Fermi motion parameter $p_F$ of $B$ meson from relativistic quark model

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Abstract

The Fermi motion parameter $p_F$ is the most important parameter of AC-CMM model, and the value $p_F \sim 0.3$ GeV has been used without clear theoretical or experimental evidence. So, we attempted to calculate the value for $p_F$ theoretically in the relativistic quark model using quantum mechanical variational method. We obtained $p_F \sim 0.5$ GeV, which is somewhat larger than 0.3 GeV, and we also derived the eigenvalue of $E_B \simeq 5.5$ GeV, which is in reasonable agreement with $m_B = 5.28$ GeV. We also recalculated $|V_{ub}/V_{cb}|$ as a function of $p_F$.

The simplest model for the semileptonic $B$-decay is the spectator model which considers the decaying $b$-quark in the $B$-meson as a free particle. The spectator model is usually used with the inclusion of perturbative QCD radiative corrections. The decay width of the process $B \to X_q \ell \nu$ is given by

$$
\Gamma_B(B \to X_q \ell \nu) \simeq \Gamma_b(b \to q\ell\nu) = |V_{bq}|^2 \left( \frac{G_F^2 m_b^5}{192 \pi^3} \right) f(m_q) \left[ 1 - \frac{2}{3} \alpha_s \left( \frac{m_q}{m_b} \right)^2 \right],
$$

where $m_q$ is the mass of the $q$-quark decayed from $b$-quark. The decay width of the spectator model depends on $m_b^5$, therefore small difference of $m_b$ would change the decay width significantly.

Altarelli et al. [1] proposed their ACCMM model for the inclusive $B$-meson semileptonic decays. This model incorporates the bound state effect by treating the $b$-quark as a virtual state particle, thus giving momentum dependence to the $b$-quark mass. The virtual state $b$-quark mass $W$ is given by

$$
W^2(p) = m_B^2 + m_{sp}^2 - 2m_B \sqrt{p^2 + m_{sp}^2},
$$

in the $B$-meson rest frame, where $m_{sp}$ is the spectator quark mass, $m_B$ the $B$-meson mass, and $p$ is the momentum of the $b$-quark.

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1 Talk given by D.S. Hwang, at the Int. Workshop on B-Physics, Nagoya, Japan, on Oct. 26-28, 1994. Proceedings published by World Scientific, Singapore, edited by A. Sanda and S. Suzuki
For the momentum distribution of the virtual $b$-quark, Altarelli et al. considered the Fermi motion inside the $B$-meson with the Gaussian momentum distribution

$$\phi(p) = \frac{4}{\sqrt{\pi} p_F^3} e^{-p^2/p_F^2},$$

(3)

where the Fermi motion parameter $p_F$ is treated as a free parameter. And the decay width is given by integrating the width $\Gamma_b$ in (1) with the weight $\phi(p)$. Then the lepton energy spectrum of the $B$-meson semileptonic decay is given by

$$\frac{d\Gamma_B}{dE_l}(p_F, m_{sp}, m_q, m_B) = \int_0^{p_{max}} dp \, p^2 \phi(p) \frac{d\Gamma_b}{dE_l}(m_b = W, m_q),$$

(4)

where $p_{max}$ is the maximum kinematically allowed value of $p = |p|$. The ACCMM model, therefore, introduces a new parameter $p_F$ for the Gaussian momentum distribution of the $b$-quark inside $B$-meson instead of the $b$-quark mass of the spectator model. In this way the ACCMM model incorporates the bound state effect and reduces the strong dependence on $b$-quark mass in the decay width of the spectator model.

The Fermi motion parameter $p_F$ is the most essential parameter of the ACCMM model as we see in the above. However, the experimental determinations of its value from the lepton energy spectrum have been very ambiguous until now because various parameters, such as $p_F$, $m_q$ and $m_{sp}$, are fitted all together from the lepton energy spectrum, and because the perturbative QCD corrections are sensitive in the end point region of the spectrum. We think that the value $p_F \sim 0.3$, which has been widely used in experimental analyses, has no theoretical or experimental clean justification, even though there has been recently an assertion that the BSUV model \cite{2} is approximately equal to ACCMM model at $p_F \simeq 0.3$. Therefore, it is strongly required to determine the value of $p_F$ more firmly when we think of the importance of its role in experimental analyses. The better determination of $p_F$ is also interesting theoretically since it has its own physical correspondence related to the Fermi motion inside $B$-meson. In this context we are going to determine theoretically the value of $p_F$ in the relativistic quark model using quantum mechanical variational method.

We consider the Gaussian probability distribution function $\phi(p)$ in (3) as the absolute square of the momentum space wave function $\chi(p)$ of the bound state $B$-meson, i.e.,

$$\phi(p) = 4\pi |\chi(p)|^2, \quad \chi(p) = \frac{1}{(\sqrt{\pi} p_F)^{3/2}} e^{-p^2/p_F^2}.$$

(5)

The Fourier transform of $\chi(p)$ gives the coordinate space wave function $\psi(r)$, which is also Gaussian,

$$\psi(r) = \frac{p_F}{\sqrt{\pi}} r^{3/2} e^{-r^2 p_F^2/2}.$$

(6)

Then we can approach the determination of $p_F$ in the framework of quantum mechanics. For the $B$-meson system we treat the $b$-quark non-relativistically, but the $u$- or $d$-quark relativistically with the Hamiltonian

$$H = M + \frac{p^2}{2M} + \sqrt{p^2 + m^2} + V(r),$$

(7)
where $M = m_b$ is the $b$-quark mass and $m = m_{sp}$ is the $u$- or $d$- quark mass. We apply the variational method the Hamiltonian (7) with the trial wave function
\[
\psi(r) = \left(\frac{\mu}{\sqrt{\pi}}\right)^{3/2} e^{-\mu^2 r^2/2},
\]
where $\mu$ is the variational parameter. The ground state is given by minimizing the expectation value of $H$,
\[
\langle H \rangle = \langle \psi | H | \psi \rangle = E(\mu), \quad \frac{d}{d\mu} E(\mu) = 0 \quad \text{at} \quad \mu = \bar{\mu},
\]
and then $\bar{\mu} = p_F$ and $\bar{E} \equiv E(\bar{\mu})$ approximates $m_B$. The value of $\mu$ or $p_F$ corresponds to the measure of the radius of the two body bound state as can be seen from $\langle r \rangle = 2 \sqrt{\frac{1}{\mu}}$ and $\langle r^2 \rangle^{1/2} = \frac{3}{2 \mu}$.

In (7) we take the Cornell potential which is composed of the Coulomb and linear potentials,
\[
V(r) = -\frac{\alpha_c}{r} + K r.
\]
For the values of the parameters $\alpha_c \equiv \frac{4}{3} \alpha_s$, $K$, and the $b$-quark mass $m_b$, we use the values given by Hagiwara et al. [3],
\[
\alpha_c = 0.47 \quad (\alpha_s = 0.35), \quad K = 0.19 \text{ GeV}^2, \quad m_b = 4.75 \text{ GeV}, \quad \alpha_s = 0.35, \quad K = 0.19 \text{ GeV}^2, \quad m_b = 4.75 \text{ GeV},
\]
which have been determined by the best fit of the $(c\bar{c})$ and $(b\bar{b})$ bound states. For comparison we will also consider $\alpha_c = 0.32 \quad (\alpha_s = 0.24)$, which corresponds to $\alpha_s(Q^2 = m_B^2)$.

Before applying our variational method with the Gaussian trial wave function to the $B$-meson system, let us check the method by considering the $\Upsilon(b\bar{b})$ system. The Hamiltonian of the $\Upsilon(b\bar{b})$ system can be approximated by the non-relativistic Hamiltonian
\[
H \simeq 2m_b + \frac{p^2}{m_b} + V(r).
\]
With the parameters in (11) (or with $\alpha_c = 0.32$), our variational method with the Gaussian trial wave function (8) gives $p_F = \bar{\mu} = 1.1$ GeV and $\bar{E} = E(\bar{\mu}) = 9.49$ GeV. Here $p_F = 1.1$ GeV corresponds to the radius $R(\Upsilon) = 0.2$ fm, and $\bar{E}(\Upsilon) = 9.49$ GeV is within 0.3 % error compared with the experimental value $E_{exp} = m_\Upsilon = 9.46$ GeV. Therefore, the variational method with the non-relativistic Hamiltonian (12) gives fairly accurate results for the $\Upsilon$ ground state.

However, since the $u$- or $d$- quark in the $B$-meson is very light, the non-relativistic description can not be applied to the $B$-meson system. For example, when we apply the variational method with the non-relativistic Hamiltonian to the $B$-meson, we get the results
\[
p_F = 0.29 \text{ GeV}, \quad \bar{E} = 5.92 \text{ GeV} \quad \text{for} \quad \alpha_s = 0.35, \quad (13)
p_F = 0.29 \text{ GeV}, \quad \bar{E} = 5.97 \text{ GeV} \quad \text{for} \quad \alpha_s = 0.24. \quad (14)
\]
The above masses $\tilde{E}$ are much larger compared to the experimental value $m_B = 5.28$ GeV, and moreover the expectation values of the higher terms in the non-relativistic perturbative expansion are bigger than those of the lower terms. Therefore, we can not apply the variational method with the non-relativistic Hamiltonian to the $B$-meson system.

Let us come back to our Hamiltonian (7) of the $B$-meson system. In our variational method the trial wave function is Gaussian both in the coordinate space and in the momentum space, so the expectation value of $H$ can be calculated in either space from $\langle H \rangle = \langle \psi(r) | H | \psi(r) \rangle = \langle \chi(p) | H | \chi(p) \rangle$. Also, the Gaussian function is a smooth function and its derivative of any order is square integrable, thus any power of the Laplacian operator $\nabla^2$ is a hermitian operator at least under Gaussian functions. Therefore, analyzing the Hamiltonian (7) with the variational method can be considered as reasonable even though solving the eigenvalue equation of the differential operator (7) may be confronted with the mathematical difficulties because of the square root operator in (7).

With the Gaussian trial wave function (5) or (8), the expectation value of the Hamiltonian (7) can be calculated easily besides the square root operator,

$$\langle p^2 \rangle = \langle \psi(r) | p^2 | \psi(r) \rangle = \langle \chi(p) | p^2 | \chi(p) \rangle = \frac{3}{2} \mu^2, \quad (15)$$

$$\langle V(r) \rangle = \langle \psi(r) | -\frac{\alpha_e}{r} + K r | \psi(r) \rangle = \frac{2}{\sqrt{\pi}} (-\alpha_e \mu + K/\mu). \quad (16)$$

Now let us consider the expectation value of the square root operator in the momentum space

$$\langle \sqrt{p^2 + m^2} \rangle = \langle \chi(p) | \sqrt{p^2 + m^2} | \chi(p) \rangle = \left( \frac{\mu}{\sqrt{\pi}} \right)^3 \int_0^\infty e^{-p^2/\mu^2} \sqrt{p^2 + m^2} \, d^3p$$

$$= \frac{4\mu}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \sqrt{x^2 + (m/\mu)^2} \, x^2 \, dx. \quad (17)$$

The integral (17) can be given as a series expansion by the following procedure. First, define

$$I(s) \equiv \int_0^\infty \sqrt{x^2 + s} \, x^2 e^{-x^2} \, dx = s^2 \int_0^\infty \sqrt{t^2 + 1} \, t^2 e^{-st^2} \, dt, \quad (18)$$

$$I_0(s) \equiv \int_0^\infty \sqrt{x^2 + s} \, e^{-x^2} \, dx = s \int_0^\infty \sqrt{t^2 + 1} \, e^{-st^2} \, dt. \quad (19)$$

Next, from (18) and (19), we find the following differential relations

$$\frac{d}{ds} \left( \frac{I_0}{s} \right) = -\frac{1}{s^2} I, \quad \frac{dI}{ds} = -\frac{1}{2} I_0 + I. \quad (20)$$

Combining two equations in (20), we get a second order differential equation for $I(s)$,

$$sI''(s) - (1 + s)I'(s) + \frac{1}{2} I(s) = 0. \quad (21)$$
The series solution to (21) is given as
\[ I(s) = c_1 I_1(s) + c_2 I_2(s), \]
\[ I_1(s) = s^2 F(s; \frac{3}{2}, 3) = s^2 \left\{ 1 + \frac{1}{2} s + \frac{5}{32} s^2 + \frac{7}{192} s^3 + \frac{7}{1024} s^4 + \cdots \right\}, \tag{22} \]
\[ I_2(s) = I_1(s) \int \frac{se^s}{[I_1(s)]^2} ds = -\frac{1}{16} s^2 \ln s \left( 1 + \frac{1}{2} s - \frac{5}{32} s^2 + \frac{7}{192} s^3 + \frac{7}{1536} s^4 + \cdots \right) \]
\[ - \frac{1}{2} \left( 1 + \frac{1}{2} s - \frac{5}{32} s^2 + \frac{7}{192} s^3 + \frac{7}{1536} s^4 + \cdots \right), \]
where \( F(s; \frac{3}{2}, 3) \) is the confluent hypergeometric function which is convergent for any finite \( s \), and the integral constants \( c_1 \approx -0.095, c_2 = -1 \). See Appendix for the derivation of these numerical values for \( c_i \).

Finally, collecting (15), (16) and (17), the expectation value of \( H \) is written as
\[ \langle H \rangle = M + \frac{1}{2M} \left( \frac{3}{2} \mu^2 \right) + \frac{2}{\sqrt{\pi}} (-\alpha c \mu + K/\mu) \]
\[ + \frac{2\mu}{\sqrt{\pi}} \left[ 1 + \frac{1}{2} (m/\mu)^2 + \left( \frac{5}{32} - 2c_1 \right) (m/\mu)^4 + \frac{1}{4} (m/\mu)^4 \ln(m/\mu) \right], \tag{23} \]
up to \((m/\mu)^4\).

With the input value of \( m = m_{sp} = 0.15 \) GeV, we minimize \( \langle H \rangle \) of (23), and then we obtain
\[ p_F = \bar{\mu} = 0.54 \text{ GeV}, \quad m_B = \bar{E} = 5.54 \text{ GeV} \quad \text{for } \alpha_s = 0.35, \tag{24} \]
\[ \bar{\mu} = 0.49 \text{ GeV}, \quad \bar{E} = 5.63 \text{ GeV} \quad \text{for } \alpha_s = 0.24. \]

Here let us check how much sensitive our calculation of \( p_F \) is by considering the case where \( m = m_{sp} = 0 \) for comparison. For \( m_{sp} = 0 \) the integral in (17) is done easily and we obtain the following values of \( \bar{\mu} = p_F \) by the above variational method.
\[ \bar{\mu} = 0.53 \text{ GeV}, \quad \bar{E} = 5.52 \text{ GeV} \quad \text{for } \alpha_s = 0.35, \tag{25} \]
\[ \bar{\mu} = 0.48 \text{ GeV}, \quad \bar{E} = 5.60 \text{ GeV} \quad \text{for } \alpha_s = 0.24. \]

As we see in (25), the results are similar to those in (24) where \( m_{sp} = 0.15 \) GeV. We could expect this insensitivity of the value of \( p_F \) to that of \( m_{sp} \), because the value of \( m_{sp} \), which should be small in any case, can not affect the integral in (23) significantly.

The calculated values of the \( B \)-meson mass, \( \bar{E} \), are much larger than the measured value of 5.28 GeV. The large values for the mass are originated partly because the Hamiltonian (23) does not take care of the correct spin dependences for \( B \) and \( B^* \). The difference between the pseudoscalar meson and the vector meson is given arise to by the chromomagnetic hyperfine splitting, which is given by
\[ V_s = \frac{2}{3Mm} \vec{s}_1 \cdot \vec{s}_2 \nabla^2 (-\frac{\alpha c}{r}). \tag{26} \]
Then the expectation values of \( V_s \) are given by
\[
\langle V_s \rangle = -\frac{2}{\sqrt{\pi}} \frac{\alpha_s \mu^3}{Mm} \quad \text{for } B, \quad \langle V_s \rangle = \frac{2}{3\sqrt{\pi}} \frac{\alpha_s \mu^3}{Mm} \quad \text{for } B^*,
\]
and we treat \( \langle V_s \rangle \) only as a perturbation. Then we get for \( B \) meson
\[
\begin{align*}
p_F &= 0.54 \text{ GeV}, & E_B = 5.42 \text{ GeV} & \text{for } \alpha_s = 0.35, \\
p_F &= 0.49 \text{ GeV}, & E_B = 5.56 \text{ GeV} & \text{for } \alpha_s = 0.24,
\end{align*}
\]
and for \( B^* \)
\[
\begin{align*}
p_F &= 0.54 \text{ GeV}, & \bar{E}_{B^*} = 5.58 \text{ GeV} & \text{for } \alpha_s = 0.35, \\
p_F &= 0.49 \text{ GeV}, & \bar{E}_{B^*} = 5.65 \text{ GeV} & \text{for } \alpha_s = 0.24.
\end{align*}
\]

The calculated values of the \( B \)-meson mass, 5.42 GeV (\( \alpha_s = 0.35 \)) and 5.56 GeV (\( \alpha_s = 0.24 \)) are in reasonable agreement compared to the experimental value of \( m_B = 5.28 \) GeV; the relative errors are 2.7% and 5.3%, respectively. However, for Fermi motion parameter \( p_F \), the calculated values, 0.54 GeV (\( \alpha_s = 0.35 \)) and 0.49 GeV (\( \alpha_s = 0.24 \)), are somewhat larger than the value 0.3 GeV, which has been widely used in the experimental analyses of energy spectrum of semileptonic \( B \)-meson decay. The value \( p_F = 0.3 \) GeV corresponds to the \( B \)-meson radius \( R_B \sim 0.66 \) fm, which seems too large. On the other hand, the value \( p_F = 0.5 \) GeV corresponds to \( R_B \sim 0.39 \) fm, which looks in reasonable range.

If we use \( p_F = 0.5 \) GeV, instead of \( p_F = 0.3 \) GeV, in the experimental analysis of the end point region of lepton energy spectrum, the value of \( |V_{ub}/V_{cb}| \) becomes significantly changed. Using the experimental data of the end point region, \( i.e. \) 2.3 GeV < \( E_\ell < 2.6 \) GeV of the CLEO result [4], we can find the relation
\[
\frac{|V_{ub}|^2}{|V_{cb}|^2}_{p_F=0.5} = \frac{|V_{ub}|^2}{|V_{cb}|^2}_{p_F=0.3} \times \frac{\bar{\Gamma}(0.3)}{\bar{\Gamma}(0.5)} = \frac{|V_{ub}|^2}{|V_{cb}|^2}_{p_F=0.3} \times 1.81,
\]
where \( \bar{\Gamma}(p_F) \equiv \int_{2.3}^{2.6} dE_\ell \frac{d\Gamma}{dE_\ell}(p_F) \) with \( |V_{ub}| = 1 \), and we assume the value of \( |V_{cb}| \) is determined independently from the other analyses. We numerically calculated \( \bar{\Gamma}(0.3)/\bar{\Gamma}(0.5) \) by using [4] with \( m_{sp} = 0.15 \) GeV, \( m_q = 0.15 \) GeV and \( m_B = 5.28 \) GeV, to get its value as 1.81. Previously the CLEO analyzed with \( p_F = 0.3 \) GeV the end point lepton energy spectrum to get [4]
\[
10^2 \times |V_{ub}/V_{cb}|^2 = 0.57 \pm 0.11 \quad (\text{ACCMM [1]}) \\
= 1.02 \pm 0.20 \quad (\text{ISGW [3]}).
\]
As can be seen, those values are in large disagreement. However, if we use \( p_F = 0.5 \) GeV, the result of the ACCMM model becomes 1.03 from [30], and these two models are in good agreement for the value of \( |V_{ub}/V_{cb}| \). Finally we show the values of \( |V_{ub}(p_F)/V_{ub}(p_F = 0.3)| \) as a function of \( p_F \) in Fig. 1.
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Appendix

The integration constants \( c_1 \) and \( c_2 \) in (22) are given by the following relations,

\[
I(0) = -\frac{1}{2}c_2 = \int_0^\infty x^3 e^{-x^2} \, dx = \frac{1}{2},
\]

\[
I''(s \approx 0) = 2c_1 + c_2(-\frac{1}{8} \ln s - \frac{11}{32})
= -\frac{1}{4} \int_0^\infty x^2(x^2 + s)^{-3/2} e^{-x^2} \, dx \quad \text{at} \quad s \approx 0.
\]

Then, from (32), we get

\[
c_2 = -1.
\]

The integral in (33) can be expanded as

\[
J(s = a^2) = \int_0^\infty x^2(x^2 + a^2)^{-3/2} e^{-x^2} \, dx
= \int_0^\infty x^2[(x + a)^2 - 2ax]^3/2 e^{-x^2} \, dx
= \int_0^\infty x^2(x + a)^{-3} \left[1 - \frac{2ax}{(x + a)^2}\right]^{-3/2} e^{-x^2} \, dx
= \sum_{n=0}^\infty \frac{(2n+1)!a^n}{2^n(n!)^2} \int_0^\infty \frac{x^{n+2}}{(x + a)^{2n+3}} e^{-x^2} \, dx.
\]

Next the integral in (33) is obtained by successive differentiations of an integral,

\[
\int_0^\infty \frac{x^{n+2}}{(x + a)^{2n+3}} e^{-x^2} \, dx = \frac{1}{(2n+2)!} \left(\frac{\partial}{\partial a}\right)^{2n+2} \int_0^\infty \frac{x^{n+2}}{x + a} e^{-x^2} \, dx.
\]

Again the integral in (36) is related to another integral, for a small value of \( a \),

\[
\int_0^\infty \frac{x^{n+2}}{x + a} e^{-x^2} \, dx = \sum_{k=0}^{n+1} \frac{(-a)^k}{2} \left(\frac{n-k}{2}\right)! + (-a)^{n+2} \int_0^\infty \frac{e^{-x^2}}{x + a} \, dx.
\]

The integral in (37) can be expanded in a similar way as the series (22) was obtained by making use of differential equations. For a small value of \( a \),

\[
\int_0^\infty \frac{e^{-x^2}}{x + a} \, dx = -\frac{1}{2} e^{-a^2} (2 \ln a + \gamma + a^2 + \frac{1}{2} a^4 + \cdots) + \sqrt{\pi} e^{-a^2} (a + \frac{1}{3} a^3 + \frac{1}{5} a^5 + \cdots),
\]

where \( \gamma \sim 0.5772 \) is the Euler's constant. In this way the constant \( c_1 \) is given by an infinite series,

\[
c_1 = -\frac{3}{64} + \frac{\gamma}{16} - \frac{1}{8} \sum_{n=1}^\infty \frac{1}{n2^n} \approx -0.0975.
\]
Fig.1 $|V_{ub}(p_F)/V_{ub}(p_F = 0.3)|$ as a function of $p_F$.  

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