WEAK SOLVABILITY OF FRACTIONAL VOIGT MODEL OF VISCOELASTICITY

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To Professor Rafael de la Llave

Abstract. In the present paper we establish the existence of weak solutions to one fractional Voigt type model of viscoelastic fluid. This model takes into account a memory along the motion trajectories. The investigation is based on the theory of regular Lagrangean flows, approximation of the problem under consideration by a sequence of regularized Navier-Stokes systems and the following passage to the limit.

1. Introduction. It is well known the Cauchy momentum equation of a fluid which occupies a bounded domain $\Omega$ in $\mathbb{R}^N$, $N = 2, 3$, $\partial \Omega \in C^2$, (see [9]) has the form:

$$\rho(\partial v/\partial t + v_i \partial v/\partial x_i) = -\nabla p + \text{Div} \sigma + \rho f(t, x) \quad (t, x) \in Q_T = [0, T] \times \Omega. \quad (1)$$

Here $v(t, x) = (v_1(t, x), \ldots, v_N(t, x))$ is the velocity vector of a particle at the point $x$ of $\Omega$ at time $t$, $\rho(t, x)$ is the fluid density (which is supposed to be equal to 1), $p = p(t, x)$ is the pressure of the fluid at the point $x$ at time $t$, $\sigma(t, x)$ is the deviator of the stress tensor, $f(t, x)$ is the density of external forces acting on the fluid; $\text{Div} \sigma(t, x)$ is the vector, coordinates of which are divergences with respect to $x$ of the rows of matrix $\sigma(t, x)$.

The rheological relation determines the type of a continuum (fluid) (see eg. [4], [22], [23] and the references therein). A wide range of continua is determined by rheological relation of the form

$$\sum_{k=0}^{n} \sum_{i=0}^{s} b_{ki} D_{lt}^{k+\beta_{ki}} \sigma = \sum_{k=0}^{m} \sum_{i=0}^{r} a_{ki} D_{lt}^{k+\beta_{ki}} \varepsilon, \quad 0 \leq \beta_{ki} < 1, \quad (2)$$

where $D_{lt}^{\alpha}$ denotes some fractional derivative and $\varepsilon$ is the strain tensor.

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Models with integer derivatives ($\beta_{ki} = 0$) are a particular case of model (2). The models of Newton, Maxwell, Voigt, Jeffreys, etc. (see eg. [23], [22] and the references therein) are among them.

Description of a wide range of polymers caused introduction of models with fractional derivatives. Such a models reflect the influence of creep and relaxation effects.

Scott-Blair, Zener, Burgers, generalized Maxwell and Kelvin-Voigt fractional models describe above mentioned polymers. In [12] there are given a mechanical interpretations of these models and a good bibliographical review.

It has been noted in [10] that many authors used in (2) various types of fractional derivatives, for example the fractional derivatives of Grunwald-Letnikov, Liouville, Caputto-Liouville, Riemann-Liouville etc.

The use of Caputo derivatives gives (see [13]) the rheological relation of the form

$$\sum_{k=0}^{n} \sum_{i=0}^{s} b_{ki} D_{qt}^{\beta_{ki}} D^k \sigma = \sum_{k=0}^{m} \sum_{i=0}^{r} a_{ki} D_{ot}^{\beta_{ki}} D^k \varepsilon, \quad 0 \leq \beta_{ki} < 1.$$  

(3)

The simplest fractional rheological model $\sigma = \mu D_0^{\alpha} \varepsilon, \mu > 0$, was introduced by Scott-Blair (see [17]). A. N. Gerasimov proposed (see [8]) a rheological relation of the form (2) for $t \in (-\infty, +\infty)$ to study an anomalous dynamical behavior of viscoelastic materials.

A similar model to (2) with Caputo fractional derivatives for $t \in (-\infty, +\infty)$ and for $t \in (0, +\infty)$ was proposed by Caputo and Mainardi in [2], [3].

Though there exists a lot of fractional models, as far as we know there are no nonlocal existence theorems of weak solutions to the corresponding initial-boundary value problems.

This is caused by the presence of singularities in integral representations of the fractional derivatives and integrals, in contrast to the integer models (see [15]).

The experiments demonstrate that the usage of full derivatives instead of time ones in (3) give a more precise description of nonlinear effects in fluids (see [23], [22]).

In some models it allows to express explicitly $\sigma$ in terms of $v$ along trajectories of the vector field $v$ (see eg. [15]). This implies the appearance of an integral term along the trajectories of the field $v$ in the motion equation. This means that the Cauchy problem (in integral form)

$$z(\tau; t, x) = x + \int_{t}^{\tau} v(s, z(s; t, x)) \, ds, \quad 0 \leq t, \tau \leq T, \quad x \in \Omega;$$  

(4)

determining the trajectories $z = z(\tau; t, x)$ of the field $v$ has to be added to the momentum equation.

Let us mark that the presence of the integral along trajectories in the momentum equation means the presence of a memory in the fluid.

The presence of $z$ in the momentum equation requires the unique solvability of (4). However, the existence of solutions to (4) for fixed $v$ is known only in the case of $v \in L_1(0, T; C^{1}(\Omega)^{N})$ and this is the unique solution if $v \in L_1(0, T; C^{1}(\Omega)^{N})$, $v(0, \cdot) = 0$ (see e.g. [14]). But even for strong solutions ($v \in L_2(0, T; W^{2,2}(\Omega)^{N})$) equation (4) is generally speaking not unique solvable and consequently the trajectories $z$ are not determined uniquely.

One possible way out of this situation is a regularization of the velocity field (see [20]).
In the study of weak solvability of equations of the form (1) it is usual that \( v \in L^2(0, T; W^{1,2}(\Omega)^N) \). But this is insufficient for classical solvability of irregularized Cauchy problem (4). Recently (see eg. [6]-[7]) the unique solvability of the Cauchy problem (4) in the case of \( v \) belonging to a Sobolev space was established in the class of Regular Lagrangian Flows, a generalization of the concept of classical solutions.

In the present work, this allowed to proof the existence of weak solutions without a regularization of \( v \) in equation (4).

Below we consider the special case of model (2) which is some fractional equivalent to the Voigt model, a rheological relation of which has the form (see [13])

\[
\sigma = \mu_0 \dot{\varepsilon} + \mu_1 D_{0+}^\alpha \dot{\varepsilon}, \ 0 < \alpha < 1.
\]

The structure of the work is as follows. In section 2 auxiliary assertions are given. In section 3 the fractional model under consideration is discussed. In section 4 we formulate the main results. Section 5 is devoted to the study of regularized problems and consists of 4 subsections. In subsection 5.1 we consider \( \varepsilon \)-regularized of the original problem. In subsection 5.2 we investigate regularized Navier-Stokes system. In subsections 5.3-5.4 we construct a sequence of approximations for \( \varepsilon \)-regularization and establish their solvability and estimates of solutions for small \( T \). In section 6 using passage to the limit in the approximating problems we establish the solvability of \( \varepsilon \)-regularization for small \( T \). In section 7 we prove a priori estimates for \( \varepsilon \)-regularization which are necessary for the proof of the solvability of \( \varepsilon \)-regularization for arbitrary \( T \). In section 8 we prove the solvability of \( \varepsilon \)-regularization for arbitrary \( T \). In section 9 the solvability of the main problem is obtained via passage to the limit as \( n \to +\infty \) in 1/n-regularization.

Constants in inequalities and chains of inequalities which do not depend on significant parameters are denoted by a single letter \( M \).

2. Basic definitions and auxiliary results. Functional spaces. Let \( C_0^\infty(\Omega)^N \) be the set of infinitely differentiable compactly supported \( R^N \)-valued functions on \( \Omega \). Let \( \mathcal{V} = \{ v : v \in C_0^\infty(\Omega)^N, \text{ div } v = 0 \} \). Denote by \( H \) and \( V \) the closures of \( \mathcal{V} \) w.r.t. norms of \( L^2(\Omega)^N \) and \( W^{1,2}(\Omega)^N \), respectively. Let \( V^{-1} \) denote the conjugate to \( V \) space.

Denote by \( \mathcal{E}(v) \) the matrix with components \( \mathcal{E}_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i) \).

The space \( V \) is a Hilbert space with the scalar product \( (v, u)_V = \int_\Omega \mathcal{E}(u) : \mathcal{E}(v) \, dx \) (where \( \mathcal{E}(u) = \sum_{i,j=1}^N \mathcal{E}_{ij}(u) \mathcal{E}_{ij}(v) \)) and the corresponding norm. This norm in the space \( V \) is equivalent to the norm of \( W^{1,2}(\Omega)^N \). Denote by \( \langle f, v \rangle \) the action functional \( f \) from the adjoint to \( V \) space \( V^{-1} \) on a function \( v \) from \( V \).

The norms in the spaces \( H \) and \( L^2(\Omega)^N \) we denote by \( | \cdot |_0 \), while in \( V \) and \( W^{1,2}(\Omega)^N \) by \( | \cdot |_1 \). The norms in \( L^2(0, T; H) \) and \( L^2(0, T; L^2(\Omega)^N) \) are denoted by \( \| \cdot \|_0 \), the norms in \( L^2(0, T; V) \) and \( L^2(0, T; W^{1,2}(\Omega)^N) \) as \( \| \cdot \|_{0,1} \) and the norm in the space \( L^2(0, T; V^{-1}) \) by \( \| \cdot \|_{0,-1} \).

The sign \( \langle \cdot, \cdot \rangle \) stands for the scalar product in Hilbert spaces \( L^2(\Omega)^N \), \( H \), \( L^2(\Omega)^N \) and \( L^2(\Omega)^{N \times N} \). From a context it is clear what the space is meant.

The identification of the Hilbert space \( H \) with its conjugate space \( H^{-1} \) and the theorem of Riesz lead to the continuous embedding \( V \subset H = H^{-1} \subset V^{-1} \). In addition, for \( u, w \in V \) the relation \( \langle u, w \rangle = (u, w) \) is valid with the scalar product in \( H \).

Fractional Riemann-Liouville integrals and derivatives. Recall some facts about fractional derivatives and integrals (see [12],[16]). The fractional integrals of
fractional integration: 

$$D_0^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds,$$  

(5)

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} \, dt$ is the Euler’s Gamma function.

Fractional Riemann-Liouville derivative of order $\alpha > 0$ of the function $y(t)$ on $[0, T]$ is determined by the formula

$$D_0^\alpha y(t) = \frac{d^n}{dt^n} I_0^{1-\alpha} y(t), \quad t > 0, \quad n = [\alpha] + 1.$$

In particular, if $0 < \alpha < 1$ then $D_0^\alpha y(t) = \frac{1}{\Gamma(-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s) \, ds$, and when $\alpha = n > 0$ is integer, then $D_0^\alpha y(t) = \frac{d^n}{dt^n} y(t)$ is the usual derivative of order $n$.

The fractional differential operator $D_0^\alpha$ is inverse to the left-side operator of fractional integration: $D_0^\alpha I_0^{1-\alpha} y(t) = y(t)$.

**Regular Lagrangian Flows.** Consider the Cauchy problem (in integral form)

$$z(\tau; t, x) = x + \int_t^{\tau} v(s, z(s; t, x)) \, ds, \quad 0 \leq t, \tau \leq T, \quad x \in \overline{\Omega}.$$  

(6)

In the case of $v \in L_1(0, T; C^1(\Omega))$ with the zero condition on the boundary the problem (6) has a unique solution in the classical sense (see [14]). However, in the case of only summable with respect to $t$ vector-function $v$, the situation is much more complicated and one has to use a more general concept of the solution to (6).

**Definition 2.1.** Associated to $v$ Regular Lagrangian flow (RLF) is the function $z(\tau; t, x), (\tau; t, x) \in [0, T] \times [0, T] \times \overline{\Omega}$ which satisfies the following conditions:

1) for a.a. $x$ and any $t \in [0, T]$ the function $\gamma(\tau) = z(\tau; t, x)$ is absolutely continuous and satisfies equation (6);  

2) for any $t, \tau \in [0, T]$ and an arbitrary Lebesgue measurable set $B \subseteq \overline{\Omega}$ with measure $m(B)$ the relation $m(z(\tau; t, B)) = m(B)$ is valid;  

3) for all $t_i \in [0, T], \ i = 1, 2, 3$ and a.a. $x \in \overline{\Omega}$

$$z(t_3; t_1, x) = z(t_3; t_2, z(t_2; t_1, x)).$$  

(7)

The definition of RLF can be found, for example, in [1], [5], [7]. Here the definition of RLF is given in the particular case of a bounded domain $\Omega$ and divergence free function $v$.

Let us recall some results on RLF.

Let $D = [0, T] \times [0, T]$ and $L$ be the set of measurable on $D$ functions which is considered as a metric space with the metric

$$d(f, g) = \int_{Q_T} |f(t, x) - g(t, x)| (1 + |f(t, x) - g(t, x)|)^{-1} \, dt \, dx.$$  

Let $v_x$ be the Jacobian matrix of a vector function $v$.

**Theorem 2.2 (see [7]).** Let $v \in L_1(0, T; W^{1,p}(\Omega)^N), 1 \leq p \leq +\infty$, $\div v(t, x) = 0$ and $v|_{[0, T] \times \partial \Omega} = 0$. Then there exists a unique RLF $z \in C(D; L^N)$ associated to $v$. Moreover, $z(s; t, \cdot) \in W^{1,1}(\Omega)^N$ and

$$\frac{\partial}{\partial \tau} z(\tau; t, x) = v(\tau, z(\tau; t, x)), \quad t, \tau \in [0, T], \quad a.a. \ x \in \Omega,$$

(8)

$$z(\tau; t, \overline{\Omega}) \subset \overline{\Omega} \quad \text{(up to a set of zero measure).}$$  

(9)
Theorem 2.3. Let \( v, v^m \in L_1(0,T;W^{1,p}(\Omega)^N) \), \( m = 1, 2, \ldots \) for some \( p > 1 \). Let \( \text{div} \, v = 0 \), \( \text{div} \, v^m = 0 \), \( v^m|_{[0,T] \times \partial \Omega} = 0 \), \( v|_{[0,T] \times \partial \Omega} = 0 \). Let inequalities
\[
\|v_x\|_{L_1(0,T;L_p(\Omega)^N \times N)} + \|v\|_{L_1(0,T;L_1(\Omega)^N)} \leq M,
\]
\[
\|v^m_x\|_{L_1(0,T;L_p(\Omega)^N \times N)} + \|v^m\|_{L_1(0,T;L_1(\Omega)^N)} \leq M.
\]
are valid. Let \( v^m \) converges to \( v \) in \( L_1(Q_T) \) as \( m \to +\infty \). Let \( z^m(\tau;t,x) \) and \( z(\tau;t,x) \) be RLF associated to \( v^m \) and \( v \), respectively. Then the sequence \( z^m \) converges (up to a subsequences) to \( z \) w.r.t. Lebesgue measure on the set \([0,T] \times \Omega\) uniformly on \( t \in [0,T] \).

In a more general formulation this result is proved in [5], Corollaries 3.6, 3.7, 3.9.

3. Fractional Voigt model. This model has a mechanical interpretation in the form of the parallel connection of Newton and Scott-Blair elements (see [12]). Indeed, a Newton element \( N \) is determined by the rheological relation \( \sigma_1 = \nu_1 \dot{\varepsilon}_1 \) and a Scott-Blair element \( SB \) is determined by the rheological relation \( \sigma_2 = \nu_2 D_{0t}^\alpha \varepsilon_2 \).

For the parallel connection \( N \parallel SB \) of elements \( N \) and \( SB \) the relations \( \sigma = \sigma_1 + \sigma_2 \) and \( \varepsilon = \varepsilon_1 = \varepsilon_2 \) are valid where \( \sigma \) is the deviator of the stress tensor and \( \varepsilon \) is the strain tensor of the element \( N \parallel SB \).

It follows that \( \sigma = \nu \dot{\varepsilon}_1 + \nu_2 D_{0t}^\alpha \varepsilon_2 = \nu \dot{\varepsilon} + \nu_2 D_{0t}^\alpha \varepsilon \).

Consider a viscoelastic fluid with the rheological relation \( \sigma = \nu_1 \dot{\varepsilon}_1 + \nu_2 D_{0t}^\alpha \varepsilon_2 = \nu_1 \dot{\varepsilon} + \nu_2 D_{0t}^\alpha \varepsilon \). Assuming \( \dot{\varepsilon} = \mathcal{E}(v) \) we deduce
\[
\sigma = \mu_0 \mathcal{E}(v) + \mu_1 I_{0t}^{1-\alpha} \mathcal{E}(v), \quad 0 < \alpha < 1. \tag{11}
\]
Here \( \mathcal{E}(v) \) is the strain rate tensor and
\[
I_{0t}^{1-\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} y(s) \, ds
\]
is a fractional Riemann-Liouville integral of order \( 1 - \alpha \).

Model takes into account the history of the fluid motion along the spatial variable \( x \). However, more realistic are models which take into account the history of the fluid motion along the trajectories of fluid motion. In the case of rheological relation (11) such a model has the form
\[
\sigma = \mu_0 \mathcal{E}(v) + \mu_1 \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \mathcal{E}(v)(s,z(s;t,x)) \, ds. \tag{12}
\]
Here \( z(\tau;t,x) \) is the solution to the Cauchy problem (6).

Substituting (12) in (1) we obtain the initial-boundary value problem
\[
\partial v/\partial t + \sum_{i=1}^N v_i \partial v/\partial x_i - \mu_0 \Delta v - \mu_1 \frac{1}{\Gamma(1-\alpha)} \text{Div} \int_0^t (t-s)^{-\alpha} \mathcal{E}(v)(s,z(s;t,x)) ds \nonumber
\]
\[
+ \nabla p = f(t,x), \quad (t,x) \in Q_T; \tag{13}
\]
\[
\text{div} \, v(t,x) = 0, \quad (t,x) \in Q_T; \tag{14}
\]
\[
z(\tau;t,x) = x + \int_0^\tau v(s,z(s;t,x)) \, ds, \quad t, \tau \in [0,T], \quad x \in \overline{\Omega}; \tag{15}
\]
\[
v(0,x) = v_0(x), \quad x \in \Omega, \quad v|_{[0,T] \times \partial \Omega} = 0. \tag{16}
\]
of (13)-(16) we consider the dependent on $\varepsilon > 0$. Regularized problems. Let $u(t) \in L_2(a, b; V)$. Due to continuity of embeddings $V \subset H \subset V^{-1}$ one can consider $u$ as $V^{-1}$-valued function. Function $g(t) \in L_1(a, b; V^{-1})$ is called the derivative of function $u$ in the sense of distributions, if it satisfies the identity $\frac{d}{dt}\langle u(t), \varphi \rangle = \langle g(t), \varphi \rangle$ for all $\varphi \in V$, a.a. $t \in [a, b]$ and is denoted as $u'$. By this the action of functional $u(t) \in V^{-1}$ on $\varphi \in V$ is determined as follows $\langle u(t), \varphi \rangle = \langle u(t), \varphi \rangle$.

It will be useful for us to consider a solution $v(t, x)$ to problem (13)-(16) as a function $v \in L_2(0, T; V)$ of variable $t$ with values in the space $V$ (of variable $x$).

Introduce functional space

$$W_1(a, b) = \{v : v \in L_2(a, b; V) \cap L_{\infty}(a, b; H), \, v' \in L_1(a, b; V^{-1})\}.$$  

**Definition 4.1.** Let $f \in L_2(0, T; V^{-1})$, $v^0 \in H$. A weak solution to problem (13)-(16) is a function $v \in W_1(0, T)$ satisfying initial condition (16) and the identity

$$d(v, \varphi)/dt - \sum_{i=1}^{N}(v_i v, \partial \varphi / \partial x_i) + \mu_0\langle E(v), \mathcal{E}(\varphi) \rangle$$

$$+ \mu_1 \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \langle E(v)(s, z(s; t, x)), \mathcal{E}(\varphi) \rangle \, ds = \langle f(t, \cdot), \varphi(\cdot) \rangle \quad (17)$$

for any $\varphi \in V$ and a.a. $t \in [0, T]$. Here $z$ is associated to $v$ RLF.

**Remark 4.1.** Note that for any $v \in L_2(0, T; V)$ due to Theorem 4.2 the Cauchy problem (15) has a unique solution $z(\tau; t, x)$ in the class of RLF.

**Remark 4.2.** As a weak solution $v$ to problem (13)-(16) belongs to the space $W_1(0, T)$, it is known (see [19], Lemma III.1.4) that $W_1(0, T) \subset C_{weak}(0, T, H)$, so the initial condition (16) has sense.

Let us formulate the main result.

**Theorem 4.2.** Let $f \in L_2(0, T; V^{-1})$, $v^0 \in H$. Then problem (13)-(16) has at least one weak solution.

5. **Regularized problems.**

5.1. **$\varepsilon$-regularization of problem (13)-(16).** In order to establish the solvability of (13)-(16) we consider the dependent on $\varepsilon > 0$ auxiliary regularized problem

$$\partial v / \partial t + K_\varepsilon v - \mu_0 \Delta v - \mu_1 \frac{1}{\Gamma(1-\alpha)} \text{Div} \int_0^t (t-s)^{-\alpha} E(v)(s, z(s; t, x)) \, ds$$

$$+ \nabla p = f(t, x), \quad (t, x) \in Q_T; \quad (18)$$

$$\text{div } v(t, x) = 0, \quad (t, x) \in Q_T; \quad (19)$$

$$z(\tau; t, x) = x + \int_\tau^t v(s, z(s; t, x)) \, ds, \quad t, \tau \in [0, T], x \in \overline{\Omega}; \quad (20)$$

$$v(0, x) = v^0(x), \quad x \in \Omega, \quad v|_{[0, T] \times \partial \Omega} = 0. \quad (21)$$
Here \( z(\tau; t, x) \) is the solution to the Cauchy problem (20) and \( K_\varepsilon \) is determined by the formula
\[
K_\varepsilon(v) = \sum_{i=1}^{N} \partial(v_i(1 + \varepsilon|v|^2)^{-1}v)/\partial x_i, \; \varepsilon \geq 0, \; \text{for} \; v \in V.
\]

**Definition 5.1.** Let \( f \in L_2(0, T; V^{-1}), \; v^0 \in H, \; \varepsilon > 0 \). A weak solution to problem (18)-(21) is a function \( v \in W_1(0, T) \) satisfying the initial condition (21) and the identity
\[
\frac{d(v, \varphi)}{dt} - \sum_{i=1}^{N} (v_i v(1 + \varepsilon|v|^2)^{-1}, \partial \varphi/\partial x_i) + \mu_0(E(v), E(\varphi)) + \mu_1 \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} E(v)(s, z(s;t,\cdot))ds, E(\varphi)(\cdot) \right) = \langle f(t, \cdot), \varphi(\cdot) \rangle \quad (22)
\]
for a.a. \( t \in [0, T] \) and any \( \varphi \in V \). Here \( z \) is associated to \( v \) RLF.

To prove the solvability of problem (18)-(21) let us consider the successive approximations \( v^n, \; n = 1, 2, 3, ... \) defined as the solution of the auxiliary regularized problems
\[
\frac{\partial v^n}{\partial t} + K_\varepsilon(v^n) - \mu_0 \Delta v^n + \nabla p^n = w^n; \quad \text{(23)}
\]
\[
\text{div } v^n = 0; \quad \text{(24)}
\]
\[
v^n(0, x) = v^0(x), \; v^n|_{[0,T] \times \partial \Omega} = 0. \quad \text{(25)}
\]
Here
\[
z^{n-1}(\tau; t, x) = x + \int_{t}^{\tau} v^{n-1}(s, z^{n-1}(s; t, x)) \, ds, \; x \in \Omega, \; t, \tau \in [0, T], \quad \text{(26)}
\]
\[
w^n = f + \mu_1 \frac{1}{\Gamma(1-\alpha)} \text{Div} \int_0^t (t-s)^{-\alpha} E(v^{n-1})(s, z^{n-1}(s; t, x)) \, ds, \quad \text{(27)}
\]
and \( v^1 \) is chosen as \( v^1 = 0 \).

Below we show that weak solutions \( v^n \) of problem (23)-(27) converge to a weak solution of problem (18)-(21) as \( n \to +\infty \).

Let us first recall some facts on the regularized Navier-Stokes system.

### 5.2. Regularized Navier-Stokes system and properties of its solutions.

Problem (23)-(25) is a regularization for the Navier-Stokes system of the form
\[
\frac{\partial v}{\partial t} + K_\varepsilon(v) - \mu_0 \Delta v + \nabla p = F; \; \text{div } v = 0; \; v(0, x) = v^0(x); \; v|_{[0,T] \times \partial \Omega} = 0. \quad (28)
\]

**Definition 5.2.** Let \( \varepsilon \geq 0 \). Let \( F \in L_2(0, T; V^{-1}), \; v^0 \in H \). A weak solution of problem (28) is a function \( v \in W_1(0, T) \) satisfying the initial condition (28) and for a.a. \( t \in [0, T] \) the identity
\[
\frac{d}{dt}(v, \varphi) - \sum_{i=1}^{N} (v_i v(1 + \varepsilon|v|^2)^{-1}, \partial \varphi/\partial x_i) + \mu_0(E(v), E(\varphi)) = \langle F, \varphi \rangle \quad \text{(29)}
\]
for any \( \varphi \in V \).
In [19], Theorem 3.1, Chapter III, the weak solvability of problem (28) was established for \( \varepsilon = 0 \) (the Navier-Stokes system) and for any \( F \in L_2(0,T;V^{-1}), \) \( v^0 \in H \) in the class \( W_1(0,T) \). It is obvious that the regularized problem (28) \( (\varepsilon > 0) \) also has a solution in \( W_1(0,T) \). We show that for \( \varepsilon > 0 \) the solution possesses better properties, namely it belongs to \( W(0,T) \). The space \( W(0,T) \) is defined by the formula

\[
W(a,b) = \{ v : v \in L_2(a,b;V) \cap L_\infty(a,b;H), \ v' \in L_2(a,b;V^{-1}) \}.
\]

**Theorem 5.3.** For any \( F \in L_2(0,T;V^{-1}), \) \( v^0 \in H \) and \( \varepsilon > 0 \) problem (28) has a weak solution \( v \in W(0,T) \). Furthermore, for any weak solution \( v \in W(0,T) \) the inequalities

\[
\sup_{0 \leq t \leq T} |v(t,\cdot)|_0 + \|v\|_{0,1} \leq M_0(\|F\|_{0,-1} + |v^0|_0), \tag{30}
\]

\[
\|v'\|_{L_1(0,T;V^{-1})} \leq M_0(1 + \|F\|_{0,-1} + |v^0|_0)^2 \tag{31}
\]

hold true with independent on \( \varepsilon \) constant \( M_0 \).

**Proof of Theorem 5.3.** Using the terms of (29) we introduce functional on \( V \) and hence the map

\[
A : V \to V^{-1}, \quad \langle A(u), h \rangle = \langle \mathcal{E}(u), \mathcal{E}(h) \rangle, \quad u, h \in V.
\]

We introduce the operator \( \mathbf{K}_\varepsilon : V \to V^{-1}, \)

\[
\langle \mathbf{K}_\varepsilon(u), h \rangle = \sum_{i,j=1}^N (u_{ij}/(1 + \varepsilon|u|^2), \partial h_{ij}/\partial x_j), \quad u, h \in V.
\]

For a function \( v \in W_1(0,T) \) the relation is valid (see [19], Chapter III, Lemma 1.1)

\[
\langle v'(t,\cdot), \varphi(\cdot) \rangle = \frac{d}{dt} \langle v(t,\cdot), \varphi(\cdot) \rangle \quad \forall \varphi \in V.
\]

Then the problem (28) can be rewritten in the operator form (see [19], Section III.3.1)

\[
v' + \mu_0 A(v) - \mathbf{K}_\varepsilon(v) = F, \quad v(0) = v^0. \tag{32}
\]

Let \( v \) be a weak solution to problem (28) from \( W_1(0,T) \). Then from (32) it follows that

\[
v' = F + \mathbf{K}_\varepsilon(v) - \mu_0 A(v). \tag{33}
\]

In [20] it is established that for any \( \varepsilon > 0 \)

\[
\|\mathbf{K}_\varepsilon(v)\|_{0,-1} \leq M \varepsilon^{-1} \|v\|^2_{0,1}, \quad \|\mathbf{K}_\varepsilon(v)\|_{L_1(0,T;V^{-1})} \leq M \|v\|^2_{0,1},
\]

\[
\|A(v)\|_{0,-1} \leq M \|v\|_{0,1}. \tag{34}
\]

It follows that all summands in the right-hand side part of equation (32) belong to \( L_2(0,T;V^{-1}) \) and therefore \( v' \in L_2(0,T;V^{-1}) \).

Thus, \( v \in W(0,T) \).

Let us apply both sides of (32) (which belong to \( V^{-1} \)) on \( v \in V \). We get

\[
\langle v', v \rangle + \mu_0 \langle A(v), v \rangle - \langle \mathbf{K}_\varepsilon(v), v \rangle = \langle F, v \rangle.
\]

On the strength of Lemma 1.2, Chapter III in [19] for \( v \in W(0,T) \) the following relation is valid

\[
\frac{d}{dt} \langle v(t,\cdot), v(t,\cdot) \rangle = \frac{1}{2} \frac{d}{dt} \|v(t,\cdot)\|^2_0. \tag{35}
\]
Using (35), definition of operator $A$ and having in mind that $(K_x(u), u) = 0$ for $u \in V$ (see [20], [23], p. 208) we have
\[
\frac{d}{dt}[v(t, \cdot)]^2 + 2\mu_0|\mathcal{E}(v)(t, \cdot)|^2 = (F, v).
\]

From Korn’s inequality it follows that $|\mathcal{E}(u)|_0 \geq m|u|_1$, $m > 0$ for $u \in V$. Using this fact and elementary transformations we find that for arbitrary $\delta > 0$ the following inequality holds
\[
\frac{d}{dt}|v(t, \cdot)|^2 + 2\mu_0|v(t, \cdot)|^2 \leq M||F||_{-1}|v|_1 \leq M(\delta)|F|^2_1 + \delta|v|^2_1.
\]
\[
(36)
\]
Choosing $\delta > 0$ small enough, shifting the last term in (36) to the left side and integrating by $t$, by simple arguments we obtain the inequality (30).

Let us establish estimate (31). From equation (35), the second estimate (34), inequality (30) and monotonicity of the $L_p$ norms w.r.t. $p$ it follows that
\[
\|v''\|_{L_1(0,T;V^{-1})} \leq \|F\|_{L_1(0,T;V^{-1})} + \|K_x(v)\|_{L_1(0,T;V^{-1})} + \|A(v)\|_{L_1(0,T;V^{-1})}
\]
\[
\leq M(\|F\|_{0,-1} + \|v\|^2_{0,1} + \|v\|_{0,1}) \leq M(1 + \|F\|_{0,-1} + |v|_0^2).
\]

Estimate (31) is proved. 

Theorem 5.3 is proved. \[\square\]

**Remark 5.1.** Note that the statement of Theorem 5.3 is obviously true for problem (28) on $Q_{T'}$ for any $T' \leq T$ with the change of $T$ by $T'$ in (30) and (31).

Properties of solutions of the regularized Navier-Stokes system will be used below.

5.3. **Solvability of system (23)-(27) and properties of its solutions.** By Theorem 5.3 the problem (23)-(27) has for any $w^n \in L_2(0,T;V^{-1})$ at least one solution $v^n \in W(0, T)$ satisfying estimates of the form (30)-(31). We will show that the right-hand side part of (23) $w^n$ which is determined by (27) in fact belongs to $L_2(0,T;V^{-1})$ for $n = 2, 3, \ldots$ and, hence, problem (23)-(27) has solution $v^n \in W(0, T)$ for $n = 2, 3, \ldots$.

First, note that for $v^{n-1} \in L_2(0,T;V)$ the Cauchy problem (26) defines a unique RLF $z^{n-1}(\tau; t, x)$ due to Theorem 4.2.

Establish the following fact.

**Lemma 5.4.** Let $v^{n-1} \in L_2(0,T;V)$ and $z^{n-1}(\tau; t, x)$ be RLF associated to the Cauchy problem (26). Then for
\[
G(v^{n-1}) = \int_0^t (t-s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s; t, x)) \, ds
\]
the inequality holds
\[
||\text{Div } G(v^{n-1})||_{0,-1} \leq M_1 T^{1-\alpha} ||v^{n-1}||_{0,1}.
\]
\[
(37)
\]
**Proof.** Let $A : \Omega \to R^{N \times N}$ be a matrix function and $A \in L_2(\Omega)^{N \times N}$. It is easy to show (see e.g. [11], [18]) that
\[
||\text{Div } A(x)||_{-1} \leq M ||A||_{L_2(\Omega)^{N \times N}}.
\]
\[
(38)
\]
Let
\[
J = ||\mathcal{E}(v)(s, z(s; t, \cdot))||^2_{L_2(\Omega)^{N \times N}} = \sum_{i,j=1}^N \int_{\Omega} |E_{ij}(s, v_i(s, z(s; t, x)))|^2 \, dx.
\]

It is obviously that

\[ J \leq M \sum_{i,j=1}^{N} J_{ij}, \]

where

\[ J_{ij} = \int_{\Omega} \left| \partial v_i^{n-1}(s, z^{n-1}(s;t,x)) / \partial x_j \right|^2 \, dx. \]

Let’s make the change of variable \( x = z^{n-1}(t;y) \) in integral \( J_{ij} \) (the inverse change is \( y = z^{n-1}(s;t,x) \)). Since \( RLF \ z^{n-1}(s;t,x) \) is associated to divergence free \( v^{n-1} \) then for Jacobian matrix \( z^{n-1}_x(s;t,x) \) the relation \( \det z^{n-1}_x(s;t,x) = 1 \) is valid.

It follows

\[ J \leq \sum_{i,j=1}^{N} \int_{\Omega} \left| \partial v_i^{n-1}(s, z^{n-1}(s;t,x)) / \partial x_j \right|^2 \, dx \]

\[ = \sum_{i,j=1}^{N} \int_{\Omega} \left| \partial v_i^{n-1}(s,y) / \partial y_j \right|^2 \, dy \leq M|v^{n-1}(s,)|^2. \] (39)

In virtue of (38) and (39) and easily checking inequality \( |\mathcal{E}(w)|_0 \leq M|w|_1 \) for \( w \in V \) we have

\[ |\text{Div} \ G(v^{n-1})|_{-1} = |\text{Div} \int_{0}^{t} (t-s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s;t,·)) \, ds |_{-1} \]

\[ \leq M \left\| \int_{0}^{t} (t-s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s;t,·)) \, ds \right\|_{L_2(\Omega)^N \times N} \]

\[ \leq M \int_{0}^{t} (t-s)^{-\alpha} \left\| \mathcal{E}(v^{n-1})(s, z^{n-1}(s;t,·)) \right\|_{L_2(\Omega)^N \times N} \, ds \]

\[ \leq M \int_{0}^{t} (t-s)^{-\alpha} |v^{n-1}(s,·)|_1 \, ds. \] (40)

Since

\[ \| \int_{0}^{t} (t-s)^{-\alpha} \varphi(s) \, ds \|_{L_p(0,T)} \]

\[ \leq MT^{1-\alpha} \| \varphi(s) \|_{L_p(0,T)}, \ \varphi(s) \in L_p(0,T), \ 1 \leq p < +\infty \] (41)

(see [16], Theorem 2.6), then it follows that

\[ \| \text{Div} \ G(v^{n-1})\|_{0,-1} \leq MT^{1-\alpha} \| \mathcal{E}(v^{n-1}) \|_{L_2(0,T;L_2(\Omega)^N \times N)} \]

\[ \leq MT^{1-\alpha} \| v^{n-1} \|_{0,1}. \]

Lemma 5.4 is proved.

\[ \square \]

**Lemma 5.5.** Function \( w^n \) belongs to \( L_2(0,T;V^{-1}) \) for any \( n \geq 1 \).
Proof. Obviously, if \( v^{n-1} \in W(0, T) \) then the following inequality holds

\[
|w^n(t, \cdot)|_{-1} \leq |f(t, \cdot)|_{-1} + \mu_1 \frac{1}{\Gamma(1 - \alpha)} |\text{Div} \int_0^t (t-s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s; t, \cdot)) ds|_{-1}.
\]

Estimates (42) and (37) imply that

\[
\|w^n\|_{0, -1} \leq \|f\|_{0, -1} + \mu_1 \frac{1}{\Gamma(1 - \alpha)} M_1 T^{1-\alpha} \|v^{n-1}\|_{0, 1}
\]

\[
\leq \|f\|_{0, -1} + M_2 T^{1-\alpha} \|v^{n-1}\|_{0, 1}, \quad M_2 = \mu_1 \frac{1}{\Gamma(1 - \alpha)} M_1.
\]

Lemma 5.4 is proved.

Thus, \( w^n \in L_2(0, T; V^{-1}) \). Consequently, functions \( v^n \) are defined correctly and \( v^n \in W(0, T) \).

Establish estimates for functions \( v^n \).

5.4. Estimates of weak solutions of regularized problem (23)-(27) for small \( T \).

**Theorem 5.6.** Let \( T \) be small. For solutions \( v^n \) to the regularized problem (23)-(27) the uniform w.r.t. \( n \) estimates

\[
\sup_{0 \leq t \leq T} |v^n(t, \cdot)|_0 + \|v^n\|_{0, 1} \leq M_3(\|f\|_{0, -1} + |v^0|_0),
\]

\[
\|v^n\|_{L_1(0, T; V^{-1})} \leq M_3(1 + \|f\|_{0, -1} + |v^0|_0)^2
\]

hold. Here \( M_3 \) doesn't depend on \( n \).

**Proof of Theorem 5.6.** From Theorem 5.3 and (43), it follows that

\[
\sup_{0 \leq t \leq T} |v^n(t, \cdot)|_0 + \|v^n\|_{0, 1} \leq M_0(\|f\|_{0, -1} + \mu_1 \frac{1}{\Gamma(1 - \alpha)} |\text{Div} \int_0^t (t-s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s; t, \cdot)) ds|_{0, -1} + |v^0|_0)
\]

\[
\leq M_0(\|f\|_{0, -1} + M_1 T^{1-\alpha} \|v^{n-1}\|_{0, 0} + |v^0|_0)
\]

\[
=M_0(\|f\|_{0, -1} + |v^0|_0) + M_0 M_1 T^{1-\alpha} \|v^{n-1}\|_{0, 1} = M(f, v^0) + q \|v^{n-1}\|_{0, 1}.
\]

Here \( M(f, v^0) = M_0(\|f\|_{0, -1} + |v^0|_0) \), \( q = M_0 M_1 T^{1-\alpha} \).

Let \( T \) be such that \( q < 1 \). Using estimate (46) step by step we obtain

\[
\|v^n\|_{0, 1} \leq M(f, v^0) + q(M(f, v^0) + q |v^{n-2}|_{0, 1})
\]

\[
=M(f, v^0) + M(f, v^0) q + q^2 \|v^{n-2}\|_{0, 1}
\]

\[
\leq M(f, v^0)(1 + q) + q^2 \|M(f, v^0) + q |v^{n-3}|_{0, 1}
\]

\[
=M(f, v^0)(1 + q + q^2) + q^3 \|v^{n-3}\|_{0, 1}
\]

\[
\leq M(f, v^0) \sum_{k=0}^{n-1} q^k \leq M(f, v^0)(1 - q^{-1}).
\]

From (46), (47) it follows that for all \( n \) the inequality

\[
\sup_{0 \leq t \leq T} |v^n(t, \cdot)|_0 + \|v^n\|_{0, 1} \leq M_4(\|f\|_{0, -1} + |v^0|_0)
\]

where

\[
M_4 = \frac{M_0 M_1 T^{1-\alpha}}{1 - q^{-1}}.
\]
is valid. Here $M_4$ does not depend on $n$ but depend on $T$. From (48) it follows the validity of (44).

Next, $w^n \in L_2(0,T;V^{-1})$ due to Lemma 5.5. It follows then from Theorem 5.3 that $v^n \in W(0,T)$.

Let us prove estimate (45). It is obviously that $v^n$ satisfies equation

$$(v^n)' + \mu_0 A(v^n) - K(v^n) = w^n,$$

where $w^n$ is defined by (27).

On the strength of estimate (31) it follows from here that

$$\|((v^n)')_{L_1(0,T;V^{-1})} \leq M_3(1 + \|w^n\|_{0,1} + |v^0|_0)^2. \tag{49}$$

Using estimates (43) and (44) we easily get

$$\|w^n\|_{0,-1} \leq \|f\|_{0,-1} + M_2 T^{1-\alpha}\|v^{n-1}\|_{0,1} \leq M(\|f\|_{0,1} + |v^0|_0). \tag{50}$$

Estimates (49) and (50) imply inequality (44).

Theorem 5.6 is proved.

\[ \square \]

6. Solvability of (18)-(21) for small $T$.

**Theorem 6.1.** Let $f \in L_2(0,T;V^{-1})$, $v^0 \in H$. Let $T$ be small. Then problem (18)-(21) has at least one weak solution $v \in W(0,T)$ satisfying the uniform on $\varepsilon$ estimates

$$\sup_{0 \leq t \leq T} |v(t,\cdot)|_0 + \|v\|_{0,1} \leq M_6(\|f\|_{0,-1} + |v^0|_0), \tag{51}$$

$$\|v'\|_{L_1(0,T;V^{-1})} \leq M_6(1 + \|f\|_{0,-1} + |v^0|_0)^2. \tag{52}$$

Here $M_6$ doesn’t depend on $\varepsilon$.

**Proof of Theorem 6.1.** From estimates (44) it follows that the sequence $v^n$ is bounded in Hilbert space $L_2(0,T;V)$ and hence weakly compact in $L_2(0,T;V)$.

From Remark 2.1 ([19, p.223]) it follows that $W^{1,1}(0,T;V^{-1}) \cap L_2(0,T;V) \subset L_2(0,T;H)$ is compact. Hence, in virtue of estimates (44) and (45) it follows that the sequence $v^n$ is compact in $L_2(0,T;H)$.

We will assume that $v^n$ converges (with up to a subsequence) to some $v$ weakly in $L_2(0,T;V)$ and strongly in $L_2(0,T;H)$.

From estimates (30), it follows that due to the boundedness in $L_\infty(0,T;H)$ of the sequence $v^n$ it converges (with up to a subsequence) to $v$ weakly in $L_\infty(0,T;H)$.

Thus $v \in L_\infty(0,T;H) \cap L_2(0,T;V)$.

Next, estimates from Theorem 2.3 imply that the sequence $z^n$ converges to the associated to $v RLF z(\tau;t,x)$ w.r.t. Lebesgue measure on the set $[0,T] \times \Omega$ uniformly on $t \in [0,T]$.

We will show that $v$ is a weak solution of problem (13)-(16).

To do this, pass to the limit in problem (23)-(27).

From the definition of weak solutions of problem (23)-(27) there follows validity of the identity

$$(v^n(T, \cdot), \varphi(\cdot)) - \sum_{i=1}^N \int_0^T (v^n(1 + \varepsilon|v^n|^2)^{-1}, \partial \varphi/\partial x_i)$$

$$+ \mu_0 \int_0^T (E(v^n)(s, \cdot), E(\varphi(\cdot))) ds$$
\[ + \mu_1 \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^t (t - s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s; t, \cdot)) ds, \mathcal{E}(\varphi)(\cdot) \, dt \]

\[ = \int_0^T (f, \varphi) \, ds + (v^0, \varphi), \quad \varphi \in V. \]  

Let

\[ I_1(n) = (v^n(T, \cdot), \varphi(\cdot)), \quad I_2(n) = \sum_{i=1}^N \int_0^T (v^n_i^n v^n(1 + \varepsilon |v^n|^2)^{-1}, \partial \varphi / \partial x_i) \, ds, \]

\[ I_3(n) = \int_0^T (\mathcal{E}(v^n)(s, \cdot), \mathcal{E}(\varphi)(\cdot)) \, ds, \]

\[ I_4(n) = \int_0^T \left( \int_0^t (t - s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s; t, \cdot)) ds, \mathcal{E}(\varphi)(\cdot) \right) \, dt. \]

We rewrite the identity (53) in the form

\[ I_1(n) - I_2(n) + \mu_0 I_3(n) + \mu_1 \frac{1}{\Gamma(1-\alpha)} I_4(n) = \int_0^T (f, \varphi) \, ds + (v^0, \varphi) \quad (54) \]

and pass to the limit in (53) and (54) as \( n \to +\infty \).

Let \( \varphi \) be smooth. Estimate (48) implies the boundedness of \( v^n \) in \( L_2(0, T; V) \). From (48) and the weak continuity of \( v(t, \cdot) \) there follows the boundedness of \( v^n(T, \cdot) \) in \( H \). Without loss of generality we assume that \( v^n \) converges weakly to \( v \) in \( L_2(0, T; H) \) and \( v^n(T, \cdot) \) weakly converges to \( v(T, \cdot) \) in \( H \). Therefore,

\[ \lim_{n \to \infty} I_1(n) = (v(T, \cdot), \varphi(\cdot)), \quad \lim_{n \to \infty} I_2(n) = \int_0^T (\mathcal{E}(v)(s, \cdot), \mathcal{E}(\varphi)(\cdot)) \, dt. \quad (55) \]

Weak convergence of \( v^n \) to \( v \) in \( L_2(0, T; V) \) and strong in \( L_2(0, T; H) \) suggests (see [20]) that

\[ \lim_{n \to \infty} I_2(n) = \sum_{i=1}^N \int_0^T (v_i^n v(1 + \varepsilon |v|^2)^{-1}, \partial \varphi / \partial x_i) \, ds. \quad (56) \]

Consider \( I_4(n) \). Let us recall that

\[ \int_\Omega \mathcal{E}(u) : \mathcal{E}(w) \, dx = \sum_{i,j=1}^N \int_\Omega \mathcal{E}_{ij}(u) \mathcal{E}_{ij}(w) \, dx = (\mathcal{E}(u), \mathcal{E}(w)) \]

for \( u, v \in V \).

Making the change of variable \( y = z^n(s, t, x) \) ( \( x = z^n(t, s, y) \) is the inverse change), using (7) and the fact that \( \det z^n_{xx}^{-1}(s; t, x) = 1 \) for the Jacobian matrix \( z^n_{xx}^{-1}(s, t, x) \) since \( v^n_{xx} \) is divergence free, we find that

\[ \int_\Omega \int_0^t (t - s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s; t, \cdot)) \, ds : \mathcal{E}(\varphi)(x) \, dy = \int_0^t \int_\Omega (t - s)^{-\alpha} \mathcal{E}(v^{n-1})(s, y) : \mathcal{E}(\varphi)(z^{n-1}(t; s, y)) \, ds \, dy. \]
Using this relation and changing the integration order we have

\[
I_4(n) = \int_0^T \left( \int_0^t (t-s)^{-\alpha} \mathcal{E}(v^{n-1})(s, z^{n-1}(s; t, \cdot)) ds, \mathcal{E}(\varphi)(x) \right) dt \\
= \int_0^T \int_0^\Omega \mathcal{E}(v^{n-1})(s, y) : \int_s^T (t-s)^{-\alpha} \mathcal{E}(\varphi)(y) dt dy ds \\
= \int_0^T \int_\Omega \mathcal{E}(v^{n-1})(s, y) : \psi^n(s, y) dy ds
\]

where

\[
\psi^n(s, y) = \int_s^T (t-s)^{-\alpha} \mathcal{E}(\varphi)(z^{n-1}(t, s, y)) dt.
\]

Consider \(\psi^n\). Since the sequence \(z^n(t, s, y)\) converges to \(z(t, s, y)\) w.r.t. \((t, y)\) measure we will assume that \(z^n\) converges to \(z(t, s, y)\) a.e. (up to a subsequence). Due to the smoothness of \(\varphi\) the function \(\mathcal{E}(\varphi)(z^n(t, s, y))\) is bounded and \(\mathcal{E}(\varphi)(z^n(t, s, y))\) converges a.e. on \(Q_T\) to the bounded function \(\mathcal{E}(\varphi)(z(t, s, y))\). In virtue of the Lebesgue Theorem the uniformly bounded sequence \(\psi^n(s, y)\) converges a.e. on \(Q_T\) to the bounded function \(\psi(s, y) = \int_s^T (t-s)^{-\alpha} \mathcal{E}(\varphi)(z(s, t, y)) dt\).

Thus, in the integrand of

\[
I_4(n) = \int_0^T \int_\Omega \mathcal{E}(v^{n-1})(s, y) : \psi^n(s, y) dy ds \quad (57)
\]

the first factor converges weakly in \(L_2(Q_T)^{N \times N}\) while the second one a.e. in \(Q_T\). This implies that in (57) one can pass to the limit as \(n \to +\infty\) and get

\[
I_4 = \lim_{n \to +\infty} I_4(n) = \int_0^T (\mathcal{E}(v)(s, y), \psi(s, y)) ds \\
= \int_0^T \int_s^T (t-s)^{-\alpha} (\mathcal{E}(v)(s, y), \mathcal{E}(\varphi)(z(t, s, y))) dt ds.
\]

Changing the integration order and making the change of variable \(y = z(s, t, x)\), we get

\[
I_4 = \int_0^T \int_0^t (\mathcal{E}(v)(s, z(s, t, x)), \mathcal{E}(\varphi)(x)) ds dt. \quad (58)
\]

From the established convergence of the summands \(I_1(n)\) it follows that the function \(v(t, x)\) satisfies

\[
(v(T, \cdot), \varphi(\cdot)) - \int_0^T (v, v(1 + \varepsilon|v|^2)^{-1}, \partial \varphi/\partial x_1) ds + \mu_0 \int_0^T (\mathcal{E}(v)(t, \cdot), \mathcal{E}(\varphi)(\cdot)) dt \\
+ \mu_1 \frac{1}{\Gamma(1 - \alpha)} \int_0^T \int_0^t (t-s)^{-\alpha} (\mathcal{E}(v)(s, z(s; t, \cdot)), \mathcal{E}(\varphi)(\cdot)) ds dt \\
= \int_0^T \langle f, \varphi \rangle dt \tag{59}
\]

for any smooth \(\varphi\).

Let \(\varphi \in V\) be arbitrary. Choose a sequence of smooth \(\varphi_m \in V, m = 1, 2, \ldots\) such that \(\varphi_m\) converges in \(V\) to \(\varphi\) as \(m \to +\infty\). Taking \(\varphi = \varphi_m\) in (59) and passing to the limit as \(m \to +\infty\) we get (59) for arbitrary \(\varphi \in V\). The passage to the limit
is possible, since the convergence of $\varphi_m$ to $\varphi$ in $V$ implies the convergence of $E(\varphi^m)$ to $E(\varphi)$ in $L^2(\Omega)^{N\times N}$ and, in addition, the scalar products in (59) are continuous w.r.t. to its factors.

It is easy to show that (59) is true for any $t \in (0, T)$ instead of $T$.

Hence, using differentiation w.r.t. $t$ we obtain that $v$ satisfies the identity (22).

Let us show that $v \in W(0, T)$. Similarly to the case of problem (23)-(27) rewrite problem (18)-(21) in the operator form

$$v' + \mu_0 A(v) - K_\varepsilon(v) = w, \ v(0) = v^0, \tag{60}$$

where

$$w = f - \mu_1 \frac{1}{\Gamma(1-\alpha)} \text{Div} G(v), \ G(v) = \int_0^t (t-s)^{-\alpha} E(v)(s, z(s; t, x)) \, ds.$$ 

Since $v \in L^2(0, T; V)$ then from Lemma 5.4 we get $\text{Div} G(v) \in L^2(0, T; V^{-1})$. This and $f \in L^2(0, T; V^{-1})$ implies $w \in L^2(0, T; V^{-1})$. The inclusion $v \in W(0, T)$ now follows from Theorem 5.3.

Let us establish estimates (51) and (52). Since $v$ is a weak solution to regularized Navier-Stokes problem for known $w$, then due to Theorem 5.3 the following estimates are valid:

$$\sup_{0 \leq t \leq T} |v(t, \cdot)|_0 + \|v\|_{0,1} \leq M(\|w\|_{0,-1} + |v^0|_0), \tag{61}$$

$$\|v'\|_{L^1(0, T; V^{-1})} \leq M(1 + \|w\|_{0,-1} + |v^0|_0)^2 \tag{62}$$

with independent on $\varepsilon$ constant.

In the same way as in the proof of (43) it is shown that

$$\|w\|_{L^2(0, T; V^{-1})} \leq \|f\|_{0,1} + M_2 T^{1-\alpha} \|v\|_{0,1}. \tag{63}$$

From estimates (61) and (63) it follows that for small $T$ inequality (51) holds true.

Using (62), (63) and (51) as in the proof of inequality (45) we obtain inequality (52).

Theorem 6.1 and consequently the solvability of problem (18)-(21) for small $T$ is proved.

Establish the solvability of problem (18)-(21) for arbitrary $T$.

For this we need a priori estimates of weak solutions to problem (32).

7. A priori estimates of weak solutions for $\varepsilon$-regularization. The following fact is valid.

**Theorem 7.1.** Let $f \in L^2(0, T; V^{-1})$, $v^0 \in H$, $\varepsilon > 0$. Then if the solution of the problem (32) $v \in W(0, T)$, then it satisfies the estimates

$$\sup_t |v(t, \cdot)|_0 + \|v\|_{0,1} \leq M_7(\|f\|_{0,-1} + |v^0|_0), \tag{64}$$

$$\|v'\|_{L^1(0, T; V^{-1})} \leq M_7(\|f\|_{0,-1}^2 + |v^0|_0^2) \tag{65}$$

with independent on $\varepsilon$ constant $M_7$. 
Proof of Theorem 7.1. For a function \( v \in W(0, T) \) relation (35) is valid. Applying both sides of equation (60) to \( v \in V \), using the obtained relation and \( (K_v(u), u) = 0 \) for \( u \in V \) (see [20], [23], p. 208) by means of standard transformations with usage of Lemma 1.2 (see [19]) we obtain the identity

\[
\frac{1}{2} \frac{d}{dt} |v(t, \cdot)|^2_0 + \mu_0 \mathcal{E}(v)(t, \cdot) = (f(t, \cdot), v(t, \cdot))
\]

\[
- \mu_1 \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \mathcal{E}(v)(s, z(s; t, \cdot)) ds, \mathcal{E}(v)(t, \cdot). \tag{66}
\]

Changing \( t \) by \( s \) in (66), integrating w.r.t. \( s \) over interval \([0, t] \subset [0, T]\) we have

\[
\frac{1}{2} |v(t, \cdot)|^2_0 + \mu_0 \int_0^t |v(s, \cdot)|^2_1 ds = \int_0^t (f(s, \cdot), v(s, \cdot)) ds
\]

\[
- \mu_1 \frac{1}{\Gamma(1 - \alpha)} \int_0^t (s - \tau)^{-\alpha} \mathcal{E}(v)(\tau, z(\tau; s, \cdot)) d\tau, \mathcal{E}(v)(s, \cdot)) ds + \frac{1}{2} |v^0|^2_0. \tag{67}
\]

For the first term on the right-hand side in (67) we get for sufficiently small \( \eta > 0 \)

\[
\left| \int_0^t (f(s, \cdot), v(s, \cdot)) ds \right| \leq \int_0^t |f(s, \cdot)|_{-1} |v(s, \cdot)|_1 ds
\]

\[
\leq C_\eta \int_0^t |f(s, \cdot)|^2_2 ds + \eta \int_0^t |v(s, \cdot)|^2_1 ds \leq \frac{1}{2} C_\eta \|f\|^2_0 + \eta \|v\|^2_{L_2(0, t; V)}, \tag{68}
\]

Next, it is easy to see that due to Korn’s inequality

\[
m_1 \|v\|^2_{L_2(0, t; V)} \leq \int_0^t |\mathcal{E}(v)(t, \cdot)|^2_0 ds \leq m_2 \int_0^t |v(s, \cdot)|^2_1 ds
\]

\[
= m_2 \|v\|^2_{L_2(0, t; V)}, \quad m_i > 0. \tag{69}
\]

Consider the last term in (67) and rewrite it in the form

\[
\mu_1 \frac{1}{\Gamma(1 - \alpha)} \int_0^t (s - \tau)^{-\alpha} \mathcal{E}(v)(\tau, z(\tau; s, \cdot)) d\tau, \mathcal{E}(v)(s, x)) ds = \int_0^t Z(s) ds \tag{70}
\]

where

\[
Z(t) = \mu_1 \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \mathcal{E}(v)(s, z(s; t, \cdot)) ds, \mathcal{E}(v)(t, \cdot)). \tag{71}
\]

Making elementary transformations we have for arbitrary \( \eta > 0 \)

\[
|Z(t)| \leq M \int_0^t (t - s)^{-\alpha} |v(s, z(s; t, \cdot))|_1 ds |v(t, \cdot)|_1
\]
\[ \leq C(\eta) \left( \int_0^t (t-s)^{-\alpha} |v(s, z(s; t, \cdot))|_1^2 ds \right)^2 + \eta \|v(t, \cdot)\|_1^2. \quad (72) \]

Let
\[ I^2 = |v(s, z(s; t, \cdot))|_1^2 = \int_\Omega |v_x(s, z(s; t, x))|^2 dx. \]

Making in \( I \) the change of variables
\[ x = z(t; s, y), \quad y = z(s; t, x). \]

and having in mind that \( \det z_x(s; t, x) = 1 \) in the case of divergence free \( v \) we get
\[ |v(s, z(s; t, x))|^2 \leq M |v_x(s, z(s; t, x))|^2_0 = M \int_\Omega |v_x(s, z(s; t, x))|^2 dx \]
\[ = M \int_\Omega |v_x(s, y)|^2 dy = M |v_x(s, \cdot)|_0^2 \leq M |v(s, \cdot)|_1^2. \quad (74) \]

Thus, from (72) and (74) it follows that
\[ \int_0^t Z(s) ds \leq C(\eta) \int_0^t (s-\tau)^{-\alpha} |v(\tau, \cdot)|_1^2 d\tau + \eta \int_0^t |v(\tau, \cdot)|_1^2 ds. \quad (75) \]

Using (68), (75), (69) and choosing \( \eta \) small enough, we obtain from (67)
\[ \frac{1}{2} \|v(t, \cdot)\|_0^2 + \mu_0 \int_0^t |v(s, \cdot)|_1^2 ds \]
\[ \leq M(\|f\|_{L_2(0,t;V^{-1})}^2 + |v|_0^2 + \int_0^t \int_0^s (s-\tau)^{-\alpha} |v(\tau, \cdot)|_1^2 d\tau d\tau). \quad (76) \]

Denoting the last summand via \( Z_2(t) \), we have
\[ Z_2(t) = \int_0^t \int_0^s (s-\tau)^{-\alpha} |v(\tau, \cdot)|_1^2 d\tau d\tau = \|\int_0^s (s-\tau)^{-\alpha} |v(\tau, \cdot)|_1^2 d\tau\|_{L_2(0,t)}^2. \quad (77) \]

Denote by \( \bar{v}(\tau, x) \) the extension by zero of \( v(\tau, x) \) from \([0, t]\) on \((-\infty, +\infty)\). Let \( K(\xi) = \xi^{-\alpha}\) for \( t > \xi > 0, \ K(\xi) = 0 \) for \( \xi \notin (0, t) \). Then using the change of variable \( \xi = s-\tau \) we get
\[ \int_0^s (s-\tau)^{-\alpha} |v(\tau, \cdot)|_1 d\tau = \int_{-\infty}^{+\infty} K(s-\tau) |\bar{v}(\tau, \cdot)|_1 d\xi = \int_{-\infty}^{+\infty} K(\xi) |\bar{v}(s-\xi, \cdot)|_1 d\xi. \quad (78) \]

Using the Minkowski integral inequality in (77) and the invariance of \( L_2(-\infty, +\infty) \) norms w.r.t. shift we have
\[ Z_2(t) \leq \|\int_{-\infty}^{+\infty} K(\xi) |\bar{v}(s-\xi, \cdot)|_1 d\xi\|_{L_2(-\infty, +\infty)}^2 \]
\[ \leq \left( \int_{-\infty}^{+\infty} K(\xi) \|\bar{v}(s-\xi, \cdot)|_1\|_{L_2(-\infty, +\infty)}^2 \right)^2. \]
\[ \leq (\| \tilde{v}(s, \cdot) \|_{L^2(-\infty, +\infty)} t^+ \int_{-\infty}^{+\infty} K(\xi) \, d\xi)^2 \]

Thus,

\[ = (\| \tilde{v}(s, \cdot) \|_{L^2(-\infty, +\infty)} t^+ \int_{0}^{t} \xi^{-\alpha} \, d\xi)^2 \]

\[ \leq M((1 - \alpha)^{-1} t^{1-\alpha})^2 \| v(s, \cdot) \|_{L^2(0, t; V)}^2. \]

From the relations (76) and (79) it follows that for \( 0 < t \leq t_0 \) where \( t_0 > 0 \) is small enough, the inequality

\[ |v(t, \cdot)|_0^2 + \int_0^t |v|^2 ds \leq M(\| f \|_{0,-1}^2 + \| v \|_{0}^2) \]  

(80)

is valid.

Estimate (64) in the case \( 0 < T \leq t_0 \) follows from (80). Now consider the case of arbitrary \( T > t_0 \). Let \( t > t_0 \).

Represent \( Z_2(t) \) for \( t > t_0 \) in the form

\[ Z_2(t) = Z_{21} + Z_{22} \]  

(81)

where

\[ Z_{21} = \int_{t_0}^{t} (\int_{0}^{s} (s - \tau)^{-\alpha} |v(\tau, \cdot)|_{1} \, d\tau)^2 \, ds, \quad Z_{22} = \int_{t_0}^{t} (\int_{0}^{s} (s - \tau)^{-\alpha} |v(\tau, \cdot)|_{1} \, d\tau)^2 \, ds. \]

It is clear that \( Z_{21} = Z_2(t_0) \) and from (79), it follows that

\[ Z_{21} \leq M_1 t_0^2 (1-\alpha) \| v(s, \cdot) \|_{L^2(0, t_0; V)}^2. \]  

(82)

Consider \( Z_{22} \) and rewrite it in the form

\[ Z_{22} = \int_{t_0}^{t} (\int_{t_0}^{s} (s - \tau)^{-\alpha} |v(\tau, \cdot)|_{1} \, d\tau \]

\[ + \int_{s-t_0}^{s} (s - \tau)^{-\alpha} |v(\tau, \cdot)|_{1} \, d\tau)^2 \, ds \leq 2(\int_{t_0}^{t} (\int_{0}^{s} (s - \tau)^{-\alpha} |v(\tau, \cdot)|_{1} \, d\tau)^2 \, ds \]  

(83)

\[ + \int_{t_0}^{t} (\int_{t_0}^{s} (s - \tau)^{-\alpha} |v(\tau, \cdot)|_{1} \, d\tau)^2 \, ds = 2(Z_{221} + Z_{222}). \]

For \( Z_{221} \) we have

\[ Z_{221} \leq t_0^{-2\alpha} \int_{t_0}^{t} (\int_{0}^{s-t_0} |v(\tau, \cdot)|_{1} \, d\tau)^2 \, ds \leq (t - t_0) t_0^{-2\alpha} \int_{t_0}^{t} (\int_{0}^{s} |v(\tau, \cdot)|_{1}^2 \, d\tau) \, ds \]

\[ \leq M_1 \int_{t_0}^{t} \int_{0}^{s} |v(\tau, \cdot)|_{1}^2 \, d\tau \, ds. \]  

(84)
Let us estimate $Z_{222}$. Using the change of variable $\xi = s - \tau$ we get

$$Z_{222} \leq M \int_{t_0}^{t} (\int_0^{t} \xi^{-\alpha} |v(s - \xi, \cdot)|_1 d\xi)^2 ds = M \int_{t_0}^{t} \xi^{-\alpha} |v(s - \xi, \cdot)|_1 d\xi \|v\|^2_{\mathcal{L}_2(t_0, t)}.$$  

In the same way as in the derivation of estimates (80), using the Minkowski integral inequality one has

$$Z_{222} \leq (\int_0^{t} \xi^{-\alpha} d\xi)^2 \|v(\cdot, \cdot, \cdot)|^2_{\mathcal{L}_2(0, t; V)} \leq M(1 - \alpha)^{-2} t_0^2 (1 - \alpha) \|v\|^2_{\mathcal{L}_2(0, t; V)}.$$  

From estimates (82), (84) and (86) it follows that for $t > t_0$

$$Z_2(t) \leq M_1 \int_{t_0}^{t} \int_0^{s} |v(\tau, \cdot)|^2_1 d\tau ds + M(1 - \alpha)^{-2} t_0^2 (1 - \alpha) \int_0^{t} |v(\tau, \cdot)|^2_1 d\tau.$$  

Using the inequality (87) for estimation of the last term in (76) and supposing $t_0$ small enough, simple transformations yields the inequality

$$|v(t, \cdot)|^2_1 + \int_0^{t} |v(\tau, \cdot)|^2_1 d\tau \leq M(|f|_{\mathcal{L}_2} + |v^0|_1) + M_1 \int_{t_0}^{t} \int_0^{s} |v(\tau, \cdot)|^2_1 d\tau ds.$$  

Dropping in (88) the first summand, we get the Gronwall inequality for $\varphi(t) = \int_0^{t} |v(\tau, \cdot)|^2_1 d\tau$, from which it follows that

$$\varphi(t) \leq M_2 (|f|_{\mathcal{L}_2} + |v^0|_1), \quad t_0 \leq t \leq T.$$  

From (88) and (89) there follows (64).

Estimate (65) follows from equation (32), second estimate (34) and estimate (64). Theorem 7.1 is proved.

8. Solvability of regularized problem (18)-(21) for arbitrary $T$.

**Theorem 8.1.** Let $f \in L_2(0, T; V^{-1})$, $v^0 \in H$. Then problem (18)-(21) has at least one weak solution $v \in W(0, T)$ that satisfies estimates (51)-(52) with independent on $\varepsilon$ constant $M_\varepsilon$.

**Proof of Theorem 8.1.** Solvability of problem (18)-(21) in the case of some small $T$ (denote it by $T_0$) was established in the Section 6. Assuming without loss of generality $k_0 = T/T_0$ to be integer and $T_k = T_0 k$, consider the sequence of regularized problems on $[0, T_k]$ for $k = 1, 2 \ldots k_0$:

$$\frac{\partial v}{\partial t} + K_v(v) - \mu_0 \Delta v - \mu_1 \frac{1}{\Gamma(1 - \alpha)} \text{Div} \int_0^{t} (t - s)^{\alpha - 1} \mathcal{E}(v)(s, z(s; t, x)) ds$$

$$+ \mathcal{V} p = f(t, x), \quad (t, x) \in Q_k = [0, T_k] \times \Omega; \quad (90)$$

$$\text{div} v(t, x) = 0, \quad (t, x) \in Q_k; \quad (91)$$

$$z(\tau; t, x) = x + \int_{t}^{\tau} v(s, z(s; t, x)) ds, \quad t, \tau \in [0, T_k], x \in \overline{\Omega}; \quad (92)$$

$$v(0, x) = v^0(x), \quad x \in \Omega; \quad v_{|[0, T_k] \times \partial \Omega} = 0. \quad (93)$$
Supposing the solution of problem (18)-(21) to be known on \([0, T_k]\), extend it on \([T_k, T_{k+1}]\).

Let \(\bar{v}(t, x) \in W(0, T_k)\) be a solution to problem (90)-(93) on \([0, T_k]\). Let \(\bar{z}\) be RLF associated to \(\bar{v}\). Construct a continuation of \(\bar{v}(t, x)\) on \([T_k, T_{k+1}]\).

By Theorem 5.6 \(\bar{v}(t, x) \in W(0, T_k)\) and satisfies the estimates

\[
\sup_{0 \leq t \leq T_k} |\bar{v}(t, \cdot)|_0 + \|\bar{v}\|_{L_2(0, T_k; V)} \leq M_3(\|f\|_{L_2(0, T_k; V^{-1})} + |v^0|_0),
\]

(94)

\[
\|\bar{v}\|_{L_1((0, T_k; V^{-1})} \leq M_3(1 + \|f\|_{L_2(0, T_k; V^{-1})} + |v^0|_0)^2.
\]

(95)

Consider on \([T_k, T_{k+1}]\) the problem

\[
\partial v/\partial t + K_z(v) - \mu_0 \Delta v -
\]

\[
\mu_1 \frac{1}{\Gamma(1-\alpha)} \text{Div} \int_0^t (t-s)^{\alpha-1} \mathcal{E}(v)(s, z(s, t, x)) ds + \nabla p = F,
\]

(96)

\[
div v(t, x) = 0, \quad (t, x) \in \Omega;
\]

(97)

\[
z(\tau; t, x) = x + \int_{t}^{\tau} v(s, z(s, t, x)) ds, \quad t, \tau \in [T_k, T_{k+1}], x \in \Omega;
\]

(98)

\[
v(T_k, x) = \bar{v}(T_k, x), \quad x \in \Omega; \quad v|_{[T_k, T_{k+1}] \times \partial \Omega} = 0;
\]

(99)

\[
F = f(t, x) + \mu_1 \frac{1}{\Gamma(1-\alpha)} \text{Div} \int_0^T (t-s)^{\alpha-1} \mathcal{E}(v)(s, z(s, t, x)) ds
\]

(100)

for \((t, x) \in Q_{k,k+1} = [T_k, T_{k+1}] \times \Omega\).

A weak solution to problem (96)-(99) is a function \(v \in W(T_k, T_{k+1})\) satisfying the appropriate identity and the initial condition.

In the same way as for problem (13)-(16) on \([0, T_0]\) for given \(\bar{v}(T_k, x) \in H\) and \(F \in L_2(T_k, T_{k+1}; V^{-1})\) there are established the solvability of (96)-(99) in the class \(W(T_k, T_{k+1})\) on \([T_k, T_{k+1}]\) and estimates

\[
\|v\|_{L_2(T_k, T_{k+1}; V)} \leq M(\|F\|_{L_2(T_k, T_{k+1}; V^{-1})} + |\bar{v}(T_k, \cdot)|_0),
\]

(101)

\[
\|v\|_{L_1(T_k, T_{k+1}; V^{-1})} \leq M(1 + \|F\|_{L_2(T_k, T_{k+1}; V^{-1})} + |\bar{v}(T_k, \cdot)|_0)^2.
\]

(102)

Let us denote

\[
f_1 = \text{Div} \int_0^{T_k} (t-s)^{\alpha-1} \mathcal{E}(v)(s, z(s, t, x)) ds, \quad (t, x) \in Q_{k,k+1} = [T_k, T_{k+1}] \times \Omega
\]

and consider the problem (96)-(99) on \([T_k, T_{k+1}]\) for \(F = f + f_1\).

For its solvability it is enough to show that \(f_1 \in L_2(T_k, T_{k+1}; V^{-1})\) and therefore \(F \in L_2(T_k, T_{k+1}; V^{-1})\).

Lemma 8.2. The function \(f_1\) belongs to \(L_2(T_k, T_{k+1}; V^{-1})\) and estimate

\[
\|f_1\|_{L_2(T_k, T_{k+1}; V^{-1})} \leq M(\|f\|_{L_2(0, T; V^{-1})} + |v^0|_0)
\]

(103)

is valid.
Proof. Denote by \( \hat{v}(t, x) \) the continuation of \( \bar{v}(t, x) \) by zero from \([0, T_k]\) on \((\infty, +\infty)\) and by \( R(s) \) the continuation of \( s^{-\alpha} \) by zero from \([0, T_k]\) on \((\infty, +\infty)\). Then

\[
f_1 = \Div \int_0^{T_k} (t - s)^{\alpha - 1} \mathcal{E}(\bar{v})(s, x) \, ds = \Div \int_{-\infty}^{+\infty} R(t - s) \mathcal{E}(\bar{v})(s, x) \, ds \nonumber
\]

\[
= \Div \int_{-\infty}^{+\infty} R(\xi) \mathcal{E}(\hat{v})(t - \xi x) \, d\xi. \tag{104}
\]

Using standard transformations, (38), the Minkowski integral inequality and the invariance of the \( L_2(\infty, +\infty) \) norms w.r.t. shifts, we have

\[
\|f_1\|_{L_2(T_k, T_{k+1}; V^{-1})} \leq M \| \int_{-\infty}^{+\infty} R(\xi) \mathcal{E}(\hat{v})(t - \xi x) \, d\xi \|_{L_2(T_k, T_{k+1}; L_2(\Omega))} \nonumber
\]

\[
\leq M \| \int_{-\infty}^{+\infty} R(\xi) |\hat{v}(t - \xi, \cdot)|_1 \, d\xi \|_{L_2(T_k, T_{k+1})} \nonumber
\]

\[
\leq M \int_{-\infty}^{+\infty} R(\xi) \| |\hat{v}(t - \xi, \cdot)|_1 \|_{L_2(T_k, T_{k+1})} \, d\xi \nonumber
\]

\[
\leq M \int_{-\infty}^{+\infty} R(\xi) \| |\hat{v}(t - \xi, \cdot)|_1 \|_{L_2(T_k, T_{k+1})} \, d\xi \nonumber
\]

\[
\leq M \int_{-\infty}^{+\infty} R(\xi) \| |\hat{v}(t - \xi, \cdot)|_1 \|_{L_2(T_k, T_{k+1})} \, d\xi \nonumber
\]

\[
\leq M \int_{-\infty}^{+\infty} R(\xi) \| |\hat{v}(t - \xi, \cdot)|_1 \|_{L_2(T_k, T_{k+1})} \, d\xi \nonumber
\]

\[
\leq MT_k^{\frac{1}{1 - \alpha} - 1} \| |\hat{v}(t - \xi, \cdot)|_1 \|_{L_2(T_k, T_{k+1}; V)}. \tag{105}
\]

Using the estimate (95) on \([0, T_k]\) we get here

\[
\|f_1\|_{L_2(T_k, T_{k+1}; V^{-1})} \leq MT_k^{\frac{1}{1 - \alpha} - 1} \| |\hat{v}(t - \xi, \cdot)|_1 \|_{L_2(0, T_k; V)} \nonumber
\]

\[
\leq MT_k^{\frac{1}{1 - \alpha} - 1} (\|f\|_{L_2(0, T_k; V^{-1})} + |v_0|_0) \leq M(\|f\|_{L_2(0, T_k; V^{-1})} + |v_0|_0). \tag{106}
\]

From this estimate (103) follows.

Lemma 8.2 is proved. \( \square \)

From Lemma 8.2 it follows that \( F \in L_2(T_k, T_{k+1}; V^{-1}) \) and hence the problem (90)-(93) on \([T_k, T_{k+1}]\) with

\[
F = f(t, x) + \mu_1 \frac{1}{\Gamma(1 - \alpha)} \Div \int_0^{T_k} (t - s)^{\alpha - 1} \mathcal{E}(\bar{v})(s, z(s; t, x)) \, ds \nonumber
\]

\[
= f(t, x) + \mu_1 \frac{1}{\Gamma(1 - \alpha)} f_1(t, x), \quad (t, x) \in Q_{k, k+1} = [T_k, T_{k+1}] \times \Omega \nonumber
\]

has a solution \( \hat{v} \). Continuing the function \( \bar{v} \) on \([0, T_k]\) by function \( \bar{v} \in W(0, T_{k+1}) \) on \([T_k, T_{k+1}]\) and denoting the continuation by \( v \) we get solution \( v \in W(0, T_{k+1}) \) to problem (18)-(21) on \([0, T_{k+1}]\).
By Theorem 7.1 for $v$ estimates (51) and (52) on $[0, T_{k+1}]$ hold true. It is obvious that carrying out the finite process of continuation step by step we obtain a weak solution to problem (18)-(21) on $[0, T]$ for which estimates (64)-(65) are valid.

Theorem 8.1 is proved. \hfill \Box

9. **Proof of Theorem 4.2.** Establish solvability of irregularized problem (13)-(16). For this we consider the sequence of regularized problems

\[
\frac{\partial v^n}{\partial t} + K_{1/n}(v^n) - \mu_0 \Delta v^n = \frac{1}{\Gamma(1-\alpha)} \text{Div} \int_0^t (t-s)^{-\alpha} \mathcal{E}(v^n)(s, z^n(s; t, x)) ds
\]

\[+ \nabla p^n = f(t, x), \ (t, x) \in Q_T; \quad (107)\]

\[
\text{div} v^n(t, x) = 0, \quad (t, x) \in Q_T; \quad (108)\]

\[
z^n(\tau; t, x) = x + \int_0^\tau v^n(s, z(s; t, x)) ds, \quad t, \tau \in [0, T], x \in \overline{\Omega}; \quad (109)\]

\[
v^n(0, x) = v^0(x), \quad x \in \Omega; \quad v^n|_{\partial\Omega} = 0. \quad (110)\]

From Theorem 8.1 there follows for any fixed $n$ the existence of weak solution $v^n$ to problem (107)-(110) and validity of estimates

\[
\sup_{0 \leq t \leq T} \|v(t, \cdot)|_0 + \|v^n\|_{0, 1} \leq M_3(\|f\|_{0, -1} + \|v^0\|_0), \quad (111)\]

\[
\|v'\|_{L_1(0, T; V^{-1})} \leq M_3(1 + \|f\|_{0, -1} + \|v^0\|_0)^2, \quad (112)\]

where $M_3$ doesn’t depend on $n$.

From the definition of weak solutions to problem (107)-(110) there follows the validity of the identity

\[
(v^n(T, \cdot), \varphi(\cdot)) - \sum_{i=1}^N \int_0^T ((v^n_i v^n(1 + n^{-1}|v^n|^2)^{-1}, \partial \varphi/\partial x_1) dt
\]

\[+ \mu_0 \int_0^T (\mathcal{E}(v^n)(s, \cdot), \mathcal{E}(\varphi)(\cdot)) ds
\]

\[+ \mu_1 \frac{1}{\Gamma(1-\alpha)} \int_0^T (t-s)^{-\alpha} \mathcal{E}(v^n)(s, z^n(s; t, \cdot)) ds, \mathcal{E}(\varphi)(\cdot)) dt
\]

\[= \int_0^T \langle f, \varphi \rangle ds + (v^0, \varphi), \quad \varphi \in V. \quad (113)\]

Let

\[
I_1(n) = (v^n(T, \cdot), \varphi(\cdot)), \quad I_2(n) = \sum_{i=1}^N \int_0^T (v^n_i v^n(1 + n^{-1}|v^n|^2)^{-1}, \partial \varphi/\partial x_1) ds,
\]

\[
I_3(n) = \int_0^T (\mathcal{E}(v^n)(s, \cdot), \mathcal{E}(\varphi)(\cdot)) ds,
\]

\[
I_4(n) = \int_0^T (t-s)^{-\alpha} \mathcal{E}(v^n)(s, z^n(s; t, \cdot)) ds, \mathcal{E}(\varphi)(\cdot)) ds dt.
\]
Let us write the identity (113) in the form

\[ I_1(n) - I_2(n) + \mu_0 I_3(n) + \mu_1 \frac{1}{\Gamma(1 - \alpha)} I_4(n) = \int_0^T \langle f, \varphi \rangle \, ds + \langle v^0, \varphi \rangle \]  

(114)

and pass to the limit in (114) (or in (113), that is the same) as \( n \to +\infty \).

Here the justification of the passage to the limit is the same as in the proof of Theorem 6.1, with the exception of term \( I_2(n) \). However, weak convergence of \( v^n \) to \( v \) in \( L_2(0, T; \mathcal{V}) \) and strong in \( L_2(0, T; \mathcal{H}) \) suggests (see [20]) that

\[ \lim_{n \to \infty} I_2(n) = \sum_{i=1}^N \int_0^T (v_i v, \partial_\varphi / \partial x_i) \, ds. \]

Thus, the limit function \( v \) satisfies the identity

\[ (v(T, \cdot), \varphi(\cdot)) - \int_0^T (v v, \partial_\varphi / \partial x_i) \, ds + \mu_0 \int_0^T (\mathcal{E}(v)(t, \cdot), \mathcal{E}(\varphi)(\cdot)) \, dt \\
+ \mu_1 \int_0^T \int_0^t (t - s)^{-\alpha} (\mathcal{E}(v)(s, z(s; t, \cdot)), \mathcal{E}(\varphi)(\cdot)) \, ds \, dt = \int_0^T \langle f, \varphi \rangle \, dt \]  

(115)

for any \( \varphi \in \mathcal{V} \).

Hence, in the same way as in the proof of Theorem 6.1 it is deduced that \( v \) is a weak solution to (13)-(16).

Theorem 4.2 is proved.

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