Selecting Gauge Theories on an Interval
by 5D Gauge Transformations

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Gauge symmetry breaking by boundary conditions is studied in a general warped geometry in five dimensions. It has been suggested that a wider class of boundary conditions is allowed by requiring only vanishing surface terms when deriving the field equations for gauge theories on an interval (i.e., employing a variational principle), in comparison to the twist in orbifolding with automorphisms of the Lie algebra. We find that there are classes of boundary conditions allowed by the variational principle which violate the Ward-Takahashi identity and give four-point tree amplitudes that increase with energy in channels that have not yet been explored, leading to cross sections that increase as powers of the energy (which violates the tree level unitarity). We also find that such boundary conditions are forbidden by the requirement that the definitions of the restricted class of five-dimensional (5D) gauge transformations be consistent.

§1. Introduction

In models with extra dimensions, there are various possibilities for gauge fields. The initial proposal for the large extra dimensions assumes that all particles in the standard model are localized on a brane with a four-dimensional (4D) world volume.1)–4) Formulating the localization of gauge fields on a wall is a challenging problem, which can be realized in certain models.5),6) However, other interesting possibilities arise if the gauge fields are propagating in the higher-dimensional bulk spacetime. The extra-dimensional component of gauge fields can act as a Higgs scalar field to break the gauge symmetry.7) The Wilson line dynamics can provide another source of gauge symmetry breaking, namely, the Hosotani mechanism.8) If the extra dimensions are compactified on a topologically nontrivial manifold, such as $S^1$, twisting can be realized, and the Scherk-Schwarz symmetry breaking mechanism thereby appears.9) The key to these mechanisms can be summarized as a nontrivial holonomy along a nontrivial cycle, which can also be understood as vacuum expectation values of adjoint scalar fields coming from gauge field components along the extra dimensions. If orbifolds are introduced, one can also impose boundary conditions at the fixed points of the orbifold to break all or part of a gauge group, usually using the automorphisms of the Lie algebra.10)–15) Combined with the Wilson lines, the orbifold models have recently gained much attention.16)–20) A class of boundary conditions wider than the orbifolding with automorphisms has been pursued to obtain more realistic models, in particular, to reduce the rank of the gauge group.12) One notable proposal is to consider gauge theories on an interval and to require

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that the surface terms must vanish in order for the variational principle\textsuperscript{21}) to give field equations. By imposing boundary conditions, part of the five-dimensional (5D) gauge invariance is explicitly broken, although the 5D gauge invariance is preserved in the bulk. It is commonly believed that in 4D, gauge invariance is vital to guarantee the Ward-Takahashi identity and unitarity. Therefore it is important to determine whether or not the wider range of boundary conditions allowed by the variational principle introduces difficulties in regard to these matters. It should also be useful to examine if a certain restricted class of 5D gauge transformations (with gauge transformation functions restricted by certain boundary conditions) can be consistently defined as the invariance of the theories with boundaries.

Higgs bosons are needed in 4D to cancel terms that increase with energy in the scattering amplitudes of longitudinal massive gauge bosons.\textsuperscript{23}) The unitarity relation between $\langle n |$ and $| n \rangle$ corresponding to a single state,

$$- i\langle n |(T - T^\dagger)| n \rangle = \sum_m \langle n|T^\dagger|m\rangle\langle m|T|n\rangle, \quad (1.1)$$

gives the imaginary part of the elastic scattering amplitude as a sum of cross sections of various channels, including elastic ($m = n$) and inelastic ones ($m \neq n$). Any two-to-two elastic or inelastic scattering amplitudes that increase with energy lead to an increasing elastic scattering amplitude, because the contributions of each channel in Eq.(1.1) are nonnegative. In this case, the unitarity bound for the elastic scattering amplitude is violated. This is called tree level unitarity, which can be tested even in higher-dimensional gauge theories. It has been shown that scattering amplitudes that increase with energy are canceled by the exchange of Kaluza-Klein (KK) gauge bosons in the higher-dimensional gauge theories compactified on the torus $S^1$, and thus the tree level unitarity is maintained, in spite of absence of the explicit Higgs scalars.\textsuperscript{24}) Even with the wider class of boundary conditions allowed by the variational principle, it has been shown that the contributions of the elastic scattering of longitudinal massive gauge bosons in the same KK excitation level do not increase with energy. Therefore, these boundary conditions pass the consistency test of the tree level unitarity, at least for the contributions from elastic scattering.\textsuperscript{21})

The purpose of this paper is to point out that the variational principle allows certain simple classes of boundary conditions which violate the Ward-Takahashi identity and the tree level unitarity.\textsuperscript{**} In contrast to previous works,\textsuperscript{13,21,24,31,32} we compute scattering amplitudes in various channels, including the massless modes and the KK modes at various levels, in particular inelastic scattering amplitudes.

\textsuperscript{**} The authors have learned that Masaharu Tanabashi also knew of examples of boundary conditions which are allowed by the variational principle, and violate the equivalence theorem, and consequently tree level unitarity.
involving different excitation levels. We also show that the condition that the definition of the restricted class of 5D gauge transformations be consistent forbids such classes of boundary conditions.

The 5D gauge transformations have been examined by consistently gauging the global symmetry. In 4D, BRST invariance is useful as a sophisticated formulation of gauge theories after the gauge fixing. The BRST approach for higher-dimensional gauge theories has been used to study the deconstruction approach and orbifold models. We find that the gauge transformations to select boundary conditions can be consistently defined in terms of the BRST formulation as well.

In §2.1, the variational principle is briefly reviewed. In §2.2, we compute the scattering amplitudes and find that terms that violate the Ward-Takahashi identity and the tree level unitarity can be reduced to functions of the values of the mode functions at the boundaries. In §2.3, we demonstrate that the variational principle allows boundary conditions that violate both the Ward-Takahashi identity and unitarity. We refer to such boundary conditions as the coset-N/subgroup-D boundary conditions. In §3, we show that the coset-N/subgroup-D boundary conditions do not allow a consistent definition of a restricted class of 5D gauge transformations. We also discuss the BRST formulation. Useful formulas for mode functions are summarized in Appendix A. Some details of the computation of scattering amplitudes are given in Appendix B.

§2. Variational principle and scattering amplitudes

2.1. Boundary conditions obtained from the variational principle

To a good approximation, our 4D spacetime is flat (except for a very small positive cosmological constant). Assuming flat 4D slices in 5D spacetime, we parameterize the generic metric in terms of two arbitrary functions, $W(y)$ and $g_{55}(y)$, of the extra-dimensional coordinate $x^5 \equiv y$:

$$ds^2 = g_{MN}dx^M dx^N = e^{-4W(y)}\eta_{\mu\nu}dx^\mu dx^\nu + g_{55}(y)dy^2. \quad (2.1)$$

Here, the upper-case Latin indices $M, N, \cdots = 0, 1, 2, 3, 5$ run over the five spacetime dimensions, $g_{MN}$ is the 5D metric, and the 4D spacetime is flat: $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$ with $\mu, \nu = 0, 1, 2, 3$. We assume the extra dimension to consistent of an interval, $0 \leq y \leq \pi R$, and consider appropriate boundary conditions at $y = 0$ and $y = \pi R$.

As a simple illustrative example, we consider pure gauge theory, and choose the $SU(N)$ gauge group in the case that we need to specify the gauge group. Introducing

\footnote{Because the problem of defining the gauge transformations arises at the tree level rather than the loop level, using the BRST formulation in selecting the boundary conditions is equivalent to using the classical gauge transformations.}

\footnote{The extra-dimensional coordinate $y$ can be reparameterized such that either one of the two positive functions $e^{-4W(y)}$ or $g_{55}(y)$ is constant; for example, we could choose $g_{55}(y) = 1$. However, we leave these two functions unfixed, in order to to accommodate various coordinate systems.}
the gauge fixing term with the parameter $\xi$, we obtain the action in the $R_\xi$ gauge as

$$S + S_{GF} = \int d^4x \int_0^{\pi R} dy \sqrt{-g(y)} \left( -\frac{1}{4} F_{MN}^a F_{PQ}^a g^{MP} g^{NQ} - \frac{1}{2\xi} (G^a)^2 \right).$$  \hspace{1cm} (2.2)

The field strength is given by $F_{MN}^a = \partial_M A_{N}^a - \partial_N A_{M}^a + g_5 f^{abc} A_M^b A_N^c$, where $f^{abc}$ is the structure constant of the gauge group, and $g_5$ is the 5D gauge coupling. We choose the gauge fixing function $G^a$ so as to cancel the cross terms between the 4D components $A_{\mu}^a$ and the extra-dimensional component $A_5^a$ (4D scalar):

$$G^a = e^{4W} \left( \eta^{\mu\nu} \partial_\mu A_5^a + \frac{1}{\sqrt{g_5}} \partial_5 e^{-4W} \frac{1}{\sqrt{g_5}} A_5^a \right).$$  \hspace{1cm} (2.3)

We omit the corresponding ghost fields, as they do not appear in tree level scattering amplitudes.

Let us briefly review the variational principle used to determine the boundary conditions.\hspace{1cm} (2.2) The action (2.2) should realize its minimum with respect to variation about the configuration satisfying the field equations in the bulk. To obtain the field equations, we must perform integrations by parts, which results in the boundary terms

$$(\delta S + \delta S_{GF})_{\text{boundary}} = -\int d^4x \left[ e^{-4W} \sqrt{g_5} \eta^{\mu\nu} F_{\mu\nu}^a A_5^a + e^{-8W} \sqrt{g_5} G^a \delta A_5^a \right]_{y=\pi R} - \left[ e^{-4W} \sqrt{g_5} \eta^{\mu\nu} F_{\mu\nu}^a A_5^a + e^{-8W} \sqrt{g_5} G^a \delta A_5^a \right]_{y=0}. \hspace{1cm} (2.4)$$

These boundary terms must vanish for the variational principle to be well-defined. The simplest way to satisfy this condition is to use the Neumann, $\partial_5 A_{\mu}^a|_{y=0} = 0$, or Dirichlet, $A_{\mu,5}^a|_{y=0} = 0$, boundary conditions. It has been found that the following three forms of the Neumann and/or Dirichlet boundary conditions provide the solution:

$$A_{\mu}^0|_{y=0} = 0, \quad A_{\mu}^0|_{y=\pi R} = 0, \hspace{1cm} (2.5)$$

$$A_5^a|_{y=0} = 0, \quad \partial_5 e^{-4W} \sqrt{g_5} A_5^a|_{y=0} = 0, \hspace{1cm} (2.6)$$

$$\partial_5 A_{\mu}^a|_{y=0} = 0, \quad A_5^a|_{y=0} = 0. \hspace{1cm} (2.7)$$

The Neumann conditions $\partial_5 A_{\mu}^a|_{y=0} = 0$ for the 4D vector $A_{\mu}^a$ at both boundaries, $y = 0$ and $\pi R$, are necessary for the existence of a zero mode with the color $a$ in the low-energy effective theory. It is also known that the choice of these three types of boundary conditions can be made independently for each color $a$ and for the 4D vector $A_{\mu}^a$ and the 4D scalar $A_5^a$. The 4D Lorentz invariance also implies that the gauge generators of the 4D massless gauge fields $A_{\mu}^a$ form a group, as argued in Ref.\hspace{1cm} (2.1):

$$\partial_5 A_{\mu}^a|_{y=0} = 0, \quad a \in H \subseteq G, \quad y = 0, \pi R, \hspace{1cm} (2.8)$$

where $H$ is a (sub)group of the gauge group $G$.\n
In the case of the second choice of the boundary conditions, (2.6), and the third choice, (2.7), mode functions for $A_\mu$ and $A_5$ satisfy simple relations, such as (A.9) in Appendix A. This simplicity plays an important role in the investigation of possible violations of the Ward-Takahashi identity and unitarity. In the case of the first choice, (2.5), the mode functions for $A_\mu$ and $A_5$ satisfy a different relation, and this results in the necessity of a separate treatment. Therefore, we consider the second and third choices, (2.6) and (2.7), in this section and discuss the first choice, (2.5), only briefly in §3. We denote the Neumann or Dirichlet boundary conditions for the 4D vector $A^a_\mu$ at each boundary with the color $a$ as $D(a)$. (The opposite boundary conditions for the scalar $A^a_5$ are implied.) Using these boundary conditions, we can obtain the $n$-th mode functions $f_n^{D(a)}(y)$ and $g_n^{D(a)}(y)$ for the vector $A^a_\mu(x, y)$ and scalar $A^a_5(x, y)$, respectively, and can decompose them into the KK effective 4D fields $A^a_\mu(x)$ and $A^a_5(x)$ as

$$A^a_\mu(x, y) = \sum_{n=0}^{\infty} A^a_{\mu n}(x)f_n^{D(a)}(y), \quad (2.9)$$

$$A^a_5(x, y) = \sum_{n=0}^{\infty} A^a_{5n}(x)g_n^{D(a)}(y). \quad (2.10)$$

Important properties of the mode functions are given in Appendix A.

2.2. Scattering amplitudes

Now, we examine scattering amplitudes for various choices of Dirichlet and/or Neumann boundary conditions at each boundary and for each color. We consider the gauge boson scattering $A^a_nA^b_m \rightarrow A^c_lA^d_m$, where the subscripts $l, n$ and $m$ indicate the KK levels, and the superscripts $a, b, c$ and $d$ indicate the colors, as illustrated in Fig. 1. For simplicity, we choose the KK level and boundary conditions for the gauge bosons with the colors $b$ and $d$ identical. In the center-of-mass frame, the scattering angle and the total energy are denoted as $\theta$ and $E = E_n + E_m$, respectively. To examine the possible growth of scattering amplitudes at a fixed scattering angle as the total energy $E$ increase, we choose polarization vectors $\epsilon$ for gauge bosons to be longitudinal, except for the gauge boson $A^b_l$, as tabulated in Table I. We examine both longitudinal and transverse polarizations and massive as well as massless modes.

![Fig. 1. $A^a_nA^b_m \rightarrow A^c_lA^d_m$](image-url)
for $A_0^f$. Some details of the scattering amplitude calculation with arbitrary gauge


| $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|-------|-------|-------|-------|
| $(E_n,0,0,p)$ | $(E_m,0,0,-p)$ | $(E_l,p^*\sin \theta,0,p^*\cos \theta)$ | $(E_m,-p^*\sin \theta,0,-p^*\cos \theta)$ |
| $\epsilon(p_1) = (p/M_n,0,0,E_n/M_m)$ | $\epsilon(p_2) = (p/M_m,0,0,-E_m/M_m)$ | $\epsilon(p_3) = \text{Eq. (2.15) for transverse or Eq. (2.18) for longitudinal}$ | $\epsilon(p_4) = (p^*/M_m,-E_m^*/\sin \theta/M_m,0,-E_m^*/\cos \theta/M_m)$ |

parameters $\xi$ are given in Appendix B. To test the tree level unitarity, we study the scattering amplitude at energies high enough that the total energy $E$ is much larger than any mass $M_k$ for the intermediate states as well as the external states. Leaving the polarization $\epsilon(p_3)$ for $A_0^f$ unspecified, the terms that increase with $E$ in the invariant matrix element for the gauge parameter $\xi \ll E^2/M_k^2$ are given by

$$
\begin{align*}
ig^2 f^{abc} f^{cde} \frac{E^3}{8M_nM_m} & \epsilon^*(p_3) \left\{ F^{D(cde)}_{lmk} F^{D_0(abc)}_{nmk} A^\mu \\
+ \frac{2}{E^2} \left( M_l T^D_{mkl} - M_m T^D_{lkm} \right) \left( M_n T^D_{mkn} - M_m T^D_{nkm} \right) (0, -\sin \theta, 0, 1 - \cos \theta)^\mu \right\}, \\
\end{align*}
$$

(2.11)

with the four-vector $A^\mu$ whose components are $A^\mu=2 = 0$ and

$$
\begin{align*}
A^0 &= 1 + \frac{3M_n^2 - 3M_m^2 + 2M_l^2 - 4M_k^2}{E^2} - \left( 1 + \frac{M_n^2 + M_m^2}{E^2} \right) \cos \theta, \\
A^1 &= \left( 1 + \frac{M_n^2 - 7M_m^2 - 2M_l^2 + 2M_k^2}{E^2} \right) \sin \theta \\
&\quad - \left( 1 + \frac{M_n^2 + M_m^2 - 2M_l^2}{E^2} \right) \sin \theta \cos \theta, \\
A^3 &= \frac{2M_n^2 + 4M_m^2 + 2M_l^2 - 6M_k^2}{E^2} + \left( 1 + \frac{M_n^2 - 7M_m^2 - 2M_l^2 + 2M_k^2}{E^2} \right) \cos \theta \\
&\quad - \left( 1 + \frac{M_n^2 + M_m^2 - 2M_l^2}{E^2} \right) \cos^2 \theta.
\end{align*}
$$

(2.12, 2.13, 2.14)

Here, we have ignored terms which do not increase with energies [to $O((E/M_k)^0)$]. The overlap functions $F^{D(cde)}_{lmk}$ and $T^D_{mkl}$ are given in Eq. (A.8) in Appendix A.

Let us consider the case with a massive mode for the external $A_0^f$ boson. Then, the transverse polarization is given by

$$
\epsilon_\mu(p_3) = (0, \cos \theta, 0, -\sin \theta) \quad \text{or} \quad (0, 0, 1, 0).
$$

(2.15)

The latter polarization gives a vanishing result upon multiplication by $A^\mu$. For the former polarization, the invariant matrix element (2.11) becomes

$$
-ig^2 f^{abc} f^{cde} \frac{E \sin \theta}{4M_nM_m^2} K,
$$

(2.16)
where
\[
\mathcal{K} = \sum_k \left\{ (M_n^2 + 2M_m^2 - 3M_k^2 + M_t^2) F_{nmk}^{D(cde)} F_{nml}^{D(abe)} + (M_l T_{nml}^{D(dec)} - M_m T_{nkl}^{D(ced)}) \left( M_n T_{mkn}^{D(boa)} - M_m T_{nlm}^{D(amb)} \right) \right\}
\]
\[
= -3 \sum_k \left[ \frac{e^{-4W}}{\sqrt{55}} g_k \left( f_m^{D(b)} f_n^{D(c)} \right) \right]^{\pi R}_0 \left[ \frac{e^{-4W}}{\sqrt{55}} g_k \left( f_m^{D(d)} f_n^{D(a)} \right) \right]^{\pi R}_0 + 2 \left[ \frac{e^{-4W}}{\sqrt{55}} \left( f_n^{D(a)} f_m^{D(b)} f_m^{D(d)} f_l^{D(c)} \right) \right]^{\pi R}_0 .
\] (2.17)

For the longitudinal polarization, which is given by
\[
\epsilon^*(p_3) = (p'_3/M_l, E_l \sin \theta/M_l, 0, E_l \cos \theta/M_l)
\]
the invariant matrix element is
\[
-ig_5^2 f^{abc} f^{cde} E^2 (1 - \cos \theta) \frac{\pi R}{8 M_n M_m^2 M_t} \mathcal{K}.
\] (2.19)

The amplitude that increases with energy has the coefficient \(\mathcal{K}\) for both transverse and longitudinal polarizations. We emphasize that the coefficient \(\mathcal{K}\) in Eq. (2.17) reduces to a function of the values of the mode functions at the boundaries. Therefore, it is directly determined by the choices of the boundary conditions. If the coefficient \(\mathcal{K}\) of the elastic (inelastic) scattering amplitudes does not vanish, the unitarity relation \((1.1)\) gives the imaginary part of the elastic amplitude that increases with energy and violates the unitarity bound.

Equation (2.16) is applicable also for \(l \to 0\). If we choose the zero mode \((l = 0)\) for the gauge boson \(A_c^l\), the scattering amplitude should satisfy the (on-shell) Ward-Takahashi identity: The amplitude must vanish if we substitute the momentum of the zero mode for the polarization vector \(\epsilon\). Making the substitution
\[
\epsilon^*(p_3) \to p_3 = (p'_3(1, \sin \theta, 0, \cos \theta),
\]
we obtain the amplitude \(\ref{eq2.11}\) as
\[
ig_5^2 f^{abc} f^{cde} (1 - \cos \theta) \frac{E^2}{8 M_n M_m^2} f^{D(e)}_{l=0} \left[ \frac{e^{-4W}}{\sqrt{55}} \left( f_m^{D(b)} f_m^{D(d)} f_n^{D(a)} - f_n^{D(a)} f_m^{D(b)} f_m^{D(d)} \right) \right]^{\pi R}_0 ,
\] (2.21)
which also reduces to a function of the values of the mode functions at the boundaries.

We thus find that the violation of both the Ward-Takahashi identity and the tree level unitarity are proportional to the functions given in \(\ref{eq2.21}\) and \(\ref{eq2.17}\), which are functions of the values of the mode functions at the boundaries. In the next subsection, we will present explicit examples of boundary conditions that are allowed by the variational principle and give nonvanishing functions \(\ref{eq2.21}\) that lead to the violation of the Ward-Takahashi identity and \(\ref{eq2.17}\) that lead to the violation of tree level unitarity.
2.3. The coset-N/subgroup-D boundary conditions

In order to illustrate boundary conditions that violate the Ward-Takahashi identity and tree level unitarity, we consider the $SU(N)$ gauge group as an example. To obtain a semi-realistic symmetry breaking pattern, let us consider the symmetry breaking in two steps: two different boundary conditions at $y = 0$ and $y = \pi R$, such as $SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)$. The warp factor $e^{-4W(y)}$ and $g_{55}(y)$ can provide two vastly different mass scales for the boundaries $y = 0$ and $y = \pi R$. Without loss of generality, let us take one of the boundaries, say $y = 0$, to be associated with the high energy scale, $M_{GUT}$. At this high energy scale, we break the gauge group as

$$SU(N) \rightarrow SU(N_1) \times SU(N - N_1) \times U(1) \quad (2.22)$$

by imposing the Neumann boundary conditions (2.7) for the 4D vectors $A_a^\mu$ in the subgroups $SU(N_1)$, $SU(N - N_1)$ and $U(1)$, and by imposing the Dirichlet boundary conditions (2.6) for the coset $SU(N)/[SU(N_1) \times SU(N - N_1) \times U(1)]$. The boundary conditions are written in matrix form as

$$(\begin{array}{c|c|c}
N & D & \hline 
D & N
\end{array})^{N_1}_{N - N_1} \quad \text{at} \quad y = 0,$$

where $N$ and $D$ denote the Neumann and Dirichlet boundary conditions, respectively. This set of boundary conditions violates neither the Ward-Takahashi identity nor tree level unitarity. It is realizable with automorphisms of the Lie algebra in orbifoldings and is often used. We denote the generators of the subgroups and the coset by

$$G \in SU(N_1), \quad W \in SU(N - N_1), \quad B \in U(1), \quad X \in SU(N)/[SU(N_1) \times SU(N - N_1) \times U(1)]. \quad (2.23)$$

Now we consider the boundary condition at $y = \pi R$, which is associated with the lower energy scale, $M_W$. In order to preserve gauge invariance with respect to $SU(N_1)$ and $U(1)$, we impose the Neumann boundary condition for the generators $G \in SU(N_1)$ and $B \in U(1)$ at both boundaries, $y = 0$ and $\pi R$. To break the gauge group $SU(N - N_1)$ by means of boundary conditions, we impose the Dirichlet boundary condition (2.6) for $W \in SU(N - N_1)$ generators.

The remaining generators belong to the coset $X \in SU(N)/[SU(N_1) \times SU(N - N_1) \times U(1)]$. Usually, the Neumann boundary conditions are not imposed on 4D vectors in the coset at both boundaries, because massless gauge fields have to form a group, as indicated in Eq.(2.8). We have the freedom of choosing the Neumann boundary conditions here, as it is clear that the 4D vectors with colors in the coset are massive at the high energy scale $M_{GUT}$ by the boundary condition at $y = 0$. Thus, we can assign the boundary condition in matrix form as

$$(\begin{array}{c|c|c}
N & N & \hline 
N & D
\end{array})^{N_1}_{N - N_1} \quad \text{at} \quad y = \pi R,$$

with unbroken $U(1)$ satisfying the Neumann boundary conditions. Let us refer to this assignment of the Neumann boundary conditions for coset and the Dirichlet
boundary conditions for a subgroup as the coset-N/subgroup-D boundary conditions. We show that these coset-N/subgroup-D boundary conditions violate both the Ward-Takahashi identity and tree level unitarity. We also show in §3 that this choice of boundary conditions is forbidden if we require the compatibility with the restricted class of 5D gauge transformations.

Nonvanishing values of Eqs. (2.16) and (2.19) violate tree level unitarity, whereas nonvanishing values of Eq. (2.21) violate the Ward-Takahashi identity. Note that all of them have a common factor of $f^{abc} f^{cde}$. Also note that our decomposition in Eq. (2.23) has the following commutation relations, as follows from the group structure

$$[G, G] = G, \quad [W, W] = W, \quad [X, X] = G + W + B.$$  (2.24)

This implies the following nonvanishing structure constants:

$$f^{GGG}, \quad f^{WWW}, \quad f^{XXG}, \quad f^{XXW}, \quad f^{XXX}.$$  (2.25)

Therefore, the nonvanishing group theory factor $f^{abc} f^{cde}$ can be classified into the following three types: among the labels $a, b, c, d$ and $e$ the coset generators $X$ appear nowhere, only in three labels, $(a, c, e)$, $(a, d, e)$, $(b, c, e)$ or $(b, d, e)$, or only in four labels, $a, b, c$ and $d$.

We first examine the violation of the Ward-Takahashi identity, which is given by Eq. (2.21) and is proportional to

$$f^{abc} f^{cde} f^{D(c)} e^{-4W} \sqrt{g_{55}} \sum_k \left[ e^{-4W} g_k D^{(e)} \right] \pi R \not= 0, \quad f^{D(a=W)} \not= 0, \quad f^{D(b=X)} \not= 0, \quad f^{D(c=B,G)} \not= 0, \quad f^{D(d=X)} \not= 0.$$  (2.26)

Let us consider the processes $WX \to BX$ and $WX \to GX$, with the zero mode $l = 0$ for the massless gauge bosons $B$ and $G$, by choosing the colors $(a, b, c, d) = (W, X, B, X)$ and $(W, X, G, X)$. We find that the coset generator $X$ can contribute to the intermediate state $e = X$ and that the coset-N/subgroup-D boundary conditions at $y = \pi R$ give

$$f^{WXX} f^{BX} \not= 0, \quad f^{D(a=W)} \not= 0, \quad f^{D(b=X)} \not= 0, \quad f^{D(c=B,G)} \not= 0, \quad f^{D(d=X)} \not= 0.$$  (2.27)

The first term in Eq. (2.26) does not vanish, whereas the second term vanishes for these boundary conditions. Therefore, the Ward-Takahashi identity is violated by the coset-N/subgroup-D boundary conditions in these processes, $WX \to BX$ and $WX \to GX$.

We next consider the coefficient $K$ in Eq. (2.17), which appears in Eqs. (2.16) and (2.19), for the violation of tree level unitarity. The coefficient $f^{abc} f^{cde} K$ consists of two pieces,

$$f^{abc} f^{cde} \sum_k \left[ e^{-4W} g_k D^{(e)} \right] \pi R \not= 0, \quad f^{D(a=W)} \not= 0, \quad f^{D(b=X)} \not= 0, \quad f^{D(c=B,G)} \not= 0, \quad f^{D(d=X)} \not= 0.$$  (2.28)
Fig. 2. Typical Feynman diagrams contributing to the violation (2.21) of the Ward-Takahashi identity and the violation (2.17) of tree level unitarity.

\[
\left. \frac{e^{-4W}}{\sqrt{g_{55}}} \left( f_{n}^{D(a)} f_{m}^{D(b)} f_{m}^{D(d)} f_{l}^{I(c)} \right) \right|_{0}^{\pi R} \tag{2.29}
\]

Let us choose the same processes, \(WX \to BX\) and \(WX \to GX\), as in the case of the violation of the Ward-Takahashi identity, without restriction to the zero mode for \(B\) and \(G\) in this case. For these processes, the first term, (2.28), vanishes, but the second term, (2.29), does not vanish because of Eq. (2.27). This establishes that the coset-N/subgroup-D boundary conditions violate tree level unitarity in the processes \(WX \to BX\) and \(WX \to GX\).

We illustrate typical Feynman diagrams in Fig. 2 that contribute to the violation (2.21) of the Ward-Takahashi identity and the violation (2.17) of tree level unitarity. The colors of the external gauge bosons with momentum assignments in Fig. 1 are chosen as follows. The gauge boson with momentum \(p_1\) is assigned to \(W \in SU(N - N_1)\), those with \(p_2, p_4\) to \(X \in SU(N)/[SU(N_1) \times SU(N - N_1) \times U(1)]\), and that with \(p_3\) to \(B \in U(1)\) or \(G \in SU(N_1)\). We can choose the color of the internal gauge bosons to be \(X \in SU(N)/[SU(N_1) \times SU(N - N_1) \times U(1)]\).

Thus, we conclude that the coset-N/subgroup-D boundary conditions give a four gauge boson scattering amplitude that increases as powers of the center-of-mass energies. More specifically, this breakdown of tree level unitarity occurs if we take one of the external gauge bosons in a subgroup with the Dirichlet boundary conditions, the other two gauge bosons in the coset with the Neumann boundary conditions, and another gauge boson in an unbroken subgroup with the Neumann boundary conditions. The Ward-Takahashi identity is also broken with this choice of the boundary conditions.

Note that we should not use the unitary gauge in the calculation of the tree diagram. As we have used the condition \(\xi \ll E^2/M_k^2\) in computing the scattering amplitude, our calculation is not valid for the unitary gauge, for which \(\xi \to \infty\). The part of the scattering amplitude that increase with the energy does not depend on the gauge parameter \(\xi\) for finite values of \(\xi\). For the unitary gauge, i.e. \(\xi = \frac{e^{-4W}}{\sqrt{g_{55}}} \left( f_{n}^{D(a)} f_{m}^{D(b)} f_{m}^{D(d)} f_{l}^{I(c)} \right) \right|_{0}^{\pi R} \)

\(^{*)\) The first term, (2.28), is nonvanishing with the coset-N/subgroup-D boundary conditions if we consider the process \(XX \to XX\). In order to ascertain whether tree level unitarity is indeed violated in this process, we need to take account of another s-channel Feynman diagram, in addition to those computed in Appendix B. We compute inelastic scattering \(XX \to XX\) with different KK levels for all \(X\), in contrast to Ref. 21, where only the elastic scattering is computed with identical KK levels for all gauge bosons \(X\).
∞, the extra-dimensional component of the gauge boson drops out, as seen from the propagator \[B.2\]. Then the breakdown of tree level unitarity and the Ward-Takahashi identity seems to be invisible, even for the choice of the coset-N/subgroup-D boundary conditions.

The breakdown of tree level unitarity and the Ward-Takahashi identity is a serious problem for gauge theories in higher dimensions. For this reason, we look for an additional requirement which forbids such a choice of boundary conditions that violate tree level unitarity and the Ward-Takahashi identity in the scattering amplitudes. It is most likely that this problem is related to the gauge transformation property of gauge fields near the boundary. We examine the 5D gauge transformations in the next section.

§3. 5D gauge transformations and boundary conditions

5D gauge transformations

Let us consider 5D gauge transformations of the 4D vector component \(A^a_\mu(x, y)\), which are expressed as

\[
\delta A^a_\mu(x, y) = \partial_\mu \epsilon^a(x, y) + g_5 f^{abc} A^b_\mu(x, y) \epsilon^c(x, y).
\]  

(3.1)

The first term on the right-hand side is inhomogeneous in the gauge field and is independent of the gauge coupling constant \(g_5\). The second nonlinear term is first order in \(g_5\). In the bulk, the ordinary 5D gauge invariance is preserved, and the transformation function \(\epsilon^a\) is an arbitrary function of \(x\) and \(y\). By contrast, on the boundaries, \(\epsilon^a\), as well as \(A^a_\mu\) and \(A^a_5\), is subject to certain boundary conditions. Only if appropriate boundary conditions are chosen is the theory invariant under the transformation (3.1), with the transformation function restricted by the boundary conditions. We call such transformations the restricted class of 5D gauge transformations.

We first examine the relation between the boundary conditions for \(A^a_\mu\) and \(A^a_5\) to leading order (i.e., in the limit of a vanishing gauge coupling, \(g_5 \to 0\)). The first (inhomogeneous) term on the right-hand side of Eq. (3.1) implies that the gauge transformation function \(\epsilon^a(x, y)\) has the same boundary condition as the corresponding 4D vector component, \(A^a_\mu(x, y)\). The gauge transformation of the 4D scalar component \(A^a_5(x, y)\) is given by

\[
\delta A^a_5(x, y) = \partial_5 \epsilon^a(x, y) + g_5 f^{abc} A^b_5(x, y) \epsilon^c(x, y).
\]  

(3.2)

The first (inhomogeneous) term implies that the boundary condition for \(A^a_5(x, y)\) is the same as \(\partial_5 \epsilon^a\). In other words, the Neumann (Dirichlet) boundary conditions for \(A^a_5(x, y)\) imply the Dirichlet (Neumann) boundary conditions for \(\epsilon^a(x, y)\).

\* In §3 we show that the 5D gauge transformations are ill-defined if the coset-N/subgroup-D boundary conditions are used. Therefore, the different “gauge choice” may not give the same physical results.
First choice of the boundary conditions, (2.5)

We can now examine the first choice of the boundary conditions, (2.5). The restricted class of 5D gauge transformations for \( A^a_\mu \) in Eq. (3.1) implies that the boundary conditions for \( A^a_\mu \) and for the gauge transformation function \( \epsilon^a \) are the same, whereas the restricted class of 5D gauge transformations for \( A^a_5 \) in Eq. (3.2) implies that the boundary conditions for \( A^a_5 \) are opposite to those for \( \epsilon \). Therefore, the restricted class of 5D gauge transformations (at the order of the inhomogeneous term) implies that the boundary conditions for \( A^a_\mu \) and \( A^a_5 \) are opposite. The first choice of boundary conditions, (2.5), allowed by the variational principle contradicts the gauge transformation properties of \( A^a_\mu (x, y) \) and \( A^a_5 (x, y) \). Specifically, the first choice of boundary conditions does not allow a consistent definition of the 5D gauge transformation functions \( \epsilon^a (x, y) \).

Coset-N/subgroup-D boundary conditions

Next, let us study whether the coset-N/subgroup-D boundary conditions considered in § 2.3 are consistent with the restricted class of 5D gauge transformations (3.1) and (3.2). We need to examine the nonlinear terms with the structure constant \( f^{abc} \) in these gauge transformations. The group structure given in Eq. (2.25) implies the following. If \( a \) is in a subgroup, either both \( b \) and \( c \) belong to the subgroup or both belong to the coset. If \( a \) is in a coset, then either \( b \) or \( c \) belongs to the subgroup and the other to the coset.

Let us consider the restricted class of 5D gauge transformations for the generators in the coset \( X \in SU(N)/[SU(N_1) \times SU(N - N_1) \times U(1)] \) with the Neumann boundary conditions for \( A^a_\mu (x, y) \) (and the Dirichlet boundary conditions for \( A^a_5 (x, y) \)). The nonlinear term \( g_5 f^{abc} A^b_\mu \epsilon^c \) in the restricted class of 5D gauge transformations \( \delta A^a_\mu \) in Eq. (3.1) for the 4D vector contains contributions with \( A^b_\mu \) in the subgroup (coset) and \( \epsilon^c \) in the coset (subgroup). The boundary conditions for \( A^b_\mu \) and \( \epsilon^c \) with the same color \( b \) are the same, whereas the generators in the subgroup and in the coset have opposite boundary conditions. Hence, one of the following two cases is realized:

\[
A^b_\mu (x, y = \pi R) = 0, \quad \partial_5 \epsilon^c (x, y = \pi R) = 0, \quad (3.3)
\]

\[
\partial_5 A^b_\mu (x, y = \pi R) = 0, \quad \epsilon^c (x, y = \pi R) = 0. \quad (3.4)
\]

Then the nonlinear term \( g_5 f^{abc} A^b_\mu \epsilon^c \) consists of a product of functions with the Dirichlet and the Neumann boundary conditions:

\[
\partial_5 \left( A^b_\mu (x, y) \epsilon^c (x, y) \right) \bigg|_{y=\pi R} = \partial_5 A^b_\mu (x, y = \pi R) \epsilon^c (x, y = \pi R) + A^b_\mu (x, y = \pi R) \partial_5 \epsilon^c (x, y = \pi R) \neq 0. \quad (3.5)
\]

Since this nonlinear term does not satisfy the Neumann boundary conditions, the boundary conditions for \( A^a_\mu \) are not satisfied.

\[^{35}\] The same conclusion was reached previously with a different argument using the unitary gauge.
Similarly, we can also examine the restricted class of 5D gauge transformations for the generators in the subgroup $W \in SU(N - N_1)$ with the Neumann boundary conditions for $A_5^a(x, y)$ (and the Dirichlet boundary conditions for $A_5^a(x, y)$). The nonlinear term $g_5 f^{abc} A_5^b \epsilon^c$ in the restricted class of 5D gauge transformations $\delta A_5^a$ in Eq. (3.2) for 4D scalar contains contributions from $b$ and $c$ in the coset. The boundary conditions in the coset are Neumann for the gauge function $\epsilon$ and Dirichlet for the 4D scalar $A_5: A_5^a(x, y = \pi R) = 0$, $\partial_5 \epsilon(x, y = \pi R) = 0$. Therefore, the product does not satisfy the Neumann boundary condition.

In both cases, there is no way to define the gauge transformation function $\epsilon^a(x, y)$ consistently. Therefore, the coset-N/subgroup-D boundary conditions do not allow a consistent definition of the restricted class of 5D gauge transformations.

Mode expansions and 4D gauge symmetry

To identify the 4D gauge symmetry, it is useful to perform mode expansions for the 5D gauge transformation functions $\epsilon^a(x, y)$ as well as the 5D gauge fields $A_5^a(x, y)$. Once the boundary conditions for the gauge transformation functions and the gauge fields are defined consistently, we can perform mode expansions of the restricted class of 5D gauge transformations of the 4D vector component $A_5^a(x, y)$ in Eq. (3.1) as

$$
\sum_{n=0}^{\infty} \delta A_5^a(x) f_n^{D(a)}(y) = \sum_{n=0}^{\infty} \partial_\mu \epsilon^a_n(x) f_n^{D(\mu)}(y)
+ g_5 f^{abc} \sum_{n=0}^{\infty} A_5^b(x) f_n^{D(b)}(y) \sum_{m=0}^{\infty} \epsilon^c_m(x) f_m^{D(c)}(y). \quad (3.6)
$$

If the zero mode is present for the gauge transformation function $\epsilon^a(x, y)$, the 4D gauge invariance is maintained, and the corresponding 4D gauge field $A_5^a(x)$ contains a massless mode. If the zero mode is absent for the gauge transformation function $\epsilon^a(x, y)$, the 4D gauge invariance is broken, and the modes of the corresponding 4D gauge field $A_5^a(x)$ are all massive with no zero mode.

Orbifoldings

We now confirm that the boundary conditions realized as the (inner) automorphism of the Lie algebra in orbifoldings are consistent with the restricted class of 5D gauge transformations. We refer to the automorphism at the boundary $y = 0$ as $P_0$

$$
A_\mu(x, -y) = P_0 A_\mu(x, y) P_0^{-1}, \quad (3.7)
$$

with $P_0 = P_0^\dagger = P_0^{-1}$. Similarly, the automorphism $P_1$ is defined at $y = \pi R$. To obtain the Dirichlet or Neumann boundary conditions, we restrict ourselves to diagonal matrices for the automorphisms. Then we never obtain boundary conditions

---

*More generally, Dirichlet boundary conditions for $A_5^a$ in the (non-Abelian) subgroup give a similar inconsistency for $\delta A_5^a$, although it does not show up in Eqs. (2.17) and (2.21) for the 2-to-2 elastic (inelastic) scattering amplitudes.*
that correspond to the reduction of the rank of the group. 

Combining the automorphism with the gauge transformation

$$\delta A_\mu(x, -y) = \partial_\mu \epsilon(x, -y) - ig_5 [A_\mu(x, -y), \epsilon(x, -y)],$$

we find that the gauge transformation function transforms under the automorphism

$$\epsilon(x, -y) = P_0 \epsilon(x, y) P_0^{-1}.$$ 

On the other hand, the gauge transformation dictates the transformation of $A_5$ under the automorphism

$$\delta A_5(x, -y) = - \partial_5 \epsilon(x, -y) - ig_5 [A_5(x, -y), \epsilon(x, -y)],$$

$$A_5(x, -y) = - P_0 A_5(x, y) P_0^{-1}.$$ 

We can repeat the same analysis at $y = \pi R$. This shows that a consistent transformation property under the automorphism can be assigned for the gauge transformation functions. This consideration is in accord with the previous result: The automorphism of the Lie algebra in orbifolding satisfies tree level unitarity. 

**BRST formulation**

BRST transformations replace the classical gauge transformations after the gauge fixing. The BRST invariance is equivalent to the classical gauge invariance and gives no extra advantage at the tree level, which we are considering. We briefly consider the BRST formulation in order to see that we can obtain the same result as in the case of the gauge transformations. As long as the boundary conditions for the 5D gauge transformation function $\epsilon^a$ in

$$\delta A_M^a(x, y) = \partial_M \epsilon^a(x, y) + g_5 f^{abc} A_M^b(x, y) \epsilon^c(x, y)$$

are defined consistently with the boundary conditions for the gauge fields $A_M^a$, (the restricted class of 5D) BRST transformations can be defined straightforwardly by promoting the gauge transformation function $\epsilon^a(x, y)$ to the ghost field $c^a(x, y)$ multiplied by an anti-commuting parameter $\varepsilon$. We must also add the anti-ghost field $\bar{c}^a(x, y)$ and the Nakanishi-Lautrup field $B^a(x, y)$. The restricted class of 5D BRST transformations $\delta$ are given by

$$\varepsilon \delta A_M^a(x, y) = \varepsilon \left[ \partial_M \epsilon^a(x, y) + g_5 f^{abc} A_M^b(x, y) \epsilon^c(x, y) \right],$$

$$\varepsilon \delta c^a(x, y) = - \frac{1}{2} g_5 \varepsilon f^{abc} c^b(x, y) c^c(x, y),$$

$$\varepsilon \delta \bar{c}^a(x, y) = i \varepsilon B^a(x, y),$$

$$\varepsilon \delta B^a(x, y) = 0,$$

where the functions $c^a(x, y)$ satisfy the same boundary conditions as the corresponding $\epsilon^a(x, y)$. More precisely, here we are considering inner automorphisms of the Lie algebra excluding the possibility of outer automorphisms. 

---

\[\text{Footnotes:} 30) \text{Footnotes:} 31) \text{Footnotes:} 32)\]
Summary and prospects

Summarizing our results, we have examined the choice of the Neumann and Dirichlet boundary conditions to break the gauge symmetry in 5D gauge theory. The variational principle allows (coset-N/subgroup-D) boundary conditions that violate the Ward-Takahashi identity and tree level unitarity. We have observed that the 5D gauge transformation functions $\epsilon(x, y)$, in addition to the gauge fields $A_M(x, y)$, must be given boundary conditions that are consistent with the restricted class of 5D gauge transformations. This condition provides a stringent constraint and forbids the coset-N/subgroup-D boundary conditions.

In considering orbifoldings, it is possible to use nondiagonal automorphisms of the Lie algebra which give boundary conditions that are more involved than Neumann or Dirichlet conditions. More generally, there are possibilities of nontrivial Wilson lines which break the gauge group. It has been found that different boundary conditions are sometimes related by gauge transformations. Under such circumstances, the effective potential at the 1-loop level determines the value of the Wilson lines and the symmetry breaking pattern. The elucidation of such a possibility is an interesting problem.

We have considered pure gauge theory without matter fields in 5D. We need to re-analyze the boundary conditions in the case that there exist matter fields, especially if they are localized on the boundary.

The deconstruction approach employs 4D gauge theories to build higher-dimensional gauge models in a discretized version. Various boundary conditions, such as the automorphisms of the Lie algebra in the orbifolding, can be obtained as appropriate limits from the deconstruction. It has been shown that these boundary conditions satisfy tree level unitarity automatically. The deconstruction can provide realistic models if some amount of fine tuning is allowed. It is an interesting problem to determine whether boundary conditions that reduce the rank of the group can be forbidden as a limit of the discretized gauge theories as in the deconstruction approach.

It is also an interesting problem to build a realistic model using the 5D gauge theories with boundaries. Left-right symmetric models have been extensively studied in model building. It would be interesting to obtain a reduction of the rank of the gauge group with such a left-right symmetric model. Although they employ linear combinations of generators of the factor group, it should be examined carefully whether the rank reduction is indeed compatible with the restricted class of 5D gauge transformations.

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We choose them to be orthonormal,

\[ 1 \]

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**Appendix A**

**Formulas for Mode Functions and Overlap Integrals**

In this appendix, we summarize properties of the mode functions \( f_n^{D(a)}(y) \) and \( g_m^{D(b)}(y) \) and overlap functions constructed from them.

The mode functions \( f_n^{D(a)}(y) \) and \( g_n^{D(a)}(y) \) are defined by the eigenvalue equations with the mass eigenvalue \( M_n \) as

\[
\begin{align*}
\frac{1}{\sqrt{g_{55}}} \partial_5 e^{-4W} \partial_5 f_n^{D(a)}(y) &= M_n^2 f_n^{D(a)}(y), \\
\partial_5 \frac{1}{\sqrt{g_{55}}} e^{-4W} g_n^{D(a)}(y) &= M_n^2 g_n^{D(a)}(y),
\end{align*}
\]

which can be rewritten as the following coupled first-order differential equations

\[
\begin{align*}
\partial_5 f_n^{D(a)} &= M_n g_n^{D(a)}, \\
-\partial_5 \frac{1}{\sqrt{g_{55}}} e^{-4W} g_n^{D(a)} &= M_n f_n^{D(a)}.
\end{align*}
\]

We choose them to be orthonormal,

\[
\int_0^{\pi R} dy \sqrt{g_{55}} f_n^{D(a)} f_m^{D(a)} = \delta_{nm}, \quad \int_0^{\pi R} dy \sqrt{g_{55}} -1 e^{-4W} g_n^{D(a)} g_m^{D(a)} = \delta_{nm},
\]

and assume them to be complete

\[
\begin{align*}
\sum_n f_n^{D(a)}(y) f_n^{D(a)}(y') &= \frac{1}{\sqrt{g_{55}}} \delta(y - y'), \\
\sum_n g_n^{D(a)}(y) g_n^{D(a)}(y') &= e^{4W} \sqrt{g_{55}} \delta(y - y').
\end{align*}
\]

Using the KK decompositions in Eqs. (2.9) and (2.10) and integrating over \( y \), we obtain the action for the field strengths without the gauge fixing term in Eq. (2.22) as

\[
S = \int d^4x \left[ -\frac{1}{4} \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) (\partial^\mu A_\nu^a - \partial^\nu A_\mu^a) - \frac{1}{2} M_n^2 A_\mu^a A_\mu^a + M_n (\partial_\mu A_5^a) A_\mu^a \\
- g_5 f_{abc} A_\mu^b A_\nu^c \partial_\mu A_\nu^a f_{D(bca)} + \frac{1}{4} g_5 2 f_{abc} f_{ade} A_\mu^b A_\mu^e A_\lambda^d A_k^e F_{mnkl}^{D(bde)} \\
- \frac{1}{2} \partial_\mu A_5^a \partial_\nu A_5^a - g_5 f_{abc} A_\mu^a A_\nu^5 \partial_\mu A_5^a T_{mnkl}^{D(bca)} \\
+ M_5 g_5 f_{abc} A_\mu^b A_5^c A_\mu^a T_{mnkl}^{D(bca)} - \frac{1}{2} g_5 2 f_{abc} f_{ade} A_\mu^b A_5^c A_\mu^d A_5^e T_{mnkl}^{D(bde)} \right].
\]

(A.7)
where the overlap integrals are given by

\[
F_{nml}^{D(bca)} = \int_0^{\pi R} dy \sqrt{g_{55}} f_n^{D(b)} f_m^{D(c)} f_l^{D(a)},
\]

\[
T_{nml}^{D(bca)} = \int_0^{\pi R} dy e^{-4W} \sqrt{g_{55}} f_n^{D(b)} g_m^{D(c)} g_l^{D(a)},
\]

\[
F_{mnlk}^{D(bcde)} = \int_0^{\pi R} dy \sqrt{g_{55}} f_m^{D(b)} g_n^{D(c)} f_l^{D(d)} f_k^{D(e)},
\]

\[
T_{mnlk}^{D(bcde)} = \int_0^{\pi R} dy e^{-4W} \sqrt{g_{55}} f_m^{D(b)} g_n^{D(c)} f_l^{D(d)} g_k^{D(e)}.
\]

(A.8)

By repeatedly using the defining equations (A.3), we obtain the useful identity

\[
M_m (M_n T_{mnmm}^{D(bda)} - M_m T_{nmmn}^{D(abd)}) = (M_n^2 - M_m^2) F_{nmnm}^{D(abd)} + \left[ e^{-4W} (f_m^{D(b)} f_m^{D(d)} f_n^{D(a)r} - f_n^{D(a)} f_m^{D(b)} f_m^{D(d)r}) \right]_{0}^{\pi R}.
\]

(A.9)

Appendix B

Calculation of Scattering Amplitude

In this appendix, we give some details concerning the calculation of the scattering amplitude \( A_n^a A_m^b \rightarrow A_c^e A_m^d \) at tree level with the high energy approximation, in which the total energy \( E \) is sufficiently large compared to the mass of any external or intermediate state. The tree level diagrams are presented in Fig. 3.

![Fig. 3. The tree level diagrams: The \( A_\mu \) and \( A_5 \) components of the gauge bosons are represented by wavy lines and straight lines, respectively.](image)

In the \( R_\xi \) gauge, the propagators for \( A_\mu \) and \( A_5 \) have a \( \xi \) dependent mass:

\[
\langle A_{\mu m}^{a} A_{\nu k}^{b} \rangle = \frac{-i \delta^{ab} \delta_{mk}}{p^2 + M_k^2} \left( \eta_{\mu \nu} - \left( 1 - \xi \right) \frac{p_{\mu} p_{\nu}}{p^2 + \xi M_k^2} \right),
\]

(B.1)

\[
\langle A_{5 m}^{a} A_{5 k}^{b} \rangle = \frac{-i \delta^{ab} \delta_{mk}}{p^2 + \xi M_k^2}.
\]

(B.2)

The vertices can be read off of the KK decomposed action (A.3), as given in Fig. 4. We use an expansion in powers of \( M_k/E \). As seen from the propagators, this ex-

*) Because we are interested in the case in which the gauge boson \( A_l^c \) can have a zero mode \( (l = 0) \), we list only the diagrams appropriate for that case.
pansion requires the condition $\xi \ll E^2/M^2_k$, which is possible except in the unitary
gauge, where we have $\xi \to \infty$.

Assuming $\xi \ll E^2/M^2_k$, the contribution of each diagram in Fig. 3 to the invariant
matrix element is listed in Table II. Summing all the contributions in Table II, we
obtain the total invariant matrix element as

$$
ig_5 f_{abc} f_{nm} \left[ \eta^\mu (k-p)^\rho + \eta^\rho (p-q)^\mu + \eta^\mu (q-k)^\nu \right]
$$

$-ig_5^2 F_{mnk} \left[ f_{abe} f_{cde} (\eta^\mu \eta^\sigma \eta^\nu \eta^\rho - \eta^\mu \eta^\nu \eta^\sigma \eta^\rho) + f_{ace} f_{bde} (\eta^\mu \eta^\sigma \eta^\nu \eta^\rho - \eta^\mu \eta^\nu \eta^\sigma \eta^\rho) + f_{ade} f_{bce} (\eta^\mu \eta^\nu \eta^\sigma \eta^\rho - \eta^\mu \eta^\nu \eta^\sigma \eta^\rho) \right]
$$

$$
ig_5 f_{abc} \left( m_n T_{nm} - m_n T_{mn} \right) \eta^\mu
$$

Fig. 4. Vertices.
where \( F_{lmnm}^{D(cde)} = \sum_k F_{lmnk}^{D(dce)} F_{nmk}^{D(abe)} \). We have not yet specified the polarization \( \epsilon_\mu(p_3) \) of the external \( A_5^l \) boson so that we are free to choose a transverse or longitudinal polarization as well as a massless or massive \( A_5^l \) in the following.

Let us first consider the case in which the \( A_5^l \) boson is a massive KK mode.

Table II. Contribution from each diagram to the invariant matrix element in the case \( \xi \ll E^2/M^2_5 \).

Each contribution should be multiplied by the factor \(-i g^2 f^{abc} f^{cde} E^2/(8 M_n M_m^2) \epsilon_\mu(p_3)\). Here, the Jacobi identity \( f^{ade} f^{cbe} = f^{abc} f^{cde} \) has been used.

\[
-c \left| F_{lmnm}^{D(cde)} \left( 1 - \frac{M_l^2}{E^2} \right) \left( 1 - \frac{M_m^2}{E^2} \right) \left( 3 - \frac{4 M_n^2 - M_m^2}{E^2} \right) \cos \theta, -2 \left( 1 - \frac{M_n^2}{E^2} \right) \sin \theta, 0, -3 + \frac{2 M_n^2 + 7 M_m^2}{E^2} - \left( \frac{2 M_n^2 + M_m^2}{E^2} \right) \cos \theta \right)^\mu.
\]

\[
s -2 \sum_k F_{lmnk}^{D(dce)} F_{nmk}^{D(abe)} \left( 1 - \frac{M_l^2}{E^2} \right) \left( \frac{M_n^2 - M_m^2}{E^2} + \frac{4 M_m^2}{E^2} \cos \theta, \frac{M_n^2 - M_m^2}{E^2} \sin \theta, 0, -2 \left( 1 - \frac{M_m^2 - M_k^2}{E^2} \right) + \frac{M_n^2 - M_m^2}{E^2} \cos \theta \right)^\mu.
\]

\[
u - \sum_k F_{mnk}^{D(dae)} F_{mlk}^{D(bce)} \left( 1 - \frac{M_l^2}{E^2} \right)
\times \left( 2 + \frac{M_n^2 - 2 M_m^2 + 3 M_l^2 - 4 M_k^2}{E^2} + \frac{5 M_n^2 + 8 M_m^2 + M_l^2}{E^2} \right) \cos \theta,
- \left( 1 - \frac{M_n^2 - 5 M_m^2 - M_k^2 + 2 M_l^2}{E^2} \right) \sin \theta - \left( 1 + \frac{M_n^2 + M_m^2 - M_l^2}{E^2} \right) \sin \theta \cos \theta, 0,
1 + \frac{4 M_n^2 + 5 M_m^2 + 2 M_l^2 - 2 M_k^2}{E^2} \left( \frac{M_n^2 - 4 M_m^2 - M_l^2 + 2 M_k^2}{E^2} \right) \cos \theta
- \left( 1 + \frac{M_l^2 + M_m^2 - M_n^2}{E^2} \right) \cos \theta \right)^\mu.
\]

\[
s5 \left| -\frac{2}{E^2} \sum_k \left( M_l T_{mkkl}^{D(dae)} - M_m T_{lkm}^{D(dae)} \right) \left( M_n T_{mkn}^{D(bca)} - M_m T_{nkln}^{D(bca)} \right) \left( 1, - \sin \theta, 0, - \cos \theta \right)^\mu.
\]

\[
u5 \left| -\frac{2}{E^2} \sum_k \left( M_n T_{mkn}^{D(dae)} - M_m T_{nkln}^{D(dae)} \right) \left( M_l T_{mkkl}^{D(bce)} - M_m T_{lkm}^{D(bce)} \right) \left( -1, 0, 0, 1 \right)^\mu.
\]
Then, the on-shell polarization can be transverse, as in Eq. (2.15), or longitudinal, as in Eq. (2.18). If we choose the transverse polarization \( \epsilon^*(p_3) = (0, \cos \theta, 0, -\sin \theta) \), we find that the invariant matrix element \( B_{3} \) reduces to Eq. (2.16), whereas the other transverse polarization, \( \epsilon^*(p_3) = (0, 0, 1, 0) \), makes the invariant matrix element vanish. If we choose the longitudinal polarization in Eq. (2.18), we find that the invariant matrix element \( B_{3} \) reduces to Eq. (2.19).

Let us next consider the case in which the external \( A^c_l \) boson is massless. Then, only the state with \( M_k = M_m \) is allowed in the intermediate state, since \( F^{D(ced)}_{lmk} \) and \( T^{D(ced)}_{lkmm} \) reduce to \( f^{D(c)}_l \). Before taking the high energy limit, we obtain the result for arbitrary \( \xi \) by substituting the polarization of \( A^c_l \) with its momentum,

\[
ig_5^2 f^{abe} f^{cde} \times \left[ f^{D(c)}_l \left( (M_n^2 - M_m^2) F^{D(abd)}_{mmnm} - M_m (M_n T^{D(bda)}_{mmnm} - M_m T^{D(abd)}_{mmnm}) \right) \frac{p_3^\sigma \eta^\mu \nu}{(p_3 + p_4)^2 + \xi M_m^2} \right]
\]

Equation (A.9) implies that this amplitude is proportional to a function of the values of the mode functions at the boundaries. Moreover, it has \( \xi \) in its denominator. Therefore, the unitary gauge, for which \( \xi \to \infty \), clearly misses this possible violation of the Ward-Takahashi identity. Returning to the high energy approximation, we find by using Eq. (A.9) that the invariant matrix element in Eq. (B.3) reduces to Eq. (2.21).

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