Research Article

Modified Robust Ridge M-Estimators in Two-Parameter Ridge Regression Model

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The methods of two-parameter ridge and ordinary ridge regression are very sensitive to the presence of the joint problem of multicollinearity and outliers in the y-direction. To overcome this problem, modified robust ridge M-estimators are proposed. The new estimators are then compared with the existing ones by means of extensive Monte Carlo simulations. According to mean squared error (MSE) criterion, the new estimators outperform the least square estimator, ridge regression estimator, and two-parameter ridge estimator in many considered scenarios. Two numerical examples are also presented to illustrate the simulation results.

1. Introduction

The matrix form of the multiple linear regression model is

\[ Y = X\beta + \epsilon, \]

where \( Y \) is the vector of the response variable, \( X \) is the matrix of predictor variables, \( \beta \) is the vector of unknown regression coefficients, and \( \epsilon \) is the vector of disturbance term, such that \( \epsilon \sim N(0, \sigma^2) \). The ordinary least square (OLS) estimates of \( \beta \) is defined as:

\[ \hat{\beta}_{OLS} = (X'X)^{-1}X'Y. \]

The estimator \( \hat{\beta} \) is unbiased and has minimum variance among all the linear unbiased estimators. However, the performance of this estimator is poor in the presence of multicollinearity, such that it is statistically insignificant with large variance [1]. To cope with this issue, several alternatives have been developed. The first method is proposed by Ref. [2] and is defined as

\[ \hat{\beta}(k) = (X'X + kI)^{-1}X'Y = T_k\hat{\beta}, \]

where \( I \) is the identity matrix, \( k \geq 0 \) and \( T_k = (X'X + kI)^{-1}X' \). To handle the problem of outliers, Ref.[3] derived a new estimator known as M-estimator (ME). M-estimator is defined as the solution of the equations

\[ \sum \psi(e_i/s) = 0 \quad \text{and} \quad \sum \psi(e_i/s)z_i = 0 \]

with \( e_i = y_i - z_i\hat{\beta}_M \) being scale estimator for errors and \( \psi(\cdot) \) being a suitably chosen function.

Ref. [4] illustrated that ridge regression (RR) is sensitive to outliers in the y-direction, hence developed a new robust ridge M-estimator (MRE) defined as
\[
\tilde{\beta}_M(k) = T_k \tilde{\beta}_M,
\]

where \( \tilde{\beta}_M \) is M-estimator.

According to Ref. [5], the quality of fit for RR is not good as compared to OLS. To overcome this deficiency, they developed a two-parameter ridge estimator (TPR) that always performs better than the ordinary RR. Also, TPR has good orthogonal properties between the residuals and predicted values of dependent variables. They defined TPR as

\[
\tilde{\beta}_q(k) = q^T \tilde{\beta},
\]

where

\[
q = \frac{y'X(X'X + kl)^{-1}X'y}{y'X(X'X + kl)^{-1}X'X(X'X + kl)^{-1}X'y}.
\]

Later on, many researchers worked on TPR, see e.g., [6–13]. The selection of ridge M-estimators plays an important role to reduce the MSE of TPR in the presence of multicollinearity and outliers. Different ridge M-estimators have been proposed by various researchers. Some of them are Refs. [4, 8, 14–17]; and recently Ref. [18]. In case of near singularity and large number of outliers, the existing estimators do not perform well in terms of MSE. Therefore, the aim of this article was to continue the series of work on the selection of ridge M-estimator in TPR. Motivated by the work of Ref. [8] and following the idea of Ref. [1], we proposed the modified ridge M-estimators in TPR. The developed M-estimators provide the minimum MSE than OLS, RR, and existing TPR estimators for different levels of correlation, sample size, error variance, and outliers.

The organization of this article is as follows: Section 2 gave the review of estimators included in this study, new developed estimators for the selection of k and their comparison criterion. Section 3 included the simulation design that we have adopted in this article together with the discussion of simulation results and numerical examples. Concluding remarks are given in section 4.

2. Methodology

The canonical form of the model given in equation (1) can be written as

\[
Y = Z\alpha + \epsilon,
\]

where \( Z = XT, \alpha = T'\beta, \) and \( T'\) is \( p \times p \) where \( T \) is the orthogonal matrix with the columns constituting the eigenvectors of \( X'X \) and \( \lambda_j \), \( j = 1, 2, \ldots, p \) are the ordered eigenvalues of \( X'X \). The estimators in canonical form are

\[
\tilde{\beta} = \Lambda^{-1}Z'y,
\]

\[
\tilde{\alpha} = \Lambda^{-1}Z'y,
\]

\[
\alpha(k) = (\Lambda + kl)^{-1}Z'y
\]

\[
\tilde{\alpha}_M(k) = \tilde{T}_k \tilde{\alpha}_M,
\]

\[
\tilde{\alpha}_q(k) = q^T \tilde{T}_k \tilde{\alpha},
\]

\[
\tilde{\alpha}_qm(k) = q^T \tilde{T}_k \tilde{\alpha}_M,
\]

where \( q^* = y'Z(\Lambda + kl)^{-1}Z'y/\max(\Lambda^2/\tilde{\alpha}_M, \alpha) (\Lambda + kl)^{-1}Z'y, \) \( \tilde{T}_k = (\Lambda + kl)^{-1}\Lambda \) and \( k > 0 \).

2.1. Existing Estimators

(i) RE \( \longrightarrow \tilde{\alpha}_{HK} = \tilde{\alpha}^{s}/\tilde{\alpha}_{max}^{2} \) [2].

(ii) RME RME \( \longrightarrow \tilde{\alpha}_M = p\tilde{\alpha}^2/\tilde{\alpha}_M\tilde{\alpha}_M \) [4], where \( \tilde{\alpha}_M^2 = s^2(n-p)^{-1} \sum_{i=1}^{p} (\psi(e_i/s))^2/ \sum_{i=1}^{p} (1/n) \psi'(e_i/s)^2 \) [3].

(iii) TRME1 \( \longrightarrow \tilde{\alpha} = \max(\tilde{\alpha}_M^2/\tilde{\alpha}_Q, \tilde{\alpha}) = \sum_{i=1}^{p} (\tilde{\alpha}_M^2/\tilde{\alpha}_Q, \tilde{\alpha}) / \sum_{i=1}^{p} (\Lambda^2/\tilde{\alpha}_Q, \tilde{\alpha}) / (\Lambda + kl) \) \[8\].

(iv) TRME2 \( \longrightarrow \) iterative method defined in the following Algorithm 1: [8].

In general, ridge M-estimators available in the literature may not fully address the simultaneous occurrence of high multicollinearity and outliers in data. To resolve this issue, we propose some new ridge M-estimators in TPR that perform generally better than other existing estimators in most of the considered situations.

2.2. Performance Criterion. To examine the performance of our developed estimators with the existing estimators, we used the MSE criterion defined as

\[
\text{MSE}(\tilde{\beta}) = E[(\tilde{\beta} - \beta)'(\tilde{\beta} - \beta)] = \text{tr}(\text{var}(\tilde{\beta})) + [\text{bias}(\tilde{\beta})]'[\text{bias}(\tilde{\beta})],
\]

where

\[
\text{var}(\tilde{\beta}) = E[(\tilde{\beta} - E(\tilde{\beta}))(\tilde{\beta} - E(\tilde{\beta}))'],
\]

\[
\text{bias}(\tilde{\beta}) = E(\tilde{\beta}) - \beta.
\]

The MSE of the above defined estimators is

\[
\text{MSE}(\tilde{\alpha}) = \sigma^2 \sum_{j=1}^{p} \frac{1}{\lambda_j}.
\]

\[
\text{MSE}(\tilde{\alpha}_M) = \sum_{j=1}^{p} \Omega_{jj},
\]

\[
\text{MSE}(\tilde{\alpha}_q(k)) = \sigma^2 \sum_{j=1}^{p} \frac{\lambda_j}{\lambda_j + k} + \sum_{j=1}^{p} \frac{k^2 \alpha_j^2}{(\lambda_j + k)^2}.
\]
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(i) Calculate $\hat{k} > \max(\Lambda^2/\tilde{a}_M^2)$.
(ii) Estimate $\tilde{q} = \sum_{j=1}^{p} \tilde{a}_{Mj}^2/\lambda_j/\lambda_j + k/\sum_{j=1}^{p} \tilde{a}_{Mj}^2/\lambda_j + \tilde{a}_M^2/\lambda_j + k^2$ using $\tilde{k}$ in (i).
(iii) Obtain $k = (1/p) \sum_{j=1}^{p} (\tilde{q} \lambda_j + (q-1)\lambda_j^2/\tilde{a}_M^2/\lambda_j + \tilde{a}_M^2/\lambda_j + k)$ using $\tilde{q}$ in (ii).
(iv) $M1 \rightarrow M1 = \hat{k}_M = p/\sum_{j=1}^{p} \tilde{a}_{Mj}^2/(\sqrt{\lambda_j \tilde{a}_M^2/\lambda_j + \tilde{a}_M^2/\lambda_j + \hat{k}^2}$ [14].

Algorithm 1: Iterative algorithm for modified two-parameter ridge estimators.

\[
\text{MSE}(\hat{\alpha}_M(k)) = \sum_{j=1}^{p} \lambda_j^2 \Omega_{jj} + \sum_{j=1}^{p} k^2 \alpha_j^2,
\]

\[
\text{MSE}(\alpha_q(k)) = q^2 \sigma^2 \sum_{j=1}^{p} \lambda_j^2 \Omega_{jj} + \sum_{j=1}^{p} \alpha_j^2^2,
\]

\[
\text{MSE}(\alpha_Q(k)) = q^2 \sum_{j=1}^{p} \lambda_j^2 \Omega_{jj} + \sum_{j=1}^{p} \alpha_j^2^2,
\]

where $\sigma^2$ is error variance and $\Omega = \text{cov}(\hat{\alpha}_M)$ where $\Omega_{jj}$ are the diagonal elements of $\Omega$. Ref. [8] proved that if $\Omega_{jj} < \sigma^2 \lambda_j^{-1}$ for $j = 1, 2, \ldots, p$, then MSE($\alpha_q(k)$) < MSE($\alpha_q(k)$) for $k > 0$ and MSE($\alpha_q(k)$) < MSE($\alpha_q(k)$).

New Estimators. According to Ref. [8], the TPR is also sensitive to outliers in the $y$-direction as RR is. Thus, here we suggest modified ridge $M$-estimators (MTPM) in TPR. In a similar manner to TPR, the primary focus in MTPM is to find the suitable value of biasing parameter, which minimizes the MSE. By adopting the idea of Ref. [1], we multiply a quantity $V_{Mj} = \lambda_j/\tilde{a}_M$ with $\tilde{k}$ as suggested by Ref. [8]. Hence, the modified biasing parameter is

\[
\hat{k}_{Mj} = V_{Mj} \tilde{k},
\]

where $\tilde{k} = \tilde{q} \Lambda^2 \lambda_j + (\tilde{q}-1) \lambda_j^2 \tilde{a}_M^2/\lambda_j + \tilde{a}_M^2/\lambda_j + \lambda_j^2$ and $\tilde{q}$ defined in TRME1.

As $\hat{k}_j$ is based on correlation, an increase in the degree of correlation causes an increase in the value of $V_{Mj}$. This increase in $V_{Mj}$ will lead to the larger value of $\hat{k}_{Mj}$. Since many existing estimators did not provide a large enough value of $\hat{k}_{Mj}$, this increase is required to obtain the suitable value of $\hat{k}_{Mj}$ to solve the problem of near singularity. The term $\tilde{a}_M$ is used to deal with the outliers. Here, we have used Huber’s $M$-estimator.

We proposed three new methods by taking arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM) of $\hat{k}_{Mj}$, denoted by MTPM1, MTPM2, and MTPM3, respectively, and defined as

\[
k_{AM}^* = \frac{\sum_{j=1}^{p} (\hat{k}_{Mj})}{p},
\]

\[
k_{GM}^* = \left( \prod_{j=1}^{p} \hat{k}_{Mj} \right)^{1/p},
\]

\[
k_{HM}^* = \frac{p}{\sum_{j=1}^{p} 1/\hat{k}_{Mj}}.
\]

Hence, the new modified two parameter ridge $M$-estimator is defined in the canonical form as

\[
\tilde{a}_{QM}(k) = q^2 T_k^\ast \tilde{a}_M,
\]

where

\[
\tilde{q} = \frac{\sum_{j=1}^{p} (\tilde{a}_M^2/\lambda_j + \tilde{a}_M^2/\lambda_j + k^2)}{\sum_{j=1}^{p} ((\Lambda^2 \lambda_j + \tilde{a}_M^2/\lambda_j + \tilde{a}_M^2)^2)/\lambda_j + \tilde{a}_M^2)^2},
\]

and $k^* = k_{AM}^*, k_{GM}^*$ and $k_{HM}^*$.

Furthermore, through Algorithm 1, we proposed the modified iterative two-parameter ridge estimators. The new modified iterative TPR is defined as

\[
\hat{k}_{(I)Mj} = V_{Mj} \tilde{k},
\]

where $\tilde{k}$ is from algorithm of TRME2. Now by taking the AM, GM, and HM of $\hat{k}_{(I)Mj}$, three new estimators denoted by MTPM4, MTPM5, and MTPM6 are obtained and defined as

\[
k_{(I)AM}^* = \frac{\sum_{j=1}^{p} (\hat{k}_{(I)Mj})}{p},
\]

\[
k_{(I)GM}^* = \left( \prod_{j=1}^{p} (\hat{k}_{(I)Mj}) \right)^{1/p},
\]

\[
k_{(I)HM}^* = \frac{p}{\sum_{j=1}^{p} 1/\hat{k}_{(I)Mj}}.
\]

The new modified iterative two parameter ridge $M$-estimator is defined in the canonical form as
\[
\tilde{\alpha}_{(j)\text{GM}}(k) = \tilde{q}_j T_j^* \tilde{\alpha}_M,
\]

where
\[
\tilde{q}_j = \frac{\sum_{j=1}^{p} (\tilde{\alpha}_{M,j}^2 \lambda_j (\lambda_j + k_j^*))}{\sum_{j=1}^{p} (\tilde{\alpha}_{M,j}^2 (\lambda_j + k_j^*))^2}, \quad T_j^* = (\Lambda + k_j^* I)^{-1} \Lambda,
\]

and \(k_j^* = k_{(j)\text{AM}}, k_{(j)\text{GM}}\) and \(k_{(j)\text{HM}}\).

### 3. Simulation Study

In this section, a simulation study is taken to check the performance of new and existing estimators.

#### 3.1. Simulation Design

By following the simulation design of Refs. [8, 15], predictors are generated as

\[
x_{i,j} = (1 - \delta^2)^{\frac{1}{2}} z_{i,j} + \delta z_{i,p}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, p,
\]

where \(\delta^2\) shows the correlation between two predictor variables and \(z_{i,j}\) are pseudo random numbers generated using standard normal distribution. The response variable is generated as

\[
y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_p x_{i,p} + u_i, \quad i = 1, 2, \ldots, n,
\]

where \(\beta_0\) is set to be zero and \(u_i \sim N(0, \sigma^2)\). This simulation experiment is carried out by randomly generating different factors that we consider in this study. The details are given below:

- \(p\) (number of predictors) = 4 and 10, \(n\) (sample size) = 20, 50 and 100,
- \(\sigma^2\) (error variance) = 0.1, 1, 5 and 10,
- \(\delta^2\) (levels of correlation) = 0.85, 0.95, 0.99 and 0.999.

To check the robustness of the newly proposed estimators against outliers, different percentages of outliers (10%, 20%, and 30%) in the y-direction are generated using an error term \(u_i \sim N(50, \sigma^2)\), see Refs. [19, 20]. These simulation results based on 5000 replications and estimated MSE is calculated as

\[
\text{MSE}(\tilde{\alpha}_j) = \frac{1}{5000} \sum_{k=1}^{5000} (\tilde{\alpha}_{jk} - \alpha_j) (\tilde{\alpha}_{jk} - \alpha_j).
\]

#### 3.2. Performance of New Proposed Estimators

In view of the results from Tables 1–18, we can get some conclusions:

1. The estimated MSE of all considered estimators increases, as \(\sigma^2\) increases. In general, MTPM1 performs well as compared to other estimators.

2. For all sample sizes, MSE of all estimators decreases with increasing sample size from 20 to 100. For \(n = 20\), MTPM1 performs better, but as \(n\) increases, MTPM4 also performs better than the existing estimators.

3. Estimated MSE of all estimators increases in accordance with the increase in the degree of correlation. Newly developed estimators MTPM1 and MTPM6 perform better in terms of smaller MSEs. There are few cases where M1 and TRME2 have better performance than the rest of the estimators.

4. The estimated MSE of all estimators increases with regard to increase in the number of predictors (\(p\)).

5. As the percentage of outliers increases in the data, the estimated MSE of newly developed estimators decreases. MTPM1 outperforms the other estimators.

6. When there are multicollinearity and outliers in data, MTPM1 performs better than the other considered estimators. There are some cases in which M1 and TRME2 are better alternatives.

7. From these simulation results, we can conclude that in the presence of multicollinearity and outliers, MTPM1 and MTPM4 are best alternatives of the existing estimators.

#### 3.3. Real-Life Applications

**Example 1.** We consider the Tobacco data of Ref. [21] to show the performance of newly modified estimators. The data contain four predictor variables with 30 observations. Condition number is 1892.33 which shows severe multicollinearity. Considering the following linear model:

\[
Y = \beta_0 + \beta_1 X_{1} + \beta_2 X_{2} + \beta_3 X_{3} + \beta_4 X_{4} + e_i, \quad i = 1, 2, \ldots, n.
\]

The eigenvalues are \(\lambda_1 = 3.9739, \lambda_2 = 0.0176, \lambda_3 = 0.0064,\) and \(\lambda_4 = 0.0021\). The calculated value of error variance is 0.223. The correlation among the predictor variables is shown in Table 19. The data contain two outliers in the y-direction. Estimated MSE and regression coefficients for tobacco data are presented in Table 20. From the result, it is noticed that MTPM3 has the smallest MSE among all the considered estimators.

**Example 2.** The second example is of water quality data taken from the Pakistan Council of Research in Water Resources (PCRWR) for the year 2014–2015. We consider four predictors each with 31 observations. Predictor variables are HCO3, SO4, Na, and EC, while response variable is TDS. The estimated error variance is 0.111 and eigenvalues are \(\lambda_1 = 3.3024, \lambda_2 = 0.6599, \lambda_3 = 0.0210,\) and \(\lambda_4 = 0.0166\). Condition number is 157.257, which shows strong multicollinearity. Table 21 shows the correlation among the predictors. The outliers are present in the y-direction. The estimated MSE and regression coefficients are shown in
| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS   | M     | RE    | MRE   | M1    | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|------------------------|-----------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                   | 0.1                         | 29.9274 | 0.0502 | 23.2404 | 0.0261 | 0.0216 | 0.0106 | **0.0083** | 0.0085 | 0.0231 | 0.0089 | 0.0084 | 0.0084 | 0.0086 |
|                        | 1                           | 29.9445 | 0.1417 | 23.2420 | 0.0632 | 0.0495 | 0.0288 | **0.0093** | 0.0131 | 0.0774 | 0.0127 | 0.0094 | 0.0103 | 0.0127 |
|                        | 5                           | 30.4298 | 3.3915 | 23.1925 | 1.7838 | 1.5059 | 1.4681 | **0.6818** | 1.4220 | 2.2634 | 0.7886 | 0.7381 | 0.9542 | 1.3006 |
|                        | 10                          | 32.4434 | 13.1172 | 23.6350 | 7.1911 | 6.0915 | 6.2093 | **3.3007** | 7.0652 | 8.8103 | 3.3957 | 3.6493 | 4.8688 | 6.4071 |
| 0.95                   | 0.1                         | 97.1115 | 0.1181 | 74.8629 | 0.0517 | 0.0401 | 0.0204 | **0.0064** | 0.0099 | 0.0752 | 0.0087 | 0.0064 | 0.0069 | 0.0091 |
|                        | 1                           | 97.2380 | 0.4375 | 74.9289 | 0.1990 | 0.1578 | 0.1251 | **0.0313** | 0.0915 | 0.2865 | 0.0499 | 0.0323 | 0.0428 | 0.0737 |
|                        | 5                           | 98.3760 | 10.7958 | 74.2675 | 5.6184 | 4.7191 | 4.8034 | **1.8124** | 6.1413 | 7.1779 | 2.3344 | 2.0383 | 3.3799 | 5.1130 |
|                        | 10                          | 105.4989 | 42.3045 | 76.3297 | 22.4123 | 18.9711 | 19.9321 | **8.0798** | 27.0598 | 28.2246 | 9.4586 | 9.7093 | 17.0666 | 23.7698 |
| 0.99                   | 0.1                         | 494.5338 | 0.6391 | 378.7278 | 0.4060 | 0.3627 | 0.4209 | **0.1843** | 0.4405 | 0.4367 | 0.2326 | 0.1867 | 0.2453 | 0.3455 |
|                        | 1                           | 494.9122 | 2.3451 | 378.7337 | 1.3656 | 1.1914 | 1.3549 | **0.5190** | 1.5744 | 1.5693 | 0.6876 | 0.5316 | 0.8023 | 1.2228 |
|                        | 5                           | 503.2993 | 57.3636 | 378.2001 | 30.9619 | 26.3464 | 28.6807 | **10.2081** | 38.8394 | 37.9620 | 13.7355 | 11.1733 | 23.5614 | 34.3006 |
|                        | 10                          | 532.7590 | 213.6679 | 381.2322 | 114.2745 | 97.1373 | 105.5232 | **37.3810** | 143.7024 | 141.0203 | 48.2938 | 43.5950 | 101.2758 | 134.1682 |
| 0.999                  | 0.1                         | 4969.1344 | 5.7404 | 3794.8173 | 3.1926 | 2.7530 | 3.1148 | **1.0561** | 3.9311 | 3.7796 | 1.4735 | 1.0616 | 1.6793 | 3.0110 |
|                        | 1                           | 4973.9821 | 22.5688 | 3796.9397 | 12.1422 | 10.3665 | 11.4408 | **3.7689** | 15.1519 | 14.7099 | 5.3805 | 3.8051 | 7.1422 | 12.6832 |
|                        | 5                           | 4995.4508 | 540.7746 | 3730.1907 | 281.9896 | 239.2546 | 258.5691 | **83.8542** | 353.4930 | 350.4758 | 118.9266 | 87.5022 | 245.6556 | 339.7473 |
|                        | 10                          | 5377.6750 | 2152.1231 | 3860.4020 | 1142.1380 | 967.5422 | 1063.4883 | **321.1969** | 1428.4656 | 1422.1817 | 453.0490 | 438.1524 | 1126.7981 | 1402.4136 |
Table 2: Estimated MSE for \( n = 50, p = 4, \) and 10% outlier in the y-direction.

| Correlation (\( \delta^2 \)) | Error variance (\( \sigma^2 \)) | OLS    | M  | RE  | MRE | M1  | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|-----------------------------|---------------------------------|--------|----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                        |                                 | 0.7585 | 0.0307 | 0.2913 | 0.0280 | 0.0274 | 0.0296 | 0.0251 | 0.0269 | 0.0305 | **0.0249** | 0.0255 | 0.0263 | 0.0271 |
|                             |                                 | 0.7609 | 0.0344 | 0.2929 | 0.0264 | 0.0256 | 0.0273 | 0.0236 | 0.0240 | 0.0256 | **0.0214** | 0.0236 | 0.0238 | 0.0240 |
|                             |                                 | 0.8413 | 0.1519 | 0.3525 | 0.0554 | 0.0503 | 0.0404 | 0.0302 | 0.0316 | 0.0399 | **0.0241** | 0.0303 | 0.0308 | 0.0313 |
| 0.85                        | 10                              | 1.1015 | 0.5261 | 0.5286 | 0.1806 | 0.1635 | 0.1351 | 0.0954 | 0.1060 | 0.1487 | **0.0710** | 0.0960 | 0.0989 | 0.1023 |
| 0.95                        | 0.1                             | 1.3792 | 0.0322 | 0.3545 | 0.0264 | 0.0256 | 0.0282 | **0.0215** | 0.0229 | 0.0286 | 0.0228 | 0.0216 | 0.0221 | 0.0229 |
|                             |                                 | 1.3922 | 0.0411 | 0.3660 | 0.0257 | 0.0245 | 0.0233 | 0.0189 | 0.0195 | 0.0233 | **0.0188** | 0.0190 | 0.0192 | 0.0195 |
|                             |                                 | 1.6427 | 0.3583 | 0.5588 | 0.1269 | 0.1141 | 0.0921 | **0.0552** | 0.0711 | 0.1229 | 0.0574 | 0.0557 | 0.0602 | 0.0665 |
|                             | 10                              | 2.4292 | 1.3496 | 1.0185 | 0.4754 | 0.4299 | 0.3539 | **0.2224** | 0.3183 | 0.4887 | 0.2289 | 0.2257 | 0.2545 | 0.2882 |
| 0.99                        | 0.1                             | 4.6979 | 0.0500 | 1.3155 | 0.0382 | 0.0369 | 0.0384 | **0.0330** | 0.0359 | 0.0393 | 0.0348 | 0.0331 | 0.0337 | 0.0351 |
|                             |                                 | 4.7376 | 0.1024 | 1.3298 | 0.0631 | 0.0604 | 0.0610 | **0.0490** | 0.0570 | 0.0628 | 0.0526 | 0.0491 | 0.0507 | 0.0538 |
|                             |                                 | 6.0250 | 1.7027 | 1.9035 | 0.7320 | 0.6728 | 0.6155 | **0.3970** | 0.6860 | 0.6729 | 0.4576 | 0.4005 | 0.4770 | 0.5607 |
|                             | 10                              | 9.7760 | 6.3620 | 3.6970 | 2.5692 | 2.3461 | 2.0915 | **1.3364** | 2.5054 | 2.2630 | 1.5449 | 1.3606 | 1.7775 | 2.0897 |
| 0.999                       | 0.1                             | 39.3635 | 0.2032 | 11.3236 | 0.1081 | 0.1015 | 0.0993 | **0.0738** | 0.1099 | 0.0837 | 0.0833 | 0.0739 | 0.0784 | 0.0858 |
|                             |                                 | 39.7880 | 0.7061 | 11.4896 | 0.3210 | 0.2949 | 0.2736 | **0.1767** | 0.3275 | 0.2496 | 0.2088 | 0.1778 | 0.2029 | 0.2431 |
|                             |                                 | 53.4644 | 16.2923 | 16.6556 | 6.6495 | 5.9977 | 5.3450 | **3.1234** | 6.2694 | 5.1622 | 3.8264 | 3.1480 | 4.5246 | 5.4647 |
|                             | 10                              | 94.6233 | 65.2268 | 35.3048 | 26.3465 | 23.7320 | 21.2039 | **12.3585** | 23.8011 | 20.5982 | 15.0928 | 12.5423 | 19.6824 | 22.9686 |
Table 3: Estimated MSE for $n = 100$, $p = 4$, and 10% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS   | M    | RE   | MRE  | M1   | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|----------------------------|-------|------|------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                     |                            |       |      |      |      |      |       |       |       |       |       |       |       |       |
| 0.1                      | 0.9275                     | 0.0329| 0.4340| 0.0315| 0.0310| 0.0323| **0.0272**| 0.0286| 0.0310| 0.0293| 0.0274| 0.0281| 0.0290|       |
| 1                        | 0.9297                     | 0.0382| 0.4355| 0.0342| 0.0333| 0.0354| **0.0302**| 0.0308| 0.0322| 0.0311| 0.0303| 0.0306| 0.0309|       |
| 5                        | 0.9916                     | 0.1694| 0.4769| 0.1004| 0.0964| 0.0997| 0.0723| 0.0773| 0.0963| **0.0683**| 0.0725| 0.0737| 0.0754|       |
| 10                       | 1.1843                     | 0.5491| 0.6033| 0.2787| 0.2637| 0.2602| 0.1722| 0.1950| 0.2716| **0.1620**| 0.1732| 0.1783| 0.1852|       |
| 0.95                     |                            |       |      |      |      |      |       |       |       |       |       |       |       |       |
| 0.1                      | 3.2560                     | 0.0369| 1.6067| 0.0339| 0.0328| 0.0347| **0.0279**| 0.0293| 0.0333| 0.0305| 0.0280| 0.0284| 0.0292|       |
| 5                        | 3.4721                     | 0.4553| 1.7521| 0.2347| 0.2190| 0.2165| **0.1244**| 0.1589| 0.2488| 0.1404| 0.1249| 0.1312| 0.1427|       |
| 10                       | 4.1057                     | 1.6973| 2.1547| 0.7843| 0.7212| 0.6706| **0.3434**| 0.5347| 0.8508| 0.3996| 0.3468| 0.3822| 0.4411|       |
| 0.99                     |                            |       |      |      |      |      |       |       |       |       |       |       |       |       |
| 0.1                      | 17.5629                    | 0.0449| 8.9316| 0.0257| 0.0237| 0.0208| **0.0164**| 0.0171| 0.0275| 0.0177| 0.0164|       |       | 0.0169|
| 5                        | 18.6528                    | 2.0265| 9.6328| 0.7783| 0.6836| 0.5296| **0.2091**| 0.5426| 0.9257| 0.2877| 0.2103| 0.2485| 0.3383|       |
| 10                       | 21.5402                    | 8.0274| 11.4839| 3.1954| 2.8252| 2.3491| **0.9360**| 2.8802| 3.6792| 1.2658| 0.9457| 1.2364| 1.7895|       |
| 0.999                    |                            |       |      |      |      |      |       |       |       |       |       |       |       |       |
| 0.1                      | 180.6319                   | 0.2170| 93.1727| 0.0806| 0.0695| 0.0493| **0.0217**| 0.0520| 0.1022| 0.0281| 0.0217| 0.0229| 0.0280|       |
| 5                        | 190.1303                   | 20.1457| 99.0496| 8.1435| 7.1653| 6.0414| **2.5724**| 8.9018| 8.9304| 3.4805| 2.5801| 3.5699| 5.6702|       |
| 10                       | 225.9094                   | 83.8263| 122.3470| 34.7554| 30.6328| 26.4175| **10.7838**| 38.4951| 37.9785| 14.7022| 10.8521| 17.2603| 27.6589|       |
| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS | M | RE | MRE | M1 | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|------------------------|-----------------------------|-----|---|----|-----|----|-------|-------|-------|-------|-------|-------|-------|-------|
|                        |                             | 0.1 | 1 | 5  | 10  | 0.85|       |       |       |       |       |       |       |       |
|                        |                             | 11.812 | 0.0703 | 5.181 | 0.0271 | 0.0258 | 0.0183 | **0.0109** | 0.0127 | 0.0293 | 0.0102 | 0.0110 | 0.0116 | 0.0129 |
|                        |                             | 11.8312 | 0.2131 | 5.156 | 0.0684 | 0.0648 | 0.0495 | **0.0203** | 0.0359 | 0.0806 | 0.0247 | 0.0216 | 0.0271 | 0.0368 |
|                        |                             | 12.3381 | 4.4321 | 5.722 | 1.6831 | 1.6024 | 1.3815 | 0.9410 | 1.5166 | 1.8005 | **0.9368** | 1.0517 | 1.2858 | 1.4946 |
|                        |                             | 14.1711 | 11.3115 | 6.85 | 4.6942 | 4.3209 | 3.4207 | 2.6380 | 4.1682 | 5.2327 | **2.4187** | 2.8576 | 3.4128 | 3.9952 |
|                        |                             | 0.1  | 31.422 | 0.1506 | 12.8667 | 0.0488 | 0.0459 | 0.0330 | **0.0095** | 0.0254 | 0.0592 | 0.0127 | 0.0099 | 0.0141 | 0.0247 |
|                        |                             | 1    | 31.5119 | 0.5661 | 12.9202 | 0.1919 | 0.1813 | 0.1525 | **0.0536** | 0.1545 | 0.2046 | 0.0728 | 0.0593 | 0.1021 | 0.1516 |
|                        |                             | 5    | 33.0406 | 12.432 | 14.139 | 4.9587 | 4.6676 | 4.054 | **2.4146** | 4.979 | 4.6419 | 2.5501 | 2.9027 | 4.1804 | 4.6898 |
|                        |                             | 10   | 39.2961 | 32.8099 | 18.3596 | 13.6878 | 12.5347 | 9.8595 | **6.7084** | 14.1981 | 14.2217 | 6.7705 | 7.8309 | 11.1966 | 13.0773 |
|                        |                             | 0.1  | 140.3318 | 0.7587 | 55.4464 | 0.3901 | 0.3693 | 0.3591 | **0.2165** | 0.3966 | 0.2790 | 0.2614 | 0.2223 | 0.3210 | 0.3728 |
|                        |                             | 1    | 140.619 | 2.7976 | 55.5895 | 1.2968 | 1.2261 | 1.1633 | **0.6398** | 1.2700 | 0.9187 | 0.7727 | 0.6750 | 1.0989 | 1.2139 |
|                        |                             | 5    | 149.5943 | 61.198 | 63.6205 | 26.3927 | 24.7399 | 21.9077 | **12.547** | 24.2947 | 19.807 | 13.8506 | 14.862 | 24.7173 | 23.900 |
|                        |                             | 10   | 178.399 | 158.4206 | 81.7183 | 69.2094 | 63.4528 | 51.7558 | **32.143** | 70.2289 | 62.6102 | 34.6014 | 37.8464 | 64.5082 | 67.7464 |
|                        |                             | 0.1  | 1315.525 | 6.6983 | 512.4102 | 3.0299 | 2.831 | 2.6747 | **1.2864** | 2.5462 | 1.8925 | 1.6494 | 1.3083 | 2.6109 | 2.8092 |
|                        |                             | 1    | 1318.783 | 26.1026 | 518.4871 | 11.4704 | 10.7198 | 10.2012 | **4.6638** | 8.9297 | 7.1542 | 6.009 | 4.8315 | 10.6363 | 10.1794 |
|                        |                             | 5    | 1387.456 | 565.0867 | 597.5064 | 243.5939 | 226.9272 | 205.5404 | **96.7687** | 181.1837 | 165.0595 | 119.0271 | 109.1384 | 236.7054 | 197.67 |
|                        |                             | 10   | 1720.931 | 1556.154 | 800.0413 | 678.0977 | 620.283 | 492.9333 | **269.8147** | 619.7776 | 591.5999 | 319.3605 | 307.8377 | 675.9897 | 642.3477 |
Table 5: Estimated MSE for $n = 50$, $p = 4$, and 20% outlier in the y-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS  | M  | RE  | MRE | M1  | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|------|----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                     |                             |      |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 1.0046                      | 0.0316 | 0.3436 | 0.0268 | 0.0262 | 0.0292 | 0.0246 | 0.0260 | 0.0294 | 0.0228 | 0.0249 | 0.0255 | 0.0261 |
| 1                        | 1.007                       | 0.0374 | 0.3446 | 0.0241 | 0.0233 | 0.0236 | 0.0200 | 0.0203 | 0.0217 | 0.0176 | 0.0201 | 0.0202 | 0.0204 |
| 5                        | 1.0861                      | 0.2182 | 0.4073 | 0.0558 | 0.0501 | 0.0352 | 0.0255 | 0.0269 | 0.0357 | 0.0202 | 0.0256 | 0.0261 | 0.0268 |
| 10                       | 1.3473                      | 0.7871 | 0.5802 | 0.2088 | 0.1902 | 0.1516 | 0.1173 | 0.1276 | 0.1704 | 0.0843 | 0.1178 | 0.1214 | 0.1253 |
| 0.95                     |                             |      |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 1.5121                      | 0.0325 | 0.4332 | 0.0236 | 0.023 | 0.0235 | 0.0199 | 0.0249 | 0.0196 | 0.0190 | 0.0194 | 0.0199 |
| 1                        | 1.5258                      | 0.0441 | 0.4379 | 0.0213 | 0.0202 | 0.0173 | 0.0136 | 0.0139 | 0.0168 | 0.0135 | 0.0136 | 0.0137 | 0.0139 |
| 5                        | 1.7731                      | 0.4573 | 0.5311 | 0.1284 | 0.1154 | 0.0924 | 0.0616 | 0.0768 | 0.1191 | 0.0629 | 0.0621 | 0.0673 | 0.0743 |
| 10                       | 2.564                       | 1.7584 | 0.8909 | 0.5447 | 0.4991 | 0.431 | 0.3200 | 0.4149 | 0.5251 | 0.3082 | 0.3232 | 0.3593 | 0.3965 |
| 0.99                     |                             |      |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 5.3332                      | 0.0582 | 1.2647 | 0.045 | 0.0442 | 0.0452 | 0.0421 | 0.0439 | 0.0445 | 0.0421 | 0.0421 | 0.0427 | 0.0436 |
| 1                        | 5.3712                      | 0.1278 | 1.275 | 0.081 | 0.0788 | 0.0793 | 0.0710 | 0.0766 | 0.0721 | 0.0710 | 0.0728 | 0.0756 |
| 5                        | 6.6603                      | 2.2004 | 1.6411 | 0.9615 | 0.9000 | 0.8569 | 0.6656 | 0.9338 | 0.7362 | 0.7125 | 0.6700 | 0.7681 | 0.8245 |
| 10                       | 10.4667                     | 8.2752 | 3.0704 | 3.361 | 3.1245 | 2.9375 | 2.2455 | 3.2649 | 2.3121 | 2.4149 | 2.2762 | 2.7646 | 2.899 |
| 0.999                    |                             |      |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 53.052                      | 0.267 | 9.8224 | 0.1463 | 0.1395 | 0.1371 | 0.1155 | 0.1155 | 0.1490 | 0.0911 | 0.1249 | 0.1155 | 0.1221 | 0.1239 |
| 1                        | 53.4613                     | 0.9327 | 9.9287 | 0.4316 | 0.4036 | 0.3828 | 0.2943 | 0.4230 | 0.2473 | 0.3308 | 0.2947 | 0.3296 | 0.344 |
| 5                        | 66.7318                     | 21.3094 | 14.1345 | 8.7511 | 8.0179 | 7.3517 | 5.2700 | 6.9837 | 4.7084 | 6.061 | 5.3091 | 6.6861 | 6.5022 |
| 10                       | 107.0157                    | 84.4225 | 30.361 | 34.0915 | 31.1721 | 28.6126 | 20.3848 | 24.8393 | 18.3989 | 23.3445 | 20.6612 | 26.7314 | 24.6425 |
Table 6: Estimated MSE for \( n = 100, \; p = 4, \) and 20% outlier in the \( y \)-direction.

| Correlation (\( \delta^2 \)) | Error variance (\( \sigma^2 \)) | OLS | M | RE | MRE | M1 | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|-----------------------------|-----------------------------|-----|---|----|-----|----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                        |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                         | 2.7119                      | 0.0372 | 1.5777 | 0.0355 | 0.0348 | 0.0361 | **0.0309** | 0.0315 | 0.0327 | 0.0324 | **0.0309** | 0.0313 | 0.0317 |
| 1                           | 2.7142                      | 0.0502 | 1.5791 | 0.0446 | 0.0433 | 0.0458 | **0.0388** | 0.0402 | 0.0422 | 0.0411 | 0.0389 | 0.0396 | 0.0404 |
| 5                           | 2.774                       | 0.3607 | 1.6103 | 0.2152 | 0.2049 | 0.2205 | **0.1473** | 0.1652 | 0.2208 | 0.1577 | 0.1484 | 0.1535 | 0.1605 |
| 10                          | 2.9626                      | 1.2603 | 1.7144 | 0.6673 | 0.6256 | 0.6446 | **0.4047** | 0.4881 | 0.6967 | 0.4284 | 0.4100 | 0.4329 | 0.4625 |
| 0.95                        |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                         | 9.859                       | 0.0472 | 6.0528 | 0.0429 | 0.0414 | 0.0436 | **0.0350** | 0.0338 | 0.0424 | 0.0386 | 0.0351 | 0.0362 | 0.0379 |
| 1                           | 9.8708                      | 0.0839 | 6.0613 | 0.0662 | 0.0634 | 0.0692 | **0.0501** | 0.057 | 0.0697 | 0.0554 | 0.0503 | 0.0523 | 0.0556 |
| 5                           | 10.806                      | 1.0901 | 6.1683 | 0.5738 | 0.5259 | 0.548 | **0.2891** | 0.4148 | 0.6306 | 0.3519 | 0.2917 | 0.3177 | 0.3633 |
| 10                          | 10.7315                     | 4.1578 | 6.5088 | 2.0137 | 1.8176 | 1.8164 | **0.8818** | 1.555 | 2.2562 | 1.0912 | 0.8988 | 1.0543 | 1.2925 |
| 0.99                        |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                         | 54.3701                     | 0.0681 | 34.4309 | 0.0271 | 0.0224 | 0.0106 | **0.0078** | 0.0078 | 0.0319 | 0.0079 | **0.0078** | 0.0078 | 0.0078 |
| 1                           | 54.3824                     | 0.2078 | 34.4236 | 0.0714 | 0.0573 | 0.0296 | **0.0082** | 0.0132 | 0.0985 | 0.0128 | **0.0082** | 0.0086 | 0.0103 |
| 5                           | 55.5214                     | 5.1604 | 34.9726 | 2.1149 | 1.8064 | 1.5603 | **0.5777** | 1.7436 | 2.6499 | 0.8481 | 0.5826 | 0.7399 | 1.1057 |
| 10                          | 58.1486                     | 20.5163 | 36.0667 | 8.7298 | 7.5339 | 6.7873 | **2.6340** | 8.7871 | 10.6035 | 3.7859 | 2.6744 | 3.8553 | 5.909 |
| 0.999                       |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                         | 565.474                     | 0.5198 | 36.4029 | 0.1889 | 0.1535 | 0.1021 | **0.0289** | 0.1250 | 0.2703 | 0.0473 | **0.0289** | 0.0326 | 0.0522 |
| 1                           | 566.133                     | 2.1078 | 363.8108 | 0.8372 | 0.7005 | 0.5722 | **0.1843** | 0.8139 | 1.1006 | 0.2873 | 0.1846 | 0.222 | 0.3697 |
| 5                           | 574.9528                    | 53.0487 | 366.5572 | 22.8981 | 19.6015 | 17.8107 | **7.0759** | 26.7947 | 27.6164 | 10.1589 | 7.1089 | 10.9766 | 18.5534 |
| 10                          | 613.5392                    | 216.4098 | 387.9505 | 95.2347 | 81.8515 | 75.8252 | **29.854** | 112.8437 | 113.6536 | 42.6099 | 30.1315 | 55.3014 | 88.4608 |
| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS | M | RE | MRE | M1 | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.85 | 0.1 | 19.6349 | 17.6440 | 9.9596 | 7.3550 | 7.3387 | 7.4691 | 6.0729 | 7.9097 | 6.1804 | 5.8016 | 6.9749 | 7.7877 | 7.6471 |
| | 1 | 19.6548 | 17.8135 | 9.9702 | 7.4407 | 7.4144 | 7.5191 | 6.1374 | 8.0019 | 6.2894 | 5.8733 | 7.0315 | 7.8576 | 7.7398 |
| | 5 | 20.1568 | 19.9328 | 10.4139 | 8.8980 | 8.6554 | 8.4084 | 6.9604 | 9.4876 | 8.2301 | 6.7105 | 7.8577 | 8.9484 | 9.2204 |
| | 10 | 21.9848 | 23.1124 | 11.8408 | 10.9145 | 10.3371 | 9.2462 | 7.9231 | 10.7291 | 10.6007 | 7.3025 | 6.0931 | 9.8156 | 10.3140 |
| 0.95 | 0.1 | 56.9964 | 52.2204 | 28.2251 | 21.8769 | 21.6163 | 22.5320 | 15.4020 | 21.0251 | 16.0025 | 14.4870 | 20.2660 | 23.1706 | 19.6896 |
| | 1 | 57.0936 | 52.6418 | 28.4642 | 22.0848 | 21.7950 | 22.6299 | 15.5887 | 21.2996 | 16.2955 | 14.6583 | 20.4086 | 23.3535 | 19.9663 |
| | 5 | 58.5930 | 59.0152 | 30.7393 | 26.2820 | 25.3331 | 24.9819 | 17.9565 | 26.6148 | 21.9210 | 16.8300 | 22.5822 | 27.0839 | 25.3281 |
| | 10 | 64.9236 | 69.9297 | 35.3306 | 32.6575 | 30.6076 | 27.3927 | 20.7521 | 32.9312 | 29.6569 | 19.1308 | 25.0495 | 31.0582 | 31.2033 |
| 0.99 | 0.1 | 273.872 | 255.3598 | 146.4735 | 112.4649 | 110.0771 | 117.1600 | 65.2409 | 81.5832 | 73.2479 | 68.5016 | 92.2531 | 111.2077 | 83.1597 |
| | 1 | 274.0876 | 257.4551 | 146.4241 | 113.3492 | 110.8365 | 117.4684 | 66.0318 | 82.8040 | 74.4629 | 69.1385 | 93.0513 | 111.7953 | 84.3836 |
| | 5 | 282.9705 | 294.5439 | 154.3337 | 135.9912 | 130.1752 | 129.6193 | 77.9042 | 113.4549 | 104.1344 | 78.8442 | 104.7876 | 137.0343 | 115.4497 |
| | 10 | 311.5896 | 343.4734 | 172.5693 | 163.8457 | 152.9535 | 138.5359 | 91.0598 | 146.0081 | 135.7390 | 88.0053 | 116.9538 | 160.5102 | 146.0775 |
| 0.999 | 0.1 | 267.6728 | 2471.9678 | 1497.3005 | 1084.9156 | 1057.3681 | 1136.9282 | 515.1581 | 687.3880 | 678.4272 | 627.2289 | 652.7648 | 961.5072 | 710.4567 |
| | 1 | 267.9734 | 2492.3767 | 1497.2785 | 1094.1867 | 1065.4237 | 1141.4125 | 520.5305 | 700.4929 | 691.5172 | 633.1840 | 661.4029 | 972.2813 | 723.6880 |
| | 5 | 274.6362 | 2820.2333 | 1544.0392 | 1294.6657 | 1235.9941 | 1253.7144 | 596.6097 | 962.0827 | 950.2034 | 709.5270 | 761.9032 | 1228.7157 | 991.5623 |
| | 10 | 3083.7823 | 3405.6089 | 1751.0325 | 1623.6314 | 1507.2433 | 1359.6274 | 723.7783 | 1350.1377 | 1330.7578 | 810.9957 | 906.9225 | 1594.6897 | 1387.0441 |
Table 8: Estimated MSE for $n = 50$, $p = 4$, and 30% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS | M  | RE  | MRE | M1  | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|-----|----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
|                          | 0.85                        |     |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 1.2168                      | 0.0323 | 0.3411 | 0.0239 | 0.0235 | 0.0268 | 0.0226 | 0.0233 | 0.0257 | **0.0207** | 0.0228 | 0.0230 | 0.0234 |
| 1                        | 1.2192                      | 0.0406 | 0.3420 | 0.0195 | 0.0191 | 0.0174 | 0.0154 | 0.0155 | 0.0163 | **0.0143** | 0.0154 | 0.0155 | 0.0156 |
| 5                        | 1.2971                      | 0.3032 | 0.3849 | 0.0458 | 0.0431 | 0.0301 | 0.0249 | 0.0258 | 0.0322 | **0.0185** | 0.0249 | 0.0253 | 0.0258 |
| 10                       | 1.5608                      | 1.0696 | 0.5451 | 0.1946 | 0.1879 | 0.1516 | 0.1336 | 0.1400 | 0.1699 | **0.0845** | 0.1339 | 0.1363 | 0.1388 |
|                          | 0.95                        |     |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 2.0589                      | 0.0331 | 0.3605 | 0.0193 | 0.0190 | 0.0199 | **0.0161** | 0.0166 | 0.0196 | 0.0168 | **0.0161** | 0.0163 | 0.0166 |
| 1                        | 2.0726                      | 0.0483 | 0.3686 | 0.0149 | 0.0144 | 0.0117 | **0.0102** | 0.0103 | 0.0119 | 0.0102 | **0.0102** | 0.0103 | 0.0104 |
| 5                        | 2.3211                      | 0.5841 | 0.5057 | 0.1120 | 0.1086 | 0.0946 | **0.0798** | 0.0894 | 0.1033 | 0.0729 | 0.0799 | 0.0838 | 0.0882 |
| 10                       | 3.1149                      | 2.1995 | 0.8700 | 0.5326 | 0.5217 | 0.4819 | **0.4226** | 0.4844 | 0.4809 | 0.3706 | 0.4238 | 0.4499 | 0.4682 |
|                          | 0.99                        |     |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 6.7615                      | 0.0642 | 1.1660 | 0.0504 | 0.0500 | 0.0509 | 0.0493 | 0.0503 | 0.0492 | **0.0462** | 0.0493 | 0.0487 | 0.0503 |
| 1                        | 6.7974                      | 0.1461 | 1.1706 | 0.0928 | 0.0921 | 0.0934 | 0.0890 | 0.0932 | **0.0838** | 0.0848 | 0.0890 | 0.0905 | 0.0916 |
| 5                        | 8.0998                      | 2.5763 | 1.5128 | 1.0540 | 1.0344 | 1.0352 | 0.9373 | 1.0453 | **0.6220** | 0.9343 | 0.9405 | 1.0019 | 0.9511 |
| 10                       | 11.8850                     | 9.4826 | 2.8141 | 3.5394 | 3.4506 | 3.4307 | 3.0802 | 3.2887 | **1.7573** | 3.0532 | 3.1035 | 3.3144 | 2.9752 |
|                          | 0.999                       |     |    |     |     |     |       |       |       |       |       |       |       |       |
| 0.1                      | 57.3371                     | 0.3033 | 10.2082 | 0.1671 | 0.1634 | 0.1656 | 0.1533 | 0.1636 | **0.0768** | 0.1594 | 0.1534 | 0.1577 | 0.1392 |
| 1                        | 57.6005                     | 1.0574 | 10.2039 | 0.4793 | 0.4646 | 0.4680 | 0.4190 | 0.4234 | **0.1716** | 0.4413 | 0.4194 | 0.4337 | 0.3546 |
| 5                        | 70.9714                     | 23.7989 | 12.8283 | 9.1058 | 8.7374 | 8.7753 | 7.6743 | 5.2785 | **2.5667** | 8.1664 | 7.7158 | 6.9566 | 4.8686 |
| 10                       | 112.5459                    | 92.3967 | 25.6295 | 34.2080 | 32.6806 | 32.7241 | 28.3661 | 17.4819 | **10.4795** | 30.3034 | 28.6226 | 24.2442 | 16.9600 |
Table 9: Estimated MSE for $n = 100$, $p = 4$, and 30% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS  | M    | RE   | MRE  | M1   | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                     | 0.1                         | 3.3449 | 0.0418 | 1.7731 | 0.0389 | 0.0379 | 0.0393 | 0.0336 | 0.0341 | 0.0350 | 0.0347 | 0.0337 | 0.0339 | 0.0342 |
|                          | 1                           | 3.3474 | 0.0645 | 1.7750 | 0.0534 | 0.0518 | 0.0558 | 0.0453 | 0.0478 | 0.0515 | 0.0492 | 0.0455 | 0.0467 | 0.0483 |
|                          | 5                           | 3.4047 | 0.6348 | 1.8109 | 0.3317 | 0.3177 | 0.3447 | 0.2137 | 0.2523 | 0.3508 | 0.2415 | 0.2171 | 0.2301 | 0.2470 |
|                          | 10                          | 3.5908 | 2.2707 | 1.9313 | 1.0627 | 1.0067 | 1.0485 | 0.6170 | 0.8006 | 1.1405 | 0.6929 | 0.6341 | 0.6941 | 0.7667 |
| 0.95                     | 0.1                         | 12.3445 | 0.0596 | 6.9830 | 0.0508 | 0.0488 | 0.0526 | 0.0397 | 0.0450 | 0.0523 | 0.0444 | 0.0399 | 0.0420 | 0.0453 |
|                          | 1                           | 12.3572 | 0.1280 | 6.9927 | 0.0889 | 0.0850 | 0.0958 | 0.0615 | 0.0736 | 0.0969 | 0.0709 | 0.0699 | 0.0654 | 0.0716 |
|                          | 5                           | 12.5641 | 2.0742 | 7.1144 | 0.9556 | 0.8793 | 0.9328 | 0.4314 | 0.7235 | 1.0716 | 0.5641 | 0.4398 | 0.5121 | 0.6309 |
|                          | 10                          | 13.2232 | 7.9378 | 7.5122 | 3.4172 | 3.1097 | 3.1602 | 1.3608 | 2.8742 | 3.9144 | 1.8204 | 1.4142 | 1.8437 | 2.4253 |
| 0.99                     | 0.1                         | 68.7407 | 0.1169 | 40.4274 | 0.0362 | 0.0291 | 0.0102 | 0.0050 | 0.0051 | 0.0491 | 0.0055 | 0.0050 | 0.0050 | 0.0050 |
|                          | 1                           | 68.7558 | 0.4077 | 40.4263 | 0.1263 | 0.1035 | 0.0614 | 0.0123 | 0.0291 | 0.1834 | 0.0241 | 0.0123 | 0.0138 | 0.0192 |
|                          | 5                           | 69.8943 | 10.4866 | 41.0566 | 3.9028 | 3.3826 | 3.0807 | 0.9863 | 3.6712 | 5.0762 | 1.5841 | 1.0018 | 1.4625 | 2.4341 |
|                          | 10                          | 72.4442 | 40.9857 | 42.3274 | 15.7245 | 13.7219 | 12.8923 | 4.3473 | 17.1075 | 19.1979 | 6.7860 | 4.4743 | 7.7631 | 12.4983 |
| 0.999                    | 0.1                         | 718.4494 | 1.0575 | 430.4856 | 0.3515 | 0.2914 | 0.2122 | 0.0519 | 0.2859 | 0.5187 | 0.0953 | 0.0519 | 0.0622 | 0.1139 |
|                          | 1                           | 719.1863 | 4.3493 | 430.9832 | 1.5823 | 1.3432 | 1.1660 | 0.3260 | 1.6995 | 2.1610 | 0.5637 | 0.3266 | 0.4242 | 0.8104 |
|                          | 5                           | 727.5877 | 108.7339 | 434.5422 | 42.1996 | 36.4626 | 34.3498 | 11.5190 | 52.2224 | 53.8210 | 18.3461 | 11.6078 | 21.7009 | 38.9226 |
|                          | 10                          | 766.9152 | 432.6482 | 458.3362 | 171.0409 | 148.1112 | 143.3046 | 47.0839 | 214.1686 | 216.5307 | 74.8107 | 47.8085 | 110.5502 | 177.7827 |
| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS  | M    | RE   | MRE  | M1   | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                     | 0.1                         | 90.3028 | 0.1260 | 52.6964 | 0.0722 | 0.0668 | 0.0433 | **0.0185** | 0.0352 | 0.0787 | 0.0220 | 0.0194 | 0.0241 | 0.0327 |
|                          | 1                           | 90.0911 | 0.4226 | 52.4878 | 0.2051 | 0.1844 | 0.0922 | **0.0349** | 0.0879 | 0.2428 | 0.0479 | 0.0379 | 0.0531 | 0.0783 |
|                          | 5                           | 92.8369 | 10.4365 | 54.1176 | 4.5927 | 4.0380 | 1.8522 | **0.8838** | 2.7471 | 6.0351 | 1.2195 | 1.1600 | 1.7581 | 2.4192 |
|                          | 10                          | 103.4848 | 41.0851 | 61.1427 | 18.2693 | 15.9582 | 7.5356 | **3.9093** | 12.6529 | 24.2898 | 5.0297 | 5.2433 | 8.4338 | 11.1894 |
| 0.95                     | 0.1                         | 276.0997 | 0.3408 | 160.5305 | 0.1649 | 0.1470 | 0.0709 | **0.0201** | 0.0928 | 0.2081 | 0.0324 | 0.0209 | 0.0378 | 0.0678 |
|                          | 1                           | 276.6652 | 1.3403 | 161.0217 | 0.5994 | 0.5227 | 0.2322 | **0.0652** | 0.3721 | 0.8082 | 0.1871 | 0.0721 | 0.1612 | 0.2740 |
|                          | 5                           | 285.2570 | 32.8959 | 165.9873 | 14.1830 | 12.2513 | 5.5580 | **2.1726** | 11.9823 | 19.6288 | 3.4422 | 2.8621 | 6.8468 | 9.3847 |
|                          | 10                          | 317.2783 | 124.7314 | 186.5438 | 53.5069 | 46.2205 | 20.9994 | **9.1732** | 50.3107 | 74.7270 | 13.3553 | 13.2954 | 30.0157 | 39.8692 |
| 0.99                     | 0.1                         | 1331.9345 | 1.6007 | 771.5345 | 0.7000 | 0.6068 | 0.2533 | **0.0728** | 0.5988 | 0.9596 | 0.1295 | 0.0752 | 0.2202 | 0.3786 |
|                          | 1                           | 1332.3341 | 6.3826 | 771.5778 | 2.7037 | 2.3343 | 0.9786 | **0.3198** | 2.5805 | 3.7825 | 0.5602 | 0.3409 | 1.1295 | 1.6962 |
|                          | 5                           | 1369.4252 | 156.6853 | 791.6950 | 66.8823 | 57.3543 | 24.9037 | **10.2312** | 78.6769 | 92.5953 | 15.7728 | 12.6029 | 41.8408 | 55.9108 |
|                          | 10                          | 1525.2393 | 605.6192 | 891.6240 | 256.1311 | 220.7198 | 98.9894 | **43.9823** | 324.4023 | 356.2406 | 63.4203 | 61.1637 | 189.0302 | 249.2287 |
| 0.999                    | 0.1                         | 11754.4582 | 15.6950 | 6662.8779 | 6.7219 | 5.7320 | 2.4027 | **0.8240** | 8.7512 | 9.4396 | 1.3550 | 0.8411 | 3.4754 | 5.0458 |
|                          | 1                           | 11776.6281 | 64.1466 | 6680.3665 | 27.4920 | 23.3474 | 9.9847 | **3.6566** | 36.5974 | 38.7906 | 5.8203 | 3.7990 | 16.6504 | 23.3558 |
|                          | 5                           | 12209.4252 | 1561.7432 | 6952.7963 | 665.2483 | 565.5803 | 246.8486 | **101.4521** | 920.6514 | 939.0568 | 151.7494 | 116.8905 | 539.8239 | 720.1992 |
|                          | 10                          | 13442.1080 | 5949.3244 | 7723.1463 | 2552.5134 | 2175.1664 | 956.6620 | **409.2780** | 3538.7896 | 3571.3962 | 586.9131 | 519.9193 | 2347.1318 | 2984.5567 |
Table 11: Estimated MSE for \( n = 50, \ p = 10, \) and 10\% outlier in the \( y \)-direction.

| Correlation (\( \delta^2 \)) | Error variance (\( \sigma^2 \)) | OLS | M | RE | MRE | M1 | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|-----------------------------|---------------------------------|-----|---|---|-----|----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                        |                                 |     |   |   |     |     |       |       |       |       |       |       |       |       |
| 0.1                         | 3.1400                          | 0.0305 | 1.6863 | 0.0238 | 0.0231 | 0.0236 | **0.0194** | 0.0198 | 0.0225 | 0.0208 | 0.0195 | 0.0196 | 0.0198 |
| 1                           | 3.1561                          | 0.0462 | 1.6920 | 0.0270 | 0.0258 | 0.0221 | 0.0190 | 0.0192 | 0.0219 | **0.0188** | 0.0190 | 0.0191 | 0.0192 |
| 5                           | 3.4663                          | 0.5723 | 1.8953 | 0.2111 | 0.1961 | 0.1109 | 0.0722 | 0.0833 | 0.1536 | **0.0716** | 0.0732 | 0.0765 | 0.0804 |
| 10                          | 4.5277                          | 2.2567 | 2.5827 | 0.8310 | 0.7702 | 0.4243 | **0.2653** | 0.3517 | 0.6845 | 0.2713 | 0.2734 | 0.2992 | 0.3265 |
| 0.95                        |                                 |     |   |   |     |     |       |       |       |       |       |       |       |       |
| 0.1                         | 9.9214                          | 0.0462 | 5.3267 | 0.0330 | 0.0316 | 0.0284 | **0.0221** | 0.0238 | 0.0324 | 0.0267 | **0.0221** | 0.0225 | 0.0231 |
| 1                           | 9.9619                          | 0.1034 | 5.3420 | 0.0558 | 0.0528 | 0.0396 | **0.0285** | 0.0331 | 0.0533 | 0.0360 | 0.0286 | 0.0296 | 0.0310 |
| 5                           | 11.1307                         | 1.8869 | 6.1136 | 0.7269 | 0.6728 | 0.3477 | **0.1833** | 0.3891 | 0.7390 | 0.2784 | 0.1882 | 0.2399 | 0.2992 |
| 10                          | 14.3022                         | 7.2575 | 8.1562 | 2.6836 | 2.4817 | 1.2027 | **0.6021** | 1.6919 | 2.8017 | 0.9410 | 0.6360 | 0.9607 | 1.2574 |
| 0.99                        |                                 |     |   |   |     |     |       |       |       |       |       |       |       |       |
| 0.1                         | 52.3089                         | 0.1172 | 27.5676 | 0.0488 | 0.0456 | 0.0263 | **0.0178** | 0.0303 | 0.0541 | 0.0230 | **0.0178** | 0.0192 | 0.0216 |
| 1                           | 52.3859                         | 0.3925 | 27.5835 | 0.1424 | 0.1313 | 0.0595 | **0.0300** | 0.0943 | 0.1611 | 0.0469 | 0.0301 | 0.0386 | 0.0519 |
| 5                           | 57.5278                         | 9.3499 | 31.3616 | 3.3018 | 3.0374 | 1.2681 | **0.5155** | 2.9513 | 3.5869 | 0.9249 | 0.5322 | 1.3274 | 1.9247 |
| 10                          | 75.2018                         | 37.5897 | 42.6667 | 13.3190 | 12.2854 | 5.2417 | **2.1703** | 12.6431 | 14.2020 | 3.9275 | 2.3468 | 7.1939 | 9.4193 |
| 0.999                       |                                 |     |   |   |     |     |       |       |       |       |       |       |       |       |
| 0.1                         | 537.2870                        | 0.9777 | 280.3796 | 0.3461 | 0.3176 | 0.1309 | **0.0547** | 0.3453 | 0.3722 | 0.0896 | 0.0549 | 0.1133 | 0.1798 |
| 1                           | 539.5618                        | 3.9135 | 281.1094 | 1.3699 | 1.2561 | 0.5174 | **0.2026** | 1.4159 | 1.4404 | 0.3520 | 0.2040 | 0.5772 | 0.8806 |
| 5                           | 592.0523                        | 95.1437 | 321.0047 | 33.9110 | 31.0736 | 12.9534 | **5.2710** | 35.3561 | 33.9454 | 8.9894 | 5.4539 | 23.6814 | 28.8843 |
| 10                          | 763.2005                        | 383.4488 | 430.3889 | 136.8452 | 125.9769 | 54.0759 | **21.1317** | 141.4128 | 135.6074 | 36.0725 | 22.6733 | 111.2358 | 127.3903 |
Table 12: Estimated MSE for $n = 100$, $p = 10$, and 10% outlier in the y-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS  | M   | RE  | MRE | M1  | TRME2 | MTPM1 | MTPM2 | MTPM3 | TRME1 | MTPM4 | MTPM5 | MTPM6 |
|-------------------------|-----------------------------|------|-----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
|                         |                             | 0.85 |     |     |     |     |       |       |       |       |       |       |       |       |
|                         |                             | 0.01 | 1.5457 | 0.0281 | 0.8512 | 0.0251 | 0.0248 | 0.0270 | 0.0207 | 0.0217 | 0.0244 | 0.0223 | 0.0209 | 0.0212 | 0.0217 |
|                         |                             | 1    | 1.5497 | 0.0346 | 0.8562 | 0.0255 | 0.0248 | 0.0251 | 0.0200 | 0.0203 | 0.0223 | 0.0198 | 0.0201 | 0.0202 | 0.0203 |
|                         |                             | 5    | 1.7168 | 0.2697 | 0.9977 | 0.0962 | 0.0902 | 0.0486 | 0.0339 | 0.0349 | 0.0567 | 0.0290 | 0.0340 | 0.0343 | 0.0347 |
|                         |                             | 10   | 2.1653 | 0.9986 | 1.2864 | 0.3254 | 0.3030 | 0.1496 | 0.1047 | 0.1109 | 0.2071 | **0.0850** | 0.1053 | 0.1071 | 0.1098 |
|                         |                             | 0.95 | 0.01 | 5.1422 | 0.0328 | 2.6793 | 0.0255 | 0.0248 | 0.0255 | 0.0171 | 0.0183 | 0.0242 | 0.0203 | **0.0171** | 0.0174 | 0.0177 |
|                         |                             | 1    | 5.1501 | 0.0553 | 2.6826 | 0.0305 | 0.0293 | 0.0223 | 0.0158 | 0.0165 | 0.0257 | 0.0185 | **0.0158** | 0.0159 | 0.0162 |
|                         |                             | 5    | 5.6167 | 0.8043 | 3.0691 | 0.2498 | 0.2339 | 0.0889 | 0.0478 | 0.0620 | 0.2163 | 0.0715 | 0.0482 | 0.0516 | 0.0573 |
|                         |                             | 10   | 7.1428 | 3.2113 | 4.0873 | 0.9756 | 0.9110 | 0.3503 | **0.1878** | 0.3013 | 0.9685 | 0.2858 | 0.1909 | 0.2190 | 0.2623 |
|                         |                             | 0.99 | 0.01 | 27.6845 | 0.0726 | 14.4324 | 0.0373 | 0.0356 | 0.0290 | 0.0246 | 0.0273 | 0.0400 | 0.0295 | **0.0246** | 0.0250 | 0.0259 |
|                         |                             | 1    | 27.7221 | 0.1985 | 14.4506 | 0.0755 | 0.0710 | 0.0459 | 0.0360 | 0.0456 | 0.0852 | 0.0464 | **0.0360** | 0.0375 | 0.0403 |
|                         |                             | 5    | 30.3951 | 4.3259 | 16.2291 | 1.2978 | 1.2031 | 0.4901 | **0.2851** | 0.8326 | 1.6593 | 0.4525 | 0.2878 | 0.3917 | 0.5359 |
|                         |                             | 10   | 37.9948 | 16.9806 | 21.2871 | 5.0412 | 4.6765 | 1.8132 | **0.9805** | 3.6825 | 6.4484 | 1.6190 | 1.0014 | 1.7239 | 2.4475 |
|                         |                             | 0.999| 0.01 | 286.8901 | 0.4413 | 149.7851 | 0.1403 | 0.1313 | 0.0479 | **0.0221** | 0.1158 | 0.1758 | 0.0344 | **0.0221** | 0.0296 | 0.0462 |
|                         |                             | 1    | 287.7825 | 1.7449 | 150.3169 | 0.5105 | 0.4746 | 0.1564 | **0.0661** | 0.4344 | 0.6642 | 0.1126 | 0.0663 | 0.1185 | 0.2018 |
|                         |                             | 5    | 312.5711 | 42.2094 | 165.7353 | 12.3487 | 11.4621 | 3.9620 | **1.8308** | 12.3250 | 15.2596 | 3.0848 | 1.8535 | 5.7130 | 8.1303 |
|                         |                             | 10   | 391.1042 | 169.4866 | 218.8966 | 49.4887 | 45.8882 | 16.2240 | **7.7598** | 52.6821 | 60.4953 | 12.6323 | 7.9611 | 28.5068 | 36.5864 |
| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS | M | RE | MRE | M1 | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|------------------------|-----------------------------|-----|---|----|-----|----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                   |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.85                   | 0.1                         | 273.4793 | 156.7689 | 206.1094 | 86.1653 | 77.0995 | 43.6674 | **21.6971** | 87.8649 | 113.8645 | 36.1357 | 40.114 | 67.8668 | 86.8745 |
|                        | 1                           | 273.0539 | 157.4089 | 205.5957 | 86.2185 | 77.1741 | 42.9368 | **21.7251** | 87.907 | 114.0479 | 35.8279 | 40.0981 | 67.7972 | 86.823 |
|                        | 5                           | 275.4729 | 157.4634 | 205.4779 | 85.6156 | 76.837 | 37.7866 | **20.6376** | 80.175 | 110.7684 | 31.3609 | 35.0601 | 58.2451 | 75.305 |
|                        | 10                          | 286.5291 | 182.217 | 209.3173 | 98.923 | 87.2395 | 43.8661 | **23.8501** | 90.6212 | 127.5594 | 33.3307 | 39.2086 | 64.0506 | 83.212 |
| 0.95                   |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.95                   | 0.1                         | 819.1836 | 455.426 | 612.785 | 246.591 | 221.5231 | 127.5565 | **53.7494** | 315.4418 | 325.2237 | 92.1506 | 105.1981 | 226.5454 | 284.7621 |
|                        | 1                           | 819.9241 | 459.3204 | 613.2985 | 247.6881 | 222.4191 | 125.7323 | **54.2189** | 317.7502 | 327.5723 | 90.9346 | 107.0494 | 228.1556 | 286.9329 |
|                        | 5                           | 827.7403 | 462.5611 | 613.419 | 248.1087 | 222.9635 | 109.3106 | **55.1827** | 301.6026 | 321.9096 | 81.5994 | 100.7558 | 208.9254 | 263.933 |
|                        | 10                          | 860.5311 | 534.6709 | 623.4981 | 283.2951 | 249.8283 | 120.6435 | **60.6973** | 337.4342 | 368.3171 | 85.6961 | 109.0555 | 225.2578 | 288.8946 |
| 0.99                   |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.99                   | 0.1                         | 3901.296 | 2150.331 | 2901.249 | 1165.627 | 1051.513 | 624.9954 | **231.795** | 1534.714 | 1517.236 | 434.0425 | 400.8136 | 1227.798 | 1467.73 |
|                        | 1                           | 3901.224 | 2167.916 | 2899.996 | 1170.029 | 1055.25 | 614.6947 | **233.3161** | 1544.155 | 1526.87 | 427.0813 | 407.4048 | 1238.502 | 1477.911 |
|                        | 5                           | 3930.612 | 2169.253 | 2892.813 | 1168.533 | 1054.01 | 531.9327 | **250.3522** | 1499.776 | 1491.422 | 371.725 | 431.5145 | 1200.767 | 1410.728 |
|                        | 10                          | 4093.132 | 2522.274 | 2942.757 | 1339.398 | 1184.609 | 580.171 | **276.6247** | 1721.755 | 1716.665 | 388.6839 | 475.891 | 1319.492 | 1580.715 |
| 0.999                  |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.999                  | 0.1                         | 34369.11 | 20559.64 | 2521.145 | 11310.83 | 10175.82 | 6176.982 | **2128.203** | 14561.13 | 14550.91 | 4296.599 | 2775.708 | 13400.44 | 14661.01 |
|                        | 1                           | 34395.97 | 20732.63 | 2522.64 | 11355.68 | 10121.66 | 6092.753 | **2131.213** | 14663.9 | 14653.83 | 4227.387 | 2803.213 | 13509.58 | 14757.11 |
|                        | 5                           | 34854.44 | 20888.84 | 2534.785 | 11428.23 | 10263.13 | 5268.545 | **2313.528** | 14461.07 | 14455.43 | 3601.397 | 3343.458 | 13395.13 | 14349.82 |
|                        | 10                          | 35984.14 | 24046.08 | 2557.888 | 13055.78 | 11492.34 | 5758.728 | **2525.172** | 16536.65 | 16525.08 | 3696.633 | 3712.092 | 14860.09 | 16222.84 |
Table 14: Estimated MSE for $n = 50$, $p = 10$, and 20% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS | M | RE | MRE | M1 | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|-----|---|----|-----|----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                     |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                      | 2.7761                      | 0.0323 | 0.9977 | 0.0245 | 0.0239 | 0.0246 | **0.0210** | 0.0213 | 0.0235 | 0.0222 | 0.0211 | 0.0212 | 0.0213 |
| 1                        | 2.7892                      | 0.0508 | 1.0011 | 0.0287 | 0.0278 | 0.0257 | 0.0228 | 0.0230 | 0.0254 | **0.0215** | 0.0228 | 0.0229 | 0.0230 |
| 5                        | 3.112                       | 0.6421 | 1.1505 | 0.2058 | 0.1932 | 0.1229 | 0.0917 | 0.101 | 0.1536 | **0.0804** | 0.0926 | 0.0955 | 0.099 |
| 10                       | 4.1454                      | 2.4681 | 1.7115 | 0.7513 | 0.7032 | 0.4159 | 0.2974 | 0.3598 | 0.5942 | **0.2503** | 0.3038 | 0.3244 | 0.3461 |
| 0.95                     |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                      | 8.2761                      | 0.0455 | 3.1829 | 0.0278 | 0.0267 | 0.0237 | **0.0194** | 0.0204 | 0.027 | 0.0225 | **0.0194** | 0.0196 | 0.0200 |
| 1                        | 8.3088                      | 0.1045 | 3.1933 | 0.0446 | 0.0423 | 0.0302 | **0.0230** | 0.0258 | 0.0406 | 0.0277 | 0.0231 | 0.0237 | 0.0247 |
| 5                        | 9.3876                      | 1.9919 | 3.5665 | 0.603 | 0.5612 | 0.2727 | **0.1556** | 0.3109 | 0.5801 | 0.2308 | 0.1596 | 0.2017 | 0.2503 |
| 10                       | 12.7455                     | 7.7504 | 5.2954 | 2.2809 | 2.1229 | 0.9919 | **0.5577** | 1.4008 | 2.2157 | 0.8113 | 0.5865 | 0.8586 | 1.0926 |
| 0.99                     |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                      | 42.0201                     | 0.1267 | 15.5703 | 0.0503 | 0.0476 | 0.0313 | **0.0234** | 0.0370 | 0.0520 | 0.0300 | **0.0234** | 0.0255 | 0.0287 |
| 1                        | 42.1635                     | 0.4275 | 15.5843 | 0.1385 | 0.1295 | 0.0675 | **0.0414** | 0.1046 | 0.1412 | 0.0613 | 0.0416 | 0.0524 | 0.0679 |
| 5                        | 47.4737                     | 9.9848 | 17.7272 | 2.9076 | 2.698 | 1.1729 | **0.5721** | 2.6098 | 2.8145 | 0.9911 | 0.5947 | 1.3803 | 1.8684 |
| 10                       | 65.0568                     | 39.608 | 26.7775 | 11.4639 | 10.6656 | 4.5255 | **2.2619** | 10.5127 | 10.5867 | 3.9538 | 2.4476 | 6.9581 | 8.5596 |
| 0.999                    |                             |     |   |    |     |    |       |       |       |       |       |       |       |       |
| 0.1                      | 425.9581                    | 1.0527 | 150.9938 | 0.3227 | 0.2994 | 0.1376 | **0.0726** | 0.3115 | 0.2868 | 0.1116 | 0.0729 | 0.1471 | 0.2104 |
| 1                        | 427.7258                    | 4.1515 | 151.3241 | 1.2111 | 1.1211 | 0.4808 | **0.2377** | 1.1672 | 1.0543 | 0.3849 | 0.2395 | 0.6441 | 0.8837 |
| 5                        | 481.2639                    | 100.9849 | 175.9382 | 29.432 | 27.2082 | 11.4594 | **5.4706** | 27.0912 | 24.1158 | 9.0196 | 5.7011 | 22.719 | 25.6489 |
| 10                       | 654.8952                    | 403.9386 | 269.8719 | 117.5676 | 108.6369 | 47.1495 | **21.5542** | 104.9628 | 95.8291 | 36.0268 | 23.3956 | 98.8043 | 105.13 |
Table 15: Estimated MSE for $n = 100$, $p = 10$, and 20% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS | M | RE | MRE | M1 | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|-----|---|----|-----|----|-------|-------|-------|-------|-------|-------|-------|-------|
|                          | 0.85                        |     |   |    |     |    |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 3.2451 | 0.0253 | 2.7555 | 0.0203 | 0.0198 | 0.0173 | **0.0133** | 0.0134 | 0.0141 | 0.0114 | 0.0133 | 0.0133 | 0.0134 |
|                          | 1                           | 3.2509 | 0.0346 | 2.7591 | 0.0201 | 0.0189 | 0.0176 | **0.0041** | 0.0041 | 0.0047 | 0.0056 | 0.0041 | 0.0041 | 0.0041 |
|                          | 5                           | 3.4173 | 0.4886 | 2.8446 | 0.2264 | 0.2042 | 0.1099 | **0.0765** | 0.0813 | 0.1237 | 0.1068 | 0.077 | 0.0788 | 0.0814 |
|                          | 10                          | 3.8593 | 1.9541 | 3.0336 | 0.9563 | 0.8704 | 0.547 | **0.4315** | 0.4649 | 0.6705 | 0.5311 | 0.4344 | 0.4469 | 0.4638 |
|                          | 0.95                        |     |   |    |     |    |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 9.3741 | 0.0312 | 7.8276 | 0.0197 | 0.0188 | 0.0088 | **0.0058** | 0.0056 | 0.0076 | 0.0053 | 0.0058 | 0.0058 | 0.0057 |
|                          | 1                           | 9.3873 | 0.0676 | 7.8342 | 0.0304 | 0.0277 | 0.0066 | **0.0021** | 0.0022 | 0.0116 | 0.0048 | 0.0021 | 0.0021 | 0.0021 |
|                          | 5                           | 9.847 | 1.4819 | 8.0207 | 0.6893 | 0.6234 | 0.3324 | **0.2351** | 0.3129 | 0.576 | 0.3881 | 0.2372 | 0.2609 | 0.296 |
|                          | 10                          | 11.3908 | 5.951 | 8.6758 | 2.8854 | 2.6214 | 1.5357 | **1.1631** | 1.6898 | 2.5681 | 1.8101 | 1.1803 | 1.3608 | 1.5744 |
|                          | 0.99                        |     |   |    |     |    |       |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 46.4089 | 0.129 | 37.4722 | 0.0879 | 0.0828 | 0.0702 | **0.0652** | 0.0713 | 0.0839 | 0.0801 | 0.0652 | 0.0666 | 0.0691 |
|                          | 1                           | 46.4645 | 0.3853 | 37.4911 | 0.2299 | 0.2131 | 0.1648 | **0.1452** | 0.1820 | 0.2131 | 0.1879 | 0.1454 | 0.1538 | 0.1668 |
|                          | 5                           | 49.1191 | 8.1003 | 38.5063 | 4.1501 | 3.783 | 2.4759 | **1.9925** | 3.6376 | 3.4776 | 2.8234 | 2.0099 | 2.6322 | 3.1064 |
|                          | 10                          | 56.8298 | 31.572 | 41.4607 | 15.7501 | 14.3453 | 8.9881 | **7.0750** | 13.9437 | 12.5323 | 10.1609 | 7.2163 | 10.8469 | 12.3447 |
|                          | 0.999                       |     |   |    |     |    |       |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 464.2142 | 0.7394 | 359.9263 | 0.3234 | 0.2896 | 0.1319 | **0.0779** | 0.2865 | 0.2568 | 0.1186 | 0.0780 | 0.1203 | 0.1793 |
|                          | 1                           | 464.8062 | 3.0143 | 359.8676 | 1.3679 | 1.2316 | 0.6496 | **0.4420** | 1.2111 | 0.9862 | 0.6321 | 0.4432 | 0.7364 | 0.9597 |
|                          | 5                           | 491.269 | 77.486 | 369.5471 | 37.6975 | 34.0229 | 19.8836 | **15.0287** | 30.0781 | 23.7421 | 20.0757 | 15.2075 | 29.5357 | 30.8367 |
|                          | 10                          | 569.9381 | 305.4332 | 398.1864 | 148.8783 | 134.5564 | 80.6066 | **61.1582** | 92.1612 | 81.2925 | 62.5972 | 120.4841 | 119.1042 |
Table 16: Estimated MSE for $n = 20$, $p = 10$, and 30% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS  | M   | RE  | MRE | M1  | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|------|-----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
|                          | 0.85                        |      |     |     |     |     |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 205.7834 | 141.6378 | 99.9801 | 67.8697 | 65.0790 | 48.3363 | 21.8213 | 72.1238 | 79.2728 | 32.6643 | 38.8749 | 61.2424 | 72.6874 |
|                          | 1                           | 205.4415 | 142.9469 | 99.7031 | 67.8909 | 65.1966 | 47.7858 | 21.8601 | 72.0202 | 79.1161 | 32.7874 | 38.9244 | 61.2286 | 72.5918 |
|                          | 5                           | 207.9705 | 159.4720 | 101.7499 | 70.6918 | 68.3491 | 37.2622 | 21.3410 | 70.1729 | 80.8271 | 32.1374 | 35.9264 | 57.0197 | 68.6325 |
|                          | 10                          | 218.9210 | 183.6254 | 110.5344 | 81.2402 | 77.2286 | 37.2726 | 23.1308 | 75.0286 | 94.6082 | 32.1748 | 33.2026 | 57.1618 | 70.2639 |
|                          | 0.95                        | 613.7642 | 410.5140 | 293.4535 | 195.3906 | 187.8605 | 140.7899 | 52.7384 | 234.7824 | 222.7477 | 85.7302 | 101.9153 | 194.9430 | 223.8902 |
|                          | 1                           | 614.4291 | 416.7934 | 293.9398 | 196.3216 | 188.8401 | 138.7500 | 52.8830 | 235.7429 | 223.9087 | 86.2712 | 102.4222 | 195.6178 | 224.7375 |
|                          | 5                           | 622.5571 | 488.0620 | 300.3344 | 205.5732 | 198.4699 | 105.9158 | 55.6603 | 238.6753 | 232.3588 | 82.7148 | 103.2003 | 191.6737 | 222.0138 |
|                          | 10                          | 654.9862 | 541.4709 | 326.3308 | 233.9611 | 222.5775 | 103.3222 | 60.4464 | 263.8341 | 270.1863 | 84.3826 | 106.2509 | 198.8060 | 236.4139 |
|                          | 0.99                        | 2926.8009 | 1939.7028 | 1385.7818 | 930.7545 | 894.3956 | 680.1648 | 226.2364 | 1063.6066 | 1031.5670 | 490.5610 | 388.3974 | 1004.8222 | 1089.1009 |
|                          | 1                           | 2927.0006 | 1969.3180 | 1385.9845 | 933.6852 | 898.4241 | 666.3663 | 226.7199 | 1067.6564 | 1035.2800 | 412.0722 | 390.7977 | 1007.4889 | 1092.2706 |
|                          | 5                           | 2959.4666 | 2210.8292 | 1414.5044 | 971.8431 | 938.4028 | 505.7410 | 246.4459 | 1102.8516 | 1070.2247 | 376.2312 | 436.2568 | 1017.0801 | 1100.7011 |
|                          | 10                          | 3119.7524 | 2572.0106 | 1535.7347 | 1108.7157 | 1054.0945 | 497.7095 | 278.9844 | 1283.1639 | 1254.6899 | 380.3297 | 484.7797 | 1106.7342 | 1229.2880 |
|                          | 0.999                       | 26020.0836 | 18382.5162 | 12070.5841 | 9037.5094 | 8646.2487 | 6631.3230 | 2074.9325 | 9955.2759 | 9938.1800 | 4074.8508 | 2703.3372 | 10325.2734 | 10357.6674 |
|                          | 1                           | 26044.2082 | 18859.4079 | 12090.5235 | 9064.0106 | 8681.5909 | 6482.2998 | 2075.2290 | 10002.0712 | 9984.8118 | 4091.8701 | 2710.7318 | 10357.7464 | 10418.9171 |
|                          | 5                           | 26505.2319 | 21286.4174 | 12462.2558 | 9508.9101 | 9121.2575 | 5013.9659 | 2242.3954 | 10510.2215 | 10491.8444 | 3651.4717 | 3270.2347 | 10607.1266 | 10747.1725 |
|                          | 10                          | 27685.9453 | 24442.7212 | 13413.6787 | 10782.8538 | 10169.5093 | 4884.2933 | 2535.0076 | 12190.7112 | 12176.7742 | 3581.6182 | 3848.4375 | 11736.1248 | 12228.1869 |
Table 17: Estimated MSE for $n = 50$, $p = 10$, and 30% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS | M  | RE  | MRE | M1  | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|-----|----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
|                          | 0.85                        |     |    |     |     |     |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 2.8034 | 0.0418 | 0.6265 | 0.0301 | 0.0296 | 0.0295 | 0.0278 | 0.0279 | 0.0296 | 0.0284 | 0.0278 | 0.0279 | 0.0279 |
|                          | 1                           | 2.8161 | 0.0769 | 0.6433 | 0.0447 | 0.0442 | 0.0436 | 0.0409 | 0.0413 | 0.0442 | 0.0377 | 0.0410 | 0.0411 | 0.0413 |
|                          | 5                           | 3.1386 | 1.0538 | 0.8365 | 0.4012 | 0.3874 | 0.3254 | 0.2883 | 0.3042 | 0.3662 | 0.2372 | 0.2895 | 0.2952 | 0.3015 |
|                          | 10                          | 4.1862 | 3.7763 | 1.3657 | 1.3162 | 1.2622 | 0.9793 | 0.8468 | 0.9418 | 1.1957 | 0.7012 | 0.8548 | 0.8891 | 0.9234 |
|                          | 0.95                        |     |    |     |     |     |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 8.2824 | 0.0457 | 1.8604 | 0.0205 | 0.0191 | 0.0104 | 0.0083 | 0.0085 | 0.0124 | 0.0090 | 0.0083 | 0.0083 | 0.0084 |
|                          | 1                           | 8.3198 | 0.1265 | 1.8804 | 0.0393 | 0.0357 | 0.0123 | 0.0074 | 0.0088 | 0.0242 | 0.0111 | 0.0074 | 0.0077 | 0.0084 |
|                          | 5                           | 9.4177 | 2.9551 | 2.4566 | 0.9021 | 0.8366 | 0.4332 | 0.3113 | 0.5347 | 0.7875 | 0.4501 | 0.3170 | 0.3876 | 0.4582 |
|                          | 10                          | 12.7200 | 11.1474 | 4.1154 | 3.4953 | 3.2611 | 1.8381 | 1.3945 | 2.5522 | 3.0008 | 1.8856 | 1.4409 | 1.8891 | 2.1855 |
|                          | 0.99                        |     |    |     |     |     |       |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 42.1446 | 0.2038 | 8.7628 | 0.0938 | 0.0901 | 0.0757 | 0.0665 | 0.0886 | 0.0802 | 0.0836 | 0.0666 | 0.0721 | 0.0783 |
|                          | 1                           | 42.2714 | 0.6839 | 8.9779 | 0.2669 | 0.2526 | 0.1863 | 0.1555 | 0.2511 | 0.2103 | 0.2115 | 0.1562 | 0.1855 | 0.2105 |
|                          | 5                           | 47.4247 | 15.4249 | 12.0844 | 5.1698 | 4.8172 | 2.9603 | 2.2944 | 4.5383 | 3.3348 | 3.3375 | 2.3533 | 3.7645 | 4.0758 |
|                          | 10                          | 65.1868 | 58.1920 | 21.0159 | 18.9935 | 17.7031 | 10.4142 | 7.9401 | 15.6632 | 11.4621 | 11.6153 | 8.3931 | 14.5739 | 14.8844 |
|                          | 0.999                       |     |    |     |     |     |       |       |       |       |       |       |       |       |       |
|                          | 0.1                         | 428.4407 | 1.6629 | 94.3298 | 0.6042 | 0.5652 | 0.3815 | 0.3048 | 0.4817 | 0.3127 | 0.3931 | 0.3058 | 0.4654 | 0.4918 |
|                          | 1                           | 430.4556 | 6.4901 | 94.8235 | 2.1981 | 2.0435 | 1.2675 | 0.9711 | 1.5786 | 1.1062 | 1.2889 | 0.9778 | 1.7135 | 1.7367 |
|                          | 5                           | 483.3848 | 154.7984 | 122.9701 | 50.4991 | 46.7159 | 27.1745 | 20.1056 | 29.6981 | 24.2687 | 27.0737 | 20.8408 | 38.9822 | 35.5692 |
|                          | 10                          | 657.2009 | 589.6618 | 211.8361 | 190.5493 | 176.2327 | 100.3782 | 74.6610 | 106.6794 | 92.7202 | 100.7135 | 79.9923 | 143.9978 | 128.5776 |
Table 18: Estimated MSE for $n = 100$, $p = 10$, and 30% outlier in the $y$-direction.

| Correlation ($\delta^2$) | Error variance ($\sigma^2$) | OLS    | M     | RE    | MRE   | M1    | TRME1 | MTPM1 | MTPM2 | MTPM3 | TRME2 | MTPM4 | MTPM5 | MTPM6 |
|--------------------------|-----------------------------|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.85                     | 0.1                         | 8.4826 | 0.0394| 5.1948| 0.0276| 0.0271| 0.0258| 0.0189| 0.0194| 0.0231| 0.0146| 0.0189| 0.0191| 0.0193|
|                          | 1                            | 8.4855 | 0.0890| 5.1965| 0.0405| 0.0395| 0.0177| 0.0091| 0.0093| 0.0171| 0.0082| 0.0092| 0.0092| 0.0093|
|                          | 5                            | 8.6780 | 1.8400| 5.3301| 0.5553| 0.5256| 0.1567| 0.0969| 0.1057| 0.3502| 0.0901| 0.0980| 0.1011| 0.1057|
|                          | 10                           | 9.0954 | 6.5097| 5.6027| 1.9497| 1.8389| 0.6384| 0.4447| 0.5022| 1.4624| 0.3945| 0.4512| 0.4702| 0.4969|
| 0.95                     | 0.1                         | 28.4759| 0.0782| 17.2240| 0.0374| 0.0369| 0.0178| 0.0080| 0.0082| 0.0262| 0.0080| 0.0080| 0.0080| 0.0080|
|                          | 1                            | 28.4706| 0.2509| 17.2181| 0.0836| 0.0807| 0.0157| 0.0038| 0.0045| 0.0632| 0.0070| 0.0038| 0.0039| 0.0042|
|                          | 5                            | 28.8961| 6.0107| 17.4879| 1.6654| 1.5710| 0.3755| 0.1987| 0.3657| 1.9400| 0.3453| 0.2040| 0.2518| 0.3237|
|                          | 10                           | 30.4591| 21.5668| 18.5572| 5.9924| 5.6367| 1.5212| 0.9041| 1.9778| 7.7118| 1.4577| 0.9413| 1.2574| 1.6509|
| 0.99                     | 0.1                         | 150.1793| 0.3654| 90.3545| 0.1156| 0.1078| 0.0650| 0.0561| 0.0706| 0.1629| 0.0715| 0.0562| 0.0591| 0.0640|
|                          | 1                            | 150.1569| 1.3263| 90.3260| 0.3740| 0.3491| 0.1522| 0.1194| 0.2077| 0.5651| 0.1684| 0.1199| 0.1366| 0.1627|
|                          | 5                            | 153.0834| 32.0451| 92.3445| 8.6056| 8.0419| 2.4032| 1.5535| 5.8182| 14.0976| 2.5146| 1.5992| 2.9589| 4.1416|
|                          | 10                           | 160.5987| 115.0369| 97.2960| 31.1223| 29.0699| 8.2828| 5.1017| 22.8452| 51.1951| 8.4474| 5.4155| 12.4860| 16.8958|
| 0.999                    | 0.1                         | 1527.9423| 3.1511| 917.9334| 0.8519| 0.8059| 0.1444| 0.0547| 0.6849| 1.4054| 0.1093| 0.0550| 0.1570| 0.3072|
|                          | 1                            | 1529.1376| 12.8061| 918.7779| 3.3980| 3.1954| 0.6557| 0.3171| 3.1369| 5.7548| 0.5610| 0.3201| 1.0239| 1.6817|
|                          | 5                            | 1554.1906| 319.8589| 935.8012| 85.5438| 80.3752| 19.8591| 11.3013| 111.3797| 143.5321| 18.2631| 11.2743| 49.1241| 65.9755|
|                          | 10                           | 1630.1649| 1130.8124| 985.2088| 304.0284| 285.2449| 74.4466| 42.7931| 428.2197| 507.4851| 67.0750| 46.0316| 206.8277| 271.0353|
4. Concluding Remarks

In this article, modified robust ridge M-estimators for two parameter ridge regression model are proposed to overcome the joint problem of multicollinearity and outliers in the y-direction. We proposed six new estimators as an alternate to TRME. A simulation study is conducted to investigate the performance of new estimators on the basis of MSE. The simulation results indicated that the performance of new modified robust ridge M-estimators is better than the other considered estimators. It is also noticed that proposed estimators MTPM1 and MTPM4 and in some cases MTPM3 performed well in the presence of multicollinearity and outliers. The benefits of the new estimators are also shown through the two different numerical examples. Therefore, on the basis of these results, we recommend the use of proposed estimators in the considered scenarios.

Data Availability

Data used in this research were taken from the website available at [21]. All the results reported in this research are carried out on R-environment, a user-friendly statistical analysis tool. Furthermore, research code will be available on request from the corresponding author upon acceptance of this research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest or personal relationships that could have appeared to influence the research work presented in this article.

Authors’ Contributions

S. Yasin conceptualized the study and wrote of the manuscript. S. Sultan and S. Kamal did the critical review. M. Suhail developed the methodology, reviewed and edited the article. Y. A. Khan provided the software and performed validation and formal analysis. S. Sultan and Y. A. Khan wrote the original draft. H. Ayed, S. Sultan, and M. Suhail reviewed and edited the article and performed visualization.

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