FROM JACK TO DOUBLE JACK POLYNOMIALS
VIA THE SUPERSYMMETRIC BRIDGE

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Abstract. The Calogero-Sutherland (CS) model occurs in a large number of physical contexts, either directly or via its eigenfunctions, the Jack polynomials. The supersymmetric counterpart of the CS model, although much less ubiquitous, has an equally rich structure. In particular, its eigenfunctions, the Jack superpolynomials, appear to share the very same remarkable properties as their non-supersymmetric versions. These super-functions are parametrized by superpartitions with fixed bosonic and fermionic degrees. Now, a truly amazing feature pops out when the fermionic degree is sufficiently large: the Jack superpolynomials stabilize and factorize. Their stability is with respect to their expansion in terms of an elementary basis where, in the stable sector, the expansion coefficients become independent of the fermionic degree. Their factorization is seen when the fermionic variables are stripped off in a suitable way which results in a product of two ordinary Jack polynomials (somewhat modified by plethystic transformations), dubbed the double Jack polynomials. The corresponding double CS Hamiltonian involves not only the expected CS pieces but also combinations of the generators of an underlying affine $\hat{sl}_2$ algebra.

This article is dedicated to Luc Vinet on the occasion of his 60th birthday

1. Introduction

The AGT correspondence [3], that relates conformal blocks to the $U(2)$ Nekrasov instanton partition function [20], has generated a boost of interest for Jack polynomials. Indeed, the latter have been shown to be key components of a new AGT-motivated basis of states in 2d-CFT [2]. More precisely, the Jack polynomials appear there in a generalized version which is indexed by a pair of partitions and decomposes into product of two Jacks with different arguments [2, 19].

Here we present a somewhat analogous type of generalization of the Jack polynomials also labelled by two partitions. These new generalized Jacks arise directly from the construction of the supersymmetric counterparts of the Jack polynomials, the Jack superpolynomials [9,1]. The latter are eigenfunctions of the supersymmetric version of the Calogero-Sutherland (CS) model [24]. It turns out that for excited states with large fermionic degree, the eigenfunctions acquire an unexpected stability behavior. More remarkably, in this stability sector, these eigenfunctions (after a minor transformation) factorize into a product of two Jack polynomials. This factorization is highly non-trivial: there is a sort of twisting in the coupling constant (the free parameter $\alpha$), which is different for the two constituent Jacks, and a reorganization of the variables (technically: a plethystic transformation). The factorized form of the eigenfunctions is referred to as the “double Jack polynomials”. We stress that the non-trivial structure of these double Jacks is inherited from the supersymmetric construction, which thus serves as a bridge linking the Jacks to their double version.

These peculiar properties of stability and factorization have first been observed at the level of the Macdonald generalization of the Jack superpolynomials [3]. Here we make explicit the one-parameter limit characterizing the Jacks. In addition, we unravel their underlying integrable structure by constructing the Hamiltonian for which these are eigenfunctions. Somewhat unexpectedly, this Hamiltonian is built in part from the generators of the nonnegative modes of an $\hat{sl}_2$ algebra.

1We also use the terminology “Jack polynomials in superspace”.

2By contrast, the AGT-type double Jack polynomials [2,19] are composed of two Jacks with different variables (albeit corresponding to a less radical plethystic transformation), but the same coupling constant.

3We note that the latter aspect would have been very difficult to study for the Macdonald case given the complexity of the supersymmetric form of the corresponding Ruisjenaars-Schneider model [7].
The article is organized as follows. In Section 2, we briefly review the Calogero-Sutherland model and their eigenfunctions, emphasizing a Fock space representation to be used throughout the article. The supersymmetric CS model is introduced in Section 3, together with the Jack superpolynomials. For a sufficiently high fermionic degree, the supersymmetric Hamiltonian eigenvalues are shown to be decomposable into two independent parts. This points toward the splitting of the Hamiltonian into two independent CS Hamiltonians and the corresponding factorization of its eigenfunctions. The resulting double Jack polynomials are defined formally in Section 4 and exemplified for simple cases, while their corresponding Hamiltonian is derived in Section 5.

2. The Calogero-Sutherland model and Jack polynomials

The CS model describes a system of $N$ identical particles of mass $m$ lying on a circle of circumference $L$ and interacting pairwise through the inverse of chord distance squared. Setting $m = \hbar = 1$ and $L = 2\pi$, the Hamiltonian reads \[25\]:

$$H^{CS} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{1 \leq i < j \leq N} \frac{\beta (\beta - 1)}{4 \sin^2 \frac{1}{2} (x_i - x_j)}, \quad (2.1)$$

where $\beta$ is a dimensionless real coupling constant and $[x_j, p_k] = i\delta_{jk}$ To the ground state correspond the following wavefunction and eigenvalue:

$$\psi_0(x) = \prod_{j<k} \left| \sin \frac{1}{2} (x_j - x_k) \right|^\beta \quad \text{with} \quad E_0 = \frac{\beta^2 N (N^2 - 1)}{24}. \quad (2.2)$$

It is convenient to define $z_j = e^{ix_j}$ and to factor out the contribution of the ground state by redefining a gauged Hamiltonian as $\psi^{-1}_0 (H^{CS} - E_0) \psi_0 / \beta$ and to set $\beta = 1/\alpha$:

$$\mathcal{H}^{(\alpha)} = \alpha \sum_{i=1}^{N} (z_i \partial_{z_i})^2 + \sum_{1 \leq i < j \leq N} \left( \frac{z_i + z_j}{z_i - z_j} \right) (z_i \partial_{z_i} - z_j \partial_{z_j}) \quad (2.3)$$

This is our starting point.

The symmetric and triangular eigenfunctions of (2.3) are known as the Jack polynomials $J_{\lambda}^{(\alpha)}(z) \quad [14]$ where the index $\lambda$ stands for a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$, with the $\lambda_i$’s being non-negative integers such that $\lambda_i \geq \lambda_{i+1}$. Their eigenvalues are

$$\varepsilon_{\lambda}^{(\alpha)} = 2\alpha n(\lambda') - 2n(\lambda) + (N - 1 + \alpha) |\lambda|, \quad (2.4)$$

where [10]:

$$n(\lambda) = \sum_i (i - 1) \lambda_i = \sum_i \left( \frac{\lambda_i}{2} \right). \quad (2.5)$$

Here $\lambda'$ is the conjugate of $\lambda$ obtained from $\lambda$ by replacing rows by columns in its diagrammatic representation, and $|\lambda| = \sum_i \lambda_i$ is the degree of $\lambda$. We will be interested in the behavior of the wavefunction when $N$ is large. It is thus preferable to remove the dependency in $N$ in the eigenvalue. For this, we note that $J_{\lambda}^{(\alpha)}(z)$ is homogeneous in the $z_i$’s, so that it is an eigenfunction of the momentum operator $\mathcal{P}$:

$$\mathcal{P} J_{\lambda}^{(\alpha)}(z) = \sum_i z_i \partial_{z_i} J_{\lambda}^{(\alpha)}(z) = |\lambda| J_{\lambda}^{(\alpha)}(z). \quad (2.6)$$

Our task is achieved by redefining the Hamiltonian as

$$\mathcal{H}^{(\alpha)} \rightarrow \hat{\mathcal{H}}^{(\alpha)} = \mathcal{H}^{(\alpha)} - (N - 1 + \alpha) \sum_i z_i \partial_{z_i} \quad (2.7)$$

Jack polynomials $J_{\lambda}^{(\alpha)}(z)$ are thus eigenfunctions of $\hat{\mathcal{H}}^{(\alpha)}$ with eigenvalues

$$\hat{\varepsilon}^{(\alpha)}_{\lambda} = 2\alpha n(\lambda') - 2n(\lambda). \quad (2.8)$$

\[4\]See [13] for an extensive and very clear presentation of the CS model.

\[5\]For a physicist introduction to the Jack polynomials, we refer to [13] [12]. A more mathematical presentation can be found in [16].
In the large $N$ limit, it is convenient to rewrite the Hamiltonian in terms of power sums $p_k = z^k + \frac{z^{2^k}}{2} + \cdots$. Since $\hat{H}^{(\alpha)}$ is a differential operator of order two, it is sufficient to determine its action on the product $p_m p_n$. A direct computation gives \[ (\alpha - 1) \sum_{n \geq 1} (n^2 - n) p_n \partial p_n + \sum_{n, m \geq 1} [(m + n) p_m p_n \partial p_{m+n} + \alpha mn p_{m+n} \partial p_n \partial p_m] \] (2.9)

This naturally leads to the Fock space representation
\[
\alpha \hat{H}^{(\alpha)} = (\alpha - 1) \sum_{\ell \geq 1} (\ell - 1) a_\ell \partial a_\ell + \sum_{k, \ell \geq 1} [a_k^\dagger a_k^\dagger a_{k+\ell} + a_k^\dagger a_{k+\ell} a_k a_{k+\ell}] \]
(2.10)

where
\[
[a_k, a^\dagger_\ell] = k \alpha \delta_{k, \ell} \quad \text{and} \quad [a_k, a_\ell] = [a_k^\dagger, a_\ell^\dagger] = 0. \]
(2.11)

The correspondence with symmetric functions, together with $|0\rangle \longleftrightarrow 1$, is
\[
a_k^\dagger \longleftrightarrow p_k \quad \text{and} \quad a_k \longleftrightarrow k \alpha \partial p_k. \]
(2.12)

This correspondence preserves the commutation relations. In this representation, the eigenfunctions take the form of a combination of states
\[
J^{(\alpha)}_\lambda(a_1^\dagger, a_2^\dagger, a_3^\dagger, \cdots) |0\rangle. \]
(2.13)

For instance, up to a multiplicative constant\[^6\]
\[
J^{(\alpha)}_{(3,1)}(0) \propto [(a_1^\dagger)^4 + (3\alpha - 1) a_2^\dagger (a_1^\dagger)^2 + 2\alpha (\alpha - 1) a_3^\dagger a_1^\dagger a_2^\dagger a_2^\dagger - \alpha (a_2^\dagger)^2 - 2\alpha^2 a_4^\dagger] |0\rangle. \]
(2.14)

As a side remark, we point out that it is through the correspondence \((2.12)\) that the connection between Virasoro singular vectors and Jack polynomials is established \[17,4,23,21\]. The technology of Jack polynomials can even be used to derive the spectrum of the Virasoro minimal models \[26,21\]. These applications have recently been lifted to the \(\mathfrak{sl}(2)\) WZW model at fractional level \[22\].

3. Supersymmetric version

In order to supersymmetrize the CS model, we need to introduce anticommuting variables $\theta_1, \ldots, \theta_N$ and extend the CS Hamiltonian $H$ in the following way:
\[
\hat{H}^{(\alpha)} \rightarrow \hat{H}^{(\alpha)}_{\text{susy}} = \{Q, Q^\dagger\} = \hat{H}^{(\alpha)} + \text{terms depending on} \ \theta_i, \]
(3.1)

for two fermionic charges $Q$ and $Q^\dagger$ of the form $Q = \sum_i \theta_i A_i(x, p)$ and $Q^\dagger = \sum_i \partial \theta_i A_i^\dagger(x, p)$, where $A_i$ and $A_i^\dagger$ are fixed by the requirement of reproducing the $\hat{H}^{(\alpha)}$ term on the rhs of the above equation. This construction leads to
\[
\hat{H}^{(\alpha)}_{\text{susy}} = \hat{H}^{(\alpha)} - 2 \sum_{1 \leq i < j \leq N} \frac{z_i z_j}{(z_i - z_j)^2} (\theta_i - \theta_j) \partial \theta_i - \partial \theta_j. \]
(3.2)

This operator is part of the tower of conserved quantities $H_n, 1 \leq n \leq N$ ($P = H_1$ and $H^{(\alpha)}_{\text{susy}} = H_2$) that reduce to the usual (gauged) CS conservation laws in the absence of anticommuting variables. But given that there are $2N$ degrees of freedom in the supersymmetric version, there are $N$ extra conserved charges that vanish when all $\theta_i = 0$ \[9\]. The first nontrivial representative of this second tower is
\[
\mathcal{T}^{(\alpha)}_{\text{susy}} = \alpha \sum_{i=1}^N z_i \theta_i \partial z_i - \partial \theta_i + \sum_{1 \leq i < j \leq N} \frac{z_i \theta_j + z_j \theta_i}{z_i - z_j} (\partial \theta_i - \partial \theta_j). \]
(3.3)

\[^6\]In the monic normalization $J^{(\alpha)}_{\lambda} = m_\lambda + \text{lower terms}$, where $m_\lambda$ is the monomial symmetric function, this coefficient is $1/(2(1+\alpha)^2)$. 
As a side remark, we mention that both expressions can be represented in the Fock space of a free boson, described by the modes \(a_k, a_k^\dagger\) (with \(k \geq 1\), i.e., without the zero mode) and a free fermion, whose modes are denoted \(b_k, b_k^\dagger\).

\[
\alpha \mathcal{H}_\text{susy}^{(\alpha)} = (\alpha - 1) \sum_{\ell \geq 1} (\ell - 1) a_\ell^\dagger a_\ell + \sum_{k, \ell \geq 1} \left[ a_k^\dagger a_k^\dagger a_{k+\ell} + a_{k+\ell}^\dagger a_k a_\ell \right] \\
+ \alpha (\alpha - 1) \sum_{\ell \geq 1} (\ell^2 - \ell) b_\ell^\dagger b_\ell + \alpha \sum_{k, \ell \geq 1} 2\ell \left[ a_k^\dagger b_\ell^\dagger b_{k+\ell} + b_{k+\ell}^\dagger b_\ell a_k \right]
\]  

(3.4)

and

\[
\mathcal{I}_\text{susy}^{(\alpha)} = (\alpha - 1) \sum_{\ell \geq 0} \ell b_\ell^\dagger b_\ell + \sum_{\ell \geq 0, k \geq 1} \left[ b_{\ell+k}^\dagger b_\ell a_k + a_k b_{\ell+k}^\dagger b_\ell \right]
\]  

(3.5)

The fermionic modes are governed by the anticommutation relations:

\[
\{b_k, b_j^\dagger\} = \delta_k, \ell \quad \{b_k, b_j\} = \{b_k^\dagger, b_j^\dagger\} = 0, 
\]

and their correspondence with symmetric functions is:

\[
b_k^\dagger \leftrightarrow \bar{p}_k = \theta_1 z^k_1 + \theta_2 z^k_2 + \cdots \quad \text{and} \quad b_k \leftrightarrow \partial p_k.
\]  

(3.7)

Now, assuming a natural triangularity condition, the common eigenfunctions of \(\mathcal{H}_\text{susy}^{(\alpha)}\) and \(\mathcal{I}_\text{susy}^{(\alpha)}\) are the Jack polynomials in superspace, or Jack superpolynomials, denoted by \(J^{(\alpha)}(z, \theta)\). They are homogeneous in \(z\) and \(\theta\) and invariant under the exchange of pairs \((z_i, \theta_i) \leftrightarrow (z_j, \theta_j)\). Their labelling index \(\Lambda\) is a superpartition. Before displaying the eigenvalues, some notation related to superpartitions is required.

A superpartition \(\Lambda\) is a pair of partitions

\[
\Lambda = (\Lambda^a; \Lambda^s) \quad \text{such that} \quad \begin{cases} 
\Lambda^s & \text{is an ordinary partition} \\
\Lambda^a & \text{is a partition with no repeated parts}
\end{cases}
\]

(3.8)

Note that the last part of \(\Lambda^a\) is allowed to be zero. We denote by \(\Lambda^*\) the partition obtained by reordering in non-increasing order the entries of \(\Lambda^a\) and \(\Lambda^s\) concatenated. The diagrammatic representation of \(\Lambda\) is obtained by putting dots at the end of the rows that come from \(\Lambda^a\) (in such a way thatdots never lie under an empty cell). Here is an example:

\[
\Lambda = (4, 2, 0; 3, 2, 1, 1) \quad \leftrightarrow \quad (4, 3, 2, 2, 1, 1, 0) \quad \leftrightarrow \quad .
\]

(3.9)

A superpartition is equally well described by the pair \(\Lambda^*\) and \(\Lambda^\circ\), where the latter is the partition obtained by replacing dots by boxes, e.g., in the example above,

\[
\Lambda^* \quad \quad \Lambda^\circ
\]

(3.10)

Finally, the bosonic degree of a superpartition is the number of boxes of \(\Lambda^*\) and the fermionic degree, generally denoted by \(m\), is the number of dots in the diagonal of \(\Lambda\), that is, the number of parts of \(\Lambda^a\).

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Footnotes:

3 In a supersymmetric context, the modes of the partner free fermion should pertain to the Neveu-Schwarz sector, hence be half-integers. This can be achieved by redefining \((b_k, b_k^\dagger)\) as \((\bar{b}_{k+1/2}, b_{k+1/2}^\dagger)\) in the relation (3.7) below. However, this precision is not required in the present context.

4 Via such free-field representation, the Jack superpolynomials have been shown to be related to the super-Virasoro singular vectors.
We are now in position to give the eigenvalues of $\mathcal{H}_{\text{susy}}^{(\alpha)}$ and $\mathcal{J}_{\text{susy}}^{(\alpha)}$ corresponding to the eigenfunction $J_{\Lambda}^{(\alpha)}$. These are respectively

$$
\varepsilon_{\Lambda}^{(\alpha)} = 2\alpha n(\Lambda^*) - 2n(\Lambda^*) \quad \text{and} \quad \varepsilon_{\Lambda}^{(\alpha)} = \alpha |\Lambda^*| - |\Lambda^m|.
$$

(3.11)

In the supersymmetric case, we are not only interested in the large $N$ limit but also in the large $m$ limit (actually, in the large $m$ and $N - m$ limits). We thus want to extract from the above two eigenvalues, their dependence on $m$ which is somewhat hidden. For this, we first notice that when $m$ is large (relative to the size of $\Lambda$, an estimation that is made precise in [422]), there are circles in every possible position in the diagram of $\Lambda$. As such, the circles can be ignored and we observe that $\Lambda^*$ differs slightly from its core $\delta^{(m)} = (m - 1, m - 2, \ldots, 1, 0)$. In the diagrammatic representation of $\Lambda^*$, the deviations to the core are located at the top right and at the bottom left of the diagram. We thus see that the superpartition can be disentangled into its fermionic core plus two small partitions $\lambda$ and $\mu$ such that $\Lambda = (\lambda + \delta^{(m)}; \mu)$ [9][10]. For instance, for $m = 8$, we have

\[
\Lambda = \begin{array}{cccccccc}
\| & | & | & | & | & | & | & | \\
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\end{array} \quad \longleftrightarrow \quad \Lambda = \begin{array}{cccccccc}
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\end{array}
\]

\[
\varepsilon_{\Lambda}^{(\alpha)} = 2\alpha n(\Lambda^*) - 2n(\Lambda^*) \quad \longleftrightarrow \quad \varepsilon_{\Lambda}^{(\alpha)} = 2\alpha n(\Lambda^*) - 2n(\Lambda^*)
\]

(3.12)

It is clear that $\Lambda$ is fully characterized by $m$ and the pair $(\lambda, \mu)$ (whose total degree is much less than that of $\Lambda$). The main advantage of this diagrammatic decomposition is that it implies readily that when $m$ is large the conjugate of $\Lambda$ is $\Lambda^* = (\mu + \delta^{(m)}; \lambda^*)$.

Let us reformulate the eigenvalues in terms of the data $\lambda, \mu$ and $m$. For the $\mathcal{J}_{\text{susy}}^{(\alpha)}$ eigenvalue, the computation is easy and yields

$$
\varepsilon_{\Lambda,\mu}^{(\alpha)} \overset{\text{large}}{\longrightarrow} \varepsilon_{\lambda,\mu}^{(\alpha)} = \alpha |\lambda| - |\mu| + (\alpha - 1)m(m - 1)/2
$$

(3.13)

We can easily remove the dependency in $m$ in the eigenvalue by redefining $\mathcal{J}_{\text{susy}}^{(\alpha)}$ as follows:

$$
\mathcal{J}_{\text{susy}}^{(\alpha)} \longrightarrow \mathcal{J}_{\text{susy}}^{(\alpha)} = \mathcal{J}_{\text{susy}}^{(\alpha)} - (\alpha - 1)\mathcal{M}(\mathcal{M} - 1)/2 \quad \text{where} \quad \mathcal{M} = \sum_{i} \theta_i \partial_{\theta_i}.
$$

(3.14)

This subtraction is well defined since $\mathcal{M}$ is also a conserved quantity. The modified eigenvalue reads then

$$
\varepsilon_{\lambda,\mu}^{(\alpha)} = \alpha |\lambda| - |\mu|.
$$

(3.15)

The eigenvalues of $\mathcal{H}_{\text{susy}}^{(\alpha)}$ can also be reformulated in terms of $\lambda, \mu$ and $m$, again keeping in mind that this is valid only for sufficiently large $m$. Observe that \[9\]

$$
\Lambda = (\lambda + \delta^{(m)}) \cup \mu \quad \Rightarrow \quad \Lambda^* = \{\lambda_j + m - j \mid 1 \leq j \leq m\} \cup \{\mu_k \mid 1 \leq k \leq \ell(\mu)\},
$$

(3.16)

and similarly for $\Lambda^* = (\mu + \delta^{(m)}) \cup \lambda^*$. The statement is always true for instance when $m \geq |\lambda| + |\mu| + 1$. Since $\ell(\mu) = |\lambda| + |\mu| + 1 \geq |\lambda| + 1 + \mu_1$, the statement is always true for instance when $m \geq |\lambda| + |\mu| + 1$.

\[10\] For the $+$ operation, the parts add up. For example, we have $(3, 1) + (4, 2, 2) = (7, 3, 2)$.

\[11\] For the $\cup$ operation, the rows of the second partition are inserted into the first one; for instance $(3, 1) \cup (4, 2, 2) = (4, 3, 2, 1)$.
and \( j, k \) in the above notation if we use the second expression of \( n(\lambda) \) given in (2.5). Let us first consider:

\[
\begin{align*}
n(\Lambda^*) &= \sum_i \left( \Lambda_i^* \right) = \sum_{i=1}^m \left( \lambda_i + m - i \right) + \sum_{i=1}^{\ell(\mu)} \left( \mu_i \right) \\
&= \sum_{i=1}^m \left[ \left( \lambda_i \right) + \left( \frac{m - i}{2} \right) + \lambda_i(m - i) \right] + n(\mu') \\
&= n(\lambda') + n(\mu') + \sum_{i=1}^{\ell(\lambda)} \lambda_i[(m - 1) - (i - 1)] + m(m - 1)(m - 2)/6
\end{align*}
\]

where in the last step, we use the first expression in (2.5). For the computation of \( n(\Lambda^*) \), we simply replace \( \lambda \) and \( \mu \) by \( \mu' \) and \( \lambda' \) respectively in the previous expression to get:

\[
n(\Lambda^*) = n(\mu') + n(\lambda) + (m - 1)(\mu') - n(\mu') + m(m - 1)(m - 2)/6.
\]

Combining these two expressions yields

\[
\varepsilon^{(\alpha)}_\Lambda \xrightarrow{\text{large}} \varepsilon^{(\alpha)}_{\lambda,\mu} = (\alpha + 1)^2 \varepsilon^{(\alpha/(\alpha+1))} + \varepsilon^{(\alpha+1)} + 2(m - 1)^2 \varepsilon^{(\alpha)}_{\mu,\mu} + (\alpha - 1)m(m - 1)(m - 2)/3,
\]

where \( \varepsilon^{(\alpha)}_\mu \) is defined in (2.8). We can thus remove the dependency in \( m \) in the eigenvalue by redefining \( \mathcal{H}^{(\alpha)}_{\text{susy}} \) as

\[
\mathcal{H}^{(\alpha)}_{\text{susy}} \rightarrow \hat{\mathcal{H}}^{(\alpha)}_{\text{susy}} = \mathcal{H}^{(\alpha)}_{\text{susy}} - 2(M - 1)^2 \mathcal{H}^{(\alpha)}_{\text{susy}} - (\alpha - 1)M(M - 1)(M - 2)/3.
\]

The \( \hat{\mathcal{H}}^{(\alpha)}_{\text{susy}} \) eigenvalue is then simply

\[
\varepsilon^{(\alpha)}_{\lambda,\mu} = (\alpha + 1)^2 \varepsilon^{(\alpha/(\alpha+1))} + \varepsilon^{(\alpha+1)}.
\]

That the \( m \)-dependence of the eigenvalues can be removed is an indication of the stability property of the eigenfunctions. On the other hand, the decoupling of the eigenvalue \( \varepsilon^{(\alpha)}_{\lambda,\mu} \) into two independent sectors \( \lambda \) and \( \mu \) with modified coupling constant is a clear hint that, in the large \( m \) limit, the eigenfunction \( J^{(\alpha)}_\lambda \) should somehow factorize into a product of the form \( J^{(\alpha/(\alpha+1))}_\lambda \) times \( J^{(\alpha+1)}_\mu \) (with which the expression (3.15) is compatible).

These two expectations are indeed verified: the eigenfunctions both stabilize and factorize (after a certain transformation that will be explained in eq. (3.24) for \( \Gamma \) and

\[
m \geq |\lambda| + |\mu|
\]

Let us consider a simple example. For \( (\lambda, \mu) = (\Gamma, \Gamma) \), the \( m = 1, 2, 3, 4 \) eigenfunctions read respectively

\[
\begin{align*}
&\left[ b_0(a_1) + a_1 b_0(a_2) \right] 0 \\
&\left[ b_0 b_0(a_1) + \alpha b_0 b_1 a_2 \right] 0 \\
&\left[ b_1 b_1 b_0(a_1) + (\alpha - 1) b_1 b_1 b_{10} \right] 0 \\
&\left[ b_1 b_1 b_0 a_1 b_1 b_1 b_0 a_2 b_1 b_1 b_0 \right] 0
\end{align*}
\]

Clearly, the \( m = 1 \) \((< |\lambda| + |\mu| = 2)\) wavefunction does not belong to the stable sector. For \( m \geq 2 \), the coefficients and the \( a^\dagger \) content of each term are always the same. This is the stability property.

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\[\text{When the Jack superpolynomial is expressed in terms of the variables (x, \theta) rather than in modes, the transformation is simply}\]

\[\Delta_m(x)^{-1} \partial_{\theta_m} \cdots \partial_{\theta_1} J^{(\alpha)}_\lambda(x, \theta) \quad \text{where} \quad \Delta_m(x) = \prod_{1 \leq i < j \leq m} (x_i - x_j).\]
Although they stabilize, the eigenfunctions still depend on \( m \). However, consider the map

\[
\begin{align*}
(A^\alpha; \Lambda^\gamma) & \longleftrightarrow (\lambda, \mu) : \quad b_{\lambda, \alpha}^1 a_{\lambda, \alpha}^0 |0\rangle & \longleftrightarrow s_\lambda(y) p_\mu(y, z),
\end{align*}
\]

where

\[
p_\mu = p_{\mu_1} \cdots p_{\mu_\ell} \quad \text{with} \quad p_n(y, z) = \sum_{i=1}^m y_i^n + \sum_{i=1}^{N-m} z_i^n,
\]

and \( s_\lambda \) is the Schur function. Observe that \( p_n(y, z) \) is simply \( p_n \) in the variables \( y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{N-m} \). This maps the above eigenfunctions corresponding to the values \( m = 2, 3, 4 \) to (inserting the proper normalization)

\[
J^{(\alpha)}_{(1), (1)}(y, z) = \frac{1}{(1 + \alpha)} [p_1(y, z)^2 + (\alpha - 1) s_1(y) p_1(y, z) - \alpha s_{1,1}(y) - \alpha s_2(y)]
\]

The stability has now been lifted to the full structure of the eigenfunction.

But in addition, the map (3.24) captures the factorization property suggested by the form of the eigenvalues. Using the Pieri rule for Schur functions [5, 16] to express the sum of the last two terms in a product form,

\[
s_{1,1}(y) + s_2(y) = s_1(y) s_1(y),
\]

we see that \( J^{(\alpha)}_{(1), (1)} \) can also be written in a product form

\[
J^{(\alpha)}_{(1), (1)}(y, z) = \frac{1}{(1 + \alpha)} \big( p_1(y, z) + \alpha s_1(y) \big) \big( p_1(y, z) - s_1(y) \big).
\]

This is a simple illustration of the announced factorization.

4. Double Jack Polynomials

In general, the action of the map (3.24) at the level of Jack polynomials is [6]

\[
J^{(\alpha)}_{\lambda}(b_1, b_2, \cdots, a_1, a_2, \cdots) |0\rangle \longleftrightarrow J^{(\alpha)}_{\lambda, \mu}(y, z)
\]

where

\[
J^{(\alpha)}_{\lambda, \mu}(y, z) = J^{(\alpha/(\alpha+1))}_{\lambda} \left[ Y + \frac{1}{\alpha + 1} Z \right] J^{(\alpha+1)}_{\mu}(z).
\]

Here we use the plethystic notation (see e.g., [5, 13]). In our case, it simply means that if \( J^{(\alpha/(\alpha+1))}_{\lambda}(p_1, p_2, p_3, \ldots) \) is the expansion of \( J^{(\alpha/(\alpha+1))}_{\lambda}(z) \) in terms of power-sums by

\[
\sum_{i=1}^{N-m} \left[ Y + \frac{1}{\alpha + 1} Z \right] J^{(\alpha/(\alpha+1))}_{\lambda}(z) = J^{(\alpha/(\alpha+1))}_{\lambda} \left( p_1(z) + \frac{1}{\alpha + 1} p_2(z), p_2(z) + \frac{1}{\alpha + 1} p_3(z), \ldots \right)
\]

that is, \( J^{(\alpha/(\alpha+1))}_{\lambda} \left[ Y + Z/(1 + \alpha) \right] \) is obtained from the expansion of \( J^{(\alpha/(\alpha+1))}_{\lambda}(z) \) in terms of power-sums by replacing \( p_n \) by \( p_n(y) + \frac{1}{\alpha + 1} p_n(z) \).

Let us recover (3.28) from the general expression (4.2). This is a particularly simple case given that \( J^{(\alpha)}_{(1)} = s_{(1)} = m_{(1)} = p_1 \). With \( p_1(y, z) = p_1(y) + p_1(z), J^{(\alpha)}_{(1), (1)}(y, z) \) becomes

\[
J^{(\alpha)}_{(1), (1)}(y, z) = \left( p_1(y) + \frac{1}{\alpha + 1} p_1(z) \right) p_1(z),
\]

which is indeed of the form (4.2). Here is a slightly more complicated example:

\[
J^{(\alpha)}_{(2), (0)}(y, z) = \frac{1}{(1 + \alpha)(1 + 2\alpha)} \left[ p_1(y, z)^2 + \alpha p_2(y, z) + 2\alpha s_{(1)}(y) p_1(y, z) + 2\alpha^2 s_{(2)}(y) \right].
\]

With \( s_{(2)} = (p_1^2 + p_2^2)/2 \), simple algebra yields

\[
J^{(\alpha)}_{(2), (0)}(y, z) = \frac{(1 + \alpha)}{(1 + 2\alpha)} \left( p_1[X]^2 + \frac{\alpha}{\alpha + 1} p_2[X] \right) = J^{(\alpha/(\alpha+1))}_{(2)}[X]
\]
with \( X = Y + Z/(\alpha + 1) \) and where in the last step, we used the expression \( J^{(\alpha)}_{(2)} = (p_1^2 + \alpha p_2)/(1 + \alpha) \).

A more formal characterization of \( J^{(\alpha)}_{\lambda,\mu}(y, z) \), which we call the double Jack polynomials, is as follows \([6]\). They are the unique bi-symmetric functions such that

\[
J^{(\alpha)}_{\lambda,\mu}(y, z) = s_\lambda(y)s_\mu(z) + \text{smaller terms} \tag{4.7}
\]

and

\[
\left\langle J^{(\alpha)}_{\lambda,\mu}(y, z), J^{(\alpha)}_{\nu,\kappa}(y, z) \right\rangle = 0 \quad \text{if} \quad (\lambda, \mu) \neq (\nu, \kappa) \tag{4.8}
\]

The triangularity condition that specifies the “smaller terms” refers to the double version of the dominance ordering:

\[
(\lambda, \mu) \preceq (\nu, \kappa) \iff |\lambda| + |\mu| = |\nu| + |\kappa|, \quad \sum_{i=1}^\ell (\lambda_i - \nu_i) \geq 0 \quad \text{and} \quad |\lambda| - |\nu| + \sum_{j=1}^{\ell}(\mu_j - \kappa_j) \geq 0 \quad \forall \ell \tag{4.9}
\]

while the orthogonality condition refers to the scalar product

\[
\left\langle s_\lambda(y) p_\mu(z), s_\nu(y) p_\kappa(z) \right\rangle = \delta_{\lambda\nu} \delta_{\mu\kappa} z_\mu \alpha^{\ell(\mu)} \tag{4.10}
\]

with \( z_\mu = \prod_{i \geq 1} i^{n_\mu(i)} n_\mu(i)! \), \( n_\mu(i) \) being the multiplicity of the part \( i \) in \( \mu \). Observe that this scalar product has the form

\[
\left\langle \bullet, \bullet \right\rangle = \left\langle \cdot, \cdot \right\rangle_{\text{Schur}}^{y} \left\langle \cdot, \cdot \right\rangle_{\text{Jack}}^{y,z} \tag{4.11}
\]

where \( \left\langle \cdot, \cdot \right\rangle_{\text{Schur}}^{y} \) is the scalar product with respect to which the Schur functions \( s_\lambda(y) \) are orthonormal while \( \left\langle \cdot, \cdot \right\rangle_{\text{Jack}}^{y,z} \) is the scalar product with respect to which the Jack polynomials \( J^{(\alpha)}_\lambda(y,z) \) are orthogonal (\( J^{(\alpha)}_\lambda(y,z) \) being the usual Jack polynomials in the variables \( y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{N-m} \)).

5. The double CS model and an emerging \( \tilde{\mathfrak{sl}}_2 \)

Let us now unravel the integrable model whose eigenfunctions are the double Jack polynomials. The factorized expression \([4.2]\) of these polynomials and the splitting of the eigenvalue displayed in \([5.21]\) readily indicate that the underlying Hamiltonian \( \mathcal{H}_D \) is a sum of two CS Hamiltonians, albeit with modified coupling constants and involving unusual variables:

\[
\mathcal{H}_D = (\alpha + 1)\mathcal{H}_1 + \mathcal{H}_2 \tag{5.1}
\]

where

\[
\mathcal{H}_1 = \hat{\mathcal{H}}^{(\alpha/(\alpha+1))} \quad \text{with} \quad \left\{ \begin{array}{l} p_n \mapsto p_n[X] \\ \partial_{p_n} \mapsto \partial_{p_n[X]} \end{array} \right. \quad \text{where} \quad X = Y + (\alpha + 1)^{-1}Z, \tag{5.2}
\]

and

\[
\mathcal{H}_2 = \hat{\mathcal{H}}^{(\alpha+1)} \quad \text{with} \quad \left\{ \begin{array}{l} p_n \mapsto p_n(z) \\ \partial_{p_n} \mapsto \partial_{p_n(z)} \end{array} \right. \tag{5.3}
\]

Note that \( p_n[X] \) and \( p_n(z) \) are considered to be independent. Being the sum of two independent integrable Hamiltonians, \( \mathcal{H}_D \) trivially characterizes a new integrable model.

However, the above splitting of \( \mathcal{H}_D \) is not very interesting since it is hard to give a physical meaning to the power-sums \( p_n[X] \), \( p_n(z) \) and their derivatives. The structure of the scalar product \([4.10]\) points toward a more interesting choice of variables, namely \( p_n(y) \) and \( p_n(y,z) \), whose adjoints are \( n\partial_{p_n(y)} \) and \( n\alpha\partial_{p_n(y,z)} \) respectively. With

\[
X = Y + \frac{1}{\alpha + 1}Z = \frac{\alpha}{\alpha + 1}Y + \frac{1}{\alpha + 1}(Y + Z), \tag{5.4}
\]

the change of variables is thus

\[
p_n[X] = \frac{\alpha}{\alpha + 1}p_n(y) + \frac{1}{\alpha + 1}p_n(y, z)
\]

\[
p_n(z) = p_n(y, z) - p_n(y), \tag{5.5}
\]
which gives (using the chain rule in two variables)

\[ \partial p_n(x) = \partial p_n(y) + \partial p_n(y,z) \]
\[ \partial p_n(z) = \frac{\alpha}{\alpha + 1}\partial p_n(y,z) - \frac{1}{\alpha + 1}\partial p_n(y). \]  

(5.6)

These expressions are readily checked by verifying that they satisfy the commutation relations:

\[ [\partial p_n(x), p_m(x)] = \delta_{n,m}, \quad [\partial p_n(z), p_m(z)] = \delta_{n,m}, \]
\[ [\partial p_n(x), p_m(z)] = 0 \quad \text{and} \quad [\partial p_n(z), p_m(x)] = 0. \]  

(5.7)

(5.8)

For these manipulations, we stress that \( p_n(y) \) and \( p_n(y,z) \) are considered to be independent, meaning:

\[ [\partial p_n(y), p_m(y,z)] = [\partial p_n(y,z), p_m(y)] = 0. \]  

(5.9)

We then substitute (5.5) and (5.6) into \((\alpha + 1)\mathcal{H}_1 + \mathcal{H}_2\). The result, obtained after straightforward manipulations, is best rewritten in terms of two independent sets of bosonic modes defined as

\[ A_n^\dagger = p_n(y) \quad \text{and} \quad A_n = n\partial p_n(y) \quad (\Rightarrow [A_k, A_n^\dagger] = k\delta_{k,n}). \]  

(5.10)

Together with

\[ a_n^\dagger = p_n(y,z) \quad \text{and} \quad a_n = n\alpha \partial p_n(y,z) \quad (\Rightarrow [a_k, a_n^\dagger] = k\alpha \delta_{k,n}). \]  

(5.11)

The resulting form of \( \mathcal{H}_D \) is

\[ \alpha \mathcal{H}_D = \sum_{k,\ell \geq 1} \left[ a_k^\dagger a_{k+\ell} + a_k^\dagger a_{k+\ell} a_k a_{\ell} \right] + (\alpha - 1) \sum_{\ell \geq 1} (\ell - 1) a_{\ell} a_{\ell} - \alpha \sum_{\ell \geq 1} (\ell - 1) \left[ a_{\ell}^\dagger A_{\ell} + A_{\ell}^\dagger a_{\ell} \right] \]
\[ + \alpha \sum_{k,\ell \geq 1} \left[ 2 a_k^\dagger A_k A_{k+\ell} + a_k^\dagger a_{k+\ell} a_k A_{\ell} \right] + \alpha \sum_{k,\ell \geq 1} \left[ A_k^\dagger A_{k+\ell} a_k A_{\ell} + 2 A_k^\dagger a_{k+\ell} A_{k} A_{\ell} \right] \]
\[ + \alpha (\alpha - 1) \sum_{k,\ell \geq 1} \left[ A_k^\dagger A_{k+\ell} A_{k+\ell} + A_k^\dagger a_{k+\ell} A_{k} A_{\ell} \right]. \]  

(5.12)

It turns out that \( \mathcal{H}_D \) can be reexpressed as

\[ \mathcal{H}_D = \mathcal{H}_{y,z}^{(\alpha)} + (\alpha - 1)\mathcal{H}_y^{(1)} + [Q_1, \mathcal{H}_y^{(1)}] - \frac{1}{2}[Q_1, Q_2] \]  

(5.13)

where

\[ Q_1 = \sum_{\ell \geq 1} \frac{1}{\ell} \left[ a_{\ell}^\dagger A_{\ell} - A_{\ell}^\dagger a_{\ell} \right] \quad \text{and} \quad Q_2 = \sum_{\ell \geq 1} (\ell - 1) \left[ A_{\ell}^\dagger A_{\ell} - \frac{1}{\alpha} a_{\ell}^\dagger a_{\ell} \right]. \]  

(5.14)

Note that \( \mathcal{H}_{y,z}^{(\alpha)} \) is \( \mathcal{H}^{(\alpha)} \) in the variables \( y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{N-m} \), so that:

\[ \alpha \mathcal{H}_{y,z}^{(\alpha)} = \sum_{k,\ell \geq 1} \left[ a_k^\dagger a_{k+\ell} + a_k^\dagger a_{k+\ell} a_k a_{\ell} \right] + (\alpha - 1) \sum_{\ell \geq 1} (\ell - 1) a_{\ell}^\dagger a_{\ell}. \]  

(5.15)

Similarly, \( \mathcal{H}_y^{(1)} \) is \( \mathcal{H}^{(\alpha)} \) in the variables \( y_1, y_2, \ldots, y_m \) but evaluated at \( \alpha = 1 \):

\[ \alpha (\alpha - 1)\mathcal{H}_y^{(1)} = \alpha (\alpha - 1) \sum_{k,\ell \geq 1} \left[ A_k^\dagger A_k A_{k+\ell} + A_k^\dagger a_{k+\ell} A_{k} A_{\ell} \right]. \]  

(5.16)

Next, it is simple to check that \( \alpha [Q_1, \mathcal{H}_y^{(1)}] \) yields the second line in (5.12). Therefore, parts of the constituents of \( \mathcal{H}_D \) have a direct interpretation in terms of variables. However, this is not the case for \( Q_1 \) and \( Q_2 \). Note that the action of \( Q_1 \) amounts to exchanging the \( a \) and \( A \) modes (which thereby appears to be a remnant of the action of a supersymmetric charge). Nevertheless, it turns out that \( Q_1 \) and \( Q_2 \) have a nice Lie algebraic interpretation. More precisely, both are combinations of the generators of an underlying affine \( \mathfrak{sl}_2 \) algebra (whose existence is not surprising in the presence of two independent infinite sets of bosonic modes). It is straightforward to verify that the operators

\[ e^{(k)} = \frac{1}{\sqrt{\alpha}} \sum_{\ell \geq 1} \ell^{k-1} A_{\ell}^\dagger a_{\ell}, \quad f^{(k)} = \frac{1}{\sqrt{\alpha}} \sum_{\ell \geq 1} \ell^{k-1} a_{\ell}^\dagger A_{\ell} \quad \text{and} \quad h^{(k)} = \sum_{\ell \geq 1} \ell^{k-1} \left[ A_{\ell}^\dagger A_{\ell} - \frac{1}{\alpha} a_{\ell}^\dagger a_{\ell} \right] \]  

(5.17)
do satisfy the $\hat{\mathfrak{sl}}_2$ commutation relations
\[
[e^{(k)}, f^{(\ell)}] = h^{(k+\ell)} , \quad [h^{(k)}, e^{(\ell)}] = 2e^{(k+\ell)} , \quad [h^{(k)}, f^{(\ell)}] = -2f^{(k+\ell)} .
\] (5.18)

We thus get that
\[
Q_1 = \sqrt{\alpha}(f^{(0)} - e^{(0)}) \quad \text{and} \quad Q_2 = h^{(1)} - h^{(0)}
\] (5.19)
and, as such, $\mathcal{H}_D$ is built from a special intertwining of $\mathcal{H}^{(\alpha)}_{y,z}$ and $\mathcal{H}^{(1)}_y$ with the generators $e^{(0)}$, $f^{(0)}$ and $h^{(1)}$ of the nonnegative part of $\hat{\mathfrak{sl}}_2$.

This intertwining pattern is expected to hold for all the conserved quantities of the double CS model. Consider for instance the two conserved quantities of degree 1
\[
\mathcal{I}_D = (\alpha - 1) \sum_{\ell \geq 1} A_\ell^+ A_\ell + \sum_{\ell \geq 1} \left[ a_\ell^+ A_\ell + A_\ell^+ a_\ell \right] ,
\] (5.20)
\[
\mathcal{P}_D = \sum_{\ell \geq 1} \left[ A_\ell^+ A_\ell + \frac{1}{\alpha} a_\ell^+ a_\ell \right] ,
\] (5.21)
whose eigenvalues are respectively (3.15) and $|\lambda| + |\mu|$. As for $\mathcal{H}_D$, the conserved quantity $\mathcal{I}_D$ can be written in terms of the usual conserved quantities of the two CS models specified by $\mathcal{H}^{(\alpha)}_{y,z}$ and $\mathcal{H}^{(1)}_y$, and the (nonnegative-mode) generators of $\hat{\mathfrak{sl}}_2$:
\[
\mathcal{I}_D = (\alpha - 1) \mathcal{P}_y + [Q_1 , \mathcal{P}_y] ,
\] (5.22)
where $\mathcal{P}_y$ is the momentum operator $\mathcal{P}$ in the variables $y_1, \ldots, y_m$. Similarly, we have
\[
\mathcal{P}_D = \mathcal{P}_y + \mathcal{P}_{y,z} .
\] (5.23)

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