Analytic Solutions of the Regge-Wheeler Equation
and the Post-Minkowskian Expansion

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Abstract
Analytic solutions of the Regge-Wheeler equation are presented in the form of series of hypergeometric functions and Coulomb wave functions which have different regions of convergence. Relations between these solutions are established. The series solutions are given as the Post-Minkowskian expansion with respect to a parameter \( \epsilon \equiv 2M\omega \), \( M \) being the mass of black hole. This expansion corresponds to the post-Newtonian expansion when they are applied to the gravitational radiation from a particle in circular orbit around a black hole. These solutions can also be useful for numerical computations.

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1 Introduction

In our previous work\cite{1}, we presented analytic solutions of Regge-Wheeler (RW) equation in the form of series of hypergeometric functions. We proved that recurrence relations among hypergeometric functions as given in Appendix A in this text and showed that coefficients of series are systematically determined in the power series of $\epsilon \equiv 2M\omega$, $M$ being the mass of black hole. We also presented analytic solutions in the form of series of Coulomb wave functions which turn out to be the same as those given by Leaver\cite{2}. We found that the series of solutions are characterized by the renormalized angular momentum which turns out to be identical. Then, we obtained a good solution by matching these two types of solutions.

This method can be extended for Teukolsky equation\cite{3} in Kerr geometry. In this case, the coefficients of series of hypergeometric functions and also those of series of Coulomb wave functions series satisfy the three term recurrence relations. About these recurrence relations, Otchik\cite{4} made an important observation that the recurrence relation for both series are identical, which enabled to relate these two solutions\cite{4}. Following the discussion by Otchik\cite{4}, Mano, Suzuki and Takasugi\cite{5} extended our analysis to Teukolsky equation in Kerr geometry and reported analytic solutions. We discussed the convergence regions of these series and the relation between two solutions of difference regions of convergences. The series are expressed in the $\epsilon$ expansion which corresponds to the Post-Minkowskian expansion and also to the post-Newtonian expansion when they are applied to the gravitational radiation from a particle in circular orbit around a black hole.

In this paper, we present analytic solutions of RW equation and discuss the analytic properties of these solutions by reorganizing our previous work\cite{1} following our work on Teukolsky equation\cite{5}. Solutions are given in the form of series of hypergeometric functions and Coulomb wave functions and the convergence of these series are studied.

\footnote{In his paper, the relation between the series of hypergeometric functions and the series of Coulomb wave functions are related in the intermediate region where both series converge, though the series which he treated are not the solutions of Teukolsky equation.}
The relation of these solutions with different regions of convergence are discussed. The \( \epsilon \) expansions of solutions are given up to the second order. It is not difficult to obtain solutions in the expansion of \( \epsilon \) which are used to estimate physical quantities in the gravitational wave astrophysics as discussed by Poisson and Sasaki\[6\]. It is interesting to compare solutions given in \( \epsilon \) expansion and numerical solutions to test the accuracy of the \( \epsilon \) expansion. Since the convergence of series is fast, solutions can also be used for numerical computations.

The Regge-Wheeler equation is given by\[7\]

\[
X'' + \left[ \frac{1}{z - \epsilon} - \frac{1}{z} \right] X' + \left[ 1 + \frac{2\epsilon}{z - \epsilon} + \frac{\epsilon^2}{(z - \epsilon)^2} - \frac{l(l + 1)}{z(z - \epsilon)} + \frac{3\epsilon}{z^2(z - \epsilon)} \right] X = 0,
\]

where \( \epsilon = 2M\omega, z = \omega r \), \( M \) being the mass of Schwarzschild black hole and \( \epsilon \) the angular frequency. In the following, we summarize our result.

In Sec.II, we derive analytic solutions of RW equation in the form of series of hypergeometric functions. They are \( X_{\nu}^0 \) and \( X_{-\nu-1}^0 \) which satisfy the incoming boundary condition and the outgoing boundary condition on the horizon. These solutions are labeled by the renormalized angular momentum \( \nu = l + O(\epsilon) \) which is determined by requiring the convergence of the series. It can be shown that \( -\nu - 1 \) is also the renormalized angular momentum. Solutions \( X_{\nu}^0 \) and \( X_{-\nu-1}^0 \) are invariant under the exchange of \( \nu \) and \( -\nu - 1 \). These solutions can be expressed as combinations of other type of independent solutions \( X_{\nu}^0 \) and \( X_{-\nu-1}^0 \). These series solutions are convergent except for infinity as shown in Appendix B.

In Sec.III, solutions are given in the form of series of Coulomb wave functions. They are \( X_{C}^\nu \) and \( X_{C}^{-\nu-1} \). They are shown to be convergent for \( z > \epsilon \) as shown in Appendix B. These solutions can be expressed by linear combination of solutions which satisfy the incoming and outgoing boundary conditions, \( X_{C}^\nu_{in} \) and \( X_{C}^\nu_{out} \). Because the three term recurrence relation among coefficients is the same of the hypergeometric function series, the renormalized angular momentum for the Coulomb wave function series is the same as for the hypergeometric case.
In Sec.IV, we show that $X^\nu_C$ is proportional to $X^\nu_0$. This is due to the fact that both series are specified by the same renormalized angular momentum $\nu$. The same is true for and $X^{-\nu-1}_C$ and $X^{-\nu-1}_0$. Now we have analytic solutions both of which is expressed in the form of series of hypergeometric functions which is convergent except infinity and is expressed in the form of Coulomb wave functions which is convergent for $z > \epsilon$.

In Sec.V, we discuss the relation between solutions of RW equation and solutions of Teukolsky equation. For some solutions, the relations can be explicitly proved.

In Sec.VI, the Post-Minkowski ($\epsilon$) expansion of solutions are given up to the second order of $\epsilon$. The renormalized angular momentum $\nu$ is expressed as a form of $\epsilon$ expansion. We present the second order result.

Summary and discussions are given in Sec.VII and the derivation of the three term recurrence relations among hypergeometric functions and among Coulomb wave functions which are crucial to derive the three term recurrence relation among coefficients are given.

2 Analytic solution in the form of series of hypergeometric functions

The RW equation has two regular singularities at $r = 0, 2M$ and an irregular singularity at $r = \infty$. In order to obtain the solution in the form of series of hypergeometric functions, we have to deal with these regular singularities. In particular, we eliminate the terms of $1/z^2$ and $1/(z-\epsilon)^2$ in Eq.(1.1) by taking the parameterization

$$X^\nu_m = e^{i\epsilon(x-1)}(-x)^{-i\epsilon}(1-x)^{-1}p^\nu_m(x),$$

(2.1)

where $x = 1 - r/2M = 1 - z/\epsilon$ and $\epsilon = 2M\omega$. Then, the RW equation becomes

$$x(1-x)p^{\nu''}_m + [1 - 2i\epsilon + 2(1+i\epsilon)x]p^{\nu'}_m + (l - 1 - i\epsilon)(l + 2 + i\epsilon)p^\nu_m$$

$$= -i\epsilon[2x(1-x)p^{\nu'}_m + 2xp^\nu_m + (1+i\epsilon)p^\nu_m],$$

(2.2)
We obtain the solution of this equation in the form of series of hypergeometric functions. For this, we introduce a parameter $\nu$ and seek the solution in the form

$$ p_\nu(x) = \sum_{n=-\infty}^{\infty} a_n^\nu p_{n+\nu}(x), \quad (2.3) $$

where

$$ p_{n+\nu}(x) = \frac{\Gamma(n + \nu - 1 - i\epsilon)\Gamma(-n - \nu - 2 - i\epsilon)}{\Gamma(1 - 2i\epsilon)} \times F(n + \nu - 1 - i\epsilon, -n - \nu - 2 - i\epsilon; 1 - 2i\epsilon; x). \quad (2.4) $$

As an initial condition, we set that $a_n^\nu = 1$ for $n = 0$ and 0 for others so that $\nu = l + O(\epsilon)$.

In order for the coefficients of series (2.3) to be solved, it is essential that the coefficients $a_n^\nu$'s satisfy the three term recurrence relation. For this, it is essential that terms such as $x(1-x)p'_{n+\nu}$ and $xp_{n+\nu}$ are expressed as linear combinations of $p_{n+\nu+1}$, $p_{n+\nu}$ and $p_{n+\nu-1}$. This turns out to be true and the recurrence relations and the derivation of these relations are given in Appendix A.

By substituting the form in Eq.(2.3) into the radial RW equation (2.2) and using the recurrence relations in Eqs.(A.7) and (A.9) in Appendix A, we find that $p_\nu(x)$ becomes a solution if the following recurrence relation among the coefficients is satisfied:

$$ \alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0, \quad (2.5) $$

where

$$ \alpha_n^\nu = -\frac{i\epsilon(n + \nu - 1 + i\epsilon)(n + \nu - 1 - i\epsilon)(n + \nu + 1 - i\epsilon)}{(n + \nu + 1)(2n + 2\nu + 3)}, \quad (2.6) $$

$$ \beta_n^\nu = (n + \nu)(n + \nu + 1) - l(l + 1) + 2\epsilon^2 + \frac{\epsilon^2(4 + \epsilon^2)}{(n + \nu)(n + \nu + 1)}, \quad (2.7) $$

$$ \gamma_n^\nu = \frac{i\epsilon(n + \nu + 2 + i\epsilon)(n + \nu + 2 - i\epsilon)(n + \nu + i\epsilon)}{(n + \nu)(2n + 2\nu - 1)}. \quad (2.8) $$

By introducing the continued fractions

$$ R_n(\nu) = \frac{a_n^\nu}{a_{n-1}^\nu}, \quad L_n(\nu) = \frac{a_n^\nu}{a_{n+1}^\nu}, \quad (2.9) $$
we find
\[
R_n(\nu) = -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}(\nu)}, \quad L_n(\nu) = -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}(\nu)}.
\] (2.10)

From these equations, we can evaluate the coefficients by taking the initial condition \(a_0^\nu = 1\). The parameter \(\nu\) is determined by requiring that the coefficients obtained by using \(R_n(\nu)\) agree with those by using \(L_n(\nu)\). This is the condition of the convergence of the series. This condition gives the transcendental equation for \(\nu\)
\[
R_n(\nu)L_{n-1}(\nu) = 1. \tag{2.11}
\]

The solution \(\nu\) of this equation is called the renormalized angular momentum. By using this \(\nu\), the series is shown to be convergent in all the complex plane of \(x\) except for \(x = \infty\) as discussed in Appendix B.

The normalization of \(X_n^\nu\) is given on the horizon \((x = \epsilon)\) as
\[
X_n^\nu \rightarrow \left(e^{-i\epsilon} \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(n + \nu - 1 - i\epsilon)\Gamma(-n - \nu - 2 - i\epsilon)}{\Gamma(1 - 2i\epsilon)}\right) e^{-i\epsilon\ln(-x)}. \tag{2.12}
\]

Now we can show that \(-\nu+1\) satisfies Eq.(2.13) so that \(-\nu-1\) is also the renormalized angular momentum\[5\]. As for the recurrence relation (2.8), we find \(\alpha_{-n}^{-\nu-1} = \gamma_n^\nu\) and \(\gamma_{-n}^{-\nu-1} = \alpha_n^\nu\) so that \(a_{-n}^{-\nu-1}\) satisfies the same recurrence relation as \(a_n^\nu\) does. Thus if we choose \(a_0^\nu = a_0^{-\nu-1} = 1\), we have
\[
a_n^\nu = a_{-n}^{-\nu-1}. \tag{2.13}
\]

By using the formula
\[
p_{n+\nu}(x) = \frac{\Gamma(-n - \nu - 2 - i\epsilon)\Gamma(2n + 2\nu + 1)}{\Gamma(n + \nu + 3 - i\epsilon)(1 - x)^{n+\nu+2+i\epsilon}}
\times \frac{\Gamma(n + \nu - 1 - i\epsilon)\Gamma(-2n - 2\nu - 1)}{\Gamma(-n - \nu + 2 - i\epsilon)(1 - x)^{-n-\nu+1+i\epsilon}}
\times F(-n - \nu - 2 - i\epsilon, -n - \nu + 2 - i\epsilon; 2n + 2\nu + 2; \frac{1}{1 - x})
\]
\[
+ \frac{\Gamma(n + \nu - 1 - i\epsilon)\Gamma(-2n - 2\nu - 1)}{\Gamma(-n - \nu + 2 - i\epsilon)(1 - x)^{-n-\nu+1+i\epsilon}}
\times F(n + \nu - 1 - i\epsilon, n + \nu + 3 - i\epsilon; 2n + 2\nu + 2; \frac{1}{1 - x}), \tag{2.14}
\]
we can show
\[ X_\nu^\nu = X_0^\nu + X_0^{-\nu-1}, \]  
(2.15)

where
\[ X_0^\nu = e^{i\epsilon(x-1)}(-x)^{-i\epsilon} (1-x)^{\nu+1+i\epsilon} \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(-n-\nu-2-i\epsilon)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+3-i\epsilon)} \times (1-x)^n F(-n-\nu-2-i\epsilon,-n-\nu+2-i\epsilon;-2n-2\nu;\frac{1}{1-x}). \]  
(2.16)

We consider the solution which satisfies the outgoing boundary condition on the horizon. The outgoing solution is parametralized by
\[ X^\nu_{\text{out}} = e^{i\epsilon(x-1)}(-x)^{i\epsilon} (1-x)^{-1} p^\nu_{\text{out}}. \]  
(2.17)

Then \( p^\nu_{\text{out}} \) should satisfy the following equation,
\[ x(1-x)p''_{\text{out}} + [1 + 2i\epsilon + 2(1-i\epsilon)x]p'_{\text{out}} + (l - 1 + i\epsilon)(l + 2 - i\epsilon)p_{\text{out}} = -i\epsilon[2x(1-x)p'_{\text{out}} + 2(1-2i\epsilon)x p_{\text{out}} + (1+5i\epsilon) p_{\text{out}}], \]  
(2.18)

Now we expand \( p^\nu_{\text{out}} \) as
\[ p^\nu_{\text{out}}(x) = \sum_{n=-\infty}^{\infty} \tilde{a}_n^\nu \tilde{p}_{n+\nu}(x), \]  
(2.19)

where
\[ \tilde{p}_{n+\nu}(x) = \frac{\Gamma(n+\nu-1+i\epsilon)\Gamma(-n-\nu-2+i\epsilon)}{\Gamma(1+2i\epsilon)} \times F(n+\nu-1+i\epsilon,-n-\nu-2+i\epsilon;1+2i\epsilon;x). \]  
(2.20)

Similarly to the solution satisfying the incoming boundary condition, we find that the above series becomes a solution if \( \tilde{a}_n^\nu \) satisfies the same three term recurrence relation as \( a_n^\nu \) does. Then, by choosing \( \tilde{a}_0^\nu = \tilde{a}_{-\nu-1} = 1 \), we find
\[ \tilde{a}_0^\nu = a_0^\nu. \]  
(2.21)

By using Eq.(2.21), the outgoing solution is given by
\[ X^\nu_{\text{out}} = e^{i\epsilon(x-1)}(-x)^{i\epsilon} (1-x)^{-1} \sum_{n=-\infty}^{\infty} a_n^\nu \tilde{p}_{n+\nu}(x) \]
\[ = A_\nu X_0^\nu + A_{-\nu-1} X_0^{-\nu-1}, \]  
(2.22)
where
\[ A_\nu = \frac{\sin \pi (\nu + i\epsilon)}{\sin \pi (\nu - i\epsilon)}. \] (2.23)

This relation also shows that \( X_0^\nu \) and \( X_0^{-\nu - 1} \) are independent solutions of Eq.(2.1).

### 3 Analytic solutions in the form of series of Coulomb wave functions

Analytic solution in the form of series of Coulomb wave functions are given by Leaver\(^2\). Here, we parameterization to remove the singularity at \( r = 2M \). By using a variable \( z = \omega r = \epsilon(1 - x) \), we take the following form

\[ X_C^\nu = \left(1 - \frac{\epsilon}{z}\right)^{-i\epsilon} f_\nu(z). \] (3.1)

Then, we find

\[
\begin{align*}
z^2 f''_\nu + \left[z^2 + 2\epsilon z - l(l + 1)\right] f_\nu &= \epsilon \left[z(f''_\nu + f_\nu) - (1 - 2i\epsilon)f'_\nu - \frac{1}{z}(3 + 2i\epsilon + \epsilon^2)f_\nu - \epsilon f_\nu\right].
\end{align*}
\] (3.2)

If we consider \( \nu \) to be \( \nu = l + O(\epsilon) \), the right-hand side of Eq.(3.2) is the quantity of order \( \epsilon \) so that this equation is a suitable one to obtain the solution in the expansion of \( \epsilon \).

Here we shall obtain the exact solution by expanding \( f_\nu(z) \) in terms of Coulomb functions with the renormalized angular momentum \( \nu \),

\[
f_\nu = \sum_{n=-\infty}^{\infty} i^n \frac{\Gamma(n + \nu - 1 - i\epsilon)\Gamma(n + \nu + 1 - i\epsilon)}{\Gamma(n + \nu + 1 + i\epsilon)\Gamma(n + \nu + 3 + i\epsilon)} b_n^\nu F_{n+\nu}(z),
\] (3.3)

where \( F_{n+\nu} \) is the unnormalized Coulomb wave function,

\[
F_{n+\nu} = e^{-iz(2z)^{n+\nu}} z^{\frac{\Gamma(n + \nu + 1 + i\epsilon)}{\Gamma(2n + 2\nu + 2)}} \Phi(n + \nu + 1 + i\epsilon, 2n + 2\nu + 2; 2iz).
\] (3.4)

By substituting Eq.(3.3) into Eq.(3.2) and using the recurrence relations in Eqs.(A.18) and (A.19) in Appendix A, we find that the coefficients \( b_n^\nu \) satisfy the same three term
recurrence relation as \( a'_n \) does. Now we choose
\[
b'_n = a'_n.\tag{3.5}
\]

The fact that the recurrence relation for the Coulomb expansion case is the same to the one for the hypergeometric case means that the renormalized angular momenta \( \nu \) are the same for both solutions. This is important for these two solutions to be related each other. Therefore, \(-\nu - 1\) is also the renormalized angular momentum. As a result, another independent solution is obtained by replacing \( \nu \) with \(-\nu - 1\). Thus two independent solutions are \( X_C^\nu \) and \( X_{C}^{-\nu-1} \) and \( X_C^\nu \) is given by
\[
X_C^\nu = \left(1 - \frac{\epsilon}{z}\right)^{-i\epsilon} \sum_{n=-\infty}^{\infty} i^n \frac{\Gamma(n + \nu - 1 - i\epsilon) \Gamma(n + \nu + 1 - i\epsilon) a_n^\nu \text{F}_{\nu+n}(z)}{\Gamma(n + \nu + 1 + i\epsilon) \Gamma(n + \nu + 3 + i\epsilon)} a_n^\nu F_{\nu+n}(z). \tag{3.6}
\]

Another forms of these solutions suitable for examining the asymptotic behavior are
\[
X_C^\nu = X_C^{\nu \text{in}} + X_C^{\nu \text{out}}, \tag{3.7}
\]
\[
X_{C}^{-\nu-1} = X_{C \text{in}}^{-\nu-1} + X_{C \text{out}}^{-\nu-1}
\]
\[
= i\epsilon^{-i\pi(\nu+1)} \frac{\sin \pi(\nu + i\epsilon)}{\sin \pi(\nu - i\epsilon)} X_C^\nu - i\epsilon^{i\pi(\nu+1)} X_C^{\nu \text{out}}, \tag{3.8}
\]
where
\[
X_C^{\nu \text{out}} = e^{iz \nu + 1} \left(1 - \frac{\epsilon}{z}\right)^{-i\epsilon} 2^{\nu} e^{-\pi \epsilon} e^{-i\pi(\nu+1)}
\]
\[
\times \sum_{n=-\infty}^{\infty} i^n \frac{\Gamma(n + \nu - 1 - i\epsilon) \Gamma(n + \nu + 1 - i\epsilon) a_n^\nu (-2z)^n}{\Gamma(n + \nu + 1 + i\epsilon) \Gamma(n + \nu + 3 + i\epsilon)} a_n^\nu (-2z)^n
\]
\[
\times \Psi(n + \nu + 1 - i\epsilon, 2n + 2\nu + 2; -2iz), \tag{3.9}
\]
\[
X_C^{\nu \text{in}} = e^{-iz \nu + 1} \left(1 - \frac{\epsilon}{z}\right)^{-i\epsilon} 2^{\nu} e^{-\pi \epsilon} e^{i\pi(\nu+1)}
\]
\[
\times \sum_{n=-\infty}^{\infty} i^n \frac{\Gamma(n + \nu - 1 - i\epsilon) \Gamma(n + \nu + 1 - i\epsilon) a_n^\nu (-2z)^n}{\Gamma(n + \nu + 3 + i\epsilon)} a_n^\nu (-2z)^n \Psi(n + \nu + 1 + i\epsilon, 2n + 2\nu + 2; 2iz). \tag{3.10}
\]

In the end of this section, we present another set of solutions as
\[
\tilde{X}_C^\nu = \left(1 - \frac{\epsilon}{z}\right)^{i\epsilon} \sum_{n=-\infty}^{\infty} i^n \frac{\Gamma(n + \nu - 1 + i\epsilon) \Gamma(n + \nu + 1 - i\epsilon) a_n^\nu \text{F}_{\nu+n}(z)}{\Gamma(n + \nu + 1 + i\epsilon) \Gamma(n + \nu + 3 + i\epsilon)} a_n^\nu \text{F}_{\nu+n}(z). \tag{3.11}
\]
where \( F_{\nu+n} \) is defined in Eq.(3.4). In this parameterization, \( \tilde{X}_C^\nu \) and \( \tilde{X}_C^{-\nu-1} \) are independent solutions. There are two sets of independent solutions, \( \{X_C^\nu, X_C^{-\nu-1}\} \) and \( \{\tilde{X}_C^\nu, \tilde{X}_C^{-\nu-1}\} \). In the following, we consider a set \( \{X_C^\nu, X_C^{-\nu-1}\} \) which are related directly to a set of hypergeometric type solutions \( \{X_0^\nu, X_0^{-\nu-1}\} \) given in the proceeding section.

4 The relation between two solutions

First we notice that \( R_0^\nu \) and \( R_C^\nu \) are solutions of Teukolsky equation. Second we see that if we expand these solutions in Laurent series of \( 1 - x = z/\epsilon \kappa \), both solutions give the series with the same characteristic exponent at \( x \to \infty \). Thus, \( R_0^\nu \) must be proportional to \( R_C^\nu \),

\[
X_0^\nu = K^\nu X_C^\nu. \tag{4.1}
\]

The constant factor \( K^\nu \) is determined by comparing like terms of these series. We find

\[
K^\nu = -\frac{\pi i^2 2^{-\nu-r}\epsilon^{-\nu-r-1}}{\Gamma(1+r+\nu+i\epsilon)\Gamma(-1+r+\nu+i\epsilon)\Gamma(3+r+\nu+i\epsilon)\sin\pi(\nu+i\epsilon)} \times \sum_{n=r}^{\infty} \frac{\Gamma(n+\nu-1+i\epsilon)\Gamma(n+r+2\nu+1)}{(n-r)!\Gamma(n+\nu+3-i\epsilon)} a^\nu_n \\
\times \left[ \sum_{n=-\infty}^{r} \frac{\Gamma(n+\nu-1-i\epsilon)\Gamma(n+\nu+1-i\epsilon)\Gamma(n+\nu+3+i\epsilon)\Gamma(n+r+2\nu+2)}{(r-n)!\Gamma(n+\nu+3+i\epsilon)} a^\nu_n \right]^{-1}, \tag{4.2}
\]

where \( r \) is an arbitrary integer.

By using these relations, \( R_{in}^\nu \) can be written by using the Coulomb expansion solutions as

\[
X_{in}^\nu = (K^\nu X_{cin}^\nu + K_{-\nu-1}X_{cin}^{-\nu-1}) + (K^\nu X_{cout}^\nu + K_{-\nu-1}X_{cout}^{-\nu-1}) \\
= \left[ K^\nu + ie^{-i\pi(\nu+1)\sin\pi(\nu+i\epsilon)/\sin\pi(\nu-i\epsilon)}K_{-\nu-1} \right] X_{cin}^\nu + (K^\nu - ie^{i\pi(\nu+1)\sin\pi(\nu-i\epsilon)}/K_{-\nu-1}) X_{cout}^\nu. \tag{4.3}
\]

The asymptotic behavior at \( z \to \infty \) is

\[
X_{in}^\nu = A_{out}^\nu e^{iz} + A_{in}^\nu e^{-iz}, \tag{4.4}
\]
where $A_{\text{out}}^\nu$ and $A_{\text{in}}^\nu$ are amplitudes of the outgoing and incoming waves at infinity of the solution which satisfy the incoming boundary condition at the outer horizon. They are given by

$$A_{\text{out}}^\nu = e^{-\frac{\pi}{2}2^{-1+i\epsilon}}\left[K_\nu(-i)^{\nu+1} + K_{-\nu-1}(-i)^{\nu}\right] \sum_{n=-\infty}^{\infty} b_n^\nu(-i)^n, \quad (4.5)$$

and

$$A_{\text{in}}^\nu = e^{-\frac{\pi}{2}2^{-1-i\epsilon}}\left[K_\nu i^{\nu+1} + K_{-\nu-1}(-i)^{\nu}\frac{\sin\pi(\nu + i\epsilon)}{\sin\pi(\nu - i\epsilon)}\right] \times \sum_{n=-\infty}^{\infty} b_n^\nu i^n \frac{\Gamma(n + \nu + 1 + i\epsilon)}{\Gamma(n + \nu + 1 - i\epsilon)}. \quad (4.6)$$

One application of these amplitudes is to derive the absorption coefficients. The absorption coefficient $\Gamma$ can be expressed in terms of $A_{\text{in}}^\nu$ and $A_{\text{out}}^\nu$ as follows;

$$\Gamma^\nu = 1 - \left|\frac{A_{\text{out}}^\nu}{A_{\text{in}}^\nu}\right|^2 \quad (4.7)$$

The upgoing solution which satisfy the outgoing boundary condition at infinity is given by $X_{\text{Cout}}^\nu$. This solution is expressed in terms of $X_0^\nu$ and $X_0^{-\nu-1}$ as follows

$$X_{\text{up}}^\nu \equiv \frac{\Gamma(\nu - 1 - i\epsilon)\Gamma(\nu + 1 - i\epsilon)}{\Gamma(\nu + 1 + i\epsilon)\Gamma(\nu + 3 + i\epsilon)} X_{\text{Cout}}^\nu = \left[\frac{\sin\pi(\nu + i\epsilon)}{\sin\pi(\nu - i\epsilon)}(K_\nu)^{-1}X_0^\nu - ie^{i\pi\nu}(K_{-\nu-1})^{-1}X_0^{-\nu-1}\right]$$

$$\times \left[e^{2i\pi\nu} + \frac{\sin\pi(\nu + i\epsilon)}{\sin\pi(\nu - i\epsilon)}\right]^{-1}. \quad (4.8)$$

5 Relation between solutions of Teukolsky and Regge-Wheeler equations

Let us present some solutions of Teukolsky equation in Ref.1 for spin $s = -2$ in the Schwarzschild limit ($a \to 0$):
The solution which satisfies the incoming boundary condition on the horizon is expressed as a series of hypergeometric functions as

\[ R^\nu_{\text{in}(-2)} = e^{i\epsilon x}(-x)^{2-i\epsilon} \sum_{n=-\infty}^{\infty} a^\nu_{n,T} F(n + \nu + 1 - i\tau, -n - \nu - i\tau; 3 - 2i\epsilon; x) \] (5.1)

and coefficients \( a^\nu_{n,T} \) are determined by the three term recurrence relation,

\[
\alpha^\nu_{n,T} a^\nu_{n+1} + \beta^\nu_{n} a^\nu_{n} + \gamma^\nu_{n} a^\nu_{n-1} = 0, \tag{5.2}
\]

\[
\alpha^\nu_{n,T} = i\epsilon (n + \nu - 1 + i\epsilon)(n + \nu - 1 - i\epsilon)(n + \nu + 1 + i\epsilon) \frac{1}{(n + \nu + 1)(2n + 2\nu + 3)}, \tag{5.3}
\]

\[
\gamma^\nu_{n} = -i\epsilon (n + \nu + 2 + i\epsilon)(n + \nu + 2 - i\epsilon)(n + \nu - i\epsilon) \frac{1}{(n + \nu)(2n + 2\nu - 1)}, \tag{5.4}
\]

and most importantly \( \beta^\nu_{n} \) is the same as the one in Eq.(2.9). By comparing the recurrence relation in Eq.(2.5) and (5.4), we find with the condition with \( a^\nu_{0,T} = a^{-\nu-1,T} = 1 \) that

\[
a^\nu_{n,T} = (-1)^n \frac{\Gamma(n + \nu + 1 + i\epsilon)\Gamma(n + \nu + 1 - i\epsilon)}{\Gamma(n + \nu + 1 - i\epsilon)\Gamma(n + \nu + 1 + i\epsilon)} a^\nu_{n}. \tag{5.5}
\]

A solution valid at infinity is given by

\[ R^\nu_{C(-2)} = z \left(1 - \frac{x}{z}\right)^{i\epsilon} \sum_{n=-\infty}^{\infty} (-i)^n \]

\[
\times \frac{\Gamma(n + \nu + 1 + i\epsilon)\Gamma(n + \nu + 1 - i\epsilon)}{\Gamma(n + \nu + 3 + i\epsilon)\Gamma(n + \nu + 3 - i\epsilon)} a^\nu_{n,T} F^T_{n+\nu}(z), \tag{5.6}
\]

where

\[
F^T_{n+\nu} = e^{-iz}(2z)^{n+\nu} \frac{\Gamma(n + \nu + 3 + i\epsilon)}{\Gamma(2n + 2\nu + 2)} \Phi(n + \nu + 3 + i\epsilon, 2n + 2\nu + 2; 2i\epsilon). \tag{5.7}
\]

Another solution is \( R^{-\nu-1}_{C(-2)} \). These solutions are given in the expansion around the origin and are different from the ones given in our previous paper [1] which corresponds to solutions expanded around the horizon.

Let us first consider the following relation between a solution \( R \) of the Teukolsky equation for \( s = -2 \) and a solution \( X \) of the RW equation,

\[
R = \Delta \left( \frac{d}{dr^*} + i\omega \right) \frac{r^2}{\Delta} \left( \frac{d}{dr^*} + i\omega \right) rX(z), \tag{5.8}
\]
where \( r^* = r + 2M \ln(r/2M - 1) \) and \( \Delta = r^2 - 2Mr \). We substitute the incoming solution \( X_{in}^{\nu} \) in Eq.(2.1) for \( X \) in Eq.(5.1) and find

\[
R = \frac{\epsilon}{\omega} (-x)^2 \left( \frac{d}{dx} - i\epsilon + \frac{i\epsilon}{x} \right) e^{i\epsilon(x-1)} (-x)^{-i\epsilon} p_{in}^{\nu} \\
= \frac{\epsilon}{\omega} e^{i\epsilon(x-1)} (-x)^{2-i\epsilon} \frac{d}{dx} p_{in}^{\nu} \\
= \frac{\epsilon}{\omega} e^{i\epsilon(x-1)} (-x)^{2-i\epsilon} \sum_{n=-\infty}^{\infty} \frac{\Gamma(n + \nu + 1 - i\epsilon) \Gamma(-n - \nu - i\epsilon)}{\Gamma(3 - 2i\epsilon)} a_n^{\nu} \\
\times F(n + \nu + 1 - i\epsilon, -n - \nu - i\epsilon; 3 - 2i\epsilon; x),
\]

(5.9)

where \( p_{in}^{\nu} \) is defined in Eq.(2.4). By substituting Eq.(5.6) into (5.9), we find that \( R \) in Eq.(5.8) is proportional to \( R_{in(-2)}^{\nu} \). We find

\[
\Delta \left( \frac{d}{dr^*} + i\omega \right) R^{\nu} \frac{d}{dr^*} + i\omega \right) rX_{in}^{\nu} = e^{-i\epsilon} \frac{\epsilon}{\omega} \frac{\Gamma(n + \nu - 1 - i\epsilon) \Gamma(-n - \nu - i\epsilon)}{\Gamma(3 - 2i\epsilon)} R_{in(-2)}^{\nu}. \quad (5.10)
\]

We next consider the inverse relation

\[
X = \frac{r^5}{c_0\Delta} \left( \frac{d}{dr^*} - i\omega \right) \frac{d^2}{dr^*} + i\omega \right) R \frac{r^2}{r^2}. \quad (5.11)
\]

For this relation, we consider a solution of the Teukolsky equation given in Eq.(5.4).

Now, we substitute \( R_{C(-2)} \) into Eq.(5.7) and find

\[
X = \frac{\omega}{c_0} \frac{z^3}{(dz)} - i - \frac{i\epsilon}{z - \epsilon} \right)^2 \left( \frac{R_{C(-2)}^{\nu}}{z^2} \right) \\
= \frac{\omega}{c_0} \frac{z^3}{(dz)} - i - \frac{i\epsilon}{z - \epsilon} \right)^2 e^{i\epsilon(z - \epsilon)} \left[ e^{-i\epsilon(z - \epsilon)^{-i\epsilon} z^{-2} R_{C(-2)}^{\nu}} \right] \\
= \frac{\omega}{c_0} \frac{\Gamma(n + \nu + 1 + i\epsilon)}{\Gamma(n + \nu + 1 - i\epsilon)} z^3 e^{i\epsilon(z - \epsilon)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(n + \nu + 1 + i\epsilon) \Gamma(n + \nu - 1 - i\epsilon)}{\Gamma(n + \nu + 3 - i\epsilon) \Gamma(n + \nu + 3 - i\epsilon) \Gamma(2n + 2\nu + 2)} a_n^{\nu} \\
\times \frac{d^2}{dx^2} e^{-2i\epsilon z^{-2} -i\epsilon} (2z)^n \Phi(n + \nu + 3 + i\epsilon, 2n + 2\nu + 2; 2iz) \\
= \frac{\omega}{c_0} \frac{\Gamma(n + \nu + 1 + i\epsilon)}{\Gamma(n + \nu + 1 - i\epsilon)} \frac{1 - \frac{1}{2} \epsilon}{z} \sum_{n=-\infty}^{\infty} \frac{\Gamma(n + \nu + 1 + i\epsilon) \Gamma(n + \nu + 1 - i\epsilon)}{\Gamma(n + \nu + 1 + i\epsilon) \Gamma(n + \nu + 3 - i\epsilon)} a_n^{\nu} F_{n+\nu},
\]

(5.12)

where \( F_{n+\nu} \) is the Coulomb wave function defined in Eq.(3.4). The third equality is obtained by substituting \( a_n^{\nu T} \) in Eq.(5.4) and the final equality is derived by using the
following relation

\[ \frac{d^2}{dx^2}[e^{-x}x^{-a+1} \Phi(a, c; x)] = (c-a)(c-a+1)e^{-x}x^{-a-1} \Phi(a-2, c; x). \]  

(5.13)

By comparing Eq.(5.16) and Eq.(3.21), we find \( X \) is equal to \( \tilde{X}_C^\nu \) up to the numerical factor. Thus, we obtain the relation

\[ \frac{r^5}{c_0 \Delta} \left( \frac{d}{dr^*} - i\omega \right) \frac{r^2}{\Delta} \left( \frac{d}{dr^*} - i\omega \right) \frac{R_C^\nu}{r^2} = \frac{\omega}{c_0} \frac{\Gamma(\nu + 1 + i\epsilon)}{\Gamma(\nu + 1 - i\epsilon)} \tilde{X}_C^\nu. \]  

(5.14)

Equations (5.12) and (5.20) show the relations between a solution of Teukolsky equation with \( s = -2 \) and a solution of the RW equation.

### 6 Post-Minkowskian expansions of solutions

In this section, we discuss how to derive the solution in the expansion of the small parameter \( \epsilon = 2M\omega \). In order to find the solution in Eqs.(2.1) and (4.3) up to some power of \( \epsilon \), we have to calculate \( \nu \) and \( a_\nu^\nu \) to that order by using Eq.(2.12) and (2.13) with the condition (2.14) and \( a_\nu^\nu = a_0^\nu \). Other coefficients \( b_\nu \) can be calculated from \( a_\nu^\nu \) by using the formula (3.10).

For \( a_\nu^\nu \) with \( n \geq 1 \), the equation for \( R_n(\nu) \) is useful. Since \( \alpha_\nu^\nu, \gamma_\nu^\nu \sim O(\epsilon) \) and and \( \beta_\nu^\nu \sim n(n+2l+1) \sim O(1) \), we find

\[ R_n(\nu) \sim O(\epsilon) \]  

(6.1)

for all positive integer \( n \). As a result with \( a_0^\nu = 1 \), we find

\[ a_\nu^\nu \sim O(\epsilon^n) \quad \text{for} \quad n \geq 1. \]  

(6.2)

Before discussing the coefficients for \( n < 0 \), we derive the renormalized angular momentum \( \nu \) up to \( O(\epsilon^2) \). For this, it is convenient to use the constraint for \( n = 1 \), \( R_1(\nu)L_0(\nu) = 1 \). We notice that \( R_1(\nu) \sim O(\epsilon) \) so that \( L_0(\nu) \) must behave as \( O(1/\epsilon) \), which requires that \( \beta_0^\nu + \gamma_0^\nu L_{-1}(\nu) \sim O(\epsilon^2) \) because \( a_0^\nu \sim O(\epsilon) \). In order to obtain \( \nu \) up to \( O(\epsilon) \), we need to know the information of \( \beta_0^\nu \) up to \( O(\epsilon^2) \) where the second order
term of $\nu$ involves. Thus, we need the information about $R_1(\nu)$, $L_{-1}(\nu)$, $\alpha_0^\nu$ and $\gamma_0^\nu$ up to $O(\epsilon)$. Here we assume that $L_{-2}(\nu) \sim O(\epsilon)$ whose validity will be discussed later. In this situation, $R_1(\nu)$, $L_{-1}(\nu)$, $\alpha_0^\nu$ and $\gamma_0^\nu$ can be calculated immediately. By substituting these to the constraint equation $R_1(\nu)L_0(\nu) = 1$ to find

$$\nu = l + \frac{1}{2l + 1} \left[ -2 + \frac{-4}{l(l + 1)} + \frac{(l - 1)^2(l + 3)^2}{(2l + 1)(2l + 2)(2l + 3)} - \frac{(l - 2)^2(l + 2)^2}{(2l - 1)(2l)(2l + 1)} \right] \epsilon^2 + O(\epsilon^3).$$ (6.3)

In general, we can show that $\nu$ is a even function of $\epsilon$. We see that the continued fractions $R_n(\nu)$ and $L_n(\nu)$ are determined by $\beta_n^\nu$ and $\alpha_n^\nu \gamma_n^\nu + 1$. Since these quantities are even functions, the parameters in the transcendental equation $R_n(\nu)L_{n-1}(\nu) = 1$ aside from $\nu$ are even functions of $\epsilon$ which concludes that $\nu$ an even function of $\epsilon$. Thus, the next correction term in $\nu$ enters in the 4-th order term of $\epsilon$.

The fact that the correction term of $\nu$ starts from the second order term of $\epsilon$ simplifies the calculation of the coefficients up to $O(\epsilon^2)$.

Now we discuss the coefficients for negative integer $n$ which are derived by using the equation for $L_n(\nu)$. For large negative value of $|n|$, $L_n(\nu) \simeq i\epsilon/2n$. Most of the negative integer value of $n$, $L_n(\nu) \sim O(\epsilon)$. There arise some exceptions for certain values of $n$ because the denominator of $\alpha_n^\nu$ vanishes at $n = -l - 1$ and also $\beta_n^\nu$ vanishes at $n = -2l - 1$ in the zeroth order of $\epsilon$. There is a speciality of Regge-Wheeler solutions that $\alpha_{-t+1}^\nu \sim O(\epsilon^3)$. Because of these, we find

$$L_{-t+1}(\nu) \sim O(\epsilon^3),$$

$$L_{-t-1}(\nu) \sim O(1),$$

$$L_{-2l-1}(\nu) \sim O(1/\epsilon),$$

$$L_n(\nu) \sim O(\epsilon) \quad \text{for all others.}$$ (6.4)

From these observations, we find

$$a_n^\nu \sim O(\epsilon^{|n|}), \quad \text{for} \quad -1 \geq n \geq -l + 2,$$

$$a_{-t+1}^\nu \sim O(\epsilon^{t+1}).$$
\[ a_{-l} \sim a_{-l-1} \sim O(\epsilon^{l+2}) \]
\[ a_\nu \sim O(\epsilon^{|n|+1}), \quad \text{for} \quad -l - 2 \geq n \geq -2l, \]
\[ a_\nu \sim O(\epsilon^{|n|-1}), \quad \text{for} \quad -2l - 1 \geq n. \]  
(6.5)

With the above order estimates, we see that how many terms should be needed to calculate the coefficients with the specified accuracy of \( \epsilon \).

Comming back to \( \nu \), we assumed that \( L_{-2}(\nu) \sim O(\epsilon) \) which is valid since \( l \geq 2 \).

The coefficients \( a_\nu \) and also \( b_\nu \) up to \( O(\epsilon^2) \) are obtained explicitly by

\[
a_1 = \frac{-i(l + 3)^2}{2(l + 1)(2l + 1)} \epsilon + \frac{(l + 3)^2}{2(l + 1)^2(2l + 1)} \epsilon^2 + O(\epsilon^3),
\]  
(6.6)

\[
a_2 = \frac{-(l + 3)^2(l + 4)^2}{4(l + 1)(2l + 1)(2l + 3)^2} \epsilon^2 + O(\epsilon^3),
\]  
(6.7)

\[
a_{-1} = \frac{-i(l - 2)^2}{2l(2l + 1)} \epsilon - \frac{(l - 2)^2}{2l^2(2l + 1)} \epsilon^2 + O(\epsilon^3),
\]  
(6.8)

\[
a_{-2} = \frac{-(l - 3)^2(l - 2)^2}{4l(2l - 1)^2(2l + 1)} \epsilon^2 + O(\epsilon^3),
\]  
(6.9)

By using these coefficients, we can evaluate the incoming and the outgoing amplitudes in infinity. From Eq.(4.2), we find by taking \( r = 0 \) that \( K_\nu \sim O(\epsilon^{-l-1}) \). On the other hand, the estimate of \( K_{-\nu-1} \) needs some care. By taking into account of the singular behaviors of gamma functions and the fact that the deviation of \( \nu \) from \( l \) starts from the second order of \( \epsilon \), we find that \( K_{-\nu-1} \sim O(\epsilon^{l-1}) \). Thus we obtain

\[
\frac{K_{-\nu-1}}{K_\nu} \sim O(\epsilon^{2l}),
\]  
(6.10)

which concludes that \( K_{-\nu-1} \) term contributes at most the order \( \epsilon^4 \) ( \( l \geq 2 \)). In the approximation up to \( O(\epsilon^2) \), we can safely neglect \( K_{-\nu-1} \) term.

Thus, we get the simple expressions for the outgoing and the incoming amplitudes as follows;

\[
A_{\text{out}}^\nu = B_0^\nu K_\nu \sum_{n=-2}^{2} \frac{\Gamma(n + \nu - 1 - i\epsilon)\Gamma(n + \nu + 1 - i\epsilon)}{\Gamma(n + \nu + 1 + i\epsilon)\Gamma(n + \nu + 3 + i\epsilon)} a_\nu^n,
\]  
(6.11)
and

\[ A^\nu_{\text{in}} = (B^\nu_0)^* K_\nu \sum_{n=-2}^{2} (-1)^n \frac{\Gamma(n+\nu-1-i\epsilon)}{\Gamma(n+3+i\epsilon)} a^\nu_n, \quad (6.12) \]

where

\[ B_0^\nu = (-i)^{\nu+1} 2^{1+i\epsilon} e^{-\pi\epsilon/2}. \quad (6.13) \]

By substituting the coefficients, we can easily calculate the amplitudes up to the order \( \epsilon^2 \). Since the explicit expressions are complicated, we present the amplitudes up to \( O(\epsilon) \) explicitly. We find

\[ A^\nu_{\text{out}} = B^\nu_0 K_\nu \frac{\Gamma(\nu-1-i\epsilon)\Gamma(\nu+1-i\epsilon)}{\Gamma(\nu+1+i\epsilon)\Gamma(\nu+3+i\epsilon)} \left\{ 1 - i\epsilon \left( \frac{(l-1)(l+3) + (l-2)(l+1)(l+2)}{2l(l+1)(2l+1)} \right) - \epsilon^2 \frac{2}{l^2(l+1)^2} \right\}, \]

\[ A^\nu_{\text{in}} = \frac{(B^\nu_0)^*}{B^\nu_0} \frac{\Gamma(\nu+1+i\epsilon)}{\Gamma(\nu+1-i\epsilon)} A^\nu_{\text{out}}. \quad (6.14) \]

The above result shows that the absorption coefficient \( \Gamma \) in Eq.(4.7) is zero up to the order of \( \epsilon \) for Schwarzschild black hole.

7 Summary and Remarks

We presented analytical solutions of Regge-Wheeler equation in the form of series of hypergeometric functions and Coulomb wave functions. The series are characterized by the renormalized angular momentum which turns out to be identical for both series which enabled us to relate these two series in the intermediate region where both series converge. The relation between solutions of Regge-Wheeler equation and those of Teukolsky solutions. This shows the consistency between present solutions and solutions presented in our previous paper[3].
Solutions which we found will be used for the gravitational wave astrophysics, the gravitational wave emitted from a Schwarzschild black hole and also emitted from the particle rotating along a circular orbit around a black hole. Since solutions are given in simpler forms than those of Teukolsky equation, these solutions can be obtained in the $\epsilon$ expansion in much higher accuracy so that these solutions can be used to test the $\epsilon$ expansion. Solutions will be useful for numerical computations because the series converges fast. We expect that our solutions will become a powerful weapon to construct the theoretical template towards LIGO and VIRGO projects.

Acknowledgment

We would like to thank to M. Sasaki, M. Shibata and T. Tanaka for comments and encouragements. This work is supported in part by the Japanese Grant-in-Aid for Scientific Research of Ministry of Education, Science, Sports and Culture, No. 06640396.
Appendix A: Proof of the recurrence relations among hypergeometric functions and Coulomb wave functions

(a) Proof of the recurrence relations among hypergeometric functions

Let us define

\[ A_{L,n} = \frac{\Gamma(n + L - 1 - i\epsilon)\Gamma(n - L - 2 - i\epsilon)}{\Gamma(n + 1 - 2i\epsilon)n!}, \]

then

\[ p_L = \sum_{n=0}^{\infty} A_{L,n}x^n. \]  

(A.1)

We parameterize \( P_{L+1} \) as

\[ p_{L+1} = \sum_{n=0}^{\infty} \frac{\Gamma(n + L - i\epsilon)\Gamma(n - L - 3 - i\epsilon)}{\Gamma(n + 1 - 2i\epsilon)n!}x^n 
= \sum_{n=0}^{\infty} \left[ 1 + \frac{2(L+1)}{n - L - 3 - i\epsilon} \right] A_{L,n}x^n, \]

so that we find

\[ \sum_{n=0}^{\infty} \frac{A_{L,n}x^n}{n - L - 3 - i\epsilon} = \frac{1}{2(L+1)}(p_{L+1} - p_L). \]

(A.3)

Similarly, we find

\[ \sum_{n=0}^{\infty} \frac{A_{L,n}x^n}{n + L - 2 - i\epsilon} = \frac{1}{2L}(p_L - p_{L-1}). \]

(A.4)

Now we rewrite \( xp_L \) as

\[ xp_L = \sum_{n=0}^{\infty} \frac{\Gamma(n + L - 1 - i\epsilon)\Gamma(n - L - 2 - i\epsilon)}{\Gamma(n + 1 - 2i\epsilon)n!}x^{n+1} 
= \sum_{n=0}^{\infty} \frac{n(n - 2i\epsilon)}{(n - L - 3 - i\epsilon)(n + L - 2 - i\epsilon)} A_{L,n}x^n 
= \sum_{n=0}^{\infty} \left[ 1 + \frac{(L + 3 + i\epsilon)(L + 3 - i\epsilon)}{(2L+1)(n - L - 3 - i\epsilon)} - \frac{(L - 2 + i\epsilon)(L - 2 - i\epsilon)}{(2L+1)(n + L - 2 - i\epsilon)} \right] A_{L,n}x^n. \]

(A.6)
By using Eqs. (A.4) and (A.5), we obtain

\[ xp_{n+\nu} = \frac{(n + \nu + 3 + i\epsilon)(n + \nu + 3 - i\epsilon)}{2(n + \nu + 1)(2n + 2\nu + 1)} p_{n+\nu+1} \]

\[ + \frac{1}{2} \left[ 1 - \frac{4 + \epsilon^2}{(n + \nu)(n + \nu + 1)} \right] p_{n+\nu} \]

\[ + \frac{(n + \nu - 2 + i\epsilon)(n + \nu - 2 - i\epsilon)}{2(n + \nu)(2n + 2\nu + 1)} p_{n+\nu-1}. \]  

(A.7)

Similarly, we rewrite \( x(1-x)p'_{L} \) as

\[ x(1-x)p'_{L} = \sum_{n=0}^{\infty} nA_{L,n}(x^n - x^{n+1}) \]

\[ = \sum_{n=0}^{\infty} \left[ 1 - \frac{(n - 1)(n - 2i\epsilon)}{(n - L - 3 - i\epsilon)(n + L - 2 - i\epsilon)} \right] A_{L,n}x^n \]

\[ = -\sum_{n=0}^{\infty} \left[ \frac{(L + 3 - i\epsilon)(L + 2 + i\epsilon)}{(2L+1)(n - L - 3 - i\epsilon)} + \frac{(L - 2 + i\epsilon)(L - 1 - i\epsilon)}{(2L+1)(n + L - 2 - i\epsilon)} \right] A_{L,n}x^n \]

\[ = -\sum_{n=0}^{\infty} \left[ \frac{4 + (L + 3 + i\epsilon)(L + 3 - i\epsilon)(L + 2 + i\epsilon)}{2L+1} - \frac{(L - 2 + i\epsilon)(L - 2 - i\epsilon)(L - 1 - i\epsilon)}{2L+1} \right] A_{L,n}x^n. \]  

(A.8)

By using Eqs. (A.4) and (A.5), we find

\[ x(1-x)p'_{n+\nu} = -\frac{(n + \nu + 3 + i\epsilon)(n + \nu + 3 - i\epsilon)(n + \nu + 2 + i\epsilon)}{2(n + \nu + 1)(2n + 2\nu + 1)} p_{n+\nu+1} \]

\[ + \frac{1}{2} \left[ -2 + i\epsilon + \frac{(4 + \epsilon^2)(1 + i\epsilon)}{(n + \nu)(n + \nu + 1)} \right] p_{n+\nu} \]

\[ + \frac{(n + \nu - 2 + i\epsilon)(n + \nu - 2 - i\epsilon)(n + \nu - 1 - i\epsilon)}{2(n + \nu)(2n + 2\nu + 1)} p_{n+\nu-1}. \]  

(A.9)

(b) Proof of the recurrence relations among Coulomb wave functions

Let us define

\[ B_{L} = e^{-iz(2z)^{L}} z, \quad C_{L,n} = \frac{\Gamma(n + L + 1 + i\epsilon)}{\Gamma(n + 2L + 2)n!}. \]  

(A.10)
then $F_L$ is written by

$$F_L = B_L \sum_{n=0}^{\infty} C_{L,n}(2iz)^n. \quad (A.11)$$

Now let us consider the decomposition of $F_{L+1}$ as

$$F_{L+1} = B_L \frac{1}{i} \sum_{n=1}^{\infty} \frac{\Gamma(n + L + 1 + i\epsilon)}{\Gamma(n + 2L + 3)(n-1)!} (2iz)^n$$

$$= B_L \frac{1}{i} \sum_{n=0}^{\infty} \frac{n}{n + 2L + 2} C_{L,n}(2iz)^n$$

$$= \frac{1}{i} \left[ F_L - 2(L + 1)B_L \sum_{n=0}^{\infty} \frac{C_{L,n}(2iz)^n}{n + 2L + 2} \right]. \quad (A.12)$$

Thus we have

$$B_L \sum_{n=0}^{\infty} \frac{C_{L,n}(2iz)^n}{n + 2L + 2} = \frac{1}{2(L+1)}(F_L - iF_{L+1}). \quad (A.13)$$

Similarly,

$$F_{L-1} = B_L i \sum_{n=0}^{\infty} \frac{\Gamma(n + L + i\epsilon)}{\Gamma(n + 2L)(n)!} (2iz)^{n-1}$$

$$= B_L i \left[ \frac{\Gamma(L + i\epsilon)}{\Gamma(2L)} \frac{1}{2iz} + \sum_{n=0}^{\infty} \frac{n + 2L + 1}{n + 1} C_{L,n}(2iz)^n \right]. \quad (A.14)$$

Then we find

$$B_L \sum_{n=0}^{\infty} \left[ \frac{\Gamma(L + i\epsilon)}{\Gamma(2L + 1)} \frac{1}{2iz} + \frac{C_{L,n}(2iz)^n}{n + 1} \right] = -\frac{1}{2L}(F_L + iF_{L-1}). \quad (A.15)$$

Now we evaluate $F_L/z$ as

$$\frac{F_L}{z} = 2iB_L \left[ \sum_{n=0}^{\infty} \frac{\Gamma(n + L + 1 + i\epsilon)}{\Gamma(n + 2L + 2)n!} (2iz)^n \right]$$

$$= 2iB_L \left[ \frac{\Gamma(L + 1 + i\epsilon)}{\Gamma(2L + 2)} \frac{1}{2iz} + \sum_{n=0}^{\infty} \frac{n + L + 1 + i\epsilon}{(n + 2L + 2)(n + 1)} C_{L,n}(2iz)^n \right]$$

$$= 2iB_L \left\{ \frac{L + i\epsilon}{2L + 1} \left[ \frac{\Gamma(L + i\epsilon)}{\Gamma(2L + 1)} \frac{1}{2iz} + \sum_{n=0}^{\infty} \frac{C_{L,n}(2iz)^n}{n + 1} \right] \right. \right.$$
By using Eqs. (A.13) and (A.14), we find

\[
\frac{1}{z} F_{n+\nu} = \frac{(n + \nu + 1 - i\epsilon)}{(n + \nu + 1)(2n + 2\nu + 1)} F_{n+\nu+1} + \frac{\epsilon}{(n + \nu)(n + \nu + 1)} F_{n+\nu} \\
+ \frac{(n + \nu + i\epsilon)}{(n + \nu)(2n + 2\nu + 1)} F_{n+\nu-1}.
\]  

(A.17)

Similarly,

\[
F'_{L} = -iF_{L} + \left(\frac{L+1}{z} + 2iB_{L}\right) \sum_{n=0}^{\infty} \frac{n+L+1+i\epsilon}{n+2L+2} C_{L,n} (2iz)^n.
\]  

(A.18)

By using Eqs. (A.13) and (A.14), we obtain

\[
F'_{n+\nu} = -\frac{(n + \nu)(n + \nu + 1 - i\epsilon)}{(n + \nu + 1)(2n + 2\nu + 1)} F_{n+\nu+1} + \frac{\epsilon}{(n + \nu)(n + \nu + 1)} F_{n+\nu} \\
+ \frac{(n + \nu + 1)(n + \nu + i\epsilon)}{(n + \nu)(2n + 2\nu + 1)} F_{n+\nu-1}.
\]  

(A.19)

Appendix B: The convergence region of solutions

From Eq. (2.13), we find

\[
\lim_{n \to \infty} n \frac{a_{n}^{\nu}}{a_{n-1}^{\nu}} = -\lim_{n \to -\infty} n \frac{a_{n}^{\nu}}{a_{n+1}^{\nu}} = -\frac{i\epsilon}{2}.
\]  

(B.1)

By combining the large \( n \) behavior of hypergeometric functions\[8\], we find

\[
\lim_{n \to \infty} \frac{na_{n}^{\nu}p_{n+\nu}(x)}{a_{n-1}^{\nu}p_{n+\nu-1}(x)} = -\lim_{n \to -\infty} \frac{na_{n}^{\nu}p_{n+\nu}(x)}{a_{n+1}^{\nu}p_{n+\nu+1}(x)} = \frac{i\epsilon}{2} [1 - 2x + ((1 - 2x)^2 - 1)^{1/2}].
\]  

(B.2)

Thus the hypergeometric series converges in all the complex plane of \( x \) except for \( x = \infty \).

As for the convergence of Coulomb series in Eq. (3.3), we find\[8\]

\[
\lim_{n \to \infty} \frac{F_{n+\nu}(z)}{nF_{n+\nu-1}(z)} = \lim_{n \to -\infty} \frac{F_{n+\nu}(z)}{nF_{n+\nu+1}(z)} = \frac{2}{z},
\]  

(B.3)

that

\[
\lim_{n \to \infty} \frac{a_{n}^{\nu}F_{n+\nu}(z)}{a_{n-1}^{\nu}F_{n+\nu-1}(z)} = \lim_{n \to -\infty} \frac{a_{n}^{\nu}F_{n+\nu}(z)}{a_{n+1}^{\nu}F_{n+\nu+1}(z)} = -\frac{i\epsilon}{z}.
\]  

(B.4)

Thus, the series converges for \( z > \epsilon \) or \( |x| > 1 \).

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