Global smoothness for a 1D supercritical transport model with nonlocal velocity

Lucas C. F. Ferreira\textsuperscript{1,*} Valter V. C. Moitinho\textsuperscript{2}

\textsuperscript{1,2} State University of Campinas (Unicamp), IMECC-Department of Mathematics, Rua Sérgio Buarque de Holanda, 651, CEP 13083-859, Campinas, SP, Brazil.

Abstract

We are concerned with a nonlocal transport 1D-model with supercritical dissipation $\gamma \in (0, 1)$ in which the velocity is coupled via the Hilbert transform. This model arises in fluid mechanics linked to vortex-sheet problems, and as a lower dimensional model for the 2D dissipative quasi-geostrophic equation. It is known that solutions can blow up in finite time when $\gamma \in (0, 1/2)$. On the other hand, in the supercritical subrange $\gamma \in [1/2, 1)$ it is an open problem to know whether solutions are globally regular, as stated by Kiselev (2010). We show global smoothness of solutions in a supercritical subrange (close to 1) that depends on the non-negative initial data. More precisely, for each smooth non-negative initial data (without smallness condition) the model has a unique global smooth solution provided that $\gamma \in [\gamma_1, 1)$, where $\gamma_1$ depends on the initial data size. Our approach is inspired on that of Coti Zelati and Vicol (2016).

AMS MSC: 35Q35; 35B65; 76D03

Keywords: 1D transport model; Nonlocal velocity; Hilbert transform; Global regularity; Supercritical dissipation

1 Introduction

We consider the initial value problem for the 1D transport equation with nonlocal velocity

\[
\begin{aligned}
\partial_t \theta - \mathcal{H}\theta_x + \Lambda^\gamma \theta &= 0 \quad \text{in} \quad \mathbb{T} \times (0, \infty) \\
\theta(x, 0) &= \theta_0(x) \quad \text{in} \quad \mathbb{T},
\end{aligned}
\]

where $\gamma \in (0, 2)$, $\mathbb{T}$ is the 1D torus, $\Lambda = (-\Delta)^{1/2}$ and $\mathcal{H}$ denotes the Hilbert transform. In the literature, the equation (1.1) arises in the context of fluid mechanics linked to vortex sheet evolution (Birkhoff-Rott equations) and as an one-dimensional model for the 2D dissipative quasi-geostrophic equation (2DQG) (see e.g. [1, 6]). For more details and results about the 2DQG see [4, 7, 8, 11] and references therein.

\*Corresponding author.

E-mail addresses: lcff@ime.unicamp.br (L.C.F. Ferreira), valtermoitinho@live.com (V.V.C Moitinho)

LCF Ferreira was supported by FAPESP and CNPq, Brazil.

V.V.C Moitinho was supported by CNPq, Brazil.
In view of the transport structure of (1.1), any sufficiently regular solution satisfies the following maximum principle

$$\|\theta(\cdot, t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty(T)}.$$  \hspace{1cm} (1.2)

For non-negative initial data $\theta_0$, one can show that the solution $\theta$ is also non-negative. The condition $\theta_0 \geq 0$ is required to obtain

$$\|\theta(\cdot, t)\|_{L^2(T)} \leq \|\theta_0\|_{L^2(T)}$$ \hspace{1cm} (1.3)

for all $t \geq 0$, by means of energy methods.

The IVP (1.1) has three basic cases: subcritical $1 < \gamma \leq 2$, critical $\gamma = 1$ and supercritical $0 < \gamma \leq 1$. The global smoothness problem in the critical and subcritical cases have already been solved. In [6], Córdoba, Córdoba and Fontelos proved global regularity for non-negative initial data $\theta_0 \in H^2$ for $1 < \gamma \leq 2$. In the critical case, they obtained global $H^1$-solutions by assuming smallness condition on $L^\infty$-norm of the non-negative initial data. Also, in the inviscid case of (1.1), i.e., without the viscous term $\Lambda^n\theta$, they showed blow-up of solutions for compactly supported, even and non-negative $C^{1+\varepsilon} (\mathbb{R})$-initial data such that $\max_{x \in \mathbb{R}} \theta_0(x) = \theta_0(0) = 1$. In [10], Dong showed global well-posedness of (1.1) for arbitrary initial data in $H^{s_0}$ where $s_0 = \max\{\frac{3}{2} - \gamma, 0\}$ and $1 \leq \gamma \leq 2$ (critical and subcritical cases). In the supercritical case, he assumed a smallness condition on the initial data.

The global regularity problem for solutions of (1.1) in the supercritical case remains an open problem. In the part $0 < \gamma < \frac{1}{2}$ of the supercritical range, Li and Rodrigo [13] proved blow-up of solutions in finite time for non-positive, smooth, even and compactly supported initial data satisfying $\theta_0(0) = 0$ and a suitable weighted integrability condition. In [12], still in the same range, Kiselev showed blow-up of solutions in finite time for even, positive, bounded and smooth initial data $\theta_0$ satisfying $\max_{x \in \mathbb{R}} \theta_0(x) = \theta_0(0)$ and suitable integrability conditions. In the supercritical case, he assumed a smallness condition on the data.

In the range $\frac{1}{2} \leq \gamma < 1$, to the best of our knowledge, the formation of singularity in finite time or global smoothness is an open problem (stated by [12, p.251]). In [9], for $0 < \gamma < 1$, T.D. Do obtained eventual regularization of solutions for non-negative initial data. He also obtained global regularity for a modified 1D model with $\Lambda^n\theta$ replaced by

$$\mathcal{L}(\theta) = -\frac{\Lambda}{\log(1 - \Lambda)} \theta,$$

which can be understood as supercritical dissipation in a log-sense. In [14], Silvestre and Vicol provided four essentially different proofs of blow-up of solutions in the inviscid case. Moreover, they conjectured that solutions obtained as vanishing viscosity approximations could be bounded in $C^{1/2}$, for all $t > 0$, which would possibly yield Hölder regularization effects for the case $1/2 \leq \gamma < 1$ and then would solve the global regularity conjecture in [12, p.251] (see Conjectures 7.1 and 7.2 in [14]). In [2], Bae, Granero-Belinchón and Lazar considered the inviscid case and developed a theory of global weak super solutions for (1.1) with non-negative data $\theta_0 \in L^1 \cap L^2$.

In this paper we focus on supercritical values of $\gamma$ contained in the range $\frac{1}{2} \leq \gamma < 1$ (in fact, close to 1) and prove existence of global classical solutions for (1.1). More precisely, we show existence of $H^2$-strong solution for arbitrary non-negative initial data $\theta_0 \in H^2$ and $\gamma_1 \leq \gamma < 1$, where $\gamma_1$ depends on the $H^2$-norm of $\theta_0$. Indeed, due to standard regularization
of $H^2$-solutions, our solutions are $C^\infty$-smooth for $t > 0$. For that matter, first we obtain an eventual regularization result with an explicit control on the regularization time $T^*$, namely

$$T^* = C\alpha^{\frac{1}{\gamma - 1}} \|\theta_0\|^{\frac{1}{1 - \gamma}}_{L^\infty(\mathbb{T})}.$$  

Afterwards, we provide an estimate for the existence time of $H^2$-solutions and then compare it with $T^*$. It is worth mentioning that, in consonance with the conjectures in [14], we obtain boundedness of solutions in $C^\alpha$ for $\alpha > 1 - \gamma$.

Our approach follows the spirit of Coti Zelati and Vicol [8] who showed existence and uniqueness of global classical solutions for the 2DQG with supercritical values of $\gamma$.

Our main result reads as follows.

**Theorem 1.1.** Let $R > 0$ be arbitrary. If $\theta_0 \in H^2(\mathbb{T})$ is non-negative and satisfies

$$\|\theta_0\|_{H^2(\mathbb{T})}^{1 - \frac{2\gamma}{T}} \|\theta_0\|^{\frac{2\gamma}{T}}_{L^2(\mathbb{T})} \leq R,$$

then there exists $\gamma_1 = \gamma_1(R) \in (0, 1)$ such that for each $\gamma \in [\gamma_1, 1]$ the IVP (1.1) has a unique global (classical) $H^2$-solution.

For each $\gamma \in (0, 1]$, let $R_\gamma$ be the supremum of all $R > 0$ such that, for any $\theta_0 \in H^2$ with $\|\theta_0\|_{H^2(\mathbb{T})}^{1 - \frac{2\gamma}{T}} \|\theta_0\|^{\frac{2\gamma}{T}}_{L^2(\mathbb{T})} \leq R$, the unique $H^2$-solution of (1.1) with initial data $\theta_0$ does not blow up in finite time. In view of arguments in the proof of Theorem 1.1, we have that $R_\gamma \to \infty$ as $\gamma \to 1^{-}$. This paper is organized as follows. In Section 2 we recall some definitions, notations and properties for Hilbert transform and fractional Laplacian operator. Section 3 is devoted to the eventual regularity property. Finally, in Section 4 we prove Theorem 1.1.

## 2 Preliminaries

Let $1 \leq p \leq \infty$ and denote the norm of $L^p(\mathbb{T})$ by $\|\cdot\|_{L^p}$. For $s \in \mathbb{R}$ the norms of the homogeneous Sobolev space $H^s(\mathbb{T})$ and its nonhomogeneous counterpart $H^s(\mathbb{T})$ are denoted by $\|\cdot\|_{H^s} = \|\cdot\|_{L^2}$ and $\|\cdot\|_{H^s} = \|\cdot\|_{L^2} + \|\cdot\|_{L^2}$, respectively. In turn, for each $\alpha \in (0, 1)$, the Hölder space $C^\alpha(\mathbb{T})$ is endowed with the norm $\|\phi\|_{C^\alpha(\mathbb{T})} = [\phi]_{C^\alpha(\mathbb{T})} + \|\phi\|_{L^\infty(\mathbb{T})}$, where the seminorm $[\phi]$ is given by

$$[\phi]_{C^\alpha(\mathbb{T})} = \sup_{x, y \in \mathbb{T}, x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha}. \quad (2.1)$$

We recall that the periodic Hilbert transform $\mathcal{H}$ is defined by means of Fourier transform as

$$\widehat{\mathcal{H}\phi}(m) = -i \text{ sign}(m) \hat{\phi}(m)$$

for all $m \in \mathbb{Z}$ and $\phi \in C^\infty(\mathbb{T})$. Alternatively, in original variables we have the expression

$$\mathcal{H}\phi(x) = \frac{1}{2\pi} P.V. \int_{\mathbb{T}} \frac{\phi(y)}{\tan(\frac{\pi}{2} x - y)} dy, \quad (2.2)$$
which can be equivalently written as (see [3])

\[
\mathcal{H}\phi(x) = \frac{1}{\pi} P.V. \int_{\mathbb{T}} \frac{\phi(y)}{x - y} dy + \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} \phi(y) \left( \frac{1}{x - y - 2k\pi} + \frac{1}{2k\pi} \right) dy \\
= \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{\phi(y)}{x - y} dy.
\]

(2.3)

In the last integral in (2.3), recall that \(P.V.\) is defined by

\[
P.V. \int_{\mathbb{R}} \frac{\phi(y)}{x - y} dy = \lim_{\epsilon \to 0} \int_{\epsilon < |x-y| < \frac{1}{\epsilon}} \frac{\phi(y)}{x - y} dy.
\]

Hilbert transform commutates with derivatives and in particular we have that

\[
\partial_x \mathcal{H}(\phi)(x) = \mathcal{H}(\partial_x \phi)(x).
\]

For \(0 < \gamma < 2\) and \(\phi \in C^\infty(\mathbb{T})\), the fractional Laplacian \(\Lambda^\gamma\) is defined by the following singular integral (see [7] for more details)

\[
\Lambda^\gamma \phi(x) = c_\gamma \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} \frac{\phi(x) - \phi(x+y)}{|y - 2\pi k|^{1+\gamma}} dy = c_\gamma P.V. \int_{\mathbb{R}} \frac{\phi(x) - \phi(x+y)}{|y|^{1+\gamma}} dy
\]

where \(c_\gamma\) is a normalization constant. For \(\gamma \in (\gamma_0, 1)\), the constant \(c_\gamma\) can be bounded from below and above by using \(\gamma_0\) and some universal constants \(C\). The exact expression of the constant \(c_\gamma\) is not necessary for our ends.

The next lemma contains a property of the fractional Laplacian (see [7] for more details and a proof in the two-dimensional case).

**Lemma 2.1.** Assume that \(0 < \gamma < 2\) and \(\phi \in C^\infty(\mathbb{T})\). Then, we have the pointwise equality

\[
2\phi(x) \Lambda^\gamma \phi(x) = \Lambda^\gamma (\phi)^2(x) + D_\gamma(\phi)(x) \text{ in } \mathbb{T},
\]

where

\[
D_\gamma(\phi)(x) = c_\gamma \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} \frac{(\phi(x) - \phi(x+y))^2}{|y - 2\pi k|^{1+\gamma}} dy = c_\gamma P.V. \int_{\mathbb{R}} \frac{(\phi(x) - \phi(x+y))^2}{|y|^{1+\gamma}} dy.
\]

We finish this section with a technical lemma that will be useful for our purposes (see [4, Lemma B.1]).

**Lemma 2.2.** Let \(\mathcal{K} \subset \mathbb{R}^n\) be compact and \(T > 0\). Consider a function

\[
f : \mathcal{K} \times (0, T) \to [0, \infty)
\]

and assume that the functions

\[
f_\lambda(\cdot) = f(\lambda, \cdot) : (0, T) \to [0, \infty) \quad \text{and} \quad f'_\lambda(\cdot) = (\partial_t f)(\lambda, \cdot) : (0, T) \to \mathbb{R}
\]

are continuous, for each \(\lambda \in \mathcal{K}\). Additionally, assume that the following properties hold true:
(i) The families \( \{ f_\lambda \}_{\lambda \in \mathcal{K}} \) and \( \{ f'_\lambda \}_{\lambda \in \mathcal{K}} \) are uniformly equicontinuous with respect to \( t \);

(ii) For every \( t \in (0, T) \), the functions

\[
 f(\cdot, t) : \mathcal{K} \to [0, \infty) \quad \text{and} \quad (\partial_t f)(\cdot, t) : \mathcal{K} \to \mathbb{R}
\]

are continuous. Moreover, define

\[ F(t) = \sup_{\lambda \in \mathcal{K}} f_\lambda(t). \]

Then, for almost every \( t \in (0, T) \) the function \( F \) is differentiable at \( t \) and there exists \( \lambda_* = \lambda_*(t) \in \mathcal{K} \) such that the following equalities hold simultaneously:

\[
 F(t) = f_{\lambda_*}(t) \quad \text{and} \quad F'(t) = f'_{\lambda_*}(t).
\]

### 3 Eventual regularization

In this section, we show an “eventual regularization” result for solutions of \((1.1)\) in which we provide an explicit control on the eventual regularity time \( T^* \).

Firstly, in [10], we can find the following local existence result for \((1.1)\): let \( \gamma \in (0, 1) \) and \( \theta_0 \in H^{2-\gamma}(\mathbb{T}) \). Then, there exists \( T > 0 \) such that the IVP \((1.1)\) has a unique strong solution \( \theta \in C([0, T); H^{2-\gamma}(\mathbb{T})) \cap L^2((0, T); H^{\frac{2\gamma}{1+\gamma}}(\mathbb{T})). \)

Using regularity techniques for the solution obtained in [10] (see Section 4 and estimates (4.4)-(4.5)), we can show that there exists \( 0 < T_1 \leq T \) such that

\[
 \theta \in C([0, T_1); H^2(\mathbb{T})) \quad \text{(3.2)}
\]

provided that \( \theta_0 \in H^2(\mathbb{T}) \). Moreover,

\[
 \theta \in C^1((0, T_1); H^1(\mathbb{T})) \quad \partial_t \theta \in L^\infty((0, \tilde{T}); H^1(\mathbb{T})) \quad \text{and} \quad \theta \in L^2((0, \tilde{T}); H^{2+\frac{2\gamma}{1+\gamma}}(\mathbb{T})),
\]

for all \( 0 < \tilde{T} < T_1 \). In what follows, we state our “eventual regularization” result.

**Theorem 3.1.** Suppose that \( \gamma \in (1/2, 1) \) and \( \theta_0 \in L^\infty(\mathbb{T}) \) is non-negative. Let \( \alpha \in (1-\gamma, 1/2] \) and define

\[
 T^* = C \alpha^{-1} \| \theta_0 \|_{L^\infty(\mathbb{T})} \gamma^{-1}, \quad \text{(3.3)}
\]

where \( C = \gamma^{-1}k_1k_2^{1-\gamma} > 0 \) with \( k_1 \) and \( k_2 \) being independent of \( \alpha, \gamma \) and \( \theta_0 \). Let \( \theta \) be a solution of \((1.1)\) in the class \((3.2)\) with existence time \( 0 < T_1 < \infty \). If \( T^* < T_1 \), then \( \theta \in C^\infty(\mathbb{T} \times (T^*, T_1]) \).

**Remark 3.2.** Let us observe that the expression “eventual regularization” is used in the literature in the context of weak solutions and \( T_1 = \infty \). Nevertheless, in our range of \( \gamma \), it is not known whether \((1.1)\) has global weak solution and then we need to adapt this kind of result to our context but borrowing the same expression.
In next lemma we recall a well-known result that assures that the control of high-order H"older norms is sufficient to obtain smoothness. This result essentially follows from [9, Theorem 2.1] which extended the results of [5] about 2DQG to (1.1).

**Lemma 3.3.** Let $\theta$ be a solution of (1.1) in the class (3.2) with non-negative initial data $\theta_0$. If $0 < t_0 < t_1 \leq T_1$ and
$$\theta \in L^\infty((t_0, t_1); C^\alpha(\mathbb{T}))$$
with $0 < \gamma < 1$ and $1 - \gamma < \alpha$, then
$$\theta \in C^\infty(\mathbb{T} \times (t_0, t_1]).$$

### 3.1 Proof of Theorem 3.1

In view of Lemma 3.3, we need to show that
$$\theta \in L^\infty((T^*, T_1); C^\alpha(\mathbb{T})), \quad (3.4)$$
where $\alpha > 1 - \gamma$ and $T^*$ is as in (3.3). For that, we denote
$$\delta_h \theta(x, t) = \theta(x + h, t) - \theta(x, t)$$
and define
$$v(x, t, h) = \frac{\delta_h \theta(x, t)}{(\xi^2(t) + |h|^2)^{\frac{\gamma}{2}}}, \quad (3.5)$$
where $\xi : [0, \infty) \to [0, \infty)$ is a bounded decreasing differentiable function which will be determined later. Notice that it is sufficient to estimate $\|v(t)\|_{L^\infty}$ when $\xi(t) = 0$ in order to control the seminorm (2.1) of $\theta$ in $C^\alpha(\mathbb{T})$.

We start by providing estimates for $L^2$ where $L$ is the operator of the corresponding equation satisfied by $\delta_h \theta$. We split the proof into a sequence of lemmas. Taking the differences in (1.1) evaluated in $x + h$ and $x$, it follows that
$$\partial_t \delta_h \theta - \mathcal{H} \theta \partial_x \delta_h \theta - \delta_h \mathcal{H} \theta \partial_h \delta_h \theta + \Lambda^\gamma \delta_h \theta = 0, \quad (3.6)$$
which gives $L = \partial_t - u \partial_x - \delta_h u \partial_h + \Lambda^\gamma$ with $u = \mathcal{H} \theta$. Combining (3.6) and Lemma 2.1 we obtain that $v^2$ satisfies
$$L v^2 + \frac{1}{(\xi^2(t) + |h|^2)^{\frac{\alpha}{2}}} D^\gamma(\delta_h \theta) = -2\alpha \xi' \frac{\xi}{\xi^2 + |h|^2} v^2 + 2\alpha \frac{h}{\xi^2 + |h|^2} \delta_h \mathcal{H} \theta v^2. \quad (3.7)$$

Next we estimate the term $D^\gamma(\delta_h \theta)$.

**Lemma 3.4.** Let $0 < \gamma_0 \leq \gamma < 1$ and $\alpha \in (1 - \gamma, 1)$. Then, there exists a positive constant $c_0 = c_0(\gamma_0)$ such that
$$D^\gamma(\delta_h \theta)(x) \geq \frac{1}{c_0|h|^{\gamma}} \left( \frac{\|v(x, h)\|_{L^\infty}}{\|v\|_{L^\infty}} \right)^{\frac{\gamma}{2\alpha}} |\delta_h \theta(x)|^2,$$
for all $x, h \in \mathbb{T}$.
Proof. Let \( \chi \) be a smooth radially non-decreasing cutoff function such that \( \chi \) vanishes for \(|x| \leq 1\), \( \chi \) is identically 1 for \(|x| \geq 2\) and its derivative verifies \(|\chi'| \leq 2\). For \( R \geq 4|h| \), we can estimate
\[
D_\gamma(\delta_h \theta)(x) \geq c_\gamma \int_\mathbb{R} \frac{(\delta_h \theta(x) - \delta_h \theta(x + y))^2}{|y|^{1+\gamma}} \chi \left( \frac{|y|}{R} \right) \, dy \\
\geq c_\gamma |\delta_h \theta(x)|^2 \int_{|y| \geq 2R} \frac{1}{|y|^{1+\gamma}} \, dy - 2c_\gamma |\delta_h \theta(x)| \int_\mathbb{R} \frac{\delta_h \theta(x + y)}{|y|^{1+\gamma}} \chi \left( \frac{|y|}{R} \right) \, dy \\
\geq c_\gamma |\delta_h \theta(x)|^2 \left( \frac{1}{R^\gamma} - 2c_\gamma |\delta_h \theta(x)| \right) \int_\mathbb{R} (\theta(x + y) - \theta(x)) \delta_{-h} \left( \frac{|y|}{R} \right) \, dy, \quad (3.8)
\]

Denoting \( g(y) = \chi \left( \frac{|y|}{R} \right) |y|^{-(1+\gamma)} \) and using mean value theorem, we obtain
\[
|\delta_{-h} g(y)| \leq |h| \max_{0 \leq \lambda \leq 1} |g'(y - \lambda h)| \leq c_1 \frac{|h|}{|y|^{1+\gamma}} 1 \{ \frac{|y|}{R} \leq |y| \},
\]
for some constant \( c_1 \geq 1 \). Hence the integral in (3.8) can be estimated as
\[
\left| \int_\mathbb{R} (\theta(x + y) - \theta(x)) \delta_{-h} \left( \frac{\chi \left( \frac{|y|}{R} \right)}{|y|^{1+\gamma}} \right) \, dy \right| \leq c_1 |h| \int_{|y| \geq 2R} \frac{|\delta_y \theta(x)|}{|y|^{1+\gamma}} \left( \frac{\xi^2 + |y|^2}{2} \right)^{\frac{\gamma}{2}} \, dy \\
\leq c_1 |h| \|v\|_{L^\infty} \int_{|y| \geq 2R} \left( \frac{\xi^2 + |y|^2}{2} \right)^{\frac{\gamma}{2}} |y|^{-2+\gamma} \, dy \\
\leq c_1 c_2 \frac{|h| \|v\|_{L^\infty}}{R^\gamma} \left( \frac{\xi^2}{R^\gamma} + \frac{1}{R^{1-\alpha}} \right), \quad (3.9)
\]
for some constant \( c_2 \geq 1 \).

Now choose \( R > 0 \) defined by
\[
R = \left[ \frac{8c_1 c_2 \|v\|_{L^\infty}}{|v(x, t)|} \right]^{\frac{1}{1-\alpha}} |h|. \quad (3.10)
\]
Since \( c_1 c_2 \geq 1 \) and \(|v(x, t)| \leq \|v\|_{L^\infty}\), it follows that \( R \geq 8^{\frac{1}{1-\alpha}} |h| \geq 4|h| \).

Using (3.10), we can rewrite estimate (3.9) as
\[
\left| \int_\mathbb{R} (\theta(x + y) - \theta(x)) \delta_{-h} \left( \frac{\chi \left( \frac{|y|}{R} \right)}{|y|^{1+\gamma}} \right) \, dy \right| \leq \frac{1}{R^\gamma} \left[ c_1 c_2 \xi^\alpha \|v\|_{L^\infty} \left( \frac{|v(x, t)|}{\|v\|_{L^\infty}} \right)^{\frac{1}{1-\alpha}} + \frac{|v(x, t)| |h|^\alpha}{8} \right] \\
\leq \frac{|\delta_h \theta(x)|}{8R^\gamma} = \frac{|\delta_h \theta(x)|}{4R^\gamma}, \quad (3.11)
\]
because \( a^{\frac{1}{1-\alpha}} \leq a \) for all \( a \leq 1 \).

Combining estimates (3.8) and (3.11) and using the definition of \( R \), we arrive at
\[
D_\gamma(\delta_h \theta)(x) \geq c_\gamma \frac{|\delta_h \theta(x)|^2}{2R^\gamma} \\
\geq c_\gamma \frac{1}{2(8c_1 c_2)^{\frac{1}{1-\alpha}}} \frac{|v(x, t)|^{\frac{1}{1-\alpha}}}{|h|^\gamma} \frac{|v(x, t)|^{\frac{1}{1-\alpha}}}{\|v\|_{L^\infty}} |h|^\gamma |\delta_h \theta(x)|^2, \quad (3.12)
\]
as required.

\[ \text{Remark 3.5.} \] Notice that the condition \( 0 < \gamma_0 \leq \gamma < 1 \) arises from the need of controlling terms with the power \( \frac{1}{1-\alpha} \) in (3.10) and (3.12).

In the next lemma, we define \( \xi \) by an ordinary differential equation and obtain an estimate for the first term on the right hand side of (3.7).

\[ \text{Lemma 3.6.} \] Let \( \gamma_0 > 0 \), \( \gamma \in [\gamma_0, 1) \) and \( \alpha \in (1 - \gamma, 1) \). There exists a positive constant \( k_1 = k_1(\gamma_0) \) such that if
\[
\xi' = -\frac{1}{\alpha k_1} \xi^{1-\gamma}, \tag{3.13}
\]
then
\[
-2\alpha \xi' \frac{\xi}{\xi^2 + |h|^2} v^2 \leq \frac{1}{8 c_0 |h|^\gamma} v^2, \text{ for all } x, h \in \mathbb{T}, \tag{3.14}
\]
where \( c_0 \) is the same constant appearing in Lemma 3.4.

\[ \text{Proof.} \] Substituting (3.13) on the left hand side of (3.14), we conclude that
\[
-2\alpha \xi' \frac{\xi}{\xi^2 + |h|^2} v^2 \leq \frac{2(k_1)^{-1} \xi^{2-\gamma}}{\xi^2 + |h|^2} v^2 \leq \frac{2(k_1)^{-1} (\xi^2 + |h|^2)^{1-\frac{\gamma}{2}}}{\xi^2 + |h|^2} v^2 \leq \frac{2(k_1)^{-1}}{(\xi^2 + |h|^2)^{\frac{\gamma}{2}}} \leq \frac{2(k_1)^{-1}}{|h|^\gamma} v^2.
\]

Just choosing \( k_1 = 16 c_0 \), we are done.

\[ \text{Now we need to estimate the term in (3.7) that depends on the Hilbert transform. We do that in the next two lemmas. First, we work with the factor } \delta_h \mathcal{H}(x) \text{ in the second term of the right hand side of (3.7).} \]

\[ \text{Lemma 3.7.} \] Let \( \gamma \in (1/2, 1) \) and \( \alpha \in (1 - \gamma, 1/2] \). If \( \rho \geq 4|h| \), then
\[
|\delta_h \mathcal{H}(x)| \leq C \left\{ \frac{\rho^\gamma}{p} (D \gamma(\delta_h \theta)(x))^\frac{\gamma}{2} + \|v\|_{L^\infty} \left( \frac{|h|\xi^\alpha}{p} + \frac{|h|}{\rho^{1-\alpha}} \right) \right\}, \tag{3.15}
\]
for all \( x, h \in \mathbb{T} \).

\[ \text{Proof.} \] Let \( \chi \) be a smooth radially non-decreasing cutoff function that vanishes for \(|x| \leq 1\) and is equal 1 for \(|x| \geq 2\), and such that the derivative satisfies \(|\chi'| \leq 2\). From (2.3), we obtain
\[
\delta_h(\mathcal{H}(\theta))(x) = \mathcal{H}(\delta_h \theta)(x) = -\frac{1}{\pi} P.V \int_{\mathbb{R}} \frac{\delta_h \theta(x + y)}{y} dy.
\]

For \( \epsilon > 0 \), it follows that
\[
\int_{\epsilon \leq |y| \leq \frac{1}{\epsilon}} \frac{\delta_h \theta(x + y)}{y} dy = \int_{\epsilon \leq |y| \leq \frac{1}{\epsilon}} \frac{\delta_h \theta(x + y) - \delta_h \theta(x)}{y} dy := I + J, \tag{3.16}
\]
where
\[
I = \int_{|y| \leq \frac{1}{\epsilon}} \left( 1 - \chi\left( \frac{|y|}{\rho} \right) \right) \frac{\delta_h \theta(x + y) - \delta_h \theta(x)}{y} dy
\]
and
\[
J = \int_{|y| \leq \frac{1}{\epsilon}} \chi\left( \frac{|y|}{\rho} \right) \frac{\delta_h \theta(x + y) - \delta_h \theta(x)}{y} dy.
\]

Applying Hölder inequality and taking \( \epsilon \leq \frac{1}{2\rho} \), we can estimate \( I \) as follows
\[
|I| \leq \int_{|y| \leq 2\rho} \frac{|\delta_h \theta(x + y) - \delta_h \theta(x)|}{|y|} dy 
\leq \left( \int_{|y| \leq 2\rho} \frac{1}{|y|^{1-\gamma}} dy \right)^{\frac{1}{\gamma}} \left( \int_{|y| \leq 2\rho} \frac{|\delta_h \theta(x + y) - \delta_h \theta(x)|^2}{|y|^{1+\gamma}} dy \right)^{\frac{1}{2}}
\leq C\rho^{\frac{\gamma}{2}} \left( \int_{|y| \leq 1} \frac{|\delta_h \theta(x + y) - \delta_h \theta(x)|^2}{|y|^{1+\gamma}} dy \right)^{\frac{1}{2}} \quad (3.17)
\]

For \( J \), we have that
\[
|J| \leq \int_{|y| \leq \frac{1}{\epsilon}} \left| \delta_{-h} \left( \chi\left( \frac{|y|}{\rho} \right) \right) \right| |\theta(x + y) - \theta(x)| dy. \quad (3.18)
\]

Taking \( g(y) = \chi\left( \frac{|y|}{\rho} \right) \cdot y^{-1} \) and applying mean value theorem, we arrive at
\[
|\delta_{-h}g(y)| \leq |h| \max_{0 \leq \lambda \leq 1} |g'(y - \lambda h)| \leq C|h| \frac{1}{|y|^2}. \quad (3.19)
\]

Substituting (3.19) into (3.18) and taking \( \epsilon \leq \frac{3\rho}{4} \), we obtain
\[
J \leq C|h| \int_{\frac{3\rho}{4} \leq |y| \leq \frac{1}{\epsilon}} \frac{|\theta(x + y) - \theta(x)|}{|y|^2} dy 
\leq C|h| \|v\|_{L^\infty} \int_{\frac{3\rho}{4} \leq |y|} \frac{(\xi^2 + |y|^2)^{\frac{\gamma}{2}}}{|y|^2} dy
\leq C|h| \|v\|_{L^\infty} \left( \frac{\xi^\alpha}{\rho} + \frac{1}{\rho^{1-\alpha}} \right). \quad (3.20)
\]

The estimate (3.15) follows by inserting (3.17) and (3.20) in (3.16) and making \( \epsilon \to 0 \).

Next we provide a condition on the initial data of \( \xi \) to relate (3.15) to the estimates of the other terms.

**Lemma 3.8.** Let \( \gamma \in (1/2, 1) \) and \( \alpha \in (1 - \gamma, 1/2] \), and assume that
\[
\|v\|_{L^\infty} \leq \frac{4 \|\theta_0\|_{L^\infty}}{\xi_0^\alpha}. \quad (3.21)
\]
There exists a constant $k_2 = k_2(\gamma_0) \geq 1$ such that if

$$\xi_0 = (k_2\alpha \| \theta_0 \|_{L^\infty})^{\frac{1}{1-\gamma}},$$

then

$$2\alpha \frac{|h|}{\xi^2 + |h|^2} |\delta_h \mathcal{H} \theta| v^2 \leq \frac{1}{2(\xi^2 + |h|^2)^\alpha} D_\gamma(\delta_h \theta)(x) + \frac{1}{8c_0|h|^\gamma} v^2,$$

for all $x, h \in \mathbb{T}$ with $|h| \leq \xi_0$.

**Proof.** From (3.15) and Young’s inequality for products it follows that

$$2\alpha \frac{|h|}{\xi^2 + |h|^2} |\delta_h \mathcal{H} \theta| v^2 \leq \frac{|h|}{2(\xi^2 + |h|^2)^\alpha} D_\gamma(\delta_h \theta)(x) + C\alpha \frac{|h|^2}{\xi^2 + |h|^2} \left[ \frac{\alpha \rho^\gamma v^2}{(\xi^2 + |h|^2)^{1-\alpha}} + \|v\|_{L^\infty} \left( \frac{\xi^\alpha}{\rho} + \frac{1}{\rho^{1-\alpha}} \right) \right] v^2.$$  

It is sufficient to show that

$$C\alpha \left[ \frac{\alpha \rho^\gamma v^2}{(\xi^2 + |h|^2)^{1-\alpha}} + \|v\|_{L^\infty} \left( \frac{\xi^\alpha}{\rho} + \frac{1}{\rho^{1-\alpha}} \right) \right] \leq \frac{1}{8c_0|h|^\gamma}. \quad (3.22)$$

Choose

$$\rho = 4(\xi^2 + |h|^2)^{\frac{1}{2}}, \quad (3.23)$$

and note that $\rho \geq 4|h|$. Combining (3.21) and (3.23), we obtain

$$\frac{\alpha \rho^\gamma v^2}{(\xi^2 + |h|^2)^{1-\alpha}} \leq C\alpha \frac{\|\theta_0\|_{L^\infty}^2 (\xi^2 + |h|^2)^{\frac{\gamma}{2}}}{\xi_0^{2\alpha}(\xi^2 + |h|^2)^{1-\alpha}}. \quad (3.24)$$

For $|h| \leq \xi_0$, $1 - \gamma < \alpha < 1$ and $\xi \leq \xi_0$, we deduce that

$$\frac{(\xi^2 + |h|^2)^{\frac{\gamma}{2}}}{\xi_0^{2\alpha}(\xi^2 + |h|^2)^{1-\alpha}} \leq C\alpha \frac{(\xi^2 + |h|^2)^{\gamma + \alpha - 1}}{\xi_0^{2\alpha}(\xi^2 + |h|^2)^{1-\alpha}} \leq C\alpha \frac{1}{\xi_0^{2(1-\gamma)}|h|^\gamma}. \quad (3.25)$$

Estimates (3.24) and (3.25) yield

$$\frac{\alpha \rho^\gamma v^2}{(\xi^2 + |h|^2)^{1-\alpha}} \leq C\alpha \frac{\|\theta_0\|^2_{L^\infty}}{\xi_0^{2(1-\gamma)}|h|^\gamma}. \quad (3.26)$$

Proceeding similarly, one also can show that

$$\frac{\|v\|_{L^\infty}^\xi \|v\|_{L^\infty}}{\rho^{\xi_0}} \leq C\frac{\|\theta_0\|_{L^\infty}}{\xi_0^{\alpha}} \left( \frac{\xi^\alpha}{(\xi^2 + |h|^2)^{\frac{\gamma}{2}}} + \frac{1}{(\xi^2 + |h|^2)^{\frac{1-\alpha}{2}}} \right) \leq C\frac{\|\theta_0\|_{L^\infty}}{\xi_0^{\alpha}} \left( \frac{(\xi^2 + |h|^2)^{\frac{\gamma + \alpha - 1}{2}}}{|h|^\gamma} \right) \leq C\frac{\|\theta_0\|_{L^\infty}}{\xi_0^{1-\gamma}|h|^\gamma}. \quad (3.27)$$
Adding (3.26) and (3.27), we get
\[
\frac{\alpha \rho^\gamma v^2}{(\xi^2 + |h|^2)^{1-\alpha}} + \|v\|_{L^\infty} \left( \frac{\xi^\alpha}{\rho} + \frac{1}{\rho^{1-\alpha}} \right) \leq C \left( \frac{\alpha \|\theta_0\|^2_{L^\infty}}{\xi_0^{2(1-\gamma)}} + \frac{\|\theta_0\|_{L^\infty}}{\xi_0^{1-\gamma}} \right) \frac{1}{|h|^\gamma}. \tag{3.28}
\]

In view of (3.28), notice that we only need to show that
\[
C \alpha \left( \frac{\alpha \|\theta_0\|^2_{L^\infty}}{\xi_0^{2(1-\gamma)}} + \frac{\|\theta_0\|_{L^\infty}}{\xi_0^{1-\gamma}} \right) \leq \frac{1}{8c_0}. \tag{3.29}
\]

For that, we simply choose \( \xi_0 \) satisfying
\[
\frac{\|\theta_0\|_{L^\infty}}{\xi_0^{1-\gamma}} \leq \frac{1}{16 \hat{C} c_0 \alpha},
\]
for some constant \( \hat{C} \) such that \( \hat{C} \geq C \) and \( \hat{C} c_0 \geq 1 \). Combining (3.28) and (3.29), we conclude (3.22).

Finally, we are ready to conclude the proof of Theorem 3.1. For \( 1 - \gamma < \alpha \leq 1/2 \), let
\[
M = \frac{4 \|\theta_0\|_{L^\infty}}{\xi_0^\alpha}
\]
and define
\[
t_* = \sup \{0 \leq t < T_1 : \|v(\tau)\|_{L^\infty} < M \text{ for all } \tau \in [0, t] \}. \tag{3.30}
\]
Note that \( t_* \) is well-defined since (3.5) provides
\[
\|v(0)\|_{L^\infty} \leq \frac{M}{2}.
\]
We show that \( t_* = T_1 \). Suppose by contradiction that \( t_* < T_1 \). Since \( v \) is continuous and periodic in \( x \) and \( h \) there exists \( (x_0, h_0) \in \mathbb{T} \times \mathbb{T} \) such that \( |v(x_0, t_*, h_0)| = M \). We claim that \( |h_0| \leq \xi_0 \). Indeed, if \( |h_0| \geq \xi_0 \), then
\[
|v(x_0, t_*, h_0)| \leq \frac{2 \|\theta\|_{L^\infty}}{|h_0|^\alpha} \leq \frac{2 \|\theta_0\|_{L^\infty}}{\xi_0^\alpha} \leq \frac{M}{2}.
\]

Applying Lemmas 3.6 and 3.8 in (3.7) for \( t \in (0, t_*] \), we obtain the estimate
\[
Lv^2 + \frac{1}{(\xi^2 + |h|^2)\alpha} D_\gamma (\delta h \theta) \leq \frac{1}{2(\xi^2 + |h|^2)\alpha} D_\gamma (\delta h \theta) + \frac{1}{4c_0|h|^\gamma} v^2, \tag{3.31}
\]
for all \( x, h \in \mathbb{T} \) with \( |h| \leq \xi_0 \). Using Lemma 3.4 we can rewrite (3.31) as follows
\[
Lv^2 + \frac{1}{4c_0|h|^\gamma} \left[ \frac{|v(x, h)|}{\|v\|_{L^\infty}} \right]^{\frac{\gamma}{1-\alpha}} - 1 v^2 + \frac{1}{4c_0|h|^\gamma} \left( \frac{|v(x, h)|}{\|v\|_{L^\infty}} \right)^{\frac{\gamma}{1-\alpha}} v^2 \leq 0. \tag{3.32}
\]
Now consider \( \epsilon > 0 \) such that
\[
\|v(t)\|_{L^\infty} \geq \frac{7M}{8}, \text{ for all } t \in [t_* - \epsilon, t_*]. \tag{3.33}
\]
Given \( t \in [t_\ast - \epsilon, t_\ast) \), consider \((x_t, h_t) \in \mathbb{T} \times \mathbb{T} \) such that the function \((x, h) \rightarrow v^2(x, t, h)\) reaches its maximum. At this point, we have that \( \partial_x v^2 = \partial_h v^2 = 0 \), \( \Lambda^\gamma v^2 \geq 0 \) and \(|h_t| \leq \xi_0\), which leads us to
\[
(\partial_t v^2)(x_t, t, h_t) \leq L v^2(x_t, t, h_t). \tag{3.34}
\]
Using (3.33) and \(|h_t| \leq \xi_0\), we deduce that
\[
\frac{49M}{256c_0\xi_0^\gamma} \leq \frac{v^2(x_t, t, h_t)}{4c_0|h_t|^\gamma}. \tag{3.35}
\]
Next, adding (3.34) and (3.35), we conclude
\[
(\partial_t v^2)(x_t, t, h_t) + \frac{49M}{256c_0\xi_0^\gamma} \leq L v^2(x_t, t, h_t) + \frac{v^2(x_t, t, h_t)}{4c_0|h_t|^\gamma}. \tag{3.36}
\]
Combining estimate (3.32) at the point \((x_t, t, h_t)\) with (3.36) and using that \(|v(x_t, t, h_t)| = \|v(t)\|_{L^\infty}\), it follows that
\[
(\partial_t v^2)(x_t, t, h_t) \leq -\frac{49M}{256c_0\xi_0^\gamma}, \tag{3.37}
\]
for all \( t \in [t_\ast - \epsilon, t_\ast) \).

Lemma 2.2 with \( f(t, \lambda) = v(x, t, h)^2 \) and \( \lambda = (x, h) \in \mathcal{K} = \mathbb{T} \times \mathbb{T} \) yields
\[
\frac{d}{dt} \|v(t)\|_{L^\infty}^2 \leq (\partial_t v^2)(x_t, t, h_t) \leq -\frac{49M}{256c_0\xi_0^\gamma}, \tag{3.38}
\]
for all \( t \in [t_\ast - \epsilon, t_\ast) \). Integrating (3.38), we arrive at
\[
\|v(t_\ast)\|_{L^\infty} < M,
\]
which contradicts (3.30). Consequently \( t_\ast = T_1 \) and \( v \in L^\infty(\mathbb{T} \times (0, T_1)) \).

Next, taking \( \xi_0 = (k_2 \alpha \|\theta_0\|_{L^\infty})^{\frac{1}{\gamma}} \) as in Lemma 3.8, then the solution of (3.13) is given by
\[
\xi(t) = \begin{cases} 
[\xi_0^\gamma - \frac{\gamma}{\alpha k_1} t]^{\frac{1}{\gamma}}, & \text{if } 0 \leq t \leq T_\ast \\
0, & \text{if } T_\ast < t < T_1
\end{cases}
\]
where
\[
T_\ast = C_\alpha^{\frac{1}{\gamma}} \|\theta_0\|_{L^\infty}^{\frac{1}{\gamma}} \quad \text{with } C = \gamma^{-1} k_1 k_2^{\frac{1}{\gamma}}.
\]
Since \( \xi(t) = 0 \) for \( T_\ast < t < T_1 \), it follows that
\[
[\theta(\cdot, t)]_{C^\alpha} \leq C \|v(t)\|_{L^\infty} \leq M, \quad \text{for all } T_\ast < t < T_1,
\]
and we are done.
4 Proof of Theorem 1.1

Firstly we obtain an explicit lower bound of the local existence time with $H^2$-initial data. For that, we need an \textit{a priori} estimate of $H^2$-norm of the solution and after compare with the eventual regularization time $T^*$ (3.3).

Formally applying $\partial_{xx}$ in (1.1) and then multiply by $\partial_{xx}\theta$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\theta_{xx}\|_{L^2}^2 + \left\| \Lambda^2 \theta_{xx} \right\|_{L^2}^2 = \int_{\mathbb{T}} (\mathcal{H}\theta_x)_{xx} \theta_{xx} dx \\
= 2 \int_{\mathbb{T}} (\mathcal{H}\theta)_{x} (\theta_{xx})^2 dx + \int_{\mathbb{T}} \mathcal{H}\theta_{xx} \theta_{xx} \theta_x dx \\
+ \int_{\mathbb{T}} (\mathcal{H}\theta)_{xx} \theta_{xx} \theta_x dx.
$$

(4.1)

An integration by parts leads us to

$$
\int_{\mathbb{T}} \mathcal{H}\theta_{xx} \theta_{xx} \theta_x dx = - \int_{\mathbb{T}} (\mathcal{H}\theta_{xx})_{x} \theta_{xx} dx
$$

$$
= - \int_{\mathbb{T}} (\mathcal{H}\theta)_{x} (\theta_{xx})^2 dx - \int_{\mathbb{T}} \mathcal{H}\theta_{xx} \theta_{xx} \theta_x dx.
$$

(4.2)

Thus, using (4.2), we can estimate the right hand side of (4.1) as

$$
\frac{1}{2} \frac{d}{dt} \|\theta_{xx}\|_{L^2}^2 + \left\| \Lambda^2 \theta_{xx} \right\|_{L^2}^2 = \frac{3}{2} \int_{\mathbb{T}} (\mathcal{H}\theta)_{x} (\theta_{xx})^2 dx \\
+ \int_{\mathbb{T}} (\mathcal{H}\theta)_{xx} \theta_{xx} \theta_x dx \\
\leq \frac{3}{2} \|\mathcal{H}\theta_{x}\|_{L^\infty} \|\theta_{xx}\|_{L^2}^2 \\
+ \|\theta_{x}\|_{L^\infty} \|\mathcal{H}\theta_{xx}\|_{L^2} \|\theta_{xx}\|_{L^2} \\
\leq \frac{3}{2} (\|\mathcal{H}\theta_{x}\|_{L^\infty} + \|\theta_{x}\|_{L^\infty}) \|\theta_{xx}\|_{L^2}^2.
$$

(4.3)

Since $\|\mathcal{H}\theta_{x}\|_{L^\infty} + \|\theta_{x}\|_{L^\infty} \leq C_1 \|\theta\|_{\dot{H}^2}$, we conclude that

$$
\frac{1}{2} \frac{d}{dt} \|\theta\|_{H^2}^2 + \left\| \Lambda^2 \theta \right\|_{H^2}^2 \leq C_1 \|\theta\|^3_{H^2},
$$

(4.4)

which, in particular, gives

$$
\|\theta\|_{H^2} \leq \frac{\|\theta_0\|_{\dot{H}^2}}{1 - C_1 \|\theta_0\|_{\dot{H}^2} t} \leq 2 \|\theta_0\|_{H^2}, \text{ for all } 0 \leq t \leq \frac{1}{2 C_1 \|\theta_0\|_{\dot{H}^2}}.
$$

(4.5)

\textit{A priori} estimate (4.5) together with (1.3) yield

$$
\|\theta(\cdot, t)\|_{H^2} \leq 2 \|\theta_0\|_{H^2}, \text{ for } 0 \leq t \leq \left(2C_1 \|\theta_0\|_{\dot{H}^2}\right)^{-1}
$$

and then $\|\theta(\cdot, t)\|_{H^2}$ does not blow up until

$$
T_1 = \frac{1}{C_0 \|\theta_0\|_{\dot{H}^2}},
$$

(4.6)
where $C_0$ is a constant independent of $\gamma$, $\alpha$ and $\theta_0$ and satisfying $C_0 \geq 2C_1$. Thus, in 3.2 we can consider $T_1$ as in (4.6).

On the other hand, using Gagliardo-Nirenberg inequality, we can estimate $T^*$ in (3.3) as

$$T^* = C\alpha^{\frac{1}{1-\gamma}} \left\| \theta_0 \right\|_{L^\infty(T)} \leq t_\gamma$$

where

$$t_\gamma = Ck_0^{\frac{\gamma}{1-\gamma}} \alpha^{\frac{1}{1-\gamma}} \left\| \theta_0 \right\|_{H^2(T)}^{\frac{\gamma(1-\gamma)}{4}} \left\| \theta_0 \right\|_{L^2(T)}^{\frac{3\gamma}{4(1-\gamma)}}$$

with $k_0$ independent of $\gamma$, $\alpha$ and $\theta_0$.

We claim that for $\gamma$ sufficiently close to 1 we can choose $\alpha \in (1-\gamma, 1/2]$ such that $t_\gamma < T_1$. Taking $C_2 = C\ k_0^{\frac{1}{1-\gamma}}$, this is equivalent to

$$C_0C_2\alpha^{\frac{1}{1-\gamma}} \left\| \theta_0 \right\|_{H^2(T)}^{\frac{\gamma(1-\gamma)}{4}} \left\| \theta_0 \right\|_{L^2(T)}^{\frac{3\gamma}{4(1-\gamma)}} \leq 1.$$ 

Taking $C_3 = C_0C_2$ and assuming (1.4), it is sufficient to have that

$$C_3\alpha^{\frac{1}{1-\gamma}} R^{\frac{1}{1-\gamma}} \leq 1$$

or, equivalently,

$$\alpha \leq R^{-1} C_3^{-(1-\gamma)}.$$  \hfill (4.7)

Choosing $\alpha = \min \left\{ 2(1-\gamma), \frac{1}{2} \right\}$, it follows from (4.7) that there exists $\gamma_1 := \gamma_1(R) \in [\gamma_0, 1)$ such that $T^* \leq t_\gamma < T_1$ for all $\gamma \in [\gamma_1, 1)$, which gives the claim.

Next, let $T_{\text{max}}$ be the maximal existence time for the solution (3.2) of (1.1). Assume by contradiction that $T_{\text{max}} < \infty$. We have that $\theta \in C([0, T_{\text{max}}); H^2(T))]$ with $T^* < T_1 \leq T_{\text{max}}$. Then, by Theorem 3.1, $\theta \in C^\infty(T \times (T^*, T_{\text{max}}))$ and, in particular, $\theta(T_{\text{max}}) \in H^2(T)$. So, by using standard arguments and the local-existence of [10], we can extend $\theta$ in the class (3.2) to a time-interval $[0, T_2)$ with $T_{\text{max}} < T_2$, which is a contradiction. It follows that $T_{\text{max}} = \infty$ and $\theta$ is a global $H^2$-solution (which is classical) for (1.1).

References

[1] G. R. Baker, X. Li and A. C. Morlet, Analytic structure of 1D-transport equations with nonlocal fluxes, Physica D 91 (1996), 349–375.

[2] H. Bae, R. Granero-Belinchón and O. Lazar, On the local and global existence of solutions to 1D transport equations with nonlocal velocity, arXiv:1806.01011, 2018.

[3] A. Calderón and A. Zygmund, Singular integrals and periodic functions, Studia Math. 14 (1954), 249–271.

[4] P. Constantin, A. Tarfulea and V. Vicol, Long time dynamics of forced critical SQGV, Commun. Math. Phys. 335 (2015), 93–141.
[5] P. Constantin and J. Wu, Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation, Annales de l’Institut Henri Poincare Non Linear Analysis 25 (6) (2008), 1103–1110.

[6] A. Córdoba, D. Córdoba and M. Fontelos, Formation of Singularities for a Transport Equation with Nonlocal Velocity, Annals of Mathematics, 162 (3) (2005) 1377–1389.

[7] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Commun. Math. Phys. (2004) (249) 511–528.

[8] M. Coti Zelati and V. Vicol, On the global regularity for the supercritical SQG equation. Indiana Univ. Math. J. 65 (2) (2016), 535–552.

[9] T. Do, On a 1D transport equation with nonlocal velocity and supercritical dissipation, Journal of Differential Equations, 256 (9) (2014), 3166–3178.

[10] H. Dong, Well-posedness for a transport equation with nonlocal velocity, Journal of Functional Analysis 255 (11) (2008), 3070–3097.

[11] A. Kiselev, Nonlocal maximum principles for active scalars, Advances in Mathematics, 227 (5) 2011, 1806–1826.

[12] A. Kiselev, Regularity and blow up for active scalars. Math. Model. Nat. Phenom. 5 (4) (2010), 225–255.

[13] D. Li and J. Rodrigo, Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation, Advances in Mathematics, 217 (6) (2008), 2563–2568.

[14] L. Silvestre and V. Vicol, On a transport equation with nonlocal drift, Trans. Amer. Math. Soc. 368 (2016), 6159–6188