INDUCTION AND ABSORPTION OF REPRESENTATIONS
AND AMENABILITY OF BANACH *-ALGEBRAIC

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Abstract. Given a Fell bundle \( \mathcal{B} \) (saturated or not) over \( G \) and a closed subgroup \( H \subset G \), we prove that any *-representation of the reduction \( \mathcal{B}_H \) can be induced to \( \mathcal{B} \). We observe that Exel-Ng’s reduced cross sectional C*-algebra \( C^*_r(\mathcal{B}) \) is universal for the *-representations induced from \( B_e = \mathcal{B}_e \{e\} \) and construct a cross sectional C*-algebra of \( \mathcal{B} \), \( C^*_H(\mathcal{B}) \), that is universal for the *-representations induced from \( \mathcal{B}_H \).

We prove an absorption principle for \( C^*_H(\mathcal{B}) \) with respect to tensor products of *-representations of \( \mathcal{B} \) and *-representations of \( G \) induced from \( H \). Using this principle we show, among other results, that given closed normal subgroups of \( G \), \( H \subset K \), there exists a quotient map \( q_{KB} : C^*_K(\mathcal{B}_K) \rightarrow C^*_H(\mathcal{B}_H) \) which is a C*-isomorphism if and only if \( q_{KB} : C^*(\mathcal{B}_K) \rightarrow C^*_H(\mathcal{B}_K) \) is a C*-isomorphism. We also prove the two conditions above hold if \( q_{KB} : C^*_K(G) \rightarrow C^*_H(G) \) is a C*-isomorphism. All the constructions are performed using Banach *-algebraic bundles having a strong approximate unit.

Contents

Introduction 1
1. An addenda to Fell’s book on Banach *-algebraic bundles 4
2. Induction of *-representations 8
2.1. Integration and disintegration of *-representations 8
2.2. Positive *-representations and induction 10
2.3. Weak containment and Fell’s absorption principle 19
2.4. Induction in stages 27
3. Amenability 29
3.1. Amenability and reductions to normal subgroups 34
4. C*-completions of Banach *-algebraic bundles 38
4.1. Cross sectional bundles, C*-completions and induction 41
References 48

Introduction

The (universal) crossed product of a C*-dynamical system \((A, G, \alpha)\) is a C*-algebra \( A \rtimes_\alpha G \) which is universal for the covariant pairs representing the system [10, 2.6]. Given a closed subgroup \( H \) of the locally compact and Hausdorff (LCH) group \( G \), one may construct a crossed product \( A \rtimes_{H\alpha} G \) inducing a faithful *-representation from the crossed product \( A \rtimes_{\alpha|_H} H \) associated to the restriction \((A, H, \alpha|_H)\) (see, for example, [10, 7.2]). The reduced crossed product of \((A, G, \alpha)\), \( A \rtimes_{r\alpha} G \), is just the crossed product corresponding to the subgroup \( \{e\}, A \rtimes_{\{e\}\alpha} G \). By the induction in stages [10, Theorem 5.9], for every inclusion \( H \rightarrow K \) of subgroups of \( G \) we have a

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The crossed product \( \tilde{C} \) of \( B \) and \( A \) is the reduced cross sectional C*-algebra \( C^* \) of \( B \). The reduced cross sectional C*-algebra \( L^2_\alpha(B) \) is defined as the image of \( C^*(B) \) under the integrated form \( \Lambda^B : C^*(B) \to \mathbb{B}(L^2_\alpha(B)) \) they call the regular representation of \( B \). Reduced cross sectional C*-algebra \( L^2_\alpha(B) \) is defined as the image of \( C^*(B) \) under the integrated form \( \Lambda^B : C^*(B) \to \mathbb{B}(L^2_\alpha(B)) \) of \( B \). Quite interestingly, all the *-representations of \( B = \mathbb{B} \) are \( \mathbb{B} \)-positive by [7, XI 8.9]. Then one can induce any non degenerate *-representation \( \pi : B \to \mathbb{B}(Y) \) via Feller's concrete induction process [7, XI 9.24] to produce the (concretely) induced *-representation \( \text{Ind}_{B \equiv \mathbb{B}}(\pi) : B \to \mathbb{B}(Z) \). The *-representation abstractly induced by \( \pi \) is (by construction [7, XI 9.25])
\[
\Lambda^B \otimes_{\pi} 1 : B \to \mathbb{B}(L^2(B) \otimes_{\pi} Y), \ b \mapsto \Lambda^B_b \otimes_{\pi} 1,
\]
and it is unitary equivalent to \( \text{Ind}_{B \equiv \mathbb{B}}(\pi) \) by [7, XI 9.26]. One may then think of \( C^*(B) \) as the universal C*-algebra for the (integrated forms of) *-representations induced from the trivial subgroup \( \{e\} \). and write \( C_\pi^*(B), C^*(B_{\{e\}}) \) and \( L^2_\pi(B) \) instead of \( C^*(B), B \) and \( L^2(B) \) (respectively).

Given a saturated Fell bundle \( B = \{B_t\}_{t \in G} \) and a closed subgroup \( H \subset G \), all the *-representations of \( B_H \) are \( \mathbb{B} \)-positive and, after some time spent consulting [7] and [6], one can produce a right \( C^*(B_H) \)-Hilbert module \( L^2_H(B) \) and a *-representation \( \Lambda^{HB} : B \to \mathbb{B}(L^2_H(B)) \) in such a way that give a *-representation \( T : B_H \to \mathbb{B}(Y) \) with integrated form \( \tilde{T} : C^*(B) \to \mathbb{B}(Y) \), the *-representation abstractly induced by \( T \) becomes
\[
\Lambda^{HB} \otimes_{\tilde{T}} 1 : B \to \mathbb{B}(L^2_H(B) \otimes_{\tilde{T}} Y), \ b \mapsto \Lambda^{HB}_b \otimes_{\tilde{T}} 1.
\]
It is then natural to natural to define the \( H \)-cross sectional C*-algebra of \( B, C^*_H(B) \), as the image of \( C^*(B) \) under the integrated form of \( \Lambda^{HB} : C^*(B) \to \mathbb{B}(L^2_H(B)) \).

As a particular case of the construction above one may consider the trivial bundle over \( G \) with constant fibre \( C, \mathcal{T}_G \). Then, for every closed subgroup \( H \subset G \), the identities
\[
C^*(G) = C^*(\mathcal{T}_G) \quad \quad \quad \quad \quad \quad \quad \quad C^*_H(G) = C^*_H(\mathcal{T}_G) \quad \quad \quad \quad \quad \quad \quad \quad (\mathcal{T}_G)_H = \mathcal{T}_H
\]
hold either by definition or by construction. It is natural to define \( C^*_H(G) := C^*_H(\mathcal{T}_G) \) and to say \( G \) is \( H \)-amenable if the canonical quotient map \( \varphi^H_G : C^*(G) \to C^*_H(G) \) is a C*-isomorphism. The induction in stages [10, Theorem 5.9] implies that \( G \) is \( H \)-amenable for every closed subgroup \( H \subset G \) if and only if it is \( \{e\} \)-amenable (i.e. amenable in the usual sense).

A Fell bundle \( B = \{B_t\}_{t \in G} \) is amenable (in Exel-Ng's sense [6]) if the natural quotient map \( \varphi^B \equiv \tilde{\Lambda}^B : C^*(B) \to C^*_H(B) \) is faithful; which is the case if \( G \) is amenable. One may then say \( B \) is amenable with respect to the closed subgroup \( H \subset G \) (or just \( H \)-amenable) if the natural quotient map \( \varphi^B_H : C^*(B) \to C^*_H(B) \) is a C*-isomorphisms.
Fell’s induction in stages [1] XI 12.15] then implies that a saturated Fell bundle $\mathcal{B} = \{B_t\}_{t \in G}$ is $H$–amenable for every closed subgroup $H \subset G$ if and only if it is amenable.

Exel’s version of Fell Absorption Principle (see [3] Section 18] and [6] Corollary 2.15]) states that given a Fell bundle $\mathcal{B} = \{B_t\}_{t \in G}$, a non degenerate *-representation $T: \mathcal{B} \to \mathbb{B}(Y)$ and letting $\mathfrak{t}t: G \to \mathbb{B}(L^2(G))$ be the left regular representation; then the integrated form of the *-representation

$$T \otimes \mathfrak{t}t: G \to \mathbb{B}(Y \otimes L^2(G)), \ (b \in B_t) \mapsto T_b \otimes \mathfrak{t}t_t$$

is weakly contained in the integrated form of $\Lambda^B \otimes_\pi 1 = \text{Ind}_{\{e\} \cap G}(\kappa)$ and $L^2(G) = L^2(G) \otimes_\kappa \mathbb{C}$. One may then replace $\{e\}$ with any closed subgroup $H \subset G$ and $\kappa$ with any *-representation $V$ of $H$ in the paragraph above and ask if the resulting statement holds. Let’s be more precise about this. Take a Fell bundle $\mathcal{B}$ over $G$, a closed subgroup $H \subset G$, a *-representation $T: \mathcal{B} \to \mathbb{B}(Y)$ and a unitary *-representation $V: H \to \mathbb{B}(Z)$. Let $\text{Ind}_H^G(V): G \to \mathbb{B}(W)$ be the unitary representation induced by $V$ and $T \otimes \text{Ind}_H^G(V): \mathcal{B} \to \mathbb{B}(Z \otimes W)$ the *-representation mapping $b \in B_t$ to $T_b \otimes \text{Ind}_H^G(V)(\lambda)$. The question is whether or not $T \otimes \text{Ind}_H^G(V)$ is weakly contained in a set of *-representations induced from $\mathcal{B}_H$.

To give a partial answer we consider the regular $\mathbb{B}(W)$—projection–valued Borel measure on $G/H$ induced by $V$, which we denote $P$ and is part of a system of imprimitivity for $G$ (see [7] XI 10 & XI 14.3]. Then $(T \otimes \text{Ind}_H^G(V), 1 \otimes P)$ is a system of imprimitivity for $\mathcal{B}$ and, in case $\mathcal{B}$ is saturated, we may use Fell’s Imprimitivity Theorem [7] XI 14.18] to deduce $T \otimes \text{Ind}_H^G(V)$ is induced from a *-representation of $\mathcal{B}_H$. Thus a general form of Exel’s Absorption Principle holds for saturated Fell bundles.

Things get much more complicated without the saturation hypothesis, mainly because one has something like [7] XI 14.18] in this situation; the closest result being [7] XI 14.17], which is not very helpful as the example given in [7] XI 14.24] shows. We think the solution to this problem is hiding in the constructions used to prove [6] Corollary 2.15] and it is our intention to reveal it. Once this is solved, one may proceed as in [5] to prove $C^*_H(\mathcal{B})$ is (isomorphic to) a $C^*$-subalgebra of $\mathbb{B}(C^*(\mathcal{B}) \otimes C^*_H(G))$ (we use minimal tensor products here). The inclusion should be given in such a way that identifying $C^*_H(G) = C^*(G)$ via the canonical quotient map, then $C^*(\mathcal{B}) = C^*_H(\mathcal{B})$ in $\mathbb{B}(C^*(\mathcal{B}) \otimes C^*(G))$. We will not be able to do this in full generality, but only when $H$ is normal in $G$.

The outline of this article is as follows. We start with a section in which we introduce most of our notation and recall Fell’s definition of positivity of *-representations with respect to a Banach *-algebraic bundle. The main result of this section states that if $\mathcal{B} = \{B_t\}_{t \in G}$ is a Fell bundle (saturated or not) and $H \subset G$ a closed subgroup, then any *-representation of $\mathcal{B}_H$ is $\mathcal{B}$–positive. This answers a question raised by Fell in [7] XI 11.10] and implies any *-representation of $\mathcal{B}_H$ can be induced to $\mathcal{B}$. As already noticed by Fell, the affirmative answer gives a characterization of $\mathcal{B}$–positivity which is much easier to work with than the original one and (in short) implies there is no loss in generality if one only considers Fell bundles instead of Banach *-algebraic bundles when developing a theory of induction of *-representations. We will go that far only in case we need it, mainly because certain grade of generality will be useful in Section 4.

In Section 2 we construct the Hilbert modules $L^2_H(\mathcal{B})$ using Fell’s abstract induction process and the theory of Hilbert modules (as presented in [8]). This way of presenting
the induction process uses several key facts of [7]. We then prove two results we will refer to as FExell’s Absorption Principle, these being the main tools we will use in the rest of the article. The last part of Section 2 is dedicated to prove *-representation (on Hilbert modules) of Banach *-algebraic modules can be induced in stages, which is just an adaptation of [11 XI 12.15].

The third section of the article is dedicated to present and solve some amenability questions. For example, we will prove that if $B = \{B_t\} \subseteq G$ is a Fell bundle and is $H \subset G$ a closed normal subgroup, then $B$ is $H$-amenabe if $G$ is so. Moreover, we will prove that given a closed normal subgroup $K$ such that $H \subset K \subset G$, there exists a canonical quotient map $q_{K,H}^B : C^*_K(B) \to C^*_H(B)$ and that this map is a $C^*$-isomorphism whenever it’s analogue $q_{K,H}^G : C^*_K(G) \to C^*_H(G)$ is a $C^*$-isomorphism. FExell’s Absorption Principle will be of key importance in this section.

In the final part of this article we prove the Fell bundle version of the well known fact that given a closed normal subgroup $N$ of $G$, $G$ is amenable if and only if both $N$ and $G/N$ are amenable. If now one is given a Fell bundle $B$ over $G$, then the rôles of $G$, $N$ and $G/N$ are played by $B$, $B_N$ and the bundle $C^*$-completion of the partial cross-sectional bundle over $G/N$ derived from $B$ [7 VIII 6 & 16]. We will also construct other $C^*$-completions of this last partial cross-sectional bundle and relate it’s reduced cross sectional $C^*$-algebras to the $C^*$-algebras $C^*_H(B)$ for closed normal subgroups $H \subset N$.

1. An addenda to Fell’s book on Banach *-algebraic bundles

Almost all the definitions in this work are taken or adapted from [7]. To start with we will use the definitions of Banach *-algebraic bundles, $C^*$-algebraic bundles (which we will call Fell bundles) and *-representations of Banach *-algebraic bundles. We recall also the definition of strong approximate unit [7 VIII 2.11] and the fact that every Fell bundle has one [7 VIII 16.3].

When we say $B = \{B_t\} \subseteq G$ is a Banach *-algebraic bundle we will be implicitly assuming $G$ is LCH, and when we say $H \subset G$ is a subgroup we will actually mean $H$ is a closed subgroup of the LCH group $G$. The unit of any group will be denoted $e$ (or 0 if the group is abelian). Integration with respect to a left invariant Haar measure on $G$ will be represented by $dt$ or $d\Delta t$ and the modular function of $G$ will be denoted $\Delta$ or $\Delta_G$. The letter $B$ will also represent the disjoint union of the family of fibers $\{B_t\} \subseteq G$. Recall that $B$ itself is a topological space and that the topology of $B_t$ relative to $B$ is the norm topology of $B_t$.

Given a Banach *-algebraic bundle $B = \{B_t\} \subseteq G$ and an open or closed subset $C \subset G$, the reduction $B_C := \{B_t\} \subseteq C$ is a Banach *-algebraic bundle with the topology and vector space operations inherited from $B$. If $C$ is a subgroup (and $B$ a Fell bundle), then $B_H$ is a Banach *-algebraic bundle (a Fell bundle, respectively).

The set of continuous cross sections (with compact support) of $B$ will be denoted $C(B)$ (respectively, $C_c(B)$). The $L^1$—cross sectional *-algebra of $B$ will be denoted $L^1(B)$, as in [7 VIII 5]. This algebra is the completion of the normed *-algebra $C_c(B)$ equipped with the norm $\| \cdot \|_1$, the convolution product $(f, g) \mapsto f \ast g$ and the involution $f \mapsto f^*$; which are determined by

$$\|f\|_1 = \int_G \|f(t)\| \, dt \quad f \ast g(t) = \int_G f(r) g(r^{-1}t) \, dr \quad f^*(t) = \Delta(t)^{-1} f(t)^*.$$
The (universal) cross sectional $C^*$-algebra of $\mathcal{B}$, denoted $C^*(\mathcal{B})$ and defined in [7, VIII 17.2], is the enveloping $C^*$-algebra of $L^1(\mathcal{B})$ and the canonical morphism of $*$-algebras from $L^1(\mathcal{B})$ to $C^*(\mathcal{B})$ will be denoted $\chi^B: L^1(\mathcal{B}) \to C^*(\mathcal{B})$. In case $\mathcal{B}$ is a Fell bundle, $\chi^B$ is injective by [7, VIII 16.4] and we view $L^1(\mathcal{B})$ as a dense $*$-subalgebra of $C^*(\mathcal{B})$. By [7, VIII 5.11] the existence of a strong approximate unit of $\mathcal{B}$ guarantees the existence of an approximate unit of $L^1(\mathcal{B})$.

Given a Banach $*$-algebraic bundle $\mathcal{B} = \{B_t\}_{t \in G}$ and a subgroup $H \subset G$, the generalized restriction map $p: C_c(\mathcal{B}) \to C_c(\mathcal{B}_H)$ is defined in [7, XI 8.4] and it is given by

$$p(f)(t) = \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} f(t), \ \forall f \in C_c(\mathcal{B}), \ t \in H.$$  

A $*$-representation $T: \mathcal{B}_H \to \mathbb{B}(Y)$ is $\mathcal{B}$–positive in the sense of [7, XI 8] if for all $f \in C_c(\mathcal{B})$ it follows that $\tilde{T}_p(f^* f) \geq 0$ in $\mathbb{B}(Y)$; with $\tilde{T}: L^1(\mathcal{B}) \to \mathbb{B}(Y)$ being the integrated form of $T$ [7, VIII 11]. The $C^*$-integrated form of $T$ is the unique $*$-representation $\chi^B_T: C^*(\mathcal{B}) \to \mathbb{B}(Y)$ such that $\chi^B_T \circ \chi^B = \tilde{T}$. In case $\mathcal{B}$ is a Fell bundle we think of $\chi^B_T$ as an extension of $\tilde{T}$ and just write $\tilde{T}$ instead of $\chi^B_T$.

**Notation 1.2.** In the paragraph above it is implicit that $Y$ is a Hilbert space and that we denote $\mathbb{B}(Y)$ the set of bounded linear operators on $Y$. In forthcoming sections we will need to use $*$-representations of Banach $*$-algebraic bundles by adjointable operators on (right) Hilbert modules; so it will be convenient to think of Hilbert spaces as right Hilbert modules over the complex field $\mathbb{C}$. Hence, our inner products (even those of Hilbert spaces) are assumed to be linear in the second variable and the formulas from [7] should be modified accordingly. When we say $Y_A$—is a Hilbert module we will be meaning that $A$ is a $C^*$-algebra and that $Y_A$ is a right $A$–Hilbert module. The $C^*$-algebra of adjointable operators on $Y_A$ will be denoted $\mathbb{B}(Y_A)$.

The goal of the following result is to recall (part of) Fell’s characterization of positivity [7, XI 8.9] and to answer affirmatively a question raised by him in [7, XI 11.10].

**Theorem 1.3.** Let $\mathcal{B} = \{B_t\}_{t \in G}$ be a Banach $*$-algebraic bundle, $H \subset G$ a subgroup and $T: \mathcal{B}_H \to \mathbb{B}(Y)$ a $*$-representation. Then the following are equivalent:

1. $T$ is $\mathcal{B}$–positive.
2. For every coset $\alpha \in G/H$, every positive integer $n$, all $b_1, \ldots, b_n \in B_\alpha$, and all $\xi_1, \ldots, \xi_n \in Y$, $\sum_{i,j=1}^n \langle \xi_i, T_{b_i} \xi_j \rangle \geq 0$.
3. The restriction $T|_{B_\xi}: B_\xi \to \mathbb{B}(Y)$ is $\mathcal{B}$–positive, that is to say $\langle \xi, T_{b_\xi} \xi \rangle \geq 0$ for all $b \in \mathcal{B}$ and $\xi \in Y$.

Besides, the three conditions above hold if $\mathcal{B}$ is a Fell bundle.

**Proof.** The equivalence between (1) and (2) is part of the content of [7, XI 8.9] and (2) clearly implies (3). Note also that if $\mathcal{B}$ is a Fell bundle then claim (3) does hold because $b^* b \geq 0$ in $B_\xi$ for all $b \in \mathcal{B}$ and the restriction $T|_{B_\xi}$ is a $*$-representation of $B_\xi$. As indicated by Fell at the end of [7, XI 11.10], to prove our statement it is enough to show that (2) does hold under the (only) assumption of $\mathcal{B}$ being a Fell bundle.

Assume $\mathcal{B}$ is a Fell bundle, take a coset $\alpha \in G/H$, $b_1, \ldots, b_n \in B_\alpha$ and $\xi_1, \ldots, \xi_n \in Y$. Let $t_1, \ldots, t_n \in \alpha$ be such that $b_j \in B_{t_j}$ (for every $j = 1, \ldots, n$) and set $t := (t_1, \ldots, t_n)$. Define the matrix space

$$\mathbb{M}_t(\mathcal{B}) := \{(M_{i,j})_{i,j=1}^n: M_{i,j} \in B_{t_i^{-1}t_j}, \ \forall i,j = 1, \ldots, n\}$$

as in [4, Lemma 2.8]. Then $\mathbb{M}_t(\mathcal{B})$ is a $C^*$-algebra with usual matrix multiplication as product and $*$–transpose as involution.
The matrix $M := (b_i^* b_j)_{i,j=1}^n$ belongs to $\mathcal{M}_t(\mathcal{B})$ and, regarding $\mathcal{B}$ as a $\mathcal{B} - \mathcal{B}$-equivalence bundle, it follows from the proof of [3] Lemma 2.8 that $M$ is positive in $\mathcal{M}_t(\mathcal{B})$. If $N \in \mathcal{M}_t(\mathcal{B})$ is the positive square root of $M$, then all the entries of $N$ belong to $\mathcal{B}_H$ and

$$\sum_{i,j=1}^n \langle \xi_i, T_{b_i^* b_j} \xi_j \rangle = \sum_{i,j,k=1}^n \langle \xi_i, T_{N_{k,i} \cdot N_{k,j}} \xi_j \rangle = \sum_{k=1}^n \sum_{i=1}^n T_{N_{k,i}} \xi_i \sum_{j=1}^n T_{N_{k,j}} \xi_j \geq 0;$$

proving that $T$ is $\mathcal{B}$-positive. \qed

In [7, VIII 16] Fell starts with a Banach $^*$-algebraic bundle $\mathcal{B} = \{B_t\}_{t \in G}$ and constructs a Fell bundle $\mathcal{C} = \{C_t\}_{t \in G}$, which he calls the bundle $C^*$-completion of $\mathcal{B}$, with equivalent representations theories. To put this in precise terms we introduce the following.

**Definition 1.4.** We say $\{\rho_t\}_{t \in G} : \{B_t\}_{t \in G} \to \{C_t\}_{t \in G}$ is a morphism of Banach $^*$-algebraic bundles if

1. $\mathcal{B} \equiv \{B_t\}_{t \in G}$ and $\mathcal{C} \equiv \{C_t\}_{t \in G}$ are Banach $^*$-algebraic bundles.
2. $\rho_t : B_t \to C_t$ is a linear function, for all $t \in G$.
3. The map $\rho : \mathcal{B} \to \mathcal{C}$, sending $b \in B_t$ to $\rho_t(b)$, is continuous.
4. There exists $M \geq 0$ satisfying $\|\rho(b)\| \leq M\|b\|$ for all $b \in \mathcal{B}$.
5. For all $b, c \in \mathcal{B}$ it follows that $\rho(bc) = \rho(b)\rho(c)$ and $\rho(b^*) = \rho(b)^*$.

We call the $M$ of (4) an upper bound for $\rho$ and denote $\|\rho\|$ the least upper bound of $\rho$. The composition of $\rho \equiv \{\rho_t\}_{t \in G}$ with $\pi \equiv \{\pi_t\}_{t \in G} : \{C_t\}_{t \in G} \to \{D_t\}_{t \in G}$ is defined by $\pi \circ \rho := \{\pi_t \circ \rho_t\}_{t \in G} : \{B_t\}_{t \in G} \to \{D_t\}_{t \in G}$.

The integrated form of the morphism $\rho : \mathcal{B} \to \mathcal{C}$ is the unique morphism of Banach $^*$-algebras $\tilde{\rho} : L^1(\mathcal{B}) \to L^1(\mathcal{C})$ such that $\tilde{\rho}(f) = \rho \circ f$ for all $f \in C_c(\mathcal{B})$. If $T : \mathcal{C} \to \mathbb{B}(Y)$ is a $^*$-representation, then so it is $T \circ \rho : \mathcal{B} \to \mathbb{B}(Y)$ and $(T \circ \rho^*) = \tilde{T} \circ \tilde{\rho}$. The $C^*$-integrated form of $\rho$, $\chi^\rho : C^*(\mathcal{B}) \to C^*(\mathcal{C})$, is the unique morphism of $^*$-algebras such that $\chi^\rho \circ \chi^\mathcal{B} = \chi^\mathcal{C} \circ \tilde{\rho}$.

In this article we think of Banach $^*$-algebras as Banach $^*$-algebraic bundles over the trivial group $\{e\}$. Hence, by the Definition above, morphism between Banach $^*$-algebras are continuous. Recall that if $\rho : \mathcal{B} \to \mathcal{C}$ is a morphism of $^*$-algebras with $B$ a Banach $^*$-algebra and $C$ a $C^*$-algebra, then $\rho$ is contractive and so a morphism of Banach $^*$-algebras. In case the bundle $\mathcal{C}$ of Definition 1.3 is a Fell bundle it follows that $\|\rho\| \leq 1$ because $\rho_c : B_t \to C_t$ is contractive and for all $b \in \mathcal{B}$ we have $\|\rho(b)\| = \|\rho(b^*)\|^{1/2} \leq \|b^*b\|^{1/2} \leq \|b\|$.}

Fell constructs the $C^*$-completion $\mathcal{C} = \{C_t\}_{t \in G}$ of $\mathcal{B} = \{B_t\}_{t \in G}$ by considering a norm $\|e\|$ on the fibers of $\mathcal{B}$. He defines $C_t$ as the norm completion of the quotient of $B_t$ by $\{b \in B_t : \|b\| = 0\}$. This gives a canonical morphism $\rho : \mathcal{B} \to \mathcal{C}$ which we will refer to as the canonical morphism; this construction motivates Definition 1.5 below (which will be used intensively in Section 3).

By a $C^*$-completion of a $^*$-algebra $C$ we mean a morphism of $^*$-algebras $\iota : B \to C$ such that $C$ is a $C^*$-algebra and $\iota(B)$ is dense in $C$. If there is need to specify $\iota$ we just say $C$ is a $C^*$-completion of $B$. A $^*$-algebra $B$ is called reduced if it has a $C^*$-completion $\iota : B \to A$ such that $\iota$ is faithful (i.e. $\iota(b) = 0$ implies $b = 0$). This is equivalent to say $B$ has a faithful $^*$-representation as bounded operators on a Hilbert space. We now want to extend these ideas to the realm of Banach $^*$-algebraic bundles.

**Definition 1.5** (c.f. [7, XI 12.6]). We say $\rho : \mathcal{B} \to \mathcal{C}$ is a $C^*$-completion (of $\mathcal{B} = \{B_t\}_{t \in G}$) if it is a morphism of Banach $^*$-algebraic bundles, $\mathcal{C}$ is a Fell bundle and $\rho(B_t)$ is dense in $C_t$ for all $t \in G$. Given another $C^*$-completion of $\mathcal{B}$, $\kappa : \mathcal{B} \to \mathcal{D}$, we say $\rho : \iota \to \kappa$...
is a morphism if \( \rho: \mathcal{C} \to \mathcal{D} \) is a morphism of Banach \(*\)-algebraic bundles such that
\( \rho \circ \iota = \kappa \). The composition of morphism is the composition of functions (and this defines isomorphisms).

**Remark 1.6.** Given a \( \mathcal{C}^*\)-completion \( \rho: \mathcal{B} \to \mathcal{C} \), the set of sections \( \hat{\rho}(\mathcal{C}_c(\mathcal{B})) \) is pointwise dense in the sense that \( \{ f(t): f \in \hat{\rho}(\mathcal{C}_c(\mathcal{B})) \} \) is dense in \( C_t \), for all \( t \in G \). Besides, \( C_t(\Gamma) \subset \Gamma \) and by [7, II 14.6] these properties imply \( \hat{\rho}(\mathcal{C}_c(\mathcal{B})) \) is dense in \( C_c(\mathcal{C}) \) in the inductive limit topology.

We now present a Banach \(*\)-algebraic bundle whose unit fibre is a \( \mathcal{C}^*\)-algebra but it’s only \( \mathcal{C}^*\)-completion is the zero (or null) one.

**Example 1.7.** Let \( \mathcal{B} = \mathbb{C} \times \mathbb{Z}_2 \) be the trivial bundle with constant fiber \( \mathbb{C} \) over the additive group \( \mathbb{Z}_2 = \{0, 1\} \). Define the involution by \((\lambda, r)^* := (\overline{\lambda}, r)\) and the product
\[
(\lambda, r)(\mu, s) = \begin{cases}
(\lambda \mu, r + s) & \text{if } r \neq 1 \text{ or } s \neq 1, \\
(-\lambda \mu, 0) & \text{if } r = s = 1.
\end{cases}
\]

If \( \rho: \mathcal{B} \to \mathcal{C} \) is a \( \mathcal{C}^*\)-completion, then either \( \rho_0: \mathcal{B}_c \to \mathcal{C}_0 \) is a \(*\)-isomorphism or either \( C_0 = \{0\} \). The first case is excluded because \((-1, 0) = (1, 1)^*(1, 1)\) is negative in \( B_0 \), so we must have \( C_0 = \{0\} \) and this forces \( C_1 = \{0\} \). It is interesting to notice that the norm of \( \mathcal{B} \) satisfies \( \|b^*b\| = \|b\|^2 \), but \( \mathcal{B} \) is not a Fell bundle.

The construction of the bundle \( \mathcal{C}^*\)-completion \( \mathcal{C} \) of the Banach \(*\)-algebraic bundle \( \mathcal{B} \) is performed in such a way that if \( \rho: \mathcal{B} \to \mathcal{C} \) is the canonical morphism, then for any \(*\)-representation \( T: \mathcal{B} \to \mathbb{B}(Y) \) there exists a unique \(*\)-representation \( T^\rho: \mathcal{C} \to \mathbb{B}(Y) \) such that \( T^\rho \circ \rho = T \). We now extend this property to \( \mathcal{B} \)-positive \(*\)-representations of reductions of \( \mathcal{B} \) to subgroups.

**Proposition 1.8.** Let \( \mathcal{B} = \{ B_t \}_{t \in G} \) be a Banach \(*\)-algebraic bundle, \( H \subset G \) a subgroup, \( \mathcal{C} \) the bundle \( \mathcal{C}^*\)-completion of \( \mathcal{B} \) and \( \rho: \mathcal{B} \to \mathcal{C} \) the canonical morphism. Then for every \( \mathcal{B} \)-positive \(*\)-representation \( T: \mathcal{B}_H \to \mathbb{B}(Y) \) there exists a unique \(*\)-representation \( T^\rho: \mathcal{C}_H \to \mathbb{B}(Y) \) such that \( T^\rho \circ \rho = T \). Reciprocally, for every \(*\)-representation \( S: \mathcal{C}_H \to \mathbb{B}(Z) \) the composition \( T := S \circ (\rho|_{\mathcal{B}_H}): \mathcal{B}_H \to \mathbb{B}(Z) \) is a \( \mathcal{B} \)-positive \(*\)-representation and \( T^\rho = S \).

**Proof.** Let \( T: \mathcal{B}_H \to \mathbb{B}(Y) \) be a \( \mathcal{B} \)-positive \(*\)-representation and denote \( Y^T \) the essential space of \( T \), i.e. the closed linear span of \( \{ T_b \xi: b \in \mathcal{B}, \xi \in Y \} \). Then the \(*\)-representation
\[
T^\rho: \mathcal{B}_H \to \mathbb{B}(Y^T)
\]
given by \( T^\rho_b \xi = T_b \xi \) is non degenerate and \( \| T^\rho_b \| = \| T_b \| \) for all \( b \in \mathcal{B} \). By [7, XI 11.3] there exists a \(*\)-representation \( T^{\rho'}: \mathcal{B} \to \mathbb{B}(W) \) such that
\[
\| T^\rho_b \| \leq \| T^{\rho'}_b \| \quad \text{for all } b \in \mathcal{B}_c.
\]
Thus the construction above and the construction of the bundle \( \mathcal{C}^*\)-completion of \( \mathcal{B} \) imply that for all \( b \in \mathcal{B} \), \( \| T^\rho_b \| = \| T^{\rho'}_b \|^{1/2} \leq \| T^{\rho''}_b \|^{1/2} = \| T^\rho_b \| = \| T^\rho_{\rho(b)} \| \leq \rho(b) \). Hence, the construction of \( \mathcal{C} \) implies the existence of a unique \(*\)-representation \( T^\rho: \mathcal{C}_H \to \mathbb{B}(Y) \) such that \( T^\rho_{\rho(b)} = T_b \) for all \( b \in \mathcal{B}_H \). We suggest to consult [7, VIII 16] to see how Fell shows this last claim when \( G = H \). Uniqueness of \( S \) follows from property (1) of \( \rho: \mathcal{B} \to \mathcal{C} \).

Now take a \(*\)-representation \( S: \mathcal{C}_H \to \mathbb{B}(Z) \). Then \( S \geq 0 \) for all positive \( c \in \mathcal{C}_c \) and, since \( \rho(b^*b) = \rho(b)^*\rho(b) \in C^+_c \) for all \( b \in \mathcal{B} \), we get that \( S \circ \rho(b^*b) \geq 0 \) for all \( b \in \mathcal{B} \). This implies \( T: \mathcal{B}_H \to \mathbb{B}(Z), b \mapsto S_{\rho(b)} \), is \( \mathcal{B} \)-positive and \( T_b = S_{\rho(b)} = T^\rho_{\rho(b)} \) for all \( b \in \mathcal{B}_H \). Thus property (1) implies \( T^\rho = S \). \( \square \)
2. Induction of *-representations

Proposition 1.8 reveals that if one is only interested in studying the representation theory of a Banach *-algebraic bundle, then one is allowed to work with the bundle C*-completion instead. One can do so even when working with induced representations [7, XI 12.6]. But the theory works nicely enough if we only assume the existence of strong approximate units. Thus from now on we adopt the following conventions, which we will refer to as “conventions (C)” or just “(C):"

(C1) By a group we mean a LCH topological group and by a (normal) subgroup we mean a (normal and) closed subgroup.

(C2) When we say “B is a Banach *-algebraic bundle” we actually mean that “B is a Banach *-algebraic bundle over a (LCH topological) group and B has a strong approximate unit”.

Any other hypothesis will be stated explicitly.

2.1. Integration and disintegration of *-representations. We now adapt Fell’s integration and disintegration theory [7, VIII] to representations on Hilbert modules.

Definition 2.1. A *-representation of the Banach *-algebraic bundle B on the right A–Hilbert module Y_A is a function T: B → B(Y_A) which is linear when restricted to any fibre; is multiplicative (T_ab = T_a T_b); preserves the involution (T_a* = T_a*) and, for all \( \xi, \eta \in Y_A \), the function \( B \to A, b \mapsto \langle T_b \xi, \eta \rangle \), is continuous. We say T is non degenerate if the essential space of T, defined as \( Y_A^T := \text{span}\{T_b \xi: b \in B, \xi \in Y_A\} \), equals \( Y_A \). A vector \( \xi \) is cyclic for T if \( Y_A = \text{span}\{T_b \xi: b \in B\} \).

By Cohen-Hewitt’s Theorem [7, V 9] and (C) we have \( Y_A^T = \{T_b \xi: b \in B, \xi \in Y_A\} \). The essential space is then a closed A–submodule of \( Y_A \). By the essential part of T we mean the *-representation \( T': B \to B(Y_A^T) \) such that \( T_b' \xi = T_b \xi \) for all \( b \in B \) and \( \xi \in Y_A \). Note \( T' \) is non degenerate.

Notation 2.2. When we say \( T: B \to B(Y) \) is a *-representation we will be meaning that \( Y = Y_A \) is a Hilbert space. For general *-representations on Hilbert modules we will write \( T: B \to B(Y_A) \). The only exception of this rule being the case when it is clear from the context that \( Y = A \) is a C*-algebra regarded as a right A–Hilbert module.

Given any *-representation \( T: B \to B(Y_A) \), the restriction \( T|_{B_a}: B_a \to B(Y_A) \) is contractive. Then for all \( b \in B \) we have

\[
\|T_b\| = \|T_b\|^{1/2} \leq \|b^* b\|^{1/2} \leq \|b\|.
\]

Note also that given any \( \xi \in Y_A \) we have

\[
\lim_{a \to b} \|T_a \xi - T_b \xi\|^2 = \lim_{a \to b} \|\langle T_{a^* a} \xi, \xi \rangle + \langle T_{b^* b} \xi, \xi \rangle - \langle T_{b^* a} \xi, \xi \rangle - \langle T_{a^* b} \xi, \xi \rangle\| = 0.
\]

Thus the function \( B \to Y_A, b \mapsto T_b \xi \), is continuous and given \( f \in C_c(B) \) it makes sense to define a function

\[
\tilde{T}_f: Y_A \to Y_A, \tilde{T}_f \xi := \int_G T_{f(t)} \xi \, dt.
\]

Moreover,

\[
\|\tilde{T}_f\| \leq \int_G \|T_{f(t)}\|\|\xi\| \, dt = \|f\|_1 \|\xi\|
\]

for every \( f \in C_c(B) \) and \( \xi \in Y_A \).
Proposition 2.4. For every *-representation $T: B \to \mathcal{B}(Y_A)$ there exists a unique *-representation $\tilde{T}: L^1(B) \to \mathcal{B}(Y_A)$ such that for all $f \in C_c(B)$ and $\eta \in Y_A$, $\tilde{T}_f(\eta) = \int_c T_f(g) \eta \, dt$. Moreover, the essential spaces of $T$ and $\tilde{T}$ agree and $\xi \in Y_A$ is cyclic for $T$ if and only if it is cyclic for $\tilde{T}$.

Proof. The comments preceding the statement imply $T$ is a Banach representation in the sense of [7, VIII 8.2], then it is integrable (in the sense of [7, VIII 11.2]) to a representation of $\tilde{T}$ of $C_c(B)$ by [7, VIII 11.3]. Then we can adapt the proof of [7, VIII 11.4] to show that $\tilde{T}: C_c(B) \to \mathcal{B}(Y_A)$ is a *-representation, which is contractive by (2.8). Thus $\tilde{T}: C_c(B) \to \mathcal{B}(Y_A)$ admits a unique continuous extension $\tilde{T}: L^1(B) \to \mathcal{B}(Y_A)$, which is also a *-representation.

The claim about non degeneracy can be proved as in [7, 11.10], which amounts to show that for any $b \in B$, $f \in C_c(B)$ and $\xi \in Y_A$ it follows that $\tilde{T}_f(\xi) \in \overline{\text{span}}\{\tilde{T}_g(\xi) : g \in C_c(B)\}$ and that $\tilde{T}_f(\xi) \in \overline{\text{span}}\{\tilde{T}_c(\xi) : c \in B\}$, for every $\xi \in Y_A$, $b \in B$ and $f \in C_c(B)$. □

Definition 2.5. The $L^1$-integrated form of the *-representation $T: B \to \mathcal{B}(Y_A)$ is the *-representation $\tilde{T}: L^1(B) \to \mathcal{B}(Y_A)$ given by the Proposition above. The $C^*$-integrated form of $T$ is the unique *-representation $\chi^T_B: C^*(B) \to \mathcal{B}(Y_A)$ such that $\chi_B^T \circ \chi_B^T = \tilde{T}$. If it is convenient to do so, we will write $T$ instead of $\tilde{T}$.

Remark 2.6. If $T: B \to \mathcal{B}(Y_A)$ and $\pi: A \to \mathcal{B}(Z_C)$ are *-representations and we form the $\pi-$balanced tensor product $Y_A \otimes_{\pi} Z_C$, then there exists a unique *-representation $T \otimes_{\pi} 1: B \to \mathcal{B}(Y_A \otimes_{\pi} Z_C)$ such that $(T \otimes_{\pi} 1)_a(\xi \otimes_{\pi} \eta) = (T_a(\xi)) \otimes_{\pi} \eta$. The integrated form of $T \otimes_{\pi} 1$ is $(T \otimes_{\pi} 1)_B = \tilde{T} \otimes_{\pi} 1$.

To disintegrate *-representations on Hilbert modules we adapt the ideas of [7, VIII 13.2]. The key to do this is the following extension of [7, VI 19.11].

Proposition 2.7. Let $B$ be a Banach *-algebra, $I$ a (not necessarily closed) *-ideal of $B$, $Y_A$ a Hilbert module and $\pi: I \to \mathcal{B}(Y_A)$ a *-representation (i.e. a morphism of *-algebras). Then $\pi$ is contractive with respect to the norm of $I$ inherited from $B$. If $\pi$ is non degenerate, that is to say $Y_A = \overline{\text{span}}{\pi(b)\xi : b \in I, \xi \in Y_A}$, then it admits a unique extension $\pi'$ to a *-representation of $\tilde{B}$. In case $A = C$ and $\pi$ is degenerate, it also admits an extension to a *-representation of $B$.

Proof. Given $a \in I$, $\xi \in Y_A$ and a state $\varphi$ of $A$ define $p: B \to \mathbb{C}$ by $p(b) = \varphi(\langle \pi(ab^a)\xi, \xi \rangle)$. Then $p$ is positive in the sense of [7, VI 18] and, by [7, VI 18.14], it satisfies $p(b^a b) \leq \|c\|p(b^a b)$ for all $c \in B$ and $b \in I$. Thus we obtain, for all $c, b \in I$

$$\varphi(\langle \pi(c)\pi(b)\pi(a)\xi, \pi(b)\pi(a)\xi \rangle) \leq \|c\|\varphi(\langle \pi(b)\pi(a)\xi, \pi(b)\pi(a)\xi \rangle).$$

The closure of $\pi(I)$ is a $C^*$-subalgebra of $\mathcal{B}(Y_A)$, so there exists a net $\{b_i\}_{i \in J}$ of self adjoint elements such that $\pi(b_i)_{i \in J}$ is bounded and $\lim_i \pi(c)\pi(b_i) = \lim_i \pi(b_i)\pi(c) = \pi(c)$ for all $c \in I$. Putting $c = ab^a$ and $b = b_i$ in (2.8) and taking limit we obtain that

$$\|aa^*\|^2 = \pi(aa^*) \geq 0 \Rightarrow \|aa^*\|\|\pi(a)\|^2 \geq \|\pi(a)\|^4 \Rightarrow \|aa^*\|^{1/2} \geq \|\pi(a)\|$$

for all $a \in I$. Then $\pi$ is bounded because $\|aa^*\|^{1/2} \leq \|a\|$ for all $a \in I$.

Assume $\pi$ is non degenerate. In such a case the uniqueness of $\pi'$ is immediate and, as in [7, VI 19.11], it’s existence is equivalent to the fact that given $a \in B$, $b_1, \ldots, b_n \in I$ and $\xi_1, \ldots, \xi_n \in Y_A$ it follows that

$$\|\sum_{j=1}^n \pi(ab_j)\xi_j\| \leq \|a\|\|\sum_{j=1}^n \pi(b_j)\xi_j\|.\tag{2.9}$$
To prove the identity above take a faithful and non degenerate *-representation \( \rho: A \to \mathbb{B}(Z) \), form the tensor product \( Y \otimes_\rho Z \); and consider the *-representation \( \pi \otimes_\rho 1: I \to \mathbb{B}(Y \otimes_\rho Z) \) such that \( (\pi \otimes_\rho 1)_{\pi}(\xi \otimes \xi) = \pi(b)\xi \otimes \xi \). Then \( \pi \otimes_\rho 1 \) is non degenerate and admits a unique extension \( (\pi \otimes_\rho 1)': B \to \mathbb{B}(Y \otimes_\rho Z) \) by \([7, VI 19.11]\).

If given \( a \in B, b_1, \ldots, b_n \in I, \xi_i, \ldots, \xi_n \in Y_A \) and \( \xi \in Z \) we set \( u := \sum_{j=1}^n \pi(b_j)\xi_j \) and \( v := \sum_{j=1}^n \pi(ab_j)\xi_j \), then we have

\[
\langle \xi, \rho((v, v)_A)\xi \rangle = \langle \xi \otimes v, v \otimes \xi \rangle = \| (\pi \otimes 1)(a) (u \otimes \xi) \|^2 \leq \| a \|^2 \| u \otimes \xi \|^2
\]

This implies \( \langle v, v \rangle_A \leq \| a \|^2 \langle u, u \rangle_A \) and \((2.13)\) follows (taking norms and square roots).

In case \( A = \mathbb{C} \) and \( \pi \) is non degenerate we may consider the essential space \( Y_\pi := \text{span}_\pi(I)Y \) of \( \pi \) and consider the extension \( \pi': B \to \mathbb{B}(Y_\pi) \subset \mathbb{B}(Y) \) of \( \pi: I \to \mathbb{B}(Y_\pi) \) as a *-representation of \( B \) on \( \mathbb{B}(Y) \).

**Proposition 2.10.** Let \( \mathcal{B} \) be a Banach *-algebraic bundle. Then for every non degenerate *-representation \( \pi: L^1(\mathcal{B}) \to \mathbb{B}(Y_A) \) there exists a unique non degenerate *-representation \( T: \mathcal{B} \to \mathbb{B}(Y_A) \) such that \( \pi = T \).

**Proof.** Follow the ideas of \([7, VI 19.11]\) noticing that \( \pi \) can be extended to the bounded multiplier algebra of \( L^1(\mathcal{B}) \) by Proposition 2.7. \( \square \)

### 2.2. Positive *-representations and induction.

Fell’s induction process of representations \([7, XI]\) is intimately related to the notion of positivity of *-representations. To extend this process to *-representations on Hilbert modules one needs to choose whether to extend Fell’s “concrete” or “abstract” induction processes (see \([7, XI 9.26]\)).

The two approaches are equivalent for *-representations on Hilbert modules, the main difference being the machinery required to develop each one of them. The concrete approach uses Hilbert spaces and is appropriate when fine surgery is required. Meanwhile, the abstract approach uses (the nowadays well known theory of) Hilbert modules and induction of *-representations: this being the main reason why we have decided to adopt the abstract approach. This being said, we will not hesitate on changing to the concrete approach when necessary or to use Fell’s profound understanding of the (concrete) induction process.

Let \( \mathcal{B} = \{ B_I \}_{I \in G} \) be a Banach *-algebraic bundle and \( H \) a subgroup of \( G \). In order to induce *-representations from \( \mathcal{B}_H \) to \( \mathcal{B} \), Fell equips \( C_c(\mathcal{B}) \) with a right \( C_c(\mathcal{B}_H) \)—valued conditional expectation.

The action of \( u \in C_c(\mathcal{B}_H) \) on \( f \in C_c(\mathcal{B}) \) (on the right) produces the element \( fu \in C_c(\mathcal{B}) \) determined by

\[
(fu)(t) = \int_H f(ts)u(s^{-1}) \Delta_G(s)^{1/2} \Delta_H(s)^{-1/2} ds, \quad t \in G.
\]

The generalized restriction map \( p: C_c(\mathcal{B}) \to C_c(\mathcal{B}_H) \) is that of \((1.1)\). If, for notational convenience, it is necessary to specify the groups \( G \) and \( H \), we will denote \( p_H^G \) the generalized restriction map from \( C_c(\mathcal{B}) \) to \( C_c(\mathcal{B}_H) \).

**Remark 2.12.** If \( H \) is normal in \( G \) then \( \Delta_H \) is the restriction of \( \Delta_G \) and \( p \) is exactly the restriction map. In particular, if \( H = \{ e \} \) then \( p(f) = f(e) \).

The fundamental properties of \( p \) and the action of \( C_c(\mathcal{B}_H) \) on \( C_c(\mathcal{B}) \) are the following (see \([7, XI 8.4]\)):

\[
(2.13) \quad p(f)^* = p(f^*) \quad p(f) \ast u = p(fu) \quad (f \ast g)u = f \ast (gu) \quad f(u \ast v) = (fu)v
\]
where the identities above hold for all \( f, g \in C_c(\mathcal{B}) \) and \( u, v \in C_c(\mathcal{B}_H) \).

The following is the natural extension of Fell’s definition of positivity \cite{7, XI 8}.

**Definition 2.14.** A \(*\)-representation \( T: \mathcal{B}_H \to \mathfrak{B}(Y_A) \) is \( \mathcal{B} \)-positive if \( \langle \xi, \tilde{T}_{p(f'*f)} \xi \rangle \geq 0 \) for all \( \xi \in Y_A \) and \( f \in C_c(\mathcal{B}) \).

Two straightforward remarks are in order. Firstly, conventions (C) imply a \(*\)-representation \( T \) of \( \mathcal{B}_H \) is \( \mathcal{B} \)-positive if and only if the essential part of \( T \) is \( \mathcal{B} \)-positive. Secondly, if \( H = G \) then \( p(f) = f \) and \( \langle \xi, \tilde{T}_{p(f'*f)} \xi \rangle = \langle \tilde{T}_f \xi, \tilde{T}_f \xi \rangle \geq 0 \); proving that every \(*\)-representation of \( \mathcal{B}_G \equiv \mathcal{B} \) is \( \mathcal{B} \)-positive.

**Theorem 2.15.** Let \( \mathcal{B} = \{ B_1 \}_{i \in G} \) be a Banach \(*\)-algebraic bundle, \( H \subset G \) a subgroup, \( T: \mathcal{B}_H \to \mathfrak{B}(Y_A) \) a \(*\)-representation and \( \pi: A \to \mathfrak{B}(Z) \) a \(*\)-representation. Consider the following claims

(1) \( T \) is \( \mathcal{B} \)-positive.

(2) For every coset \( \alpha \in G/H \), every positive integer \( n \), all \( b_1, \ldots, b_n \in B_\alpha \), and all \( \xi_1, \ldots, \xi_n \in Y_A \), \( \sum_{i,j=1}^{n} \langle \xi_i, T_{b_i^* b_j} \xi_j \rangle \geq 0 \).

(3) The restriction \( T|_{B_\alpha}: B_\alpha \to \mathfrak{B}(Y_A) \) is \( \mathcal{B} \)-positive, that is to say \( \langle \xi, T_{b_i^* b_j} \xi \rangle \geq 0 \) for all \( b \in \mathcal{B} \) and \( \xi \in Y_A \).

(4) The \(*\)-representation \( T \otimes \pi: \mathcal{B}_H \to \mathfrak{B}(Y_A \otimes Z) \) of Remark \ref{rmk:tensor} is \( \mathcal{B} \)-positive (and, consequently, this claim can be replaced with any of the equivalent ones for \( T \otimes \pi \) given by Theorem \ref{thm:tensor}.

Then the first three are equivalent and imply the fourth, the converse holds if \( \pi \) is faithful. All the claims hold if \( \mathcal{B} \) is a Fell bundle.

**Proof.** Let \( \rho: \mathfrak{B}(Y_A) \to \mathfrak{B}(Y_A \otimes Z) \) be the \(*\)-representation such that \( \rho(M)(\xi \otimes \eta) = M \xi \otimes \eta \). Then \( \rho \circ T = T \otimes \pi \) and \( \rho \circ \tilde{T} = \tilde{T} \otimes \pi \) by Remark \ref{rmk:identity}. Since \( \rho \) maps the positive cone of \( \mathfrak{B}(Y_A) \), \( \mathfrak{B}(Y_A)^+ \), into \( \mathfrak{B}(Y_A \otimes Z)^+ \), it follows that (1) implies (3). In case \( \pi \) is faithful then so it is \( \rho \) and \( \mathfrak{B}(Y_A)^+ = \rho^{-1}(\mathfrak{B}(Y_A \otimes Z)^+) \). Hence, (4) implies (1) in case \( \pi \) is faithful. All we need to do now is to prove claims (1) to (3) are equivalent, and to do this we may assume \( \pi \) is faithful.

Clearly, (1) and (4) are equivalent and (2) implies (3). By Theorem \ref{thm:tensor} (4) is equivalent to say \( \rho \circ T|_{B_\alpha} = T \otimes \pi|_{B_\alpha} \) is \( \mathcal{B} \)-positive, meaning that \( T|_{B_\alpha} \) is \( \mathcal{B} \)-positive (this is claim (3)) because \( \rho \) is faithful. At this point we know (1), (3) and (4) are equivalent and we only need to prove they imply (2).

Assume (4) holds and take \( \alpha \in G/H, n, b_1, \ldots, b_n \in B_\alpha \) and \( \xi_1, \ldots, \xi_n \in Y_A \) as in (2). To prove that \( \sum_{i,j=1}^{n} \langle \xi_i, T_{b_i^* b_j} \xi_j \rangle \geq 0 \) is suffices to show that

\[
\langle \eta, \rho(\sum_{i,j=1}^{n} \langle \xi_i, T_{b_i^* b_j} \xi_j \rangle) \eta \rangle \geq 0
\]

for all \( \eta \in Z \). Given any \( \eta \in Z \) we have, by Theorem \ref{thm:tensor}

\[
\langle \eta, \rho(\sum_{i,j=1}^{n} \langle \xi_i, T_{b_i^* b_j} \xi_j \rangle) \eta \rangle = \sum_{i,j=1}^{n} \langle \xi_i \otimes \eta, (T \otimes \pi)_{b_i^* b_j} (\xi_j \otimes \eta) \rangle \geq 0.
\]

This shows (2). \( \square \)

In the theorem below we prove the existence of a universal C*-algebra for the \( \mathcal{B} \)-positive \(*\)-representations of \( \mathcal{B}_H \). It will turn out to be the quotient of \( C^*(\mathcal{B}_H) \) by the C*-ideal generated by the kernel of the integrated forms of all the \( \mathcal{B} \)-positive \(*\)-representations of \( \mathcal{B}_H \). Well, this is in fact an alternative definition of the C*-algebra \( C^*(\mathcal{B}_H) \) we construct below.
Theorem 2.16. Let $\mathcal{B} = \{B_t\}_{t \in G}$ be a Banach *-algebraic bundle and $H$ a subgroup of $G$. Then there exists a $C^*$-completion $\pi^C: L^1(B_H) \to C$ such that

1. For every $f \in C_c(B)$, $\pi^C(p(f^* f)) \geq 0$.
2. Given any $\mathcal{B}$--positive and non degenerate *-representation $T: B_H \to \mathbb{B}(Y)$, there exists a *-representation $\pi^C_T: C \to \mathbb{B}(Y)$ such that $\pi^C_T \circ \pi^C = T$.

Moreover,

(i) Given any two $C^*$-completions $\pi^C_1, \pi^C_2: L^1(B_H) \to C$ and $\pi^D_1, \pi^D_2: L^1(B_H) \to D$ (both satisfying conditions (1) and (2)), there exists a unique $C^*$-isomorphism $\Phi: C \to D$ such that $\Phi \circ \pi^C = \pi^D$.

(ii) The map $\pi^C_T$ in claim (2) is unique and, even with this addition, (2) holds even if $T$ is a (possibly degenerate) *-representation on a Hilbert module.

(iii) A *-representation $T: B_H \to \mathbb{B}(Y_A)$ is $\mathcal{B}$--positive if and only if there exists a morphism of *-algebras $\pi^C_T: C \to \mathbb{B}(Y_A)$ such that $\pi^C_T \circ \pi^C = \tilde{T}$.

(iv) Given a non degenerate *-representation $\mu: C \to \mathbb{B}(Y_A)$ there exists a unique $\mathcal{B}$--positive *-representation $T: B_H \to \mathbb{B}(Y_A)$ such that $\mu = \pi^C_T$.

Proof. Let $C$ be the bundle $C^*$-completion of $\mathcal{B}$ and $\rho: \mathcal{B} \to C$ the canonical morphism. Set $C := C^*(C_H)$ and let $\pi^C: L^1(\mathcal{B}) \to C^*(C_H)$ be the composition of the integrated form of $\rho_H: B_H \to C_H$, $b \mapsto \rho(b)$, with the inclusion $L^1(C_H) \hookrightarrow C^*(C_H)$.

Take a $\mathcal{B}$--positive (and possibly degenerate) *-representation $T: B_H \to \mathbb{B}(Y_A)$ and let $\pi: A \to \mathbb{B}(Z)$ be a faithful and non degenerate *-representation. Then $T \otimes_\pi 1$ is $\mathcal{B}$--positive by Theorem 2.15 and, by Proposition 1.8, it follows that $\|T_b\| = \|(T \otimes_\pi 1)_b\| \leq \|\rho(b)\|$ for all $b \in B_H$. By repeating the proof of Proposition 1.8, we can produce a non degenerate *-representation $T': C_H \to \mathbb{B}(Y_A)$ such that $T' \circ \rho_H = T$. If $\pi^C_T: C^*(C_H) \to \mathbb{B}(Y)$ is the $C^*$-integrated form of $T'$, then the identity $\pi^C_T \circ \pi^C = \tilde{T}$ follows by construction.

Let $\mu: C^*(C_H) \to \mathbb{B}(Z)$ be a non degenerate *-representation. Then $\mu|_{L^1(C_H)}$ is the integrated form of a unique *-representation $S: C_H \to \mathbb{B}(Z)$. If we set $T := S \circ \rho_H$, then $S = T'$ and it follows that $\pi^C_T = \mu$. At this point it is convenient to denote the generalized restrictions maps for $\mathcal{B}$ and $C$ with different letters, so we write $p^B_H: C_c(\mathcal{B}) \to C_c(B_H)$ instead of $p^C_H$, and proceed analogously with $C$. We have

$$\mu(\pi^C(p^B_H(f^* f))) = \tilde{S}_{p^B_H((\rho \circ \pi)(f^* f))} \geq 0$$

for all $f \in C_c(\mathcal{B})$. Since $\mu$ can be arranged to be faithful, it follows that $\pi^C(p^B_H(f^* f)) \geq 0$ for all $f \in C_c(\mathcal{B})$.

We have managed to produce a $C^*$-completion $\pi^C: L^1(B_H) \to C$ for which claims (1), (2), (ii) and (iv) hold. Assume we are given another $C^*$-completion $\pi^D: L^1(B_H) \to D$ satisfying (1) and (2) and let $\mu, S$ and $T$ be as before, with the additional requirement of $\mu$ being faithful. Then there exists a *-representation $\pi^D_T: D \to \mathbb{B}(Z)$ such that $\pi^D_T \circ \pi^D = \tilde{T}$. Note $\pi^D_T(D) = \pi^D_T \circ \pi^D(L^1(B_H)) = \mu(C)$, then there exists a unique morphism of $C^*$-algebras $\Psi: D \to C$ such that $\mu \circ \Psi = \pi^D_T$. Moreover, $\mu(\Psi \circ \pi^D(f)) = \tilde{T}_f = \pi^D_T(\pi^C(f)) = \mu(\pi^C(f))$ for all $f \in L^1(B_H)$. Thus $\Psi \circ \pi^D = \pi^C$ and it follows that $\Psi$ is surjective. In order to prove that $\Psi$ is faithful, take a non degenerate and faithful *-representation $\nu: D \to \mathbb{B}(W)$. Then $\nu \circ \pi^D$ is the integrated form of a unique non degenerate *-representation $R: B_H \to \mathbb{B}(W)$, which turns out to be $\mathcal{B}$--positive by condition (1). So $\pi^R_H \circ \Psi \circ \pi^D = \pi^R_H \circ \pi^C = \tilde{R} = \nu \circ \pi^D$ and it follows that $\pi^R_H \circ \Psi = \nu$. Consequently, the condition $\Psi(x) = 0$ implies $\nu(x) = 0$ and this forces $x = 0$; proving that $\Psi$ is a $C^*$-isomorphism. The $C^*$-isomorphism of claim (i) is $\Psi^{-1}$. 
Assume we are given a *-representation \( T : B_H \to \mathbb{B}(Y_A) \) for which there exists a *-representation \( \kappa : C \to \mathbb{B}(Y_A) \) such that \( \kappa \circ \pi^C = \tilde{T} \). Then condition (1) implies, for all \( \eta \in Y_A \) and \( f \in C_c(B) \), that
\[
\langle \eta, \tilde{T}_{p(f^* \ast f)} \rangle = \langle \eta, \kappa \circ \pi^C (p(f^* \ast f)) \rangle \eta \geq 0.
\]
Hence \( T \) is \( B \)-positive. This implies (iii). \( \square \)

**Definition 2.17.** Given a Banach *-algebraic bundle \( B \) over \( G \) and a subgroup \( H \) of \( G \), the \( B^+ \)-completion of \( L^1(B_H) \) is the \( C^* \)-algebra \( C = C^*(C_H) \) we constructed in the proof of Theorem 2.16 and \( \chi^{B_H^+} : L^1(B_H) \to C^*(B_H^+) \) is just \( \pi^C : L^1(B_H) \to C \). The algebra \( C^*(B_H^+) \) will be called the \( B^+ \)-cross sectional \( C^* \)-algebra of \( B_H \). The map \( \chi^{B_H^+} : C^*(B_H) \to C^*(B_H^+) \) is, by definition, the unique morphism of *-algebras such that \( \chi^{B_H^+} \circ \chi^{B_H} = \chi^{B_H^+} \).

If the bundle \( B \) in the definition above is a Fell bundle, then the identity \( C^*(B_H^+) = C^*(B_H) \) holds by construction and by the fact that every Fell bundle is its own bundle \( C^* \)-completion \([7, VIII 16.10] \).

Most of the time we will think of \( C^*(B_H^+) \) as an universal object and not as the concrete \( C^* \)-algebra we constructed. Consequently, any time we know \( \chi^{B_H^+} \) is faithful we will identify \( C^*(B_H) \) with \( C^*(B_H^+) \).

**Corollary 2.18.** In the conditions of the Definition above and of Theorem 2.16 the following are equivalent:

1. \( \chi^{B_H^+} \) is faithful, and hence a \( C^* \)-isomorphism, between \( C^*(B_H) \) and \( C^*(B_H^+) \).
2. Every *-representation of \( B_H \) is \( B \)-positive.
3. Every cyclic *-representation of \( B_H \) on a Hilbert space is \( B \)-positive.

In particular, \( C^*(B) = C^*(B_G^+) \). In case \( B \) is a Fell bundle all the equivalent conditions above hold.

**Proof.** Is left to the reader. \( \square \)

To describe the induction process in terms of Hilbert modules we construct "the inducing" module for the \( B \)-positive *-representations. Let \( B = \{ B_i \}_{i \in G} \) be a Banach *-algebraic bundle and \( H \) a subgroup of \( G \). Consider the \( B^+ \)-completion of \( L^1(B_H) \), \( \chi^{B_H^+} : L^1(B_H) \to C^*(B_H^+) \), and define the *-algebra \( C_0 := \chi^{B_H^+}(C_c(B_H)) \), which is dense in \( C^*(B_H^+) \).

We consider \( C_c(B) \) with the action of \( C_c(B_H) \) on the left given by \((2.11)\) and define the \( C_0 \)-valued sesquilinear form
\[
(C_c(B) \times C_c(B) \to C_0, (f, g) \mapsto [f, g]_p := \chi^{B_H^+}(p(f^* \ast g))).
\]
This form is linear in the second variable, positive semidefinite and \([f, g]_p^* = [g, f]_p^* \).

Given any state \( \varphi \) of \( C^*(B_H^+) \) the function \( C_c(B) \times C_c(B) \to \mathbb{C} \), \( (f, g) \mapsto \varphi([f, g]_p) \), is a pre-inner product. It then follows that
\[
|\varphi([f, g]_p)| \leq \varphi([f, f]_p)^{1/2} \varphi([g, g]_p)^{1/2}.
\]
Note this implies \([f, g]_p = 0\) for all \( g \in C_c(B) \) if and only if \([f, f]_p = 0 \).

Define \( I := \{ f \in C_c(B) : [f, f]_p = 0 \} \) and note the quotient space \( C_c(B)/I \) has a unique \( C_0 \)-valued sesquilinear form
\[
(C_c(B)/I \times (C_c(B)/I) \to C_0, (f + I, g + I) \mapsto \langle f + I, g + I \rangle_p := [f, g]_p,
\]
which is linear in the second variable, positive definite and satisfies \((f + I, g + I)^* = \langle g + I, f + I \rangle_p \).
If we are given \( f \in C_\cdot(\mathcal{B}) \) and \( u \in C_\cdot(\mathcal{B}_H) \), then

\[
0 \leq [fu, fu]_p = \chi^{B_H}(u)^* [f, f]_p \chi^{B_H}(u) \leq \|[f, f]_p\| \chi^{B_H}(u) \chi^{B_H}(u).
\]

In particular, if either \( f \in I \) or \( \chi^{B_H}(u) = 0 \). It then follows that there exists a unique \( C \)-valued \( \cdot \)-action of \( C \) on \( C_\cdot(\mathcal{B})/I \),

\[
(C_c(\mathcal{B})/I) \times C_0 \to C_c(\mathcal{B})/I, \quad (f + I, c) \mapsto (f + I)c,
\]

such that \((f + I)\chi^{B_H}(u) = fu + I\).

**Definition 2.20.** The \( C^*(\mathcal{B}_H^\perp) \)-Hilbert module obtained by completing \( C_c(\mathcal{B})/I \) (as indicated in \cite[Lemma 2.16]{[8]}) with respect to the \( C \)-valued \( \cdot \)-inner product described above will be denoted \( L^2_H(\mathcal{B}) \). It’s \( \cdot \)-inner product will be denoted \( \langle \cdot, \cdot \rangle_p \).

**Remark 2.21.** The map \( \iota: C_c(\mathcal{B}) \to L^2_H(\mathcal{B}), f \mapsto f + I \), is linear, continuous in the inductive limit topology and has dense range. Indeed, the linearity and density claims follow immediately by construction. To prove the continuity claim fix a compact set \( D \subset G \) and take \( f \in C_c(\mathcal{B}) \) with support contained in \( D \). Let \( \alpha_D \) be the measure of \( D \) in \( G \); \( \beta_D \) the measure of \( (D^{-1}D) \cap H \) in \( H \) and \( \gamma_D := \max\{\Delta_H(s)^{1/2}\Delta_H(s)^{-1/2} : s \in (D^{-1}D) \cap H\} \). Then

\[
\|f + I\| = \|\chi^{B_H}(p(f^* f))\|^{1/2} \leq \|p(f^* f)\|^{1/2}
\]

\[
\leq \left( \int_H \Delta_H(s)^{1/2}\Delta_H(s)^{-1/2} \int_G \|f(r)\| \|\|f(s)\| \|\|dGr \| dh \right)^{1/2}
\]

\[
\leq (\alpha_D \beta_D \gamma_D)^{1/2} \|f\|_\infty.
\]

Thus \( \iota \) is continuous in the inductive limit topology by \cite[II 14.3]{[7]}.

**Remark 2.22.** If \( \mathcal{B} \) is a Fell bundle and \( H = \{e\} \), then \( \mathcal{B}_H = B_e \) and the generalized restriction map \( p: C_c(\mathcal{B}) \to B_e \) is just the evaluation map \( f \mapsto f(e) \). It then follows that for all \( f, g \in C_c(\mathcal{B}) \) we have \([fg]_p = (f^* g)(e) = \int_G f(t^* g(t)) \, dt \). This is exactly the \( B_e \)-valued \( \cdot \)-inner product used by Exel and Ng in \cite[Lemma 2.1]{[6]} to construct the \( B_e \)-Hilbert module \( L^2_e(\mathcal{B}) \), so \( L^2_e(\mathcal{B}) = L^2(\mathcal{B}_{\{e\}}^\perp) \).

**Example 2.23.** If \( \mathcal{B} \) is a Fell bundle over \( G \) and \( H = G \), then \( C^*(\mathcal{B}_G^\perp) = C^*(\mathcal{B}) \) and \( L^2_G(\mathcal{B}) \) is the \( \mathcal{B}^* \)-algebra \( C^*(\mathcal{B}) \) regarded as a right \( C^*(\mathcal{B}) \)–Hilbert module.

**Example 2.24.** If \( \mathcal{B} \) is again a Fell bundle over \( G \) and \( H = G \), then \( C^*(\mathcal{B}_G) = C^*(\mathcal{B}) \) and \( L^2_G(\mathcal{B}) \) is the \( \mathcal{B}^* \)-algebra \( C^*(\mathcal{B}) \) regarded as a right \( C^*(\mathcal{B}) \)–Hilbert module.

**Remark 2.25.** Our conventions (C) imply \( L^2_H(\mathcal{B}) \) is \( C^*(\mathcal{B}_H^\perp) \)-full. To prove this take any \( f \in C_c(\mathcal{B}_H) \). By \cite[II 14.8]{[7]} there exists \( g \in C_c(\mathcal{B}) \) such that \( p(g) = f \). Now take the approximate unit \( \{h_\lambda\}_{\lambda \in \Lambda} \subset C_c(\mathcal{B}) \) of \( L^1(\mathcal{B}) \) constructed in \cite[VIII 5.11]{[7]}. Then \( \{h_\lambda * g\}_{\lambda \in \Lambda} \) converges to \( g \) in the inductive limit topology of \( C_c(\mathcal{B}) \) and, since \( p \) is continuous with respect to this topology, the net \( \{p(h_\lambda * g)\}_{\lambda \in \Lambda} \) converges to \( p(g) = f \) in the inductive limit topology. It thus follows that \( \{\langle h_\lambda^* + I, g \rangle_p\}_{\lambda \in \Lambda} \) converges to \( \chi^{B_H}(f) \), implying that \( L^2_H(\mathcal{B}) \) is full.

We want a \( \mathcal{B} \)-representation \( \rho: L^1(\mathcal{B}) \to \mathcal{B}(L^2_H(\mathcal{B})) \) such that \( \rho(f)(g + I) = f * g + I \) for all \( f, g \in C_c(\mathcal{B}) \). If \( H = \{e\} \) and \( \mathcal{B} \) is Fell bundle, the existence of such a \( \mathcal{B} \)-representation is guaranteed by \cite[Proposition 2.6]{[6]}. If now \( H \) is the whole group \( G \) and \( \mathcal{B} \) is still a Fell bundle, then \( \rho \) is just the natural inclusion of \( L^1(\mathcal{B}) \) in \( C^*(\mathcal{B}) \subset \mathcal{B}(C^*(\mathcal{B})) = \mathcal{B}(L^2_G(\mathcal{B})) \).
One can then suspect the “disintegrated form” of $\rho$ has something to do with the disintegrations of $*$-representations of $L^1(\mathcal{B})$ and may consult [4] to get the formula for the disintegrated form of $\rho$, those formulas being expressions (10) and (11) in [7, VIII 5.8]. So we define, for every $r \in G$, $b \in B_r$, and $f \in C_c(\mathcal{B})$; the function $bf \in C_c(\mathcal{B})$ by $(bf)(s) := bf(r^{-1}s)$. Then $\mathcal{B} \times C_c(\mathcal{B}) \rightarrow C_c(\mathcal{B})$, $(b, f) \mapsto bf$, is a multiplicative and associative action of $\mathcal{B}$ on $C_c(\mathcal{B})$ with the additional property that $(bf)^*(gf) = f^* (bg)$. Moreover, for any fixed $f \in C_c(\mathcal{B})$ the function $\mathcal{B} \rightarrow C_c(\mathcal{B})$, $b \mapsto bf$, is continuous in the inductive limit topology.

**Proposition 2.26.** Let $\mathcal{B} = \{B_i\}_{i \in I}$ be a Banach $*$-algebraic bundle, $H$ a subgroup of $G$ and $\chi^B_H \colon L^1(\mathcal{B}_H) \rightarrow C^*(\mathcal{B}_H^+)$ the $\mathcal{B}^+$-completion of $L^1(\mathcal{B}_H)$. Construct the $C^*(\mathcal{B}_H^+)$—Hilbert module $L^2_H(\mathcal{B})$ as explained before. Then there exists a unique $*$-representation $\Lambda^H \colon \mathcal{B} \rightarrow \mathcal{B}(L^2_H(\mathcal{B}))$ such that $\Lambda^H(f + I) = bf + I$ for all $b \in \mathcal{B}$ and $f \in C_c(\mathcal{B})$. Moreover,

1. $\Lambda^H$ is non degenerate, and so it is its integrated form.

2. The integrated form $\bar{\Lambda}^H \colon L^1(\mathcal{B}) \rightarrow B(L^2_H(\mathcal{B}))$ is the unique $*$-representation of $L^1(\mathcal{B})$ such that $\bar{\Lambda}^H(g + I) = f \ast g + I$, for all $f, g \in C_c(\mathcal{B})$.

**Proof.** Uniqueness follows immediately. To prove existence take $f \in C_c(\mathcal{B})$ and a state $\varphi$ of $C^*(\mathcal{B}_H^+)$. Let $\pi : C^*(\mathcal{B}_H^+) \rightarrow B(Y)$ be the GNS construction for $\varphi$, with cyclic vector $\xi$ or norm one. There exists a unique non degenerate $*$-representation $T : \mathcal{B}_H \rightarrow B(Y)$ such that $\pi \circ \chi^B_H = T$, this representation is $\mathcal{B}$-positive by Theorem 2.16. The functional $\mu : B_e \rightarrow \mathbb{C}$ given by $\mu(b) := \varphi \circ \chi^B_H \circ p(f^* \ast (bf))$ is positive for all $b \in \mathcal{B}$ we have $\mu(b^*af) = \xi, \overline{T} \varphi(p((bf)^* \ast (bf)))\xi \geq 0$. Using [7, VI 18.14] we deduce that $|\mu(b^*af)| \leq \|a\|\mu(b^*b)$ for all $a, b \in B_e$. Hence, for all $b \in B_e$, $a \in \mathcal{B}$, $f \in C_c(\mathcal{B})$ and every state $\varphi$ of $C^*(\mathcal{B}_H^+)$ we have

\begin{equation}
0 \leq \varphi \circ \chi^B_H \circ p((bf)^* \ast (bf)) \leq \|a\|^2 \varphi \circ \chi^B_H \circ p((bf)^* \ast (bf)).
\end{equation}

Let $\{u_i\}_{i \in I}$ be a strong approximate unit of $\mathcal{B}$. Then the net $\{\varphi \circ \chi^B_H \circ p((uf)^* \ast (uf))\}_{i \in I}$ converges in the inductive limit to $f^* \ast f$. On the other hand, $p : C_c(\mathcal{B}) \rightarrow C_c(\mathcal{B}_H)$ is continuous in the inductive limit topology. Thus

\begin{equation}
\lim_i \|a\|^2 \varphi \circ \chi^B_H \circ p((fu_i)^* \ast (fu_i)) = \|a\|^2 \varphi \circ \chi^B_H \circ p(f^* \ast f).
\end{equation}

A similar argument implies that

\begin{equation}
\lim_\lambda \varphi \circ \chi^B_H \circ p((ae_\lambda f)^* \ast (ae_\lambda f)) = \varphi \circ \chi^B_H \circ p((af)^* \ast (af)).
\end{equation}

Then by (2.27),

\begin{equation}
0 \leq \langle af + I, af + I \rangle_p \leq \|a\|^2 \langle f + I, f + I \rangle_p
\end{equation}

for all $f \in C_c(\mathcal{B})$ and $a \in \mathcal{B}$.

By (2.27) above, for every $a \in \mathcal{B}$ there exists a unique linear and bounded map $\Lambda^H : L^2_H(\mathcal{B}) \rightarrow L^2_H(\mathcal{B})$ such that $\Lambda^H(f + I) = af + I$. Note that

\begin{equation}
\langle \Lambda^H_a(f + I), g + I \rangle_p = \chi^B_H(p((af)^* \ast g)) = \chi^B_H(p(f^* \ast (ag))) = \langle (f + I, \Lambda^H_a(g + I)) \rangle_p
\end{equation}

so it follows that $\Lambda^H_a$ is adjointable with adjoint $\Lambda^H_a$.

Define $\Lambda^H : \mathcal{B} \rightarrow B(L^2_H(\mathcal{B}))$ by $a \mapsto \Lambda^H_a$. It is straightforward to verify that $\Lambda^H$ is multiplicative and linear on each fibre, and we showed $\Lambda^H$ preserves the involution. Note that, given $f, g \in C_c(\mathcal{B})$, the function $a \mapsto \langle \Lambda^H_a(f + I), g + I \rangle_p = \chi^B_H(p((af)^* \ast g))$ is continuous because the function $\mathcal{B} \rightarrow C_c(\mathcal{B}_H)$ given by $a \mapsto p((af)^* \ast g)$ is continuous in the inductive limit topology. Since $C_c(\mathcal{B})/I$ is dense in $L^2_H(\mathcal{B})$, it follows that $\Lambda^H$ is a $*$-representation such that $\Lambda^H_a(f + I) = af + I$. 


Given any $f \in C_c(B)$, if $\{u_i\}_{i \in I}$ is a strong approximate unit of $B$ then
\[
\lim_{\lambda} \| \Lambda_{u_i}^{HB} (f + I) - (f + I) \|^2 = \lim_{\lambda} \| \chi_{B_H^+}(p((u_i f - f) \ast (u_i f - f))) \|^2 = 0
\]
because $\{(u_i f - f) \ast (u_i f - f)\}_{i \in I}$ converges to 0 in the inductive limit topology of $C_c(B)$. This, together with Proposition 2.34, implies both $\Lambda^{HB}$ and its integrated form are non degenerate.

To prove claim (2) take $f, g, h \in C_c(B)$. The function $G \to C_c(B)$, $t \mapsto h^\ast (g(t)f)$, has compact support and is continuous in the inductive limit topology. Moreover, there exists a compact set $U \subset G$ such that $\text{supp}(h^\ast (g(t)f)) \subset U$ for all $t \in G$ and the integral $\int_G h^\ast (g(t)f) \, dt$ makes sense in $C_c(B)$ with respect to the inductive limit topology. In fact the integral $\int_G g(t)h^\ast (g(t)f) \, dt$ also makes sense in this topology and $\int_G h^\ast (g(t)f) \, dt = h^\ast \int_G g(t)h^\ast (g(t)f) \, dt$. The construction of the product of $C_c(B)$ performed in [7] implies $\int G g(t)h^\ast (g(t)f) \, dt = g \ast f$. After this we can deduce that
\[
\langle h + I, \tilde{\Lambda}_{g}^{HB} (f + I) \rangle_p = \int_G \langle h + I, g(t)f + I \rangle_p \, dt = \chi_{B_H^+}^{G}(p \left( \int_G h^\ast (g(t)f) \, dt \right))
\]
and the identity $\tilde{\Lambda}_{g}^{HB}(f + I) = g \ast f + I$ follows for all $f, g \in C_c(B)$. \qed

**Definition 2.29.** Given a Banach *-algebraic bundle $B$ over $G$ and a subgroup $H$ of $G$, the $H$–regular *-representations of $B$ and $L^1(B)$ are, respectively, the *-representation $\Lambda^{HB}: B \to \mathcal{B}(L^2_H(B))$ and the integrated form $\tilde{\Lambda}^{HB}: L^1(B) \to \mathcal{B}(L^2_H(B))$ given by Proposition 2.26. The $H$–cross sectional C*-algebra for $B$, $C_H^*(B)$, is the closure of $\tilde{\Lambda}^{HB}(L^1(B))$. We define $\varphi_{HB}^B: C^*(B) \to C^*_H(B)$ as the unique morphism of *-algebras such that $\varphi_{HB}^B \circ \chi_B = \tilde{\Lambda}^{HB}$.

**Remark 2.30.** By construction $C^*_H(B)$ is a non degenerate C*-subalgebra of $\mathcal{B}(L^2_H(B))$, thus we may regard $\mathcal{B}(C^*_H(B))$ as the C*-subalgebra of $\mathcal{B}(L^2_H(B))$ formed by those $M \in \mathcal{B}(L^2_H(B))$ such that both $MC^*_H(B)$ and $C^*_H(B)M$ are contained in $C^*_H(B)$. It is then clear that the image of $\Lambda^{HB}$ is contained in $\mathcal{B}(C^*_H(B))$ and, when convenient, we will regard $\Lambda^{HB}$ as the non degenerate *-representation $\Lambda^{HB}: B \to \mathcal{B}(C^*_H(B))$.

In case $B$ is a Fell bundle over $G$, $C^*(B) := C^*_c(B)$ is just the usual cross sectional C*-algebra of $B$. The reduced cross sectional C*-algebra of $B$, denoted $C^*_r(B)$ and defined in [5], is $C^*_r(B_{(e)}(B))$. These two claims hold by Examples 2.23 and 2.24

Let’s continue working under the hypotheses of Proposition 2.26 and take a $B$–positive *-representation $S: \mathcal{B}_H \to \mathcal{B}(Y_A)$. By Theorem 2.16 there exists a unique *-representation $\chi_{S}^{B_H^+}: C^*(B_H^+) \to \mathcal{B}(Y_A)$ such that $\chi_{S}^{B_H^+} \circ \chi_{B_H^+} = \tilde{S}$. We can now form the balanced tensor product
\[
L^2_H(B) \otimes_S Y_A := L^2_H(B) \otimes_{\chi_{S}^{B_H^+}} Y_A.
\]
The image of the (algebraic) elementary elementary tensor $(f + I) \otimes \xi \in (C_c(B)/I) \otimes Y_A$ (with $f \in C_c(B)$ and $\xi \in Y_A$) in $L^2_H(B) \otimes_S Y_A$ will be denoted $f \otimes_S \xi$.

The map
\[
\mathcal{B}(L^2_H(B)) \to \mathcal{B}(L^2_H(B) \otimes_S Y_A), \quad T \mapsto T \otimes S 1
\]
is a morphism of $C^*$-algebras and we can compose it with $\Lambda^{HB}: B \to \mathcal{B}(L^2_H(B))$ to get the (abstractly) induced *-representation
\[
(2.31) \quad \text{Ind}_{H}^{B}(S): B \to \mathcal{B}(L^2_H(B) \otimes_S Y_A), \quad a \mapsto T_a \otimes_S 1,
\]
which has integrated form

\[(2.32) \quad \widetilde{\text{Ind}}_H^B(S) : B \to \mathbb{B}(L^2_H(B) \otimes_S Y_A), \quad f \mapsto \mathcal{T}_f \otimes_S 1,\]

Note induced representations are always non degenerate by conventions (C), and also that \(\text{Ind}_H^B(S)\) agrees with the representation induced by the essential part of \(S\).

**Remark 2.33.** If \(Y_A\) is a Hilbert space, a close examination of the construction of \(L^2_H(B) \otimes_S Y_A\) in terms of \(C_0(B)\) and \(S\) reveals that \(\text{Ind}_H^B(S)\) is (in Fell’s terms \([7, \text{XI} 9.26]\)) the abstractly induced *-representation \(\text{Ind}_{L^1(B_H) \uparrow L^1(B)}(S)\). Now \([7, \text{XI} 9.26]\) tells us that \(\text{Ind}_H^B(S)\) is unitary equivalent to Fell’s concretely induced *-representation \(\text{Ind}_{B_H \uparrow B}(S)\).

**Remark 2.34.** If \(\rho : A \to \mathbb{B}(Z_C)\) is a *-representation, then by the associativity of balanced tensor products there exists a unique unitary

\[U : L^2_H(B) \otimes_{T_{\otimes_1}} (Y_A \otimes_{\rho} Z_C) \to (L^2_H(B) \otimes_T Y_A) \otimes_{\rho} Z_C\]

such that \(U(f \otimes_{T_{\otimes_1}} (\xi \otimes_{\rho} \eta)) = (f \otimes_T \xi) \otimes_{\rho} \eta\). By Remark 2.6 using \(U\) we get the unitary equivalence of *-representations

\[\text{Ind}_H^B(T \otimes_{\rho} 1) \cong \text{Ind}_H^B(T) \otimes_{\rho} 1, \quad \text{Ind}_H^B(T \otimes_{\rho} 1) \cong \tilde{\text{Ind}}_H^B(T) \otimes_{\rho} 1.\]

**Example 2.35** (The regular *-representation). Let \(B = \{B_t\}_{t \in G}\) be a Fell bundle and think of the identity map \(\text{id} : B_e \to B_e\) as a map from \(B_e\) to the multiplier algebra \(\mathbb{B}(B_e)\). Let’s denote \(\Lambda^{eB}\) and \(\tilde{\Lambda}^{eB}\) the \(\{e\}\)-regular *-representations of \(B\) and \(L^1(B)\), respectively. If we take the universal *-representation of \(B_e\), \(\rho : B_e \to \mathbb{B}(Y)\), then the *-representation used in \([7, \text{VIII} 16.4]\) to show \(L^1(B)\) is reduced may be regarded as a subrepresentation of \(\tilde{\text{Ind}}_e(\rho) = \tilde{\Lambda}^{eB} \otimes_{\rho} 1\). Hence \(\tilde{\Lambda}^{eB} : L^1(B) \to \mathbb{B}(L^2_e(B))\) is a faithful *-representation.

Part of our next Theorem is expressed in terms of Fell’s induced systems of imprimitivity \([7, \text{XI} 14.3]\). It may be considered as the formalization of the fact that \(L^2_H(B)\) is the the Hilbert module behind Fell’s induction process.

**Notation 2.36.** Given a group \(G\) and a subgroup \(H \subset G\), the natural action of \(G\) on \(C_0(G/H)\) will be denoted \(\sigma^{HG}\). More precisely, \(\sigma^{HG}(f)(tH) = f(s^{-1}tH)\) for all \(f \in C_0(G/H)\) and \(s, t \in G\).

**Theorem 2.37.** Assume \(B = \{B_t\}_{t \in G}\) is a Banach *-algebraic bundle and \(H \subset G\) a subgroup. Then there exists a *-representation \(\psi^{HB} : C_0(G/H) \to \mathbb{B}(L^2_H(B))\) which is non degenerate and

1. \(\Lambda^t^{HB} \psi^{HB}(f) = \psi^{HB}(\sigma^{HG}_t(f)) \Lambda^t^{HB}\) for all \(b \in B_t, \ t \in G \text{ and } f \in C_0(G/H)\).
2. Given a \(B\)-positive *-representation \(S : B_H \to \mathbb{B}(Y_A)\), the *-representation \(\psi^S : C_0(G/H) \to \mathbb{B}(L^2_H(B) \otimes_S Y_A), \ f \mapsto \psi^{HB}(f) \otimes_S 1,\)

is non-degenerate and for all \(b \in B_t, \ t \in G \text{ and } f \in C_0(G/H)\),

\[\text{Ind}_H^B(S)_b \psi^S(f) = \psi^S(\sigma^{HG}_t(f)) \text{Ind}_H^B(S)_b.\]

3. Assuming \(Y_A\) is a Hilbert space, \(S\) is non degenerate, using the terminology of \([4, \text{XI} 14.3 \text{pp-1181}]\) and the unitary equivalence mentioned in Remark 2.33, \(\psi^S\) is the integrated form of the projection-valued measure \(P\) induced by \(S\) and \(\langle \text{Ind}_H^B(S), P \rangle\) is the system of Imprimitivity induced by \(S\).
Proof. We will, as usual, denote the generalized restriction and the $B^+$—completion of $L^1(B_H)$ by $p: C_c(B) \to C_c(B_H)$ and $\chi^B_H: L^1(B) \to C^*(B_H^p)$, respectively.

Given $f \in C_b(G/H)$ and $g \in C_c(B)$ define $fg \in C_c(B)$ by $fg(t) := f(tH)g(t)$. If $S: B_H \to B(Z)$ is a $B$—positive $*$—representation, then for all $\xi \in Z$ we have

$$\|f\|\|\xi, S_p(g)\xi\| - \langle \xi, S_p(g)\xi\rangle = \langle \xi, S_p(\|fg\|\xi)\rangle \geq 0.$$ 

It then follows that

$$\langle fg + I, fg + I\rangle_p \leq \|f\|\langle g + I, g + I\rangle_p$$

and, consequently, for every $f \in C_b(G/H)$ there exists a unique bounded operator $\psi^{HB}(f): L^2_H(B) \to L^2_H(B)$ such that $\psi^{HB}(f)(g + I) = fg + I$.

The identity $p((fg)^*h) = p(g^*(f^*g))$ holds for all $f, g, h \in C_c(B)$. Thus $\langle \psi^{HB}(f)u, v\rangle_p = \langle u, \psi^{HB}(f^*)v\rangle_p$ for all $u, v \in C_c(B)/I$ and it follows that $\psi^{HB}(f)$ is adjointable with adjoint $\psi^{HB}(f^*)$. We leave to the reader the verification of the fact that $\psi^{HB}: C_0(G/H) \to B(L^2_H(B))$ is a unital $*$—representation. Note the representation $\psi^{HB}$ of the statement is in fact the restriction to $C_0(G/H)$ of the $\psi^{HB}$ we have just constructed; for that reason we use the same symbol to denote both representations.

To give a direct proof of the non degeneracy of $\psi^{HB}|_{C_0(G/H)}$ one may proceed as follows. Given $g \in C_c(B)$ let $f \in C_0(G/H)$ be such that $f(tH) = 1$ if $t \in \text{supp}(g)$. Then $fg = g$ and $\psi^{HB}(f)(g + I) = g + I$, implying that $C_c(B)/I$ is contained in the essential space of $\psi^{HB}$.

Take $t \in G$, $b \in B_t$, $f \in C_b(G/H)$ and $g \in C_c(B)$. Then for all $r \in G$

$$b(fg)(r) = b(fg)(t^{-1}r) = b(t^{-1}rH)g(t^{-1}r) = \sigma^H_f(f)(rH)(bg)(r) = (\sigma^H_r(f)(bg))(r);$$

implying that

$$\Lambda^H_b\psi^{HB}(f)(g + I) = \psi^{HB}(\sigma^H_r(f))\Lambda^H_b(g + I).$$

Hence (1) follows.

Claim (2) follows at once from (1) after one recalls that $\text{Ind}^B_H(S) = \Lambda^H \otimes_{S} 1$. To prove (3) we assume $Y \equiv Y_A$ is a Hilbert space and $S$ is non degenerate. In such a situation we know (by Remark 2.33) that $\text{Ind}^B_H(S)$ if unitary equivalent to Fell’s concretely induced representation $\text{Ind}_{B_H,B}(S)$. We then need to specify the unitary equivalence mentioned in Remark 2.33.

We start (exactly) as in [7, XI 9], so we fix a continuous and everywhere positive $H$—rho function $\rho$ on $G$ (see [7, III 14.5]) and denote $\rho^\#$ the regular Borel Measure on $G/H$ constructed from $\rho$ (as in [7, III 13.10]). Fell’s Hilbert space $X \equiv X(S)$ is our $Y$. Since our inner products are linear in the second variable, we will be forced to adapt Fell’s formulas to our situation.

Let $\mathcal{Y} = \{Y_a\}_{a \in G/H}$ be the Hilbert bundle over $G/H$ induced by $S$ and let $L^2(\rho^\#, \mathcal{Y})$ be the corresponding cross sectional Hilbert space (see [7, XI 9.7] and [7, II 15.12]). The unitary operator $E: L^2_H(B) \otimes_S \mathcal{Y} \to L^2(\rho^\#, \mathcal{Y})$ of [7, XI 9.8] intertwines the abstract and concrete induced representations. The identity $E(\psi_S(f)(g \otimes_S \xi)) = fE(g \otimes_S \xi)$ follows at once for all $f \in C_b(G/H)$ and every elementary tensor $g \otimes_S \xi$. So $E$ intertwines the integrated form of the $P$ mentioned in claim (3) and $\psi^S$.

Theorem 2.37 motivates the following.

**Definition 2.38.** Let $B = \{B_t\}_{t \in G}$ be a Banach $*$—algebraic bundle and $H \subset G$ a subgroup. An integrated system of $H$—imprimitivity for $B$ is a tern $\Xi := (Y_A, T, \psi)$ such that:
Theorem 2.40. Let $G$ be a LCH topological group and $H$ a subgroup of $G$. A necessary and sufficient condition for a unitary $*$-representation $U: G \to \mathbb{B}(Y)$ to be

1. $T: \mathcal{B} \to \mathbb{B}(Y_A)$ and $\psi: C_0(G/H) \to \mathbb{B}(Y_A)$ are $*$-representations and the essential space of $\psi$ contains that of $T$.
2. For all $b \in B$, $f \in C_0(G/H)$ and $t \in G$ it follows that $T_b \psi(f) = \psi(\sigma^H_t(f))T_b$. A system $(Y_A, T, \psi)$ is non degenerate if $T$ (and hence $\psi$) is non degenerate.

Remark 2.39. Condition (2) implies the essential space of $T$, $Y^T$, is invariant under the action of $C_0(G/H)$ by $\psi$. Thus there exists a unique non degenerate integrated system of $H$–imprimitivity $(Y^T, T^T, \psi^T)$ such that $T_b^T \xi = T_b \xi$ and $\psi^T(f) \xi = \psi(f) \xi$ for all $b \in B$, $f \in C_0(G/H)$ and $\xi \in Y^T$. We call this system the essential part of $(Y_A, T, \psi)$.

Fell’s approach to imprimitivity systems is through projection-valued Borel measures of $G/H$. This tool is available if one works with Hilbert spaces because projections abound in $\mathbb{B}(Y)$. But if $Y_A$ is an $A$–Hilbert module and $\mathbb{B}(Y_A)$ is not a von Neumann algebra, it may not be possible to speak of “projection valued measures” on $\mathbb{B}(Y_A)$ and it is more convenient to consider “integrated forms of the projection valued measures”, i.e. $*$-representations of $C_0(G/H)$. This approach is justified by [7, VIII 18.7 & 18.8]. One may “disintegrate” a $*$-representations of $C_0(G/H)$ on $\mathbb{B}(Y_A)$ by taking a faithful and non degenerate $*$-representation $\rho: A \to \mathbb{B}(Z)$ and considering the faithful $*$-representation $\mathbb{B}(Y_A) \to \mathbb{B}(Y_A \otimes \rho Z)$, $M \mapsto M \otimes \rho 1$. 

2.3. Weak containment and Fell’s absorption principle. Given a LCH topological group $G$ one may view the representation theory of $G$ as the representation theory of the trivial Fell bundle over $G$ with constant fibre $C$, which we denote $\mathcal{T}_G = \{C \delta_t\}_{t \in G}$. If $H \subset G$ is a subgroup, then $(\mathcal{T}_G)_H \equiv \mathcal{T}_H$ and the induction of $*$-representations from $H$ to $G$ may be regarded as the induction of $*$-representations from $C^*(H) := C^*(\mathcal{T}_H)$ to $C^*(G) := C^*(\mathcal{T}_G)$ via the $C^*(H)$–Hilbert module $L^2_H(G) := L^2_H(\mathcal{T}_G)$. The $C^*$-algebra $C^*_H(G)$ is defined as $C^*_H(\mathcal{T}_G) \subset \mathbb{B}(L^2_H(G))$ and the map $q^T_H: C^*_H(G) \to C^*_H(G)$ will be denoted $q^{T}_H$. Theorem 2.15 implies any $*$-representation of $H$ can be induced to a $*$-representation of $G$ (c.f. [9, Theorem 4.4]).

Given a non degenerate integrated system of $H$–imprimitivity for $\mathcal{T}_G$, $(Y_A, T, \psi)$, the map $U: G \to \mathbb{B}(Y_A)$, $U^T_t := T_{1b^t}$, is a unitary representation of $G$ such that $U^T_t \psi(f) = \psi(\sigma^H_t(f))U^T_t$ for all $t \in G$ and $f \in C_0(G/H)$. It is then natural to say a tern $(Z_C, U, \phi)$ is a system of $H$–imprimitivity for $G$ if

1. $Z_C$ is a $C$–Hilbert module.
2. $U: G \to \mathbb{B}(Z_C)$ is a unitary $*$-representation.
3. $\psi: C_0(G/H) \to \mathbb{B}(Z_C)$ is a non degenerate $*$-representation.
4. For all $t \in G$ and $f \in C_0(G/H)$, $U_t \phi(f) = \phi(\sigma^H_t(f))U_t$.

Note the process $(Y_A, T, \psi) \sim (Y_A, U^T, \phi)$ establishes a bijective correspondence between the non degenerated systems of $H$–imprimitivity of $\mathcal{T}_G$ and the systems of $H$–imprimitivity for $G$.

Definition 2.40. The $H$–regular representation of $G$ is $U^{HG} := U^{\Lambda^{HT_G}}$, with $\Lambda^{HT_G}$ being that of Theorem 2.17. The system of $H$–imprimitivity for $G$ associated to $U^{HG}$ is $(L^2_H(G), U^{HG}, \psi^{HG})$ with $\psi^{HG} := \psi^{HT_G}$.

Fell’s Imprimitivity Theorem [7, XI 14.18] gives Mackey’s Imprimitivity Theorem because $\mathcal{T}_G$ is saturated, see also [9, Theorem 7.18] and the references therein. We state it here for future reference.

Theorem 2.41. Let $G$ be a LCH topological group and $H$ a subgroup of $G$. A necessary and sufficient condition for a unitary $*$-representation $U: G \to \mathbb{B}(Y)$ to be
(unitary equivalent to) a \(*\)-representation induced by a \(*\)-representation of \(H\) is the existence of a non degenerate \(*\)-representation \(\psi: C_0(G/H) \to \mathbb{B}(Y)\) such that \(U_t \psi(f) = \psi(\sigma^H_t(f))U_t\) for all \(t \in G\) and \(f \in C_0(G/H)\).

Let \(\mathcal{B} = \{B_t\}_{t \in G}\) be Banach \(*\)-algebraic bundle and \(H \subseteq G\) a subgroup. In general, the reductions \(B_{tH^{-1}}\) may be quite different for different \(t\). As an example of this consider a discrete group \(G\) with a subgroup \(H\) for which there exists \(t \in H\) such that \(H \cap (tHt^{-1}) = \{e\}\). Let \(\mathcal{B} = \{B_t\}_{t \in G}\) be the subbundle of \(T_G\) such that \(B_t = \mathbb{C}\) if \(t \in H\) and \(B_t = \{0\}\) if \(t \notin H\). Then \(B_H = T_H\) but all the fibers of \(B_{tH^{-1}}\) are \(\{0\}\), except for \(B_e = \mathbb{C}\). Note also that \(C^*(B_H) = C^*(H)\) and \(C^*(B_{tH^{-1}}) = C^*_t(H)\), thus \(C^*_H(\mathcal{B})\) and \(C^*_{tH^{-1}}(\mathcal{B}) = C^*_t(G)\) may be quite different from each other.

**Theorem 2.42.** Let \(\mathcal{B} = \{B_t\}_{t \in G}\) be a Banach \(*\)-algebraic bundle, \(H \subseteq G\) a subgroup, fix \(s \in G\) and set \(K := sHs^{-1}\). Consider the Haar measure on \(K\), \(d_Kt\), such that \(d_K(t) = d_H(s^{-1}ts)\). Assume at least one of the following conditions hold:

1. \(H\) is normal in \(G\).
2. \(\mathcal{B}\) is saturated.
3. \(\mathcal{B}\) has a unitary multiplier of order \(s\) \([7, \text{VIII 3.9}]\).

Then

\[
(2.43) \quad \|\tilde{\Lambda}^{HB}_f\| = \|\tilde{\Lambda}^{KB}_f\| \quad \forall f \in L^1(\mathcal{B}).
\]

In particular, there exists a unique morphism of \(*\)-algebras \(\rho: C^*_H(\mathcal{B}) \to C^*_K(\mathcal{B})\) such that \(\rho \circ \tilde{\Lambda}^{HB} = \tilde{\Lambda}^{KB}\). In fact \(\rho\) is a \(*\)-isomorphism and it is also the unique non degenerate \(*\)-representation \(\rho: C^*_H(\mathcal{B}) \to \mathbb{B}(C^*_K(\mathcal{B}))\) such that \(\overline{\rho} \circ \Lambda^{HB} = \Lambda^{KB}\).

**Proof.** The statement is immediate if \(H\) is normal in \(G\). Assume \(\mathcal{B}\) is saturated, in which case we adopt Fell’s notation (and construction) on conjugated representations \([7, \text{XI 16}]\). Let \(T: B_H \to \mathbb{B}(Y)\) be a non degenerate \(\mathcal{B}\)-positive \(*\)-representation such that \(\|\text{Ind}^B_H(T)|_f\| = \|\tilde{\Lambda}^{HB}_f\|\) for all \(f \in L^1(\mathcal{B})\). By \([7, \text{XI 16.7}]\) the conjugated \(*\)-representation \(^{\ast}T: B_K \to \mathbb{B}(Z)\) is \(\mathcal{B}\)-positive and, by \([7, \text{XI 16.19}]\), \(\text{Ind}^B_K(^{\ast}T)\) is unitary equivalent to \(\text{Ind}^B_H(T)\). Thus, for all \(f \in L^1(\mathcal{B})\),

\[
\|\tilde{\Lambda}^{HB}_f\| = \|\text{Ind}^B_H(T)|_f\| = \|\text{Ind}^B_K(^{\ast}T)|_f\| \leq \|\tilde{\Lambda}^{KB}_f\|.
\]

By symmetry we obtain \((2.43)\).

Now assume \(\mathcal{B}\) has a unitary multiplier \(u\) of order \(s\). In this situation we may proceed as in \([7, \text{XI 16.16}]\), we repeat the construction to show the reader the saturation hypothesis is not really needed. Let \(T: B_H \to \mathbb{B}(Y)\) be a non degenerate \(\mathcal{B}\)-positive \(*\)-representation for which the \(*\)-representation \(\chi^B_T\) is faithful (recall \(\chi^B_{T^*} \circ \chi^B_T = \tilde{T}\)). Define \(^{\ast}uT: B_K \to \mathbb{B}(Y)\) by \(^{uT}_T := T_{urba}\), and note that \(^u\) \(^{\ast}T\) = \(T\). Set \(S := ^{\ast}uT\) and given any \(f \in C_c(\mathcal{B})\) define \([fu] \in C_c(\mathcal{B})\) by \([fu](t) = \Delta_G(s)^{-1/2}f(ts^{-1})u\).

For all \(f, g \in C_c(\mathcal{B})\) and \(\xi, \eta \in Y\):

\[
\langle [fu] \otimes_T \xi, [fu] \otimes_T \eta \rangle = \langle \xi, \tilde{T}_{fu}^H((fu)^*|gu)\rangle = \int_H \int_G \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} \langle \xi, T_{[fu](r)}^*|gu|v(r)\rangle \, dG \, d_H t
\]

\[
= \int_H \int_G \Delta_G(s)^{-1} \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} \langle \xi, T_{fu(r)^{-1}}^*|g(rts^{-1})u\rangle \, dG \, d_H t
\]

\[
= \int_K \int_G \Delta_G(t)^{1/2} \Delta_K(t)^{-1/2} \langle \xi, S_{f(r)^{-1}}^*|g(r)\rangle \, dG \, d_K t = \langle f \otimes \xi, g \otimes \eta \rangle.
\]
Thus there exists a unique unitary operator $U : L^2_K(B) \otimes_S Y \to L^2_H(B) \otimes_T Y$ such that $U(f \otimes_S \xi) = [fu] \otimes_T \xi$.

Given $t \in G$, $b \in B_t$, $f \in C_c(B)$ and $\xi \in Y$ we have

$$ (b[uf])(r) = b[fu](t^{-1}r) = \Delta_G(s)^{-1/2}bf(t^{-1}rs^{-1})u = [(bf)u](r), $$

for all $r \in G$. Thus,

$$ U^* \text{Ind}_H^B(T) b U(f \otimes_S \xi) = U^*(b[fu] \otimes_T \xi) = U^*([(bf)u] \otimes_T \xi) = \text{Ind}_K^B(S)b(f \otimes_S \xi); $$

implying that $U$ intertwines $\text{Ind}_H^B(T)$ and $\text{Ind}_K^B(S)$ (and their integrated forms).

Our choice of $T$ guarantees that for all $f \in L^1(B)$

$$ \|\tilde{\Lambda}^{HB}_f\| = \|\text{Ind}_H^B(T)f\| = \|\text{Ind}_K^B(S)f\| \leq \|\tilde{\Lambda}^{KB}_f\|. $$

Since $u^*$ is a unitary multiplier of order $s^{-1}$ and $s^{-1}Ks = H$, we conclude that (2.43) holds.

At this point we know that any of the conditions (1), (2) or (3) imply (2.43). Since the ranges of both $\tilde{\Lambda}^{HB}$ and $\tilde{\Lambda}^{KB}$ are dense *-subalgebras of $C^*_H(B)$ and $C^*_K(B)$, respectively, the existence of $C^*$-isomorphism $\rho : C^*_H(B) \to C^*_K(B)$ such that $\rho \circ \tilde{\Lambda}^{HB} = \tilde{\Lambda}^{KB}$ follows. We leave the rest of the proof to the reader. \hfill \square

Our version of the Absorption Principle will be sated in terms of minimal tensor products of *-representations of Hilbert modules. We now briefly present some standard tensor product constructions in convenient way.

Given two right Hilbert modules, $Y_A$ and $Z_C$, the (minimal) tensor product $Y_A \otimes Z_C$ is a right $A \otimes C$—Hilbert module with inner product determined by the condition

$$ \langle \xi \otimes \eta, \zeta \otimes \nu \rangle = \langle \xi, \zeta \rangle \otimes \langle \eta, \nu \rangle; $$

which implies $\|\xi \otimes \eta\| = \|\xi\|\|\eta\|$. The construction of $Y_A \otimes Z_C$ may be performed inside the tensor product between the linking algebras $\mathbb{L}(Y_A) \otimes \mathbb{L}(Z_C)$, in which case the $A \otimes C$—valued inner product of $Y_A \otimes Z_C$ is just the restriction of the natural inner product of $\mathbb{L}(Y_A) \otimes \mathbb{L}(Z_C)$. The results of [3, Section 5.2] imply this construction yields the minimal tensor product of Hilbert modules.

The description of minimal tensor products we have just exposed implies that in case $Y_A = A_A$ and $Z_C = C_C$, the tensor product $A_A \otimes C_C$ is just the $C^*$-algebra $A \otimes C$ regarded as a Hilbert module, this is $A_A \otimes C_C = (A \otimes C)_{A\otimes C}$.

If $Z_C = Z$ is a Hilbert space ($C = C$), we identify $A \otimes C$ with $A$ and regard $Y_A \otimes Z$ as an $A$—Hilbert module. This module is also the balanced tensor product $Z \otimes_\rho Y_A$ for the trivial representation $\rho : C \to \mathbb{B}(Y_A)$, $\lambda \mapsto \lambda 1$. Thus, if both $A$ and $C$ are the complex field $\mathbb{C}$, $Y_A \otimes Z_C$ is just the usual tensor product of Hilbert spaces.

Every $T \in \mathbb{B}(Y_A)$ defines an adjointable map $\mathbb{L}(T) \in \mathbb{B}(\mathbb{L}(Y_A))$ that, in the usual matrix representation of $\mathbb{L}(Y_A)$, is given by

$$ \mathbb{L}(T) \left( \begin{array}{cc} R & \xi \\ \bar{\eta} & a \end{array} \right) := \left( \begin{array}{cc} TR & T\xi \\ \bar{\eta} & a \end{array} \right). $$

By faithfully representing both $\mathbb{L}(Y_A)$ and $\mathbb{L}(Z_C)$ on a Hilbert space we can prove the existence of a unique operator $\mathbb{L}(T) \otimes 1 \in \mathbb{B}(\mathbb{L}(Y_A) \otimes \mathbb{L}(Z_C))$ mapping $\xi \otimes \eta$ to $\mathbb{L}(T)\xi \otimes \eta$. The subspace $Y_A \otimes Z_C$ is invariant under both $\mathbb{L}(T)$ and $\mathbb{L}(T^*) = \mathbb{L}(T)^*$ and the restriction of $\mathbb{L}(T) \otimes 1$ to this invariant space is the (unique) operator $T \otimes 1 \in \mathbb{B}(Y_A \otimes Z_C)$ mapping $\xi \otimes \eta$ to $T\xi \otimes \eta$. In this very same way one can construct, for each $S \in \mathbb{B}(Z_C)$, the (unique) operator $1 \otimes S \in \mathbb{B}(Y_A \otimes Z_C)$ mapping $\xi \otimes \eta$ to $\xi \otimes S\eta$. The composition
$T \otimes S := (T \otimes 1) \circ (1 \otimes S) = (1 \otimes S) \circ (T \otimes 1)$ is the unique adjointable operator of $Y_A \otimes Z_C$ mapping $\xi \otimes \eta$ to $T\xi \otimes S\eta$, and it’s adjoint is $T^* \otimes S^*$.

Putting together several facts from [7] (with some standard tricks) and extending the ideas of [6] we got the theorem below. The reader should notice that if one takes $H = \{e\}$ (this is Exel-Ng’s situation [6]) then one may use the trivial representation $H \rightarrow C$, $t \mapsto 1$, instead of the $*$-representation $V$ we will use in the proof below. In doing so our proof is becomes a modified version of that of Exel and Ng.

**Theorem 2.44** (FExell’s Absorption Principle I). Assume $B = \{B_t\}_{t \in G}$ is a Banach $*$-algebraic bundle and $H \subset G$ is a subgroup. Given $t \in G$ and non degenerate $*$-representations $T: B \rightarrow \mathbb{B}(Y_A)$ and $V: H \rightarrow \mathbb{B}(Z_C)$ define $H^t := tHt^{-1}$ and the $*$-representations $V^t: H^t \rightarrow \mathbb{B}(Z_C)$, $V^t_r := V_r^{-1}t$ and $S^t: B_{H^t} \rightarrow \mathbb{B}(Y_A)$, $S^t_b := T_b$. If $U := \text{Ind}^G_H(V)$ and $W_C := L^2_H(G) \otimes_V Z_C$, then the functions

\begin{equation}
T \otimes U: B \rightarrow \mathbb{B}(Y_A \otimes W_C), \quad (b \in B_r) \mapsto T_b \otimes U_r
\end{equation}

\begin{equation}
S^t \otimes V^t: B_{H^t} \rightarrow \mathbb{B}(Y_A \otimes Z_C), \quad (b \in B_r) \mapsto S^t_b \otimes V^t_r
\end{equation}

are non degenerate $*$-representations, $S^t \otimes V^t$ is $B$-positive and

\begin{equation}
\| (T \otimes U)_f \| = \sup \{ \| \text{Ind}^B_H(S^t \otimes V^t)_f \| : t \in G \} \quad \forall \, f \in L^1(B).
\end{equation}

**Proof.** The verification of the facts that both $T \otimes U$ and $S^t \otimes V^t$ are non degenerate $*$-representations are left to the reader. Note the restriction $(S^t \otimes V^t)|_{B_r} = (T \otimes 1)|_{B_r}$ is $B$-positive. Hence Theorem 2.43 implies $S^t \otimes V^t$ is $B$-positive.

Define the $A \otimes C$-Hilbert module $Y_A \otimes^G W_C := \ell^2(G) \otimes (Y_Z \otimes W_C)$, which we regard as the direct sum of $G$-copies of $Y_A \otimes Z_C$. Similarly, we define $*$-representation

$T \otimes^G U: B \rightarrow \mathbb{B}(Y_A \otimes^G W_C), \quad (b \mapsto 1_{\ell^2(G)} \otimes (T \otimes U)_b),$

which is the composition of $T \otimes U$ with the unital and faithful $*$-representation

$\Theta: \mathbb{B}(Y_A \otimes W_C) \rightarrow \mathbb{B}(Y_A \otimes^G W_C), \quad \Theta(R) = 1_{\ell^2(G)} \otimes R.$

Note that $(T \otimes^G U) = \Theta \circ (T \otimes U)$. Thus it suffices to prove

\begin{equation}
\| (T \otimes^G U)_f \| = \sup \{ \| \text{Ind}^B_H(S^t \otimes V^t)_f \| : t \in G \} \quad \forall \, f \in L^1(B).
\end{equation}

We claim there exists a unique linear map

$L: C_c(G; Y_A \otimes^G Z_C) \rightarrow Y_A \otimes^G W_C$

which is continuous in the inductive limit topology and

\begin{equation}
L(f \circ [\delta_t \otimes \xi \otimes \eta]) = \delta_t \otimes (\xi \otimes [f \circ V \eta])
\end{equation}

for all $f \in C_c(G)$, $\xi \in Y_A$, $\eta \in Z_C$ and $t \in G$, where:

- We regard $C_c(G)$ as $C_c(T_G)$ and think of $L^2_H(G)$ as a completion of $C_c(G)$.
- If $w \in Y_A \otimes^G Z_C$, $f \circ w \in C_c(G; Y_A \otimes^G Z_C)$ is given by $f \circ w(r) = f(r)w$.

Uniqueness of $L$ follows from the fact that the functions of the form $f \circ (\delta_t \otimes \xi \otimes \eta)$ span a dense subset of $C_c(G; Y_A \otimes^G Z_C)$ which is dense in the inductive limit topology (see [7] II 14.6). Take functions $u, v \in C_c(G; Y_A \otimes^G Z_C)$ that can be expressed as elementary tensors

$u = f \circ (\delta_r \otimes \xi \otimes \eta) \quad \quad v = g \circ (\delta_s \otimes \zeta \otimes \kappa)$
as explained before. If \( \delta_{r,s} \) is the Kronecker delta (equals 1 if and only if \( r = s \)) then

\[
\langle \delta_r \otimes (\xi \otimes [f \otimes V \eta]), \delta_s \otimes (\zeta \otimes [g \otimes V \kappa]) \rangle = \\
= \delta_{r,s} \langle \xi, \zeta \rangle \otimes \int_H \int_G \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} f(zt) g(zt) \langle \eta, V \kappa \rangle \, dG z dH t \\
= \int_H \int_G \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} f(z) g(zt) \langle \delta_r \otimes \xi \otimes \eta, (1 \otimes 1 \otimes V_t)(\delta_s \otimes \zeta \otimes \kappa) \rangle \, dG z dH t \\
= \int_H \int_G \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} \langle u(z), (1 \otimes 1 \otimes V_t) v(zt) \rangle \, dG z dH t.
\]

Fix a compact set \( D \) and denote \( C_D(G, Y_A \otimes^G Z_C) \) the set formed by those \( f \in C_c(G, Y_A \otimes^G Z_C) \) with support contained in \( D \); this set is in fact a Banach space with the norm \( \| \cdot \|_D \). Let \( C_D^0 \) be the subspace of \( C_D(G, Y_A \otimes^G Z_C) \) spanned by the functions of the form \( f \otimes (\delta_r \otimes \xi \otimes \eta) \) with \( f \in C_D(G), t \in G, \xi \in Y_A \) and \( \eta \in Z_C \). We clearly have \( C_c(G) C_D^0 \subset C_D^0 \). If, for every \( t \in G \), we define \( C_D^0(t) \) as the closure of \( \{ u(t) : u \in C_D^0 \} \), then \( C_D^0(t) = Y_A \otimes^G Z_C \) if \( t \) is in the interior of \( D \) and \( \{0\} \) otherwise. By [1] Lemma 5.1, the closure of \( C_D^0 \) in \( C_c(G, Y_A \otimes^G Z_C) \) with respect to the inductive limit topology is \( \{ f \in C_c(G, Y_A \otimes^G Z_C) : f(t) \in C_D^0(t) \forall t \in G \} = C_D(G, Y_A \otimes^G Z_C) \).

Take any \( u, v \in C_D(G, Y_A \otimes^G Z_C) \). By the preceding paragraph there exists sequences \( \{ u_n \}_{n \in \mathbb{N}} \) and \( \{ v_n \}_{n \in \mathbb{N}} \) in \( C_D^0 \) converging uniformly to \( u \) and \( v \), respectively. Then for all \( n \in \mathbb{N} \) there exists a positive integer \( m_n \) and (for each \( j = 1, \ldots, m_n \)) elements \( f_{n,j}, g_{n,j} \in C_D(G), r_{n,j}, \xi_{n,j}, \gamma_{n,j} \in G, \xi_{n,j}, \zeta_{n,j} \in Y_A \) and \( \zeta_{n,j}, \kappa_{n,j} \in Z_C \) such that

\[
u_n = \sum_{j=1}^{m_n} f_{n,j} \otimes (\delta_{r_{n,j}} \otimes \xi_{n,j} \otimes \eta_{n,j}) \quad \text{and} \quad v = \sum_{j=1}^{m_n} g_{n,j} \otimes (\delta_{r_{n,j}} \otimes \xi_{n,j} \otimes \eta_{n,j}).
\]

Let \( \alpha_D \) be the measure of \( D \) with respect to \( d_G s \) and \( \beta_D \) that of \( H \cap (D^{-1} D) \) with respect to \( d_H t \). If \( \gamma_D := \sup \{ |\Delta_G(t)^{1/2} \Delta_H(t)^{-1/2}| : t \in H \cap (D^{-1} D) \} \), then (2.50) implies that for all \( p, q \in \mathbb{N} \) we have

\[
\left\| \sum_{j=1}^{m_n} \delta_{r_{p,j}} \otimes (\xi_{p,j} \otimes [f_{p,j} \otimes V \eta_{p,j}]) \right\|^2 = \\
= \left\| \int_H \int_G \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} \langle (u_p - v_p)(z), (1 \otimes 1 \otimes V_t)(u_p - v_p)(zt) \rangle \, dG z dH t \right\| \\
\leq \alpha_D \beta_D \gamma_D \left\| u_p - v_p \right\|^2_{\infty}.
\]

Several conclusion arise from the inequality above:

(a) If \( u = v \) and \( v_q = u_q \), it follows that \( \left\{ \sum_{j=1}^{m_n} \delta_{r_{p,j}} \otimes (\xi_{p,j} \otimes [f_{p,j} \otimes V \eta_{p,j}]) \right\}_{p \in \mathbb{N}} \) is a Cauchy sequence in \( Y_A \otimes^G W_C \) and hence has a limit \( L_D(\{u_n\}_{n \in \mathbb{N}}) \).

(b) If \( u = v \) and we take limit in \( p \) and \( q \), it follows that \( L_D(\{u_n\}_{n \in \mathbb{N}}) = L_D(\{v_n\}_{n \in \mathbb{N}}) \).

Hence \( L_D(\{u_n\}_{n \in \mathbb{N}}) \) depends only on \( u \) and it makes sense to define a function \( L_D : C_D(G, Y_A \otimes^G Z_C) \to Y_A \otimes^G W_C, f \mapsto L_D(f) \), that can be computed using the procedure we have described before.

(c) Taking limit in \( p \) and \( q \) we obtain \( \left\| L_D(u) - L_D(v) \right\| \leq \sqrt{\alpha_D \beta_D \gamma_D} \left\| u - v \right\|_{\infty} \), so \( L_D \) is continuous.

(d) \( L_D(f \otimes (\delta_t \otimes \xi \otimes \eta)) = \delta_t \otimes \xi \otimes (f \otimes V \eta) \) and \( L_D \) is linear when restricted to \( C_D^0 \).

Thus \( L_D \) is linear.

(e) By (2.50) and the continuity of \( L_D \),

\[
(2.51) \quad \langle L_D(u), L_D(v) \rangle = \int_G \int_H \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2} \langle u(s), (1 \otimes 1 \otimes V_t)v(st) \rangle \, dH t dG s.
\]
Clearly, if $E \subset G$ is a compact containing $D$, $L_E$ is an extension of $L_D$. Then there exists a unique function $L: C_c(G, Y_A \otimes^G Z_C) \to Y_A \otimes^G W_C$ extending all the functions $L_D$. This extension is linear and continuous in the inductive limit topology, by [7, II 14.3]. Note also that $L$ satisfies (2.49) and so has dense range.

Given $t \in G$, $f \in C_c(B)$, $\xi \in Y_A$ and $\eta \in Z_C$ we define $[t, f, \xi, \eta] \in C_c(G, Y_A \otimes^G Z_C)$ by

$$[t, f, \xi, \eta](r) = \Delta_G(t)^{-1/2} \delta_t \otimes T_{f(r^{-1})} \xi \otimes \eta$$

(2.52)

We claim there exists a unique linear and continuous map

$$I: \bigoplus_{t \in G} L^2_{\mathcal{H}}(B) \otimes_{S^t \otimes V^t} (Y_A \otimes Z_C) \to Y_A \otimes^G W_C$$

such that for all $t \in G$, $f \in C_c(B) \subset L^2_{\mathcal{H}}(B)$, $\xi \in Y_A$ and $\eta \in Z_C$,

$$I(f \otimes_{S^t \otimes V^t} (\xi \otimes \eta)) = L([t, f, \xi, \eta]).$$

(2.53)

The direct summand $L^2_{\mathcal{H}}(B) \otimes_{S^t \otimes V^t} (Y_A \otimes Z_C)$ of $\bigoplus_{t \in G} L^2_{\mathcal{H}}(B) \otimes_{S^t \otimes V^t} (Y_A \otimes Z_C)$ is generated by elements $f \otimes_{S^t \otimes V^t} (\xi \otimes \eta)$, with $f \in C_c(B)$, $\xi \in Y_A$ and $\eta \in Z_C$. Note the “t” in the tensor product indicates the direct summand the tensor belongs to.

Take vectors $f \otimes_{S^t \otimes V^t} (\xi \otimes \eta)$ and $g \otimes_{S^t \otimes V^t} (\zeta \otimes \kappa)$. If $\Psi_r(w) := \Delta_G(w)^{-1/2} \Delta_{H^r}(w)^{-1/2}$, then by (2.51) we have

$$\langle L([r, f, \xi, \eta]), L([t, g, \zeta, \kappa]) \rangle =$$

$$= \int_H \int_G \Psi_r(w) \Delta_G(rt)^{-1/2} \delta_r \otimes T_{f(zr^{-1})} \xi \otimes \eta, \delta_t \otimes T_{g(zw^{-1})} \zeta \otimes \kappa \rangle d_G z d_H w$$

$$= \int_H \int_G \Psi_r(w) \Delta_G(r^{-1}) \delta_{r,t} \langle T_{f(zr^{-1})} \xi \otimes \eta, T_{g(zw^{-1})} \zeta \otimes \kappa \rangle \rangle d_G z d_H w$$

$$= \int_H \int_G \Psi_r(w) \delta_{r,t} \langle T_{f(zr^{-1})} \xi \otimes \eta, T_{g(zw^{-1})} \zeta \otimes \kappa \rangle \rangle d_G z d_H w$$

$$= \int_H \int_G \Psi_r(w) \delta_{r,t} \langle \xi \otimes \eta, T_{f(zr^{-1})} \zeta \otimes \kappa \rangle \rangle d_G z d_H w$$

$$= \int_H \int_G \Psi_r(w) \delta_{r,t} \langle \xi \otimes \eta, (S^t \otimes V^t)_{\rho^t_{g}(f \otimes \delta_t)} (\zeta \otimes \kappa) \rangle \rangle$$

$$= \langle f \otimes_{S^t \otimes V^t} (\xi \otimes \eta), g \otimes_{S^t \otimes V^t} (\zeta \otimes \kappa) \rangle.$$

Thus there exists a linear isometry $I$ satisfying both (2.53) and (2.51). Besides, $I$ preserves inner products.

Adapting the ideas of [6], we define

$$\rho: G \to \mathcal{B}(Y_A \otimes^G W_C), \rho(t) := 1t_t \otimes 1Y_A \otimes 1W_C,$$

where $1t_t: G \to \mathcal{B}(l^2(G))$ is the left regular representation, which we recall is determined by the condition $1t_t(\delta_s) = \delta_{ts}$. Note $\rho$ and $\Theta$ have commuting ranges, so the range of $\rho$ commutes with that of $T \otimes^G U$ and $(T \otimes^G U)$ (and the respective closures, of course).

Let $K$ be the image of the map $I$ of (2.53). We claim that $G \cdot K := \text{span}\{\rho(t)K : t \in G\}$ is dense in $Y_A \otimes^G W_C$. To prove this we define, for each $t \in G$, the function

$$\mu(t): C_c(G, Y_A \otimes^G Z_C) \to C_c(G, Y_A \otimes^G Z_C), (\mu(t)f)(z) = (1t_t \otimes 1Y_A \otimes 1Z_C)f(z).$$

In particular, $\mu(t)(f \otimes (\delta_t \otimes \xi \otimes \eta)) = f \otimes (\delta_t \otimes \xi \otimes \eta)$. Hence,

$$L \circ \mu(t)(f \otimes (\delta_t \otimes \xi \otimes \eta)) = L(f \otimes (\delta_t \otimes \xi \otimes \eta)) = \delta_t \otimes \xi \otimes [f \otimes V \eta] =$$

$$= \rho(t)(\delta_t \otimes \xi \otimes [f \otimes V \eta]) = \rho(t)L(f \otimes (\delta_t \otimes \xi \otimes \eta)).$$
Since both $L \circ \mu(t)$ and $\rho(t) \circ L$ are linear and continuous in the inductive limit topology and agree on a dense set, it follows that $L \circ \mu(t) = \rho(t) \circ L$. Thus $G \cdot \mathcal{K}$ contains the image under $L$ of

\[ K_0 := \text{span}\{\mu(t)[r, f, \xi, \eta] : r, t \in G, f \in C_c(\mathcal{B}), \xi \in Y_A, \eta \in Z_C\} \subset C_c(G, Y_A \otimes^G Z_C). \]

Note $C(G)K_0 \subset K_0$. Besides,

\[ \mu(r)[t, f, \xi, \eta](z) = \Delta_G(t)^{-1/2} \delta_t \otimes T_{f(zt^{-1})} \xi \otimes \eta. \]

Fixing $z \in G$ and varying $r, t \in G, \xi \in Y_A, \eta \in Z_C$ and $f \in C_c(\mathcal{B})$, the elements we obtain on the right hand side of the displayed equation above are all those of the form $\delta_s \otimes T_b \xi \otimes \eta$, for arbitrary $s \in G$, $b \in \mathcal{B}$, $\xi \in Y_A$ and $\eta \in Z_C$. Since $T$ is non degenerate, this last type of vectors span $Y_A \otimes^G Z_C$, and we conclude (using [7, II 14.3]) that $K_0$ is dense in $C_c(G, Y_A \otimes^G Z_C)$ in the inductive limit topology. Hence $G \cdot \mathcal{K}$ contains the dense set $L(K_0)$ and it follows that $G \cdot \mathcal{K} = Y_A \otimes^G W_C$.

Our next goal is to show that defining the $*$-representation

\[ R := \bigoplus_{r \in G} \text{Ind}^B_r (S^r \otimes V^r) \]

the identity

\[ (T \otimes^G U)_b \circ I = I \circ R_b \]

obtains for all $b \in \mathcal{B}$. To prove this claim we fix $r, p, q \in G$, $b \in B_r$, $f \in C_c(\mathcal{B})$, $g \in C_c(G)$, $\xi, \zeta \in Y_A$ and $\eta, \kappa \in Z_C$. For convenience we denote $u$ and $v$ the tensors $f \otimes_{S^p \otimes V^p} (\xi \otimes \eta)$ and $g \otimes (\delta_q \otimes \zeta \otimes \kappa)$, respectively. Recalling (2.51) we get

\[
\langle (T \otimes^G U)_b \circ I(u), L(v) \rangle = \\
= \langle L([p, f, \xi, \eta]), (T \otimes^G U)_b \cdot (\delta_q \otimes \zeta \otimes (g \otimes V^r \kappa)) \rangle \\
= \langle L([p, f, \xi, \eta]), \delta_q \otimes T_{b \cdot \zeta} \otimes (\Lambda^{H_1^G}(g) \otimes V^r \kappa) \rangle \\
= \langle L([p, f, \xi, \eta]), L(\Lambda^{H_1^G}(g) \otimes (\delta_q \otimes T_{b \cdot \zeta} \otimes \kappa)) \rangle \\
= \int_H \int_G \Psi_e(w)\langle [p, f, \xi, \eta](z), \delta_q \otimes T_{b \cdot \zeta} \otimes V_{w \kappa}\rangle g(rzw) \, d_G zd_H w \\
= \int_H \int_G \Psi_e(w)\Delta_G(p)^{-1/2} \delta_{p, q} \langle T_{bf(r^{-1}z^{-1})} \xi \otimes \eta, \delta_q \otimes T_{b \cdot \zeta} \otimes V_{w \kappa}\rangle g(zw) \, d_G zd_H w \\
= \int_H \int_G \Psi_e(w)\Delta_G(p)^{-1/2} \delta_{p, q} \langle T_{bf} \otimes (\zeta \otimes \eta), \delta_q \otimes (\delta_q \otimes \zeta \otimes \kappa) \rangle \rangle \rangle (I(bf \otimes_{S^p \otimes V^r} (\xi \otimes \eta)), L(v)) \\
= \langle I \circ R_b(u), L(v) \rangle.
\]

Since $u$ and $v$ are arbitrary, $L$ has dense range and the elements like $u$ and $v$ span dense subspaces of their containing spaces, by linearity and continuity we get that $(T \otimes^G U)_b \circ I = I \circ R_b$; which implies that

\[ (2.55) \quad (T \otimes^G U)_f \circ I = I \circ \tilde{R}_f, \quad \forall f \in L^1(\mathcal{B}). \]

Consider the $*$-representations

\[ \Omega_T : C^*(\mathcal{B}) \rightarrow \mathbb{B}(Y_A \otimes^G W_C) \]

\[ \Omega_R : C^*(\mathcal{B}) \rightarrow \mathbb{B}(\bigoplus_{r \in G} L^2_{H_r}(G) \otimes_{S^r \otimes V^r} (Y_A \otimes Z_C)) \]
such that $\Omega_T \circ \tilde{\Lambda}^{GB} = (T \otimes^G U)\widetilde{\cdot}$ and $\Omega_R \circ \tilde{\Lambda}^{GB} = \widetilde{R}$. Since the image of $\tilde{\Lambda}^{GB}$ is dense in $C^*(\mathcal{B})$, \textcolor{red}{(2.55)} implies $\Omega_T(f) \circ I = I \circ \Omega_R(f)$ for all $f \in C^*(\mathcal{B})$. Besides, the image of $\Omega_T$ is the closure of that of $(T \otimes^G U)\widetilde{\cdot}$, which commutes with the image of $\rho$. Thus the images of $\Omega_T$ and $\rho$ commute.

Assume we are given $f \in C^*(\mathcal{B})$ such that $\Omega_R(f) = 0$. Then, for all $t \in G$ and $u \in \bigoplus_{r \in G} L^2(H_r(G)) \otimes S^r \otimes V^r (Y_A \otimes Z_C)$,

$$
\Omega_T(f) \circ \rho(t) \circ I(u) = \rho(t) \circ \Omega_T(f) \circ I(u) = \rho(t) \circ I \circ \Omega_R(f)(u) = 0.
$$

Recalling that $I(u) \in K$ and that $G \cdot K$ spans a dense subset of $Y_A \otimes^G W_C$, we deduce that $\Omega_T(f) = 0$.

By thinking of the image of $\Omega_R$ as the quotient of $C^*(\mathcal{B})$ by the kernel of $\Omega_R$, we can define a morphism of $*-$algebras $\Phi: \Omega_R(C^*(\mathcal{B})) \to \Omega_T(C^*(\mathcal{B}))$ such that $\Psi \circ \Omega_R = \Omega_T$. Since $\Psi$ is contractive, it follows that $\|\Omega_R(f)\| \geq \|\Omega_T(f)\|$ for all $f \in C^*(\mathcal{B})$. The inequality $\|\Omega_R(f)\| \leq \|\Omega_T(f)\|$ is trivial because $I$ is an isometry and $\Omega_T(f) \circ I = I \circ \Omega_R(f)$. Then $\Phi$ is isometric and, consequently, a C*-isomorphism. It thus follows that for all $f \in L^1(\mathcal{B})$,

$$
\|(T \otimes^G U)f\| = \|\tilde{\Lambda}^{GB}_f\| = \|\tilde{\Lambda}^{GB}_f\| = \|\tilde{R}_f\| = \sup_{r \in G} \|\text{Ind}_H^B(S^r \otimes V^r)f\|,
$$

showing (2.48) holds and completing the proof. □

**Corollary 2.56** (FExell’s Absorption Principle II). Let $\mathcal{B} = \{B_t\}_{t \in G}$ be a Banach $*$-algebraic bundle, $H \subset G$ a subgroup, $T: \mathcal{B} \to \mathbb{B}(Y_A)$ a non degenerate $*$-representation and $(Z_C, U, \Psi)$ an integrated system of $H$-imprimitivity for $G$. Assume at least one of the following conditions holds

1. $H$ is normal in $G$.
2. $H$ is saturated.
3. There exists a set $S \subset G$ such that $\{tHt^{-1}: t \in S\} = \{tHt^{-1}: t \in G\}$ and for every $t \in S$ there exists a unitary multiplier of $\mathcal{B}$ of order $t$.

Then there exists a unique $*$-representation $\pi_{TU}: C^*_H(\mathcal{B}) \to \mathbb{B}(Y_A \otimes Z_C)$ such that $\pi_{TU} \circ \tilde{\Lambda}^{HB} = (T \otimes U)\widetilde{\cdot}$. Moreover, $\pi_{TU}$ is also the unique non degenerate $*$-representation $\rho: C^*_H(\mathcal{B}) \to \mathbb{B}(Y_A \otimes Z_C)$ such that $\overline{\rho} \circ \tilde{\Lambda}^{HB} = T \otimes U$, with $\overline{\rho}: \mathbb{B}(C^*_H(\mathcal{B})) \to \mathbb{B}(Y_A \otimes Z_C)$ being the unique $*$-representation extending $\rho$.

**Proof.** The key point of the proof is to show that

\begin{equation}
(2.57) \quad \|(T \otimes^G U)f\| \leq \|\tilde{\Lambda}^{GB}_f\| \quad \forall f \in L^1(\mathcal{B}).
\end{equation}

Take faithful and non degenerate $*$-representations $\mu: A \to \mathbb{B}(V)$ and $\nu: C \to \mathbb{B}(W)$. Then,

$$
S: \mathcal{B} \to \mathbb{B}( (Y_A \otimes W_C) \otimes_{\mu \otimes \nu} (V \otimes W)), \quad (b \in B_t) \mapsto (T_b \otimes U_t) \otimes_{\mu \otimes \nu} (1_V \otimes 1_W),
$$

is a $*$-representation and for all $f \in L^1(\mathcal{B})$ it follows that $\|S_f\| = \|(T \otimes U)f\|$. But $S$ is unitary equivalent to $(T \otimes_{\mu} 1_V) \otimes (U \otimes_{\nu} 1_W)$. Thus, to prove (2.57), we may replace $T$ with $T \otimes_{\mu} 1_V$ and $U$ with $U \otimes_{\nu} 1_W$. In doing so one replaces $Y_A$ and $Z_C$ with Hilbert spaces, thus to prove (2.57) we may assume $Y_A$ and $Z_C$ are Hilbert spaces to start with (and forget $\mu$ and $\nu$).
By Makey’s Theorem 2.41 the system \((Z, U, \psi)\) is induced from some unitary \(*\)-representation \(V: H \to \mathbb{B}(W)\). Then, by Theorem 2.42 and the construction of induced representations
\[
\| (T \otimes U) \tilde{\lambda} \| \leq \sup \{ \| \tilde{\lambda}^G_f \| : t \in G \} \quad \forall \ f \in L^1(B).
\]
Thus Theorem 2.42 together with any of the conditions (1), (2) or (3) gives (2.57).

Clearly, condition (2.57) is no other thing than the claim of the existence of a unique morphism of \(*\)-algebras \(\pi^0_{TU}: \Lambda^{HB}(\mathbb{B}_G) \to \mathbb{B}(Y_A \otimes Z_C)\) such that: (i) \(\pi^0_{TU} \circ \Lambda^{HB} = (T \otimes U)\tilde{\lambda}\), and (ii) \(\pi^0_{TU}\) is contractive with respect to the C*-norm inherited from \(C^*_H(B)\).

We define \(\pi^{0}_{TU}\) as the unique continuous extension of \(\pi^0_{TU}\). This extension is a non degenerate \(*\)-representation because its image contains the non degenerate algebra \((T \otimes U)^{(L^1(B))}\).

Note \(\pi^{0}_{TU} \circ \Lambda^{HB}: B \to \mathbb{B}(Y_A \otimes Z_C)\) is a non degenerate \(*\)-representation with integrated form \((T \otimes U)^{\tilde{\lambda}}\). Thus the uniqueness of integrated forms implies \(\pi^{0}_{TU} \circ \Lambda^{HB} = T \otimes U\). Assume, conversely, that \(\rho: C^*_H(B) \to \mathbb{B}(Y_A \otimes Z_C)\) is a non degenerate \(*\)-representation such that \(\rho \circ \Lambda^{HB} = T \otimes U\). Then
\[
\pi_{TU} \circ \tilde{\Lambda}^{HB} = (T \otimes U)^{\tilde{\lambda}} = (\rho \circ \Lambda^{HB})^{\tilde{\lambda}} = \rho \circ \tilde{\Lambda}^{HB} = \rho \circ \tilde{\lambda}^{HB},
\]
implying that \(\pi_{TU}\) and \(\rho\) agree on a dense set. Thus \(\rho = \pi_{TU}\).

**Corollary 2.58.** Let \(B = \{B_t\}_{t \in G}\) be a Banach \(*\)-algebraic bundle, \(H \subset G\) a normal subgroup, \(T: B \to \mathbb{B}(Y_A)\) a non degenerate \(*\)-representation and \(V: H \to \mathbb{B}(Z_C)\) a unitary \(*\)-representation. Then the \(*\)-representation \((T|_{B_H}) \otimes V\) is \(B\)-positive and there exists a unique \(*\)-representation \(\pi_{TV}: C^*_H(B) \to \mathbb{B}(Y_A \otimes Z_C)\) such that \(\pi_{TV} \circ \tilde{\Lambda}^{HB} = (T \otimes \text{Ind}_{H}^G(V))\).

Moreover,

1. \(\pi_{TV}\) is also the unique non degenerate \(*\)-representation \(\rho: C^*_H(B) \to \mathbb{B}(Y_A \otimes Z_C)\) such that \(\rho \circ \Lambda^{HB} = T \otimes \text{Ind}_{H}^G(V)\).
2. If the integrated form of \((T|_{B_H}) \otimes V\) factors via \(\chi^{B_H}_{B_H}: L^1(B_H) \to C^*(B_H^+)\) through a faithful \(*\)-representation of \(C^*(B_H^+)\), then \(\pi_{TV}\) is faithful. To accomplish this one may consider, for example, the trivial representation \(V: H \to C\) and a \(*\)-representation \(T\) like the one we will construct before stating Corollary 2.59.

**Proof.** The existence and uniqueness of \(\pi_{TV}\) follows from FExell’s Absorption Principle II because \(U := \text{Ind}_{H}^G(V)\) is, by construction, part of an integrated system of \(H\)-imprimitivity. If condition (2) holds, by FExell’s Absorption Principle I we have
\[
\| \tilde{\Lambda}^G_f \| \geq \| \pi_{TV}(\tilde{\Lambda}^H_f) \| = \sup \{ \| \tilde{\text{Ind}}^B_{H}(\tilde{\lambda}^H_f) \otimes V \| : t \in G \} \geq \| \tilde{\text{Ind}}^B_{H}(\tilde{\lambda}^H_f) \otimes V \| = \| \tilde{\Lambda}^H_f \|.
\]
It thus follows that \(\pi_{TV}\) is a isometry and hence faithful.

**2.4. Induction in stages.** In [7, XI 12.15] Fell shows that \(*\)-representations may be induced in stages. For general \(*\)-representations on Hilbert modules this becomes the result below.

**Theorem 2.59.** Let \(B = \{B_t\}_{t \in G}\) be a Banach \(*\)-algebraic. Consider two subgroups, \(H\) and \(K\), with \(H \subset K \subset G\) and a \(*\)-representation \(S: B_H \to \mathbb{B}(Y_A)\). Then the following are equivalent:

1. \(S\) is \(B\)-positive.
2. \(S\) is \(B_K\)-positive and \(\text{Ind}_{H}^{K}(S)\) is \(B\)-positive.
If the conditions above hold, then there exists a unique unitary

$$U: L^2_K(B) \otimes \text{Ind}_H^K(S) (L^2_K(B_K) \otimes_S Y_A) \to L^2_H(B) \otimes_S Y_A$$

mapping $f \otimes \text{Ind}_H^K(S) (u \otimes_S \xi)$ to $fu \otimes_S \xi$ for all $f \in C_c(B)$ and $u \in C_c(B_K)$. Moreover, this unitary intertwines $\text{Ind}_H^K(\text{Ind}_H^K(S))$ and $\text{Ind}_H^K(S)$.

Proof. If $Y$ where a Hilbert space the proof would follow at once by [7, XI 12.15]. We will present a complete proof instead of indicating how to modify Fell’s proof to the general case, mainly because we think this is shorter and more convenient to the reader.

The generalized restriction maps will be denoted indicating the groups as follows:

$$p^K_B: C_c(B_K) \to C_c(B_H), \quad p^K_B(f)(t) = \Delta_K(t)^{1/2}\Delta_H(t)^{-1/2}f(t).$$

A direct computation shows that $p^K_B \circ p^S_B = p^K_H$.

If condition (1) holds, then $S_{B_K}$ is $B-$positive and hence $B_K-$positive. By Theorem 2.15 this implies $S$ is $B_K-$positive. Thus to prove the equivalence between (1) and (2) we can assume $S$ is $B_K-$positive and consider the induced $^*$-representation $\text{Ind}_H^K(S): B_K \to \mathfrak{B}(L^2_B(B_K) \otimes_S Y_A)$. Given any $f, g \in C_c(B)$, $u, v \in C_c(B_K)$ and $\xi, \eta \in Y_A$ we have:

$$\langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle$$

Assume $S$ is $B-$positive. To show that $\text{Ind}_H^K(S)$ is $B-$positive it suffices to show that $\langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle$.

$$\langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle = \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*g)v \otimes_S \eta \rangle$$

proving that (1) implies (2).

We now assume claim (2) holds. To prove that $S$ is $B-$positive we take $f \in C_c(B)$ and $\xi \in Y_A$. By [7, VIII 5.11] and conventions (C) $L^1(B_K)$ has an approximate unit $\{u_\lambda\}_{\lambda \in \Lambda} \subset C_c(B_K)$ such that both $\{u_\lambda * v\}_{\lambda \in \Lambda}$ and $\{v * u_\lambda\}_{\lambda \in \Lambda}$ converge to $v$ in the inductive limit topology, for all $v \in C_c(B_K)$. Moreover, by adapting the proof of [7, VIII 5.11] it can be shown that $\{u_\lambda * v * u_\lambda\}_{\lambda \in \Lambda}$ converges to $v$ in the inductive limit topology. With $v = p^K_B(f^*f)$, the continuity of the generalized restrictions with respect to the inductive limit topologies and (2.61) imply

$$\langle \xi, \text{Ind}_H^K(S)p^K_B(f^*f)v \otimes_S \eta \rangle = \lim\lambda \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*f)v \otimes_S \eta \rangle = \lim\lambda \langle \xi, \text{Ind}_H^K(S)p^K_B(f^*f)v \otimes_S \eta \rangle \geq 0.$$

Hence (2) implies (1).

From now on we assume $S$ is $B-$positive. In this situation (2.61) implies the existence of a unique linear isometry $U$ as in (2.60). In fact (2.61) can now be interpreted as the fact that $U$ preserves inner products. Thus $U$ is a unitary if and only if it is surjective. To prove this is the case fix $f \in C_c(B)$, $\xi \in Y_A$ and take an approximate
unit \{u_\lambda\}_{\lambda \in \Lambda} \subset C_c(\mathcal{B}_K)$ as the one we considered before. Then some straightforward arguments show that
\[
\lim_\lambda \|f \otimes S \xi - U(f \otimes \text{Ind}_{\Lambda}^\mathcal{B}_K(S) u_\lambda \otimes S \xi)\|^2 = \lim_\lambda \| (f - fu_\lambda) \otimes S \xi \|^2
\]
\[
= \lim_\lambda \| (\xi, S_{\tilde{\Lambda}^\mathcal{B}_K} p_{\tilde{\Lambda}^\mathcal{B}_K}(f^* + f) + u_\lambda^* p_{\tilde{\Lambda}^\mathcal{B}_K}(f^* + f) \ast u_\lambda - u_\lambda^* p_{\tilde{\Lambda}^\mathcal{B}_K}(f^* + f) \ast u_\lambda \xi)\| = 0.
\]
Thus $U$ has dense range and is in fact surjective because it is an isometry.

Finally, for all $b \in \mathcal{B}$, $f \in C_c(\mathcal{B})$, $u \in C_c(\mathcal{B}_K)$ and $\xi \in Y_A$ we have
\[
U^* \text{Ind}_{\mathcal{B}_K}(S)_b U(f \otimes \text{Ind}_{\Lambda}^\mathcal{B}_K(S) u \otimes S \xi) = U^*(bf u \otimes S \xi) = U^*((bf) u \otimes S \xi)
\]
\[
= (bf) \otimes \text{Ind}_{\Lambda}^\mathcal{B}_K(S) u \otimes S \xi
\]
\[
= \text{Ind}_{\mathcal{B}_K}(\text{Ind}_{\Lambda}^\mathcal{B}_K(S))_b (f \otimes \text{Ind}_{\Lambda}^\mathcal{B}_K(S) u \otimes S \xi).
\]
Thus the proof follows by the density of elementary tensor products.

When specialized to Fell bundles the induction in stages is a statement about regular *-representations of subgroups. Recall that after Theorem 2.15 there is no need to check positivity of *-representations when working with Fell bundles.

**Corollary 2.62.** Assume $\mathcal{B}$ is a Fell bundle over $G$ and consider two subgroups of $G$, $H$ and $K$, such that $H \subset K \subset G$. If $\tilde{\Lambda}^{\mathcal{B}K} : \mathcal{B}_K \to \mathcal{B}(L^2_H(\mathcal{B}_K))$ is the $H$-regular *-representation of $\mathcal{B}_K$, then there exists a unitary
\[
U : L^2_K(\mathcal{B}) \otimes_{\Lambda^{\mathcal{B}K}} L^2_H(\mathcal{B}_K) \to L^2_H(\mathcal{B})
\]
mapping $f \otimes \Lambda^{\mathcal{B}K} u$ to $fu$, for all $f \in C_c(\mathcal{B})$ and $u \in C_c(\mathcal{B}_K)$. Moreover, this unitary intertwines $\text{Ind}_{\mathcal{B}_K}(\tilde{\Lambda}^{\mathcal{B}K})$ and $\Lambda^{\mathcal{B}}$.

**Proof.** Follows directly from Theorem 2.59. \hfill \Box

**Corollary 2.63.** If $\mathcal{B}$ is a Fell bundle over $G$ and $H$ is a subgroup of $G$, then the $H$-regular *-representation $\tilde{\Lambda}^{\mathcal{B}} : L^1(\mathcal{B}) \to \mathcal{B}(L^2_H(\mathcal{B}))$ is faithful.

**Proof.** It suffices to show that $\tilde{\Lambda}^{\mathcal{B}} \otimes \chi^\text{reg} H : L^1(\mathcal{B}) \to \mathcal{B}(L^2_H(\mathcal{B}))$ is faithful. By construction $\tilde{\Lambda}^{\mathcal{B}} \otimes \chi^\text{reg} H$ is $\text{Ind}_{\mathcal{B}}(\Lambda^{\mathcal{B}})$, and Corollary 2.62 implies this *-representation is unitary equivalent to $\Lambda^{\mathcal{B}} : L^1(\mathcal{B}) \to \mathcal{B}(L^2(\mathcal{B}))$; the last being faithful by Remark 2.35. \hfill \Box

### 3. Amenableability

Recall from Definition 2.29 that given a Banach *-algebraic bundle $\mathcal{B}$ over $G$ and a subgroup $H \subset G$, the $H$-cross sectional $C^*$-algebra $C^*_H(\mathcal{B}) \subset \mathcal{B}(L^2_H(\mathcal{B}))$ is the image of the *-representation $q_H^\mathcal{B} : C^*(\mathcal{B}) \to \mathcal{B}(L^2_H(\mathcal{B}))$, this representation being the unique making the diagram below a commutative one

\[
\begin{array}{ccc}
L^1(\mathcal{B}) & \xrightarrow{\tilde{\Lambda}^\mathcal{B}} & C^*_H(\mathcal{B}) \\
& \chi^\mathcal{B} \downarrow & \downarrow q_H^\mathcal{B} \\
C^*(\mathcal{B}) & & \end{array}
\]

If $\mathcal{B}$ is a Fell bundle and $H = \{e\}$, we write $q_e^\mathcal{B}$ instead of $q_H^\mathcal{B}$, and note $C^*_e(\mathcal{B})$ is Exel-Ng’s reduced cross sectional $C^*$-algebra $C^*_r(\mathcal{B})$; and also that $q_e^\mathcal{B} : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ is the canonical quotient map. In this setting $\mathcal{B}$ is said to be amenable if $q_e^\mathcal{B}$ is injective.
Proposition 3.1. Let $\mathcal{B}$ be a Banach *-algebraic bundle over $G$ and consider subgroups $H \subset K \subset G$. Then there exists a unique morphism of *-algebras $q_{KH}^{B}$ making the following a commutative diagram

Moreover, $q_{KH}^{B}$ is surjective and $q_{H}^{B}$ is a C*-isomorphism if and only if both $q_{KH}^{B}$ and $q_{K}^{B}$ are C*-isomorphisms. In case $\mathcal{B}$ is a Fell bundle and $q_{KH}^{B}: C^{*}(\mathcal{B}_{K}) \rightarrow C_{H}^{*}(\mathcal{B}_{K})$ is a C*-isomorphism, then $q_{KH}^{B}$ is a C*-isomorphism.

Proof. Since $q_{K}^{B}$ is surjective, $q_{KH}^{B}$ is unique if it exists and to prove it’s existence it suffices to show that $\|q_{H}^{B}(f)\| \leq \|q_{K}^{B}(f)\|$ for all $f \in C^{*}(\mathcal{B})$.

Take a non degenerate and faithful *-representation $\pi: C^{*}(\mathcal{B}_{K}) \rightarrow \mathbb{B}(Y)$. Then there exists a *-representation $T: \mathcal{B}_{H} \rightarrow \mathbb{B}(Y)$ such that $\pi = \chi_{T_{H}}^{B}$. By Theorem 2.59 for all $f \in L^{1}(\mathcal{B}_{K})$ we have

$$\|q_{H}^{B}(f)\| = \|q_{H}^{B}(f) \otimes_{T} 1\| \equiv \|q_{H}^{B}(f) \otimes_{T} 1\| = \|q_{K}^{B}(f) \otimes_{\text{Ind}_{H}^{K}} 1 \otimes_{T} 1\| \leq \|q_{K}^{B}(f)\|.$$  

In case $\mathcal{B}$ is a Fell bundle and $q_{H}^{B}$ is a C*-isomorphism, Corollary 2.62 implies the *-representation $q_{K}^{B} \otimes_{q_{H}^{B}} 1: C^{*}(\mathcal{B}) \rightarrow \mathbb{B}(L_{K}^{2}(\mathcal{B}) \otimes q_{K}^{B}(L_{H}^{2}(\mathcal{B}_{K})))$ is unitary equivalent to $q_{H}^{B}$. Then for all $f \in C^{*}(\mathcal{B})$ we have

$$\|q_{KH}^{B}(q_{K}^{B}(f))\| = \|q_{H}^{B}(f)\| = \|q_{K}^{B}(f) \otimes_{q_{H}^{B}} 1\| = \|q_{K}^{B}(f)\|;$$

and this implies $q_{KH}^{B}$ is a C*-isomorphism.

Note $q_{K}^{B}$, $q_{K}^{B}$ and $q_{KH}^{B}$ are surjective. In case $q_{H}^{B} = q_{KH}^{B} \circ q_{K}^{B}$ is a C*-isomorphism, then $q_{K}^{B}$ must be faithful and hence a C*-isomorphism and this forces $q_{KH}^{B} = q_{H}^{B} \circ (q_{K}^{B})^{-1}$ to be a C*-isomorphism.

Notation 3.2. If $\mathcal{B} = \mathcal{T}_{G}$ we will write $q_{H}^{T_{G}}: C^{*}(\mathcal{G}) \rightarrow C_{H}^{*}(\mathcal{G})$ and $q_{K}^{C}: C_{K}^{*}(\mathcal{G}) \rightarrow C_{H}^{*}(\mathcal{G})$ instead of $q_{H}^{T_{G}}: C^{*}(\mathcal{T}_{G}) \rightarrow C_{H}^{*}(\mathcal{T}_{G})$ and $q_{K}^{T_{G}}: C_{K}^{*}(\mathcal{T}_{G}) \rightarrow C_{H}^{*}(\mathcal{T}_{G})$, respectively.

After the Corollary above, for a Fell bundle $\mathcal{B}$ over $G$ the diagram (on the left below) of inclusions of subgroups of $G$ gives the commutative diagram of surjective morphism of C*-algebras on the right

We do not know (in general) if $q_{KH}^{B}$ is a C*-isomorphism if and only if $q_{H}^{B}$ is so, but we will be able to prove this assuming both $H$ and $K$ are normal in $G$ and $\mathcal{B}$ is a Fell bundle. As a consequence of this we will get that if $\mathcal{B}$ is amenable then so it is $\mathcal{B}_{K}$. A lot of work will be needed to prove this claim.
We begin with a Lemma saying that the induction process followed by a restriction increases the norm of integrated forms, the precise form of this claim being the following one.

**Lemma 3.4** (c.f. [7 XI 11.3]). Let $B = \{B_i\}_{i \in G}$ be a Banach $^*$-algebraic, $H \subset G$ a subgroup and $S: Bh \to \mathbb{B}(Y_A)$ a non degenerate $B$--positive $^*$-representation. If $T: Bh \to \mathbb{B}(L^2_H(B) \otimes_SY_A)$ is the restriction of $\text{Ind}^B_H(S): B \to \mathbb{B}(L^2_H(B) \otimes_SY_A)$, then $\|\tilde{T}_f\| \leq \|T_f\| \leq \|\text{Ind}^B_H(f)\|$ for all $f \in L^1(B_H)$.

**Proof.** If $\rho: A \to \mathbb{B}(Z)$ is a faithful and non degenerate $^*$-representation, then

$$\text{Ind}^B_H(S \otimes \pi 1) = \text{Ind}^B_H(S) \otimes \pi 1$$

and this implies $T \otimes \pi 1$ is the restriction of $\text{Ind}^B_H(S \otimes \pi 1)$ to $B_H$. Besides, $\|\tilde{T}_f\| = \|(S \otimes \pi 1)f\|$ and $\|T_f\| = \|(T \otimes \pi 1)f\|$. Hence, by replacing $S$ with $S \otimes \pi 1$, we may assume $Y_A$ is a Hilbert space.

Since $Y \equiv Y_A$ is a Hilbert space, we may use the concretely induced $^*$-representation $\text{Ind}^B_H(B)(S)$ instead of $\text{Ind}^B_H(S)$. This has the advantage of describing $L^2_H(B) \otimes_SY_A$ as $L^2(f^*,Y)$ (exactly as in the proof of Theorem 2.37). In [7 XI 9.9 - 9.20] Fell describes a (continuous) action $\tau: B \times Y \to Y$, $(b,y) \mapsto \tau_by$, with the following properties:

1. For all $s,t \in G$, $b \in B_s$, and $y \in Y_{stH}$, $\tau_by \in Y_{sth}$.
2. There exists a continuous function $\zeta: G \times G/H \to \mathbb{R}$ such that for all $s,t \in G$, $b \in B_s$ and $y \in Y_{stH}$, $\|\tau_by\| \leq \|b\||y||\zeta(s,tH)$.
3. For all $s,t \in G$ and $b \in B_s$ the function $B_s \times Y_{th} \to Y_{sth}$, $(b,y) \mapsto \tau_by$, is bilinear.
4. For all $f \in C_c(Y)$, $s,t \in G$ and $b \in B_s$: $\|\text{Ind}^B_H(B)(f)(tH)\| = \tau_tg(s^{-1}tH)$.
5. The $^*$-representation $S': B_H \to \mathbb{B}(Y_H)$, $b \mapsto \tau_0|_{Y_H}$, is unitary equivalent to $S$.

Fix $f \in C_c(B_H)$ and $g \in C_c(Y)$. Given $\varphi \in C_c(G/H)^+$ such that $\int_{G/H} \varphi^2 \, d\rho^# = 1$ we have

$$\|\langle \varphi \zeta, \tilde{T}_f \varphi \zeta \rangle - \langle g(H), \tilde{S}_fg(H) \rangle\| \leq \|\langle \varphi \zeta, \tilde{T}_f \varphi \zeta \rangle - \langle g(H), \tilde{S}_fg(H) \rangle\|$$

Let $N$ be the set of compact neighbourhoods of $H \subset G/H$ ordered by decreasing inclusion: $U \leq V$ if an only if $V \subset U$. Choose, for each $U \in N$, $\varphi_U \in C_c(G/H)^+$ with support contained in $U$ and $\int_{G/H} \varphi_U^2 \, d\rho^# = 1$.

We claim that

$$\lim_{U} \sup \{\varphi(x)|\varphi(t^{-1}x) - \varphi(x)||g||\infty|f(t)||\zeta(t,t^{-1}x): t \in G, x \in U\} = 0.$$
to a point \( t \in \text{supp}(f) \). Then the left hand side of (3.7) converges to

\[
\varphi(H)|\varphi(t^{-1}H) - \varphi(H)||g||\|f(t)||\zeta(t, t^{-1}H);
\]
which is zero because \( t^{-1}H = H \). This proves (3.6).

Adapting the arguments of the paragraph above we can show that

\[
\lim_{U} \sup \{ \varphi(x^2)|\langle g(x), \tau_{f(t)}(g(t^{-1}x)) \rangle - \langle g(H), \tau_{f(t)}g(H) \rangle : t \in G, x \in U \} = 0.
\]

It then follows from (3.5) that

\[
U
\]

and a similar argument implies \( \lim_{U} \langle \varphi_Ug, \varphi_Ug \rangle = \langle g(H), g(H) \rangle \).

We then get that for all \( f \in C_c(B_H) \) and \( g \in C_c(Y) : \)

\[
\|S_{f}g(H)\|^2 = \langle g(H), S_{f^*f}g(H) \rangle = \lim_{U} \langle \varphi_Ug, \tilde{T}_{f^*f}\varphi_Ug \rangle \leq \lim_{U} \|\tilde{T}_{f}\|^2 \lim_{U} \langle \varphi_Ug, \varphi_Ug \rangle \leq \|\tilde{T}_{f}\|^2 \lim_{U} \langle g(H), g(H) \rangle = \|\tilde{T}_{f}\|^2 \|g(H)\|^2.
\]

This implies that \( \|S_{f}\| \leq \|\tilde{T}_{f}\| \) because the fact that \( Y \) is a Banach bundle implies \( Y_H = \{ g(H) : g \in C_c(B) \} \).

Note \( T|B_H \) is \( B \)-positive because \( (T|B_H)|B_c = T|B_c \) is \( B \)-positive. Then Theorem 2.16 implies \( \|\tilde{T}_{f}\| \leq \|\pi_{F}^{B}(\chi^{B_{H}}(f))\| \leq \|\chi^{B_{H}}(f)\| \) for all \( f \in L^{1}(B_H) \). \( \square \)

The following result implies that that given a subgroup \( K \subset G \), \( B(C_{K}(B)) \) contains all the \( C^n \)-algebras \( C^*(B_{H}) \) for closed subgroups \( H \subset K \).

**Corollary 3.8.** Let \( B \) be a Banach \( * \)-algebraic bundle over \( G \) and consider subgroups \( H \subset K \subset G \). If \( S : B_{H} \rightarrow B(L_{2}^{K}(B)) \) is the restriction of \( \Lambda^{KB} : B \rightarrow B(L_{2}^{K}(B)) \) and \( \chi_{S}^{B_{H}} : C^*(B_{H}) \rightarrow B(L_{2}^{K}(B)) \) is the unique \( * \)-representation such that \( \chi_{S}^{B_{H}} \circ \chi^{B_{H}} = \tilde{S} \), then \( \chi_{S}^{B_{H}} \) is faithful. Moreover, if we extend the inclusion \( C_{K}(B) \subset B(L_{2}^{K}(B)) \) to an inclusion \( B(C_{K}(B)) \subset B(L_{2}^{K}(B)) \), then \( \chi_{S}^{B_{H}}(C^*(B_{H})) \subset B(C_{K}(B)) \) and we may think \( C^*(B_{H}) \subset B(C_{K}(B)) \subset B(L_{2}^{K}(B)) \).

**Proof.** First of all note \( S \) is \( B \)-positive because it is the restriction of a \( * \)-representation of \( B \), so the existence of \( \chi_{S}^{B_{H}} \) is guaranteed by the universal property of \( C^*(B_{H}) \). Writing \( S^{H_{K}} : B_{H} \rightarrow B(L_{2}^{K}(B)) \) instead of \( S : B_{H} \rightarrow B(L_{2}^{K}(B)) \) and using the maps \( q \) provided by Proposition 3.1 we get \( q_{K}^{B_{H}} \circ S^{K_{H}} = S^{H_{H}} \) and, in integrated forms, \( q_{K}^{B_{H}} \circ \tilde{S}^{H_{H}} = \tilde{S}^{H_{H}} \). Here \( q_{K}^{B_{H}} : B(C_{K}(B)) \rightarrow B(C_{H}(B)) \) is the natural extension of \( q_{K}^{B_{H}} \). We then get that

\[
q_{K}^{B_{H}} \circ \chi_{S_{K}H}^{B_{H}} = \chi_{S_{K}H}^{B_{H}}
\]

and this implies that \( \chi_{S_{K}H}^{B_{H}} \) is faithful if \( \chi_{S_{K}H}^{B_{H}} \) is so. Hence to prove this Corollary we may assume \( K = H \) and, in doing so, write \( S \) instead of \( S^{H_{H}} \).

Let \( \pi : C^*(B_{H}) \rightarrow B(Y) \) be a non degenerate and faithful \( * \)-representation. Then there exists a \( B \)-positive \( * \)-representation \( T : B_{H} \rightarrow B(Y) \) such that \( \pi \circ \chi^{B_{H}} = \tilde{T} \), meaning that \( \pi = \chi_{T}^{B_{H}} \). By Theorem 2.53, \( S \otimes 1 \) is unitary equivalent to the restriction to \( B_{H} \) of \( \text{Ind}_{B_{H}}^{B_{K}}(T) \). Thus for all \( f \in L^{1}(B_H) \) we have

\[
\|\chi_{B_{H}}^{B_{H}}(f)\| = \|\tilde{T}_{f}\| \leq \|(S \otimes 1)\| = \|\tilde{S}_{f}\| \leq \|\chi_{S_{K}H}^{B_{H}} \circ \chi_{B_{H}}^{B_{H}}(f)\| \leq \|\chi_{B_{H}}^{B_{H}}(f)\|.
\]
It is then clear that $\chi_{S^*}^{B^+}$ is an isometry when restricted to the dense subalgebra $\chi_{S^*}^{B^+}(L^1(B_H))$ and this implies $\chi_{S^*}^{B^+}$ is faithful. \(\square\)

In the proof above we produced a $\ast$-representation $R := \text{Ind}_{B^+}^B(T)$ of $B$ on a Hilbert space $Z := L^2_{B^+}(\mathcal{B}) \otimes_T Y$ such that setting $R' := R|_{B^H}$, $R'$ is $B^-$-positive and the respective $\ast$-representation $\chi_{R'}^{B^+} : C^*(B_H^+) \to \mathcal{B}(Z)$ is faithful. In case $B$ is a Fell bundle, $C^*(B_H^+) \equiv C^*(B_H)$ and we regard $\chi_{R'}^{B^+} \equiv \chi_{R'}^{B^+}$ as the extension of $\tilde{R}' : L^1(B_H) \to \mathcal{B}(Z)$ to $C^*(B_H)$. In short, if $B$ is a Fell bundle then one may get a faithful $\ast$-representation of $C^*(B_H)$ by integrating a restriction of a $\ast$-representation of $B$.

The following result gives an inclusion $C^*_H(B) \subset \mathcal{B}(C^*(B) \otimes C^*_H(G))$, in fact one may take $C^*_K(B)$ instead of $C^*(B)$, provided that $H \subset K$.

**Corollary 3.9** (c.f. [1] Propositions 18.6 & 18.7). Assume $\mathcal{B}$ be a Banach $\ast$-algebraic bundle over $G$, we have closed subgroups $H \subset K \subset G$ and at least one of the conditions (1), (2) or (3) of Corollary 2.50 holds. Consider the integrated (universal) systems of imprimitivity $(L^2_K(B), \Lambda^{KB}, \psi^{KB})$ and $(L^2_H(G), U^{HG}, \psi^{HG})$ for $B$ and $G$, respectively. Then the $\ast$-representation

$$\Psi_{K_H}^B := \pi_{\Lambda^{KB}U^{KG}} : C^*_H(B) \to \mathcal{B}(C^*_K(B) \otimes C^*_H(G))$$

provided by Corollary 2.50 is faithful.

**Proof.** Let $q_K^B : C^*_K(B) \to C^*_H(B)$ be the map provided by Proposition 3.1. Then $q_K^B \otimes 1_{C^*_H(G)} \circ \Psi_{K_H}^B = \Psi_{H}^B$ and it suffices to prove

$$\Psi_{H}^B : C^*_H(B) \to \mathcal{B}(C^*_H(B) \otimes C^*_H(G))$$

is faithful.

We denote $V$ the trivial representation of $G$ on $C(V_1 = 1)$ and $V'$ the restriction of $V$ to $H$. Take a non degenerate $\ast$-representation $R : \mathcal{B} \to \mathcal{B}(Z)$ as the one we constructed before stating the corollary we are trying to prove; we also set $R' := R|_{B^H}$. Then $R' \otimes V'$ is unitary equivalent to $R'$ and this implies $\chi_{R'}^{B^+} \otimes V'$ is unitary equivalent to $\chi_{R'}^{B^+}$ and so it is also faithful. Hence FExell’s Absorption Principle I implies, for all $f \in L^1(B)$,

$$\|\tilde{\Lambda}^{HB}_f\| = \|\Lambda^{HB}_f \otimes R' \otimes V'\| = \|\text{Ind}_{H}^B(R' \otimes V')_f\| \leq \|(R \otimes \text{Ind}_{H}^G(V'))_f\|$$

(3.10)

Consider the (unique) $\ast$-representation $\rho : C^*_H(B) \to \mathcal{B}(Z)$ such that $\rho \circ \tilde{\Lambda}^{HB} = \tilde{R}$ (recall $R$ is induced from a $\mathcal{B}$—positive $\ast$-representation of $B_H$). Notice that $\rho$ is non degenerate and $\tilde{\pi} \circ \Lambda^{HB} = R$. Let $\mu : C^*_H(G) \to \mathcal{B}(L^2_H(G) \otimes V' \otimes C)$ the unique $\ast$-representation such that $\mu \circ U^{HG} = \text{Ind}_{H}^G(V')$; which is also the unique non degenerate one such that $\tilde{\pi} \circ U^{HG} = \text{Ind}_{H}^G(V')$. By construction, the non degenerate $\ast$-representation

$$\rho \otimes \mu : C^*_H(B) \otimes C^*_H(G) \to \mathcal{B}(Z \otimes (L^2_H(G) \otimes V', B))$$

satisfies, for all $r \in G$ and $b \in B_r$,

$$\rho \otimes \mu \circ \Psi_{H}^B(\Lambda^{HB}_f) = \rho \otimes \mu (\Lambda^{HB}_b \otimes U^{HG}_r) = R_b \otimes \text{Ind}_{H}^G(V')_r = (R \otimes \text{Ind}_{H}^G(V'))_b.$$ 

Then, for all $f \in L^1(B)$,

$$\rho \otimes \mu \circ \Psi_{H}^B(\tilde{\Lambda}^{HB}_f) = (R \otimes \text{Ind}_{H}^G(V'))_f.$$ 

This last identity together with (3.10) gives

$$\|\tilde{\Lambda}^{HB}_f\| \leq \|\rho \otimes \mu \circ \Psi_{H}^B(\tilde{\Lambda}^{HB}_f)\| \leq \|\Psi_{H}^B(\tilde{\Lambda}^{HB}_f)\| \leq \|\tilde{\Lambda}^{HB}_f\|$$

for all $f \in L^1(B)$. Thus $\Psi_{H}^B$ is an isometry and the proof is complete. \(\square\)
It is important to note the defining properties of the maps $\Psi_{HK}^B$ are the non degeneracy and that fact that
$$\Psi_{HK}^B(\Lambda_B^{HB}) = \Lambda_B^{KB} \otimes U_s^{HG} \forall b \in B_s, \ s \in G.$$  

**Corollary 3.11.** Let $\mathcal{B}$ be a Banach *-algebraic bundle over $G$ and consider subgroups $H \subset K \subset G$. If $q_H^{B|K} = \text{injective and either (a) } \mathcal{B} \text{ is saturated or (b) both } H \text{ and } K \text{ are normal, then } q_H^{B|K} \text{ is faithful. In particular, if } \mathcal{B} \text{ is a Fell bundle over } G \text{ and we put } H = \{e\} \text{ and } K = G, \text{ this says } \mathcal{B} \text{ is amenable if } G \text{ is amenable.}$$

**Proof.** The defining property for the $\Psi$ maps and the notation adopted in 3.2 implies the diagram

$$\begin{array}{c}
C_K^*(\mathcal{B}) \xrightarrow{\Psi_{HK}^B} \mathbb{B}(C^*(\mathcal{B}) \otimes C_K^*(G)) \\
\downarrow q_{KH}^B \leftarrow \downarrow 1 \otimes q_{KH}^B \\
C_H^*(\mathcal{B}) \xrightarrow{\Psi_{GH}^B} \mathbb{B}(C^*(\mathcal{B}) \otimes C_H^*(G))
\end{array}$$

commutes. The horizontal arrows are isometries and the vertical arrow on the right is an isomorphism, thus $q_{KH}^B$ is a $C^*$-isomorphism. $\square$

3.1. Amenability and reductions to normal subgroups. The main goal of this section is to show that if $\mathcal{B}$ is a Fell bundle over $G$ and both $H$ and $K$ are normal subgroups of $G$ with $H \subset K$, then $q_{HK}^B: C^*(\mathcal{B}_K) \to C^*_H(\mathcal{B}_K)$ is a $C^*$-isomorphism if and only if $q_{KH}^B: C_K(\mathcal{B}) \to C^*_H(\mathcal{B})$ is a $C^*$-isomorphism. To do this we first show the following.

**Theorem 3.12.** Let $\mathcal{B}$ be a Fell bundle over $G$ and consider two subgroups of $G$ such that $H \subset K$. If we define $S := \Lambda^{HB}|_{\mathcal{B}_K}: \mathcal{B}_K \to \mathbb{B}(L_H^2(\mathcal{B}))$ and there exists a $\ast$-representation $\pi: C_H^*(\mathcal{B}_K) \to \mathbb{B}(L_H^2(\mathcal{B}))$ such that

$$L^1(\mathcal{B}_K) \xrightarrow{\chi_{\mathcal{B}_K}} \mathbb{B}(L_H^2(\mathcal{B})) \xrightarrow{\pi} C_H^*(\mathcal{B}_K) \xrightarrow{q_{HK}^B} C^*_H(\mathcal{B}_K)$$

commutes, then $\pi$ is unique, faithful, non degenerate, $\pi(C_H^*(\mathcal{B}_K)) \subset \mathbb{B}(C^*(\mathcal{B}))$ and $\pi$ is the unique non degenerate $\ast$-representation such that $\pi(\Lambda^{HB}_b) = \Lambda^{HB}_b$ for all $b \in \mathcal{B}_K$, with $\pi: \mathbb{B}(C_H^*(\mathcal{B}_K)) \to \mathbb{B}(C^*(\mathcal{B}))$ being the unique $\ast$-representation extending $\pi$. Such a map $\pi$ exists if both $H$ and $K$ are normal in $G$.

**Proof.** By Theorem 2.15 $C^*(\mathcal{B}_H) = C^*(\mathcal{B}_H^K)$, $C^*(\mathcal{B}_K) = C^*(\mathcal{B}_K^H)$ and $C^*(\mathcal{B}_H^K) = C^*(\mathcal{B}_H)$. Assume $\pi$ exists. Then (3.13) determines $\pi$ in the dense set $q_{HK}^B(\chi_{\mathcal{B}_K}(L(\mathcal{B}_K)))$ and so it is unique. The diagram also implies the image of $\pi$ is the closure of $S(L(\mathcal{B}_K))$, which is a non degenerate $\ast$-subalgebra of $\mathbb{B}(L_H^2(\mathcal{B}))$ because $S$ is non degenerate. Thus $\pi$ is a $\ast$-subalgebra of $\mathcal{B}_K$ in $\mathbb{B}(C_H^*(\mathcal{B}_K))$.

To prove the identity $\pi(\Lambda^{HB}_b) = \Lambda^{HB}_b$ for all $b \in \mathcal{B}$ we name $P$ the $\ast$-representation $\mathcal{B}_K \to \mathbb{B}(L_H^2(\mathcal{B}))$ given by $b \mapsto \pi(\Lambda^{HB}_b)$. Then

$$\begin{align*}
\tilde{P} &= \pi \circ (\Lambda^{HB})|_{\mathcal{B}_K} \sim \pi \circ q_{HK}^B \circ \chi_{\mathcal{B}_K} = \pi \circ q_{HK}^B \circ \chi_{\mathcal{B}_K} = \tilde{S}.
\end{align*}$$

Thus $P$ and $S$ have the same integrated form and $P = S$, this being the identity we wanted to prove. In fact (3.14) can also be used to show that any non degenerate
Theorem 2.44. Recall that \( \pi \) will also satisfy \( \pi(q^B_H \circ \chi^B(f)) = \tilde{S} \), making (3.13) a commutative diagram.

To show that \( \pi \) is faithful recall it is contractive and that the induction in stages for Fell bundles \( 2.62 \) gives \( S = \Lambda^H |_{B_K} = \text{Ind}_K^B(\Lambda^H |_{B_K}) \). Then by Lemma 3.4 we have, for all \( f \in L^1(B_K) \),

\[
\| \tilde{\Lambda}^H_{f,K} \| = \| q^B_H \circ \chi^B(f) \| \geq \| \pi(q^B_H \circ \chi^B(f)) \| = \| \tilde{S}_f \| \geq \| \tilde{\Lambda}^H_{f,K} \|.
\]

Thus \( \pi \) is isometric on a dense set, and it must be faithful.

The discussion above implies that \( \pi \) exists if and only if \( \| \tilde{S}_f \| \leq \| \tilde{\Lambda}^H_{f,K} \| \) for all \( f \in C_c(B_K) \). This is what we will now prove assuming both \( H \) and \( K \) are normal in \( G \).

Since \( K \) is normal in \( G \) we have \( \Delta_G(s) = \Delta_K(s) \) for all \( s \in K \), in particular this holds for all \( s \in H \). Besides, \( H \) is normal in \( G \) and \( K \), so \( \Delta_G(s) = \Delta_K(s) = \Delta_H(s) \) for all \( s \in H \). Let \( \Gamma_{GK} : G \to (0, +\infty) \) be the function such that for all \( a \in C_c(K) \) and \( r \in G \), \( \int_K a(\text{tr}^{-1}) d_K r = \Gamma_{GK}(t) \int_K a(r) d_K r \) (see [7, III.8.3]). We define \( \Gamma_{GH} \) analogously, viewing \( H \) as a normal subgroup of \( G \). We consider the left invariant Haar measures of \( G, K \) and \( G/K \) have been chosen so that

\[
\int_G a(r) d_{G/K} = \int_{G/K} a(r) d_K t d_{G/K}(rK), \quad \forall a \in C_c(G).
\]

Fix \( f \in C_c(B_K) \) and take \( g \in C_c(B) \). Recall that \( L^2_H(B) \) is constructed out of \( C_c(B) \) by using the \( C_c(B) \)-valued pre inner product \( \langle g, h \rangle = p^G_H(g^* h) \) (with \( g, h \in C_c(B) \)), see Remark 2.22. We want to prove that

\[
(\xi, \rho(\tilde{S}_f, \tilde{g})) \leq \| \tilde{\Lambda}^H_{f,K} \| (g, \rho(g, g)) \xi.
\]

Take a non degenerate *-representation \( T : B \to \mathbb{B}(Y) \) such that the integrated form \( \tilde{T} : L^1(B) \to \mathbb{B}(Y) \) factors via a faithful *-representation \( \chi^B : C^*(B) \to \mathbb{B}(Y) \). Recall that \( L^2_G(B) \equiv C^*(B) \) and this identification gives a unitary equivalence \( L^2_G(B) \to Y \approx Y \). Now Corollary 3.8 says the restriction \( T |_{B_K} : B_K \to \mathbb{B}(Y) \) integrates to a *-representation \( L^1(B_K) \to \mathbb{B}(Y) \) that factors through a faithful *-representation of \( \rho : C^*(B_K) \to \mathbb{B}(Y) \). Thus to prove (3.15) it suffices to show that

\[
\langle \xi, \rho(\tilde{S}_f, \tilde{g}) \rangle \leq \| \tilde{\Lambda}^H_{f,K} \| \langle \xi, \rho(g, g) \rangle \xi,
\]

for all \( \xi \in Y \). By the definition of \( \rho \) the identity above is equivalent to

\[
\int_K \int_{G/K} \langle \xi, T^G_{\rho(g^* f(t) g)}(s) \rangle d_K t d_{H/K} \xi \leq \| \tilde{\Lambda}^H_{f,K} \| \int_K \int_{G/K} \langle \xi, T^G_{\rho(g^* g)}(s) \rangle d_K t d_{H/K} \xi.
\]

By construction the left hand side of (3.12) is

\[
\int_K \int_{G/K} \int_G \langle T^G_{\rho(g^* f(t) g)}(s) \rangle d_K t d_{H/K} d_H s =
\]

\[
= \int_{G/K} \int_{G/K} \int_{G/K} \int_H \int_K \langle T^G_{\rho(g^* f(t) g)}(s) \rangle d_K t d_{H/K} d_H s d_{K/K} d_{G/K}(rK)
\]

\[
= \int_{G/K} \int_{G/K} \int_{G/K} \Gamma_{GK}(r) \Gamma_{GH}(r) \langle T^G_{\rho(g^* f(t) g)}(s) \rangle d_K t d_{H/K} d_H s d_{K/K} d_{G/K}(rK).
\]

What we want to do now is to describe the inner triple integral in the last term above as an inner product. To do this we fix a coset \( rK \), we even consider \( r \) fixed.

Let \( \kappa : H \to \mathbb{B} \) be the trivial representation, set \( W := L^2_H(K) \otimes_{\kappa} C \), \( U := \text{Ind}_H^K(\kappa) \) and consider the map \( L : C_c(K, Y \otimes K C) \to \mathbb{B}(Y \otimes K W) \) constructed in the proof of Theorem 2.44. Recall that \( Y \otimes K C = \ell^2(K) \otimes Y \otimes C \). Given any \( u, v \in C_c(K) \), \( \eta, \zeta \in Y \).
and $q, z \in K$ we consider the elements $\varphi := u \odot (\delta_z \otimes \eta \otimes 1)$ and $\theta := v \odot (\delta_z \otimes \zeta \otimes 1)$ of $C_c(K, Y \otimes K \mathbb{C})$. Then

$$
\langle L(\varphi), (1 \otimes \rho(K) \otimes (T \otimes U))_{f^* f(t)} L(\theta) \rangle = \\
= \langle \delta_q \otimes \eta \otimes (u \otimes_{\eta} 1), (1 \otimes \rho(K) \otimes (T \otimes U))_{f^* f(t)}(\delta_z \otimes \zeta \otimes (v \otimes_{\eta} 1)) \rangle \\
= \langle \delta_q \otimes \eta \otimes (u \otimes_{\eta} 1), \delta_z \otimes T_{f^* f(t)}(\zeta \otimes (U_t^{HK}(v) \otimes_{\eta} 1)) \rangle \\
= \langle \delta_q, \delta_z \rangle \langle \eta, T_{f^* f(t)}(\zeta) \int_K \int_H \bar{\pi}(p)v(t^{-1}p^{-1}s)\,d_Kp\,d_Hs \rangle \\
= \int_K \int_H \langle \varphi(p), (1 \otimes \rho(K) \otimes T_{f^* f(t)} \otimes 1)\theta(t^{-1}p^{-1}s) \rangle \,d_Kp\,d_Hs.
$$

The first and last terms of the identities above are additive and continuous in the inductive limit topology with respect to the variables $\varphi$ and $\theta$. Thus by continuity we get that

$$
\langle L(\varphi), (1 \otimes \rho(K) \otimes (T \otimes U))_{f^* f(t)} L(\theta) \rangle = \\
= \int_K \int_H \langle \varphi(p), (1 \otimes \rho(K) \otimes T_{f^* f(t)} \otimes 1)\theta(t^{-1}p^{-1}s) \rangle \,d_Kp\,d_Hs
$$

for all $\varphi, \theta \in C_c(K, Y \otimes K \mathbb{C})$.

Define $h_r \in C_c(K, Y \otimes K \mathbb{C})$ by $h_r(s) = \delta_r \otimes T_{g(sr)}(\xi \otimes \Gamma_{GK}^{1/2}(r)\Gamma_{GH}^{1/2}(r))$. Then the identity above and Corollary 2.56 imply

$$
\int_K \int_H \int_K \Gamma_{GK}(r)\Gamma_{GH}(r)\langle T_{g(pr)}(\xi), T_{f^* f(t)}(T_{g(t^{-1}psr)}(\xi) d_Kt \,d_Hs \,d_Kp = \\
= \int_K \int_H \int_K \langle h_r(p), (1 \otimes \rho(K) \otimes T_{f^* f(t)} \otimes 1)h_r(t^{-1}ps) \rangle \,d_Kp\,d_Hs\,d_Kt \\
= \int_K \langle L(h_r), (1 \otimes \rho(K) \otimes (T \otimes U))_{f^* f(t)} L(h_r) \rangle \,d_Kt \\
= \langle L(h_r), (1 \otimes \rho(K) \otimes (T \otimes U))_{f^* f(t)} L(h_r) \rangle \\
\leq \| (T \otimes U)_{f^* f}\| \| L(h_r), L(h_r) \| \leq \| \tilde{\Lambda}_{f^* f}^{HK} \| \| L(h_r), L(h_r) \| \\
\leq \| \tilde{\Lambda}_{f^* f}^{HK} \| \int_K \int_H \Gamma_{GK}(r)\Gamma_{GH}(r)\langle T_{g(pr)}(\xi), T_{g(t^{-1}psr)}(\xi) \,d_Hs \,d_Kp.
$$
If we now go back to (3.18) and (3.17) and use the inequality we obtained before we get

\[
\langle \xi, \rho(g, \tilde{f}^{*}s_{f}(g))\rangle = \int_{H} \int_{K} \int_{H} \int_{K} \Gamma_{GH}(r) \Gamma_{GH}(r) \langle T_{g(pr)} \xi, T_{f^{*}s_{f}(g^{*})} T_{g(t^{-1}prs)} \xi \rangle \, dH \, dK \, dK \, dH
\]

\[
\leq \| \tilde{\Lambda}_{f^{*}s_{f}} \| \int_{G/K} \int_{K} \int_{H} \Gamma_{GH}(r) \Gamma_{GH}(r) \langle T_{g(pr)} \xi, T_{g(t^{-1}prs)} \xi \rangle \, dH \, dK \, dH
\]

\[
\leq \| \tilde{\Lambda}_{f^{*}s_{f}} \| \int_{G/K} \int_{K} \int_{H} \Gamma_{GH}(r) \Gamma_{GH}(r) \langle T_{g(rr^{-1}pr)} \xi, T_{g(rr^{-1}prs)} \xi \rangle \, dH \, dK \, dH
\]

\[
\leq \| \tilde{\Lambda}_{f^{*}s_{f}} \| \int_{G/K} \int_{H} \langle T_{g(r)} \xi, T_{g(r)s} \xi \rangle \, dH \, dr \leq \| \tilde{\Lambda}_{f^{*}s_{f}} \| \int_{H} \langle \xi, T_{g^{*}s_{f}(g^{*})} \xi \rangle \, dH
\]

Thus (3.16) holds and the proof is complete. \qed

We now can prove one of the main results of this article.

**Theorem 3.19.** Let \( \mathcal{B} \) be a Fell bundle over \( G \) and take normal subgroups \( H \subset K \subset N \subset G \). Define the maps \( q_{KH}^{B} : C_{K}^{*}(\mathcal{B}) \to C_{H}^{*}(\mathcal{B}) \) and \( q_{KH}^{BN} : C_{K}^{*}(\mathcal{B}_{N}) \to C_{H}^{*}(\mathcal{B}_{N}) \) as in Proposition 3.4. If \( q_{KH}^{B} \) is a C*-isomorphism then \( q_{KH}^{BN} \) is so (and the converse holds, by Proposition 3.4, if \( K = N \)). In particular,

1. If \( H = \{e\} \), then the fact of the canonical map \( q_{KH}^{B} : C_{K}^{*}(\mathcal{B}) \to C_{H}^{*}(\mathcal{B}) \) being a C*-isomorphism implies \( q_{KH}^{BN} : C_{K}^{*}(\mathcal{B}_{N}) \to C_{H}^{*}(\mathcal{B}_{N}) \) is so.

2. If \( \mathcal{B} \) is amenable then so is \( \mathcal{B}_{N} \).

**Proof.** By Theorem 3.12 there exists non degenerate and faithful *-representations \( \pi_{K} : C_{K}^{*}(\mathcal{B}_{N}) \to \mathcal{B}(C_{K}^{*}(\mathcal{B})) \) and \( \pi_{H} : C_{H}^{*}(\mathcal{B}_{N}) \to \mathcal{B}(C_{H}^{*}(\mathcal{B})) \) such that \( \pi_{H}(\Lambda_{b}^{KB}) = \Lambda_{b}^{KB} \) and \( \pi_{H}(\Lambda_{b}^{HB}) = \Lambda_{b}^{HB} \) for all \( b \in \mathcal{B}_{N} \). We claim the diagram below commutes:

\[
\begin{array}{ccc}
C_{K}^{*}(\mathcal{B}_{N}) & \xrightarrow{\pi_{K}} & \mathcal{B}(C_{K}^{*}(\mathcal{B})) \\
q_{KH}^{BN} \downarrow & & \downarrow \pi_{KH} \\
C_{H}^{*}(\mathcal{B}_{N}) & \xrightarrow{\pi_{H}} & \mathcal{B}(C_{H}^{*}(\mathcal{B}))
\end{array}
\]

Indeed, for all \( f \in L^{1}(\mathcal{B}_{N}) \) we have

\[
q_{KH}^{-1} \circ \pi_{K}(\chi_{B_{N}}^{\tilde{f}}(f)) = \pi_{KH}(\tilde{\Lambda}_{f}^{KB}) = (q_{KH}^{-1} \circ \Lambda_{b}^{KB})_{f} = \tilde{\Lambda}_{f}^{HB} = (\pi_{H} \circ \Lambda_{b}^{HB})_{f} = \pi_{H} \circ q_{KH}^{BN}(\chi_{B_{N}}^{\tilde{f}}(f)).
\]

Thus \( q_{KH}^{-1} \circ \pi_{K} = \pi_{H} \circ q_{KH}^{BN} \) because both maps are continuous and agree on a dense set.

Now, if \( q_{KH}^{B} \) is a C*-isomorphism then \( \pi_{H} \circ q_{KH}^{BN} \) is faithful. This implies \( q_{KH}^{BN} \) is faithful and also a C*-isomorphism (recall it is surjective). \qed
4. C*-completions of Banach *-algebraic bundles

In this section we deal with C*-completions of a Banach *-algebraic bundle (other that the bundle C*-completion). We will use Definitions 1.4 and 1.5, the notation we adopted after them and the definition of the integrated form of a morphism of Banach *-algebraic bundles.

Proposition 4.1. Let \( \mathcal{B} = \{ B_t \}_{t \in G} \) be a Banach *-algebraic bundle and consider two C*-completions, \( \iota : \mathcal{B} \to \mathcal{A} \) and \( \kappa : \mathcal{B} \to \mathcal{C} \). Then there exists a morphism \( \rho : \iota \to \kappa \) if and only if \( \| \kappa(b) \| \leq \| \iota(b) \| \) for all \( b \in \mathcal{B} \). In fact the morphism is unique (if it exists) and it is an isomorphism if and only if \( \| \iota(b) \| = \| \kappa(b) \| \) for all \( b \in \mathcal{B} \).

Proof. If \( \rho : \iota \to \kappa \) is a morphism then \( \| \rho \| \leq 1 \) and this gives \( \| \kappa(b) \| = \| \rho(\iota(b)) \| \leq \| \iota(b) \| \) for all \( b \in \mathcal{B} \).

Assume, conversely, that \( \| \kappa(b) \| \leq \| \iota(b) \| \) for all \( b \in \mathcal{B} \). Then for all \( b \in \mathcal{B} \) we have

\[
\| \kappa(b) \| = \| \kappa(b) \|^{1/2} \leq \| \iota(b) \|^{1/2} = \| \iota(b) \|.
\]

This last inequality, together with the fact that \( \iota(B_t) \) is a dense subspace of \( A_t \), implies the existence (for all \( t \in G \)) of a unique continuous linear map \( \rho_t : A_t \to C_t \) such that \( \rho_t(\iota(b)) = \kappa(b) \) for all \( b \in B_t \).

Let \( \rho = \{ \rho_t \}_{t \in G} : \mathcal{B} \to \mathcal{C} \) be the unique extension of all the maps \( \rho_t \). Then Remark 1.6 together with [7, II 13.16] implies \( \rho \) is continuous. It is also multiplicative and preserves the involution when restricted to the dense set \( \iota(\mathcal{B}) \), thus \( \rho : \iota \to \kappa \) is a morphism and it is unique because the condition \( \rho \circ \iota = \kappa \) determines \( \iota \) in the dense set \( \iota(\mathcal{B}) \). The isomorphism claim follows immediately.

The Proposition above motivates the following Definition.

Definition 4.2. Given a Banach *-algebraic bundle \( \mathcal{B} = \{ B_t \}_{t \in G} \) and a C*-completion \( \rho : B_e \to A \), a \( \rho \)-completion of \( \mathcal{B} \) is a C*-completion \( \iota : \mathcal{B} \to \mathcal{A} \) such that \( \iota_e = \rho \).

We now combine the induction process with the existence of particular C*-completion of Banach *-algebraic bundles.

Theorem 4.3. Let \( \mathcal{B} = \{ B_t \}_{t \in G} \) be a Banach *-algebraic bundle and let \( \pi : B_e \to A \) be a C*-completion, which we regard as a \( \pi \)-representation \( \pi : B_e \to A \subset \mathcal{B}(A) \). Then the following are equivalent:

1. There exists a \( \pi \)-completion of \( \mathcal{B} \).
2. \( \pi \) is \( \mathcal{B} \)-positive and \( \pi(c^*b^*bc) \leq \pi(b^*b)\pi(c^*c) \) for all \( b,c \in \mathcal{B} \).
3. For every \( b,c \in \mathcal{B} \) it follows that \( 0 \leq \pi(c^*b^*bc) \leq \pi(b^*b)\pi(c^*c) \).
4. \( \pi \) is \( \mathcal{B} \)-positive and \( \| \text{Ind}_B^\mathcal{B} (\pi)_b \| \leq \| \pi(b) \| \) for all \( b \in \mathcal{B}_e \).
5. \( \pi \) is \( \mathcal{B} \)-positive and \( \| \text{Ind}_B^\mathcal{B} (\pi)_b \| = \| \pi(b) \| \) for all \( b \in \mathcal{B}_e \).
6. There exists a non degenerate *-representation \( T : \mathcal{B} \to \mathcal{B}(Y) \) (\( Y \) being a Hilbert space) such that \( \| \pi(b) \| = \| T_b \| \) for all \( b \in \mathcal{B}_e \).
7. There exists a *-representation \( T : \mathcal{B} \to \mathcal{B}(Y_A) \) such that \( \| \pi(b) \| = \| T_b \| \) for all \( b \in \mathcal{B}_e \).

Proof. By Theorem 2.15 \( \pi \) is \( \mathcal{B} \)-positive if and only if \( \pi(b^*b) \geq 0 \) for all \( b \in \mathcal{B} \). Thus (2) implies (3) and the existence of approximate units guarantees the converse.

If (1) holds, then \( \pi \) is \( \mathcal{B} \)-positive because for all \( b \in \mathcal{B} \) we have \( \pi(b^*b) = \iota(b)^*\iota(b) \geq 0 \). If we replace the topology of \( G \) by the discrete one, then \( \mathcal{A} \) becomes a Fell bundle over a discrete group and using [5, Lemma 17.2] we deduce that \( a^*b^*ba \leq \| b^*b \| a^*a \) (in \( A_e \)) for all \( a,b \in \mathcal{A} \) (no matter which topology we consider on \( G \)). Thus for all \( b,c \in \mathcal{B} \) we have

\[
\pi(c^*b^*bc) = \iota(c)^*\iota(b)^*\iota(b)\iota(c) \leq \| b^*b \| \| c^*c \| \| \iota(b) \|^2 = \| \pi(b^*b) \| \| \pi(c^*c) \|.
\]

This shows (1) implies both (2) and (3).
We now assume (2) holds and will prove (4). Clearly π is B-positive. Noticing that the inner product of $L^2(B) \otimes A$ is given by $(f \otimes a, g \otimes b) = \int_B a^* \pi(f(t))^* g(t) b dt$, using (2) it follows that for all $\zeta = \sum_{j=1}^n f_j \otimes a_j \in L^2(B) \otimes A$ with $a_j \in B_e$ and all $b \in B$

\[
\langle \text{Ind}_c^B(\pi)b\zeta, \text{Ind}_c^B(\pi)b\zeta \rangle = \int_B \pi((\sum_{j=1}^n a_j f_j(t))^* b^* b(\sum_{k=1}^n a_k f_k(t))) \leq \|\pi(b^* b)\| \langle \zeta, \zeta \rangle.
\]

Since $\pi(B_e)$ dense in $A$ it follows that $\|\text{Ind}_c^B(\pi)b\|^2 \leq \|\pi(b^* b)\|$.

If we now assume (4) holds and take a non degenerate and faithful *-representation $\rho: A \to \mathbb{B}(Z)$, then recalling that the abstract induced representation $\text{Ind}_c^B(\pi \otimes \rho 1)$ is unitary equivalent to the concretely induced representation $\text{Ind}_{B_e \triangleright B}(\pi \otimes 1)$ and using [7 XI 11.3] we get that

\[
\|\pi(a)\| = \|\pi \otimes 1\| = \|\text{Ind}_{B_e \triangleright B}(\pi \otimes 1)_a\| = \|\text{Ind}_c^B(\pi \otimes \rho 1)_a\| = \|\text{Ind}_c^B(\pi)_a\|
\]

for all $a \in B_e$. Then (5) follows and it implies (6) with $T = \text{Ind}_c^B(\pi \otimes \rho 1)$; which in turn trivially implies (7).

Finally, assume (7). To prove (1) follow the procedure described in [7 VIII 16.7] but using the norm $\|T_b\|$ instead of the norm $\|\|$, used by Fell. When Fell says “form the Hilbert direct sum $T$ of enough *-representations of $B$ so that $\|T_b\| = \|b\|_{c^*}$” just consider the *-representation $T$ given by claim (7). This produces a $C^*$-completion $\iota: B \to C$ in such a way that $C_e$ is naturally isomorphic to the closure of $T(B_e)$. But claim (7) implies the existence of a unique isomorphism of $C^*$-algebras $\phi: A \to C_e$ such that $\phi \circ \pi = \iota|_{B_e}$. Then we may replace $C_e$ by $A$ in $C = \{C_t\}_{t \in G}$ and $\iota_e$ by $\pi$ in $\iota = \{\iota_t\}_{t \in G}$ to get a $C^*$-completion as in claim (1).

Remark 4.4. The Theorem above implies that if $B$ is a Fell bundle then there exists a *-representation $T: B \to \mathbb{B}(Y)$ with $\|T_b\| = \|b\|$ for all $b \in B_e$. Hence, for all $b \in B$, $\|T_b\| = \|T_b\|^{1/2} = \|b\|^{1/2} = \|b\|$. This is the property used by Fell in [7 VIII 16.10] to prove every Fell bundle is its own bundle $C^*$-completion.

Remark 4.5 (Universal property of the bundle $C^*$-completion). Let $B = \{B_t\}_{t \in G}$ be a Banach *-algebraic bundle and let $\iota: B \to C$ be it’s bundle $C^*$-completion. For any other $C^*$-completion $\kappa: B \to A$ there exists a *-representation $T: A \to \mathbb{B}(Z)$ such that $\|T_a\| = \|a\|$ for all $a \in A$. Consequently, $T \circ \kappa$ is a *-representation of $B$ and the construction of $C$ implies $\|\kappa(b)\| = \|T \circ \kappa(b)\| \leq \|\iota(b)\|$ for all $b \in B_e$. Then Proposition 4.4 implies the existence of a unique morphism $\rho: C \to A$. So the bundle $C^*$-completion is the universal $C^*$-completion of $B$, and we denote it $\iota^*: B \to B^*$.

Then follows immediately that every Fell bundle is it’s own bundle $C^*$-completion (as already noticed by Fell).

Considering a fixed group $G$ and a subgroup $H$ of $G$, the proposition below can be used to construct functors

\[
(\mathcal{B} \xrightarrow{\rho} \mathcal{C}) \mapsto \left( C^*(\mathcal{B}_H^+) \xrightarrow{\chi_H^*} C^*(\mathcal{C}_H^+) \right)
\]

\[
(\mathcal{B} \xrightarrow{\rho} \mathcal{C}) \mapsto \left( C^*(\mathcal{B}) \xrightarrow{\chi_H^*} C^*(\mathcal{C}) \right)
\]

from the category of Banach *-algebraic bundles over $G$ to the category of $C^*$-algebras.

Proposition 4.6. Let $B = \{B_t\}_{t \in G}$ and $C = \{C_t\}_{t \in G}$ be Banach *-algebraic bundles, $H \subset G$ a subgroup and $\rho: \mathcal{B} \to \mathcal{C}$ a morphism of Banach *-algebraic bundles. Denote $\rho_H: \mathcal{B}_H \to \mathcal{C}_H$ the morphism such that $\rho_H(b) = \rho(b)$ for all $b \in \mathcal{B}_H$. If either $\mathcal{B}$ is
saturated or $H$ is normal in $G$, then there exists unique morphism of *-algebras $\chi^\rho_H$ and $\chi^\rho_H^+$ such that the following diagrams commute

$$
\begin{array}{ccc}
L^1(B_H) & \xrightarrow{\tilde{\rho}_H} & L^1(C_H) \\
\chi^\rho_H & \downarrow & \chi^\rho_H^+ \\
C^*(B_H^+) & \xrightarrow{\chi^\rho_H^+} & C(C_H^+)
\end{array}
\quad
\begin{array}{ccc}
C^*(B) & \xrightarrow{x^\rho} & C^*(C) \\
\chi^\rho_H & \downarrow & \chi^\rho_H^+ \\
C^*(B_H) & \xrightarrow{\chi^\rho_H^+} & C^*(C_H)
\end{array}
$$

In case $\chi^\rho_H^+$ is faithful then so it is $\chi^\rho_H$. If $\rho: B \to C$ is a $C^*$-completion then both $\chi^\rho_H^+$ and $\chi^\rho_H$ are surjective maps and they are $C^*$-isomorphism if $\rho$ is the canonical map from a Banach *-algebraic bundle to it’s bundle $C^*$-completion.

If both $B$ and $C$ are Fell bundles and $H = \{e\}$, the statements above imply the unique extension of $\tilde{\rho}: L^1(B) \to L^1(C)$ to a morphism of $C^*$-algebras $C^*_\iota(B) \to C^*_\iota(C)$ is faithful whenever $\rho_e: B_e \to C_e$ is so (because $\rho_e = \chi^\rho_H$).

**Proof.** The uniqueness of the morphism follows from the commutativity of the diagrams and the fact that the both $\chi^B\rho_H$ and $\chi^B_H$ have dense ranges. To prove the existence of $\chi^B\rho_H$ we may regard $C^*(C^+_H)$ as a non degenerate $C^*$-subalgebra of $B\langle Y \rangle$, for some Hilbert space $Y$. This entails the existence of a non degenerate and $C$–positive *-representation $T: C_H \to B\langle Y \rangle$ such that the *-representation $\chi^C\rho_H^+ : C^*(C^+_H) \to B\langle Y \rangle$ is just the inclusion map. Then $\tilde{T} \circ \rho_H$ is $B$–positive by Theorem \[1\] and there exists a morphism of *-algebras $\tilde{\chi}^B_H : C^*_\iota(B_H) \to B\langle Y \rangle$ such that $\chi^B_H \circ \tilde{\chi}^B_H = (T \circ \rho_H) \tilde{T} \circ \rho_H$. Thus $\tilde{\chi}^B_H(C^*(C^+_B)) \subset C^*(C^+_H)$ and it suffices to set $\chi^H : = \chi^B_H$ to prove the existence of the morphism $\chi^B\rho_H$ with the desired properties.

By Lemma \[3\] there exists a non degenerate *-representation $T: C \to B\langle Y \rangle$ such that the restriction $S := T_{|C_H}$ is non degenerate and $\chi^C_S : C^*(C^+_H) \to B\langle Y \rangle$ is faithful. Recall that, by construction, $\chi^C_S \circ \chi^C_H = \tilde{S}$. Let $\kappa : H \to \mathbb{C}$ be the trivial representation, set $U := \text{Ind}_{\kappa}^C(S \otimes \kappa)$ and consider the *-representation $\pi_{T\kappa} : C^*_\iota(C) \to B\langle Y \otimes (L^2(H) \otimes \kappa) \rangle$ of Corollary \[2\]. Then, by Theorem \[2\],

$$
\|\tilde{\Lambda}_f^H\| = \|\tilde{\Lambda}_f^H \otimes_{S \otimes \kappa} 1\| = \|\text{Ind}_f^C(S \otimes \kappa)\| \leq \|\pi_{T\kappa}(\tilde{\Lambda}_f^HC)\| \leq \|\tilde{\Lambda}_f^HC\|
$$

for all $f \in L^1(C)$. We conclude that $\pi_{T\kappa}$ is faithful.

Although the composition $T \circ \rho : B \to B\langle Y \rangle$ may be degenerate, by restricting to it’s essential space and applying Corollary \[2\] we can construct a *-representation $\pi_{T\circ \rho}\kappa : C^*_\iota(B) \to B\langle Y \otimes (L^2(H) \otimes \kappa) \rangle$ such that $\pi_{T\circ \rho}\kappa \circ \tilde{\Lambda}^HB = (T \circ \rho \otimes U) \tilde{T} \circ \rho_H$. Since $\pi_{T\kappa} \circ \tilde{\Lambda}^HC \circ \tilde{\rho} = (T \circ \rho \otimes U)$, it follows that $\tilde{\rho} \circ \pi_{T\kappa}(\tilde{\Lambda}_f^H) = \|\tilde{\Lambda}_f^H \circ \tilde{\rho}(f)\| = \|\pi_{T\circ \rho}\kappa(\tilde{\Lambda}_f^H)\| \leq \|\tilde{\Lambda}_f^H\|,

Then we can construct a morphism of *-algebras $\tilde{\Lambda}^H_B(L^1(B)) \to C^*_\iota(C)$ mapping $\tilde{\Lambda}_f^H$ to $\tilde{\Lambda}_f^H \circ \tilde{\rho}(f)$ and this morphism if contractive with respect to the $C^*$-norm inherited from $C^*_\iota(B)$. Thus the morphism has a unique extension to a morphism of $C^*$-algebras $\chi^\rho_H : C^*_\iota(B) \to C^*_\iota(C)$ such that $\chi^\rho_H \circ \tilde{\Lambda}^H_B = \tilde{\Lambda}^HC \circ \tilde{\rho}$. Hence, for all $f \in L^1(B)$,

$$
\chi^\rho_H \circ q_H^B(\tilde{\Lambda}_f^GB) = \chi^\rho_H \circ \tilde{\Lambda}_f^H = \tilde{\Lambda}^HC \circ \tilde{\rho}(f) = q_H^C(\tilde{\Lambda}_f^G) = q_H^C \circ \chi^\rho(\tilde{\Lambda}_f^GB),
$$

and it follows that $\chi^\rho_H \circ q_H^B = q_H^C \circ \chi^\rho$. 

Assume $\chi^\alpha_{tN}$ is faithful. Then $X^{tN}_{\alpha} \circ X^{\rho}_{H} \circ \chi^\beta_{tN} = X^{tN}_{\alpha} \circ X^{\rho}_{H} \circ \tilde{\rho} = \tilde{S} \circ \tilde{\rho} = (T \circ \rho |_{H})$, and this implies $\|X^\beta_{tN}(f)\| = \|[(T \circ \rho |_{H}) \otimes \kappa]|f\|$ for all $f \in L^1(B_H)$. This implies the inequality in (4.8) is an equality and, in this case, our construction of $\chi^\rho_{tN}$ implies it is faithful (to prove this claim follow our proof of the fact that $\pi_{T\kappa}$ is faithful).

Suppose $\rho: B \to C$ is a $C^*$-completion. Then both $\tilde{\rho}$ and $\tilde{\rho}_H$ have dense images and this implies $\chi^\rho_{tH}$ and $\chi^\rho_{tN}$ have dense images and hence are surjective maps. □

4.1. Cross sectional bundles, $C^*$-completions and induction. Let’s fix, for the rest to this section, a Banach $^*$-algebraic bundle $B = \{B_t\}_{t \in G}$ (with a strong approximate unit, of course) and also a closed normal subgroup $N$ of $G$. We will briefly recall the main properties of the $L^1$—cross sectional bundle over $G/N$ derived from $B$, the detailed construction can found in [7, VIII 6].

We regard the quotient group $G/N = \{tN: t \in G\}$ as a LCH topological group with the quotient topology. Given a coset $\alpha = tN \ (t \in G)$ let $\nu_\alpha$ be the regular Borel measure on $tN$ such that $\int_{tN} f(x)\,d\nu_\alpha(x) = \int_{tN} f(tx)\,d_Nx$ for all $f \in C_c(tN)$. Here $d_Nx$ is the integration with respect to a (fixed) left invariant Haar Measure of $N$. There is no ambiguity in the definition of $\nu_\alpha$ because the left invariance of $d_N$ implies the function $tN \to \mathbb{R}, \ r \mapsto \int_{tN} f(rx)\,d_Nx$, is constant.

For every function $f \in C_c(G)$ we define $f^0: G \to C$ as $f^0(t) := \int_{tN} f(tx)\,d_Nx = \int_{tN} f(x)\,d_{N^t}$. It can be shown that $f^0$ is continuous and constant in the cosets, so it defines a function $f^{00} \in C(G/N)$ that vanishes outside the projection of supp$(f)$ on $G/N$. Then, by construction,

$$\int_{G/N} f^{00}(x)\,d_{G/N}x = \int_{G/N} d_{G/N} tN \int_N f(ts)\,d_Ns.$$ 

Throughout this work we assume the left invariant Haar measures of $G$, $N$ and $G/N$ are normalized in such a way that for all $f \in C_c(G)$

$$(4.9) \quad \int_{G} f(t)\,d_Gt = \int_{G/N} d_{G/N} tN \int_N f(ts)\,d_Ns,$$

exactly as in [7, VIII 6.7]. In case $G$ is a product $H \times K$, $N = H$ and $K = (H \times K)/H$; we meet this requirement by considering the product measure $d_G(r,s) = d_Hr \times d_Ks$.

By [7, VIII 6.5] there exists a unique continuous homomorphism $\Gamma: G \to (0, + \infty)$ such that

$$\int_N f(xy^{-1})\,d_Ny = \Gamma(x) \int_N f(y)\,d_Ny, \ \forall \ x \in G, \ f \in C_c(N).$$

In case $G$ is a product group $H \times K$ and $N = H$ one has $\Gamma(r,s) = \Delta_H(r)$, where $\Delta_H$ is the modular function of $H$.

The $L^1$—partial cross sectional bundle over $G/N$ derived from $B$, $C = \{C_\alpha\}_{\alpha \in G/N}$, is determined by the following properties:

- For every $\alpha \in G/N$, if $B_\alpha = \{B_t\}_{t \in \alpha}$ is the reduction of $B$ to $\alpha$, then $C_\alpha$ is the completion of $C_c(B_\alpha)$ with respect to the norm $\|f\| = \int_\alpha \|f(t)\|\,d\nu_\alpha(t)$.
- For every $r,s \in G$, $f \in C_c(B_{rsN})$ and $g \in C_c(B_{sN})$, the product $f \ast g \in C_c(B_{rsN}) \subset C_{rsN}$ and the involution $f^* \in C_c(B_{rsN})$ are determined by $f \ast g(x) = \int_{rN} f(y)g(y^{-1}x)\,d_{rN}(y)$ and $f^*(z) = \Gamma(z)^{-1}f(z^{-1})^*$, for all $x \in rsN$ and $z \in r^{-1}N$.
- Given $f \in C_c(B)$, if $f|: G/N \to C$ is given by the restriction $f|(\alpha) := f|_\alpha$, then $f|$ is a continuous cross section.
**Notation 4.10.** The $L^1$—partial cross sectional bundle over $G/N$ derived from $\mathcal{B}$ will be denoted $L^1(\mathcal{B}, N) = \{L^1(\mathcal{B}_\alpha)\}_{\alpha \in G/N}$. This is more than just a notation because if given the coset $\alpha \in G/H$ we denote by $\mathcal{B}_\alpha$ the reduction of $\mathcal{B}$ to $\alpha$, then $\mathcal{B}_\alpha$ is a Banach bundle and $L^1(\mathcal{B}_\alpha)$ is (constructively and symbolically) the completion of $C_c(\mathcal{B}_\alpha)$ with respect to $\| \cdot \|_1$. The usual $L^1$—cross sectional algebra of $\mathcal{B}$ may be regarded (notationally and concretely) as $L^1(\mathcal{B}, G)$ and the bundle $\mathcal{B}$ itself as $L^1(\mathcal{B}, \{e\})$.

**Remark 4.11.** Our conventions (C) state that $\mathcal{B}$ has a strong approximate unit, this is also the case for $L^1(\mathcal{B}, N)$ by [7, VIII 6.9].

**Remark 4.12.** The set $\{f: f \in C_c(\mathcal{B})\} \subset C_c(L^1(\mathcal{B}, N))$ is dense in $C_c(L^1(\mathcal{B}, N))$ with respect to the inductive limit topology by [7, II 14.6] and, consequently, it is dense in $L^1(L^1(\mathcal{B}, N))$.

**Remark 4.13.** It is shown in [7, VIII 6.7] that there exists a unique isometric isomorphism of Banach *-algebras $\Phi: L^1(\mathcal{B}) \to L^1(L^1(\mathcal{B}, N))$ such that $\Phi(f) = \| f \|$, for all $f \in C_c(\mathcal{B})$.

Given a *-representation $T: \mathcal{B} \to \mathbb{B}(Y_A)$ we can follow [7, VIII 15.9] and construct a (unique) *-representation

$$L^1(T, N): L^1(\mathcal{B}, N) \to \mathbb{B}(Y_A)$$

such that for every coset $\alpha \in G/N$, $f \in C_c(\mathcal{B}_\alpha)$ and $\xi \in Y_A$,

$$L^1(T, N)f\xi = \int_G T(f(\alpha))\xi d\nu_\alpha(t).$$

If $\Phi: L^1(\mathcal{B}) \to L^1(L^1(\mathcal{B}, N))$ is the isomorphism of Remark 4.13 then $\tilde{L}^1(T, N) \circ \Phi = \tilde{T}$; $\tilde{L}^1(T, N)$ being the integrated form of $L^1(T, N)$.

In case we are given a non degenerate *-representation $S: L^1(\mathcal{B}, N) \to \mathbb{B}(Y_A)$, the composition $\tilde{S} \circ \Phi: L^1(\mathcal{B}) \to \mathbb{B}(Y_A)$ is a non degenerate *-representation that can be disintegrated (uniquely) to a *-representation $T: \mathcal{B} \to \mathbb{B}(Y_A)$. Thus we obtain $\tilde{T} = \tilde{S} \circ \Phi$, implying both $\tilde{T}$ and $T$ are non degenerate. Moreover, $L^1(T, N) \circ \Phi = \tilde{T} = \tilde{S} \circ \Phi$. Thus $\tilde{L}^1(T, N) = \tilde{S}$ and this implies $L^1(T, N) = S$.

Any subgroup of $G/N$ can be expressed as $H/N$ for a unique subgroup $H$ of $G$ containing $N$. Then we obtain the identity of Banach *-algebraic bundles

$$L^1(\mathcal{B}_H, N) = \{L^1(\mathcal{B}_\alpha)\}_{\alpha \in H/N} = L^1(\mathcal{B}, N)_{H/N};$$

and we have an isometric isomorphism of Banach *-algebras

$$\Phi^H: L^1(\mathcal{B}_H) \to L^1(L^1(\mathcal{B}_H, N)) \equiv L^1(L^1(\mathcal{B}, N)_{H/N});$$

which for $H = G$ is just the isomorphism $L^1(\mathcal{B}) \cong L^1(L^1(\mathcal{B}, N))$ we have considered before.

The equivalence of the representations theories of $\mathcal{B}$ and $L^1(\mathcal{B}, N)$ now reduces to a correspondence $T \rightsquigarrow L^1(T, N)$ between non degenerate *-representations of $\mathcal{B}_H$ and $L^1(\mathcal{B}_H, N) \equiv L^1(\mathcal{B}, N)_{H/N}$.

Given a *-representation $T: \mathcal{B}_H \to \mathbb{B}(Y_A)$ and a faithful *-representation $\rho: A \to \mathbb{B}(Z)$ we have $L^1(T, N) \otimes_\rho 1 = L^1(T \otimes_\rho 1, N)$. Hence by Theorem 2.15 and [7, XI 12.7] the following are equivalent:

1. $L^1(T, N)$ is $L^1(\mathcal{B}, N)$—positive.
2. $L^1(T \otimes_\rho 1, N)$ is $L^1(\mathcal{B}, N)$—positive.
3. $T \otimes_\rho 1$ is $\mathcal{B}$—positive.
4. $T$ is $\mathcal{B}$—positive.
The equivalence of the claims above should be kept in mind, as we will use it quite frequently without explicit mention.

We want to stress two points. Firstly, when determining positivity of *-representations we may restrict to the essential space (and assume non degeneracy) because we are assuming \( B \) has a strong approximate unit. Secondly, [7, XI 12.7] is stated for non degenerate *-representation and so can be extended to our context by the previous comment. In fact Fell begins the proof of [7, XI 12.7] adopting “Rieffel’s formulation of the inducing process” meaning he is using the abstract inducing process, i.e. the process we have been working with. Hence, all the computations of [7, XI 12.7] hold verbatim replacing \( \text{Ind}_{B \uparrow B} \) by \( \text{Ind}_{B}^{B} \) (concrete by abstract) and even considering representations on Hilbert modules. In particular equation (6) in [7, pp 1164] becomes (in our notation)

\[
(4.15) \quad T \left( \frac{L^{1}(T, N)}{P_{H/N}(\Phi(f)^{*} \Phi(g))} \right) = T \left( \frac{L^{1}(T, N)}{P_{H/N}(\Phi(f)^{*} \Phi(g))} \right) = \tilde{T}_{H}^{G}(f^{*} g);
\]

and holds for every *-representation \( T: B_{H} \rightarrow \mathbb{B}(Y) \) and \( f, g \in C_{c}(B) \). Here the \( p \) functions are the generalized restrictions for the groups indicated in the super and sub indexes.

Given a non degenerate \( L^{1}(B, N) \)–positive *-representation \( S: L^{1}(B, N)_{H/N} \rightarrow \mathbb{B}(Y) \) let \( T: B_{H} \rightarrow \mathbb{B}(Y) \) be the (non degenerate) *-representation such that \( S = L^{1}(T, N) \).

We then get the following commutative diagram

\[
L^{1}(L^{1}(B, N)_{H/N}) \overset{\Phi}{\longrightarrow} L^{1}(B_{H}) \overset{\pi}{\longrightarrow} \mathbb{B}(Y) \quad \text{and} \quad C^{*}(B_{H}^{\perp}) \quad \text{by} \quad \pi_{T}
\]

We can arrange the \( T \) above so that \( \pi_{T} \) is faithful. In such a situation we know (by [7, XI 12.7] or (4.15)) that \( \pi \circ \Phi^{-1} \left( \frac{G^{*}}{H/N}(f^{*} f) \right) \geq 0 \) for all \( f, g \in C_{c}(L^{1}(B, N)) \). Then

\[
\pi \circ \Phi^{-1}: L^{1}(L^{1}(B, N)_{H/N}) \rightarrow C^{*}(B_{H}^{\perp})
\]

is a C*-completion satisfying conditions (1) and (2) of Theorem [2.16] meaning that

\[
C^{*}(L^{1}(B, N)_{H/N}) \cong C^{*}(B_{H}^{\perp}).
\]

By restricting the isometric *-isomorphism \( \Phi: L^{1}(B) \rightarrow L^{1}(L^{1}(B, N)) \) to \( C_{c}(B) \) we obtain a unitary operator

\[
U: L_{H}^{2}(B) \rightarrow L_{H/N}^{2}(L^{1}(B, N))
\]

mapping \( f + I \in C_{c}(B)/I \) to \( \Phi(f) + I' \), \( I' \) being the null space of \( C_{c}(L^{1}(B, N)) \) with respect to the \( C^{*}(L^{1}(B, N)_{H/N}) \)–valued pre inner product of \( C_{c}(L^{1}(B, N)) \). This claim holds, ultimately, by (4.15). It then follows that the conjugation by \( U \) gives the unitary equivalence of C*-algebras

\[
(4.16) \quad C^{*}_{H}(B) \cong C^{*}_{H/N}(L^{1}(B, N)) \cong C^{*}_{H/N}(C^{*}(B, N)),
\]

where the C*-isomorphism on the right is that given by Proposition [4.6].

We can now re-interpret (and extend) [7, XI 12.7] as the fact that for every \( B \)–positive *-representation \( T: B_{H} \rightarrow \mathbb{B}(Y) \) there exits a unitary

\[
U \otimes 1: L_{H}^{2}(B) \otimes_{T} Y_{A} \rightarrow L_{H/N}^{2}(L^{1}(B, N)) \otimes_{L^{1}(T, N)} Y_{A},
\]
mapping $f \otimes_T \xi$ to $\Phi(f) \otimes L^1(T, N) \xi$, for every elementary tensor $f \otimes_T \xi$. This operator establishes the unitary equivalence of $*$-representations

$$L^1(\text{Ind}_B^B(T), N) \cong \text{Ind}_{B/N}(L^1(T, N)).$$

The first consequence of the identity above is the following.

**Proposition 4.18.** Assume $B$ is a Fell bundle over $G$, $N \subset G$ a normal subgroup and $\chi^{B_N}$: $L^1(\mathcal{B}_N) \to C^*(\mathcal{B}_N)$ the universal $C^*$-completion. Then the bundle $C^*$-completion of $L^1(B, N)$, $\mathcal{C} = \{C_\alpha\}_{\alpha \in G/N}$, is (isomorphic to) the $\chi^{B_N}$-completion of $L^1(B, N)$ in the sense of Definition 4.2.

**Proof.** Let $\rho = \{\rho_\alpha\} : L^1(\mathcal{B}, N) \to C$ be the canonical morphism. By Proposition 4.11 it suffices to show that $\|\rho_N(f)\| = \|\chi^{B_N}(f)\|$ for all $f \in L^1(\mathcal{B}_N)$. The inequality $\|\rho_N(f)\| \leq \|\chi^{B_N}(f)\|$ holds for all $f \in L^1(\mathcal{B}_N)$ because $C_N$ is a $C^*$-algebra and $\rho_N : L^1(\mathcal{B}_N) \to C_N$ a morphism of $*$-algebras.

By Theorem 2.13 we can induce $\chi^{B_N}$ to a $*$-representation $S := \text{Ind}_{e}(L^1(\mathcal{B}, N)(\chi^{B_N})$ of $L^1(B, N)$. Then Lemma 3.1 and the construction of the bundle $C^*$-completion imply, for all $f \in L^1(\mathcal{B}_N)$, that

$$\|\chi^{B_N}(f)\| = \|Sf\| \leq \|\rho_N(f)\|.$$

Hence $\|\chi^{B_N}(f)\| \leq \|\rho_N(f)\| \leq \|\chi^{B_N}(f)\|$ and the proof is complete. \qed

It now makes sense to adopt the following.

**Notation 4.19.** In the situation of the Proposition above, the bundle $C^*$-completion of $L^1(\mathcal{B}, N)$ will be denoted $C^*(\mathcal{B}, N) = \{C^*(\mathcal{B}_\alpha)\}_{\alpha \in G/N}$. This makes sense because with this notation the unit fibre of $C^*(\mathcal{B}, N)$ is, both symbolically and concretely, $C^*(\mathcal{B}_N)$.

Previously in this section we obtained a $C^*$-isomorphism $C^*_H(\mathcal{B}) \cong C^*_{H/N}(L^1(\mathcal{B}, N))$, provided that $H \subset G$ is a subgroup containing $N$. We now relate this fact to the amenability of $L^1(\mathcal{B}, N)$.

**Proposition 4.20.** Consider a $*$-Banach algebraic bundle $\mathcal{B} = \{B_t\}_{t \in G}$ and subgroups $N \subset H \subset K \subset G$ with $N$ normal in $G$. Let $\Phi : L^1(\mathcal{B}) \to L^1(\mathcal{B}, N)$ be the isomorphism of Remark 4.12. $\psi_H : C^*_H(\mathcal{B}) \to C^*_{H/N}(L^1(\mathcal{B}, N))$ and $\psi_K : C^*_K(\mathcal{B}) \to C^*_{K/N}(L^1(\mathcal{B}, N))$ the $C^*$-isomorphism of (4.10) and consider the quotient maps $q$ for $\mathcal{B}$ and $L^1(\mathcal{B}, N)$ of Proposition 4.4. Then the following diagram commutes

\[
\begin{array}{ccc}
L^1(\mathcal{B}) & \xrightarrow{\Phi} & C^*_K(\mathcal{B}) \\
L^1(\mathcal{B}, N) & \xrightarrow{\chi^{B_N}} & C^*_K/N(\mathcal{B}(\mathcal{B}, N)) \\
\end{array}
\]

In particular, $q^B_{KH}$ is a $C^*$-isomorphism if and only if $q_{K/N}(\mathcal{B}(H/N))$ is so. Setting $K = G$ and $H = N$ the preceding claim becomes: $q^B_N : C^*(\mathcal{B}) \to C^*_N(\mathcal{B})$ is a $C^*$-isomorphism if and only if $q_{e}(L^1(\mathcal{B}, N)) : C^*(L^1(\mathcal{B}, N)) \to C^*_e(L^1(\mathcal{B}, N))$ is a $C^*$-isomorphism (i.e. $L^1(\mathcal{B}, N)$ is amenable).

**Proof.** The outer and inner left rectangles of (4.21) commute by the construction of the $\psi$ maps and Proposition 3.1. Besides, the compositions of the $q$ and $\chi$ maps in the inner left diagram have dense ranges. This forces the inner right diagram to commute. The rest of the proof follows immediately. \qed
At this point we have a complete understanding of the relation between the C*-algebras $C^*_H(B)$ and $C^*_K(L^1(B, N))$ provided that $N \subset H$ and $K \subset G/N$ are subgroups. What about the C*-algebras $C^*_H(B)$ for $H \subset N$?

The following result may be used in conjunction with Proposition 3.11 Corollary 3.11 or Theorem 3.19 to describe $C^*_H(L^1(B, N))$ as $C^*_H(B)$ for some subgroup $H \subset N$.

**Proposition 4.22.** Let $B = \{B_t\}_{t \in G}$ be a Banach *-algebraic bundle and $N \subset G$ a normal subgroup. Then for any subgroup $H \subset N$ there exists a unique morphism of *-algebras $\pi_H: C^*_H(L^1(B, N)) \to C^*_H(B)$ such that the diagram below commutes

\[
\begin{array}{ccc}
L^1(L^1(B, N)) & \xrightarrow{\tilde{\lambda}^1(L^1(B, N))} & C^*_H(B) \\
\downarrow & & \downarrow \pi_H \\
C^*_H(L^1(B, N)) & \xrightarrow{\pi_H} & C^*_H(B)
\end{array}
\]

Moreover, $\pi_H$ is surjective and it is faithful if and only if $q^B_N: C^*_N(B) \to C^*_H(B)$ is so. In particular, if $B$ is a Fell bundle, $H = \{e\}$ and we identify $C^*_e(L^1(B, N))$ with $C^*_e(C^*(B, N))$ as in Proposition 2.6, then $\pi_e: C^*_e(C^*(B, N)) \to C^*_e(B)$ is a C*-isomorphism if and only if $B_N$ is amenable.

**Proof.** It is implicit in the claim of the proposition that the image of

\[\tilde{\lambda}^1(L^1(B, N)): L^1(L^1(B, N)) \to B(C^*_H(B))\]

is contained in $C^*_H(B)$. This is so because the image of $\tilde{\lambda}^1: L^1(B) \to B(C^*_H(B))$ is contained in $C^*_H(B)$ and the diagram

\[
\begin{array}{ccc}
L^1(B) & \xrightarrow{\Phi} & \tilde{\lambda}^1(L^1(B, N)) \\
\downarrow & & \downarrow \tilde{\lambda}^1(L^1(B, N)) \\
L^1(L^1(B, N)) & \xrightarrow{\pi_H} & B(C^*_H(B))
\end{array}
\]

commutes, with $\Phi$ being the isomorphism of Remark 4.13. By (4.21) and Proposition 3.1 the following diagram commutes:

\[
\begin{array}{ccc}
L^1(L^1(B, N)) & \xrightarrow{\chi^L(B,N)} & L^1(\Lambda_{NB},N) \\
\downarrow & & \downarrow \chi^L_{S} \circ \chi^L_{S_1}(N) \circ \chi^L_{S_1}(N) \\
C^*(L^1(B, N)) & \xrightarrow{q^L_N(B,N)} & C^*_H(L^1(B, N))
\end{array}
\]

and $q^L_{e}(B,N) \circ \chi^L(B,N) = \tilde{\lambda}^e L^1(B, N)$. Thus we may define $\pi_H := q^B_N \circ (\psi_N)^{-1}$ to make (4.23) a commutative diagram. It is the unique with such property because (4.23) determines $\pi_H$ in a dense set. Besides, $\pi_H$ is surjective and our construction implies it is faithful if and only if $q^B_N$ is so.

If $B$ is a Fell bundle and $H = \{e\}$ then, by Theorem 3.11 $\pi_H$ is a C*-isomorphism $\iff q^B_N$ is a C*-isomorphism $\iff B_N$ is amenable. $\square$
Theorem 4.24. Let $\mathcal{B} = \{ B_t \}_{t \in G}$ be a Fell bundle and consider normal subgroups of $G$, $H \subset N \subset G$. Then the C*-completion $q_H^B \circ \chi^B_N : L^1(\mathcal{B}_N) \to C^*_H(\mathcal{B}_N)$ satisfies the equivalent conditions of Theorem 4.3 when considered as *-representation of the unit fibre of $L^1(\mathcal{B}, N)$. If $\kappa : L^1(\mathcal{B}, N) \to C^*_H(\mathcal{B}, N)$ is the $q_H^B \circ \chi^B_N$-completion of $L^1(\mathcal{B}, N)$, then there exists a unitary $U : L^2_H(\mathcal{B}) \to L^2_N(C^*_H(\mathcal{B}, N))$ with the following properties:

1. $U(f) = \tilde{\kappa} \circ \Phi(f)$, for all $f \in C_c(\mathcal{B})$, with $\Phi : L^1(\mathcal{B}) \to L^1(L^1(\mathcal{B}, N))$ being that $L^1(\mathcal{B}, N)$.
2. $C^*_H(\mathcal{B}) = \{ U^* MU : M \in C^*_H(\mathcal{B}, N) \}$.
3. If $\varphi : C^*_H(\mathcal{B}) \to C^*_H(\mathcal{B}, N)$ is given by $\varphi(M) = UMU^*$, then the diagram

\[
\begin{array}{ccc}
C^*_H(\mathcal{B}) & \xrightarrow{\pi_H} & C^*_H(\mathcal{B}, N) \\
\varphi \downarrow & & \downarrow U^* \\
C^*_H(\mathcal{B}) & \xrightarrow{\chi^*_N} & C^*_H(\mathcal{B}, N)
\end{array}
\]

commutes; with $\chi^*_N : C^*_H(\mathcal{B}, N) \to C^*_H(\mathcal{B}, N)$ being the map provided by Proposition 4.22 and $\pi_H$ the map of Proposition 4.22.

Besides, the following are equivalent:

(i) $q_H^B : C^*(\mathcal{B}) \to C^*_H(\mathcal{B})$ is a C*-isomorphism.

(ii) $q_H^B : C^*(\mathcal{B}) \to C^*_N(\mathcal{B})$ and $q_H^B : C^*(\mathcal{B}_N) \to C^*_H(\mathcal{B}_N)$ are C*-isomorphisms.

(iii) $C^*_H(\mathcal{B}, N)$ is amenable and $q_H^B$ is a C*-isomorphism.

(iv) $C^*(\mathcal{B}, N)$ is amenable and $q_H^B$ is a C*-isomorphism.

(v) The morphism between the C*-completions $\rho : C^*(\mathcal{B}, N) \to C^*_H(\mathcal{B}, N)$, provided by Remark 4.13, is an isomorphism and $C^*_H(\mathcal{B}, N)$ is amenable.

Proof. If in Theorem 3.12 we put $K = N$, then the *-representation $S$ of that Theorem is

\[ (\Lambda^{HB}|_{\mathcal{B}_N}) = L^1(\Lambda^{HB}, N)|_{L^1(\mathcal{B}_N)}. \]

Thus [3.13] gives the identity $\| q_H^B \circ \chi^B_N(f) \| = \| L^1(\Lambda^{HB}, N)f \|$ for all $f \in L^1(\mathcal{B}_N)$. It is then clear that $q_H^B \circ \chi^B_N$ satisfies condition (7) of Theorem 4.3. Let then $\kappa : L^1(\mathcal{B}, N) \to C^*_H(\mathcal{B}, N)$ be the $q_H^B \circ \chi^B_N$-completion of $L^1(\mathcal{B}, N)$.

We now will prove the existence of the unitary $U : L^2_H(\mathcal{B}) \to L^2_N(C^*_H(\mathcal{B}, N))$ such that $Uf = \tilde{\kappa} \circ \Phi(f)$ for all $f \in C_c(\mathcal{B})$. Recall that we may view $L^2_H(\mathcal{B})$ as a completion of $C_c(\mathcal{B})$ (see Remark 2.22). To prove the existence of $U$ take $f, g \in C_c(\mathcal{B})$ and note that:

\[ \langle \tilde{\kappa} \circ \Phi(f), \tilde{\kappa} \circ \Phi(g) \rangle_{L^2(C^*_H(\mathcal{B}, N))} = (\tilde{\kappa} \circ \Phi(f)^*) \ast (\tilde{\kappa} \circ \Phi(g)) (N) = \tilde{\kappa} \circ \Phi(f^* \ast g) (N) \]

\[ = p(f^* \ast g|N) = f^* \ast g|N = p_B^N(f^* \ast g) = \langle f, g \rangle_{L^2_H(\mathcal{B})}; \]

where the identity $f^* \ast g|N = p_B^N(f^* \ast g)$ holds because $\Delta_C|N = \Delta_N$. It is then clear that there exists a unique linear map $U : L^2_H(\mathcal{B}) \to L^2_N(C^*_H(\mathcal{B}, N))$ that preserves inner products and when restricted to $C_c(\mathcal{B})$ is given by $f \mapsto \tilde{\kappa} \circ \Phi(f)$. Note that $U$ is surjective because it is an isometry and its image contains the set $\tilde{\kappa} \circ \Phi(C_c(\mathcal{B}))$, which is dense in the inductive limit topology in $C_c(C^*_H(\mathcal{B}, N))$ and hence dense in $L^2_N(C^*_H(\mathcal{B}, N))$ by Remark 2.21.

For all $f, g \in C_c(\mathcal{B})$ we have

\[ \tilde{\kappa} \circ C^*_H(\mathcal{B}, N) \tilde{\kappa} \circ \Phi(f) = (\tilde{\kappa} \circ \Phi(f))^* \ast (\tilde{\kappa} \circ \Phi(g)) = \tilde{\kappa} \circ \Phi(f^* \ast g) = U \tilde{\Lambda}^{HB} g. \]

Thus $\tilde{\kappa} \circ C^*_H(\mathcal{B}, N)(\tilde{\kappa} \circ \Phi(f)) = U \tilde{\Lambda}^{HB} U^*$ for all $f \in L^1(\mathcal{B})$ and it follows that

\[ \tilde{\kappa} \circ C^*_H(\mathcal{B}, N)(L^1(\mathcal{B}, N)) = U \tilde{\Lambda}^{HB} (L^1(\mathcal{B})) U^*. \]
Claim (2) of the present Theorem follows by taking closures on both sides of the identity above.

The construction of \( \pi_H, \varphi \) and \( \chi^\circ \) imply that for all \( f \in L^1(\mathcal{B}) \):
\[
\pi_H(\tilde{\Lambda}^{eL^1(\mathcal{B}, N)}_{\Phi(f)}) = \tilde{L}^1(\Lambda^{H_B}, N) \circ \Phi(f) = \tilde{\Lambda}^{H_B}_{\kappa \circ \Phi(f)} = U^* \tilde{\Lambda}^{C^*_\mathcal{B}(\mathcal{B}, N)}_{\kappa \circ \Phi(f)} U = \varphi(\tilde{\Lambda}^{C^*_\mathcal{B}(\mathcal{B}, N)}_{\kappa \circ \Phi(f)})
\]
meaning that \( \pi_H \) and \( \varphi \circ \chi^\circ \) agree when restricted to the image of \( \tilde{\Lambda}^{eL^1(\mathcal{B}, N)} \). Thus, by continuity, \( \pi_H = \varphi \circ \chi^\circ \). Having shown claims (1) to (3) we now turn to prove the equivalence of the claims (i) to (v).

By Proposition 3.1 claim (i) is equivalent to say both \( q^B_H \colon C^*(\mathcal{B}) \to C^*_H(\mathcal{B}) \) and \( q^B_{KH} \colon C^*_K(\mathcal{B}) \to C^*_H(\mathcal{B}) \) are C*-isomorphisms; which in turn is equivalent to (ii) by Theorem 4.18 implies the C*-completions \( C^*_H(\mathcal{B}, N) \) and \( C^*(\mathcal{B}, N) \) of the unit fibre \( L^1(\mathcal{B}, N) \) of \( L^1(\mathcal{B}, N) \) agree, thus the C*-completions \( C^*_H(\mathcal{B}, N) \) and \( C^*(\mathcal{B}, N) \) agree by Proposition 4.1. By using the identifications \( C^*(\mathcal{B}) \equiv C^*_H(\mathcal{B}, N) \) and \( C^*_N(\mathcal{B}) \equiv C^*_H(C^*(\mathcal{B}, N)) \) of Proposition 4.20 it follows that (ii) implies (iii). As explained before, the fact of \( q^B_H \) being a C*-isomorphism implies \( C^*_H(\mathcal{B}, N) = C^*(\mathcal{B}, N) \). Thus (iii) and (iv) are equivalent and both imply (v). Finally, by Propositions 4.11 and 4.18 claim (v) implies \( q^B_H \) is a C*-isomorphism and, by Proposition 4.21 that \( q^B_H \) is a C*-isomorphism. Hence (v) implies (ii) and the proof is complete.

As a particular case of the Theorem above one may consider \( H = \{e\} \) and, as usual, write \( C^*_r \) instead of \( C^*_{\{e\}} \). In doing so one obtains the equivalence of the following claims:

1. \( \mathcal{B} \) is amenable.
2. \( C^*_r(\mathcal{B}, N) \) and \( \mathcal{B}_N \) are amenable.
3. \( C^*_r(\mathcal{B}, N) \) and \( \mathcal{B}_N \) are amenable.
4. The morphism of C*-completions \( \rho \colon C^*_r(\mathcal{B}, N) \to C^*_r(\mathcal{B}, N) \) is an isomorphism and \( C^*_r(\mathcal{B}, N) \) is amenable.

At this point there is no much room left for applications and consequences of the theory we have developed. Still, we want to give some example of how to use our theory. We start with to Corollaries that are known to hold for the subgroups \( H = \{e\} \) and \( H = G(13) \). These results are an extension to Fell bundles of the compatibility between the induction from subgroups and any Morita equivalence of crossed products coming from a Morita equivalence of actions (see the motivating examples of [1]).

**Corollary 4.25.** Let \( \mathcal{B} = \{B_t\}_{t \in G} \) be a Fell bundle and \( \mathcal{A} \subset \mathcal{B} \) a Fell subbundle. If \( N \subset G \) is a normal subgroup and \( \mathcal{A}_N \) is hereditary in \( \mathcal{B}_N \) in the sense of [4], then \( C^*_N(\mathcal{A}) \) is C*-isomorphic to the closure of \( q^B_N(L^1(\mathcal{A})) \) in \( C^*_N(\mathcal{B}) \).

**Proof.** By [3] we know \( C^*(\mathcal{A}_N) \) is \( (C^*_r(\mathcal{B}, N)) \)-isomorphic to the closure of \( L^1(\mathcal{A}_N) \) in \( L^1(\mathcal{B}_N) \). If we regard \( L^1(\mathcal{A}, N) \) as a Banach *-algebraic subbundle of \( L^1(\mathcal{B}, N) \) and then use Proposition 4.18 to construct the C*-completions \( C^*_r(\mathcal{A}, N) \) and \( C^*(\mathcal{B}, N) \), we then can view \( C^*_r(\mathcal{A}, N) \) as a Fell subbundle of \( C^*(\mathcal{B}, N) \). It follows from [1] Proposition 3.2 that the inclusion \( L^1(C^*_r(\mathcal{A}, N)) \hookrightarrow L^1(C^*(\mathcal{B}, N)) \) extends to an inclusion \( C^*_r(C^*_r(\mathcal{A}, N)) \hookrightarrow C^*(\mathcal{B}, N) \). If we view \( L^1(\mathcal{A}) = L^1(L^1(\mathcal{A}, N)) \) as a *-subalgebra of \( L^1(C^*_r(\mathcal{A}, N)) \) and identify \( C^*_N(\mathcal{A}) \) with \( C^*_r(C^*(\mathcal{A}, N)) \) (and do the same thing for \( \mathcal{B} \)) we see the inclusion \( L^1(\mathcal{A}) \hookrightarrow L^1(\mathcal{B}) \) extends to an inclusion \( C^*_N(\mathcal{A}) \hookrightarrow C^*_N(\mathcal{B}) \), the image of which is \( q^B_N(L^1(\mathcal{A})) \).
Corollary 4.26. Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over $G$ which are (Morita) equivalent in the sense of [2]. Then for every normal subgroup $N \subset G$ the $C^*$-algebras $C^*_N(\mathcal{A})$ and $C^*_N(\mathcal{B})$ are Morita equivalent.

Proof. Let $\mathcal{X}$ be an $\mathcal{A}-\mathcal{B}$-equivalence bundle and form the linking bundle $\mathcal{L}(\mathcal{X})$ as explained in [3]. Then both $\mathcal{A}$ and $\mathcal{B}$ are hereditary Fell subbundles of $\mathcal{L}(\mathcal{X})$, so both $\mathcal{A}_N$ and $\mathcal{B}_N$ are hereditary in $C^*_N(\mathcal{L}(\mathcal{X}))$ and we may identify $C^*_N(\mathcal{A}) \cong q^L_N(C^*(\mathcal{A}))$ and $C^*_N(\mathcal{B}) \cong q^L_N(C^*(\mathcal{B}))$. Define, as in [4], $C^*(\mathcal{X})$ as the closure of $L^1(\mathcal{X}) \subset L^1(\mathcal{L}(\mathcal{X}))$ in $C^*(\mathcal{L}(\mathcal{X}))$. Then $C^*(\mathcal{X})$ is a $C^*(\mathcal{A})-C^*(\mathcal{B})$-equivalence bimodule with bimodule structure inherited from the canonical $C^*(\mathcal{L}(\mathcal{X}))-C^*(\mathcal{L}(\mathcal{X}))$-equivalence module structure of $C^*(\mathcal{L}(\mathcal{X}))$. Since $q^L(\mathcal{X}) : C^*(\mathcal{L}(\mathcal{X})) \to C^*_N(\mathcal{L}(\mathcal{X}))$ is a surjective and a morphism of $C^*$-algebras, it follows that $C^*_N(\mathcal{X}) := q^L_N(C^*(\mathcal{X}))$ is a $C^*_N(\mathcal{A})-C^*_N(\mathcal{B})$-equivalence bimodule.

We close this article with a Corollary relating the weak approximation property (WAP) of [2] and our constructions. This last result is intended to give an idea of how to combine our characterization $C^*_N(\mathcal{B}) = C^*_r(C^*(\mathcal{B},N))$ with the WAP. The reader may use the ideas we expose to produce other results of this sort.

Corollary 4.27. Let $\mathcal{B}$ be a Fell bundle over a discrete group $G$ and $N \subset G$ a normal subgroup. Then the following are equivalent.

1. $C^*(\mathcal{B})$ is nuclear.
2. $C^*_N(\mathcal{B})$ is nuclear.
3. $C^*_r(\mathcal{B})$ is nuclear.
4. $\mathcal{B}$ has the WAP and $B_e$ is nuclear.
5. $C^*(\mathcal{B}_N)$ is nuclear and $C^*(\mathcal{B},N)$ has the WAP.
6. $C^*_r(\mathcal{B}_N)$ is nuclear and $C^*_r(\mathcal{B},N)$ has the WAP.
7. Both $\mathcal{B}_N$ and $C^*(\mathcal{B},N)$ have the WAP and $B_e$ is nuclear.
8. Both $\mathcal{B}_N$ and $C^*_r(\mathcal{B},N)$ have the WAP and $B_e$ is nuclear.

If the conditions above hold, then both $\mathcal{B}$ and $\mathcal{B}_N$ are amenable.

Proof. Assume (1) holds, then (2) does so because $C^*_N(\mathcal{B})$ is a quotient of a nuclear $C^*$-algebra. Recalling that $C^*_r(\mathcal{B}) = C^*_r(\mathcal{B})$ is a quotient of $C^*_N(\mathcal{B})$, we conclude (2) implies (3). If (3) holds, then the existence of a conditional expectation from $C^*_r(\mathcal{B})$ to $B_e$ [5] implies $B_e$ is nuclear and we then can use [2, Proposition 7.3] to conclude $\mathcal{B}$ has the WAP. The same Proposition can be used to prove (4) implies (1), then the the first four claims are equivalent. This equivalence together with the fact that $C^*(\mathcal{B})$ is $C^*$-isomorphic to $C^*(C^*(\mathcal{B},N))$ (and $C^*_r(\mathcal{B})$ to $C^*_r(C^*(\mathcal{B},N))$) implies the first six claims are equivalent. Finally, this last equivalence can be used to prove all the eight claims are equivalent. The proof ends after one recalls from [2] that the WAP implies amenability. □

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