Sinkhorn normal form for unitary matrices

MARTIN IDEL, MICHAEL M. WOLF
Zentrum Mathematik, Technische Universität München

Abstract

Sinkhorn proved that every entry-wise positive matrix can be made doubly stochastic by multiplying with two diagonal matrices. In this note we prove a recently conjectured analogue for unitary matrices: every unitary can be decomposed into two diagonal unitaries and one whose row- and column sums are equal to one. The proof is non-constructive and based on a reformulation in terms of symplectic topology. As a corollary, we obtain a decomposition of unitary matrices into an interlaced product of unitary diagonal matrices and discrete Fourier transformations. This provides a new decomposition of linear optics arrays into phase shifters and canonical multi-ports described by Fourier transformations.

1. Introduction

For every \( n \times n \) matrix with positive entries there exist two diagonal matrices \( L, R \) such that \( LAR \) is doubly stochastic, i.e. the entries of each column and row sum up to one. This result was first obtained by Sinkhorn [Sin64], who also gave an algorithm how to compute \( L \) and \( R \) by iterated left and right multiplication of diagonal matrices.

Recently, De Vos and De Baerdemacker studied the same problem for unitary matrices [DVBL4]. They conjectured that for every \( n \times n \) unitary \( U \) there exist two unitary diagonal matrices \( L, R \) such that \( LUR \) has all row and column sums equal to one. To support their conjecture, they construct an algorithm similar to the iteration procedure for matrices with positive entries from [Sin64, SK67]. They also provide numerical evidence that the algorithm always converges to a unitary matrix with row and column sums equal to one.

The goal of this paper is to prove the conjecture of De Vos and De Baerdemacker that such a normal form always exists by reformulating the problem in terms of symplectic topology. It turns out that the reformulated problem is a special case of the Arnold (sometimes Arnold-Givental) conjecture on the intersection of Lagrangian submanifolds [MS98], which was solved for this case in [BEP04, Cho04]. More precisely, in section 2 we show:

**Theorem 2.** For every unitary matrix \( U \in U(n) \) there exist two diagonal unitary matrices \( L, R \in U(n) \) such that \( A := LUR \) satisfies \( \sum_j A_{ji} = \sum_j A_{ij} = 1 \) for all \( i = 1, \ldots, n \).
For a given unitary $U \in U(n)$ the pair $(L, R)$ is certainly not unique, since multiplying $L$ by a global phase and $R$ by its inverse does not change $A$. Hence, it makes sense to consider the decomposition $U = e^{i\varphi} L'A R'$, where $L', R'$ are unitary diagonal such that $L_{11}' = R_{11}' = 1$ and $\varphi \in [0, 2\pi)$. This decomposition is not unique, either, but for the generic case, there is only a finite number of decompositions. In particular, for $U(2)$, a simple complete solution was given in [DVB14] from which one can see that for every nondiagonal matrix, there are only two different decompositions. The reformulation in terms of symplectic topology gives further insight into the freedom of the decomposition. In addition to the Sinkhorn-type normal form above, in section 3 we give several reformulations that might be interesting for applications, for instance regarding the decomposition of general $2n$-port linear optics devices into canonical multiports and phase shifters.

2. Sinkhorn-type normal form

In order to prove the decomposition theorem, we reformulate the problem of rescaling a unitary matrix into a problem in symplectic topology. For the reader’s convenience, necessary results including elementary calculations and definitions are included in appendix [A]. We only repeat the most important definitions for our reformulation. Recall that the complex projective space $\mathbb{C}P^n$ consists of all equivalence classes of $\mathbb{C}^{n+1} \setminus \{0\}$ w.r.t. $x \sim y \iff x = \lambda y$ with $\lambda \in \mathbb{C} \setminus \{0\}$.

**Definition 1.** The Clifford Torus is the $n$-dimensional torus embedded in $\mathbb{C}P^n$, i.e. the set of points

$$T^n := \{[w_0, \ldots, w_n] \in \mathbb{C}P^n | |w_0| = |w_1| = \ldots = |w_n|\}.$$ 

This torus, as shown in the appendix in proposition 4, is a Lagrangian submanifold of the symplectic manifold $\mathbb{C}P^n$. We obtain the following connection to our normal form:

**Lemma 1.** For any unitary $U \in U(n)$, there exist diagonal unitaries $L$ and $R$ such that $A := LUR$ has row and column sums equal to one if and only if the Clifford torus $T^{n-1} \subset \mathbb{C}P^{n-1}$ fulfills $T^{n-1} \cap UT^{n-1} \neq \emptyset$.

**Proof.** Let $U \in U(n)$ be arbitrary but fixed. We first consider the usual torus $T^n \subset \mathbb{C}^n$, i.e. the set of all vectors for which each component has modulus one:

$$T^n := \{e^{i\phi_1}, \ldots, e^{i\phi_n} \subset \mathbb{C}^n | \phi_j \in \mathbb{R}\}$$

Let us first show that the existence of a normal form is equivalent to $T^n \cap UT^n \neq \emptyset$. For one direction, let $\varphi \in T^n$ such that $U \varphi \in T^n$, i.e. $\varphi \in T^n \cap UT^n$. Define the two diagonal matrices $R^{-1} := \text{diag}(\varphi_1, \ldots, \varphi_n) \in U(n)$ and $L^{-1} := \text{diag}((U \varphi)_1^{-1}) = \text{diag}((U \varphi)_i) \in U(n)$. With $A := L^{-1} U R^{-1}$ and $e := (1, \ldots, 1)^T$ we obtain:

$$Ae = L^{-1} U \varphi = e$$

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since \((U\varphi)_i(U\varphi)_i = |(U\varphi)_i| = 1\) for all \(i \in \{1, \ldots, n\}\) due to \(U\varphi \in \mathbb{T}^n\). Likewise, since \(Ae = Ae\) and \(A\) is unitary, we obtain
\[
A^T e = A^T Ae = e.
\]
so that columns and rows of \(A\) sum up to one.

For the other direction, suppose \(U = LAR\) is a decomposition as proposed. Then \(\varphi := R^{-1}e \in \mathbb{T}^n\) and
\[
U\varphi = LAR\varphi = LAe = Le \in \mathbb{T}^n
\]
hence \(U\varphi \in \mathbb{T}^n \cap UT^n\).

The next step is to reformulate the problem using the Clifford torus. Clearly, \(T^{n-1} \cap UT^{n-1} \neq \emptyset\) iff \((\lambda \mathbb{T}^n) \cap UT^n \neq \emptyset\) for some \(\lambda \in \mathbb{C} \setminus \{0\}\). Since \(U\) is norm preserving, any intersection requires \(|\lambda| = 1\) so that
\[
T^{n-1} \cap UT^{n-1} \neq \emptyset \quad \leftrightarrow \quad \mathbb{T}^n \cap UT^n \neq \emptyset.
\]

One of the main conjectures in symplectic topology, the Arnold or Arnold-Givental conjecture, states that a Lagrangian submanifold and its image under a Hamiltonian isotopy intersect at least as often as the sum of the \(\mathbb{Z}_2\)-Betti-numbers. For \(T^n\), this sum is not zero, thus, using proposition 5, Arnold’s conjecture states in particular that \(T^n\) should intersect with \(UT^n\) at least once. While the Arnold conjecture is wrong in all generality and most cases are unknown, there is a positive result to the weaker question whether the torus intersects with its displaced version (c.f. [BEP04, Cho04]). In order to formulate this result, we need the following:

**Definition 2.** Let \((M, \omega)\) be a closed symplectic manifold with Hamiltonian symplectomorphisms \(\text{Ham}(M)\). A Lagrangian submanifold \(L \subset M\) is called displaceable by a Hamiltonian diffeomorphism, if there exists a \(\psi \in \text{Ham}(M)\) such that
\[
L \cap \psi L = \emptyset.
\]

The definition is slightly different from the one in [BEP04], where the authors only consider nonempty open sets such that the restriction of \(\omega\) to these sets is exact. However, they prove that the torus \(T^n\) is displaceable in the above definition, if and only if there exists an open neighbourhood \(V \supset T^n\) such that \(\omega|_V\) is exact and \(V\) is displaceable. With this we can state the final and crucial ingredient in the proof of the normal form:

**Theorem 1** ([BEP04] theorem 1.3). The Clifford torus \(T^n \subset \mathbb{CP}^n\) cannot be displaced from itself by a Hamiltonian isotopy.

Because every unitary matrix defines a Hamiltonian isotopy (see proposition 5 in the appendix), the theorem tells us in particular \(T^n \cap UT^n \neq \emptyset\) for all unitaries \(U \in U(n)\) so that together with lemma 1, this proves the sought normal form:

**Theorem 2.** For every unitary matrix \(U \in U(n)\) there exist two diagonal unitary matrices \(L, R \in U(n)\) such that \(A := LUR\) fulfills \(\sum_{j} A_{ji} = \sum_{j} A_{ij} = 1\) for all \(i = 1, \ldots n\).
3. Equivalent normal forms for unitary matrices

To obtain equivalent normal forms, consider the \( n \times n \) dimensional complex matrix \( F_n \) with entries \( (F_n)_{kl} := \frac{1}{\sqrt{n}} \exp\left(\frac{2\pi i}{n} kl\right) \) with \( k, l \in \{0, \ldots, n-1\} \), which is known as the discrete Fourier transformation. It is easy to see that \( F_n^{-1} = F_n^\dagger \), hence \( F_n \in U(n) \). If we denote the standard basis of \( \mathbb{C}^n \) by \( \{e_i\}_{i=0}^{n-1} \) and \( e := (1, \ldots, 1)^T \), then

\[
F_n e_0 = F_n^\dagger e_0 = \frac{e}{\sqrt{n}}.
\]

Now let \( A \in U(n) \) be such that \( Ae = A^T e = e \). Then \( F_n^\dagger AF_n e_0 = e_0 \) and similarly, \( (F_n^\dagger AF_n)^T e_0 = F_n A^T F_n^\dagger e_0 = e_0 \), which shows that

\[
F_n^\dagger AF_n = \begin{pmatrix} 1 & 0_n^{-1} \\ 0_{n-1} & \tilde{U} \end{pmatrix}
\]

where \( 0_{n-1} := 0 \in \mathbb{C}^{n-1} \) and \( \tilde{U} \in U(n-1) \). Thus, given a unitary \( U \in U(n) \), we know that there exists a decomposition

\[
U = LF_n \begin{pmatrix} 1 & 0_n^{-1} \\ 0_{n-1} & \tilde{U} \end{pmatrix} F_n^\dagger R
\]

with \( \tilde{U} \in U(n-1) \) and diagonal \( L, R \in U(n) \). We can now iterate the procedure by applying it to the \((n-1) \times (n-1)\)-dimensional submatrix \( \tilde{U} \) and obtain the corollary:

**Corollary 1.** Let \( U \in U(n) \), then there exist diagonal unitary matrices \( D_1, \ldots, D_n \) and \( \tilde{D}_1, \ldots, \tilde{D}_{n-1} \) and a \( \varphi \in [0, 2\pi) \) such that the first \( i-1 \) entries in each \( D_i, \tilde{D}_i \) are equal to one and

\[
U = D_1 F_n D_2 (1_2 \oplus F_{n-1}) D_3 (1_2 \oplus F_{n-2}) \cdots D_{n-1} (1_{n-2} \oplus F_2) D_n (1_{n-2} \oplus F_1) \tilde{D}_{n-1} \cdots (1_1 \oplus F_1) \tilde{D}_2 F_2^\dagger \tilde{D}_1 e^{i\varphi}.
\]

In other words any unitary can be decomposed into diagonal unitaries and discrete Fourier transformations in this way. This has an immediate application in quantum optics, where any \( n \times n \) unitary corresponds to a passive transformation on \( n \) modes or a \( 2n \)-multiport. In this scenario a diagonal unitary corresponds to a set of phase shifters, which are applied to the modes individually and the discrete Fourier transformation is known as canonical \( 2n \)-multiport [MMW+95], which may be implemented by a symmetric fibre coupler. The structure if the corresponding decomposition is graphically depicted in Figure [1].

Another version of the normal form is found by using that \( D \) is a diagonal matrix iff \( FDF^\dagger \) is a circulant matrix, i.e. \( (FDF^\dagger)_{ij} =: \alpha_{i-j} \in \mathbb{C} \). Since the diagonal matrices form a group, so do the circulant matrices and we denote the group of \( n \times n \) circulant matrices by Circ(\( n \)). Then:
Figure 1: In quantum optics, passive transformations on $n$ modes are in one-to-one correspondence with $n \times n$ unitaries. Up to an overall phase, each unitary $U$ admits a decomposition into $2(n-1)$ canonical multiports (which are independent of $U$ and described by discrete Fourier transformations [blue]) surrounded by $2n-1$ layers of single-mode phase shifters [grey]. Here, this is exemplified for $n = 4$.

**Corollary 2.** Let $U \in U(n)$, then there exist $C_1, C_2 \in \text{Circ}(n)$ and $\tilde{U} \in U(n-1)$ such that

$$U = C_1 \text{diag}(1, \tilde{U}) C_2.$$ 

Let us finally discuss the question of uniqueness of these decompositions and to this end come back to the original normal form

$$U = e^{i\varphi}D_1 AD_2,$$ (1)

where $D_1, D_2$ are unitary diagonal with $(D_i)_{11} = 1$ and $A$ has row and column sums equal to 1. Counting parameters, using that the matrices $A$ are isomorphic to $U(n-1)$ as proven above, we have:

$$1 + (n - 1) + (n - 1)^2 + (n - 1) = n^2$$

parameters (c.f. [DVB14]). Hence, the number of parameters matches exactly the dimension of $U(n)$. Given a unitary $U = e^{i\varphi}D_1 AD_2$ as above, this means that as long as there does not exist a diagonal unitary $D$ which commutes with $U$ (and thus with $A$), there should be only a discrete set of different decompositions. The exact number of different decompositions can easily be seen to be two for the case $n = 2$ (c.f. [DVB14]), but already for $n = 3$ and $n = 4$, there is only a conjectured bound (6 and 20, c.f. [Shc13]).

In [Cho04] it is proven that if $T^n$ and $UT^n$ intersect transversally, their number of distinct intersection points must be at least $2^n$, which follows from general results in Floer-homology theory when applied to Lagrangian intersection theory. Since transversality
is a generic property for intersections, one might therefore conjecture that for a generic unitary $U \in U(n)$ [Cho04] implies a lower bound $2^{n-1}$ on the number of different normal forms.

### 4. Conclusion

We have studied variants of a Sinkhorn type normal form for unitary matrices. Its existence was conjectured in [DVB14] and we give a nonconstructive proof. This means in particular that the question, whether the algorithm presented in [DVB14] always converges for any set of starting conditions, remains open. Also, it would be nice to have an elementary proof of the fact that for any unitary matrix $U$ we have $T^n \cap UT^n \neq \emptyset$. By counting parameters, it becomes clear that for generic unitaries, there exists no continuous family of decompositions. We suggested an argument that it might grow exponentially in the dimension, however this lower bound relies on a lower bound on Lagrangian intersections which holds only for transversal intersections.

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A. Symplectic Preliminaries

This section introduces the definitions and results from symplectic topology beyond the first chapters of [MS98] needed to understand the basic reductions of the proof of theorem [1] in [BEPT].

A.1. Notation and basic definitions

To fix notation, a symplectic manifold will always be denoted by $M$ and its symplectic form will be called $\omega$. The group of symplectomorphisms of a symplectic manifold $(M, \omega)$ will be denoted by $\text{Symp}(M)$ and its Hamiltonian symplectomorphisms (i.e. all symplectomorphisms which are elements of the flow of a Hamiltonian vector field) will be denoted by $\text{Ham}(M)$. We have the following characterization ([MS98], chapter 10):

**Proposition 3.** Let $(M, \omega)$ be a closed symplectic manifold. If the manifold is simply connected (i.e. every loop is contractible)

$$\text{Ham}(M) = \text{Symp}_0(M)$$

where $\text{Symp}_0(M)$ denotes the connected component of the identity of the whole group of symplectomorphisms.

In principle, the result also holds for arbitrary symplectic manifolds. One has to be more careful with non-compactly supported functions, but we can safely ignore these subtleties, since our manifold of interest will be closed.

Furthermore, let us recall that a Lagrangian submanifold $L$ of a $2n$-dimensional symplectic manifold $(\mathcal{M}, \omega)$ is a smooth $n$-dimensional submanifold of $M$ such that

$$T_pL^c := \{ X \in T_p\mathcal{M} | \omega(X, Y) = 0 \ \forall Y \in T_pL \} = T_pL \ \forall p \in L$$

A.2. The Clifford-torus as a Lagrangian submanifold

We now study the Clifford torus as a special case of the Lagrangian submanifold of interest for our result.

Before proving that the Clifford torus is a Lagrangian submanifold, we need to specify the symplectic structure on $\mathbb{C}P^n$: Consider the map $\Phi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{S}^{n+1} \subset \mathbb{C}^{n+1}$ via $z \mapsto z/|z|$. We will show that the pullback $\Phi^*\omega$ of the standard symplectic structure $\omega$ on $\mathbb{C}^{n+1}$ descends to a symplectic form $\omega_{FB}$ on $\mathbb{C}P^n$, the standard symplectic structure or Fubini-Study form of the complex projective space.

**Proposition 4.** $\mathbb{C}P^n$, equipped with the Fubini-Study form is a $2n$-dimensional symplectic manifold and the Clifford Torus is a Lagrangian submanifold thereof.

**Proof.** Let us go through the construction in more detail and see, how it defines a symplectic form, e.g. a non-degenerate and closed 2-form on $\mathbb{C}P^n$. Throughout, we will consider the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$. 

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Note that if \((x_0, y_0, \ldots, x_n, y_n)\) are the real coordinates of \(\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}\), we can use \((z_0, \bar{z}_0, \ldots, z_n, \bar{z}_n)\) as coordinates for any point \((z_0, \ldots, z_n)\) \(\in \mathbb{C}^{n+1}\) as well. Then the standard symplectic form reads

\[
\omega = \sum_j dx^j \wedge dy^j = \frac{i}{2} \sum_j dz^j \wedge d\bar{z}^j
\]

Considering the action of \(\mathbb{C}^*\) on \(\mathbb{C}^{n+1}\), we obtain \(\omega_{\lambda, z} = \frac{i}{2} \sum_j d(\lambda \cdot z^j) \wedge d(\bar{\lambda} \cdot \bar{z}^j) = |\lambda|^2 \frac{i}{2} \sum_j dz^j \wedge d\bar{z}^j = |\lambda|^2 \omega_z\). Hence, if \(\Phi : \mathbb{C}^{n+1} \setminus \{0\} \to S^{2n+1}\) is given by \(z \mapsto z/|z|\), then \(\Phi^* \omega\) will be invariant under the action of \(\mathbb{C}^*\). This shows that \(\Phi^* \omega\) descends to a well-defined 2-form \(\omega_{FS}\) on \(\mathbb{C}P^n\), by defining:

\[
(\omega_{FS})_{\pi(p)}(d\pi X_{\pi(p)}, d\pi Y_{\pi(p)}) = (\Phi^* \omega)_p(X, Y)
\]

The next step is to show non-degeneracy. For this, note that \(\Phi^* \omega(X, Y) = 0\) \(\forall Y\) if and only if \(d\Phi X = 0\) pointwise, since \(\omega\) is non-degenerate. But \(d\Phi X = 0\) implies in particular \(d\pi X = 0\) and hence, \(\omega_{FS}\) as defined above is a non-degenerate 2-form.

Finally, we need to prove closedness. This can either be computed directly by considering coordinates, or by considering local sections of the projection \(\pi\). Let \(\{U_i\}_i\) be a cover of \(\mathbb{C}P^n\) such that there exist local section \(\sigma_i : U_i \to \mathbb{C}^{n+1} \setminus \{0\}\). On each \(U_i\) we have \(\omega_{FS} = \sigma_i^* \Phi^* \omega\). But then

\[
d\omega_{FS} = d(\sigma_i^* \Phi^* \omega) = (\sigma_i \Phi)^* d\omega = 0
\]

since \(d\) commutes with pullbacks and \(\omega\) is closed. Since this holds on any patch \(U_i\), \(d\omega_{FS} = 0\) globally.

In addition, we need to see that the Clifford torus is a Lagrangian submanifold. It is easy to see that the Clifford torus is a submanifold of (real) dimension \(n\), hence we only need to prove \((T_pT^n)\epsilon = T_pT^n \forall p \in T^n\). Given the canonical projection \(\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n\), \(T^n\) is the image of \(\pi\) of the torus

\[
T^n := \{(z_0, \ldots, z_n)|z_0| = |z_1| = \ldots = |z_n| = 1\}
\]

By inspection, we obtain for \(p = (p_0, \ldots, p_n) \in \mathbb{C}^{n+1} \setminus \{0\}\):

\[
T_p T^n = \text{span}\{p_i \partial_{p_i} - p_i \partial_{\bar{p}_i}|i = 0, \ldots, n\} =: \text{span}\{X_p^i|i = 0, \ldots, n\}
\]

Then \(T_{\pi(p)} T^n\) will be spanned by \(d\pi X_{\pi(p)}^i\).

Now, since already on the level of \(\omega\), we have \(\omega_p(X_p^i, X_p^j) = 0\) for all \(i, j \in \{0, \ldots, n\}\) and all \(p \in \mathbb{C}^{n+1} \setminus \{0\}\), it is immediate that also \((\omega_{FS})_{\pi(p)}(\pi_* X_p^i, \pi_* X_p^j) = 0\) for all \(i, j\) and for all \(\pi(p) \in \mathbb{C}P^n\). Hence we have that \((T_p T^n)\epsilon \supseteq T_p T^n \forall p \in T^n\). Since equality then has to hold by dimensional analysis, we have \(T^n\) is a Lagrangian submanifold. 

Now consider the standard action of \(U \in U(n+1)\) on \(\mathbb{C}^{n+1}\). Note that \(U\) leaves \(\omega\) invariant, since \(\sum_i d(U z)^i \wedge d\bar{U} \bar{z}^i = \sum_{ijk} U_{ij} \bar{U}_{ik} dz^j \wedge d\bar{z}^k = \sum_i dz^i \wedge d\bar{z}^i\). Furthermore,
since $U$ leaves the norm invariant by definition, we have that $U^*\omega_{FS} = \omega_{FS}$, where $U^*$ is the pullback associated with the map $U$. This means that any unitary $U \in U(n + 1)$ corresponds to a symplectomorphism of $\mathbb{C}P^n$. Since it is well-known that the complex projective space is simply connected and closed, its Hamiltonian symplectomorphism correspond to its symplectomorphism. Hence:

**Proposition 5.** We have $U(n + 1) \subset \text{Ham}(\mathbb{C}P^n, \omega_{FS})$, where the identification is achieved by considering the standard action of $U$ on $\mathbb{C}^{n+1}$. 