Dynamic Factor Models with Sparse VAR Idiosyncratic Components

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Abstract

We reconcile the two worlds of dense and sparse modeling by exploiting the positive aspects of both. We employ a dynamic factor model and assume the idiosyncratic term follows a sparse vector autoregressive model (VAR) which allows for cross-sectional and time dependence. The estimation is articulated in two steps: first, the factors and their loadings are estimated via principal component analysis and second, the sparse VAR is estimated by regularized regression on the estimated idiosyncratic components. We prove consistency of the proposed estimation approach as the time and cross-sectional dimension diverge. In the second step, the estimation error of the first step needs to be accounted for. Here, we do not follow the naïve approach of simply plugging in the standard rates derived for the factor estimation. Instead, we derive a more refined expression of the error. This enables us to derive tighter rates. We discuss the implications to forecasting and semi-parametric estimation of the inverse of the spectral density matrix and we complement our procedure with a joint information criteria for the VAR lag-length and the number of factors. The finite sample performance is illustrated by means of an extensive simulation exercise. Empirically, we assess the performance of the proposed method for macroeconomic forecasting using the FRED-MD dataset.

Keywords: Dynamic Factor Model, Sparse & Dense, High-dimensional VARs

JEL codes: C55, C53, C32

1. Introduction

With the increased availability of large dimensional datasets and the need of techniques able to handle them, the econometrics literature has adapted and rapidly grown over the last years. Datasets containing large amounts of variables (\(N\)) with respect to the sample size (\(T\)) are loaded with information. This represents a great potential to be exploited (the blessings) but it also carries several troubles for the statistician to deal with (the curses). The parameter space of standard (low-dimensional) models expands at fast pace with increasing dimension and its elements to estimate soon start to be too many for the sample information available to reliably estimate them. Overfitting and surging variance cause failure of standard methods designed for settings where \(N\) is small relative to \(T\). In one way or another, the core idea to get away from the curses is: dimensionality reduction. The econometric literature mostly polarizes
on either factor models\(^1\) or sparse penalized regression techniques\(^2\). The former assumes the behavior of an economic variable is sensibly decomposed into a component driven by few unobservable (latent) factors, which are common to many other economic variables but load differently on each of them, and a variable specific idiosyncratic component. Factors are particular linear combination of the original data. The dimensionality reduction occurs as few factors can summarise the original variables well enough by preserving as much as possible the original variability. Therefore, the *dense* label is attributed to the avoidance of any sparsity assumption over the underlying data generating process. On the other hand, sparse penalized regression techniques yield dimensionality reduction via model selection and they work upon assumption of an exact (or approximately) *sparse* underlying model. Sparsity effectively limits the number of direct channels that a variable has to affect other variables. Both factor models and sparse-regression techniques are widely employed in practice both for prediction and inference. However, usually one excludes the other. The reason being the radically different approach in putting a structure over the model, namely *dense* versus *sparse*. Neither of the two are free of criticism: sparsity is often seen as a too strong assumption to be realistic in practice, especially in macroeconomics; on the other hand, factor models need some consistent criterion to select a certain – not too large – amount of factors and the interpretation of the factors might not be straightforward in many empirical contexts. Many papers have compared empirically the performances of both the approaches, especially in terms of macroeconomic forecasting, among others: Smeekes & Wijler (2018), Coulombe et al. (2020), Medeiros et al. (2021). None of these works can in general strongly conclude in favour of one or the other approach and the choice ends up being adaptive with respect to the relevant empirical context.

In this paper we reconcile the two worlds of dense and sparse modeling by focusing on exploiting the positive aspects of both. We work within the framework of a Dynamic Factor Model and assume the idiosyncratic component follows a high-dimensional vector autoregressive model (VAR) model. This allows cross-sectional and time dependence in the idiosyncratic term. We employ principal component analysis (PCA) to estimate the factors and high-dimensional penalized VAR through the adaptive lasso in order to estimate the idiosyncratic components. We show consistent estimation of both the sparse VAR model driving the idiosyncratic components and the factors as both the cross-sectional and time dimensions grow large. A naive approach would be to simply plug in the standard rates derived for the factor estimation. This lead to a suboptimal rate. Instead, an important contribution of our work is to

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\(^1\)Factor models are widely used in economics. For instance, the literature on Dynamic Stochastic General Equilibrium (DSGE) models and real business cycle (see e.g. Sargent et al., 1977) assumes few common forces to drive the whole economy. Factor models were originally envisioned for cross-sectional data and their time-series extension, broadly referred to as Dynamic Factor Models (DFM), was first proposed by Geweke (1977). DFMs are nowadays ubiquitous in economics, their applications range over from: macroeconomics forecasting (see among others: Stock & Watson, 1999, Koopman & van der Weel, 2013, Marcellino et al., 2016), real-time monitoring *(nowcasting)* (see among others: Giannone et al., 2008, Aruoba & Diebold, 2010), international business cycle (see among others: Kim & Nelson, 1998, Lee, 2013), construction of leading indicators (see among others: Stock & Watson, 1989, Forni et al., 2001), to monetary policy applications (see among others: Bernanke et al., 2005, Forni & Gambetti, 2010)

\(^2\)Sparse penalized regression techniques aims at selecting and estimating only the subset of truly relevant variables. The (adaptive) lasso, elastic net, scad are only few among the plethora of techniques developed to tradeoff variance with bias and thus get away from the curse of dimensionality. Many more refinements of such procedures have acquired a great deal of space and interest in the statistics and econometrics literatures. Econometrics applications of these procedures see among others: instrumental variables estimation (see among others: Belloni et al., 2012, 2014), treatment effect models (see among others: Li & Bell, 2017, Ju et al., 2020), time series models (see among others: Kock & Callot, 2015, Medeiros & Mendes, 2016).
derive detailed expressions of the occurring errors. This enables us to obtain tighter rates for the second step and employ a semi-parametric estimator for the inverse of the spectral density matrix. Additionally, these expressions can be helpful for deriving inference results with the de-biased/de-sparsified lasso. We also propose a joint information criteria which combines the approach of Bai & Ng (2002) with an extra penalty allowing for simultaneous lag-length estimation of the VAR model. The benefit of our proposed procedure is confirmed through extensive simulations where different levels of sparsity, number of factors, lag-length of the VARs and idiosyncratic covariance matrix are considered for different sample sizes and dimensions. We also compare our combined procedure with the standard high-dimensional forecasting methods which fully rely on either a sparse or a dense procedure. Empirically, we also show our method can be deemed beneficial for macroeconomic forecasting. We indeed show how the combined procedure helps in refining forecasts for many key macroeconomic variables within the FRED-MD.

There already exists applications in the literature combining dynamic factor models with sparse vector autoregressive models, see e.g., Barigozzi & Hallin (2017), Barigozzi & Brownlees (2019). However, Barigozzi & Hallin (2017), Barigozzi & Brownlees (2019) do not present theoretical results about the combined approach. In the non-dynamic set-up, Kneip et al. (2011), Fan et al. (2020) combine factors with regularized models. Since regularized methods as the lasso have difficulties with strongly correlated regressors, especially in the context of model selection, their aim is to decorrelate the regressors by adjusting for the factors. Furthermore, Fan et al. (2021) provide hypothesis tests to test whether after removing factors (as well as trends in a first step) the regressors possess some pre-defined weakly correlated structure or not. Fan et al. (2020, 2021) allow for time-dependent regressors, however they do not consider nor allow that the idiosyncratic part follows a sparse vector autoregressive model where the cross-sectional sparsity can grow with the sample size. In the context of high-dimensional VAR models, another approach is to consider the slope matrices as a combination of a low-rank matrix and a sparse matrix as done in Basu et al. (2019). The low-rank part takes here a similar role as the common component of DFM's. The combination of low-rank plus sparse has been also explored in the context of an approximate factor model by Lin & Michailidis (2020). However, these approaches differ from ours and their results are more useful in quantifying the overall estimation and prediction error whereas in this paper also the prediction error for one component of the multivariate system can be quantified. A more detailed discussion can be found in Remark 3.

The remainder of the paper is organized as follows: Section 2 introduces the dynamic factor model with sparse VAR idiosyncratic components and reports few standard assumptions defining its behavior. Section 3 is devoted to describe the two-step procedure used to estimate the DFM with sparse VAR idiosyncratic components and prove its consistency. Theorem 1 derives a representation of the idiosyncratic components estimation error while Theorem 2 is the main result establishing bounds for the estimation error for the second step of the estimation procedure, i.e., for the lasso on the sample estimates of the idiosyncratic component. The same two-step procedure with mild additional assumptions can also be employed in estimating the spectral density of the process and Theorem 4 derives the relative estimation error bounds. Section 4 considers the problems of: estimating the number of factors, determining the lag-length in the VAR and tuning the penalty parameter for the lasso. Section 5 reports simulation results for our proposed
method under different VAR data generating processes in terms of design and sparsity. In Section 6 the proposed DFM with sparse VAR idiosyncratic components is used to forecast macroeconomic variables within the FRED-MD dataset. The forecasting performances are compared with several other techniques using either only sparse or dense approaches. Partial coherence networks of idiosyncratic components are also computed, demonstrating the presence of cross-sectional idiosyncratic dependence and hence the need for its modeling, as advocated. Finally, Section 7 concludes.

A few words on notation. Throughout the paper we use boldface characters to indicate vectors and boldface capital characters for matrices. For any n-dimensional vector \( x \), we let \( \| x \|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \) denote the \( \ell_p \)-norm and \( e_j = (0,\ldots,0,1,0,\ldots,0)^\top \) denotes a unit vector of appropriate dimension with the one appearing in the \( j \)th position. Furthermore, for a \( r \times s \) matrix \( A = (a_{i,j})_{i=1,\ldots,r; j=1,\ldots,s} \), \( \| A \|_1 = \max_{1 \leq j \leq s} \sum_{i=1}^r |a_{i,j}| = \max_j \| Ae_j \|_1 \), \( \| A \|_\infty = \max_{1 \leq i \leq r} \sum_{j=1}^s |a_{i,j}| = \max_i \| e_i^T A \|_1 \) and \( \| A \|_{\text{max}} = \max_{i,j} |a_{i,j}| \). \( A^i \) denotes the \( i \)th matrix power of \( A \) and \( A^{(i)} \) refers to the \( i \)th element of a sequence of matrices. Denote the largest absolute eigenvalue of a square matrix \( A \) by \( \sigma_{\text{max}}(A) \) and let \( \| A \|_2^2 = \sigma_{\text{max}}(AA^\top) \). We denote the smallest eigenvalue of a matrix \( A \) by \( \sigma_{\text{min}}(A) \). For any index set \( S \subseteq \{1,\ldots,n\} \), let \( x_S \) denote the sub-vector of \( x_t \) containing only those elements \( x_i \) such that \( i \in S \). \( \| x \|_0 \) denotes the number of non-zero elements of \( x \).

2. The Model

We work with the Dynamic Factor Model where both factors and idiosyncratic components are allowed to be stationary stochastic processes and the loadings are static\(^3\). To elaborate, let \( x_t = (x_{1,t},\ldots,x_{N,t})^\top \), \( t = 1,\ldots,T \), be a \( N \times T \) rectangular data array representing a finite realization of an underlying real-valued stochastic process \( \{x_{i,t}\} \). Assume that for each \( t \), \( x_t \) can be decomposed into a sum of a common component \( \chi_t = (\chi_{1,t},\ldots,\chi_{N,t})^\top \) and an idiosyncratic component \( \xi_t = (\xi_{1,t},\ldots,\xi_{N,t})^\top \), both latent. Then the factor model decomposition takes the following usual form

\[
  x_t = \chi_t + \xi_t. \tag{1}
\]

Furthermore, the common component \( \chi_t \) is driven linearly by an \( r \)-dimensional vector of common factors \( f_t = (f_{1,t},\ldots,f_{r,t})^\top \), where \( r \) is considered as fixed as both the cross sectional dimension \( N \) and the time series dimension \( T \) grow large and \( r \ll N \).\(^4\) That is, the common components \( \chi_{i,t} \) can be expressed as the following linear combination

\[
  \chi_{i,t} = \ell_{i,1} f_{1,t} + \ell_{i,2} f_{2,t} + \cdots + \ell_{i,r} f_{r,t} = \Lambda^i_1 f_t, \tag{2}
\]

where \( \ell_{i,t} \) are denoted as loadings. Note that \( \chi_{i,t} \) is uniquely defined. But since for some rotation matrix \( H, \chi_{i,t} = \Lambda^i_1 H H^{-1} f_t \) is a valid linear combination as well, \( \Lambda^i_1 \cdot f_t \) are only identified up to some

\(^3\)Even if the factors are dynamic, the relationship between \( x_t \) and \( f_t \) is assumed to be static. This differs from the framework of Forni et al. (2000) which assumes a pervasive dynamics of the common factors where \( x_t \) is set to also depend on \( f_t \) with lags in time. However, in some cases it is possible to transform a dynamic relationship into a stacked static relationship, see among others Section 2.1.2 in Stock & Watson (2016).

\(^4\)This is a reasonable claim: assuming \( r \) to be a strictly increasing function in \( N \) or \( T \) would be tantamount to assume that all the eigenvalues of a large dimensional covariance matrix would necessarily diverge as the dimensions increase which would clearly not be reasonable.
arbitrary rotation.

As we are in the context of a Dynamic Factor Model we do not assume that the factors $f_t$ nor the idiosyncratic component $\xi_t$ are independent and identically distributed but we allow them to be dynamic. To elaborate, we consider that the factors are given by a one-sided linear process, see Assumption 2 below. This includes the cases that the factors are driven by a stable vector autoregressive (VAR) model. Additionally, the idiosyncratic component $\xi_t$ is considered to follow a sparse VAR model of order $p$, where $p$ is the lag-length, as

$$\xi_t = \sum_{j=1}^{p} A^{(j)} \xi_{t-j} + v_t = \sum_{j=0}^{\infty} B^{(j)} v_{t-j}, \quad (3)$$

for $v_t$ being a white noise process, $A^{(j)}$ the sparse slope matrices and $B^{(j)}$ the moving average matrices, see Assumption 1 and 2 below for details on sparsity and moment conditions.

To aid the intuition of a VAR modeling of the idiosyncratic component, one can think about the asset pricing models designed to explain asset returns through several factors of risk. The common components would represent here the systematic, unobserved, part of information explaining the asset return, in other words: those risk components which, systemically interconnected, decide the level of the asset return. The idiosyncratic part instead, is the whole remaining non-systematic or individual part of information pertaining to the single assets which also contributes in driving the return of the asset. It follows that in order to model the linear dependence among the idiosyncratic part, the most reasonable choice is the VAR.

In the following Assumptions 1, 2, and 3, the sparsity and stability conditions, the factors, moment conditions, and loadings are further specified.

**Assumption 1. (Sparsity and stability)**

(i) Let $A$ denote the stacked (companion) VAR matrix of (3). Let $k$ denote the row-wise sparsity of $A$ with approximate sparsity parameter $q \in [0, 1)$, i.e.,

$$\max_i \sum_{s=1}^{p} \sum_{j=1}^{N} |A^{(s)}_{i,j}|^q = \max_i \sum_{j=1}^{Np} |A_{i,j}|^q \leq k.$$

(ii) The VAR process is considered as stable such that for a constant $\rho \in (0, 1)$ we have independently of the sample size $T$: $\|A^T\|_2 = \sqrt{\lambda_{\max}(A^T A)} \leq M\rho^j$, where $M$ is some finite constant. Additionally, we have $\|\Gamma_{\xi}(0)\|_\infty \leq k_2 M$, where $\Gamma_{\xi}(0) = \text{Var}(\xi_t)$ and $\sigma_{\min}(\text{Var}((\xi_{t}^\top, \ldots, \xi_{t-p+1}^\top)) > \alpha$. The sparsity parameter $k$ as well as $k_\xi$ are allowed to grow with the sample size.

**Assumption 2. (Factor dynamic and moments)**

The factors are given by a one-sided linear filter with geometrically decaying coefficients, that is:

$$f_t = \sum_{j=0}^{\infty} D^{(j)} u_{t-j},$$

and $\|D^{(j)}\|_2 \leq K\rho^j$, where $K$ is some positive constant and $\rho \in (0, 1)$. Furthermore, $\{(u_t^\top, v_t^\top)^T, t \in \mathbb{Z}\}$ is an i.i.d. sequence and $\text{Cov}(u_t, v_t) = 0$. Let $\zeta > 8$ be the number of finite moments of $\{(u_t^\top, v_t^\top)^T, t \in \mathbb{Z}\}$, i.e., $E|u_{t,j}|^\zeta \leq M$ and $\max_{\|w\|_2 \leq 1} E|w^\top v_t|^\zeta \leq M$. We denote $\Sigma_u := \text{Var}(u_t)$ and $\Sigma_v := \text{Var}(v_t)$.
Assumption 3. (Factors and loadings)

Let $M$ be some finite constant, then

1. $\lim_{T \to \infty} 1/T \sum_{t=1}^{T} f_i f_i^\top = E[f_i f_i^\top] = \Sigma_F \in \mathbb{R}^{r \times r}$ positive definite and $\|\Sigma_F\|_2 \leq M$.

2. $\lim_{N \to \infty} 1/N \sum_{i=1}^{N} A_i A_i^\top = \Sigma_A \in \mathbb{R}^{r \times r}$, positive definite with largest eigenvalue $\sigma_{\lambda, \max} < M < \infty$ and smallest eigenvalue $\sigma_{\lambda, \min} > 1/M > 0$, $\|1/N \sum_{i=1}^{N} A_i A_i^\top\|_2 \leq M$ for all $N$.

3. All eigenvalues of $\Sigma_F$, $\Sigma_A$ are distinct.

Some comments on the above assumptions are in order. Assumption 1 and 2 imply that $\{\xi_t\}$ is stationary and let the autocovariance function be given by $\Gamma_\xi(s-t) = \text{Cov}(\xi_s, \xi_t)$. Furthermore, Assumption 2 implies that the factors are also a stationary process such that $\{x_t\}$ itself is indeed stationary. In order to quantify the dependence of stochastic processes, we use the concept of functional dependence, see Wu (2005). Since this is only necessary for the proofs, we do not introduce the notation here and refer to Remark 4 in the appendix. Let us emphasize that the weaker assumption of approximate sparsity instead of exact sparsity (i.e., $q = 0$), is used throughout. In the context of forecasting, the assumption of a stable and row-wise sparse VAR model is a standard assumption in the literature of sparse VAR models, see among others Kock & Callot (2015), Han et al. (2015), Masini et al. (2019). When the focus is on the estimation of the dependency structure, e.g., spectral density matrices, additional column-wise sparsity seems unavoidable, see Krampe & Paparoditis (2021) for a discussion of different sparsity concepts for VAR models. Within the paper, when it comes to forecasting in Section 3.1, we will require only row-wise sparsity, and in Section 3.2, where spectral density estimation is discussed, we also require column-wise sparsity.

In the established literature on factor estimation, a common assumption is to restrict the growth of the linear dependence of the idiosyncratic component. For instance, Assumption 3C in Bai (2003) states that the absolute sum of all covariances of the idiosyncratic component grow with order $N$. Here, we quantify the linear dependence of the idiosyncratic component by the condition $\|\Gamma_\xi(0)\|_\infty = \max_i \sum_{j=1}^{N} |\text{Cov}(x_{i,t}, x_{j,t})| \leq k_\xi$ and the object $k_\xi$. Since $\|\Gamma_\xi(0)\|_\infty \leq \sqrt{N} \|\Gamma_\xi(0)\|_2$, $\sqrt{N}$ is an upper bound for the growth rate of $k_\xi$. We do not specify here a rate for $k_\xi$ but a rate smaller than $\sqrt{N}$ seems most realistic and would be more in line with established assumptions in the factor literature. The reason for this is that a growth rate of $\sqrt{N}$ would allow that the absolute sum of all covariances could grow with rate $N^{3/2}$. This would violate, for instance, the previously mentioned Assumption 3C in Bai (2003). Note further, that if the maximal absolute row sum of the covariance of the idiosyncratic component is growing way faster than the average row sum, we may end up in the context of weak factors, see among others Onatski (2012). Furthermore, Assumption 1 gives the upper bound $k^2 \|\Sigma_v\|_\infty$ for the growth rate of $k_\xi$, where $\Sigma_v = \text{Var}(v_t)$ is the variance matrix of the residuals of the idiosyncratic component. But since this requires a parameter for $\|\Sigma_v\|_\infty$ and this is not sharp, we work with $k_\xi$. The moment condition in Assumption 2 refers to the situation in which only a finite number of moments, here $\zeta$, are finite. Hence, we do not assume sub-Gaussian processes or similar, which is often assumed for sparse VAR processes, see Basu & Michailidis (2015), Kock & Callot (2015), Han et al. (2015). For sub-Gaussian processes the polynomial terms depending on $\zeta$ would vanish, which would result in tighter error bounds obtained later on. The reason for only assuming $\zeta$ finite moments is to be more in line with the classical factor literature,
see among others Bai (2003), Stock & Watson (2002), Forni et al. (2000, 2017). E.g., Bai (2003) derived inferential results for factor models under 8th finite moments of the idiosyncratic part and 4th finite moments of the factors. Note that the filter in Assumption 2 can be the one-sided representation of a stable VARMA model as in Assumption 2 in Forni et al. (2017). Assumption 3 is a standard assumption in the context of strong factor models, see Stock & Watson (2002), Bai (2003). It implies that each of the factors provides a non-negligible contribution to the variance of each component of \( \{x_t\} \). We would like to point out here that the time and cross-sectional dependence of the idiosyncratic component is only limited by assuming that it follows a sparse VAR model. Furthermore, it is not clear if assuming a sparse VAR model for the idiosyncratic part is a special case of the assumptions to time and cross-section dependence in the factor literature, see among others Bai (2003), Assumption C or Forni et al. (2017), Assumption 4. The reason for this is that the sparsity is not fixed but it can grow with the sample size. Nevertheless, the error bounds obtained later on requires that the sparsity cannot grow too fast with increasing dimension.

3. Estimation

In this section we propose a two-step approach to estimate a DFM with sparse VAR idiosyncratic components and prove its consistency. For this, let \( x_t, t = 1, \ldots, T \) be some observations and let \( X = \chi + \Xi \) denote the \( T \times N \) matrix form of (1). Furthermore, \( \Lambda \) denotes the \( N \times r \) matrix of loadings and \( F \) denotes the \( T \times r \) matrix of factors such that \( \chi = F\Lambda^\top \) is the matrix counterpart of (2). Then, an estimation of the factor decomposition can be obtained by using Principal Components Analysis (PCA), see among others Bai (2003), Bai & Ng (2020). To elaborate, let

\[
X/\sqrt{NT} = U_{NT}D_{NT}V_{NT}^\top,
\]

denote a singular value decomposition of \( X/\sqrt{NT} \) such that \( D_{NT} \) is a diagonal matrix with the singular values arranged in descending order on its diagonal. \( U_{NT} \) and \( V_{NT} \) are the corresponding left and right singular vectors, respectively. This can be further written as

\[
U_{NT}D_{NT}V_{NT}^\top = U_{NT,r}D_{NT,r}V_{NT,r}^\top + U_{NT,N-r}D_{NT,N-r}V_{NT,N-r}^\top,
\]

where \( D_{NT,r} \) is a diagonal matrix with the first \( r \) largest singular values, \( d_{NT,1}, \ldots, d_{NT,r} \), arranged in descending order on its diagonal, \( D_{NT,N-r} \) is a diagonal matrix with the remaining \( N-r \) largest singular values, and \( U_{NT,r}, U_{NT,N-r}, V_{NT,r}, V_{NT,N-r} \) are the corresponding left and right singular vectors. Then, the estimators of a rotated version of \( F \) and \( \Lambda \) are given by

\[
\hat{F} = \sqrt{T}U_{NT,r} \text{ and } \hat{\Lambda} = \sqrt{N}V_{NT,r}D_{NT,r},
\]

such that \( \hat{\chi} = \hat{F}\hat{\Lambda}^\top \) and \( \hat{\xi} = x_t - \hat{\chi} \). This uses the normalization \( \hat{F}^\top \hat{F}/T = I_r \) and \( \hat{\Lambda}^\top \hat{\Lambda} \) is a diagonal matrix. Consider the estimated idiosyncratic components \( \{\hat{\xi}_t\} \). As it is assumed that \( \{\xi_t\} \) follows a sparse vector autoregressive model, we estimate this sparse VAR on \( \{\hat{\xi}_t\} \) by regularized methods such as the (adaptive) lasso. This idea leads to the following two-step estimation procedure:
1. Perform a singular value decomposition of

\[ X/\sqrt{NT} = U_{NT,r}D_{NT,r}V_{NT,r}^T + U_{NT,N-r}D_{NT,N-r}V_{NT,N-r}^T, \]

where \( U_{NT,r}, D_{NT,r}, V_{NT,r} \) corresponds to the first \( r \) singular values.

Set \( \bar{F} = \sqrt{T}U_{NT,r}, \bar{A} = \sqrt{N}V_{NT,r}, \) and \( \bar{\xi} = x_t - \bar{F} \bar{A}^T. \)

2. Let \( \hat{\xi}_t^i = (\bar{\xi}_t^i, \ldots, \bar{\xi}_t^i(\ell_p))^T. \) Then, an adaptive lasso estimator for \( \beta^{(j)} \) i.e., the \( j \)th row of \((A^{(1)}, \ldots, A^{(p)})\), is given by

\[
\hat{\beta}^{(j)} = \arg \min_{\beta \in \mathbb{R}^p} \frac{1}{T-p} \sum_{t=p+1}^{T} (\hat{\xi}_t^j - \beta^T \hat{\xi}_{t-1}^j)^2 + \lambda \sum_{j=1}^{N} |g_i \beta_i|, \quad j = 1, \ldots, N, \tag{4}
\]

where \( \lambda \) is a non-negative tuning parameter which determines the strength of the penalty and \( g_i, i = 1, \ldots, N, \) are weights. For instance, \( g_i = 1 \) leads to the standard lasso. Let \((\hat{A}^{(1)}, \ldots, \hat{A}^{(p)})\) be matrices corresponding to stacking \( \beta^{(j)}, j = 1, \ldots, N. \)

By estimating the factors \( f_t \) through standard Principal Components Analysis (PCA) and the sparse VAR models of the idiosyncratic components \( \xi_t \) via sparse penalized regression techniques, we combine a dense estimation approach with a sparse one. This can possibly better capture and disentangle both the dependence coming from the diverging eigenvalues of \( \mathbb{E}(XX^T) \), i.e., the factors, as well as the dependence coming from the non-diverging eigenvalues of \( \mathbb{E}(XX^T) \), i.e., the idiosyncratic components.

The estimation of factors and loadings via PCA is a well established method in the literature, see among others Stock & Watson (2002), Bai (2003), and the common and idiosyncratic component can be estimated with rate \( O_P(\max(1/\sqrt{T}, 1/\sqrt{N})) \). Since it is no different in this setting, we focus our presentation on the second estimation step. For a sparse stationary VAR model, deviation bounds and restricted eigenvalue conditions can be established, see among others Basu & Michailidis (2015), Kock & Callot (2015). Given these, the consistency of the lasso can be derived and, under additional Gaussianity assumption, one obtains a rate of \( O_P(s \sqrt{\log(N)/T}) \). However, as the idiosyncratic component \( \{\xi_t\} \) is not observed in our setting and hence needs to be estimated, the regression in Step 2 is performed only with the estimated idiosyncratic component. Consequently, the aforementioned results cannot be applied here.

Before analyzing the second step, we in fact need to quantify the estimation error \( w_t := \hat{\xi}_t - \xi_t \) arising from the first step. For the consistency of the lasso, this means quantifying the estimation error \( w_t \) in quantities such as \( \|1/T \sum_{t=1}^{T}(\xi_t + w_t)(\xi_t + w_t)^T\|_{max}. \) If we simply apply the rate derived in the literature for approximate factor models, see among others Stock & Watson (2002), Bai (2003) which derive \( w_t = O_P(\max(1/\sqrt{T}, 1/\sqrt{N})) \), we would obtain \( \|1/T \sum_{t=1}^{T}(\xi_t + w_t)(\xi_t + w_t)^T\|_{max} = \|1/T \sum_{t=1}^{T} \xi_t \xi_t^T\|_{max} + O_P(\max(1/\sqrt{T}, 1/\sqrt{N})). \) This may lead to a rate for the second step of \( O_P(s \sqrt{\log(N)/T + s/\sqrt{N}}) \). However, this can be improved if we analyze the estimation error \( w_t \) more closely. For this, we follow the idea of the decomposition in eq. (6) in Bai & Ng (2020). To elaborate, we have \( 1/(NT)XX^T \hat{F} = \bar{F} \bar{D}_{NT,r}^2. \) Plugging in (1), and using the rotation matrix

\[
H_{NT} = (A^T A/N)(P^T \hat{F}/T)D_{NT,r}^2, \tag{5}
\]

we obtain the following representation for the error between the estimated factors and a rotated version
of the true factors
\[ \hat{f}_t - H_{NT}f_t = \frac{1}{NT} \left[ \sum_{i=1}^{N} \sum_{s=1}^{T} f_i^\top \Lambda_i \xi_{i,s} \hat{f}_s + \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,s} \Lambda_i f_s^\top \hat{f}_s + \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,s} f_s^\top \hat{f}_s \right] D_{NT,r}^{-2}. \]  \tag{6}

Similarly, we obtain by symmetry for the loadings
\[ (H_{NT}^\top)^{-1} \Lambda_i - \Lambda_i = \frac{1}{T} \left[ \sum_{s=1}^{T} H_{NT} f_s \xi_{i,s} + \sum_{s=1}^{T} (\hat{f}_s - H_{NT} f_s) \xi_{i,s} + \sum_{s=1}^{T} H_{NT} f_s (\hat{f}_s - H_{NT} f_s)^\top (H_{NT}^\top)^{-1} \Lambda_i \right. \]
\[ \left. \sum_{s=1}^{T} (\hat{f}_s - H_{NT} f_s) (\hat{f}_s - H_{NT} f_s)^\top (H_{NT}^\top)^{-1} \Lambda_i \right]. \]

These representations can be used to derive the order of the estimation error for the factors and loadings as it is done with a slightly different rotation matrix in Bai (2003). However, as our focus is on \( w_t := \hat{\xi}_t - \xi_t \), we use these results to derive a simpler representation of \( w_t \). With the obtained representation for \( w_t \), we can analyze more closely the estimation error of the second step. For this, note first that \( \|1/T \sum_{t=1}^{T} (\xi_t + w_t) (\xi_t + w_t)^\top \|_{\max} \leq \|1/T \sum_{t=1}^{T} (\xi_t)^\top (\xi_t)^\top \|_{\max} + \|1/T \sum_{t=1}^{T} (w_t) (w_t)^\top \|_{\max} \). Bounds for the latter objects and the representation of \( w_t \) are given in the following Theorem 1.

**Theorem 1.** Under Assumption 1, 2, and 3, we have for \( t = 1, \ldots, T, j = 1, \ldots, N \) and
\[ w_{j,t} := \hat{\xi}_{j,t} - \xi_{j,t} = \Lambda_t^\top H_{NT}^{-1} \frac{1}{NT} \left[ \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,s} \Lambda_i f_s^\top H_{NT} f_s + \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,s} \Lambda_i f_s^\top H_{NT} f_s \right] D_{NT,r}^{-2}, \]
\[ + f_t^\top H_{NT}^{-1} \frac{1}{T} \left[ \sum_{s=1}^{T} H_{NT} f_s \xi_{j,s} \right] + \text{Error}_j, \]  \tag{7}

where
\[ \max_j |\text{Error}_j| = O_P \left( \frac{\log(N)}{T} + \frac{k_\xi}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N,T,\zeta) \right), \]

with
\[ g(N,T,\zeta) = (NT)^{\zeta/3} \left( \frac{1}{\sqrt{NT}} + \frac{1}{T^{3/2}} + (NT)^{\zeta/3} \frac{1}{T^2} \right). \]

Furthermore, we have for \( k \in \{0,1\} \)
\[ \left\| \frac{1}{T} \sum_{t=1}^{T} w_t \xi_{i-k}^\top \right\|_{\max} = O_P \left( \frac{\log(N)}{T} + \frac{k_\xi}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N,T,\zeta) \right), \]
\[ \left\| \frac{1}{T} \sum_{t=1}^{T} w_t \xi_{i-k}^\top \right\|_{\max} = O_P \left( \frac{\log(N)}{T} + \frac{k_\xi}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N,T,\zeta) \right). \]

If \( N = T^a \) and \( a \leq \zeta - 4 \), we have \( g(N,T,\zeta) \leq 1/\sqrt{NT} + 1/T \) which means \( g(N,T,\zeta) \) could be dropped in the above \( O_P \) terms.

We focus here on the lasso itself but the above theorem is also helpful for obtaining rates for the de-sparsified/de-biased lasso in this framework. As mentioned previously, if we just plug-in the rate for \( w_t \) we would obtain the slower rate of \( O_P(\max(1/\sqrt{T}, 1/\sqrt{N})) \). With the results above we can establish bounds for the estimation error of the second step, see the following Theorem 2.
Theorem 2. Under Assumption 1, 2, and 3 we have for $j = 1, \ldots, N$

$$
\|\hat{\beta}^{(j)} - \beta^{(j)}\|_1 = O_P\left(k \left[ \frac{\sqrt{\log(N)}}{\sqrt{T}} + \frac{(NpT)^{2/\zeta}}{T} + k\left(\frac{k_\xi}{N} + \frac{\sqrt{\log(Np)}}{\sqrt{NT}} + (p)^{2/\zeta}g(N,T,\zeta)\right)\right]^{1-q}\right)
$$

(8)

$$
\|\hat{\beta}^{(j)} - \beta^{(j)}\|_2 = O_P\left(\sqrt{k} \left[ \frac{\sqrt{\log(N)}}{\sqrt{T}} + \frac{(NpT)^{2/\zeta}}{T} + k\left(\frac{k_\xi}{N} + \frac{\log(Np)}{T} + \frac{\sqrt{\log(Np)}}{\sqrt{NT}} + (p)^{2/\zeta}g(N,T,\zeta)\right)\right]^{1-q/2}\right)

+ k^{3/2} \left[ \frac{\sqrt{\log(N)}}{\sqrt{T}} + \frac{(NpT)^{2/\zeta}}{T} + k\left(\frac{k_\xi}{N} + \frac{\log(Np)}{T} + \frac{\sqrt{\log(Np)}}{\sqrt{NT}} + g(N,T,\zeta)\right)\right]^{(3-q)/2}.
$$

(9)

Let us have a closer look on the bound $\|\hat{\beta}^{(j)} - \beta^{(j)}\|_1$ and consider $N = T^a, p = T^b$ for some $a, b > 0$ and let $\zeta \geq 4(1 + a + b)$. Then, for $j = 1, \ldots, N$

$$
\|\hat{\beta}^{(j)} - \beta^{(j)}\|_1 = O_P\left(k\left[\sqrt{\log(Np)/T} + kk_\xi/N + k\sqrt{\log(Np)/(NT)}\right]^{1-q}\right),
$$

and $\|\hat{\beta}^{(j)} - \beta^{(j)}\|_2 = O_P\left(\sqrt{k}\left[\sqrt{\log(Np)/T} + kk_\xi/N + k\sqrt{\log(Np)/(NT)}\right]^{1-q/2} + k^{3/2} \left[\sqrt{\log(Np)/T} + kk_\xi/N + k\sqrt{\log(Np)/(NT)}\right]^{(3-q)/2}\right)$. Note that if sparsity is considered such that $\|\hat{\beta}^{(j)} - \beta^{(j)}\|_1 = o_P(1)$, then the terms with $k^{3/2}$ upfront in the error bound $\|\hat{\beta}^{(j)} - \beta^{(j)}\|_2$ are of higher order and negligible. That means, if $\|\hat{\beta}^{(j)} - \beta^{(j)}\|_1 = o_P(1)$, we have

$$
\|\hat{\beta}^{(j)} - \beta^{(j)}\|_2 = O_P\left(\sqrt{k}\left[\sqrt{\log(Np)/T} + kk_\xi/N + k\sqrt{\log(Np)/(NT)}\right]^{1-q/2}\right).
$$

For an even closer look on the bound $\|\hat{\beta}^{(j)} - \beta^{(j)}\|_1$, let $k = T^c, k_\xi = T^d$ for some $c, d > 0$. Then, we can simplify the error bound in $O$-notation and obtain

$$
\|\hat{\beta}^{(j)} - \beta^{(j)}\|_1 = O_P(\log(T)^{(1-q)/2}T^{c- (1-q)/2} + T^{c+(1-q)(c+d-a)}).
$$

That means a consistent estimation is obtained if $c < 1/2(1 - q)$ and $c/(1 - q) + c + d < a$. The first condition, i.e., this upper bound on the sparsity in relation to sample size $T$, is standard for approximately sparse models, see among others Corollary 2.4 in van de Geer (2016). In contrast, the second condition is not standard for approximately sparse models and appears due to the estimation error of the first step. This condition reflects the error occurring in factor models which are due to the introduced dependency of the VAR model, not exact but only approximate. Hence, the time and cross-sectional dependency of the idiosyncratic component is quantified by $k$ and $k_\xi$, i.e., $c$ and $d$. This dependence cannot be too strong in relation to the dimension such that it can be averaged out and a satisfactory estimation of the common and idiosyncratic component can be obtained.

Remark 1 (Estimation with Strong Idiosyncratic Components). In the error bounds in Theorem 2, the factor $k_\xi/N$ plays an important role. $k_\xi = \|\text{Var}(\xi_\ell)\|_\infty$ quantifies the serial dependence of the idiosyncratic component. If this is large, the estimation in all steps suffers. Motivated by Generalized Least Squares (GLS), Boivin & Ng (2006) proposes to weight the data such that the serial dependence of the idiosyncratic component can be decreased. This approach is also denoted generalized principal component analysis and it is analyzed in more detail in Choi (2012). Let $W \in \mathbb{R}^{N \times N}$ be a matrix of weights, then the factors are
estimated using the weighted data $XW$. Note that we have $\text{Var}(XW) = WW^\top \Sigma \Lambda \Sigma^\top W + W^\top \xi(0) W$. Hence, the factors can be estimated by a PCA of $XW$ whereas the loadings are obtained by regressing $X$ onto the estimated factors. Since non-diagonal weighting schemes are seldom feasible without sparsity constraints, Boivin & Ng (2006) suggest different diagonal weighting schemes. With the additional assumption that the $\Sigma_v$ -- the variance matrix of the idiosyncratic innovation $v_t$ -- is sparse, we suggest to use the VAR structure of the idiosyncratic component to obtain a more refined weighting scheme. To elaborate, we have that $\text{Var}(\xi_t) = \Gamma(0) = \sum_{j=0}^\infty B^{(j)} \Sigma_v (B^{(j)})^\top$, where $(B^{(j)})_{r,c} = (A^{(j)})_{r,c} \in \mathbb{R}^{N \times N}$. Hence, $\Gamma(0)$ is given by $A^{(1)}, \ldots, A^{(p)}$, $\Sigma_v$, and it can be estimated by plugging in estimators, see among others Theorem 5 in Krampe & Paparoditis (2021). Let us denote this estimator as $\hat{\Gamma}(0)$. Depending on whether sparsity constraints on $\Sigma_v$ or $\Sigma_v^{-1}$ are more realistic, estimators are given by thresholding of the empirical covariance matrix (Bickel & Levina, 2008, Cai & Liu, 2011) or by component-wise regularized regression (Friedman et al., 2008, Cai et al., 2016a,b). The weighting matrix is then given as $W = \hat{\Gamma}(0)^{-1/2}$. Consequently, the “new” $k_\xi$ is given by $\|\hat{\Gamma}(0)^{-1/2} \Gamma(0) \hat{\Gamma}(0)^{-1/2}\|_\infty$ which can be considerably smaller if the used estimators give reasonable results. Since the weighting leads also to a new estimation of the idiosyncratic component, it might be helpful to apply this approach more than once.

**Remark 2** (*AR filtering*). Ignoring persistency in the time series within a context of PCA-estimated factor models introduces some issues. PCA exploits contemporaneous correlations and not time correlations of the data and it is in fact well suited for settings where observations are independent in time. Assuming $X$ to be defined by an MA($\infty$) process as $X = \mu + \sum_{j=0}^\infty \Psi_j \epsilon_{t-j}$ for $\mu \in \mathbb{R}^N$, $\Psi_j$ real $N \times N$ coefficient matrices and $\epsilon_t$ an $N$-dimensional white noise process, autocorrelation in $X$ has the effect of increasing the variance of the principal components of $X$, making it larger than the variance of the principal components of $\epsilon_t$ (see Zamprogno et al., 2020, for a proof of this fact). Furthermore, in practice, as observed in our simulations in Section 5, whenever the time series length is not sufficiently large, weak dependence that belongs to the idiosyncratic components can be misinterpreted as dependence due to the factor structure (Greenaway-McGrevy et al., 2012), thus deteriorating the advantages of our modeling of the idiosyncratic components for the forecast. While this is not the case for our macroeconomic application in Section 6, given the time dimension is as large as $T = 743$, it is nonetheless of practical relevance. For these reasons, when using our method we suggest to pre-apply a linear AR filter to the series before computing the forecasts and select the lag-length via BIC.

**Remark 3** (*Similarity and Differences to Low-Rank plus Sparse Models*). As mentioned in Section 1, the low-rank plus sparse VAR model discussed in Basu et al. (2019), as well as the linear dynamical system discussed in Lin & Michailidis (2020), are related to the model proposed here. We now stress the similarities and differences of these models beginning with the model of Basu et al. (2019). The low-rank plus sparse VAR model of order $p$ is given by $x_t = \sum_{j=1}^p \Theta^{(j)} x_{t-j} + \epsilon_t$, where the coefficient matrix can be decomposed as $\Theta^{(j)} = L^{(j)} + S^{(j)}$, and rank$(L^{(j)}) = r \ll N$. $\epsilon_t$ is some white-noise process, $L^{(j)}$ is a low-rank matrix, and $S^{(j)}$ possesses some type of sparsity structure. The low-rank matrix takes here the role of the common component, see also Bai & Ng (2019). Thus, this approach also combines a dense and a sparse approach. However, there are two major differences to the approach presented in this paper. First,
note that while the low-rank plus sparse VAR model is some special form of a VAR(p) model, a dynamic factor model is instead in general a VAR(∞) process even if the factors and idiosyncratic components follow finite order VAR processes. Second and most important, with the approach presented here we can derive estimation error bounds for a single time series, see Theorem 3. This is in contrast to the results derived in Basu et al. (2019). They impose sparsity constraints on vec(S^(j))^5 and they do not estimate the VAR system row-wise as in (4). Instead, all regression equations are combined using the Frobenius norm, the VAR slope matrices are considered as a sum of two matrices where the first matrix is regularized using the nuclear norm – this imposes a low-rank structure – and the second matrix is regularized using the l_1 norm on the vectorized matrix – this imposes a sparse structure. They derive error bounds only regarding the Frobenius norm. That means they consider only the overall estimation error. In connection with the sparsity constraints on vec(S^(j)) this is too restrictive or too less detailed for the row-wise estimation error which is helpful for a forecast of a single time series. For a more detailed discussion of the different sparsity concepts and their implication regarding estimation error bounds we refer to Section 2 in Krampe & Paparoditis (2021). For the model discussed in Lin & Michailidis (2020) note first that a dynamic factor model x_t = Λf_t + ξ_t whose idiosyncratic component follows a VAR(p) model, ξ_t = \sum_{j=1}^{p} A^{(j)}ξ_{t-j} + v_t can be written as x_t = Λf_t - \sum_{j=1}^{p} A^{(j)}A^{(j)}x_{t-j} + v_t. The component Λf_t - \sum_{j=1}^{p} A^{(j)}A^{(j)}f_{t-j} can be considered as the common component of a general dynamic factor model as in Forni et al. (2000) and it is low-rank. Lin & Michailidis (2020) assume that the matrices A^{(j)}, j = 1, ..., p, are sparse and they impose sparsity constraints on vec(S^(j)). Furthermore, they combine all regression equations using the Frobenius norm and the low-rank part is handled by regularization of its nuclear norm. Similarly to Basu et al. (2019), they derive error bounds only regarding the Frobenius norm. That means for a forecast of a single time series the same drawbacks described above apply. Let us note here that it is not clear if the approach of handling the low-rank part by regularization of its nuclear norm can be also done equation-wise such that the error bounds are more helpful for a forecast of a single time series.

3.1. Forecasting

One of the main application of dynamic factor models is forecasting. In this paper, the temporal and cross-sectional dependence is not only modelled in the common component but also in the idiosyncratic component. Consequently, the forecast of both components can contribute to an accurate forecast of the original time series. We present here the forecast method for a one-step-ahead prediction. An h-step-ahead prediction can be done recursively^6. Based on the estimation method proposed in the previous section, the approach is as follows. First, a standard linear one-step-ahead prediction is computed based on the estimated factors. Combining this prediction with the loadings gives a prediction of the common component. Second, the sparse VAR model is used to get a prediction of the idiosyncratic component. Finally, the sum of these two predicted components gives the prediction of the original process x_t. To elaborate, consider first that the factors and idiosyncratic component are observed. Then,

^5Basu et al. (2019) consider also a group-sparse structure for S^(j). For this sparsity concept the discussion is quite similar.

^6If the main interest is on an h-step ahead forecast, a direct h-step-ahead can be more accurate, see among others Smeekes & Wijler (2018). A direct h-ahead forecast can be obtain by changing the regression equation from t to t + h.
let \( f_{T+1}^{(1,p)} = \sum_{j=1}^{p_j} \Pi_{j}^{(p)} f_{T+1-j} \) be the linear one-step-ahead prediction based on \( f_T, \ldots, f_{T-p_f} \), where \( \sum_{j=1}^{p_j} \Pi_{j}^{(p)} \Gamma_f(i-j) = \Gamma_f(i), i = 1, \ldots, p_f \) and \( \Gamma_f(i-j) = \mathbb{E} f_{T+i} f_{T+1-j}^{\top} \), see among others Section 11.4 in Brockwell & Davis (1991). Furthermore, since \( \{\xi_t\} \) follows a VAR\((p)\) model, \( \xi_T^{(1)} = \sum_{j=1}^{p} A(j) \xi_{T-j} \) is the one-step-ahead prediction for the idiosyncratic component. That means, \( X_T^{(1,p)} = \Lambda f_T^{(1,p)} + \xi_T^{(1)} \) is the joint one-step-ahead prediction for \( X_T+1 \) with the prediction error \( \text{Var}(X_T+1 - X_T^{(1,p)}) = \Lambda \text{Var}(f_T^{(1,p)} - f_T) \Lambda^\top + \Sigma_e \) and for a single variable \( j \) we have \( \text{Var}(e_T^j (X_T+1 - X_T^{(1,p)})) = \Lambda_j^\top \text{Var}(f_T^{(1,p)} - f_T) \Lambda_j + e_T^j \Sigma_e e_T^j. \) If \( \{f_t\} \) follows a VAR\((p_f)\) model, this simplifies to \( \text{Var}(X_T+1 - X_T^{(1,p)}) = \Lambda \Sigma_a \Lambda^\top + \Sigma_b. \)

Since the parameters are unknown and the factors and idiosyncratic component are latent, this approach is unfeasible but the results of Theorem 1 and 2 help to obtain a feasible approach. For this, we construct feasible counterparts of the prediction approach above. Let \( \hat{f}_{T+1}^{(1,p_f)} = \sum_{j=1}^{p_f} \hat{\Pi}_{j}^{(p_f)} \hat{f}_{T+1-j} \) be the linear one-step-ahead prediction based on \( f_T, \ldots, f_{T-p_f} \), where \( \sum_{j=1}^{p_f} \hat{\Pi}_{j}^{(p_f)} \hat{\Gamma}_f(i-j) = \hat{\Gamma}_f(i), i = 1, \ldots, p_f \), and \( \hat{\Gamma}_f(i-j) = 1/n \sum_{t=1+i}^{T-i+j} \hat{f}_{t+i} \hat{f}_{t-j} \). Furthermore, let \( \hat{\xi}_T^{(1)} = \hat{A}(\hat{\xi}_T^T, \ldots, \hat{\xi}_{T-p}^T) \) be the one-step-ahead prediction for the idiosyncratic component. Then, \( \hat{X}_T^{(1,p_f)} = \hat{\Lambda} \hat{f}_T^{(1,p_f)} + \hat{\xi}_T^{(1)} \) is the joint and feasible one-step-ahead prediction for \( X_T+1 \). Even though a high-dimensional time series system is considered, the interest is often in the prediction of some key time series. We quantify in the following Theorem 3 the estimation error between the feasible and unfeasible approach for a single time series.

**Theorem 3.** Under Assumption 1, 2, and 3 we have for \( j = 1, \ldots, N \)

\[
e_T^j (X_T^{1,p_f} - X_T^{(1,p_f)}) = O_P \left( \frac{1}{\sqrt{N}} + k \left( \frac{\sqrt{\log(Np)}}{\sqrt{T}} + \frac{(NpT)^{2/3}}{T} + k \left( k_p \frac{1}{N} + \frac{\sqrt{\log(Np)}}{\sqrt{NT}} + (p)^{2/3} g(N,T,\zeta) \right)^{1-q} \right) \right).
\]

In relation to the error bound for \( \|\hat{\beta}^{(j)} - \beta^{(j)}\|_1 \) derived in Theorem 2 only an additional \( 1/\sqrt{N} \) appears which arises due to the estimation of the factors.

**3.2. Estimation of the Spectral Density**

Spectral analysis offers a series of powerful tools for analyzing the second order (and beyond) properties of multiple time series. Parameters like the coherence or the partial coherence describe in a comprehensive way the linear relations between the components of the vector time series taking into account all lead and lag relations as well as the distinction between direct and indirect effects. Starting point for such an analysis are quite often an estimation of the spectral density matrix. When the dimension of the time series is small, the spectral density matrix is often estimated by non-parametric approaches as lag-window estimators or smoothed periodograms, respectively, see among others Brillinger (2001), Koopmans (1995), Hannan (1970). In a high dimensional set-up, the problem of estimating the spectral density matrix or its inverse has been extensively investigated in the literature during the last decade. One approach is here to combine the non-parametric lag-window estimators with regularization techniques developed for the covariance and precision matrix estimation, see among others Sun et al. (2018), Fiecas et al. (2019), Zhang & Wu (2021). Such approaches work under the assumption that the spectral density matrix or its inverse is sparse. However, a direct sparsity assumption on the spectral density matrix or its inverse is contradicting the assumption of the existence of strong factors. That means the factors need to be taken into account in the estimation of the spectral density matrix. The procedure developed in the
previous section can be used to obtain (under slightly modified assumptions) a consistent estimator of the inverse of the spectral density matrix. Since the VAR structure of the idiosyncratic component is used, we obtain a semi-parametric estimator for the inverse of the spectral density matrix. Our focus here is on the estimation of the inverse of the spectral density matrix. Let us mention, that the factors can be also taken into account by using a low-rank plus sparse approach applied to a smooth periodogram, see Barigozzi & Farnè (2021). This estimator differs, however, to the ones presented here in several aspects. First, the low-rank plus sparse approach describes in finite samples a different model then the approach used here, see also the discussion of low-rank plus sparse structures in Remark 3. Second, they focus on consistency results regarding \( \| \cdot \|_2 \) whereas we present here also row- and column-wise consistency results meaning consistency with respect to \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \).

Let us begin with defining the spectral density matrix of the time series \( \{X_t\} \) given by (1). The spectral density matrix of the factor process specified in Assumption 2 is given by

\[
f_f(\omega) = \left[ \sum_{j=0}^{\infty} D^{(j)} \exp(-ij\omega) \right] \Sigma_u \left[ \sum_{j=0}^{\infty} D^{(j)} \exp(ij\omega) \right]^\top, \quad \omega \in [0, 2\pi]
\]

and for the idiosyncratic component driven by a VAR\((p)\) we have

\[
f_\xi(\omega) = \left[ I_N - \sum_{j=1}^{p} A^{(j)} \exp(-ij\omega) \right]^{-1} \Sigma_u \left[ \left[ I_N - \sum_{j=1}^{p} A^{(j)} \exp(ij\omega) \right]^{-1} \right]^\top,
\]

with the inverse

\[
f_\xi(\omega)^{-1} = \left[ I_N - \sum_{j=1}^{p} A^{(j)} \exp(-ij\omega) \right] \Sigma_u^{-1} \left[ I_N - \sum_{j=1}^{p} A^{(j)} \exp(ij\omega) \right].
\]

That is, the spectral density of the process \( \{X_t\} \) is given by

\[
f_X(\omega) = \Lambda f_f(\omega) \Lambda^\top + f_\xi(\omega), \tag{10}\]

and its inverse using the Sherman–Morrison–Woodbury formula is given by

\[
f_X(\omega)^{-1} = f_\xi^{-1}(\omega) - f_\xi^{-1}(\omega) \Lambda \left[ f_f^{-1}(\omega) + \Lambda^\top f_\xi^{-1}(\omega) \Lambda \right]^{-1} \Lambda^\top f_\xi^{-1}(\omega). \tag{11}\]

We estimate \( f_X(\omega)^{-1} \) by estimating \( f_f^{-1} \) and \( f_\xi^{-1} \) separately. Note that the factors lead to an unbounded \( \| f_X(\omega) \|_2 \) for growing dimension but the inverse is stable, i.e., \( \| f_X(\omega)^{-1} \|_2 \) is bounded.

The spectral density \( f_f \) or its inverse, respectively, can be estimated by classical methods such as non-parametric lag-window estimators as it is of fixed dimension \( r \). For this, let \( K \) be a kernel fulfilling Assumption 1 in Wu & Zaffaroni (2018). That is, \( K \) is an even and bounded function with bounded support in \( (-1, 1) \), continuous in \( (-1, 1) \), \( K(0) = 1, \kappa = \int_{-1}^{1} K^2(u) \text{d}u < 1 \), and \( \sum_{l \in \mathbb{Z}} \sup_{|s-l|<1} |K(l\omega) - K(s\omega)| = O(1) \) as \( \omega \to 0 \). Furthermore, let \( B_T = T^b, b \in (0, 1) \) be the lag-window size fulfilling Assumption 2 in Wu & Zaffaroni (2018). Then, a spectral density estimator is given by

\[
f_f(\omega) = \frac{1}{2\pi} \sum_{h=-T+1}^{T-1} K\left( \frac{h}{B_T} \right) \exp(-ih\omega) \hat{F}(h), \tag{12}\]

where \( \hat{F}(h) \) is the sample autocovariance function \( \hat{F}(h) = 1/T \sum_{t=1}^{T} \hat{f}_{t+h} \hat{f}_t^\top \). Based on observations
results can be established for the graphical lasso estimator. Then, we construct the following estimator used. In the proofs we consider the CLIME method and denote this estimator by $\hat{f}$, Friedman et al. (2008) or (A)CLIME Cai et al. (2011, 2016a,b) can be procedures like graphical lasso. Friedman et al. (2008) or (A)CLIME Cai et al. (2011, 2016a,b) can be feasible) estimator of $A$. Such a column-wise consistency requires additional sparsity assumptions, see also Krampe & Paparoditis (2021) for a discussion. Furthermore, a parametric estimation of the spectral density matrix of the residual process $\{v_t\}$. Since our focus is on the estimation of inverse of the spectral density matrix, we estimate the precision matrix and formulate sparsity assumption on this matrix. See Assumption 4 for the exact definition of the additional sparsity assumptions.

Assumption 4. (Sparsity and stability)

(i) The VAR process is row- and column-wise approximately sparse with approximate sparsity parameter $q_0 \in (0, 1)$, i.e.,

$$\max_{j=1}^p \max_{l=1}^N |A_{i,j}^{(l)}| \leq k_0,$$  

(ii) As in Assumption 1 (ii) and $\sup_{\omega} \|f_\omega\|_\infty \leq k_0 M$.

(iii) The precision matrix $\Sigma_v^{-1} = \text{Var}(v_1)^{-1}$ of the VAR innovations $\{v_t\}$ is positive definite and approximately sparse and $\|\Sigma_v^{-1}\|_2 \leq M$. Let $q_v \in [0, 1)$ denote the approximate sparsity parameter and $k_v$ the sparsity. Then,

$$\max_{j=1}^p \sum_{i=1}^N |(\Sigma_v^{-1})_{i,j}|^{q_v} \leq k_v.$$

As mentioned, the precision matrix of the residuals $\{v_t\}$ needs to be estimated. The residuals can be estimated by $\hat{v}_t = \hat{\xi}_t - \sum_{j=1}^p \hat{A}^{(j)} \hat{\xi}_{t-j}, t = p + 1, \ldots, T$. Then, based on these estimated residuals, procedures like graphical lasso Friedman et al. (2008) or (A)CLIME Cai et al. (2011, 2016a,b) can be used. In the proofs we consider the CLIME method and denote this estimator by $\hat{\Sigma}_v^{-1,\text{CLIME}}$ but similar results can be established for the graphical lasso estimator. Then, we construct the following estimator for $f_\omega^{-1}$

$$f_\omega^{-1} = \left[I_N - \sum_{j=1}^p \hat{A}^{(thr,j)} \exp(i j \omega)\right]^T \hat{\Sigma}_v^{-1,\text{CLIME}} \left[I_N - \sum_{j=1}^p \hat{A}^{(thr,j)} \exp(-i j \omega)\right].$$

}\end{equation}

where $\hat{A}^{(thr,j)} = (\text{THR}_{\lambda^j} \hat{A}^{(j)})$ and THR is a thresholding function with threshold parameter $\lambda^j$ fulfilling the conditions (i) to (iii) in Section 2 in Cai & Liu (2011). For instance, such a thresholding function can be the adaptive lasso thresholding function given by $\text{THR}_{\lambda^j}(z) = z(1 - |\lambda/z|^{\nu}),$ with $\nu \geq 1$. Soft thresholding ($\nu = 1$) and hard thresholding ($\nu = \infty$) are boundary cases of this function. This thresholding functions act by thresholding every element of the matrix $\hat{A}^{(j)}$ and it results in a row- and column-wise consistent estimation of the VAR slope matrices. In Lemma A.4 in the appendix, we present the error bounds $\|f_\omega^{-1} - f_\omega^{-1}\|_\infty$ and $\|f_\omega^{-1} - f_\omega^{-1}\|_2$. Finally, replacing in (11) all quantities
with the estimators discussed above, leads to our final estimator of the inverse of the spectral density matrix of \( \{X_t\} \). Its error bounds are given in the following Theorem 4. We only present here explicitly the rate for a simplified case. In the general case, an explicit rate can be obtained by inserting the results of Lemma A.4 and Theorem 1. Since it leads to a lengthy and not insightful expression, we omit it here. The rate is dominated by the estimation error of the sparse VAR and it is similar to the one in Theorem 2. However, the rate is more affected by the sparsity parameter in the sense that a sparse process is required to obtain a consistent estimation.

**Theorem 4.** Under Assumption 2,3,4 and Assumption 1 and 2 in Wu & Zaffaroni (2018) (conditions on the used kernel and lag-window of the non-parametric estimator) we have the following

\[
\|f_X(\omega)^{-1} - \hat{f}_X(\omega)^{-1}\|_1 = O_P(k_\xi \|\hat{f}_\xi^{-1}(\omega) - f_\xi^{-1}(\omega)\|_\infty + k_\xi^2 \|\hat{A} - \Lambda H_{NT}^{-1}\|_{\max}), l \in [1, \infty]
\]

and

\[
\|f_X(\omega)^{-1} - \hat{f}_X(\omega)^{-1}\|_2 = O_P(\|\hat{f}_\xi^{-1}(\omega) - f_\xi^{-1}(\omega)\|_2 + \|\hat{A} - \Lambda H_{NT}^{-1}\|_{\max}).
\]

If \( N = T^a, p = T^b \) for some \( a, b > 0, \zeta \geq (1 + a + b) \) and \( k = o(\sqrt{T/\log(Np)}) \), these error bounds simplify to

\[
\|f_X(\omega)^{-1} - \hat{f}_X(\omega)^{-1}\|_1 = O_P\left(k_\nu \|\Sigma^{-1}_w\|_1 \left[k_\nu \left[\sqrt{(\log(N)/T)} + k \frac{k_\xi}{N} + \frac{\log(N)}{\sqrt{NT}}\right] + \frac{\log(N)}{\sqrt{NT}}\right]^{1-\nu} + \right.
\]

\[
\sqrt{k} \left[\sqrt{\log(N)/T} + k \frac{k_\xi}{N} + k \frac{\log(N)}{\sqrt{NT}}\right]^{1-\nu/2}, l \in [1, \infty],
\]

\[
\|f_X(\omega)^{-1} - \hat{f}_X(\omega)^{-1}\|_2 = O_P\left(k_\nu \|\Sigma^{-1}_w\|_1 \left[k_\nu \left[\sqrt{(\log(N)/T)} + k \frac{k_\xi}{N} + \frac{\log(N)}{\sqrt{NT}}\right] + \frac{\log(N)}{\sqrt{NT}}\right]^{1-\nu} + \right.
\]

\[
+k^{3/2} \left[\sqrt{\log(N)/T} + k \frac{k_\xi}{N} + k \frac{\log(N)}{\sqrt{NT}}\right]^{1-\nu/2},
\]

4. Number of Factors, Lag-length and Penalty Tuning

4.1. Number of Factors

The seminal work of Bai & Ng (2002) (BN henceforth) introduced a focus over a data-driven correct specification of the number of factors. The idea for how to tackle estimation of the number of common factors is induced by the amount of eigenvalues of the covariance matrix of the data diverging to infinity. Finding a threshold able to clearly separate among finite (bounded) and infinite eigenvalues solves the problem. BN designed precisely this, namely an information criteria able to threshold the diverging eigenvalues. Their key contribution lies in allowing both the cross-sectional dimension as well as the time dimension to (jointly) diverge. In fact, their framework calls for the overfitting penalty to be a function of both \( N \) and \( T \). Their approach in estimating the common factors is non-parametric in nature where asymptotic principal components is used. The asymptotic behavior then yields consistent estimation of the number of factors. However, in practice the BN criterion suffers from a penalty identification issue
which can return non-robust results: the number of factors can arbitrarily be overestimated or underestimated. As observed in Hallin & Liška (2007), Alessi et al. (2010) (HLA henceforth), for respectively the cases of dynamic and static factor models, the consistency of the estimated number of factors via the BN criterion still holds if the penalty parameter of the criterion is multiplied by an arbitrary real, positive constant \( c \). HLA then proposed to pre-multiply the penalty function by a positive real number \( c \), in such a way that the number of factors becomes a monotonic function in \( c \). The tuning of \( c \) then allows for either no penalization when \( c = 0 \), underpenalization when \( c \) is positive but small and overpenalization when \( c \) is positive but large. It follows that \( c \) is optimized whenever its value let the number of factors to be a stable function over the batch.

4.2. Lag-length

Until now we assumed the lag-length \( p \) to be given. Clearly, in practice, \( p \) needs to be estimated. Theoretically, one can safely assume the VAR lag-length \( p \) to be reasonably small in a high-dimensional framework (for a justification, see e.g., Hecq et al., 2016). As standard criteria do not work for high-dimensional settings, a data driven approach to estimate a (global) lag-length is discussed e.g., in Hecq et al. (2021) where the VAR is marginalized into a sequence of univariate AR(\( p \)) regressions and the lag-length is selected by minimizing an approximated Bayesian Information Criterion (BIC) on the residual covariance matrix. Consistency of the BIC up to a multiplicative, slowly diverging constant \( C_N \) (as long as \( C_N N \log(T)/T \to 0 \)) in the penalty term has been proved in Wang et al. (2009). Under few technical conditions on the divergence speed of the model dimension and the size of non-zero coefficients, it is shown how this slightly modified BIC can identify the true model consistently even when the dimension diverges. Alternatively, one can directly apply regularizations of the lasso type. However, there seems to be not much gain, both computationally and in terms of possible erratic behaviors due to high-correlations, to use these regularization techniques for the purpose of lag-length selection. In fact, by doing so one only shifts the model selection problem to the choice of the tuning parameter in the regularization technique (see also Remark 4.4). Selection consistency of these shrinkage estimators in fact crucially relies on how one tunes the penalty parameter. For instance, although widely employed in practice, the generalized cross validation method does not consistently recover the true model even in fixed dimension settings (see e.g., Wang et al., 2007, Wang & Leng, 2007, for examples with SCAD and adaptive lasso).

4.3. A Combined Approach for Single Time Series

In light of the previous two paragraphs, we seek for a unified procedure able at the same time to consistently estimate the lag-length as well as the number of factors. For a given lag-length and number of factors, the penalty parameter can be chosen with the approaches discussed in the next subsection and we take this for granted here. This means the VAR models for the different lags are considered as sparse here. Furthermore, in some applications the focus is more on forecasting a small subset of time series.

\[ A(L)\xi_t = v_t. \]

Each cross-sectional equation in this latter expression then follows an ARMA(\( N_p, (N - 1)p \)) which is of very large orders already e.g., for a setting as \( N = 100 \) and \( p = 2 \).
series of the system. To take this into account we present here two approaches. A global approach which gives a single lag-length and number of factors for the entire system and a local approach in which the lag-length or number of factors may differ across the time series. We present the two approaches first and then discuss their differences.

We consider that the factors are driven by a VAR model that is \( \mathbf{f}_t = \sum_{j=1}^{p_f} \Pi_j \mathbf{f}_{t-j} + \mathbf{v}_{t-j} \). That means we have two lag-lengths to choose: \( p \) and \( p_f \). The one-step ahead forecast error of model (1) for the \( i \)th component is given by

\[
\text{Var}(\mathbf{x}_{i,t} - \sum_{j=1}^{p_f} \Lambda_j^{(i)} \Pi_j \mathbf{f}_{t-j} - \sum_{j=1}^{p} \mathbf{e}_{i,t}^\top \mathbf{A}^{(i)} \mathbf{\xi}_{t-j}^{(i)}).
\]

If we treat the factors and idiosyncratic components as known, we have to estimate for all components the \( \sum_{j=1}^{p_f} \Pi_j \mathbf{f}_{t-j} \) parameters. That means in total we have \( \sum_{i=1}^{r} (N \times r) + \sum_{j=1}^{p_f} \Pi_j \mathbf{f}_{t-j} \) parameters for all components. For a single component, we treat \( \Lambda_j, \Pi_j \) as \( r \)-dimensional vectors which gives in total for the \( j \)th component \( r p_f + \sum_{j=1}^{p_f} \| \mathbf{e}_{i,t}^\top \mathbf{A}_j \|_0 \) parameters. The sparsity of the idiosyncratic component has the important implication that \( \sum_{j=1}^{p_f} \| \mathbf{e}_{i,t}^\top \mathbf{A}_j \|_0 \) grow much slower than \( N p \). To be precise, the error bounds in Theorem 2 imply that only rates slower than \( \sqrt{T} \) are reasonable. Hence, the number of parameters considered grow slower than the sample size and consequently, this fits into the framework of Wang et al. (2009) and their modified BIC. Note however, that the results of Wang et al. (2009) are derived under an i.i.d. set-up and also the pre-selection of the penalty parameter \( \lambda_n \) is not taken into account here. In this modified BIC set-up \( C_T \) denotes a slowly diverging series which is discussed shortly. This motivates the following global information criteria

\[
IC^{(global)}_{T,N} := \min_{r,p,p_f} \frac{1}{NT} \log \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \mathbf{x}_{i,t} - \sum_{j=1}^{p_f} \hat{\Lambda}_j^{(i)} \hat{\Pi}_j \mathbf{f}_{t-j} - \sum_{j=1}^{p} \hat{\mathbf{e}}_{i,t}^\top \hat{\mathbf{A}}^{(i)} \hat{\mathbf{\xi}}_{t-j}^{(i)} \right)^2 + \left( r (p_f + N) + \sum_{j=1}^{p} \| \hat{\mathbf{A}}^{(i)} \|_0 \right) \frac{\log(T)}{NT} C_T. \tag{14}
\]

For the \( i \)th component we obtain the following local information criteria

\[
IC^{(i)}_{T,N} := \min_{r,p,p_f} \frac{1}{T} \log \frac{1}{T} \sum_{t=1+\max(p,p_f)}^{T} \left( \mathbf{x}_{i,t} - \sum_{j=1}^{p_f} \hat{\Lambda}_j^{(i)} \hat{\Pi}_j \mathbf{f}_{t-j} - \sum_{j=1}^{p} \hat{\mathbf{e}}_{i,t}^\top \hat{\mathbf{A}}^{(i)} \hat{\mathbf{\xi}}_{t-j}^{(i)} \right)^2 + \left( r p_f + \sum_{j=1}^{p} \| \hat{\mathbf{e}}_{i,t}^\top \hat{\mathbf{A}}^{(i)} \|_0 \right) \frac{\log(T)}{T} C_T. \tag{15}
\]

In practice, the minimum is evaluated over a finite grid. That means one sets a maximal number of factors \( r_{\max} \) and maximal lag-lengths \( p_{\max}, p_f, \max \). If one sets \( r_{\max} = 0 \) or \( p_{\max} = 0 \), this criteria can also be used to fit plain sparse VAR models or plain factor models, respectively. The series \( C_T \) can be diverging very slowly and Wang et al. (2009) suggest for instance, \( \log(\log(T)) \). We would like to consider the diverging dimension as well and follow a similar route as Bai & Ng (2002). So we set \( C_T = c \frac{\log(NT)/(N+T)}{\log(T)} \) with \( c = 1/2 \). Note that for the global approach the factors are penalized by \( (p_f + N) \frac{\log(NT)/(N+T))}{(NT)} \). This also implies that this series fits into the penalization function framework of Theorem 2 in Bai & Ng (2002) required to obtain a consistent estimation of the number of
factors, i.e., this series converges to 0 for \( N, T \to \infty \) and diverges if scaled by \( \min(N, T) \).

Some remarks to these two information criteria. First, the local approach requires for the \( i \)th component only an estimation of \( e_i^\top \hat{\mathbf{A}}^{(j)} \). If the interest is only in some time series of the system, this reduces the computational burden. Second, if the number of factors differs among the time series, the entire system cannot be written as a DFM with maximal number of factors and maximal number of lags. Third, the local approach takes into account that large data sets come as a – in some sense arbitrary – collection of series and it is most likely that some series are not driven by factors or a small lag-length is sufficient. However, the additional cross-section average in the global approach also leads to more stable results. In simulations as well as for the real data set considered, the local approach outperforms the global approach, see Section 5 for further discussion.

4.4. Penalty Tuning

Another important aspect to consider within the framework of \( \ell_1 \)-regularizations is the choice of the tuning parameter \( \lambda \). The latter should be set in order to balance between the fit of the model and its complexity, thus trading off bias with variance. Whenever the tuning parameter is large in its magnitude, the consequence is strong variable selection, i.e., many potentially relevant variables might be set to zero by the regularization technique (e.g., lasso), thus implying a larger estimation bias. In parallel, when instead \( \lambda \approx 0 \), no variable selection is performed and thus regularization techniques such as lasso converge in the limit to the standard OLS estimator. Among the most popular techniques to tune \( \lambda \) is cross-validation (CV). While CV have seen a surge of applications in statistics in the last decades, it can suffer from some shortcomings. It is often computationally demanding, especially in high dimensions, given it has to recursively train and validate on batches of the sample. Also, it needs to be adapted for different data, e.g., \( K \)-fold cross validation needs to be used for time series.

Alternatively, a fast and reliable way of tuning \( \lambda \) is by minimizing an information criterion (IC). Let \( \xi_{t,S} \) be the subvector containing those columns of \( \xi_t \) belonging to the set \( S \). Let further \( \hat{S} \) be the active set identified by the lasso for a given \( \lambda \). Then the value \( \lambda^{IC} \) chosen by information criteria is found as

\[
\lambda^{IC} = \arg \min_{\lambda} \left( \ln \left( \frac{1}{T-p+1} \sum_{t=p+1}^{T} \left( \xi_{t,t} - \sum_{j=1}^{p} \beta_{S(\lambda),j}^{(t)} \xi_{t,S(\lambda)} \right)^2 + \left( \frac{1}{T-p+1} \right) C_T df \right) \right),
\]

where \( df \) represents the degrees of freedom after the penalization, i.e., the cardinality of the estimated active set. \( C_T \) is the penalty specific to each criterion, where the most popular choices are: \( C_T = 2 \), the Akaike information criterion (AIC) by Akaike (1974); \( C_T = \log(T) \), the Bayesian information criterion (BIC) by Schwarz et al. (1978).\(^8\) As for the case of the lag-length selection in Remark 4.2, the slight modification of the BIC proposed in Wang et al. (2009) also holds for penalized estimators as the lasso, thus making it consistent asymptotically in both \( N \) and \( T \).

5. Numerical Results

5.1. Simulation Set-up

\(^8\)Note: for non-Gaussian distributions the residual sum is often used as a proxy for the likelihood.
All results presented in this section are based on implementations in R (R Core Team, 2020). We generate the data generating processes (DGPs) at random and consider the following model class: \( \mathbf{x}_t = \mathbf{\Lambda} \mathbf{f}_t + \xi_t, \mathbf{f}_t = \sum_{j=1}^{p_f} \mathbf{\Pi}^{(j)} \mathbf{f}_{t-j} + \mathbf{u}_t, \xi_t = \sum_{j=1}^{p} \mathbf{A}^{(j)} \xi_{t-j} + \mathbf{v}_t \). The innovations \( \{\mathbf{u}_t\}, \{\mathbf{v}_t\} \) are generated as Gaussian processes and \( \Sigma_u = \text{Var}(\mathbf{u}_t) \) is generated as a positive definite matrix with eigenvalues in the range 1 to 10, using the implementation of the package clusterGeneration (Qiu & Joe., 2020). If not denoted otherwise, sparsity of a matrix is obtained by setting entries – beginning with the absolute smallest values – to zero such that the specified amount of sparsity is obtained. The entries of \( \mathbf{A}^{(j)} \) are generated randomly using a \( t \) distribution with 3 degrees of freedom. After sparsifying, the matrices are rescaled to fit the eigenvalue conditions of 0.8. In real data it is often observed that for a component of a multivariate time series the history of the component itself is quite an important predictor. That means that the diagonals of \( \mathbf{A}^{(j)}, j = 1, \ldots, p \) are (at least for one \( j \)) often non-sparse. To take this into account we put more weight onto the diagonal of \( \mathbf{A}^{(1)} \) by adding 0.4\( I_N \) before sparsifying the randomly generated matrix. This results in a much more dominant diagonal and the diagonal of \( \mathbf{A}^{(1)} \) is more dominant the smaller \( p \) is.

Furthermore, we consider the following specifications:

- The number of factors is given by \( r \in \{0, 2, 4, 6\} \).
- The sample size is given by \( T \in \{100, 200\} \).
- The dimension is given by \( N \in \{50, 100, 250\} \).
- The lag-length of the VAR driving the factors is given by \( p_f \in \{0, 1, 2\} \). The slope matrices are generated at random and the maximal absolute eigenvalue of the stacked VAR matrix is 0.8.
- The lag-length of the VAR driving the idiosyncratic component is given by \( p \in \{0, 1, 3\} \). The slope matrices are generated at random with a row-wise and column-wise sparsity of \( k \in \{5, 10, \min(N, 100)\} \) and the maximal absolute eigenvalue of the stacked VAR matrix is 0.8.
- \( \Sigma_v = \text{Var}(\mathbf{v}_t) \) is generated as a positive definite matrix with eigenvalues in the range 1 to 10 and sparsity of \( k_{\Sigma} \in \{N/10, N\} \).
- The loadings \( \mathbf{\Lambda} \in \mathbb{R}^{N \times r} \) are generated by random sampling from a Uniform\([-1, 1]\) distribution with a column-wise sparsity of \( k_{\Lambda} \in \{N, N/2, N/2^*\} \). \( N/2^* \) refers to a setting in which the lower left and upper right part are zero. For this setting, also the the lower left and upper right part of \( \mathbf{\Pi}^{(j)} \) for \( j = 1, \ldots, p_f \) and \( \Sigma_u \) are set to zero.

Note that if the sparsity parameter is of similar size as the dimension, we have no sparsity. Dropping unnecessary combinations, e.g., varying the sparsity for \( p = 0 \), we end up in 2352 different set-ups for the DGP. We run each set-up 100 times. To evaluate the performance, we consider the average one-step ahead prediction error of the first ten time series. To compute the one-step ahead prediction error a test set of 10000 time points is used. That is, the one-step ahead prediction error of component \( i \) is given by \( \text{MSFE}_{x_i} = \left[ \frac{1}{10000} \sum_{t=1}^{10000} (\hat{x}_{i,T+t}^{(1)} - x_{i,T+t}^{(1)})^2 \right] \). These are then averaged over the 10 components as well as over the 100 set-ups.
We consider the following models to predict:

- **AR**\(_{BIC}\): Univariate ARs which lag-length is chosen by BIC.

- **L\(_{sel}\)**: A sparse VAR which is estimated by a row-wise adaptive lasso and the penalty parameter is chosen by BIC. The lag-length is chosen by the local information criteria of Section 4.3 with maximal number of factors equal to zero.

- **F\(_{BNAR}\)**: A DFM with a VAR for the factors and univariate AR for the idiosyncratic component. The number of factors is chosen by information criteria of Bai & Ng (2002) with the first penalty function, i.e., \(g(N,T) = (N + T)/(NT/(N + T))\). The lag-length for the VAR is chosen by BIC and the lag-lengths of the univariate ARs are chosen by AIC.

- **F\(_{Lsel}\)**: The approach presented in this paper, i.e., a DFM with a VAR for the factors and a sparse VAR for the idiosyncratic component. The number of factors and lag-length are chosen by the local information criteria of Section 4.3. The sparse VAR is estimated by a row-wise adaptive lasso and the penalty parameter is chosen by BIC.

- **F\(_{ARfilt BN}\)**: A univariate AR\((p)\) filter, where \(p\) is chosen by BIC, is applied first. Then **F\(_{BNAR}\)** without univariate AR for the idiosyncratic component is applied.

- **F\(_{ARfilt Lsel}\)**: A univariate AR\((p)\) filter, where \(p\) is chosen by BIC, is applied first. Then **F\(_{Lsel}\)** is used.

**F\(_{BNAR}\)** and **F\(_{ARfilt BN}\)** use the information criteria of Bai & Ng (2002) to determine the number of factors. Let us mention that in preliminary simulations we also considered the method of Alessi et al. (2010) as well as the global information criteria of Section 4.3 to determine the number of factors. The obtained results are almost identical to the ones with the information criteria of Bai & Ng (2002). So for these two specifications we focus the presentation on the criteria of Bai & Ng (2002) only. Furthermore, for **L\(_{sel}\)**, **F\(_{Lsel}\)**, and **F\(_{ARfilt Lsel}\)** we also considered the global information criteria of Section 4.3 but do not present its result here. The findings here can be summarized as follows. The local information criteria outperforms the global criteria and the differences are larger for small sample sizes and dimensions.

In the following, we present the MSFE-results in relation to the MSFE of **F\(_{ARfilt Lsel}\)**. That means values larger than 1 indicate a performance worse than **F\(_{ARfilt Lsel}\)** and values smaller than 1 vice versa. The overall performance is summarized in Table 1.

| \(T\)  | \(AR\(_{BIC}\)\) | \(L\(_{sel}\)\) | **F\(_{BNAR}\)** | **F\(_{Lsel}\)** | **F\(_{ARfilt BN}\)** |
|------|----------------|-------------|------------------|-----------------|-------------------|
| 100  | 1.01           | 1.08        | 1.03             | 1.08            | 0.99              |
| 200  | 1.09           | 1.06        | 1.09             | 1.03            | 1.06              |

Table 1: Overall performance measured in MSFE and in relation to **F\(_{ARfilt Lsel}\)**.

The relative performance over all 2352 different DGP set-ups is displayed in Figure 1-3. Each dot represents the relative MSFE for one DGP set-up averaged over the runs. The set-ups are sorted by sample size\((T)\), lag-length of the idiosyncratic part\((p)\), sparsity\((k)\), lag-length of the factors\((p_f)\) and number of factors\((r)\). The obtained groups for \(T, p,\) and \(k\) are highlighted by vertical bars and the specific parameter values are given at the bottom of the figure. This sorting is chosen because these specification parameters matters the most in the sense that the results can differ substantially among different specification of the parameter values.
Figure 1: The relative performance of $L_{sel}$ over all 2352 different DGP set-ups. Each dot represents the relative MSFE for one DGP set-up. The set-ups are sorted by sample size, lag-length of the idiosyncratic part $p$, and sparsity. The obtained groups are highlighted by vertical bars and the specific parameter values are given at the bottom of the figure. Note that a sparsity of 100 implies in principle here no sparsity at all.

Figure 2: The relative performance of $AR_{BIC}$ over all 2352 different DGP set-ups. Each dot represents the relative MSFE for one DGP set-up. The set-ups are sorted by sample size, lag-length of the idiosyncratic part $p$, and sparsity. The obtained groups are highlighted by vertical bars and the specific parameter values are given at the bottom of the figure. Note that a sparsity of 100 implies in principle here no sparsity at all.

Let us discuss the three Figures 1 to 3 starting with the performance relation of $AR_{BIC}$ and $FL_{sel}^{ARfilt}$. If the sample size is small ($T = 100$), $FL_{sel}^{ARfilt}$ outperform $AR_{BIC}$ only in the very sparse cases (sparsity = 5). In all other cases it performs equally good or even slightly worse (no sparsity or no dependence in the idiosyncratic component). This behavior changes for the larger sample size settings. Here, $FL_{sel}^{ARfilt}$ performs equally well in the cases of no sparsity or no dependence and clearly outperforms $AR_{BIC}$ in the other cases with a smaller MSFE of 10% or more. $L_{sel}$ is slightly outperforming $FL_{sel}^{ARfilt}$ in the case when the dependence is driven purely by the idiosyncratic part. In all other cases $FL_{sel}^{ARfilt}$ is clearly outperforming $L_{sel}$ where the reduction in MSFE is larger for the smaller sample size. For $P_{BN}^{ARfilt}$ and if the sample size is small ($T = 100$), the overall performance is equal. If the idiosyncratic component has no dependency or no sparsity, $F_{BN}^{ARfilt}$ outperforms $FL_{sel}^{ARfilt}$, whereas it is vice versa if a strong sparsity (sparsity=5) is present. The picture changes if the sample size is increased to $T = 200$. Then, except for the case where the idiosyncratic component has no dependency or no sparsity, $FL_{sel}^{ARfilt}$ strongly outperforms $P_{BN}^{ARfilt}$ with a better 10% to 40% smaller MSFE. To conclude, for the smaller sample size ($T = 100$) the additional modelling of the idiosyncratic parts of
Figure 3: The relative performance of $F_{BN}^{ARfilt}$ over all 2352 different DGP set-ups. Each dot represents the relative MSE for one DGP set-up. The set-ups are sorted by sample size, lag-length of the idiosyncratic part $p$, and sparsity. The obtained groups are highlighted by vertical bars and the specific parameter values are given at the bottom of the figure. Note that a sparsity of 100 implies in principle here no sparsity at all.

**FL**$_{sel}^{ARfilt}$ does not always pay off but for the larger sample size ($T = 200$) $FL_{sel}^{ARfilt}$ does perform best among all competitors and if there is dependence in the idiosyncratic component the gain can be quite substantial.

6. Macroeconomic Forecasting: a comparison of approaches

In this section we are going to put our proposed dynamic factor model with sparse VAR idiosyncratic component into practice and forecast real macroeconomic data. The dataset we employ is the FRED-MD (see McCracken & Ng, 2016) which contains a large number of U.S. macroeconomic series sampled at monthly frequency and it is regularly updated every month. FRED-MD is extremely popular in applied macroeconomics and econometrics applications given it is among the best free sources of high-dimensional macroeconomic data and it comes with complementary transformations to render the time series stationary as well as automatic imputation of missing data and outliers handling. After the necessary cleaning of the dataset due to missings, we are left with 123 macroeconomic series for a time span ranging from January 1959 until December 2019. We intentionally truncate the last few years to exclude the Covid-19 crisis. Over a total of 60 years of data, the first 40 are used to compute the forecasts and the remaining 20 years constitute the validation sample. Even though the series within FRED-MD are stationary-transformed, persistence is still clearly visible by simple analysis of the ACF plots of its series. If nothing, the stationary transformation might even introduce negative serial dependence. Furthermore, breaks are surely included in the time span. Among others, the financial crisis of 2008 surely represents one such structural breaks. Therefore, as justified in Remark 2, we apply a linear AR($p$) filter to the series before computing the forecasts, where $p$ is selected via BIC. We also compare the results to the set-up where no AR filtering is applied, thus showing empirically the relevance of applying such filter. Obviously, this filtering transformation will increase the favours of the different forecasting models considered with respect to an autoregressive benchmark, which we anyway report and show it is still able to outperform the other competitors on few occasions. Many of the series within FRED-MD are aggregates where the disaggregated sources of information are combined in a single indicator, for instance the
monetary aggregates. Such aggregation can result in favoring a simple AR forecast rather than a more sophisticated methodology. This is an important limitation but it is important to stress as it underlines where the more sophisticated methodologies scores better.

First, we are going to compare our model’s forecast of all the 123 series in the FRED-MD with the forecasts obtained via the other methods listed in Section 5. In Table 2 we report the number of series where each method scores either first, second or third according to its MSFE.

| Score | AR_{BIC} | L_{sel} | F_{BN\text{AR}} | FL_{sel} | F_{BN\text{ARfilt}} | FL_{sel}\text{ARfilt} |
|-------|----------|---------|-----------------|----------|-------------------|-------------------|
| First | 22       | 12      | 20              | 15       | 30                | 24                |
| Second| 14       | 16      | 7               | 22       | 33                | 31                |
| Third | 20       | 27      | 17              | 18       | 16                | 25                |

Table 2: Ranking of MSFE scores over all FRED-MD variables.

First off, the benefits of the AR filter is evident in our method where throughout the different ranks the pre-filtering ($FL_{sel}\text{ARfilt}$) scores better for the majority of the FRED-MD series than when no filtering is applied ($FL_{sel}$). An AR(p) pre-filter is also applied to the plain factor model where the factors are selected using the usual Bai & Ng (2002) information criterion ($F_{BN\text{ARfilt}}$). The global outcomes show how over all the 123 series in FRED-MD, forecasting using a plain factor model after a pre-filtering step of the series ($F_{BN\text{ARfilt}}$) returns the most accurate outcomes. This reflects the well known fact that the majority of the macroeconomic aggregates within FRED-MD are fully driven by few factors. As a consequence, a more complicated framework that also attempt at modeling the idiosyncratic components ends up in over-specifying. However, a clear benefit in modeling the idiosyncratic components emerges from Table 2. In fact, our pre-filtered DFM with sparse VAR idiosyncratic components outperforms the competitors in 24 out of 123 series, very close to the 30 series where a pre-filtered plain factor model is found superior. If no pre-filtering of the series is applied ($FL_{sel}$), then our approach outperforms the competitors in only 15 of the 123 series, even less than the autoregressive benchmark ($AR_{BIC}$). The benefit of modeling the idiosyncratic components for certain series is also clear from the column $F_{BN\text{AR}}$ where combining an unfiltered plain factor model with a simple AR modeling of the idiosyncratic components ends up in being the best choice in forecasting 20 of the 123 series.

Second, we focus our attention on eight specific macroeconomic series within FRED-MD. Four of them are measures of real economics activity: real production income (RPI); total industrial production (INDPRO); real manufacturing and trade industry sales (CMRMTSPLx); number of employees on non-agricultural payrolls (PAYEMS). The remaining four series are price indices: the producer index for finished goods (WPSFD49207); the consumer price index (CPIAUCSL); the consumer price index less food (CPIULFSL); the personal consumption expenditure implicit price deflator (PCEPI). Several other works in the literature have focused on the same or similar series (see among others Smeekes & Wijler, 2018, Kristensen, 2017, Ludvigson & Ng, 2009). In Table 3 we report the MSFE’s of the various competing methods relative to the AR(p) benchmark.
## Series

| Series       | $L_{sel}$ | $F_{BN AR}$ | $FL_{sel}$ | $F_{BN ARfilt}$ | $FL_{sel}$ |
|--------------|-----------|-------------|------------|----------------|------------|
| RPI          | 0.927     | 1.047       | 0.962      | 1.161          | 1.194      |
| INDPRO       | 0.993     | 0.868       | 0.983      | 0.926          | 0.901      |
| CMRMTSPLx    | 1.171     | 1.097       | 1.144      | 0.999          | 1.021      |
| PAYEMS       | 0.969     | 1.148       | 0.960      | 0.844          | 1.049      |
| WPSFD49207   | 1.031     | 1.068       | 1.036      | 0.977          | 0.926      |
| CPIAUCLSL    | 0.990     | 1.118       | 0.991      | 0.966          | 0.866      |
| CPIULFSL     | 0.983     | 1.120       | 0.980      | 0.976          | 0.886      |
| PCEPI        | 1.022     | 1.151       | 1.032      | 0.966          | 0.869      |

Table 3: RMSFE relative to $AR_{BIC}$ benchmark for specific FRED-MD series.

The results displayed in Table 3 tell a very similar story to the global results reported in Table 2. AR pre-filtering is deemed necessary given the high degree of persistence of some series. However, when persistence is an artifact of some structural breaks, as it is the case for both RPI and PAYEMS, the AR filtering is not beneficial. We report the CUSUM statistic and its p-value based on OLS AR(1) residuals in Table 4.

| Series   | CUSUM | p-value |
|----------|-------|---------|
| RPI      | 1.558 | 0.015   |
| INDPRO   | 1.298 | 0.068   |
| CMRMTSPLx| 1.033 | 0.235   |
| PAYEMS   | 1.831 | 0.002   |
| WPSFD49207| 0.241 | 1       |
| CPIAUCLSL| 0.308 | 1       |
| CPIULFSL | 0.323 | 0.999   |
| PCEPI    | 0.299 | 1       |

Table 4: CUSUM statistic based on OLS AR(1) residuals & p-value.

When no structural break is present within the series, our pre-filtered DFM with sparse idiosyncratic components ($FL_{sel}$) always outperforms its non-pre-filtered counterpart ($FL_{sel}$). On a one-to-one comparison between the pre-filtered plain factor model ($F_{BN ARfilt}$) and our approach ($FL_{sel}$), the latter outperforms $F_{BN ARfilt}$ in five out of the eight series considered. Furthermore, two of the three series where $F_{BN ARfilt}$ outperforms our method are RPI and PAYEMS which suffer from instability problems as highlighted in Table 4. The remaining series is “real manufacturing and trade industry sales” (CMRMTSPLx) for which however the RMSFE differential between the two competing methods is only 0.022.

We now turn to investigate FRED-MD idiosyncratic partial coherence networks. Parameters like the coherence or the partial coherence, describe in a comprehensive way the linear relations between the components of the vector time series by taking into account all lead and lag relations as well as the distinction between direct and indirect linear effects. We base the analysis on the partial coherence ($R$) computed from the estimated idiosyncratic spectral density within our proposed DFM with sparse VAR idiosyncratic components, as described in Section 3.2. This helps in shedding further light on the presence and strength of cross sectional dependence among idiosyncratic components in such a macroeconomic context. To elaborate, the partial coherence is given by

$$R_{u,v}(\omega) = |\rho_{u,v}(\omega)|, \quad \text{where} \quad \rho_{u,v}(\omega) = -f_{u,v}^{-1}(\omega)\sqrt{f_{u,u}^{-1}(\omega)f_{v,v}^{-1}(\omega)},$$

where $f_{u,v}^{-1}(\omega)$ denotes the $(u, v)$th element of the inverse of the spectral density matrix at frequency $\omega$. We
take a look here at \( \sup_{\omega} R_{u,v}(\omega) \) evaluated over a grid of Fourier frequencies. The inverse of the spectral density matrix is estimated as described in Section 3.2 and as the focus is here not on forecasting, we determine the number of factors using the criteria of Bai & Ng (2002) with the penalty function 
\[
g(N,T) = \frac{(N + T)}{(NT)} \log \left( \frac{NT}{(N + T)} \right)
\]
. The lag-length of the sparse VAR for the idiosyncratic component is selected by the information criteria (14) directly applied on the estimated idiosyncratic component with \( r_{\text{max}} = 0 \). We consider three settings for our sample: the entire 1959-2019 sample as considered for the previous forecasting exercise and the two halves of the same sample, namely January 1960 until December 1989 and January 1990 until December 2019. The aim is to also observe whether or not the (amount of) connections are roughly comparable throughout the sample. We also consider three lower-bound levels of partial coherence: \( R > 0.05, R > 0.25, \) and \( R > 0.5 \). For the sake of brevity we only report figures for the case of \( R > 0.05 \) but still comment on the results for \( R > 0.25 \) and \( R > 0.5 \). As the focus is now shifted from macroeconomic forecasting to the estimation of macroeconomic partial coherence networks, we prioritize unbiasedness and do not pre-filter the data as done earlier. The reason for this is that filtering in the spectral domain has the effect of removing low frequencies while accentuating the high-frequency components. This would result in masking many (weak) connections making networks noticeably sparser. In the figures, the labels of the macroeconomic variables are accompanied by a number within square brackets which refers to the group they belong to according to FRED-MD. More specifically, group 1 is “Output and Income”, group 2 is “Labor Market”, group 3 is “Housing”, group 4 is “Consumption, Orders and Inventories”, group 5 is “Money and Credit”, group 6 is “Interest and Exchange rates”, group 7 is “Prices” and finally group 8 is “Stock Market”. Active vertices i.e., vertices that are connected with at least one of the others, are reported in red while the non-active ones are in light-blue.

In Figure 4, the highest number of connections is observed on the second half of the sample, in panel (4c). 60 active vertices are found compared to the 42 active in the first half of the sample in panel (4b) and the overall 50 active vertices retrieved over the full sample in panel (4a). “Interest and Exchange Rates” group 6 is the most active group of vertices across the different (sub)samples: 17 of its variables are connected in both the full and second half of the sample, while 13 are active in the first half of the sample. Group 2, 3 and 5 i.e., respectively: “Labor Market”, “Housing” and “Money and Credit” are also particularly active. In fact, in the second half of the sample 17 vertices belonging to group 2 are found, compared to the 11 in the full sample and only 3 in the first half. 10 active vertices within group 3 are found in the first half of the sample compared to 9 in the full sample and 6 in the second half. Vertices belonging to group 5 are 7 for both the full and first half of the sample and 9 for the second half. Across the different samples the overlapping active vertices are 26, among which 12 belong to group 6, 6 belong to group 3, 5 belong to group 5 and the remainings belong to group 1, 2, 8. One can perhaps speculate that the higher amount of connections in the second half of the sample is due to the financial crisis which amplified the idiosyncratic components of macroeconomic indicators. However, many of such connections seem to be rather weak when it comes to the level of partial coherence considered. In fact,

\[^9\text{Networks for the cases of } R > 0.25 \text{ and } R > 0.5 \text{ are available upon request.}\]
when $R > 0.25$, the most amount of active vertices is observed over the full sample and not anymore on the second half. Indeed, 35 active vertices are found for the full sample against 30 for the second half and 20 for the first half. The overall picture is similar to the case of $R > 0.05$ when it comes to groups of active vertices. In fact, “Interest and Exchange Rates” group 6 is the most active group of vertices, especially in the full sample and the second half, directly followed by “Housing” group 3 and “Labor Market” group 2. When the lower bound on the partial coherence is set to $R > 0.5$, the overall amount of connections does not change much for the full sample which only decreases to 32 while the first and second half of the sample both decrease to a total of 14 active vertices. The number of overlapping active vertices for the settings of $R > 0.25$ and $R > 0.5$ is, however, considerably smaller: 6 overlapping active vertices are identified where the majority belong again to group 6, and only 1 overlapping active vertex corresponding to the FRED series “Capacity Utilization: Manufacturing” (CUMFNS) of group 1: “Output and Income”. In Table 5, we report the five series with highest scores of eigenvector centrality (reported in round brackets) for the $R > 0.05$ adjacency matrices of the three (sub)samples.
As the eigenvector centrality measures the importance of a vector to the degree of it being connected with others highly connected vertices, it seems that the speculation about the financial crisis to resonate in the macroeconomy is supported by the results. The second half of the sample i.e., the one containing the 2008 subprime crisis, has 4 out of five vertices with highest eigenvector centrality belonging to “Interest and Exchange Rates” group 6. Concerning the first part of the sample, “Housing” group 3 takes up all the first five positions in terms of eigenvector centrality, likewise for the full sample.

7. Conclusions

We blend the dense dimensionality reduction of factor models with the one of sparsity-inducing high-dimensional VARs. Hence, we propose a dynamic factor model whose factors and relative loadings are estimated via standard principal components while its idiosyncratic components are assumed to follow a high-dimensional sparse VAR model and are thus estimated via $\ell_1$—norm regularization techniques such as the adaptive lasso. The estimation is articulated in two steps: first the factors and their loadings are estimated via standard singular value decomposition and the estimated idiosyncratic components are obtained as estimated residuals. Second, an adaptive lasso is estimated on the previously obtained idiosyncratic components. As the second step is performed on the estimated idiosyncratic term from the first step, in order to derive the consistency of the lasso we first derive a simple representation of the idiosyncratic estimation error. This helps to obtain tighter rates for the deviation bound in the second step. The simple representation is derived from the decomposition idea in Bai & Ng (2020). This also allows to obtain an expression for the error between estimated factors/loadings and a rotated version of the true factors/loadings and hence derive as well the order of their estimation errors. Through this blended approach we are able to disentangle the dependence among the factors, i.e., the diverging eigenvalues of the covariance matrix, and the dependence among the idiosyncratic components, i.e., the bounded ones, while allowing for both cross-sectional and time dependence in the idiosyncratic term. In order to choose the number of factors and the lag-length of the VAR, we also propose a unified procedure able to simultaneously estimate both. We consider the factors being driven by a VAR($p_f$) while idiosyncratic components follow a VAR($p$). We then set-up an information criteria minimizing the one-step ahead forecast error of the model over a grid of both $p, p_f$ and the number of factors. The composite penalty of the criterion allows the joint selection of lags and number of factors. We corroborate our results with a thorough simulation exercise in which several data generating processes are considered both for different sparsity patterns as well as for different specifications of the numbers of factors and lag-lengths. We compare the performance of our proposed method with several workhorse forecasting models in the literature and find that the procedure proposed here performs well. Empirically, we investigate the performances of our method for macroeconomic forecasting using the popular FRED-MD.
dataset. We show our approach can improve forecast accuracy and we compare it against several other benchmarks. Partial coherence networks of idiosyncratic components are also built. They demonstrate how idiosyncratic cross-sectional dependence is present even in a macroeconomic context and hence it is relevant to account for it as we propose.

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A. Proofs and additional Lemmas

In order to quantify the dependence of the stochastic processes, we use the concept of functional dependence, see Wu (2005), and concentration inequalities derived under this concept of dependence, see among others Liu et al. (2013), Wu et al. (2016). In the following remark 4 we summarize the main notation of this dependence concept.

Remark 4 (Functional Dependence Measure). Let $Y_{t,i} = G_i(\xi_t, \xi_{t-1}, \ldots, i = 1, \ldots, N, t \in \mathbb{Z}$, be some process generated causally by the i.i.d. processes $\{\xi_t\}$ for some measurable function $G = (G_1, \ldots, G_N)$. Furthermore, denote by $Y^{(k)}_{t,i} = G_i(\xi_t, \xi_{t-1}, \ldots, \xi_{t-k+1}, \varepsilon_{t-k}^i, \xi_{t-k-1}, \varepsilon_{t-k-2}, \ldots)$ the process where $\varepsilon_{t-k}$ is replaced by an i.i.d. copy $\varepsilon_{t-k}^i$. We follow Wu (2005), Wu et al. (2016) and define the physical/functional dependence coefficients in the following way. Let $||\xi_{i,t}||_{E,q} := (E|\xi_{i,t}|^q)^{1/q} < \infty, q \geq 1$. Furthermore, let the functional dependence measure be defined as $\delta_{k,q,i} = ||Y_{0,i} - Y^{(k)}_{0,i}||_{E,q}, k \geq 0$. In order to account for the dependence in the process $Y_{t,i}$ let $\Delta_{m,q,i} = \sum_{k=m}^\infty \delta_{k,q,i}$ such that the dependence adjusted norm is defined as $||Y_{t,i}||_{q,\alpha} = \sup_{m \geq 0} (m+1)^{\alpha} \Delta_{m,q,i}$. As we work in a high-dimensional setting, in order to take this into account we need a uniform dependence adjusted norm $\Psi_{q,\alpha} = \max_{1 \leq i \leq N} ||Y_{t,i}||_{q,\alpha}$, and an overall dependence adjusted norm $\psi_{q,\alpha} = (\sum_{i=1}^{N} \sup_{m \geq 0} (m+1)^{\alpha} \Delta_{m,q,i})^{1/q}$. Furthermore, define for the $N$ dimensional stationary process $Y_{t,i}$ the $L^\infty$ functional dependence measure with its corresponding dependence adjusted norm: $\omega_{k,q} = |||Y_{t,i} - Y^{(k)}_{t,i}||_{E,q}, |||Y_{t,i}||_{\infty,q} = \sup_{m \geq 0} (m+1)^{\alpha} \Delta_{m,q,i}$ for $\Omega_{m,q} = \sum_{k=m}^\infty \omega_{k,q}$. Finally, let $\nu_q = \sum_{j=1}^{\infty} (j^{q/2-1} \omega_{k,q})^{1/(q+1)}$.

Assumption 1 implies that $||e^i B^{(j)}||_2 \leq M \rho^j$. Hence, it follows by Example 3 in Wu et al. (2016) and the moment condition in Assumption 2 that $\max_j ||\xi_{j,t}||_{\zeta,\alpha} < \infty$ for all $\alpha > 0$. Since $\{f_i\}$ is a linear processes of fixed dimension $r$, we also have $\max_j ||X_{j,t}||_{\zeta,\alpha} < \infty$ for all $\alpha > 0$. Hence, we have by the Minkowski-inequality $\max_j ||x_{j,t}||_{\zeta,\alpha} < \infty$, see also the Proof of Proposition 5 in Forni et al. (2017). Additionally, we have by the Cauchy-Schwarz-inequality for some $q > 2$, $\max_j ||\{\xi_{j,t}\}||_{q,\alpha} \leq C(\max_j ||\xi_{j,t}||_{2q,\alpha} + \max_j ||f_{j,t}||_{2q,\alpha} + \max_j ||\xi_{j,t}||_{2q,\alpha} \max_j ||f_{j,t}||_{2q,\alpha})$, where $C$ is some constant depending on $q$ only.

In the following lemma, we derive the order of several expression. Key ingredient of the proof of this lemma is the Nagaev’s inequality for dependent processes, see Section 2.1 in Wu et al. (2016). To abbreviate the expression, we display all results here in $O_p$-notation. The proof of all lemmas presented here can be found in the supplementary material. Also, recall

$$g(N,T,\zeta) = (NT)^{2/\zeta} \left( \frac{1}{\sqrt{NT}} + \frac{1}{T^{3/2}} + (NT)^{2/\zeta} \frac{1}{T^2} \right).$$

Lemma A1. Let $C_1, C_2, C_3$ be constants depending only on $q$ and $\alpha$. Under Assumption 1,2,3 we have the following:

A) $\|D^2_{NT,r}\|_2 = O_P(1)$ and $\|D^2_{NT,r}\|_2 = O_P(1)$.

B) For $i = 1, \ldots, N$ and $j = 1, \ldots, r$, we have

$$\max_{t,j} \frac{1}{T} \sum_{s=1}^{T} \xi_{i,s} f_{j,s} = O_P(\sqrt{\log(N)/T}) + N^{2/\zeta} T^{2/\zeta - 1}.$$
C) For each $j = 1, \ldots, r$, we have $1/T \sum_{s=1}^{T} |f_{s,j}|^2 = \Sigma F_{j,j} + O_P(1/\sqrt{T})$ and \( \max_j 1/T \sum_{s=1}^{T} f_{s,j} \hat{f}_{s,j} | \leq (M + O_P(1/\sqrt{T}))^{1/2}. \)

D) For each $j, j_2 = 1, \ldots, N$, we have

$$\max_{j, j_2} |1/T \sum_{s=1}^{T} \xi_{s,j_1}, \xi_{s,j_2} | \leq M + O_P(\sqrt{(\log(N)/T)} + N^{2/\zeta T^2/\zeta - 1}).$$

E) For each $k = 1, \ldots, r$ we have $e_k^T \Lambda \Gamma(0) \Lambda e_k / N \leq M^4 / (1 - \rho^2) < \infty$ and

$$1/T \sum_{s=1}^{T} (1/\sqrt{N} \sum_{i=1}^{N} \xi_{i,k,\xi_{i,s}})^2 = O_P(1).$$

F) We have

$$\max_{j, k} |1/T \sum_{s=1}^{T} e_k^T \xi_{j}^T \Lambda^T e_k | = O_P(k_\zeta + \sqrt{\log(N)/T} + N^{2/\zeta T^2/\zeta - 1}).$$

G) We have

$$\max_{j, l} \left| 1/T \sum_{s=1}^{T} e_l^T (\hat{f}_s - H_{NT} f_s) \xi_{i,s} \right| = O_P \left( \frac{\log(N)}{T} + \frac{k_\zeta}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N, T, \zeta) \right).$$

H) We have for $t \in \mathbb{Z}$

$$\hat{f}_t - H_{NT} f_t = \frac{1}{NT} \left[ \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,t} \Lambda_i f_s^T H_{NT} f_s + \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,t} \Lambda_i H_{NT} f_s \right] D^{-2}_{NT,r}$$

$$+ O_P \left( \frac{\log(N)}{T} + \frac{k_\zeta}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N, T, \zeta) \right).$$

I) We have

$$\max_{j, l} \left| 1/T \sum_{s=1}^{T} f_{j,s} [\hat{f}_s - H_{NT} f_s]^T e_l \right| = O_P \left( \frac{\log(N)}{T} + \frac{k_\zeta}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N, T, \zeta) \right).$$

J) For each $j, l = 1, \ldots, r$

$$\frac{1}{T} \sum_{s=1}^{T} e_l^T [\hat{f}_s - H_{NT} f_s] [\hat{f}_s - H_{NT} f_s]^T e_l = O_P \left( \frac{\log(N)}{T} + \frac{k_\zeta}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N, T, \zeta) \right).$$

K) We have

$$(H_{NT})^{-1} \Lambda_i - \hat{\Lambda}_i = \frac{1}{T} \sum_{s=1}^{T} H_{NT} f_s \xi_{i,s} + Error_i,$$

where $\max_i |Error_i| = O_P \left( \frac{\log(N)}{T} + \frac{k_\zeta}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + g(N, T, \zeta) \right).$ Furthermore,

$$\|(H_{NT})^{-1} \Lambda - \hat{\Lambda}\| = O_P(\sqrt{\log(N)/T} + (NT)^{2/\zeta}/T + k_\zeta/N).$$
Proof of Theorem 1. First note that we have the following representation
\[
w_{t,t} = \Lambda_{t}^\top f_{t} - \Lambda_{t}^\top \hat{f}_{t} = \Lambda_{t}^\top H_{NT}^{-1}[H_{NT} f_{t} - \hat{f}_{t}] + [(H_{NT}^{-1})^{-1} \Lambda_{t} - \hat{\Lambda}_{t}]^\top H_{NT} f_{t} + [(H_{NT}^{-1})^{-1} \Lambda_{t} - \hat{\Lambda}_{t}]^\top [H_{NT} f_{t} - \hat{f}_{t}],
\]
Then, the first assertion in (7) follows by inserting the orders derived in Lemma A.1.

For the second and third assertion of Theorem 1, namely for the orders of \(1/T \sum_{t=1}^{T} w_{t} \xi_{t} \|_{\max} \) and \(1/T \sum_{t=1}^{T} w_{t} w_{t}^\top \|_{\max} \) respectively, we focus on the case \(k = 0 \). The case \(k = 1 \) follows by the same arguments. For the second assertion, we have by Lemma A.1
\[
\begin{align*}
\| \frac{1}{T} \sum_{t=1}^{T} w_{t} \xi_{t} \|_{\max} &\leq \| \Lambda \|_{\max} \| H_{NT} \|_{\max} \| D_{f_{t},\xi_{t}} \|_{\max} \left[ \frac{1}{N T} \sum_{t=1}^{T} \Lambda_{t}^\top \xi_{t} \xi_{t}^\top \right]_{\max} + \max_{i} |\text{Error}_i| + \frac{1}{T} \sum_{t=1}^{T} f_{t}^\top \xi_{t} \|_{\max} \left( \frac{1}{T} \sum_{t=1}^{T} \xi_{t} \xi_{t}^\top - \Gamma_{\xi}(0) \right) \|_{\max} + \| \Gamma_{\xi}(0) \|_{\infty}/N \right) + \frac{1}{T} \sum_{t=1}^{T} f_{t}^\top \xi_{t}^2 \|_{\max} \\
&= O_P \left( \frac{k_{\xi}}{N} + \frac{\sqrt{\log(N)}}{N \sqrt{T}} + (NT)^{2/\xi - 1} + \frac{\log(N)}{T} + (NT)^{4/\xi}/T^{2} + \frac{k_{\xi}}{N} \left( \frac{\sqrt{\log(N)/T}}{N} + N^{2/\xi} T^{2/\xi - 1} \right) \right) \\
&+ O_P \left( \frac{\log(N)}{T} + \frac{k_{\xi}}{N} + \frac{\sqrt{\log(N)}}{N \sqrt{T}} + g(N, T, \xi) \right).
\end{align*}
\]
For the third assertion, we have by Lemma A.1
\[
\begin{align*}
\| \frac{1}{T} \sum_{t=1}^{T} w_{t} w_{t}^\top \|_{\max} &\leq \| \Lambda \|_{\max} \| H_{NT} \|_{\max} \| D_{f_{t},\xi_{t}} \|_{\max} \left[ \frac{1}{N T} \sum_{t=1}^{T} f_{t}^\top \xi_{t} \xi_{t}^\top \right]_{\max}^2 + \| \frac{1}{T} \sum_{t=1}^{T} f_{t}^\top f_{t} \|_{\max} + \frac{1}{T} \sum_{t=1}^{T} f_{t}^\top \xi_{t} \|_{\max}^2 \\
&+ \frac{1}{T} \sum_{t=1}^{T} f_{t}^\top \xi_{t} \xi_{t}^\top \|_{\max} \left( \frac{1}{T} \sum_{t=1}^{T} \xi_{t} \xi_{t}^\top - \Gamma_{\xi}(0) \|_{\max} + \| \Gamma_{\xi}(0) \|_{\infty}/N \right) + \frac{1}{T} \sum_{t=1}^{T} f_{t}^\top \xi_{t}^3 \|_{\max} \\
&= O_P \left( \frac{1}{N} + \frac{\sqrt{\log(N)}}{\sqrt{T}} + (NT)^{2/\xi}/T \left( \frac{k_{\xi}}{N} + \frac{\sqrt{\log(N)}}{\sqrt{T}} + (NT)^{2/\xi}/T \right) \right) \\
&+ O_P \left( \frac{\log(N)}{T} + \frac{k_{\xi}}{N} + \frac{\sqrt{\log(N)}}{N \sqrt{T}} + g(N, T, \xi) \right) \\
&= O_P \left( \frac{k_{\xi}}{N} + \frac{\log(N)}{T} + \frac{\sqrt{\log(N)}}{N \sqrt{T}} + g(N, T, \xi) \right).
\end{align*}
\]

\[\square\]

Lemma A.2. Under Assumption 1 and if
\[
\| \frac{1}{T-p} \sum_{t=p+1}^{T} (\hat{\xi}_{t} - \beta^\top \hat{\xi}_{t-1}) \hat{\xi}_{t-1}^\top \|_{\max} \leq \lambda/4,
\]
and
\[
\Theta^\top \frac{1}{T-p} \sum_{t=p+1}^{T} \hat{\xi}_{t-1} (\hat{\xi}_{t-1}^\top) \Theta \geq a \| \Theta \|_{2}^2 - \hat{\theta} \| \Theta \|_{2}^2 \vee \Theta \in \mathbb{R}^{np},
\]

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we have
\[ \|\hat{\beta}(j) - \beta(j)\|_2 \leq 16 \max\left( \sqrt{k(\lambda/\alpha)^{1-\eta/2}}, \sqrt{\tau}s(\lambda/\alpha)^{1-\eta} \right), \]
and
\[ \|\hat{\beta}(j) - \beta(j)\|_1 \leq \max(68k(\lambda/\alpha)^{1-\eta}, 64\sqrt{\tau}k^{3/2}(\lambda/\alpha)^{1-3/2\eta} + 4k(\lambda/\alpha)^{1-\eta}). \]

**Proof of Theorem 2.** The idea is to determine the order of the quantities \( \hat{\lambda}_T \) and \( \hat{\tau} \) in Lemma A.2. For this, first note that since \((\xi_{j,t} - \beta_{j}^\top\xi_{t-1}^\top)\xi_{t-1} = v_j^\top\xi_{t-1}\) and \(\|v_j\|_{\xi_{t-1}} = 0\), we have \(\|1/p \sum_{t=p+1}^T (\xi_{j,t} - \beta_{j}^\top\xi_{t-1}^\top)\xi_{t-1}\|_{\max} = O_P(\sqrt{\log(Np)/T} + (NpT)^{2/\zeta}/T)\) by the same arguments as in the proof of Lemma A.1 E and note that \(\xi_{t-1}^\top\) is of dimension \(Np\). Additionally, we have \(\max_j \|\beta_j\|_1 \leq M^{1-\eta}k\)
and
\[ \hat{\lambda}_T = \left\| \frac{1}{T-p} \sum_{t=p+1}^T (\xi_{j,t} - \beta_{j}^\top\xi_{t-1}^\top)\xi_{t-1}\right\|_{\max} \leq \left\| \frac{1}{T-p} \sum_{t=p+1}^T (v_j)^\top\xi_{t-1}\right\|_{\max} + \frac{1}{T-p} \sum_{t=p+1}^T (v_j)^\top w_{t-1} \leq \left\| \frac{1}{T-p} \sum_{t=p+1}^T v_j w_{t-1}\right\|_{\max} \]

**Proof of Theorem 3.** First note that \(e_{1/p}^\top(\hat{X}^1_{T+1} - X^1_{T+1}) = \beta_{j}^\top\hat{\xi}_T - \beta_{j}^\top\xi_T + \hat{\Lambda}_{j}^\top H_{NT}^1 H_{NT}^1 \hat{\Pi}_{j}^{(p)} H_{NT} H_{NT}^1 f_{T+1-i} \)
and \(\Lambda_{j}^\top H_{NT}^1 H_{NT}^1 \sum_{i=1}^{p-2} \hat{\Pi}_{j}^{(p)} H_{NT} f_{T+1-i} \rightarrow \Lambda_{j}^\top H_{NT}^1 H_{NT}^1 \sum_{i=1}^{p-2} \Pi_{j}^{(p)} H_{NT} H_{NT} f_{T+1-i} \). Then, the results derived in Theorem 1, 2 and Lemma A.1 can be plugged in. Note further that due to due to Assumption 2 and 3 we have \(A\xi_{t}/N = \|A\|_{2/N}A/\|A\|_{2} \hat{\xi}_t = O_P(1/\sqrt{N})\) appearing in \(f_t \sim H_{NT} f_t\) and the assertion follows.

**Lemma A.3.** Under Assumption 2, 3, 4 and Assumption 1 and 2 in Wu & Zaffaroni (2018) we have the following
\[ \|f_t(\omega) - H_{NT} f_t(\omega) H_{NT}\|_{\max} = O_P\left( \sqrt{B_T/T} + \frac{1}{B_T} \left( \log(N) + \frac{kT}{N} + \frac{\log(N)}{\sqrt{N}} \right) \right) \]
and
\[ \|f_t^{-1}(\omega) - (H_{NT} f_t(\omega) H_{NT}^{-1})^{-1}\|_l = O_P\left( \sqrt{B_T/T} + \frac{1}{B_T} \left( \log(N) + \frac{kT}{N} + \frac{\log(N)}{\sqrt{N}} \right) \right) \]
for \(l \in [1, \infty]. \)
Lemma A.4. Under Assumption 2, 3, 4 we have the following:

\[
\|\Sigma^{-1}_{CLIME} - \Sigma^{-1}\|_1 = O_P\left(k_e\|\Sigma^{-1}\|_1\left[\sqrt{\log(N)/T} + N^{2/3}T^{2/3} - 1 + k \left(\frac{k_e}{N} + \frac{\log(N)}{T} + \frac{\sqrt{\log(N)}}{\sqrt{NT}}\right)\right]\right)\],
\[
(NT)^{2/3 - 1}k_e + \left(\frac{(NT)^{2/3}}{T} + (Np)^{2/3}T^{2/3} - 1\right)\left(\frac{k_e}{N} + \frac{\log(N)}{T} + \frac{1}{\sqrt{NT}} + \frac{1}{T^{3/2}} + (NpT)^{2/3}/T\right)^{1-q_o})^{1-q_o}, l \in [1, \infty],
\]

\[
\|f_\xi(\omega)^{-1} - \hat{f}_\xi(\omega)^{-1}\|_1 = O_P\left(k^2\left(\|\Sigma^{-1} - \hat{\Sigma}^{-1}_{CLIME}\|_1 + \max_k\|\hat{\beta}^{(s)} - \beta^{(s)}\|_2\right)\|\Sigma^{-1}\|_1\right), l \in [1, \infty],
\]

\[
\|f_\xi(\omega)^{-1} - \hat{f}_\xi(\omega)^{-1}\|_2 = O_P\left(\|\Sigma^{-1} - \hat{\Sigma}^{-1}_{CLIME}\|_1 + k\max_k\|\hat{\beta}^{(s)} - \beta^{(s)}\|_2\right), l \in [1, \infty],
\]

If \(N = T^a, p = T^b\) for some \(a, b > 0, \zeta \geq 4(1 + a + b)\) and \(k\) such that \(\|\hat{\alpha} - \alpha\|_\infty = o_P(1)\), these error bounds simplify to

\[
\|\Sigma^{-1}_{CLIME} - \Sigma^{-1}\|_1 = O_P\left(k_e\|\Sigma^{-1}\|_1\left[\sqrt{\log(N)/T} + k \left(\frac{k_e}{N} + \frac{\log(N)}{T} + \frac{\sqrt{\log(N)}}{\sqrt{NT}}\right)\right]\right)\], l \in [1, \infty],
\]

\[
\|f_\xi(\omega)^{-1} - \hat{f}_\xi(\omega)^{-1}\|_1 = O_P\left(k^2\left(\|\Sigma^{-1}\|_1\left(\frac{k_e}{N} + \frac{\log(N)}{T} + \frac{\sqrt{\log(N)}}{\sqrt{NT}}\right)\right)^{1-q_o}\right)\right) + \sqrt{k_1\left(\frac{\log(N)}{T} + k\|\Sigma^{-1}\|_1\right)^{1-q_o} + k\left(\frac{k_k}{N} + \frac{\log(N)}{T} + \frac{\sqrt{\log(N)}}{\sqrt{NT}}\right)^{1-q_o}}, l \in [1, \infty],
\]

\[
\|f_\xi(\omega)^{-1} - \hat{f}_\xi(\omega)^{-1}\|_2 = O_P\left(k^2\left(\|\Sigma^{-1}\|_1\left[\sqrt{\log(N)/T} + k \left(\frac{k_e}{N} + \frac{\log(N)}{T} + \frac{\sqrt{\log(N)}}{\sqrt{NT}}\right)\right]\right)^{1-q_o}\right) + k^{1/2}\left(\frac{\log(N)}{T} + k\|\Sigma^{-1}\|_1\right)^{1-q_o}, l \in [1, \infty],
\]

Proof of Theorem 4. We have

\[
\|f_\xi(\omega)^{-1} - \hat{f}_\xi(\omega)^{-1}\|_1 \leq \|f_\xi^{-1}(\omega) - \hat{f}_\xi^{-1}(\omega)\|_1 + \|\hat{f}_\xi^{-1}(\omega)\hat{A}\left(\frac{f_\xi^{-1}(\omega)}{N} + \hat{\lambda}^T/\sqrt{N}\right)^{-1}\hat{\lambda}^T/Nf_\xi^{-1}(\omega)
\]
\[
- f_\xi^{-1}(\omega)\hat{A}\left(\frac{1}{\hat{H}_{NT}^T}\right)^{-1}f_\xi^{-1}(\omega)\frac{H_{NT}^{-1}}{N} + \left(\frac{H_{NT}^{-1}}{N}\right)^{-1}\hat{\lambda}^T/\sqrt{N}f_\xi^{-1}(\omega)\hat{A}\left(\frac{1}{\hat{H}_{NT}^T}\right)^{-1}\hat{\lambda}^T/Nf_\xi^{-1}(\omega)\|_1,
\]

Let \(G = (\hat{H}_{NT}^{-1})f_\xi^{-1}(\omega)H_{NT}^{-1}/N + (\hat{H}_{NT}^{-1})^T\hat{\lambda}^T/\sqrt{N}f_\xi^{-1}(\omega)\hat{A}\hat{H}_{NT}^{-1}/\sqrt{N}\) and \(\hat{G} = (\hat{f}_\xi^{-1}(\omega)/N + \hat{\lambda}^Tf_\xi^{-1}(\omega)\hat{A})^{-1}/N\).

Lemma A.4 gives a rate for \(\|f_\xi^{-1}(\omega) - \hat{f}_\xi^{-1}(\omega)\|_1\). Furthermore, the second term on the right hand side

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of \((A.1)\) is smaller or equal to:

\[
\|f^{-1}_\xi'(\omega)\hat{\Lambda} - f^{-1}_\xi'(\omega)\Lambda(H^{-1}_{NT})\|^2 \leq \|G\|^2 \|\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\| + \|f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/N\| \|G - \hat{\Lambda}\| \|\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\| + \|f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/N\| \|G\| \|\hat{\Lambda} - \Lambda\| \|\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\| + \|f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/N\| \|G\| \|\hat{\Lambda} - \Lambda\| \|\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\| + \|f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/N\| \|G\| \|\hat{\Lambda} - \Lambda\| \|\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\|
\]

\(G\) is of fixed dimension \(r \times r\) and we first show that \(\|G\| = O(1), l \in [1, \infty)\). For this, we have

\[
\|G\|_2 \leq \left(\sigma_{\min}(\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega))H^{-1}_{NT}/N) + \sigma_{\min}(\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/N)\right)^{-1}.
\]

Note that Lemma A.1, A) implies that \(1/\sigma_{\min}(H_{NT}) = O(1)\) and \(1/\sigma_{\min}(H^{-1}_{NT}) = O(1)\) and we have for symmetric matrices \(A, B, 1/\sigma_{\min}(AB) \leq 1/(\sigma_{\min}(A)\sigma_{\min}(B))\). Hence, \(\sigma_{\min}(\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega))H^{-1}_{NT}/N) = O(1/N)\). Furthermore, let \(\hat{\Lambda} = (\Lambda^T \Lambda/N)^{-1/2} \Lambda\). Note that \(\Lambda^T \Lambda/N = \Sigma_{\Lambda} + o(1)\) and \(\Sigma_{\Lambda}\) is positive definite by Assumption 3 and \(\sigma_{\min}(\Lambda^T \Lambda/N) > 1/M > 0\). Then, \(\hat{\Lambda}^T \hat{\Lambda}/N = I_r\) and we have by Poincare's separation theorem \(\sigma_{\min}(\left(H^{-1}_{NT}\right)^T \Lambda^T f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/N) \geq \sigma_{\min}(\left(H^{-1}_{NT}\right)^2\sigma_{\min}(\Lambda^T \Lambda/N)^{-1})\sigma_{\min}(f^{-1}_\xi'(\omega))\). Thus, \(\|G\|_2 = O(1)\) and since it is of fixed dimension, we also have \(\|G\| = O(1), l \in [1, \infty)\). Since \(f_X\) is hermitian, we can focus on \(l = \infty\). We have by Assumption 3 and 4 \(\|f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/N\|_\infty \leq \|f^{-1}_\xi'(\omega)\|_\infty \|\Lambda\|_\infty \|\Lambda^{-1}\|_\infty \leq O(\kappa_\xi).\) Note that \(\Lambda \in N \times r\) which means \(\|\Lambda\|_\infty \leq r\|\Lambda\|_{\max} = O(1)\).

Similarly, since \(\|\Lambda^T / N\|_\infty \leq N/N\|\Lambda\|_{\max}\), we have \(\|\left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\|_\infty = O(\kappa_\xi).\) By similar arguments, we have \(\|f^{-1}_\xi'(\omega)\hat{\Lambda} - f^{-1}_\xi'(\omega)\Lambda(H^{-1}_{NT})\|_\infty = O_P(\|f^{-1}_\xi'(\omega) - f^{-1}_\xi'(\omega)\|_\infty \|\Lambda\|_\infty \|\hat{\Lambda} - \Lambda H^{-1}_{NT}\|_{\max}) = O_P(\|f^{-1}_\xi'(\omega) - f^{-1}_\xi'(\omega)\|_\infty + k\xi_1\sqrt{\log(N)/T} + (NT)^{1/2}/T + k\xi_2/N)\) and \(\|\Lambda^T / N f^{-1}_\xi'(\omega) - \left(H^{-1}_{NT}\right)^T \Lambda^T / N f^{-1}_\xi'(\omega)\|_\infty = O_P(\|f^{-1}_\xi'(\omega) - f^{-1}_\xi'(\omega)\|_\infty + k\xi_1\sqrt{\log(N)/T} + (NT)^{1/2}/T + k\xi_2/N)\). We have further \(\|G - \hat{G}\|_2 \leq \|G\|_2 \|\hat{G}\|_2 \|G^{-1} - \hat{G}^{-1}\|_2 \leq \|\left(H^{-1}_{NT}\right)^T f^{-1}_\xi'(\omega)\Lambda^{-1}/N - f^{-1}_\xi'(\omega)/N\|_2 + \|\left(H^{-1}_{NT}\right)^{-1} f^{-1}_\xi'(\omega)\Lambda H^{-1}_{NT}/\sqrt{N - \hat{\Lambda}^T / \sqrt{F}} f^{-1}_\xi'(\omega)\hat{\Lambda}^{-1}/\sqrt{N}\|_2\). Note \(\|\left(H^{-1}_{NT}\right)^{-1} \Lambda^T - \hat{\Lambda}^T / \sqrt{N}\|_2 \leq \|\left(H^{-1}_{NT}\right)^{-1} \Lambda^T - \hat{\Lambda}^T\|_{\max} = O_P(\sqrt{\log(N)/T} + (NT)^{1/2}/T + k\xi_1/N)\), \(\|\left(H^{-1}_{NT}\right)^{-1} \Lambda^T\|_2 = O(1)\) and \(\|f^{-1}_\xi'(\omega)\|_2 = O(1)\).

Hence, these results and Lemma A.3 we have \(\|G^{-1} - \hat{G}^{-1}\|_2 = O_P(\sqrt{\log(N)/T} + (NT)^{1/2}/T + k\xi_1/N + \|f^{-1}_\xi'(\omega) - \hat{f}^{-1}_\xi'(\omega)\|_2) = O_P(\sqrt{\log(N)/T} + (NT)^{1/2}/T + k\xi_1/N + \|\Sigma_{\varepsilon_1} - \Sigma_{\varepsilon_1}^{\text{CLIMME}}\|_2 + k\max_{\beta, \vartheta} \|\beta^{(v)}(\omega) - \vartheta^{(v)}\|_2)\) which is faster than \(\|f^{-1}_\xi'(\omega)\hat{\Lambda} - f^{-1}_\xi'(\omega)\Lambda(H^{-1}_{NT})\|_\infty\). That means \(\|f_X(\omega)^{-1} - \hat{f}_X(\omega)^{-1}\|_\infty = O_P(\kappa_\xi \|f^{-1}_\xi'(\omega) - f^{-1}_\xi'(\omega)\|_\infty + k\xi_1\|\hat{\Lambda} - \Lambda H^{-1}_{NT}\|_{\max}).\) Since \(\|\Lambda H^{-1}_{NT}/\sqrt{N}\|_2 = O(1)\) and \(\|f_X(\omega)^{-1}\|_2 = O(1)\), we have further \(\|f_X(\omega)^{-1} - \hat{f}_X(\omega)^{-1}\|_2 = O_P(\|f^{-1}_\xi'(\omega) - f^{-1}_\xi'(\omega)\|_2 + \|\hat{\Lambda} - \Lambda H^{-1}_{NT}\|_{\max}).\)

The assertions follows after inserting the rates of Lemma A.4. 

\(\square\)
Supplementary Material

In this supplementary material we collect the proofs of the lemmas.

Proof of Lemma A.1. First note that under Assumption 2 and Remark 4 we have for $\alpha > 0.5, q \geq 4$ \[ \max_j \|\{\xi_{j,t}\}\|_{q,\alpha} < \infty, \max_i \|\{f_{i,t}\}\|_{q,\alpha} < \infty \text{ and } \max_{j,s} \|\{\xi_{j,t,f_{i,t}}\}\|_{q/2,\alpha} < \infty. \] Furthermore, since \{f_t\} and \{\xi_t\} are linear processes and \[ \|B^{(j)}\|_2 \leq M\rho^j, \] we have \[ \max_{w \in \mathbb{R}^r, \|w\|_2 \leq 1} \|\{w^T \xi_t\}\|_{q,\alpha} < \infty = \max_{w \in \mathbb{R}^r, \|w\|_2 \leq 1} \|\{w^T B^{(j)} \xi_t\}\|_{q,\alpha} = \max_{w \in \mathbb{R}^r, \|w\|_2 \leq 1} (m+1)^\alpha \sum_{k=0}^\infty \|w^T B^{(k)}(e_0 - e_0')\|_{q,\alpha} \leq \sup_{m \geq 0} (m+1)^\alpha C\rho^m < \infty \] and similarly \[ \max_j \|\{\xi_{j,t}\}\|_{q/2,\alpha} < \infty \] and \[ \max_j \|\{f_{i,t}^2\}\|_{q/2,\alpha} < \infty. \]

For part A, we have \[ X^T X/NT = A\tilde{F}^T A\tilde{F}^T /NT + \Xi^T \Xi/NT + \Xi^T \tilde{F}^T /NT + \tilde{F}^T \Xi/NT. \] Assumption 3, i.e. $\Sigma_A > 0, \Sigma_F > 0$, implies for $N, T$ large enough that $F/\sqrt{T}$ and $\Lambda/\sqrt{N}$ have rank $r$ and that all $r$ eigenvalues are strictly positive. Furthermore, note that we have by part D and Assumption 1 \[ \|\Xi^T \Xi/NT\|_F^2 = (1/N^2 \sum_{i_1,i_2} (1/T \sum_{t=1}^T \xi_{t,i_1} \xi_{t,i_2})^2) = (1/N^2 \sum_{i_1,i_2} (e_{i_1}^T \Gamma_0 e_{i_2} + 1/T \sum_{t=1}^T \xi_{t,i_1} \xi_{t,i_2} - e_{i_1}^T \Gamma_0 e_{i_2})^2) = O_P(k_q / N + \log(N)/T) + (NT)^{1/2}/T. \] Additionally, by part B we have \[ \|\tilde{F}^T \Xi/NT\|_F^2 = \|\Xi^T \tilde{F}^T /NT\|_F^2 = 1/N^2 \sum_{i_1,i_2} (1/T \sum_{t=1}^T \xi_{t,i_1} \xi_{t,i_2})^2 = O_P(\log(N)/T + (NT)^{1/2}/T). \] That means for $N, T$ large the eigenvalues of \[ X^T X/NT \] are approximately those of \[ A\tilde{F}^T A\tilde{F}^T /NT. \] Hence, for $N, T$ large, \[ X^T X/NT \] possesses $r$ positive eigenvalues which implies that \[ D_{2NT,r}^2 > O_P(1). \] Since by Assumption 3 \[ \lim_T \|F/\sqrt{T}\|_2 \leq M \] and \[ \lim_T \|\Lambda/\sqrt{N}\|_2 \leq M, \] we also have \[ D_{2NT,r}^2 = O_P(1). \]

For the part B, note first that \[ \sum_{s=1}^T \xi_{i,s} f_{j,s} = 0 \] due to Assumption 2. Furthermore, since \[ P(\max_{i,j} |\sum_{s=1}^T \xi_{i,s} f_{j,s}| \geq x) \leq \sum_{i,j} P(\sum_{s=1}^T \xi_{i,s} f_{j,s} \geq x), \] we have Assumption 2 and Theorem 2 in Wu et al. (2016) for $q > 2$ and $C_1, C_2, C_3$ some constants depending only on $q$ and $\alpha$ \[ P \left( \sum_{s=1}^T \xi_{i,s} f_{j,s} \geq x \right) \leq C_1 \frac{T \max_{i,j} \|\{\xi_{i,s} f_{j,s}\}\|_{q,\alpha}}{x^q} + C_2 \exp \left( - \frac{C_3 x^2}{T \max_{i,j} \|\{\xi_{i,s} f_{j,s}\}\|_{q,\alpha}^2} + \log(N) \right). \] Furthermore, \[ \max_{i,j} \|\sum_{s=1}^T \xi_{i,s} f_{j,s}\|_{q,\alpha} \leq C_1 N \frac{T \max_{i,j} \|\{\xi_{i,s} f_{j,s}\}\|_{q,\alpha}}{x^q} + C_2 \exp \left( - \frac{C_3 x^2}{T \max_{i,j} \|\{\xi_{i,s} f_{j,s}\}\|_{q,\alpha}^2} + \log(N) \right). \] This implies \[ \max_{i,j} |1/T \sum_{s=1}^T \xi_{i,s} f_{j,s}| = O_P(\sqrt{\log(N)/T}) + N^{2/\zeta} T^{1/2}/T^{1-\zeta}. \] Since $f_{j,s} = \Sigma_{j,k} \xi_{i,k} \xi_{j,s} = e_{j,k}^T \Gamma_k \xi_{j,s},$ and $r$ is fixed, Part C and D follow by the same arguments. Note also that for some vectors $\mathbf{u}, \mathbf{v}$ and some symmetric matrix $\Gamma,$ we have $\mathbf{u}^T \Gamma \mathbf{v} \leq \mathbf{u}^T \Gamma \mathbf{v} + \mathbf{u}^T \Gamma \mathbf{v}$. That is why for max_{j,s} |\sum_{s=1}^T (\xi_{j,s} \xi_{j,s} - e_{j,k}^T \Gamma_k \xi_{j,s})| it is sufficient to look at max_{j} |\sum_{s=1}^T (\xi_{j,s} \xi_{j,s} - e_{j,k}^T \Gamma_k \xi_{j,s})|.

For the part E, note that $\Gamma_\xi(0) = \sum_{j=0}^{\infty} B^{(j)} \Sigma_v (B^{(j)})^T$ and $1/\sqrt{N} \sum_{i,k} \xi_{i,k} \xi_{i,s} = 1/\sqrt{N} e_k^T \Lambda \xi_s,$ where $1/\sqrt{N} e_k^T \Lambda \in \mathbb{R}^N$ and $\|1/\sqrt{N} e_k^T \Lambda\|_2 \leq M.$ Since $\|B^{(j)}\|_2 \leq M\rho^j$ and $\|\Sigma_v\|_2 \leq M$ by Assumption 1.2, we have $\|\Gamma_\xi(0)\|_2 \leq M^3/(1 - \rho^2)$ and the assertions follows then by Assumption 3 and part D.
For Part F we obtain by similar arguments for \( q \geq 2 \),
\[
P \left( \max_{j,k} \left| \frac{1}{T} \sum_{t=1}^{T} e_j^T \xi_j \xi_t^T \Gamma(0) \Lambda^T e_k \right| \geq x \right) \leq C_1 \frac{N T \max_i \|\xi_i^2\|_{\xi,\alpha}}{x^q} + C_2 \exp \left( - \frac{C_3 x^2}{T \max_i \|\xi_i^2\|_{\xi,\alpha}^2} + \log(N) \right).
\]
Since \( \|\Gamma(0)\Lambda\|_{\max} \leq \|\Gamma(0)\|_{\infty} \|\Lambda\|_{\max} \leq M^2 k_\xi \), we have \( \max_{j,k} |1/T \sum_{s=1}^{T} e_j^T \xi_j \xi_t^T \Lambda^T e_k| = O_P(k_\xi + \sqrt{\log(N)/T} + N^{2/\zeta} T^{2/\zeta - 1}) \).

For Part G, first note that we have by Cauchy-Schwarz and the previous parts of this lemma
\[
\max_j \left| \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{s,t=1}^{T} \xi_{j,t} f_i^T \Lambda_i \xi_{i,s} \hat{f}_{s,t} \right| = O_P(\sqrt{\log(N)/NT} + N^{2/\zeta - 1/2} T^{2/\zeta - 3/2}),
\]
\[
\max_{j} \left| \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{s,t=1}^{T} \xi_{j,t} \xi_{i,s} \Lambda_i \hat{f}_{s,t} \right| \leq \left( \sum_{k=1}^{r} \max_j \left( \frac{1}{NT} \sum_{t=1}^{T} e_j^T \xi_j \xi_t^T \Lambda^T e_k \right) \right)^{1/2} \left( \sum_{k=1}^{r} \left( \frac{1}{NT} \sum_{s=1}^{T} f_{k,s}^2 \right) \right)^{1/2} \leq \frac{k_\xi}{N} \left( \sum_{k=1}^{r} \max_j \left( \frac{1}{NT} \sum_{t=1}^{T} e_j^T \xi_j \xi_t^T \Lambda^T e_k \right) \right)^{1/2} \left( \sum_{k=1}^{r} \left( \frac{1}{NT} \sum_{s=1}^{T} f_{k,s}^2 \right) \right)^{1/2} = O_P \left( \frac{k_\xi}{N} + \frac{\sqrt{\log(N)}}{\sqrt{NT}} + N^{2/\zeta - 1} T^{2/\zeta - 1} \right),
\]
\[
\max_j \left| \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{s,t=1}^{T} \xi_{j,t} \xi_{i,s} f_{s,t} \right| = \max_j \left| \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{T} \sum_{t=1}^{T} \xi_{j,t} \xi_{i,t} - e_j \Gamma(0) e_i + e_j \Gamma(0) e_i \right) \right| \left( \frac{1}{T} \sum_{s=1}^{T} \xi_{i,s} f_{s,t} \right) \left( \frac{1}{T} \sum_{s=1}^{T} \xi_{j,s} f_{s,t} \right) \leq \frac{\log(N)}{T} + (NT)^{1/\zeta} / T^2 + k_\xi \left( \sqrt{\log(N)/T} + N^{2/\zeta} T^{2/\zeta - 1} \right),
\]
and
\[
\max_j \left| \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{s,t=1}^{T} \xi_{j,t} \xi_{i,s} \hat{f}_{s,t} \right| = O_P \left( \frac{\log(N)}{T} + N^{4/\zeta} T^{4/\zeta - 2} + k_\xi \frac{\sqrt{\log(N)}}{\sqrt{NT}} + N^{2/\zeta - 1} T^{2/\zeta - 1} \right).
\]
Then, we have further

\[
\frac{1}{NT^2} \sum_{t=1}^{N} \sum_{s=1}^{T} \xi_{t,s}^T \xi_{t,s} f_{t,s} = \\
= \sum_{k=1}^{r} \left( \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,s}^T \xi_{i,s} f_{k,s} \right) (e_i^T H_{NT} e_i) + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,s}^T \xi_{i,s} (f_{i,s} - e_i^T H_{NT} f_s) = I_j + I_{J_j},
\]

\[
\max_j |I_j| = O_P \left( \left( \frac{\sqrt{\log(N)}}{T} + N^{2/\zeta} T^{2/\zeta - 3/2} + \frac{k_{\xi}}{N} \left( \frac{\sqrt{\log(N)}}{T} + N^{2/\zeta} T^{2/\zeta - 1} \right) \right) \right).
\]

Furthermore, we have

\[
\max_j |I_{J_j}| =
= \max_j \left| \sum_{i=1}^{N} \sum_{s=1}^{T} \xi_{i,s}^T \xi_{i,s} f_{t,s} \right|
= \max_j \left| \frac{1}{NT} \sum_{t=1}^{T} \xi_{i,t}^T \xi_{i,t} f_{t,s} \right|
\]

\[
\leq \frac{1}{\sqrt{N}} \max_j \left( \frac{1}{T} \sum_{t=1}^{T} \xi_{i,t}^T \xi_{i,t} f_{t,s} \right) \left( \max_{i,j} \frac{1}{NT} \sum_{t=1}^{T} \xi_{i,t}^T \xi_{i,t} f_{t,s} \right)^{1/2} \max_j \left( \frac{1}{T} \sum_{t=1}^{T} f_{t,s} \right) \left( D_{NT}^{-2} \max_j r^2 \right)
\]

\[
= O_P \left( \left( \frac{\sqrt{\log(N)}}{T} + N^{2/\zeta} T^{2/\zeta - 1} \right) \left( \frac{\sqrt{\log(N)}}{T} + N^{2/\zeta} T^{2/\zeta - 1} \right) \right) + III
\]

\[
= O_P \left( \left( \frac{\log(N)}{T} + N^{4/\zeta} T^{4/\zeta - 2} + \frac{k_{\xi}}{N} + \frac{\log(N)}{\sqrt{T N}} + N^{2/\zeta - 1} T^{2/\zeta - 1} \right) \right),
\]

where \( III \leq 1/NT^2 (\sum_{t=1}^{T} \xi_{i,t}^T - \Gamma_{\xi}(k)|\xi_{i,t}|^2) \) is the combination of the previous results.

For part \( H \), first note that we have from part \( G \).

\[
\left| \frac{1}{NT} \sum_{t=1}^{T} \xi_{i,t}^T \xi_{i,t} (f_{t,s} - H_{NT} f_s) \right| \leq \frac{1}{N} \sum_{s=1}^{T} \xi_{i,s}^T \xi_{i,s} \max_{j,l} \left| \frac{1}{T} \sum_{t=1}^{T} e_i^T (f_{t,s} - H_{NT} f_s) \xi_{i,t} \right|
\]

\[
= O_P \left( \left( \frac{\log(N)}{T} + \frac{k_{\xi}}{N} + \frac{\log(N)}{\sqrt{T N}} + g(N, T, \zeta) \right) \right).
\]

Furthermore, from part \( B \) and Assumption 3. we have

\[
\left| \frac{1}{NT} \sum_{t=1}^{T} \sum_{s=1}^{T} e_i^T f_{t,s}^T A_{i,s} (f_{s} - H_{NT} f_s) \right|
\]

\[
= O_P \left( \left( \frac{\log(N)}{T} + \frac{k_{\xi}}{N} + \frac{\log(N)}{\sqrt{T N}} + g(N, T, \zeta) \right) \right).
\]

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Then, the assertion follows by the same arguments as in part G. Note that in each assertion in part G \( \xi_{j,t} \) is replaced with \( \|e_j^T A^T\|_2/N(e_j^T A^T/e_j^T A^T\|_2\xi) \). Since \( \|e_j^T A^T\|_2/N = O(1/\sqrt{N}) \) the assertion follows by part B,E.

For part I, we have for \( j,l = 1, \ldots, r \)

\[
\left| \frac{1}{T} \sum_{t=1}^{T} f_{j,t} \left( x_t - H_NTJ_s \right)^T \mathbf{D}_{J_s} f_{j,t} \right| = \frac{1}{NT^2} \left| \sum_{i=1}^{N} \sum_{t=1}^{T} f_{j,t} \xi_i, A_s f_i^T \tilde{f}_{j,t} + \frac{T}{NT^2} \sum_{i=1}^{T} f_{j,t} \xi_i, \xi_i \right| \\
+ O_P \left( \frac{\log(N)}{T} + \frac{k_{\xi}}{N} + \frac{\sqrt{\log(N)}}{N^2} + (NT)^{2/\kappa} \left( \frac{1}{\sqrt{NT}} + \frac{1}{T^{\kappa/2}} + (NT)^{2/\kappa} \frac{1}{T^2} \right) \right) \\
\leq \left( \frac{1}{TN} \sum_{i=1}^{T} \left( \frac{1}{T} \sum_{t=1}^{T} \left( e_i^T A^T \xi_i f_{j,t} \right)^2 \right)^{1/2} \left( \frac{1}{T} \sum_{t=1}^{T} \left( \sum_{i=1}^{T} f_{j,t} \right)^2 \right) \right)^{1/2} \\
+ \left( \left( \frac{1}{T} \sum_{t=1}^{T} f_{j,t} \xi_i^2 \right) \left( \frac{1}{T} \sum_{t=1}^{T} \xi_i \xi_i^T \right) \left( \frac{1}{T} \sum_{t=1}^{T} \xi_i \xi_i^T \right) \right)^{1/2} \\
+ O_P \left( \frac{\log(N)}{T} + \frac{k_{\xi}}{N} + \frac{\sqrt{\log(N)}}{N^2} + (NT)^{2/\kappa} \left( \frac{1}{\sqrt{NT}} + \frac{1}{T^{\kappa/2}} + (NT)^{2/\kappa} \frac{1}{T^2} \right) \right) \\
= O_P \left( \frac{\log(N)}{T} + \frac{k_{\xi}}{N} + \frac{\sqrt{\log(N)}}{N^2} + (NT)^{2/\kappa} \left( \frac{1}{\sqrt{NT}} + \frac{1}{T^{\kappa/2}} + (NT)^{2/\kappa} \frac{1}{T^2} \right) \right).
\]

Part J follows by part B,E, and H. For Part K, note that \( \|A\|_{\max} \leq M \). Then, this part follows by G,I, and J.

\[ \square \]

**Proof of Lemma A.2.** This proof follows ideas of the Proof of Proposition 4.1 in Basu & Michailidis (2015) as well as the Proof of Corollary 3 in Negahban et al. (2012). Let \( \gamma = 1/(T - p) \sum_{t=p+1}^{T} \xi_{j,t} \xi_{j,t-1}^T \) and \( \tilde{\gamma} = 1/(T - p) \sum_{t=p+1}^{T} \xi_{j,t} \xi_{j,t-1}^T \). Let \( \beta^* := \beta_j, \beta := \tilde{\beta}_j \) and \( v = \tilde{v} - \beta^* \). Furthermore, let for some threshold \( \eta > 0 \), \( J' = \hat{J}_\eta = \{ j \in \{1, \ldots, np\} | e_j^T \beta^* > \eta \} \) denote the set of indices for which \( \beta^* \) is absolutely greater than the threshold \( \eta \), \( \beta_{a,u} \) refers to the hard thresholded vector with threshold \( \eta \) and for some vector \( u, u_j, u_{jc} \) denotes the vector obtained by the indices in \( J, J' \), respectively. We have by Assumption 1 \( \|\beta_j^* - \beta^*\|_1 \leq \eta^{-q} k \). Furthermore, \( |J| \leq \eta^{-q} k \).

Since \( \beta_j \) is the minimum given in (4), we have \( -\beta^T \gamma + \beta^T \tilde{\gamma} + \lambda \parallel \beta \parallel_1 \leq 2 \beta^* \gamma + (\beta^*)^T \tilde{\gamma} \beta^* + \lambda \parallel \beta^* \parallel_1 \). This gives further \( v^T \tilde{\gamma} v \leq 2 v^T (\gamma - \tilde{\gamma} \beta^*) + \lambda (\parallel \beta^* \parallel_1 - \parallel \beta^* + v \parallel_1) \leq 2 v^T (\gamma - \tilde{\gamma} \beta^*) + \lambda (\parallel v_j \parallel_1 - \parallel v_{jc} \parallel_1 + 2 \eta^{-q} k) \). This implies with the condition \( \parallel \frac{1}{T-p} \sum_{t=p+1}^{T} \xi_{j,t} \parallel_{\max} \leq \Lambda/4 \) that \( v^T \tilde{\gamma} v \leq 3/2 \lambda \parallel v_j \parallel_1 - 1/2 \parallel v_{jc} \parallel_1 \leq 2 \lambda \parallel v \parallel_1 + 24 \eta^{-q} k \). Hence, \( \parallel v_{jc} \parallel_1 \leq 3 \parallel v_j \parallel_1 + 4 \eta^{-q} k \) and since \( |J| \leq \eta^{-q} k \), \( \parallel v \parallel_1 \leq 4 \sqrt{k} \eta^{-q/2} \parallel v \parallel_2 + 4 \eta^{-q} \).

Then, with the condition \( \Theta^T \frac{1}{T-p} \sum_{t=p+1}^{T} \xi_{j,t} \xi_{j,t-1}^T \Theta \geq \alpha \parallel \Theta \parallel_2^2 - \gamma \parallel \Theta \parallel_2 \forall \Theta \in \mathbb{R}^{np} \) we obtain that \( \alpha \parallel v \parallel_2^2 - \gamma \parallel v \parallel_1 \leq 8 \lambda \parallel v \parallel_2 \sqrt{k} \eta^{-q/2} + 10 \lambda k \eta^{-q} \). Set \( \eta = \Lambda / \alpha \). Then, with the bound for \( \parallel v \parallel_1 \) and dropping minor terms in the maximum we obtain \( \parallel v \parallel_2 \leq 16 \max(\sqrt{E(\Lambda/\alpha)^{-q/2}}, \sqrt{E(\Lambda/\alpha)^{-q}}) \).

Furthermore, \( \parallel v \parallel_1 \leq \max(68k(\Lambda/\alpha)^{-q/2}, 64 \sqrt{\kappa} k^{3/2} (\Lambda/\alpha)^{1-3/2q} + 4k(\Lambda/\alpha)^{-q}) \).

\[ \square \]

**Proof of Lemma A.3.** We have \( \|\tilde{f}_j(\omega) - H_NT J_s f_j(\omega) H_NT \|_{\max} \leq \|\tilde{f}_j(\omega) - H_NT J_s f_j(\omega) H_NT \|_{\max} + \|\tilde{f}_j(\omega) - f_j(\omega) \|_{\max} \| H_NT J_s(\omega) \|_{H_NT(\omega)} \). We have by Lemma A.1 \( \| H_NT(\omega) \|_1 = O(1) = \| H_NT(\omega) \|_{\infty} \) and by Theorem 3 in Wu & Zaffaroni (2018) \( \|\tilde{f}_j(\omega) - f_j(\omega)\|_{\max} = O_P(\sqrt{B_T/T}) \). Note that the dimension \( r \) of the process \( \{ f_t \} \) is fixed. Furthermore, we have \( \| f_j(\omega) - H_NT J_s(\omega) H_NT \|_{\max} = \)
by Theorem 1 in Krampe & Paparoditis (2021) that under Assumption 4 $\|f\|_{\infty} + \{\text{Fuk-Nagaev inequality}\}$ for $l_i, j_i$ the estimated residuals given by $\hat{\xi}_t = \xi_t - \sum_{j=1}^p \hat{A}^{(j)} \xi_{t-j}$. This gives the sample covariance $\hat{\Sigma}_v = 1/T \sum_t v_t v_t^\top$. We have $\hat{\Sigma}_v = \hat{\Sigma}_v = 1/T \sum_t (v_t - \hat{v}_t) v_t^\top + v_t (v_t - \hat{v}_t)^\top + (v_t - \hat{v}_t) (v_t - \hat{v}_t)^\top$. Furthermore, $v_t - \hat{v}_t = w_t + \sum_{j=1}^p A^{(j)} w_{t-j} + \sum_{j=1}^p (A^{(j)} - \hat{A}^{(j)}) \xi_{t-j} + \sum_{j=1}^p (A^{(j)} - \hat{A}^{(j)}) w_{t-j}$. Hence, following the arguments of Theorem 1 we have $\|\hat{\Sigma}_v - \Sigma_v\|_{\infty} = O_P(\|\hat{\xi}\|_{\infty} 1/T \sum_{t=1}^T w_t \xi_t\|_{\infty} + \sum_{j=1}^p \|\hat{A}^{(j)} - A^{(j)}\|_{\infty} \Omega_{t-j} + \sum_{j=1}^p (A^{(j)} - \hat{A}^{(j)}) w_{t-j}$. Since $\Omega_{t-j} = 0$, we have by the arguments of Lemma A.1 $1/T \sum_{t=1}^T \Omega_{t-j} v_t\|_{\infty} = O_P(\sqrt{\log(N)}/T + (Np)^{2/\zeta} T^{2/\zeta_-1})$. Together with Theorem 1 and $\|\hat{\Sigma}_v - \Sigma_v\|_{\infty}$ this lead to the following:

$$\|\hat{\Sigma}_v - \Sigma_v\|_{\infty} = O_P(\sqrt{\log(N)/T} + N^{2/\zeta_-1} + k [k_\xi N \log(N)/T^2 + \sqrt{\log(N)}/\sqrt{NT} + (NT)^{2/\zeta_-1} + k_\xi + (NT)^{4/\zeta}/T^2] +$$

$$(\sqrt{\log(N)/T} + (Np)^{2/\zeta_-1}) \left[ k_\xi N \log(N)/T + (Np)^{2/\zeta}/T + k_\xi + \sqrt{\log(N)p}/\sqrt{NT} +$$

$$\sqrt{\log(N)/T} + Np^2/\zeta + (Np)^{2/\zeta_1}/T^2 + (Np)^{2/\zeta_1}/T^2) \right] 1^{-\alpha}$$

Setting up a CLIME estimator on $\{v_t\}$ leads to $\hat{\Sigma}_v^{-1, \text{CLIME}}$ and following now the arguments of Cai et al. (2011) gives us that the CLIME estimator fulfill $\|\hat{\Sigma}_v^{-1, \text{CLIME}} - \hat{\Sigma}_v^{-1, \text{CLIME}}\|_l = O_P(k_\xi (\|\hat{\Sigma}_v^{-1, \text{CLIME}} - \hat{\Sigma}_v\|_{\infty}^{1-q}))$ for $l \in [1, \infty]$.

We have by Theorem 2 that $\|\hat{\Sigma}_v - \hat{\Sigma}_v\|_{\infty} = O_P(\max_{\hat{\beta}, \beta} \|\hat{\beta} - \beta\|_2)$. Consequently, we obtain by Theorem 1 in Krampe & Paparoditis (2021) that under Assumption 4 $\sum_{j=1}^p \|\hat{A}^{(j,\tau)} - A^{(j)}\|_l = O(k \max_{\hat{\beta}, \beta} \|\hat{\beta} - \beta\|_2^{1-q})$. Then, we have by Theorem 6 in Krampe & Paparoditis (2021) $\|\hat{f}(\omega)^{-1} - \hat{f}(\omega)^{-1}\|_l = O_P(\sum_{j=1}^p \|\hat{A}^{(j)}\|_l \|\hat{\Sigma}_v^{-1, \text{CLIME}}\|_l + \sum_{j=1}^p \|\hat{A}^{(j,\tau)} - A^{(j)}\|_l \|A^{(j)}\|_l \|\hat{\Sigma}_v\|_l)$.