \sigma(c) & \sigma(d) \\ \sigma(b) & \sigma(a) \end{array} \right] \begin{vmatrix} c \\ d \\ b \\ a \end{vmatrix} = W \begin{vmatrix} c \\ d \\ b \\ a \end{vmatrix}, \quad \sigma(\omega_{\mu}) = \omega_{\mu+1}.
\]

The free field representation (4.8) is not invariant under \( \sigma \)-transformation, so that we have other free field representations:
\[
\phi_{i+\mu} (v) = \oint_{\gamma} \prod_{j=1}^{\mu} \frac{dz_j}{2\pi \sqrt{-1}z_j} U_{\alpha_{i}}(v) U_{-\alpha_{i}}(v_{i}) \cdots U_{-\alpha_{n}}(v_{\mu}) \\
\times \prod_{j=0}^{\mu-1} f(v_{j+1} - v_j, \pi_{i+j+\mu}) \prod_{j=0}^{n-1} [\pi_{i+j+\mu}]^{-1} \\
= (-1)^\mu \oint_{\gamma} \prod_{j=1}^{\mu} \frac{dz_j}{2\pi \sqrt{-1}z_j} U_{-\alpha_{i}}(v_{\mu}) \cdots U_{-\alpha_{i}}(v_{i}) U_{\alpha_{i}}(v) \\
\times \prod_{j=0}^{\mu-1} f(v_j - v_{j+1}, 1 - \pi_{i+j+\mu}) \prod_{j=0}^{n-1} [\pi_{i+j+\mu}]^{-1}, \tag{4.14}
\]

where \( v_0 = v \) and \( z_j = x^{2\nu_j} \), and the integral contours are the same one as (4.8). In this representation the space of states \( \mathcal{H}_{\ell,k}^{(i)} \) should be identified with \( \mathcal{F}_{\sigma^{-i}(\ell), \sigma^{-i}(k)} \).
4.4. Free field realization of CTM Hamiltonian

Let

$$H_F = \sum_{m=1}^{\infty} \frac{[rm]}{[(r-1)m]} \sum_{j=1}^{m-1} \chi^{(2j-2\lambda+1)m}B_m^j (B_m^j - B_m^{j+1}) + \frac{1}{2} \sum_{j=1}^{m-1} P_{\alpha_j} P_{\alpha_j}$$

be the CTM Hamiltonian on the Fock space $\mathcal{F}_{1, k}$ [14]. Then we have the homogeneity relation

$$\phi_\mu(z) = q^{H_F} \phi_\mu(q^{-1}z)$$

and

$$\text{tr}_{\mathcal{F}_{1, k}}(x^{2n} H_F G_a) = \frac{x^{n|\beta k + \beta l||^2}}{(x^{2n} x^{2\omega})^{a-1}} G_a.$$  \hspace{1cm} (4.17)

By comparing (3.20) and (4.17), we conclude that $\rho^{(i)}_{1, k} = G_a x^{2n} H_F$ and $\mathcal{T}_{1, k}^{(i)} = \mathcal{F}_{1, k}$, where $k = \alpha + \rho$.

The relation between $\rho^{(i)}$ and $\rho^{(j)}$ is as follows:

$$\rho^{(i)} = \sum_{k=\alpha + \mu \mod (\phi)} T(u) \phi^{(j)}_{\mu} / b_j T(u)^{\lambda a}. \hspace{1cm} (4.18)$$

4.5. Free field realization of tail operators

Consider (3.37) for $(c, b, a) \rightarrow (a, a + \varepsilon_0 + \varepsilon_\mu, a - \varepsilon_\mu)$, where $\mu \neq 0$. The coefficient $L$ diverges when $u \rightarrow v$, so that we obtain the following necessary condition:

$$\prod_{j=1}^{n-1} [a_{\beta j} \Phi(v)^a_{\mu - \varepsilon_\mu} \Lambda(v)^{a - \varepsilon_\mu}_{\mu - \varepsilon_\mu} + \prod_{j=1}^{n-1} [a_{\mu j} \Phi(v)^a_{\mu - \varepsilon_\mu} \Lambda(v)^{a - \varepsilon_\mu}_{\mu - \varepsilon_\mu} = 0. \hspace{1cm} (4.19)$$

By solving (4.19), we obtain

$$\Lambda(u)^{a - \varepsilon_\mu}_{a - \varepsilon_\mu} = G_{\pi \phi} \Phi \prod_{j=1}^{\mu} \frac{d\pi_j}{2\pi \sqrt{-1} \pi_j} U_{-a_1}(v_1) \cdots U_{-a_\mu}(v_\mu) \prod_{j=0}^{\mu-1} f(v_{j+1} - v_j, \pi_j) G_{\pi - 1}, \hspace{1cm} (4.20)$$

where

$$G_{\pi} := \prod_{k<\lambda} [\pi_{\lambda_k}].$$

Note that a free field representation of $\Lambda(u)^{a - \varepsilon_\mu}_{a - \varepsilon_\mu}$ for $v > 0$ can be constructed on $\mathcal{F}_{(a - (i), a - (k))}$.

In the following section, we need a tail operator $\Lambda(u)^{a - \sum_{j=1}^{n} \varepsilon_j}_{a - \sum_{j=1}^{n} \varepsilon_j}$ in order to calculate n-point functions. This type tail operator can be represented in terms of free bosons. In order to show this fact, let us introduce the symbol $\lesssim$ as follows. We say $\mu \lesssim v$ if $0 \leq \mu_0 \leq v_0 \leq n - 1$ and $\mu = \mu_0 \mod n$, $v = v_0 \mod n$.

It is clear that there exists $0 \leq i \leq n - 1$ such that

$$\lesssim (j|v_j + i \lesssim 0) > 0,$$
and
\[ n \{ j | \mu_j + i \leq m \} \leq n \{ j | v_j + i \leq m \}, \]
for every \( 0 \leq m \leq n - 1 \). In this case a free field representation of the tail operator \( \Lambda(u) \sum_{j=1}^{n-1} \lambda_j \) can be constructed on \( \mathcal{F}_{\sigma^{-1}(j), \sigma^{-1}(k)} \).

5. Correlation functions

5.1. General formulae

Consider the local state probability (LSP) such that the state variable at \( j \)th site is equal to \( \mu_j \) \( (1 \leq j \leq N) \), under a certain fixed boundary condition. In order to obtain LSP, it is convenient to divide the lattice into four transfer matrices and 2\( N \) vertex operators as follows:

Here, the incoming vertex operator \( \Phi'_{\mu}(v) \) should be distinguished from the outgoing vertex operator \( \Phi_{\mu}(v) \).

Let us consider the normalized partition function with fixed \( \mu_1, \ldots, \mu_N \):
\[
P^{(i)}(\mu_1, \ldots, \mu_N, v_1, \ldots, v_N) \coloneqq \frac{1}{\chi(i)} \tr_{\mathcal{F}^{(i)}} \left( A^{(i)}(v) \Phi'_{\mu_1}(v_1) \cdots \Phi'_{\mu_N}(v_N) \right)
\times A^{(i)}(v) A^{(i)}(v) A^{(i)}(v) A^{(i)}(v) \cdots \Phi_{\mu_1}(v) \Phi_{\mu_N}(v).
\]

(5.1)

In the vertex operator approach [10], the LSP can be given by \( P^{(i)}(\mu_1, \ldots, \mu_N)(v_1, \ldots, v_N) = P^{(i)}(\mu_1, \ldots, \mu_N, v_1, \ldots, v_N) \big|_{v_1=\cdots=v_N=v} \).

In what follows, we denote \( P^{(i)}(\mu_1, \ldots, \mu_N) = P^{(i)}(\mu_1, \ldots, \mu_N, v_1, \ldots, v_N) \big|_{v_1=\cdots=v_N=v} \).

The YBE and the crossing symmetry imply the following relation [8]:
\[
\Phi'_{\mu}(v') A^{(i)}(v) A^{(i)}(v) = \Phi_{\mu}(v) A^{(i)}(v) A^{(i)}(v) \Phi'_{\mu}(v').
\]

(5.2)

Thus, one-point local state probability of the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model can be given by
\[
P^{(i)}_{\mu_1, \ldots, \mu_N} = \frac{1}{\chi^{(i)}(\mu_1, \ldots, \mu_N, v_1, \ldots, v_N)} \left( T(u)^{\mu}(v) \Phi_{\mu}(v) \Phi_{\mu}(v) T(u)^{\mu}(v) \right).
\]

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by performing traces and where the second sum on the second line should be taken such that 

\[ \Phi^*(v)_a^{\varepsilon_j} \Phi(v)_a^{\varepsilon_j} \Lambda(u)_a^{\varepsilon_j} \frac{\rho_{i,k}^{(i)}}{b_i} \).  

(5.3)

Here in the third equality of (5.3), we use (3.35) and the fact that \( \Phi^*(v)_a^{\varepsilon_j} \) and \( t^*_j(v)_a^{\varepsilon_j} \) are given by the fusion of \( n-1 \Phi^*(v)_a^{\varepsilon_j} \)'s, \( \Phi(v)_a^{\varepsilon_j} \)'s and \( t^*_j(v)_a^{\varepsilon_j} \)'s, respectively.

In general, \( N \)-point local state possibility of this model can be given by

\[
P_{j_1 \cdots j_N}^{(i)} = \frac{1}{\chi^{(i)}} \text{tr}_{H^+_a} \left( \Phi^{\ast}(v)_a^{i_1} \cdots \Phi^{\ast}(v)_a^{i_N} \Phi^{\ast}(v)_a^{j_1} \cdots \Phi^{\ast}(v)_a^{j_N} \rho^{(i)}_v \right)
\]

\[
= \frac{1}{\chi^{(i)}} \sum_{\mu, \nu} \sum_{\substack{k \in \mathbb{Z} + \nu_0 \subseteq Q \atop \mu \subseteq Q}} t_j^*(v - u)_a^{\varepsilon_j} \cdots t_j^*(v - u)_a^{\varepsilon_j} (v - u)_a^{\varepsilon_j} \cdots (v - u)_a^{\varepsilon_j - 1} \nabla \times \text{tr}_{H_{a_1}} \left( \Phi(v)_a^{i_1} \cdots \Phi(v)_a^{i_2} \Phi(v)_a^{i_3} \cdots \Phi(v)_a^{i_N - 1} \Lambda(u)_a^{i_N} \frac{\rho_{i,k}^{(i)}}{b_i} \right),
\]

(5.4)

where the second sum on the second line should be taken such that \( (a, a_N), \ldots, (a_2, a_1) \) and \( (a'_1, a_1), \ldots, (a'_N, a'_{N-1}) \) are all admissible.

5.2. Spontaneous polarization

In this section, we reproduce the expression for spontaneous polarization [8]:

\[
\langle g \rangle^{(i)} = \sum_{j=0}^{n-1} \alpha^j P_j^{(i)} = \alpha^{n-1} \frac{(x^2; x^2)_{\infty}}{(x^{2r}; x^{2r})_{\infty}} \frac{(\omega x^{2r}; x^{2r})_{\infty}(\omega^{-1} x^{2r}; x^{2r})_{\infty}}{(\omega x^2; x^2)_{\infty}(\omega^{-1} x^2; x^2)_{\infty}},
\]

(5.5)

by performing traces and \( n \)-fold integrals on (5.3). In [8], expression (5.5) was obtained by solving a system of difference equations, the quantum Knizhnik–Zamolodchikov equations of level \(-2\nu\).

First we replace \( a + \varepsilon_v \) by \( a \) for simplicity:

\[
P_j^{(i)} = \frac{1}{\chi^{(i)}} \sum_{\mu, \nu} \sum_{k \in \mathbb{Z} + \nu_0 \subseteq Q \atop \mu \subseteq Q} t_j^*(v - u)_a^{\varepsilon_j} \nabla \times \text{tr}_{H_{a_1}} \left( \Phi(v)_a^{i_1} \cdots \Phi(v)_a^{i_2} \Phi(v)_a^{i_3} \cdots \Phi(v)_a^{i_N - 1} \Lambda(u)_a^{i_N} \frac{\rho_{i,k}^{(i)}}{b_i} \right).
\]

(5.6)

We note that

\[
\sum_{j=0}^{n-1} \alpha^j t_j^*(v - u)_a^{\varepsilon_j} \nabla \times \text{tr}_{H_{a_1}} \left( \Phi(v)_a^{i_1} \cdots \Phi(v)_a^{i_2} \Phi(v)_a^{i_3} \cdots \Phi(v)_a^{i_N - 1} \Lambda(u)_a^{i_N} \frac{\rho_{i,k}^{(i)}}{b_i} \right).
\]

(5.7)

where

\[
[u]_a = x^{-\frac{\nu}{2}} \Theta_{x^2} (\omega x^{2r}).
\]

Thus, the spontaneous polarization can be reduced as

\[
\langle g \rangle^{(i)} = \frac{1}{\chi^{(i)}} \sum_{\mu=0}^{n-1} \langle g \rangle^{(i)}_\mu,
\]

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where
\[
(g)_{0,\mu}^{(i)} = \sum_{k=0}^{n-1} \sum_{v=0}^{n-1} \frac{[v-u+a_{\mu v}]}{[v-u]} \prod_{j=0}^{n-1} \frac{[a_{\mu j}]}{[a_{\mu j}]} \text{tr}_{H_{v+k}^{(i)}} \times \left( \Phi(v)^{a_{\mu v}}_b \Lambda(v)^{a_{\mu v}}_b G_{a-b} \Phi^*(v+n)^{a_{\mu v}}_b \right).
\]

When \(\mu = 0\), in order to calculate the operator product \(\Phi(v)^{a_{\mu v}}_b \Lambda(v)^{a_{\mu v}}_b G_{a-b} \Phi^*(v+n)^{a_{\mu v}}_b\), the following operator product formulae are useful:
\[
c^{n-1}_v U_{w_0}(v) U_{w_0}(v_1) \cdots U_{w_{n-1}}(v_{n-1}) U_{w_{n-1}} \left( v + \frac{n}{2} \right) \]
\[
= x^{-\frac{n}{2}} \prod_{j=0}^{n-1} (x^2; x^2)^{a_{\mu j}} (x^2; x^2)^{a_{\mu j}} \prod_{j=0}^{n-1} \int z_j \frac{(x^{2n-1} z_j; x^2)_\infty}{(x z_j; x^2)_\infty} \times : U_{w_0}(v) U_{w_0}(v_1) \cdots U_{w_{n-1}}(v_{n-1}) U_{w_{n-1}} \left( v + \frac{n}{2} \right), \tag{5.8}
\]
where \(z_0 = z\) and \(z_n = x^n z\). Using (5.8) we have the following trace formulae:
\[
\text{tr}_{H_{v+k}^{(i)}} \left( c^{n-1}_v U_{w_0}(v) U_{w_0}(v_1) \cdots U_{w_{n-1}}(v_{n-1}) U_{w_{n-1}} \left( v + \frac{n}{2} \right) \right) \]
\[
= x^n (\frac{1}{(2k)!} \frac{1}{(2|\mu|)!}) \chi \frac{1}{2} \sum_{j=0}^{n-1} \frac{a_{\mu j}}{(v_j + 1 - v_j)} \frac{2+2n-3}{\chi^2 + 2n - 2; x^2, x^2} \prod_{j=0}^{n-1} (x^{2n-1} z_j; x^2)_\infty \]
\[
\times \prod_{j=0}^{n-1} \frac{(x^{2n-1} z_j; x^2)_\infty}{(x z_j; x^2)_\infty} \times A^{\mu}_{1, k}(v; v_1, \ldots, v_{n-1}). \tag{5.9}
\]
Let us denote the rhs of (5.9) by \(A^{\mu}_{1, k}(v_1, \ldots, v_{n-1})\). Then we have
\[
(g)_{0,\mu}^{(i)} = \frac{1}{b_1} \sum_{k=0}^{n-1} \prod_{0 \leq j < k} [a_{\mu j}] \sum_{v=0}^{n-1} \frac{[v-u+a_{\mu v}]}{[v-u]} \int_{C_v} \prod_{j=0}^{n-1} \frac{dz_j}{2\pi\sqrt{|z_j|}} \]
\[
\times \prod_{j=0}^{n-1} \frac{[a_{\mu j}]}{[a_{\mu j}]} f(v_j+1 - v_j, 1 - a_{\mu j}) A^{\mu}_{1, k}(v_1, \ldots, v_{n-1}). \tag{5.10}
\]
Here, the integral contour \(C_v\) is chosen such that

\[
|z_j| = \begin{cases} 
\chi^j (|z| + j \varepsilon) & (1 \leq j \leq v) \\
\chi^j (|z| - (n-j) \varepsilon) & (v+1 \leq j \leq n-1),
\end{cases}
\]
where \(\varepsilon > 0\) is a very small positive number.

Let us denote the rhs of (5.10) by \(H_{v+k}^{(i)}\). As noted in the previous section, the trace on \(H_{v+k}^{(i)}\) should be taken on \(F_{\sigma^{-\prime}, \sigma^{-\prime}}(J, J)\) for \(\mu > 0\). Thus, \((g)_{0,\mu}^{(i)}\) can be reduced to \(H_{\sigma^{-\prime}}^{(i)}\). Let
\[
B^{(i)}_{1, k}(v, u) := x^n (\frac{1}{(2k)!} \frac{1}{(2|\mu|)!}) \chi \frac{1}{2} \sum_{j=0}^{n-1} \frac{a_{\mu j}}{(v_j + 1 - v_j)} \frac{2+2n-3}{\chi^2 + 2n - 2; x^2, x^2} \prod_{j=0}^{n-1} (x^{2n-1} z_j; x^2)_\infty \]
\[
\times \prod_{j=0}^{n-1} \frac{(x^{2n-1} z_j; x^2)_\infty}{(x z_j; x^2)_\infty} \times \tilde{G}_v \equiv \prod_{j=0}^{n-1} [a_{\mu j}] \prod_{0 \leq j < k} [a_{\mu j}].
\]
Consider the following sum,
\[
S^{(i)}(v, u) := \frac{[0]}{[v-u]} \sum_{\mu=0}^{n-1} \sum_{k=0}^{n-1} \sum_{\mu=0}^{n-1} B^{(i)}_{\sigma^{-\prime}, \sigma^{-\prime}}(v, u),
\]
where
and take the limit \( u \to v^2 \). Then we have

\[
\lim_{u \to v} S^{(i)}(v, u) = \omega_i^{+1} b_l (\omega x^{2r}; x^{2r})_\infty (\omega^{-1} x^{2r}; x^{2r})_\infty (x^{2r}; x^{2r})^n_\infty (\omega; x^{2r})_\infty (\omega^{-1} x^{2r}; x^{2r})^n_\infty .
\]  

(5.11)

This can be confirmed by comparing the series expansion in \( x \) of both sides order by order.

Here we cite the sum formula from [2]:

\[
\sum_{\nu=0}^{n-1} \prod_{j=0}^{n-1} [\nu_j + 1 - \nu_j, 1 - \pi_j] = 0.
\]  

(5.12)

This can be derived by applying Liouville's second theorem to the following elliptic function:

\[
F(w) = \prod_{j=0}^{n-1} \left[ \frac{v_j + 1 - v_j}{2} + w - \pi_j \right].
\]

On equation (5.10), the contour for \( z_1 \)-integral is common for all \( \nu \) except for \( \nu = 0 \). Thus, by using (5.12), \( H_i^{(i)} \) can be evaluated by the residue at \( z_1 = x^{1+2u} \to x^{1} z \). The resulting \((n-2)\)-fold integral has the same structures of both the integrand and the contour as the original \((n-1)\)-fold one, except for the number of integral variables by one. Thus, we can repeat this evaluation procedure \( n-1 \) times to find

\[
H_i^{(i)} \sim \frac{1}{[v - u] b_l (x^{2r}; x^{2r})^n_\infty},
\]  

(5.13)

at \( u \sim v \). Substituting (5.11) and (5.13) into

\[
\langle g \rangle^{(i)} = \frac{1}{\chi^{(i)}} \sum_{\mu=0}^{n-1} H_i^{(i)} \sigma^{(i)}_{\sigma = \mu},
\]  

(5.14)

we reproduce the expression for the spontaneous polarization (5.5) originally obtained in [8].

6. Concluding remarks

In this paper, we constructed a free field representation method in order to obtain correlation functions of Belavin's \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model. The essential point was to find a free field representation of the tail operator \( \Lambda^{(i)}_\sigma \), the nonlocal operator which intertwines the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model and \( A^{(i)}_\sigma \) model. As a consistency check, we perform \((n-1)\)-fold integrals and traces for the one-point function to reproduce the expression of the spontaneous polarization originally obtained in [8].

There are some related works concerning the eight-vertex model and its higher spin version. A bootstrap approach for the eight-vertex model was presented in [15]. The vertex operators of the eight-vertex model with some special values of \( r \) were directly bosonized in [16]. A free field representation method for form factors of the eight-vertex model was constructed in [17]. A higher spin generalization of the free field representation method was achieved in [18]. As for the \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model, it is important to consider the extension to the form factor problem or the application to the fused model. We wish to address these problems in future.

2 Belavin’s \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model does not have the parameter \( u \) so that all the physical quantities should be independent of \( u \). Thus, we set \( u \to v \) here, in order to avoid some difficulty.
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