INTERMEDIATE EXTENSIONS AND CRYSTALLINE DISTRIBUTION ALGEBRAS

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Abstract. Let $G$ be a connected split reductive group over a complete discrete valuation ring of mixed characteristic. We use the theory of intermediate extensions due to Abe-Caro and arithmetic Beilinson-Bernstein localization to classify irreducible modules over the crystalline distribution algebra of $G$ in terms of overconvergent isocrystals on locally closed subspaces in the (formal) flag variety of $G$. We treat the case of $SL_2$ as an example.

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1. Introduction

Let $\mathfrak{o}$ denote a complete discrete valuation ring of mixed characteristic $(0, p)$, with fraction field $K$ and perfect residue field $k$. Let $G$ be a connected split reductive group over $\mathfrak{o}$ with $K$-Lie algebra $\mathfrak{g} = \text{Lie}(G) \otimes \mathbb{Q}$.

In [26] we have introduced and studied the crystalline distribution algebra $D^\dagger(\mathcal{G})$ associated to the $p$-adic completion $\mathcal{G}$ of $G$. It is a certain weak completion of the classical universal enveloping algebra $U(\mathfrak{g})$. The interest in the algebra $D^\dagger(\mathcal{G})$ comes at least from two sources. On the one hand, it has the universal property to act as global arithmetic differential operators (in the sense of Berthelot [4]) on any formal $\mathfrak{o}$-scheme which has a $\mathcal{G}$-action. On the other hand, $D^\dagger(\mathcal{G})$ is canonically isomorphic to Emerton’s analytic distribution algebra $D^\text{an}(\mathcal{G}^\circ)$ as introduced in [16]. Here, $\mathcal{G}^\circ$ equals the rigid-analytic generic fibre of the formal completion of $G$ along its unit section. Analytic distribution algebras can be useful tools in the locally analytic representation theory of $G$-modules, a combination of the Beilinson-Bernstein localization theorem over the flag variety of $G$ and let $\mathcal{G}$ be a connected split reductive group over $\mathfrak{o}$. Let $\mathcal{P}$ be a Borel subgroup scheme. In [27] we have established an analogue of the Beilinson-Bernstein theorem for arithmetic differential operators on the formal completion $\mathcal{P}$ of the flag scheme $P = G/B$ of $G$: one has a canonical isomorphism $H^0(\mathcal{P}, \mathcal{G}_p) \simeq D^\dagger(\mathcal{G})_{\theta_0}$ and the global sections functor furnishes an equivalence between the category of coherent arithmetic $\mathcal{G}_p$-modules and coherent $D^\dagger(\mathcal{G})_{\theta_0}$-modules respectively.

Let $\theta$ denote a $G$-valued point of $\mathfrak{g}$ and let $\mathcal{P}$ be a connected split reductive group over $\mathfrak{o}$. Let $\text{Rep}_\theta^\circ(G(K))$ be the full subcategory of admissible representations $V$ of character $\theta$, which are generated by their $G(n)^\circ$-analytic vectors. Let $\text{Coh}(D^\text{an}(G(n)^\circ)_\theta)$ be the category of coherent modules over the central reduction $D^\text{an}(G(n)^\circ)_{\theta}$. The formation $V \mapsto (V_{G(n)^\circ-\text{an}})'$ is a faithful and exact functor

$$\text{Rep}_\theta^\circ(G(K)) \to \text{Coh}(D^\text{an}(G(n)^\circ)_\theta),$$

which detects irreducibility: if the module $(V_{G(n)^\circ-\text{an}})'$ is irreducible, then $V$ is an irreducible object in $\text{Rep}_\theta^\circ(G(K))$.

In this article, we only consider the simplest case: representations of level zero and with trivial infinitesimal character $\theta_0$. We then propose to study the irreducible modules over the ring $D^\dagger(\mathcal{G})_{\theta_0}$. Our approach will be geometric through some crystalline version of localization, similar to the classical procedure of localizing $U(\mathfrak{g})$-modules. Recall that, in the classical setting of $U(\mathfrak{g})$-modules, a combination of the Beilinson-Bernstein localization theorem over the flag variety of $\mathfrak{g}$ together with the formalism of intermediate extensions [2] [10] [23] produces a geometric classification of many irreducible modules, namely those which localize to $D$-modules which are holonomic.

Let in the following $B \subset G$ be a Borel subgroup scheme. In [27] we have established an analogue of the Beilinson-Bernstein theorem for arithmetic differential operators on the formal completion $\mathcal{P}$ of the flag scheme $P = G/B$ of $G$: one has a canonical isomorphism $H^0(\mathcal{P}, \mathcal{G}_p) \simeq D^\dagger(\mathcal{G})_{\theta_0}$ and the global sections functor furnishes an equivalence between the category of coherent arithmetic $\mathcal{G}_p$-modules and coherent $D^\dagger(\mathcal{G})_{\theta_0}$-modules respectively.
An explicit quasi-inverse is given by the adjoint functor $\mathcal{L}oc(M) = \mathcal{D}_P^! \otimes_{\mathcal{D}^!(\mathcal{G})} M$. This allows to pass back and forth between modules over $\mathcal{D}^!(\mathcal{G})_{\theta_0}$ and sheaves on $\mathcal{P}$.

On the other hand, Abe-Caro have recently developed a theory of weights in $p$-adic cohomology [11] building on the six functor formalism for Caro’s overholonomic complexes [12]. On the way, they also defined an intermediate extension functor for arithmetic $\mathcal{G}$-modules and investigated some of its properties. We then use a combination of Abe-Caro’s theory, specialized to the flag variety, and localization to obtain classification results for irreducible $\mathcal{D}^!(\mathcal{G})$-modules, in analogy to the classical setting of $U(\mathfrak{g})$-modules.

Our main result is the following: we call a nonzero $\mathcal{D}^!(\mathcal{G})_{\theta_0}$-module $M$ geometrically $F$-holonomic, if its localization $\mathcal{L}oc(M)$ has a Frobenius structure and is holonomic. We then consider the parameter set of pairs $(Y, \mathcal{E})$ where $Y \subset \mathcal{P}_s$ is a connected smooth locally closed subvariety of the special fibre $\mathcal{P}_s$, $X$ its Zariski closure, and $\mathcal{E}$ is an irreducible overconvergent $F$-isocrystal on the couple $\mathbb{Y} = (Y, X)$. Two pairs $(Y, \mathcal{E})$ and $(Y', \mathcal{E}')$ are equivalent if $X = X'$ and the two isocrystals $\mathcal{E}, \mathcal{E}'$ coincide on an open dense subset of $X$. Given such a pair $(Y, \mathcal{E})$ we put

$$\mathcal{L}(Y, \mathcal{E}) := v_{1+}(\mathcal{E})$$

where $v : \mathbb{Y} \to \mathbb{P} = (\mathcal{P}_s, \mathcal{P}_s)$ is the immersion of couples associated with $Y$ and $v_{1+}$ is the corresponding intermediate extension functor. We then have, cf. Thm. 4.3.

**Theorem 1.** The correspondence $(Y, \mathcal{E}) \mapsto H^0(\mathcal{P}, \mathcal{L}(Y, \mathcal{E}))$ induces a bijection

$$\{\text{equivalence classes of pairs } (Y, \mathcal{E})\} \xrightarrow{\simeq} \{\text{irreducible } F\text{-holonomic } \mathcal{D}^!(\mathcal{G})_{\theta_0}\text{-modules}\}/\simeq$$

For example, each couple $\mathbb{Y}$ is equipped with the constant overconvergent $F$-isocrystal $\mathcal{O}_Y$. If $Z$ is a divisor in $\mathcal{P}_s$ and $\mathcal{U} = \mathcal{P} \setminus Z$ with $\mathbb{Y} = (\mathcal{U}_s, \mathcal{P}_s)$, then $\mathcal{O}_Y = \mathcal{O}_{\mathcal{P}, \mathcal{G}}(\mathcal{U}^! Z)$, i.e. functions on $\mathcal{U}$ with overconvergent singularities along $Z$. In general, if $Y$ admits a formal lift with connected rigid-analytic generic fibre, then $\mathcal{O}_Y$ is irreducible and corresponds therefore to an irreducible $F$-holonomic $\mathcal{D}^!(\mathcal{G})_{\theta_0}$-module.

We expect that many $\mathcal{D}^!(\mathcal{G})_{\theta_0}$-modules, in particular those which come from admissible $G(K)$-representations, are in fact geometrically $F$-holonomic. As an example, we treat the case of highest weight modules (but there are many more, already in dimension one, cf. Theorem 3 below). We show that the central block of the classical BGG category $\mathcal{O}_0$ embeds, via the base change $U(\mathfrak{g}) \to \mathcal{D}^!(\mathcal{G})$, fully faithfully into the category of coherent $\mathcal{D}^!(\mathcal{G})$-modules (cf. Thm. 5.7). It is well-known that the irreducible modules in $\mathcal{O}_0$ are parametrized by the Weyl group elements $w \in W$ via $L(w) := L(-w(\rho) - \rho)$ where $\rho$ denotes half the sum over the positive roots and where $L(-w(\rho) - \rho)$ denotes the unique irreducible quotient of the Verma module with highest weight $-w(\rho) - \rho$. We write

$$L^!(w) := \mathcal{D}^!(\mathcal{G}) \otimes_{U(\mathfrak{g})} L(w)$$
for its crystalline counterpart. On the other hand, let
\[ Y_w := BwB/B \subset P = G/B \]
be the Bruhat cell in \( P \) associated with \( w \in W \) and let \( X_w \) be its Zariski-closure, a Schubert scheme. We have the couple \( Y_w = (Y_{w,s}, X_{w,s}) \) and the immersion \( v : Y_w \to \mathbb{P} \).

Our second main result is the following, cf. Thm. 5.9:

**Theorem 2.** One has a canonical isomorphism of \( D_p \)-modules
\[ \mathcal{L}oc(L^\dagger(w)) \simeq v_+ (\mathcal{O}_{Y_w}). \]

In particular, the modules \( L^\dagger(w) \) are geometrically \( F \)-holonomic for all \( w \in W \).

This result is in analogy with the classical result identifying the localization of the irreducible \( U(\mathfrak{g}) \)-module \( L^\dagger(w) \) with the intermediate extension of the constant connection on the complex Bruhat cell associated with \( w \), cf. [2, 10].

In the final section, we discuss somewhat detailed the example \( G = SL_2 \). In this case, \( P \) equals the projective line over \( \mathbb{O} \) and any irreducible \( D_p \)-module is holonomic. Moreover, theorem 1 gives a classification in terms of irreducible overconvergent \( F \)-isocrystals \( E \) on either a closed point \( y \) of \( \mathbb{P}_k^1 \) or an open complement \( Y \) of finitely many closed points \( Z = \{ y_1, ..., y_n \} \) of \( \mathbb{P}_k^1 \). In the first case, the point is a complete invariant. For example, the point \( y = \infty \) corresponds to the highest weight module \( L^\dagger(-2\rho) \). In the second case, the empty divisor \( Z = \emptyset \) corresponds to the trivial representation. For a non-empty \( Z \), we may suppose that all its points \( y_1, ..., y_n \) are \( k \)-rational with \( y_1 = \infty \). There are then two extreme cases
\[ Y = \mathbb{A}_k^1 \quad \text{and} \quad Y = \mathbb{P}_k^1 \setminus \mathbb{P}_k^1(k), \]
the affine line and Drinfeld’s upper half plane, respectively. We illustrate the two by means of two ”new” examples. In the case \( Y = \mathbb{A}_k^1 \) we assume that \( K \) contains the \( p \)-th roots of unity \( \mu_p \) and we choose an element \( \pi \in \mathfrak{o} \) with \( \text{ord}_p(\pi) = 1/(p-1) \). We let \( \mathcal{L}_\pi \) be the coherent \( D_p \)-module defined by the Dwork overconvergent \( F \)-isocrystal on \( Y \) associated with \( \pi \). On the other hand, we let \( n = K.e \) be the nilpotent radical of \( \text{Lie}(B) \), where \( e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) . Let \( \eta : n \to K \) be a nonzero character and consider Kostant’s standard Whittaker module
\[ W_{\theta_0,\eta} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}) \otimes U(n)} K_{\theta_0,\eta} \]
with character \( \eta \) and infinitesimal character \( \theta_0 \) [29 (3.6.1)]. It is an irreducible \( U(\mathfrak{g}) \)-module [29 Thm. 3.6.1], but not a highest weight module, i.e. it does not lie in \( \mathcal{O}_0 \). We write
\[ W^\dagger_{\theta_0,\eta} := D^\dagger(\mathcal{G}) \otimes_{U(\mathfrak{g})} W_{\theta_0,\eta} \]
for its crystalline counterpart. Our third main result is the following, cf. 5.10:

**Theorem 3.** Consider the character \( \eta \) defined by \( \eta(e) := \pi \). There is a canonical isomorphism
\[ \mathcal{L}oc(W^\dagger_{\theta_0,\eta}) \xrightarrow{\simeq} \mathcal{L}_\pi \]
as left $\mathcal{D}_F$-modules. In particular, $W_{\theta_0,\eta}^\dagger$ is geometrically $F$-holonomic.

The theorem shows, in particular, that the Dwork isocrystal $\mathcal{L}_\pi$ is algebraic in the sense that it comes from an algebraic $\mathcal{D}_{\mathbb{P}^1_K}$-module, namely $\text{Loc}(W_{\theta_0,\eta})$, by extension of scalars $\mathcal{D}_{\mathbb{P}^1_K} \to \mathcal{D}_F$. The holonomic $\mathcal{D}_{\mathbb{P}^1_K}$-module $\text{Loc}(W_{\theta_0,\eta})$, however, is not regular, but has an irregular singularity at infinity.

We discuss an example in the Drinfeld case, where $Y = \mathbb{P}^1_k \setminus \mathbb{P}^1(k)$. We identify $k = \mathbb{F}_q$. We assume that $K$ contains the cyclic group $\mu_{q+1}$ of $(q+1)$-th roots of unity. The space $Y$ admits a distinguished unramified Galois covering $u : Y' \to Y$ with Galois group $\mu_{q+1}$, given by the so-called Drinfeld curve

$$Y' = \left\{ (x, y) \in \mathbb{A}^2_k \mid xy^q - x^qy = 1 \right\}.$$

The latter admits a smooth and projective compactification $\overline{Y'}$. The covering map $u$ extends to a smooth and tamely ramified morphism $u : \overline{Y'} \to \mathbb{P}^1_k$ which maps the boundary bijectively to $Z = \mathbb{P}^1(k)$. We denote by $u : \overline{Y'} \to \overline{Y}$ the morphism of couples induced by $u$ in this situation and we let

$$\mathcal{E} = \mathbb{R}^* u_{\text{rig}},, \mathcal{O}_{\overline{Y}}$$

be the relative rigid cohomology sheaf. Using results of Grosse-Klönne [18], we show that $\mathcal{E}$ admits an isotypic decomposition into overconvergent $F$-isocrystals $\mathcal{E}(j)$ on $\overline{Y}$ of rank one. In particular, each pair $(Y, \mathcal{E}(j))$ corresponds in the classification of theorem 1 to an irreducible geometrically $F$-holonomic $D^!(\mathcal{G})_{\theta_0}$-module $H^0(\mathcal{P}, v_\dagger \mathcal{E}(j))$.

We do not know whether the modules $H^0(\mathcal{P}, v_\dagger \mathcal{E}(j))$ are algebraic, in the sense that they arise by base change from irreducible $U(\mathfrak{g})$-modules. If algebraic, to which class do they belong? We recall that irreducible $U(\mathfrak{g})$-modules fall into three classes: highest weight modules, Whittaker modules and a third class whose objects (with a fixed central character) are in bijective correspondence with similarity classes of irreducible elements of a certain localization of the first Weyl algebra [8]. We plan to come back to these questions in future work.

Notations and Conventions. In this article, $\mathfrak{o}$ denotes a complete discrete valuation ring of mixed characteristic $(0, p)$. We let $K$ be its fraction field and $k$ its residue field, which is assumed to be perfect. We suppose that there exists a lifting of the Frobenius of $k$ to $\mathfrak{o}$. We denote by $\varpi$ a uniformizer of $\mathfrak{o}$. All formal schemes $\mathfrak{X}$ over $\mathfrak{o}$ are assumed to be locally noetherian and such that $\varpi \mathcal{O}_X$ is an ideal of definition. Without further mentioning, all occurring modules will be left modules.

2. Overholonomic modules and intermediate extensions

For a smooth formal $\mathfrak{o}$-scheme $\mathfrak{X}$ we denote by $\mathcal{D}_\mathfrak{X}^\dagger$ the sheaf of arithmetic differential operators on $\mathfrak{X}$. We refer to [4] for the basic features of the category of $\mathcal{D}_\mathfrak{X}^\dagger$-modules.
2.1. Overholonomic modules. We introduce the framework of overholonomic complexes of arithmetic \( \mathcal{D} \)-modules with Frobenius structure, following Abe-Caro [1].

Recall that a frame \((Y, X, \mathcal{P})\) is the data consisting of a separated and smooth formal scheme \( \mathcal{P} \) over \( \mathfrak{a} \), a closed subvariety \( X \) of its special fibre \( \mathcal{P}_s \), and an open subscheme \( Y \) of \( X \). A morphism between two such frames is the data \( u = (b, a, f) \) consisting of morphisms \( b : Y' \to Y, a : X' \to X, f : \mathcal{P}' \to \mathcal{P} \) such that \( f \) induces \( b \) and \( a \). A l.p. frame \((Y, X, \mathcal{P}, \mathcal{Q})\) is the data of a proper and smooth formal scheme \( \mathcal{Q} \) over \( \mathfrak{a} \), an open formal subscheme \( \mathcal{P} \subset \mathcal{Q} \) such that \((Y, X, \mathcal{P})\) is a frame. A morphism of l.p frames is defined in analogy to a morphisms of frames. It is called complete if the morphism \( a : X' \to X \) is proper.

A couple \( \mathcal{Y} \) is the data \((Y, X)\) consisting of a \( k \)-variety \( X \) and an open subscheme \( Y \subset X \) such that there exists a l.p. frame of the form \((Y, X, \mathcal{P}, \mathcal{Q})\). A morphism of couples is the data \( u = (b, a) \) consisting of morphisms \( b : Y' \to Y, a : X' \to X \) such that \( b \) is induced by \( a \). It is called complete if \( a \) is proper. Let \( P \) be a property of morphisms of schemes. One says that \( u \) is \( c-P \) if \( u \) is complete and \( b \) satisfies \( P \). For all this, cf. [1, 1.1].

Denote by \( \mathcal{P} \) a smooth and proper formal scheme over \( \mathfrak{a} \).

We denote by \( \mathcal{D}^b_{ovhol}(\mathcal{P}_p) \) the triangulated category of complexes of overholonomic \( \mathcal{D}^b_{P} \)-modules introduced by Caro [12, 3.1]. Let \( Z \) be a closed subset of \( \mathcal{P}_s \), the special fibre of \( \mathcal{P} \). There are two functors \( R\Gamma^+_Z \) and \((^tZ) \) defined on \( \mathcal{D}^b_{ovhol}(\mathcal{P}_p) \) giving rise to a localization triangle

\[
R\Gamma^+_Z(\mathcal{E}) \to \mathcal{E} \to (^tZ)\mathcal{E} \quad \approx \quad \eta
\]

for \( \mathcal{E} \in \mathcal{D}^b_{ovhol}(\mathcal{P}_p) \), cf. [1, 1.1.5].

Let now \( \mathcal{Y} = (Y, X) \) be a couple such that \((Y, X, \mathcal{P}, \mathcal{Q})\) is a l.p. frame. By abuse of notation, we will sometimes denote the frame \((Y, X, \mathcal{P})\) (or even the l.p. frame \((Y, X, \mathcal{P}, \mathcal{Q})\)) by \( \mathcal{Y} \), too. This should not cause confusion.

The couple \( \mathcal{P} \) is obtained by taking \( Y = X = \mathcal{P}_s \).

Let \( Z = X \setminus Y \). For \( \mathcal{E} \in \mathcal{D}^b_{ovhol}(\mathcal{P}_p) \) one sets

\[
\mathbb{R}\Gamma^+_Z(\mathcal{E}) := \mathbb{R}\Gamma^+_X \circ (^tZ)(\mathcal{E}).
\]

The category \( \mathcal{D}^b_{ovhol}(\mathcal{Y}/K) \) of overholonomic complexes of arithmetic \( \mathcal{D} \)-modules on \( \mathcal{Y} \) is defined to be the full subcategory of \( \mathcal{D}^b_{ovhol}(\mathcal{P}_p) \) formed by objects \( \mathcal{E} \) such that there is an isomorphism \( \mathcal{E} \cong \mathbb{R}\Gamma^+_Z(\mathcal{E}) \) [1, 1.1.5]. Of course, \( \mathcal{D}^b_{ovhol}(\mathcal{P}/K) = \mathcal{D}^b_{ovhol}(\mathcal{P}_p) \).

Abe-Caro introduce a canonical \( t \)-structure on \( \mathcal{D}^b_{ovhol}(\mathcal{Y}/K) \) in the following way [1, 1.2]. Write \( \mathcal{U} = \mathcal{P}_s \setminus Z \) for the open complement of \( Z \) in \( \mathcal{P} \). Then \( \mathcal{D}^b_{ovhol}(\mathcal{Y}/K) \) is defined to be the strictly full subcategory of objects \( \mathcal{E} \in \mathcal{D}^b_{ovhol}(\mathcal{Y}/K) \) such that

\[
\mathcal{E}_{|\mathcal{U}} \in \mathcal{D}^{\geq 0}(\mathcal{D}_\mathcal{U}^1)
\]
(analogously for $\leq 0$). The truncation functors relative to the couple $\mathbb{Y}$ are defined to be

$$
\tau_{\geq 0}^\mathbb{Y} = (^! Z) \circ \tau_{\geq 0} \quad \text{resp.} \quad \tau_{\leq 0}^\mathbb{Y} = (^! Z) \circ \tau_{\leq 0},
$$

where $\tau_{\geq 0}$ resp. $\tau_{\leq 0}$ are the usual truncation functors. The functors $\tau_{\geq 0}^\mathbb{Y}$ and $\tau_{\leq 0}^\mathbb{Y}$ define a $t$-structure on $F\text{-}
\mathcal{D}_{\text{ovhol}}(\mathbb{Y}/K)$ whose heart is denoted by $F\text{-}\text{Ovhol}(\mathbb{Y}/K)$ \cite{1}. The latter is an abelian category which is noetherian and artinian \cite{1.2.13}. When $Y$ is smooth, the category $F\text{-}\text{Ovhol}(\mathbb{Y}/K)$ contains a full subcategory $F\text{-}\text{Isoc}_t(\mathbb{Y}/K)$ which is equivalent to the category of overconvergent $F$-isocrystals on $\mathbb{Y}$, the usual coefficients of rigid cohomology \cite{1.2.14}.

We recall an important key lemma.

**Lemma 2.1.** Let $\mathcal{U} = \mathcal{P}\backslash \mathcal{Z}$ and let $\alpha$ be a morphism in $D^b_{\text{ovhol}}(\mathbb{Y}/K)$. Then $\alpha$ is an isomorphism in $D^b_{\text{ovhol}}(\mathbb{Y}/K)$ if and only if $\alpha|_{\mathcal{U}}$ is an isomorphism in $D^b(D^b(\mathcal{U}))$.

**Proof.** This is \cite{1.1.2.3}. \hfill \Box

Main examples: (i) In the case where $\mathbb{Y} = \mathbb{P}$, the category $F\text{-}\text{Ovhol}(\mathbb{P}/K)$ is the usual category of overholonomic arithmetic $F\text{-}\mathcal{D}^b$-modules on $\mathcal{P}$.

(ii) If $Z$ is a divisor in $\mathcal{P}_s$ with open complement $Y = \mathcal{P}_s\backslash Z$ and $\mathbb{Y} = (Y, \mathcal{P}_s, \mathcal{P})$, then $F\text{-}\text{Ovhol}(\mathbb{Y}/K)$ is the usual category of overholonomic $F\text{-}\mathcal{D}^b(1^! Z)$-modules.

2.2. **Intermediate extensions.** We keep the notation of the previous subsection. We introduce the intermediate extension functor for arithmetic $\mathcal{D}$-modules following Abe-Caro \cite{1}.

Let

$$
u : \mathbb{Y} \longrightarrow \mathbb{Y}'$$

be a complete morphism of couples. There is a canonical homomorphism

$$
\theta_{\nu, \mathcal{E}} : \nu_! \mathcal{E} \longrightarrow \nu_+ \mathcal{E}
$$

for any complex $\mathcal{E} \in F\text{-}\mathcal{D}_{\text{ovhol}}(\mathbb{Y})$, cf. \cite{1.1.3.4}. The morphism is compatible with composition in the following sense: if $w = u_2 \circ u_1$, where $u_1$ and $u_2$ are $c$-complete morphisms of couples, then

$$
\begin{array}{c}
\xymatrix{ u_2 ! \circ u_1 ! & u_2 ! \circ u_1 + \mathcal{E} \ar[ll]^{w_2 ! (\theta_{u_1})} \ar[rr]_{\theta_{u_2 \circ u_1}} & & u_2 + \circ u_1 + \mathcal{E} }
\end{array}
$$

by \cite{1 Prop. 1.3.7}. We denote by an exponent $(-)^0 = \mathcal{H}_t^0$ the application of the first cohomology sheaf $\mathcal{H}_t^0 = \tau_{\leq 0}^{\mathbb{Y}} \tau_{\leq 0}$ relative to the $t$-structure on $F\text{-}\mathcal{D}_{\text{ovhol}}(\mathbb{Y}/K)$ (and similar for $\mathbb{Y}'$). If $\nu$ is a $c$-immersion, and if $\mathcal{E} \in F\text{-}\text{Ovhol}(\mathbb{Y})$, then the intermediate extension of $\mathcal{E}$ on $\mathbb{Y}'$ is defined to be

$$
\nu_+ (\mathcal{E}) := \text{im}(\theta_{\nu, \mathcal{E}}^0 : \nu^0_! \mathcal{E} \longrightarrow \nu^0_+ \mathcal{E}).
$$
Note that if \( w \) is a c-affine immersion, then \( u_+ \) and \( u_! \) are \( t \)-exact by [1, Remark 1.4.2], so that the definition simplifies to
\[
    u_+(\mathcal{E}) = \text{im}(\theta_{u_\mathcal{E}} : u_\mathcal{E} \rightarrow u_+\mathcal{E}).
\]

2.3. A classification result. We keep the notation of the previous subsections. In particular, \( \mathcal{P} \) still denotes a smooth and proper formal scheme over \( \mathfrak{a} \). Our goal here is to classify the irreducible overholonomic \( F-\mathcal{D}_\mathcal{P}^! \)-modules, up to isomorphism. This is in close analogy to the classical setting of algebraic \( D \)-modules on complex varieties, e.g. [23, 3.4].

We will only consider couples that arise from a smooth locally closed subvariety \( Y \subseteq \mathcal{P}_s \) by taking its Zariski closure \( X = \bar{Y} \) in \( \mathcal{P}_s \). Then \( (Y, X, \mathcal{P}) \) is a frame and \( (Y, X, \mathcal{P}, \mathcal{P}) \) is a l.p. frame and \( \bar{Y} = (Y, X) \) is a couple. By abuse of notation, we will sometimes denote the frame \( (Y, X, \mathcal{P}) \) (or even the l.p. frame \( (Y, X, \mathcal{P}, \mathcal{P}) \)) by \( \bar{Y} \) too. This should not cause confusion.

For such a couple \( \bar{Y} = (Y, X) \), we consider the corresponding c-locally closed immersion
\[
v : \bar{Y} \rightarrow \mathcal{P}.
\]

The associated intermediate extension functor
\[
v_+ : F-\text{Ovhol}(\bar{Y}/K) \rightarrow F-\text{Ovhol}(\mathcal{P}/K) = \{ \text{overholonomic } F-\mathcal{D}_\mathcal{P}^! \text{-modules} \}
\]
is given by
\[
v_+(\mathcal{E}) := \text{Im}(\theta_{v_\mathcal{E}} : v_0\mathcal{E} \rightarrow v_+\mathcal{E}).
\]

Suppose for a moment that \( Y \subseteq \mathcal{P}_s \) is closed and there exists a \( \mathfrak{a} \)-smooth closed formal subscheme \( \bar{Y} \subseteq \mathcal{P}_s \), defined by some coherent ideal sheaf in \( \mathcal{O}_\mathcal{P} \), which lifts the closed immersion \( Y \subseteq \mathcal{P}_s \). Then \( \text{Ovhol}(\bar{Y}) \) identifies with the category of overholonomic \( \mathcal{D}_\mathcal{Y}^! \)-modules and the functor \( v_+ \) coincides with the direct image functor appearing in Kashiwara’s equivalence [7, 25]. By the latter equivalence, the functor \( v_+ \) induces a bijection between the (isomorphism classes of) irreducible \( \mathcal{D}_\mathcal{Y}^! \)-modules and irreducible \( \mathcal{D}_\mathcal{P}^! \)-modules supported on \( \mathcal{Y} \).

The case of a closed immersion generalizes as follows.

**Lemma 2.2.** Let \( \mathcal{M} \) be an irreducible overholonomic \( F-\mathcal{D}_\mathcal{P}^! \)-module. There is an open dense smooth subscheme \( U \subseteq \mathcal{P}_s \) with the property: if \( u : U = (\bar{U}, \mathcal{P}_s, \mathcal{P}) \rightarrow \mathcal{P} \) denotes the corresponding c-open immersion, then \( u^!\mathcal{M} \) is an overconvergent \( F \)-isocrystal on \( \bar{U} \).

**Proof.** By [1, 1.4.9(i)], we know that there is an open dense smooth subscheme \( U \subseteq \mathcal{P}_s \) and an overconvergent \( F \)-isocrystal \( \mathcal{E} \) on \( \bar{U} \) such that \( \mathcal{M} = u_+\mathcal{E} \). By left \( t \)-exactness of \( u^! \) we obtain \( u^!\mathcal{M} \subseteq u^!u_+\mathcal{E} = \mathcal{E} \). Hence, \( u^!\mathcal{M} \) is an overconvergent \( F \)-isocrystal. \( \square \)

Since any overholonomic \( \mathcal{D}_\mathcal{P}^! \)-module \( \mathcal{M} \) is coherent [12, 3.1], we may view its support \( \text{Supp}(\mathcal{M}) \) as a closed (reduced) subvariety of \( \mathcal{P}_s \).
Proposition 2.3. Let \( \mathcal{M} \) be an irreducible overholonomic \( F-\mathcal{D}_P^! \)-module. There is an open dense smooth affine subscheme of an irreducible component of \( \text{Supp}(\mathcal{M}) \) with the property: if \( v : Y \to \mathbb{P} \) denotes the corresponding immersion, then \( \mathcal{E} := v^!\mathcal{M} \) is an irreducible overconvergent \( F \)-isocrystal on \( Y \). Moreover, \( v_+(\mathcal{E}) = \mathcal{M} \).

Proof. According to the preceding lemma, we may choose an open dense subscheme \( U \subset \mathcal{P}_s \) over which \( \mathcal{M} \) becomes an overconvergent isocrystal. We may choose an open dense smooth affine subscheme \( Y \) of an irreducible component of \( \text{Supp}(\mathcal{M}) \), which is contained in \( U \). Let \( \mathcal{E} := v^!\mathcal{M} \). If \( k \) denotes the c-closed immersion \( Y \to \mathcal{U} \), then Abe-Caro’s version of Kashiwara’s theorem \([\text{I} \ 1.3.2(iii)]\) together with \([\text{I} \ 1.4.9(ii)]\) imply that \( \mathcal{E} = k^!u^!\mathcal{M} \) is irreducible. Moreover, by adjointness \([\text{I} \ 1.1.10]\)
\[
\text{Hom}(v_!\mathcal{E}, \mathcal{M}) = \text{Hom}(\mathcal{E}, v^!\mathcal{M}) \neq 0
\]
and there is therefore a non-zero morphism \( v_!\mathcal{E} \to \mathcal{M} \). In other words, \( \mathcal{M} \) is a quotient of \( v_!\mathcal{E} \). But \( v_!\mathcal{E} = v_0^!\mathcal{E} \), since \( Y \) is affine, and \( v_+\mathcal{E} \) is the unique irreducible quotient of \( v_0^!\mathcal{E} \). We therefore must have \( v_+\mathcal{E} = \mathcal{M} \). \( \square \)

Consider now a pair \((Y, \mathcal{E})\) where \( Y \subset \mathcal{P}_s \) is a connected smooth locally closed subvariety and \( \mathcal{E} \) is an irreducible overconvergent \( F \)-isocrystal on \( Y = (Y, X) \). We write
\[
\mathcal{L}(Y, \mathcal{E}) := v_+(\mathcal{E}) \in F-\text{Ovhol}(\mathbb{P}).
\]

Proposition 2.4. The overholonomic \( F-\mathcal{D}_P^! \)-module \( \mathcal{L}(Y, \mathcal{E}) \) is irreducible, has support \( \overline{Y} \) and satisfies \( v^!\mathcal{L}(Y, \mathcal{E}) = \mathcal{E} \).

Proof. The irreducibility statement and the fact that \( 0 \neq v^!\mathcal{L}(Y, \mathcal{E}) \subset v^!v_+^!\mathcal{E} = \mathcal{E} \) follow from \([\text{I} \ 1.4.7(i)]\) and its proof. Since \( \mathcal{E} \) is irreducible, \( v^!\mathcal{L}(Y, \mathcal{E}) = \mathcal{E} \) as claimed. Finally, if \( k : Y \to \mathcal{U} \) is a c-closed immersion and \( u : \mathcal{U} \to \mathbb{P} \) a c-open immersion such that \( v = u \circ k \), then \( v_+ = u_+ \circ k_+ \). \([\text{I} \ 1.4.5(i)]\). The support of \( k_+\mathcal{E} = k_+\mathcal{E} \) equals \( Y \) and the support of \( \mathcal{L}(Y, \mathcal{E}) = u_+k_+\mathcal{E} \) equals \( \overline{Y} \). \( \square \)

Two pairs \((Y, \mathcal{E})\) and \((Y', \mathcal{E}')\) are said to be equivalent if \( \overline{Y} = \overline{Y}' \) and there is an open dense \( U \subset \overline{Y} \) contained in the intersection \( Y \cap Y' \) such that \( u^!\mathcal{E} \simeq u'^!\mathcal{E}' \). Here \( u \) denotes the c-open immersion \( U = (U, \overline{Y}, \mathcal{P}) \to \overline{Y} \) and similarly for \( u' \). This defines an equivalence relation on the set of couples.

Theorem 2.5. The correspondence \((Y, \mathcal{E}) \mapsto \mathcal{L}(Y, \mathcal{E})\) induces a bijection
\[
\{\text{equivalence classes of pairs } (Y, \mathcal{E})\} \overset{\simeq}{\to} \{\text{irreducible overholonomic } F-\mathcal{D}_P^!\text{-modules}\}.\]

Proof. Let us show that the map in question is well-defined. Let \((Y, \mathcal{E})\) and \((Y', \mathcal{E}')\) be two equivalent couples. Choose an open dense \( U \subset \overline{Y} \) contained in the intersection \( Y \cap Y' \) such that \( u^!\mathcal{E} \simeq u'^!\mathcal{E}' \). Note that \( v \circ u = v' \circ u' \). Define \( \mathcal{F} = (v \circ u)^!\mathcal{L}(Y, \mathcal{E}) \) and similarly for \( \mathcal{L}(Y', \mathcal{E}') \). Then \( (v \circ u)_+^!\mathcal{F} = \mathcal{L}(Y, \mathcal{E}) \) according to \([2.3]\) and \( \mathcal{F} = u^!u'^!\mathcal{L}(Y, \mathcal{E}) = u'^!\mathcal{E} \) according to \([2.4]\) Hence, \( \mathcal{F} \simeq \mathcal{F}' \) and we obtain \( \mathcal{L}(Y, \mathcal{E}) \simeq \mathcal{L}(Y', \mathcal{E}') \).

Let us next show that the map is injective. So suppose that \( \mathcal{L}(Y, \mathcal{E}) \simeq \mathcal{L}(Y', \mathcal{E}') \) for two couples \((Y, \mathcal{E})\) and \((Y', \mathcal{E}')\). Then \([2.4]\) implies \( \overline{Y} = \overline{Y}' \) and moreover, if \( U \subset \overline{Y} \) is open dense
and contained in the intersection $Y \cap Y'$, then $(v \circ u)^\dagger \mathcal{L}(Y, \mathcal{E}) = u^\dagger \mathcal{E}$. Since $v \circ u = v' \circ u'$, we obtain $u^\dagger \mathcal{E} \simeq u'^\dagger \mathcal{E}'$ as desired. This proves the injectivity. The surjectivity of the map is a direct consequence of [2.3].

Let $Y \subset \mathcal{P}_s$ be a smooth locally closed subvariety and $\mathcal{Y} = (Y, X)$.

**Definition 2.6.** Let $d := \dim(\mathcal{P}_s) - \dim(Y)$. We define the constant overholonomic module on the frame $\mathcal{Y}$ to be

$$\mathcal{O}_\mathcal{Y} = R\Gamma(\mathcal{O}_{\mathcal{P}_s}, \mathcal{Y})[d].$$

**Proposition 2.7.** Suppose that $Y$ is connected and there exists a smooth formal scheme $\mathfrak{Y}$ over $\mathfrak{o}$, so that the immersion $Y \to \mathcal{P}$ lifts to some morphism of formal schemes $\mathfrak{Y} \to \mathcal{P}$. The module $\mathcal{O}_\mathfrak{Y}$ lies in $F\text{-Isoc}^\dagger(\mathcal{Y}/K)$. If the rigid-analytic generic fiber $\mathcal{Y}_K$ is connected, then $\mathcal{O}_\mathfrak{Y}$ is an irreducible object in the category $\text{Ovhol}(\mathfrak{Y})$.

**Proof.** Denote $Z = X \setminus Y$ and $\mathcal{U} = \mathcal{P} \setminus Z$. We have the closed immersion of smooth formal schemes $v : \mathfrak{Y} \hookrightarrow \mathcal{U}$. Then, by [3] Proposition 1.4, we see that

$$\mathcal{O}_{\mathfrak{YU}} = R\Gamma(\mathcal{O}_{\mathcal{U}, \mathcal{Y}})[d] \simeq v_+v^!\mathcal{O}_{\mathcal{U}, \mathcal{Y}}[d] = v_+\mathcal{O}_\mathfrak{Y}.$$

This coincides with $\text{sp}_+\mathcal{O}_\mathfrak{Y}$ and hence lies in the category $F\text{-Isoc}^\dagger(\mathcal{Y}, \mathcal{U}, K)$, in the notation of [1.2.14]. This shows $\mathcal{O}_\mathfrak{Y} \in F\text{-Isoc}^\dagger(\mathcal{Y}/K)$.

The irreducibility statement is based on the following

**Lemma 2.8.** Let $\mathcal{Q}$ be a connected smooth formal scheme over $\mathfrak{o}$ and $\mathcal{Q}_K$ its generic fiber (as rigid analytic space). Assume furthermore that $\mathcal{Q}_K$ is connected.

(i) The constant isocrystal $\mathcal{O}_{\mathcal{Q}_K}$ is irreducible in the category of convergent isocrystals.

(ii) The coherent $\mathcal{D}^+_{\mathcal{Q}}$-module $\mathcal{O}_{\mathcal{Q}, \mathcal{Q}}$ is irreducible in the category of $\mathcal{D}^+_{\mathcal{Q}}$-modules.

**Proof.** We begin by (i). Let $E$ be a subobject of $\mathcal{O}_{\mathcal{Q}_K}$ in the abelian category of convergent isocrystals over $\mathcal{Q}_K$, and $E' = \mathcal{O}_{\mathcal{Q}_K}/E$ be the quotient. As convergent isocrystals over $\mathcal{Q}_K$, $E$ and $E'$ are locally free $\mathcal{O}_{\mathcal{Q}_{K}}$-modules so that there exists an admissible cover by affinoids $\mathcal{U}_i$ ($i \in I$) such that $E_{|\mathcal{U}_i}$ and $E'_{|\mathcal{U}_i}$ are free $\mathcal{O}_{\mathcal{U}_i}$-modules for each $i$. Fix $i_0$ and denote by $A = \Gamma(\mathcal{U}_{i_0}, \mathcal{O}_{\mathcal{U}_{i_0}})$. Since $\mathcal{U}_{i_0}$ is affinoid, we have an exact sequence of free $A$-modules

$$0 \to \Gamma(\mathcal{U}_{i_0}, E) \to A \to \Gamma(\mathcal{U}_{i_0}, E') \to 0.$$

Take $x$ a point of $\mathcal{U}_{i_0}$, and $K(x)$ its residue field, then the previous exact sequence remains exact after tensoring by $K(x)$, meaning that $\Gamma(\mathcal{U}_{i_0}, E)$ is either equal to 0 or to $A$. Assume for example that this is equal to 0, so that $E_{|\mathcal{U}_{i_0}} = 0$ by Tate’s acyclicity theorem. By Zorn’s lemma there is a maximal subset $J \subset I$ such that $E_{|\mathcal{U}_i} = 0$ for each $i \in J$. Assume that $J \neq I$ then $J' = I \setminus J$ is not empty. By connectedness, the union $\bigcup_{i \in J} \mathcal{U}_i$ intersects the union $\bigcup_{i \in J'} \mathcal{U}_i$, thus there exist $i \in J'$, $i \in J$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$. Since $E_{|\mathcal{U}_i}$ is either equal to 0 or to $\mathcal{O}_{\mathcal{U}_i}$, we see that it is zero by restricting to $\mathcal{U}_i \cap \mathcal{U}_j$, which contradicts the fact that $J \neq I$. This proves (i).
For (ii) we use then that the abelian category of convergent isocrystals over the generic fiber $Q_K$ of the formal scheme $Q$ is equivalent to the category of coherent $D^+_Q$-modules, that are coherent $O_{Q,Q}$-modules ([4, 4.1.4]). The functors $sp_*$ and $sp^*$ realize this equivalence of categories. Let $\mathcal{E}$ be a non-zero coherent $D^+_Q$-submodule of $O_{Q,Q}$, then $E = sp^*\mathcal{E}$ is a convergent isocrystal, that is a subobject of $O_{Q,K}$. By (i), it is either 0 or equal to the constant convergent isocrystal $O_{Q_K}$. Thus $\mathcal{E}$ is either 0 or $O_{Q,Q}$ and this proves (ii). \hfill $\square$

Let us come back to the proof of the proposition. Let $\alpha : \mathcal{E} \rightarrow O_Y$ be an injective morphism in the category $Ovhol(Y)$. As remarked in the beginning of the proof, $O_{Y,U} = R\Gamma_Y(O_{U,Q})[d] \simeq v_+v'O_{U,Q}[d] = v_+O_Y$.

By Kashiwara’s theorem for the closed immersion $v : Y \rightarrow U$ [7, 25] and the previous lemma, $v_+O_Y$ is irreducible in the category of coherent $D^+_U$-modules with support in $Y$, so that $\mathcal{E}_{|U}$ is either 0 or equal to $v_+O_Y$. Using 2.1 we conclude that $\mathcal{E}$ is either 0 or equal to $O_Y$. \hfill $\square$

Example: If $Z$ is a divisor in $\mathcal{P}_s$, $U = \mathcal{P}\setminus Z$, $Y = (U_s, \mathcal{P}_s, \mathcal{P})$ then $F = Ovhol(Y)$ is the usual category of overholonomic $F - D^{+}_P(1Z)$-modules. In this case, if $\mathcal{U}$ and its generic fiber $U_K$ are connected, then the constant overholonomic module $O_Y = O_{\mathcal{P},Q}(1Z)$ is an irreducible $D^{+}_P(1Z)$-module by the previous proposition applied to $Y = U_s$, the special fiber of $\mathcal{U}$.

Proposition 2.9. The intermediate extension $v_+(O_Y)$ is an irreducible overholonomic $F-D^{+}_P$-module.

Proof. This follows from the theorem 2.5 and the above proposition. \hfill $\square$

3. Some Compatibility Results Between Generic and Special Fibre

We keep the notations introduced in the preceding section. In this section, we place ourselves into certain integral situations involving schemes over $\mathfrak{o}$ and establish various compatibilities between the classical intermediate extensions on generic fibres and Abe-Caro intermediate extensions arising after reduction on the special fibre. We will focus in particular on the cases of open immersions and proper morphisms.

The results of this section are then applied in the final section 5 in the case of the flag variety, in order to compare intermediate extensions over the Bruhat cells, both in the generic and in the special fibre, cf. prop. 5.8 and thm. 5.9

3.1. Notations. Let us begin with some notations: if $X$ denotes a $\mathfrak{o}$-scheme, then $X_s$ will be its special fiber, $X_Q$ its generic fiber, $X_i = X \times \text{Spec} \mathfrak{o}/\mathfrak{m}^{i+1}, \mathfrak{X}$ (or $\mathfrak{X}$) the associated formal scheme obtained after $p$-adic completion, and $\mathfrak{X}$ is the frame $\mathfrak{X} = (X_s, X_s, \mathfrak{X})$. Moreover, if $f : X \rightarrow Y$ is a morphism of $\mathfrak{o}$-schemes, then $f_s$ (resp. $f_Q$, $\hat{f}$, $f_i$) will denote the induced morphism $X_s \rightarrow Y_s$ (resp. $X_Q \rightarrow Y_Q$, $\mathfrak{X} \rightarrow \mathfrak{Y}$, $X_i \rightarrow Y_i$), and $F = (f_s, f_s, \hat{f})$ will denote the morphism of frames between the frames $\mathfrak{X}$ and $\mathfrak{Y}$.
3.2. **Open immersions.** Let $P$ be a smooth scheme over $\mathfrak{o}$. The closed immersions $P_i \hookrightarrow \mathcal{P}$ give rise to a canonical morphism of ringed spaces

$$\alpha : \mathcal{P} = \lim_{\longrightarrow} P_i \rightarrow P.$$ 

It comes with the diagram

$$\mathcal{P} \xrightarrow{\alpha} P \xrightarrow{j} P_Q,$$

which will be our basic underlying structure in the following. We have the following first result.

**Lemma 3.1.**

(i) There is a canonical isomorphism

$$\mathcal{O}_{P,Q} \simeq j_* \mathcal{O}_{P_Q}.$$ 

(ii) There is a canonical isomorphism

$$\mathcal{D}^{(m)}_{P,Q} \simeq j_* \mathcal{D}_{P_Q}$$ 

for any $m$.

**Proof.** We consider the canonical morphism of quasi-coherent $\mathcal{O}_P$-sheaves $\mathcal{O}_P \rightarrow j_* \mathcal{O}_{P_Q}$. After tensoring with $\mathbb{Q}$, we get a morphism $\mathcal{O}_{P,Q} \rightarrow j_* \mathcal{O}_{P_Q}$. If $P = \text{Spec} \ A$ is affine, then this morphism is the identity of $A_Q = \Gamma(P, \mathcal{O}_{P,Q}) = \Gamma(P, j_* \mathcal{O}_{P_Q})$. This proves (i). For (ii), we start with the canonical morphism

$$\mathcal{D}^{(m)}_P \rightarrow j_* \mathcal{D}_{P_Q} \simeq j_* \mathcal{D}^{(m)}_{P_Q}.$$ 

It induces a morphism $\mathcal{D}^{(m)}_{P,Q} \rightarrow j_* \mathcal{D}_{P_Q}$. In order to prove that this morphism is an isomorphism, we may assume that $P$ is affine with local coordinates $x_1, \ldots, x_M$. In this case, both sheaves are free $\mathcal{O}_{P,Q}$-modules with basis $\mathcal{E}^b$ and we conclude using (i). $\square$

For a quasi-coherent $\mathcal{O}_{P_Q}$-module $\mathcal{E}$, we set

$$\overline{\mathcal{E}} := \alpha^{-1} j_* \mathcal{E}.$$ 

**Lemma 3.2.** The formation $\mathcal{E} \mapsto \overline{\mathcal{E}}$ is an exact functor from the category of quasi-coherent $\mathcal{O}_{P_Q}$-modules to the category of $\overline{\mathcal{O}}_{P_Q}$-modules.

**Proof.** This statement comes from the fact that the functor $j_*$ is exact on quasi-coherent $\mathcal{O}_{P_Q}$-sheaves, since $j$ is affine, as well as $\alpha^{-1}$. The functor $\mathcal{E} \mapsto \overline{\mathcal{E}}$ is thus exact as the composition of two exact functors. $\square$

According to the lemma, the functor $\mathcal{E} \mapsto \overline{\mathcal{E}}$ passes directly to derived categories and gives a functor

$$D^b_{\text{qcoh}}(\mathcal{O}_{P_Q}) \longrightarrow D^b(\overline{\mathcal{O}}_{P_Q}).$$

We now consider the sheaf of rings $\overline{\mathcal{D}}_{P_Q}$ on $\mathcal{P}$.

**Lemma 3.3.** There is an injective flat morphism of sheaves of rings

$$\overline{\mathcal{D}}_{P_Q} \hookrightarrow \overline{\mathcal{D}}_P.$$
Proof. If \( U \subset P \) is an open affine of \( P \) with local coordinates \( x_1, \ldots, x_M \), then the following description
\[
\Gamma(U, \alpha_*(\mathcal{D}_P^t)) = \left\{ \sum_{\nu} a_{\nu} x_\nu^{[\nu]} | a_{\nu} \in \mathcal{O}_U \otimes \mathbb{Q} | \exists c > 0, \eta < 1, ||a_{\nu}|| \leq c\eta^{[\nu]} \right\}
\]
and
\[
\Gamma(U, \mathcal{D}_{P,Q}) = \left\{ \sum_{\nu, \text{finite}} a_{\nu} x_\nu^{[\nu]} | a_{\nu} \in \mathcal{O}_U \otimes \mathbb{Q} \right\}.
\]
This gives the inclusion. Next, the ring \( \mathcal{D}_P^t \) is flat over \( \hat{\mathcal{D}}_{P,Q}^{(0)} \) [4, Cor. 3.5.4]. Moreover the sheaf \( \hat{\mathcal{D}}_{P,Q}^{(0)} \) is flat over \( \mathcal{D}_P^t \), by completion, so that \( \mathcal{D}_P^t \) is indeed flat over \( \mathcal{D}_{P,Q} \). \( \square \)

The proof of the following lemma is easy and left to the reader.

Lemma 3.4. \((i) \) Let \( E \) be a coherent \( \mathcal{D}_{P,Q} \)-module, then \( \mathcal{D}_P^t \otimes_{\mathcal{D}_{P,Q}} E \) is a coherent \( \mathcal{D}_P^t \)-module. 
\((ii) \) Let \( E \in D^b_{coh}(\mathcal{D}_{P,Q}) \), then \( \mathcal{D}_P^t \otimes_{\mathcal{D}_{P,Q}} E \in D^b_{coh}(\mathcal{D}_P^t) \).

The following proposition is due to Virrion and shows that duality commutes with scalar extension. Her formulation involves perfect complexes, but since the scheme \( P_Q \) is smooth, the sheaf \( \mathcal{D}_{P,Q} \) has finite cohomological dimension and the category \( D^b_{coh}(\mathcal{D}_{P,Q}) \) coincides with the category of perfect complexes of \( \mathcal{D}_{P,Q} \)-modules.

Proposition 3.5. Let \( E \in D^b_{coh}(\mathcal{D}_{P,Q}) \), then
\[
\mathcal{D}_P^t \otimes_{\mathcal{D}_{P,Q}} \mathcal{D}_{P,Q}(E) \simeq \mathcal{D}_{P,Q}(\mathcal{D}_P^t \otimes_{\mathcal{D}_{P,Q}} E).
\]

Proof. This is [38, 1.4,4.4]. \( \square \)

Let us recall that, if \( d = \dim(P_Q) \),
\[
\mathcal{D}_{P,Q}(E) = R\text{Hom}_{\mathcal{D}_{P,Q}}(E, \mathcal{D}_{P,Q}[d]) \otimes_{\mathcal{D}_{P,Q}} \omega_{P_Q}.
\]

Definition 3.6. Let \( P \) be a smooth \( \mathfrak{o} \)-scheme and \( Z \subset P \) a divisor. We say that \( Z \) is a transversal divisor if \( Z_s \) and \( Z_Q \) are divisors respectively of \( P_s \) and \( P_Q \).

Let \( Z \) be a transversal divisor and let \( j : P\setminus Z \hookrightarrow P \) be the inclusion of its open complement. For any coherent \( \mathcal{D}_{P,Q} \)-module \( E \), we put
\[
(*Z_Q)E := \mathcal{D}_{P,Q}(*Z_Q) \otimes_{\mathcal{D}_{P,Q}} E
\]
so that \( (*Z_Q)E = j_Q j_Q^! E \). In the same spirit, we define for any coherent \( \mathcal{D}_P^t \)-module \( E \),
\[
(^tZ_s)E := \mathcal{D}_P^t(^tZ_s) \otimes_{\mathcal{D}_P^t} E.
\]
Let $Y = P \setminus Z$ with immersion $j : Y \hookrightarrow P$ and consider the morphism of frames

$$J : \mathcal{Y} := (Y_s, P_s, \mathcal{P}) \to \mathcal{P} := (P_s, P_s, \mathcal{P}).$$

Then $(\uparrow Z_s)\mathcal{E} = J_+ J^* \mathcal{E}$. Note that, in this situation, $J_+$ is just the forgetful functor from the category $\text{Ovh}(\mathcal{Y}/K)$ to the category $\text{Ovh}(\mathcal{P}/K)$. Moreover the functor $j_{Q+}$ is exact since $\mathcal{Z}_Q$ is a divisor of $P_Q$, and induces an equivalence of categories between coherent $\mathcal{D}_{P_Q}(\uparrow \mathcal{Z}_Q)$-modules and coherent $\mathcal{D}_{Y_Q}$-modules. Of course, at the level of sheaves of $\mathcal{O}_{Y_Q}$-modules, we have $j_{Q+} = j_{Q*}$. Recall also that, in this situation, objects of $\text{Ovh}(\mathcal{Y}/K)$ consist of degree zero complexes of $\mathcal{D}_{P}(\uparrow \mathcal{Z}_s)$-modules by [1] Remark 1.2.7 (iii).

**Proposition 3.7.** Let $\mathcal{E} \in D^b_{\text{hol}}(\mathcal{Y}_Q)$ and suppose that $\mathcal{F} := \mathcal{D}_P \otimes_{\mathcal{O}_{P_Q}} j_{Q+} \mathcal{E} \in D^b_{\text{ovhol}}(\mathcal{Y})$. Then there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{j_{Q+}} & J_{Q+} \mathcal{D}_{Y_Q} \circ \mathcal{D}_{Y_Q}(\mathcal{E}) \\
\downarrow{1 \otimes id_{j_{Q+}, \mathcal{E}}} & & \downarrow{1} \\
J_{+}(\mathcal{D}_P \otimes_{\mathcal{O}_{P_Q}} j_{Q+} \mathcal{E}) & \xrightarrow{J_* C} & J_{+} \mathcal{D}_{Y} \circ \mathcal{D}_{Y}(\mathcal{D}_P \otimes_{\mathcal{O}_{P_Q}} j_{Q+} \mathcal{E}).
\end{array}
$$

Here, $c_Q$ is the canonical isomorphism $\mathcal{E} \simeq \mathcal{D}_{Y_Q} \circ \mathcal{D}_{Y_Q}(\mathcal{E})$ and $C$ is the canonical isomorphism $\mathcal{F} \simeq \mathcal{D}_{Y} \circ \mathcal{D}_{Y}(\mathcal{F})$.

**Proof.** Let us first remark that the sheaf $\mathcal{D}_P(\uparrow \mathcal{Z}_s)$ is flat over $\mathcal{D}_{P_Q}$ and thus flat over $\mathcal{D}_{P}$.

This explains that no derived tensor product appears in the stated diagram. Moreover it will be enough to prove the statement for a single holonomic $\mathcal{D}_{Y_Q}$-module $\mathcal{E}$ such that $\mathcal{F} := \mathcal{D}_P \otimes_{\mathcal{O}_{P_Q}} j_{Q+} \mathcal{E}$ is an overholonomic module over $\mathcal{Y}$. In this case, all complexes are single modules in degree zero. The top horizontal arrow of the diagram is induced by the following map:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\text{Hom}_{\mathcal{D}_{Y_Q}}(\mathcal{H}om_{\mathcal{D}_{Y_Q}}(\mathcal{E}, \mathcal{D}_{Y_Q}))} & \mathcal{D}_{Y_Q} \circ \mathcal{D}_{Y_Q}(\mathcal{E}) \\
x & \xrightarrow{ev_x(\varphi) = \varphi(x)} &.
\end{array}
$$

Recall that $C : \mathcal{F} \to \mathcal{D}_{Y} \mathcal{D}_{Y}\mathcal{F}$ is defined as follows in our case: as $\mathcal{F}$ is overholonomic over $\mathcal{Y}$, we have the identifications

$$
\mathcal{F} \simeq (\uparrow \mathcal{Z}_s)\mathcal{F} = \mathcal{D}_P(\uparrow \mathcal{Z}_s) \otimes_{\mathcal{D}_P} \mathcal{F}.
$$

We therefore deduce, using the base change result [38 1.4.4.4], that

$$
\mathcal{D}_{Y}(\mathcal{F}) = \mathcal{D}_P(\uparrow \mathcal{Z}_s) \otimes_{\mathcal{D}_P} \mathcal{D}_{P}(\mathcal{F}) = R\mathcal{H}om_{\mathcal{D}_P(\uparrow \mathcal{Z}_s)}(\mathcal{F}, \mathcal{D}_P(\uparrow \mathcal{Z}_s)[d]) \otimes_{\mathcal{O}_{P}} \omega_{P},
$$

$$
= R\mathcal{H}om_{\mathcal{D}_P(\uparrow \mathcal{Z}_s)}(\mathcal{F}, \mathcal{D}_P(\uparrow \mathcal{Z}_s)[d]) \otimes_{\mathcal{O}_{P}(\uparrow \mathcal{Z}_s)} \omega_{P}(\uparrow \mathcal{Z}_s),
$$
and the canonical map $C$ is then given by the following composition

\[ \mathcal{F} \xrightarrow{\text{Hom}_{D_p(Z_s)}(\mathcal{F}, D_p(Z_s))} \mathbb{D}_Y \circ \mathbb{D}_Y(\mathcal{F}). \]

Note that $C$ is an isomorphism since it is an isomorphism when restricted to $P \backslash Z$ by [4, 4.3.10]. Moreover one has a canonical isomorphism

\[ \mathcal{F} \cong D_p(Z_s) \otimes_{\mathcal{O}_P} D_p(Z_s) \overset{\text{can}}{\longrightarrow} \mathbb{D}_Y(\mathcal{F}). \]

so that we can use again [38, 1.4, 4.4], to get the following identification

\[ \mathbb{D}_Y(\mathcal{F}) \cong D_p(Z_s) \otimes_{\mathcal{O}_P} D_p(Z_s) \overset{\text{can}}{\longrightarrow} \mathbb{D}_Y(\mathcal{F}). \]

Here, we identify

\[ j_{Q+} \mathcal{E} = RHom_{D_p(Z_s)}(j_{Q+} \mathcal{E}, D_p(Z_s)[d]) \otimes_{\mathcal{O}_P} \omega_{P_q}(Z). \]

Using again [38, 1.4, 4.4] applied to the sheaves $\mathcal{D}_{P_q}(Z)$ and $D_p(Z_s)$, we find a canonical isomorphism

\[ \mathbb{D}_Y(\mathcal{F}) \cong D_p(Z_s) \otimes_{\mathcal{O}_P} D_p(Z_s) \overset{\text{can}}{\longrightarrow} \mathbb{D}_Y(\mathcal{F}). \]

that allows us to write the diagram of the statement in the following way

\[ \begin{array}{ccc}
J_{Q+} \mathcal{E} & \xrightarrow{\text{can}} & j_{Q+} \mathcal{E} \\
|^{1 \otimes id} & & |^{1 \otimes id} \\
J_+ (D_p(Z_s) \otimes_{\mathcal{O}_P} D_p(Z_s)) & \xrightarrow{J_{Q+} \mathcal{E}} & D_p(Z_s) \otimes_{\mathcal{O}_P} D_p(Z_s) \overset{\text{can}}{\longrightarrow} \mathbb{D}_Y(\mathcal{F}).
\end{array} \]

The commutativity of this diagram comes then from the fact that if $x$ is a local section of $J_{Q+} \mathcal{E}$, we have $(1 \otimes id)(ev_x) = ev(1 \otimes x)$. \qed

Under the same hypothesis as in the previous proposition ($Z$ is a transversal divisor of $P$) and with the same notations, we have the

**Corollary 3.8.** Let $\mathcal{E} \in D_{\text{hol}}^b(Y_Q)$ and suppose that $\mathcal{F} = D_p \otimes_{\mathcal{O}_P} \mathcal{E} \in D_{\text{hol}}^b(Y)$. There is a commutative diagram

\[ \begin{array}{ccc}
\mathbb{D}_Y(\mathcal{F}) & \xrightarrow{\text{can}} & \mathbb{D}_Y(\mathcal{F}) \\
|^{1 \otimes id} & & |^{1 \otimes id} \\
\mathbb{D}_Y(\mathcal{F}) & \xrightarrow{\text{can}} & \mathbb{D}_Y(\mathcal{F}).
\end{array} \]
Proof. We have the following equality as functors on $D^b_{\text{hol}}(Y_Q)$
\[
j_Q^! j_Q^! = j_Q^! \mathbb{D}_{P,Q} j_Q^! \mathbb{D}_{Y_Q} \\
= \mathbb{D}_{Y_Q} j_Q^! j_Q^! \mathbb{D}_{Y_Q} \\
= \mathbb{D}_{Y_Q} \mathbb{D}_{Y_Q} \simeq \text{id}.
\]
On the other hand, let us notice that $J^! = \mathcal{D}_P^! (\dual Z_s) \otimes_{\mathcal{O}_P} (-)$ is a scalar extension, so that again by [38, 1.4.4], $J^! \mathbb{D}_P = \mathbb{D}_Y J^!$. Moreover for $\mathcal{F} \in D^b_{\text{whol}}(Y)$ one has
\[
J^! J^* \mathcal{F} = \mathcal{D}_P^! (\dual Z_s) \otimes_{\mathcal{O}_P} \mathcal{F} \simeq \mathcal{F},
\]
essentially by definition of the objects of $D^b_{\text{whol}}(Y)$. Using these remarks, we compute
\[
J^! J^! = J^! \mathbb{D}_P J^* \mathbb{D}_Y \\
= \mathbb{D}_Y J^! J^* \mathbb{D}_Y \\
= \mathbb{D}_Y \mathbb{D}_Y \simeq \text{id},
\]
so that the diagram of the corollary is the same as the diagram of the previous proposition [3.7]. □

We next give another compatibility statement.

**Proposition 3.9.** Let $\mathcal{E} \in D^b_{\text{hol}}(P_Q)$ and suppose that $\mathcal{F} = \mathcal{D}_P^! \otimes_{\mathcal{O}_{P,Q}} j_Q^! \mathcal{E} \in D^b_{\text{whol}}(P)$. Let $\text{can} : \mathcal{E} \to j_Q^! j_Q^! \mathcal{E}$ and $\text{CAN} : \mathcal{F} \to J^! J^* \mathcal{F}$ be the canonical morphisms. Then the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\text{can}} & j_Q^! j_Q^! \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{D}_P^! \otimes_{\mathcal{O}_{P,Q}} \mathcal{E} & \xrightarrow{\text{CAN}} & J^* (\mathcal{D}_P^! \otimes_{\mathcal{O}_{P,Q}} \mathcal{E}).
\end{array}
\]

**Proof.** Writing out explicitly all functors, we see that the above diagram comes down to the commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\text{can}} & \mathcal{D}_{P,Q} (\dual Z_Q) \otimes_{\mathcal{O}_{P,Q}} \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{D}_P^! \otimes_{\mathcal{O}_{P,Q}} \mathcal{E} & \xrightarrow{\text{CAN}} & \mathcal{D}_P^! (\dual Z_s) \otimes_{\mathcal{O}_{P,Q}} \mathcal{E}.
\end{array}
\]

□

We recall the following result of Berthelot [3, 4.3.2]. Recall that a relative normal crossing divisor is transversal.
Proposition 3.10. Let \( Z \subset P \) be a relative normal crossing divisor. Then one has
\[
\mathcal{O}_{P,Q}(\uparrow Z_s) \simeq \mathcal{D}_P^! \otimes_{\mathcal{D}_Q P} j_{Q*} \mathcal{O}_{Y_Q}.
\]
Note that the sheaf \( j_{Q*} \mathcal{O}_{Y_Q} \) is equal to \( \mathcal{O}_{P,Q}(\ast Z_Q) \) and the isomorphism is given by the canonical inclusion of sheaves of rings \( \mathcal{O}_{P,Q}(\ast Z_Q) \hookrightarrow \mathcal{O}_{P,Q}(\uparrow Z_s) \), sending 1 to 1. This allows us to identify \( \mathcal{O}_{P,Q}(\uparrow Z_s) \) with \( \mathcal{D}_P^! \otimes_{\mathcal{D}_Q P} j_{Q*} \mathcal{O}_{Y_Q} \).

Proposition 3.11. Let \( Z \subset P \) be a relative normal crossing divisor.

(i) \( \mathcal{D}_Q \mathcal{O}_{Y_Q} = \mathcal{O}_{Y_Q} \), \( \mathcal{D}_Y \mathcal{O}_{Y} = \mathcal{O}_{Y} \).

(ii) there is a canonical isomorphism \( j_! \mathcal{O}_{Y} \simeq \mathcal{D}_P^! \otimes_{\mathcal{D}_Q P} j_{Q*} \mathcal{O}_{Y_Q} \).

Proof. The fact that \( \mathcal{D}_Q \mathcal{O}_{Y_Q} = \mathcal{O}_{Y_Q} \) is classical and comes from the fact that on the smooth scheme \( Y_Q \), the \( \mathcal{D}_Q \)-module \( \mathcal{O}_{Y_Q} \) admits a resolution by the Spencer complex, and that this latter complex is auto-dual. To see the second statement, we use the fact proved in [31, Lemme 4.2.1] that \( \mathcal{O}_{Y} = \mathcal{O}_{P,Q}(\uparrow Z_s) \) also admits a resolution by a Spencer complex (with \( d = \dim Y_Q \))
\[
0 \to \mathcal{D}_P^!(\uparrow Z_s) \otimes_{\mathcal{D}_P} \Lambda^d \mathcal{T}_P \to \cdots \to \mathcal{D}_P^!(\uparrow Z_s) \otimes_{\mathcal{O}_P} \Lambda^1 \mathcal{T}_P \to \mathcal{D}_P^!(\uparrow Z_s)
\]
that is auto-dual for the functor \( \mathcal{D}_Y = R\mathcal{H}om_{\mathcal{D}_P^!(\uparrow Z_s)}(\cdot, \mathcal{D}_P^!(\uparrow Z_s)[d]) \otimes_{\mathcal{O}_P} \omega_P \). This proves (i). Then we compute
\[
\begin{align*}
\mathcal{D}_P^!(\uparrow Z_s) & \otimes_{\mathcal{D}_P} \Lambda^d \mathcal{T}_P \\
& \simeq \mathcal{D}_P^!(\uparrow Z_s) \\
& \simeq j_{Q*} \mathcal{O}_{Y_Q} \\
& \simeq j_{Q*} \mathcal{O}_{Y_Q} \\
& \simeq \mathcal{D}_P^!(\uparrow Z_s) \otimes_{\mathcal{D}_Q P} j_{Q*} \mathcal{O}_{Y_Q}
\end{align*}
\]
which proves (ii).

3.3. Proper morphisms. Before giving compatibility results for direct images relative to proper morphisms we establish the following auxiliary lemmas.

Lemma 3.12. Let \( f : P \to Q \) be a morphism of smooth \( \mathcal{O} \)-schemes and \( \mathcal{F} \in D_{qcoho}^+(\mathcal{O}_P) \) with \( \mathcal{F} \in D^+(\mathcal{O}_P) \). There is a natural map \( Rf_* \mathcal{F} \to R\hat{f}_* \mathcal{F} \).

Proof. We have the following diagram
\[
\begin{array}{ccc}
P & \xrightarrow{j} & P_Q \\
\downarrow f & & \downarrow j_Q \\
Q & \xrightarrow{j} & Q_Q,
\end{array}
\]

\[
\begin{array}{ccc}
P & \xrightarrow{j} & P_Q \\
\downarrow f & & \downarrow j_Q \\
Q & \xrightarrow{j} & Q_Q,
\end{array}
\]
Lemma 3.13. Let \( f : P \to Q \) be a morphism of smooth \( \mathfrak{o} \)-schemes and \( \mathcal{E} \in D_{\mathrm{coh}}^b(\mathcal{O}_P) \). Then there is a canonical morphism in \( D^+(\mathcal{O}_Q) \)

\[
\mathcal{D}_Q^1 \otimes_{\mathcal{O}_Q} f_{Q+}(\mathcal{E}) \to \hat{f}_+ \left( \mathcal{D}_P^1 \otimes_{\mathcal{O}_P} \mathcal{E} \right).
\]

Proof. It is enough to prove that there is a map \( \overline{Rf_{Q*}(\mathcal{F})} \to \hat{f}_+ \left( \mathcal{D}_P^1 \otimes_{\mathcal{O}_P} \mathcal{E} \right) \).

Let us introduce the transfer sheaves \( \mathcal{D}_{Q-r} = \omega_{P/Q} \otimes_{\mathcal{O}_P} f_{r*} \mathcal{D}_Q \), and

\[
\Gamma_{P-Q}^{(m)} = \lim_{\longrightarrow} f_{i*} \mathcal{D}_{Q-i}, \quad \Gamma_{Q-P}^{(m)} = \omega_{P/Q} \otimes_{\mathcal{O}_P} \Gamma_{P-Q}^{(m)}, \quad \Gamma_{Q-P}^1 = \lim_{\longrightarrow} \Gamma_{Q-P}^{(m)}.
\]

Recall that

\[
f_{Q+}(\mathcal{E}) = Rf_{Q*} \left( \mathcal{D}_{Q-r} \otimes_{\mathcal{O}_P} \mathcal{E} \right), \quad \hat{f}_+ \left( \mathcal{D}_P^1 \otimes_{\mathcal{O}_P} \mathcal{E} \right) = R\hat{f}_* \left( \Gamma_{Q-P}^1 \otimes_{\mathcal{O}_P} \mathcal{E} \right).
\]

Note that we have

\[
\overline{f_{Q+}(\mathcal{E})} \simeq \alpha^{-1} f^{-1}(\mathcal{D}_Q \otimes \mathcal{O}_Q) = \hat{f}^{-1} \alpha^{-1}(\mathcal{D}_Q \otimes \mathcal{O}_Q),
\]

and for all \( m \) we have maps \( \alpha^{-1}(\mathcal{D}_Q) \to \Gamma_{P-Q}^{(m)} \). This gives maps \( \hat{f}^*(\alpha^{-1}(\mathcal{D}_Q)) \to \Gamma_{P-Q}^{(m)} \), that give rise to maps of transfer sheaves \( \Gamma_{Q-P} \to \Gamma_{Q-P}^1 \).

Take \( \mathcal{E} \in D_{\mathrm{coh}}^b(\mathcal{O}_P) \), then, observing that \( \mathcal{D}_{Q-r} \) is a quasi-coherent \( \mathcal{O}_P \)-module, we see that

\[
\mathcal{D}_{Q-r} \otimes_{\mathcal{O}_P} \mathcal{E} \in D_{\mathrm{coh}}^b(\mathcal{O}_P),
\]

so that we can apply 3.12 to this complex of sheaves and we finally get the map of the statement by the following composition

\[
Rf_{Q*} \left( \mathcal{D}_{Q-r} \otimes_{\mathcal{O}_P} \mathcal{E} \right) \to R\hat{f}_* \left( \Gamma_{Q-P} \otimes_{\mathcal{O}_P} \mathcal{E} \right) \to R\hat{f}_* \left( \Gamma_{Q-P}^1 \otimes_{\mathcal{O}_P} \mathcal{E} \right).
\]
Assume from now on and for the rest of this subsection that \( f : P \to Q \) is a proper map of smooth \( \mathfrak{a} \)-schemes. Both sheaves \( \mathcal{D}_P \) and \( \mathcal{D}_P^l \) have finite cohomological dimension [6 4.4.8], as well as \( RF_* \) since \( f \) is proper. Let \( * \in \{-, b\} \) and \( E \in D^*_\text{coh}(\mathcal{D}_P) \), then \( f_{Q+}(E) \) (resp. \( f_{Q+}(\mathcal{D}_P \otimes_{\mathcal{D}_P} \mathcal{E}) \)) are objects of \( D^*_\text{coh}(\mathcal{D}_{Q_0}) \), (resp. \( D^*_\text{coh}(\mathcal{D}^l_{Q_0}) \)), and thanks to the lemma, there is a map

\[
\mathcal{D}_P^l \otimes_{\mathcal{D}_{Q_0}} f_{Q+}(\mathcal{E}) \to f_{Q+}(\mathcal{D}_P \otimes_{\mathcal{D}_P} \mathcal{E}).
\]

Our goal (see 3.17) is to prove that this map is an isomorphism when \( P \) and \( Q \) are projective \( \mathfrak{a} \)-schemes. As usual, we will factorize \( f \) as a closed immersion and a projection. We first deal with the case of a closed immersion.

Let \( i : P \hookrightarrow Q \) be a closed immersion of smooth \( \mathfrak{a} \)-schemes, defined by a sheaf of ideals \( I \subset \mathcal{O}_Q \). We then have the following compatibility result for closed immersions.

**Proposition 3.14.** Let \( * \in \{-, b\} \) and let \( E \in D^*_\text{coh}(\mathcal{D}_P) \). Then there is a canonical isomorphism in \( D^*_\text{coh}(\mathcal{D}_Q) \)

\[
\mathcal{D}_Q^l \otimes_{\mathcal{D}_{Q_0}} i_{Q+}(\mathcal{E}) \cong i_{Q+}(\mathcal{D}_P \otimes_{\mathcal{D}_P} \mathcal{E}).
\]

**Proof.** It is well known that \( i_{Q+} \) sends \( D^*_\text{coh}(\mathcal{D}_P) \) to \( D^*_\text{coh}(\mathcal{D}_{Q_0}) \), resp. \( i_{Q+} \) sends \( D^*_\text{coh}(\mathcal{D}^l_P) \) to \( D^*_\text{coh}(\mathcal{D}^l_{Q_0}) \). Finally both functors send \( D^*_\text{coh}(\mathcal{D}_P) \) to \( D^*_\text{coh}(\mathcal{D}^l_P) \). The map from the left-hand side to the right-hand side is the one given in the previous lemma 3.13. Since \( i \) is a closed immersion, \( i \) is affine, it has finite cohomological dimension and both functors are way out left in the sense of [22 1.7]. Proving that the map is an isomorphism is a local question on \( Q \), so that we can assume that \( P \) and \( Q \) are affine. In this case any coherent \( \mathcal{D}_P \)-module is a quotient of a finite free \( \mathcal{D}_P \)-module, and using a standard dévissage argument for way out left functors [22 1.7, (iv)] we are reduced to prove the statement in the case where \( \mathcal{E} = \mathcal{D}_P \). In this case, we have the following formulas

\[
i_{Q+}(\mathcal{D}_P) = i_*(\mathcal{D}_{Q_0} \hookrightarrow P_0), \quad i_{Q+}(\mathcal{D}^l_P) = i_*(\mathcal{D}^l_{Q_0} \hookrightarrow P_0).
\]

As \( i \) is a quasi-compact morphism, \( R\hat{i}_* = \hat{i}_* \) commutes with inductive limits so that

\[
\hat{i}_*(\mathcal{D}^l_{Q \hookrightarrow P}) = \lim_m \hat{i}_*(\mathcal{D}^{(m)}_{Q_0 \hookrightarrow P_0}^l).
\]

Let us fix an integer \( m \). We have to show that

\[
(3.3.14) \quad \hat{i}_*(\mathcal{D}^{(m)}_{Q_0 \hookrightarrow P_0}) \cong \hat{\mathcal{D}}^{(m)}_{Q_0 \hookrightarrow P_0} \otimes_{\mathcal{D}_{Q_0}} i_*(\mathcal{D}_{Q_0 \hookrightarrow P_0}).
\]

We first compute the left hand side of this formula. By [7 Thm.3.5.3], we know that \( \hat{i}_*(\mathcal{D}^{(m)}_P) \) is a coherent \( \hat{\mathcal{D}}^{(m)}_Q \)-module, and as \( i \) is affine, we have by [19 prop. 13.2.3]

\[
(3.3.15) \quad \hat{i}_*(\mathcal{D}^{(m)}_P) \cong \lim_i \hat{i}_*(\mathcal{D}^{(m)}_{Q_i \hookrightarrow P_i}).
\]

We now come to the right hand side of the formula 3.3.14. Let us first recall the following
Lemma 3.16. The sheaf $i_+ (\mathcal{D}_P)$ is a coherent $\mathcal{D}_Q$-module.

Proof. We have
\[ i_+ \mathcal{D}_P = i_* \left( i^* \omega_Q^{-1} \otimes \mathcal{O}_Q \otimes \mathcal{D}_Q \mathcal{O}_P \omega_P \right) \]
\[ \cong \omega_Q^{-1} \otimes \mathcal{O}_Q \mathcal{D}_Q \mathcal{O}_P \omega_P \] by the projection formula,

the left $\mathcal{D}_Q$-module structure being given by the one of $\omega_Q^{-1} \otimes \mathcal{O}_Q \mathcal{D}_Q$, that is by the right structure on $\mathcal{D}_Q$ twisted on the left, which makes this left $\mathcal{D}_Q$-module a coherent module. □

Consider now the following $\mathcal{D}_Q$-module, that is coherent by the previous lemma,
\[ \mathcal{M} = \mathcal{D}_Q \otimes \alpha^{-1} \mathcal{D}_Q \alpha^{-1} i_+ \mathcal{D}_P. \]

As $\mathcal{M}$ is coherent, by [4, 3.2.4], we have
\[ \mathcal{M} \cong \lim_i \mathcal{D}_Q \otimes \mathcal{D}_Q \alpha^{-1} i_+ \mathcal{D}_P. \]

As $\mathcal{M}_Q$ coincides with $\mathcal{D}_Q \otimes \mathcal{D}_Q \alpha^{-1} i_+ (\mathcal{D}_P)$, this module is isomorphic with the right-hand side of 3.3.14. We finally obtain our result by comparing with the left-hand side of 3.3.14 which is computed using 3.3.15 and gives exactly the same thing. □

As before, let $* \in \{b, -, \}$. 

**Proposition 3.17.** Let $P$, $Q$ be smooth and projective $\mathfrak{a}$-schemes, let $f : P \to Q$ be a proper morphism with formal completion $\hat{f} : \mathcal{P} \to \mathcal{Q}$. For any $\mathcal{E} \in \mathcal{D}^*_{\text{coh}}(\mathcal{D}_P)$, there is a natural isomorphism in $\mathcal{D}^*_{\text{coh}}(\mathcal{D}_Q)$$\mathcal{D}_Q \otimes \mathcal{D}_Q \hat{f}_Q^+ (\mathcal{E}) \cong \hat{f}_+ (\mathcal{D}_P \otimes \mathcal{D}_Q \mathcal{E})$.

Proof. We already noticed that both functors send objects of $\mathcal{D}^*_{\text{coh}}(\mathcal{D}_P)$ to objects of $\mathcal{D}^*_{\text{coh}}(\mathcal{D}_Q)$, as $f$ has finite cohomological dimension. Moreover both functors are way out left in the sense of [22, I.7]. The map from left-hand side to the right-hand side was defined in 3.13. Using [36, Tag 0C4Q], we see that the morphism $f$ is projective. Then, using the previous compatibility result 3.14 for closed immersions, it is enough to prove the statement when $P$ is a relative projective space over $Q$, say $P = \mathbb{P}_Q^M$ and $f : \mathbb{P}_Q^M \to Q$ is the canonical map. Since the question is local on $Q$, we can (and we do) assume that $Q$ is affine and smooth with coordinates $t_1, \ldots, t_s$. Let $\mathcal{E}$ be a coherent $\mathcal{D}_{P,Q}$-module. As $P_Q$ is a noetherian space, $\mathcal{E}$ is an inductive limit of its sub $\mathcal{O}_{P_Q}$-coherent sheaves, so that there is a $\mathcal{O}_{P_Q}$-coherent sheaf $\mathcal{E}'$ and a surjection of $\mathcal{D}_{P_Q}$-modules $\mathcal{D}_{P_Q} \otimes \mathcal{O}_{P_Q} \mathcal{E}' \to \mathcal{E}$, where the $\mathcal{D}_{P_Q}$-module structure on the left hand side is given by the one of $\mathcal{D}_{P_Q}$. By Serre’s theorem, for
some $a, r \in \mathbb{N}$, there is a surjection of coherent $\mathcal{O}_{P_Q}$-modules $\mathcal{O}_{P_Q}(-a)^r \to \mathcal{E}'$, and we see that there is a surjection of coherent $\mathcal{D}_{P_Q}$-modules $\mathcal{D}_{P_Q}(-a)^r \to \mathcal{E}$. Iterating this process, we see that each coherent $\mathcal{D}_{P_Q}$-module has some resolution by $\mathcal{D}_{P_Q}$-modules of the type $\mathcal{D}_{P_Q}(-a)^r$. Finally using again the dévissage argument for way out left functors of $\mathcal{D}_{P_Q}$, we are reduced to prove the proposition for a projective morphism $f : P = \mathbb{P}^M_Q \to Q$, with $Q$ affine, endowed with coordinates, and $E = \mathcal{D}_{P_Q}(-a)$, with $a \in \mathbb{N}$. Let us assume this from now on. Since the functor $Rf_* \text{ commutes with inductive limits, because } \mathbb{P}^M_Q$ and $Q$ are quasi-compact, it is also enough to prove that, for all $m$, we have

(3.3.17) $\hat{\mathcal{G}}_Q^{(m)} \otimes_{\mathcal{D}_{Q}} f_+(\mathcal{D}_{P_Q}(-a)) \cong f_+(\hat{\mathcal{G}}_{P,Q}^{(m)} \otimes_{\mathcal{D}_{P_Q}} \mathcal{D}_{P_Q}(-a))$.

The following lemma therefore completes the proof of the proposition. \hfill $\square$

**Lemma 3.18.** Assertion [3.3.17] is true for any $m$.

**Proof.** Let $\mathcal{F} = \mathcal{D}_{P}^{(m)}(-a)$, we have

\[
\begin{align*}
f_+(\mathcal{F}) &= Rf_* \left( \mathcal{D}_{Q}^{(m)} \otimes_{\mathcal{D}_{P}} \mathcal{D}_{P}^{(m)}(-a) \right) \\
&= Rf_* \left( f^* \mathcal{D}_{Q}^{(m)} \otimes_{\mathcal{O}_P} \omega_{P/Q}(-a) \right),
\end{align*}
\]

where the left $\mathcal{D}_{Q}^{(m)}$-module structure is given by the left structure of $\mathcal{D}_{Q}^{(m)} \otimes_{\mathcal{O}_Q} \omega_{Q}^{-1}$, obtained by twisting the right structure of $\mathcal{D}_{Q}^{(m)}$. As $\omega_{Q}$ is free of rank 1,

$$\omega_{P} \simeq \omega_{P} \otimes_{\mathcal{O}_P} f^* \omega_{Q}^{-1} \simeq \omega_{P/Q} \simeq \mathcal{O}_P(-M - 1),$$

where $M := \dim P_Q - \dim Q_Q$. We refer for example to [21, III, thm. 5.1] for the computation of $Rf_*(\mathcal{O}_P(-M - 1))$ over any affine base $Q$, which is a complex of finite free $\mathcal{O}_Q$-modules. More precisely, denote

$$d = \max\{\text{rank}(H^0(P, \mathcal{O}_P(-a - M - 1))), \text{rank}(H^M(P, \mathcal{O}_P(-a - M - 1)))\}.$$

There are several cases:

(i) If $a < -M - 1$, then $f_+(\mathcal{F}) \simeq \mathcal{D}_{Q}^{(m)} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q$ is concentrated in degree 0,

(ii) if $a \geq 0$, then $f_+(\mathcal{F}) \simeq \mathcal{D}_{Q}^{(m)} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q[-M]$ is concentrated in degree $M$,

(iii) if $-M \leq a < -1$, then $f_+(\mathcal{F})=0$.

Note also that we have the following isomorphism of (twisted) left $\hat{\mathcal{G}}_{Q,Q}^{(m)}$-modules

$$\hat{\mathcal{G}}_{Q,Q}^{(m)} \otimes_{\mathcal{D}_{Q}} \mathcal{D}^{(m)}_Q \otimes_{\mathcal{D}_{Q}} \omega_{Q}^{-1} \simeq \hat{\mathcal{G}}_{Q,Q}^{(m)} \otimes_{\mathcal{O}_Q} \omega_{Q}^{-1}.$$

We will first compute the left-hand side of [3.3.17]. Let us denote

$$A := \hat{\mathcal{G}}_{Q}^{(m)} \otimes_{\alpha^{-1} \hat{\mathcal{G}}_{Q}^{(m)}} \alpha^{-1} f_+(\mathcal{F}) \in D_{\text{coh}}(\hat{\mathcal{G}}_{Q}^{(m)})$$

$$= (\hat{\mathcal{G}}_{Q}^{(m)} \otimes_{\mathcal{O}_Q} \omega_{Q}^{-1}) \otimes_{\alpha^{-1} \mathcal{O}_Q} \alpha^{-1} Rf_*(\omega_{P}(-a)).$$
This is a complex concentrated in at most one degree, where it is isomorphic to a direct sum of \(d\) copies of \(\mathcal{D}_Q^{(m)}\). In particular, by [7], 3.2.1, it satisfies \(A \simeq R\lim_i (\mathcal{D}_Q^{(m)} \otimes L \mathcal{D}_Q^{(m)} A)\).

Since \(A\) is a complex of finite free \(\mathcal{D}_Q^{(m)}\)-modules, we have that
\[
\mathcal{D}_Q^{(m)} \otimes \mathcal{D}_Q^{(m)} A = \mathcal{D}_Q^{(m)} \otimes \mathcal{D}_Q^{(m)} A \simeq \mathcal{D}_Q^{(m)} \otimes \mathcal{O}_{Q_i} \omega_{Q_i}^{-1} \otimes \mathcal{O}_{Q} A^{-1} Rf_*(\omega_P(-a)),
\]
is a complex either in degree \(M\) or 0, where it is isomorphic to a direct sum of \(d\) copies of \(\mathcal{D}_Q^{(m)}\). Finally we have
\[
A \simeq R\lim_i \left( \mathcal{D}_Q^{(m)} \otimes \mathcal{O}_{Q_i} \omega_{Q_i}^{-1} \otimes \mathcal{O}_{Q} A^{-1} Rf_*(\omega_P(-a)) \right).
\]

To compute the right-hand side of 3.3.17, we introduce \(B = \hat{f}_+ (\mathcal{D}_P^{(m)}(-a))\), so that we have
\[
B \simeq R\hat{f}_* \left( \hat{f}^* (\mathcal{D}_Q^{(m)} \otimes \mathcal{O}_Q \omega_{Q}^{-1}) \otimes \mathcal{O}_P \omega_P(-a) \right)
\]
\[
\simeq R\hat{f}_* R\lim_i \left( \hat{f}_i^* (\mathcal{D}_Q^{(m)} \otimes \mathcal{O}_{Q_i} \omega_{Q_i}^{-1}) \otimes \mathcal{O}_{P_i} \omega_{P_i}(-a) \right)
\]
\[
\simeq R\lim_i Rf_{i*} \left( f_i^* (\mathcal{D}_Q^{(m)} \otimes \mathcal{O}_{Q_i} \omega_{Q_i}^{-1}) \otimes \mathcal{O}_{P_i} \omega_{P_i}(-a) \right) \quad \text{by } [36] \text{ Tag 0BKP}
\]
\[
\simeq R\lim_i \left( \mathcal{D}_Q^{(m)} \otimes \mathcal{O}_{Q_i} \omega_{Q_i}^{-1} \otimes \mathcal{O}_{Q} Rf_{i*} \omega_{P_i}(-a) \right).
\]

Again, by using the computation of [21] Thm. III.5.1, we see that
\[
Rf_{i*} \omega_{P_i}(-a) \simeq Rf_{i*} \mathcal{O}_{P_i}(-a - M - 1)
\]
is a complex concentrated in only one degree and
\[
Rf_{i*} \omega_{P_i}(-a) \simeq \mathcal{O}_{Q_i} \otimes \mathcal{O}_{Q} Rf_{i*} \omega_P(-a).
\]
This finally shows that \(B\) is isomorphic to \(A\). This implies the lemma. \(\square\)

### 3.4. Compatibility for intermediate extensions of constant coefficients

We now come to the main application of our previous compatibility results. For this we place ourselves in the following axiomatic situation (S):

(i) \(Y\) is an affine and smooth scheme over \(\mathfrak{a}\).

(ii) there is an immersion \(v : Y \hookrightarrow P\) into a smooth projective scheme \(P\) over \(\mathfrak{a}\). Let \(X := \overline{Y}\) be the Zariski closure of \(Y\) in \(P\) and \(Z := X \backslash Y\).

(iii) There is a smooth and projective \(\mathfrak{a}\)-scheme \(X'\), a surjective morphism
\[
b : X' \rightarrow X
\]
inducing an isomorphism \( Y' := b^{-1}Y \simeq Y \), such that \( Z' = X' \cap Y' \) is a transversal divisor as defined in [3.6] with normal crossings. We have the open immersion \( j' : Y \simeq b^{-1}Y \hookrightarrow X' \).

As usual, \( \mathfrak{X}, \mathfrak{Y}, \ldots \) denote the formal schemes obtained from these schemes by \( p \)-adic completion, and \( X_s, Y_s, \ldots \) denote their special fiber. For simplicity, we also write \( v \) for the morphism of frames

\[
v : Y = (Y_s, X_s, \mathcal{P}) \longrightarrow (P_s, P_s, \mathcal{P}) = \mathbb{P}
\]

induced by the immersion \( v : Y \hookrightarrow P \). Let us introduce the composite morphism

\[
g : X' \overset{b}{\longrightarrow} X \hookrightarrow P.
\]

By \( p \)-adic completion we obtain a morphism \( \hat{g} : \mathfrak{X}' \to \mathfrak{P} \), and a morphism of frames

\[
u = (Id_{Y_s}, b_s, \hat{g}) : Y' = (Y_s, X'_s, \mathfrak{X}') \longrightarrow Y = (Y_s, X_s, \mathcal{P}).
\]

Denoting \( G = (g_s, g_s, \hat{g}) \) and \( J' = (j'_s, \text{id}_{X'_s}, \text{id}_{\mathfrak{X}'}) \), we then have the basic commutative diagram of frames:

\[
\begin{array}{ccc}
Y' = (Y_s, X'_s, \mathfrak{X}') & \xrightarrow{J'} & (X'_s, X'_s, \mathfrak{X}') = \mathfrak{X}' \\
\downarrow \quad u & & \downarrow G \\
Y = (Y_s, X_s, \mathcal{P}) & \xrightarrow{v} & (P_s, P_s, \mathcal{P}) = \mathbb{P}.
\end{array}
\]

The frame morphism \( u \) is \( c \)-affine, and the first morphism of this frame is equal to the identity, so that by [1.2.8], we know that \( u_t \) and \( u_+ \) are \( t \)-exact, and that we have \( u_t = u_+ \), as functors of abelian categories \( F\text{-Ovhол}(\mathbb{Y}') \to F\text{-Ovhол}(\mathbb{Y}) \), and that these functors are in fact equal to \( \mathcal{H}_0^0 u_+ = \mathcal{H}_0^0 u_t \). Let

\[
Q := v \circ u = G \circ J',
\]

which is a \( c \)-affine immersion, (in particular \( Y_s \hookrightarrow P_s \) is an immersion). Note that we have

\[
J'_{+}\mathcal{O}_{\mathfrak{Y}'} = \mathcal{O}_{\mathfrak{X}', \mathcal{Q}}(\mathfrak{Z}'_s),
\]

and that in our case \( J'_{+} \) is the forgetful functor \( F\text{-Ovhол}(\mathbb{Y}') \to F\text{-Ovhол}(\mathfrak{X}') \). Let \( v_\mathcal{Q} \) be the immersion \( Y_\mathcal{Q} \hookrightarrow P_\mathcal{Q} \). We now fix once and for all the following notations:

\[
\begin{align*}
\theta_{v_\mathcal{Q}} & := \theta_{v_\mathcal{Q}, \mathcal{O}_{\mathcal{Y}_\mathcal{Q}}} : v_\mathcal{Q} \mathcal{O}_{\mathcal{Y}_\mathcal{Q}} \to v_{\mathcal{Q}+} \mathcal{O}_{\mathcal{Y}_\mathcal{Q}}, & \text{resp.} & \theta_{J'_{+}} := \theta_{J'_{+}, \mathcal{O}_{\mathcal{Y}_\mathcal{Q}}} \\
\theta_\mathcal{Q} := \theta^0_{\mathcal{Q}, \mathcal{O}_{\mathfrak{Y}'}} & = \theta_{\mathcal{Q}, \mathcal{O}_{\mathfrak{Y}'}} : Q_+ \mathcal{O}_{\mathfrak{Y}'} \to Q_+ \mathcal{O}_{\mathfrak{Y}'}, & \text{resp.} & \theta_{\mathfrak{Y}'} := \theta_{\mathfrak{Y}'}, \mathcal{O}_{\mathfrak{Y}'}.
\end{align*}
\]

We also need the two morphisms

\[
\theta_{v_\mathcal{Q}}^{alg} := \overline{\theta}_{v_\mathcal{Q}}, \quad \text{resp.} \quad \theta_{J'_{+}}^{alg} := \overline{\theta}_{J'_{+}}.
\]

Our goal is to describe the relation between the classical intermediate extension \( v_{\mathcal{Q}+} \mathcal{O}_{\mathcal{Y}_\mathcal{Q}} \) on the generic fibre and the Abe-Caro intermediate extension \( v_{\mathcal{Q}+} \mathcal{O}_{\mathfrak{Y}} \) on the special fibre.
We start with the following lemma.

**Lemma 3.19.** We have the following commutative diagram in $F$-Ovhol($X'$)

$$
\begin{array}{ccc}
\mathcal{D}_{X'}^+ \otimes_{\mathcal{X}'_{\mathbb{Q}^c}} j'_{Q!} \mathcal{O}_{Y_0} & \longrightarrow & \mathcal{D}_{X'}^+ \otimes_{\mathcal{X}'_{\mathbb{Q}^c}} j'_{Q+} \mathcal{O}_{Y_0} \\
\downarrow \cong & & \downarrow \cong \\
J'_i \mathcal{O}_{Y'} & \longrightarrow & J'_i \mathcal{O}_{Y'}
\end{array}
$$

where all maps are canonical and where the upper horizontal arrow equals $\mathcal{D}_{X'}^+ \otimes \theta'^{alg}_{j'}$.

**Proof.** The diagram of the statement can be completed by the following diagram

$$
\begin{array}{ccc}
\mathcal{D}_{X'}^+ \otimes_{\mathcal{X}'_{\mathbb{Q}^c}} j'_{Q!} \mathcal{O}_{Y_0} & \longrightarrow & \mathcal{D}_{X'}^+ \otimes_{\mathcal{X}'_{\mathbb{Q}^c}} j'_{Q+} j'_{Q!} \mathcal{O}_{Y_0} \\
\downarrow \cong & & \downarrow \cong \\
C & \longrightarrow & J'_i \mathcal{O}_{Y'}
\end{array}
$$

Let us prove that both squares of this diagram are commutative. The isomorphism (3) is given by Berthelot’s result 3.10. The right square of this diagram is commutative by 3.8 and the horizontal maps of this square are isomorphisms, so that (2) is an isomorphism as well. The left square of this diagram is commutative by 3.9 applied to $j'_{Q!} \mathcal{O}_{Y_0}$ and 3.11. Moreover (ii) of 3.11 tells us that (1) is an isomorphism. We conclude that the external square is commutative with vertical arrows being isomorphisms. □

Recall that $Q = G \circ J' = v \circ u$, $v_Q = g_Q \circ j'_Q$, $\theta'^{alg}_{j'} = \overline{\theta}_{v_Q} \mathcal{O}_{Y_0}$, and $\theta_Q = \theta_{Q!, \mathcal{O}_{Y'}}$.

**Corollary 3.20.** There is a commutative diagram (with canonical vertical maps) in $F$-Ovhol($P$)

$$
\begin{array}{ccc}
\mathcal{D}_P^+ \otimes_{\mathcal{X}'_{P_{\mathbb{Q}^c}}} v_Q! \mathcal{O}_{Y_0} & \longrightarrow & \mathcal{D}_P^+ \otimes_{\mathcal{X}'_{P_{\mathbb{Q}^c}}} v_Q+ \mathcal{O}_{Y_0} \\
\downarrow \cong & & \downarrow \cong \\
Q_! \mathcal{O}_{Y'} & \longrightarrow & Q_+ \mathcal{O}_{Y'}
\end{array}
$$

where the upper horizontal arrow equals the map $\mathcal{D}_P^+ \otimes \theta^{alg}_{v}$.

**Proof.** As $G$ is c-proper, $G_+ = G_1$, and using 2.2.1 we can see that $\theta_Q = G_+ \circ \theta_{J'}$. Similarly, we have the equality $v_Q = g_Q \circ j'_Q$, and as $g_Q$ is proper, $\theta_{v_Q} = g_{Q+} \circ \theta_{j'_Q}$. We finally use the compatibility for projective morphisms 3.17, and we observe that we obtain the diagram of the corollary after applying $\hat{g}_+$ to the previous diagram 3.19. □
Remark: We have the identifications \( \theta_{1_{\mathcal{Y}}} (\mathcal{O}_{\mathcal{Y}_q}) \simeq \mathcal{O}_{\mathcal{X}_q} \) and \( \theta_{1_{\mathcal{X}'}} \mathcal{O}_{\mathcal{X}',\mathcal{Q}}(1\mathcal{Z}'_s) \simeq \mathcal{O}_{\mathcal{X}',\mathcal{Q}} \), and that

\[
\mathcal{D}_\mathcal{X}' \otimes_{\mathcal{Y}_q} \mathcal{O}_{\mathcal{X}'_q} \simeq \mathcal{O}_{\mathcal{X}',\mathcal{Q}}.
\]

We have the constant overholonomic modules on \( \mathcal{Y} \) resp. \( \mathcal{Y}' \)

\[
\mathcal{O}_\mathcal{Y} = R\Gamma_\mathcal{Y}(\mathcal{O}_{\mathcal{P},\mathcal{Q}})[d] \quad \text{resp.} \quad \mathcal{O}_\mathcal{Y}' = R\Gamma_\mathcal{Y}'(\mathcal{O}_{\mathcal{X}',\mathcal{Q}}) = \mathcal{O}_{\mathcal{X}',\mathcal{Q}}(1\mathcal{Z}'_s)
\]
as defined in \( \text{2.6} \) where \( d = \dim \mathcal{P}_s - \dim \mathcal{Y}_s \).

**Lemma 3.21.** There are canonical isomorphisms

1. \( u^! \mathcal{O}_\mathcal{Y} \simeq \mathcal{O}_\mathcal{Y}' \)
2. \( u_+ \mathcal{O}_\mathcal{Y} \simeq \mathcal{O}_\mathcal{Y} \)

**Proof.** By \( \text{11.1.2.8} \), \( u^! \) and \( u_+ \) are exact functors of the categories \( \text{Ovhol} \mathcal{Y} \) and \( \text{Ovhol} \mathcal{Y}' \), and quasi-inverse, so that (ii) is a direct consequence of (i). Recall also that, by \( \text{11.2.6.1,2.2.8,2.2.14} \), \( R\Gamma_\mathcal{Y} \circ R\Gamma_\mathcal{Y}' = R\Gamma_\mathcal{Y}' \). We compute

\[
u^! \mathcal{O}_\mathcal{Y} = R\Gamma_\mathcal{Y} \circ \hat{g}^!(\mathcal{R}_\mathcal{X}_s(1\mathcal{Z}_s)(\mathcal{O}_{\mathcal{P},\mathcal{Q}})[d]) \\
\simeq R\Gamma_\mathcal{Y} \circ R\Gamma_\mathcal{X}_s \circ (1\mathcal{Z}_s)\hat{g}^!(\mathcal{O}_{\mathcal{P},\mathcal{Q}})[d] \quad \text{[11 Théorème 2.2.18]} \\
\simeq R\Gamma_\mathcal{Y} \circ R\Gamma_\mathcal{Y}' \mathcal{O}_{\mathcal{X},\mathcal{Q}} \\
\simeq R\Gamma_\mathcal{Y}' \mathcal{O}_{\mathcal{X}',\mathcal{Q}}.
\]

We come to the main result, which describes the relation between the classical intermediate extension \( \nu_{\mathcal{Q}+} \mathcal{O}_{\mathcal{Y}_q} \) on the generic fibre and the Abe-Caro intermediate extension \( \nu_+ \mathcal{O}_\mathcal{Y} \) on the special fibre.

**Theorem 3.22.** There is a canonical isomorphism

\[
\mathcal{D}_\mathcal{X} \otimes_{\mathcal{P}_{\mathcal{Q}}} \nu_{\mathcal{Y}+}(\mathcal{O}_{\mathcal{Y}_q}) \simeq \nu_+(\mathcal{O}_\mathcal{Y}).
\]

**Proof.** Again, by \( \text{1.1.2.8} \), \( u_+ = u_+ \), and \( \theta_\mathcal{Q} = \theta_\mathcal{Q} \circ u_+ \). By previous lemma 3.21 \( u_+ \mathcal{O}_\mathcal{Y}' \simeq \mathcal{O}_\mathcal{Y} \) and we have a commutative diagram

\[
\begin{array}{ccc}
\nu_+ \mathcal{O}_\mathcal{Y} & \xrightarrow{\theta_\mathcal{Q}} & \nu_1 \mathcal{O}_\mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{Q}_+ \mathcal{O}_\mathcal{Y}' & \xrightarrow{\theta_\mathcal{Q}} & \mathcal{Q}_1 \mathcal{O}_\mathcal{Y}'.
\end{array}
\]
Now we have
\[ v_+(O_Q) = \text{im}(\theta_v) \]
\[ \simeq \text{im}(\theta_Q) \]
\[ \simeq \mathcal{D}_P \otimes \mathcal{D}_{P, Q} \text{ im}(\theta_{PV}) \text{ by } \text{3.20} \]
\[ \simeq \mathcal{D}_P \otimes \mathcal{D}_{P, Q} v_{Q+}(O_{Y_Q}). \]

\[ \square \]

4. Localization theory on the flag variety

We specialize the above theory to the case where \( P \) is the (formal) flag variety of a connected split reductive group \( G \) over \( \mathfrak{o} \). Such a space is coherently \( \mathcal{D} \)-affine and its algebra of global differential operators \( H^0(P, \mathcal{D}_P) \) identifies with (a central reduction of) the crystalline distribution algebra of \( G \). Truly in the spirit of classical localization theory [2], this allows us to analyze geometrically the module theory of the distribution algebra.

4.1. Crystalline distribution algebras. Let \( G \) be a connected split reductive group scheme over \( \mathfrak{o} \). Let \( I \) be the kernel of the morphism \( \mathfrak{o} \)-algebras \( \epsilon_G : \mathfrak{o}[G] \to \mathfrak{o} \) which represents \( 1 \in G \). Then \( I/I^2 \) is a free \( \mathfrak{o} = \mathfrak{o}[G]/I \)-module of finite rank. Let \( t_1, \ldots, t_N \in I \) whose classes modulo \( I^2 \) form a base of \( I/I^2 \). The \( m \)-PD-envelope of \( I \) is denoted by \( P_{(m)}(G) \). This algebra is a free \( \mathfrak{o} \)-module with basis

\[ \mathfrak{o} \cdot t_i^{(k)} = t_1^{(k_1)} \cdots t_N^{(k_N)}, \]

where \( q_i^{(k)} = t_i^{k_i} \) with \( i = p^m q_i + r \) et \( r < p^m \) [4 1.5]. The algebra \( P_{(m)}(G) \) has a descending filtration by the ideals

\[ I^{(n)} = \bigoplus_{|k| \geq n} \mathfrak{o} \cdot t_i^{(k)}. \]

The quotients \( P_{(m)}(G) := P_{(m)}(G)/I^{(n+1)} \) are generated, as \( \mathfrak{o} \)-module, by the elements \( \mathfrak{o} \cdot t_i^{(k)} \) where \( |k| \leq n \) and there is an isomorphism \( P_{(m)}(G) \simeq \bigoplus_{|k| \leq n} \mathfrak{o} t_i^{(k)} \) as \( \mathfrak{o} \)-modules. There are canonical surjections \( p_{n+1,n} : P_{(m)}(G) \to P_{(m)}(G) \).

We note

\[ \text{Lie}(G) := \text{Hom}_\mathfrak{o}(I/I^2, \mathfrak{o}). \]

The Lie-algebra \( \text{Lie}(G) \) is a free \( \mathfrak{o} \)-module with basis \( \xi_1, \ldots, \xi_N \) dual to \( t_1, \ldots, t_N \). For \( m' \geq m \), the universal property of divided power algebras gives homomorphisms of filtered algebras \( \psi_{m,m'} : P_{(m')}(G) \to P_{(m)}(G) \) which induce on quotients homomorphisms
of algebras $\psi_{m,m'}^n : P_{(m')}^n(G) \to P_{(m)}^n(G)$. The module of distributions of level $m$ and order $n$ is $D_{n}^{(m)}(G) := \text{Hom}_\mathfrak{o}(P_{(m)}^n(G), \mathfrak{o})$. The algebra of distributions of level $m$ is defined to be $D^{(m)}(G) := \lim_n D_{n}^{(m)}(G)$ where the limit is taken with respect to the maps $\text{Hom}_\mathfrak{o}(pr^{n+1}, \mathfrak{o})$.

For $m' \geq m$, the algebra homomorphisms $\psi_{m,m'}^n$ give dually linear maps $\Phi_{m,m'}^n : D^{(m)}(G) \to D^{(m')}_{n}(G)$ and finally a morphism of filtered algebras $\Phi_{m,m'} : D^{(m)}(G) \to D^{(m')}_{m}(G)$. The direct limit

$$\text{Dist}(G) = \lim_m D^{(m)}(G)$$

equals the classical distribution algebra of the group scheme $G$ \cite[I. §4.6.1]{15}.

Let now $G$ be the completion of $G$ along its special fibre. We write $G_i = \text{Spec } \mathfrak{o}[G]/\pi^{i+1}$. The morphism $G_{i+1} \hookrightarrow G_i$ induces $D^{(m)}(G_{i+1}) \to D^{(m)}(G_i)$. We put

$$\hat{D}^{(m)}(G) := \lim_i D^{(m)}(G_i).$$

If $m' \geq m$, one has the morphisms $\hat{\Phi}_{m,m'} : \hat{D}^{(m)}(G) \to \hat{D}^{(m')}_{m}(G)$ and the crystalline distribution algebra is defined to be

$$D^\dagger(G) := \lim_m \hat{D}^{(m)}(G) \otimes \mathbb{Q}.$$

Note, as for differential operators, that this dagger-algebra appears with coefficients tensored by $\mathbb{Q}$. For more details on the basic theory of the algebra $D^\dagger(G)$ we refer to \cite{26,27}.

For a character $\theta : Z(g) \to K$ of the center $Z(g)$ of the universal enveloping algebra of the $K$-Lie algebra $g = \text{Lie}(G) \otimes \mathbb{Q}$, we will always denote by

$$D^\dagger(G)_\theta := D^\dagger(G)_0 \otimes_{Z(g),\theta} K$$

the corresponding central reduction of $D^\dagger(G)$. The trivial character is the character $\theta_0$ with $\ker \theta_0 = Z(g) \cap (U(g)g)$.

4.2. The localization theorem and holonomicity. Let in the following $\theta = \theta_0$ be the trivial character. Our goal is to analyze the central block of the category of $D^\dagger(G)$-modules, i.e. the category of $D^\dagger(G)_{\theta_0}$-modules. We keep the notation from the preceding section.

We let $B \subset G$ be a Borel subgroup containing a maximal split torus $T$, with unipotent radical $N$. Denote by

$$P := G/B$$
the flag scheme. It is a smooth and projective scheme over $\mathfrak{a}$. We denote by $\mathcal{P}$ its formal completion. The $G$-action on $P$ by translations endows $\mathcal{P}$ with a $G$-action. We recall the localization theorem for arithmetic $\mathcal{D}$-modules on the flag variety.

**Theorem 4.1.** (a) The global section functor induces an equivalence of categories between coherent $\mathcal{D}_P$-modules and coherent $H^0(\mathcal{P}, \mathcal{D}_P)$-modules. A quasi-inverse is given by the functor

$$\mathcal{L}oc(M) = \mathcal{D}_P \otimes_{H^0(\mathcal{P}, \mathcal{D}_P)} M.$$  

(b) The $G$-action on $\mathcal{P}$ induces an algebra isomorphism

$$D^1(G)_{\theta_0} \xrightarrow{\sim} H^0(\mathcal{P}, \mathcal{D}_P).$$

**Proof.** This summarizes the main results of [27] and [32]. □

**Remark:** Sarrazola-Alzate has extended the above theorem to the case of an arbitrary central character $\theta$ using a twisted version of the sheaf $\mathcal{D}_P$, as in the classical setting, cf. [33].

We specialize the classification result 2.3 to the case of the flag variety $\mathcal{P}$. First of all, $\mathcal{P}$ is quasi-projective (in fact projective) over $\mathfrak{a}$. According to the main result of [13] the notions overholonomic and holonomic coincide for $F-\mathcal{D}_P$-modules. Hence we have the equality

$$F-\text{Ovhol}(\mathbb{P}/K) = \{ \text{holonomic } F-\mathcal{D}_P\text{-modules} \}$$

inside the category of coherent $\mathcal{D}_P$-modules. This motivates the following definition.

**Definition 4.2.** A $D^1(G)_{\theta_0}$-module $M$ is called geometrically $F$-holonomic if $\mathcal{L}oc(M) \in F-\text{Ovhol}(\mathbb{P}/K)$.

Recall from [24] the set of equivalence classes of pairs $(Y, \mathcal{E})$ where $Y \subset \mathcal{P}$ is a connected smooth locally closed subvariety and $\mathcal{E}$ is an irreducible overconvergent $F$-isocrystal on $Y = (Y, X)$. We put $\mathcal{L}(Y, \mathcal{E}) := v_+ v_\times (\mathcal{E}) \in F-\text{Ovhol}(\mathbb{P})$ where $v : \mathbb{Y} \to \mathbb{P}$ is the immersion of couples associated with $Y$.

**Theorem 4.3.** The correspondence $(Y, \mathcal{E}) \mapsto H^0(\mathcal{P}, \mathcal{L}(Y, \mathcal{E}))$ induces a bijection

$$\{ \text{equivalence classes of pairs } (Y, \mathcal{E}) \} \xrightarrow{\sim} \{ \text{irreducible } F\text{-holonomic } D^1(G)_{\theta_0}\text{-modules} \}/\sim.$$  

**Proof.** This follows from the classification theorem [2.5] together with [4.1] □

We point out a related interesting property of the category of holonomic $F-\mathcal{D}_P$-modules.

It is conjectured by de Jong that, if $X$ is a connected smooth projective variety over an algebraically closed field of characteristic $p > 0$ with trivial étale fundamental group, then any isocrystal on $X$ is constant. This conjecture is proved under certain additional assumptions by Esnault-Shiho in [17]. In our case, the fibration $G \to G/B = P$ is a separable proper morphism with geometrically connected fibre between locally noetherian
connected schemes. To compute the fundamental group of $P_s$, we may pass to a simply connected cover of the semisimple derived group of $G_s$. The homotopy exact sequence \cite[Exp. 10 Cor. 1.4]{20} implies then that étale fundamental group of $P_s$ is trivial. Here is a short representation-theoretic proof of de Jong’s conjecture for the flag variety $P_s$.

**Proposition 4.4.** Any convergent isocrystal on $P_s$ is constant.

*Proof.* Any convergent isocrystal $\mathcal{E}$ may be viewed as a coherent $\mathcal{D}_P$-module which is coherent over $\mathcal{O}_{P, \mathbb{Q}}$ \cite[Prop. (4.1.4)]{1}. Then $H^0(\mathcal{P}, \mathcal{E})$ is a finite dimensional representation of the reductive $K$-Lie algebra $\mathfrak{g}$ and hence completely reducible (semisimple). In addition, it has central character $\theta_0$. But the trivial one dimensional representation is the only irreducible $\mathfrak{g}$-representation of finite dimension and with central character $\theta_0$. Since the trivial representation localizes to the trivial connection $\mathcal{O}_{P, \mathbb{Q}}$ and since localization commutes with direct sums, the isocrystal $\mathcal{E}$ must be constant. \hfill $\square$

5. **Highest weight representations and the rank one case**

We keep the notation from the preceding section. We assume from now on that the field $K$ is locally compact.

5.1. **Highest weight representations.** We establish a crystalline version of the central block of the classical BGG category $\mathcal{O}$ and show that its objects are geometrically $F$-holonomic. We then compute their associated parameters $(Y, \mathcal{E})$ in the geometric classification \cite{3}.

Let $\Delta$ be the set of simple roots in $\Phi^+$. We fix a (Chevalley) basis for Lie($G$) compatible with its root space decomposition. In particular, we obtain a $\mathfrak{a}$-basis $t_1, ..., t_n$ of Lie($T$) which is made up from a $K$-basis of the center of $\mathfrak{g}$ and finitely many elements $t_\alpha$, indexed by $\alpha \in \Delta$, such that $\beta(t_\alpha) \in \mathbb{Z}$ for all $\beta \in \Phi$. Let $\Gamma := \mathbb{Z}_{\geq 0} \Phi^+ \subset \mathbb{Q} \Phi =: \Lambda_r \subseteq \Lambda$ where $\Lambda_r$ and $\Lambda$ are the root lattice and the integral weight lattice respectively.

For $w \in W$ we let $\lambda_w = -w(\rho) - \rho$. These are $|W|$ pairwise different elements of $\Lambda_r$.

Let $\mathcal{O}_0$ be the central block of the classical BGG category, e.g. \cite{24}. This is a full abelian subcategory of finitely generated $U(\mathfrak{g})_{\theta_0}$-modules which is noetherian and artinian. Its irreducible objects are given by the unique irreducible quotients $M(\lambda_w) \rightarrow L(\lambda_w)$ where

$$M(\lambda_w) := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}), \lambda_w} K$$

is the Verma module with highest weight $\lambda_w$ for $w \in W$.

\footnote{The homotopy exact sequence implies in the same manner that the generic fibre $P_K$ has trivial étale fundamental group. By Chern-Weil theory and Grothendieck’s theorem on formal functions, the de Rham Chern classes on $P_K$ become trivial after tensoring with $\mathbb{Q}$. But these classes correspond to the rational crystalline classes on $P$ via the comparison theorem between de Rham and crystalline cohomology, from which one may deduce the conjecture. We thank H. Esnault for explaining this general argument to us.}
To define a crystalline variant of the category $\mathcal{O}_0$ we follow the constructions given in [34] in the case of the Arens-Michael envelope of $U(\mathfrak{g})$. In order to do so, we need the field $K$ to be locally compact.

By the discussion in [26, 5.3] the algebra $D^t(\mathcal{G}) = \varprojlim_{m} \hat{D}^{(m)}(\mathcal{G}) \otimes \mathbb{Q}$ is an inductive limit of Hausdorff locally convex $K$-vector spaces with injective and compact transition maps. According to [35, 7.19/16.9/16.10] it is therefore Hausdorff, complete and barrelled.

The framework of diagonalisable modules over suitable commutative topological $K$-algebras as described in [34, sec. 2] applies therefore to the $K$-algebra $D^t(\mathcal{T})$. Note that it contains the universal enveloping algebra $U(t)$ as a dense subalgebra. A $K$-valued weight $\lambda$ of $D^t(\mathcal{T})$ is a $K$-algebra homomorphism $D^t(\mathcal{T}) \to K$. A set of weights $Y$ is called relatively compact if its image under the injective map $\lambda \mapsto (\lambda(t_1), \ldots, \lambda(t_n))$ has a compact closure in $K^n$. Let $\lambda$ be weight and $M$ some topological $D^t(\mathcal{T})$-module. A nonzero $m \in M$ is called a $\lambda$-weight vector if $h.m = \lambda(h).m$ for all $h \in D^t(\mathcal{T})$. In this case $\lambda$ is called a weight of $M$. The closure $M_\lambda$ in $M$ of the $K$-vector space generated by all $\lambda$-weight vectors is called the $\lambda$-weight space of $M$. The module $M$ is called $D^t(\mathcal{T})$-diagonalisable if there is a set of weights $\Pi(M)$ with the property: to every $m \in M$ there exists a family $\{m_\lambda \in M_\lambda\}_{\lambda \in \Pi(M)}$ converging cofinitely against zero in $M$ and satisfying

$$m = \sum_{\lambda \in \Pi(M)} m_\lambda.$$ 

Given a diagonalisable module $M$ we may form $M^{ss} = \bigoplus_{\lambda \in \Pi(M)} M_\lambda$ (depending on the choice of $\Pi(M)$).

**Definition 5.1.** The category $\mathcal{O}_0^\dagger$ equals the full subcategory of $D^t(\mathcal{G})_{\theta_0}$-modules $M$ satisfying:

1. $M$ is a coherent $D^t(\mathcal{G})_{\theta_0}$-module.
2. $M$ is $D^t(\mathcal{T})$-diagonalisable with $\Pi(M)$ contained in the union of the cosets $\lambda_w - \Gamma$.
3. All weight spaces $M_\lambda$, $\lambda \in \Pi(M)$, are finite dimensional over $K$.

By definition, given $M \in \mathcal{O}_0^\dagger$, then any finitely generated $U(\mathfrak{g})$-submodule of $M^{ss}$ lies in $\mathcal{O}_0$. In particular, $M^{ss}$ contains a maximal vector, i.e. a nonzero $m \in M_\lambda$ (of some weight $\lambda$) such that $n.m = 0$. We will make precise the relation between the two categories $\mathcal{O}_0$ and $\mathcal{O}_0^\dagger$ below.

We list some basic properties of the category $\mathcal{O}_0^\dagger$.

**Proposition 5.2.**

1. The direct sum of two modules of $\mathcal{O}_0^\dagger$ is in $\mathcal{O}_0^\dagger$.
2. The (co)kernel and (co)image of an arbitrary $D^t(\mathcal{G})_{\theta_0}$-linear map between objects in $\mathcal{O}_0^\dagger$ is in $\mathcal{O}_0^\dagger$.
3. The sum of two coherent submodules of an object in $\mathcal{O}_0^\dagger$ is in $\mathcal{O}_0^\dagger$.
4. Any finitely generated submodule of an object in $\mathcal{O}_0$ is in $\mathcal{O}_0^\dagger$.
5. $\mathcal{O}_0^\dagger$ is an abelian category.
Proof. This can be proved using a variant of the proof of [34, Prop. 3.6.3]. Note that any Π(ℳ) which is contained in the union of the cosets λₜ – Γ is relatively compact. Indeed, Γ is relatively compact its closure being contained in the compact subset ℤ_p]| of K^n, cf. [34, Lem. 3.6.1].

We exhibit Verma type modules in O_0^T. The main difference between the case of the crystalline distribution algebra and the case of the Arens-Michael envelope treated in [34] is that not every weight t → K extends to a weight of D^T(T). The following lemma is sufficient for our purposes.

Lemma 5.3. Any linear form λ : Lie(T) → o such that λ(h_i) ∈ ℤ_p for all i = 1,...,n extends canonically to a K-algebra homomorphism D^T(T) → K.

Proof. Recall that the distribution algebra Dist(G_m) of the o-group scheme G_m is generated as an o-module by the elements (δ_k^i) for k ∈ N where δ_i is a generator of Lie(G_m), cf. [28, Part I.7.8]. Our choice of Chevalley basis implies an isomorphism of group schemes T ≃ ℓ_p_{i=1,...,n}G_m such that the basis element h_i becomes the generator of the i-th copy Lie(G_m). Since (λ(h_i)) ∈ ℤ_p, the associated K-algebra homomorphism λ : U(t) → K restricts to an o-algebra homomorphism Dist(T) → o. Since Dist(T) = lim_m D^m(T), this extends then to a K-algebra homomorphism D^T(T) → K. □

We may apply the lemma to any weight λₜ and hence consider the D^T(G)-module

M^T(λₜ) := D^T(G) ⊗_{D^T(T),λₜ} K.

Proposition 5.4. The module M^T(λₜ) lies in O_0^T. We have

M^T(λₜ)^ss = M(λₜ) and M^T(λₜ) = D^T(G) ⊗_{U(g)} M(λₜ).

There is a canonical inclusion preserving bijection between subobjects of M^*(λₜ) and abstract U(g)-submodules of M(λₜ). In particular, M^*(λₜ) admits a unique maximal subobject and hence a unique irreducible quotient L^*(λₜ). The latter satisfies L^*(λₜ)^ss = L(λₜ).

Proof. This can be proved as in [34, Prop. 3.7.1]. Note that the triangular decomposition D^m(G) = D^m(N^-) ⊗_o D^m(T) ⊗_o D^m(N), cf. [27, 2.2], implies that M^*(w) ≃ D^*(N^-) as a left D^*(N^-)-module. This implies the first displayed identity. Moreover, M^*(λₜ) equals the quotient of D^*(G) by the left ideal generated by ker(λₜ), which implies the second displayed identity. Note also that the nonzero quotient morphism M^*(λₜ) → L^*(λₜ) yields a nonzero quotient morphism M^*(λₜ)^ss → L^*(λₜ)^ss since (-)^ss is faithful and exact [34, Prop. 2.0.2]. Hence M^*(λₜ)^ss = M(λₜ) implies L^*(λₜ)^ss = L(λₜ). □

Corollary 5.5. The modules L^*(λₜ) exhaust, up to isomorphism, all the irreducible objects in O_0^T.

Proof. Let L be an irreducible object in O_0^T. Take a maximal vector m ∈ L^ss of some weight λ. Then U(g)m is a highest weight module in O of weight λ, cf. [24, 1.2]. Hence
$Z(\mathfrak{g})$ acts on the maximal vector $m$ via the central character $\theta_\lambda$ associated to $\lambda$ via the Harish-Chandra homomorphism [24, 1.7]. But $U(\mathfrak{g})m \subseteq L$ whence $\theta_\lambda = \theta_0$ and so $\lambda = \lambda_w$ for some $w \in W$. We obtain a nonzero $D^\dagger(\mathcal{G})$-linear map $M^\dagger(\lambda_w) \to L, 1 \otimes 1 \mapsto m$. So $L$ is an irreducible quotient of $M^\dagger(\lambda_w)$, i.e., $L \cong L^\dagger(\lambda_w)$.

**Corollary 5.6.** The category $\mathcal{O}_0^\dagger$ is artinian and noetherian.

**Proof.** This can be deduced similarly to [34] Prop.4.2.2. In fact, let $M \in \mathcal{O}_0^\dagger$ and consider the finite-dimensional $K$-vector space $V := \sum_w M_{\lambda_w}$. Suppose $N' \not\subseteq N \subseteq M$ are two subobjects. Let $m \in N \setminus N'$ be a maximal vector of some weight $\lambda$. As in the preceding proof we deduce from the action of $Z(\mathfrak{g})$ on $m$ that $\lambda = \lambda_w$ for some $w \in W$. So $m \in N \cap V$ whence $\dim_K N \cap V > \dim_K N' \cap V$. This implies that $M$ has finite length. \qed

Given a module $M \in \mathcal{O}_0$, we can define the coherent $D^\dagger(\mathcal{G})_{\theta_0}$-module

$$M^\dagger := D^\dagger(\mathcal{G}) \otimes_{U(\mathfrak{g})} M.$$  

**Theorem 5.7.** The functor $F : M \mapsto M^\dagger$ is exact and induces an equivalence of abelian categories

$$\mathcal{O}_0 \cong \mathcal{O}_0^\dagger.$$  

A quasi-inverse is given by the functor $(-)^{ss}$.

**Proof.** The ring extension $U(\mathfrak{g}) \to D^\dagger(\mathcal{G})$ is flat [27, Lem. 4.1]. We already now that $F(M(\lambda_w)) = M^\dagger(\lambda_w)$. Since any object $M \in \mathcal{O}_0$ admits a finite composition series with irreducible constituents of the form $L(\mathfrak{g})$, there is a surjection $\oplus_w M(\lambda_w) \to M$. Since $F$ commutes with direct sums, we see that $F(M)$ equals the quotient of $\oplus_w M^\dagger(\lambda_w)$ modulo a finitely generated submodule and so lies in $\mathcal{O}_0^\dagger$, according to parts (iii)-(v) of 5.2. We therefore have an exact functor $F : \mathcal{O}_0 \to \mathcal{O}_0^\dagger$. Given $M \in \mathcal{O}_0^\dagger$ we have a functorial morphism $M \to F(M)^{ss}, m \mapsto 1 \otimes m$ which is bijective for irreducible $M$ according to 5.4. By d\'evissage, we obtain $M \cong F(M)^{ss}$ in general. Let $M \in \mathcal{O}_0^\dagger$. To obtain $M^{ss} \in \mathcal{O}_0$ we use induction on the length of $M$ and suppose that $N \subseteq M$ is a maximal submodule, i.e., $M/N \cong L^\dagger(\lambda_w)$ for some $w$, such that $N^{ss} \in \mathcal{O}_0$. Exactness of $(-)^{ss}$ and $L^\dagger(\lambda_w)^{ss} = L(\lambda_w)$ implies that $M^{ss}$ is an extension of two finitely generated $U(\mathfrak{g})$-modules and hence itself finitely generated. So $M^{ss} \in \mathcal{O}_0$. We may now deduce that $(-)^{ss}$ is also a right quasi-inverse to $F$. Indeed, for any $M \in \mathcal{O}_0^\dagger$, there is a natural morphism $F(M^{ss}) \to M$ in $\mathcal{O}_0^\dagger$ which is bijective for irreducible $M$ according to 5.4. By d\'evissage, we obtain $F(M^{ss}) \cong M$ in general. \qed

To finish this section, we will show that the irreducible modules $L^\dagger(\lambda_w)$ are all geometrically $F$-holonomic.

To do this, fix $w \in W$ and let

$$Y_w := BwB/B \subseteq P = G/B.$$
be the Bruhat cell in $P$ associated with $w \in W$. Let $v : Y_w \hookrightarrow P$ be the corresponding immersion over $\mathfrak{o}$ and let $v_Q : Y_{wQ} \hookrightarrow P_Q$ be the corresponding immersion on the level of $K$-algebraic varieties. It is well-known (e.g. [23, Prop. 12.3.2]) that there is a canonical isomorphism of $D_{P_Q}$-modules

$$\text{Loc}(L(\lambda_w)) := \mathcal{D}_{P_Q} \otimes_{U(g)} L(\lambda_w) \simeq v_{Q!+}(\mathcal{O}_{Y_{wQ}}).$$

Let $X_w \subset P$ be the Zariski closure of the Bruhat cell $Y_w$ in $P$, a Schubert scheme. We let

$$X'_w \longrightarrow X_w$$

be its Demazure desingularization, which is defined at the level of $\mathfrak{o}$-schemes [28, II, 13.6]. We are then in the axiomatic situation (S), the point of departure for subsection 3.4, so that all the results of this subsection apply. In particular, we have the frame $\mathcal{Y}_w = (Y_{w,s}, X_{w,s}, \mathcal{P})$ together with its $c$-locally closed immersion

$$v : \mathcal{Y}_w \longrightarrow \mathcal{P}$$

and the constant overholonomic module $\mathcal{O}_{\mathcal{Y}_w}$ on $\mathcal{Y}_w$. Its intermediate extension $v_{Q!+}(\mathcal{O}_{Y_{wQ}})$ is an irreducible holonomic $F$-$\mathcal{D}_P$-module. In this situation, the main theorem 3.22 implies directly the following result.

**Proposition 5.8.** There is a canonical isomorphism of $\mathcal{D}_P$-modules

$$\mathcal{D}_P \otimes_{\mathcal{D}_{PQ}} v_{Q!+}(\mathcal{O}_{Y_{wQ}}) \simeq v_{Q!+}(\mathcal{O}_{Y_{wQ}}).$$

Now consider the localization

$$\mathcal{L}\text{oc}(L^!(\lambda_w)) = \mathcal{D}_P \otimes_{D^!(G)} L(\lambda_w)^!.$$ 

**Theorem 5.9.** One has a canonical isomorphism of $\mathcal{D}_P$-modules

$$\mathcal{L}\text{oc}(L^!(\lambda_w)) \simeq v_{Q!+}(\mathcal{O}_{\mathcal{Y}_w}).$$

The modules $L^!(\lambda_w)$ are geometrically $F$-holonomic for all $w \in W$.

**Proof.** We write $L(w)$ resp. $L^!(w)$ for $L(\lambda_w)$ resp. $L^!(\lambda_w)$. Since $L^!(w) = D^!(G) \otimes_{U(g)} L(w)$, associativity of tensor products yields a canonical isomorphism

$$\mathcal{D}_P \otimes_{D^!(G)} L(w)^! \simeq \mathcal{D}_P \otimes_{U(g)} L(w) \simeq \mathcal{D}_P \otimes_{\mathcal{D}_{PQ}} (\mathcal{D}_{PQ} \otimes_{U(g)} L(w)) \simeq \mathcal{D}_P \otimes_{\mathcal{D}_{PQ}} \text{Loc}(L(w)).$$

Since $\text{Loc}(L(w)) \simeq v_{Q!+}(\mathcal{O}_{Y_{wQ}})$, the asserted isomorphism follows now in combination with 5.8. Since $v_{Q!+}(\mathcal{O}_{Y_{wQ}})$ is a holonomic $F$-$\mathcal{D}_P$-module, the module $L^!(w)$ is seen to be geometrically $F$-holonomic. □
5.2. The $\text{SL}_2$-case. We suppose $G = \text{SL}_2$. We let $B$ be the subgroup of upper triangular matrices and $T \subset B$ be the subgroup of diagonal matrices. We identify $\Lambda = \mathbb{Z}$ so that $\Delta = \{\alpha\}$ with $\alpha = 2$. We identify $P = G/B = \mathbb{P}^1_\mathfrak{o}$ with the projective line $\mathbb{P}^1_\mathfrak{o}$ over $\mathfrak{o}$. We choose an affine coordinate $t$ around zero. The group $G$ acts by fractional transformations

$$
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot (t) = \left( \begin{array}{c} at + b \\ ct + d \end{array} \right)
$$

in the usual way. The stabiliser of the point $\infty \in \mathbb{P}^1_\mathfrak{o}$ is $B$.

We note that any irreducible $\mathbb{D}^1_{\mathfrak{p}}$-module is holonomic, since $\mathfrak{p}$ has dimension one. In this setting, the theorem 4.3 amounts to a classification of all irreducible $\mathbb{F}^1_{\mathfrak{p}}$-modules in terms of irreducible overconvergent $\mathbb{F}$-isocrystals $\mathcal{E}$ on couples $Y = (Y, X)$ where $Y$ is either:

1. a closed point of $\mathbb{P}^1_k$ or
2. an open complement of finitely many closed points $Z = \{y_1, \ldots, y_n\}$ of $\mathbb{P}^1_k$.

In case (1), the point is a complete invariant, since we have necessarily $\mathcal{E} = \mathcal{O}_Y$ in this case. Suppose that the point is $k$-rational. Since the (finitely many) $k$-rational points $\mathbb{P}^1(k)$ of $\mathbb{P}^1_k$ form a single orbit under the natural action of the (finite) group $G(k)$ of $k$-rational points of $G$, it suffices to consider the point

$$\{\infty\} = Y_{1,s} = X_{1,s}$$

in $\mathbb{P}^1_k$. According to 5.9 the global sections of $\mathfrak{m}_+(\mathcal{O}_Y)$ are equal to the $D^1(\mathcal{G})_{\mathfrak{m}_0}$-module $L^1(-2)$, the crystalline version of the classical anti-dominant Verma module $M(-2) = L(-2)$.

Suppose now that the the point is $k'$-rational for a finite extension field $k'/k$. Let $M = H^0(\mathcal{P}, \mathfrak{m}_+(\mathcal{O}_Y))$. Let $\mathfrak{o}'$ be a finite extension of $\mathfrak{o}$ with residue field $k'$ and quotient field $K'$. The base change $M_{K'} = M \otimes_K K'$ has the same geometric parameter, but now considered a rational point of the special fibre of $\mathcal{P} \times_{\mathfrak{o}} \mathfrak{o}'$. This means that $M$ is a twisted form of the module $L^1(-2)$, with respect to the field extension $K'/K$.

We come to case (2). For $Z = \emptyset$ and hence $Y = \mathbb{P}^1_k$ we obtain the trivial representation, i.e. the augmentation character $D^1(\mathcal{G}) \to K$. Indeed, there are no convergent $\mathbb{F}$-isocrystals on $\mathcal{P}$ besides the constant one, cf. prop. 4.4. Let $n > 0$. Modulo the appearance of twisted forms (see the above argument), we may assume that all points $y_1, \ldots, y_n$ are $k$-rational and $y_1 = \infty$. There are then two extreme cases

$$Y = \mathbb{A}^1_k \text{ resp. } Y = \mathbb{P}^1_k \setminus \mathbb{P}^1(k),$$

the affine line and so-called Drinfeld’s upper half plane, respectively.
We discuss an interesting example in the case $Y = \mathbb{A}^1_K$. For this, we assume that $K$ contains the $p$-th roots of unity $\mu_p$ and we choose an element $\pi \in \mathfrak{o}$ with $\text{ord}_p(\pi) = 1/(p-1)$. We have the affine coordinate $t$ on $\mathbb{A}^1_K$ and we let $\partial = d/dt$. We let $\mathcal{L}_\pi$ be the coherent $\mathcal{D}_p$-module defined by the Dwork overconvergent $F$-isocrystal $L_\pi$ on $\mathbb{Y}$ associated with $\pi$, i.e. $\mathcal{L}_\pi = \nu_{1,+} L_\pi$ where $\nu : \mathbb{Y} \to \mathbb{P}$. Recall that the underlying $\mathcal{O}_{\mathbb{P}, \mathbb{Q}}$-module of $\mathcal{L}_\pi$ is $\mathcal{O}_{\mathbb{P}, \mathbb{Q}}(x)$, endowed with a compatible $\mathcal{D}_p$-module structure for which $\partial(1) = -\pi$, [7, 4.5.5].

Write $n = K.e$ with $e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $\eta : n \to K$ be a nonzero character and consider Kostant’s standard Whittaker module

$$W_{\theta_0, \eta} := U(\mathfrak{g}) \otimes_{\mathbb{Z}(\mathfrak{g}) \otimes U(\mathfrak{n})} K_{\theta_0, \eta}$$

with character $\eta$ and central character $\theta_0$, [29, (3.6.1)]. It is an irreducible $U(\mathfrak{g})$-module, but does not lie in $\mathcal{O}_0$. In fact, the restriction of the $\mathcal{D}_{\mathcal{P}_1}$-module $\text{Loc}(W_{\theta_0, \eta})$ to $\mathbb{A}^1_K$ has an irregular singularity at $\infty$, [30, 4.4].

Let $W_{\theta_0, \eta}^! := D^!(\mathcal{G}) \otimes_{U(\mathfrak{g})} W_{\theta_0, \eta}$.

**Theorem 5.10.** Consider the character $\eta$ defined by $\eta(e) := \pi$. There is a canonical isomorphism

$$\mathcal{L}_{\text{oc}}(W_{\theta_0, \eta}^!) \xrightarrow{\sim} \mathcal{L}_\pi$$

as left $\mathcal{D}_p$-modules. In particular, $W_{\theta_0, \eta}^!$ is geometrically $F$-holonomic.

**Proof.** The standard Whittaker module $W_{\theta_0, \eta}$ admits the presentation

$$W_{\theta_0, \eta} = U(\mathfrak{g})/U(\mathfrak{g})(e - \eta(e)) = U(\mathfrak{g})/U(\mathfrak{g})(e - \pi)$$

whence $W_{\theta_0, \eta}^! = D^!(\mathcal{G})/D^!(\mathcal{G})(e - \pi)$. The canonical morphism $U(\mathfrak{g}) \to \mathcal{D}_{\mathbb{P}_1, K}$ maps $e$ to $-\partial$, cf. [23, 11.2.1], and the isomorphism of part (b) in theorem 4.1 is compatible with this morphism. We obtain

$$\mathcal{L}_{\text{oc}}(W_{\theta_0, \eta}^!) = \mathcal{D}_p^! / \mathcal{D}_p^!(\partial + \pi)$$

which coincides with the standard presentation of the $\mathcal{D}_p^!$-module $\mathcal{L}_\pi$, [3, Prop. 5.2.3].

**Remark:** It is interesting to note that the Dwork isocrystal $\mathcal{L}_\pi$ is algebraic in the sense that it comes from an algebraic $\mathcal{D}_{\mathbb{P}_1}$-module, namely $\text{Loc}(W_{\theta_0, \eta})$, by extension of scalars $\mathcal{D}_{\mathbb{P}_1}^! \to \mathcal{D}_p^!$.

We discuss an example in the second case, where $Y = \mathbb{P}^1_K \setminus \mathbb{P}^1(k)$. We identify $k = \mathbb{F}_q$. We assume that $K$ contains the cyclic group $\mu_{q+1}$ of $(q+1)$-th roots of unity. We consider the so-called Drinfeld curve

$$Y' = \left\{ (x, y) \in \mathbb{A}^2_K \mid xy^q - x^q y = 1 \right\}.$$
It is an affine smooth irreducible curve and the map \((x, y) \mapsto [x : y]\) is an unramified Galois covering

\[ u : Y' \longrightarrow Y \]

with Galois group \(\mu_{q+1}\). The group \(\mu_{q+1}\) acts by homotheties \(\zeta \cdot (x, y) = (\zeta x, \zeta y)\). We have a smooth projective compactification

\[ Y' = \left\{ \left[ x : y : z \right] \in \mathbb{P}_k^2 \mid xy^q - x^q y = z^{q+1} \right\} \]

and the covering extends to a smooth (and tamely ramified) morphism

\[ u : Y' \longrightarrow \mathbb{P}_k^1, \]

given by \([x : y : z] \mapsto [x : y]\). The boundary \(Z' = Y' \setminus Y\) is mapped bijectively to \(Z = \mathbb{P}_k^1(k)\) and the ramification index at each point in \(Z\) is \(q + 1\). For more details the reader may consult [9, chap. 2]. We denote by \(u : Y' \longrightarrow Y\) the morphism of couples induced by \(u\). We let \(E = \mathbb{R}^*_{rig} \mathcal{O}_{Y'}\) be the relative rigid cohomology sheaf which, in our situation, is just the direct image of \(\mathcal{O}_Y\) under the morphism \(u\) endowed with the Gauss-Manin connection.

**Proposition 5.11.** The relative rigid cohomology sheaf, as an overconvergent \(F\)-isocrystal on \(Y\), admits a decomposition \(E = \bigoplus_{j=0}^q E(j)\), where \(E(j)\) is the isotypic subspace (of rank one) on which \(\mu_{q+1}\) acts by the character \(\zeta \mapsto \zeta^j\). In particular, each pair \((Y, E(j))\) corresponds to an irreducible geometrically \(\mathcal{D}^\mathfrak{g}\)-holonomic \(\mathcal{D}^\mathfrak{g}\)-module \(H^0(\mathcal{P}, v_1 E(j))\).

**Proof.** The cover \(u : Y' \longrightarrow Y\) is an abelian prime-to-\(p\) Galois covering as considered in [18]. The relative rigid cohomology, as an overconvergent \(F\)-isocrystal on the base \(Y\) (denoted there by \(E^1\)) together with its decomposition \(E^1 = \bigoplus_j E^1(j)\) is constructed in [18, sec. 2]. Note that \(u : Y' \longrightarrow Y\) is even equal to (one of the \(q - 1\) connected components of) the Deligne-Lusztig torsor for the nonsplit torus \(\mu_{q+1}\) in the finite group \(G_k\). A special situation considered in [18, sec. 4].

Are the modules \(H^0(\mathcal{P}, v_1 E(j))\) algebraic in the sense that they arise from irreducible \(U(\mathfrak{g})\)-modules, by extension of scalars \(U(\mathfrak{g}) \rightarrow \mathcal{D}^\mathfrak{g}\)?) Let us remark that the *théorème d’algébrisation* of Christol-Mebkhout [14] thm. 5.0-10] implies that any overconvergent \(F\)-isocrystal on the open \(Y\) is algebraic, i.e. comes from an algebraic connection on a characteristic zero lift of \(Y\). However, this does not imply (at least a priori) that the intermediate extensions preserve this algebraicity. To our knowledge, the most general result in this direction at the moment is our theorem 3.22 above.

If the modules \(H^0(\mathcal{P}, v_1 E(j))\) are algebraic, to which class do they belong? We recall that irreducible \(U(\mathfrak{g})\)-modules fall into three classes: highest weight modules, Whittaker modules and a third class whose objects (with a fixed central character) are in bijective correspondence with similarity classes of irreducible elements of a certain localization of the first Weyl algebra [8]. We plan to come back to these question in future work.
We finish this paper with the remark, still in the case (2), that if we concentrate on the subcategory of overconvergent \( F \)-isocrystals on \( Y = \mathbb{P}_k^1 \setminus Z \) which are unit-root, then work of Tsuzuki [37, Thm. 7.2.3] shows that this category is equivalent to the category of \( p \)-adic representations of the étale fundamental group \( \pi^\text{et}_1(Y) \) with finite monodromy (i.e. representations such that for each \( y \in Z \) the inertia subgroup at \( y \) acts through a finite quotient). Of course, the trivial representation corresponds to the constant isocrystal \( \mathcal{O}_Y \).

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