QUADRATIC HECKE SUMS AND MASS EQUIDISTRIBUTION

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Abstract. We consider the analogue of the quantum unique ergodicity conjecture for holomorphic Hecke eigenforms on compact arithmetic hyperbolic surfaces. We show that this conjecture follows from nontrivial bounds for Hecke eigenvalues summed over quadratic progressions. Our reduction provides an analogue for the compact case of a criterion established by Luo–Sarnak for the case of the non-compact modular surface. The novelty is that known proofs of such criteria have depended crucially upon Fourier expansions, which are not available in the compact case. Unconditionally, we establish a twisted variant of the Holowinsky–Soundararajan theorem involving restrictions of normalized Hilbert modular forms arising via base change.

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1. Introduction

1.1. Context. Let $B$ be a quaternion algebra over $\mathbb{Q}$. We assume that $B$ splits over $\mathbb{R}$, and fix an identification of the real completion $B_\infty := B \otimes_{\mathbb{Q}} \mathbb{R}$ with the matrix algebra $M_2(\mathbb{R})$. Let $R$ be a maximal order in $B$. Let $\mathbb{H}$ denote the upper half-plane. Let $R^{(1)}$ denote the group of norm one units in $R$, regarded as a subgroup of $\text{SL}_2(\mathbb{R})$, and write $Y := R^{(1)} \backslash \mathbb{H}$ for the corresponding finite volume arithmetic hyperbolic surface.

The space $Y$ exhibits an important dichotomy according to whether $B$ is split (over $\mathbb{Q}$). In the split case, we may identify $B$ with the $2 \times 2$ matrix algebra $M_2(\mathbb{Q})$ and choose $R = M_2(\mathbb{Z})$, so that $R^{(1)} = \text{SL}_2(\mathbb{Z})$. The quotient $Y = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is then non-compact, and modular forms on $Y$ enjoy Fourier expansions $\sum a_n e(nz)$ ($e(z) := e^{2\pi i z}$) corresponding to their invariance under the substitution $z \mapsto z + 1$. 

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generating the stabilizer of the cusp at $\infty$. In the non-split case, the quotient $Y$ is compact, and such expansions are not available. For analytic problems involving such quotients $Y$, the split case is often more technically complicated due to the non-compactness of $Y$ and the continuous part of the spectral decomposition of $L^2(Y)$, but this technical complication is compensated for by the existence of Fourier expansions, which have proven to be a useful analytic tool. This work continues a series of works [Ne3, Ne4, Ne5, Ne7] studying the non-split case of problems that had previously been understood only in the split case by means of Fourier expansions.

We turn to the main subject matter of this paper. Lindenstrauss [Li] and Soundararajan [So1], addressing a case of the quantum unique ergodicity conjecture of Rudnick–Sarnak [RS], showed that cuspidal Hecke–Laplace eigenfunctions on $Y$ have equidistributed $L^2$-mass in the large eigenvalue limit. We consider here the analogous problem for holomorphic forms. Let $(\phi_k)$ be a sequence, indexed by a sequence of large enough even integers $k$, consisting of (nonzero) cuspidal holomorphic Hecke eigenforms $\phi_k$ on $Y$ of weight $k$.

**Conjecture 1.1.** For each bounded continuous $\Psi : Y \to \mathbb{C}$, we have

$$\frac{\int_Y |\phi_k(z)|^2 \Psi(z) \, dx \, dy}{\int_Y |\phi_k(z)|^2 \, dx \, dy} \to \frac{\int_Y \Psi(z) \, dx \, dy}{\int_Y \, dx \, dy}$$

as $k \to \infty$.

By Watson’s formula [Wa], this conjecture follows from the generalized Lindelöf hypothesis (or indeed, from any subconvex bound for the associated family of triple product $L$-functions (1.26)). Sarnak [Sa] established the special case of this conjecture in which the $\phi_k$ are dihedral. The general split case (formulated by Luo–Sarnak [LS]) is a celebrated result of Holowinsky–Soundararajan [HS]. The general non-split case remains open.

The work of Holowinsky–Soundararajan synthesizes two complementary methods developed independently by Soundararajan [So2] and Holowinsky [HS]. The method of Soundararajan applies just as well to the non-split case, while the method of Holowinsky does not. The latter method departs by reducing the problem to suitable estimates for shifted convolution sums

$$\sum_{n} f(n/k) \lambda_{\phi_k}(n) \lambda_{\phi_k}(n+\ell).$$

Here $f \in C_c^\infty(\mathbb{R}_+^\times)$ is a fixed test function, $\ell$ is a fixed integer, and $\lambda_{\phi_k} : \mathbb{N} \to \mathbb{C}$ describes the Hecke eigenvalues of $\phi_k$, normalized so that the Deligne bound reads $|\lambda_{\phi_k}(p)| \leq 2$ for primes $p$. The following criterion, established by Luo–Sarnak [LS, Cor 2.2], clarifies the relationship between Conjecture 1.1 and bounds for such sums.

**Theorem 1.2** (Luo–Sarnak). Assume that $B$ is split. Then Conjecture 1.1 holds if and only if for each $f \in C_c^\infty(\mathbb{R}_+^\times)$ and $\ell \in \mathbb{Z}$,

$$\lim_{k \to \infty} \frac{\zeta(2)}{kL(ad \phi_k, 1)} \sum_{n \in \mathbb{N}} f(n/k) \lambda_{\phi_k}(n) \lambda_{\phi_k}(n+\ell) = \int_0^\infty f(y) \, dy.$$  (1.3)

Here and henceforth $L(\cdot \cdot \cdot, s)$ denotes the finite part of an $L$-function, excluding archimedean factors.
The proofs of Theorem 1.2 and its variant due to Holowinsky [HS, Thm 1] are based upon an analysis of Fourier expansions and the associated Poincaré series. Indeed, for a certain Poincaré series Ψ attached to ℓ and f, the left hand sides of (1.3) and (1.1) are asymptotic, while the right hand sides are equal. In particular, those proofs are fundamentally limited to the split case. It is natural to ask whether some criterion analogous to Theorem 1.2 might exist in the non-split case. As a first hint, we note that by Hecke multiplicativity and Möbius inversion, estimates for the shifted sums (1.2) are substantially equivalent to those for the expressions

\[ \sum_{n} f(n/k)\lambda_{\varphi_k}(Q(n)) \]  

when Q is a reducible quadratic polynomial of the form Q(n) = n(n + ℓ). By analogy, one might speculate that the non-split case of Conjecture 1.1 should be related, somehow, to the sums (1.4) for irreducible quadratic polynomials Q. It is perhaps less clear how one might prove such a relationship.

1.2. Results. The main purpose of this article is to confirm the speculation indicated above. Informally, our first main result reduces the non-split case of Conjecture 1.1 to upper bounds for (1.4) of the shape

\[ o_{k \to \infty}(kL(ad \varphi_k, 1)), \]  

for irreducible Q, with “polynomial dependence” upon Q and f.

Definition 1.3. An integer-valued quadratic polynomial we mean a polynomial Q ∈ ℚ[x] of the form Q(x) = ax² + bx + c, with a ≠ 0, such that Q(n) ∈ ℤ for all n ∈ ℤ. Such a polynomial is irreducible precisely when it has no rational roots. We set \( ||Q|| := \max(|a|, |b|, |c|) \).

Definition 1.4. For \( f \in C^\infty_c(\mathbb{R}^\times) \), we write \( f' \) for the derivative, \( Df(y) := yf'(y) \) for the invariant derivative, and \( S_N(f) \) for the Sobolev norms defined for \( N \in \mathbb{Z}_{\geq 0} \) by the formula

\[ S_N(f) := \max_{j \leq N} \sup_{y \in \mathbb{R}^\times} (y + 1/y)^N |D^j f(y)|. \]  

We recall a special case of a result of Templier–Tsimerman [TT, Thm 1], generalizing and extending earlier work of Blomer [Bl] and Templier [Te].

Theorem 1.5 (Templier–Tsimerman). Let π be a non-dihedral cuspidal automorphic representation of GL₂. Write L(π, s) = \( \sum_{n \geq 1} \lambda_{\varphi}(n)/n^s \). Let Q be a non-split polynomial of the form Q(n) = n² − d for some (non-zero, non-square) integer d. Assume that f is supported in the interval (1, 2) and that \( S_N(f) \ll N \) for each fixed N. Then for \( \varepsilon > 0 \) and \( X \gg |d|^{1/2} \),

\[ \sum_{n} \lambda_{\varphi}(Q(n))f(n/X) \ll_{\varphi, \varepsilon} X^{1/2 + \varepsilon}|d|^\delta \left( \frac{X}{|d|^{1/2}} \right)^\delta. \]  

The exponents \( \vartheta, \delta \) are described in [TT, Thm 1]; they quantify bounds towards the Ramanujan conjecture in both integral and half-integral weight, for which the latest records are \( \vartheta \leq 7/64 \) [Ki, BB] and \( \delta \leq 1/6 \) [CI, Pe, Ne].

Because the estimate (1.7) depends in an unspecified manner upon the eigenform \( \varphi \), it does not apply to sums like (1.4) in which the eigenform and the length of summation vary simultaneously. We might nevertheless expect, in the absence of
obvious evidence to the contrary, that the sums (1.4) enjoy square-root cancellation. In particular, we might extrapolate Theorem 1.5 to the following conjecture.

**Conjecture 1.6.** Assume that the \( \varphi_k \) are non-dihedral.\(^1\) There exists \( A \geq 0 \) so that for all sequences of irreducible integer-valued quadratic polynomials \( Q_k \) and test functions \( f_k \in C^\infty(\mathbb{R}_+^\times) \) satisfying \( S_N(f_k) \ll_N 1 \) for each fixed \( N \),

\[
\lim_{k \to \infty} \frac{\sum_n \lambda_{\varphi_k}(|Q_k(n)|) f_k(n/k)}{k L(\text{ad} \varphi_k, 1)\|Q_k\|^A} = 0. \tag{1.8}
\]

The quantities \( Q_k \) and \( f_k \) in Conjecture 1.6 are allowed to vary with \( k \), but standard estimates for \( |\lambda_\varphi| \) and \( L(\text{ad} \varphi_k, 1) \) show that the desired estimate (1.8) is nontrivial only if they vary at most mildly. Conjecture 1.6 should thus be regarded as a mild strengthening of the true analogue of (1.3) obtained by taking \( Q_k \) and \( f_k \) independent of \( k \). For instance, the Deligne bound \( |\lambda_\varphi(n)| \leq \tau(n) \) (here \( \tau \) denotes the divisor function), the divisor bound \( \tau(n) \ll n^\varepsilon \) and the Hoffstein–Lockhart bound \( L(\text{ad} \varphi_k, 1) \gtrsim \varepsilon^{-1} k^{-\varepsilon} \) give what one might call the trivial bound

\[
\frac{\sum_n |\lambda_{\varphi_k}(|Q_k(n)|) f_k(n/k)|}{k L(\text{ad} \varphi_k, 1)} \ll \varepsilon k^\varepsilon. \tag{1.9}
\]

Using (1.9), or even the much weaker estimate obtained by replacing \( k^\varepsilon \) with \( k^{O(1)} \), we may reduce the verification of (1.8) to the case that for all \( \varepsilon > 0 \),

\[
\|Q_k\| \ll \varepsilon k^\varepsilon, \quad \text{supp}(f_k) \subseteq [k^{-\varepsilon}, k^{\varepsilon}] \text{ for } k \geq k_0(\varepsilon).
\]

We may reduce further using the following more precise estimate:

**Theorem 1.7 (Sieve bound).** Under the hypotheses of Conjecture 1.6, we have

\[
\frac{\sum_n |\lambda_{\varphi_k}(|Q_k(n)|) f_k(n/k)|}{k L(\text{ad} \varphi_k, 1)} \ll \varepsilon \|Q_k\|^{O(1)} (\log k)^{1/2+\varepsilon}, \tag{1.10}
\]

where all implied constants are absolute.

The proof is recorded in §9, following that of Proposition 9.3. This bound allows us to reduce the verification of (1.8) to the case that

\[
\|Q_k\| \ll \varepsilon (\log k)^\varepsilon, \quad \text{supp}(f_k) \subseteq [(\log k)^{-\varepsilon}, (\log k)^\varepsilon] \text{ for } k \geq k_0(\varepsilon). \tag{1.11}
\]

We do not know how to improve upon (1.10) unconditionally. We expect the following stronger estimate.

**Conjecture 1.8.** There exists \( \delta > 0 \) so that under the hypotheses of Conjecture 1.6,

\[
\frac{\sum_n |\lambda_{\varphi_k}(|Q_k(n)|) f_k(n/k)|}{k L(\text{ad} \varphi_k, 1)} \ll \varepsilon \|Q_k\|^{O(1)} (\log k)^{-\delta}. \tag{1.12}
\]

Conjecture 1.8 is motivated by hypothetical uniform forms of the Sato–Tate conjecture (see Remark 1.14 and [Ho1]). Clearly Conjecture 1.8 implies Conjecture 1.6.

We may now state our first main result:

**Theorem A.** Conjecture 1.6 implies Conjecture 1.1.

\(^1\)In fact, since we have restricted to \( \varphi_k \) of full level, the non-dihedrality assumption is automatic. The conjecture generalizes to the case of higher levels, where such an assumption is relevant.
Remark 1.9. It should be possible to formulate an analogue of Conjecture 1.6 in dihedral cases by incorporating a suitable main term (as in [TT]) and in reducible cases by adapting (1.3).

Remark 1.10. The proof of Theorem A shows that to deduce Conjecture 1.1, it is not necessary to know that (1.8) holds for every $Q := Q_k$, but rather for “sufficiently many” $Q$ satisfying the condition

$$Q \text{ is irreducible at every place at which } B \text{ does not split,}$$

and no further splitting conditions. For instance, when $B$ is split (over $\mathbb{Q}$), the condition (1.13) is empty and it suffices to consider reducible $Q$, as follows from the Luo–Sarnak criterion. When $B$ is non-split, the condition (1.13) forces $Q$ to be irreducible.

Remark 1.11. We expect that Conjecture 1.6 also implies the generalization of Conjecture 1.1 to higher fixed level (e.g., taking for $R$ an Eichler order) and to definite quaternion algebras (as in the “QUE on the sphere” problem considered in [BSS]). Conversely, we expect that mildly generalized and strengthened forms of Conjecture 1.1 and Conjecture 1.6 are equivalent, but we do not attempt to formulate such an equivalence here. The analogue of Conjecture 1.6 for the “level $q \to \infty$” aspect as in [Ne1, NPS, Hu] should involve quadratic polynomials that vary considerably with $q$, e.g., $n \mapsto n^2 - q^2 \ell$.

1.3. Applicability of the Holowinsky–Soundararajan method. We address here the tantalizing question of whether Theorem A and the Holowinsky–Soundararajan method suffice to resolve Conjecture 1.1. We will observe a significant discrepancy between the split and non-split cases, arising ultimately from dichotomies of the following sort:

- $n(n + 1)$ almost always has at least two prime divisors, but
- $n^2 + 1$ is expected to be prime infinitely often.

Soundararajan’s results (see [So2], [HS, Lem 2] and [Ne2, §1.3]) show that for $\Psi$ a cuspidal Hecke–Maass form, the conclusion of Conjecture 1.1 holds provided that the Hecke eigenvalues $\lambda := \lambda_{\psi_k}$ satisfy the estimate

$$\frac{\sum_{p \leq k} |\lambda(p)|^2 / p}{\sum_{p \leq k} 1/p} \geq 1/2 + \delta$$

for some fixed $\delta > 0$ (1.14) for large enough $k$. In seeking to prove Conjecture 1.1 for such $\Psi$, we may thus assume without loss of generality (after passing to a subsequence if necessary) that the condition (1.14) fails. By the triangle inequality for the $\ell^2$-norm, we then have

$$\frac{\sum_{p \leq k} (1 - |\lambda(p)|)^2 / p}{\sum_{p \leq k} 1/p} \geq \delta$$

or indeed, for any fixed $\delta < (1 - 1/\sqrt{2})^2$.

Remark 1.12. The condition (1.14) is expected to hold, since a sufficiently uniform form of the Sato–Tate conjecture would imply that $\sum_{p \leq k} |\lambda(p)|^2 / p \sim \sum_{p \leq k} 1/p$, but this expectation seems difficult to establish unconditionally.

We now recall how Holowinsky’s approach [Ho2] establishes the Luo–Sarnak criterion (1.3) for $\ell \neq 0$ under the assumption (1.15). (The case $\ell = 0$ requires an additional “$Y$-thickening” technique (see [Ho2, §3.1] or [Ne3, Lem 5.4]) which
we do not discuss here.) Holowinsky bounds the Hecke eigenvalues in magnitude, forfeiting any potential cancellation in the sums (1.2), and appeals to sieve-theoretic bounds. For simplicity, take $\ell = 1$. We must verify that

$$ \frac{\sum_{n \leq k} |\lambda(n)\lambda(n+1)|}{kL(\Ad \varphi_k, 1)} \tag{1.16} $$

tends to zero as $k \to \infty$. On the one hand, it follows from [HS, Lem 2] that $L(\Ad \varphi_k, 1)$ is bounded from below (possibly up to a $(\log \log k)^{O(1)}$ factor, negligible for the present aims) by $\exp \sum_{p \leq k} (|\lambda(p)|^2 - 1)/p$. On the other hand, a sieve bound due to Nair [Na] gives the estimate

$$ \frac{1}{k} \sum_{n \leq k} |\lambda(n)\lambda(n+1)| \ll \exp \sum_{p \leq k} \frac{2|\lambda(p)| - 2}{p}. \tag{1.17} $$

Thus (1.16) is majorized by

$$ \frac{\exp \sum_{p \leq k} (2|\lambda(p)| - 2)/p}{\exp \sum_{p \leq k} (|\lambda(p)|^2 - 1)/p} = \exp \left( - \sum_{p \leq k} \frac{(1 - |\lambda(p)|)^2}{p} \right). \tag{1.18} $$

If (1.15) holds, then (1.18) decays. Holowinsky–Soundararajan [HS] established the split case of Conjecture 1.1 via similar arguments.

To establish the non-split case of Conjecture 1.1 via Theorem A and the Holowinsky–Soundararajan method would seem to require verifying that if (1.14) fails, then the non-split shifted convolution sums such as

$$ \frac{\sum_{n \leq k} |\lambda(n^2 + 1)|}{kL(\Ad \varphi_k, 1)} \tag{1.19} $$

tend to zero as $k \to \infty$. The sieve bound [Na] analogous to (1.17) reads

$$ \frac{1}{k} \sum_{n \leq k} |\lambda(n^2 + 1)| \ll \exp \sum_{p \leq k : p \equiv 1(4)} \frac{2|\lambda(p)| - 2}{p}, \tag{1.20} $$

so (1.19) is majorized by the ratio

$$ \frac{\exp \sum_{p \leq k : p \equiv 1(4)} (2|\lambda(p)| - 2)/p}{\exp \sum_{p \leq k} (|\lambda(p)|^2 - 1)/p}, \tag{1.21} $$

which we may rewrite up to bounded multiplicative error as

$$ \exp \left( - \sum_{p \leq k : p \equiv 1(4)} \frac{(1 - |\lambda(p)|)^2}{p} + \sum_{p \leq k : p \equiv 3(4)} \frac{1 - |\lambda(p)|^2}{p} \right). \tag{1.22} $$

Unfortunately, we see no way to deduce that such expressions decay. For instance, we see no way to rule out unconditionally that for $p \leq k$,

$$ |\lambda(p)| \approx 1 \text{ for } p \equiv 1(4), \quad |\lambda(p)| \approx 0 \text{ for } p \equiv 3(4), \tag{1.23} $$

in which case (1.14) fails and (1.22) does not decay. Even if the statistical behavior of $\lambda(p)$ is sufficiently unbiased by the residue class of $p$ modulo 4 that we may approximate (1.22) by

$$ \exp \left( \sum_{p \leq k} \frac{(1 - |\lambda(p)|)^2}{2p} + \frac{1 - |\lambda(p)|^2}{2p} \right) = \exp \left( \sum_{p \leq k} \frac{|\lambda(p)|}{p} - \frac{|\lambda(p)|^2}{p} \right), \tag{1.24} $$
then there remain hypothetical problem cases such as when
\[ |\lambda(p)| \approx 1/\sqrt{2} \text{ for } p \leq k. \] (1.25)
Thus some new idea seems necessary to establish Conjecture 1.1 via Theorem A.

Remark 1.13. Holowinsky’s approach differs significantly from that of most literature on the shifted convolution problem (split or non-split), see for instance [Mi, §4.4], Theorem 1.5 and [BH, Bl, Te, TT]. The cited works seek to achieve power savings estimates for sums like (1.2) and (1.4) by exploiting cancellation coming from the variation of the sign of the Hecke eigenvalues, but with the automorphic form \( \varphi_k \) held essentially fixed (i.e., independent of \( k \)) as the length of the sum increases. The estimates obtained in this way are not sufficiently uniform with respect to \( \varphi_k \) to broach the Luo–Sarnak criterion (1.3) or Conjecture 1.6.

Remark 1.14. A sufficiently uniform form of the Sato–Tate conjecture would suggest that for non-dihedral \( \varphi_k \), the collection of Hecke eigenvalues \( \{\lambda(p) : p \leq k\} \) behaves like the random variable \( 2 \cos \theta \), where \( \theta \) is sampled from \([0, \pi]\) with respect to the probability measure \( \frac{2}{\pi} \sin^2 \theta \, d\theta \). In particular, we expect that
\[ \sum_{p \leq k} \frac{|\lambda(p)| - |\lambda(p)|^2}{p} \approx c \log \log k, \quad c := \frac{2}{\pi} \int_0^\pi \left( |2 \cos \theta| - |2 \cos \theta|^2 \right) \sin^2 \theta \, d\theta = \frac{8}{3\pi} - 1 \approx -0.151. \]
Since \( c < 0 \), we do in fact expect the non-split sums (1.19) to decay (like \( (\log k)^c \approx (\log k)^{-0.151} \)), but unlike for the split sums (1.16), there is no apparent way to verify anything approaching this expectation. Compare with [Ho1].

1.4. Main ideas of the proof. We now discuss the proof of Theorem A. We may assume that \( \mathcal{B} \) is non-split and that \( \Psi \) is a cuspidal (i.e., non-constant) Hecke–Maass eigenform. The basic difficulty, relative to existing methods, is that the automorphic forms appearing in the integral on the LHS of (1.1) do not admit Fourier expansions. We aim to relate those integrals to other integrals of automorphic forms that do admit Fourier expansions. This can be achieved using the theta correspondence as in [Ne3, Ne4, Ne5, Ne7], but since we are concerned here only with the magnitude of the integrals, it is more direct to work with \( L \)-functions and period formulas. Watson’s triple product formula relates the squared magnitude of the LHS of (1.1) to the central triple product \( L \)-value
\[ L(\varphi \times \varphi \times \Psi, 1/2), \] (1.26)
so our task is to estimate that \( L \)-value in terms of integrals of automorphic forms that admit Fourier expansions and then to relate such integrals to the Hecke eigenvalues of our original forms.

Naively, one might hope to achieve this aim by simply replacing \( \varphi \) and \( \Psi \) by their Jacquet–Langlands lifts to the \( \text{PGL}_2(\mathbb{Q}) \), but then Prasad’s uniqueness theorem [Pr1, Thm 1.2] implies that the corresponding triple product integrals vanish identically for local reasons, hence carry no information about the \( L \)-value (1.26). (We discuss this point at more length in Remark 5.4.) We must thus look outside the triple product setting.

We indicate two approaches to the problem. The first approach is not developed in detail in this paper, but seems to us to convey most efficiently that Conjectures 1.1 and 1.6 should be connected at all; it is related to our existing work [Ne3, Ne4, Ne5, Ne7]. The second approach is the basis of our proof of Theorem A
(and also of Theorem B, stated below). We expect that the two approaches may be related to one another via a seesaw identity as in [IC1, Prop 5.2] or [Q, Proof of Thm 1].

(1) By the factorization
\[ L(\varphi \times \varphi \times \Psi, 1/2) = L(\text{ad} \varphi \times \Psi, 1/2)L(\Psi, 1/2). \] (1.27)

it suffices to estimate the \( L \)-value \( L(\text{ad} \varphi \times \Psi, 1/2) \). That \( L \)-value appears in Shimura-type integral representations on the metaplectic double cover of \( \text{SL}_2 \) (see [Q, Thm 4.5] and [Ne8]) roughly of the shape
\[ L(\text{ad} \varphi \times \Psi, 1/2) \approx | \int \varphi' k \theta \tilde{\Psi} |^2, \] (1.28)

where
- \( \varphi'_k(z) = \sum_{m \geq 1} \lambda_{\varphi_k}(m)m^{(k-1)/2}e(mz) \) denotes the Jacquet–Langlands lift of \( \varphi_k \) to \( \text{PGL}_2 \),
- \( \theta \) is an elementary theta function, e.g., \( \theta(z) = \sum_{n \in \mathbb{Z}} e(n^2z) \), and
- \( \tilde{\Psi} \) is the Maass–Shintani–Waldspurger theta lift of \( \Psi \).

Since \( \tilde{\Psi} \) is fixed, it suffices to estimate the corresponding integrals obtained by replacing \( \tilde{\Psi} \) by a Poincaré series. Those integrals unfold naturally in terms of the Fourier coefficients of the product
\[ \varphi'_k(z)\theta(z) = \sum_{m \geq 1} \lambda_{\varphi_k}(m)m^{(k-1)/2}e(-m\bar{z}) \sum_{n \in \mathbb{Z}} e(n^2z). \] (1.29)

The \( \ell \)th Fourier coefficient of that product has the shape
\[ \sum_{n} \lambda_{\varphi_k}(n^2 - \ell)f(n). \] (1.30)

It in fact suffices to consider the restricted class of Poincaré series indexed by non-square \( \ell \) together with the elementary theta functions. We thereby encounter sums roughly as in Conjecture 1.6. To implement this strategy would require an analysis of the test vector problem for the integrals (1.28), which we do not address here.

Remark 1.15. By applying the steps outlined here in reverse, it may be possible to establish a “converse” to Theorem A of the sort suggested in Remark 1.11.

(2) Let \( D \) be a non-square quadratic fundamental discriminant. We assume, to eliminate some case analysis, that \( D \) is positive. Let \( (\varphi_k)_D \) denote the weight \( (k, -k) \) Hilbert modular form for \( \text{PGL}_2(\mathbb{Q}(\sqrt{D})) \) obtained from \( \varphi_k \) by quadratic base change (see §4 for details). By restriction, it defines a modular form \( \text{res}((\varphi_k)_D) \) for \( \text{PGL}_2(\mathbb{Q}) \). Let \( \Psi' \) denote the Jacquet–Langlands lift of \( \Psi \) to a newform on \( \text{PGL}_2(\mathbb{Q}) \). Then the twisted triple product formula (§5) relates the squared restriction period
\[ | \int \text{res}((\varphi_k)_D)\Psi'|^2 \]
to the twisted Asai \( L \)-value \( L(\text{asai}((\varphi_k)_D) \times \Psi, 1/2) \), which factors as \( L(\text{ad} \varphi_k) \times \Psi, 1/2)L(\Psi \otimes \chi_D, 1/2) \). Crucially, we may choose \( D \) so that the proportionality constant in this period formula is nonzero and so that
$L(Ψ \otimes χ_D, 1/2)$ is nonzero (§3). We thereby reduce Conjecture 1.1 to suitable bounds for restriction periods. By an “approximate functional equation” for such periods (§6), we may relate them to quadratic sums of Hecke eigenvalues (§7, §8), hence to Conjecture 1.6.

1.5. A twisted variant of arithmetic quantum unique ergodicity. We now describe our second main result, whose relation to Theorem A will be clarified below.

Fix a cuspidal Hecke–Maass eigenform $Ψ$ on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ and a real quadratic field $\mathbb{Q}(\sqrt{D})$ of discriminant $D$. For even $k \geq 12$, let $ϕ_k$ be a (non-dihedral) cuspidal holomorphic Hecke eigenform on $SL_2(\mathbb{Z})$ of weight $k$. Let $(ϕ_k)_D$, as in the proof sketch above, denote the quadratic base change lift of $ϕ_k$ to a weight $(k, -k)$ cuspidal Hilbert modular newform on $PGL_2(\mathbb{Q}(\sqrt{D}))$, (§4). We define the $L^2$-norm $\|((ϕ_k)_D)\|$ by integrating over $PGL_2(\mathbb{Q}(\sqrt{D})) \backslash PGL_2(A_\mathbb{Q}(\sqrt{D}))$ with respect to some Haar measure. (The measure normalization is not important for our purposes.) By restriction, we obtain a function $\text{res}((ϕ_k)_D)$ on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$. We define the restriction period $\int \text{res}((ϕ_k)_D)Ψ$ by integrating over $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ with respect to the standard measure $dx dy / y^2$.

**Theorem B.** For fixed $\varepsilon > 0$, we have

$$\frac{\int \text{res}((ϕ_k)_D)Ψ}{\|((ϕ_k)_D)\|} \ll_\varepsilon (\log k)^{-1/8+\varepsilon}.$$  

(1.31)

Note that if we apply the same construction but with $D = 1$ and $\mathbb{Q}(\sqrt{D})$ replaced by $\mathbb{Q} \times \mathbb{Q}$, then $\text{res}((ϕ_k)_D)(z) = y^k |ϕ_k|^2(z)$ and the left hand sides of (1.1) and (1.31) coincide for suitable measure normalizations. Thus the estimate (1.31) may be understood as a twisted variant of the result [HS, Thm 1 (i)] of Holowinsky–Soundararajan.

The proof of Theorem B is essentially identical to that of Theorem A except that in the final steps, we are left with sums normalized not like (1.19) but instead like

$$\sum_{n \leq k} |λ(n^2 + 1)|$$

(1.32)

(or more precisely their real quadratic analogues involving $χ_D$ and polynomials such as $n^2 - D$). We verify that the Holowinsky–Soundararajan method successfully applies to such sums.

1.6. Plan for this paper. §2–§8 are devoted to the proof of Theorem A, following the sketch indicated in §1.4. §9 gives the proof of Theorem B, borrowing many results from the previous sections.

2. Notation and preliminary reductions

We adopt the setting of Conjecture 1.1. To simplify notation, we drop the subscripts $k$, thus $ϕ := ϕ_k$.

By the Holowinsky–Soundararajan theorem, we may and shall assume that $B$ is non-split. The span of the constant functions and the Hecke–Maass cusp forms is then dense in the space of continuous functions on $Y$ equipped with the supremum norm, so it suffices to consider the case that $Ψ$ is a Hecke–Maass cusp form.
For any prime \( p \), the Hecke operator \( T_p \) is defined as follows. For a function \( f : Y \to \mathbb{C} \), we set \( T_p f(z) := \sum_{\gamma \in \Gamma} R_p(\gamma) f(\gamma z) \), where \( R_p(\gamma) := \{\gamma \in R : \det(\gamma) = p\} \).

Let \( \text{ram}(B) \) denote the set of finite primes at which the quaternion algebra \( B \) does not split. Since \( B \) splits at \( \infty \), we know that \( \text{ram}(B) \) is a finite set of even cardinality. For \( p \in \text{ram}(B) \), the corresponding Hecke operator \( T_p \) on \( Y \) is an involution. Each such involution acts on the eigenform \( \varphi \) by some sign \( \pm 1 \), hence leaves the measure \( y^k |\varphi(z)|^2 \) invariant. The operator \( T_p \) on \( L^2(Y) \) is self-adjoint and acts on \( \Psi \) by some sign, so the LHS of (1.1) vanishes identically unless

\[
T_p \Psi = \Psi \quad \text{for all} \quad p \in \text{ram}(B),
\]

as we henceforth assume.

The eigenform \( \varphi \) (resp. \( \Psi \)) generates a cuspidal automorphic representation \( \pi^B \) (resp. \( \sigma^B \)) of \( \text{PB}^\times(\mathbb{A}) \). By the Jacquet–Langlands lift, we obtain a cuspidal automorphic representation \( \pi \) (resp. \( \sigma \)) of \( \text{PGL}_2(\mathbb{A}) \).

Let \( p \in \text{ram}(B) \). The evenness condition (2.1) implies that the local component \( \sigma_p^B \) is the trivial representation, hence that \( \sigma_p \) is the Steinberg representation of \( \text{PGL}_2(\mathbb{Q}_p) \) (see [BH, §56.2]). The local component \( \pi_p \) is either the Steinberg representation or its twist by the nontrivial unramified quadratic character of \( \mathbb{Q}_p^\times \).

For a finite prime \( p \notin \text{ram}(B) \), the local components \( \sigma_p = \sigma_p^B \) and \( \pi_p = \pi_p^B \) are unramified principal series representations of \( \text{PGL}_2(\mathbb{Q}_p) \cong B_p/\mathbb{Q}_p^\times \).

We record a special case of Watson’s formula [Wa, Thm 3].

**Proposition 2.1.** The squared magnitude of the LHS of (1.1) is equal to

\[
\frac{\zeta(\pi \times \pi \times \sigma, 1/2)}{\Lambda(\text{ad} \pi, 1)^2},
\]

where \( c \geq 0 \) depends only upon \( \Psi \).

Here and henceforth \( \Lambda(\cdot, \cdot) \) denotes a completed \( L \)-function, including the archimedean local factor \( L(\cdot, s) \), while \( L(\cdot, s) \) denotes the finite part of an \( L \)-function, given for \( \text{Re}(s) \) large enough by a convergent Euler product \( \prod_p L_p(\cdot, s) \) with \( p \) running over the finite primes of \( \mathbb{Q} \). We note that, e.g., \( L(\pi \times \pi \times \sigma, 1/2) \) was denoted \( L(\varphi \times \varphi \times \Psi, 1/2) \) in §1.4. We denote by (e.g.) \( \varepsilon(\sigma) \) the global \( \varepsilon \)-factor, evaluated at the central point \( s = 1/2 \); it factors as the product of \( \varepsilon_p(\sigma) = \varepsilon(\sigma_p) \) over all places \( p \) over \( \mathbb{Q} \), finite or infinite.

By the result of Sarnak [Sa] noted in §1, we may and shall assume that \( \pi \) is non-dihedral. This assumption is actually a consequence of our earlier assumption that \( \varphi \) is full level, which forces each local component of \( \pi \) at a finite place to be spherical or an unramified twist of Steinberg. We prefer to invoke the full level assumption only when it genuinely simplifies our discussion.

By the factorization

\[
\Lambda(\pi \times \pi \times \sigma, s) = \Lambda(\text{ad} \pi \times \sigma, s) \Lambda(\sigma, s),
\]

we see that the RHS of (2.2) vanishes unless \( L(\sigma, 1/2) \neq 0 \), in which case

\[
\varepsilon(\sigma) = 1.
\]

For the proof of Theorem A, there is thus no loss in assuming (2.4). However, because some of the discussion to follow will be used also in the proof of Theorem B, we do not impose (2.4) as a blanket assumption.
By a nontrivial fundamental discriminant we mean the discriminant of a quadratic field extension of \( \mathbb{Q} \). Recall that by class field theory, the following are in natural bijection:

- Nontrivial fundamental discriminants \( D \).
- Nontrivial quadratic characters \( \chi_D \) of \( \mathbb{A}^\times / \mathbb{Q}^\times \).
- Quadratic field extensions \( \mathbb{Q}(\sqrt{D}) \) of \( \mathbb{Q} \).

For each nontrivial fundamental discriminant \( D \), we write

- \( \mathcal{O}_D \) for the ring of integers in \( \mathbb{Q}(\sqrt{D}) \),
- \( \mathcal{N}_D \) for the monoid of integral ideals in \( \mathcal{O}_D \),
- \( \mathfrak{d} = (\sqrt{D}) \) for the different ideal, and
- \( \mathcal{N}(\mathfrak{a}) \) for the absolute norm of a fractional ideal \( \mathfrak{a} \) of \( \mathcal{O}_D \).

We recall that \( \mathcal{O}_D \) consists of all elements \( (n + \ell\sqrt{D})/2 \in \mathbb{Q}(\sqrt{D}) \) for which \( \ell \in \mathbb{Z} \) and

\[
\begin{cases} 
  n \in 2\mathbb{Z} & \text{if } D \text{ is even}, \\
  n \in \ell + 2\mathbb{Z} & \text{if } D \text{ is odd}. 
\end{cases}
\]  

(2.5)

When \( D \) is clear from context, we say that a rational prime \( p \) is split, inert or ramified according to its behavior with respect to \( \mathcal{O}_D \). We say more generally that a natural number \( n \in \mathbb{N} \) is split or inert or ramified if it is a product of primes with the indicated property. When \( D \) is positive, we fix an ordering on the archimedean places (i.e., real embeddings) \( \infty_1, \infty_2 \) of \( \mathbb{Q}(\sqrt{D}) \), with \( \infty_1 \) the standard embedding with respect to which \( \sqrt{D} \) is positive.

### 3. Choice of quadratic character

In this section we construct a family of quadratic characters \( \chi_D \) relevant for the proof of Theorem A.

**Proposition 3.1.** Assume (2.4). Then there are infinitely many nontrivial quadratic characters \( \chi_D \) of \( \mathbb{A}^\times / \mathbb{Q}^\times \) with the following properties:

(i) \( L(\sigma \otimes \chi_D, 1/2) \neq 0 \).

(ii) The archimedean local component \( (\chi_D)_{\infty} \) is trivial.

(iii) For each \( p \in \text{ram}(B) \), the local component \( (\chi_D)_p \) is the nontrivial unramified quadratic character of \( \mathbb{Q}^\times_p \).

**Proof.** A result of Friedberg–Hoffstein [FH, Thm B] reduces our task to verifying the existence of at least one quadratic character \( \chi_D \) satisfying conditions (ii), (iii) and for which

\[
\varepsilon(\sigma \otimes \chi_D) = 1. 
\]  

(3.1)

But our assumption (2.4) implies that (3.1) holds for all such \( \chi_D \). Indeed, we may factor the sign \( \varepsilon(\sigma) \) as a product of local signs \( \varepsilon(\sigma_p) \), with \( p \) running over the places of \( \mathbb{Q} \) (finite or infinite), and similarly for \( \varepsilon(\sigma \otimes \chi_D) \). We compute for each such \( p \) the ratio

\[
\frac{\varepsilon(\sigma_p \otimes (\chi_D)_p)}{\varepsilon(\sigma_p)}. 
\]  

(3.2)

- For \( p = \infty \), the component \( (\chi_D)_p \) is trivial, so the ratio is 1.
- For \( p \in \text{ram}(B) \), the component \( (\chi_D)_p \) is the nontrivial unramified quadratic character, while \( \sigma_p \) is the Steinberg representation. By [Sch, Prop 3.1.2, Thm 3.2.2], the ratio is \(-1\).
4. Base change

Let $D$ be a nontrivial fundamental discriminant. Recall that $π$ is (assumed) non-dihedral. By quadratic base change (see [GL, §5.3]), we obtain from $π$ a cuspidal automorphic representation $π_D$ of $\text{PGL}_2(\mathbb{A}_{\mathbb{Q}(\sqrt{D})})$.

As in §1, let $λ := λ_π : \mathbb{N} → \mathbb{C}$ denote the multiplicative function describing the normalized Hecke eigenvalues of $π_B$, so that

$$L(π, s) = \sum_{n \in \mathbb{N}} \frac{λ(n)}{n^s}. \quad (4.1)$$

The normalized Hecke eigenvalues of $π_D$ are then described by the $\text{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})$-invariant multiplicative function $λ_D : \mathbb{N}_D → \mathbb{C}$ characterized by the relation

$$L(π_D, s) := \sum_{a \in \mathbb{N}_D} \frac{λ_D(a)}{N(a)^s} = L(π, s)L(π ⊗ χ_D, s). \quad (4.2)$$

This relation yields explicit formulas for $λ_D$.

Lemma 4.1. For a prime $p$ of $\mathcal{O}_D$ lying over a rational prime $p$, we have

$$λ_D(p^n) = λ(N(p^n)) \quad (4.3)$$

except when $p$ is inert and $p \notin \text{ram}(B)$, in which case $λ_D(p^n) = α^{2n} + α^{2n-2} + \cdots + α^{-2n}$ for any $α ∈ \mathbb{C}^\times$ with $α + α^{-1} = λ(p)$.

Proof. The main point is that the local factors $L_p(π, s)$ at primes $p \in \text{ram}(B)$ have degree one. For convenience, we record the details. Let $p$ be a rational prime. By taking Euler factors at $p$ of both sides of (4.2), we obtain

$$L_p(π_D, s) = L_p(π, s)L_p(π ⊗ χ_D, s).$$

We view this identity as one of formal power series in the variable $X := p^{-s}$. We argue case-by-case:

• Suppose that $p$ is split. Let $p$ be any prime of $\mathcal{O}_D$ lying over $p$. Since $λ_D$ is Galois-invariant, we have $L_p(π_D, s) = \left(\sum_{n≥0} λ_D(p^n)X^n\right)^2$. On the other hand, since $p$ is split, the local component $(χ_D)_p$ is trivial, and so
Suppose next that $p$ is inert and that $p \notin \text{ram}(B)$. Let $\alpha$ be as in the statement, so that $\{\alpha, \alpha^{-1}\}$ is the multiset of Satake parameters for $\pi$ at $p$. Then $L_p(\pi, s) = (1 - \alpha X)^{-1}(1 - \alpha^{-1}X)^{-1}$, while $L_p(\pi \otimes \chi_D, s) = (1 + \alpha X)^{-1}(1 + \alpha^{-1}X)^{-1}$, hence

$$
\sum_{n \geq 0} \lambda_D(p^n)X^{2n} = L_p(\pi_D, s) = \frac{1}{(1 - \alpha^2X^2)(1 - \alpha^{-2}X^2)}.
$$

By expanding the RHS as a product of geometric series and comparing coefficients of $X^{2n}$, the required identity follows.

□

For future reference, we deduce some consequences of this description. For a nonzero element $x$ of $\mathcal{O}_D$ we abbreviate $\lambda_D(x) := \lambda_D((x))$.

Lemma 4.2. Suppose that $\mathfrak{a} \in \mathbb{N}_D$ is not divisible by any rational prime $p$ that is either split or inert. Let $d_1, d_2 \in \mathbb{N}$, with $d_1$ split and $d_2$ inert. Then $\lambda_D(d_1d_2\mathfrak{a}) = \lambda(d_1)\lambda_D(d_2)\lambda(d_1)\mathcal{N}(\mathfrak{a})$.

Proof. Our hypothesis implies that we may write $\mathfrak{a}$ as a product $\prod_p p(p)^{n(p)}$, where

- $p$ runs over non-inert rational primes,
- $p(p)$ is a prime of $\mathcal{O}_D$ lying over $p$, hence of degree one (as is the case for any split or ramified prime), and
- $n(p) \in \mathbb{Z}_{\geq 0}$.

To see this, we observe first that for each inert rational prime $p$, no prime $p$ lying over $p$ divides $\mathfrak{a}$. Otherwise, since $p$ is generated by $p$, we deduce that $\mathfrak{a}$ is divisible by $p$, contrary to hypothesis. We observe next that if the rational prime $p$ splits in $\mathcal{O}_D$ as the product of prime ideals $p_1p_2$, then at most one of the primes $p_1$ or $p_2$ may divide $\mathfrak{a}$, since otherwise we would again deduce that $p$ divides $\mathfrak{a}$. It follows that $\mathfrak{a}$ is divisible by at most one element of each Galois orbit of primes of $\mathcal{O}_D$. The required product decomposition follows.
Write $d_i = \prod p\,p^{m_i(p)}$. For $p$ inert, set $n(p) := 0$. Then in all cases, $N(p^n(p)) = p^{n(p)}$. By multiplicativity, the required formula reads

$$\prod_p \lambda_D(p^{m_1(p) + m_2(p)}(p^n(p))) = \prod_p \lambda(p^{m_1(p)})\lambda_D(p^{m_2(p)}(p^{m_1(p) + n(p)})).$$

We verify this formula one rational prime $p$ at a time. For notational simplicity, we abbreviate $m := m_1(p)$, $\mathfrak{p} := (p(p)$ and $n := n(p)$.

- Suppose that $p$ is split. Then $m_2 = 0$, so we must check that

$$\lambda_D(p^{m_1}p^n) = \lambda(p^{m_1})\lambda(p^{m_1+n}).$$

Let $\bar{p}$ denote the conjugate of $p$, so that $\{p, \bar{p}\}$ is the set of primes lying over $p$. Then $p^{m_1}p^n = \bar{p}^{m_1+n}p^{m_1}$. By the multiplicativity of $\lambda_D$, it follows that $\lambda_D(p^{m_1}p^n) = \lambda_D(p^{m_1+n})\lambda_D(\bar{p}^{m_1})$. The required identity then follows from (4.3).

- If $p$ is inert, then $m_1 = n = 0$, so the required identity is the tautology $\lambda_D(p^{m_2}) = \lambda_D(p^{m_2}).$

- Suppose that $p$ is ramified. Then $m_1 = m_2 = 0$, so we must check that $\lambda_D(p^n) = \lambda(p^n)$, which again follows from (4.3).

\[
\square
\]

**Lemma 4.3.** Let $x = (n + \ell\sqrt{D})/2$ be a nonzero element of $O_D$. Let $d_1$ (resp. $d_2$) denote the largest split (resp. inert) natural number dividing $x$. Then

$$\lambda_D(x) = \lambda(d_1)\lambda_D(d_2)\lambda \left( \frac{|n^2 - D\ell^2|}{4d_1d_2^2} \right).$$

(4.4)

**Proof.** We apply lemma 4.2 to $a := (x/d_1d_2)$.

\[
\square
\]

Assume for the remainder of this section that $D$ is positive, so that $\mathbb{Q}((\sqrt{D}))$ splits at $\infty$. Each archimedean local component $(\pi_D)_{\infty_1}, (\pi_D)_{\infty_2}$ of $\pi_D$ is then isomorphic to the archimedean local component $\pi_{\infty}$ of $\pi$, which is the holomorphic discrete series representation of $\text{PGL}_2(\mathbb{R})$ with weights

$$\{\ldots, -k - 2, -k, k, k + 2, k + 4, \ldots\},$$

(4.5)

each occurring with multiplicity one. In particular,

$$L_{\infty}(ad(\pi_D), s) = L_{\infty}(ad(\pi), s)^2.$$  

(4.6)

The pure tensors $\varphi_D \in \pi_D$ for which

- the local component $\langle \varphi_D \rangle_p$ at each finite place $p$ of $\mathbb{Q}((\sqrt{D}))$ is a newvector $[\text{Cas}]$, and

- the archimedean component $\langle \varphi_D \rangle_\infty$ has weight $(k, -k)$

span a one-dimensional space. We normalize a specific element $\varphi_D$ of this space, as follows. We will use the notation

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) := \begin{pmatrix} y \\ 1 \end{pmatrix}$$

(4.7)

to describe elements of $\text{PGL}_2(\mathbb{R})$ over a ring. Let $\psi_D$ denote the nontrivial unitary character of $\text{A}_{\mathbb{Q}((\sqrt{D}})/\mathbb{Q}((\sqrt{D}))$ whose infinite component $\langle \psi_D \rangle_\infty$ is given by $(x_1, x_2) \mapsto e(x_1 + x_2)$. For each finite prime $p$ of $\mathbb{Q}((\sqrt{D}))$, the local component $\langle \psi_D \rangle_p$ is then trivial on the local component $\mathcal{D}^{-1}_p$ of the inverse different, but not on any larger fractional $(O_D)_p$-ideal. We have the $\psi_D$-Whittaker expansion
\[ \varphi_D(g) = \sum_{\xi \in \mathbb{Q}(\sqrt{D})} W(a(\xi)g), \quad \text{where } W \text{ is the } \psi_D^-\text{-Whittaker function given by the integral } W(g) = \int_{x \in \mathbb{A}_\mathbb{Q}(\mathbb{Q}(\sqrt{D}))} \varphi_D(n(x)g)\psi_D(-x) \, dx \text{ with respect to the probability Haar } dx. \] Since \( \varphi_D \) is a pure tensor, we may factor \( W(g) \) as a product \( \prod_p W_p(g_p) \) over all places \( p \) of \( \mathbb{Q}(\sqrt{D}) \). We normalize \( \varphi_D \) in terms of its Whittaker function \( W \), as follows:

- Let \( y = (y_p)_{p < \infty} \) be an element of the finite ideles of \( \mathbb{Q}(\sqrt{D}) \) with corresponding fractional ideal \( a \). Thus the local component \( a_p \) is the fractional \( (O_D)_p\)-ideal generated by \( y_p \). We require that \( \prod_p W_p(a(y_p)) = 0 \) unless the \( \mathfrak{d}a \) is an integral ideal, in which case
  \[ \prod_p W_p(a(y_p)) = \lambda_D(\mathfrak{d}a)/\mathcal{N}(\mathfrak{d}a)^{1/2}, \tag{4.8} \]

where \( \lambda := \lambda_\varphi : \mathbb{N} \to \mathbb{C} \) describes the normalized Hecke eigenvalues of \( \varphi \) as in §1. For this normalization to make sense, we need to know the local \((\psi_D)_p\)-Whittaker function \( W_p \) described in the \((\psi_D)_p\)-Kirillov model by

\[ y_p \mapsto \begin{cases} \lambda_D(\mathfrak{d}p\mathfrak{a}_p)/\mathcal{N}(\mathfrak{d}p\mathfrak{a}_p)^{1/2} & \text{if } \mathfrak{d}p\mathfrak{a}_p \text{ is integral}, \\ 0 & \text{otherwise} \end{cases} \tag{4.9} \]

is a newvector. By translating by a generator of \( \mathfrak{d}_p \), we may assume that \( \mathfrak{d}_p = 1 \) and \((\psi_D)_p\) is unramified. We observe then that \( \varphi_D \) defines an \((O_D)_\infty\)-invariant function on \( \mathbb{Q}(\sqrt{D})_\infty = \mathbb{R}^\times \times \mathbb{R}^\times \) whose Mellin transform is the local Euler factor \( L_p(\pi_D, s) = \sum_{k \geq 0} \lambda_D(p^k)/\mathcal{N}(p)^{ks} \) of \( (4.2) \). We then appeal to the fact [JPSS] that when \((\psi_D)_p\) is unramified, any vector with these properties is a newvector.

- For an element \( y = (y_1, y_2) \) of \( \mathbb{Q}(\sqrt{D})_\infty = \mathbb{R}^\times \times \mathbb{R}^\times \), we require that
  \[ \prod_{j=1,2} W_{\infty,j}(a(y_j)) = W_k(y_1)W_k(-y_2), \tag{4.10} \]

where \( W_k : \mathbb{R}^\times \to \mathbb{C} \) denotes the \( L^2 \)-normalized Whittaker function

\[ W_k(y) := 1_{y > 0} \Gamma(k)^{-1/2}(4\pi y)^{k/2}e^{-2\pi y}. \tag{4.11} \]

For this normalization to make sense, we need to know that \( y \mapsto W_k(\pm y) \) defines a vector of weight \( \pm k \) in the holomorphic discrete series representation of \( \text{PGL}_2(\mathbb{R}) \) with weights \( (4.5) \). For this fact, we refer to [Go, (80), (82)].

By restricting \( \varphi_D \) to the identity component of \( \text{PGL}_2(\mathbb{A}_{\mathbb{Q}(\sqrt{D})}) \), we obtain a function on \( \text{PGL}_2(\mathbb{R})^+ \times \text{PGL}_2(\mathbb{R})^+ \) that identifies with a Hilbert modular form of weight \( k, -k \). We denote that Hilbert modular form again by \( \varphi_D : \mathbb{H} \times \mathbb{H} \to \mathbb{C} \). Explicitly, for \( z_j = x_j + iy_j \) we set \( \varphi_D(z_1, z_2) := \varphi_D(g) \), where \( g_p = 1 \) for finite \( p \) and \( g_\infty = n(x_j)a(y_j) \). The Fourier expansion of this Hilbert modular form reads

\[ \varphi_D(z_1, z_2) = \sum_{0 \neq m \in \mathfrak{o}^{-1}} e(m_1x_1 + m_2x_2)W_k(m_3y_1)W_k(-m_3y_2)\frac{\lambda_D(m\mathfrak{o})}{\mathcal{N}(m\mathfrak{o})^{1/2}}, \tag{4.12} \]

where

- \( \mathfrak{o}^{-1} \) is the inverse different, i.e., the fractional ideal \( (1/\sqrt{D}) \), and
- \( m_1, m_2 \) denote the images of \( m \) under the real embeddings of \( \mathbb{Q}(\sqrt{D}) \).
We define the $L^2$-norm $\|\varphi_D\|$ by integrating over $\text{PGL}_2(\mathbb{Q}(\sqrt{D}) \setminus \text{PGL}_2(\mathbb{A}_{\mathbb{Q}(\sqrt{D})})$ with respect to a Haar measure.

Lemma 4.4.

$$\|\varphi_D\|^2 = cL(\pi_D, 1),$$

(4.13) where $c > 0$ depends only upon $B, D$ and the choice of Haar measure.

Proof. This follows from the normalization $\int_{y \in \mathbb{R} \times |W_k(y)|^2} = 1$ and a standard formula obtained via Rankin–Selberg theory; see for instance [Ne5, §3.2.2] or [MV, Lem 2.2.3]. □

5. Twisted triple products

We first choose an element $\Psi'$ of the Jacquet–Langlands lift $\sigma$ of $\sigma_B$. Recall that for a finite place $p$ of $\mathbb{Q}$, the local component $\sigma_p$ is unramified for $p \notin \text{ram}(B)$ and is the Steinberg representation for $p \in \text{ram}(B)$. We fix a nonzero pure tensor $\Psi' \in \sigma$ whose local component at each finite place is a newvector and whose archimedean component has weight 0. It is invariant for each prime $p$ by the action of the unit group of the order $(\mathbb{Z}_p \mathbb{Z}_p d_B \mathbb{Z}_p \mathbb{Z}_p)$, where $d_B := \prod_{p \in \text{ram}(B)} p$ denotes the reduced discriminant of $B$. We write also $\Psi': \mathbb{H} \to \mathbb{C}$ for the cuspidal Hecke–Maass eigenform on $\Gamma_0(d_B) \setminus \mathbb{H}$ given by $\Psi'(x + iy) := \Psi'(g)$ with $g_p = 1$ for finite $p$ and $g_\infty = n(x) a(y)$. The Fourier expansion of $\Psi'$ may be written

$$\Psi'(x + iy) = \sum_{0 \neq \ell \in \mathbb{Z}} \frac{\rho(\ell)}{|\ell|^{1/2}} W_\Psi(\ell y)e(\ell x),$$

(5.2)

where $\rho : \mathbb{N} \to \mathbb{C}$ denotes the normalized Hecke eigenvalue and $W_\Psi : \mathbb{R}^\times \to \mathbb{C}$ the Whittaker function. For mild convenience, we may and shall assume that $\Psi'$ is real-valued. We write $\|\Psi'\|$ for the $L^2$-norm, defined with respect to some Haar measure.

Let $D$ be a nontrivial fundamental discriminant. Let $\varphi_D$ be as in §4. We denote by $\text{res}(\varphi_D)$ the restriction of $\varphi_D$ to $\text{PGL}_2(\mathbb{A})$. We may form the integral

$$\int \text{res}(\varphi_D) \Psi'$$

taken over $\text{PGL}_2(\mathbb{Q}) \setminus \text{PGL}_2(\mathbb{A})$ with respect to some Haar measure.

We record a specialized form of Ichino’s twisted triple product formula [Ic2]. The statement involves Asai $L$-functions and their Rankin–Selberg convolutions. For a summary of the relevant properties of these, we refer to [Ch, §4.1] and its references. For our purposes, what matters is just the factorization

$$\Lambda(\text{asai}(\pi_D) \times \sigma, s) = \Lambda(\text{ad}(\pi) \times \sigma, s) \Lambda(\sigma \otimes \chi_D, s).$$

(5.3)

Proposition 5.1. Retain, as usual, the assumptions of §2. Let $D$ be a positive nontrivial fundamental discriminant. There is a finite subset $c \subseteq \mathbb{R}_{>0}$, depending only upon $B, D$ and the choices of Haar measure, so that

$$\frac{\left| \int \text{res}(\varphi_D) \Psi' \right|^2}{\|\varphi_D\|^2 \|\Psi'\|^2} = c \frac{\Lambda(\text{asai}(\pi_D) \times \sigma, 1/2)}{\Lambda(\text{ad}(\pi_D), 1)}$$

(5.4)
for some \( c \in c \). If \( D \) satisfies the local conditions (ii) and (iii) of Proposition 3.1, then \( c > 0 \).

**Proof.** Ichino’s formula tells us that (5.4) holds with \( c \) the multiple by a nonzero constant of a (finite) product \( \prod_p I_p \) over all places \( p \) of \( \mathbb{Q} \) of normalized local integrals \( I_p \). To describe the local integrals, we fix unitary factorizations \( \pi_D = \otimes_p (\pi_D)_p \) and \( \sigma = \otimes_p \sigma_p \). Thus \( (\pi_D)_p \) is the tensor product of the components \( (\pi_D)_p \) taken over all places \( p \) of \( \mathbb{Q}(\sqrt{D}) \) lying over \( p \). We obtain corresponding factorizations \( \varphi_D = \otimes_p (\varphi_D)_p \) and \( \Psi' = \otimes_p \Psi'_p \) of our vectors. Then

\[
I_p = L^{-1}_p \int_{g \in \text{PGL}_2(\mathbb{Q}_p)} \frac{\langle g(\varphi_D)_p, (\varphi_D)_p \rangle \langle g \Psi'_p, \Psi'_p \rangle}{\langle (\varphi_D)_p, (\varphi_D)_p \rangle \langle \Psi'_p, \Psi'_p \rangle} dg,
\]

(5.5)

where

\[
L_p = \frac{\zeta_{\mathbb{Q}(\sqrt{D}) \otimes \mathbb{Q}_p}(2) \zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(2)} \frac{L_p(\text{asai}(\pi_D) \times \sigma, 1/2)}{L_p(\text{ad}(\pi_D), 1)L_p(\text{ad}(\sigma), 1)}.
\]

(5.6)

For precise normalizations we refer to [Ic2]. These local integrals have all already been computed in the literature, so our task is just to assemble the relevant computations. We rely primarily upon the works of Chen–Cheng [CC] and Cheng [Ch].

It is shown in [Ch, Prop 6.14] that \( I_{\infty} \) is a nonzero constant. (Alternatively, we may reduce to Watson’s calculations and the comparison of local integrals proved in [Ic2] by noting that \( I_{\infty} \) is the same local integral that appears in the setting of Proposition 2.1.)

Let \( p \) be a finite prime not in \( \text{ram}(B) \). Then \( (\pi_D)_p \) and \( \sigma_p \) are unramified. By appeal to

- [Ic2, Lem 2.2] or [CC, Prop 4.5, part (1)] if \( \mathbb{Q}(\sqrt{D})_p \) is the split extension \( \mathbb{Q}_p \times \mathbb{Q}_p \) or an unramified quadratic extension, and
- [CC, Prop 4.7, part (1)] if it is a ramified quadratic extension,

we see that \( I_p = 1 \).

Let \( p \in \text{ram}(B) \). There are only finitely many possibilities for the splitting behavior of \( p \) and for the isomorphism classes of \( \pi_p \) and \( \sigma_p \), hence only finitely many possibilities for \( I_p \). Suppose now that \( D \) satisfies the indicated local conditions. Then \( p \) is inert in \( \mathcal{O}_D \). In particular, there is a unique prime \( p \) lying over \( p \). We have noted already that \( \sigma_p \) is the Steinberg representation and that \( \pi_p \) is the twist of the Steinberg representation by some (possibly trivial) unramified quadratic character \( \eta \) of \( \mathbb{Q}_p^\times \). The local component \( (\pi_D)_p = (\pi_D)_p \) is the local base change of \( \pi_p \). Since \( \mathbb{Q}(\sqrt{D})_p/\mathbb{Q}_p \) is unramified, the character \( \eta \) restricts trivially to the image \( \mathbb{Q}_p^\times \) of the norm map from \( \mathbb{Q}(\sqrt{D})_p^\times \). By [GL, Prop 2 (b)], we deduce that \( (\pi_D)_p \) is the (untwisted) Steinberg representation of \( \text{PGL}_2(\mathbb{Q}(\sqrt{D})_p) \). Under these conditions, an exact formula for \( I_p \) is given by [CC, Prop 4.8, part (2)], confirming in particular that \( I_p \neq 0 \). Thus (5.4) holds with \( c > 0 \).

\[ \square \]

**Remark 5.2.** The key feature of Proposition 5.1 is that for \( D \) as in Proposition 3.1, the constant \( c \) is nonzero. This property is an analytic incarnation of the existence of a nonzero \( \text{PGL}_2(\mathbb{A}) \)-invariant functional on \( \pi_D \otimes \sigma \). It relies crucially on our choice of \( D \), specifically on the local conditions at primes \( p \in \text{ram}(B) \).

To summarize, we record a preliminary result towards Theorem A.
Theorem 5.3. Assume for some positive nontrivial fundamental discriminant $D$ satisfying the conclusions of Proposition 3.1 that

$$\frac{\int \text{res}(\varphi_D)\Psi'}{L(\text{ad } \pi, 1)} \to 0$$

(5.7)

as $k \to \infty$. Then the conclusion of Conjecture 1.1 holds.

Proof. By Watson’s triple product formula (Proposition 2.1), we reduce to estimating the ratio (2.2) involving triple product $L$-functions. By comparing the factorizations (5.3) and (2.3), we see that

$$\Lambda(\pi \times \pi \times \sigma, s) = \frac{\Lambda(\sigma, s)}{\Lambda(\sigma \otimes \chi_D, s)} \Lambda(\text{asai}(\pi_D) \times \sigma, s).$$

(5.8)

By (5.8) and the nonvanishing of $L(\sigma \otimes \chi_D, 1/2)$, we reduce to estimating the ratio on the RHS of (5.4) involving twisted triple product $L$-functions. By applying the twisted triple product formula (5.4) in reverse, together with the formulas (4.13) for $||\varphi_D||^2$ and (4.6) for $L_\infty(\text{ad}(\pi_D), 1)^2$, we reduce to estimating the LHS of (5.7). \[\square\]

Remark 5.4. Let us summarize what we have done thus far. We started with automorphic representations $\pi^B$ and $\sigma^B$ of the non-split group $\text{PB}^x(A)$. Our aim was to estimate certain $\text{PB}^x(A)$-invariant period integrals defined on $\pi^B \otimes \pi^B \otimes \sigma^B$ in terms of other period integrals taken over split quotients. The triple product period on $\text{PGL}_2$ was not a suitable candidate because, in view of Prasad’s uniqueness theorem [Pr1, Thm 1.2], $\pi \otimes \pi \otimes \sigma$ is not locally distinguished: it does not admit a nonzero $\text{PGL}_2(A)$-invariant trilinear functional. The twisted triple product periods for $\pi_D \otimes \sigma$ were natural candidates: the corresponding $L$-values share a degree six Euler factor with those attached to our original periods, and the freedom to vary $D$ allowed us to arrange that $\pi_D \otimes \sigma$ be locally distinguished. The local distinction of $\pi_D \otimes \sigma$ entered into our analysis implicitly, through the proof of Proposition 5.1; it may be verified alternatively using the $\varepsilon$-factor criterion of [Pr2, Thm D] (see also [PS, §5], [Gan]) and explicit calculations as in the proof of Proposition 3.1.\[2\]

6. Approximate functional equation for periods

Let $D$ be a positive nontrivial fundamental discriminant. We aim to evaluate the integrals $\int \text{res}(\varphi_D)\Psi'$ in terms of the Fourier coefficients of $\varphi_D$ and $\Psi'$.

It will be convenient first to rewrite those integrals classically. The function $\text{res}(\varphi_D)\Psi' : \text{PGL}_2(Q) \setminus \text{PGL}_2(A) \to \mathbb{C}$ is right-invariant under the action of $\text{SO}(2)$ at the infinite place and, for each prime $p$, the action of the unit group of the order $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ d_B \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. By strong approximation as in [Ne3, Lem B.1], we deduce that with

\[\text{We sketch the details here. The cited criterion says that } \pi_D \otimes \sigma \text{ is locally distinguished if and only if for each place } p \text{ of } Q, \text{ one has } \varepsilon_p(\pi_D \otimes \sigma) = (\chi_D)_p(-1); \text{ moreover, } \pi_D \otimes \sigma \text{ fails to be \text{locally distinguished at } } p \text{ only if the local components of } \pi_D \text{ and } \sigma \text{ belong to the discrete series. At } p = \infty, \text{ the local component of } \sigma \text{ belongs to the principal series, so local distinction holds. At a finite prime } p \not\in \text{ram}(B), \text{ the local components of } \pi \text{ and } \sigma \text{ are unramified, and we compute using } [\text{Sch}, \text{Prop 3.1.1}] \text{ that } \varepsilon_p(\pi_D \otimes \sigma) = \varepsilon_p(\sigma \otimes \chi_D) = (\chi_D)_p(-1). \text{ At a finite prime } p \in \text{ram}(B), \text{ the local component of } \chi_D \text{ is the unramified quadratic character, thus } (\chi_D)_p(-1) = 1; \text{ using that } \pi_p \text{ is an unramified twist of Steinberg and that } \sigma_p \text{ is (untwisted) Steinberg, we compute as in the proof of Proposition 5.1 that } \varepsilon_p(\pi_D \otimes \sigma) = \varepsilon_p(\sigma \otimes \chi_D) = 1.\]
suitable normalization of Haar measure,
\[
\int \text{res}(\varphi_D)\Psi' = \int_{z \in \Gamma \setminus \mathbb{H}} \varphi_D(z, z)\Psi'(z) \frac{dz}{y^2}.
\] (6.1)

We would like to evaluate (or at least estimate) such integrals in terms of the Fourier coefficients of \(\varphi_D\) and of \(\Psi'\). To address this problem, we might be tempted to apply Holowinsky’s “\(Y\)-thickening technique” [Ho2, §3.1]. Unfortunately, to apply that technique effectively here seems to require more \textit{a priori} control over \(\text{res}(\varphi_D)\) than is available. For instance, to estimate the analogue of the quantities “\(R_q(Y)^*\)” considered in [Ho2, Lem 3.1a] seems to require a sharp bound for the \(L^1\)-norm of \(\text{res}(\varphi_D)\), which seems difficult to achieve. By contrast, for the split analogue of our discussion (\(D\) is a square, \(Q(\sqrt{D}) = \mathbb{Q} \times \mathbb{Q}\), and \(\varphi_D = \varphi \otimes \bar{\varphi}\)), the \(L^1\)-norm of \(\text{res}(\varphi_D)\) is simply the squared \(L^2\)-norm of \(\varphi\), which we may normalize to be 1. We instead appeal to the following “approximate functional equation” for integrals of automorphic forms.

\textbf{Proposition 6.1.} Let \(\Gamma\) be a finite index subgroup of \(\text{SL}_2(\mathbb{Z})\) with \(-1 \in \Gamma\). Let \(\phi : \Gamma \setminus \mathbb{H} \to \mathbb{C}\) be a bounded continuous function satisfying \(|\phi(\tau z)| \leq Cy^{-\alpha}\) for some fixed \(C, \alpha > 0\), all \(z = x + iy\) with \(y \geq 1\), and all \(\tau \in \text{SL}_2(\mathbb{Z})\). Set
\[
a_0(y) := \int_{x \in \mathbb{R}/\mathbb{Z}} \sum_{\tau \in \Gamma \setminus \text{SL}_2(\mathbb{Z})} \phi(\tau (x + iy)) \, dx
\] (6.2)
and, for \(\text{Re}(s) > 1\),
\[
\tilde{a}_0(s) = \int_{y \in \mathbb{R}^\times} a_0(y) y^s \frac{dy}{y^2}.
\] (6.3)

Then for \(\delta > 0\),
\[
\int_{\Gamma \setminus \mathbb{H}} \phi(z) \frac{dx \, dy}{y^2} = \int_{\text{Re}(s) = 1 + \delta} (2s - 1)2\xi(2s)\tilde{a}_0(s) \frac{ds}{2\pi i}.
\] (6.4)

where \(\xi(s) := \Gamma_\mathbb{R}(s)\zeta(s)\), \(\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)\) denotes the completed Riemann zeta function.

\textbf{Proof.} This is the special case \(H(s) = s\) of [Ne3, Thm 5.6] (corrected by requiring that \(-1 \in \Gamma\)). \(\square\)

\textbf{Remark 6.2.} We refer to [Ne3, §5] for some discussion (motivated by numerical applications, but relevant for analytic ones) of the relationship between “\(Y\)-thickening” and Proposition 6.1, and to [Col] and [Coh, §4] for further applications.

We apply this result to \(\Gamma = \Gamma_0(d_B)\) and \(\phi(z) := \varphi_D(z, z)\Psi'(z)\). The main point in evaluating \(a_0(y)\) is then the calculation
\[
\int_{x \in \mathbb{R}/\mathbb{Z}} \phi(x + iy) \, dx = \sum_{\begin{subarray}{c}0 \neq \ell \in \mathbb{Z}, \\ 0 \neq m \in \mathbb{Z}^{-1}: \\ \operatorname{tr}(m) = \ell \end{subarray}} \rho(\ell) \frac{\lambda_D(m\mathfrak{d})}{|\ell|^{1/2} N(m\mathfrak{d})^{1/2}} W_\phi(\ell y) W_k(m_1 y) W_k(-m_2 y),
\] (6.5)

which follows by opening the Fourier series (4.12), (5.2) and using that \(\Psi'\) is real-valued. By combining this with similar calculations at the other cusps of \(\Gamma_0(d_B)\), we will verify the following.
Proposition 6.3. We have
\[
\int \text{res}(\varphi_D) \Psi' = \sum_{0 \neq \ell \in \mathbb{Z}, \atop 0 \neq m \in V^{-1}; \atop \text{tr}(m) = \ell} \frac{\rho(\ell)}{\ell^{1/2}} \frac{\lambda_D(m)\mathcal{D}(m\mathcal{D})^{1/2}}{\mathcal{N}(m\mathcal{D})^{1/2}} V_k(\ell, m)
\]  
(6.6)

where
\[
V_k(\ell, m) := \int_{y \in \mathbb{R}^+} h(y) W_\Psi(\ell y) W_k(m_1 y) W_k(-m_2 y) \frac{dy}{y^2}
\]
(6.7)

with \(h \in C^\infty(\mathbb{R}^+)\) defined by the rapidly-convergent Mellin integral
\[
h(y) := \int_{\text{Re}(s) = 1 + \delta} (2s - 1)\zeta(2s) \left( \sum_{d \mid d_0} d^s \right) y^s \frac{ds}{2\pi i}.
\]
(6.8)

Proof. For the quotient \(\Gamma_0(d_B) \setminus \text{SL}_2(\mathbb{Z})\), we take the coset representatives \(w(d)n(j)\), where \(d\) traverses the set of positive divisors of \(d_B\), \(j\) runs over \(\mathbb{Z}/d\), and \(w(d), n(j) \in \text{SL}_2(\mathbb{Z})\) are described by
\[
w(d) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mod d, \quad w(d) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod d_B/d, \quad n(j) := \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}.
\]
(6.9)

Let \(p\) be a prime divisor of \(d_B\). Then \(p\) remains prime in \(\mathcal{O}_D\), and the local components of \(\pi_D\) and \(\sigma\) at \(p\) are Steinberg representations, hence have local Atkin–Lehner eigenvalue \(-1\) (see [Sch, Prop 3.1.2]). It follows that
\[
\varphi_D(w(d)z, w(d)z) = \mu(d)\varphi_D(z/d, z/d), \quad \Psi'(w(d)z) = \mu(d)\Psi'(z/d),
\]
(6.10)

where \(\mu\) denotes the Möbius function. Using that \(\mu(d)^2 = 1\) and \(n(j)(x + iy) = x + j + iy\), we deduce that
\[
a_0(y) = \sum_{d \mid d_B} \int_{x \in \mathbb{R}/d} \sum_{j \in \mathbb{Z}/d} \phi\left( \frac{x + j + iy}{d} \right) dx = \sum_{d \mid d_B} \int_{x \in \mathbb{R}/d} \phi\left( \frac{x + iy}{d} \right) dx,
\]
(6.11)

thus
\[
\hat{a}_0(s) = \left( \sum_{d \mid d_B} d^s \right) \int_{y \in \mathbb{R}^+} \left( \int_{x \in \mathbb{R}/d} \phi(x + iy) dx \right) y^s \frac{dy}{y^2}.
\]
(6.12)

We evaluate the inner integral over \(x\) as in (6.5) and insert the resulting formula for \(\hat{a}_0(s)\) into (6.4), giving the formula
\[
\int \text{res}(\varphi_D) \Psi' = \int_{\text{Re}(s) = 1 + \delta} \int_{y \in \mathbb{R}^+} \sum_{m \in \mathcal{O}_D} \mathcal{R}(s) \mathcal{T}(m, y) y^s \frac{dy}{y^2} \frac{ds}{2\pi i},
\]
(6.13)

where
\[
\mathcal{R}(s) := (2s - 1)\zeta(2s) \left( \sum_{d \mid d_B} d^s \right),
\]
\[
\mathcal{T}(m, y) := 1_{m \neq 0} \frac{\rho(\ell)}{\ell^{1/2}} \frac{\lambda_D(m\mathcal{D})}{\mathcal{N}(m\mathcal{D})^{1/2}} W_\Psi(\ell y) W_k(m_1 y) W_k(-m_2 y).
\]

We may shift the contour to \(\text{Re}(s) = c\) with \(c\) large enough but fixed. We observe the following estimates:
• $\mathcal{R}(s) \ll_{c,A} (1+|s|)^{-A}$ for each fixed $A$, due to the rapid decay coming from the $\Gamma$-factor in $\xi(2s)$.

• $|y^s| \ll y^c$.

• $T(m,y) \ll_{A,c} (1+|y|)^{-A}(1+|m_1 y|)^{-A}(1+|m_2 y|)^{-A}$ by the Hecke bound for the Hecke eigenvalues, trivial bounds for the Whittaker functions near zero, and the rapid decay of the Whittaker functions near infinity.

• $\sum_{m \in \mathfrak{o}^{-1}} |T(m,y)| \ll A y^{-2} + y^{-A}$ by the previous estimate and the bound $O(X^2)$ for the number of $m \in \mathfrak{o}^{-1}$ with $|m_1|, |m_2| \leq X$.

Using these estimates, we deduce that the three-fold iterated sum/integral on the RHS of (6.13) converges absolutely. We may thus rearrange it as $\sum_{m} \int_{y} \int_{s} (\cdots)$. Shifting the contour back to $\Re(s) = 1 + \delta$ then yields the required formula. □

7. Asymptotics of archimedean integrals

We retain the setting of §6. Let $\ell$ be a nonzero integer, and let $m \in \mathfrak{o}^{-1}$ with trace$(m) = \ell$. Since $\mathfrak{o}^{-1}$ is the fractional $O_D$-ideal generated by $1/\sqrt{D}$, we may write

$$m = \frac{\ell + n/\sqrt{D}}{2}$$

for some $n$ satisfying (2.5). In this way, we may view the RHS of (6.6) as a sum over integers $\ell$ and $n$, with $\ell$ nonzero, satisfying (2.5). Since $\infty_1$ is the standard embedding $\mathbb{Q}(\sqrt{D}) \hookrightarrow \mathbb{R}$ with respect to which $\sqrt{D}$ is positive, we may assume that $n$ is positive, since otherwise $V_k(\ell,m)$ vanishes due to the support condition on $W_k$.

We turn now to estimates. The following conventions concerning asymptotic notation and terminology will be in effect for the remainder of the paper. We say that a quantity is fixed if it is independent of our sequence parameter $k$. For instance, the eigenform $\Psi$ is fixed. We let $\varepsilon > 0$ and $N \in \mathbb{Z}_{\geq 0}$ denote fixed quantities, with $\varepsilon$ sufficiently small and $N$ sufficiently large. We use the notation $A = O(B)$ or $A \ll B$ to denote that $|A| \leq C|B|$ for some fixed $C \geq 0$, which we allow to depend upon any previously mentioned fixed quantities. In particular, such implied constants $C$ may depend upon the fixed quantities $\varepsilon, N$ and $\Psi$.

**Proposition 7.1.** We have

$$\int \text{res}(\varphi_D)\Psi' = \sum_{\ell,n|\ell|<k^\varepsilon} \lambda_D \left( \frac{n + \ell \sqrt{D}}{2} \right) f_\ell(n/k) \frac{f_\ell(n/k)}{k} + O(k^{-1+\varepsilon}),$$

(7.2)

where

- the symbol $\sum$ indicates that $\ell$ and $n$ are integers, with $\ell$ nonzero and $n$ positive, satisfying the congruence condition (2.5), and
- the $f_\ell$ are smooth functions on $\mathbb{R}_+^\times$ satisfying, for $S_N$ as in Definition 1.4, the estimates

$$S_N(f_\ell) \ll |\ell|^{-N}.$$  

(7.3)

The proof occupies the remainder of §7.

Set

$$h_\ell(y) := h(y)W_\Psi(\ell y)/y.$$  

(7.4)
By substituting the definition (4.11) of $W_k$ and executing the change of variables $y \mapsto y\sqrt{D}/2\pi n$, we see that $V_k(\ell, m) = 0$ unless $n^2 - \ell^2 D > 0$, in which case

$$V_k(\ell, m) = \frac{(1 - D\ell^2/n^2)^{k/2}}{\Gamma(k)} \int_{y \in \mathbb{R}^+_+} h_\ell \left( \frac{\sqrt{D}}{2\pi n} y \right) y^k e^{-y} dy / y. \quad (7.5)$$

We observe that the function $h_\ell(y)$ and its derivatives decay rapidly with respect to both $y$ (tending either to 0 or $\infty$) and $\ell$:

**Lemma 7.2.** We have

$$S_N(h_\ell) \ll |\ell|^{-N}. \quad (7.6)$$

**Proof.** By shifting contours in the definition (6.8) of $h$, we see that $h(y) \ll y^N$ as $y \to 0$ and $h(y) = c + O(y^{-N})$ as $y \to \infty$ for some fixed $c > 0$. On the other hand, the Whittaker function $W_k(y)$ is $\ll y^{1/2-1/64} \ll 1$ for small $y$ and decays exponentially for large $y$. By these and similar estimates for derivatives, the required conclusion follows.

Using what amounts to the rapid decay of $y^k e^{-y}$ for large $k$ near both 0 and $\infty$, we verify that $V_k(\ell, m)$ is small unless $n$ is of size $k$:

**Lemma 7.3.** If $n \geq k^{1+\varepsilon}$ or $n \leq k^{1-\varepsilon}$, then

$$V_k(\ell, m) \ll (kn|\ell|)^{-N} \quad (7.7)$$

**Proof.** By (7.6), we have

$$h_\ell \left( \frac{\sqrt{D}}{2\pi n} y \right) \ll |\ell|^{-N} \min \left( \frac{y}{n}, \frac{n}{y} \right)^N.$$

Since $(y/n)^N \leq (n/k)^{-N}(y/k)^N$ and $(n/y)^N \leq (n/k)^N(k/y)^N$, it follows that

$$V_k(\ell, m) \ll |\ell|^{-N} \left( \frac{n}{k} \right)^{-N} \frac{\Gamma(k+N)}{k^N\Gamma(k)}$$

and

$$V_k(\ell, m) \ll |\ell|^{-N} \left( \frac{n}{k} \right)^{-N} \frac{k^N\Gamma(k-N)}{\Gamma(k)}.$$

We apply the first of these estimates when $n \geq k^{1+\varepsilon}$ and the second when $n \leq k^{1-\varepsilon}$. Since $\Gamma(k+N) \ll k^N\Gamma(k)$, and $k^N\Gamma(k-N) \ll \Gamma(k)$, we may conclude by appealing to our hypothesis on $n$ and replacing $N$ with something sufficiently large in terms of $N$ and $\varepsilon$.

These estimates imply already that for each $\ell$, the contribution to the sum on the RHS of (6.6) from $n$ outside the interval $(k^{1-\varepsilon}, k^{1+\varepsilon})$ is negligible, i.e., of size $O(k^{-N}(1+|\ell|)^{-N})$. The contribution to the remaining sum from $|\ell| \geq k^\varepsilon$ is likewise negligible. We are left with

$$\int \text{res}(\varphi_D) \Psi' = \sum_{\ell, m:} \sum_{|\ell|<k^\varepsilon, k^{1-\varepsilon}<n<k^{1+\varepsilon}} \rho(|\ell|) \frac{\lambda_D(m\delta)}{|\ell|^{3/2} N(m\delta)^{1/2}} V_k(\ell, m) + O(k^{-N}) \quad (7.8)$$

with the symbol $\Psi'$ as in the statement of Proposition 7.1.

We may analyze the remaining sum by expanding $h_\ell$ via Mellin transform and appealing to asymptotic formulas for $\Gamma(s+k)/\Gamma(s)$, exactly as in Luo–Sarnak [LS,
For completeness and variety of presentation, we record an alternative argument using the following elementary estimate due to Iwaniec [Iw, Lem C].

**Lemma 7.4.** For \( f \in C_c^\infty(\mathbb{R}_+^2) \) and \( s > 0 \), the integral

\[
J(s) := \frac{1}{\Gamma(s)} \int_{\mathbb{R}_+^2} f(y) y^s e^{-y} \frac{dy}{y}
\]

satisfies the estimate

\[
|J(s) - f(s)| \leq \frac{s}{2} \|f''\|_{L^\infty}
\]

**Proof.** We appeal to Taylor’s theorem with remainder in the form

\[
|f(y) - f(s) - (y - s)f'(s)| \leq \frac{1}{2}(y - s)^2 \|f''\|_{L^\infty}
\]

and note that

\[
\int_{\mathbb{R}_+^2} (y - s)^2 y^s e^{-y} \frac{dy}{y} = \Gamma(s + 2) - 2s\Gamma(s + 1) + s^2\Gamma(s) = s\Gamma(s).
\]

It follows that

\[
V_k(\ell, m) = \left(1 - \frac{D\ell^2}{n^2}\right)^{k/2} h_\ell \left(\frac{k\sqrt{D}}{2\pi n}\right) + O\left(\frac{k}{n^2} (1 + |\ell|)^{-N}\right).
\]

Let \( \ell, n, m \) be as in (7.8). Then

\[
\left(1 - \frac{D\ell^2}{n^2}\right)^{k/2} = 1 + O(k^{4\varepsilon - 1}),
\]

\[
N(m\mathfrak{d})^{1/2} = \frac{\sqrt{n^2 - \ell^2D}}{2} = \frac{n}{2} + O(k^{-1+\varepsilon}),
\]

and

\[
\lambda_D(m\mathfrak{d}) = \lambda_D\left(\frac{n + \ell\sqrt{D}}{2}\right),
\]

where for a nonzero element \( x \) of \( \mathcal{O}_D \) we abbreviate \( \lambda_D(x) := \lambda_D((x)) \). For convenience, we may use Deligne’s results to bound \( \lambda_D(m\mathfrak{d}) \) by the number of divisors of the ideal \( m\mathfrak{d} \), which in turn is \( O(k^e) \). (The averaged form of this bound following from Rankin–Selberg theory would also suffice for our purposes.) It follows that

\[
\int \text{res}(\varphi_D)\Psi' \sum_{\ell, n: |\ell| < k^e, k^{1+\varepsilon} < n < k^{1+\varepsilon}} \lambda_D\left(\frac{n + \ell\sqrt{D}}{2}\right) f_\ell(n/k) + O(k^{-1+10\varepsilon})
\]

with

\[
f_\ell(u) := \rho(|\ell|) \frac{2}{|\ell|^{1/2}} h_\ell \left(\frac{\sqrt{D}}{2\pi u}\right)
\]
The estimates (7.3) for \( f_\ell \) are satisfied, so we incur negligible error in removing the summation condition \( k^{1-\varepsilon} < n < k^{1+\varepsilon} \) and restricting the sum further to \( |\ell| < k^{\varepsilon/10} \). After renaming \( \varepsilon \), we obtain the conclusion of Proposition 7.1.

8. Proof of Theorem A

We assume Conjecture 1.6 and must deduce Conjecture 1.1. We retain the asymptotic notation and terminology of \( \S 7 \). We may assume the \( \varepsilon \)-factor condition (2.4) and may thus find a positive nontrivial fundamental discriminant \( D \) satisfying the conclusion of Proposition 3.1. By Theorem 5.3, we reduce to verifying (5.7). We use the lower bound \( L(\ad \pi, 1) \gg k^{-\varepsilon} \), known in stronger form by [HL], and the asymptotic formula (7.2) for \( \int \res(\varphi_D) \Psi' \). We evaluate \( \lambda_D \) using Lemma 4.3. The natural numbers \( d_1, d_2 \) defined in that lemma depend only upon the congruence class \( a \pmod{2\ell} \) of \( n \). Let us write \( d_j = d_j(a) \) to indicate that dependence. The restrictions on \( n \) implied by the \( \sum_n^\varepsilon \) notation likewise depend only upon \( a \), so let us accordingly write \( \sum_n^\varepsilon \). We obtain

\[
\frac{\int \res(\varphi_D) \Psi'}{L(\ad \pi, 1)} = \sum_{0 \neq |\ell| < k^{\varepsilon}} \sum_{a \in \mathbb{Z}/2\ell} \lambda(d_1(a))\lambda_D(d_2(a))S(\ell, a) + O(k^{-1+\varepsilon}),
\] (8.1)

where

\[
S(\ell, a) := \frac{1}{kL(\ad \pi, 1)} \sum_{n \equiv a(2\ell)} \lambda \left( \frac{n^2 - D\ell^2}{4d_1d_2^2} \right) f_\ell \left( \frac{n}{k} \right).
\] (8.2)

By our assumption of Conjecture 1.6, we may find some fixed \( N \in \mathbb{Z}_{\geq 0} \) so that for each fixed \( \varepsilon > 0 \), we have for large enough \( k \) the inequality

\[
|S(\ell, a)| \leq \varepsilon |\ell|^N S_N(f_\ell).
\] (8.3)

Using the trivial estimate \( |\lambda(d_1)\lambda_D(d_2)| \leq 10|\ell|^{10} \), we deduce that the LHS of (8.1) is bounded in magnitude by

\[
20\varepsilon \sum_{0 \neq |\ell| < k^{\varepsilon}} |\ell|^{N+11} S_N(f_\ell) + O(k^{-1+\varepsilon}).
\] (8.4)

By the estimate (7.6) (applied with a larger value of \( N \)), we see that the sum over \( \ell \) in (8.4) is bounded by some fixed quantity depending only upon \( N \). Taking \( \varepsilon \) sufficiently small in terms of \( N \), we conclude that (8.4) can be made arbitrarily small. This completes the required deduction of (5.7).

9. Proof of Theorem B

We adopt the setting of \( \S 1.5 \). We set \( B := M_2(\mathbb{Q}) \), so that the discussion of \( \S 2 \) applies. The set \( \text{ram}(B) \) is empty, so any conditions concerning \( p \in \text{ram}(B) \) hold tautologically. We drop subscripts as before: \( \varphi := \varphi_k \), \( \varphi_D := (\varphi_k)_D \). We again abbreviate \( \lambda := \lambda_\varphi \) and write \( \pi \) and \( \sigma \) for the cuspidal automorphic representations generated by \( \varphi \) and \( \Psi \), respectively, and \( \pi_D \supset \varphi_D \) for the base change of \( \pi \). We assumed in \( \S 2 \) that \( \varepsilon(\sigma) = 1 \), but do not impose that assumption here; we had invoked that assumption above only in \( \S 3 \) and do not refer here to any results depending upon that section. We use asymptotic notation and terminology as in \( \S 7 \). In particular, \( \varepsilon > 0 \) (resp. \( N \in \mathbb{Z}_{\geq 0} \)) are sufficiently small (resp. large) and fixed. We regard \( D \) as fixed, so that implied constants may freely depend upon it.
The adjectives “split” and “inert” refer to \( \mathbb{Q}(\sqrt{D}) \). The discussion of \S 5 applies to \( \Psi' = \Psi \).

We begin with the twisted analogue of [HS, Lem 2].

**Lemma 9.1.** We have

\[
1/\|\varphi_D\| \asymp L(\text{ad}(\pi_D), 1)^{-1/2} \ll (\log \log k)^{O(1)} \exp \sum_{\text{split } p \leq k} \frac{1 - |\lambda(p)|^2}{p}. \tag{9.1}
\]

**Proof.** The first estimate was noted already in Lemma 4.4, so our task is to prove the second estimate.

Recall that \( \pi \) is (assumed) non-dihedral. In particular, the adjoint lift \( \text{ad}(\pi) \) (and its twist \( \text{ad}(\pi) \otimes \chi_D \)) are cuspidal. We appeal to the factorization\(^3\)

\[
L(\text{ad}(\pi_D), s) = L(\text{ad}(\pi), s)L(\text{ad}(\pi) \otimes \chi_D, s). \tag{9.2}
\]

In view of the identity \( \lambda(p^2) = |\lambda(p)|^2 - 1 \), the proof of [HS, Lem 2] gives

\[
L(\text{ad}(\pi), 1) \gg (\log \log k)^{-3} \exp \sum_{p \leq k} \frac{|\lambda(p)|^2 - 1}{p}, \tag{9.3}
\]

and

\[
L(\text{ad}(\pi) \otimes \chi_D, s) \gg (\log \log k)^{-3} \exp \sum_{p \leq k} \chi_D(p) \frac{|\lambda(p)|^2 - 1}{p}, \tag{9.4}
\]

where by abuse of notation we write \( \chi_D(p) \) for the value taken at \( p \) by the quadratic Dirichlet character corresponding to the idele class character \( \chi_D \), thus \( \chi_D(p) = 0 \) unless \( p \nmid D \), in which case \( \chi_D(p) \in \{ \pm 1 \} \) is the value taken at the uniformizer \( p \in \mathbb{Q}_p^\times \) of the local component \( (\chi_D)_p \), which in turn is +1 for split \( p \) and -1 for inert \( p \). The required conclusion follows from the identity

\[
\frac{1 + \chi_D(p)}{2} = \begin{cases} 1 & \text{if } p \text{ is split,} \\ 0 & \text{otherwise.} \end{cases} \tag{9.5}
\]

\[\square\]

We turn next to the twisted analogue of Soundararajan’s estimate [HS, Thm 3].

**Proposition 9.2.** We have

\[
\left\| \text{res}(\varphi_D) \Psi \right\| \ll \exp \sum_{\text{split } p \leq k} \frac{\varepsilon - |\lambda(p)|^2}{p}. \tag{9.6}
\]

\(^3\)We verify the claimed factorization. By the work of Gelbart–Jacquet and Arthur–Clozel, each factor in the claimed identity is known to be the \( L \)-function attached to an isobaric automorphic representation. By strong multiplicity one [JS], it is enough to compare local factors at almost all rational primes \( p \). If \( p \) splits \( \mathbb{Q}(\sqrt{D}) \), then both \( L_p(\text{ad}(\pi_D), s) \) and \( L_p(\text{ad}(\pi), s)L_p(\text{ad}(\pi) \otimes \chi_D, s) \) are equal to \( L_p(\text{ad}(\pi_p), s)^2 \). Suppose that \( p \) is inert and that \( \pi \) and \( \chi \) are unramified at \( p \). Write \( E = Q(\sqrt{D})_p \) for the associated quadratic field extension of \( Q_p \). Let \( \{ \alpha, \alpha^{-1} \} \) denote the Satake parameters for \( \pi \) at \( p \). Then the local component at \( p \) of \( \text{ad}(\pi) \) (resp. \( \text{ad}(\pi) \otimes \chi_D \)) is the unramified representation of \( \text{GL}_3(Q_p) \) with Satake parameters \( \{ \alpha^2, 1, \alpha^{-2} \} \) (resp. \( \{ -\alpha^2, -1, -\alpha^{-2} \} \)). The local component of \( \pi_D \) (resp. \( \text{ad}(\pi_D) \)) at \( p \) is the unramified representation of \( \text{GL}_2(E) \) (resp. \( \text{GL}_3(E) \)) with Satake parameters \( \{ \alpha^2, \alpha^{-2} \} \) (resp. \( \{ \alpha^4, 1, \alpha^{-4} \} \)). The required factorization follows from the identity of polynomials

\[
(1 - \alpha^4 X)(1 - X)(1 - \alpha^{-4} X) = (1 - \alpha^2 X)(1 - X)(1 - \alpha^{-2} X) \cdot (1 + \alpha^2 X)(1 + X)(1 + \alpha^{-2} X).
\]
Proof. By the proof of Proposition 5.1, we have
\[
\frac{|\int \text{res}(\varphi_D)\Psi|^2}{\|\varphi_D\|^2} \ll \frac{\Lambda(\text{asai}(\pi_D) \times \sigma, 1/2)}{\Lambda(\text{ad}(\pi_D), 1)}.
\] (9.7)
The $\Gamma$-factors appearing on the RHS of (9.7) are exactly as in the untwisted case considered by Holowinsky–Soundararajan, so by an application of Stirling’s formula as in [So2, p1476],
\[
\frac{\Lambda(\text{asai}(\pi_D) \times \sigma, 1/2)}{\Lambda(\text{ad}(\pi_D), 1)} \asymp \frac{k^{-1}L(\text{asai}(\pi_D) \times \sigma, 1/2)}{L(\text{ad}(\pi_D), 1)}.
\] (9.8)
We appeal to the same consequence of Soundararajan’s weakly subconvex bounds [So2] as in the untwisted case:
\[
k^{-1}L(\text{ad}(\pi) \times \sigma, 1/2) \ll (\log k)^{-\epsilon}.
\] (9.9)
We conclude by (9.1) and the estimate $(\log k)^{1/2} \asymp \exp \sum_{\text{split } p \leq k} 1/p$. □

We turn to the twisted analogue of Holowinsky’s estimate [HS, Thm 2].

**Proposition 9.3.** For $k \leq x \leq k^{1+\epsilon}$ and $|\ell| \leq k^{\epsilon}$,
\[
x^{-1} \sum_{n \leq x} \lambda_D \left(\frac{n + \ell\sqrt{D}}{2}\right) \ll |\ell|^{-O(1)} \exp \sum_{\text{split } p \leq x} \frac{2(|\lambda(p)| - 1) + \epsilon}{p}.
\] (9.10)
where the notation $\sum^\dagger$ is as in the statement of Proposition 7.1.

Proof. By summing over $n$ in arithmetic progressions modulo $2\ell$ and evaluating $\lambda_D$ as in §8, we reduce to verifying that for every irreducible quadratic polynomial $Q$ with discriminant in the same square-class as $D$, every nonnegative multiplicative function $f$ bounded by the divisor function, and all $x \geq 1$, we have
\[
x^{-1} \sum_{n \leq x} f(|Q(n)|) \ll \|Q\|^{-O(1)} \exp \sum_{\text{split } p \leq x} \frac{2(f(p) - 1) + \epsilon}{p}.
\] (9.11)
Such estimates follow readily (in stronger form) from arguments of Nair [Na], but the results stated by Nair are not uniform enough to deduce (9.11).4 For completeness, we record a proof as in [Ho2, §4]. We may assume that $x$ is sufficiently large and that $\|Q\| \leq x^{\epsilon}$, since otherwise the required estimate is trivial. We choose $\alpha > 0$ fixed but sufficiently small in terms of $\epsilon$, and set
\[
y := x^\alpha, \quad z := x^{1/\alpha \log \log x}.
\] (9.12)
Recall that a natural number $a$ is called $z$-smooth if every prime divisor $p$ of $a$ satisfies $p \leq z$. We say that a natural number $b$ is $z$-rough if every prime divisor $p$ of $b$ satisfies $p > z$. Given a natural number $n \leq x$, we may factor $|Q(n)|$ uniquely as a product $ab$, where $a$ is $z$-smooth and $b$ is $z$-rough. Then $f(|Q(n)|) = f(a)f(b)$. Let $\Omega(b)$ denote the number of prime factors of $b$, counted with multiplicity. We have $|Q(n)| \leq 3\|Q\|x^2 \leq 3x^{2+\epsilon}$, so $\Omega(b) \leq 3\|Q\|x^2 \leq 3x^{2+\epsilon}$, so $\Omega(b) \leq

4We note that we were likewise unable to deduce (9.11) from refinements of Nair’s results given by Nair–Tenenbaum [NT] and Henriot [He1, He2]. The issue in applying the latter work is that we require $Q$ to be non-primitive, or equivalently, $f$ to satisfy a weaker condition than multiplicativity, which seems to require some uniformity with respect the parameters “$A, B$” in [He1].
We thereby reduce to establishing the bound

\[ x^{-1} \sum_{\text{smooth } a \leq x} f(a) \# N(a) \ll \|Q\|^{O(1)} \exp \sum_{\text{split } p \leq x} \frac{2(f(p) - 1)}{p}, \tag{9.13} \]

where \( N(a) \) denotes the set of all \( n \leq x \) admitting a factorization \( |Q(n)| = ab \) as above. Clearly \( \# N(a) \leq x/a + 1 \leq x/a \). From the estimates \( \sum_{n \leq x} f(n) \ll x \log(x) \) and

\[ \sum_{\text{smooth } a : y < a \leq x} 1/a \ll x/\log(x)^N \tag{9.14} \]

(see, e.g., [Ne2, (108)]) we deduce that the contribution to (9.13) from \( a > y \) is negligible for the purposes of proving (9.13). On the other hand, for \( a \leq y \), a sieve estimate as in [Na, p264] gives

\[ \# N(a) \ll x \frac{\rho(a)}{a} \prod_{p \leq z : p \nmid a} \left( 1 - \frac{\rho(p)}{p} \right), \tag{9.15} \]

where \( \rho(n) \) denotes the number of roots of \( Q \) in \( \mathbb{Z} / n \). Let \( \Xi \) denote the set of primes that either divide the discriminant of \( Q \) or for which \( \rho(p) = p \). Then for \( p \notin \Xi \), we have \( \rho(p) = 2 \) or 0 according as \( p \) is split or inert in \( \mathbb{Q}(\sqrt{D}) \). Discarding wastefully the factors in (9.15) indexed by \( p \in \Xi \), we obtain

\[ \# N(a) \ll x \prod_{p \leq z \notin \Xi} \left( 1 - \frac{2}{p} \right) \prod_{p \leq z} \frac{\rho(a)}{a} \prod_{p \nmid a} \left( 1 + \frac{2}{p} \right). \tag{9.16} \]

Thus the LHS of (9.13) is majorized by

\[ \prod_{\text{split } p \leq z : p \notin \Xi} \left( 1 - \frac{2}{p} \right) \prod_{p \leq z} \left( 1 + \left( 1 + \frac{2}{p} \right) \left( \frac{\rho(p)f(p)}{p} + \frac{\rho(p^2)f(p^2)}{p^2} + \ldots \right) \right). \tag{9.17} \]

Let \( d \) denote the greatest common divisor of the coefficients of \( Q \). We observe that each prime divisor of \( d \) lies in \( \Xi \), that \( \rho(p^a) \ll \gcd(d, p^a) \) for \( p \in \Xi \) with \( p \mid d \), and that \( \rho(p) \leq 2 \) and \( \rho(p^a) \leq 2 \gcd(p^a, D)^{1/2} \) for \( p \in \Xi \) with \( p \mid d \); these last assertions are consequences of Hensel’s lemma. Writing \( \tau(d) \) for the number of divisors of \( d \), we deduce that (9.17) is majorized by

\[ \tau(d)^{O(1)} \prod_{p \mid D} \left( 1 + \frac{1}{p} \right)^{O(1)} \prod_{\text{split } p \leq z : p \notin \Xi} \left( 1 + \frac{2(f(p) - 1)}{p} \right)^{O(1)}, \tag{9.18} \]

with an absolute implied constant. We conclude by the following crude consequence of the divisor bound:

\[ \tau(d) \prod_{p \mid D} \left( 1 + \frac{1}{p} \right) \ll \|Q\|^8 \ll Q^{O(1)}. \]

We pause to record the proof of Theorem 1.7:
Proof. By combining the estimates (9.3) and (9.13), we see that for all $Q = Q_k$ and $f = f_k$ as in the hypotheses of Conjecture 1.6,
\[
\sum_n \frac{\vert\lambda(Q(n))\vert f(n/k)}{kL(\text{ad} \varphi, 1)} \ll \|Q\|^{O(1)} (\log k)^{\varepsilon} \exp\left(\sum_{p \leq k} \frac{1 - \vert\lambda(p)\vert^2}{p} + \sum_{\text{split } p \leq k} \frac{2(\vert\lambda(p)\vert - 1)}{p}\right).
\]
(We refer to [Ne2, Proof of Thm 3.1] for complete details concerning a closely related argument.) We have
\[
\sum_{p \leq k} \frac{1 - \vert\lambda(p)\vert^2}{p} + \sum_{\text{split } p \leq k} \frac{2(\vert\lambda(p)\vert - 1)}{p} = \sum_{\text{split } p \leq k} \frac{2\vert\lambda(p)\vert - \vert\lambda(p)\vert^2}{p} - \sum_{\text{inert } p \leq k} \frac{\vert\lambda(p)\vert^2}{p} + O(\log D),
\]
with $D$ the discriminant of $Q$ and with “split” and “inert” referring to $\mathbb{Q}(\sqrt{D})$. Using the trivial lower bound $\vert\lambda(p)\vert^2 \geq 0$ for inert $p$ and the upper bound $2\vert\lambda(p)\vert - \vert\lambda(p)\vert^2 \leq 1$ for split $p$, we deduce that the above is at most $(1/2) \log \log k + O(\log D)$. Since $D \ll \|Q\|^{O(1)}$, the required bound (1.10) follows. \hfill \Box

Returning to the proof of Theorem B, we note the following consequence of the preceding estimates:

**Corollary 9.4.**
\[
\frac{\int \text{res} (\varphi_D) \Psi}{\|\varphi_D\|} \ll \exp\left(\sum_{\text{split } p \leq x} \frac{\varepsilon - (1 - \vert\lambda(p)\vert^2)}{p}\right). \tag{9.19}
\]

**Proof.** We insert the estimate (9.10) into the asymptotic formula (7.2) for $\int \text{res} (\varphi_D) \Psi$. In view of the decay properties of the test functions $f_k$ occurring in that formula, we obtain
\[
\int \text{res} (\varphi_D) \Psi \ll \exp\sum_{\text{split } p \leq k} \frac{2(\vert\lambda(p)\vert - 1) + \varepsilon}{p}. \tag{9.20}
\]
(see [Ne2, Proof of Thm 3.1] for complete details concerning a closely related argument). We then appeal to (9.1) and the identity
\[
2(\vert\lambda(p)\vert - 1) + (1 - \vert\lambda(p)\vert^2) = -(1 - \vert\lambda(p)\vert)^2 \tag{9.21}
\]
(compare with [HS, (1.2)]). \hfill \Box

We now complete the proof of Theorem B. By the Deligne bound $\vert\lambda(p)\vert \leq 2$, we may write
\[
\frac{\sum_{\text{split } p \leq k} \vert\lambda(p)\vert^2 / p}{\sum_{\text{split } p \leq k} 1/p} = c^2 \tag{9.22}
\]
for some $c \in [0, 2]$. It follows then from the Minkowski inequality (i.e., the triangle inequality for $\ell^2(\mathbb{Z})$) that
\[
\frac{\sum_{\text{split } p \leq k} (1 - \vert\lambda(p)\vert^2)^2 / p}{\sum_{\text{split } p \leq k} 1/p} \geq (1 - c)^2, \tag{9.23}
\]
hence by Proposition 9.2, Proposition 9.3 and the estimate $\exp \sum_{\text{split } p \leq k} 1/p \asymp (\log k)^{1/2}$ that
\[
\frac{\int \text{res} (\varphi_D) \Psi}{\|\varphi_D\|} \ll (\log k)^{- \max(c^2, (1-c)^2)/2 + \varepsilon}. \tag{9.24}
\]
The quantity \( \max(c^2, (1-c)^2) \) is minimized when \( c^2 = (1-c)^2 \), i.e., for \( c = 1/2 \), in which case \( \max(c^2, (1-c)^2)/2 = 1/8 \). The proof of Theorem B is thus complete.

**Remark 9.5.** In the split case, it seems likely that similar arguments yield for the LHS of (1.1) the estimate \( \ll (\log k)^{-\delta+\epsilon} \) with \( \delta = \min_{c \in [0,2]} \max(c^2-1/2, (1-c)^2) = 1/16 \) (for \( \Psi \) cuspidal), improving upon the exponent \( \delta = 1/30 \) obtained in [HS, Thm 1 (i)].

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