Hermitian composition operators on Hardy-Smirnov spaces

Abstract: Let $\Omega$ be an open simply connected proper subset of the complex plane and $\phi$ an analytic self map of $\Omega$. If $f$ is in the Hardy-Smirnov space defined on $\Omega$, then the operator that takes $f$ to $f \circ \phi$ is a composition operator. We show that for any $f$, analytic self maps that induce bounded Hermitian composition operators are of the form $\phi(w) = aw + b$ where $a$ is a real number. For certain $\Omega$, we completely describe values of $a$ and $b$ that induce bounded Hermitian composition operators.

Keywords: Composition operator, Hermitian operator, Hardy-Smirnov space

MSC: 47B33, 30H10

1 Introduction

Let $\Omega$ be an open simply connected proper subset of the complex plane and $\gamma$ be a Riemann map from the open unit disc onto $\Omega$. The Hardy-Smirnov space $H^2(\Omega)$ is the Hilbert space of functions $F$ analytic on $\Omega$ such that the integrals of $|F|^2$ over the images of the circles $|z| = r$, $0 < r < 1$, under $\gamma$ are uniformly bounded; see [5, Chapter 10] or [10, p. 63]. We will refer to the map $\gamma$ as the underlying Riemann map of $H^2(\Omega)$.

If $f \in H^2(\Omega)$, and $\phi$ is an analytic self map of $\Omega$, then the composition operator induced by $\phi$ on $H^2(\Omega)$, denoted by $C_\phi$, is the linear operator defined by

$$C_\phi(f) = f \circ \phi.$$ 

Such operators on $H^2(\Omega)$ are studied in [10]. Also, composition operators on a Hardy space of a half-plane are studied in [7],[8], and [9].

In this paper we study bounded Hermitian composition operators on $H^2(\Omega)$. In Theorem 3.3 we show that if $C_\phi$ is bounded and Hermitian, then $\phi(w) = aw + b$ where $a \in \mathbb{R}$. However, the self map $\phi$ having the form $aw + b, a \in \mathbb{R}$, is not always sufficient for $C_\phi$ to be Hermitian. Some sufficient conditions are given in Theorem 3.3 in order for $C_\phi$ to be Hermitian. We also show that if a constant map induces a Hermitian composition operator, then $\Omega$ is a disc.

The sufficient condition in Theorem 3.3 requires the computation of $\gamma^{-1} \circ \phi \circ \gamma$ which could be complicated for some $\gamma$. Therefore in section 4, for some domains, we provide sufficient conditions that do not require the computation of $\gamma^{-1} \circ \phi \circ \gamma$. In Lemma 4.7 it is proved that if $\phi(w) = -w + b$ is a self map of $\Omega$, then $C_\phi$ is bounded and Hermitian on $H^2(\Omega)$. In Theorem 4.10 we show that if the boundary of $\Omega$ has infinite length, $\phi$ has a fixed point in $\Omega$ and $a^2 \neq 1$, then $C_\phi$ is not Hermitian on $H^2(\Omega)$.

Some concrete examples are provided in section 5. First we consider the strip $\Omega = \{z : -1 < Im(z) < 1\}$, where $Im(z)$ denotes the imaginary part of $z$. It is proved in Theorem 5.2 that the composition operator $C_\phi$ is
bounded, non-trivial and Hermitian on $H^2(\Omega)$ if and only if $\phi(w) = -w + b$ where $b \in \mathbb{R}$. Next we consider $\Omega = \{ z : -1 < Im(z) < 1 \} \cup \{ z : -1 < Re(z) < 1 \}$, where the real part of $z$ is denoted by $Re(z)$. In Theorem 5.3 it is proved that the only non-trivial bounded Hermitian composition operator $C_\phi$ on $H^2(\Omega)$ is induced by $\phi(w) = -w$.

## 2 Background material

Throughout this paper $\Omega$ will represent an open simply connected proper subset of the complex plane. The underlying Riemann map associated with $H^2(\Omega)$ is denoted by $\gamma$.

### Notation
- The set of real numbers is denoted by $\mathbb{R}$.
- The complex plane is denoted by $\mathbb{C}$.
- The open unit disc $\{ z : |z| < 1 \}$ is denoted by $D$.
- The real part of the complex number $z$ is denoted by $Re(z)$.
- The imaginary part of the complex number $z$ is denoted by $Im(z)$.

### Hardy-Smirnov spaces

Let $\gamma$ be a Riemann map that takes $D$ onto $\Omega$. For $0 < r < 1$, let $\Gamma_r$ be the curve in $\Omega$ defined by $\Gamma_r = \gamma(|z| = r)$. The set of functions analytic on $\Omega$ for which

$$\sup_{0<r<1} \int_{\Gamma_r} |f(w)|^2|dw| < \infty$$

is a Hardy-Smirnov space on $\Omega$. We denote this space by $H^2(\Omega)$ and refer to it simply as a Hardy space on $\Omega$. The functional $\| \cdot \|_\Omega$ defined on $H^2(\Omega)$ by $\| f \|_\Omega = \left( \sup_{0<r<1} \left( \frac{1}{2\pi} \int_{\Gamma_r} |f(w)|^2|dw| \right)^{1/2} \right) \in H^2(\Omega)$ (see [10, p. 63]).

### Hardy space of the unit disc $H^2$

The classical Hardy space $H^2$ of analytic functions on the open unit disc corresponds to the choice $\Omega = D$ and $\gamma$ the identity map. Thus $H^2$ is the set of analytic functions on the open unit disc for which

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$ 

For an arbitrary $\Omega$ it turns out that $H^2(\Omega)$ is isometrically isomorphic to $H^2$ and the details of the isomorphism are given in Theorem A below.

**Theorem A.** Suppose $f$ is holomorphic on $\Omega$. Then $f \in H^2(\Omega)$ if and only if $(f \circ \gamma)(\gamma')^{1/2} \in H^2$. The map $V$ given by $V(f) = (f \circ \gamma)(\gamma')^{1/2}$ is a linear isometry from $H^2(\Omega)$ onto $H^2$.

For a proof see [10, p. 63] or [5, p. 169]. Using $V$, we can define a function $\langle \cdot, \cdot \rangle_\Omega : H^2(\Omega) \times H^2(\Omega) \rightarrow \mathbb{C}$ by

$$\langle f,g \rangle_\Omega = \langle V(f), V(g) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $H^2$. Since $V$ is an isometry it follows that $\langle f,f \rangle_\Omega = \| V(f) \|_{H^2}^2 = \| f \|_{\Omega}^2$. Thus $H^2(\Omega)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_\Omega$. If $g \in H^2$, then it is easy to see that

$$V^{-1}(g) = \frac{1}{(\gamma' \circ \gamma^{-1})^{1/2}} g \circ \gamma^{-1}.$$ (1)
Suppose that \( C_\phi \) is a composition operator on \( H^2(\Omega) \). If \( g \in H^2 \) and \( z \in D \), then

\[
VC_\phi V^{-1}(g)(z) = \left( \frac{y'(z)}{y'(\psi(z))} \right)^{1/2} g(\psi(z)),
\]

where \( \psi = \gamma^{-1} \circ \phi \circ \gamma \). Let \( \psi = (y'/y')^{1/2} \). Then

\[
VC_\phi V^{-1}(g) = C_{\psi', \psi}(g),
\]

where \( C_{\psi, \psi}(g) = \psi \cdot g \circ \psi \). Such an operator is called a weighted composition operator. From the discussion above, \( C_\phi \) on \( H^2(\Omega) \) is isometrically similar to the weighted composition operator \( C_{\psi, \psi} \) on \( H^2 \) (see [10, p. 66]).

Next we cite Theorem 2.1 of [3] which characterizes maps \( \psi \) and \( \phi \) when \( C_{\psi, \psi} \) is Hermitian on \( H^2 \). The following theorem can also be deduced from Theorem 6 of [4].

**Theorem B.** If the weighted composition operator \( C_{\psi', \psi} \) is bounded and Hermitian on \( H^2 \), then

\[
\psi(z) = a_0 + \frac{a_1 z}{1 - a_0 z} \quad \text{and} \quad \psi(z) = \frac{c}{1 - a_0 z}
\]

where \( a_1 \) and \( c \) are real numbers.

Conversely, suppose that \( a_0 \in \mathbb{D} \) and \( c, a_1 \in \mathbb{R} \). If \( \psi(z) = a_0 + a_1 z/(1 - \overline{a_0}z) \) maps the unit disc into itself and \( \psi(z) = c/(1 - \overline{a_0}z) \), then the weighted composition operator \( C_{\psi', \psi} \) is Hermitian and bounded on \( H^2 \).

We cite the following theorem from [6] that will be used later.

**Theorem C.** Let \( \phi \) be an analytic self map of \( \Omega \). The composition operator \( C_\phi \) is unitary on \( H^2(\Omega) \) if and only if \( \phi \) is an automorphism of \( \Omega \) that takes the form \( \phi(w) = e^{i\theta}w + \lambda \) for some real number \( \theta \) and a complex number \( \lambda \).

### 3 Hermitian operators

We begin our work by studying bounded Hermitian composition operators induced by constant maps. Notice that it is elementary to prove that a Riemann map from the unit disc onto another disc is a linear fractional transformation.

**Lemma 3.1.** Let \( \phi \) be a constant self map of \( \Omega \). If \( C_\phi \) is a bounded Hermitian operator on \( H^2(\Omega) \), then \( \Omega \) is a disc.

Conversely, let \( \Omega \) be an open disc and \( \gamma(z) = (az + b)/(cz + d) \), where \( a, b, c, d \in \mathbb{C} \), be the underlying Riemann map of \( H^2(\Omega) \). If \( \phi \) is a constant map that takes the value \((b\overline{d} - a\overline{c})/(|d|^2 - |c|^2)\), then \( C_\phi \) is bounded and Hermitian.

**Proof.** First assume that \( \phi \) is constant and \( C_\phi \) is a bounded Hermitian operator. For \( z \in D \) let

\[
\psi(z) = \gamma^{-1} \circ \phi \circ \gamma(z) \quad \text{and} \quad \psi(z) = \left( \frac{y'(z)}{y'(\psi(z))} \right)^{1/2}.
\]

Next let \( V : H^2(\Omega) \to H^2 \) be given by \( V(f) = (f \circ \gamma)(\gamma')^{1/2} \) (see Theorem A). From (3) it follows that \( VC_\phi V^{-1} \) is the weighted composition operator \( C_{\psi', \psi} \). Since \( V \) is a linear isometry \( C_{\psi', \psi} \) is bounded and Hermitian on \( H^2 \). Therefore from Theorem B it follows that

\[
\psi(z) = a_0 + \frac{a_1 z}{1 - a_0 z} \quad \text{and} \quad \psi(z) = \frac{k}{1 - a_0 z}
\]

where \( k, a_1 \in \mathbb{R} \). Since \( \phi \) is a constant map \( \psi \) is a constant self map of \( D \). Hence \( a_1 = 0 \) and \( |a_0| < 1 \). Then \( \psi(z) = a_0 \) and from (4) it follows that

\[
\frac{y'(z)}{y'(a_0)} = \frac{k^2}{(1 - a_0 z)^2}.
\]

Since \( \Omega = \gamma(D) \) and \( \Omega \) is an open set \( \gamma \) cannot be a constant map. Thus, \( k \neq 0 \).
Next we consider the two cases \( a_0 = 0 \) and \( a_0 \neq 0 \).

First assume that \( a_0 = 0 \). Then, \( \gamma'(z) = k^2 \gamma'(0) \). Thus \( \gamma(z) = k^2 \gamma'(0)z + j \) for some \( j \in \mathbb{C} \). Since \( k^2 \gamma'(0) \) is nonzero \( \gamma(D) \) is a disc.

Next assume that \( a_0 \neq 0 \). Then, \( \gamma'(z) = k^2 \gamma'(a_0)/(1 - \overline{a_0}z)^2 \) and it easily follows that

\[
\gamma(z) = \frac{k^2 \gamma'(a_0)}{a_0(1 - \overline{a_0}z)} + j
\]

for some \( j \in \mathbb{C} \). Therefore \( \gamma \) is a linear fractional map without any poles on the closed unit disc. Thus \( \gamma(D) \) is a disc.

Next we prove the converse;

From direct computations we get \( \gamma^{-1} \circ \phi \circ \gamma(z) = -c/d \) and

\[
\frac{\gamma'(z)}{\gamma'(z)} = \frac{|d|^2 - |c|^2)^2}{|d|^4 (1 + (c/d)z)^2}.
\]

Now let \( \varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z) \) and \( \psi(z) = (\gamma'(z)/\gamma'(\varphi(z)))^{1/2} \). Since \( \gamma(D) \) is a bounded set, \(|c/d| < 1\). From Theorem B it follows that \( C_{\psi, \omega} \) is bounded and Hermitian on \( H^2 \). Since \( C_{\phi} = V^{-1}C_{\psi, \omega}V \) and \( V \) is an isometry \( C_{\phi} \) is bounded and Hermitian on \( H^2(\Omega) \).

The domain \( \Omega \) is said to be symmetric about the point \( p \) if \(- (\Omega - p) = \Omega - p\).

**Lemma 3.2.** Let \( \phi \) be an analytic self map of \( \Omega \) which is neither a constant nor the identity. Suppose that \( C_{\phi} \) is bounded and Hermitian on \( H^2(\Omega) \) and \( \gamma^{-1} \circ \phi \circ \gamma(0) = 0 \). Then the following are true.

1. If \( (\gamma^{-1} \circ \phi \circ \gamma)'(0) = -1 \), then \( \Omega \) is symmetric about \( \gamma(0) \).
2. If \( (\gamma^{-1} \circ \phi \circ \gamma)'(0) \neq -1 \), then \( \Omega \) is a disc.

**Proof.** Let \( \varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z) \) and \( \psi(z) = (\gamma'(z)/\gamma'(\varphi(z)))^{1/2} \). In \( \Omega \). Let \( V : H^2(\Omega) \rightarrow H^2 \) be given by \( V(f) = (f \circ \gamma)(\gamma')^{1/2} \) (see Theorem A). From (3) it follows that \( VC_{\phi}V^{-1} \) is the weighted composition operator \( C_{\psi, \omega} \). Since \( V \) is a linear isometry \( C_{\psi, \omega} \) is bounded and Hermitian on \( H^2 \). Therefore from Theorem B it follows that

\[
\varphi(z) = a_0 + \frac{a_1z}{1 - \overline{a_0}z} \text{ and } \psi(z) = \frac{c}{1 - \overline{a_0}z}
\]

where \( c, a \in \mathbb{R} \). Since \( a_0 = \varphi(0) \), and \( \varphi = \gamma^{-1} \circ \phi \circ \gamma \) it is easy to see that \( \varphi(z) = a_1z, \psi(z) = c \). Therefore

\[
\gamma(z) = \gamma'(a_1z)c^2.
\]

By letting \( z = 0 \) in the equation above it can be readily seen that \( c^2 = 1 \). Notice that \( a_1 = (\gamma^{-1} \circ \phi \circ \gamma)'(0) \).

(1) If \( a_1 = -1 \), from equation (6) it easily follows that \( \gamma(z) = -\gamma(-z) + 2y(0) \). Thus \( \gamma(z) - \gamma(0) = -(\gamma(-z) - \gamma(0)) \). Since \( \{\gamma(z) : z \in D\} = \{\gamma(-z) : z \in D\} \), it can be easily seen that \( \Omega \) is symmetric about \( \gamma(0) \).

(2) Let \( \gamma^{(n)} \) denote the derivative of order \( n \) of \( \gamma \). It follows from equation (6) that \( \gamma^{(n)}(0) = a_1^{n-1}a_1^{(n)}(0) \)

where \( n \geq 2 \). Since \( \phi \) is not the identity map \( \varphi \) is also not the identity thus, \( \phi \neq \gamma \). Since \( \phi \) is real and \( |a_1| \neq 1 \), it easily follows that \( \gamma^{(n)}(0) = 0 \) for \( n \geq 2 \). Since \( \Omega \) is a nonempty open set \( \gamma \) cannot be a constant thus \( \gamma \) is a polynomial of degree 1. Therefore \( \gamma(D) \) is a disc.

**Theorem 3.3.** Let \( \phi \) be an analytic self map of \( \Omega \).

1. If \( \phi \) is a nonconstant map. If the composition operator \( C_{\phi} \) is bounded and Hermitian on \( H^2(\Omega) \), then \( \phi(w) = aw + b \) for some \( a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{C} \).

2. Suppose that \( \phi(w) = aw + b, a \in \mathbb{R}, b \in \mathbb{C} \). Let \( \varphi = \gamma^{-1} \circ \phi \circ \gamma \). If

\[
\psi(z) = a_0 + \frac{a_1z}{1 - \overline{a_0}z} \left( \frac{\gamma'(z)}{\gamma'(\psi(z))} \right)^{1/2} = \frac{c}{1 - \overline{a_0}z}
\]

with \( c, a_1 \in \mathbb{R}, a_0 \in \mathbb{D}, \text{and } -1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|^2)^2 \), then \( C_{\phi} \) is a bounded Hermitian operator on \( H^2(\Omega) \).
Proof. Let \( \varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z) \) and \( \psi(z) = (\gamma'(z)/\gamma'(\varphi(z)))^{1/2}, z \in D \). Let \( V : H^2(\Omega) \to H^2 \) be given by \( V(f) = (f \circ \gamma)(\gamma'')^{1/2} \) (see Theorem A). From (3) it follows that \( VC_\phi V^{-1} \) is the weighted composition operator \( C_{\psi,\varphi} \). Since \( V \) is a linear isometry, \( C \) is Hermitian on \( H^2 \). Therefore from Theorem B it follows that 
\[
\varphi(z) = a_0 + \frac{a_1 z}{1 - \bar{a}_0 z} \quad \text{and} \quad \psi(z) = \frac{c}{1 - \bar{a}_0 z}
\]
where \( c, a_1 \in \mathbb{R} \). Thus,
\[
\frac{\gamma'(z)}{\gamma'(\psi(z))} = \frac{c^2}{(1 - \bar{a}_0 z)^2}.
\]
Since \( \Omega \) is an open set \( \gamma \) cannot be a constant map. Thus \( c \neq 0 \). The map \( \phi \) is nonconstant therefore \( \varphi \) is a nonconstant map. Hence \( a_1 \neq 0 \). Since \( \varphi'(z) = a_1/(1 - \bar{a}_0 z)^2 \) it follows that
\[
\gamma'(z) = \frac{c^2}{a_1} \gamma'(\varphi(z)) \varphi'(z).
\]
Thus,
\[
\gamma(z) = \frac{c^2}{a_1} \gamma(\varphi(z)) + p
\]
for some constant \( p \). But \( \gamma(\varphi(z)) = \phi(\gamma(z)) \), hence \( \gamma(z) = (c^2/a_1) \phi(\gamma(z)) + p \). Now let \( w = \gamma(z) \). Then,
\[
\phi(w) = \frac{a_1}{c^2} w - \frac{a_1}{c^2} p
\]
Proof of (2):

Since \( -1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2 \), from Corollary 2.3 of [3] we get that \( \varphi \) maps the open unit disc into itself. Then from Theorem B it follows that \( C_{\psi,\varphi} \) is bounded and Hermitian on \( H^2 \). Since \( VC_\phi V^{-1} = C_{\psi,\varphi} \) and \( V \) is an isometry, \( C_\phi \) is bounded and Hermitian on \( H^2(\Omega) \). \( \square \)

Let \( \Pi \) denote the upper half plane and take \( \gamma(z) = i(1 + z)/(1 - z) \) to be the underlying Riemann map. The following result, which is also in [9] can be obtained using the theorem above.

The composition operator \( C_\phi \) is bounded and Hermitian on \( H^2(\Pi) \) if and only if
\[
\phi(w) = w + i k, k \geq 0.
\]

4 Geometry of \( \Omega \)

Throughout Section 4 the symbol \( \Omega \) will represent an open, simply connected proper subset of the complex plane. The underlying Riemann map of \( H^2(\Omega) \) from \( D \) onto \( \Omega \) is denoted by \( \gamma \).

There are domains with simple geometric descriptions whose Riemann maps are complicated. For such domains computing \( \gamma^{-1} \circ \phi \circ \gamma \) and \( \gamma'/\gamma' \circ \varphi \) as required by Theorem 3.3 could be difficult. Thus in this section we find some conditions for \( C_\phi \) to be Hermitian that does not involve the computation of \( \gamma^{-1} \circ \phi \circ \gamma \) and \( \gamma'/\gamma' \circ \varphi \).

The following lemma describes some geometric properties of the linear fractional transformation \( \varphi \) when \( C_{\psi,\varphi} \) is Hermitian on \( H^2 \).

Lemma 4.1. Let \( \varphi(z) = a_0 + \frac{a_1 z}{1 - \bar{a}_0 z} \) where \( 0 < |a_0| < 1 \) and \( a_1 \in \mathbb{R} \). Then the following are true.
1. \( a_0 = \varphi(0) \) and \( a_1 = \varphi'(0) \).
2. \( \varphi \) maps the unit disc into itself if and only if \(-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2 \).
3. If \(-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2 \), then the closure of \( \varphi(D) \) is contained in \( D \), and \( \varphi \) has a fixed point in \( D \).
4. If \( r_1, r_2 \) are the fixed points of \( \varphi \) in \( C \), then \( |r_1 r_2| = 1 \).
5. If \(-1 + |a_0|^2 = a_1 \), then \( \varphi(z) = \frac{a_0 - z}{1 - \bar{a}_0 z} \) and the fixed point of \( \varphi \) inside \( D \) is \( (1 - \sqrt{1 - |a_0|^2})/\bar{a}_0 \).
6. If \( a_1 = (1 - |a_0|)^2 \), then \( \varphi \) has no fixed points inside \( D \) and has only one fixed point on the unit circle. The fixed point on the unit circle is \( |a_0|/\bar{a}_0 \) and \( \varphi'(|a_0|/\bar{a}_0) = 1 \).
Proof. (1) A routine computation shows that \( a_0 = \varphi(0) \) and \( a_1 = \varphi'(0) \).

(2) See Corollary 2.3 of [3].

(3) See the first paragraph on page 5778 of [3].

To prove the rest fixed points of \( \varphi \) must be investigated. Solutions of \( \varphi(z) = z \) are the fixed points of \( \varphi \). It is easy to see that the solutions of \( \varphi(z) = z \) are the roots of the quadratic equation

\[
\overline{a_0}z^2 + (a_1 - 1 - |a_0|^2)z + a_0 = 0. 
\tag{7}
\]

(4) If \( r_1, r_2 \) are the roots of the equation (7), then \( r_1 r_2 = a_0/\overline{a_0} \).

(5) If \( -1 + |a_0|^2 = a_1 \), then a routine computation yields that \( \varphi(z) = a_0 - \overline{a_0}z \). It is easy to see that \( r_1 = (1 - \sqrt{1 - |a_0|^2})/\overline{a_0} \) is a root of the equation (7) and \(|r_1| < 1\). From part (4) it follows that the other fixed point must lie outside the closed unit disc.

(6) If \( a_1 = (1 - |a_0|^2) \), then the equation (7) has only one root and it is readily seen that the root is \( |a_0|/\overline{a_0} \). A routine computation shows that \( \varphi'(z) = (1 - |a_0|^2)/(1 - \overline{a_0}z)^2 \), hence \( \varphi'(|a_0|/\overline{a_0}) = 1 \).

The following lemma describes how the different values of \( \varphi(0) \) and \( \varphi'(0) \) affect the operator theoretic properties of the Hermitian operator \( C_{\varphi, \psi} \).

**Lemma 4.2.** Suppose that \( C_{\varphi, \psi} \) is bounded and Hermitian on \( H^2 \). Then the following are true,

(1) If \( -1 + |\varphi(0)|^2 < \varphi(0) < (1 - |\varphi(0)|)^2 \), then \( C_{\varphi, \psi} \) is compact and \( \varphi \) has a fixed point inside the open unit disc.

(2) If \( -1 + |\varphi(0)|^2 = \varphi(0) \), then \( C_{\varphi, \psi} \) is an isometry and \( \varphi \) has a fixed point inside the open unit disc.

For a proof of the lemma above see the third paragraph on page 5778 of [3].

**Lemma 4.3.** Suppose that \( \phi(w) = rw + k \), where \( r \in \mathbb{R} \) and \( k \in \mathbb{C} \) maps \( \Omega \) into itself. Assume that \( C_{\phi} \) is bounded and Hermitian on \( H^2(\Omega) \). Let \( \varphi = \gamma^{-1} \circ \phi \circ \gamma \). Then the following are true.

(1) \( \varphi'(0) = r \gamma'(0)/\gamma' \varphi(0) \).

(2) If \( \varphi(p) = p \) and \( p \in \mathcal{D} \), then \( r = \varphi'(p) \).

(3) Suppose that \( \varphi(p) = p \) and \( |p| = 1 \). If \( \gamma' \) exists at \( p \) and \( \gamma'(p) \neq 0 \), then \( r = 1 \).

Proof. Let \( \psi = (\gamma'/\gamma' \circ \varphi)^{1/2} \). From (3) it follows that \( C_{\psi, \varphi} = V C_{\phi} V^{-1} \). Let \( z \in \mathcal{D} \). Since \( \gamma \circ \varphi = \phi \circ \gamma \), we have, \( \gamma \varphi(z) = r \gamma(z) + k \). Now it follows that

\[
\gamma'(z) \varphi'(z) = r \gamma'(z). 
\tag{8}
\]

To prove (1) let \( z = 0 \) in (8).

To prove (2) let \( z = p \) in (8).

Proof of (3); since \( C_{\phi} \) is bounded and Hermitian \( \varphi \) is continuous on the closed unit disc (see Theorem B). Since \( \gamma' \) exist at \( p \) we let \( z = p \) in (8). Then \( r = \varphi'(p) \). From part (6) of Lemma 4.1 it follows that \( \varphi'(p) = 1 \).

**Fixed points**

Suppose that \( \phi(w) = aw + b, a \in \mathbb{R} \setminus \{1\}, b \in \mathbb{C} \) is a self map of \( \Omega \). It is easy to see that \( \phi \) has a natural extension \( \hat{\phi} \) to the whole complex plane. Since \( \hat{\phi}(b/(1-a)) = b/(1-a) \), and \( \hat{\phi} \) clearly does not have more than one fixed point in \( \mathbb{C} \) it can be readily seen that \( \phi \) has a fixed point in \( \Omega \) if and only if \( b/(1-a) \in \Omega \).

**Proposition 4.4.** Suppose that \( \Omega \) is unbounded. Let \( \phi(w) = aw + b, a \in \mathbb{R}, b \in \mathbb{C} \) be a nonautomorphic self map of \( \Omega \) without any fixed points in \( \Omega \). Further suppose that \( C_{\phi} \) is bounded and Hermitian on \( H^2(\Omega) \). Let \( \varphi = \gamma^{-1} \circ \phi \circ \gamma \). Then, \( \varphi \) has a fixed point \( \xi \) on the unit circle and \( \lim_{z \to \xi} \gamma(z) \) does not exist.

Proof. From Lemma 3.1 and Theorem B it follows that \( \varphi \) is a nonconstant linear fractional self map of \( \mathcal{D} \). Hence \( \varphi(\mathcal{D}) \) must be a disc. Since the map \( \phi \) is not an automorphism of \( \Omega \) the map \( \varphi \) is not an automorphism of \( \mathcal{D} \). Thus, \( \varphi(\mathcal{D}) \) must be properly contained in \( \mathcal{D} \).
Since \( \phi \) does not possess a fixed point in \( \Omega \) it follows that \( \psi \) does not fix any points in \( \mathcal{D} \). But, \( \psi \) is an analytic self map of the unit disc, therefore it has a fixed point on the unit circle (see page 59 of [2]). Since \( \psi(\mathcal{D}) \) is a disc properly contained in \( \mathcal{D} \) it follows that \( \psi \) has exactly one fixed point on the unit circle. Let \( \zeta \) be the fixed point of \( \psi \).

To prove the remaining part suppose that \( \lim_{z \to \zeta} \gamma(z) \) exist finitely. Then, there is a disc \( U \) centered at \( \zeta \) such that \( \gamma(\mathcal{D} \cap U) \) is a bounded set. The closure of \( \psi(\mathcal{D}) \setminus U \) is contained in \( \mathcal{D} \). Therefore, the set \( \gamma(\psi(\mathcal{D}) \setminus U) \) is bounded. Since

\[
\gamma(\psi(\mathcal{D})) = (\gamma(\psi(\mathcal{D}) \setminus U)) \cup (\gamma(\psi(\mathcal{D}) \cap U))
\]

it easily follows that the set \( \gamma(\psi(\mathcal{D})) \) is bounded. Clearly, \( \gamma \circ \psi(z) = \phi \circ \gamma(z) \), for \( z \in \mathcal{D} \), hence

\[
\gamma(z) = \frac{1}{d}(\gamma(\psi(z)) - b).
\]

Note that since \( \psi \) is a nonconstant \( a \) is nonzero. Since \( \gamma(\psi(\mathcal{D})) \) is a bounded set, from the equation above it follows that the set \( \gamma(\mathcal{D}) \) is bounded. But \( \Omega \) is an unbounded set, hence our assumption that \( \lim_{z \to \zeta} \gamma(z) \) exist finitely must be false. \( \square \)

**Bergman space of the unit disc \( A^2 \)**

The set of analytic functions on the open unit disc for which

\[
\int_{\mathcal{D}} |f(w)|^2 dA < \infty
\]

where \( dA \) is the Lebesgue area measure is known as the Bergman space of the unit disc. This space is denoted by \( A^2 \).

Let \( w \in \mathcal{D} \). If \( K_w(z) = 1/(1 - \overline{w}z)^2 \) for \( z \in \mathcal{D} \), then \( K_w \) is the reproducing kernel function at \( w \) in \( A^2 \) and

\[
\int_{\mathcal{D}} f(z) K_w(z) dA = f(w), \text{ for any } f \in A^2.
\]

In the following result we look at \( \gamma \) whose derivative is in the Bergman space.

**Proposition 4.5.** Suppose that \( \phi(w) = aw + b, a \in \mathcal{R}, b \in \mathcal{C} \), is a self map of \( \Omega \) and \( \phi \) has no fixed points in \( \Omega \). If \( \gamma' \) is contained in the Bergman space of the unit disc \( A^2 \), then \( C_{\phi} \) is not a bounded Hermitian operator on \( H^2(\Omega) \).

**Proof.** We prove this by contradiction. Assume that \( C_{\phi} \) is bounded and Hermitian on \( H^2(\Omega) \). Let \( \psi = \gamma^{-1} \circ \phi \circ \gamma \) and \( \psi = (\gamma'/\gamma') \circ \phi \)^{1/2}. Then \( C_{\psi, \psi} = V C_{\phi} V^{-1} \) (see (3)). Since \( V \) is an isometry \( C_{\psi, \psi} \) is bounded and Hermitian on \( H^2 \), hence

\[
\varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0} z} \quad \text{and} \quad \psi(z) = \frac{c}{1 - \overline{a_0} z}
\]

for some \( a_1, c \in \mathcal{R} \) (see Theorem B). Clearly \( \gamma'(\psi(z)) = \phi(\gamma(z)) \). Hence

\[
\gamma(\psi(z)) = \alpha \gamma(z) + b.
\]

Now it follows that

\[
\gamma'(\psi(z))\psi'(z) = \gamma'(z).
\]

Notice that \( \psi'(z) = a_1/(1 - \overline{a_0} z)^2 \) is a scalar multiple of the reproducing kernel function at \( a_0 \) in \( A^2 \). Thus, from Theorem 6 of [4] it follows that the weighted composition operator \( C_{\psi', \psi} \) is Hermitian on \( A^2 \). Equation (9) yields that

\[
C_{\psi', \psi}(\gamma') = \gamma'.
\]

Therefore, \( \gamma' \) is an eigenvector of \( C_{\psi', \psi} \). Since \( \psi \) is a self map of the unit disc \( -1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2 \), (see part (2) of Lemma 4.1). Hence one of the following 3 cases must be true:

1. \(-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2\)
2. \(a_1 = -1 + |a_0|^2\)
3. \(a_1 = (1 - |a_0|)^2\)
Since the map $\phi$ does not have any fixed points in $\Omega$ the map $\psi$ does not have any fixed points in $D$. If the case (1) or case (2) above is true, then $\psi$ has a fixed point in the open unit disc (see parts (3) and (5) of Lemma 4.1). Therefore, $a_1 = (1 - |a_0|)^2$. Now, from corollaries 2 and 15 of [4] it follows that $C_{\psi, \phi}$ does not have any eigenvectors. This is the desired contradiction. 

If $C_{\psi}$ is the identity operator on the Hardy space of the unit disc $H^2$, then it is very easy to see that $\psi$ is the identity function on the unit disc. Next we prove for that for any $\Omega$, if $C_{\phi}$ is the identity on $H^2(\Omega)$, then $\phi$ is the identity map on $\Omega$.

**Lemma 4.6.** Let $\tau$ be an analytic self map of $\Omega$. If the composition operator $C_{\tau}$ is the identity operator on $H^2(\Omega)$, then $\tau(w) = w$ for all $w \in \Omega$.

**Proof.** If $\psi = \gamma^{-1} \circ \tau \circ \gamma$ and $\psi = (\gamma'/\gamma \circ \phi)^1/2$, from (2) it follows that

$$C_{\psi, \phi} = VC_{\tau}V^{-1}.$$ 

Therefore, if $C_{\tau}$ is the identity operator on $H^2(\Omega)$ it easily follows that $C_{\psi, \phi}$ is the identity operator on $H^2$. If $f \in H^2$, then

$$C_{\psi, \phi}(f) = \psi \cdot f \circ \phi$$

(10) Substitute the constant function $f(z) = 1$ in (10) and it follows immediately that $\psi(z) = 1$ for all $z \in D$. Thus, for all $f \in H^2$,

$$C_{\psi, \phi}(f) = f \circ \psi.$$ 

(11) Next substitute the function $f(z) = z$ in (11) and it yields $\psi(z) = z$ for $z \in D$. Since $\psi = \gamma^{-1} \circ \phi \circ \gamma$, it follows that $\phi(w) = w$ for all $w \in \Omega$. 

For self maps $\phi(w) = aw + b$ of $\Omega$ with $a \in \mathcal{R}, b \in \mathcal{C}$, next we look at $a = -1, a = 1$ and $|a| \neq 1$ separately.

**Lemma 4.7.** If $\phi(w) = -w + r, r \in \mathcal{C}$ maps $\Omega$ into itself, then $C_{\phi}$ is bounded and Hermitian on $H^2(\Omega)$.

**Proof.** If $w \in \Omega$, then it is easy to see that

$$\phi \circ \phi(w) = -(-w + r) + r = w.$$ 

Thus $\phi$ is an automorphism of $\Omega$. From Theorem C it follows that $C_{\phi}$ is unitary. Hence

$$C_{\phi}^{-1} = C_{\phi}^*.$$ 

If $f \in H^2(\Omega)$, then $C_{\phi}(f) = f \circ \phi$ and it follows that $C_{\phi}C_{\phi}(f) = f \circ \phi \circ \phi$. Hence $C_{\phi}C_{\phi}$ is the identity operator on $H^2(\Omega)$. Therefore $C_{\phi} = C_{\phi}^{-1}$ and now it follows easily that

$$C_{\phi} = C_{\phi}^*.$$ 

Next we look at Hermitian $C_{\phi}$ where $\phi(w) = w + r$.

**Lemma 4.8.** Suppose that $\phi(w) = w + b, b \in \mathcal{C} \setminus \{0\}$ is a self map of $\Omega$. If $\phi$ is an automorphism of $\Omega$, then $C_{\phi}$ is not a bounded Hermitian operator on $H^2(\Omega)$.

**Proof.** Suppose that $C_{\phi}$ is bounded and Hermitian. Since $\phi$ is an automorphism of $\Omega$, from Theorem C it follows that $C_{\phi}$ is unitary. Now, $C_{\phi}$ is both Hermitian and unitary therefore, $C_{\phi}C_{\phi}$ is the identity operator on $H^2(\Omega)$. Let $f \in H^2(\Omega)$. Since $C_{\phi}(f) = f \circ \phi$, it is easy to see that $C_{\phi}C_{\phi}(f) = f \circ \phi \circ \phi$. Thus,

$$C_{\phi}C_{\phi} = C_{\phi \circ \phi}.$$ 

Now, from Lemma 4.6 it follows that $\phi \circ \phi(w) = w$ for $w \in \Omega$. But, $\phi \circ \phi(w) = w + 2b$, and this leads to a contradiction since $b \neq 0$. 


Next we look at domains with infinitely long boundaries.

**Lemma 4.9.** Suppose that one-dimensional Hausdorff measure of the boundary of $\Omega$ is infinite. Let $\phi(w) = aw + b, a \in \mathbb{R}, b \in \mathbb{C}$ be a self map of $\Omega$. If $a^2 \neq 1$ and $b/(1-a) \in \Omega$, then $C_\phi$ is not a bounded Hermitian operator.

**Proof.** Suppose that $C_\phi$ is bounded and Hermitian on $H^2(\Omega)$. For $z \in \mathcal{D}$, let $\varphi(z) = y^{-1} \circ \phi \circ \gamma(z)$ and $\psi(z) = (y'(z)/y'((\varphi(z))))^{1/2}$. Then $VC_\phi V^{-1} = C_{\psi,\psi}$ (see (3)). Since $V$ is an isometry $C_{\psi,\psi}$ is bounded and Hermitian on $H^2$. Now from Theorem B it follows that $\varphi(z) = a_0 + a_1 z/(1-\overline{a}_0 z)$ where $a_1 \in \mathbb{R}$. Clearly $\varphi$ is a self map of $\mathcal{D}$, hence from Lemma 4.1 it follows that

$$-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2.$$ 

Therefore, one of the following 3 cases must be true:

1. $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$
2. $a_1 = -1 + |a_0|^2$
3. $a_1 = (1 - |a_0|)^2$

Next we will show that none of these cases are possible.

Case (1): if $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$, then $C_{\psi,\psi}$ is compact (Lemma 4.2). Thus $C_\phi$ is compact. Since the length of the boundary of $\Omega$ is infinite, from Theorem 1.5 of [10] it follows that $H^2(\Omega)$ does not possess any compact composition operators. Thus, case (1) is not possible.

Case (2): if $-1 + |a_0|^2 = a_1$, then $C_{\psi,\psi}$ is an isometry (Lemma 4.2). Since $V$ is an isometry $C_\phi$ is also an isometry. Thus,

$$C_\phi^*C_\phi = I$$

where $I$ is the identity operator on $H^2(\Omega)$. Since $C_\phi$ is Hermitian we get $C_\phi C_\phi = I$. For $g$ in $H^2(\Omega)$ it is readily seen that $C_\phi C_\phi (g) = g \circ \phi \circ \phi$. Hence $C_\phi C_\phi = C_{\phi \circ \phi \circ \phi}$. Thus, $C_{\phi \circ \phi \circ \phi}$ is the identity operator on $H^2(\Omega)$ and from Lemma 4.6 it follows that $\phi \circ \phi (w) = w$ for $w \in \Omega$. A direct computation shows that for $w \in \Omega$,

$$\phi \circ \phi (w) = a^2 w + (ab + b)$$

and since $a^2 \neq 1$ it is clear that $\phi \circ \phi (w) \neq w$. Therefore case(2) is not possible.

Case (3): if $a_1 = (1 - |a_0|)^2$, then $\varphi$ does not have any fixed points inside the open unit disc (part 6 of Lemma 4.1). Let $w_0 = b/(1-a)$. It can be readily seen that $\phi$ fixes $w_0$. Therefore, $\varphi$ fixes $\gamma^{-1}(w_0)$ which is inside $\mathcal{D}$. This shows that case(3) is not possible.

Since none of the three cases are possible it follows that our assumption, $C_\phi$ is bounded and Hermitian is false.

The next theorem summarizes results above when boundary of $\Omega$ is infinite and $\phi$ has a fixed point inside $\Omega$.

**Theorem 4.10.** Suppose that one-dimensional Hausdorff measure of the boundary of $\Omega$ is infinite. Let $\phi(w) = aw + b, a \in \mathbb{R}, b \in \mathbb{C}$ be a self map of $\Omega$ which is neither the identity nor a constant. If $\phi$ has a fixed point in $\Omega$, then the following are true.

1. If $a^2 \neq 1$, then $C_\phi$ is not a bounded Hermitian operator on $H^2(\Omega)$.
2. If $a = -1$, then $C_\phi$ is a bounded Hermitian operator on $H^2(\Omega)$.

The proof of the theorem above easily follows from Lemma 4.7 and Lemma 4.9.

## 5 Examples

### 5.1 Strip

Throughout the subsection 5.1 the set $\{z : -1 < Im(z) < 1\}$ is denoted by $\Omega$ and the underlying Riemann map of $H^2(\Omega)$ is denoted by $\gamma$. 
Suppose that \( \phi(w) = aw + b, \ a \in \mathbb{R} \) and \( b \in \mathbb{C} \) is a self map of \( \Omega \). It is easy to see that if \( |a| > 1 \), then \( \phi(\Omega) \not\subseteq \Omega \). Thus \(-1 \leq a \leq 1 \).

**Lemma 5.1.** Let \( \phi(w) = aw + b, \ -1 < a < 1, \ b \in \mathbb{C} \) be a self map of \( \Omega \). If \( C_\phi \) is a bounded Hermitian operator on \( H^2(\Omega) \), then \( \phi \) has a fixed point in \( \Omega \).

**Proof.** Recall that \( \phi \) has a fixed point in \( \Omega \) if and only if \( b/(1-a) \) is in \( \Omega \). From Lemma 3.1 it follows that \( a \neq 0 \).

First consider \(-1 < a < 0 \). Since \( \phi(0) = b \) it follows that \( b \in \Omega \). Clearly \(-1 - a > 1 \), and \( \Omega \) is a convex region that contains 0 therefore, \( b/(1-a) \in \Omega \).

Now consider \( 0 < a < 1 \). Let \( b = a + i\beta \). If \( w \in \Omega \), then \( Im(\phi(w)) = aIm(w) + \beta \). Thus

\[
-1 < aIm(w) + \beta < 1.
\]

Since \( Im(w) \) can take any value in the interval \((-1, 1)\) it follows that

\[
a + \beta \leq 1.
\]

We will next show that the assumption \( a + \beta = 1 \), leads to a contradiction. If \( a + \beta = 1 \), then \( \beta/(1-a) = 1 \). Since the imaginary part of \( b/(1-a) \) is \( \beta/(1-a) \) it follows that \( b/(1-a) \) is on the boundary of \( \Omega \). Let

\[
\varphi = \gamma^{-1} \circ \phi \circ \gamma.
\]

Since \( C_\phi \) is a bounded Hermitian operator \( \varphi(z) = a_0 + (a_1 z)/(1-\overline{a_0}z) \) where \( a_1 \in \mathbb{R} \) and \( a_0 \in \mathbb{D} \) (see (3) and Theorem B). Clearly \( \varphi \) is a self map of \( \mathbb{D} \), hence from Lemma 4.1 it follows that \(-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2 \).

Therefore, one of the following 3 cases must be true;

1. \(-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2 \)
2. \( a_1 = -1 + |a_0|^2 \)
3. \( a_1 = (1 - |a_0|)^2 \)

If case (1) or (2) above is true, then from parts (3) and (5) of Lemma 4.1 it follows that \( \varphi \) has a fixed point in \( \mathbb{D} \). Since the point \( b/(1-a) \) is on the boundary of \( \Omega \) the map \( \varphi \) does not fix any points in \( \Omega \). Thus \( \varphi \) does not have any fixed points in the open unit disc. Therefore cases (1) or (2) above cannot be true: hence

\[
a_1 = (1 - |a_0|)^2.
\]

Then from part (6) of Lemma 4.1 it follows that \( \varphi \) has exactly one fixed point \( \zeta \) on the unit circle. The map \( \varphi \) is not an automorphism of the unit disc (see the second paragraph of page 5778 of [3]). Therefore \( \phi \) cannot be an automorphism of \( \Omega \).

Let \( \{w_n\} \) be a sequence in \( \Omega \) that converges to \( b/(1-a) \). Clearly \( \gamma^{-1}(w_n) \) has a subsequence \( \gamma^{-1}(w_{m_n}) \) that converges to some \( \mu \) in the closed unit disc. From Theorem 2 of chapter 6 in [1] it follows that \( \mu \) is on the unit circle. Let \( z_n = \gamma^{-1}(w_{m_n}) \). If \( z_n \to \zeta \), from Proposition 4.4 it follows that \( \{\gamma(z_n)\} \) diverges. Thus \( \mu \neq \zeta \). Let \( \gamma_0 = \varphi(\mu) \). Since \( \varphi \) is a linear fractional self map of \( \mathbb{D} \) which is not an automorphism it is easy to see that \( \gamma_0 \in \mathbb{D} \). The continuity of \( \varphi \) at \( \mu \) yields that \( \varphi(z_n) \to \gamma_0 \). From equation (13) it follows that

\[
\gamma(\varphi(z_n)) = a\gamma(z_n) + b.
\]

Letting \( n \) tend to infinity in equation (14) we get

\[
\gamma(\gamma_0) = \frac{b}{1-a}.
\]

But \( \gamma(\gamma_0) \in \Omega \) and \( b/(1-a) \not\in \Omega \) hence our assumption that \( a + \beta = 1 \), cannot be true. Thus \( a + \beta < 1 \).

From inequality (12) it also follows that \(-1 \leq -a + \beta \). Therefore, using a method similar to the one used above it can be proved that \(-1 < -a + \beta \). Therefore

\[
-1 < \beta/(1-a) < 1
\]

Since the imaginary part of \( b/(1-a) \) is \( \beta/(1-a) \) it follows that \( \Omega \) contains the point \( b/(1-a) \). Hence \( \phi \) has a fixed point in \( \Omega \).
The next theorem characterizes bounded Hermitian operators on $H^2(\Omega)$ when $\Omega$ is the strip \{x + iy : -1 < y < 1\}.

**Theorem 5.2.** Let $\Omega = \{z : -1 < Im(z) < 1\}$. The composition operator $C_\phi$ is bounded, non-trivial and Hermitian on $H^2(\Omega)$ if and only if $\phi(w) = -w + b$ where $b \in \mathbb{R}$.

**Proof.** First assume that $C_\phi$ is bounded and Hermitian on $H^2(\Omega)$. From Theorem 3.3 it follows that $\phi(w) = aw + b$ where $a \in \mathbb{R}$ and $b \in \mathbb{C}$. Since $\phi$ is a self map of $\Omega$ it is easy to see that $-1 \leq a \leq 1$.

If $-1 < a < 1$, then from Lemma 5.1 it follows that $\phi$ has a fixed point in $\Omega$. Then, the part 1 of Theorem 4.10 says that $C_\phi$ is not a bounded Hermitian operator. Thus $a = \pm 1$.

Since $\phi$ maps $\Omega$ to itself, if $a = \pm 1$, then it is clear that $Im(b) = 0$.

If $a = 1$, and $b$ is a real number, then $\phi$ is an automorphism of $\Omega$. Now from Lemma 4.8 it follows that $b = 0$.

Since $C_\phi$ is not the identity operator it can now be concluded that $a = -1$. Thus, $\phi(w) = -w + b$ for some $b \in \mathbb{R}$.

Now assume that $\phi(w) = -w + b$ where $b$ is a real number. It is readily seen that $\phi$ is a self map of $\Omega$. Then from Lemma 4.7 it follows that $C_\phi$ is bounded and Hermitian on $H^2(\Omega)$. \qed

### 5.2 Cross

Throughout the subsection 5.2 the set $\Omega = \{z : -1 < Im(z) < 1\} \cup \{z : -1 < Re(z) < 1\}$ is denoted by $\Omega$ and the underlying Riemann map of $H^2(\Omega)$ is denoted by $\gamma$.

**Theorem 5.3.** The only non-trivial bounded Hermitian composition operator $C_\phi$ on $H^2(\Omega)$ is induced by $\phi(w) = -w$.

The proof of Theorem 5.3 is similar to the proof of Theorem 5.2 therefore we will only provide an outline.

The map $\phi(w) = -w$ is a self map of $\Omega$, therefore from Lemma 4.7 it follows that $C_\phi$ is bounded and Hermitian on $H^2(\Omega)$.

If $C_\phi$ is bounded and Hermitian on $H^2(\Omega)$, then $\phi(w) = aw + b$ where $a \in \mathbb{R}, b \in \mathbb{C}$ (see Theorem 3.3).

If $|a| > 1$, then $\Omega$ cannot contain $\phi(\Omega)$. Thus $|a| \leq 1$. Using a proof similar to the proof of Lemma 5.1 it can be proved that $\phi \not\subset (-1, 1)$. Hence $a = \pm 1$. If $\phi(w) = w + b$ maps $\Omega$ into itself then it is clear that $b = 0$. Thus $a = 1$ results in the identity operator. If $\phi(w) = -w + b$ is a self map of $\Omega$ it is not difficult to see that $b = 0$. Therefore $\phi(w) = -w$.

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