A generalized Kraichnan like model for simulating stratified flows

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Abstract. A generalization to the Kraichnan gaussian, white noise in time model of turbulent velocity field is proposed. The generalization is designed to mimic the effects of stratification in the atmospheric boundary layer by introducing different scaling exponents for horizontal and vertical velocity components, resulting in strong vertical anisotropy. The inevitable compressibility of the model leads one to take into account also the density statistics of the fluid, which will immediately lead to a boundary layer like structure in the flow statistics and non zero time correlations. Possible applications for passive scalar, tracer and inertial particle statistics and their anomalous scaling behavior will be discussed.

1. Introduction
Compared to homogeneous, isotropic turbulence, perhaps the most significant difference in a stably stratified atmospheric boundary layer is the strong vertical anisotropy manifesting as a $k^{-3}$ mesoscale spectrum (corresponding to $r^2$ spatial behavior in the structure function) of the vertical velocity components, as opposed to the usual Kolmogorov-Obukhov $k^{-5/3}$ spectrum of the horizontal components (see e.g., [4] and references therein). The vertical velocities are "smoothed" by the effect of gravity pulling the atmosphere toward the ground. Recently, much understanding on the nature of intermittency and anomalous scaling (or multiscaling) has been acquired by the so called Kraichnan model, where one assumes the turbulent velocity statistics to be known and completely determined by a gaussian, white in time random velocity field (see e.g. [5] for a review). The purpose of the present and future work is to modify the Kraichnan model to mimic the behavior of stratified flows and to ultimately apply the model to the passive scalar and tracers[2].

2. Kraichnan model
The Kraichnan model (see e.g. [5, 6]) is a gaussian, mean zero, white in time random field which can be completely defined via the pair correlation function

$$\langle v_\mu(t, x)v_\nu(t', x') \rangle = \delta(\Delta t)D_{\mu\nu}(\Delta x),$$

where $\delta(\Delta t) = \delta(t - t')$ is the Dirac delta function and $D_{\mu\nu}$ is an incompressible tensor field, which may be defined as

$$D_{\mu\nu}(r) = D_0 \int d^3p \frac{\delta_{\mu\nu} - p_\mu p_\nu/p^2}{(p^2 + L^{-2})^{3/2 + \xi/2}} e^{ip\cdot r}.$$
The constant $L$ is the integral scale of the flow and $0 < \xi < 2$ describes the spatial roughness of the velocity field and appears as a scaling exponent in the velocity structure function in the limit of infinite $L$ as

$$\langle |v(t, x) - v(t', x')|^2 \rangle \propto \delta(\Delta t) |\Delta x|^\xi.$$  \hfill (3)

One should note that although the infinite $L$ limit of the structure function is finite, the tensor field of (2) diverges as $L^\xi$.

The behavior of a passive scalar density (e.g. dye in fluid) under the Kraichnan flow can be described by the equation

$$\partial_t \theta - \kappa \Delta \theta + \mathbf{v} \cdot \nabla \theta = f.$$  \hfill (4)

The forcing $f = f(t, r)$ is usually defined as a gaussian, white in time random field with a pair correlation function

$$\langle f(t, r) f(t', r') \rangle \equiv \delta(\Delta t) C(\Delta r/L_f)$$  \hfill (5)

where $L_f$ is the scale of forcing or dye injection[5]. The role of the forcing is to counter molecular dissipation and to thus maintain a non-equilibrium steady state. The name “passive” refers to an approximation where the backreaction of the scalar on the velocity is disregarded, which is an acceptable assumption in many realistic cases.

In the case of an isotropic, large scale forcing, i.e. $C \rightarrow C(0)$ as $L_f \rightarrow \infty$, the passive scalar equal time structure functions scale as

$$\langle [\theta(t, x + r) - \theta(t, x)]^N \rangle \propto r^{\frac{\xi}{2}(2 - \xi) - \Delta N}$$  \hfill (6)

with $\Delta_N = \frac{N(N-1)}{2(d+2)} \xi + \mathcal{O}(\xi^2)$ for small $\xi$ (see [5] and references therein). The simple gaussian model of turbulence has therefore lead to an intermittency effect manifesting as multiscaling of the passive scalar structure functions.

3. Modified Kraichnan model with stratification

The idea of the present work is to study the effect of stratification in the context of a Kraichnan like turbulence model. For example, one would like to determine if and how the stratification affects the anomalous scaling exponents $\Delta_N$ above. We define a stratified generalization of the Kraichnan model as

$$D_{i\nu} \equiv \delta_{\nu i} \int d^3 \mathbf{p} \frac{P_i(\mathbf{p}) e^{-lp}}{(p^2 + L^{-2})^{3/2 + \xi/2}} e^{i\mathbf{p} \cdot \mathbf{r}} = D_{\nu i}$$  \hfill (7)

for the non-vertical components with $i \in 1, 2, \mu \in 1, 2, 3$ and $r_3$ is the vertical component. The vertical component is taken as

$$D_{33} \equiv \delta_{\nu 3} L^{\xi-\eta} \int d^3 \mathbf{p} \frac{P_3(\mathbf{p}) e^{-lp}}{(p^2 + (\alpha L)^{-2})^{3/2 + \eta/2}} e^{i\mathbf{p} \cdot \mathbf{r}}.$$  \hfill (8)

Here $P_{i\mu}(\mathbf{p}) = \delta_{i\mu} - \mathbf{p}_i \mathbf{p}_\mu$ is the usual incompressibility tensor, $0 \leq \xi < \eta \leq 2$ are the horizontal and vertical scaling exponents, $\alpha \leq 1$ is the aspect ratio of vertical to horizontal integral scales and the term $L^{\xi-\eta}$ in the prefactor ensures correct dimensionality. In an atmospheric boundary layer one would have $\eta \approx 2$, but we will keep $\eta$ as a free parameter. Also included is an ultraviolet
cutoff term $\propto e^{-lp}$, where the length scale $l$ corresponds to a viscous dissipative scale. It is often omitted in the Kraichnan model, since one is usually interested in scales larger than the viscous dissipation and because the model is finite as $l$ is taken to zero. Here the viscous scale $l$ must be kept finite (for now) because the model is not incompressible due to the different horizontal and vertical scaling. Specifically, the quantity $\langle (\nabla \cdot \mathbf{v})^2 \rangle$ diverges as $l$ approaches zero. The diffusivity ratio behaves as $D_{33}/D_u \propto \alpha^3$, which implies vertically inhibited diffusion for small aspect ratios, as is known to occur in real stratified turbulence (see e.g. [7]).

Consider now the passive scalar equation in the presence of inhomogeneous density[7],

$$\partial_t (\rho \theta) + \nabla \cdot (\mathbf{v} \rho \theta) - \kappa \nabla \cdot (\rho \nabla \theta) = 0. \tag{9}$$

As usual, the above equation should be interpreted as a stochastic PDE of Stratonovich (mid point) type [8]. Taking this carefully into account and going from Stratonovich to Itô prescription\(^1\), using the mass continuity equation

$$\partial_t \rho + \nabla \cdot (\mathbf{v} \rho) = 0 \tag{10}$$

and finally taking the limit of vanishing molecular diffusivity $\kappa$, we obtain

$$\partial_t \theta + [v_\mu + D_{\mu\nu}(0)\partial_\nu \chi] \partial_\mu \theta - \frac{1}{2} \tilde{D}_0 \theta = 0, \tag{11}$$

where $\chi \equiv \log (\rho/\rho_0)$ (with a reference density $\rho_0$), henceforth referred to as log-density, and notation $\tilde{D}_0 \equiv D_{\mu\nu}(0)\partial_\mu \partial_\nu$. The passive scalar is therefore advected by an effective velocity field $\mathbf{u} \equiv \mathbf{v} + D(0) \cdot \nabla \chi$. Although the turbulent velocity field $\mathbf{v}$ is compressible, the effective velocity field $\mathbf{u}$ is in fact incompressible[2]. It is easy to understand this physically: the effect of compressibility is that fluid flowing into a given region does not match the outgoing flow all the time. This results in temporary regional increase or decrease of the flow density, as dictated by the mass continuity equation (10). The advection of the passive scalar however depends also on the gradient of the density, which cancels the compressible effects of the turbulent velocity field $\mathbf{v}$.

The density field is therefore another random field in the model which plays an important role in e.g. passive scalar and inertial particle behavior. We will rewrite also the mass continuity equation (10) in terms of the log-density $\chi$. The equation for the log-density follows by similar use of the rules of stochastic calculus, resulting in the equation

$$\partial_t \chi + \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \chi - \frac{1}{2} \tilde{D}_0 \chi + \gamma = 0, \tag{12}$$

with notation $\gamma \equiv \langle (\nabla \cdot \mathbf{v})^2 \rangle = -D_{\mu\nu,\mu\nu}(0)$ (symbols after a comma denote derivatives). In addition to the velocity field pair correlation functions, we therefore need to consider also the correlators $\langle \chi \rangle$, $\langle \mathbf{v} \chi \rangle$, $\langle \chi \chi \rangle$ etc. One should note that as the log-density is determined by a first order stochastic differential equation in time, it is evident that the above correlators will exhibit finite time correlations. This will also imply that the effective velocity field $\mathbf{u}$ is not decorrelated.

Using the above SPDE, we write the steady state equations for the mean log-density $g(z) \equiv \langle \chi \rangle$ (which we assume to depend only on the vertical coordinate $z = x_3$),

$$\tilde{D}_0 g = 2\gamma, \tag{13}$$

\(^1\) Sketchily, $\mathbf{v}(t) \equiv \text{d}V(t)/\text{d}t$, where $V(t)$ is an infinite dimensional Wiener process satisfying $\text{d}V(t)\text{d}V(t') = D(t - t')\text{d}t$. Then we can go from Stratonovich to Itô prescription by $\text{d}V_{\nu}(t)\theta (t + \text{d}t/2) - \text{d}V_{\nu}(t)\theta (t) = \frac{1}{2} \text{d}V_{\nu}(t)\theta (t + \text{d}t) - \frac{1}{2} \text{d}V_{\nu}(t)\theta (t + \text{d}t) - \frac{1}{2} \text{d}V_{\nu}(t)\nabla \cdot (\text{d}V(t)\theta (t)) = \frac{1}{2} \text{d}\theta_{\nu} (D_{\mu\nu}(0)\theta (t))$. 

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which can be readily solved with suitable boundary conditions, yielding
\[ \langle \chi \rangle = g(z) = \frac{1}{2} \omega z \left( \omega z - \frac{\pi}{2} \right), \]  
(14)

where \( \omega^2 \equiv 2\langle (\nabla \cdot \mathbf{v})^2 \rangle / \langle \mathbf{v}^2 \rangle \propto \alpha^{-\eta} (l/L)^\xi l^{-2} \) is a compressibility like parameter. The usual definition of compressibility \( p = \langle (\nabla \cdot \mathbf{v})^2 \rangle / \langle (\nabla \mathbf{v})^2 \rangle \) would here give \( p = \text{const.} \) for \( r \ll l \) and \( p \propto (l/r)^{2-\xi} \) for \( r \gg l \). Moreover, it turns out that the effects of compressibility cancel with contributions from the density fluctuations in e.g. the passive scalar equation[2]. The solution will therefore tend to zero as \( L \) approaches infinity. It is perhaps more instructive to consider the mean density rather than the mean log-density. The solution for mean density is
\[ \langle \rho \rangle = \rho_0 \cos (\omega z), \]  
(15)

which leads one to consider the "height" of the boundary layer as \( h \equiv \pi/2\omega \propto L^{\xi/2} \). The gradient of the mean log-density (14) implies that the mean effective velocity has a nonzero vertical component, pointing towards the mean height. This implies that passive quantities will be concentrated at height \( h/2 \). The situation is illustrated in Fig. (1). Clearly this is not a completely realistic density profile, as one might expect since the velocity correlation function does not exhibit any height dependence in this approximation.

The equation for the flux like correlator \( F_\mu (t - t', \mathbf{x} - \mathbf{x}') = \langle \chi(t, \mathbf{x}) v_\mu(t', \mathbf{x}') \rangle \) is
\[ \partial_t F_\mu = -\delta(t - t') \left[ D_{\mu\nu,\nu} + g'(z) D_{\mu\lambda} \right] + \frac{1}{2} \hat{D}_0 F_\mu. \]  
(16)

Since it is the gradient of the log-density that appears e.g. in the passive scalar equation, we take a derivative of the solution, which at leading order as \( L \to \infty \) is approximately
\[ \langle \partial_\mu \chi(t, \mathbf{x}) v_i(t', \mathbf{x}') \rangle \approx -\theta(\Delta t) e^{\frac{1}{2} \Delta t} \hat{D}_0 \partial_\mu \left[ D_{\mu\nu} g'(z) + D_{\mu\nu} \right] \]  
\[ \approx \theta(\Delta t) \left[ c_{1\mu}^1 + c_{1\mu}^2 (r/L)^2 + c_{1\mu}^3 \Delta t / L^2 - \xi \right], \]  
(18)
where the $c^i$ are some anisotropic unit tensors and $r = |\Delta x|$. If we further assume that $\Delta t/L^{2-\xi} \gg 1$, we obtain $\langle \chi v'_i \rangle \propto (\Delta t)(\Delta t)^{-(5-\xi)/2}$, i.e. the long time time correlation is a power law. Similar result for the vertical velocity component can be derived but with $\xi$ replaced by $\eta$.

For the equal time two point function $G_0(x, x') \equiv \langle \chi(t, x) \chi(t, x') \rangle$ we have

$$
\frac{1}{2} D_{\mu\nu}(0) \left( \partial_\mu \partial_\nu + 2 \partial_\mu \partial'_\nu + \partial'_\mu \partial'_\nu \right) G_0 - d_{\mu\nu} \partial_\mu \partial'_\nu G_0 - D_{\mu\nu,\mu\nu} - \gamma (g(z) + g(z')) = 0. \tag{19}
$$

At leading order the last term vanishes. We can then assume homogeneity, which results also in the vanishing of the first term, yielding the leading order solution (again with gradients of $\chi$)

$$
\langle \partial_\mu \chi(t, x) \partial_\nu \chi(t, x') \rangle \approx -\partial_\mu \partial_\nu \tilde{d}^{-1} D_{\alpha\beta,\alpha\beta}(x - x') \tag{20}.
$$

$$
\approx c^4_{\mu\nu} r^{-2}, \tag{21}
$$

with another anisotropic unit tensor.

### 4. Concluding remarks and future perspectives

The next step in the process would naturally be to consider e.g. the passive scalar and tracer or inertial particles in a flow defined by the above model. From the approximative expressions above, one can already gain some insight into some properties of these passive quantities. First, since there is now a mean effective velocity field pointing toward the height $h/2$, this is where one may expect the particles and passive scalar fields to be concentrated. Second, it was demonstrated that the log-density gradient field exhibits a time-like correlation, as opposed to the white noise in time correlation of the Kraichnan model. The effect of finite time correlation on clustering of inertial particles can be understood intuitively as follows: the orbits of a turbulent (incompressible) flow consist of closed, nonintersecting curves around the eddies, along which tracer particles would traverse. Inertial particles however deviate from these orbits due to centrifugal forces, and therefore tend to cluster on the boundaries of eddies. A delta correlation in time, such as in the Kraichnan model, corresponds to eddies changing so rapidly in time that the inertial particles simply do not have enough time to cluster on the boundaries of eddies. It should be noted however that clustering phenomena are possible also for decorrelated, compressible models (see e.g. [3] and references therein).
It is therefore foreseeable that such clustering occurs for this model, simply due to the anisotropic density fluctuations. Also, a common critique toward extensions of the Kraichnan model to models with time-like correlations is that they violate Galilean invariance. Here such a problem does not arise since the time correlation originates from the density field in the mass continuity equation, which of course is Galilean invariant by construction.

The extent to which the model can be considered a realistic description of stably stratified turbulent flows is afflicted by the same pathologies as the Kraichnan model, namely time decorrelation, gaussianity and (therefore) lack of intermittency. Although the presence of density fluctuations introduces an effective time correlation, the velocity field itself is still decorrelated. At large spatial and timelike scales the time decorrelation is a decent approximation, but gaussianity really is not. However, as evidenced by the Kraichnan passive scalar, such non-intermittent models may still successfully describe some interesting behavior of advection and transport in real turbulent flows[5]. The presence of compressibility seemingly poses another challenge, especially from numerical perspective. However, as mentioned above, although we have \( \gamma \doteq \langle (\nabla \cdot \mathbf{v})^2 \rangle > 0 \), it turns out that the effective velocity \( \mathbf{u} \) is incompressible, i.e. \( \langle (\nabla \cdot \mathbf{u})^2 \rangle = 0 \) [2].

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References
[1] H. Arponen (2009). Anomalous scaling and anisotropy in models of passively advected vector fields, Phys. Rev. E 79, pp 056303. doi:10.1103/PhysRevE.79.056303
[2] H. Arponen. Passive scalar in a stably stratified modification of the Kraichnan model (to be published)
[3] J. Bec, M. Cencini, R. Hillerbrand, K. Turitsyn, Stochastic suspensions of heavy particles, Physica D: Nonlinear Phenomena, Volume 237, Issues 14-17, Euler Equations: 250 Years On - Proceedings of an international conference, 15 August 2008, Pages 2037-2050, ISSN 0167-2789, DOI: 10.1016/j.physd.2008.02.022. (http://www.sciencedirect.com/science/article/pii/S0167278908000638)
[4] G. Brethouwer, P. Billant, E. Lindborg and J.-M. Chomaz (2007). Scaling analysis and simulation of strongly stratified turbulent flows. Journal of Fluid Mechanics, 585, pp 343-368, doi:10.1017/S0022112007006854
[5] G. Falkovich, G. Gawędzki, K. and Vergassola, M. (2001) Particles and fields in fluid turbulence. Rev. Mod. Phys., Vol. 73, pp 913-975, doi: 10.1103/RevModPhys.73.913
[6] Robert H. Kraichnan. Anomalous scaling of a randomly advected passive scalar. Phys. Rev. Lett., 72(7):10161019, Feb 1994.
[7] A. Vincent, G. Michaud, and M. Meneguzzi (1996) On the turbulent transport of a passive scalar by anisotropic turbulence. Phys. Fluids 8, 1312, doi:10.1063/1.868912
[8] J. Zinn-Justin (1996), Quantum Field Theory and Critical Phenomena 3rd ed. Oxford University Press