THE CONSTANT WIDTH MEASURE SET,
THE SPHERICAL MEASURE SET
AND ISOPERIMETRIC EQUALITIES FOR PLANAR OVALS

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ABSTRACT. In this paper we introduce and study sets: the Constant Width Measure Set and the Spherical Measure Set, which measure the constant width property and the spherical property of planar regular simple curves, respectively.

Using these sets and the Wigner caustic, we find the exact relations between the length and the area bounded by a closed regular simple convex planar curve $M$. Namely, the following equalities are fulfilled:

\begin{align}
L_M^2 &= 4\pi A_M + 8\pi \left| \tilde{A}_{E_1^2}(M) \right| + \pi \left| \tilde{A}_{CWMS}(M) \right|, \\
L_M^2 &= 4\pi A_M + 4\pi \left| \tilde{A}_{SMS}(M) \right|
\end{align}

where $L_M$, $A_M$, $\tilde{A}_{E_1^2}(M)$, $\tilde{A}_{CWMS}(M)$, $\tilde{A}_{SMS}(M)$ are the length of $M$, the area bounded by $M$, the oriented area of the Wigner caustic of $M$, the oriented area of the Constant Width Measure Set of $M$ and the oriented area of the Spherical Measure Set of $M$, respectively. The equality (0.2) holds also if $M$ is a regular closed simple curve.

We also present a new proof of Four–Vertex Theorem for generic ovals based on the number of singular points of the SMS.

1. Introduction

The classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$ states that:

**Theorem 1.1.** (Isoperimetric inequality) Let $M$ be a simple closed curve of the length $L_M$, enclosing a region of the area $A_M$, then

\begin{equation}
L_M^2 \geq 4\pi A_M,
\end{equation}

and the equality (1.1) holds if and only if $M$ is a circle.

This inequality was already known by the ancient Greeks. The first mathematical proof of this famous statement was given in the nineteenth century by Steiner [39]. After that, there have been many new proofs, generalizations, and applications of this theorem, see [3][13][16][22][23][27][32][39][40], and the literature therein. In [46] recently we prove the improved isoperimetric inequality, which also gives the isoperimetric equality for convex curves of constant width.

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Theorem 1.2. (Improved isoperimetric inequality 1) Let $M$ be a closed regular simple convex curve. Then
\[ L_M^2 \geq 4\pi A_M + 8\pi \left| \tilde{A}_{E_{\frac{1}{2}}(M)} \right|, \]
where $\tilde{A}_{E_{\frac{1}{2}}(M)}$ is the oriented area of the Wigner caustic of $M$, and the equality holds if and only if $M$ is a curve of constant width.

An affine equidistant is the set of points of chords connecting points on $M$ where tangent lines to $M$ are parallel, which divide the chord segments between the base points with a fixed ratio $\lambda$. If the ratio $\lambda$ is equal to $\frac{1}{2}$ then this set is also known as the Wigner caustic of $M$. The Wigner caustic was first introduced by Berry, in his celebrated 1977 paper [1] on the semiclassical limit of Wigner’s phase-space representation of quantum states. There are many papers considering affine equidistants and in particular the Wigner caustic, see [4, 7, 8, 9, 10, 11, 18, 20, 29, 38, 55, 46], and the literature therein. The Wigner caustic is also known as the area evolute ([4, 18]) or as the symmetry defect ([29]). Singularities of the Wigner caustic for ovals occur exactly from an antipodal pair (the tangents at the two points are parallel and the curvatures are equal), the well-known Blaschke-Süss theorem states that there are at least three pairs of antipodal points on an oval ([21, 33]). The construction of the Wigner caustic also leads to one of the two constructions of bi-dimensional improper affine spheres, which can be generalized to higher even dimensions ([5]).

Offset curves and surfaces are well-known geometric objects in the field of mathematics and computer aided geometric design, possibly because they give a powerful tool in many applications (see [12, 19, 28, 31, 36]). The Spherical Measure Set is an offset.

The classical four–vertex theorem states that the curvature function of a simple, closed, smooth plane curve has at least four local extrema. Four–Vertex Theorem was first proved for ovals in 1909 by Mukhopadhyaya ([34]) and in general by Kneser in 1912 using a projective argument ([30]), see also ([35] and the literature therein. We present a new proof of Four–Vertex Theorem for generic ovals and the proof is a simply consequence of the number of singular points of the Spherical Measure Set of an oval – see Corollary ([4, 8]).

There are many important inequalities in the convex geometry and differential geometry, such as the isoperimetric inequality, the Brunn–Minkowski inequality, Aleksandrov–Fenchel inequality, Gage’s inequality. The stability property of them are of great interest in geometric analysis, see [15, 16, 24, 27, 37, 40, 41] and the literature therein.

The paper is organized as follows.

In Section 2 we present geometric quantities, their Fourier series and affine equidistants, including the Wigner caustic.

Section 3 contains a definition of the Constant Width Measure Set and properties of this set for planar ovals.

In Section 4 we introduce the Spherical Measure Set and present properties of this set for planar simple regular closed curves.

Section 5 is devoted to prove the main results, isoperimetric equalities for planar simple regular closed curves.
In Section 6 we study the stability property of the isoperimetric inequality involving the Constant Width Measure Set and thanks to it we find the lower bounds of the absolute value of the oriented area of the Wigner caustic, the Constant Width Measure Set and the Spherical Measure Set.

2. Geometric quantities, affine equidistants and the Fourier series

Let $M$ be a smooth planar curve, i.e. the image of the $C^\infty$ smooth map from an interval to $\mathbb{R}^2$. A smooth curve is called closed if it is the image of a $C^\infty$ smooth map from $S^1$ to $\mathbb{R}^2$. A smooth curve is called regular if its velocity does not vanish. A regular closed curve is called convex if its signed curvature has a constant sign. An oval is a smooth regular closed convex curve which is simple, i.e. it has no self-intersections.

**Definition 2.1.** Let $M$ be a regular closed curve. A pair $a, b \in M \ (a \neq b)$ is called the parallel pair if tangent lines to $M$ at points $a, b$ are parallel.

**Definition 2.2.** A chord passing through a pair $a, b \in M$, is the line:
$$l(a, b) = \{ \lambda a + (1 - \lambda)b \mid \lambda \in \mathbb{R} \}.$$

**Definition 2.3.** An affine $\lambda$-equidistant is the following set.
$$E_{\lambda}(M) = \{ \lambda a + (1 - \lambda)b \mid a, b \text{ is a parallel pair of } M \}.$$

The set $E_{\frac{1}{2}}(M)$ will be called the Wigner caustic of $M$.

Note that, for any given $\lambda \in \mathbb{R}$ we have $E_{\lambda}(M) = E_{1-\lambda}(M)$. Thus, the case $\lambda = \frac{1}{2}$ is special. In particular $E_0(M) = E_1(M) = M$.

**Remark 2.4.** It is well known that if $M$ is a generic oval, then for a generic $\lambda$ the set $E_{\lambda}(M)$ is a smooth closed curve with at most cusp singularities [11][18], the number of cusps of $E_{\frac{1}{2}}(M)$ is odd and not smaller than 3 [11][18] and the number of cusps of $E_{\lambda}(M)$ for a generic value of $\lambda \neq \frac{1}{2}$ is even [11].
Definition 2.5. An oval $M$ is said to have constant width if the distance between every pair of parallel tangent lines to $M$ is constant. This constant is called the width of the curve and we denote it by $w_M$.

Definition 2.6. An average width of an oval $M$ is $w_M = \frac{L_M}{\pi}$.

When an oval $M$ has the constant width property, then from Barbier’s theorem one can get that $w_M = w_M$.

Let us recall some basic facts about plane ovals which will be used later. The details can be found in the classical literature [23, 26].

Let $M$ be a positively oriented oval. Take a point $\Theta$ inside $M$ as the origin of our frame. Let $p$ be the oriented perpendicular distance from $\Theta$ to the tangent line at a point on $M$, and $\theta$ the oriented angle from the positive $x_1$-axis to this perpendicular ray. Clearly, $p$ is a single-valued periodic function of $\theta$ with period $2\pi$ and the parameterization of $M$ in terms of $\theta$ and $p(\theta)$ is as follows:

$$
\gamma(\theta) = \left(\gamma_1(\theta), \gamma_2(\theta)\right) = \left(p(\theta) \cos \theta - p'(\theta) \sin \theta, p(\theta) \sin \theta + p'(\theta) \cos \theta\right).
$$

(2.1)

The couple $(\theta, p(\theta))$ is usually called the polar tangential coordinate on $M$, and $p(\theta)$ its Minkowski support function.

Then, the curvature $\kappa$ of $M$ is in the following form:

$$
\kappa(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0,
$$

(2.2)
or equivalently, the radius of a curvature $\rho$ of $M$ is given by:

$$
\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta).
$$

Let $L_M$ and $A_M$ be the length of $M$ and the area it bounds, respectively. Then one can get that

$$
L_M = \int_M ds = \int_0^{2\pi} p(\theta)d\theta = \int_0^{2\pi} p(\theta)d\theta,
$$

(2.3)

and

$$
A_M = \frac{1}{2} \int_M p(\theta)ds = \frac{1}{2} \int_0^{2\pi} p(\theta) [p(\theta) + p''(\theta)] d\theta = \frac{1}{2} \int_0^{2\pi} \left[p^2(\theta) - p'^2(\theta)\right] d\theta.
$$

(2.4)

The formulas (2.3) and (2.4) are known as Cauchy’s formula and Blaschke’s formula, respectively.

Since the Minkowski support function of $M$ is smooth bounded and $2\pi$–periodic, its Fourier series is in the form:

$$
p(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).
$$

(2.5)

Differentiation of (2.5) with respect to $\theta$ gives

$$
p'(\theta) = \sum_{n=1}^{\infty} n(-a_n \sin n\theta + b_n \cos n\theta).
$$

(2.6)
By (2.5), (2.6) and the Parseval identity one can express $L_M$ and $A_M$ in terms of Fourier’s coefficients of $p(\theta)$ in the following way:

(2.7) $L_M = 2\pi a_0,$

(2.8) $A_M = \pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1)(a_n^2 + b_n^2).$

One can notice that $\gamma(\theta), \gamma(\theta + \pi)$ is a parallel pair of $M$, hence $\gamma_\lambda$ - the parameterization of $E_\lambda(M)$ is as follows:

(2.9) $\gamma_\lambda(\theta) = \begin{pmatrix} \gamma_{\lambda,1}(\theta), \gamma_{\lambda,2}(\theta) \end{pmatrix} = \lambda \gamma(\theta) + (1 - \lambda)\gamma(\theta + \pi) = (P_\lambda(\theta) \cos \theta - P'_\lambda(\theta) \sin \theta, P_\lambda(\theta) \sin \theta + P'_\lambda(\theta) \cos \theta),$

where $P_\lambda(\theta) = \lambda p(\theta) - (1 - \lambda)p(\theta + \pi)$, $\theta \in [0, 2\pi]$.

**Remark 2.7.** If $\lambda = \frac{1}{2}$, then the map $M \ni \gamma(\theta) \mapsto \gamma_{\frac{1}{2}}(\theta) \in E_{\frac{1}{2}}(M)$ for $\theta \in [0, 2\pi]$ is the double covering of the Wigner caustic of $M$.

Let us also notice that in particular

(2.10) $P_{\frac{1}{2}}(\theta) = \frac{1}{2}(p(\theta) - p(\theta + \pi))$

(2.11) $= \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right).$

Let $L_{E_\lambda(M)}, \bar{A}_{E_\lambda(M)}$ denote the length of $E_\lambda(M)$ and the oriented area of $E_\lambda(M)$, respectively.

In [11] we show that $2L_{\frac{1}{2}}(M) \leq L_M$ if $M$ is an oval.

One can check that the oriented area of the Wigner caustic has the following formula in terms of coefficients of the Fourier series of the Minkowski support function $p$.

(2.12) $2\bar{A}_{E_{\frac{1}{2}}(M)} = \frac{1}{2} \int_0^{2\pi} \left[P_{\frac{1}{2}}^2(\theta) - P'_{\frac{1}{2}}^2(\theta)\right] d\theta$

$= -\frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1)(a_n^2 + b_n^2).$

**Remark 2.8.** Let us notice that $M$ is an oval of constant width if and only if coefficients $a_{2n}, b_{2n}$ for $n \geq 1$ in the Fourier series of the Minkowski support function $p$ are all equal to zero [14, 23].

3. The Constant Width Measure Set

**Definition 3.1.** Let $M$ be a positively oriented oval. The *Constant Width Measure Set* of $M$ is the following set:

(3.1) $\text{CWMS}(M) = \left\{ a - b + \nu_M \cdot \nu(a) \mid a, b \text{ is a parallel pair of } M \right\},$
where $\overline{w}_M$ is a average width of $M$ and $n(a)$ is a continuous unit normal vector field to $M$ at $a$ compatible with the orientation of $M$. We treat CWMS($M$) as a subset of a vector space $V = \mathbb{R}^2$.

To visualize CWMS($M$) and $M$ in the same picture let us assume that $\Theta$ is the origin of $V$.

In Fig. 2 there is an example of a vector which belongs to CWMS($M$) and in Fig. 3 there is an example of CWMS($M$) of an oval $M$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{An oval $M$, a parallel pair $a, b$, tangent lines to $M$ at $a$ and $b$, vectors $a - b, \overline{w}_M \cdot n(a)$ and $a - b + \overline{w}_M \cdot n(a)$.}
\end{figure}

**Theorem 3.2.** $M$ is an oval of constant width if and only if CWMS($M$) = $\{\Theta\}$.

*Proof.* If CWMS($M$) = $\{\Theta\}$, then $b - a = \overline{w}_M \cdot n(a)$ for all parallel pairs $a, b$. This gives that the length of $b - a$ is always constant and equal to $\overline{w}_M$ and consequently $M$ is an oval of constant width.

If $M$ is an oval of constant width $w_M$, then one can notice that if $a, b$ is a parallel pair of $M$, then $a - b$ is perpendicular to the tangent lines to $M$ at $a$ and $b$. This gives $a - b + w_M \cdot n(a) = \Theta$. \hfill $\square$

**Theorem 3.3.** Let $M$ be a positively oriented oval. Let $\kappa_M(a)$ denote the curvature of $M$ at $a$. Let $a, b$ be a parallel pair of $M$ and let $q = a - b + \overline{w}_M \cdot n(a)$ be a non-singular point of CWMS($M$). Then

(i) the tangent line to CWMS($M$) at $q$ is parallel to the tangent lines to $M$ at $a$ and $b$. 

(ii) the curvature of CWMS(M) at q is equal

\[ \kappa_{\text{CWMS}(M)}(q) = \frac{\kappa_M(a) \cdot \kappa_M(b)}{[\kappa_M(a) + \kappa_M(b) - w_M \cdot \kappa_M(a) \cdot \kappa_M(b)]}. \]

Proof. Let (2.1) be the parameterization of M and p(θ) its Minkowski support function.
One can check that the Minkowski support function of CWMS(M) is in the following form:

\[ p_{\text{CWMS}(M)}(θ) = p(θ) + p(θ + \pi) - w_M. \]

The formula (3.3) in terms of coefficients of a Fourier series of p(θ) is as follows:

\[ p_{\text{CWMS}(M)}(θ) = 2 \sum_{n=2, n \text{ is even}}^{\infty} (a_n \cos nθ + b_n \sin nθ). \]

Hence the parameterization of CWMS(M) is in the following form:

\[ \gamma_{\text{CWMS}(M)}(θ) = \left(p_{\text{CWMS}(M)}(θ) \cos θ - p'_{\text{CWMS}(M)}(θ) \sin θ, p_{\text{CWMS}(M)}(θ) \sin θ + p''_{\text{CWMS}(M)}(θ) \cos θ\right). \]

Then:

\[ \gamma'(θ) = (p(θ) + p''(θ)) \cdot (-\sin θ, \cos θ), \]

\[ \gamma'_{\text{CWMS}(M)}(θ) = (p_{\text{CWMS}(M)}(θ) + p''_{\text{CWMS}(M)}(θ)) \cdot (-\sin θ, \cos θ). \]
Therefore the tangent line to $M$ at $\gamma(\theta)$ is parallel to the tangent line to CWMS($M$) at non-singular point $\gamma_{\text{CWMS}(M)}(\theta)$.

By (2.5) one can get that
\[
\rho_M(\theta) = p(\theta) + p''(\theta)
= a_0 - \sum_{n=1}^{\infty} (n^2 - 1)(a_n \cos n\theta + b_n \sin n\theta),
\]
\[
\rho_M(\theta + \pi) = a_0 - \sum_{n=1}^{\infty} (-1)^n(n^2 - 1)(a_n \cos n\theta + b_n \sin n\theta).
\]

By (3.4), (3.6), (3.7) one can get that
\[
\rho_{\text{CWMS}(M)}(\theta) = |p_{\text{CWMS}(M)}(\theta) + p''_{\text{CWMS}(M)}|
= \left| -2 \sum_{n=2, n \text{ is even}}^{\infty} (n^2 - 1)(a_n \cos n\theta + b_n \sin n\theta) \right|
= |\rho_M(\theta) + \rho_M(\theta + \pi) - 2a_0|
= |\rho_M(\theta) + \rho_M(\theta + \pi) - \overline{w}_M|.
\]

The signed curvature of CWMS($M$) at $\gamma_{\text{CWMS}(M)}(\theta)$ is in the following form:
\[
\kappa_{\text{CWMS}(M)}(\theta) = \frac{1}{\rho_{\text{CWMS}(M)}(\theta)}
= \frac{\kappa_M(\theta)\kappa_M(\theta + \pi)}{|\kappa_M(\theta) + \kappa_M(\theta + \pi) - \overline{w}_M \cdot \kappa_M(\theta) \cdot \kappa_M(\theta + \pi)|}.
\]

By Theorem 3.3 one can get the following corollaries.

**Corollary 3.4.** Let $M$ be a positively oriented oval. Then the curvature of CWMS($M$) is positive on each regular connected component of CWMS($M$).

**Corollary 3.5.** Let $M$ be a positively oriented oval of an average width equal to $\overline{w}_M$. Let $n$ be a continuous unit normal vector field to $M$. Let $a, b$ in $M$ be a parallel pair. Let $\rho_M(a), \rho_M(b)$ denote the radius of curvature of $M$ at $a, b$, respectively. Then CWMS($M$) is singular at the point $a - b + \overline{w}_M \cdot n(a)$ if and only if
\[
\rho_M(a) + \rho_M(b) = \overline{w}_M.
\]

**Theorem 3.6.** Let $M$ be a generic oval. Then CWMS($M$) is a connected smooth curve with cusp singularities and the number of cusps is a positive multiple of 4.

**Proof.** CWMS($M$) is a connected smooth curve because of (3.4). From Corollary 3.5 and from theory of Thom (1975) [44] one can get that generically CWMS($M$) has cusp singularity when (3.9) holds. By (3.8) the condition (3.9) can be written in terms of the Fourier series of the Minkowski support function of $M$ in the following way:
\[
\rho_{\text{CWMS}(M)}(\theta) = 0 \iff \sum_{n=2, n \text{ is even}}^{\infty} (n^2 - 1)(a_n \cos n\theta + b_n \sin n\theta) = 0.
\]
Therefore the number of cusps of CWMS(M) is the number of zeros of (3.10) in the range 0 to 2π. Since ρ_{CWMS(M)}(0) = ρ_{CWMS(M)}(π), the number of cusps in the range from 0 to π must be even. The function ρ_{CWMS(M)}(θ) is π-periodic, hence the number of cusps in the range from 0 to 2π is a multiple of 4. Moreover, the number of cusps must be at least 4, since the Fourier series in (3.10) begins with terms in cos 2θ and sin 2θ. □

Definition 3.7. The tangent line to the CWMS(M) at a cusp point p is the limit of a sequence of T_{q_n} M for any sequence q_n of regular points of CWMS(M) converging to p.

It is easy to see that this definition does not depend on the choice of a converging sequence of regular points. From Theorem 3.3(i) we can see that the tangent line to CWMS(M) at the cusp point a − b + w_M · n(a) is parallel to the tangent lines to M at a and b.

It is easy to see that if M is a generic oval, then for any line l in \( \mathbb{R}^2 \) there exists exactly two points a, b ∈ M in which tangent lines to M at a and b are parallel to l. Therefore we get the following corollary.

Corollary 3.8. Let M be a generic oval. Then for any line l in \( \mathbb{R}^2 \) there exists exactly two points p, q ∈ CWMS(M) such that tangent lines to CWMS(M) at p and q are parallel to l.

We can define the normal vector to CWMS(M) at any its point a − b + w_M · n(a) in the following way.

Definition 3.9. The normal vector to CWMS(M) at a − b + w_M · n(a) is n(a).

Let us notice that the continuous normal vector field to CWMS(M) at regular and cusp points is perpendicular to the tangent line to CWMS(M). Using this fact and the above definition we define the rotation number in the following way.

Definition 3.10. The rotation number of a smooth closed curve M with at most cusp singularities is the number of rotation of its continuous normal vector field.

Proposition 3.11. Let M be a generic oval. Then the absolute value of the rotation number of CWMS(M) is equal to one.

Proof. It is a consequence of Theorem 3.6 and Corollary 3.8. □

Proposition 3.12. If M is an oval, then CWMS(M) has the center of symmetry.

Proof. Let a, b be a parallel pair of M, then \( p = a − b + w_M \cdot n(a) \in CWMS(M) \) and \( q = b − a + w_M \cdot n(b) \in CWMS(M), \) and because \( n(a) = −n(b) \) one can notice that Θ is the center of symmetry of CWMS(M). □

Proposition 3.13. There exists oval M for which the Constant Width Measure Set has exactly 4n cusp singularities.

Proof. One can check that \( \rho_{4n}(θ) = 4n^2 + 2 + \cos 2nθ \) is a Minkowski support function of an oval \( M_{4n} \) and CWMS(\( M_{4n} \)) has exactly 4n cusps. □

In Fig. 4 there is an example of an oval M for which the Constant Width Measure Set has exactly 12 cusp singularities.
Theorem 3.14. Let $M$ be an oval. Then

$$L_{CWMS(M)} \leq 4L_M.$$

Proof. Let $p(\theta)$ be the Minkowski support function of $M$ and let $p_{CWMS(M)}$ (like in (3.3)) be the support function of CWMS($M$). Then

$$L_{CWMS} = \int_0^{2\pi} |\gamma'_{CWMS(M)}(\theta)|d\theta$$

$$= \int_0^{2\pi} |p_{CWMS(M)}(\theta) + p''_{CWMS(M)}(\theta)|d\theta$$

$$= \int_0^{2\pi} |p(\theta) + p''(\theta) + p(\theta + \pi) + p''(\theta + \pi) - \bar{w}|d\theta$$

$$\leq \int_0^{2\pi} |p(\theta) + p''(\theta)|d\theta + \int_0^{2\pi} |p(\theta + \pi) + p''(\theta + \pi)|d\theta + \int_0^{2\pi} |\bar{w}|d\theta$$

$$= L_M + L_M + 2L_M = 4L_M.$$

4. The Spherical Measure Set

We recall definition of an offset (this set is also known as a parallel set).

Definition 4.1. Let $M$ be a regular positively oriented simply closed curve. An $\alpha$-offset of $M$ is the following set

$$F_{\alpha}(M) = \{ a + \alpha \cdot \mathbf{n}(a) \mid a \in M \},$$

where $\mathbf{n}(a)$ is a continuous unit normal vector field to $M$ at $a$ compatible with the orientation of $M$.
It is well known that offsets generically admit at most cusp singularities, singularities of all offsets of $M$ form an evolute of $M$ and set of all points of self–intersections of offsets forms a medial axis of $M$. See [12, 19, 28, 31, 36] and the literature therein.

**Definition 4.2.** The Spherical Measure Set of a regular positively oriented simply closed curve $M$ is an offset at level $\frac{L_M}{2\pi}$,

\begin{equation}
\text{SMS}(M) = \mathcal{F}_{\\frac{L_M}{2\pi}}(M) = \{ a + \frac{L_M}{2\pi} \cdot n(a) \mid a \in M \},
\end{equation}

where $n$ is the unit normal vector field compatible with the orientation of $M$.

**Remark 4.3.** If $M$ is an oval, then $\frac{L_M}{2\pi} = \frac{w_M}{2}$.

See Fig. 5 and Fig. 6 for examples of SMS.

**Figure 5.** An oval $M$ and SMS($M$).

**Figure 6.** A regular simple closed curve $M$ and SMS($M$).
Proposition 4.4. Let $M$ be a regular positively oriented simply closed curve. Then $\text{SMS}(M) = \{x\}$ if and only if $M$ is a circle and $x$ is its center.

Proof. If $M$ is a circle and $x$ is its center, then it easy to see that $\frac{1}{2}x$ is a radius of $M$ and $\text{SMS}(M) = \{x\}$.

Let $w = \frac{L_M}{\pi}$. Let us assume that $\text{SMS}(M) = \{x\}$. Let $a, b$ be a parallel pair of $M$, then $w \cdot n(a) = 2x - 2a$ and $w \cdot n(b) = 2x - 2b$. If $n(a) = n(b)$, then $a = b$, otherwise if $n(a) = -n(b)$, then $|a - b| = w$ and $a + b = 2x$, so $M$ is a curve of constant width and $x$ is the center of symmetry of $M$, hence $M$ is a circle. □

Proposition 4.5. Let $M$ be a positively oriented regular closed curve. Let $\kappa_M$ denote the curvature of $M$ at $a$. Then

(i) a point $a + \frac{L_M}{2\pi} \cdot n(a)$ is a singular point of $\text{SMS}$ if and only if

\[ L_M \kappa_M(a) = 2\pi, \]

(ii) the tangent line to $\text{SMS}(M)$ at non-singular point $a + \frac{L_M}{2\pi} \cdot n(a)$ is parallel to the tangent line to $M$ at a point $a$.

(iii) the curvature of $\text{SMS}(M)$ at non-singular point $q = a + \frac{L_M}{2\pi} \cdot n(a)$ is equal

\[ \kappa_{\text{SMS}(M)}(q) = \frac{2\pi \kappa_M(a)}{|2\pi - L_M \kappa_M(a)|}. \]

Proof. Let $\gamma(s)$ be the arc length parameterization of $M$ and let $t, b$ be its Frenet frame. Then a parameterization of $\text{SMS}(M)$ is as follows:

\[ \gamma_{\text{SMS}(M)}(s) = \gamma(s) + \frac{L_M}{2\pi} n(s). \]

Then:

\[ \gamma'_{\text{SMS}(M)}(s) = t(s) - \frac{L_M}{2\pi} \cdot \kappa_M(s) t(s) \]

\[ = \frac{1}{2\pi} (2\pi - L_M \kappa_M(s)) t(s). \]

Therefore (i), (ii) hold.

The curvature of $\text{SMS}(M)$ at non-singular $\gamma_{\text{SMS}(M)}(s)$ is equal to

\[ \kappa_{\text{SMS}(M)}(s) = \frac{\det (\gamma'_{\text{SMS}(M)}(s), \gamma''_{\text{SMS}(M)}(s))}{|\gamma'_ {\text{SMS}(M)}(s)|^3} \]

\[ = \frac{2\pi \kappa_M(s)}{|2\pi - L_M \kappa_M(s)|}. \]

□

Theorem 4.6. Let $M$ be a generic regular closed curve. Then the number of cusp singularities of $\text{SMS}(M)$ is even and not smaller than 2. If $M$ is a generic oval, then the number of cusp singularities of $\text{SMS}(M)$ is not smaller than 4.

Proof. Let us assume that $M$ is positively oriented.

By theory of Thom (1975) [44] one can get that generically $\text{SMS}(M)$ has cusp singularity when (4.2) holds.
Figure 7. The cusp singularity with a continuous normal vector field. Vectors in upper regular component of a curve are directed outside the cusp, others are directed inside the cusp.

A continuous normal vector field to the germ of a curve with the cusp singularity is directed outside the cusp on the one of two connected regular components and is directed inside the cusp on the other component as it is showed in Fig. 7.

Without loss of generality, let us assume that $M$ is positively oriented. It is easy to see that the rotation number of $\text{SMS}(M)$ is equal to the rotation number of $M$, which is an integer. Thus, the number of cusps of $\text{SMS}(M)$ is even.

Let us assume that $\text{SMS}(M)$ is regular. Let $[0,L_M] \ni s \mapsto \gamma(s) \in \mathbb{R}^2$ be the arc length parameterization of $M$. Then by Proposition 4.5(i) we get that the curvature at each point of $M$ is smaller or greater than $\frac{2\pi}{L_M}$. If $\kappa_M(s) < \frac{2\pi}{L_M}$ for all $s$, then

$$\int_0^{L_M} \kappa_M(s) ds < \int_0^{L_M} \frac{2\pi}{L_M} ds.$$

We get the same result if we assume that $\kappa_M(s) > \frac{2\pi}{L_M}$ for all $s$, therefore $\text{SMS}(M)$ got at least one cusp singularity.

Let $M$ be a generic oval.

Let (2.1) be the parameterization of $M$ and $p(\theta)$ its Minkowski support function.

By Remark 2.4 there are at least 3 cusps of the Wigner caustic of $M$, then there exist $\theta_1 < \theta_2 < \theta_3$ such that $\rho_M(\theta_i) = \rho_M(\theta_i + \pi)$ for $i = 1, 2, 3$ and by Corollary 3.5 there are at least 4 cusps of the Constant Width Measure Set of $M$, then there exists $\varphi_1 < \varphi_2$ such that $\rho_M(\varphi_j) + \rho_M(\varphi_j + \pi) = \overline{w}_M$ for $j = 1, 2$, therefore it is easy to see that there are at least four values of $\theta \in S^1$ such that $\rho_M(\theta) = \frac{1}{2} \overline{w}_M$, hence there are at least 4 cusps of the Spherical Measure Set of $M$.

Remark 4.7. The minimal number of cusp singularities of a $\text{SMS}(M)$ of a generic regular closed simple curve $M$ cannot be 4 - see an example in Fig. 8.

By Theorem 4.6 we get another proof of the classical Four–Vertex Theorem for ovals.

Corollary 4.8. Let $M$ be a generic oval. Then $M$ has at least four vertices.
Proof. It is a consequence of Roll’s theorem and the fact that there are at least 4 points on \( M \) in which the curvature is equal to \( \frac{2\pi}{L_M} \) - see Proposition 4.5(i) and Theorem 4.6.

**Proposition 4.9.** If \( M \) is an oval of constant width, then \( SMS(M) = E_\frac{1}{2}(M) \). Thus, the map \( M \ni \gamma(\theta) \mapsto \gamma_{SMS(M)}(\theta) \in SMS(M) \) is the double covering of \( SMS(M) \).

If \( M \) is a curve with the center of symmetry, then \( CWMS(M) \) and \( SMS(M) \) are similar curves and the ratio of symmetry between them is equal to 2.

Proof. Let \( p \) be the Minkowski support function of \( M \). Then one can check that the Minkowski support function of \( SMS(M) \) is in the form:

\[
p_{SMS(M)}(\theta) = p(\theta) - \frac{1}{2} w_M.
\]

(4.4)

In terms of coefficients of the Fourier series of \( p(\theta) \):

\[
p_{SMS(M)}(\theta) = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).
\]

(4.5)

Then Proposition 4.9 is a consequence of the support functions of \( E_\frac{1}{2}(M) \), \( CWMS(M) \), \( SMS(M) \) in terms of coefficients of the Fourier series of \( p(\theta) \) - see (2.11), (3.4), (4.5).
Corollary 4.10. Let $M$ be an oval such that $\text{SMS}(M)$ is a curve with at most cusp singularities. If $M$ is not a curve of constant width, then the absolute value of the rotation number of $\text{SMS}(M)$ is equal to one, otherwise it is equal to $\frac{1}{2}$.

Proposition 4.11. There exists oval $M$ for which the Constant Width Measure Set has exactly $n$ cusp singularities for $n \geq 3$.

Proof. Let $n \geq 3$ and let

$$p_n(\theta) = \begin{cases} n^2 + 2 + \cos n\theta & \text{if } n \equiv 1(\text{mod } 2) \\ (\frac{n}{2})^2 + 2 + \cos \frac{n\theta}{2} & \text{if } n \equiv 0(\text{mod } 4) \\ n^2 + 2 + \cos \frac{(n-2)\theta}{2} + \cos \frac{n\theta}{2} & \text{if } n \equiv 2(\text{mod } 4) \end{cases}$$

be a Minkowski support function of an oval $M_n$. Then one can check that $\text{SMS}(M_n)$ has exactly $n$ cusp singularities. \hfill \square

In Fig. 9 there is an example of an oval $M$ for which $\text{SMS}(M)$ has exactly 10 cusp singularities.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_oval}
\caption{An oval $M_{10}$ and $\text{SMS}(M_{10})$ which has 10 cusp singularities. The support function of $M_{10}$ is $p_{10}(\theta) = 102 + \cos 4\theta + \cos 5\theta$.}
\end{figure}

Theorem 4.12. Let $M$ be an oval. Then

\begin{equation}
L_{\text{SMS}(M)} \leq 2L_M
\end{equation}

and if $M$ is an oval of constant width, then

\begin{equation}
L_{\text{SMS}(M)} \leq L_M
\end{equation}

Proof. Let $p(\theta)$ be the Minkowski support function of $M$ and let $p_{\text{SMS}(M)}$ (like in (4.4)) be the support function of $\text{SMS}(M)$. Let $M$ is not an oval of constant width.
Then
\begin{align*}
L_{\text{SMS}} &= \int_0^{2\pi} |\gamma'_{\text{SMS}}(\theta)| \, d\theta \\
&= \int_0^{2\pi} |p_{\text{SMS}}(\theta) + p''_{\text{SMS}}(\theta)| \, d\theta \\
&= \int_0^{2\pi} |p(\theta) - \frac{1}{2} M_M' + M_M''| \, d\theta \\
&\leq \int_0^{2\pi} |p(\theta) + M_M''| \, d\theta + \frac{1}{2} \int_0^{2\pi} M_M' \, d\theta \\
&= L_M + L_M = 2L_M.
\end{align*}

By (4.8) and Proposition 4.9 the inequality (4.7) holds. \(\square\)

5. Isoperimetric equalities

**Theorem 5.1.** (Isoperimetric equality 1) Let \(M\) be a closed regular simple convex planar curve. Then
\begin{align*}
L_M^2 &= 4\pi A_M + 8\pi \left|\bar{A}_{E_2}(M)\right| + \pi \left|\bar{A}_{\text{CWMS}}(M)\right|,
\end{align*}
where \(L_M, A_M, \bar{A}_{E_2}(M), \bar{A}_{\text{CWMS}}(M)\) are the length of \(M\), the area bounded by \(M\), the oriented area of the Wigner caustic of \(M\) and the oriented area of the Constant Width Measure Set of \(M\), respectively.

**Proof.** Let \(p(\theta)\) be the Minkowski support function of \(M\) and let (2.5) be its Fourier series. Then (3.4) is the Fourier series of the Minkowski support function \(p_{\text{CWMS}}(M)(\theta)\) of \(\text{CWMS}(M)\).

The oriented area of \(\text{CWMS}(M)\) in terms of coefficients of the Fourier series of \(p\) is equal
\begin{align*}
\bar{A}_{\text{CWMS}}(M) &= \frac{1}{2} \int_0^{2\pi} \left(p^2_{\text{CWMS}}(\theta) - p'^2_{\text{CWMS}}(\theta)\right) \, d\theta \\
&= -2\pi \sum_{n=2, \text{n is even}}^{\infty} (n^2 - 1)(\alpha_n^2 + \beta_n^2).
\end{align*}

Then by (2.7), (2.8), (2.12), (5.2) one can easily verify that the equality (6.9) holds. \(\square\)

**Theorem 5.2.** (Isoperimetric equality 2) Let \(M\) be a closed simple regular planar curve. If \(M\) is not a curve of constant width, then
\begin{align*}
L_M^2 &= 4\pi A_M + 4\pi \left|\bar{A}_{\text{SMS}}(M)\right|,
\end{align*}
otherwise
\begin{align*}
L_M^2 &= 4\pi A_M + 8\pi \left|\bar{A}_{\text{SMS}}(M)\right|,
\end{align*}
where \(L_M, A_M, \bar{A}_{\text{SMS}}(M)\) are the length of \(M\), the area bounded by \(M\) and the oriented area of the Spherical Measure Set of \(M\), respectively.
Proof. Let $M$ be not a curve of constant width.

Let $[0,L_M) \ni s \mapsto \gamma(s) \in M$ be an arc length parameterization of $M$. Then $[0,L_M) \ni s \mapsto \gamma_{SMS}(M)(s) = \gamma(s) + \frac{L_M}{2\pi} \cdot n(s) \in SMS(M)$ is a parameterization of $M$.

Let $t,n,b$ be the Frenet frame of $M$.

By Green’s theorem the oriented area of $SMS(M)$ is equal to

$$A_{SMS(M)} = \frac{1}{2} \int_0^{L_M} (\gamma_{SMS}(M)(s) \times \gamma'_{SMS}(M)(s)) \cdot b(s) ds$$

$$= \frac{1}{2} \int_0^{L_M} (\gamma(s) + \frac{L_M}{2\pi} \cdot n(s)) \times \left(1 - \frac{L_M}{2\pi} \kappa(s)\right) t(s) \cdot b(s) ds$$

$$= \frac{1}{2} \int_0^{L_M} \gamma(s) \times t(s) \cdot b(s) ds - \frac{L_M}{2\pi} \int_0^{L_M} b(s) \cdot b(s) ds +$$

$$+ \frac{1}{2} \cdot \frac{L_M^2}{4\pi^2} \int_0^{L_M} \kappa(s) ds$$

$$= A_M - \frac{L_M}{2\pi} \cdot L_M + \pi \cdot \frac{L_M^2}{4\pi^2} = A_M - \frac{L_M^2}{4\pi} \quad \square$$

From the isoperimetric equalities one can get following corollaries.

**Corollary 5.3.** [10] A closed regular simple convex planar curve $M$ is a curve of constant width if and only if

$$L_M^2 = 4\pi A_M + 8\pi \left| A_{E_{1/2}}(M) \right|,$$

where $L_M$, $A_M$, $A_{E_{1/2}}(M)$ are the length of $M$, the area bounded by $M$ and the oriented area of the Wigner caustic of $M$, respectively.

**Corollary 5.4.** A closed regular simple convex planar curve $M$ is a curve which has the center of symmetry if and only if

$$L_M^2 = 4\pi A_M + \pi \left| A_{CWMS(M)} \right|,$$

where $L_M$, $A_M$, $A_{CWMS(M)}$ are the length of $M$, the area bounded by $M$ and the oriented area of the Constant Width Measure Set of $M$, respectively.

**Corollary 5.5.** Let $M$ be a closed simple convex planar curve. If $M$ is not a curve of constant width then

$$4 \left| A_{SMS(M)} \right| = 8 \left| A_{E_{1/2}}(M) \right| + \left| A_{CWMS(M)} \right|,$$

otherwise

$$SMS(M) = E_{1/2}(M), \quad CWMS(M) = \{\Theta\},$$

where $A_{E_{1/2}}(M)$, $A_{CWMS(M)}$, $A_{SMS(M)}$ are the oriented area of the Wigner caustic of $M$, the oriented area of the Constant Width Measure Set of $M$ and the oriented area of the Spherical Measure Set of $M$, respectively.
Proposition 5.6. Let $M$ be a closed simple convex planar curve. If $M$ is not a curve of constant width then

\[(5.8) \quad L_{\text{SMS}}(M) \leq \frac{1}{2} L_{\text{CWMS}}(M) + 2L_{E_{\frac{1}{2}}}(M).\]

Proof. By (2.10), (3.3), (4.4) and Remark 2.7 we get the following:

\[L_{\text{SMS}}(M) = \int_0^{2\pi} |p_{\text{SMS}}(\theta)| d\theta = \int_0^{2\pi} |p(\theta) - \frac{1}{2} w_M| d\theta \]

\[= \frac{1}{2} \int_0^{2\pi} |p(\theta) + p(\theta + \pi) - w_M + p(\theta) - p(\theta + \pi)| d\theta \]

\[= \frac{1}{2} \int_0^{2\pi} |p_{\text{CWMS}}(\theta) + 2p_{E_{\frac{1}{2}}}(\theta)| d\theta \]

\[\leq \frac{1}{2} \int_0^{2\pi} |p_{\text{CWMS}}(\theta)| d\theta + \int_0^{2\pi} |p_{E_{\frac{1}{2}}}(\theta)| d\theta \]

\[= \frac{1}{2} L_{\text{CWMS}}(M) + 2L_{E_{\frac{1}{2}}}(M) \]

\[\square\]

Proposition 5.7. Let $M$ be a generic regular simple closed curve. Then the orientation of $\text{SMS}(M)$ is reversed against the orientation of $M$. If $M$ is a generic oval, then the orientation of $\text{CWMS}(M)$ is reversed against the orientation of $M$.

Proof. Let $M$ be positively oriented. Then the negatively orientation of $\text{CWMS}(M)$ and $\text{SMS}(M)$ is a consequence of (5.2) and (5.5). \[\square\]

In Fig. 10 there is an example of an oval $M$ for which

\[p(\theta) = 115 + 10 \cos 2\theta + \frac{1}{3} \cos 3\theta + \sin 4\theta - 3 \sin 5\theta\]

is its Minkowski support function. Then $E_{\frac{1}{2}}(M)$ has 5 cusp singularities, $\text{CWMS}(M)$ has 4 cusp singularities, $\text{SMS}(M)$ has 10 cusp singularities and their support functions are as follows:

\[p_{E_{\frac{1}{2}}}(\theta) = \frac{1}{3} \cos 3\theta - 3 \sin 5\theta,\]

\[p_{\text{CWMS}}(\theta) = 10 \cos 2\theta + \sin 4\theta,\]

\[p_{\text{SMS}}(\theta) = 10 \cos 2\theta + \frac{1}{3} \cos 3\theta + \sin 4\theta - 3 \sin 5\theta.\]

6. The Stability of Improved Isoperimetric Inequalities

An $n$-dimensional convex body is a bounded convex subset in $\mathbb{R}^n$ which is closed and has interior points. Let $\mathcal{C}^n$ denote the set of all $n$-dimensional convex bodies.

An inequality in the convex geometry can be written as

\[(6.1) \quad \Phi(K) \geq 0,\]
Figure 10. An oval $M$ and $E_2(M)$ (dashed line, 5 cusps), CWMS($M$) (bold line, 4 cusps), SMS($M$) (normal line, 10 cusps). The Minkowski support function of $M$ is equal to $p(\theta) = 115 + 10 \cos 2\theta + \frac{1}{3} \cos 3\theta + \sin 4\theta - 3 \sin 5\theta$. 
where \( \Phi : C^n \to \mathbb{R} \) is a function and the inequality (6.1) holds for all \( K \) in \( C^n \). Let \( C^n_\Phi \) be a subset of \( C^n \) for which equality in (6.1) holds.

Let \( L_{\partial K} \) denote the length of the boundary of \( K \), let \( A_{\partial K} \) denote the area enclosed by \( \partial K \) (i.e. the area of \( K \)).

For example, let \( n = 2 \) and let \( \Phi(K) = L_{\partial K}^2 - 4\pi A_{\partial K} \). Then the inequality \( \Phi(K) \geq 0 \) is the classical isoperimetric inequality in \( \mathbb{R}^2 \). In this case \( C^2_\Phi \) is a set of disks.

In this section we will study stability properties associated with (6.1). We ask if \( K \) must be close to a member of \( C^n_\Phi \) whenever \( \Phi(K) \) is close to zero.

Let \( d : C^n \times C^n \to \mathbb{R} \) denote in some sense the deviation between two convex bodies. It should satisfy two following conditions:

(i) \( d(K,L) \geq 0 \) for all \( K,L \in C^n \),

(ii) \( d(K,L) = 0 \) if and only if \( K = L \).

If \( \Phi, C^n_\Phi \) and \( d \) are given, then the stability problem associated with (6.1) is as follows.

Find positive constants \( c, \alpha \) such that for each \( K \in C^n \), there exists \( N \in C^n_\Phi \) such that

\[
\Phi(K) \geq cg^\alpha(K,N).
\]

Let \( p_{\partial K} \) and \( p_{\partial N} \) be support functions of convex bodies \( K \) and \( N \), respectively.

Usually to measure the deviation between \( K \) and \( N \) one can use the Hausdorff distance,

\[
d_{\infty}(K,N) = \max_{\theta} \left| p_{\partial K}(\theta) - p_{\partial N}(\theta) \right|.
\]

Another such measure is the measure that corresponds to the \( L_2 \)-metric in the function space. It is defined by

\[
d_2(K,N) = \left( \int_0^{2\pi} \left| p_{\partial K}(\theta) - p_{\partial N}(\theta) \right|^2 \, d\theta \right)^{\frac{1}{2}}.
\]

It is easy to see that \( d_{\infty}(K,N) = 0 \) (or \( d_2(K,N) = 0 \)) if and only of \( K = N \).

**Definition 6.1.** [46] Let \( p_M \) be the Minkowski support function of a positively oriented oval \( M \) of length \( L_M \). Then

\[
p_{W_M}(\theta) = \frac{L_M}{2\pi} + \frac{p_M(\theta) - p_M(\theta + \pi)}{2}
\]

will be the support function of a curve \( W_M \) which will be called the Wigner caustic type curve associated with \( M \).

**Proposition 6.2.** [46] Let \( W_M \) be the Wigner caustic type curve associated with an oval \( M \). Then \( W_M \) has the following properties:

(i) \( W_M \) is an oval of constant width,

(ii) \( L_{W_M} = L_M \),

(iii) \( E_2(W_M) = E_2(M) \),

(iv) \( A_{W_M} \geq A_M \) and the equality holds if and only if \( M \) is a curve of constant width,

(v) \( W_M = M \) if and only if \( M \) is a curve of constant width.

In [46] we show the stability properties of Theorem 1.2.
Theorem 6.3. Let $K$ be a strictly convex domain of area $A_{\partial K}$ and perimeter $L_{\partial K}$ and let $\tilde{A}_{E_2}(\partial K)$ denote the oriented area of the Wigner caustic of $\partial K$. Let $W_K$ denote the convex body for which $\partial W_K$ is the Wigner caustic type curve associated with $\partial K$. Then
\begin{equation}
L_{\partial K}^2 - 4\pi A_{\partial K} - 8\left|\tilde{A}_{E_2}(\partial K)\right| \geq 4\pi^2 d_\infty^2(K, W_K),
\end{equation}
where equality holds if and only if $\partial K$ is a curve of constant width.

Theorem 6.4. Under the same assumptions of Theorem 6.3, one gets
\begin{equation}
L_{\partial K}^2 - 4\pi A_{\partial K} - 8\pi \left|\tilde{A}_{E_2}(\partial K)\right| \geq 6\pi d_2^2(K, W_K),
\end{equation}
where equality holds if and only if $\partial K$ is a curve of constant width, or the Minkowski support function of $\partial K$ is in the form
\begin{equation*}
p_{\partial K}(\theta) = a_0 + a_2 \cos 2\theta + b_2 \sin 2\theta + \sum_{n=1, n \text{ is odd}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).
\end{equation*}

From Theorem 6.3, Theorem 6.8 and 5.1 we can get the following result.

Corollary 6.5. Under the same assumptions of Theorem 6.3, one gets
\begin{equation}
\left|\tilde{A}_{\text{CWMS}}(\partial K)\right| \geq \max\left\{ 4\pi d_\infty^2(K, W_K), 6\pi d_2^2(K, W_K) \right\}.
\end{equation}

From Theorem 6.6, we can easily obtain the second improved isoperimetric inequality (see Theorem 6.6), study its stability property and obtain similar results like these in Corollary 6.5, but for the Wigner caustic of an oval.

Theorem 6.6. (Improved isoperimetric inequality 2) Let $M$ be a closed regular simple convex planar curve. Then
\begin{equation}
L_M^2 \geq 4\pi A_M + \pi \left|\tilde{A}_{\text{CWMS}}(M)\right|,
\end{equation}
where $L_M, A_M, \tilde{A}_{\text{CWMS}}(M)$ are the length of $M$, the area bounded by $M$ and the oriented area of the Constant Width Measure Set of $M$, respectively. The equality in (6.9) holds if and only if $M$ is a centrally symmetric.

From this point let us assume that $n = 2$ and by Theorem 6.6 let
\begin{equation}
\Phi(K) = L_{\partial K}^2 - 4\pi A_{\partial K} - \pi \left|\tilde{A}_{\text{CWMS}}(\partial K)\right|.
\end{equation}

From Theorem 6.6 one can see that $C_4^2$ consists of centrally symmetric bodies.

Let $K$ be the convex body in $\mathbb{R}^2$. Let $\vec{u}$ be a unit vector in $\mathbb{R}^2$. Then the Steiner point which can be defined in terms of the Minkowski support function of $\partial K$ is as follows:
\begin{equation}
\vec{s}(K) = \frac{1}{\pi} \int_{0}^{2\pi} \vec{u}(\theta) \cdot p_{\partial K}(\theta) d\theta.
\end{equation}

In terms of coefficients of the Fourier series of $p_{\partial K}$ one can notice that $\vec{s}(K) = (a_1, b_1)$.

For more details on the Steiner point of a convex body see [25][41][42].
**Definition 6.7.** Let $K$ be the convex body. Then $S_K = \frac{1}{2}(K + (-K))$ is centrally symmetric convex body, where $+$ is the Minkowski addition and $-K = \{-k \mid k \in K\}$. Often $S_K$ is called the Steiner symmetral of $K$, and the process of generating the set from $K$ is known as symmetrisation. Let as translate $S_K$ such that the center of $S_K$ becomes the Steiner point of $K$.

The support function of $\partial S_K$ is equal to

$$p_{\partial S_K}(\theta) = s(K) + \frac{p_{\partial K}(\theta) + p_{\partial K}(\theta + \pi)}{2}$$

and in terms of coefficients of the Fourier series of $p_{\partial K}$:

$$p_{\partial S_K}(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + \sum_{n=2, \text{n is even}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

**Theorem 6.8.** Let $K$ be strictly convex domain of the area $A_{\partial K}$ and the perimeter $L_{\partial K}$ and let $\tilde{A}_{CWMS}(\partial K)$ denote the oriented area of the Constant Width Measure Set of $\partial K$. Let $S_K$ be the Steiner symmetral of $K$. Then

$$L_{\partial K}^2 - 4\pi A_{\partial K} - \pi |\tilde{A}_{CWMS}(\partial K)| \geq 8\pi^2 d_\infty^2(K, S_K),$$

where equality holds if and only if $\partial K$ is centrally symmetric.

**Proof.** By (2.5), (2.6), (2.3), (2.4), one can get the Fourier series of $\Phi$ (see (6.10)):

$$\Phi(K) = L_{\partial K}^2 - 4\pi A_{\partial K} - 8\pi |\tilde{A}_{CWMS}(\partial K)|$$

$$= 2\pi^2 \sum_{n=3, \text{n is odd}}^{\infty} (n^2 - 1)(a_n^2 + b_n^2).$$

One can check that $|a_n \cos n\theta + b_n \sin n\theta| \leq \sqrt{a_n^2 + b_n^2}$ and then by (6.3) and Hölder’s inequality:

$$d_\infty(K, S_K) = \max_\theta \left| p_{\partial K}(\theta) - p_{\partial S_K}(\theta) \right|$$

$$= \max_\theta \left| \sum_{n=3, \text{n is odd}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \right|$$

$$\leq \max_\theta \left( \sum_{n=3, \text{n is odd}}^{\infty} |a_n \cos n\theta + b_n \sin n\theta| \right)$$

$$\leq \sum_{n=3, \text{n is odd}}^{\infty} \frac{1}{\sqrt{n^2 - 1}} \cdot \sqrt{n^2 - 1} \sqrt{a_n^2 + b_n^2}$$
\[\\sqrt{\sum_{n=3, n \text{ is odd}}^{\infty} \frac{1}{n^2 - 1}} \cdot \sqrt{\sum_{n=3, n \text{ is odd}}^{\infty} (n^2 - 1)(a_n^2 + b_n^2)}\\]

\[= \sqrt{\frac{1}{4}} \cdot \sqrt{\Phi(K)}.\]

And the equality holds if and only if \(a_2m+1 = b_2m+1 = 0\) for all \(m \in \mathbb{N}\), so \(\partial K\) is centrally symmetric. \qed}

**Theorem 6.9.** Under the same assumptions of Theorem 6.8 one gets

\[(6.16) \quad L_{\partial K}^2 - 4\pi A_{\partial K} - 8 \left| \widetilde{A}_{\text{CWMS}}(\partial K) \right| \geq 16\pi d_2^2(K, S_K),\]

where equality holds if and only if \(\partial K\) is centrally symmetric or the Minkowski support function of \(\partial K\) is in the form

\[p_{\partial K}(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_3 \cos 3\theta + b_3 \sin 3\theta + \sum_{n=2, n \text{ is even}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).\]

**Proof.** By (6.13) and (6.15)

\[d_2^2(K, S_K) = \int_0^{2\pi} \left| p_{\partial K}(\theta) - p_{\partial S_K}(\theta) \right|^2 d\theta = \int_0^{2\pi} \left( \sum_{n=3, n \text{ is odd}}^{\infty} (a_n \cos n\theta + b_n \sin \theta) \right)^2 d\theta = \pi \sum_{n=3, n \text{ is odd}}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{16\pi} \cdot 2\pi^2 \sum_{n=3, n \text{ is odd}}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) = \frac{1}{16\pi} \Phi(K).\]

And the equality holds if and only if \(a_2m+1 = b_2m+1 = 0\) for all \(m \in \mathbb{N}\), so \(\partial K\) is a centrally symmetric curve, or \(p_{\partial K}(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_3 \cos 3\theta + b_3 \sin 3\theta + \sum_{n=2, n \text{ is even}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).\) \qed

**Corollary 6.10.** Under the same assumptions of Theorem 6.8 one gets

\[(6.17) \quad \left| \widetilde{A}_{E_{2}(\partial K)} \right| \geq \max \left\{ \pi d_\infty^2(K, S_K), 2d_2^2(K, S_K) \right\},\]
\[ A_{\text{SMS}}(\partial K) \geq \max \left\{ 2\pi d^2_\infty(K, S_K), 2\pi d^2_\infty(K, W_K), 4d^2_2(K, S_K) + \pi d^2_\infty(K, W_K), 4d^2_2(K, S_K) + \frac{3}{2} d^2_2(K, W_K) \right\}, \]

where (6.18) holds whenever \( \partial K \) is not a curve of constant width.

Proof. It is an easy consequence of isoperimetric inequalities (see Theorem 5.1–5.2 and Corollary 5.5) and stability properties of improved isoperimetric inequalities (see Corollary 6.5 and Theorem 6.8–6.9).

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