Superhorizon Curvaton Amplitude in Inflation and Pre-Big Bang Cosmology

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Abstract

We follow the evolution of the curvaton on superhorizon scales and check that the spectral tilt of the curvaton perturbations is unchanged as the curvaton becomes non-relativistic. Both inflation and pre-big bang cosmology can be treated since the curvaton mechanism within the two scenarios works the same way. We also discuss the amplitude of the density perturbations, which leads to some interesting constrains on the pre-big bang scenario. It is shown that within a $SL(3,R)$ non-linear sigma model one of the three axions has the right coupling to the dilaton and moduli to yield a flat spectrum with a high string scale, if a quadratic non-perturbative potential is generated and an intermediate string phase lasts long enough.

1 Introduction

The curvaton mechanism is an alternative way to generate the initial adiabatic density perturbations leading to the observed CMB anisotropies. It has recently generated some interest both in the context of pre-big bang (PBB) cosmology [1–5] and inflation [6–10]. The mechanism was first discussed more than ten years ago in [11], but also later briefly in [12]. The curvaton is an effectively massive scalar particle that only gets to dominate energy density after the beginning of the first radiation dominated era. As it decays, the initial isocurvature perturbations of the curvaton is converted into adiabatic ones. The name “curvaton” was first suggested in [6]. In inflation models the mechanism has received interest as an alternative to the usual scenario where the density perturbations are created by the quantum fluctuation of the inflaton, which can lead to a significant component of both isocurvature modes and non-Gaussian fluctuations. Within the PBB scenario, the mechanism is even more interesting

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since it might be needed for the scenario to yield the observed flat spectrum of adiabatic density perturbations.

It has been known for some time that the adiabatic perturbations created in the PBB phase has strong spectral tilt $\Delta n = 3$ where $\Delta n = 0$ is scale invariant [13] unless an exponential potential for the dilaton is included [14, 15]. Non standard matching conditions like those used in the ekpyrotic scenario has been applied also to PBB by Durrer and Vernizzi [16], who find $\Delta n = -1$, although there are some arguments that the original matching conditions are correct [17, 18]. However, a flat spectrum of isocurvature perturbations is generated by the axion fields, which do not contribute to the background in the PBB dynamics. Since the axion spectrum is flat, as opposed to the very blue spectrum of the dilaton, they will dominate the energy density fluctuation spectrum. By means of the curvaton mechanism, those isocurvature density fluctuations can be converted into adiabatic ones.

In this paper we shall concentrate on some aspects of the curvaton mechanism, which we believe have not been treated in detail before. It can be argued very simply [6], that any evolution of the curvaton field perturbation on superhorizon scales only leads to an overall scale independent factor, which will conserve the flatness of the spectrum. Here we confirm the validity of the argument as we follow the evolution of the curvaton on superhorizon scales till after the modes has become non-relativistic. We show explicitly that the spectrum stays flat after the curvaton starts to oscillate in its quadratic potential. The argument is similar to work done in connection to the “seed mechanism” in [23].

It has also been noted that if the curvaton of the PBB scenario is the NS-NS axion, then with a quadratic potential, the field fluctuations of the curvaton seems to be too large to be compatible with a high string scale of order $M_s \sim 10^{-2} m_p$ [1, 2]. It was first suggested that a periodic potential usually expected for the axion could damp the fluctuation to the right level [1], but it has also been suggested that a break in the spectrum can do the job [2]. We will investigate this further and show that an intermediate string phase can help provided the coupling of the axion to the dilaton and the moduli satisfy certain requirements. We give an explicit example where the requirements are satisfied.

2 The curvaton model

During the inflationary or PBB era the curvaton field is constant and does not contribute to the background. However, the canonically normalized quantum fluctuation of the curvaton is amplified. In this epoch the curvaton field contributes to the total energy density like a cosmological constant. After the universe has entered the first radiation dominated era at conformal time $\eta = \eta_r$ and when the Hubble rate subsequently reaches $H_{osc} \simeq m$, the curvaton starts to oscillate in its potential. From this point on, the energy density of the curvaton will behave like matter and falls off like $\rho_\sigma \propto a^{-3}$. This implies that the curvaton will soon dominate energy density and at this point the Hubble rate is $H_{dom} \sim \Omega m \sigma_r^4$. Here $\Omega$ is a conformal factor relevant only in the PBB scenario and $\sigma_r$ is the value of the curvaton
field at the beginning of the first radiation dominated epoch. Finally the curvaton decays into radiation, and the curvaton fluctuations induces the initial adiabatic density perturbations in the cosmic fluid leading to the CMB anisotropies.

From requiring that the curvaton only dominates energy density after it has started to oscillate in its potential* we must constrain ourselves to $\sigma_r/m_p < 1/\Omega$ ($m_p$ is the Planck mass).

Since the curvaton models within the PBB and the inflationary scenarios are very similar, we will treat them both. In this section we will set up the notation and, as we will confine ourself to a curvaton with a quadratic potential, parameterize the action for the curvaton $\sigma$ in the following way

$$ S = \frac{1}{2} \int d^4x \sqrt{-g} e^{l\phi} e^{mb} \left[ (\nabla \sigma)^2 + m^2 \sigma^2 \right]. \quad (1) $$

Where $g$ is the determinant of the metric, $b$ is the moduli of the internal dimension and $\phi$ is the dilaton. The inflationary case is obtained by taking $l = m = 0$, while in the PBB set-up $l, m$ can in principle take any positive or negative value of order one.

We will be interested in the inhomogeneous fluctuations of the curvaton field around a homogeneous background field $\sigma = \langle \sigma \rangle + \delta \sigma$. In the following $\sigma$ denotes only the background value of the curvaton field. It is useful to define a pump field $S = a \exp(l\phi/2 + mb/2)$. Then the canonically normalized perturbation field is $\psi_k = S \delta \sigma(k)$ and the perturbation equation can be written as

$$ \psi_k'' + \left( k^2 - \frac{S''}{S} + a^2 m^2 \right) \psi_k = 0. \quad (2) $$

Here $'$ denotes derivative with respect to the conformal time $\eta$. In accordance with the curvaton scenario, we will assume that the curvaton does not affect the background evolution until after the field has started to oscillate in its potential. We will also assume that the curvature perturbations at the beginning of the first radiation dominated era are negligible, so we can ignore the coupling to other perturbations at least until the curvaton dominates energy density.

Below we will remind about some well known features of inflation and PBB, paving the way for the discussion in the following sections.

### 2.1 Inflation

In an inflationary set-up the pump field is just $S = a$. During inflation, when the mass of the curvaton can be neglected†, the perturbation equation becomes

$$ \psi_k'' + \left( k^2 - \frac{a''}{a} \right) \psi_k = 0. \quad (3) $$

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*If this is not the case, we are not really considering a curvaton model, since the curvaton will then lead to some additional inflation [2].

†The mass can be neglected until at least a few Hubble times after Horizon exit, where the curvaton fluctuations are frozen.
In case of de Sitter inflation we have \( a''/a = 2/\eta^2 \). If we define \( \lambda(\lambda - 1)/\eta^2 \equiv a''/a \), then the solution to the perturbation equation normalized to a vacuum fluctuation in the infinite past is
\[
\psi_k = \frac{\sqrt{\pi}}{2} \sqrt{\eta} H_\mu^{(2)}(|k\eta|) , \quad \mu = |\lambda - 1/2| .
\] (4)

Thus, for de Sitter inflation \( \mu = 3/2 \). \( H_\mu^{(2)} \) is the Hankel function of the second kind. At horizon exit \( k = a_* H_* = 1/\eta_* \) and with \( \mu = 3/2 \) we obtain, by using the small argument expansion of the Hankel function,
\[
\mathcal{P}_{\delta\sigma}^{1/2}(k) = \frac{k^{3/2}}{2\pi} \eta H_* , \quad \mu = 3/2
\] (5)

where \( H_* \) is the Hubble parameter at horizon exit. Since \( H \) is approximately constant during inflation \( H_* \) is equal to the Hubble parameter at the beginning of the first radiation dominated epoch \( H_r \). The power-spectrum in equation (5) was applied in [6].

### 2.2 Pre-Big Bang

The PBB scenario [19–21] is based on a four-dimensional effective tree-level string theory action, which in Einstein frame can be written as
\[
S_{\text{eff}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \mathcal{R} + \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}(\nabla b)^2 + \mathcal{L}_{\text{matter}} \right] ,
\] (6)

where \( \kappa^2 = 8\pi/M_s^2 \) and \( M_s \) is the string mass. Here \( \phi \) and \( b \) are the universal four-dimensional moduli fields. This action can be derived from 10-dimensional string theory compactified on some 6-dimensional compact internal manifold. The matter Lagrangian, \( \mathcal{L}_{\text{matter}} \), is composed of gauge fields and axions, which we assume do not to contribute to the background.

The solution in Einstein frame to the equations of motion for the background fields can in the spatially flat case be parameterized as [22]
\[
a = a_r \left| \frac{\eta}{\eta_r} \right|^{1/2} , \quad e^\phi = e^{\phi_r} \left| \frac{\eta}{\eta_r} \right| \sqrt{3} \cos \zeta , \quad e^b = e^{b_r} \left| \frac{\eta}{\eta_r} \right| \sqrt{3} \sin \zeta .
\] (7)

As mentioned in the introduction, in the PBB scenario we identify the curvaton with the axion field. Let us split out the part of the matter Lagrangian that contains the axion field \( \mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{axion}} \), where we will assume that the axion Lagrangian has the general form
\[
\int d^4x \sqrt{-g} \mathcal{L}_{\text{axion}} = \frac{1}{2} \int d\eta d^3x a^4 e^{i\phi} e^{mb} [ (\nabla \sigma)^2 + m^2 \sigma^2] .
\] (8)

In the PBB era we can (like in the inflationary phase) ignore the mass term in the resulting perturbation equation and the solution to the perturbation equation normalized to a vacuum fluctuation in the infinite past is like in the previous subsection
\[
\psi_k = \frac{\sqrt{\pi}}{2} \sqrt{\eta} H_\mu^{(2)}(|k\eta|) , \quad \mu = |\lambda - 1/2| \quad (9)
\]
with $\lambda(\lambda - 1)/\eta^2 = S''/S$. The only difference being that $\mu$ now depends on the PBB background dynamics through the non-trivial pump field $S = a \exp(l\phi/2 + mb/2)$ and in principle can take any value depending on the values of $l$, $m$ and $\zeta$.

Having set up the notation, in such a way that the curvaton perturbation spectrum takes the same form in the inflationary and PBB scenarios, we are ready to discuss the superhorizon curvaton amplitude during the first radiation dominated epoch.

3 Superhorizon curvaton amplitude

In this section we calculate the amplitude of the curvaton perturbation outside the horizon in the non-relativistic regime by matching its solution to the spectrum obtained at the onset of the first radiation dominated epoch.

Assume that the perturbation spectrum of the curvaton at the beginning of the first radiation dominated phase is given by

$$\psi_k = \frac{\sqrt{\pi}}{2} \sqrt{\eta H^{(2)}_{\mu}(|k\eta|)}.$$  \hspace{1cm} (10)

Let $\eta_r$ denote the beginning of the radiation dominated phase. In the radiation dominated era $\eta > \eta_r$, the scale factor is proportional to the conformal time $a \sim \eta$. Since at this point in PBB cosmology the dilaton and moduli fields are frozen to a constant value, then for both inflation and PBB cosmology we can ignore the effective potential term $S''/S$ in the perturbation equation (2), and it simply reads

$$\psi_k'' + (k^2 + a^2 m^2)\psi_k = 0 , \quad \eta > \eta_r . \hspace{1cm} (11)$$

Following [23] and defining $\alpha = mH_{r}a^2$, where $a = a_r\eta/\eta_r$ and $H_r = 1/(a_r\eta_r)$, it is useful to note that $\alpha^2\eta^2 = a^2 m^2$, such that the equation (11) becomes

$$\psi_k'' + (k^2 + \alpha^2 \eta^2)\psi_k = 0 , \quad \eta > \eta_r . \hspace{1cm} (12)$$

Finally, by defining $x = \eta(2\alpha)^{1/2}$ and $-b = k^2/(2\alpha)$ one obtains

$$\frac{d^2\psi_k}{dx^2} + \left(\frac{x^2}{4} - b\right)\psi_k = 0 . \hspace{1cm} (13)$$

This equation can be solved exactly by parabolic cylinder functions

$$\psi_k = Ay_1(b, x) + By_2(b, x) \hspace{1cm} (14)$$

where $y_1$ and $y_2$ are the even/odd parts of the parabolic cylinder functions.

We are now in position to match (10) and (14) at the beginning of the radiation dominated era $\eta = \eta_r$. Let $H_m = 1/(a_m\eta_m) \sim m$, then $k_m = 1/\eta_m$ marks the scale that re-enters just as
it becomes non-relativistic. At the matching time \( x \sim \eta/\eta_m < 1 \) and for modes outside the horizon \( x^2 b < < 1 \), we can use the expansion

\[
y_1(b, x) = 1 + b \frac{x^2}{2!} + \ldots \\
y_2(b, x) = x + b \frac{x^2}{2!} + \ldots
\]

This implies \([23]\)

\[
\psi_k = C(\mu)(2\eta, \alpha)^{-1/2} |k\eta_r|^{-\mu} y_2(b, x) .
\]

with

\[
C(\mu) = \sqrt{2} \frac{\Gamma(\mu) \Gamma(3/2)^{2/3}}{\Gamma(1/4) \Gamma(3/2)^{2/3}} .
\]

The curvaton perturbation spectrum in equation (16) was also applied at the beginning of the radiation dominated era in [2], where we can use \( x << 1 \) i.e. \( y_2 \sim x \) to yield

\[
\delta \sigma(k) = C(\mu) a^{-3/2} H^{-1/2} \left( \frac{k}{k_r} \right)^{-\mu} , \quad \eta < \eta_{osc}
\]

thus, before the curvaton starts to oscillate in its potential \( \sigma \) as well as \( \delta \sigma(k) \) are constant as we expected.

But, as in [23], we can follow the superhorizon evolution of the perturbation spectrum to the matter dominated era. After the curvaton has started to oscillate in the potential \( \eta > \eta_m \) i.e. \( x > 1 \) we get

\[
\psi_k = \frac{A}{(2\alpha \eta)^{1/2}} \left( \frac{k^2 \eta_r}{2\alpha} \right)^{1/4} |k\eta_r|^{-\mu-1/2} \sin \left( \frac{m}{H} + \frac{1}{8\pi} \right)
\]

where \( A = \sqrt{2} C(\mu) \sqrt{\Gamma(1/4)/\Gamma(3/4)} \). By using \( k^2/\alpha = k^2 H_r/(k_r^2 m) \) we find straightforwardly that the amplitude of the curvaton oscillation is

\[
\delta \sigma(k) = \frac{\tilde{A}}{a \sqrt{am}} \left( \frac{H_r}{m} \right)^{1/4} \left( \frac{k}{k_r} \right)^{-\mu} , \quad \eta > \eta_{osc}
\]

where

\[
\tilde{A} = 2^{-3/4} \Omega^{-1} A .
\]

For the inflationary scenario \( \tilde{A} = 2^{-3/4} A \), while for PBB \( \tilde{A} = 2^{-3/4} \exp(-l\phi/2 - mb/2) A \).

It is now easy to see that

\[
\delta \sigma(k) = \left( \frac{a_{osc}}{a} \right)^{3/2} \delta \sigma_{osc}(k) , \quad \eta > \eta_{osc}
\]

and using the fact that the universe is radiation dominated until the curvaton starts to oscillate in its potential, we may use also \( m = H_{osc} = H_r (a_r/a_{osc})^2 \) and we obtain

\[
\delta \sigma_{osc}(k) = \frac{\tilde{A}}{C(\mu)} \left( \frac{a_r}{a_{osc}} \right)^{3/2} \left( \frac{H_r}{m} \right)^{3/4} \delta \sigma_r(k) = \frac{\tilde{A}}{C(\mu)} \delta \sigma_r(k) .
\]

As expected, the curvaton field fluctuations are constant until the curvaton starts to oscillate in the quadratic potential and the field fluctuations begin to fall off as \( a^{-3/2} \), just like the background field \( \sigma \).
3.1 Perturbation amplitude at decay

There are now two cases to investigate. The case with a non-vanishing background value of the curvaton field, and the case of vanishing background field. Let us first consider the more interesting first case\(^\dagger\). In this case we can write

\[ \rho_{\sigma} = \frac{1}{2} \Omega^2 m^2 \sigma^2 \quad \delta \rho_{\sigma}(k) = m^2 \Omega^2 \sigma \delta \sigma(k), \]

(24)

where \( \Omega = \exp(\ell \phi/2 + mb/2) \). The curvaton density perturbation becomes

\[ \delta \equiv \frac{\delta \rho_{\sigma}(k)}{\langle \rho_{\sigma} \rangle} = 2 \frac{\delta \sigma(k)}{\sigma}. \]

(25)

So after the axion has started to oscillate in its potential we get

\[ \frac{\delta \sigma(k)}{\sigma} = \frac{\tilde{A}}{C(\mu)} \frac{\delta \sigma_r(k)}{\sigma_r} \]

(26)

where we used that \( \sigma_{osc} = \sigma_r \) and after the curvaton has started to oscillate in the potential its amplitude falls off like \( a^{-3/2} \). It is interesting to note that even if the curvaton field fluctuations depend on \( \Omega \), the fluctuations in the curvaton energy density itself does not. This peculiarity has also been noted earlier, see for instance [24]. This also implies that the relation in equation (26) implicitly depends on \( \Omega \) through \( \tilde{A} \).

Finally

\[ \frac{k^{3/2} \delta \sigma(k)}{\sigma} = k^{3/2} \frac{\tilde{A}}{C(\mu)} \frac{\delta \sigma_r(k)}{\sigma_r} = \frac{\tilde{A} H_r}{\sigma_r} \left( \frac{k}{k_r} \right)^{3/2 - \mu}, \quad \eta > \eta_{osc} \]

(27)

which was obtained for the PBB curvaton model in [1, 2] and for inflation in [6] up to the numerical factor \( \tilde{A} \).

For completeness we mention that, if the background field is vanishing (\( \sigma(k) = \delta \sigma(k) \)), the power spectrum of density perturbations is given by

\[ \int \frac{d^3k}{(2\pi k)^3} e^{i \hat{k} \cdot (\vec{x} - \vec{x}')} P_{\rho}(k) = \langle \rho_x(\sigma) \rho_{x'}(\sigma) \rangle - \langle \rho_x(\sigma) \rangle^2 \]

\[ \Omega^4 \frac{m^4}{4} \left( \langle \sigma_x^2 \sigma_{x'}^2 \rangle - \langle \sigma_x^2 \rangle^2 \right) = \Omega^4 \frac{m^4}{4} \int \frac{d^3k}{(2\pi)^3} e^{i \hat{k} \cdot (\vec{x} - \vec{x}')} \int \frac{d^3p}{(2\pi)^3} |\sigma_p|^2 |\sigma_{k-p}|^2. \]

(28)

Thus,

\[ P_{\rho}(k) = \Omega^4 \frac{\tilde{A}^4}{16\pi} m H_1 \left( \frac{k}{k_r} \right)^3 \left( \frac{k_r}{a} \right)^6 \int \frac{dp}{p} \left( \frac{p}{k_r} \right)^4 \frac{|k - p|}{k_r} n_p n_{k-p} \]

(29)

where \( n_k = |k \eta_r|^{-2\mu-1} \). The integral was calculated in [23] and it implies the following power spectrum

\[ \frac{P_{\rho}(k)}{\rho_c^2} = \Omega^4 \frac{\tilde{A}^2 B}{16\pi^2(\mu - 1)} \frac{1}{9} \left( \frac{H_r}{m_p} \right)^4 \frac{m}{H_r} \left( \frac{H_r}{H} \right)^4 \left( \frac{a}{a_r} \right)^6 \left( \frac{k}{k_r} \right)^{6-4\mu}. \]

(30)

\(^\dagger\)A vanishing background field leads to non-Gaussian perturbations, which are ruled out by observations [6, 7].
where
\[ B = \frac{2^{4\mu-4}}{\sqrt{\pi}} \Gamma(2 - 2\mu) \Gamma(2\mu - 3/4) \cos 2\pi(\mu - 1) - 1 \] .

(31)

Note that also the spectrum in equation (30) is constant in the matter dominated era. At times \( \eta > \eta_{eq} \), one gets
\[ \frac{P_\rho(k)}{\rho_c^2} = \frac{\Omega^4 \tilde{A}^2 B}{16\pi^2(\mu - 1)} \frac{9}{9} \left( \frac{H_r}{m_p} \right)^4 \left( \frac{m}{H_{eq}} \right) \left( \frac{k}{k_{r}} \right)^{6 - 4\mu} . \]

(32)

4 \ COBE normalization

If the curvaton starts to dominate energy density only after is has started to oscillate in the potential we must require \( \Omega \sigma_r < 1 \) in Planck units. Otherwise a short era of inflation will arise [2]. Normalizing to the COBE observations, we must require
\[ 10^{-5} \simeq P_\zeta^{1/2}(k_0) \simeq \frac{k_0^{3/2} \delta \rho_r(k_0)}{\rho_r} . \]

(33)

which implies that a flat spectrum of CMB perturbations is only possible if [6]
\[ H_r < 10^{-5} m_p . \]

(34)

This is more or less the same as the constrain on the Hubble scale during inflation obtained in [25] from COBE observations. However for PBB the situation is more interesting. In the PBB scenario we expect \( M_s \simeq H_r \) and a flat spectrum would require
\[ M_s < 10^{-5} m_p \]

(35)

as observed in [1, 2]. This is a very low value for the string scale. It looks like the amplitude of the curvaton field fluctuations is too large to be in agreement with the standard string scale and the COBE normalization. However, there are several ways how one can circumvent this problem. In [1] we suggested that the curvaton is to be identified with an axion with a periodic potential \( ^5 \), such that the periodic potential will damp the density fluctuations to the right level. For a quadratic curvaton potential it was suggested in [2] that a kink in the spectrum might also do the job of lowering the curvaton field fluctuations to the right level.

There also exists another interesting possibility; the curvaton might decay before it dominates the energy density\( ^6 \). The curvature perturbation is then given by [6]
\[ \zeta = \frac{\rho_\sigma}{4 \rho_{rad} + 3 \rho_\sigma \rho_\sigma} . \]

(36)

\( ^5 \)One might fear that a periodic potential will lead to formation of topological defects. However, if the potential has only one degenerate minimum i.e. if the anomaly factor is one \( (N = 1) \), then the topological defects decay and do not pose a cosmological problem [26].

\( ^6 \)This was first suggested by Lyth and Wands [6].
If $\rho_{\text{rad}} \gg \rho_\sigma$ at the decay, we get from equation (36) and (26)

$$\zeta_d \simeq \frac{1}{2} \left( \frac{\rho_\sigma}{\rho_{\text{rad}}} \right)_d \frac{\delta \sigma_r(k)}{\sigma_r},$$

(37)

where $\rho_{\text{rad}}$ is the energy density of radiation and subscript $d$ denotes the point when the curvaton decays. Thus, if the PBB curvaton field decays when it only contributes $r = 10^{-3}$ of the total energy density, then by means of equation (37) we would get $M_s = 10^{-2}m_p$. This is within the current constrains on non-Gaussianity from COBE data, which yield $r > 6 \times 10^{-4}$ (see [7] for details on the non-Gaussian aspects of the curvature perturbations). In this case the curvaton can not be the axion suggested in [1], but must have a stronger interaction with the photons than what an axion can offer. This possibility is exciting since $r \simeq 10^{-3}$ can already be ruled out or be confirmed by the MAP and PLANCK satellites. Note, that it was recently shown by Lyth, Ungarelli and Wands [7] that if $r << 1$ then CDM cannot be created before the curvaton decays or by the curvaton decay itself because large correlated or anti-correlated isocurvature perturbations are produced.

In the next subsection we will see how an intermediate string phase might also do the job of lowering the curvaton field fluctuations to the right level in order to be in agreement with a high string scale and the COBE normalization.

### 4.1 Intermediate string phase

Assume that there is an intermediate string phase $\eta_s < \eta < \eta_r$ during which the curvaton field fluctuations are frozen on super horizon scales. We parameterize our ignorance about the string phase like in [27]. We let $z_s = a_r/a_s$ denote the ratio of the scale factor at the end of the string phase and at the beginning of the string phase. Likewise, we denote by $g_r = \exp(\phi(\eta_r)/2)$ and $g_s = \exp(\phi(\eta_s)/2)$ the string coupling constant at respectively the end and the beginning of the string phase. In order to have enough inflation in the pre-big bang era, we must assume $1 < z_s < 10^{20}$ [27]. It is also natural to assume $g_r/g_s > 1$.

Since it is $\delta \sigma(k)$ which is constant on superhorizon scales and not the canonical normalized field $\psi_k$, we should match $\delta \sigma(k)$ at $\eta_s$ and $\eta_r$. Using equation (10) and (14) the matching leads to

$$S_r \delta \sigma(k) = C(\mu) \frac{S_r}{S_s} \left( \frac{\eta_s}{\eta_r} \right)^{-\mu+1/2} \frac{1}{(2\eta_s\alpha)^{-1/2}} |k\eta_r|^{\mu} y_2(b, x), \quad \eta > \eta_r. \quad (38)$$

By comparing to equation (16) we see that the curvaton fluctuation spectrum is multiplied by an additional factor of\(^\dagger\)

$$\frac{S_r}{S_s} \left( \frac{\eta_s}{\eta_r} \right)^{-\mu+1/2} = z_s^{-\mu+3/2} \left( \frac{H_r}{H_s} \right)^{-\mu+1/2} \left( \frac{g_r}{g_s} \right)^{1/2} \left( \frac{\beta_r}{\beta_s} \right)^m \quad (39)$$

\(^\dagger\)If we write $S_r/S_s = (k_r/k_s)^{-\delta}$, then $\delta$ agrees with the parameterization of the break in the spectrum discussed by Bozza et al. in [2].

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where we defined $\beta(\eta) \equiv \exp(b(\eta)/2)$. When normalizing to the COBE observations, this leads to a non trivial dependence on $l, m$.

As an example of a model in which the parameters $l, m$ are non-trivial, we consider a non-linear sigma model in Einstein gravity where the scalar fields parameterize a $SL(3, \mathcal{R})/SO(3)$ coset [28]. In particular we consider the following lowest order action in 4-D Einstein frame [22]

$$
S = \frac{1}{2\kappa^2} \int d^4\sqrt{-g} \left[ R - \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}(\nabla b)^2 - \frac{1}{2}e^{\sqrt{3}b+\phi}(\nabla \sigma_3)^2 - \frac{1}{2}e^{-\sqrt{3}b+\phi}(\nabla \sigma_2)^2 - \frac{1}{2}e^{2\phi}(\nabla \sigma_1 - \sigma_3 \nabla \sigma_2)^2 \right],
$$

where $\kappa^2 = 8\pi/m_p^2$, $\phi$ is the 4-D dilaton, $b$ is the moduli, $\sigma_1$ is the NS-NS axion and $\sigma_2, \sigma_3$ are RR axions. This model has been discussed in great detail in [22, 24, 28–30]. The action contains no potential terms, but one expects them to be generated non-perturbatively. The perturbations of the axion with the smallest tilt will dominate the energy density perturbation spectra. It has been demonstrated that there are regions of parameter space where either of the axions can be the dominating one and with a flat scale invariant spectrum [24]. We note that the conformal factors for the three axion fields are [24]

$$
\Omega_i^2 = \begin{cases} 
  e^{2\phi} & \text{for } \sigma_1 \\
  e^{\phi - \sqrt{3}b} & \text{for } \sigma_2 \\
  e^{\phi + \sqrt{3}b} & \text{for } \sigma_3 
\end{cases}
$$

Curiously, for the third axion field $l = 1$, $m = \sqrt{3}$. For a flat spectrum with $\mu = 3/2$, we obtain from equation (33)

$$
10^{-5} > \frac{g_r H_s}{g_s H_r} \left( \frac{\beta_s}{\beta_r} \right)^{\sqrt{3}} \frac{M_s}{m_p}
$$

which, depending of the detailed dynamics of the string phase, can relax the bound in equation (35) if for instance** $\beta_s/\beta_r \sim g_r/g_s >> 1, H_s/H_r \lesssim 1$. But if we like in [31] take the intermediate string phase to be a period of constant curvature, frozen internal dimensions and linearly growing dilaton in the string frame, then in the Einstein frame the curvature scale will not be constant but vary like the string coupling $H_s/H_r \approx g_s/g_r$, so even a very long string phase of this kind will not improve on the bound in equation (35) for $\sigma_3$ and for the Kalb-Ramond axion $\sigma_1$ with $l = 2$ it gets worse. A dual dilaton phase (also considered in [31]) with frozen internal dimensions and decreasing curvature scale in the string frame might even be problematic since it would change the bound in equation (35) in the wrong direction. However, it is clear that an intermediate string phase of accelerated expansion and increasing curvature scale (as considered in [33]) only have to be very short in order to relax the bound in equation (35), with the curvature scale growing only two orders of magnitude.

**It is assumed that the extra dimensions are contracting in the PBB phase as well as in the intermediate string phase such that $\beta_s >> \beta_r$.**
If the internal dimensions are contracting, we can obtain a high string scale even with a constant curvature scale during the intermediate string phase (in string frame) with $\sigma_3$. If the internal dimensions contract fast enough the $\sigma_3$ field might even work as a curvaton if we have a decelerating dual dilaton phase.

The discussion above is of course just an example. Generally we expect as many as 15 different axion fields in a $SL(6, \mathcal{R})$ invariant model, which is the maximal invariance for toroidal compactification from 10 to 4 dimensions, all with different conformal factors [32]. Note that in the simplest case of sudden transition without an intermediate string phase, all the different axion fields have the same amplitude for modes crossing the Hubble scale at the start of the post big bang era $k \approx k_r$. On larger scales the amplitudes are fixed by the different spectral tilts and the field with the smallest tilt yields the largest density perturbations. The increased symmetry group will enlarge the part of parameter space where the spectral tilt of the dominating axion is close to flat and where $l, m$ has non trivial values. In the maximal symmetric case, the 14 other axions will have a more blue spectrum and only contribute fractionally to the density perturbations. It might be possible that some of the fields are stable and can act as cold dark matter, leading also to an isocurvature component in the density fluctuations. However, such scenarios depends on a more detailed understanding of how the non-perturbative potentials are generated.

In this light it might be possible to obtain a flat spectrum in the PBB curvaton scenario with the standard string scale, but a more detailed understanding especially of the nature of the graceful exit is still needed.

### 4.2 Bounds on mass and reheat temperature

From requiring that the energy density contributed by the curvaton at the beginning of the radiation dominated epoch is less than critical, we obtain an upper bound on the curvaton mass and the reheat temperature. From

$$\rho_\sigma < \rho_c, \quad \eta = \eta_r$$

we have

$$\frac{1}{2} \Omega^2 m^2 \sigma^2 r < 3 \eta^2 r H^2$$

which leads to

$$10^{-5} \simeq \mathcal{P}_\zeta^{1/2} \simeq \frac{k^{3/2} \delta \rho r}{\rho r} > \frac{m}{m_p} \, .$$

This is consistent also with the bound obtained in "the simplest curvaton model" of Bartolo and Liddle [9].

To get an upper bound on the reheat temperature, let us assume that the curvaton interaction is suppressed by some mass-scale $M$ and define $R = M/m_p$. Then the lifetime of the curvaton is given by

$$\tau = M^2 / m^3$$

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and the reheat temperature is approximately given by

\[ T_{RH} \simeq \frac{m_p^{1/2}}{\tau^{1/2}}. \]  

(47)

Combined with our bound on the curvaton mass in equation (45), this leads to

\[ T_{RH} < \frac{1}{R} 10^{-7} m_p. \]  

(48)

The reheat temperature is thus likely to be low enough to avoid excessive production of gravitinos and monopoles, as was also the case in the explicit example in [1], provided that \( R \approx 1 \), which means that the curvaton decays only gravitationally.

5 Summary

We have followed the dynamics of the curvaton with a quadratic potential till after the perturbation modes have become non-relativistic. We found explicitly, that the spectral tilt of the curvaton stays the same throughout the dynamic evolution as was suggested in [6]. The case is similar to that of the ”seed mechanism” of the PBB scenario [23, 33].

The inflationary scale is constrained by observations to less than five orders of magnitude smaller than the Planck scale, which is in agreement with also the constrain from the curvaton mechanism by normalizing to the COBE observations [6]. However, in the PBB scenario the curvature scale at the beginning of the first radiation dominated epoch is supposed to be near the string scale. This leads to an inconsistency, when requiring both a flat spectrum and normalization to the COBE observations [1, 2]. It has been suggested before that either a periodic potential [1] or a break in the spectrum [2] can solve this problem. If the curvaton decays faster it could also help in this direction. This an interesting possibility which could be verified or excluded already by observations from the MAP and PLANCK satellites and which leads to severe theoretical constrains on the CDM due to the possible large isocurvature modes [7]. But, depending of the specific model, one can get potential problems with a too high reheat temperature if the curvaton couples too strongly to other fields.

It should be noted that if the string scale is of order the GUT scale, then the perturbations have naturally the amplitude required by observations. Also, with a quadratic potential we obtain a slightly blue spectrum for the curvaton, but with the periodic potential considered in [1] it is also possible to have a slightly red spectrum even with a high string scale. In any case there is still many things to be understood. To obtain more precise predictions from the PBB curvaton scenario, we need to understand more about the details of the non-perturbative potential of the axion and especially we will need to know more about the nature of the graceful exit.

Finally, we investigated the conditions under which an intermediate string phase leading can lead to the observed level of density fluctuations with a high string scale. The intermediate string phase is supposed to facilitate the graceful exit and various types of such phases has
been considered in the literature. We found that if the curvature scale is frozen during this phase the Kalb-Ramond axion has wrong coupling, while within a $SL(3, \mathcal{R})$ non-linear sigma model the $\sigma_3$ field due to its coupling to the contracting internal dimensions is a consistent candidate as a curvaton. If the intermediate string phase is a phase of accelerated expansion even the Kalb-Ramond axion might be consistent with a high string scale, while a dual dilaton phase of decelerated expansion could be a problem for the Kalb-Ramond axion leading to a small string scale. In any case, if the internal dimensions contract fast enough, the $\sigma_3$ remains a consistent candidate.

We showed that within a $SL(3, \mathcal{R})$ non-linear sigma model one of the three axions has the right coupling to the dilaton to yield a flat spectrum even with a high string scale, as long as the intermediate phase is not that of a decelerated expansion of the external dimensions and frozen internal dimensions.

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