NEW PROOFS FOR SOME FUNDAMENTAL RESULTS OF TOPOLOGY

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Abstract. In this paper, it is shown that a topological space \( X \) is quasi-compact iff every maximal ideal of the power set ring \( \mathcal{P}(X) \) is Zariski convergent to a point of \( X \). Then as an application, simple and ring-theoretic proofs are provided for the Tychonoff theorem and Alexander subbase theorem. Finally, we give a new proof to the fact that a topological space is a profinite space iff it is compact and totally disconnected.

1. Introduction

Tychonoff theorem is one of the deep results of topology. Many of the mathematicians consider it as the single most important result in general topology. This result has been investigated in the literature over the years from various point of view, see e.g. [1]-[6].

In this paper, an interesting ring-theoretic characterization for the quasi-compactness of a topological space is given, see Theorem 3.1. Using this, then Tychonoff theorem and Alexander subbase theorem are easily deduced. Finally, we give a ring-theoretic proof to the fact that a topological space is a profinite space iff it is compact and totally disconnected, see Theorem 3.5. Most of the proofs are greatly based on the Zariski convergent notion and on the significant using of the power set ring.

2. Preliminaries

If \( X \) is a set then its power set \( \mathcal{P}(X) \) together with the symmetric difference \( A + B = (A \cup B) \setminus (A \cap B) \) as the addition and the intersection \( A \cdot B = A \cap B \) as the multiplication form a commutative ring whose zero and unit are respectively the empty set and the whole set \( X \). The ring

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\( \mathcal{P}(X) \) is called the \textit{power set ring} of \( X \). If \( f : X \to Y \) is a function then the map \( \mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X) \) defined by \( A \mapsto f^{-1}(A) \) is a morphism of rings. In fact, the assignments \( X \mapsto \mathcal{P}(X) \) and \( f \mapsto \mathcal{P}(f) \) form a faithful contravariant functor from the category of sets to the category of commutative rings. We call it the \textit{power set functor}.

A ring is called a Boolean ring if every element is idempotent. It is easy to see that every Boolean ring is a commutative ring, and in a Boolean ring every prime ideal is a maximal ideal. The power set ring \( \mathcal{P}(X) \) is a typical example of Boolean rings.

If \( A \in \mathcal{P}(X) \) then \( \mathcal{P}(A) \) is an ideal of \( \mathcal{P}(X) \). In fact, \( \mathcal{P}(A) = (A) \) is a principal ideal. For every finite number \( A_1, ..., A_n \) of members of \( \mathcal{P}(X) \) then \( (A_1, ..., A_n) = \mathcal{P}(\bigcup_{i=1}^{n} A_i) \). For each \( x \in X \) then clearly \( m_x := \mathcal{P}(X \setminus \{x\}) \) is a maximal ideal of \( \mathcal{P}(X) \). It is also easy to see that \( X \) is a finite set iff every maximal ideal of \( \mathcal{P}(X) \) is of the form \( m_x \).

If \( f \) is a member of a ring \( R \) then \( D(f) = \{ p \in \text{Spec}(R) : f \notin p \} \).

\textbf{Definition 2.1.} Let \( X \) be a topological space, \( x \in X \) and \( M \) a maximal ideal of \( \mathcal{P}(X) \). Then we say that \( M \) is \textit{convergent} (or, \textit{Zariski convergent}) to the point \( x \) if \( U \) is an open of \( X \) containing \( x \) then \( M \in D(U) \).

Let \( \varphi : X \to Y \) be a continuous map of topological spaces. If a maximal ideal \( M \) of \( \mathcal{P}(X) \) is convergent to some point \( x \in X \) then clearly the maximal ideal \( \mathcal{P}(\varphi)^{-1}(M) \) of \( \mathcal{P}(Y) \) is convergent to \( \varphi(x) \).

A quasi-compact and Hausdorff topological space is called a compact space.

\textbf{Definition 2.2.} Let \( (X_i, \varphi_{i,j}) \) be a projective (inverse) system of finite discrete spaces over a poset \( (I, \prec) \). The projective (inverse) limit \( X = \lim_{\underset{i \in I}{\longrightarrow}} X_i \) with the induced product topology, as a subset of \( \prod_{i \in I} X_i \), is called a \textit{profinite space}.
3. Main results

**Theorem 3.1.** A topological space $X$ is quasi-compact if and only if every maximal ideal $M$ of $\mathcal{P}(X)$ is convergent to a point of $X$.

**Proof.** Let $X$ be quasi-compact. The collection of closures $\overline{A}$ with $A \in S := \mathcal{P}(X) \setminus M$ has the finite intersection property. Therefore $\bigcap_{A \in S} \overline{A} \neq \emptyset$ since $X$ is quasi-compact. Thus we may choose some point $x$ in the intersection. If $U$ is an open of $X$ containing $x$, then $B := X \setminus U \in M$ since otherwise $x \notin \overline{B} = B$ which is a contradiction. This yields that $U \notin M$ since otherwise $1 = B + U \in M$ which is a contradiction. Thus $M$ is convergent to the point $x$. Conversely, let $(E_i)_{i \in I}$ be a family of closed subsets $X$ with the finite intersection property. It suffices to show that $\bigcap_{i \in I} E_i \neq \emptyset$. The ideal $\mathfrak{J}$ generated by the elements $E_i^c = X \setminus E_i$ is not the whole ring $\mathcal{P}(X)$, since otherwise we may find a finite number $E_1, ..., E_n$ of the family such that $\mathcal{P}(X) = (E_1^c, ..., E_n^c) = \mathcal{P}(\bigcup_{i=1}^n E_i^c)$, thus $X = \bigcup_{i=1}^n E_i^c$ and so $\bigcap_{i=1}^n E_i = \emptyset$ which is a contradiction. Thus there exists a maximal ideal $M$ of $\mathcal{P}(X)$ such that $\mathfrak{J} \subseteq M$. By the hypothesis, $M$ is convergent to a point $x \in X$. It follows that $x \in \bigcap_{i \in I} E_i$. $\square$

**Corollary 3.2.** (Tychonoff Theorem) Let $(X_i)_{i \in I}$ be a family of quasi-compact topological spaces. Then $X = \prod_{i \in I} X_i$ with the product topology is quasi-compact.

**Proof.** Let $M$ be a maximal ideal of $\mathcal{P}(X)$. Setting $M_i := \mathcal{P}(\pi_i)^{-1}(M)$ where $\pi_i : X \to X_i$ is the canonical projection map. For each $i \in I$ then by Theorem 3.1 the maximal ideal $M_i$ is convergent to a point $x_i \in X_i$. To prove the assertion, by Theorem 3.1 it suffices to show that $M$ is convergent to the point $x = (x_i)$. Let $U$ be an open of $X$ containing $x$. Then there exists a basis open $V = \prod_{i \in I} V_i$ of $X$ such that $x \in V \subseteq U$ where each $V_i$ is an open of $X_i$ and $V_i = X_i$ for all but a finite number of indices $i$. Let $J$ be the set of all $i \in I$ such that $V_i \neq X_i$. Then $V = \bigcap_{i \in J} \pi_i^{-1}(V_i)$. Clearly $V_i \notin M_i$ and so $\pi_i^{-1}(V_i) \notin M$ for all $i$. Thus $V \notin M$ since $J$ is a finite set. Therefore $U \notin M$. $\square$
Corollary 3.3. \textit{(Alexander Subbase Theorem)} Let $X$ be a topological space and let $\mathcal{D}$ be a subbasis of $X$ such that every covering of $X$ by elements of $\mathcal{D}$ has a finite refinement. Then $X$ is quasi-compact.

\textbf{Proof.} Let $M$ be a maximal ideal of $\mathcal{P}(X)$. By Theorem 3.1, it will be enough to show that $M$ is convergent to a point of $X$. By the hypothesis, $X \neq \bigcup_{D \in M \cap \mathcal{D}} D$ since otherwise we may find a finite number $D_1, \ldots, D_n \in M \cap \mathcal{D}$ such that $X = \bigcup_{i=1}^{n} D_i$ and so $\mathcal{P}(X) = (D_1, \ldots, D_n) \subseteq M$ which is a contradiction. Hence, we may choose some $x \in X$ such that $x \notin \bigcup_{D \in M \cap \mathcal{D}} D$. If $U$ is an open of $X$ containing $x$ then there exists a finite number $D'_1, \ldots, D'_s \in \mathcal{D}$ such that $x \in \bigcap_{k=1}^{s} D'_k \subseteq U$. But $\bigcap_{k=1}^{s} D'_k \notin M$. Therefore $U \notin M$. □

Let $R$ be a Boolean ring. Then $\{0, 1\}$ is a subring of $R$. If $A$ is a subring of $R$ and $f \in R$ then $A[f] = \{a + bf : a, b \in A\}$. In particular, if $A$ is a finite subring of $R$ and $f_1, \ldots, f_n \in R$ then $A[f_1, \ldots, f_n]$ is also a finite subring of $R$. Let $\{R_i : i \in I\}$ be the set of all finite subrings of $R$. We define $j < i$ if $R_j$ is a proper subset of $R_i$. Then the poset $(I, <)$ is directed, because if $A$ and $B$ are finite subrings of $R$ then we observed in the above that $A[B]$ is a finite subring of $R$ and $A, B \subseteq A[B]$.

\textbf{Lemma 3.4.} Let $R$ be a Boolean ring. Then $\text{Spec}(R)$ is a profinite space.

\textbf{Proof.} Let $\{R_i : i \in I\}$ be the set of all finite subrings of $R$. Then the $\text{Spec}(R_i)$ together with the $\varphi_{i,j} : \text{Spec}(R_i) \to \text{Spec}(R_j)$ induced by the inclusions $R_j \subseteq R_i$, as the transition morphisms, form a projective system of finite discrete spaces over the poset $(I, <)$. We show that $\text{Spec}(R)$ together with the canonical maps $p_i : \text{Spec}(R) \to \text{Spec}(R_i)$, induced by the inclusions $R_i \subseteq R$, is the projective limit of the above system. By the universal property of the projective limits, there exists a (unique) continuous map $\varphi : \text{Spec}(R) \to X = \lim_{i \in I} \text{Spec}(R_i)$ such that $p_i = \pi_i \circ \varphi$ for all $i$, where each $\pi_i : X \to \text{Spec}(R_i)$ is the canonical projection. Therefore $\varphi(M) = (M \cap R_i)$. The map $\varphi$ is clearly a closed map because $\text{Spec}(R)$ is quasi-compact and $X$ is Hausdorff. It remains to show that it is bijective. Suppose $\varphi(M) = \varphi(N)$. If $f \in R$ then $A = \{0, 1, f, 1 + f\}$ is a subring of $R$ and so $M \cap A = N \cap A$. Thus
Finally, take \((M_i) \in X\) where each \(M_i\) is a maximal ideal of \(R_i\). Let \(M\) be the ideal of \(R\) generated by the subset \(\bigcup_{i \in I} M_i \subseteq R\). We show that \(M\) is a maximal ideal of \(R\). Clearly \(M\) is a proper ideal of \(R\). If not, then there exists a finite subset \(J \subseteq I\) such that the ideal generated by the subset \(\bigcup_{i \in J} M_i\) is the whole ring \(R\). But we may find some \(k \in I\) such that \(i \leq k\) for all \(i \in J\), since \((I, \prec)\) is directed. We have \(\bigcup_{i \in J} M_i \subseteq M_k\) because \(M_i = R_i \cap M_k\) for all \(i \in J\). This yields that \(M_k = R_k\), which is a contradiction. If \(f, g \in R\) such that \(fg \in M\) then similarly above there exists some \(k \in I\) such that \(f, g \in R_k\) and \(fg \in M_k\). Thus either \(f \in M_k\) or \(g \in M_k\). Therefore \(M\) is a prime ideal of \(R\). Clearly \(M_i \subseteq M \cap R_i\) and so \(M_i = M \cap R_i\) for all \(i \in I\). □

**Theorem 3.5.** A topological space is a profinite space iff it is compact and totally disconnected.

**Proof.** Let \(X\) be a profinite space. It is clearly Hausdorff and totally disconnected. To see the quasi-compactness we use Theorem 3.4. So let \(M\) be a maximal ideal of \(P(X)\). By taking into account the notations of Definition 2.2, setting \(M_i := P(\pi_i)^{-1}(M)\) where \(\pi_i : X \to X_i\) is the canonical projection. Then clearly \(M_i = P(X_i \setminus \{x_i\})\) for some \(x_i \in X_i\). We prove that \(M\) is convergent to \(x = (x_i)\). First we have to show that \(x \in X\). If \(j \leq i\) then \(\pi_j = \varphi_{i,j} \circ \pi_i\). It follows that \(M_j = P(\varphi_{i,j})^{-1}(M_i)\). This yields that \(M_j\) is convergent to \(\varphi_{i,j}(x_i)\) because \(M_i\) is obviously convergent to \(x_i\). Thus the open \(\{\varphi_{i,j}(x)\}\) is not a member of \(M_j\) and so \(\varphi_{i,j}(x_i) = x_j\). Hence, \(x \in X\). Now let \(U\) be an open of \(X\) containing \(x\). There exists a basis open \(V = \prod_{i \in I} V_i\) in the product topology such that \(x \in X \cap V \subseteq U\). Thus there exists a finite subset \(J \subseteq I\) such that \(X \cap V = \bigcap_{i \in J} \pi_i^{-1}(V_i)\) and \(\pi_i^{-1}(V_i) \notin M\) for all \(i\). Therefore \(X \cap V\) and so \(U\) are not in \(M\). Hence, \(M\) is convergent to \(x\). Conversely, let \(X\) be a compact totally disconnected space. Let \(R = \text{Clop}(X)\) be the set of all clopen (both open and closed) subsets of \(X\). Clearly \(R\) is a subring of \(P(X)\). It is well known that for each point \(x\) in a compact space \(X\) then the connected component \([x]\) of \(X\) is the intersection of all \(A \subseteq R\) such that \(x \in A\). Using this fact, then it can be shown that the map \(X \to \text{Spec}(R)\) given by \(x \mapsto m_x \cap R\) is a homeomorphism where \(m_x = P(X \setminus \{x\})\). Therefore by Lemma 3.4 \(X\) is a profinite space. □
Corollary 3.6. Let $R$ be a commutative ring. Then $\text{Spec}(R)$ equipped with the patch topology is a profinite space.

Proof. It is well known that the patch topology is compact and totally disconnected. Then apply Theorem 3.5. □

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