A categorical view of varieties of ordered algebras

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Abstract

It is well known that classical varieties of $\Sigma$-algebras correspond bijectively to finitary monads on $\text{Set}$. We present an analogous result for varieties of ordered $\Sigma$-algebras, that is, categories of algebras presented by inequations between $\Sigma$-terms. We prove that they correspond bijectively to strongly finitary monads on $\text{Pos}$. That is, those finitary monads which preserve reflexive coinserters. We deduce that strongly finitary monads have a coinserter presentation, analogous to the coequalizer presentation of finitary monads due to Kelly and Power. We also show that these monads are liftings of finitary monads on $\text{Set}$. Finally, varieties presented by equations are proved to correspond to extensions of finitary monads on $\text{Set}$ to strongly finitary monads on $\text{Pos}$.

Keywords: Ordered algebras; varieties; strongly finitary monads.

Dedicated to John Power,
from whom we have learned so much,
on the occasion of his 60th birthday

1. Introduction

Varieties of ordered algebras, that is, classes of ordered $\Sigma$-algebras (for a finitary signature $\Sigma$) presented by inequations between $\Sigma$-terms, play an important role in universal algebra and computer science; for example, ordered monoids with bottom $e$ as the unit are presented by the inequation $e \leq x$ and the usual equations for classical monoids. For every variety $\mathcal{V}$ free algebras exist on all posets and the forgetful functor $\mathcal{V} \to \text{Pos}$ thus has a left adjoint. The corresponding monad $T$ on $\text{Pos}$ will be proved to be strongly finitary, which means that its underlying endofunctor $T$ is enriched (i.e., locally monotone) and preserves

1. filtered colimits, and
2. coinserters of reflexive pairs.

In the above example of ordered monoids, $T$ is a lifting of the word monad (of monoids) on $\text{Set}$. For every poset $X$, we have the poset $TX = X^*$ with the following order: a word $x_0 \ldots x_{n-1}$ is smaller or equal to a word $w$ iff $w$ decomposes as $w = w_0 \ldots w_{n-1}$ and each $w_i$ contains a letter $y_i \in X$ with $x_i \leq y_i$ in $X$.

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Conversely, given a strongly finitary monad \( T \) on \( \text{Pos} \), its Eilenberg–Moore category \( \text{Pos}^T \) will be proved to be isomorphic to a variety of ordered algebras. This leads to the following main result of our paper:

**Theorem.** The category of varieties of ordered algebras (with concrete functors as morphisms) is dually equivalent to the category of strongly finitary monads on \( \text{Pos} \).

We thus obtain a bijective correspondence between varieties of ordered algebras and strongly finitary monads on \( \text{Pos} \). This is analogous to the well-known correspondence between (classical) varieties and finitary monads on \( \text{Set} \), up to natural isomorphism.

Moreover, every variety of ordered algebras is a lifting of a classical variety. This follows from the above bijective correspondence and the fact we prove that every strongly finitary monad \( T \) on \( \text{Pos} \) is a lifting of a finitary monad \( \tilde{T} \) on \( \text{Set} \): for every poset \( X \) (with the underlying set \(|X|\)), we have \(|TX| = \tilde{T}|X|\), and the underlying maps of \( \eta_X \) and \( \mu_X \) are \( \tilde{\eta}_{|X|} \) and \( \tilde{\mu}_{|X|} \), respectively. Naturally, one classical variety can have many liftings, consider, for example, ordered monoids (a “minimal” lifting of the variety of monoids), compared with our example above. Finally, we characterize varieties presented by *equations*: their monads are extensions of finitary monads on \( \text{Set} \).

**Related results**
The bijective correspondence between varieties of ordered algebras and strongly finitary monads has been established already by Kurz and Velebil (2017). However, the proof there was derived from technically involved results concerning the exactness (in \( \text{Pos} \)-enriched sense) of these varieties. Our present proof is much simpler. A completely different proof has been presented, after the submission of our paper, by Rosický (2012).

Strongly finitary monads on enriched categories were studied by Kelly and Lack (1993). When specialized to \( \text{Pos} \) (enriched over itself as a cartesian closed category), their results yield a bijection between strongly finitary monads and equationally (!) presented classes of \( \Sigma_1 \)-algebras. However, they use a much more complex concept of a signature, following the paper of Kelly and Power (1993): let \( \text{Pos}_f \) be a set of finite posets representing all of them up to isomorphism. The signatures in \( \text{Pos} \) introduced in Kelly and Power (1993) are collections \( \Sigma = (\Sigma_n)_{n \in \text{Pos}_f} \) of posets \( \Sigma_n \). In the recent paper (Adámek et al., 2021), finitary (ordinary as well as enriched) monads on \( \text{Pos} \) are studied. They are related to inequationally specified classes of \( \Sigma \)-algebras for signatures \( \Sigma \) that present a compromise between the classical signatures (used in the present paper) and those of Kelly and Power: they are collections of sets \( \Sigma_n \) indexed by \( n \in \text{Pos}_f \). In the recent work (Adámek et al., 2021), a connection between such generalized varieties of ordered algebras and monads on \( \text{Pos} \) is investigated.

There is a different categorical perspective of varieties of ordered algebras, using discrete Lawvere theories over \( \text{Pos} \). In the pioneering thesis of Lawvere (2004), a bijective correspondence between classical varieties and algebraic theories was established. Power introduced a generalization of algebraic theories called discrete Lawvere theories in Power (2005). And Lack and Rosický proved that varieties of ordered algebras bijectively correspond to the (finitary version of) discrete Lawvere theories on \( \text{Pos} \), see Lack and Rosický (2011). A connection between theories and monads is also investigated in the recent paper of Ford et al. (2021).

### 2. Finitary and strongly finitary functors
In the present section, we recall finitary and strongly finitary endofunctors of \( \text{Pos} \). We observe that the latter means that the functor is enriched, finitary and preserves reflexive coinserters.
Remark 1. (1) Throughout the paper, we view $\text{Pos}$ as the cartesian closed category with the hom-sets $\text{Pos}(X, Y)$ ordered pointwise. All categories are understood to be enriched over $\text{Pos}$. That is, hom-sets carry partial orders such that composition is monotone. All functors, limits, colimits, and adjunctions are understood as enriched over $\text{Pos}$.

Thus, when we say “endofunctor $H$ of $\text{Pos}$,” we automatically mean that it is locally monotone. Its underlying ordinary functor is denoted by $H_o$.

(2) Colimits are understood to be weighted. Let us recall that for a given scheme, that is, a small category $\mathcal{D}$, a weight is a functor $\varphi : \mathcal{D}^{op} \to \text{Pos}$. For example, given a poset $X$ and a diagram $D : \mathcal{D} \to \text{Pos}$, the functor $\text{Pos}(D-, X) : \mathcal{D}^{op} \to \text{Pos}$ is a weight. The category of all weights with scheme $\mathcal{D}$ is simply the functor category $[\mathcal{D}^{op}, \text{Pos}]$.

A weighted colimit of a diagram $D : \mathcal{D} \to \text{Pos}$ of weight $\varphi$ is a poset $\varphi \ast D$ together with an isomorphism:

$$\text{Pos}(\varphi \ast D, X) \cong [\mathcal{D}^{op}, \text{Pos}](\varphi, \text{Pos}(D-, X))$$

of posets, natural in $X \in \text{Pos}$.

(3) Conical colimits are weighted colimits with the trivial weight, the constant functor of value 1. If the ordinary category underlying $\mathcal{D}$ is filtered, we speak about filtered (conical) colimits.

(4) Every set is considered as a poset with the discrete order. In particular, every natural number $n$ is the discrete poset on the set $\{0, 1, \ldots, n-1\}$.

Notation 2. The full subcategory of $\text{Pos}$ on natural numbers is denoted by $\mathcal{N}$, and the full embedding by:

$$I : \mathcal{N} \to \text{Pos}.$$  

Example 3. Coinserters are colimits of the scheme $\mathcal{D}$ given by a parallel pair:

$$\begin{array}{ccc}
  & & 1 \\
X & \downarrow & Y \\
& & \downarrow 0
\end{array}$$

weighted by $\varphi : \mathcal{D}^{op} \to \text{Pos}$ defined as follows:

$$\begin{array}{ccc}
\varphi 0 & \varphi Y & \varphi 1 \\
\varphi X & & \\
& & \bullet
\end{array}$$

Thus, a diagram in $\text{Pos}$ is a parallel pair:

$$f_0, f_1 : A \to B$$

of monotone maps (considered as an ordered pair $(f_0, f_1)$, of course). And the coinserter is a morphism $c : B \to C$ universal w.r.t. the inequality $c \cdot f_0 \leq c \cdot f_1$:
That is,

(1) for every morphism \( u : B \to D \) with \( u \cdot f_0 \leq u \cdot f_1 \), there exists a unique morphism \( v : C \to D \) with \( u = v \cdot c \), and

(2) the map \( u \mapsto v \) is monotone: given \( u' = v' \cdot c \), then \( u \leq u' \) implies \( v \leq v' \).

**Remark 4.** Every finite poset \( P \) is a canonical co inserter of a parallel pair:

\[
\begin{array}{ccc}
  k & \xrightarrow{p_1} & n \\
  \downarrow{p_0} & & \downarrow{} \\
  0 & \xrightarrow{} & n
\end{array}
\]

of morphisms in \( \mathcal{N} \). Let \( n \) be the number of elements of \( P \) and \( k \), the number of comparable pairs in \( P \). Thus, we can assume that \( P \) has elements \( 0, \ldots, n-1 \), and we can index all comparable pairs as follows:

\[
p_0(t) \leq p_1(t) \text{ for } t = 0, \ldots, k-1.
\]

This defines functions \( p_0, p_1 : k \to n \). The co inserter of this pair is carried by the identity map:

\[
\begin{array}{ccc}
  k & \xrightarrow{p_1} & n \\
  \downarrow{p_0} & & \downarrow{id} \\
  0 & \xrightarrow{} & P
\end{array}
\]

**Notation 5.** Denote by

\( J : \text{Pos}_f \hookrightarrow \text{Pos} \)

a full embedding of a subcategory \( \text{Pos}_f \) representing all finite posets up to isomorphism. We choose \( \text{Pos}_f \) so that it contains \( \mathcal{N} \) as a full subcategory.

**Remark 6.** (1) \( \text{Pos} \) is a free completion of \( \text{Pos}_f \) under filtered colimits. In the realm of ordinary categories, this follows from Adámek and Rosický (1994, Theorem 1.46), since \( \text{Pos} \) is a locally finitely presentable category with finite posets precisely the finitely presentable objects. Thus, given an ordinary category \( \mathcal{N} \) with filtered colimits, for every ordinary functor \( H : \text{Pos}_f \to \mathcal{N} \), there exists an extension \( H' : \text{Pos} \to \mathcal{N} \) preserving filtered colimits, unique up to natural isomorphism. Filtered colimits in \( \text{Pos} \) are conical: given a colimit cocone \( c_i : C_i \to C \) \((i \in I)\), then for two morphisms \( u, v : C \to X \) we have \( u \leq v \) iff \( u \cdot c_i \leq v \cdot c_i \) for all \( i \in I \). It follows that \( H' \) is locally monotone whenever \( H \) is. Thus, the statement above holds also in the enriched sense.

(2) Following Kelly (1982), we call an endofunctor of \( \text{Pos} \) finitary iff its underlying ordinary endofunctor is finitary (i.e., preserves ordinary filtered colimits).

We now turn to strongly finitary functors.

**Definition 7.** (Kelly and Lack 1993). An endofunctor \( H \) of \( \text{Pos} \) is called strongly finitary if it is the left Kan extension of its restriction to \( \mathcal{N} \). More precisely:

\[
H = \text{Lan}_I H \cdot I.
\]

**Remark 8.** In ordinary categories, sifted colimits are colimits of diagrams whose schemes \( \mathcal{D} \) are (small) sifted categories. This means categories such that colimits of diagrams \( D : \mathcal{D} \to \text{Set} \) commute with finite products.
In our enriched setting, sifted colimits are introduced analogously. A weight \( \varphi : \mathcal{D}^{\text{op}} \to \text{Pos} \) is called \textit{sifted} if the colimit functor \( \varphi \ast - : [\mathcal{D}, \text{Pos}] \to \text{Pos} \) preserves finite (conical) products. Sifted colimits then are colimits weighted by sifted weights.

**Example 9.**

1. Filtered colimits are clearly sifted: the ordinary category \( \text{Pos} \) is locally finitely presentable and filtered colimits thus commute with finite (conical) limits (see Adámek and Rosický 1994, Example 1.10 and Proposition 1.59).

2. Reflexive coinserter is sifted, as we prove next. Recall that a parallel pair \( f_0, f_1 : X \to Y \) is called \textit{reflexive} if there exists a morphism \( d : Y \to X \) with \( f_i \cdot d = \text{id}_Y \) for \( i = 0, 1 \). **Reflexive coinserter**, that is, coinserter of reflexive pairs, are weighted colimits with the scheme \( \mathcal{D} \) given by:

\[
\begin{array}{c}
n_1 \\
\downarrow \downarrow \\
\downarrow \downarrow \\
n_0 \\
X \xleftarrow{d} Y
\end{array}
\]

(where \( 1 \cdot d = \text{id}_Y = 0 \cdot d \)) and the weight \( \psi : \mathcal{D}^{\text{op}} \to \text{Pos} \) obtained from the weight \( \varphi \) of Example 3 by defining \( \psi d : \psi X \to \psi Y \) to be the constant map.

We now verify that the weight \( \psi \) is sifted. In other words, reflexive coinserter commute with (conical) finite products in \( \text{Pos} \). The proof is analogous to the proof that reflexive coequalizers commute with finite products in \( \text{Set} \) (Adámek et al. 2010, Example 1.2).

**Proposition 10.** In \( \text{Pos} \), reflexive coinserter commute with finite conical products.

**Proof.**

1. We first describe the coinserter of a reflexive pair \( u_0, u_1 : X \to Y \) in \( \text{Pos} \). We define a preorder \( \sqsubseteq \) on \( Y \) for which the posetal reflection \( e : (Y, \sqsubseteq) \to (Y, \sqsubseteq)/\sim \), where \( y \sim \gamma \) iff \( y \sqsubseteq \gamma \sqsubseteq y \), is the desired coinserter. Put \( y \sqsubseteq \gamma \) iff there exist \( x_0, \ldots, x_n \in X \) such that \( y = u_0(x_0), \gamma = u_1(x_n) \) and

\[
u_i(x_i) \leq u_0(x_{i+1}) \quad \text{for } i = 0, \ldots, n - 1.
\]

Having a monotone map \( d : Y \to X \) with \( u_0 \cdot d = \text{id}_Y = u_1 \cdot d \), we see that \( \sqsubseteq \) is a preorder whose posetal reflection is the coinserter of \( u_0 \) and \( u_1 \). Moreover, given a sequence \( x_0, \ldots, x_m \) for \( y \) and \( \gamma \) as above, we can extend it for every \( m > n \) to a corresponding sequence \( x_0, \ldots, x_m \): put \( x_i = d(\gamma) \) for \( i = n + 1, \ldots, m \).

2. For every other reflexive pair \( u_0', u_1' : X' \to Y' \) with the corresponding coinserter \( e' : (Y', \sqsubseteq') \to (Y', \sqsubseteq')/\sim \), it is our task to prove that we have the following coinserter:

\[
\begin{array}{ccc}
X \times X' & \xrightarrow{u_0 \times u_1'} & Y \times Y' \\
\downarrow{u_0 \times u_0'} & & \downarrow{e \times e'} \\
Y \times Y' & \xrightarrow{(\sim, \sim)} & (Y'/\sim) \times (Y'/\sim)
\end{array}
\]

Indeed, the pair \( u_0 \times u_0' \) and \( u_1 \times u_1' \) is clearly reflexive, thus its coinserter is described as in (1) above. Let \( \leq \) be the resulting preorder on \( Y \times Y' \) and let elements \((y, y')\) and \((\gamma, \gamma')\) be given. Whenever a sequence \((x_0, x_0'), \ldots, (x_n, x_n')\) in \( X \times X' \) demonstrates that \((y, y') \preceq (\gamma, \gamma')\), then we have \( y \sqsubseteq \gamma \) and \( y' \sqsubseteq \gamma' \): use the sequences in \( X \) and \( X' \), respectively, obtained by the projection of \((x_0, x_0'), \ldots, (x_n, x_n')\).

It remains to prove the converse: from \( y \sqsubseteq \gamma \) and \( y' \sqsubseteq \gamma' \), we derive \((y, y') \preceq (\gamma, \gamma')\). Here, we use that the sequence in (1) can be prolonged to any larger length. Thus, for \( y \sqsubseteq \gamma \), we have a sequence \( x_0, \ldots, x_n \) in \( X \) as in (1), and for \( y' \sqsubseteq \gamma' \), we have a sequence of the same
Theorem 11. (Bourke 2010, Corollary 8.45). The following conditions are equivalent for endofunc-
 tors $H$ of $\text{Pos}$:

1. $H$ is strongly finitary,
2. $H$ preserves sifted colimits,
3. $H$ is finitary and preserves reflexive coinserter.

Proof. Every poset is a filtered colimit of its finite subposets, each of which is a coinerter as in
Remark 4.

Consequently, starting with the subcategory $\mathcal{N}$, we obtain all of $\text{Pos}$ by reflexive coinserter
and filtered colimits. Moreover, every functor $\text{Pos}(\mathbb{N}, -)$ preserves these colimits (see Remark 6
and Proposition 10). In the terminology of Kelly (2005, Theorem 5.29), this states that the
embedding $I : \mathcal{N} \to \text{Pos}$ has a codensity presentation formed by filtered colimits and reflexive
coinserter. By that theorem, properties (1)–(3) are equivalent. □

Remark 12. In (3), we can substitute reflexive coinserter by canonical coinserter, as is clear from
the above proof.

Remark 13. The above theorem is completely analogous to the fact proved in Adámek et al.
(2010) for ordinary endofunctors of categories with finite coproducts: preservation of sifted
colimits is equivalent to the preservation of filtered colimits and reflexive coequalizers.

Example 14.

1. The endofunctor $X \mapsto X^m$ (for $m \in \mathbb{N}$) of $\text{Pos}$ is strongly finitary: it clearly preserves filtered
colimits, and we verify that it also preserves the canonical coinserter of Remark 4. Suppose
$m = 2$. Then, a comparable pair in $P \times P$ is a pair $(a, b)$ where the left-hand components of
$a$ and $b$ are comparable in $P$, and thus have the form $x_{p_0(i)} \leq x_{p_1(i)}$ for some $i \leq k - 1$. And
the right-hand components have the form $x_{p_0(j)} \leq x_{p_1(j)}$ for some $j \leq k - 1$. Thus, the only
comparable pairs of $P \times P$ are $(x_{p_0(i)}, x_{p_0(j)})$, $(x_{p_1(i)}, x_{p_1(j)})$. We conclude that the canonical
coinserter of the poset $P \times P$ is given by $p_0 \times p_0, p_1 \times p_1 : k \times k \to n \times n$. The argument is
analogous for $m > 2$.

2. Coproducts of strongly finitary endofunctors are strongly finitary. For example, given a
signature $\Sigma$, the corresponding polynomial functor $H_{\Sigma}X = \bigsqcup_{m \in \mathbb{N}} \Sigma^m \times X^m$ is strongly
finitary.

3. (Weighted) colimits of strongly finitary endofunctors are strongly finitary.

4. A composite of strongly finitary endofunctors is strongly finitary.

Remark 15. Every strongly finitary endofunctor $H$ of $\text{Pos}$ generates a free monad whose under-
lying functor $\hat{H}$ is also strongly finitary. Indeed, following Trnková et al. (1975), $\hat{H}$ is a conical
colimit in $[\text{Pos}, \text{Pos}]$ of the following $\omega$-chain:

$$
\begin{array}{c}
\text{Id} \xrightarrow{w_0} H + \text{Id} \xrightarrow{w_1} H(H + \text{Id}) + \text{Id} \xrightarrow{w_2} \ldots
\end{array}
$$

That is, the chain $W : \omega \to [\text{Pos}, \text{Pos}]$ has objects:

$W_0 = \text{Id}$ and $W_{n+1} = H W_n + \text{Id}$.
and morphisms:

\[ w_0 : \text{Id} \to H + \text{Id} \]

the coproduct injection

and

\[ w_{n+1} = Hw_n + \text{id}. \]

Thus, if \( H \) is strongly finitary, so is each \( W_n \) (by the preceding example). Consequently, \( \hat{H} = \text{colim} \ W_n \) is strongly finitary.

**Notation 16.** A monad whose endofunctor is strongly finitary is called a strongly finitary monad. We denote by:

\[ \text{Mnd}_{sf}(\text{Pos}) \]

the ordinary category of strongly finitary monads and monad morphisms.

**Example 17.** The endofunctor \( H_\Sigma \) generates the following free monad \( T_\Sigma = (T_\Sigma, \mu_\Sigma, \eta_\Sigma) \) on \( \text{Pos} \): to every poset \( X \) (of variables), it assigns the poset \( T_\Sigma X \) of \( \Sigma \)-terms with variables from \( X \). That is, the underlying set is the smallest set containing \( X \) and such that for every \( \sigma \in \Sigma_n \) and every \( n \)-tuple \( t_i \) in \( T_\Sigma X \), we have \( \sigma(t_i) \) in \( T_\Sigma X \). This yields a structure of a \( \Sigma \)-algebra on \( T_\Sigma X \). The ordering of \( T_\Sigma X \) is the smallest one such that \( T_\Sigma X \) contains \( X \) as a subposet, and all operations are monotone. Thus, \( \sigma(t_i) \leq \sigma'(t'_i) \) holds iff \( \sigma = \sigma' \) and \( t_i \leq t'_i \) for each \( i \). And \( x \in X \) is incompatible with \( \sigma(t_i) \). It follows from Remark 15 that \( T_\Sigma \) is strongly finitary.

**Remark 18.** The category \( \text{Mnd}_{sf}(\text{Pos}) \) has (weighted) colimits. To show this, we can use the argument from Lack (1999) that the category of finitary monads on a locally finitely presentable category is monadic. (The same argument works for strongly finitary monads after minor changes.) For a direct argument, see Theorem 38 in Bourke and Garner (2019) showing that the underlying functor \( U : \text{Mnd}_{sf}(\text{Pos}) \to [\mathcal{N}, \text{Pos}] \) is monadic and that \( \text{Mnd}_{sf}(\text{Pos}) \) is locally presentable. (In our special case of their general theory, the essential functor \( K : \mathcal{A} \to \mathcal{E} \) is the inclusion of \( \mathcal{N} \) into \( \text{Pos} \), specifying our arities of interest to be finite and discrete.)

### 3. From Varieties of Ordered Algebras to Strongly Finitary Monads

In the current section, we prove that the free-algebra monad of every variety is strongly finitary, and that it has a coinserter presentation via free monads on signatures.

**Notation 19.** Let \( \Sigma \) be a signature, that is, a collection of sets \( \Sigma_n \) (of \( n \)-ary operation symbols) indexed by \( n \in \mathbb{N} \). An ordered \( \Sigma \)-algebra is a poset \( A \) together with a monotone map \( \sigma_A : A^n \to A \) for every \( n \in \mathbb{N} \) and \( \sigma \in \Sigma_n \). The category of ordered \( \Sigma \)-algebras and homomorphisms (i.e., monotone functions preserving the given operations) is denoted by \( \text{Alg}(\Sigma) \).

**Remark 20.**

1. \( \text{Alg}(\Sigma) \) is clearly isomorphic to the category of algebras for \( H_\Sigma \) (see Example 14), that is, pairs \( (A, \alpha) \) where \( A \) is a poset and \( \alpha : H_\Sigma A \to A \) is a monotone function. (Morphisms are monotone maps making the obvious square commutative.)
2. It follows from Barr (1970) that the category of algebras for an ordinary endofunctor \( H \) is equivalent to the category of Eilenberg–Moore algebras for the free monad \( \hat{H} \) (see Remark 15 (2)). The same result holds for enriched endofunctors. In particular, we conclude

\[ H_\Sigma - \text{Alg} \cong \text{Alg}(\Sigma) \simeq \text{Pos}^{T_\Sigma}. \]
(3) The poset $T_X \Sigma \mathcal{X}$ of terms (Example 17) is a free $\Sigma$-algebra on $\mathcal{X}$ and $\eta_X : X \to T_X \Sigma \mathcal{X}$ is the inclusion of variables: for every $\Sigma$-algebra $A$ and every monotone function $f : X \to A$ the unique extension to a homomorphism $f^\Sigma : T_X \Sigma \mathcal{X} \to A$ is given by:

$$f^\Sigma(\sigma(t_i)) = \sigma_A(f^\Sigma(t_i)).$$

**Definition 21.** Let $V = \{x_n \mid n \in \mathbb{N}\}$ be a countably infinite set of variables (considered as a discrete poset). An ordered pair of terms in $T_V \Sigma$ is called an inequation and is written as $u \leq v$. A $\Sigma$-algebra $(A, \alpha)$ satisfies $u \leq v$ if and only if every map $f : V \to A$ (interpretation of variables) fulfills $f^\Sigma(u) \leq f^\Sigma(v)$.

By a variety of ordered $\Sigma$-algebras, we understand a full subcategory of $\text{Alg}(\Sigma)$ specified by a set $E$ of inequations. That is, a $\Sigma$-algebra belongs to the variety iff it satisfies every inequation from $E$.

**Remark 22.**

(1) Every $n$-ary operation symbol $\sigma$ can be identified with the term $\sigma(x_0, \ldots, x_{n-1})$.

(2) Every equation $u = v$ is identified with the pair:

$$u \leq v \text{ and } v \leq u.$$

**Example 23.**

(1) Bounded posets (with a least element $0$ and a largest element $1$) form a variety with $\Sigma$ given by nullary operations $0,1$ and the variety is presented by the inequations:

$$0 \leq x \text{ and } x \leq 1.$$

This is a lifting of the variety of non-ordered algebras with two nullary operations.

(2) Ordered semigroups, a.k.a. posemigroups (see Blyth 2005), are ordered algebras on one associative binary operation. The corresponding monad $T$ assigns to a poset $X$ the poset $TX = X^+$ of finite nonempty words over $X$, ordered componentwise:

$$x_0 \ldots x_{n-1} \leq y_0 \ldots y_{m-1} \text{ iff } n = m \text{ and } x_i \leq y_i \text{ for } i = 0, \ldots, n - 1.$$

(3) Analogously for ordered monoids: the signature is $\Sigma = \{\cdot, e\}$ with $\cdot$ binary and $e$ unary. They are presented by the usual monoid equations and yield the monad $TX = X^*$ of finite words, ordered componentwise.

(4) If we add to the equations above, the inequation:

$$x \leq x \cdot y$$

we obtain the variety of ordered monoids with $e$ the smallest element. That is, the above inequation is equivalent to

$$e \leq y.$$

Indeed the first inequation yields the latter one by putting $x = e$. Conversely, from $e \leq y$ we get $x = x \cdot e \leq x \cdot y$.

The corresponding monad is the lifting of the word monad:

$$TX = X^*$$

ordered as follows:

$$x_0 x_1 \ldots x_{n-1} \leq w \text{ iff } w = w_0 w_1 \ldots w_{n-1} \text{ and } w_i \text{ contains } y_i \text{ with } x_i \leq y_i \text{ (}i < n\text{).}$$

(5) Ordered positive semirings (see Golan 2003). A (non-ordered) semiring is an algebra of signature $\Sigma = \{+, 0, \cdot, 1\}$ satisfying the monoid equations for $(+, 0)$ as well as for $(\cdot, 1)$, the commutative law for $+$, the equations $0 \cdot a = 0 = a \cdot 0$ and the left and right distributive
laws for $+$ and $\cdot$. An ordered positive semiring is an ordered $\Sigma$-algebra satisfying the above equations and the inequation $x \geq 0$.

(6) In contrast, ordered groups or rings are not ordered algebras in our sense: the operation $x \mapsto -x$ is usually not monotone.

**Remark 24.** Every variety of $\Sigma$-algebras is a reflective subcategory of $\text{Alg}(\Sigma)$ with surjective reflections. In fact:

1. Every variety is ordinarily reflective in $\text{Alg}(\Sigma)$. Indeed, since $H_{\Sigma}$ is a finitary endofunctor on a locally finitely presentable category, $\text{Alg}(\Sigma) \cong H_{\Sigma}\text{-Alg}$ is also locally finitely presentable (Adámek and Rosický, 1994, Remark 2.78). In particular, it is complete and cowellpowered. The factorization system $(\text{epi}, \text{embedding})$ on $\text{Pos}$ lifts, since $H_{\Sigma}$ preserves epimorphisms, to $\text{Alg}(\Sigma)$. Since a variety $\mathcal{V}$ is easily seen to be closed under products and subalgebras carried by embeddings, the surjective reflections follow Adámek et al. (1990, Theorem 16.8).

2. Since $\mathcal{V}$ is closed in $\text{Alg}(\Sigma)$ under powers of the poset $2 = \{0 \leq 1\}$, it follows from Kelly (2005) (Section 1.11) that the above ordinary reflection is in fact enriched.

**Construction 25.** (See Bloom 1976). For every variety $\mathcal{V}$ of ordered algebras, the free algebra $T_{\mathcal{V}}X$ of $\mathcal{V}$ on a poset $X$ can be constructed as follows.

For every poset $X$, let $\mathcal{E}_X$ be the collection of all inequations $s \leq t$ satisfied by all algebras of $\mathcal{V}$, where $s, t \in T_{\Sigma}X$ are terms in variables from $|X|$. Then $\mathcal{E}_X$ is a preorder, that is, a reflexive and transitive relation on the underlying set $|T_{\Sigma}X|$ of $T_{\Sigma}X$. Moreover, it is admissible: given an $n$-ary symbol $\sigma \in \Sigma$ and $n$ pairs $s_i \leq t_i$ ($i < n$) in $\mathcal{E}_X$, it follows that the pair $\sigma(s_i) \leq \sigma(t_i)$ also lies in $\mathcal{E}_X$. Indeed, given an algebra $(A, \omega) \in \mathcal{V}$ and an interpretation $f : X \rightarrow A$, we know that the homomorphism $f^\mathcal{E} : T_{\Sigma}X \rightarrow A$ fulfills $f^\mathcal{E}(s_i) \leq f^\mathcal{E}(t_i)$ for all $i$, thus

$$f^\mathcal{E}(\sigma(s_i)) = \sigma_A(f^\mathcal{E}(s_i)) \leq \sigma_A(f^\mathcal{E}(t_i)) = f^\mathcal{E}(\sigma(t_i)).$$

Consequently, for the induced equivalence relation:

$$\mathcal{E}_X^0 = \mathcal{E}_X \cap \mathcal{E}_X^{-1}$$

we obtain a $\Sigma$-algebra $T_{\mathcal{V}}X$ on the quotient set:

$$|T_{\mathcal{V}}X| = |T_{\Sigma}X|/\mathcal{E}_X^0$$

(of all equivalence classes $[t]$ of terms $t \in T_{\Sigma}X$). The operations are as expected:

$$\sigma_{T_{\mathcal{V}}X}([t_0], \ldots, [t_{n-1}]) = [\sigma(t_0, \ldots, t_{n-1})]$$

for every $n$-ary $\sigma$ and all $n$-tuples $t_0, \ldots, t_{n-1} \in T_{\Sigma}X$. Finally, we consider $T_{\mathcal{V}}X$ as a poset via

$$[s] \leq [t] \text{ iff } (s, t) \in \mathcal{E}_X.$$

The following theorem was stated by Bloom (1976, Theorem 2.2). We present a full proof since we need it later, and the original proof was only a sketch.

**Theorem 26.** The above ordered algebra $T_{\mathcal{V}}X$ is a free algebra of the variety $\mathcal{V}$ on the poset $X$ w.r.t. $\eta_X : X \rightarrow T_{\mathcal{V}}X$ given by $x \mapsto [x]$.

**Proof.**

1. $T_{\mathcal{V}}X$ is a well-defined ordered $\Sigma$-algebra. This follows easily from the fact that $\mathcal{E}_X$ is an admissible preorder.
(2) \(V\) has a free algebra on \(X\) which is given by an admissible preorder \(\sqsubseteq\) on \(T_\Sigma X\) (i.e., for the induced equivalence relation \(\sim\), the underlying poset is \(|T_\Sigma X|/\sim\) and the operations are induced by those of \(T_\Sigma X\)). This statement follows from Remark 24, which implies that a free algebra \(T_\gamma X\) exists, and the unique homomorphism:

\[ e_\gamma : T_\Sigma X \to T_\gamma X \]

extending the universal arrow is epic. Indeed, the desired preorder is simply

\[ s \sqsubseteq t \text{ iff } e_\gamma(s) \leq e_\gamma(t). \]

(3) The preorder \(\mathcal{E}_X\) of the above construction coincides with \(\sqsubseteq\) of (2). Indeed, if \((s, t) \in \mathcal{E}_X\), then the algebra \(T_\gamma X\) satisfies \(s \leq t\) (since it lies in \(V\)) and taking the universal map \((\eta_\gamma)_X : X \to T_\gamma X\) as the interpretation, we have

\[ e_\gamma = (\eta_\gamma)_X^\sharp \]

(because \(e_\gamma\) is a \(\Sigma\)-homomorphism). Since \(e_\gamma(s) \leq e_\gamma(t)\), we conclude that \(s \sqsubseteq t\).

Conversely, if \(s \sqsubseteq t\), which means \(e_\gamma(s) \leq e_\gamma(t)\), we verify that every algebra \(A \in V\) satisfies \(s \leq t\). Let \(f : X \to A\) be an interpretation, then the corresponding homomorphism \(f^\sharp : T_\Sigma X \to A\) factorizes through the reflection of \(T_\Sigma X\) in \(V\) in \(\text{Alg}(\Sigma)\):

\[
\begin{array}{ccc}
T_\Sigma X & \xrightarrow{e_X} & T_\gamma X \\
\downarrow f^\sharp & & \downarrow h \\
A & & A
\end{array}
\]

Since \(h\) is monotone, the inequality \(e_\gamma(s) \leq e_\gamma(t)\) implies \(f^\sharp(s) \leq f^\sharp(t)\), as required.

\[ \square \]

**Notation 27.** For every variety \(V\) of \(\Sigma\)-algebras, we denote by

\[ c_V : T_\Sigma \to T_V \]

the monad morphism whose components are the canonical quotient maps:

\[ |T_\Sigma X| \to |T_\Sigma X|/\mathcal{E}_X^0. \]

**Lemma 28.** For every variety \(V\), the forgetful functor to \(\text{Pos}\) is strictly monadic: the comparison functor \(K : V \to \text{Pos}^V\) is an isomorphism.

**Proof.** For classical varieties, see Mac Lane (1998), Theorem VI.8.1. The proof for varieties of ordered algebras can be used verbatim (with the equation \(\lambda_B = \mu_B\) being replaced by \(\lambda_B \leq \mu_B\)).

\[ \square \]

**Remark 29.** In the proof below that \(T_\gamma\) is a strongly finitary monad, we use a technique developed by Kelly (2005) which enables us to identify algebras of a monad \(T\) on a given poset \(A\) with monad morphisms from \(T\) to a certain monad \((A, A)\) that we now recall. See Kelly (2005) for the proof of the following facts:
(1) The continuation monad \( \langle A, A \rangle \) on \( \text{Pos} \) associated with a poset \( A \) assigns to a poset \( X \) the power of \( A \) to the set \( \text{Pos}_o(X, A) \) of all monotone maps \( f : X \to A \):

\[
\langle A, A \rangle X = \prod_{\text{Pos}_o(X, A)} A.
\]

This is the monad resulting from the adjunction given by the right adjoint \( \text{Pos}(-, A) : \text{Pos} \to \text{Pos}^{op} \). Denote by \( \pi_f : \langle A, A \rangle X \to A \) the projection corresponding to \( f : X \to A \). To every morphism \( h : X \to Y \), the monad assigns the morphism \( \langle A, A \rangle h \) determined by the following commutative triangles:

\[
\begin{array}{c}
\prod_{\text{Pos}_o(X, A)} A \\
\downarrow \pi_f \\
A
\end{array} \xrightarrow{\langle A, A \rangle h} \begin{array}{c}
\prod_{\text{Pos}_o(Y, A)} A \\
\downarrow \pi_f \\
A
\end{array} \quad f \in \text{Pos}_o(Y, A).
\]

The unit is \( \langle f \rangle \in \text{Pos}_o(X, A) \), and the multiplication \( \mu_X \) is determined by the following commutative triangles:

\[
\begin{array}{c}
\prod_{\text{Pos}_o((A, A), X, A)} A \\
\downarrow \pi_{\pi_f} \\
A
\end{array} \xrightarrow{\mu_X} \begin{array}{c}
\prod_{\text{Pos}_o(X, A)} A \\
\downarrow \pi_f \\
A
\end{array} \quad f \in \text{Pos}_o(X, A).
\]

(2) It follows from Dubuc (1970) that for every monad \( T \) and every poset \( A \), there is a bijection between monad morphisms \( T \to \langle A, A \rangle \) and algebras of \( \text{Pos}^T \) on \( A \). This bijection assigns to an algebra \( \alpha : TA \to A \) the monad morphism:

\[
\hat{\alpha} : T \to \langle A, A \rangle
\]

with components \( \hat{\alpha}_X \) determined by the following commutative squares:

\[
\begin{array}{c}
TX \\
\downarrow Tf \\
TA \\
\downarrow \alpha \\
\end{array} \xrightarrow{\hat{\alpha}_X} \begin{array}{c}
(A, A)X \\
\downarrow \pi_f \\
A
\end{array} \quad f \in \text{Pos}_o(X, A).
\]

Thus, if \( T = T_{\Sigma} \), then \( \hat{\alpha}_X \) assigns to a term \( t \in T_{\Sigma}X \) the tuple \( \hat{\alpha}^2(t)_{f:X \to A} \).

(3) Let \( b : S \to T \) be a monad morphism. Every algebra \( (A, \alpha) \) in \( \text{Pos}^T \) then yields an algebra \( (A, \alpha \cdot b_A) \) in \( \text{Pos}^S \). The following triangle:

\[
\begin{array}{c}
S \\
\xrightarrow{b} \\
\text{Pos}_o(X, A)
\end{array} \xrightarrow{\hat{\alpha}} \begin{array}{c}
(A, A) \\
\downarrow \alpha \\
T
\end{array} \quad \text{commutes. Indeed, for every poset } X \text{ and every } f \in \text{Pos}_o(X, A), \text{ we have}
\]

\[
\pi_f(\hat{\alpha}_X \cdot b_X) = \alpha \cdot Tf \cdot b_X = \alpha \cdot b_A \cdot Sf.
\]
The same result is obtained by:
\[ \pi f((\alpha \cdot b)\hat{A})X = \alpha \cdot b\cdot A \cdot Sf. \]

(4) In particular, let \( T = T_\Sigma \) for a signature \( \Sigma \). Given a term \( u \) in \( T_\Sigma n \), it corresponds to a monad morphism:
\[ \tilde{u} : T_{\Omega n} \to T_\Sigma \]
where \( \Omega_n \) is a signature of a single operation \( \omega \) of arity \( n \). Its component \( \tilde{u}_X : T_{\Omega n}X \to T_\Sigma X \) assigns to a term \( t \) over \( X \) (containing the unique operation symbol \( \omega \)) the \( \Sigma \)-term obtained by replacing each \( \omega \) by the term \( u \). Thus, if a \( \Sigma \)-algebra \( (A, \alpha) \) satisfies an inequation \( u_0 \leq u_1 \), the inequation \( (\tilde{\alpha} \cdot \tilde{u}0)_X \leq (\tilde{\alpha} \cdot \tilde{u}1)_X \) holds for all posets \( X \). Shortly: \( \tilde{\alpha} \cdot \tilde{u}0 \leq \tilde{\alpha} \cdot \tilde{u}1 \).

Construction 30. We describe the free-algebra monad of the variety given by a single inequation \( u_0 \leq u_1 \) in signature \( \Sigma \). Let \( u_0, u_1 \) be terms with variables \( x_0, \ldots, x_{n-1} \). For the signature \( \Omega_n \) of a single operation of arity \( n \), they can be viewed (via Yoneda lemma) as natural transformations:
\[ u_0, u_1 : H_{\Omega_n} \to T_{\Sigma}. \]
The corresponding monad morphisms:
\[ \tilde{u}_0, \tilde{u}_1 : T_{\Omega n} \to T_{\Sigma} \]
have a coinserter in \( \text{Mnd}_{sf}(\text{Pos}) \) which we denote as follows:
\[ T_{\Omega_n} \quad \tilde{u}_1 \quad T_{\Sigma} \quad \tilde{u}_0 \quad c \quad T \]

We verify that this is precisely \( c_\mathcal{V} \) of Notation 27 for the variety presented by \( u_0 \leq u_1 \):

Proposition 31. The above monad \( T \) is the free-algebra monad of the variety presented by the inequation \( u_0 \leq u_1 \).

Proof. The variety \( \mathcal{V} \) presented by \( u_0 \leq u_1 \) yields a free-algebra monad \( T_\mathcal{V} \). The proposition will be proved by verifying that \( c_\mathcal{V} \) is a coinserter of \( \tilde{u}_0, \tilde{u}_1 \) in \( \text{Mnd}_{sf}(\text{Pos}) \). From the definition of \( c_\mathcal{V} \), we conclude
\[ c_\mathcal{V} \cdot \tilde{u}_0 \leq c_\mathcal{V} \cdot \tilde{u}_1. \]
(a) Given a strongly finitary monad \( S = (S, \mu, \eta) \) and a monad morphism \( b : T_\Sigma \to S \) with
\[ b \cdot \tilde{u}_0 \leq b \cdot \tilde{u}_1, \]
we prove that \( b \) factorizes through \( c_\mathcal{V} \) via a monad morphism.
For every poset \( X \), the free algebra \( (S X, \mu_X) \) for \( S \) yields, since \( b \) is a monad morphism, the following algebra for \( T_\Sigma \) on \( S X \):
\[ \beta_X : T_\Sigma S X \xrightarrow{b_{SX}} S S X \xrightarrow{\mu_X} S X \]
From \( \alpha_X \cdot (\tilde{u}_0)_X \leq \alpha_X \cdot (\tilde{u}_1)_X \) we deduce, using Remark 29 (4), that the \( \Sigma \)-algebra \( (S X, \beta_X) \) satisfies the inequation \( u_0 \leq u_1 \). Since the free algebra \( (T X, \mu_X^T) \) of \( \mathcal{V} \) on \( X \) corresponds to the \( \Sigma \)-algebra:
\[ T_\Sigma TX \xrightarrow{(c_\mathcal{V})TX} TTX \xrightarrow{\mu_X^T} TX, \]
we obtain a unique $\Sigma$-homomorphism $\bar{b}_X$ with $\bar{b}_X \cdot \eta^T_X = \eta^S_X$:

We verify that these morphisms $\bar{b}_X$ form a monad morphism:

$$\bar{b} : T \to S$$ with $b = \bar{b} \cdot c_{\psi}$.

1. The equality $b_X = \bar{b}_X \cdot (c_{\psi})_X : T\Sigma X \to SX$ holds because both sides are homomorphisms of $\Sigma$-algebras and we have

$$b_X \cdot \eta^T_X = \eta^S_X = \bar{b}_X \cdot \eta^T_X = \bar{b}_X \cdot (c_{\psi})_X \cdot \eta^T_X.$$

2. $\bar{b}_X$ is natural in $X$. In fact, every morphism $f : X \to Y$ yields a $\Sigma$-homomorphism:

$$Tf : (TX, \mu^T_X \cdot (c_{\psi})_TX) \to (TY, \mu^T_Y \cdot (c_{\psi})_TY)$$

Thus, $\bar{b}_Y \cdot Tf$ is also a $\Sigma$-homomorphism, and so is $Sf \cdot \bar{b}_X : (TX, \mu^T_X \cdot (c_{\psi})_TX) \to (SY, \alpha_Y)$. Since the domain of both composites is a free algebra of $\psi$ on $X$, for proving that they are equal we just need to verify

$$\bar{b}_Y \cdot Tf \cdot \eta^T_X = Sf \cdot \bar{b}_X \cdot \eta^T_X.$$ See the following diagram:

(3) The equality $\bar{b} \cdot \eta^T = \eta^S$ follows from the right-hand triangle in the diagram defining $\bar{b}_X$ above.
(4) We finally prove $\bar{b} \cdot \mu^T = \mu^S \cdot S\bar{b} \cdot bT$. Consider the following diagram:

$$
\begin{array}{cccc}
T \Sigma TX & \xrightarrow{(c \gamma)_TX} & TTX & \xrightarrow{\mu^T_X} & TX \\
\downarrow b_{TX} & & \downarrow b_{TX} & & \downarrow \bar{b}_X \\
T \Sigma \bar{b}_X & & STX & & \bar{b}_X \\
\downarrow s\bar{b}_X & & \downarrow s\bar{b}_X & & \downarrow \bar{b}_X \\
T \Sigma SX & \xrightarrow{b_{SX}} & SSX & \xrightarrow{\mu^S_X} & SX
\end{array}
$$

The outward rectangle is the definition of $\bar{b}_X$. The left-hand parts commute by (1) and (2). Consequently, the desired right-hand square commutes since it does when precomposed by the epimorphism $(c \gamma)_TX$.

(b) Finally for every monad morphism $b' : T \Sigma \rightarrow S$ factorized as $b' = \bar{b'} \cdot c \gamma$ we are to verify that

$$b \leq b' \text{ implies } \bar{b} \leq \bar{b'}$$

This is trivial since the components of $c \gamma$ are surjective.

\[\square\]

**Construction 32.** The above proposition immediately generalizes to sets of inequations. For every variety $\mathcal{V}$ of $\Sigma$-algebras, the free-algebra monad $T \mathcal{V}$ is a canonical quotient $c \gamma : T \Sigma \rightarrow T \mathcal{V}$ of the free-$\Sigma$-algebra monad, see Notation 27. We construct monad morphisms $\tilde{u}_0, \tilde{u}_1 : T \Omega \rightarrow T \Sigma$ for some signature $\Omega$ forming a coinserter in $Mndsf(\text{Pos})$ as follows:

$$T \Omega \xrightarrow{\tilde{u}_1} T \Sigma \xrightarrow{c \gamma} T \mathcal{V}$$

Given a set

$$u^i \leq u'^i, \quad i \in I$$

of inequations specifying the variety $\mathcal{V}$, let $n_i$ be the number of variables on both sides. We define a signature $\Omega = \{ \gamma_i \}_{i \in I}$, where $\gamma_i$ has arity $n_i$. By Yoneda lemma, we obtain natural transformations $u_0, u_1 : H_\Omega \rightarrow T \Sigma$, since we have $H_\Omega \cong \bigsqcup_{i \in I} \text{Pos}(n_i, \rightarrow)$. Let $\tilde{u}_0, \tilde{u}_1 : T \Omega \rightarrow T \Sigma$ be the corresponding monad morphisms. In the category $Mndsf(\text{Pos})$, we form a coinserter:

$$T \Omega \xrightarrow{\tilde{u}_1} T \Sigma \xrightarrow{c} T$$

**Proposition 33.** For every variety $\mathcal{V}$ of ordered algebras, the above monad $T$ is the corresponding free-algebra monad $T \mathcal{V}$.

The proof is completely analogous to the proof of Proposition 31.
**Corollary 34.** The free-algebra monad $T_{\mathcal{V}}$ of a variety of ordered algebras is strongly finitary.

Indeed, this follows from the above co inserter in $\mathbf{Mnd}_{df}(\mathbf{Pos})$ via $T_{\mathcal{V}} = T$.

**Example 35.** A finitary monad on $\mathbf{Pos}$ need not be strongly finitary. (In contrast, every finitary monad on $\mathbf{Set}$ is strongly finitary in the sense of preserving reflexive coequalizers, see Kurz and Rosický 2012.)

Denote by $\mathcal{V}$ the category of partial algebras $(A, \alpha)$ where $A$ is a poset and $\alpha$ a monotone function assigning to every pair $a_0 \leq a_1$ in $A$ an element of $A$. Morphisms to $(B, \beta)$ are monotone functions $h : A \to B$ such that $h\alpha(a_0, a_1) = \beta(h(a_0), h(a_1))$ holds for all $a_0 \leq a_1$. This is a “variety in context” as introduced in Adámek et al. (2021), from which it follows that the forgetful functor $U : \mathcal{V} \to \mathbf{Pos}$ is finitary monadic, see Theorem 3.24 in op. cit. The corresponding monad $T$ assigns to a poset $X$ the poset $TX$ defined by induction as follows:

1. elements of $X$ are terms; they are ordered as in $X$, and
2. given terms $u_0 \leq u_1$, then $\alpha(u_0, u_1)$ is a term and the ordering is pointwise: for terms $v_0 \leq v_1$ we have $\alpha(u_0, u_1) \leq \alpha(v_0, v_1)$ iff $u_i \leq v_i$ for $i = 0, 1$.

This monad is not strongly finitary because for the 2-chain $P$ given by $x_0 \leq x_1$ it does not preserve its canonical reflexive co inserter (recall Remark 4):

Indeed, every co inserter is surjective, whereas $Tc$ is not: the element $\alpha(x_0, x_1)$ of $TP$ does not lie in the image of $Tc$.

**4. From Strongly Finitary Monads to Varieties**

We now prove that the results of Section 3 can be reversed: for every strongly finitary monad $T$, a variety whose free-algebra monad is $T$.

Recall that, given a monad $T$, every morphism $f : X \to TY$ yields a homomorphism $f^* : (TX, \mu_X) \to (TY, \mu_Y)$ by $f^* = \mu_Y \cdot Tf$. Recall from Remark 22 that we associate with every $n$-ary operation symbol $\sigma$ the term $\sigma(x_i)_{i \leq n}$ over $V$.

**Definition 36.** For every monad $T$ on $\mathbf{Pos}$, the associated variety $\mathcal{V}_T$ has the signature $\Sigma$ whose $n$-ary symbols are the elements of $Tn$ ($n \in \mathbb{N}$). The variety is presented by inequations as follows (with $n$ and $m$ ranging over $\mathbb{N}$):

1. $\sigma(x_i) \leq \tau(x_i)$ for all $\sigma \leq \tau$ in $Tn$;
2. $k^*(\sigma)(x_i) = \sigma(k_0(x_i), \ldots, k_{m-1}(x_i))$ for all $m$-tuples $k : m \to Tn$, $k = (k_0, \ldots, k_{m-1})$ and all $\sigma \in Tm$;
3. $x_i = \eta_n(x_i)$ for all $i = 0, \ldots, n - 1$. 

Construction 37. Every algebra $\alpha : TA \to A$ in $\text{Pos}^T$ yields a $\Sigma$-algebra in $\forall T$: given an $n$-ary symbol $\sigma \in Tn$ and an $n$-tuple $f : n \to A$, let $f^+ = \alpha \cdot Tf : (Tn, \mu_n) \to (A, \alpha)$ be the corresponding homomorphism for $T$. We define $\sigma_A : A^n \to A$ by:

$$\sigma_A(f) = f^+(\sigma).$$

To verify that this $\Sigma$-algebra satisfies (1) in Definition 36, observe that for every $n$-tuple $f : n \to A$ the corresponding $\Sigma$-homomorphism $f^\sigma : T\Sigma V \to A$ fulfills

$$f^\sigma(\sigma(x_i)) = f^+(\sigma) \text{ for all } \sigma \in Tn.$$  \hfill (5)

This equality holds since $\sigma(x_i)$ is the result of the operation $\sigma$ in the algebra $T\Sigma n$ (Example 17) applied to $(x_i)$, thus, $f^\sigma(\sigma(x_i)) = \sigma_A(f(x_i))$. Given $\sigma \leq \tau$ in $Tn$, then $f^+(\sigma) \leq f^+(\tau)$ since $f^+ = \alpha \cdot Tf$ is monotone, thus $f^\sigma(\sigma(x_i)) \leq f^\sigma(\tau(x_i))$ holds.

To verify (2), we need to prove

$$f^\sigma(k^*\sigma)(x_i)) = f^\sigma(k_0(x_i), \ldots, k_{m-1}(x_i))$$

for every $n$-tuple $f : n \to A$. Due to (5) above, the left-hand side is

$$f^+(k^*\sigma).$$

Since $f^\sigma$ is a homomorphism, the right-hand side is

$$\sigma_A(f^\sigma(k_0(x_i)), \ldots, f^\sigma(k_{m-1}(x_i)))$$

which due to (5) is equal to

$$\sigma_A(f^+ \cdot k) = (f^+ \cdot k)^+(\sigma)$$

Thus, we only need to observe that:

$$f^+ \cdot k^* = (f^+ \cdot k)^+ : (Tn, \mu_n) \to (A, \alpha).$$  \hfill (6)

Indeed, both sides are homomorphisms in $\text{Pos}^T$, and they are equal when precomposed with the universal map:

$$f^+ \cdot k^* \cdot \eta_n = f^+ \cdot k = (f^+ \cdot k)^+ \cdot \eta_n.$$  

The verification of (3) is trivial: for $\sigma = \eta_n(x_i)$ we have (due to $\alpha \cdot \eta_A = \text{id}$) that $\sigma_A$ is the $i$-th projection: given an $n$-tuple $f : n \to A$, we get

$$\sigma_A(f) = \alpha \cdot Tf \cdot \eta_n(x_i) = \alpha \cdot \eta_A \cdot f(x_i) = f(x_i).$$

Remark 38. We can thus consider $\text{Pos}^T$ as a full subcategory of $\forall T$. Indeed, given two algebras $(A, \alpha)$ and $(B, \beta)$ in $\text{Pos}^T$, then a monotone map $h : A \to B$ is a homomorphism in $\text{Pos}^T$ iff it is a $\Sigma$-homomorphism:

1. Let $h \cdot \alpha = \beta \cdot Th$. Then

$$h \cdot f^+ = (h \cdot f)^+ : (Tn, \mu_n) \to (A, \alpha)$$

because both sides are homomorphisms of $\text{Pos}^T$ extending $h \cdot f$. For every $\sigma \in \Sigma_n$ and every $n$-tuple $f : n \to A$ we have

$$h(\sigma_A(f)) = h \cdot f^+(\sigma), \text{ by definition of } \sigma_A$$

$$= (h \cdot f)^+, \text{ as } h \cdot f^+ = (h \cdot f)^+$$

$$= \sigma_B(h \cdot f), \text{ by definition of } \sigma_B.$$  

Thus, $h$ is a $\Sigma$-homomorphism.
(2) Let \( h \) be a \( \Sigma \)-homomorphism. To prove that \( h \) is a homomorphism of \( T \)-algebras, consider the diagram below for an arbitrary \( n \in \mathbb{N} \) and \( f : n \to A \). (Recall that \( n \) is the discrete poset on \( \{0, \ldots, n-1\} \).) Since \( T \) is finitary, it is sufficient to show that the desired square commutes when precomposed by \( Tf \):

\[
\begin{array}{ccc}
Tn & \xrightarrow{Tf} & TA \\
\downarrow{Th} & & \downarrow{h} \\
TB & \xrightarrow{\beta} & B
\end{array}
\]

Indeed, given \( \sigma \in Tn \) we have

\[
\beta \cdot Th \cdot Tf(\sigma) = (h \cdot f)^+(\sigma), \text{ by definition of } (h \cdot f)^+ \\
= \sigma_B(h \cdot f), \text{ by definition of } \sigma_B \\
= h(\sigma_A(f)), \text{ since } h \text{ is a } \Sigma \text{-homomorphism} \\
= h(f^+(\sigma)), \text{ by definition of } \sigma_A \\
= h(\alpha \cdot Tf(\sigma)), \text{ by definition of } f^+.
\]

**Theorem 39.** Every strongly finitary monad on \( \text{Pos} \) is the free-algebra monad of the associated variety \( \mathcal{V}_T \).

**Proof.**

(1) For every poset \( X \), we prove that the free algebra \((TX, \mu_X)\) on \( X \) in \( \text{Pos}^T \), considered as a \( \Sigma \)-algebra, is free on \( X \) in \( \mathcal{V}_T \) w.r.t. \( \eta_X \) as the universal map. To verify this, we can restrict ourselves to finite posets \( X \). Then, this holds for all posets since \( T \) preserves filtered colimits: express \( X = \operatorname{colim} X_i \) as a filtered colimit of finite posets, then \( TX = \operatorname{colim} TX_i \), and from Remark 38 we conclude that the \( \Sigma \)-algebra \( TX \) is a filtered colimit of \( TX_i \) \((i \in I)\) in \( \text{Alg}(\Sigma) \). Thus, from \( TX \) being a free \( \Sigma \)-algebra on \( X \) in \( \mathcal{V}_T \), we conclude that \( TX \) is a free \( \Sigma \)-algebra on \( X \).

(2) We first consider the discrete poset \( n \). Given an algebra \( A \) and an \( n \)-tuple \( f : n \to A \), we prove that for \((Tn, \mu_n)\) considered as a \( \Sigma \)-algebra there exists a unique homomorphism \( \bar{f} : Tn \to A \) with \( f = \bar{f} \cdot \eta_n \). Existence: define \( \bar{f}(\sigma) = \sigma_A(f(i))_{i < n} \) for every \( \sigma \in Tn \). Then, \( f = \bar{f} \cdot \eta_n \) follows from \( A \) fulfilling the equations (3): the operation \((\eta_n(x_i))_A\) is the \( i \)-th projection, thus \( \bar{f}(\eta_n(x_i)) = f(x_i) \). Equations (1) guarantee that \( \bar{f} \) is monotone. Finally, \( \bar{f} \) is a \( \Sigma \)-homomorphism: \( \bar{f} \cdot \tau_{Tn} = \tau_A \cdot \bar{f} m \) holds for all \( m \) and all \( \tau \in Tm \). That is, given \( k : m \to Tn \) we verify

\[
\bar{f} \cdot \tau_{Tn}(k) = \tau_A \cdot \bar{f}(k).
\]

The right-hand side is \( \tau_A((k(x_i))_A(f(x_i))) \) due to the definition of \( \bar{f} \). Since \( A \) satisfies equations (2) when applied to \( k \) and \( \tau \), we see that the right-hand side is \( k^n(\tau)_A(f(x_i)) \). The left-hand side is, since \( \tau_{Tn}(k) = k^n(\tau) \), the same due to the definition of \( \bar{f} \). Uniqueness: let \( \bar{f} \) be a \( \Sigma \)-homomorphism with \( f = \bar{f} \cdot \eta_n \). Recalling that \( \eta_n^* = \text{id}_{Tn} \), for every \( \sigma \in Tn \) we get \( \sigma_{Tn}(\eta_n(x_i)) = \sigma \cdot (f(x_i)) \), as above.
(3) We now consider an arbitrary finite poset \( P \) on the set \( \{x_0, \ldots, x_{n-1}\} \). Then its canonical coinserter (Remark 4) yields, since \( T \) is strongly finitary, the following coinserter:

\[
\begin{array}{c}
T_k \\
\downarrow T_{p_0} \\
T_n \\
\downarrow id \\
TP
\end{array}
\]

The free algebras \( T_k \) and \( T_n \) of \( \text{Pos}^T \) are by (2) also free \( \Sigma \)-algebras in \( V^T \). Given an algebra \( A \) of \( V^T \) and a monotone function \( f : P \to A \), we have a unique \( \Sigma \)-homomorphism \( f' : T_n \to A \) with \( f = f' \cdot \eta_n \). To prove that \( f' \) is also a \( \Sigma \)-homomorphism \( f' : TP \to A \), it is sufficient to verify

\[
f' \cdot T_{p_0} \leq f' \cdot T_{p_1} : Tk \to A.
\]

Thus, we need to prove that for each \( x \in Tk \) we have \( f'(T_{p_0}(x)) \leq f'(T_{p_1}(x)) \). Indeed, this holds for all \( x = \eta_k(y_i) \) with variables \( y_i \in k \):

\[
f' \cdot T_{p_0}(\eta_k(y_i)) = f(p_0(y_i)), \quad \text{by the diagram above}
\leq f(p_1(y_i)), \quad \text{by } f \text{ being monotone}
= f' \cdot T_{p_1}(\eta_k(y_i)), \quad \text{by the diagram above}.
\]

And thus, we only need to observe that the set of all \( x \in Tk \) with the desired property is closed under the \( \Sigma \)-operations. For every \( \sigma \in \Sigma_n \) and every \( n \)-tuple \( (x_i)_{i<n} \) with \( f' \cdot T_{p_0}(\sigma_{Tk}(x_i)) \leq f' \cdot T_{p_1}(\sigma_{Tk}(x_i)) \), we have (since \( T_{p_i} \) are homomorphisms of \( \text{Pos}^T \)):

\[
f' \cdot T_{p_0}(\sigma_{Tk}(x_i)) = f'(\sigma_{Tk}(T_{p_0}(x_i))), \quad \text{by Remark 38}
= \sigma_A(f'(T_{p_0}(x_i))), \quad \text{since } f' \text{ is a } \Sigma \text{-homomorphism}
\leq \sigma_A(f'(T_{p_1}(x_i))), \quad \text{since } \sigma_A \text{ is monotone}
= f' \cdot T_{p_1}(\sigma_{Tk}(x_i)) \text{ as above.}
\]

(4) The full embedding \( E : \text{Pos}^T \to \mathcal{V}_T \) of Remark 38 is concrete. That is, if \( U : \text{Pos}^T \to \text{Pos} \) and \( V : \mathcal{V}_T \to \text{Pos} \) denote the forgetful functors, the triangle:

\[
\begin{array}{c}
\text{Pos}^T \\
\downarrow U \\
\downarrow V \\
\text{Pos}
\end{array}
\]

commutes. Both \( U \) and \( V \) are monadic functors by Lemma 28. It follows from (1) that the corresponding monads are isomorphic.

\( \square \)
**Notation 40.** Let $\text{Var}(\text{Pos})$ denote the ordinary category of varieties of ordered algebras and concrete functors. These are functors $F : \mathcal{V}_1 \to \mathcal{V}_2$ which commute (strictly) with the forgetful functors $U_i : \mathcal{V}_i \to \text{Pos}$:

\[
\begin{array}{ccc}
\mathcal{V}_1 & \xrightarrow{F} & \mathcal{V}_2 \\
\downarrow U_1 & & \downarrow U_2 \\
\text{Pos} & & \text{Pos}
\end{array}
\]

**Theorem 41.** The ordinary category of varieties is dually equivalent to the ordinary category of strongly finitary monads:

$$\text{Var}(\text{Pos}) \simeq \text{Mnd}_{sf}(\text{Pos})^{op}.$$ 

**Proof.** (1) Let $F : \mathcal{V}_1 \to \mathcal{V}_2$ be a concrete functor. The comparison functors $K_i : \mathcal{V}_i \to \text{Pos}^{\mathcal{V}_i}$ are isomorphisms of categories by Lemma 28. These isomorphisms are concrete: if $U_i' : \text{Pos}^{\mathcal{V}_i} \to \text{Pos}$ denotes the underlying functor, then $U_i = U_i' : K_i$. From $F$, we thus obtain a concrete functor:

$$F = K_2 \cdot F \cdot K_1^{-1} : \text{Pos}^{\mathcal{V}_1} \to \text{Pos}^{\mathcal{V}_2}.$$

The passage $F \mapsto F$ is bijective (with the inverse passage $K_2^{-1} \cdot (\cdot) \cdot K_1$) and preserves composition and identity morphisms.

(2) Given monads $T_1, T_2$, monad morphisms $\rho : T_2 \to T_1$ bijectively correspond to concrete functors from $\text{Pos}^{T_1}$ to $\text{Pos}^{T_2}$: the bijection takes $\rho$ to $H_\rho : \text{Pos}^{T_1} \to \text{Pos}^{T_2}$ assigning to an algebra $\alpha : T_1 A \to A$ in $\text{Pos}^{T_1}$ the algebra:

$$T_2 A \xrightarrow{\rho_A} T_1 A \xrightarrow{\alpha} A$$

in $\text{Pos}^{T_2}$. This passage $\rho \mapsto H_\rho$ moreover preserves composition and identity morphisms. See Barr and Wells (1985), Theorem 3.6.3.

(3) Define a functor:

$$R : \text{Var}(\text{Pos}) \to \text{Mnd}_{sf}(\text{Pos})^{op}$$

on objects by

$$R(\mathcal{V}) = T_\mathcal{V}$$
and on morphisms $F : \mathcal{V}_1 \to \mathcal{V}_2$ by the following rule:

$$R(F) = \rho \text{ iff } H_\rho = F.$$ 

It follows from (1) and (2) that $R$ is a well-defined full and faithful functor. Theorem 39 tells us that every strongly finitary monad is isomorphic to $R(\mathcal{V})$ for some variety $\mathcal{V}$. Therefore, $R$ is an equivalence of categories.

\[ \square \]

5. Lifting and Extending Finitary Monads from $\text{Set}$ to $\text{Pos}$

The examples of varieties of ordered algebras presented so far are all liftings of varieties of classical algebras (over $\text{Set}$). In the present section, we prove that this is no coincidence: there are no other examples. Since varieties of ordered algebras are in a bijective correspondence with strongly finitary monads on $\text{Pos}$ (and varieties of classical algebras are in a bijective correspondence with finitary monads on $\text{Set}$), an equivalent statement is the theorem below. (To lighten up the notation, we write $\text{Set}$ instead of $\text{Set}_o$ when talking about ordinary categories below.)

**Definition 42.** Denote by $U : \text{Pos}_o \to \text{Set}$ the ordinary forgetful functor and by $D \dashv U$ its left ordinary adjoint $DX = (X, =)$.

(1) A lifting of an ordinary monad $\tilde{T} = (\tilde{T}, \tilde{\mu}, \tilde{\eta})$ on $\text{Set}$ to $\text{Pos}$ is a monad $T = (T, \mu, \eta)$ on $\text{Pos}$ such that the square:

$$
\begin{array}{ccc}
\text{Pos}_o & \xrightarrow{T_o} & \text{Pos}_o \\
\downarrow U & & \downarrow U \\
\text{Set} & \xrightarrow{T} & \text{Set}
\end{array}
$$

commutes, and the equalities $U\eta = \tilde{\eta}U$ and $U\mu = \tilde{\mu}U$ hold.

(2) An extension of $\tilde{T}$ is a monad $T$ on $\text{Pos}$ such that the square:

$$
\begin{array}{ccc}
\text{Pos}_o & \xrightarrow{T_o} & \text{Pos}_o \\
\downarrow D & & \downarrow D \\
\text{Set} & \xrightarrow{\tilde{T}} & \text{Set}
\end{array}
$$

commutes, and the equalities $\eta D = D\tilde{\eta}$ and $\mu D = D\tilde{\mu}$ hold.

**Remark 43.**

(1) There is a less strict concept of a lifting of an ordinary monad $\tilde{T}$ on $\text{Set}$: A monad $T$ on $\text{Pos}$ is a *non-strict lifting* of $\tilde{T}$ iff there is a natural isomorphism $\varphi$:

$$
\begin{array}{ccc}
\text{Pos}_o & \xrightarrow{T_o} & \text{Pos}_o \\
\downarrow U & & \downarrow U \\
\text{Set} & \xrightarrow{\tilde{T}} & \text{Set}
\end{array}
$$
such that the following diagrams commute

\[
\begin{align*}
\begin{tikzcd}
U & 
\tilde{T}U \\
UT_o & 
\tilde{T}UT_o
\end{tikzcd}
\end{align*}
\]

(2) Given \(\varphi\) as above, \(T\) is isomorphic to a monad \(T_o\) on Pos which is a strict lifting of \(\tilde{T}\) (i.e., for which the conditions in the above definition hold). Indeed, define \(T_o = (T_o, \mu_o, \eta_o)\) by letting \(T_oX\) be the unique poset on the set \(\tilde{T}|X|\) for which \(\varphi_X\) carries an isomorphism \(TX \cong T_oX\) in Pos. Analogously define \(T_o\) on morphisms \(f: X \to Y\): the underlying map of \(T_o f\) is such that the square:

\[
\begin{align*}
\begin{tikzcd}
TX & 
TY \\
T_oX & 
T_oY
\end{tikzcd}
\end{align*}
\]

commutes. The unit of \(T_o\) has components \(\varphi_X \cdot \eta_X: X \to T_oX\) and the multiplication \((\mu_o)_X: T_oT_oX \to T_oX\) is the unique monotone map for which the square:

\[
\begin{align*}
\begin{tikzcd}
TTX & 
TX \\
T_oT_oX & 
T_oX
\end{tikzcd}
\end{align*}
\]

commutes. It is easy to see that \(T_o = (T_o, \eta_o, \mu_o)\) is a well-defined monad on Pos isomorphic to \(T\) via \(\varphi\), and that it is a strict lifting of \(\tilde{T}\).

**Theorem 44.** Every strongly finitary monad on Pos is a lifting of an ordinary finitary monad on Set.

**Proof.** In view of Theorem 41 it is sufficient to present, for every variety \(\mathcal{V}\) of ordered \(\Sigma\)-algebras, a variety \(\tilde{\mathcal{V}}\) of non-ordered algebras such that \(T_{\mathcal{V}}\) is a lifting of \(T_{\tilde{\mathcal{V}}}\). Here, \(T_{\mathcal{V}}\) is the ordinary \(\mathcal{V}\)-free-algebra monad on Pos, and \(T_{\tilde{\mathcal{V}}}\) the \(\tilde{\mathcal{V}}\)-free-algebra (ordinary) monad on Set. Recall that we consider an arbitrary set as the poset with the trivial order.

\(1\) For our standard set \(V = \{x_0, x_1, x_2, \ldots\\}\) of variables in Definition 21, we have defined a set \(\mathcal{E}_V\) of equations in Construction 25: they are those equations \(s = t\) which every
ordered algebra in \( \mathcal{V} \) satisfies. (Since this is equivalent to satisfying both \( s \leq t \) and \( t \leq s \).) We denote by \( \mathcal{V} \) the variety of non-ordered algebras presented by \( \mathcal{E}^0_X \). This clearly implies that every algebra in \( \mathcal{V} \) satisfies, for every set \( X \), all equations \( s = t \) for pairs in \( \mathcal{E}^0_X \). Moreover, \( \mathcal{E}^0_X \) is clearly a congruence on the non-ordered \( \Sigma \)-algebra \( T_\Sigma X \) of all \( \Sigma \)-terms on \( X \).

(2) Denote by \( T_\mathcal{V} X \) the free algebra of \( \mathcal{V} \) on the set \( X \). It can be constructed as the quotient of the non-ordered algebra \( T_\Sigma X \) modulo the congruence \( \mathcal{E}^0_X \):

\[
T_\mathcal{V} X = T_\Sigma X / \mathcal{E}^0_X.
\]

The proof is completely analogous to that of Theorem 26. We thus conclude that for an arbitrary poset \( X \) our choice of \( T_\mathcal{V} X \) and \( T_\mathcal{V} |X| \) can be such that the underlying set of \( T_\mathcal{V} X \) is \( T_\mathcal{V} |X| \) and all operations are equal. The universal arrows \( (\eta_\mathcal{V})_X : X \to |T_\mathcal{V} X| \) and \( (\eta_\mathcal{V})_X : |X| \to T_\mathcal{V} |X| \) are both given by forming the equivalence classes of \( x \in X \) modulo \( \mathcal{E}^0_X \), thus \( \eta_\mathcal{V} \) is the underlying map of \( \eta_\mathcal{V} \). The multiplication \( (\mu_\mathcal{V})_X : T_\mathcal{V} T_\mathcal{V} X \to T_\mathcal{V} X \) is an interpretation of every term \( t \in T_\mathcal{V} X \) over the poset \( T_\mathcal{V} X \) of \( \Sigma \)-terms as a term \( \mu_\mathcal{V}(t) \) over \( X \) modulo \( \mathcal{E}^0_X \). This interpretation is independent of the ordering of \( X \), shortly, the underlying function of \( (\mu_\mathcal{V})_X \) is the corresponding interpretation \( (\tilde{\mu}_\mathcal{V})_|X| \) of terms modulo \( \mathcal{E}^0_X \) w.r.t. \( \tilde{T} \).

\[\Box\]

**Definition 45.** A variety \( \mathcal{V} \) of ordered algebras is called a lifting of a variety \( \tilde{\mathcal{V}} \) of classical (non-ordered) algebras if a concrete functor from \( \mathcal{V} \) to \( \tilde{\mathcal{V}} \) over Set exists which takes the free algebra on any poset \( X \) to the free algebra on \( |X| \).

**Corollary 46.** Every variety of ordered algebras is a lifting of some classical variety.

Indeed, given a variety \( \mathcal{V} \), let \( \tilde{T} \) be an ordinary monad of Set such that \( T_\mathcal{V} \) is a lifting of it. The comparison functor is an isomorphism \( K : \mathcal{V} \to \text{Pos}^{T_\mathcal{V}} \) over Pos (Lemma 28). And we have a classical variety \( \tilde{\mathcal{V}} \) with an analogous concrete isomorphism \( \tilde{K} : \tilde{\mathcal{V}} \to \text{Set}^{\tilde{T}} \) over Set. Define a concrete functor \( H : \text{Pos}^{T} \to \text{Set}^{\tilde{T}} \) over Set by the obvious rule: it sends an algebra \( \alpha : TA \to A \) to \( \alpha : \tilde{T}|A| \to |A| \). The desired functor is \( \tilde{K}^{-1} \circ H \circ K : \mathcal{V} \to \tilde{\mathcal{V}} \).

**Example 47.** The monad of ordered monoids (Example 23 (3)) is not only a lifting of \( \tilde{T}X = X^e \), it is also its extension: for every set \( X \) the free ordered monoid \( T(DX) \) is discrete, thus \( TD = DT \); and clearly \( \eta D = \tilde{D} \eta \) and \( \mu D = \tilde{D} \mu \) also hold.

In contrast, ordered monoids with \( e \) the least element (Example 23 (4)) yield a monad \( T \) which is not an extension of \( \tilde{T} \): for every set \( X \neq \emptyset \) the monoid \( T(DX) \) is not discrete. We now demonstrate that this is not a coincidence.

**Definition 48.** By an equational variety of ordered algebras is meant a variety specified by equations between terms.

**Theorem 49.** The following categories are dually equivalent:

1. equational varieties (and concrete functors) and
2. strongly finitary monads on Pos which are extensions of monads on Set (and monad morphisms).

**Proof.** Using the equivalence of Theorem 41, it is sufficient to prove that a variety \( \mathcal{V} \) of ordered algebras is equational iff the monad \( T_\mathcal{V} \) is an extension of a finitary monad on Set.
(1) Let $T_V$ be a strongly finitary extension of a finitary monad $\tilde{T}$. If an inequation $u \leq v$ holds for all algebras of $V$, we prove that $v \leq u$ also holds. Then $V$ is presented by the corresponding equations $u = v$. Let $X$ be the set of variables that occur in $u$ or $v$. Then $T_V(DX)$ is discretely ordered, since $T_V D = DT$, and it satisfies $u \leq v$. The trivial interpretation $\eta_{DX}: DX \rightarrow T_V(DX)$, with $\eta_{DX} = id$, thus implies $u \leq v$ in $T_V(DX)$, hence $u = v$. Since $T_V(DX)$ is the free algebra on $DX$, this implies that every algebra of $V$ satisfies $u = v$.

(2) Let $\mathcal{V}$ be a variety of ordered algebras specified by equations. For every set $X$, the free algebra $T_V(DX)$ is discretely ordered. Indeed, this is an ordered algebra in $\mathcal{V}$ and we have the discretely ordered $\Sigma$-algebra $A$ underlying $T_V(DX)$. Now $A$ satisfies every equation $u = v$ satisfied by $T_V(DX)$: given an interpretation $f: V \rightarrow |T_V(DX)|$ of variables, the corresponding homomorphism $f^\mathcal{V}: T_V V \rightarrow T_V(DX)$ is the same for $A$. Thus $A$ lies in $\mathcal{V}$. This implies $A = T_V(DX)$. In other words, free algebras of $\mathcal{V}$ on discrete posets are discretely ordered, and $\eta_{DX} = D\tilde{\eta}_X$. From this, $\mu_{DX} = D\tilde{\mu}_X$ easily follows.

Example 50. A monad on $\text{Pos}$ which is both an extension and a lifting of a finitary monad on $\text{Set}$ need not be strongly finitary. Let $\Sigma = \{+, \ast\}$ be a signature of two binary operations. The full subcategory on all ordered algebras for which the implication:

$$x \leq y \implies x + y \leq x \ast y$$

holds yields the following monad $T$ on $\text{Pos}$. Given a poset $X$, the poset $TX$ consists of all terms with variables in $X$ using $+$ and $\ast$, where the order on $TX$ is the smallest one such that

1. $x \leq y$ in $X$ implies $x \leq y$ in $TX$,
2. $+$ and $\ast$ are monotone, and
3. $t + s \leq t \ast s$ for all terms $t \leq s$ in $TX$.

Thus, $T$ is a lifting of the monad $\tilde{T}$ on $\text{Set}$ of free $\Sigma$-algebras. If $X$ is discretely ordered, then so is $TX$, and thus we see that $T$ is also an extension of $\tilde{T}$.

The monad $T$ is not strongly finitary. For example, it does not preserve the canonical co inserter (recall Remark 4) of the chain $2$ given by $0 < 1$:

$$\begin{array}{ccc}
3 & \xrightarrow{p_1} & 2 \\
\circlearrowleft & & \circlearrowright \\
p_0 & \xrightarrow{id} & 2
\end{array}$$

Indeed, in $T2$ we have $0 + 1 < 0 \ast 1$. In contrast, this does not hold in the co inserter of $Tp_0$ and $Tp_1$. We can describe the order of that co inserter as the smallest one that, besides conditions (1)-(3) above, also fulfills $t \leq s$ for terms such that $s$ is obtained by changing some $0$ in $t$ to $1$. The down-set of the term $0 \ast 1$ in that co inserter consists of the following terms:

$$0 + 0 < 0 \ast 0 < 0 \ast 1.$$ 

Thus, $T$ does not preserve the co inserter of $p_0$ and $p_1$.

6. Conclusions

Kelly and Power proved that every finitary monad $T$ on $\text{Pos}$ has a presentation as a coequalizer of a parallel pair of monad morphisms between free monads on generalised signatures, see Kelly
and Power (1993). In the present paper, we derive an analogous result for strongly finitary monads: each such monad has a presentation as a co inserter of a parallel pair of monad morphisms between free monads $T_{\Sigma}$ on (classical) signatures $\Sigma$, see Construction 32. The move from coequalizers to coininserters is needed since the signatures used in Kelly and Power (1993) were substantially more general than the classical ones we use here: they were collections $\Sigma = (\Sigma_\Gamma)_{\Gamma \in \operatorname{Pos}}$ of posets $\Sigma_\Gamma$ indexed by finite posets. However, the proof method we use is closely related to that in Kelly and Power (1993).

We have proved that for (classical) varieties of ordered $\Sigma$-algebras, the corresponding free-algebra monad on $\operatorname{Pos}$ is strongly finitary, that is, enriched, finitary and preserving reflexive coininserters. Using this, we proved that the category of varieties of ordered algebras is dually equivalent to the category of strongly finitary monads on $\operatorname{Pos}$.

In the future, we plan extending our results to strongly finitary monads on more general $\mathcal{V}$-categories for closed monoidal categories $\mathcal{V}$, for example, the category of small categories. For general $\mathcal{V}$, it is interesting to know how to characterize strongly finitary functors. But the main question is whether strongly finitary monads correspond again to “naturally” defined varieties of algebras in $\mathcal{V}$.

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