Variable mesh non polynomial spline method for singular perturbation problems exhibiting twin layers

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Abstract. In this paper, we descend a variable mesh finite difference scheme based on non polynomial spline approximation for the solution of singular perturbation problems with twin boundary layers. We develop the discretization equation for the problem using the condition of continuity for the first order derivatives of the variable mesh non polynomial spline at the interior nodes. The discrete invariant imbedding algorithm is utilized to solve the tridiagonal system obtained by the method. Endeavor examples are illustrated and maximum absolute errors in comparison to the other methods in the literature are shown to vindicate the method.

1. Introduction

We consider a second order linear singularly perturbed two – point boundary value problem of the form:

$$\varepsilon y''(x) = p(x)y(x) + q(x)$$

with boundary conditions

$$y(0) = \gamma_1, \quad y(1) = \gamma_2$$

where $\gamma_1, \gamma_2$ are given constants, $\varepsilon$ is a small positive parameter such that $0 < \varepsilon << 1$ and $p(x), q(x)$ are bounded continuous functions. It is known that the above problem exhibits boundary layers at both ends of the interval depending upon the properties of $p(x)$. These problems arise in many areas of engineering and applied mathematics. Examples of these are heat transport problem with Peclet numbers and Navier Stokes flows with large Reynold number. Because of the presence of boundary layers, difficulties are experienced in solving these types of problems using numerical methods with uniform mesh. In order to get a good approximation, a fine mesh is required in the boundary layer region. In this paper we derived a variable mesh finite difference method based on non polynomial spline approximation that gives third order approximation to the solution of (1) - (2).

A wide variety of splines are described in the text books [1, 2]. Several numerical methods have been developed based on splines for the numerical solution of singular perturbed boundary value problems, in particular to the problems having boundary layers at one or both the ends of the interval. Kadalbajoo and Rajesh Bava [3, 4] used variable mesh Splines of the third and second order convergence methods for singularly perturbed boundary value problems. Tariq Aziz, Arshad Khan [5] used uniform mesh spline methods and Khan et. Al. [6] used variable mesh approximation method using cubic spline in tension for solving such type of problems. Also the application of splines for the numerical solution of singularly perturbed boundary value problems has been described in many papers [7,8,9,10,11,12,13,14,15].
In this present paper, we have derived a uniformly convergent variable mesh finite difference scheme using non polynomial spline for the solution of above problem. The main idea is to use the condition of continuity of the first order derivatives of the variable mesh non polynomial spline at the interior nodes as a discretization equation for the problem. The advantage of our method is higher accuracy with the same computational effort and easy to implement in computer. The paper is organized as follows: In section 2, we define the non polynomial spline method. In section 3, we describe the numerical method for solving singular perturbed singular two – point boundary value problem. In Section 4, the truncation error and classification of various orders of the proposed method are given. In section 5 we discuss convergence analysis of the method. Finally, numerical results and comparison with other methods are given in final section.

2. Non polynomial spline method for variable mesh

Let \( 0 = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = 1 \) be a sub division of an interval \([0, 1]\) with variable step size \( h_i = x_i - x_{i-1} \) for \( i = 1 \) to \( n \) and \( h_{i+1} = \sigma_i h_i \).

Let \( y(x) \) be the exact solution and \( y_i \) be an approximation to \( y(x_i) \) obtained by the non polynomial cubic spline \( S_i(x) \) passing through the points \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\). The polynomial \( S_i(x) \) satisfies interpolatory conditions at \( x_i \) and \( x_{i+1} \), also the continuity of first derivative at the common nodes \((x_i, y_i)\). For each \( i^{th} \) segment, the cubic non-polynomial spline function \( S_i(x) \) has the form

\[
S_i(x) = a_i + b_i(x-x_i) + c_i \sin \tau(x-x_i) + d_i \cos \tau(x-x_i), \quad i = 0, 1, 2, \ldots, n-1.
\]

where \( a_i, b_i, c_i \) and \( d_i \) are constants and \( \tau \) is a free parameter.

A non-polynomial function \( S(x) \) of class \( C^2[0,1] \) interpolates \( y(x) \) at the grid points \( x_i \), \( i = 0, 1, 2, \ldots, N \) depends on a parameter \( \tau \) and reduces to ordinary cubic spline \( S(x) \) in \([a, b]\) as \( \tau \to 0 \). To derive an expression for the coefficients of Eq. (3) in term of \( y_i, y_{i+1}, M_i \) and \( M_{i+1} \), we first define

\[
S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1},
\]

\[
S_i'(x_i) = M_i, S_i'(x_{i+1}) = M_{i+1}.
\]

Then by algebraic manipulation, we get the following expressions for the unknowns:

\[
a_i = y_i + \frac{M_i}{\tau^2},
\]

\[
b_i = \frac{y_{i+1} - y_i}{h_i} + \frac{M_{i+1} - M_i}{\tau \theta},
\]

\[
c_i = \frac{M_i \cos \theta - M_{i+1}}{\tau^2 \sin \theta},
\]

\[
d_i = -\frac{M_i}{\tau^2}.
\]

where \( \theta = \tau h_{i+1} \), for \( i = 0, 1, 2, \ldots, n-1 \).

Using the continuity of the first derivative at \((x_i, y_i)\), i.e. \( S_i'(x_i) = S_i'(x_{i+1}) \), we obtain the following relation for \( i = 1, 2, \ldots, n-1 \).

\[
\sigma y_{i-1} - (1+\sigma) y_i + y_{i+1} = h_i h_{i+1} \left[ a_i M_{i+1} + \beta (1+\sigma) M_i + \alpha_2 \sigma M_{i+1} \right]
\]

where

\[
\alpha_1 = -\frac{1}{\tau^2 h_i^2} + \frac{1}{\tau h_i \sin(\tau h_i)}, \quad \beta = \frac{1}{\tau h_i \theta} + \frac{\sin \tau(h_i + h_{i+1})}{\tau (h_i + h_{i+1}) \sin(\tau h_i) \sin \theta}, \quad \alpha_2 = -\frac{1}{\theta^2} + \frac{1}{\theta \sin \theta}.
\]
\[ M_j = y'(x_j), \quad j = i, i \pm 1 \text{ and } \theta = \tau h_{i+1}. \]

Note that, the nonpolynomial spline relation (4) is consistent with the usual polynomial cubic spline if \( \tau \to 0 \) that is \( \alpha_1 = \alpha_2 = \frac{1}{6}, \beta = \frac{1}{3}. \)

3. Description of the Numerical method

At the grid points \( x_i \), the differential equation (1) may be discretized by

\[ e y''(x) = p(x) y' + q(x) \]

by using Spline’s second derivatives, we have

\[ e M_j = p(x_j) y_j + q(x_j) \quad \text{for } j = i, i \pm 1 \tag{5} \]

Substituting the above equations in equation (4), we get the following tridiagonal finite difference scheme:

\[
\begin{align*}
(-e\sigma + \alpha_1 h_{i+1} p_{i+1}) y_{i+1} + (e(1 + \sigma) + \beta(1 + \sigma) h_{i+1} p_i) y_i + (-e + \sigma \alpha_2 h_{i+1} p_{i+1}) y_{i+1} \\
&= -h_i h_{i+1} (\alpha_1 q_{i+1} + \beta(1 + \sigma) q_i + \sigma \alpha_2 q_{i+1}) 
\end{align*}
\]

We solve the tridiagonal system (6) for \( i = 1, 2, \ldots, N-1 \) to get the approximations \( y_1, y_2, \ldots, y_{N-1} \) of the solution \( y(x) \) at \( x_1, x_2, \ldots, x_{N-1} \) by using the discrete invariant imbedding algorithm.

Remark 1: For \( e = 1 \) (regular problem), \( \sigma = 1 \) (uniform mesh), we get

\[
\begin{pmatrix}
-1 + \frac{h_i^2}{12} p_{i+1} & \frac{2 + 10^2 h_i}{12} p_i & \frac{-1 + \frac{h_i^2}{12} p_{i+1}}{12} y_{i+1} & \frac{h_i^2}{12} (q_{i+1} + 10 q_i + q_{i+1})
\end{pmatrix}
\]

This is well known fourth order Numerov method for the regular problem \( y''(x) = p(x) y'(x) + q(x) \).

4. Truncation error

The local truncation error \( T_i(h) \) associated with our scheme (4) is

\[
T_i(h) = \begin{pmatrix}
-\frac{\sigma}{2} (1 + \sigma) + \sigma (\alpha_1 + (1 + \sigma) \beta + \sigma \alpha_2) y_i^{(3)} h_i^2 + \left[ -\frac{\sigma}{6} (1 + \sigma) - \sigma (\alpha_1 - \sigma^2 \alpha_2) \right] y_i^{(5)} h_i^3 \\
&+ \left[ -\frac{\sigma}{24} (1 + \sigma) + \frac{\sigma}{2} (\alpha_1 + \sigma^2 \alpha_2) \right] y_i^{(4)} h_i^4 + \left[ -\frac{\sigma}{120} (1 + \sigma) - \frac{\sigma}{6} (\alpha_1 - \sigma^2 \alpha_2) \right] y_i^{(5)} h_i^5 + o(h^6)
\end{pmatrix}
\]

If we choose \( \alpha_1 = \frac{1 + \sigma - \sigma^2}{12} \), \( \alpha_2 = \frac{\sigma^2 + \sigma - 1}{12 \sigma^2} \) and \( \beta = \frac{\sigma^3 + 4 \sigma^2 + 4 \sigma + 1}{12 \sigma (1 + \sigma)} \) the local truncation error of the scheme reduces to \( T_i(h) = O(h^5) \).

5. Convergence analysis

Now we consider the convergence analysis of the non-polynomial spline method described in Section 3 & 4 for the problem (1). Incorporating the boundary conditions we obtain the system of equations in the matrix form as
where
\[
\left[ \begin{array}{ccccccc}
(1 + \sigma)e & -\varepsilon & 0 & 0 & \ldots & 0 \\
-\varepsilon & (1 + \sigma)e & -\varepsilon & 0 & \ldots & 0 \\
0 & -\varepsilon & (1 + \sigma)e & -\varepsilon & \ldots & 0 \\
-\varepsilon & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right]
\]

\[
D = \left[ -\sigma e, (1 + \sigma)e, -\varepsilon \right]
\]

and
\[
P = \left[ z_i, v_i, w_i \right] = \left[ \begin{array}{ccccccc}
v_1 & w_1 & 0 & 0 & \ldots & 0 \\
z_2 & v_2 & w_2 & 0 & \ldots & 0 \\
z_3 & v_3 & w_3 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right]
\]

where
\[
z_i = \alpha_i h_i h_{i+1} p_{i+1}, \quad v_i = \beta_i h_i h_{i+1} (1 + \sigma) p_{i+1}, \quad w_i = \sigma \alpha_i h_i h_{i+1} p_{i+1}
\]

\[Q = \left[ q_1 + \left( -\varepsilon \sigma + z_i \right)^T 0, q_2, q_3, \ldots, q_{N-2}, q_{N-1} + \left( -\varepsilon + w_{N-1} \right)^T \right]
\]

\[T(h) = 0(h^3) \quad \text{for} \quad \alpha_1 = \frac{1 + \sigma - \sigma^2}{12}, \quad \alpha_2 = \frac{\sigma^2 + \sigma - 1}{12\sigma^2}, \quad \beta = \frac{\sigma^3 + 4\sigma^2 + 4\sigma + 1}{12\sigma(1 + \sigma)}
\]

and \(Y = [Y_1, Y_2, \ldots, Y_{N-1}]^T\), \(T(h) = [T_1, T_2, \ldots, T_{N-1}]^T\), \(O = [0, 0, \ldots, 0]^T\)

are associated vectors of equation (7).

Let \(y = [y_1, y_2, \ldots, y_{N-1}]^T \cong Y\) which satisfies the equation
\[
(D + P)y + Q = 0 \tag{8}
\]

Let \(e_i = y_i - Y_i, i = 1(1)N - 1\) be the discretization error so that \(E = [e_1, e_2, \ldots, e_{N-1}]^T = y - Y\).

Subtracting (7) from (8), we get the error equation
\[
(D + P)E = T(h) \tag{9}
\]

Let \(|p(x)| \leq C_1\), where \(C_1\) a positive constant. If \(P_{i,j}\) be the \((i, j)^{th}\) element of \(P\), then
\[
|P_{i,j}| = |w_i| \leq \varepsilon + \alpha_i \sigma^2 h_i^2 C_1, \quad i = 1, 2, 3, 4, \ldots, N - 2
\]
\[
|P_{i,j}| = |z_i| \leq \varepsilon \sigma + \sigma \alpha_i h_i^2 C_1, \quad i = 2, 3, \ldots, N - 1
\]

Thus for sufficiently small \(h_i\),

we have \(|P_{i,j}| < \varepsilon, \quad i = 1, 2, \ldots, N - 2\) \(\tag{10}\)

and \(|P_{i,j}| < \varepsilon, \quad i = 2, 3, \ldots, N - 1\) \(\tag{11}\)

Hence \((D + P)\) is irreducible (see Ref. [17]).

Let \(\overline{s}_i\) be the sum of the elements of the ith row of the matrix \((D + P)\), then we have
\[ S_1 = \sum_{j=1}^{N-1} m_{ij} = e\sigma + \alpha h_i^2 (\beta (1 + \sigma) p_{i1} + \alpha_2 p_2) \]
\[ S_i = \sum_{j=1}^{N-1} m_{ij} = h_i h_{i+1} (\alpha_1 p_{i+1} + \beta (1 + \sigma) p_i + \alpha_3 q_{i+1}) = h_i^2 B_i, \quad i = 2 \quad N - 2 \]

where \( B_i = \sigma (\alpha_1 p_{i+1} + \beta (1 + \sigma) p_i + \alpha_3 q_{i+1}) \)
\[ S_{N-1} = \sum_{j=1}^{N-1} m_{N-1,j} = e + \alpha h_{N-1}^2 (\alpha_1 p_{N-2} + \beta (1 + \sigma) p_{N,1}) \]

We find that for sufficiently small \( h \)
\[ S_i > h_i^2 [\sigma (1 + \sigma) p_{i1} + \alpha_2 \sigma^2 p_2] \quad \text{for } i = 1 \]
\[ S_i > h_i^2 [\alpha_1 q_{N,1} + \beta (1 + \sigma) q_{N,1}] \quad \text{for } i = N - 1 \]
\[ S_i > h_i^2 B_i \quad \text{for } i = 2, 3, 4, \ldots, n - 2 \]

Let \( C_i = \min_{1 \leq i \leq N} |p(x)| \) and \( C_i^* = \max_{1 \leq i \leq N} |p(x)| \), Since \( 0 < \varepsilon << 1 \) and \( \varepsilon \propto O(h_i) \), it is verify that for sufficiently small \( h_i \), \( (D + P) \) is monotone \([16, 17]\). Hence \((D + P)^{-1}\) exists and \((D + P)^{-1} \geq 0\).

Thus from Eq. (9), we get
\[ \|E\| \leq \left\| (D + P)^{-1} \right\| \|f\| \]
(12)

Let \((D + P)^{-1}_{i,k}\) be the \((i,k)^{th}\) element of \((D + P)^{-1}\) and we define
\[ \left\| (D + P)^{-1} \right\| = \max_{1 \leq i \leq N-1} \sum_{k=1}^{N-1} (D + P)^{-1}_{i,k} \] and \[ \|f(h)\| = \max_{1 \leq i \leq N-1} \|T(h_i)\| \]
(13)

Since \((D + P)^{-1}_{i,k} \geq 0\) and \( \sum_{k=1}^{N-1} (D + P)^{-1}_{i,k} S_i = 1 \quad \text{for } i = 1, 2, 3, \ldots, N - 1 \).

Hence \((D + P)^{-1}_{i,k} \leq \frac{1}{S_i} \leq \frac{1}{h_i^2 [\sigma (1 + \sigma) p_{i1} + \alpha_2 \sigma^2 p_2]} \]
(14)
\[(D + P)^{-1}_{i,k} \leq \frac{1}{S_{N-1}^2} \leq \frac{1}{h_i^2 [\alpha_1 q_{N,1} + \beta (1 + \sigma) q_{N,1}]} \]
(15)

Furthermore, \( \sum_{k=1}^{N-2} (D + P)^{-1}_{i,k} \leq \frac{1}{\min_{2 \leq i \leq N-2} S_i} \leq \frac{1}{h_i^2 B_i} \quad i = 2, 3, 4, \ldots, N - 2 \)
(16)

By the help of Eqs. (13)-(16), from (12), we obtain
\[ \|E\| \leq O(h_i^4) \]
(17)

Hence the method \( (6) \) is third order convergent for
\[ \alpha_1 = \frac{1 + \sigma - \sigma^2}{12}, \quad \alpha_2 = \frac{\sigma^2 + \sigma - 1}{12\sigma^2}, \quad \beta = \frac{\sigma^2 + 4\sigma^2 + 4\sigma + 1}{12\sigma (1 + \sigma)} \]

6. Numerical illustration

In order to test the viability of the proposed method, we compared the results with the method of Kadalbajoo and Bava \([4], [3]\) (for \( \lambda = 1 \)) and A. Khan, I. Khan, T. Aziz & Stojanovic \([6]\). The maximum absolute errors at the nodal point are tabulated for different values of \( n, \sigma, \varepsilon \). In order to get a good approximation, a finite mesh is required at near the ends of the interval. For this we consider the half interval \([0, 0.5]\), choose \( \sigma_i = \sigma, \forall i \) and take \( h_i = 0.5(\sigma - 1)/(\sigma^{1/2} - 1) \), \( h_{i+1} = \sigma h_i \) for \( i = 1 \ldots n/2 - 1 \) with \( \sigma > 1 \) where \( n \) (even) is the total number of mesh points. This ensure more
mesh points near the left end and we take the mirror image of \([0, 0.5]\) in the other half interval \([0.5, 1]\).

The value of \(y_n = y(x = \frac{1}{2})\) is obtained by the solution of reduced problem.

**Example 1.** Consider the non-homogeneous singular perturbation problem

\[
x^2 y''(x) - y(x) = \cos^2 \pi x + 2e^{\pi^2} \cos 2\pi x; \quad x \in [0, 1]\text{ with } y(0) = 0 \text{ and } y(1) = 0.
\]

The exact solution is given by

\[
y(x) = \left( e^{-(x-1)/\sqrt{\varepsilon}} + e^{-(x+1)/\sqrt{\varepsilon}} \right) \left( 1 + e^{-1/\sqrt{\varepsilon}} \right)^{-1} \cos^2 \pi x.
\]

The maximum absolute errors are presented in Table 1 and Table 2.

**Example 2.** Consider the singular perturbation problem

\[
x^2 y'' = y \quad \text{with } y(-1) = 1, \quad y(1) = 2
\]

The exact solution is given by

\[
y(x) = \frac{e^{x/\sqrt{\varepsilon}} - 2e^{(x+1)/\sqrt{\varepsilon}} + 2e^{-x/\sqrt{\varepsilon}} - e^{(2-x)/\sqrt{\varepsilon}}}{(e^{-1/\sqrt{\varepsilon}} - e^{3/\sqrt{\varepsilon}})}
\]

The maximum absolute errors are presented in Table 3.

**7. Conclusion**

We possess presented a variable mesh non-polynomial spline method for singular perturbation problems exhibiting twin layers. We have implemented the present method on standard test problems because they have been widely discussed in literature and exact solutions are available for comparison. We have presented maximum absolute errors and compared the results with other methods to support the method. The convergence analysis of the proposed method has been discussed. It is observed from the results that the present method approximate the accurate solution really advantageously for smaller value of \(\varepsilon\) also.

**TABLE 1. Maximum absolute errors for Example-1**

| \(\sigma\) | \(\varepsilon = 10^{-5}\) | \(\varepsilon = 10^{-8}\) | \(\varepsilon = 10^{-10}\) | \(\varepsilon = 10^{-12}\) |
|---|---|---|---|---|
| \(n = 100\) | \(n = 200\) | \(n = 250\) | \(n = 300\) |
| Present method: | | | | |
| 1.00 | 2.6200e-002 | 1.0040e-001 | 1.0100e-001 | 1.0100e-001 |
| 1.05 | 9.9431e-003 | 6.4000e-003 | 5.8200e-002 | 8.8700e-002 |
| 1.10 | 1.2162e-004 | 4.7803e-006 | 4.6446e-006 | 4.5196e-006 |
| 1.15 | 1.2672e-004 | 1.5115e-006 | 1.5002e-005 | 1.5052e-005 |
| M.K. Kadalbajoo & Bava [4] | | | | |
| 1.00 | 1.1971e-001 | 2.6683e-001 | 2.6793e-001 | 2.6794e-001 |
| 1.05 | 9.4616e-003 | 5.9746e-002 | 1.9204e-001 | 2.5196e-001 |
| 1.10 | 9.2068e-004 | 4.1627e-004 | 4.0608e-004 | 3.9659e-004 |
| 1.15 | 7.3534e-004 | 6.4327e-004 | 6.4209e-004 | 6.4059e-004 |
| A.Khan et.al. [6] | | | | |
| 1.05 | 1.27e-003 | 8.64e-003 | 6.03e-002 | 4.01e-004 |
| 1.10 | 6.72e-004 | 4.08e-004 | 4.01e-004 | |
| \(\alpha, \beta\) | Uniform mesh [5] | | | |
| 1/18,4/9 | 1.4446e-003 | 6.2234e-002 | 6.2780e-002 |
| 1/14,3/7 | 1.5282e-002 | 8.3364e-002 | 8.3911e-002 |
| 1/24,11/24 | 1.0061e-002 | 4.5070e-002 | 4.5541e-002 |
| 1/30,14/30 | 1.6707e-002 | 3.5299e-002 | 3.5752e-002 |
### TABLE 2. Maximum absolute errors for Example 1

| n  | σ     | Present method | M.K. Kadalbajoo & Bava [3] (for λ = 1) |
|----|-------|----------------|---------------------------------------|
|    |       |                | for ε = 10^{-3}                        |
| 16 | 1.00  | 8.0000e-003    | 8.0341e-003                           |
|    | 1.02  | 6.2000e-003    | 5.8312e-003                           |
|    | 1.05  | 3.9000e-003    | 3.2642e-003                           |
| 20 | 1.00  | 4.0000e-003    | 3.9714e-003                           |
|    | 1.02  | 2.7000e-003    | 5.8326e-003                           |
|    | 1.05  | 1.3000e-003    | 9.3692e-003                           |
| 24 | 1.00  | 2.1000e-003    | 2.1064e-003                           |
|    | 1.02  | 1.3000e-003    | 1.1663e-003                           |
|    | 1.05  | 1.4000e-003    | 6.6318e-004                           |
| 28 | 1.00  | 1.2000e-005    | 1.1882e-003                           |
|    | 1.02  | 6.1859e-004    | 2.1674e-003                           |
|    | 1.05  | 1.6000e-003    | 4.4452e-003                           |

| n  | σ     | Present method | M.K. Kadalbajoo & Bava [3] (for λ = 1) |
|----|-------|----------------|---------------------------------------|
|    |       |                | for ε = 10^{-4}                        |
| 32 | 1.00  | 2.5600e-002    | 1.6951e-002                           |
|    | 1.02  | 1.7800e-002    | 1.7381e-003                           |
|    | 1.05  | 8.8000e-003    | 7.5573e-003                           |
| 36 | 1.00  | 1.9800e-002    | 2.9531e-001                           |
|    | 1.02  | 1.2500e-002    | 1.4518e-003                           |
|    | 1.05  | 5.0000e-003    | 4.5667e-002                           |
| 40 | 1.00  | 1.5300e-002    | 8.2226e-003                           |
|    | 1.02  | 8.7000e-003    | 1.2263e-003                           |
|    | 1.05  | 2.8000e-003    | 2.2994e-003                           |

### TABLE 3. Maximum absolute errors for Example 2

| n  | σ     | Present method | M.K. Kadalbajoo & Bava [3] (for λ = 1) |
|----|-------|----------------|---------------------------------------|
|    |       |                | for ε = 10^{-3}                        |
| 16 | 1.00  | 3.9300e-002    | 7.8638e-002                           |
|    | 1.02  | 3.4400e-002    | 6.6361e-002                           |
|    | 1.05  | 2.7200e-002    | 4.9231e-002                           |
| 20 | 1.00  | 2.6200e-002    | 5.2392e-002                           |
|    | 1.02  | 2.1100e-002    | 6.5389e-002                           |
|    | 1.05  | 1.4400e-002    | 8.5546e-002                           |
| 24 | 1.00  | 1.7400e-002    | 3.4881e-002                           |
|    | 1.02  | 1.2800e-002    | 2.4274e-002                           |
|    | 1.05  | 7.2000e-003    | 1.2376e-002                           |
| 28 | 1.00  | 1.1700e-002    | 2.3841e-002                           |
|    | 1.02  | 7.8000e-003    | 3.4747e-002                           |
|    | 1.05  | 3.5000e-003    | 5.4983e-002                           |

| n  | σ     | Present method | M.K. Kadalbajoo & Bava [3] (for λ = 1) |
|----|-------|----------------|---------------------------------------|
|    |       |                | for ε = 10^{-4}                        |
| 32 | 1.00  | 6.7500e-002    | 1.3491e-001                           |
|    | 1.02  | 5.7500e-002    | 1.1267e-001                           |
|    | 1.05  | 4.1400e-002    | 7.6951e-002                           |
| 36 | 1.00  | 6.0800e-002    | 1.3491e-001                           |
|    | 1.02  | 4.9100e-002    | 1.4462e-001                           |
|    | 1.05  | 3.1200e-002    | 7.6951e-002                           |
| 40 | 1.00  | 5.4300e-002    | 1.0857e-001                           |
|    | 1.02  | 4.1100e-002    | 7.9811e-002                           |
|    | 1.05  | 2.2500e-002    | 4.0294e-002                           |
| 44 | 1.00  | 4.8200e-002    | 9.6453e-002                           |
|    | 1.02  | 3.4000e-002    | 1.2598e-001                           |
|    | 1.05  | 1.5700e-002    | 1.6162e-001                           |
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