One-to-one mapping between steering and joint measurability problems

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Quantum steering refers to the possibility for Alice to remotely steer Bob’s state by performing local measurements on her half of a bipartite system. Two necessary ingredients for steering are entanglement and incompatibility of Alice’s measurements. In particular, it has been recently proven that for the case of pure states of maximal Schmidt rank the problem of steerability for Bob’s assemblage is equivalent to the problem of joint measurability for Alice’s observables. We show that such an equivalence holds in general, namely, the steerability of any assemblage can always be formulated as a joint measurability problem, and vice versa. We use this connection to introduce steering inequalities from joint measurability criteria and develop quantifiers for the incompatibility of measurements.

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Introduction.— Steering is a quantum effect by which one experimenter, Alice, can remotely prepare an ensemble of states for another experimenter, Bob, by performing local measurement on her half of a bipartite system and communicating the results to Bob. Introduced by Schrödinger in 1935 [1], quantum steering is a form of quantum correlation intermediate between Bell nonlocality and entanglement. It has recently attracted increasing interest [2–7], both from a theoretical and experimental perspective, and it has been recognized as a resource for different tasks such as one-sided device-independent quantum key distribution [8, 9] and subchannel discrimination [10]. In addition, the question which quantum states can be used for steering can be addressed with efficient numerical techniques, contrary to the notion of steering inequalities from joint measurability criteria and develop quantifiers for the incompatibility of measurements.

A successful implementation of a steering protocol involves different elements, e.g., entangled states and incompatible measurements, and therefore steering has been investigated under different perspectives. On the one hand, allowing for an optimization over all possible quantum states or, equivalently, considering the maximal entangled state, steering has been identified with the lack of joint measurability of Alice’s local observables [13, 14], similarly to the case of nonlocality [15]. On the other hand, if an optimization over all possible measurements for Alice has been considered, steering has been identified with a property of the state allowing for optimal subchannel discrimination when one is restricted to local measurements and one-way classical communication [16]. In addition, a very natural and interesting framework for steering is that of one-sided device-independent (1SDI) quantum information processing. In the case of device-independent quantum information processing, both parties are untrusted, hence no assumption is made on the system and the measurement apparatuses and the only resources are the observed (nonlocal) correlations. Similarly, in 1SDI scenarios, where only one party (Bob) is trusted, it is natural to identify the resources for information processing tasks with the ensemble of states Bob obtains as a consequence of Alice’s measurement (see also Ref. [16] for a discussion of this point).

Taking the above perspective, we are able to prove that any steerability problem can be translated into a joint measurability problem, and vice versa. This result connects the well-known theory of joint measurements [17, 18] and uncertainty relations [19–22] to the relatively new research direction of steering. This is done by mapping any state ensemble for Bob in a corresponding steering-equivalent positive operator valued measure (POVM). This simple technique is shown to give an intuitive way of generalizing the known results [13–14]. Moreover, the power of the technique is demonstrated by mapping joint measurement uncertainty relations [19] into steering inequalities, and discussing the role of known steering monotones as monotones for incompatibility.

Preliminary notions.— Given a quantum state \( \rho \), i.e., a positive operator with trace one, an ensemble \( E = \{ \rho_a \} \) for \( \rho \) is a collection of positive operators such that \( \sum_a \rho_a = \rho \). An assemblage \( \mathcal{A} = \{ \mathcal{E}_x \}_x \) is a collection of ensembles for the same state \( \rho \), i.e., \( \sum_a \rho_{a|x} = \rho \) for all \( x \). Similarly, a measurement assemblage \( \mathcal{M} = \{ M_{a|x} \}_{a,x} \) is a collection of operators \( M_{a|x} \geq 0 \) such that \( \sum_a M_{a|x} = 1 \) for all \( x \). Each subset \( \{ M_{a|x} \}_a \) is called a positive-operator-valued measure (POVM), and it gives the outcome probabilities for a general quantum measurement via the formula \( P(a|x) = \text{tr}[M_{a|x}\rho] \).

A measurement assemblage \( \mathcal{M} = \{ M_{a|x} \}_{a,x} \) is defined to be jointly measurable (JM) [23] if there exist numbers \( p_M(a|x, \lambda) \) and positive operators \( \{ G_\lambda \} \) such that

\[
M_{a|x} = \sum_{\lambda} p_M(a|x, \lambda) \ G_\lambda,
\]
with \( \sum_{\lambda} G_{\lambda} = \mathbb{1}, \) \( \rho_M(a|x, \lambda) \geq 0, \) and \( \sum_{\lambda} \rho_M(a|x, \lambda) = 1. \) Physically, this means that all the measurements in the assembly can be measured jointly by performing the measurement \( \{G_{\lambda}\} \) and doing some post-processing of the obtained probabilities.

In a steering scenario, a bipartite state \( \rho_{AB} \) is shared by Alice and Bob. Alice performs measurements on her system with possible settings \( a \) and possible outcomes \( x \), that is, the measurement assembly \( \{A_{a|x}\}_{a,x} \). As a result of her measurement with the setting \( x \), Bob obtains the reduced state \( \rho(a|x) \) with probability \( P(a|x) \). Such a collection of reduced states and probabilities defines the state assembly \( \{\rho_{a|x}\}_{a,x} \), where

\[
\rho_{a|x} = \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}],
\]

with \( P(a|x) = \text{tr}[(A_{a|x} \otimes \mathbb{1})\rho_{AB}] = \text{tr}_B[\rho_{a|x}] \) and \( \rho(a|x) = \rho_{a|x}/P(a|x) \). In particular, elements of the assembly satisfy

\[
\rho_B = \sum_a \rho_{a|x} = \sum_a \rho_{a|x'}, \text{ for all settings } x, x',
\]

where \( \rho_B = \text{tr}_A[\rho_{AB}] \). This expresses the fact that Alice cannot signal to Bob by choosing her measurement \( x \).

A state assembly \( \{\rho_{a|x}\}_{a,x} \) is called unsteerable if there exists a local hidden state (LHS) model, namely, numbers \( p_{\rho}(a|x, \lambda) \geq 0 \) and positive operators \( \{\sigma_{\lambda}\} \) such that

\[
\rho_{a|x} = \sum_{\lambda} p_{\rho}(a|x, \lambda) \sigma_{\lambda},
\]

with \( \text{tr}[\sum_{\lambda} \sigma_{\lambda}] = 1 \). A state assembly is called steerable if it is not unsteerable. The physical interpretation is the following: If the assembly has a LHS model, then Bob can interpret his conditional states \( \rho_{a|x} \) as coming from the pre-existing states \( \sigma_{\lambda} \), where only the probabilities are changed due to the knowledge of Alice’s measurement and result. Contrary, if no LHS model is possible, then Bob must believe that Alice can remotely steer the states in his lab by making measurements on her side.

**Steerability as a joint-measurability problem.**— We now prove the main results of the paper, namely, that the steerability properties of a state assembly can always be translated in terms of joint measurability properties of a measurement assembly.

Let \( \{\rho_{a|x}\}_{a,x} \) be a state assembly and \( \rho_B \) the corresponding total reduced state for Bob. We define \( \Pi_B : \mathcal{H}_B \rightarrow \mathcal{K}_{\rho_B} \subset \mathcal{H}_B \) as the projection on the subspace \( \mathcal{K}_{\rho_B} := \text{range}(\rho_B) \), i.e., \( \Pi_B\rho_B = \mathbb{1}_{\mathcal{K}_{\rho_B}} \) and \( \Pi_B\Pi_B = \Pi_B \) is a Hermitian projector in \( \mathcal{L}(\mathcal{H}_B) \).

Since \( \rho_{a|x} \) are positive operators, Eq. (3) implies \( \text{range}(\rho_{a|x}) \subset \text{range}(\rho_B) \) for all \( a, x \) [24]. Hence, we can define the restriction of our assembly elements to the subspace \( \mathcal{K}_{\rho_B} \) as \( \hat{\rho}_{a|x} = \Pi_B\rho_{a|x}\Pi_B^* \) and \( \hat{\rho}_B = \Pi_B\rho_B\Pi_B^* \), preserving the positivity of the operators. Such a restriction is needed in order to define \( (\hat{\rho}_B)^{-\frac{1}{2}} \) (see below).

Then, we define Bob’s steering-equivalent (SE) observables \( B_{a|x} \in \mathcal{L}(\mathcal{K}_{\rho_B}) \) as

\[
B_{a|x} = (\hat{\rho}_B)^{-\frac{1}{2}} \rho_{a|x} (\hat{\rho}_B)^{-\frac{1}{2}}.
\]

These operators are clearly positive and, by Eq. (3), \( \sum_x B_{a|x} = \mathbb{1}_{\mathcal{K}_{\rho_B}} \), hence \( \{B_{a|x}\}_a \) forms a POVM. We can formulate the first equivalence:

**Theorem 1.** The state assembly \( \{\rho_{a|x}\}_{a,x} \) is unsteerable if and only if the measurement assembly \( \{B_{a|x}\}_{a,x} \) as defined by Eq. (5) is jointly measurable.

**Proof.** First, notice that it is sufficient to discuss the existence of a LHS model for \( \{\rho_{a|x}\}_{a,x} \). From Eqs. (4) and (1), one can easily see that from a LHS for \( \{\rho_{a|x}\}_{a,x} \) one can construct a joint observable for \( \{B_{a|x}\}_{a,x} \) and vice versa. The corresponding LHS model and joint observable are obtainable via the relation

\[
G_{\lambda} = (\hat{\rho}_B)^{-\frac{1}{2}} \hat{\sigma}_{\lambda} (\hat{\rho}_B)^{-\frac{1}{2}},
\]

where \( \hat{\sigma}_{\lambda} \) denotes the elements of the LHS for \( \hat{\rho}_{a|x} \). □

The above theorem shows that every steerability problem can be recast as a joint measurability problem. The other direction is trivial, since every joint measurability problem corresponds, up to a multiplicative constant, to a steerability problem with \( \rho_B = \mathbb{1}/d \). We can then state the main result:

**Theorem 2.** The steerability problem of any state assembly \( \{\rho_{a|x}\}_{a,x} \) can be translated into a joint measurability problem for a measurement assembly \( \{M_{a|x}\}_{a,x} \), and vice versa.

It is now interesting to discuss the interpretation of Bob’s SE observables. Let \( \rho = \sum_{i,j=1}^n \lambda_i \lambda_j |i\rangle \langle j| \) be a pure state on a finite-dimensional Hilbert \( \mathcal{H}_A \otimes \mathcal{H}_B \), where \( \{|i\rangle_A\}_{i=1}^d \otimes \{|i\rangle_B\}_{i=1}^d \) are the local bases associated with the above Schmidt decomposition of \( \rho \), \( n \leq \min\{d_A, d_B\}, \lambda_i > 0 \), and \( \text{tr}[\rho] = \sum_i \lambda_i^2 = 1 \).

The reduced states for Alice and Bob have in such basis an identical form, namely, \( \rho_X = \sum_{i=1}^n \lambda_i^2 |i\rangle \langle i| \) with \( X = A, B \), hence their ranges, \( \mathcal{K}_{\rho_X}, \mathcal{K}_{\rho_B} \) are isomorphic through the obvious mapping \( |i\rangle_A \leftrightarrow |i\rangle_B \). Using that, we can formally write

\[
\rho_{a|x} = \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho]\rho_B = \sum_{i,j=1}^n \lambda_i \lambda_j |A_{a|x}|i\rangle |j\rangle = \rho_A^{1/2} A_{a|x}^i \rho_B^{1/2} A_{a|x}^j,
\]

recovering a similar relation as in Eq. (5). The only missing step is to invert the relation by projecting on \( \mathcal{K}_{\rho_B} \) and writing the inverse \( \rho_B^{-1/2} \). Hence, for any pure state, Theorem 1 gives us a clear interpretation of Bob’s SE observables that generalizes the result given in Refs. [13, 14], namely, that for Schmidt rank \( d \) state it is sufficient for Alice to use non jointly measurable observables in order to demonstrate steering.
Remark. For a pure bipartite state, in order for Alice to demonstrate steering, her observable must be not jointly measurable even when restricted to the subspace where her reduced state, $\rho_A$, does not vanish.

Notice that the above remark holds also for pure separable states, however, since the corresponding subspace $\mathcal{K}_{\rho_A}$ is one-dimensional, joint measurability of Alice’s observables is always trivially achieved.

For the case of mixed states, a straightforward generalization of the above argument, e.g., via convex combinations, is not possible. Hence, the physical interpretation of Bob’s SE observable for mixed states remains an open problem.

Steering inequalities. — We use the above result to give new steering inequalities for an assemblage arising from two and three dichotomic measurements for Alice when Bob’s system is a qubit. We begin with the assemblage arising from two dichotomic measurements.

Given the assemblage $\{\rho_{a|x}\}$, with $a = \pm$ and $x \in \{1, 2\}$, written in terms of Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ as

$$\rho_{\pm|x} = t_{\pm}^x \mathbb{1} + \vec{s}_{\pm} \cdot \vec{\sigma},$$

with $\vec{s}_{\pm} = (s_{1\pm}, s_{2\pm}, s_{3\pm})$, the only nontrivial case corresponds to a reduced state $\rho_B = \sum_{a=\pm} \rho_{a|x}$ of rank 2, otherwise the total state would be separable.

Then, the SE observables for Bob can be written as

$$B_{\pm|x} = \frac{1}{2}((1 + \alpha_x) \mathbb{1} + \vec{r}_{\pm} \cdot \vec{\sigma}), \quad B_{-|x} = \mathbb{1} - B_{\pm|x},$$

with $\alpha_x$ and $\vec{r}_{\pm} = (r_{1\pm}, r_{2\pm}, r_{3\pm})$ being functions of the assemblage $\{\rho_{a|x}\}$, the explicit forms of these functions is given in the Supplemental Material. For such observables Busch et al. [19] have defined the degree of incompatibility to be the amount of violation of the following inequality

$$\|\vec{r}_1 + \vec{r}_2\| + \|\vec{r}_1 - \vec{r}_2\| \leq 2.$$

(10)

This inequality is a measurement uncertainty relation for joint measurements and as such it is a necessary condition for the joint measurability of two observables on a qubit (see also Ref. [13]). A violation of this inequality means that the SE observables of Bob are not jointly measurable and hence the setup is steerable. However, it has been shown that the degree of incompatibility does not capture all incompatible observables and a more fine-tuned version of this inequality, providing necessary and sufficient conditions has been derived [23]:

$$(1 - F_i^2 - F_{-i}^2) \left(1 - \frac{\alpha_i^2}{F_i^2} \frac{\alpha_{-i}^2}{F_{-i}^2}\right) \leq (\vec{r}_1 \cdot \vec{r}_2 - \alpha_1 \alpha_{-i})^2,$$

with $F_i = \frac{1}{2} \sqrt{(1 + \alpha_i)^2 - \|\vec{r}_i\|^2} + \sqrt{(1 - \alpha_i)^2 - \|\vec{r}_{-i}\|^2}$, for $i = 1, 2$

With the above definition, we can see the difference in the steerable assemblages detected by the steering inequality [10], which provides only a necessary condition, and inequality [11], which completely characterizes steerability. Consider an ensemble of two reduced states along the $z$ axis and symmetric with respect to the origin, i.e., $\rho_{\pm|1} = \frac{1}{2} (|1 \pm \lambda \sigma_z\rangle \langle 1|)$.

Given another ensemble $\rho_{\pm|2}$, by Eq. [8] only one of the two reduced states can be chosen freely, say $\rho_{+|2} = t_+^2 + \vec{s}_{+} \cdot \vec{\sigma}$, with the conditions $t_+^2 \leq 1/2$ and $\|\vec{s}_{+}\| \leq t_+^2$. The steerability detected by Eqs. [10] [11] is plotted in Fig. 1 for different values of the parameters $\lambda, r := \|\vec{s}_{+}\|$, and the angle $\theta$ between $\vec{s}_{+}$ and the $z$ axis.

Finally, for the case of three dichotomic measurements on Alice’s side (and Bob holding a qubit) we get three steering equivalent observables of the form Eq. [9]. For this case a joint measurement uncertainty relation and hence a steering inequality is given by [26]:

$$\sum_{i=1}^{4} \|\vec{R}_i - \vec{R}_F\| \leq 4,$$

(12)

where $\vec{R}_1 = \vec{r}_1 + \vec{r}_2 + \vec{r}_3$, $\vec{R}_i = 2\vec{r}_{i-1} - \vec{R}_1$ $(i = 2, 3, 4)$, and $\vec{R}_F$ is the Fermat-Torricelli point of the vectors $\vec{R}_i$, i.e. the point which minimizes the left hand side of Eq. [12]. Analogously to the case of Eq. [10], Eq. [12] provides a necessary condition for the unsteerability of the state assemblage.

Steering monotones. — The previously known connection between joint measurability and steering [13] [14] has inspired the definition of incompatibility monotones, i.e.,
measures of incompatibility that are non increasing under local channels, based on steering monotones \[27\] or associated with steering tasks \[28\].

Following the same spirit and in light of Theorem \[3\] we introduce a incompatibility monotone based on a recently proposed steering monotone, i.e., the steering robustness \[10\]. Given a measurement assemblage \(\{M_{a|x}\}_{a,x}\) we define the incompatibility robustness \(\mathcal{IR}\) as the minimum \(t\) such that there exist another measurement assemblage \(\{N_{a|x}\}_{a,x}\) such that \(\{(M_{a|x} + t N_{a|x})/(1 + t)\}_{a,x}\) is jointly measurable. The idea is to quantify the robustness of the incompatibility properties of the measurement assemblage under the most general form of noise. It is easily proven that \(\mathcal{IR}\) can be computed as a semidefinite program and that it is monotone under the action of a quantum channel (cf. Supplemental Material).

It is interesting to discuss the relation with previously proposed incompatibility monotones. In Ref. \[27\], the incompatibility weight \(\mathcal{IW}\) of Ref. \[3\] was defined for a set of POVMs \(\{M_{a|x}\}_{a,x}\) as the minimum positive number \(\lambda\) such that the decomposition \(M_{a|x} = \lambda O_{a|x} + (1 - \lambda) N_{a|x}\) holds for assemblage \(\{N_{a|x}\}_{a,x}\) and jointly measurable assemblage \(\{O_{a|x}\}_{a,x}\). From the definition it is clear that the \(\mathcal{IW}\) suffer from a similar problem as \(\mathcal{SW}\), namely that whenever the elements of the (state or measurement) assemblage are rank-1, such weight is maximal. As a consequence, each pair of projective measurements, e.g., on a qubit, even along arbitrary close directions, are maximally incompatible according to \(\mathcal{IW}\), and, similarly, the state assemblage arising from a bipartite pure state, even with arbitrary small entanglement, is maximally steerable according to \(\mathcal{SW}\) (see also the discussion in Ref. \[10\]).

Another monotone has been proposed by Heinosaari et al. \[28\], based on noise robustness of the incompatibility with respect to mixing with white biased noise. This definition can be obtained from \(\mathcal{IR}\), with the substitution \(N_{a|x} \mapsto \frac{I}{d}\) (white noise) and, for the corresponding coefficient \(\lambda := t/(1 + t)\), the substitution \(\lambda \mapsto (1 + ab)\lambda\), in the case of dichotomic measurements, i.e., \(a = \pm 1\). The notions of biasedness refers to the possibility of having a different disturbance for different outcomes.

As a consequence, \(\mathcal{IR}\) is always a lower bound to the white noise tolerance. It is interesting to discuss such differences in a simple example. Consider a mixing of a measurement assemblage \(\{M_{a|x}\}_{a,x}\) with white or general noise

\[\mathcal{M}_g = \{(1 - \lambda_g)M_{a|x} + \lambda_g N_{a|x}\}_{a,x} ,\]

\[\mathcal{M}_w = \{(1 - \lambda_w)M_{a|x} + \lambda_w \frac{I}{d}\}_{a,x} .\]

If we choose in a qubit case \(M_{a|x} = \frac{1}{2} (I + \sum_{i} |i\rangle \langle i| \cdot \sigma_i)\) and \(N_{a|x} = \frac{1}{2} (I - \sum_{i} |i\rangle \langle i| \cdot \sigma_i)\) we end up with the mixings \(\mathcal{M}_g = \frac{1}{2} (I + (1 - 2\lambda) \sum_{i} |i\rangle \langle i| \cdot \sigma_i)\) and \(\mathcal{M}_w = \frac{1}{2} (I + \lambda \sum_{i} |i\rangle \langle i| \cdot \sigma_i)\). It is then clear that in this case the noise robustness for general noise is always smaller than half the noise robustness with respect to white noise, namely,

\[\min\{\lambda_g | \mathcal{M}_g\} \leq \frac{1}{2} \min\{\lambda_w | \mathcal{M}_w\} \text{ is JM } .\]

Explicit calculations (plotted in Fig. \[2\]) show that the above choice for \(N_{a|x}\) is not always the optimal one. The same noise robustness, for the case of orthogonal steering monotones, has been calculated in Ref. \[29\].

The case of biased white noise corresponds to the substitution in Eq. \[14\] \(\lambda \mapsto \lambda (1 + ab)\) for the case of binary measurements, i.e., \(a = \pm 1\). For the simplest case, i.e., two sharp projective measurement on a qubit, the noise robustness for for mixing with general noise or with white noise plus a bias is plotted in Fig. \[2\].

Conclusions.— We have proven that every steerability problem can be recast as a joint measurability problem, and vice versa. As opposed to previous results \[13\] \[14\], our approach does not include any assumption on the state of the system, but it is applicable knowing solely Bob’s state assemblage. This is arguably the most natural resource for steering, especially for one-sided device-independent quantum information protocols, where only Bob’s side is characterized \[16\].

Our work connects the relatively new field of quantum steering with the much older topic of joint measurability. As we showed with concrete examples, that this connection allows to translate results from one field to the other. On the one hand, we were able to derive new steering inequalities for the two simplest steering scenarios based on joint measurability criteria for qubit observables. As opposed to previously defined steering inequalities based on

![FIG. 2. Plot of noise robustness for white and general noise for two sharp qubit measurements separated by an angle \(\theta\). The line denoted by \(g\) corresponds to the parameter \(\lambda_g\) of Eq. \[14\], whereas lines denoted by \(b\) to the parameter \(\lambda_w\) of Eq. \[14\] for different level of bias, namely, \(b = 0, 0.5, 0.8, 1\) (see main text). The plot shows that the white noise tolerance is always at least double than the general noise tolerance \(\lambda_g\). Moreover, the introduction of biased noise, quantified by the parameter \(b\), with \(b = 0\) corresponding to unbiased white noise, only increases the noise tolerance.](image-url)
Our inequalities are not defined in terms of an optimization for a specific assemblage, but are valid in general. For example, Eq. (11) gives a complete analytical characterization of the simplest steering scenario for any state assemblage.

On the other hand, our result allowed to introduce a new incompatibility monotone based on a steering monotone. This opens a connection to entanglement theory: Similar quantities as the incompatibility monotone have been used to quantify entanglement [30][32]. So, for future work it would be very interesting to use ideas from entanglement theory to characterize the incompatibility of measurements.

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Appendix

Explicit form of Bob’s SE observables for a qubit and tight steering inequality

Given the assemblage \{ρ_{a|x}\}, with \(a = \pm\) and \(x \in \{1, 2\}\), written in terms of Pauli matrices \(\vec{s} = (σ_1, σ_2, σ_3)\) as

\[
ρ_{±|x} = t_x^± \mathbb{1} + \vec{s}_x^± \cdot \vec{σ},
\]

with \(\vec{s}_x^± = (s_{1x}^±, s_{2x}^±, s_{3x}^±)\), the only nontrivial case corresponds to a reduced state \(ρ_B = \sum_{a=±} ρ_{a|x}\) of rank 2, otherwise the total state would be separable.

Since \(ρ_B\) is full rank, we can directly compute first the square root \((ρ_B)^½\) and then its inverse \((ρ_B)^-½\) as a function of \(\vec{s}_x^±\), either via a tedious direct calculation or with the aid of symbolic mathematical computation program.

Then the SE observables for Bob can then be obtained from the equation

\[
B_{±|x} = (ρ_B)^-½ \rho_{±|x} (ρ_B)^-½,
\]

as

\[
B_{+|x} = \frac{1}{2}((1 + \alpha_x) \mathbb{1} + \vec{r}_x \cdot \vec{σ}), \quad B_{-|x} = \mathbb{1} - B_{+|x},
\]

with \(\vec{r}_x = (r_{1x}, r_{2x}, r_{3x})\) and the substitutions

\[
\alpha_x = -1 + (2t_x^+ \rho_0^2 - 4s_{1x}^+ \rho_0 \beta_3 + 2t_x^+ \beta_3^2)/Γ^2, \quad (19)
\]

\[
r_{1x} = (2s_{1x}^+ \rho_0^2 - 4s_{1x}^+ \rho_0 \beta_2 + 2s_{1x}^+ \beta_2^2)/Γ^2, \quad (20)
\]

\[
r_{2x} = 2(s_{1x}^+ \rho_0^2 + 2s_{1x}^+ \rho_0 \beta_1 - s_{1x}^+ \beta_1^2)/Γ^2, \quad (21)
\]

\[
r_{3x} = (2s_{1x}^+ \rho_0^2 + 2t_x^+ \rho_0 \beta_1 - 2s_{1x}^+ \beta_1^2)/Γ^2, \quad (22)
\]

\[
Γ = (β_0^2 - |β|^2), \quad (23)
\]

\[
β = \frac{λ}{8β_0} (s_1^+ + s_2^+), \quad (24)
\]

\[
β_0 = \frac{1}{2} \sqrt{1 - \sqrt{1 - λ^2}}, \quad (25)
\]

\[
λ = |s_1^+ + s_2^+|, \quad (26)
\]

Notice that \(λ\) can be computed both from \(s_1^±\) and \(s_2^±\), it corresponds to the norm of the Bloch vector associated with Bob’s reduced state.

Incompatibility robustness as a semidefinite program

The following construction is almost identical to the one presented in Ref. [10], we discuss it here for completeness. By definition

\[
\mathcal{IR} = \min \left\{ t ≥ 0 \mid \frac{M_{a|x} + tN_{a|x}}{1 + t} := O_{a|x} \text{ are JM}, \right. \}
\]

\[
\{ N_{a|x} \}_a,x \text{ measurement assemblage} \}. \quad (27)
\]

We can then write

\[
N_{a|x} = \frac{(1 + t)O_{a|x} - M_{a|x}}{t} ≥ 0, \quad (28)
\]

where \(≥\) denotes a positive semidefiniteness condition. Eq. (28) is satisfied whenever

\[
(1 + t)O_{a|x} - M_{a|x} ≥ 0, \quad (29)
\]

which can be rewritten, using the joint measurability properties of \(\{O_{a|x}\}_a,x\), i.e., \(O_{a|x} = \sum_\lambda p_M(a|x, \lambda)G_\lambda\) for all \(a, x\), as

\[
(1 + t)\sum_\lambda p_M(a|x, \lambda)G_\lambda ≥ M_{a|x} \forall a, x. \quad (30)
\]

By incorporating the factor \(1 + t\) in the definition of \(G_\lambda\), one can easily see that the value of \(1 + \mathcal{IR}\) can be obtained via the following SDP:

\[
\text{minimize: } \frac{1}{d} \sum_\lambda \text{tr}[G_\lambda]
\]

subject to: \(\sum_\lambda p_M(a|x, \lambda)G_\lambda ≥ M_{a|x} \forall a, x,\)

\[
G_\lambda ≥ 0.
\]

\[
\sum_\lambda G_\lambda = \frac{1}{d} \left( \sum_\lambda \text{tr}[G_\lambda] \right),
\]
where the last equation encode the fact that $G$, up to the correct normalization, must be an observable. In addition, the postprocessing can be chosen, without loss of generality, as the deterministic strategy $p_M(a|x, \lambda) = \delta_{a, \lambda}$, where $\lambda := (\lambda_x)_x$ and $\lambda_x$ is the hidden variable associated with the setting $x$, taking as value the possible outcomes $a$.

It can be easily proven that the program is strictly feasible (e.g., take $G_1 = 1$) and bounded from below, i.e., the optimal value is always larger or equal one.

**Monotonocity of the incompatibility robustness under local channels**

To prove monotonocity of $\mathcal{I}[\mathcal{R}]$ under the action of a quantum channel $\Lambda$ it is sufficient to prove that
\[
\left\{ \frac{M_{a|x} + tN_{a|x}}{1 + t} \right\}_{a,x} \text{ is JM} \\
\Rightarrow \left\{ \Lambda \left( \frac{M_{a|x} + tN_{a|x}}{1 + t} \right) \right\}_{a,x} \text{ is JM}. \tag{32}
\]

Let us denote again $O_{a|x} := (M_{a|x} + tN_{a|x})/(1 + t)$, with $\{O_{a|x}\}_{a,x}$ admitting a joint measurement, i.e., $O_{a|x} = \sum_{\lambda} p_M(a|x, \lambda) G_{\lambda}$. It is sufficient to check that $\{\Lambda(O_{a|x})\}_{a,x}$ again admits a joint measurement $\Lambda(O_{a|x}) = \sum_{\lambda} p_M(a|x, \lambda) \Lambda(G_{\lambda})$. That $\Lambda(G_\lambda)$ is a POVM follows directly the properties of the channel $\Lambda$, since
\[
\Lambda(G_\lambda) \geq 0, \\
\sum_{\lambda} \Lambda(G_\lambda) = \Lambda \left( \sum_{\lambda} G_\lambda \right) = \Lambda(\mathbb{I}) = \mathbb{I}. \tag{33}
\]

Notice that, since we are looking for the transformation of the observables, we use the channel in the Heisenberg picture, hence the fact that the map is trace preserving when acting on states (Schrödinger picture) corresponds to its adjoint (Heisenberg picture) being unital.

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[1] E. Schrödinger, Mathematical Proceedings of the Cambridge Philosophical Society 31, 555 (1935).
[2] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. 98, 140402 (2007).
[3] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, Phys. Rev. Lett. 112, 180404 (2014).
[4] M. F. Pusey, Phys. Rev. A 88, 032313 (2013).
[5] S. Jevtic, M. Pusey, D. Jennings, and T. Rudolph, Phys. Rev. Lett. 113, 020402 (2014).
[6] A. Milne, S. Jevtic, D. Jennings, H. Wiseman, and T. Rudolph, New Journal of Physics 16, 083017 (2014).
[7] I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acín, and G. Adesso, arXiv:1507.04164 (2015).
[8] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, Phys. Rev. A 85, 010301 (2012).
[9] Q. Y. He and M. D. Reid, Phys. Rev. Lett. 111, 250403 (2013).
[10] M. Piani and J. Watrous, Phys. Rev. Lett. 114, 060404 (2015).
[11] T. Moroder, O. Gittsovich, M. Huber, and O. Gühne, Phys. Rev. Lett. 113, 050404 (2014).
[12] T. Vértesi and N. Brunner, Nat. Comm. 5, 5297 (2014).
[13] R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014).
[14] M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014).
[15] M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. 103, 230402 (2009).
[16] R. Gallego and L. Aolita, arXiv:1409.5804 (2013).
[17] P. Busch, Int. J. Theor. Phys. 24, 63 (1985).
[18] P. Busch, Phys. Rev. D 33, 2253 (1986).
[19] P. Busch, Phys. Rev. A 89, 012129 (2014).
[20] P. Busch, T. Heinonen, and P. Lahti, Phys. Reports 452, 155 (2007).
[21] P. Busch, P. Lahti, and R. F. Werner, Phys. Rev. Lett. 111, 160405 (2013).
[22] P. Busch, P. Lahti, and R. F. Werner, Rev. Mod. Phys. 86, 1261 (2014).
[23] S. Ali, C. Carmeli, T. Heinosaari, and A. Toigo, Foundations of Physics 39, 593 (2009).
[24] It is sufficient to notice that $\text{Ker}(A_p) = \cap_{a,x} \text{Ker}(O_{a|x})$ and range $A = \text{Ker}A^\perp$ for any Hermitian operator $A$.
[25] S. Yu, N.-l. Liu, L. Li, and C. H. Oh, Phys. Rev. A 81, 062116 (2010).
[26] S. Yu and C. Oh, arXiv:1312.6470 (2013).
[27] M. F. Pusey, J. Opt. Soc. Am. B 32, A56 (2015).
[28] T. Heinosaari, J. Kinkus, and D. Reitzner, arXiv:1501.04554 (2015).
[29] E. Haapasalo, arXiv:1502.04881 (2015).
[30] G. Vidal and R. Tarrach, Phys. Rev. A 59, 141 (1999).
[31] F. C. S. L. Brandão, Phys. Rev. A 72, 022310 (2005).
[32] D. Cavalcanti, Phys. Rev. A 73, 044302 (2006).