Virtual pull-backs

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Abstract

We propose a generalization of Gysin maps for DM-type morphisms of stacks \( F \to G \) that admit a perfect relative obstruction theory \( E^*_{F/G} \). We prove functoriality properties of the generalized Gysin maps. As applications, we analyze Gromov-Witten invariants of blow-ups and we give a short proof of Costello’s push-forward formula.

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1 Introduction

Given a map \( f : F \to G \) of DM-stacks that admit virtual classes in the sense of [2] one would like to have a morphism \( f^! : A_*(G) \to A_*(F) \) that sends the virtual class of \( G \) to the virtual class of \( F \). This suggests one should look for a generalized version of Gysin pull-backs. The idea is to try to “treat \( f \) as if it was regular” whenever this is possible. It will be seen that in some cases this expectation is hopeless, although the virtual codimension of \( F \) in \( G \) is always constant.

It should be said that the idea is not entirely new, although we did not find this approach in the literature. The main inspiration point was the “functoriality property of the Behrend-Fantechi class” in [12]. Also, a similar
situation appears in [17] in shape of the \([M, N]\)_{\text{virt}}-construction.

Recently, we have been informed of the existence of [8]. The ideas there are basically the same, with a slightly different flavor. We hope, however, that our point of view may contribute to a clear understanding of this subject.

The main idea of the first section is to replace the normal bundle \(N_{F/G}\) with a “virtual normal bundle”. The appropriate context for this is given by obstruction theories. Precisely, if \(f\) admits a perfect relative obstruction theory \(E^\bullet_{F/G}\), then we take the virtual normal bundle to be \(h^1/h^0((E^\vee_{F/G})^\bullet)\).

The key tool is Kresch’s “deformation to the normal cone” for (Artin) stacks. In section 2 we show that the generalized Gysin pull-back satisfies the usual compatibility conditions. Moreover, we prove that subject to a very natural compatibility relation between obstructions the construction gives a map that sends the virtual class to the virtual class. The statement may also be seen as a generalization of the functoriality property in [2] and [12].

As an application we provide the answer to a very natural question. Given a smooth projective variety \(X\) and its blow-up \(p: \tilde{X} \to X\) along some smooth projective subvariety, we would like to know when do the GW-invariants agree. More precisely, if we start with a given homology class \(\beta \in A_1(X)\) and a collection of cohomology classes \(\gamma_i \in A^*(X)\), then we can associate a “lifted” homology class in \(A_1(\tilde{X})\) (see Definition 6 for a precise definition) and cohomology classes \(p^*\gamma_i \in A^*(\tilde{X})\). One could expect that the GWI associated to these data are equal. This was first analyzed in [6] where \(X\) was some projective space and \(Y\) a point and in [9], [11] where it was treated the blow-up along points, curves and surfaces. Recently, it was shown in [8] that (subject to a minor condition) the expectation is true for genus zero GW-invariants of blow-ups along subvarieties with convex normal bundles.

Our idea is to show the equality of rational GWI for \(X\) convex and then “pull the relation back” (see Proposition 1). The statement we get should be compared with Theorem 1.6 in [8].

We also give a short proof of Costello’s push-forward formula.

**Notation and conventions.** Unless otherwise stated we denote all inclusions by \(i\) and all projections by \(p\).

We work over a fixed ground field.

An Artin stack is an algebraic stack in the sense of [16] of finite type over the ground field.

We will usually denote schemes by \(X, Y, Z\), etc, DM-stacks by \(F, G, H\), etc. and Artin stacks by gothic letters \(M, E, F\), etc.

Unless otherwise stated, by a perfect obstruction theory in the sense of [2] we will always mean perfect in \([-1, 0]\).

By cones we mean cone-stacks in the sense of [2].

If \(X\) is a smooth projective complex variety and \(\beta \in A_1(X)\), then \(\overline{M}_{g,n}(X, \beta)\)
denotes the moduli stack of \( n \)-pointed stable maps of genus \( g \) to \( X \) of class \( \beta \) (see [4]).

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2 Construction

2.1 Generalized Gysin pull-backs

Let \( f : F \to G \) be a DM-type morphism of algebraic stacks and, moreover, let us assume \( F \) is a DM-stack. In this section, we define a pull-back morphism \( A^*_\ast(G) \xrightarrow{f^\ast} A^*_\ast(F) \) depending on some vector bundle stack \( \mathcal{E} \), in the same way as in [5] Chapter 6, but we replace the condition "\( f \) is regular" by a weaker condition. Precisely, if \( C_{F/G} \) denotes the normal cone stack to \( f \) introduced in [14], we require the following

\* there exists a vector bundle stack \( \mathcal{E} \), and a closed immersion \( C_{F/G} \hookrightarrow \mathcal{E} \).

Let us recall the basic definitions.

**Definition 1.** A morphism \( f : F \to G \) of Artin stacks is called of Deligne-Mumford type (or shortly of DM-type) if for any morphism \( V \to G \), with \( V \) a scheme, \( F \times_G V \) is a Deligne-Mumford stack.

Let us now give Kresch’s definition of \( C_{F/G} \). We refer at [14], section 5.1. where the author treats the case \( f \) is representable and locally separated. As remarked in [12], proof of Proposition 1, the construction copies for \( f \) of DM-type. Let us briefly sketch the construction. Let \( f : F \to G \) be a DM-type morphism of Artin stacks. Then one can construct a commutative
diagram (not unique)

\[
\begin{array}{ccc}
U & \xrightarrow{\tilde{f}} & V := M \times W \\
\downarrow \alpha & & \downarrow \delta \\
F \times_G W & \xrightarrow{\varepsilon} & W \\
\downarrow & & \downarrow \\
F & \xrightarrow{f} & G
\end{array}
\]  

(1)

where \(U\) and \(V\) are schemes, the vertical arrows are smooth surjective and the top arrow \(U \to V\) is a closed immersion.

Let now \(R := U \times_F U\) and \(S := V \times_G V\) with \(p_1, p_2 : R \to U\) and \(q_1, q_2 : S \to V\) be the obvious projections. Moreover, \(U\) and \(V\) in diagram (1) can be taken such that the natural map \(R \to S\) is a locally closed immersion. Therefore, this map gives rise to a normal cone \(C_{R/S}\) (see [18]). Let \(s_1, s_2\) be the maps obtained by composing

\[
C_{R/S} \Rightarrow (C_{U \times_V S/S}) \times_{U \times_V S} R \simeq C_{U/V} \times_U R \to C_{U/V},
\]

where the first ones are induced by the maps \(R = U \times_F U \to U \times_G U \to U \times_G V \simeq U \times_V S\), the last isomorphism depending on \(q_i\). By [14], section 5.1, the groupoid \([C_{R/S} \Rightarrow C_{U/V}]\) defines a stack that is denoted by \(C_{F/G}\).

In a completely analogous manner one can define a groupoid \([N_{R/S} \Rightarrow N_{U/V}]\), where \(N_{R/S}, N_{U/V}\) are the normal sheaves (see e.g. [2] section 1, for the definition of the normal sheaf of a closed embedding and see below for locally closed embeddings). This groupoid defines a stack that we denote \(N_{F/G}\).

**Definition 2.** We call the cone \(C_{F/G}\) the normal cone of \(f\) and \(N_{F/G}\) the normal sheaf of \(f\).

Let us now compare these objects with the ones in [2]. Let us first recall the definitions.

**Definition 3.** Let \(f : F \to G\) be a DM-type morphism and let \(L_{F/G} \in \text{ob} \ D(\mathcal{O}_F)\) be the cotangent complex. Then we denote the stack \(h^1/h^0(L_{F/G})^\vee\) by \(\mathcal{N}_{F/G}\) and we call it the intrinsic normal sheaf.

We denote by \(\mathcal{C}_{F/G}\) the unique subcone of \(\mathcal{N}_{F/G}\) that for any diagram (1) with \(U \to F\) étale, \(\mathcal{C}_{F/G}|U = [C_{U/V}/\tilde{f}^*T_{V/G}]\) and we call it the intrinsic normal sheaf.

The following Lemma in probably well-known to experts, but as we did not find it in the literature, we give a detailed proof for completeness.
**Lemma 1.** The cone stack $C_{F/G}$ of Definition 2 is canonically isomorphic to the intrinsic normal cone $C_{F/G}$ of Definition 3.

**Proof.** We divide the proof in several cases. In what follows we use the notation “$=$” for canonical isomorphisms.

**Case 1.** If $f$ is a closed embedding of schemes the statement is trivial.

**Case 2.** If $f$ is a locally closed embedding of stacks, then $N_{F/G}$ and $C_{F/G}$ are obtained by descent on $F$ (see [18]) and hence it suffices to check the statement locally. This shows the statement follows by the first case.

**Case 3.** If $f$ factors as

$$
W := G \times M
$$

with $i$ a locally closed embedding and $M$ a smooth scheme, then $\mathfrak{M}_{F/G} = N_{F/W/i^*T_M}$. Let us take $U$, $V$ étale covers of $F$ and $G \times M$ such that $U \to V$ is a closed embedding of schemes. Then, it suffices to show we have an isomorphism

$$
N_{U/V} \times_{N_{F/W/T_M}} N_{U/V} \simeq N_{U \times_F U/V \times G V}
$$

compatible with the groupoid structure. For this, we see the first term is isomorphic to $p^*i^*T_M \times N_{U/V} \times N_{F/W} N_{U/V}$ and using $V \to W$ is étale we obtain the first term is isomorphic to $p^*i^*T_M \times N_{U \times_F U/V}$. On the other hand, we know by the previous case that $N_{U \times_F U/V \times G V}$ is canonically isomorphic to $\mathfrak{M}_{U \times_F U/V \times G V}$ for which we know it is isomorphic to $\mathfrak{M}_{U \times_F U/V} \times p^*i^*T_{V/G}$. This shows $\mathfrak{M}_{F/G} = N_{F/G}$.

**Case 4.** In general, we show $\mathfrak{M}_{F/G} = N_{F/G}$. The proof is very similar to Case 3, above. Let us first give an explicit description of $\mathfrak{M}_{F/G}$. For this, we take $W$ to be a smooth atlas of $G$ and $U$ an affine atlas of $F \times G W$. Taking $M$ a smooth scheme such that $U$ embeds in $M$, we obtain diagram (1). Now, the diagonal map $g : U \to G$ is as in Case 3 above and therefore

$$
N_{U/G} = \mathfrak{M}_{U/G} = N_{U/V} / T_{V/G}
$$

In order to analyze the lower triangle, we consider the distinguished triangle

$$
p^*L_{F/G} \to L_{U/G} \to L_{U/F} \to p^*L_{F/G}[1]
$$

of relative cotangent complexes. As $p : U \to F$ is smooth it is easy to see that we are in the conditions of Proposition 2.7 in [2] and thus we get a
short exact sequence of intrinsic normal sheaves

\[ 0 \to \mathfrak{N}_{U/F} \to \mathfrak{N}_{U/G} \to p^* \mathfrak{N}_{F/G} \to 0. \tag{4} \]

By (3) and (4), in a similar way as before we get local isomorphisms. Moreover, the same equations (3) and (4) give a smooth morphism of abelian cone stacks \( N_{U/V} \to \mathfrak{N}_{F/G} \) and in a completely analogous fashion we get morphisms of abelian cone stacks \( N_{U/V} \to N_{U/G} \times \mathfrak{N}_{F/G} N_{U/V} \). This shows we obtain an isomorphism \( N_{U/V} \times \mathfrak{N}_{F/G} N_{U/V} \simeq N_{U/V} \times \tilde{f}^* T_{V/G} \). Checking the diagram below is commutative

\[
\begin{array}{ccc}
N_{U \times F \times U \times G V} & \xrightarrow{\sim} & N_{U/V} \\
\downarrow & & \downarrow \\
N_{U/V} \times \mathfrak{N}_{F/G} N_{U/V} & \xrightarrow{\sim} & N_{U/V}
\end{array}
\]

we obtain an isomorphism of groupoids and therefore the conclusion.

**Case 5.** By Case 4 above, it is enough to check that \( C_{F/G} \) is canonically isomorphic to the relative intrinsic normal cone \( \mathfrak{C}_{F/G} \) locally. For this, we look at the groupoid \( [C_{U/V \times G V} \Rightarrow C_{U/V}] \) with the two maps obtained by replacing \( F \) with \( U \). It is easy to see that \( N_{U/V \times G V} \) is isomorphic to \( N_{U/V} \times \tilde{f}^* T_{V/G} \). Via this isomorphism, the two maps defining the groupoid are the projection and the natural action of \( \tilde{f}^* T_{V/G} \) on \( C_{U/V} \). This shows \( C_{F/G} \) is locally isomorphic to \( [C_{U/V} / \tilde{f}^* T_{V/G}] \).

**Remark 1.** By the above Lemma we are allowed to identify the normal cone to a morphism with the intrinsic normal cone. In particular, the above Lemma shows that Definition \( \mathfrak{C}_{F/G} \) is independent of the choice of \( U \) and \( V \) in diagram (1). Although normal cones are cone stacks, we will use for simplicity the notation \( C_{F/G} \) instead of \( \mathfrak{C}_{F/G} \).

**Theorem 1.** Let \( F \to G \) be a DM-type morphism of Artin stacks. One can define a deformation space \( M_F^G \to \mathbb{P}^1 \) (i.e. a flat morphism) with general fibre \( G \) and special fibre the normal cone \( C_{F/G} \). Moreover, for any cartesian diagram

\[
\begin{array}{ccc}
F' & \to & G' \\
\downarrow & & \downarrow \\
F & \to & G
\end{array}
\]

there exists an induced morphism \( M_{F'}^G \to M_F^G \) that fits into a cartesian
Proof. By [14] in the representable locally separated case (or [15] for locally closed immersions) and by [12] for the DM-type case.

Example 1. (i) Let $G$ be a DM-stack, $E$ a vector bundle on $G$ and $G \to E$ the zero section. Let $V$ be an étale atlas of $G$ and $E_V$ the pull-back of $E$ to $V$, then we can construct a commutative diagram as above and $C_{G/E}$ is obtained by descent from $C_{V/E_V} \simeq E_V$. This shows that $C_{G/E}$ is precisely $E$.

(ii) Let $F \to G$ be a DM-type morphism and $p : E \to G$ a vector bundle on $G$. If $i : G \to E$ is the zero section, then we have a morphism of distinguished triangles corresponding to $f$ and $i$ respectively

$$f^*i^*L_E \to L_F \to L_{F/E} \to f^*i^*L_E[1].$$

Using $p$ instead of $i$ we obtain in the same way a morphism $L_{F/G} \to L_{F/E}$ and thus we get a morphism $f^*L_{G/E} \oplus L_{F/G} \to L_{F/E}$. To show it is an isomorphism it suffices to show the statement locally. As we may assume $G$ is an affine scheme, it is easy to see that $i^*L_E = L_G \oplus E$. On the other hand, $L_{G/E} = [E \to 0]$, where $E$ stays in degree $-1$ and therefore we reduced the problem to showing the triangle

$$f^*L_G \oplus E \to L_F \to L_{F/G} \oplus [E \to 0]$$

is distinguished. But this follows trivially from the definition of the mapping cone. This shows that $h^1/h^0(L_{F/E}^\psi) \simeq h^1/h^0(L_{F/G}^\psi) \times_F h^1/h^0(f^*L_{G/E}^\psi)$. We have thus obtained $C_{F/E}$ is isomorphic to $C_{F/G \times_F f^*E}$.

(iii) Let $F \to G$ be DM-type morphism and $\mathcal{E} := E^1/E^0$ a vector bundle stack on $G$. It is easy to see that $C_{F/E} = C_{F/E^1/E^0}$. Using (ii) above for $C_{F/E^1}$, we obtain that the normal cone of $F$ in $\mathcal{E}$ is isomorphic to $C_{F/G \times_F f^*\mathcal{E}}$.

(iv) Let $F \to G$ be a smooth morphism of DM-stacks. Let $W$ be an étale atlas of $F$. Then we can take $U = V := W$. As above, locally the normal cone $C_{F/G}$ is given by $W/T_{W/G}$, that is a vector bundle stack.

(v) Let $X \to Y$ be a morphism of smooth schemes. Then, $U$ and $V$ above
can be taken to be $X$ and $X \times Y$ as below

\[
\begin{array}{c}
X \\ \downarrow \id \times f \\
X \times Y \\ \downarrow \pi_2 \\
X \\ Y
\end{array}
\]

where $\pi_2$ is the projection on $Y$. It is then easy to see that the normal cone is $[N_{X/X \times Y}/T_X]$ that is a vector bundle stack.

**Lemma 2.** Let

\[
\begin{array}{c}
F' \\ p \\
F \\ f \\
\downarrow \\
G' \\
q \\
G
\end{array}
\]

be a cartesian diagram of Artin stacks with $f$ of DM-type. Then, there exists a closed immersion $C_{F'/G'} \hookrightarrow p^*C_{F/G}$.

**Proof.** By either [2] Proposition 7.1 or [14]. ☐

**Remark 2.** Theorem 1 shows that whenever $G$ is purely dimensional of dimension $r$, $C_{F/G}$ is again purely dimensional of dimension $r$.

**Construction 1.** Let $F$ be a DM-stack and $F \to G$ be a DM-type morphism of algebraic stacks. Given a vector bundle stack $E$ of (virtual) rank $n$ on $F$ and a closed immersion $C_{F/G} \hookrightarrow E$, we construct a pull-back map $i^!_{E} : A_*(G) \to A_{*-n}(F)$ as the composition

\[
A_*(G) \xrightarrow{\sigma} A_*(C_{F/G}) \xrightarrow{i_*} A_*(E) \xrightarrow{s^*} A_{*-n}(F).
\]

The first map is defined on the level of cycles by $\sigma(\sum n_i[V_i]) = \sum n_i[C_{W \cap V_i}/V_i]$ and is a consequence of Lemma 1 (see [14]). The second map is just the push-forward via the closed immersion $i$ and the last is the morphism of [14], Proposition 5.3.2.

Going further, for any cartesian diagram

\[
\begin{array}{c}
F' \\ p \\
F \\ f \\
\downarrow \\
G' \\
q \\
G
\end{array}
\]

let $f^!_{E} : A_*(G') \to A_{*-n}(F')$ be the composition

\[
A_*(G') \xrightarrow{\sigma} A_*(C_{F'/G'}) \xrightarrow{i_*} A_*(C_{F/G} \times_F F') \xrightarrow{i_*} A_*(p^*E) \xrightarrow{s^*} A_{*-n}(F').
\]
Remark 3. Note that in case $X$, $Y$ are schemes such that $X$ is regularly embedded in $Y$, then the normal bundle of $X$ in $Y$ satisfies condition $\star$ and $i^!_{N_X/Y}$ is precisely the Gysin pull-back. We remark that the pull-back depends on the chosen bundle. For example, if $E$ is a bundle satisfying condition $(\star)$ we can construct $i^!_{E\oplus E'}$, where $E'$ is any other vector bundle. These morphisms will be obviously different from each other.

Remark 4. If $X \rightarrow Y$ is a smooth morphism of schemes, then by Example 1 (iv) $C_{X/Y}$ is a vector bundle stack and hence we can construct its generalized Gysin pull-back. We will show later that our definition agrees with the usual flat pull-back.

2.2 Obstruction Theories

The purpose of this section is to produce non-trivial examples. Precisely, when the normal cone of $F \rightarrow G$ is not a vector bundle stack we will find under extra assumptions a vector bundle stack that contains it.

By a relative obstruction theory we mean a relative obstruction theory in the sense of [2].

Corollary 1. If $F \rightarrow G$ is a DM-type morphism and there exists a perfect relative obstruction theory $E^\bullet_{F/G}$, then condition $(\star)$ is fulfilled.

Proof. By Lemma 1 the normal sheaf $N_{F/G}$ is nothing but $\mathcal{N}_{F/G}$. On the other hand we know that $N_{F/G}$ embeds in $E^\bullet_{F/G} := h^1/\pi^0(E^\bullet_{F/G})$ ([2], Proposition 2.6.). Our condition on the relative obstruction theory ensures $E^\bullet_{F/G}$ is a vector bundle stack ([ibid.]).

Construction 2. Let us now assume $F$ and $G$ are DM-stacks and have relative obstruction theories with respect to some smooth Artin stack $\mathcal{M}$. Let us denote them by $E^\bullet_{F/\mathcal{M}}$ and $E^\bullet_{G/\mathcal{M}}$ respectively. Given a morphism $\varphi : f^*E^\bullet_{G/\mathcal{M}} \rightarrow E^\bullet_{F/\mathcal{M}}$ commuting with $f^*L_{G/\mathcal{M}} \rightarrow L_{F/\mathcal{M}}$, we construct a relative obstruction theory $E^\bullet_{F/G}$.

The morphism $f : F \rightarrow G$ induces a distinguished triangle of cotangent complexes

$$f^*L_{G/\mathcal{M}} \rightarrow L_{F/\mathcal{M}} \rightarrow L_{F/G} \rightarrow f^*L_{G/\mathcal{M}}[1].$$

Similarly, $\varphi$ gives rise to a distinguished triangle

$$f^*E^\bullet_{G/\mathcal{M}} \varphi \rightarrow E^\bullet_{F/\mathcal{M}} \rightarrow E^\bullet_{F/G} \rightarrow f^*E^\bullet_{G/\mathcal{M}}[1]$$  \hspace{1cm} (5)

hence we have a morphism of distinguished triangles that induces the following morphism in cohomology

$$
\begin{align*}
h^{-1}(f^*E^\bullet_{G/\mathcal{M}}) &\rightarrow h^{-1}(E^\bullet_{F/\mathcal{M}}) \rightarrow h^{-1}(E^\bullet_{F/G}) \\
h^0(f^*E^\bullet_{G/\mathcal{M}}) &\rightarrow h^0(E^\bullet_{F/\mathcal{M}}) \rightarrow h^0(E^\bullet_{F/G}) \\
h^{-1}(f^*L^\bullet_{G/\mathcal{M}}) &\rightarrow h^{-1}(L^\bullet_{F/\mathcal{M}}) \rightarrow h^{-1}(L^\bullet_{F/G}) \\
h^0(f^*L^\bullet_{G/\mathcal{M}}) &\rightarrow h^0(L^\bullet_{F/\mathcal{M}}) \rightarrow h^0(L^\bullet_{F/G})
\end{align*}
$$
We know that the first vertical arrows are surjective and by the definition of obstruction theories we get by a simple diagram chase that \( E_{F/G} \) is also an obstruction theory.

**Example 2.** A special case of this construction is when \( F \to G \) is a locally closed immersion and \( G \) is taken to be smooth over \( \mathfrak{M} \). Then, \( h^{-1}(f^*E_{G/\mathfrak{M}}) = 0 \) and this shows that \( h^{-2}(E_{F/G}) = 0 \). This makes \( E_{F/G} \) into a perfect obstruction theory concentrated in degree \(-1\) and consequently \( \mathfrak{E} \) into a vector bundle.

**Remark 5.** The condition we impose seems to be very restrictive. For example if we consider \( \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(\mathbb{P}^r,i_*\beta) \) the above construction applies without further conditions only in genus zero because in general \( h^{-2}(E_{\overline{M}_{g,n}(X,\beta)}/\overline{M}_{g,n}(\mathbb{P}^r,i_*\beta)) \) might not vanish. In general, the relative obstruction theory induced by \( i \) will not be perfect in higher genus. Let us consider \( \mathbb{P}^r \hookrightarrow \mathbb{P}^r \times \mathbb{P}^s \). Then we have an induced map \( \overline{M}_{g,n}(\mathbb{P}^r,d_1) \hookrightarrow \overline{M}_{g,n}(\mathbb{P}^r \times \mathbb{P}^s,(d_1,0)) \). A simple argument (see 4.2) shows that the dual relative obstruction theory we obtain is \( R^\bullet \pi_{*} f^* \mathcal{O}_{\mathbb{P}^r/\mathbb{P}^r \times \mathbb{P}^s} \).

We have that the normal bundle \( \mathcal{N}_{\mathbb{P}^r/\mathbb{P}^r \times \mathbb{P}^s} \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^s} \). Since the map \( H^0(\mathcal{O}_{\mathbb{P}^r}^{\oplus s}) \to H^1(\mathcal{O}_{\mathbb{P}^s}^{\oplus s}) \) is obviously not surjective for \( g \geq 2 \), the (dual) relative obstruction theory will never be perfect.

**Definition 4.** In the notation above, we call \( f_{E_{F/G}}^!: A_*(G) \to A_*(F) \) a virtual pull-back. When there is no risk of confusion we will omit the index.

Let us motivate the definition. For this, let us assume \( E_{F/G} \) and \( E_{G/G} \) are perfect in \([-1,0] \). Then on \( F \) and \( G \) we have well defined virtual classes \([F]^\text{virt} \) and \([G]^\text{virt} \) respectively and we will show in the following that \( f_{E_{F/G}}^! \) sends the virtual class of \( G \) to the virtual class of \( F \). As remarked in the previous example, the situation is particularly nice when \( G \) is taken to be smooth over \( \mathfrak{M} \).

**Example 3.** The basic case.

In the notation above, let us suppose \( G \) is smooth and \( F \hookrightarrow G \) is a closed substack and there exists a morphism \( f^*L_G \to E_F \). For simplicity we take \( \mathfrak{M} \) to be \( \text{Spec} \ k \). Then we have

(i) \((C_{F/G}, E_{F/G}) \) induces the same virtual class on \( F \) as \((C_F, E_F) \).

(ii) The pull back defined by \( E_{F/G} \) respects the relation

\[ i^![G] = [F]^{\text{virt}}. \]

**Proof.** As \( G \) is smooth, the intrinsic normal cone \( \mathfrak{C}_F \) defined in \([2]\) is nothing but \([C_{F/G}/T_G] \). Moreover, \( i^*L_G \) can be represented by a complex concentrated in \( 0 \) and \( E_{F/G} \) by a complex concentrated in \(-1\). By abuse of notation, we will indicate the corresponding sheaves by \( i^*L_G \) and \( E_{F/G} \) respectively.
Taking the long exact cohomology sequence of the exact triangle \((5)\), we see that \(E^*_F\) is quasi isomorphic to \([E_{F/G} \to i^*L_G]\). Therefore the vector bundle stack \(\mathcal{E}_F := h^1/h^0((E^*_F)^\vee)\) is equal to \([E_{F/G}^\vee/T_G]\). Thus we have the diagram with cartesian faces

\[
\begin{array}{c}
F \\
\downarrow \\
E_{F/G}
\end{array}
\quad
\begin{array}{c}
C_F \\
\downarrow \\
C_{F/G}
\end{array}
\quad
\begin{array}{c}
\mathcal{E}_F \\
\downarrow \\
\end{array}
\quad
\begin{array}{c}
F \to E_{F/G}
\end{array}
\]

that is another way of saying that the morphism \(A_*(\mathcal{E}_F) \to A_*(F)\) factorizes through \(A_*(E_{F/G})\) as follows:

\[
A_*(\mathcal{E}_F) \to A_*(E_{F/G}) \to A_*(F) \quad [C_F] \mapsto [C_{F/G}] \mapsto [F]^{\text{virt}}.
\]

For the second statement, we just have to note that by our definition \(i^!G = s^*([C_{F/G}])\), and by (i) is precisely \([F]^{\text{virt}}\) as defined in [2].

\[\Box\]

### 2.3 Generalization

**Restricted virtual pull-backs.** As we already remarked, perfect relative obstruction theories are not likely to exist in general. However, if we restrict ourselves to a smaller group, then under fair assumptions we can still define a pullback. Let us make this precise.

Let \(F, G\) be DM-stacks as in Corollary [4] that admit perfect obstruction theories to some (smooth) Artin stack \(\mathfrak{M}\) and let us denote the image of the virtual pull-back \(f^{1}_{\mathcal{E}/\mathfrak{M}} A_*(\mathfrak{M}) \to A_*(G)\) by \(A_{\mathfrak{M}}(G)\). For any \(\alpha \in A_{\mathfrak{M}}(G)\) there exists \(\beta \in A_*(\mathfrak{M})\) such that

\[
\alpha = p^1_{\mathcal{E}/\mathfrak{M}}(\beta).
\]

Then we define the **restricted virtual pull-back** \(f^1 : A_{\mathfrak{M}}(G) \to A_*(F)\) by

\[
f^1(\alpha) = p^1_{\mathcal{E}/F}(\beta).
\]

### 3 Basic properties

Once we have defined a “pull-back”, we want to show it has the usual properties. Due to the geometric properties of the normal cone \([1]\), the proofs follow essentially in the same way as the ones in \([5]\). The fact that our pull-back defines a bivariant class is analogous to Example 17.6.4 in \([5]\). The only
point we need to be careful at is the functoriality property, where we need a compatibility condition between the vector bundle (stacks) that replace the normal bundles.

**Theorem 2.** Consider a fibre diagram of Artin stacks

\[
\begin{array}{ccc}
F'' & \longrightarrow & G'' \\
\downarrow q & & \downarrow p \\
F' & \overset{f'}{\longrightarrow} & G' \\
\downarrow g & & \downarrow h \\
F & \overset{f}{\longrightarrow} & G
\end{array}
\]

and let us assume \( f \) is a DM-type morphism and \( \mathcal{E} \) is a vector bundle stack of rank \( d \) that satisfies condition \((\ast)\) for \( f \).

(i) (Push-forward) If \( p \) is proper and \( \alpha \in A_k(G'') \), then \( f'_*\mathcal{E}(\alpha) = q_* f'^_* \mathcal{E}(\alpha) \) in \( A_{k-d}(F') \).

(ii) (Pull-back) If \( p \) is flat of relative dimension \( n \) and \( \alpha \in A_k(G') \), then \( f'^*_\mathcal{E}(\alpha) = q^* f'_* \mathcal{E}(\alpha) \) in \( A_{k+n-d}(F'') \)

(iii) (Compatibility) If \( \alpha \in A_k(G'') \), then \( f'_* \mathcal{E}(\alpha) = f'^*_\mathcal{E}(\alpha) \) in \( A_{k-d}(F'') \).

**Proof.** (i) It is enough to show that the diagram of groups commutes

\[
\begin{array}{ccc}
A_*(G'') & \longrightarrow & A_*(C_{F''/G''}) \\
\downarrow p_* & & \downarrow Q \\
A_*(G') & \longrightarrow & A_*(C_{F'/G'})
\end{array}
\]

where \( Q \) is the map induced by the map between the deformation spaces \( M^\circ_{F''} \rightarrow M^\circ_{F'} \). But this follows in the same way as Prop 4.2 in [5].

(ii) Let us choose a cycle \([Z]\) representing \( \alpha \) and let \( U \) be the fiber product \( p^{-1}(Z) \times_{G'} F' \). We need to show that \( q^{-1} C_{F'/G'} = C_{U/p^{-1}Z} \), which is clear.

(iii) Is obvious. \( \square \)

**Remark 6.** As remarked before, the generalized Gysin pull-back is well-defined for smooth pull-backs. Let us show that the two definitions agree. By (i) above, it is enough to prove the claim for \( \alpha = [G] \), for which it follows trivially by construction.

**Theorem 3.** (Commutativity) Consider the fiber diagram of Artin stacks

\[
\begin{array}{ccc}
F'' & \overset{v}{\longrightarrow} & G'' \\
\downarrow q & & \downarrow u \\
F' & \longrightarrow & G' \\
\downarrow g & & \downarrow h \\
F & \overset{f}{\longrightarrow} & G
\end{array}
\]
Let us assume \( f \) and \( g \) are morphisms of DM-type and let \( \mathfrak{E} \) and \( \mathfrak{F} \) be vector bundle stacks of rank \( d \), respectively \( e \) that satisfy condition \((\star)\) for \( f \), respectively \( g \). Then for all \( \alpha \in A_k(G') \),

\[
g_\mathfrak{F}^! f_\mathfrak{E}^!(\alpha) = f_\mathfrak{E}^! g_\mathfrak{F}^!(\alpha)
\]
in \( A_{k-d-e}(F'^v) \).

**Proof.** Using Theorem [2] we may assume \( \alpha = [G'] \). We see that the pull-back of \( g_\mathfrak{F}^! f_\mathfrak{E}^![G'] \) to \( p^*q^* \mathfrak{E} \oplus v^*u^* \mathfrak{F} \) is equal to \( C_{C_{F'/G'} \times G'/G'} \) and the pull-back of \( f_\mathfrak{E}^! g_\mathfrak{F}^![G'] \) to \( p^*q^* \mathfrak{E} \oplus v^*u^* \mathfrak{F} \) is equal to \( C_{C_{G'/G'} \times G'/G'} \). By Vistoli’s rational equivalence (see [14] or [15])

\[
[C_{C_{F'/G'} \times G'/G'}] = [C_{C_{G'/G'} \times G'/G'}]
\]
in \( A_*(C_{F'/G'} \times G'/G') \). This equivalence pushes forward to \( A_*(p^*q^* \mathfrak{E} \oplus v^*u^* \mathfrak{F}) \) and therefore the conclusion. \( \square \)

**Corollary 2.** In notations as in Construction 2, \( f \) defines a bivariant class in the sense of [3], Definition 17.1.

**Corollary 3.** (Projection formula) Let \( F \overset{f}{\to} G \) a proper morphism of DM-type that satisfies \((\star)\). Then for any \( \alpha \in A^*(G) \) and \( \beta \in A_*(F) \)

\[
\alpha \cdot f^*(\beta) = f^*(\alpha) \cdot \beta
\]
in \( A_*(G) \).

**Proof.** By Corollary 2 all we need to do is to use the projection formula for bivariant classes in [18]. \( \square \)

**Definition 5.** Let \( F \overset{f}{\to} G \overset{g}{\to} \mathfrak{M} \) morphisms of stacks. If there exists a distinguished triangle of relative obstruction theories which are perfect in \([-1, 0]\]

\[
g^*E_{G/\mathfrak{M}} \xrightarrow{\varphi} E_{F/\mathfrak{M}} \to E_{F/G} \to g^*E_{G/\mathfrak{M}}[1]
\]

with a morphism to the distinguished triangle

\[
g^*L_{G/\mathfrak{M}} \to L_{F/\mathfrak{M}} \to L_{F/G} \to g^*L_{G/\mathfrak{M}}[1],
\]

then we call \((E_{F/G}^\bullet, E_{G/\mathfrak{M}}^\bullet, E_{F/\mathfrak{M}}^\bullet)\) a compatible triple.

**Remark 7.** As in Construction 2 if there is a morphism \( E_{F/G}^\bullet \overset{\psi}{\to} g^*E_{G/\mathfrak{M}}^\bullet[1] \) compatible with the corresponding morphism between the cotangent complexes, then \( \psi \) determines a complex \( E_{F/\mathfrak{M}}^\bullet \) that fits in a distinguished triangle as above. Moreover, \( E_{F/\mathfrak{M}}^\bullet \) defines a relative obstruction theory. If \( E_{F/G}^\bullet \) and \( E_{G/\mathfrak{M}}^\bullet \) are perfect, then \( E_{F/\mathfrak{M}}^\bullet \) is perfect.
Lemma 3. Consider a fibre diagram

\[
\begin{array}{ccc}
F' & \overset{f'}{\longrightarrow} & G' \\
\downarrow p & & \downarrow q \\
F & \overset{f}{\longrightarrow} & G \\
\downarrow r & & \downarrow \theta
\end{array}
\]

with \( f \) a morphism of DM-type, \( F \) a vector bundle stack and \( F' \) its pullback to \( G' \). Let us assume there exists a vector bundle stack \( E' \) that satisfies condition (\( \star \)) for \( f \). Then

\[
C_{F/\theta} \to E := E' \oplus f'^*\theta
\]

and for any \( \alpha \in A_k(\theta') \)

\[
(0 \circ f')^1[\theta'] = f'^1(0_\theta^1(\alpha)).
\]

Proof. For the first part it suffices to show that \( C_{F'/\theta'} \) is canonically isomorphic to \( C_{F'/G'} \times_{F'} (C_{G/\theta} \times G') \), that is example (iii).

The equality follows in the same way as in [ibid.] Let us notice that by theorem (i) and the homotopy property for vector bundle stacks (13) we may assume \( \alpha \) to be represented by \( \theta' \) and \( G' \) can be taken to be irreducible.

Now, the problem reduces to

\[
(0 \circ f')^1[\theta'] = f'^1[G'].
\]

If \( \pi_1 : E \to p^*E' \) and \( \pi_2 : E' \to F' \) are the natural projections, then we have by the above

\[
[C_{F'/\theta'}] = [\pi_1^* C_{F'/G'}] \in A_*(E).
\]

From the construction of Gysin pull-backs

\[
[C_{F'/G'}] = \pi_2^* f'^1[G'] \in A_*(p^*E')
\]

and

\[
[C_{F'/\theta'}] = (\pi_1 \pi_2)^* (0f'^1)[\theta'] \in A_*(E).
\]

Combining the three equalities we get equality (6) above, and therefore the conclusion.

Theorem 4. (Functoriality) Consider a fibre diagram

\[
\begin{array}{ccc}
F' & \overset{f'}{\longrightarrow} & G' \\
\downarrow p & & \downarrow q \\
F & \overset{f}{\longrightarrow} & G \\
\downarrow r & & \downarrow \theta
\end{array}
\]

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Let us assume \( f, g \) and \( g \circ f \) have perfect relative obstruction theories \( E'^\bullet \), \( E''^\bullet \) and \( E^\bullet \) respectively and let us denote the associated vector bundle stacks by \( \mathcal{E}' \), \( \mathcal{E}'' \) and \( \mathcal{E} \) respectively. If \( (E'^\bullet, E''^\bullet, E^\bullet) \) is a compatible triple, then for any \( \alpha \in A_k(\mathcal{M}') \)
\[
(g \circ f)^!_\mathcal{E}(\alpha) = f^!_{\mathcal{E}'}(g^!_{\mathcal{E}''}(\alpha)).
\]

**Proof.** We argue as in the proof of Theorem 1 in [12] (or Theorem 6.5 of [5]). In the same way as in the proof of the previous lemma \( \mathcal{M}' \) may be assumed irreducible and reduced and \( \alpha = [\mathcal{M}'] \).

Consider the vector bundle stacks: \( \rho : \mathcal{E} \to \mathcal{F} \), \( \pi : \mathcal{E}'' \to \mathcal{G} \) and \( \sigma : \mathcal{E}' \oplus f^* \mathcal{E}'' \to \mathcal{F} \).

By definition
\[
(g \circ f)^1 \mathcal{M}' = (\rho^*)^{-1}([C_F'/\mathcal{M}'])
\]
\[
g^1 \mathcal{M}' = (\pi^*)^{-1}([C_G'/\mathcal{M}']).
\]

Let us now look at the cartesian diagram

\[
\begin{array}{ccc}
F' & \xrightarrow{f'} & G' \\
\downarrow & & \downarrow \\
F'' & \xrightarrow{f''} & G''
\end{array}
\]

From the definition of the pull-back we know that \( f^1(g^1 \mathcal{M}') \) is equal to \( f^1(0^! [C_G'/\mathcal{M}']) \) and by the previous lemma
\[
f^1(0^! [C_G'/\mathcal{M}']) = (0 \circ f)^1 [C_G'/\mathcal{M}'].
\]

If we denote \( C_{G'/\mathcal{M}} \) by \( C_0 \), then the above shows that \( f^1(g^1 \mathcal{M}') \) is represented in \( \mathcal{E}_{F'/G} \oplus f^* \mathcal{E}_{G'/\mathcal{M}} \) by the cycle \( [C_{F'/C_0}] \). The construction respects equivalence in Chow groups and so we are reduced to showing
\[
(\sigma^*)^{-1}([C_{F'/C_0}]) = (\rho^*)^{-1}([C_{F'/\mathcal{M}}])
\]

in \( A_*(F') \).

Introduce the double deformation space \( M' := M^\circ_{F'} \times \mathbb{P}^1 / M^\circ_{G'/\mathcal{M}'} \to \mathbb{P}^1 \times \mathbb{P}^1 \) with general fiber \( M^\circ_{G'/\mathcal{M}'} \) and special fibre \( C_{F'/\mathbb{P}^1 / M^\circ_{G'/\mathcal{M}'}} \) over \( \{0\} \times \mathbb{P}^1 \) (see [12], proof of Theorem 1). Restricting to this special fibre and considering the rational equivalence on the second \( \mathbb{P}^1 \) we see that
\[
[C_{F'/C_0}] \sim [C_{F'/\mathcal{M}}]
\]
in $A_s(C_{F \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x})$.

In a completely analogous fashion there exists a double deformation space $M := M_{F \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x}$. If we consider the map $w : F' \times \mathbb{P}^1 \overset{p \times 1_{\mathbb{P}^1}}{\rightarrow} F \times \mathbb{P}^1$, then the general fibers of $M$ and $M'$ are related by the cartesian diagram

$$
\begin{array}{ccc}
F' \times \mathbb{P}^1 & \rightarrow & M_{G/\mathfrak{g}/\mathfrak{m}^\circ} \\
\downarrow w & & \downarrow \\
F \times \mathbb{P}^1 & \rightarrow & M_{G/\mathfrak{g}/\mathfrak{m}^\circ}.
\end{array}
$$

This implies $C_{F' \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x} \overset{i}{\rightarrow} (p \times 1_{\mathbb{P}^1})^*C_{F \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x}$ is a closed immersion and consequently we can push forward relation (\ref{equation:relation}) in $A_s(w^*C_{F \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x})$.

Now, by Proposition 1, in \cite{[12]}, we have a morphism

$$
A_s(C_{F \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x}) \overset{i^*}{\rightarrow} A_s(h^1/h^0(c(u)^\vee))
$$

where $u := (T \cdot id, U \cdot can)$ is the map

$$
f^*L_{G/\mathfrak{g}/\mathfrak{m}} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \overset{u}{\rightarrow} f^*L_{G/\mathfrak{g}/\mathfrak{m}} \oplus L_{F/\mathfrak{g}/\mathfrak{m}}
$$

in $\mathcal{D}(F \times \mathbb{P}^1)$ and $c(u)$ its mapping cone. Here we denoted by $T$ and $U$ the homogeneous coordinates on $\mathbb{P}^1$. Let us consider the closed immersion

$$w^*i : A_s(w^*C_{F \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x}) \hookrightarrow A_s(w^*h^1/h^0(c(u)^\vee)).$$

Then pushing forward via $w^*i$ the equivalence relation we have in $A_s(w^*C_{F \times \mathbb{P}^1 \times M_{G/\mathfrak{g}/\mathfrak{m}^\circ}^x})$, we obtain the equivalence relation (\ref{equation:relation}) in $A_s(w^*h^1/h^0(c(u)^\vee))$.

Let us now use the notation of Construction \cite{[2]}. Consider the morphism $v := (T \cdot id, U \cdot \cdot) : f^*E_{G/\mathfrak{g}/\mathfrak{m}} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow f^*E_{G/\mathfrak{g}/\mathfrak{m}} \oplus E_{F/\mathfrak{g}/\mathfrak{m}}$ in $\mathcal{D}(F \times \mathbb{P}^1)$.

The morphism of distinguished triangles in Definition \ref{definition:distinguished_triangle} gives a morphism of distinguished triangles

$$
\begin{array}{ccc}
(fp)^*E_{G/\mathfrak{g}/\mathfrak{m}}(-1) & \overset{w^*v}{\rightarrow} & (fp)^*E_{G/\mathfrak{g}/\mathfrak{m}} \oplus p^*E^* \overset{w^*c(v)}{\rightarrow} (fp)^*E_{G/\mathfrak{g}/\mathfrak{m}}(-1)[1] \\
\downarrow & & \downarrow \\
(fp)^*L_{G/\mathfrak{g}/\mathfrak{m}}(-1) & \overset{w^*u}{\rightarrow} & (fp)^*L_{G/\mathfrak{g}/\mathfrak{m}} \oplus p^*L_{F/\mathfrak{g}/\mathfrak{m}} \overset{w^*c(u)}{\rightarrow} (fp)^*L_{G/\mathfrak{g}/\mathfrak{m}}(-1)[1]
\end{array}
$$

over $F' \times \mathbb{P}^1$. Dualizing and taking $h^1/h^0$ of the map $w^*c(v) \rightarrow w^*c(u)$, we obtain a morphism of Picard stacks $w^*h^1/h^0(c(u)^\vee) \rightarrow w^*h^1/h^0(c(v)^\vee)$ that is a closed immersion. Therefore, we can push forward the rational equivalence (\ref{equation:relation}) on $w^*h^1/h^0(c(v)^\vee)$ that is a vector bundle stack on $F' \times \mathbb{P}^1$. The fact that the above map between cone stacks is a closed immersion follows from Prop 2.6 in \cite{[2]} and the fact that the maps in cohomology induced by the vertical maps in the above diagram are isomorphisms in degree 0 and
surjective in degree $-1$.

Let us now conclude the proof. We have obtained $[C_{F'/C_0}] \sim [C_{F'/\mathbb{P}^1}]$ in $A_*(w^*h^1/h^0(c(v)^\vee))$. Looking at $w^*h^1/h^0(c(v)^\vee) \to \mathbb{P}^1$, we see that $w^*h^1/h^0(c(v)^\vee)$ restricts to $F_0 := p^*\mathcal{E}' \oplus p^*f^*\mathcal{E}''$ and $F_1 := p^*\mathcal{E}$ in $F' \times \{0\}$ respectively $F' \times \{1\}$. As the map

$$A_*(w^*h^1/h^0(c(v)^\vee)) \to A_*(F_i) \to F'$$

does not depend on $i$ we deduce equality (7).

**Corollary 4.** In the notation of Construction 2, let us suppose we have a commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{f} & G \\
\downarrow p & & \downarrow p \\
\mathcal{M} & \xrightarrow{g} & \mathcal{M} \\
\end{array}
$$

with $\mathcal{M}$ purely dimensional. If $E_{F/G}^\bullet$ is perfect, then

$$f^!_{E_{F/G}}[G]^\text{virt} = [F]^\text{virt}.$$

**Proof.** By the definition of virtual classes we have

$$[F]^\text{virt} = p^!_{E_{F/\mathcal{M}}}[\mathcal{M}]$$

$$[G]^\text{virt} = p^!_{E_{G/\mathcal{M}}}[\mathcal{M}].$$

Moreover, by the construction of $E_{F/G}$ we are in the hypotheses of Theorem 4 and therefore

$$(p \circ f)^!_{E_{F/\mathcal{M}}}[\mathcal{M}] = f^!_{E_{F/G}}p^!_{E_{G/\mathcal{M}}}[\mathcal{M}].$$

The two equations above show that $f^!_{E_{F/G}}[G]^\text{virt} = [F]^\text{virt}.$

**Remark 8.** Let us consider a cartesian diagram of DM stacks

$$
\begin{array}{ccc}
F' & \xrightarrow{g} & G' \\
\downarrow & & \downarrow f \\
F & \xrightarrow{i} & G \\
\end{array}
$$

with obstruction theories $E_{F'}^\bullet$, $E_{G'}^\bullet$, $E_{F'}^\bullet$, $E_{G'}^\bullet$ and let us assume $E_{F/G}^\bullet$ and $E_{F'/G'}^\bullet$ exist and are perfect. If $g^*\mathcal{E}_{F/G} = \mathcal{E}_{F'/G'}$, then $i'^![G']^\text{virt} = [F']^\text{virt}.$

This is a version of Proposition 5.10 in [2] and Theorem 1 in [12]. The advantage is that looking at obstruction theories is much easier than looking at cotangent complexes, that in general are difficult to compute.
4 Applications

In this section we collect some applications of the virtual pull-back we defined. We take the ground field to be $\mathbb{C}$.

4.1 Pulling back divisors

Let $\mathbb{P}$ be a convex variety and $d \in A_1(\mathbb{P})$ be the class of a curve. If $X \hookrightarrow \mathbb{P}$ is an embedding of smooth projective varieties, then $i$ induces a morphism $\overline{M}_{g,n}(X,d) \xrightarrow{i} \overline{M}_{g,n}(\mathbb{P},d)$ where we made the convention that $\overline{M}_{0,n}(X,d)$ is the union of all $\overline{M}_{0,n}(X,\beta)$ such that $i_*\beta = d$. Let $D_\mathbb{P} := D_\mathbb{P}(g_1, n_1, d_1 | g_2, n_2, d_2)$ be a boundary divisor in $\overline{M}_{g,n}(\mathbb{P},d)$ that comes with a virtual class obtained by pull-back along the obvious forgetful morphism $D_\mathbb{P} \to \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1}$ and analogously we have a boundary divisor $D_X := D_X(g_1, n_1, d_1 | g_2, n_2, d_2)$ in $\overline{M}_{g,n}(X,d)$ equipped with a virtual fundamental class. Constructing the following cartesian diagram

$$
\begin{array}{ccc}
D_X & \rightarrow & D_\mathbb{P} \\
\downarrow & & \downarrow \\
\overline{M}_{g,n}(X,d) & \xrightarrow{i} & \overline{M}_{g,n}(\mathbb{P},d) \\
\downarrow & & \downarrow \\
\overline{M} & \rightarrow & \overline{M}
\end{array}
$$

we get

$$i^! [D_\mathbb{P}(g_1, n_1, d_1 | g_2, n_2, d_2)]^{\text{virt}} = [D_X(g_1, n_1, d_1 | g_2, n_2, d_2)]^{\text{virt}}.$$ 

Indeed, in genus zero the obstructions are compatible and in higher genus the claim is tautological.

Remark 9. The above shows that for any $X \hookrightarrow \mathbb{P}$ pulling-back the WDVV equations on $\overline{M}_{g,n}(\mathbb{P},d)$ gives the WDVV equations on $\overline{M}_{g,n}(X,d)$. In particular, it is hopeless to expect we can compute the rational GWI of $X$ with $K_X$ “negative enough” for any arbitrary $X$ (see for example [7], Chapter 3) by simply ”pulling back” relations from $\overline{M}_{g,n}(\mathbb{P},d)$ (see for example [7], Proposition 1.3.10).

4.2 Blow-up

Let $X$ be a smooth $r$-dimensional projective variety, $Y \subseteq X$ a smooth $r'$-codimensional subvariety and $p : \tilde{X} \to X$ the blow-up of $X$ in $Y$, with exceptional divisor $E$.

Definition 6. For every blow up $p : \tilde{X} \to X$ and every class $\beta \in A_1(X)$ we call the class $p^*\beta$ the lifting of $\beta$ and we denote it by $\tilde{\beta}$. 

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Remark 10. The lifting of $\beta$ satisfies two basic properties that follow trivially from the projection formula, namely $p_* \tilde{\beta} = \beta$ and $\tilde{\beta} \cdot E = 0$.

Lemma 4. The moduli space of stable maps to $\tilde{X}$ of class $\tilde{\beta}$ and the moduli space of stable maps to $X$ of class $\beta$ have the same virtual dimension.

Proof. By [5] we know that

$$K_{\tilde{X}} = p^* K_X + (r' - 1)E$$

and therefore the virtual dimension of $\mathcal{M}_{g,n}(\tilde{X}, \tilde{\beta})$ is

$$\text{vdim}(\mathcal{M}_{g,n}(\tilde{X}, \tilde{\beta})) = (1 - g)(r - 3) - K_{\tilde{X}} \cdot \tilde{\beta} + n$$

$$= (1 - g)(r - 3) - [p^* K_X + (r' - 1)E] \cdot \tilde{\beta} + n$$

$$= (1 - g)(r - 3) - p^* K_X \cdot \tilde{\beta} + n$$

$$= (1 - g)(r - 3) - K_X \cdot \beta + n$$

$$= \text{vdim}(\mathcal{M}_{g,n}(X, \beta)).$$

Lemma 5. If $P$ is convex then $p_*[\mathcal{M}_{0,n}(\tilde{P}, \tilde{\beta})]^\text{virt} = [\mathcal{M}_{0,n}(P, \beta)]^\text{virt}.$

Proof. The proof is a straightforward generalization of [6], Proposition 2.2. Let us sketch here the argument for completeness. Since $P$ is convex the stack $\mathcal{M}_{0,n}(P, \beta)$ is smooth of expected dimension $d$. Let $Z_1, ..., Z_k$ the connected components of $\mathcal{M}_{0,n}(P, \beta)$. As $\mathcal{M}_{0,n}(P, \beta)$ has expected dimension $d$ we have

$$p_*[\mathcal{M}_{0,n}(\tilde{P}, \tilde{\beta})]^\text{virt} = \alpha_1[Z_1] + ... + \alpha_k[Z_k]$$

for some $\alpha_i \in \mathbb{Q}$. If we show that $p$ is a local isomorphism around a generic point $C := (C, x_1, ..., x_n, f) \in Z_i$ for some $1 \leq i \leq k$ then by [6] we have

$$p_*[\mathcal{M}_{0,n}(\tilde{P}, \tilde{\beta})]^\text{virt} = [Z_1] + ... + [Z_k] = \mathcal{M}_{0,n}(P, \beta).$$

Let us suppose that for some generic $C \in Z_i$, $p^{-1}(C)$ is not a point. As $C$ is generic, we may assume that $C$ is irreducible. Then $f(C)$ must intersect the blown up locus and the subscheme $M$ of $\mathcal{M}_{0,n}(P, \beta)$ consisting of such maps must have dimension $d$. But $P$ is convex and $Y$ has codimension at least two, hence $M$ has codimension at least 1. This leads to a contradiction.

Proposition 1. Let $X$ be a smooth projective subvariety of some smooth projective convex variety $P$ and $Z$ a smooth subvariety of $P$, such that $X$ and $Z$ intersect transversely. Then for any non-negative integer $n$ and any $\beta \in A_1(\tilde{X})$ with lifting $\tilde{\beta} \in A_*(\tilde{X})$

$$p_*[\mathcal{M}_{0,n}(\tilde{X}, \tilde{\beta})]^\text{virt} = [\mathcal{M}_{0,n}(X, \beta)]^\text{virt}. $$
Proof. If \( Y = X \cap Z \), \( \tilde{X} \) is the blow-up of \( X \) along \( Y \) and \( \tilde{P} \) is the blow-up of \( \mathbb{P} \) along \( Z \) then the diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{i} & \tilde{P} \\
p & & \downarrow p \\
X & \xrightarrow{i} & \mathbb{P}
\end{array}
\]

is cartesian. Also, the rotated diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{p} & X \\
i & & i \\
\tilde{P} & \xrightarrow{p} & \mathbb{P}
\end{array}
\]

is cartesian and therefore we have

\[
p \ast \bar{\beta} = \beta
\]

and

\[
\tilde{i} \ast \beta = i \ast \bar{\beta}.
\]

We are interested in analyzing the moduli spaces of stable maps with these varieties as target spaces. The inclusions above induce morphisms between the moduli spaces of genus-zero stable maps with \( n \) marked points \( \overline{M}_{0,n}(\tilde{X}, \tilde{\beta}) \xrightarrow{i} \overline{M}_{0,n}(\tilde{P}, \tilde{i} \ast \bar{\beta}) \) and \( \overline{M}_{0,n}(X, \beta) \xrightarrow{i} \overline{M}_{0,n}(\mathbb{P}, i \ast \beta) \). Putting all together, we get a diagram of DM-stacks

\[
\begin{array}{ccc}
\overline{M}_{0,n}(\tilde{X}, \tilde{\beta}) & \xrightarrow{i} & \overline{M}_{0,n}(\mathbb{P}, \tilde{i} \ast \bar{\beta}) \\
p & & \downarrow p \\
\overline{M}_{0,n}(X, \beta) & \xrightarrow{i} & \overline{M}_{0,n}(\mathbb{P}, i \ast \beta).
\end{array}
\]

A closer look at the groupoid defining the spaces of stable maps shows that \( \overline{M}_{0,n}(\tilde{X}, \tilde{\beta}) = \overline{M}_{0,n}(X, \beta) \times_{\overline{M}_{0,n}(\mathbb{P}, i \ast \beta)} \overline{M}_{0,n}(\tilde{P}, \tilde{i} \ast \bar{\beta}) \). In order to apply the virtual push-forward machinery to this diagram, we first need to analyze the obstruction theories involved.

Let us denote the moduli space of genus-zero (pre-stable) curves with \( n \) marked points by \( \mathfrak{M} \). Then it is a well-known fact that the relative obstruction theories are \( E_{\overline{M}_{0,n}(\mathbb{P}, i \ast \beta)}/\mathfrak{M} \) := \( (\mathcal{R}^i\pi_\ast ev^\ast T_\mathbb{P})^\vee \) and \( E_{\overline{M}_{0,n}(X, \beta)}/\mathfrak{M} \) := \( (\mathcal{R}^i\pi_\ast ev^\ast T_X)^\vee \). A standard cohomology and base change argument shows we have a natural map \( \varphi : i^\ast E_{\overline{M}_{0,n}(\mathbb{P}, i \ast \beta)}/\mathfrak{M} \rightarrow E_{\overline{M}_{0,n}(X, \beta)}/\mathfrak{M} \). Moreover, the smoothness of \( \overline{M}_{0,n}(\mathbb{P}, i \ast \beta) \) implies the existence of a perfect relative obstruction theory \( E_{\overline{M}_{0,n}(X, \beta)}/\overline{M}_{0,n}(\mathbb{P}, i \ast \beta) \). Precisely, if we denote by \( N_{X/\mathbb{P}} \)
the normal bundle of $X$ in $\mathbb{P}$ then $E^•_{\overline{M}_{0,n}(X,\beta)/\overline{M}_{0,n}(\mathbb{P},i_*\beta)}$ is $(\mathcal{R}^0\pi_*ev^*N_X/\mathbb{P})^\vee$ viewed as a complex concentrated in $-1$. By Construction 2 and Corollary 4 we have

$$i'[\overline{M}_{0,n}(\mathbb{P},i_*\beta)]^{\text{virt}} = [\overline{M}_{0,n}(X,\beta)]^{\text{virt}}.$$ (9)

For the upper inclusion, we argue again by cohomology and base change. The 3-terms relative obstruction we are interested in is

$$[(\mathcal{R}^1\pi_*ev^*N_X/\mathbb{P})^\vee \to (\mathcal{R}^0\pi_*ev^*N_X/\mathbb{P})^\vee \to 0].$$

In order to prove it is perfect we compare it with $p^*E^•_{\overline{M}_{0,n}(X,\beta)/\overline{M}_{0,n}(\mathbb{P},i_*\beta)}$, that we already know is perfect. The first ingredient is $p^*N_X/\mathbb{P} = N_{\overline{X}/\mathbb{P}}$. Then, all we need to show is

$$p^*(\mathcal{R}^i\pi_*ev^*N_X/\mathbb{P}) = (\mathcal{R}^i\pi_*ev^*p^*N_X/\mathbb{P})$$ (10)

in the derived category of $\overline{M}_{0,n}(\overline{X},\overline{\beta})$. As usual, we replace $ev^*N_X/\mathbb{P}$ by a quasi-isomorphic complex of vector bundles on $\overline{M}_{0,n+1}(X,\beta)$, $K$ such that $\mathcal{R}^i\pi_*K$ is again a complex of vector bundles (see [1]). But now $\mathcal{R}^i\pi_*p^*K = p^*\mathcal{R}^i\pi_*K$, therefore we can conclude equality (10) holds. Finally, we apply Proposition 2 (iii) and we get

$$i'[\overline{M}_{0,n}(\tilde{\mathbb{P}},i_*\beta)]^{\text{virt}} = [\overline{M}_{0,n}(\tilde{X},\tilde{\beta})]^{\text{virt}}.$$ (11)

On the other hand, $p$ is proper and Proposition 2 (i) gives

$$i^!p_*[\overline{M}_{0,n}(\tilde{\mathbb{P}},i_*\beta)]^{\text{virt}} = p_*i^![\overline{M}_{0,n}(\tilde{\mathbb{P}},i_*\beta)]^{\text{virt}}.$$ (12)

On the left hand side we can apply Proposition 5 and we obtain

$$i^!p_*[\overline{M}_{0,n}(\tilde{\mathbb{P}},i_*\beta)]^{\text{virt}} = i^![\overline{M}_{0,n}(\tilde{\mathbb{P}},i_*\beta)]^{\text{virt}}.$$ (13)

Gathering all together, equations 9, 11, 12 translate in

$$p_*[\overline{M}_{0,n}(\tilde{X},\tilde{\beta})]^{\text{virt}} = [\overline{M}_{0,n}(X,\beta)]^{\text{virt}}.$$

The projection formula gives the following Corollary.

**Corollary 5.** Let $X$ and $Y$ as above, and let $\gamma \in A_*(X)\otimes^n$ be any $n$-tuple of classes such that $\sum \text{codim}(\gamma_i) = \text{vdim}\overline{M}_{0,n}(X,\beta)$. Then, $I_{\overline{X}}^{\overline{X},0,n,\beta}(p^*\gamma) = I_{0,n,\beta}^X(\gamma)$. 

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Remark 11. This result was obtained in [8] in a more general context. Lai starts with $X$ and $Y$ such that $N_{Y/X}$ is convex and he analyzes the map $\overline{M}_{0,n}(\tilde{X}, \tilde{\beta}) \rightarrow \overline{M}_{0,n}(X, \beta)$. Under this hypothesis the relative obstruction theory induced by $p$ is perfect that in our language means that $p$ admits a virtual pull-back. We should stress however, that we cannot use the usual relative obstruction theories to $\overline{M}$ in order to obtain the previous result because the diagram in Corollary 4 is not commutative. In [8], Lai uses the absolute obstruction theories and he shows they are compatible with the relative one. In our language this means that $p!\left[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})\right]_{\text{virt}} = \left[\overline{M}_{0,n}(X, \beta)\right]_{\text{virt}}$. From now on, the main problem is that $p^*\left[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})\right]_{\text{virt}}$ is not necessarily equal to $\left[\overline{M}_{0,n}(X, \beta)\right]_{\text{virt}}$ and in order to fix it Lai needs to impose an extra slight condition (see Theorem 4.11).

Remark 12. If $X$ is the zero-locus of a section $s \in H^0(\mathbb{P}, V)$, for some convex vector bundle $V$ on $\mathbb{P}$ and $Y$ respects the hypothesis of Proposition [4] then the equality $p_*\left[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})\right]_{\text{virt}} = \left[\overline{M}_{0,n}(X, \beta)\right]_{\text{virt}}$ follows trivially from [12]. Indeed, using the notations of [ibid.] we have

$$i_*\left[\overline{M}_{0,n}(X, \beta)\right]_{\text{virt}} = c_{\text{top}}(\mathcal{R}^0_\pi ev^*V) \cdot \left[\overline{M}_{0,n}(\mathbb{P}, i_*\beta)\right]_{\text{virt}}.$$  

Again, using equality [10] we get the same relation with blow-ups, namely,

$$i_*\left[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})\right]_{\text{virt}} = c_{\text{top}}(p^*\mathcal{R}^0_\pi ev^*V) \cdot \left[\overline{M}_{0,n}(\tilde{\mathbb{P}}, i_*\tilde{\beta})\right]_{\text{virt}}.$$  

All we have to do now, is to apply the projection formula.

### 4.3 Costello’s push-forward formula

The push-forward formula in [3] is a consequence of the basic properties of virtual pull-backs (push-forward and functoriality). We recall the set-up. Let us consider a cartesian diagram

$$\begin{array}{ccc}
F & \xrightarrow{f} & G \\
\downarrow{p_1} & & \downarrow{p_2} \\
\mathcal{M}_1 & \xrightarrow{g} & \mathcal{M}_2 
\end{array}$$

such that

1. $f$ is a proper morphism;
2. $g$ is DM-type morphism of degree $d$;
3. $\mathcal{M}_1$ and $\mathcal{M}_2$ are Artin stacks of the same pure dimension;
4. $F$ and $G$ are DM-stacks equipped with perfect relative obstruction theories $E_{F/\mathcal{M}_1}$ and $E_{G/\mathcal{M}_2}$ inducing virtual classes $[F]_{\text{virt}}$ and $[G]_{\text{virt}}$;
5. \( E_{F/\mathfrak{M}} = f^*E_{G/\mathfrak{M}} \).

**Proposition 2.** (Costello, [3], Theorem 5.0.1.) Under the assumptions above, \( f_*(F)^{\text{virt}} = d(G)^{\text{virt}}. \)

**Proof.** As \( E_{F/\mathfrak{M}} \) and \( E_{G/\mathfrak{M}} \) are perfect, \( p_1 \) and \( p_2 \) induce pull-back morphisms and \( E_{F/\mathfrak{M}} = f^*E_{G/\mathfrak{M}} \) implies \( p_1^! \) is induced by \( p_2^! \). Applying Theorem 2 (i) we get \( f_*p_1^! [\mathfrak{M}] = p_2^! g_*[\mathfrak{M}] \). Using the fact that \( g_*[\mathfrak{M}] = d[\mathfrak{M}] \) and the definition of virtual classes we get
\[
f_*(F)^{\text{virt}} = d(G)^{\text{virt}}.
\]

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