On the intrinsic origin of 1/f noise

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Abstract

The problem of the intrinsic origin of 1/f noise is considered. Currents and signals consisting of a sequence of pulses are analysed. It is shown that intrinsic origin of 1/f noise is a random walk of the average time between subsequent pulses of the pulse sequence, or interevent time. This results in the long-memory process for the pulse occurrence time and in 1/f type power spectrum of the signal.

1 Introduction

Ubiquitous of signals and processes with 1/f power spectral density at low frequencies has led to speculations that there might exist some generic mechanism underlying production of 1/f noise. The generic origins of two popular noises: white noise (no correlation in time, \(S(f) \propto 1/f^0\)) and Brownian noise (no correlation between increments, \(S(f) \propto 1/f^2\)) are very well known. It should be noted, that Brownian motion is the integral of white noise and that operation of integration of the signal increases the exponent by 2 while the inverse operation of differentiation decreases it by 2. Therefore, 1/f noise can not be obtained by simple procedure of integration or differentiation of the convenient signals. There are no simple, even stochastic, equations generating signals with 1/f noise. Note in this context also to the concept of the fractional Brownian motion and to the half-integration of a white noise signals used for generation of processes with 1/f noise \[1\]. These and similar mathematical algorithms, procedures and models for generation of the processes with 1/f noise \[2,3\] are, however, sufficiently specific, formal or unphysical. They can not, as a rule, be solved analytically and they do not reveal the origin as well as the necessary and sufficient conditions for appearance of 1/f type fluctuations. Physical models of 1/f noise in some physical systems are usually very specialized, complicated and they do not explain the internal origin of the omnipresent processes with 1/f^δ spectrum.
A lot of contributions are available in the literature concerning the origin of 1/f noise. On the web (http://linkage.rockefeller.edu/wli/1fnoise), Wentain Li is collective bibliography of flicker noise. Sufficiently comprehensive bibliography of the contributions concerning the modeling of 1/f noise may be find in references [3–7].

This work is a continuation of series of papers devoted to the modeling 1/f noise in simple systems [4, 5] and search of necessary conditions for appearance of the signals with power spectrum at low frequencies like \( S(f) \propto 1/f^{\delta} \) (\( \delta \simeq 1 \)) [6–9]. In papers [4, 5] an analysis of the necessary conditions for appearance of 1/f type fluctuations in the simple systems consisting of few or even one particle and affected by random perturbations is presented. Later, a simple analytically solvable model of 1/f noise has been proposed [6], analyzed [7] and generalized [8]. The model reveals main features, and parameter dependencies of the power spectrum of 1/f noise.

Here considering signals and currents as consisting of pulses generalizations and development of the model [6] are presented. The paper includes derivation of the expression for the correlation function, analysis of the examples of different signals and exhibition of the necessary conditions for appearance of 1/f type power spectrum in the signals consisting of pulses. It is shown that intrinsic origin of 1/f noise is a Brownian motion of the pulse interevent time.

## 2 The model

Let us consider currents or signals represented as sequences of random (but correlated) pulses \( A_k(t - t_k) \). Function \( A_k(t - t_k) \) represents the shape of the \( k \)-pulse of the signal in the region of the pulse occurrence time \( t_k \). The signal or intensity of the current of particles in some space cross section may, therefore, be expressed as

\[
I(t) = \sum_k A_k(t - t_k). \tag{1}
\]

It is easy to show that the shapes of the pulses mainly influence the high frequency, \( f \geq \Delta t_p \) with \( \Delta t_p \) being the characteristic pulse length, power spectrum while fluctuations of the pulse amplitudes result, as a rule, in the white or Lorentzian but not 1/f noise [10]. Therefore, we restrict our analysis to the noise due to the correlations between the pulse occurrence times \( t_k \). In such an approach we can replace the function \( A_k(t - t_k) \) by \( a\delta(t - t_k) \). Here \( \delta(t - t_k) \) is the Dirac delta function, \( a = \int_{-\infty}^{+\infty} A_k(t - t_k) \) is the average area of the pulse and the brackets \( \langle \ldots \rangle \) denote the averaging over realizations of the process. In such an approach the signal (1) my be expressed as

\[
I(t) = a \sum_k \delta(t - t_k). \tag{2}
\]

This model also corresponds to the flow of identical point objects: electrons, photons, cars and so on. On the other hand, fluctuations of the amplitudes \( A_k \) may result in the additional noise but can not reduce 1/f noise we are looking for.
2.1 Power spectrum

Power spectral density of the signal $I(t)$ is

$$S(f) = \lim_{T \to \infty} \left\langle \frac{2}{T} \left| \int_{t_i}^{t_f} I(t) e^{-i2\pi ft} dt \right|^2 \right\rangle$$

(3)

where $T = t_f - t_i$ is the observation time.

We can also introduce the autocorrelation function

$$\Phi(s) = \left\langle \frac{1}{T} \int_{t_i}^{t_f-s} I(t) I(t+s) dt \right\rangle$$

(4)

and use the Wiener-Khintchine relations:

$$S(f) = 4 \lim_{T \to \infty} \int_{0}^{T} \Phi(s) \cos(2\pi fs) ds,$$

(5)

$$\Phi(s) = \int_{0}^{\infty} S(f) \cos(2\pi fs) df.$$

Substitution of Eq. (2) into Eq. (3) results in the power spectral density of the signal expressed as a sequence of pulses

$$S(f) = \lim_{T \to \infty} \left\langle \frac{2\alpha^2}{T} \left| \sum_{k=k_{\text{min}}}^{k_{\text{max}}} e^{-i2\pi ft_k} \right|^2 \right\rangle$$

$$= \lim_{T \to \infty} \left\langle \frac{2\alpha^2}{T} \sum_{k=k_{\text{min}}}^{k_{\text{max}}} \sum_{q=k_{\text{min}}-k}^{k_{\text{max}}-k} e^{i2\pi f\Delta(k;q)} \right\rangle$$

(6)

where $\Delta(k;q) \equiv t_{k+q} - t_k$ is the difference of pulse occurrence times $t_{k+q}$ and $t_k$ while $k_{\text{min}}$ and $k_{\text{max}}$ are minimal and maximal values of index $k$ in the interval of observation $T = t_f - t_i$.

For a stationary process Eq. (6) yields

$$S(f) = \frac{2\alpha^2}{T} \sum_{q=-N}^{N} \left( N + 1 - |q| \right) \left\langle e^{i2\pi f\Delta(q)} \right\rangle.$$

Here $N = k_{\text{max}}-k_{\text{min}}$, the brackets $\langle \ldots \rangle$ denote the averaging over realizations of the process and over the time (index $k$) and a definition $\Delta(q) = -\Delta(-q) = \Delta(k;q)$ is introduced. For abbreviation of the equations we have omitted the mark of the limit $\lim_{T \to \infty}$ here and further on in expressions for the power spectrum $S(f)$.

3
The average over the distribution of $\Delta (q)$ may be expressed as

$$
\langle e^{i2\pi f \Delta(q)} \rangle \equiv \int_{-\infty}^{+\infty} e^{i2\pi f \Delta(q)} \Psi_{\Delta} (\Delta (q)) \, d\Delta (q) = \chi_{\Delta(q)} (2\pi f).
$$

Here $\Psi_{\Delta} (\Delta (q))$ is the distribution density of $\Delta (q)$, and $\chi_{\Delta(q)} (2\pi f)$ is the characteristic function of the distribution $\Psi_{\Delta} (\Delta (q))$. Therefore

$$
S(f) = 2a^2 \sum_{q=-N}^{N} \left( \bar{\nu} - \frac{|q|}{T} \right) \chi_{\Delta(q)} (2\pi f) \quad (7)
$$

where $\bar{\nu} \equiv \lim_{T \to \infty} \frac{N+1}{T}$ is the mean number of pulses per unit time.

When the sum $\sum_{q=-N}^{N} |q| \chi_{\Delta(q)} (2\pi f)$ converges and $T \to \infty$ we have the power spectrum from Eq. (7) in the form

$$
S(f) = 2aT \sum_{q=-N}^{N} \chi_{\Delta(q)} (2\pi f) \quad (8)
$$

Here $T \equiv \langle T(t) \rangle = \bar{\nu} a$ is the average current.

### 2.2 Correlation function

Substitution of Eq. (2) into Eq. (4) yields the correlation function of the signal (2)

$$
\Phi(s) = \left\langle \frac{a^2}{T} \sum_{k,q} \delta(t_{k+q} - t_k - s) \right\rangle.
$$

After summation over index $k$ we have

$$
\Phi(s) = \bar{I}a \sum_{q} \langle \delta (\Delta (q) - s) \rangle
$$

where the brackets $\langle \ldots \rangle$ denote again the averaging over realizations of the process and over the time (index $k$), as well. Such averaging coincides with the averaging over the distribution of the time difference $\Delta (q)$

$$
\Phi(s) = \bar{I}a \sum_{q} \int_{-\infty}^{+\infty} \psi_{\Delta}(\Delta) \delta(\Delta - s) \, d\Delta
$$

$$
\Phi(s) = \bar{I}a \delta(s) + \bar{I}a \sum_{q \neq 0} \psi_{\Delta}(s) \quad (9)
$$

Here $\psi_{\Delta}(\Delta)$ is the distribution density of $\Delta (q)$. Substitution of Eq. (9) into Eq. (5) yields expressions (7) and (8).
3 Examples

Consider some examples of the signals represented by Eq. (2).

(i) Periodic signal expressed as
\[ I(t) = a \sum_k \delta(t - k\tau) \]
generates the power spectrum
\[ S(f) = 2a^2 \lim_{T \to \infty} \frac{\sin^2(\pi fNT)}{T \sin^2(\pi fT)} \Rightarrow 2T^2 \delta(f), \quad f \ll \tau^{-1}. \]

(ii) Perturbed periodic signal represented by Eq. (2) with the time series expressed as recurrence equations
\[ t_k - t_{k-1} \equiv \tau_k = \tau + \sigma \varepsilon_k \]
where \( \{ \varepsilon_k \} \) is a sequence of uncorrelated normally distributed random variables with zero expectation and unit variance and \( \sigma \) being the standard deviation of this white noise [5]. For this model we have
\[ \Delta(q) = q\tau + \sigma \sum_{l=k+1}^{k+q} \varepsilon_l, \]
and the characteristic function
\[ \chi_{\Delta(q)}(2\pi f) = e^{i2\pi f(\Delta(q)) - \frac{1}{2}(2\pi f)^2 \sigma_{\Delta}^2} \] (10)
where \( \langle\Delta(q)\rangle = q\tau \) and the variance \( \sigma_{\Delta}^2 \) of the time difference \( \Delta(q) \) equals
\[ \sigma_{\Delta}^2 \equiv \left\langle \Delta(q)^2 \right\rangle - \left\langle \Delta(q) \right\rangle^2 = \sigma^2 |q|. \]
Substitution of Eq. (10) into Eq. (8) yields the Lorentzian spectrum
\[ S(f) = T^2 \frac{4\tau_{rel}}{1 + \tau_{rel}^2 \omega^2} \] (11)
where \( \omega = 2\pi f \) and \( \tau_{rel} = \sigma^2/2\tau. \)

(iii) Time difference
\[ \Delta(q) = \sum_{l=k+1}^{k+q} \tau_l \]
as a sum of uncorrelated interevent times \( \tau_l. \) According to Eqs. (6)–(8) we have in this case

\[ S(f) = 2aT \left[ 1 + 2 \text{Re} \sum_{q=1}^{N} \left\langle e^{i2\pi f\tau} \right\rangle^q \right] \]

\[ S(f) = 2aT \left[ 1 + 2 \text{Re} \frac{\chi_{\tau}(\omega)}{1 - \chi_{\tau}(\omega)} \right]. \] (12)

For instance, substitution at \( f \ll \tau^{-1} \) and \( f \ll \sigma^{-1} \) of Eq. (10) with \( q = 1 \) into Eq. (12) results in Eq. (11).

(iv) For the Poisson process

\[ \chi_{\tau}(2\pi f) = \frac{1}{1 - i2\pi f\tau}, \quad \text{Re} \frac{\chi_{\tau}(\omega)}{1 - \chi_{\tau}(\omega)} = 0 \]
and we have from Eq. (12) only the shot noise
\[ S(f) = 2aT = S_{\text{shot}}. \] (13)

(v) Brownian motion of the interevent time \( \tau_k \) with some restrictions, e.g., with the relaxation to the average value \( \bar{\tau}, \)
\[ \tau_k = \tau_{k-1} - \gamma (\tau_{k-1} - \bar{\tau}) + \sigma \varepsilon_k, \] (14)
when the pulse occurrence times \( t_k \) are expressed as
\[
t_k = t_{k-1} + \tau_k. \tag{15}
\]
According to Eq. (6) the power spectrum of the signal (2) with the pulse occurrence times \( t_k \) generated by Eqs. (14) and (15) for sufficiently small parameters \( \sigma \) and \( \gamma \) in any desirably wide range of frequencies, \( f_1 = \gamma / \pi \sigma_\tau < f < f_2 = 1 / \pi \sigma_\tau \), is 1/f-like \([6, 7, 8, 9]\), i.e.,
\[
S(f) = \sqrt{2 \over \pi} \exp \left(-\frac{\tau^2}{2 \sigma_\tau^2}\right) \sigma_\tau f. \tag{16}
\]
Here \( \sigma_\tau^2 = \sigma^2 / 2 \gamma \) is the variance of the interevent time \( \tau_k \).

## 4 Origin of 1/f noise

The origin for appearance of 1/f fluctuations in the model described in Eqs. (14) and (15) is related with the relatively slow, Brownian, fluctuations of the pulse interevent time. For this reason, the variance \( \sigma_\Delta^2 \) of the time difference \( \Delta (k; q) \) for \( |q| \ll \gamma^{-1} \) is a quadratic function of the time difference and, consequently, of the difference \( q \) of the pulse serial numbers \( k \) \([6, 7, 8, 9]\), i.e.,
\[
\sigma_\Delta^2 (k; q) = \sigma_\tau^2 (k) q^2. \tag{17}
\]
Substitution of Eqs. (10) and (17) into Eq. (8) yields 1/f spectrum (16).

### 4.1 Generalization

For slowly fluctuating interevent time, the time difference \( \Delta (k; q) \) may be expressed as \([6, 7, 8, 9]\)
\[
\Delta (q) = \sum_{l=k+1}^{k+q} \tau_l \simeq q \tau \tag{18}
\]
where \( \tau = (t_{k+q} - t_k) / q \) is the average interevent time in the time interval \( (t_k, t_{k+q}) \), a slowly fluctuating function of the arguments \( k \) and \( q \). In such an approach, the power spectrum according to Eq. (6) is
\[
S(f) = 2 \overline{I} \sigma \sum_q \langle e^{i2\pi f q \tau} \rangle \tag{19}
\]
where
\[
\langle e^{i2\pi f q \tau} \rangle \equiv \int_{-\infty}^{+\infty} e^{i2\pi f q \tau} \psi_\tau (\tau) d\tau = \chi_\tau (2\pi f q)
\]
is the characteristic function of the distribution density \( \psi_\tau (\tau) \) of the interevent time \( \tau \). Therefore, the power spectrum according to Eq. (19) may be expressed as
\[
S(f) \simeq 2 \overline{I}^2 \tau \Psi_\tau (0) / f. \tag{20}
\]
Here the property \( \int_{-\infty}^{+\infty} \chi_\tau (x) dx = 2\pi \Psi_\tau (0) \) of the characteristic function has been used.
4.2 Correlation function of $1/f$ noise

The correlation function of $1/f$ noise in the approximation (18) may be calculated according to Eq. (9), i.e.,

$$\Phi (s) = \bar{I}a \sum_{q}^{\infty} \int_{-\infty}^{\infty} \psi_{\tau}(\tau)\delta (q\tau - s) \, d\tau = \bar{I}a\delta (s) + \bar{I}a \sum_{q \neq 0} \psi_{\tau}(\frac{s}{q}) \frac{1}{|q|}. \quad (21)$$

For the Gaussian distribution of the interevent time $\tau$

$$\psi_{\tau}(\tau) = \frac{1}{\sqrt{2\pi\sigma_{\tau}}} \exp \left( -\frac{(\tau - \bar{\tau})^2}{2\sigma_{\tau}^2} \right)$$

the correlation function (21) reads as

$$\Phi (s) = \bar{I}a \sqrt{2\pi\sigma_{\tau}} \sum_{q} e^{-\frac{(s - q\bar{\tau})^2}{\sigma_{\tau}^2}} \frac{1}{|q|}. \quad (22)$$

It should be noted that the deviation of the variance $\sigma_{\tau}^2$ for large $q$ from the quadratic dependence (17) and approach to the linear function $\sigma_{\tau}^2 = 2D_{tk}|q|$ ensures the convergence of sums (21) and (22) and, consequently, results in the Lorentzian power spectrum (11) at $f \to 0$ [6, 7, 8, 9]. Here $D_{tk}$ is the ”diffusion” coefficient of the pulse occurrence time $t_k$, related with the variance $\sigma_{t_k}^2$ of the pulse occurrence time as $\sigma_{t_k}^2 = 2D_{tk}$. For the model (14)–(15) $D_{tk} = \sigma^2/2\gamma^2$.

The power spectra calculated according to Eq. (5) with the correlation functions (21) and (22) are expressed as Eq. (20) and Eq. (16), respectively.

5 Conclusions

From the above analysis we can conclude that the intrinsic origin of $1/f$ noise is a Brownian fluctuations of the interevent time of the signal pulses, similarly to the Brownian fluctuations of the signal amplitude resulting in $1/f^2$ noise. The random walk of the interevent time in the time axis is a property of the randomly perturbed or complex systems with the elements of selforganization.

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