On the upper bound of the maximal absolute projection constant providing the simple proof of Grünbaum conjecture.

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Abstract

Let $\lambda_K(m)$ denote the maximal absolute projection constant over the subspaces of dimension $m$. Apart from the trivial case for $m = 1$, the only known value of $\lambda_K(m)$ is for $m = 2$ and $K = \mathbb{R}$. In 1960, B. Grünbaum conjectured that $\lambda_{\mathbb{R}}(2) = \frac{4}{3}$ and in 2010, B. Chalmers and G. Lewicki proved it. In 2019, G. Basso delivered the alternative proof of this conjecture. Both proofs are quite complicated, and there was a strong belief that providing an exact value for $\lambda_K(m)$ in other cases will be a tough task. In our paper, we present an upper bound of the value $\lambda_K(m)$, which becomes an exact value for the numerous cases. This bound was first stated in [H. König, N. Tomczak-Jaegermann, Norms of minimal projections, J. Funct. Anal. 119/2 (1994), 253–280], but with an erroneous proof. The crucial idea of our proof will be an application of some results from the articles [B. Bukh, C. Cox, Nearly orthogonal vectors and small antipodal spherical codes, Isr. J. Math. 238, 359–388 (2020)] and [G. Basso, Computation of maximal projection constants, J. Funct. Anal. 277/10 (2019), 3560–3585.], for which simplified proofs will be given.

Keywords:
maximal absolute projection constant, maximal relative projection constant, quasimaximal relative projection constant, equiangular tight frames

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1. Introduction

Let \( X \) be a Banach space over \( K \) (where \( K = \mathbb{R} \) or \( K = \mathbb{C} \)) and let \( Y \subset X \) be a finite-dimensional subspace. Let \( \mathcal{P}(X, Y) \) denote the set of all linear and continuous projections from \( X \) onto \( Y \), recalling that an operator \( P : X \to Y \) is called a projection onto \( Y \) if \( P|_Y = \text{Id}_Y \). We define the relative projection constant of \( Y \) by

\[
\lambda_K(Y, X) := \inf \{ \| P \| : P \in \mathcal{P}(X, Y) \}
\]

and the absolute projection constant of \( Y \) by

\[
\lambda_K(Y) := \sup \{ \lambda(Y, X) : Y \subset X \},
\]

(1)

and finally the maximal absolute projection constant, by

\[
\lambda_K(m) := \sup \{ \lambda(Y) : \dim(Y) = m \}.
\]

To calculate this value, it suffices to take the supremum over finite-dimensional \( l_\infty(K) \) superspaces (see e.g. \([13, \text{III.B.5}]\)). Therefore, the latter can be defined as a supremum over maximal relative projection constants for \( N \geq m \)

\[
\lambda_K(m, N) := \sup \{ \lambda(Y, l_\infty^{(n)}(K)) : \dim(Y) = m \text{ and } Y \subset l_\infty^{(n)}(K) \}.
\]

Our estimation of \( \lambda_K(m, N) \) will rely on the following result proved in \([4, \text{combining Theorem 2.2 and Theorem 2.1}]\) (the simplified version of the proof can be found in \([8, \text{Appendix A}]\)).

**Theorem 1.1.** For integers \( n \geq m \), we have

\[
\lambda_K(m, N) = \max \left\{ \sum_{i,j=1}^N t_i t_j |U^*U|_{ij} : t \in \mathbb{R}_+^n, \| t \| = 1, U \in K^{m \times n}, \ UU^* = I_m \right\}.
\]

Since computing this value is difficult, the literature deals with its lower bound called the quasimaximal relative projection constant, which arises when choosing a vector \( t \) with equal coordinates. To be more precise, for \( N \geq m \)

\[
\mu_K(m, N) := \max \left\{ \frac{1}{N} \sum_{i,j=1}^N |U^*U|_{ij} : U \in K^{m \times N}, \ UU^* = I_m \right\}.
\]

(2)
Analogously as for the maximal relative projection constant, we define the *quasi-maximal absolute projection constant* by

\[ \mu_K(m) = \sup \{ \mu_K(m, N) : N \geq m \}. \quad (3) \]

In fact, it appears that \( \mu_K(m) \) is an equivalent definition of \( \lambda_K(m) \). In the real case, it was proved by Basso [1, Proof of Theorem 1.2]. In our paper, we give an elementary proof of this fact, which is also valid in the complex case (Theorem 2.2). The best known bound for the maximal relative projection was given in [11]. We present it in the form stated in [8, Theorem 5], where also an easier proof of this result was provided.

**Theorem 1.2.** For integers \( n \geq m \), the maximal relative projection constant \( \lambda_K(m, n) \) is upper bounded by

\[ \delta_{m,n} := \frac{m}{n} \left( 1 + \sqrt{\frac{(n-1)(n-m)}{m}} \right). \]

Moreover, the following properties are equivalent:

1. There is an equiangular tight frame consisting of \( n \) vectors in \( \mathbb{K}^m \).
2. \( \mu_K(m, n) = \frac{m}{n} \left( 1 + \sqrt{\frac{(n-1)(n-m)}{m}} \right) \).
3. \( \lambda_K(m, n) = \frac{m}{n} \left( 1 + \sqrt{\frac{(n-1)(n-m)}{m}} \right) \).

Recall that a system of vectors \( (u_1, \ldots, u_N) \) in \( \mathbb{K}^m \) is called a *tight frame* if there exists a constant \( \alpha > 0 \) such that one of the following equivalent conditions holds:

- \( \|x\|^2 = \alpha \sum_{k=1}^{N} |\langle x, u_k \rangle|^2 \) for all \( x \in \mathbb{K}^m \).
- \( x = \alpha \sum_{k=1}^{N} \langle x, u_k \rangle u_k \) for all \( x \in \mathbb{K}^m \).
- \( UU^* = \frac{1}{\alpha} I_m \), where \( U \) is the matrix with columns \( u_1, \ldots, u_N \).

The system \( (u_1, \ldots, u_n) \) of unit vectors is called an *equiangular tight frame* ETF\((m, N)\) if it is tight and the value of \(|\langle u_i, u_j \rangle|\) is constant over all \( i \neq j \).

It is well known (see e.g. [7, Theorem 5.7]) that if \( u_1, \ldots, u_N \in \mathbb{K}^N \) is an ETF then

\[ \varphi_{m,n} := |\langle u_i, u_j \rangle| = \sqrt{\frac{n-m}{m(n-1)}} \quad \text{for all} \ i, j \in \{1, \ldots, N\}, \ i \neq j, \quad (4) \]
and the cardinality $N$ cannot exceed $\frac{m(m+1)}{2}$ in the real case and $m^2$ in the complex case (see e.g. [7, Theorem 5.10]). An ETF that realizes this upper bound is called the maximal ETF.

2. Main Result.

We start with the theorem, which is a special case of [3, Lemma 5] that was stated for isotropic measures. In our case, the proof given by Bukh and Cox can be streamlined.

**Theorem 2.1.** Let $1 < m \leq N$. Then the following inequalities holds

$$\mu_R(m, N) \leq \delta_{m, \frac{m(m+1)}{2}} = \frac{2}{m+1} \left(1 + \frac{m-1}{2}\sqrt{m+2}\right)$$

$$\mu_C(m, N) \leq \delta_{m,m^2} = \frac{1}{m} \left(1 + (m-1)\sqrt{m+1}\right)$$

**Proof.** Let $U \in \mathbb{K}^{m \times N}$ be such that $UU^* = I_m$. Denote by $u_i$ the i-th column of the matrix $U$. Observe that for matrix $\tilde{U} \in \mathbb{K}^{m \times \tilde{N}}$ created from only nonzero columns (since $u_1, \ldots, u_N$ form a tight frame, they span $\mathbb{K}^m$ and $\tilde{N} \geq m$) we have

$$\frac{1}{N} \sum_{i,j = 1}^{N} |U^* U|_{ij} \leq \frac{1}{\tilde{N}} \sum_{i,j = 1}^{\tilde{N}} |\tilde{U}^* \tilde{U}|_{ij} \quad \text{and} \quad \tilde{U} \tilde{U}^* = I_m .$$

So, without loss of generality, we can assume that all $u_i$ are nonzero vectors. Now, we are ready to define the operators on $\mathbb{K}^m$ associated with the vectors $u_i$ by

$$L_i(v) = \|u_i\|^{-\frac{3}{2}} \langle v, u_i \rangle u_i . \quad (5)$$

Denote by $G$ the Gram matrix of the system $(L_1, \ldots, L_N)$ with respect to the Frobenius inner product. Then,

$$G_{ij} = (L_i, L_j)_F = \text{tr}(L_i L_j^*) = \text{tr}(L_i L_j) = \sum_{k=1}^{m} \langle L_i L_j(e_k), e_k \rangle$$

$$= \sum_{k=1}^{m} \langle L_j(e_k), L_i(e_k) \rangle = \|u_i\|^{-\frac{3}{2}} \|u_j\|^{-\frac{3}{2}} \sum_{k=1}^{m} \langle e_k, u_j \rangle \langle e_k, u_i \rangle \langle u_j, u_i \rangle$$

$$= \|u_i\|^{-\frac{3}{2}} \|u_j\|^{-\frac{3}{2}} \left( \sum_{k=1}^{m} \langle u_i, e_k \rangle e_k, u_j \right) \langle u_j, u_i \rangle = \|u_i\|^{-\frac{3}{2}} \|u_j\|^{-\frac{3}{2}} \|\langle u_i, u_j \rangle\|^2$$

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Now observe that for all \( i \in \{1, \ldots, N\} \), since \((u_1, \ldots, u_N)\) is a tight frame with the constant \( \alpha \) equal to 1

\[
\|u_i\|^{1/2} = \|u_i\|^2 \cdot \|u_i\|^2 = \sum_{k=1}^{N} \|u_i\|^2 \cdot \|u_k\|^2 = \sum_{k=1}^{N} \|u_k\|^2 G_{ik}.
\]

Let \( g_k \) denote the \( k \)-th column of the matrix \( G \), then for fixed \( i \in \{1, \ldots, N\} \)

\[
\sum_{k=1}^{N} \|u_i\|^2 \|u_k\|^2 g_k = \sum_{k=1}^{N} \sum_{l=1}^{N} \|u_i\|^2 \|u_k\|^2 \frac{1}{2} G_{il} [G_{1k}, \ldots, G_{Nk}]^T
\]

\[
= \sum_{l=1}^{N} \|u_l\|^2 g_l [\sum_{k=1}^{N} \|u_k\|^2 G_{1k}, \ldots, \sum_{k=1}^{N} \|u_k\|^2 G_{Nk}]^T
\]

\[
= \|u_i\|^2 \|u_1\|^2, \ldots, \|u_i\|^2 \|u_N\|^2)^T.
\]

Therefore, for any constants \( a, b \in \mathbb{R} \) the matrix \( A \) with the entries given by \( A_{ij} := aG_{ij} - b\|u_i\|^2 \|u_j\|^2 \) has the rank less than or equal to \( \text{rk}(G) \). Let \( \varphi > 0 \).

Then combining the Cauchy-Schwarz inequality and \( \text{tr}(AA^*) \geq \frac{(\text{tr}(A))^2}{\text{rk}(A)} \) (see e.g. [2, Fact 7.12.13]) we get

\[
\sum_{i,j=1}^{N} \frac{(|\langle u_i, u_j \rangle - \varphi \|u_i\|^2 \|u_j\|^2|)^2}{\|u_i\|^2 \|u_j\|^2} = \sum_{i,j=1}^{N} \frac{(|\langle u_i, u_j \rangle|^2 - \varphi^2 \|u_i\|^2 \|u_j\|^2)^2}{\|u_i\|^2 \|u_j\|^2 (|\langle u_i, u_j \rangle| + \varphi \|u_i\|^2 \|u_j\|^2)^2}
\]

\[
\geq \sum_{i,j=1}^{N} \frac{(|\langle u_i, u_j \rangle|^2 - \varphi^2 \|u_i\|^2 \|u_j\|^2)^2}{(1 + \varphi^2 \|u_i\|^2 \|u_j\|^2)^2} = \sum_{i,j=1}^{N} \left( \frac{G_{ij}}{1 + \varphi^2 \|u_i\|^2 \|u_j\|^2} \right)^2
\]

\[
\geq \frac{1}{\text{rk}(G)} \left( 1 - \varphi \sum_{i=1}^{N} \|u_i\|^2 \right)^2.
\]

Squaring the addends of the left-sided sum and rearranging the latter inequality gives

\[
2\varphi \sum_{i=1}^{N} |U^* U|_{ij} \leq \sum_{i,j=1}^{N} \frac{|\langle u_i, u_j \rangle|^2}{\|u_i\|^2 \|u_j\|^2} + \left( \varphi^2 - \frac{(1 - \varphi)^2}{\text{rk}(G)} \right) \left( \sum_{i=1}^{N} \|u_i\|^2 \right)^2.
\]

The rank of the Gram matrix of the system of vectors is equal to the dimension of the space spanned by these vectors. Now, we have to consider the real and complex
cases separately. When \( K = \mathbb{R} \), then the operators \( L_1, \ldots, L_N \) live in the space of symmetric operators, whose dimension is equal to \( m(m+1)/2 \). Therefore, \( \text{rk}(G) \leq m(m+1)/2 \). Setting \( \varphi = \frac{1}{\sqrt{m+2}} \) (notice that if ETF(\( m, m(m+1)/2 \)) exists \( \varphi = \varphi_{m, m(m+1)/2} \)), (6) reads

\[
\frac{2}{\sqrt{m+2}} \sum_{i=1}^{N} |U^*U|_{ij} \leq \sum_{i,j=1}^{N} \frac{|\langle u_i, u_j \rangle|^2}{\|u_i\| \|u_j\|} + \frac{(m-3)\sqrt{m+2} + 4}{m(m+1)\sqrt{m+2}} \left( \sum_{i=1}^{N} \|u_i\| \right)^2
\]

Using the Cauchy - Schwarz inequality and the tightness of the vectors \( u_1, \ldots, u_N \), we obtain

\[
\sum_{i,j=1}^{N} \frac{|\langle u_i, u_j \rangle|^2}{\|u_i\| \|u_j\|} \leq \sqrt{\sum_{i,j=1}^{N} \frac{|\langle u_i, u_j \rangle|^2}{\|u_i\|^2} \sum_{i,j=1}^{N} \frac{|\langle u_i, u_j \rangle|^2}{\|u_j\|^2}} = \sum_{i=1}^{N} \frac{1}{\|u_i\|^2} \sum_{j=1}^{N} |\langle u_i, u_j \rangle|^2
\]

and

\[
\left( \sum_{i=1}^{N} \|u_i\|^2 \right)^2 = \langle [1, \ldots, 1], [\|u_1\|, \ldots, \|u_N\|] \rangle^2 \leq N \sum_{i=1}^{N} \|u_i\|^2 = N \text{tr}(UU^*)
\]

\[
= N \text{tr}(U^*U) = N m.
\]

Combining all these inequalities, we derive

\[
\frac{1}{N} \sum_{i=1}^{N} |U^*U|_{ij} \leq \frac{2}{m+1} \left( 1 + \frac{m-1}{2} \sqrt{m+2} \right),
\]

which after taking the maximum over all \( U \) leads to the announced upper bound of \( \mu_G(m, N) \). As the set of Hermitian operators is not a vector space, in the complex case we only know that \( \text{rk}(G) \leq m^2 \). Fix \( \varphi = \frac{1}{\sqrt{m+1}} \). If ETF(\( m, m^2 \)) exists, then \( \varphi = \varphi_{m^2} \). Now, inequality (6) can be expressed as

\[
\frac{2}{\sqrt{m+1}} \sum_{i=1}^{N} |U^*U|_{ij} \leq \sum_{i,j=1}^{N} \frac{|\langle u_i, u_j \rangle|^2}{\|u_i\| \|u_j\|} + \frac{(m-2)\sqrt{m+1} + 2}{m^2 \sqrt{m+1}} \left( \sum_{i=1}^{N} \|u_i\| \right)^2.
\]

Adapting the reasoning of the real case, we arrive at

\[
\mu_C(m, N) \leq \delta_{m^2} = \frac{1}{m} \left( 1 + (m-1)\sqrt{m+1} \right)
\]
as desired.

Now to establish the upper bound for $\lambda_K(m)$ it is sufficient to prove the equality of the maximal and quasimaximal absolute projection constants. In the real case, Basso proved it in [1, Proof of Theorem 1.2]. We propose an alternative argument, which is also valid in the complex case.

**Theorem 2.2.** Let $m \geq 1$, then

$$\lambda_K(m) = \mu_K(m).$$  \hspace{1cm} (8)

**Proof.** Since for every $N > m$ we have $\mu_k(m, N) \leq \lambda_k(m, N)$, it is enough to show that $\lambda_k(m, N) \leq \mu_k(m)$. We choose $U_0 \in \mathbb{R}^{m \times N}$ and $t^0 \in \mathbb{R}_+^n$ that realize $\lambda_k(m, N)$. Let $S_N^+ := \{t \in \mathbb{R}_+^N : \|t\| = 1\}$. Next we define the function

$$f : S_N^+ \ni t \mapsto \sum_{i,j=1}^{N} t_i t_j |U_0^* U_0|_{ij}.$$  

Since $f$ is a continuous function, for every $\varepsilon > 0$ there exists $t^\varepsilon \in S_N^+ \cap \mathbb{Q}^N$ such that $\lambda_k(m, N) = f(t^0) \leq f(t^\varepsilon) + \varepsilon$. Taking the common denominator, we know that there exists $q, n_1, \ldots, n_N \in \mathbb{N}$ such that $t^\varepsilon = \frac{1}{q}(n_1, \ldots, n_N)$. Since $\|t^\varepsilon\| = 1$, $q = \sqrt{n_1^2 + \cdots + n_N^2}$. Let $u_i$ denote the $i$-th column of the matrix $U_0$. Now we consider the matrix $U_\varepsilon \in \mathbb{R}^{m \times q^2}$ defined, in block notation, by

$$U_\varepsilon := \left[ \begin{array}{c|c|c} \frac{1}{n_1} u_1 \mathbb{1}^\top_{n_1} & \cdots & \frac{1}{n_N} u_N \mathbb{1}^\top_{n_N} \end{array} \right],$$

where $\mathbb{1}_k$ denotes the $k$-dimensional vector with all entries equal to 1. Notice that

$$U_\varepsilon U_\varepsilon^* = \frac{1}{n_1^2} u_1^* \mathbb{1}_{n_1^2} \mathbb{1}_{n_1^2} u_1^* + \cdots + \frac{1}{n_N^2} u_N^* \mathbb{1}_{n_N^2} \mathbb{1}_{n_N^2} u_N^* = u_1 u_1^* + \cdots + u_N u_N^*$$

$$= U_0 U_0^* = I_m.$$  

In turn,

$$\frac{1}{q^2} \sum_{i,j=1}^{q^2} |U_\varepsilon^* U_\varepsilon|_{ij} = \frac{1}{q^2} \sum_{i,j=1}^{N} \left| \frac{1}{n_i} u_i, \frac{1}{n_j} u_j \right| n_i^2 n_j^2 = \sum_{i,j=1}^{N} |\langle u_i, u_j \rangle| \frac{n_i n_j}{q^2} = f(t^\varepsilon).$$

Putting everything together yields

$$\mu_k(m) \geq \mu_k(m, q^2) \geq \lambda_k(m, N) - \varepsilon.$$
Letting $\varepsilon$ to 0 we establish the desired result.

In view of Theorem 1.2 and Theorem 2.1, the latter leads to an immediate corollary.

**Theorem 2.3.** Let $m > 1$ then

i) $\lambda_R(m) \leq \delta_{m, m(m+1)/2} = \frac{2}{m+1} \left(1 + \frac{m-1}{2} \sqrt{m+2}\right)$

ii) $\lambda_C(m) \leq \delta_{m, m^2} = \frac{1}{m} \left(1 + (m-1) \sqrt{m+1}\right)$.

Furthermore, the inequality becomes an equality if there exists a maximal ETF in $\mathbb{K}^m$.

The above was first stated in [10], with the proof based on an erroneous lemma, as pointed out in [4]. In the real case, maximal ETFs seem to be rare objects. The only known cases are for $m$ equal to 2, 3, 7 and 23. Applying the Theorem 2.3 we get

**Theorem 2.4.**

- $\lambda_R(2) = \frac{4}{3}$. It is known as the Grünbaum conjecture, which was stated in [9]. The first complete proof was given in [3]. Recently, a new proof was presented in [4];
- $\lambda_R(3) = \frac{1 + \sqrt{5}}{2}$;
- $\lambda_R(7) = \frac{5}{2}$;
- $\lambda_R(23) = \frac{14}{3}$.

Unlike the real case, numerous examples of complex maximal ETF are known. For example, for $m \in \{1, \ldots, 17, 19, 24, 28, 35, 48\}$ (see, e.g. [6]). Hence, we have the following.

**Theorem 2.5.** For $m \in \{1, \ldots, 17, 19, 24, 28, 35, 48\}$ we have

$$\lambda_C(m) = \frac{1}{m} \left(1 + (m-1) \sqrt{m+1}\right).$$

In fact, it is conjectured that there is a complex maximal ETF in every dimension (Zauner’s conjecture [12]), which allows us to state the following conjecture.

**Conjecture 2.1.** For every $m \geq 1$

$$\lambda_C(m) = \frac{1}{m} \left(1 + (m-1) \sqrt{m+1}\right).$$
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