Abstract: We generalise a result of R. Thomas to establish the non-vanishing of the first $\ell^2$ Betti number for a class of finitely generated groups.

2010 Mathematics Subject Classification: 20J06.

Key words: finitely generated groups, orbifolds, cohomology, Euler characteristic.

In this note we give the following generalisation of a result of Richard Thomas [8].

Theorem 1. Let $G$ be a finitely generated group given by the presentation

$$\langle x_1, \ldots, x_d : u_1^{m_1}, \ldots, u_r^{m_r} \rangle$$

such that each relator $u_i$ has order $m_i$ in $G$.

1. If $G$ is finite then $1 - d + \sum_{i=1}^{r} \frac{1}{m_i} > 0$ and $|G| \geq \frac{1}{1 - d + \sum_{i=1}^{r} \frac{1}{m_i}}$.

2. If the first $\ell^2$ Betti number $\beta_1^2(G)$ of $G$ is zero, then

$$1 - d + \sum_{i=1}^{r} \frac{1}{m_i} \geq 0.$$ 

In particular, the case when all the exponents $m_i$ in the presentation are equal to 1 yields the well known observation that when the first $\ell^2$ Betti number is zero the deficiency of the presentation $d - r$ must be at most 1. The vanishing of the first $\ell^2$ Betti number of a group $G$ holds for example if $G$ is finite, if it satisfies Kazhdan’s property (T) or if it admits an infinite normal amenable subgroup (in particular if it is infinite amenable). We refer to [4] for other interesting examples. We obtain as a corollary:

Corollary 2. Let $G$ be a finitely generated group given by the presentation

$$\langle x_1, \ldots, x_d : u_1^{m_1}, \ldots, u_r^{m_r} \rangle$$

Research partially supported by EPSRC grant EP/F031947/1.
such that each relator $u_i$ has order $m_i$ in $G$. If $d > 1 + \sum_{i=1}^{\infty} \frac{1}{m_i}$, then $G$ is infinite, does not satisfy Kazhdan’s property (T) and has no amenable infinite normal subgroups.

Thomas established the inequality in (1) above by providing a simple but elegant computation of the dimension of the $\mathbb{F}_2$-vector space of 1-cycles of the cellular chain complex of the Cayley graph of $G$ (Thomas refers to this space as the cycle space of $\Gamma$.) If $\Gamma$ has $d$ edges and $v$ vertices then the dimension of this vector space is $d - v + 1$. An alternative approach, yielding information about the classical first Betti number of $G$ and its finite index subgroups is explored by Allcock in [1].

We generalise this idea to give the additional inequality in (2) above by using elementary observations about the $\ell^2$ Betti numbers $\beta_i^2$ of the orbihedral presentation 2-complex of $G$. For an introduction to $\ell^2$ Betti numbers, we refer the reader to [3]. The first $\ell^2$ Betti number vanishes for all finite groups. Cheeger and Gromov have shown that if a group $G$ is amenable then $\beta_1^2(G) = 0$ [2, Theorem 0.2]. More generally, $\beta_1^2(G)$ is zero for any group $G$ which contains an infinite normal amenable subgroup.

Remark 3. Theorem 1 can be derived from deeper results of Peterson and Thom; in particular, Equation (3) yields the inequality $\beta_1^2(G) \geq \frac{1}{|G|} + d - 1 - \sum_{i \in I} \frac{1}{m_i}$ from [7]. Here, $|G|$ denotes the size of $G$ and $\frac{1}{|G|}$ is understood to be zero when $G$ is infinite.

Finitely generated but not finitely presented groups. Lück has defined $\ell^2$ Betti numbers for any countable discrete group. The notion agrees with the cellular $\ell^2$ Betti numbers for finitely presented groups and the basic properties including a generalised Euler-Poincaré formula for $G$-CW complexes may be found in Chapter 6 of [6]. Working in this context and arguing as in the proof of Theorem 1, we obtain the following generalisation.

Theorem 4. Suppose a group $G$ is given by the presentation

$$G = \langle x_1, \ldots, x_d : u_i^{m_i}, i \in I \rangle$$

where $I$ is a countable set and each relator $u_i$ has order $m_i$ in $G$. If $\sum_{i \in I} \frac{1}{m_i}$ converges then $\beta_1^2(G) \geq \frac{1}{|G|} + d - 1 - \sum_{i \in I} \frac{1}{m_i}$. In particular if $\beta_1^2(G) = 0$ then $\sum_{i \in I} \frac{1}{m_i} - d + 1 \geq 0$.

Before we embark on the proof of Theorem 1, we need a short lemma which says that the orbihedral Euler characteristic of a $G$-CW complex $Y$ may be computed from its $\ell^2$ Betti numbers. The lemma is well known and may be found in [6].
Lemma 5 ([6, Theorem 6.80]). If $G$ acts on a connected CW complex $\tilde{Y}$ with finite quotient $Y$ such that stabilisers of cells are finite, then the $\ell^2$-Euler characteristic of $Y$ is equal to the orbihedral Euler characteristic of $Y$. More precisely, if for each $i$, $\Sigma_i$ is a choice of representatives for the orbits of $i$-cells in $\tilde{Y}$ and the stabiliser of a cell $\sigma$ in $G$ is written $G_\sigma$, then

$$
\sum_i (-1)^i \beta_i^2(Y) = \sum_i (-1)^i \sum_{\sigma \in \Sigma_i} \frac{1}{|G_\sigma|}.
$$

We now proceed with the proof of Theorem 1.

Proof of Theorem 1: Let $G$ be a group given by the presentation $\langle x_1, \ldots, x_d : u_{1}^{m_{1}}, \ldots, u_{r}^{m_{r}} \rangle$ where each relator $u_i$ has order $m_i$ in $G$. The orbihedral presentation 2-complex of $G$, which we will denote by $\mathcal{P}$, has one vertex and $d$ edges forming a bouquet of $d$ circles. Identifying each of the circles with one of the generators $x_i$ we identify the fundamental group of this bouquet with the free group on $\{x_1, \ldots, x_d\}$. Attached to this are $r$ discs, $D_1, \ldots, D_r$. For each $i = 1, \ldots, r$, the disc $D_i$ is endowed with a cone point of cone angle $\frac{2\pi}{m_i}$ and its boundary is attached by a degree 1 map along the loop in the bouquet of circles corresponding to the element $u_i$.

Attaching the corresponding stabilisers to cells we obtain, in the language of Haefliger [5], a developable complex of groups, meaning that the orbihedral universal cover $X$ of $\mathcal{P}$ exists. In fact, $X$ has a simple description in terms of the Cayley graph $\mathcal{C}$ of $G$. The 1-skeleton of the orbihedral universal cover is the Cayley graph of $G$ with respect to the generating set $\{x_1, \ldots, x_d\}$, while the 2-skeleton is obtained from the 2-skeleton of the topological universal cover of the presentation 2-complex by collapsing stacks of relator discs having common boundaries. Specifically, the relator $u_{i}^{m_{i}}$ corresponds to a loop $\gamma_i$ in $\mathcal{P}$ bounding a disc and there is a unique lift $\tilde{\gamma}_i$ of $\gamma_i$ based at the identity vertex in $\mathcal{C}$. In the topological universal cover of the presentation 2-complex there are additional copies of this disc (glued along the same loop) based at the elements $u_i, \ldots, u_i^{m_i-1}$ and the action of the subgroup $\langle u_i \rangle$ permutes these discs so that each has trivial stabiliser. In contrast, these copies are identified in the orbihedral cover to give a single disc and it is preserved by the element $u_i$. The hypothesis that $u_i$ has order $m_i$ controls the order of the cell stabiliser.

We now apply the identity in (1) to our complex $X$. The action of $G$ on the vertices and the edges of $X$ is both free and transitive. On the other hand, by hypothesis, the stabiliser of a lift of a 2-cell $D_i$ has
order \( m_i \). Hence, 
\[ \beta_0^2(\mathcal{P}) - \beta_1^2(\mathcal{P}) + \beta_2^2(\mathcal{P}) = 1 - d + \sum_i \frac{1}{m_i}. \]
We also know that 
\[ \beta_0^2(\mathcal{P}) = \frac{1}{|\mathcal{G}|} \]
where \( \frac{1}{|\mathcal{G}|} \) is understood to be zero when \( \mathcal{G} \) is infinite. Therefore,

\[ \frac{1}{|\mathcal{G}|} - \beta_1^2(\mathcal{P}) + \beta_2^2(\mathcal{P}) = 1 - d + \sum_i \frac{1}{m_i}. \]  

Finally we remark that the first \( \ell^2 \) Betti number of the group \( \mathcal{G} \) may be computed as the first \( \ell^2 \) Betti number of the orbihedral presentation complex used above. By definition, \( \beta_1^2(\mathcal{G}) \) is the von Neumann dimension of the first \( \ell^2 \) homology group of \( Y \) with coefficients in the von-Neumann algebra of \( \mathcal{G} \), where \( Y \) is the universal cover of the (topological) presentation 2 complex for \( \mathcal{G} \). Since both \( X \) and \( Y \) are simply connected we deduce from Theorem 6.54(3) of [6] that \( \beta_1^2(\mathcal{G}) = \beta_1^2(\mathcal{P}) \). Therefore, Equation (2) becomes

\[ \frac{1}{|\mathcal{G}|} - \beta_1^2(\mathcal{G}) + \beta_2^2(\mathcal{P}) = 1 - d + \sum_i \frac{1}{m_i}. \]

Now assume that \( \beta_1^2(\mathcal{G}) = 0 \). Since \( \beta_2^2(\mathcal{P}) \geq 0 \), we get the identity we are looking for, namely

\[ 1 - d + \sum_{i=1}^{r} \frac{1}{m_i} \geq \frac{1}{|\mathcal{G}|}. \]

In particular, if \( \mathcal{G} \) is finite, then the \( \ell^2 \) cohomology of \( \mathcal{G} \) is just the group cohomology with real coefficients, and this vanishes so we obtain Thomas’s result that \( 1 - d + \sum_{i=1}^{r} \frac{1}{m_i} > 0 \) and \( |\mathcal{G}| \geq \frac{1}{1-d+\sum_{i=1}^{r} \frac{1}{m_i}} \).

On the other hand, if \( \mathcal{G} \) is infinite and its first \( \ell^2 \) Betti number is zero, in particular if \( \mathcal{G} \) is an infinite amenable group, then we obtain the inequality \( 1 - d + \sum_{i=1}^{r} \frac{1}{m_i} \geq 0 \), as required. \(\square\)

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Primera versió rebuda el 28 de març de 2011,
darrera versió rebuda el 9 de novembre de 2011.