Blazing a Trail via Matrix Multiplications: A Faster Algorithm for Non-shortest Induced Paths

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Abstract

For vertices $u$ and $v$ of an $n$-vertex graph $G$, a $uv$-trail of $G$ is an induced $uv$-path of $G$ that is not a shortest $uv$-path of $G$. Berger, Seymour, and Spirkl [Discrete Mathematics 2021] gave the previously only known polynomial-time algorithm, running in $O(n^{18})$ time, to either output a $uv$-trail of $G$ or ensure that $G$ admits no $uv$-trail. We reduce the complexity to the time required to perform a polylogarithmic number of multiplications of $n^2 \times n^2$ Boolean matrices, leading to a largely improved $O(n^{4.75})$-time algorithm.

1 Introduction

Let $G$ be an $n$-vertex simple, finite, undirected, and unweighted graph. Let $V(G)$ (respectively, $E(G)$) denote the vertex (respectively, edge) set of $G$. For any subgraph $H$ of $G$, let $G[H]$ be the subgraph of $G$ induced by $V(H)$. A subgraph $H$ of $G$ is induced if $G[H] = H$. That is, an induced subgraph of $G$ is a subgraph of $G$ that can be obtained by deleting a set of vertices together with its incident edges from $G$. Various kinds of induced subgraphs are involved in the deepest results of graph theory and graph algorithms. One of the most prominent examples concerns “perfect graphs”. A graph $G$ is perfect if every induced subgraph $H$ of $G$ has chromatic number equal to its clique number. A graph is odd (respectively, even) if it has an odd (respectively, even) number of edges. A hole of $G$ is an induced cycle of $G$ having at least four edges. The seminal Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour, and Thomas [18, 23], conjectured by Berge in 1960 [4, 5, 6], states that a graph is perfect if and only if it has no odd hole or complement of an odd hole, implying that a perfect graph $G$ can be recognized by detecting an odd hole in $G$ or its complement $\bar{G}$. Based on the theorem, the first known polynomial-time algorithms for recognizing perfect graphs take $O(n^{18})$ [32] and $O(n^9)$ [15] time. The $O(n^9)$-time version can be implemented to run in $O(n^{8.373})$ time via Boolean matrix multiplications [55, §6.2].

Detecting a class of induced subgraphs can be more difficult than detecting its counterpart that need not be induced [36]. For instance, detecting a path spanning three prespecified vertices is tractable (via, e.g., [53, 62]), but the three-in-a-path problem that detects an induced path spanning three prespecified
vertices is NP-hard (see, e.g., [46, 55]). Cycle detection has a similar situation. Detecting a cycle of length three, which has to be induced, is the classical triangle detection problem that can also be solved efficiently by Boolean matrix multiplications. Although it is tractable to detect a cycle of length at least four spanning two prespecified vertices (also via, e.g., [53, 62]), the two-in-a-cycle problem that detects a hole spanning two prespecified vertices is NP-hard (and so are the corresponding one-in-an-even-cycle and one-in-an-odd-cycle problems) [8, 9]. See, e.g., [61, §3.1] for graph classes on which the two-in-a-cycle problem is tractable. Detecting a tree spanning an arbitrary set of prespecified vertices is easy via computing the connected components of G. Detecting an induced tree spanning an arbitrary set of prespecified vertices is NP-hard [45]. The three-in-a-tree problem that detects an induced tree spanning three prespecified vertices was first shown to be solvable in \(O(n^4)\) time [22] and then in \(O(n^2 \log^2 n)\) time [55]. The tractability of the corresponding k-in-a-tree problem for any fixed \(k \geq 4\) is unknown. See [57] for an \(O(n^4)\)-time algorithm for the k-in-a-tree problem in a graph of girth at least \(k\).

As for subgraph detection without the requirement of spanning prespecified vertices, detecting a cycle is straightforward. Even and odd cycles are also long known to be efficiently detectable (see, e.g., [3, 34, 67]). While a hole can be detected (i.e., recognizing chordal graphs) in \(O(n^2)\) time [63, 64, 65] and found in \(O(n^4)\) time [59], detecting an odd (respectively, even) hole is more difficult. There are early \(O(n^3)\)-time algorithms for detecting odd and even holes in planar graphs [49, 60], but the tractability of detecting an odd hole was open for decades (see, e.g., [24, 26, 29]) until the recent major breakthrough of Chudnovsky, Scott, Seymour, and Spirkl [21] showing that an odd hole can be detected in \(O(n^9)\) time. Via the Strong Perfect Graph Theorem this yields another \(O(n^9)\)-time algorithm to recognize perfect graphs. Their \(O(n^9)\)-time algorithm is later implemented to run in \(O(n^8)\) time [55] and recently improved to run in \(O(n^7)\) time [13], implying that the state-of-the-art algorithm for recognizing perfect graphs runs in \(O(n^7)\) time. It is also known that a shortest odd hole can be found in \(O(n^4)\) time [20] and \(O(n^{13})\) time [13]. As for detecting even holes, the first polynomial-time algorithm, running in about \(O(n^{40})\) time, appeared in 1997 [25, 27, 28]. It takes a line of intensive efforts to bring down the complexity to \(O(n^{31})\) [16], \(O(n^{19})\) [33], \(O(n^{11})\) [10], and finally \(O(n^9)\) [55]. The tractability of finding a shortest even hole, open for 16 years [16, 51], is resolved by [12] showing with more careful analysis that the \(O(n^{31})\)-time algorithm of [16] actually outputs a shortest even hole if there is one. The complexity is recently reduced to \(O(n^{19})\) [13]. See [19] (respectively, [31]) for detecting an odd (respectively, even) hole with a prespecified length lower bound. See [1, 17] for the first polynomial-time algorithm for finding an independent set of maximum weight in a graph having no hole of length at least five. See [35] for upper and lower bounds on the complexity of detecting an \(O(1)\)-vertex induced subgraph.

The two-in-a-path problem that detects an induced path spanning two prespecified vertices is equivalent to determining whether the two vertices are connected. On the other hand, the corresponding two-in-an-odd-path and two-in-an-even-path problems are NP-hard [8, 9], although each of them admits an \(O(n^7)\)-time algorithm when \(G\) is planar [52]. See [38, 40, 58] for how an induced even uv-path of \(G\) affects whether \(G\) is perfect. See [54] for a conjecture by Erdős on how an induced uv-path of \(G\) affects the connectivity between \(u\) and \(v\) in \(G\). Finding a longest uv-path in \(G\) that has to (respectively, need not) be induced is NP-hard [42, GT23] (respectively, [42, ND29]). See [44, 50] for longest or long induced paths in special graphs. The presence of long induced paths in \(G\) affects the tractability of coloring \(G\) [43]. See also [1] for the first polynomial-time algorithm for finding a minimum vertex set intersecting all cycles of a graph having no induced path of length at least five. Detecting a non-shortest uv-path in \(G\) is easy. A k-th shortest uv-path in \(G\) can also be found in near linear time [37].
Figure 1: The red \( uv \)-path \( P \) is the only \( uv \)-trail of the \( uv \)-straight graph \( G \). The twist pair of \( P \) is \((c, b)\). The twist of \( P \) is 6. \( P[a^*, c] \) and \( P[b, d^*] \) form a pair of wings for the quadruple \((a, b, c, d)\) of \( V(G) \) in \( G \).

See \cite{[47]} for algorithms that list induced paths and induced cycles. See \cite{[11], §4} for the parameterized complexity of detecting an induced path with a prespecified length. Detecting an induced \( uv \)-path in a directed graph \( G \) is NP-complete (even if \( G \) is planar) \cite{[39]} and \( W[1] \)-complete \cite{[46]}. However, the tractability of detecting a non-shortest induced \( uv \)-path in an undirected graph \( G \) was unknown until the recent result of Berger, Seymour, and Spirkl \cite{[7]}.

Let \( \|G\| \) denote the number of edges in \( G \). A path with end-vertices \( u \) and \( v \) is a \( uv \)-path. If \( P \) is a path with \( \{u, v\} \subseteq V(P) \), then let \( P[u, v] \) denote the \( uv \)-path of \( P \). A \( uv \)-path \( P \) of \( G \) is shortest if \( G \) admits no \( uv \)-path \( Q \) with \( \|Q\| < \|P\| \), so each shortest \( uv \)-path of \( G \) is induced. We call an induced \( uv \)-path of \( G \) that is not a shortest \( uv \)-path of \( G \) a \( uv \)-trail of \( G \). See Figure 1 for an example. A graph admitting no \( uv \)-trail is \( uv \)-trailless. Berger, Seymour, and Spirkl \cite{[7]} gave the formerly only known polynomial-time algorithm, running in \( O(n^{18}) \) time, to either output a \( uv \)-trail of \( G \) or ensure that \( G \) is \( uv \)-trailless. Their result leads to an \( O(n^{21}) \)-time algorithm \cite{[30]} to determine whether all holes of \( G \) have the same length. The \( \tilde{O} \) notation hides an \( O(\log^2 n) \) factor and \( \omega < 2.373 \) denotes the exponent of square-matrix multiplication \cite{[2, 56, 66]} throughout the paper. We improve the time of finding a \( uv \)-trail to \( O(n^{4.75}) \) as summarized in the following theorem, which immediately reduces the \( O(n^{21}) \) time \cite{[30]} of recognizing a graph with all holes the same length to \( O(n^{7.75}) \).

**Theorem 1.** For any two vertices \( u \) and \( v \) of an \( n \)-vertex graph \( G \), it takes \( \tilde{O}(n^{2\omega}) \) time to either obtain a \( uv \)-trail of \( G \) or ensure that \( G \) is \( uv \)-trailless.

**Technical overview** Berger et al.’s and our algorithms are based on the following “guess-and-verify” approach. A subroutine \( B \) taking an \( \ell \)-tuple of \( V(G) \) as the only argument is a \( uv \)-trailblazer of degree \( \ell \) for \( G \) if running \( B \) on all \( \ell \)-tuples of \( V(G) \) always reports a \( uv \)-trail of \( G \) unless \( G \) is \( uv \)-trailless. We call an \( \ell \)-tuple of \( V(G) \) on which \( B \) reports a \( uv \)-trail of \( G \) a trail marker for \( B \). An \( O(f(n)) \)-time \( uv \)-trailblazer of degree \( \ell \) for \( G \) immediately implies the following \( O(n^\ell \cdot f(n)) \)-time trailblazing algorithm for \( G \):

For each \( \ell \)-tuple \((a_1, \ldots, a_\ell) \) of \( V(G) \),

- if \( B(a_1, \ldots, a_\ell) \) returns a \( uv \)-trail \( P \) of \( G \), then output \( P \) and halt the algorithm;
- otherwise, knowing that \((a_1, \ldots, a_\ell) \) is not a trail marker for \( B \), proceed to the next iteration.

Report that \( G \) is \( uv \)-trailless.

A graph \( H \) is \( uv \)-straight \cite{[7]} if \( \{u, v\} \subseteq V(H) \) and each vertex of \( H \) belongs to at least one shortest \( uv \)-path of \( H \). For instance, the graph in Figure 1 is \( uv \)-straight. Berger et al.’s algorithm starts with an \( O(n^3) \)-
time preprocessing step (see Lemma 1) that either reports a $uv$-trail of $G$ or obtains a $uv$-straight graph $H$ with $V(H) \subseteq V(G)$ such that

- a $uv$-trail of $G$ can be obtained from a $uv$-trail of $H$ in $O(n^2)$ time and
- if $H$ is $uv$-trailless, then so is $G$.

If no $uv$-trail is reported by the preprocessing, then the main procedure runs an $O(n^{18})$-time trailblazing algorithm on the $uv$-straight graph $H$ based on an $O(n^4)$-time degree-14 $uv$-trailblazer for $H$. As for postprocessing, if a $uv$-trail of $H$ is obtained by the main procedure, then report a $uv$-trail of $G$ obtainable in $O(n^2)$ time as ensured by the preprocessing. Otherwise, report that $G$ is $uv$-trailless.

Our $O(n^{4.75})$-time algorithm adopts the preprocessing and postprocessing steps of Berger et al., while reducing the preprocessing time from $O(n^3)$ to $O(n^6)$ (see Lemma 5). For the benefit of the main procedure, we run a second preprocessing step, taking $O(n^{4.75})$ time via the witness-matrix technique of Galil and Margalit [41], to compute a static data structure from which a pair of “wings” that are some disjoint paths in $H$, if any, for each quadruple of $V(H)$ can be obtained in $O(n)$ time (see Lemma 6). Our main procedure is also a trailblazing algorithm, based on a faster $uv$-trailblazer of a lower degree for $H$: We reduce the time from $O(n^4)$ to $O(n^2 \log^2 n)$ and largely bring down the degree from 14 to 2. Thus, the main procedure runs in $O(n^4 \cdot \log^2 n)$ time, even faster than the second preprocessing step.

The key to our improved $uv$-trailblazer is a new observation, described by Lemma 4, on any shortest $uv$-trail $P$ of a $uv$-straight graph $G$. Specifically, Berger et al.’s degree-14 $uv$-trailblazer seeks in $O(n^4)$ time a $uv$-trail $P$ of $G$ that consists of

- a shortest $us$-path $S$ of $G$ containing 7 prespecified vertices and a shortest $tv$-path $T$ of $G$ containing another 7 prespecified vertices such that $S$ and $T$ are disjoint and nonadjacent in $G$ and
- a shortest $st$-path $Q$ of $G_{S,T} = G - (N_G[S \cup T] \setminus N_G[\{s, t\}])$.

The 14 vertices are to ensure that $s$ and $t$ are connected in $G_{S,T}$. Lemma 4 implies that much fewer prespecified vertices on $S$ and $T$ suffice to guarantee that $s$ and $t$ are connected in $G_{S,T}$. To illustrate the usefulness of Lemma 4, we show in §2 that three lemmas of Berger et al. [7] (i.e., Lemmas 1, 2, and 3) together with Lemma 4 already yield an $O(n^2)$-time $uv$-trailblazer of degree 4 for $G$, leading to a simple $O(n^6)$-time trailblazing algorithm on $G$. More precisely, as in the example of Figure 1, let $\{a, b, c, d\} \subseteq V(P)$ for a shortest $uv$-trail $P$ of $G$ with $d_P(u, a) \leq d_P(u, c) < d_P(u, b) \leq d_P(u, d)$, $d_G(u, a) = d_G(u, b)$, and $d_G(u, c) = d_G(u, d)$ such that $n \cdot (d_G(u, c) - d_G(u, a)) + d_P(a, b) + d_P(c, d)$ is maximized. Due to the symmetry between $u$ and $v$ in $G$, Lemma 4 guarantees an $O(n^2)$-time obtainable $uv$-trail of $G$ that contains the precomputed pair of “wings” for the 4-tuple $(a, b, c, d)$ (see Lemma 2), implying that $(a, b, c, d)$ is a trail marker for an $O(n^2)$-time $uv$-trailblazer for $G$.

Our proof of Theorem 1 in §3 further displays the usefulness of Lemma 4. We show that the aforementioned vertices $a$ and $b$ in $P$ actually form a trail marker $(a, b)$ for an $O(n^2 \log^2 n)$-time $uv$-trailblazer for $G$. Dropping both $c$ and $d$ from the trail marker $(a, b, c, d)$ of §2 inevitably increases the time of the $uv$-trailblazer for $G$. We manage to keep the time of a degree-two $uv$-trailblazer as low as $O(n^2 \log^2 n)$ via the dynamic data structure of Holm, de Lichtenberg, and Thorup [48] supporting efficient edge updates and connectivity queries for $G$ (see Lemma 7). To make our proof of Theorem 1 in §3 self-contained, a simplified proof of Lemma 3 is included in §2. Since Lemmas 1 and 2 are implied by Lemmas 5 and 6, which are proved in §3, our proof for the $O(n^6)$-time algorithm in §2 is also self-contained.
2 A simple $O(n^6)$-time algorithm

Let $G$ be a connected graph containing vertices $u$ and $v$. For any vertices $x$ and $y$ of $G$, let $d_G(x,y) = ||P||$ for a shortest $xy$-path $P$ of $G$. Let $h(x) = d_G(u,x)$ be the height of a vertex $x$ in $G$. If $xy$ is an edge of $G$, then $|h(x) - h(y)| \leq 1$. For an $H \subseteq G$, (i) let $G - H = G[V(G) \setminus V(H)]$, (ii) let $N_G(H)$ consist of the vertices $y \in V(G - H)$ adjacent to at least one vertex of $H$ in $G$, and (iii) let $N_G[H] = N_G(H) \cup V(H)$. For an $x \in V(G)$, let $G - x = G - \{x\}$, then $N_G(x) = N_G(\{x\})$, and let $N_G[x] = N_G(\{x\})$. $X$ and $Y$ are adjacent (respectively, anticomplete) in $G$ if $N_G(X) \cap V(Y) \neq \emptyset$ (respectively, $N_G[X] \cap V(Y) = \emptyset$).

**Lemma 1** (Berger et al. [7, Lemma 2.2]). For any vertices $u$ and $v$ of an $n$-vertex connected graph $G_0$, it takes $O(n^3)$ time to obtain (1) a uv-trail of $G_0$ or (2) a uv-straight graph $G$ with $V(G) \subseteq V(G_0)$ such that

(a) a uv-trail of $G_0$ is $O(n^3)$-time obtainable from that of $G$ and
(b) if $G$ is uv-trailless, then so is $G_0$.

A path of $G$ is monotone [7] if all of its vertices have distinct heights in $G$. A monotone $xy$-path of $G$ is a shortest $xy$-path of $G$. The converse may not hold. A shortest $xy$-path of a uv-straight graph $G$ with $\{x,y\} \cap \{u,v\} \neq \emptyset$ is monotone. A monotone $ca^*$-path $W_1$ of $G$ containing a vertex $a$ and a monotone $bd^*$-path $W_2$ of $G$ containing a vertex $d$ with

$$h(a^*) + 1 = h(a) = h(b) \leq h(c) = h(d) = h(d^*) - 1$$

form a pair $(W_1,W_2)$ of wings for the quadruple $(a,b,c,d)$ of $V(G)$ in $G$ if

$$d_G(W_1 \cup W_2)(a^*,d^*) > ||W_1|| + ||W_2||,$$

that is, $W_1 - c$ (respectively, $W_1$) and $W_2$ (respectively, $W_2 - b$) are anticomplete in $G$. A quadruple $(a,b,c,d)$ of $V(G)$ is winged in $G$ if $G$ admits a pair of wings for $(a,b,c,d)$. See Figure 1 for an example.

**Lemma 2** (Implicit in Berger et al. [7, Lemma 2.1]). It takes $O(n^5)$ time to compute a data structure from which the following statements hold for any quadruple $(a,b,c,d)$ of $V(G)$ for an $n$-vertex graph $G$:

1. It takes $O(1)$ time to determine whether $(a,b,c,d)$ is winged in $G$.
2. If $(a,b,c,d)$ is winged in $G$, then it takes $O(n)$ time to obtain a pair of wings for $(a,b,c,d)$ in $G$.

We comment that the proof of [7, Lemma 2.1] is easily adjustable into one for Lemma 2. Moreover, see §3 for the proof of Lemma 6, which implies and improves upon Lemma 2.

Let $P$ be a uv-path of a uv-straight graph $G$. The twist pair of $P$ is the vertex pair $(s,t)$ of $P$ such that $P[u,s]$ and $P[t,v]$ are the maximal monotone prefix and suffix of $P$. The twist [7] of $P$ is $h(s) - h(t)$ for the twist pair $(s,t)$ of $P$. See also Figure 1 for an example. If $(s,t)$ is the twist pair of a uv-path $P$ of $G$, then $P[u,s]$ and $P[t,v]$ are disjoint if and only if $P$ is a non-shortest uv-path of $G$. The next lemma is also needed in §3. To make our proof of Theorem 1 in §3 self-contained, we include a proof of Lemma 3 simplified from that of Berger et al. [7, Lemma 2.3].

**Lemma 3** (Berger et al. [7, Lemma 2.3]). If $(s,t)$ is the twist pair of a shortest uv-trail $P$ of a uv-straight graph $G$, then $h(s) \geq h(x) \geq h(t)$ holds for each vertex $x$ of $P[s,t]$.

**Proof.** Let $I = V(P[s,t]) \setminus \{s,t\}$. Let $s^*$ (respectively, $t^*$) be the neighbor of $s$ (respectively, $t$) in $P[s,t]$. By definition of $(s,t)$, we have $h(s^*) \leq h(s)$ and $h(t^*) \geq h(t)$. If $I = \emptyset$, then $(s^*,t^*) = (t,s)$ implies the lemma. Otherwise, it suffices to prove $h(s) \geq h(x) \geq h(t)$ for each $x \in I$. If $h(x) > h(s)$ were true for the
x \in I \text{ maximizing } n \cdot h(x) + d_{P, x}(x, t), \text{ then the concatenation of } P, x \text{ and a shortest } xv\text{-path of } G \text{ is a } uv\text{-trail of } G \text{ shorter than } P. \text{ If } h(x) < h(t) \text{ were true for the } x \in I \text{ maximizing } n \cdot h(x) + d_{P, x}(x, t), \text{ then the concatenation of a shortest } ux\text{-path of } G \text{ and } P, x, v \text{ is a } uv\text{-trail of } G \text{ shorter than } P. \qedhere

Let } P \text{ be a } uv\text{-trail of a } uv\text{-straight graph } G \text{ with twist pair } (s, t). \text{ Lemma 3 implies } h(t) \leq h(s). \text{ A monotone } uc\text{-path } S \text{ of } G \text{ with } h(c) = h(s) \text{ is a sidetrack for } P \text{ if the following } \text{Conditions } S \text{ hold.}

\textbf{S1: } G \text{ contains a monotone } tv\text{-path } T \text{ with } d_{G[S \cup T]}(u, v) > \|S\| + \|T\|.

\textbf{S2: } S \text{ contains the vertex } a \text{ of } P, u, s \text{ with } h(a) = h(t).

We comment that existence and unique of } a \text{ in Condition S2 follows from } h(t) \leq h(s) \text{ and the fact that } P, u, s \text{ is monotone by definition of twist pair } (s, t). \text{ Condition S1 is equivalent to the statement that } S - c \text{ (respectively, } S) \text{ and } T \text{ (respectively, } T - t) \text{ are anticomplete in } G. \text{ Let } a^* \text{ be the vertex of the monotone } uc\text{-path } S \text{ with } h(a^*) = h(a) - 1. \text{ Let } d^* \text{ be the edge of the } tv\text{-path } T \text{ with } h(s) = h(d) = h(d^*) - 1. \text{ Condition S1 implies that } S[a^*, c] \text{ and } T[t, d^*] \text{ form a pair of wings for } (a, t, c, d) \text{ in } G. \text{ See Figure 2 for an example. The key to our largely improved } uv\text{-trailblazers in } \S 2 \text{ and } \S 3 \text{ is the following lemma, whose proof is illustrated in Figure 3.}

\textbf{Lemma 4. } If } S \text{ is a sidetrack for a shortest } uv\text{-trail } P \text{ of a } uv\text{-straight graph } G \text{ with twist pair } (s, t), \text{ then }

\[ d_{G[S \cup P]}(u, t) \geq d_{P}(u, t). \]

\textbf{Proof. } Let } T \text{ be a monotone } tv\text{-path of } G \text{ with } d_{G[S \cup T]}(u, v) > \|S\| + \|T\| \text{ by Condition S1. Assume for contradiction that there exists a shortest } ut\text{-path } Q \text{ of } G[S \cup P] \text{ with }

\[ \|Q\| < d_{P}(u, t), \tag{1} \]

implying } \|R\| < \|P\| \text{ for a shortest } uv\text{-path } R \text{ of } G[S \cup T]. \text{ Since } R \text{ is an induced } uv\text{-path of } G \text{ shorter than } P, R \text{ is monotone. Since Condition S1 implies } t \notin V(S), \text{ there exists an edge } x, y \text{ of } Q \text{ with } x \in V(S) \text{ and } y \in V(P[s, t]) \text{ that minimizes } d_{P}(y, t). \text{ We have } \{x, y\} \subseteq V(R) \text{ or else } R \text{ deviates from the}
induced path \( Q \) at a vertex \( q \) of \( Q[u, x] = S[u, x] \) and enters \( T \) (by \( V(R) \subseteq V(Q \cup T) \)) at a vertex \( b \) with height \( h(q) + 1 \) (by monotonicity of \( R \)), violating Condition S1. Since \( d_R(u, x) < d_R(u, y) \), we have

\[
h(x) + 1 = h(y). \tag{2}
\]

By Equation (1), a shortest \( uv \)-path \( R' \) of \( G[Q \cup T'] \) with \( T' = P[t, v] \) is monotone. Hence,

\[
h(x') + 1 = h(y') \tag{3}
\]

holds for an edge \( x'y' \) of \( R' \) with \( x' \in V(Q - t) \) and \( y' \in V(T') \), implying \( x' \not\in V(P[s, t]) \) by Lemma 3. Thus, \( x' \in V(S[u, x]) \). Condition S1 implies \( y' \neq t \). Condition S2 and Equation (3) imply \( h(y') \neq h(t) + 1 \). Therefore, we have \( h(y') \geq h(t) + 2 \), implying \( h(x) \geq h(t) + 1 \) (by Equation (3)) and

\[
d_Q(t', y') \geq 2 \tag{4}
\]

for the vertex \( t' \in V(P[y, t]) \) with \( h(t') = h(t) \) that minimizes \( d_P(t', y) \). Since \( h(x) \geq h(t) + 1 \) and by choices of \( y \) and \( t' \), the concatenation \( Q' \) of

(i) a shortest \( ut' \)-path of \( G \) and
(ii) \( P[t', y] \)

is an induced \( uy \)-path of \( G \) with \( ||Q'|| \leq ||P[u, a]|| + ||P[s, t]|| \). By Equations (2), (3), and (4), a shortest \( xv \)-path \( Q'' \) of \( G[S[x, x'] \cup P[y, v]] \) satisfies \( ||Q''|| \leq ||P[a, s]|| + ||P[t, v]|| - 2 \). Thus, \( P' = Q' \cup yx \cup Q'' \) is a \( uv \)-path of \( G \) with \( ||P'|| < ||P|| \). By definitions of the \( ut \)-path \( Q \) and the edge \( xy \) of \( Q', Q' \) is anticomplete to \( S[x', x] - x \) in \( G \). By Equation (4), \( Q' \) is anticomplete to \( P[y', v] \) in \( G \). Hence, \( P' \) is an induced \( uv \)-path of \( G \). By Equation (2) and \( d_P(u, x) > d_P(u, y) \), \( P' \) is a \( uv \)-trail of \( G \) shorter than \( P \), contradiction.

\[\square\]

We are ready to describe and justify an \( O(n^6) \)-time algorithm that either reports a \( uv \)-trail of \( G \) or ensures that \( G \) is \( uv \)-trailless. Let \( G \) be connected without loss of generality.
Our $O(n^6)$-time algorithm

Let $G_0$ be the input $n$-vertex graph. Apply Lemma 1 in $O(n^3)$ time to either report a $uv$-trail of $G_0$ as stated in Lemma 1(1) or obtain a $uv$-straight graph $G$ satisfying Conditions (a) and (b) of Lemma 1(2). If no $uv$-trail is reported in the previous step, then apply Lemma 2 to obtain the data structure $D$ for the winged quadruples of $G$ in $O(n^6)$ time. By Conditions (a) and (b) of Lemma 1(2), it remains to show an $O(n^2)$-time degree-4 $uv$-trailblazer for the $uv$-straight graph $G$, which immediately leads to an $O(n^6)$-time trailblazing algorithm that either reports a $uv$-trail of $G$ or ensures that $G$ is $uv$-trailless.

Let $B$ be the following $O(n^2)$-time subroutine, taking a quadruple $(a, b, c, d)$ of $V(G)$ as the argument: Determine in $O(1)$ time from the data structure $D$ whether $(a, b, c, d)$ is winged in $G$. If not, then exit. Otherwise, obtain in $O(n)$ time from $D$ a pair $(W_1, W_2)$ of wings for $(a, b, c, d)$ in $G$. Since $G$ is $uv$-straight, each monotone $xy$-path $Q$ of $G$ with $h(x) \leq h(y)$ is contained by the monotone $uv$-path $P \cup Q \cup R$, where $P$ is an arbitrary monotone $ux$-path of $G$ and $R$ is an arbitrary monotone $vy$-path of $G$. Thus, it takes $O(n^2)$ time to obtain a monotone $uc$-path $S$ of $G$ containing $W_1$ and a monotone $bv$-path $T$ of $G$ containing $W_2$. Obtain in $O(n^2)$ time the subgraph $G_{c,b}$ of $G$ induced by

$$\{x \in V(G) : h(b) \leq h(x) \leq h(c)\} \setminus ((N_G[S - c] \cup N_G[T - b]) \setminus \{c, b\}).$$

If $c$ and $b$ are not connected in $G_{c,b}$, then exit. Otherwise, report the concatenation $P_{c,b}$ of (i) the $uc$-path $S$, (ii) a shortest $cb$-path of $G_{c,b}$, and (iii) the $bv$-path $T$.

By definition of $S$, $T$, and $G_{c,b}$, the $uv$-path $P_{c,b}$ of $G$ reported by $B(a, b, c, d)$ is induced in $G$, which is not monotone by $h(b) \leq h(c)$. Thus, $P_{c,b}$ is a $uv$-trail of $G$.

Let $P$ be an arbitrary unknown shortest $uv$-trail of $G$ with twist pair $(s, t)$. By Lemma 3, we have $h(t) \leq h(s)$. Let $a$ (respectively, $d$) be the vertex of the monotone $P[u, s]$ (respectively, $P[t, v]$) with $h(a) = h(t)$ (respectively, $h(d) = h(s)$), whose existence and uniqueness are due to the fact that $P[u, s]$ and $P[t, v]$ are monotone. See Figure 4 for an illustration. The rest of the section shows that $(a, t,s,d)$ is a trail marker for $B$.

Let $a^*$ be the neighbor of $a$ in $P[u,a]$ and $d^*$ be the neighbor of $d$ in $P[d,v]$. $P[a^*,s]$ and $P[t,d^*]$ form a pair of wings for $(a, t,s,d)$ in $G$. Thus, the quadruple $(a, t,s,d)$ is winged in $G$. Let $(W_1, W_2)$ be a pair of wings for $(a, t,s,d)$. The monotone $us$-path $S$ of $G$ containing $W_1$ is a sidetrack for $P$, since the monotone $tv$-path $T$ of $G$ containing $W_2$ satisfies Conditions S1 and S2 for $T$. Due to the symmetry between $u$ and $v$ in the undirected $uv$-straight graph $G$ with respect to the height function that maps each vertex to its distance to $v$ in $G$, the monotone $vt$-path $T$ of the $vu$-straight graph $G$ is also a sidetrack for the shortest $vu$-trail $P$ of $G$ with twist pair $(t,s)$, since the monotone $su$-path $S$ of $G$ satisfies Conditions S1 and S2 for $T$. Lemma 3 guarantees $h(t) \leq h(x) \leq h(s)$ for each vertex $x$ of $P[s,t]$. By Lemma 4, $P[s,t]-\{s,t\}$ is anticomplete to both $S-s$ and $T-t$, implying that $P[s,t]$ is a path of $G_{s,t}$. Since $s$ and $t$ are connected in $G_{s,t}$, the subroutine call $B(a,t,s,d)$ outputs a $uv$-trail $P_{s,t}$ of $G$ in $O(n^2)$ time. Hence, $(a, t,s,d)$ is indeed a trail marker of $B$.

As a matter of fact, $P_{s,t}$ is a shortest $uv$-trail of $G$ due to the fact that $\|P_{s,t}\| = \|P\|$. Since the preprocessing and postprocessing may ruin the shortestness of the reported $uv$-trail, we have an $O(n^6)$-time algorithm on an $n$-vertex general (respectively, $uv$-straight) graph $G$ that either reports a general (respectively, shortest) $uv$-trail of $G$ or ensures that $G$ is $uv$-trailless.
Figure 4: An illustration for the proof that $B$ is a $uv$-trailblazer of degree four. The red path denotes a shortest $uv$-trail of the $uv$-straight graph $G$. The blue and green dotted paths denote a monotone $u$s-path and a monotone $t$v-path of $G$ containing a precomputed pair of wings for $(a, t, s, d)$ that need not coincide with $P$ except at $a$, $t$, $s$, and $d$.

3 An $O(n^{4.75})$-time algorithm

This section gives a self-contained proof of Theorem 1. The product of $m \times m$ Boolean matrices $A$ and $B$ is the $m \times m$ Boolean matrix $C$ such that $C(i, k) = true$ if and only if $A(i, j) = B(j, k) = true$ holds for an index $j$. The following lemma implies and improves upon Lemma 1, which states that it takes $O(n^3)$ time to obtain a $uv$-trail of $G$ from a $uv$-trail of $H$.

Lemma 5. For any vertices $u$ and $v$ of an $n$-vertex connected graph $G$, it takes $O(n^{\omega})$ time to obtain (1) a $uv$-trail of $G$ or (2) a $uv$-straight graph $H$ with $V(H) \subseteq V(G)$ such that (a) a $uv$-trail of $G$ can be obtained from a $uv$-trail of $H$ in $O(n^2)$ time and (b) if $H$ is $uv$-trailless, then so is $G$.

Proof. We adopt the proof of Berger et al. [7, Lemma 2.2] with slight simplification and improvement. It takes $O(n^2)$ time via, e.g., breadth-first search to obtain a maximal set $F \subseteq V(G)$ such that $G[F]$ is $uv$-straight. If $F = V(G)$, the lemma is proved by returning $H = G$. The rest of the proof assumes $F \subset V(G)$. It takes $O(n^{\omega})$ time to determine whether some connected component $K$ of $G - F$ admits an $(x, y) \subseteq N_G(K)$ with $xy \notin E(G)$ and $h(x) < h(y)$ via finding a triangle intersecting exactly two vertices of $F$ in the following graph $A$ such that each vertex of $A$ corresponds to a vertex $x$ in $F$ or a connected component $K$ of $G - F$:

- A vertex $x$ in $F$ is adjacent to a connected component $K$ of $G - F$ if and only if $x \in N_G(K)$.
- Distinct connected components of $G - F$ are nonadjacent in $A$.
- Distinct vertices $x$ and $y$ of $F$ are adjacent in $A$ if and only if $xy \notin E(G)$ and $h(x) \neq h(y)$.

If there is such a $(K, x, y)$, then a shortest $uv$-path of $G[P_x \cup K \cup P_y]$ for any monotone $ux$-path $P_x$ and $vy$-path $P_y$ of $G$ is a $uv$-trail of $G$ (by the maximality of $F$) obtainable in $O(n^2)$ time, proving the lemma. Otherwise, let $H$ be the union of the $uv$-straight graph $G[F]$ and the $O(n^{\omega})$-time obtainable graph $H'$ with $V(H') = F$ (via contracting each connected component of $G - F$ into a single vertex and then squaring the adjacency matrix) such that distinct vertices $x$ and $y$ are adjacent in $H'$ if and
only if \( \{x, y\} \subseteq N_G(K) \) holds for a connected component \( K \) of \( G - F \). Observe that each edge \( xy \) of \( H' \) with \( h(x) \neq h(y) \) is also an edge of \( G[F] \). By \( |h(x) - h(y)| \leq 1 \) for all edges \( xy \) of \( H' \), \( H \) remains \( uv \)-straight and \( d_H(u, x) = h(x) \) holds for each \( x \in F \). To see Condition (a), for any given \( uv \)-trail \( Q \) of \( H \), let \( P \) be an \( O(n^2) \)-time obtainable non-monotone \( uv \)-path of \( G \) obtained from \( Q \) by replacing each edge \( xy \) of \( Q \) not in \( G[F] \) with a shortest \( xy \)-path \( P_{xy} \) of \( G - (F \setminus \{x, y\}) \). If \( P \) were not induced, then there is an edge \( zz' \) of \( P[|P|] \) not in \( P \) with \( z \in V(P_{xy}) \) and \( z' \in V(P_{x'y'}) \) for distinct edges \( xy \) and \( x'y' \) of \( Q \) that are not in \( G[F] \). Thus, \( \{x, y, x', y'\} \subseteq N_G(K) \) holds for some connected component \( K \) of \( G - F \). By definition of \( H' \), \( H[\{x, y, x', y'\}] \) is complete, contradicting that \( Q \) is an induced path of \( H \). Thus, \( P \) is a \( uv \)-trail of \( G \), proving Condition (a). As for Condition (b), let \( P \) be a \( uv \)-trail of \( G \). For any distinct vertices \( x \) and \( y \) of \( P \) such that \( P[x, y] \) is a maximal subpath of \( P \) contained by \( G[\{x, y\} \cup K] \) for some connected component \( K \) of \( G - F \), \( P[x, y] \) is an induced \( xy \)-path of \( G[\{x, y\} \cup K] \). The path \( Q \) obtained from \( P \) by replacing each such \( P[x, y] \) by the edge \( xy \) of \( H' \) is an induced \( uv \)-path of \( H \). If \( Q \) were a shortest \( uv \)-path of \( H \), then \( |h(x) - h(y)| = 1 \) holds for each edge \( xy \) of \( Q \), implying that each edge \( xy \) of \( Q \) is an edge of \( P \), contradicting that \( P \) is a \( uv \)-trail of \( G \). \( \square \)

The bottleneck of our algorithm for Theorem 1 comes from the following lemma, which implies and improves upon Lemma 2 that takes \( O(n^6) \) time.

**Lemma 6.** It takes \( \bar{O}(n^{2\omega}) \) time to compute a data structure from which the following statements hold for any quadruple \( (a, b, c, d) \) of \( V(G) \) for an \( n \)-vertex graph \( G \):

1. It takes \( O(1) \) time to determine whether \( (a, b, c, d) \) is winged in \( G \).
2. If \( (a, b, c, d) \) is winged in \( G \), then it takes \( O(n) \) time to obtain a pair of wings for \( (a, b, c, d) \) in \( G \).

**Proof.** The lemma holds clearly for the quadruples \( (a, b, c, d) \) of \( V(G) \) with \( h(c) \leq h(a) + 1 \). The rest of the proof handles those with \( h(a) + 2 \leq h(c) \). A pair of wings for such an \( (a, b, c, d) \) must be anticomplete in \( G \). It takes \( O(n^4) \) time to obtain the \( n^2 \times n^2 \) Boolean matrix \( A \) such that \( A((a, b), (c, d)) = true \) if and only if (i) \( h(a) = h(b) \leq h(c) = h(d) \leq h(a) + 1 \) and (ii) \( G \) admits a pair of anticomplete wings for \( (a, b, c, d) \). The transitive closure \( C = A^i \) of \( A \) can be obtained in \( O(n^{2\omega} \log n) \) time via obtaining \( A^{2i} \) in the \( i \)-th iteration. That is, for each \( (a, b, c, d) \), we have \( C((a, b), (c, d)) = true \) if and only if (i) \( h(a) = h(b) \leq h(c) = h(d) \) and (ii) \( G \) admits a pair of anticomplete wings for \( (a, b, c, d) \) in \( G \). Statement 1 is proved. Statement 2 is immediate from the \( \bar{O}(n^{2\omega}) \)-time obtainable \( n^2 \times n^2 \) witness matrix \( W \) for \( C \) by, e.g., Galil and Margalit [41]: if \( C((a, b), (c, d)) = true \) and \( h(a) + 2 \leq h(c) \), then \( W((a, b), (c, d)) \) is a vertex pair \( (x, y) \) with \( h(a) < h(x) < h(c) \) and \( C((a, b), (x, y)) = C((x, y), (c, d)) = true \). \( \square \)

The next dynamic data structure for a graph supports efficient edge updates and connectivity queries.

**Lemma 7** (Holm, de Lichtenberg, and Thorup [48]). There is a data structure for an initially empty \( n \)-vertex graph that supports each edge insertion and edge deletion in amortized \( O(\log^2 n) \) time and answers whether two vertices are connected in \( O(\log n / \log \log n) \) time.

We are ready to prove Theorem 1. Assume without loss of generality that \( G \) is connected.

**Our \( O(n^{4.75}) \)-time algorithm** Apply Lemma 5 in \( O(n^6) \) time to either report a \( uv \)-trail of \( G \) as in Lemma 5(1) or make \( G \) a \( uv \)-straight graph satisfying Conditions (a) and (b) of Lemma 5(2) with respect to the original \( G \). If no \( uv \)-trail is reported in the previous step, then apply Lemma 6 in \( \bar{O}(n^{2\omega}) \) time to
obtain the data structure $D$ for the winged quadruples of $V(G)$ in $G$. It remains to show an $O(n^2 \log^2 n)$-time degree-two $uv$-trailblazer for the $uv$-straight graph $G$ based on the precomputed $D$ which already spends $O(n^{4.75})$ time. We proceed in two phases. Phase 1 shows that we already have an $O(n^3)$-time degree-two $uv$-trailblazer for $G$. Phase 2 then reduces the time to $O(n^2 \log^2 n)$ via Lemma 7.

**Phase 1** Let $B_1$ be the $O(n^3)$-time subroutine, taking a pair $(a, b)$ of $V(G)$ as the only argument, that runs the following $O(n^2)$-time procedure for each vertex $c$ of $G$: Determine from $D$ in $O(n)$ time whether $G$ admits a winged quadruple $(a, b, c, d_c)$ of $V(G)$ for some $d_c$. If not, then exit. Otherwise, obtain from $D$ in $O(n)$ time a pair $(W_1, W_2)$ of wings for an arbitrary winged $(a, b, c, d_c)$. Since $G$ is $uv$-straight, it takes $O(n^2)$ time to obtain a monotone $uc$-path $S_c$ of $G$ containing $W_1$ and a monotone $bv$-path $T_c$ of $G$ containing $W_2$. Obtain in $O(n^2)$ time the subgraph $G_c$ of $G$ induced by

$$\{x \in V(G) : h(b) \leq h(x) \leq h(c) \} \cup (N_G[S_c - c] \setminus \{c\}) \cup V(T_c).$$

If the vertices $c$ and $b$ are not connected in $G_c$, then exit. Otherwise, report the $O(n^2)$-time obtainable concatenation $P_c$ of the $uc$-path $S_c$ of $G$ and a shortest $cv$-path of $G_c$.

By definition of $S_c$, $T_c$, and $G_c$, the $uv$-path $P_c$ of $G$ reported by $B_1(a, b)$ for any $c$ is induced in $G$. Since the height of each neighbor of $c$ in $G_c$ is at most $h(c)$, $P_c$ is not monotone. Thus, $P_c$ is a $uv$-trail of $G$. Let $P$ be an arbitrary unknown shortest $uv$-trail of $G$ with twist pair $(s, t)$. Let $a$ (respectively, $e$) be the vertex of the monotone $P[u, s]$ (respectively, $P[t, v]$) with $h(a) = h(t)$ (respectively, $h(e) = h(s)$). See Figure 5 for an illustration. To ensure that $B_1$ is an $O(n^3)$-time $uv$-trailblazer of degree 2 for $G$, the rest of the phase proves that $(a, t)$ is a trail marker for $B_1$ by showing that the iteration with $c = s$ reports a $uv$-trail $P_s$ of $G$.

Let $a^*$ be the neighbor of $a$ in the monotone $P[u, a]$, implying $h(a^*) = h(t) - 1$. Let $e^*$ be the neighbor of $e$ in the monotone $P[e, v]$, implying $h(e^*) = h(s) + 1$. Since $P[a^*, s]$ and $P[t, e^*]$ form a pair of wings for $(a, t, s, e)$ in $G$, there is a $d_c$ such that $(a, t, s, d_c)$ is winged in $G$. Let $(W_1, W_2)$ be the pair of wings

![Figure 5: An illustration for the proof that $B_1$ is a $uv$-trailblazer of degree two. The red path denotes a shortest $uv$-trail $P$ of the $uv$-straight graph $G$. The blue and green dotted paths denote a monotone $uc$-path $S_c$ and a monotone $tv$-path $T_c$ of $G$ containing a precomputed pair of wings for $(a, t, c, d_c)$ that need not coincide with $P$ except at $a$ and $t$.](image)
for \((a, t, s, d_i)\) in \(D\) obtained from \(D\). The monotone \(us\)-path \(S_c\) of \(G\) containing \(W_i\) is a sidetrack for \(P\), since the monotone \(tv\)-path \(T_v\) of \(G\) containing \(W_2\) satisfies Conditions S1 and S2 for \(S_v\). By Lemma 3, each vertex \(x\) of \(P[s, t]\) satisfies \(h(t) \leq h(x) \leq h(s)\). By Lemma 4, \(S_v - s\) and \(P[s, t] - s\) are anticomplete in \(G\), implying that \(P[s, t]\) is a path of \(G\). Since \(s\) and \(t\) are connected in \(G_s\), the subroutine call \(B_1(a, t)\) outputs a \(uv\)-trail \(P_s\) of \(G\) in the iteration with \(c = s\). Hence, \((a, t)\) is indeed a trail marker of \(B\). One can verify that \(P_s\) is also a shortest \(uv\)-trail of the \(uv\)-straight graph \(G\), although \(d_i\) need not be \(e\). Thus, we have an \(O(n^5)\)-time algorithm on an \(n\)-vertex general (respectively, \(uv\)-straight) graph \(G\) that either reports a general (respectively, shortest) \(uv\)-trail of \(G\) or ensures that \(G\) is \(uv\)-trailless.

**Phase 2** Since many prefixes of a long sidetrack for a shortest \(uv\)-trail \(P\) of \(G\) remain sidetracks for \(P\), an edge can be deleted and then inserted back \(O(n)\) times in Phase 1. Phase 2 avoids the redundancy by processing the sidetracks in the decreasing order of their lengths. Let \(B_2\) be the following subroutine that takes a pair \((a, b)\) of \(V(G)\) as the only argument. Obtain in overall \(O(n^2)\) time from \(D\) each set \(C_i\) with \(0 \leq i \leq h(v)\) that consists of the vertices \(c\) of \(G\) with \(h(c) = i\) such that \(G\) admits a winged quadruple \((a, b, c, d_i)\) for some vertex \(d_i\). Let \(C\) be the union of all \(C_i\) with \(0 \leq i \leq h(v)\). Obtain in overall \(O(n^2)\) time from \(D\) for each vertex \(c \in C\) (i) a monotone \(uc\)-path \(S_c\) of \(G\) containing \(a\) and (ii) a monotone \(bv\)-path \(T_c\) with

\[
d_{G[S_c \cup T_c]}(u, v) > \|S_c\| + \|T_c\|.
\]

Obtain the subgraph \(H\) of \(G\) induced by the vertices with heights at least \(h(a)\) in \(O(n^2 \log^2 n)\) time by the dynamic data structure of Lemma 7. Iteratively perform the following steps in the decreasing order of the indices \(i\) with \(h(a) \leq i < h(v)\):

1. Delete from \(H\) the incident edges of \(N_G[S_c - c] \setminus \{c\}\) in \(G\) for all \(c \in C_i\).
2. Insert to \(H\) the incident edges of \(C_i\) in \(G\).
3. Delete from \(H\) all edges \(xy\) of \(G\) with \(h(x) = i\) and \(h(y) = i + 1\).
4. If \(b\) is not connected to any \(c \in C_i\) in \(H\), then proceed to the next iteration. Otherwise, let \(c\) be an arbitrary vertex of \(C_i\) that is connected to \(b\) in \(H\). Exit the loop and report the \(O(n^2)\)-time obtainable concatenation \(P_c\) of \(S_c\) and a shortest \(cv\)-path of \(G[H \cup T_c]\).

Since \(S_c - c\) and \(T_c - b\) are anticomplete in \(G\) and the height of each neighbor of \(c\) in \(H\) is at most \(h(c)\), any arbitrary reported \(uv\)-path \(P_c\) of \(G\) is a \(uv\)-trail of \(G\).

Throughout all iterations, the incident edges of each vertex of \(G\) is deleted \(O(1)\) times by Step 1, each edge of \(G\) is updated \(O(1)\) times by Steps 2 and 3, and each vertex \(c \in C\) is queried \(O(1)\) times by Step 4. Thus, each subroutine call \(B_2(a, b)\) runs in \(O(n^2 \log^2 n)\) time.

Let \(P\) be an arbitrary shortest \(uv\)-trail of \(G\) with twist pair \((s, t)\). As in Phase 1, let \(a\) (respectively, \(e\)) be the vertex of the monotone \(P[u, s]\) (respectively, \(P[t, v]\)) with \(h(a) = h(t)\) (respectively, \(h(e) = h(s)\)). The rest of the phase proves that \((a, t)\) is a trail marker for \(B_2\) by showing that an iteration with \(i \geq h(s)\) in the loop of the subroutine call \(B_2(a, t)\) reports a \(uv\)-trail \(P_c\) of \(G\). See Figure 6 for an illustration.

If an iteration of \(B_2(a, t)\) with \(i \geq h(s) + 1\) reports a \(uv\)-trail of \(G\) (that need not be shortest), then we are done. Otherwise, we show that the iteration with \(i = h(s)\) has to report a \(uv\)-trail of \(G\). For each \(c \in C\) with \(h(c) \geq i\), let \(s_c\) be the unknown vertex of \(S_c\) with \(h(s_c) = i\). \(S_c[u, s_c]\) remains a sidetrack for \(P\), since \(T_c\) still satisfies Conditions S1 and S2 for \(S_c[u, s_c]\). Thus, \(s_c \in C_i\). By Lemma 4, \(S_c[u, s_c] - s_c\) and \(P[s, t] - s\) are anticomplete in \(G\) even if \(S_c[u, s_c]\) need not be \(S_c\). As a result, \(P[s, t] - NP_{[s, t]}[s]\) is a path of the \(H\) at the completion of Step 1 in the \(i\)-th iteration. By \(s \in C_i\) and Lemma 3, \(P[s, t]\) is a path of the graph \(H\) at
Figure 6: An illustration for the proof that $B_2$ is a $uv$-trailblazer of degree two. The red path denotes a shortest $uv$-trail $P$ of the $uv$-straight graph $G$. The blue and green dotted paths denote a monotone uc-path $S_c$ and a monotone tv-path $T_c$ of $G$ containing a precomputed pair of wings for $(a, t, c, d_c)$ that need not coincide with $P$ except at $a$ and $t$. $S_c[u, s_c]$ remains a sidetrack for $P$.

the completion of Step 3 in the $i$-th iteration. Therefore, $s$ is a $c \in C_i$ that is connected to $t$ in $H$. Step 4 in the $i$-th iteration has to output a (shortest) $uv$-trail $P_c$ of $G$ for some $c \in C_i$ that need not be $s$. Thus, we have an $O(n^{4.75})$-time algorithm that either obtains a $uv$-trail of $G$ or ensures that $G$ is $uv$-trailless. A reported $uv$-trail of $G$ by this $O(n^{4.75})$-time algorithm need not be a shortest $uv$-trail of $G$, since we cannot afford to spend $O(n^2)$ time, as in Phase 1, for each $c \in C$ that is connected to $t$ in the graph $H$ at the $h(c)$-th iteration to obtain a shortest $cv$-path of $G[H \cup T_c]$.

4 Concluding remarks

We show an $O(n^{4.75})$-time algorithm for computing a $uv$-trail of an $n$-vertex undirected unweighted graph $G$ with $\{u, v\} \subseteq V(G)$. The key to our improved algorithm is the observation regarding an arbitrary shortest $uv$-trail of a $uv$-straight graph $G$ described by Lemma 4. The inequality of Lemma 4 is stronger than the condition that $S - c$ and $P[s, t] - s$ are anticomplete in $G$. As a matter of fact, the latter suffices for our $uv$-trailblazers in §2 and §3. Thus, a further improved $uv$-trailblazer might be possible if the wings for a winged quadruple can be obtained more efficiently. As mentioned in Phase 1 of §3, a shortest $uv$-trail, if any, of a $uv$-straight $G$ can be obtained by our $B_1$-based trailblazing algorithm in $O(n^5)$ time. Detecting a $uv$-trail with length at least $2d_G(u, v)$ is NP-complete [7, Theorem 1.6]. It is of interest to see if a shortest $uv$-trail or a $uv$-trail having length at least $d_G(u, v) + k$ for a positive $k = O(1)$ in a general $G$ can be obtained in polynomial time. It is also of interest to see whether the one-to-all (respectively, all-pairs) version of the problem can be solved in time much lower than $O(n^{5.75})$ (respectively, $O(n^{6.75})$).

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