STRONG SOLUTIONS FOR STOCHASTIC POROUS MEDIA EQUATIONS WITH JUMPS

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Abstract. We prove global well-posedness in the strong sense for stochastic generalized porous media equations driven by a square integrable martingale with stationary independent increments.

1. Introduction

The purpose of this note is to establish well-posedness in the strong sense for a class of nonlinear stochastic PDEs driven by Lévy noise. More precisely, we shall consider the following stochastic porous media equation

\[ dX(t) - \Delta \beta(X(t)) \, dt = B(X(t^-)) \, dM(t), \]

where \( M \) is a square integrable martingale with stationary independent increments taking values in a Hilbert space \( K \), and the diffusion coefficient \( B \) satisfies a Lipschitz assumption. Full details on the data of the problem are given below. The present paper is a continuation of and should be read together with [2], where (1.1) is studied assuming that \( M \) is a Wiener process.

Let us just briefly mention that the deterministic counterpart of (1.1), i.e. with \( B \equiv 0 \), has been extensively studied both for its physical importance and as a model nonlinear PDE (see e.g. [4, 10] for systematic treatments). Porous media equations perturbed by Wiener noise have also been intensively investigated in the past few years (see references in [2]).

It should be said that without polynomial growth assumptions on \( \beta \), one cannot apply variational methods (see e.g. [5, 8]), and since the drift contains no linear term, the semigroup approach does not apply either (see e.g. [9]). We have also been unable to find in the literature results on well-posedness in the strong or mild sense for nonlinear SPDEs with discontinuous noise that do not fall into any of the two mentioned settings.

Here we show that (1.1) admits a unique solution which depends continuously on the initial datum, thus extending the results of [2] to the case of a jump noise.

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Moreover, we prove that the solution of (1.1) lives in a “better” space than the one used in [2], and we introduce a concept of generalized solution that allows to remove some restrictions on the coefficient \( B \) used in [2].

Let us conclude this introductory section with a few words about notation. Given two separable Hilbert spaces \( H, K \) we shall denote the space of Hilbert-Schmidt operators from \( H \) to \( K \) by \( \mathcal{L}_2(H,K) \), and the space of trace-class operators on \( H \) by \( \mathcal{L}_1(H) \). Moreover, \( \mathcal{L}^+_1 \) stands for the subset of \( \mathcal{L}_1 \) of positive operators. Given a self-adjoint operator \( Q \in \mathcal{L}^+_1(K) \), we denote by \( \mathcal{L}_Q^2(K,H) \) the set of all (possibly unbounded) operators \( B : \mathcal{L}^{1/2}(H) \to K \) such that \( BQ^{1/2} \in \mathcal{L}_2(H,K) \).

The norm in \( \mathcal{L}_Q^2(K,H) \) will be denoted by \( | \cdot |_Q \). We shall denote the space of weakly continuous functions defined on the interval \( I \subseteq \mathbb{R} \) and taking values in a Banach space \( X \) by \( C^w(I,X) \). Throughout the paper, \( \Xi \) will be an open bounded subset of \( \mathbb{R}^d \), \( d > 1 \), with smooth boundary \( \partial \Xi \), and \( Q_t := (0,t) \times \Xi, \partial Q_t := (0,t) \times \partial \Xi, t > 0 \).

We denote by \( H^{-1} := H^{-1}(\Xi) \) the dual of the Sobolev space \( H^1_0(\Xi) \), endowed with the scalar product \( \langle f,g \rangle_{-1} = \langle (-\Delta)^{-1}f,g \rangle \), where \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( H^1_0(\Xi) \) and \( H^{-1}(\Xi) \), and \( \Delta \) stands for the Laplacian with Dirichlet homogeneous boundary conditions. The inner product in \( L^2(\Xi) \) will be denoted by \( \langle \cdot, \cdot \rangle_2 \). Whenever no misunderstanding can arise, we shall write all functional spaces without explicitly indicating the domain \( \Xi \), e.g. \( H^{-1} = H^{-1}(\Xi) \), etc.

2. Main result

Let us first recall a few facts from the theory of stochastic integration in Hilbert spaces. For all unexplained notation we refer to [6]. Denoting the space of locally square integrable martingales on \( K \) by \( \mathcal{M}^2_{\text{loc}}(K) \), let \( M \in \mathcal{M}^2_{\text{loc}}(K) \) and \( T > 0 \) a fixed (deterministic) time. Appealing to [6, §22], one can construct stochastic integrals of the type

\[
G \cdot M(t) := \int_{(0,t]} G(s) \, dM(s), \quad t \in [0,T],
\]

for a class of operator valued predictable processes \( G \). In particular, according to the results of [6, 7], there exists a predictable \( \mathcal{L}^+_1(K) \)-valued process \( (Q_M(t))_{t \in [0,T]} \) such that

\[
\langle M \rangle(t) = \int_0^t Q_M(s) \, d\langle M \rangle(s), \quad t \in [0,T],
\]

and the stochastic integral (2.1) is well defined for all predictable processes \( G : [0,T] \times \Omega \to \mathcal{L}^2_{\text{loc}}(K,H) \) such that

\[
\mathbb{E} \int_0^T |G(s)Q_M(s)\|^2_{\mathcal{L}^2(K,H)} \, d\langle M \rangle(s) < \infty.
\]
The set of all such processes will be denoted by \( \mathcal{G}(H) \). Here and everywhere in the following we shall write, with a slight abuse of notation, \( \int_0^t \) instead of \( \int_{[0,t]} \).

**Remark 2.1.** If \( M \) has stationary independent increments (i.e. \( M \) is also a Lévy process), then \( Q_M = (\text{Tr} Q)^{-1} Q \), where \( Q \) is the covariance operator of \( M \), and \( \langle M \rangle(t) = t \text{Tr} Q \), \( \langle \langle M \rangle \rangle(t) = t(\text{Tr} Q)^{-1} Q \) for all \( t \in [0,T] \). Moreover, \( Q \) is a deterministic operator.

We shall study existence, uniqueness and regular dependence on the initial datum for the following stochastic Cauchy problem:

\[
\begin{cases}
    dX(t) - \Delta \beta(X(t)) \, dt = B(X(t-)) \, dM(t) & \text{in } Q_T \\
    \beta(X(t)) = 0 & \text{in } \partial Q_T \\
    X(0) = x & \text{in } \Xi
\end{cases}
\]  

under a set of assumptions on \( \beta \), \( B \) and \( M \) precised below. In particular, we shall assume that the diffusion coefficient is of the form

\[
B : H^{-1} \to L^Q_2(K, D(( - \Delta)^{\gamma})), \quad \gamma > d/2,
\]

and satisfies

\[
|B(x)|^2_Q \leq k(1 + |x|^2), \quad |B(x) - B(y)|^2_Q \leq k|x - y|^2,
\]

for some constant \( k > 0 \). Assumption (2.4) will be relaxed in the last section.

The (multivalued) function \( \beta : \mathbb{R} \to 2^{\mathbb{R}} \) is assumed to be a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) such that \( 0 \in \beta(0) \), \( D(\beta) = R(\beta) = \mathbb{R} \), and

\[
\limsup_{|x| \to \infty} \frac{j(-x)}{j(x)} < \infty,
\]

where \( j : \mathbb{R} \to \mathbb{R} \) is a convex function such that \( \beta = \partial j \) (such a function always exists and is unique modulo addition of constants – see e.g. \[1\], \[3\]). Here \( \partial \) stands for the subdifferential in the sense of convex analysis.

**Definition 2.2.** An adapted weakly càdlàg process \( X \in L^1((0,T) \times \Xi \times \Omega) \) is a strong solution of (1.1) if there exists an adapted process \( \eta \in L^1((0,T) \times \Xi \times \Omega) \) such that \( \eta(t,\xi) \in \beta(X(t,\xi)) \) a.e. in \( Q_T \), and

\[
(2.6) \quad t \mapsto \int_0^t \eta(s,\xi) \, ds \in C^w([0,T], H_0^1),
\]

\[
(2.7) \quad X(t) - \Delta \int_0^t \eta(s) \, ds = x + \int_0^t B(X(s-)) \, dM(s) \quad \forall t \in [0,T],
\]

and \( j(X), j^*(\eta) \in L^1((0,T) \times \Xi \times \Omega) \). All statements are meant to hold \( \mathbb{P} \)-a.s..
We first establish a well-posedness result for the SPDE with additive noise. Let us denote by $H_2(T)$ and $\mathbb{H}_2(T)$ the spaces of adapted processes $u : [0,T] \to H^{-1}$ such that
\[
\sup_{t \leq T} \mathbb{E}|u(t)|^2_{-1} < \infty \quad \text{and} \quad \mathbb{E}\sup_{t \leq T} |u(t)|^2_{-1} < \infty,
\]
respectively. We shall also denote by $H_2$ the space of $H^{-1}$-valued random variables with finite second moment.

The following intermediate result amounts to saying that the SPDE with additive noise is globally well-posed. Note that we do not yet need to assume that $M$ has stationary independent increments, as the latter assumption will be used only in the proof of theorem 2.4.

**Theorem 2.3.** If $G \in \mathcal{G}(D((-\Delta)^\gamma))$, $\gamma > d/2$, then for each $x \in H_2$ there exists a unique strong solution to the equation
\[
\begin{cases}
  dY(t) - \Delta \beta(Y(t)) \, dt = G(t) \, dM(t) & \text{in } Q_T \\
  \beta(Y(t)) = 0 & \text{in } \partial Q_T \\
  Y(0) = x & \text{in } \Xi
\end{cases}
\]
Moreover, for $G_1, G_2 \in \mathcal{G}(D((-\Delta)^\gamma))$ and $y_1, y_2 \in H_2$, denoting by $Y(t, y_i, G_i)$, $i = 1, 2$, the solutions of (2.8) with $G = G_i$ and $Y(0) = y_i$, respectively, the following estimate holds:
\[
\mathbb{E}|Y(t, y_1, G_1) - Y(t, y_2, G_2)|^2_{-1} \leq \mathbb{E}|y_1 - y_2|^2_{-1} + \mathbb{E}\int_0^t |G_1(s) - G_2(s)|^2_{Q_M} \, d\langle M \rangle(s).
\]
Finally, the solution map $x \mapsto Y$ is a contraction from $H_2$ to $H_2(T)$.

Our main result is the following.

**Theorem 2.4.** Assume that $M$ has stationary independent increments. Then for each $x \in H_2$ there exists a unique strong solution of (2.3). Moreover, the solution map $x \mapsto X$ is Lipschitz from $H_2$ to $H_2(T)$.

3. Auxiliary results

Since the stochastic integral (2.1) is a locally square integrable martingale for any $G$ satisfying (2.2), Doob’s inequality yields the following simple result.

**Lemma 3.1.** Let $M \in \mathcal{M}^2_{loc}(K)$ and $G \in \mathcal{G}(H)$. Then
\[
\mathbb{P}\left(\sup_{t \leq T} |G \cdot M(t)|_H < \infty \right) = 1
\]

**Proof.** It is enough to note that, by Cauchy-Schwartz’ inequality,
\[
\mathbb{E}\sup_{t \leq T} |G \cdot M(t)|_H \leq (\mathbb{E}\sup_{t \leq T} |G \cdot M(t)|^2_H)^{1/2},
\]
and that (since $|G \cdot M|^2_H$ is a submartingale) Doob’s inequality yields
\[
\mathbb{E} \sup_{t \leq T} |G \cdot M(t)|^2_H \leq 4\mathbb{E}|G \cdot M(T)|^2_H = 4\mathbb{E} \int_0^T |G(s)|^2_{Q_M(s)} d\langle M \rangle(s) < \infty.
\]
The last step follows by the isometric formula (see [7]) and (2.2).

We shall also need an Itô’s formula for the square of the norm of strong solutions to (2.8).

**Lemma 3.2.** Let $Y$ be a strong solution of (2.8). Then one has
\[
|Y(t)|^2_{-1} = |Y(0)|^2_{-1} - 2 \int_0^t \langle Y(s), \eta(s) \rangle ds + 2 \int_0^t \langle Y(s-), G(s) dM(s) \rangle_{-1} + [G \cdot M](t)
\]
for all $t \in [0, T]$, $\mathbb{P}$-a.s..

**Proof.** Let us set, for $m \in \mathbb{N}$ such that $m > 2 \vee (d + 2)/4$,
\[
Y_\varepsilon(t) = (1 - \varepsilon \Delta)^{-m}Y(t), \quad \eta_\varepsilon = (1 - \varepsilon \Delta)^{-m}\eta(t),
\]
\[
G_\varepsilon(t) = (1 - \varepsilon \Delta)^{-m}G(t), \quad x_\varepsilon = (1 - \varepsilon \Delta)^{-m}Y(0).
\]
Then we have by (2.8) that
\[
dY_\varepsilon(t) = \Delta \eta_\varepsilon(t) dt + G_\varepsilon(t) dM(t), \quad Y_\varepsilon(0) = x_\varepsilon, \quad \eta_\varepsilon = 0 \text{ on } \partial \Xi,
\]
and Itô’s formula for $|Y_\varepsilon(t)|^2_{-1}$ yields
\[
\int_0^t \langle Y_\varepsilon(s-), \eta_\varepsilon(s) \rangle ds + 2 \int_0^t \langle Y_\varepsilon(s-), G_\varepsilon(s) dM(s) \rangle_{-1}
\]
\[
\int_0^t d\langle G_\varepsilon \cdot M \rangle^c(s) + \sum_{s \leq t} |\Delta(G_\varepsilon \cdot M)(s)|^2_{-1}
\]
\[
= |x_\varepsilon|^2_{-1} - 2 \int_0^t \langle Y_\varepsilon(s-), \eta_\varepsilon(s) \rangle ds + 2 \int_0^t \langle Y_\varepsilon(s-), G_\varepsilon(s) dM(s) \rangle_{-1}
\]
\[
+ [G_\varepsilon \cdot M](t)
\]
$\mathbb{P}$-a.s., where we have used the identity $\text{Tr}[Z] = [Z]$, which holds for any semimartingale $Z$. We clearly have $|x_\varepsilon|^2_{-1} \uparrow |Y(0)|^2_{-1} \mathbb{P}$-a.s. as $\varepsilon \to 0$. Moreover, we have
\[
\int_0^t \langle Y_\varepsilon(s-), \eta_\varepsilon(s) \rangle ds = \int_0^t \langle Y_\varepsilon(s), \eta_\varepsilon(s) \rangle ds \to \int_0^t \langle Y(s), \eta(s) \rangle ds,
\]
\[
(3.3)
\]
as it follows from lemma 3.1 of [2]. In fact, recalling that
\( Y_\varepsilon(s) = y_\varepsilon(s) + G_\varepsilon \cdot M(s) \),
where \( y_\varepsilon \) is weakly continuous and \( G_\varepsilon \cdot M \) is càdlàg, we have
\[
\int_0^t \langle y_\varepsilon(s), \eta_\varepsilon(s) \rangle_2 \, ds = \int_0^t \langle y_\varepsilon(s), \eta(s) \rangle_2 \, ds
\]
by weak continuity of \( y_\varepsilon \), and
\[
\int_0^t \langle G_\varepsilon \cdot M(s) - G_\varepsilon \cdot M(s^-), \eta_\varepsilon(s) \rangle_2 \, ds = 0
\]
because the times of discontinuity of càdlàg processes are at most countable, hence
with Lebesgue measure zero.

For the last term on the right hand side of (3.2) we can write
\[
\mathbb{E}[G_\varepsilon \cdot M](t) = \mathbb{E}(G_\varepsilon \cdot M)(t) \leq \mathbb{E} \int_0^t |G_\varepsilon(s)|^2 \mathbb{Q}_M \, d\langle M \rangle(s) \leq \mathbb{E} \int_0^t |G(s)|^2 \mathbb{Q}_M \, d\langle M \rangle(s) < \infty,
\]
hence by monotone convergence we have \( \mathbb{E}[G_\varepsilon \cdot M](t) \rightarrow \mathbb{E}[G \cdot M](t) \), and \( |G_\varepsilon \cdot M|(t) \rightarrow |G \cdot M|(t) \) \( \mathbb{P} \)-a.s. for all \( t \in [0, T] \), passing to a subsequence if necessary.

Let us now consider the third term on the right hand side of (3.2). We can write
\[
\mathbb{E} \left| \int_0^t \langle Y_\varepsilon(s), G_\varepsilon(s) dM(s) \rangle_1 - \int_0^t \langle Y(s), G(s) dM(s) \rangle_1 \right|
\leq \mathbb{E} \left| \int_0^t \langle Y(s), (G_\varepsilon(s) - G(s)) dM(s) \rangle_1 \right|
+ \mathbb{E} \left| \int_0^t \langle Y_\varepsilon(s) - Y(s), G_\varepsilon(s) dM(s) \rangle_1 \right| =: I_1 + I_2.
\]
Davis’ inequality and Cauchy-Schwartz’ inequality yield (here we adopt the usual notation \( Y^*(t) := \sup_{s \leq t} |Y(s)|_1 \))
\[
I_1 \leq \mathbb{E} \left[ (G_\varepsilon - G) \cdot M \right]^{1/2}(t) \leq \mathbb{E} Y^*(t)[(G_\varepsilon - G) \cdot M]^{1/2}(t)
\leq (\mathbb{E} Y^*(t)^2)^{1/2} \mathbb{E}[(G_\varepsilon - G) \cdot M](t)^{1/2}
= (\mathbb{E} Y^*(t)^2)^{1/2} \mathbb{E}[(G_\varepsilon - G) \cdot M](t)^{1/2}
= (\mathbb{E} Y^*(t)^2)^{1/2} \mathbb{E} \left[ \int_0^t |G_\varepsilon(s) - G(s)|^2 \mathbb{Q}_M d\langle M \rangle(s) \right]^{1/2},
\]
which converges to zero as \( \varepsilon \rightarrow 0 \) by (2.2) and dominated convergence, provided we can show that \( \mathbb{E}(Y^*(t))^2 < \infty \). By (3.2), recalling that \( (1 - \varepsilon \Delta)^{-m} \) is a contraction, we have
\[
\mathbb{E} Y^*_\varepsilon(T)^2 \leq \mathbb{E}|Y(0)|^2 + 2 \mathbb{E} \sup_{s \leq T} \left| \int_0^s \langle Y_\varepsilon(r), G_\varepsilon(r) dM(r) \rangle_{1} \right| + |G \cdot M|(T),
\]
where we have used the inequality
\begin{equation}
\int_0^t \langle Y_\varepsilon(s), \eta_\varepsilon(s) \rangle_2 \, ds \geq 0 \quad \mathbb{P}\text{-a.s..}
\end{equation}

The latter holds true by the following argument: since $\eta(t, x) \in \beta(Y(t, x))$ for almost all $(t, x) \in Q_T$ \(\mathbb{P}\text{-a.s.}\), then $Y(t, x) \eta(t, x) \geq 0$ a.e. in $Q_T$ \(\mathbb{P}\text{-a.s.}\) by monotonicity of $\beta$. Since $(1 - \varepsilon \Delta)^{-1}$ preserves the sign, thus so does also $(1 - \varepsilon \Delta)^{-m}$, one infers that $Y_\varepsilon(t, x) \eta_\varepsilon(t, x) \geq 0$ a.e. in $Q_T$ \(\mathbb{P}\text{-a.s.}\), which implies (3.4). Setting $Z_\varepsilon = G_\varepsilon \cdot M$, Davis’ inequality and the elementary inequality $ab \leq \delta a^2 + \frac{b^2}{\delta}$ yield
\[
\mathbb{E} \sup_{t \leq T} \left| \int_0^t \langle Y_\varepsilon(s), G_\varepsilon(s) \rangle_2 \, dM(s) \right|_1 = \mathbb{E} \sup_{t \leq T} \left| \int_0^t \langle Y_\varepsilon(s), G_\varepsilon(s) \rangle_2 \, dZ_\varepsilon(s) \right|_1
\leq 3 \mathbb{E} [Y_\varepsilon \cdot Z_\varepsilon](T)^{1/2} \leq 3 \mathbb{E} Y_\varepsilon^*(T)[Z_\varepsilon](T)^{1/2}
\leq 3 \delta \mathbb{E} Y_\varepsilon^*(T)^2 + \frac{3}{\delta} \mathbb{E} \int_0^T |G(s)|_Q M \, d(M)(s),
\]

because
\[
\mathbb{E}[Z_\varepsilon](T) = \mathbb{E}[Z_\varepsilon](T) \leq \mathbb{E} \int_0^T |G(s)|_Q M \, d(M)(s).
\]
Thus we obtain
\[
\mathbb{E} \sup_{s \leq T} |Y_\varepsilon(s)|_2^2 \leq N \mathbb{E} \int_0^T |G(s)|_Q^2 \, d(M)(s) < \infty,
\]
where $N$ is a constant independent of $\varepsilon$. Since $\varepsilon \mapsto \mathbb{E} Y_\varepsilon^*(T)^2$ is an increasing bounded sequence, Fatou’s lemma allows to conclude that $\mathbb{E} Y_\varepsilon^*(T)^2 < \infty$, hence finally that $I_1 \to 0$ as $\varepsilon \to 0$. A similar reasoning shows that $I_2 \to 0$ as $\varepsilon \to 0$. We thus have, by Chebyshev’s inequality, that
\[
\int_0^t \langle Y_\varepsilon(s), G_\varepsilon(s) \rangle_2 \, dM(s) \rightarrow \int_0^t \langle Y(s), G(s) \rangle_2 \, dM(s)
\]
\(\mathbb{P}\text{-a.s.}\) and for all $t \in [0, T]$, at least on a subsequence of $\varepsilon$, still denoted by $\varepsilon$, as $\varepsilon \to 0$. \(\square\)

4. Proof of theorem 2.3

The proof will be sketched only, underlying the differences with respect to the corresponding proof in [2].

Let us consider the approximating SPDE (in integral form)
\begin{equation}
X(t) = x + \int_0^t \Delta (\beta_\lambda(X(s)) + \lambda X(s)) \, ds + G \cdot M(t),
\end{equation}
where $\beta_\lambda = \lambda^{-1}(I - (I + \lambda \beta)^{-1})$, $\lambda > 0$, is the Yosida approximation of $\beta$.

Then one has the following result.
Lemma 4.1. The SPDE (4.1) admits a unique càdlàg adapted solution $X_\lambda$ such that

$$X_\lambda, \beta_\lambda(X_\lambda) \in L^2([0, T], H^1_0).$$

Proof. Equation (4.1) can be equivalently rewritten as the deterministic PDE with random coefficients

$$y' = \Delta \tilde{\beta}_\lambda(y + G \cdot M),$$

setting $y = X - G \cdot M$ and $\tilde{\beta}_\lambda(x) := \beta_\lambda(x) + \lambda x$. Moreover, for any fixed $\omega \in \Omega$, the time-dependent operator

$$A(t) : H^1_0 \to H^{-1}$$

$$x \mapsto -\Delta \tilde{\beta}_\lambda(x + G \cdot M)$$

satisfies the assumptions of Theorem III.4.2 in [1], hence (4.2) admits a unique solution $y_\lambda \in C([0, T], L^2) \cap L^2([0, T], H^1_0)$, with $y'_\lambda \in L^2([0, T], H^{-1})$. Moreover, $y_\lambda$ depends continuously on $G \cdot M$ with respect to pathwise convergence in $H^{-1}$, hence $X_\lambda := y_\lambda + G \cdot M$ is an adapted càdlàg solution of (4.1), as required.

□

Remark 4.2. Since $\beta_\lambda$ is Lipschitz, one can immediately conclude by [6, Thm 24.7] that (4.1) has a unique càdlàg (strong) solution taking values in $H^{-1}$. It does not seem immediate to obtain also that the solution belongs to $L^2([0, T], H^1_0)$.

We shall need some a priori estimates for $z_\lambda := (1 + \lambda \beta)^{-1} X_\lambda$ and $\eta_\lambda := \beta_\lambda(X_\lambda)$.

Lemma 4.3. There exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every fixed $\omega \in \Omega_0$, one has

$$\int_{Q_T} (j(z_\lambda) + j^*(\eta_\lambda)) \, d\xi \, ds \leq N_1 (1 + |x|_{-1}^2),$$

$$\int_{Q_T} |X_\lambda - z_\lambda|^2 \, d\xi \, ds \leq 2 \lambda N_1 (1 + |x|_{-1}^2),$$

where $N_1$ is a positive constant that may depend on $\omega$.

Proof. By lemma 3.1, Sobolev’s embedding theorem $D((-\Delta)^\gamma) \subset L^\infty$, $\gamma > d/2$, and the hypothesis that $\mathbb{E}|x|_{-1}^2 < \infty$, it follows that there exists $\Omega_0 \subset \Omega$, $\mathbb{P}(\Omega_0) = 1$, such that

$$\sup_{t \leq T} |G \cdot M(t)|_{L^\infty} < \infty, \quad |x(\omega)|_{-1}^2 < \infty \quad \forall \omega \in \Omega_0.$$

Using this estimate in place of the corresponding one for $W_G$ in [2 §3.1], the claim follows.

□

The above estimates allow us to pass to the limit as $\lambda \to 0$, as in [2 §3.2], obtaining the following result.
Lemma 4.4. There exist \( y \in C^w([0, T], H^{-1}) \cup L^1(Q_T) \) and \( \eta \in L^1(Q_T) \cap L^\infty([0, T], H^1_0) \) such that

\[
y(t) + A \int_0^t \eta(s) \, ds = x. \tag{4.5}
\]

One then continues proving that \( \eta \in \beta(y + G \cdot M) \) a.e. in \( Q_T \), and that such \( y \) and \( \eta \) are unique. Hence the above convergence results hold \( \mathbb{P}\)-a.s. for any choice of the sequence \( \lambda \). In particular, \( y \) and \( \eta \) are adapted processes. Moreover, since \( G \cdot M \) is càdlàg and \( y \) is weakly continuous, it follows that \( Y(t) = y(t) + G \cdot M(t) \) is an \( H^{-1}\)-valued weakly càdlàg process such that

\[
Y(t) - \Delta \int_0^t \eta(s) \, ds = x + G \cdot M(t) \quad \forall t \in [0, T] \quad \mathbb{P}\text{-a.s.,}
\]

i.e. \( Y \) solves (2.8).

Once existence has been established, we need to prove uniqueness and continuous dependence on the initial datum. This can be achieved with the help of Lemma 3.2. In particular, taking into account that the second term on the right-hand side of (3.1) is negative because \( \eta(s) \in \beta(Y(s)) \) \( \mathbb{P}\)-a.s. for a.a. \( s \in [0, T] \), we have, by Lemma 3.2

\[
|Y_1(t) - Y_2(t)|_{-1}^2 \leq |y_1 - y_2|_{-1}^2 + 2 \int_0^t \langle Y_1(s-), Y_2(s-), (G_1(s) - G_2(s)) dM(s) \rangle_{-1}
+ [(G_1 - G_2) \cdot M](t), \tag{4.6}
\]

where we set, for simplicity of notation, \( Y_i := Y(\cdot, y_i, G_i), i = 1, 2 \). Taking expectation on both sides, we are left with

\[
\mathbb{E}|Y_1(t) - Y_2(t)|_{-1}^2 \leq \mathbb{E}|y_1 - y_2|_{-1}^2 + \mathbb{E} \int_0^t \mathbb{E}(G_1(s) - G_2(s))^2_{Q_M} \, d(M)(s). \tag{4.7}
\]

Similarly, if \( G_1 = G_2 \), (4.6) immediately yields

\[
\mathbb{E} \sup_{t \leq T} |Y(t, y_1) - Y(t, y_2)|_{-1}^2 \leq \mathbb{E}|y_1 - y_2|_{-1}^2. \]

5. Proof of theorem 2.4

Consider the equation

\[
dY(t) = \Delta \beta(Y(t)) \, dt + B(X(t-)) \, dM(t), \quad t \in [0, T], \tag{5.1}
\]

and define the operator \( \Phi : X \mapsto Y \) that associates to \( X \in \mathcal{H}_2(T) \) the solution \( Y \) of (5.1). We are going to prove that \( \Phi \) is an endomorphism of \( \mathcal{H}_2(T) \) and is a contraction. Moreover, since \( t \mapsto B(X(t-)) \) is predictable, we know by theorem
that $Y$ is adapted and weakly càdlàg. Let us first obtain two estimates that hold for any quasi-left-continuous $M \in \mathcal{M}^2_{loc}(K)$. Itô’s formula yields

$$|Y(t)|^2_{-1} + 2 \int_0^t \langle Y(s), \eta(s) \rangle_2 \, ds =
|Y(0)|^2_{-1} + 2 \int_0^t \langle Y(s-), B(X(s-)) \rangle dM(s)_{-1} + [B(X_-) \cdot M](t),$$

where $B(X_-)$ stands for $t \mapsto B(X(t-))$. Since $(Y(s), \eta(s)) \geq 0 \, \mathbb{P}$-a.s. for all $s \leq t$, we can write

$$(5.2) \quad \mathbb{E} \sup_{t \leq T} |Y(t)|^2_{-1} \leq 2 \mathbb{E} \sup_{t \leq T} \left| \int_0^t \langle Y(s-), (B(X(s-)) \rangle dM(s)_{-1} \right| + \mathbb{E} \int_0^T |B(X(s))|_{Q_M}^2 \, d\langle M \rangle(s).$$

Following in a completely similar way as we have done in the last paragraph of the proof of Lemma 3.2, we obtain

$$(5.3) \quad (1 - 6\varepsilon)|Y|_{\mathbb{H}_2(T)}^2 \leq (6/\varepsilon + 1)\mathbb{E} \int_0^T |B(X(s))|_{Q_M}^2 \, d\langle M \rangle(s).$$

Similarly, writing

$$\begin{cases}
    dY_1 = \Delta \beta(Y_1) \, dt + B(X_{1-}) \, dM \\
    dY_2 = \Delta \beta(Y_2) \, dt + B(X_{2-}) \, dM,
\end{cases}$$

with $Y_1(0) = Y_2(0)$, using again Itô’s formula (see lemma 3.2), in complete analogy to the above derivation, we obtain the estimate

$$(5.4) \quad (1 - 6\varepsilon)|Y_1 - Y_2|_{\mathbb{H}_2(T)}^2 \leq (6/\varepsilon + 1)\mathbb{E} \int_0^T |B(X_1(t)) - B(X_2(t))|_{Q_M}^2 \, d\langle M \rangle(t).$$

If $M \in \mathcal{M}^2_{loc}(K)$ has also stationary independent increments, then, in view of remark 2.1, we have

$$\mathbb{E} \int_0^T |B(X(s))|_{Q_M}^2 \, d\langle M \rangle(s) = \mathbb{E} \int_0^T |B(X(s))|_{Q}^2 \, ds \leq k\mathbb{E} \int_0^T (1 + |X(s)|^2) \, ds \leq kT(1 + |X|_{\mathbb{H}_2(T)}^2) < \infty,$$

hence, choosing $\varepsilon < 1/6$, $|Y|_{\mathbb{H}_2(T)}^2 < \infty$, by virtue of (5.3). This proves that the image of $\Phi$ is contained in $\mathbb{H}_2(T)$. Let us now show that $\Phi$ is a contraction. In fact, (5.4) and assumption (2.5) yield

$$|Y_1 - Y_2|_{\mathbb{H}_2(T)}^2 \leq \frac{1 + 6/\varepsilon}{1 - 6\varepsilon} \, kT|X_1 - X_2|_{\mathbb{H}_2(T)}^2.$$
i.e. $\Phi$ is a contraction on $H_2(\mathbb{T})$ whenever
\begin{equation}
T < \frac{1 - 6\varepsilon}{1 + 6/\varepsilon} \frac{1}{k}.
\end{equation}
Then, by the Banach fixed point theorem, there exists a unique solution of (1.1). If $T$ does not satisfy (5.5), then one proceeds in a classical way considering intervals $[0, T_0], [T_0, 2T_0], \text{etc.}$, with suitably small $T_0$, such that $\Phi$ is a contraction on $H_2(T_0)$.

In order to prove Lipschitz continuity of the solution map, note that we have
\begin{equation}
(1 - 6\varepsilon)|Y(\cdot, y_1) - Y(\cdot, y_2)|_{H_2(\mathbb{T})} \leq (6/\varepsilon + 1)kT|Y(\cdot, y_1) - Y(\cdot, y_2)|_{H_2(\mathbb{T})}^2 + |y_1 - y_2|_{H_2}^2,
\end{equation}
hence for any $T_0$ such that
\begin{equation}
1 - 6\varepsilon - kT_0(6/\varepsilon + 1) > 0
\end{equation}
we have
\begin{equation}
|Y(\cdot, y_1) - Y(\cdot, y_2)|_{H_2(T_0)} \leq N_0|y_1 - y_2|_{H_2},
\end{equation}
where
\begin{equation}
N_0 = (1 - 6\varepsilon - kT_0(6/\varepsilon + 1))^{-1/2}.
\end{equation}
Considering intervals of length $T_0$ covering $[0, T]$ one finally gets
\begin{equation}
|Y(\cdot, y_1) - Y(\cdot, y_2)|_{H_2(T)} \leq N|y_1 - y_2|_{H_2},
\end{equation}
where $N = N(k, T)$.

6. Generalized solutions

In this section we introduce a concept of generalized solution for equation (1.1), which allows to replace the assumption (2.4) with
\begin{equation}
B : H^{-1} \to L^2(K, H^{-1}).
\end{equation}
As we did before, we start with the case of additive noise and general $M \in \mathcal{M}_{\text{loc}}^2(K)$.

**Definition 6.1.** Let $G \in \mathcal{G}(H^{-1})$. An adapted process $Y$ is called a $\mathcal{H}$-generalized solution of (2.8) if there exists a sequence $\{G_n\}_{n \in \mathbb{N}} \subset \mathcal{G}(D((-\Delta)^\gamma))$ with
\begin{equation}
\lim_{n \to \infty} \mathbb{E} \int_0^T |G_n(s) - G(s)|_{Q_M}^2 d(M)(s) = 0
\end{equation}
such that the solution $Y_n$ to
\begin{equation}
dY(t) = \Delta \beta(Y(t)) dt + G_n(t) dM(t),
\end{equation}
equipped with the same initial and boundary conditions of (2.8), converges to $Y$ in $\mathcal{H}_2(T)$. If the convergence is in $\mathbb{H}_2(T)$, $X$ is called $\mathbb{H}$-generalized solution.

It is clear that a $\mathbb{H}$-generalized solution is also a $\mathcal{H}$-generalized solution. In the following we shall refer to $\mathbb{H}$-generalized solutions simply as generalized solutions.
Theorem 6.2. Let $G \in \mathcal{G}(H^{-1})$. Then (2.8) admits a unique generalized solution. Moreover, the solution map $x \mapsto Y$ is a contraction from $H_2$ to $H_2(T)$.

For the proof of the theorem we need the following approximation procedure for elements of the space $H^{-1}$. Let $f \in H^{-1}$. Then there exists $F \in H_0^1$ such that $f = \Delta F$, and $|f|^2_1 = |\nabla F|^2_2$. Set $F_n = \zeta_n * F$ and $f_n = \Delta F_n$, where $\{\zeta_n\}_{n \in \mathbb{N}}$ is a standard sequence of mollifiers (here we have considered an extension of $F$ to $H^{-1}(\mathbb{R}^d)$, still denoted by $F$). In particular, since $F_n \in C^\infty$, then $f_n \in C^\infty(\Xi) \subset L^\infty(\Xi)$. Recalling that $\nabla(\zeta_n * F) = \zeta_n * \nabla F$, we have

$$\lim_{n \to \infty} E \int_0^T |G_n(s) - G(s)|_{Q_M}^2 d\langle M \rangle(s) = 0.$$ 

The claim follows by (6.2), the dominated convergence theorem and the inequality

$$|G_n(s)_{Q_M}^{1/2} e_k|_{-1} \leq |G(s)_{Q_M}^{1/2} e_k|_{-1} \quad \mathbb{P}\text{-a.s.},$$

which holds for all $s \in [0, T]$ and all $k \in \mathbb{N}$, where $(e_k)_{k \in \mathbb{N}}$ is a basis of $K$, as it follows by (6.3).

Now (4.7) implies that

$$\sup_{t \leq T} E |Y_n(t) - Y_m(t)|^2_{-1} \leq E \int_0^T |G_n(s) - G_m(s)|_{Q_M}^2 d\langle M \rangle(s),$$

that is $\{Y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $H_2(T)$, which converges to a $\mathcal{H}$-generalized solution $Y$ of (2.8).

Let us show that $\{Y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence also in $H_2(T)$, which proves the existence of a $\mathcal{H}$-generalized solution. In fact, using an argument based on Itô’s formula and Davis’ inequality completely analogous to the one leading to (5.4), we obtain

$$\sup_{t \leq T} E |Y_n(t) - Y_m(t)|^2 \leq N E \int_0^T |G_n(t) - G_m(t)|_{Q_M}^2 d\langle M \rangle(t),$$

where $N$ is a positive constant. \hfill \square

It is now possible to extend the result to equations with multiplicative noise.
Theorem 6.3. Assume that $M$ has stationary independent increments and $B$ is as in (6.1). Then (2.3) admits a unique generalized solution. Moreover, the solution map $x \mapsto X$ is Lipschitz from $\mathcal{H}_2$ to $\mathbb{H}_2(T)$.

Proof. The argument is an extension of that used in the proof of theorem 2.4, using the previous theorem. In fact, let $X \in \mathbb{H}_2(T)$ and consider equation (5.1), which admits a unique generalized solution by theorem 6.2. Since estimates (5.3) and (5.4) hold also for generalized solutions (by a now obvious limiting procedure), the map associating $Y$ to $X$, as defined in the proof of theorem 2.4, is a contraction in $\mathbb{H}_2(T_0)$ for a suitably small $T_0$. The rest of the proof is identical to that of theorem 2.4. □

REFERENCES

1. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Leyden, 1976. MR MR0390843 (52 #11666)
2. V. Barbu, G. Da Prato, and M. Röckner, *Existence of strong solutions for stochastic porous media equation under general monotonicity conditions*, Ann. Probab. 37 (2009), no. 2, 428–452. MR MR2510012
3. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam, 1973. MR MR0348562 (50 #1060)
4. P. Daskalopoulos and C. E. Kenig, *Degenerate diffusions*, European Mathematical Society, 2007. MR MR2338118 (2009b:35214)
5. I. Gyöngy, *On stochastic equations with respect to semimartingales. III*, Stochastics 7 (1982), no. 4, 231–254.
6. M. Métivier, *Semimartingales*, Walter de Gruyter & Co., Berlin, 1982. MR MR688144 (84i:60002)
7. M. Métivier and G. Pistone, *Une formule d’isométrie pour l’intégrale stochastique hilbertienne et équations d’évolution linéaires stochastiques*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 33 (1975/76), no. 1, 1–18. MR MR0383527 (52 #4408)
8. ______. *Sur une équation d’évolution stochastique*, Bull. Soc. Math. France 104 (1976), no. 1, 65–85. MR MR0420854 (54 #8866)
9. Sz. Peszat and J. Zabczyk, *Stochastic partial differential equations with Lévy noise*, Cambridge University Press, Cambridge, 2007. MR MR2356959
10. J. L. Vázquez, *The porous medium equation*, Oxford University Press, Oxford, 2007. MR MR2286292

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