TRIPLE-CROSSING NUMBER, THE GENUS OF A KNOT
AND TORUS KNOTS

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Abstract. We show that the triple-crossing number of any knot is greater or equal to twice its (canonical) genus. As an application we show that this bound is strong enough to obtain the (up to now unknown) triple-crossing number of all torus knots, their connected sums and of many more knots.

1. Introduction

In general position of planar diagrams of knots and links, two strands meet at every crossing. It is known since [1] that any knot and every link has a diagram where, at each its multiple point in the plane, exactly three strands are allowed to cross (pairwise transversely). Such triple-point diagrams has been studied in several recent papers, such as [1, 2, 3, 5, 8].

The triple-crossing number of a knot or a link $K$, denoted $c_3(K)$, is defined analogously to the classical (double-crossing) number as the least number of triple-crossings for any triple-crossing diagram of $K$. There are lower bounds for the triple-crossing number: in terms of double-crossing number $c_3(K) \geq \frac{1}{3}c_2(K)$, and if $K$ is an alternating knot then $c_3(K) \geq \frac{1}{2}c_2(K)$ (see [1]); in terms of double-crossing braid index $c_3(K) \geq \beta_2(K) - 1$ (see [8]); We will prove the following bound of the triple-crossing number $c_3$ by the canonical genus $g_c$.

Theorem 1. Let $K$ be a knot. Then $c_3(K) \geq 2 \cdot g_c(K)$.

The paper is organized as follows. In Section 3 of this paper we give a lower bound for the triple-crossing number of a given knot in terms of the canonical genus (and hence also the genus) of that knot. As an application we show in Section 4 that the triple-crossing number of torus $T(p,q)$ knot is equal to $(p - 1)(q - 1)$ and describe its minimal triple-crossing diagram. In Section 5 we calculate the triple-crossing number of additional knots.

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2. Definitions

The projection of a knot or a link $K \subset \mathbb{R}^3$ is its image under the standard projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ (or into a 2-sphere) such that it has only a finite number of self-intersections, called multiple points, and in each multiple point each pair of its strands are transverse.

If each multiple point of a projection has multiplicity three then we call this projection a triple-crossing projection. The triple-crossing is a three-strand crossing with the strand labeled $T,M,B$, for top, middle and bottom. The triple-crossing diagram is a triple-crossing projection such that each of its triple point is a triple-crossing, such that $\pi^{-1}$ of the strand labeled $T$ (in the neighborhood of that triple point) is on the top of the strand corresponding to the strand labeled $M$, and the latter strand is on the top of the strand corresponding to the strand labeled $B$ (see Figure 1).

![Figure 1. A deconstruction/construction of a triple-crossing.](image)

The triple-crossing number of a knot or link $K$, denoted $c_3(K)$, is the least number of triple-crossings for any triple-crossing diagram of $K$. The classical double-crossing number invariant we will denote by $c_2$. The minimal triple-crossing diagram of a knot or link $K$ is a triple-crossing diagrams of $K$ that has exactly $c_3(K)$ triple-crossings.

A natural orientation (see an equivalent definition in [3]) on a triple-crossing diagram is an orientation of each component of that link, such that in each crossing the strands are oriented in-out-in-out-in-out, as we encircle the crossing. We begin with an interesting notice.

**Lemma 2** ([3]). Every orientation of the triple-crossing diagram obtained from an oriented knot is the natural orientation.

Given any knot, one can obtain a Seifert surface (i.e. an embedded orientable in $\mathbb{R}^3$ surface whose boundary equals the knot) by applying Seiferts algorithm to an oriented double-crossing of its projection. We split the diagram at each crossing and reglue so that the orientations of the resulting strands match. Then each of the disjoint simple closed curves that result is spanned with a disk. Half-twisted bands are attached to the boundaries of the disk at each crossing. We call a surface obtained in this manner a canonical Seifert surface for the knot.
3. Canonical genus and triple-crossing number of a knot

A canonical genus of a knot $K$, denoted here by $g_c(K)$, is the least genus of any canonical Seifert surface for $K$. A genus of a knot $K$, denoted by $g(K)$, is the least genus of any Seifert surface for $K$. This definition and Theorem 1 immediately imply the lower bound of the triple-crossing number $c_3$ by the genus $g$, since for any knot $K$ we have $g_c(K) \geq g(K)$.

![Figure 2. Deconstructing a triple-crossing and performing the resolution.](image)

**Proof of Theorem 1.** Let $K$ be any knot and let $D$ be a minimal triple-crossing diagram of the knot $K$, and let $n$ be the number of the set of triple-crossings of $D$ (denoted $\text{vertices}(D)$). The set of edges of $D$, denoted $\text{edges}(D)$, contains $3n$ elements. For a planar (or a spherical) diagram we know, from Euler characteristic, that the number of elements in $\text{faces}(D)$ (i.e. regions of the complement of $D$) equals $\#\text{faces}(D) = 2 + \#\text{edges} - \#\text{vertices} = 2 + 3n - n = 2n + 2$. We can color elements of $\text{faces}(D)$ in a checkerboard fashion (see 3). Without loss of generality assume that at least half of elements from $\text{faces}(D)$ are colored white and are elements of the set $\text{whitefaces}(D)$, i.e. $\#\text{whitefaces}(D) \geq \frac{1}{2}(2n + 2) = n + 1$.

We now deconstruct every triple-crossing to a three double-crossings in a way such that we (locally) increase the number of white faces by one (see Figure 2 where the white regions are marked by the letter $W$, and the flat crossing can have arbitrary crossing information). This figure cover all cases since, from Lemma 2 the diagram $D$ has the natural orientation. In the end we obtain a double-crossing diagram $\bar{D}$ such that $\text{vertices}(\bar{D}) = 3n$, and $\#\text{whitefaces}(\bar{D}) \geq (n + 1) + n = 2n + 1$. 
The Seifert algorithm performed on $\overline{D}$ gives us a canonical Seifert surface $F$ of genus $g(F) = \frac{1}{2}(1 + \text{vertices}(D) - \#\text{whitefaces}(D)) \leq \frac{1}{2}(1 + 3n - (2n + 1)) = \frac{n}{2}$.

Hence, $c_3(K) = n \geq 2 \cdot g(F) \geq 2 \cdot g_c(K)$.

\[\square\]

4. Torus knots and their minimal triple-crossing diagrams

Torus knots, denoted $T(p, q)$ for coprime positive integers $p < q$ are known to have their minimal double-crossing diagrams as a closure of the braid word $(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$ (see [7]). In this infinite family of knots, up to now, the triple-crossing number of nontrivial torus knots has been known only for the trefoil and the cinquefoil.

**Proposition 3.** Let $K$ be a torus $T(p, q)$ knot (for coprime positive integers $p, q$). Then $c_3(K) = (p - 1)(q - 1)$.

**Proof.** It is well-known (see [4]) that the genus of the torus knot $T(p, q)$ equals $g(T(p, q)) = \frac{1}{2}(p - 1)(q - 1)$. From Theorem \[\square\] we have now that $c_3(T(p, q)) \geq 2 \cdot g_c(T(p, q)) \geq 2 \cdot g(T(p, q)) = (p - 1)(q - 1)$. To finish the proof it suffice to find a triple-crossing diagram of $T(p, q)$ with $(p - 1)(q - 1)$ triple-crossings. Assume $p < q$, we done this by first putting $p - 1$ circles covering all double-crossings, one between each neighboring strands of the closure of $(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$ braid word. Then performing the folding operation on each circle (described in [1]), obtaining triple-crossing diagram of $T(p, q)$ knot with $(p - 1)q - (p - 1) = (p - 1)(q - 1)$ triple-crossings. \[\square\]

**Remark 4.** By analogy the same statement as in the above proposition holds for the mirror image of those knots and their diagrams.

**Corollary 5.** Let $K_1$ and $K_2$ be torus knots. Then $c_3(K_1) + c_3(K_2) = c_3(K_1\#K_2)$.

**Proof.** From Theorem \[\square\] Proposition \[\square\] and the additivity of the genus (in respect to the connected sum $\#$) we have: $c_3(K_1) + c_3(K_2) \geq c_3(K_1\#K_2) \geq 2 \cdot g(K_1\#K_2) = 2 \cdot g(K_1) + 2 \cdot g(K_2) = c_3(K_1) + c_3(K_2)$. \[\square\]

5. Other knots

We have the following upper bound on triple-crossing number.

**Lemma 6 (7).** Let $K$ be a nontrivial knot or a nontrivial link. If $K \neq T(2, n)$ for any $n \in \mathbb{Z}$, then $c_3(K) \leq c_2(K) - 2$.

**Remark 7.** From Lemma \[\square\] Theorem \[\square\] and \[\square\] we obtain an exact values of (up to now unknown) triple-crossing numbers for many more knots, such as: $8_2, 8_5, 8_7, 8_9, 8_{10}, 8_{16}, 8_{17}, 8_{18}, 10_2, 10_5, 10_9, 10_{17}, 10_{46}, 10_{47}, 10_{48}, 10_{62}, 10_{64}, 10_{79}, 10_{82}, 10_{85}, 10_{91}, 10_{94}, 10_{99}, 10_{100}, 10_{104}, 10_{106}, 10_{109}, 10_{112}, 10_{116}, 10_{118}, 10_{123}, 10_{139}, 10_{152}, K12a_{146}, K12a_{369}, K12a_{576}, K12a_{716}, K12a_{722}, K12a_{805}, K12a_{815}, K12a_{819}, K12a_{824},$
K12a₈₃₅, K12a₈₃₈, K12a₈₅₀, K12a₈₅₉, K12a₈₆₄, K12a₈₆₉, K12a₈₇₈, K12a₈₉₈, K12a₉₀₉, K12a₉₁₆, K12a₉₂₀, K12a₉₈₁, K12a₉₈₄, K12a₉₉₉, K12a₁₀₀₂, K12a₁₀₁₁, K12a₁₀₁₃, K12a₁₀₂₇, K12a₁₀₄₇, K12a₁₀₅₁, K12a₁₁₁₄, K12a₁₁₂₀, K12a₁₁₂₈, K12a₁₁₃₄, K12a₁₁₆₈, K12a₁₁₇₆, K12a₁₁₉₁, K12a₁₁₉₉, K12a₁₂₀₃, K12a₁₂₀₉, K12a₁₂₁₀, K12a₁₂₁₁, K12a₁₂₁₂, K12a₁₂₁₄, K12a₁₂₁₅, K12a₁₂₁₈, K12a₁₂₁₉, K12a₁₂₂₀, K12a₁₂₂₁, K12a₁₂₂₂, K12a₁₂₂₃, K12a₁₂₂₅, K12a₁₂₂₆, K12a₁₂₂₇, K12a₁₂₂₉, K12a₁₂₃₀, K12a₁₂₃₁, K12a₁₂₃₃, K12a₁₂₃₅, K12a₁₂₃₈, K12a₁₂₄₆, K12a₁₂₄₈, K12a₁₂₄₉, K12a₁₂₅₀, K12a₁₂₅₃, K12a₁₂₅₄, K12a₁₂₅₅, K12a₁₂₅₈, K12a₁₂₆₀, K12a₁₂₇₃, K12a₁₂₈₃, K12a₁₂₈₈, K12n₂₄₂, K12n₄₇₂, K12n₅₇₄, K12n₆₇₉, K12n₆₈₈, K12n₇₂₅, K12n₈₈₈.

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