§0. INTRODUCTION. The analytic methods of $L^2$ estimates of $\bar{\partial}$ and multiplier ideal sheaves provide a powerful new approach to a number of long outstanding problems in algebraic geometry. Besides effective results on problems related to the Fujita conjecture and the Matsusaka big theorem, the deformational invariance of the plurigenera was proved by such an approach first for the case of general type [Siu 1998] and then for the general algebraic case [Siu 2002].

The techniques developed for the deformational invariance of the plurigenera were intended to prove the finite generation of the canonical ring. The extension result on pluricanonical sections from the method of the deformational invariance of the plurigenera opens up the possibility of using restriction to hypersurfaces and induction on dimension to prove the finite generation of the canonical ring. However, from an analytic viewpoint the technical details arising from the various singular situations are quite daunting.

The deformational invariance of the plurigenera was proved by using the techniques of (i) the global generation of multiplier ideal sheaves (A.1), (ii) the extension theorem of Ohsawa-Takegoshi [Ohsawa-Takegoshi 1987] (which in this setting can be replaced by the vanishing theorem for multiplier ideal sheaves [Kawamata 1982, Viehweg 1982, Nadel 1990] ), and (iii) the use of one of the canonical bundle inside the pluricanonical bundle as the volume form to be used in the $L^2$ estimates of $\bar{\partial}$. The proof of the finite generation of the canonical ring can more easily be handled by directly applying the above three techniques from the proof of the deformational invariance of the plurigenera.

In this article we give an overview of the analytic proof of the following theorem on the finite generation of the canonical ring for the case of general type.

\footnote{Partially supported by a grant from the National Science Foundation.}
Theorem. Let $X$ be a compact complex algebraic manifold of general type. Then the canonical ring

$$R(X, K_X) = \bigoplus_{m=1}^{\infty} \Gamma (X, mK_X)$$

is finitely generated.

Details of techniques for the analytic proof of the finite generation of the canonical ring for the case of general type were posted in [Siu 2006, Siu 2007]. An algebraic proof was posted in [Birken-Cascini-Hacon-McKernan 2006].

In this overview we focus on the formulation using the notion of a discrepancy subspace, which measures the extent of failure of achieving stable vanishing order in terms of uniformity in $m$ for all $m$-canonical bundles. It highlights more clearly how the analytic method handles the problem of infinite number of interminable blow-ups in the intuitive approach to prove the finite generation of the canonical ring.

Toward the end of this overview we discuss how our situation is similar to what is needed for a proof of the abundance conjecture. An adaptation of the argument here for use in a proof of the abundance conjecture would require an analytic argument of controlling the estimates in passing to limit, which is analogous to the situation of extending the proof of the deformational invariance of the plurigenera for the case of general type [Siu 1998] to the general algebraic case without the general type assumption [Siu 2002].

The notations $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$, and $\mathbb{N}$ denote respectively the complex numbers, the rational numbers, the integers, and the positive integers. The reduced structure sheaf of a complex space $W$ is denoted by $\mathcal{O}_W$. The maximum ideal of a point $P$ of a complex manifold $Y$ is denoted by $\mathfrak{m}_P$. The ideal sheaf of a subvariety $Z$ in a complex manifold $Y$ is denoted by $\mathcal{I}_Z$. The canonical section of the line bundle associated to a complex hypersurface $V$ in a complex manifold $Y$ is denoted by $s_V$. The multiplier ideal sheaf of a metric $e^{-\psi}$ is denoted by $\mathcal{I}_\psi$ (see (A.7)). The canonical line bundle of a complex manifold $Y$ is denoted by $K_Y$. A multi-valued holomorphic section $\sigma$ of a holomorphic line bundle $E$ over a complex manifold $Y$ means that $\sigma^m$ is a holomorphic section of $mE$ over $Y$ for some positive integer $m$. 
§1. Setup of Discrepancy Subspace. In this overview of the analytic proof of the finite generation of the canonical ring, for the sake of clarity and transparency of the core arguments we skip all arguments involving diophantine approximations so that certain real numbers which need to be proved by diophantine arguments to be rational are just assumed to be known to be rational. In order not to disrupt the main line of the core arguments, we move to the Appendix many side arguments and the listing of known facts and techniques together with some simple adaptations needed for our purpose. Sometimes the condition of a positive integer being sufficiently divisible is simply stated sloppily as being sufficiently large.

(1.1) Metric of Minimum Singularity and Its Truncation. Let $X$ be a compact complex algebraic manifold of general type. Let $s_1^{(m)}, \ldots, s_q^{(m)} \in \Gamma (X, mK_X)$ be a $C$-linear basis and let $\{\varepsilon_m\}_{m \in \mathbb{N}}$ be a sequence of positive constants decreasing so rapidly monotonically to zero that

$$\Phi = \sum_{m=1}^{\infty} \varepsilon_m \sum_{j=1}^{q_m} |s_j^{(m)}|^{2m}$$

converges on $X$. Let $\varphi = \log \Phi$ so that $\frac{1}{\Phi} = e^{-\varphi}$ is a metric for $K_X$ which we call the metric of $K_X$ of minimum singularity. (It actually is not unique, but depends on the choice of the basis $s_1^{(m)}, \ldots, s_q^{(m)}$ and the sequence $\{\varepsilon_m\}_{m \in \mathbb{N}}$ of positive numbers.) For $N \in \mathbb{N}$ let

$$\Phi_N = \sum_{m=1}^{N} \varepsilon_m \sum_{j=1}^{q_m} |s_j^{(m)}|^{\frac{2m}{N}}$$

We call the metric $\frac{1}{\Phi_N}$ for $K_X$ the $N$-th truncation of the metric of minimum singularity.

(1.2) Stable Vanishing Order. For a point $P$ of $X$ we say that the stable vanishing order is achieved at $P$ if there exists some $N \in \mathbb{N}$ such that the two functions $\Phi_N$ and $\Phi$ are comparable on some open neighborhood $U$ of $P$ in $X$ in the sense that $\Phi \leq C\Phi_N$ for some positive constant $C$ on $U$. 
When we say that the generic stable vanishing order across some subvariety $Y$ of $X$ is achieved, we mean that at a generic point of $Y$ the vanishing order of $\Phi$ across $Y$ is the same as the vanishing order of $\Phi_N$ across $Y$ for some $N \in \mathbb{N}$.

It follows from a direct application of Skoda’s result on ideal generation that when the stable vanishing order is achieved at every point of $X$, the canonical ring is finitely generated (see Appendix (A.4)). So the problem of the finite generation of the canonical ring is reduced to proving the achievement of stable vanishing order everywhere on $X$.

The most natural approach is to consider the subvariety $Z$ of $X$ where the stable vanishing order is not achieved and then try to prove that if $Z$ is nonempty, we can show that at one of its points the stable vanishing order is achieved, giving a contradiction.

However, it turns out that in deriving a contradiction for not achieving the stable vanishing order everywhere, it is more efficient to get a contradiction for a statement incorporating the extent (or multiplicity) of the failure of the achievement of the stable vanishing order. We introduce now the notion of discrepancy subspace which measures the extent (or multiplicity) of the failure of the achievement of the stable vanishing order.

(1.3) Definition of Discrepancy Subspace. Let $\mathcal{J}$ be a coherent ideal sheaf on $X$. The stable vanishing order of the canonical line bundle of $X$ is said to be precisely achieved modulo the subspace of $X$ defined by $\mathcal{J}$ if there exist some positive integer $m_{\mathcal{J}}$ and some positive constant $C_{m,k,\mathcal{J}}$ for $k, m \in \mathbb{N}$ with $m \geq m_{\mathcal{J}}$ such that the inequality

\[(\dagger) \quad |s_{\mathcal{J}}|^2 \sum_{j=1}^{q_k} |s_j^{(k)}|^{2m_{\mathcal{J}}} \leq C_{m,k,\mathcal{J}} \sum_{j=1}^{q_m} |s_j^{(m)}|^2\]

holds on $X$ for all $k, m \in \mathbb{N}$ with $m \geq m_{\mathcal{J}}$. Here the notation $|s_{\mathcal{J}}|^2$ means the following. For a coherent ideal sheaf $\mathcal{I}$ on $X$ generated locally by holomorphic function germs $\tau_1, \cdots, \tau_\ell$, we define

\[|s_{\mathcal{I}}|^2 = \sum_{j=1}^\ell |\tau_j|^2.\]

Let $Z$ be the zero-set of the coherent ideal sheaf $\mathcal{J}$. We call the ringed space $(Z, O_X/\mathcal{J})$ a discrepancy subspace. We call the coherent ideal sheaf $\mathcal{J}$ a discrepancy ideal sheaf.
Remark. The main feature of the formulation is that a discrepancy subspace not only specifies the set of points (which is the zero-set of the ideal sheaf $J$) where the failure of achieving the stable vanishing order occurs but also describes the extent of the failure by providing the ideal sheaf $J$, simultaneously for all $mK_X$ for sufficiently large $m$, so that if one adds the vanishing order of this ideal sheaf the stable vanishing order is no less than that given by a finite number of pluricanonical sections. The vanishing order of an ideal sheaf is formulated here in terms of the sum of the absolute-value-squares of its local generators. This description of the extent of the failure by the ideal sheaf $J$ makes the induction process easier to handle. The simultaneity for all $mK_X$ for sufficiently large $m$ holds the key to understanding the reason for the termination of the infinite process of blow-ups in the intuitive approach of blowing up successively to prove the finite generation of the canonical ring.

Formulation in Terms of Metric of Minimum Singularity. In the formulation $(\dagger)$ the constant $C_{m,k,J}$ is allowed to depend on $m$ and $k$, because we are free to choose the rapidly decreasing sequence of positive numbers $\varepsilon_\ell$ used in the definition of

$$\Phi = \sum_{\ell=1}^{\infty} \varepsilon_\ell \sum_{j=1}^{q_\ell} |s_j^{(\ell)}|^2$$

in order to obtain the inequality $\Phi \leq C\Phi_N$ for some positive number $C$ and some positive integer $N$ with

$$\Phi_N = \sum_{\ell=1}^{N} \varepsilon_\ell \sum_{j=1}^{q_\ell} |s_j^{(\ell)}|^2.$$

Another way to formulate $(\dagger)$ is that for any fixed $m \geq m_J$ and for an $m$-dependent appropriate choice of the rapidly decreasing sequence of positive numbers $\varepsilon_\ell$ used in the definition of

$$\Phi = \sum_{\ell=1}^{\infty} \varepsilon_\ell \sum_{j=1}^{q_\ell} |s_j^{(\ell)}|^2$$

the inequality

$$(\dagger)_m^b \quad |s_J|^2 \Phi^m \leq \tilde{C}_{m,J} \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$
holds on $X$ for some constant $\tilde{C}_{m,J}$. In other words, we use the inequality
\[
|s_J|^2 (\Phi_m)^m \leq \tilde{C}_{m,J} \sum_{j=1}^{q_m} |s_j^{(m)}|^2,
\]
where
\[
\Phi_m = \sum_{k=1}^{\infty} \varepsilon_{k,m} \sum_{j=1}^{q_k} |s_j^{(k)}|^2
\]
for some positive constants $\varepsilon_{k,m}$.

(1.5) **Transformation of Discrepancy Subspace in Blow-Up and Conductor.** In a blow-up $\tilde{X} \to X$ of $X$, the rôle played by the adjunction formula is canceled by its effect on both sides of $(\dagger)$ and $|s_J|^2$ simply transforms as a lifting of a local function from $X$ to $\tilde{X}$. This enables us to assume that, after replacing $X$ by its blowup, $J$ is the ideal sheaf of a divisor whose components are in normal crossing. With the blow-up, we can use the technique of the minimum center of log canonical singularity [Kawamata 1985, Shokurov 1985]. Also note that from the inequality
\[
|s_J|^2 \Phi^m \leq \tilde{C}_{m,J} \sum_{j=1}^{q_m} |s_j^{(m)}|^2
\]
in $(\dagger)_b$, if
\[
\int \frac{|f|^2}{\Phi^m}
\]
is finite for some local holomorphic function germ $f$, then
\[
\int \frac{|s_J|^2 |f|^2}{\sum_{j=1}^m |s_j^{(m)}|^2} \leq \tilde{C}_{m,J} \int \frac{|f|^2}{\Phi^m}
\]
is also finite. This means that
\[
\mathcal{J} \mathcal{I}_m \log \Phi \subset \mathcal{I} \log \sum_{j=1}^m |s_j^{(m)}|^2
\]
and the key point is that the conductor $\mathcal{J}$ is independent of $m$. In particular, we are able to locate the minimum center of log canonical singularity (or irreducible subspace of minimum discrepancy) in a way which works for all $mK_X$ with $m$ sufficiently large.
Intersection of Discrepancy Subspaces and Minimum Discrepancy Subspace. Though for the precise achievement of stable vanishing order we need only show that $\Phi^m$ and $\sum_{j=1}^{q_m} |s_j^{(m)}|^2$ are comparable for some $m \in \mathbb{N}$, the reason why we need $(\dagger)_m$ for all $m \geq m_\mathcal{J}$ and not just for some single $m$ is that we use induction to reduce the subspace defined by $\mathcal{J}$ step-by-step, which means that by replacing $m_\mathcal{J}$ by an appropriately larger $m_\tilde{\mathcal{J}}$ in the comparison between $\Phi^m$ and $\sum_{j=1}^{q_m} |s_j^{(m)}|^2$ we seek to replace $\mathcal{J}$ by a strictly bigger ideal sheaf $\tilde{\mathcal{J}}$ in the inequality $(\dagger)_m^\flat$. Since we need to use a bigger $m_\tilde{\mathcal{J}}$ in every one of the finite-step inductive process, the inequality $(\dagger)_m^\flat$ has to be formulated to hold for all $m \geq m_\tilde{\mathcal{J}}$. Another way to look at this is as follows. Suppose we have, not only

$$(\dagger)_\mathcal{J} \quad |s_\mathcal{J}|^2 \sum_{j=1}^{q_k} |s_j^{(k)}|^{2m} \leq C_{m,k,\mathcal{J}} \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$

on $X$ for all $k, m \in \mathbb{N}$ with $m \geq m_\mathcal{J}$, but also

$$(\dagger)_{\tilde{\mathcal{J}}} \quad |s_{\tilde{\mathcal{J}}}|^2 \sum_{j=1}^{q_k} |s_j^{(k)}|^{2m} \leq C_{m,k,\tilde{\mathcal{J}}} \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$

on $X$ for all $k, m \in \mathbb{N}$ with $m \geq m_{\tilde{\mathcal{J}}}$. Then we can define $\mathcal{K}$ as the sum of $\mathcal{J}$ and $\tilde{\mathcal{J}}$ and set $m_{\mathcal{K}}$ as the maximum of $m_\mathcal{J}$ and $m_{\tilde{\mathcal{J}}}$ and set $C_{m,k,\mathcal{K}}$ as the sum of $C_{m,k,\mathcal{J}}$ and $C_{m,k,\tilde{\mathcal{J}}}$ and get

$$(\dagger)_\mathcal{K} \quad |s_\mathcal{K}|^2 \sum_{j=1}^{q_k} |s_j^{(k)}|^{2m} \leq C_{m,k,\mathcal{K}} \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$

on $X$ for all $k, m \in \mathbb{N}$ with $m \geq m_{\mathcal{K}}$.

On the other hand, if we have only

$$|s_\mathcal{J}|^2 \Phi^m \leq \tilde{C}_{m,\mathcal{J}} \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$

for some constant $\tilde{C}_{m,\mathcal{J}}$ for one single $m = m_\mathcal{J}$, then having

$$|s_{\tilde{\mathcal{J}}}|^2 \Phi^m \leq \tilde{C}_{m,\tilde{\mathcal{J}}} \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$
for some constant $\tilde{C}_{m,\tilde{J}}$ for one single $m = m_{\tilde{J}}$ would not be sufficient to give us

$$|s_K|^2 \Phi^m \leq \tilde{C}_{m,K} \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$

for some constant $\tilde{C}_{m,K}$ for one single $m = m_K$ when $K$ is the sum of $J$ and $\tilde{J}$. Even, if we take the least common multiple $\tilde{m}$ of $m_J$ and $m_{\tilde{J}}$ so that $\tilde{m} = m_J p_J$ and $\tilde{m} = m_{\tilde{J}} p_{\tilde{J}}$, then we can only get

$$|s_{\tilde{K}}|^2 \Phi^\tilde{m} \leq \tilde{C} \sum_{j=1}^{q_{\tilde{m}}} |s_j^{(\tilde{m})}|^2$$

for some $\tilde{C} > 0$, where $\tilde{K}$ is generated by $J p_J$ and $\tilde{J} p_{\tilde{J}}$. In this case, in general $\tilde{K}$ is not equal to $K$.

Since we can take the sum of two coherent ideal sheaves for discrepancy subspaces and since we have the Noetherian property for a nondecreasing chain of coherent ideal sheaves, we can get a minimum discrepancy ideal sheaf. Moreover, by blowing up, we can assume that the maximum discrepancy ideal sheaf is the ideal sheaf for a divisor whose components are nonsingular hypersurfaces in normal crossing.

(1.7) Non-Achievement of Stable Vanishing Order and Discrepancy Subspace. Suppose $J$ is the maximum discrepancy ideal sheaf and $\tilde{J}$ is equal to the ideal sheaf of the divisor

$$D = \sum_{j=1}^\ell \alpha_j D_j$$

with $\{D_j\}_{1 \leq j \leq \ell}$ composed of nonsingular hypersurfaces in normal crossing and $\ell \in \mathbb{N}$ and $\alpha_j \in \mathbb{N}$ for $1 \leq j \leq \ell$. It does not mean that at points of $D_1 - \cup_{j=2}^\ell D_j$ the generic stable vanishing order definitely cannot be achieved across $D_1$. The reason is the following. For simplicity let us explain this in the case of $\ell = 1$. Logically it may happen that there are coherent ideal sheaves $I_m$ on $X$ for $m \in \mathbb{N}$ whose zero-set $Z_m$ is a proper subvariety of $D_1$ and the inequality

$$|s_{I_m}|^2 \left( \Phi_m \right)^m \leq C_m \sum_{j=1}^{q_m} |s_j^{(m)}|^2$$
holds for some positive constant $C_m^\sharp$, where

$$\Phi_m = \sum_{k=1}^{\infty} \varepsilon_{k,m} \sum_{j=1}^{q_k} \left| s_j^{(k)} \right|^2$$

for some positive constants $\varepsilon_{k,m}$, but $\mathcal{I}_{m+1}$ is a proper ideal subsheaf of $\mathcal{I}_m$ for $m \in \mathbb{N}$ so that there is no coherent ideal $\mathcal{I}$ on $X$ independent of $m$ such that $\mathcal{I}$ is contained in each $\mathcal{I}_m$ for $m \in \mathbb{N}$ and the zero-set of $\mathcal{I}$ is a proper subvariety $Z$ of $D_1$. What may happen is that $\mathcal{J}$ may be equal to the intersection $\cap_{m=1}^{\infty} \mathcal{I}_m$ and, though the zero-set of each $\mathcal{I}_m$ is the proper subvariety $Z_m$ of $D_1$, yet the zero-set of the intersection $\cap_{m=1}^{\infty} \mathcal{I}_m$ is $D_1$. In a way this formulation of using discrepancy subspaces differs from the simple precise achievement of generic stable vanishing order in that it removes the difficulty of the ever-changing setting as the $m$ in $mK_X$ increases but it also makes the task of decreasing the discrepancy subspace so much harder.

(1.7.1) *Remark.* When we try to decrease the discrepancy ideal subspace, the situation may seem simpler if the generic stable vanishing order is achieved at some point of $D$. However, it is not exactly the case, because the discrepancy measures the difference between bigness and some appropriate ampleness and does not just measure the difference between bigness and numerical effectiveness. When the generic stable vanishing order $\gamma$ across $Y$ is achieved, the $\mathbb{Q}$-line bundle $K_X - \gamma Y$ may be locally numerically effective at some affine open subset $\Omega_Y$ of $Y$, yet for $K_X - \gamma Y$ at points of $\Omega_Y$ there may still be a difference between the local numerical effectiveness and the required ampleness at the points of $\Omega_Y$.

(1.8) *Geometric Construction of Minimum Discrepancy Subspace.* There is a more constructive way to identify the minimum discrepancy subspace. For fixed $k$ and $m$ let $\mathcal{A}_{k,m}$ be the ideal sheaf consisting of all holomorphic function germs $f$ on $X$ at a point $P$ of $X$ such that

$$(\dagger)_{k,m} \quad |f|^2 \sum_{j=1}^{q_k} \left| s_j^{(k)} \right|^2 \leq C_{f,P} \sum_{j=1}^{q_m} \left| s_j^{(m)} \right|^2$$

on some open neighborhood $U$ of $P$ for some positive constant $C_{f,P}$. By using blow-ups of $X$, it is clear that $\mathcal{A}_{k,m}$ is coherent. Let $\mathcal{B}_m$ be the largest coherent ideal sheaf on $X$ such that $\mathcal{B}_m$ is contained in $\cap_{k,\ell \geq m} \mathcal{A}_{k,\ell}$. Clearly $\mathcal{B}_m$ is contained in $\mathcal{B}_{m+1}$, because $\cap_{k,\ell \geq m} \mathcal{A}_{k,\ell}$ is contained in $\cap_{k,\ell \geq m+1} \mathcal{A}_{k,\ell}$. It
follows from the Noetherian property of a convergent power series ring and
the compactness of $X$ that there exists some $m_0 \in \mathbb{N}$ such that $\mathcal{B}_{m_0} = \mathcal{B}_m$ for
$m \geq m_0$. The coherent ideal sheaf $\mathcal{B}_{m_0}$ is equal to the maximum discrepancy
ideal sheaf $\mathcal{J}$ with $m_{\mathcal{J}} = m_0$.

Note that the discrepancy subspace defined by $\mathcal{J}$ must be inside $D$ when
$aK_X = A + D$ (for some $a \in \mathbb{N}$ and some ample $A$ and some effective divisor $D$) and thus inside some common zero-set of pluricanonical sections.
We can actually identify elements of $\mathcal{J}$ by looking at pluricanonical sections
whose vanishing order is above the stable vanishing order by some appropriate amount.

(1.9) Too Strongly Formulated Condition. The inequality (†) is weaker than

\[(†)^{\sharp} \quad |s_{\mathcal{J}}|^2 \sum_{j=1}^{q_k} |s_j^{(k)}|^2 \leq C_{m,k,\mathcal{J}} \sum_{j=1}^{q_m} |s_j^{(m)}|^\frac{2k}{m}\]

for all $m,k \geq m_{\mathcal{J}}$ which states that the common vanishing order of all
$m$-canonical sections raised to the power $\frac{k}{m}$ is no more than the common vanishing order of all $k$-canonical sections multiplied by local generators of $\mathcal{J}$. By Skoda’s result on ideal generation this stronger inequality (†)$^{\sharp}$ would imply immediately the finite generation of the canonical ring. See the Appendix (A.5). We do not use this formulation, because it is too strongly formulated and is much more difficult to verify.

§2. Constructing and Decreasing Discrepancy Subspace. The proof of the finite generation of the canonical ring is done by first constructing the initial codimension-one discrepancy subspace and then decreasing the discrepancy subspace until the stable vanishing order is achieved everywhere. The decreasing of the discrepancy subspace is done by imitating the construction of the initial codimension-one discrepancy subspace and using the family of subvarieties associated to a closed positive $(1,1)$-current which is motivated by the intuition of getting strict positive lower bound for the current along the normal directions of the subvarieties with the modified restriction of the current to the subvariety being of the special form (that is, in the second case of the dichotomy (A.10)).

(2.1) Construction of Initial Codimension-One Discrepancy Subspace. As the first step we now construct the initial codimension-one discrepancy subspace.
We do this by using the technique of the global generation of the multiplier ideal sheaf (Appendix (A.1)) and the decomposition of $K_X$ as a sum of an ample $\mathbb{Q}$-line bundle and an effective $\mathbb{Q}$-divisor from the general type property of $X$.

Let $A$ be an ample line bundle on $X$ which is ample enough for the global generation of multiplier ideal sheaves (see Appendix (A.1)). We write $aK_X = A + D$, where $D$ is an effective divisor in $X$ and $a$ is a positive integer. We use the metric

$$
\frac{1}{\Phi^m |s_D|^2}
$$

for the line bundle

$$
mK_X + D = (m + a) K_X - A,
$$

where $s_D$ is the canonical section of the line bundle $D$ so that the divisor of $s_D$ is precisely $D$. Let $\mathcal{I}^{(m)}$ be the multiplier ideal sheaf of the metric

$$
\frac{1}{\Phi^m |s_D|^2}.
$$

Then the multiplier ideal sheaf $\mathcal{I}^{(m)}$ is generated by elements of

$$
\Gamma \left( X, \mathcal{I}^{(m)} (mK_X + D + A) \right) = \Gamma \left( X, \mathcal{I}^{(m)} ((m + a) K_X) \right) \subset \Gamma \left( X, (m + a) K_X \right).
$$

From the Lemma on sup norm domination of metric by generators of multiplier ideal (see Appendix (A.6)) we conclude that

$$
(\%) \quad \left| s_j^{(k)} \right|^{2m} \left| s_D \right|^2 \leq C_{k,j} \sum_{j=1}^{q_m+a} \left| s_j^{(m+a)} \right|^2
$$

for $k \in \mathbb{N}$. This shows that we can choose $\mathcal{J}$ to be the ideal sheaf generated by $s_D$ and choose $m_{\mathcal{J}}$ as $a + 1$.

(2.2) **Decreasing Discrepancy Subspace by Imitating the Argument of Constructing Initial Codimension-One Discrepancy Subspace.** We are going to decrease the discrepancy subspace by imitating the argument of constructing the initial codimension-one discrepancy subspace given in (2.1). Let us reinterpret and recast the argument of (2.1) so that we can more easily explain how we imitate it and adapt it to decrease the discrepancy subspace.
Let $q \in \mathbb{N}$. Suppose $A$ is a holomorphic line bundle on $X$ which is sufficiently ample so that for any point $P$ of $X$ there exist

$$\tau_1, \cdots, \tau_r \in \Gamma(X, A)$$

with the property that the multiplier ideal sheaf of the metric

$$\frac{1}{\sum_{j=1}^r |\tau_j|^2}$$

is in a neighborhood of $P$ in $X$ equal to $B$ with $B \subset (\mathfrak{m}_P)^q$ and $P$ being an isolated zero of $B$. In other words, we are able to get elements of $\Gamma(X, A)$ which could give metrics whose multiplier ideal sheaves have the desired properties of isolated zeroes at prescribed points. We now have $aK_X = A + D$. We can interpret it as $aK_X - D = A$, which means that we can get elements of $\Gamma(X, \mathcal{J}(aK_X)) = \Gamma(X, aK_X - D)$ which could give metrics whose multiplier ideal sheaves have the desired properties of additional isolated zeroes at prescribed points, where $\mathcal{J}$ is the multiplier ideal sheaf for the metric $\frac{1}{|s_{D}|^2}$. More precisely, for a point $P$ in $X$ we can find elements $\sigma_1, \cdots, \sigma_\ell$ of $\Gamma(X, \mathcal{J}(aK_X))$ such that the multiplier ideal sheaf of the metric

$$\frac{1}{\sum_{j=1}^\ell |\sigma_j|^2}$$

in a neighborhood of $P$ is of the form $A\mathcal{J}$ where $A \subset (\mathfrak{m}_P)^q$ and $P$ is an isolated zero of $A$. The ideal sheaf $A$ is what we mean by additional isolated zeroes. Note that $\mathcal{J}$ is generated by the element $s_D s_A \in \Gamma(X, aK_X)$.

What we need is actually not the global sections $\sigma_1, \cdots, \sigma_\ell$ but the metrics

$$\frac{1}{\sum_{j=1}^\ell |\sigma_j|^2}$$

for $aK_X$ whose multiplier ideal sheaves have the desired properties of additional isolated zeroes at prescribed points. Of course, the use of such a metric is to enable us to consider the metric

$$e^{-m\varphi} \frac{1}{\sum_{j=1}^\ell |\sigma_j|^2}$$

of $(m + a)K_X$ so that the desired properties of additional isolated zeroes at prescribed points would enable us to conclude that $D$ can be used as a discrepancy subspace.
Another important observation is that we can use the same argument if we have such a metric not for $aK_X$ but for a $\mathbb{Q}$-line bundle $aK_X + \delta B$ for some fixed ample line bundle $B$ and for a sufficiently small positive rational number $\delta$. The reason is the following. Let us denote by $h$ this metric for $aK_X + \delta B$. In the argument what matters is the multiplier ideal sheaves and not the metrics themselves. We can write $K_X = \alpha B + E$ for some rational positive number $\alpha$ and some effective $\mathbb{Q}$-divisor $E$ and we can replace the use of the metric $e^{-m\varphi}$ of $mK_X$ by the use of the metric

$$
\frac{e^{-(m-\frac{\delta}{\alpha})\varphi}}{|s_E|^{2\alpha}}
$$

of $mK_X - \delta B$ to form the metric

$$
\frac{he^{-(m-\frac{\delta}{\alpha})\varphi}}{|s_E|^{2\alpha}}
$$

for $(m + a)K_X$ by putting together the metric $(&)$ of $mK_X - \delta B$ and the metric $h$ of $aK + \delta B$. This is the argument of absorbing a sufficiently small ample summand by the bigness of the canonical line bundle (which is also described in Appendix (A.12) for convenience of referral in some other steps of this overview). This is the part which needs the general type assumption of $X$. It is the same argument of absorption of small ample summand by the bigness of the canonical line bundle which makes the proof of the plurigenera for the case of general type so much easier, because of the avoidance of a laborious estimation process in analysis. This finishes our re-interpreting and recasting of the construction of the initial codimension-one discrepancy subspace given in (2.1).

We now continue with the process of decreasing the discrepancy subspace. For the step of decreasing the discrepancy subspace, as observed in (1.5), after blowing up we can assume without loss of generality that the discrepancy ideal sheaf $\mathcal{J}$ is the ideal sheaf of an effective $\mathbb{Z}$-divisor $Y = \sum b_j Y_j$ whose components are nonsingular hypersurfaces $\{Y_j\}_j$ in normal crossing. To make the argument simpler to understand, we look at the special case of $Y$ being a single nonsingular hypersurface $Y$ of $X$ with multiplicity $1$. Let $\gamma$ be the generic stable vanishing order across $Y$. Because of the skipping of diophantine arguments, we assume $\gamma$ to be rational. We will replace $K_X$ on $X$ by $(K_X - \gamma Y)|_Y$ on $Y$. From the way the discrepancy subspace is defined, we
can assume without loss of generality that there is some $m_1$-canonical section $s^*$ of $X$ such that at every point of some Zariski open subset of $Y$ the multiplicity of $s^*$ is strictly greater than $m_1\gamma$. For the general case of $Y = \sum_j b_jY_j$ a modification of the argument works to reduce one of the positive coefficients $b_j$ by 1. Now for this special case we have to look at $Y$ instead of $X$. There are a number of modifications needed from construction of the initial codimension-one discrepancy subspace given in (2.1).

First of all, the $\mathbb{Q}$-line bundle $(K_X - \gamma Y)|_Y$ may not be big on $Y$. Let $L = (K_X - \gamma Y)|_Y$. Let $\Theta_Y$ be the curvature current on $Y$ constructed from the metric $e^{-\phi} \gamma$ of $K_X - \gamma Y$ on $X$. Some modification is needed to define $\Theta_Y$ on $Y$ (see Appendix (A.11)). It may happen that we have the extreme case of the closed positive $(1,1)$-current $\Theta_L$ being of the special form (that is, in the second case of the dichotomy (A.10)). In general, there exist some nonnegative integer $\kappa \leq n - 1$ and a complex submanifold $V$ of complex dimension $\kappa$ in $Y$ such that

(i) the restriction of $\Theta_L$ to $V$ dominates some strictly positive smooth $(1,1)$-form $\omega_V$ on $V$ (which we can assume without loss of generality to be closed also), and

(ii) there exists a holomorphic family $\{W_s\}_{s \in S}$ of subvarieties $W_s$ of pure dimension $n - 1 - \kappa$ such that the “modified restrictions” of $\Theta_Y$ to each $W_s$ are of the special form.

To explain “modified restriction”, for the sake of simplicity let us consider the case $\kappa = 1$ so that each $W_s$ is a hypersurface in $Y$. In this case the modified restriction of $\Theta_Y$ to $W_s$ is the restriction of $\Theta_Y - \gamma_{W_s}W_s$ to $W_s$, where $\gamma_{W_s}$ is the generic Lelong number of $\Theta_Y$ at points of $W_s$ [Siu 1974]. For the case of a general $\kappa$, when $W_s$ is cut out by branches of divisors of pluricanonical sections, $W_s$ is a hypersurface in a subvariety one dimension higher, cut out by the branches of a subset of such divisors, which is in another subvariety one dimension higher, cut out by the branches of another subset of such divisors, et cetera until we get to $Y$ and the modified restriction is defined inductively.

To define “modified restriction”, instead of the divisor of a pluricanonical section, we also allow ourselves the use of a subspace defined by a metric of a pluricanonical line bundle with nonnegative curvature current. For the general case the assumption about “modified restrictions” of $\Theta_Y$ to $W_s$ means that any “modified restriction” of $\Theta_Y$ to $W_s$ so defined must be of the special
form (that is, in the second case of the dichotomy). Since our purpose here is just to explain the main argument, in order to avoid non-illuminating very complicated descriptions and notations, so far as the assumption of the “modified restriction” of \( \Theta_Y \) to \( W_s \) is concerned, we will confine ourselves to the case of \( \kappa = 1 \).

For \( Y \) the subvariety \( W_s \) will play the role of a point \( P \) in \( X \) in our argument to decrease the discrepancy subspace. The complex submanifold \( V \) of \( Y \) is related to the parameter space \( S \) for the holomorphic family \( \{ W_s \}_{s \in S} \) of subvarieties in \( Y \) in that there is a holomorphic finite-fibered map from \( V \) to \( S \). Intuitively, \( V \) gives the directions in \( Y \) where the current \( \Theta_Y \) has a strictly positive lower bound and each \( W_s \) gives the directions where there is no longer any positive lower bound for \( \Theta_Y \). The complex submanifold \( V \) and the family \( \{ W_s \}_{s \in S} \) are constructed from taking a fixed sufficiently ample line bundle \( B \) on \( X \) and using elements of \( \Gamma (Y, mL + B) \) for \( m \) sufficiently large so that

(i) the dimension of the restriction \( \Gamma (Y, mL + B) | V \) of \( \Gamma (Y, mL + B) \) to \( V \) is of the order \( cm^\kappa \) for some positive constant \( c > 0 \) as \( m \to \infty \),

(ii) \( W_s \) is obtained as the limit as \( m \to \infty \) of the multiplier ideal sheaves defined by appropriate roots of sums of squares of elements of \( \Gamma (Y, mL + B) | V \) which vanish at a prescribed point to a high order depending on \( m \).

For more details about \( W \) and \( \{ W_s \}_{s \in S} \) see Appendix (A.13).

Instead of using \( D \) in \( aK_X = A + D \) for \( X \), we construct our new divisor \( D_Y \) in \( Y \) by using the zero-set \( \tilde{D}_Y \) of the multiplier ideal sheaf of the closed positive \((1, 1)\)-current \((\Theta_Y | V) - \omega_V\) on \( V \) (see Appendix (A.7)) and the set of all \( W_s \) such that \( W_s \) intersects \( \tilde{D}_Y \). We end up with a coherent ideal sheaf \( \mathcal{K} \) on \( X \) which contains the ideal sheaf \( \mathcal{I}_Y \) of \( Y \) such that for each \( s \in S \) the metric of \( \frac{m}{p_m}L + \frac{1}{p_m}B \) of the form

\[
\frac{1}{\sum_{j=1}^\ell |\sigma_j|^{\frac{1}{p_m}}}
\]

for some appropriate

\[ \sigma_1, \ldots, \sigma_\ell \in \Gamma (Y, \mathcal{K} (mL + B)) \]

(and some \( m \in \mathbb{N} \) and \( p_m \in \mathbb{N} \) with \( p_m \) sufficiently large for the purpose of applying the technique in Appendix (A.9)) gives the multiplier ideal sheaf
\( \mathcal{K} \mathcal{I}_{W_s} \) with additional zeroes given by \( \mathcal{I}_{W_s} \). Now regard \( \sigma_1, \ldots, \sigma_\ell \) as elements of \( \Gamma (Y, \mathcal{I}_{m \varphi_Y} (mL + B)) \) (where \( e^{-\varphi_Y} = \frac{e^{-\varphi}}{|s_Y|^2} \)) and extend them to

\[
\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell \in \Gamma (X, \mathcal{I}_{m \varphi_Y} (mL + B)).
\]

This is possible by Appendix (A.9) when \( B \) is sufficiently ample (which we assume to be the case).

Also the ideal sheaf \( \mathcal{K} \) can be obtained as the multiplier ideal sheaf of a metric \( \frac{\hat{m}}{\hat{p}_m} L + \frac{1}{\hat{p}_m} B \) of the form

\[
\frac{1}{\sum_{j=1}^\ell \left| \tau_j \right|^{\frac{1}{\hat{p}_m}}}
\]

for some appropriate

\[
\tau_1, \ldots, \tau_\ell \in \Gamma (Y, \mathcal{I}_{\hat{m} \varphi_Y} (\hat{m}L + B))
\]

(and some \( \hat{m} \in \mathbb{N} \) and \( \hat{p}_m \in \mathbb{N} \) with \( \hat{p}_m \) sufficiently large for the purpose of apply the technique of in Appendix (A.9)). Now extend \( \tau_1, \ldots, \tau_\ell \) to

\[
\hat{\tau}_1, \ldots, \hat{\tau}_\ell \in \Gamma (X, \mathcal{I}_{\hat{m} \varphi_Y} (\hat{m}L + B)).
\]

Again this is possible by Appendix (A.9) when \( B \) is sufficiently ample (which we assume to be the case). Note that \( \hat{\tau}_1, \ldots, \hat{\tau}_\ell \) can be regarded as the analog of \( s_{D_X} \in \Gamma (X, aK_X) \) in the construction of the initial codimension-one discrepancy subspace done in (2.1) and re-interpreted and recast above. Note that whenever the zero-set of \( \mathcal{K} \) intersects some \( W_s \), it contains all of \( W_s \) such that \( W_s \) intersects \( D_Y \) and the set of all \( W_s \) such that \( W_s \) intersects \( D_Y \).

We now use the technique of constructing sections from curvature currents of the special form on the subvariety \( W_s \) and extending them to all of \( X \) by vanishing theorems from multiplier ideal sheaves of the metrics described above. We conclude that \( \mathcal{K} \mathcal{I}_{m \varphi_Y} \) is generated by elements of \( \Gamma (X, mL) \) for all \( m \) sufficiently large. Now we use Appendix (A.6) to conclude that \( \mathcal{K} \) is a discrepancy ideal sheaf which is strictly bigger than \( J \).

There are a number of subtle points which we gloss over in our overview of the argument to decrease the discrepancy subspace. Since it is easier to explain those subtle points in isolation, we will do some of them in Appendix (A.14) and (A.15).
(2.2.1) **Remark.** The above argument of decreasing the discrepancy subspace by using the special form of the current current and extension is a more streamlined version of the argument in [Siu 2006] of slicing by ample divisors into curves and using the second case of the dichotomy. Instead of using ample divisors to slice, we simply use a metric of the ample divisor with nonnegative curvature current whose multiplier ideal sheaf gives the ideal sheaf of a subvariety where the modified restriction of the curvature current is of special form. Special form here means the second case of the dichotomy in [Siu 2006].

(2.3) **Abundance Conjecture.** For a compact complex algebraic manifold $W$ and an ample line bundle $A$ on $W$, the abundance conjecture which compares

$$\limsup_{m \to \infty} \frac{\log \dim \Gamma(W, mK_W + A))}{\log m}$$

and

$$\limsup_{m \to \infty} \frac{\log \dim \Gamma(W, mK_W)}{\log m}$$

deals with a situation similar to what is done in the argument (2.2) to decrease the discrepancy subspace by using a holomorphic family of subvarieties identified by directions of strict positive lower bound for a closed positive $(1,1)$-current. Unfortunately the argument of absorption of small ample line bundle discussed in (A.12) is used here, making it impossible to use directly the argument for a proof of the abundance conjecture. To adapt the argument for use in a proof of the abundance conjecture, we encounter the situation similar to adapting the proof of the deformational invariance of the plurigenera for the case of general type [Siu 1998] to the general algebraic case without the general type assumption [Siu 2002], requiring an analytic argument of controlling the estimates in passing to limit.

**APPENDIX**

(A.1) **Statement on Global Generation of Multiplier Ideal Sheaves.** Let $L$ be a holomorphic line bundle over an $n$-dimensional compact complex manifold $Y$ with a Hermitian metric which is locally of the form $e^{-\xi}$ with $\xi$ plurisubharmonic. Let $\mathcal{I}_\xi$ be the multiplier ideal sheaf of the Hermitian metric $e^{-\xi}$. Let $A$ be an ample holomorphic line bundle over $Y$ such that for every point
of $Y$ there are a finite number of elements of $\Gamma(Y, A)$ which all vanish to order at least $n + 1$ at $P$ and which do not simultaneously vanish outside $P$. Then $\Gamma(Y, I \xi \otimes (L + A + K_Y))$ generates $I \xi \otimes (L + A + K_Y)$ at every point of $Y$.

(A.2) Skoda’s Result on Ideal Generation [Skoda 1972]. Let $\Omega$ be a domain spread over $\mathbb{C}^n$ which is Stein. Let $\psi$ be a plurisubharmonic function on $\Omega$, $g_1, \ldots, g_p$ be holomorphic functions on $\Omega$, $\alpha > 1$, $q = \min(n, p - 1)$, and $f$ be a holomorphic function on $\Omega$. Assume that

$$\int_{\Omega} \frac{|f|^2 e^{-\psi}}{\left(\sum_{j=1}^{p} |g_j|^2\right)^{\alpha q + 1}} < \infty.$$ 

Then there exist holomorphic functions $h_1, \ldots, h_p$ on $\Omega$ with $f = \sum_{j=1}^{p} h_j g_j$ on $\Omega$ such that

$$\int_{\Omega} \frac{|h_k|^2 e^{-\psi}}{\left(\sum_{j=1}^{p} |g_j|^2\right)^{\alpha q}} \leq \frac{\alpha}{\alpha - 1} \int_{\Omega} \frac{|f|^2 e^{-\psi}}{\left(\sum_{j=1}^{p} |g_j|^2\right)^{\alpha q + 1}}$$

for $1 \leq k \leq p$.

(A.3) Multiplier-Ideal Version of Skoda’s Result on Ideal Generation. Let $X$ be a compact complex algebraic manifold of complex dimension $n$, $L$ be a holomorphic line bundle over $X$, and $E$ be a holomorphic line bundle on $X$ with metric $e^{-\psi}$ such that $\psi$ is plurisubharmonic. Let $k \geq 1$ be an integer, $G_1, \ldots, G_p \in \Gamma(X, L)$, and $|G|^2 = \sum_{j=1}^{p} |G_j|^2$. Let $I = I_{(n+k+1)\log |G|^2 + \psi}$ and $J = I_{(n+k)\log |G|^2 + \psi}$. Then

$$\Gamma(X, I \otimes ((n + k + 1)L + E + K_X)) = \sum_{j=1}^{p} G_j \Gamma(X, J \otimes ((n + k)L + E + K_X)).$$

(A.4) Finite Generation of Canonical Ring From Achievement of Stable Vanishing Order. Suppose the stable vanishing orders are achieved at every point of $X$ for some $m_0 \in \mathbb{N}$. Denote $(m_0)!$ by $m_1$. Then the canonical ring

$$\bigoplus_{m=1}^{\infty} \Gamma(X, mK_X)$$







is generated by
\[ \bigoplus_{m=1}^{(n+2)m_1} \Gamma(X, mK_X) \]
and hence is finitely generated by the finite set of elements
\[ \left\{ s_j^{(m)} \right\}_{1 \leq m \leq m_1, 1 \leq j \leq q_m} \]

**Proof.** Let \( e^{-\varphi} = \frac{1}{\Phi} \). For \( m > (n+2)m_1 \) and any \( s \in \Gamma(X, mK_X) \) we have
\[ \int_X |s|^2 e^{-(m-(n+2)m_1-1)\varphi} \left( \sum_{j=1}^{q_m} |s_j^{(m)}|^2 \right)^{n+2} < \infty, \]
because \( |s|^2 \leq \tilde{C} \Phi^m \) on \( X \) for some \( \tilde{C} \). By Skoda’s theorem on ideal generation ((A.2) and (A.3)) there exist
\[ h_1, \ldots, h_{q_m} \in \Gamma(X, (m-m_1)K_X) \]
such that \( s = \sum_{j=1}^{q_m} h_j s_j^{(m)} \). If \( m-(n+2)m_1 \) is still greater than \( (n+2)m_1 \), we can apply the argument to each \( h_j \) instead of \( s \) until we get
\[ h_1^{(j_1, \ldots, j_\nu)}, \ldots, h_{q_m}^{(j_1, \ldots, j_\nu)} \in \Gamma(X, (m-m_1(\nu+1))K_X) \]
for \( 1 \leq j_1, \ldots, j_\nu \leq q_m \) with \( 0 \leq \nu < N \), where \( N = \left\lfloor \frac{m}{m_1} \right\rfloor \), such that
\[ s = \sum_{1 \leq j_1, \ldots, j_N \leq q_m} h_1^{(j_1, \ldots, j_{N-1})} \prod_{\lambda=1}^{N} s_j^{(m)} \]

Q.E.D.

(A.5) *Too Strongly Formulated Version of Discrepancy Subspace.* One can introduce a more strongly formulated version of the discrepancy subspace by using the inequality
\[ (\tilde{\tau})_{k,m}^{s,J} \left( \sum_{j=1}^{q_k} s_j^{(k)} \right)^2 \leq C_{m,k,J} \sum_{j=1}^{q_m} |s_j^{(m)}|^{\frac{2k}{m}} \]

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for all $m, k \geq m_J$, which states that the common vanishing order of all $m$-canonical sections raised to the power $\frac{k}{m}$ is no more than the common vanishing order of all $k$-canonical sections multiplied by local generators of $J$. This notion turns out to be too strong for our purpose. It actually gives immediately the finite generation of the canonical ring. We now verify that the inequality $(\dagger)_{k,m}^\sharp$ for all $m, k \geq m_J$ implies right away the finite generation of the canonical ring by Skoda’s result on ideal generation. Take any $\lambda \in \mathbb{N}$. From

$$(\sum_{j=1}^{q_k} |s_j^{(k)}|^2)^\lambda \leq C_{\lambda,k}^\ast \sum_{j=1}^{q_k} |s_j^{(\lambda k)}|^2$$

and

$$|s_J|^2 \leq C_k^* |s_J|^\frac{2}{k}$$

and from taking the $k$-root of $(\dagger)^2_{k,\lambda, m}$ that

$$|s_J|^2 \left( \sum_{j=1}^{q_k} |s_j^{(k)}|^2 \right)^\frac{1}{\lambda k} \leq C_k^* \left( C_{\lambda,k}^\ast C_{m,\lambda,k,J} \sum_{j=1}^{q_m} |s_j^{(m)}|^{2\lambda k/m} \right)^\frac{1}{\lambda k},$$

which implies that for fixed $m$ and $\lambda$ we can find a positive constant $\tilde{C}_{m,\lambda}$ and some $(m, \lambda)$-dependent rapidly decreasing sequence of positive numbers $\varepsilon_{\ell}$ used in the definition of

$$\Phi = \sum_{\ell=1}^{\infty} \varepsilon_{\ell} \sum_{j=1}^{q_\ell} |s_j^{(\ell)}|^2$$

such that

$$(\&^\prime) \quad |s_J|^2 \Phi^\lambda \leq \tilde{C}_{m,\lambda} \left( \sum_{j=1}^{q_m} |s_j^{(m)}|^2 \right)^\lambda.$$

Fix $m_0 \geq m_J$. Choose $\ell \in \mathbb{N}$ sufficiently large so that $|s_J|^2$ is locally integrable on $X$. Replacing $\lambda$ by $\ell \lambda$ and taking the $\ell$-th root of $(\&)$, we get

$$(\&^\prime)^\prime \quad \frac{\Phi^\lambda}{\left( \sum_{j=1}^{q_m} |s_j^{(m)}|^2 \right)^\lambda} \leq \left( \tilde{C}_{m,\ell \lambda} \right)^\frac{1}{\ell}. $$

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This implies that, for \( \nu \geq (n + 2)m \),

\[
\int_X \Phi^{- (n + 2)m} \left( \sum_{j=1}^{q_m} |s_j^{(m)}|^2 \right)^{(n + 2)m} \leq \int_X \Phi^\nu \left( \sum_{j=1}^{q_m} |s_j^{(m)}|^2 \right)^{(n + 2)m} \leq \int_X \left( \varepsilon_{\nu+1} \right)^{\nu+1} \left( \sum_{j=1}^{q_m} |s_j^{(m)}|^2 \right)^{(n + 2)m} < \infty,
\]

because

\[
|s_k^{(\nu+1)}|^2 \leq \frac{\Phi^{\mu+1}}{(\varepsilon_{\nu+1})^{\mu+1}}.
\]

We now apply the Multiplier-Ideal Version of Skoda’s Theorem on Ideal Generation (A.3) with the following choices.

(i) \( k = 1, L = mK_X, E = (\nu - (n + 2)m)K_X \),

(ii) \( G_j = s_j^{(m)} \) for \( 1 \leq j \leq q_m \).

(iii) \( e^{-\psi} = \frac{1}{\Phi^{\nu-(n+1)m}} \).

We get

\[
s_k^{(\nu+1)} = \sum_{j=1}^{m} \sigma_{k,j}^{(\nu+1-m)} s_j^{(m)}
\]

for some \( \sigma_{k,j}^{(\nu+1-m)} \in \Gamma (X, (\nu + 1 - m)K_X) \) for \( 1 \leq j \leq q_m \). We can now apply the same argument to \( \sigma_{k,j}^{(\nu+1-m)} \) instead of \( s_k^{(\nu+1)} \) and continue inductively. This would give the finite generation of the canonical ring.

(A.6) Lemma on Sup Norm Domination of Metric by Generators of Multiplier Ideal. Let \( f_j \) be holomorphic functions on some open neighborhood \( U \) of the origin in \( \mathbb{C}^n \). Let \( \varepsilon_j > 0 \) and \( m_j \in \mathbb{N} \) so that

\[
\Psi = \sum_{j=1}^{\infty} \varepsilon_j |f_j|^{m_j}
\]

converges uniformly on compact subsets of \( U \). Let \( \mathcal{J} \) be the multiplier ideal sheaf of the metric \( \frac{1}{\Psi} \) and \( g_1, \ldots, g_\ell \) be holomorphic function germs on \( \mathbb{C}^n \).
at the origin such that the stalk of $\mathcal{J}$ at the origin is generated by $g_1, \cdots, g_\ell$ over $\mathcal{O}_{\mathbb{C}^n,0}$. Then there exists an open neighborhood $W$ of the origin in $\mathbb{C}^n$ where $g_1, \cdots, g_\ell$ are defined and there exists a positive constant $C_j$ such that

$$|f_j|^2 \leq C_j \sum_{k=1}^\ell |g_k|^2$$

on $W$.

(A.6.1) *Remark.* The geometric reason for this lemma is that the minimum of the orders of the zeros of the generators of a multiplier ideal $\mathcal{J}$ should be no more than the order of the pole of the metric $\frac{1}{\Psi}$. A proof, for example, is given in [Demailly 1992].

(A.7) *Multiplier Ideal Sheaves of Closed Positive $(1,1)$-Currents.* For a plurisubharmonic function $\varphi$ on some open subset $\Omega$ of $\mathbb{C}^n$, the multiplier ideal sheaf $\mathcal{I}_\varphi$ is defined as consisting of all holomorphic function germs $f$ on $\Omega$ such that $|f|^2 e^{-\varphi}$ is locally integrable.

When we have two metrics $e^{-\varphi}$ and $e^{-\psi}$, in general we do not have the relation $\mathcal{I}_\varphi \mathcal{I}_\psi = \mathcal{I}_{\varphi+\psi}$. In many situations considered in this article the two ideal sheaves $\mathcal{I}_\varphi \mathcal{I}_\psi$ and $\mathcal{I}_{\varphi+\psi}$ are very close. To emphasize the closeness in such situations, we also use the notation $\mathcal{I}_{\varphi} \mathcal{I}_{\psi}$ to denote $\mathcal{I}_{\varphi+\psi}$ or even go to the extreme of simply using the inaccurate expression $\mathcal{I}_{\varphi} \mathcal{I}_{\psi}$.

Let $\Theta$ be a closed positive $(1,1)$-current on $\Omega$. Locally there is a potential function $\psi$ for $\Theta$ in the sense that

$$\Theta = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi$$

locally. The multiplier ideal sheaf $\mathcal{I}_\Theta$ is defined as consisting of all holomorphic function germs $f$ on $\Omega$ such that $|f|^2 e^{-\psi}$ is locally integrable. The ideal sheaf $\mathcal{I}_\Theta$ is independent of the choice of the local potential function $\psi$, because if $\tilde{\psi}$ is another local potential function for $\Theta$, then the difference $\psi - \tilde{\psi}$ must be pluriharmonic and therefore must be smooth.

(A.8) *Existence of Global Sections of Amply Twisted Multiple and Metric of Nonnegative Curvature Current.* For a line bundle $L$ over a compact complex manifold $X$, the following two statements are equivalent.
(a) There exists a metric $e^{-\psi}$ along the fibers of $L$ such that the curvature current of $e^{-\psi}$ is a closed positive $(1,1)$-current.

(b) For any sufficiently ample line bundle $A$ (which depends on $X$ but independent of $L$) there is a nonzero element of $\Gamma (X, mL + A)$ for any $m \in \mathbb{N}$.

The condition of sufficient ampleness of $A$ in Condition (b) is satisfied if $A - K_X$ is sufficiently ample for the global generation of multiplier ideal sheaves on $X$ (Appendix (A.1)). The implication of Condition (b) by Condition (a) simply comes from the fact that $\mathcal{I}_{m\psi} (mL + K_X + (A - K_X))$ is globally generated over $X$ for any $m \in \mathbb{N}$ and, as a consequence, the subspace of $\Gamma (X, \mathcal{I}_{m\psi} (mL + K_X + (A - K_X)))$ of $\Gamma (X, mL + A)$ is nonzero.

For the other direction we take any nonzero section $\sigma_m \in \Gamma (X, mL + A)$ for $m \in \mathbb{N}$ and consider the closed positive $(1,1)$-current 

$$\Theta_m = \frac{1}{m} \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |\sigma_m|^2$$

which represents the class of $L + \frac{1}{m} A$. Let $\omega_A$ be the positive curvature form of a smooth metric $h_A$ of the line bundle $A$. The total mass of $\Theta_m$ with respect to the Kähler metric $\omega_A$ is given by

$$\int_X \Theta_m \wedge \omega_A^{n-1} = \left( L + \frac{1}{m} A \right) A^{n-1}$$

which is uniformly bounded for all $m \in \mathbb{N}$. We can select a subsequence $m_\nu$ so that $\Theta_{m_\nu}$ is weakly convergent to $\Theta_\infty$ as $\nu \to \infty$. The closed positive $(1,1)$-current $\Theta_\infty$ represents the class of $L$ and, as a consequence, we can find a metric $e^{-\psi}$ of $L$ whose curvature current is equal to $\Theta_\infty$.

(A.9) Extension of Global Twisted Sections of Multiplier Ideal Sheaves. Let $X$ be a compact complex algebraic manifold and $Y$ be a nonsingular hypersurface of $X$. Then there exists an ample line bundle $A$ with the property that, for any holomorphic line bundle $L$ on $X$ with metric $e^{-\psi}$ such that $\psi$ is locally plurisubharmonic, the map

$$\Xi : \Gamma (X, \mathcal{I}_\psi (L + A)) \to \Gamma (Y, (\mathcal{I}_\psi / \mathcal{I}_\psi \mathcal{I}_Y) (L + A))$$

is surjective.
Proof. Choose an ample holomorphic line bundle $A_0$ on $X$ so that the ideal sheaf of $Y$ is generated by elements $\sigma_1, \ldots, \sigma_k \in \Gamma(X, A_0)$. Choose an ample line bundle $A_2$ on $X$ with smooth positively curved metric $h_2$ such that $A_2 - K_X$ is ample with smooth positively curved metric $h_3$. Let $A = A_1 + A_2 + A_3$. Use the metric

$$\frac{e^{-\psi} h_2 h_3}{\sum_{j=1}^k |\sigma_j|^2}$$

of $L + A - K_X$ whose multiplier ideal sheaf is $I_{\psi I_Y}$. Then the vanishing of $H^1(X, I_{\psi I_Y} (L + A))$ implies the surjectivity of the map $\Xi$. Q.E.D.

(A.9.1) Remark. The important point is that the line bundle $A$ on $X$ needs only to be sufficiently ample and this sufficient ampleness depends only on $Y$ and is independent of $L$ and $e^{-\psi}$. Note that the comparison of $I_{\psi}/I_{\psi I_Y}$ and $I_{\psi|Y}$ can be made by using the extension theorem of Ohsawa-Takegoshi [Ohsawa-Takegoshi 1987] if the local plurisubharmonic function $\psi$ is not identically equal to $-\infty$ on $Y$.

(A.10) Canonical Decomposition of Closed Positive (1,1)-Current. Let $\Theta$ be a closed positive (1,1)-current on a complex manifold $X$. Then $\Theta$ admits a unique decomposition of the following form

$$\Theta = \sum_{j=1}^J \gamma_j [V_j] + R,$$

where $\gamma_j > 0$, $J \in \mathbb{N} \cup \{0, \infty\}$, $V_j$ is a complex hypersurface and the Lelong number of the remainder $R$ is zero outside a countable union of subvarieties of codimension $\geq 2$ in $X$ [Siu 1974]. We consider the dichotomy into two cases. The first case is either $R \neq 0$ or $J = \infty$. The second case is both $R = 0$ and $J$ is finite. We also say that the current $\Theta$ is of the special form if it is in the second case of the dichotomy.

(A.11) Modified Restriction of Closed Positive (1,1)-Current. Let $X$ be a compact complex algebraic manifold of general type and let $e^{-\varphi} = \frac{1}{\varphi}$ be the metric of minimum singularity as defined in (1.1). Let

$$\Theta_\varphi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$$
be the curvature current of the metric $e^{-\varphi}$ of $K_X$. Let $Y$ be a nonsingular hypersurface in $X$. The generic stable vanishing order $\gamma$ across $Y$ means the Lelong number of $\Theta_\varphi$ at a generic point of $Y$ [Siu 1974]. Because of the skipping of diophantine arguments, we assume $\gamma$ to be rational. The number $\gamma$ is also the infimum of the generic vanishing order of $\left(s_j^{(m)}\right)^{\frac{1}{m}}$ across $Y$ for $m \in \mathbb{N}$ and $1 \leq j \leq q_m$.

We consider the question of how to define the restriction to $Y$ of the closed positive $(1,1)$-current $\Theta - \gamma [Y]$. We call such a restriction to $Y$ the \textit{modified restriction} of $\Theta$ to $Y$, because we are restricting $\Theta$ after we modify it by subtracting $\gamma [Y]$ from it. Let $s_Y$ be the canonical section of the line bundle associated to $Y$ so that the divisor of $s_Y$ is precisely $Y$. We know that $s_j^{(m)}$ vanishes to order at least $m\gamma$ across $Y$ for each $m \in \mathbb{N}$ and each $1 \leq j \leq q_m$, but the multi-valued fraction

$$\frac{\left(s_j^{(m)}\right)^{\frac{1}{m}}}{s_Y^\gamma}$$

may still be identically zero on $Y$ for each $m \in \mathbb{N}$ and each $1 \leq j \leq q_m$ so that the sum

$$\sum_{m=1}^{\infty} \varepsilon_m \sum_{j=1}^{q_m} \left| \frac{\left(s_j^{(m)}\right)^{\frac{1}{m}}}{s_Y^\gamma} \right|^2$$

may be identically zero on $Y$, making it impossible to consider the metric

$$\frac{1}{\sum_{m=1}^{\infty} \varepsilon_m \sum_{j=1}^{q_m} \left| \frac{\left(s_j^{(m)}\right)^{\frac{1}{m}}}{s_Y^\gamma} \right|^2}$$

of the $\mathbb{Q}$-line bundle $(L - \gamma Y)|_Y$ and to use its curvature current as the restriction of the closed positive $(1,1)$-current $\Theta - \gamma [Y]$ to $Y$. The following modification is needed in the process of constructing the restriction of the closed positive $(1,1)$-current $\Theta - \gamma [Y]$ to $Y$. For $k \in \mathbb{N}$ let $\gamma_k$ be the infimum of the vanishing order of the multi-valued section $\left(s_j^{(m)}\right)^{\frac{1}{m}}$ across $Y$ for $1 \leq
Consider the metric
\[ \frac{1}{\sum_{m=1}^{k} \varepsilon_m \sum_{j=1}^{q_m} \left| \frac{(s_j^{(m)})^{1/\pi}}{s_Y^{1/\pi}} \right|^2} \]
of the \( \mathbb{Q} \)-line bundle \((L - \gamma_k Y)|_Y\) on \( Y \) and its curvature current
\[ \Theta_k = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \sum_{m=1}^{k} \varepsilon_m \sum_{j=1}^{q_m} \frac{(s_j^{(m)})^{1/\pi}}{s_Y^{1/\pi}} \]
which is a closed positive \((1,1)\)-current on \( Y \). We know that the sequence \( \gamma_k \) is non-increasing and its limit is \( \gamma \) as \( k \to \infty \) so that the \( \mathbb{Q} \)-line bundle \((L - \gamma_k Y)|_Y\) on \( Y \) approaches the \( \mathbb{Q} \)-line bundle \((L - \gamma Y)|_Y\) on \( Y \) as \( k \to \infty \). The restriction of the closed positive \((1,1)\)-current \( \Theta - \gamma [Y] \) to \( Y \) can be defined as the (weak) limit of \( \Theta_k \) (or its subsequence) as \( k \to \infty \).

\textbf{(A.12) Absorption of Small Ample Line Bundle and Small Modification of Metric.} Let \( X \) be a compact complex algebraic manifold of general type. Let \( B \) be an ample line bundle on \( X \). Then \( K_X = \alpha B + E \) for some rational positive number \( \alpha \) and some effective \( \mathbb{Q} \)-divisor \( E \). Let \( e^{-\phi} \) be the metric of minimum singularity for \( K_X \). Consider the metric
\[ h_\delta : = e^{-(m-\frac{\delta}{\pi})\phi} \frac{1}{|s_E|^{2/\pi}} \]
of \( mK_X - \delta B \). Let \( \delta > 0 \) and \( p \in \mathbb{N} \) and \( e^{-\psi} \) be a metric of nonnegative curvature current for \( pK_X + \delta B \). We form the metric \( h_\delta e^{-\psi} \) of \((p + m)K_X \) and describe it as the absorption of the small ample line bundle \( \delta B \). Let \( h_B \) be a strictly positively curved smooth metric of \( B \). We also call the metric \( h_\delta (h_B)^\delta \) of \( mK_X \) a small modification of the metric \( e^{-m\phi} \) of \( mK_X \).

\textbf{(A.13) Family of Subvarieties Associated to a Closed Positive \((1,1)\)-Current.} Let \( X \) be a compact complex algebraic manifold and \( \Theta \) be a closed positive \((1,1)\)-current on \( X \). We are going to associate to \( \Theta \) a family of subvarieties. The motivation is that there is a lower bound of positivity for \( \Theta \) in the normal
directions of the subvarieties and the subvarieties are minimum with respect to this property. Let \( \theta \) be the \((1,1)\)-class (an element of \( H^1(X, \Omega_X^1) \)) which is defined by \( \Theta \). The class \( \theta \) may not be an integral class (that is, it may not come from \( H^2(X, \mathbb{Z}) \)).

By the simultaneous approximation of a finite collection of real numbers by rational numbers (see, for example, [Hardy-Wright 1960, p.170, Th.200]) we can find elements \( \phi_m \) of \( H^1(X, \Omega_X^1) \) and \( \delta > 0 \) such that \( m(\theta + \phi_m) \) comes from \( H^2(X, \mathbb{Z}) \) and \( m^{1+\delta} \phi_m \to 0 \) in \( H^1(X, \Omega_X^1) \) as \( m \to \infty \). Let \( L_m \) be the holomorphic line bundle on \( X \) which corresponds to the integral \((1,1)\)-class \( m(\theta + \phi_m) \). We choose an ample line bundle \( A_1 \). Since \( \delta \) is positive and \( m^{1+\delta} \phi_m \to 0 \) as \( m \to \infty \), there exists \( m_0 \in \mathbb{N} \) such that for \( m \geq m_0 \) we can find \( q_m \in \mathbb{N} \) and a metric \( e^{-\varphi_m} \) for the \( \mathbb{Q} \)-line bundle \( L_m + \frac{q_m}{m} A_1 \) such that \( \varphi_m \) is locally plurisubharmonic and its curvature current \( \Theta_m \) approaches \( \Theta \) and \( \frac{q_m}{m} \to 0 \) as \( m \to \infty \).

Let \( A \) be a holomorphic line bundle on \( X \) such that \( A - K_X \) is ample enough for the global generation of multiplier ideal sheaves on \( X \) (Appendix (A.1)). Define the number \( \kappa \) by

\[
\limsup_{m \to \infty} \frac{\log \dim \mathcal{C} \Gamma(X, \mathcal{I}_{\varphi_m}(L_m + A))}{\log m}.
\]

This is a way of measuring the lower bound of positivity, because we have global generation of the multiplier ideal sheaf when we add \( A \) to \( L_m \) to consider \( \Gamma(X, \mathcal{I}_{\varphi_m}(L_m + A)) \).

Suppose in a neighborhood \( U \) of a point \( P \) of \( X \) the closed positive \((1,1)\)-current \( \Theta \) dominates a strictly positive smooth \((1,1)\)-current \( \omega \) for \( m \) sufficiently large. Then \( \Theta_m \) dominates \( \frac{q_m}{m} \omega \). We can then use a local coordinate system \( z = (z_1, \cdots, z_n) \) centered at \( P \) so that the unit ball \( B \) is relatively compact in \( U \). We can consider the function \( \tilde{\chi} \) on \( X \) which is equal to \( \frac{1}{|z|} \) on \( B \) and equal to 1 on \( X - B \) and we smooth \( \chi \) slightly near the boundary of \( B \) to get \( \chi \) which is smooth and positive on \( X - \{P\} \) and is equal to \( \tilde{\chi} \) outside a very small neighborhood of the boundary of \( B \) in \( U \). Then for some \( p_0 \in \mathbb{N} \) sufficiently large

\[
\frac{p_0}{2} \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \chi
\]

is a positive \((1,1)\)-current on \( U \). Thus the curvature current of the metric \( e^{-\varphi_m} \chi^q \) is a positive \((1,1)\)-current on all of \( X \) for \( m \geq p_0q \).
By Skoda’s result (A.2), we know that the multiplier ideal sheaf of \( e^{-\varphi_m} \chi^q \) is contained in 
\[(m_P)^{q-n-1} \mathcal{I}_{\varphi_m} \]
for \( m \geq p_0 q \). From the surjectivity of 
\[\Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A)) \to \mathcal{I}_{\varphi_m} / ((m_P)^{q-n-1} \mathcal{I}_{\varphi_m}) \]
for \( m \geq p_0 q \) we conclude that 
\[\dim \Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A)) \geq \dim \mathcal{O}_X / ((m_P)^{q-n-1}) = \binom{q-2}{n} \]
for \( m \geq p_0 q \) and 
\[\dim \Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A)) \geq \left\lfloor \frac{m}{p_0} \right\rfloor - 2 \]
This implies that 
\[\limsup_{m \to \infty} \frac{\log \dim \Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A))}{\log m} \geq n\]
and \( \kappa \geq n \).

We now consider another case. Suppose \( V \) is a complex submanifold of complex dimension \( d \) in \( X \) and \( \Theta|_V \) dominates some strictly positive smooth \((1,1)\)-current \( \omega_V \) on the neighborhood of some point \( P \) in \( V \). Assume that \( A \) is sufficiently ample so that the map 
\[\Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A)) \to \Gamma (V, \mathcal{I}_{\varphi_m}|_V (L_m + A))\]
is surjective (A.9). Since our earlier argument gives 
\[\dim \Gamma (V, \mathcal{I}_{\varphi_m}|_V (L_m + A)) \geq cm^d\]
for some \( c > 0 \) and for all \( m \in \mathbb{N} \) sufficiently large, it follows that 
\[\limsup_{m \to \infty} \frac{\log \dim \Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A))}{\log m} \geq d\]
and \( \kappa \geq d \).
Conversely, suppose we have the growth rate and we would like to conclude about the positive lower bound of the curvature current when restricted to some submanifold. We again choose some transversal submanifold $V$ of complex dimension $\kappa$ which is independent of $m$ in the sense that

$$\dim \Gamma (V, \mathcal{I}_{\varphi_m}|_V (L_m + A)) \geq cm^\kappa$$

for some $c > 0$ and for all $m \in \mathbb{N}$ sufficiently large. The choice of $V$ simply means that it is transversal to proper subvarieties of $X$ defined by homogeneous polynomials of elements of $\Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A))$. Those subvarieties are from a countable number of holomorphic families of subvarieties. We consider elements in $\Gamma (X, \mathcal{I}_{\varphi_m} (L_m + A))$ whose restrictions to $V$ vanish to order $q$ at a point $P$ of $V$. We now consider their common zero-set $W$. If the vanishing order at $P$ or any other points of $W$ is substantially less than $\beta q$ for some positive $\beta$ sufficiently close to 0, for example $\beta < q^{-\delta}$ for some $0 < \delta < 1$, then we can create a metric for $L_m$ (by raising $m$ to a high multiple first and then taking root of the metric later) so that the restriction to some $\tilde{V}$ with complex dimension greater than that of $V$ gives a metric of isolated high order singularity at a point of $\tilde{V}$. This would mean that the growth order is greater than $\kappa$ which is a contradiction. For every $P$ we can form the subvariety $W$ which depends on $P$ and which we denote by $W_P$. As $P$ varies inside $V$, we can get a holomorphic family of subvarieties $\{W_s\}_{s \in S}$.

\textbf{(A.14) Additional Vanishing and Minimal Center of Log Canonical Singularity for the Second Case of Dichotomy.} Let $X$ be a compact complex algebraic manifold and $Y$ be a nonsingular complex hypersurface in $X$. Let $\gamma$ be the generic stable vanishing order across $Y$ which we assume to be rational, because we are skipping the diophantine arguments. Let $\frac{1}{\Theta} = e^{-\varphi}$ be the metric of $K_X$ of minimum singularity with curvature current $\Theta_{\varphi}$. We have the modified restriction $\Theta_{\varphi} - \gamma [Y]$ defined in Appendix (A.11), which we denote by $\Theta_Y$. We assume that the current $\Theta_Y$ is of the special form (\textit{i.e.}, in the second case of the dichotomy according to the terminology of [Siu 2006]), which means that there are a finite number of complex hypersurfaces $V_j$ in $Y$ for $1 \leq j \leq J$ such that

$$\Theta_Y = \sum_{j=1}^{J} \alpha_j [V_j].$$

Again because we are skipping the diophantine argument, we assume that each $\alpha_j$ is rational. Take a sufficiently divisible positive integer $\tilde{m}$ so that
\( \tilde{m}\gamma \) and \( \tilde{m}\alpha_j \) are positive integers. We have a holomorphic section

\[
\sigma := \prod_{j=1}^{J} (s_{V_j})^{\tilde{m}\alpha_j}
\]

over \( Y \) of the tensor product of the holomorphic line bundle \( \tilde{m}(K_X - \gamma Y) \) and some flat bundle over \( Y \). The flat bundle situation is handled by a modification of the technique of Shokurov [Shokurov 1985] using the theorem of Riemann-Roch and the vanishing theorem for multiplier ideal sheaves. By Shokurov’s technique we can get a section of \( \tilde{m}(K_X - \gamma Y) \) over \( Y \) with the same divisor as \( \sigma \). Here we are interested in explaining a technique of minimum center of log canonical singularity and we will not go into Shokurov’s technique. We would like to extend \( \sigma \) to an element of \( \Gamma (X, \hat{m}(K_X - \gamma Y)) \). The extension is done by using the vanishing theorem for multiplier ideal sheaves.

Let \( L = K_X - \gamma Y \). If the generic stable vanishing order \( \gamma \) across \( Y \) is achieved by some \( \tau \in \Gamma (X, \hat{m}K_X) \), then

\[
\frac{\tau}{(s_Y)^{\tilde{m}\gamma}}
\]

would be a holomorphic section of \( \hat{m}L = \hat{m}(K_X - \gamma Y) \) over \( Y \) and its divisor must be \( \hat{m}\sum_{j=1}^{J} \alpha_j V_j \), because the curvature current \( \Theta_Y \) is of the special form. There is nothing more to do and there is no need for any further discussion.

What we are interested in is the case when the generic stable vanishing order \( \gamma \) across \( Y \) is not achieved, which we now assume to be the case. For \( N \in \mathbb{N} \) consider the \( N \)-truncation

\[
\Phi_N = \sum_{m=1}^{N} \sum_{j=1}^{m} \left| s_{V_j}^{(m)} \right|^{\frac{1}{m}}
\]

Then no matter how large \( N \) is, the vanishing order \( \tilde{\gamma}_N \) of \( \Phi_N \) across \( Y \) is still strictly greater than \( \gamma \). Because the vanishing theorem adds one \( K_X \), in order to get \( L - Y = K_Y - (1 + \gamma) Y \) (which is to replace the new copy of \( K_X \) by \( L \) and to get one order of vanishing across \( Y \) needed for the vanishing of the first cohomology to extend sections from \( Y \) to \( X \) from \( K_X \) we have
to get an extra vanishing order $1 + \gamma$ across $Y$. For the extension of $\sigma$ to an element of $\Gamma (X, \tilde{m} (K_X - \gamma Y))$, the ideal situation is to get the vanishing of $H^1 (X, \mathcal{I}_Y (\tilde{m}L))$ so that we have the surjectivity of the map

$$\Gamma (X, \tilde{m}L) \rightarrow \Gamma (Y, \tilde{m}L).$$

The cohomology group $H^1 (X, \mathcal{I}_Y (\tilde{m}L))$ is the same as $H^1 (X, (\mathcal{I}_Y)^{\tilde{m}\gamma+1} (\tilde{m}K_X)).$

In order to get its vanishing by using multiplier ideal sheaf, because of the addition of one $K_X$, the ideal situation is to get a metric for $(\tilde{m} - 1) K_X$ whose multiplier ideal sheaf is $(\mathcal{I}_Y)^{\tilde{m}\gamma+1}$. If we use the metric $e^{-\psi_{\delta}} := \left( \frac{1}{\Phi^{1-\delta} \Phi^N} \right)^{\tilde{m}-1}$ for $(\tilde{m} - 1) K_X$ with $0 < \delta < 1$ determined by

$$(\tilde{m} - 1) ((1 - \delta) \gamma + \delta \tilde{\gamma}_N) = \tilde{m}\gamma + 1,$$

which means

$$\delta = \frac{1 + \gamma}{(\tilde{m} - 1) (\tilde{\gamma}_N - \gamma)}.$$

To get $0 < \delta < 1$, by replacing $\tilde{m}$ by a high multiple, we can assume that

$$\tilde{m} > \frac{\tilde{\gamma}_N - \gamma}{1 + \gamma}.$$

The main difficulty is that the multiplier ideal sheaf $\mathcal{I}_{\psi_{\delta}}$ of $e^{-\psi_{\delta}}$ may be smaller than $(\mathcal{I}_Y)^{\tilde{m}\gamma+1}$ so that we can only get

$$H^1 (X, \mathcal{I}_{\psi_{\delta}} (\tilde{m}L)) = 0$$

and the surjectivity of

$$\Gamma (X, \tilde{m}L) \rightarrow \Gamma (X, (\mathcal{O}_X / \mathcal{I}_{\psi_{\delta}}) (\tilde{m}L)).$$

Since $\sigma$ is only an element of $\Gamma (Y, \tilde{m}L)$ and may not be an element of $\Gamma (X, (\mathcal{O}_X / \mathcal{I}_{\psi_{\delta}}) (\tilde{m}L))$, in general we would have trouble extending $\sigma$ to all of $X$. We call this difficulty *additional vanishing*, because the vanishing order of $(\Phi^{1-\delta} \Phi^N)^{\tilde{m}-1}$ is more than the desired order of $\tilde{m}\gamma + 1$ across $Y$ and there is additional vanishing. We do not even know that the zero-set of $\mathcal{I}_{\psi_{\delta}}$ is just $Y$. There may even be zeroes of $\mathcal{I}_{\psi_{\delta}}$ outside $Y$. 
Here we only want to explain one special technique which handles the difficulty of additional vanishing for the case when the zero-set of $I_{\psi_\delta}$ is contained in $Y$. The section $\sigma$ can be considered as an element of $\Gamma \left( Y, \left( \mathcal{I}_Y \right)^{\tilde{m}\gamma} / \left( \mathcal{I}_Y \right)^{\tilde{m}\gamma+1} \right) \left( \tilde{m}K_X \right)$.

The difficulty of additional vanishing means that $I_{\psi_\delta}$ does not contain $\left( \mathcal{I}_Y \right)^{\tilde{m}\gamma+1}$ and as a consequence $\sigma$ may not be an element of $\Gamma \left( Y, \left( \mathcal{O}_X / \mathcal{I}_{\psi_\delta} \right) \left( \tilde{m}K_X \right) \right)$. The technique is to decrease $\delta$ to $\tilde{\delta}$ such that $\sigma$ can induce a well-defined and not non-identically-zero element $\tilde{\sigma}$ of $\Gamma \left( Y, \left( \mathcal{O}_X / \mathcal{I}_{\psi_\tilde{\delta}} \right) \left( \tilde{m}K_X \right) \right)$. Then from

$$H^1 \left( X, \mathcal{I}_{\psi_\tilde{\delta}} \left( \tilde{m}L \right) \right) = 0$$

and the surjectivity of

$$\Gamma \left( X, \tilde{m}L \right) \to \Gamma \left( X, \left( \mathcal{O}_X / \mathcal{I}_{\psi_\tilde{\delta}} \right) \left( \tilde{m}L \right) \right)$$

we would be able to extend $\tilde{\sigma}$ to an element of $\tilde{\sigma}$ of $\Gamma \left( X, \tilde{m}L \right)$. Since the current $\Theta_Y$ is of the special form and since $\tilde{\sigma}$ restricts to $\tilde{\sigma}$ on $\left( \mathcal{O}_X / \mathcal{I}_{\psi_\tilde{\delta}} \right) \left( \tilde{m}L \right)$, it follows that $\tilde{\sigma}$ must induce $\sigma$ as an element of

$$\Gamma \left( Y, \left( \mathcal{I}_Y \right)^{\tilde{m}\gamma} / \left( \mathcal{I}_Y \right)^{\tilde{m}\gamma+1} \right) \left( \tilde{m}K_X \right)$$

Let us explain the above procedure with the following local concrete example. Assume that $\gamma = 0$ so that $\tilde{m}\gamma = 0$. Locally at the origin $P_0$ of a local coordinate system of $(z_1, \cdots, z_n)$ of $X$ we suppose that $Y$ is given by $z_1 = 0$ and $\Theta_Y$ is just the hypersurface $V_1$ given $z_2 = 0$ with coefficient 1. Assume that $\psi_\delta$ is generated by $z_1^2$ and $z_1z_2$. Near $P_0$ the ringed space $(Y, \mathcal{O}_X / \mathcal{I}_{\psi_\delta})$ is reduced except along $V$. Assume that near $P_0$ the divisor $\sigma$ is locally the hypersurface given by $z_2 = 0$ with coefficient 1. When we consider the element in $\mathcal{O}_X / \mathcal{I}_{\psi_\delta}$ induced by $\sigma$ locally near $P_0$ we simply get a section of the conormal bundle of $V$ in $Y$ which is non-identically-zero (from differentiating $\sigma$ in the normal direction of $V$).

(A.15) Positive Lower Bound of Curvature Current in Ambient Space. Let $X$ be a compact complex manifold of general type with metric $\Phi$ of minimum singularity and its $N$-th truncation $\frac{1}{\Phi_N}$. Let $Y$ be a nonsingular complex hypersurface in $X$ whose generic stable vanishing order $\gamma$ is not achieved.
We choose two appropriate sufficiently large positive integers $m$ and $N$ and construct two interpolations

\[ e^{-\psi_\nu} = \left( \frac{1}{\Phi^{1-\delta_\nu} \Phi_N^{\delta_\nu}} \right)^m \text{ for } \nu = 1, 2 \]

with $0 < \delta_1 < \delta_2 < 1$ very close to each other and separated by some critical value for integrability at a generic point of $Y$ (also after appropriate small modifications (A.12) if necessary) so that the two multiplier ideal sheaves $\mathcal{I}_{\psi_1}$ and $\mathcal{I}_{\psi_2}$ satisfy the relation $\mathcal{I}_{\psi_2} = \mathcal{I}_Y \mathcal{I}_{\psi_1}$. From $H^p (X, \mathcal{I}_{\psi_\nu} ((m + 1)K_X)) = 0$ for $p \geq 1$ it follows that

\[ H^p (Y, \mathcal{I}_{\psi_1} \mathcal{O}_Y ((m + 1)K_X)) = 0 \text{ for } p \geq 1, \]

where $\mathcal{I}_{\psi_1} \mathcal{O}_Y = \mathcal{I}_{\psi_1} / (\mathcal{I}_Y \mathcal{I}_{\psi_1})$. When $E$ is a holomorphic line bundle on $X$ with metric $e^{-\chi}$ of nonnegative curvature current, we can also conclude in the same way after a careful choice of $\delta_1$ and $\delta_2$ and appropriate small modifications that

\[ H^p \left( Y, \overline{\mathcal{I}_X \mathcal{I}_{\psi_1}} \mathcal{O}_Y (E + (m + 1)K_X) \right) = 0 \text{ for } p \geq 1. \]

(See (A.7) for the meaning of the symbol $\overline{\mathcal{I}_X \mathcal{I}_{\psi_1}}$.) The point is that we get a vanishing result on $Y$ from vanishing results on $X$. The vanishing theorem for multiplier ideal sheaves requires strictly positive lower bound for the curvature current. Sometimes we have such strictly positive lower bound for its curvature current on $X$, but not on the restriction to $Y$. In such a case this method enables us to get a vanishing result by using the strictly positive lower bound for the curvature current from the ambient space $X$. For example, though $X$ is of general type, $Y$ may not be of general type. Also this is another way to handle additional vanishing discussed in (A.14) so that we deal with the reduced structure of $Y$ instead of an unreduced structure for $Y$ (which comes from the additional vanishing).

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