A pathwise interpretation of the Gorin-Shkolnikov identity

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Abstract

In a recent paper by Gorin and Shkolnikov (2016), they have found, as a corollary to their result relevant to random matrix theory, that the area below a normalized Brownian excursion minus one half of the integral of the square of its total local time, is identical in law with a centered Gaussian random variable with variance $1/12$.

In this paper, we give a pathwise interpretation to their identity; Jeulin’s identity connecting normalized Brownian excursion and its local time plays an essential role in the exposition.

Keywords: normalized Brownian excursion; local time; Jeulin’s identity.

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1 Introduction

Let $r = \{r_t\}_{0 \leq t \leq 1}$ be a normalized Brownian excursion, that is, it is identical in law with a standard 3-dimensional Bessel bridge, which has the duration $[0, 1]$, and starts from and ends at the origin; see e.g., [1, Section (2.2)] and references therein for the definition of normalized Brownian excursion and its equivalence in law with standard 3-dimensional Bessel bridge. We denote by $l = \{l_x\}_{x \geq 0}$ the total local time process of $r$; namely, by the occupation time formula, two processes $r$ and $l$ are related in particular via

$$H(x) := \int_0^1 1_{\{r_t \leq x\}} \, dt = \int_0^x l_y \, dy \quad \text{for all } x \geq 0, \text{ a.s.} \quad (1.1)$$

In a recent paper [3], Gorin and Shkolnikov have found the following remarkable identity in law as a corollary to one of their results:

**Theorem 1.1** ([3], Corollary 2.15). The random variable $X$ defined by

$$X := \int_0^1 r_t \, dt - \frac{1}{2} \int_0^\infty (l_x)^2 \, dx$$

is a centered Gaussian random variable with variance $1/12$.

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In [3, Proposition 2.14], they show that the expected value of the trace of a random operator indexed by $T > 0$, arising from random matrix theory, admits the representation
\[ \sqrt{\frac{2}{\pi T^3}} \mathcal{E} \left[ \exp \left( -\frac{T^{3/2}}{2} X \right) \right] \]
for any $T > 0$; in comparison of this expression with the existing literature asserting that the expected value is equal to $\frac{\sqrt{2}}{\pi T^3 \sqrt{\pi T}} \exp \left( \frac{T^{3/2}}{96} \right)$ for every $T > 0$, they obtain Theorem 1.1 by the analytic continuation and the uniqueness of characteristic functions.

In this paper, we give a proof of Theorem 1.1 without relying on random matrix theory; Jeulin’s identity in law ([6, p. 264]):
\[ \{ r_t \}_{0 \leq t \leq 1} \overset{(d)}{=} \left\{ \frac{1}{2} l_{H^{-1}(t)} \right\}_{0 \leq t \leq 1} \]
(1.2)
with
\[ H^{-1}(t) := \inf \{ x \geq 0; H(x) \geq t \}, \]
plays a central role in the proof. For the identity (1.2), we also refer to [1, Proposition 3.6 and Théorème (5.3)].

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 and provide some relevant results.

Proof of Theorem 1.1. Recall from the representation of $r$ by means of a stochastic differential equation (see, e.g., [7, Chapter XI, Exercise (3.11)]) that the process $W = \{ W_t \}_{0 \leq t \leq 1}$ defined by
\[ W_t := r_t - \int_0^t \left( \frac{1}{r_s} - \frac{r_s}{1-s} \right) ds \]
(2.1)
is a standard Brownian motion. We integrate both sides over $[0, 1]$; since
\[ \int_0^1 \left| \frac{1}{r_s} - \frac{r_s}{1-s} \right| ds < \infty \quad \text{a.s.} \]
(2.2)
(see Remark 2.1 (3) below), we may use Fubini’s theorem on the right-hand side to have the identity
\[ \int_0^1 W_t dt = \int_0^1 r_t dt - \int_0^1 ds \left( \frac{1}{r_s} - \frac{r_s}{1-s} \right) \int_s^1 dt, \]
which entails
\[ \frac{1}{2} \int_0^1 W_t dt = \int_0^1 r_t dt - \frac{1}{2} \int_0^1 \frac{1-t}{r_s} dt. \]
(2.3)
Note that the left-hand side is a centered Gaussian random variable with variance $1/12$.

As to the right-hand side, there holds the identity in law
\[ \left( \int_0^1 r_t dt, \frac{1}{2} \int_0^1 \frac{1-t}{r_t} dt \right) \overset{(d)}{=} \left( \frac{1}{2} \int_0^\infty (l_x)^2 dx, \int_0^1 r_t dt \right). \]
(2.4)
Indeed, by Jeulin’s identity (1.2), the left-hand side of (2.4) is identical in law with
\[ \left( \frac{1}{2} \int_0^1 l_{H^{-1}(t)} dt, \int_0^1 \frac{1-t}{l_{H^{-1}(t)}} dt \right), \]
which is equal, by changing variables with \( t = H(x) \), \( x \geq 0 \), to

\[
\left( \frac{1}{2} \int_0^{\infty} l_x H'(x) \, dx \right) \frac{1}{l_x} H'(x) \, dx \\
= \left( \frac{1}{2} \int_0^{\infty} (l_x)^2 \, dx \, \int_0^1 dt \, 1_{\{r_t > x\}} \right) \\
= \left( \frac{1}{2} \int_0^{\infty} l_x^2 \, dx \, \int_0^1 r_t \, dt \right),
\]

where the second line follows from the definition (1.1) of \( H \) and the third from Fubini’s theorem. Therefore combining (2.3) and (2.4) yields

\[
\frac{1}{2} \int_0^1 W_t dt = \frac{1}{2} \int_0^{\infty} l_x^2 \, dx - \int_0^1 r_t \, dt
\]

and concludes the proof. \( \square \)

We give a remark on the proof. In what follows we denote \( M(r) = \max_{0 \leq t \leq 1} r_t \).

**Remark 2.1.** (1) We see from (1.1) that a.s.,

\[
\int_0^{\infty} l_y \, dy = \int_0^{M(r)} l_y \, dy = 1.
\]

Therefore, to be more specific, the second integral in (2.5) should be written as

\[
\int_0^{M(r)} \frac{1 - H(x)}{l_x} H'(x) \, dx.
\]

(2) By the time-reversal \( \{ r_{1-t} \}_{0 \leq t \leq 1} \overset{(d)}{=} \{ r_t \}_{0 \leq t \leq 1} \), two-dimensional random variables in (2.4) are also identical in law with

\[
\left( \int_0^1 r_t \, dt, \frac{1}{2} \int_0^1 \frac{t}{r_t} \, dt \right)
\]

These identities in law indicate in particular that the following four random variables have the same law:

\[
\int_0^1 r_t \, dt, \quad \frac{1}{2} \int_0^{\infty} (l_x)^2 \, dx, \quad \frac{1}{2} \int_0^1 \frac{1 - t}{r_t} \, dt, \quad \frac{1}{2} \int_0^1 r_t \, dt.
\]

As to the equivalence in law between the first two random variables, see [2, Theorem 2.1] for its generalization involving an independent uniform random variable on \((0,1)\). The Laplace transform of the law of \( \int_0^1 r_t \, dt \) is given in [4, Lemma 4.2] and [1, Proposition (5.5)] in terms of a series expansion.

(3) The a.s. finiteness (2.2) may be deduced from the proof of [5, Théorème (6,40) b)]. For the reader’s convenience, we give a proof of (2.2) here, which will be done in a stronger statement that

\[
\mathbb{E} \left[ \int_0^1 \frac{ds}{r_s} \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ \int_0^1 \frac{r_s}{1 - s} \, ds \right] < \infty.
\]

To this end, recall the identity in law:

\[
\{(r_t)^2\}_{0 \leq t \leq 1} \overset{(d)}{=} \left\{ (b^1_t)^2 + (b^2_t)^2 + (b^3_t)^2 \right\}_{0 \leq t \leq 1},
\]

[http://www.imstat.org/ecp/](http://www.imstat.org/ecp/)
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where \( b^i = \{b^i_t\}_{0 \leq t \leq 1}, i = 1, 2, 3, \) are independent, standard Brownian bridges. Noting that \( a^{-1} = \int_0^\infty \left( e^{-\lambda a^2} / \sqrt{\pi \lambda} \right) d\lambda \) for any \( a > 0 \), we have by Fubini’s theorem and independence of \( b^i \)'s,

\[
E \left[ \frac{1}{r_s} \right] = \int_0^\infty \frac{d\lambda}{\sqrt{\pi \lambda}} \prod_{i=1}^3 E \left[ \exp \left\{ -\lambda (b^i_s)^2 \right\} \right] \\
= \int_0^\infty \frac{d\lambda}{\sqrt{\pi \lambda}} \left( \frac{1}{\sqrt{2\lambda s(1-s)}} + 1 \right)^3 \\
= \sqrt{\frac{2}{\pi s(1-s)}},
\]

for every \( 0 < s < 1 \), where the last line may be seen by changing variables with \( 2\lambda s(1-s) = \tan^2 \theta, 0 < \theta < \pi/2 \). Therefore by Fubini’s theorem,

\[
E \left[ \int_0^1 ds \frac{1}{r_s} \right] = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \sqrt{2\pi}
\]

(cf. [7, Chapter XI, Exercise (3.9)]). In particular, we have the former finiteness in (2.6). As to the latter, we argue along the same lines as in the proof of [5, Proposition (6.37)] to see, by Schwarz’s inequality and (2.7), that

\[
E \left[ r_s \right] \leq \sqrt{3s(1-s)}
\]

for any \( 0 \leq s \leq 1 \), and hence by Fubini’s theorem,

\[
E \left[ \int_0^1 \frac{r_s}{1-s} ds \right] \leq \int_0^1 \sqrt{\frac{3s}{1-s}} ds,
\]

which is finite as claimed.

Using the same reasoning as the proof of Theorem 1.1, we may extend Theorem 1.1 to

**Proposition 2.2.** For every positive integer \( n \), the random variable

\[
2 \int_{[0,1]^n} \min \{r_1, \ldots, r_n\} \ dt_1 \cdots dt_n - \frac{n+1}{2} \int_0^\infty (1-H(x))^{n-1} (L_x)^2 dx
\]

has the Gaussian distribution with mean zero and variance \( 1/(2n+1) \).

**Proof.** For each fixed \( n \), we multiply both sides of (2.1) by \( (1-t)^{n-1} \) and integrate them over \([0,1]\). Then using Fubini’s theorem, we obtain

\[
\int_0^1 (1-t)^{n-1} W_t \ dt = \frac{n+1}{n} \int_0^1 (1-t)^{n-1} r_t \ dt - \frac{1}{n} \int_0^1 \frac{(1-t)^n}{r_t} \ dt.
\]

Since the left-hand side may be expressed as \( (1/n) \int_0^1 (1-t)^n dW_t \), we see that it is a centered Gaussian random variable with variance

\[
\frac{1}{n^2} \int_0^1 (1-t)^{2n} \ dt = \frac{1}{n^2(2n+1)}.
\]
On the other hand, for the right-hand side of (2.8), we have

\[
\left( \int_0^1 (1-t)^{n-1} r_t \, dt, \int_0^1 \frac{(1-t)^n}{r_t} \, dt \right) \overset{\text{d}}{=} \left( \frac{1}{2} \int_0^\infty (1-H(x))^{n-1} (l_x)^2 \, dx, 2 \int_{[0,1]^n} \min \{r_{t_1}, \ldots, r_{t_n} \} \, dt_1 \cdots dt_n \right). \tag{2.9}
\]

Indeed, Jeulin’s identity (1.2) entails that the left-hand side of (2.9) has the same law as

\[
\left( \frac{1}{2} \int_0^1 (1-t)^{n-1} l_{H^{-1}(t)} \, dt, 2 \int_0^1 (1-t)^n \, dt \right) = \left( \frac{1}{2} \int_0^\infty (1-H(x))^{n-1} (l_x)^2 \, dx, 2 \int_0^{M(r)} (1-H(x))^n \, dx \right).
\]

By (1.1), we may rewrite the integral in the second coordinate of the last expression as

\[
\int_0^{M(r)} dx \left( \int_0^1 dt \, 1_{\{r_t > x\}} \right)^n = \int_0^{M(r)} dx \int_{[0,1]^n} dt_1 \cdots dt_n \prod_{i=1}^n 1_{\{r_{t_i} > x\}}
\]

\[
= \int_{[0,1]^n} \min \{r_{t_1}, \ldots, r_{t_n} \} \, dt_1 \cdots dt_n,
\]

where we used Fubini’s theorem for the second equality. Therefore we obtain (2.9). Combining (2.8) and (2.9) leads to the conclusion.

We end this paper with a comment on a relevant fact which is deduced from the proof of Proposition 2.2 and which, as far as we know, has not ever been pointed out.

**Remark 2.3.** It is known (see, e.g., [1, Equation (5d)]) that

\[
M(r) \overset{\text{d}}{=} \frac{1}{2} \int_0^1 \frac{dt}{r_t};
\]

indeed, Jeulin’s identity (1.2) entails that

\[
\frac{1}{2} \int_0^1 \frac{dt}{r_t} \overset{\text{d}}{=} \int_0^{M(r)} \frac{1}{r_x} \times l_x \, dx = M(r).
\]

Combining this fact with a part of the proof of Proposition 2.2, one sees that the sequence \( \{X_n\}_{n=0}^\infty \) of random variables given by

\[
X_0 = M(r) \quad \text{and} \quad X_n = \int_{[0,1]^n} \min \{r_{t_1}, \ldots, r_{t_n} \} \, dt_1 \cdots dt_n \quad \text{for } n \geq 1,
\]

is identical in law with

\[
\frac{1}{2} \int_0^1 \frac{(1-t)^n}{r_t} \, dt, \quad n = 0, 1, 2, \ldots,
\]

as well as with

\[
\frac{1}{2} \int_0^1 \frac{t^n}{r_t} \, dt, \quad n = 0, 1, 2, \ldots
\]

by the time-reversal. As an application, one finds that

\[
\int_{[0,1]^2} |r_{t_1} - r_{t_2}| \, dt_1 \, dt_2 = 2 (X_1 - X_2)
\]

\[
\overset{\text{d}}{=} \int_0^1 \frac{t(1-t)}{r_t} \, dt.
\]
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