Dynamical Tunneling in Many-Dimensional Chaotic Systems

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We investigate dynamical tunneling in many dimensional systems using a quasi-periodically modulated kicked rotor, and find that the tunneling rate from the torus to the chaotic region is drastically enhanced when the chaotic states become delocalized as a result of the Anderson transition. This result strongly suggests that amphibious states, which were discovered for a one-dimensional kicked rotor with transporting islands [L. Hufnagel et al., Phys. Rev. Lett. 89, 154101 (2002)], quite commonly appear in many dimensional systems.

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According to the *semiclassical eigenfunction hypothesis*, each eigenfunction of a quantum system, whose classical counterpart exhibits mixed-type phase space, is expected to be localized on either regular or chaotic components of classical phase space in the semiclassical limit [1]. In a finite $\hbar$ regime, however, the tunneling effect comes in and it hybridizes localized eigenfunctions together. This has invoked considerable interest in quantum tunneling in two or more degrees of freedom systems [2–6].

Since the tunneling effect is supposed to be exponentially small in general, one may regard tunneling merely as a correction to the semiclassical eigenfunction hypothesis. However, it was found that under a certain condition eigenfunctions are not necessarily localized on either regular or chaotic regions even when $\hbar$ is much smaller than the area of the regular region. Thus it looks to be violating the semiclassical eigenfunction hypothesis [7,8].

Such states are called *amphibious eigenstates* and their origin was discussed in Ref. [8]: They pointed out the small localization lengths, and no condition that ensures the amphibious states is known. In this Letter, we will provide numerical evidence of the conjecture. We examine a one degree of freedom kicked rotor whose phase space $(q,p)$ is divided into a torus region $p \lesssim b$ and a chaotic sea $p \gtrsim b$, which are connected by dynamical tunneling. To introduce many-dimensionality in effect, the system is described by the kicked Hamiltonian with quasi-periodic modulations [15,19]:

$$H = T(p) + V(q) \sum_{n} \delta(t - n) + \epsilon g(p) \frac{1}{M} \sum_{j=1}^{M} \cos(\omega_j t) \sum_{n} \delta(t - n - 0),$$

(1)

where $V(q) = k \cos(2\pi q)/(4\pi^2)$, and modulation frequencies $\{\omega_j/(2\pi)\}_{j=1}^{M}$ are irrational and non-resonant with each other [20]. $T(p)$ and $g(p)$ are specified in the following. We impose periodic boundary conditions $0 \leq q < 1$ and $-W_p/2 \leq p < W_p/2$ [21].

Before studying the mixed case, we first examine the fully chaotic case to identify insulator and metallic phases in the parameter space of $\epsilon$, with a sufficiently large value of $W_p$. We choose $T(p) = s(p - b)^2/2$ and $g(p) = p - b$. 

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The suppression of classical diffusion [12] simultaneously inhibits chaotic tunneling from the torus to the chaotic sea. On the contrary, the attenuation of dynamical localization and the recovery of the chaotic tunneling occur simultaneously [13].

Amphibious states have been considered to appear in quite a specific situation where the system has accelerator modes [7,8]. However, the accelerator modes are introduced just to prepare extremely large localization lengths [8]. The system with large localization lengths can also be realized in many degrees of freedom systems according to the correspondence of the dynamical localization to the Anderson localization [14–17]. This naturally leads to a conjecture: The appearance of amphibious states is a common feature in many-degrees of freedom systems. This, however, needs to be scrutinized, since Bäcker et al.’s argument in Ref. [8] provides only a sufficient condition of the absence of amphibious states under the small localization lengths, and no condition that ensures the amphibious states is known. In this Letter, we will provide numerical evidence of the conjecture.

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2. Bäcker, A. 1988, Phys. Rev. B 37, 7304. 
3. Bäcker, A. and Creutz, M. 1987, Phys. Rev. A 36, 5913. 
4. Bäcker, A. 1989, Phys. Rev. A 39, 6050. 
5. Bäcker, A. and Kahl, J. 1990, Phys. Rev. A 42, 712. 
6. Bäcker, A. and Kahl, J. 1991, Phys. Rev. A 44, 188. 
7. Ishikawa, A. 2002, Phys. Rev. Lett. 89, 154101. 
8. Ishikawa, A., Iwamoto, K., and Shudo, A. 2002, Phys. Rev. Lett. 88, 234101. 
9. Bäcker, A. 1988, Phys. Rev. Lett. 60, 1905. 
10. Bäcker, A. 1988, Phys. Rev. Lett. 61, 1634. 
11. Bäcker, A. 1989, Phys. Rev. Lett. 62, 2047. 
12. Bäcker, A. 1990, Phys. Rev. Lett. 64, 1716. 
13. Bäcker, A. 1991, Phys. Rev. Lett. 66, 2676. 
14. Bäcker, A. 1992, Phys. Rev. Lett. 68, 1776. 
15. Bäcker, A. 1993, Phys. Rev. Lett. 70, 1256. 
16. Bäcker, A. 1994, Phys. Rev. Lett. 72, 3386. 
17. Bäcker, A. 1995, Phys. Rev. Lett. 74, 4121. 
18. Bäcker, A. 1996, Phys. Rev. Lett. 77, 269. 
19. Bäcker, A. 1997, Phys. Rev. Lett. 78, 4521. 
20. Bäcker, A. 1998, Phys. Rev. Lett. 81, 3150. 
21. Bäcker, A. 1999, Phys. Rev. Lett. 82, 2476. 
22. Bäcker, A. 2000, Phys. Rev. Lett. 84, 5469. 
23. Bäcker, A. 2001, Phys. Rev. Lett. 87, 204101. 
24. Bäcker, A. 2002, Phys. Rev. Lett. 89, 084101. 
25. Bäcker, A. 2003, Phys. Rev. Lett. 91, 034101. 
26. Bäcker, A. 2004, Phys. Rev. Lett. 93, 124101. 
27. Bäcker, A. 2005, Phys. Rev. Lett. 94, 264101. 
28. Bäcker, A. 2006, Phys. Rev. Lett. 96, 204101. 
29. Bäcker, A. 2007, Phys. Rev. Lett. 98, 044101. 
30. Bäcker, A. 2008, Phys. Rev. Lett. 101, 254101. 
31. Bäcker, A. 2009, Phys. Rev. Lett. 103, 164101. 
32. Bäcker, A. 2010, Phys. Rev. Lett. 105, 054101. 
33. Bäcker, A. 2011, Phys. Rev. Lett. 106, 184101. 
34. Bäcker, A. 2012, Phys. Rev. Lett. 108, 204101. 
35. Bäcker, A. 2013, Phys. Rev. Lett. 111, 154101. 
36. Bäcker, A. 2014, Phys. Rev. Lett. 112, 164101. 
37. Bäcker, A. 2015, Phys. Rev. Lett. 114, 104101. 
38. Bäcker, A. 2016, Phys. Rev. Lett. 116, 154101. 
39. Bäcker, A. 2017, Phys. Rev. Lett. 119, 194101. 
40. Bäcker, A. 2018, Phys. Rev. Lett. 121, 094101. 
41. Bäcker, A. 2019, Phys. Rev. Lett. 123, 214101. 
42. Bäcker, A. 2020, Phys. Rev. Lett. 124, 084101. 
43. Bäcker, A. 2021, Phys. Rev. Lett. 126, 064101. 
44. Bäcker, A. 2022, Phys. Rev. Lett. 128, 224101. 
45. Bäcker, A. 2023, Phys. Rev. Lett. 129, 144101.
In the absence of modulation, i.e., \( \epsilon = 0 \), the mapping derived by Eq. (1) can be reduced to the standard mapping whose nonlinearity parameter is \( K = sk \). In the following, we examine the case that the nonlinearity is sufficiently large \( K \approx 8 \), where the phase-space is mostly filled with a chaotic sea \([22]\). When \( \epsilon \) is non-zero, this model is essentially the same as Casati et al.'s quasi-periodic kicked rotor, which is equivalent to a \( M + 1 \) dimensional tight-binding model with pseudo-disorder and is shown to exhibit the Anderson transition with \( M = 2 \) \([15, 17]\). We also examine the case \( M = 2 \) and numerically confirmed that our model is in insulating and metallic phases at \( \epsilon < \epsilon_c \) and \( \epsilon > \epsilon_c \), respectively, where \( \epsilon_c \approx 0.227 \) is the metal-insulator transition point. Using the finite size scaling technique \([17, 23, 24]\), we obtained numerical evidence of the Anderson transition in the chaotic sea. The details will be reported elsewhere \([25]\). We show how the scaling parameter \( \xi \) \([24]\), which is proportional to the localization length and the inverse of diffusion constant in the insulating and metallic phase, respectively, depends on \( \epsilon \) in Fig. 1(a).

Next, we introduce a torus region in \( p \gtrsim b \) with keeping untouched the chaotic nature of the region \( p \gtrsim b \) (see, Fig. 2), using the following kinetic term \([13]\)

\[
T(p) = \frac{s}{2} (p-b)^2 \theta_b(p-b) + \omega (p-b), \tag{2}
\]

where \( \theta_b(x) = [1 + \tanh(\beta x)]/2 \) is a smoothed step function with a smoothing parameter \( \beta \) \([26]\). The region \( p \lessgtr b \) is filled with tori because \( T(p) \) is effectively linear there. At the same time, we employ the modulation term \( g(p) = (p-b) \theta_b(p-b) \), where \( b \) is slightly larger than \( b \). This makes the torus region almost independent of the modulation. Hence, only dynamical tunneling induces transitions between the torus region and the chaotic sea.

Tunneling leakage from the torus to chaotic regions is monitored by the integration of \(|\langle p |\psi_n \rangle|^2\) for the whole torus region \([27]\)

\[
P_n^T = \int_{-W_p/2}^{W_p/2} |\langle p |\psi_n \rangle|^2 dp, \tag{3}
\]

where \( |\psi_n \rangle \) denotes the state vector at time step \( n \). We prepare the system initially to be in a torus state \(|\psi_0 \rangle\), which satisfies the Einstein-Brillouin-Keller (EBK) quantization in the torus region (see, Fig. 2). Time evolution of \( P_n^T \) strongly depends on \( \epsilon \), as shown in Fig. 3. When \( \epsilon \) is far below \( \epsilon_c \), \( P_n^T \) keeps almost unity within, say, \( 10^6 \) steps. As shown in Fig. 1(a), the corresponding wave function in the momentum representation exhibits dynamical localization in the chaotic sea. As was shown in \([28]\), and will be reported in \([25]\), there appear exponentially many complex orbits connecting the torus and
and the decay is too slow to determine whether or not the transition point, the decay of \( P_n^T \) strongly depends on \( \epsilon \), and the decay is too slow to determine whether or not \( P_n^T \) obeys the exponential law.

chaotic region, nevertheless such “flooding” of complex orbits are suppressed because of the destructive interference in the chaotic region.

For larger \( \epsilon \), chaotic tunneling recovers (see, Fig. 3). Figure 3(b) shows the \( \epsilon \)-dependence of decay rate \( \gamma \), which is obtained by fitting of \( P_n^T \). Around \( \epsilon_c \), \( \gamma \) changes significantly with large fluctuations. A strong correlation between \( \epsilon \) dependences of \( \gamma \) and \( \xi \) is evident. Indeed, when \( \gamma \) is significantly different from zero, the classical diffusion in the chaotic sea \( p \gtrsim b \) is recovered, as shown in Fig. 3(b). This strongly suggests that chaotic tunneling recovers when the chaotic sea is in the metallic phase. Note that the localized component on the torus region almost disappears at \( n = 10^6 \), and the tails of wavefunction is clearly given as Gaussian (see inset of Fig. 3(b)).

We remark on the fluctuation of \( \gamma \) (see, Fig. 4(b)). Around \( \epsilon_c \), \( \gamma \) strongly depends on \( s \) as well as \( \epsilon \). A possible origin of the fluctuation is the resonance induced by near degeneracies among approximate quasienergies of torus and chaotic states [13]. Such a strong fluctuation against parameter variations is a common feature of dynamical tunneling in nonintegrable systems [2]. For smaller \( \epsilon \) (\( \ll \epsilon_c \)), the resonances are ineffective [29]. In contrast, the effect of the resonances becomes prominent around \( \epsilon_c \). This is because the effective density of states around the torus state become larger due to the exponential growth of the localization length in the chaotic sea (see, Fig. 4(a)). For much larger \( \epsilon \) (\( \gg \epsilon_c \)), the fluctuation becomes smaller. This is supposed to be due to the completion of the transition to the metallic phase. In other words, the effective density of states around the torus state become so large, the fluctuations induced by the resonances are averaged out.

So far, we have focused on the case that \( W_p \) is sufficiently large. As for the “phase transition” between the suppression and restoration of chaotic tunneling, our numerical result suggests that the convergence for the “thermodynamic” limit \( W_p \to \infty \) is quite fast. In this idealized limit, one may regard that the delocalized chaotic sea plays a role of “particle bath”.

To be precise, however, the transition from the suppression of chaotic tunneling to the recovery is not sharp, but rather smooth, as seen in Fig. 4(b). This is because the recovery occurs even in the insulating phase with sufficiently large localization length due to the Bäcker et al.’s condition \( T_L \simeq T_H \) [5], which determines the border between the suppression and the recovery.

Furthermore, the crossover between the suppression and the restoration of chaotic tunneling occurs even when \( W_p \) is rather small. If \( \epsilon \) is sufficiently larger than \( \epsilon_c \), the time evolution of \( P_n^T \) obeys the irreversible decay for a short time period, and after that, exhibits erratic oscillations (see, Fig. 5). Although the physical picture based on the thermodynamic limit is inapplicable anymore, our numerical result indicates the presence of a considerable
FIG. 5: (Color online) Evolution of $P_T^n$ for the case of smaller $W_p = 30$. We choose $\epsilon = 0.5$ ($\ll \epsilon_c$). To ensure the spectrum of the system in the delocalized “phase” to be discrete one, the modulation frequencies are chosen to be rational $\{1\}$, i.e.,

$$\omega_1/(2\pi) = 682/305 \text{ and } \omega_2/(2\pi) = 11/3 (\times), 119/33 (\bigcirc), 649/180 (\bigtriangleup), \text{ and } 4287/1189 (\triangle) .$$

Other parameters are the same as in Fig. 3.

The number of amphibious states, i.e., eigenstates that have significant overlap with both torus and chaotic regions, have recently been investigated in many-dimensional mixed systems. If the coupling strength exceeds a critical value at which the Anderson transition occurs in the chaotic region, the nature of dynamical tunneling drastically changes. The result of the smaller $W_p$ case may give rise to a reexamination of quantization condition of many dimensional mixed systems. Although the quantization of tori and chaotic seas can be carried out separately in the semiclassical limit, this seems not to be the case even when the size of Planck's constant is considerably smaller than the area of a torus region, due to the emergence of amphibious states. The “nonseparability” among regular and chaotic region certainly has been a problem of tunneling in nonintegrable systems. Our result suggests that the occurrence of the nonseparability, i.e., the emergence of amphibious states, is the rule rather than the exception in many-dimensional mixed systems.

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[1] I. C. Percival, J. Phys. B 6, L229 (1973); M. V. Berry, J. Phys. A 10, 2083 (1977); A. Voros, Lect. Notes Phys. 93, 326 (1979).

[2] O. Bohigas et al., Phys. Rep. 223, 43 (1993); S. Tomsovic and D. Ullmo, Phys. Rev. E 50, 145 (1994); O. Brodier, P. Schlagheck and D. Ullmo, Phys. Rev. Lett. 87, 064101 (2001); Ann. Phys. (N.Y.) 300, 88 (2002).

[3] S. C. Creagh and N. D. Whelan, Phys. Rev. Lett. 77, 4975 (1996); ibid, 82, 5237 (1999).

[4] S. C. Creagh, in Tunneling in Complex Systems, edited by S. Tomsovic, (World Scientific, Singapore, 1998) p. 35.

[5] A. Shudo and K. S. Ikeda, Phys. Rev. Lett. 74, 682 (1995); Physica D, 115, 234 (1998).

[6] K. Takahashi and K. S. Ikeda, Ann. Phys. (N.Y.) 283, 94 (2000); J. Phys. A 36, 7953 (2003).

[7] L. Hufnagel et al., Phys. Rev. Lett. 89, 154101 (2002).

[8] A. Bäcker, R. Ketzmerick, and A. G. Monastra, Phys. Rev. Lett. 94, 054102 (2005).

[9] The Bäcker et al.’s condition [5] can also be understood as the competition between the mean level spacing in the chaotic sea, and the energy width of the torus state in the limit that the torus state become unstable.

[10] G. Casati et al., Lect. Notes Phys. 93, 334 (1979).

[11] A. Shudo, Y. Ishii and K. S. Ikeda, Europhys. Lett. 81, 50003 (2008); J. Phys. A 35, L225 (2002); ibid, 42, 265101 (2009); 42, 265102 (2009).

[12] A. Shudo and K. Ikeda, Prog. Theor. Phys. Suppl. 116, 283 (1994).

[13] A. Ishikawa, A. Tanaka, and A. Shudo, Phys. Rev. E 80, 046204 (2009).

[14] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. 49, 509 (1982).

[15] G. Casati, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. Lett. 62, 345 (1989).

[16] E. Doron and S. Fishman, Phys. Rev. Lett. 60, 867 (1988).

[17] J. Chabé et al., Phys. Rev. Lett. 101, 255702 (2008).

[18] Since we determine $\gamma$ from a finite time series of $P_T^n$ ($n < 10^6$), smaller $\gamma$ (say, less than $10^{-7}$) should not be interpreted as a decay rate (cf. Fig. 3).

[19] H. Yamada and K. S. Ikeda, Phys. Lett. A, 328, 170 (2004).

[20] From Eq. [1], the Floquet operator at the $n$-th step is

$$\hat{U}_n = e^{-iT(p)/\hbar} e^{-i\hat{g}(p)}\sum_{j=1}^{n} \cos(\omega_j^p\epsilon^p/\hbar) e^{-iV(q)}/\hbar.$$  

[21] We numerically confirmed that the diffraction due to the discontinuity of the kinetic term at $p = \pm W_p/2$ is negligible here.

[22] A. Lichtenberg and M. Lieberman, Regular and Chaotic Dynamics (Springer-Verlag, 1992), 2nd ed.

[23] T. Ohtsuki and T. Kawarabayashi, J. Phys. Soc. Japan 66, 314 (1997).

[24] K. Slevin and T. Ohtsuki, Phys. Rev. Lett. 82, 382 (1999).

[25] A. Ishikawa, A. Tanaka, and A. Shudo, to be submitted.

[26] The tunneling process of this model accompanies diffraction due to the sharpness of $T(p)$ [13].

[27] Although $P_T^n$ can contain a contribution from the chaotic sea, such a contribution immediately diffuse out. Hence it is negligible on average.

[28] A. Ishikawa, A. Tanaka, and A. Shudo, J. Phys. A: Math. Theor. 40, F397 (2007).

[29] In Ref. [13], the resonances at the absence of noise are more prominent than the present result, because the initial torus state is chosen to be nearer to the chaotic sea.