Estimates of non-optimality of quantum measurements and a simple iterative method for computing optimal measurements

Jon Tyson*
Harvard University
Feb 14, 2008

Abstract
We construct crude estimates for non-optimality of quantum measurements in terms of their violation of Holevo’s simplified minimum-error optimality conditions. As an application, we show that a modification of Barnett and Croke’s proof of the optimality conditions yields a convergent iterative scheme for computing optimal measurements.

*jonetyson@X.Y.Z, where X=post Y=harvard, and Z=edu
1 Introduction

The minimum-error quantum detection problem arose in the 1960’s in the design of optical detectors [1] and has been of recent importance in the subjects of quantum information [2, 3, 4, 5] and quantum computation [6, 7, 8, 9, 10, 11]:

If an unknown state \( \rho_k \) is randomly chosen from a known ensemble of quantum states, what is the chance that the value of \( k \) will be discovered by an optimal measurement?

Barnett and Croke [12] have recently provided a simple operator-theoretic proof of the necessity of the standard Yuen-Kennedy-Lax & Holevo (YKLH) optimality conditions [13, 14] for the minimum-error quantum detection problem. Their proof may be shortened, since Holevo [15] had previously shown that an intermediate step of their proof (positivity of the operators \( \hat{G}_j \) defined by equation (10) of [12]) provides a simplified necessary and sufficient condition for minimum-error quantum detection.

1.1 Results

This note gives a more robust version of Holevo’s simplified optimality condition (condition II of Theorem 2, below), by estimating non-optimality in terms of quantitative violation of this condition. These bounds are used to show that the perturbative method of Barnett and Croke may be converted into a convergent iterative algorithm for computing optimal measurements, adding to the list [16, 17, 18, 19] of algorithms for this purpose. This iteration converges even for countably-infinite ensembles in an infinite-dimensional Hilbert space.

2 Conditions for minimum-error quantum discrimination

A precise description of the minimum-error quantum measurement problem is given by:

**Definition 1** Let \( \mathcal{E} = \{\rho_k\}_{k \in K} \) be an ensemble of mixed quantum states \( \rho_k \), which are represented as positive semidefinite operators on a Hilbert space \( \mathcal{H} \) normalized by a-priori probability: \( \text{Tr} \rho_k = p_k \) with \( \sum p_k = 1 \). The **support** \( \text{supp} (\mathcal{E}) \) is the closure of the span of the ranges of the \( \rho_k \). A **positive operator-valued measurement (POVM)** is a collection of positive semidefinite operators \( \{M_k\} \) satisfying \( \sum M_k = 1 \). The corresponding **Lagrange operator** is given by

\[
L = \sum M_k \rho_k.
\] (1)
The minimum-error quantum discrimination problem consists of finding a POVM maximizing the success probability

\[ P_{\text{succ}}(\{M_k\}) = \text{Tr} \sum_k M_k \rho_k = \text{Tr} L \]  

of correctly distinguishing an element blindly drawn from the ensemble \( \mathcal{E} \). (We will often abuse notation by writing \( P_{\text{succ}}(M_k) \) instead of \( P_{\text{succ}}(\{M_k\}) \).)

Holevo's simplified optimality conditions are given by property II of 1

**Theorem 2 (Holevo [15], Yuen-Kennedy-Lax [13], )** Let \( \{M_k\}_{k=1,...,m} \) be a POVM for distinguishing the ensemble \( \mathcal{E} \). Then the following are equivalent:

I. \( \{M_k\} \) maximizes \( P_{\text{succ}} \).

II. \( \left( L + L^\dagger \right) / 2 \geq \rho_k \) for all \( k \). \(^2\)

III. There exists a self-adjoint operator \( G \) satisfying \( G \geq \rho_k \) and \( (G - \rho_k) M_k = 0 \) for all \( k \).

Furthermore, under these equivalent conditions \( L = L^\dagger = G \), and \( L \) is the unique self-adjoint operator of minimal trace satisfying \( L \geq \rho_k \) for all \( k \).

The above optimality conditions were first proved in the infinite-dimensional case by Holevo, since earlier proofs worked only in finite dimensions. The inequalities in properties II-III use the standard order on self-adjoint matrices: \( A \geq B \) iff \( A - B \) is positive semidefinite. The LHS of condition II is commonly referred to as the real part:

\[ \text{Re} (L) := \left( L + L^\dagger \right) / 2. \] \(^3\)

### 3 Mathematical background

**Definition 3** Let \( A \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \) with spectral decomposition \( A = \sum \lambda_k \Pi_k \). The **positive part** of \( A \) is given by

\[ [A]_+ = \sum_{\lambda_k > 0} \lambda_k \Pi_k. \] \(^4\)

The **positive projection** is given by

\[ \chi_+(A) = \sum_{\lambda_k > 0} \Pi_k. \] \(^5\)

The **trace norm** of an operator \( B : \mathcal{H} \to \mathcal{H} \) is given by \( \|B\|_1 = \text{Tr} \sqrt{B^\dagger B} \). The **operator norm** is given by \( \|B\|_\infty = \sup_{\|\psi\|=1} \|B\psi\| \).

---

1. Another interesting optimality condition is given by Theorem 3 on page 39 of [20].
2. Earlier formulations of condition II [13, 14] were that \( L = L^\dagger \) and \( L \geq \rho_k \) for all \( k \), equivalently stated as \( L = L^\dagger \) and \( \text{Re}(L) \geq \rho_k \). (The self-adjointness condition is redundant in the latter form.)
We collect some simple mathematical facts. We will frequently use the inequalities
\[ |\text{Tr} A| \leq \|A\|_1 \] (6)
\[ \|BC\|_1 \leq \|B\|_1 \|C\|_\infty, \] (7)
which may be found in [21]. For positive semidefinite operators \( P_1, P_2 \geq 0 \) such that \( P_1 P_2 \) is trace class, one has
\[ \text{Tr} P_1 P_2 \geq 0, \] (8)
with equality iff \( P_1 P_2 = 0 \) [13] and
\[ A_1 \geq A_2 \Rightarrow C^\dagger A_1 C \geq C^\dagger A_2 C \] (9)
for all operators \( C \) and self-adjoint \( A_1, A_2 \).

4 Estimates of near- and non-optimality

Our next goal is to strengthen condition II of Theorem 2 by giving quantitative bounds in the case that condition II fails to hold. As a first step, note that in the finite-dimensional case if
\[ \text{Re} (L) \geq \rho_k - \alpha \] (10)
for some scalar \( \alpha > 0 \), then by inequality (8)
\[ P_{\text{succ}} (M_k) = \text{Tr} \text{Re} (L) = \text{Tr} \sum_k \text{Re} (L) M_k^{\text{opt}} \geq \text{Tr} \sum_k (\rho_k - \alpha) M_k^{\text{opt}} = P_{\text{succ}} (M_k^{\text{opt}}) - \alpha \dim \mathcal{H}, \] (11)
where \( M_k^{\text{opt}} \) is some optimal POVM.

In order to control dimensional factors (and to consider ensembles on infinite-dimensional Hilbert spaces) it is useful to introduce the following concept:

**Definition 4** Let \( \mathcal{E} = \{\rho_k\} \) be the ensemble of definition 1, and let \( p \in [0,1] \). The \( p \)-dimension \( \dim_p (\mathcal{E}) \) is the minimum dimension of a subspace \( \Lambda \) for which
\[ \sum_k \| (1 - \Pi_\Lambda) \rho_k \|_1 \leq p, \] (13)
where \( \Pi_\Lambda \) is the orthogonal projection onto \( \Lambda \).

**Remark:** Note that the inequality (13) implies that
\[ \text{Tr} (1 - \Pi_\Lambda) \sum_k \rho_k \leq \left\| \sum_k (1 - \Pi_\Lambda) \rho_k \right\|_1 \leq \sum_k \| (1 - \Pi_\Lambda) \rho_k \|_1 \leq p. \]
Lemma 5 For fixed $\mathcal{E}$, the function $p \mapsto \dim_p(\mathcal{E})$ is finite on $(0, 1]$ and monotonically-decreasing on $[0, 1]$.

Proof. The monotonicity of $p \mapsto \dim_p(\mathcal{E})$ is immediate from the definition. To prove finiteness for $p > 0$, take spectral decompositions $\rho_k = \sum \lambda_{k\ell} |\psi_{k\ell}\rangle \langle \psi_{k\ell}|$. For a finite subset $S$ of the $(k, \ell)$, let $\Pi_S$ be the projection onto the linear span of the $\psi_{k\ell}$ with $(k, \ell) \in S$. By the triangle inequality

$$\sum_k \|(1 - \Pi_S) \rho_k\|_1 \leq \sum_{k\ell} \|(1 - \Pi_S) \lambda_{k\ell} |\psi_{k\ell}\rangle \langle \psi_{k\ell}|\|_1 \leq \sum_{(k, \ell) \notin S} \lambda_{k\ell}$$

Since $\sum_{(k, \ell) \in S} \lambda_{k\ell} = 1$, we may take a finite subset $S$ of the $(k, \ell)$ such that the right-hand side may be made smaller than $p$. □

We may now state a robust version of Theorem 2:

Theorem 6 Let $\{M_k\}$ be a POVM for distinguishing $\mathcal{E}$, let $L = \sum M_k \rho_k$, and let $\{M_k^{\text{opt}}\}$ be an optimal measurement. Then

1. Assume that $\alpha > 0$ is a scalar such that

$$\text{Re}(L) \geq \rho_k - \alpha$$

for all $k$. Then for $p \in [0, 1/4)$

$$P_{\text{succ}}(M_k) \geq P_{\text{succ}}(M_k^{\text{opt}}) - \alpha \dim_p(\mathcal{E}) - 4p.$$ (15)

2. Suppose that $\text{Re}(L) \not\geq \rho_\ell$ for some $\ell$. Then

$$P_{\text{succ}}(M_k) \leq P_{\text{succ}}(M_k^{\text{opt}}) - \left(\text{Tr}(\rho_\ell - \text{Re}(L))_+\right)^2,$$ (16)

where $[\cdot]_+$ is the positive part, defined in definition 3.

4.1 Discussion of Theorem 6

The small-$\alpha$ case of Part 1 addresses the case where $\{M_k\}$ nearly-satisfies condition II. In particular, (15) implies that $P_{\text{succ}}(M_k) \geq P_{\text{succ}}(M_k^{\text{opt}}) - \varepsilon$ if

$$\alpha < \sup_{p \in [0, \varepsilon/4]} \frac{\varepsilon - 4p}{\dim_p(\mathcal{E})}.$$ (17)

The following example shows that the dependence of this expression on $\mathcal{E}$ may not be removed except (in the finite-dimensional case) by introducing dimensional factors:

Example 7 Let $m$ be a positive integer, and let $\mathcal{E}$ be the $m$-state ensemble on $\mathbb{C}^m$ defined by $\rho_k = |k\rangle \langle k|/m$. Set $M_k = |k+1\rangle \langle k+1|$, using addition mod $m$. Then one has $P_{\text{succ}}(M_k) = 0$ and $P_{\text{succ}}(M_k^{\text{opt}}) = 1$, but inequality (14) holds for $\alpha = 1/m$, which approaches 0 as $m \to \infty$. 5
4.2 Proof of part 1 of Theorem 6

Proof. Let $\Pi$ be an orthogonal projection, and set $\Pi^\perp = \mathbb{1} - \Pi$. Then

$$P_{\text{succ}} (M_k) = \text{Tr} (\Pi \text{Re} (L) \Pi) + \text{Tr} (\Pi^\perp \text{Re} (L) \Pi^\perp).$$

Using equations (16)-(19) to estimate the first term,

$$\text{Tr} \Pi \text{Re} (L) \Pi = \text{Tr} \sum_k \text{Re} (L) \times \Pi M_k^{\text{opt}} \Pi \geq \text{Tr} \sum_k (\rho_k - \alpha) \times \Pi M_k^{\text{opt}} \Pi \geq P_{\text{succ}} (M_k^{\text{opt}}) - \alpha \text{Tr} (\Pi) - \sum \|\rho_k - \Pi \rho_k \Pi\|_1.$$

But

$$\sum_k \|\rho_k - \Pi \rho_k \Pi\|_1 = \sum_k \|\Pi^\perp \rho_k + \rho_k \Pi^\perp + \Pi^\perp \rho_k \Pi^\perp\|_1 \leq 3 \sum_k \|\Pi^\perp \rho_k\|_1.$$

Using (7) to estimate the second term of (18),

$$|\text{Tr} (\Pi^\perp \text{Re} (L) \Pi^\perp)| \leq \frac{1}{2} \left|\sum \|\Pi^\perp \rho_k M_k \Pi^\perp + \Pi^\perp \rho_k \Pi^\perp\|_1 \leq \sum \|\Pi^\perp \rho_k\|_1$$

Putting (18) − (21) together gives

$$P_{\text{succ}} (M_k) \geq P_{\text{succ}} (M_k^{\text{opt}}) - \alpha \text{Tr} (\Pi) - 4 \sum \|\Pi^\perp \rho_k\|_1.$$

The bound (15) follows by picking $\Pi$ to minimize $\text{Tr} (\Pi)$ when the last term of (22) is constrained to be less than $p$. (By Lemma 3, such $\Pi$ of finite rank always exist.)

4.3 Proof of part 2 of Theorem 6

Definition 8 Let $\{M_k\}$ be a POVM for distinguishing the ensemble $E$ of definition 4, let $X \leq 2 \times \mathbb{1}$ be a positive semidefinite operator on $\mathcal{H}$, and let $\ell \in K$. Then the Barnett-Croke modification of $\{M_k\}$ is defined by

$$M_k (X, \ell) = (1 - X) M_k (1 - X) + \delta_{k\ell} \left(2X - X^2\right).$$

Remark: Note that since $0 \leq 2X - X^2$ for $0 \leq X \leq 2 \times \mathbb{1}$, for each $\ell$ the set $\{M_k (X, \ell)\}$ forms a POVM. Barnett and Croke [12] considered the case $X = \varepsilon |\psi\rangle \langle \psi|$, where $|\psi\rangle$ is a unit vector satisfying the eigenvalue equation

$$(\rho_\ell - \text{Re} (L)) |\psi\rangle = -\lambda |\psi\rangle,$$
with $\lambda > 0$. They showed that

$$\frac{d}{d\varepsilon} \left| \left| \varepsilon = 0 \right. \right. \right. \right. \right. P_{\text{succ}} (M_k (X, \ell)) = 2\lambda > 0.$$  

In order to complete the proof of part 2 of Theorem 6, it suffices to turn this perturbative argument into an estimate.

**Proof of part 2 of Theorem 6.** Let $\Pi_+$ be the positive projection

$$\Pi_+ = \chi_+ (\rho_\ell - \text{Re} (L)).$$  

Then for $\alpha \in [0, 2]$,

$$P_{\text{succ}} (M_k (\alpha \Pi_+, \ell)) = P_{\text{succ}} (M_k) + 2\alpha \text{Tr} (\rho_\ell - \text{Re} (L) \times \Pi_+)$$

$$- \alpha^2 \text{Tr} (\Pi_+ \rho_\ell) + \alpha^2 \text{Tr} \sum \Pi_+ M_k \Pi_+ \rho_k$$

$$\geq P_{\text{succ}} (M_k) + 2\alpha \text{Tr} (\rho_\ell - \text{Re} (L)) - \alpha^2, \quad \quad \quad \text{(25)}$$

where we have used cyclicity of the trace and (8) - (9).

Note that if $\text{Tr} (\rho_\ell - \text{Re} (L)) > 1$ then

$$P_{\text{succ}} (M_k (\Pi_+, \ell)) = P_{\text{succ}} (M_k) + 2 \text{Tr} (\rho_\ell - \text{Re} (L)) - 1 > 1,$$

giving a contradiction. In particular, we may set

$$\alpha = \text{Tr} (\rho_\ell - \text{Re} (L)) \in [0, 1], \quad \quad \quad \text{(26)}$$

maximizing the RHS of (25) over $\alpha \in [0, 1]$. This gives

$$P_{\text{succ}} (M_k (\alpha \Pi_+, \ell)) \geq P_{\text{succ}} (M_k) + (\text{Tr} (\rho_\ell - \text{Re} (L)))^2. \quad \quad \quad \text{(27)}$$

5 **Barnett-Croke iteration**

In this section we show how to convert Barnett and Croke’s perturbative proof into an algorithm for computing optimal measurements. Although the success rate of poorly-chosen iterations might fail to actually converge to that of an optimal measurement the following sequence does not exhibit this malady:

**Definition 9** Let $\{M_k\}$ be a POVM for distinguishing the ensemble $\mathcal{E}$ of definition 1, and chose $\ell$ to maximize

$$\alpha = \text{Tr} (\rho_\ell - \text{Re} (L)). \quad \quad \quad \text{(28)}$$

3In is asserted in [17] that the algorithm of [10] suffers this fate.
Then the iterate of \( \{ M_k \} \) is the POVM

\[
M_k^+ = M_k (\alpha \chi_+ (\rho_\ell - \text{Re} (L)), \ell),
\]

(29)

where \( [\cdot]_+ \) and \( \chi_+ \) are defined in (4) – (7). For a given measurement \( \{ M_k^{(0)} \} \), recursively define the iterative series \( \{ M_k^{(n)} \}_{n \geq 1} \) by

\[
M_k^{(n+1)} = \left( M_k^{(n)} \right)^+ .
\]

(30)

Remark: An index \( \ell \) maximizing (28) exists using minimax principle (Theorem XIII.1 of [22]) and the fact that \( \text{Tr} \sum \rho_\ell = 1 \).

The proof of part II of Theorem 6 actually proved the following stronger result:

**Theorem 10** The above iteration monotonically increases success rate. In particular, for an arbitrary POVM \( \{ M_k \} \) the set \( \{ M_k^+ \} \) is a well-defined POVM, and

\[
P_{\text{succ}} (M_k^+) \geq P_{\text{succ}} (M_k) + \max_{\ell} \left( \text{Tr} \left( \left[ \rho_\ell - \text{Re} (L) \right]_+ \right) \right)^2.
\]

(31)

We now show that the iterative scheme of definition 9 approaches optimality:

**Theorem 11** Let \( M_k^{(0)} \) be an arbitrary starting POVM for the iterative series (30). Then

\[
\lim_{n \to \infty} P_{\text{succ}} (M_k^{(n)}) = P_{\text{succ}} (M_k^{\text{opt}}),
\]

(32)

where \( M_k^{\text{opt}} \) is an optimal measurement.

**Proof.** Let \( \varepsilon > 0 \) be arbitrary. We seek an \( N > 0 \) such that

\[
n > N \Rightarrow P_{\text{succ}} \left( M_k^{(n)} \right) \geq P_{\text{succ}} (M_k^{\text{opt}}) - \varepsilon.
\]

(33)

Set

\[
L^{(n)} = \sum_k M_k^{(n)} \rho_k.
\]

By equation (17) and the monotonicity of \( n \mapsto P_{\text{succ}} (M_k^{(n)}) \), it suffices to find a \( n \leq N \) such that

\[
\text{Re} (L^{(n)}) \geq \rho_\ell - \Delta
\]

(34)

for all \( \ell \), where \( \Delta \) is any real number satisfying\(^5\)

\[
0 < \Delta \leq \sup_{p \in [0, \varepsilon/4]} \frac{\varepsilon - 4p}{\dim_p (\mathcal{E})}.
\]

\(^4\)Faster convergence can be obtained by replacing \( \alpha \) by \( \beta \) in equation (29), where \( \beta \in [0, 2] \) is chosen to maximize \( P_{\text{succ}} (M_k^+) \), which is quadratic in \( \beta \).

\(^5\)In finite dimensions, one may take \( \Delta = \varepsilon / \dim \mathcal{H} \leq \varepsilon / \dim (\text{supp} (\mathcal{E})) \), corresponding to \( p = 0 \).
We claim that $N = \Delta^{-2}$ suffices. Assume that

$$\max T \left( [\rho_\ell - \operatorname{Re} \left( L^{(n)} \right)]_+ \right) > \Delta$$

for all $n \leq N$. By Theorem 10,

$$P_{\text{succ}} \left( M_k^{[N]+1} \right) > N \times \Delta^2 \geq 1,$$

yielding a contraction.

It follows that

$$\max T \left( [\rho_\ell - \operatorname{Re} \left( L^{(n)} \right)]_+ \right) \leq \Delta$$

for some $n \leq N$. The inequality \[\text{3.1}\] follows from the observation that

$$A \leq \operatorname{Tr} ([A]_+ \times 1),$$

for $A = \rho_\ell - \operatorname{Re} \left( L^{(n)} \right)$.  

6 Conclusion

Using non-optimality estimates in terms of quantitative violation of Holevo's simplified optimal measurement condition, we have converted Barnett and Croke's perturbative proof into a conceptually-simple iterative scheme for computing optimal measurements. This iteration approaches the optimal success rate even in the case of infinite-dimensions and infinite ensemble cardinality. It would be interesting to try to improve the non-optimality bounds of Theorem 6 and to study the convergence rate of this iteration in more detail.

Acknowledgements: The author would like to thank A. Holevo for pointing out a useful reference, and Arthur Jaffe for his encouragement.

References

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York 1976).

[2] P. Hausladen, R. Josza, B. Schumacher, M. Westmoreland, and W. K. Wootters, “Classical information capacity of a quantum channel,” Phys Rev A 54, 1869 (1996).

[3] B. Schumacher and M. D. Westmoreland, “Sending classical information via noisy quantum channels,” Phys Rev A 56, 131 (1997).

[4] A. S. Holevo, “The capacity of the quantum channel with general signal states,” IEEE Trans. Inf. Theory 44, 269 (1998).
[5] R. Koenig, R. Renner, and C. Schaffner, “The operational meaning of min- and max-entropy,” e-print arXiv:0807.1338.

[6] L. Ip, “Shor’s algorithm is optimal,” http://lawrenceip.com/papers/hspdsdpabstract.html (2003).

[7] D. Bacon, A. M. Childs, and W. van Dam, “Optimal measurements for the dihedral hidden subgroup problem,” Chicago J. of Theoret. Comput. Sci. 2006, (2006); e-print arXiv: quant-ph/0501044.

[8] D. Bacon, A. M. Childs, and W. van Dam, “From optimal measurement to efficient quantum algorithms for the hidden subgroup problem over semidirect product groups,” Proceedings of the 46th IEEE Symp. Foundations of Computer Science, (IEEE, Los Alamitos, CA, 2005), pp. 469-478 (2005).

[9] A. M. Childs and W. van Dam, “Quantum algorithm for a generalized hidden shift problem,” Proceedings of the 18th ACM-SIAM Symp. Discrete Algorithms, (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2007), pp. 1225-1234; e-print arXiv: quant-ph/0507190.

[10] C. Moore and A. Russell, “For Distinguishing Hidden Subgroups, the Pretty Good Measurement is as Good as it Gets,” Quantum Inform. Comput. 7, 752 (2007); e-print arXiv: quant-ph/0501177.

[11] D. Bacon and T. Decker, “The optimal single-copy measurement for the hidden-subgroup problem,” Phys. Rev. A 77, 032335 (2008); e-print arXiv:0706.4478.

[12] S. M. Barnett and S. Croke, “On the conditions for discrimination between quantum states with minimum error,” J. Phys. A: Math. Theor. 42 062011 (2009): arxiv:0810.1919.

[13] H. P. Yuen, R. S. Kennedy, and M. Lax, “Optimum testing of multiple hypotheses in quantum detection theory,” IEEE Trans. Inf. Theory, IT-21, 125 (1975).

[14] A. S. Holevo, “Statistical Decision Theory for Quantum Systems,” J. Multivariate Anal. 3, 337 (1973).

[15] A. S. Holevo, “Remarks on optimal measurements,” Problems of Information Transmission 10, no.4 317-320 (1974); Translated from Problemy Peredachi Informatsii, 10 no. 4, 51-55 (1974).

[16] C. W. Helstrom, “Bayes-Cost reduction algorithm in quantum hypothesis testing,” IEEE Transactions on Information Theory IT-28 no.2, March 1982 359-366. Note: It is asserted in footnote [36] of [17] that this algorithm generally only gives a lower bound on $P_{\text{succ}}$. 

10
[17] M. Ježek, J. Řeháček, and J. Fiurášek, “Finding optimal strategies for
minimum-error quantum state discrimination,” Physical Review A 65,
060301(R) (2002); quant-ph/0201109.

[18] Z. Hradil, J. Řeháček, J. Fiurášek, and M. Ježek, “Maximum-Likelihood
Methods in Quantum Mechanics,” Lecture Notes in Physics 649, 59-112
(2004).

[19] Y. C. Eldar, A. Megretski, and G. C. Verghese, “Designing Optimal Quan-
tum Detectors Via Semidefinite Programming,” IEEE Trans. Inf. The-
ory, 49, 1007 (2003). Note: The reported implementation appears to
have mild numerical inaccuracies in the case of optimal measurement
operators which are identically zero. In particular, the numerical exam-
ple reported in equation 40 has the unique exact solution \( \mu_1 = (0,0) \),
\( \mu_2 = N_2^{-1} \left( \frac{1 + \sqrt{5}}{2},1 \right) \), and \( \mu_3 = N_3^{-1} \left( \frac{1 - \sqrt{5}}{2},1 \right) \), where \( N_{2,3} \) are
normalization factors. (The identity \( |\mu_1 \rangle \langle \mu_1| = 0 \) follows from the invert-
ibility of \( L - \rho_1 \), by an application of condition III above.)

[20] V. P. Belavkin and V. Maslov, “Design of Optimal Dynamic Analyzer:
Mathematical Aspects of Wave Pattern Recognition” In Mathematical As-
pects of Computer Engineering, Ed V. Maslov, 146-237, Mir, Moscow 1987;
quant-ph/0412031

[21] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Func-
tional Analysis (Academic, New York, 1980).

[22] M. Reed and B. Simon, Methods of Modern Mathematical Physics IV: Anal-
ysis of Operators, Academic Press, Boston (1978).