NORMAL FORMS OF MODULES OVER ADMISSIBLE ALGEBRAS WITH FORMAL TWO-RAY MODULES

GRZEGORZ BOBIŃSKI

Abstract. The aim of the paper is to classify the indecomposable modules and describe the Auslander–Reiten sequences for admissible algebras with formal two-ray modules.

Introduction

Throughout the paper $k$ is a fixed algebraically closed field. All considered categories are additive $k$-categories and all functors are $k$-functors.

One of the aims of the representation theory of finite-dimensional algebras is a description of indecomposable modules and homomorphism spaces between them. A guiding example is that of special biserial algebras, for which a full description of the indecomposable modules and the Auslander–Reiten sequences was given by Wald and Waschbüsch [22] (see also [10]). Homomorphism spaces between indecomposable modules were also investigated (see for example [12]). Another class of algebras whose representation theory is described is formed by clannish algebras (or more generally, clan problems) introduced by Crawley-Boevey [14] (see also [6,15]). Homomorphism spaces and Auslander–Reiten sequences for this class of problems were studied by Geiß [17] (see also [18] for a description of the Auslander-Reiten components).

According to Drozd’s Tame and Wild Theorem [16] (see also [11]) one may hope to obtain classifications like these above only for so called tame algebras. First examples of tame algebras are provided by the representation-finite algebras, for which there are only finitely many isomorphism classes of indecomposable modules. The representation theory of the representation-finite algebras has been intensively studied (see for example [1,7–9]) and seems to be well-understood. One knows that an algebra is representation-finite if and only if its infinite radical vanishes.

The first level in the hierarchy of representation-infinite algebras is occupied by the domestic algebras, for which in each dimension all but finitely many indecomposable modules can be parameterized by
finitely many lines (see also [13] for a different characterization of the domestic algebras). Schröer’s work [20] on the infinite radical of special biserial algebras gives hope to characterize the domestic algebras in terms of the infinite radical. In [3] (continued by [2, 5]), we initiated the study of a new class of domestic algebras, which may be seen as a test class for this characterization. The results obtained so far concern the Auslander–Reiten theory. In order to deal with the infinite radical one needs to have a more precise knowledge about indecomposable modules and homomorphisms spaces between them. In this paper we make a first step in this direction, namely we give a description of the indecomposable modules. This description resembles the description obtained for clans, thus one may hope that the corresponding results about homomorphisms can be also transferred.

The paper is organized as follows. In Section 1 we present the main result of the paper, in Section 2 we recall necessary information about vector space categories, and in final Section 3 we prove the main theorem. The paper was written during the author held a one year post-doc position at the University of Bern. Author gratefully acknowledges the support from the Schweizerischer Nationalfonds and the Polish Scientific Grant KBN No. 1 P03A 018 27.

1. Strings, the corresponding modules and the main result

In this section we first introduce notation, which is necessary to formulate the main result of the paper given at the end of the section.

1.1. In the paper, by $\mathbb{Z}$ (respectively, $\mathbb{N}_0$, $\mathbb{N}$) we denote the set of (nonnegative, positive) integers. If $m$ and $n$ are integers, then by $[m,n]$ we denote the set of all integers $l$ such that $m \leq l \leq n$. For a sequence $f : [1,n] \rightarrow \mathbb{N}$, $n \in \mathbb{N}_0$, of positive integers we denote $n$ by $|f|$. We identify finite subsets of $\mathbb{N}$ with the corresponding increasing sequences of positive integers. In particular, if $F$ is a finite subset of $\mathbb{N}$ and $i \in [1,|F|]$, then $F_i$ denotes the $i$-th element of $F$ with respect to the usual order of integers.

1.2. By a quiver $Q$ we mean an oriented graph, i.e., a set of vertices $Q_0$, a set of arrows $Q_1$ and two maps $s_Q, t_Q : Q_1 \rightarrow Q_0$, which assign to an arrow $\alpha$ in $Q$ its starting and terminating vertex, respectively. If $\alpha \in Q_1$, $s_Q(\alpha) = x$ and $t_Q(\alpha) = y$, then we write $\alpha : x \rightarrow y$. By a path in $Q$ we mean a sequence $\rho = \alpha_1 \cdots \alpha_n$ of arrows in $Q$ such that $t_Q(\alpha_{i+1}) = s_Q(\alpha_i)$ for all $i \in [1,n-1]$. The number $n$ is called the length of $\rho$ and denoted $|\rho|$. We write $s_Q(\rho)$ for $s_Q(\alpha_n)$ and $t_Q(\rho)$ for $t_Q(\alpha_1)$, and we say that $\rho$ starts at $s_Q(\rho)$ and terminates at $t_Q(\rho)$. For each vertex $x$ of $Q$ we denote also by $x$ the path of length 0 at vertex $x$ ($s_Q(x) = x = t_Q(x)$). For paths $\rho = \alpha_1 \cdots \alpha_n$ and $\rho' = \alpha'_1 \cdots \alpha'_m$ in $Q$
such that \( s_Q(\rho) = t_Q(\rho') \), we denote by \( \rho \rho' \) the path \( \alpha_1 \cdots \alpha_n \alpha'_1 \cdots \alpha'_{m} \). In particular, \( \rho s_Q(\rho) = \rho = t_Q(\rho) \rho \).

1.3. By a defining system we mean a quadruple \((p, q, S, T)\), where \( p \) and \( q \) are sequences of positive integers such that \( |q| = |p| \) and \( \sum_{i=1}^{|p|} p_i \geq 2 \), and \( S = (S_i)_{i=1}^{|p|} \) and \( T = (T_i)_{i=1}^{|p|} \) are families of finite subsets of \( \mathbb{N} \) such that for each \( i \in [1, |p|] \) hold: \( T_i \subseteq S_i \subseteq [2, p_i + |T_i|] \), if \( j \in S_i \) then \( j + 1 \notin S_i \), and \( p_i + |T_i| \notin T_i \). We write \( T_{i,j} \) instead of \((T_i)_j\) for \( i \in [1, |p|] \) and \( j \in [1, |T_i|] \). Throughout the rest of the section \((p, q, S, T)\) is a fixed defining system.

1.4. We define a quiver \( Q \) by

\[
Q_0 = \{ x_{i,j} \mid i \in [1, |p|], \ j \in [0, p_i + |T_i|] \} \\
\cup \{ y_{i,j} \mid i \in [1, |p|], \ j \in [1, q_i - 1] \} \\
\cup \{ z_{i,j} \mid i \in [1, |p|], \ j \in S_i \}
\]

and

\[
Q_1 = \{ \alpha_{i,j} : x_{i,j} \to x_{i,j-1} \mid i \in [1, |p|], \ j \in [1, p_i + |T_i|] \} \\
\cup \{ \beta_{i,j} : y_{i,j} \to y_{i,j-1} \mid i \in [1, |p|], \ j \in [1, q_i] \} \\
\cup \{ \gamma_{i,j} : z_{i,j} \to x_{i,j} \mid i \in [1, |p|], \ j \in S_i \} \\
\cup \{ \xi_{i,j} : x_{i,p_i+j} \to z_{i,T_i,j} \mid i \in [1, |p|], \ j \in [1, |T_i|] \},
\]

where \( y_{i,0} = x_{i+1,0} \) (with \( x_{p_i+1,0} = x_{1,0} \)) and \( y_{i,q_i} = x_{i,p_i} \) for \( i \in [1, |p|] \).

Let \( A \) be the path algebra of \( Q \) bounded by relations

\[
\alpha_{i,j-1} \alpha_{i,j} \gamma_{i,j}, \ i \in [1, |p|], \ j \in S_i,
\]
\[
\beta_{i,q_i} \alpha_{i,p_i+1}, \ i \in [1, |p|] \text{ such that } |T_i| > 0,
\]
\[
\xi_{i,j-1} \alpha_{i,p_i+j}, \ i \in [1, |p|], \ j \in [2, |T_i|],
\]

and

\[
\alpha_{i,T_i,j} \gamma_{i,T_i,j} \xi_{i,j} - \alpha_{i,T_i,j} \alpha_{i,T_i,j+1} \cdots \alpha_{i,p_i+j-1} \alpha_{i,p_i+j}, \ i \in [1, |p|], \ j \in [1, |T_i|].
\]

Recall that by [2] Theorem 1.1 the class of algebras defined in the above way coincides with the class of admissible algebras with formal two-ray modules introduced in [3].

In order to clarify a bit the above definitions we give a simple example. If \( p = (6, 3), \ q = (2, 2), \ S = (\{2, 4, 6, 8\}, \{2\}) \) and \( T = (\{4, 6\}, \emptyset) \),
then $A$ is the path algebra of the quiver

bounded by relations

$\alpha_{1,1}\alpha_{1,2}\gamma_{1,2}$, $\alpha_{1,3}\alpha_{1,4}\gamma_{1,4}$, $\alpha_{1,5}\alpha_{1,6}\gamma_{1,6}$, $\alpha_{1,7}\alpha_{1,8}\gamma_{1,8}$, $\alpha_{2,1}\alpha_{2,2}\gamma_{2,2}$, $\beta_{1,2}\alpha_{1,7}$, $\xi_{1,1}\alpha_{1,8}$, $\alpha_{1,2}\alpha_{1,3}\alpha_{1,4}\alpha_{1,5}\alpha_{1,6}\alpha_{1,7} - \alpha_{1,2}\gamma_{1,2}\xi_{1,1}$, $\alpha_{1,6}\alpha_{1,7}\alpha_{1,8} - \alpha_{1,6}\gamma_{1,6}\xi_{1,2}$.

1.5. Let

$Q'_1 = \{\alpha_{i,j} : x_{i,j} \rightarrow x_{i,j-1} \mid i \in [1,|p|], j \in [1,p_i + |T_i|]\}$

and $Q''_1 = Q_1 \setminus Q'_1$. Let $Q^*$ be the quiver with same set of vertices and arrows as $Q$, but with the arrows from $Q''_1$ reversed, i.e., $Q'_0 = Q_0$, $Q'_1 = Q_1$ and

$s_{Q^*}(\alpha) = \begin{cases} s_Q(\alpha) & \alpha \in Q'_1, \\ t_Q(\alpha) & \alpha \in Q''_1, \end{cases}$ and $t_{Q^*}(\alpha) = \begin{cases} t_Q(\alpha) & \alpha \in Q'_1, \\ s_Q(\alpha) & \alpha \in Q''_1. \end{cases}$

By a string in $Q$ we mean a path in $Q^*$ which does not contain a subpath $\alpha_{i,T_i,j}\alpha_{i,T_i,j+1} \cdots \alpha_{i,p_i+j}$ for $i \in [1,|p|]$ and $j \in [1,|T_i|]$. For formal reasons we also introduce the empty string denoted by $\emptyset$. By convention the length of $\emptyset$ is $-1$, the maps $s_{Q^*}$ and $t_{Q^*}$ are not defined for $\emptyset$ and it cannot be composed with other strings. If $C$ is a string
and $C = C'C''$ for strings $C'$ and $C''$, then $C'$ is called a terminating substring of $C$ and $C''$ is called a starting substring of $C$.

If $C = c_1 \cdots c_n$ is a string and $x \in Q_0$, then we put

$$J^x_C = \{ i \in [0, n-1] \mid t_{Q^x}(c_{i+1}) = x \}$$

and

$$I^x_C = \begin{cases} J^x_C \cup \{ n \}, & s_{Q^x}(c_n) = x, \\ J^x_C, & s_{Q^x}(c_n) \neq x. \end{cases}$$

In particular, $J^x_y = \emptyset$ for all $y \in Q_0$, $I^x_0 = \{ 0 \}$, and $I^x_y = \emptyset$ if $y \neq x$.

1.6. For each vertex $x$ of $Q$ we denote by $\omega_x$ (respectively $\mu_x$) the longest string terminating at $x$ and consisting only of elements of $Q'_1$ ($Q''_1$). Similarly, by $\pi_x$ (respectively $\nu_x$) we denote the longest string starting at $x$ and consisting only of elements of $Q'_1$ ($Q''_1$).

Let

$$Q'_0 = \{ x_{i,j} \mid i \in [1, |p|], j \in S_i \},$$

and

$$Q''_0 = \{ x_{i,j} \mid i \in [1, |p|], j \in T_i \}.$$ 

For $x \in Q'_0$, $x = x_{i,j}$, we denote $\alpha_{i,j}$ by $\alpha_x$ and $\gamma_{i,j}$ by $\gamma_x$.

Let $x \in Q''_0$, $x = x_{i,T_{i,j}}$. We put

$$B_x = \alpha_{i,T_{i,j}+1} \cdots \alpha_{i,j} \xi_{i,j} \gamma_{i,j}.$$ 

For a string $C$ terminating at $x$ we denote by $p_C$ the maximal integer $p \geq 0$ such that $B_x^p$ is a terminating substring of $C$, where $B_x^p$ denotes the $p$-fold composition of $B_x$ with itself (with the convention that $B_x^0 = x$). If $x \in Q'_0 \setminus Q''_0$ then we set $B_x = x$ and $p_C = 0$ for each string $C$ terminating at $x$.

1.7. For a given vertex $x$ of $Q$ we introduce a linear order in the set of all strings terminating at $x$. Let $C$ and $C'$ be two strings terminating at $x$ and let $C_0$ be the longest string which is both a terminating substring of $C$ and a terminating substring of $C'$. Then $C < C'$ if and only if either $C = C_0 \beta D$ for $\beta \in Q'_1$ and a string $D$ or $C' = C_0 \alpha D'$ for $\alpha \in Q'_1$ and a string $D'$. Note that the maximal string terminating at $x$ is $\omega_x$ and the minimal one is $\mu_x$.

If $C \neq \omega_x$ is a string terminating at $x$, then there exists a direct successor $C_+$ of $C$, which can be described in the following way. If there exists $\alpha \in Q'_1$ such that $C \alpha$ is a string, then $C_+ = C \alpha \mu_{s_{Q^x}(\alpha)}$. Otherwise, there exist a string $C'$ and $\beta \in Q''_1$ such that $C = C' \beta \omega_{Q(\beta)}$. In this case $C_+ = C'$. We also put $(\omega_x)_+ = \emptyset$.

Similarly, we may define a string $+_C$, which is a direct successor of $C$ with respect to the appropriate order in the set of all strings starting at $s_{Q^x}(C)$. Since this order will play no role in the sequel, we only give a
description of \( C \). If there exists \( \beta \in Q_1'' \) such that \( \beta C \) is a string, then 
\[ +C = \pi_{\sigma(\beta)}\beta C. \]
Otherwise, 
\[ +C = C'' = \nu t_{Q(\alpha)}\alpha C'' \]
for \( \alpha \in Q_1' \) and a string \( C'' \), or 
\[ +C = \emptyset \text{ if } C = \nu x. \]

Let \( C \) be a string such that \( |C_+| + |+C| \geq |C| \) (this is equivalent to saying that \( C \neq \nu_x \omega_x \) for a vertex \( x \) of \( Q \)). Then we define \( +C_+ \) by

\[ +C_+ = \begin{cases} 
+(C_+) & C_+ \neq \emptyset, \\
(+/C) & +C \neq \emptyset.
\end{cases} \]

One easily verifies that the above definition is correct and \( +C_+ \neq \emptyset \).

We also put \( +(\nu_x \omega_x)_+ = \emptyset \) for \( x \in Q_0 \).

1.8. Let \( S \) be the set of all strings in \( Q \). For \( x \in Q_0' \) we denote by \( S_x \) the set of all strings \( C \) terminating at \( x \) such that \( \alpha_x C' \) is a string, where 
\[ C = B_x^{\nu c} C'' \quad (S_x \text{ is the set of all strings terminating at } x \text{ if } x \in Q_0' \setminus Q_0''). \]

Let \( P_x \) be the set all pairs \( (C, C') \) of \( C, C' \in S_x \) such that \( C < C' \) and, if \( x \in Q_0', C' \neq B_x C \). Finally, we put 
\[ B' = \{ B_x \mid x \in Q_0'' \} \text{ and } B = \{ B_0 \} \cup B', \]
where 
\[ B_0 = \alpha_{1,1} \cdots \alpha_{1,p_1} \beta_{1,1} \cdots \beta_{1,1} \cdots \alpha_{p,1} \cdots \alpha_{p,p} \beta_{p,1} \cdots \beta_{p,1}. \]

1.9. Let \( B = b_1 \cdots b_n \in B \), \( \lambda \in k^* \) and \( m \in \mathbb{N} \). We define a representation \( R(B, \lambda, m) \) of \( Q \) as follows:

\[ R(B, \lambda, m)_y = \bigoplus_{j \in [1,m]} \bigoplus_{i \in j_B'} kv_i^{(j)} \]

and

\[ R(B, \lambda, m)_{\alpha}(v_i^{(j)}) = \begin{cases} 
v_i^{(j)} & \alpha \in Q_1', \alpha = b_i, i \in [1, n-1], \\
v_i^{(j+1)} & \alpha \in Q_1'', \alpha = b_{i+1}, i \in [0, n-2], \\
\lambda v_0^{(j)} + v_0^{(j+1)} & \alpha = b_n, i = n-1, j \in [1, m-1], \\
\lambda v_0^{(m)} & \alpha = b_n, i = n-1, j = m, \\
0 & \text{otherwise}. \end{cases} \]

We also put \( R(B, \lambda, 0) = 0 \).

1.10. Let \( x \in Q_0'' \), \( B = b_1 \cdots b_n = B_x \) and \( m \in \mathbb{N} \). We define a representation \( Q(B, m) \) of \( Q \) as follows:

\[ Q(B, m)_y = \begin{cases} 
k v' \oplus \bigoplus_{j \in [1,m]} \bigoplus_{i \in j_C'} kv_i^{(j)} & y = t_Q(\alpha_x), \\
\bigoplus_{j \in [1,m]} \bigoplus_{i \in j_C'} kv_i^{(j)} & \text{otherwise}, \end{cases} \]
We also put

\[ Q(B, m)_{\alpha}(v_i^{(j)}) = \begin{cases} v' & \alpha = \alpha_x, i = 0, j = 1, \\
v_{i-1}^{(j)} & \alpha \in Q'_1, \alpha = b_i, i \in [1, n-1], \\
v_{i+1}^{(j)} & \alpha \in Q'_1, \alpha = b_{i+1}, i \in [0, n-2], \\
v_0^{(j)} + v_0^{(j+1)} & \alpha = b_n, i = n - 1, j \in [1, m-1], \\
v_0^{(m)} & \alpha = b_n, i = n - 1, j = m, \\
0 & \text{otherwise}, \end{cases} \]

and

\[ Q(B, m)_{\alpha}(v') = 0. \]

1.11. Let \( C = c_1 \cdots c_n \in S \). We define a representation \( M(C) \) of \( Q \) as follows:

\[ M(C)_y = \bigoplus_{i \in I^\alpha_C} kv_i \]

and

\[ M(C)_{\alpha}(v_i) = \begin{cases} v_{i-1} & \alpha \in Q'_1, \alpha = c_i, i \in [1, n], \\
v_{i+1} & \alpha \in Q'_1, \alpha = c_{i+1}, i \in [0, n-1], \\
0 & \text{otherwise.} \end{cases} \]

In particular, \( M(x) \) is the simple representation of \( Q \) at \( x \). We also put \( M(\emptyset) = 0 \).

1.12. Let \( x \in Q'_0 \) and \( C = c_1 \cdots c_n \in S_x \). We define a representation \( N(C) \) of \( Q \) as follows:

\[ N(C)_y = \begin{cases} kv' \oplus \bigoplus_{i \in I^\alpha_C} kv_i & y = t_Q(\alpha_x), \\
kv'' \oplus \bigoplus_{i \in I^\alpha_C} kv_i & y = s_Q(\gamma_x), \\
\bigoplus_{i \in I^\alpha_C} kv_i & \text{otherwise}, \end{cases} \]

\[ N(C)_{\alpha}(v_i) = \begin{cases} v' & \alpha = \alpha_x, i = p|B_x|, p \in [0, p_C], \\
v_{i-1} & \alpha \in Q'_1, \alpha = c_i, i \in [1, n], \\
v_{i+1} & \alpha \in Q'_1, \alpha = c_{i+1}, i \in [0, n-1], \\
0 & \text{otherwise}, \end{cases} \]

\[ N(C)_{\alpha}(v') = 0, \]

and

\[ N(C)_{\alpha}(v'') = \begin{cases} v_0 & \alpha = \gamma_x, \\
0 & \text{otherwise.} \end{cases} \]

We also put \( N(\emptyset) = M(s_Q(\gamma_x)) \) (more precisely, we should write \( N_x(\emptyset) \), but we omit the vertex if it causes no confusion).
1.13. Let \( x \in Q''_0 \) and \( C = c_1 \cdots c_n \in \mathcal{S}_x \) be such that \( p_C > 0 \). We define a representation \( L(C) \) of \( Q \) as follows:

\[
L(C)_y = \begin{cases} 
kv' \oplus \bigoplus_{i \in I_C^y} kv_i & y = t_Q(\alpha_x), \\
\bigoplus_{i \in I_C^y} kv_i & \text{otherwise}, 
\end{cases}
\]

\[
L(C)_{\alpha}(v_i) = \begin{cases} 
v' \quad \alpha = \alpha_x, \ i = p|B_x|, \ p \in [0, p_C], \\
v_{i-1} \quad \alpha \in Q'_1, \ \alpha = c_i, \ i \in [1, n], \\
v_{i+1} \quad \alpha \in Q''_1, \ \alpha = c_i, \ i \in [0, n - 1], \\
0 \quad \text{otherwise}, 
\end{cases}
\]

and

\[
L(C)_{\alpha}(v') = 0.
\]

1.14. Let \( x \in Q'_0 \) and \( (C = c_1 \cdots c_n, C' = c'_1 \cdots c'_m) \in \mathcal{P}_x \). We define a representation \( N(C, C') \) of \( Q \) as follows:

\[
N(C, C')_y = \begin{cases} 
kv' \oplus \bigoplus_{i \in I_C^y} kv_i \oplus \bigoplus_{i \in I_C^{y'}} kv_i' & y = t_Q(\alpha_x), \\
kv'' \oplus \bigoplus_{i \in I_C^y} kv_i \oplus \bigoplus_{i \in I_C^{y'}} kv_i' & y = s_Q(\gamma_x), \\
\bigoplus_{i \in I_C^y} kv_i \oplus \bigoplus_{i \in I_C^{y'}} kv_i' & \text{otherwise}, 
\end{cases}
\]

\[
N(C, C')_{\alpha}(v_i) = \begin{cases} 
v' \quad \alpha = \alpha_x, \ i = p|B_x|, \ p \in [0, p_C], \\
v_{i-1} \quad \alpha \in Q'_1, \ \alpha = c_i, \ i \in [1, n], \\
v_{i+1} \quad \alpha \in Q''_1, \ \alpha = c_{i+1}, \ i \in [0, n - 1], \\
0 \quad \text{otherwise}, 
\end{cases}
\]

\[
N(C, C')_{\alpha}(v'_i) = \begin{cases} 
v' \quad \alpha = \alpha_x, \ i = p|B_x|, \ p \in [0, p_{C'}], \\
v'_{i-1} \quad \alpha \in Q'_1, \ \alpha = c'_i, \ i \in [1, m], \\
v'_{i+1} \quad \alpha \in Q''_1, \ \alpha = c'_{i+1}, \ i \in [0, m - 1], \\
0 \quad \text{otherwise}, 
\end{cases}
\]

\[
N(C, C')_{\alpha}(v'') = 0,
\]

and

\[
N(C, C')_{\alpha}(v''') = \begin{cases} 
v_0 \quad \alpha = \gamma_x, \\
0 \quad \text{otherwise}.
\end{cases}
\]

We also put \( N(C, \emptyset) = M(\gamma_x C) \), \( N(C, C) = N(C) \oplus M(C) \) and, if \( x \in Q'_1 \), \( N(C, B_x C) = L(B_x C) \oplus M(\gamma_x C) \).

1.15. Let

\[
\mathcal{S}' = \mathcal{S} \setminus \{ \nu x_0 \omega x \mid x \in Q_0 \} \cup \{ C \mid C \in \mathcal{S}_x, \ x \in Q'_0 \} \\
\quad \cup \{ \alpha x C \mid C \in \mathcal{S}_x, \ x \in Q'_0 \} \cup \{ \gamma x C \mid C \in \mathcal{S}_x, \ x \in Q''_0 \}.
\]
Observe, that \( \nu_x \omega_x \in \mathcal{S} \) for all \( x \in Q_0 \), and \( \alpha_x C \in \mathcal{S} \) for all \( x \in Q_0' \setminus Q_0'' \) and \( C \in \mathcal{S}_x \). Moreover, if \( x \in Q_0'' \) and \( C \in \mathcal{S}_x \), then \( \gamma_x C \in \mathcal{S} \), but \( \alpha_x C \in \mathcal{S} \) if and only if \( \omega_x \) is not a terminating substring of \( C \).

The following theorem is the main result of the paper.

**Theorem.** Let \((p,q,S,T)\) be a defining system and let \( A \) be the corresponding algebra.

1. **Representations**
   
   \[
   R(B, \lambda, m), B \in \mathcal{B}, \lambda \in k^*, m \in \mathbb{N},
   \]
   
   \[
   Q(B, m), B \in \mathcal{B}', m \in \mathbb{N},
   \]
   
   \[
   M(C), C \in \mathcal{S},
   \]
   
   \[
   N(C), C \in \mathcal{S}_x, x \in Q_0',
   \]
   
   \[
   L(B_x C), C \in \mathcal{S}_x, x \in Q_0'',
   \]
   
   \[
   N(C, C'), (C, C') \in \mathcal{P}_x, x \in Q_0',
   \]
   
   form a complete set of pairwise nonisomorphic indecomposable modules over \( A \).

2. **Sequences**

   \[
   0 \rightarrow R(B, \lambda, m) \rightarrow R(B, \lambda, m + 1) \oplus R(B, \lambda, m - 1) \rightarrow R(B, \lambda, m)
   \]

   \[
   \rightarrow 0, (B, \lambda, m) \in \mathcal{B} \times k^* \times \mathbb{N}, B = B_0 \text{ or } \lambda \neq 1,
   \]
   
   \[
   0 \rightarrow R(B, 1, m) \rightarrow Q(B, m + 1) \oplus R(B, 1, m - 1) \rightarrow Q(B, m) \rightarrow 0,
   \]

   \[
   B \in \mathcal{B}', m \in \mathbb{N},
   \]
   
   \[
   0 \rightarrow Q(B, m) \rightarrow R(B, m) \oplus Q(B, m - 1) \rightarrow R(B, 1, m - 1) \rightarrow 0,
   \]

   \[
   B \in \mathcal{B}', m \in \mathbb{N}, m > 1,
   \]
   
   \[
   0 \rightarrow M(C) \rightarrow M(C_+ \oplus M(C_+)) \rightarrow M(C_+ \oplus C_+) \rightarrow 0, C \in \mathcal{S}',
   \]
   
   \[
   0 \rightarrow M(C) \rightarrow M(C_+ \oplus N(\mu_x, C_+)) \rightarrow N(\mu_x, C_+) \rightarrow 0,
   \]

   \[
   C = \alpha_x C', C'' \in \mathcal{S}_x, x \in Q_0',
   \]
   
   \[
   0 \rightarrow M(C) \rightarrow N(C, C_+) \rightarrow N(C_+) \rightarrow 0, C \in \mathcal{S}_x, x \in Q_0',
   \]
   
   \[
   0 \rightarrow M(\gamma_x C) \rightarrow N(C_+, B_x C) \rightarrow L(B_x C_+) \rightarrow 0, C \in \mathcal{S}_x, x \in Q_0'',
   \]
   
   \[
   0 \rightarrow N(C) \rightarrow N(C, C_+) \rightarrow M(C_+) \rightarrow 0, C \in \mathcal{S}_x, x \in Q_0', C \neq \omega_x,
   \]
   
   \[
   0 \rightarrow L(B_x C) \rightarrow N(C_+, B_x C) \rightarrow M(\gamma_x C_+) \rightarrow 0, C \in \mathcal{S}_x, x \in Q_0'',
   \]
   
   \[
   0 \rightarrow N(C, C') \rightarrow N(C, C'_+) \oplus N(C_+, C') \rightarrow N(C_+, C'_+) \rightarrow 0,
   \]

   \[
   (C, C') \in \mathcal{P}_x, x \in Q_0',
   \]

   form a complete list of Auslander–Reiten sequences in \( \text{mod} A \).

We finish this section with some remarks concerning the above theorem. First of all, if \( x \in Q_0' \) then \( \omega_x \in \mathcal{S}_x \) if and only if \( x \notin Q_0'' \). If \( x \in Q_0'', C \in \mathcal{S}_x \) and \( \alpha_x C \in \mathcal{S} \), then \( \alpha_x C = C \) and \( \alpha_x C_+^+ = C_+ \). Moreover, if \( C \neq \omega_x \), then \( (\alpha_x C)_+ = \alpha_x C_+ \). Finally, if \( x \in Q_0' \setminus Q_0'' \), then \( (\alpha_x \omega)_+ = \emptyset \).
2. Vector space categories

In this section we describe vector space categories and subspace categories needed in the proof of our main result.

2.1. Following \[21, \text{Section 17.1}\] (see also \[19, \text{Section 2.4}\]) by a vector space category we mean a pair $\mathbb{K} = (\mathcal{K}, \mathcal{V})$, where $\mathcal{K}$ is a Krull–Schmidt category and $| - | : \mathcal{K} \to \text{mod } k$ is a faithful functor. For a vector space category $\mathbb{K}$ we consider the subspace category $\mathcal{U}(\mathbb{K})$ of $\mathbb{K}$. The objects of $\mathcal{U}(\mathbb{K})$ are triples $V = (V_0, V_1, \gamma_V)$ with $V_0 \in \mathcal{K}$, $V_1 \in \text{mod } k$ and $\gamma_V : V_1 \to |V_0|$ a $k$-linear map. If $V = (V_0, V_1, \gamma_V)$ and $W = (W_0, W_1, \gamma_W)$ are two objects of $\mathcal{U}(\mathbb{K})$, then a morphism $f : V \to W$ in $\mathcal{U}(\mathbb{K})$ is a pair $f = (f_0, f_1)$, where $f_0 : V_0 \to W_0$ is a morphism in $\mathcal{K}$, $f_1 : V_1 \to W_1$ is a $k$-linear map and the condition $|f_0|\gamma_V = \gamma_W f_1$ is satisfied. By $\overline{0}$ we denote the triple $(0, k, 0)$ in $\mathcal{U}(\mathbb{K})$.

2.2. An ordered set $I$ is called semi-admissible, if the order is linear and for each element of $I$ which is not maximal there exists a direct successor. If in addition, there exist a minimal and a maximal elements in $I$, then we call $I$ admissible. If $I$ is a semi-admissible ordered set and $\gamma \in I$ is not maximal in $I$, then by $\gamma_+$ we denote the direct successor of $\gamma$ in $I$.

If $I_1$ and $I_2$ are two semi-admissible ordered sets, then we introduce the order in $I_1 \times I_2$ by saying that $(x_1, y_1) \leq (x_2, y_2)$ if either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 \leq y_2$, for $x_1, x_2 \in I_1$ and $y_1, y_2 \in I_2$. If $(x, y) \in I_1 \times I_2$, then we put $(x, y)^+ = (x+, y)$. If in addition $I_1$ and $I_2$ are disjoint, then by $I_1 + I_2$ we denote the ordered set $I_1 \cup I_2$ with the elements of $I_1$ smaller than the elements of $I_2$.

If $I$ is an admissible ordered set, then we denote by $I_-$ the set $\{\ast\} + I$, where $\ast \not\in I$. Note that in this case $\ast = \min I_-$ and $\ast_+ = \min I$. Similarly, we put $I_+ = I + \{\ast\}$ (thus in this case $\ast = \max I_+ = (\max I)_+$). Finally, we denote by $I'$ the ordered set $I \setminus \{\max I\}$.

2.3. Let $I_1, \ldots, I_{r+1}, r \in \mathbb{N}_0$, be a family of admissible ordered sets. Let $\mathcal{K}$ be the Krull–Schmidt category, whose indecomposable objects are

- $X_\gamma, \gamma \in I_p, p \in [1, r + 1]$,
- $X_{\max I_p}, X_{\max I_p}', p \in [1, r]$,

and all indecomposable objects of $\mathcal{K}$ are one-dimensional, i.e., for each indecomposable object $X$ of $\mathcal{K}$, $\dim_k |X| = 1$. If $U$ and $V$ are indecomposable objects of $\mathcal{K}$, then $\text{Hom}_\mathcal{K}(U, V) \neq 0$ if and only if one of the following conditions holds:

- $U = X_\gamma, V = X_{\gamma''}, \gamma' \in I_p, \gamma'' \in I'_q, (p, \gamma') \leq (q, \gamma'')$,
- $U = X_\gamma, V = X''_{\max I_q}, \gamma \in I'_p, p \leq q$,
- $U = X_\gamma, V = X''_{\max I_q}, \gamma \in I'_p, p \leq q$,
- $U = X'_{\max I_p}, V = X_\gamma, \gamma \in I'_q, p < q,$
• $U = X'_{\text{max} I_p}, V = X'_{\text{max} I_q}, p \leq q$,
• $U = X''_{\text{max} I_p}, V = X''_{\text{max} I_q}, p < q$,
• $U = X''_{\text{max} I_p}, V = X'_{\text{max} I_q}, p < q$,
• $U = X''_{\text{max} I_p}, V = X''_{\text{max} I_q}, p \leq q$.

By $\mathbb{K}_{I_1, \ldots, I_{r + 1}}$ we denote the vector space category $(\mathcal{K}, | - |)$, where $| - | : \mathcal{K} \to \text{mod } k$ is the forgetful functor.

2.4. Let $I$ be an admissible ordered set. Let $\mathcal{L}$ be the Krull–Schmidt category, whose indecomposable objects are

• $X_\gamma$, $\gamma \in I$,
• $Y_\gamma$, $\gamma \in I$,

and all indecomposable objects of $\mathcal{L}$ are one-dimensional. If $U$ and $V$ are indecomposable objects of $\mathcal{L}$, then $\text{Hom}_\mathcal{L}(U, V) \neq 0$ if and only if one of the following conditions holds:

• $U = X'_\gamma$, $V = X''_\gamma$, $\gamma' \leq \gamma''$,
• $U = X'_\gamma$, $V = Y''_\gamma$, $\gamma' \leq \gamma''$,
• $U = Y'_\gamma$, $V = Y''_\gamma$, $\gamma' \leq \gamma''$.

By $\mathbb{L}_I$ we denote the vector space category $(\mathcal{L}, | - |)$, where $| - | : \mathcal{L} \to \text{mod } k$ is the forgetful functor.

2.5. We have the following description of the indecomposable objects and the Auslander–Reiten sequences in $\mathcal{U}(\mathbb{L}_I)$. For definitions of the relevant objects and the proof we refer to [3, Section 3].

Proposition. Let $I$ be an admissible ordered set.

(1) Objects

$M_{\text{min} I, \gamma} = X_\gamma$, $\gamma \in I$,
$M_{\gamma', \gamma''} = \overline{\gamma} X_{\gamma''}$, $\gamma', \gamma'' \in I$, $\gamma' < \gamma''$,
$M_{\gamma, \text{max} I_+} = \overline{\gamma}$, $\gamma \in I$,
$M'_{\gamma, \gamma} = Y_\gamma$, $\gamma \in I$,
$M''_{\gamma, \gamma} = \overline{X}_\gamma$, $\gamma \in I$,
$M''_{\text{max} I_+ \text{max} I_+} = \overline{0}$,

form a complete set of pairwise nonisomorphic indecomposable objects in $\mathcal{U}(\mathbb{L}_I)$.

(2) Sequences

$0 \to M_{\gamma', \gamma''} \to M'_{\gamma, \gamma''} \oplus M''_{\gamma', \gamma''} \to M_{\gamma', \gamma''} \to 0$, $\gamma', \gamma'' \in I_-$, $\gamma' < \gamma''$,

$0 \to M''_{\gamma, \gamma} \to M_{\gamma, \gamma} \to M''_{\gamma, \gamma} \to 0$, $\gamma \in I$,

$0 \to M''_{\gamma, \gamma} \to M_{\gamma, \gamma} \to M''_{\gamma, \gamma} \to 0$, $\gamma \in I'$,
form a complete list of Auslander–Reiten sequences in $U(L_1)$, where

\[ M_{\gamma, \gamma} = M'_{\gamma, \gamma} \oplus M''_{\gamma, \gamma}, \quad \gamma \in I, \]

\[ M_{\min I_-, \max I_+} = 0. \]

2.6. Let $I_0, \ldots, I_{r+1}$, $r \in \mathbb{N}$, be a family of admissible ordered sets. Let $L$ be the Krull–Schmidt category, whose indecomposable objects are

- $X_{\gamma}$, $\gamma \in I'$, $p \in [0, r+1]$,
- $X'_{\max I_p}$, $X''_{\max I_p}$, $p \in [0, r]$,
- $Y_{\gamma}$, $\gamma \in I'_0$,
- $Z$.

If $U$ is an indecomposable object of $L$, then

\[ \dim_k |U| = \begin{cases} 
2 & U = X_{\min I_1}, \\
1 & \text{otherwise}. 
\end{cases} \]

If $U$ and $V$ are indecomposable objects of $L$, then $\dim_k \Hom_L(U, V) \leq 2$, $\Hom_L(U, V) \neq 0$ if and only if one of the following conditions holds:

- $U = X_{\gamma}$, $V = X_{\gamma''}$, $\gamma' \in I'_p$, $\gamma'' \in I'_{q'}$, $(p, \gamma') \leq (q, \gamma'')$,
- $U = X_{\gamma}$, $V = X'_{\max I_q}$, $\gamma \in I'_p$, $p \leq q$,
- $U = X_{\gamma}$, $V = X''_{\max I_q}$, $\gamma \in I'_p$, $p \leq q$,
- $U = X'_{\gamma'}$, $V = Y_{\gamma''}$, $\gamma' \in I'_0$, $\gamma' \leq \gamma''$,
- $U = X_{\gamma}$, $V = Z$, $\gamma \in I'_0$,
- $U = X'_{\max I_p}$, $V = X_{\gamma}$, $\gamma \in I'_q$, $p < q$,
- $U = X'_{\max I_p}$, $V = X''_{\max I_q}$, $p \leq q$,
- $U = X''_{\max I_p}$, $V = X'_{\max I_q}$, $p < q$,
- $U = X''_{\max I_p}$, $V = X''_{\max I_q}$, $p \leq q$,
- $U = X''_{\max I_0}$, $V = Z$,
- $U = Y_{\gamma}$, $V = X_{\min I_1}$,
- $U = Y_{\gamma'}$, $V = Y_{\gamma''}$, $\gamma' \leq \gamma''$,
- $U = Y_{\gamma}$, $V = Z$,
- $U = Z$, $V = Z$,

and $\dim_k \Hom_L(U, V) = 2$ if and only if $U = X_{\gamma}$, $\gamma \in I'_0$, $V = X_{\min I_1}$.

By $L_{I_0, \ldots, I_{r+1}}$ we denote the vector space category $(L, | - |)$, where $| - | : L \to \text{mod } k$ is the forgetful functor. We refer the reader to [4, Section 1] for pictures presenting vector space categories of the above type, and in particular explaining how the forgetful functor $| - |$ is defined on $\Hom_L(X_{\gamma}, X_{\min I_1})$ for $\gamma \in I'_0$.

2.7. We describe the indecomposable objects and the Auslander–Reiten sequences in $U(L_{I_0, \ldots, I_{r+1}})$. We refer to [1] for definitions of the objects listed in the below proposition and its proof.
Proposition. Let $I_0, \ldots, I_{r+1}, r \in \mathbb{N}$, be admissible ordered sets. Put

$$I_p'' = \begin{cases} I_1' \setminus \{\min I_1\} & p = 1, \\ I_p' & p \in [2, r + 1], \end{cases}$$

(1) Objects

$$M_{(-1, \max I_0), (0, \gamma)} = X_\gamma, \gamma \in I_0'$$

$$M_{(0, \gamma'), (0, \gamma'')} = Y_\gamma X'_\gamma X''_\gamma$$

$$M_{(n-1, \max I_0), (n, \gamma)} = Y_\gamma X'_\gamma X''_\gamma X'_{\max I_0} X''_{\max I_0} X_{\min I_1}^{2n-1} 2^n$$

$$M_{(n, \gamma), (n, \max I_0)} = Y_\gamma X'_\gamma X''_\gamma X'_{\max I_0} X''_{\max I_0} X_{\min I_1}^{2n+1}$$

$$M_{(n, \gamma'), (n+1, \gamma')} = Y_\gamma Y_\gamma X'_\gamma X''_\gamma X'_{\max I_0} X''_{\max I_0} X_{\min I_1}^{2n+2}$$

$$\gamma', \gamma'' \in I_0', \gamma' < \gamma'', n \in \mathbb{N}_0,$$

$$M_{(n, \gamma'), (n, \gamma'')} = Y_\gamma Y_\gamma X'_\gamma X''_\gamma X'_{\max I_0} X''_{\max I_0} X_{\min I_1}^{2n-1} 2^n$$

$$\gamma', \gamma'' \in I_0', \gamma' < \gamma'', n \in \mathbb{N}_0,$$

$$M'_{(n, \gamma), (n, \gamma)} = X_{\min I_1}^n$$

$$M''_{(0, \gamma), (0, \gamma)} = X_\gamma$$

$$M''_{(n, \max I_0), (n, \max I_0)} = X_{\min I_1}^n X_{\max I_0}^n X''_{\max I_0} X_{\min I_1}^{n+1}$$

$$M''_{(n, \gamma), (n, \gamma)} = Y_\gamma X'_\gamma X''_\gamma X'_{\max I_0} X''_{\max I_0} X_{\min I_1}^{n-1} 2^{n+1}$$

$$M''_{(n-1, \max I_0), (n, \max I_0)} = X_{\max I_0} X''_{\max I_0} X_{\min I_1}^n$$

$$M'_{(n, \gamma), (n+1, \gamma)} = Y_\gamma X'_\gamma X''_\gamma X'_{\max I_0} X''_{\min I_1}^n$$

$$M''_{(n-1, \max I_0), (n, \max I_0)} = X''_{\max I_0} X_{\min I_1}^n$$

$$M''_{(n, \gamma), (n+1, \gamma)} = Y_\gamma X''_\gamma X'_{\max I_0} X''_{\min I_1}^n$$

$$R^n_\lambda = X_{\min I_1}^n (\lambda), \lambda \in k^*, \lambda \neq 1, n \in \mathbb{N},$$

$$R^n_{2n-1} = X'_{\max I_0} X''_{\min I_1}^n$$

$$R^n_{2n} = X_{\min I_1}^n (1), n \geq 1,$$

$$R^n_{2n-1, 0} = X_{\max I_0} X''_{\min I_1}^{n-1}$$

$$R^n_{2n-1, 1} = X_{\min I_1}^{n-1} Z^{n-1}$$

$$R^n_{2n, 0} = X_{\min I_1}^{\neg i} (\infty), n \in \mathbb{N},$$

$$R^n_{2n, 1} = X_{\max I_0} X''_{\min I_1}^{n-1} Z$$

$$S_{p, (n-1, \max I_0), (m-1, \max I_0)} = X_{\min I_1}^n X_{\max I_0} X'_{p} X''_{\max I_0} X_{\min I_1}^{\neg i + m},$$

$$p \in [1, r], n, m \in \mathbb{N}_0, n < m,$$
\[ S_p(n, \gamma),(m-1, \text{max } I_0) = Y_{\gamma} X^n_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1}^{n+m+1}, \]
\[ p \in [1, r], \gamma \in I'_0, n, m \in \mathbb{N}_0, n < m, \]
\[ S_p(n-1, \text{max } I_0),(m, \gamma) = Y_{\gamma} X^n_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1} Y_{\gamma}^{n+m+1}, \]
\[ p \in [1, r], \gamma \in I'_0, n, m \in \mathbb{N}_0, n \leq m, \]
\[ S_p(n, \gamma'),(m, \gamma'') = Y_{\gamma'} X^n_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1} Y_{\gamma'}^{n+m+2}, \]
\[ p \in [1, r], \gamma, \gamma' \in I'_0, n, m \in \mathbb{N}_0, (n, \gamma') < (m, \gamma''), \]
\[ S_p'(n-1, \text{max } I_0),(n-1, \text{max } I_0) = X^n_{\text{min } I_1} X^r_{\text{max } I_p}^n, p \in [1, r], n \in \mathbb{N}_0, \]
\[ S_p'(n, \gamma),(n, \gamma) = Y_{\gamma} X^n_{\text{min } I_1} X^r_{\text{max } I_p}^n, p \in [1, r], \gamma \in I'_0, n \in \mathbb{N}_0, \]
\[ S_{p}'(n-1, \text{max } I_0),(n-1, \text{max } I_0) = X^n_{\text{min } I_1} X^r_{\text{max } I_p}^n, p \in [1, r], n \in \mathbb{N}_0, \]
\[ T_{p, \gamma},(m-1, \text{max } I_0) = X^m_{\text{min } I_1} X^r_{\gamma}^n, p \in [1, r+1], \gamma \in I'_p, m \in \mathbb{N}_0, \]
\[ T_{p, \gamma'},(m, \gamma'') = Y_{\gamma'} X^n_{\text{min } I_1} X^r_{\gamma'}^{n+1}, p \in [1, r+1], \gamma' \in I'_p, \gamma'' \in I'_0, m \in \mathbb{N}_0, \]
\[ T_{r+1, \text{max } I_{r+1},(m-1, \text{max } I_0)} = X^m_{\text{min } I_1}^m(0), m \in \mathbb{N}, \]
\[ T_{r+1, \text{max } I_{r+1},(m, \gamma)} = Y_{\gamma} X^m_{\text{min } I_1}^{m+1}, \gamma \in I'_0, m \in \mathbb{N}_0, \]
\[ U_{p,2n,(m-1, \text{max } I_0)} = X^m_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1}^{n+m+1}, \]
\[ p \in [1, r], n, m \in \mathbb{N}_0, \]
\[ U_{p,2n,(m, \gamma)} = Y_{\gamma} X^n_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1}^{n+m+2}, \]
\[ p \in [1, r], \gamma \in I'_0, n, m \in \mathbb{N}_0, \]
\[ U_{p,2n+1,(m-1, \text{max } I_0)} = X^m_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1} Z^{n+m+1}, \]
\[ p \in [1, r], m \in \mathbb{N}_0, \]
\[ U_{p,2n+1,(m, \gamma)} = Y_{\gamma} X^n_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1} Z^{n+m+2}, \]
\[ p \in [1, r], \gamma \in I'_0, n, m \in \mathbb{N}_0, \]
\[ V_{p,2n, \gamma} = X^n_{\text{min } I_1} X^r_{\gamma}^{n+1}, p \in [1, r+1], \gamma \in I'_p, n \in \mathbb{N}_0, \]
\[ V_{p,2n+1, \gamma} = X^n_{\text{min } I_1} X^r_{\gamma} Z^{n+1}, p \in [1, r+1], \gamma \in I'_p, n \in \mathbb{N}_0, \]
\[ V_{r+1,2n, \text{max } I_{r+1}} = X^n_{\text{min } I_1}^{n+1}, n \in \mathbb{N}_0, \]
\[ V_{r+1,2n+1, \text{max } I_{r+1}} = X^n_{\text{min } I_1} Z^{n+1}, n \in \mathbb{N}_0, \]
\[ W_{p,2n,2m} = X^n_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1}^{n+m+2}, \]
\[ p \in [1, r], n, m \in \mathbb{N}_0, m < n, \]
\[ W_{p,2n+1,2m} = Z X^n_{\text{min } I_1} X^r_{\text{max } I_p} X^n_{\text{max } I_p} X^m_{\text{min } I_1}^{n+m+2}, \]
\[ p \in [1, r], n, m \in \mathbb{N}_0, m \leq n, \]
\[
W_{p,2n,2m+1} = X^n_{\min I_1} X'_{\max I_p} X''_{\max I_p} X^m_{\min I_1} Z^{n+m+2},
\]
\[p \in [1, r], n, m \in \mathbb{N}_0, m < n,
\]
\[
W_{p,2n+1,2m+1} = Z X^n_{\min I_1} X'_{\max I_p} X''_{\max I_p} X^m_{\min I_1} Z^{n+m+2},
\]
\[p \in [1, r], n, m \in \mathbb{N}_0, m < n,
\]
\[
W'_{p,2n,2n} = X^n_{\min I_1} X'_{\max I_p} Z^{n+1}, p \in [1, r], n \in \mathbb{N}_0,
\]
\[
W'_{p,2n+1,2n+1} = X^n_{\min I_1} X'_{\max I_p} Z^{n+1}, p \in [1, r], n \in \mathbb{N}_0,
\]
\[
W''_{p,2n,2n} = X^n_{\min I_1} X'_{\max I_p} Z^{n+1}, p \in [1, r], n \in \mathbb{N}_0,
\]
\[
W''_{p,2n+1,2n+1} = X^n_{\min I_1} X'_{\max I_p} Z^{n+1}, p \in [1, r], n \in \mathbb{N}_0,
\]

form a complete list of indecomposable objects in \(\mathcal{U}(\mathbb{L}_{I_0, \ldots, I_{r+1}})\).

(2) Sequences

0 → \(M_{\gamma', \gamma''} \rightarrow M_{\gamma', \gamma''} + M_{\gamma', \gamma''} \rightarrow M_{\gamma', \gamma''} \rightarrow 0\),
\(\gamma', \gamma'' \in \mathbb{Z} \times I_0, (-1, \max I_0) \leq \gamma' < \gamma'' < (\gamma')' +1\),
0 → \(M'_{\gamma, \gamma} \rightarrow M_{\gamma+1}, \gamma_+ \rightarrow M''_{\gamma+1, \gamma_+} \rightarrow 0, \gamma \in \mathbb{N}_0 \times I_0\),
0 → \(M''_{\gamma, \gamma_+} \rightarrow M_{\gamma+1, \gamma_+} \rightarrow M'_{\gamma+1, \gamma_+} \rightarrow 0, \gamma \in \mathbb{N}_0 \times I_0, (-1, \max I_0) \leq \gamma\),
0 → \(M''_{\gamma, \gamma_+} \rightarrow M_{\gamma+1, \gamma_+} \rightarrow M'_{\gamma+1, \gamma_+} \rightarrow 0, \gamma \in \mathbb{Z} \times I_0, (-1, \max I_0) \leq \gamma\),
0 → \(R^0_\lambda \rightarrow R^0_{n+1} \oplus R^0_{n-1} \rightarrow R^0_\lambda \rightarrow 0, \lambda \in K^* \neq 1, n \in \mathbb{N}\),
0 → \(R^0_{n+1} \rightarrow R^1_{n+2} \oplus R^1_{n-1} \rightarrow R^1_\lambda \rightarrow 0, n \in \mathbb{N}\),
0 → \(R^0_{n,i} \rightarrow R^0_{n+1,i} \oplus R^0_{n-1,i+1} \rightarrow R^0_{n+1,i+1} \rightarrow 0, i \in \mathbb{Z}_2, n \in \mathbb{N}\),
0 → \(S_{p, \gamma, \gamma'} \rightarrow S_{p, \gamma, \gamma'} \rightarrow S_{p, \gamma, \gamma'} \rightarrow S_{p, \gamma, \gamma'} \rightarrow 0, p \in [1, r], \gamma', \gamma'' \in \mathbb{Z} \times I_0, (-1, \max I_0) \leq \gamma' < \gamma''\),
0 → \(S'_{p, \gamma, \gamma} \rightarrow S_{p, \gamma, \gamma} \rightarrow S''_{p, \gamma, \gamma} \rightarrow 0, p \in [1, r], \gamma \in \mathbb{Z} \times I_0, (-1, \max I_0) \leq \gamma\),
0 → \(S''_{p, \gamma, \gamma} \rightarrow S_{p, \gamma, \gamma} \rightarrow S'_{p, \gamma, \gamma} \rightarrow 0, p \in [1, r], \gamma \in \mathbb{Z} \times I_0, (-1, \max I_0) \leq \gamma\),
0 → \(T_{r+1, \max I_{r+1}, (0, \max I_0^r)} \rightarrow T_{r+1, \max I_{r+1}, (0, \max I_0^r)} \rightarrow T_{0, \min I_{r+1}, (0, \max I_0^r)} \rightarrow 0, p \in [1, r+1], \gamma' \in (I_p^0)^-, \gamma'' \in \mathbb{Z} \times I_0, (-1, \max I_0) \leq \gamma'\),
0 → \(U_{p, n, \gamma} \rightarrow U_{p, n, \gamma} \oplus U_{p, n-1, \gamma} \rightarrow U_{p, n-1, \gamma} \rightarrow 0, p \in [1, r], n \in \mathbb{N}, \gamma \in \mathbb{N}_0 \times I_0\),
0 → \(V_{p, n, \gamma} \rightarrow V_{p, n, \gamma} \oplus V_{p, n-1, \gamma} \rightarrow V_{p, n-1, \gamma} \rightarrow 0, p \in [1, r+1], n \in \mathbb{N}, \gamma \in (I_p^0)^-\).
0 → \( W_{p,n,m} \to W_{p,n-1,m} \oplus W_{p,n,m-1} \to W_{p,n-1,m-1} \to 0, \)
\( p \in [1, r], n, m \in \mathbb{N}, m < n, \)

0 → \( W'_{p,n,n} \to W_{p,n-1,n} \to W''_{p,n-1,n-1} \to 0, p \in [1, r], n \in \mathbb{N}, \)

0 → \( W'_{p,n,n} \to W_{p,n-1,n} \to W''_{p,n-1,n-1} \to 0, p \in [1, r], n \in \mathbb{N}, \)

form a complete list of Auslander–Reiten sequences in the category \( \mathcal{U}(\mathbb{L}_{I_0, \ldots, I_{r+1}}) \), where

\[
M_{\gamma,\gamma} = M'_{\gamma,\gamma} \oplus M''_{\gamma,\gamma}, \ \gamma \in \mathbb{N}_0 \times I_0,
\]
\[
M_{\gamma,\gamma^+} = M'_{\gamma,\gamma^+} \oplus M''_{\gamma,\gamma^+}, \ \gamma \in \mathbb{Z} \times I_0, (−1, \max I_0) \leq \gamma,
\]
\[
R_{0,0}^\lambda = 0, \ \lambda \in k^*,
\]
\[
R_{0,0}^\infty = 0, i \in \mathbb{Z}_2,
\]
\[
S_{\gamma,\gamma} = S'_{\gamma,\gamma} \oplus S''_{\gamma,\gamma}, \ \gamma \in \mathbb{N}_0 \times I_0,
\]
\[
T_{1, \min(t_{r+1}^-),\gamma} = T_{r+1, \max I_{r+1},\gamma^+}, \ \gamma \in \mathbb{Z} \times I_0, (−1, \max I_0) \leq \gamma,
\]
\[
T_{p, \min(t_{r}^-),\gamma} = U_{p−1,0,\gamma}, p \in [2, r+1], \ \gamma \in \mathbb{Z} \times I_0, (−1, \max I_0) \leq \gamma,
\]
\[
T_{p, \max I_{p,0},\gamma} = S_{p,(−1,\max I_0),\gamma}, p \in [1, r], \ \gamma \in \mathbb{Z} \times I_0, (−1, \max I_0) \leq \gamma,
\]
\[
T_{r+1, \max I_{r+1},(−1,\max I_0)} = 0,
\]
\[
V_{1,n, \min(t_{r}^-)} = V_{r+1,n, \max I_{r+1}}, n \in \mathbb{N}_0,
\]
\[
V_{p,n, \min(t_{r}^-)} = W_{p−1,n,0}, p \in [2, r+1], n \in \mathbb{N}_0,
\]
\[
V_{p,n, \max I_{p}} = U_{p,n,(−1,\max I_0)}, p \in [1, r], n \in \mathbb{N}_0,
\]
\[
W_{p,n,n} = W''_{p,n,n} \oplus W'_p, p \in [1, r], n \in \mathbb{N}_0.
\]

3. Proof of the main result

In this section we present the proof of the main theorem of the paper.

3.1. Let \( A \) be an algebra and let \( R \) be an \( A \)-module. By \( A[R] \) we denote the one-point extension of \( A \) by \( R \) defined as
\[
\begin{bmatrix}
A & R \\
0 & k
\end{bmatrix}.
\]
The category of \( A[R] \)-modules is equivalent to the category of triples \((V_0, V_1, \gamma_V)\), with \( V_0 \in \text{mod } A, V_1 \in \text{mod } k \) and \( \gamma_V : V_1 \to \text{Hom}_A(R, V_0) \) is a \( k \)-linear map (see [19, 2.5(8)]).

Let \( \text{Hom}(R, \text{mod } A) \) be the vector space category \((\mathcal{K}, |−|)\), where \( \mathcal{K} = \text{mod } A / \text{Ker } \text{Hom}_A(R, −) \) and \( |−| : \mathcal{K} \to \text{mod } k \) is the functor induced by \( \text{Hom}_A(R, −) \). It follows from the above remark that we may view the objects of \( \mathcal{U}(\text{Hom}(R, \text{mod } A)) \) as objects of \( \text{mod } A[R] \). Consequently, if \( X \) is an indecomposable \( A[R] \)-module then either \( X \in \text{mod } A \) or \( X \in \mathcal{U}(\text{Hom}(R, \text{mod } A)) \). Moreover, each Auslander–Reiten sequence in \( \text{mod } A[R] \) is either of the form
\[
0 \to (X, \text{Hom}_A(R, X), \text{Hom}_A(R, \text{Id}_X))
\]
for an Auslander–Reiten sequence $0 \to X \xrightarrow{f} Y \to Z \to 0$ in $\text{mod } A$, or comes from an Auslander–Reiten sequence in $U(\text{Hom}(R, \text{mod } A))$.

3.2. From now on we assume that $(p, q, S, T)$ is a fixed defining system. We also use notation introduced in Section 1.

A vertex $x$ of $Q$ is called admissible if one of the following possibilities holds:

- $x = x_{i,j}, i \in [1, |p|], j \in [2, \mu_i + |T_i|], j - 1, j, j + 1 \notin S_i$,
- $x = z_{i,j}, i \in [1, |p|], j \in S_i \cap [T_i |T_i| + 2, p_i + |T_i|]$.

For an admissible vertex $x$ of $Q$ we define a defining system $(p, q, S^x, T^x)$ by:

- if $x = x_{i_0,j_0}$, then
  $$S^x_i = \begin{cases} S_{i_0} \cup \{j_0\} & i = i_0, \\ S_i & i \neq i_0, \end{cases} \text{ and } T^x = T,$$
- if $x = z_{i_0,j_0}$, then
  $$S^x = S \text{ and } T^x_i = \begin{cases} T_{i_0} \cup \{j_0\} & i = i_0, \\ T_i & i \neq i_0. \end{cases}$$

A defining system $(p, q, S, T)$ is called fundamental if $S_i = \emptyset = T_i$ for all $i$. The following observation allows us to perform inductive proofs: each defining system is an iterated extension of a fundamental one by admissible vertices.

3.3. For $x \in Q_0$, $x = x_{i,j}$, let $X_x = M(\mu_x), I_x = M(\omega_x)$ and $R_x = M(\alpha_x \mu_x)$. Similarly, if $x = z_{i,j}$, then $X_x = M(\gamma_y \mu_y)$ and $R_x = N(\mu_y, \omega_y)$, where $y = x_{i,j}$.

For a vertex $x$ of $Q$ let $C_x$ denote the set of all strings terminating at $x$ ordered by the relation introduced in 1.7. Recall that $C'_x = C_x \setminus \{\omega_x\}$. We prove the main theorem inductively together with the following series of lemmas.

**Lemma 3.3.1.** Let $x$ be an admissible vertex of $Q$.

1. If $x = x_{i_0,j_0}$, then the assignment
   $$X_C \mapsto M(\alpha_x C), \ C \in C_x,$$
   $$Y_C \mapsto M(C), \ C \in C_x,$$
   induces an equivalence between $\mathbb{L}_{C_x}$ and $\text{Hom}(R_x, \text{mod } A)$.

2. If $x = z_{i_0,j_0}$, let $\{j_1 < \cdots < j_r\} = S_{i_0} \cap [j_0 + 1, p_{i_0} + |T_{i_0}|]$ and $j_{r+1} = p_{i_0} + |T_{i_0}| + 1$. The assignment
   $$X_C \mapsto N(C, \omega_{x_{i_0,j_p}}), \ C \in C'_{x_{i_0,j_p}}, \ p \in [0, r],$$
   $$X_{j_p} \mapsto M(\gamma_{i_0,j_p} \omega_{x_{i_0,j_p}}), \ p \in [0, r],$$
\[ X_j \mapsto M(\omega_{x_{i,0,j}}), \quad j \in [j_p + 1, \ldots, j_{p+1} - 1], \quad p \in [0, r], \]
\[ X'_{x_{i,0,j}} \mapsto M(\omega_{x_{i,0,j}}), \quad p \in [0, r], \]
\[ X''_{x_{i,0,j}} \mapsto N(\omega_{x_{i,0,j}}), \quad p \in [0, r], \]
\[ Y_C \mapsto M(\gamma_{i_0,j_0} C), \quad C \in C'_{x_{i,0,j}}, \]
\[ Z \mapsto M(x), \]
induces an equivalence between
\[ \mathbb{L}_{C_{x_{i,0,j}, [j_0,j_1-1]+C_{x_{i,0,j_1}, \ldots,[j_{r-1},j_{r-1}]+C_{x_{i,0,j_r}, [j_r,j_{r+1}+1]}}} \text{ and } \text{Hom}(R_x, \text{mod } A). \]

**Lemma 3.3.2.** Let \( x \) be an admissible vertex of \( Q \). The assignment
\[ X_C \mapsto M(C), \quad C \in C_x, \]
induces an equivalence between \( \mathbb{K}_{C_x} \) and \( \text{Hom}(X_x, \text{mod } A) \).

**Lemma 3.3.3.** Let \( x = x_{i_0,j_0} \) be such that \( j_0 \in [T_{i_0}, T_{i_0}] + 1, p_{i_0} + [T_{i_0}] \setminus S_{i_0} \). Let \( \{j_1 < \cdots < j_r\} = S_{i_0} \cap [j_0 + 1, p_{i_0} + [T_{i_0}] \text{ and } j_{r+1} = p_{i_0} + [T_{i_0}] + 1 \). The assignment
\[ X_C \mapsto N(C, \omega_{x_{i,0,j}}), \quad C \in C'_{x_{i,0,j}}, \quad p \in [1, r], \]
\[ X_{j_0} \mapsto M(\omega_{x_{i,0,j_0}}), \]
\[ X_p \mapsto M(\gamma_{i_0,j_p} \omega_{x_{i,0,j_p}}), \quad p \in [1, r], \]
\[ X_j \mapsto M(\omega_{x_{i,0,j}}), \quad j \in [j_p + 1, j_{p+1} - 1], \quad p \in [0, r], \]
\[ X'_{x_{i,0,j}} \mapsto M(\omega_{x_{i,0,j}}), \quad p \in [1, r], \]
\[ X''_{x_{i,0,j}} \mapsto N(\omega_{x_{i,0,j}}), \quad p \in [1, r], \]
induces an equivalence between
\[ \mathbb{K}_{[j_0,j_1-1]+C_{x_{i,0,j_1}, \ldots,[j_{r-1},j_{r-1}]+C_{x_{i,0,j_r}, [j_r,j_{r+1}+1]}]} \text{ and } \text{Hom}(I_x, \text{mod } A). \]

### 3.4

If \((p, q, S, T)\) is a fundamental defining system, then Theorem 1.15 and Lemmas 3.3 are easy exercises in the representation theory of a hereditary algebra of type \( \tilde{A}_{p,q} \).

From now on we assume that we Theorem 1.15 and Lemmas 3.3 have been proved for \((p, q, S, T)\). Let \( x \) be an admissible vertex of \( Q \). We will show that Theorem 1.15 and Lemmas 3.3 hold for \((p, q, S^x, T^x)\).

By \( Q^x \) (respectively, \( A^x \)) we will denote the quiver (algebra) associated with \((p, q, S^x, T^x)\). We also define \( R_x^x \), \( X_x^x \) and \( I_x^x \) in the analogous way as the corresponding modules for \((p, q, S, T)\).

### 3.5

Assume first that \( x = x_{i_0,j_0} \). Let \( \gamma = \gamma_{i_0,j_0} \) be the new arrow of \( Q^x \) and \( z = z_{i_0,j_0} \) be the new vertex of \( Q^x \). Theorem 1.15 for \((p, q, S^x, T^x)\) follows from the induction hypothesis (Theorem 1.15 and Lemma 3.3.1 for \((p, q, S, T)\)), Proposition 2.5 and the following isomorphisms
\[ M(\alpha_x C) \simeq N(C), \quad C \in C_x, \]
remains to consider the case $i$ satisfying the hypothesis of Lemma 3.3.3. We have $1$ permissible vertex of $Q$. Assume now that $x = x_{i,j}$. Consider first the case $x' = x_{i,j}$. Then either $i \neq i_0$ or $i = i_0$ and $|j - j_0| > 1$, hence it is easily seen that in this case we also have $R_x^p = R_x$ and $\text{Hom}(R_x^p, \text{mod} \ A) = \text{Hom}(R_x, \text{mod} \ A)$, thus the claim follows.

Let now $x' = z_{i,j}$ for $(i,j) \neq (i_0,j_0)$. In this case also $R_x^p = R_x$. Moreover, if $i \neq i_0$ or $i = i_0$ and $j_0 < j$, then $\text{Hom}(R_x^p, \text{mod} \ A) = \text{Hom}(R_x, \text{mod} \ A)$. If $j < j_0$, then the claim about $\text{Hom}(R_x^p, \text{mod} \ A)$ follows by observing that its indecomposable objects are the indecomposable objects of $\text{Hom}(R_x, \text{mod} \ A)$ and

$$\bar{M}(C)M(\alpha x \omega_x), \ C \in C_x, \ M(\alpha x \omega_x), \ M(\bar{\omega_x}).$$

Finally, let $x' = z$. Then $R_x^p = I_x$ and the claim follows from Lemmas 3.3.2 and 3.3.3.

3.7. In order to show Lemma 3.3.2 we have to consider the cases analogous to the ones considered above. If $x' = x_{i,j}$ or $x' = z_{i,j}$, then $X_{x'} = X_x$ and $\text{Hom}(X_{x'}, \text{mod} \ A) = \text{Hom}(X_x, \text{mod} \ A)$. Thus it remains to consider the case $x' = z$. In this case $X_{x'} = \overline{X_x}$ and the description of $\text{Hom}(X_{x'}, \text{mod} \ A)$ follows easily from the description of $\text{Hom}(X_x, \text{mod} \ A)$.

3.8. It remains to show Lemma 3.3.3. Let $x' = x_{i,j}$ be the vertex of $Q$ satisfying the hypothesis of Lemma 3.3.3. We have $I_{x'} = I_x$. If $i \neq i_0$ or $i = i_0$ and $j_0 < j$, then also $\text{Hom}(I_{x'}, \text{mod} \ A) = \text{Hom}(I_x, \text{mod} \ A)$. If $j < j_0$, then we have observed that indecomposable objects of $\text{Hom}(I_{x'}, \text{mod} \ A)$ are the indecomposable objects of $\text{Hom}(I_x, \text{mod} \ A)$ and

$$\bar{M}(C)M(\alpha x \omega_x), \ C \in C_x, \ M(\alpha x \omega_x), \ M(\bar{\omega_x}).$$

3.9. Assume now that $x = z_{i_0,j_0}$. Let $\{j_1 < \cdots < j_r\} = S_{i_0} \cap [j + 1, p_{i_0} + |T_{i_0}|]$ and $j_{r+1} = p_{i_0} + |T_{i_0}| + 1$. Put

$$C = C_{x_{i_0,j_0}}, \quad C_p = C_{x_{i_0,j_p}}, \quad p \in [1, r],$$

$$\gamma = \gamma_{i_0,j_0}, \quad \gamma_p = \gamma_{i_0,j_p}, \quad p \in [1, r],$$

$$\omega = \omega_{x_{i_0,j_0}}, \quad \omega_j = \omega_{x_{i_0,j}}, \quad j \in [j + 1, p_{i_0} + |T_{i_0}|],$$

$$\alpha = \alpha_{i_0,p_{i_0}+|T_{i_0}|+1}, \quad \xi = \xi_{i_0,|T_{i_0}|+1},$$
and
\[ B = \omega \alpha \xi \gamma, \quad B_j = \omega_j \alpha \xi \gamma, \ j \in [j + 1, p_0 + |T_0|]. \]

Finally, let \[ z = x_{i_0, j} + 1. \]

3.10. In this case Theorem 1.15 for \((p, q, S^x, T^x)\) follows from the induction hypothesis, Proposition 2.7 and the following isomorphisms

\begin{align*}
M(\gamma C')N(C'', \omega) & \simeq N(C'', BC'), C', C'' \in C', C' < C'', \\
M(\gamma C')M(\omega)N(\omega)M(\gamma \omega)^{n-2n} & \simeq N(B^n C, B^n \omega), C \in C', n \in \mathbb{N}, \\
M(\gamma C')M(\omega)N(\omega)M(\gamma \omega)^{2n+1} & \simeq N(B^n \omega, B^{n+1} C'), C \in C', n \in \mathbb{N}, \\
M(\gamma C')M(\gamma C')M(\omega)N(\omega)M(\gamma \omega)^{2n+2} & \simeq N(B^{n+1} C', B^{n+1} C''), C', C'' \in C', C' < C'', n \in \mathbb{N}_0,
\end{align*}

\begin{align*}
M(\gamma C')M(\gamma C')M(\omega)N(\omega)M(\gamma \omega)^{2n+1} & \simeq N(B^n C'', B^{n+1} C'), C', C'' \in C', C' < C'', n \in \mathbb{N}, \\
M(\gamma C')M(\gamma C') & \simeq M(\gamma B^n C), C \in C', n \in \mathbb{N}, \\
M(\gamma \omega)^{n+1} & \simeq M(\gamma B^n \omega), n \in \mathbb{N}, \\
N(\omega) & \simeq L(BC), C \in C', \\
M(\omega)N(\omega)M(\gamma \omega)^{n+1} & \simeq L(B^{n+1} \omega), n \in \mathbb{N}_0,
\end{align*}

\begin{align*}
M(\gamma C')M(\omega)N(\omega)M(\gamma \omega)^{n-1} & \simeq L(B^{n+1} C), C \in C', n \in \mathbb{N}, \\
M(\omega)M(\gamma \omega)^{n-1} & \simeq M(B^n \omega), n \in \mathbb{N}, \\
M(\gamma C')M(\omega)M(\gamma \omega)^{n+1} & \simeq M(B^{n+1} C), C \in C', n \in \mathbb{N}_0, \\
N(\omega)M(\gamma \omega)^n & \simeq N(B^n \omega), n \in \mathbb{N}, \\
M(\gamma C')M(\omega)M(\gamma \omega)^{n+1} & \simeq N(B^{n+1} C), C \in C', n \in \mathbb{N}_0,
\end{align*}

\begin{align*}
M(\gamma \omega)^n(\lambda) & \simeq R(B, \lambda, n), n \in \mathbb{N}, \lambda \in k^*, \\
N(\omega)M(\gamma \omega)^{n+1} & \simeq Q(B^{n+1}), n \in \mathbb{N}_0,
\end{align*}

\begin{align*}
M(\omega)M(\gamma \omega)^{n+1} & \simeq M(B^n \omega \alpha), n \in \mathbb{N}_0, \\
M(\gamma \omega)^nM(x)^n & \simeq M(\gamma B^{n-1} \omega \alpha \xi), n \in \mathbb{N}, \\
M(\gamma \omega)^n(\infty) & \simeq M(\gamma B^{n-1} \omega \alpha), n \in \mathbb{N}, \\
M(\omega)M(\gamma \omega)^{n+1} & \simeq M(B^n \omega \alpha \xi), n \in \mathbb{N}_0,
\end{align*}

\begin{align*}
M(\omega p)N(\omega p)M(\gamma \omega)^{mn} & \simeq N(\omega p, B_{jp} B^{m-1} \omega), p \in [1, r], m \in \mathbb{N}, \\
M(\gamma \omega)^nM(\omega p)N(\omega p)M(\gamma \omega)^{m+n} & \simeq N(B_{jp} B^{n-1} \omega, B_{jp} B^{m-1} \omega), p \in [1, r], n, m \in \mathbb{N}, n < m,
\end{align*}
\[M(\gamma C)M(\gamma^\omega)^n M(\omega_{jp})N(\omega_{jp}) M(\gamma^\omega)^{m+n+1}\]
\[\simeq N(B_{jp}B^n C, B_{jp}B^{m-1}\omega), \quad p \in [1, r], \quad C \in C', \quad n, m \in N_0, \quad n < m,\]
\[M(\omega_{jp})N(\omega_{jp}) M(\gamma^\omega)^m M(\gamma C)^{m+1} \simeq N(\omega_{jp}, B_{jp} B^m C), \]
\[p \in [1, r], \quad C \in C', \quad m \in N_0,\]
\[M(\gamma^\omega)^n M(\omega_{jp})N(\omega_{jp}) M(\gamma^\omega)^m M(\gamma C)^{m+n+1}\]
\[\simeq N(B_{jp}B^{n-1}\omega, B_{jp} B^m C), \quad p \in [1, r], \quad C \in C', \quad m, n \in N, \quad n \leq m,\]
\[M(\gamma C')M(\gamma^\omega)^n M(\omega_{jp})N(\omega_{jp}) M(\gamma^\omega)^m M(\gamma C')^{m+n+1}\]
\[\simeq N(B_{jp}B^n C', B_{jp} B^m C''), \quad p \in [1, r], \quad C', C'' \in C', \quad n, m \in N_0, \quad (n, C') < (m, C''),\]
\[M(\gamma^\omega)^n M(\omega_{jp}) \simeq M(B_j B^{n-1}\omega), \quad j \in [j_0 + 1, j_{r+1} - 1], \quad n \in N,\]
\[M(\gamma C)M(\gamma^\omega)^n M(\omega_{jp}) \simeq M(B_j B^n C), \quad j \in [j_0 + 1, j_{r+1} - 1], \quad n \in N_0,\]
\[M(\gamma^\omega)^n M(\omega_{jp}) \simeq N(B_{jp} B^{n-1}\omega), \quad p \in [1, r], \quad n \in N,\]
\[M(\gamma C)M(\gamma^\omega)^n M(\omega_{jp}) \simeq N(B_{jp} B^{n-1}\omega), \quad p \in [1, r], \quad m \in N,\]
\[M(\gamma^\omega)^n M(\omega_{jp}) \simeq N(C, B_{jp} B^{n-1}\omega), \quad C \in C', \quad p \in [1, r], \quad m \in N,\]
\[M(\gamma C)M(\gamma^\omega)^n M(\omega_{jp}) \simeq M(\gamma B_{jp} B^{m-1}\omega), \quad C \in C', \quad p \in [1, r], \quad m \in N_0,\]
\[M(\gamma^\omega)^n M(\omega_{jp}) \simeq N(C'', B_{jp} B^m C''), \quad C'' \in C', \quad n, m \in N_0,\]
\[M(\gamma C)M(\gamma^\omega)^n (0) \simeq M(\xi B^{m-1}\omega), \quad m \in N,\]
\[M(\gamma C)M(\gamma^\omega)^{m+1} \simeq M(\xi B^m C), \quad C \in C', \quad m \in N_0,\]
\[M(\omega_{jp})N(\omega_{jp}) \simeq N(\omega_{jp}, \omega_{jp} \alpha), \quad p \in [1, r],\]
\[M(\gamma^\omega)^n M(\omega_{jp})N(\omega_{jp}) \simeq N(B_{jp} B^{m-1}\omega, \omega_{jp} \alpha), \quad p \in [1, r], \quad m \in N,\]
\[M(\omega_{jp})N(\omega_{jp}) M(\gamma^\omega)^n \simeq N(\omega_{jp}, B_{jp} B^{n-1}\omega\alpha), \quad p \in [1, r], \quad n \in N,\]
\[M(\gamma^\omega)^n M(\omega_{jp})N(\omega_{jp}) M(\gamma^\omega)^{n+m+1} \simeq N(B_{jp} B^{m-1}\omega, B_{jp} B^{n-1}\omega\alpha), \quad p \in [1, r], \quad n, m \in N,\]
\[M(\gamma C)M(\gamma^\omega)^n M(\omega_{jp})N(\omega_{jp}) \simeq N(B_{jp} B^m C, \omega_{jp} \alpha), \quad C \in C', \quad p \in [1, r], \quad m \in N_0,\]
\[M(\gamma C)M(\gamma^\omega)^n M(\omega_{jp})N(\omega_{jp}) M(\gamma^\omega)^{n+m+2} \]
\[\simeq N(B_{jp} B^m C, B_{jp} B^{n-1}\omega\alpha), \quad C \in C', \quad p \in [1, r], \quad n \in N, \quad m \in N_0,\]
\[ M(\omega_j)N(\omega_j)M(x)^1 \simeq N(\omega_j, \omega_j, \alpha \xi), \quad p \in [1, r], \]
\[ M(\gamma \omega)^m M(\omega_j)N(\omega_j)M(x)^{m+1} \simeq N(B_{j_p} B^{m-1} \omega, \omega_j, \alpha \xi), \quad p \in [1, r], \quad m \in \mathbb{N}, \]
\[ M(\omega_j)N(\omega_j)M(\gamma \omega)^n M(x)^{n+1} \simeq N(\omega_j, B_{j_p} B^{n-1} \omega \alpha \xi), \quad p \in [1, r], \quad n \in \mathbb{N}, \]
\[ M(\gamma \omega)^m M(\omega_j)N(\omega_j)M(\gamma \omega)^n M(x)^{n+m+1} \]
\[ \simeq N(B_{j_p} B^{m-1} \omega, B_{j_p} B^{n-1} \omega \alpha \xi), \quad p \in [1, r], \quad n, m \in \mathbb{N}, \]
\[ M(\gamma C) M(\gamma \omega)^m M(\omega_j)N(\omega_j)M(x)^{m+2} \simeq N(B_{j_p} B^{m} C, \omega_j, \alpha \xi), \quad C \in C', \quad p \in [1, r], \quad n \in \mathbb{N}, \quad m \in \mathbb{N}_0, \]
\[ M(\gamma C) M(\gamma \omega)^m M(\omega_j)N(\omega_j)M(\gamma \omega)^n M(x)^{n+m+2} \]
\[ \simeq N(B_{j_p} B^{m} C, B_{j_p} B^{n-1} \omega \alpha \xi), \quad C \in C', \quad p \in [1, r], \quad n \in \mathbb{N}, \quad m \in \mathbb{N}_0, \]
\[ M(\gamma \omega)^{1} \simeq M(\gamma \omega_j, \alpha), \quad p \in [1, r], \]
\[ M(\gamma \omega)^{n+1} \simeq M(\gamma B_{j_p} B^{n-1} \omega \alpha), \quad p \in [1, r], \quad n \in \mathbb{N}, \]
\[ M(\omega_j)^{1} \simeq M(\omega_j \alpha), \quad j \in [j_0 + 1, j_{r+1} - 1], \]
\[ M(\gamma \omega)^{n+1} \simeq M(B_j B^{n-1} \omega \alpha), \quad j \in [j_0 + 1, j_{r+1} - 1], \quad n \in \mathbb{N}, \]
\[ N(C, \omega_j)^{1} \simeq N(C, \omega_j, \alpha), \quad C \in C_p, \quad p \in [1, r], \]
\[ N(\gamma \omega)^{n+1} \simeq N(C, B_j B^{n-1} \omega \alpha), \quad C \in C_p, \quad p \in [1, r], \quad n \in \mathbb{N}, \]
\[ M(\gamma \omega_j) M(x)^{1} \simeq M(\gamma \omega_j, \alpha \xi), \quad p \in [1, r], \]
\[ M(\gamma \omega)^{n+1} \simeq M(\gamma B_{j_p} B^{n-1} \omega \alpha \xi), \quad p \in [1, r], \quad n \in \mathbb{N}, \]
\[ M(\omega_j)^{1} \simeq M(\omega_j \alpha \xi), \quad j \in [j_0 + 1, j_{r+1} - 1], \]
\[ M(\gamma \omega)^{n+1} \simeq M(B_j B^{n-1} \omega \alpha \xi), \quad j \in [j_0 + 1, j_{r+1} - 1], \]
\[ N(C, \omega_j)^{1} \simeq N(C, \omega_j, \alpha \xi), \quad C \in C_p, \quad p \in [1, r], \]
\[ M(\gamma \omega)^{n+1} \simeq N(C, B_j B^{n-1} \omega \alpha \xi), \quad C \in C_p, \quad p \in [1, r], \quad n \in \mathbb{N}, \]
\[ N(\gamma \omega_j) M(x)^{1} \simeq M(\gamma \omega_j, \alpha \xi), \]
\[ N(\gamma \omega)^{n+1} \simeq M(\gamma \xi B^{n-1} \omega \alpha), \quad n \in \mathbb{N}, \]
\[ M(x)^{1} \simeq M(\xi), \]
\[ M(\gamma \omega)^{n+1} \simeq M(\xi, \gamma B^{n-1} \omega \alpha \xi), \quad n \in \mathbb{N}, \]
We now prove Lemma 3.3.1. Let $x \prec j < j$.

\[ M(\gamma \omega)^n M(\omega_j) N(\omega_{j_0}) \overset{n+2}{\sim} N(B_{j_0} B^{n-1} \omega \alpha, \omega_\beta), \quad p \in [1, r], \quad n \in \mathbb{N}, \]

\[ M(\gamma \omega)^n M(\omega_j) N(\omega_{j_0}) M(\gamma \omega)^m \overset{n+m+2}{\sim} N(B_{j_0} B^{n-1} \omega \alpha, B_{j_0} B^{m-1} \omega \alpha), \quad p \in [1, r], \quad n, m \in \mathbb{N}, \quad m < n, \]

\[ M(x) M(\omega_j) N(\omega_{j_0}) \overset{2}{\sim} N(\omega_j, \omega_\beta), \quad p \in [1, r], \]

\[ M(\gamma \omega)^n M(\omega_j) N(\omega_{j_0}) \overset{n+2}{\sim} N(B_{j_0} B^{n-1} \omega \alpha \beta, \omega_\beta), \quad p \in [1, r], \quad n \in \mathbb{N}, \]

\[ M(\gamma \omega)^n M(\omega_j) N(\omega_{j_0}) M(x) \overset{n+m+2}{\sim} N(B_{j_0} B^{n-1} \omega \alpha \beta, \omega_\beta), \quad p \in [1, r], \quad n \in \mathbb{N}, \]

\[ M(x) M(\omega_j) N(\omega_{j_0}) M(\gamma \omega)^m \overset{n+m+2}{\sim} N(B_{j_0} B^{n-1} \omega \alpha \beta, B_{j_0} B^{m-1} \omega \alpha \beta), \quad p \in [1, r], \quad n, m \in \mathbb{N}, \quad m < n, \]

\[ M(x) M(\omega_j) N(\omega_{j_0}) M(x) \overset{n+2}{\sim} N(B_{j_0} B^{n-1} \omega \alpha \beta, \omega_\beta), \quad p \in [1, r], \quad n \in \mathbb{N}, \]

\[ M(\gamma \omega)^n M(\omega_{j_0}) \overset{1}{\sim} N(\omega_{j_0}, \omega_\beta), \quad p \in [1, r], \]

\[ N(\omega_{j_0}) \overset{1}{\sim} N(\omega_{j_0}, \omega_\beta), \quad p \in [1, r], \quad n \in \mathbb{N}, \]

\[ N(\omega_j) M(x) \overset{1}{\sim} N(B_{j_0} B^{n-1} \omega \alpha \beta), \quad p \in [1, r], \quad n \in \mathbb{N}, \]

\[ M(\gamma \omega)^n M(\omega_{j_0}) M(x) \overset{n+1}{\sim} N(B_{j_0} B^{n-1} \omega \alpha \beta), \quad p \in [1, r]. \]

3.11. We now prove Lemma 3.3.1. Let $x'$ be an admissible index of $Q^r$. Let first $x' = x_{i,j}$, $x' \neq z$. In this case $R^r_{x'} = R^r_{x}$. If $i \neq i_0$ or $i = i_0$ and $j < j_0$, then also $\text{Hom}(R^r_{x'}, \text{mod } A^r) = \text{Hom}(R^r_{x}, \text{mod } A^r)$ (if $i = i_0$ and $j_0 < j$), then the indecomposable objects of $\text{Hom}(R^r_{x'}, \text{mod } A^r)$ are the indecomposable objects of $\text{Hom}(R^r_{x}, \text{mod } A^r)$ and

\[ \overline{M(\gamma \omega)^n M(\omega_{j_0})} \overset{n}{\sim} M(\gamma \omega)^n M(\omega_{j_0}), \quad n \in \mathbb{N}, \]

\[ M(\gamma C) M(\omega_j) M(\omega_{j_0}) \overset{n+1}{\sim} M(\gamma C) M(\omega_j) M(\omega_{j_0})^{n+1}, \quad C \in C', \quad n \in \mathbb{N}, \]

\[ \overline{M(\gamma \omega)^n M(\omega_{j_0})} \overset{n+1}{\sim} M(\gamma \omega)^n M(\omega_{j_0})^{n+1}, \quad n \in \mathbb{N}, \]

and

\[ \overline{M(\gamma \omega)^n M(\omega_{j_0}) M(x)} \overset{n+1}{\sim} M(\gamma \omega)^n M(\omega_{j_0}) M(x)^{n+1}, \quad n \in \mathbb{N}. \]
Assume now that $x' = z_{i,j}$. If $i \neq i_0$, then again $R_{x'}^x = R_{x'}$ and \( \text{Hom}(R_{x'}^x, \text{mod } A^x) = \text{Hom}(R_{x'}^x, \text{mod } A) \). If $i = i_0$ then $j = j_p$ for $p \in [1, r]$. Moreover, $R_{x'}^x = \overline{R}_{x'}$ and the indecomposable objects of $\text{Hom}(R_{x'}^x, \text{mod } A^x)$ are

$$M(\gamma_p C), C \in C_p, M(x'),$$

$$\overline{N(C, \omega_{j_q})}, \overline{M(\gamma_q \omega_{j_q})}, \overline{N(\omega_{j_q})}, C \in C_{j_q}, q \in [p, r],$$

$$M(\omega_l), l \in [j_p, \ldots, j_{r+1} - 1], \overline{0},$$

$$\overline{M(\gamma \omega)^n M(\gamma_q \omega_{j_p})}, \overline{M(\gamma \omega)^{n-1} M(\omega_{j_q}) N(\omega_{j_q})}, n \in \mathbb{N}, q \in [p, r],$$

$$\overline{M(\gamma C) M(\gamma \omega)^n M(\gamma_q \omega_{j_p})^{n+1}}, C \in C', n \in \mathbb{N}_0,$$

$$\overline{M(\gamma C) M(\gamma \omega)^n M(\omega_{j_q}) N(\omega_{j_q})^{n+2}}, C \in C_{j_q}', n \in \mathbb{N}_0, q \in [p, r],$$

and

$$\overline{M(\gamma \omega)^n M(\omega_{j_q}) N(\omega_{j_q}) M(x)^{n+2}}, n \in \mathbb{N}_0, q \in [p, r].$$

Finally, assume that $x' = z$ (it is possible, if $p_{i_0} + |T_{i_0}| \notin S_{i_0}$). In this case $R_{x'}^x = \overline{X}_x M(x_{i_0, p_{i_0} + |T_{i_0}|})$. It follows that the indecomposable objects of $\text{Hom}(R_{x'}^x, \text{mod } A^x)$ are

$$\overline{M(C) M(x_{i_0, p_{i_0} + |T_{i_0}|})}, \overline{M(C)}, C \in C_x,$$

and

$$\overline{M(x_{i_0, p_{i_0} + |T_{i_0}|})}, \overline{0}.$$

3.12 Now we indicate how to prove Lemma \[3.3.2\]. Let $x'$ be an admissible index of $Q^x$. If $x' = x_{i,j}$, $x' \neq z$, then $X_{x'}^x = X_{x'}$. If in addition, $i \neq i_0$ or $i = i_0$ and $j < j_0$, then $\text{Hom}(X_{x'}^x, \text{mod } A^x) = \text{Hom}(X_{x'}^x, \text{mod } A)$. Let $i = i_0$ and $j_0 < j$. Then the indecomposable objects of $\text{Hom}(X_{x'}^x, \text{mod } A^x)$ are the indecomposable objects of $\text{Hom}(X_{x'}^x, \text{mod } A)$ and

$$\overline{M(\gamma \omega)^{n+1} M(\omega_{j_q})^{n+1}}, \overline{M(C) M(\gamma \omega)^n M(\omega_{j_q})^{n+1}}, C \in C', n \in \mathbb{N}_0,$$

and

$$\overline{M(\gamma \omega)^n M(\omega_{j_q})^{n+1}}, \overline{M(\gamma \omega)^n M(\omega_{j_q}) M(x)^{n+1}}, n \in \mathbb{N}_0.$$

Assume now that $x' = z_{i,j}$. Again $X_{x'}^x = X_{x'}$ and if $i \neq i_0$ then $\text{Hom}(X_{x'}^x, \text{mod } A^x) = \text{Hom}(X_{x'}^x, \text{mod } A)$. Let $i = i_0$. Then $j = j_p$ for $p \in [1, r]$. The indecomposable objects of $\text{Hom}(X_{x'}^x, \text{mod } A^x)$ are the indecomposable objects of $\text{Hom}(X_{x'}^x, \text{mod } A)$ and

$$\overline{M(\gamma \omega)^{n+1} M(\gamma_q \omega_{j_p})^{n+1}}, \overline{M(C) M(\gamma \omega)^n M(\gamma_q \omega_{j_p})^{n+1}}, C \in C', n \in \mathbb{N}_0,$$
and
\[ M(\gamma \omega)^n M(\gamma p \omega_j p)^{n+1}, \quad M(\gamma \omega)^n M(\gamma p \omega_j p) M(x)^{n+1}, \quad n \in \mathbb{N}_0. \]

Finally, let \( x' = z \). In this case \( X_x x' = \overline{X_x} \) and the indecomposable objects of \( \text{Hom}(X_x x', \mod A^x) \) are
\[ M(C), \quad C \in C_x, \quad \text{and} \quad \overline{0}. \]

3.13. It remains to give the proof of Lemma 3.3. Let \( x' = x_{i,j} \) be the vertex of \( Q^x \) satisfying the hypothesis of Lemma 3.3. If \( i \neq i_0 \) then \( I_{x'} = I_{x'} x' \) and \( \text{Hom}(I_{x'} x', \mod A^x) = \text{Hom}(I_{x'} x', \mod A) \). Assume now that \( i = i_0 \). Then \( j \in [j_0 + 1, j_{r+1}] \). Let \( p = \min \{ q \in [1, r+1] \mid j \leq j_q \} \). First consider the case \( j \neq j_{r+1} \). Then \( I_{x'} = I_{x'} x' \) and the indecomposable objects of \( \text{Hom}(I_{x'} x', \mod A^x) \) are
\[ N(C, \omega_j q), \quad M(\gamma q \omega_j q), \quad N(\omega_j q), \quad C \in C'_j q, \quad q \in [p, r], \]
\[ M(\omega l), \quad l \in [j, \ldots, j_{r+1} - 1], \quad \overline{0}, \]
\[ M(\gamma \omega)^{n-1} M(\omega_j q) N(\omega_j q)^n, \quad n \in \mathbb{N}, \quad q \in [p, r], \]
\[ M(\gamma C) M(\gamma \omega)^n M(\omega_j q) N(\omega_j q)^{n+2}, \quad C \in C'_j q, \quad n \in \mathbb{N}_0, \quad q \in [p, r], \]
and
\[ M(x) M(\gamma \omega)^n M(\omega_j q) N(\omega_j q)^{n+2}, \quad n \in \mathbb{N}_0, \quad q \in [p, r]. \]
If \( j = j_{r+1} \) then the claim is clear.

REFERENCES

[1] R. Bautista, P. Gabriel, A. V. Roıter, and L. Salmerón, Representation-finite algebras and multiplicative bases, Invent. Math. 81 (1985), 217–285.
[2] G. Bobiński, Characterization of admissible algebras with formal two-ray modules, Comm. Algebra (in press).
[3] G. Bobiński, P. Draxler, and A. Skowroński, Domestic algebras with many nonperiodic Auslander-Reiten components, Comm. Algebra 31 (2003), 1881–1926.
[4] G. Bobiński and A. Skowroński, On a family of vector space categories, Cent. Eur. J. Math. 1 (2003), 332–359.
[5] ———, Domestic iterated one-point extensions of algebras by two-ray modules, Cent. Eur. J. Math. 1 (2003), 457–476.
[6] V. M. Bondarenko, Representations of bundles of semichained sets and their applications, St. Petersburg Math. J. 3 (1992), 973–996.
[7] K. Bongartz, A criterion for finite representation type, Math. Ann. 269 (1984), 1–12.
[8] K. Bongartz and P. Gabriel, Covering spaces in representation-theory, Invent. Math. 65 (1981/82), 331–378.
[9] O. Bretscher and P. Gabriel, *The standard form of a representation-finite algebra*, Bull. Soc. Math. France **111** (1983), 21–40.

[10] M. C. R. Butler and C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra **15** (1987), 145–179.

[11] W. W. Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56** (1988), 451–483.

[12] ———, *Maps between representations of zero-relation algebras*, J. Algebra **126** (1989), 259–263.

[13] ———, *Tame algebras and generic modules*, Proc. London Math. Soc. (3) **63** (1991), 241–265.

[14] ———, *Functorial filtrations. II. Clans and the Gel’fand problem*, J. London Math. Soc. (2) **40** (1989), 9–30.

[15] B. Deng, *On a problem of Nazarova and Roiter*, Comment. Math. Helv. **75** (2000), 368–409.

[16] Yu. A. Drozd, *Tame and wild matrix problems*, Representation Theory, II, Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 242–258.

[17] Ch. Geiß, *Maps between representations of clans*, J. Algebra **218** (1999), 131–164.

[18] Ch. Geiß and J. A. de la Peña, *Auslander-Reiten components for clans*, Bol. Soc. Mat. Mexicana (3) **5** (1999), 307–326.

[19] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Mat., vol. 1099, Springer, Berlin, 1984.

[20] J. Schröer, *On the infinite radical of a module category*, Proc. London Math. Soc. (3) **81** (2000), 651–674.

[21] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra, Logic and Applications, vol. 4, Gordon and Breach Science Publishers, Montreux, 1992.

[22] B. Wald and J. Waschbüscher, *Tame biserial algebras*, J. Algebra **95** (1985), 480–500.

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

E-mail address: gregbob@mat.uni.torun.pl