A Method for Reducing the Complexity of Model Predictive Control in Robotics Applications

Michael Muehlebach and Raffaello D’Andrea

Abstract—This article describes an approach for parametrizing input and state trajectories in model predictive control. The parametrization is designed to be invariant to time shifts, which enables warm-starting the successive optimization problems and reduces the computational complexity of the online optimization. It is shown that in certain cases (e.g., for linear time-invariant dynamics with input and state constraints) the parametrization leads to inherent stability and recursive feasibility guarantees without additional terminal set constraints. Due to the fact that the number of decision variables are greatly reduced through the parametrization, while the warm-starting capabilities are preserved, the approach is suitable for applications where the available computational resources (memory and CPU-power) are limited.

I. INTRODUCTION

Model predictive control (MPC) has become a well-known and widely used control strategy for solving challenging control problems. Unlike many other approaches, MPC addresses input and state constraints in a systematic way. It is based on repeatedly solving an optimal control problem, including the actual state as an initial condition and a prediction of the system’s evolution. This leads naturally to an implicit feedback law, providing robustness against modeling errors and disturbances, [1].

Model predictive control has been successfully applied to many robotic systems: For example, the authors of [2] present an MPC-based steering controller for an autonomous vehicle. The controller is shown to perform complex steering maneuvers in an emergency scenario that includes a double lane change on snow. A real-time path planner for an all-terrain vehicle that is relying on MPC is presented in [3], the MPC-based control architecture of an autonomous wheelchair is discussed in [4], and MPC is used to control an unmanned rotorcraft in [5].

In many applications, however, computational resources are limited, and therefore it is often desirable to simplify the resulting optimization problems that are solved at every time step. This can be done, for example, by exploiting the specific system’s structure, [6], or by a careful design of the control structure, [7]. In this work we propose a different strategy, that is based on systematically reducing the degrees of freedom by parametrizing input and state trajectories with basis functions. We will show that the proposed parametrization evolves naturally from the requirement that the resulting trajectories should be invariant to time shifts, meaning that a previously calculated input and/or state trajectory can be used to warm start the optimization at the next time step.

In addition, it will be argued that by choosing exponentially decaying basis functions the truncation of the prediction horizon can be avoided, which, combined with the time-shift property, leads to straightforward recursive feasibility and closed-loop stability guarantees.

Related works: Several strategies have been suggested to reduce the complexity of MPC. For instance, move-blocking describes the general idea of fixing the input or its derivative to be constant over several time steps, [8]. Many different flavors and variants exist, ranging from simply clamping certain inputs together to more elaborate, time-dependent strategies, such as moving window blocking, that are designed to preserve closed-loop stability and recursive feasibility guarantees, [9]. A different point of view is adopted in [10], [11], and [12], for example, where the input is parametrized with Laguerre and/or Kautz basis functions.

In [13], multiresolution analysis is used for parametrizing the input trajectory. However, the approach is mainly applicable to open-loop stable systems, where the impulse response is assumed to be negligible after a certain time horizon. For dealing with unstable systems, the proposed approach would require additional terminal constraints on the unstable modes. Similarly, the authors from [14] apply the wavelet transformation for simplifying the control laws obtained with explicit model predictive control. They show that the resulting simplified control law is everywhere feasible and quantifiable the suboptimality.

Our contribution is twofold: We show that requiring the parametrization to be invariant with respect to time-shifts imposes a specific structure on the basis functions used for representing the input (and/or state) trajectories. We will highlight that the proposed parametrization captures the classical input description, where no restriction on the input sequence is imposed as a special case. Likewise, our parametrization can be chosen such that Laguerre functions are obtained as a special case, and for example described in [15]. In addition, we show that choosing the basis functions to be decaying, an infinite prediction horizon can be retained, yielding inherent closed-loop stability and recursive feasibility guarantees.

The input and state parametrizations that will be introduced in the following can be viewed as approximations to the underlying constrained linear quadratic regulator problem. This point of view has been explored in the technical report [16], where various approximation results (including convergence) are discussed. Preliminary results appeared in

1Both authors are with the Institute for Dynamic Systems and Control, ETH Zurich, Switzerland. The contact author is Michael Muehlebach, michaelm@berkeley.edu. This work was supported by ETHGrant ETH-48 15-1.
the conference papers [17] and [18]. In [19], and [20] a
similar approach has been applied to control a flying vehicle
actuated by ducted fans. Unlike the earlier contributions,
this article presents a general and flexible parametrization
strategy for reducing the computational complexity of MPC,
when controlling nonlinear systems. It adopts a discrete-
time point of view, and presents previous stability and
recursive feasibility results in a unified way. The discrete-
time formulation enables an exact description of the sample-
and-hold process, which is not possible with the earlier
continuous-time results. The effectiveness of the approach
is demonstrated by performing a swing up of the inverted
pendulum-on-a-cart system, which requires, unlike earlier
applications, to fully account for the nonlinear dynamics. In
addition, a principled way of choosing the basis functions
that parametrize input and state trajectories is discussed,
and a comparison is in terms of execution time and closed-loop
performance. We also comment on the feasibility of the
resulting optimization problems.

Outline: Sec. II introduces the problem formulation
and the notation that is used throughout the article. The
parametrization of the input and state trajectories is discussed
in Sec. III. Sec. IV covers the representation of the dynamics,
and-hold process, which is not possible with the earlier
time formulation enables an exact description of the sample-
and-hold process, which is not possible with the earlier
continuous-time results. The effectiveness of the approach
is demonstrated by performing a swing up of the inverted
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and a comparison is in terms of execution time and closed-loop
performance. We also comment on the feasibility of the
resulting optimization problems.

II. PROBLEM FORMULATION

We consider the following optimization problem as a
starting point

\[
\inf \sum_{k=0}^{\infty} l(k, x(k), u(k)) \tag{1}
\]

\[
\text{s.t. } x(k+1) = f(k, x(k), u(k)), \quad g(x(k), u(k)) \leq 0, \quad k \in \mathbb{Z}^+,
\]
\[
x(0) = x_0,
\]

where the function \( l : \mathbb{Z}^+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) describes the
running cost, \( g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) the constraints, and \( f : \mathbb{Z}^+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) the dynamics. The set of all
nonnegative integers is denoted by \( \mathbb{Z}^+ \), the real numbers
by \( \mathbb{R} \). The integer \( n \) describes the state dimension, \( m \) the
input dimension, and \( n_k \) the number of constraints. The initial
condition is denoted by \( x_0 \). The above formulation encodes
finite horizon problems as a special case, as the running cost
can be chosen to be zero for all time indices larger than the
given horizon.

III. THE PARAMETRIZATION

We will parametrize the input and state trajectory using
basis functions, that is,

\[
\tilde{x}(k) = (I_n \otimes \tau(k))^T \eta_x, \quad \tilde{u}(k) = (I_m \otimes \tau(k))^T \eta_u, \tag{2}
\]

where \( \otimes \) refers to the Kronecker product, and \( \eta_x \in \mathbb{R}^{n_s} \)
and \( \eta_u \in \mathbb{R}^{m_s} \) to the parameter vectors. The basis functions
are captured with the vectors \( \tau(k) \in \mathbb{R}^s, k \in \mathbb{Z}^+ \), where
the integer \( s > 0 \) refers to the number of basis functions.
We use the same basis functions to describe input and state
trajectories, but this not necessarily needs to be the case. The
choice is motivated by the fact that, as a result, linear time-
invariant dynamics can be expressed by a very simple linear
relationship between the parameters \( \eta_x \) and \( \eta_u \). Throughout
the article we will denote parametrized trajectories with a
tilde. We will make the following fundamental assumption:

Assumption A1) The basis functions \( \tau(k), k \in \mathbb{Z}^+ \)
are linearly independent.

This assumption is necessary for the solutions in (1), when
\( x(k) \) and \( u(k) \) are replaced with \( \tilde{x}(k) \) and \( \tilde{u}(k) \), to be unique
(provided that they exist).

In MPC, (1) will be optimized (up to a given tolerance) at
every time step, subject to changing initial conditions \( x_0 \). In
order to facilitate computation, we would like to warmstart
the optimization at the next time step with the trajectories that
were computed at the previous time step. This implies that
for any \( \eta \in \mathbb{R}^s \), there exists a parameter vector \( \hat{\eta}(\eta) \in \mathbb{R}^s \),
such that

\[
\tau(k+1)^T \eta = \tau(k)^T \hat{\eta}(\eta), \quad \forall k \in \mathbb{Z}^+. \tag{3}
\]

Due to the fact that the basis functions are linearly independent
there are \( s \) time indices \( I := \{k_1, k_2, \ldots, k_s\} \) such that
the matrix

\[
T := \begin{bmatrix} \tau(k_1)^T \\ \vdots \\ \tau(k_s)^T \end{bmatrix} \tag{4}
\]

is full rank. Thus, (3) implies that

\[
[\tau(k+1)^T]_{k \in I} \eta = T \hat{\eta}(\eta), \tag{5}
\]

which concludes that \( \hat{\eta} \) is a linear function of \( \eta \) (multiplica-
tion of both sides with \( T^{-1} \)). Combined with the fact that
(5) is required to hold for all \( \eta \in \mathbb{R}^s \), we obtain, by taking
the derivative of (5) with respect to \( \eta \)

\[
\tau(k+1) = M \tau(k), \quad \forall k \in \mathbb{Z}^+, \tag{6}
\]

where the constant matrix \( M \in \mathbb{R}^{s \times s} \) contains the partial
derivative of \( \hat{\eta} \) with respect to \( \eta \). This concludes that if the
basis function are required to fulfill relation (3), they will
automatically satisfy (6) for some matrix \( M \in \mathbb{R}^{s \times s} \).

Conversely, if the basis functions satisfy (3), for some
matrix \( M \in \mathbb{R}^{s \times s} \), (3) is satisfied for all \( \eta \in \mathbb{R}^s \) with
\( \hat{\eta} = M^T \eta \). The result is summarized by the following
proposition.

Proposition 3.1: The basis functions satisfy (3) for some
matrix \( M \in \mathbb{R}^{s \times s} \) if and only if they are invariant under
time shifts in the sense of (5).
zero, leading to the requirement that the dynamic system should be asymptotically stable, i.e. the matrix $M$ should have all eigenvalues strictly within the unit circle. Moreover, we assume that the function $g$ describing the constraints is continuous and satisfies $g(0,0) < 0$. The case where the equilibrium lies right on the boundary of the constraint manifold has limited practical importance, since, due to measurement noise, the resulting MPC algorithm is likely to become infeasible, even for initial conditions that are in a neighborhood of the equilibrium. The regularity assumptions on $g$ are needed for the closed-loop stability and recursive feasibility statements in Sec. VII where they guarantee existence of the minimizer of (1), for example.

Summarizing, we thus impose the following requirements on the basis functions:

A1) The basis functions $\tau(k), k \in \mathbb{Z}^+$ are linearly independent.

A2) The basis functions $\tau(k), k \in \mathbb{Z}^+$ satisfy the relation for some matrix $M \in \mathbb{R}^{n \times s}$ that has all its eigenvalues within the unit circle.

### A. Examples

There are numerous examples of basis functions that comply with Assumptions A1 and A2.

- The choice

  $$M = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \tau(0) = (1, 0, 0, \ldots)^T, \quad (7)$$

  yields the classical approach, c.f. [1], where, in this case, the first $s$ inputs can be chosen arbitrarily. More precisely, if $\tilde{u}(k) = \tau(k)^T \eta$, then $\tilde{u}(0) = \eta_1$, $\tilde{u}(1) = \eta_2$, and so on, until $\tilde{u}(s) = 0$, where $\eta = (\eta_1, \eta_2, \ldots, \eta_s)^T$.

- The choice

  $$M = e^{M_1 T}, \quad M_c = \begin{pmatrix} -\nu & 0 & 0 & \ldots \\ -2\nu & -\nu & 0 & \ldots \\ -2\nu & -2\nu & -\nu & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (8)$$

  $\tau(0) = \sqrt{2\nu} (1, 1, \ldots)^T$, yields so-called Laguerre functions, [21]. These are essentially exponentially decaying polynomials, in the following sense

  $$\tau(k) \in e^{-\nu k T} \text{span}(1, k T, (k T)^2, \ldots), \quad (9)$$

  where $\nu$ is the decay rate ($s^{-1}$) and $T$, a sampling time.

- In an analogous way, the basis functions could be chosen such that

  $$\tau(k) \in e^{-\nu k T} \text{span}(1, \sin(\omega k T), \cos(\omega k T), \sin(2\omega k T), \cos(2\omega k T), \ldots), \quad (10)$$

  where $\omega$ denotes the frequency (rad/s).

- In case the dynamics in [1] are linear time-invariant, the cost is quadratic, and constraints are absent ($g = 0$), the choice $M = A + BK$, where $A$ and $B$ refer to the system dynamics, and $K$ to the infinite-horizon linear quadratic regulator gain, recovers the solutions to the infinite-horizon linear quadratic regulator problem (see [22] for the continuous-time analogue).

Any superposition of the above choices is valid as well, and is obtained by a blockdiagonal choice of the matrix $M$. In Sec. VIII we present a systematic choice for Laguerre basis functions (according to (8)) based on optimizing the closed-loop performance.

### IV. Dynamics

We propose to encode the dynamics using a variational formulation. We first note that the dynamics given in (1) can be restated as

$$\sum_{k=0}^{\infty} \delta p(k)(x(k+1) - f(k, x(k), u(k))) = 0, \quad (11)$$

for all variations $\delta p(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}^+$. We apply the Galerkin approach to obtain a nonlinear equality constraint that approximates the dynamics. More precisely, we require the variations to be spanned by the basis functions and insert the parametrizations for the input and state trajectories. This yields

$$\sum_{k=0}^{\infty} (I_n \otimes \tau(k))(\tilde{x}(k+1) - f(k, \tilde{x}(k), \tilde{u}(k))) = 0, \quad (12)$$

Note that the above equation defines a nonlinear equality constraint between the parameter vectors $\eta_x$ and $\eta_u$ that parametrize input and state trajectories. If $f$ is continuous about $x = 0, u = 0$ uniformly in $k$, then the above sum is guaranteed to exist, due to the exponential decay of the basis functions.

For gaining additional insights we will now highlight two special cases.

#### A. Linear time-invariant dynamics

In this case $f(k, x(k), u(k)) = Ax(k) + Bu(k)$, where $A$ and $B$ are real matrices of appropriate size. As a result, (12) reduces to

$$\sum_{k=0}^{\infty} \sum_{i=0}^{n_x} \sum_{i=0}^{n_u} \sum_{k=0}^{\infty} \tau(k)(\tau(k)^T) (I_n \otimes M)^{\otimes k} (A \otimes I_s) \eta_x (B \otimes I_s) \eta_u = 0, \quad (13)$$

where the property (6) of the basis functions and the properties of the Kronecker product have been exploited. By the linear independence of the basis functions, the matrix $I_n \otimes \sum_{k=0}^{\infty} \tau(k)(\tau(k)^T)$ has full rank and thus the above

\footnote{The matrix is positive definite, as shown by a contradiction argument: If there existed a vector $\eta \neq 0$ such that $\eta^T \sum_{k=0}^{\infty} \tau(k)(\tau(k)^T) \eta = 0$, this would imply that $\tau(k)^T \eta = 0 \, \forall k \in \mathbb{Z}^+$, contradicting the fact that the basis functions are linearly independent.}
The Cayley-Hamilton theorem implies that $M \in \mathbb{R}^{n \times n}$ is such that $M^n = 0$. The linear constraint (14) implies that the state and input trajectories fulfill
\[ \tilde{x}(k+1) = A\tilde{x}(k) + B\tilde{u}(k), \quad \forall k \in \mathbb{Z}^+, \tag{15} \]
that is, the dynamics are fulfilled exactly (as can be seen by multiplying (14) with $(I_n \otimes \tau(k)^T)$ from the left). Linear independence of the basis functions can be used to conclude the converse. We will not cover the details, since the argument parallels Sec. IIIA, hinging on the fact that the matrix (4) is full rank. The result is summarized by the following proposition.

**Proposition 4.1:** Provided that the basis functions fulfill Assumptions A1 and A2, and that the dynamics are linear time-invariant, (14) and (15) are equivalent.

**B. Basis functions chosen according to (7)**

In case the basis functions are chosen according to (7), the first $s$ variations are fully decoupled. As result, (12) reduces to
\[ \tilde{x}(k+1) = f(k, \tilde{x}(k), \tilde{u}(k)), \quad k \in \{0, 1, \ldots, s\}, \tag{16} \]
that is, the dynamics are fulfilled exactly for the first $s$ time steps.

**V. CONSTRAINTS**

The following section discusses the implementation of the inequality constraints,
\[ g(\tilde{x}(k), \tilde{u}(k)) \leq 0, \quad \forall k \in \mathbb{Z}^+. \tag{17} \]
It turns out that in many cases the constraint needs only to be checked at finitely many time instances, due to the decaying nature of the basis functions. We will relate the implementation of the inequality constraints to the well-studied problem of computing the maximum output admissible set for an autonomous linear time-invariant system. To that extent we define the variable $\tilde{z}(k)$ as
\[ \tilde{z}(k) := (\tilde{x}(k), \tilde{u}(k)) = (I_{n+m} \otimes \tau(k))^T \eta_z, \tag{18} \]
where $\eta_z := (\eta_x, \eta_u)$ is obtained by stacking $\eta_x$ and $\eta_u$. For any time-shift $q \in \mathbb{Z}^+$, we can express $\tilde{z}(k+q)$ as
\[ \tilde{z}(k+q) = (I_{n+m} \otimes \tau(k))^T (I_{n+m} \otimes M^q)^T \eta_z. \tag{19} \]

The Cayley-Hamilton theorem implies that $M^s$ can be expressed as $M^s = \sum_{j=0}^{s-1} c_j M^j$, where the constants $c_j \in \mathbb{R}$, $j = 0, 1, \ldots, s-1$ are related to the characteristic polynomial of the matrix $M$. This fact can be combined with (19), enabling us to rewrite $\tilde{z}(k+s)$ as
\[ \tilde{z}(k+s) = \sum_{j=0}^{s-1} c_j \tilde{z}(k+j). \tag{20} \]

In other words, the property (6) of the basis functions implies that the trajectories $\tilde{x}(k), \tilde{u}(k)$ match the trajectories of the autonomous linear time-invariant system,
\[ \tilde{z}(k+1) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_0 & c_1 & \cdots & c_{s-1} \end{pmatrix} \otimes I_{n+m} \tilde{z}(k), \tag{21} \]
given the initial condition $\tilde{z}(0) = (\tilde{z}(0), \ldots, \tilde{z}(s-1))$, and where $\tilde{z}(k) := (\tilde{z}(k), \ldots, \tilde{z}(k+s-1))$. As a result, from a certain point of view, the constraint (17) describes an output admissible set of the autonomous system (21). In [23], an algorithm is derived that computes a finite dimensional representation (or a close approximation) of the output admissible set of a given autonomous linear time-invariant system. Applied to our case this yields the following result: Under favorable circumstances (see the next paragraph for a precise statement), the constraint (17) is equivalent to
\[ g(\tilde{x}(k), \tilde{u}(k)) \leq 0, \quad \forall k \in \{0, 1, \ldots, N_{\max}\}, \tag{22} \]
where $N_{\max}$ can be computed by the recursive algorithm given in Alg. 1. For a general function $g$, the optimization that has to be carried out at each step of the algorithm is non-convex and there are therefore no guarantees that the constants $J_i$ are correctly computed. In case $g$ is affine, the optimization simplifies to a linear program that can be solved efficiently and the constants $J_i$ are guaranteed to be computed exactly (up to a given tolerance).

As remarked in [23] it is difficult to state general conditions on the existence of $N_{\max}$ (that is, if the algorithm converges). According to [23, Thm. 4.1] an $N_{\max} \in \mathbb{Z}^+$ is guaranteed to exist if $g$ is affine, $g(0,0) \neq 0$, and the set of all $(x,u)$ such that $g(x,u) \leq 0$ is bounded.

**Data:** $M, \tau(0), g$

**Result:** $N_{\max} \in \mathbb{Z}^+$

**while not converged**

**do**

**Compute** $J_i$ for all $i = 1, 2, \ldots, n_c$:

$J_i := \sup_{\eta_x, \eta_u} g_i(\tilde{x}(j+1), \tilde{u}(j+1)) \tag{23}$

s.t. $g(\tilde{x}(k), \tilde{u}(k)) \leq 0, \quad \forall k \in \{0, \ldots, j\}$.

**if** $J_i \leq 0, i = 1, \ldots, n_c$ **then**

$N_{\max} = j$, converged.

**else**

$j \leftarrow j + 1$

**end**

**Algorithm 1:** Algorithm for computing $N_{\max}$ (if it exists).

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2The observability condition, which is additionally required in [23] is met by construction.
VI. RESULTING OPTIMIZATION

Combining the previous results, we propose to approximate (1) in the following way

\[ \inf_{\eta_x \in \mathbb{R}^{n_x}, \eta_u \in \mathbb{R}^{n_u}} \sum_{k=0}^{\infty} l(k, \hat{x}(k), \hat{u}(k)) \quad \text{s.t.} \quad (24) \]

where the matrices \( I_n \otimes \tau(k) ) \) is replaced by \( \hat{x}(k) = (I_n \otimes \tau(k)) \eta_x \) and \( \hat{u}(k) \) by \( \hat{u}(k) = (I_m \otimes \tau(k))^T \eta_u \) for all \( k \in \mathbb{Z}^+ \).

Proposition 7.1: Provided that the optimization (24) is feasible at time 0, it remains feasible for all time steps. The resulting model predictive control algorithm asymptotically stabilizes the origin in the sense of Lyapunov.

VIII. SIMULATION RESULTS

The next section discusses the application of the proposed MPC algorithm to a quadruple integrator system. The aim is to highlight and discuss a strategy for choosing the matrix \( M \) and to compare the approach to the MPC strategy presented in [10, Ch. 3] that is likewise based on the parametrization of the input.

The dynamics of the quadruple integrator are given by \( x_i^{(4)}(t) = u_i(t) \), where \( x_i(t) \) denotes the integrator state, the superscript (4) denotes the fourth derivative with respect to time, and \( u_i(t) \) the input. The input is constrained to \( u_i(t) \in [-0.5, 0.5] \) for all \( t \in [0, \infty) \). A discretization with zero-order hold and a sampling time of 20ms yields discrete-time dynamics of the form (1) that are used as a starting point for the proposed MPC strategy. The running cost is chosen to be

\[ l(x_1, ..., x_i, u_i) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 0.05u_i^2. \] (26)

The penalty on the input is deliberately chosen to be small in order generate control signals that are likely to hit the constraints. The dynamics and the running cost are motivated by the fact that the resulting optimization problem (24) reduces to a convex quadratic program. As a result, the optimization algorithm that solves (24) (respectively (25)) is guaranteed to return a solution close to the global minimum, irrespective of the initial guess. This enables a rigorous study with randomized initial conditions. Moreover, the approach presented in [10] is based on linear time-invariant dynamics and a quadratic running cost, which is thus required for a comparison.

The following strategy is used for choosing the matrix \( M \): We start with a parametrization given by (8) and change the value of \( \nu \) from 0.5s\(^{-1}\) to 2s\(^{-1}\) in steps of 0.1s\(^{-1}\). For each value of \( \nu \), we compute the averaged closed-loop cost that is achieved when running the MPC algorithm in closed-loop for 2000 steps (which amounts to 40s), starting at 100 randomized initial conditions. The initial conditions are sampled from a uniform distribution over \([-0.5, 0.5]^4\) and the value \( N_{\text{max}} \) is computed for each different choice of \( M \) according to Alg. 1. For the choice \( s = 8 \), the resulting averaged closed-loop cost as a function of \( \nu \) is shown in Fig. 1. If \( \nu \) is increased above 1.8s\(^{-1}\), not all initial conditions are feasible. The graph reveals that the averaged closed-loop cost is a non-convex function that has multiple local minima. Due to the lack of robustness as \( \nu \) increases above 1.8s\(^{-1}\), the choice \( \nu = 0.8s^{-1} \) is more favorable than \( \nu = 1.8s^{-1} \), although the averaged closed-loop cost is slightly lower for \( \nu = 1.8s^{-1} \). The resulting closed-loop trajectory starting from \( x_i(0) = \dot{x}_i(0) = \ddot{x}_i(0) = \dddot{x}_i(0) = 0.5 \) is depicted in Fig. 2 where the bang-bang behavior of the input is clearly visible.
Next, the proposed approach is compared to the approach presented in [10, Ch. 3], where the MATLAB scripts given in [10, p. 109, p. 110, p. 123] are used for the implementation. The approach from [10, Ch. 3] is based on augmenting the dynamics, to parametrize the input increments (instead of the input) with Laguerre basis functions, and to eliminate the state trajectory. The value $N_{\text{max}}$, computed according to Alg. [1] is used as the length of the prediction horizon that is required for the approach from [10, Ch. 3]. In both cases, the resulting quadratic program is solved with the optimization routine qpOASES-v.3.2.1, [25] (with the default options). Fig. 3 compares the closed-loop performance, averaged over the 100 randomized initial conditions with the average (over a single simulation) execution time, when increasing the number of basis functions $s$ from 8 to 12 (for the approach presented herein) and from 5 to 12 (for the approach presented in [10, Ch. 3]). For values below $s = 8$ not all trajectories are stabilized by the approach presented herein and for values below $s = 5$ the performance of the approach presented in [10, Ch. 3] exceeds 5. The approach presented in [10, Ch. 3] is based on Laguerre basis functions, and therefore the same basis functions are used for the approach presented herein. The decay rate $\nu$ is set to $1$ s$^{-1}$. It can be concluded that approach presented herein achieves a smaller cost at similar execution times. Note that in the approach from [10, Ch. 3] no terminal state-constraint and terminal cost is added, and hence closed-loop stability is not guaranteed, which contrasts the approach presented herein (see Prop. [7]). It is observed that the approach from [10, Ch. 3] is still able to stabilize all initial conditions even for very low values of $s$ (we conducted experiments reducing $s$ up to 2). In contrast, the optimization problem (25) is infeasible for some initial conditions for such low values of $s$ (values below 8). This needs, however, to be put in perspective with the additional recursive feasibility and closed-loop guarantees that are inherent to (25), which restrict the region where (25) is feasible.

In addition, results that are obtained with a tailored optimization routine exploiting the structure of (25) are also included in Fig. 3. The source code is available in [26].

**IX. Experimental results**

The next section discusses the application of the proposed MPC algorithm to a pendulum-on-a-cart system. The aim is to swing up the pendulum from its hanging equilibrium to its upright position, while accounting for the input and state constraints. A detailed description of the experimental setup can be found in App. [II].

The cost function is chosen to be time-invariant and quadratic,

$$ l(x,u) = 20x_c^2 + 2\dot{x}_c^2 + 50\dot{\phi}^2 + 2\dot{\phi}^2 + 10u^2, $$

where relatively large penalties are given to the cart’s position $x_c$, the pendulum angle $\phi$, and the input $u$. Note that the physical units are omitted for simplicity in (27). The cart is attached to a rail of 0.9m length, and the input (voltage applied to the motor driving the cart) is limited to $\pm24V$, hence $x_c \in [-0.45m, 0.45m]$ and $u \in [-24V, 24V]$. 

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**Fig. 1.** The plot shows the closed-loop cost, averaged over 100 randomized initial conditions as a function of the parameter $\nu$ describing the decay rate of the basis functions. If $\nu$ is increased above $1.8$s$^{-1}$ certain initial conditions are infeasible, and cannot be stabilized by the MPC algorithm.

**Fig. 2.** The plot shows the closed-loop trajectories of a quadruple integrator system that is controlled with the MPC approach presented herein, where the basis functions are chosen according to (8) with $\nu = 0.8$s$^{-1}$.

**Fig. 3.** The plot shows a comparison between the approach presented herein (full marks) and the approach presented in [10, Ch. 3] (crosses), where the number of basis functions is varied form $s = 8$ to $s = 12$ (full marks) and from $s = 5$ to $s = 12$ (crosses). The results obtained from a dedicated MPC solver that exploits the structure of the optimization problem (25) are shown with empty marks. The standard deviation of the execution time (over the different randomized initial conditions) is indicated with dashed lines.
The basis functions are chosen to be a combination of the standard parametrization according to [7] and Laguerre functions according to [8], that is,

$$ M = \begin{pmatrix} M_d & 0 & 0 \\ 0 & \hat{\tau}(0) & M_d \end{pmatrix} \in \mathbb{R}^{19 \times 19}, $$

(28)

$$ \tau(0) = (1, 0, 0, \ldots)^T \in \mathbb{R}^{19}, $$

where $M_d \in \mathbb{R}^{12 \times 12}$ is chosen according to [7], and $M_d \in \mathbb{R}^{7 \times 7}$ and $\hat{\tau}(0)$ are chosen according to [8] with $\nu = 14\text{s}^{-1}$. The choice is motivated by the fact that a typical swing-up trajectory (see Fig. 4) involves a bang-bang control input for a bit more than 0.2s (hence roughly 12 time steps) followed by smaller inputs that guide the pendulum to its upright position. These smaller inputs are well approximated by Laguerre basis functions with a relatively small time constant of $1/(14)s$. A small time constant is desirable due to the fact that the resulting value of $N_{\text{max}}$, and hence the number of inequalities in the corresponding optimization problem, is relatively small. The basis functions are chosen to be orthogonal, that is, they are normalized such that $\sum_{k=0}^{\infty} \tau(k)\tau(k)^T = I_s$. Running algorithm Alg. 1 subject to the input (limited voltage) and state constraint (limited rail length) results in $N_{\text{max}} = 30$.

A rough initial guess for the swing-up trajectories (state and input) is generated via simulation, where the input $\hat{u}(k)$ is obtained by combining step inputs with a linear quadratic regulator for catching the pendulum in upright position. The input trajectory is shown in Fig. 4 (dashed line). The trajectories are then represented using the basis functions. In case of the input trajectory, the resulting parameter vector is given by $\hat{\eta}_u = \sum_{k=0}^{\infty} \tau(k)\hat{u}(k)$, and the resulting trajectory $\tau(k)^T\hat{\eta}_u$ is shown in Fig. 4 (solid line). The representation of the state trajectories are obtained analogously. The state constraint (limited rail length) is not considered for the generation of the initial guess.

The dynamics are imposed using the variational formulation given by (12), where the infinite sum is truncated after $k = 150$. The truncation is motivated by the fact that the contributions for $k > 42$ are negligible, as the basis function decay with a time constant of $1/\nu = 0.071$s (that is, 4 samples result in a decay of roughly 67%).

The nonlinear optimization problem is solved with Ipopt, [27] (version 3.12.10) and interfaced through C++. The functions for evaluating the derivatives of the Lagrangian and the equality constraints are generated with Casadi, [28] (version 3.4.4). Ipopt is run with the default settings, a relative tolerance of 1e-6, and with a maximum number of 8 iterations (during warmstarts). The time-shift property of the basis functions is exploited for warm-starting the optimization during the swing up. The nominal open-loop swing-up trajectory (as obtained by solving the optimization at the first time instant) is shown in Fig. 5.

The resulting swing-up trajectories are shown in Fig. 6. After roughly 1s the pendulum reaches its upright position. During the whole swing-up trajectory the input and state constraints are accounted for, and the execution time of the optimization remains below 0.02s.

Note the difference between the planned open-loop trajectory and resulting closed-loop trajectory. For instance, the constraint at $x_c = 0.45$m is reached during the open-loop trajectory, whereas in closed-loop $x_c = 0.45$m is never reached. This discrepancy is most likely due to modeling errors. In particular, the dry friction between the cart and the rail, which is substantial, is difficult to model accurately and to compensate for. Nevertheless, the MPC controller is able to perform the swing up. The robustness of the optimization with respect to the initial guess is further discussed and quantified in App. II.

The above choice of basis functions enables a reduction of the degrees of freedom. In case a standard parametrization according to (7) is used, 250 optimization variables are needed to capture a prediction horizon of 1s, which is roughly required for performing a swing up. The parametrization allows to capture about the same time horizon with only 85 optimization variables.
Fig. 6. Trajectories resulting from a successful swing up.

X. CONCLUSION

The article discussed a method for reducing the complexity of the optimization algorithms encountered in MPC by parametrizing input and state trajectories with basis functions. The basis functions are chosen to be invariant to time-shifts, which enables warm-starting of consecutive optimization problems. In addition, in the case of linear time-invariant dynamics and a quadratic cost, closed-loop stability and recursive feasibility results are obtained without the addition of a terminal set constraint and a terminal cost. The method is applied to the swing-up of an inverted pendulum system, where it is shown that the number of degrees of freedom can be greatly reduced with the parametrization. The resulting MPC algorithm runs with a sampling time of 20ms and is able to swing the inverted pendulum up while accounting for input and state constraints.

REFERENCES

[1] M. Morari and J. H. Lee, “Model predictive control: past, present and future,” Computers and Chemical Engineering, vol. 23, no. 4, pp. 667–682, 1999.
[2] F. Borrelli, P. Falcone, T. Keviczky, J. Asgari, and D. Hrovat, “MPC-based approach to active steering for autonomous vehicle systems,” International Journal of Vehicle Autonomous Systems, vol. 3, no. 2-4, pp. 265–291, 2005.
[3] L. Bascetta, D. Cucci, G. Magnani, M. Matteucci, D. Osmankovic, and A. Tahirovic, “Towards the implementation of a MPC-based planner on an autonomous All-Terrain Vehicle,” International Conference on Intelligent Robots and Systems, pp. 1–7, 2012.
[4] G. Bardaro, L. Bascetta, E. Ceravolo, M. Farina, M. Gabellone, and M. Matteucci, “MPC-based control architecture of an autonomous wheelchair for indoor environments,” Control Engineering Practice, vol. 78, no. 1, pp. 160–174, 2018.
[5] T. Templeton, D. H. Shin, C. Geyer, and S. S. Sastry, “Autonomous Vision-based Landing and Terrain Mapping Using an MPC-controlled Unmanned Rotorcraft,” International Conference on Robotics and Automation, pp. 1349–1356, 2007.
[6] M. W. Mueller, M. Hehn, and R. D’Andrea, “A computationally efficient motion primitive for quadrotor trajectory generation,” IEEE Transactions on Robotics, vol. 31, no. 6, pp. 1294–1310, 2015.
[7] M. Bangura and R. Mahony, “Real-time model predictive control for quadrotors.” Proceedings of the 19th World Congress of the International Federation of Automatic Control, pp. 11 773–11 780, 2014.
[8] R. Gondhalekar and J. Imura, “Least-restrictive move-blocking model predictive control,” Automatica, vol. 46, no. 7, pp. 1234–1240, 2010.
[9] R. Cagniard, P. Grieder, E. Kerrigan, and M. Morari, “Move blocking strategies in receding horizon control,” Journal of Process Control, vol. 17, no. 6, pp. 563–570, 2007.
[10] L. Wang, Model Predictive Control System Design and Implementation using MATLAB. Springer, 2009.
[11] J. A. Rossiter, L. Wang, and G. Valencia-Palomo, “Efficient algorithms for trading off feasibility and performance in predictive control,” International Journal of Control, vol. 83, no. 4, pp. 789–797, 2010.
[12] B. Khan and J. A. Rossiter, “Alternative parametrisation within predictive control: a systematic selection,” International Journal of Control, vol. 86, no. 8, pp. 1397–1409, 2013.
[13] J. H. Lee, Y. Chikkula, Z. Yu, and J. C. Kantor, “Improving computational efficiency of model predictive control algorithm using wavelet transformation,” International Journal of Control, vol. 61, no. 4, pp. 859–883, 1995.
[14] S. Summers, C. N. Jones, J. Lygeros, and M. Morari, “A multi-resolution approximation method for fast explicit model predictive control,” IEEE Transactions on Automatic Control, vol. 56, no. 11, pp. 2530–2541, 2011.
[15] J. A. Rossiter and L. Wang, “Exploiting Laguerre functions to improve the feasibility/performance compromise in MPC,” Proceedings of the International Conference on Decision and Control, pp. 4737–4742, 2008.
[16] M. Muehlebach and R. D’Andrea, “On the approximation of constrained linear quadratic regulator problems and their application to model predictive control,” ETH Zurich Research Collection, 2018. [Online]. Available: https://doi.org/10.3929/ethz-b-000292793
[17] ——, “Parametrized infinite-horizon model predictive control for linear time-invariant systems with input and state constraints,” Proceedings of the American Control Conference, pp. 2669–2674, 2016.
[18] ——, “Approximation of continuous-time infinite-horizon optimal control problems arising in model predictive control,” Proceedings of the IEEE Conference on Decision and Control, pp. 1464–1470, 2016.
[19] M. Hofer, M. Muehlebach, and R. D’Andrea, “Application of an approximate model predictive control scheme on an unmanned aerial vehicle,” Proceedings of the International Conference on Robotics and Automation, pp. 2952–2957, 2016.
[20] M. Muehlebach, C. Sferrazza, and R. D’Andrea, “Implementation of a parametrized infinite-horizon model predictive control scheme with stability guarantees,” Proceedings of the International Conference on Robotics and Automation, pp. 2723–2730, 2017.
[21] B. Khan, G. Valencia-Palomo, J. A. Rossiter, C. N. Jones, and R. Gondhalekar, “Long horizon input parametrisations to enlarge the region of attraction of MPC,” Optimal Control Applications and Methods, vol. 37, no. 1, pp. 139–153, 2016.
[22] M. Muehlebach and R. D’Andrea, “Basis functions design for the approximation of constrained linear quadratic regulator problems encountered in model predictive control,” Proceedings of the International Conference on Decision and Control, pp. 6189–6196, 2017.
[23] E. G. Gilbert and K. T. Tan, “Linear systems with state and control constraints: The theory and application of maximal output admissible sets,” IEEE Transactions on Automatic Control, vol. 36, no. 9, pp. 1008–1020, 1991.
[24] R. T. Rockafellar and R. J.-B. Wets, Variational Analysis. Springer, 1998.
[25] H. J. Ferreau, C. Kirches, A. Potschka, H. G. Bock, and M. Diehl, “qpOASES: A parametric active-set algorithm for quadratic program-
We consider the optimal cost to be a function of the initial condition $x_0$, 

$$ J(x_0) := \min_{\eta_x \in \mathbb{R}^n, \eta_u \in \mathbb{R}^n} \sum_{k=0}^{\infty} l(\tilde{x}(k), \tilde{u}(k)) \quad \text{s.t.} \quad (29) $$

$$ (I_n \otimes M^T - A \otimes I_n) \eta_x - (B \otimes I_n) \eta_u = 0, $$

$$ \dot{x}(0) = x_0, $$

$$ g(\tilde{x}(k), \tilde{u}(k)) \leq 0, \quad \forall k \in \{0, 1, \ldots, N_{\text{max}}\}, $$

for all $x_0 \in X$, where $X$ denotes the set of all $x_0$ such that the above optimization is feasible. For $x_0 \in X$, the assumptions on $l$ and $g$ (it is continuous, $l(0, 0) = 0$, and bounded below by a quadratic function, $g$ is continuous) assert that the above minimum exists and is attained, see [24, p. 12, Ex. 1.11]. They further imply that $J$ is lower bounded,

$$ J(x_0) \geq l(x_0, \tilde{u}(0)) \geq \sigma|x_0|^2, \quad \forall x_0 \in X. \quad (30) $$

Due to the fact that the inequality constraints in (29) describe a closed set that includes $\eta_x = 0$, $\eta_u = 0$ in its interior ($g$ is assumed to be continuous and $g(0, 0) < 0$), the regularity of equality constraints and the quadratic upper bound on the running cost imply that the inequality constraints are not active for small enough $x_0$. According to [24, p. 16, Thm.1.17], $J(x_0)$ is therefore continuous at zero.

We now consider the trajectory of the closed-loop system $x(k), k \in \mathbb{Z}^+$ starting at any $x_0 \in X$. According to Prop. 4.1, the trajectories $\tilde{x}(k)$ and $\tilde{u}(k)$ resulting from solving the optimization (29) at time 0 satisfy the dynamics and the constraints for all times. Due to the time-shift property of the basis functions, the trajectories $\tilde{x}(k + 1)$ and $\tilde{u}(k + 1)$ lie likewise in the span of the basis functions, and therefore, they represent feasible candidates for the optimization at the next time step, i.e. for $k = 1$. Moreover, the optimal cost at the next time step ($k = 1$), $J(x(1))$, is therefore bounded by

$$ J(x(1)) \leq \sum_{k=0}^{\infty} l(\tilde{x}(k + 1), \tilde{u}(k + 1)). \quad (31) $$

The right-hand side can be expressed in terms of the cost $J(x(0))$, that is,

$$ \sum_{k=0}^{\infty} l(\tilde{x}(k), \tilde{u}(k)) - l(\tilde{x}(0), \tilde{u}(0)) = J(x(0)) - l(\tilde{x}(0), \tilde{u}(0)), $$

which yields, (using $\tilde{x}(0) = x(0)$ and $\tilde{u}(0) = u(0)$),

$$ J(x(1)) - J(x(0)) \leq -l(x(0), u(0)) \quad (32) $$

It follows by induction that (29) is recursively feasible, and that, for any $k \geq 0$, 

$$ J(x(k + 1)) - J(x(k)) \leq -l(x(k), u(k)) \leq -\sigma|x(k)|^2. \quad (33) $$

Hence, the cost $J(x(k))$ forms a monotonically decreasing sequence (in $k \in \mathbb{Z}^+$) that is bounded below by 0 and therefore converges. In the limit as $k \to \infty$, (33) reduces to 

$$ 0 \leq \lim_{k \to \infty} -\sigma|x(k)|^2, $$

implying that $x(k) \to 0$ as $k \to \infty$. This shows that the origin is attractive in $X$.

It remains to show that the origin is stable in the sense of Lyapunov. Note that attractivity alone does not imply stability. For proving stability we pick any $\epsilon > 0$. Due to the continuity of $J(x_0)$, there exists a $\delta > 0$ such that $|x_0| < \delta$ implies $J(x_0) < \epsilon^2$ (and $x_0 \in X$). As argued in the previous paragraph, the cost $J(x(k))$ of any closed-loop trajectory $x(k)$ starting at $|x_0| < \delta$ is monotonically decreasing. This concludes that $\sigma|x(k)|^2 \leq J(x(k)) \leq J(x_0) < \epsilon^2$, that is, $|x(k)| < \epsilon$.

Remarks

The section concludes with the two remarks that indicate further extensions and generalizations of the recursive feasibility and closed-loop stability results:

1) The results can be extended to a running cost that includes a discount, that is,

$$ l(k, x, u) = \rho_x^k x^T Q x + \rho_u^k u^T R u, \quad (35) $$

where $\rho_x, \rho_u \in \mathbb{R}$ are in the interval $(0, 1)$, and $Q \in \mathbb{R}^{n \times n}$ is positive definite and $R \in \mathbb{R}^{m \times m}$ is positive semi-definite.

The results can also be extended to the trajectory-tracking case. More precisely, provided that the dynamics are linear and time-invariant, and that the reference input and state trajectories are spanned by the basis functions, the arguments of Prop. 7.1 can be used to conclude recursive feasibility and asymptotic stability of the tracking error.

2) If the dynamics are nonlinear, recursive feasibility and closed-loop stability are no longer inherent to the problem formulation. However, our approach could be modified to enforce the nonlinear dynamics exactly (for example by eliminating the state trajectory) and to include similar stabilizing terminal conditions as in a finite-horizon non-parametrized approach, see e.g. [29, Ch. 5]. In that case, the arguments presented in [29, Ch. 5] would apply with minor modifications. However, the computational complexity of the resulting MPC algorithm would most likely increase. In all these stability arguments, the fact that the parameterization is invariant to time-shifts is of paramount importance.
APPENDIX II
EXPERIMENTAL RESULTS

A. Hardware setup and first-principles model

Fig. 7 shows the experimental setup. The cart is actuated by an electrical motor via a transmission belt. The voltage that can be applied to the electrical motor represents the input \( u \) to the system and is limited to \( \pm 24 \text{V} \). The cart is attached to a rail of 0.9m length. The position of the cart and the angle of pendulum are measured with encoders. An extended Kalman filter is used to estimate the state \( x := (x_c, \dot{x}_c, \varphi, \dot{\varphi}) \). The control and estimation algorithms are run on a laptop (Intel Core i7, 2.6GHz, 8GB random access memory) that is interfacing the pendulum system through a serial communication. All the control and estimation algorithms run at 50Hz.

The following first-principles model is used

\[
\begin{align*}
\ddot{x}_c &= \frac{F_u}{m} - g \sin(\varphi) \cos(\varphi) + l \dot{\varphi}^2 \sin(\varphi), \\
\ddot{\varphi} &= -\frac{F_u}{ml} \cos(\varphi) + \frac{M + m}{m} g \sin(\varphi) - \varphi^2 \sin(\varphi) \cos(\varphi), \\
F_u &= \frac{k_m}{r_m} (u - \frac{\dot{x}_c}{k_m r_n}),
\end{align*}
\]

where \( F_u \) denotes the force driving the pendulum, \( x_c \in [-0.45 \text{m}, 0.45 \text{m}] \) the cart position, and \( \varphi \) the pendulum angle (\( \varphi = 0 \) corresponds to the upright equilibrium). The parameters are listed in Tab. I. Friction is neglected in the model used for the MPC controller, but dry friction between the cart and rail is compensated with feedforward. The parameters of the first-principles model are identified by fitting different step-responses about the lower equilibrium.

B. Robustness with respect to the initial guess

The robustness with respect to the initial guess that starts the optimization at each time step is studied in simulations. In general, an initial guess that is close to a successful swing-up trajectory is required to start the optimization (such as the one given by Fig. 4). In order to quantify the robustness of the optimization with respect to the initial guess, we included the following artificial disturbance of the state variable, while performing (closed-loop) swing-up maneuvers in simulation,

\[
x(k + 1) = f(x(k), u(k)) + \begin{pmatrix} 0.02 \text{m} \\ 0.005 \text{m/s} \\ 1 \text{deg} \\ 0.5 \text{deg/s} \end{pmatrix} n(k), \tag{37}
\]

where \( f(x(k), u(k)) \) are the nominal dynamics and \( n(k) \) is a disturbance, uniformly distributed in \([-n_{\text{max}}, n_{\text{max}}]\) and independent across time. The optimization is thus warm-started with the solution obtained from the previous time step, but has no information about the perturbation \( n(k) \). The resulting closed-loop cost as a function of the disturbance amplitude perturbing the state at each time step.

![Fig. 8](image)

**Fig. 8.** The plot shows how the closed-loop cost varies as a function of the disturbance amplitude perturbing the state at each time step.

**TABLE I**

| parameter | value     | description             |
|-----------|-----------|-------------------------|
| \( m \)  | 0.17kg    | pendulum mass           |
| \( M \)  | 0.74kg    | cart mass (lumped)      |
| \( l \)  | 0.30m     | pendulum length         |
| \( k_m \)| 0.011Nm/A | torque constant         |
| \( k_n \)| 20.62V/s  | speed constant          |
| \( R_m \)| 0.30\( \Omega \) | resistance            |
| \( r_m \)| 0.018m    | belt wheel radius       |

In our experience, it is important that the optimization is able to handle these disturbances well, as these also occur in the experimental setup due to model mismatch.