Multiple D0-branes in Weakly Curved Backgrounds

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Abstract

We investigate further our recent proposal for the form of the matrix theory action in weak background fields. Using Seiberg’s scaling argument we relate the matrix theory action to a low-energy system of many D0-branes in an arbitrary but weak NS-NS and R-R background. The resulting multiple D0-brane action agrees with the known Born-Infeld action in the case of a single brane and gives an explicit formulation of many additional terms which appear in the multiple brane action. The linear coupling to an arbitrary background metric satisfies the nontrivial consistency condition suggested by Douglas that the masses of off-diagonal fields are given by the geodesic distance between the corresponding pair of D0-branes. This agreement arises from combinatorial factors which depend upon the symmetrized trace ordering prescription found earlier for higher moments of the matrix theory stress-energy tensor. We study the effect of a weak background metric on two graviton interactions and find that our formalism agrees with the results expected from supergravity. The results presented here can be T-dualized to give explicit formulae for the operators in any D-brane world-volume theory which couple linearly to bulk gravitational fields and their derivatives.

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1 Introduction

Over the past few years our understanding of string theory has developed considerably. We now know that the five superstring theories as well as low-energy 11-dimensional supergravity are related through an intricate series of dualities and it has been argued that all these theories are limits of an underlying 11-dimensional theory called “M-theory” \([1]\) whose microscopic description is as yet unknown. It has been found that in addition to one-dimensional stringlike excitations there are higher-dimensional branes in each of these theories which may in some regimes be considered to be just as fundamental as the strings. In the five superstring theories there are D-branes of various dimensions \([2]\) as well as the fundamental string and NS5-branes. In M-theory there are M2-branes and M5-branes which are related to the branes of the superstring theories through various duality transformations.

A fundamental class of problems is the identification of the world-volume action for the various branes appearing in the six theories of interest. This problem can be posed in a number of contexts of differing degrees of complexity. The simplest problem is to find the low-energy action for a single brane in a flat background metric with no nontrivial background supergravity fields. A more difficult problem is to find the action for a single brane in an arbitrary background metric and field configuration which satisfies the supergravity equations of motion. The problem can be made still harder by considering systems of many branes, either in a flat or general background. Even for a single fundamental superstring the action in a general background including arbitrary R-R fields is not yet well understood; for recent work in this direction see, for example, \([3]\) and references therein. For single D-branes the situation is somewhat better. The action for a single D-brane moving in a general background is the Born-Infeld action \([4]\), which reduces to the maximally supersymmetric \(U(1)\) super Yang-Mills theory on the world volume in the case where the brane is almost flat and has only low-energy excitations. This action is supplemented by Chern-Simons type couplings to background R-R fields \([5, 6]\). Even for the single D-brane, however, there are subtle issues involved in giving a world-volume supersymmetric description of the Born-Infeld theory. For systems of \(N\) D-branes, it is known that the low-energy action for parallel branes in a flat background is given by the supersymmetric \(U(N)\) super Yang-Mills theory found by dimensional reduction from 10D \([7]\). So far there has been little progress in describing the action governing systems of many D-branes in a general background. This problem is due in part to the absence of a nonabelian generalization of the Born-Infeld action (although one proposal for such an action was made in \([8]\)), and in part to ordering problems which arise even in the low-energy theory in the presence of general backgrounds.

In this paper we consider the simplest system of many D-branes in a general background: low-energy configurations of many D0-branes moving in an arbitrary but weak background of type IIA supergravity. According to an argument of Seiberg \([9]\) (see also \([10]\)), the action for such a system of D0-branes should be related to the DLCQ description of M-theory in an associated 11-dimensional supergravity background. This generalizes the BFSS matrix theory conjecture \([11, 12]\), which states that supersymmetric matrix quantum mechanics
(the low-energy theory of $N$ D0-branes in flat space) gives a light-front description of M-theory in a flat background. In a previous paper [13] we used a matrix theory formulation of the multipole moments of the supercurrent components in 11D supergravity (derived in [14, 13, 13]) to propose an explicit description of the matrix theory action up to terms linear in the background fields, as well as an algorithm for using higher-loop calculations in matrix theory to find the higher order terms in the matrix theory action in general backgrounds. In this paper, we use our proposal for the general background Matrix theory action and follow the arguments of Seiberg to deduce the leading terms in the action for multiple D0-branes in weak type IIA supergravity backgrounds. We then perform some simple tests of the Matrix theory action and the related multiple D0-brane action. In the D0-brane case, we show that our prescription satisfies a constraint originally suggested by Douglas [16] that the masses of off-diagonal matrix elements between a pair of separated D0-branes agree with the minimal geodesic length between the D0-branes. This property holds also in the Matrix theory case where the separated objects are a pair of gravitons, and we use it to show that the leading order potential between a pair of gravitons in a weakly curved Ricci-flat background is correctly reproduced by the proposed general background Matrix theory action.

The paper is organized as follows. In Section 2 we review our proposal for the linear terms in the general background Matrix theory action. Then, using this action, we follow the arguments of Seiberg to deduce leading terms in the action for multiple D0-branes in an arbitrary weak type IIA supergravity background. In section 3 we describe tests of the IIA and matrix theory actions. We conclude in section 4 with a discussion of related issues and comments on further directions.

2 Linear coupling to backgrounds

In subsection 2.1 we recall the proposal made in [13] for the terms in the action of matrix theory which are linear in the background fields. In subsection 2.2 we use the approach of Seiberg to relate this matrix theory action to an action for multiple D0-branes in IIA background fields. This allows us to deduce the leading terms in the multiple D0-brane action, which to the best of our knowledge have not been previously described.

2.1 Backgrounds in matrix theory

In [13] we proposed that the linear effects of a general matrix theory background with metric $g_{IJ} = \eta_{IJ} + h_{IJ}$ and 3-form field $A_{IJK}$ could be described by supplementing the flat space matrix theory action

$$S_{\text{flat}} = -\frac{1}{2R} \int d\tau \left\{ -D_x X_i D_x X_i + \frac{1}{2} [X_i, X_j][X_i, X_j] + \Theta_a D_x \Theta_\alpha - \Theta_\alpha \gamma^a_{\alpha\beta}[X_i, \Theta_\beta] \right\}$$

(1)

with additional terms of the form

$$S_{\text{weak}} = \int d\tau \sum_{n=0}^{\infty} \sum \frac{1}{n!} \left( \frac{1}{2} T^{IJK(i_1 \ldots i_n)} \partial_{i_1} \cdots \partial_{i_n} h_{IJ}(0) + J^{IJK(i_1 \ldots i_n)} \partial_{i_1} \cdots \partial_{i_n} A_{IJK}(0) \right)$$
\[ + M^{IJKLMN(i_1 \cdots i_n)} \partial_{i_1} \cdots \partial_{i_n} A^D_{IJKLMN}(0) + \text{fermion terms} \]  

(2)

where \( A^D \) is the dual 6-form field which satisfies at linear order

\[ dA^D = * (dA). \]

In (3) the matrix expressions \( T^{IJ(i_1 \cdots i_n)}, J^{JK(i_1 \cdots i_n)}, M^{IJKLMN(i_1 \cdots i_n)} \) are the matrix theory forms of the multipole moments of the stress-energy tensor, membrane current and 5-brane current of 11D supergravity. Explicit forms for the parts of these moments depending only on the 9 bosonic transverse matrices \( X^i \) were given in [15], and the terms quadratic in the fermions were given in [16], as well as some terms quartic in the fermions. For example, the zeroth moments of the components of the stress-energy tensor are given by

\[
T^{++} = \frac{1}{R} \text{STr} \left( \mathbb{1} \right) \\
T^{+i} = \frac{1}{R} \text{STr} \left( D_i X^i \right) \\
T^{+-} = \frac{1}{R} \text{STr} \left( \frac{1}{2} D_i X^i D_t X_t + \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \Theta \gamma^i [X^i, \Theta] \right) \\
T^{ij} = \frac{1}{R} \text{STr} \left( D_t X_t D_t X_j + F_{ik} F_{kj} - \frac{1}{4} \Theta \gamma^i [X^i, \Theta] - \frac{1}{4} \Theta \gamma^j [X^j, \Theta] \right) \\
T^{-i} = \frac{1}{R} \text{STr} \left( \frac{1}{2} D_t X_t D_t X_j D_t X_j + \frac{1}{4} D_t X_t F_{jk} F_{jk} + F_{ij} F_{jk} D_t X_k \right) \\
T_{ij} = \frac{1}{4R} \text{STr} \left( F_{ab} F_{bc} F_{cd} F_{da} - \frac{1}{4} F_{ab} F_{cd} F_{de} F_{ef} + \Theta \Gamma^b \Gamma^c \Gamma^d F_{ab} F_{cd} D_\Theta + O(\Theta^4) \right)
\]

(3)

where \( \text{STr} \) indicates a trace which is symmetrized over all orderings of the forms \( F_{ab}, \Theta \) and \( [X^i, \Theta] \), indices \( i(a) \) run from 1 (0) through 9, and we have defined \( F_{0i} = D_t X^i, F_{ij} = i[X^i, X^j] \). There are two types of terms which contribute to higher moments of these components of the stress-energy tensor

\[ T^{IJ(i_1 i_2 \cdots i_n)} = \text{Sym} \left( T^{IJ, X^{i_1}, X^{i_2}, \ldots, X^{i_n}} \right) + T^{IJ(i_1 i_2 \cdots i_n)}_{\text{fermion}} \]

(4)

The contributions \( \text{Sym} \left( \text{STr} \left( Y \right); X^{i_1}, \ldots, X^{i_n} \right) \) are defined as the symmetrized average over all possible orderings when the matrices \( X^{i_n} \) are inserted into the trace of any product \( Y \) of the forms \( F_{ab}, \Theta, [X^i, \Theta] \). In general there are additional fermionic contributions of arbitrary order to the higher multipole moments, of which the simplest example is the spin contribution to the angular momentum

\[ T^{ij}_{\text{fermion}} = \frac{1}{8R} \text{Tr} \left( \Theta \gamma^{[ij]} \Theta \right) \]

(5)
The precise form of these fermionic contributions will not be important to us in this paper, for reasons which will be discussed in section 3.1.3.

The results of \cite{13} for the matrix membrane and 5-brane currents are reproduced in the Appendix for convenience. With these definitions, (2) gives a formulation of matrix theory in a weak background metric to first order in the metric \( h_{IJ} \), the 3-form \( A_{IJK} \), and all their higher derivatives. It was argued in \cite{13} that if the matrix theory conjecture is true in flat space, this formulation must be correct at least to order \( \partial^4 h, \partial^4 A \) for a class of backgrounds which can be produced as the long range fields around supergravity sources described by matrix theory objects. We conjectured further that this form may work to all orders and in a general background. It should be emphasized, however, that this formulation can only be given for M-theory backgrounds with a global \( U(1) \) symmetry around a compact direction, as we do not know how to incorporate dependence of the background on the compact coordinate \( x^- \). We only expect this action to be part of a consistent all-orders matrix theory action in a general background when the background satisfies the equations of motion. The derivation of this action also depended upon a particular choice of gauge for the graviton, so that it may be necessary to restrict attention to background fields satisfying the linearized gauge

\[
\partial^I h_{IJ} = \partial^I (h_{IJ} - \frac{1}{2} \eta_{IJ} h^K_K) = 0. \tag{6}
\]

2.2 Backgrounds for D0-branes

We now investigate how the Matrix theory action described in the previous section is related to the action for multiple D0-branes in background type IIA supergravity fields.

In the case of a flat background, the Matrix theory action may be derived by showing an equivalence between the DLCQ limit of M-theory in a flat background with \( N \) units of momentum around the circle and a particular limit of type IIA string theory with \( N \) D0-branes \cite{9}. In this limit, the only remaining degrees of freedom are the lowest energy modes of open strings ending on the \( N \) D0-branes. The dynamics of these modes are in general described by a non-abelian generalization of the Born-Infeld action whose complete form is not known. However, in the appropriate limit of type IIA string theory, most of the terms in this action vanish, and we find that the dynamics of DLCQ M-theory in a flat background are described by an action equivalent to the dimensional reduction of D=10 super Yang-Mills theory to 0+1 dimensions.

The action for Matrix theory with background supergravity fields given in the previous section has been derived completely within the context of Matrix theory. However, in principle, one should be able to apply Seiberg’s arguments to this case also and derive the same action as a limit of the action for D0-branes in type IIA string theory with background supergravity fields. Again, only particular terms in the D-brane action will survive in the appropriate limit, but unlike the flat space case, not even these terms are known except in the case of a single brane. Hence, in the case \( N = 1 \), we should be able to rederive our result from previously known facts about D-branes, but more importantly, we will be able
to apply the arguments in reverse for \( N > 1 \) to derive previously unknown leading terms in the action for multiple D0-branes in an arbitrary weak type IIA supergravity background. Using T-duality, our result may be extended to give leading terms in the actions for all other types of D-branes.

### 2.2.1 Relationship between DLCQ and type IIA backgrounds

We now review the steps taken in [9] as they apply in the case of weak backgrounds to make precise the relationship between the matrix theory action and the multiple D0-brane actions. In particular, we must determine the relationship between the D=11 supergravity fields appearing in the Matrix theory action (2) and the type IIA supergravity fields appearing in the related D0-brane action.

We start by considering M-theory with background metric

\[
g_{IJ} = \eta_{IJ} + h_{IJ}
\]

in a frame with a compact coordinate \( x^- \) of size \( R \) which is light-like in the flat space limit \( h_{IJ} = 0 \). This theory can be described as a limit of a family of space-like compactified theories. Defining an \( \hat{M} \)-theory with background metric

\[
\hat{g}_{IJ} = \eta_{IJ} + \hat{h}_{IJ}
\]

in a frame with a spacelike compact direction \( x^{10} \) of size \( R_s \), the DLCQ limit of the original M-theory can be found by boosting the \( \hat{M} \)-theory in the \( x^{10} \) direction with boost parameter

\[
\gamma = \sqrt{\frac{R^2}{2R_s^2} + 1}
\]

and then taking a limit \( R_s \to 0 \). The metric \( \hat{g}_{IJ} \) in the \( \hat{M} \)-theory is related to that of the original M-theory by

\[
\hat{h}_{ij} = h_{ij}, \quad \hat{h}_{0i} = \frac{\alpha}{\sqrt{2}} h_+ i + \frac{1}{\alpha \sqrt{2}} h_- i, \\
\hat{h}_{10i} = \frac{\alpha}{\sqrt{2}} h_+ i - \frac{1}{\alpha \sqrt{2}} h_- i, \quad \hat{h}_{00} = h_{++} + \frac{\alpha^2}{2} h_{++} + \frac{1}{2\alpha^2} h_{--}, \\
\hat{h}_{1010} = -h_{--} + \frac{\alpha^2}{2} h_{++} + \frac{1}{2\alpha^2} h_{--}, \quad \hat{h}_{010} = \frac{\alpha^2}{2} h_{++} - \frac{1}{2\alpha^2} h_{--}
\]
where we have defined

$$\alpha = \gamma (1 - v) = \gamma - \sqrt{\gamma^2 - 1} = \frac{R_s}{R\sqrt{2}} + \mathcal{O}((R_s/R)^3)$$

M-theory on a small spacelike circle of radius $R_s$ is equivalent to type IIA string theory with background fields given to leading order by:

$$h_{\mu\nu}^{IIA} = \hat{h}_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \hat{h}_{1010}$$

$$C_\mu = \hat{h}_{10\mu}$$

$$\phi = \frac{3}{4} \hat{h}_{1010}$$

and parameters

$$g_s = (R_s M_p)^{3/2}, \quad M_s = R_s^{1/2} M_p^{3/2}$$

where $M_p$ is the eleven-dimensional Planck mass. Here we have defined $g_s$ to be a constant and chosen the dilaton $\phi$ so that $\phi = 0$ in the case of a circle of constant size $R_s$ with $h_{1010} = 0$. Thus the effective string coupling is given locally by the combination

$$g_s(\vec{x}) = g_s e^\phi.$$

Combining the two equivalences, we conclude that DLCQ M-theory with $N$ units of momentum on a lightlike circle of size $R$ and background metric $g_{IJ}$ is equivalent to the $R_s \to 0$ limit of type IIA string theory with $N$ D0-branes, parameters

$$g_s = (R_s M_p)^{3/2}, \quad M_s = R_s^{1/2} M_p^{3/2}$$

and background fields

$$h_{00}^{IIA} = \frac{3}{2} h_{-+} + \frac{\alpha^2}{4} h_{++} + \frac{1}{4\alpha^2} h_{--}$$

$$h_{0i}^{IIA} = \frac{\alpha}{\sqrt{2}} h_{+i} + \frac{1}{\alpha\sqrt{2}} h_{-i}$$

$$h_{ij}^{IIA} = h_{ij} + \frac{1}{2} \delta_{ij} (-h_{++} + \frac{\alpha^2}{2} h_{++} + \frac{1}{2\alpha^2} h_{--})$$

$$\phi = -\frac{3}{4} h_{++} + \frac{3\alpha^2}{8} h_{++} + \frac{3}{8\alpha^2} h_{--}$$

$$C_0 = \frac{\alpha^2}{2} h_{++} - \frac{1}{2\alpha^2} h_{--}$$

$$C_i = \frac{\alpha}{\sqrt{2}} h_{+i} - \frac{1}{\alpha\sqrt{2}} h_{-i}$$

(7)
At first glance, such a limit seems problematic. In particular, it appears that for fixed finite values of the DLCQ metric components, the background fields of the equivalent type IIA theory diverge in the limit $R_s \to 0$ since $1/\alpha \to \infty$. However, recall from the flat space case that without a further rescaling of the parameters in the type IIA picture, the energies of the states we are interested in go to 0 like $R_s$. As we shall see, the appropriate rescaling of parameters which makes the energies we are interested in finite without changing the physics also ensures that the apparent divergences of background field components do not lead to divergent terms in the final action.

Another feature of this action is that after the appropriate rescaling the characteristic length scale $L$ associated with the structure of the metric becomes much smaller than the string length $1/M_s$. While this may seem unusual, it is precisely what is needed for the physics of the system to be completely captured by the open string theory describing the D0-brane theory at substring scales studied in \cite{17}. Indeed, for compact manifolds such as tori, it is this effect which makes it possible for the wrapped string modes corresponding to momentum excitations on the dual space to become physically relevant \cite{18,9}.

2.2.2 $N = 1$ actions

We now use the correspondence just discussed to make an explicit comparison between the matrix theory and IIA descriptions of a system of $N$ 0-branes in a weak background field. We begin with the case $N = 1$. Here, both the Matrix theory and D0-brane actions are known, so we would like to check that the Matrix theory action may be derived from the D0-brane action before proceeding to the case $N > 1$ where the D0-brane action is not known. For the case $N = 1$, the Matrix theory action (1,2) reduces to

$$S = \frac{1}{R} \int dt \left\{ \frac{\dot{x}^2}{2} + \frac{1}{2} h_{++}(\vec{x}) + h_{+i}(\vec{x})\dot{x}^i + \frac{1}{2} h_{ij}(\vec{x})\dot{x}^i\dot{x}^j \\
+ \frac{1}{2} h_{++}(\vec{x})\dot{x}^2 + \frac{1}{2} h_{-i}(\vec{x})\dot{x}^2\dot{x}^i + \frac{1}{8} h_{-i}(\vec{x})\dot{x}^4 \right\}. \quad (8)$$

In this case we expect the action to describe a single graviton in curved space with unit momentum along the lightlike circle. Such an action was derived from supergravity in \cite{14}; expression (8) is indeed identical to the supergravity result given by equation (13) in that paper.

The world-volume action for a single D0-brane moving in a general type IIA background supergravity fields is given by

$$S_{IIA} = -\tau_0 \int d\xi e^{-\phi} \sqrt{g_{\mu\nu}(dx^\mu/d\xi)(dx^\nu/d\xi)} + \tau_0 \int C_\mu dx^\mu \quad (9)$$

where $\phi$, $g_{\mu\nu}$, and $C_\mu$ are the background dilaton, metric, and R-R one-form fields, and the parameter $\tau_0$ is the D0 mass, given by

$$\tau_0 = \frac{M_s}{g_s}.$$
One can also consider background R-R three form $C_{\mu\nu\lambda}$ and NS-NS antisymmetric tensor $B_{\mu\nu}$ fields, but these do not couple to a single zero-brane.

According to the equivalence presented in the previous section, the Matrix theory action (8) should arise from the D0-brane action (9) by rewriting the type IIA background fields in terms of the desired DLCQ supergravity background using the relations (7), then rescaling parameters and taking the limit $R_s \to 0$. We will now verify this explicitly. Choosing a gauge in which the coordinate time $x^0$ is identified with the worldvolume time $\xi$ we first expand the D0-brane action to leading order in the background fields, giving

$$S = -\tau_0 \int d\xi \left \{ (1 - v^2)^{1/2}(1 - \phi) - \frac{1}{2}(1 - v^2)^{-1/2}(h_{00}^{IIA} + 2h_{0i}v^i + h_{ij}v^i v^j) - C_0 - C_i v^i \right \}$$

(10)

where $v^i \equiv \dot{x}^i$. We now write the IIA background fields in terms of the background fields in the equivalent DLCQ M-theory using (7). Keeping only the leading term in $R_s/R$ for each of the components of the metric $h_{IJ}$, we find

$$S = \frac{1}{R_s} \int d\xi \left \{ -1 + \frac{1}{2}\frac{R_s^2}{R} h_{++}(\vec{x}) + \frac{R_s}{R} h_{++}(\vec{x}) v^i + \frac{1}{2} h_{++}(\vec{x}) v^i v^i + \frac{1}{2} h_{++}(\vec{x}) v^2 + \frac{1}{2} \frac{R_s^2}{R_0} h_{--}(\vec{x}) v^2 v^i + \frac{1}{8} \frac{R_s^2}{R_0^2} h_{--}(\vec{x}) v^4 \right \}. $$

Many of these terms seem to diverge in the $R_s \to 0$ limit we are interested in. However, as mentioned above, this scaling is deceptive, since we must rescale parameters in the theory so that the energies of the states we are interested in remain finite rather than going to zero in the limit. Indeed, from the fact that the conjugate momentum has a leading term of order $v/R_s$ it can be seen that all the terms in (11) which are linear in the background contribute to the Hamiltonian at order $R_s$. Thus, as we need for the Seiberg limit, the energy of the states of interest scale as $R_s$.

We may now perform the rescaling of (11) by replacing

$$R \to \left(\frac{R_s}{R}\right)^{1/2} R, \quad \vec{x} \to \left(\frac{R_s}{R}\right)^{1/2} \vec{x}, \quad h(\vec{x}) \to h(\vec{x}).$$

Note that the change of variables in the second replacement combines with the rescaling of dimensionful coefficients in the expansion of $h$ to leave $h(\vec{x})$ unchanged, as suggested by the final replacement.

With these redefinitions the action (11) becomes

$$S = \int d\xi \left \{ -\frac{1}{R_s} + \frac{1}{R} \left( \frac{1}{2} h_{++}(\vec{x}) + h_{++}(\vec{x}) v^i + \frac{1}{2} h_{ij}(\vec{x}) v^i v^j \right) + \frac{1}{2} h_{--}(\vec{x}) v^2 + \frac{v^2}{2} + \frac{1}{2} \frac{R_s^2}{R_0} h_{--}(\vec{x}) v^2 v^i + \frac{1}{8} \frac{R_s^2}{R_0^2} h_{--}(\vec{x}) v^4 \right \}. $$

The first term is divergent in the $R_s \to 0$ limit and arises from the BPS energy of the single 0-brane; this term also appears in the flat space theory and is discounted in the matrix theory.
limit. Dropping this term gives precisely the matrix theory action described by (5) in the 
\( N = 1 \) case. Thus, we have shown that the known Born-Infeld action for a single D0-brane 
correctly reproduces the matrix theory action in a weak background when the proper limit 
is taken.

2.2.3 \( N > 1 \) actions

We now turn to the case \( N > 1 \). Here, the appropriate action for multiple D0-branes is not 
known, but by requiring that it reproduces the general background Matrix theory action in 
the Seiberg limit, we will be able to deduce its leading terms.

We first write down the D0-brane action in terms of the unknown quantities coupling to 
the background fields. We define quantities \( I_x \) coupling linearly to each of the background 
fields, so that to leading order in the background fields, the action for \( N \) D0 branes is

\[
S = S_{\text{flat}} + \int dt \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{1}{2} \left( \partial_{k_1} \cdots \partial_{k_n} h^{I A}_{\mu \nu} \right) I^{\mu(k_1 \cdots k_n)}_{I A} + \left( \partial_{k_1} \cdots \partial_{k_n} \phi \right) I^{(k_1 \cdots k_n)}_{\phi} \right. \\
\left. + \left( \partial_{k_1} \cdots \partial_{k_n} C^\mu \right) I^0_{0(k_1 \cdots k_n)} + \left( \partial_{k_1} \cdots \partial_{k_n} \tilde{C}^\mu_{\lambda \rho \sigma \tau \zeta} \right) I^1_{\mu \lambda \rho \sigma \tau \zeta(k_1 \cdots k_n)} \right] \\
+ \left( \partial_{k_1} \cdots \partial_{k_n} B_{\mu \nu} \right) I^2_{\mu \nu(k_1 \cdots k_n)} + \left( \partial_{k_1} \cdots \partial_{k_n} \tilde{B}_{\mu \lambda \rho \sigma \tau} \right) I^3_{\mu \lambda \rho \sigma \tau(k_1 \cdots k_n)} \\
+ \left( \partial_{k_1} \cdots \partial_{k_n} C^{(3)} \right) I^4_{2 \mu \lambda(k_1 \cdots k_n)} + \left( \partial_{k_1} \cdots \partial_{k_n} \tilde{C}^{(3)} \right) I^5_{4 \mu \lambda \rho \sigma \tau \zeta(k_1 \cdots k_n)} \right].
\]

Here, \( S_{\text{flat}} \) is the flat space action for \( N \) D0-branes, whose leading terms are the dimensional 
reduction of D=10 SYM theory to 0+1 dimensions. The complete form of the higher order 
terms in the flat space action is not known, but these terms vanish in the Matrix theory 
limit. We assume that the background satisfies the source-free IIA supergravity equations 
of motion so that the dual fields \( \tilde{C}, \tilde{B}, \tilde{C}^{(3)} \) are well-defined 7-, 6- and 5-form fields given (at 
linear order) by

\[
d\tilde{C} = *dC, \quad d\tilde{B} = *dB, \quad d\tilde{C}^{(3)} = *dC^{(3)}.
\]

The sources \( I_{2n} \) are associated with Dirichlet \( 2n \)-brane currents, while the sources \( I_s \) and 
\( I_5 \) are associated with fundamental string and NS5-brane currents respectively. It may seem 
surprising that a system of D0-branes can give rise to a nonzero D2-brane, D4-brane or 
D6-brane charge. Indeed, the integrated higher brane charges must vanish for a system 
containing a finite number \( N \) of D0-branes. Even for \( N = 2 \), however, a D0-brane configuration 
can have nonvanishing multipole moments of higher D-brane charges. This essentially 
arises as the T-dual of the result that the \( n \)th power of the curvature form \( F \) on a Dirichlet 
Dp-brane carries \( (p - 2n) \)-brane charge \([13, 3]\); see \([20]\) and references therein for a further 
discussion of this issue.

The problem we address in this subsection is the determination of the IIA currents \( I_x \) 
under the assumption that this action reproduces the matrix theory action (2) in the Seiberg 
limit. As we will see, the leading terms in all the currents other than \( I_5 \) can be determined 
and are related to the matrix theory supercurrent components tabulated in the Appendix.
For the case $N = 1$ we see from (10) that the nonvanishing source components $I_x$ are

\begin{align*}
I^{00(k_1 \cdots k_n)}_h &= \frac{1}{R_s} (1 - \dot{x}^2)^{-1/2} \dot{x}^{k_1} \cdots \dot{x}^{k_n} \\
I^{0i(k_1 \cdots k_n)}_h &= \frac{1}{R_s} (1 - \dot{x}^2)^{-1/2} \dot{x}^i \dot{x}^{k_1} \cdots \dot{x}^{k_n} \\
I^{ij(k_1 \cdots k_n)}_h &= \frac{1}{R_s} (1 - \dot{x}^2)^{-1/2} \dot{x}^i \dot{x}^j \dot{x}^{k_1} \cdots \dot{x}^{k_n} \\
I^{\phi(k_1 \cdots k_n)}_0 &= \frac{1}{R_s} (1 - \dot{x}^2)^{1/2} x^{k_1} \cdots x^{k_n} \\
I^0(i(k_1 \cdots k_n)}_0 &= \frac{1}{R_s} \dot{x}^{i k_1} \cdots \dot{x}^{k_n} \\
I^0_{i(k_1 \cdots k_n)} &= \frac{1}{R_s} \dot{x}^{i} \dot{x}^{k_1} \cdots \dot{x}^{k_n}
\end{align*}

(14)

In the nonabelian case $N > 1$, the quantities $I_x$ will be some complicated functions of the $N \times N$ hermitian matrices $X^i$ as well as the fermionic matrices $\Theta$. For each $I$, we can make an expansion analogous to expanding in velocities for the $N = 1$ case. We write

$$I_x = \sum I_{x[n]}$$

where $n$ counts the dimension of a function of the matrices $X, \Theta$, giving $X$ dimension 1, $\dot{X}$ dimension 2 and $\Theta$ dimension 3/2. If we do a similar expansion for the flat space action $S_0$, we find that it is precisely the $n = 4$ terms that remain in the Matrix theory limit, the higher order terms being scaled to zero. In the general background case, the Matrix theory action will arise from terms $I_n$ with $n \leq 8$ (and their higher moments), so it is these terms that our analysis will determine.

We now proceed just as in the $N = 1$ case. We begin by working through the details of the analysis for those terms coupling to the IIA graviton, dilaton and R-R 1-form field. These terms are the most complicated. The analysis for the remaining bosonic fields is described at the end of this section.

By the Seiberg equivalences, the Matrix theory action with background supergravity fields should result from replacing the IIA background fields in (12) with their DLCQ counterparts (13), rescaling parameters as above, and taking the limit $R_s \to 0$. Before rescaling, we find that the D0-brane action becomes

\begin{align*}
S &= S_{\text{flat}} - \frac{1}{2} \int dt \left[ (\sqrt{2}\alpha)^{-2} h_{-\sigma} \left\{ \frac{1}{2} I^{00}_h + \frac{1}{2} I^{ii}_h + \frac{3}{2} I^{\phi}_0 - 2 I^0_0 \right\} \\
&\quad + (\sqrt{2}\alpha)^{-1} h_{-\sigma} \left\{ 2 I^{0i}_h - 2 I^i_0 \right\} \\
&\quad + h_{ij} \left\{ I^{ij}_h \right\} \\
&\quad + h_{+-} \left\{ \frac{3}{2} I^{00}_h - \frac{1}{2} I^{ii}_h - \frac{3}{2} I^{\phi}_0 \right\} \\
&\quad + (\sqrt{2}\alpha) h_{+i} \left\{ I^{0i}_h + I^i_0 \right\} \\
&\quad + (\sqrt{2}\alpha)^2 h_{++} \left\{ \frac{1}{8} I^{00}_h + \frac{1}{8} I^{ii}_h + \frac{3}{8} I^{\phi}_0 + \frac{1}{2} I^0_0 \right\} \\
&\quad + \{ \text{higher moment terms} \} \right]
\end{align*}

(15)

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The higher moment terms have exactly the same form as the terms written, for example the full set of terms linear in \( h_{-i} \) would be

\[
\sum_{n=0}^{\infty} \frac{R}{R_s} \partial_{k_1} \cdots \partial_{k_n} h_{-i} \{ 2f_h^{(k_1 \cdots k_n)} - 2f_0^{(k_1 \cdots k_n)} \}
\]

Because the distance scale associated with the metric is rescaled along with the transverse coordinates in the rescaling of [9], the rescaling of the partial derivatives in these expressions cancel the scaling of the moment indices. Thus, the higher moment terms which remain in the \( R_s \to 0 \) limit are precisely those corresponding to 0th moments which remain in the limit. The only terms in (15) which remain in the limit other than the leading divergent D0-brane energy term should be finite terms corresponding to the matrix theory action (2). All terms which are linear in the background and carry positive powers of \( R/R_s \) in the limit must cancel for the IIA action to agree with the matrix theory action. This gives a number of restrictions on the parts of the IIA currents with particular scaling dimensions. The constraints arising in this fashion for the integrated (monopole) currents are

\[
\begin{align*}
\left( \frac{1}{2} I_{h}^{00} + \frac{1}{2} I_{h}^{ii} + \frac{3}{2} I_{\phi} - 2f_0^{0} \right)_0 &= 0 \\
\left( \frac{1}{2} I_{h}^{00} + \frac{1}{2} I_{h}^{ii} + \frac{3}{2} I_{\phi} - 2f_0^{0} \right)_4 &= 0 \\
\left( \frac{1}{2} I_{h}^{00} + \frac{1}{2} I_{h}^{ii} + \frac{3}{2} I_{\phi} - 2f_0^{0} \right)_8 &= T^{--} \\
(I_{h}^{0i} - I_{h}^{i0})_2 &= 0 \\
(I_{h}^{0i} - I_{h}^{i0})_6 &= T^{-i} \\
(I^{ij})_0 &= 0 \\
(I^{ij})_4 &= T^{ij} \\
\left( \frac{3}{2} I_{h}^{00} - \frac{1}{2} I_{h}^{ii} - \frac{3}{2} I_{\phi} \right)_0 &= 0 \\
\left( \frac{3}{2} I_{h}^{00} - \frac{1}{2} I_{h}^{ii} - \frac{3}{2} I_{\phi} \right)_4 &= 2T^{++} \\
(I_{h}^{0i} + I_{h}^{i0})_2 &= 2T^{+i} \\
\left( \frac{1}{8} I_{h}^{00} + \frac{1}{8} I_{h}^{ii} + \frac{3}{8} I_{\phi} + \frac{1}{2} f_0^{0} \right)_0 &= T^{++}
\end{align*}
\]

We will assume that the degrees at which a given current \( I_x \) has nonvanishing contributions are the same as in the \( N = 1 \) case. These are the terms for which we have explicitly written constraints in (15). This assumption agrees with what we know about the nonabelian Born-Infeld action. If this assumption is incorrect, there may be additional terms appearing in the IIA currents at other orders which do not contribute to the matrix theory action. Identical relations to (15) must hold for the quantities coupling to higher order terms in the Taylor expansion of the metric.

Solving the constraints (15), we find

\[
I_{h}^{00} = T^{++} + T^{+i} + (I_{h}^{00})_8 + \mathcal{O}(v^6)
\]
\[ I_{h}^{0i} = T^{+i} + T^{-i} + \mathcal{O}(v^{5}) \]

\[ I_{h}^{ij} = T^{ij} + (I_{h}^{ij})_{8} + \mathcal{O}(v^{6}) \]

\[ I_{\phi} = T^{++} - \frac{1}{3} T^{+-} - \frac{1}{3} T^{ii} + (I_{\phi})_{8} + \mathcal{O}(v^{6}) \]

\[ I_{0}^{0} = T^{++} \]

\[ I_{0}^{i} = T^{+i} \]

Here, we have assumed that the one-form field components \( C_{0} \) and \( C_{i} \) should couple to the D0-brane charge \( N/R = T^{++} \) and spatial current \( \text{Tr} (\hat{X}^{i})/R = T^{+i} \) so that the last two expressions are exact. The fourth order quantities \( (I_{h}^{00})_{8}, (I_{h}^{ij})_{8}, \) and \( (I_{\phi})_{8} \) are not completely determined by our analysis, but they must obey the relation

\[ \left( \frac{1}{2} I_{h}^{00} + \frac{1}{2} I_{h}^{ii} + \frac{3}{2} I_{\phi} \right)_{8} = T^{--} \]

From the \( N = 1 \) results \( \mathbf{[14]} \) we expect that

\[ (I_{h}^{00})_{8} = \frac{3}{2} T^{--} + (I_{h}^{00})_{8c} \]

\[ (I_{h}^{ii})_{8} = 2 T^{--} + (I_{h}^{ii})_{8c} \]

\[ (I_{\phi})_{8} = -\frac{1}{2} T^{--} + (I_{\phi})_{8c} \]

where \( (I)_{8c} \) are quantities of order \( v^{4} \) which contain commutators or fermions and which vanish in the \( N = 1 \) case of a single spinless graviton considered in the previous subsection.

Additional information about the currents should follow from the conservation of the D0 brane stress-energy tensor \( I_{h}^{\mu\nu}(\vec{x}) \) which is defined in terms of the moments \( I_{h}^{\mu\nu(k_{1},...,k_{n})} \). As discussed in \( \mathbf{[21]} \), the relation \( D_{\mu} I_{h}^{\mu\nu} = 0 \) implies

\[ \partial_{\mu} I_{h}^{\mu(k_{1},...,k_{n})} = I_{h}^{k_{1}\mu(k_{2},...,k_{n})} + \ldots + I_{h}^{k_{n}\mu(k_{1},...,k_{n-1})} \]

In particular, \( (I_{h}^{ij})_{8} \) should be precisely determined by

\[ (I_{h}^{ij})_{8} = (\partial_{t} T^{+i(j)} + \partial_{t} T^{-i(j)})_{8}. \]

So far, we have only dealt with the case of a background metric \( h \), dilaton field \( \phi \) and R-R 1-form field \( C \). The same sort of analysis can be applied for backgrounds having nonvanishing 2-form \( B \) or 3-form \( C^{(3)} \) fields and their duals, and in fact the analysis is simpler in these cases. In order to describe nontrivial background antisymmetric tensor fields, we must generalize the relations \( \mathbf{[7]} \) which connect the IIA background fields to the 11D background 3-form field through the Seiberg limit. The \( B \) and \( C^{(3)} \) fields are related to components of the 3-form field through

\[ B_{0i} = \hat{A}_{100i} = A_{+-i} \]

\[ B_{ij} = \hat{A}_{10ij} = \frac{\alpha}{\sqrt{2}} A_{+ij} - \frac{1}{\alpha \sqrt{2}} A_{-ij} \]
\[ C^{(3)}_{0ij} = \hat{A}_{0ij} = \frac{\alpha}{\alpha \sqrt{2}} A_{+ij} + \frac{1}{\alpha \sqrt{2}} A_{-ij} \]
\[ C^{(3)}_{ijk} = \hat{A}_{ijk} = A_{ijk} \]

The constraints on the string and D2-brane currents analogous to (16) are then
\[(I^{0i})_0 = (I^{0i})_0 = (3I^{0ij}_2 - I^{ij}_2) = 0 \]
\[(I^{0i})_4 = 3J^{+-i} \]
\[(I^{ijk})_4 = J^{ijk} \]
\[(3I^{0ij}_2 + I^{ij}_2)_2 = 6J^{+ij} \]
\[(3I^{0ij}_2 - I^{ij}_2)_6 = 3J^{-ij} \]

from which we can determine
\[ I^{0i}_s = 3J^{+-i} + O(v^4) \]
\[ I^{ijk}_2 = J^{ijk} + O(v^4) \]
\[ I^{0ij}_2 = J^{+ij} + (I^{0ij}_0)_6 + O(v^5) \]
\[ I^{ij}_s = 3J^{+ij} + (I^{ij}_0)_6 + O(v^5) \]

where the terms \((I)_6\) on the last two lines must satisfy the final relation in (18). In addition, conservation relations for \(I_s\) suggest that
\[ \partial_t I^{0i(j)}_s = I^{j}_s = -I^{ij}_s \]
from which it follows that
\[ (I^{ij}_s)_6 = -3\partial_t J^{+-i(j)} = -3J^{-ij}, \quad (I^{0ij}_2)_6 = 0 \]

There are a number of comments worth making about the identifications (19). First, note that the factor of 3 appearing in the currents \(I_s\) arises from our somewhat unconventional choice of normalization for the couplings \(A_p J^p\) between a \(p\)-form field and its associated current. Often, a factor of \(1/p!\) is included in the definition of this coupling. With that redefinition of the currents, the factors of 3 in our relations would disappear. We have chosen our convention for the coupling to conform with previous literature on the subject.

Next, we briefly discuss the physical interpretation of the leading terms in (19). The leading term \(J^{+ij}\) in \(I^{0ij}_2\) is the total membrane charge of the D0-brane system. This result is the T-dual of the statement that \(\int F\) on a \(p\)-brane is the total \((p - 2)\)-brane charge coupling to the \((p - 1)\)-form R-R field [5]. Although one might think that this should be the only contribution to the D2-brane charge of the system, additional contributions may arise from the geometry of the brane embedding [22, 23, 24, 25]. Note that while for finite \(N\) the integrated membrane charge \(J^{+ij} = \text{Tr} [X^i, X^j]\) vanishes identically (since a finite size system can have no net membrane charge) the higher moments of the D2-brane charge can be nonvanishing and will couple to the derivatives of the \(C^{(3)}\) field.
The leading term in $I_s^i$ is the net string winding charge in direction $i$; this is simply the T-dual of the Poynting vector giving momentum on a dual D-brane. The leading term in the current $I_s^j$ is perhaps somewhat surprising. Although this term itself vanishes for finite size matrices, as mentioned above, the first moment is nonvanishing. The existence of this term indicates that there will be a coupling in the multiple D0-brane action of the form

$$\partial_\eta B_{jk} \text{Tr} (X^i X^j X^k \text{+ fermions}).$$

We are rather confused as to the physical origin of this term. Indeed, this term plays a puzzling role in several related situations. For example, after compactification on $T^3$, the term $J^{+ij}$ should be related by a duality transformation to the NS5-brane charge of the IIA theory [26], which we discuss below. It may be possible to understand the role of this term in the theory by studying a T-dual system such as a dual multiple D3-brane configuration. We discuss the connection with the dual theory briefly in the last section of this paper.

Now let us consider the currents coupling to the dual fields $\tilde{B}$ and $\tilde{C}^{(3)}$. These currents can be derived in a fashion precisely analogous to the above argument by considering the fields related to the dual 6-form $\tilde{A}$ of the 11D theory. We find

$$I_{ijkl}^0 = 6M^{+-ijkl} + O(v^4)$$
$$I_{ijklmn}^5 = M^{ijklmn} + O(v^4)$$
$$I_{ijkl}^5 = M^{ijkl} + (I_{ijkl}^0)_6 + O(v^5)$$
$$I_{ijklm}^5 = 6M^{ijklm} + (I_{ijklm}^0)_6 + O(v^5)$$

where

$$6(I_{ijklm}^0)_6 - (I_{ijklm})_6 = 6M^{ijklm}. $$

Just as for $I_s$, conservation relations for $I_4$ suggest that

$$\partial_t I_{ijkl}^0 = I_{ijklm}$$

from which it follows that

$$(I_{ijklm})_6 = 6\partial_t M^{+-ijkl} = -6M^{-ijkl}, \quad (I_{ijklm}^0)_6 = 0. $$

The leading term in $I_{ijkl}^0$ is the net D4-brane charge [27, 28]. This is the dual of the instanton number on a D$p$-brane. Unfortunately, we only know from matrix theory the components of the 5-brane current $M^{-1^{JKLM}}$ with one $-$ index. Thus, we can only determine the leading term in the components $I_{ijkl}^0$ of the D4-brane current, and we have no information about the leading terms in the remaining components of $I_4$ or any components of the NS5-brane current $I_5$. The absence of any known operator for the transverse 5-brane charge $M^{ijklm}$ in matrix theory is an infamous problem. No operator of this form appears in the supersymmetry algebra [27] or in the one-loop effective potential [13]. Nonetheless, we should expect higher moments of this operator to appear, corresponding to local transverse
5-brane charge. It has been argued that in a T-dual 3-brane picture the desired operator is S-dual to the charge of a D5-brane perpendicular to the 3-brane [26] (described by the operator $J^{+ij}$ mentioned above), although no explicit description of this dual operator has been given. It would be nice to have a better understanding of these terms in the multiple D0-brane action.

Finally, we consider the currents coupling to the dual field $\tilde{C}$. We expect the IIA 6-brane current to couple to this field. Unlike the other branes whose currents we have considered, the IIA 6-brane does not arise in a simple fashion from the dimensional reduction of the membrane or the 5-brane of 11D M-theory. Rather, the IIA 6-brane represents a nontrivial metric of Kaluza-Klein monopole form in the 11D theory. Nonetheless, in [13] we found a matrix theory description of interactions between such metrics and 0-branes. This appeared in the form of a term in the 2-body matrix theory potential which coupled the 0-brane stress tensor to a 10D 6-brane current $S^{\mu \nu \rho \sigma \tau \upsilon}$. Since this current already has an essentially 10D form, it is natural to map it directly to the 6-brane current we expect in the IIA theory. Thus, we believe that the 6-brane current of a system of many 0-branes which couples to the background $\tilde{C}$ field will be given by

$$I^{ijklmn}_6 = S^{ijklmn} + O(v^5)$$

$$I^{ijklmp}_6 = S^{ijklmp} + O(v^6)$$

where the matrix theory form of the 6-brane current is given in the Appendix.

### 2.3 Summary of results for multiple D0-brane action in IIA

We summarize here our results for the terms in the multiple D0-brane action which couple linearly to the background fields of type IIA supergravity and their derivatives. The full action including all linear terms is given by (12)

$$S = S_{\text{flat}} + \int \frac{dt}{\sqrt{-g}} \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{1}{2} (\partial_{k1} \cdots \partial_{kn} h_{\mu \nu}^{(IA)}) I^{\mu(k1 \cdots k_n)}_h + (\partial_{k1} \cdots \partial_{kn} \phi) I^{(k1 \cdots k_n)}_\phi \right. $$

$$\left. + (\partial_{k1} \cdots \partial_{kn} C_\mu) I^{\mu(k1 \cdots k_n)}_0 + (\partial_{k1} \cdots \partial_{kn} \tilde{C}_{\mu \nu \lambda \rho \sigma \tau \xi}) I^{\mu \nu \lambda \rho \sigma \tau \xi(k1 \cdots k_n)}_6 \right.$$ \hspace{1cm} (21)

$$\left. + (\partial_{k1} \cdots \partial_{kn} B_{\mu \nu}) I^{\mu \nu(k1 \cdots k_n)}_s + (\partial_{k1} \cdots \partial_{kn} \tilde{B}_{\mu \nu \lambda \rho \sigma \tau}) I^{\mu \nu \lambda \rho \sigma \tau(k1 \cdots k_n)}_5 \right.$$ \hspace{1cm} (19)

$$\left. + (\partial_{k1} \cdots \partial_{kn} C_{(3)}^{(3)}) I^{\mu \nu \lambda(k1 \cdots k_n)}_2 + (\partial_{k1} \cdots \partial_{kn} \tilde{C}_{(3)}^{(3)}) I^{\mu \nu \lambda \rho \sigma \tau(k1 \cdots k_n)}_4 \right.$$ \hspace{1cm} (17)

The multipole moments of the stress tensor $I_h$ and currents coupling to the background dilaton and R-R 1-form field have leading terms given by (22). The currents coupling to the NS-NS antisymmetric 2-form field and the R-R 3-form field have leading terms given by (19). Of the currents coupling to the duals of these two fields, we have only been able to identify leading term in the the moments of the 4-brane current component $I^{ijkl}_4$ as described in (21). We believe that the leading terms in the components of the 6-brane current coupling to the dual of the 1-form field are as given in (22).
We have derived these results based on our proposal in \cite{13} for the form of the matrix theory action in weak background fields and Seiberg’s scaling argument. Our results agree with the known terms in the $N = 1$ Born-Infeld action, and with the known BPS charges of the multiple D0-brane system. In the following section we give a simple test of the results for the terms coupling to the background metric. Further possible tests, applications, and extensions of these results are discussed in the concluding section.

3 Tests of the action

In this section, we test our proposals for the general background actions through two related calculations. First, we consider the D0-brane action in a background describing two separated branes in a curved space ($h_{ij} \neq 0$). We determine the masses of off diagonal bosonic and fermionic fields and show that these exactly match the geodesic distance to leading order in $h$ in agreement with the constraint suggested by Douglas. Next, we consider the analogous background in matrix theory and compute the leading order one-loop potential between two gravitons in a curved transverse space, showing that curved-space supergravity predictions are reproduced.

3.1 The geodesic length criterion

One of the earliest discussions of the problem of formulating a low-energy theory for many D0-branes moving in a curved space was given in \cite{16}. In that paper, Douglas argued that one of the most basic conditions which such an action must satisfy is that in a background corresponding to a pair of D0-branes living at points $x$ and $y$ there should be light fields with masses equal to the geodesic length between these points. This condition, together with additional axiomatic assumptions, was used by Douglas, Kato and Ooguri in \cite{28} to give the first few terms in the D0-brane action on a general Calabi-Yau 3-fold which preserves some supersymmetry. In this section we show that our formulation of the linearized coupling to a weak background in the multi-D0-brane action satisfies Douglas’s geodesic length criterion.

3.1.1 Setup

We wish to consider a pair of D0-branes at separated points in a weakly curved space. We assume that the transverse metric is described by a small perturbation about a flat background, $g_{ij} = \delta_{ij} + h_{ij}$, while the other components of the metric and the other background fields are trivial. Without loss of generality, we choose coordinates so that one brane is at the origin, while the other has transverse coordinates $r^i$. The situation is described by the multi-D0-brane action (\cite{12}) with non-zero $h_{ij}$ and fields expanded about background matrices as

$$X^i = \begin{pmatrix} r^i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \zeta^i \\ \bar{z}^i \\ \bar{\zeta}^i \end{pmatrix}$$

(23)
\[ \Theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \eta^i & \chi^i \\ \bar{\chi}^i & \bar{\eta}^i \end{pmatrix}, \]

Here, the fields \( \zeta, z, \eta, \) and \( \chi \) represent fluctuations about the background. The geodesic length condition formulated by Douglas states that the masses of the off-diagonal fields \( z \) and \( \chi \), which arise from strings stretched between the separated branes, should precisely match the geodesic length measured by the metric \( h_{ij} \) between the points 0 and \( r^i \). We will now compute both the geodesic length and the oscillator masses and show that they agree.

### 3.1.2 Geodesic distance

We begin with the geodesic length between points 0 and \( r^i \). This geodesic length is the minimum value of the length

\[ \int_{\gamma} ds = \int_{0}^{r^i} \sqrt{g_{ij}dx^idx^j} \quad (24) \]

taken over all paths \( \gamma \) between the two points. Because the geodesic path is an extremum of this functional, the variation of the length under a small variation \( \delta \gamma \) of the path is of order \( (\delta \gamma)^2 \). Since we are interested in changes in the length which are linear in the background metric, we can therefore neglect effects from the change of the geodesic path and simply evaluate the change in the geodesic length by integrating (24) along the straight line which is the geodesic in the flat metric. Thus, we take

\[ x^i(\lambda) = r^i \lambda \]

and find

\[ d(0, r^i) = \int_{0}^{1} d\lambda \sqrt{g_{ij}(\bar{x}(\lambda))\dot{x}^i\dot{x}^j} \]

\[ = \int_{0}^{1} d\lambda \sqrt{r^2 + h_{ij}(\bar{r}\lambda)r^i r^j} \]

\[ = \int_{0}^{1} d\lambda \{ r + \frac{1}{2r} h_{ij}(\bar{r}\lambda)r^i r^j + \mathcal{O}(h^2) \} \]

\[ = r + \frac{1}{2r} r^i r^j H_{ij} \]

where

\[ H_{ij} = \int_{0}^{1} d\lambda h_{ij}(\lambda \bar{r}) \]

\[ = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (r \cdot \partial)^n h_{ij}(0) \]

This gives us the geodesic length between the two points to linear order in the background metric. In the following section, it will be most convenient to compare squared oscillator masses with the squared geodesic length, given by

\[ d^2 = r^2 + r^i r^j H_{ij} + \mathcal{O}(h^2) \quad (25) \]
3.1.3 Oscillator masses

We now calculate the masses of the off-diagonal fields. From (12), we find that the $N = 2$ D0-brane action in the case of a transverse background metric has leading terms

$$S = \frac{1}{2R} \int dt \left( \frac{1}{2} [X^i, X^j][X^i, X^j] + i \Theta D_t \Theta - \Theta [\mathcal{X}, \Theta] \right)$$

$$+ \frac{1}{2} \int dt \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{k_1} \cdots \partial_{k_n} h_{ij} T^{ij(k_1 \cdots k_n)}$$

(26)

Here,

$$T^{ij(k_1 \cdots k_n)} = \frac{1}{R} \text{STr} \left( \left( D_t X^i D_t X^j - [X^i, X^k][X^k, X^j] \right) \right.$$ 

$$\left. + \frac{1}{4} \Theta \gamma^j [X^j, \Theta] - \frac{1}{4} \Theta \gamma^j [X^i, \Theta] \right) X^{k_1} \cdots X^{k_n}$$

(27)

We note here that this is precisely the action for Matrix theory in a background $h_{ij}$ to leading order in the metric. In the next section we will use exactly this action to calculate the one-loop potential between two gravitons, taking the same background (23), though allowing $\vec{r}$ to be a function of time. Such a calculation is simplest using a gauge fixed version of the action in which we choose the background field gauge, adding a term

$$S_{fix} = \frac{1}{R} \int (-D_t X^0 + i [B_i, X^i])^2$$

plus the appropriate ghost terms. For later convenience, we will analyze this gauge fixed version, keeping in mind both the Matrix theory interpretation and D0-brane interpretations.

Unlike the Matrix theory action, the complete D0-brane action contains additional terms both in the background independent part and coupled to $h_{ij}$. However these cannot contribute to the quadratic action since they contain more than two matrices (eg $\dot{X}^i$, $F_{ij}$, $\Theta$) in which there are no entries depending only on background fields. Similarly, the terms $\tilde{T}^{ij(k_1 \cdots k_n)}$, whose form has not been worked out for $n > 1$ involve at least two fermions $\Theta$ and one power of $\dot{X}$ or $[X^i, X^j]$ and so contribute only cubic and higher order terms to the action, irrelevant for determining the oscillator masses or computing the one loop potential.

We now replace $X$ and $\Theta$ in the action with the matrices given in (23), and write down the terms in the action quadratic in the off diagonal fields $z$ and $\chi$. It turns out that the symmetrization prescription for ordering the matrices in $T^{ij(k_1 \cdots k_n)}$ is very important here, since most of the orderings give no contribution to the quadratic terms we are interested in. For example, the first term in (27) contains a term

$$\frac{1}{2n!} (\partial_{k_1} \cdots \partial_{k_n} h_{ij}) \text{STr} (\dot{X}^i \dot{X}^j X^{k_1} \cdots X^{k_n})$$

$$= \frac{1}{2n!} (\partial_{k_1} \cdots \partial_{k_n} h_{ij}) \frac{1}{n+1} \sum_{m=0}^{n} \text{Tr} (\dot{X}^i X^{k_1} \cdots X^{k_m} \dot{X}^j X^{k_{m+1}} \cdots X^{k_n})$$
for which only the $m = 0$ and $m = n$ terms contribute to the quadratic action in the off diagonal field $z$. Summing over $n$, this contribution gives

$$
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\partial_{k_1} \ldots \partial_{k_n} h_{ij})(\dot{z}^i \dot{z}^j) r^{k_1} \ldots r^{k_n}
$$

$$
= \dot{z}^i \dot{z}^j \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (r \cdot \partial)^n h_{ij}
$$

$$
\equiv H_{ij} \dot{z}^i \dot{z}^j
$$

Note that the quantity $H_{ij}$ is simply the function $h_{ij}$ integrated over the straight line trajectory between 0 and $\vec{r}$,

$$
H_{ij} = \int_0^1 h_{ij}(\lambda \vec{r}) d\lambda
$$

which also appeared at first order in the geodesic distance formula (25). Using this definition, it is straightforward to write down the remaining terms in the quadratic actions for each of the off diagonal fields.

**Bosonic Terms**

In exactly the same way as for the terms just derived, we find that the complete set of terms to leading order in $h$ for the nine transverse bosonic fields is

$$
S_B = -\dot{z}^i \{(\partial_t^2 + r^2)(\delta_{ij} + H_{ij}) + r^k r^l H_{kl} \delta_{ij} - r^i r^k H_{kj} - H_{ik} r^k \rangle \} z^j
$$

Note that the matrix $(\delta_{ij} + H_{ij})$ multiplies all terms not containing $h$. Since the remaining terms are already of order $h$, if we redefine $z^i$ to eliminate this factor in the first terms, the remaining terms will only be changed at second order in $h$. After such a field redefinition, the kinetic term is proportional to the identity, and the squared oscillator masses are therefore given by the eigenvalues of the constant matrix

$$
M = r^2 \begin{pmatrix}
0 & \cdots & H_{1r} \\
\vdots & \ddots & \vdots \\
0 & \cdots & H_{8r} \\
H_{1r} & \cdots & H_{8r}
\end{pmatrix} + O(h^2)
$$

Here, to simplify the formulae, we have made a rotation so that $\vec{r}$ lies in the 9 direction, which we refer to using the index $r$. It is straightforward to solve directly for the eigenvalues and eigenvectors of this mass matrix, and one finds to this order that the masses are

$$
m_1^2 = \cdots = m_7^2 = r^2(1 + H_{rr}), \quad m_8^2 = r^2(1 + \sqrt{H_{rr}^2 + H_{ri} H_{ir}}), \quad m_9^2 = r^2(1 - \sqrt{H_{rr}^2 + H_{ri} H_{ir}})
$$

where the index $i$ is summed from 1 to 8. The oscillators with masses $m_9$ and $m_8$ correspond to directions lying in the plane defined by the $r$ direction and the perpendicular vector $H_{ir}$. The remaining oscillators correspond to the directions perpendicular to these and have
masses which precisely match the geodesic distance \( (25) \) to leading order in \( h \). The agreement between the masses of these perpendicular oscillators and the geodesic distance is precisely the criterion used by Douglas et al. in \([16, 28]\) to constrain the leading terms in multiple D0-brane action on certain classes of manifolds. The fact that the oscillators corresponding to fields not perpendicular to the separation have different masses is also expected. In the non-gauge fixed theory, the off-diagonal fields in the direction of the separation between the branes simply give a combination of a gauge rotation of the system and a relative motion of the D0-branes along a flat direction. This effect explains the failure of the masses \( m_8, m_9 \) to satisfy Douglas’s criterion.

**Gauge Field**

For a time independent \( \vec{r} \), there is no mixing between gauge field and the other bosonic oscillators in the quadratic action, and we find that the quadratic terms involving the off diagonal field \( z_0 \) are simply

\[
S_A = -\bar{z}_0 \{ \partial_t^2 + r^2 + r^2 H_{rr} \} z_0
\]

Hence, the off diagonal gauge field also has mass equal to the geodesic distance to leading order in \( h \),

\[
m_0^2 = r^2 + H_{rr}.
\]

**Ghost Fields**

Since our gauge fixing term does not depend on the background metric, the off diagonal ghost fields will have a mass given by \( m_g^2 = r^2 \), as in the flat space case, with action

\[
S_G = -\bar{c} \{ \partial_t^2 + r^2 \} c
\]

**Fermionic Fields**

Proceeding in the same way for the quadratic fermion action, we find that the action quadratic in the off diagonal field \( \chi \) is

\[
S_F = -\bar{\chi}_\alpha \{ i \partial_t + \gamma^{i\beta}_\alpha (r^i + \frac{1}{2} H_{ij} r^j) \} \chi_\beta
\]

Thus, in the presence of a background metric, the quadratic fermion action is only changed by a shift

\[
r^i \rightarrow r^i + \frac{1}{2} H_{ij} r^j
\]

so the sixteen fermion fields \( \chi \) have a mass squared matrix given by

\[
M_f^2 = \mathbb{1} (r^2 + H_{ij} r^i r^j + \mathcal{O}(h^2)) = \mathbb{1} r^2 (1 + H_{rr} + \mathcal{O}(h^2))
\]
We see that all the fermionic oscillators have a mass which reproduces the geodesic distance to leading order in $\hbar$.

**Summary**

To summarize, we have found eight complex bosons (including the gauge field) and sixteen real fermions with masses equal to the geodesic distance between 0 and $\vec{r}$. Additionally, there are complex bosons with $m^2 = r^2(1 + \sqrt{H^2_{rr} + H_{ri}H_{ir}})$ and $m^2 = r^2(1 - \sqrt{H^2_{rr} + H_{ri}H_{ir}})$ and two complex ghosts with $m^2 = r^2$.

Thus, the sum of the squared masses weighted by the number of degrees of freedom is identical for the fermions and the bosons (including the ghosts with negative weight), and for both sets of fields, the average mass squared per degree of freedom is exactly the geodesic distance \((23)\). We have now seen that the geodesic distance criterion is precisely realized in the proposed multi-D0-brane action.

### 3.2 Graviton interactions

In this section, we use the general background Matrix theory action to study the interactions between two gravitons with unit momentum around a lightlike circle in a weakly curved space. In various cases, we will compute the one-loop effective action to first order in the metric and compare with the interactions expected from DLCQ supergravity with a curved background.

The relevant matrix theory action and background are exactly the same as those considered for D0-branes in the previous section. In this case, however, we do not wish to restrict to gravitons which are fixed in the transverse space, so we allow $\vec{r}$ to be a function of time.

As for the case of flat space Matrix theory, we should require that our background matrices satisfy the equations of motion. For block diagonal backgrounds, the equations of motion decouple for each block, so for our case, we require that $\vec{r}(t)$ satisfy the equations of motion derived from the $U(1)$ action \((3)\). For a metric which is non-trivial only in the transverse directions, the equations of motion are

$$r^i = r^k r^j g^{im}(\vec{r}) \left\{ \frac{1}{2} \partial_m g_{kl}(\vec{r}) - \partial_k g_{ml}(\vec{r}) \right\} \quad (28)$$

This is just the equation for a free non-relativistic particle moving in a curved space. We will consider two simple cases of trajectories which trivially satisfy these equations of motion. First, we may consider the static case $\vec{r}(t) = \vec{r}(0)$. The second case is one for which the metric has a flat direction $i$ (so that $h_{ij} = \partial_i g_{jk} = 0$) and we take the particle to have some velocity in this direction. In this case, the right hand side of \((28)\) vanishes, so that $r^i(t) = r^i + v^i t$. 


3.2.1 Supergravity predictions

Before proceeding with the matrix theory calculation, we would like to see what supergravity predicts for the interaction potential between two gravitons in a weakly curved space.

As above, we start with a static metric \( g_{ij}(\vec{x}) = \delta_{ij} + h_{ij}(\vec{x}) \) which is assumed to satisfy the source-free Einstein equations

\[
0 = R_{ij} = \frac{1}{2} (\Delta h_{ij} - \partial_i \partial_k h_{kj} - \partial_j \partial_k h_{ki} + \partial_i \partial_j h_{kk}) + \mathcal{O}(h^2) \tag{29}
\]

We may find the potential between a pair of gravitons in this space by treating one as a source for a perturbation about the metric \( g \) and reading off the potential from the probe action (8), keeping only terms arising from the perturbation in the original metric due to the presence of the source graviton.

For our source, we choose the graviton which sits at the origin of the transverse space with unit momentum in the compact direction. The stress-energy tensor for this particle still has only a single non-vanishing component,

\[
T_{++} = T_{--} = \frac{1}{2\pi R^2} \delta(\vec{x}),
\]

The presence of this graviton will result in a perturbation of the metric \( g_{ij} \) which we denote by \( \gamma_{ij} \). The fact that \( T_{--} \) is the only non-vanishing component of the stress-energy tensor simplifies things considerably, and as with the flat space case, we may solve the Einstein equation taking only the component \( \gamma_{--} \) to be non-zero. In this case, the condition that the perturbed metric \( g + \gamma \) should continue to satisfy the Einstein equation with source \( T \) reduces at leading order in \( \gamma \) to the covariant Laplace equation,

\[
g^{ij} \nabla_i \nabla_j \gamma_{--} = g^{ij} (\partial_i \partial_j - \Gamma^k_{ij} \partial_k) \gamma_{--} = 2\kappa^2_{11} T_{--} \tag{30}
\]

(see, for example [29]). We are only interested in the solution at leading order in \( h \) (the original background metric), so we expand

\[
\gamma_{--} \equiv \gamma_0 + \gamma_1 + \mathcal{O}(h^2)
\]

Here \( \gamma_0 \) is the part independent of \( h \), equal to the flat space solution (ignoring non-numerical constants)

\[
\gamma_0 = \frac{15}{2r^7}
\]

while \( \gamma_1 \) is the part linear in \( h \).

In the case where \( g_{ij} \) is the metric corresponding to some choice of coordinates on flat space, the exact solution to (30) must be given by a covariant version of (31), replacing \( r \) with the geodesic distance \( d \) between 0 and \( r \). In this case, using (25), we have

\[
\gamma = \frac{15}{2} \left\{ \frac{1}{r^7} - \frac{7r^4 r^j}{2r^9} \int_0^1 d\lambda h_{ij}(\lambda \vec{r}) \right\} + \mathcal{O}(h^2) \tag{32}
\]
In fact, this solves (30) to leading order in $h$ in any case where the metric $g$ is Ricci-flat as may be verified explicitly by substitution. The $h$ independent part of (30) reads
\[ \partial^2 \gamma_0 = 2\kappa_{11}^2 T_-, \]
and is just the statement that $\gamma_0$ is the solution for flat space. The linear terms in $h$ in equation (30) read
\[ \partial^2 \gamma_1 = h_{ij}\partial_i\partial_j\gamma_0 + (\partial_i h_{ik} - \frac{1}{2}\partial_k h_{ii})\partial_k\gamma_0 \]
and substituting for $\gamma_0$ and $\gamma_1$ from (32) it is not hard to check that this holds, making use of the Ricci-flatness condition (29), the identity
\[ \partial_\lambda h(\lambda \vec{r}) = (r \cdot \partial)h(\lambda \vec{r}),\]
and various integrations by parts. This is done most easily by choosing coordinates so that $h$ satisfies the harmonic gauge condition in which
\[ \partial_i h_{ij} = \frac{1}{2}\partial_j h_{ii}, \quad R_{ij} = \partial^2 h_{ij} + O(h^2) \]

Thus, to leading order in the backgrounds we are considering, the metric perturbation due to the presence of a graviton at the origin is simply
\[ \gamma_{--}(\vec{x}) = \frac{15}{2d^7(\vec{x})} \]
where $d$ is the geodesic distance between 0 and $\vec{x}$. Recalling the action (8) for the probe graviton moving in the metric produced by this source, we see that to leading order in the background, the curved space graviton-graviton potential is simply
\[ V = -\frac{15}{16} \frac{v^4}{d^7} \]

In the next sections, we will carry out the graviton potential calculation in Matrix theory to compare with this supergravity prediction.

3.2.2 Static case

First, we consider the static case in which both gravitons have zero transverse velocity. In this case, as in flat space, we expect the potential to vanish at leading order in the transverse metric, as shown above.

For this case, we may directly apply the results of section 3.1. The complete action quadratic in the off diagonal fields is
\[ S = S_B + S_A + S_F + S_G \]
\[
S_B = -\bar{z}^i \{(\partial_t^2 + r^2)(\delta_{ij} + H_{ij}) \\
+ (r^k r^l H_{kl}) \delta_{ij} - r^i r^k H_{kj} - H_{ik} r^k r^j \} z^j
\]
\[
S_A = -\bar{z}_0 \{\partial_t^2 + r^2 + r^i r^j H_{ij}\} z_0
\]
\[
S_F = -\bar{\chi}_\alpha \{i \partial_t + \gamma^i_{\alpha \beta}(r^i + \frac{1}{2} H_{ij} r^j)\} \chi_\beta
\]
\[
S_G = -\bar{c} \{\partial_t^2 + r^2\} c
\]

As argued above, this leads to eight complex bosons (including the gauge field) and sixteen real fermions with masses equal to the geodesic distance between 0 and \( \vec{r} \) as well as complex bosons with \( m^2 = r^2 (1 + \sqrt{H_{rr}^2 + H_{ri} H_{ir}}) \) and \( m^2 = r^2 (1 - \sqrt{H_{rr}^2 + H_{ri} H_{ir}}) \) and two complex ghosts with \( m^2 = r^2 \).

The vanishing of the one loop potential to leading order in \( h \) is ensured by the fact that the sum of the squared masses weighted by number of degrees of freedom is identical for the fermions and bosons (including ghosts weighted by -1). This follows since the one loop effective action depends only on the oscillator masses and is given by:

\[
e^{i \Gamma_{\text{1 loop}}} = \det^{-8}(\partial_t^2 + r^2 (1 + H_{rr})) \det^8(\partial_t^2 + r^2 (1 + H_{rr})) \det^2(\partial_t^2 + r^2) \\
\det^{-1}(\partial_t^2 + r^2 (1 + \sqrt{H_{rr}^2 + H_{ri} H_{ir}})) \det^{-1}(\partial_t^2 + r^2 (1 - \sqrt{H_{rr}^2 + H_{ri} H_{ir}})) \\
= 1 + O(h^2)
\]

Hence, in the static case, we find agreement with our expectations from supergravity.

3.2.3 Velocity dependent potential

We now consider the case of two gravitons with relative velocity. Here, we assume that the particle with initial position \( \vec{r} \) moves in a direction \( \vec{v} \) which is perpendicular to \( \vec{r} \) and in which the metric is flat. In this case, we have shown above that supergravity predicts a potential

\[
-\frac{15 v^4}{16 d^7}
\]

where \( d \) is the geodesic separation distance.

To simplify our calculations, we rotate coordinates so that \( \vec{r} \) and \( \vec{v} \) lie in coordinate directions which we denote by indices \( r \) and \( v \). Thus

\[
g_{rv}(\vec{x}) = 1, \quad g_{vr}(\vec{x}) = g_{vr}(\vec{x}) = \partial_v g_{ij}(\vec{x}) = 0
\]

These equations ensure that the matrix theory equations of motion (28) are satisfied for the trajectory \( \vec{r}(t) = \vec{r} + \vec{v} t \) that we are considering. Expanding the action (33) about this background, we find that the quadratic action for the off diagonal fields is equal to the action for the static case (where \( r^i \) is interpreted as \( r^i(t) = r^i + v^i t \) plus extra terms:

\[
S_v = 2i \bar{z}^i v^i z^0 - 2i \bar{z}^0 v^i z^i
\]
Note that these terms (in which the background appears as \( \tilde{\tau} \) rather than just \( \tau \)) come only from the flat space action, since in the metric dependent terms, \( \dot{X}^i \) only appears coupled to \( h_{ij} \) while \( \dot{r}^i h_{ij} = 0 \). As a result, the remaining calculation is almost identical to the flat space calculation, performed in [17]. To see this, we note that the complete action in this case may be written (eliminating a factor \( (\delta_{ij} + H_{ij}) \) as above to diagonalize the boson kinetic term)

\[
S = S_B + S_A + S_v + S_F + S_G
\]

\[
S_B = -\bar{z}^i \{(\partial_i^2 + d^2 + v^2 t^2)\} z^i + \bar{z}^r \{r^2 H_{ri}\} z^i + \bar{z}^v \{H_{ri}rvt\} z^i + \bar{z}^i \{H_{iv}rvt\} z^v
\]

\[
S_A = -\bar{z}_0 \{(\partial_i^2 + d^2 + v^2 t^2)\} z_0
\]

\[
S_v = 2i \bar{z}^i v^0 - 2i \bar{z}^0 v^i z^i
\]

\[
S_F = -\bar{\chi}_\alpha \{i \partial_i + \gamma^i_{\alpha\beta}(d^i + v^i t)\} \chi_\beta
\]

\[
S_G = -\bar{c} \{(\partial_i^2 + d^2)\} c + \bar{c} \{H_{ri}\} c
\]

where we have defined

\[
d^i = r^i + \frac{1}{2} H_{ij} r^j
\]

so that \( d^2 \) is the squared geodesic distance. Apart from the terms with explicit factors of \( H \) in the boson and ghost actions, this is exactly the flat space action with \( r^i \) replaced by \( d^i \). Recalling the flat space calculation, we see that at zeroth order in \( H \), the oscillators \( z^i, i = 1, \cdots, 7 \), and \( z^r \) are degenerate with mass \( d^2 + v^2 t^2 \) while \( z^v \) and \( z^0 \) combine into oscillators with non-degenerate masses \( (d^2 + v^2 t^2 \pm 2v) \). Adding the perturbation

\[
\bar{z}^r \{r^2 H_{ri}\} z^i + \bar{z}^i \{r^2 H_{iv}\} z^r + \bar{z}^v \{H_{ri}rvt\} z^i + \bar{z}^i \{H_{iv}rvt\} z^v
\]

we note that the last two terms do not affect the masses at leading order in \( h \), since if we change coordinates to diagonalize the leading order mass matrix, these contribute only to non-diagonal matrix elements connecting eigenvectors of different mass. (Recall from basic perturbation theory that for eigenvectors \( |A_1\rangle, \cdots |A_n\rangle \) with degenerate zeroth order eigenvalues and eigenvectors \( |B_1\rangle, \cdots |B_m\rangle \) with non-degenerate zeroth order eigenvalues that the first order shift in the eigenvalues for \( |B_i\rangle \) come only from the matrix element \( \langle B_i|M|B_i\rangle \), while the first order eigenvalues for the space spanned by \( |A_i\rangle \) are determined only by the submatrix \( \langle A_i|M|A_j\rangle \).

The first two terms in (34) have the same effect as they did in the static case, to shift two of the degenerate boson masses by

\[
\Delta m^2 = -H_{rr} \pm \sqrt{H_{rr}^2 + H_{ri} H_{ir}}
\]

\[
\equiv -H_{rr} \pm G
\]

Meanwhile, the perturbation in the ghost action shifts the two ghost masses by

\[
\Delta m_g^2 = -H_{rr}
\]
Defining
\[ F(x) = \det(\partial^2_x + (d^2 - x) + v^2 t^2) \]
we find that since all of the mass shifts are time independent, the one loop effective action is simply
\[ e^{i\Gamma_d} = e^{i\Gamma_{d'}} \frac{F(H_{rr})F(H_{rr})}{F(H_{rr} + G)F(H_{rr} - G)} = e^{i\Gamma_{d'}}(1 + \mathcal{O}(h^2)) \]
where \( \Gamma_d \) is the flat-space potential with \( r \) replaced by the geodesic length \( d \). We conclude that the leading order one loop potential is simply given by
\[ V = -\frac{15 \, v^4}{16 \, d^7} \]
as predicted by supergravity.

4 Discussion

In this paper we have derived the leading terms in the multiple D0-brane action in a weakly curved background metric, dilaton, and antisymmetric tensor background fields. This action was derived by using Seiberg’s scaling arguments on an action we recently proposed for the M(atrix) model of M-theory in weak background fields. We found explicit forms for the IIA stress-energy tensor of a multiple D0-brane system, as well as the components of D2-brane, D4-brane, D6-brane and fundamental string currents which couple to the background R-R and B fields of the IIA theory. We tested our action by verifying that it satisfies Douglas’s geodesic length criterion. We also showed that the corrections to the one-loop effective potential between a pair of individual 0-branes correctly reproduce curved space supergravity results.

The results presented in this paper give for the first time a systematic description of the linear coupling between a system of multiple D-branes and background supergravity fields. There are a number of previous discussions of this sort of action in the literature to which our results can be related. In [28], Douglas, Kato and Ooguri used Douglas’s geodesic length criterion and other axioms including an assumption of supersymmetry to constrain the form of the multiple D3-brane action on a transverse Kähler manifold. They compute the first few correction terms in terms of the curvature tensor \( R_{ijkl} \) of the background. They show that the first term, corresponding to our coupling
\[ (\partial_{k_1 k_2} h_{ij})_{l_i h_{k_1 k_2}} \]
is uniquely determined by the geodesic length criterion. The linear part of the term they find indeed has the same form \( \partial^2 h \text{Tr} \ X^2 (F^2 + \dot{X}^2) \) as our result for this term. Their approach is not able to uniquely determine the higher order terms, but our results are compatible
with the general structure of the linear parts of the structure they find at higher order. It is interesting that they are able to determine the form of some of the quadratic couplings in their work as well as the linear terms.

It is natural to try to extend the results of this paper to determine the quadratic and higher order couplings between a system of many D-branes and the supergravity background fields. In [13] it was suggested that the coupling to $n$th order terms in the background could be determined by a $n$-loop calculation in matrix theory. The results of Okawa and Yoneya on 3-graviton scattering in matrix theory [30, 31] seem to indicate that this may work at least to quadratic order in general backgrounds, although there are indications [32, 33, 34] that there may be problems with extending this approach to higher order. In any case, even the two-loop calculation would be quite challenging to work out for completely general background configurations, so it would be nice to find a simpler approach. The terms coupling the open string fields of the D-brane system to any number of bulk supergravity fields can of course in principle be determined by a perturbative string calculation. These calculations are quite complicated, however, even for the linear coupling terms discussed in this paper. (For recent work computing the curvature squared terms in the single D-brane action see [35]). Furthermore, the results we have given here extend to arbitrary derivatives in the background fields and contain a correspondingly arbitrary number of D-brane fields. It is difficult to imagine reproducing such results from perturbative string theory. Despite these difficulties inherent in a systematic derivation of the higher order coupling terms, it may be that the symmetries of the theory and the geodesic length criterion are sufficient to determine the structure of some of the higher-order terms. The results of [28] on Kähler manifolds indicate that it is indeed possible to learn something about the higher order terms using this approach. In this paper we have found that the combinatorial structure of the higher moment terms is crucial in fixing the masses of off-diagonal strings in accord with the geodesic length criterion. This indicates that this condition will place strong constraints on the possible form of the higher order couplings to the background. Another approach which might help extend the results here to higher order is to find the symmetry principle which corresponds to general coordinate invariance for the multiple D0-brane system. Because the coordinates enter the theory as matrices, there are ordering ambiguities in determining how the operators describing the D0-brane system transform. If this symmetry could be understood in a systematic fashion, it would quite possibly uniquely determine the higher order couplings of the multiple D-brane system to general backgrounds.

The approach we have taken here to describing matrix theory and multiple D0-brane systems in curved backgrounds is to assume that the action in a weak background can be written as a matrix quantum mechanics theory with a systematic expansion in powers of the background field strength. This approach certainly seems valid for linear couplings to the background. It is not clear, however, that such an approach can be extended to all orders. In [36, 37], in fact, it was argued that even on simple manifolds like K3 or ALE spaces it may not be possible to describe DLCQ M-theory using a finite size matrix quantum mechanics theory.
We are not sure at this point how the approach we have taken here fits with the results of those papers. One possibility is that the perturbation expansion we are considering in powers of the background field will not converge to a well-defined theory when higher order terms are taken into account and the background is of the form considered in [36, 37]. Indeed, the results of those papers indicate that the expansion in weak backgrounds may break down at quadratic order in the curvature of the background. Another possibility is that either the restriction to light-front coordinates in M-theory or the fact that in the Seiberg limit the scale of the metric structures of interest becomes smaller than the string length may have a subtle effect on the relationship between the supergravity and open string descriptions of graviton interactions, leading to some modification in the conclusions of [36, 37]. It is clearly a very important question whether a good low-energy description of D0-branes can be given which maps to M-theory in the Seiberg limit, but we leave a further resolution of this issue to further work.

In this paper we have focused on the action for a system of D0-branes in a weak supergravity background. It is possible, however, to T-dualize the action we have given here to get an action for Dp-branes of arbitrary dimension in a weak background. One particularly interesting case is that of D3-branes. The T-duals of the currents $I_x$ we have determined in this paper are linear combinations of the operators in the $N = 4, D = 3+1$ super Yang-Mills theory on the world volume of a system of many D3-branes which lie in the short representations of the $SU(2, 2|4)$ superconformal symmetry group of the theory. These operators, which couple linearly to the supergravity background fields, play a crucial role in the simplest version of Maldacena’s AdS/CFT correspondence [38]. For the fields associated with the lowest partial waves of bulk fields in the AdS space, some of the corresponding operators were found from the Born-Infeld action in [39, 40, 41, 42]. All the other lowest partial wave operators are in principle determined by supersymmetry and group theory from the weight 2 operator $\text{Tr} X^{(i} X^{j)}$, following [43, 44, 45, 46] and related work. Although these operators have played a fundamental role in understanding the detailed structure of the AdS/CFT correspondence, only a few of these operators have been described explicitly in terms of the component fields in the D3-brane world-volume theory. In addition, to date no general method for understanding the structure of the higher partial wave operators (which correspond to the higher moments of our currents $I^{(\cdots)}_x$ through T-duality and which are related through supersymmetry to the higher weight chiral primary operators $\text{Tr} X^{(i_1 \cdots i_k)}$) has been presented in the literature, although some discussion of particular operators of this type was given in, for example, [40, 47]. In a separate paper we will discuss more details of the connection between the results presented here and the operators which are used in the AdS/CFT correspondence for 3-branes.

A feature which emerges from the results of [44, 45, 46] and the extension of this work in the present paper is the characteristic form of the higher moments of the currents $I_x$ which couple to the derivatives of the background supergravity fields. In general, we find that the

\footnote{Thanks to H. Ooguri for correspondence on this point.}
nth moment of the current $I_x$ has contributions of the form

$$I_x^{(i_1 \cdots i_n)} = \text{Sym} \left( I_x; X^{i_1}, X^{i_2}, \ldots X^{i_n} \right) + I_x^{(i_1 \cdots i_n)}_{(\text{fermion})}$$

(36)

where the first term on the RHS is a bosonic contribution to the higher moment given by a symmetrized trace of the polynomial giving the monopole moment of $I_x$ with a product of $n$ $X$’s, and the second term contains fermionic contributions to the higher moment. The fact that all the monopole moments of the currents as well as their higher moments can be expressed in a symmetrized trace form is reminiscent of Tseytlin’s suggestion [8] for using the symmetrized trace to resolve ordering ambiguities in the nonabelian Born-Infeld action. Indeed, the components $T^{--}$ of the 11D stress tensor are precisely the symmetrized $F^4$ terms appearing at fourth order in the nonabelian Born-Infeld action proposed by Tseytlin. This structure may be helpful in trying to predict the form of higher-order terms in the action without doing explicit matrix theory or string theory calculations. This structure is also helpful in understanding previous work in which higher partial waves of operators on the D3-brane play a role. In particular, in [40] the rate of absorption of higher partial waves of minimally coupled scalars in an extremal D3-brane background was computed in supergravity and compared to a D3-brane world-volume calculation. While the analogous calculations for s-wave absorption can be matched precisely including numerical coefficients, for partial waves with $l > 1$ a discrepancy was found in [40] between the results of these two calculations. The authors suggested that this discrepancy might arise because the higher partial wave operators were not correctly normalized. The results we have given here suggest by T-duality that they indeed used the correct normalization, but that the operators they used should have a symmetrized trace with respect to orderings of the fields. This additional information seems to help resolve the discrepancy found in [40]. A more detailed discussion of the resolution of this problem will be described in a future publication.

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**A Supercurrents from matrix theory**

We reproduce here for convenience the matrix theory forms of the multipole moments of the 11D supercurrent found in [15, 13]. The stress tensor $T^{IJ}$, membrane current $J^{IJK}$ and

\footnote{Thanks to Steve Gubser and Igor Klebanov for discussions on this point}
5-brane current $M^{ijklmn}\,$ have integrated components

\[
T^{++} = \frac{1}{R} \text{Str} \left( \mathbf{1} \right) \\
T^{+i} = \frac{1}{R} \text{Str} \left( \dot{X}_i \right) \\
T^{-i} = \frac{1}{R} \text{Str} \left( \frac{1}{2} \ddot{X}_i \dddot{X}_i + \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \theta \gamma^i [X^i, \theta] \right) \\
T^{ij} = \frac{1}{R} \text{Str} \left( \dot{X}_i \dot{X}_j + F_{ik} F_{kj} - \frac{1}{4} \theta \gamma^i [X_j, \theta] - \frac{1}{4} \theta \gamma^j [X_i, \theta] \right) \\
T^{ii} = 1 \\
J^{ij} = \frac{1}{6 R} \text{Str} \left( F_{ij} \right) \\
J^{i-j} = \frac{1}{6 R} \text{Str} \left( F_{ij} \dot{X}_j - \frac{1}{2} \theta [X_i, \theta] + \frac{1}{4} \theta \gamma^i [X_j, \theta] \right) \\
J^{ijk} = \frac{1}{6 R} \text{Str} \left( \dot{X}_i F_{jk} + \dot{X}_j F_{ki} + \dot{X}_k F_{ij} - \frac{1}{4} \theta \gamma^{[ijk]} [X_i, \theta] \right) \\
J^{ij} = \frac{1}{6 R} \text{Str} \left( + \dot{X}_i \dot{X}_k F_{kj} - \dot{X}_j \dot{X}_k F_{ki} - \frac{1}{2} \dot{X}_k \ddot{X}_k F_{ij} + \frac{1}{4} F_{ij} F_{kl} F_{kl} + F_{ik} F_{kl} F_{lj} \right) \\
+ \frac{1}{24 R} \text{Str} \left( \theta \alpha \dot{X}_k [X_m, \theta] \right) \left\{ \gamma^{[km]} + \gamma^{[jm]} \delta_{k_l} - \gamma^{[jl]} \delta_{k_m} + 2 \delta_{jm} \delta_{k_l} - 2 \delta_{jm} \delta_{k_m} \right\} \alpha \beta \\
+ \frac{1}{8} \text{Str} \left( \theta \alpha F_{kl} [X_m, \theta] \right) \left\{ \gamma^{[kl]} \delta_{mi} - \gamma^{[kl]} \delta_{mj} + 2 \gamma^{[li]} \delta_{km} + 2 \gamma^{[lj]} \delta_{km} - 2 \gamma^{[lj]} \delta_{km} \right\} \alpha \beta \\
+ \frac{i}{48 R} \text{Str} \left( \theta \gamma^{[kij]} \theta \gamma^{[kl]} \theta \gamma^{[lj]} \theta \theta \theta \right) \\
M^{+ijkl} = \frac{1}{12 R} \text{Str} \left( F_{ij} F_{kl} + F_{ik} F_{lj} + F_{il} F_{jk} + \theta \gamma^{ijkl} [X^i, \theta] \right) \\
M^{-ijklm} = \frac{5}{4 R} \text{Str} \left( X_i F_{jk} F_{lm} - \frac{1}{3} \theta \dot{X}^i \gamma^{ijkl} [X^m, \theta] - \frac{1}{6} \theta F^{ijklm} \gamma^i [X^i, \theta] \right) .
\]

Time derivatives are taken with respect to Minkowski time $t$. All expressions have been written in a gauge with $A_0 = 0$. Gauge invariance can be restored by replacing $\dot{X}$ with $D_t X$. Indices $i, j, \ldots$ run from 1 through 9, while indices $a, b, \ldots$ run from 0 through 9. In these expressions we have used the definitions $F_{0i} = \dot{X}^i, F_{ij} = i [X^i, X^j]$. We do not know of a matrix form for the transverse 5-brane current components $M^{+ijklm}, M^{-ijklmn}$. 
The higher multipole moments of these currents are given by

\[
T^ij_{(i_1 \ldots i_k)} = \text{Sym}(T^{ij}; X^{i_1}, \ldots, X^{i_k}) + T^ij_{\text{fermion}}
\]

\[
J^{ijk}_{(i_1 \ldots i_k)} = \text{Sym}(J^{ijk}; X^{i_1}, \ldots, X^{i_k}) + J^{ijk}_{\text{fermion}}
\]

\[
M^{ijklmn}_{(i_1 \ldots i_k)} = \text{Sym}(M^{ijklmn}; X^{i_1}, \ldots, X^{i_k}) + M^{ijklmn}_{\text{fermion}}
\]

where some simple examples of the two-fermion contribution to the first moment terms are

\[
T^{+i(j)}_{\text{fermion}} = \frac{1}{8R} \text{Tr}(\theta \gamma^{ij} \theta)
\]

\[
T^{+-i(i)}_{\text{fermion}} = \frac{1}{16R} \text{Tr}(-i \theta F_{kl} \gamma^{[kli]} \theta + 2i \theta \dot{X}_{l} \gamma^{[kli]} \theta)
\]

\[
J^{+ij(k)}_{\text{fermion}} = \frac{i}{48R} \text{Tr}(\theta \gamma^{ijk} \theta)
\]

\[
J^{+-i(j)}_{\text{fermion}} = \frac{1}{48R} \text{STr}(i \theta \dot{X}_k \gamma^{[kij]} \theta + i \theta F_{ik} \gamma^{[kj]} \theta)
\]

\[
M^{+ijkl(m)}_{\text{fermion}} = -\frac{i}{16R} \text{STr}(\theta F^{[jk} \gamma^{ilm]} \theta)
\]

The remaining two-fermion contributions to the first moments and some four-fermion terms are also determined by the results in [13].

There are also fermionic components of the supercurrent which couple to background fermion fields in the supergravity theory. We have not discussed these couplings in this paper, but the matrix theory form of the currents is determined in [13].

There is also a 6-brane current appearing in matrix theory related to nontrivial 11D background metrics. The components of this current as well as its first moments are

\[
S^{ijklmn} = \frac{1}{R} \text{STr}(F_{ij} F_{kl} F_{mn})
\]

\[
S^{ijklmn(p)} = \frac{1}{R} \text{STr}(F_{ij} F_{kl} F_{mn} X_p - \theta F_{[kl} F_{mn} \gamma_{ppq]} \theta)
\]

\[
S^{ijklmnp} = \frac{7}{R} \text{STr}(F_{ij} F_{kl} F_{mn} \dot{X}_p + (\theta^2, \theta^4 \text{ terms}))
\]

\[
S^{ijklmnp(q)} = \frac{7}{R} \text{STr}(F_{ij} F_{kl} F_{mn} \dot{X}_p X_q - \theta \dot{X}_{ij} F_{kl} F_{mn} \gamma_{ppq} \theta + \frac{i}{2} \theta F_{jk} F_{lm} F_{np} \gamma_{qr} \theta)
\]

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