DYNAMICS OF AXISYMMETRIC TRUNCATED DYNAMO MODELS

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Summary: An important question regarding the study of mean field dynamo models is how to make precise the nature of their underlying dynamics. This is difficult both because relatively little is known about the dynamical behaviour of infinite dimensional systems and also due to the numerical cost of studying the related partial differential equations.

As a first step towards their understanding, it is useful to consider the corresponding truncated models. Here we summarise some recent results of the study of a class of truncated axisymmetric mean field dynamo models. We find conclusive evidence in these models for various types of intermittency as well as multiple attractors and final state sensitivity.

We also find that the understanding of the underlying dynamics of such dynamo models requires the study of a new class of dynamical systems, referred to as the non-normal systems. Current work demonstrates that these types of systems are capable of a novel type of intermittency and also of relevance for the understanding of the full axisymmetric PDE dynamo models.

Key words: Axisymmetric mean field dynamos, intermittency, dynamical systems

1. INTRODUCTION

There is proxy evidence suggesting variability in the solar magnetic field over intermediate time scales of order of $10^2$ years (Eddy, 1976). This type of variability is also thought to be shared by solar type stars (Baliunas et al., 1995). Now given that there are no natural mechanisms with similar time scales operating in the Sun and solar type stars (Gough, 1990) the question arises as to the possible mechanisms responsible for such variabilities.

There is some observational evidence for the presence of non-linear phenomena in stellar and solar magnetic activity. As a result one of the explanations put forward over the last two decades is that variability on these intermediate time scales may be a natural outcome of the nonlinear regimes operative in such stars (Tavakol, 1978; Zeldovich et al., 1983; Weiss et al., 1984; Spiegel et al., 1993).

A great deal of effort has subsequently gone into the study of magnetohydrodynamical dynamos operating in the stellar interiors which are thought to give rise to such variability. The equations modelling these dynamo regimes are nonlinear partial differential equations (PDE). A number of approaches have been employed to study the behaviour of such systems, including numerical studies of these equations (Brandenburg et al., 1989a, 1989b; Brandenburg et al., 1990; Brandenburg et al., 1995), as well as their finite dimensional truncations (Zeldovich et al., 1983; Weiss et al., 1984; Spiegel et al., 1993). The difficulties with the former models are twofold: firstly the limits imposed by the numerical cost of their integration and
secondly the fact that such numerical results do not immediately make the dynamical mechanisms underlying them transparent. Such understanding is crucial if one hopes to construct precise statistical measures in order to make comparisons with observations.

Now given that dynamical systems theory is well developed for finite dimensional flows, the latter (despite their approximate nature and hence their limited direct physical applicability) are of potential value in making precise the underlying dynamics of such PDE models.

Here we summarise some recent results (Covas et al., 1997a, b, c, d) which employ such truncations of the mean field dynamo equations.

2. MODEL

Our starting point is the mean field equation describing the evolution of the mean magnetic field $B$ in the form

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B + \alpha B - \eta_t \nabla \times B),$$

(1)

where $u$ is the mean velocity and $\eta_t$ is the turbulent magnetic diffusivity. The $\alpha$–effect arises from the correlation of small scale turbulent velocities and magnetic fields and is important in maintaining the dynamo action by relating the mean electrical current arising in helical turbulence to the mean magnetic field (Krause and Rädler, 1980).

In our studies we have considered the dynamical case where $\alpha$ can be divided into a hydrodynamic ($\alpha_h$) and a magnetic part ($\alpha_m$), where the magnetic part satisfies an explicitly time dependent diffusion type equation with a nonlinear forcing, where the nonlinearity is quadratic in the magnetic field, given by Zeldovich et al. (1983) and Kleeorin and Ruzmaikin (1982) (see Covas et al., 1997a, b; 1998a for details).

To get an understanding of the underlying dynamics of these models we looked, using a spectral expansion, at the truncations of these equations, which in a symbolic form can be written as

$$\frac{dx}{dt} = F(x, D, \nu)$$

(2)

where $x \in \mathbb{R}^{T_d}$, $T_d$ is the truncation dimension, $D$ is the dynamo number, $\nu$ is the ratio of the turbulent diffusivity and the turbulent magnetic diffusivity and $F$ is a differentiable function which depends on the nature of the equations and $T_d$.

A great deal of effort has gone into the study of the truncated dynamo models (see for example Zeldovich et al., 1983; Weiss et al., 1984; Spiegel et al., 1993). Our aim has been to take a systematic look at such truncated models firstly in order to find out explicitly what types of complicated nonlinear modes of behaviour can occur in these models and secondly to find the general features of the appropriate theoretical framework necessary for their study which we hope will in turn throw some light on the the study of the full PDE dynamo models.
In the following section we summarise recent work from Covas et al. (1997a,b; 1998a). In particular we concentrate on two types of nonlinear phenomena we have observed in such truncations, which could in principle have potential relevance for observed behaviour of solar type stars: various types of intermittency and final state sensitivity.

3. RESULTS

Our studies of the above truncated models involved a range of truncation levels $T_d$. We concentrated on two families of truncated models which had the least number of dimensions for which a similar asymptotic behaviour existed (see Covas et al., 1997a):

1. Model I: a six dimensional ($T_d = 6$) truncated model of type (2) which is antisymmetric with respect to the equator and
2. Model II: a twelve dimensional ($T_d = 12$) truncated model of type (2) which has the Model I as its antisymmetric invariant subspace.

The following is a brief discussion of results.

3.1 Various forms of intermittency

**Crisis-induced Intermittency.** Consider a system of type (2) and let $D = D_c$ be a critical dynamo number such that for $D < D_c$ the system possesses two chaotic attractors. If as $D$ is increased the attractors enlarge such that at $D = D_c$ they simultaneously touch the boundary separating their basins, then for $D$ slightly greater than $D_c$ a typical orbit will spend long periods of time in each of the regions where the two previous attractors existed and intermittently switch between them. This form of dynamical behaviour is referred to as crisis-induced intermittency (Grebogi et al., 1987).

We have found the presence of such a behaviour in the Model I with $D_c = 204.2796$ and $\nu = 0.4$. In these regimes, the distribution of the average time of reversals scales as a power law with $(D - D_c)$. We have been able to substantiate this scaling in this model (Covas and Tavakol, 1997).

**Type I Intermittency.** This type of behaviour is produced in the neighbourhood of parameter values where a periodic orbit is destroyed by collision with an unstable orbit in a saddle node bifurcation. We have found the presence of such a behaviour in the Model II at $D = 170.0$ and $\nu = 0.5$. A time plot of the energy demonstrating this type of behaviour is depicted in Figure 1.

**On-Off Intermittency.** An important feature of systems with symmetry (as in the case of solar and stellar dynamos) is the presence of invariant submanifolds. It may happen that attractors in such invariant submanifolds may become unstable in transverse directions. When this happens, one possible outcome could be that the trajectories can come arbitrarily close to this submanifold but also have intermittent large deviations from it. This form of intermittency is referred as on-off intermittency (Platt et al., 1993). Recent work has shown that this is in fact a generalised...
form of on–off intermittency which we refer to as in–out intermittency (Ashwin et al., 1998).

We have found the presence of such a behaviour in the Model II at \( D = 177.70 \) and \( \nu = 0.47 \) (Covas et al., 1997d).

3.2 Final state sensitivity

An important feature of nonlinear systems is that they can possess multiple attractors at same parameter values. In that case, the final outcome would depend upon the precise values of the initial conditions. It may turn out that the basins of attraction could have a fractal or riddled nature, in which case very small changes in the initial conditions can change the attractor, hence changing the final outcome of the dynamics. In that case the system is said to have final state sensitivity (Alexander et al., 1992; Lai and Grebogi 1996; Lai et al., 1996; Ott et al., 1994).

We have found the presence of multiple attractors and fractal basins for the Model I at \( D = 204.2327 \) and \( \nu = 0.5 \). This is depicted in Figure 2. Since it is not possible
to determine the initial conditions precisely, this can lead to parameter regions where only probabilistic statements may be made regarding the behaviour of such stars.

4. DISCUSSION

As a step towards understanding the underlying the dynamics of solar and stellar variability and their corresponding PDE models, we have studied the detailed dynamical behaviour using truncated axisymmetric mean field dynamo models. We have found conclusive evidence for the occurrence of a number of novel types of behaviour which could be of potential importance for the full dynamo models. These are

1. Various types of intermittency. If repeated in real dynamos, these types of behaviour could be of significance in understanding the Maunder type Minima. We
note that similar types of behaviour have also been observed in numerical solutions of mean field dynamo models (Tworkowski et al., 1998; Covas et al., 1998a,b; Brooke, 1997; Brooke et al., 1998).

2. Final state sensitivity. The presence of such sensitivity in real stars could be of significance in that, for example, it could potentially lead to stars of same spectral type, rotational period, age and compositions showing different modes of dynamical behaviour. There is some evidence for similar types of behaviour in PDE models (Tavakol et al., 1995; Covas et al., 1998a).

3. Our studies show that an important feature of dynamo models and their truncations is that their parameters are non-normal, i.e. they vary the dynamics within the invariant subspace as well as outside it. In fact we expect such non-normality to be a generic property of parameters in general truncations of physical systems. A framework for understanding the dynamics of such systems is developed elsewhere (Covas et al., 1997; Ashwin et al., 1998) and a detailed study of the relationship between these phenomena and the PDE models is to appear elsewhere (Covas et al., 1998b).

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