A Note on Specializations of Grothendieck Polynomials

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Abstract

Buch and Rimányi proved a formula for a specialization of double Grothendieck polynomials based on the Yang-Baxter equation related to the degenerate Hecke algebra. A geometric proof was found by Yong and Woo by constructing a Gröbner basis for the Kazhdan-Lusztig ideals. In this note, we give an elementary proof for this formula by using only divided difference operators.

1 Introduction

Let $S_n$ denote the symmetric group of permutations of $\{1, 2, \ldots, n\}$. For a permutation $w \in S_n$, the double Grothendieck polynomial $G_w(x; y)$ introduced by Lascoux and Schützenberger [12] is the polynomial representative of the class of the Schubert variety for $w$ in the equivariant $K$-theory of the flag manifold. Write a permutation $v \in S_n$ in one-line notation, that is, write $v = v(1)v(2)\cdots v(n)$. The specialization

$$G_w(y_v; y) := G_w(y_{v(1)}, \ldots, y_{v(n)}; y)$$

(1.1)

of $G_w(x; y)$ obtained by replacing $x_i$ with $y_{v(i)}$ gives the restriction of this class to the fixed point corresponding to $v$. Buch and Rimányi [4] proved a formula for $G_w(y_v; y)$ based on the Yang-Baxter equation related to the degenerate Hecke algebra. Buch and Rimányi [4] also pointed out various important applications of this formula. By constructing a Gröbner basis for the Kazhdan-Lusztig ideals, Yong and Woo [15] found a geometric explanation for the Buch-Rimányi formula.

In this note, we give an elementary proof of the Buch-Rimányi formula by using only divided difference operators. As observed by Buch and Rimányi [4, Corollary 2.3], the classical pipe dream (or, RC-graph) formula of $G_w(x; y)$ (see for example [10, Corollary 5.4], [13, Theorem 6.3]) can be directly obtained from the specialization $G_w(y_v; y)$. Hence our approach implies that the pipe dream formula for double Grothendieck polynomials can be derived directly from divided difference operators.
2 The Buch-Rimányi formula

Fix a nonnegative integer $n$. For $1 \leq i < j \leq n$, let $t_{ij}$ denote the transposition $(i, j)$ in $S_n$. So, if $w \in S_n$, then $w t_{ij}$ is the permutation obtained from $w$ by interchanging $w(i)$ and $w(j)$, while $t_{ij}w$ is obtained from $w$ by interchanging the values $i$ and $j$. For example, for $w = 2143$, we have $w t_{13} = 4123$ and $t_{13}w = 2341$. Write $s_i$ for the adjacent transposition $(i, i+1)$. Each permutation can be written as a product of adjacent transpositions. The length $\ell(w)$ of a permutation $w$ is the minimum $k$ such that $w = s_i s_{i+1} \cdots s_{ik}$, and in this case, $(s_i, s_{i+1}, \ldots, s_{ik})$ is called a reduced word of $w$. It is well known that the length $\ell(w)$ is equal to the number of pairs $(i, j)$ such that $i < j$ and $w(i) > w(j)$:

$$\ell(w) = \# \{(i, j) : 1 \leq i < j \leq n, \; w(i) > w(j)\}.$$ 

Hence, it is clear that $\ell(ws_i) = \ell(w) + 1$ if and only if $w(i) < w(i+1)$, while $\ell(ws_i) = \ell(w) - 1$ if and only if $w(i) > w(i+1)$.

Let $\mathbb{Z}[x^\pm, y^\pm]$ denote the ring of Laurent polynomials in the $2n$ commuting indeterminates $x_1, \ldots, x_n, y_1, \ldots, y_n$. For a Laurent polynomial $f(x, y) \in \mathbb{Z}[x^\pm, y^\pm]$, the divided difference operator $\partial_i$ acting on $f(x, y)$ is defined by

$$\partial_i f = (f - s_i f) / (x_i - x_{i+1}),$$

where $s_i f$ is obtained from $f$ by interchanging $x_i$ and $x_{i+1}$. It is easy to check that $\partial_i f$ is still a Laurent polynomial. Let $w_0 = n \cdots 21$ be the longest permutation in $S_n$. Set

$$\mathcal{G}_{w_0}(x; y) = \prod_{i+j \leq n} \left(1 - \frac{y_j}{x_i}\right). \quad (2.1)$$

For $w \neq w_0$, choose an adjacent transposition $s_i$ such that $\ell(ws_i) = \ell(w) + 1$. Let $\pi_i = \partial_i x_i$ and define

$$\mathcal{G}_w(x; y) = \pi_i \mathcal{G}_{w s_i}(x; y) = \frac{x_i \mathcal{G}_{w s_i}(x; y) - x_{i+1} \mathcal{G}_{w s_i}(\ldots, x_{i+1}, x_i; \ldots, y)}{x_i - x_{i+1}}. \quad (2.2)$$

The above definition is independent of the choice of $s_i$ since the operators $\pi_i$ satisfy the Coxeter relations: $\pi_i \pi_j = \pi_j \pi_i$ for $|i - j| > 1$, and $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$, see for example [14] (2.14).

We remark that there are other equivalent definitions for double Grothendieck polynomials. The definition adopted here implies that $\mathcal{G}_w(x; y)$ are Laurent polynomials. The double Grothendieck polynomials $\mathcal{L}_w^{(-1)}(y; x)$ defined in [5] are legitimate polynomials, which can be obtained from $\mathcal{G}_w(x; y)$ by replacing $x_i$ and $y_i$ respectively with $\frac{1}{1-x_i}$ and $1 - y_i$. It should also be noticed that $\mathcal{G}_w(x^{-1}; y^{-1})$ are the double Grothendieck polynomials used in [9], and $\mathcal{G}_w(x^{-1}; y)$ are the double Grothendieck polynomials appearing in [10]. It is worth mentioning that the double Schubert polynomial $\mathcal{G}_w(x; y)$ is the lowest degree homogeneous component of $\mathcal{L}_w^{(-1)}(y; x)$, see [12][6][7][11] for combinatorial constructions of Schubert polynomials.

To describe the Buch-Rimányi formula, consider the left-justified array $\Delta_n$ with $n - i$ squares in row $i$. Let $w = w(1)w(2) \cdots w(n) \in S_n$. For $1 \leq i \leq n$, let

$$I(w, i) = \{w(j) : j > i, \; w(j) < w(i)\}$$

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be the set of entries in $w$ that are smaller than $w(i)$ but appear to the right of $w(i)$. Set $c(w, i) = |I(w, i)|$. It is clear that $0 \leq c(w, i) \leq n - i$. Let $D(w)$ be the subset of $\Delta_n$ consisting of the first $c(w, i)$ squares in the $i$-th row of $\Delta_n$, where $1 \leq i \leq n$. Note that $D(w)$ corresponds to the bottom RC-graph of $w$, as defined by Bergeron and Billey [1]. Assume that the values in $I(w, i)$ are

$$w(j_1) < w(j_2) < \cdots < w(j_{c(w, i)}).$$

For a square $B \in D(w)$ in row $i$ and column $k$, equip $B$ with the weight

$$\text{wt}(B) = 1 - \frac{y_{w(j_k)}}{y_{w(i)}},$$

see Figure 2.1 for an illustration.

![Figure 2.1: Weights of squares of $D(w)$ for $w = 2157634$.](image)

Given a subset $D$ of $D(w)$, one can generate a word, denoted $\text{word}(D)$, as follows. Label the square of $D(w)$ in row $i$ and column $k$ by the simple transposition $s_{i+k-1}$, see Figure 2.2 for an illustration. Then $\text{word}(D)$ is obtained by reading off the labels of the squares in $D$ along the rows from top to bottom and right to left. For example, for the diagram $D = D(w)$ in Figure 2.2 we have

$$\text{word}(D) = (s_1, s_4, s_3, s_6, s_5, s_4, s_6, s_5).$$

![Figure 2.2: Labels of the squares of $D(w)$ for $w = 2157634$.](image)
A word \((s_1, s_2, \ldots, s_m)\) is called a Hecke word of a permutation \(u\) of length \(m\) if
\[
((s_1 * s_2) * s_3) * \cdots * s_m = u,
\]
where, for a permutation \(w\), we define \(w * s_i\) to be \(w\) if \(\ell(ws_i) < \ell(w)\) and \(ws_i\) otherwise. For example, \((s_1, s_2, s_1, s_2)\) is a Hecke word of \(u = 321\) of length 4 since
\[
((s_1 * s_2) * s_1) * s_2 = ((s_2s_1) * s_1) * s_2 = (s_2s_1s_2) * s_2 = s_1s_2s_1 = 321.
\]

We note in passing that the operation \(*\) can be extended to an associative operation on the whole \(S_n\); this latter operation is the multiplication in the Hecke algebra associated to \(S_n\) at \(q = 0\), see \[8, Chapter 7.4\]. Hence \(*\) satisfies the associative property. This means that the set of permutations in \(S_n\) forms a monoid structure (0-Hecke monoid) under the operation \(*\).

Write \(\text{Hecke}(D) = u\) if word(D) is a Hecke word of a permutation \(u\). Notice that a Hecke word of \(u\) of length \(\ell(u)\) is a reduced word of \(u\). Note that for any \(w \in S_n\), the word word\((D(w))\) is a reduced word of \(w\), and therefore, if we multiply the letters of \(\text{word}(D(w))\) using either the \(*\) product or the usual product of \(S_n\), then we get \(w\). That is, \(\text{Hecke}(D(w)) = w\).

For any \(u, v \in S_n\), let \(H(u, v) = \{D \subseteq D(v) \mid \text{Hecke}(D) = u\}\).

For a subset \(D\) of \(D(v)\), let
\[
\text{wt}(D) = \prod_{B \in D} \text{wt}(B).
\]

**Theorem 2.1** (Buch-Rimányi \[4, Theorem 2.1\]). For permutations \(u, v \in S_n\), we have
\[
\mathcal{G}_u(y_v; y) = \sum_{D \in H(u,v)} (-1)^{|D| - \ell(u)} \text{wt}(D),
\]
where empty sums are interpreted as 0.

We remark that in \[4\], formula (2.4) is described in terms of the notation \(C(D_v)\) and FK-graphs for \(u\) with respect to \(D_v\). With the notation in this note, \(D(v)\) can be obtained from \(C(D_v)\) by first reflecting along the main diagonal and then left-justifying the crossing positions. This operation also establishes a weight preserving bijection between the set \(H(u,v)\) and the set of FK-graphs for \(u\) with respect to \(D_v\).

## 3 Elementary proof of Theorem 2.1

We need several lemmas which follow directly from the definition of \(\mathcal{G}_u(x;y)\).

**Lemma 3.1.** Let \(v = v's_i\) and \(\ell(v) > \ell(v')\). If \(\ell(us_i) < \ell(u)\), then
\[
\mathcal{G}_u(y_v; y) = \frac{y_{v'(i)}}{y_{v'(i+1)}} \mathcal{G}_u(y_{v'}; y) + \left(1 - \frac{y_{v'(i)}}{y_{v'(i+1)}}\right) \mathcal{G}_u(y_{us_i}; y).
\]
Lemma 3.2. Let \( v = v's_i \). If \( \ell(us_i) > \ell(u) \), then
\[
\mathcal{G}_u(y_v; y) = \mathcal{G}_u(y_{v'}; y). \tag{3.2}
\]

Proof. Applying (2.2) to \( w = us_i \) and substituting \( x_j \) with \( y_{v'(j)} \), we have
\[
\mathcal{G}_{us_i}(y_{v'}; y) = \frac{y_{v'(i)}\mathcal{G}_u(y_{v'}; y) - y_{v'(i+1)}\mathcal{G}_u(y_{v'}; y)}{y_{v'(i)} - y_{v'(i+1)}},
\]
which is equivalent to (3.1).

Lemma 3.3. We have \( \mathcal{G}_u(y_v; y) = 0 \) whenever \( u \not\leq v \) in the Bruhat order.

Proof. The idea in the proof of [11, (2.22)] for double Schubert polynomials applies to double Grothendieck polynomials, and we include a proof here for the reader’s convenience. Use descending induction on \( \ell(u) \). The initial case is \( u = w_0 \). Since \( u \not\leq v \), we have \( v \neq w_0 \). It is easily checked from (2.1) that \( \mathcal{G}_{w_0}(y_v; y) = 0 \).

We now consider the case \( u \neq w_0 \). Choose a position \( i \) such that \( u(i) < u(i+1) \). Note that \( u < us_i \). Since \( u \not\leq v \), we must have \( us_i \not\leq v \). We further claim that \( us_i \not\leq vs_i \). This can be seen as follows. We have either \( vs_i < v \) or \( v < vs_i \) (depending on which of \( \ell(vs_i) \) and \( \ell(v) \) is larger). If \( vs_i < v \), then it is clear that \( us_i \not\leq vs_i \) since otherwise there would hold \( u \leq v \). It remains to verify the case \( v < vs_i \). Suppose to the contrary that \( us_i \leq vs_i \). Then \( u < vs_i \). Since \( vs_i > v \) and \( us_i > u \), applying the Lifting Property (see [3, Proposition 2.2.7]) to \( u^{-1} \) and \( (vs_i)^{-1} \), we obtain that \( u \leq v \), leading to a contradiction. Now, by the definition in (2.2) and by the induction hypothesis,
\[
\mathcal{G}_u(y_v; y) = \frac{y_{v(i)}\mathcal{G}_{us_i}(y_v; y) - y_{v(i+1)}\mathcal{G}_{us_i}(y_{vs_i}; y)}{y_{v(i)} - y_{v(i+1)}} = 0,
\]
as desired.

Lemma 3.4. Let \( u \in S_n \) and \( u' = us_i \) for some \( i \) such that \( \ell(us_i) < \ell(u) \). Then,
\[
\mathcal{G}_u(y_u; y) = \left( 1 - \frac{y_{u(i+1)}}{y_{u(i)}} \right) \mathcal{G}_{u'}(y_{u'}; y).
\]
Proof. Apply Lemma 3.1 to \( v = u \) and \( v' = u' \). The first addend on the right side vanishes due to Lemma 3.3.

Lemma 3.5 (Buch-Rimányi [4, Corollary 2.6]). For each \( u \in S_n \), we have

\[
\mathfrak{G}_u(y_u; y) = \prod_{i < j, u(i) > u(j)} \left( 1 - \frac{y_u(j)}{y_u(i)} \right).
\]

Proof. Make descending induction on \( \ell(u) \). The induction base for \( u = w_0 \) is a restatement of (2.1). Assume that \( u \neq w_0 \). Then there exists some \( 1 \leq k < n \) such that \( \ell(us_k) > \ell(u) \). Let \( u' = us_k \). It is easy to see that the set

\[
\{(u'(i), u'(j)) \mid i < j, \ u'(i) > u'(j)\}
\]

is the union of the two disjoint sets

\[
\{(u(i), u(j)) \mid i < j, \ u(i) > u(j)\} \cup \{(u(k), u(k+1))\}.
\]

The proof follows by induction together with Lemma 3.4.

Proof of Theorem 2.1. The proof is by induction on \( \ell(v) \). Let us first consider the case \( \ell(v) = 0 \), that is, \( v \) is the identity permutation \( e \). If \( u = e \), then it follows from Lemma 3.5 (applied to \( u = e \)) that \( \mathfrak{G}_e(y_e; y) = 1 \). If \( u \neq e \), then Lemma 3.3 forces that \( \mathfrak{G}_u(y_e; y) = 0 \). So (2.4) holds for \( \ell(v) = 0 \).

Assume now that \( \ell(v) > 0 \). Let \( s_r \) be the last descent of \( v \), that is, \( r \) is the largest index such that \( v(r) > v(r+1) \). Write \( v = v's_r \). Clearly, the bottom row of \( D(v) \) lies in row \( r \) of \( \Delta_n \). The leftmost square in the bottom row of \( D(v) \), denoted \( B_0 \), has weight

\[
\text{wt}(B_0) = 1 - \frac{y_v(r+1)}{y_v(r)} = 1 - \frac{y_{v'}(r)}{y_{v'}(r+1)}.
\]

Let \( u = u's_r \). There are two cases.

**Case 1.** \( s_r \) is a descent of \( u \). By Lemma 3.1 and by induction hypothesis, we have

\[
\mathfrak{G}_u(y_u; y) = \frac{y_{v'}(r)}{y_{v'}(r+1)} \mathfrak{G}_u(y_{v'}; y) + \left( 1 - \frac{y_{v'}(r)}{y_{v'}(r+1)} \right) \mathfrak{G}_{u'}(y_{v'}; y)
\]

\[
= (1 - \text{wt}(B_0)) \sum_{D \in \mathcal{H}(u, v')} (-1)^{|D|-\ell(u)} \text{wt}(D) + \text{wt}(B_0) \sum_{D \in \mathcal{H}(u', v')} (-1)^{|D|-\ell(u')} \text{wt}(D)
\]

\[
= \sum_{D \in \mathcal{H}(u, v')} (-1)^{|D|-\ell(u)} \text{wt}(D) - \text{wt}(B_0) \sum_{D \in \mathcal{H}(u, v')} (-1)^{|D|-\ell(u')} \text{wt}(D)
\]

\[
+ \text{wt}(B_0) \sum_{D \in \mathcal{H}(u', v')} (-1)^{|D|-\ell(u')} \text{wt}(D) \tag{3.3}
\]

To proceed, note that there is an obvious bijection \( \phi \) between \( D(v') \) and \( D(v) \setminus \{B_0\} \). Since \( s_r \) is the last descent of \( v \), we have \( c(v', r) = 0 \), \( c(v', r+1) = c(v, r) - 1 \), and \( c(v', i) = c(v, i) \).
for \( i \neq r, r + 1 \). Let \( B \in D(v') \). If \( B \) lies above row \( r \), then set \( \phi(B) = B \). Assume that \( B \) lies in row \( r + 1 \) and column \( j \), then let \( \phi(B) \) be the square of \( D(v) \setminus \{B_0\} \) in row \( r \) and column \( j + 1 \). By construction, \( B \) and \( \phi(B) \) are labeled by the same simple transposition. Moreover, it is easy to see that \( \phi \) preserves the weight and words, namely, \( \text{wt}(B) = \text{wt}(\phi(B)) \) and \( \text{word}(\phi(D)) = \text{word}(D) \) for all \( D \subseteq D(v') \). Thus \( \text{Hecke}(\phi(D)) = \text{Hecke}(D) \) for all \( D \subseteq D(v') \).

We claim that \( \mathcal{H}(u, v) \) is the disjoint union of the following sets:

\[
S_1 = \{ \phi(D) : D \in \mathcal{H}(u, v) \},
\]
\[
S_2 = \{ \phi(D) \cup \{B_0\} : D \in \mathcal{H}(u, v') \},
\]
\[
S_3 = \{ \phi(D) \cup \{B_0\} : D \in \mathcal{H}(u', v') \}.
\]

This can be easily seen as follows. Keep in mind that \( B_0 \) is labeled by \( s_r \). Let \( D \in \mathcal{H}(u, v) \). If \( B_0 \notin D \), then \( D \in S_1 \). If \( B_0 \in D \), then \( \text{word}(D) \) is obtained from \( \text{word}(D \setminus \{B_0\}) \) by appending the letter \( s_r \) at the end, and thus we have \( \text{Hecke}(D) = \text{Hecke}(D \setminus \{B_0\}) \ast s_r \), and therefore either \( \text{Hecke}(D \setminus \{B_0\}) = u \) or \( \text{Hecke}(D \setminus \{B_0\}) = u' \). Hence either \( D \in S_2 \) or \( D \in S_3 \). Conversely, any \( D \in S_1 \cup S_2 \cup S_3 \) belongs to \( \mathcal{H}(u, v) \), since \( u \ast s_r = u' \ast s_r = u \). By the above claim and in view of (3.3), we obtain that

\[
\mathfrak{G}_u(y_0; y) = \sum_{D \in S_1 \cup S_2 \cup S_3} (-1)^{|D| - \ell(u)} \text{wt}(D) = \sum_{D \in \mathcal{H}(u, v)} (-1)^{|D| - \ell(u)} \text{wt}(D).
\]

**Case 2.** \( s_r \) is not a descent of \( u \). Let \( D \in \mathcal{H}(u, v) \). We claim that \( B_0 \notin D \). Suppose otherwise that \( B_0 \in D \). Consider \( D' = D \setminus \{B_0\} \). If \( s_r \) is a descent of \( \text{Hecke}(D') \), then \( \text{Hecke}(D) = \text{Hecke}(D') \ast s_r \), while if \( s_r \) is not a descent of \( \text{Hecke}(D') \), then \( \text{Hecke}(D) = \text{Hecke}(D') \ast s_r \). In both cases, \( s_r \) is a descent of \( u = \text{Hecke}(D) \), leading to a contradiction. Therefore, we see that \( \mathcal{H}(u, v) = \{ \phi(D) : D \in \mathcal{H}(u, v') \} \). By Lemma 3.2 and by induction hypothesis,

\[
\mathfrak{G}_u(y_0; y) = \mathfrak{G}_u(y_0'; y) = \sum_{D \in \mathcal{H}(u, v')} (-1)^{|D| - \ell(u)} \text{wt}(D) = \sum_{D \in \mathcal{H}(u, v')} (-1)^{|D| - \ell(u)} \text{wt}(D).
\]

This completes the proof.

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**References**

[1] N. Bergeron and S. Billey, RC-graphs and Schubert polynomials, Experiment. Math. 2 (4) (1993), 257–269.

[2] S. Billey, W. Jockusch and R.P. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (1993), 345–374.

[3] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, Grad. Texts in Math., Vol. 231, Springer, New York, 2005.
[4] A. Buch and R. Rimányi, Specializations of Grothendieck polynomials, C. R. Acad. Sci. Paris, Ser. I 339 (2004), 1–4.

[5] S. Fomin and A.N. Kirillov, Grothendieck polynomials and the Yang-Baxter equation, Proc. Formal Power Series and Alg. Comb. (1994), 183–190.

[6] S. Fomin and A.N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, in: Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), Discrete Math. 153 (1996), 123–143.

[7] S. Fomin and R.P. Stanley, Schubert polynomials and the NilCoxeter algebra, Adv. Math. 103 (1994), 196–207.

[8] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, No. 29, Cambridge Univ. Press, Cambridge, 1990.

[9] A. Knutson and E. Miller, Gröbner geometry of Schubert polynomials, Ann. Math. 161 (2005), 1245–1318.

[10] A. Knutson and E. Miller, Subword complexes in Coxeter groups, Adv. Math. 184 (2004), 161–176.

[11] T. Lam, S. Lee and M. Shimozono, Back stable Schubert calculus, arXiv:1806.11233v1.

[12] A. Lascoux and M.-P. Schützenberger, Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck dune variété de drapeaux, C. R. Acad. Sci. Paris 295 (1982), 629–633.

[13] C. Lenart, S. Robinson and F. Sottile, Grothendieck polynomials via permutation patterns and chains in the Bruhat order, Amer. J. Math. 128 (2006), 805–848.

[14] I.G. Macdonald, Notes on Schubert Polynomials, Laboratoire de combinatoire et d’informatique mathématique (LACIM), Université du Québec à Montréal, Montreal, 1991.

[15] A. Yong and A. Woo, A Gröbner basis for the Kazhdan-Lusztig ideals, Amer. J. Math. 134 (2012), 1089–1137.