A FORMULA FOR THE FIRST EIGENVALUE OF THE DIRAC OPERATOR ON COMPACT SPIN SYMMETRIC SPACES

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Abstract. Let $G/K$ be a simply connected spin compact inner irreducible symmetric space, endowed with the metric induced by the Killing form of $G$ sign-changed. We give a formula for the square of the first eigenvalue of the Dirac operator in terms of a root system of $G$. As an example of application, we give the list of the first eigenvalues for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure.

1. Introduction

Let $G/K$ be a compact, simply-connected, $n$-dimensional irreducible symmetric space with $G$ compact and simply-connected, endowed with the metric induced by the Killing form of $G$ sign-changed. Assume that $G$ and $K$ have same rank and that $G/K$ has a spin structure. In a previous paper, cf. [Mil04], we proved that the first eigenvalue $\lambda$ of the Dirac operator verifies

\[ \lambda^2 = 2 \min_{1 \leq k \leq p} \|\beta_k\|^2 + n/8, \]

where $\beta_k$, $k = 1, \ldots, p$, are the $K$-dominant weights occurring in the decomposition into irreducible components of the spin representation under the action of $K$, and where $\| \cdot \|$ is the norm associated to the scalar product induced by the Killing form of $G$.

The proof was based on a lemma of R. Parthasarathy in [Par71], which allows to express the result in the following way.

Let $T$ be a fixed common maximal torus of $G$ and $K$. Let $\Phi$ be the set of non-zero roots of $G$ with respect to $T$. Let $\Phi_G^+$ be the set of positive roots of $G$, $\Phi_K^+$ be the set of positive roots of $K$, with respect to a fixed lexicographic ordering in $\Phi$. Let $\delta_G$, (resp. $\delta_K$) be the half-sum of the positive roots of $G$, (resp. $K$). Then the square of the first eigenvalue of the Dirac operator is given by

\[ \lambda^2 = 2 \min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 + n/8, \]

where $W$ is the subset of the Weyl group $W_G$ defined by

\[ W := \{ w \in W_G : w \cdot \Phi_G^+ \supset \Phi_K^+ \}. \]

In order to avoid the determination of the subset $W$ for applications, we prove in the following that the square of the first eigenvalue of the Dirac operator is indeed given by

\[ \lambda^2 = 2 \min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 + n/8. \]
We then give a different expression to use the formula for explicit computations. We obtain

$$\lambda^2 = 2 \|\delta_G - \delta_K\|^2 + 4 \sum_{\theta \in \Lambda} <\theta, \delta_K> + n/8,$$

where \(\Lambda\) is the set

\[\Lambda := \{\theta \in \Phi_G^+ ; <\theta, \delta_K> < 0\} .\]

As an example of application of the above formula, we obtain the list of the first eigenvalues of the Dirac operator for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure. By definition, a Riemannian manifold has a quaternion-Kähler structure if its holonomy group is contained in the group \(\text{Sp}_m \text{Sp}_1\). In [Wol65], J. Wolf gave the following classification of compact quaternion-Kähler symmetric spaces:

| \(G\)   | \(K\)       | \(G/K\)                   | \(\text{dim} \ G/K\) | \(\text{Spin structure}\) |
|---------|-------------|---------------------------|------------------------|----------------------------|
| \(\text{Sp}_{m+1}\) | \(\text{Sp}_m \times \text{Sp}_1\) | Quaternionic projective space \(\mathbb{H}^m\) | \(4m\) \((m \geq 1)\)   | Yes (unique)                |
| \(\text{SU}_{m+2}\) | \(\text{SU}_m \times \text{U}_2\) | Grassmannian \(\text{Gr}_2(\mathbb{C}^{m+2})\) | \(4m\) \((m \geq 1)\)   | iff \(m\) even, unique in that case |
| \(\text{Spin}_{m+4}\) | \(\text{Spin}_m \text{Spin}_4\) | Grassmannian \(\text{Gr}_4(\mathbb{R}^{m+4})\) | \(4m\) \((m \geq 3)\)   | iff \(m\) even, unique in that case |
| \(G_2\)   | \(\text{SO}_4\)  | 8                         |                        | Yes (unique)                |
| \(F_4\)   | \(\text{Sp}_3 \text{SU}_2\) | 28                        |                        | No                          |
| \(E_6\)   | \(\text{SU}_6 \text{SU}_2\) | 40                        |                        | Yes (unique)                |
| \(E_7\)   | \(\text{Spin}_{12} \text{SU}_2\) | 64                        |                        | Yes (unique)                |
| \(E_8\)   | \(\text{E}_7 \text{SU}_2\)   | 112                       |                        | Yes (unique)                |

Note furthermore that all the symmetric spaces in that list are “inner”. Endowing each symmetric space with the metric induced by the Killing form of \(G\) sign-changed, we obtain the following table
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$$G/K$$ | Square of the first eigenvalue of $$D$$  
---|---  
$$\mathbb{H}P^n = \text{Sp}_{m+1}/(\text{Sp}_m \times \text{Sp}_1)$$ | $$\frac{m+3}{2} \cdot \frac{m+3}{4} = \frac{m+3}{m+2} \cdot \frac{Scal}{4}$$  
$$\text{Gr}_2(\mathbb{C}^{m+2}) = \text{SU}_{m+2}/(\text{SU}_m \times \text{U}_2)$$  
(m even) | $$\frac{m+4}{2} \cdot \frac{m+4}{4} = \frac{m+4}{m+2} \cdot \frac{Scal}{4}$$  
$$\widetilde{\text{Gr}}_4(\mathbb{R}^{m+4}) = \text{Spin}_{m+4}/(\text{Spin}_m \text{Spin}_4)$$  
(m even) | $$\frac{m^2+6m-4}{m(m+2)} \cdot \frac{m^2+6m-4}{4} = \frac{Scal}{4}$$  
$$\text{G}_2/\text{SO}_4$$ | $$\frac{3}{2} = \frac{3}{2} \cdot \frac{Scal}{4}$$  
$$\text{E}_6/(\text{SU}_6 \text{SU}_2)$$ | $$\frac{41}{6} = \frac{41}{30} \cdot \frac{Scal}{4}$$  
$$\text{E}_7/(\text{Spin}_{12} \text{SU}_2)$$ | $$\frac{95}{9} = \frac{95}{72} \cdot \frac{Scal}{4}$$  
$$\text{E}_8/(\text{E}_7 \text{SU}_2)$$ | $$\frac{269}{15} = \frac{269}{210} \cdot \frac{Scal}{4}$$

| TABLE I |

The result was already known for quaternionic projective spaces $$\mathbb{H}P^n$$, [Mil92], for the Grassmannians $$\text{Gr}_2(\mathbb{C}^{m+2})$$, [Mil93], and for the symmetric space $$G_2/\text{SO}_4$$, [See99]. Up to our knowledge, the other results are new.

2. Proof of formula (4)

With the notations of the introduction, and since the scalar product is $$W_G$$-invariant, one has for any $$w \in W_G$$

$$\|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \cdot \langle w \cdot \delta_G, \delta_K \rangle$$,

hence

$$\min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W} \langle w \cdot \delta_G, \delta_K \rangle$$,

and

$$\min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle$$,
So we have to prove that
\[
\max_{w \in W} < w \cdot \delta_G, \delta_K > = \max_{w \in W} < w \cdot \delta_G, \delta_K > .
\]
Let
\[
\Pi_G := \{ \theta_1, \ldots, \theta_r \} \subset \Phi_G^+,
\]
be the set of \(G\)-simple roots and let
\[
\Pi_K := \{ \theta'_1, \ldots, \theta'_r \} \subset \Phi_K^+,
\]
be the set of \(K\)-simple roots.

Let \(w_0 \in W_G\) such that
\[
< w_0 \cdot \delta_G, \delta_K > = \max_{w \in W} < w \cdot \delta_G, \delta_K > .
\]
Suppose that \(w_0 \notin W\). Then we claim that there exists a \(K\)-simple root \(\theta'_i\) such that \(w_0^{-1} \cdot \theta'_i \notin \Phi_G^+\). Otherwise, if for any \(K\)-simple root \(\theta'_i\), \(w_0^{-1} \cdot \theta'_i \in \Phi_G^+\), then since any \(K\)-positive root is a linear combination with non-negative coefficients of \(K\)-simple roots, we would have \(\forall \theta' \in \Phi_K^+, w_0^{-1} \cdot \theta' \in \Phi_G^+\), contradicting the assumption made on \(w_0\).

Now let \(\sigma'_i\) be the reflection across the hyperplane \(\theta'^+_i\). Since \(\sigma'_i \cdot \delta_K = \delta_K - \theta'_i\), (cf. for instance Corollary of Lemma B, §10.3 in [Hum72]), one gets by the \(W_G\)-invariance of the scalar product
\[
< \sigma'_i w_0 \cdot \delta_G, \delta_K > = < w_0 \cdot \delta_G, \sigma'_i \cdot \delta_K > = < w_0 \cdot \delta_G, \delta_K - \theta'_i > = < w_0 \cdot \delta_G, \delta_K > - < \delta_G, w_0^{-1} \cdot \theta'_i > .
\]
But since \(w_0^{-1} \cdot \theta'_i\) is a negative root of \(G\), one has
\[
w_0^{-1} \cdot \theta'_i = \sum_{j} k_j \theta_j, \quad k_j \in \mathbb{N}.
\]
Since for any \(G\)-simple root \(\theta_j\), \(\sigma_j \cdot \delta_G = \delta_G - \theta_j\), where \(\sigma_j\) is the reflection across the hyperplane \(\theta_j^+\), one has \(< \theta_j, \delta_G > = 2 < \theta_j, \theta_j > > 0\), so
\[
- < \delta_G, w_0^{-1} \cdot \theta'_i > = \sum_{j} k_j < \delta_G, \theta_j > > 0,
\]
hence
\[
< \sigma'_i w_0 \cdot \delta_G, \delta_K > = w_0 \cdot \delta_G, \delta_K > - < \delta_G, w_0^{-1} \cdot \theta'_i > ,
\]
but that is in contradiction with the definition \(10\) of \(w_0\), hence \(w_0 \in W\) and
\[
\max_{w \in W_G} < w \cdot \delta_G, \delta_K > = < w_0 \cdot \delta_G, \delta_K > = \max_{w \in W} < w \cdot \delta_G, \delta_K > = \max_{w \in W} < w \cdot \delta_G, \delta_K > ,
\]
hence the result.

3. Proof of formula \(15\)

In order to obtain the formula we will use the following result

**Lemma 3.1.** For any element \(w\) of the Weyl group \(W_G\)
\[
w \cdot \delta_G = \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta, \quad k_\theta = 0 \text{ or } 1.
\]
Proof. Let \( w \in W_G \). With the same notations as in the above proof, we write \( w \) in reduced form
\[
(11) \quad w = \sigma_i \cdots \sigma_k ,
\]
where \( \sigma_i \) is the reflection across the hyperplane \( \theta_i^\perp, \theta_i \in \Pi_G \), and \( k \) is minimal.
Since \( \sigma_k \cdot \delta_G = \delta_G - \theta_k \), one has
\[
w \cdot \delta_G = \sigma_i \cdots \sigma_{k-1} (\sigma_k \cdot \delta_G) = \sigma_i \cdots \sigma_{k-1} (\delta_G) - \sigma_i \cdots \sigma_{k-1} (\theta_k) .
\]
Now, since the expression of \( w \) is reduced, \( w(\theta_k) \) is a negative root, cf. for instance corollary of Lemma C, § 10.3 in [Hum72]. But \( w(\theta_k) = -\sigma_i \cdots \sigma_{k-1} (\theta_k) \), hence \( \sigma_i \cdots \sigma_{k-1} (\theta_k) \) is a positive root.

Now the element \( \sigma_i \cdots \sigma_{k-1} \in W_G \) is written in reduced form, otherwise the expression \((11)\) of \( w \) would not be reduced. Hence we may conclude as above that
\[
(11) \quad \sigma_i \cdots \sigma_{k-1} (\delta_G) = \sigma_i \cdots \sigma_{k-2} (\delta_G) - \sigma_i \cdots \sigma_{k-2} (\theta_{k-1}) ,
\]
where \( \sigma_i \cdots \sigma_{k-2} (\theta_{k-1}) \) is a positive root.

Proceeding inductively we get
\[
w \cdot \delta_G = \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta , \quad k_\theta \in \mathbb{N} .
\]

In order to conclude, we have to prove that if a \( G \)-positive root \( \theta \) appears in the above sum, then it appears only once.
Suppose that a \( G \)-positive root appears at least twice in the above sum, then there exist two integers \( p \) and \( q \), \( 1 \leq p < q \leq k-1 \) such that
\[
\sigma_i \cdots \sigma_{q} (\theta_{p+1}) = \sigma_i \cdots \sigma_q (\theta_{q+1}) .
\]
applying \( \sigma_{p+1} \sigma_{p+2} \cdots \sigma_i \) to the two members of the above equation, we get
\[
\begin{align*}
-\theta_{p+1} &= \sigma_{p+2} \cdots \sigma_{q} (\theta_{q+1}) , & \text{if } p + 1 < q , \\
-\theta_q &= \theta_{q+1} , & \text{if } p + 1 = q .
\end{align*}
\]
So we get a contradiction, even in the first case, since \( \sigma_{p+2} \cdots \sigma_{q} (\theta_{q+1}) \in W_G \) is expressed in reduced form (otherwise the expression \((11)\) of \( w \) would not be reduced), hence \( \sigma_{p+2} \cdots \sigma_{q} (\theta_{q+1}) \) is a positive root. \( \square \)

From the above result we deduce

**Lemma 3.2.** Let \( \Lambda \) be the set
\[
(12) \quad \Lambda := \{ \theta \in \Phi_G^+ ; < \theta, \delta_K > < 0 \} .
\]
One has
\[
\max_{w \in W_G} < w \cdot \delta_G, \delta_K > = < \delta_G, \delta_K > - \sum_{\theta \in \Lambda} < \theta, \delta_K > ,
\]
(setting \( \sum_{\theta \in \Lambda} < \theta, \delta_K > = 0 \), if \( \Lambda = \emptyset \)).

**Proof.** Suppose \( \Lambda \neq \emptyset \). We first prove that there exists \( w_0 \in W_G \) such that
\[
w_0 \cdot \delta_G = \delta_G - \sum_{\theta \in \Lambda} \theta .
\]
Let
\[
\Phi_n^+ := \Phi_G^+ \setminus \Phi_K^+ .
\]
We first remark that any root in \( \Lambda \) belongs to \( \Phi_n^+ \). Otherwise, if there exists \( \theta \in \Lambda \cap \Phi_K^+ \), then since \( \theta \) is a combination with non-negative coefficients of simple
$K$-roots, and since $\langle \delta_K, \theta_i' \rangle > 0$, for any $K$-simple root $\theta_i'$, we would have $\langle \delta_K, \theta \rangle \geq 0$, contradicting the fact that $\theta \in \Lambda$.

Now, consider

$$\delta_n := \frac{1}{2} \sum_{\theta \in \Phi_n^+} \theta = \delta_G - \delta_K.$$ 

Then

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \left( \delta_n - \sum_{\theta \in \Lambda} \theta \right).$$

But,

$$\beta := \delta_n - \sum_{\theta \in \Lambda} \theta,$$

is a weight of the decomposition of the spin representation under the action of $K$, cf. § 2 in [Par71]: the weights are just the elements of the form $\delta_n - \sum_{\theta \in \Upsilon} \theta$, where $\Upsilon$ is a subset of $\Phi_n^+$. 

In fact $\beta$ is the highest weight of an irreducible component in the decomposition, otherwise we would have

$$\beta + \alpha = \delta_n - \sum_{\theta \in \Upsilon} \theta,$$

where $\alpha$ is a $K$-positive root and $\Upsilon$ is a subset of $\Phi_n^+$. 

Hence setting $\Lambda' := \Lambda \setminus \Upsilon$ and $\Upsilon' := \Upsilon \setminus \Lambda$, we would have

$$-\sum_{\theta \in \Lambda'} \theta + \alpha = -\sum_{\theta \in \Upsilon'} \theta.$$

But since $\Lambda' \subset \Lambda$ and $\alpha$ is a $K$-positive root

$$\langle -\sum_{\theta \in \Lambda'} \theta + \alpha, \delta_K \rangle > 0,$$

whereas since $\Upsilon' \subset \Phi_n^+ \setminus \Lambda$

$$\langle -\sum_{\theta \in \Upsilon'} \theta, \delta_K \rangle \leq 0,$$

hence a contradiction.

Now by the result of lemma 2.2 in [Par71], any highest weight in the decomposition of the spin representation has the form

$$w \cdot \delta_G - \delta_K,$$

where $w$ belongs to the subset $W$ of $W_G$ defined in [8]. Hence there exists a $w_0 \in W$ such that

$$\beta = w_0 \cdot \delta_G - \delta_K,$$

hence

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \beta = w_0 \cdot \delta_G,$$

hence the result.
Now let \( w \) be any element in \( W_G \). By the above lemma, \( w \cdot \delta_G = \delta_G - \sum_{\vartheta \in \Phi_G^+} k_\vartheta \vartheta \), where \( k_\vartheta = 0 \) or \( 1 \).

Hence by the definition of \( \Lambda \)
\[
< w \cdot \delta_G, \delta_K > \leq < \delta_G - \sum_{\vartheta \in \Lambda} k_\vartheta \vartheta, \delta_K >.
\]
Thus
\[
\max_{w \in W_G} < w \cdot \delta_G, \delta_K > \leq < \delta_G - \sum_{\vartheta \in \Lambda} k_\vartheta \vartheta, \delta_K > = < w_0 \cdot \delta_G, \delta_K > \leq \max_{w \in W_G} < w \cdot \delta_G, \delta_K >,
\]
hence the result.

Now going back to formula (4), we get immediately from (6)

**Corollary 3.3.** The first eigenvalue \( \lambda \) of the Dirac operator verifies
\[
\lambda^2 = 2 \| \delta_G - \delta_K \|^2 + 4 \sum_{\vartheta \in \Lambda} < \vartheta, \delta_K > + n/8.
\]

**4. Proof of the results of Table I**

In the following, we note for any integer \( n \geq 1 \), \((e_1, \ldots, e_n)\), the standard basis of \( K^n \), \( K = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \). The space of \((n, n)\) matrices with coefficients in \( K \) is denoted by \( \text{M}_n(\mathbb{K}) \).

**4.1. Quaternionic projective spaces \( \mathbb{H}P^n \).** Here \( G = \text{Sp}_{m+1} \) and \( K = \text{Sp}_m \times \text{Sp}_1 \). The decomposition of the spin representation into irreducible components under the action of \( K \) is given in [Mil92], so we may conclude with formula (1).

However the result may be also simply concluded with formula (5).

The space \( \mathbb{H}^{n+1} \) is viewed as a right vector space on \( \mathbb{H} \) in such a way that \( G \) may be identified with the group
\[
\left\{ A \in \text{M}_{m+1}(\mathbb{H}) ; \quad t A A^t = I_{m+1} \right\},
\]
acting on the left on \( \mathbb{H}^{n+1} \) in the usual way. The group \( K \) is identified with the subgroup of \( G \) defined by
\[
\left\{ A \in \text{M}_{m+1}(\mathbb{H}) ; \quad A = \begin{pmatrix} B & 0 \\ 0 & q \end{pmatrix}, \quad t B B^t = I_m, \quad q \in \text{Sp}_1 \right\}.
\]

Let \( T \) be the common torus of \( G \) and \( K \)
\[
T := \left\{ \begin{pmatrix} e^{i\beta_1} & & \\ & \ddots & \\ & & e^{i\beta_{m+1}} \end{pmatrix} , \quad \beta_1, \ldots, \beta_{m+1} \in \mathbb{R} \right\},
\]
where
\[
\forall \beta \in \mathbb{R}, \quad e^{i\beta} := \cos(\beta) + \sin(\beta) i,
\]
\((1, i, j, k)\) being the standard basis of \( \mathbb{H} \).
The Lie algebra of $T$ is
\[
\mathfrak{T} = \left\{ \left( \begin{array}{c} \beta_1 \\ \vdots \\ \beta_{m+1} \end{array} \right) ; \beta_1, \beta_2, \ldots, \beta_{m+1} \in \mathbb{R} \right\}.
\]

We denote by $(x_1, \ldots, x_{m+1})$ the basis of $\mathfrak{T}^*$ given by
\[
x_k \cdot \left( \begin{array}{c} i \beta_1 \\ \vdots \\ i \beta_{m+1} \end{array} \right) = \beta_k.
\]

A vector $\mu \in i \mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{m+1} \mu_k \hat{x}_k$, in the basis $(\hat{x}_k \equiv i x_k)_{k=1,\ldots,m+1}$, is denoted by
\[
\mu = (\mu_1, \mu_2, \ldots, \mu_{m+1}).
\]

The restriction to $\mathfrak{T}$ of the Killing form $B$ of $G$ is given by
\[
\forall X \in \mathfrak{T}, \forall Y \in \mathfrak{T}, \quad B(X, Y) = 4 (m + 2) \Re \left( \text{tr}(XY) \right).
\]

It is easy to verify that the scalar product on $i \mathfrak{T}^*$ induced by the Killing form sign changed is given by
\[
\forall \mu = (\mu_1, \ldots, \mu_{m+1}) \in i \mathfrak{T}^*, \forall \mu' = (\mu'_1, \ldots, \mu'_{m+1}) \in i \mathfrak{T}^*,
\]
\[
< \mu, \mu' > = \frac{1}{4(m + 2)} \sum_{k=1}^{m+1} \mu_k \mu'_k.
\]

Now, considering the decomposition of the complexified Lie algebra of $G$ under the action of $T$, it is easy to verify that $T$ is a common maximal torus of $G$ and $K$, and that the respective roots are given by
\[
\begin{cases}
\pm (\hat{x}_i + \hat{x}_j), & 1 \leq i < j \leq m + 1, \\
\pm (\hat{x}_i - \hat{x}_j), & 1 \leq i < j \leq m, \\
\pm \hat{x}_i, & 1 \leq i \leq m + 1 \quad \text{for } G,
\end{cases}
\]
\[
\begin{cases}
\pm (\hat{x}_i + \hat{x}_j), & 1 \leq i < j \leq m, \\
\pm (\hat{x}_i - \hat{x}_j), & 1 \leq i < j \leq m, \\
\pm 2 \hat{x}_i, & 1 \leq i \leq m + 1 \quad \text{for } K.
\end{cases}
\]

We consider as sets of positive roots
\[
\Phi^+_G = \left\{ \begin{array}{c} \hat{x}_i + \hat{x}_j, \\ \hat{x}_i - \hat{x}_j, \end{array} ; 1 \leq i \leq j \leq m + 1 ; 2 \hat{x}_i, 1 \leq i \leq m + 1 \right\},
\]
and
\[
\Phi^+_K = \left\{ \begin{array}{c} \hat{x}_i + \hat{x}_j, \\ \hat{x}_i - \hat{x}_j, \end{array} ; 1 \leq i \leq j \leq m ; 2 \hat{x}_i, 1 \leq i \leq m + 1 \right\}.
\]

Then
\[
\delta_G = \sum_{k=1}^{m+1} (m + 2 - k) \hat{x}_k = (m + 1, m, \ldots, 2, 1),
\]
and
\[ \delta_K = \sum_{k=1}^{m} (m + 1 - k) \hat{x}_k + \hat{x}_{m+1} = (m, m-1, \ldots, 1, 1). \]

Hence
\[ \delta_G - \delta_K = \sum_{k=1}^{m} \hat{x}_k = (1, 1, \ldots, 1, 0), \]
so
\[ \|\delta_G - \delta_K\|^2 = \frac{m}{4(m+2)}. \]

On the other hand, it is easy to verify that the set
\[ \Lambda := \{ \theta \in \Phi_G^+; <\theta, \delta_K> < 0 \}, \]
is empty, hence by formula (5), the square of the first eigenvalue \( \lambda \) of the Dirac operator is given by
\[ \lambda^2 = m \frac{m}{2(m+2)} + m = \frac{m+3}{2(m+2)}. \]

4.2. Grassmannians \( Gr_2(C^{m+2}), \) \( m \) even \( \geq 2. \) Here \( G = SU_{m+2} \) and \( K \) is the subgroup \( S(U_m \times U_2) \) defined below. Here again, the decomposition into irreducible components of the spin representation under the action of \( K \) is known, \[ Mil98, \]

hence the result may be obtained from formula (1). However the result may be also simply concluded with formula (5).

The group \( G \) is identified with
\[ \{ A \in M_{m+2}(\mathbb{C}); \; ^tAA = I_{m+2} \; \text{and} \; \det A = 1 \}. \]

The group \( K \) is the group
\[ S(U_m \times U_2) = \{ A \in M_{m+2}(\mathbb{C}); \; A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \; B \in U_m, \; C \in U_2; \; \det A = 1 \}. \]

Let \( T \) be the common torus of \( G \) and \( K \)
\[ T := \left\{ \begin{pmatrix} e^{i\beta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{i\beta_{m+2}} \end{pmatrix}; \; \beta_1, \ldots, \beta_{m+2} \in \mathbb{R}, \; \sum_{k=1}^{m+2} \beta_k = 0 \right\}. \]

The Lie algebra of \( T \) is
\[ \mathfrak{T} = \left\{ \begin{pmatrix} i\beta_1 \\ \vdots \\ i\beta_{m+2} \end{pmatrix}; \; \beta_1, \beta_2, \ldots, \beta_{m+2} \in \mathbb{R}, \; \sum_{k=1}^{m+2} \beta_k = 0 \right\}. \]

We denote by \((x_1, \ldots, x_{m+1})\) the basis of \( \mathfrak{T}^* \) given by
\[ x_k \cdot \begin{pmatrix} i\beta_1 \\ \vdots \\ i\beta_{m+2} \end{pmatrix} = \beta_k. \]

A vector \( \mu \in i\mathfrak{T}^* \) such that \( \mu = \sum_{k=1}^{m+1} \mu_k \hat{x}_k, \) in the basis \((\hat{x}_k \equiv i\beta_k)_{k=1, \ldots, m+1}, \) is denoted by
\[ \mu = (\mu_1, \mu_2, \ldots, \mu_{m+1}). \]
The restriction to $\mathfrak{T}$ of the Killing form $B$ of $G$ is given by
\[ B(X, Y) = 2(m + 2) \Re (\text{tr}(XY)) \]
for all $X, Y \in \mathfrak{T}$. It is easy to verify that the scalar product on $i\mathfrak{T}^*$ induced by the Killing form sign changed is given by
\[ <\mu, \mu'> = \frac{1}{2(m + 2)} \sum_{k=1}^{m+1} \mu_k \mu'_k - \frac{1}{2(m + 2)^2} \left( \sum_{k=1}^{m+1} \mu_k \right) \left( \sum_{k=1}^{m+1} \mu'_k \right). \]

Considering the decomposition of the complexified Lie algebra of $G$ under the action of $T$, it is easy to verify that $T$ is a common maximal torus of $G$ and $K$, and that the respective roots are given by
\[ \pm (\hat{x}_i - \hat{x}_j), \ 1 \leq i < j \leq m + 1, \quad \pm (\hat{x}_i + \sum_{k=1}^{m+1} \hat{x}_k), \ 1 \leq i \leq m + 1, \quad \text{for } G, \]
\[ \pm (\hat{x}_i - \hat{x}_j), \ 1 \leq i < j \leq m, \quad \pm (\hat{x}_{m+1} + \sum_{k=1}^{m+1} \hat{x}_k), \quad \text{for } K. \]

We consider as sets of positive roots
\[ \Phi_G^+ = \left\{ \hat{x}_i - \hat{x}_j, \ 1 \leq i \leq m + 1; \hat{x}_i + \sum_{k=1}^{m+1} \hat{x}_k, \ 1 \leq i \leq m + 1 \right\}, \]
and
\[ \Phi_K^+ = \left\{ \hat{x}_i - \hat{x}_j, \ 1 \leq i \leq m; \hat{x}_{m+1} + \sum_{k=1}^{m+1} \hat{x}_k \right\}. \]

Then
\[ \delta_G = \sum_{k=1}^{m+1} (m + 2 - k) \hat{x}_k = (m + 1, m, \ldots, 2, 1), \]
and
\[ \delta_K = \frac{1}{2} \left( \sum_{k=1}^{m} (m + 2 - 2k) \hat{x}_k + 2 \hat{x}_{m+1} \right) = \frac{1}{2} (m, m - 2, m - 4, \ldots, 2 - m, 2). \]

Hence
\[ \delta_G - \delta_K = \frac{1}{2} (m + 2) \sum_{k=1}^{m} \hat{x}_k = \frac{1}{2} (m + 2)(1, 1, \ldots, 1, 0), \]
so
\[ \|\delta_G - \delta_K\|^2 = \frac{m}{4}. \]
We now determine the set
\[ \Lambda := \{ \theta \in \Phi_G^+ ; <\theta, \delta_K> < 0 \}. \]
Recall that from the proof of lemma 3.2, if $\Lambda$ is non empty, then any $\theta \in \Lambda$ belongs to $\Phi^+ G \setminus \Phi^+ K$. It is then easy to verify that the elements of $\Lambda$ are

\[ \hat{x}_j - \hat{x}_{m+1}, \quad \frac{m}{2} + 1 \leq j \leq m, \quad < \hat{x}_j - \hat{x}_{m+1}, \delta_K > = \frac{1}{2(m+2)} \left( \frac{m}{2} - j \right), \]

\[ \hat{x}_j + \sum_{k=1}^{m+1} \hat{x}_k, \quad \frac{m}{2} + 2 \leq j \leq m, \quad < \hat{x}_j + \sum_{k=1}^{m+1} \hat{x}_k, \delta_K > = \frac{1}{2(m+2)} \left( \frac{m}{2} + 1 - j \right). \]

So

\[ \sum_{\theta \in \Lambda} < \theta, \delta_K > = - \frac{m^2}{8(m+2)}. \]

Hence, by formula (5), the square of the first eigenvalue $\lambda$ of the Dirac operator is given by

\[ \lambda^2 = \frac{m^2}{2} - \frac{m^2}{2(m+2)} + \frac{m}{2} = \frac{m+4}{m+2} \cdot \frac{m}{2}. \]

4.3. Grassmannians $\overline{\text{Gr}}_4(\mathbb{R}^{m+4})$, $m$ even $\geq 4$. Here $G = \text{Spin}_{m+4}$ and, identifying $\mathbb{R}^m$ with the subspace of $\mathbb{R}^{m+4}$ spanned by $e_1, \ldots, e_m$, and $\mathbb{R}^4$ with the subspace spanned by $e_{m+1}, \ldots, e_{m+4}$, $K$ is the subgroup of $G$ defined by

\[ \text{Spin}_m \text{Spin}_4 := \{ \psi \in \text{Spin}_{m+4} ; \psi = \varphi \phi, \varphi \in \text{Spin}_m, \phi \in \text{Spin}_4 \}. \]

We consider the common torus of $G$ and $K$ defined by

\[ T = \left\{ \sum_{k=1}^{m+2} (\cos(\beta_k) + \sin(\beta_k)e_{2k-1} \cdot e_{2k}) ; \beta_1, \ldots, \beta_{m+2} \in \mathbb{R} \right\}. \]

The Lie algebra of $T$ is

\[ \mathfrak{T} = \left\{ \sum_{k=1}^{m+2} \beta_k e_{2k-1} \cdot e_{2k} ; \beta_1, \ldots, \beta_{m+2} \in \mathbb{R} \right\}. \]

We denote by $(x_1, \ldots, x_{m+2})$ the basis of $\mathfrak{T}^*$ given by

\[ x_k \cdot \sum_{j=1}^{m+2} \beta_j e_{2j-1} \cdot e_{2j} = \beta_k. \]

We introduce the basis $(\hat{x}_1, \ldots, \hat{x}_{m+2})$ of $i \mathfrak{T}^*$ defined by

\[ \hat{x}_k := 2i x_k, \quad k = 1, \ldots, \frac{m}{2} + 2. \]

A vector $\mu \in i \mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{m+2} \mu_k \hat{x}_k$, is denoted by

\[ \mu = (\mu_1, \mu_2, \ldots, \mu_{m+2}). \]

The restriction to $\mathfrak{T}$ of the Killing form $B$ of $G$ is given by

\[ B(e_{2k-1} \cdot e_{2k}, e_{2l-1} \cdot e_{2l}) = -8(m+2) \delta_{kl}. \]
It is easy to verify that the scalar product on $i \mathfrak{X}^*$ induced by the Killing form sign changed is given by

$$\forall \mu = (\mu_1, \ldots, \mu_{m^2 + 2}) \in i \mathfrak{X}^*, \forall \mu' = (\mu'_1, \ldots, \mu'_{m^2 + 2}) \in i \mathfrak{X}^*, \quad <\mu, \mu'> = \frac{1}{2(m + 2)} \sum_{k=1}^{m^2 + 2} \mu_k \mu'_k.$$  

(15)

Considering the decomposition of the complexified Lie algebra of $G$ under the action of $T$, it is easy to verify that $T$ is a common maximal torus of $G$ and $K$, and that the respective roots are given by

$$\pm (\hat{x}_i + \hat{x}_j), \pm (\hat{x}_i - \hat{x}_j), \quad 1 \leq i < j \leq \frac{m}{2} + 2,$$

for $G$,  

$$\pm (\hat{x}_i + \hat{x}_j), \pm (\hat{x}_i - \hat{x}_j), \quad 1 \leq i < j \leq \frac{m}{2},$$

$$\pm (\hat{x}_{m^2 + 1} + \hat{x}_{m^2 + 2}), \pm (\hat{x}_{m^2 + 1} - \hat{x}_{m^2 + 2}),$$

for $K$.

We consider as sets of positive roots

$$\Phi^+_G = \{\hat{x}_i + \hat{x}_j, \hat{x}_i - \hat{x}_j, \quad 1 \leq i < j \leq \frac{m}{2} + 2\},$$

and

$$\Phi^+_K = \{\hat{x}_i + \hat{x}_j, \hat{x}_i - \hat{x}_j, \quad 1 \leq i < j \leq \frac{m}{2}, \hat{x}_{m^2 + 1} + \hat{x}_{m^2 + 2}, \hat{x}_{m^2 + 1} - \hat{x}_{m^2 + 2}\}.$$

Then

$$\delta_G = \sum_{k=1}^{m^2 + 2} \left(\frac{m}{2} + 2 - k\right) \hat{x}_k = \left(\frac{m}{2} + 1, \frac{m}{2}, \ldots, 1, 0\right),$$

and

$$\delta_K = \sum_{k=1}^{m^2 + 2} \left(\frac{m}{2} - k\right) \hat{x}_k + \hat{x}_{m^2 + 1} = \left(\frac{m}{2} - 1, \frac{m}{2} - 2, \ldots, 1, 0\right).$$

Hence

$$\delta_G - \delta_K = 2 \sum_{k=1}^{m^2} \hat{x}_k = 2 (1, 1, \ldots, 1, 0),$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{m + 2}.$$  

On the other hand, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi^+_G; <\theta, \delta_K > < 0\},$$

has only one element, namely

$$\hat{x}_{m^2} - \hat{x}_{m^2 + 1}, \text{ with } <\hat{x}_{m^2} - \hat{x}_{m^2 + 1}, \delta_K > = -1.$$  

Hence, by formula (15), the square of the first eigenvalue $\lambda$ of the Dirac operator is given by

$$\lambda^2 = \frac{2m}{m + 2} - \frac{2}{m + 2} + \frac{m}{2} = \frac{m^2 + 6m - 4}{2(m + 2)}.$$
4.4. The four exceptional cases. Note first that since all the groups \( G \) we consider are simple, their roots system are irreducible so, up to a constant, there is only one \( W_G \)-invariant scalar product on the subspace generated by the set of roots, cf. for instance Remark (5.10), § V in [BDS85]. We use the description of root systems given in [BMP85]. Those root systems are expressed in the simple root basis \( (\alpha_i) \). Note that the \( W_G \)-invariant scalar product \( ( , ) \) used there is such that \( (\alpha, \alpha) = 2 \) for any long root \( \alpha \). In order to compare it with the scalar product \(< , >\) induced by the Killing form sign-changed, we use the “strange formula” of Freudenthal and de Vries, (cf. 47-11 in [FdV69]):

\[
< \delta_G, \delta_G > = \frac{1}{24} \dim G.
\]

To determine the set of \( K \)-positive roots, we use theorem 13, theorem 14 and the proof of theorem 18 in [CG88]. By those results, the set \( \Phi_K^+ \) may be defined as follows. Let \( \theta = \sum m_i \alpha_i \) be the highest root. In all cases considered, there exists an index \( j \) such that \( m_j = 2 \). Then

\[
\Phi_K^+ = \left\{ \sum n_i \alpha_i ; n_j \neq 1 \right\}.
\]

4.4.1. The symmetric space \( G_2/\text{SO}_4 \). Using the results of pages 18 and 64 in [BMP85], we get

\[
\delta_G = 3 \alpha_1 + 5 \alpha_2.
\]

By the expression of the Cartan matrix, the scalar product matrix is, in the basis \( (\alpha_1, \alpha_2), \begin{pmatrix} 2 & -1 \\ -1 & 2/3 \end{pmatrix} \), hence

\[
||\delta_G||^2_{( , )} = \frac{14}{3}.
\]

On the other hand, by the formula of Freudenthal and de Vries,

\[
||\delta_G||^2_{< , >} = \frac{7}{12},
\]

so

\[
< , > = \frac{1}{8} ( , ) .
\]

The set of \( K \)-positive roots is

\[
\Phi_K^+ = \{ 2 \alpha_1 + 3 \alpha_2, \alpha_2 \},
\]

hence

\[
\delta_K = \alpha_1 + 2 \alpha_2,
\]

so

\[
\delta_G - \delta_K = 2 \alpha_1 + 3 \alpha_2.
\]

Hence

\[
||\delta_G - \delta_K||^2_{< , >} = \frac{1}{8} ||\delta_G - \delta_K||^2_{( , )} = \frac{1}{4}.
\]

Finally, it is easy to verify that the set

\[
\Lambda := \{ \theta \in \Phi_G^+ ; < \theta, \delta_K > < 0 \},
\]

is empty, hence by formula (5), the square of the first eigenvalue \( \lambda \) of the Dirac operator is given by

\[
\lambda^2 = \frac{1}{2} + 1 = \frac{3}{2}.
\]
4.4.2. *The symmetric space* \( \mathbb{E}_6/(SU_6SU_2) \). Using the results of pages 14 and 60 in [BMP85], we get

\[
\delta_G = 8 \alpha_1 + 15 \alpha_2 + 21 \alpha_3 + 15 \alpha_4 + 8 \alpha_5 + 11 \alpha_6.
\]

Since all roots have same length equal to 2, we may introduce the fundamental weight basis \((\omega_i)\) because

\[
(\omega_i, \alpha_j) = \delta_{ij}.
\]

Since \(\delta_G = \sum \omega_i\), we get

\[
\|\delta_G\|_{\langle , \rangle}^2 = 78,
\]

whereas by the formula of Freudenthal and de Vries,

\[
\|\delta_G\|_{\langle , \rangle}^2 = \frac{78}{24},
\]

so

\[
\langle , \rangle = \frac{1}{24} \langle , \rangle.
\]

The set of \(K\)-positive roots may be defined by

\[
\Phi^+_K = \left\{ \sum_{i=1}^6 n_i \alpha_i ; n_6 \neq 1 \right\}.
\]

Then

\[
\delta_K = 3 \alpha_1 + 5 \alpha_2 + 6 \alpha_3 + 5 \alpha_4 + 3 \alpha_5 + \alpha_6
\]

\[
= \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 - 4 \omega_6.
\]

Hence

\[
\delta_G - \delta_K = 5 \alpha_1 + 10 \alpha_2 + 15 \alpha_3 + 10 \alpha_4 + 5 \alpha_5 + 10 \alpha_6 = 5 \omega_6.
\]

So

\[
\|\delta_G - \delta_K\|_{\langle , \rangle}^2 = \frac{1}{24} \|\delta_G - \delta_K\|_{\langle , \rangle}^2 = \frac{25}{12}.
\]

On the other hand it is easy to verify that the set

\[
\Lambda := \{ \theta \in \Phi^+_G ; \langle \theta, \delta_K \rangle < 0 \},
\]

has 7 elements and that

\[
\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = \frac{1}{24} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{7}{12}.
\]

So by formula (9), the square of the first eigenvalue \(\lambda\) of the Dirac operator is given by

\[
\lambda^2 = \frac{50}{12} - \frac{28}{12} + 5 = \frac{41}{6}.
\]
4.4.3. The symmetric space $E_7/(\text{Spin}_{12}\text{SU}_2)$. By the results of pages 15 and 61 in [BMP85], we get

$$\delta_G = \frac{1}{2} (34 \alpha_1 + 66 \alpha_2 + 96 \alpha_3 + 75 \alpha_4 + 52 \alpha_5 + 27 \alpha_6 + 49 \alpha_7).$$

Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis $(\omega_i)$. We get

$$\|\delta_G\|_2^2 = \frac{399}{2},$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{\langle, \rangle}^2 = \frac{133}{24},$$

so

$$\langle, \rangle = \frac{1}{36} \langle, \rangle.$$

The set of $K$-positive roots may be defined by

$$\Phi^+_K = \left\{ \sum_{i=1}^{7} n_i \alpha_i ; n_i \neq 1 \right\}.$$

Then

$$\delta_K = \frac{1}{2} (2 \alpha_1 + 18 \alpha_2 + 32 \alpha_3 + 27 \alpha_4 + 20 \alpha_5 + 11 \alpha_6 + 17 \alpha_7)$$

$$= -7 \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7.$$

Hence

$$\delta_G - \delta_K = 16 \alpha_1 + 24 \alpha_2 + 32 \alpha_3 + 24 \alpha_4 + 16 \alpha_5 + 8 \alpha_6 + 16 \alpha_7 = 8 \omega_6.$$

So

$$\|\delta_G - \delta_K\|_{\langle, \rangle}^2 = \frac{1}{36} \|\delta_G - \delta_K\|_2^2 = \frac{32}{9}.$$

On the other hand it can be verified that the set

$$\Lambda := \{ \theta \in \Phi^+_G ; \langle \theta, \delta_K \rangle < 0 \}$$

has 13 elements and that

$$\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = \frac{1}{36} \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = -\frac{41}{36}.$$

So by formula (5), the square of the first eigenvalue $\lambda$ of the Dirac operator is given by

$$\lambda^2 = \frac{64}{9} - \frac{41}{9} + 8 = \frac{95}{9}.$$
4.4.4. The symmetric space $E_8/(E_7SU_2)$. By the results of pages 16, 62 and 63 in [BMP85], we get
\[ \delta_G = 29\alpha_1 + 57\alpha_2 + 84\alpha_3 + 110\alpha_4 + 135\alpha_5 + 91\alpha_6 + 46\alpha_7 + 68\alpha_8. \]
Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis $(\omega_i)$. We get
\[ \|\delta_G\|_2^2 = 620, \]
whereas by the formula of Freudenthal and de Vries,
\[ \|\delta_G\|_{<,>}^2 = \frac{248}{24} = \frac{31}{3}, \]
so
\[ <, > = \frac{1}{60} (, ,). \]
The set of $K$-positive roots may be defined by
\[ \Phi_K^+ = \left\{ \sum_{i=1}^{8} n_i \alpha_i : n_1 \neq 1 \right\}. \]
Then
\[ \delta_K = \alpha_1 + 15\alpha_2 + 28\alpha_3 + 40\alpha_4 + 51\alpha_5 + 35\alpha_6 + 18\alpha_7 + 26\alpha_8 \]
\[ = -13\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7 + \omega_8. \]
Hence
\[ \delta_G - \delta_K = 28\alpha_1 + 42\alpha_2 + 56\alpha_3 + 70\alpha_4 + 84\alpha_5 + 56\alpha_6 + 28\alpha_7 + 42\alpha_8 = 14\omega_6. \]
So
\[ \|\delta_G - \delta_K\|_{<,>}^2 = \frac{1}{60} \|\delta_G - \delta_K\|_{(,)}^2 = \frac{98}{15}. \]
On the other hand it can be verified that the set
\[ \Lambda := \{ \theta \in \Phi_K^+ : <\theta, \delta_K > < 0 \}, \]
has 25 elements and that
\[ \sum_{\theta \in \Lambda} <\theta, \delta_K > = \frac{1}{60} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{137}{60}. \]
So by formula (5), the square of the first eigenvalue $\lambda$ of the Dirac operator is given by
\[ \lambda^2 = \frac{196}{15} - \frac{137}{15} + 14 = \frac{269}{15}. \]

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