Polynomial Realization of $s\ell_q(2)$ and Fusion Rules at Exceptional Values of $q$

D. Karakhanyan$^a$ & Sh. Khachatryan$^a$

$^a$ Yerevan Physics Institute, Br.Alikhanian st.2, 375036, Yerevan, Armenia.

Abstract

Representations of the $s\ell_q(2)$ algebra are constructed in the space of polynomials of real (complex) variable for $q^N = 1$. The spin addition rule based on eigenvalues of Casimir operator is illustrated on few simplest cases and conjecture for general case is formulated.

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$^1$e-mail: karakhan@lx2.yerphi.am
$^2$e-mail: shah@moon.yerphi.am
1 Introduction

The quantum groups were invented \(1\) by L.D.Faddeev and the Leningrad school on inverse scattering method in order to solve integrable models. Quantum groups have links with mathematical fields such as Lie groups, algebras and their representations, special functions, knot theory operator algebras, non-commutative geometry and many others and have a lot of interrelations with physics: quantum inverse scattering method, theory of integrable systems, conformal and quantum field theory, etc \(4, 5, 8, 3, 7\). It is expected that quantum groups will lead to a deeper understanding of the concept of symmetry in physics. The quantum symmetry bears close similarity with non-deformed classical symmetries, especially in field of representation theory. However that similarity occurs only at general values of deformation parameter and ends for so called exceptional values of \(q (q^N = 1)\).

Degeneracy in the case when \(q\) is given by a root of unity is accompanied by enlarging of the center of symmetry group and changing the structure of representation space of theory. Physically it expressed in the fact that XX model has more wide symmetry than XXZ Heisenberg model.

The new features in representation theory which appear under quantum deformation of Lie groups with parameter \(q (q^N = 1)\) were studied by V. Pasquier and H. Saleur \(2\) who introduced the notion of indecomposable representation, by D. Arnaudon \(4\) who classified all representations of quantum groups by types \(A\) and \(B\) and some other authors \(5\).

In this article we propose an explicit realization of representations in the space of polynomials, which is very useful in practical calculations. This approach is especially fruitful in the context of Universal R-matrix \(6\). Such approach allows to give an explicit operator realization of the Universal R-matrix in the space of polynomials. Based on this approach it is possible to give a heuristic illustration of basic regulations appearing in case \(q^N = 1\).

2 \(s\ell_q(2)\) algebra and co-product

The representation of algebra generators as linear function of derivatives acting on polynomials of a certain number of variables is known in mathematical literature \(9\). Physicists are more familiar, however, with the matrix representation and it seems more useful to explain the relation between them.

The simplest differential realization of the Lie algebras is built on the homogeneous polynomials. The case of \(s\ell(2)\) algebra is the most convenient for pedagogical purposes. The fundamental representation is given by Pauli matrices:

\[
S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

corresponding doublet representation is

\[
R_2 = \{ |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \},
\]

The Hermitean conjugation has to be introduced in order to formulate a non-trivial orthogonality condition for these elements. We consistently work only with the ket-states avoiding introduction of the Hermitean conjugation and bra-states. Since we would like to construct a realization on
polynomials, we introduce two independent variables, $x$ and $y$ and identify them with elements of the doublet representation,

$$|↑⟩ = y, \quad |↓⟩ = x.$$  

(3)

Then (1) and (2) imply

$$S^+ x = y, \quad S^+ y = 0$$

$$S^+ y = 0, \quad S^- y = x,$$

$$S^+ x = -\frac{1}{2} x, \quad S^+ y = \frac{1}{2} y.$$  

(4)

These relations immediately allow to realize the generator $s$ in the differential form:

$$S^+ = y \partial_x, \quad S^- = x \partial_y, \quad S^z = \frac{1}{2}(y \partial_y - x \partial_x).$$  

(5)

Now one can construct all higher representations of the type:

$$R_j = (R^{(2)} \times R^{(2)} \times \ldots \times R^{(2)})_{symmetrized}.$$  

(6)

This is very important general feature of all differential realizations with the group generators which are linear in derivatives: if realization is found for some representation, it can be applied to the product. Indeed, for any representation $R_j = R_1 \times R_2 \times \ldots$ the action of generators $S^a$, by definition, must reduce to:

$$S^a R_j = (S^a R_1) \times R_2 \times \ldots + R_1 \times (S^a R_2) \times \ldots + \ldots ,$$

which exactly coincides with the standard rules of differentiation. This property holds due to linearity of generators with respect to derivatives. The symmetrization mentioned above is obtained automatically because all doublets are identical. In $s\ell(2)$ case symmetrized product (6) covers all possible representations. This is not so for higher rank groups.

The elements of (6) are the homogeneous expressions of form $x^k y^{n-k}$, where $n$ is number of factors in (3).

Homogeneous realization of generators is not very suitable. More convenient is inhomogeneous one:

$$(R^{(2)} \times R^{(2)} \times \ldots \times R^{(2)}) = \{x^n, x^{n-1} y, x^{n-2} y^2, \ldots, y^n\} \equiv x^n\{1, \xi, \xi^2, \ldots, \xi^n\}, \quad \xi = \frac{y}{x}.$$  

(7)

Now we can rewrite $s\ell(2)$ generators so that they will act on $\{1, \xi, \xi^2, \ldots, \xi^n\}$ thus mixing different powers of $\xi$:

$$S^+ x^n\{(1, \xi) \times (1, \xi) \times \ldots\} = (S^+ x^n)\{(1, \xi) \times (1, \xi) \times \ldots\} + x^n \sum_{k=1}^{n} \{(1, \xi) \times \ldots \times (T^+(1, \xi)) \times \ldots \times (1, \xi)\}$$

$$= n x^n \xi \{(1, \xi) \times (1, \xi) \times \ldots\} + x^n \sum_{k=1}^{n} \{(1, \xi) \times \ldots \times ((0,-\xi^2)) \times \ldots \times (1, \xi)\},$$  

(8)

and similar relations for $S^-$ and $S^z$. 
Omitting the overall factor $x^n$ one can deduce from these relations that

$$
S^+ = n\xi - \xi^2 \frac{\partial}{\partial \xi}, \quad S^z = -\frac{n}{2} + \xi \frac{\partial}{\partial \xi}, \quad S^- = \frac{\partial}{\partial \xi}.
$$

(9)

Here $j = n/2$ has sense of spin of the representation.

This realization can be extended to the case of other symmetry groups as well as to the case of deformed symmetry $\mathfrak{sl}_q(2)$. The $\mathfrak{sl}_q(2)$ algebra is defined by generators $e = S^-$, $f = S^+$, $k = q^{2S^z}$, $k^{-1} = q^{-2S^z}$ and commutation relations:

$$
kfk^{-1} = q^2f, \quad kek^{-1} = q^{-2}e, \quad kk^{-1} = 1 = k^{-1}k, \quad [f, e] = \frac{k - k^{-1}}{q - q^{-1}}.
$$

The scaling or dilatation symmetry, generated by $S^z$ survives under this deformation. Hence polynomial structure of the lowest weight representations can be kept as well as explicit form of $S^z$. Another two generators then are given by following expressions:

**Definition 1.** \{Homogeneous\}

$$
S^z = \frac{1}{2}(y\partial_y - x\partial_x), \quad S^- = x/y \frac{q^{\frac{1}{2}(y^2 \partial_y - x^2 \partial_x)} - q^{-\frac{1}{2}(x^2 \partial_x - y^2 \partial_y)}}{q - q^{-1}}, \quad S^+ = y/x \frac{q^{\frac{1}{2}(x^2 \partial_x - y^2 \partial_y)} - q^{-\frac{1}{2}(y^2 \partial_y - x^2 \partial_x)}}{q - q^{-1}},
$$

(10)

and the representation space is given by the set of homogeneous expressions $\{x^{n-k}y^k\}_{k=0}^n$. It is also possible to pass to inhomogeneous representation $\{\xi^n\}_{k=0}^n$. The generator of translations $S^-$ goes to the q-derivative. Remaining generators are also given by finite-difference operators and we have come to

**Definition 2.** \{Inhomogeneous\}

$$
e = S^- = \frac{\xi^{-1}}{q - q^{-1}}(q^{\xi\partial_x} - q^{-\xi\partial_x}).
$$

(11)

$$
k = q^{2\xi\partial_x - n}, \quad f = S^+ = \frac{\xi}{q - q^{-1}}(q^n - \xi\partial_x - q^n\xi\partial_x - n).
$$

Here we follow the notations and conventions ref.[4] and define the co-product to be:

$$
\Delta(k) = k \otimes k,
$$

$$
\Delta(e) = e \otimes 1 + k \otimes e,
$$

(12)

$$
\Delta(f) = f \otimes k^{-1} + 1 \otimes f,
$$

which is a little bit simpler than usual definition

$$
\Delta(q^{S^z}) = q^{S^z} \otimes q^{S^z},
$$

$$
\Delta(S^+) = S^+ \otimes q^{S^z} + q^{S^z} \otimes S^+,
$$

(13)

except for the transformation rules under exchange $q \leftrightarrow kq^{-1}$. This rule allows an unambiguous definition of $\mathfrak{sl}_q(2)$ generators for higher tensor products as well. Let one has triple tensor product. Then there exist two possible definitions:

$$
k = k_{12}k_3, \quad e = e_{12} + k_{12}e_3, \quad f = f_{12}k_3^{-1} + f_3,
$$
and
\[ k = k_1k_2, \quad e = e_1 + k_1e_23, \quad f = f_1k_2^{-1} + f_2, \]
however result in both cases is the same:
\[ k = k_1k_2k_3, \quad e = e_1 + k_1e_2 + k_1k_2e_3, \quad f = f_1k_2^{-1}k_3^{-1} + f_2k_3^{-1} + f_3. \]
The indices here denote the numbers of the spaces. In tensor product notations they would be rewrite as
\[
\Delta(A) = \sum_i B_i \otimes C_i, \quad A, B_i, C_i \in \mathfrak{sl}_q(2), \quad \Delta\Delta(A) = \sum_i \Delta(B_i) \otimes C_i = \sum_i B_i \otimes \Delta(C_i).
\]
This relation reflects on the co-associativity property of the co-product for Hopf algebras.
\[
(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta.
\]
Then it is obvious by induction that result is consistent for higher tensor products too.

3 Representations of \( \mathfrak{sl}_q(2) \) at exceptional values of deformation parameter.

The representations of \( \mathfrak{sl}_q(2) \) at general values of \( q \) have the same structure as in non-deformed (classical) case. However when \( q \) takes exceptional values, i.e. is given by roots of unity \( (q^N = 1) \), there appear many differences. The center of \( \mathfrak{sl}_q(2) \) is enlarged for these values of \( q \): in addition to conventional quadratic Casimir operator
\[
\mathcal{C} = fe + (q - q^{-1})^{-2}(q^{-1}k + qk^{-1}) = ef + (q - q^{-1})^{-2}(qk + q^{-1}k^{-1}),
\]
there appear also new Casimirs:
\[
e^N, \quad f^N, \quad k^\pm N, \quad N = \begin{cases} N = N, & N \text{ is odd} \\ N = \frac{1}{2}N, & N \text{ is even} \end{cases}, \quad q^N = \pm 1.
\]
It is not hard to establish the following relation:
\[
f^Ne^N = \prod_{n=0}^{N-1} \left( \mathcal{C} - \frac{q^n - 2 + q^{-n}}{(q - q^{-1})^2} \right) + \frac{(k^N - 1)(k^{-N} - 1)}{(q - q^{-1})^{2N}}.
\]
There appear [4] the representations of new \( \mathcal{B} \) type or cyclic ones, which have no classic counterpart. They defined as
\[
e^N \neq 0, \quad \text{or/and} \quad f^N \neq 0.
\]
Here we are interested in \( \mathcal{A} \) type or lowest weight representations, which also differ from corresponding ones for general values of \( q \). They defined as
\[
e^N = 0, \quad f^N = 0, \quad k^N = \pm 1.
\]
In general case representation is parameterized by eigenvalues of four Casimir operator, which are related by constraint (16) and one can formulate the following
Proposition. The most general differential expression for generators consistent with $s\ell_q(2)$ algebra contains three arbitrary parameters:

$$
f \equiv S^+ = q^{\lambda/2} x \frac{q^{\alpha-x\partial} - q^{x\partial-\alpha}}{q - q^{-1}}, \quad e \equiv S^- = \frac{q^{-\lambda/2} q^{x\partial-\beta} - q^{\beta-x\partial}}{x q - q^{-1}},
$$

$$
k \equiv q^{2S^+} = q^{-\alpha-\beta} q^{2x\partial}.
$$

This differential realization is very convenient to describe lowest weight representations: there always exists lowest weight vector $\omega_0$. It annihilated by lowering generator $e$. Other vectors of representation can be obtained by repeatedly acting of rising generator $f$. Now, it becomes obvious that dimension of such representation then bounded by $N$, because after $N$ steps it will repeat itself due to $f^N \sim 1$. So it consists of not more than $N$ elements, otherwise it will be reducible. Generally irrep. of type $A$ with spin $j$ has dimension $2j + 1$. Indeed substituting (19) into lowest and highest weight conditions $e \cdot 1 = 0$ and $f \cdot x^{2j} = 0$ one can obtain that general form of generators of spin $j$ representation are:

**Definition 3.**

$$
e_{(j)}^A = \varepsilon_{(j)} x^{-1} \frac{q^{x\partial} - q^{-x\partial}}{q - q^{-1}}, \quad f_{(j)}^A = x \frac{q^{2j-x\partial} - q^{x\partial-2j}}{q - q^{-1}}, \quad k_{(j)}^A = \varepsilon_{(j)} q^{2x\partial-2j},
$$

and representation space is given by polynomials which powers do not exceed $2j$. We shall denote it $P_j$.

4 Tensor product of spins one and one half representations and fusion rules.

The co-multiplication of quantum algebra enables us to define tensor product of representations.

- Consider at first tensor product of two representations of spin $\frac{1}{2}$ in polynomial spaces with variables $x_1, x_2$. According to the definition of co-product $s\ell_q(2)$ generators in this case take the form:

$$
e = \frac{\varepsilon}{q - q^{-1}} \left[ \frac{1}{x_1} (q^{x_1} \partial_1 - q^{-x_1} \partial_1) + \frac{q^{2x_1} \partial_1 - q^{x_1} \partial_1}{x_2} (q^{x_2} \partial_2 - q^{-x_2} \partial_2) \right],
$$

$$
f = \frac{1}{q - q^{-1}} \left[ q^{1-2x_2} \partial_2 x_1 (q^{1-x_1} \partial_1 - q^{x_1} \partial_1) + x_2 (q^{1-x_2} \partial_2 - q^{x_2} \partial_2 - 1) \right],
$$

$$
k = \varepsilon q^{2x_1 \partial_1 + 2x_2 \partial_2 - 2},
$$

and act in $P_{\frac{1}{2} \frac{1}{2}} = \{1, x_1, x_2, x_1 x_2\}$ as follows:

$$
e \cdot 1 = 0, \quad e \cdot x_1 = \varepsilon, \quad e \cdot x_2 = \varepsilon q^{-1}, \quad e \cdot x_1 x_2 = \varepsilon (q x_1 + x_2)
$$

$$
k \cdot 1 = \varepsilon q^{-2} \cdot 1, \quad k \cdot x_i = \varepsilon x_i, \quad i = 1, 2, \quad k \cdot x_1 x_2 = \varepsilon q^2 x_1 x_2,
$$

$$
f \cdot 1 = (q x_1 + x_2), \quad f \cdot x_1 = x_1 x_2, \quad f \cdot x_2 = q^{-1} x_1 x_2, \quad f \cdot x_1 x_2 = 0
$$

Then one deduces that Casimir operators act on these vectors according to the formulae:

$$
e^2 = 0 = f^2, \quad k^2 = 1.
\[
\mathbb{C} \cdot 1 = \varepsilon \frac{q^3 + q^{-3}}{(q - q^{-1})^2}, \quad \mathbb{C} \cdot x_1 = (qx_1 + x_2) + \varepsilon \frac{q^3 + q^{-3}}{(q - q^{-1})^2} x_1, \\
\mathbb{C} \cdot x_2 = \varepsilon(x_1 - qx_2) + \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2} x_2, \quad \mathbb{C} \cdot x_1x_2 = \varepsilon \frac{q^3 + q^{-3}}{(q - q^{-1})^2} x_1x_2.
\]

So one can see that tensor product of two spin \( \frac{1}{2} \) representations decomposes according to eigenvalues of Casimir operator \( \mathbb{C} \):

\[
c_1 = \varepsilon \frac{q^3 + q^{-3}}{(q - q^{-1})^2},
\]

on triplet of vectors: \( \{1, (x_1 + q^{-1}x_2), x_1x_2\} \) and

\[
c_3 = \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2},
\]

on singlet \( x_1 - qx_2 \). That means that tensor product of two spin \( \frac{1}{2} \) decomposes in direct sum of spin one and spin zero for any \( N (q^N = 1) \) except for \( N = 4 \), which corresponds to XX Heisenberg model. So for \( N \neq 4 \) spin addition rule is not deformed: couple of spins \( \frac{1}{2} \) decomposes to spin one and spin zero. Indeed spin one representation can be obtained as symmetrized part of mentioned above tensor product by setting spin zero component \( x_1 - qx_2 = 0 \), i.e. \( x_2 = q^{-1}x_1 \), then representation space consists of quadratic polynomials of one variable \( x = x_1 \): \( P^\text{sym} \sim P_1 \).

Then derivative \( \partial_2 \) vanishes and generators take the form:

\[
e = \frac{\varepsilon}{q - q^{-1}}(q^{2x_0} - q^{-2x_0}), \quad f = \frac{1}{q - q^{-1}}(q^{2-2x_0} - q^{2x_0-2}), \quad k = \varepsilon q^{2x_0-2},
\]

standard for representation of spin one.

The case \( N = 4 \) require separate consideration. When deformation parameter takes values \( q = \pm i \) degeneracy of Casimir’s eigenvalues takes place, \( c_1 = c_2 = 0 \) and vectors of triplet and singlet are unified into one multiplet. Moreover, two eigenvectors which are linear with respect to \( x \)’s \( (x_1 + qx_2 \text{ and } x_1 - q^{-1}) \) coincide each to other when \( q = \pm i \). In other words the eigenvectors of Casimir operator do not longer form a basis in \( P_{+2} \) and have to be completed by one additional vector. The visual evidence of representations unification in this case is provided by matrix representation: setting

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix},
\]

one obtains that in this basis operator \( \mathbb{C} \) has form:

\[
\mathbb{C} = \varepsilon \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

which obviously cannot turn to diagonal form by linear transformation of basis. In other words representation space \( I^{(4)} \) cannot be decomposed on invariant subspaces according to eigenvalues of Casimir operator \( \mathbb{C} \). Representation \( I^{(4)} \) is called indecomposable, because it is neither
reducible (generators mix representation vectors and do not leave any invariant subspace) nor irreducible (being initially introduced as tensor product).

- Consider next tensor product of three representations. It has form:

\[ e = \frac{\varepsilon}{q - q^{-1}} \left[ \frac{1}{x_1} (q_{x_1} - q_{-x_1}) + \frac{q_{2x_1} - q_{-2x_1}}{x_2} (q_{x_2} - q_{-x_2}) + \frac{q_{2x_1} + 2q_{-2x_1} - 2j_2}{x_3} (q_{x_3} - q_{-x_3}) \right], \]

\[ f = \frac{1}{q - q^{-1}} \left[ q_{2j_2 + 2j_3 - 2x_2} (q_{2j_2 - x_1} - q_{x_1} - 2j_1) + q_{2j_3 - 2x_3} x_2 (q_{2j_2 - x_2} - q_{x_2} - 2j_2) + x_3 (q_{2j_3 - x_3} - q_{x_3} - 2j_3) \right]. \]

First consider the case \( j_1 = j_2 = j_3 = \frac{1}{2} \). On representation vectors of \( P_{3 \times 3} \), Casimir operator acts as follows:

\[ \mathbb{C} \cdot 1 = \varepsilon \frac{q^4 + q^{-4}}{(q - q^{-1})^2} \cdot 1, \quad \mathbb{C} \cdot x_1 = \varepsilon (q^2 x_1 + q x_2 + x_3 + \frac{q^2 + q^{-2}}{(q - q^{-1})^2} x_1), \]

\[ \mathbb{C} \cdot x_2 = \varepsilon (q x_1 + x_2 + q^{-1} x_3 + \frac{q^2 + q^{-2}}{(q - q^{-1})^2} x_2), \quad \mathbb{C} \cdot x_3 = \varepsilon (x_1 + q^{-1} x_2 + q^{-2} x_3 + \frac{q^2 + q^{-2}}{(q - q^{-1})^2} x_3), \]

\[ \mathbb{C} \cdot x_1 x_2 = \varepsilon ((1 + q + \frac{2}{(q - q^{-1})^2}) x_1 x_2 + q x_1 x_3 + x_2 x_3), \]

\[ \mathbb{C} \cdot x_1 x_3 = \varepsilon (q x_1 x_2 + 2(1 + \frac{1}{(q - q^{-1})^2}) x_1 x_3 + q^{-2} x_2 x_3), \]

\[ \mathbb{C} \cdot x_2 x_3 = \varepsilon (x_1 x_2 + q^{-1} x_1 x_3 + (1 + q^{-2} + \frac{2}{(q - q^{-1})^2}) x_2 x_3), \quad \mathbb{C} \cdot x_1 x_2 x_3 = \varepsilon \frac{q^4 + q^{-4}}{(q - q^{-1})^2} \cdot x_1 x_2 x_3, \]

- From these relations one can deduce that Casimir has two eigenvalues

\[ c_1 = \varepsilon \frac{q^4 + q^{-4}}{(q - q^{-1})^2}, \quad (27) \]

on quartet of vectors:

\[ \phi_0 \equiv 1, \quad \phi_1 \equiv (x_1 + q^{-1} x_2 + q^{-2} x_3), \quad \phi_2 \equiv (x_1 x_2 + q^{-1} x_1 x_3 + q^{-2} x_2 x_3), \quad \phi_3 \equiv x_1 x_2 x_3 \]

and

\[ c_2 = \varepsilon \frac{q^2 + q^{-2}}{(q - q^{-1})^2}, \quad (28) \]

on another quartet:

\[ \varphi_1^{(1)} \equiv (x_1 - q x_2), \quad \varphi_1^{(2)} \equiv (x_1 - q^2 x_3), \quad \varphi_2^{(1)} \equiv (x_1 x_2 - q x_1 x_3), \quad \varphi_2^{(2)} \equiv (x_1 x_2 - q^2 x_2 x_3). \]
Now it can be easily checked that:

\[ e_0 = 0, \quad e_1 = \varepsilon (1 + q^{-2} + q^{-4}) \phi_0, \quad e_2 = \varepsilon (q + q^{-1}) \phi_1, \quad e_3 = q^2 \phi_2, \]

\[ f_0 = q^2 \phi_1, \quad f_1 = (q + q^{-1}) \phi_2, \quad f_2 = (1 + q^{-2} + q^{-4}) \phi_3, \quad f_3 = 0. \]

and

\[ e_{\varphi_1}^{(1)} = 0, \quad e_{\varphi_1}^{(2)} = 0, \quad e_{\varphi_2}^{(1)} = \varepsilon q^{-1}(\varphi_2^{(1)} - \varphi_1^{(1)}), \quad e_{\varphi_2}^{(2)} = \varepsilon q^{-1}(\varphi_1^{(1)} + \varphi_2^{(1)}), \]

\[ f_{\varphi_1}^{(1)} = q^{-1}(\varphi_2^{(1)} - \varphi_1^{(1)}), \quad f_{\varphi_1}^{(2)} = (q - q^{-1})\varphi_2^{(1)} + q^{-1}\varphi_2^{(1)}, \quad f_{\varphi_2}^{(1)} = 0, \quad f_{\varphi_2}^{(2)} = 0. \]

From these relations one can deduce that first quartet corresponds to four dimensional representation of spin \( \frac{3}{2} \), while second one constitutes two spin \( \frac{1}{2} \) representations, when \( \mathcal{N} > 3 \). In other words, when \( g \) is a root of unity higher degree tensor product of three spin one half representation decomposes in the same way as for general values of \( q \) or in a classical case.

The low values of \( \mathcal{N} \) require separate consideration.

When \( \mathcal{N} = 2 \), one has \( \varepsilon^2 = 0 = f^2 \) on vectors \( \phi_1 \) and \( \varphi_1^{(a)} \) due to \( q + q^{-1} = 0 \) and mentioned vectors can be combined into pairs: \( (\phi_0, \phi_1), (\phi_2, \phi_3), (\varphi_1^{(1)}, \varphi_2^{(1)}), (\varphi_2^{(1)}, \varphi_1^{(1)}) \), which constitute four spin \( \frac{1}{2} \) representation spaces. Here we denoted \( \phi_1 = \varphi_1^{(2)} - \varphi_1^{(1)} \) and \( \phi_2 = \varphi_2^{(2)} - \varphi_2^{(1)} \). So the tensor product of three spin \( \frac{1}{2} \) decomposes for \( q = \pm i \) into the sum of four spin \( \frac{1}{2} \) irreps.

Now let \( \mathcal{N} = 3 \), i.e. \( N = 3 \) or \( N = 6 \). Then one has \( k^2 = 1 \) and \( \varepsilon^3 = 0 = f^3 \) due to \( 1 + q^{-2} + q^{-4} = 0 \) and eigenvalues of Casimir operator \( C \) become degenerate:

\[ c_1 = c_2 = -\frac{\varepsilon \cos \left(\frac{4\pi k}{3}\right)}{2 \sin^2 \left(\frac{2\pi k}{3}\right)} = \frac{\varepsilon}{\sqrt{3}} \]

due to \( q^2 + 1 + q^{-2} = 0 \) and one can easily check that triplets \( \{\phi_1, \varphi_1^{(1)}, \varphi_1^{(2)}\} \) and \( \{\phi_2, \varphi_2^{(1)}, \varphi_2^{(2)}\} \) become linearly dependent and two quartets of vectors are unified into one multiplet - sextet, which is an indecomposable representation like a quartet in the case \( q = \pm i \). Eigenvectors do not form a basis in representation space and have to be completed by another two vectors. It is easy to establish relations:

\[ \mathcal{C}(x_1 + \alpha x_2 + \beta x_3) = \frac{\varepsilon}{3}(x_1 + \alpha x_2 + \beta x_3) + (q^2 + q\alpha + \beta)(x_1 + q^{-1}x_2 + q^{-2}x_3), \]

and

\[ \mathcal{C}(x_1 x_2 + ax_1 x_3 + bx_2 x_3) = \varepsilon(x_1 x_2 + ax_1 x_3 + bx_2 x_3) + (q^2 + q\alpha + b)(x_1 x_2 + q^{-1}x_1 x_3 + q^{-2}x_2 x_3). \]

Now one can see that upon "symmetrization" with respect \( 1 \leftrightarrow 2 \) or \( 1 \leftrightarrow 3 \) "antisymmetric" doublet \( (\phi_1^{(2)}, \phi_2^{(2)}) \): \( (x_1 - q x_2, x_1 x_3 - q x_2 x_3) \) or \( (x_1 - q^2 x_3, x_1 x_2 - q^2 x_2 x_3) \), corresponding to spin \( \frac{1}{2} \) representation decouples:

\[ e\phi_1^{(2)} = 0, \quad f\phi_1^{(2)} = \phi_2^{(2)}, \quad e\phi_2^{(2)} = \varepsilon\phi_1^{(2)}, \quad f\phi_2^{(2)} = 0, \]

while remaining six vectors are unified into an indecomposable representation \( \phi_1^{(6)} = \{\phi_1^{(6)}\} \):

\[ \phi_1^{(6)} = 1, \quad \phi_2^{(6)} = x_1 + q^{-1}x_2 + q^{-2}x_3, \quad \phi_3^{(6)} = x_1 + qx_2 + q^2 x_3, \]

(32)
\[ \phi_4^{(6)} = x_1 x_2 + q^{-1} x_1 x_3 + q^{-2} x_2 x_3, \quad \phi_5^{(6)} = x_1 x_2 + q x_1 x_3 + q^2 x_2 x_3, \quad \phi_6^{(6)} = x_1 x_2 x_3. \]

Representing these vectors as columns \((\phi_i^{(6)})^j = \delta_i^j\) one obtains that Casimir operator \(C\) acts as

\[
C = \frac{\varepsilon}{3} + 3q^2 \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

which again cannot be turned to diagonal form. Hence corresponding sextet \(I_1^{(6)}\) is an indecomposable representation. \(I_1^{(6)}\) can be obtained directly as tensor product of spins \(\frac{1}{2}\) and 1 while in present case one has one additional doublet: \((\{\frac{1}{2}\} \circ \{\frac{1}{2}\}) \circ \{\frac{1}{2}\} = \{\{1\} \circ \{0\} \circ \{\frac{1}{2}\}\} = \{I_1^{(6)}\} \circ \{\frac{1}{2}\}\). So one can deduce that representations with spins higher than 1 (i.e. 3/2 etc.) are not allowed when \(N = 3\). According to classification given by D. Arnaudon in \([4]\) there exists another indecomposable sextet, which appears in tensor product of two spin 1 representations or in the quartic product of spins \(\frac{1}{2}\).

- For the product of two spin 1 representations one can obtain:

\[
e = \frac{\varepsilon}{q - q^{-1}} \left( \frac{1}{x_1} (q^{x_1 \partial_1} - q^{-x_1 \partial_1}) + \frac{q^{2x_1 \partial_1 - 2}}{x_2} (q^{x_2 \partial_2} - q^{-x_2 \partial_2}) \right), \quad k = \varepsilon q^{2x_1 \partial_1 + 2x_2 \partial_2 - 4},
\]

\[
f = \frac{1}{q - q^{-1}} \left( x_1 q^{2 - x_2 \partial_2} (q^{2 - x_1 \partial_1} - q^{-x_1 \partial_1 - 2}) + x_2 (q^{2 - x_2 \partial_2 - 2} - q^{-x_2 \partial_2}) \right),
\]

On representation space \(P_{11}\) Casimir operator \(C\) acts as follows:

\[
\begin{align*}
C \cdot 1 &= \varepsilon \frac{q^5 + q^{-5}}{(q - q^{-1})^2}, \\
C x_1 &= \varepsilon \left( (q + q^{-1})(q^2 x_1 + x_2) + \frac{q^3 + q^{-3}}{(q - q^{-1})^2} x_1 \right), \\
C x_2 &= \varepsilon \left( (q + q^{-1})(x_1 + q^{-2} x_2) + \frac{q^3 + q^{-3}}{(q - q^{-1})^2} x_2 \right), \\
C x_1^2 &= \varepsilon (q + q^{-1}) \left( q^2 x_1^2 + (q + q^{-1})x_1 x_2 + \frac{x_1^2}{(q - q^{-1})^2} \right), \\
C x_1 x_2 &= \varepsilon \left( q^2 x_1^2 + x_2^2 + 2(q + q^{-1})x_1 x_2 + \frac{q + q^{-1}}{(q - q^{-1})^2} \right), \\
C x_2^2 &= \varepsilon (q + q^{-1}) \left( q^2 x_2^2 + (q + q^{-1})q^{-2} x_1 x_2 + \frac{x_2^2}{(q - q^{-1})^2} \right), \\
C x_1^2 x_2 &= \varepsilon (q + q^{-1}) \left( x_1 x_2^2 + (1 + q^2)x_1 x_2 + \frac{x_1 x_2^2}{(q - q^{-1})^2} \right), \\
C x_1 x_2^2 &= \varepsilon (q + q^{-1}) \left( x_1^2 x_2 + (1 + q^2)x_1^2 x_2 + \frac{x_1^2 x_2^2}{(q - q^{-1})^2} \right), \\
C \cdot x_1^2 &= \varepsilon \frac{q^5 + q^{-5}}{(q - q^{-1})^2} x_1^2 x_2.
\end{align*}
\]
Using these relations one can deduce that \( \mathbb{C} \) has in general only three eigenvalues:

\[
c_1 = \varepsilon \frac{q^5 + q^{-5}}{(q - q^{-1})^2},
\]

on vectors

\[
\{ \varphi_i^{(1)} \}_{i=1}^5 = \{1, (x_1 + q^{-2}x_2), (x_1^2 + q^{-3}(q - q^{-1})^2x_1x_2 + q^{-4}x_2^2), (x_1^2 + q^{-2}x_1x_2^2), x_1x_2^2 \},
\]

\[
c_2 = \varepsilon \frac{q^3 + q^{-3}}{(q - q^{-1})^2},
\]

on the set

\[
\{ \varphi_i^{(2)} \}_{a=1}^3 = \{ (x_1 - q^2x_2), (x_1^2 + (q^{-3} - q)x_1x_2 - x_2^2), x_1^2x_2 - q^2x_1x_2^2 \},
\]

and third eigenvalue

\[
c_3 = \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2},
\]

on combination

\[
\{ \varphi_i^{(3)} \} = x_1^2 - (q + q^{-1})x_1x_2 + q^2x_2^2.
\]

According to these relations tensor product of two spin one representations decomposes on quintet, triplet and singlet: \( \{1\} \otimes \{1\} = \{2\} \oplus \{1\} \oplus \{0\} \) as it takes place for general values of deformation parameter. As we already learned above, the case when \( q = \text{root of unity of low degree} \) has to be studied carefully, because eigenvalues \( c_i \) of Casimir operator \( \mathbb{C} \) become degenerate. The simplest case \( \mathcal{N} = 2 \) i.e. \( q = \pm i \) is now excluded because spin 1 representation is not allowed for these \( q \).

When \( \mathcal{N} = 3 \) i.e. \( q^3 = 1 \) or \( q^6 = 1 \) one sees that \( \varepsilon^3 = 0 = f^3 \), \( c_1 = c_3 = \frac{\varepsilon}{3} \) and quadratic with respect to \( x_i \) eigenvectors become linearly dependent:

\[
x_1^2 + (q^{-3} - q)x_1x_2 - x_2^2 = x_1^2 - (q + q^{-1})x_1x_2 + q^2x_2^2,
\]

due to \( q^{-2} + q^{-4} = -1 \) and \( q^{-4} = q^2 \) when \( q^6 = 1 \). The triplet \( \{ \varphi_i^{(2)} \}_{a=1}^3 \) corresponding to spin 1 decouples. Adding to the set \( \{ \varphi_i^{(1)} \}_{i=1}^5 \) one more quadratic with respect to \( x_1 \), \( x_2 \) vector \( \varphi^{(1)}_6 = x_1^2 + (2 - \frac{5}{3}q)x_1x_2 + (\frac{1}{2}q^{-2} - 3q^{-3})x_2^2 \) to complete basis in representation space one obtains another indecomposable sextet \( I_2^{(6)} = \{ \varphi_i^{(2)} \}_{i=1}^6 \). The Casimir operator after exchange \( \varphi^{(1)}_6 \leftrightarrow \varphi^{(1)}_4 \) acts on this set as \( 6 \times 6 \) matrix:

\[
\mathbb{C} = \frac{\varepsilon}{3} - \left( \frac{5}{6} + q \right) \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{36}
\]

i.e. it again cannot be made diagonal by linear transformation of vectors of representation.

Another value of deformation parameter which leads to degeneracy of Casimir’s eigenvalues corresponds to \( q^8 = 1 \). In this case one obtains relations: \( \varepsilon^4 = 0 = f^4 \), \( k^2 = 1 \),
\( c_1 = c_2 = -\frac{\sqrt{7}}{2} = -c_3 \). Then corresponding vectors become linearly dependent and has to be completed to form basis of an eight-dimensional indecomposable representation \( f_1^{(8)} \).

- Let us now turn to the tensor product of four representations of spin \( \frac{1}{2} \). Generators, acting on \( P_{1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}} \) can be represented in following form:

\[
f = \frac{1}{q - q^{-1}} \left[ q^{3 - 2x_4\partial_4 - 2x_3\partial_3 - 2x_2\partial_2} x_1 (q^{1 - x_1\partial_1} - q^{-1}x_1\partial_1) + q^{2 - 2x_4\partial_4 - 2x_3\partial_3} x_2 (q^{1 - x_2\partial_2} - q^{-1}x_2\partial_2) + \\
q^{1 - 2x_4\partial_4} x_3 (q^{1 - x_3\partial_3} - q^{-1}x_3\partial_3) + x_4 (q^{1 - x_4\partial_4} - q^{-1}x_4\partial_4) \right],
\]

\[k = \varepsilon q^{2x_1\partial_1 + 2x_2\partial_2 + 2x_3\partial_3 + 2x_4\partial_4 - 4},\]

\[e = \frac{\varepsilon}{q - q^{-1}} \left[ \frac{1}{x_1} (q^{x_1\partial_1} - q^{-x_1\partial_1}) + \frac{q^{2x_1\partial_1 - 1}}{x_2} (q^{x_2\partial_2} - q^{-x_2\partial_2}) + \\
\frac{q^{2x_1\partial_1 + 2x_2\partial_2 - 2}}{x_3} (q^{x_3\partial_3} - q^{-x_3\partial_3}) + \frac{q^{2x_1\partial_1 + 2x_2\partial_2 + 2x_3\partial_3 - 3}}{x_4} (q^{x_4\partial_4} - q^{-x_4\partial_4}) \right].\]

Casimir operator acts on \( P_{1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}} \) as follows:

\[\mathbb{C} \cdot 1 = \varepsilon \frac{q^5 - q^{-5}}{(q - q^{-1})^2}, \quad \mathbb{C} \cdot x_1 = \varphi_1 + \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2} x_1, \quad \mathbb{C} \cdot x_2 = q^{-1} \varphi_1 + \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2} x_2, \]

\[\mathbb{C} \cdot x_3 = q^{-2} \varphi_1 + \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2} x_3, \quad \mathbb{C} \cdot x_4 = q^{-3} \varphi_1 + \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2} x_4,\]

where

\[\varphi_1 = \varepsilon (q^3 x_1 + q^2 x_2 + q x_3 + x_4),\]

\[\mathbb{C} \cdot x_1 x_2 = \varepsilon \left( q + q^3 + \frac{q + q^{-1}}{(q - q^{-1})^2} \right) x_1 x_2 + \varepsilon (q^2 x_1 x_3 + q x_1 x_4 + q x_2 x_3 + x_2 x_4),\]

\[\mathbb{C} \cdot x_1 x_3 = \varepsilon q^2 x_1 x_2 + \varepsilon \left( 2q + \frac{q + q^{-1}}{(q - q^{-1})^2} \right) x_1 x_3 + \varepsilon (q x_1 x_4 + x_2 x_3 + x_3 x_4),\]

\[\mathbb{C} \cdot x_1 x_4 = \varepsilon (q x_1 x_2 + x_1 x_3) + \varepsilon \left( q + q^{-1} + \frac{q + q^{-1}}{(q - q^{-1})^2} \right) x_1 x_4 + \varepsilon (q x_2 x_4 + q^{-1} x_3 x_4),\]

\[\mathbb{C} \cdot x_1 x_3 = \varepsilon (q x_1 x_2 + x_1 x_3) + \varepsilon \left( q + q^{-1} + \frac{q + q^{-1}}{(q - q^{-1})^2} \right) x_2 x_3 + \varepsilon (x_2 x_4 + q^{-1} x_3 x_4),\]

\[\mathbb{C} \cdot x_2 x_4 = \varepsilon (x_1 x_2 + x_1 x_3 + x_2 x_3) + \varepsilon \left( 2q^{-1} + \frac{q + q^{-1}}{(q - q^{-1})^2} \right) x_2 x_4 + \varepsilon x_3 x_4,\]

\[\mathbb{C} \cdot x_2 x_4 = \varepsilon (x_1 x_3 + q^{-1} x_1 x_4 + q^{-1} x_2 x_3 + q^{-2} x_2 x_4) + \varepsilon \left( q^{-3} + q^{-1} + \frac{q + q^{-1}}{(q - q^{-1})^2} \right) x_3 x_4,\]

\[\mathbb{C} \cdot x_1 x_3 = \varphi_3, \quad \mathbb{C} \cdot x_1 x_4 = q^{-1} \varphi_3, \quad \mathbb{C} \cdot x_2 x_4 = q^{-2} \varphi_3, \quad \mathbb{C} \cdot x_3 x_4 = q^{-3} \varphi_3, \quad \mathbb{C} \cdot x_1 x_2 x_3 x_4 = 0,\]

where

\[\varphi_3 = \varepsilon (q^3 x_1 x_2 + q^2 x_1 x_2 x_4 + q x_1 x_3 x_4 + x_2 x_3 x_4).\]
It follows from these relations that Casimir operator has three different eigenvalues:

\[ c_1 = \varepsilon \frac{q^5 + q^{-5}}{(q - q^{-1})^2}, \]

in maximally "symmetric" sector:

\[ \{ \varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4 \}, \]

where \( \varphi_0 = 1 \), \( \varphi_2 = q^4x_1x_2 + q^3x_1x_3 + q^2x_1x_4 + q^2x_2x_3 + qx_2x_4 + x_3x_4 \), \( \varphi_4 = x_1x_2x_3x_4 \)

\[ c_2 = \varepsilon \frac{q^3 + q^{-3}}{(q - q^{-1})^2}, \]

on vectors

\[ \{(x_1 - qx_2), (x_1 - q^2x_3), (x_1 - q^3x_4), (x_1x_4 - x_2x_3), \]

\( (x_1x_2 + (q^{-1} - q)x_1x_3 + (q^{-2} - 1)x_1x_4 - x_3x_4), (x_1x_3 + (q^{-1} - q)x_1x_4 - x_2x_4), \)

\( (x_1x_2x_3 - qx_1x_2x_4), (x_1x_2x_3 - q^2x_1x_3x_4), (x_1x_2x_3 - q^3x_2x_3x_4) \}\]

and

\[ c_3 = \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2}, \]

on vectors

\[ \{(x_1x_2 + (q^{-1} - q)x_1x_3 - x_1x_4 - x_2x_3 + q^2x_3x_4), (x_1x_3 - qx_1x_4 - qx_2x_3 + q^2x_2x_4) \}. \]

These eigenvalues become degenerate just for the same values of \( q \) as considered above spin 1 \( \times \) spin 1 case. Indeed tensor product of two spin \( \frac{1}{2} \) spaces differs from spin 1 by trivial one-dimensional space corresponding to spin zero. However in this case values \( q = \pm i \) are allowed too. Then one can establish relations

\[ e^2 = 0 = f^2, \quad k^2 = 1, \]

and

\[ c_1 = c_2 = c_3 = 0, \]

on \( P_{\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}} \). One can see that for \( q = \pm i \) eigenvectors of Casimir operator \( C \) do not longer form a basis in \( P_{\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}} \); in sectors linear and trilinear with respect to \( x_i \) only three vectors from four ones are independent, while in bilinear sector one has four independent vectors instead of six. In this way one obtains that after appropriate completion the set of eigenvectors of Casimir, it will take block-diagonal form with four 4 \( \times \) 4 blocks, i.e. tensor product of two indecomposable representations decomposes into direct sum of indecomposable ones: \( I(4) \otimes I(4) = I(4) \oplus I(4) \oplus I(4) \oplus I(4) \).
5 Conclusion and summary

Here we summarize some conclusions which follow from the considerations of previous section. As it is known the spin addition law or more generally representation fusion rule when \( q \) is a root of unity \( (q^N = 1) \) is not deformed if root index \( N \) is large enough, more precisely if \( m < 2N \), where \( m = m_1 \times m_2 \times \ldots, m_i \) are dimensions of representations in tensor product. Dimension of an irrep is not exceed \( N (q^N = \pm 1) \). Tensor product of irreps decomposes into direct sum of irreps and indecomposable representations of dimension \( 2N \) if dimension of tensor product exceed \( 2N \). The eigenvalues of Casimir operator \( \mathbb{C} \) play the key role in this decomposition. For remaining Casimirs one can obtain: \( f^N = 0 = e^N \) and \( k^N = (-\varepsilon)^N, \varepsilon = \pm 1 \) for representations of \( A \) type.

So, when \( q \) is given by a root of unity there exists the maximal value of spin \( j_{max} = \frac{N-1}{2} \).

Another notation has crucial importance for physical applications: when deformation parameter takes exceptional value \( q = \pm i \) the fundamental two-dimensional representation appears with property \( e^2 = 0 = f^2 \). Fusion of such representation naturally leads to the indecomposable representation with the same property. It means that value \( q = i \) which is specific for XX Heisenberg model ensures realization of Pauli principle peculiar to free fermions.

Let us summarize the fusion rules for the cases with lowest values of \( N, N = 2, 3 \), considered above and give decompositions of tensor products of all the allowed classic-like irreps and indecomposable representations arising here. From the discussions of previous section complete fusion rules are followed for the tensor products of spin \( \frac{1}{2} \) and 1 irreps.

The case \( N = 2 \). The only \( A \) type irrep is one-half spin (\( \frac{1}{2} \)) (besides of the one-dimensional zero spin representation, on which all generators act trivially, and quadric Casimir is 0), and from the fusion emerges one four-dimensional indecomposable representation in accordance to [4].

\[
\frac{1}{2} \otimes \frac{1}{2} = I^{(4)},
\frac{1}{2} \otimes I^{(4)} = 4 \oplus \frac{1}{2}, \quad I^{(4)} \otimes I^{(4)} = 4 \oplus I^{(4)}. \tag{39}
\]

From these relations follows a general rule

\[
\begin{align*}
\otimes \frac{1}{2} & = \oplus k I^{(4)}, & k = 2^{2(n-1)}, \\
\otimes \frac{1}{2} & = \oplus \frac{1}{2}, & k = 2^{2n}, \\
\otimes I^{(4)} & = \oplus k I^{(4)}, & k = 4^{n-1}, \\
\otimes I^{(4)} \otimes \frac{1}{2} & = \oplus k I^{(4)}, & k = 4^{n+r-1}, \\
\otimes I^{(4)} \otimes \frac{1}{2} & = \oplus \frac{1}{2}, & k = 4^{n+r}. \tag{40}
\end{align*}
\]
For the case $N = 3$ the $A$ type irreps are three - with spins zero, one-half and one: $(0), (\frac{1}{2}), (1)$, and from their fusions two six-dimensional indecomposable representations are arising: $Ind_A(j = 0), Ind_A(j = 1)$ in the classification [4]. The fusion rules are

$$
\begin{align*}
\frac{1}{2} \otimes \frac{1}{2} &= 1 \oplus 0, \\
\frac{1}{2} \otimes 1 &= I_1^{(6)}, \\
1 \otimes 1 &= I_2^{(6)} \oplus 1, \\
\frac{1}{2} \otimes I_1^{(6)} &= I_2^{(6)} \oplus 1 \oplus 1, \\
\frac{1}{2} \otimes I_2^{(6)} &= I_1^{(6)} \oplus 1 \oplus 1, \\
1 \otimes I_1^{(6)} &= 1 \otimes I_2^{(6)} = \bigoplus_2 I_1^{(6)} \bigoplus_2 1, \\
I_1^{(6)} \otimes I_2^{(6)} &= \bigoplus_2 I_1^{(6)} \bigoplus_4 I_2^{(6)} \bigoplus_1 1.
\end{align*}
$$

The generalization for the higher tensor products is obvious, all they consist of both of spin-irreps and indecomposable representations. For illustration let us draw for small values of $N$, $N = 2, 3$, the extended Bratteli diagrams (the decomposition rules for the tensor products of $n$ copies of similar representations) for both of irreducible and indecomposable representations.

The tensor product of the finite dimensional representations of $s\ell_q(2)$ is reduced into a linear combination

$$
\underbrace{V_i \otimes V_i \otimes \cdots \otimes V_i}_{n} = \sum_k w_n^{ki} V_k.
$$

Here the $w_n^{ki}$ are the multiplicities of the $V_k$ representations (irreducible and indecomposable representations). In the Bratteli diagrams (see figures) these numbers are consistent with the numbers of paths coming to the respective representations (dots in the figures) from the origin. The diagrams for the $s\ell_q(2)$ representations contain multiple links of $r$ times (in cases $q^3 = 1, q^3 = \pm 1$, the $r = 2, 4$). The path which is passed such link, must be multiplied by $r$.

The next steps of the towers ($n \geq 6$ in Fig.1, and $n \geq 5$ in Fig.2(b) and Fig.3), contain the same representations already appeared for the lower n-s. The multiple links are drawn either by $r$ parallel lines or by thick lines with label $(\times r)$. For comparison in Fig.2a we represent the case for $s\ell(2)$ algebra. As it is expected [4, 2] the fusions of the $A$ type representations form closed ring.

We can do some remark about values of $N$ higher than 3. The maximal allowed spin representation with dimension $N$ has spin $j_{\text{max}} = \frac{N-1}{2}$. The tensor product of $j_{\text{max}}$ with $\frac{1}{2}$ is an indecomposable representation with dimension $2N$:

$$
\begin{align*}
j_{\text{max}} \otimes \frac{1}{2} &= I_1^{2N}, \\
I_1^{2N} \otimes \frac{1}{2} &= j_{\text{max}} \otimes \frac{1}{2} \otimes \frac{1}{2} = j_{\text{max}} \otimes (1 \oplus 0) = j_{\text{max}} \oplus (j_{\text{max}} \otimes 1).
\end{align*}
$$
We expect that for general case also \((j_{\text{max}} \otimes 1)\) expands to the sum of \(j_{\text{max}}\) and another indecomposable representation \(I_2^{2N}\).

\[
\begin{align*}
  j_{\text{max}} \otimes 1 &= j_{\text{max}} \oplus I_2^{2N}.
\end{align*}
\]

So the representation with maximum spin together with indecomposable representations appears in decomposition of the tensor product of an indecomposable representation with any other \([5]\). By definition \([2]\) these are states with zero q-dimension (for irreps \(\dim_q \rho_j = [2j+1]_q\)).

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Figure 3: Extended Bratteli diagrams for fusions of indecomposable representations \((I)^n\) in cases: a) \(N = 2\), b) \(N = 3\).

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