GRADED CHARACTERS OF MODULES
SUPPORTED IN THE CLOSURE OF
A NILPOTENT CONJUGACY CLASS

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Abstract. This is a combinatorial study of the Poincaré polynomials of iso-
typic components of a natural family of graded \(GL(n)\)-modules supported in
the closure of a nilpotent conjugacy class. These polynomials generalize
the Kostka-Foulkes polynomials and are \(q\)-analogues of Littlewood-Richardson
coefficients. The coefficients of two-column Macdonald-Kostka polynomials
also occur as a special case. It is conjectured that these \(q\)-analogues are the
generating function of so-called catabolizable tableaux with the charge statis-
tic of Lascoux and Schützenberger. A general approach for a proof is given,
and is completed in certain special cases including the Kostka-Foulkes case.
Catabolizable tableaux are used to prove a characterization of Lascoux and
Schützenberger for the image of the tableaux of a given content under the
standardization map that preserves the cyclage poset.

1. Introduction

For a partition \(\mu\) of \(n\) let \(X_\mu\) be the Zariski closure of the conjugacy class in
\(gl(n, \mathbb{C})\) of the nilpotent Jordan matrix with blocks of sizes given by the conjugate
or transpose partition \(\mu^t\) of \(\mu\). Since \(X_\mu\) is a cone that is stable under the action
of \(GL(n, \mathbb{C})\) given by matrix conjugation, its coordinate ring \(\mathbb{C}[X_\mu]\) is graded and
affords an action of \(GL(n, \mathbb{C})\) that respects the grading. The natural category of
modules over \(\mathbb{C}[X_\mu]\) is the family of finitely generated graded \(\mathbb{C}[X_\mu]\)-modules (that
is, \(\mathbb{C}[gl(n)]\)-modules supported in \(X_\mu\)) that afford the action of \(GL(n, \mathbb{C})\) that is
compatible with the graded \(\mathbb{C}[X_\mu]\)-module structure.

For any permutation \(\eta\) of the parts of \(\mu\), there is a Springer desingularization
\(Z(\eta) \to X_\mu\). Corresponding to any \(GL(n)\)-weight \(\gamma \in \mathbb{Z}^n\), there is an \(O_{Z(\eta)}\)-module
\(M_{\eta, \gamma}\) with Euler characteristic \(\chi_{\eta, \gamma}\), which can be viewed as an element of the
Grothendieck group \(K'_0(\mathbb{C}[X_\mu])\) of the aforementioned category of \(\mathbb{C}[X_\mu]\)-modules.
In \([8]\) it is shown that the classes of the \(\chi_{\eta, \gamma}\) graded \(GL(n)\)-modules \(\chi_{\eta, \gamma}\) generate
the group \(K'_0(\mathbb{C}[X_\mu])\).

The purpose of this paper is to investigate the combinatorial properties of the
family of polynomials \(K_{\lambda, \gamma, \eta}(q)\) given by the isotypic components of the virtual
graded \(GL(n)\)-modules \(\chi_{\eta, \gamma}\).

The polynomials \(K_{\lambda, \gamma, \eta}(q)\) are \(q\)-analogues of Littlewood-Richardson (LR) co-
efficients. Special cases include the Kostka-Foulkes polynomials (Lusztig’s \(q\)-analogues
of weight multiplicities in type A) and coefficients of two-column Macdonald-Kostka
polynomials.

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Many formulas involving the Kostka-Foulkes polynomials have suitable generalizations for the polynomials $K_{\lambda,\gamma,\eta}(q)$, such as the $q$-Kostant partition function formula [18] and Morris’ recurrence [20]. We derive these formulas directly from the definition of the twisted modules $M_{\eta,\gamma}$.

The Kostka-Foulkes polynomials also have several combinatorial descriptions, including two beautiful formulas of Lascoux and Schützenberger involving the $q$-enumeration of two sets of tableaux [10] [13]. The main focus of this paper is a common generalization of these formulas. We define the notion of a catabolizable tableau and give a conjectural interpretation of $K_{\lambda,\gamma,\eta}(q)$ as the generating function over such tableaux with the charge statistic (Conjecture 26). A general approach for a proof is given, using a sign-reversing involution that cancels terms from the generalized Morris recurrence. The missing ingredient in the general case is to prove that the involution preserves catabolizability in a certain sense. In special cases this can be shown, so the conjecture holds in those cases.

We also develop properties of catabolizable tableaux and their intimate relationship with the cyclage poset [10] [13]. As an application, we supply a proof of a formula of Lascoux [10] for the cocharge Kostka-Foulkes polynomials.

In the special case corresponding to LR coefficients associated with products of rectangular shapes, the polynomials $K_{\lambda,\gamma,\eta}(q)$ seem to coincide with yet another $q$-analogue of LR coefficients given by the combinatorial objects known as rigged configurations. This connection is pursued in [9].

Lascoux, Leclerc and Thibon have defined a $q$-analogue of certain LR coefficients using ribbon tableaux [11]. This family of polynomials appears to contain the polynomials $K_{\lambda,\gamma,\eta}(x;q)$ as a subfamily, but the reason for this is unclear.

2. The polynomials $K_{\lambda,\gamma,\eta}(q)$

The first goal is to derive an explicit formula for the polynomials $K_{\lambda,\gamma,\eta}(q)$. By definition this polynomial is the coefficient of the irreducible character $s_\lambda(x)$ of highest weight $\lambda$ in the formal character $H_{\eta,\gamma}(x;q)$ of the virtual graded $GL(n)$-module $\chi_{\eta,\gamma}$. A nice bialternant formula for $H_{\eta,\gamma}(x;q)$ is obtained by expressing the Euler characteristic $\chi_{\eta,\gamma}$ in terms of Euler characteristics of $GL(n)$-equivariant line bundles over the flag variety and applying Bott’s theorem for calculating the latter. Some readers may prefer to take the formulas (2.2) and (2.4) as the definitions of $H_{\eta,\gamma}(x;q)$ and $K_{\lambda,\gamma,\eta}(x;q)$ respectively.

The rest of this section derives various properties of the polynomials $K_{\lambda,\gamma,\eta}(x;q)$, including positivity conditions, specializations to known families of polynomials, the interpretation as a $q$-analogue of an LR coefficient, and various symmetry and monotonicity properties.

2.1. The Poincaré polynomial $K_{\lambda,\gamma,\eta}(q)$. We review the definition of $\chi_{\eta,\gamma}$ given in [8] and derive a formula for its graded character $H_{\eta,\gamma}(x;q)$ and its coefficient polynomials $K_{\lambda,\gamma,\eta}(x;q)$.

Let $\mu$ be a partition of a fixed positive integer $n$ and $X_\mu$ the nilpotent adjoint orbit closure defined in the introduction. Let $\eta = (\eta_1, \ldots, \eta_t)$ be a reordering of the parts of the partition $\mu$. For each such $\eta$ there is a Springer desingularization $q_\eta: Z_\eta \to X_\mu$ defined as follows. Consider the variety of partial flags of dimensions given by the sequence $0 = d_0 < d_1 < \cdots < d_t < d_{t+1} = n$, where

$$d_i = n - (\eta_1 + \cdots + \eta_i) \quad \text{for} \quad 0 \leq i \leq t,$$
whose typical element is
\[ F_i = (0 = F_{d_i} \subset F_{d_{i-1}} \subset \cdots \subset F_{d_0} = \mathbb{C}^n) \]
where \( F_j \) is a subspace of dimension \( j \). This flag variety is realized by the homogeneous space \( G/P \), where \( G = \text{GL}(n, \mathbb{C}) \) and \( P = P_\eta \) is the parabolic subgroup that stabilizes the partial flag whose subspaces have the form \( \mathbb{C}^d \) for \( 0 \leq i \leq t \), where \( \mathbb{C}^d \) denotes the span of the last \( j \) standard basis vectors of the standard left \( G \)-module \( \mathbb{C}^n \), viewed as the space of column vectors. In other words, \( P \) is the subgroup of lower block triangular matrices with block sizes given in order by \( \eta_i \) through \( \eta_t \). Let \( g = \text{Lie}(G) \) and \( p = p_\eta = \text{Lie}(P) \).

Define the incidence variety
\[ Z = Z_0 := \{(A, F) \in g \times G/P \mid AF_{d_{i-1}} \subset F_i \text{ for } 1 \leq i \leq t\}. \]
Let \( q : Z \to g \) and \( p : Z \to G/P \) be the restriction to \( Z \) of the first and second projections of \( g \times G/P \). The map \( q \) is a desingularization of its image \( X_\mu \).

Next we recall the definition of the \( \mathcal{O}_Z \)-modules \( \mathcal{M}_{\eta, \gamma} \). Let \( B \subset G \) be the standard lower triangular Borel subgroup, \( H \subset B \) the subgroup of diagonal matrices, and \( \Delta^+ \) the set of positive roots, which are chosen to be in the opposite Borel to \( B \). Let \( b = \text{Lie}(B) \) and \( b = \text{Lie}(H) \). Let \( W \) be the Weyl group, the symmetric group on \( n \) letters, which shall be identified with the permutation matrices in \( G \). The character group of \( H \) (and the integral weights) may be identified with \( \mathbb{Z}^n \) such that \((\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n \) is identified with the character \((\text{diag}(x_1, \ldots, x_n)) \mapsto \prod_{i=1}^n x_i^{\gamma_i} \) where \( \text{diag}(x_1, \ldots, x_n) \) is the diagonal matrix with diagonal entries \( x_i \). According to these conventions, a weight \( \gamma \in \mathbb{Z}^n \) is dominant if \( \gamma_i \geq \gamma_{i+1} \) for \( 1 \leq i \leq n - 1 \).

Let \( \mathbb{C}_\gamma \) be the one-dimensional \( B \)-module of weight \( \gamma \) and \( \mathcal{L}_\gamma := G \times B \mathbb{C}_\gamma \) the \( G \)-equivariant line bundle over \( G/B \) given by the orbit space
\[ G \times B \mathbb{C}_\gamma := (G \times \mathbb{C}_\gamma)/B \]
with bundle map \((g, v)B \mapsto gB \). Let \( \phi = \phi_\eta : G/B \to G/P \) be the canonical projection. Define the \( \mathcal{O}_Z \)-module
\[ \mathcal{M}_{\eta, \gamma} = \mathcal{O}_Z \otimes p^* \phi_*(\mathcal{L}_\gamma). \]
Define the elements \( \chi_{\eta, \gamma} \) and \( M_{\eta, \gamma} \) in \( K_0^p(\mathbb{C}[X_\mu]) \) by
\[ \chi_{\eta, \gamma} = \sum_{i \geq 0} (-1)^i [\mathcal{R}^i q_*(\mathcal{M}_{\eta, \gamma})] \]
\[ M_{\eta, \gamma} = q_*(\mathcal{M}_{\eta, \gamma}). \]

Next, the element \( \chi_{\eta, \gamma} \) is expressed in terms of Euler characteristics of \( G \)-equivariant line bundles over \( G/B \). Since \( q : Z \to g \) is a morphism to an affine variety,
\[ \mathcal{R}^i q_*(\mathcal{M}_{\eta, \gamma}) \cong H^i(Z, \mathcal{M}_{\eta, \gamma}) \quad \text{for } i \geq 0. \]
Since
\[ \mathcal{R}^i p_*(\mathcal{M}_{\eta, \gamma}) = 0 \quad \text{for } i > 0, \]
we have
\[ H^i(Z, \mathcal{M}_{\eta, \gamma}) \cong H^i(G/P, p_*(\mathcal{M}_{\eta, \gamma})) \quad \text{for } i \geq 0. \]
By the projection formula,
\[ p_*(\mathcal{M}_{\eta, \gamma}) = p_*(\mathcal{O}_Z \otimes p^*\phi_*\mathcal{L}_\gamma) = p_*(\mathcal{O}_Z) \otimes \phi_*\mathcal{L}_\gamma. \]
As sheaves over \( G/P \), \( p_*(\mathcal{O}_Z) \) may be identified \( T := G \times^P S(g/p) \) where \( S \) is the symmetric algebra. Thus we have
\[ p_*(\mathcal{O}_Z) \otimes \phi_*\mathcal{L}_\gamma = T \otimes \phi_*\mathcal{L}_\gamma. \]
Let \( T' := G \times B S((g/p)/p) \). Then \( T' = \phi^*T \) and
\[ R^i\phi_*(T' \otimes \mathcal{L}_\gamma) = 0 \quad \text{for} \ i > 0. \]
Then
\[ H^i(G/B, T' \otimes \mathcal{L}_\gamma) = H^i(G/B, \phi^*(T) \otimes \mathcal{L}_\gamma) = H^i(G/P, \phi_*(\phi^*(T) \otimes \mathcal{L}_\gamma)). \]
Putting this all together, we have
\[ (2.1) \quad \chi_{\eta, \gamma} = \sum_{i \geq 0} (-1)^i[H^i(G/B, T' \otimes \mathcal{L}_\gamma)]. \]

The homogeneous degree is given by the degree in the polynomial ring \( S(g/p) \). Each homogeneous component of \( T' \otimes \mathcal{L}_\gamma \) has a filtration whose successive quotients are \( G \)-equivariant line bundles over \( G/B \). By the additivity of Euler characteristic the calculation reduces to Bott’s formula for the Euler characteristic of a line bundle over \( G/B \).

This allows the explicit calculation of the formal character \( H_{\eta, \gamma}(x; q) \) of \( \chi_{\eta, \gamma} \). The formal character of a finite dimensional rational \( H \)-module \( M \) is the Laurent polynomial given by
\[ \text{ch}(M) = \text{tr}(x|M), \]
the trace of the action of \( x = \text{diag}(x_1, \ldots, x_n) \) on \( M \).

Let \( \rho = (n-1, n-2, \ldots, 1, 0) \) and \( J \) and \( \pi \) the operators on \( \mathbb{C}[x_1, \ldots, x_n][\det(x)^{-1}] \) given by
\[ J(f) = \sum_{w \in W} \omega w f \]
\[ \pi(f) = J(x^\rho)^{-1} J(x^\rho f). \]
The operator \( \pi \) is the Demazure operator \( \pi_{w_0} \) in the notation of [18], where \( w_0 \) is the longest element in \( W \). Bott’s formula states that
\[ \text{ch}(\chi(G/B, \mathcal{L}_\alpha)) = \pi(x^\alpha) \]
for \( \alpha \in \mathbb{Z}^n \). In particular, if \( \alpha \) is a dominant integral weight (resp. partition) then \( \pi(x^\alpha) = s_\alpha(x) \) is the irreducible character of highest weight \( \alpha \) (resp. Schur polynomial).

Consider the weights of the adjoint action of \( P \) on \( g/p \), which is indexed by the set \( \text{Roots}_\eta \) of matrix positions above the block diagonal given by the parts of \( \eta \), that is,
\[ \text{Roots}_\eta = \{(i, j) \mid 1 \leq i \leq \eta_1 + \cdots + \eta_r, j \leq \eta \leq n \text{ for some } r\}. \]

Example 1. 1. If \( \eta = (n) \) then \( \text{Roots}_\eta \) is empty.
2. If \( \eta = (1^n) \) then \( \text{Roots}_\eta = \{(i, j) : 1 \leq i < j \leq n \} \).

Keeping track of the degree in \( S(g/p) \) by powers of the variable \( q \), let the formal power series \( B_\eta(x; q) \) (resp. \( H_{\eta, \gamma}(x; q) \)) be the formal character of the graded \( B \)-module \( S(g/p) \) (resp. the \( G \)-module \( \chi_{\eta, \gamma} \)). By (2.1) and Bott’s formula, these can be written

\[
B_\eta(x; q) = \prod_{(i, j) \in \text{Roots}_\eta} \frac{1}{1 - q x_i / x_j} 
\]

(2.2)

\[
H_{\eta, \gamma}(x; q) = \pi(x^\gamma B_\eta(x; q)).
\]

(2.3)

Then by definition,

\[
H_{\eta, \gamma}(x; q) = \sum_\lambda K_{\lambda, \gamma, \eta}(q)s_\lambda(x),
\]

(2.4)

where \( \lambda \) runs over the dominant integral weights in \( \mathbb{Z}^n \). \( B_\eta(x; q) \) and \( H_{\eta, \gamma}(x; q) \) should be viewed as formal power series in \( q \) with coefficients in the ring of formal Laurent polynomials in the \( x_i \). It is shown later in Proposition 3 that \( K_{\lambda, \gamma, \eta}(q) \) is a polynomial with integer coefficients.

**Example 2.** Let \( n = 2, \eta = (1, 1) \) and \( \gamma = (0, 0) \). Then

\[
H_{\eta, \gamma}(x; q) = \sum_{k \geq 0} q^k s_{(k, -k)}(x_1, x_2)
\]

\[
= \sum_{k \geq 0} q^k (x_1 x_2)^{-k} s_{(2k, 0)}(x_1, x_2)
\]

So \( K_{(k, -k), (0, 0), (1, 1)}(q) = q^k \) for all \( k \in \mathbb{N} \).

2.2. **Normalization.** Many of the polynomials \( K_{\lambda, \gamma, \eta}(q) \) coincide for different sets of indices. We show that certain simplifying assumptions can be made on the indices, and define another notation that explicitly indicates the connection with LR coefficients.

First, observe that \( K_{\lambda, \gamma, \eta}(q) = 0 \) unless \( |\lambda| = |\gamma| \), where \( |\gamma| := \sum_{i=1}^n \gamma_i \). To see this, consider Bott’s formula for the operator \( \pi \) acting on a Laurent monomial. For \( \alpha \in \mathbb{Z}^n \), let \( \alpha^+ \) be the unique dominant weight in the \( W \)-orbit of \( \alpha \), and let \( w_\alpha \in W \) be the shortest element such that \( w_\alpha \alpha^+ = \alpha \). Then

\[
\pi(x^\alpha) = \begin{cases} 
0 & \text{if } \alpha + \rho \text{ has a repeated part} \\
(-1)^{w_\alpha \rho} s_{(\alpha + \rho)^+ - \rho}(x) & \text{otherwise}
\end{cases}
\]

(2.5)

Every Laurent monomial \( x^\alpha \) in \( x^\gamma B_\eta(x; q) \) satisfies \( |\alpha| = |\gamma| \), and if \( \pi(x^\alpha) \) is nonzero then it equals \( \pm s_\lambda(x) \) where \( |\alpha| = |\lambda| \), proving the assertion.

Next, it may be assumed that the weights \( \lambda \) and \( \gamma \) have nonnegative parts. To see this, let \( \gamma + k \) denote the weight obtained by adding the integer \( k \) to each part of \( \gamma \). Since \( (x_1 \ldots x_n)^k \) is \( W \)-symmetric, it follows that

\[
s_{\lambda+k}(x) = (x_1 x_2 \ldots x_n)^k s_\lambda(x)
\]

\[
H_{\eta, \gamma+k}(x; q) = (x_1 x_2 \ldots x_n)^k H_{\eta, \gamma}(x; q)
\]

\[
K_{\lambda+k, \gamma+k, \eta}(q) = K_{\lambda, \gamma, \eta}(q)
\]

Given the pair \( (\eta, \gamma) \), let \( R = R(\eta, \gamma) = (R_1, R_2, \ldots, R_t) \) be the sequence of weights where \( R_1 \) is the \( GL(\eta_1) \)-weight given by the first \( \eta_1 \) parts of \( \gamma \), \( R_2 \) the \( GL(\eta_2) \)-weight given by the next \( \eta_2 \) parts of \( \gamma \), etc. Then it may be assumed that
$R_i$ is a dominant weight of $GL(\eta_i)$ for each $i$. For this, we require a few properties of the isobaric divided difference operators, whose proofs are easily derived from [19, Chapter II].

Let $I$ be a subinterval of the set $[n] = \{1, 2, \ldots, n\}$ and $\pi_I$ the isobaric divided difference operator indexed by the longest element in the symmetric group on the set $I$. Then

1. $\pi_I f = f$ provided that $f$ is symmetric in the variables $\{x_j : j \in I\}$.
2. $\pi \pi_I \pi = \pi$.

Let $A_1$ be the first $\eta_1$ numbers in $[n]$, $A_2$ the next $\eta_2$ numbers, etc. That is, for $1 \leq i \leq t$.

\begin{equation}
A_i = [\eta_1 + \cdots + \eta_{i-1} + 1, \eta_1 + \eta_2 + \cdots + \eta_i]
\end{equation}

Using the appropriate symmetry properties of $B_\eta(x; q)$, we have

\begin{align*}
H_{\eta, \gamma}(x; q) &= \pi (x^\gamma B_\eta(x; q)) \\
&= \pi \pi_{A_i} (x^\gamma B_\eta(x; q)) \\
&= \pi B_\eta(x; q) \pi_{A_i} x^\gamma.
\end{align*}

By two applications of (2.5) applied to this subset of variables, it follows that $H_{\eta, \gamma}(x; q)$ is either zero, or (up to sign) equal to $H_{\eta, \gamma'}(x; q)$ for some weight $\gamma'$ such that the associated $GL(\eta_i)$-weights $R'_i$ are dominant.

**Remark 3.** To summarize, the polynomial $K_{\lambda, \gamma, \eta}(q)$ is either zero or (up to sign) equal to another such polynomial where

1. $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition with $n$ parts (some of which may be zero);
2. The weight $R_i$ is a partition with $\eta_i$ parts (some of which may be zero) for all $i$;
3. $|\lambda| = \sum_{i=1}^t |R_i|$.

In this situation we introduce an alternative notation

\begin{equation}
K_{\lambda, R}(q) = K_{\lambda, \gamma, \eta}(q)
\end{equation}

where $R$ stands for the sequence of partitions $R = R(\eta, \gamma) = (R_1, R_2, \ldots, R_t)$. From now on we will use either notation as is convenient. Say that $R$ is dominant if $\gamma$ is.

**2.3. Examples.** In each of the following examples, the sequence of partitions $R$ consists entirely of rectangular partitions.

1. (Kostka-Foulkes) Let $\gamma$ and $\lambda$ be partitions of $N$ of length at most $n$ and $\eta = (1^n)$, so that the partition $R_i$ is a single row of length $\gamma_i$. Then

\begin{equation}
K_{\lambda; R}(q) = K_{\lambda, \gamma}(q),
\end{equation}

the Kostka-Foulkes polynomial (defined in [18, III.6]).

2. (Cocharge Kostka-Foulkes) Let $\eta$ be arbitrary, $\gamma = (1^n)$, and $\lambda$ a partition of $n$, so that $R_i = (1^{\eta_i})$ is a single column of length $\eta_i$. Then

\begin{equation}
K_{\lambda; R}(q) = K_{\lambda, \eta^+}(q)
\end{equation}

is the cocharge Kostka-Foulkes polynomial (defined in [18, III.7]), where $\lambda^t$ is the transpose of the partition $\lambda$.

3. (Nilpotent orbit) Let $\eta$ be arbitrary, $k$ a positive integer, $\gamma = (k^n)$, and $\lambda$ a partition of $kn$ with at most $n$ parts, so that $R_i = (k^{\eta_i})$ is a rectangle with $\eta_i$ rows and $k$ columns. Then $K_{\lambda; R}(q)$ is the Poincaré polynomial of the $(\lambda - k)$-th isotypic component of the coordinate ring $\mathbb{C}[X_\mu]$ where $\mu = \eta^+$.
4. (Two column Macdonald-Kostka) Let $2r \leq n$. Stembridge [28] (see also [1]) showed that the two-column Macdonald-Kostka polynomial can be written

$$K_{\lambda,(2r,1^{n-2r})}(q,t) = \sum_{k=0}^{r} q^k \binom{r}{k} M^k_{n-k}(t)$$

where $\binom{r}{k}$ is the usual $t$-binomial coefficient and $M^d_m(t)$ is a polynomial in $t$ defined by the recurrence

$$M^0_m(t) = K_{\lambda,(2m,1^{n-2m})}(t)$$
$$M^1_m(t) = M^d_m(t) - t^{n-2m-1}M^d_{m+1}(t)$$

Using this defining recurrence for $M^d_m(t)$ it can be shown using the methods of [8] that

$$M^d_m(t) = K_{\lambda,(2m,1^{n-2m})}(t),$$

which involves only rectangles with at most two cells. The right hand side of (2.7) satisfies the defining recurrence for $M^d_m(t)$, since the following sequence of modules is exact, where $M^d_m := M(1^{2m},1^{n-m-2d},1,0^{n-2m},(-1)^m)$.

$$0 \to M(1^{2m},1^{n-m-2d},1,0^{2d},1,0^{n-2d-2(m+1)},(-1)^m) \to M^d_m \to M^d_{m+1} \to 0.$$  

The notation $[r]$ indicates a shift in homogeneous degree. The proof is completed by establishing the graded character identity

$$H(1^{2m},1^{n-m-2d},1,0^{2d},1,0^{n-2d-2(m+1)},(-1)^m) = q^{m}H(1^{m+1},2^{d},1^{n-(m+1)-2d},1^{m+1},0^{n-2(m+1)},(-1)^{m+1})$$

When $m = 0$ the exact sequence is an explicit resolution of the ideal of the nilpotent orbit closure $X_{(2d+1,1^{n-2d+1})}$ over the coordinate ring of the minimally larger one $X_{(2d+1,1^{n-2d})}$ [9].

2.4. Positivity conjecture. Broer has conjectured the following sufficient condition that the polynomials $K_{\lambda,\gamma,\eta}(q)$ have nonnegative integer coefficients.

**Conjecture 4.** [8] If $\gamma$ is dominant then in the notation of section 2.4.

$$H^i(G/B, T' \otimes L_\gamma) = 0$$

for $i > 0$. In particular, the polynomial $K_{\lambda,\gamma,\eta}(q)$, being the Poincaré polynomial of an isotypic component of the graded module $M_{r,\gamma}$, has nonnegative integer coefficients.

This was verified by Broer in the case that the vector bundle $\phi_* L_\gamma$ is a line bundle [5]. In our case this means that each of the partitions $R_i$ is a rectangle. We adopt a combinatorial approach to positivity in section 3.8.

**Example 5.** 1. Let $n = 2$, $\lambda = (1,1)$, $\gamma = (0,2)$ and $\eta = (1,1)$. Then $K_{\lambda,\gamma,\eta}(q) = q - 1.$  
2. For $n = 2$, $\lambda = (1,0)$, $\gamma = (0,1)$ and $\eta = (1,1)$, $K_{\lambda,\gamma,\eta}(q) = q$.  
3. Let $n = 3$, $\lambda = (2,1,0)$, $\gamma = (0,2,1)$ and $\eta = (1,1,1)$. Then $K_{\lambda,\gamma,\eta}(q) = q^3 + q^2 - q.$
2.5. \textit{q-Kostant formula.} The following formula is a direct consequence of formulas (2.2), (2.4), and (2.5). Let \( \epsilon_i \) be the \( i \)-th standard basis vector in \( \mathbb{Z}^n \).

**Proposition 6.**

\[
K_{\lambda, \gamma, \eta}(q) = \sum_{w \in \mathcal{W}} (-1)^w \prod_{m \in \text{Roots}_n \rightarrow \mathbb{N}} q^m \sum_{(i,j) \in \text{Roots}_n} m(i,j)
\]

where \( m \) satisfies

\[
\sum_{(i,j) \in \text{Roots}_n} m(i,j) (\epsilon_i - \epsilon_j) = w^{-1}(\lambda + \rho) - (\gamma + \rho)
\]

In the Kostka-Foulkes special case, this is the formula for Lusztig’s \( q \)-analogue of weight multiplicity in type \( A \) [18, Ex. III.6.4].

2.6. \textit{Generalized Morris recurrence.} We now derive a defining recurrence for the polynomials \( K_{\lambda, R}(q) \). In the Kostka-Foulkes case this is due to Morris [20] and in the nilpotent orbit case, to Weyman [27, (6.6)]. Following Remark 3, let us assume that each of the weights \( R_i \) is a partition.

If \( \eta = (\eta_1) \) consists of a single part, then \( \text{Roots}_n \) is the empty set and \( H_{\eta, \gamma}(x; q) = \delta_{\gamma, \eta}(x) \). In other words,

\[
K_{\lambda; (R_i)}(q) = \delta_{\lambda, R_i},
\]

where \( \delta \) is the Kronecker symbol.

Otherwise suppose that \( \eta \) has more than one part. Write \( m = \eta_1 \) and

\[
\begin{align*}
\widehat{\eta} &= (\eta_2, \eta_3, \ldots, \eta_t) \\
\widehat{\gamma} &= (\gamma_{m+1}, \gamma_{m+2}, \ldots, \gamma_n) \\
\widehat{R} &= (R_2, R_3, \ldots, R_t)
\end{align*}
\]

For convenience let \( y_i = x_i \) for \( 1 \leq i \leq m \) and \( z_i = x_{m+i} \) for \( 1 \leq i \leq n - m \). Let \( W_y \) and \( W_z \) denote the subgroups of \( W \) that act only on the \( y \) variables and \( z \) variables respectively. Let \( \pi_x, \pi_y, \) and \( \pi_z \) be the isobaric divided difference operators for the longest element in \( W \), \( W_y \), and \( W_z \) respectively.

Let \( \lambda^* \) denote the highest weight of the contragredient dual \( (V_\lambda)^* \) of \( V_\lambda \). It is given explicitly by \( \lambda^* := w_0(-\lambda) = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) \). We have

\[
s_\lambda(x^*) = s_{\lambda^*}(x)
\]

where \( x^* = (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \). Now

\[
x^* B_\eta(x; q) = y^{R_z} z^{\widehat{\gamma}} B_\eta(z; q) \prod_{1 \leq i \leq m, j \leq n} (1 - q x_i/x_j)^{-1}
\]

\[
= y^{R_z} z^{\widehat{\gamma}} B_\eta(z; q) \prod_{1 \leq j \leq n - m} (1 - q y_i/z_j)^{-1}
\]

\[
= y^{R_z} z^{\widehat{\gamma}} B_\eta(z; q) \sum_\nu q^{\nu} s_\nu(y) s_{\nu^*}(z)
\]

using the definition of \( B_\eta(x; q) \) and Cauchy’s formula. The index variable \( \nu \) runs over partitions of length at most \( \min(m, n - m) \).
Applying the operator $\pi_x = \pi_x \pi_y \pi_z$ to $x^\gamma B_\eta(x; q)$, we have
\[
H_{\eta, \gamma}(x; q) = \pi_x \pi_y \pi_z y^{R_1} z^{\gamma} B_\eta(z; q) \sum_\nu q^{\nu|\nu|} s_\nu(y) s_{\nu^*}(z)
\]
\[
= \pi_x s_{R_1}(y) H_{\hat{\eta}, \hat{\gamma}}(z; q) \sum_\nu q^{\nu|\nu|} s_\nu(y) s_{\nu^*}(z)
\]
\[
= \pi_x s_{R_1}(y) \sum_\nu q^{\nu|\nu|} s_\nu(y) s_{\nu^*}(z) \sum_\sigma K_{\alpha, \beta}(q) s_\sigma(z)
\]
where $\alpha$ runs over the partitions of length at most $m$, $\beta$ and $\sigma$ run over the dominant integral weights with $n - m$ parts, and

\[
(2.9)
\]
\[
LR_{ab}^c = \dim \text{Hom}_{GL(n)}(V_c, V_a \otimes V_b).
\]
are the Littlewood-Richardson coefficients for the dominant integral weights $a$, $b$, and $c$.

Taking the coefficient of $s_\lambda(x)$ on both sides of (2.9) and applying (2.7) we have
\[
(2.10)
\]
\[
K_{\lambda, R}(q) = \sum_{w \in W/(W_a \times W_b)} (-1)^w q^{\alpha(w)|\alpha(w)| - |R_1|} \sum_\sigma K_{\alpha, \beta}(q) s_\sigma(z)
\]
where $w$ runs over the minimal length coset representatives and $\alpha(w)$ and $\beta(w)$ are the first $m$ and last $n - m$ parts of the weight $w^{-1}(\lambda + \rho) - \rho$. The restriction on $w$ is due to the fact that the formula (2.7) is being applied only to Laurent monomials of the form $y^\alpha z^\beta$, where $\alpha$ and $\beta$ are dominant weights having $m$ and $n - m$ parts respectively. The $w$-th summand is understood to be zero unless all the parts of $\alpha(w)$ are nonnegative and $\alpha(w) \geq R_1$. Note that $\beta(w)$ is always a partition since $\lambda$ is. Finally, the LR coefficients can be simplified. Note that
\[
LR_{ab}^c = \dim (V_c^* \otimes V_a \otimes V_b)^{GL(n)}
\]
where $(V_c)^* \cong V_c^*$ is the contragredient dual of $V_c$. Applying duality and the definitions, one obtains
\[
(2.11)
\]
\[
LR_{ab}^c = LR_{ba}^c = LR_{a^* b^*} = LR_{b^* c}^a.
\]
It follows that
\[
(2.12)
\]
\[
\sum_\nu LR_{\alpha(w)}^{\sigma(w)} LR_{\beta(w)}^{\sigma(w)} = \sum_\nu LR_{\alpha(w)}^{\sigma(w)} LR_{\beta(w)}^{\sigma(w)}
\]
where $V_{\lambda/\mu}$ is the $GL(n-m)$-module whose formal character is the skew Schur polynomial $s_{\lambda/\mu}(z)$. Then (2.11) can be expressed as
\[
(2.13)
\]
\[
K_{\lambda, R}(q) = \sum_{w \in W/(W_a \times W_b)} (-1)^w q^{\alpha(w)|\alpha(w)| - |R_1|} \sum_\sigma LR_{\alpha(w)}^{\sigma(w)} R_{\beta(w)}^{\sigma(w)} K_{\alpha, \beta}(q)
\]
Clearly the recurrence (2.13) together with the initial condition (2.8) uniquely defines the polynomials $K_{\lambda;R}(q)$.

### 2.7. $q$-analogue of LR coefficients

Let $\langle , \rangle$ denote the Hall inner product on the ring $\mathbb{C}[x]^W$ of $W$-symmetric polynomials. We employ the notation of Remark 3. The following result does not assume that $R$ is dominant.

**Proposition 7.**

\begin{equation}
K_{\lambda;R}(1) = LR_{\lambda;R} := \langle s_{\lambda}(x), s_{R_1}(x)s_{R_2}(x) \ldots s_{R_t}(x) \rangle
\end{equation}

**Proof.** The proof proceeds by induction on $t$, the number of partitions in $R$. For $t = 1$ the result holds by (2.8). Suppose that $t > 1$. By (2.13) and induction we have

\[
K_{\lambda;R}(1) = \sum_{w \in W/(W_y \times W_z)} (-1)^w \sum_{\sigma} K_{\sigma;\bar{R}}(1) LR_{\alpha(w)/R_1,\beta(w)}
\]

\[
= \sum_{w \in W/(W_y \times W_z)} (-1)^w \sum_{\sigma} \langle s_{\sigma}, s_{R_2}s_{R_3} \ldots s_{R_t} \rangle \langle s_{\sigma}, s_{\alpha(w)/R_1}s_{\beta(w)} \rangle
\]

\[
= \langle s_{R_2}s_{R_3} \ldots s_{R_t}, \sum_{w \in W/(W_y \times W_z)} (-1)^w s_{\alpha(w)/R_1}s_{\beta(w)} \rangle
\]

The Jacobi-Trudi formula for the skew Schur polynomial $s_{\lambda/\mu}$ is given by the determinant

\[
s_{\lambda/\mu} = \det h_{\lambda_i-i-(\mu_j-j)}(x)
\]

Using Laplace’s expansion in terms of $m \times m$ minors involving the first $m$ columns, we have

\[
s_{\lambda/R_1} = \sum_{w \in W/(W_y \times W_z)} (-1)^w s_{\alpha(w)/R_1}s_{\beta(w)}
\]

from which (2.14) follows. 

**Remark 8.** Proposition 7 can be proven using the definition of the module $M_{\eta,\gamma}$ together with an algebraic reciprocity theorem.

### 2.8. Contragredient duality

The following symmetry of the Poincaré polynomial $K_{\lambda,\gamma,\eta}$ is an immediate consequence of the formulas (2.2) and (2.4).

**Proposition 9.**

\begin{equation}
K_{\lambda,\gamma,\eta}(q) = K_{\lambda^{**},\gamma^{**},(\eta_1,\ldots,\eta_t)}(q).
\end{equation}

This gives a $q$-analogue of the “box complement” duality of LR coefficients. Suppose that $\lambda$ and $\gamma$ are partitions and $R = R(\eta,\gamma)$ is the associated sequence of partitions. Let $M$ be a positive integer such that $M \geq \max(\lambda_1,\gamma_1)$. Let $\bar{\lambda}$ be the partition obtained by taking the complement of the diagram of $\lambda$ inside the $n \times M$ rectangle and rotating it 180 degrees. That is, $\bar{\lambda} = \lambda^* + M$ in the notation of section 3. Define $\bar{R}_i$ similarly, except that the latter is complemented inside the $\eta_i \times M$ rectangle. Then

\begin{equation}
K_{\lambda;R}(q) = K_{\lambda;\bar{R}_1,\ldots,\bar{R}_t}(q)
\end{equation}
2.9. Symmetry.

**Proposition 10.** Let \( R = R(\eta, \gamma) \) be dominant and \( R' \) a dominant reordering of \( R \). Then for any \( \lambda \),

\[
K_{\lambda; R}(q) = K_{\lambda; R'}(q).
\]

**Proof.** From the assumptions it follows that \( R' \) can be written \( R' = R(\eta', \gamma) \). The dominance condition implies that \( R' \) is obtained from \( R \) by a sequence of exchanges of adjacent rectangular partitions that have the same number of columns. A series of reductions can be made. It may be assumed that \( R \) and \( R' \) differ by one such exchange, so that \( R' \) is obtained from \( R \) by exchanging \( R_i \) and \( R_{i+1} \), say. If \( i > 1 \) then by the generalized Morris-Weyman recurrence (2.13), the first common partition \( R_1 = R_1' \) may be removed, so that by induction it may be assumed that \( i = 1 \). On the other hand, by applying the “box complement” formula (2.14) it may be assumed that \( R \) and \( R' \) differ by exchanging their last two partitions. As before the common partitions at the beginning may be removed, so that one may assume that \( R = (R_1, R_2) \) and \( R' = (R_2, R_1) \) where \( R_1 \) and \( R_2 \) are rectangles having the same number of columns.

Without loss of generality it may be assumed that \( \eta_1 > \eta_2 \), so that \( \eta = (\eta_1, \eta_2) \) is a partition and \( \eta' = (\eta_2, \eta_1) \). By (2.2) it is enough to show that

\[
H_{\eta,(0,0)}(x; q) = H_{\eta',(0,0)}(x; q).
\]

These are the formal characters of the Euler characteristics of the structure sheafs of the desingularizations \( Z(\eta) \) and \( Z(\eta') \) of \( X_\eta \). By [1] the higher direct images of the corresponding desingularization maps (call them \( q \) and \( q' \) vanish, so

\[
\chi_{\eta,(0,0)} = q_*(\mathcal{O}_{Z_\eta}) = \mathcal{O}_{X_\eta} = q'_*(\mathcal{O}_{Z_{\eta'}}) = \chi_{\eta',(0,0)},
\]

and we are done. The case that \( R \) has two partitions, may also be verified by the explicit formula (3.6) derived below.

This result is not at all obvious when looking directly at the definition of \( H_{\eta, \gamma}(x; q) \) and \( H_{\eta', \gamma}(x; q) \), since the sets of weights Roots\( _\eta \) and Roots\( _{\eta'} \) differ.

2.10. Monotonicity. If \( \mu \) and \( \nu \) are partitions such that \( \mu \succeq \nu \) (that is, \( \mu_1 + \cdots + \mu_i \geq \nu_1 + \cdots + \nu_i \) for all \( i \)) then \( X_\mu \subset X_\nu \), so that restriction of functions gives a natural graded \( G \)-module epimorphism \( \mathbb{C}[X_\nu] \to \mathbb{C}[X_\mu] \). There are similar epimorphisms for the twisted modules \( M_{\mu, \gamma} \). These yield inequalities for the Poincaré polynomials of their isotypic components.

Unfortunately the Poincaré polynomial \( K_{\lambda, \gamma, \eta}(q) \) comes from the Euler characteristic \( \chi_{\eta, \gamma} \) and not not just the term \( M_{\eta, \gamma} \) in cohomological degree zero. So whenever Conjecture 3 holds (for example, for dominant sequences of rectangles), then one has a corresponding inequality for the polynomials \( K_{\lambda, \gamma, \eta}(q) \). Here are two such inequalities.

**Conjecture 11.** Let \( R = R(\eta, \gamma) \) be dominant and \( R' = R(\eta', \gamma) \), where \( \eta' \) is obtained from \( \eta \) by replacing some part \( \eta_i \) by another sequence that sums to \( \eta_i \). Then for any \( \lambda \),

\[
K_{\lambda; R}(q) \leq K_{\lambda; R'}(q)
\]

coefficientwise.
Conjecture 12. Suppose that \( R = R(\eta, \gamma) \) is dominant and \( R' \) is obtained from \( R \) by replacing some subsequence of the form
\[
((k^{\alpha_1}), \ldots, (k^{\alpha_l}))
\]
by
\[
((k^{\beta_1}), \ldots, (k^{\beta_l}))
\]
where \(|\alpha| = |\beta|\) and \(\alpha^+ \succeq \beta^+\). Then for any \( \lambda \),
\[
K_{\lambda; R}(q) \leq K_{\lambda; R'}(q)
\]
coefficientwise.

In the cocharge Kostka-Foulkes case this is a well-known monotonicity property for Kostka-Foulkes polynomials \([4,2]\) that will be discussed at length in section 4.

3. Combinatorics

In this section we give a conjectural combinatorial description (Conjecture 26) of the polynomials \( K_{\lambda; R}(q) \) as well as a general approach for its proof, which is shown to succeed in special cases. These require considerable combinatorial preliminaries. The new material begins in subsection 3.7.

3.1. Tableaux and RSK. We adopt the English convention for partition diagrams and tableaux. The Ferrers diagram of a partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0) \) is the set of ordered pairs of integers \( D(\lambda) = \{(i, j) : 1 \leq j \leq \lambda_i\} \). A skew shape \( \lambda/\mu \) is the set difference \( D(\lambda) - D(\mu) \) of Ferrers diagrams of partitions. A (skew) tableau \( T \) is a function \( T : D \to \mathbb{N}_+ \) from a skew shape \( D \) to the positive integers. The domain of a tableau \( T \) is called its shape. A tableau \( T \) of shape \( D \) is depicted as a partial matrix whose \((i, j)\)-th position contains the value \( T(i, j) \) for \((i, j) \in D\). A tableau is column strict if it weakly increases from left to right within each row and strictly increases from top to bottom within each column. The row-reading word of a (skew) tableau \( T \) is \( \ldots w^2 w^1 \), where \( w^i \) is the word comprising the \( i \)-th row of \( T \), read from left to right. The content of a word \( w \) (or tableau \( T \)) is the sequence \((c_1, c_2, \ldots)\) where \( c_i \) is the number of occurrences of the letter \( i \) in \( w \) (or \( T \)). Say that a word of length \( n \) is standard if it has content \((1^n)\). A tableau whose shape consists of \( n \) cells is said to be standard if it is column strict and has content \((1^n)\).

Example 13. Let \( D = \lambda/\mu \) where \( \lambda = (6, 5, 3, 3) \) and \( \mu = (3, 2) \). A column strict tableau \( T \) of shape \( D \) is depicted below.

\[
T = \begin{array}{cccc}
\times & \times & \times & 1 & 2 & 2 \\
\times & \times & 1 & 3 & 2 & 3 \\
2 & 3 & 3 & & & \\
4 & 4 & 5 & & & 
\end{array}
\]

The row-reading word of \( T \) is
\[
445233123122
\]
and the content of \( T \) is \((2, 4, 3, 2, 1)\).
The Knuth equivalence is the equivalence relation \( \sim_K \) on words that is generated by the relations of the following form, where \( u \) and \( v \) are arbitrary words and \( x, y, \) and \( z \) are letters:

\[
  uxzyv \sim_K uxyzv \quad \text{for} \quad x \leq y < z
\]

\[
  uyxzv \sim_K uyxzv \quad \text{for} \quad x < y \leq z
\]

For the word \( u \), let \( P(u) \) be Schensted’s \( P \)-tableau, that is, the unique column strict tableau of partition shape whose row-reading word is Knuth equivalent to the word \( u \).

We establish some notation for the column insertion version of the Robinson-Schensted-Knuth (RSK) correspondence. For each \( i \geq 1 \), let \( u^i \) be a weakly increasing word (almost all empty). The column insertion RSK correspondence is the bijection from the set of such sequences of words, to pairs of column strict tableaux \( (P,Q) \) of the same shape, defined by \( P = P(\ldots u^2 u^1) \) and \( \text{shape}(Q[i]) = \text{shape}(P(u^i u^{i-1} \ldots u^1)) \) for all \( i \), where \( Q[i] \) denotes the restriction of the tableau \( Q \) to the letters in the set \( [i] \). In other words, to produce \( P \) one performs the column insertion of the word \( \ldots u^2 u^1 \) starting from the right end, and to produce \( Q \) one records the insertions of letters in the subword \( u^i \) by the letter \( i \) in \( Q \). In particular, if \( \alpha \) is the length \( |u^1| \) of the word \( u^i \), then \( Q \) has content \( \alpha \).

### 3.2. Crystal operators

We recall the definitions of the \( r \)-th crystal reflection, raising, and lowering operators \( s_r, e_r, \) and \( f_r \) on words. These are due to Lascoux and Schützenberger. Let \( r \) be a positive integer and \( u \) a word. Ignore all letters of \( u \) which are not in the set \( \{r, r+1\} \). View each occurrence of the letter \( r \) (resp. \( r+1 \)) as a right (resp. left) parenthesis. Perform the usual matching of parentheses. Say that an occurrence of a letter \( r \) or \( r+1 \) in \( u \) is \( r\)-paired if it corresponds to a matched parenthesis. Otherwise call that letter \( r\)-unpaired. It is easy to see that the subword of \( r\)-unpaired letters of \( u \) has the form \( r^p r+1^q \) where \( r^k \) denotes the word consisting of \( p \) occurrences of the letter \( r \).

Consider the three operators \( s_r, e_r, \) and \( f_r \) on words, which are called the \( r \)-th crystal reflection, raising, and lowering operators respectively. Each is applied to the word \( u \) by replacing the \( r \)-unpaired subword \( r^p r+1^q \) of \( u \) by another subword of the same form. Below each operator is listed, together with the subword that it uses to replace \( r^p r+1^q \) in \( u \).

1. For \( s_r u \) use \( r^p r+1^q \). This operator clearly switches the number of \( r \)’s and \( r+1 \)’s in \( u \).
2. For \( e_r u \), use \( r^{p+1} r+1^{q-1} \). This only makes sense if \( q > 0 \), that is, there is an \( r \)-unpaired letter \( r+1 \) in \( u \).
3. For \( f_r u \), use \( r^{p-1} r+1^{q+1} \). This makes sense when \( p > 0 \), that is, there is an \( r \)-unpaired \( r \) in \( u \).

**Example 14.** For \( r = 2 \), the \( r \)-th crystal operators are calculated on the word \( u \).

The \( r \)-unpaired letters are underlined.

\[
  u = 124312234234333131234223
\]

\[
  s_2 u = 12431223422234333131234223
\]

\[
  e_2 u = 124312234234333131234223
\]

\[
  f_2 u = 12431223342343333131234223
\]

(3.1)
These operators may be defined on skew column strict tableaux by acting on the row-reading word. Each produces a column strict tableau of the same shape as the original (skew) tableau. It is proven in [14] that \( s_1, s_2, \ldots \) satisfy the Moore-Coxeter relations as operators on words, so that one may define an action of the infinite symmetric group on words by

\[
wu = s_{i_1}s_{i_2}\ldots s_{i_p}u
\]

where \( u \) is a word, \( w \) is a permutation and \( w = s_{i_1}\ldots s_{i_p} \) is a reduced decomposition of \( w \).

We say that two words (or tableaux) are in the same \( r \)-string if each can be obtained from the other by a power of \( e_r \) or \( f_r \), or equivalently, they differ only in their \( r \)-unpaired subwords.

3.3. Lattice property. Let \( \mu \) be a partition. Say that the word \( u \) is \( \mu \)-lattice if the sum of \( \mu \) and the content of every final subword of \( u \) is a partition. Say that a (skew) column strict tableau is \( \mu \)-lattice if its row-reading word is. Say that a word or tableau is lattice if it is \( \mu \)-lattice when \( \mu \) is the empty partition.

The following formulation of the Littlewood-Richardson rule is well-known. See [5], [24] and [29].

**Theorem 15.** (LR rule) The LR coefficient

\[
\langle \lambda/\mu, \sigma/\tau \rangle
\]

is given by the number of column strict tableaux of shape \( \sigma/\tau \) of content \( \lambda - \mu \) that are \( \mu \)-lattice.

The following is a reformulation of a result of D. White [28].

**Theorem 16.** Let \( \{u^i\} \) be a sequence of weakly increasing words and \((P, Q)\) the corresponding tableau pair under column RSK. Then there is a column-strict tableau of shape \( \lambda/\mu \), whose \( i \)-th row is given by the word \( u^i \) for all \( i \), if and only if the content of \( Q \) is \( \lambda - \mu \) and \( Q \) is \( \mu \)-lattice.

The proof of the following result is straightforward and left to the reader; see also [23] and [26].

**Lemma 17.**

1. A word is \( \mu \)-lattice if and only if the number of \( r \)-unpaired \( r+1 \)'s is at most \( \mu_r - \mu_{r+1} \) for all \( r \).

2. There is an involution on the set of words that are not \( \mu \)-lattice, given by \( u \mapsto s_r e_r^{\mu_r - \mu_{r+1} + 1} u \), where \( r+1 \) is the rightmost letter in \( u \) where \( \mu \)-latticeness fails.

3.4. Evacuation. Let \( T \) be a column strict tableau of content \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and partition shape. The evacuation \( ev_{[n]}(T) \) of \( T \) with respect to the alphabet \([n]\) is the unique column strict tableau in the alphabet \([n]\) such that

\[
\text{shape}(ev_{[n]}(T))_{[\alpha]} = \text{shape}(P(T)_{[n+1-i,n]}).
\]

Clearly \( ev(T) \) has content \((\alpha_n, \ldots, \alpha_1)\) and the same shape as \( T \).

The main result on evacuation is the following.

**Theorem 18.** [14] Let \( \{u^i : 1 \leq i \leq n\} \) be a collection of weakly increasing words in the alphabet \([N]\) and \((P, Q)\) the corresponding pair of tableaux under column RSK. Let \( v^i \) be the reverse of the word obtained from \( u^{n+1-i} \) by complementing
each letter in the alphabet $[N]$. Then the sequence of words $\{v^i\}$ corresponds under column RSK to the tableau pair $(ev_{[N]}(P), ev_{[N]}(Q))$.

3.5. Two row jeux-de-taquin. There is a duality between the crystal operators and jeux-de-taquin on two-row skew column strict tableaux. This is described below.

Define the overlap of the pair $(v, u)$ of weakly increasing words to be the length of the second row in the tableau $P(vu)$, or equivalently, the maximum number of columns of size two among the skew column strict tableaux with first row $u$ and second row $v$.

Lemma 19. Let $(P, Q)$ be the tableau pair obtained by column RSK from the sequence of words $\{v^i\}$. Then the overlap of the pair of words $(v^{r+1}, v^r)$ is equal to the number of $r$-pairs in $Q$.

Proof. (Sketch) In the case $r = 1$ this is easy to check directly. So suppose $r > 1$. Define a new sequence of weakly increasing words $\{w^i\}$ by $w^i = v^i$ for $i \geq r$ and letting $w^i$ be empty for $i < r$. Let $(\hat{P}, \hat{Q})$ be the resulting tableau pair. Then it can be shown (see [24]) that $\hat{Q} = P(Q_{[r,n]})$, that is, $\hat{Q}$ is obtained from $Q$ by removing all letters strictly less than $r$ and then taking the Schensted $P$-tableau. Moreover it is straightforward to show that the number of $r$-pairs is invariant under Knuth equivalence; this is equivalent to the well-known fact (see [24]) that the lattice property is invariant under Knuth equivalence. This reduces the proof to the case $r = 1$.

Lemma 20. Let $\{v^i\}$ and $(P, Q)$ be as in Lemma 19 and let $\{v'^i\}$ be another sequence of weakly increasing words with corresponding tableau pair $(P', Q')$. The following are equivalent.

1. $P = P'$, and $Q$ and $Q'$ are in the same $r$-string.
2. $v'^i = v^i$ for $i \not\in [r, r+1]$ and $P(v'^{r+1}) = P(v^{r+1})$.

Proof. (Sketch) Immediately one may reduce to the case that $r + 1$ is the largest letter in $Q$. If $r = 1$ then the proof is trivial, since any two column strict tableaux of the same partition shape in the alphabet $[1, 2]$ are in the same 1-string. Otherwise suppose $r > 1$. Let $N$ be the largest letter of $P$ and $u^i$ the weakly increasing word given by the reverse of the complement (in the interval $[N]$) of the word $v^{r+2-i}$ for $1 \leq i \leq r + 1$. By Theorem 13 the corresponding tableau pair under column RSK is given by $(ev_{[N]}(P), ev_{[r+1]}(Q))$. It can be shown (see the proof of Lemma 63 in [21]) that

$$ev_{[r+1]} \circ f_i = f_{r+1-i} \circ ev_{[r+1]}$$

$$ev_{[r+1]} \circ e_i = e_{r+1-i} \circ ev_{[r+1]}$$

for any $1 \leq i \leq r$. Setting $i = 1$, the proof may be reduced to the case $r = 1$.

3.6. Charge. Following [14] we review the definition of the charge, an $N$-valued function on words. There are three parts to the definition: words of standard content, partition content, and arbitrary content.

Suppose first that $u$ is a word of content $(1^n)$. Affix an index $c_i$ to the letter $i$ in $u$ according to the rule that $c_1 = 0$ and $c_i = c_{i-1}$ if $i$ appears to the left of $i - 1$ in $u$ and $c_i = c_{i-1} + 1$ if $i$ appears to the right of $i - 1$ in $u$. Let

$$\text{charge}(u) = c_1 + c_2 + \cdots + c_n$$
If \( u \) has partition content \( \mu \), then define
\[
\text{charge}(u) = \text{charge}(u^1) + \text{charge}(u^2) + \cdots
\]
where \( u \) is partitioned into disjoint standard subwords \( u^j \) of length \( \mu^j \) using the following left circular reading. To compute \( u^1 \), start from the right end of \( u \) and scan to the left. Choose the first 1 encountered, then the first 2 that occurs to the left of the selected letter 1, etc. If at any point there is no \( i + 1 \) to the left of the selected letter \( i \), circle around to the right end of \( u \) and continue scanning to the left. This process selects the subword \( u^1 \) of \( u \). Erase the letters of \( u^1 \) from \( u \) and repeat this process, obtaining the subword \( u^2 \). Continue until all the letters of \( u \) have been exhausted.

**Example 21.** The charge is calculated on the word \( u \). The words \( u, u^1, \) and \( u^2 \) appear below.

\[
\begin{align*}
  u &= 4 \ 3 \ 2 \ 3 \ 4 \ 1 \ 1 \ 2 \ 5 \ 5 \\
  u^1 &= 4 \ 3 \ 2 \ 3 \ 4 \ 1 \ 1 \ 2 \ 5 \\
  u^2 &= 4 \ 3 \ 2 \ 3 \ 4 \ 1 \ 2 \ 5
\end{align*}
\]

The charges of the subwords \( u^1 \) and \( u^2 \) are calculated. Each index \( c_i \) is written below the letter \( i \).

\[
\begin{align*}
  4 & 3 2 1 5 \quad 3 4 1 2 5 \\
  0 & 0 0 0 1 \quad 1 2 0 1 3
\end{align*}
\]

So
\[
\text{charge}(u) = \text{charge}(u^1) + \text{charge}(u^2) = 1 + 7 = 8.
\]

Finally, if the word \( u \) has content \( \alpha \), define \( \text{charge}(u) = \text{charge}((w_\alpha)^{-1}u) \) where the permutation \( (w_\alpha)^{-1} \) acts on the word \( u \) as in (3.2).

**Theorem 22.**

\[
K_{\lambda,\mu}(q) = \sum_T q^{\text{charge}(T)}
\]

where \( T \) runs over the set of column strict tableaux of shape \( \lambda \) and content \( \mu \).

The charge has the following intrinsic characterization.

**Theorem 23.** The charge is the unique function from words to \( \mathbb{N} \) such that:

1. For any word \( u \) and any permutation \( w \), \( \text{charge}(wu) = \text{charge}(u) \) where \( wu \) is defined in (3.2).
2. The charge of the empty word is zero.
3. If \( u \) is a word of partition content \( \mu \) of the form \( v1^{\mu_1} \), then
   \[
   \text{charge}(u) = \text{charge}(v)
   \]
   where \( v \) is regarded as a word of partition content \( \mu_2, \mu_3, \ldots \) in the alphabet \( \{2,3,\ldots\} \).
4. Let \( a > 1 \) be a letter and \( x \) a word such that the word \( ax \) has partition content. Then
   \[
   \text{charge}(xa) = \text{charge}(ax) + 1.
   \]
5. The charge is constant on Knuth equivalence classes.

Define the charge of a (skew) column strict tableau to be the charge of its row-reading word.
3.7. Catabolizable tableaux. We now generalize the definition of a catabolizable tableau given in [26], which was inspired by the catabolism construction of Lascoux and Schützenberger [10] [14].

Let \( R = (R_1, R_2, \ldots, R_t) \) be a sequence of partitions determined by the pair \((\eta, \gamma)\) as in Remark 3. Let \( Y_i \) be the tableau of shape \( R_i \) whose \( j \)-th row is filled with the \( j \)-th largest letter of the subinterval \( A_i \). As before we write \( \eta = (m, \hat{\eta}) \).

**Example 24.** Let \( n = 5, \eta = (2, 2, 1), \gamma = (3, 2, 2, 1, 1) \), so that \( m = 2 \) and \( R = ((3, 2), (2, 1), (1)) \). Then \( A_1 = [1, 2], A_2 = [3, 4], \) and \( A_3 = [5, 5] \). The tableaux \( Y_i \) are given by

\[
Y_1 = \begin{array} {c c c}
1 & 1 & 1 \\
2 & 2 &
\end{array} \quad Y_2 = \begin{array} {c c c}
3 & 3 & \\
4 &
\end{array} \quad Y_3 = 5
\]

Given a (possibly skew) column strict tableau \( T \) and index \( r \), let \( H_r(T) = P(T_nT_s) \) where \( T_n \) and \( T_s \) are the north and south subtableaux obtained by slicing \( T \) horizontally between its \( r \)-th and \( (r+1) \)-st rows.

Let \( S \) be a column strict tableau of partition shape in the alphabet \([n] \). Suppose the restriction \( S|_{A_1} \) of \( S \) to the subalphabet \( A_1 \) is the tableau \( Y_1 \). In this case the \( R_1 \)-catabolism of \( S \) is defined to be the tableau \( \text{cat}_{R_1}(S) = H_m(S - Y_1) \). The notion of a \( R \)-catabolizable tableau is uniquely defined by the following rules.

1. If \( R \) is the empty sequence, then the unique \( R \)-catabolizable tableau is the empty tableau.
2. Otherwise, \( T \) is \( R \)-catabolizable if and only if \( T|_[m] = Y_1 \) and \( \text{cat}_{R_1}(T) \) is \( \hat{R} \)-catabolizable in the alphabet \([m+1, n] \).

Denote by \( CT(\lambda; R) \) the set of \( R \)-catabolizable tableaux of shape \( \lambda \).

**Example 25.** Continuing the previous example, let \( \lambda = (5, 3, 1, 0, 0) \). The four \( R \)-catabolizable tableaux of shape \( \lambda \) are:

\[
\begin{array} {c c c c c}
1 & 1 & 1 & 3 & 4 \\
2 & 2 & 5 &
\end{array} \quad \begin{array} {c c c c c}
1 & 1 & 1 & 3 & 3 \\
2 & 2 & 4 &
\end{array} \quad \begin{array} {c c c c c}
1 & 1 & 1 & 4 & 5 \\
2 & 2 & 3 &
\end{array} \quad \begin{array} {c c c c c}
1 & 1 & 1 & 3 & 5 \\
2 & 2 & 4 &
\end{array} \quad \begin{array} {c c c c c}
3 & 5 &
\end{array}
\]

Let \( S \) be the last tableau. It is shown to be \( R \)-catabolizable as follows.

\[
\begin{array} {c c c c c}
1 & 1 & 1 & 3 & 5 \\
2 & 2 & 4 &
\end{array} \quad \times \times \times 3 5 \quad \times \times \times S = \begin{array} {c c c c c}
1 & 1 & 1 & 3 & 5 \\
2 & 2 & 4 &
\end{array} \quad S_n = \times \times 4 \quad S_s = \times \times
\]

\[
\text{cat}_{R_1}(S) = P(435 3) = \begin{array} {c c c}
3 & 3 &
\end{array} \quad \begin{array} {c c c}
4 & 5 &
\end{array}
\]

Now \( \text{cat}_{R_1}(S) \) contains \( Y_2 \) and \( \text{cat}_{R_2} \text{cat}_{R_1}(S) = 5 \). The latter tableau contains \( Y_3 \) and \( \text{cat}_{R_3} \text{cat}_{R_2} \text{cat}_{R_1}(S) \) is the empty tableau. Thus \( S \) is \( R \)-catabolizable.

It is clear that any \( R \)-catabolizable tableau has content \( \gamma \).

**Conjecture 26.** For \( R \) dominant,

\[
K_{\lambda; R}(q) = \sum_S q^{\text{charge}(S)}
\]

where \( S \) runs over the set of \( R \)-catabolizable tableaux of shape \( \lambda \).
Example 27. For the previous example, \(K_{\lambda; R}(q) = q^3 + 3q^4\). This can be verified by computing the polynomial two ways: 1) using the recurrence (2.13) and 2) evaluating the charge statistic on the four tableaux above. The charges of the tableaux are (in order) 3, 4, 4, 4.

We prove Conjecture 26 in the following cases.
1. \(\eta\) is a hook partition, that is, \(\eta_i = 1\) for \(i > 1\). An important subcase is the Kostka-Foulkes case, where \(\eta = (1^n)\) and \(R_i\) is a single row of length \(\gamma_i\). Then an \(R\)-catabolizable tableau is simply a column strict tableau of content \(\gamma\), and Conjecture 26 reduces to a theorem of Lascoux and Schützenberger [13]. Even in this special case our proof differs from that of Lascoux and Schützenberger, whose gaps were bridged by Butler [3].
2. In the cocharge Kostka-Foulkes case, where \(R_i\) is a single column of length \(\eta_i\) and \(\eta\) is a partition, Conjecture 26 reduces to a formula that is very nearly the same as one of Lascoux [10] for the cocharge Kostka-Foulkes polynomials. However the equivalence of the combinatorial definitions of the corresponding sets of standard tableaux can only be resolved with considerable effort. This is done in section 3.3.
3. \(\eta\) has two parts.

3.8. Proof strategy for Conjecture 26. We give a general approach for a proof of Conjecture 26 using a sign-reversing involution. For this purpose it is convenient to modify the recurrence (2.13) to include more terms. The complete expansion of the Jacobi-Trudi determinant for the skew Schur polynomial \(s_{\lambda/R, 1}(q)\) yields

\[
K_{\lambda; R}(q) = \sum_{w \in W} (-1)^w q^{(|\alpha(w)| - |R_1|)} \sum_{\sigma} K_{\sigma; (\alpha(w) - R_1, \beta(w))} K_{\sigma; \hat{R}}(q)
\]

where \(\alpha(w)\) and \(\beta(w)\) are the first \(m\) and last \(n - m\) parts of the weight

\[
\xi(w) := w^{-1}(\lambda + \rho) - \rho
\]

and \(K_{\lambda, \alpha}\) is the Kostka number [13, I.6], the number of column strict tableaux of shape \(\lambda\) and content \(\alpha\).

We now combinatorialize both sides of (3.3). Let \(S'\) be the set of triples \((w, T, U)\) where \(w \in W\) and \(T\) and \(U\) are column strict tableaux of the same partition shape where \(T\) has content \((0^m, \tilde{\gamma})\) and \(U\) has content \((\beta(w), \alpha(w) - R_1)\). Let \(S \subset S'\) be the subset of triples \((w, T, U)\) such that \(T\) is \(\hat{R}\)-catabolizable in the alphabet \([m + 1, n]\). The reordering of parts of the content of \(U\) is justified since the Kostka number is symmetric in its second index. Define a sign and weight on \(S'\) by

\[
\text{sign}(w, T, U) = (-1)^w \quad \text{weight}(w, T, U) = q^{(|\alpha(w)| - |R_1|) + \text{charge}(T)}
\]

where \(T\) is regarded as a tableau of partition content in the alphabet \([m + 1, n]\). By induction the right hand side of (3.3) is given by

\[
\sum_{(w, T, U) \in S} \text{sign}(w, T, U) \text{weight}(w, T, U)
\]

Example 28. Let \(n = 8\), \(\eta = (2, 2, 2, 1, 1)\), and \(\gamma = (3^8)\), so that \(m = 2\) and

\[
R = ((3, 3), (3, 3), (3, 3), (3), (3))
\]

\[
\hat{R} = ((3, 3), (3, 3), (3), (3)).
\]
Let \( w = 32154678 \) and \( \lambda = (6, 5, 5, 5, 2, 1, 0, 0) \) so that \( \xi(w) = (3, 5, 8, 1, 6, 1, 0, 0) \), \( \alpha(w) = (3, 5) \) and \( \beta(w) = (8, 1, 6, 1, 0, 0) \). Let \( T \) and \( U \) be given by

\[
T = \begin{array}{cccccccc}
3 & 3 & 3 & 5 & 6 & 7 & 7 & 7 \\
4 & 4 & 4 & 6 & 8 & 8 & 8 & 8 \\
5 & 5 & 8 & 8 & 8 & 8 & 8 & 8 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
U = \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

Example 29. Continuing the above example, the words \( u \) that the length of \( u \) has content \( (\lambda; R) \). Note that the contents of \( P \) and \( Q \) are \( \gamma \) and \( \xi(w) \) respectively.

Example 30. In the above example we have \( v^1 = 111, v^2 = 22256, \) and \( v^i = u^i \) for \( 2 < i \leq n \).

Example 31. The tableaux \( P \) and \( Q \) are given by

\[
P = \begin{array}{cccccccc}
1 & 1 & 1 & 5 & 5 & 6 & 7 & 7 \\
2 & 2 & 2 & 6 & 6 & 7 & 7 & 7 \\
3 & 3 & 3 & 8 & 8 & 8 & 8 & 8 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
Q = \begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
2 & 2 & 2 & 3 & 3 & 5 & 5 & 5 \\
3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
\]

Example 30. In the above example we have \( v^1 = 111, v^2 = 22256, \) and \( v^i = u^i \) for \( 2 < i \leq n \).

Finally, let \( (P, Q) \) be the pair of tableaux given by the image under column RSK, of the sequence of words \( \{v^i\} \), so that \( P(v^n v^{n-1} \ldots v^1) = P \). Note that the contents of \( P \) and \( Q \) are \( \gamma \) and \( \xi(w) \) respectively.
Define the map \( \Phi : S' \to \Phi(S') \) given by \( \Phi(w, T, U) = (w, P, Q) \). It is clear from the definitions that \( \Phi \) is injective. Define a sign and weight on \( \Phi(S') \) by \( \text{sign}(w, P, Q) = (-1)^w \) and \( \text{weight}(w, P, Q) = q^{\text{charge}(P)} \). By definition \( \Phi \) is sign-preserving. We now show that \( \Phi \) preserves weight.

**Lemma 32.** In the above notation,
\[
\text{charge}(P) = \text{charge}(T) + |\alpha(w)| - |R_1|
\]
provided that \( \gamma \) is a partition.

**Proof.** Using Theorem 23 we compute the charge of \( P \).
\[
\text{charge}(P) = \text{charge}(u^n v^{n-1} \ldots v^1)
= \text{charge}(u^n u^{n-1} \ldots u^{m+1} m^{\gamma_m} u^m (m-1)^{\gamma_{m-1}} u^{m-1} \ldots 1^{\gamma_1} u^1)
= |u^1| + \text{charge}(u^1 u^n u^{n-1} \ldots u^{m+1} m^{\gamma_m} u^m (m-1)^{\gamma_{m-1}} u^{m-1} \ldots 2^{\gamma_2} u^2)
= |u^2| + |u^1| + \text{charge}(u^2 u^1 u^n \ldots u^{m+1} m^{\gamma_m} u^m \ldots 3^{\gamma_3} u^3)
= \vdots
= |u^n| + \ldots + |u^1| + \text{charge}(u^m u^1 u^n \ldots u^{m+1})
= |\alpha(w)| - |R_1| + \text{charge}(T).
\]

We define a sign-reversing, weight-preserving involution \( \theta' \) on the set \( \Phi(S') \). Let \( (w, P, Q) \in \Phi(S') \). There are two cases.

1. \( Q \) is lattice. It follows that \( Q \) has partition content and \( w \) is the identity. Define \( \theta'(w, P, Q) = (w, P, Q) \).

2. \( Q \) is not lattice. Let \( r + 1 \) be the rightmost letter in the row-reading word of \( Q \) that violates the lattice condition. Define \( \theta'(w, P, Q) = (w', P', Q') \) where \( w' = ws_r, P' = P, \) and \( Q' = |_se_rQ \).

**Example 33.** In computing \( \theta'(w, P, Q) = (w', P', Q') \), the first violation of latticeness in the row-reading word of \( Q \) occurs at the cell \((1, 8)\) so \( r = 2 \). We have \( w' = 31254678, P' = P \) and
\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 \\
2 & 2 & 2 & 3 & 3 & 5 \\
3 & 3 & 3 & 5 & 5 \\
4 & 5 & 5 \\
5 & \\
6 \\
\end{array}
\]

\( Q' \)

**Lemma 34.** \( \theta' \) is a sign-reversing, weight-preserving involution on the set \( \Phi(S') \).

**Proof.** By definition \( \theta' \) is sign-reversing and weight-preserving. By Lemma 17 \( \theta' \) is an involution. It remains to show that \( \theta' \) stabilizes the set \( \Phi(S') \). Let \( (w, P, Q) \in \Phi(S') \) and \( \theta'(w, P, Q) = (w', P', Q') \). We may assume \( (w, P, Q) \) is not a fixed point of \( \theta' \). Let \( v' \) be to \((P', Q')\) as \( v \) is to \((P, Q)\) in the definition of \( \Phi \). It is enough to show that \((v')^i\) starts with the subword \( i^\gamma_i \) for every \( 1 \leq i \leq m \), since the other steps in the map \( \Phi \) are invertible by definition.
Since $Q$ and $Q' = s_r e_r(Q)$ are in the same $r$-string, Lemma 20 applies. There is nothing to prove unless $r \leq m$. Suppose first that $r < m$. We need only check that $v' r$ starts with $r v^r$ and $v' r + 1$ starts with $(r + 1) v^r + 1$. Since $v^r = r v^r u^r$ and $v'^r + 1 = (r + 1) v^r + 1 u^r + 1$ where all the letters of $u^r$ and $u^r + 1$ are strictly greater than $m$, it follows that

$$P(v' r + 1 r | [r, r + 1]) = P(v' r + 1 | [r, r + 1]) = P((r + 1) v^r + 1 r v^r)$$

This, together with the fact that $v^r + 1$ and $v'^r$ are weakly increasing words, implies that all of the letters $r + 1$ must precede all of the letters $r$ in the word $v' r + 1 v' r$, that is, $v' r$ starts with $i v^r$ for $i \in \{r, r + 1\}$.

The remaining case is $r = m$. Then $v^r = r v^r u^r$ and $v'^r + 1 = u^r + 1$. Let us calculate $v^r + 1$ and $v'^r$ using a two-row jeu-de-taquin. Let $R$ (resp. $R'$) be the (skew) two-row tableau with first row $v^r$ (resp. $v'^r$) and second row $v^r + 1$ (resp. $v'^r + 1$) in which the two rows achieve the maximum overlap. The overlaps of $R$ and $R'$ are equal by Lemma 19 and the fact that $Q$ and $Q'$ are in the same $r$-string and hence have the same $r$-paired letters. Furthermore this common overlap is at least $\gamma_r$. To see this, note that the overlap weakly exceeds the minimum of $\gamma_r$ and $|u^r + 1|$ since all of the letters in $u^r + 1$ have values in the alphabet $[m + 1, n] = [r + 1, n]$ and there are $\gamma_r$ copies of $r$ in $u^r$. On the other hand, $|u^r + 1| > \gamma_r$, for otherwise by Lemma 19 all of the letters $r + 1$ in $Q$ would be $r$-paired, contradicting the choice of $r$.

We calculate $R'$ from $R$ in two stages. Let $R''$ be the two-row skew tableau (whose rows have maximum overlap) such that $P(R'') = P(R)$, where the first row of $R''$ is one cell longer than that of $R$. By Lemma 21 this tableau exists since $Q$ has an $r$-unpaired letter $r + 1$; the corresponding recording tableau is $e_r Q$. Furthermore $R''$ is obtained by sliding the “hole” in the cell just to the left of the first letter in the first row of $R$, into the second row. By the same reasoning as above, $R''$ has the same overlap that $R$ does. Finally we calculate $R'$ from $R''$ by another two-row jeu-de-taquin. If the first row of $R''$ is shorter than the second, we are done, for in this case the first row $v'' r$ of $R'$ contains the first row of $R''$, which in turn contains the first row of $R$, which contains $r v^r$. So suppose the second row of $R''$ is shorter than the first, by $p$ cells, say. Now $p$ is less than or equal to the number of cells on the right end of the first column of $R''$ that have no cell of $R''$ below them. Since $R''$ has maximum overlap it follows that when $p$ holes are slid from the second row of $R''$ to the first, they all exchange with numbers lying in the portion of the first row of $R''$ that extends properly to the right of the second. Thus the subword $r v^r$ remains in the first row of $R'$, and we are done.

**Example 35.** In $v'$ all the subwords are the same as in $v$ except that $v_2' = 2225677$ and $v_3' = 333567$. In this example $r = m$. The tableaux $R$, $R''$, and $R'$ are given below.

$$R = \begin{array}{c|c|c|c|c|c|c|c|c|c} \times & \times & \times & 2 & 2 & 2 & 5 & 6 \\ 3 & 3 & 3 & 5 & 6 & 7 & 7 & 7 \end{array}$$

$$R'' = \begin{array}{c|c|c|c|c|c|c|c|c|c} \times & \times & 2 & 2 & 2 & 5 & 6 & 7 \\ 3 & 3 & 3 & 5 & 6 & 7 & 7 \end{array}$$

$$R' = \begin{array}{c|c|c|c|c|c|c|c|c|c} 2 & 2 & 2 & 5 & 6 & 7 & 7 \\ 3 & 3 & 3 & 5 & 6 & 7 \end{array}$$
Thus we may define a sign-reversing, weight-preserving involution \( \theta \) on \( S' \) by \( \theta = \Phi^{-1} \circ \theta' \circ \Phi \). By definition the fixed points of \( \theta' \) are the triples \((w, P, Q)\) where \( Q \) is the unique column strict tableau of shape and content \( \lambda \), \( w \) is the identity, and \( P \) is a column strict tableau of content \( \gamma \) such that \( P|_{A_1} = Y_1 \).

**Lemma 36.** Conjecture \[\text{Corollary 24}\] holds provided that the involution \( \theta' \) stabilizes the subset \( \Phi(S) \) of \( \Phi(S') \).

**Proof.** Suppose that \( \theta' \) stabilizes \( \Phi(S) \). Equivalently, the involution \( \theta \) stabilizes \( S \). Since \( \Phi \) is sign and weight-preserving, the generating function of the fixed points \( S^\theta \) of the restriction of \( \theta \) to \( S \) is the same as that of the set \( \Phi(S)^\theta \). But it is easy to see that \( S^\theta \) is precisely the triples \((id, P, Q)\) where \( Q \) is the unique column strict tableau of shape and content \( \lambda \) and \( P \) is \( R \)-catabolizable. \( \square \)

In fact, the proof of Conjecture \[\text{Corollary 26}\] reduces to the case of a non-fixed point \((w, T, U)\) with \( r = m \).

**Lemma 37.** In the notation of Lemma \[\text{Lemma 36}\] and the definition of \( \theta \), if \( r \neq m \) then \( T' = T \).

**Proof.** Let \( u' \) be to \((T', U')\) as \( u \) is to \((T, U)\) in the definition of \( \Phi \). It follows from Lemma \[\text{Lemma 24}\] that \( u'^i = u^i \) for \( i \not\in [r, r+1] \) and \( P(u'^r) = P(u^r) \). Since \( r \neq m \) we have

\[
P(u^m \ldots u^1) = P(u^m \ldots u^1) \\
P(u^m \ldots u^{m+1}) = P(u^m \ldots u^{m+1})
\]

which implies that

\[
T' = P(u^m \ldots u^1 u^m \ldots u^{m+1}) \\
= P(u^m \ldots u^1 u^m \ldots u^{m+1}) = T
\]

\( \square \)

We believe that \( T' \) is always \( \tilde{R} \)-catabolizable, but can only prove it in the following cases, where the definition of catabolizability becomes quite simple. The following result is an immediate consequence of the definitions.

**Proposition 38.** Let \( \eta \) be a hook partition (that is, \( \eta_i = 1 \) for \( i > 1 \)). Then \( S \) is \( R \)-catabolizable if and only if \( S|_{A_1} = Y_1 \) and \( S \) has content \( \gamma \).

**Corollary 39.** Conjecture \[\text{Corollary 26}\] holds when \( \eta = (m, 1^{n-m}) \).

**Proof.** By Proposition \[\text{Lemma 38}\] the tableau \( T \) is \( \tilde{R} \)-catabolizable in the alphabet \([m+1, n]\) if and only if \( T \) has content \((0^n, \gamma)\). In this case the sets \( S \) and \( S' \) coincide. \( \square \)

**Corollary 40.** Conjecture \[\text{Corollary 26}\] holds when \( \eta \) has two parts.

**Proof.** Consider the first recurrence \[\text{(2.13)}\], which by \[\text{(2.3)}\] takes the form

\[
K_{\lambda,(R_1,R_2)}(q) = \sum_{w \in W/(W_y \times W_z)} (-1)^w q^{\alpha(w) - |R_1|} \sum_{\sigma} LR_{\alpha(w)/R_1, \beta(w)}^\sigma \delta_{R_2} R_2
\]

\[
= \sum_{w \in W/(W_y \times W_z)} (-1)^w q^{\alpha(w) - |R_1|} LR_{\alpha(w)/R_1, \beta(w)}^R_2 R_2
\]

where \( \alpha(w) \) is the content of \( w \) and \( |R_1| \) is the length of \( R_1 \).
Suppose that the \(w\)-th term is nonzero and \(w\) is not the identity. Necessarily \(\alpha(w) \supseteq R_1\). By examining the Jacobi-Trudi determinant for \(s_{\lambda/R_1}\), it must be the case that \(\lambda_{m+1} > \gamma_m\). On the other hand, we have \(\beta(w)_1 > \beta(id)_1 = \lambda_{m+1}\). Putting the inequalities together, \(\beta(w)_1 > \gamma_m \geq \gamma_{m+1}\) by the dominance of the weight \(\gamma\). But \(\gamma_{m+1}\) is the first part of the partition \(R_2\). This means that \(R_2\) cannot contain \(\beta(w)\), so that the LR coefficient vanishes, contradicting our assumption on \(w\).

Equation \((3.5)\) now takes the form
\[
K_{\lambda,(R_1,R_2)}(q) = q^{|\alpha|-|R_1|}LR_{\alpha/R_1,\beta}^R_2
\]
where \(\alpha = \alpha(id)\) and \(\beta = \beta(id)\) are the first \(m\) and last \(n-m\) parts of \(\lambda\) respectively.

Note that the skew shape \(\lambda/R_1\) consists of two disconnected skew shapes, namely \(\alpha/R_1\) and \(\beta\). Let \(P\) be an \(R\)-catabolizable tableau of shape \(\lambda\) and \(Q\) the unique column strict tableau of shape and content \(\lambda\). Let \(T\) and \(U\) be the tableaux such that \(\Phi(id,T,U) = (id,P,Q)\). For \((id,P,Q)\) to be in the image of \(\Phi\), it must be shown that \(v^i\) starts with the subword \(i^{n_i}\) for \(1 \leq i \leq m\); but this holds since \(v^i\) is the \(i\)-th row of the tableau \(P\), and \(P|\lambda_1 = Y_1\) by the \(R\)-catabolizability of \(P\). By Lemma \((2)\), charge\((P) = |\alpha| - |R_1|\), since \(T = Y_2\) by the \(R\)-catabolizability of \(P\) and the fact that Yamanouchi tableaux have zero charge. It follows from Theorem \((1)\) that the map \(P \mapsto U\) is a bijection from \(CT(\lambda;R)\) to a set of tableaux of cardinality \(LR_{R_1,\beta}^R\). The result follows.

4. An Application of Catabolizability

We prove a beautiful formula for the cocharge Kostka-Foulkes polynomials stated by Lascoux \([10]\). The proof also settles Conjecture \((2)\) in the case that \(\gamma = (1^t)\) and \(\eta\) is a partition, that is, when all rectangles are single columns of weakly decreasing heights. This entails a deeper study of catabolizability.

4.1. Lascoux’ Formula. For the pair of partitions \(\lambda\) and \(\mu\), the cocharge Kostka-Foulkes polynomial \(\bar{K}_{\lambda,\mu}(q)\) is defined by
\[
\bar{K}_{\lambda,\mu}(q) = q^{n(\mu)}K_{\lambda,\mu}(q)
\]
where \(n(\mu) = \sum_i (i-1)\mu_i\). For a combinatorial description, define the cocharge of a word \(u\) of partition content \(\mu\) by
\[
\text{cocharge}(u) = n(\mu) - \text{charge}(u).
\]
Then Theorem \((2)\) can be rephrased as
\[
\bar{K}_{\lambda,\mu}(q) = \sum_T q^{\text{cocharge}(T)}
\]
where \(T\) runs over the set of column strict tableaux of shape \(\lambda\) and content \(\mu\).

These polynomials satisfy the monotonicity property
\[
\bar{K}_{\lambda,\mu}(q) \leq \bar{K}_{\lambda,\nu}(q)
\]
coefficientwise provided that \(\mu \supseteq \nu\).

To exhibit a combinatorial proof of the inequality \((2.2)\), Lascoux associates to each standard tableau \(S\) a partition cattype\((S)\) and asserts that
\[
\bar{K}_{\lambda,\mu}(q) = \sum_S q^{\text{cocharge}(S)}
\]
where \(S\) runs over the set of standard tableaux of shape \(\lambda\) and cattype\((S) \supseteq \mu\).
Define the tableau operator \( \text{Cat}(S) = H_1(S) \), which may be computed by column-inserting the first row of \( S \) into the remainder of \( S \). Let \( d_1(S) \) be the maximum number \( i \) such that the numbers 1 through \( i \) are all in the first row of \( S \). Then \( \text{cattype}(S) \) is the sequence of integers whose first part is \( d_1(S) \) and whose \( i \)-th part is given by \( d_1(\text{Cat}^i(S)) - d_1(\text{Cat}^{i-1}(S)) \). It can be shown that \( \text{cattype}(S) \) is a partition.

**Example 41.** The powers of \( \text{Cat} \) on the standard tableau \( S \) are given below.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 7 \\
S & = & 5 & 6 & 9 \\
8 & & & & & \\
\end{array}
\quad
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 9 \\
\text{Cat}(S) & = & & & & \\
& & & & & \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\text{Cat}^2(S) & = & & & & & & \\
& & & & & & & \\
\end{array}
\quad
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{Cat}^3(S) & = & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

So the sequence of \( d_1 \) of the powers of \( \text{Cat} \) on \( S \) is \((4, 6, 8, 9)\) and \( \text{cattype}(S) = (4, 2, 2, 1) \).

Let \( \alpha \) be a sequence of nonnegative integers and \( T(\alpha) \) the set of column strict tableaux of content \( \alpha \) and arbitrary partition shape. To prove Lascoux’s formula, in light of (4.1) and (4.3) it clearly suffices to exhibit an embedding \( \theta_\mu : T(\mu) \rightarrow T((1^n)) \) (where \( n = |\mu| \)) that is shape- and cocharge-preserving, and satisfies the additional property that \( S \) is in the image of \( \theta_\mu \) if and only if \( \text{cattype}(S) \succeq \mu \).

Lascoux defined such an embedding but did not give a proof of the characterization of the image of \( \theta_\mu \). We supply a proof of this last fact.

**4.2. Cyclage and canonical embeddings.** The embeddings \( \theta_\mu \) are best understood in terms of the cyclage poset structure [10] [14]. The cyclage is the covering relation of a graded poset structure on \( T(\alpha) \). It is defined as follows.

Suppose first that \( \alpha \) is a partition \( \mu \). For two tableaux \( T \) and \( S \) in \( T(\mu) \), say that \( T \) covers \( S \) if there exists a letter \( a > 1 \) and a word \( x \) such that \( P(ax) = T \) and \( P(xa) = S \). Equivalently, \( T \) covers \( S \) if there is a corner cell \( s \) of \( T \) such that the reverse column insertion on \( T \) at \( s \) results in a letter \( a > 1 \) and a tableau \( U \), and the row insertion of the letter \( a \) into \( U \) produces \( S \).

The above relation (called the cyclage) is the covering relation of a partial order since \( \text{cocharge}(T) = \text{cocharge}(S) + 1 \) when \( T \) covers \( S \), by Theorem [23].

**Example 42.** Starting with the tableau \( T \) and an underlined cell \( s \), the pair \((x, U)\) and the tableau \( S \) are computed.

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 & 3 \\
T & = & 2 & 3 & 4 \\
& & & & & \\
& & & & & \\
\end{array}
\quad
\begin{array}{cccc}
x & = & 2 \\
& & & & \\
& & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 1 & 2 & 3 \\
U & = & 3 & 4 \\
& & & & \\
& & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 1 & 2 & 2 \\
S & = & 3 & 3 \\
4 & 4 & & \\
\end{array}
\]

Now let us return to the case of arbitrary content \( \alpha \). Let \( \mu = \alpha^+ \) and \( w = (w_\alpha)^{-1} \) the permutation such that \( w\alpha = \mu \). Then \( w \) defines a bijection \( T(\alpha) \rightarrow T(\mu) \) given by \( T \mapsto wT \) (see [12]) that is shape-preserving.

Say that \( T \succeq S \) for \( T, S \in T(\alpha) \) if \( wT \succeq wS \), that is, the partial order on \( T(\alpha) \) is defined to make \( w \) into a poset isomorphism.
Theorem 43. \([4]\) The cyclage endows \(T(\alpha)\) with the structure of a graded poset with grading given by cocharge. The unique bottom element of \(T(\alpha)\) is the one-row tableau of content \(\alpha\).

The family of posets \(\{T(\alpha)\}\) for varying \(\alpha\), is equipped with functorial grade-preserving embeddings. For two compositions \(\alpha\) and \(\beta\), say that \(\alpha \geq \beta\) if \(\alpha^+ \geq \beta^+\). Suppose \(\alpha \geq \beta\). There is an embedding of posets \(\theta_\alpha^\beta : T(\alpha) \to T(\beta)\) that can be defined as follows.

First, if \(\alpha^+ = \beta^+\), choose any \(w \in W\) such that \(w\alpha = \beta\) and let \(\theta_\alpha^\beta = w\).

Second, if \(\beta_i = \alpha_i\) for all \(i > 2\), \(\beta_1 = \alpha_1 - 1\) and \(\beta_2 = \alpha_2 + 1\) where \(\alpha_1 > \alpha_2 + 1\), then let \(\theta_\alpha^\beta = f_1\), the crystal lowering operator, which in this case merely changes the rightmost letter 1 in a tableau to a 2.

Now let \(\alpha \geq \beta\). Then there is a sequence \(\alpha = \gamma^0, \gamma^1, \ldots, \gamma^p = \beta\) of compositions, where either \((\gamma^i)^+ = (\gamma^{i+1})^+\) or \(\gamma^{i+1} = \gamma^i + (-1, 1, 0, \ldots)\). Define

\[
\theta_\alpha^\beta = \theta_{\gamma^{p-1}} \circ \cdots \circ \theta_{\gamma^1}.
\]

Theorem 44. \([10]\) Suppose \(\alpha \geq \beta\). The map \(\theta_\alpha^\beta\) is independent of the sequence of compositions \(\{\gamma^i\}\) and is shape-preserving and an embedding of graded posets. Furthermore if \(\beta \geq \gamma\) then \(\theta_\alpha^\gamma = \theta_\beta^\alpha \circ \theta_\alpha^\beta\).

For \(|\alpha| = n\), denote by \(\theta_\alpha = \theta_\alpha^{(1^n)}\) the embedding of \(T(\alpha)\) into the standard tableaux \(T((1^n))\). Lascoux gives the following characterization of the image of \(\theta_\mu\).

Theorem 45. Let \(\mu\) be a partition of \(n\). Then the image of \(\theta_\mu\) is the set of standard tableaux \(S\) such that \(\text{cattype}(S) \geq \mu\).

4.3. Proof of Theorem 43. Our proof uses several reformulations of the condition \(\text{cattype}(S) \geq \mu\). Let \(Z_m\) be the one-row standard tableau given by the numbers from 1 to \(m\). Suppose the standard tableau \(S\) contains \(Z_m\). Define \(\text{Cat}_m(S) = H_1(S - Z_m)\). Write \(\mu = (m, \bar{\mu})\). The following definition is not consistent with the previous notion of \(R\)-catabolizability for any \(R\) since it slices the tableau in the wrong direction. Say that \(S\) is \(\mu\)-catabolizable if \(S\) and \(\mu\) are both empty, or if \(S\) contains \(Z_m\) and \(\text{Cat}_m(S)\) is \(\bar{\mu}\)-catabolizable in the alphabet \([m+1, n]\).

Proposition 46. \(S\) is \(\mu\)-catabolizable if and only if \(\text{cattype}(S) \geq \mu\).

Proof. This is trivial if \(\mu\) has one part. So let \(\mu = (m, \bar{\mu})\). We may assume that \(S\) contains \(Z_m\) for otherwise both conditions on \(S\) fail. From the definitions one has \(\text{Cat}(S) = P(Z_m\text{Cat}_m(S))\). In particular, since all the letters of \(Z_m\) are smaller than all of those in \(\text{Cat}_m(S)\), one obtains \(\text{Cat}(S)\) from \(\text{Cat}_m(S)\) by pushing the first row to the right by \(m\) cells and placing \(Z_m\) in the vacated positions. In view of this, it is clear that \(d_i(\text{Cat}^{(\text{Cat}_m(S))}) = d_i(\text{Cat}^{i+1}(S))\) for all \(i\). So if we write \(\text{cattype}(S) = \nu\) then \(\text{cattype}(\text{Cat}_m(S)) = (\nu_1 - \mu_1 + \nu_2, \nu_3, \nu_4, \ldots)\). By induction \(\text{Cat}_m(S)\) is \(\bar{\mu}\)-catabolizable if and only if \(\text{cattype}(\text{Cat}_m(S)) \geq \bar{\mu}\), which is equivalent to \(\nu \geq \mu\).

The next reformulation of the condition \(\text{cattype}(S) \geq \mu\) is precisely the transpose of the condition of \(R\)-catabolizability where \(R_i = (1^{\mu_i})\).

Define the vertical slicing operator \(V_v\) as follows. For the (skew) tableau \(T\), let \(V_v(T) = P(T_eT_w)\) where \(T_e\) and \(T_w\) are the east and west subtableaux obtained by slicing \(T\) vertically between the \(c\)-th and \((c+1)\)-st columns.
Suppose that $S$ contains $Z_m$. Define the operator $\text{CCat}_m(S) = V_m(S - Z_m)$. Say that $S$ is $\mu$-column catabolizable if $S$ and $\mu$ are empty, or if $S$ contains $Z_{\mu_1}$ and $\text{CCat}_m(S)$ is $\mu$-column catabolizable.

The following equivalence is far from obvious.

**Proposition 47.** A standard tableau is $\mu$-catabolizable if and only if it is $\mu$-column catabolizable.

To prove this it is necessary to consider the following restrictions of the cyclage relation for standard tableaux. Let us use the notation $T \geq S$ for the cyclage partial order. Say that $T$ covers $S$ in the partial order $\geq_{(r)}$ if, $T$ covers $S$ in the order $\geq$ and (in the notation of the definition of $\geq$) the “starting cell” $s$ lies in a row strictly below the $r$-th.

Say that $T$ covers $S$ in the order $\geq_{(c)}$ if $T$ covers $S$ in the order $\geq$ and the “ending cell” $s'$ (defined by the difference of the shapes of $S$ and the intermediate tableau $U$) is in a column strictly to the right of the $c$-th. Another viewpoint is to reverse the cyclage construction (call this cocyclage). One starts with a cell $s'$, performs a reverse row insertion on $S$ at $s'$ to produce a letter $x$ and a tableau $U$, then column inserts $x$ into $U$ to produce the tableau $T$. Then if $T \geq_{(c)} S$, the starting cell $s'$ of the cocyclage must start at a cell in a column strictly right of the $c$-th.

These orders are compatible with the two notions of catabolizability.

**Lemma 48.** If $T$ is $\mu$-catabolizable and $T \geq_{(1)} S$, then $S$ is also.

**Proof.** Without loss assume that $T \geq_{(1)} S$ is a covering relation. Let $\mu = (m, \hat{\mu})$. Let $s$ be the cell where the cyclage starts, $U$ and $a$ as in the definition of cyclage. By definition we have $T = P(aU)$ and $S = P(Ua)$. Let $\hat{T}$ be obtained by removing the first row from $T$. Define $\hat{S}$ and $\hat{U}$ similarly. Since $T$ is $\mu$-catabolizable, the first row of $T$ has the form $Z_m x$, where $x$ is a word. Since $s$ is not in the first row and the bumping paths of reverse column insertions move weakly south, it follows that the first rows of $T$ and $U$ coincide and $a\hat{U} \sim_K \hat{T}$. There are two cases:

1. $xa$ is a weakly increasing word. In this case the first row of $S$ is equal to $Z_m x a$ and $\hat{S} = \hat{U}$, so that

$$H_1(S - Z_m) \sim_K x a \hat{S} = xa\hat{U} \sim_K x U \sim_K H_1(T - Z_m).$$

But $H_1(T - Z_m)$ is $\hat{\mu}$-catabolizable by definition.

2. $xa$ is not weakly increasing. Here the letter $a$ is strictly smaller than some letter of $x$. Let $y$ the weakly increasing word and $b$ the letter such that $xa \sim_K by$. Then $Z_m y$ is the first row of $S$ and $\hat{S} \sim_K \hat{U} b$. We have

$$H_1(T - Z_m) \sim_K x \hat{T} \sim_K x a \hat{U} \sim_K by \hat{U}$$

$$H_1(S - Z_m) \sim_K y \hat{S} \sim_K y \hat{U} b$$

By induction it is enough to show that $P(by\hat{U}) \geq_{(1)} P(y\hat{U} b)$.

Let $s''$ be the cell where the column insertion of $b$ into $P(y\hat{U})$ ends. It is enough to show that $s''$ is not in the first row. Let $y = cz$, where $c$ is the first letter of $y$ and $z$ the remainder of $y$. Since $xa$ is not weakly increasing and $xa \sim_K by$, we have $b > c$. Thus the bumping path of the column insertion of $b$ into $P(cz\hat{U})$ is strictly south of that of the column insertion of $c$ into $P(z\hat{U})$. It follows that $s''$ is not in the first row.
Lemma 49. If \( S \) is \( \mu \)-column catabolizable and \( T \geq_{(m)} S \), then \( T \) is also.

Proof. Assume that \( T \geq_{(m)} S \) is a covering relation. Let \( s' \) be the starting cell of the cocyclage from \( S \) to \( T \), and \( a \) and \( U \) as in the definition of cyclage. We have \( S = P(Ua) \) and \( T = P(aU) \). Let \( \mu = (m, \hat{\mu}) \). Let \( S_e \) and \( S_w \) be the east and west subtableaux obtained by slicing \( S \) vertically just after its \( m \)-th column. Define \( T_e \), \( T_w \), \( U_e \), and \( U_w \) similarly. Then \( S_e \sim_K U_e a \). Now the bumping path of a reverse row insertion moves weakly east, so \( S_w = U_w \). There are two cases.

1. The column insertion of \( a \) into \( U \) ends weakly west of the \( m \)-th column. In this case \( T_w \sim_K aU_w \) and \( T_e = U_e \). It follows that

\[
V_m(T - K) \sim_K T_e T_w \sim_K U_e aU_w \sim_K S_e S_w \sim_K V_m(S - K).
\]

Thus \( \text{CCat}_m(T) = \text{CCat}_m(S) \) and \( T \) is \( \mu \)-column catabolizable.

2. The column insertion of \( a \) into \( U \) ends in a column strictly east of the \( \mu_1 \)-th.

Let \( b \) be the letter that is bumped from the \( \mu_1 \)-th column to the \( \mu_1 + 1 \)-st during this column insertion. Then \( aU_w \sim_K T_w b \) and \( bU_e \sim_K T_e \). We have

\[
V_m(T - K) \sim_K T_e T_w \sim_K bU_e T_w
\]

\[
V_m(S - K) \sim_K S_e S_w \sim_K U_e aU_w \sim_K U_e T_w b
\]

It is enough to show that \( P(bU_e T_w) \geq_{(m)} P(U_e T_w b) \). Let \( s'' \) be the cell where the row insertion of \( b \) into \( P(U_e T_w) \) ends. It is enough to show that \( s'' \) lies strictly east of the \( \mu_1 \)-th column. By the assumption of this case, \( T_w b \) is a tableau of partition shape. By \( \ref{27} \) it follows that \( s'' \) lies strictly to the east of all of the cells of the skew shape given by \( \text{shape}(P(U_e T_w))/\text{shape}(U_e) \). But this skew shape contains a horizontal strip of length \( m \) (since the first row of \( T_w \) has length \( m \)), so it follows that \( s'' \) is strictly east of the \( m \)-th column.

\[\square\]

Lemma 50. Suppose \( T \) contains \( Z_m \). Then

\[
\text{Cat}_m(T) \leq_{(1)} \text{CCat}_m(T)
\]

\[
\text{Cat}_m(T) \leq_{(m)} \text{CCat}_m(T)
\]

Proof. Slice \( T \) horizontally between the first and second rows and vertically between the \( m \)-th and \( m + 1 \)-st columns. Let the northwest, northeast, southwest, and southeast subtableaux be denoted by \( Z_m \), \( T_{ne} \), \( T_{sw} \), and \( T_{se} \) respectively. By the definitions, we have

\[
\text{Cat}_m(T) = \text{H}_1(T - Z_m) \sim_K T_{nw} T_{sw} T_{sc}
\]

\[
\text{CCat}_m(T) = V_m(T - Z_m) \sim_K T_{sc} T_{ne} T_{sw}.
\]

It is enough to show that the relation

\[
P(T_{ne} T_{sw} T_{sc}) \leq P(T_{sc} T_{ne} T_{sw})
\]

(which holds since all of the numbers in the tableau \( T_{sc} \) are strictly greater than \( 2m \)), also holds in the orders \( \leq_{(m)} \) and \( \leq_{(1)} \).

If the tableau \( T_{sc} \) is column inserted into the tableau \( T_{ne} \), all of the bumping paths end in rows strictly south of the first. It follows that if one first row inserts \( T_{sw} \) into \( T_{ne} \), and then column inserts tableau \( T_{sc} \), the column insertions are pushed weakly to the south and hence must still end in rows strictly south of the first. That
means there is a sequence of cocyclages starting in rows strictly south of the first, that prove the relation
\[ P(T_{sc}T_{ne}T_{sw}) \geq_{(1)} P(T_{ne}T_{sw}T_{se}) \]
The proof for \( \geq_{(m)} \) is similar.

Finally the proof of Proposition \(\[\text{Proposition} 47\] \) is given.

Proof. Each of the statements implies the next, using induction and the above lemmas.

1. \( T \) is \( \mu \)-column catabolizable.
2. \( T \) contains \( Z_m \) and \( \text{CCat}_m(T) \) is \( \tilde{\mu} \)-column catabolizable.
3. \( T \) contains \( Z_m \) and \( \text{CCat}_m(T) \) is \( \tilde{\mu} \)-catabolizable.
4. \( T \) contains \( Z_m \) and \( \text{Cat}_m(T) \) is \( \tilde{\mu} \)-catabolizable.
5. \( T \) is \( \mu \)-catabolizable.

Conversely, replace (3) with the following: \( T \) contains \( Z_m \) and \( \text{Cat}_m(T) \) is \( \tilde{\mu} \)-column catabolizable. Then each statement implies the previous one.

For the rest of the proof of Theorem \(\[\text{Theorem} 45\] \), a few more lemmas are needed.

Lemma 51. Suppose \( S' \in \mathcal{T}(\mu) \) with first row \( 1^m \) and remainder \( \tilde{S}' \) (of content \( (0,\mu_2,\mu_3,\ldots) \)). Let \( S = \theta_{\mu}'(S') \). Then the first row of \( S \) is the tableau \( Z_m \) and the rest is given by \( \tilde{S} \), where \( \tilde{S} = \theta_{\tilde{\mu}}'(\tilde{S}') \) and \( \theta_{\tilde{\mu}}' = \theta_{(0^m,1^{n-m})} \).

Proof. For a tableau \( T \) let \( k + T \) denote the tableau obtained by adding the integer \( k \) to every letter in \( T \). The lemma is proven by careful explicit calculation of the map \( \theta_{\mu} \) and \( \theta_{\tilde{\mu}}' \). We compute the map \( \theta_{\mu} \) on the tableau \( S' \) by the composition of the maps that change content as follows:

\[ \mu \to (\tilde{\mu},0^{n-m-1},m) \to (1^{n-m},m) \to (m,0^{m-1},1^{n-m}) \to (1^m,1^{n-m}). \]

The first and third are given by crystal permutation operators and the second and fourth by \( \theta_{\tilde{\mu}} \) and \( \theta_{(m)} \) respectively. By direct computation the image of \( S' \) under the first map is obtained by placing the number \( (n - m + 1) \) at the bottom of each of the first \( m \) columns of the tableau \( -1 + \tilde{S}' \). Then the second map leaves the letters \( (n - m + 1) \) alone and applies \( \theta_{\tilde{\mu}} \) to the letters of the subtableau \( -1 + \tilde{S}' \). Again by direct computation the third map produces the tableau with first row \( 1^m \) and remainder \( m + \theta_{\tilde{\mu}}(-1 + \tilde{S}') \). The fourth map leaves this remainder alone and changes the first row from \( 1^m \) to \( Z_m \). In particular \( \tilde{S} = m + \theta_{\tilde{\mu}}(-1 + \tilde{S}') \). Now the map \( \theta_{\tilde{\mu}}' \) on the tableau \( \tilde{S}' \) can be computed using the composition of the maps that change content as follows:

\[ (0,\tilde{\mu}) \to \tilde{\mu} \to (1^{n-m}) \to (0^m,1^{n-m}). \]

Now the first map is the crystal permutation operator, that produces the tableau \( -1 + \tilde{S}' \). This kind of trivial relabelling occurs when a crystal reflection operator \( s_r \) acts on a word that has no \( r \)'s or no \( (r+1) \)'s. The second map is \( \theta_{\tilde{\mu}} \), which produces \( \theta_{\tilde{\mu}}(-1 + \tilde{S}') \). The third map is another crystal permutation operator that produces \( m + \theta_{\tilde{\mu}}(-1 + \tilde{S}') \). We have shown that

\[ \tilde{S} = m + \theta_{\tilde{\mu}}(-1 + \tilde{S}') = \theta_{\tilde{\mu}}'(\tilde{S}'). \]

\[ \blacksquare \]
Lemma 52. Suppose $X' \in \mathcal{T}(0, \mu)$, $X \in \mathcal{T}(0^m, 1^{n-m})$, $S' = P(X'1^m)$, and $S = P(XZ_m)$. Then in the notation of the previous lemma, $\theta_\mu(X') = X$ if and only if $\theta_\mu(S') = S$.

Proof. The proof proceeds by induction on charge. Let $X_e$ and $X_w$ be the east and west subtableaux obtained by slicing $X$ vertically between the $m$-th and $(m+1)$-st columns. Make similar notation for the tableaux $X'$, $S$, and $S'$. Since the letters of the word $1^m$ (resp. $Z_m$) are strictly smaller than those in $X'$ (resp. $X$), it follows that $S'$ (resp. $S$) is obtained from $X'$ (resp. $X$) by pushing the subtableau $S'_w$ (resp. $S_w$) down by a row, placing the word $1^m$ (resp. $Z_m$) in the vacated cells, and leaving the other subtableau $S'_e$ (resp. $S_e$) in place.

So without loss we may assume that $X'$ and $X$ have the same shape (and hence $S'$ and $S$ do as well). Suppose first that $X'$ has at most $m$ columns. Then $X' = \hat{S}'$ and $X = \hat{S}$ in the notation of Lemma 51, which applies to settle this case. So assume that $X'$ has strictly more than $m$ columns. Let $s'$ be a corner cell of $X'$ of the form $(r, c)$ where $c > m$. Note that $s'$ is also a corner cell of $X$, $S'$ and $S$. Let $Y' \geq X'$, $Y \geq X$, $T' \geq S'$ and $T \geq S$ be the cyclages that end at $s'$. These cyclages exist, since the bumping paths of the reverse row insertions on the tableaux $X', X', S', S$ starting at $s'$, move weakly east so that the cycled letters are always strictly greater than one. It is not hard to see that the cocycle on $X'$ (resp. $X$) at $s'$, commutes with the row insertion of the word $1^m$ (resp. $Z_m$). So $T' = P(Y'1^m)$, $T = P(YZ_m)$. By induction, the lemma can be applied to $Y', Y, T', T$ so that $\theta_\mu(Y') = Y$ if and only if $\theta_\mu(T') = T$. But the $\theta$ operators commute with cyclage, so we are done.

Finally the proof of Theorem 47 is given.

Proof. In view of Propositions 46 and 17, it is enough to show that the standard tableau $S$ is in the image of $\theta_\mu$, if and only if $S$ is $\mu$-column catabolizable.

By Lemma 24 we may assume that $S$ contains $Z_m$. Let $S_e$ and $S_w$ (resp. $S'_e$ and $S'_w$) be the east and west tableaux obtained by slicing the skew tableau $S - Z_m$ (resp. $S' - (1^m)$) vertically between the $m$-th and $(m+1)$-st columns. Let $X' = P(S'_eS'_w)$ and $X = P(S_eS_w)$. The conditions for Lemma 52 are satisfied, so $\theta_\mu(S') = S$ if and only if $\theta_\mu(X') = X$. The following are equivalent.

1. $\theta_\mu(S') = S$.
2. $\theta_\mu(X') = X = \text{CCat}_m(S)$.
3. $\text{CCat}_m(S)$ is $\hat{\mu}$-column catabolizable.
4. $S$ is $\mu$-column catabolizable.

(1) $\iff$ (2) has just been shown, (2) $\iff$ (3) follows by induction, and (3) $\iff$ (4) holds by definition.

4.4. The cocharge Kostka special case of Conjecture 26.

Proposition 53. Conjecture 26 holds when $\eta$ is a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ and $\gamma = (1^t)$.

Proof. Let $R = (R_1, \ldots, R_t)$ be the sequence of partitions corresponding to the pair of weights $\gamma$ and $\eta = \mu$. Then $R_t = (1^{\mu_t})$, the partition consisting of a single column of height $\mu_t$. It is not hard to see that a standard tableau $S$ is $R$-catabolizable if and only if the transpose tableau $S^t$ is $\mu$-column catabolizable. Also it is easy to
see that \( \text{charge}(S) = \text{cocharge}(S^t) \). The result follows from the cocharge Kostka-Foulkes special case in subsection 2.3 and the results of the previous section.

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