Quantum Theory and Beyond: Is Entanglement Special?

Borivoje Dakić¹ and Časlav Brukner¹,²

¹Faculty of Physics, University of Vienna, Boltzmannngasse 5, A-1090 Vienna, Austria
²Institute of Quantum Optics and Quantum Information, Austrian Academy of Sciences, Boltzmannngasse 3, A-1090 Vienna, Austria

Quantum theory makes the most accurate empirical predictions and yet it lacks simple, comprehensible physical principles from which the theory can be uniquely derived. A broad class of probabilistic theories exist which all share some features with quantum theory, such as probabilistic predictions for individual outcomes (indeterminism), the impossibility of information transfer faster than speed of light (no-signaling) or the impossibility of copying of unknown states (no-cloning). A vast majority of attempts to find physical principles behind quantum theory either fall short of deriving the theory uniquely from the principles or are based on abstract mathematical assumptions that require themselves a more conclusive physical motivation. Here, we show that classical probability theory and quantum theory can be reconstructed from three reasonable axioms: (1) (Information capacity) All systems with information carrying capacity of one bit are equivalent. (2) (Locality) The state of a composite system is completely determined by measurements on its subsystems. (3) (Reversibility) Between any two pure states there exists a reversible transformation. If one requires the transformation from the last axiom to be continuous, one separates quantum theory from the classical probabilistic one. A remarkable result following from our reconstruction is that no probability theory other than quantum theory can exhibit entanglement without contradicting one or more axioms.

I. INTRODUCTION

The historical development of scientific progress teaches us that every theory that was established and broadly accepted at a certain time was later inevitably replaced by a deeper and more fundamental theory of which the old one remains a special case. One celebrated example is Newtonian (classical) mechanics which was superseded by quantum mechanics at the beginning of the last century. It is natural to ask whether in a similar manner there could be logically consistent theories that are more generic than quantum theory itself. It could then turn out that quantum mechanics is an effective description of such a theory, only valid within our current restricted domain of experience.

At present, quantum theory has been tested against very specific alternative theories that, both mathematically and in their concepts, are distinctly different. Instances of such alternative theories are non-contextual hidden-variable theories [1], local hidden-variable theories [2], crypto-nonlocal hidden-variable theories [3, 4], or some nonlinear variants of the Schrödinger equation [5, 6, 7, 8]. Currently, many groups are working on improving experimental conditions to be able to test alternative theories based on various collapse models [9, 10, 11, 12, 13]. The common trait of all these proposals is to suppress one or the other counter-intuitive feature of quantum mechanics and thus keep some of the basic notions of a classical world view intact. Specifically, hidden-variable models would allow to preassign definite values to outcomes of all measurements, collapse models are mechanisms for restraining superpositions between macroscopically distinct states and nonlinear extensions of the Schrödinger equation may admit more localized solutions for wave-packet dynamics, thereby resembling localized classical particles.

In the last years the new field of quantum information has initialized interest in generalized probabilistic theories which share certain features – such as the no-cloning and the no-broadcasting theorems [14, 15] or the trade-off between state disturbance and measurement [16] – generally thought of as specifically quantum, yet being shown to be present in all except classical theory. These generalized probabilistic theories can allow for stronger than quantum correlations in the sense that they can violate Bell’s inequalities stronger than the quantum Cirel’son bound (as it is the case for the celebrated “non-local boxes” of Popescu and Rohrlich [17]), though they all respect the “non-signaling” constraint according to which correlations cannot be used to send information faster than the speed of light.

Since the majority of the features that have been highlighted as “typically quantum” are actually quite generic for all non-classical probabilistic theories, one could conclude that additional principles must be adopted to single out quantum theory uniquely. Alternatively, these probabilistic theories indeed can be constructed in a logically consistent way, and might even be realized in nature in a domain that is still beyond our observations. The vast majority of attempts to find physical principles behind quantum theory either fail to single out the theory uniquely or are based on highly abstract mathematical assumptions without an immediate physical meaning (e.g. [18]).

On the way to reconstructions of quantum theory from foundational physical principles rather than purely mathematical axioms, one finds interesting examples coming from an instrumentalist approach [19, 20, 21], where the focus is primarily on primitive laboratory operations such as preparations, transformations and measurements. While these reconstructions are based on a short set of simple axioms, they still partially use mathematical language in their formulation.

Evidently, added value of reconstructions for better understanding quantum theory originates from its power of explanation where the structure of the theory comes from. Candidates for foundational principles were proposed giving a basis for an understanding of quantum theory as a general theory of information supplemented by several information-theoretic constraints [22, 23, 24, 25, 26]. In a wider context these ap-
proaches belong to attempts to find an explanation for quantum theory by putting primacy on the concept of information or on the concept of probability which can again be seen as a way of quantifying information [27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. Other principles were proposed for separation of quantum correlations from general non-signaling correlations, such as that communication complexity is not trivial [37, 38], that communication of \( m \) classical bits causes information gain of at most \( m \) bits (“information causality”) [39], or that any theory should recover classical physics in the macroscopic limit [40].

In his seminal paper, Hardy [19] derives quantum theory from “five reasonable axioms” within the instrumentalist framework. He sets up a link between two natural numbers, \( d \) and \( N \), characteristics of any theory. \( d \) is the number of degrees of freedom of the system and is defined as the minimum number of real parameters needed to determine the state completely. The dimension \( N \) is defined as the maximum number of states that can be reliably distinguished from one another in a single shot experiment. A closely related notion is the information carrying capacity of the system, which is the maximal number of bits encoded in the system, and is equal to \( \log N \) bits for a system of dimension \( N \).

Examples of theories with an explicit functional dependence \( d(N) \) are classical probability theory with the linear dependence \( d = N - 1 \), and quantum theory with the quadratic dependence for which it is necessary to use \( d = N^2 - 1 \) real parameters to completely characterize the quantum state [65]. Higher-order theories with more general dependencies \( d(N) \) might exist as illustrated in Figure 1. Hardy’s reconstruction resorts to a “simplicity axiom” that discards a large class of higher-order theories by requiring that for each given \( N \), \( d(N) \) takes the minimum value consistent with the other axioms. However, without making such an \textit{ad hoc} assumption the higher-order theories might be constructed in agreement with the rest of the axioms. In fact, an explicit quartic theory for which \( d = N^4 - 1 \) [41], and theories for generalized bit (\( N = 2 \)) for which \( d = 2^r - 1 \) and \( r \in N \) [42], were recently developed, though all of them are restricted to the description of individual systems only.

It is clear from the previous discussion that the question on basis of which physical principles quantum theory can be separated from the multitude of possible generalized probability theories is still open. A particularity interesting unsolved problem is whether the higher-order theories of Refs. [19, 31, 42] can be extended to describe non-trivial, i.e. entangled, states of composite systems. Any progress in theoretical understanding of these issues would be very desirable, in particular because experimental research efforts in this direction have been very sporadic. Although the majority of experiments indirectly verify also the number of the degrees of freedom of quantum systems [66], there are only few dedicated attempts at such a direct experimental verification. Quaternionic quantum mechanics (for which \( d = 2N^2 - N - 1 \)) was tested in a suboptimal setting [45] in a single neutron experiment in 1984 [43, 44], and more recently, the generalized measure theory of Sorkin [46] in which higher order interferences are predicted was tested in a three-slit experiment with photons [47].

Both experiments put an upper bound on the extent of the observational effects the two alternative theories may produce.

II. BASIC IDEAS AND THE AXIOMS

Here we reconstruct quantum theory from three reasonable axioms. Following the general structure of any reconstruction we first give a set of physical principles, then formulate their mathematical representation, and finally rigorously derive the formalism of the theory. We will only consider the case where the number of distinguishable states is finite. The three axioms which separate classical probability theory and quantum theory from all other probabilistic theories are:

\textbf{Axiom 1. (Information capacity)} An elementary system has the information carrying capacity of at most one bit. All systems of the same information carrying capacity are equivalent.

\textbf{Axiom 2. (Locality)} The state of a composite system is completely determined by local measurements on its subsystems and their correlations.

\textbf{Axiom 3. (Reversibility)} Between any two pure states there exists a reversible transformation.

A few comments on these axioms are appropriate here. The most elementary system in the theory is a two-dimensional system. All higher-dimensional systems will be built out of two-dimensional ones. Recall that the dimension is defined...
as the maximal number of states that can be reliably distin-
guished from one another in a single shot experiment. Under
the phrase “an elementary system has an information capac-
ity of at most one bit” we precisely assume that for any state
(pure or mixed) of a two-dimensional system there is a mea-
surement such that the state is a mixture of two orthogonal states
(i.e. states perfectly distinguishable in a single shot experiment), e.g.
y = i\eta x + (1 - i\eta) x². This is not fulfilled in the toy world, but is satisfied
in a theory in which the entire circle represents the pure states and
where measurements can distinguish all pairs of orthogonal states.

Axiom 2 assumes that a specification of the probabilities for
a complete set of local measurements for each of the subsys-
tems plus the joint probabilities for correlations between these
measurements is sufficient to determine completely the global
state. Note that this property does hold in both quantum theory
and classical probability theory, but not in quantum theory
formulated on the basis of real or quaternionic amplitudes in-
stead of complex. A closely related formulation of the axiom
was given by Barrett [16].

Finally, axiom 3 requires that transformations are re-
versible. This is assumed alone for the purposes that the set of
transformations builds a group structure. It is natural to
assume that a composition of two physical transformations is
again a physical transformation. It should be noted that this
axiom could be used to exclude the theories in which “non-
local boxes” occur, because there the dynamical group is triv-
ial, in the sense that it is generated solely by local operations
and permutations of systems with no entangling reversible
transformations (that is, non-local boxes cannot be prepared
from product states) [49].

If one requires the reversible transformation from our axi-
om 3 to be continuous:

**Axiom 3’. (Continuity) Between any two pure states there ex-
ists a continuous reversible transformation,**

which separates quantum theory from classical probability
theory. The same axiom is also present in Hardy’s reconstruc-
tion. By a continuous transformation is here meant that every
transformation can be made up from a sequence of transforma-

tions only infinitesimally different from the identity.

A remarkable result following from our reconstruction is
that **quantum theory is the only probabilistic theory in which
one can construct entangled states and fulfill the three axioms.**
In particular, in the higher-order theories of Refs. [15-41-42]
composite systems can only enjoy trivial separable states. On
the other hand, we will see that axiom 1 alone requires en-
tangled states to exist in all non-classical theories. This will
allow us to discard the higher-order theories in our reconstruc-
tion scheme without invoking the simplicity argument.

As a by product of our reconstruction we will be able to
answer why in nature only “odd” correlations (i.e. (1,1,-1),
(1,-1,1), (-1,1,1) and (-1,-1,-1)) are observed when two
maximally entangled qubits (spin-1/2 particles) are both mea-
sured along direction x, y and z, respectively. The most famil-
Our reconstruction will be given in the framework of typical experimental situation an observer faces in the laboratory. While this instrumentalist approach is a useful paradigm to work with, it might not be necessary. One could think about axioms 1 and 3 as referring to objective features of elementary constituents of the world which need not necessarily be related to laboratory actions. In contrast, axiom 2 seems to acquire a meaning only within the instrumentalist approach as it involves the word “measurement”. Even here one could follow a suggestion of Grinbaum [48] and rephrase the axiom to the assumption of “multiplicability of the information carrying capacity of subsystems”.

Concluding this section, we note that the conceptual groundwork for the ideas presented here has been prepared most notably by Weizsäcker [50], Wheeler [51] and Zeilinger [23] who proposed that the notion of the elementary yes-no alternative, or the “Ur”, should play a pivotal role when reconstructing quantum physics.

III. BASIC NOTIONS

Following Hardy [19] we distinguish three types of devices in a typical laboratory. The preparation device prepares systems in some state. It has a set of switches on it for varying the state produced. After state preparation the system passes through a transformation device. It also has a set of switches on it for varying the transformation applied on the state. Finally, the system is measured in a measurement apparatus. It again has switches on it with which help an experimenter can choose different measurement settings. This device outputs classical data, e.g. a click in a detector or a spot on a observation screen.

We define the state of a system as that mathematical object from which one can determine the probability for any conceivable measurement. Physical theories can have enough structure that it is not necessary to give an exhaustive list of all probabilities for all possible measurements, but only a list of probabilities for some minimal subset of them. We refer to this subset as fiducial set. Therefore, the state is specified by a list of \( d \) (where \( d \) depends on dimension \( N \)) probabilities for a set of fiducial measurements: \( \mathbf{p} = (p_1, \ldots, p_d) \). The state is pure if it is not a (convex) mixture of other states. The state is mixed if it is not pure. For example, the mixed state \( \mathbf{p} \) generated by preparing state \( \mathbf{p}_1 \) with probability \( \lambda \) and \( \mathbf{p}_2 \) with probability \( 1 - \lambda \), is \( \mathbf{p} = \lambda \mathbf{p}_1 + (1 - \lambda) \mathbf{p}_2 \).

When we refer to an \( N \)-dimensional system, we assume that there are \( N \) states each of which identifies a different outcome of some measurement setting, in the sense that they return probability one for the outcome. We call this set a set of basis or orthogonal states. Basis states can be chosen to be pure. To see this assume that some mixed state identifies one outcome. We can decompose the state into a mixture of pure states, each of which has to return probability one, and thus we can use one of them to be a basis state. We will show later that each pure state corresponds to a unique measurement outcome.

If the system in state \( \mathbf{p} \) is incident on a transformation device, its state will be transformed to some new state \( U(\mathbf{p}) \). The transformation \( U \) is a linear function of the state \( \mathbf{p} \) as it needs to preserve the linear structure of mixtures. For example, consider the mixed state \( \mathbf{p} \) which is generated by preparing state \( \mathbf{p}_1 \) with probability \( \lambda \) and \( \mathbf{p}_2 \) with probability \( 1 - \lambda \). Then, in each single run, either \( \mathbf{p}_1 \) or \( \mathbf{p}_2 \) is transformed and thus one has:

\[
U(\lambda \mathbf{p}_1 + (1 - \lambda) \mathbf{p}_2) = \lambda U(\mathbf{p}_1) + (1 - \lambda)U(\mathbf{p}_2).
\]

It is natural to assume that a composition of two or more transformations is again from a set of (reversible) transformations. This set forms some abstract group. Axiom 3 states that the transformations are reversible, i.e. for every \( U \) there is an inverse group element \( U^{-1} \). Here we assume that every transformation has its matrix representation \( U \) and that there is an orthogonal representation of the group: there exists an invertible matrix \( S \) such that \( O = SUS^{-1} \) is an orthogonal matrix, i.e. \( O^T O = \mathbf{1} \), for every \( U \) (We use the same notation both for the group element and for its matrix representation). This does not put severe restrictions to the group of transformations, as it is known that all compact groups have such a representation (the Schur-Auerbach lemma [55]). Since the transformation keeps the probabilities in the range \([0, 1]\), it has to be a compact group [19]. All finite groups and all continuous Lie groups are therefore included in our consideration.

Given a measurement setting, the outcome probability \( P_{\text{meas}} \) can be computed by some function \( f \) of the state \( \mathbf{p} \).

\[
P_{\text{meas}} = f(\mathbf{p}). \tag{2}
\]

Like a transformation, the measurement cannot change the mixing coefficients in a mixture, and therefore the measured probability is a linear function of the state \( \mathbf{p} \):

\[
f(\lambda \mathbf{p}_1 + (1 - \lambda) \mathbf{p}_2) = \lambda f(\mathbf{p}_1) + (1 - \lambda) f(\mathbf{p}_2). \tag{3}
\]

IV. ELEMENTARY SYSTEM: SYSTEM OF INFORMATION CAPACITY OF 1 BIT

A two-dimensional system has two distinguishable outcomes which can be identified by a pair of basis states \( \{ \mathbf{p}, \mathbf{p}^\dagger \} \). The state is specified by \( d \) probabilities \( \mathbf{p} = (p_1, \ldots, p_d) \) for \( d \) fiducial measurements, where \( p_i \) is probability for a particular outcome of the \( i \)-th fiducial measurement (the dependent probabilities \( 1 - p_i \) for the opposite outcomes are omitted in the state description). Instead of using the probability vector \( \mathbf{p} \) we will specify the state by its Bloch representation \( \mathbf{x} \) defined as a vector with \( d \) components:

\[
x_i = 2p_i - 1. \tag{4}
\]

The mapping between the two different representations is an invertible linear map and therefore preserves the structure of the mixture \( \lambda \mathbf{p}_1 + (1 - \lambda) \mathbf{p}_2 \mapsto \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \).
It is convenient to define a **totally mixed state** $E = \frac{1}{N} \sum_{i \in S_{\text{pure}}} \mathbf{x}$, where $S_{\text{pure}}$ denotes the set of pure states and $N$ is the normalization constant. In the case of a continuous set of pure states the summation has to be replaced by a proper integral. It is easy to verify that $E$ is a totally invariant state. This implies that every measurement and in particular the fiducial ones will return the same probability for all outcomes. In the case of a two-dimensional system this probability is $1/2$. Therefore, the Bloch vector of the totally mixed state is the zero-vector $\mathbf{E} = \mathbf{0}$.

The transformation $U$ does not change the totally mixed state, hence $U(\mathbf{E}) = \mathbf{0}$. The last condition together with the linearity condition (1) implies that any transformation is represented by some $d \times d$ real invertible matrix $U$. The same reasoning holds for measurements. Therefore, the measured probability is given by the formula:

$$ P_{\text{meas}} = \frac{1}{2} (1 + \mathbf{r}^T \mathbf{x}). \quad (5) $$

The vector $\mathbf{r}$ represents the outcome for the given measurement setting. For example, the vector $(1, 0, 0, \ldots)$ represents one of the outcomes for the first fiducial measurement.

According to axiom 1 any state is a classical mixture of some pair of orthogonal states. For example, the totally mixed state is an equally weighted mixture of some orthogonal states $\mathbf{0} = \frac{1}{2} \mathbf{x} + \frac{1}{2} \mathbf{x}^\perp$. Take $\mathbf{x}$ to be the reference state. According to axiom 3 we can generate the full set of states by applying all possible transformations to the reference state. Since the totally mixed state is invariant under the transformations, the pair of orthogonal states is represented by a pair of antiparallel vectors $\mathbf{x}^\perp = -\mathbf{x}$. Consider the set $S_{\text{pure}} = \{ U \mathbf{x} \mid U \}$ of all pure states generated by applying all transformations to the reference state. If one uses the orthogonal representation of the transformations, $U = S^{-1}OS$, which was introduced above, one maps $\mathbf{x} \mapsto S \mathbf{x}$ and $U \mapsto O$. Hence, the transformation $U/\mathbf{x} \mapsto S/\mathbf{U} \mathbf{x}$ is norm preserving. We conclude that all pure states are points on a $d$-dimensional ellipsoid described by $\|S\mathbf{x}\| = c$ with $c > 0$.

Now, we want to show that any vector $\mathbf{x}$ satisfying $\|S\mathbf{x}\| = c$ is a physical state and therefore the set of states has to be the whole ellipsoid. Let $\mathbf{x}$ be some vector satisfying $\|S\mathbf{x}\| = c$ and $\mathbf{x}(t) = t \mathbf{x}$ a line trough the origin (totally mixed state) as given in Figure 3(left). Within the set of pure states we can always find $d$ linearly independent vectors $\{ \mathbf{X}_i \}$ and $\mathbf{x}(t) = t \mathbf{x}$ a line trough the origin (totally mixed state) as given in Figure 3(left). Within the set of pure states we can always find $d$ linearly independent vectors $\{ \mathbf{X}_i \}$. For each state $\mathbf{x}_i$ there is a corresponding orthogonal state $\mathbf{x}_i^\perp = -\mathbf{x}_i$ in a set of states. We can expand a point on the line into a linearly independent set of vectors: $\mathbf{x}(t) = t \sum \mathbf{x}_i$. For sufficiently small $t$ we can define a pair of non-negative numbers $\lambda_i(t) = \frac{1}{2} (x_i^2 + t c_i)$ and $\lambda_i^\perp(t) = \frac{1}{2} (x_i^2 - t c_i)$ with $\sum \lambda_i(t) + \lambda_i^\perp(t) = 1$ such that $\mathbf{x}(t)$ is a mixture $\mathbf{x}(t) = \sum \lambda_i(t) \mathbf{x}_i + \lambda_i^\perp(t) \mathbf{x}_i^\perp$ and therefore is a physical state. Then, according to axiom 1 there exists a pair of basis states $\{ \mathbf{x}_0, -\mathbf{x}_0 \}$ such that $\mathbf{x}(t)$ is a mixture of them

$$ \mathbf{x}(t) = \mathbf{r} \mathbf{x} = \alpha \mathbf{x}_0 + (1 - \alpha)(-\mathbf{x}_0), \quad (6) $$

where $\alpha = \frac{1}{2} t$ and $\mathbf{x} = \mathbf{x}_0$. This implies that $\mathbf{x}$ is a pure state and therefore all points of the ellipsoid are physical states.

For every pure state $\mathbf{x}$, there exists at least one measurement setting with the outcome $\mathbf{r}$ such that the outcome probability is one, hence $\mathbf{r}^T \mathbf{x} = 1$. Let us define new coordinates $\mathbf{y} = \frac{1}{\lambda} S \mathbf{x}$ and $\mathbf{m} = c S^{-1} \mathbf{r}$ in the orthogonal representation. The set of pure states in the new coordinates is a $(d - 1)$-sphere $S^{d-1} = \{ \mathbf{y} \mid \|\mathbf{y}\| = 1 \}$ of the radius. The probability rule (5) remains unchanged in the new coordinates:

$$ P_{\text{meas}} = \frac{1}{2} (1 + \mathbf{m}^T \mathbf{y}). \quad (7) $$

Thus, one has $\mathbf{m}^T \mathbf{y} = 1$. Now, assume that $\mathbf{m} \neq \mathbf{y}$. Then $\|\mathbf{m}\| > 1$ and the vectors $\mathbf{m}$ and $\mathbf{y}$ span a two-dimensional plane as illustrated in Figure 5(right). The set of pure states within this plane is a unit circle. Choose the pure state $\mathbf{y}'$ to be parallel to $\mathbf{m}$. Then the outcome probability is $P_{\text{meas}} = \frac{1}{2} (1 + \|\mathbf{m}\|)$ which is non-physical, hence $\mathbf{m} = \mathbf{y}$. Therefore, to each pure state $\mathbf{y}$, we associate a measurement vector $\mathbf{m} = \mathbf{y}$ which identifies it. Equivalently, in the original coordinates, to each $\mathbf{x}$ we associate a measurement vector $\mathbf{r} = D \mathbf{x}$, where $D = \frac{1}{\lambda} S^{-1} S$ is a positive, symmetric matrix. A proof of this relation for the restricted case of $d = 3$ can be found in Ref. [19].

From now one, instead of the measurement vector $\mathbf{r}$ we will use the pure state $\mathbf{x}$ which identifies it. When we say that the measurement along the state $\mathbf{x}$ is performed we mean the measurement given by $\mathbf{r} = D \mathbf{x}$. The measurement setting is given by a pair of measurement vectors $\mathbf{r}$ and $-\mathbf{r}$. The measured probability when the state $\mathbf{x}_1$ is measured along the state $\mathbf{x}_2$...
follows from formula (5):

\[ P(x_1, x_2) = \frac{1}{2}(1 + x_1^T D x_2). \]  

(8)

We can choose orthogonal eigenvectors of the matrix \( D \) as the fiducial set of states (measurements):

\[ D x_i = a_i x_i, \]  

(9)

where \( a_i \) are eigenvalues of \( D \). Since \( x_i \) are pure states, they satisfy \( x_i^T D x_j = \delta_{ij} \). The set of pure states becomes a unit sphere \( S^{d-1} = \{ x \mid ||x|| = 1 \} \) and the probability formula is reduced to

\[ P(x_1, x_2) = \frac{1}{2}(1 + x_1^T x_2). \]  

(10)

This corresponds to a choice of a complete set of mutually complementary measurements (i.e. mutually unbiased basis sets) for the fiducial measurements. The states identifying outcomes of complementary measurements satisfy \( P(x_i, x_j) = \frac{1}{d} \) for \( i \neq j \). Two observables are said to be mutually complementary if complete certainty about one of the observables (one of two outcomes occurs with probability one) precludes any knowledge about the others (the probability for both outcomes is 1/2). Given some state \( x \), the \( i \)-th fiducial measurement returns probability \( p_i = \frac{1}{d}(1 + x_i) \). Therefore, \( x_i \) is a mean value of a dichotomic observable \( b_i = +1 x_i - 1 x_i^\perp \) with two possible outcomes \( b_i = \pm 1 \).

A theory in which the state space of the generalized bit is represented by a \((d-1)\)-sphere has \( d \) mutually complementary observables. This is a characteristic feature of the theories and they can be ordered according to their number. For example, classical physics has no complementary observables, real quantum mechanics has two, complex (standard) quantum mechanics has three (e.g. the spin projections of a spin-1/2 system along three orthogonal directions) and the one based on quaternions has five mutually complementary observables. Note that higher-order theories of a single generalized bit are such that the qubit theory can be embedded in them in the same way in which classical theory of a bit can be embedded in qubit theory itself.

Higher-order theories can have even better information processing capacity than quantum theory. For example, the computational abilities of the theories with \( d = 2^r \) and \( r \in \mathbb{N} \) in solving the Deutsch-Josza type of problems increases with the number of mutually complementary measurements [42]. It is likely that the larger this number is the larger the error rate would be in secret key distribution in these theories, in a similar manner in which the 6-state is advantageous over the 4-state protocol in (standard) quantum mechanics. In the first case one uses all three mutually complementary observables and in the second one only two of them. (See Ref. [52] for a review on characterizing generalized probabilistic theories in terms of their information-processing power and Ref. [53] for investigating the same question in much more general framework of compact closed categories.)

A final remark on higher-order theories is of more speculative nature. In various approaches to quantum theory of gravity one predicts at the Planck scale the dimension of space-time to be different from \( 3 + 1 \) [54]. If one considers directional degrees of freedom (spin), then the \( d - 1 \)-sphere (Bloch sphere) might be interpreted as the state space of a spin system embedded in real (ordinary) space of dimension \( d \), in general different than 3 which is the special case of quantum theory.

The reversible transformation \( R \) preserves the purity of state \( ||R x|| = ||x|| \) and therefore \( R \) is an orthogonal matrix. We have shown that the state space is the full \((d - 1)\)-sphere. According to axiom 3 the set of transformations must be rich enough to generate the full sphere. If \( d = 1 \) (classical bit), the group of transformations is discrete and contains only the identity and the bit-flip. If \( d > 1 \), the group is continuous and is some subgroup of the orthogonal group \( O(d) \). Every orthogonal matrix has determinant either 1 or -1. The orthogonal matrices with determinant 1 form a normal subgroup of \( O(d) \), known as the special orthogonal group \( SO(d) \). The group \( O(d) \) has two connected components: the identity component which is the \( SO(d) \) group, and the component formed by orthogonal matrices with determinant -1. Since every two points on the \((d - 1)\)-sphere are connected by some transformation, the group of transformations is at least the \( SO(d) \) group. If we include even a single transformation with determinant -1, the set of transformations becomes the entire \( O(d) \) group. (Later we will show that only some \( d \) are in agreement with our three axioms and for these \( d \)'s the set of physical transformations will be shown to be the \( SO(d) \) group.)

V. COMPOSITE SYSTEM AND THE NOTION OF LOCALITY

We now introduce a description of composite systems. We assume that when one combines two systems of dimension \( L_1 \) and \( L_2 \) into a composite one, one obtains a system of dimension \( L_1 L_2 \). Consider a composite system consisting of two generalized bits and choose a set of \( d \) complementary measurements on each subsystem as fiducial measurements. According to axiom 2 the state of the composite system is completely determined by a set of real parameters obtainable from local measurements on the two generalized bits and their correlations. We obtain \( 2d \) independent real parameters from the set of local fiducial measurements and additional \( d^2 \) parameters from correlations between them. This gives altogether \( d^2 + 2d = (d + 1)^2 - 1 \) parameters. They are the components \( x_i, y_i, i \in \{1, ..., d\} \), of the local Bloch vectors and \( T_{ij} \) of the correlation tensor:

\[ x_i = p^{(i)}(A = 1) - p^{(i)}(A = -1), \]
\[ y_j = p^{(j)}(B = 1) - p^{(j)}(B = -1), \]
\[ T_{ij} = p^{(i)}(AB = 1) - p^{(i)}(AB = -1). \]

(11, 12, 13)

Here, for example, \( p^{(i)}(A = 1) \) is the probability to obtain outcome \( A = 1 \) when the \( i \)-th measurement is performed on the first subsystem and \( p^{(i)}(AB = 1) \) is the joint probability to obtain correlated results (i.e. either \( A = B = +1 \) or \( A = B = -1 \)) when the \( i \)-th measurement is performed on the first subsystem and the \( j \)-th measurement on the second one.
Note that axiom 2 “The state of a composite system is completely determined by local measurements on its subsystems and their correlations” is formulated in a way that the non-signaling condition is implicitly assumed to hold. This is because it is sufficient to speak about “local measurements” alone without specifying the choice of measurement setting on the other, potentially distant, subsystem. Therefore, $x_i$ does not depend on $j$, and $y_j$ does not depend on $i$.

We represent a state by the triple $\psi = (x, y, T)$, where $x$ and $y$ are the local Bloch vectors and $T$ is a $d \times d$ real matrix representing the correlation tensor. The product (separable) state is represented by $\psi_p = (x, y, T)$, where $T = xy^T$ is of product form, because the correlations are just products of the components of the local Bloch vectors. We call the pure state entangled if it is not a product state.

The measured probability is a linear function of the state $\psi$. If we prepare totally mixed states of the subsystems $(0, 0, 0)$, the probability for any outcome of an arbitrary measurement will be 1/4. Therefore, the outcome probability can be written as:

$$P_{\text{meas}} = \frac{1}{4}(1 + (r, \psi)),$$

(14)

where $r = (r_1, r_2, K)$ is a measurement vector associated to the observed outcome and $(\ldots, \ldots)$ denotes the scalar product:

$$(r, \psi) = r_1^T x + r_2^T y + \text{Tr}(K^T T).$$

(15)

Now, assume that $r = (r_1, r_2, K)$ is associated to the outcome which is identified by some product state $\psi_p = (x_0, y_0, T_0)$. If we perform a measurement on the arbitrary product state $\psi = (x, y, T)$, the outcome probability has to factorize into the product of the local outcome probabilities of the form $\text{10}$:

$$P_{\text{meas}} = \frac{1}{4}(1 + r_1^T x + r_2^T y + x^T K y)$$

(16)

$$= P_1(x_0, y_0) P_2(y_0, y)$$

(17)

$$= \frac{1}{2}(1 + x_0^T x) \frac{1}{2}(1 + y_0^T y)$$

(18)

$$= \frac{1}{4} (1 + x_0^T x + y_0^T y + x^T x_0 y_0^T y),$$

(19)

which holds for all $x, y$. Therefore we have $r = \psi_p$. For each product state $\psi_p$ there is a unique outcome $r = \psi_p$ which identifies it. We will later show that correspondence $r = \psi_p$ holds for all pure states $\psi$.

If we perform local transformations $R_1$ and $R_2$ on the subsystems, the global state $\psi = (x, y, T)$ is transformed to

$$(R_1, R_2) \psi = (R_1 x, R_2 y, R_1 T R_2^T).$$

(20)

$T$ is a real matrix and we can find its singular value decomposition $\text{diag}[t_1, \ldots, t_d] = R_1 T R_2^T$, where $R_1, R_2$ are orthogonal matrices which can be chosen to have determinant 1. Therefore, we can choose the local bases such that correlation tensor $T$ is a diagonal matrix:

$$(R_1, R_2)(x, y, T) = (R_1 x, R_2 y, \text{diag}[t_1, \ldots, t_d]).$$

(21)

The last expression is called Schmidt decomposition of the state.

The local Bloch vectors satisfy $\|x\|, \|y\| \leq 1$ which implies a bound on the correlation $\|T\| \geq 1$ for all pure states. The following lemma identifies a simple entanglement witness for pure states. The proof of this and all subsequent lemmas is given in the Appendix.

**Lemma 1.** The lower bound $\|T\| = 1$ is saturated, if and only if the state is a product state $T = xy^T$.

Recall that for every transformation $U$ we can find its orthogonal representation $U = S O S^{-1}$ (the Schur-Auerbach lemma), where $S$ is an invertible matrix and $O^T O = I$. The matrix $S$ is characteristic of the representation and should be the same for all transformations $U$. If we choose some local transformation $U = (R_1, R_2)$, $U$ will be orthogonal and thus we can choose to set $S = I$. The representation of transformations is orthogonal, therefore they are norm preserving. By applying simultaneously all (local and non-local) transformations $U$ to some product state (the reference state) $\psi$ and to the measurement vector which identifies it, $r = \psi$, we generate the set of all pure states and corresponding measurement vectors. Since we have $T = P(r = \psi, \psi) = P(U r, U \psi)$, correspondence $r = \psi$ holds for any pure state $\psi$. Instead of the measurement vector $r$ in formula $1(14)$ we use the pure state which identifies it. If the state $\psi_1 = (x_1, y_1, T_1)$ is prepared and measurement along the state $\psi_2 = (x_2, y_2, T_2)$ is performed, the measured probability is given by

$$P_{12}(\psi_1, \psi_2) = \frac{1}{4}(1 + x_1^T x_2 + y_1^T y_2 + \text{Tr}(T_1^T T_2)).$$

(22)

The set of pure states obeys $P_{12}(\psi, \psi) = 1$. We can define the normalization condition for pure states $P_{12}(\psi, \psi) = \frac{1}{4}(1 + \|x\|^2 + \|y\|^2 + ||T||^2) = 1$ where $||T||^2 = \text{Tr}(T^T T)$. Therefore we have:

$$\|x\|^2 + \|y\|^2 + ||T||^2 = 3,$$

(23)

for all pure states.

An interesting observation can be made here. Although seemingly axiom 2 does not imply any strong prior restrictions to $d$, we surprisingly have obtained the explicit number 3 in the normalization condition $2(23)$. As we will see soon this relation will play an important role in deriving $d = 3$ as the only non-classical solution consistent with the axioms.

**VI. THE MAIN PROOFS**

We will now show that only classical probability theory and quantum theory are in agreement with the three axioms.

**A. Ruling out the $d$ even case**

Let us assume the total inversion $Ex = -x$ being a physical transformation. Let $\psi = (x, y, T)$ be a pure state of composite system. We apply total inversion to one of the subsystems
and obtain the state \( \psi' = (E, \mathbb{1})(x, y, T) = (-x, y, -T) \). The probability

\[
P_{12}(\psi, \psi') = \frac{1}{4} (1 - ||x||^2 - ||y||^2 - ||T||^2) \quad (24)
\]

\[
P_{12}(\psi, \psi') = \frac{1}{2} (||y||^2 - 1) \quad (25)
\]

has to be nonnegative and therefore we have \( ||y|| = 1 \). Similarly, we apply \((\mathbb{1}, E)\) to \( \psi \) and obtain \( ||x|| = 1 \). Since the local vectors are of the unit norm we have \( ||T|| = 1 \) and thus, according to lemma 1, the state \( \psi \) is a product state. We conclude that no entangled states can exist if \( E \) is a physical transformation. As we will soon see, according to axiom 1 entangled states must exist. Thus, \( E \) cannot represent a physical transformation. We will now show that this implies that \( d \) has to be odd. Recall that the set of transformations is at least the \( SO(d) \) group. \( d \) cannot be even since \( E \) would have unit determinant and would belong to \( SO(d) \). \( d \) has to be odd in which case \( E \) has determinant -1. The set of physical transformations is the \( SO(d) \) group.

### B. Ruling out the \( d > 3 \) case.

Let us define one basis set of two generalized bit product states:

\[
\psi_1 = (e_1, e_1, T_0 = e_1 e_1^T) \quad (26)
\]

\[
\psi_2 = (-e_1, -e_1, T_0) \quad (27)
\]

\[
\psi_3 = (e_1, -e_1, -T_0) \quad (28)
\]

\[
\psi_4 = (e_1, -e_1, -T_0) \quad (29)
\]

with \( e_1 = (1, 0, \ldots, 0)^T \). Now, we define two subspaces \( S_{12} \) and \( S_{34} \) spanned by the states \( \psi_1, \psi_2, \psi_3, \psi_4 \), respectively. Axiom 1 states that these two subspaces behave like one-bit spaces, therefore they are isomorphic to the \( (d - 1) \)-sphere \( S_{12} \cong S_{34} \cong S_{d-1} \). The state \( \psi \) belongs to \( S_{12} \) if and only if the following holds:

\[
P_{12}(\psi, \psi_1) + P_{12}(\psi, \psi_2) = 1. \quad (30)
\]

Since the \( \psi_1, \ldots, \psi_4 \) form a complete basis set, we have

\[
P_{12}(\psi, \psi_3) = 0, \quad P_{12}(\psi, \psi_4) = 0. \quad (31)
\]

A similar reasoning holds for states belonging to the \( S_{34} \) subspace. Since the states \( \psi \in S_{12} \) and \( \psi' \in S_{34} \) are perfectly distinguishable in a single shot experiment, we have \( P_{12}(\psi, \psi') = 0 \). Therefore, \( S_{12} \) and \( S_{34} \) are orthogonal subspaces.

Axiom 1 requires the existence of entangled states as it is apparent from the following Lemma 2.

**Lemma 2.** The only product states belonging to \( S_{12} \) are \( \psi_1 \) and \( \psi_2 \).

We define a local mapping between orthogonal subspaces \( S_{12} \) and \( S_{34} \). Let the state \( \psi = (x, y, T) \in S_{12} \), with \( x = (x_1, x_2, \ldots, x_d)^T \) and \( y = (y_1, y_2, \ldots, y_d)^T \). Consider the one-bit transformation \( R \) with the property \( Re_1 = -e_1 \). The local transformation of this type maps the state from \( S_{12} \) to \( S_{34} \) as shown by the following lemma:

**Lemma 3.** If the state \( \psi \in S_{12} \), then \( \psi' = (R, \mathbb{1})\psi \in S_{34} \) and \( \psi'' = (\mathbb{1}, R)\psi \in S_{34} \).

Let us define \( T_i^{(x)} = (T_{i1}, \ldots, T_{id}) \) and \( T_i^{(y)} = (T_{i1}, \ldots, T_{id})^T \). The correlation tensor can be rewritten in two different ways:

\[
T = \begin{pmatrix} T_1^{(x)} & \cdots & T_d^{(x)} \\ T_1^{(y)} & \cdots & T_d^{(y)} \\ \vdots & \ddots & \vdots \\ T_1^{(y)} & \cdots & T_d^{(y)} \end{pmatrix} \quad (32)
\]

Consider now the case \( d > 3 \). We define local transformations \( R_i \) flipping the first and \( i \)-th coordinate and \( R_{jkl} \) flipping the first, \( j \)-th, \( k \)-th, and \( l \)-th coordinate with \( j \neq k \neq l \neq i \). Let \( \psi = (x, y, (T_1^{(x)}, \ldots, T_d^{(x)})^T) \) belong to \( S_{12} \). According to Lemma 2, the states \( \psi_i = (R_i, \mathbb{1})\psi \) and \( \psi_{jkl} = (R_{jkl}, \mathbb{1})\psi \) belong to \( S_{34} \), therefore \( P_{12}(\psi, \psi_i) = 0 \) and \( P_{12}(\psi, \psi_{jkl}) = 0 \). We have:

\[
0 = P_{12}(\psi, \psi_i) \quad (33)
\]

\[
1 - x_1^2 + x_2^2 + \cdots - x_i^2 + \cdots + x_d^2 + ||y||^2 \quad (34)
\]

\[
-||T_1^{(x)}||^2 + ||T_2^{(x)}||^2 + \cdots - ||T_i^{(x)}||^2 + \cdots + ||T_d^{(x)}||^2 \quad (35)
\]

\[
= 1 - 2x_1^2 - 2x_2^2 - 2||T_1^{(x)}||^2 - 2||T_2^{(x)}||^2 + ||x||^2 + ||y||^2 + ||T||^2 \quad (36)
\]

Similarly, we expand \( P_{12}(\psi, \psi_{jkl}) = 0 \) and together with the last equation we obtain:

\[
x_1^2 + x_2^2 + ||T_1^{(x)}||^2 + ||T_2^{(x)}||^2 = 2 \quad (37)
\]

\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 + ||T_1^{(x)}||^2 + ||T_2^{(x)}||^2 + ||T_3^{(x)}||^2 + ||T_4^{(x)}||^2 = 2. \quad (38)
\]

Since this has to hold for all \( i, j, k, \) \( l \) we have:

\[
x_2 = x_3 = \cdots = x_d = 0 \quad (39)
\]

\[
T_2^{(x)} = T_3^{(x)} = \cdots = T_d^{(x)} = 0. \quad (40)
\]

We repeat this kind of reasoning for the transformations \((\mathbb{1}, R_i)\) and \((\mathbb{1}, R_{jkl})\) and obtain:

\[
y_1^2 + y_2^2 + ||T_1^{(y)}||^2 + ||T_2^{(y)}||^2 = 2 \quad (41)
\]

\[
y_1^2 + y_2^2 + y_3^2 + y_4^2 + ||T_1^{(y)}||^2 + ||T_2^{(y)}||^2 + ||T_3^{(y)}||^2 + ||T_4^{(y)}||^2 = 2. \quad (42)
\]

The only non-zero element of the correlation tensor is \( T_{11} \) and it has to be exactly 1, since \( ||T|| \geq 1 \). This implies that \( \psi \) is a product state, furthermore \( \psi = \psi_1 \) or \( \psi = \psi_2 \).

This concludes our proof that only the cases \( d = 1 \) and \( d = 3 \) are in agreement with our three axioms. To distinguish between the two cases, one can invoke the continuity axiom (3’) and proceed as in the reconstruction given by Hardy [19].
VII. “TWO” QUANTUM MECHANICS

We now obtain two solutions for the theory of a composite system consisting of two bits in the case when \( d = 3 \). One of them corresponds to the standard quantum theory of two qubits, the other one to its “mirror” version in which the states are obtained from the ones from the standard theory by partial transposition. Both solutions are regular as far as one considers composite systems of two bits, but the “mirror” one cannot be consistently constructed already for systems of three bits.

Two conditions (30) and (31) put the constraint to the form of \( \psi \):

\[
    x_1 = -y_1, \quad T_{11} = 1. \tag{43}
\]

The subspace \( S_{12} \) is isomorphic to the sphere \( S^2 \). Let us choose \( \psi \) complementary to the one bit basis \( \{ \psi_1, \psi_2 \} \) in \( S_{12} \). We have \( P_{12}(\psi_1, \psi_1) = P_{12}(\psi_2, \psi_2) = 1/2 \) and thus \( x_1 = y_1 = 0 \). For simplicity we write \( \psi \) in the form:

\[
    \psi = \begin{pmatrix} 0 \\ x \\ y \\ T_x \\ T_y \end{pmatrix}. \tag{44}
\]

with \( x = (x_2, x_3)^T, \ y = (y_2, y_3)^T, \ T_x = (T_{12}, T_{13})^T, \ T_y = (T_{21}, T_{31})^T \) and \( T = \begin{pmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{pmatrix} \).

Let \( R(\phi) \) be a rotation around the \( e_1 \) axis. This transformation keeps \( S_{12} \) invariant. Now, we show that the state \( \psi \) as given by equation (44) cannot be invariant under local transformation (\( \mathbb{I}, R(\phi) \)). To prove this by reductio ad absurdum suppose the opposite, i.e. that (\( \mathbb{I}, R(\phi) \))\( \psi = \psi \). We have three conditions:

\[
    R(\phi) \mathbf{y} = \mathbf{y}, \quad T_y^R(\phi) = T_y^T, \quad TR^T(\phi) = T, \tag{45}
\]

which implies \( \mathbf{y} = 0, T_y^T = 0 \) and \( T = 0 \) thus

\[
    \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ T_2 \\ T_3 \end{pmatrix}. \tag{46}
\]

According to equations (37) and (40) we can easily check that \( ||x|| = 1 \), and thus \( \psi \) is locally equivalent to the state:

\[
    \psi' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \tag{47}
\]

Let \( \chi_1 = (-e_1, e_1, -e_1 e_1^T) \) and \( \chi_2 = (-e_3, -e_3, e_3 e_3^T) \). The two conditions \( P(\psi', \chi_1) \geq 0 \) and \( P(\psi', \chi_2) \geq 0 \) become

\[
    \frac{1}{4}(1 - 1 - T_2^y) = -\frac{1}{4}T_2^y \geq 0 \tag{48}
\]

\[
    \frac{1}{4}(1 - 1 + T_2^y) = \frac{1}{4}T_2^y \geq 0 \tag{49}
\]

and thus \( T_2^y = 0 \). The normalization condition (23) gives \( T_2^z = \pm 1 \). The state \( \psi' \) is not physical. This can be seen when one performs the rotation (\( R, \mathbb{I} \)) where

\[
    R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{50}
\]

The transformed correlation tensor has a component \( \sqrt{2} \) which is non-physical. Therefore, the transformation (\( \mathbb{I}, R(\phi) \)) draws a full circle of pure states in a plane orthogonal to \( \psi_1 \) within the subspace \( S_{12} \). Similarly, the transformation (\( R(\phi), \mathbb{I} \)) draws the same set of pure states when applied to \( \psi \). Hence, for every transformation (\( \mathbb{I}, R(\phi) \)) there exists a transformation (\( R(\phi), \mathbb{I} \)) such that (\( \mathbb{I}, R(\phi) \))\( \psi \) = (\( R(\phi), \mathbb{I} \))\( \psi \). This gives us a set of conditions:

\[
    R(\phi) \mathbf{x} = \mathbf{x} \tag{51}
\]

\[
    R(\phi) \mathbf{y} = \mathbf{y} \tag{52}
\]

\[
    \mathbf{y} = \mathbf{T} \tag{53}
\]

\[
    T_y^R(\phi_1) = T_y^T \tag{54}
\]

\[
    R(\phi_2)T = TR^T(\phi_1) \tag{55}
\]

which are fulfilled if \( \mathbf{x} = \mathbf{y} = \mathbf{T} \) and \( \mathbf{T} = \text{diag}[1, 1, 1] \). We finally end up with two different solutions:

\[
    \psi_{\text{QM}} = (0, 0, 0, \text{diag}[1, 1, -1]) \quad \text{or} \quad \psi_{\text{QM}} = (0, 0, \text{diag}[1, 1, 1]). \tag{56}
\]

The first “M” in \( \psi_{\text{QM}} \) stands for “mirror”. The two solutions are incompatible and cannot coexist within the same theory. The first solution corresponds to the triplet state \( \psi^* \) of ordinary quantum mechanics. The second solution is a totally invariant state and has a negative overlap with, for example, the singlet state \( \psi^- \) for which \( T = \text{diag}[1, 1, -1, -1] \). That is, if the system were prepared in one of the two states and the other one was measured, the probability would be negative. Nevertheless, both solutions are regular at the level of two bits. The first belongs to ordinary quantum mechanics with the singlet in the “antiparallel” subspace \( S_{34} \) and the second solution is “the singlet state in the parallel subspace” \( S_{12} \). We will show that one can build the full state space, transformations and measurements in both cases. The states from one quantum mechanics can be obtained from the other by partial transposition \( \psi_{\text{QM}} = \psi_{\text{QM}} \). In particular, the four maximal entangled states (Bell states) from “mirror quantum mechanics” have correlations of the opposite sign of those from the standard quantum mechanics (see Figure 4).

Now we show that the theory with “mirror states” is physically inconsistent when applied to composite system of three bits.
bits. Let us first derive the full set of states and transformations for two qubits in standard quantum mechanics. We have seen that the state $\psi_{QM}$ belongs to the subspace $S_{12}$, and furthermore, that it is complementary (within $S_{12}$) to the product states $\psi_1$ and $\psi_2$. The totally mixed state within the $S_{12}$ subspace is $E_{12} = \frac{1}{2} \psi_1 + \frac{1}{2} \psi_2$. The states $\psi_1$ and $\psi_{QM}$ span one two-dimensional plane, and the set of pure states within this plane is a circle:

$$\psi(x) = E_{12} + \cos x (\psi_1 - E_{12}) + \sin x (\psi_{QM} - E_{12}) \quad (57)$$

$$= (\cos x \, e_i, \cos x \, e_i, \text{diag}[1, -\sin x, \sin x]). \quad (58)$$

We can apply a complete set of local transformations to the set $\psi(x)$ to obtain the set of all pure two-qubit states. Let us represent a pure state $\psi = (x, y, T)$ by the $4 \times 4$ Hermitian matrix $\rho$:

$$\rho = \frac{1}{4} \left( I \otimes \mathcal{P}_i \otimes I + \sum_{i=1}^{3} y_i \mathcal{P}_i \otimes \mathcal{P}_i + \sum_{i,j=1}^{3} T_{ij} \mathcal{P}_i \otimes \mathcal{P}_j \right), \quad (59)$$

where $\mathcal{P}_i, i \in \{1, 2, 3\}$, are the three Pauli matrices. It is easy to show that the set of states (57) corresponds to the set of one-dimensional projectors $|\psi(x)\rangle \langle \psi(x)|$, where $|\psi(x)\rangle = \cos \frac{\sqrt{2}}{2} (00) + \sin \frac{\sqrt{2}}{2} (11)$. The action of local transformations $(R_1, R_2)\psi$ corresponds to local unitary transformation $U_1 \otimes U_2|\psi(U)| \otimes U_1^T$, where the correspondence between $U$ and $R$ is given by the isomorphism between the groups $SU(2)$ and $SO(3)$:

$$U\rho U^T = \frac{1}{2} \left( I \otimes \mathcal{P}_i \otimes I + \sum_{i=1}^{3} \sum_{j=1}^{3} R_{ij} x_{ij} \mathcal{P}_i \otimes \mathcal{P}_j \right). \quad (60)$$

Here $R_{ij} = \text{Tr}(\sigma_i U \sigma_j U^T)$ and $x_i = \text{Tr}(\sigma_i \rho)$. When we apply a complete set of local transformations to the states $|\psi(x)\rangle$ we obtain the whole set of pure states for two qubits. The group of transformations is the set of unitary transformations $SU(4)$.

The set of states from “mirror quantum mechanics” can be obtained by applying partial transposition to the set of quantum states. Formally, partial transposition with respect to subsystem 1 is defined by action on a set of product operators:

$$PT_1(\rho_1 \otimes \rho_2) = \rho_1^T \otimes \rho_2. \quad (61)$$

where $\rho_1$ and $\rho_2$ are arbitrary operators. Similarly, we can define the partial transposition with respect to subsystem 2, $PT_2$. To each unitary transformation $U$ in quantum mechanics we define the corresponding transformation in “mirror mechanics”, e.g. with respect to subsystem 1: $PT_1 U PT_1$. Therefore, the set of transformations is a conjugate group $PT_1 SU(4) PT_1 := \{ PT_1 U PT_1 \mid U \in SU(4) \}$. Note that we could equally have chosen to apply partial transposition with respect to subsystem 2, and would obtain the same set of states. In fact, one can show that $PT_1 U PT_1 = PT_2 U^* PT_2$, where $U^*$ is a conjugate unitary transformation (see Lemma 4 in the Appendix). Therefore, the two conjugate groups are the same $PT_1 SU(4) PT_1 = PT_2 SU(4) PT_2$. We can generate the set of “mirror states” by applying all the transformations $PT_1 U PT$ to some product state, regardless of which particular partial transposition is used.

Now, we show that “mirror mechanics” cannot be consistently extended to composite systems consisting of three bits. Let $\psi_p = (x, y, z, T_{12}, T_{13}, T_{23}, T_{23})$ be some product state of three bits, where $x$, $y$ and $z$ are local Bloch vectors, $T_{12}$, $T_{13}$, $T_{23}$ and $T_{23}$ are two- and three-body correlation tensors, respectively. We can apply the transformations $PT_1 U PT$ to a composite system of $i$ and $j$, and we are free to choose with respect to which subsystem ($i$ or $j$) to take the partial transposition. Furthermore, we can combine transformations in 12 and 13 subsystems such that the resulting state is genuine three-partite entangled, and we can choose to partially transpose subsystem 2 in both cases. We obtain the transformation

$$U_{123} = PT_2 U_{12} PT_2 PT_3 U_{23} PT_2 \quad (62)$$

$$= PT_2 U_{12} U_{23} PT_2. \quad (63)$$

When we apply $U_{123}$ to $\psi_p$ we obtain the state $PT_2 U_{12} U_{23} \phi_p$, where $\phi_p = PT_2 \psi_p$ is again some product state. The state $U_{123} U_{23} \phi_p$ is a quantum three qubit state. Since states $\psi_p$ and $\phi_p$ are product states and do belong to standard quantum states, we can use the formalism of quantum mechanics and denote them as $|\psi_p\rangle$ and $|\phi_p\rangle$. Furthermore, since the state $|\psi_p\rangle$ is an arbitrary product state, without loss of generality we set $|\phi_p\rangle = |0\rangle|0\rangle|0\rangle$. We can choose $U_{12}$ and $U_{23}$ such that:

$$U_{12}|0\rangle|0\rangle = |0\rangle|0\rangle \quad (64)$$

$$U_{23}|0\rangle|1\rangle = \frac{1}{\sqrt{2}} (|0\rangle|1\rangle + |1\rangle|0\rangle) \quad (65)$$

$$U_{23}|0\rangle|0\rangle = \frac{1}{\sqrt{3}} (|0\rangle|1\rangle + |1\rangle|0\rangle) \quad (66)$$

This way we can generate the $W$-state

$$|W\rangle = U_{12} U_{23} |0\rangle|0\rangle|0\rangle = \frac{1}{\sqrt{3}} (|0\rangle|0\rangle|1\rangle + |0\rangle|1\rangle|0\rangle + |1\rangle|0\rangle|0\rangle). \quad (67)$$

When we apply partial transposition with respect to subsystem 2, we obtain the corresponding “mirror W-state” which we denote as $W_{QM}$-state, $W_{QM} = PT_2 W$. The local Bloch vectors and two-body correlation tensors for the $W$ state are

$$x = y = z = (0, 0, \frac{1}{\sqrt{3}})^T \quad (68)$$

$$T_{12} = T_{13} = T_{23} = \text{diag} [\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}] \quad (69)$$

$$T_{12} = T_{13} = T_{23} = \text{diag} [\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}] \quad (70)$$

The asymmetry in the signs of correlations in the tensors $T_{12}, T_{23}$ and $T_{13}$ leads to inconsistencies because they define three different reduced states $\psi_{ij} = (x_i, x_i, T_{ij}), \ ij \in \{12, 23, 13\}$, which cannot coexist within a single theory. The states $\psi_{12}$ and $\psi_{23}$ belong to “mirror quantum mechanics”, while the state $\psi_{13}$ belongs to ordinary quantum mechanics.
To see this, take the state $\psi = (0, 0, \text{diag}(-1, -1, 1))$ which is locally equivalent to state $\psi_{\text{MOM}} = (0, 0, 1)$. The overlap (measured probability) between the states $\psi_{13}$ and $\psi$ is negative

$$P(\psi, \psi_{13}) = \frac{1}{4} \left( 1 - \frac{2}{3} - \frac{2}{3} - \frac{1}{3} \right) = -\frac{1}{6}. \quad (74)$$

We conclude that “mirror quantum mechanics” – while being a perfectly regular solution for a theory of two bits – cannot be consistently extended to also describe systems consisting of many bits. This also answers the question why we find in nature only four types of correlations as given in the table (Figure 4) on the left, rather than all eight logically possible ones.

**VIII. HIGHER-DIMENSIONAL SYSTEMS AND STATE UPDATE RULE IN MEASUREMENT**

Having obtained $d = 3$ for a two-dimensional system we have derived quantum theory of this system. We have also reconstructed quantum mechanics of a composite system consisting of two qubits. Further reconstruction of quantum mechanics can be proceeded as in Hardy’s work [19]. In particular, the reconstruction of higher-dimensional systems from the two-dimensional ones and the general transformations of the state after measurement are explicitly given there. We only briefly comment on them here.

In order to derive the state space, measurements and transformations for a higher-dimensional system, we can use quantum theory of a two-dimensional system in conjunction with axiom 1. The axiom requires that upon any two linearly independent states one can construct a two-dimensional subspace that is isomorphic to the state space of a qubit (2-sphere). The state space of a higher dimensional system can be characterized such that if the state is restricted to any given two dimensional subspace, then it behaves like a qubit. The fact that all other (higher-dimensional) systems can be built out of two-dimensional ones suggests that the latter can be considered as fundamental constituents of the world and gives a justification for the usage of the term “elementary system” in the formulations of the axioms.

When a measurement is performed and an outcome is obtained, our knowledge about the state of the system changes and its representation in form of the probabilities must be updated to be in agreement with the new knowledge acquired in the measurement. This is the most natural update rule present in any probability theory. Only if one views this change as a real physical process conceptual problems arise related to discontinuous and abrupt “collapse of the wave function”. There is no basis for any such assumption. Associated with each outcome is the measurement vector $p$. When the outcome is observed the state after the measurement is updated to $p$ and the measurement will be a certain transformation on the initial state. Update rules for more general measurements can accordingly be given.

**IX. WHAT THE PRESENT RECONSTRUCTION TELLS US ABOUT QUANTUM MECHANICS**

It is often said that reconstructions of quantum theory within an operational approach are devoid of ontological commitments, and that nothing can be generally said about the ontological content that arises from the first principles or about the status of the notion of realism. As a supporting argument one usually notes that within a realistic world view one would anyway expect quantum theory at the operational level to be deducible from some underlying theory of “deeper reality”. After all, we have the Broglie-Bohm theory [58] which is a nonlocal realistic theory in full agreement with the predictions of (non-relativistic) quantum theory. Having said this, we cannot but emphasize that realism does stay “orthogonal” to the basic idea behind our reconstruction.

Be it local or nonlocal, realism asserts that outcomes correspond to actualities objectively existing prior to and independent of measurements. On the other hand, we have shown that the finiteness of information carrying capacity of quantum systems is an important ingredient in deriving quantum theory. This capacity is not enough to allow assignment of definite values to outcomes of all possible measurements. The elementary system has the information carrying capacity of one bit. This is signified by the possibility to decompose any state of an elementary system (qubit) in quantum mechanics in two orthogonal states. In a realistic theory based on hidden variables and an “epistemic constraint” on an observer’s knowledge of the variables’ values one can reproduce this feature at the level of the entire distribution of the hidden variables [59]. That this is possible is not surprising if one bears in mind that hidden-variable theories were at the first place introduced to reproduce quantum mechanics and yet give a more complete description [67]. But any realism of that kind at the same time assumes an infinite information capacity at the level of hidden variables. Even to reproduce measurements on a single qubit requires infinitely many orthogonal hidden-variable states [60, 61, 62]. It might be a matter of taste whether or not one is ready to work with this “ontological access baggage” [60] not doing any explanatory work at the operational level. But it is certainly conceptually distinctly different from the theory analyzed here, in which the information capacity of the most elementary systems – those which are by definition not reducible further – is fundamentally limited.

To further clarify our position consider the Mach-Zehnder interferometer in which both the path information and interference observable are dichotomic, i.e. two-valued observables. It is meaningless to speak about “the path the particle took in the interferometer in the interference experiment” because this would already require to assign 2 bits of information to the system, which would exceed its information capacity of 1 bit [63]. The information capacity of the system is simply not enough to provide definite outcomes to all possible measurements. Then, by necessity the outcome in some experiments must contain an element of randomness and there must be observables that are complementarity to each other. Entanglement and consequently the violation of Bell’s inequality (and thus of local realism) arise from the possibility
to define an abstract elementary system carrying at most one bit such that correlations (“00” and “11” in a joint measurement of two subsystems) are basis states.

X. CONCLUSIONS

Quantum theory is our most accurate description of nature and is fundamental to our understanding of, for example, the stability of matter, the periodic table of chemical elements, and the energy of the sun. It has led to the development of great inventions like the electronic transistor, the laser, or quantum cryptography. Given the enormous success of quantum theory, can we consider it as our final and ultimate theory? We have to define an abstract elementary system carrying at most one bit such that correlations (“00” and “11” in a joint measurement of two subsystems) are basis states.

Quantum theory has caused much controversy in interpreting what its philosophical and epistemological implications are. At the heart of this controversy lies the fact that the theory makes only probabilistic predictions. In recent years it was however shown that some features of quantum theory that one might have expected to be uniquely quantum, turned out to be highly generic for generalized probabilistic theories. Is there any reason why the universe should obey the laws of quantum theory, as opposed to any other possible probabilistic theory?

In this work we have shown that classical probability theory and quantum theory — the only two probability theories for which we have empirical evidences — are special in a way that they fulfill three reasonable axioms on the systems’ information carrying capacity, on the notion of locality and on the reversibility of transformations. The two theories can be separated if one restricts the transformations between the pure states to be continuous [19]. An interesting finding is that quantum theory is the only non-classical probability theory that can exhibit entanglement without conflicting one or more axioms. Therefore — to use Schrödinger’s words [64] — entanglement is not only “the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought”, but also the one that enforces the departure from a broad class of more general probabilistic theories.

Acknowledgments

We thank M. Aspelmeyer, J. Kofler, T. Paterek and A. Zeilinger for discussions. We acknowledge support from the Austrian Science Foundation FWF within Project No. P19570-N16, SFB and CoQuS No. W1210-N16, the European Commission Project QAP (No. 015848) and the Foundational Question Institute (FQXi).

XI. APPENDIX

In this appendix we give the proofs of the lemmas from the main text.

Lemma 1. The lower bound \( \|T\| = 1 \) is saturated, if and only if the state is a product state \( T = xy^T \).

Proof. If the state is a product state then \( \|T\|^2 = \|x\|^2 \|y\|^2 = 1 \). On the other hand, assume that the state \( \psi = (x, y, T) \) satisfies \( \|T\| = 1 \). Normalization (25) gives \( \|x\| = \|y\| = 1 \). Let \( \phi_p = (-x, -y, T_0 = xy^T) \) be a product state. We have \( P(\psi, \phi_p) \geq 0 \) and therefore

\[
1 - \|x\|^2 - \|y\|^2 + \text{Tr}(T^T T_0) = -1 + \text{Tr}(T^T T_0) \geq 0. \quad (75)
\]

The last inequality \( \text{Tr}(T^T T_0) \geq 1 \) can be seen as \( (T, T_0) \geq 1 \) where \( (\cdot, \cdot) \) is the scalar product in Hilbert-Schmidt space. Since the vectors \( T, T_0 \) are normalized, \( \|T\| = \|T_0\| = 1 \), the scalar product between them is always \( (T, T_0) \leq 1 \). Therefore, we have \( (T, T_0) = 1 \) which is equivalent to \( T = T_0 = xy^T \).

QED

Lemma 2. The only product states belonging to \( S_{12} \) are \( \psi_1 \) and \( \psi_2 \).

Proof. Let \( \psi_p = (x, y, xy^T) \in S_{12} \). We have

\[
1 = P_{12}(\psi_p, \psi_1) + P_{12}(\psi_p, \psi_2) \quad (76)
\]

\[
= \frac{1}{4} \left( (1 + x e_1 + e_y + (x e_1)(y e_1)) + (1 - x e_1 - e_y + (x e_1)(y e_1)) \right) \quad (77)
\]

\[
= \frac{1}{2} (1 + (x e_1)(y e_1)) \quad (78)
\]

\[
\Rightarrow e_x = e_y = 1 \quad \vee \quad e_x = e_y = -1 \quad (80)
\]

\[
\Leftrightarrow x = y = e_1 \quad \vee \quad x = y = -e_1. \quad (81)
\]

QED

Lemma 3. If the state \( \psi \in S_{12} \), then \( \psi' = (R, \mathbb{I})\psi \in S_{34} \) and \( \psi'' = (\mathbb{I}, R)\psi \in S_{34} \).

Proof. If \( \psi \in S_{12} \) we have

\[
1 = P_{12}(\psi, \psi_1) + P_{12}(\psi, \psi_2) \quad (82)
\]

\[
= P_{12}((R, \mathbb{I})\psi, (R, \mathbb{I})\psi_1) + P_{12}((R, \mathbb{I})\psi, (R, \mathbb{I})\psi_2) \quad (83)
\]

\[
= P_{12}(\psi', \psi_3) + P_{12}(\psi', \psi_4). \quad (84)
\]

Similarly, one can show that \( (\mathbb{I}, R)\psi \in S_{34} \).

QED

Lemma 4. Let \( U \) be some operator with the following action in the Hilbert-Schmidt space: \( U(\rho) = U \rho U^\dagger \), and \( PT_1 \) and \( PT_2 \) are partial transpositions with respect to subsystems 1 and 2, respectively. The following identity holds: \( PT_1 \ UPT_1 = PT_2 \ U^\dagger \ UPT_2 \), where \( U^* \) is the complex-conjugate operator.

Proof. We can expand \( U \) into some product basis in the Hilbert-Schmidt space \( U = \sum_{ijkl} u_{ijkl} A_i^k \otimes B_j^l \). We have

\[
PT_1 \ UPT_1 (\rho_1 \otimes \rho_2) = PT_1 [U \rho_1^T \otimes \rho_2 U^\dagger] \quad (85)
\]

\[
= \sum_{ijkl} u_{ijkl} (A_i^k \rho_1 A_i^k) \otimes (B_j^l \rho_2 B_j^l) \quad (86)
\]

\[
= PT_2 \sum_{ijkl} u_{ijkl} (A_i^k \otimes B_j^l) (\rho_1 \otimes \rho_2) (A_i^k \otimes B_j^l) \quad (87)
\]

\[
= PT_2 \ U^\dagger \ UPT_2 (\rho_1 \otimes \rho_2), \quad (88)
\]

QED
for arbitrary operators $\rho_1$ and $\rho_2$.

\section*{QED}

\begin{enumerate}
\item S. Kochen and E.P. Specker, \textit{The Problem of Hidden Variables in Quantum Mechanics}, J. Math. Mech. 17, 59 (1967).
\item J.S. Bell, \textit{On the Einstein-Podolsky-Rosen paradox}, Physics 1, 195-200 (1964); reprinted in J.S. Bell, “Speakable and Unspeakable in Quantum Mechanics” (Cambridge Univ. Press, Cambridge, 1987).
\item A.J. Leggett, \textit{Nonlocal Hidden-Variables Theories and Quantum Mechanics: An Incompatibility Theorem}, Found. Phys. 33, 1469 (2003).
\item S. Popescu and D. Rohrlich, \textit{Quantum Nonlocality and Contextuality}, (1994) (arXiv:quant-ph/9403037).
\item R. Spekkens, \textit{Contextual objectivity: a realistic interpretation of quantum mechanics}, Eur. J. Phys. 23, 823–844 (2002) (arXiv:quant-ph/0205039).
\item A. Grinbaum, \textit{Quantum Mechanics and Hilbert Space}, (Cambridge University Press, Cambridge UK), (arXiv:0807.4383).
\item C.A. Fuchs and R. Schack, \textit{Informational origins of quantum theory}, Int. J. Quantum Inform. 7(8), 1165 (2009). (arXiv:0901.2492).
\end{enumerate}
[40] M. Navascues and H. Wunderlich, *A glance beyond the quantum model* (arXiv:0907.0372).

[41] K. Zyczkowski, *Quartic quantum theory: an extension of the standard quantum mechanics*, J. Phys. A 41, 355302 (2008).

[42] T. Paterek, B. Dakic, C. Brukner, *Theories of systems with limited information content* (2008) (arXiv:0804.1423).

[43] A. Peres, *Proposed test for complex versus quaternion quantum theory*, Phys. Rev. Lett. 42, 683 (1979).

[44] H. Kaiser, E.A. George, and S.A. Werner, *Neutron interferometric search for quaternions in quantum mechanics*, Phys. Rev A 29, 2276 (1984).

[45] A. Peres, *Quaternionic quantum interferometry*, in “Quantum Interferometry”, Eds. F. De Martini et al., (VCH Publ., 1996), 431–437 (arXiv:quant-ph/9605024).

[46] R. D. Sorkin, *Quantum Mechanics as Quantum Measure Theory*, Mod. Phys. Lett. A 9, 3119 (1994) (arXiv:gr-qc/9401003).

[47] U. Sinha, C. Couteau, Z. Medendorp, I. Söllner, R. Laflamme, R. Sorkin and G. Weihs, *Testing Born’s Rule in Quantum Mechanics with a Triple Slit Experiment*, (2008) (arXiv:0811.2068). Submitted to the proceedings of Foundations of Probability and Physics-5, Vaxjo, Sweden, August 2008.

[48] A. Grinbaum, *Reconstruction of Quantum Theory*, Brit. J. Phil. Sci. 8, 387 (2007).

[49] D. Gross, M. Mueller, R. Colbeck, and O.C.O. Dahlsten, *All reversible dynamics in maximally non-local theories are trivial* (2009) (arXiv:0910.1840). O.C.O. Dahlstein, private communication.

[50] C.F. von Weizsäcker, 1958, *Aufbau der Physik* (Carl Hanser, München,1958).

[51] J.A. Wheeler, *Law without Law in Quantum Theory and Measurement*, Eds. J.A. Wheeler and W.H. Zurek (Princeton University Press, Princeton, 1983) 182.

[52] H. Barnum and A. Wilce, *Information processing in convex operational theories*, to be published in DCM/QPL (Developments in Computational Models / Quantum Programming Languages) (Oxford University, 2009) (arXiv:0908.5352).

[53] S. Abramsky and B. Coecke, *A categorical semantics of quantum protocols*, Proc. 19th IEEE Conference on Logic in Computer Science, 415–425 (IEEE Computer Science Press, 2004).

[54] See, for example, J. Ambjorn, J. Jurkiewicz and R. Loll, *Reconstructing the Universe*, Phys. Rev. D 72 064014 (2005).

[55] H. Boerner, *Representations of groups*, (North- Holland publishing company, Amsterdam 1963).

[56] R.A. Horn and C.R. Johnson, *Matrix Analysis*, (Cambridge University Press, Chapter 8, 1990).

[57] R. Spekkens, private communication.

[58] D. Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of “Hidden Variables” I*, Phys. Rev. 85, 166 (1952). D. Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of “Hidden Variables” II*, Phys. Rev. 85, 180 (1952).

[59] See Ref [35] for a local version of such hidden-variable theory in which quantum mechanical predictions are partially reproduced.

[60] L. Hardy, *Quantum Ontological Excess Baggage*, Stud. Hist. Philos. Mod. Phys. 35, 267 (2004).

[61] A. Montina, *Exponential growth of the ontological space dimension with the physical size*, Phys. Rev. A 77, 022104 (2008).

[62] B. Dakic, M. Suvakov, T. Paterek, and Č. Brukner, *Efficient Hidden-Variable Simulation of Measurements in Quantum Experiments*, Phys. Rev. Lett. 101, 190402 (2008).

[63] Č. Brukner and A. Zeilinger, *Young’s experiment and the finiteness of information*, Phil. Trans. R. Soc. Lond. A 360, 1061 (2002).

[64] E. Schrödinger, *Discussion of Probability Relations Between Separated Systems*, Proceedings of the Cambridge Philosophical Society 31 (1935) 555-563; 32 (1936): 446-451

[65] Hardy considers unnormalized states and for that reason takes $K = d + 1$ (in his notation) as the number of degrees of freedom.

[66] As noted by Zyczkowski [41] it is thinkable that within the time scales of standard experimental conditions “hyper-decoherence” may occur which cause a system described in the framework of the higher-order theory to specific properties and behavior according to predictions of standard (complex) quantum theory.

[67] That this cannot be done without allowing nonlocal influences from space-like distant regions is a valid point for itself, which we do not want to follow here further.