A UNIQUENESS THEOREM FOR HOLOMORPHIC MAPPINGS IN THE DISK SHARING TOTALLY GEODESIC HYPERSURFACES

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Abstract. In this paper, we prove a Second Main Theorem for holomorphic mappings in a disk whose image intersects some families of nonlinear hypersurfaces (totally geodesic hypersurfaces with respect to a meromorphic connection) in the complex projective space $\mathbb{P}^k$. This is a generalization of Cartan’s Second Main Theorem. As a consequence, we establish a uniqueness theorem for holomorphic mappings which intersects $O(k^3)$ many totally geodesic hypersurfaces.

1. Introduction and main results

Let $f$ and $g$ be two nonconstant meromorphic functions defined on $\mathbb{C}$. We say that $f$ and $g$ share a complex number $b$ CM (IM) if $f - b$ and $g - b$ have the same zeros in $\mathbb{C}$, counting with the same multiplicities (ignoring multiplicities). In 1926, Nevanlinna proved a uniqueness theorem for meromorphic functions, based on his Second Main Theorem (SMT). This is the famous Nevanlinna Five Point Theorem, which says that any two nonconstant meromorphic functions $f$ and $g$ sharing five values IM must be identical. As each meromorphic function could be seen as a holomorphic curve from $\mathbb{C}$ into the projective space $\mathbb{P}^1$, it is natural to extend Nevanlinna’s uniqueness theorem to holomorphic curves in higher dimensional complex projective spaces.

In 1974, Fujimoto [8] showed that for two linearly non-degenerated meromorphic mappings $f$ and $g$ from $\mathbb{C}$ into $\mathbb{P}^k$, if they have the same inverse images of $3k + 2$ hyperplanes counted with multiplicities in $\mathbb{P}^k$ in a general position, then $f \equiv g$. In 1983, Smiley [12] obtained the uniqueness theorem of the meromorphic mappings sharing $3k + 2$ hyperplanes without counting multiplicity. Later on, in 2009, Chen and Yan [3] reduced the number $3k + 2$ to $2k + 3$, which is the smallest number of sharing hyperplanes in the uniqueness theorems so far; this
type of result was extended and developed by many authors, such as those of [5], and others. For example, one may naturally consider the problem of sharing hypersurfaces. In 2008, Dulock and Ru [6] investigated for the first time the case of sharing hypersurfaces without counting multiplicities, and they proved the following theorem by using a result of An and Phuong in [2]:

**Theorem 1** (Dulock-Ru [6]). Let \( \{Q_j\}_{j=1}^q \) be hypersurfaces of degree \( d_j \) in \( \mathbb{P}^k \) in a general position. Let \( d_0 = \min\{d_1, \ldots, d_q\} \), \( d = \text{lcm}\{d_1, \ldots, d_q\} \), and \( M = 2d[2^{k-1}(k+1)kd(d+1)]^k \). Suppose that \( f \) and \( g \) are algebraically non-degenerated meromorphic mappings of \( \mathbb{C} \) into \( \mathbb{P}^k \) such that \( f(z) = g(z) \) for any \( z \in \bigcup_{j=1}^q \{f^{-1}(Q_j) \cup g^{-1}(Q_j)\} \). If \( q > (k+1) + \frac{2M}{d_0} + \frac{1}{2} \), then \( f \equiv g \). Here \( f^{-1}(Q_j) \) means the zero set of \( Q_j \circ f \).

The proof of Theorem 1 is based on a Second Main Theorem of An and Phuong [2], and also the following lemma of Dulock and Ru [7]:

**Lemma 1** ([7]). Let \( H \) be a hyperplane line bundle on \( \mathbb{P}^k \). For \( m = 1, 2 \), we let \( \pi_m : \mathbb{P}^k \times \mathbb{P}^k \to \mathbb{P}^k \) be the canonical projection mappings. Let \( f \times g : \mathbb{C} \to \mathbb{P}^k \times \mathbb{P}^k \) be a holomorphic map such that \( f \not\equiv g \), there exists a section \( s \) of \( H' := \pi_1^*H \otimes \pi_2^*H \) so that the diagonal \( \Delta \) of \( \mathbb{P}^k \times \mathbb{P}^k \) is contained in \( \text{Supp}(s) \), but the image \( (f \times g)(\mathbb{C}) \) is not contained in \( \text{Supp}(s) \).

We notice that the number of sharing hypersurfaces in Dulock-Ru’s result is of order \( k^{2k} \), which is much bigger than \( 3k^2 + 2 \) or \( 2k + 3 \) in the hyperplane case. As an improvement of the truncated version of Ru’s Second Main Theorem [9], an expected smaller number of hypersurfaces can be found in Theorem 2 (below) of Quang and An [1]. However, this number is still much bigger than \( 3k^2 + 2 \) or \( 2k + 3 \). Therefore, it would be interesting to try to get a uniqueness theorem for holomorphic curves sharing fewer hypersurfaces.

Let \( V \) be a complex projective subvariety of \( \mathbb{P}^k \) of dimension \( m(\leq k) \). Let \( d \) be a positive integer. We denote by \( I(V) \) the ideal of homogeneous polynomials in \( \mathbb{C}[X_0, \ldots, X_k] \) defining \( V \), and by \( H_d \) the \( \mathbb{C} \)-vector space of all homogeneous polynomials in \( \mathbb{C}[X_0, \ldots, X_k] \) of degree \( d \). Define

\[
I_d(V) = \frac{H_d}{I(V) \cap H_d} \quad \text{and} \quad H_V(d) = \dim I_d(V).
\]

Then \( H_V(d) \) is the Hilbert function of \( V \). Each element of \( I_V(d) \) can be represented by \( [Q] \) for some \( Q \in H_d \). In the case where \( V \) is a linear space of dimension \( m \) and \( d = 1 \), we have that \( H_V(d) = m + 1 \). For the
general case, it is easy to see that \( H_V(d) \leq \frac{(k + d)!}{k!d!} \sim \frac{(k + d)^d}{d!} \) for a fixed \( d \) and \( k \) sufficiently large.

**Definition 1.** Let \( f : \mathbb{C} \to V \) be a holomorphic mapping of \( \mathbb{C} \) into \( V \). Then \( f \) is said to be degenerated over \( I_d(V) \) if there exists a non-zero \([Q] \in I_d(V)\) such that \( Q(f) \equiv 0\). Otherwise, we say that \( f \) is non-degenerated over \( I_d(V) \). One can see that if \( f \) is algebraically non-degenerated, then \( f \) is non-degenerated over \( I_d(V) \) for \( d \geq 1 \).

In 2017, Quang and An first established a truncated version of the Second Main Theorem involving \( H_V(d) \) and as an application of this, they improved Dulock-Ru’s result (Theorem 1) and obtained the following uniqueness theorem for holomorphic curves sharing a possibly smaller number of hypersurfaces in \( \mathbb{P}^k \):

**Theorem 2** ([1]). Let \( V \) be a complex projective subvariety of \( \mathbb{P}^k \) of dimension \( m (m \leq k) \). Let \( \{Q_i\}_{i=1}^q \) be hypersurfaces in \( \mathbb{P}^k \) in an \( N \)-subgeneral position with respect to \( V \) and \( \deg Q_i = d_i \) \((1 \leq i \leq q)\). Let \( d = \text{lcm}(d_1, \ldots, d_q) \). Let \( f \) and \( g \) be holomorphic curves of \( \mathbb{C} \) into \( V \) which are non-degenerated over \( I_d(V) \). Assume that \( f(z) = g(z) \) for any \( z \in \bigcup_{i=1}^q \{f^{-1}(Q_i) \cup g^{-1}(Q_i)\} \). Then the following assertions hold:

(a) if \( q > \frac{2(H_V(d) - 1)}{d} + \frac{(2N - m + 1)H_V(d)}{m + 1} \), then \( f \equiv g \);

(b) if \( q > \frac{2(2N - m + 1)H_V(d)}{m + 1} \), then there exist \( N + 1 \) hypersurfaces \( (Q_{i_0}), \ldots, (Q_{i_N}) \), \( 1 \leq i_0 < \cdots < i_N \leq q \) such that

\[
\frac{Q_{i_0}(f)}{Q_{i_0}(g)} = \cdots = \frac{Q_{i_N}(f)}{Q_{i_N}(g)}.
\]

**Remark 1.** Part (b) of Theorem 2 implies Chen-Yan’s result [3] (see Corollary 1 in [1]).

Although the number of sharing hypersurfaces in Theorem 2 is much smaller than the one in Dulock-Ru’s result (Theorem 1), the number \( H_V(d) \) is not easy to explicitly estimate and is bounded by \( O(k^d) \) depending on the degree \( d \) of the hypersurfaces. In this paper, we would like to give an explicit estimation (around \( O(k^3) \) at independent over the degree \( d \)) of the number of shared special hypersurfaces.

So far, the tools to solve the unicity problem of holomorphic curves have been various versions of the Second Main Theorem. In 2012, Tiba [13] made use of Demailly’s [4] meromorphic partial projective connection (see Definition 2) to prove a Second Main Theorem for a holomorphic curve in \( \mathbb{P}^k \) crossing totally geodesic hypersurfaces (see Definition 3). As a consequence, one can obtain a uniqueness theorem of holomorphic curves intersecting totally geodesic hypersurfaces; the
required number of hypersurfaces is smaller than the one in Dulock-Ru’s result (Theorem 1), and more precise than the one in Quang-An’s result (Theorem 2).

To formulate our result, we have to introduce the definition of meromorphic partial projective connections first provided by Siu [11], and that of totally geodesic hypersurfaces on a complex projective algebraic manifold $X$. One can refer to Demailly [4], Section 11 or Tiba [13], Section 3 for the details.

Let $\{U_j\}_{1 \leq j \leq N}$ be an affine open covering of $X$.

**Definition 2 (Meromorphic partial projective connection).** A meromorphic partial projective connection $\Lambda$, relative to an affine open covering $\{U_j\}_{1 \leq j \leq N}$ of $X$, is a collection of meromorphic connections $\Lambda_j$ on $U_j$ satisfying

$$\Lambda_j - \Lambda_k = \alpha_{jk} \otimes Id_{TX} + Id_{TX} \otimes \beta_{jk}$$
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on $U_j \cap U_k$ for all $1 \leq j, k \leq N$, where $\alpha_{jk}$ and $\beta_{jk}$ are meromorphic one-forms on $U_j \cap U_k$. We write $\Lambda = \{(\Lambda_j, U_j)\}_{1 \leq j \leq N}$.

Let $S_j$ be the smallest subvariety of $X$ such that $\Lambda_j$ is a holomorphic connection on $U_j \setminus S_j \cap U_j$. We set $\text{supp}(\Lambda)_\infty := \bigcup_{1 \leq j \leq N} S_j$ and call it the polar locus of $\Lambda$.

Let $D$ be a reduced effective divisor of a $k$-dimensional complex projective algebraic manifold $X$, and let $\Lambda$ be a meromorphic connection. Consider the holomorphic function $s$ on an open set $U \subset X$ such that $D|_U = (s)$, and fix a local coordinate system $(z_1, \ldots, z_n)$ on $U$. In particular, if $X = \mathbb{P}^k$, one can always construct a meromorphic partial projective connection from some given homogenous polynomials (see Demailly [4] or Tiba [13]).

Let $S_0, \ldots, S_k$ be homogenous polynomials of degree $d$ in $\mathbb{C}[X_0, \ldots, X_k]$ such that

$$\det \left( \frac{\partial S_\mu}{\partial X_i} \right)_{0 \leq \mu, i \leq k} \neq 0.$$ 

Then one can construct the meromorphic connection $\tilde{\Lambda} = d + \tilde{\Gamma}$ on $\mathbb{C}^{k+1}$ defined by

$$\sum_{0 \leq \lambda \leq k} \frac{\partial S_\mu}{\partial X_\lambda} \tilde{\Gamma}^{i\lambda}_{ij} = \frac{\partial^2 S_\mu}{\partial X_i \partial X_j}.$$ 

This meromorphic connection induces the meromorphic partial projective connection $\Lambda$ on $\mathbb{P}^k$ by the following lemma from Demailly [4]:

**Lemma 2 ([4]).** Let $\tilde{\Lambda} = d + \tilde{\Gamma}$ on $\mathbb{C}^{k+1}$. Let $\zeta = \sum z_j \frac{\partial}{\partial z_j}$ be the Euler vector field on $\mathbb{C}^{k+1}$ and let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ be the canonical projection. Then $\tilde{\Lambda}$ induces a meromorphic partial projective connection $\Lambda$ on $\mathbb{P}^k$ provided that
(i) the Christoffel symbols $\tilde{\Gamma}_{j\mu}^\lambda$ are homogeneous rational functions of degree $-1$ (i.e. the total degree of its numerator minus that of its denominator is equal to $-1$);
(ii) there are meromorphic functions $\alpha, \beta$ and meromorphic 1-forms $\gamma, \eta$ on $\mathbb{C}^{k+1} \setminus \{0\}$ such that $\tilde{\Gamma} \cdot < \varsigma, v > = \alpha v + \gamma(v) \varsigma$ and $\tilde{\Gamma} \cdot < w, \varsigma > = \beta w + \eta(w) \varsigma$ for all vector fields $v, w$.

**Definition 3** (Totally geodesic hypersurface). A hypersurface $D$ is said to be totally geodesic with respect to $\Lambda$ on $U$ if there exist meromorphic one-forms

$$
a = \sum_{j=1}^{k} a_j dz_j, \quad b = \sum_{j=1}^{k} b_j dz_j
$$

and a meromorphic two-form

$$
c = \sum_{1 \leq j, \mu \leq k} c_{j\mu} dz_j \otimes dz_\mu
$$

such that no polar locus of $a_j, b_j$, or $c_{j\mu}$ ($1 \leq j, \mu \leq k$) contains $\text{supp} \, D|_U$, and

$$
\Lambda^\star(ds) = d^2s - ds \circ \Gamma = a \otimes ds + ds \otimes b + sc
$$

in $U$, where $\Lambda^\star$ is the induced connection on the cotangent bundle $T X^*$ of $X$.

Now, we are ready to present our uniqueness theorem for holomorphic curves intersecting totally geodesic hypersurfaces.

**Theorem 3.** Let $S_0, S_1, \ldots, S_k$ be homogeneous polynomials of degree $d$ in $\mathbb{C}[X_0, \ldots, X_k]$ satisfying

$$
\det \left( \frac{\partial S_\mu}{\partial X_i} \right)_{0 \leq \mu, i \leq k} \neq 0. \quad (1.2)
$$

Assume that $\sigma_j, 1 \leq j \leq q$ are elements of the linear system $Y_\alpha = \{\alpha_0 S_0 + \alpha_1 S_1 + \cdots + \alpha_k S_k = 0\}$ such that the hypersurfaces $Y_j = (\sigma_j), 1 \leq j \leq q$ are smooth and in a general position. Let $f$ and $g$ be two holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^k$ such that their images are neither contained in the support of an element of the linear system $Y_\alpha$ nor contained in the polar locus of the meromorphic partial projective connection $\Lambda$ induced by (1.1). Suppose that $f(z) = g(z)$ for all $z \in S$, where

$$
S := \bigcup_{j=1}^{q} \{ f^{-1}(\sigma_j) \cup g^{-1}(\sigma_j) \}.
$$

Then the following assertions hold:

(i) if $q > \frac{3k+1}{d} + \frac{1}{2d}(k-1)k(k+1)(d-1)$, then $f \equiv g$;
(ii) if $q > 2 \left( \frac{k+1}{d} + \frac{1}{2d}(k-1)k(k+1)(d-1) \right)$, then there exist $k+1$ hypersurfaces $Y_{i_0}, \ldots, Y_{i_k}$, $1 \leq i_0 < \cdots < i_k \leq q$ such that

$$
\frac{\sigma_{i_0}(f)}{\sigma_{i_0}(g)} = \cdots = \frac{\sigma_{i_k}(f)}{\sigma_{i_k}(g)}.
$$
Remark 2. The number of hypersurfaces required in Theorem 3 becomes much smaller than for the previous results (cf. [1, 6]).

In the case \( d = 1 \), from Theorem 3(ii), we can also recover the uniqueness theorem for linearly non-degenerated holomorphic curves sharing \( 2k + 3 \) hyperplanes in \( \mathbb{P}^k \) in a general position; this was first obtained by Chen and Yan [3].

Indeed, when \( d = 1 \), let \( S_i = X_i \), the meromorphic partial projective connection in \( \mathbb{P}^k \) is the flat connection, and it is easy to see that any hyperplane \( H \) is an element of the linear system \( H_\alpha = \{ \alpha_0 S_0 + \cdots + \alpha_k S_k = 0 \} \), and always smooth. We choose \( q \geq 2k + 3 \) hyperplanes \( H_i, 1 \leq i \leq q \) to be in a general position. From Theorem 3(ii), there exist \( k + 1 \) hyperplanes \( H_{i_0}, \ldots, H_{i_k} \) such that

\[
m = \frac{H_{i_0}(f)}{H_{i_0}(g)} = \cdots = \frac{H_{i_k}(f)}{H_{i_k}(g)}.
\]

As \( H_1, \ldots, H_q \) are in a general position, for any \( k + 1 \) hyperplanes \( \{ H_{i_0}, \ldots, H_{i_k} \} \), the determinant of \( (H_{i_0}, \ldots, H_{i_k}) \) is nonzero, hence \( f_i = m g_i \), and therefore \( f \equiv g \).

Recently, Ru and Sibony [10] defined a growth index of a holomorphic map \( f \) from a disc \( \mathbb{D}_R \) centred at zero with radius \( R \) to a complex manifold and generalized the classical value distribution theory for holomorphic curves on the whole complex plane.

**Definition 4.** Let \( M \) be a complex manifold with a positive (1, 1) form \( \omega \) of finite volume. Let \( 0 < R \leq \infty \) and let \( f : \mathbb{D}_R \to M \) be a holomorphic map. The growth index of \( f \) with respect to \( \omega \) is defined as

\[
c_{f,\omega} := \inf \left\{ c > 0 \left| \int_0^R \exp(c T_{f,\omega}(r))dr = \infty \right. \right\},
\]

where

\[
T_{f,\omega}(r) := \int_0^r \frac{dt}{t} \int_{|z| < t} f^* \omega, \quad 0 < r < R
\]

is the characteristic function of \( f \) with respect to \( \omega \). If \( M = \mathbb{P}^k \) and \( \omega \) is the Fubini-Study form, then we simply denote \( c_{f,\omega} \) by \( c_f \).

Notice that if \( R = \infty \), then \( c_{f,\omega} = 0 \). Therefore, we can also extend Theorem 3 to the hyperbolic case.

**Theorem 4.** Let \( S_0, S_1, \ldots, S_k \) be homogeneous polynomials of degree \( d \) in \( \mathbb{C}[X_0, \ldots, X_k] \) satisfying

\[
\det \left( \frac{\partial S_\mu}{\partial X_i} \right)_{0 \leq \mu, i \leq k} \neq 0.
\]  

Assume that \( \sigma_j, 1 \leq j \leq q \) are elements of the linear system \( Y_\alpha = \{ \alpha_0 S_0 + \alpha_1 S_1 + \cdots + \alpha_k S_k = 0 \} \) such that the hypersurfaces \( Y_j = (\sigma_j), 1 \leq j \leq q \) are smooth and in a general position. Let \( f \) and \( g \) be
two holomorphic maps from \( \mathbb{D}_R \) into \( \mathbb{P}^k \) with \( c_f < \infty \) and \( c_g < \infty \) such that their images are neither contained in the support of an element of the linear system \( Y_\alpha \) nor contained in the polar locus of \( \Lambda \). Suppose that \( f(z) = g(z) \) for all \( z \in S \), where

\[
S := \bigcup_{j=1}^{q} \{ f^{-1}(\sigma_j) \cup g^{-1}(\sigma_j) \}.
\]

Then the following assertions hold:

(i) if \( q > \frac{3k+1}{d} + \frac{1}{2d}(k-1)k(k+1)(d-1) + \frac{k^2(k+1)^2}{2} \max\{c_f, c_g\} \), then \( f \equiv g \);

(ii) if \( q > 2 \left( \frac{k+1}{d} + \frac{1}{2d}(k-1)k(k+1)(d-1) \right) + \frac{k^2(k+1)^2}{d} \max\{c_f, c_g\} \), then there exist \( k + 1 \) hypersurfaces \( Y_{i_0}, \ldots, Y_{i_k}, 1 \leq i_0 < \cdots < i_k \leq q \) such that

\[
\frac{\sigma_{i_0}(f)}{\sigma_{i_0}(g)} = \cdots = \frac{\sigma_{i_k}(f)}{\sigma_{i_k}(g)}.
\]

To prove the above result, we need to obtain a result analogous to that of Tibá’s SMT for holomorphic mappings on a disk.

**Theorem 5.** Let \( X \) be a \( k \)-dimensional complex projective algebraic manifold with a positive \((1, 1)\) form \( \omega \), and let \( L_i := (Q_i) \ (1 \leq i \leq q) \) be a smooth holomorphic line bundle on \( X \) with the holomorphic section \( Q_i \) such that \( \sum_{1 \leq i \leq q} L_i \) is a simple normal crossing divisor of \( X \). Let \( \Lambda = \{(\Lambda_j, U_j)\}_{1 \leq j \leq N} \) be a meromorphic partial projective connection relative to an affine covering \( \{U_j\} \) of \( X \), and let \( \beta \) be a holomorphic section of the holomorphic line bundle \( L \) on \( X \) such that \( \beta \Lambda_j \) is holomorphic on \( U_j \) for all \( j \). Let \( f = (f_1, \ldots, f_k) : \mathbb{D}_R \to X \) be a holomorphic map with \( c_{f,\omega} < \infty \) such that \( f(\mathbb{D}_R) \) is not contained in the polar locus \( \text{supp}(\Lambda)_{\infty} \) of \( \Lambda \) and we have the Wronskian

\[
W_{\Lambda}(f) := f' \wedge \Lambda f' \wedge \cdots \wedge \Lambda^{(k-1)} f' \neq 0,
\]

where \( \Lambda^{(m)} := \Lambda \circ \cdots \circ \Lambda \) \( m \)-times. Suppose that \( \text{supp}(Q_i) \) is not contained in \( \text{supp}(\Lambda)_{\infty} \) and \( L_i \) is totally geodesic with respect to \( \Lambda \) for \( 1 \leq i \leq q \). Then

\[
\sum_{1 \leq i \leq q} T_f(r, L_i) + T_f(r, K_X) - \frac{1}{2} k(k-1) T_f(r, L) \leq \sum_{1 \leq i \leq q} N_k(r, f^*(Q_i)) + S_f(r),
\]

where, for any \( \epsilon > 0 \),

\[
S_f(r) = O(\log^+ r + \log^+ T_f(r) + \sum_{1 \leq m \leq k} \log^+ T_{f_m}(r)) + \frac{N_k^2(k+1)}{2}(1+\epsilon)(c_{f,\omega}+\epsilon)T_f(r) \|.
\]
Here $\|\|$ means that the inequality holds outside a set $E \subset (0, R)$ with
\[ \int_E \exp((c_f + \epsilon)T_f(r))dr < \infty. \]

If $X = \mathbb{P}^k$, then we have the following result:

**Theorem 6.** For each $j = 1, \ldots, q$, let $\sigma_j$ be an element of the linear system $|\{S_0, \ldots, S_k\}|$ such that the divisor $(\sigma_j)$ is smooth and $\sum_{1 \leq j \leq q} (\sigma_j)$ is a simple normal crossing divisor. Let $f = [f_0 : f_1 : \cdots : f_k]: \mathbb{D}_R \to \mathbb{P}^k$ be a non-constant holomorphic map with $c_f < \infty$ such that its image is neither contained in the support of an element of linear system $|\{S_0, \ldots, S_n\}|$ nor contained in the polar locus of $\Lambda$. Then we have
\[ \left( q - \frac{k + 1}{d} - \frac{1}{2d}(k - 1)k(k + 1)(d - 1) \right) T_f(r, dH) \leq \sum_{1 \leq j \leq q} N_k(r, f^*(\sigma_j)) + S_f(r), \]
where $H$ is a hyperplane bundle on $\mathbb{P}^k$, and, for any $\epsilon > 0$,
\[ S_f(r) = O(\log^+ T_f(r) + \log^+ r) + \frac{k^2(k + 1)^2}{2}(1 + \epsilon)(c_f + \epsilon)T_f(r) \| \]

**Remark 3.** When the degree of $S_i$ is equal to 1, one can choose $(S_i)$ to be the coordinate hyperplane, and then the meromorphic connection constructed becomes the flat connection. Thus any hyperplane is a totally geodesic hypersurface with respect to the flat connection on $\mathbb{P}^k$ and the holomorphic curve $f$ is linearly non-degenerated if and only if $W_{\nabla}(f) \not\equiv 0$. Here $W_{\nabla}(f)$ is actually equivalent to the classical Wronskian of $f$. Hence Theorem 6 can be regarded as a generalization of Cartan’s Second Main Theorem.

### 2. Notations and Preliminaries

In this section, we introduce basic definitions and results for the Nevanlinna theory.

Let $f: \mathbb{D}_R \to \mathbb{P}^k$ be a nonconstant holomorphic map. Let $E = \sum m_j P_j$ be an effective divisor on $\mathbb{D}_R$, where $P_j$ is a set of discrete points in $\mathbb{D}_R$ and $m_j$ are positive integers. For $0 < r < R$, put $n_k(r, E) = \sum_{|P_j| < r} \min\{k, m_j\}$. We define the counting function of $E$ by
\[ N_k(r, E) = \int_0^r \frac{n_k(t, E)}{t} dt. \]

Let $X$ be a complex projective algebraic manifold with dimension $k$ (e.g. $X = \mathbb{P}^k$), and let $D$ be an effective divisor on $X$. Put $L = \mathcal{O}(D)$, where $\mathcal{O}(D)$ denotes the line bundle associated with $D$. Let $\sigma$ be a holomorphic section of $L$ such that $D = \{ \sigma = 0 \}$. We define the proximity function of $f$ with respect to $D$ under the assumption that
\( f(\mathbb{D}_R) \not\subset D \), by
\[
m_f(r, D) = \int_0^{2\pi} \log \frac{1}{\| \sigma(f(re^{i\theta}) \|_L} \frac{d\theta}{2\pi},
\]
where \( \| \cdot \|_L \) is a hermitian metric in \( L \). In particular, if \( \sigma \) is a homogeneous polynomial of degree \( d \), then the proximity function could be defined by
\[
m_f(r, D) = \int_0^{2\pi} \log \frac{\| f(re^{i\theta}) \|_L^d}{\| \sigma(f(re^{i\theta}) \|_L^d} \frac{d\theta}{2\pi},
\]
where \( \| f(re^{i\theta}) \| = \max\{ |f_0(re^{i\theta})|, \ldots, |f_k(re^{i\theta})| \} \), and \( f = (f_0, \ldots, f_k) \) a reduced representation. The integrated counting function and truncated counting functions of \( f \) with respect to \( D \) are defined, under the assumption that \( f(\mathbb{D}_R) \not\subset D \), by
\[
N(r, f^*D) = \int_0^r \frac{n(t, f^*D) - n(0, f^*D)}{t} dt + n(0, f^*D) \log r,
\]
\[
N_k(r, f^*D) = \int_0^r \frac{n_k(t, f^*D) - n_k(0, f^*D)}{t} dt + n_k(0, f^*D) \log r,
\]
where \( n(t, f^*D) \) equals the number of points of \( f^{-1}(D) \) in the disc \( |z| < t \), counting multiplicity, \( n(0, f^*D) = \lim_{t \to 0} n(t, f^*D) \), and \( n_k(t, f^*D) = \min\{k, n(t, f^*D)\} \). Similarly, for a set \( S \subset \mathbb{D}_R \) with \( \#(S \cap \{|z| < r\}) < \infty \) for every \( 0 < r < R \), we denote by \( n(r, S) \) the number of elements of \( S \) in the disk with a center at zero and radius \( r \), and
\[
N(r, S) = \int_0^r \frac{n(t, S) - n(0, S)}{t} dt + n(0, S) \log r.
\]
For a line bundle associated with \( D \), we define the characteristic function of \( T_f(r, \mathcal{O}(D)) \) by
\[
T_f(r, \mathcal{O}(D)) = \int_0^r \frac{dt}{t} \int_{|z| \leq t} f^*c_1(\mathcal{O}(D)),
\]
where \( c_1(\mathcal{O}(D)) \) is the Chern form of \( \mathcal{O}(D) \). If the line bundle is a hyperline line bundle \( H \), we set
\[
T_f(r) = T_f(r, H) = \int_0^r \frac{dt}{t} \int_{|z| \leq t} f^*\omega_{FS} = \frac{1}{2\pi} \int_0^{2\pi} \log \| f(re^{i\theta}) \| d\theta + O(1),
\]
where \( \omega_{FS} \) is the Fubini-Study form on \( \mathbb{P}^k \).

3. Proof of Theorem 5 and 6

To prove our results, we need the following lemmata provided by Ru and Siibony [10]:

**Lemma 3** (Calculus Lemma [10]). Let \( 0 < R \leq \infty \) and let \( \gamma(r) \) be a nonnegative function defined on \((0, R)\) with \( \int_0^R \gamma(r) dr = \infty \). Let \( h \) be a nondecreasing function of class \( C^1 \) defined on \((0, R)\). Assume that
\[ \lim_{r \to R} h(r) = \infty \text{ and } h(r_0) \geq c > 0. \] Then, for every \( 0 < \delta < 1 \), the inequality

\[ h'(r) \leq h^{1+\delta}(r)\gamma(r) \]

holds for all \( r \in (0, R) \) outside a set \( E \) with \( \int_E \gamma(r)dr < \infty \).

**Lemma 4** (Logarithmic Derivative Lemma [10]). Let \( R \) and \( \gamma(r) \) be as in Lemma 3. Let \( f \) be a meromorphic function on \( \mathbb{D}_R \). Then, for \( l \geq 1 \) and \( \delta > 0 \), the inequality

\[
\int_0^{2\pi} \log^+ \left| \frac{f^{(j)}(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \leq (1 + \delta)l \log \gamma(r) + \delta l \log r \\
+ O(\log T_f(r) + \log \log \gamma(r) + \log \log r)
\]

holds outside a set \( E \subset (0, R) \) with \( \int_E \gamma(r)dr < \infty \).

The following lemma was obtained by Demailly [4]:

**Lemma 5** ([4], [13]). Assume that \( D = (s) \) is totally geodesic with respect to \( \Lambda \) on \( U \). Let \( \beta \) be a holomorphic function on \( U \) such that \( \beta_a, \beta_b, \beta_c \) are holomorphic forms, where \( a, b, c \) are defined as in Definition 3. Let \( \gamma(r) \) be a holomorphic function on \( U \). Let \( f : V \to U \) be a holomorphic map. Then, for \( m \in \mathbb{N} \), we have that

\[
(s \circ f)^{(m)} = \gamma_m s \circ f + \sum_{0 \leq l \leq m-2} \gamma_{l,m} ds \cdot \Lambda^{(l)} f' + ds \cdot \Lambda^{(m-1)} f'
\]

in \( V \). Here \( \gamma_m \) and \( \gamma_{l,m} \) are meromorphic on \( V \) such that \( \beta^{m-1}(f) \gamma_m \) and \( \beta^{m-l-1}(f) \gamma_{l,m} \) are holomorphic on \( V \).

### 3.1. Proof of Theorem 5

We follow the argument of [13]. Let \( \{V_j\}_{1 \leq j \leq N} \) be an open covering of \( X \) such that the topological closure \( \overline{V_j} \) is contained in \( U_j \) and \( \overline{V_j} \) is compact. Then, we have a partition of unity \( \{\phi_j\}_{1 \leq j \leq N} \) subordinate to the covering \( \{V_j\} \). Take holomorphic function \( z_1, \ldots, z_k \) on \( U_j \) such that \( dz_1, \ldots, dz_k \) are linearly independent and

\[ U_j \cap \bigcup_{i=1}^{q} \text{supp}(Q_i) = \{ w \in U_j | z_1(w) \cdots z_p(w) = 0 \} \]

for some \( p, \ 0 \leq p \leq k \). Set

\[ f_l = z_l \circ f, \quad (\Lambda^{(m)} f')_l = dz_l \cdot \Lambda^{(m)} f'. \]
Then we have

\[
\phi_j(f) \log^+ \frac{\|W_\Lambda(f)\|_{\Lambda^r TX} \|\beta(f)\|^{k(k-1)/2}_L}{\prod_{i=1}^p \|Q_i(f)\|_{L_i}} = \phi_j(f) \log^+ (\Phi_j(f) \|\beta(f)\|^{k(k-1)/2}_L)
\]

\[
\begin{vmatrix}
\frac{f_1'}{f_1} & \cdots & \frac{f_p'}{f_p} \\
\frac{f_1'}{f_1} & \cdots & \frac{f_p'}{f_p} \\
\vdots & \ddots & \vdots \\
\frac{f_1'}{f_1} & \cdots & \frac{f_p'}{f_p}
\end{vmatrix}
\]

on \( f^{-1}(U_j) \), where \( \Phi_j \) is a \( C^\infty \)-function on \( U_j \). By Lemma 5,

\[
((A_j)^{m_i}_{m'})_i = \sum_{0 \leq i \leq m+1} a_{i,i,m}(z) \frac{d^{i}f_i}{dz^i}(z)
\]

for \( 1 \leq i \leq p \), where \( a_{i,i,m} \) are meromorphic functions on \( f^{-1}(U_j) \) such that \( a_{i,i,m}(\beta \circ f(z))_m \) is a holomorphic function. Hence it follows that, for \( 0 < r < R \),

\[
K_j := \int_0^{2\pi} \phi_j(f) \log^+ \frac{\|W_\Lambda(f)\|_{\Lambda^r TX} \|\beta(f)\|^{k(k-1)/2}_L}{\prod_{i=1}^p \|Q_i(f)\|_{L_i}} (r e^{i\theta}) d\theta
\]

\[
\leq \int_0^{2\pi} \Phi(f)(r e^{i\theta}) d\theta + \sum_{1 \leq m \leq p} \sum_{1 \leq l \leq k} \int_0^{2\pi} \log^+ \left| \frac{f_{m,l}^{(l)}}{f_m^{(l)}} (r e^{i\theta}) \right| d\theta
\]

\[
+ \sum_{p+1 \leq m \leq k} \sum_{1 \leq l \leq k} \int_0^{2\pi} \Psi(f)(r e^{i\theta}) \log^+ |f_{m,l}^{(l)}| (r e^{i\theta}) \frac{d\theta}{2\pi},
\]

where \( \Phi \) and \( \Psi \) are bounded \( C^\infty \)-functions on \( X \). By using the Logarithmic Derivative Lemma (Lemma 4) with \( \gamma(r) := \exp((c f_\omega + c) T_f(r)) \), it follows that

\[
\int_0^{2\pi} \log^+ \left| \frac{f_{m,l}^{(l)}}{f_m^{(l)}} (r e^{i\theta}) \right| d\theta \leq S_f(r,l)
\]

\[
\int_0^{2\pi} \Psi(f)(r e^{i\theta}) \log^+ |f_{m,l}^{(l)}(r e^{i\theta})| \frac{d\theta}{2\pi}
\]

\[
\leq \int_0^{2\pi} \Psi(f)(r e^{i\theta}) \log^+ |f_{m,l}^{(l)}(r e^{i\theta})| \frac{d\theta}{2\pi} + \int_0^{2\pi} \log^+ \left| \frac{f_{m,l}^{(l)}}{f_m^{(l)}} (r e^{i\theta}) \right| d\theta
\]

\[
\leq \int_0^{2\pi} \Psi(f)(r e^{i\theta}) \log^+ |f_{m,l}^{(l)}(r e^{i\theta})| \frac{d\theta}{2\pi} + S_f(r,l - 1),
\]
Also,

\[
\int_0^{2\pi} \Psi(f(re^{i\theta})) \log^+ |f'_m(re^{i\theta})| \frac{d\theta}{2\pi} = \frac{1}{2} \int_0^{2\pi} \Psi(f(re^{i\theta})) \log^+ |f'_m(re^{i\theta})| \frac{d\theta}{2\pi} \leq \frac{1}{2} \int_0^{2\pi} \log^+ \|f'(re^{i\theta})\|^2 \frac{d\theta}{2\pi} + O(1),
\]

where \( \| \cdot \|_{TX} \) is a hermitian metric of \( TX \).

By the Calculus Lemma with \( \gamma(r) := \exp((c_{f,\omega} + \epsilon)T_f(r)) \) and the concavity of log, we have that

\[
\frac{1}{2} \int_0^{2\pi} \log^+ \|f'(re^{i\theta})\|^2 \frac{d\theta}{2\pi} \leq \frac{1}{2} \int_0^{2\pi} \log(||f'(re^{i\theta})||^2_{TX} + 1) \frac{d\theta}{2\pi} + O(1)
\]

\[
\leq \frac{1}{2} \log(1 + \int_0^{2\pi} \|f'(re^{i\theta})||^2_{TX} \frac{d\theta}{2\pi} + O(1))
\]

\[
\leq \frac{1}{2} \log(1 + \frac{1}{2\pi r dr} \int_{\mathbb{B}_r} \|f'(z)||^2_{TX} \sqrt{-1} \frac{dz \wedge d\bar{z}}{2}) + O(1)
\]

\[
\leq \frac{1}{2} \log(1 + \frac{1}{2\pi r} \left( \int_{\mathbb{B}_r} \|f'(z)||^2_{TX} \sqrt{-1} \frac{dz \wedge d\bar{z}}{2} \right)^{1+\epsilon}) + O(1) \parallel \ (\text{Lemma 3})
\]

\[
= \frac{1}{2} \log(1 + \frac{r^\epsilon}{2\pi} \left( \frac{d}{dr} \int_1^r \frac{dt}{t} \int_{\mathbb{B}_1} \|f'(z)||^2_{TX} \sqrt{-1} \frac{dz \wedge d\bar{z}}{2} \right)^{1+\epsilon}) + O(1) \parallel \ (\text{Lemma 3})
\]

\[
\leq \frac{1}{2} \log(1 + \epsilon \log^+ r + (1 + \epsilon)^2 \gamma(r)^{2+\epsilon}) + O(1) \parallel \ (\text{Lemma 3})
\]

\[
\leq \epsilon \log^+ r + (1 + \epsilon)^2 \log^+ T_f(r) + (1 + \epsilon)(c_{f,\omega} + \epsilon)T_f(r) + O(1) \parallel .
\]

Therefore,

\[
K_j \leq O(1) + \sum_{1 \leq m \leq p} \sum_{1 \leq l \leq k} S_f(r, l) + \sum_{p+1 \leq m \leq k} \sum_{1 \leq l \leq k} S_f(r, l - 1)
\]

\[
+ \sum_{p+1 \leq m \leq k} \sum_{1 \leq l \leq k} (\epsilon \log^+ r + (1 + \epsilon)^2 \log^+ T_f(r) + (1 + \epsilon)(c_{f,\omega} + \epsilon)T_f(r))
\]

\[
\leq O(\log^+ r + \log^+ T_f(r) + \sum_{1 \leq m \leq k} \log^+ T_{f_m}(r)) + \frac{k^2(k + 1)}{2}(1 + \epsilon)(c_{f,\omega} + \epsilon)T_f(r) \parallel .
\]
Then we have that
\[
\int_0^{2\pi} \log^+ \frac{\|W_\Lambda(f)\|_{\Lambda^*TX} \|\beta(f)\|_L^{k(1-k)/2}}{\prod_{i=1}^N \|Q_i(f)\|_{L_j}} (re^{i\theta}) \frac{d\theta}{2\pi} = \sum_{j=1}^N K_j
\]
\[
\leq O(\log^+ r + \log^+ T_f(r) + \sum_{1 \leq m \leq k} \log^+ T_{f_m}(r)) + \frac{Nk^2(k+1)}{2}(1 + \epsilon)(c_{f,\omega} + \epsilon)T_f(r).
\]

We denote by \(\text{ord}_z(Q_j \circ f)\) the order of zeros of \(Q_j \circ f\) at the point of \(z \in \mathbb{D}_R\), and by \(\text{ord}_z(\beta(f))^{k(k-1)/2}W_\Lambda(f)\) the order of zero of \(\beta(f)^{k(k-1)/2}W_\Lambda(f)\) at the point \(z \in \mathbb{D}_R\). If \(\text{ord}_z(Q_j \circ f) \geq k + 1\) for \(z \in \mathbb{D}_R\), then
\[
\text{ord}_z(\beta(f)^{k(k-1)/2}W_\Lambda(f)) \geq \text{ord}_z(Q_j \circ f) - k.
\]

By Lemma 5 and the First Main Theorem, we have that
\[
\sum_{1 \leq i \leq q} T(r, L_i) - \sum_{i} N_k(r, f^*(Q_i)) - T_f(r, \Lambda^*TX) - \frac{k(k-1)}{2}T_f(r, L) \leq \int_0^{2\pi} \log^+ \frac{\|W_\Lambda(f)\|_{\Lambda^*TX} \|\beta(f)\|_L^{k(1-k)/2}}{\prod_{i=1}^N \|Q_i(f)\|_{L_j}} (re^{i\theta}) \frac{d\theta}{2\pi} \leq S_f(r).
\]

This completes the proof of Theorem 5.

3.2. Proof of Theorem 6. Let \([X_0 : \cdots : X_k]\) be a homogeneous coordinate system of \(\mathbb{P}^k\). Then by the same method used in Section 3 of Tiba [13], one can construct the meromorphic partial projective connection \(\Lambda = \{(\Lambda_j, U_j)\}_{0 \leq j \leq k}\) on \(\mathbb{P}^k\), where \(U_j = \{[X_0 : \cdots : X_k] \in \mathbb{P}^k| X_j \neq 0\}\). By Cramer’s rule, the solutions are of the form
\[
\Gamma^\Lambda_{i,j} = \frac{\delta^\Lambda_{i,j}}{\delta},
\]
where \(\delta = \det(\partial S_\mu/\partial X_\lambda)_{0 \leq \mu, \lambda \leq k}\), and \(\delta^\Lambda_{i,j}\) is the determinant replacing the column of index \(\lambda\) in \(\delta\) by the column of \(\partial^2 S_\mu/\partial X_i \partial X_j\) for \(0 \leq \mu \leq k\).

Since \(\partial S_\mu/\partial X_\lambda\) is a homogeneous polynomial of degree \(d - 1\), \(\delta\) is a homogeneous polynomial of degree \((k + 1)(d - 1)\). This implies that the degree of the polar divisor of each \(\Lambda_j\) is less than or equal to \((k + 1)(d - 1)\), hence \(T(r, L) \leq (k + 1)(d - 1)T_f(r)\). In this case, \(N = k + 1\) and \(T_{f,\mu}(r) \leq T_f(r) + O(1)\). Thus, Theorem 6 follows from Theorem 5, the First Main Theorem and \(K_{\mathbb{P}^k} = -(k + 1)H\).

4. Proof of Theorem 4

To prove Theorem 4, we need the following important proposition from [6]:

...
Proposition 1 ([6]). Let $\sigma_j, 1 \leq j \leq q$ be the smooth hypersurfaces defined in Theorem 4. Let $f$ and $g$ be two holomorphic maps from $\mathbb{D}_R$ into $\mathbb{P}^k$ with $c_f < \infty$ and $c_g < \infty$. Suppose that $f(z) = g(z)$ for all $z \in S$, where

$$S := \bigcup_{j=1}^{q} \{ f^{-1}(\sigma_j) \cup g^{-1}(\sigma_j) \}.$$ 

If $f \not= g$, we have that

$$N(r, S) \leq T_f(r) + T_g(r) + O(1).$$

Proof. Let $\pi_1, \pi_2$ be the projective maps from $\mathbb{P}^k \times \mathbb{P}^k$ into the first $\mathbb{P}^k$ and the second one, respectively. If $f \not= g$, then from the proof of Lemma 1 it is not hard to see that the diagonal $(f \times g)(S)$ is in Supp$(s)$, but the image $(f \times g)(\mathbb{D}_R)$ is not contained in Supp$(s)$, where $s$ is a section of $H' := \pi_1 H \otimes \pi_2 H$ determined by a non-constant polynomial with complex coefficients

$$P([z_0, \ldots, z_k], [w_0, \ldots, w_k]) = \sum_{0 \leq m < l \leq k} a_{ml}(z_m w_l - z_l w_m).$$

Thus, we have that

$$N(r, S) \leq N(r, (f \times g)^*(s)).$$

By the First Main Theorem for Line Bundles,

$$N(r, (f \times g)^*(s)) \leq T_{f \times g}(r, H') + O(1) \leq T_f(r) + T_g(r) + O(1).$$

Hence, we obtain the result. \[\square\]

4.1. Proof of Theorem 4(i). The proof is by contradiction. Suppose that $f \not= g$. Theorem 6 gives that

$$\left( q - \frac{k+1}{d} - \frac{1}{2d} k(k+1)(d-1) \right) T_f(r) \leq d^{-1} \sum_{1 \leq i \leq q} N_k(r, f^*(\sigma_i)) + S_f(r),$$

$$\left( q - \frac{k+1}{d} - \frac{1}{2d} k(k+1)(d-1) \right) T_g(r) \leq d^{-1} \sum_{1 \leq i \leq q} N_k(r, g^*(\sigma_i)) + S_g(r).$$

Thus,

$$\left( q - \frac{k+1}{d} - \frac{1}{2d} k(k+1)(d-1) \right) (T_f(r) + T_g(r)) \leq d^{-1} \sum_{1 \leq i \leq q} (N_k(r, f^*(\sigma_i)) + N_k(r, g^*(\sigma_i))) + (S_f(r) + S_g(r)).$$
From the condition \( f(z) = g(z) \) on \( S \), we have that
\[
\sum_{1 \leq i \leq q} (n_k(r, f^*(\sigma_i)) + n_k(r, g^*(\sigma_i))) \leq k \sum_{1 \leq i \leq q} (n_1(r, f^*(\sigma_i)) + n_1(r, g^*(\sigma_i))) \\
\leq 2kn(r, S).
\]
Hence,
\[
\left(q - \frac{k + 1}{d} - \frac{1}{2d}(k - 1)k(k + 1)(d - 1)\right)(T_f(r) + T_g(r)) \\
\leq \frac{2k}{d}N(r, S) + (S_f(r) + S_g(r)).
\]
Let \( c_{f,g} = \max\{c_f, c_g\} \). Applying Proposition 1 gives
\[
\left(q - \frac{k + 1}{d} - \frac{1}{2d}(k - 1)k(k + 1)(d - 1)\right)(T_f(r) + T_g(r)) \\
\leq \frac{2k}{d}(T_f(r) + T_g(r)) + (S_f(r) + S_g(r)) \\
\leq \frac{2k}{d}(T_f(r) + T_g(r)) + \frac{k^2(k + 1)^2}{2}(1 + \epsilon)(c_{f,g} + \epsilon)(T_f(r) + T_g(r)) \\
+ O(\log^+ r + \log^+ T_f(r) + \log^+ T_g(r))
\]
Thus
\[
q \leq \frac{3k + 1}{d} + \frac{1}{2d}(k - 1)k(k + 1)(d - 1) + \frac{k^2(k + 1)^2}{2}c_{f,g},
\]
which is in contradiction to the assumption that
\[
q > \frac{3k + 1}{d} + \frac{1}{2d}(k - 1)k(k + 1)(d - 1) + \frac{k^2(k + 1)^2}{2}\max\{c_f, c_g\}.
\]
Hence \( f \equiv g \).

4.2. **Proof of Theorem 4(ii).** We follow the method of Chen and Yan [3]. Suppose that the assertion does not hold. By changing indices if necessary, we may assume that
\[
\frac{\sigma_1(f)}{\sigma_1(g)} = \ldots = \frac{\sigma_{v_1}(f)}{\sigma_{v_1}(g)} \neq \frac{\sigma_{v_1+1}(f)}{\sigma_{v_1+1}(g)} = \ldots = \frac{\sigma_{v_2}(f)}{\sigma_{v_2}(g)} \neq \ldots \neq \frac{\sigma_{v_{s-1}+1}(f)}{\sigma_{v_{s-1}+1}(g)} = \ldots = \frac{\sigma_{v_s}(f)}{\sigma_{v_s}(g)},
\]
where \( v_s = q \).

Since the assertion (ii) does not hold, the number of elements of each group is at most \( k \). For each \( 1 \leq i \leq q \), we set
\[
p(i) = \begin{cases} i + k & \text{if } i + k \leq q \\ i + k - q & \text{if } i + k > q \end{cases}
\]
and \( P_i = \sigma_i(f)\sigma_{p(i)}(g) - \sigma_i(g)\sigma_{p(i)}(f) \). Then \( \frac{\sigma_i(f)}{\sigma_i(g)} \) and \( \frac{\sigma_{p(i)}(f)}{\sigma_{p(i)}(g)} \) are from two distinct groups, and hence \( P_i \neq 0 \) for every \( 1 \leq i \leq q \). Since \( f(z) = g(z) \) for all \( z \in S \), note that \( n_f(r, \sigma_j) = n(r, 0, \sigma_j(f)) \) for all \( j = 1, \ldots, q \). For convenience, we denote by \( n_E(r, 0, f) \) the number of zeros of \( f \) in \( D_r \cap E \). Recall that

\[
S = \bigcup_{j=1}^{q} \{ f^{-1}(\sigma_j) \cup g^{-1}(\sigma_j) \}.
\]

Let \( S_1 = S \setminus \{ \sigma_i(f)\sigma_{p(i)}(f) = 0 \} \); we consider the following cases:

**Case 1:** If \( z \in S_1 \), then \( \sigma_j(f)(z) \neq 0 \) for \( j = i, p(i) \), and hence

\[
n_{S_1}(r, 0, P_i) \geq \sum_{j=1, j \neq i, p(i)}^{q} \min\{1, n_{S_1}(r, 0, \sigma_j(f))\}.
\]

**Case 2:** If \( z \notin S_1 \), then \( \sigma_j(f)(z) \neq 0 \) for \( j \neq i, p(i) \) and \( \sigma_i(f)(z)\sigma_{p(i)}(f)(z) = 0 \), hence the form of \( P_i \) gives that

\[
n_{S_1}(r, 0, P_i) \geq \min\{n_{S_1}(r, 0, \sigma_i(f)), n_{S_1}(r, 0, \sigma_i(g))\} + \min\{n_{S_1}(r, 0, \sigma_{p(i)}(f)), n_{S_1}(r, 0, \sigma_{p(i)}(g))\},
\]

where \( S_1^c \) means the complement of \( S_1 \) in \( \mathbb{D}_r \).

Combining Case 1, Case 2 and the fact that \( n_f(r, \sigma_i) = n_{S_1}(r, 0, \sigma_i(f)) + n_{S_1}(r, 0, \sigma_i(f)) \), as well as \( \min\{a, b\} \geq \min\{a, k\} + \min\{b, k\} - k \); we deduce that

\[
n_{P_i}(r, 0) \geq \min\{n_f(r, \sigma_i), n_g(r, \sigma_i)\} + \min\{n_f(r, \sigma_{p(i)}), n_g(r, \sigma_{p(i)})\}
\]

\[
+ \sum_{j=1, j \neq i, p(i)}^{q} \min\{1, n_f(r, \sigma_j)\}
\]

\[
\geq \sum_{j=i, p(i)}^{q} \left( \min\{n_f(r, \sigma_j), k\} + \min\{n_g(r, \sigma_j), k\} - k \min\{n_f(r, \sigma_j), 1\} \right)
\]

\[
+ \sum_{j=1, j \neq i, p(i)}^{q} \min\{1, n_f(r, \sigma_j)\}.
\]

Integrating both sides of this inequality, we obtain that

\[
N_{P_i}(r, 0) \geq \sum_{j=i, p(i)}^{q} \left( N_k(r, f^*(\sigma_j)) + N_k(r, g^*(\sigma_j)) - kN_1(r, f^*(\sigma_j)) \right)
\]

\[
+ \sum_{j=1, j \neq i, p(i)}^{q} N_1(r, f^*(\sigma_j)).
\]

Notice that

\[
\frac{P_i}{\sigma_i(g)\sigma_{p(i)}(f)} = \frac{\sigma_{p(i)}(g)}{\sigma_i(g)} - \frac{\sigma_{p(i)}(f)}{\sigma_i(f)}
\]
and that $\sigma_i(g)$, $\sigma_i(f)$ are entire functions. Then, for $0 < r < R$,
\[
N_{P_i}(r, 0) \leq T \left( r, \frac{\sigma_{p(i)}(g)}{\sigma_i(g)} - \frac{\sigma_{p(i)}(f)}{\sigma_i(f)} \right) \leq d(T_f(r) + T_g(r)) + O(1).
\]
Thus, we have that
\[
\sum_{i=1}^{q} d(T_f(r) + T_g(r)) + O(1) \geq \sum_{i=1}^{q} \sum_{j=i, p(i)} \left( N_k(r, f^*(\sigma_j)) + N_k(r, g^*(\sigma_j)) \right)
\]
\[
-kN_1(r, f^*(\sigma_j)) + \sum_{i=1}^{q} \sum_{j \neq i, p(i)} N_1(r, f^*(\sigma_j))
\]
\[
\geq 2 \sum_{i=1}^{q} \left( N_k(r, f^*(\sigma_j)) + N_k(r, g^*(\sigma_j)) \right) + (q - 2(k + 1)) \sum_{j=1}^{q} N_1(r, f^*(\sigma_j)).
\]
By Theorem 6 and $q \geq 2(k + 1)$, it follows that
\[
qd(T_f(r) + T_g(r)) + 2(S_f(r) + S_g(r)) + O(1)
\]
\[
\geq 2d \left( q - \frac{k + 1}{d} - \frac{1}{2d} (k - 1)k(k + 1)(d - 1) \right) (T_f(r) + T_g(r)).
\]
Since $S_f(r) + S_g(r) \leq O(\log^+ r + \log^+ T_f(r) + \log^+ T_g(r)) + \frac{k^2(k + 1)^2}{2}(1 + \epsilon)(c_{f,g} + \epsilon)(T_f(r) + T_g(r))$, it follows that
\[
q \leq 2 \left( \frac{k + 1}{d} + \frac{1}{2d} (k - 1)k(k + 1)(d - 1) \right) + \frac{k^2(k + 1)^2}{d} \max(c_f, c_g),
\]
which is in contradiction to the assumption in Theorem 4(ii). This completes the proof.

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