Truthfulness in Repeated Predictions

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ABSTRACT

Proper scoring rules elicit truth-telling when making predictions, or otherwise revealing information. However, when multiple predictions are made of the same event, telling the truth is in general no longer optimal, as agents are motivated to distort early predictions to mislead competitors. We demonstrate this, and then prove a significant exception: In a multi-agent prediction setting where all agent signals belong to a jointly multivariate normal distribution, and signal variances are common knowledge, the (proper) logarithmic scoring rule will elicit truthful predictions from every agent at every prediction, regardless of the number, order and timing of predictions. The result applies in several financial models.

1. INTRODUCTION

Publicly-traded companies release data on earnings in the previous annual or quarterly period, at dates set well in advance. These announcements are major drivers of stock prices, and so receive much attention from investors. Stock market analysts, presenting themselves as experts on the matter, issue public predictions of future earnings announcements. Major companies typically have tens of analysts following them, with each typically issuing several revisions of their estimates in the run-up to the event. Is there an incentive scheme that will motivate all these analysts to always state their true beliefs?

This is part of a wider question, of eliciting opinions about uncertain events, which is crucial for good decision making. Proper scoring rules have been proposed as a mechanism to incentivize an expert to accurately disclose his probability assessment of an uncertain event. The logarithmic scoring rule and quadratic scoring rule are two widely-used examples of proper scoring rules. Proper scoring rules indeed make telling the truth optimal for each prediction in isolation. However, when the expert also considers his reward for future predictions, the literature has described scenarios where bluffing and reticence in revealing information is optimal, and frankness is not. Therefore, proper scoring rules are sometimes described as myopically incentive compatible. Had our earnings-estimating analysts only one shot at predicting estimates, any proper scoring rule would be the answer. As the analysts can and do make revisions to their predictions, the question is open.

In this paper we first confirm with some examples the potential for manipulation and reticence which comes with repeated predictions. The examples seemingly suggest that this potential is inherent in the sequential nature of repeated predictions. However, our main result shows that this, in fact, is not true: By making an eminently reasonable assumption on expert signals, that they belong to a jointly multivariate normal (or log-normal) distribution, the proper logarithmic scoring rule incentivizes all experts to make truthful predictions no matter the prediction schedule. In the Discussion, we build a model for earnings predictions where analysts are motivated to state their true beliefs.

These distributions are of importance in finance. Indeed, geometric Brownian motion is a widely-used model of stock price behavior, including in the Black and Scholes (1973) option pricing model. The varying quantity in such a process belongs to a multivariate log-normal distribution.

1.1 The Prediction Paradigm

The prediction paradigm we consider is one where a public is interested in the value of a future event, and is advised by several experts who make public predictions in the timespan remaining to the event. The predictions take the form of distributions of the event. The public, as well as all experts, draw all possible Bayesian conclusions from public and private information. As a result, the public has a running probability distribution for the event, which may change by an expert making public a new prediction. A proper scoring rule incrementally rewards experts for their predictions, i.e. the public’s distribution is scored when the outcome is known, and each expert is awarded the (positive or negative) difference his prediction made to the public prediction’s score.

Our paradigm is equivalent to a prediction market incentivized by a market scoring rule. In this configuration, due to [Hanson 2003], the market price (actually a price distribution) replaces our public prediction, and traders replace experts. Instead of making public predictions, the traders/experts bet on their private price by changing the market price to their own. The effect is the same as in our paradigm, where a rational market will adopt the latest expert prediction using pure Bayesian inference. An automated market maker rewards the traders/experts in a mech-
anism called LMSR (Logarithmic Market Scoring Rule), which is, in effect, the incremental logarithmic scoring rule. Chen and Pennock (2010) say “LMSR has become the de facto market maker mechanism for prediction markets. It is used by many companies including InKing Markets, Consensus Point, Yahoo!, and Microsoft”.

1.2 Related Literature

Scoring rules have a very long history, going back to De Finetti (1937), Brier (1950) and Good (1952), and are studied in much subsequent work (Sanders (1963), Winkler (1969), Savage (1971), Gneiting and Raftery (2007)). Proper scoring rules are often used for incentive-compatible belief elicitation of risk-neutral agents (Nyarko and Schotter (2002), Armantier and Treich (2013), Offerman et al (2009), Rutström and Wilcox (2009)).

The limitations of proper scoring rules and their “myopic” nature have been discussed, e.g., in Chen et al (2010), Othman and Sandholm (2010) and Chen and Kash (2011).

The multivariate normal distribution has been extensively studied. We use Tong (2012)’s textbook as reference.

1.3 Paper Organization

The rest of this paper is organized as follows: Section 2 specifies the general prediction model and preliminaries. Section 3 demonstrates where truthfulness is not best policy. In Section 4 we prove that truth is optimal for multivariate normal and log-normal distributions. In Section 5 we summarize and offer concluding remarks.

2. PRELIMINARIES

2.1 A GeneralPrediction Model

At a future time $T$, a discrete or continuous random variable $X_T$ will be realized and have a value (the outcome) of $x_T = x$, randomly taken from a known prior distribution $X_0$. There are $m$ sources, $1, 2, …, m$ on the outcome. A source is either nature, or the judgment of an expert. Every source $k \in [m]$ receives a private time-varying signal of the outcome, and based on that signal (alone, ignoring information available from other sources), at every time $t < T$ has a belief distribution $Y_{k,t}$ of the outcome.

There exists a common-knowledge model of how the sources’ signals are stochastically interrelated, so that a rational Bayesian expert who is aware of a set of private signals $A = \{Y_{1,3}, Y_{1,6}, Y_{2,2}, Y_{1,10}\}$ will consistently form a posterior predictive distribution $M(A)$ of the outcome. The inference function $M(\cdot)$’s argument may also have elements that are posterior distributions themselves, e.g. $M(\{Y_{1,1}, M(Y_{1,1}, Y_{2,2})\})$ is the distribution inferred from source 1 announcing a prediction of $Y_{1,1}$ at time 1, and then source 2 announcing his prediction at time 2, which is itself inferred from 2’s private signal and 1’s previous prediction.

The sources publicly announce their belief distribution according to some arbitrary schedule, in which each source makes zero, one, or several pronouncements. The schedule consists of points in time when a source should announce a prediction, or alternatively, is allowed to announce a prediction or pass at its discretion. The schedule is fixed in advance at time 0 (i.e. we do not consider schemes in which

W.l.o.g. we assume time is discrete, integral, and positive a source’s right to make a prediction is affected by his own or others’ prediction decisions and values).

The source announcements influence the belief of the public. The public has no knowledge of its own. Let $A_t$ be the set of predictions announced up to $t$. If source $k$ makes a prediction at $t$, it will either reveal its private signal $Y_{k,t}$ (for nature sources), or (for expert sources) its best estimate of the outcome $M(A_{t-1} \cup \{Y_{k,t}\})$, the distribution inferred from all previously announced data and its own private signal.

Nature sources are always truthful. Expert sources, on the other hand, are motivated to maximize their payoff, and cannot be trusted to report their distributions truthfully. The payoff is provided to the public, who, post factum awards each expert prediction by its increment to a proper scoring rule $S(\Delta_r)$. A scoring rule $S$ rewards a prediction distribution $\Delta$ an amount $S(\Delta_r)$ when the outcome is $r$. A proper scoring rule is one for which $E_r [S(\Delta_r)] \geq E_r [S(\Delta_r)]$ for every distribution $\Delta$.

2.2 Time and Expectation Notation

We use the notation $E[Z]$ to denote the expectation of a random variable $Z$ according to the distribution known at $t$. This is shorthand for $E[Z]_{X_t}$ when referring to the public’s expectation, or for $E[Z]_{X_{k,t}}$ when referring to the an expert’s expectation. Which of the two is meant will either be clear from the context or explicitly stated.

2.3 Prediction Score Expectation

Here we calculate the expected score of normally distributed predictions with the logarithmic scoring rule.

Assume an expert makes a prediction. Let the public prediction prior to the expert prediction be $X_\sim \sim N(\mu_+, \sigma_+^2)$ with density $f_-$ and let the posterior public prediction be $X_+ \sim N(\mu_+, \sigma_+^2)$ with density $f_+$. Let expert’s reward be denoted by $W$, then

$$W = \log \frac{f_+(x)}{f_-(x)}$$

$$= \log \frac{1}{\sqrt{2\pi \sigma_+^2}} e^{-\frac{(x-\mu_+)^2}{2\sigma_+^2}} - \frac{(x-\mu_+)^2}{2\sigma_+^2}$$

$$= \log \frac{\sigma_+}{\sigma_-} + \frac{(x-\mu_+)^2}{2\sigma_+^2} - \frac{(x-\mu_-)^2}{2\sigma_-^2}$$

(1)
As the reward depends on \( x \), its value is known at time \( T \). The expert can calculate his reward expectation when making it, based on his belief about the distribution of \( x \).

Consider the case that the expert prediction is truthful, so that after it, his beliefs of \( x \) are identical to the public's.

**Lemma 1.** If the public’s prediction before an expert prediction at \( t \) is \( X_{t-1} \sim N(\mu_-, \sigma_-^2) \), and after it is \( X_t \sim N(\mu_+, \sigma_+^2) \), then if the prediction is truthful the expert’s expected reward is positive and equals the Kullback-Leibler divergence \( D_{KL}(X_t||X_{t-1}) \).

\[
\mathbb{E}[W] = D_{KL}(X_t||X_{t-1}) = \frac{(\mu_+ - \mu_-)^2}{2\sigma_+^2} + \frac{\sigma_+^2}{2\sigma_-^2} - \frac{\sigma_-^2}{\sigma_+^2} \tag{2}
\]

**Proof.** As the second moment of the normal distribution \( N(\mu, \sigma^2) \) is \( \mu^2 + \sigma^2 \), and since the expert’s distribution translates to

\[
x - \mu_- \sim N(\mu_+ - \mu_-, \sigma_+^2)
\]

\[
x - \mu_+ \sim N(0, \sigma_-^2)
\]

we get by taking expectations from (1)

\[
\mathbb{E}[W] = \mathbb{E}[\mathbb{E}[W|X_{t-1} = x, Y_{t-1} = y, p_1, p_2]] = \mathbb{E}[\log \sigma_+^2 (2\sigma_-^2)\mathbb{E}[\sigma_+^2 - \frac{(\mu_+ - \mu_-)^2}{2\sigma_+^2} - \frac{\sigma_+^2}{\sigma_-^2} - \frac{\sigma_-^2}{\sigma_+^2}]]
\]

This is positive, because for every \( x < 1 \), \( \log(1 - x) \leq -x \.

3. WHEN TRUTH IS NOT INCENTIVE COMPATIBLE

3.1 Binary Predictions

**Example 1.** Let \( x_1 = x \) be the outcome of a binary prediction, with uniform prior. There are \( m = 2 \) expert sources. The schedule is for expert 1 to predict at \( t = 1 \), expert 2 to predict at \( t = 2 \), and then expert 1 predicts again at \( t = 3 \). Predictions are scored with the logarithmic scoring rule \( S(p, r) = \log p r \).

The \( Y_{t,1} \)'s here have Bernoulli distribution. Let \( p_1 = \mathbb{E}[Y_{1,1}], p_2 = \mathbb{E}[Y_{1,2}], q = \mathbb{E}[Y_{2,2}] \), and \( p_2 \) are last predictions, so can be trusted to be truthful, but \( p_1 \) is not. Assume expert 1 announces a Bernoulli distribution with mean \( \hat{p}_1 \).

The inference function \( M(\cdot) \) depends on how the distributions are inter-dependent. Whatever form \( M(\cdot) \) takes, its mean is clearly monotonic in \( p_1, p_2 \) and \( q \), and so it is invertible, and \( \hat{p}_1, q, p_2 \) may be inferred from \( M(A_1), M(A_2), M(A_3) \), respectively.

We can therefore define

\[
m(\hat{p}_1, q) = \mathbb{E}[M(A_1)]
\]

\[
m(p_2, q) = \mathbb{E}[M(A_2)]
\]

\[
m(x, y) \text{ is the mean of a Bernoulli distribution inferred (by some unspecified rule) from expert 1 having mean } p \text{ and expert 2 having mean } q. \text{ Note that } M(A_1) \sim B(\hat{p}_1) \text{ regardless of } M.
\]

Expert 1’s total score for both predictions is

\[
W(\hat{p}_1, p_2, q) = S(M(A_1), x) - S(x_0, x) + S(M(p_2), q, x) - S(M(\hat{p}_1), q, x)
\]

with expectation (under the true outcome distribution)

\[
x \sim B(p_1) \mathbb{E}[W(\hat{p}_1, p_2, q)] = p_1 \log \hat{p}_1 + 1 - p_1 \log(1 - \hat{p}_1) - \frac{1}{1 - m(\hat{p}_1, q)} + p_1 \log m(p_2, q) + (1 - p_1) \log(1 - m(p_2, q)) - p_1 \log m(\hat{p}_1, q) \tag{3}
\]

Differentiating with respect to \( \hat{p}_1 \)

\[
\frac{d}{dp_1} \mathbb{E}[W(\hat{p}_1, p_2, q)] = \frac{p_1 - 1 - p_1}{1 - p_1} - \frac{d}{dp_1} m(p_1, q) \log \frac{p_1 - 1 - m(\hat{p}_1, q)}{1 - m(\hat{p}_1, q)}
\]

For truthfulness to be optimal, we must have \( \frac{d}{dp_1} \mathbb{E}[W(\hat{p}_1, p_2, q)] = 0 \) at \( \hat{p}_1 = p_1 \). Substituting \( \hat{p}_1 = p_1 \) in (3), this occurs only when

\[
p_1 = m(\hat{p}_1, q)
\]

or

\[
\frac{d}{dp_1} m(p_1, q) = 0
\]

In other words, telling the truth is optimal for expert 1 only if the inference function ignores one of the expert signals. This implies that the one of the experts knows everything the other knows. (Formally, his distribution is conditional on the other’s). Clearly, that is a special case, and if it is not the case here, telling the truth is not optimal.

3.2 Normal Distributions

We now show an example with a continuous value prediction, with normally (but not jointly multivariate normal) distributed signals, and common-knowledge variances. Again it does not lead to truth-telling.

**Example 2.** Let \( x_1 = x \) be the outcome of a continuous prediction, with normal prior \( X_0 = N(0, \sigma_0) \). There are \( m = 2 \) expert sources. The schedule is for expert 1 to predict at \( t = 1 \), expert 2 to predict at \( t = 2 \), and then expert 1 predicts again at \( t = 3 \). Predictions are scored with the logarithmic scoring rule \( S(p, r) = \log p r \).

At \( t = 1 \), expert 1’s signal is \( Y_{1,1} \sim N(y_{1,1}, 1) \) with \( y_{1,1} \) randomly taken from a \( N(x, 1) \) distribution.

At \( t = 2 \), expert 2’s signal is \( Y_{2,2} \sim x + (Y_{1,1} - x) W + V_2 \), with \( V_2 \sim N(0, \epsilon) \) for some \( \epsilon > 0 \), and with \( W \) a random variable, \( W \sim 1 \) with probability \( \frac{1}{2} \), and \( W = -1 \) with probability \( \frac{1}{2} \). \( Y_{1,1}, V_2, and \ W \) are pairwise independent.

At \( t = 3 \), expert 1’s signal is \( Y_{1,3} = Y_{1,1} \) (no new information).

The signal set here is \( \{X_0, Y_{1,1}, Y_{2,2}, Y_{1,3}\} \). While it does not have a multivariate normal distribution, \( Y_{2,2} \)’s marginal distribution is normal with mean \( x \) and variance \( 1 + \epsilon \), since \( V_2 \) is zero-mean normal and independent of \( Y_{2,2} - V_2 \), and \( Y_{2,2} - V_2 \) is distributed as \( Y_{1,1} \), because, for every real \( y \)

\[
Pr[Y_{2,2} - V_2 \leq x + y] = \frac{1}{2} Pr[Y_{1,1} \leq x + y] + \frac{1}{2} Pr[Y_{1,1} \geq x - y] = Pr[Y_{1,1} \leq x + y]
\]

Requiring \( \epsilon > 0 \) avoids the certainties in inferences and resulting infinities in scores that \( \epsilon = 0 \) entails.
Assume the strategy of both experts is to be truthful. We show that this strategy is not in equilibrium when the experts are rewarded by the logarithmic scoring rule:

- At $t = 1$, expert 1 announces his prediction of $N(y_{1,1}, 1)$.
- At $t = 2$, expert 2 compares his signal $Y_{2,2}$ with 1’s signal $Y_{1,1}$. Define $\delta := \frac{|y_{1,1} - E[Y_{2,2}]|}{\epsilon}$. He will attribute probability $2 - 2\Phi(\delta)$ to $W = 1$, in which case his posterior distribution is the same as the public’s (and 1’s). He attributes probability $2\Phi(\delta) - 1$ to $W = -1$, in which case his prediction is $\frac{1}{2}(y_{1,1} + E[Y_{2,2}]) = x + \frac{1}{2}V$, with variance $\epsilon/4$. His prediction will be this mixture of these two normal distributions.
- At $t = 3$, expert 1, having no new signal, will repeat the current public prediction.

Analyzing this scenario, we see that expert 1’s expected reward at $t = 1$ is 1.

For the $t = 1$ prediction, $D_{KL}(Y_{1,1} \| X_0)$ by Lemma 1.

2. For the $t = 3$ prediction, zero.

Expert 1’s total reward is therefore fixed, and does not depend on $\epsilon$. We shall compare this to what happens if expert 1 distorts his first prediction by an amount $c^2$.

- Expert 1 can expect to lose from not telling the truth at $t = 1$, but this loss does not depend on $\epsilon$.

At $t = 2$, expert 2 will have reward expectation $\to -\infty$ as $\epsilon \to 0$. This is because expert 2 attributes probability close to 1 that $W = -1$ if $\delta \gg 1$. But with 1’s distortion, this happens with high probability also when $W = 1$. Therefore expert 2 attributes probability $2\Phi(\delta) - 1 \to 1$ as $\epsilon \to 0$ to a prediction with variance $\epsilon/4 \to 0$. Furthermore, his calculated prediction mean $\frac{1}{2}(y_{1,1} + E[Y_{2,2}])$ is, with high probability, wrong, because $y_{1,1}$ is distorted by $c$.

- Predicting a distribution with wrong mean, and with high confidence (vanishingly small variance $\epsilon/4$), will be penalized by the logarithmic scoring rule with unbounded negative scores as $\epsilon \to 0$.

The incremental scoring means that expert 2’s loss will become expert 1’s gain at $t = 3$, when expert 1 reverts to his truthful prediction of $Y_{1,3} = Y_{1,1}$, with unbounded positive score as $\epsilon \to 0$.

Therefore, for a small enough $\epsilon$, or for a large enough distortion $c$, truthfulness is not expert 1’s optimal strategy.

4. WHEN TRUTH IS BEST POLICY

4.1 In the Multivariate Normal Distribution

We now show that for multivariate normal signals, experts will be truthful when compensated by the (incremental) logarithmic scoring rule.

We assume that signal variances are common knowledge. This prevents an expert from lying about his prediction variance, effectively posing as being more or less accurate than he really is. In the context of a normal distribution, this leaves the expert the option of misreporting his mean.

The multivariate normal distribution has been extensively researched. We remind of the following established facts about it, as well as of the univariate normal distribution:

- When a (univariate) normal random variable has two independent observations with known error variances: $\mu_0$ with variance $\sigma_0^2$, and $\mu$ with variance $\sigma^2$, its posterior distribution is

$$N\left(\frac{\mu_0 + \mu}{\sigma_0^2 + \sigma^2}, \frac{1}{\sigma_0^2 + \sigma^2}\right)$$

(4)

- Every linear combination of the components of a multivariate normal distribution is normal (Tong (2012), Ch. 1.1).

- The conditional distribution of a multivariate normal distribution is a multivariate normal distribution (ibid, Ch. 1.1).

- Components of a multivariate normal distribution are independent iff they are uncorrelated (ibid, Ch. 1.1).

- Moreover, the joint distributions of a subset of components $Q$ and a subset of components $R$ are independent iff for every component pair $Q \in Q, R \in R, Q$ and $R$ are uncorrelated (ibid, Theorem 3.3.2).

- In a bivariate normal distribution of $(X_1, X_2)$ with covariance matrix $\begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$, the conditional distribution of $X_1$ given $X_2 = x_2$ is normal with

$$E[X_1|X_2 = x_2] = E[X_1] + \frac{\rho \sigma_1}{\sigma_2} (x_2 - E[X_2])$$

(5)

(ibid, Theorem 2.1.1).

We shall need the following lemma.

**Lemma 2.** If $X, Y$ are components of a multivariate normal distribution, and $Y$ is inferred from $X$, i.e., $Y$ is conditional on a subset of components including $X$, then $Cov(X, Y) = Var Y$.

**Proof.** Clearly, $Y|X = Y$, i.e., conditioning on $X$ again does not change $Y$. Define $Z = \alpha X + (1 - \alpha)Y$. If for any real $\alpha$ there would hold $Cov(Z, Y) = 0$ and hence $Z$ and $Y$ would be independent, then by (3) they would have a posterior distribution different from $Y$. I.e., we would have $E[Y|Z] = E[Y|X] \neq Y$, a contradiction.

In fact, $Cov(Z, Y) = \alpha Cov(X, Y) + (1 - \alpha)V ar Y = 0$ always has a solution for $\alpha$ unless $Cov(X, Y) = Var Y$. The lemma follows.

From the lemma we conclude that, when a public prediction is known, all earlier signals are immaterial in forming a later public prediction. Formally

**Lemma 3.** With signals of a multivariate normal distribution, let $X_{\tau}, X_t$ be the public’s prediction at two points in time $\tau < t$. Then, conditional on $X_{\tau}, X_t$ is independent of every signal prior to $\tau$.

**Proof.** $X_t$ is inferred from $X_{\tau}$, as well as from every signal $Z$ earlier than $\tau$. Therefore by Lemma 2 $Cov(X_{\tau}, X_t) = Cov(X_t, X_{\tau}) = Var X_t$. By the linearity of covariance $Cov(Z - X_{\tau}, X_t) = 0$. By the same reasoning, $Cov(Z - X_t, X_{\tau}) = 0$. Therefore (Tong (2012), Th. 3.3.2) $Z = X_t$ is independent of each of $X_{\tau}, X_{\tau}$ as well as of their joint distribution.
given $X_t$ there is a one-to-one correspondence between $Z$ and $Z_t$,
\[
\Pr[X_t|X_t, Z] = \frac{\Pr[X_t, Z_t, (Z - Z_t)]}{\Pr[X_t|(Z - Z_t)]} = \frac{\Pr[X_t, Z_t]{(Z - Z_t)}\Pr[X_t]}{\Pr[X_t|(Z - Z_t)]}
\]
It follows that $X_t$ is conditionally independent of $Z$ given $X_t$.

We now state the truthfulness result.

**Proposition 4.** Let $X_0$ and $Y_{k,t} \sim N(y_{k,t}, \sigma_{k,t}^2)$ for every $k \in [m], t \in [T]$ all belong to a multivariate normal distribution $N(\mu, \Sigma)$, with all variances $\sigma_{k,t}^2$ common knowledge, and with prediction schedule set in advance. Then, when experts are rewarded for their incremental contributions to a public prediction by the logarithmic scoring rule, each expert’s optimal reward expectation is attained by making truthful predictions.

**Proof.** If the expert makes a single prediction, the result follows from the fact that the logarithmic scoring rule is proper. Therefore, assume more than one prediction, and let us focus on any consecutive pair of predictions.

Assume that an expert makes two consecutive predictions at times $\tau$ and $\tau > \tau$. The timing of the latter prediction at $t$ need not be known at $\tau$. It may depend on the experts’ strategies and signals revealed after $\tau$. Given $S$, the ensemble of experts’ strategies, at $\tau$, $t \in \Delta([\tau - 1])$ is a random variable whose realization is conditioned on signals known after $\tau$.

$X_\tau, X_{t-1}$ are, as before, the public predictions at $\tau$ and $t - 1$, whether the expert was truthful or not at $\tau$. On the assumption that the expert made a truthful prediction at $\tau$, let the public’s prediction be $Z_{\tau} \sim N(\mu_{\tau}, \sigma_{\tau}^2)$, and the public’s prediction at $t - 1$ be $Z_{t-1} \sim N(\mu_{t-1}, \sigma_{t-1}^2)$.

**Lemma 5.** Assume that the expert misrepresents his prediction mean at $\tau$ by an amount $c_\tau$ and that, as a result, the public’s prediction mean at $t - 1$, inferred from the (wrong) assumption that the expert is truthful, is distorted by $c_{t-1}$. Then the expert’s net gain/loss expectation (at $\tau$) from the misrepresentation is

$$\mathbb{E}[(\Delta W)_{\tau}] = \mathbb{E}\left[\frac{c_{\tau - 1}^2}{2\sigma_{\tau - 1}^2} - \frac{c_{\tau}^2}{2\sigma_{\tau}^2}\right]$$

**Proof.** To recap the times $\tau < t < T$ involved in this lemma:

- $\tau$ is when an expert makes a prediction, distorting his mean by an amount $c_\tau$.
- $t$ is when the same expert makes his next prediction. $t - 1$ is the time immediately before it.
- $T$ is when the outcome $x$ becomes known.

As the expert’s prediction variance is common knowledge, it is unaffected by truthfulness. Therefore $\text{Var}X_\tau = \text{Var}Z_{\tau} = \sigma_{\tau}^2$ and $\text{Var}X_{t-1} = \text{Var}Z_{t-1} = \sigma_{t-1}^2$ (the $Z$ distributions resulting from a truthful prediction, while the $X$ distributions resulting from a possibly untruthful one).

It is here that the assumption that prediction schedule is fixed is used. If not, $t$ may conditionally not exist.

Noting that, in general, later public predictions are conditional on (i.e., are inferences of) earlier ones, $X_{t-1}$ is a conditional distribution on $X_{t}$.

By Lemma 2

$$\text{Cov}(X_\tau, X_{t-1}) = \text{Var}X_{t-1} = \sigma_{t-1}^2 \quad (6)$$

The correlation of $X_{t-1}$ and $X_{t}$, is by definition (6)

$$\rho := \rho(X_\tau, X_{t-1}) = \frac{\text{Cov}(X_\tau, X_{t-1})}{\sqrt{\text{Var}X_\tau \text{Var}X_{t-1}}} = \frac{\sigma_{t-1}}{\sigma_{\tau}} \quad (7)$$

Note that as the correlation is at most 1, this implies $\sigma_{t-1} \leq \sigma_{\tau}$.

From (6), distorting one variable’s mean has a proportional distortion on the mean of an inferred variable. The constant of proportionality is given by differentiating (6)

$$\frac{d}{dx_2} \mathbb{E}[X_\tau | X_2 = x_2] = \rho \frac{\sigma_{t-1}}{\sigma_{\tau}} \quad (8)$$

Therefore, from (7)

$$c_{t-1} = \rho \frac{\sigma_{t-1}}{\sigma_{\tau}} c_\tau - \frac{\sigma_{t-1}^2}{\sigma_{\tau}^2} c_\tau \quad (9)$$

As the expert makes a new prediction at $t$, a false prediction at $\tau$ has no effect on the expert’s reward for any prediction made later than $t$ (by Lemma 3 as the prediction at $t$ will become $X_t$). Furthermore, for the two affected rewards at $\tau, t$, only some terms are affected: Referring to (11), a false prediction at time $\tau$ affects only the term $\frac{(x - \mu_{\tau})^2}{2\sigma_{\tau}^2}$ at $\tau$, and only the term $\frac{(x - \mu_{\tau})^2}{2\sigma_{\tau}^2}$ at $t$.

We calculate the expectation $\mathbb{E}[\cdot]$ of the difference the false prediction makes to each of these two affected terms, i.e., the expectation at $\tau$ taken over the expert’s true distribution.

- **Difference to reward at $\tau$:**
  In the affected term $\frac{(x - \mu_{\tau})^2}{2\sigma_{\tau}^2}$, note that $\mu_{\tau} = \mu_{\tau}$ if the expert predicted truthfully, while $\mu_{\tau} = \mu_{\tau} + c_\tau$ if he lied about his mean.

  The reward expectation difference, according to the expert’s distribution at $\tau$, is therefore

  $$\mathbb{E}\left[\frac{(x - \mu_{\tau} + c_\tau)^2}{2\sigma_{\tau}^2} - \frac{(x - \mu_{\tau})^2}{2\sigma_{\tau}^2}\right] = -\frac{c_\tau^2}{2\sigma_{\tau}^2}$$

- **Difference to reward at $t$:**
  In the affected term $\frac{(x - \mu_{\tau})^2}{2\sigma_{\tau}^2}$, note that $\mu_{\tau} = \mu_{t-1}$ if the expert predicted truthfully, while $\mu_{\tau} = \mu_{t-1} + c_{t-1}$ if he lied about his mean.

  The reward expectation difference, according to the expert’s distribution at $\tau$, is therefore

  $$\mathbb{E}\left[\frac{(x - \mu_{t-1} + c_{t-1})^2}{2\sigma_{t-1}^2} - \frac{(x - \mu_{t-1})^2}{2\sigma_{t-1}^2}\right] = \frac{c_{t-1}^2}{2\sigma_{t-1}^2}$$

From (9), $\frac{c_{t-1}^2}{\sigma_{t-1}^2} = \frac{c_\tau^2}{\sigma_{\tau}^2}$ is a non-random constant (given information known at $\tau$). Therefore

$$\mathbb{E}\left[\frac{c_{t-1}(x - \mu_{t-1})}{\sigma_{t-1}^2}\right] = \frac{c_\tau}{\sigma_{\tau}} \mathbb{E}[x - \mu_{t-1}]$$
To evaluate $\mathbb{E}[x - \mu_{t-1}]$, observe that $X_{t-1}$ is a conditional distribution on $X_r$. By the Law of Total Expectation

$$\mathbb{E}[\mathbb{E}[X_{t-1} | X_r]] = \mathbb{E}[X_r]$$

i.e.,

$$\mathbb{E}[\mu_{t-1}] = \mu_r$$

since $\mathbb{E}[x] = \mu_r$, we conclude $\mathbb{E}[x - \mu_{t-1}] = 0$, and we get from (11)

$$\mathbb{E}\left[\frac{(x - \mu_{t-1} - c_{t-1})^2}{2\sigma_{t-1}^2} - \frac{(x - \mu_{t-1})^2}{2\sigma_{t-1}^2}\right] = \mathbb{E}\left[\frac{c_{t-1}^2}{2\sigma_{t-1}^2}\right] \quad (12)$$

Adding (10) and (12) the lemma follows. □

Fix any $t \in \{\tau + 1, T - 1\}$. As $p \leq 1$, by (7) $\sigma_{t-1} \leq \sigma_r$. Then by (10)

$$\frac{\sigma_{t-1}^2}{2\sigma_r^2} - \frac{\sigma_r^2}{2\sigma_r^2} = \frac{\sigma_r^2}{2\sigma_r^2} (\sigma_{t-1}^2 - \sigma_r^2) \leq 0$$

By the lemma

$$\mathbb{E}[\Delta W] = \mathbb{E}\left[\frac{\sigma_r^2}{2\sigma_r^2} \frac{\sigma_{t-1}^2}{2\sigma_{t-1}^2}\right]$$

Since the expression under expectation is non-positive, so is the expectation, i.e., $\mathbb{E}[\Delta W] \leq 0$. We conclude that for every $\tau$, for any ensemble of expert strategies $S$, and independently of any other prediction the expert has made or will make, the reward expectation for distorting a prediction at $\tau$ is not positive. If the prediction schedule is fixed and unaffected by the value of the expert’s prediction, the expert maximizes his multi-prediction benefit expectation by making a truthful prediction at this particular prediction, and therefore at all predictions.

This completes the proof of Proposition 4. □

4.2 In the Multivariate Log-Normal Distribution

If $X$ has a multivariate normal distribution, then $Y = \exp(X)$ has a multivariate log-normal distribution. Truthfulness in this distribution is a consequence of truthfulness in the multivariate normal distribution.

COROLLARY 6. Let $X_0$ and $Y_{k,t} \sim \text{ln} N(y_{k,t}, \sigma_{k,t}^2)$ for every $k \in [m], t \in \{T\}$ all belong to a multivariate log-normal distribution $\text{ln} N(\mu, \Sigma)$, with all variances $\sigma_{k,t}^2$ common knowledge, and with prediction schedule set in advance. Then, when experts are rewarded for their incremental contributions to a public prediction by the logarithmic scoring rule, each expert’s optimal reward expectation is attained by making truthful predictions.

**Proof.** If there is a possibility of lying advantageously about a random variable in a log-normal multivariate distribution $Y$, then there also exists a possibility of advantageously lying about its logarithm, a random variable in the normal multivariate distribution $\text{ln}(Y)$, contradicting Proposition 4. □

5. DISCUSSION

5.1 Conclusions

We have shown that repeated, sequential predictions of a result in general do not lead to truth-telling even when incentivized by a proper scoring rule, including when that rule is the logarithmic scoring rule, and all signal variances are common knowledge.

Nevertheless, when signals belong to a multivariate normal distribution, and their variances are common knowledge, the logarithmic scoring rule elicits truth-telling by all agents whatever the number, order and timing of their predictions. The same is true for signals of a multivariate log-normal distributions.

5.2 A Truthful Model for Earnings Predictions

In the Introduction, we asked whether incentives can be designed to make multiple earning predictions truthful. In the following model, our result answers this question positively.

**Example 3.** Let $x_T = x$ be the actual earnings announced by the company at time $T$, and let there be $m > 1$ sources. Source 1 is nature, and the others expert analysts.

The nature source receives signals $Y_{1,t}$ at every $t \leq T$, which together constitute a Gaussian random walk with standard normal (i.e., $\sim N(0, 1)$) i.i.d. steps $Z_{1,t}$, ending at the outcome $x$. I.e.,

$$Y_{1,t} = x + \sum_{i=1}^{T-1} Z_{1,i}$$

The nature source announces $Y_{1,t}$ every $t \leq T$.

Similarly, every expert $k > 1$ receives signals $Y_{k,t}$ at every $t \leq T$, which together constitute a Gaussian random walk with i.i.d. steps $B_{k,t} \sim N(0, 1-q_k)$, where $q_k \in [0,1]$, ending at the outcome $x$. I.e.,

$$Y_{k,t} = x + \sum_{i=1}^{T-1} B_{k,i}$$

Furthermore for each $k > 1, t < T$, $A_{k,t} := Z_{t} - B_{k,t}$ is independent of $B_{k,t}$, and is normally distributed $N(0, q_k)$.

All random walk steps $Z_{1,t}$ for every $t$, and $B_{k,t}$ for every $k > 1$, are jointly multivariate normal (and therefore so are $A_{k,t}$ and $Y_{k,t}$). The covariance matrix of $(Z_{1}, Z_{2}, \ldots, Z_{m}, B_{m})$ is common knowledge for each $t$.

Experts get a round-robin option of predicting. I.e., for every $k > 1$ expert $k$ may announce a prediction, or pass, at every $t \equiv k - 1 \mod m - 1$.

In the model above, the $B_{k,t}$ represent uncertainty, while the $A_{k,t}$ represent knowledge. I.e., nature “imposes” an uncertainty in the form of a Gaussian random walk with variance 1 increments. The uncertainty gradually dissipates as the event approaches. Each expert “knows” a part of that uncertainty $\sum_{i=t}^{T-1} A_{k,i}$, and remains ignorant of the rest, in the form of a Gaussian random walk with variance $1 - q_k$ increments. $q_k$ therefore measures the “quality” of the expert. (An expert with $q_k = 1$ knows the outcome with certainty, while an expert with $q_k = 0$ has no meaningful signal).

The model may plausibly arise if, for example, the outcome is determined by a set $U$ of hidden variables, with each unknown hidden variable contributing to uncertainty. Every expert $k$ knows the values of $K_k \subseteq U$ and is ignorant of the rest $U_k := U \setminus K_k$. The sets $K_k$ determine expert qualities $q_k$, and the intersections of the $U_k$’s determine covariances.
Clearly the model can be generalized in several directions without disturbing the result: The uniform time that a random walk inherently assumes can be waived to a non-uniform-time martingale. The discrete random walk may be replaced by a continuous Brownian motion process, and so on.

5.3 Open Problems

Our analysis does not lead to a full characterization of when incentive compatibility of truthful predictions is maintained by the logarithmic scoring rule. Nor does it show whether proper scoring rules other than the logarithmic induce truthfulness in significant distribution classes. In the multivariate normal distribution, we did not analyze the case where signal variances are not common knowledge, with the result that they may be misreported.

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