Critical points of 2d disordered Dirac fermions: the Quantum Hall Transitions revisited

André LeClair

Newman Laboratory, Cornell University, Ithaca, NY

(Dated: October 2007, Revised May 2008)

Abstract

We propose a resolution of the renormalization group flow for the disordered Dirac fermion theories describing the quantum Hall transition (QHT) and spin Quantum Hall transition (SQHT), which previously revealed no perturbative fixed points at 1-loop and higher. The approach involves carrying out the flow in 2 stages, the first stage utilizing a new form of super spin-charge separation to flow to \( gl(1|1)_N \) and \( osp(2|2)_{-2N} \) supercurrent algebra theories, where \( N \) is the number of copies. This fixed point breaks the copy symmetry. In the second stage, additional forms of disorder are incorporated as dimension zero logarithmic operators, and the resulting actions have explicit forms in terms of two scalar fields and a symplectic fermion. Multi-fractal exponents are computed with the result \( q(1 - q)/4 \) and \( q(1 - q)/8 \) for the QHT and SQHT respectively, in agreement with numerical estimates.
I. INTRODUCTION

Disordered Dirac fermions in $2+1$ dimensions have many important applications in condensed matter physics. They are theoretically interesting since they can represent new universality classes of Anderson localization/delocalization transitions. Perhaps the most important is the Chalker-Coddington network model for the quantum Hall transition (QHT)\cite{1}, which can be mapped onto disordered Dirac fermions\cite{2,3}. A partial list of other applications includes to dirty superconductors \cite{4,5,6,7}, and studies of hopping models on bipartite lattices \cite{8}. More recent applications are to graphene \cite{9,10}, where the Dirac fermions are present from the start. The possible universality classes of disordered Dirac fermions were classified according to their discrete symmetries in \cite{11}. The latter classification contains 13 classes and is thus a minor refinement of Altland-Zirnbauer’s classification which does not assume the Dirac structure\cite{12}.

A number of new theoretical techniques have been developed over the last decade to study these problems; a partial list includes for instance \cite{13,14,15,16,17,18,19}. For the most part, a proper understanding of the critical points for generic disorder is still lacking. A notable exception is the spin quantum Hall transition (SQHT). Its network model\cite{21} can also be mapped onto disordered Dirac fermions\cite{20}. Remarkably, the equivalent spin chain was mapped onto 2D classical percolation by Gruzberg, Ludwig and Read\cite{22,23,24}, and this leads to the exact knowledge of the correlation length exponent $\nu_{\text{perc}} = 4/3$ and density of states exponent $\rho(E) \sim E^{1/7}$.

For the QHT, one should also mention the replica sigma model approach of Pruisken\cite{25}. Although it appears to have the right ingredients as outlined in \cite{26}, it has proved too difficult to solve thus far, so it remains unknown whether it really does have the correct critical point. There is also the later proposal of Zirnbauer\cite{27} which uses supersymmetry. Based on symmetry and various other requirements the critical point for the QHT was proposed to be described by a sigma model of WZNW type based on the supergroup $PSL(2|2)$. The model was further studied in \cite{28}. The main problem with this proposal is that the level $k$ of the $PSL(2|2)$ WZNW model is an exactly marginal perturbation so that the model actually has a line of fixed points depending on $k$. This would lead to the prediction of non-universality in the QHT, which is contrary to the numerical evidence. (For a recent review, see \cite{29,32}.) It was pointed out recently by Tsvelik that the value $k = 8$ gives
very reasonable exponents\cite{33}. (The identical exponents were actually already speculated in \cite{34,37}.) Unfortunately, it was noted that there are no known constructive arguments leading to $k = 8$ based on the $PSL(2|2)$ approach. In the work we present here, $PSL(2|2)$ will not play a rôle, but rather the simpler superalgebra $gl(1|1)$ will be central, and we will describe a precise mechanism for obtaining higher integer levels $k$ based on a new form of super spin-charge separation\cite{35}.

Due to an extensive effort over the last few decades, vast classes of conformally invariant $2D$ critical points can be constructed\cite{30} and many of these mathematical constructions can be extended to theories with supergroup symmetries. However in the study of disordered Dirac fermions, it is important that the possible critical point is supposed to be reached by renormalization group flow in the effective disorder-averaged effective field theory. There are a very limited number of known mechanisms for obtaining a fixed point from a renormalization group flow, and unless disordered systems depend on some new mechanism, it is helpful to identify the known mechanisms as a guide:

(i) Non-linear sigma models like the $O(N)$ model. Due to the Mermin-Wagner theorem, these models are only critical for $-2 < N < 2$.

(ii) Non-linear sigma models with topological terms, the primary example being the $O(3)$ non-linear sigma model with $\theta = \pi$.

(iii) Certain relevant perturbations that induce flows between minimal models.

(iv) Decoupling of massive degrees of freedom using spin-charge separation in the case of marginal current-current interactions. Here the primary example is the $1d$ Hubbard model which has $SU(2) \otimes SU(2)$ symmetry. One $SU(2)$ sector is marginally relevant, the other marginally irrelevant. Thus one $SU(2)$ is gapped out in the flow, and the fixed point is the $SU(2)$ current algebra theory (WZNW model)\cite{31}.

For the sake of comparison with our work, Pruisken’s model is based on the scheme (ii), whereas Zirnbauer’s model is mathematically constructed directly at the fixed point so it is unknown under which renormalization group flow scheme it can be realized. In the approach pursued in the present work, disorder averaging of Dirac fermions is known to yield marginal current-current interactions so naturally the mechanism for obtaining a fixed point will be (iv), i.e. based on super version of spin-charge separation.

The starting point of the present work is the detailed disordered Dirac fermion theories for the Chalker-Coddington network model for the QHT and its variant for the SQHT.
Based on extensive numerical evidence\textsuperscript{32}, there is no doubt that these specific models have a critical point. Performing the disorder averaging using Efetov’s supersymmetric method\textsuperscript{36} is known to lead to marginal anisotropic left-right current-current interactions of the underlying super current algebra. The problem with such marginal perturbations is that they typically do not have perturbative fixed points at finite values of the coupling constants. In particular, the coupled beta functions do not have any non-trivial zeros at one loop. Higher loop corrections to the renormalization group (RG) beta functions were computed for the network models in \textsuperscript{34, 37} based on the general proposal in \textsuperscript{38} and also did not reveal any perturbative fixed points. These analyses were nevertheless useful for understanding whether any new couplings were generated under RG. In these studies it is significant that the couplings flow to a singular point in a finite RG time, which suggests an incomplete resolution of the flow rather than the lack of a fixed point. It was pointed out that the higher order beta functions are possibly not exact due to some contributions that were missed at 4-loops\textsuperscript{39}, however it seems unlikely that this could resolve the issue in a constructive manner.

Since a perturbative fixed point of the beta functions is unlikely, one must identify the correct non-perturbative mechanism that singles out the expected fixed point. In this paper we propose to resolve the RG flow in two stages. We first focus on the important symmetries of the N-copy theory \textit{before disorder averaging}, i.e. we identify the relevant symmetries that are present for any realization of the disorder. This leads to the special rôle of the symmetries corresponding to the current algebras $gl(1|1)_N \otimes su(N)_0$ for the QHT and $osp(2|2)_{-2N} \otimes sp(2N)_0$ for the SQHT. (Our nomenclature is that $g_k$ refers to the current algebra for the finite (super) Lie algebra $g$ at level $k$.) The disorder averaged effective actions have several couplings which correspond to the strengths (variances) of the various random potentials. Rather than study the simultaneous flow of all couplings, in the first stage we set some of the couplings to zero and carry out the RG flow for a subset of the couplings corresponding to the above symmetries. A new form of the super spin-charge separation obtained in \textsuperscript{35} is then used to argue that in the first stage one flows to the fixed point $gl(1|1)_N$ for the QHT. This result indicates that the locality constraints studied in \textsuperscript{13} for the $gl(1|1)_k$ theory, which led to $k$ being an inverse integer, are too restrictive. For the N-copy SQHT, the analogous flow is to $osp(2|2)_{-2N}$. This kind of flow for 1-copy of the SQHT was studied in \textsuperscript{17} where it was viewed as a fine-tuning of the model. In this paper our point of view
is that the first stage of the RG flow identifies the proper degrees of freedom that are the most important for the actual critical point.

In the second stage of the flow, we restore the additional kinds of disorder that were possibly initially present in the model as additional relevant perturbations. The possible operators which appear in this second stage are dictated by the quantum numbers of the original fields and the super spin-charge separation. Another new aspect of the present work is that we use the results in [35] to explicitly construct the operators corresponding to the additional kinds of disorder. In particular, \( gl(1|1)_k \) at any level \( k \) has a simple free field representation in terms of two scalar fields and a symplectic fermion. The additional kinds of disorder correspond to logarithmic operators of scaling dimension zero. For the QHT one obtains a \( gl(1|1) \) generalization of the sine-Gordon theory, where \( N \), the number of copies, appears as a coupling. It can also be viewed as radius of compactification \( R = \sqrt{N} \).

The important feature of these kinds of perturbations is that they do not drive the theory to a new fixed point, but rather just lead to logarithmic corrections to the correlation functions [35, 40], and this explains why for example the \( osp(2|2)_{-2N} \) current algebra contains the correct exponents for the SQHT.

As models of disordered Dirac fermions, the QHT and SQHT are not so different in their formulations, and if the methods are general enough, they should be subject to the same kind of analysis. It is therefore very instructive to work out both cases in parallel, since some exact results are known for the SQHT. This also avoids idiosyncratic proposals for special cases. All the remaining sections of this paper have subsections treating the QHT and SQHT cases.

Our results are presented as follows. In sections II and III we review the definitions of the models and the supersymmetric method for disorder averaging, introducing a convenient notation to deal with the profusion of fields in the N-copy theories. The symmetries of the models for any realization of disorder are studied in section IV. In section V we consider a subgroup of these symmetries that commutes with the permutation of the \( N \) copies, which leads to \( gl(1|1)_N \) and \( osp(2|2)_{-2N} \). The properties of these super current algebras that we need are reviewed in section VI. In section VII we describe our 2-stage strategy for resolving the RG flow. Since we focus on symmetries that are present for any realization of disorder, the analysis does not depend strongly on any assumed distributions of the random potentials. On the other hand the arguments rely strongly on super spin-charge separation and
some simple 1-loop beta function arguments concerning the marginal relevance/irrelevance of operators in the disorder averaged effective action. The additional perturbations in the second stage of the RG flow are constrained by the quantum numbers of the fields after gapping out the \( su(N)_0 \) and \( sp(2N)_0 \) “copy” symmetries. Typically one obtains perturbations by logarithmic operators which were explicitly constructed in \[35\]. This results in some relatively simple lagrangians involving the two scalar fields and a symplectic fermion. Under some assumptions, the multi-fractal exponents are computed in section VIII. Our results agree favorably (within about 1%) with the numerical simulations in \[41, 42, 43, 44\]. In section IX we discuss the localization length exponents.

II. DEFINITION OF THE MODELS.

A. Chalker-Coddington network model.

The Chalker-Coddington network model can be mapped to the following 2d hamiltonian \[2, 3\]

\[
H = \begin{pmatrix}
V + M & -i\partial z + A_z \\
-i\partial \bar{z} + A_{\bar{z}} & V - M
\end{pmatrix}
\]

where \( z, \bar{z} \) are euclidean light-cone coordinates, \( z = (x + iy)/\sqrt{2}, \bar{z} = z^* \) with \( x, y \) the 2d spacial coordinates. \( A_z, \bar{z} \) is a \( u(1) \) gauge field, \( A_z = (A_x + iA_y)/\sqrt{2} \). The hamiltonian is hermitian if \( A_x, A_y, V, \) and \( M \) are real. The hamiltonian is first order in derivatives and operates on a 2-component wave-function. It thus corresponds to a universality class of disordered Dirac fermions, class \( A=GUE \) in \[12\], or class \( 0 \) according to the more specific classification in \[11\]. All the potentials \( A, M, V \) depend on \( x, y \) and are random variables. The model in \[16\] on the other hand is in the chiral \( GUE \), i.e. class \( 2 \).

B. Spin network model.

The network model for the SQHT is also a model of disordered Dirac fermions, but in class \( C \) \[12\] (class \( 4_- \) according to \[11\]). The hamiltonian is

\[
H = \begin{pmatrix}
2\vec{\alpha} \cdot \vec{\sigma} + M & -i\partial \bar{z} + A_{\bar{z}} \\
-i\partial z + A_z & 2\vec{\alpha} \cdot \vec{\sigma} - M
\end{pmatrix}
\]
where \( \vec{\sigma} \) are Pauli matrices and \( A \) is an \( su(2) \) gauge field, \( A = \vec{A} \cdot \vec{\sigma} \). The hamiltonian thus operates on a 4-component wave-function. Again, all the potentials \( \vec{\alpha}, M, \) and \( A \) depend on \( x, y \) and are random.

### III. SUPERSYMMETRIC DISORDER AVERAGING.

Since the hamiltonians describe non-interacting fermions, the disorder averaged correlation functions can be studied with Efetov’s supersymmetric method\(^{36}\).

#### A. QHT

Let us denote the 2-component wave-functions as follows:

\[
\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi^* = (\psi_-, \psi_-)
\]  

(3)

For simplicity, let us refer to all the disordered potentials simply as “\( V \)”. The Green functions can be defined with respect to a functional integral with the action

\[
S(\psi; V) = i \int \frac{d^2x}{2\pi} \psi^* H(V) \psi
\]

(4)

Above, the fields are taken to be fermionic.

The supersymmetric method is a trick to cancel the fermionic determinant \( Z(V) \) where \( Z \) is the partition function at fixed disorder \( V \). One introduces ghost partners \( \beta, \beta^* \) to the \( \psi \)'s and considers the action

\[
S_{\text{susy}} = S(\psi; V) + S(\beta; V)
\]

(5)

where \( S(\beta; V) \) is identical to \( S(\psi; V) \) but with the replacement \( \psi \rightarrow \beta \). The \( \beta \)-fields are bosonic. The effective action upon disorder averaging is then defined as

\[
e^{-S_{\text{eff}}(\psi, \beta)} = \int DV \mathcal{P}[V] e^{-S_{\text{susy}}}
\]

(6)

where \( \mathcal{P}[V] \) is the probability distribution of the random potentials. If \( \mathcal{P}[V] \) is taken to be gaussian, then \( S_{\text{eff}} \) contains quartic interactions among the fermions and ghosts. As we will see, many of our arguments are independent of the specific form of these probability distributions.
In order to clearly display the symmetries of $S_{\text{susy}}$ and $S_{\text{eff}}$, it will prove convenient to introduce the following notation. Let $\Psi_{\pm}$ denote 2-component fields built out of $\psi_{\pm}, \beta_{\pm}$, and similarly for $\overline{\Psi}_{\pm}$:

$$\Psi_{\pm} = (\psi_{\pm}, \beta_{\pm}), \quad \overline{\Psi}_{\pm} = (\overline{\psi}_{\pm}, \overline{\beta}_{\pm}) \quad (7)$$

The index that runs over the two components of $\Psi_{\pm}$ will be denoted as \(r\), \(r = 1, 2\):

$$\Psi_{1,\pm} = \psi_{\pm}, \quad \Psi_{2,\pm} = \beta_{\pm}.$$  

We will also be interested in computing disorder averaged moments of correlation functions. To compute averages of \(N\)-th moments such as

$$\langle \psi(x)\psi(0) \rangle \langle \psi(x)\psi(0) \rangle \ldots \langle \psi(x)\psi(0) \rangle \quad (8)$$

we need to introduce \(N\)-copies of the models. Namely, we introduce fields $\Psi_{\pm}^{\alpha}$, $\alpha = 1, \ldots, N$, so that the complete set of fields is $\Psi_{\pm}^{r,\alpha}$ and $\overline{\Psi}_{\pm}^{r,\alpha}$, \(r = 1, 2\). Thus $\Psi_{\pm}$ refers to \(2N\) different fields.

At a fixed realization of disorder, $S_{\text{susy}}$ can be expressed in the compact form:

$$S_{\text{susy}} = \int \frac{d^2x}{2\pi} \left[ \overline{\Psi}_{-}(\partial_z - iA_z(x))\overline{\Psi}_{+} + \Psi_{-}(\partial_z - iA_{\overline{z}}(x))\Psi_{+} - iV(x) \left( \overline{\Psi}_{-}\Psi_{+} + \Psi_{-}\overline{\Psi}_{+} \right) 
- iM(x) \left( \overline{\Psi}_{-}\Psi_{+} - \Psi_{-}\overline{\Psi}_{+} \right) \right] \quad (9)$$

where for example

$$\overline{\Psi}_{-}\Psi_{+} = \sum_{r,\alpha} \overline{\Psi}_{-}^{r,\alpha}\Psi_{+}^{r,\alpha}.$$  

**B. SQHT**

For the spin-network model, one needs to introduce an additional $su(2)$ index \(i\) and consider fields $\Psi_{\pm}^{r,\alpha, i}$, i.e. there are \(4N\) fields in $\Psi_{+}$ for example. The action is then

$$S_{\text{susy}} = \int \frac{d^2x}{2\pi} \left[ \overline{\Psi}_{-}(\partial_z - iA_z(x))\overline{\Psi}_{+} + \Psi_{-}(\partial_z - iA_{\overline{z}}(x))\Psi_{+} - i\overline{\sigma}(x) \cdot (\overline{\Psi}_{-}\sigma\Psi_{+} + \Psi_{-}\overline{\sigma}\overline{\Psi}_{+}) 
- iM(x) \left( \overline{\Psi}_{-}\Psi_{+} - \Psi_{-}\overline{\Psi}_{+} \right) \right] \quad (10)$$

Above, the Pauli matrices, including the ones in \(A\), operate on the index \(i\) so that for example

$$\overline{\Psi}_{-}\overline{\sigma}_{ij}\Psi_{+} = \sum_{r,i,j,\alpha} \overline{\Psi}_{-}^{r,\alpha, i}\sigma_{ij}\Psi_{+}^{r,\alpha}.$$
IV. SYMMETRIES AT FIXED DISORDER.

A. QHT

First consider all disordered potentials set to zero in $S_{\text{susy}}$. The result is a free conformal field theory of Dirac fermions and ghosts which has total central charge equal to zero:

$$S_{\text{free}} = \int \frac{d^2x}{2\pi} \sum_{\alpha=1}^{2N} \left( \overline{\psi}_- \partial_z \psi_+^\alpha + \psi_+^\alpha \partial_z \overline{\psi}_+^\alpha + \overline{\beta}_- \partial_z \beta_+^\alpha + \beta_+^\alpha \partial_z \overline{\beta}_+^\alpha \right)$$

(11)

The two point functions are

$$\langle \psi_-(z) \psi_+(w) \rangle = \langle \psi_+(z) \psi_-(w) \rangle = \langle \beta_+(z) \beta_-(w) \rangle = -\langle \beta_-(z) \beta_+(w) \rangle = \frac{1}{z-w}$$

(12)

and similarly for the right-movers, $\langle \overline{\psi}_-(\overline{z}) \overline{\psi}_+(\overline{w}) \rangle = 1/(\overline{z}-\overline{w})$, etc. (For a review of 2D conformal field theory see [30, 45].) In the sequel we will not display the right-moving counterparts if they are the obvious duplications of the left.

For notational simplicity let us group the $r, \alpha$ indices of $\Psi_{\pm}^{r,\alpha}$ into a single index $a$ and refer to these fields as $\{\Psi_a^{\pm}\}, a = 1, \ldots, 2N$. The extra minus sign in the above two point functions can be accounted for by introducing a grade $[a] = 0$ for bosonic components and $[a] = 1$ for fermionic ones. One then has

$$\Psi_+^a(z) \Psi_-^b(0) \sim \frac{1}{z} \delta^{ab}, \quad \Psi_-^a(z) \Psi_+^b(0) \sim \frac{1}{z} (-)^{[a]+1} \delta^{ab}$$

(13)

The complete set of chiral currents are then

$$J_{\pm}^{ab} = \Psi_+^a \Psi_-^b, \quad H^{ab} = \Psi_+^a \Psi_-^b$$

(14)

These currents generate an $osp(2N|2N)_k$ super-current algebra at level $k = 1$, and this represents the maximal symmetry without disorder. Our conventions for all the current algebras that appear in this paper are presented in Appendix A.

Without disorder the symmetry is actually the sum of left and right, $osp(2N|2N)_L^T \oplus osp(2N|2N)_R^T$ since the theory is conformal. In the presence of disorder the conformal symmetry is broken and one does not have the full current-algebra symmetry. However one can study the global left-right diagonal symmetries generated by the charges

$$Q = \oint \frac{dz}{2\pi i} J(z) + \oint \frac{d\overline{z}}{2\pi i} \overline{J}(\overline{z})$$

(15)
where $J, \overline{J}$ are the left/right moving currents. These conserved charges are always associated to a set of left-moving currents for a current algebra and in the sequel this correspondence is implicit.

With disorder, the maximal $osp(2N|2N)$ symmetry is broken to something smaller. Consider the transformation $\psi \rightarrow \psi + \delta \psi$ which acts left-right diagonally: $\delta \psi_+ = \beta_+^{\alpha'}$, $\delta \beta_+^{\alpha'} = -\psi_+^{\alpha}$, $\delta \overline{\psi}_+ = \overline{\beta}_-^{\alpha'}$, $\delta \overline{\beta}_-^{\alpha'} = -\overline{\psi}_+^{\alpha}$, and is zero on all other fields, where $\alpha, \alpha'$ are fixed copy indices. All of the operators $\Psi_- \Psi_+, \overline{\Psi_-} \overline{\Psi}_+$, $\Psi_- \overline{\Psi}_+$ and $\overline{\Psi_-} \Psi_+$ are invariant under this transformation. There is another symmetry of this type with $+ \leftrightarrow -$. The left-moving currents that generate these two symmetries are $S_\pm^{\alpha, \alpha'} = \pm \psi_+^{\alpha} \beta_+^{\alpha'}$. The charges for these fermionic symmetries are nilpotent, $Q^2 = 0$, and the symmetry they generate can thus be thought of as a BRST symmetry. Namely, the disorder dependent part of $S_{\text{susy}}$ can be written as $\delta X$ for some $X$, and its invariance is a consequence of $\delta^2 = 0$.[14]

Consider other nilpotent symmetries with $\delta \psi_+ = \beta_+^{\alpha'}$, i.e. that flip the $u(1)$ charges. These correspond to the currents $\tilde{S}_\pm^{\alpha, \alpha'} = \psi_+^{\alpha} \beta_+^{\alpha'}$. One finds in this case that due to fermionic exchange signs, the operators $\Psi_- \Psi_+$ and $\overline{\Psi_-} \overline{\Psi}_+$ are not invariant. The only invariant is the combination $(\overline{\Psi_-} \Psi_+ - \Psi_- \overline{\Psi}_+)$. Thus, examining the action eq. (9), one sees that the only nilpotent symmetries are those corresponding to the diagonal left-right symmetry which corresponds to the left-moving currents $S_\pm^{\alpha, \alpha'}$. The operator product expansion (OPE) of these currents closes on the super-current algebra $gl(N|N)_{k=1}$ (See Appendix A.) In other words, at any fixed realization of the disorder, the model has a global $gl(N|N)$ symmetry corresponding to the current algebra $gl(N|N)_1$. We will refer to this symmetry as the BRST symmetry.

**B. SQHT**

Since there are twice as many fields in the SQHT, the maximal current algebra symmetry with zero disorder is $osp(4N|4N)_1$. Repeating the analysis above for the QHT, one sees that there are nilpotent symmetries generated by the currents $S_\pm^{\alpha, \alpha'} = \pm \sum_{i=1,2} \psi_+^{\alpha'} \beta_+^{\alpha'}$. A basic result we will use repeatedly is the following. Given two copies of the same current algebra with currents $J_1^a$ at level $k_1$ and $J_2^a$ at level $k_2$ which furthermore commute, $[J_1^a(z), J_2^b(w)] = 0$. Then $J^a = J_1^a + J_2^a$ satisfies the current algebra at level $k_1 + k_2$. Thus, since the $su(2)$ indices $i$ are summed over in $S_\pm$, these currents close on $gl(N|N)_{k=2}$ since
each copy has level 1 and the levels add.

The $gl(N|N)$ symmetry is actually enlarged due to an additional nilpotent symmetry. Introduce the matrix $\epsilon$ which acts on the $su(2)$ indices $i$: $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using $\epsilon\sigma = -\sigma\epsilon$ and $\epsilon^2 = -1$, one can verify that $S_{\text{susy}}$ is invariant under the left-right diagonal symmetry corresponding to the left-moving currents $\tilde{S}_{\pm}^{\alpha\alpha'} = \psi_\pm^\alpha \epsilon \beta_\pm^{\alpha'}$. It is important to note that this symmetry would not be valid if there were additional “$V$” type of disorder, or if the gauge field $A$ contained a $U(1)$ component. For $N = 1$ copy, the currents $\tilde{S}_\pm, S_\pm$ close on the $osp(2|2)_k$ current algebra at level $k = -2$\cite{17}. For $N$-copies this symmetry is promoted to the BRST symmetry $osp(2N|2N)_{-2}$.

V. PERMUTATION INVARIANT BRST SYMMETRIES.

The BRST symmetries discussed in the last section are rather large since their dimension depends on the number of copies $N$. Furthermore, the current algebras are only moderately interesting as possible critical points; for example the $gl(N|N)_1$ theory has only integer scaling dimensions at level 1. In this section we constrain the possible fixed point further by considering permutations in the number of copies.

Let $\mathcal{P}_N$ denote the discrete permutation group for $N$ elements. The actions $S_{\text{susy}}$ possess this symmetry where $N$ is the number of copies. It is natural then to make the hypothesis that a possible fixed point also has the permutation symmetry. The BRST symmetries of the last section do not commute with $\mathcal{P}_N$, however there is a sub-algebra that does, which we will refer to as the $\mathcal{P}_N$ invariant BRST symmetry. In section VII we will provide arguments based on super spin-charge separation that indicate how a fixed point with this restricted symmetry can arise under RG flow.

A. QHT

For the QHT the generators that commute with $\mathcal{P}_N$ are

$$H = \sum_\alpha \psi_+^\alpha \psi_-^\alpha, \quad J = \sum_\alpha \beta_+^\alpha \beta_-^\alpha, \quad S_\pm = \pm \sum_\alpha \psi_\pm^\alpha \beta_\mp^\alpha$$

The above currents satisfy the $gl(1|1)_k$ current algebra at level $k = N$:

$$H(z)H(0) \sim \frac{k}{z^2}, \quad J(z)J(0) \sim -\frac{k}{z^2}$$
\begin{align*}
H(z)S_\pm(0) &\sim J(z)S_\pm(0) \sim \pm \frac{1}{z} S_\pm \\
S_+(z)S_-(0) &\sim \frac{k}{z^2} + \frac{1}{z} (H - J)
\end{align*}

(17)

It will be important to determine any additional continuous symmetries that commute with the \(P_N\)-invariant BRST symmetry \(gl(1|1)_N\). There is obviously an \(su(N)\) symmetry which mixes the copies. Let \(L^a_\psi, L^a_\beta\) denote the \(su(N)\) currents in the separate sectors and \(L^a\) their sum:

\begin{align*}
L^a_\psi &= \psi_\alpha t^a_{\alpha\alpha'} \psi'_{\alpha'} , \\
L^a_\beta &= \beta_\alpha t^a_{\alpha\alpha'} \beta'_{\alpha'} , \\
L^a &= L^a_\psi + L^a_\beta
\end{align*}

(18)

where \(t^a\) is the \(N \times N\) dimensional matrix representation of the vector of \(su(N)\). The currents \(L^a_\psi\) satisfy \(su(N)_1\), whereas the \(L^a_\beta\) satisfy \(su(N)_{-1}\). Therefore the total currents \(L^a\) satisfy \(su(N)_0\) at level \(k = 0\). In summary, the symmetries that will play a significant rôle in the sequel is \(gl(1|1)_N \oplus su(N)_0\) and these two current algebras commute.

B. SQHT

For the SQHT the \(P_N\) invariant BRST symmetries correspond to the currents

\begin{align*}
H &= \beta_+ \beta_- , \\
J &= \psi_+ \psi_- , \\
J_\pm &= \psi_\pm \epsilon \psi_\pm \\
S_\pm &= \psi_\pm \beta_\pm , \\
\hat{S}_\pm &= \psi_\pm \epsilon \beta_\pm
\end{align*}

(19)

where \(\psi_\pm = \sum_i \psi_{+i}^\alpha \psi_{-i}^{\alpha'}\) and \(\psi_\pm \epsilon \psi_\pm = \sum_i \psi_{+i}^\alpha \epsilon_{ij} \psi_{-j}^{\alpha'}\), etc. The above currents satisfy \(osp(2|2)_{-2N}\).

The \(\hat{S}_\pm\) and \(J_\pm\) currents are invariant under \(\psi_\pm \rightarrow M \psi_\pm, \beta_\pm \rightarrow M \beta_\pm\) where \(M\) is a \(2N\) dimensional matrix satisfying \(M^T(\epsilon \otimes 1)M = \epsilon \otimes 1\). \(M\) is thus an element of \(Sp(2N)\). The currents satisfy \(sp(2N)_0\). Since \(\hat{S}_\pm\) and \(J_\pm\) close on \(osp(2|2)_{-2N}\), the \(sp(2N)_0\) commutes with \(osp(2|2)_{-2N}\).

For \(N = 1\), \(sp(2) = su(2)\), and this \(su(2)\) is the original \(su(2)\) symmetry of the spin network model. We wish to emphasize that here the \(sp(2N)_0\) symmetry is a property of the \(N\)-copy theory, which is to be contrasted with different models that have a random \(sp(2N)\) gauge field from the very beginning in the 1-copy theory, for example the models in [46].
VI. THE $gl(1|1)_k$ AND $osp(2|2)_k$ SUPER CURRENT ALGEBRAS.

In this section we summarize the main results we will need for the current algebras $gl(1|1)_k$ and $osp(2|2)_k$. For $gl(1|1)_k$ we mainly summarize our recent work\[35\], which builds on \[47, 48\]. The $osp(2|2)$ results are based on the work\[15\].

A. $gl(1|1)_k$

We will need the Sugawara stress tensor $T(z)$. The algebra $gl(1|1)$ has two independent quadratic casimirs:

$$C_2 = J^2 - H^2 + S^+ S^- - S^- S^+, \quad C'_2 = (J - H)^2$$

where it is implicit that the above operators are the zero modes of the currents. The stress tensor is fixed by the condition $T(z)J^a(0) \sim J^a(0)/z^2$, which requires it to be built out of both casimirs\[49\]:

$$T(z) = -\frac{1}{2k} \left( J^2 - H^2 + S^+ S^- - S^- S^+ \right) + \frac{1}{2k^2} (J - H)^2$$

For any level $k$ there exists a free field representation in terms of 2 scalar fields and a symplectic fermion. The free fields have the action

$$S = \frac{1}{8\pi} \int d^2x \sum_{a,b=1}^2 \left( \eta_{ab} \partial_\mu \phi^a \partial_\mu \phi^b + \epsilon_{ab} \partial_\mu \chi^a \partial_\mu \chi^b \right)$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\partial_\mu^2 = 2\partial_z \partial_{\bar{z}}$. The $\chi$ fields are Grassman: $(\chi^a)^2 = 0$ and have Virasoro central charge $c = -2$, so that the total central charge is zero. Note that the metric for the bosonic fields has indefinite signature, which will turn out to be important. The equations of motion imply that the fields can be decomposed into left and right moving parts:

$$\phi^a(z, \bar{z}) = \phi^a(z) + \bar{\phi}^a(\bar{z})$$
$$\chi^a(z, \bar{z}) = \chi^a(z) + \bar{\chi}^a(\bar{z})$$

Above, the bold face signifies local fields. The two point functions are

$$\langle \phi^a(z) \phi^b(w) \rangle = -\eta^{ab} \log(z - w), \quad \langle \chi^a(z) \chi^b(w) \rangle = -\epsilon^{ab} \log(z - w)$$
Our conventions are \( \eta^{ab} = \eta_{ab}, \epsilon^{ab} = \epsilon_{ab} \). Exponentials of the bosons have the conformal dimension:

\[
\Delta \left( e^{i(a\phi^1 + b\phi^2)} \right) = \frac{a^2 - b^2}{2}
\]  

(26)

It is straightforward to verify the following representation of the OPE’s in eq. (17):

\[
H = i\sqrt{k} \partial_z \phi^1, \quad J = i\sqrt{k} \partial_z \phi^2
\]

\[
S_+ = \sqrt{k} \partial_z \chi^1 e^{i(\phi^1 - \phi^2)}/\sqrt{k}, \quad S_- = -\sqrt{k} \partial_z \chi^2 e^{-i(\phi^1 - \phi^2)}/\sqrt{k}
\]

(27)

In the sequel, where there is no cause for confusion, we will simply write \( \partial \phi \) for \( \partial_z \phi(z) \).

The algebra \( gl(1|1) \) has 2-dimensional representations where \( H = \text{diag}(h, h - 1) \) and \( J = \text{diag}(j, j - 1) \) which will be denoted as \( \langle h, j \rangle \). (We follow the conventions in [33].)

These are so-called typical representations when \( h \neq j \). Primary fields associated with these representations have conformal dimension

\[
\Delta_{\langle h, j \rangle} = \frac{(h - j)^2}{2k^2} + \frac{(h - j)(h + j - 1)}{2k}
\]

(28)

The basic fields \( \psi_\pm, \beta_\pm \) are in the fundamental representations:

\[
(\psi_+, \beta_+) \leftrightarrow \langle 1, 0 \rangle, \quad (\beta_-, \psi_-) \leftrightarrow \langle 0, 1 \rangle
\]

(29)

and have scaling dimension 1/2 when \( k = 1 \).

The tensor product of two typical representations is

\[
\langle h_1, j_1 \rangle \otimes \langle h_2, j_2 \rangle = \langle h_1 + h_2, j_1 + j_2 \rangle \oplus \langle h_1 + h_2 - 1, j_1 + j_2 - 1 \rangle, \quad (h_1 + h_2 \neq j_1 + j_2)
\]

(30)

When \( h_1 + h_2 = j_1 + j_2 \) the tensor product gives a new reducible but indecomposable 4-dimensional representation denoted \( \langle h \rangle_{(4)} \):

\[
\langle h_1, j_1 \rangle \otimes \langle h_2, j_2 \rangle = \langle h_1 + h_2 - 1 \rangle_{(4)}
\]

(31)

These representations have \( \Delta_{\langle h \rangle_{(4)}} = 0 \), however they are logarithmic since the casimir \( C_2 \) is not diagonal.

The vertex operators \( V_{\langle h, j \rangle} \) can be explicitly constructed in the free field theory, and require the twist fields in the symplectic fermion sector [35]. As for the spin fields of the Ising model, the twist fields modify the boundary conditions of the fundamental field \( \chi \):

\[
\chi^1(e^{2\pi i z})\mu_\lambda(0) = e^{-2\pi i \lambda} \chi^1(z)\mu_\lambda(0)
\]

\[
\chi^2(e^{2\pi i z})\mu_\lambda(0) = e^{2\pi i \lambda} \chi^2(z)\mu_\lambda(0)
\]

(32)
The properties of these fields were studied in \cite{50}. It is clear from the above equation that $2\pi \lambda$ is a phase and is restricted to $-1 < \lambda < 1$. We also need the doublet of twist fields $\sigma_\lambda^a$, which arise in the OPE:

$$\partial \chi^1(z) \mu_\lambda(0) \sim \frac{\sqrt{1 - \lambda}}{z^{\lambda}} \sigma_\lambda^1, \quad \partial \chi^2(z) \mu_\lambda(0) \sim \frac{\sqrt{\lambda}}{z^{1 - \lambda}} \sigma_\lambda^2 \quad (33)$$

The conformal dimensions of the twist fields are

$$\Delta(\mu_\lambda) = \frac{\lambda(\lambda - 1)}{2} \equiv \Delta_\lambda^{(\chi)}, \quad \Delta(\sigma_\lambda^1) = \Delta_\lambda^{(\chi)} \lambda_{\lambda-1}, \quad \Delta(\sigma_\lambda^2) = \Delta_\lambda^{(\chi)} \lambda_{\lambda+1} \quad (34)$$

The vertex operator for $V_{(h,j)}$ requires twist fields with $\lambda = (h - j)/k$. For $h > j$:

$$V_{(h,j)} = (h - j)^{1/4} \left( -\mu_\lambda e^{i(h\phi^1 - j\phi^2)/\sqrt{k}} \right), \quad \lambda = \frac{h - j}{k} \quad (35)$$

whereas for $h < j$:

$$V_{(h,j)} = (j - h)^{1/4} \left( -\mu_{\lambda+1} e^{i[(h-1)\phi^1 - (j-1)\phi^2]/\sqrt{k}} \right), \quad \lambda = \frac{h - j}{k} \quad (36)$$

The vertex operator $V_{(\lambda)}^{(4)}$ for the representation $\langle h \rangle^{(4)}$ is constructed from the logarithmic field $\epsilon_{ab} \chi^a \chi^b$:

$$V_{(\lambda)}^{(4)} = \left( \begin{array}{c} \chi^1 e^{i(h+1)(\phi^1 - \phi^2)/\sqrt{k}} \\ \sqrt{k} e^{ih(\phi^1 - \phi^2)/\sqrt{k}} \\ -\mu_\lambda e^{i(h\phi^1 - j\phi^2)/\sqrt{k}} \\ \mu_{\lambda+1} e^{i[(h-1)\phi^1 - (j-1)\phi^2]/\sqrt{k}} \\ \chi^2 e^{i(h-1)(\phi^1 - \phi^2)/\sqrt{k}} \end{array} \right) \quad (37)$$

The two middle fields form a logarithmic pair with $\Delta = 0$. It is important that the above logarithmic field has a simple and explicit construction in the second-order symplectic fermion theory; this is not transparent in the minimal model description of $c = -2$, nor in the first-order description. For a review of logarithmic conformal field theory, see \cite{51, 52}.

A closed operator algebra is obtained when $k$ is an integer and the spectrum of fields $V_{(h,j)}$ is restricted to $h, j$ integers satisfying

$$-k \leq h - j \leq k \quad (38)$$

This operator algebra can be viewed as generated by OPE’s of the fundamental vertex operators $V_{(1,0)}$ and $V_{(0,1)}$. For instance:

$$V_{(1,0)}(z)V_{(0,1)}(0) \sim \frac{1}{z^{1/k^2}} V_{(0)}^{(4)} \quad (39)$$
B. \(osp(2|2)_k\).

The finite dimensional representations of \(osp(2|2)\) can be labeled by the \(su(2)\) with generators \(J, J_\pm\) and by the \(u(1)\) charge \(H\). The typical, irreducible representations will be denoted as \([b, s]^{osp}\) where \(s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ..., \}\) is an \(su(2)\) spin and \(b = H/2\). These representations are 8s dimensional. In order to describe their \(su(2) \otimes u(1)\) decomposition, let \([b, s]^{su}\) denote the \(2s + 1\) dimensional representation with \(J/2 = s_3 = -s, -s + 1, ..., s\) and \(H = 2b\). The generic decomposition is

\[
[b, s]^{osp} = [b, s]^{su} \oplus [b + \frac{1}{2}, s - \frac{1}{2}]^{su} \oplus [b - \frac{1}{2}, s - \frac{1}{2}]^{su} \oplus [b, s - 1]^{su}
\]

(40)

As for \(gl(1|1)\), there are atypical, indecomposable but reducible representations at \(b^2 = s^2\). The simplest is 8 dimensional and arises in the following tensor product

\[
[0, \frac{1}{2}]^{OSP} \otimes [0, \frac{1}{2}]^{OSP} = [0, 1]^{OSP} \oplus [8]^{OSP}
\]

(41)

The \([8]^{OSP}\) has the quantum numbers of \([\frac{1}{2}, \frac{1}{2}]^{OSP} \oplus [-\frac{1}{2}, \frac{1}{2}]^{OSP}\).

The stress tensor is built from the single quadratic casimir:

\[
T_{osp(2|2)} = \frac{1}{2(2 - k)} \left[ J^2 - H^2 - \frac{1}{2}(J_+ J_+ + J_- J_-) + (S_+ S_- - S_- S_+) + (\hat{S}_+ \hat{S}_- - \hat{S}_- \hat{S}_+) \right]
\]

(42)

and the typical representations with \(b^2 \neq s^2\) have conformal dimension

\[
\Delta_{[b,s]}^{osp} = \frac{2(s^2 - b^2)}{2 - k}
\]

(43)

At \(k = -2\) there is a free field representation with the same field content as in eq. (22)\cite{47}. This can be derived from the \(gl(1|1)_2\) embedding\cite{35}. In fact:

\[
T_{osp(2|2)_{-2}} = T_{gl(1|1)_2} = T_{gl(1|1)_{-2}}
\]

(44)

VII. DISORDERED CRITICAL POINTS: SUPER SPIN-CHARGE SEPARATION AND THE RENORMALIZATION GROUP.

A. General strategy.

If the random potentials are taken to be gaussian distributed, e.g. \(\mathcal{P}[V] = \exp(-\frac{1}{4\pi\sigma^2} \int d^2 x \ V(x)^2)\), then the functional integrals over all the random potentials can
be performed and one obtains the general form

\[ S_{\text{eff}} = S_{\text{free}} + \int \frac{d^2 x}{2\pi} \sum_A g_A O^A(x) \]  

(45)

where \( S_{\text{free}} \) is the free action of the \( \psi_\pm, \beta_\pm \) fields eq. (11), \( g_A \) are variances that measure the strength of the disorder, and \( O^A \) are marginal operators. The operators \( O^A \) can always be expressed as left-right current-current perturbations, i.e. they are of the form

\[ O^A = \sum_{a,b} d_{ab}^A J^a J^b \]  

(46)

for some bilinears \( d_{ab}^A \), and the currents \( J^a \) are those of the maximal symmetry, \( osp(2N|2N)_1 \) for the QHT and \( osp(4N|4N)_1 \) for the SQHT, and are bilinears in the fields \( \psi_\pm, \beta_\pm \). Since the models have the BRST symmetries for any realization of disorder, the operators \( O^A \) must be BRST invariant, i.e. \( S_{\text{eff}} \) has a \( gl(N|N) \) symmetry in the QHT and an \( osp(2N|2N) \) symmetry for the SQHT. The perturbations \( O^A \) can thus be viewed as anisotropic interactions of the maximal current algebra that are BRST invariant.

As discussed in the introduction, the perturbative RG for the simultaneous flow of all the couplings \( g_A \) does not reveal a fixed point. In order to resolve this difficulty, we propose to perform the RG in two stages, with special attention paid to the symmetries that exist at any realization of disorder.

We will need the following general property. Consider two commuting current algebras \( G_A \) and \( G_B \) with currents \( J_A, J_B \). Furthermore, let us suppose that the stress tensor for a given conformal theory separates as follows:

\[ T_{\text{cft}} = T_{\nu A} + T_{\nu B} \]  

(47)

Consider the perturbation of the conformal field theory by left-right current-current perturbations:

\[ S = S_{\text{cft}} + \int \frac{d^2 x}{2\pi} \left( g_A J_A \cdot J_A + g_B J_B \cdot J_B \right) \]  

(48)

where \( J \cdot J \) is the invariant built on the quadratic casimir. Since the currents commute, the RG beta-functions decouple; to 1-loop the result is:

\[ \frac{dg_A}{d\ell} = C_{\text{adj}}^A g_A^2, \quad \frac{dg_B}{d\ell} = C_{\text{adj}}^B g_B^2 \]  

(49)

where \( \ell \) is the logarithm of the length scale and \( C_{\text{adj}}^A \) is the casimir for the adjoint representation of the finite dimensional part of \( G_A \). Let us suppose that the physical regime
corresponds to positive \( g_{A,B} \). If \( \text{C}_{B} \text{adj} \) is positive, then the coupling \( g_{B} \) is marginally relevant and the flow is to infinity. This is a massive sector and the \( \mathcal{G}_{B} \) degrees of freedom are “gapped out” in the RG flow. If \( \text{C}_{A} \text{adj} \) is negative, then the coupling \( g_{A} \) is marginally irrelevant. This results in the fixed point characterized by the \( \mathcal{G}_{A} \) current algebra. If the original conformal field theory corresponds to the current algebra \( \mathcal{G}_{\text{max}} \), then the fixed point may be viewed as the coset \( \mathcal{G}_{\text{max}}/\mathcal{G}_{B} \). This scenario was proposed for generic fixed points of marginal current-current perturbations in \[34\], however what was missing in the argument was the spin-charge separation. The resulting coset is a somewhat trivial example of the GKO construction \[53\] because of the separation of the stress tensor.

In the first stage of the RG flow we consider only the disorder couplings for the \( \mathcal{P}_{N} \) invariant BRST symmetries and the additional “copy” symmetries. The essential ingredient here is the spin-charge separation \[17,35,46\]. In the first stage we will obtain a critical point corresponding to the \( \mathcal{P}_{N} \) invariant BRST symmetry. In other words, in the first stage of the RG flow we identify the massless degrees of freedom that are most important at the critical point. In the second stage of the RG flow, we reintroduce the other kinds of disorder as further relevant perturbations of these massless degrees of freedom.

### B. QHT

The \( \mathcal{P}_{N} \) invariant BRST symmetry is \( \text{gl}(1|1)_{N} \) and the copy symmetry that commutes with it is \( \text{su}(N)_{0} \). Remarkably there exists the following spin-charge separation \[35\]:

\[
T_{\text{free}}^{N-\text{copy}} = -\frac{1}{2} \sum_{\alpha=1}^{N} (\psi_{-}^{\alpha} \partial_{z} \psi_{+}^{\alpha} + \beta_{-}^{\alpha} \partial_{z} \beta_{+}^{\alpha}) = T_{\text{gl}(1|1)_{k=N}} + T_{\text{su}(N)_{0}}
\]

where the stress tensors \( T_{\text{gl}(1|1)_{k=N}} \) and \( T_{\text{su}(N)_{0}} \) are the Sugawara stress tensors for interacting theories. Simple checks of the above result are as follows. First, all of the stress tensors have \( c = 0 \). The free theory contains \( 4N \) fields \( \psi_{\pm}^{\alpha}, \beta_{\pm}^{\alpha} \). Under the \( \text{gl}(1|1) \otimes \text{su}(N) \) they transform as \( \langle (1,0) \oplus (0,1) \rangle \otimes [\text{vec}] \) where \( [\text{vec}] \) is the \( N \)-dimensional vector representation of \( \text{su}(N) \). The later has conformal dimension \( \Delta_{\text{su}(N)_{k}} = \frac{N^{2}-1}{2N(k+N)} \) at general level \( k \), whereas \( \Delta_{(1,0)} = \Delta_{(0,1)} = \frac{1}{2k^{2}} \). One sees that the dimensions add up properly: \( \Delta_{(1,0)} + \Delta_{\text{su}(N)_{0}} = 1/2 \), as is appropriate for the free \( \psi_{\pm}, \beta_{\pm} \) fields.

In the first stage of the RG flow we consider the action of the form eq. \( \text{(48)} \) where \( \mathcal{G}_{A} = \text{gl}(1|1)_{N} \) and \( \mathcal{G}_{B} = \text{su}(N)_{0} \). For \( \text{su}(N), \text{C}_{\text{adj}} > 0 \), and it is gapped out in the flow. For
$gl(1|1)_N$ current perturbations the situation is somewhat more subtle because there are two quadratic casimirs\[16\]. Consider

$$S = S_{\text{free}} + \int \frac{d^2x}{2\pi} \left[ g \left( J\overline{J} - H\overline{H} + S_+\overline{S}_- - S_-\overline{S}_+ \right) + g'(J - H)(\overline{J} - \overline{H}) \right] \quad (51)$$

Then the 1-loop beta function for $g$ is zero, whereas $dg'/d\ell = -g^2\[16\]$. (Here we fixed the sign by setting some couplings to zero in the more general result in \[34\].) Therefore these $gl(1|1)$ current interactions are irrelevant. It is important that this is in contrast to the situation for the model in the Gade-Wegner universality class which has the same beta function up to a sign\[16\], $dg'/d\ell = +g^2$, which implies the disorder in that case is marginally relevant. This difference in sign is a consequence of the detailed form of the hamiltonian in \[16\] which acts on a 4-component wavefunction rather than two. The sign of the beta function can also be flipped by considering an imaginary gauge potential. The higher loop corrections computed in \[34\] do not alter this picture. Thus in the first stage of the RG flow, one flows to the fixed point $gl(1|1)_N$.

It is important to point out that one feature of this scheme for arriving at the fixed point $gl(1|1)_N$ is that the critical exponents now depend on $N$, whereas in the original $gl(N|N)$ invariant theory they were independent of $N$. This is due to the fact that we have gapped out the $SU(N)_0$ sub-algebra of the $gl(N|N)_1$. On the other hand, the common assumption in the literature is that the critical exponents should be independent of $N$, even though an RG scenario that achieves this has not yet been proposed. We will return to this issue where it must be faced in section VIII on the multi-fractal exponents.

Additional kinds of disorder can now be incorporated as follows. The original currents for the maximal symmetry $osp(2N|2N)_1$, eq. \[14\] can be classified according to the $gl(1|1)_N \otimes su(N)_0$. Since the RG flow in the first stage gaps out the $su(N)_0$, what remains are the $gl(1|1)_N$ degrees of freedom. The possible $gl(1|1)_N$ representations the residual currents fall into follows from the fact that they are bilinears in the fields $\psi_\pm, \beta_\pm$, and the latter correspond to the $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ representations, eq. \[29\]. Using the tensor products

$$\langle 1, 0 \rangle \otimes \langle 1, 0 \rangle = \langle 2, 0 \rangle \oplus \langle 1, -1 \rangle$$
$$\langle 0, 1 \rangle \otimes \langle 0, 1 \rangle = \langle 0, 2 \rangle \oplus \langle -1, 1 \rangle$$
$$\langle 1, 0 \rangle \otimes \langle 0, 1 \rangle = \langle 0 \rangle_{(4)}$$

one sees that the $J_\pm^{ab}$ currents transform in the $\langle 2, 0 \rangle, \langle 1, -1 \rangle, \langle 0, 2 \rangle$ and $\langle -1, 1 \rangle$ representa-
tions, whereas $H^{ab}$ transform in the $\langle 0 \rangle_{(4)}$. Note that $\langle 0 \rangle_{(4)}$ has the same quantum numbers as the adjoint of $gl(1|1)$, and represents what is left of the $gl(N|N)$ after gapping out the $su(N)_0$. The conformal dimensions of these representations are

\[
\Delta_{(2,0)} = \Delta_{(-1,1)} = \frac{2 + N}{N^2}
\]

\[
\Delta_{(0,2)} = \Delta_{(1,-1)} = \frac{2 - N}{N^2}
\]

and $\Delta_{(0)_{(4)}} = 0$.

Let $\Phi_r(z, \overline{z})$ denote the $gl(1|1)$ invariant local field associated with the representation $r$ of $gl(1|1)$ with scaling dimension $2\Delta_r$. It can be expressed as a product of left-right vertex operators $V_r \cdot \overline{V_r}$. For general $N$, the fields $\Phi_{(2,0)}, \Phi_{(-1,1)}, \Phi_{(0,2)}$ and $\Phi_{(1,-1)}$ are expressed in terms of the twist fields $\mu^\lambda$ and $\sigma^a_\lambda$ with $\lambda = 2/N, 1 - 2/N$. Explicit expressions can be found in [35]. The field $\Phi_{(0)_{(4)}}$ on the other hand requires only the symplectic fermion and bosons, and for $N \leq 2$ it is the most relevant operator. If we keep only the most relevant operator, then we should consider

\[
S = S_{gl(1|1)_N} + g \int \frac{d^2x}{8\pi} \Phi_{(0)_{(4)}}
\]

\[
= \int \frac{d^2x}{8\pi} \left( \sum_{a,b=1}^{2} \eta_{ab} \partial_\mu \phi^a \partial_\mu \phi^b + \epsilon_{ab} \partial_\mu \chi^a \partial_\mu \chi^b + g \chi^1 \chi^2 \cos \left( \frac{(\phi^1 - \phi^2)/\sqrt{N}}{\sqrt{N}} \right) \right)
\]

where here, and henceforth, $\phi, \chi$ are the local fields $\phi, \chi$.

The case of $N = 2$ is distinct since both $\Phi_{(0,2)}$ and $\Phi_{(1,-1)}$ have $\Delta = 0$ which is degenerate with the dimension of $\Phi_{(0)_{(4)}}$, so that the latter is no longer the most relevant operator. The explicit forms at $N = 2$ are [35]:

\[
\Phi_{(1,-1)} - \tilde{\Phi}_{(0,2)} = 4\chi^1 \chi^2 \cos \left( \frac{(\phi^1 + \phi^2)/\sqrt{2}}{\sqrt{2}} \right) + 4(\partial_\mu \phi^1 \partial_\mu \phi^2)(\chi^1 \chi^2) \cos \left( \frac{\sqrt{2} \phi^2}{\sqrt{N}} \right)
\]

($\tilde{\Phi}_{(0,2)}$ only differs from $\Phi_{(0,2)}$ by some fermionic exchange signs.) For $N > 2$ the operators $\Phi_{(0,2)}$ become more relevant than $\Phi_{(0)_{(4)}}$ and may need to be included as additional perturbations in eq. (54).

The important feature of logarithmic perturbations such as eq. (54) is that we believe they should not drive the theory to another fixed point, but rather just give logarithmic corrections to correlation functions. General arguments were given in [40] for marginal logarithmic perturbations. Here the logarithmic perturbation has dimension zero and is thus strongly relevant so the arguments in [40] do not necessarily apply. However arguments based
on the fact that the perturbation has exactly dimension zero were given in [35] supporting
the idea that one is not driven to a new fixed point. For the concrete model eq. (54) this
is easy to see since, due to the indefinite signature of the scalar fields, the OPE of the
exponentials is regular:
\[ e^{ia(\phi^1 - \phi^2)(z)} e^{ib(\phi^1 - \phi^2)(0)} \sim \text{regular} \] (56)
This implies that in conformal perturbation theory the perturbation behaves like a mass
term \[ \chi_1 \chi_2 \] , and does not lead to contributions to the beta function at any order in pertur-
bation theory. A contribution to the beta function for \( g \) would require a singular term in
the OPE of the form \[ \Phi \langle 0 \rangle^{(4)}_x \Phi \langle 0 \rangle^{(4)}_0 \sim \Phi \langle 0 \rangle^{(4)}_x \], however there is no such term. Although
we cannot give any stronger arguments at this stage, in the sequel we will adopt the work-
ing hypothesis that dimension zero logarithmic perturbations essentially do not change the
critical exponents.

C. SQHT

The version of super spin-charge separation we need was proven in [46]:
\[ T_{\text{free-copy}}^{2N-2N} = T_{\text{osp}(2|2)-2N} + T_{\text{sp}(2N)} \] (57)
Here the check of the scaling dimensions goes as follows. The 8N fields \( \psi^{a,\pm,1}_{\pm,1}, \beta^{a,\pm,1}_{\pm,1} \) transform as \([0, 1/2]^{\text{osp}} \otimes \text{vec}\] under the \( \text{osp}(2|2)-2N \otimes \text{sp}(2N) \) where the vector representation of \( \text{sp}(2N) \)
is 2N dimensional. For general \( k \) the latter has dimension \( \Delta_{\text{sp}(2N)} = \frac{2N+1}{4(k+N+1)} \). At level
\( k = -2N \) from eq. (13) one has \( \Delta^{\text{osp}}_{[0, 1/2]} = \frac{1}{4(N+1)} \). Again one has \( \Delta^{\text{osp}}_{[0, 1/2]} + \Delta_{\text{sp}(N)} = 1/2 \), as
required.

Repeating the same arguments as for the QHT, since \( C^{\text{adj}}_{\text{sp}(2N)} > 0 \) and \( C^{\text{adj}}_{\text{osp}(2|2)} < 0 \), the
first stage of the RG flows takes us to the current algebra \( \text{osp}(2|2)-2N \). For \( N = 1 \) this RG
flow was studied in greater detail in [17, 37], where it was viewed as a fine-tuning of the
initial model. (The couplings \( g_c \) and \( g_s \) in [37] correspond to the couplings for the \( \text{osp}(2|2) \)
and \( \text{su}(2) \) currents respectively.)

As before, in the second stage we restore other kinds of disorder by examining the quan-
tum numbers of the remaining degrees of freedom. The residual currents are bilinears of
fields in the \([0, 1/2]^{\text{osp}} \) thus their quantum numbers follow from eq. (11). Since \( \Delta^{\text{osp}}_{[0, 1]} = \frac{1}{N+1} \)
and the \([8]^{\text{osp}} \) has \( \Delta = 0 \), for any \( N \) the dimension zero logarithmic field \( \Phi_{[8]} \) is the most
relevant operator. Therefore, the additional kinds of disorder should correspond to the logarithmic perturbation:

\[ S = S_{osp(2|2) - 2N} + g_8 \int \frac{d^2x}{2\pi} \Phi[8] \]  

(58)

where \( S_{osp(2|2) - 2N} \) formally represents the conformal field theory with the current algebra symmetry. For \( N = 1 \) the detailed study of the effective action indeed leads to the same conclusion \([17, 37]\).

For the special case of \( N = 1 \), \( osp(2|2) - 2 \) has a free field representation with the same content as for \( gl(1|1)_k \), so that \( S_{osp(2|2) - 2} \) has the free field form in eq. \([22] 35, 47\). The explicit form of the perturbation was given in \([35]\):

\[ \Phi[8] = 4 \chi^1 \chi^2 \left( \cosh((\phi^1 - \phi^2)/\sqrt{2}) + \cosh((\phi^1 + \phi^2)/\sqrt{2}) \right) + 4(\partial_\mu \chi^1 \partial_\mu \chi^2)(\chi^1 \chi^2) \cosh(\sqrt{2}\phi^1) \]  

(59)

For general \( N \) the free field representation requires more fields \([54]\).

Since the additional forms of disorder correspond to the above logarithmic perturbation, as we argued above, the exponents for the SQHT should be contained in the \( osp(2|2) - 2N \) theory. For the 1-copy theory, this is more transparent using the \( gl(1|1)_2 \) embedding since \( T_{gl(1|1)_2} = T_{osp(2|2) - 2} \) \([35]\). The vector representation of \( osp(2|2) \) corresponds to the \( gl(1|1)_2 \) fields \( \Phi(1,0) \) and \( \Phi(0,1) \) with scaling dimension \( 2\Delta_{(1,0)} = 1/4 \), and this determines the density of states exponent \( \rho(E) \sim E^{1/7} \), since \( 1/7 = \Delta/(1 - \Delta) \) with \( \Delta = 1/8 \). The remaining low dimension fields are \( \Phi(2,1) \) and \( \Phi(1,2) \) with \( \Delta = 5/8, -3/8 \) respectively. The other fields have dimensions which differ by an integer from the fields considered thus far. The \( \Delta = 5/8 \) field determines the correlation length exponent for percolation: \( \nu_{perc} = (2(1 - 5/8))^{-1} = 4/3 \). (In the \( osp(2|2) - 2 \) description, the \( \Delta = 5/8 \) field is a descendant of the field \([\pm 1, \frac{1}{2}]^{osp} \) with \( \Delta = -3/8 \).) Both of these exponents agree with the exact results in \([22]\). Note that the \( c = 0 \) minimal model field with \( \Delta = 1/3 \), which determines the localization length for self-avoiding walks, is not contained in the spectrum, which is consistent with observations made in \([23, 56]\). Our proposal appears to be consistent with other observations made in \([56]\), since, because of the logarithmic perturbation, the critical point is not strictly speaking a conformal current algebra, even though it has some of the same exponents. A further check will be given in the next section based on the multi-fractal exponents.

On the other hand, the above spectrum does not contain the full spectrum for the SQHT proposed in \([56]\). The latter was based on a specific mapping of the lattice model to a
Coulomb gas. This is not necessarily inconsistent with our results, since the above spectrum is based on the simplest closed operator algebra and it contains the main exponents that are known to be physically meaningful. It seems likely that other twisted $\chi$ sectors and/or bosonic $\phi$ sectors could be consistently added to our theory, however we have not studied in detail the possibility of obtaining exactly the partition functions proposed in [56].

VIII. MULTI-FRACTAL EXPONENTS.

A. Generalities.

In order to study multi-fractality in the density of states, we add an energy term in the action corresponding to $H \rightarrow H - \mathcal{E}$:

$$S_\mathcal{E} = \int \frac{d^2x}{2\pi} i\mathcal{E} (\overline{\Psi}_- \Psi_+ + \Psi_- \overline{\Psi}_+)$$

The density of states operator is then

$$\rho(x) = \overline{\Psi}_- \Psi_+ + \Psi_- \overline{\Psi}_+$$

Multi-fractal properties refer to disorder averages of q-th moments of $\rho$, $\rho^q$, and are simply related to wave-function $\psi$ multi-fractality since $\rho = \psi^\dagger \psi$. Properly normalized quantities are

$$P^{(q)} = \frac{\int d^2x \rho(x)^q}{(\int d^2x \rho(x))^q}$$

where here $\rho$ represents $\langle \rho \rangle$ at fixed disorder. At a critical point $P^{(q)}$ scales as

$$P^{(q)} \sim L^{-\tau_q}$$

where $L$ is the system size. The exponents $\tau_q$ are related to the scaling dimensions of operators as follows:

$$\tau_q = \Gamma_q - q\Gamma_1 + 2(q - 1)$$

where $\Gamma_q$ is the scaling dimension of $\rho^q$ in the effective disorder averaged theory.

For both the QHT and SQHT, there is a regime at low $q$ where $\tau_q$ is quadratic in $q$. Since $\tau_1 = 0$ and $\tau_0 = -2$, in this parabolic regime $\tau_q$ is characterized by a single parameter $\alpha_0$:

$$\tau_q = (2 - \alpha_0)q^2 + \alpha_0q - 2$$
Since the $q \Gamma_1$ term is simply a matter of normalization, it is meaningful to define

$$\tilde{\Gamma}_q \equiv \Gamma_q - q \Gamma_1 = (\alpha_0 - 2)q(1 - q)$$

(66)

For the purpose of comparing with numerical simulations, one can perform the Legendre transformation

$$f(\alpha) = \alpha q - \tau_q, \quad \alpha = \frac{d\tau_q}{dq}$$

(67)

One finds $q = (\alpha - \alpha_0)/(2(2 - \alpha_0))$, which leads to

$$f(\alpha) = -\frac{(\alpha - \alpha_0)^2}{4(\alpha_0 - 2)} + 2$$

(68)

The parameter $\alpha_0$ determines the typical density of states $\exp(\log(\rho)) \sim L^{-\alpha_0}$.

A common belief expressed in the literature is that in the supersymmetric method for disorder averaging, $q$-th moments can be calculated in the $N$-copy theory for any $N$ greater than $q$ and one should obtain a result independent of $N$. This can be proven by simple manipulations of the path integral, and this property is manifested for instance in the $N$-independence of RG beta functions. For the QHT (SQHT), it follows from properties of the $gl(N|N)$ ($osp(2N|2N)$) BRST symmetries of the theory before and after disorder averaging. This “copy-symmetry” is analogous to replica-symmetry. In the scheme proposed in this paper, this copy-symmetry is broken in the first stage of the RG flow since $gl(N|N)$ is broken to $gl(1|1)$ for the QHT and similarly for the SQHT, as emphasized in section VII. Thus, the $q$-th moments will depend on the number of copies $N$ one started with. Since this is contrary to expectations, at this stage it must be viewed as a working hypothesis. However we can provide some justification for this copy-symmetry breaking. First of all, the breaking of symmetries in the flow to a low energy fixed point is a common phenomenon, and in higher dimensions is more the rule rather than the exception. In the present context one must bear in mind the Mermin-Wagner theorem which prohibits the spontaneous breaking of symmetries. However the breaking of $SU(2) \otimes SU(2)$ to $SU(2)$ based on spin-charge separation in the $1d$ Hubbard model, which is the prototype for our RG scheme, is well-understood and known not to violate any theorems: the symmetry is broken in the RG flow but not spontaneously and there are no Goldstone bosons. Secondly, in spite of the powerful map to percolation for the SQHE, it remains unknown how to obtain the multifractal spectrum that has been found numerically, i.e. $\tilde{\Gamma}_q \approx q(1 - q)/8$ for continuous $q$ from this map. (See below.) Although it may just be a matter of time before this is eventually
understood, it could instead suggest that some basic assumptions are incorrect. The fact that we can obtain this result quite easily in the copy-symmetry broken fixed point is a positive indication. Thirdly, on the face of it, unbroken copy symmetry seems incompatible with the phenomenon of multi-fractality termination, whereas our scheme in fact relies on it to fix \( N \), as we now describe.

For \( q \) greater than some critical value \( q_c \), \( \tau_q \) is no longer parabolic. This phenomenon of multi-fractality termination is thought to be distinct from the considerations of this paper, i.e. it is a separate issue unrelated to the RG flow of the disorder couplings\(^{[57, 58, 59]} \). For the QHT, \( 2 < q_c < 3 \), whereas for the SQHT, \( 3 < q_c < 4 \). For the SQHT, the map to percolation was used to obtain \( \tau_q \) only for the integer values \( q = 1, 2, 3 \) in \([44]\) and the argument breaks down for \( q > 3 \).

If one wishes to compute then \( \hat{\Gamma}_q \) for all of \( q \) continuous and less than \( q_c \) in a single, fixed theory, then this should be possible in the \( N \) copy theory with \( N \) fixed to be the largest integer less than \( q_c \), i.e. \( N = 2 \) for the QHT and \( N = 3 \) for the SQHT. We wish to emphasize again that this new approach gives results that are different from the spectrum of multi-fractal exponents studied in \([13, 58]\), and also differs from the calculation in \([37]\). In the rest of this section we show that these assumptions appear to give results in very good agreement with numerical work.

### B. QHT

The energy operator corresponds to the \( gl(1|1) \) fields

\[
\rho = \Phi_{(1,0)} + \Phi_{(0,1)}
\]

We thus first need the \( gl(1|1) \) content of \( \rho^q \) for integer \( q \). Using

\[
\langle h, j \rangle \otimes (\langle 1, 0 \rangle \oplus \langle 0, 1 \rangle) = \langle h + 1, j \rangle \oplus \langle h, j - 1 \rangle \oplus \langle h, j + 1 \rangle \oplus \langle h - 1, j \rangle
\]

one sees that the above tensor product involves \( h + j \) new = \( h + j \pm 1 \) and \( h - j \) new = \( h - j \pm 1 \). Therefore \( \rho^q \) contains the representations \( \langle h, j \rangle \) with \(-q \leq h + j \leq q, -q \leq h - j \leq q \). Examining \( \Delta_{(h,j)} \) one finds that the most relevant operator in \( \rho^q \) has \( h = 0, j = q \). As explained above, we now set \( N = 2 = k \), and the field \( \Phi_{(0,q)} \) has dimension \( \Gamma_q = 2 \Delta_{(0,q)} = q(2 - q)/4 \), which gives

\[
\hat{\Gamma}_q = \frac{q(1 - q)}{4}
\]
i.e. \( \alpha_0 = 9/4 \). We are implicitly analytically continuing the \( j \) quantum number to continuous \( q \). This agrees very favorably with the numerical results in \[41\], \( \alpha_0 = 2.26 \pm .01 \), and in \[42\], \( \alpha_0 = 2.260 \pm .003 \).

It needs to be emphasized that the copy-breaking feature of the RG flow in our scheme implies that \( \Gamma_1 = 1/4 \) in the \( N = 2 \) copy theory is not supposed to equal the usual \( N = 1 \) result, i.e. \( \Gamma_1 = 0 \).

C. SQHT

For the SQHT the density operator corresponds to the \( osp(2|2) \) field:

\[
\rho = \Phi^{osp}_{[0, \frac{1}{2}]}
\]

We need the quantum numbers of \( \rho^q \) for \( q = 2, 3 \). We have already considered \( q = 2 \) in eq. \[41\]. Since the \([8]^{osp}\) can be viewed as \([\frac{1}{2}, \frac{1}{2}]^{osp} \oplus [-\frac{1}{2}, \frac{1}{2}]^{osp}\), taking one more tensor product and using the rules in \[15\] one obtains

\[
[0, \frac{1}{2}] \otimes [0, \frac{1}{2}] \otimes [0, \frac{1}{2}] = [0, \frac{3}{2}] \oplus 3[0, \frac{1}{2}] \oplus 2[\frac{1}{2}, 1] \oplus 2[-\frac{1}{2}, 1] \oplus [1, \frac{1}{2}] \oplus [-1, \frac{1}{2}]
\]

where \([b,s]^{osp}\) refers to \([b,s]^{osp}\). Examining the conformal dimensions \( \Delta^{osp}_{[b,s]} \), one finds that for \( q = 1, 2, 3 \) the most relevant operator in \( \rho^q \) corresponds to \([\frac{2-1}{2}, \frac{1}{2}]^{osp}\). Fixing \( N = 3 \), i.e. \( k = -6 \), as suggested above, one then has \( \Gamma_q = 2\Delta^{osp}_{[\frac{2-1}{2}, \frac{1}{2}]} = q(2 - q)/8 \) which gives

\[
\hat{\Gamma}_q = \frac{q(1 - q)}{8}
\]

In this case one has \( \alpha_0 - 2 = 1/8 \). Again \( \Gamma_1 = 1/8 \) in the \( N = 3 \) theory is not the same as the well-known \( \Gamma_1 = 1/4 \) of the \( N = 1 \) theory (see section VIIC).

In \[32, 43\] it was found numerically that \( \alpha_0 - 2 \approx 1/8 \) in the parabolic approximation (within a few percent). However it was also observed that in comparison to the QHT, for the SQHT there are more marked deviations from parabolicity. If one takes into account the non-parabolicity and simply defines \( \alpha_0 \) from the maximum of \( f(\alpha) \), then one obtains the result \( \alpha_0 - 2 = 0.137 \pm 0.003 \). These observed deviations from a parabolic regime could have a number of explanations in our model. It could be due to the effects of logarithmic corrections due to the \( \Phi_{[8]} \) perturbation. It could also be due to the large number of operators in eq. \[73\], where we took only the most relevant.

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IX. LOCALIZATION LENGTH EXPONENT.

In simulations of the network models, one needs to tune to a critical point by adjusting a parameter $\lambda$, analogous to tuning to the critical probability $p_c = 1/2$ in 2D classical percolation. In our description, this should correspond to a term in the action

$$\delta S_\nu = \int \frac{d^2x}{2\pi} \lambda O_\nu(x)$$

for some operator $O_\nu$. If $O_\nu$ has scaling dimension $\Gamma_\nu$, then $\lambda$ has dimension $2 - \Gamma_\nu$ and the correlation length diverges as $\xi_c \sim (\lambda - \lambda_c)^{-\nu}$ with $\nu = 1/(2 - \Gamma_\nu)$.

In contrast to the density of states exponents, we do not have arguments based on quantum numbers to identify the field $O_\nu$. In the SQHT we know that $\Gamma_\nu = 5/4$. In the $gl(1|1)_2$ embedding in the 1-copy theory for the SQHT the field $O_\nu$ thus corresponds to $\Phi_{(2,1)}$, which has the explicit form

$$\Phi_{(2,1)} = \mu_{1/2} \theta_{1/2} e^{i(2\phi_1 - \phi_2)/\sqrt{2}} + \sigma_{1/2} \bar{\sigma}_{1/2} e^{i\phi_1/\sqrt{2}}$$

Above, $\mu_{1/2}$ and $\sigma_{1/2}$ are twist fields with conformal dimension $-1/8$ and $3/8$ respectively.

Lacking a first-principles identification of $O_\nu$ for the QHT, we can only give plausible values based on the spectrum of dimensions in our model. It was understood long ago that one must consider at least $N = 2$ copies, since the exponent describes criticality in the conductance. The latter is related to a product of retarded/advanced 2-point Green functions, and one needs separate copies for retarded verses advanced. We have already used the $N = 2$ copy exponents to explain the multi-fractality in the density of states. Therefore, it seems likely that one needs to consider $N > 2$ copies. Let us therefore double the number of copies one more time and consider the $N = 4$ theory. As explained in section VII, when $N > 2$ there are potentially dangerous perturbations corresponding to the operators $\Phi_{(0,2)}$ and $\Phi_{(1,-1)}$ since $\Phi_{(0,1)}$ is no longer the most relevant operator. The possibility of such additional perturbations can be further investigated, however since this is beyond the original scope of this paper, let us simply assume this issue is not important, and consider the theory $gl(1|1)_4$ plus the logarithmic perturbation $\Phi_{(0,1)}$ as in eq. (54). The operator that most closely parallels $O_\nu$ for the SQHT is $\Phi_{(N,N-1)}$ with conformal dimension

$$\Delta_{(N,N-1)} = \frac{2N(N - 1) + 1}{2N^2}$$

(77)
in the $N$-copy theory, and leads to $\nu = N^2/(2N - 1)$. At $N = 4$, the operator $\Phi_{(4,3)}$ has $\Delta = 25/32$ and has the form

$$\Phi_{(4,3)} = \mu_{1/4} e^{i(\phi_1 - 3\phi_2)/2} + \sigma_{1/4} e^{i(3\phi_1 - 2\phi_2)/2}$$

(78)

Here the twist fields $\mu_{1/4}$ and $\sigma_{1/4}$ have $\Delta = -3/32$ and $5/32$ respectively. The above fields are local since $\Delta(x) = \Delta(1-x)$. If one identifies the above operator with $O_\nu$, then this gives the exponent $\nu = 16/7$, which is within $2\%$ of the numerical results $2.35 \pm 0.03$ and $2.33 \pm 0.03$ (For a survey of the various methods see [29]). The value $16/7 \approx 2.29$ is also consistent with the experimental measurement $2.3 \pm 0.1$.

X. CONCLUSIONS

In summary, by carrying out the RG flow in two stages and using a new form of super spin-charge separation, we argued that the disordered Dirac fermion theories for the QHT and SQHT are described by logarithmic perturbations of the current algebras $gl(1\mid 1)_N$ and $osp(2\mid 2)_{-2N}$ where $N$ is the number of copies. The explicit forms of the resulting actions were constructed using the recent results in [35]. We also argued that the logarithmic perturbations do not modify the exponents of the current algebra theories, however they generally lead to logarithmic corrections to correlation functions.

The unconventional outcome of our RG scheme is that the $N$-copy symmetries implicit in the $gl(N\mid N)$ symmetry of the theory before disorder averaging are broken in the flow to the low energy fixed point in the first stage since the $SU(N)_0$ is gapped out in the flow. This leads to $N$-dependence of the $q$-th moments, contrary to common expectations. Although this may seem problematic, we gave several arguments in favor of it in section VIII, and showed that it leads to a computation of the multi-fractal exponents in the parabolic regime, which agree very favorably with known numerical results. We also speculated on the localization length exponent for the QHT, and suggested one needs to consider $N = 4$ copies. After making some plausible assumptions, we were led to suggest the value $\nu = 16/7$, however more investigations of this proposal are clearly necessary.

The QHT and SQHT are the smallest members of two of the main classes of disordered Dirac fermions [11], and the chiral GUE class was already solved in [16]. It would be interesting to investigate if the methods in this paper could be extended to the other classes as
XI. ACKNOWLEDGMENTS

I would like to thank the organizers of the program *Strong fields, Integrability, and Strings* at the Isaac Newton Institute for Mathematical Sciences during which this work was begun in July 2007. I would also like to thank A. Mirlin for correspondence.

XII. APPENDIX A: SUPER-CURRENT ALGEBRAS.

Let \( g \) denote a finite dimensional super Lie algebra and \( \{ J^a \} \) its generators. Each generator \( J^a \) can be assigned a grade \( [a] = 0 \) for bosonic generators and \( [a] = 1 \) for fermionic ones. The super Lie algebra can be presented as

\[
J^a J^b - (-)^{[a][b]} J^b J^a = f^{ab}_c J^c
\]  

(79)

In the super current algebra at level \( k \), denoted \( g_k \), the above generators are promoted to fields satisfying the OPE

\[
J^a(z)J^b(0) \sim \frac{k}{z^2} \eta^{ab} + \frac{1}{z} f^{ab}_c J^c(0)
\]

(80)

The currents have the mode expansion \( J^a(z) = \sum_n J^a_n z^{-n-1} \) and the zero modes \( J^a_0 \) satisfy eq. (79).

An important construction is the Sugawara stress tensor built on the casimir:

\[
T = \kappa \sum_a J^a J^a
\]

(81)

where the coefficient \( \kappa \) is fixed by the requirement \( T(z)J^a(0) \sim J^a(0)/z^2 \). The conformal scaling dimension of a primary field in the representation \( r \) of \( g \), is given by \( \kappa C_2 \) where \( C_2 \) is the quadratic casimir for \( r \).

Our conventions for the level \( k \) are based on the free field representations of these algebras in terms of the fields \( \psi_\pm, \beta_\pm \). Let \( \Psi_\pm = (\psi_\pm, \beta_\pm) \) denote 2 component fields and \( \Psi^a_\pm \), \( a = 1, 2, \ldots, N \) the \( N \)-copy version. Let us arrange all these fields into \( 2N \) component fields \( \Psi^a_\pm, a = 1, \ldots, 2N \) and let \( [a] \) be the grade. A complete basis of currents is defined in eq. (14). OPE’s can be readily computed from the OPE’s of the \( \Psi^a_\pm \) in eq. (13) and will serve...
as our defining relations for the level $k = 1$ super current algebra. For instance,

$$H^{ab}(z)H^{cd}(0) \sim \frac{k}{z^2}(-)^{|b|+1}\delta^{bc}\delta^{ad} + \frac{1}{z}\left((-)^{|b|+1}\delta^{bc}H^{ad} + (-)^{|a|(|b|+|c|)+|b||c|}\delta^{ad}H^{bc}\right) \quad (82)$$

where $k = 1$. There are similar relations of the form $H(z)J_\pm(0) \sim J_\pm/z$ and $J_+(z)J_-(0) \sim k/z^2 + H/z$. We take the above OPE’s as the defining relations of $osp(2N|2N)_k$ for general $k$. The $H^{ab}$ form a closed subalgebra which defines $gl(N|N)_k$.

Generally the Lie super-algebra $g$ can be decomposed into its bosonic generators $g^{(0)}$ and fermionic generators $g^{(1)}$, $g = g^{(0)} \oplus g^{(1)}$. The bosonic generators form a closed subalgebra, and the fermionic generators fall into representations of $g^{(0)}$. For $osp(2N|2N)$, $g^{(0)} = so(2N) \oplus sp(2N)$ and $g^{(1)}$ corresponds to the representation $(2N, 2N)$. The dimension of $osp(2N|2N)$ is thus $8N^2$ and its rank is $2N$.

The algebra $gl(N|N)$ is $psl(N|N) \oplus u(1) \oplus u(1)$ where $psl(N|N)$ is denoted $A(N-1, N-1)/Z$ in [63]. The bosonic subalgebra is $g^{(0)} = sl(N) \oplus sl(N) \oplus u(1) \oplus u(1)$. It has dimension $4N^2$ and its rank is $2N$.

For the current algebras $gl(1|1)_k$ and $osp(2|2)_k$ we adopt a more specific notation. Again our conventions for the levels are based on the $N = 1$ copy of the fields $\psi_\pm, \beta_\pm$. Define:

$$
H = \psi_+\psi_-, \quad J = \beta_+\beta_-, \quad J_\pm = \beta_\pm^2
$$

$$
S_\pm = \pm\psi_\pm\beta_\pm, \quad \hat{S}_\pm = \psi_\pm\beta_\pm
$$

These currents satisfy the OPE’s

$$
J(z)J(0) \sim -k/z^2, \quad H(z)H(0) \sim k/z^2
$$

$$
J(z)J_\pm(0) \sim \pm 2/z J_\pm, \quad J_+(z)J_-(0) \sim 2k/z^2 - 4/z J
$$

$$
J(z)S_\pm(0) \sim \pm 1/z S_\pm, \quad J(z)\hat{S}_\pm(0) \sim \pm 1/z \hat{S}_\pm
$$

$$
H(z)S_\pm(0) \sim \pm 1/z S_\pm, \quad H(z)\hat{S}_\pm(0) \sim \mp 1/z \hat{S}_\pm
$$

$$
J_\pm(z)S_\pm(0) \sim 2/z \hat{S}_\pm, \quad J_\pm(z)\hat{S}_\pm(0) \sim -2/z S_\pm
$$

$$
S_\pm(z)\hat{S}_\pm(0) \sim \pm 1/z J_\pm
$$

$$
S_+(z)S_-(0) \sim k/z^2 + 1/z (H - J)
$$

$$
\hat{S}_+(z)\hat{S}_-(0) \sim -k/z^2 + 1/z (H + J)
$$
Since $sp(2) = su(2)$, the bosonic subalgebra generated by $H, J, J_\pm$ corresponds to $su(2) \oplus u(1)$ where the $J, J_\pm$ are $su(2)$ generators. Rescaling $J \rightarrow 2J, J_\pm \rightarrow \pm 2\sqrt{2}J_\pm$ one finds they satisfy the $su(2)$ current algebra at level $-k/2$. Thus, $osp(2|2)_k$ has an $su(2)_{-k/2}$ subalgebra.

Our conventions for $gl(1|1)$ are taken from the subalgebra of $osp(2|2)_k$ generated by $S_\pm, H, J$ and are presented in eq. (17). The stress tensors and scaling dimensions of primary fields are given in the main body of the paper.

Note that letting $H \rightarrow J, J \rightarrow H, S_\pm \rightarrow \pm S_\pm$, the new currents satisfy $gl(1|1)_{-k}$. Thus there exists an automorphism of $gl(1|1)_k$ which flips the sign of $k$, and was used in [35].

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