Pseudoinverse-free randomized block iterative algorithms for consistent and inconsistent linear systems

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Abstract

Randomized iterative algorithms have attracted much attention in recent years because they can approximately solve large-scale linear systems of equations without accessing the entire coefficient matrix. In this paper, we propose two novel pseudoinverse-free randomized block iterative algorithms for solving consistent and inconsistent linear systems. The proposed algorithms require two user-defined random matrices: one for row sampling and the other for column sampling. We can recover the well-known doubly stochastic Gauss–Seidel, randomized Kaczmarz, randomized coordinate descent, and randomized extended Kaczmarz algorithms by choosing appropriate random matrices used in our algorithms. Because our algorithms allow for a much wider selection of these two random matrices, a number of new specific algorithms can be obtained. We prove the linear convergence in the mean square sense of our algorithms. Numerical experiments for linear systems with synthetic and real-world coefficient matrices demonstrate the efficiency of some special cases of our algorithms.

Keywords. Block row sampling, block column sampling, extended block row sampling, consistent and inconsistent linear systems, linear convergence

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1 Introduction

Row-action iterative algorithms such as the Kaczmarz algorithm [13] (also known in computerized tomography as the algebraic reconstruction technique [22]) are widely used to solve a linear system of equations

$$Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m. \quad (1)$$

They do not need to compute entire matrix-vector multiplications, and each iteration only requires a sample of rows of the coefficient matrix. Numerical experiments show that using the rows of the coefficient matrix in random order rather than in their given order can often greatly improve the convergence [12, 22]. In a seminal paper [30], Strohmer and Vershynin proposed a randomized Kaczmarz algorithm which converges linearly in the mean square sense to a solution of a consistent linear system. The convergence rate of the randomized Kaczmarz algorithm depends only on the scaled condition number of the coefficient matrix. Many subsequent studies on the development and analysis of randomized iterative algorithms for consistent and inconsistent linear systems of equations have been triggered; see, for example, [15, 24, 33, 9, 10, 19, 20, 4, 7, 27, 11]. Variants based on a variety of acceleration strategies have also been proposed; see, for example, [16, 11, 23, 17, 23, 8, 20, 52, 21, 28, 15, 31].

In this paper, we propose a doubly stochastic block iterative algorithm and an extended block row sampling iterative algorithm for solving consistent and inconsistent linear systems. Our algorithms employ two user-defined discrete or continuous random matrices: one for row sampling and the other for column sampling. By choosing appropriate random matrices in our algorithms, we recover the doubly stochastic Gauss–Seidel (DSGS) algorithm [27], the randomized Kaczmarz (RK) algorithm [30], the randomized coordinate descent (RCD) algorithm [15], and the randomized extended Kaczmarz (REK) algorithm [33]. We emphasize that our algorithms are pseudoinverse-free and therefore different from

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Numerical results are reported to illustrate the efficiency of some special cases of our algorithms. For a wide range of distributions, more efficient block cases of the general algorithms can be designed.

Main theoretical results. For arbitrary initial guess \( \mathbf{x}^0 \in \mathbb{R}^n \), we define the vector
\[
\mathbf{x}^0 := \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{x}^0,
\]
which is a solution if \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is consistent, or a least squares solution if \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is inconsistent. We mention that \( \mathbf{x}^0 \) is the orthogonal projection of \( \mathbf{x}^0 \) onto the set
\[
\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b} \}.
\]
The main theoretical results of this work are as follows.

1. The block row sampling iterative algorithm (one special case of the doubly stochastic block iterative algorithm; see Section 2.1) converges linearly in the mean square sense to \( \mathbf{x}^0 \) if \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is consistent (Theorem 5) and to within a radius of \( \mathbf{x}^0 \) if \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is inconsistent (Theorem 6).

2. The block column sampling iterative algorithm (another special case of the doubly stochastic block iterative algorithm; see Section 2.2) converges linearly in the mean square sense to \( \mathbf{A}^\dagger \mathbf{b} \) if \( \mathbf{A} \) has full column rank (Theorem 7).

3. The extended block row sampling iterative algorithm converges linearly in the mean square sense to \( \mathbf{x}^0 \) for arbitrary linear system \( \mathbf{A}\mathbf{x} = \mathbf{b} \) (we make no assumptions about the dimensions or rank of the coefficient matrix \( \mathbf{A} \) and the system can be consistent or inconsistent; see Theorem 11).

Organization of the paper. In Section 2 we propose the doubly stochastic block iterative algorithm (including its special cases) and construct the convergence theory. In Section 3 we propose the extended block row sampling algorithm and prove its linear convergence for arbitrary linear systems. We report the numerical results in Section 4. Finally, we present brief concluding remarks in Section 5.

Notation. For any random variable \( \xi \), we use \( \mathbb{E}[\xi] \) to denote the expectation of \( \xi \). For an integer \( m \geq 1 \), let \( [m] := \{1, 2, 3, \ldots, m\} \). For any vector \( \mathbf{b} \in \mathbb{R}^n \), we use \( \mathbf{b}_i, \mathbf{b}^\top \) and \( ||\mathbf{b}|| \) to denote the \( i \)th entry, the transpose and the Euclidean norm of \( \mathbf{b} \), respectively. We use \( \mathbf{I} \) to denote the identity matrix whose order is clear from the context. For any matrix \( \mathbf{A} \in \mathbb{R}^{m \times n} \), we use \( \mathbf{A}_{i,j}, \mathbf{A}_{i,:}, \mathbf{A}_{:,j}, \mathbf{A}^\top, \mathbf{A}^\dagger, \|\mathbf{A}\|, \|\mathbf{A}\|_F, \text{range}(\mathbf{A}), \text{rank}(\mathbf{A}), \sigma_{\max}(\mathbf{A}) \) and \( \sigma_{\min}(\mathbf{A}) \) to denote the \((i,j)\) entry, the \( i \)th row, the \( j \)th column, the transpose, the Moore–Penrose pseudoinverse, the spectral norm, the Frobenius norm, the column space, the rank, the maximum and the minimum nonzero singular values of \( \mathbf{A} \), respectively.

If \( \text{rank}(\mathbf{A}) = r \), we also denote all the nonzero singular values of \( \mathbf{A} \) by \( \sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{A}) \geq \cdots \geq \sigma_r(\mathbf{A}) > 0 \).

For index sets \( \mathcal{I} \subseteq [m] \) and \( \mathcal{J} \subseteq [n] \), let \( \mathbf{A}_{\mathcal{I},:}, \mathbf{A}_{:,\mathcal{J}}, \) and \( \mathbf{A}_{\mathcal{I},\mathcal{J}} \) denote the row submatrix indexed by \( \mathcal{I} \), the column submatrix indexed by \( \mathcal{J} \), and the submatrix that lies in the rows indexed by \( \mathcal{I} \) and the columns indexed by \( \mathcal{J} \), respectively. We use \( |\mathcal{I}| \) to denote the cardinality of a set \( \mathcal{I} \subseteq [m] \). Given a symmetric matrix \( \mathbf{A} \), we use \( \lambda_{\max}(\mathbf{A}) \) to denote the largest eigenvalue of \( \mathbf{A} \). Given two symmetric matrices \( \mathbf{A} \) and \( \mathbf{B} \), we use \( \mathbf{A} \succeq \mathbf{B} \) to denote that \( \mathbf{A} - \mathbf{B} \) is positive semidefinite.

Preliminary. The following lemma will be used, and its proof is straightforward.

**Lemma 1.** Let \( \alpha > 0 \), \( \beta > 0 \), and \( \mathbf{A} \in \mathbb{R}^{m \times n} \) be any nonzero matrix with \( \text{rank}(\mathbf{A}) = r \). For all \( \mathbf{u} \in \text{range}(\mathbf{A}^\top) \), and \( 0 \leq i \leq k \), it holds
\[
\| (\mathbf{I} - \beta \mathbf{A}^\top \mathbf{A})^i (\mathbf{I} - \alpha \mathbf{A}^\top \mathbf{A})^{k-i} \mathbf{u} \| \leq \delta^k \| \mathbf{u} \|,
\]
where
\[
\delta = \max_{1 \leq i \leq r} \{ |1 - \alpha \sigma_i^2(\mathbf{A})|, |1 - \beta \sigma_i^2(\mathbf{A})| \}.
\]
2 The doubly stochastic block iterative algorithm

Given an arbitrary initial guess $x^0 \in \mathbb{R}^n$, the $k$th iterate of the doubly stochastic block iterative (DSBI) algorithm is defined as

$$x^k = x^{k-1} - \alpha \mathbf{T} \mathbf{T}^\top \mathbf{A}^\top \mathbf{S} \mathbf{S}^\top (A x^{k-1} - b),$$

(2)

where the stepsize parameter $\alpha > 0$, and the random parameter matrix pair $(\mathbf{S}, \mathbf{T})$ is sampled independently in each iteration from a distribution $\mathcal{D}$ and satisfies

$$\mathbb{E}[\mathbf{T} \mathbf{T}^\top \mathbf{A}^\top \mathbf{S} \mathbf{S}^\top] = \mathbf{A}^\top.$$

We note that the random parameter matrices $\mathbf{S} \in \mathbb{R}^{m \times p}$ and $\mathbf{T} \in \mathbb{R}^{n \times q}$ can be independent or not. We also emphasize that we do not restrict the numbers of columns of $\mathbf{S}$ and $\mathbf{T}$; indeed, we allow $p$ and $q$ to vary (and hence $p$ and $q$ are random variables). Let $\mathbb{E}_{k-1} [\cdot]$ denote the conditional expectation conditioned on the first $k-1$ iterations of the DSBI algorithm. We have

$$\mathbb{E}_{k-1}[x^k] = x^{k-1} - \alpha \mathbf{A}^\top (\mathbf{A} x^{k-1} - b),$$

which is the update of the Landweber iteration [14]. We note that the DSGS algorithm [27] is one special case of the DSBI algorithm. Let the index pair $(i, j)$ be randomly selected with probability $|A_{i,j}|^2 / \| \mathbf{A} \|^2_F$. Setting $\mathbf{S} = \| \mathbf{A} \|^2_F^{-1} \mathbf{I}_{i,j}$ and $\mathbf{T} = \mathbf{I}_{i,j}$ in (2), we have

$$\mathbb{E}[\mathbf{T} \mathbf{T}^\top \mathbf{A}^\top \mathbf{S} \mathbf{S}^\top] = \frac{\| \mathbf{A} \|^2_F}{\| \mathbf{A} \|^2_F} \sum_{i=1}^{m} \sum_{j=1}^{n} \| I_{i,j} \|^2 \| I_{i,j} \|^2 / \| \mathbf{A} \|^2_F = \mathbf{A}^\top,$$

and

$$x^k = x^{k-1} - \alpha \| \mathbf{A} \|^2_F^{-1} I_{i,j} (A_{i,j} x^{k-1} - b_i),$$

which is the $k$th iterate of the DSGS algorithm.

In the following, we shall present two convergence results (Theorems 2 and 4) of the DSBI algorithm: the first is the convergence of $\| \mathbb{E}[x^k] - x^0 \|$ for arbitrary linear systems, and the second is the convergence of $\| \mathbb{E}[x^k - \mathbf{A}^\dagger b]|^2 \|$ for linear systems with full column rank $\mathbf{A}$.

**Theorem 2.** For arbitrary $x^0 \in \mathbb{R}^n$, the $k$th iterate $x^k$ of the DSBI algorithm satisfies

$$\mathbb{E}[x^k - x^0] = (\mathbf{I} - \alpha \mathbf{A}^\top \mathbf{A})^k (x^0 - x^0).$$

Moreover,

$$\| \mathbb{E}[x^k] - x^0 \| \leq \left( \max_{1 \leq i \leq r} |1 - \alpha \sigma_i^2(\mathbf{A})| \right)^k \| x^0 - x^0 \|. \quad (4)$$

**Proof.** By $\mathbf{A}^\top \mathbf{A} x^0 = \mathbf{A}^\top b$ and straightforward calculations, we have

$$\mathbb{E}_{k-1}[x^k - x^0] = x^{k-1} - x^0 - \alpha \mathbb{E}[\mathbf{T} \mathbf{T}^\top \mathbf{A}^\top \mathbf{S} \mathbf{S}^\top] (\mathbf{A} x^{k-1} - b)$$

$$= x^{k-1} - x^0 - \alpha \mathbf{A}^\top (\mathbf{A} x^{k-1} - b)$$

$$= x^{k-1} - x^0 - \alpha \mathbf{A}^\top (\mathbf{A} x^{k-1} - x^0)$$

$$= (\mathbf{I} - \alpha \mathbf{A}^\top \mathbf{A}) (x^{k-1} - x^0).$$

Then, by the law of total expectation, we have

$$\mathbb{E}[x^k - x^0] = (\mathbf{I} - \alpha \mathbf{A}^\top \mathbf{A}) \mathbb{E}[x^{k-1} - x^0].$$

Unrolling the recurrence yields the formula (3). By $x^0 - x^0 = \mathbf{A}^\dagger \mathbf{A} x^0 - \mathbf{A}^\dagger b \in \text{range} (\mathbf{A}^\top)$ and Lemma [1] we obtain the estimate (4). \qed
Remark 3. In Theorem 2, no assumptions about the dimensions or rank of \( A \) are assumed, and the system \( Ax = b \) can be consistent or inconsistent. If \( 0 < \alpha < 2/\sigma_{\text{max}}^2(A) \), then \( \max_{1 \leq i \leq r} |1 - \alpha \sigma_i^2(A)| < 1 \). This means \( x^k \) is an asymptotically unbiased estimator for \( x^0 \).

Theorem 4. Let \( A \) have full column rank. For arbitrary \( x^0 \in \mathbb{R}^n \) and \( \varepsilon > 0 \), if \( 0 < \alpha < 2\sigma_{\text{min}}^2(A) \), then the \( k \)th iterate \( x^k \) of the DSBI algorithm satisfies

\[
\mathbb{E} \left[ \| x^k - A^\dagger b \|^2 \right] \leq \eta^k \| x^0 - A^\dagger b \|^2 + \frac{\alpha(1 + 1/\varepsilon)\gamma(1 - \eta^k)}{2\sigma_{\text{min}}^2(A) - (1 + \varepsilon)\alpha\beta},
\]

where

\[
\eta = 1 - 2\alpha\sigma_{\text{min}}^2(A) + (1 + \varepsilon)\alpha^2\beta, \quad \beta = \| \mathbb{E} \left[ A^\top S S^\top A T T^\top T T^\top A^\top S S^\top A \right] \|,
\]

and

\[
\gamma = \mathbb{E} \left[ \| T T^\top A^\top S S^\top (A A^\dagger b - b) \|^2 \right].
\]

Proof. Straightforward calculations yield

\[
\| x^k - A^\dagger b \|^2 = \| x^{k-1} - A^\dagger b \|^2 - 2\alpha(x^{k-1} - A^\dagger b)^\top T T^\top A^\top S S^\top (Ax^{k-1} - b) + \alpha^2 \| T T^\top A^\top S S^\top (Ax^{k-1} - b) \|^2.
\]

By (5), \( \mathbb{E} \left[ T T^\top A^\top S S^\top \right] = A^\top \), and \( A^\top A A^\dagger b = A^\dagger b \), we have

\[
\mathbb{E}_{k-1} \left[ \| x^k - A^\dagger b \|^2 \right] = \| x^{k-1} - A^\dagger b \|^2 - 2\alpha(x^{k-1} - A^\dagger b)^\top A^\top A(x^{k-1} - A^\dagger b) + \alpha^2 \mathbb{E}_{k-1} \left[ \| T T^\top A^\top S S^\top (Ax^{k-1} - b) \|^2 \right].
\]

It follows from \( A \) has full column rank that

\[
(x^{k-1} - A^\dagger b)^\top A^\top A(x^{k-1} - A^\dagger b) \geq \sigma_{\text{min}}^2(A) \| x^{k-1} - A^\dagger b \|^2.
\]

By triangular inequality and Young’s inequality, we have

\[
\| T T^\top A^\top S S^\top (Ax^{k-1} - b) \|^2 \leq \| T T^\top A^\top S S^\top (Ax^{k-1} - AA^\dagger b + AA^\dagger b - b) \|^2
\]

\[
\leq (\| T T^\top A^\top S S^\top (Ax^{k-1} - AA^\dagger b) \| + \| T T^\top A^\top S S^\top (AA^\dagger b - b) \|)^2
\]

\[
\leq (1 + \varepsilon)\| T T^\top A^\top S S^\top A(x^{k-1} - A^\dagger b) \|^2 + (1 + 1/\varepsilon)\| T T^\top A^\top S S^\top (AA^\dagger b - b) \|^2.
\]

Note that

\[
\mathbb{E}_{k-1} \left[ \| T T^\top A^\top S S^\top (Ax^{k-1} - A^\dagger b) \|^2 \right]
\]

\[
= (x^{k-1} - A^\dagger b)^\top \mathbb{E} \left[ A^\top S S^\top A T T^\top T T^\top A^\top S S^\top A \right] (x^{k-1} - A^\dagger b)
\]

\[
\leq \beta \| x^{k-1} - A^\dagger b \|^2.
\]

Combining (6), (7), (8), and (9) yields

\[
\mathbb{E}_{k-1} \left[ \| x^k - A^\dagger b \|^2 \right] \leq \| x^{k-1} - A^\dagger b \|^2 - 2\alpha\sigma_{\text{min}}^2(A) \| x^{k-1} - A^\dagger b \|^2 + (1 + \varepsilon)\alpha^2\beta \| x^{k-1} - A^\dagger b \|^2 + (1 + 1/\varepsilon)\alpha^2\gamma
\]

\[
= \eta \| x^{k-1} - A^\dagger b \|^2 + (1 + 1/\varepsilon)\alpha^2\gamma.
\]

Then the expected squared norm of the error can be bounded by

\[
\mathbb{E} \left[ \| x^k - A^\dagger b \|^2 \right] \leq \eta \mathbb{E} \left[ \| x^{k-1} - A^\dagger b \|^2 \right] + (1 + 1/\varepsilon)\alpha^2\gamma
\]

\[
\leq \eta^k \| x^0 - A^\dagger b \|^2 + (1 + 1/\varepsilon)\alpha^2\gamma \sum_{i=0}^{k-1} \eta^i
\]

\[
= \eta^k \| x^0 - A^\dagger b \|^2 + \frac{\alpha(1 + 1/\varepsilon)\gamma(1 - \eta^k)}{1 - \eta}
\]

\[
= \eta^k \| x^0 - A^\dagger b \|^2 + \frac{\alpha(1 + 1/\varepsilon)\gamma(1 - \eta^k)}{2\sigma_{\text{min}}^2(A) - (1 + \varepsilon)\alpha\beta}.
\]

This completes the proof. \( \square \)
In the last inequality, we use the facts that $\gamma = 0$, the DSBI algorithm with sufficiently small $\alpha$ converges linearly to the unique solution in the mean square sense. But, on the other hand, a small parameter $\alpha$ straightforwardly implies very slow convergence ($\eta \approx 1$). If more about the random parameter matrices $S$ and $T$ are available, then improved convergence results can be obtained. In the following subsections, we discuss the case $T = I$ and the case $S = I$, respectively.

2.1 Block row sampling

In this subsection, we consider the case $T = I$ and refer to the resulting algorithm as the block row sampling iterative (BRSI) algorithm. Given an arbitrary initial guess $x^0 \in \mathbb{R}^n$, the $k$th iterate of the BRSI algorithm is

$$x^k = x^{k-1} - \alpha_r A^\top SS^\top (Ax^{k-1} - b),$$

(10)

where the stepsize parameter $\alpha_r > 0$, and the random parameter matrix $S$ is sampled independently in each iteration from a distribution $D_r$ and satisfies $\mathbb{E}[SS^\top] = I$. Various choices for $S$ can be used, e.g., see [5].

In the following, we shall present the convergence of $\mathbb{E}[\|x^k - x^0\|^2]$ for the consistent case and the inconsistent case in Theorems 5 and 6, respectively. For the consistent case, Theorem 5 shows that the BRSI algorithm converges linearly to a solution. For the inconsistent case, Theorem 6 shows that the BRSI algorithm can only converge to within a radius (convergence horizon) of a least squares solution. Throughout, we define $\lambda^r_{\text{max}} = \max_{S \sim D_r} \lambda_{\text{max}}(A^\top SS^\top A)$.

**Theorem 5.** Assume that $0 < \alpha_r < 2/\lambda^r_{\text{max}}$. If $Ax = b$ is consistent, then for arbitrary $x^0 \in \mathbb{R}^n$, the $k$th iterate $x^k$ of the BRSI algorithm satisfies

$$\mathbb{E}[\|x^k - x^0\|^2] \leq \eta^k \|x^0 - x^0\|^2,$$

where

$$\eta_k = 1 - \alpha_r (2 - \alpha_r \lambda^r_{\text{max}}) \sigma^2_{\text{min}}(A).$$

**Proof.** It follows from $Ax^0 = b$ and

$$x^k - x^0 = x^{k-1} - x^0 - \alpha_r A^\top SS^\top (Ax^{k-1} - b)$$

(11)

that

$$\|x^k - x^0\|^2 = \|x^{k-1} - x^0\|^2 - 2\alpha_r (x^{k-1} - x^0)^\top A^\top SS^\top A (x^{k-1} - x^0) + \alpha_r^2 \|A^\top SS^\top A (x^{k-1} - x^0)\|^2.$$ 

Note that for any $H \succeq 0$, it holds $\lambda_{\text{max}}(H)H \succeq H^2$. Then we have

$$\|A^\top SS^\top A (x^{k-1} - x^0)\|^2 = (x^{k-1} - x^0)^\top (A^\top SS^\top A)^2 (x^{k-1} - x^0)$$

$$\leq \lambda_{\text{max}}(A^\top SS^\top A) (x^{k-1} - x^0)^\top A^\top SS^\top A (x^{k-1} - x^0)$$

$$\leq \lambda^r_{\text{max}} (x^{k-1} - x^0)^\top A^\top SS^\top A (x^{k-1} - x^0).$$

(12)

By $x^0 - x_* = A^\top(Ax^0 - b) \in \text{range}(A^\top)$, $A^\top SS^\top A (x^{k-1} - x^0) \in \text{range}(A^\top)$, and (11), we can prove that $x^k - x_0^* \in \text{range}(A^\top)$ by induction. Therefore,

$$\mathbb{E}_{k-1}[\|x^k - x^0\|^2] \leq \|x^{k-1} - x^0\|^2 - 2\alpha_r (x^{k-1} - x^0)^\top A^\top A (x^{k-1} - x^0) + \alpha_r^2 \lambda^r_{\text{max}} (x^{k-1} - x^0)^\top A^\top A (x^{k-1} - x^0)$$

$$\leq (1 - \alpha_r (2 - \alpha_r \lambda^r_{\text{max}}) \sigma^2_{\text{min}}(A))\|x^{k-1} - x^0\|^2.$$

In the last inequality, we use the facts that $-\alpha_r (2 - \alpha_r \lambda^r_{\text{max}}) < 0$, and for all $u \in \text{range}(A^\top)$, it holds $u^\top A^\top A u \geq \sigma^2_{\text{min}}(A)\|u\|^2$. Next, by the law of total expectation, we have

$$\mathbb{E}[\|x^k - x^0\|^2] \leq \eta_k \mathbb{E}[\|x^{k-1} - x^0\|^2].$$

Unrolling the recurrence yields the result.
According to Theorem 5, the best convergence rate \( \eta_t = 1 - \frac{\sigma^2_{\min}(A)}{\lambda_{\text{max}}^r} \) of the BRSI algorithm is achieved when \( \alpha_t = 1/\lambda_{\text{max}}^r \). However, the proof of Theorem 5 uses the worst-case estimates, so the convergence bound may be pessimistic, and in practical applications it may not precisely measure the actual convergence rate of the BRSI algorithm.

**Theorem 6.** Assume that \( \varepsilon > 0 \) and \( 0 < \alpha_t < \frac{2}{(1 + \varepsilon)\lambda_{\text{max}}^r} \). If \( Ax = b \) is inconsistent, then for arbitrary \( x^0 \in \mathbb{R}^n \), the \( \ell \)th iterate \( x^\ell \) of the BRSI algorithm satisfies

\[
\mathbb{E} \left[ \|x^\ell - x^*_\ell\|^2 \right] \leq \eta^\ell \|x^0 - x^*_0\|^2 + \frac{\alpha_t(1 + 1/\varepsilon)\gamma(1 - \eta^k)}{(2 - \alpha_t(1 + \varepsilon)\lambda_{\text{max}}^r)\sigma^2_{\min}(A)},
\]

where

\[
\eta_t = 1 - \alpha_t(2 - \alpha_t(1 + \varepsilon)\lambda_{\text{max}}^r)\sigma^2_{\min}(A)
\]

and

\[
\gamma = \mathbb{E} \left[ \|A^\top S^\top (AA^\dagger b - b)\|^2 \right].
\]

**Proof.** It follows from

\[
x^\ell - x^*_\ell = x^{\ell - 1} - x^*_0 - \alpha_t A^\top S^\top (Ax^{\ell - 1} - b)
\]

that

\[
\|x^\ell - x^*_\ell\|^2 = \|x^{\ell - 1} - x^*_0\|^2 - 2\alpha_t(x^{\ell - 1} - x^*_0)^\top A^\top S^\top (Ax^{\ell - 1} - b) + \alpha_t^2\|A^\top S^\top (Ax^{\ell - 1} - b)\|^2
\]

By \( Ax^0 = AA^\dagger b \), triangle inequality, and Young’s inequality, we have

\[
\|A^\top S^\top (Ax^{\ell - 1} - b)\|^2 = \|A^\top S^\top (Ax^{\ell - 1} - Ax^0 + AA^\dagger b - b)\|^2
\]

\[
\leq (\|A^\top S^\top A(x^{\ell - 1} - x^0)\| + \|A^\top S^\top (AA^\dagger b - b)\|)^2
\]

\[
\leq (1 + \varepsilon)\|A^\top S^\top A(x^{\ell - 1} - x^0)\|^2 + (1 + 1/\varepsilon)\|A^\top S^\top (AA^\dagger b - b)\|^2.
\]

By (12), (13), (14), and \( A^\top AA^\dagger b = A^\top b \), we have

\[
E_{k-1} \left[ \|x^\ell - x^*_\ell\|^2 \right]
\]

\[
= \|x^{\ell - 1} - x^*_0\|^2 - 2\alpha_t(x^{\ell - 1} - x^*_0)^\top A^\top A(x^{\ell - 1} - x^*_0) + \alpha_t^2 E_{k-1} \left[ \|A^\top S^\top (Ax^{\ell - 1} - b)\|^2 \right]
\]

\[
\leq \|x^{\ell - 1} - x^*_0\|^2 - 2(\alpha_t - \alpha_t^2(1 + \varepsilon)\lambda_{\text{max}}^r)(x^{\ell - 1} - x^*_0)^\top A^\top A(x^{\ell - 1} - x^*_0) + \alpha_t^2(1 + 1/\varepsilon)\gamma
\]

\[
\leq (1 - \alpha_t(2 - \alpha_t(1 + \varepsilon)\lambda_{\text{max}}^r)\sigma^2_{\min}(A))\|x^{\ell - 1} - x^*_0\|^2 + \alpha_t^2(1 + 1/\varepsilon)\gamma
\]

Then the expected squared norm of the error can be bounded by

\[
\mathbb{E} \left[ \|x^\ell - x^*_\ell\|^2 \right] \leq \eta^\ell \mathbb{E} \left[ \|x^{\ell - 1} - x^*_0\|^2 \right] + \alpha_t^2(1 + 1/\varepsilon)\gamma
\]

\[
\leq \eta^\ell \|x^0 - x^*_0\|^2 + \alpha_t^2(1 + 1/\varepsilon)\gamma \sum_{i=0}^{k-1} \eta^i
\]

\[
= \eta^\ell \|x^0 - x^*_0\|^2 + \frac{\alpha_t(1 + 1/\varepsilon)\gamma(1 - \eta^k)}{1 - \eta_t}
\]

\[
= \eta^\ell \|x^0 - x^*_0\|^2 + \frac{\alpha_t(1 + 1/\varepsilon)\gamma(1 - \eta^k)}{(2 - \alpha_t(1 + \varepsilon)\lambda_{\text{max}}^r)\sigma^2_{\min}(A)}.
\]

This completes the proof. \( \square \)
2.1.1 The randomized Kaczmarz algorithm

The RK algorithm \[30\] is one special case of the BRSI algorithm. Choosing \( S = \frac{\|A\|_F}{\|A_{i,:}\|}I_{i,:} \) with probability \( \frac{\|A_{i,:}\|^2}{\|A\|_F^2} \) in \[10\], we have

\[
\mathbb{E}[SS^\top] = \|A\|_F^2 \mathbb{E}\left[ \frac{I_{i,:}^\top(I_{i,:})}{\|A_{i,:}\|^2} \right] = \|A\|_F^2 \sum_{i=1}^m \frac{I_{i,:}^\top(I_{i,:})}{\|A_{i,:}\|^2} \|A\|_F^2 = \sum_{i=1}^m I_{i,:}^\top(I_{i,:}) = I,
\]

and recover the RK iteration

\[
x^k = x^{k-1} - \alpha_r \|A\|_F^2 A_{i,:} x^{k-1} - b_i (A_{i,:})^\top.
\]

For this case, we have \( \lambda_r = \|A\|_F^2 \). Choosing \( \alpha_r = 1/\|A\|_F^2 \) in Theorem \[5\] yields the convergence estimate of \[30\]:

\[
\mathbb{E}\left[\|x^k - x^*\|^2\right] \leq \left(1 - \frac{\sigma_{\min}^2(A)}{\|A\|_F^2}\right)^k \|x^0 - x^*\|^2.
\]

2.1.2 The block row uniform sampling algorithm

We propose one special case of the BRSI algorithm by using uniform sampling and refer to it as the block row uniform sampling (BRUS) algorithm. We note that the BRUS algorithm is also one special case of the randomized average block Kaczmarz algorithm \[23\]. Assume \( 1 \leq \ell \leq m \). Let \( \mathcal{I} \) denote the set consisting of the uniform sampling of \( \ell \) different numbers of \([m]\). Setting \( S = \sqrt{m/\ell}I_{\mathcal{I},:} \), we have

\[
\mathbb{E}[SS^\top] = \frac{m}{\ell} \sum_{\mathcal{I} \subseteq [m], |\mathcal{I}| = \ell} I_{\mathcal{I},:} I_{\mathcal{I},:}^\top = \frac{m}{\ell} \left( \frac{m}{m} \right) \left( \frac{m-1}{\ell-1} \right) I = I,
\]

and obtain the iteration

\[
x^k = x^{k-1} - \alpha_r \frac{m}{\ell} (A_{\mathcal{I},:})^\top (A_{\mathcal{I},:} x^{k-1} - b_{\mathcal{I}}).
\]

For this case, we have

\[
\lambda_r = \frac{m}{\ell} \max_{\mathcal{I} \subseteq [m], |\mathcal{I}| = \ell} \|A_{\mathcal{I},:}\|^2.
\]

By Theorem \[5\] the BRUS algorithm can have a faster convergence rate than that of the RK algorithm if there exists \( \ell \in [m] \) satisfying

\[
\frac{m}{\ell} \max_{\mathcal{I} \subseteq [m], |\mathcal{I}| = \ell} \|A_{\mathcal{I},:}\|^2 \leq \frac{\|A\|_F^2}{\|A\|_F^2}.
\]

We present the details of the BRUS algorithm with block size \( \ell \) and initial guess \( x^0 = 0 \) in Algorithm 1. We also note that the constant \( m/\ell \) is incorporated in the stepsize parameter \( \alpha_r \).

**Algorithm 1: BRUS(\ell)**

1. Initialize \( x^0 = 0 \) and a fixed \( 1 \leq \ell \leq m \)
2. for \( k = 1, 2, \ldots \) do
   1. Select randomly a set \( \mathcal{I} \) consisting of the uniform sampling of \( \ell \) numbers of \([m]\)
   2. Update \( x^k = x^{k-1} - \alpha_r (A_{\mathcal{I},:})^\top (A_{\mathcal{I},:} x^{k-1} - b_{\mathcal{I}}) \)
3. end for
2.2 Block column sampling

In this subsection, we consider the case \( S = I \) and refer to the resulting algorithm as the block column sampling iterative (BCSI) algorithm. Given an arbitrary initial guess \( x^0 \in \mathbb{R}^n \), the \( k \)th iterate of the BCSI algorithm is

\[
x^k = x^{k-1} - \alpha_c T T^\top A^\top (Ax^{k-1} - b),
\]

where the stepsize parameter \( \alpha_c > 0 \), and the random parameter matrix \( T \) is sampled independently in each iteration from a distribution \( D_c \) and satisfies \( \mathbb{E}[TT^\top] = I \).

In the following, we shall present the convergence of \( \mathbb{E}[\|A(x^k - A^\dagger b)\|^2] \) for arbitrary linear systems. Throughout, we define

\[
\lambda^c_{\text{max}} = \max_{T \sim D_c} \lambda_{\text{max}}(ATT^\top A^\top).
\]

**Theorem 7.** Assume that \( 0 < \alpha_c < 2/\lambda^c_{\text{max}} \). For arbitrary \( x^0 \in \mathbb{R}^n \), the \( k \)th iterate \( x^k \) of the BCSI algorithm satisfies

\[
\mathbb{E}[\|A(x^k - A^\dagger b)\|^2] \leq \eta_k \|A(x^0 - A^\dagger b)\|^2,
\]

where

\[
\eta_k = 1 - \alpha_c (2 - \alpha_c \lambda^c_{\text{max}}) \sigma^2_{\text{min}}(A).
\]

**Proof.** It follows from (16) and \( A^\top A A^\dagger b = A^\top b \) that

\[
A(x^k - A^\dagger b) = A(x^{k-1} - A^\dagger b - \alpha_c T T^\top A^\top (Ax^{k-1} - b)) = A(x^{k-1} - A^\dagger b) - \alpha_c T T^\top A^\top A(x^{k-1} - A^\dagger b).
\]

Then we have

\[
\|A(x^k - A^\dagger b)\|^2 = \|A(x^{k-1} - A^\dagger b)\|^2 - 2\alpha_c(x^{k-1} - A^\dagger b)^\top A^\top T T^\top A^\top A(x^{k-1} - A^\dagger b)
+ \alpha^2_c(x^{k-1} - A^\dagger b)^\top A^\top (T T^\top A^\top)^2 A(x^{k-1} - A^\dagger b).
\]

Note that

\[
(x^{k-1} - A^\dagger b)^\top A^\top (T T^\top A^\top)^2 A(x^{k-1} - A^\dagger b)
\leq \lambda_{\text{max}}(ATT^\top A^\top) (x^{k-1} - A^\dagger b)^\top A^\top ATT^\top A^\top A(x^{k-1} - A^\dagger b)
\leq \lambda^c_{\text{max}}(x^{k-1} - A^\dagger b)^\top A^\top ATT^\top A^\top A(x^{k-1} - A^\dagger b).
\]

By (17), (18), and \( A(x^{k-1} - A^\dagger b) \in \text{range}(A) \), we have

\[
\mathbb{E}_{k-1}[\|A(x^k - A^\dagger b)\|^2] \leq \|A(x^{k-1} - A^\dagger b)\|^2 - 2\alpha_c(x^{k-1} - A^\dagger b)^\top A^\top AA^\top A(x^{k-1} - A^\dagger b)
+ \alpha^2_c\lambda^c_{\text{max}}(x^{k-1} - A^\dagger b)^\top A^\top AA^\top A(x^{k-1} - A^\dagger b)
\leq (1 - \alpha_c (2 - \alpha_c \lambda^c_{\text{max}}) \sigma^2_{\text{min}}(A)) \|A(x^{k-1} - A^\dagger b)\|^2.
\]

In the last inequality, we use the facts that \( -\alpha_c (2 - \alpha_c \lambda^c_{\text{max}}) < 0 \), and for all \( u \in \text{range}(A) \), it holds \( u^\top AA^\top u \geq \sigma^2_{\text{min}}(A) \|u\|^2 \). Next, by the law of total expectation, we have

\[
\mathbb{E}[\|A(x^k - A^\dagger b)\|^2] \leq \eta_k \mathbb{E}[\|A(x^{k-1} - A^\dagger b)\|^2]
\]

Unrolling the recurrence yields the result. \( \square \)

**Remark 8.** If \( A \) has full column rank, then Theorem 7 implies that \( x^k \) of the BCSI algorithm converges linearly to \( A^\dagger b \) in the mean square sense.
2.2.1 The randomized coordinate descent algorithm

The RCD algorithm [15] is one special case of the BCSI algorithm. Choosing $T = \frac{\|A\|_F}{\|A_{\cdot,j}\|_F} I_{1:j}$ with probability $\|A_{\cdot,j}\|_F^2$ in (16), we have

$$E[TT^T] = \|A\|_F^2 E\left[ \frac{(I_{1:j})^T(I_{1:j})}{\|A_{1:j}\|_F^2} \right] = \|A\|_F^2 \sum_{j=1}^n \frac{I_{1:j}(I_{1:j})^T}{\|A_{1:j}\|_2^2} \|A_{1:j}\|_F^2 = \sum_{j=1}^n I_{1:j}(I_{1:j})^T \|A_{1:j}\|_F^2 = I,$$

and recover the RCD iteration

$$x^k = x^{k-1} - \alpha c \frac{\|Ax\|_F^2 (A_{\cdot,j})^T(Ax^{k-1} - b)}{\|A_{\cdot,j}\|_F^2} I_{1:j}. $$

For this case, we have $\lambda_{\text{max}}^c = \|A\|_F^2$. Choosing $\alpha_c = 1/\|A\|_F^2$ in Theorem 7 yields the convergence estimate of [15, 19]:

$$E[\|A(x^k - A^1b)\|_2^2] \leq \left(1 - \frac{\sigma_{\text{min}}^2(A)}{\|A\|_F^2} \right) \|A(x^0 - A^1b)\|_2^2.$$

2.2.2 The block column uniform sampling algorithm

We propose one new special case of the BCSI algorithm by using uniform sampling and refer to it as the block column uniform sampling (BCUS) algorithm. Assume $1 \leq \ell \leq n$. Let $J$ denote the set consisting of the uniform sampling of $\ell$ different numbers of $[n]$. Setting $T = \sqrt{n}/I_{1:J}$, we have

$$E[TT^T] = \frac{n}{\ell} \sum_{J \subseteq [n], |J| = \ell} I_{1:J} I_{1:J}^T = \frac{n}{\ell} \binom{n-1}{\ell-1} I = I,$$

and obtain the iteration

$$x^k = x^{k-1} - \alpha c \frac{n}{\ell} I_{1:J} (A_{\cdot,J})^T (Ax^{k-1} - b). $$

For this case, we have

$$\lambda_{\text{max}}^c = \frac{n}{\ell} \max_{J \subseteq [n], |J| = \ell} \|A_{\cdot,J}\|_F^2.$$

By Theorem 7, the BCUS algorithm can have a faster convergence rate than that of the RCD algorithm if there exists $\ell \in [n]$ satisfying

$$\frac{n}{\ell} \max_{J \subseteq [n], |J| = \ell} \|A_{\cdot,J}\|_F^2 \leq \|A\|_F^2.$$

To avoid entire matrix-vector multiplications, we introduce an auxiliary vector $r^k = b - Ax^k$ in each iteration of the BCUS algorithm. We present the details of the BCUS algorithm with block size $\ell$ and initial guess $x^0 = 0$ in Algorithm 2. We also note that the constant $n/\ell$ is incorporated in the stepsize parameter $\alpha_c$.

---

**Algorithm 2:** BCUS($\ell$)

Initialize $x^0 = 0$, $r^0 = b$, and a fixed $1 \leq \ell \leq n$

for $k = 1, 2, \ldots$ do

Select randomly a set $J$ consisting of the uniform sampling of $\ell$ numbers of $[n]$

Compute $w^k = \alpha_c (A_{\cdot,J})^T r^{k-1}$

Update $x^k_J = x^{k-1}_J + w^k$ and $r^k = r^{k-1} - A_{\cdot,J} w^k$.

end for
3 The extended block row sampling iterative algorithm

Solving $A^\top z = 0$ by the RK algorithm with initial guess $z^0 \in b + \text{range}(A)$ produces a sequence $\{z^k\}$, which converges to $b - AA^\dagger b$ (see, e.g., [7]). Zouzias and Freris [33] proved that the $k$th iterate $x^k$ (which is produced by one RK update for $Ax = b - z^k$ from $x^{k-1}$) of the REK algorithm converges to a solution of $Ax = AA^\dagger b$. We note that any solution of $Ax = AA^\dagger b$ is a solution of $Ax = b$ if it is consistent or a least squares solution of $Ax = b$ if it is inconsistent. In this section, based on the idea of the REK algorithm, we propose an extended block row sampling iterative (EBRSI) algorithm. In Appendix A, we also propose an extended block column and row sampling iterative algorithm based on the idea of the randomized extended Gauss–Seidel algorithm [19].

Given $z^0 \in b + \text{range}(A)$ and an arbitrary initial guess $x^0 \in \mathbb{R}^n$, the iterates of the EBRSI algorithm at step $k$ are defined as

$$
\begin{align*}
z^k &= z^{k-1} - \alpha_c A^\top T A^\top z^{k-1}, \\
x^k &= x^{k-1} - \alpha_r A^\top SS^\top (Ax^{k-1} - b + z^k),
\end{align*}
$$

where the random parameter matrices $S$ and $T$ are independent, and satisfy

$$E[SS^\top] = I, \quad E[TT^\top] = I.$$

We note that the iteration [19] is the BRSI algorithm for $A^\top z = 0$ with initial guess $z^0 \in b + \text{range}(A)$, and the iterate $x^k$ in (20) is one BRSI update for $Ax = b - z^k$ from $x^{k-1}$. By Theorem 5, we have

$$E[\|z^k - (I - AA^\dagger)b\|^2] \leq \eta_c^k \|z^0 - (I - AA^\dagger)b\|^2.$$

In the following, we shall present two convergence results of the EBRSI algorithm: Theorem 9 is on the convergence of $\|E[x^k] - x^0\|$, and Theorem 11 is on the convergence of $E[\|x^k - x^0\|^2]$. We emphasize that both the convergence results hold for arbitrary linear systems. Let $E_{k-1} [\cdot]$ denote the conditional expectation conditioned on $z^{k-1}$ and $x^{k-1}$. Let $E_{k-1}^\tau [\cdot]$ denote the conditional expectation conditioned on $z^k$ and $x^{k-1}$. Then, by the law of total expectation, we have

$$E_{k-1} [\cdot] = E_{k-1} [E_{k-1}^\tau [\cdot]].$$

**Theorem 9.** For arbitrary $z^0 \in \mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$, the $k$th iterate $x^k$ of the EBRSI algorithm satisfies

$$E[x^k - x^0] = (I - \alpha_r A^\top A)^k(x^0 - x^0) - \alpha_r \sum_{i=0}^{k-1} (I - \alpha_r A^\top A)^i(I - \alpha_c A^\top A)k-iA^\top z^0.$$  

Moreover,

$$\|E[x^k] - x^0\| \leq \delta^k(\|x^0 - z^0\| + k\alpha_r\|A^\top z^0\|),$$

where

$$\delta = \max_{1 \leq i \leq r} \{|1 - \alpha_r\sigma_i^2(A)|, |1 - \alpha_c\sigma_i^2(A)| \}.$$  

**Proof.** By (19), we have

$$E_{k-1} [z^k] = z^{k-1} - \alpha_c AA^\top z^{k-1} = (I - \alpha_c AA^\top)z^{k-1},$$

which, by the law of total expectation, yields

$$E[z^k] = (I - \alpha_c AA^\top)E[z^{k-1}] = \cdots = (I - \alpha_c AA^\top)^kz^0.$$  

Taking expectation conditioned on $z^{k-1}$ and $x^{k-1}$ for

$$x^k - x^0 = x^{k-1} - x^0 - \alpha_r A^\top SS^\top (Ax^{k-1} - b + z^k),$$
we obtain
\[
\mathbb{E}_{k-1} [x^k - x^0_*] = \mathbb{E}_{k-1} \left[ \mathbb{E}_{k-1} ^* \left[ x^{k-1} - x^0_* - \alpha_r A^T S S^T (A x^{k-1} - b + z^k) \right] \right] \\
= \mathbb{E}_{k-1} [x^{k-1} - x^0_* - \alpha_r A^T (A x^{k-1} - b + z^k)] \\
= x^{k-1} - x^0_* - \alpha_r (A^T A x^{k-1} - A^T b) - \alpha_r A^T \mathbb{E}_{k-1} [z^k] \\
= x^{k-1} - x^0_* - \alpha_r (A^T A x^{k-1} - A^T A x^0) - \alpha_r A^T \mathbb{E}_{k-1} [z^k] \\
= (I - \alpha_r A^T A)(x^{k-1} - x^0_*) - \alpha_r A^T \mathbb{E}_{k-1} [z^k],
\]

which, by the law of total expectation, yields
\[
\mathbb{E} [x^k - x^0_*] = (I - \alpha_r A^T A)E [x^{k-1} - x^0_*] - \alpha_r A^T E [z^k] \\
= (I - \alpha_r A^T A)E [x^{k-1} - x^0_*] - \alpha_r A^T (I - \alpha A A^T) z^0 \\
= (I - \alpha_r A^T A)E [x^{k-1} - x^0_*] - \alpha_r (I - \alpha A A^T) A^T z^0 \\
= (I - \alpha_r A^T A)^2 E [x^{k-2} - x^0_*] - \alpha_r (I - \alpha A A^T) (I - \alpha A A^T) A^T z^0 \\
- \alpha_r (I - \alpha A A^T) A^T z^0 \\
= \ldots \\
= (I - \alpha_r A^T A)^k (x^0 - x^0_*) - \alpha_r \sum_{i=0}^{k-1} (I - \alpha_r A A^T)^{i}(I - \alpha A A^T)^{k-i} A^T z^0.
\]

Taking 2-norm, by triangle inequality, \(x^0 - x^0_* \in \text{range}(A^T), A^T z^0 \in \text{range}(A^T), \) and Lemma \ref{lemma:1} we obtain the estimate \eqref{eq:22}.

\begin{remark}
In Theorem \ref{thm:9} no assumptions about the dimensions or rank of \(A\) are assumed, and the system \(A x = b\) can be consistent or inconsistent. If \(0 < \alpha_r < 2/\sigma^2_{\text{max}}(A)\) and \(0 < \alpha_c < 2/\sigma^2_{\text{max}}(A)\), then \(0 < \delta < 1\). This means \(x^k\) is an asymptotically unbiased estimator for \(x^0_*\).
\end{remark}

\begin{theorem}
Assume that \(0 < \alpha_c < 2/\lambda^c_{\text{max}}\) and \(0 < \alpha_r < 2/\lambda^r_{\text{max}}\). For arbitrary \(x^0 \in \mathbb{R}^n, z^0 \in b + \text{range}(A), \) and \(\varepsilon > 0,\) the \(k\)th iterate \(x^k\) of the EBRSI algorithm satisfies
\[
\mathbb{E} \left[ \|x^k - x^0_*\|^2 \right] \leq (1 + \varepsilon)^k \eta_r \|x^0 - x^0_*\|^2 \\
+ (1 + \varepsilon)\alpha_r^2 \lambda^r_{\text{max}} \|z^0 - (I - AA^\dagger)b\|^2 \sum_{i=0}^{k-1} \eta_c^{k-i} (1 + \varepsilon)^i \eta_r^i,
\]
where
\[
\eta_r = 1 - \alpha_r (2 - \alpha_r \lambda^r_{\text{max}}) \sigma^2_{\text{min}}(A), \quad \eta_c = 1 - \alpha_c (2 - \alpha_c \lambda^c_{\text{max}}) \sigma^2_{\text{min}}(A).
\]
\end{theorem}

\begin{proof}
We define
\[
\hat{x}^k = x^{k-1} - \alpha_r A^T S S^T (A x^{k-1} - A x^0_*),
\]
which is one BRSI update for \(A x = A x^0\) from \(x^{k-1}\). We have
\[
x^k - \hat{x}^k = \alpha_r A^T S S^T (b - A x^0_* - z^k).
\]
It follows from \(\lambda_{\text{max}}(S^T A A^T S) = \lambda_{\text{max}}(A^T S S^T A)\) that
\[
\|x^k - \hat{x}^k\|^2 = \alpha_r^2 (b - A x^0_* - z^k)^T S S^T A A^T S S^T (b - A x^0_* - z^k) \\
\leq \alpha_r^2 \lambda^r_{\text{max}} (b - A x^0_* - z^k)^T S S^T (b - A x^0_* - z^k).
\]
Taking conditional expectation conditioned on \(z^k\) and \(x^{k-1}\), by \(b - A x^0 = (I - AA^\dagger)b,\) we have
\[
\mathbb{E}_{k-1} [\|x^k - \hat{x}^k\|^2] = \mathbb{E}_{k-1} [\mathbb{E}_{k-1}^* \left[ \|x^k - \hat{x}^k\|^2 \right]] \\
\leq \alpha_r^2 \lambda^r_{\text{max}} \mathbb{E}_{k-1} [\|z^k - (I - AA^\dagger)b\|^2]
\]

Then, by the law of total expectation and the estimate (21), we have
\[ E[\|x^k - \tilde{x}^k\|^2] \leq \alpha_r^2 \lambda^2 \max \mathbb{E}[\|z^k - (I - AA^1)b\|^2] \leq \alpha_r^2 \lambda^2 \max \mathbb{E}[\|z^0 - (I - AA^1)b\|^2]. \]
By \( x^0 - x_*^0 \in \text{range}(A^\top) \) and \( A^\top SS^\top (Ax^k - b + z^k) \in \text{range}(A^\top) \), we can show that \( x^k - x_*^0 \in \text{range}(A^\top) \) by induction. Then,
\[ \|\tilde{x}^k - x_*^0\|^2 = \|x^k - x_*^0\|^2 - 2\alpha_r(x^k - x_*^0)^\top A^\top SS^\top A(x^k - x_*^0) + \alpha_r^2(x^k - x_*^0)^\top (A^\top SS^\top A)^2(x^k - x_*^0) \leq \|x^k - x_*^0\|^2 - 2\alpha_r(x^k - x_*^0)^\top A^\top SS^\top A(x^k - x_*^0) + \alpha_r^2 \lambda^2 \max(x^k - x_*^0)^\top A^\top SS^\top A(x^k - x_*^0). \]
Taking conditional expectation conditioned on \( z^k \) and \( x^k \), we have
\[ \mathbb{E}_{k-1}[\|\tilde{x}^k - x_*^0\|^2] \leq \|x^k - x_*^0\|^2 - 2\alpha_r(x^k - x_*^0)^\top A^\top SS^\top A(x^k - x_*^0) + \alpha_r^2 \lambda^2 \max(x^k - x_*^0)^\top A^\top SS^\top A(x^k - x_*^0) \leq \eta_r \|x^k - x_*^0\|^2. \]
By the law of total expectation, we have
\[ \mathbb{E}[\|\tilde{x}^k - x_*^0\|^2] \leq \eta_r \mathbb{E}[\|x^k - x_*^0\|^2]. \]
By triangle inequality and Young’s inequality, we have
\[ \|x^k - x_*^0\|^2 \leq (\|x^k - \tilde{x}^k\|^2 + \|\tilde{x}^k - x_*^0\|^2) \leq (1 + 1/\varepsilon)\|x^k - \tilde{x}^k\|^2 + (1 + \varepsilon)\|\tilde{x}^k - x_*^0\|^2. \]
Taking expectation, we have
\[ \mathbb{E}[\|x^k - x_*^0\|^2] \leq (1 + 1/\varepsilon)\mathbb{E}[\|x^k - \tilde{x}^k\|^2] + (1 + \varepsilon)\mathbb{E}[\|\tilde{x}^k - x_*^0\|^2] \leq (1 + 1/\varepsilon)\alpha_r^2 \lambda^2 \max \mathbb{E}[\|z^0 - (I - AA^1)b\|^2] + (1 + \varepsilon)\eta_r \mathbb{E}[\|x^k - x_*^0\|^2] \leq (1 + 1/\varepsilon)\alpha_r^2 \lambda^2 \max \mathbb{E}[\|z^0 - (I - AA^1)b\|^2](\eta_r^2 + \eta_r^{-1}(1 + \varepsilon)\eta_r) + (1 + \varepsilon)^2 \eta_r \mathbb{E}[\|x^k - x_*^0\|^2] \leq \cdots \]
\[ \leq (1 + 1/\varepsilon)\alpha_r^2 \lambda^2 \max \mathbb{E}[\|z^0 - (I - AA^1)b\|^2(1 + \varepsilon)\eta_r^2 + (1 + \varepsilon)(1 + \varepsilon)^2 \eta_r \mathbb{E}[\|x^k - x_*^0\|^2]. \]
This completes the proof. 

**Remark 12.** In Theorem 27, no assumptions about the dimensions or rank of \( A \) are assumed, and the system \( Ax = b \) can be consistent or inconsistent. Let \( \eta = \max\{\eta_r, \eta_r^2\} \). It follows from \( 0 < \alpha_c < 2/\lambda^2 \max \) and \( 0 < \alpha_r < 2/\lambda^2 \max \) that \( \eta < 1 \). Assume that \( \varepsilon \) satisfies \( (1 + \varepsilon)\eta < 1 \). We have
\[ \mathbb{E}[\|x^k - x_*^0\|^2] \leq (1 + \varepsilon)^k \eta_r \mathbb{E}[\|x^k - x_*^0\|^2] + (1 + \varepsilon)\alpha_r^2 \lambda^2 \max \mathbb{E}[\|z^0 - (I - AA^1)b\|^2/\varepsilon^2], \]
which shows that the EBRSI algorithm converges linearly in the mean square sense to \( x_*^0 \) with the rate \( (1 + \varepsilon)\eta \).

### 3.1 The randomized extended Kaczmarz algorithm

The REK algorithm [32] is one special case of the EBRSI algorithm. Choosing \( S = \frac{\|A\|_F}{\|A\|_F}I_{i,i} \) with probability \( \frac{\|A_{i,i}\|^2}{\|A\|_F^2} \) and \( T = \frac{\|A\|_F}{\|A\|_F}I_{i,j} \) with probability \( \frac{\|A_{i,j}\|^2}{\|A\|_F^2} \), we have
\[ \mathbb{E}[SS^\top] = I, \quad \mathbb{E}[TT^\top] = I, \]

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and obtain
\[ z^k = z^{k-1} - \alpha_c \frac{\|A\|_F^2}{\|A_{i,:}\|^2} (A_{i,:})^\top z^{k-1} A_{i,:}, \]
\[ x^k = x^{k-1} - \alpha_r \frac{\|A\|_F^2}{\|A_{i,:}\|^2} (A_{i,:})^\top (A_{i,:} x^{k-1} - b_i + z_i^k). \]

We have \( \lambda^c_{\text{max}} = \lambda^c_{\text{max}} = \|A\|_F^2. \) Setting \( \alpha_r = \alpha_c = 1/\|A\|_F^2, \) we recover the REK algorithm [33, 7].

### 3.2 The randomized extended average block Kaczmarz algorithm

The randomized extended average block Kaczmarz (REBAK) algorithm [8] is one special case of the EBRSI algorithm. Let \( \{I_1, I_2, \ldots, I_s\} \) be a partition of \([m]\) satisfying \( I_i \cap I_j = \emptyset \) for \( i \neq j \) and \( \cup_{i=1}^s I_i = [m]. \) Let \( \{J_1, J_2, \ldots, J_r\} \) be a partition of \([n]\) satisfying \( J_i \cap J_j = \emptyset \) for \( i \neq j \) and \( \cup_{j=1}^r J_j = [n]. \) Choosing \( S = \frac{\|A\|_F}{\|A_{I,:}\|^2} I_{I,:} \) with probability \( \|A_{I,:}\|_F^2 \) and \( T = \frac{\|A\|_F}{\|A_{J,:}\|^2} I_{J,:} \) with probability \( \|A_{J,:}\|_F^2, \) we have
\[ E[SS^\top] = I, \quad E[TT^\top] = I, \]
and obtain
\[ z^k = z^{k-1} - \alpha_c \frac{\|A\|_F^2}{\|A_{I,:}\|^2} A_{I,:} (A_{I,:})^\top z^{k-1}, \]
\[ x^k = x^{k-1} - \alpha_r \frac{\|A\|_F^2}{\|A_{J,:}\|^2} (A_{J,:})^\top (A_{J,:} x^{k-1} - b_J + z_J^k). \]

Setting \( \alpha_r = \alpha_c = \alpha/\|A\|_F^2, \) we recover the REBAK algorithm [8].

### 3.3 The extended block row uniform sampling algorithm

We propose one new special case of the EBRSI algorithm by using uniform sampling and refer to it as the extended block row uniform sampling (EBRSU) algorithm. Assume \( 1 \leq \ell \leq \min\{m, n\}. \) Let \( \mathcal{I} \) (resp. \( \mathcal{J} \)) denote the set consisting of the uniform sampling of \( \ell \) different numbers of \([m]\) (resp. \([n]\)). Setting \( S = \sqrt{m/\mathcal{I}_{\ell,:}} \) and \( T = \sqrt{n/\mathcal{J}_{\ell,:}} \), we have
\[ E[SS^\top] = \frac{m}{\ell} \sum_{I \subseteq [m], |I| = \ell} I_{I,:} I_{I,:}^\top = I, \quad E[TT^\top] = \frac{n}{\ell} \sum_{J \subseteq [n], |J| = \ell} I_{J,:} I_{J,:}^\top = I, \]
and obtain
\[ z^k = z^{k-1} - \alpha_c \frac{n}{\ell} A_{I,:} (A_{I,:})^\top z^{k-1}, \]
\[ x^k = x^{k-1} - \alpha_r \frac{m}{\ell} (A_{J,:})^\top (A_{J,:} x^{k-1} - b_J + z_J^k). \]

We have
\[ \lambda^c_{\text{max}} = \frac{m}{\ell} \max_{I \subseteq [m], |I| = \ell} \|A_{I,:}\|^2, \quad \lambda^c_{\text{max}} = \frac{n}{\ell} \max_{J \subseteq [n], |J| = \ell} \|A_{J,:}\|^2. \]

By Theorem [11] the EBRUS algorithm can have a faster convergence rate than that of the REK algorithm if there exists \( 1 \leq \ell \leq \min\{m, n\} \) satisfying
\[ \frac{m}{\ell} \max_{I \subseteq [m], |I| = \ell} \|A_{I,:}\|^2 \leq \|A\|_F^2, \quad \frac{n}{\ell} \max_{J \subseteq [n], |J| = \ell} \|A_{J,:}\|^2 \leq \|A\|_F^2. \]

We present the details of the EBRUS algorithm with block size \( \ell \) and initial guesses \( x^0 = 0 \) and \( z^0 = b \) in Algorithm 3. We also note that the constants \( m/\ell \) and \( n/\ell \) are incorporated in the stepsize parameters \( \alpha_r \) and \( \alpha_c \), respectively.
Algorithm 3: EBRUS(ℓ)

\begin{algorithm}
\begin{algorithmic}
\State Initialize $z^0 = b$, $x^0 = 0$ and a fixed $1 \leq \ell \leq \min\{m,n\}$
\For{$k = 1, 2, \ldots$}
\State Select randomly a set $J$ consisting of the uniform sampling of $\ell$ numbers of $[n]$
\State Update $z^k = z^{k-1} - \alpha_r A_{:,J}(A_{:,J})^\top z^{k-1}$
\State Select randomly a set $I$ consisting of the uniform sampling of $\ell$ numbers of $[m]$
\State Update $x^k = x^{k-1} - \alpha_r (A_{I,:}x^{k-1} - b_I + z_I^k)$
\EndFor
\end{algorithmic}
\end{algorithm}

4 Numerical results

In this section, we report numerical results showing the influence of different stepsize parameters and different block sizes on the convergence of the BRUS, BCUS, and EBRUS algorithms. We also compare the performance of these three algorithms with the following ten randomized algorithms: the randomized Kaczmarz (RK) algorithm [30], the greedy randomized Kaczmarz (GRK) algorithm [1], the randomized block Kaczmarz (RBK) algorithm [10], the randomized coordinate descent (RCD) algorithm [15], the greedy randomized coordinate descent (GRCD) algorithm [3], the randomized block coordinate descent (RBCD) algorithm [10], the randomized extended Kaczmarz (REK) algorithm [8], the two-subspace randomized extended Kaczmarz (TREK) algorithm [31], the randomized extended block Kaczmarz (REBK) algorithm (which is slightly different from the randomized double block Kaczmarz algorithm of [26]), and the randomized extended average block Kaczmarz (REABK) algorithm [8]. According to our theoretical results (Theorems 5, 7, and 11), and taking into consideration the computational cost of each step, we divide the thirteen algorithms into three groups for comparison: (1) RK, GRK, RBK, and BRUS for solving arbitrary consistent linear systems; (2) RCD, GRCD, RBCD, and BCUS for solving full column rank (i.e., rank($A$) = $n$) inconsistent linear systems; (3) REK, TREK, REBK, REABK, and EBRUS for solving rank-deficient (i.e., rank($A$) < $n$) inconsistent linear systems.

All experiments are performed using MATLAB R2019a on an iMac with 3.4 GHz Intel Core i5 processor and 8 GB 1600 MHz DDR3 memory. Given a matrix $A \in \mathbb{R}^{m \times n}$, a consistent system is constructed by setting $b = A \ast \text{randn}(n,1)$, and an inconsistent one is constructed by setting $b = A \ast \text{randn}(n,1) + \text{null}(A) \ast \text{randn}(n-r,1)$, where $r$ is the rank of $A$. Both synthetic data matrix and real-world data matrix are tested. All synthetic data matrices are generated as follows. Given $m$, $n$, $r = \text{rank}(A) \leq \min\{m,n\}$, and $\kappa \geq 1$, we construct $A$ by $A = UDV^\top$, where $U \in \mathbb{R}^{m \times r}$, $D \in \mathbb{R}^{r \times r}$, and $V \in \mathbb{R}^{n \times r}$ are given by $[U, \sim] = \text{qr}((\text{randn}(m,r), 0)$, $D = \text{diag}(\text{ones}(r,1) + (\kappa-1) \ast \text{rand}(r,1))$, and $[V, \sim] = \text{qr}((\text{randn}(n,r), 0)$. So the condition number of $A$, $\sigma_{\text{max}}(A)/\sigma_{\text{min}}(A)$, is bounded by $\kappa$. We use $\kappa = 5$ for all synthetic data matrices.

We note that $A \backslash b$ will be the same as $\text{pinv}(A) \ast b$ when $A$ have full column rank and usually not be the same as $\text{pinv}(A) \ast b$ when $A$ is rank-deficient. Therefore, we use MATLAB’s ‘\’ to solve the small least squares problem at each step of the RBCD algorithm, and use MATLAB’s $\text{lsqminnorm}$ (which is typically more efficient than $\text{pinv}$) to solve the small least squares problems at each step of the RBK and REBK algorithms. For all algorithms, we use $x^0 = 0$ (and $z^0 = b$ if needed) and stop if the relative error ($\text{relerr}$), defined by

$$\text{relerr} = \frac{\|x^k - A^\top b\|^2}{\|A^\top b\|^2},$$

satisfies $\text{relerr} \leq 10^{-10}$. We check this stopping criterion after each epoch. For the RK, GRK, RBK, and BRUS algorithms, an epoch consists of $m$, $m/\ell$, and $\lfloor m/\ell \rfloor$ iterations, respectively. For the RCD, GRCD, RBCD, and BCUS algorithms, an epoch consists of $n$, $n/\ell$, and $\lfloor n/\ell \rfloor$ iterations, respectively. For the REK, TREK, REBK, REABK, and EBRUS algorithms, an epoch consists of $\max\{m,n\}$, $\lfloor \max\{m,n\}/2 \rfloor$, $\lfloor \max\{m,n\}/\ell \rfloor$, and $\lfloor \max\{m,n\}/\ell \rfloor$ iterations, respectively. In all examples, the reported results are average of 10 independent trials.
Figure 1: The relative error history versus the number of epochs of the BRUS, BCUS, and EBRUS algorithms with different stepsize parameters. (a) BRUS for a consistent linear system with synthetic data matrix \((m = 1000, n = 500, r = 250)\). (b) BCUS for a full column rank inconsistent linear system with synthetic data matrix \((m = 1000, n = 500, r = 500)\). (c) EBRUS for a rank-deficient inconsistent linear system with synthetic data matrix \((m = 1000, n = 500, r = 250)\).

First we use three linear systems to investigate how different stepsize parameters affect the convergence of the BRUS, BCUS, and EBRUS algorithms. We use the block size \(\ell = 10\) and define

\[
\hat{\lambda}^I_{\text{max}} := \max_{1 \leq i \leq \ell} \| A_{I_i} \|_2, \quad \hat{\lambda}^J_{\text{max}} := \max_{1 \leq i \leq \ell} \| A_{J_i} \|_2,
\]

where \(\{I_i\}_{i=1}^\ell\) are independent sets consisting of the uniform sampling of \(\ell\) different numbers of \([m]\), and \(\{J_i\}_{i=1}^\ell\) are independent sets consisting of the uniform sampling of \(\ell\) different numbers of \([n]\).

In Figure 1, we plot the relative error history versus the number of epochs of the BRUS, BCUS, and EBRUS algorithms with different stepsize parameters for solving a consistent linear system with synthetic data matrix \((m = 1000, n = 500, r = 250)\), a full column rank inconsistent linear system with synthetic data matrix \((m = 1000, n = 500, r = 500)\), and a rank-deficient inconsistent linear system with synthetic data matrix \((m = 1000, n = 500, r = 250)\), respectively. Because the required number of epochs of each independent trial is different, the average relative error history is only available up to the minimum number of epochs. For the BRUS and BCUS algorithms, we observe that the convergence rate becomes faster as the increase of the stepsize, and then slows down after reaching the fastest rate. For the EBRUS algorithm, we observe that appropriate stepsize parameters can remarkably improve the convergence.

Figure 2: The running time versus the block size \((\ell = 5, 10, 20, 50, 100, 200)\) of the BRUS, BCUS, and EBRUS algorithms with stepsize parameters \(\alpha_r = 2/\hat{\lambda}^I_{\text{max}}\) and \(\alpha_c = 2/\hat{\lambda}^J_{\text{max}}\). (a) BRUS for a consistent linear system with synthetic data matrix \((m = 10000, n = 5000, r = 2500)\). (b) BCUS for a full column rank inconsistent linear system with synthetic data matrix \((m = 10000, n = 5000, r = 5000)\). (c) EBRUS for a rank-deficient inconsistent linear system with synthetic data matrix \((m = 10000, n = 5000, r = 2500)\).
Next we use three linear systems to investigate how different block sizes affect the performance of the BRUS, BCUS, and EBRUS algorithms. We use the block sizes $\ell = 5, 10, 20, 50, 100, 200$ and the empirical stepsize parameters $\alpha_r = 2/\hat{\lambda}_I^{max}$ and $\alpha_c = 2/\hat{\lambda}_J^{max}$. In Figure 2, we plot the running time (for all the reported results in this section, the time for computing stepsize parameters $\alpha_r$ and $\alpha_c$ is contained) versus the block size of the BRUS, BCUS, and EBRUS algorithms for solving a consistent linear system with synthetic data matrix ($m = 10000, n = 5000, r = 2500$), a full column rank inconsistent linear system with synthetic data matrix ($m = 10000, n = 5000, r = 5000$), and a rank-deficient inconsistent linear system with synthetic data matrix ($m = 10000, n = 5000, r = 2500$), respectively. For the BRUS, BCUS, and EBRUS algorithms, we observe that the running time first decreases, and then increases after reaching the minimum value with the increase of block size.

Table 1: The number of epochs (epochs), the number of iterations (iters), the relative error (relerr), and the running time (runtime) of the RK, GRK, RBK($\ell$), and BRUS($\ell$) algorithms for seven consistent linear systems (both full column rank and rank-deficient cases are included). Here $\ell$ is the block size.

| matrix   | m    | n    | rank | algorithm | epochs | iters | relerr    | runtime |
|----------|------|------|------|-----------|--------|-------|-----------|---------|
| synth_urd| 500  | 2000 | 250  | RK        | 51.2   | 25600 | 8.78E−11  | 1.89    |
|          |      |      |      | GRK       | 12.4   | 6200  | 5.78E−11  | 1.74    |
|          |      |      |      | RBK(20)   | 45.4   | 1135  | 8.30E−11  | 0.96    |
|          |      |      |      | BRUS(20)  | 42.4   | 1060  | 8.34E−11  | 0.42    |
| synth_ord| 2000 | 500  | 250  | RK        | 12.0   | 24000 | 4.62E−11  | 1.29    |
|          |      |      |      | GRK       | 2.0    | 4000  | 1.55E−11  | 1.10    |
|          |      |      |      | RBK(20)   | 10.6   | 1060  | 5.27E−11  | 0.32    |
|          |      |      |      | BRUS(20)  | 11.2   | 1120  | 4.41E−11  | 0.13    |
| synth_ofr| 2000 | 500  | 500  | RK        | 22.7   | 45400 | 6.71E−11  | 2.44    |
|          |      |      |      | GRK       | 5.0    | 10000 | 4.06E−12  | 2.72    |
|          |      |      |      | RBK(20)   | 21.6   | 2160  | 6.11E−11  | 0.63    |
|          |      |      |      | BRUS(20)  | 17.8   | 1780  | 5.28E−11  | 0.19    |
| abtah1   | 14596| 209  | 209  | RK        | 12.5   | 182450| 4.89E−11  | 39.68   |
|          |      |      |      | GRK       | 1.0    | 14596 | 4.77E−31  | 3.19    |
|          |      |      |      | RBK(20)   | 6.5    | 4745  | 4.34E−11  | 2.61    |
|          |      |      |      | BRUS(20)  | 13.2   | 9636  | 6.79E−11  | 2.88    |
| ash958   | 958  | 292  | 292  | RK        | 11.3   | 10825 | 4.44E−11  | 0.48    |
|          |      |      |      | GRK       | 2.0    | 1916  | 1.17E−14  | 0.09    |
|          |      |      |      | RBK(10)   | 11.0   | 1056  | 3.35E−11  | 0.25    |
|          |      |      |      | BRUS(10)  | 11.1   | 1066  | 4.28E−11  | 0.02    |
| lp_nug15 | 6330 | 22275| 5698 | RK        | 22.0   | 139260| 5.51E−11  | 83.12   |
|          |      |      |      | GRK       | 5.0    | 31650 | 1.16E−12  | 16.58   |
|          |      |      |      | RBK(20)   | 21.7   | 6879  | 5.77E−11  | 8.89    |
|          |      |      |      | BRUS(20)  | 46.6   | 14772 | 7.90E−11  | 8.44    |
| relat7   | 21924| 1045 | 1012 | RK        | 15.1   | 331052| 4.09E−11  | 117.69  |
|          |      |      |      | GRK       | 2.0    | 43848 | 4.02E−11  | 16.67   |
|          |      |      |      | RBK(20)   | 15.6   | 17133 | 5.78E−11  | 10.51   |
|          |      |      |      | BRUS(20)  | 14.9   | 16345 | 5.24E−11  | 4.86    |

Last we compare the thirteen algorithms (in three groups) using three synthetic data matrices (synth_urd, synth_ord, and synth_ofr) and four real-world data matrices (abtah1, ash958, lp_nug15, and relat7) from the SuiteSparse Matrix Collection (formerly known as the University of Florida Sparse Matrix Collection) [6]. The four matrices, synth_urd, synth_ord, lp_nug15, and relat7, are rank-deficient, and the other three matrices, synth_ofr, abtah1, and ash958, have full column rank.

In Table 1 we report the numerical results of the RK, GRK, RBK, and BRUS algorithms for seven
Table 2: The number of epochs (epochs), the number of iterations (iters), the relative error (relerr), and the running time (runtime) of the RCD, GRCD, RBCD(ℓ), and BCUS(ℓ) algorithms for three full column rank inconsistent linear systems. Here ℓ is the block size.

| matrix      | m   | n   | rank | algorithm | epochs | iters | relerr | runtime |
|-------------|-----|-----|------|-----------|--------|-------|--------|---------|
| synth_ofr  | 2000| 500 | 500  | RCD       | 97.8   | 48900 | 8.68E−11| 1.36    |
|             |     |     |      | GRCD      | **29.6**|       | 7.47E−11| 3.59    |
|             |     |     |      | RBCD(20)  | 90.7   | **2268**| 8.83E−11| 0.81    |
|             |     |     |      | BCUS(20)  | 125.3  | 3133  | 9.05E−11| **0.31**|
|             |     |     |      | RCD       | 769.9  | 160909| 9.79E−11| 6.46    |
|             |     |     |      | GRCD      | 29.6   |       | 3.59    |
|             |     |     |      | RBCD(20)  | 90.7   | **2268**| 8.83E−11| 0.81    |
|             |     |     |      | BCUS(20)  | 125.3  | 3133  | 9.05E−11| **0.31**|

Table 3: The number of epochs (epochs), the number of iterations (iters), the relative error (relerr), and the running time (runtime) of the REK, TREK, REBK(ℓ), REABK(ℓ), and EBRUS(ℓ) algorithms for four rank-deficient inconsistent linear systems. Here ℓ is the block size.

| matrix      | m   | n   | rank | algorithm | epochs | iters | relerr | runtime |
|-------------|-----|-----|------|-----------|--------|-------|--------|---------|
| synth_urd  | 500 | 2000| 250  | REK       | 17.6   | 35200 | 6.28E−11| 3.96    |
|             |     |     |      | TREK      | 17.7   | 17700 | 4.17E−11| 5.08    |
|             |     |     |      | REBK(20)  | 15.6   | 1560  | 5.15E−11| 1.61    |
|             |     |     |      | REABK(20) | 18.4   | 1840  | 5.48E−11| **0.57**|
|             |     |     |      | EBRUS(20) | 15.6   | 1560  | 4.29E−11| 0.67    |
|             |     |     |      | REK       | 17.6   | 35200 | 6.28E−11| 3.96    |
|             |     |     |      | TREK      | 17.7   | 17700 | 4.17E−11| 5.08    |
|             |     |     |      | REBK(20)  | 15.6   | 1560  | 5.15E−11| 1.61    |
|             |     |     |      | REABK(20) | 18.4   | 1840  | 5.48E−11| 0.41    |
|             |     |     |      | EBRUS(20) | 15.6   | 1560  | 4.22E−11| **0.31**|
| synth_ord  | 2000| 500 | 250  | REK       | 16.9   | 33800 | 5.82E−11| 2.83    |
|             |     |     |      | TREK      | 16.7   | 16700 | 4.88E−11| 3.22    |
|             |     |     |      | REBK(20)  | 15.1   | 1510  | 5.09E−11| 1.16    |
|             |     |     |      | REABK(20) | 18.0   | 1800  | 5.88E−11| 0.41    |
|             |     |     |      | EBRUS(20) | 15.2   | 1520  | 4.22E−11| **0.31**|
|             |     |     |      | REK       | 8.0    | 178200| 8.69E−12| 160.08  |
|             |     |     |      | TREK      | 8.0    | 89104 | 1.34E−11| 218.18  |
|             |     |     |      | REBK(20)  | 13.1   | 14593 | 5.31E−11| 26.18   |
|             |     |     |      | REABK(20) | 20.8   | 23171 | 5.08E−11| 18.20   |
|             |     |     |      | EBRUS(20) | 15.7   | 17490 | 5.38E−11| **10.18**|
| relat7     | 21924| 1045| 1012 | REK       | 16.9   | 370516| 4.61E−11| 154.30  |
|             |     |     |      | TREK      | 17.7   | 194027| 5.83E−11| 229.91  |
|             |     |     |      | REBK(20)  | 15.0   | 16455 | 5.89E−11| 35.36   |
|             |     |     |      | REABK(20) | 124.4  | 136467| 9.26E−11| 53.22   |
|             |     |     |      | EBRUS(20) | 17.2   | 18868 | 5.74E−11| **6.41**|
consistent linear systems (both full column rank and rank-deficient cases are included). For the BRUS algorithm, we use the empirical stepsize parameter $\alpha_r = 2/\hat{\lambda}_{\text{max}}^I$. For all cases, the GRK algorithm is the best in terms of the number of epochs. For the case of abtaha1, the RBK algorithm is the best in terms of the running time; and for the other six cases, the BRUS algorithm is the best.

In Table 2 we report the numerical results of the RCD, GRCD, RBCD, and BCUS algorithms for three full column rank inconsistent linear systems. For the BCUS algorithm, we use the empirical stepsize parameter $\alpha_c = 1/\hat{\lambda}_{\text{max}}^J$. For all cases, the GRCD algorithm is the best in terms of the number of epochs, the RBCD algorithm is the best in terms of the number of iterations, and the BCUS algorithm is the best in terms of the running time.

In Table 3 we report the numerical results of the REK, TREK, REBK, REABK, and EBRUS algorithms for four rank-deficient inconsistent linear systems. For the REABK algorithm, we use the same empirical stepsize parameter reported in [8]. For the EBRUS algorithm, we use the empirical stepsize parameters $\alpha_r = 2/\hat{\lambda}_{\text{max}}^I$ and $\alpha_c = 2/\hat{\lambda}_{\text{max}}^J$. For the case of synth_urd, the REABK algorithm is the best in terms of the running time; and for the other three cases, the EBRUS algorithm is the best.

5 Concluding remarks

We have proposed two novel pseudoinverse-free randomized block iterative algorithms for solving consistent and inconsistent linear systems of equations. Our main results show that our algorithms converge linearly in the mean square sense to a (least squares) solution of the linear system $Ax = b$. By using uniform sampling, we have designed the BRUS algorithm for solving consistent linear systems, the BCUS algorithm for solving full column rank inconsistent linear systems, and the EBRUS algorithm for solving rank-deficient inconsistent linear systems. Numerical experiments for both synthetic and real-world matrices demonstrate that the BRUS, BCUS, and EBRUS algorithms with appropriate stepsize parameters and block size can significantly outperform several existing randomized algorithms in terms of the running time.

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A Extended block column and row sampling iterative algorithm

The RCD algorithm [15] for $Az = b$ with arbitrary initial guess $z^0 \in \mathbb{R}^n$ produces a sequence $\{z^k\}$, which satisfies $Az^k \rightarrow AA^\dagger b$. Ma, Needell, and Ramdas [19] proved that the $k$th iterate $x^k$ (which is produced by one RK update for $Ax = Az^k$ from $x^{k-1}$) of the randomized extended Gauss–Seidel (REGS) algorithm converges to a solution of $Ax = AA^\dagger b$. In this section, based on the idea of the REGS algorithm, we propose an extended block column and row sampling iterative (EBCRSI) algorithm.

Given arbitrary $z^0 \in \mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$, the iterates of the EBCRSI algorithm at step $k$ are defined as

\begin{align}
    z^k &= z^{k-1} - \alpha_c TT^\top A^\top (Az^{k-1} - b), \\
    x^k &= x^{k-1} - \alpha_r A^\top SS^\top A(x^{k-1} - z^k),
\end{align}

where the random parameter matrices $S$ and $T$ are independent, and satisfy

\[
    \mathbb{E}[SS^\top] = I, \quad \mathbb{E}[TT^\top] = I.
\]
We note that the iteration \( z^k \) is the BCSI algorithm for \( Az = b \) with arbitrary initial guess \( z^0 \in \mathbb{R}^n \), and the iterate \( x^k \) in \( z^k \) is one BRSI update for \( Ax = Az^k \) from \( x^{k-1} \). By Theorem \( 7 \) we have
\[
E \left[ \|A(x^k - A^1b)\|_2^2 \right] \leq 2^k \|A(z^0 - A^1b)\|_2^2, \tag{25}
\]
In the following, we shall present two convergence results of the EBCRSI algorithms: Theorem \( 13 \) is on the convergence of \( \|E[x^k] - x^0\| \), and Theorem \( 15 \) is on the convergence of \( E[\|x^k - x^0\|^2] \). We emphasize that both the convergence results hold for arbitrary linear systems.

**Theorem 13.** For arbitrary \( z^0 \in \mathbb{R}^n \) and \( x^0 \in \mathbb{R}^n \), the \( k \)th iterate \( x^k \) of the EBCRSI algorithm satisfies
\[
E \left[ x^k - x^0 \right] = (I - \alpha A^\top A)^k(x^0 - x^0) + \alpha \sum_{i=0}^{k-1} (I - \alpha A^\top A)^i(I - \alpha_c A^\top A)^{k-i} A^\top (Az^0 - b).
\]
Moreover,
\[
\|E[x^k] - x^0\| \leq \delta^k(\|x^0 - x^0\| + k\alpha_r \|A^\top (Az^0 - b)\|), \tag{26}
\]
where
\[
\delta = \max_{1 \leq i \leq r} \{|1 - \alpha_c \sigma_i^2(A)|, |1 - \alpha_c \sigma_i^2(A)|\}.
\]

**Proof.** By \( A^\top AA^1b = A^1b \) and \( z^k \), we have
\[
Az^k - AA^1b = Az^{k-1} - AA^1b - \alpha_c ATT^\top A^\top (Az^{k-1} - AA^1b).
\]
Taking conditional expectation conditioned on \( z^{k-1} \) and \( x^{k-1} \), we have
\[
E_{k-1} \left[ Az^k - AA^1b \right] = Az^{k-1} - AA^1b - \alpha_c AA^\top (Az^{k-1} - AA^1b)
= (I - \alpha_c AA^\top)(Az^{k-1} - AA^1b).
\]
Then, by the law of total expectation, we have
\[
E[Az^k - AA^1b] = (I - \alpha_c AA^\top)E[Az^{k-1} - AA^1b]
= (I - \alpha_c AA^\top)^k(Az^0 - AA^1b).
\]
Taking expectation conditioned on \( x^{k-1} \) and \( x^{k-1} \) for
\[
x^k - x^0 = x^{k-1} - x^0 - \alpha_x A^\top SS^\top A(x^{k-1} - z^k),
\]
we obtain
\[
E_{k-1} \left[ x^k - x^0 \right] = E_{k-1} \left[ E_{k-1} \left[ x^{k-1} - x^0 - \alpha_x A^\top SS^\top A(x^{k-1} - z^k) \right] \right]
= E_{k-1} \left[ x^{k-1} - x^0 - \alpha_x A^\top A(x^{k-1} - x^0) + x^0 - z^k \right]
= (I - \alpha_x A^\top A)(x^{k-1} - x^0) + \alpha_x A^\top E_{k-1} \left[ Az^k - Ax^0 \right]
= (I - \alpha_x A^\top A)(x^{k-1} - x^0) + \alpha_x A^\top E_{k-1} \left[ Az^k - AA^1b \right],
\]
which, by the law of total expectation, yields
\[
E \left[ x^k - x^0 \right] = (I - \alpha_x A^\top A)E \left[ x^{k-1} - x^0 \right] + \alpha_x A^\top E \left[ Az^k - AA^1b \right]
= (I - \alpha_x A^\top A)E \left[ x^{k-1} - x^0 \right] + \alpha_x A^\top (I - \alpha_c AA^\top)^k(Az^0 - AA^1b)
= (I - \alpha_x A^\top A)E \left[ x^{k-1} - x^0 \right] + \alpha_x (I - \alpha_c A^\top A)k A^\top (Az^0 - b)
+ \alpha_x (I - \alpha_c A^\top A)(I - \alpha_c A^\top A)^{k-1} A^\top (Az^0 - b)
= \cdots
= (I - \alpha_x A^\top A)^k(x^0 - x^0) + \alpha_x \sum_{i=0}^{k-1} (I - \alpha_x A^\top A)^i(I - \alpha_c A^\top A)^{k-i} A^\top (Az^0 - b).
\]
Taking 2-norm, by triangle inequality, \( x^0 - x^0 \in \text{range}(A^\top) \), \( A^\top (Az^0 - b) \in \text{range}(A^\top) \), and Lemma \( 1 \), we obtain the estimate \( 26 \).
Remark 14. In Theorem 13, no assumptions about the dimensions or rank of $A$ are assumed, and the system $Ax = b$ can be consistent or inconsistent. If $0 < \alpha_0 < 2/\sigma_{\max}(A)$ and $0 < \alpha_c < 2/\sigma_{\max}(A)$, then $0 < \delta < 1$. This means $x_k^*$ is an asymptotically unbiased estimator for $x_0^*$.

Theorem 15. Assume that $0 < \alpha_c < 2/\lambda_{\max}$ and $0 < \alpha_0 < 2/\lambda_{\max}$. For arbitrary $z^0 \in \mathbb{R}^n$, $x^0 \in \mathbb{R}^n$, and $\varepsilon > 0$, the $k$th iterate $x^k$ of the EBCRSI algorithm satisfies

$$
E \left[ \|x^k - x^*_0\|^2 \right] \leq (1 + \varepsilon)^k \eta_k \|x^0 - x^0_0\|^2
$$

$$
+ (1 + 1/\varepsilon)\alpha_0^2 \lambda_{\max}^2 \|A(z^0 - A^1b)\|^2 \sum_{i=0}^{k-1} \eta_i^{k-i} (1 + \varepsilon)^i \eta_i,
$$

where

$$
\eta_k = 1 - \alpha_0 \alpha_\varepsilon \lambda_{\max}^2 \sigma_{\min}^2(A), \quad \text{and} \quad \eta_c = 1 - \alpha_c \alpha_\varepsilon \lambda_{\max}^2 \sigma_{\min}^2(A).
$$

Proof. By (24), we have

$$
x^k - x^*_0 = x^{k-1} - x^*_0 - \alpha_t A^T S S^T A(x^{k-1} - x^*_0 + x^*_0 - z^k).
$$

By triangle inequality and Young’s inequality, we have

$$
\|x^k - x^*_0\|^2 \leq \| (I - \alpha_t A^T S S^T A)(x^{k-1} - x^*_0) \| + \alpha_t \| A^T S S^T A(z^k - x^*_0) \|^2
$$

$$
\leq (1 + \varepsilon) \| (I - \alpha_t A^T S S^T A)(x^{k-1} - x^*_0) \|^2 + (1 + 1/\varepsilon) \alpha_0^2 \| A^T S S^T A(z^k - x^*_0) \|^2.
$$

By $x^0 - x^*_0 \in \text{range}(A^T)$ and $A^T S S^T A(x^k - x^*_0) \in \text{range}(A^T)$, we can show that $x^k - x^*_0 \in \text{range}(A^T)$ by induction. It follows that

$$
\| (I - \alpha_t A^T S S^T A)(x^{k-1} - x^*_0) \|^2 = \| x^{k-1} - x^*_0 \|^2 - 2 \alpha_t \langle x^{k-1} - x^*_0, A^T S S^T A(x^{k-1} - x^*_0) \rangle
$$

$$
+ \alpha_0^2 \| x^{k-1} - x^*_0 \|^2 (\alpha_t A^T S S^T A)^2(x^{k-1} - x^*_0)
$$

$$
\leq \| x^{k-1} - x^*_0 \|^2 - 2 \alpha_t \| x^{k-1} - x^*_0 \|^2 (\alpha_t A^T S S^T A)^2(x^{k-1} - x^*_0)
$$

$$
+ \alpha_0^2 \lambda_{\max}^2 \| x^{k-1} - x^*_0 \|^2 (\alpha_t A^T S S^T A)^2(x^{k-1} - x^*_0).
$$

Taking conditional expectation conditioned on $z^{k-1}$ and $x^{k-1}$, we obtain

$$
E_{k-1} \left[ \| (I - \alpha_t A^T S S^T A)(x^{k-1} - x^*_0) \|^2 \right] \leq \eta_t \| x^{k-1} - x^*_0 \|^2.
$$

Taking conditional expectation conditioned on $z^{k-1}$ and $x^{k-1}$ for

$$
\| A^T S S^T A(z^k - x^*_0) \|^2 \leq \lambda_{\max}(z^k - x^*_0) A^T S S^T A(z^k - x^*_0),
$$

and by $Ax^0 = AA^1b$, we obtain

$$
E_{k-1} \left[ \| A^T S S^T A(z^k - x^*_0) \|^2 \right] \leq \lambda_{\max}^2 E_{k-1} \left[ \| A(z^k - x^*_0) \|^2 \right] = \lambda_{\max}^2 E_{k-1} \left[ \| A(z^k - A^1b) \|^2 \right].
$$

It follows that

$$
E_{k-1} \left[ \| x^k - x^*_0 \|^2 \right] \leq (1 + \varepsilon) \eta_k \| x^{k-1} - x^*_0 \|^2 + (1 + 1/\varepsilon) \alpha_0^2 \lambda_{\max}^2 E_{k-1} \left[ \| A(z^k - A^1b) \|^2 \right].
$$

Then, by the law of total expectation and the estimate [25], we have

$$
E \left[ \| x^k - x^*_0 \|^2 \right] \leq (1 + 1/\varepsilon) \alpha_0^2 \lambda_{\max}^2 E \left[ \| A(z^0 - A^1b) \|^2 \right] + (1 + \varepsilon) \eta_k E \left[ \| x^{k-1} - x^*_0 \|^2 \right]
$$

$$
\leq (1 + 1/\varepsilon) \alpha_0^2 \lambda_{\max}^2 \eta_k \| A(z^0 - A^1b) \|^2 + (1 + \varepsilon) \eta_k E \left[ \| x^{k-1} - x^*_0 \|^2 \right]
$$

$$
\leq (1 + 1/\varepsilon) \alpha_0^2 \lambda_{\max}^2 \| A(z^0 - A^1b) \|^2 \eta_k^{2-k} \eta_k + \eta_k^{k-k} (1 + \varepsilon) \eta_k
$$

$$
+ (1 + \varepsilon)^2 \eta_k \left[ \| x^{k-2} - x^*_0 \|^2 \right]
$$

$$
\leq \cdots
$$

$$
\leq (1 + 1/\varepsilon) \alpha_0^2 \lambda_{\max}^2 \| A(z^0 - A^1b) \|^2 \sum_{i=0}^{k-1} \eta_i^{k-i} (1 + \varepsilon)^i \eta_i
$$

$$
+ (1 + \varepsilon)^k \eta_k \| x^0 - x^0_0 \|^2.
$$

This completes the proof. □
Remark 16. In Theorem [13] no assumptions about the dimensions or rank of $A$ are assumed, and the system $Ax = b$ can be consistent or inconsistent. Let $\eta = \max\{\eta_r, \eta_c\}$. It follows from $0 < \alpha_c < 2/\lambda_{\max}^c$ and $0 < \alpha_r < 2/\lambda_{\max}^r$ that $\eta < 1$. Assume that $\varepsilon$ satisfies $(1 + \varepsilon)\eta < 1$. We have

$$
E\left[\|x^k - x^0\|^2\right] \leq (1 + \varepsilon)k\eta^k(\|x^0 - x^0\|^2 + (1 + \varepsilon)\alpha_c^2\lambda_{\max}^c\|A(z^0 - A^b)\|^2/\varepsilon^2),
$$

which shows that the EBCRSI algorithm converges linearly in the mean square sense to $x_*$ with the rate $(1 + \varepsilon)\eta$.

A.1 The randomized extended Gauss–Seidel algorithm

The REGS algorithm [19] is one special case of the EBCRSI algorithm. Choosing $S = \frac{A_F}{\|A\|_F}I_{i,i}$ with probability $\frac{\|A_{i,:}\|^2}{\|A\|^2_F}$ and $T = \frac{A}{\|A\|_F}I_{i,j}$ with probability $\frac{\|A_{-i,j}\|^2}{\|A\|^2_F}$, we have

$$
E[SS^\top] = I, \quad E[TT^\top] = I,
$$

and obtain

$$
z^k = z^{k-1} - \alpha_c\frac{\|A\|^2_F}{\|A_{i,:}\|^2}(A_{i,:})^\top(Az^{k-1} - b)I_{i,j},
$$

$$
x^k = x^{k-1} - \alpha_r\frac{\|A\|^2_F}{\|A\|^2_F}A_{i,:}(x^{k-1} - z^k)(A_{i,:})^\top.
$$

We have $\lambda_{\max}^c = \lambda_{\max}^c = \|A\|^2_F$. Setting $\alpha_r = \alpha_c = 1/\|A\|^2_F$, we recover the REGS algorithm [19, 7].

A.2 The extended block column and row uniform sampling algorithm

We propose one new special case of the EBCRSI algorithm by using uniform sampling and refer to it as the extended block column and row uniform sampling (EBCRUS) algorithm. Assume $1 \leq \ell \leq \min\{m, n\}$. Let $I$ (resp. $J$) denote the set consisting of the uniform sampling of $\ell$ different numbers of $[m]$ (resp. $[n]$). Setting $S = \sqrt{m/\ell}I_{i,i}$ and $T = \sqrt{n/\ell}I_{i,j}$, we have

$$
E[SS^\top] = \frac{m}{\ell} \sum_{I \subseteq [m], |I| = \ell} I_{i,I}I_{i,I}^\top = I_m, \quad E[TT^\top] = \frac{n}{\ell} \sum_{J \subseteq [n], |J| = \ell} I_{i,J}I_{i,J}^\top = I_n,
$$

and obtain

$$
z^k = z^{k-1} - \alpha_c\frac{n}{\ell}I_{i,J}(A_{i,J})^\top(Az^{k-1} - b),
$$

$$
x^k = x^{k-1} - \alpha_r\frac{m}{\ell}(A_{I,:})^\top A_{I,:}(x^{k-1} - z^k).
$$

We have $\lambda_{\max}^c = \frac{m}{\ell} \max_{I \subseteq [m], |I| = \ell} \|A_{I,:}\|^2$, and $\lambda_{\max}^r = \frac{n}{\ell} \max_{J \subseteq [n], |J| = \ell} \|A_{i,J}\|^2$.

By Theorem [13] the EBCRUS algorithm can have a faster convergence rate than that of the REGS algorithm if there exists $1 \leq \ell \leq \min\{m, n\}$ satisfying

$$
\frac{m}{\ell} \max_{I \subseteq [m], |I| = \ell} \|A_{I,:}\|^2 \leq \|A\|^2_F, \quad \text{and} \quad \frac{n}{\ell} \max_{J \subseteq [n], |J| = \ell} \|A_{i,J}\|^2 \leq \|A\|^2_F.
$$
References

[1] Z.-Z. Bai and W.-T. Wu. On greedy randomized Kaczmarz method for solving large sparse linear systems. *SIAM J. Sci. Comput.*, 40(1):A592–A606, 2018.

[2] Z.-Z. Bai and W.-T. Wu. On relaxed greedy randomized Kaczmarz methods for solving large sparse linear systems. *Appl. Math. Lett.*, 83:21–26, 2018.

[3] Z.-Z. Bai and W.-T. Wu. On greedy randomized coordinate descent methods for solving large linear least-squares problems. *Numer. Linear Algebra Appl.*, 26(4):e2237, 15, 2019.

[4] Z.-Z. Bai and W.-T. Wu. On partially randomized extended Kaczmarz method for solving large sparse overdetermined inconsistent linear systems. *Linear Algebra Appl.*, 578:225–250, 2019.

[5] J. Chung, M. Chung, J. T. Slagel, and L. Tenorio. Stochastic Newton and quasi-Newton methods for large linear least-squares problems. [arXiv:1702.07367](https://arxiv.org/abs/1702.07367), 2017.

[6] T. A. Davis and Y. Hu. The University of Florida sparse matrix collection. *ACM Trans. Math. Software*, 38(1):Art. 1, 25, 2011.

[7] K. Du. Tight upper bounds for the convergence of the randomized extended Kaczmarz and Gauss-Seidel algorithms. *Numer. Linear Algebra Appl.*, 26(3):e2233, 14, 2019.

[8] K. Du, W.-T. Si, and X.-H. Sun. Randomized extended average block Kaczmarz for solving least squares. *SIAM J. Sci. Comput.*, 42(6):A3541–A3559, 2020.

[9] B. Dumitrescu. On the relation between the randomized extended Kaczmarz algorithm and coordinate descent. *BIT*, 55(4):1005–1015, 2015.

[10] R. M. Gower and P. Richtárik. Randomized iterative methods for linear systems. *SIAM J. Matrix Anal. Appl.*, 36(4):1660–1690, 2015.

[11] Y.-J. Guan, W.-G. Li, L.-L. Xing, and T.-T. Qiao. A note on convergence rate of randomized Kaczmarz method. *Calcolo*, 57(3):26, 11, 2020.

[12] G. T. Herman and L. B. Meyer. Algebraic reconstruction techniques can be made computationally efficient. *IEEE Trans. Medical Imaging*, 12(3):600–9, 1993.

[13] S. Kaczmarz. Angenäherte auflösung von systemen linearer gleichungen. *Bull. Intern. Acad. Polonaise Sci. Lett., Cl. Sci. Math. Nat. A*, 35:355–357, 1937.

[14] L. H. Landweber. An iteration formula for Fredholm integral equations of the first kind. *Amer. J. Math.*, 73:615–624, 1951.

[15] D. Leventhal and A. S. Lewis. Randomized methods for linear constraints: convergence rates and conditioning. *Math. Oper. Res.*, 35(3):641–654, 2010.

[16] J. Liu and S. J. Wright. An accelerated randomized Kaczmarz algorithm. *Math. Comp.*, 85(297):153–178, 2016.

[17] Y. Liu and C.-Q. Gu. Variant of greedy randomized Kaczmarz for ridge regression. *Appl. Numer. Math.*, 143:223–246, 2019.

[18] Y. Liu and C.-Q. Gu. On greedy randomized block Kaczmarz method for consistent linear systems. *Linear Algebra Appl.*, 616:178–200, 2021.

[19] A. Ma, D. Needell, and A. Ramdas. Convergence properties of the randomized extended Gauss-Seidel and Kaczmarz methods. *SIAM J. Matrix Anal. Appl.*, 36(4):1590–1604, 2015.

[20] A. Ma, D. Needell, and A. Ramdas. Iterative methods for solving factorized linear systems. *SIAM J. Matrix Anal. Appl.*, 39(1):104–122, 2018.
[21] J. D. Moorman, T. K. Tu, D. Molitor, and D. Needell. Randomized Kaczmarz with averaging. BIT, 61(1):337–359, 2021.

[22] F. Natterer. The mathematics of computerized tomography, volume 32 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Reprint of the 1986 original.

[23] I. Necoara. Faster randomized block Kaczmarz algorithms. SIAM J. Matrix Anal. Appl., 40(4):1425–1452, 2019.

[24] D. Needell. Randomized Kaczmarz solver for noisy linear systems. BIT, 50(2):395–403, 2010.

[25] D. Needell and J. A. Tropp. Paved with good intentions: analysis of a randomized block Kaczmarz method. Linear Algebra Appl., 441:199–221, 2014.

[26] D. Needell, R. Zhao, and A. Zouzias. Randomized block Kaczmarz method with projection for solving least squares. Linear Algebra Appl., 484:322–343, 2015.

[27] M. Razaviyayn, M. Hong, N. Reyhanian, and Z.-Q. Luo. A linearly convergent doubly stochastic Gauss-Seidel algorithm for solving linear equations and a certain class of over-parameterized optimization problems. Math. Program., 176(1-2, Ser. B):465–496, 2019.

[28] E. Rebrova and D. Needell. On block Gaussian sketching for the Kaczmarz method. Numer. Algorithms, 86(1):443–473, 2021.

[29] P. Richtárik and M. Takáč. Stochastic Reformulations of Linear Systems: Algorithms and Convergence Theory. SIAM J. Matrix Anal. Appl., 41(2):487–524, 2020.

[30] T. Strohmer and R. Vershynin. A randomized Kaczmarz algorithm with exponential convergence. J. Fourier Anal. Appl., 15(2):262–278, 2009.

[31] W.-T. Wu. On two-subspace randomized extended Kaczmarz method for solving large linear least-squares problems. Numer. Algorithms, to appear, 2021.

[32] J.-H. Zhang and J.-H. Guo. On relaxed greedy randomized coordinate descent methods for solving large linear least-squares problems. Appl. Numer. Math., 157:372–384, 2020.

[33] A. Zouzias and N. M. Freris. Randomized extended Kaczmarz for solving least squares. SIAM J. Matrix Anal. Appl., 34(2):773–793, 2013.