Mixed-Integer Programming Using a Bosonic Quantum Computer

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Abstract—We propose a scheme for solving mixed-integer programming problems in which the optimization problem is translated to a ground-state preparation problem on a set of bosonic quantum field modes (qumodes). We perform numerical demonstrations by simulating a circuit-based optical quantum computer with each individual qumode prepared in a Gaussian state. We simulate an adiabatic evolution from an initial mixing Hamiltonian, written in terms of the momentum operators of the qumodes, to a final Hamiltonian which is a polynomial of the position and boson number operators. In these demonstrations, we solve a variety of small non-convex optimization problems in integer programming, continuous non-convex optimization, and mixed-integer programming.

I. INTRODUCTION

Achieving a quantum advantage in solving mathematical optimization problems will tremendously broaden the range of applications of quantum computers. To this end, a great variety of approaches has been explored, from the experimental realization of quantum annealers [1] to the design of circuit model quantum algorithms for optimization [2]–[4]. Quantum annealers rely on the analog simulation of an adiabatic evolution process, which restricts the sizes of optimization problems they can solve in the limited time available before the system decoheres. Quantum circuit model algorithms for optimization typically offer an asymptotic, yet modest, quadratic speedup compared to the exponentially costly exhaustive search. This quantum speedup relies on quantum amplitude amplification and estimation techniques [5], and will likely require very large fault-tolerant quantum computers [6] to solve problems of a size practical for real-world applications. In addition, both approaches perform quantum algorithms on a register of qubits, which restricts the types of optimization problems they can solve, especially in the absence of dense couplings between the qubits.

In this paper, we use a set of bosonic quantum field modes (or “qumodes”\(^1\)) to encode a mixed-integer programming (MIP) problem. We translate integer and continuous variables to different observables of the bosonic system. Non-negative integer variables are represented by the eigenstates of the number operators \(\hat{n} = \hat{a}^\dagger \hat{a}\) of the qumodes. Here, \(\hat{a}^\dagger\) and \(\hat{a}\) are, respectively, the bosonic creation and annihilation operators. In contrast, continuous variables are encoded in the continuous spectrum of the quadrature operators \(\hat{x} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger)\) or \(\hat{p} = \frac{1}{\sqrt{2}} (\hat{a} - \hat{a}^\dagger)\), that is, the canonically conjugate position and momentum observables of quantum harmonic oscillators, respectively.

Computation using bosonic systems has been explored in various ways to date. Continuous-variable quantum computation (CVQC) [7] pertains to performing unitary transformations according to Hamiltonians that are polynomials of quadrature operators. In contrast, in boson sampling [8]–[10], quantum information is encoded in the discrete spectrum of the boson number operators. Both computing tasks are of experimental and theoretical interest in the particular case of quantum computation using Gaussian states [11], [12]. In addition, advances in measurement-based, one-way universal quantum computation using CVQC [13] and the creation of cluster states in optical settings [14], [15] provide promising prospects for experimental CVQC.

Computational significance. A mathematical programming problem is an optimization problem in which an objective function is optimized with respect to a set of variables satisfying a set of constraints. In MIP, both the objective function and constraints are polynomials of integer and continuous variables. Mixed-integer linear programming pertains to problems in which the objective function and constraints are linear functions of the variables. This is enough to obtain natural models for typical combinatorial and discrete optimization problems [16], including famous NP-hard optimization problems in constraint satisfaction, graph and network optimization, and job scheduling [17]. Mixed-integer nonlinear programming problems extend this class to a boundless domain of real-world applications in science, engineering, operations research, management, health care, decision making, and system control [18], [19]. Despite this flexible modelling power, classical algorithms for solving nonlinear and mixed-integer optimization problems scale poorly with respect to problem size [20]. For instance, solving non-convex quadratic optimization problems is itself extremely challenging, to the extent that even verifying local optimality of their feasible points is NP-hard [21].

In this paper, we recast an MIP problem as the problem...
of preparing the ground state of a quantum Hamiltonian in a multi-mode Fock space, and then investigate an adiabatic scheme for finding the ground state of the Hamiltonian. However, other state preparation protocols, such as variational quantum algorithms [22], may be envisaged. Indeed, Verdon et al. [23] implement a continuous-variable analogue to the quantum approximate optimization algorithm (QAOA) [24] by writing the problem Hamiltonian in terms of one quadrature operator and the mixing Hamiltonian in terms of the other. However, to the best of our knowledge, the use of Fock states to encode integer variables in the context of solving MIP problems, and the capability and computational significance of solving mixed-integer non-convex global optimization problems using CVQC, have not been previously explored.

Numerical demonstrations. In Section III, we investigate the optimization of integer linear programming (ILP) problems, and solve a small example instance of the NP-hard unbounded knapsack problem [25]. In Section IV, we use a quadratic binary formulation of the NP-hard MAX CLIQUE problem to show that our scheme works, without any substantial modifications, in the case of nonlinear integer programming (NLIP) problems. Then, in Section V, we use the connection between the MAX CLIQUE problem and continuous non-convex optimization, established by Motzkin and Straus [26], to explore global non-convex optimization via both continuous and integer implementations of the MAX CLIQUE problem. We solve the problem using only the quadrature operators in the case of the continuous formulation and only the number operators in the case of the integer formulation. Finally, in Section VI, we demonstrate the application of our method in solving MIP problems, using an example instance of a sparse optimization problem. Such cardinality-constrained optimization problems are also NP-hard [27] and of practical importance in compressed sensing, signal processing, and computational finance [28].

For these demonstrations, we simulate a circuit model optical device, where individual qumodes are prepared as Gaussian states. An adiabatic evolution process is executed beginning with Gaussian states, followed by homodyne or photon number resolving measurements of the output qumodes, with the goal of preparing a state that overlaps substantially with the ground state of a problem Hamiltonian.

Experimental considerations. In a gate-based optical device, qumodes can be prepared as Gaussian states, after which they are acted upon by single- or two-qumode quantum optical gates integrated in photonic or optical fibre waveguides [29]–[31]. While suffering from their own limitations, quantum optical devices have an advantage in that they can operate at room temperature and are easier to scale compared to other implementations of quantum processors [32].

Current realizations of experiments that implement photon–photon interactions restrict the practical realization of our method to solving integer programming problems that are at most quadratic in terms of integer variables. The computation is carried out by a set of gates implementing functions of the photon number operator, including the rotation gate \( \hat{R}(\phi) = \exp(i\phi \hat{n}) \), the Kerr gate \( \hat{K}(\alpha) = \exp(i\alpha \hat{n}^2) \), and the cross-Kerr gate \( \hat{C}K(\alpha) = \exp(i\alpha \hat{n}_1 \hat{n}_2) \), the last of which acts on the Fock space of two qumodes [33]. Together with the quadratic phase gate \( \hat{P}(\alpha) = \exp(i\alpha \hat{p}^2) \) and the position displacement gate \( \hat{X}(x) = \exp(ix\hat{p}) \), this gate set is sufficient for solving integer linear and quadratic programming problems. To solve MIP problems, we require combined unitary gates that are functions of \( \hat{n} \) and one of the quadrature operators; for example, \( \hat{X}^2N(\alpha) = \exp(\alpha x^2\hat{p}) \). While unitary operations that use such a combination of quadrature and photon number operators have not been realized, we believe that our results will provide the impetus for the experimental realization of such unitary operations. Toward the realization of a universal gate set for CVQC, Yanagimoto et al. [34] propose a promising deterministic implementation of the non-Gaussian cubic phase gate \( \exp(i\alpha \hat{p}^3) \), an approach worth exploring for implementing the two-qumode higher-order combined gates we have introduced.

Implementing non-Gaussian gates at a high level of fidelity remains an experimental challenge. While on-chip Mach–Zehnder interferometers based on beam splitters and thermo-optic phase shifters are efficiently fabricated for high-fidelity performance [35]–[37], the viability of creating photon–photon interactions, such as in Kerr and cross-Kerr gates, through \( \chi^{(3)} \) third-order optical nonlinearity, has been subject to skepticism due to various challenges hampering the performance of these gates [38]–[41]. However, ongoing research on photon–photon interactions points to the possibility of generating phase shifts from single-photon pulses using other methods such as quantum dots [42], Rydberg blockades [43], [44], and four-wave mixing using atomic systems [45]. These experiments show the physical possibility of creating Kerr interactions through a variety of approaches. The results presented in this paper motivate experimental efforts toward the realization of quantum unitary operations that incorporate photon–photon interactions. Here, we rely on the assumption of the existence of a viable method of implementing high-fidelity Kerr and cross-Kerr interactions (regardless of the method of their realization), which are necessary to implement the quantum operations required for MIP using CVQC.

II. ADIABATIC GROUND-STATE PREPARATION

We study the probability distribution of states prepared via adiabatic quantum computation (AQC). The time-dependent Hamiltonian considered for AQC is

\[
\hat{H}(\tau) = (1 - \tau)\hat{H}_M + \tau\hat{H}_P, \tag{1}
\]

where \( \tau = t/T \) is the normalized time and \( T \) is the total evolution time. The Hamiltonian \( \hat{H}_P \) represents the problem Hamiltonian whose ground states encode the solutions to the optimization problem of interest, and \( \hat{H}_M \) is called a mixing Hamiltonian and is required to have an easy-to-prepare ground state and not to commute with \( \hat{H}_P \).

When \( \hat{H}_M \) and \( \hat{H}_P \) have discrete spectra, the evolution time \( T \) is in \( O(\delta^{-2}) \), where \( \delta \) is the smallest spectral gap of \( \hat{H}_P \) for

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all \( \tau \) [46]. Therefore, implementing this evolution in a scalable way via analog computation can be challenging [47]. While for AQC involving Hamiltonians with continuous spectra the evolution time cannot be bounded in terms of a well-defined spectral gap, results similar to the conventional adiabatic theorem [48] suggest that the evolution in time \( T \) deviates from identity by \( O(1/T) \). However, the evolution can be discretized and implemented on a fault-tolerant circuit model quantum device. To this end, we use the common technique of Trotterization to approximate the adiabatic time evolution. The resulting circuit consists of iterative applications of unitary evolutions according to \( \hat{H}_M \) and \( \hat{H}_P \),

\[
\hat{U} = e^{-ib_H \hat{H}_M} e^{-i\alpha \hat{H}_P} \ldots e^{-ib_H \hat{H}_M} e^{-i\alpha \hat{H}_P} ,
\]

to an initial state \( |\psi_0\rangle \) prepared in the ground state of \( \hat{H}_M \). The overall number of Trotter steps, denoted by \( k \), directly pertains to the circuit depth of the algorithm. The choice of the coefficients in the exponents depends on the function describing the schedule. For a linear schedule as given in Eq. (1), \( b_j = (k - j + \frac{1}{2}) \frac{\tau}{2} \) and \( c_j = (j - \frac{1}{2}) \frac{\tau}{2} \). Note that the sum of the coefficients of both the mixing and problem Hamiltonians is equal to \( T \), and that their coefficients decrease and increase linearly with each Trotter step \( j \), respectively. In our numerical simulations, we find that the continuous AQC algorithm and its discretized version yield very similar final results for all of the problems studied. For this reason, except in the case of the feasibility problem (see Section III), we present the results only for the continuous algorithm.

The mixing Hamiltonians used in our work are of the form

\[
\hat{H}_M = \sum_{i=1}^{N} \left( \hat{p}_i^2 - 2p_0 \hat{p}_i \right) ,
\]

which is equivalent to \( \sum_{i=1}^{N} (\hat{p}_i - p_0)^2 \) whose ground state is the momentum eigenstate \( |\psi_0\rangle = |p_0\rangle \otimes \cdots \otimes |p_0\rangle \). Here, \( N \) is the number of qumodes. The quantum state \( |\psi_0\rangle \) is a negatively and infinitely squeezed coherent state along the momentum axis [49]. Although such states are not physically realizable, we can approximate them with the squeezed coherent states \( |\alpha, -r\rangle \), where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) and \( r = (r_1, r_2, \ldots, r_N) \). In all our simulations, \( \alpha_j = i p_0 j / \sqrt{2N} \) and \( r_j = r \) for all \( j \), where \( r \) is an experimentally feasible squeezing parameter, for example, taking values between 0.5 and 0.8. As the values of \( p_0 \) increase, these states overlap increasingly less with the vacuum state. This is important, as the vacuum state is an eigenstate of the number operator that is used to construct the problem Hamiltonian, and having a large overlap between the initial state and the eigenstates of the problem Hamiltonian will impair the adiabatic ground-state preparation.

In general, the initial state parameter \( p_0 \) affects the final quantum states obtained via the adiabatic scheme and can be treated as a hyperparameter that gives us direct access to a continuum of the initial mixing Hamiltonians and the associated ground states to use in our algorithm. For example, if \( \hat{H}_P \) has degenerate ground states, \( p_0 \) can be tuned to amplify the amplitude of each of those states or, alternatively, to create a uniform superposition of all of those states (see Supplementary Material, Section S2). The latter is known as fair sampling, which is equivalent to Gibbs sampling at the zero temperature limit and is therefore a desirable property in machine learning [50], [51]. In addition, for a generic problem Hamiltonian written in terms of \( \hat{n} \) and \( \hat{x} \) operators, the mixing Hamiltonian above satisfies the non-commutativity relation \( [\hat{H}_P, \hat{H}_M] \neq 0 \).

### III. INTEGER LINEAR PROGRAMMING

Every ILP problem can be written in the canonical form

\[
\text{minimize} \quad -c^T n \\
\text{subject to} \quad A^T n \leq b \quad \text{and} \quad n \geq 0 ,
\]

where \( n \) is a column vector of \( N \) integer variables, \( c \) is the matrix of coefficients describing the objective function, and the matrix \( A \) and column vector \( b \) specify a system of \( \ell \) inequality constraints on the variables. As mentioned above, we use the boson number observable for each of the involved bosonic modes to represent the corresponding entries of \( n \). Hence, the condition \( n \geq 0 \) and the integrality of these variables are automatically satisfied.

Each inequality \( a_j^T n \leq b_j \), for \( 1 \leq j \leq \ell \), can be rewritten as an equality constraint \( a_j^T n - b_j + \eta_j = 0 \) via an auxiliary variable \( \eta_j \geq 0 \). This requires the addition of \( \ell \) ancillary qumodes. Assuming \( A \) and \( b \) have fractional entries, we can multiply them by their lowest common denominator to turn all of their entries into integers. We then use the number operator of the \( j \)-th ancillary qumode to represent \( \eta_j \). Alternatively, \( \eta_j \) can be considered a continuous variable represented by the non-negative-valued observable \( x_j^2 \) of the \( j \)-th qumode. This, however, turns problem (4) into an MIP similar to those in Section VI.

To solve problem (4), we seek the ground state of the Hamiltonian

\[
\hat{H}_P = -\sum_{i=1}^{N} c_i \hat{n}_i + \lambda \sum_{j=1}^{\ell} \left( \sum_{i=1}^{N} A_{ij} \hat{n}_i \right) + \hat{n}_j - b_j ,
\]

that is, by replacing the vectors \( n \) and \( \eta \) with vectors of the photon number operators \( \hat{n} \) and \( \hat{\eta} \). Here, \( \lambda > 0 \) is a penalty coefficient tuned to suppress the infeasible subspace of the Hilbert space. We note that simulating the problem Hamiltonian (5) requires the implementation of unitary evolutions of the form \( \exp(i \alpha_i \hat{n}_i) \), \( \exp(i \alpha_i \hat{n}_i^2) \), and \( \exp(i \alpha \hat{n}_j) \), which can all be achieved using cross-Kerr, Kerr, and rotation operations, respectively.

As a first example in our study of ILP problems, we consider the simple problem of solving the equation \( n_1 + n_2 = 5 \) over the integer domain. This simple equation can be re-written in the canonical form (4) by setting

\[
A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} , \quad b = \begin{pmatrix} 5 \\ -5 \end{pmatrix} , \quad \text{and} \quad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,
\]

that is, by imposing two inequalities, \( n_1 + n_2 \leq 5 \) and \( -n_1 - n_2 \leq -5 \), to determine the feasible domain. As
such a problem does not effectively minimize an objective function but requires the finding of integer solutions only to a set of constraints, it is known as a *feasibility problem*. For this reason, we avoid reformulating the problem in the canonical form (as it would require employing four qumodes) and instead find the ground states of $H_P = (\hat{n}_1 + \hat{n}_2 - 5)^2$.

Figure 1a shows the probability distribution of the results of measuring the final state in the two-mode Fock basis prepared using AQ. The simulation is performed by solving the Schrödinger equation for the quantum states represented in the Fock basis. The Fock states are truncated to $d$ dimensions (to be reported for each example in what follows) due to a limited availability of computational resources. The probability amplitudes of the quantum states evolved over time via AQ and through its discretized form are depicted in Figs. 1b and 1c, respectively. The two methods yield similar results, with the most-likely measurement outcomes corresponding to the six degenerate ground states $|0, 5\rangle$, $|1, 4\rangle$, $|2, 3\rangle$, $|3, 2\rangle$, $|4, 1\rangle$, and $|5, 0\rangle$ representing the solutions of the ILP problem. We observe that the ground-state amplitudes are amplified by the adiabatic scheme while those of the other Fock states are suppressed. We tune $p_{\text{mix}}$ in the mixing Hamiltonian (3) to achieve fair sampling (see Section S2).

We now consider a more interesting ILP problem in the canonical form. In the *knapsack problem*, a set of items and a knapsack are considered, where each item $i$ has a value $v_i$ and a weight $w_i$ associated to it, and the knapsack has a total capacity $W$. The goal is to insert a number of items into the knapsack such that their selection maximizes the sum of the values of items in the knapsack while not exceeding its capacity. In typical logistics scenarios, however, each item is of a type $i$ and we are allowed to choose any number of items of that type. This is called the *unbounded* knapsack problem (UKP), which can be formulated in the canonical form (4) as follows:

$$\text{minimize} \quad - \sum_{i=1}^{N} v_i n_i$$

subject to

$$\sum_{i=1}^{N} w_i n_i \leq W, \quad n_i \geq 0 \text{ for all } i.$$

The integer variables $n_i$ determine how many of each item type, if any, can be included in the knapsack.

Figure 2 shows the time progress of AQ for solving the UKP instance

$$\text{minimize} \quad - n_1 - 2n_2$$

subject to

$$4n_1 + 1.5n_2 \leq 11, \quad n_1, n_2 \geq 0,$$

which has a unique optimal solution $(n_1, n_2) = (0, 7)$. We observe that the state $|\psi\rangle = |0, 7\rangle$ is amplified by AQ while all other Fock states are suppressed.

**IV. NONLINEAR INTEGER PROGRAMMING**

In NLIP problems, both the objective functions and the constraints can be nonlinear functions of the integer variables. Such problems can be converted into ILP problem instances; however, this is at the cost of requiring additional qumodes.
In this section, we show that NLIP problems can be solved directly on a bosonic quantum computer.

We use the MAXCLIQUE problem [52] as our working example. In this section, we consider a binary NLIP formulation of this problem, and in Section V we solve an integer NLIP formulation as well as a non-convex continuous formulation. In the MAXCLIQUE problem, a graph \( G = (V, E) \) is given, where \( V \) is the set of vertices of \( G \) and \( E \) is the set of edges. Each edge \( e \in E \) is a pair of vertices \( \{i, j\} \subseteq V \). The set of edges can be represented by a symmetric adjacency matrix \( A \), wherein \( A_{ij} = 1 \) if \( \{i, j\} \) is in \( E \), and \( A_{ij} = 0 \) otherwise.

The goal of the MAXCLIQUE problem is to find the largest complete subgraph of \( G \), that is, to find the largest subset \( S \subseteq V \) such that all pairs of vertices in \( S \) are connected by an edge. In what follows, we also use \( S \) to denote the subgraph it determines (see Fig. 3a for an example).

Let \( n_i \in \{0, 1\} \) represent the selection of vertex \( i \) in the maximum clique. Then, the following NLIP formulation can be used to solve the MAXCLIQUE problem:

\[
\begin{align*}
\text{minimize} & \quad - \sum_i n_i \\
\text{subject to} & \quad n^T (1 - A) n = 0, \\
& \quad n_i \in \{0, 1\} \quad \text{for all } i.
\end{align*}
\]

Here, \( 1 \) is the \( N \times N \) matrix with every element equal to 1. We thus write the following problem Hamiltonian in terms of the number operators of the qumodes:

\[
\hat{H}_p = -\sum_i \hat{n}_i + \lambda \sum_{i,j} (1 - A_{ij}) \hat{n}_i \hat{n}_j + \mu \sum_i \hat{n}_i (\hat{n}_i - 1). \tag{9}
\]

The first term corresponds to the objective function of problem (8) and the second term penalizes any violation of the adjacency constraint of problem (8). The last term restricts our search to the subspace of the Fock space corresponding to the eigenvalues 0 or 1 for the number operators of the qumodes. It is easy to see that any values of \( \lambda > \frac{1}{2} \) and \( \mu > 1 \) suffice.

We apply AQC to the MAXCLIQUE problem for the graph shown in Fig. 3a. In Fig. 3b, we plot the three largest probabilities at the end of the adiabatic evolution. The two degenerate states representing the maximum cliques correspond to the vertices \( \{1, 2, 4\} \) and \( \{1, 3, 4\} \). Thus, in Fig. 3b, the probability amplitudes of the five-mode Fock states \( |1, 1, 0, 1, 0\rangle \) and \( |1, 0, 1, 1, 0\rangle \) converge to large values.

V. NON-CONVEX GLOBAL OPTIMIZATION

Non-convex continuous functions are difficult to optimize classically even when they contain no discrete variables [20]. For instance, Motzkin and Straus [26] provide a continuous formulation of the NP-hard MAXCLIQUE problem. As in the previous section, we let \( A \) denote the adjacency matrix of a graph \( G \). Then, according to the Motzkin–Straus theorem, the optimal solutions of the continuous quadratic programming problem,

\[
\begin{align*}
\text{minimize} & \quad -x^T Ax \\
\text{subject to} & \quad \sum_i x_i = 1, \quad x_i \geq 0 \quad \text{for all } i,
\end{align*}
\]

have nonzero entries \( x_i = \frac{1}{\xi} \) for \( i \in S \), where \( S \) is a maximum clique of \( G \), and \( x_i = 0 \) for all \( i \notin S \). Here, \( \xi = |S| \) is the size of the one or possibly many maximum cliques.

We use the position observable of the qumodes to represent continuous variables, and to impose the non-negativity of each \( x_i \) we substitute \( \tilde{x}_i \) with \( \tilde{x}_i^2 \). The resulting problem Hamiltonian is

\[
\hat{H}_p = -\sum_{i,j} A_{ij} \tilde{x}_i^2 \tilde{x}_j^2 + \lambda \left( \sum_i \tilde{x}_i^2 - 1 \right)^2. \tag{11}
\]

As all entries in the optimal solutions of problem (10) are 0 or \( 1/\xi \), we may substitute each \( x_i \) with \( n_i/\xi \) for an integer variable \( n_i \) and instead solve the problem

\[
\begin{align*}
\text{minimize} & \quad -n^T A n \\
\text{subject to} & \quad \sum_i n_i = \sigma, \quad n_i \in \mathbb{Z}_{\geq 0} \quad \text{for all } i
\end{align*}
\]

using \( \sigma = \xi \). However, \( \xi \) is not a priori known, but we may perform a binary search for its value using \( O(\log(|V|)) \) choices of \( \sigma \in \{1, \ldots, |V|\} \), where \( |V| \) is the number of vertices of the graph. In addition, we can infer the maximum cliques correctly even when \( \sigma \geq \xi \) (see Section S3 for more detail). Using an approach similar to that of Section IV, the resulting problem Hamiltonian for this NLIP can be written as

\[
\hat{H}_p = -\sum_{i,j} A_{ij} \hat{n}_i \hat{n}_j + \lambda \left( \sum_i \hat{n}_i - \sigma \right)^2. \tag{13}
\]

We note that the Hamiltonians (9), (11), and (13) are all quartic in \( a \) and \( a^\dagger \) because \( \hat{n} \) and \( \hat{x}^2 \) are both quadratic in \( a \) and \( a^\dagger \).

In Fig. 4a, we show the progression of AQC using the problem Hamiltonian (11) for solving the MAXCLIQUE problem.
for the graph in Fig. 3a. The solutions are found by performing a homodyne measurement of the \( \hat{x} \) observable for each mode, respectively (i.e., by projection on the \( |x\rangle \) eigenstates). Here, \( |1\rangle_x \) denotes a state with a measured value that is greater than or equal to \( 1/\sqrt{2|V|} \) for the eigenvalue of \( \hat{x} \), while \( |0\rangle_x \) denotes a state with a measured value less than \( 1/\sqrt{2|V|} \). The probabilities are found by performing measurements 1000 times. The five qumodes are measured consecutively with conditional homodyne measurements, and a histogram of the states \( \{n_1, n_2, n_3, n_4, n_5\}_x \) is generated, where \( n_i \in \{0, 1\} \) according to the above definition. The probabilities are found from this histogram.

In Fig. 4b, we show the evolution of the probabilities of both the dominant quantum states (in this case those that have the highest final probability amplitudes) and the vacuum state, after performing AQC using the NLIP representation of the problem Hamiltonian (13). We observe that the continuous evolution shown in Fig. 4a is less performant (with a total success probability of observing a maximum clique of about 55%) than the probabilities shown in Fig. 3b and Fig. 4b for the integer formulations of the MAXCLIQUE problem (both with success probabilities greater than 90%). One reason for this is the penalty relaxation in the Hamiltonian (11) is not equivalent to the original problem (10) and instead provides a lower bound for the optimal value of problem (10). Another reason is that, in our numerical simulations, the Fock space has been truncated to five dimensions due to limitations in classical computational resources. These two facts point to there being a broader probability distribution in the \( x \) direction of the phase space, which can cause a greater variance in the \( \hat{x} \) measurements.

We also observe a greater number of oscillations over time in the ground-state amplitudes for problem (12) compared to the binary formulation (8). This can be understood by the fact that Hamiltonian (13) has larger eigenvalues than Hamiltonian (9); the evolution via Hamiltonian (13) is populated by higher-number Fock states, whereas in Hamiltonian (9) only the values 0 or 1 are allowed for the boson number observable of each qumode. The amplitude oscillations can therefore be reduced by scaling down the problem Hamiltonian by a factor that is less than one while leaving the mixing Hamiltonian unaffected. Experimentally, performing such a normalization can be helpful in situations where the parameters of the quantum optical gates must remain within specific operating ranges. Finally, we note that the difference between the probabilities of observing the different degenerate ground states in Fig. 3b and Fig. 4b can be reduced by tuning the parameters of the initial states and the mixing Hamiltonian in order to achieve more “fair” samples. See Section S2 for more detail.

VI. MIXED-INTEGER PROGRAMMING

Finally, we demonstrate the application of our method in solving MIP problems (i.e., problems that consist of both integer and continuous variables). We use sparse optimization as our example.

Consider the problem

\[
\begin{align*}
\text{minimize} & \quad f(x) = \sum_{i=1}^{3} (x_i - \mu_i)^2 \\
\text{subject to} & \quad \sum_{i=1}^{3} x_i = 3, \quad \|x\|_0 \leq 2, \\
& \quad x_i \geq 0 \text{ for all } i.
\end{align*}
\]

(14)

Here, \( \mu_i \) are constants. The objective function is a quadratic (convex) function of three continuous variables defined on the (also convex) positive two-dimensional simplex \( S = \{ (x_1, x_2, x_3) : \sum_{i=1}^{3} x_i = 3 \} \) in three dimensions. However, the feasible domain for the problem is the 2-skeleton of \( S \) consisting of those points in \( S \) that have at most two nonzero components, \( S \cap \{ x : \|x\|_0 \leq 2 \} \). As a result, the constraints impose a non-convex feasible domain on the problem. Non-convex optimization problems as such are of practical significance in regression, machine learning, and signal processing [28].

We reformulate problem (14) as an MIP problem by adding the binary decision variables \( b_i \):

\[
\begin{align*}
\text{minimize} & \quad f(x) = \sum_{i=1}^{3} (x_i b_i - \mu_i)^2 \\
\text{subject to} & \quad \sum_{i=1}^{3} x_i b_i = 3, \quad \sum_{i=1}^{3} b_i = 2, \\
& \quad b_i \in \{0, 1\} \text{ and } x_i \geq 0 \text{ for all } i.
\end{align*}
\]

(15)

The assignment \( b_i = 1 \) indicates that the corresponding continuous variable \( x_i \) is allowed to attain nonzero values.

To solve this MIP problem using our method, we use six qumodes with the operators \( \hat{X}_i^2 \), for \( i = 1, 2, 3 \), on the first three qumodes representing the continuous variables, and the

![Fig. 4: Probability evolution in solving the MaxClique problem using the Motzkin–Straus framework, for the graph shown in Fig. 3a. (a) Solving the non-convex continuous quadratic problem (10) using the problem Hamiltonian (11). (b) Solving the non-convex integer quadratic problem (12) using the problem Hamiltonian (13).](image-url)
number operators $\hat{n}_i$, for $i = 4, 5, 6$, representing the binary variables $b_{1-3}$. The MIP problem Hamiltonian is given by
\begin{equation}
\hat{H}_P = \sum_{i=1}^{3} (\hat{x}^2_i \hat{n}_{i+3} - \mu_i)^2 + \lambda_1 \left( \sum_{i=1}^{3} \hat{x}^2_i \hat{n}_{i+3} - 3 \right)^2 \\
+ \lambda_2 \left( \sum_{i=1}^{3} \hat{n}_{i+3} - 2 \right)^2 + \lambda_3 \sum_{i=1}^{3} \hat{n}_{i+3}(\hat{n}_{i+3} - 1).
\end{equation}

Performing a homodyne measurement on the first three qumodes of the ground state of this Hamiltonian returns the solutions $x_1$, while a photon number resolving measurement on the second three qumodes reveals the nonzero support of the solution; that is, $x_1$ is taken to be zero if $n_{i+3}$ is zero.

Figure 5 shows the progression of AQC in finding the ground states of the problem Hamiltonian (16) representing the MIP problem (15). Figure 5a shows the time evolution of the probabilities of the states $|n_4, n_5, n_6\rangle = |1, 0, 1\rangle$ and $p_0 = 0.558/6$, for all of the qumodes. (a) Time evolution of the probabilities of the states $|n_4, n_5, n_6\rangle = |1, 0, 1\rangle$ and $p_0 = 0.558/6$, for all of the qumodes. The average measured values of the observables $\hat{x}^2_i$ for the first three qumodes, conditional upon the most probable quantum state $|n_4, n_5, n_6\rangle = |1, 0, 1\rangle$, at the end of AQC are $(\langle \hat{x}^2_1 \rangle, \langle \hat{x}^2_2 \rangle, \langle \hat{x}^2_3 \rangle) \simeq (0.7, 0.4, 1.6)$. The values of $\langle \hat{x}^2_i \rangle$ for these qumodes are calculated by averaging 1000 homodyne measurements. (b-d) The time evolution of the probability distribution of the first three qumodes representing the continuous variables of the sparse optimization problem. The panels show the instantaneous probability distribution at (b) $t = 0$, (c) $t = T/2$, and (d) $t = T$, on the simplex. The discrepancy between the optimal solution (shown using a red dot) and the maximum of the probability distribution (blue dot) at $t = T$ is due to the truncation of the Fock space to $d = 5$ dimensions. Note that the variance of the distribution decreases throughout the adiabatic evolution.

![Figure 5: Progression of AQC in finding the ground states of the problem Hamiltonian (16) representing the MIP problem (15).](image)

VII. CONCLUSION

We have proposed a method for solving mixed-integer programming (MIP) problems using a set of programmable bosonic quantum field modes. Our approach takes advantage of the fact that the eigenspectrum of the bosonic number operators $\hat{n} = \hat{a}^\dagger \hat{a}$ consists of non-negative integers; thus, they can naturally represent integer variables. On the other hand, the quadrature operators $\hat{x} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = \frac{1}{\sqrt{2}} (\hat{a} - \hat{a}^\dagger)$ have continuous spectra and do not commute with each other or with the number operator. Therefore, the MIP problem reduces to the problem of preparing the ground state of a Hamiltonian written in terms of $\hat{n}$ and one of the quadrature operators. For instance, an adiabatic ground-state preparation scheme can be envisaged that uses a mixing Hamiltonian written in terms of the canonically conjugate quadrature operator of the one used in the Hamiltonian specifying the problem.

We have discussed the experimental realization and numerical simulation of this scheme on a continuous-variable, circuit model based quantum optical computer capable of preparing coherent Gaussian states, and performing rotation gates, Kerr gates, and cross-Kerr gates on them. At the end of the ground-state preparation scheme, homodyne and photon number resolving measurements were performed to return the optimal solutions to an MIP problem. We further analyzed a variety of linear and nonlinear MIP problems and provided numerical demonstrations on small instances of NP-hard optimization problems, specifically, the unbounded knapsack problem, the MAXCLIQUE problem, and cardinality-constrained optimization.

This work demonstrates that bosonic continuous-variable quantum computation devices can provide the natural processing capabilities required to solve MIP problems efficiently. Our results, moreover, motivate the experimental realization of a variety of non-Gaussian unitary gates on quantum optical platforms. In addition, our work illuminates an interesting direction for future theoretical research in investigating the computational complexity of the scheme proposed herein in view of adiabatic theorems for Hamiltonians with continuous spectra (see Supplementary Material, Section S4).

The simulations performed in this study rely on the assumption of an ideal (i.e., noiseless) circuit-model bosonic quantum
computing device. To solve large-scale optimization problems, our algorithm requires a large number of Trotter steps, which results in very deep quantum circuits. Such deep circuits necessitate fault-tolerant quantum computation techniques to tackle the detrimental effects of noise and errors inherently present in quantum operations and quantum state preparation and measurement. Fault-tolerant continuous-variable quantum computation using bosonic systems is an active and ongoing field of research [53]–[55], suggesting a need for a separate focused study for a fault-tolerant implementation of the MIP scheme introduced in this work.

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Supplementary Material:
Mixed-Integer Programming Using a Bosonic Quantum Computer

S1. Adiabaticity

One common metric in studying the performance of the adiabatic quantum computation (AQC) algorithm is the adiabaticity of the method. This is done by studying the performance of the algorithm with respect to different total evolution time values \( T \). Figure S1 shows the evolution of the success probabilities of solving the problem, as well as the probabilities of finding suboptimal solutions, for the integer and mixed-integer programming (MIP) problems discussed in the main manuscript. As expected, the solutions for all of the problems show the similar trend of starting at low success probabilities for a small total evolution time \( T \), and plateauing to their final success probability values for larger values of \( T \). These results indicate the optimum total evolution time for a given problem.

S2. Fair Sampling

In this section, we look at fair sampling of the degenerate ground states of the integer programming problems studied. In many applications, it is undesirable that the optimizer gives preference to one or a few of the solutions [56] instead sampling all optimal solutions uniformly. For instance, in the example graph instance shown in Fig. 3a, fair sampling in the MAXCLIQUE problem is attained when both maximum cliques \{1, 2, 4\} and \{1, 3, 4\} are found with equal probabilities by the solver.

Figure S2 shows the fair sampling in the solutions of the integer linear programming (ILP) and MAXCLIQUE problems discussed in the main manuscript. The plots show the biases of the probabilities of finding the solutions as a function of the value of \( p_0 \) defining the mixing Hamiltonian Eq. (3). All of the qumodes are initialized as the squeezed coherent states \( |\alpha, -\alpha\rangle \), where \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \), with \( \alpha_j = ip_0/\sqrt{2} \) for all \( j \). Having six degenerate solutions, the bias in the solutions of the ILP feasibility problem \( n_1 + n_2 = 5 \) is described by the standard deviation of the final probabilities of the solutions states \{0, 5\}, \{1, 4\}, \{2, 3\}, \{3, 2\}, \{4, 1\}, and \{5, 0\}. A zero value for the standard deviation means that all six states have equal final probabilities. Figure S2a shows the standard deviation of the probabilities of the six solution states (shown using a purple curve), as well as the probabilities of the three states \{0, 5\}, \{1, 4\}, and \{2, 3\}, the success probability (the sum of the probabilities of these six states), and the total probability (black curve). Due to the symmetry of the problem between \( n_1 \) and \( n_2 \) in \{\( n_1, n_2 \)\}, the probabilities of finding the other three states are the same as those for the states shown, with the boson number observables swapped. Figure S3 shows the probability distributions of the final quantum states in the two-qumode Fock basis \( \{|n_1, n_2\rangle\} \), for three different \( p_0 \) values. As shown, by changing a single parameter \( p_0 \), one is able to adjust the distribution over the probabilities of the solution states, until a “fair” distribution has been reached. Here, a value of \( p_0/\sqrt{\mathcal{H}} = 0.72 \) gives the most even probability distribution among all solution quantum states for the studied ILP feasibility problem. This is also evident from the minimum of the standard deviation of these probabilities at \( p_0/\sqrt{\mathcal{H}} = 0.72 \) in Fig. S2a.

Figure S2b and Fig. S2c show the results of our fair sampling analysis for the MAXCLIQUE problem using formulations (8) and (12). The bias value is defined as \( b = P_{s1} - P_{s2} \), where \( P_{s1} \) and \( P_{s2} \) are the final probabilities of finding the two degenerate solutions \{1, 2, 4\} and \{1, 3, 4\}, respectively. As shown using a purple curve in Fig. S2b and S2c, a bias of zero means that the two solutions have equal final probabilities.

S3. Parameters in the Motzkin–Straus Formalism of the MaxClique Problem

As mentioned earlier, for the integer representation of the Motzkin–Straus formalism (12), the parameter \( \sigma \) is set equal to the size \( \xi \) of the maximum clique(s) to find a solution with high probability. As this value is not a priori known, a binary search can be performed to find its value by solving \( \mathcal{O}(\log(|V|)) \) instances of problem (12) using different choices of \( \sigma \in \{1, \ldots, |V|\} \). In this section, we study the case when the parameter \( \sigma \) is larger than the size of the maximum clique of the problem and show that even though optimization problem (12) may no longer be equivalent to problem (10), the correct results can still be inferred.

Figure S4 shows the success probability, as well as the probability of finding either of the two solutions, as a function of the hyperparameter \( \lambda \) for the MAXCLIQUE problem shown in Fig. 3a. The probability of finding a particular solution here is defined as the sum of the probabilities of all of the states that have one or more photons in the qumodes corresponding to the maximum clique. For the maximum cliques given by the vertices \{1, 2, 4\} and \{1, 3, 4\}, the solution probabilities are given by

\[
P_{s1} = \sum_{n_1, n_2, n_4 = 1}^{d} \langle n_1, n_2, 0, n_4, 0 | \psi \rangle^2, \quad \text{(S1a)}
\]
The results in Fig. S4 indicate that the maximum number. The results in Fig. S4 indicate that the maximum avalanche detectors (SPAD), where the arrival of one or more given value of $\sigma$ choice of $p$ must be finite. As shown in Fig. S4, there is an optimal $p$ the solution states. The bias in panels (b) and (c) for the M
success probabilities, and standard deviation or bias of the probabilities of the solution states, as a function of $p$ the unbounded knapsack problem (Section III); (c) the binary non-linear integer programming formulation of the M $\overline{\text{CLIQUE}}$ problem (Section IV); (d) the integer Motzkin–Straus formulation of the M $\overline{\text{CLIQUE}}$ problem (Section V); (e) the continuous Motzkin–Straus formulation of the M $\overline{\text{CLIQUE}}$ problem (Section V); and (f) the MIP sparse optimization problem (Section VI). The plots show the evolution of the success probabilities of solving the problems, as well as the probabilities of finding suboptimal solutions, for different values of the total evolution time $T$. Note that here, for each value of $T$, each data point represents the probabilities at the end of an individual adiabatic evolution process, whereas in Fig. 1 to Fig. 5 of the main manuscript, each plot represents a single evolution. Other than the parameter $T$, the parameters used for these simulations are the same as those used in Fig. 1 to Fig. 5.

![adiabaticity plots](image)

Fig. S1: Adiabaticity plots for the problem types studied: (a) simple integer linear programming (see Section III of the main manuscript); (b) the unbounded knapsack problem (Section III); (c) the binary non-linear integer programming formulation of the M $\overline{\text{CLIQUE}}$ problem (Section IV); (d) the integer Motzkin–Straus formulation of the M $\overline{\text{CLIQUE}}$ problem (Section V); (e) the continuous Motzkin–Straus formulation of the M $\overline{\text{CLIQUE}}$ problem (Section V); and (f) the MIP sparse optimization problem (Section VI). The plots show the evolution of the success probabilities of solving the problems, as well as the probabilities of finding suboptimal solutions, for different values of the total evolution time $T$. Note that here, for each value of $T$, each data point represents the probabilities at the end of an individual adiabatic evolution process, whereas in Fig. 1 to Fig. 5 of the main manuscript, each plot represents a single evolution. Other than the parameter $T$, the parameters used for these simulations are the same as those used in Fig. 1 to Fig. 5.

![fair sampling results](image)

Fig. S2: Fair sampling results for the integer linear programming feasibility problem, and for the two non-integer linear programming formulations of the M $\overline{\text{CLIQUE}}$ problem. The success probabilities, and standard deviation or bias of the probabilities of the solution states, as a function of $p_0$, are shown using a solid orange and solid purple curve, respectively. All qumodes are initialized with $p_{01} = p_{02}$. The bias for the integer linear programming problem is evaluated by finding the standard deviation of the probabilities of the solution states. The bias in panels (b) and (c) for the M $\overline{\text{CLIQUE}}$ problem is defined as $b = \frac{P_{12} - P_{11} - P_{22}}{\sqrt{P_{11} + P_{22}}}$. The total probability (black curves) is obtained as the sum of probabilities of all quantum states within the truncated Fock space (resulting in a total probability value smaller than 1).

$$P_{2} = \sum_{n_1, n_2, n_4=1}^{d} |\langle n_1, 0, n_3, n_4, 0|\psi\rangle|^2 ,$$  

(S1b)

respectively, where $|\psi\rangle$ is the final quantum state prepared using AQC. Here, $d$ is the truncation dimension of the Fock space, which is physically infinite but in a classical simulation must be finite. As shown in Fig. S4, there is an optimal choice of $\lambda$ at which the largest success probability for a given value of $\sigma$ is reached. This method of finding the success probabilities aligns well with the functionality of single-photon avalanche detectors (SPAD), where the arrival of one or more photons is detected without the capability to resolve the photon number. The results in Fig. S4 indicate that the maximum cliques correspond to the qumodes that have nonzero photon numbers and thus can be measured using currently available SPADs.

S4. Discussion on Quantum Adiabatic Theorems for Mixed-Integer Programming Problems

Our paper presents numerical studies for solving MIP problems by adiabatic quantum computation using a bosonic system. Our studies rely on using small-sized toy-problem instances to provide proof-of-concept demonstrations for applying the quantum adiabatic algorithm to the type of bosonic Hamiltonians used to solve MIP problems. Such Hamiltonians are characterized by a mixing Hamiltonian that has a contin-

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The maximum clique size is specified by the equation $n_1 + n_2 = 5$. The results are for (a) $p_0 / \sqrt{\pi} = 0.2$, (b) $p_0 / \sqrt{\pi} = 0.5$, and (c) $p_0 / \sqrt{\pi} = 0.72$. Note that the value $p_0 / \sqrt{\pi} = 0.72$ corresponds to the most “fair” sampling of the six solution quantum states. This is also evident in Fig. S2a, from the minimum of the standard deviation of the probabilities of the solution Fock states (shown using a purple curve), at $p_0 / \sqrt{\pi} = 0.72$.

Fig. S4: Probability of solving optimization problem (12) without knowing the maximum clique size $\xi$. Shown is the probability of finding the solutions $P_{s1}$ and $P_{s2}$ (defined in Eqs. (S1b)), as well as the success probability $P_s = P_{s1} + P_{s2}$, when $\sigma > \xi$, as a function of the hyperparameter $\lambda$ in the problem Hamiltonian (13). The results are for (a) $\sigma = 4$ and (b) $\sigma = 5$ for the problem shown in Fig. 3a, for which the maximum clique size is $\xi = 3$. As shown in these plots, by tuning $\lambda$, one is still able to find the solutions to the MAXCLIQUE problem.

In the case of a purely discrete spectrum, the validity of the adiabatic approximation is typically expressed in terms of an upper bound on the rate at which the Hamiltonian can be evolved at any instance of the adiabatic schedule. The critical quantity characterizing this bound is the time-dependent spectral gap between the two lowest eigenvalues of the instantaneous Hamiltonian $[57], [58]$. As determining the instantaneous spectral gap is intractable for most Hamiltonians, the minimum spectral gap is typically used to characterize the lower bound on the overall time of the adiabatic evolution. Adiabatic theorems without a spectral gap condition have also been proposed (see, e.g., [59]), though a conclusion on the required rate of change of the Hamiltonian cannot be drawn from theorems with such propositions. The work [48] attempts to generalize the adiabatic theorem to Hamiltonians with continuous spectra by introducing a notion of spectral gap using the concept of Weyl’s eigendifferentials. While this framework can be extended to spectra with both continuous and discrete characteristics, it cannot be applied to the problems studied in this work, as it is restricted to systems that have a nondegenerate continuous spectrum.

Nevertheless, our numerical studies demonstrate that adiabatic quantum evolution can succeed in finding the ground state of Hamiltonians with degenerate continuous spectra. Our results serve to encourage the development of a more general framework for adiabatic quantum computation, that is, a quantum adiabatic theorem for Hamiltonians that have continuous or mixed discrete–continuous spectra and involve spectral degeneracy and level crossing. Examples of how analyzing the gap of quantum adiabatic evolutions can yield complexity theoretic results in the discrete case can be found in [58].

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