ON A CONJECTURE OF SCHMIDT FOR THE PARAMETRIC GEOMETRY OF NUMBERS

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Abstract. With the help of the recently introduced parametric geometry of numbers by W. M. Schmidt and L. Summerer, we prove a strong version of a conjecture of Schmidt concerning the successive minima of a lattice.

1. Introduction

Among the conjectures proposed by W. M. Schmidt in 1983, one is concerned with the parametric geometry of numbers [1, Conjecture 2]. This conjecture was proven in 2012 by N. G. Moshchevitin [1, Theorem 1]. The goal of this paper is to prove a stronger statement along the same lines and we will show that this generalization is the best possible. We start by recalling Moshchevitin’s result, using the notations of D. Roy in [2].

Fix an integer \( n \geq 2 \). For each non-zero \( \xi \in \mathbb{R}^{n+1} \), we associate the family of convex bodies

\[
\mathcal{C}_\xi(Q) := \{ x \in \mathbb{R}^{n+1} ; \| x \| \leq 1, |x \cdot \xi| \leq Q^{-1} \} \quad (Q \geq 1),
\]

where \( x \cdot y \) denotes the standard scalar product in \( \mathbb{R}^n \) and \( \| x \| = (x \cdot x)^{1/2} \) denotes the euclidean norm of \( x \). Define

\[
L_{\xi,j}(q) = \log \lambda_j (\mathcal{C}_\xi(e^q); \mathbb{Z}^{n+1}) \quad (q \geq 0; 1 \leq j \leq n+1),
\]

where \( \lambda_j (\mathcal{C}; \Lambda) \) is defined for a convex body \( \mathcal{C} \) and lattice \( \Lambda \) in \( \mathbb{R}^{n+1} \) to be the \( j \)-th minimum of \( \mathcal{C} \) with respect to \( \Lambda \), i.e. the smallest \( \lambda \geq 0 \) such that \( \lambda \mathcal{C} \) contains at least \( j \) linearly independent elements of \( \Lambda \). Clearly, we have

\[
L_{\xi,1}(q) \leq \cdots \leq L_{\xi,n+1}(q) \quad (q \geq 0).
\]
The functions $L_{\xi,j} : [0, \infty) \rightarrow \mathbb{R}$ ($1 \leq j \leq n + 1$) are continuous and piecewise linear, with slopes alternating between 0 and 1 (see [2, §2], [6, §3]). Moreover, since the volume of $C_\xi(e^q)$ is bounded below and above by multiples of $e^{-q}$, Minkowski’s theorem implies that

$$ q - \sum_{j=1}^{n+1} L_{\xi,j}(q) $$

is a bounded function in $q$, and so the average of the $L_{\xi,j}$’s is $q/(n + 1)$. If the coordinates of $\xi$ are linearly independent over $\mathbb{Q}$, then for each $j = 1, \ldots, n + 1$, there exists arbitrarily large values of $q$ such that

$$ L_{\xi,j}(q) = L_{\xi,j+1}(q) $$

(see [5, Theorem 1]). On the other hand, we have the following result.

**Theorem 1** (N. G. Moshchevitin, 2012). For each integer $k$ with $2 \leq k \leq n$, there exists $\xi \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ such that

$$ \lim_{q\to\infty} \left( L_{\xi,k-1}(q) - \frac{q}{n+1} \right) = -\infty \quad \text{and} \quad \lim_{q\to\infty} \left( L_{\xi,k+1}(q) - \frac{q}{n+1} \right) = \infty. $$

Thus, the functions $L_{\xi,k-1}(q)$ and $L_{\xi,k+1}(q)$ can diverge from each other on each side by $q/(n + 1)$. Our main result improves upon these estimates, and is stated as follows.

**Theorem 2.** For each integer $k$ with $2 \leq k \leq n$, there exist uncountably many vectors $\xi \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ such that

$$ \lim_{q\to\infty} \frac{L_{\xi,k-1}(q)}{q} = 0 \quad \text{and} \quad \liminf_{q\to\infty} \frac{L_{\xi,k+1}(q)}{q} = \frac{1}{n - k + 2}. $$

Further, this result is the best possible in the following sense.

**Theorem 3.** Let $k$ be an integer with $2 \leq k \leq n$, and suppose that $\xi$ is a point in $\mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ and which satisfies

$$ \lim_{q\to\infty} \frac{L_{\xi,k-1}(q)}{q} = 0. $$

Then, we have

$$ \liminf_{q\to\infty} \frac{L_{\xi,k+1}(q)}{q} \leq \frac{1}{n - k + 2}. $$
In the following section, we state Schmidt’s original conjecture, and we justify the above reformulation of Moshchevitin’s result. In section 3, we use the results of §4 to prove Theorem 2. Finally, section 4 provides a proof of theorem 3 by using Schmidt and Summerer’s parametric geometry of numbers.

2. Link with Schmidt’s Original Conjecture

For each $N \in \mathbb{R}$ with $N \geq 1$ and for each $\xi = (1, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, Schmidt introduced the lattice $\Lambda(\xi, N) \subset \mathbb{R}^{n+1}$ generated by the vectors

$$v_0 = (N^{-1}, N^{1/n} \xi_1, \ldots, N^{1/n} \xi_n), \quad v_1 = (0, -N^{1/n}, \ldots, 0), \quad \ldots, \quad v_n = (0, 0, \ldots, -N^{1/n}),$$

and defined

$$\mu_j(\xi, N) = \lambda_j(B; \Lambda(\xi, N)) \quad (1 \leq j \leq n+1)$$

where $B = \{(y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}; |y_i| \leq 1, i = 0, \ldots, n\}$ is the unit hypercube in $\mathbb{R}^{n+1}$.

With these notations, he conjectured the following result, later proven by Moshchevitin.

**Theorem 4** (N. G. Moshchevitin, 2012). Let $k$ be an integer with $2 \leq k \leq n$. There exists real numbers $\xi_1, \ldots, \xi_n \in [0, 1)$ such that

- $1, \xi_1, \ldots, \xi_n$ are linearly independent over $\mathbb{Q}$;
- $\lim_{N \to \infty} \mu_{k-1}(\xi, N) = 0$ and $\lim_{N \to \infty} \mu_{k+1}(\xi, N) = \infty$, where $\xi = (1, \xi_1, \ldots, \xi_n)$.

In fact, Schmidt’s original conjecture omits the linear independence condition, but as Moshchevitin mentions in his article, (see §3), the conjecture is trivial without this hypothesis.

To show the equivalence between Theorems 1 and 4, fix a point $\xi = (1, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$, and fix an integer $k$ with $2 \leq k \leq n$. In §1, Moshchevitin begins by observing that

$$\mu_j(\xi, N) = \lambda_j(K_\xi(N); \mathbb{Z}^{n+1}) \quad (N \geq 1, \ 1 \leq j \leq n+1),$$

where

$$K_\xi(N) = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}; |x_0| \leq N, |x_0 \xi_j - x_j| \leq N^{-1/n}, j = 1, \ldots, n\}.$$
Consequently, the second statement of theorem \[ \| \] can be rewritten as
\[
\lim_{N \to \infty} \lambda_{k-1}(\mathcal{K}_\xi(N); Z^{n+1}) = 0 \quad \text{and} \quad \lim_{N \to \infty} \lambda_{k+1}(\mathcal{K}_\xi(N); Z^{n+1}) = \infty.
\]

Meanwhile, Mahler’s duality theorem yields
\[
\lambda_j(\mathcal{K}_\xi(N); Z^{n+1}) \lambda_{n-j+2}(\mathcal{K}_\xi^*(N); Z^{n+1}) \asymp 1 \quad (1 \leq j \leq n+1),
\]
where
\[
\mathcal{K}_\xi^*(N) = \{ x \in \mathbb{R}^{n+1} : |x \cdot \xi| \leq N^{-1}, ||x|| \leq N^{1/n} \}
\]
is essentially the convex body dual to \( \mathcal{K}_\xi(N) \). Thus, the conditions in (1) become
\[
\lim_{N \to \infty} \lambda_{n+3-k}(\mathcal{K}_\xi^*(N); Z^{n+1}) = \infty \quad \text{and} \quad \lim_{N \to \infty} \lambda_{n+1-k}(\mathcal{K}_\xi^*(N); Z^{n+1}) = 0.
\]

On the other hand, since \( C_\xi(e^q) = e^{-q/(n+1)} \mathcal{K}_\xi^*(e^{nq/(n+1)}) \), it follows that
\[
L_{\xi,j}(q) = \frac{q}{n+1} + \log \lambda_j(\mathcal{K}_\xi^*(e^{nq/(n+1)}); Z^{n+1}) \quad (1 \leq j \leq n+1).
\]
Thus, the conditions in (2) can be rewritten as
\[
\lim_{q \to \infty} \left( L_{\xi,n+3-k}(q) - \frac{q}{n+1} \right) = \infty \quad \text{and} \quad \lim_{q \to \infty} \left( L_{\xi,n+1-k}(q) - \frac{q}{n+1} \right) = -\infty.
\]
The equivalence between theorems \[ \| \] and \[ \| \] follows.

3. Proof of the Main Result

In order to prove Theorem \[ \| \], we need to establish some preliminary results which rely on
the following basic construction.

**Proposition 1.** Let \( a, b, c, \alpha, \beta, \gamma \in (0, \infty) \) with \( a < b < c \). There exists a unique choice of
real numbers \( r, s, t, u \in (0, \infty) \) with \( r < s < t < u \) and a unique triplet of continuous and
piecewise linear functions \( (A, B, C) \) on \([r, u]\) such that the union of their graphs is as in
Figure 1, i.e.

i) for all \( q \in [r, u] \), we have
\[
A(q) \leq B(q) \leq C(q) \quad \text{and} \quad \frac{1}{\alpha} A(q) + \frac{1}{\beta} B(q) + \frac{1}{\gamma} C(q) = q;
\]

ii) the function \( A \) is constant equal to \( a \) on \([r, t]\), has slope \( \alpha \) on \([t, u]\), and satisfies
\( A(u) = b \);

iii) the function \( B \) has slope \( \beta \) on \([r, s]\), is constant equal to \( b \) on \([s, u]\), and satisfies
\( B(r) = a \);
iv) the function $C$ is constant equal to $b$ on $[r, s]$, has slope $\gamma$ on $[s, t]$, and is constant equal to $c$ on $[t, u]$.

**Figure 1.**

**Proof.** If there exist real numbers $r, s, t, u$ and functions $A, B, C$ as in the claim, then substituting $q$ by $r, s, t, u$ in the second condition of (3) yields, respectively,

$$(4) \quad r = \frac{a}{\alpha} + \frac{a}{\beta} + \frac{b}{\gamma}; \quad s = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{b}{\gamma}; \quad t = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}; \quad u = \frac{b}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma},$$

which uniquely determines them all.

Now, let $r, s, t, u$ be given by (4). Since $r < s < t < u$, there exists a unique triplet of continuous functions $(A, B, C)$ on $[r, u]$ with constant slopes on $[r, s]$, $[s, t]$ and $[t, u]$, and with

$A(r) = A(s) = A(t) = a$ and $A(u) = b$,

$B(r) = a$ and $B(s) = B(t) = B(u) = b$,

$C(r) = C(s) = b$ and $C(t) = C(u) = c$.

Thus, the function $F = \frac{1}{\alpha}A + \frac{1}{\beta}B + \frac{1}{\gamma}C$ is continuous and of constant slope on each of the interval $[r, s]$, $[s, t]$, and $[t, u]$. By construction, we have that $F(q) = q$ for $q = r, s, t, u$. Thus,

$F(q) = q$ for all $q \in [r, u]$.

Since $A$ and $C$ are constant on $[r, s]$, this implies that $B$ has slope $\beta$ on $[r, s]$. Similarly, we deduce that $C$ has slope $\gamma$ on $[s, t]$, and that $A$ has slope $\alpha$ on $[t, u]$.

\(\square\)
Proposition 2. With the same notation as above, suppose that $\frac{b}{a} < c/b$. Then, we have

\[
\max_{q \in [r, u]} \frac{A(q)}{q} = \frac{a}{r} \quad \text{and} \quad \min_{q \in [r, u]} \frac{C(q)}{q} = \frac{b}{s}.
\] (5)

Proof. First, using (4) note that

\[
\frac{a}{t} < \frac{b}{u} < \frac{a}{r} \quad \text{and} \quad \frac{b}{s} < \frac{b}{r} < \frac{c}{u} < \frac{c}{t}.
\]

Since $\frac{a}{r} < \alpha$ and $\frac{b}{s} < \gamma$, it follows that the ratio $A(q)/q$ is decreasing on $[r, t]$ and increasing on $[t, u]$, and that the ratio $C(q)/q$ is decreasing on $[r, s]$, increasing on $[s, t]$ and decreasing on $[t, u]$. The conclusion follows straightforwardly. \qed

Let $\Delta$ denote the set of sequences $(a_m)_{m \in \mathbb{Z}}$ of positive reals which satisfy

\[
1 < \frac{a_{m+1}}{a_m} < \frac{a_{m+2}}{a_{m+1}} \quad (m \in \mathbb{Z}),
\]

\[
\lim_{m \to -\infty} a_m = 0 \quad \text{and} \quad \lim_{m \to \infty} \frac{a_{m+1}}{a_m} = +\infty.
\]

The following result further extends the preceding propositions.

Proposition 3. Let $(a_m)_{m \in \mathbb{Z}} \in \Delta$ and let $\alpha, \beta, \gamma \in (0, \infty)$. Define

\[
r_m = \frac{a_m}{\alpha} + \frac{a_{m+1}}{\beta} + \frac{a_{m+2}}{\gamma} \quad (m \in \mathbb{Z}).
\] (6)

Then, there exists a unique triplet of continuous and piecewise linear functions $(A, B, C)$ on $(0, \infty)$ whose restriction to the interval $[r_m, r_{m+1}]$ fulfills the conditions of Proposition 2 with $a = a_m$, $b = a_{m+1}$ and $c = a_{m+2}$ for each $m \in \mathbb{Z}$. Moreover, we have

\[
\lim_{q \to \infty} A(q) = \infty, \quad \limsup_{q \to \infty} \frac{A(q)}{q} = 0 \quad \text{and} \quad \liminf_{q \to \infty} \frac{C(q)}{q} = \frac{\beta \gamma}{\beta + \gamma}.
\] (7)

Proof. Let $(a_m)_{m \in \mathbb{Z}} \in \Delta$, and define

\[
s_m = \frac{a_m}{\alpha} + \frac{a_{m+1}}{\beta} + \frac{a_{m+2}}{\gamma} \quad \text{and} \quad t_m = \frac{a_m}{\alpha} + \frac{a_{m+1}}{\beta} + \frac{a_{m+2}}{\gamma} \quad (m \in \mathbb{Z}).
\] (8)

By setting $a = a_m$, $b = a_{m+1}$ and $c = a_{m+2}$, Proposition 2 and (4) yield for each $m \in \mathbb{Z}$ a triplet of continuous and piecewise linear functions $(A^{(m)}, B^{(m)}, C^{(m)})$ on $[r, u] = [r_m, r_{m+1}]$.

Since the triplets $(A^{(m-1)}, B^{(m-1)}, C^{(m-1)})$ and $(A^{(m)}, B^{(m)}, C^{(m)})$ coincide at the point $r_m$ and are equal to $(a_m, a_m, a_{m+1})$ for each $m \in \mathbb{Z}$, it follows that the sequence of triplets
of functions \((A^{(m)}, B^{(m)}, C^{(m)})\) with \(m \in \mathbb{Z}\) determine a unique triplet of continuous and piecewise linear functions \((A, B, C)\) on \(\bigcup_{m \in \mathbb{Z}} [r_m, r_{m+1}] = (0, \infty)\). Now, Proposition \(\ref{prop:unique_triplet}\) gives

\[
\max_{q \in [r_m, r_{m+1}]} \frac{A(q)}{q} = \frac{a_m}{r_m} \quad \text{and} \quad \min_{q \in [r_m, r_{m+1}]} \frac{C(q)}{q} = \frac{a_{m+1}}{s_m},
\]

and so

\[
\limsup_{q \to \infty} \frac{A(q)}{q} = \lim_{m \to \infty} \frac{a_m}{r_m} = 0 \quad \text{and} \quad \liminf_{q \to \infty} \frac{C(q)}{q} = \lim_{m \to \infty} \frac{a_{m+1}}{s_m} = \frac{\beta \gamma}{\beta + \gamma}.
\]

Our next result uses the notion of generalized \((n + 1)\)-system introduced by D. Roy in \cite{Roy2018}. It provides a good approximation of the functions \(L_\xi\) for non-zero point \(\xi \in \mathbb{R}^{n+1}\) (see \cite{Roy2018} for more details). We recall here the definition.

**Definition.** Let \(I\) be a subinterval of \([0, \infty)\) with non-empty interior. A generalized \((n + 1)\)-system on \(I\) is a map \(P = (P_1, \ldots, P_{n+1}) : I \to \mathbb{R}^{n+1}\) with the following properties.

(G1) For each \(q \in I\), we have \(0 \leq P_1(q) \leq \cdots \leq P_{n+1}(q)\) and \(P_1(q) + \cdots + P_{n+1}(q) = q\).

(G2) If \(H\) is a non-empty open subinterval of \(I\) on which \(P\) is differentiable, then there are integers \(\underline{r}, \overline{r}\) with \(1 \leq \underline{r} \leq \overline{r} \leq n + 1\) such that \(P_{\underline{r}}, P_{\overline{r}+1}, \ldots, P_{\overline{r}}\) coincide on the whole interval \(H\) and have slope \(1/(\overline{r} - \underline{r} + 1)\) while any other component \(P_j\) of \(P\) is constant on \(H\).

(G3) If \(q\) is an interior point of \(I\) at which \(P\) is not differentiable, if \(\underline{r}, \overline{r}, \underline{s}, \overline{s}\) are the integers for which

\[
P_j'(q^-) = \frac{1}{\overline{r} - \underline{r} + 1} \quad (\underline{r} \leq j \leq \overline{r}) \quad \text{et} \quad P_j'(q^+) = \frac{1}{\overline{s} - \underline{s} + 1} \quad (\underline{s} \leq j \leq \overline{s})
\]

and if \(\underline{r} \leq \underline{s}\), then we have \(P_{\underline{r}}(q) = P_{\overline{r}+1}(q) = \cdots = P_{\overline{s}}(q)\).

We now combine the previous Propositions to establish the following result.

**Proposition 4.** Let \(k\) be an integer with \(2 \leq k \leq n\). With the notation of Proposition \(\ref{prop:approximation}\) suppose that \(\alpha = 1/(k - 1)\), \(\beta = 1\) and \(\gamma = 1/(n + 1 - k)\). For all \(q > 0\), let

\[
P_1(q) = \cdots = P_{k-1}(q) = A(q) \quad P_k(q) = B(q) \quad \text{and} \quad P_{k+1}(q) = \cdots = P_{n+1}(q).
\]

Then the function \(P : (0, \infty) \to \mathbb{R}^{n+1}\) defined by

\[
P(q) := (P_1(q), \ldots, P_{n+1}(q)) \quad (q > 0)
\]
is an generalized \((n + 1)\)-system on \((0, \infty)\). Moreover, we have
\[
\lim_{q \to \infty} P_1(q) = \infty, \quad \limsup_{q \to \infty} \frac{P_{k-1}(q)}{q} = 0 \quad \text{and} \quad \liminf_{q \to \infty} \frac{P_{k+1}(q)}{q} = \frac{1}{n - k + 2}.
\]

**Proof.** The components \(P_1, \ldots, P_{n+1}\) of \(P\) are continuous and piecewise linear on \((0, \infty)\).

They satisfy
\[
0 \leq P_1(q) \leq \cdots \leq P_{n+1}(q) \quad \text{and} \quad P_1(q) + \cdots + P_{n+1}(q) = q \quad (q > 0).
\]

The function \(P\) is differentiable on \((0, \infty)\) except at the points \(r_m, s_m, t_m\) given by (6) and (8). On each of the interval \([r_m, s_m]\), \([s_m, t_m]\), \([t_m, r_{m+1}]\), the components \(P_1, \ldots, P_{n+1}\) are constant except for few, say \(h\) of them, which coincide on the interval and which have slope \(1/h\). At the point \(r_m\), the slopes of \(P_1, \ldots, P_{k-1}\) go from \(1/(k-1)\) to 0, while the slope of \(P_k\) goes from 0 to 1, and all these functions take the same value, i.e.
\[
P_1(r_m) = \cdots = P_k(r_m) \quad (m \in \mathbb{Z}).
\]

At the point \(s_m\), the function \(P_k\) goes from slope 1 to slope 0, while the slopes of \(P_{k+1}, \ldots, P_{n+1}\) go from 0 to \(1/(n - k + 1)\), and similarly.
\[
P_k(s_m) = P_{k+1}(s_m) = \cdots = P_{n+1}(s_m) \quad (m \in \mathbb{Z}).
\]

Finally, at the point \(t_m\), the slopes of \(P_{k+1}, \ldots, P_{n+1}\) go from \(1/(n - k + 1)\) to 0, while the slopes of \(P_1, \ldots, P_{k-1}\) go from 0 to \(1/(k-1)\), and we have
\[
P_1(t_m) = \cdots = P_{k-1}(t_m) < P_k(t_m) < P_{k+1}(t_m) = \cdots = P_n(t_m) \quad (m \in \mathbb{Z}).
\]

Therefore, the function \(P\) is an generalized \((n + 1)\)-system on \((0, \infty)\). The second assertion of the proposition follows from (7). \(\Box\)

In [3, §4], D. Roy shows that for each generalized \((n + 1)\)-system \(P\) on \([q_0, \infty)\) with \(q_0 \geq 0\), there exists a non-zero point \(\xi\) of \(\mathbb{R}^{n+1}\) such that the difference \(L_\xi - P\) is bounded. Then, we have
\[
\limsup_{q \to \infty} \frac{L_{\xi,j}(q)}{q} = \limsup_{q \to \infty} \frac{P_j(q)}{q} \quad \text{and} \quad \liminf_{q \to \infty} \frac{L_{\xi,j}(q)}{q} = \liminf_{q \to \infty} \frac{P_j(q)}{q} \quad (1 \leq j \leq n + 1).
\]

In the context of Proposition 4, this guarantees the existence of a point \(\xi \in \mathbb{R}^{n+1}\) with
\[
\limsup_{q \to \infty} \frac{L_{\xi,k-1}(q)}{q} = 0 \quad \text{and} \quad \liminf_{q \to \infty} \frac{L_{\xi,k+1}(q)}{q} = \frac{1}{n - k + 2}.
\]
Moreover, since \( \lim_{q \to \infty} P_1(q) = \infty \), the function \( L_{\xi,1} \) is unbounded. It follows that \( \xi \) is a point whose coordinates are linearly independent over \( \mathbb{Q} \).

To finish the proof, it remains to show that one can construct uncountably many such points. For each \( \theta \in (0, \infty) \), we define
\[
a_m^{(\theta)} = \theta 2^{m^3} \quad (m \in \mathbb{Z}).
\]

Then, the sequence \( (a_m^{(\theta)})_{m \in \mathbb{Z}} \) belongs to \( \Delta \), and Propositions 3 and 4 associate to it an generalized \((n+1)\)-system \( \mathbf{P}(\theta) \) on \((0, \infty)\), and a point \( \xi^{(\theta)} \in \mathbb{R}^{n+1} \). Extending the notation in an obvious manner gives
\[
r_m^{(\theta)} = k a_m^{(\theta)} + (n - k + 1) a_{m+1}^{(\theta)} < (n + 1) a_{m+1}^{(\theta)}
\]
\[
t_m^{(\theta)} = (k - 1) a_m^{(\theta)} + a_{m+1}^{(\theta)} + (n - k + 1) a_{m+2}^{(\theta)} > a_{m+2}^{(\theta)}
\]
for all \( m \in \mathbb{Z} \), and \( t_m^{(\theta)} / r_m^{(\theta)} \) tends to infinity with \( m \). Thus, if \( \theta, \theta' \in (0, \infty) \) with \( \theta < \theta' \), then
\[
r_m^{(\theta)} < r_m^{(\theta')} = (\theta' / \theta) r_m^{(\theta)} < t_m^{(\theta)}
\]
for all sufficiently large \( m \in \mathbb{Z} \), and so
\[
\| \mathbf{P}(\theta') (r_m^{(\theta)}) - \mathbf{P}(\theta) (r_m^{(\theta)}) \| \geq |P_1(\theta') (r_m^{(\theta)}) - P_1(\theta) (r_m^{(\theta)})| = |a_m^{(\theta')} - a_m^{(\theta)}| = (\theta' - \theta) 2^{m^3}.
\]
This means that the difference \( \mathbf{P}(\theta') - \mathbf{P}(\theta) \) is unbounded. Thus, the points \( \xi^{(\theta') \prime} \) and \( \xi^{(\theta)} \) are distinct, and consequently, the map \( \theta \mapsto \xi^{(\theta)} \) is injective on \((0, \infty)\). Its image is therefore uncountable.

### 4. Proof of Theorem 3

Let \( \xi \) be a point in \( \mathbb{R}^{n+1} \) whose coordinates are linearly independent over \( \mathbb{Q} \). On the model of Schmidt and Summerer in [5, §1], we define
\[
\varphi_j(\xi) = \lim_{q \to \infty} \frac{L_{\xi, j}(q)}{q} \quad \text{and} \quad \varphi_j(\xi) = \lim_{q \to \infty} \frac{L_{\xi, j}(q)}{q} \quad (1 \leq j \leq n+1).
\]

In [5, §1], Schmidt and Summerer show that these quantities satisfy
\[
\varphi_{j+1}(\xi) \leq \varphi_j(\xi) \quad (1 \leq j \leq n).
\]
Now, suppose that $\varphi_{k-1}(\xi) = 0$ for some integer $k$ with $2 \leq k \leq n$. Since $q - \sum_{j=1}^{n+1} L_{\xi,j}(q)$ is a bounded function in $q$ on $(0, \infty)$, we have that

$$(n - k + 2)\varphi_k(\xi) \leq \limsup_{q \to \infty} \frac{1}{q} \sum_{j=k}^{n+1} L_{\xi,j}(q) = \limsup_{q \to \infty} \frac{1}{q} \left( q - \sum_{j=1}^{k-1} L_{\xi,j}(q) \right) = 1,$$

and so $\varphi_k(\xi) \leq 1/(n - k + 2)$. This yields $\varphi_{k+1}(\xi) \leq 1/(n - k + 2)$.

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