An extended Rayleigh model: Properties, regression and COVID-19 application

Gauss M. Cordeiro
Universidade Federal de Pernambuco
gausscordeiro@gmail.com

Gabriela M. Rodrigues
Universidade de São Paulo
gabrielar@usp.br

Edwin M. M. Ortega
Universidade de São Paulo
edwin@usp.br

Luís H. de Santana
Universidade Federal de Pernambuco
desantanalh@gmail.com

Roberto Vila
Universidade de Brasília
rovig161@gmail.com

Abstract

We define a four-parameter extended Rayleigh distribution, and obtain several mathematical properties including a stochastic representation. We construct a regression from the new distribution. The estimation is done by maximum likelihood. The utility of the new models is proved in two real applications.

Keywords: Censored data, generalized Rayleigh; maximum likelihood estimation; regression model, stochastic representation.

1 Introduction

The Rayleigh distribution has been employed in many areas (Johnson et al., 1994). Several of its extensions have been published so far.
The cumulative distribution function (cdf) of the generalized Rayleigh (GR) distribution (Vodă, 1976) is
\[ G_{GR}(x; \delta, \theta) = \gamma_1(\delta + 1, \theta x^2), \quad x > 0, \] (1)
where \( \delta > -1 \) and \( \theta > 0 \), \( \Gamma(p) = \int_0^\infty w^{p-1} e^{-w} dw \) and \( \gamma_1(p, x) = \Gamma(p)^{-1} \int_0^x w^{p-1} e^{-w} dw \) are the gamma function and lower incomplete gamma function ratio, respectively.

The GR distribution includes well-known sub-models. The classical Rayleigh distribution follows when \( \delta = 0 \) and \( \theta = \lambda^{-2} \). If \( \delta = 2^{-1} \) and \( \theta = (2\lambda^2)^{-1} \), it gives the Maxwell distribution. The chi-square refers to \( \theta = (2\tau^2)^{-1}, \tau > 0 \), and \( \delta = n/2 - 1, n \in \mathbb{N} \), and the half-normal to \( \delta = -2^{-1} \) and \( \theta = 2\sigma^{-2} \).

The probability density function (pdf) corresponding to (1) is
\[ g_{GR}(x; \delta, \theta) = \frac{2\theta^{\delta+1}}{\Gamma(\delta+1)} x^{2\delta+1} e^{-\theta x^2}. \] (2)

Let \( Z \sim GR(\delta, \theta) \) be a random variable with density (2). The moments of \( Z \) are easily obtained from the integral given in Section 3.478 of Gradshteyn and Ryzhik (2000)
\[ \int_0^\infty x^{\nu-1} e^{-\mu x^p} dx = \frac{\Gamma(\nu/p)}{\mu^{\nu/p}}, \]
where \( p, \nu, \mu > 0 \). Indeed, the \( s \)th ordinary moment of \( Z \) (for a positive real number \( s \)) is
\[ \mu'_s(\delta, \theta) = \frac{\Gamma(s/2 + \delta + 1)}{\theta^{s/2} \Gamma(\delta + 1)}. \] (3)

The cdf of the generalized odd log-logistic-G (GOLL-G) family follows from Cordeiro et al. (2017)
\[ F(x; \alpha, \beta, \xi) = \frac{G(x; \xi)^{\alpha\beta}}{G(x; \xi)^{\alpha\beta} + [1 - G(x; \xi)^\beta]^{\alpha}}, \] (4)
where \( \alpha > 0 \) and \( \beta > 0 \) are two extra parameters.

The generalized log-logistic (Gleaton and Lynch, 2006) and proportional reversed hazard rate (Gupta and Gupta, 2007) are special models of Equation (4). Further, the parent comes when \( \alpha = \beta = 1 \). Prataviera et al. (2018) discussed the generalized odd log-logistic flexible Weibull distribution.

If \( g(x; \xi) \) is the parent density, the pdf corresponding to (4) can be written as
\[ f(x; \alpha, \beta, \xi) = \frac{\alpha\beta g(x; \xi) G(x; \xi)^{\alpha\beta-1} [1 - G(x; \xi)^\beta]^{\alpha-1}}{\{G(x; \xi)^{\alpha\beta} + [1 - G(x; \xi)^\beta]^{\alpha}\}^2}. \] (5)

The paper is structured as follows. Section 2 defines the generalized odd log-logistic generalized Rayleigh (GOLLGR) distribution, addresses some asymptotes and quantile function (qf), gives a linear representation for the family density, reports moments and generating function, and addresses maximum likelihood estimation. A new regression is constructed in Section 3, and some simulation studies are carried out as well as residual analysis. Two lifetime data sets in Section 4 show the utility of the new models. Some conclusions are reported in Section 5.
2 The GOLLGR model, properties and estimation

The GOLLGR density is defined (for \( x > 0 \)) by inserting (1) and (2) in Equation (5)
\[
f(x) = f(x; \alpha, \beta, \delta, \theta) = \frac{2\alpha \beta \theta^{\delta+1} x^{2\delta+1} e^{-\theta x^2} \gamma_1 (\delta + 1, \theta x^2) \alpha \beta - 1 [1 - \gamma_1 (\delta + 1, \theta x^2)^\beta] \alpha - 1}{\Gamma(\delta + 1) [\gamma_1 (\delta + 1, \theta x^2)^\beta + [1 - \gamma_1 (\delta + 1, \theta x^2)^\beta] \alpha^2]},
\]
and its hazard rate function (hrf) is
\[
h(x) = \frac{2\alpha \beta \theta^{\delta+1} x^{2\delta+1} e^{-\theta x^2} \gamma_1 (\delta + 1, \theta x^2)^\alpha - 1}{\Gamma(\delta + 1) [1 - \gamma_1 (\delta + 1, \theta x^2)^\beta] \{\gamma_1 (\delta + 1, \theta x^2)^\alpha + [1 - \gamma_1 (\delta + 1, \theta x^2)^\beta] \alpha\}}.
\]
We have \( \lim_{x \to \infty} f(x) = 0 \), and
\[
\lim_{x \to 0^+} f(x) = \begin{cases} 
\frac{2\sqrt{\beta}}{\sqrt{\pi}}, & \alpha \beta - 1 = 0, \ 2\delta + 1 = 0, \\
0, & \alpha \beta - 1 > 0, \ 2\delta + 1 > 0, \\
\infty, & \alpha \beta - 1 > 0, \ 2\delta + 1 < 0, \ 0 < \delta + 1 < \frac{2\delta + 1}{2(1-\alpha \beta)}, \\
\frac{2\alpha \beta \sqrt{\beta}}{\Gamma(\delta + 1)} \left[ \frac{2(1-\alpha \beta)}{\Gamma(\delta + 1)(2\delta + 1)} \right]^{\alpha - 1}, & \alpha \beta - 1 > 0, \ 2\delta + 1 < 0, \ \delta + 1 = \frac{2\delta + 1}{2(1-\alpha \beta)}, \\
0, & \alpha \beta - 1 < 0, \ 2\delta + 1 < 0, \ \delta + 1 > \frac{2\delta + 1}{2(1-\alpha \beta)}, \\
\infty, & \alpha \beta - 1 < 0, \ 2\delta + 1 > 0, \ \delta + 1 > \frac{2\delta + 1}{2(1-\alpha \beta)}, \\
\frac{2\alpha \beta \sqrt{\beta}}{\Gamma(\delta + 1)} \left[ \frac{2(1-\alpha \beta)}{\Gamma(\delta + 1)(2\delta + 1)} \right]^{\alpha - 1}, & \alpha \beta - 1 < 0, \ 2\delta + 1 > 0, \ \delta + 1 = \frac{2\delta + 1}{2(1-\alpha \beta)}, \\
\infty, & \alpha \beta - 1 \leq 0, \ 2\delta + 1 \leq 0, \ \alpha \beta - 1 \neq 0 \text{ or } 2\delta + 1 \neq 0.
\end{cases}
\]
Further, \( \lim_{x \to \infty} h(x) = \infty \) and \( \lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} f(x) \).

Hereafter, we omit the arguments of the functions, and let \( X \sim \text{GOLLGR}(\alpha, \beta, \delta, \theta) \) be a random variable with pdf (6), The odd log-logistic GR follows when \( \beta = 1 \), and the proportional reversed hazard rate GR when \( \alpha = 1 \). Plots of the pdf and hrf of \( X \) are displayed in Figures 1 and 2, respectively. They reveal the bimodality and asymmetry of the pdf, and different shapes of the hrf.

2.1 Asymptotes and quantile function

The asymptotes below follow from Equations (4) and (5)
\[
F(x) \sim G(x)^{\alpha \beta}, \quad f(x), h(x) \sim \alpha \beta g(x)G(x)^{\alpha \beta - 1} \text{ as } x \to 0^+,
\]
and
\[ f(x) \sim \alpha \beta g(x) \left[ 1 - G(x)^\beta \right]^{\alpha - 1}, \quad h(x) \sim \frac{\alpha \beta g(x)}{1 - G(x)^\beta} \quad \text{as} \quad x \to \infty. \]

For the GOLLGR distribution, we obtain
\[
F(x) \sim \gamma_1 (\delta + 1, \theta x^2)^\alpha \beta, \\
f(x), h(x) \sim 2\alpha \beta \theta^{\delta + 1} x^{2\delta + 1} e^{-\theta x^2} \gamma_1 (\delta + 1, \theta x^2)^{\alpha \beta - 1} \quad \text{as} \quad x \to 0^+, \\
\]
and
\[
f(x) \sim 2\alpha \beta \theta^{\delta + 1} x^{2\delta + 1} e^{-\theta x^2} \left[ 1 - \gamma_1 (\delta + 1, \theta x^2) \right]^{\alpha - 1}, \\
h(x) \sim 2\alpha \beta \theta^{\delta + 1} x^{2\delta + 1} e^{-\theta x^2} \frac{1}{\Gamma(\delta + 1) \left[ 1 - \gamma_1 (\delta + 1, \theta x^2) \right]} \quad \text{as} \quad x \to \infty.
\]

By combining the inverse functions of (1) and (4), the qf of \( X \) can be expressed as
\[
x = Q(u) = Q_{GR} \left( \frac{(\frac{u}{1-u})^{1/\alpha}}{1 + (\frac{u}{1-u})^{1/\alpha}} \right)^{1/\beta}, \quad \tag{9}
\]
Figure 2: The GOLLGR hrf. (a) $\alpha = 0.1$ and $\delta = 1.5$. (b) $\alpha = 0.3$ and $\beta = 2.5$. (c) $\alpha = 0.1$.

where the GR qf $Q_{GR}(z) = G_{GR}^{-1}(z; \delta, \theta)$ comes as

$$Q_{GR}(z) = \left[\theta^{-1} \gamma^{-1}(\delta + 1; z)\right]^{1/2}.$$  

Here, $\gamma^{-1}(\delta + 1; z)$ is the gamma qf with shape $\delta + 1$ and unity scale. So, the GOLLGR variates follow easily from (9).

2.2 Stochastic representation

The pdf of a log-logistic random variable $Y \sim LL(\nu, \alpha)$ is

$$f_Y(y; \nu, \alpha) = \frac{(\alpha/\nu)(y/\nu)^{\alpha-1}}{[1 + (y/\nu)\alpha]^2},$$  

where $\nu > 0$ and $\alpha > 0$ are scale and shape, respectively.

If $X \sim GOLLGR(\alpha, \beta, \delta, \theta)$ and $Y \sim LL(1, \alpha)$, we can write from (4) and (10)

$$F(x; \alpha, \beta, \delta, \theta) = P\left(Y \leq \frac{G_{GR}(x; \delta, \theta)^{\beta}}{1 - G_{GR}(x; \delta, \theta)^{\beta}}\right)$$

$$= P\left(\left(\frac{Y}{1+Y}\right)^{1/\beta} \leq G_{GR}(x; \delta, \theta)\right) = P\left(\left[\theta^{-1} \gamma^{-1}(\delta + 1; \left(\frac{Y}{1+Y}\right)^{1/\beta})\right]^{1/2} \leq x\right),$$

where $G_{GR}(x; \delta, \theta) = \gamma_1(\delta + 1, \theta x^2), \ x > 0$. So, we have proved the following:
Proposition 1. The GOLLGR random variable \( X \) admits the stochastic representation:
\[
X = \left[ \theta^{-1} \gamma^{-1} \left( \delta + 1; \left( \frac{Y}{1+Y} \right)^{1/\beta} \right) \right]^{1/2} \quad \text{for } Y \sim LL(1, \alpha).
\]

2.3 Critical points and modality

For brevity, we define
\[
T(x) = T(x; \beta, \delta, \theta) = \frac{G_{GR}(x; \delta, \theta)^{\beta}}{1 - G_{GR}(x; \delta, \theta)^{\beta}}. \tag{13}
\]

Since \( T(x/k; \beta, \delta, \theta) = T(x; \beta, \delta, \theta/k^2), \ k > 0 \), the next result follows from (12):

Proposition 2. If \( X \sim GOLLGR(\alpha, \beta, \delta, \theta) \), then \( kX \sim GOLLGR(\alpha, \beta, \delta, \theta/k^2) \).

By differentiating (12) with respect to \( x \), we obtain
\[
f(x; \alpha, \beta, \delta, \theta) = f_Y(T(x); 1, \alpha) T'(x), \quad Y \sim LL(1, \alpha). \tag{14}
\]

Then, the derivative of \( f(x; \alpha, \beta, \delta, \theta) \) is
\[
f'(x; \alpha, \beta, \delta, \theta) = f_Y'(T(x); 1, \alpha) [T'(x)]^2 + f_Y(T(x); 1, \alpha) T''(x). \tag{15}
\]

Since
\[
f_Y'(t; 1, \alpha) = -f_Y(t; 1, \alpha) r[t], \quad \text{with } r[t] = \frac{t^{\alpha} + \alpha(t^{\alpha} - 1) + 1}{t(t^{\alpha} + 1)},
\]
Equation (15) can be expressed as
\[
f'(x; \alpha, \beta, \delta, \theta) = f_Y(T(x); 1, \alpha) \{ T''(x) - r[T(x)][T'(x)]^2 \},
\]
where
\[
T'(x) = \frac{\beta g_{GR}(x; \delta, \theta) T(x)}{G_{GR}(x; \delta, \theta)[1 - G_{GR}(x; \delta, \theta)^{\beta}]},
\]
\[
T''(x) = T'(x) \left\{ \frac{g_{GR}'(x; \delta, \theta)}{g_{GR}(x; \delta, \theta)} + g_{GR}(x; \delta, \theta) \frac{(\beta + 1)G_{GR}(x; \delta, \theta)^{\beta} + \beta - 1}{G_{GR}(x; \delta, \theta)[1 - G_{GR}(x; \delta, \theta)^{\beta}]} \right\},
\]
and
\[
\frac{g_{GR}'(x; \delta, \theta)}{g_{GR}(x; \delta, \theta)} = \frac{2 \delta + 2 \theta x^2 - 1}{x}. \tag{16}
\]
Then,

\[ f'(x; \alpha, \beta, \delta, \theta) = \]

\[ f_Y(T(x); 1, \alpha)T'(x) \left\{ \frac{g_{GR}(x; \delta, \theta)}{g_{GR}(x; \delta, \theta)} + \frac{g_{GR}(x; \delta, \theta)}{G_{GR}(x; \delta, \theta)} [T(x) + 1] \left\{ (\beta + 1)G_{GR}(x; \delta, \theta)^\beta - \alpha\beta \frac{T(x)^{\alpha - 1}}{T(x)^\alpha + 1} - 1 \right\} \right\}. \]

Equation (14) gives \( f_Y(T(x); 1, \alpha)T'(x) = f(x; \alpha, \beta, \delta, \theta) \), which is a positive function. Hence, any critical point of the pdf of \( X \) satisfies the non-linear equation:

\[ \frac{g'_{GR}(x; \delta, \theta)}{g_{GR}(x; \delta, \theta)} + \frac{g_{GR}(x; \delta, \theta)}{G_{GR}(x; \delta, \theta)} [T(x) + 1] \left\{ (\beta + 1)G_{GR}(x; \delta, \theta)^\beta - \alpha\beta \frac{T(x)^{\alpha - 1}}{T(x)^\alpha + 1} - 1 \right\} = 0. \]

The previous result can be written equivalently as:

**Proposition 3.** A critical point of the pdf of \( X \) satisfies

\[ y'' \left( \frac{y'}{y''} \right)^2 + \frac{(\beta + 1)y^{\beta}[y^{\alpha\beta} + (1 - y^\beta)^\alpha] - (\alpha\beta + 1)y^{\alpha\beta} - 2(1 - y^\beta)^\alpha}{y(1 - y^\beta)[y^{\alpha\beta} + (1 - y^\beta)^\alpha]} = 0, \]  

(17)

where \( y = y(x) = G_{GR}(x; \delta, \theta) = \gamma_1(\delta + 1, \theta x^2) \).

In the remainder of this section, we use Equation (17) and the limit in (8) to analyze the modality of the pdf of \( X \) when \( \alpha = 1 \).

For \( \alpha = 1 \), Equation (17) reduces to

\[ \frac{g_{GR}(x; \delta, \theta)}{G_{GR}(x; \delta, \theta)} = \frac{\delta + \theta x^2 - (1/2)}{x}. \]

Equivalently,

\[ G_{GR}(x; \delta, \theta) = g(x), \]  

(18)

where

\[ g(x) = x g_{GR}(x; \delta, \theta) = \frac{2\theta^{\delta+1}}{\Gamma(\delta + 1)} \frac{x^{2(\delta+1)} e^{-\theta x^2}}{\delta + \theta x^2 - (1/2)}. \]

A careful analysis shows that, on \((0, \infty)\), \( g(x) \) is decreasing/decreasing-increasing-decreasing when \( \delta < 1/2 \) or unimodal when \( \delta \geq 1/2 \), and \( \lim_{x \to 0^+} g(x) = \lim_{x \to \infty} g(x) = 0 \). Moreover, \( g(x) \) has a vertical asymptotic at \( x_{va} = \sqrt{(1/2) - \delta}/\theta \) when \( \delta < 1/2 \), with \( g(x) < 0 \) for all \( x < x_{va} \) and \( g(x) > 0 \) for all \( x > x_{va} \). Since \( G_{GR}(x; \delta, \theta) \) is increasing on \((0, \infty)\), because this one is a cdf, it is plausible to expect that, by varying the parameters, Equation (18) has at most three roots on \((0, \infty)\). So, the pdf of \( X \) has at most three critical points on \((0, \infty)\). In the following, we analyze some possible scenarios:
• If the GOLLGR pdf has a single critical point, say $x_0$, and $\beta > 1$ and $\delta \geq 1 / 2$, we have by the second limit in (8) $\lim_{x \to 0^+} f(x; \alpha, \beta, \delta, \theta) = \lim_{x \to \infty} f(x; \alpha, \beta, \delta, \theta) = 0$. Then, $x_0$ is really a maximum point, and the pdf of $X$ is unimodal.

• If the GOLLGR pdf has three single critical points, say $x_1 < x_2 < x_3$, and $\beta > 1$ and $\delta \geq 1 / 2$, again, by the second limit in (8), we have $\lim_{x \to 0^+} f(x; \alpha, \beta, \delta, \theta) = \lim_{x \to \infty} f(x; \alpha, \beta, \delta, \theta) = 0$. Hence, $x_1$ and $x_3$ are maximum points, and $x_2$ is a minimum point, and the GOLLGR pdf is bimodal.

In general, as done previously, we can use the limit in (8) and the number of critical points of the pdf of $X$ to obtain the result:

**Theorem 4.** The GOLLGR pdf is decreasing/ decreasing-increasing-decreasing/unimodal or bimodal.

### 2.4 Tail behavior

Here, we prove that, under certain constraints in the parameter space, the distribution of $X$ has thinner tails than an exponential distribution. More precisely, we prove the following two results:

**Proposition 5.** For any $\alpha \geq 1$ and any $t > 0$,

$$\lim_{x \to \infty} \frac{e^{-tx}}{1 - F(x; \alpha, \beta, \delta, \theta)} = \infty. \quad (19)$$

**Proof.** If $Y \sim LL(1, \alpha)$, by (12), $F(x; \alpha, \beta, \delta, \theta) = \mathbb{P}(Y \leq T(x))$. Moreover, it is well-known that $\mathbb{P}(Y \leq y) = y^\alpha / (1 + y^\alpha)$. Then, from the definition (13) of $T$, we have (for any $t > 0$),

$$\frac{e^{-tx}}{1 - F(x; \alpha, \beta, \delta, \theta)} = \frac{e^{-tx}}{1 - \mathbb{P}(Y \leq T(x))} = e^{-tx}[1 + T(x)^\alpha]$$

$$\geq e^{-tx}T(x)^\alpha = \frac{e^{-tx} G_{GR}(x; \delta, \theta)^{\alpha \beta}}{[1 - G_{GR}(x; \delta, \theta)^{\beta}]^\alpha}. \quad (20)$$

The L’Hôpital’s rule yields

$$\lim_{x \to \infty} \frac{e^{-tx}}{[1 - G_{GR}(x; \delta, \theta)^{\beta}]^\alpha} = \lim_{x \to \infty} \frac{t \left[ e^{-tx} g_{GR}(x; \delta, \theta) \right]}{\alpha \beta [1 - G_{GR}(x; \delta, \theta)^{\beta}]^{\alpha - 1} G_{GR}(x; \delta, \theta)^{\beta - 1}}. \quad (21)$$

Since (for $\alpha \geq 1$),

$$\lim_{x \to \infty} \frac{e^{-tx}}{g_{GR}(x; \delta, \theta)} = \left[ \frac{2\theta^{\delta+1}}{\Gamma(\delta+1)} x^{2\delta+1} e^{-\theta x^2 + t x} \right]^{-1} = \infty$$
and \( \lim_{x \to \infty} G_{GR}(x; \delta, \theta) = 1 \). We have from (21)

\[
\lim_{x \to \infty} \frac{e^{-tx}}{[1 - G_{GR}(x; \delta, \theta)]^\alpha} = \infty.
\]

By taking \( x \to \infty \) for both sides of inequality (20) and by using the above limit, it follows (19). \( \square \)

**Proposition 6.** For any \( 0 < \beta \leq 1 \) and \( \delta > 0 \), the limit (for \( t > 0 \)) (19) holds.

**Proof.** By inequality in (20), it is enough to prove

\[
\lim_{x \to \infty} \frac{e^{-tx}}{[1 - G_{GR}(x; \delta, \theta)]^\alpha} = \lim_{x \to \infty} \frac{1}{e^{tx}[1 - \gamma_1(\delta + 1, 1, \theta x^2)]^\alpha} = \infty. \tag{22}
\]

Indeed, by using the \( C_p \) inequality:

\[
\forall x, y \geq 0; \ (x + y)^p \leq C_p(x^p + y^p), \quad \text{where} \ p > 0 \text{ and } C_p = \max\{1, 2^{p-1}\};
\]

we have (for \( 0 < \beta \leq 1 \))

\[
1 - \gamma_1(\delta + 1, \theta x^2)^\beta \leq [1 - \gamma_1(\delta + 1, \theta x^2)]^\beta = \Gamma_1(\delta + 1, \theta x^2)^\beta, \tag{23}
\]

where \( \Gamma_1(p, x) = \Gamma(p)^{-1} \int_x^\infty w^{p-1} e^{-w} dw \) is the upper incomplete gamma function ratio.

By using the inequality of Natalini and Palumbo (2000): for \( a > 1, B > 1 \) and \( x > B(a - 1)/(B - 1) \),

\[
\Gamma(a, x) < B x^{a-1} e^{-x};
\]

we have (for \( x > \sqrt{B\delta/\theta(B-1)} \) and \( \delta > 0 \))

\[
\Gamma_1(\delta + 1, \theta x^2)^\beta < B^\beta \theta^{\beta \delta} \Gamma(\delta + 1)^{-\beta} x^{2\beta \delta} e^{-\beta \theta x^2}. \tag{24}
\]

By combining (23) and (24), we obtain (for any \( x > \sqrt{B\delta/\theta(B-1)} \))

\[
e^{tx}[1 - \gamma_1(\delta + 1, \theta x^2)^\beta]^\alpha < B^\alpha \theta^{\alpha \beta \delta} \Gamma(\delta + 1)^{-\alpha \beta} x^{2\alpha \beta \delta} e^{-\alpha \beta \theta x^2 + tx}.
\]

Letting \( x \to \infty \) in the above inequality, we have \( e^{tx}[1 - \gamma_1(\delta + 1, \theta x^2)^\beta]^\alpha \) tends to zero, proving the limit in (22). Thus, we complete the proof. \( \square \)
2.5 Linear Representation

First, the exponentiated-$G$ ("Exp-G") random variable $W \sim \text{Exp}^c G$ for a continuous cdf $G(x)$ and $c > 0$, has cdf $H_c(x) = G(x)^c$ and pdf $h_c(x) = cg(x)G(x)^{c-1}$. Many Exp-G properties were published in the last three decades.

We obtain after some algebra the power series

$$G(x)^{\alpha \beta} + [1 - G(x)^\beta]^{a} = \sum_{k=0}^{\infty} c_k G(x)^k,$$  \hspace{1cm} (25)

where $a_k = a_k(\alpha,\beta) = \sum_{i=0}^{\infty} (a_k)^{\alpha \beta} (i)$. and

$$c_k = c_k(\alpha, \beta) = a_k(\alpha, \beta) + \sum_{i=0}^{\infty} \sum_{j=k}^{\infty} (-1)^{i+j+k} \binom{\alpha \beta}{i} \binom{i \beta}{j} \binom{j}{k}.$$ .

Equation (4) can be rewritten from the ratio of two power series as

$$F(x) = \sum_{k=0}^{\infty} d_k G(x)^k,$$  \hspace{1cm} (26)

where $d_k = d_k(\alpha, \beta)$'s are found recursively (for $k > 0$, $d_0 = a_0/c_0$)

$$d_k = c_0^{-1} \left( a_k + \sum_{r=1}^{k} c_r d_{k-r} \right).$$

By differentiating (26) and changing indices

$$f(x) = \sum_{l=0}^{\infty} (l + 1) d_{l+1} g(x) G(x)^l.$$  

For the GR model, we obtain

$$f(x) = \sum_{l=0}^{\infty} (l + 1) d_{l+1}(\alpha, \beta) \frac{2^{\delta+1}}{\Gamma(\delta + 1)} x^{2\delta+1} e^{-\theta x^2} \gamma_1(\delta + 1, \theta x^2)^l,$$  \hspace{1cm} (27)

The power series for the incomplete gamma function ratio holds

$$\gamma_1(\delta + 1, \theta x^2) = \frac{(\theta x^2)^{\delta+1}}{\Gamma(\delta + 1)} \sum_{m=0}^{\infty} \frac{(-1)^m (\theta x^2)^m}{m! (\alpha + 1 + m)}.$$ .

Equation 0.314 in Gradshteyn and Ryzhik (2000) gives (for a natural number $l \geq 1$)

$$\left( \sum_{m=0}^{\infty} q_m x^m \right)^l = \sum_{m=0}^{\infty} e_m^{(l)} x^m.$$
where \( e_0^{(l)} = q_0^{l} \), and \( e_m^{(l)} \) (for \( l \geq 1 \)) can be found from

\[
e_m^{(l)} = 1 \frac{m}{mq_0} \sum_{i=1}^{m} [(l+1)i - m]q_i e_{m-i}^{(l)}.
\]

(28)

Then,

\[
\gamma_1(\delta+1,\theta x^2)^l = \frac{(\theta x^2)^l(\delta+1)}{\Gamma(\delta+1)^l} \sum_{m=0}^{\infty} e_m^{(l)} x^{2m},
\]

(29)

where the quantities \( e_m^{(l)} \) follow from (28) with the constants

\[
q_m = \frac{(-1)^m \theta^m}{(\delta + 1 + m)m!}
\]

for \( m = 0, 1, \ldots \).

Further, we set the conditions \( e_0^{(0)} = 1 \) and \( e_m^{(0)} = 0 \) for \( m \geq 1 \). Hence, inserting (29) in Equation (27) (under these conditions) yields

\[
f(x) = \sum_{l,m=0}^{\infty} 2^{(l+1)}d_{l+1}(\alpha, \beta) e_m^{(l)} \theta^{l(\delta+1)+m+\delta+1} x^{2l(\delta+1)+m+\delta+1} e^{-\theta x^2}
\]

and then

\[
f(x) = \sum_{l,m=0}^{\infty} w_{l,m} g_{GR}(x; \theta, \delta_{l,m}^*)
\]

(30)

where \( \delta_{l,m}^* = l(\delta+1) + m + \delta \), and the coefficients are

\[
w_{l,m} = \frac{w_{l,m}(\theta, \delta, \alpha, \beta)}{\theta^m \Gamma(\delta + 1)^{l+1}}
\]

Equation (30) is useful to obtain some properties of \( X \) from those of the GR model.

### 2.6 Properties

The \( s \)th ordinary moment of \( X \) comes from (3) and (30) as

\[
E(X^s) = \sum_{l,m=0}^{\infty} w_{l,m}(\theta, \delta, \alpha, \beta) \frac{\Gamma(s/2 + \delta_{l,m}^* + 1)}{\theta^{s/2} \Gamma(\delta_{l,m}^* + 1)}
\]
The $s$th incomplete moment of $X$ follows from (30) as
\[
m_s(x) = \sum_{l,m=0}^{\infty} \frac{w_{l,m}(\theta, \delta, \alpha, \beta)}{\Gamma(\delta_l^* + 1)} \int_0^x 2\theta^{\delta_l^*+1} t^s \theta^{2\delta_l^*+1} e^{-\theta t^2} dt
\]
\[
= \sum_{l,m=0}^{\infty} w_{l,m}(\theta, \delta, \alpha, \beta) \frac{\Gamma(\delta_l^* + s/2 + 1)}{\Gamma(\delta_l^* + 1)\theta^{s/2}} \int_0^x 2\theta^{\delta_l^*+s/2+1} \theta^{2(\delta_l^*+s/2)+1} e^{-\theta t^2} dt
\]
\[
= \sum_{l,m=0}^{\infty} w_{l,m}(\theta, \delta, \alpha, \beta) \frac{\Gamma(s/2 + \delta_l^* + 1)}{\theta^{s/2} \Gamma(\delta_l^* + 1)} \gamma_1(\delta_l^* + s/2 + 1, \theta x^2).
\]

The mean deviations and Bonferroni and Lorenz curves of $X$ are obtained from $m_1(x)$.

The generating function (gf) of $X$ can be expressed as
\[
M(t) = \int_0^\infty e^{tx} \sum_{l,m=0}^{\infty} w_{l,m}(\theta, \delta, \alpha, \beta) g_{GR}(x; \theta, \delta_l^*) \, dx,
\]
that is,
\[
M(t) = \sum_{l,m=0}^{\infty} \frac{2 w_{l,m}(\theta, \delta, \alpha, \beta)}{\Gamma(\delta_l^* + 1)} \theta^{2\delta_l^*+1} \int_0^\infty x^{2\delta_l^*+1} e^{-\theta x^2 + tx} \, dx.
\]

From Equation 2.3.15.3 in Prudnikov et al. (1986), we can write
\[
\int_0^\infty x^{\alpha-1} e^{-px^2} e^{-q x} \, dx = \frac{\Gamma(\alpha)}{(2p)^{\alpha/2}} \exp\left(\frac{q^2}{8p}\right) D_{-\alpha}\left(\frac{q}{\sqrt{2p}}\right),
\]
where $\text{Re}(\alpha), \text{Re}(p) > 0$, and
\[
D_p(y) = \frac{\exp(-y^2/4)}{\Gamma(-p)} \int_0^\infty \exp\{- (wy + w^2/2)\} w^{-(p+1)} \, dw.
\]

Thus,
\[
M(t) = \sum_{l,m=0}^{\infty} \frac{2 w_{l,m} \theta^{\delta_l^*+1}}{\Gamma(\delta_l^* + 1)} \frac{\Gamma(\delta_l^*)}{(2\theta)^{\delta_l^*}} \exp\left(\frac{t^2}{8\theta}\right) D_{-\delta_l^*}\left(-\frac{t}{\sqrt{2\theta}}\right),
\]
where $\delta_l^* = 2(\delta_l^* + 1)$. 


2.7 Estimation

We obtain the maximum likelihood estimate (MLE) of $\eta = (\alpha, \beta, \delta, \theta)^\top$ given the data $x_1, \ldots, x_n$ from the GOLLGR distribution.

The total log-likelihood function for $\eta$ is

$$l_n(\eta) = n \left[ \log(\alpha \beta) + (\delta + 1) \log \theta \right] + (2\delta + 1) \sum_{i=1}^{n} \log x_i - \theta \sum_{i=1}^{n} x_i^2$$

$$+ (\alpha \beta - 1) \sum_{i=1}^{n} \log \gamma_1(\delta + 1, \theta x_i^2) + (\alpha - 1) \sum_{i=1}^{n} \log[1 - \gamma_1(\delta + 1, \theta x_i^2)^\beta]$$

$$- 2 \sum_{i=1}^{n} \log \{\gamma_1(\delta + 1, \theta x_i^2)^{\alpha \beta} + [1 - \gamma_1(\delta + 1, \theta x_i^2)^\beta]^{\alpha}\}. \quad (31)$$

The maximization of (31) can be done using the R software or SAS (PROC NLMIXED), among others.

3 The GOLLGR regression

Recently some papers on regression models have been published, for example, see, Hashimoto et al. (2019), Prataviera et al. (2020), Silva et al. (2020) and Vasconcelos et al. (2021). Based on these papers we introduced the regression model based on the GOLLGR distribution.

The systematic components of the GOLLGR regression for $X$ are defined by

$$\theta_i = \exp(v_i^\top \lambda_1) \quad \text{and} \quad \delta_i = \exp(v_i^\top \lambda_2) - 1 \quad i = 1, \ldots, n, \quad (32)$$

where $\lambda_1 = (\lambda_{11}, \ldots, \lambda_{1p})^\top$ and $\lambda_2 = (\lambda_{21}, \ldots, \lambda_{2p})^\top$ are vectors of unknown coefficients, and $v_i^\top = (v_{i1}, \ldots, v_{ip})$ is a vector of known explanatory variables.

The survival function of $X$ comes from (4) as

$$S(x_i|v) = \frac{[1 - \gamma_1(\delta_i + 1, \theta_i x_i^2)^{\beta}]^{\alpha}}{\gamma_1(\delta_i + 1, \theta_i x_i^2)^{\alpha \beta} + [1 - \gamma_1(\delta_i + 1, \theta_i x_i^2)^\beta]^{\alpha \beta}}. \quad (33)$$

Equation (33) opens new possibilities for fitting different types of regressions. The odd log-logistic GR (OLLGR) regression follows when $\beta = 1$, the exponentiated GR (EGR) regression when $\alpha = 1$, and the GR regression when $\beta = \alpha = 1$.

Let $X_i$ be the lifetime and $C_i$ be the non-informative censoring time (assuming independent),
and $x_i = \min\{X_i, C_i\}$. The total log-likelihood function for $\eta = (a, b, \lambda_1^T)^T$ from regression (32) is

$$l(\eta) = r \log \left( \frac{\alpha \beta 2}{\Gamma(\delta_i + 1)} \right) + (\delta_i + 1) \sum_{i \in F} \log(\theta_i) + (2 \delta_i + 1) \sum_{i \in F} \log(x_i) - \sum_{i \in F} \theta_i x_i^2$$

$$+ (\alpha \beta - 1) \sum_{i \in F} \log[\gamma_1(\delta_i + 1, \theta_i x_i^2)] + (\alpha - 1) \sum_{i \in F} \log[1 - \gamma_1(\delta_i + 1, \theta_i x_i^2)^\beta]$$

$$- 2 \sum_{i \in F} \log \left\{ \gamma_1(\delta_i + 1, \theta_i x_i^2)^\alpha \beta + [1 - \gamma_1(\delta_i + 1, \theta_i x_i^2)^\beta]^\alpha \right\}$$

$$+ \sum_{i \in C} \log \left\{ \frac{[1 - \gamma_1(\delta_i + 1, \hat{\theta}_i x_i^2)^\beta]^\alpha}{\gamma_1(\delta_i + 1, \hat{\theta}_i x_i^2)^\alpha \beta + [1 - \gamma_1(\delta_i + 1, \hat{\theta}_i x_i^2)^\beta]^\alpha} \right\},$$

(34)

where $F$ and $C$ are the sets of observed lifetimes and censoring times, $r$ is the number of uncensored observations (failures). The MLE $\hat{\eta}$ of the vector of unknown parameters can be found by maximizing (34).

### 3.1 Residual analysis

The quantile residuals (qrs) (Dunn and Smith, 1996) have been adopted frequently in regression applications to verify possible deviations from the model assumptions. For the proposed regression, they are

$$q_i r = \Phi^{-1} \left\{ \frac{\gamma_1(\delta_i + 1, \hat{\theta}_i x_i^2)^\alpha \beta}{\gamma_1(\delta_i + 1, \hat{\theta}_i x_i^2)^\alpha \beta + [1 - \gamma_1(\delta_i + 1, \hat{\theta}_i x_i^2)^\beta]^\alpha} \right\},$$

(35)

where $\Phi^{-1}(\cdot)$ is the standard normal qf.

### 3.2 Simulations

In this section, two simulation studies are presented by using `gamlss` package in R software.

**The GOLLGR distribution**

First, we generate 1,000 samples from Equation (9) with $\alpha = 0.35$, $\beta = 0.55$, $\delta = -0.55$ and $\theta = 0.11$, and sample sizes $n = 50, 150$ and $500$, and calculate the MLEs. The average estimates (AEs), biases and mean squared errors (MSEs) in Table 1 show that the AEs converge to the true values and the biases and MSEs decrease when $n$ increases.

**The GOLLGR regression**
Table 1: Simulation results from the GOLLGR distribution.

| Parameter | $n = 50$ | $n = 150$ | $n = 500$ |
|-----------|----------|-----------|-----------|
|           | AE       | Bias      | MSE       | AE       | Bias      | MSE       | AE       | Bias      | MSE       |
| $\alpha$  | 0.3533   | 0.0033    | 0.0205    | 0.3487   | -0.0013   | 0.0050    | 0.3488   | -0.0012   | 0.0027    |
| $\beta$   | 0.7140   | 0.1640    | 0.3857    | 0.5861   | 0.0361    | 0.0197    | 0.5688   | 0.0188    | 0.0095    |
| $\delta$  | -0.5217  | 0.0283    | 0.1054    | -0.5512  | -0.0012   | 0.0044    | -0.5514  | -0.0014   | 0.0018    |
| $\sigma$  | 0.1484   | 0.0384    | 0.0115    | 0.1188   | 0.0088    | 0.0015    | 0.1148   | 0.0048    | 0.0008    |

The second study examines the accuracy of the MLEs in the proposed regression. The observations are generated from $X_i \sim \text{GOLLGR}(\alpha, \beta, \delta_i, \theta_i)$ and $v_{1i} \sim \text{Binomial}(1, 0.5)$, where $v_{1i}$ is taken in two groups (0 and 1). We consider 1,000 samples for $\alpha = 0.37$, $\beta = 0.61$, $\lambda_{10} = 0.55$, $\lambda_{11} = 1.75$, $\lambda_{20} = 0.65$ and $\lambda_{21} = 2.75$ and $n = 150, 350$ and 650, and the simulation process follows as:

(i) Generate $v_{1i} \sim \text{Binomial}(1, 0.5)$;
(ii) Calculate $\delta_i$ and $\theta_i$ from the systematic components: $\delta_i = \exp(\lambda_{10} + \lambda_{11}v_{1i}) - 1$ and $\theta_i = \exp(\lambda_{20} + \lambda_{21}v_{1i})$, respectively;
(iii) Generate $u_i \sim U(0, 1)$;
(iv) The previous steps yield $x_i$’s from (9).

The numbers in Table 2 reveal that the AEs converge to the true values, and the biases and MSEs decrease when $n$ increases, thus indicating the consistency of the estimators.

Table 2: Simulation results from the GOLLGR regression.

| Parameters | $n = 150$ | $n = 350$ | $n = 650$ |
|-----------|----------|-----------|-----------|
|           | AE       | Bias      | MSE       | AE       | Bias      | MSE       | AE       | Bias      | MSE       |
| $\alpha$  | 0.3736   | 0.0036    | 0.0146    | 0.3700   | 0.0000    | 0.0055    | 0.3722   | 0.0022    | 0.0028    |
| $\beta$   | 0.7324   | 0.1224    | 0.2991    | 0.6615   | 0.0515    | 0.0897    | 0.6317   | 0.0217    | 0.0332    |
| $\lambda_{10}$ | 0.5917 | 0.0417    | 0.2022    | 0.5787   | 0.0287    | 0.0973    | 0.5618   | 0.0118    | 0.0467    |
| $\lambda_{11}$ | 1.7715 | 0.0215    | 0.0535    | 1.7586   | 0.0086    | 0.0219    | 1.7553   | 0.0053    | 0.0114    |
| $\lambda_{20}$ | 0.7308 | 0.0808    | 0.1444    | 0.6955   | 0.0455    | 0.0606    | 0.6640   | 0.0140    | 0.0307    |
| $\lambda_{21}$ | 2.7400 | -0.0100   | 0.0582    | 2.7450   | -0.0050   | 0.0249    | 2.7518   | 0.0018    | 0.0127    |

4 Applications

Here, we compare the GOLLGR model and its GR, EGR, OLLGR sub-models in two applications. We determine the MLEs, and the criteria: Akaike Information Criterion (AIC), Consistent Akaike, Information Criterion (CAIC), and Bayesian Information Criterion (BIC).
4.1 Application 1: Voltage data

We consider the times of failure and running times for a field-tracking study (Meeker and Escobar, 1998). We fit the Rayleigh distribution \((\alpha = \beta = 1)\) to find initial values for \(\theta\) and \(\delta\). All computations are done through the NLMIXED subroutine in SAS. Table 3 lists the MLEs (their standard errors in parentheses), and the previous measures, which reveal that the GOLLG distribution can be chosen as the best model.

| Model   | \(\alpha\)  | \(\beta\)  | \(\delta\) | \(\theta\) | AIC       | CAIC      | BIC       |
|---------|-------------|-------------|-------------|-------------|-----------|-----------|-----------|
| GOLLGR  | 0.0437      | 0.4611      | 34.6605     | 0.0011      | 362.0     | 363.6     | 367.6     |
|         | (0.0111)    | (0.0151)    | (0.00002)   | (3.958E-6)  |           |           |           |
| OLLGR   | 0.0246      | 1           | 37.0100     | 0.0010      | 383.2     | 384.1     | 387.4     |
|         | (0.0041)    | (0.0070)    | (0.00006)   |             |           |           |           |
| EGR     | 1           | 0.0133      | 35.3427     | 0.00019     | 366.5     | 367.4     | 370.7     |
|         | (0.0014)    | (0.0051)    | (0.00003)   |             |           |           |           |
| GR      | 1           | 1           | -0.5079     | 0.000011    | 368.4     | 368.8     | 371.2     |
|         | (0.1147)    | (5.1E-6)    |             |             |           |           |           |

The likelihood ratio (LR) statistics in Table 4 indicate that the GOLLGR distribution is the best model among the others. The histogram and the plots of the estimated densities in Figure 3(a), and those of the empirical and estimated survival functions in Figure 3(b), support the previous conclusion.

| Hypotheses                     | LR statistic | p-value |
|--------------------------------|--------------|---------|
| GOLLGR vs OLLGR: \(H_0 : \beta = 1\) vs \(H_1 : \beta\) is false | 23.2         | <0.00001|
| GOLLGR vs EGR: \(H_0 : \alpha = 1\) vs \(H_1 : \alpha\) is false | 6.5          | 0.01079 |
| GOLLGR vs GR: \(H_0 : \beta = \alpha = 1\) vs \(H_1 : \beta\) is false | 10.4         | 0.0055  |

4.2 Application 2: COVID-19 data

The second application refers to lifetimes of individuals diagnosed with COVID-19 (Coronavirus Disease 1999) (Galvão and Roncalli, 2021). Since it was declared an international health emergency, many studies have been conducted to obtain information about the clinical, epidemiological and prognostic aspects of the disease; see, for example, Cordeiro et al. (2021a), Cordeiro et al. (2021b) and Marinho et al. (2021).
In Brazil, the epidemiological data are disclosed by the Health Information System (available in: https://opendatasus.saude.gov.br/en/dataset/srag-2021-e-2022. In this analysis, we work with the `gamlss` package of R.

In this study, 881 patients infected by the virus are considered, confirmed by the RT-PCR test method. The participants consisted of hospitalized patients and outpatients living in the city of Campinas (Brazil) in January and February 2021. The survival consisted of the interval between the first symptoms until the date of death due to COVID-19 (failure). Deaths due to other causes or after the is 73.6%. Equation (32) is considered with factors associated with the highest risk of death. The results are compared with the OLLGR, EGR and GR sub-regressions.

The following variables were considered for each patient \((i = 1, \ldots, 881):\)

- \(x_i: \) time until death due to COVID-19 (in days);
- \(\text{cens}_i: \) censoring indicator \((0 = \text{censored}, 1 = \text{observed lifetime});\)
- \(v_{i1}: \) age (in years);
- \(v_{i2}: \) diabetes mellitus \((0= \text{no or not reported}, 1= \text{yes}).\)

The total number of patients suffering from the comorbidity diabetes was 264 (29.97%), among whom 104 (39.39%) died. In turn, of the 617 patients (70.03%) without the disease or who did
not report it, 128 (20.75%) died. Figure 4 presents the Kaplan-Meier survival curve, showing the greater risk of death among patients suffering from diabetes.

![Kaplan-Meier survival curve](image)

Figure 4: Kaplan-Meier survival curve for the variable diabetes mellitus (1 = yes, 0 = no or not informed).

The statistics in Table 5 support that the GOLLGR regression can be chosen as the best model. Further, the LR statistics in Table 6 indicate that the wider regression yields the best fit. Table 7 reports the MLEs (SEs in parentheses) from the fitted GOLLGR regression.

Table 5: Findings from the fitted regressions to COVID-19 data.

| Model  | AIC     | BIC     | CAIC    |
|--------|---------|---------|---------|
| GOLLGR | 2222.54 | 2260.78 | 2237.66 |
| OLLGR  | 2237.73 | 2271.19 | 2262.63 |
| EGR    | 2240.17 | 2273.63 | 2265.07 |
| GR     | 2238.66 | 2267.34 | 2273.34 |

Figure 5 provides the graphs of the quantile residuals (qrs) (35). The residual index plot (Figure 5a) reveals that the qrs have a random behavior and that only four observations are outside the \([-3, 3]\) range. The normal probability plot for the qrs (Figure 5b) indicates that the residuals...
Table 6: LR statistics for COVID-19 data.

| Model            | Hypotheses                          | LR statistic | p-value  |
|------------------|--------------------------------------|--------------|----------|
| GOLLGR vs OLLGR  | $H_0 : \beta = 1$ vs $H_1 : H_0$ is false | 17.1915      | <0.00001 |
| GOLLGR vs EGR    | $H_0 : \alpha = 1$ vs $H_1 : H_0$ is false | 19.6298      | <0.00001 |
| GOLLGR vs GR     | $H_0 : \beta = \alpha = 1$ vs $H_1 : H_0$ is false | 20.1202      | <0.00001 |

Table 7: Results from the fitted GOLLGR regressions to COVID-19 data.

|           | MLE      | SE       | p-value  |
|-----------|----------|----------|----------|
| $\lambda_{10}$ | -0.3599  | 0.0289   | <0.0001  |
| $\lambda_{11}$ | -0.0028  | 0.0006   | <0.0001  |
| $\lambda_{12}$ | 0.0468   | 0.0359   | 0.1920   |
| $\lambda_{20}$ | -10.1148 | 0.1176   | <0.0001  |
| $\lambda_{21}$ | 0.0480   | 0.0022   | <0.0001  |
| $\lambda_{22}$ | 0.3331   | 0.0774   | <0.0001  |
| log($\alpha$) | -0.9748  | 0.0130   |          |
| log($\beta$)  | 2.0127   | 0.0130   |          |

follow approximately a normal distribution, which support the fitted regression. Thus, there is no evidence against the GOLLGR regression assumptions.

Figure 5: COVID-19 data: (a) Index plot of the qrs. (b) Normal probability plot of the qrs.
Some interpretations are in order. Table 7 shows that the covariable age is significant, meaning that older individuals tend to have a progressively shorter period until death due to this coronavirus. It is noted a significant difference between individuals with and without diabetes mellitus in relation to the time until death by COVID-19.

5 Conclusions

There is a clear need for extended well-known distributions and their successful applications in several areas. The Rayleigh distribution plays a crucial role in modelling and analyzing lifetime data, and several extensions of this distribution have been published in recent years. We constructed a new regression based on the four-parameter extended Rayleigh distribution, and showed its utility in the analysis of lifetime data.

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