Equivalence of a Bridged Link Calculus and Kirby’s Calculus of Links on Non-Simply Connected 3-Manifolds

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Abstract: We recall an extension of Kirby’s Calculus on non-simply connected 3-manifolds given in [FR], and the surgery calculus of bridged links from [Ke], which involves only local moves. We give a short combinatorial proof that the two calculi are equivalent, and thus describe the same classes of 3-manifolds. This makes the proofs for the validity of surgery calculi in [FR] and [Ke] interchangeable.

1.) The Extended Kirby Calculus: A popular way of presenting a closed, compact, connected, and oriented 3-manifold, $N^{(3)}$, is by use of framed links in $S^3$. To any such link, $L$, we can associate a three-fold, $S^3_L$, by doing surgery along its components. It is long been known [LW] that any closed $N^{(3)}$ is homeomorphic to $S^3_L$ for some link, $L$.

In [Ki] Kirby shows that $S^3_{L_1}$ is homeomorphic to $S^3_{L_2}$, if and only if $L_1$ can be changed into $L_2$ by a sequence of certain moves, i.e., if $L_1$ and $L_2$ are $\delta$-equivalent in the language of [Ki]. The moves are either ambient isotopies, the signature move $O_1$, or the two-handle slide $O_2$. Thickening the links into ribbons, in order to indicate their framings, the $O_2$-move is depicted in Figure 1, where $R_2$ is slid over $R_1$.

![Figure 1: $O_2$-move](image)

Hence Kirby’s calculus establishes a one-to-one correspondence between the set of homeomorphism classes of closed 3-manifolds and the combinatorial set of $\delta$-equivalence classes of links in $S^3$.

In a more general situation we can consider framed links $L$ embedded into a connected, compact, and oriented 3-manifold, $M = M^{(3)}$, which may have a boundary or be non-simply
connected. For two framed links, \( L_1 \) and \( L_2 \), in \( M \), we can again consider the correspondingly surgered manifolds \( M_{L_1} \) and \( M_{L_2} \). It was suggested by Kirby and proven by Fenn and Rourke [FR] that \( M_{L_1} \cong M_{L_2} \), if and only if \( L_1 \) is \( \alpha \)-equivalent to \( L_2 \), where the \( \alpha \)-equivalences are generated by isotopies, the \( O_1 \)-, \( O_2 \)-, and \( O_3 \)-move. The new \( O_3 \)-move (also called \( \eta \)-move in [Ke]) is given by including or deleting a two-component configuration from a link, as shown in Figure 2. It consist of an arbitrary component, \( R \), together with an annulus, \( A \), that encircles \( R \).

![Diagram](image)

**Figure 2: \( O_3 \)-move**

For the purpose of this letter let us consider the slightly smaller equivalent classes, where we omit the \( O_1 \)-move as a generator of equivalences. The space of classes of framed links in a given compact manifold, \( M \), modulo \( O_2 \)- and \( O_3 \)-moves shall be denoted \( L_M \).

It can be taken from arguments in [Ki], [FR], and [Ke] that, in fact, it suffices to choose a much smaller set of \( O_3 \)-moves by selecting two long ribbons \( R \) for each class in \([S^1, M]\), with different framings mod 2. All other \( O_3 \)-moves can be reduced to these special ones by combining them with isotopies and \( O_2 \)-moves. The \( O_3 \)-move for the trivial element in \([S^1, M]\) consists then of adding or deleting isolated Hopf links, \( 0 \circ 0, 1 \circ 0 \), with framings as indicated. Applying \( O_2 \)-moves they can both be substituted for the pair of \( O_1 \)-moves, in which \( 0 \circ 0 \) is added or deleted.

In order to see the topological interpretation of \( L_M \), recall that a framed link \( L \subset M \) defines also a four dimensional cobordism \( W_L : M \to M_L \), with \( \partial W_L \cong M \cup_{\partial M} M_L \). It is now easily extracted from [Ki] or [FR] that \( L_1 \) and \( L_2 \) are equivalent via \( O_2 \)-, and \( O_3 \)-moves, if and only if \( M_{L_1} \cong M_{L_2} \) and \( \text{sign}(W_{L_1}) = \text{sign}(W_{L_2}) \) for the signatures. The \( O_1 \)-move connects a \( \mathbb{CP}^2 \) to \( W_L \) and thus increases \( \text{sign}(W_L) \) by one. (Note, however, that connecting \( \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \) or \( S^2 \times S^2 \), as in the \( 0 \circ 0,1 \)-moves, does not change the signature).

Moreover, if \( N \) is any other compact, oriented, and connected 3-manifold with \( \partial M \cong \partial N \), then there is a framed link \( L \), such that \( M_L \cong N \), extending the given homeomorphism on the boundaries. Hence the classes in \( L_M \) are in one-to-one correspondence with pairs \((N, n)\), where \( N \) is a homeomorphism class of 3-manifolds with boundary \( \cong \partial M \), and \( n \in \mathbb{Z} \) is given by \( n = \text{sign}(W_L) \). Let us denote the set of all such pairs by \( M_{\partial M} \).

The proof in [FR], although referring to results from [Ki], is quite involved since \( \alpha \)-equivalence of two links is first expressed as \( \delta \)-equivalence together with an isomorphism between the corresponding \( \pi_1(W_L) \) and the vanishing of an additional obstruction class. A shorter proof that builds directly on the one in [Ki] was given very recently by Roberts [R]. Several central arguments used there to generalize the steps in the proof of [Ki] had already been given in [Ke] and an earlier version of this letter.
2.) **Calculus of Bridged Links** : In [Ke] we derive an alternative calculus of links, starting in a similar way as in [Ki] but following a different strategy to single out moves that generate the necessary equivalences. In order to include the possibility of index-1-surgery we consider *bridged links* \( L \subset M \) in a connected, compact, and oriented 3-manifold, \( M \). Thus \( L \) not only consists of ribbons but also pairs of balls, and a ribbon can enter one ball of a pair and reemerge at a corresponding spot at the other ball.

In contrast to the non-local moves \( O_2 \) and \( O_3 \) of the extended Kirby calculus, the inclusion of 1-handle data allows us to use instead two *local* moves. The motivation in [Ke] is that locality is essential, e.g., for efficient presentations of cobordism categories, Hopf-algebraic interpretations of elementary cobordisms, and thus constructions of TQFT’s by means of straightforward structural comparisons rather than involved calculations, see [KL].

The easier of the two moves is the *isolated cancellation move*, which we shall call here \( \mathcal{A}_1 \). As depicted in Figure 3 the \( \mathcal{A}_1 \)-move adds or deletes an isolated configuration from the rest of the link, which consists of one ribbon, and one pair of balls. The ribbon passes through the surgery spheres exactly once.

![Figure 3: \( \mathcal{A}_1 \)-move](image)

The \( \mathcal{A}_2 \)-move, more commonly called *modification* or *handle trading*, is described as follows. Any 0-framed component, \( A \), bounding a disc, can be replaced by a pair of surgery balls, \( B_1 \) and \( B_2 \), as indicated in Figure 4. Other strands of the links that run through the disc bounded by \( A \) are redirected through the surgery spheres.

![Figure 4: \( \mathcal{A}_2 \)-move](image)
Let us denote by $\mathcal{BL}_M$ the space of bridged links modulo equivalences generated by isotopies, $A_1$-, and $A_2$-moves.

In [Ke] it was shown that $\mathcal{BL}_M$ for a connected, compact, and oriented 3-manifold is, like $\mathcal{LM}$, in one-to-one correspondence with the set $\mathcal{M}_{\partial M}$ of pairs $(N,n)$, where $N$ is a homeomorphism class of 3-manifolds with $\partial N \cong \partial M$ and $n \in \mathbb{Z}$.

The correspondence is again given by the assignment $L \mapsto (M_L, \text{sign}(W_L))$, only now we obtain $M_L$ by first performing index-1-surgeries along the pairs of balls (yielding $\tilde{M} \cong M \# S^1 \times S^2 \# \ldots \# S^1 \times S^2$) before we do the usual index-2-surgeries along the remaining framed link in $\tilde{M}$. Similarly, $W_L$ is obtained, by also adding 1-handles instead of just 2-handles to $M \times [0,1]$.

Since both link spaces have the same topological interpretation in terms of $\mathcal{M}_{\partial M}$ it is obvious that there has to be a one-to-one correspondence also between $\mathcal{LM}$ and $\mathcal{BL}_M$ for a given connected $M$ as above.

The main purpose of this letter is to present a direct and concise of proof of the equivalence of $\mathcal{LM}$ and $\mathcal{BL}_M$ as formal link calculi in $M$, and thereby show, how the two calculi can be translated into each other purely combinatorially. More precisely, we prove the following:

**Theorem 1**

The identification of a framed link in $M$ as a special bridged link in $M$ factors into a map on the equivalence classes of links

$$I : \mathcal{LM} \longrightarrow \mathcal{BL}_M.$$ 

Moreover, the map $I$ is a bijection.

In the proof of Theorem 1 we shall only use the definition of $\mathcal{LM}$ and $\mathcal{BL}_M$ as sets of classes of links as given in Paragraphs 1 and 2. In particular, none of the arguments in the proof will involve the topological interpretation of the links as surgery data.

In the next theorem let us relate this combinatorial result to the results on presentations of 3-manifolds outlined previously. The first part is obvious from the fact that the moves $O_2$, $O_3$, $A_1$, and $A_2$ can be interpreted as manipulations of $W_L$, which do not change the homeomorphisms type of the boundary piece $M_L$.

**Theorem 2** Let $M$ be a compact, connected, and oriented 3-manifold.

1. The interpretations of (bridged) links as surgery prescriptions factor into maps:

$$S_{\text{KFR}} : \mathcal{LM} \longrightarrow \mathcal{M}_{\partial M} \quad \text{and} \quad S_{\text{BL}} : \mathcal{BL}_M \longrightarrow \mathcal{M}_{\partial M}$$

with

$$S_{\text{KFR}} = S_{\text{BL}} \circ I.$$ 

2. Both $S_{\text{KFR}}$ and $S_{\text{BL}}$ are bijections.

The second part reveals an interesting application of Theorem 1 in that bijectivity of $I$ allows us to exchange the proofs for the two surgery calculi. I.e., we can use the proofs in [FR] or [Ki] in order to establish the validity of the bridged link calculus, or, alternatively, infer the extended Kirby calculus starting from the results in [Ke].
4.) **Proof of Theorem 1:**

4.1) **Existence of $I$:** In order to see that $I$ is well defined, we need to show that if two links lie in the same class in $L_M$, then they are also equivalent as links in $BL_M$. For example, if two links differ by an $O_3$-move, we see from Figure 5 that they can also be related by a combination of $A_1$- and $A_2$-moves, and thus belong to the same class in $BL_M$.

![Figure 5: $O_3$ from $A_1$ and $A_2$](image)

A more general version, $A_{1}^{\text{gen}}$, of the cancellation move depicted in Figure 3 is defined by letting additional strands pass through the surgery balls. It is shown in Figure 6 that, with the $O_3$-move now found to be an equivalence in $BL_M$, also $A_{1}^{\text{gen}}$ is an equivalence in $BL_M$.

![Figure 6: $A_{1}^{\text{gen}}$ from $O_3$ and $A_2$](image)

Moreover, as depicted in Figure 7, the $O_2$-move can be expressed as a combination of a general cancellation and uncancellation $A_{1}^{\text{gen}}$, and an isotopy, where the surgery ball $B$ is pushed all the way along the second ribbon until it meets $B'$ again at the other end.

4.2) **Surjectivity of $I$:** It is easy to see that every class in $BL_M$ has an ordinary, framed link as a representative. Given an arbitrary bridged link, such a representative in the same class can be found by applying an $A_2$-move to each pair of balls after they have been moved together in the connected manifold $M$. The choice of isotopy, by which the balls are moved together, is conveniently expressed by a recombination band between the two balls in the original bridged link. See for example Figure 8, where we indicated the isotopy and $A_2$-move by the recombination band $r^B$ between $B$ and $B'$. 

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4.3) Injectivity of $I$: We verify injectivity by constructing a left inverse, $P$, for $I$. The image of $P$ of a class in $\mathcal{BL}_M$ shall be defined by taking a representing bridged link, recombining all pairs of surgery balls via $A_2$-moves, as described in 4.2), and then taking the class of that link in $\mathcal{LM}$. To see that $P$ is well defined by this prescription, we first need to show that the image in $L_M$ does not depend on the way we apply the $A_2$-moves, i.e., the choice of recombination ribbons, and is also invariant under the equivalences in $\mathcal{BL}_M$.

Consider for a pair of balls in a bridged link two choices, $r^B_1$ and $r^B_2$, of recombination ribbons. As depicted in Figure 9 we obtain either the link on the left with annulus $A_2$ or the link on the right with annulus $A_1$. To both diagrams we can apply an $O_3$-move as indicated, where the long ribbon $C$ runs along $r^B_1$ and $r^B_2$, and the short one is the respective opposite annulus. The resulting links differ only by $O_2$-slides over $C$ as depicted in the bottom row.
of Figure 9. Hence the links obtained from either recombining along $r_1^B$ or along $r_2^B$ are in the same class in $\mathcal{L}_M$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9}
\caption{Recombinations related by $\mathcal{O}_2$ and $\mathcal{O}_3$}
\end{figure}

Furthermore, if two bridged links differ by an $\mathcal{A}_2$-move, as in Figure 4, and we choose the recombination ribbon to be a straight strip between the balls, the images of the two links in Figure 4 under $P$ are tautologically the same.

Finally, suppose two bridged links differ by an $\mathcal{A}_1$-move, i.e., by an isolated configuration as in Figure 3. This configuration is mapped by $P$ to the Hopf link $0 \cup 0$ for an appropriately chosen recombination. As described in Paragraph 1 removing or adding this link is a special case of the $\mathcal{O}_3$-move.

We can thus conclude that $P$ is well defined on $\mathcal{BL}_M$. It is also obvious that $P \circ I$ is the identity in $\mathcal{L}_M$ so that $I$ must be injective. Thus $I$ is a bijection as asserted in Theorem 1.

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