Deterministic chaos for Markov chains

Marat Akhmet∗

Department of Mathematics, Middle East Technical University, 06800 Ankara, Turkey

Abstract. We find that Markov chains with finite state space are Poincaré chaotic. Moreover, finite realizations of the chains are arcs of each unpredictable orbit for sure. An illustrating example with a proper numerical simulation is provided.

1 Introduction

The main task of the present research is to find signs of deterministic chaos in Markov chains. Then, to show that the presence is maximal in the sense of probability. The method of domain structured chaos, [5], is in the basis of our study. It has been successfully applied for chaos indication in fractals [4], neural networks [5], and for Bernoulli process [2]. This time, we provide arguments for the chaos in Markov chains. At first, we formalize the realizations of the chains as conveniently constructed sequences. Then, give explanations, why the random dynamics admits the chaos features. Considering, first of all the Poincaré chaos.

The first goal of the Markov research [33] was to show that the random processes of dependent events may behave as processes with independent events. Thus, the model, simple and most effective for many applications was invented. It is impossible underestimate the role of the Markov chains and processes in development of random dynamics theory and its applications. For instance, the egrodic theorem was strictly approved at the first time under the circumstances. There are many observations that the chains are relative to symbolic dynamics and correspondingly to Bernoulli scheme. The significant step in the comprehension was done by Donald Ornstein in [35], who verified that $B$–automorphisms such as sub-shifts of finite type and Markov shifts, Anosov flows and Sinai’s billiards, ergodic automorphisms of the $n$–torus [38], and the continued fraction transform are, in fact, isomorphic. Issuing from these sources it is of great interest not only to show that different stochastic processes can be described in the terms of deterministic chaos, but that they relate equally to the chaos. Researchers have looked in both directions, for chaos in random dynamics, as well as for stochastic features in deterministic motions [13, 15, 17–21, 36, 37]. Thus, the problem of chaos in Markov chains, which is discussed in the paper, is a part of the more general project. One can apply finite [13, 15] or countable [22] partitioning to find randomness in deterministic motions. Differently, in the present research we utilize an uncountable partitioning to establish strong relations of the random processes with deterministic chaos.

In resent studies [4–7, 10], we formalize a chaos generation through specially structured sets, subduing them to the similarity map, which is in the paradigm of the Bernoulli shift [42]. This provides to us the strong comprehension that the chaotic behavior is proper to the Bernoulli scheme dynamics and then to various random processes on discrete and continuous time [1]- [3]. Moreover, it was clarified in which sense one should understand presence of deterministic chaos attributes in stochastic processes.

In the present research we are on the next step of the exploration, and show that the chaos can be recognized for Markov chains. Time-homogeneous chains on a finite state space, and chains with memory are discussed. The approach is applied such that the dynamics can be investigated, focusing not on ergodicity, but on a single motion description such that an individual geometry is better seen. Thus, we show that an isolated realization of a Markov chain behaves in time identically with a properly chosen realization of a Bernoulli scheme, and they both are identical in dynamics with the path of the similarity map in a correspondingly chosen space. The unpredictable orbit [8] as a single isolated motion, presenting the Poincaré chaos [25] in the chains, is the ultimate point of the comprehension.

Considering the signs of the chaos in Markov chains, it is important to say not about sensitivity, but individual behavior of trajectories. That we say that simulating the random process, one can not determine that it is different than an orbit of corresponding deterministic dynamics, which we can suggest for sure. But discussing the sensitivity,
one can say that it is true with some positive probability. More precisely, simulating a random process, we are
certain that there is another motion which starts arbitrary near and diverges from the first one on the distance
not less than a positive constant common for the dynamics with a positive probability. Of course, the probability
less than one can make us critical, but it does not different than that for the deterministic chaos as the points ,
which are start for the diverging can not be determined for both dynamics, deterministic as well as for the random.
But, we in our research know how to find the start point. For this we have developed the sequential test. On this
basis, one can not only evaluate the sequences, which are in the basis of the definition of the unpredictable point,
but also determine point which provide the divergence with arbitrary smallness. This everything is done on the
basis of a single orbit or let say a single simulation.

2 Preliminaries

The domain structured chaos unites models, which admit all types of theoretical chaos such as Poincaré, Li-Yorke
and Devaney [5], but the first one is most convenient to confirm that chaos is proper for the random dynamics. For
this reason, let us start with the definition of the unpredictable sequence. The notion is the strict evidence for the
Poincaré chaos [25].

Definition 1. [25] A bounded sequence \( \kappa_i, i \geq 1 \), in \( \mathbb{R}^p \) is called unpredictable if there exist a positive number \( \varepsilon_0 \)
and sequences \( \zeta_n, \eta_n, n \geq 1 \), of positive integers both of which diverge to infinity such that \( \|\kappa_i + \zeta_n - \kappa_i\| \to 0 \) as
\( n \to \infty \) for each \( i \) in bounded intervals of integers and \( \|\kappa_i + \eta_n - \kappa_\eta_n\| \geq \varepsilon_0 \) for each natural number \( n \).

For a finite metric space \((S,d)\), the last definition has the next form.

Definition 2. A bounded sequence \( s_i \in S, i \geq 0 \), is called unpredictable if there exist a positive number \( \varepsilon_0 \)
and sequences \( \zeta_n, \eta_n, n \geq 0 \), of positive integers both of which diverge to infinity such that \( s_{i+\zeta_n} = s_i \) for each bounded
interval of integers, if \( n \) is sufficiently large, and \( d(s_{\zeta_n+\eta_n}, s_{\eta_n}) \geq \varepsilon_0 \) for each natural number \( n \).

The main result of the paper is that there exists the realization, an unpredictable sequence, of the Markov
process with the finite state space, which closure in the topology of convergence on bounded intervals is the set of
all infinite realizations. Then, it implies that each finite simulation of the dynamics is an ark of the unpredictable
realization. This is why, one can confirm that the Poincaré chaos is a certain event for the Markov chain. Let
us remind that an event is certain, if it occurs at every performance of an experiment. One must specify that the
convergence, in the circumstances, is the coincidence of the unpredictable realization with each finite realization.

The closure of the unpredictable realization is said to be the quasi-minimal set [40]. It contains uncountable set
of unpredictable realizations. We have proved that a quasi-minimal set as the union of all infinite realizations of
the Markov chain is a certain event. And this is another formulation of the main result. The results on chaos presence
in the random processes have been investigated in our previous papers [2,11], but the the chaos appearance in the
dynamics with probability one is approved at the first time. We have explored that namely unpredictable orbit is
most proper candidate for the analysis.

Generally speaking, our results confirm that the Bernoulli scheme [2], Markov chains as well as abstract hyper-

tic dynamics [1] are all with the same type of chaos. It is significant that the Poincaré chaos is proper for the

otions. This provides, new opportunities, exceptionally for stochastic processes. We suppose that the research can
be complemented with similar analysis for other Bernoulli automorphisms considered by D. Ornstein [35] as well
with extension of the results for majority of stochastic processes, if proper structured domains will be constructed.

A Markov chain is a stochastic model, which describes a sequence of possible events such that the probability
of each event depends only on the state attained in the previous one [29,30]. There are many applications of the
Markov chains as statistical models of real-world processes such as studying queues or lines of customers arriving at
an airport, currency exchange rates, cruise control systems in motor vehicles and animal population dynamics [34].

Consider a discrete-time stochastic process \( X_n, n \geq 0 \), on a countable set \( S \). That is, a collection of random
variables defined on a probability space \((\Omega,F,P)\), where \( P \) is a probability measure on a family of events \( F \) in an

vent-space \( \Omega \). The set \( S \) is the state space of the process, and the value \( X_n \in S \) is the state of the process at time
\( n \).

The Markov chain, is a stochastic process such that the Markov property \( P\{X_{n+1} = s_j|X_0,...,X_n\} = P\{X_{n+1} = s_j|X_n\} \)
is true for all \( s_i, s_j \in S \) and \( n \geq 0 \), and \( P\{X_{n+1} = s_j|X_n = s_i\} = p_{ij} \), where \( p_{ij} \) is the transition probability
that the chain jumps from state \( i \) to state \( j \). The property says that, at any time \( n \), the next state \( X_{n+1} \) is conditionally independent of the past \( X_0,...,X_{n-1} \) given the present state \( X_n \). More precisely, that the transition
probabilities do not depend on the time parameter \( n \). That is, the chain is time-homogeneous. If the transition
probabilities were functions of time, the process would be a non-time-homogeneous Markov chain.
To avoid any terminological confuse, we will appeal to the dynamical interpretation of Markov chains as the following recursion

$$X_n = f(X_{n-1}, Y_n),$$

where $n \geq 0$ is the time parameter, $Y_1, Y_2, \ldots$ are independent and identically distributed and $f$ is a deterministic function. That is, the new state $X_n$ is simply a function of the last state and an auxiliary random variable. In other words, one can consider a Markov chain as a random dynamical system [14]. In the present paper realizations are considered orbits or trajectories of the corresponding random dynamics (1). Consequently, one can denote the realization as $X_n = X(n) = X(n, X_0)$, where $X_0 = X(0)$ is the initial value, which is determined randomly. This is a step of better comprehension of the stochastic dynamics through chaotic interpretation. If the parameter $n$ runs over $\mathbb{N}$ we say that a realization is infinite. Otherwise, it is finite. Thus, realizations are infinite or finite sequences of elements from the state space. We will use the set of all realizations to discuss the problem of the deterministic chaos for the Markov chains.

3 Domain structured chaos and Markov chains

Consider a finite state space $S = \{s_1, \ldots, s_m\}$, where $m$ is a natural number, not smaller than two, and a metric $d$ for the space. Denote $p_{ij} = p_j(s_i)$ the Markov probability for $s_j, j = 1, \ldots, m$, such that $\Sigma_{j=1}^{m} p_{ij} = 1$ for all $i = 1, \ldots, m$. Assume that all probabilities $p_{ij}, i, j = 1, \ldots, m$, are positive.

In the basis of our construction is the event $f_{ij}$, $i, j = 1, \ldots, m$, which consists of two elementary events, $s_i$ and $s_j$, happen successively, such that an infinite realization of the Markov chain is formalized as the infinite sequence $f_{i_1 j_1} f_{i_2 j_2} \cdots f_{i_n j_n} \cdots$, with $j_k = i_{k-1}$ for all $k = 2, 3, \ldots$. We have that $p(f_{ij}) = p_{ij}$. The formalization does not give advantages, if one consider the chains without memory, but it makes easier the discussion of the processes with memory, in what follows. Present the last sequence as the element $F_{i_1,i_2,\ldots,i_k} = 1, 2, \ldots, m, k = 1, 2, \ldots$, of the space $F$ with metric $\delta(F_{i_1,i_2,\ldots,i_k}) = \Sigma_{k=1}^{\infty} d(s_{i_k}, s_{i_k})/2^k$.

Next, we formalize Markov chains with memory of a non-zero length. We start with the length equal to two such that the element $f_{i_1 j_1}$ presents three elementary events $s_i, s_j$ and $s_k$ happen successively, and the probability for $s_k$ is equal to $p_{i_1 j_1} = p_k(s_i, s_j)$. Then the Markov chain with memory has the formal presentation $f_{i_1 j_1} f_{i_1 j_2} f_{i_1, j_2, k_3} \cdots f_{i_1, j_2, k_3, k_4} \cdots$, where $j_l = k_{l-1}, i_l = j_{l-1}$ for all $l = 2, 3, \ldots$. Accepting the last sequence as the element $F_{i_1,i_2,\ldots,i_k} = 1, 2, \ldots, m, k = 1, 2, \ldots$, of the space $F$ with metric $\delta(F_{i_1,i_2,\ldots,i_k}) = \Sigma_{k=1}^{\infty} d(s_{i_k}, s_{i_k})/2^k$ we attain the basis, common with that for the chain without memory.

At last, consider the Markov process with the length equal to arbitrary natural number $n$. Then we formalize the discussion with the elements $f_{i_1,\ldots,i_n}$, which consist of successive elementary events $s_{i_1}, \ldots, s_{i_n}$, such that the sequence $f_{i_1} f_{i_2} \cdots f_{i_n}$ with $i_l = \frac{i_l}{i_{l-1}} = i_{l-1}, j = 1, \ldots, n - 1, l = 2, 3, \ldots$, is the formalization of the chain. We obtain the structure for the dynamics research, if accept the last sequence as an element $F_{i_1,i_2,\ldots,i_k} = 1, 2, \ldots, m, k = 1, 2, \ldots$, of the space $F$, making stress on the events with the indices $i_k, k = 1, 2, \ldots$. To complete the chaos analysis we shall consider, other indexes also, namely, the spaces consisting of elements $F_{i_1,i_2,\ldots,i_k} = 1, 2, \ldots, m, k = 1, 2, \ldots$, with arbitrary fixed $j = 1, 2, \ldots, n$. This is not necessary, if one consider the dynamics from the traditional point of view, when the phenomenon has to be observed only for unbounded sequence of moments, even for continuous time. Remember, Poincaré stroboscopic approach. Nevertheless, in the research of the Markov chains with memory, that is in our present case, it is significant to precise that there are finite number of subsequences which cover with chaotic dynamics over the whole discrete time range. Thus the specific properties of the dynamics are emphasized.

It is clear that for all cases, regardless are they with memory or without memory, we have constructed one and the same space $F$ of elements $F_{i_1,i_2,\ldots,i_k} = 1, 2, \ldots, m, k = 1, 2, \ldots$, with the distance $\delta(F_{i_1,i_2,\ldots,i_k}) = \Sigma_{k=1}^{\infty} d(s_{i_k}, s_{i_k})/2^k$. This is why, the space is the object of analysis for the deterministic chaos presence, next. To complete the dynamics, we shall need the special map on the space.

Consider the map $\varphi : F \rightarrow F$ such that

$$\varphi(F_{i_1,i_2,\ldots,i_n}) = F_{i_2,i_3,\ldots,i_n}$$

(2)

for each element of the set. The map $\varphi$ is in the paradigm of the Bernoulli shift [42], known for the symbolic dynamics. It is said to be the similarity map [4] as it is convenient to describe fractals, which are determined through the self-similarity.

From the definitions of the set $F$ and map $\varphi$ it implies that the values of the map correspond to the members of the state space, which appear orderly. Consequently, if one proves that the process of appearance is chaotic in
one of the senses accepted in literature, then we must accept that the chaos is proper for the stochastic dynamics with proper arguments of probability.

The following sets are needed,

\[ F_{i_1i_2...i_n} = \bigcup_{j_k=1,2,...,m} F_{i_1i_2...i_nj_1j_2...j_k} \]  

(3)

where indices \( i_1, i_2, ..., i_n \), are fixed.

It is clear that

\[ F \supseteq F_{i_1} \supseteq F_{i_1i_2} \supseteq ... \supseteq F_{i_1i_2...i_n} \supseteq F_{i_1i_2...i_nj_1j_2...j_k} \]

(4)

that is, the sets form a nested sequence.

Considering iterations of the map, one can verify that

\[ \varphi^n(F_{i_1i_2...i_n}) = F, \]

(5)

for arbitrary natural number \( n \) and \( i_k = 1, 2, ..., m \), \( k = 1, 2, ... \). The relations (2) and (4) give us a reason to call \( \varphi \) a similarity map and the number \( n \) the order of similarity.

We will say that for the sets \( F_{i_1i_2...i_n} \) the diameter condition is valid, if

\[ \max_{i_k=1,2,...,m} \text{diam}(F_{i_1i_2...i_n}) \to 0 \text{ as } n \to \infty, \]

(6)

where \( \text{diam}(A) = \sup\{d(x, y) : x, y \in A\} \), for a set \( A \) in \( F \).

Denote the distance between two nonempty bounded sets \( A \) and \( B \) in \( F \) by \( d(A, B) = \inf\{d(x, y) : x \in A, y \in B\} \). Set \( F \) satisfies the separation condition of degree \( n \) if there exist a positive number \( \varepsilon_0 \) and a natural number \( n \) such that for arbitrary indices \( i_1i_2...i_n \) one can find indices \( j_1j_2...j_n \) such that

\[ d(F_{i_1i_2...i_n}, F_{j_1j_2...j_n}) \geq \varepsilon_0 \]

(7)

Next, we will formulate two theorems on Poincaré and Devany chaos. Verification of the assertion is given in [2]. The following theorem asserts that the similarity map \( \varphi \) possesses the three ingredients of Devaney chaos, namely density of periodic points, transitivity and sensitivity. A point \( F_{i_1i_2...i_n} \in F \) is periodic with period \( n \) if its index consists of endless repetitions of a block of \( n \) terms. If \( t \) said to be a chaotic structure for \( F \). The next theorem is a particular case of more general Theorem 3.

**Theorem 1.** [2, 5] If the diameter and separation conditions are valid, then the dynamics \((F, \delta, \phi)\) is chaotic in the sense of Devaney.

In [8, 25], Poisson stable motion is utilized to distinguish chaotic behavior from periodic motions in Devaney and Li-Yorke types. The dynamics is given the named Poincaré chaos. The next theorem shows that the Poincaré chaos is valid for the similarity dynamics.

**Theorem 2.** [2, 5] If the diameter and separation conditions are valid, then the similarity map possesses Poincaré chaos.

In addition to the Devaney and Poincaré chaos, it can be shown that the Li-Yorke chaos is also present in the dynamics of the map \( \varphi \). The proof of the theorem is similar to that of Theorem 6.35 in [23] for the shift map defined in the space of symbolic sequences.

Thus, it is proven that the dynamics admits the ingredients of all the theoretical chaos, and we say that that for the space \((F, \delta, \varphi)\) the domain structured chaos is proper. For the present research it is important that there exists an unpredictable trajectory of \( \varphi \) in the sense of Definition 2. It is Poisson stable, and the closure of the set of all trajectories is a quasi-minimal set. In our research [8] we have proved that if a Poisson stable point is additionally unpredictable, then there is the sensitivity in the set of motions. Thus, it was recognized that there is Poincaré chaos. The set \( F \) is bounded. The convergence is in the topology on bounded sets. The orbits of the map \( \varphi \) are infinite sequences.

Since of the accordance between the Markov chain on the finite state space and the set dynamics of the map \( \varphi \), one can conclude that the following assertion is valid. To formulate the result, let us fix an unpredictable realization of the chain, which can be determined as follows. Consider an unpredictable point \( F_{i_1i_2...} \) of the map \( \varphi \). Fix the sequence, \( s^* = \{s^*_k\}_k \), which is the corresponding realization. It is an unpredictable sequence. Due to the Definition 2 and Theorem 2, the following assertion is valid.
Theorem 3. Each finite realization of the Markov chain coincides with an arc of $s^*$. That is, the realization happens in each experiment of the chain, and is a certain event.

The sensitivity property is obvious, since of the finite state space to start arbitrary near means, to coincide at the start moment, and absense of the divergence means absense of the randomness.

Thus, one has to recognize that each experiment with the Markov chain (without memory) produces an arc of an unpredictable orbit. This result can be considered as the main one in this research. It is in full accordance with the principle of the ergodic theory [41] that a single trajectory proves behavior of the whole dynamics and all other trajectories. In fact, we can say that it is a fixed unpredictable orbit. In other words, this is reproduction of the chaos with probability one. This is what guaranties the irregularity of each finite sample path of the chain. As it was proved [25], the existence of an unpredictable orbit implies sensitivity. Consequently, for each infinite sample path one can find arbitrary near another realization, which definitely diverges at finite time for some positive number common in the dynamics. Additionally, our research confirms that constructing a finite realization of the chain numerically, we build a piece of the graph of unpredictable sequence, and this can be applied for definition of unpredictable functions. This was realized in our paper [11]. Finally, consider a fixed infinite realization. One can see that arbitrary near to the sequence a finite realization starts, which exists with positive probability and diverges from the fixed realization on the distance not less than a positive constant common for the chain. That is the sensitivity presents with non-zero probability. We suppose that the probability is equal to the unit, but this problem is for the future research.

Now, let us focus on the Markov chains with memory. Let the space $F$ of elements $F_{i_1i_2...i_k}$, $i_1 = 1, 2, ..., m, k = 1, 2, ...$ is given with the similarity map $\varphi$. It is clear that the Poincaré chaos with probability one is present in the discrete dynamics. The discussion is identical with that done for the chains without memory. Consequently, one can conclude that the chaos exists in the dynamics with probability one. If one objects the assertions, since we made the decision considering subsequences of realizations, our response is that this is true for many chaos research in literature. For example, we indicate chaos for continuous dynamics just by considering Poincaré sections observation. In our case, the arguments are much more strong, as we observe the chaos for all sets $F_{i_1i_2...i_k}$, $i_1 = 1, 2, ..., m, k = 1, 2, ..., j = 1, ..., m$. Consequently, we can make decision that the chaos is more strong in its presence for the chains with memory.

3.1 An example: random walk

Consider, as an example, the following Markovian chain. Let the real valued scalar dynamics $X_{n+1} = X_n + Y_n, n \geq 0$, be given such that $Y_n = \{-1, 1\}$ is a random variable, with probability distribution $P(1) = P(-1) = 1/2$, if $X_n \neq 1, 4$, and $Y_n = -1$, if $X_n = 4$, and $Y_n = 1$, if $X_n = 1$. To satisfy the construction of the present research, we will make the following agreements. First of all, denote $s_0 = 1, s_1 = 2, s_2 = 3, s_3 = 4$. Consider, the state space $S = \{s_1, s_2\}$. Introduce the following events, $f_{12} = \{s_1, s_0, s_1\}, f_{12} = \{s_1, s_2\}, f_{21} = \{s_2, s_1\}, f_{22} = \{s_2, s_3, s_2\}$. It is clear that $p_{i1} + p_{i2} = 1, i = 1, 2$, and all the probabilities are equal to the half. That is, we are in the circumstances of the theory of the present paper. Consequently, there is Poincaré chaos. To visualize an unpredictable realization, we will draw the graph of the function $\phi(t) = X_n, t = \{n/10, (n+1)/10\}, 0 \leq t \leq 60$. According to the last Theorem, it is an arc of an unpredictable sequence. The graph of the function is seen in Figure 1. It illustrates an unpredictable sequence, the sample path of the random walk. For better visibility of the dynamics the vertical lines connecting pieces of the graph are drawn.

4 Conclusion

The outcome of the research is the existence of an unpredictable sequence as a realization of the Markov chain, and the sequence appears as finite realization of each experiment of the process. That is, appearance of the sequence is a certain event for the stochastic dynamics. From this point of view one can say that the deterministic chaos is a certain event for the stochastic dynamics. This result is true for many other discrete time random processes. For instance, the Bernoulli scheme. The significant use of the investigation is that one can unite methods of deterministic chaos with those for stochastic dynamics. Many other opportunities may appear. Among the methods are controllability and synchronization of chaos [28] as well as different ways of chaos generation [26,32]. We have proved the sensitivity is present certainly. Thus, the deterministic chaos has been approved for the stochastic processes. Evidently, it is true for the Bernoulli scheme [2] and other dynamics, which can be approved for the domain structuring. Next our study will relate with Markov processes with continuous time, as well as unbounded. This also relates to many other random processes. Our results provide more lights on the Markov chains as ergodic processes, since we have
shown that there is the uncountable set of realizations, unpredictable orbits and each of them are dense in the set of all realizations.

References

[1] M. Akhmet, Abstract Hyperbolic Chaos, arXiv:2006.14700 (2020).
[2] M. Akhmet, Domain structured chaos for discrete random processes, arXiv:1912.10478. (2020).
[3] M. Akhmet, Modular chaos for random processes, arXiv:2004.08383. (2020).
[4] M. Akhmet, E. M. Alejaily, 2019, Abstract Similarity, Fractals and Chaos, Discrete and Continuous Dynamical Systems, Ser. B., doi:10.3934/dcdsb.2020191.
[5] M. Akhmet, E. M. Alejaily, 2019, Domain-Structured Chaos in a Hopfield Neural Network, Int. J. Bifurc. Chaos, 29 (14), 1950205.
[6] M. Akhmet, E. M. Alejaily, 2019, Abstract Fractals, ArXiv e-prints, arXiv:1908.04273, Discontinuity, Nonlinearity and Complexity (accepted).
[7] M. Akhmet, E. M. Alejaily, Finite dimensional space chaotification, Kazakh Mathematical Journal, 19 (4) (2019)21-26
[8] M. Akhmet, M.O. Fen, 2016, Unpredictable points and chaos. Commun. Nonlinear Sci. Numer. Simulat. 40 1-5.
[9] M. Akhmet, M.O. Fen, 2016, Poincaré chaos and unpredictable functions. Commun. Nonlinear Sci. Numer. Simulat. 48 85-94.
[10] M. Akhmet, M.O. Fen and E. M. Alejaily, 2020, Dynamics with chaos and fractals, Springer.
[11] M. Akhmet, M.O. Fen and E. M. Alejaily, A randomly determined unpredictable function, Kazakh Mathematical Journal, 20 (2) (2020) 30-36.
[12] M. Akhmet, A. Tola, Unpredictable Strings, arXiv:2006.0852 (2020).
[13] V.M. Alekseev, M.V. Yakobson, Symbolic dynamics and hyperbolic dynamic systems, Physics reports, 75 (5) (1981) 287-325.
[14] L. Arnold, 1998, Random Dynamical Systems, Springer.
[15] R. Bowen, Markov partitions for axiom A diffeomorphisms, Am. J. Math., 92, 725–747 (1970)
[16] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Mathematics Vol. 470 (Springer, 1975).

[17] R. Bowen, Markov Partitions and Minimal Sets for Axiom A Diffeomorphisms, Amer. J. Math. 92 (1970) 907—918.

[18] R. Bowen, Symbolic Dynamics for Hyperbolic Flows, Amer. J. Math. 95(1973) 429—.440.

[19] R. Bowen and D. Ruelle, The Ergodic Theory of Axiom A Flows, Invent. Math. 29 (1975) 181—202.

[20] R. Bowen, A Horseshoe with Positive Measure, Invent. Math. 29 (1975) 203—204.

[21] R. Bowen, Topological Entropy for Noncompact Sets, Trans. Amer. Math. Soc. 184 (1973) 125—136.

[22] L. Bunimovich, Ya. Sinai, Markov Partitions for Dispersed Billiards, Commun. Math. Phys., 78 (2), 247–280 (1980).

[23] G. Chen, Y. Huang, Chaotic Maps: Dynamics, Fractals and Rapid Fluctuations, Synthesis Lectures on Mathematics and Statistics. Morgan and Claypool Publishers, Texas (2011).

[24] R. L. Devaney, An Introduction to Chaotic Dynamical Systems. Addison-Wesley, Menlo Park (1987).

[25] M. Feckan, Bifurcation and Chaos in Discontinuous and Continuous Systems, HEP-Springer, New York (2011).

[26] P. A. Gagniuc, Markov Chains: From Theory to Implementation and Experimentation. USA, NJ: John Wiley and Sons, (2017).

[27] J.M. González-Miranda, Synchronization and control of chaos, Imperial College Press, London (2004).

[28] B. Hajek, Random Processes for Engineers. Cambridge University Press (2015).

[29] S. Karlin, H. E. Taylor, A First Course in Stochastic Processes. Academic Press, (2012).

[30] T. Y. Li, and J. A. Yorke, Period Three Implies Chaos. Amer. Math. Monthly, 82 (1975) 985-992.

[31] A. Luo, Periodic Flows to Chaos in Time-delay Systems, Springer, New York (2017).

[32] A. A. Markov, Extension of the limit theorems of probability theory to a sum of variables connected in a chain, reprinted in Appendix B of: R. Howard. Dynamic Probabilistic Systems, volume 1: Markov Chains. John Wiley and Sons, 1971.

[33] S. Meyn, R. L. Tweedie, Markov Chains and Stochastic Stability. Cambridge University Press (2009).

[34] D. Ornstein, Bernoulli shifts with the same entropy are isomorphic, Advances in Math. 4 (1970) 337–352.

[35] D. Ornstein, In what sense can a deterministic system be random? Decidability and predictability in the theory of dynamical systems, Chaos Solitons Fractals, 5(2) (1995) 139–141.

[36] D. Ornstein, B. Weiss, On the Bernoulli nature of systems with some hyperbolic structure, Ergodic Theory Dynam. Systems 18(2) (1998) 441–456.

[37] G. Ponce, R. Varao, An Introduction to the Kolmogorov–Bernoulli Equivalence, Springer, Cham, Switzerland (2019).

[38] C. Robinson, Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, CRC Press, NW (1999).

[39] G. Sell, Topological dynamics and ordinary differential equations, Van Nostrand Reinhold, New York, (1971).

[40] P. Walters, An Introduction to Ergodic Theory, Springer, New York (1982)

[41] S. Wiggins, Global Bifurcation and Chaos: Analytical Methods, Springer-Verlag, New York, Berlin (1988).