Fast calculation of the variance of edge crossings in random linear arrangements

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\textbf{ABSTRACT}

The interest in spatial networks where vertices are embedded in a one-dimensional space is growing. Remarkable examples of these networks are syntactic dependency trees and RNA structures. In this setup, the vertices of the network are arranged linearly and then edges may cross when drawn above the sequence of vertices. Recently, two aspects of the distribution of the number of crossings in uniformly random linear arrangements have been investigated: the expectation and the variance. While the computation of the expectation is straightforward, that of the variance is not. Here we present fast algorithms to calculate that variance in arbitrary graphs and forests. As for the latter, the algorithm calculates variance in linear time with respect to the number of vertices. This paves the way for many applications that rely on an exact but fast calculation of that variance. These algorithms are based on novel arithmetic expressions for the calculation of the variance that we develop from previous theoretical work.

\textit{Keywords:} graph; edge crossing; random linear arrangement; variance of crossings.

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1. Introduction

The study of spatial networks, networks whose vertices are embedded in some space, is experiencing a renaissance since the early studies of geographers on transportation networks in the 1960s-1970s [1]. Prototypical examples of spatial networks are streets, roads and transportation networks (e.g., subway and train networks). These are spatial networks on a space that is usually assumed to be two-dimensional. Such renaissance is mostly due to the development of the theory of spatial network in non-Euclidean geometries [1]. However, the growing interest in networks whose vertices are embedded in a one-dimensional Euclidean space cannot be neglected. Remarkable examples are syntactic dependency networks and RNA structures. As for the former, the syntactic structure of a sentence can be defined as a network where vertices are words and edges indicate syntactic dependencies (Figure 1). Syntactic dependency structures have become the de facto standard to represent the syntactic structure of sentences in computational linguistics [2] and the fuel for many quantitative studies [3]. In that setup, edges may cross when drawn above of the sentence (Figure 1). The ordering of words in the sentence defines a linear arrangement of the vertices rla, hereafter, a linear arrangement. Crossings may also occur in linear arrangements of RNA secondary structures, where vertices are nucleotides A, G, U, and C, and edges are Watson-Crick (A-U, G-C) and (U-G) base pairs [4].

By construction of the network, crossings are practically impossible in street networks [5]. Crossings are scarce in road networks and imply bridges and tunnels [6]. In syntactic dependency networks, $C$, the number of crossings, has been shown to be
low with respect to random linear arrangements of the words of the sentences [7] and predictable to a large extent by the Euclidean distance between syntactically related words: crossings are more likely for dependencies involving distant words, for the range of distances that is typically found in real sentences [8].

In a previous work two statistical properties of the distribution of $C$ in some random layout * of the vertices of a general graph have been investigated, i.e., the expectation $E_\ast [C]$ and the variance $V_\ast [C]$ [10]. In addition, the particular case of $E_{rla} [C]$ and $V_{rla} [C]$, the variance and expectation of $C$ in uniformly random linear arrangements (hereafter random linear arrangements), has also been investigated. These layouts have to satisfy three elementary properties. Firstly, that a pair of edges can only cross once. Secondly, that a pair of edges that share vertices cannot cross. And, thirdly, $e$ edges crossing at the same point incur in $\binom{e}{2}$ crossings. The notation * for random layout includes both the physical space in which the graph’s vertices are embedded and the distribution of the vertex positions. An example of the first are linear arrangements of vertices, and an example of the second are uniformly random linear arrangements.

These two properties allow one to standardize real values of $C$ using a $z$-score, defined as

$$z = \frac{C - E_\ast [C]}{\sqrt{V_\ast [C]}}.$$ 

$z$-scores have been used to detect scale invariance in empirical curves [11, 12] or motifs in complex networks [13] (see [14] for a historical overview). Thus, $z$-scores of $C$ would allow one to discover new statistical patterns involving $C$ in syntactic dependency trees, to name one example. Moreover, $z$-scores of $C$ can help aggregate or compare values of $C$ from graphs with different structural properties (number of vertices, degree distribution,...), as it happens with syntactic dependency trees, when calculating the average number of crossings in collections of sentences of a given language [7].

A prerequisite for this new research avenue are algorithms that allow one to calculate $E_\ast [C]$ and $V_\ast [C]$ exactly and efficiently. In this work we provide an arithmetic expression for $V_\ast [C]$ from which efficient algorithms can be derived to compute its exact value. We apply it to derive concrete algorithms to compute the exact value of $V_{rla} [C]$, which can be used to compute $V_{rla} [C]$ straightforwardly. The need of computing the exact value of $V_{rla} [C]$ efficiently arises from the increasingly large collections of syntactic dependency structures that are available [15]. These efficient algorithms
Figure 2.: The subgraphs of each type $\omega \in \Omega$ (Equation 3) is indicated below them. Each type is an element of $Q \times Q$. Equally-colored edges belong to the same element of $Q$. Bi-colored edges (as in types 12, 13 and 24) denote an edge of the two elements of $Q$ of the type.

would also allow one to improve tests for the significance of the real value of $C$ with respect to random linear arrangements which are based on a Monte Carlo estimation of the $p$-value [7]. Using such algorithms, an upper bound of the $p$-value could be obtained immediately using Chebyshev-like inequalities, that require knowledge of $\mathbb{E}_{rla} [C]$ and $\mathbb{V}_{rla} [C]$. If the $p$-value was below the significance level, the null hypothesis could be rejected quickly, skipping the time-consuming estimation of the $p$-value.

On the positive side, the calculation of $\mathbb{E}_s [C]$ is straightforward. It has been shown that $\mathbb{E}_s [C] = q \delta_s$, where $\delta_s$ is the probability that two independent edges cross in the given layout and $q$ is the number of independent pairs of edges, namely, the number of pairs of edges that do not share vertices [10]. $q$ can be computed in constant time given $n$, the number of vertices of the networks, $m$, its number of edges, $\langle k^2 \rangle$, the second moment of degree about zero (i.e. the mean of squared vertex degrees) because

$$ q = \frac{1}{2} \left( m(m + 1) - n \langle k^2 \rangle \right). $$

On the negative side, the calculation of $\mathbb{V}_s [C]$ is more complex [10]. Fortunately, its computation reduces to a subgraph counting problem [16, Chapter 4]. The subgraphs to count are shown in Figure 2. Then, $\mathbb{V}_s [C]$ in an arbitrary graph $G$ becomes

$$ \mathbb{V}_s [C] = \sum_{\omega \in \Omega} a_\omega n_G (F_\omega) \mathbb{E}_s (\gamma_\omega), $$

where $\Omega$ is a set of subgraphs indexed by two or three digits,

$$ \Omega = \{00, 01, 021, 022, 03, 04, 12, 13, 24\}, $$

$a_\omega$ is an integer constant, and $n_G (F_\omega)$ is the number of times the subgraph $F_\omega$ appears in the graph. $F_\omega$ can be a connected graph or disjoint unions of connected graphs. The values of $a_\omega$, $F_\omega$ and $\mathbb{E}_{rla} (\gamma_\omega)$ are summarized in Table 1. In sum, $\mathbb{V}_s [C]$ is expressed as a function of the number of subgraphs of each kind that appear in Table 1. $n_G (F_\omega)$ is the only graph-dependent term, and $\mathbb{E}_s (\gamma_\omega)$ is the only layout-dependent term. The constant $a_\omega$ depends only on the type.

Equation 2 can be seen as a particular case of the general equation to define a
Table 1.: The values of $\omega$, $F_\omega$ and $E_{rta}[\gamma_\omega]$ as function of $\omega$. $L_n$ and $C_n$ denote linear trees (or path graphs) and cycle graphs of $n$ vertices, respectively, and the operator $\oplus$ indicates the disjoint union of graphs.

| $\omega \in \Omega$ | $a_\omega$ | $F_\omega$ | $E_{rta}[\gamma_\omega]$ |
|---------------------|-----------|-----------|---------------------|
| 00                  | 6         | $L_2 \oplus L_2 \oplus L_2 \oplus L_2$ | 0                   |
| 24                  | 1         | $L_2 \oplus L_2$                | 2/9                 |
| 13                  | 2         | $L_3 \oplus L_2$                | 1/18                |
| 12                  | 6         | $L_2 \oplus L_2 \oplus L_2$    | 1/45                |
| 04                  | 2         | $C_4$                             | -1/9                |
| 03                  | 2         | $L_5$                             | -1/36               |
| 021                 | 2         | $L_4 \oplus L_2$                | -1/90               |
| 022                 | 4         | $L_3 \oplus L_3$                | 1/180               |
| 01                  | 4         | $L_3 \oplus L_2 \oplus L_2$    | 0                   |

molecular property $P$ of a graph $G$ as a summation over all subgraphs $g$, i.e. [17, 18]

$$P(G) = \sum_F \eta_F n_G(F)$$

where $\eta_F$ is the contribution of graph $F$ to the molecular property. In our case, we have that $\eta_F = a_\omega E_{s}[\gamma_\omega]$ if $F$ is in Figure 2 and $\eta_F = 0$ otherwise (in our case the summation is not restricted to connected subgraphs).

In this article we aim to develop fast algorithms to calculate the exact value of $V_{s}[C]$ in arbitrary graphs as well as ad hoc algorithms for forests. An specific algorithm for forests is useful because in RNA secondary structures the graphs are such that the maximum vertex degree is 1 [4] and are thus disconnected and acyclic. Moreover, forests are a straightforward generalization of trees and are the kind of graphs that are typically found in syntactic dependency structures. By providing algorithms for forests or more general graphs we are accommodating all the possible exceptions and variants that have been discussed to define the syntactic structure of sentences, e.g., allowing for cycles (see for instance, [19, Section 4.9 Graph-theoretic properties]). In addition, the syntactic structures in recent experiments with deep agents are forests [20].

The present article is a piece of a broader research program on the statistical properties of network measures on linear arrangements. Recently, the distribution of $D$, the sum of edge distances in random linear arrangements, has been investigated [21], and so has been the distribution of $C$ [10]. While the calculation of $E_{rta}[D]$ and $V_{rta}[D]$ is straightforward, the efficient calculation of $V_{rta}[C]$, and that of $V_s[C]$, requires further investigation. The present article can be seen as a continuation of the previous theoretical analysis [10] into the realm of practical calculation of $V_s[C]$. Here we take a step towards this direction by providing an algorithm that is more efficient than a naïve implementation based on a direct implementation of results in [10], and showing that $V_s[C]$ in forests is linear-time computable in their number of vertices.

The remainder of the article is organized as follows. Section 2 provides formal definitions, reviews the theoretical arguments allowing one to express $V_s[C]$ as in Equation
2 and outlines simple (yet inefficient) algorithms to calculate the exact value of $V_s[C]$. In Section 3 we provide arithmetic expressions for $f_\omega = a_\omega n_G(F_\omega)$ that are then used to obtain a general arithmetic expression for $V_s[C]$ via Equation 2. In Section 4, such a general expression is then translated into algorithms to calculate $V_s[C]$ in general graphs and forests. Finally, Section 5 discusses our findings and their implications, and suggests future work.

2. Theoretical background

Consider a graph $G = (V,E)$ of $n = |V|$ vertices and $m = |E|$ edges whose vertices are arranged with a function $\pi_s$ that indicates the position of every vertex in the layout. For example, in the one-dimensional layout, i.e., a linear arrangement, $\pi_{la} : V \to [n]$ is a function that indicates the position of every vertex in the linear sequence of length $n$. Throughout this article we use letters $s,t,...,z$ to indicate distinct vertices. Now we give an example of crossing in the one-dimensional layout. Consider four vertices $s,t,u,$ and $v$ forming edges $\{s,t\}$ and $\{u,v\}$. Without any loss of generality, suppose that in a linear arrangement $s$ precedes $t$, i.e. $\pi_{la}(s) < \pi_{la}(t)$, and $u$ precedes $v$, i.e. $\pi_{la}(u) < \pi_{la}(v)$. Their edges cross if and only if one of the two following conditions is met

$$\pi_{la}(s) < \pi_{la}(u) < \pi_{la}(t) < \pi_{la}(v), \text{ or}$$
$$\pi_{la}(u) < \pi_{la}(s) < \pi_{la}(v) < \pi_{la}(t).$$

Notice that whether edges are directed as in Figure 1 or not as in RNA secondary structures [4] is irrelevant for the definition of crossing. Then, $C$ is defined as the number of edge crossings in a given layout, and $E_s[C]$ and $V_s[C]$ denote, respectively, its expectation and its variance. $C$ can be defined by making explicit the dependence on a particular arrangement as

$$C(\pi_s) = \sum_{\{e_1,e_2\} \in Q} \alpha_{\pi_s}(e_1,e_2),$$

where $\alpha_{\pi_s}(e_1,e_2) = \alpha(e_1,e_2)$ is an indicator random variable equal to 1 if, and only if the edges $e_1$ and $e_2$ cross in the given arrangement $\pi_s$ embedded in the layout under consideration, and $Q$ is defined as the set of pairs of independent edges (i.e. the set of pairs of edges that may potentially cross [10]). Throughout this article we use $q = |Q|$ (Equation 1).

Thanks to recent work, the exact value of $V_s[C]$ can be computed in $O(poly(n))$ [10], which reduces its calculation to a subgraph counting problem (Equation 2). A summary of this previous work is presented next.

By definition of a random variable’s variance,

$$V_s[C] = E_s[(C - E_s[C])^2].$$

Noting that

$$C - E_s[C] = \sum_{\{e_1,e_2\} \in Q} \beta(e_1,e_2)$$



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with
\[
\beta(e_1, e_2) = \alpha(e_1, e_2) - \mathbb{E}_* [\alpha(e_1, e_2)] \\
= \alpha(e_1, e_2) - \delta_*,
\]
where \(\delta_* = \mathbb{E}_* [\alpha(e_1, e_2)]\) is the probability that two independent edges cross in a given layout since \(\alpha\) is an indicator random variable, we can express \(V_* [C]\) as
\[
V_* [C] = \mathbb{E}_* \left[ \left( \sum_{\{e_1, e_2\} \in Q} \beta(e_1, e_2) \right)^2 \right].
\]
Expanding the square in the previous expression, \(V_* [C]\) can be decomposed into a sum of \(|Q \times Q| = |Q|^2\) summands of the form \(\mathbb{E}_* [\beta(e_1, e_2) \beta(e_3, e_4)]\), i.e. \([10]\)
\[
V_* [C] = \mathbb{E}_* \left[ \sum_{\{e_1, e_2\} \in Q} \sum_{\{e_3, e_4\} \in Q} \beta(e_1, e_2) \beta(e_3, e_4) \right] \]
\[
= \sum_{\{e_1, e_2\} \in Q} \sum_{\{e_3, e_4\} \in Q} \mathbb{E}_* [\beta(e_1, e_2) \beta(e_3, e_4)].
\]
Each of the products \(\beta(e_1, e_2) \beta(e_3, e_4)\) can be classified into a type \(\omega\) based on the edges \(st, uv, wx\) and \(yz\). There are only 9 types which are labeled using codes of two or three digits. The set of all the types is \(\Omega\) (Equation 3). The features of each type of product are summarized in Table 2.
As a result of the classification above, \(V_* [C]\) can be expressed compactly as
\[
V_* [C] = \sum_{\omega \in \Omega} f_\omega \mathbb{E}_* [\gamma_\omega],
\]
where \(\mathbb{E}_* [\gamma_\omega] = \mathbb{E}_* [\beta(e_1, e_2) \beta(e_3, e_4)]\) when \(T(\{e_1, e_2\}, \{e_3, e_4\}) = \omega\), namely, when the type of the product \(\beta(e_1, e_2) \beta(e_3, e_4)\) is \(\omega\), and \(f_\omega\) is the number of products of type \(\omega\), defined as
\[
f_\omega = \sum_{\{e_1, e_2\} \in Q} \sum_{\{e_3, e_4\} \in Q} \mathbb{E}_* [\gamma_{\omega}] \quad \text{when} \quad T(\{e_1, e_2\}, \{e_3, e_4\}) = \omega.
\]
In addition,
\[
\mathbb{E}_* [\gamma_\omega] = p_{*,\omega} - \delta_*^2,
\]
where \(p_{*,\omega} = \mathbb{E}_* [\alpha(e_1, e_2) \alpha(e_3, e_4)]\) when \(T(\{e_1, e_2\}, \{e_3, e_4\}) = \omega\). In random linear arrangements, \(\delta_{rla} = 1/3\) \([10]\), and in uniformly random spherical arrangements, \(\delta_{rsa} = 1/8\) \([22]\). In the latter layout, the vertices of the graph are placed on the surface of a sphere uniformly at random, and each edge of the graph becomes the geodesic between each of its corresponding vertices. It was shown in \([10]\) that \(p_{*,00} = p_{*,01} = \delta_*^2, p_{*,24} = \delta_*\), and \(p_{rla,04} = p_{rsa,04} = 0\).
Table 2.: The classification of the elements of $Q \times Q$ into types of products abstracting away from the order of the elements of the pair. $\omega \in \Omega$ is the code that identifies the product type; these codes have two (or three digits) that results from concatenating $\tau$ and $\phi$ (except for types 021-022, where a third digit is required). $(\{e_1, e_2\}, \{e_3, e_4\})$ is an element of $Q \times Q$ where the symbols $s, t, u, v, x, y, z$ indicate distinct vertices, $|v|$ is the number of different vertices of the type, $\tau$ is the size of the intersection between $\{e_1, e_2\}$ and $\{e_3, e_4\}$, $\phi$ is the number of edge intersections, $p_{rla, \omega}$ is the probability that $\alpha(e_1, e_2)\alpha(e_3, e_4) = 1$ in a uniformly random permutation of all the vertices of the graph, and $E_{rla}[\gamma_\omega] = p_{rla, \omega} - \delta_{rla}^2$, where $\delta_{rla} = 1/3$. Source [10, Table 2, p. 18].

| $\omega \in \Omega$ | $(\{e_1, e_2\}, \{e_3, e_4\})$ | $|v|$ | $\tau$ | $\phi$ | $p_{rla, \omega}$ | $E_{rla}[\gamma_\omega]$ |
|-------------------|---------------------------------|-----|-----|-----|-----------------|------------------|
| 00                | $(\{st, uw\}, \{wx, yz\})$    | 8   | 0   | 0   | 1/9             | 0                |
| 24                | $(\{st, uv\}, \{st, uw\})$    | 4   | 2   | 4   | 1/3             | 2/9              |
| 13                | $(\{st, uv\}, \{st, uw\})$    | 5   | 1   | 3   | 1/6             | 1/18             |
| 12                | $(\{st, uv\}, \{st, wx\})$    | 6   | 1   | 2   | 2/15            | 1/45             |
| 04                | $(\{st, uv\}, \{su, tv\})$    | 4   | 0   | 4   | 0               | -1/9             |
| 03                | $(\{st, uv\}, \{su, vw\})$    | 5   | 0   | 3   | 1/12            | -1/36            |
| 021               | $(\{st, uv\}, \{su, wx\})$    | 6   | 0   | 2   | 1/10            | -1/90            |
| 022               | $(\{st, uv\}, \{sw, ux\})$    | 6   | 0   | 2   | 7/60            | 1/180            |
| 01                | $(\{st, uv\}, \{sw, xy\})$    | 7   | 0   | 1   | 1/9             | 0                |

Interestingly, $f_\omega$ is proportional to the amount of subgraphs of type $\omega$, i.e.

$$f_\omega = a_\omega n_G(F_\omega),$$

(7)

where $a_\omega$ is an integer constant and $n_G(F_\omega)$ is the number of occurrences of the subgraph $F_\omega$, in the graph under consideration (Table 1).

Figure 2 relates each type of product with the subgraph that has to be counted within the graph so as to obtain the exact value of $V_\ast[C]$. These graphs may be elementary graphs, a simple linear tree of a fixed number of vertices, e.g., $L_4$ or $L_5$, a cycle graph, e.g., $C_4$, or a combination with the operator $\oplus$, the disjoint union of graphs.

Table 1 allows one to obtain $f_\omega$ via Equation 7. In [10], these expressions were used to obtain arithmetic expressions for $f_\omega$ in certain types of graphs, which were in turn used to obtain an arithmetic expression for $V_{rla}[C]$ in those classes of graphs, e.g., complete graphs, complete bipartite graphs, cycle graphs, etc. Such arithmetic expressions depend only on the number of vertices of the graph. In arbitrary graphs, we can still use Table 1 to derive an $O(m^4)$-time algorithm to calculate the exact value of $V_\ast[C]$. Letting, $E(G)$ be the set of edges of a graph $G$, the algorithm consists of calculating $n_G(F_\omega)$ by making all subsets of $|E(F_\omega)|$ edges and checking, for each subset, if it matches $F_\omega$. This is a naive algorithm for being a direct implementation of theoretical results. Here we develop algorithms with lower asymptotic complexity for general graphs and forests. Table 3 summarizes the cost of the first approximations discussed here and that of the algorithms that are presented in subsequent sections.
Table 3.: Summary of the time and space complexity algorithms devised for computing $V^*_C$ for general graphs and forests, where $n$ is the number of vertices, $k_{max}$ is the maximum degree, $\langle k^2 \rangle$ is the second moment of degree about zero, $n_G(F)$ is the number of subgraphs isomorphic to $F$ in $G$ and $T = n_G(C_3)/n_G(L_3)$ is the so-called transitivity index [16, Chapter 4.5.1].

| Algorithm         | Time                | Space     |
|-------------------|---------------------|-----------|
| Naive algorithm   | $O(m^4)$            | $O(1)$    |
| Algorithm 4.2     | $O(k_{max}n\langle k^2 \rangle)$ | $O(n)$    |
| (general graphs)  | $o(nm^2)$           |           |
| Algorithm C.2     | $O(n + k_{max}(n\langle k^2 \rangle - n_G(C_3)))$ | $O(n\langle k^2 \rangle - n_G(C_3))$ |
| (general graphs)  |                     |           |
| Algorithm 4.3     | $O(n)$              | $O(n)$    |
| (forests)         |                     |           |

3. An arithmetic expression for $V^*_C$

In this work we further develop the algebraic expressions of the $f_\omega$’s for $\omega \in \Omega$ (Equation 3) because we focus on the $\omega$’s that actually contribute to $V^*_C$ (Table 2). We use the formalization of the $f_\omega$’s in [10] as a starting point to obtain an arithmetic expression for $V^*_C$ in general graphs. These formalizations were originally used in [10] to obtain the expressions of the form of Equation 7.

In Section 3.1, we first describe the notation that we use in later sections to obtain compact expressions. Then, in Section 3.2, we present general results that relate summations over the set $Q$ with subgraph countings that we use, later in Section 3.3, to obtain arithmetic expressions for the $f_\omega$’s. The latter are used to obtain an expression for $V^*_C$ in Section 3.4.

3.1. Preliminaries

We define $A^p = \{a_{st}^{(p)}\}$ as the $p$-th power of the adjacency matrix $A = \{a_{ij}\}$ of the graph ($A = A^1$), and $m_p$ the sum of one of half of the values of $A^p$ excluding the diagonal, i.e.

$$m_p = \sum_{s < t} a_{st}^{(p)}.$$

Notice that $m_1 = m$. The $p$-th moment of degree about zero is defined as

$$\langle k^p \rangle = \frac{1}{n} \sum_{s=1}^{n} k_s^p. \quad (8)$$
Figure 3.: The effect of removing vertices $s, t, u, v$ such that \{st, uv\} $\in$ $Q$ from a graph $G$ to produce the graph $G_{-stuv}$. Solid thick lines indicate the edges formed by these vertices in $G$, solid thin lines indicate edges in $G_{-stuv}$ and dashed lines indicate potential edges, namely that may not exist, between vertices $s, t, u, v$ and vertices in $G_{-stuv}$. Source [10, Figure 7, p. 24].

We define the sum of the degrees of the neighbors of a vertex $s$ as

$$\xi(s) = \sum_{t \in V : t \in \Gamma(s)} k_t = \sum_{st \in E} k_t,$$

(9)

the neighborhood intersection of two vertices $s$ and $t$

$$c(s, t) = \Gamma(s) \cap \Gamma(t),$$

(10)

and the sum of the degrees of the vertices in $c(s, t)$ as

$$S_{s,t} = \sum_{u \in c(s,t)} k_u.$$

(11)

Notice that if $s = t$ then $S_{s,t} = \xi(s) = \xi(t)$. We also define the sum of the product of degrees at both ends of an edge as

$$\psi = \sum_{st \in E} k_s k_t,$$

(12)

which is involved in the calculation of degree correlations [23]. $\xi(s)$, $c(s, t)$, $S_{s,t}$ and $\psi$ prove useful in making the expressions of the $f_o$’s compact as well as in deriving the algorithms to compute the exact value of $V_s[C]$.

For any undirected simple graph $G = (V, E)$, let $G_{-s}$ be the induced graph resulting from removing vertex $s$ from $G$. More generally, we define $G_{-L}$ as the induced graph resulting from the removal of the vertices in $L \subseteq V$. Unless stated otherwise, we use $Q = Q(G)$, to refer to the set of pairs of independent edges of $G$.

We denote $G_{-\{s,t,u,v\}}$ simply as $G_{-stuv}$, illustrated in Figure 3. The number of edges in a graph $G_{-stuv}$ is easy to calculate as a function of the number of edges in $G$. If $s, t, u, v$ are four distinct vertices, then

$$|E(G_{-stuv})| = m - (k_s + k_t + k_u + k_v) + a_{st} + a_{su} + a_{sv} + a_{tu} + a_{tv} + a_{uv}.$$

(13)
By default, network features refer to $G$. Therefore, $m$, $k_u$, $a_{uv}$, ... in Equation 13 refer to $G$, and not to $G_{-stuv}$. We use $\Gamma(s, -L) = \Gamma(s) \setminus L$ to denote the set of neighbors of vertices $s \in L \subseteq V$ in $V(G_{-L})$. Its size is:

$$|\Gamma(s, -L)| = k_s - |\{w \in L \mid \{w, s\} \in E\}| = k_s - \sum_{w \in L} a_{sw}. \quad (14)$$

We use $\Gamma(k, -\{s, t, u, v\})$ as an abbreviation for $\Gamma(k, -stuv)$, with $k \in \{s, t, u, v\}$.

Throughout this paper, the term $n$-path refers to a sequence of $n$ pairwise distinct vertices $v_0v_1 \cdots v_{n-1}$ such that $v_i;v_{i+1} \in E$. We consider $v_0v_1 \cdots v_{n-1}$ and $v_{n-1} \cdots v_1v_0$ to be two different paths. Lastly, we use $n_G(F)$ to denote the amount of subgraphs isomorphic to $F$ in $G$.

The calculations of the $f_\omega$’s to come require a clear notation that states the vertices shared between each pair of elements of $Q$ for an arbitrary graph $G$. Throughout this article we need to use summations of the form

$$\sum_{s,t,u,v \in V: \{st,uv\} \in Q(G)} \sum_{w,x,y,z \in V: \{wx,yz\} \in Q(G_{-stuv})} \Box,$$

where below each summation operand there is a scope on top of a condition. The “$\Box$” represents any term. For the sake of brevity, we contract them as

$$\sum_{(stuv) \in Q} \sum_{wxyz \in Q(G_{stuv})} \Box.$$

Notice that the scope is omitted in the new notation. This detail is crucial for the countings performed with the help of these compact summations. Likewise, if we want to denote when two elements of $Q$ from each of the summations share one or more vertices, we use:

$$\sum_{\{stuv\} \in Q} \sum_{\{sx,tz\} \in Q(G_{-u})} \Box = \sum_{s,t,u,v \in V: \{stuv\} \in Q(G)} \sum_{x,z \in V: \{sx,tz\} \in Q(G_{-u})} \Box.$$

This expression denotes the summation over the pairs of elements of $Q$ in which the second one shares two vertices with the first one. Again, the expression to the left is a shorthand for the one to the right. For the sake of comprehensiveness, we also present two more compact definitions:

$$\sum_{\{stuv\} \in Q} \sum_{\{sv,yz\} \in Q(G_{-tu})} \Box = \sum_{s,t,u,v \in V: \{stuv\} \in Q(G)} \sum_{y,z \in V: \{sv,yz\} \in Q(G_{-tu})} \Box,$$

$$\sum_{\{stuv\} \in Q} \sum_{\{sv,yz\} \in Q(G_{-tu})} \Box = \sum_{s,t,u,v \in V: \{stuv\} \in Q(G)} \sum_{y,z \in V: \{sv,yz\} \in Q(G_{-tu})} \Box.$$
Figure 4.: Two simple graphs. (a) the paw graph $Z$, (b) the $Y = C_3 \oplus L_2 = \overline{K}_{2,3}$ graph.

### 3.2. General results

Here we introduce some results on summations over $Q$, that prove to be very useful, in Section 3.3, to simplify expressions on the $f_\omega$’s, that are initially defined as summations over pairs of elements of $Q$ (Equation 5). These expressions pave the way towards an efficient computation of $\nabla_s [C]$, and hence of $\nabla_{rla} [C]$. The proof of each proposition can be found in Appendix A.

The first two relate the amount of $L_4$ and $L_5$ to elements of $Q$.

**Proposition 1.** The number of subgraphs isomorphic to $L_4$, namely half the amount of paths of 4 vertices, in a graph $G$ is

$$
n_G(L_4) = \sum_{\{st,uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv})$$

$$= \frac{1}{2} \sum_{s=1}^{n} \sum_{t=1}^{n} (a_{st}^{(3)} - a_{st}(2k_t - 1))$$

$$= m_3 + m_1 - n(k^2).$$

**Proposition 2.** The number of subgraphs isomorphic to $L_5$, namely half the amount of paths of 5 vertices, in a graph $G$ is

$$
n_G(L_5) = \sum_{\{st,uv\} \in Q} \left( \sum_{w_s \in \Gamma(s,-stuv)} (a_{uw_s} + a_{vw_s}) + \sum_{w_t \in \Gamma(t,-stuv)} (a_{uw_t} + a_{vw_t}) \right).$$

Similarly, in the next two propositions we relate the number of subgraphs isomorphic to the paw graph, and to $C_3 \oplus L_2$.

**Proposition 3.** The number of subgraphs isomorphic to the paw graph (Figure 4(a)), denoted as $Z$, in a graph $G$ is

$$n_G(Z) = \sum_{\{st,uv\} \in Q} (a_{su} + a_{tv})(a_{sv} + a_{tu}).$$

**Proposition 4.** The number of subgraphs isomorphic to $Y = C_3 \oplus L_2$ (Figure 4(b))
in a graph $G$ is

$$n_G(Y) = \frac{1}{3} \sum_{\{st,uv\} \in Q} \left( \sum_{w_s \in \Gamma(s, -stuv)} a_{tw_s} + \sum_{w_u \in \Gamma(u, -stuv)} a_{vw_u} \right). \quad (20)$$

The next proposition relates the number of cycles of four vertices in a graph with its set of independent edges $Q$.

**Proposition 5.** The number of cycles of 4 vertices, namely $C_4$, in a graph $G$ is

$$n_G(C_4) = \frac{1}{2} \sum_{\{st,uv\} \in Q} (a_{sv}a_{tu} + a_{su}a_{tv})$$

$$= \frac{1}{8} \left[ tr(A^4) - 4n_G(L_3) - 2n_G(L_2) \right]$$

$$= \frac{1}{8} \left[ tr(A^4) + 4q - 2m^2 \right]. \quad (21)$$

There is another useful result regarding the sum of the degrees of all vertices involved in the elements in $Q$.

**Proposition 6.** In any graph $G$,

$$K = \sum_{\{st,uv\} \in Q} (k_s + k_t + k_u + k_v) = n[(m + 1)\langle k^2 \rangle - \langle k^3 \rangle] - 2\psi \quad (23)$$

where $\langle k^p \rangle$ and $\psi$ are defined in Equations 8 and 12.

Furthermore,

**Proposition 7.** In any graph $G$,

$$\Phi_1 = \sum_{\{st,uv\} \in Q} (k_s k_t + k_u k_v) = (m + 1)\psi - \sum_{st \in E} k_s k_t (k_s + k_t) \quad (24)$$

where $\psi$ is defined in Equation 12.

Finally,

**Proposition 8.** In any graph $G$,

$$\Phi_2 = \sum_{\{st,uv\} \in Q} (k_s + k_t)(k_u + k_v)$$

$$= \frac{1}{2} \sum_{st \in E} \left[ (k_s + k_t) \left( n\langle k^2 \rangle - (\xi(s) + \xi(t)) - k_s(k_s - 1) - k_t(k_t - 1) \right) \right], \quad (25)$$

where $\langle k^2 \rangle$ and $\xi(s)$ are defined in Equations 8 and 9.
Table 4.: Expressions for the $f_\omega$'s linking them to graph theory as a function of $q$ and/or other network features. $q = |Q|$ is the number of pairs of independent edges of a graph $G = (V, E)$, $n = |V|$, $m = |E|$, $K$ is defined in Equation 23, $\Phi_1$, $\Phi_2$, $\Lambda_1$ and $\Lambda_2$ are defined in Equations 24, 25, 34 and 38. $Z$ and $Y$ denote the graphs depicted in Figure 4. $L_n$ and $C_n$ denote linear trees (path graphs) and cycle graphs of $n$ vertices, respectively.

\[
\begin{align*}
  f_{24} &= q & 26 \\
  f_{13} &= K - 4q - 2n_G(L_4) & 28 \\
  f_{12} &= 2[(m + 2)q + n_G(L_4) - K] & 30 \\
  f_{04} &= 2n_G(C_4) \\
    &= \frac{1}{2}tr(A^4) - n_G(L_3) - \frac{1}{2}n_G(L_2) & 32 \\
    &= \frac{1}{2}tr(A^4) + q - \frac{1}{2}m^2 \\
  f_{03} &= \Lambda_1 - 2n_G(L_4) - 8n_G(C_4) - 2n_G(Z) & 35 \\
  f_{021} &= 2q + (m + 5)n_G(L_4) + 8n_G(C_4) + 3n_G(Z) + \Phi_1 \\
    &\quad - n_G(Y) - \Lambda_1 - \Lambda_2 - K & 42 \\
  f_{022} &= 4q + 5n_G(L_4) + 2n_G(Z) + 4n_G(C_4) + \Phi_2 - \Lambda_2 - 2K - n_G(L_5) & 44
\end{align*}
\]

3.3. Theoretical formulae

In the following subsections, we obtain general expressions for the $f_\omega$'s based on the formalization given in [10]. These expressions are designed based on three non-exclusive principles: easing the computation of $V^*_n[C]$, linking with standard graph theory and linking with the recently emerging subfield of crossing theory for linear arrangements ([10, Section 2] and also [8]). In [10], the expressions for the $f_\omega$'s were linked with graph theory via (recall Section 2)

\[ f_\omega = a_\omega n_G(F_\omega). \]

In the coming subsections, we derive simpler arithmetic expressions for the $f_\omega$'s to help derive arithmetic expression for $V^*_n[C]$ (Section 3.4). Accordingly, we focus our work on the $f_\omega$'s that actually contribute to $V^*_n[C]$, namely those $f_\omega$ with $\omega \neq 00, 01$ because $E_\omega(\gamma_{00}) = E_\omega(\gamma_{01}) = 0$ [10]. An overview of the expressions that are derived for the $f_\omega$'s is shown in Table 4.

We use the same approach for the calculations in all the $f_\omega$'s in the coming subsections. It consist of further developing the algebraic formalizations of the $f_\omega$ given in [10], which are briefly summarized below in this article to make it more self-contained. In Appendix B, we explain the tests employed to ensure the correctness of said expressions.
3.3.1. $\tau = 2$, $\phi = 4$

It was already shown in [10] that

$$f_{24} = \sum_{\{st,uv\} \in Q} \sum_{w,x,y,z \in V: w = s, x = t, y = u, z = v} 1 = \sum_{\{st,uv\} \in Q} 1 = q. \quad (26)$$

3.3.2. $\tau = 1$, $\phi = 3$

This type deals with the pairs of edges sharing exactly one edge ($\tau = 1$) and that have 3 vertices in common ($\phi = 3$). Via Equation 5, a possible formalization of $f_{13}$ is [10]

$$f_{13} = \sum_{\{st,uv\} \in Q} \left( \sum_{\{st,uw\} \in Q(G_{-v})} 1 + \sum_{\{st,vw\} \in Q(G_{-u})} 1 + \sum_{\{uv,sw\} \in Q(G_{-t})} 1 + \sum_{\{uv,tw\} \in Q(G_{-s})} 1 \right). \quad (27)$$

The first inner summation in the previous equation denotes the amount of neighbors of $u$ in $G$ that are not $s, t, v$, the second the amount of neighbors of $v$ in $G$ that are not $s, t, u$, and so on (Figure 5 for an illustration). In that figure, and similar figures to follow, solid thick lines indicate existing edges, solid thin lines indicate real edges of a hypothetical graph and dashed lines indicate potential edges, namely edges that may not exist. Given $\{st, uv\} \in Q$, the first inner summation turns out to be [10]

$$\sum_{\{st,uw\} \in Q(G_{-v})} 1 = |\Gamma(u, -stuv)|.$$  

Likewise for the other inner summations. Then, Equation 27 becomes [10]

$$f_{13} = \sum_{\{st,uv\} \in Q} |\Gamma(u, -stuv)| + \sum_{\{st,uv\} \in Q} |\Gamma(v, -stuv)| \quad (27)$$

A more meaningful expression for $f_\omega$ is obtained applying

$$|\Gamma(u, -stuv)| = k_u - (a_{uv} + a_{us} + a_{ut})$$

into Equation 27 (recall that $\{st, uv\} \in Q$ implies $a_{st} = a_{uv} = 1$)

$$f_{13} = \sum_{\{st,uv\} \in Q} (k_s + k_t + k_u + k_v) - 4q - 2 \sum_{\{st,uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv}).$$

Finally, Propositions 1 and 6 allow one to rewrite $f_{13}$ as

$$f_{13} = K - 4q - 2n_G(L_4). \quad (28)$$
3.3.3. $\tau = 1$, $\phi = 2$

In this type, as in type 13, one edge is shared, but this time only two vertices are equal. Therefore, $f_{12}$ was formalized as [10]

$$f_{12} = \sum_{\{st, uv\} \in Q} \left( \sum_{\{st, wx\} \in Q(G - uv)} 1 + \sum_{\{uv, wx\} \in Q(G - uv)} 1 \right).$$

(29)

Since [10]

$$f_{12} = 2 \sum_{\{st, uv\} \in Q} |E(G - stuw)|.$$

Equation 13 allows one to rewrite $f_{12}$ equivalently as

$$f_{12} = 2 \left( q(m + 2) + \sum_{\{st, uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv}) - \sum_{\{st, uv\} \in Q} (k_s + k_t + k_u + k_v) \right)$$

which, thanks to Propositions 1 and 6, leads to

$$f_{12} = 2[(m + 2)q + n_G(L_4) - K],$$

(30)

3.3.4. $\tau = 0$, $\phi = 4$

All pairs of elements of $Q$ classified into this type share no edges yet have 4 vertices in common. This allowed a brief formalization for $f_{04}$ in [10]

$$f_{04} = \sum_{\{st, uv\} \in Q} \left( \sum_{\{su, tv\} \in Q} 1 + \sum_{\{sv, tu\} \in Q} 1 \right) = \sum_{\{st, uv\} \in Q} (a_{su}a_{tv} + a_{sv}a_{tu}).$$

(31)
Therefore, by Proposition 5 we rewrite Equation 31 as

\[ f_{04} = 2n_G(C_4) \]
\[ = \frac{1}{4} tr(A^4) - n_G(L_3) - \frac{1}{2} n_G(L_2) \]
\[ = \frac{1}{4} tr(A^4) + q - \frac{1}{2} m^2. \]  

(32)

3.3.5. \( \tau = 0, \phi = 3 \)

This type defines the pairs in \( Q \times Q \) that do not share an edge completely but that have three vertices in common. Given a fixed \( \{st,uv\} \in Q \), the other element of \( Q \) that makes the pair has to be of the form [10],

\[ \{su,tw\} \quad \{su,vw\} \quad \{sv,tw\} \quad \{sv,uw\} \quad \{tu,sw\} \quad \{tu,vw\} \quad \{tv,sw\} \quad \{tv,uw\} \quad w \neq s,t,u,v \]

Therefore, \( f_{03} \) can be formalized as [10]

\[ f_{03} = \sum_{\{s,t,u,v\} \subseteq Q} (\varphi_{sut} + \varphi_{svt} + \varphi_{tus} + \varphi_{tus} + \varphi_{svu} + \varphi_{tvu} + \varphi_{tuv} + \varphi_{suv}) \] 

(33)

where \( \varphi_{sut}, \varphi_{svt}, ... \) are functions over \( \{st,uv\} \in Q \). In particular these \( \varphi_{...} \) are defined as

\[ \varphi_{xyz} = a_{xy}|\Gamma(z,-stuv)|. \]

where \( x,y,z \in \{s,t,u,v\} \) are explicit distinct parameters and \( \{s,t,u,v\} \setminus \{x,y,z\} \) as implicit parameter. Examining \( \varphi_{sut}, \varphi_{tus}, \ldots, \varphi_{svu} \) separately, we see that, given \( \{st,uv\} \in Q \), \( \varphi_{sut} \) counts the amount of neighbors of \( t \) in \( G_{-stuv} \) if \( su \in E \), \( \varphi_{tus} \) counts the amount of neighbors of \( s \) in \( G_{-stuv} \) if \( tu \in E \), and so on (Figure 7).
Figure 7.: Illustration of $\varphi_{tu_2}$, that is exactly $|\Gamma^*_s| = |\Gamma(s, -stuv)|$ (the amount of neighbors of $s$ in $G_{-stuv}$), provided that the edge $tu$ exists (i.e. $a_{tu} = 1$). Solid thick lines indicate the edges formed by these vertices in $G$, solid thin lines indicate edges in $G_{-stuv}$ and dashed lines indicate potential edges, namely that may not exist between vertices $s, t, u, v$ and vertices in $G_{-stuv}$. Source [10, Figure 10, p. 30].

Our goal is to obtain a simpler expression for Equation 33. We do this by simplifying first the inner sum of all the $\varphi_{\ldots}$. For this, we apply Equation 14 to a pairwise sum of these $\varphi_{\ldots}$ so as to obtain a series of expressions that are easier to evaluate

$$
\varphi_{tu_2} + \varphi_{tvu} = \left(a_{tu} + a_{tv}\right)\left(k_s - 1 - a_{su} - a_{sv}\right),
$$
$$
\varphi_{stu} + \varphi_{svu} = \left(a_{su} + a_{sv}\right)\left(k_t - 1 - a_{tv} - a_{tu}\right),
$$
$$
\varphi_{svu} + \varphi_{tvu} = \left(a_{sv} + a_{tv}\right)\left(k_u - 1 - a_{ut} - a_{us}\right),
$$
$$
\varphi_{svu} + \varphi_{tvu} = \left(a_{su} + a_{tv}\right)\left(k_v - 1 - a_{vt} - a_{vs}\right).
$$

When adding all of them together, we can simplify the expression a bit more

$$
\left(a_{tu} + a_{tv}\right)\left(k_s - 1\right) - \left(a_{tu} + a_{tv}\right)\left(a_{su} + a_{sv}\right) + \left(a_{su} + a_{sv}\right)\left(k_t - 1\right) - \left(a_{su} + a_{sv}\right)\left(a_{tu} + a_{tv}\right) + \left(a_{su} + a_{sv}\right)\left(k_u - 1\right) - \left(a_{su} + a_{sv}\right)\left(a_{tv} + a_{us}\right)
$$
$$
= \left(a_{tu} + a_{tv}\right)\left(k_s - 1\right) + \left(a_{su} + a_{sv}\right)\left(k_t - 1\right) + \left(a_{sv} + a_{tv}\right)\left(k_u - 1\right) + \left(a_{su} + a_{tv}\right)\left(k_v - 1\right) - 2\left(a_{su} + a_{sv}\right)\left(a_{tv} + a_{us}\right).
$$

Upon expansion of the positive part of the expression, we obtain

$$
k_s(a_{tu} + a_{tv}) + k_t(a_{su} + a_{sv}) + k_u(a_{sv} + a_{tv}) + k_v(a_{su} + a_{tu}) - 2(a_{tu} + a_{tv} + a_{sv} + a_{su}),
$$

and, upon expansion of the negative part,

$$
-2\left((a_{tu} + a_{tv})(a_{su} + a_{sv}) + (a_{sv} + a_{tv})(a_{tu} + a_{us})\right)
$$
$$
= -4(a_{us}a_{tu} + a_{vt}a_{us}) - 2(a_{tu} + a_{us})(a_{su} + a_{tv}).
$$

Thanks to the results for each part, the expression for $f_{03}$ becomes

$$
f_{03} = \sum_{\{st, uv\} \in Q} \left(k_s(a_{tu} + a_{tv}) + k_t(a_{su} + a_{sv}) + k_u(a_{sv} + a_{tv}) + k_v(a_{su} + a_{tu})\right)
$$
$$
- 2 \sum_{\{st, uv\} \in Q} \left(a_{tu} + a_{tv} + a_{sv} + a_{su}\right) - 4 \sum_{\{st, uv\} \in Q} \left(a_{us}a_{tu} + a_{vt}a_{us}\right).$$
(a) (b) (c) (d)

\{sw, tx\} \in Q(G^{\sim uv}) \quad \{su, wx\} \in Q(G^{\sim tv}) \quad \{uw, vx\} \in Q(G^{\sim st}) \quad \{sv, wx\} \in Q(G^{\sim tu}) \quad \\{tu, wx\} \in Q(G^{\sim sv}) \quad \{tv, wx\} \in Q(G^{\sim su})

Figure 8.: Elements of $Q$ such that when paired with element $\{st, uv\} \in Q$, the pair is classified as type 021. Elements in (b) and (c) are symmetric. Source [10, Figure 11, p. 31].

$$-2 \sum_{\{st, uv\} \in Q} (a_{tu} + a_{us})(a_{su} + a_{tv}).$$

By introducing

$$\Lambda_1 = \sum_{\{st, uv\} \in Q} (k_s(a_{tu} + a_{tv}) + k_t(a_{su} + a_{sv}) + k_u(a_{sv} + a_{tv}) + k_v(a_{su} + a_{tu}))$$

$$= \sum_{\{st, uv\} \in Q} (a_{su}(k_t + k_v) + a_{sv}(k_t + k_u) + a_{tu}(k_s + k_v) + a_{tv}(k_s + k_u)), \quad (34)$$

and thanks to Propositions 1, 3 and 5, $f_{03}$ can be simplified further, yielding

$$f_{03} = \Lambda_1 - 2n_G(C_4) - 8n_G(C_4) - 2n_G(Z). \quad (35)$$

3.3.6. $\tau = 0, \phi = 2$, Subtype 1

$f_{021}$ can be formalized as [10]

$$f_{021} = f_{021,\varphi} + f_{021,\varepsilon}, \quad (36)$$

where

$$f_{021,\varphi} = \sum_{\{st, uv\} \in Q} (\varphi_{st} + \varphi_{uv}), \quad f_{021,\varepsilon} = \sum_{\{st, uv\} \in Q} (\varepsilon_{su} + \varepsilon_{sv} + \varepsilon_{tu} + \varepsilon_{tv}),$$

and $\varphi_{xy}$ and $\varepsilon_{xy}$ are functions over $\{st, uv\} \in Q$, being $xy$ the explicit parameters and $\{s, t, u, v\} \setminus \{x, y\}$ as implicit parameters. These functions are defined as

$$\varphi_{xy} = \sum_{w \in \Gamma(x, -stuv)} \sum_{w' \in \Gamma(y, -stuvw)} 1$$

$$= \sum_{w \in \Gamma(x, -stuv)} |\Gamma(y, -stuvw)|,$$

$$\varepsilon_{xy} = a_{xy}|E(G^{\sim stuv})|, \quad x, y \in \{s, t, u, v\}. \quad (37)$$
Figure 9.: Illustration of (a) $\varphi_{st}$, and (b) $\varepsilon_{su}$. In (a), $w_s$ represents the only neighbor of $s$ different from $t, u, v$. Therefore, in this case, $\varphi_{st}$ is exactly the amount of vertices in $G_{-stuvw_s}$ neighbors of $t$, indicated with $\Gamma_t^* = \Gamma(t, -stuvw_s)$. $\varepsilon_{su}$ requires the existence of an edge between $s$ and $u$, indicated with $a_{su}$, and is equal to the amount of edges in $G_{-stuv}$. Solid thick lines indicate the edges formed by these vertices in $G$, solid thin lines indicate edges in $G_{-stuv}$ and dashed lines indicate potential edges, namely that may not exist between vertices $s, t, u, v$ and vertices in $G_{-stuv}$. Source [10, Figure 12, p. 33].

The functions $\varphi_{st}$ and $\varphi_{uv}$ count the elements of the form of the illustrated in Figures 8(a) and 8(d). The first function counts, for each neighbor of $s$, $w_s \neq t, u, v$, the number of neighbors of $t$, $w_t \neq s, t, u, v, w_s$. Likewise for the second function. On the other hand, the values $\varepsilon_{su}$, $\varepsilon_{sv}$, $\varepsilon_{tu}$, $\varepsilon_{tv}$ count the edges $xy \in E$, $x, y \neq s, t, u, v$ such that when paired with $su$, $sv$, $tu$, $tv$ form an element of $Q$ whose form is that of those elements illustrated in Figures 8(b) and 8(c). These amounts are counted only if such edges exist in the graph, hence the $a_{su}$ for $\varepsilon_{su}$, and likewise for the other $\varepsilon_{-}$. One the one hand, Equation 37 allows one to factor out $|E(G_{-stuv})|$ giving

$$f_{021,\varepsilon} = \sum_{\{st, uv\} \in Q} (\varphi_{st} + \varphi_{uv} + (a_{su} + a_{sv} + a_{tu} + a_{tv})|E(G_{-stuv})|).$$

Equation 13 produces

$$(a_{su} + a_{sv} + a_{tu} + a_{tv})|E(G_{-stuv})| = (a_{su} + a_{sv} + a_{tu} + a_{tv})m - (a_{su} + a_{sv} + a_{tu} + a_{tv})(k_s + k_t + k_u + k_v) + (a_{su} + a_{sv} + a_{tu} + a_{tv})^2 + 2(a_{su} + a_{sv} + a_{tu} + a_{tv}).$$

Recall that for $\{st, uv\} \in Q$, $a_{st} + a_{uv} = 2$ and let

$$\Lambda_2 = \sum_{\{st, uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv})(k_s + k_t + k_u + k_v). \quad (38)$$

Then, thanks to Proposition 1, we obtain

$$f_{021,\varepsilon} = m \sum_{\{st, uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv}) - \Lambda_2$$
\[
+ \sum_{\{st,uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv})^2 + 2 \sum_{\{st,uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv})
\]

\[
= (m + 2)n_G(\mathcal{L}_4) - \Lambda_2 + \sum_{\{st,uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv})^2.
\]

On the other hand, Equation 14 leads to

\[
|\Gamma(t, -stuvw_s)| = k_t - \sum_{z \in \{s,t,u,v,w_s\}} a_{tz} = k_t - (a_{ts} + a_{tu} + a_{tv} + a_{tw_s}),
\]

\[
\varphi_{st} = \sum_{w_s \in \Gamma(s, -stuv)} (k_t - (a_{ts} + a_{tu} + a_{tv} + a_{tw_s}))
\]

\[
= (k_s - a_{su} - a_{sv} - 1)(k_t - a_{tu} - a_{tv} - 1) - \sum_{w_s \in \Gamma(s, -stuv)} a_{tw_s}. \quad (39)
\]

Likewise for \(\varphi_{uv}\)

\[
\varphi_{uv} = (k_u - a_{us} - a_{ut} - 1)(k_v - a_{vs} - a_{vt} - 1) - \sum_{w_u \in \Gamma(u, -stuv)} a_{vw_u}. \quad (40)
\]

The negative summations in \(\varphi_{st}\), Equation 39 (and \(\varphi_{uv}\), Equation 40) represent the amount of vertices from \(G_{-stuv}\) neighbors of \(s\) (of \(u\)) in \(G\) that are also neighbors of \(t\) (of \(v\)), in \(G\). Therefore, the triangles formed by vertices \(s, t, w_s\) and \(u, v, w_u\) respectively. Then,

\[
f_{021,\varphi} = \sum_{\{st,uv\} \in Q} (k_s - a_{su} - a_{sv} - 1)(k_t - a_{tu} - a_{tv} - 1)
\]

\[
+ \sum_{\{st,uv\} \in Q} (k_u - a_{us} - a_{ut} - 1)(k_v - a_{vs} - a_{vt} - 1)
\]

\[
- \sum_{\{st,uv\} \in Q} \left( \sum_{w_s \in \Gamma(s, -stuv)} a_{tw_s} + \sum_{w_u \in \Gamma(u, -stuv)} a_{vw_u} \right).
\]

Within \(f_{021,\varphi}\),

\[
(a_{su} + a_{sv} + a_{tu} + a_{tv})^2 = 2(a_{tu} + a_{sv})(a_{su} + a_{tv}) + 2(a_{su}a_{tv} + a_{sv}a_{tu}) + a_{su} + a_{sv} + a_{tu} + a_{tv}, \quad (41)
\]

and, within \(f_{021,\varphi}\),

\[
(k_s - a_{su} - a_{sv} - 1)(k_t - a_{tu} - a_{tv} - 1)
\]

\[
+ (k_u - a_{us} - a_{ut} - 1)(k_v - a_{vs} - a_{vt} - 1)
\]

\[
= (k_s k_t + k_u k_v) - (k_s + k_t + k_u + k_v)
\]

\[
- (k_s(a_{tu} + a_{tv}) + k_t(a_{su} + a_{sv}) + k_u(a_{sv} + a_{tv}) + k_v(a_{su} + a_{tu}))
\]

\[
+ (a_{su} + a_{tv})(a_{su} + a_{tv}) + 2(a_{su}a_{tv} + a_{sv}a_{tu})
\]

\[
+ 2(a_{su} + a_{sv} + a_{tu} + a_{tv}) + 2.
\]
Then Equation 34, and Propositions 3, 4 and 5 allow one to rewrite Equation 36 as

\[
f_{021} = (m + 5)n_G(L_4) + 8n_G(C_4) + 2q - K + 3n_G(Z) - n_G(Y) - \Lambda_1 - \Lambda_2 + \Phi_1 \tag{42}
\]

where \(\Phi_1\) is defined in Equation 24.

### 3.3.7. \(\tau = 0, \phi = 2\), Subtype 2

Given a fixed \(q = \{st, uv\} \in Q\), there are only four different forms of elements of \(Q\) such that when paired with \(q\) yield a pair of \(Q \times Q\) of type 022 [10]. These are shown in Figure 10. As a result of this analysis, \(f_{022}\) can be formalized as [10]

\[
f_{022} = \sum_{\{st, uv\} \in Q} (\varphi_{su} + \varphi_{sv} + \varphi_{tu} + \varphi_{tv}), \tag{43}
\]

where \(\varphi_{xy}\) is an auxiliary function defined as in Equation 37. \(\varphi_{su}\) can be understood from the case of \(\varphi_{st}\) in Figure 9(a).

We aim at deriving a useful arithmetic expression for \(f_{022}\). First, we expand, following Equations 39 and 40, the \(\varphi_{...}\) in Equation 43 as

\[
\begin{align*}
\varphi_{su} &= (k_s - a_{su} - a_{sv} - 1)(k_u - a_{us} - a_{ut} - 1) - \sum_{w_t \in \Gamma(t, -stuv)} a_{uw_t}, \\
\varphi_{sv} &= (k_s - a_{su} - a_{sv} - 1)(k_v - a_{vs} - a_{vt} - 1) - \sum_{w_t \in \Gamma(t, -stuv)} a_{vw_t}, \\
\varphi_{tu} &= (k_t - a_{tu} - a_{tv} - 1)(k_u - a_{us} - a_{ut} - 1) - \sum_{w_t \in \Gamma(t, -stuv)} a_{uw_t}, \\
\varphi_{tv} &= (k_t - a_{tu} - a_{tv} - 1)(k_v - a_{vs} - a_{vt} - 1) - \sum_{w_t \in \Gamma(t, -stuv)} a_{vw_t}.
\end{align*}
\]

Inserting these expressions into Equation 43 and taking common factors out, one obtains

\[
f_{022} = \sum_{\{st, uv\} \in Q} (k_s + k_t - a_{su} - a_{sv} - a_{tu} - a_{tv} - 2)(k_u + k_v - a_{us} - a_{ut} - a_{vs} - a_{vt} - 2)
\]

Figure 10.: Elements of \(Q\) such that when paired with element \(\{st, uv\} \in Q\), the pair is classified as type 022. (a) One of the bipartite graphs of type 022. (b) The other bipartite graph of type 022. The elements in (a) are symmetric to those of (b). Source [10, Figure 13, p. 34].
which can be further developed and then simplified using Propositions 1 and 2, and Equation 38, becoming

\[ f_{022} = 4n_G(\mathcal{L}_4) - n_G(\mathcal{L}_5) - \Lambda_2 \]

\[ + \sum_{\{st,uv\} \in Q} (k_s + k_t - 2)(k_u + k_v - 2) + \sum_{\{st,uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv})^2. \]

Using Equations 25 and 41, Propositions 3 and 5, and by expanding \((k_s + k_t - 2)(k_u + k_v - 2)\), we finally obtain

\[ f_{022} = 4q - 2K + 5n_G(\mathcal{L}_4) - n_G(\mathcal{L}_5) + 2n_G(Z) + 4n_G(\mathcal{L}_4) - \Lambda_2 + \Phi_2. \] (44)

### 3.4. Variance of the number of crossings

Now we derive an arithmetic expression for \(V_\ast [C]\) via Equation 2 applying the arithmetic expressions of the \(f_\ast\)’s above (summarized in Table 4). This arithmetic expression is as follows

\[
V_\ast [C] = q(E_\ast [\gamma_24] - 4E_\ast [\gamma_13] + 2(m + 2)E_\ast [\gamma_12] + 2E_\ast [\gamma_021] + 4E_\ast [\gamma_022]) \\
+ K(E_\ast [\gamma_13] - 2E_\ast [\gamma_12] - E_\ast [\gamma_021] - 2E_\ast [\gamma_022]) \\
+ n_G(\mathcal{L}_4)(-2E_\ast [\gamma_13] + 2E_\ast [\gamma_12] - 2E_\ast [\gamma_03] + (m + 5)E_\ast [\gamma_021] + 5E_\ast [\gamma_022]) \\
+ n_G(\mathcal{L}_5)(2E_\ast [\gamma_04] - 8E_\ast [\gamma_03] + 8E_\ast [\gamma_021] + 4E_\ast [\gamma_022]) \\
+ n_G(Z)(-2E_\ast [\gamma_03] + 3E_\ast [\gamma_021] + 2E_\ast [\gamma_022]) \\
+ \Lambda_1(E_\ast [\gamma_03] - E_\ast [\gamma_021]) \\
- \Lambda_2(E_\ast [\gamma_021] + E_\ast [\gamma_022]) \\
- n_G(\mathcal{L}_5)E_\ast [\gamma_022] - n_G(Y)E_\ast [\gamma_021] \\
+ \Phi_1E_\ast [\gamma_021] + \Phi_2E_\ast [\gamma_022],
\] (45)

where \(q\) is the number of pairs of independent edges (Equation 1), \(K\) is defined in Equation 23, \(\Phi_1\) and \(\Phi_2\) are defined in Equations 24 and 25, \(\Lambda_1\) and \(\Lambda_2\) are defined in Equations 34 and 38, and \(Z\) is the graph paw and \(Y = \mathcal{C}_3 \oplus \mathcal{L}_2\), depicted in Figure 4. The fact that forests are acyclic implies \(n_G(\mathcal{L}_4) = n_G(Y) = n_G(Z) = 0\) and allows one to express \(V_\ast [C]\) in these graphs as

\[
V_\ast [C] = q(E_\ast [\gamma_24] - 4E_\ast [\gamma_13] + 2(m + 2)E_\ast [\gamma_12] + 2E_\ast [\gamma_021] + 4E_\ast [\gamma_022]) \\
+ K(E_\ast [\gamma_13] - 2E_\ast [\gamma_12] - E_\ast [\gamma_021] - 2E_\ast [\gamma_022]) \\
+ n_G(\mathcal{L}_4)(-2E_\ast [\gamma_13] + 2E_\ast [\gamma_12] - 2E_\ast [\gamma_03] + (m + 5)E_\ast [\gamma_021] + 5E_\ast [\gamma_022]) \\
+ n_G(\mathcal{L}_5)(2E_\ast [\gamma_04] - 8E_\ast [\gamma_03] + 8E_\ast [\gamma_021] + 4E_\ast [\gamma_022]) \\
+ n_G(Z)(-2E_\ast [\gamma_03] + 3E_\ast [\gamma_021] + 2E_\ast [\gamma_022]) \\
+ \Lambda_1(E_\ast [\gamma_03] - E_\ast [\gamma_021]) \\
- \Lambda_2(E_\ast [\gamma_021] + E_\ast [\gamma_022]) \\
- n_G(\mathcal{L}_5)E_\ast [\gamma_022] \\
+ \Phi_1E_\ast [\gamma_021] + \Phi_2E_\ast [\gamma_022].
\] (46)
For the case of uniformly random linear arrangements, the result of instantiating Equation 45 is

\[ V_{rla}[C] = \frac{1}{180}[8(m + 2)q + 2K - (2m + 7)n_G(L_4) - 12n_G(C_4) + 6n_G(Z) - n_G(L_5) + 2n_G(Y) - 3\Lambda_1 + \Lambda_2 - 2\Phi_1 + \Phi_2], \]

(47)

and the result of instantiating Equation 46 is

\[ V_{rla}[C] = \frac{1}{180}[8(m + 2)q + 2K - (2m + 7)n_G(L_4) - n_G(L_5) - 3\Lambda_1 + \Lambda_2 - 2\Phi_1 + \Phi_2]. \]

(48)

We refer the reader to [10, Table 5] for a summary of expressions of \( V_{rla}[C] \) in particular graphs (complete graphs, complete bipartite graphs, linear trees, cycle graphs, one-regular, star graphs and quasi-star graphs).

4. Algorithms to compute \( V_\ast[C] \)

In this section we provide algorithms that computes the exact value of \( V_\ast[C] \) in general graphs based on Equation 45) in Section 4.1 and in forests (based on Equation 46) in Section 4.2. As the computation of \( V_\ast[C] \) reduces to a subgraph counting problem (Section 3.4), the algorithms presented in the subsequent sections consist of solving the subgraph counting problem that we face in Equations 45 and 46 and then computing \( V_\ast[C] \) with knowledge of \( E_\ast[\gamma_\omega] \).

4.1. An algorithm for general graphs

Our algorithm to calculate \( V_\ast[C] \) reduces to a simple sequential traversal of the list of edges of the graph with some extra work to be done for each edge. In general graphs, this yields an algorithm of time complexity \( O(k_{\max}n(k^2)) = o(nm^2) \). For the case of forests (Section 4.2), this approach helps us derive an algorithm that computes \( V_\ast[C] \) in such a way that the amount of work per edge has constant-time and constant-space complexity, hence yielding an \( O(n) \)-time algorithm.

The way to such a reduction is three-fold. Firstly, we have shown that the values \( q, K, \Phi_1 \) and \( \Phi_2 \) have very simple arithmetic expressions that allow us to compute them in either linear time in the number of vertices (see the definition of \( q \) in Equation 1) or in linear time in the number of edges (see the definition of \( K, \Phi_1 \) and \( \Phi_2 \) in Equations 23, 24 and 25, respectively). Secondly, the subgraphs present in Equation 47, namely \( L_4, C_4, L_5, Z \) and \( Y \) (the last two are depicted in Figure 4) could be counted straightforwardly by relying on previous work [24, 25]. For example, Alon et. al. [24], among other contributions, showed that

\[ n_G(C_3) = \frac{1}{6}tr(A^3), \]

\[ n_G(C_4) = \frac{1}{8}\left[tr(A^4) - 4\sum_{u \in V}(k_u)2 - 2m\right], \]
\[ n_G(Z) = \frac{1}{2} \left[ \sum_{i=1}^{n} a_{ii}^{(3)} (d_i - 2) \right], \]

and Movarraei [25] showed that
\[ n_G(L_4) = \frac{1}{2} \sum_{i \neq j} \left( a_{ij}^{(3)} - (2k_j - 1)a_{ij} \right), \]
\[ n_G(L_5) = \frac{1}{2} \left[ \sum_{i \neq j} (a_{ij}^{(4)} - 2a_{ij}^{(2)} (k_j - a_{ij})) - \sum_{i=1}^{n} \left( (2k_i - 1)a_{ii}^{(3)} + 6 \left( \frac{k_i}{3} \right) \right) \right]. \]

However, a direct implementation based on the previous expressions would lead to an algorithm that has quadratic space complexity since the adjacency matrix should be both computed and stored in memory, leaving aside the costs of computing powers of the adjacency matrix. For this reason, we follow an alternative combinatorial approach to keep the time and space complexity as low as possible. Thirdly, we provide an interpretation of the values represented by \( \Lambda_1 \) and \( \Lambda_2 \) (defined in Equations 34 and 38) that allow us to compute them without the need of enumerating the elements of \( Q \).

Next we detail the mathematical expressions involved in the last two ways. We use these expressions later in this section to derive Algorithm 4.2. We refer the reader to Appendix A for the less immediate proofs.

**Proposition 9.** Let \( Z \) be the paw graph, depicted in Figure 4(a). In any graph \( G \),
\[ n_G(Z) = \sum_{st \in E} \sum_{u \in c(s,t)} (k_u - 2), \] (49)
where \( c(s,t) \) is defined in Equation 10.

We already gave a result related to graphs \( C_3 \oplus L_2 \) in Proposition 4.

**Proposition 10.** Let \( Y = C_3 \oplus L_2 = \overline{K_{2,3}} \), depicted in Figure 4(b). In any graph \( G \),
\[ n_G(Y) = \frac{1}{3} \sum_{st \in E} \sum_{u \in c(s,t)} (m - k_s - k_t - k_u + 3), \] (50)
where \( c(s,t) \) is defined in Equation 10.

**Proposition 11.** In any graph \( G \),
\[ n_G(C_4) = \frac{1}{4} \sum_{st \in E} \sum_{u \in \Gamma(t) \setminus \{s\}} (|c(s,u)| - 1), \] (51)
where \( c(s,t) \) is defined in Equation 10.
Proposition 12. In any graph \( G \),
\[
n_G(L_4) = m - n(k^2) + \mu_1 - \mu_2,
\]
where
\[
\mu_1 = \frac{1}{2} \sum_{st \in E} (\xi(s) + \xi(t)), \quad \mu_2 = \sum_{st \in E} |c(s, t)|,
\]
and \( \xi(s) \) and \( c(s, t) \) are defined in Equations 9 and 10.

Proposition 13. In any graph \( G \),
\[
n_G(L_5) = \frac{1}{2} \sum_{st \in E} (g_1(s, t) + g_1(t, s)),
\]
where
\[
g_1(s, t) = \sum_{u \in \Gamma(s) \setminus \{t\}} (k_t - 1 - a_{ut})(k_u - 1 - a_{ut}) + 1 - |c(t, u)|
\]
and \( c(s, t) \) is defined in Equation 10.

Now it remains to know how to compute the values \( \Lambda_1 \) and \( \Lambda_2 \), defined in Equations 34 and 38, without summing over the elements of \( Q \). First, the reader should notice that those summations accumulate degrees of vertices of some of the vertices of all the subgraphs isomorphic to \( L_4 \). This can be easily seen by taking as reference Proposition 1, where we state that
\[
n_G(L_4) = \sum_{\{st, uv\} \in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv}).
\]
The summation in Equation 34 accumulates the degrees of the leaves of every subgraph isomorphic to \( L_4 \) whereas the summation in Equation 38 accumulates the degrees of all the vertices of each \( L_4 \). Being aware of these facts, we claim

Proposition 14. In any graph,
\[
\Lambda_1 = \sum_{st \in E} ((k_t - 1)(\xi(s) - k_t) + (\xi(t) - k_s) - 2S_{s,t}),
\]
where \( \xi(s), c(s, t), S_{s,t} \) and \( \Lambda_1 \) are defined in Equations 9, 10, 11 and 34.

Proposition 15. In any graph,
\[
\Lambda_2 = \Lambda_1 + \sum_{st \in E} (k_s + k_t)((k_s - 1)(k_t - 1) - |c(s, t)|),
\]
where \( c(s, t), \Lambda_1 \) and \( \Lambda_2 \) are defined in Equations 10, 34 and 38.
All the expressions above are integrated in Algorithm 4.2 to compute the exact value of $V_{rla}[C]$. The algorithm runs in $O(k_{\max}n\langle k^2 \rangle) = o(nm^2)$ time. We prove this algorithm’s complexity in Proposition 16.

**Proposition 16.** Let $G = (V, E)$ be a graph with $n = |V|$ and $m = |E|$. Let the graph be implemented with sorted adjacency lists, namely, the adjacency list of each vertex contains labels sorted in increasing lexicographic order. Algorithm 4.2 computes $V_{rla}[C]$ in $G$ in time $O(k_{\max}n\langle k^2 \rangle) = o(nm^2)$ and space $O(n)$.

**Proof.** The computation of $V_{rla}[C]$’s value is done by putting together all the results presented in this work that involve the terms in Equation 45. The results’ correctness has already been proved and the composition of the algorithm using them is correct.

As for the space complexity, we need $O(n)$-space to store the values of the function $\xi(s)$ (Equation 9) for each vertex $s \in V$. As for the time complexity, the cost of SETUP (Algorithm 4.1) is $O(n + m)$ and then the algorithm iterates over the set of edges and performs, for each edge, three intersection operations to compute the values $|c(t, u_1)|$, $|c(s, u_2)|$ and $|c(s, t)|$. The other computations are constant-time operations as a function of the vertices of similar intersection or as a function of the vertices of the edge. Recall that $\sum_{st \in E}(k_s + k_t) = \sum_{u \in V}k_u^2 = n\langle k^2 \rangle$. Now, since the cost of the intersection of two sorted lists $\Gamma(u)$ and $\Gamma(v)$ has cost $\Delta(u, v)$ where $\Delta(u, v) = O(\max\{k_u, k_v\})$ then the algorithm has cost

$$O\left(n + m + \sum_{st \in E}(\Delta(s, t) + \sum_{u_1 \in \Gamma(s) \setminus \{t\}}\Delta(t, u_1) + \sum_{u_2 \in \Gamma(t) \setminus \{s\}}\Delta(s, u_2))\right)$$

$$= O\left(n + m + k_{\max}\sum_{st \in E}(k_s + k_t - 1)\right)$$

$$= O\left(n + m + k_{\max}(n\langle k^2 \rangle - m)\right)$$

$$= O\left(n - m(k_{\max} - 1) + k_{\max}n\langle k^2 \rangle\right)$$

$$= O\left(n + k_{\max}n\langle k^2 \rangle\right)$$

The cost of the algorithm is expressed in terms of $k_{\max}$ and the structure of the graph, as captured by $\langle k^2 \rangle$. To obtain a cost in terms of $n$ and $m$, note that the condition $|Q| \geq 0$ (because $Q$ is the set of pairs of independent edges) on Equation 1 gives $n\langle k^2 \rangle \leq m(m + 1)$ which leads to $O(k_{\max}n\langle k^2 \rangle) = o(nm^2)$. \hfill \Box

Notice that the assumption that the graph’s adjacency list being sorted merely simplifies the algorithm. In case it was not, sorting it, prior to the algorithm’s execution, has cost $O(\sum_{u \in V}k_u \log k_u) = o(n\langle k^2 \rangle)$.

In Appendix C, we extend Algorithm 4.2 to reuse computations (Algorithm C.2) and show, using empirical results, that doing so produces an algorithm several times faster in practice. The parts we focus on reusing are marked in red in Algorithm 4.2.
Algorithm 4.1: Setup.

1 Function Setup(G) is
2   Calculate $\langle k^2 \rangle$ and $\langle k^3 \rangle$ in $O(n)$-time // Equation 8
3   $\xi(s) \leftarrow 0^n$ // Equation 9
4   Calculate $\xi(s)$ for all $s \in V$ in $O(m)$-time.
5   $\psi \leftarrow 0$ // Equation 12
6   $\Phi_1, \Phi_2 \leftarrow 0$ // Equations 24, 25
7   $n_G(Z), n_G(Y), n_G(C_4) \leftarrow 0$ // Equations 49, 50, 51
8   $\mu_1, \mu_2 \leftarrow 0$ // Equation 53
9   $n_G(L_5) \leftarrow 0$ // Equation 54
10  $\Lambda_1, \Lambda_2 \leftarrow 0$ // Equations 55, 56

4.2. An algorithm for trees and forests

$V_s[C]$ can be computed more efficiently in forests than in general graphs by exploiting the fact that they are acyclic. We already used this property when we gave its arithmetic expression in Equation 48. The algorithm we present computes the exact value of $V_s[C]$ in forests in $O(n)$-time, where $n$ is the number of vertices of the forest. We first show that the heavy components of Equation 46, namely, $n_T(L_4)$, $n_T(L_5)$, as well as $\Lambda_1$ and $\Lambda_2$ (defined in Equations 34 and 38), are simpler to calculate in a tree $T$. These expressions are given only for trees, since forests are merely a disjoint union of two or more trees and can be generalized straightforwardly. Most of these expressions do not require a formal proof. Then we prove the algorithm’s correctness and complexity in Proposition 19.

Proposition 17. In any tree $T$,

$$n_T(L_4) = \sum_{st \in E} (k_s - 1)(k_t - 1).$$  \hfill (57)

Proposition 18. In any tree $T$,

$$n_T(L_5) = \frac{1}{2} \sum_{st \in E} (g_2(s,t) + g_2(t,s))$$  \hfill (58)

where $g_2(s,t) = (k_t - 1)(\xi(s) - k_t - k_s + 1)$ and $\xi(s)$ is defined in Equation 9.

The following corollaries are immediate simplifications of Propositions 14 and 15, respectively.

Corollary 1. In any tree,

$$\Lambda_1 = \sum_{st \in E} ((k_t - 1)(\xi(s) - k_t) + (k_s - 1)(\xi(t) - k_s)),$$  \hfill (59)

where $\xi(s)$ and $\Lambda_1$ are defined in Equations 9 and 34.
Algorithm 4.2: Calculate $\mathbb{V}_s[C]$ in general graphs.

Input: $G = (V, E)$ a graph as described in proposition 16.
Output: $\mathbb{V}_s[C]$, the variance of the number of crossings.

Function VARIANCEC(G) is

1. SETUP(G) // Algorithm 4.1
2. for $\{s, t\} \in E$ do
3.   for $u_1 \in \Gamma(s) \setminus \{t\}$ do
4.     $n_G(L_5) \leftarrow n_G(L_5) - |c(t, u_1)| + (k_t - 1 - a_{tu_1})(k_u - 1 - a_{tu_1}) + 1$
5.   for $u_2 \in \Gamma(t) \setminus \{s\}$ do
6.     $n_G(L_5) \leftarrow n_G(L_5) - |c(s, u_2)| + (k_s - 1 - a_{su_2})(k_u - 1 - a_{su_2}) + 1$
7.     $n_G(L_4) \leftarrow n_G(L_4) + |c(s, u_2)| - 1$
8.   $|c(s, t)|, S_{s,t} \leftarrow 0$ // Equations 10, 11
9. for $u \in \Gamma(s) \cap \Gamma(t)$ do
10.   $|c(s, t)| \leftarrow |c(s, t)| + 1$
11. $S_{s,t} \leftarrow S_{s,t} + k_u$
12. $n_G(Z) \leftarrow n_G(Z) + S_{s,t} - 2|c(s, t)|$
13. $n_G(Y) \leftarrow n_G(Y) + (m - k_s - k_t + 3)|c(s, t)| - S_{s,t}$
14. $\psi \leftarrow \psi + k_s k_t$
15. $\Phi_1 \leftarrow \Phi_1 - k_s k_t (k_s + k_t)$
16. $\Phi_2 \leftarrow \Phi_2 + (k_s + k_t)(n\langle k^2 \rangle - (\xi(s) + \xi(t)) - k_s(k_s - 1) - k_t(k_t - 1))$
17. $\mu_1 \leftarrow \mu_1 + \xi(s) + \xi(t)$
18. $\mu_2 \leftarrow \mu_2 + |c(s, t)|$
19. $\Delta_1 \leftarrow \Delta_1 + (k_t - 1)(\xi(s) - k_t) + (\xi(t) - k_s) - 2S_{s,t}$
20. $\Delta_2 \leftarrow \Delta_2 + (k_s + k_t)((k_s - 1)(k_t - 1) - |c(s, t)|)$
21. $q \leftarrow \frac{1}{2}[m(m + 1) - n\langle k^2 \rangle]$ // Equation 1
22. $K \leftarrow (m + 1)n\langle k^2 \rangle - n\langle k^3 \rangle - 2\psi$ // Equation 23
23. $\Phi_1 \leftarrow \Phi_1 + (m + 1)\psi$ // Equation 24
24. $\Phi_2 \leftarrow \frac{1}{2}\Phi_2$ // Equation 25
25. $n_G(Y) \leftarrow \frac{1}{2}n_G(Y)$ // Equation 50
26. $n_G(L_4) \leftarrow \frac{1}{2}n_G(L_4)$ // Equation 51
27. $n_G(L_5) \leftarrow m - n\langle k^2 \rangle + \frac{1}{2}\mu_1 - \mu_2$ // Equation 52
28. $n_G(L_5) \leftarrow \frac{1}{2}n_G(L_5)$ // Equation 54
29. $\Delta_2 \leftarrow \Delta_1 + \Delta_2$ // Equation 56
30. Compute $\mathbb{V}_s[C]$ by instating Equation 45 appropriately


Corollary 2. In any tree,
\[ \Lambda_2 = \Lambda_1 + \sum_{st \in E} (k_s - 1)(k_t - 1)(k_s + k_t), \]
(60)
where \( \Lambda_1 \) and \( \Lambda_2 \) are defined in Equations 34 and 38.

Thanks to the previous results, it is possible to adapt Algorithm 4.2 to produce the particular version for forests in Algorithm 4.3. Its correctness and complexity is addressed by the following proposition.

**Algorithm 4.3:** Computing \( V_* [C] \) in forests.

**Input:** \( F = \{ T_i \}_{i=1}^r \) a forest.

**Output:** \( V_* [C] \), the variance of the number of crossings in \( F \).

1. **Function** \( \text{Variance}(F) \) is
2. \( \xi(s) \leftarrow 0^n \) // Equation 8
3. Calculate \( \xi(s) \) for each \( s \in V \) in \( O(n) \)-time
4. \( \psi \leftarrow 0 \) // Equation 12
5. \( \Phi_1, \Phi_2 \leftarrow 0 \) // Equations 24 and 25
6. \( n_F(L_4), n_F(L_5) \leftarrow 0 \) // Equations 57 and 58
7. \( \Lambda_1, \Lambda_2 \leftarrow 0 \) // Equations 59 and 60
8. **for** \( \{s, t\} \in E(F) \) **do**
9. \( \psi \leftarrow \psi + k_s k_t \)
10. \( n_F(L_5) \leftarrow n_F(L_5) + (k_t - 1)(\xi(s) - k_t - k_s + 1) + (k_s - 1)(\xi(t) - k_t - k_s + 1) \)
11. \( n_F(L_4) \leftarrow n_F(L_4) + (k_t - 1)(k_t - 1) \)
12. \( \Lambda_1 \leftarrow \Lambda_1 + (k_t - 1)(\xi(s) - k_t) + (k_s - 1)(\xi(t) - k_s) \)
13. \( \Lambda_2 \leftarrow \Lambda_2 + (k_s - 1)(k_t - 1)(k_s + k_t) \)
14. \( \Phi_1 \leftarrow \Phi_1 - k_s k_t (k_s + k_t) \)
15. \( \Phi_2 \leftarrow \Phi_2 + (k_s + k_t) n(\xi^2) - \xi(s) - \xi(t) - k_s (k_s - 1) - k_t (k_t - 1) \)
16. \( q \leftarrow \tfrac{1}{2}[m(m + 1) - n(\xi^2)] \) // Equation 1
17. \( K \leftarrow (m + 1)n(\xi^2) - n(\xi^3) - 2\psi \) // Equation 23
18. \( \Phi_1 \leftarrow \Phi_1 + (m + 1)\psi \) // Equation 24
19. \( \Phi_2 \leftarrow \tfrac{1}{2}\Phi_2 \) // Equation 25
20. \( n_F(L_5) \leftarrow \tfrac{1}{2} n_F(L_5) \) // Equation 59
21. \( \Lambda_2 \leftarrow \Lambda_2 + \Lambda_1 \) // Equation 60
22. **Compute** \( V_* [C] \) by instating Equation 46 appropriately

**Proposition 19.** Let \( F = \{ T_i \}_{i=1}^r \), with \( r \geq 1 \) and \( n = |V(F)| \), be a forest. Algorithm 4.3 computes \( V_* [C] \) in \( F \) in time and space \( O(n) \).

**Proof.** The terms \( q, K, \Phi_1 \) and \( \Phi_2 \) are defined for general graphs as sums of values that only depend on vertices that are adjacent (Equations 1, 23, 24 and 25). The terms \( n_F(L_4) \) and \( n_F(L_5) \), \( \Lambda_1 \) and \( \Lambda_2 \) can be computed in forests as easily as they can be computed in trees. Namely, the arguments in the proofs of Propositions 17, 18, 1 and 2 that led to their respective equations can be extended to forests straightforwardly.
while keeping the same conclusions thanks to the fact that forests, as their definition states, are merely a disjoint union of trees.

The time complexity of the algorithm is obviously $O(n)$ since loop in line 9 iterates over the set of edges of $F$, and the computation of the $\xi(s)$ is $O(m)$ which is asymptotically equal to $O(n)$. It is also obvious that the space complexity is $O(n)$ since we have to store the values of $\xi(s)$ for each $s \in V$.

5. Discussion

We have developed efficient algorithms to calculate $V_e[C]$ in general graphs and also in forests. Since the calculation of $V_e[C]$ reduces to a problem of counting the number of subgraphs of a certain type (Equation 2 and Table 1), one could also apply algorithms for counting graphlets or graphettes [26, 27]. Graphlets are connected subgraphs while graphettes are a generalization of graphlets to potentially disconnected subgraphs. The subgraphs for calculating the number of products of each types are indeed graphettes; only for types 03 and 04 the subgraphs are graphlets (Figure 2). As the subgraphs we are interested in have between 4 and 6 vertices (Figure 2; recall that types 00 and 01 do not matter), we could generate all subsets of 4, 5 and 6 vertices and use the look-up table provided in [27] to classify the corresponding subgraphs in constant time for each subset. However, that would produce and algorithm that runs in $\Theta(n^8)$ time while ours runs in $O(n^5)$ (recall it runs in $o(n^m^2)$, Table 3).

Our algorithms are based on formulae for the $f^e_\omega$’s, namely, the number of products of each type (Table 4). As a side-effect of such characterization, we have contributed with expressions for the number of paths of 4 and 5 vertices of a graph (Propositions 1 and 2) that are more compact than others obtained in previous work [25]. Future work should investigate simple formulae for the number of products of a type, specially types that are relevant for the calculation of variance but for whom a simple arithmetic formula is not forthcoming yet, i.e. types 03, 021 i 022 (Table 4).

Part of our work has consisted of reducing the complexity of the computation of arithmetic expressions of the $f^e_\omega$’s for all types (except types 00 and 01, since $E_e[\gamma_{00}] = E_e[\gamma_{01}] = 0$). Using Equation 47 we derived algorithms tailored to those layouts meeting the requirements summarized in section 1. One of these layouts are linear arrangements. We have alleviated the complexity of the computation of $V_e[C]$ for such layouts by several orders of magnitude with respect to the $O(m^4)$-time algorithm that can be derived from [10].

These algorithms pave the way for further statistical research on crossings in syntactic dependency trees [7]. In this context, $C$ has been shown to be significantly low with respect to random linear arrangements with the help of Monte Carlo statistical tests in syntactic dependency structures [7]. Faster algorithms for such a test could be developed with the help of Chebyshev-like inequalities that allow one to calculate upper bounds of the real $p$-values. These inequalities typically imply the calculation of $E_{rla}[C]$, which is straightforward (simply, $E_{rla}[C] = q/3$ [10]) and the calculation of $V_{rla}[C]$, that thanks to the present article, has become simpler to compute. Similar tests could be applied to determine if the number of crossings in RNA structures [4] is lower or greater than that expected by chance. For the same reasons, our algorithms allow one to calculate $z$-scores of $C$ (that are also a function of the expectation and the variance of $C$) efficiently. In other contexts, $z$-scores allowed researchers to detect scale patterns in data (invariance in empirical curves [11, 12] or motifs in complex networks [14]). In addition, $z$-scores of $C$ can help to aggregate or compare values of $C$ from
heterogeneous sources as it happens in the context syntactic dependency trees, where one has to aggregate or compare values of \( C \) of syntactic dependency trees from the same language but that differ in parameters such as size \( n \) or internal structure (e.g., different value of \( \langle k^2 \rangle \)) [7]. We hope that our algorithms promote further research on crossings in linguistic and biological sequences.

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Appendix A. Proofs

Here we give the proofs of many of the propositions given throughout this work. In particular, we give the proofs of those non-trivial propositions that are relevant enough regarding the goal of this paper.

A.1. Proof of Proposition 1

Let \( \{st, uv\} \) be a pair of edges from \( Q \). The expression \( a_{su} + a_{sv} + a_{ts} + a_{tv} \) counts how many \( \mathcal{L}_4 \) we can make with these two edges. This is trivially true. By definition of \( Q \), \( a_{st} = a_{uv} = 1 \). To these two edges, we only have to add one of the four edges in the expression (i.e., any of the edges that connect a vertex of \( st \) with another vertex of \( uv \)) to make a \( \mathcal{L}_4 \). Each of the edges in the expression produces a distinct \( \mathcal{L}_4 \) (Figure A1).

Let \( \mathcal{L}_4(q_1) \) be the set of \( \mathcal{L}_4 \) into which \( q_1 \in Q \) is mapped. The \( \mathcal{L}_4 \) of \( \mathcal{L}_4(q_1) \) follow a concrete pattern: the edges of \( q \) are at each end of the \( \mathcal{L}_4 \) (Figure A1). Equation 15 could be false for two reasons. Firstly, some \( \mathcal{L}_4 \) is not counted. Suppose that there exists a \( \mathcal{L}_4 \) with vertices \( s - t - u - v \) in \( G \) that is not counted in the summation. If this was true then the pair of independent edges we can make using its vertices (\( \{st, uv\} \)) would not be in \( Q \). But this cannot happen by definition of \( Q \). Secondly, some \( \mathcal{L}_4 \) is counted more than once. It can only happen when there exists a \( q_2 \in Q, q_1 \neq q_2 \) such that \( \mathcal{L}_4(q_1) \cap \mathcal{L}_4(q_2) \neq \emptyset \). For this to happen, \( q_2 \) must place the same edges as \( q_1 \) at the end of the \( \mathcal{L}_4 \), which is a contradiction because \( q_2 \neq q_1 \). Therefore, such a \( q_2 \) does not exist.
Equation 16 follows from [25] and can be expressed as

\[
\frac{1}{2} \sum_{s \neq t} (a_{st}^{(3)} - (2k_t - 1)a_{st}) = m_3 + m_1 - \sum_{s \neq t} a_{st}k_t = m_3 + m_1 - n(k^2),
\]

namely Equation 17. Recall \( m_1 \) is the number of edges of the graph.

### A.2. Proof of Proposition 2

The proof is similar to the proof of Proposition 1. For a given \( q = \{st, uv\} \in Q \), the inner summations of Equation 18, i.e.

\[
\sum_{w \in \Gamma(s, -stuv)} (a_{uw_s} + a_{vws}) + \sum_{w \in \Gamma(t, -stuv)} (a_{uw_t} + a_{vwt}),
\]

count the number of \( \mathcal{L}_5 \) we can make with edges \( st \) and \( uv \) of \( q \). This set of \( \mathcal{L}_5 \) is denoted as \( \mathcal{L}_5(q) \) and contains the \( \mathcal{L}_5 \) that follow a concrete pattern: each edge of \( q \) is at one end of the \( \mathcal{L}_5 \) and the edges of \( q \) are linked via a fifth vertex \( w \), such that \( w \neq s, t, u, v \) (Figure A2). For example, \( \mathcal{L}_5(q) \) may contain \( t - s - w - u - v \) if \( a_{sw} = a_{uw} = 1 \). Therefore, the graphs \( \mathcal{L}_5(q) \), for any \( q \in Q \), have 4 different forms determined by the choice of the vertex of each edge of \( q \) that will be at one of the ends of the \( \mathcal{L}_5 \).

Similarly, Equation 18 could be wrong for two reasons: some path may be counted more than once or not counted at all.

All \( \mathcal{L}_5 \) are counted: by contradiction, given \( \{st, uv\} \in Q \), suppose that there is a \( \mathcal{L}_5 \), \( s - t - w - u - v \), not counted in the inner summation. By definition of \( \mathcal{L}_5 \), \( a_{st} = a_{tw} = a_{uw} = a_{uw} = 1 \), the vertices are distinct and we have \( \{st, uv\} \in Q \). Therefore, if such \( \mathcal{L}_5 \) is not counted then \( \{st, uv\} \) would not be in \( Q \).

Some \( \mathcal{L}_5 \) may be counted more than once: if this was true then for some \( q_1 = \{st, uv\} \in Q \) there would exist a \( q_2 \in Q \), \( q_1 \neq q_2 \), such that \( \mathcal{L}_5(q_1) \cap \mathcal{L}_5(q_2) \neq \emptyset \). For this to happen, \( q_2 \) must place the same edges as \( q_1 \) at the end of the \( \mathcal{L}_5 \), which is a contradiction because \( q_2 \neq q_1 \). Therefore, such a \( q_2 \) does not exist.

As a conclusion, all paths are counted, and no path is counted more than once. Therefore, Equation 18 evaluates to exactly all \( \mathcal{L}_5 \) in \( G \).
Figure A3.: All possible paw graphs (Figure 4(a)), that can be made with \( \{st, uv\} \in Q \) given the adjacencies at the bottom of each graph.

### A.3. Proof of Proposition 3

The proof is similar to that of Proposition 1. We first show that the term in the summation

\[
(a_{tu} + a_{sv})(a_{tv} + a_{su}) = a_{tu}a_{tv} + a_{tu}a_{su} + a_{sv}a_{tv} + a_{sv}a_{su}
\]

counts the amount of subgraphs isomorphic to \( Z \) in \( G \) that we can form given \( \{st, uv\} \in Q \). This can be easily seen in Figure A3 where, for a given \( q = \{st, uv\} \in Q \), are shown all possible labeled graphs, \( Z(q) \), isomorphic to \( Z \), that can be made with \( st \) and \( uv \) assuming the existence of the pairs of edges in the summation \( a_{tu}a_{tv} + a_{tu}a_{su} + a_{sv}a_{tv} + a_{sv}a_{su} \). This means that when counting how many of these pairs of adjacencies exist we are actually counting how many subgraphs isomorphic to \( Z \) exist that have these four vertices.

Now we prove the claim in this proposition by contradiction. Since we know that the term inside the summation counts all labeled \( Z \) that can be made with every element of \( Q \), the claim can only be false for two reasons: in the whole summation of Equation 19 some \( Z \) is not counted, and/or some of these \( Z \) is counted more than once.

Firstly, it is clear that all \( Z \) are counted at least once. If one was not counted then the element of \( Q \) we can make with its vertices would not be in \( Q \) which cannot happen by definition of \( Q \).

Secondly, none is counted more than once. Let \( Z(q_1) \) be the set of labeled graphs \( q_1 = \{e_1, e_2\} \in Q \) is mapped to. A paw is a triangle with a link attached to it. The paws in \( Z(\{e_1, e_2\}) \) follow a concrete pattern: \( e_1 \) is linked to a triangle containing \( e_2 \), or the other way around, \( e_2 \) is linked to a triangle containing \( e_1 \) (Figure A3). If any \( Z \in Z(q_1) \) is counted twice then there exists a different \( q_2 \in Q \) such that \( Z(q_1) \cap Z(q_2) \neq \emptyset \). For this to happen, \( q_2 \) must place the same edges as \( q_1 \) in and outside the triangle of the paw, which is a contradiction because \( q_2 \neq q_1 \). Therefore, such a \( q_2 \) does not exist.

### A.4. Proof of Proposition 4

We use \( Y = C_3 \oplus L_2 \) for brevity. Similarly as in previous proofs we first show that the inner summation of the right hand side of Equation 20, i.e.,

\[
\sum_{w_s \in \Gamma(s, -stuv)} a_{tw_s} + \sum_{w_u \in \Gamma(u, -stuv)} a_{vw_u}
\]

(A1)

counts the amount of graphs isomorphic to \( Y \) that can be made using the edges of \( q = \{st, uv\} \in Q \), which we denote as \( Y(q) \). Graphs in \( Y(q) \) follow a concrete pattern:
Figure A4.: All the labeled graphs isomorphic to $C_3 \oplus L_2$ that can be formed using a fixed element $\{st, uv\} \in Q$. In this figure, $w_s$ ($w_u$) is one of the common neighbors of $s$ and $t$ ($u$ and $v$) different from $s, t, u, v$. The graphs in (a) correspond to the first summation of Equation A1 and the graphs in (b) correspond to the second.

if edge $st$ is part of $C_3$ then $uv$ is the $L_2$, and vice versa. The $C_3$ is completed with another vertex $w$ neighbor to the other two vertices of the $C_3$. None of the graphs in $Y(q)$ are repeated and are illustrated in Figure A4.

Let $V(Y) = \{c_1, c_2, c_3, l_1, l_2\}$ be the set of vertices of a $Y$ where the $c_i \in V(C_3)$ and $l_i \in V(L_2)$. Then we have that $\{c_1c_2, l_1l_2\}, \{c_1c_3, l_1l_2\}, \{c_2c_3, l_1l_2\} \in Q$. If a $Y$ was not counted in Equation 20 it would mean that none of these elements would be in $Q$. This cannot happen by definition of $Q$. Finally, since we can make three elements of $Q$ from $V(Y)$, every $Y$ is present in the set $Y(q_i)$ of three different $q_i$. Hence the $1/3$ factor in Equation 20.

A.5. Proof of Proposition 5

The first equality was proven in [10, section 4.4.5]. Equations 21 and 22 follow from previous results in [24, 28]. Suppose that $n_G(C_4)$ is the number of different cycles of length 4 that are contained in $G$ [24]. More technically, $n_G(C_4)$ is the number of subgraphs of $G$ that are isomorphic to $C_4$.

We have that [24]

$$n_G(C_4) = \frac{1}{8} \left[ \text{tr}(A^4) - 4n_G(H_2) - 2n_G(H_1) \right]$$

(A2)

with

$$n_G(H_1) = m, \quad n_G(H_2) = \sum_{s=1}^{n} \binom{k_s}{2}.$$

A similar formula for $n_G(C_4)$ was derived in the pioneering research by Harary and Manvel [28]. The fact that $n_G(H_1) = n_G(L_2)$ and $n_G(H_2) = n_G(L_3)$ [24], transforms Equation A2 into Equation 21. Recalling the definition of $q$ in [29]

$$q = \binom{m}{2} - \sum_{s=1}^{n} \binom{k_s}{2},$$

we may write $n_G(C_4)$ equivalently as

$$n_G(C_4) = \frac{1}{8} \left[ \text{tr}(A^4) + 4q - 2m^2 \right].$$
A.6. Proof of Proposition 6

The amount of edges of the graph induced from the removal of vertices $s$ and $t$

$$|E(G_{st})| = \sum_{uv \in E(G_{st})} 1 = q(s, t) = m + 1 - k_s - k_t, \quad (A3)$$

where $q(s, t)$ is the number of edges that the may potentially cross the edge formed by $s$ and $t$ (only independent edges can cross) [10].

The proof is straightforward. First,

$$K = \sum_{\{st, uv\} \in Q} (k_s + k_t + k_u + k_v)$$

$$= \sum_{\{st, uv\} \in Q} (k_s + k_t) + \sum_{\{st, uv\} \in Q} (k_u + k_v)$$

$$= \frac{1}{2} \left[ \sum_{st \in E} \sum_{uv \in E(G_{st})} (k_s + k_t) + \sum_{st \in E} \sum_{uv \in E(G_{st})} (k_u + k_v) \right]$$

$$= \sum_{st \in E} (k_s + k_t) \sum_{uv \in E(G_{st})} 1.$$

Thanks to Equation A3 one obtains

$$K = \sum_{st \in E} (k_s + k_t)(m + 1 - k_s - k_t)$$

$$= n[(m + 1)(k^2) - \langle k^3 \rangle] - 2\psi,$$

where $\psi$ is defined in Equation 12.

A.7. Proof of Proposition 7

The proof is simple. Notice that

$$\Phi_1 = \sum_{\{st, uv\} \in Q} k_s k_t + \sum_{\{st, uv\} \in Q} k_u k_v$$

$$= \frac{1}{2} \left[ \sum_{st \in E} \sum_{uv \in E(G_{st})} k_s k_t + \sum_{uv \in E} \sum_{st \in E(G_{st})} k_u k_v \right]$$

$$= \sum_{st \in E} \sum_{uv \in E(G_{st})} k_s k_t = \sum_{st \in E} k_s k_t \sum_{uv \in E(G_{st})} 1.$$

Thanks to Equation A3, Equation 24 follows immediately.
A.8. Proof of Proposition 8

The proof is also straightforward. We start by noticing that

$$\Phi_2 = \frac{1}{2} \sum_{st\in E} (k_s + k_t) \sum_{uv\in E(G_{-st})} (k_u + k_v).$$

For a fixed edge \(\{s,t\}\), the inner summation above sums over the set

$$E(G_{-st}) = E \setminus ( \{us \in E : u \in V \setminus \{t\}\} \cup \{ut \in E : u \in V \setminus \{s\}\} \cup \{st\}).$$

Then, for a fixed edge \(\{s,t\}\), this leads to

$$\sum_{uv\in E(G_{-st})} = \sum_{uv\in E} (k_u + k_v) = \sum_{us\in E} k_u - \sum_{us\in E} k_s - \sum_{ut\in E} k_t - (k_s + k_t)$$

$$= \sum_{u\in V} k_u^2 - \sum_{us\in E} k_u - k_s(k_s - 1) - \sum_{ut\in E} k_t - k_t(k_t - 1) - (k_s + k_t)$$

$$= n(k^2) - (\xi(s) + \xi(t)) - k_s(k_s - 1) - k_t(k_t - 1).$$

A.9. Proof of Proposition 12

Consider three vertices \(s,t,u\) inducing a path of 3 vertices in \(G\): \((s,t,u)\). We can count all induced subgraphs \(L_4\) that start with vertices \(s,t,u\) and finish at \(v \in \Gamma(u)\) by counting how many \(v\) are different from \(s\) and \(t\) in the neighborhood of \(u\), \(\Gamma(u)\). Then,

$$n_G(L_4) = \frac{1}{2} \sum_{s\in V} \sum_{t\in \Gamma(s)} \sum_{u\in \Gamma(s) \setminus \{t\}} \sum_{v\in \Gamma(u) \setminus \{s,t\}} 1.$$ 

We replace the inner-most summation with the expression \(k_u - 1 - a_{su}\). This expression, when summed over the vertices \(u \in \Gamma(t) \setminus \{s\}\), can be simplified further leading to

$$n_G(L_4) = \frac{1}{2} \sum_{s\in V} \sum_{t\in \Gamma(s)} (\xi(t) - (k_s + k_t) + 1 - c(s,t))$$

$$= \frac{1}{2} \sum_{st\in E} (\xi(s) + \xi(t) - 2(k_s + k_t) + 2 - 2|c(s,t)|).$$

Obtaining the expression in Equation 52 is now straightforward.

A.10. Proof of Proposition 13

The proof is also straightforward and similar to the proof of proposition 12.

$$n_G(L_5) = \frac{1}{2} \sum_{s\in V} \sum_{t\in \Gamma(s)} \sum_{u\in \Gamma(s) \setminus \{t\}} \sum_{v\in \Gamma(t) \setminus \{s,u\}} \sum_{w\in \Gamma(u) \setminus \{s,t,v\}} 1.$$
\[ \frac{1}{2} \sum_{s \in V} \sum_{t \in \Gamma(s)} \sum_{u \in \Gamma(t) \setminus \{s\}} (k_u - 1 - a_{ut} - a_{uv}) \]

\[ = \frac{1}{2} \sum_{s \in V} \sum_{t \in \Gamma(s)} \sum_{u \in \Gamma(t) \setminus \{s\}} ((k_t - 1 - a_{ut})(k_u - 1 - a_{ut}) + 1 - |c(t, u)|) \]

\[ = \frac{1}{2} \sum_{s \in V} \sum_{t \in \Gamma(s)} \sum_{u \in \Gamma(t) \setminus \{s\}} \left( \left( k_t - 1 \right) \left( k_u - 1 \right) - 2S_{s, t} \right), \]

where \( S_{s, t} \) is defined in Equation 11.

**A.11. Proof of Proposition 14**

First, take notice that whenever one of the adjacencies \( a_{su}, a_{sv}, a_{tu} \) or \( a_{tv} \) equals 1, the summation in Equation 34 adds the degree of the first and last vertices of the \( L_4 \) induced by the edges \( st, uv \) and the adjacencies that equal 1. Therefore, it is easy to see that

\[ \Lambda_1 = \sum_{st \in E} \sum_{u \in \Gamma(s) \setminus \{t\}} \sum_{v \in \Gamma(t) \setminus \{s\}} (k_u + k_v) \]

\[ = \sum_{st \in E} \left( (k_t - 1)(\xi(s) - k_t) + (k_u - 1)(\xi(t) - k_u) - 2S_{s, t} \right), \]

**A.12. Proof of Proposition 15**

Similarly as in Proposition 14, we can see that the summation in Equation 38 adds the degrees of the vertices of each \( L_4 \) in \( G \). Therefore, we can express Equation 38 equivalently as

\[ \Lambda_2 = \sum_{st \in E} \sum_{u \in \Gamma(s) \setminus \{t\}} \sum_{v \in \Gamma(t) \setminus \{s\}} (k_u + k_v + k_s + k_t) \]

\[ = \Lambda_1 + \sum_{st \in E} \sum_{u \in \Gamma(s) \setminus \{t\}} \sum_{v \in \Gamma(t) \setminus \{s\}} (k_u + k_v) \]

\[ = \Lambda_1 + \sum_{st \in E} (k_u + k_v)((k_s - 1)(k_t - 1) - |c(s, t)|). \]

**A.13. Proof of Proposition 18**

Any \( L_5 \) has only one centroidal vertex \( s \in V \). For any pair of different neighbors of \( s, t \in \Gamma(s) \) and \( u \in \Gamma(s) \setminus \{t\} \), the product \( (k_t - 1)(k_u - 1) \) gives the amount of \( L_5 \) with centroidal vertex \( s \) and through vertices \( t \) and \( u \). Therefore

\[ n_T(L_5) = \sum_{s \in V} \left( \frac{1}{2} \sum_{t \in \Gamma(s)} \sum_{u \in \Gamma(s) \setminus \{t\}} (k_t - 1)(k_u - 1) \right). \]
Notice that the two inner summations of equation A4 count such paths, twice. As

$$\sum_{u \in \Gamma(s) \setminus \{t\}} (k_u - 1) = -k_s + 1 + \sum_{u \in \Gamma(s) \setminus \{t\}} k_u = \xi(s) - k_t - k_s + 1$$

we finally obtain

$$n_T(L_5) = \frac{1}{2} \sum_{s \in V} \left( \sum_{t \in \Gamma(s)} g_2(s, t) = \frac{1}{2} \sum_{st \in E} (g_2(s, t) + g_2(t, s)) \right)$$

with $g_2(s, t) = (k_t - 1)(\xi(s) - k_t - k_s - 1)$.

Appendix B. Testing protocol

The derivations of the $f_\omega$'s in Section 3.3 and the algorithms to compute $V_s [C]$ presented in Section 4 have been tested thoroughly via automated tests. For these algorithms we only consider the case of uniformly random linear arrangements, i.e., $V_{rla} [C]$. Here we detail how we assessed the correctness of the work presented above.

The calculation of the $f_\omega$'s have been tested comparing three different but equivalent procedures whose results must coincide. Firstly, the $f_\omega$'s are computed by classifying all elements of $Q \times Q$ into their corresponding $\omega$ (Table 2 for the classification criteria). Secondly, the $f_\omega$'s are computed via Equation 7 after counting by brute force the amount of subgraphs corresponding to each $\omega$ (Figure 2). Finally, the $f_\omega$'s are calculated using the expressions summarized in Table 4 with a direct implementation of the corresponding arithmetic expression. Such a three-way test was performed on Erdős-Rényi random graphs $G_{n,p}$ [30, Section V] of several sizes ($1 \leq n \leq 50$) and three different probabilities of edge creation $p = 0.1, 0.2, 0.5$. The test was also formed on particular types of graphs for which formulae for the $f_\omega$'s that depend only on $n$ are known: cycle graphs, linear trees, complete graphs, complete bipartite graphs, star trees and quasi star trees [10], for values of $n \leq 100$.

The algorithms in Section 4 have been tested in three ensembles of graphs: general graphs, forest and trees. The values of $V_{rla} [C]$ are always represented as an exact rational value (the GMP library (see https://gmplib.org/) provides implementations of these numbers). In each ensemble, $V_{rla} [C]$ is computed in a certain number of different ways. The test consists of checking that all the ways give the same result. The first way, $V_{rla} [C]^{(1)}$, consists of computing $V_{rla} [C]$ by brute force, i.e., by classifying all elements in $Q \times Q$ to compute the values of the $f_\omega$'s that are in turn used to obtain $V_{rla} [C]$ via Equation 2. The second way, $V_{rla} [C]^{(2)}$, is obtained computing $V_{rla} [C]$ with a direct implementation of the derivations of the $f_\omega$'s (summarized in Table 4). $V_{rla} [C]^{(3)}$, is obtained computing $V_{rla} [C]$ via Algorithm 4.2. Finally, we also computed $V_{rla} [C]$ in forests using Algorithm 4.3, denoted as $V_{rla} [C]^{(4)}$, and in trees, denoted as $V_{rla} [C]^{(5)}$. Within in each ensemble of graphs, the details of the test are as follows:

1. General graphs. In this group we computed $V_{rla} [C]^{(i)}$ for $i = 1, 2, 3$ for all the following graphs: Erdős-Rényi graphs (for $10 \leq n \leq 50$ and $p = 0.1, \ldots, 1.0$), complete graphs ($n \leq 20$), complete bipartite graphs (all pairs of sizes with $2 \leq n_1, n_2 \leq 9$), linear trees ($2 \leq n \leq 100$), one-regular graphs (all even values
of $2 \leq n \leq 100$), quasi-star trees ($2 \leq n \leq 100$), star trees ($4 \leq n \leq 100$) and cycle graphs ($2 \leq n \leq 100$). Since the computation of $V_{rla} [C]^{(1)}$ is extremely time-consuming in dense Erdős-Rényi graphs with a high number of vertices ($n \geq 40$, $p \geq 0.6$), we computed it once for all these graphs and stored it on disk.

(2) Forests. In this group we computed $V_{rla} [C]^{(i)}$ for $i = 1, 2, 3, 4$ in all the trees listed above and also in forests of random trees. We generated these forests by joining several trees of potentially different sizes generated uniformly at random. The total size of the forest was always kept under $n \leq 270$.

(3) Trees. In this group we computed $V_{rla} [C]^{(i)}$ for $i = 1, 2, 3, 4, 5$, in all free unla-

Appendix C. Improving the algorithm by reusing computations

Algorithm C.1: Update the hash table $H$ of Algorithm C.2, if necessary.

| Function COMPUTEANDSTORE(H, G, u, v) is |
|---|
| 1 | $|c(u, v)| \leftarrow 0$ |
| 2 | $S_{u,v} \leftarrow 0$ // Equation 11 |
| 3 | if $\langle u, v \rangle \notin H$ then |
| 4 | // Compute values $|c(u, v)|$ and $S_{u,v}$ |
| 5 | for $w \in \Gamma(u) \cap \Gamma(v)$ do |
| 6 | $|c(u, v)| \leftarrow |c(u, v)| + 1$ |
| 7 | $S_{u,v} \leftarrow S_{u,v} + k_w$ |
| 8 | // Store $|c(u, v)|$ and $S_{u,v}$ in $H$ indexed with key $\langle u, v \rangle$ |
| 9 | $H \leftarrow H \cup \{\langle u, v \rangle, \{|c(u, v)|, S_{u,v}\}\}$ |
| 10 | else |
| 11 | // Retrieve $|c(u, v)|$ and $S_{u,v}$ from the table |
| 12 | $|c(u, v)| \leftarrow H(\langle u, v \rangle), |c(u, v)|$ |
| 13 | $S_{u,v} \leftarrow H(\langle u, v \rangle), S_{u,v}$ |
| 14 | return $\{|c(u, v)|, S_{u,v}\}$ |

In this section we improve Algorithm 4.2 by reusing computations. This new algorithm is detailed in Algorithm C.2. We also analyze the complexity of the new algorithm in Proposition 20 where we show how the transitivity index of a graph $T = 3n_G(C_3)/n_G(L_3)$ [16, Chapter 4.5.1] influences its time and space complexity.

Algorithm C.2 reuses the calculation of the number of common neighbors of two, not necessarily connected vertices $u, v$, i.e., $|c(u, v)|$, defined in Equation 10, and the sum of the degrees of the vertices that are neighbors of both vertices $S_{u,v}$, defined in Equation 11. These values are marked in red in Algorithm 4.2. In order to reuse them, Algorithm C.2 makes use of a hash table $H$ whose keys are unordered pairs of vertices $u$ and $v$, denoted as $\langle u, v \rangle$, and the associated values are $|c(u, v)|$ and $S_{u,v}$. Keys are made up of vertices that are either adjacent ($\{u, v\} \in E$) or there exists another vertex $w$ such that $a_{uw} = a_{wv} = 1$. Notice that these cases are not mutually exclusive: if two
vertices are both adjacent and connected via a third vertex (i.e., when \( u \) and \( v \) are vertices of a \( C_3 \)) then the values associated are computed only once and the pair \( \langle u, v \rangle \) is stored only once. The same applies if \( u \) and \( v \) are not adjacent but are connected via several other vertices.

Whenever \(|c(u, v)|\) or \(S_{u,v} \) are needed, the pair \( \langle u, v \rangle \) is first searched in \( H \). If \( H \) has such pair, its associated values are retrieved. If it does not, both \(|c(u, v)|\) and \(S_{u,v} \) are computed and stored in \( H \). This update step is detailed in Algorithm C.1. These ideas yield Algorithm C.2, where changes with respect to the original algorithm are marked in red.

### Algorithm C.2: Calculate \( \mathbb{V}_s[C] \) in general graphs reusing computations.

**Input:** \( G = (V, E) \) a graph as described in proposition 16.

**Output:** \( \mathbb{V}_s[C] \), the variance of the number of crossings.

**Function** \( \text{VarianceC}(G) \) is

1. **Setup** \( G \) // Algorithm 4.1
2. for \( \{s, t\} \in E \) do
   3. for \( u_1 \in \Gamma(s) \setminus \{t\} \) do
      4. \[ |c(t, u_1)|, \_ \leftarrow \text{ComputeAndStore}(H, G, t, u_1) \]
      5. \[ n_G(L_5) \leftarrow n_G(L_5) - |c(t, u_1)| + (k_t - 1 - a_{tu_1})(k_u - 1 - a_{tu_1}) + 1 \]
   6. for \( u_2 \in \Gamma(t) \setminus \{s\} \) do
      7. \[ |c(s, u_2)|, \_ \leftarrow \text{ComputeAndStore}(H, G, s, u_2) \]
      8. \[ n_G(L_5) \leftarrow n_G(L_5) - |c(s, u_2)| + (k_s - 1 - a_{su_2})(k_u - 1 - a_{su_2}) + 1 \]
      9. \[ n_G(C_4) \leftarrow n_G(C_4) + c_{s,u_2} - 1 \]
   10. \[ |c(s, t)|, S_{s,t} \leftarrow \text{ComputeAndStore}(H, G, s, t) \]
   11. \[ n_G(Z) \leftarrow n_G(Z) + S_{s,t} - 2|c(s, t)| \]
   12. \[ n_G(Y) \leftarrow n_G(Y) + (m - k_s - k_t + 3)|c(s, t)| - S_{s,t} \]
   13. \[ \psi \leftarrow \psi + k_s k_t \]
   14. \[ \Phi_1 \leftarrow \Phi_1 - k_s k_t (k_s + k_t) \]
   15. \[ \Phi_2 \leftarrow \Phi_2 + \left( k_s - 3 \right) (k_t - 3) \]
   16. \[ \mu_1 \leftarrow \mu_1 - k_s (k_t - 3) \]
   17. \[ \mu_2 \leftarrow \mu_2 - 2|c(s, t)| \]
   18. \[ A_1 \leftarrow A_1 + (k_t - 1)(\xi(s) - k_t) + (\xi(t) - k_s)(k_t - 1) - 2S_{s,t} \]
   19. \[ A_2 \leftarrow A_2 + (k_s - 3)(k_t - 3) \]
   20. \[ q \leftarrow \frac{1}{2} m(k_t + 1 - n) \]
   21. \[ K \leftarrow \left( m + 1 \right) n - n(k^2) - 2\psi \]
   22. \[ \Phi_1 \leftarrow \Phi_1 - \left( k_t + 1 \right) \psi \]
   23. \[ \Phi_2 \leftarrow \frac{1}{2} \Phi_2 \]
   24. \[ n_G(Y) \leftarrow \frac{1}{2} n_G(Y) \]
   25. \[ n_G(C_4) \leftarrow \frac{1}{4} n_G(C_4) \]
   26. \[ n_G(C_4) \leftarrow m - n(k^2) + \frac{1}{2} \mu_1 - \mu_2 \]
   27. \[ n_G(L_5) \leftarrow \frac{1}{2} n_G(L_5) \]
   28. \[ A_1 \leftarrow A_1 + A_2 \]
   29. \[ \text{Compute} \mathbb{V}_s[C] \text{ by instating Equation 45 appropriately} \]

**Proposition 20.** Consider a graph \( G = (V, E) \) to be as in Proposition 16. Algorithm
C.2 computes $V^* C$ in $G$ in time $O(n + k_{\text{max}}|H|)$ and space complexity $O(n + |H|)$, where $|H|$ is the size of the hash table $H$ at the end of the algorithm

$$|H| = O\left(n(k^2) - n_G(C_3)\right) = O\left(Tm + (1 - T)n(k^2)\right)$$

(C1)

where $T = 3n_G(C_3)/n_G(L_3)$ is the so-called transitivity index [16, Chapter 4.5.1].

**Proof.** The space complexity of this algorithm depends on the memory used to store the values of function $\xi(s)$ (Equation 9) and the size of the hash table $H$ at the end of its execution. The former needs $O(n)$-space. Now follows a derivation of $H$’s size.

The size of any hash table is proportional to the amount of keys plus the values associated to each of them. In our case, the keys have fixed size (a pair of vertices) and the amount of values associated to the keys is always constant (two integers), so we only need to know the amount of keys it contains at the end of the algorithm.

As explained above, pairs of vertices $\langle u,v \rangle$ are added to $H$ in two not necessarily mutually exclusive cases. However, we consider them to be so in order to obtain an upper bound of $|H|$, and define case (1) when the vertices are an edge of the graph and are not connected via a third vertex, and case (2) when the opposite happens, i.e., when the two vertices are not an edge of the graph but are connected via a third vertex. In (1), $\rho_1$ pairs are added to $H$, and in (2), $\rho_2$. Then, $|H| = O(\rho_1 + \rho_2)$.

Consider case (1). The pair can only be added to $H$ by the call to algorithm C.1 in line 11. This means that all adjacent pairs of vertices contribute to $H$ with $\rho_1$ unique entries in total.

Consider case (2). In this case, $u$ and $v$ are the vertices of an open $L_3$ (if $u$ and $v$ were adjacent then the $L_3$ would be closed, also a $C_3$). The exact value of $\rho_2$ is the amount of pairs of vertices $s, t$ such that $a_{st} = 0$ and $a_{st}^{(2)} \neq 0$.

$$\rho_2 = \sum_{\substack{s < t \\ a_{st}^{(1)} = 0 \land a_{st}^{(2)} \neq 0}} 1.$$  

An upper bound of $\rho_2$ is the number of open $L_3$ in a graph, i.e.,

$$\rho_2 \leq \text{no. open } L_3 = \text{no. open and closed } L_3 - \text{no. closed } L_3 = n_G(L_3) - 3n_G(C_3) = (1 - T)n_G(L_3).$$

Therefore, the time complexity of this algorithm is the cost of the Setup algorithm (Algorithm 4.1) plus the cost of Algorithm C.1 for those pairs of vertices that are to be added to $H$, i.e., $O(n + m + k_{\text{max}}|H|)$ and the space complexity is $O(n + |H|)$ where $|H| = O(\rho_1 + \rho_2) = O(m + n_G(L_3) - n_G(C_3))$. This size can be expressed in two different ways. First, since [31, 103]

$$n_G(L_3) = \sum_{i=1}^{n} \binom{k_i}{2} = \frac{n}{2} \langle k^2 \rangle - m,$$

(C2)

we get that

$$|H| = O\left(n\langle k^2 \rangle - n_G(C_3)\right).$$
Moreover, we can express the size of $H$ using the transitivity index $T$

$$|H| = O(m + (1 - T)n_G(L_3)).$$  \hspace{1cm} (C3)

Applying Equation C2 to Equation C3, we obtain via straightforward arithmetic operations the cost in Equation C1. Finally, since $|H| \geq m$, the cost of the algorithm is $O(n + m + k_{max}|H|) = O(n + k_{max}|H|)$.

Comparing the running time of the original algorithm (Algorithm 4.2) and the algorithm that reuses computations (Algorithm C.2), it is easy to see that the latter reduces the asymptotic upper bound of the cost by $O(k_{max}n_G(L_3))$, suggesting that the latter should be faster in denser graphs. However, this does not imply Algorithm C.2 is going to be faster in general: not only the time costs are estimates of upper bounds but also the terms $n(k^2)$ and $n_G(L_3)$ are not independent (neither are the $\rho_1$ and $\rho_2$ in the proof above). Therefore, we evaluate the potential speed-up of Algorithm C.2 with the help of Erdös-Rényi random graphs in $G_{n,p}$ [30, Section V].

Table C1 shows that Algorithm C.2 is faster for sufficiently large $n$ and $p$ as Algorithm C.1. The maximum speed up was achieved for graphs in $G_{150,0.70}$, where the algorithm that reuses computations turned out to be about 6 times faster on average.

The results confirm our expectation of an advantage in denser graphs as they have more triangles, but also indicate that such advantage is lost when the graphs is too dense. Furthermore, the denser the graph the higher is the amount of memory that Algorithm C.2 uses. These issues should be the subject of future research.
Table C1.: The ratio between the execution time of the algorithm to compute $V_{rla} [C]$ when it reuses computations (Algorithm C.2) and when it does not reuse computations (Algorithm 4.2) as a function of the parameters $n$ and $p$ of the Erdős-Rényi model. The execution time for algorithm C.2 is the average of its execution time on 10 graphs in $G_{n,p}$. The execution time for a single graph in $G_{n,p}$ is the average over $r$ executions of the algorithm on a single graph in $G_{n,p}$. Algorithm 4.2’s execution time was measured likewise. The speedup is calculated as the ratio of the two final averages. We used different values of $r$ depending on the value of $p$: for $p < 0.05$ we used $r = 1000$, for $0.05 \leq p \leq 0.15$ we used $r = 100$, and for $p \geq 0.2$ we used $r = 10$.  

| $n \setminus p$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.10 | 0.15 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 10              | 1.009 | 1.031 | 1.002 | 0.931 | 0.935 | 0.936 | 0.911 | 0.793 | 0.747 | 0.865 | 0.973 | 0.951 | 1.173 | 1.232 | 1.198 |
| 20              | 0.963 | 0.964 | 0.886 | 0.873 | 0.842 | 0.645 | 0.589 | 0.632 | 0.813 | 0.989 | 1.284 | 1.541 | 1.723 | 1.763 | 1.691 | 1.452 |
| 30              | 0.964 | 0.934 | 0.783 | 0.689 | 0.629 | 0.510 | 0.556 | 0.668 | 1.014 | 1.402 | 1.793 | 2.070 | 2.229 | 2.166 | 1.938 | 1.498 |
| 40              | 0.894 | 0.756 | 0.626 | 0.544 | 0.504 | 0.498 | 0.611 | 0.807 | 1.253 | 1.736 | 2.168 | 2.495 | 2.590 | 2.493 | 2.188 | 1.592 |
| 50              | 0.810 | 0.643 | 0.512 | 0.467 | 0.436 | 0.501 | 0.689 | 0.938 | 1.474 | 2.027 | 2.512 | 2.791 | 2.930 | 2.809 | 2.432 | 1.659 |
| 60              | 0.769 | 0.548 | 0.442 | 0.416 | 0.409 | 0.543 | 0.755 | 1.063 | 1.672 | 2.270 | 2.758 | 3.112 | 3.284 | 3.127 | 2.627 | 1.754 |
| 70              | 0.614 | 0.433 | 0.402 | 0.371 | 0.382 | 0.544 | 0.799 | 1.140 | 1.832 | 2.468 | 3.031 | 3.474 | 3.630 | 3.435 | 2.803 | 1.721 |
| 80              | 0.565 | 0.391 | 0.357 | 0.369 | 0.380 | 0.598 | 0.870 | 1.276 | 2.024 | 2.680 | 3.327 | 3.757 | 3.942 | 3.702 | 3.051 | 1.834 |
| 90              | 0.507 | 0.367 | 0.352 | 0.366 | 0.391 | 0.637 | 0.964 | 1.363 | 2.181 | 2.913 | 3.596 | 4.089 | 4.239 | 4.042 | 3.288 | 1.937 |
| 100             | 0.446 | 0.354 | 0.351 | 0.370 | 0.411 | 0.671 | 1.038 | 1.461 | 2.323 | 3.107 | 3.824 | 4.350 | 4.557 | 4.368 | 3.546 | 1.999 |
| 110             | 0.399 | 0.328 | 0.338 | 0.366 | 0.401 | 0.687 | 1.071 | 1.505 | 2.372 | 3.197 | 3.945 | 4.497 | 4.834 | 4.624 | 3.727 | 2.118 |
| 120             | 0.380 | 0.329 | 0.343 | 0.391 | 0.431 | 0.737 | 1.156 | 1.624 | 2.565 | 3.457 | 4.269 | 4.911 | 5.159 | 4.943 | 3.950 | 2.067 |
| 130             | 0.367 | 0.326 | 0.348 | 0.390 | 0.441 | 0.782 | 1.211 | 1.703 | 2.700 | 3.669 | 4.530 | 5.237 | 5.517 | 5.183 | 3.997 | 2.179 |
| 140             | 0.347 | 0.322 | 0.361 | 0.405 | 0.457 | 0.812 | 1.278 | 1.786 | 2.838 | 3.855 | 4.763 | 5.507 | 5.787 | 5.428 | 4.285 | 2.179 |
| 150             | 0.330 | 0.331 | 0.359 | 0.415 | 0.480 | 0.844 | 1.331 | 1.875 | 2.969 | 4.024 | 5.055 | 5.850 | 6.085 | 5.596 | 4.498 | 2.123 |
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