Pebble Minimization of Polyregular Functions

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Abstract

We show that a polyregular word-to-word function is regular if and only its output size is at most linear in its input size. Moreover a polyregular function can be realized by a transducer with two pebbles if and only if its output has quadratic size in its input, a transducer with three pebbles if and only if its output has cubic size in its input, etc.

Moreover the characterization is decidable and, given a polyregular function, one can compute a transducer realizing it with the minimal number of pebbles.

We apply the result to mso interpretations from words to words. We show that mso interpretations of dimension $k$ exactly coincide with $k$-pebble transductions.

CCS Concepts: • Theory of computation → Transducers.

Keywords: pebble transducers, polyregular functions, minimization, MSO interpretations

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Introduction

Regular and polyregular functions This article is about word-to-word (partial) functions, which we often call transductions. Regular functions constitute an extensively studied class of functions which is characterized by many different computation models: two-way deterministic automata with outputs, streaming string transducers [Alur and Cerný 2010], mso-transductions [Engelfriet and Hoogeboom 2001, Theorem 10], regular combinators [Alur et al. 2014, Theorem 15] and regular list functions [Bojańczyk et al. 2018, Theorem 4.3].

In this article we consider pebble transducers which were proposed in [Milo et al. 2003], following the earlier work of [Globerman and Harel 1996]. A pebble transducer is a deterministic finite state device which can mark a bounded number of positions of its input with pebbles. These pebbles follow a stack discipline, which means that only the most recently placed pebble can be moved. This restriction ensures that the model only recognizes regular languages [Globerman and Harel 1996, Theorem 4.2]. On every transition the machine may append a finite word to the output.

In [Bojańczyk 2018] the author provides three equivalent characterizations of functions realized by pebble transducers, which are called polyregular functions: an imperative programming language with for loops, a functional programming language with limited access to recursion (e.g. map but not fold), compositions of simple basic functions using a Krohn-Rhodes-like decomposition.

Another characterization, in terms of mso interpretations, is shown in [Bojańczyk et al. 2019, Theorem 7].

One difference between regular and polyregular functions is that regular functions have linear growth (i.e. the image of a word of length $n$ has length in $O(n)$) while polyregular functions may have, as the name suggests, polynomial growth. In fact, as we will see, this is the only difference.

Growth rate We study the growth of polyregular functions. One of our main motivations was the following question: can one decide if a polyregular function is regular?

The output size of a $k$-pebble transducer over an input of size $n$ is in $O(n^k)$. This can be easily seen since the number of configurations (state and positions of pebbles) is in $O(n^k)$. In particular, a regular function has linear growth since a two-way transducer is nothing more than a 1-pebble transducer.

Our first main result is to show that the converse holds as well. This means in particular that a polyregular function is regular if and only if its growth is linear.
Outline In Section 1 we give the definition of polyregular function and pebble transducer. In Section 2 we state our main results. In Section 3 we introduce the techniques used in this paper, and we show how to decide if a regular function is bounded. In Section 4 we extend these techniques to show the main technical lemma of the article, which we call the dichotomy Lemma; one can decide if a transducer with $k$-pebble transductions we obtain our third result: mso interpretations of dimension $k$ exactly capture polyregular functions of growth $O(n^k)$.

1 Polyregular functions

Polyregular functions have been shown to be characterized by many different computational models [Bojańczyk 2018; Bojańczyk et al. 2019]. The model we are interested in is that of pebble transducers which are automata that can place pebbles on a bounded number of positions following a stack discipline, meaning that only the most recently placed pebble can be moved. We consider in this section the model of pebble transducers and start with 1-pebble transducers (usually called two-way transducers) which characterize the regular functions.

1-pebble transducers A 1-pebble transducer, (usually known as a two-way transducer) over input alphabet $\Sigma$ and output alphabet $\Gamma$ is a two-way automaton (meaning that it has a reading head, called here a pebble, which can scan the word in both directions) which reads words over $\Sigma^*$, and has the ability to output words over $\Gamma^*$ on every transition. The output of a 1-pebble transducer over some word is the concatenation of the outputs of its transitions along a run. In Figure 1 we represent a configuration of a 1-pebble transducer:

**Definition 1** (1-pebble transducer). A 1-pebble transducer is a tuple $(\Sigma, \Gamma, Q, q_I, q_F, \delta)$, which consists of:

- a finite input alphabet $\Sigma$ and a finite output alphabet $\Gamma$;
- a finite set of states $Q$;
- two designated states $q_I$ and $q_F$: the initial and final one;
- a transition function of type $\delta : (\Sigma \cup \{+,-\}) \times Q \rightarrow Q \times \{-,\} \times \Gamma^*$.

The symbols $+$ and $-$ are the endmarkers of the word.

Let us define the behavior of the transducer over an input word $w \in \Sigma^*$. The transducer actually reads the word $+w-$ and we denote by $\Sigma_+$, the set $\Sigma \cup \{+, -\}$. A configuration is seen as a word over the alphabet $\Sigma_+ \times 2^Q$ such that the first letter is in $\{+, -\} \times 2^Q$, the last one in $\{+, -\} \times 2^Q$ and, only one position has a non-empty $2^Q$ component which is a singleton. In the following, in order to simplify notations, we will denote a pair $(a, \emptyset)$ simply by $a$ and a pair $(a, \{q\})$ by $\left[ \begin{array}{c} a \\ q \end{array} \right]$, keeping the braces to indicate that it is a single letter. In Figure 2 we represent a configuration.

**Figure 2.** Configuration of a 1-pebble transducer.

The successor configuration of a configuration $c$, when it exists, is obtained in the following way: apply $\delta$ to the pair $(a, q)$ corresponding to the unique letter $\left[ \begin{array}{c} a \\ q \end{array} \right]$ in the configuration, update the state and move the position of the pebble to the right or to the left accordingly depending on $\delta$. The output of $c$ is the word in $\Gamma^*$ obtained by applying $\delta$. A run on $w$ is a sequence of configurations related by the successor relation defined above. The output of a run is the word obtained by concatenating the outputs of its configurations.

A configuration in $\left[ \begin{array}{c} + \\ q_I \end{array} \right] \Sigma^*$ is called initial, and a configuration in $\left[ \begin{array}{c} - \\ q_F \end{array} \right]$ is called final. A run is accepting if the
first configuration is initial, the last one is final, and no other configuration is final. The accepting run over a word \( w \), if it exists, is the unique (thanks to determinism) accepting run starting in \( q_1 \) \( \xrightarrow{} \) \( w \). A pair \( (w, v) \) is realized by the transducer if \( v \) is the output of the accepting run of \( w \).

A (partial) function is called regular if it is realized by a 1-pebble transducer.

**Example 2.** Let us give an example of a transducer over alphabet \{\( a, b \)\} which writes in unary in \( \{ \circ \} \) the length of the prefix of \( a \)'s of a word. We draw in Figure 3 the sequence of configurations of the run over the word \( aaba \), whose image is \( \circ \circ \).

\[
\begin{array}{c}
\xrightarrow{} aaba_1 \\
\xrightarrow{} aaba_2 \\
\xrightarrow{} aaba_3 \\
\xrightarrow{} aaba_4 \\
\xrightarrow{} aaba_5 \\
\end{array}
\]

**Figure 3.** Sequence of configurations.

Nested transducers In the literature (see e.g. [Bojańczyk 2018]), a \( k \)-pebble transducer is a transducer with \( k \) reading heads. The movement of these heads is subject to a stack discipline: only the pebble on top of the stack can move. In this paper, we will work with a different yet equivalent model called \( k \)-nested transducers. Here a \( k \)-nested transducer is a collection of \( k \) distinct 1-pebble transducers. The idea is that the transducer number \( k \) can, along its run, call transducer \( k - 1 \) to run over its current configuration. Then transducer \( k - 1 \) can itself call transducer \( k - 2 \) to run over its current configuration, and so on. This is analogous of program composition: when a program \( A \) calls a program \( B \) as a subroutine, first program \( B \) is executed on the current state of program \( A \), then program \( A \) resumes its computation.

**Definition 3.** A \( k \)-nested transducer of input alphabet \( \Sigma \) and output alphabet \( \Gamma \) is a tuple \( T = \langle T_1, \ldots, T_k \rangle \) such that for every \( i \in \{1, \ldots, n\} \):

- \( T_i \) is a 1-pebble transducer, whose set of states is denoted \( Q_i \);
- The input alphabet of \( T_i \) is \( \Sigma \) with additional predicates in \( Q_{>i} \) \( (Q_{>i} = \bigcup_{j>i} Q_j) \);
- The output alphabet of \( T_i \) is \( \Gamma \cup \{ \text{call}_1, \ldots, \text{call}_{i-1} \} \).

In particular, the input alphabet of \( T_k \) is \( \Sigma \) and the output alphabet of \( T_1 \) is \( \Gamma \).

The output letter \( \text{call}_i \) is to be interpreted as the transducer calling \( T_i \) to run over its current configuration. For every \( i \in \{1, \ldots, k\} \), the sequence \( \langle T_1, \ldots, T_i \rangle \) can be seen as an \( i \)-pebble transducer, of input alphabet \( \Sigma_i = \Sigma \times 2^{Q_{>i}} \) and output alphabet \( \Gamma \). We denote this transducer by \( T_i \).

**Definition 4.** We define, by induction on \( k \), the function realized by a \( k \)-nested transducer. The case \( k = 1 \) has been treated in Definition 1.

Consider a \( k + 1 \)-nested transducer \( T = \langle T_1, \ldots, T_k, T_{k+1} \rangle \), and let \( Q_i \) be the set of states of \( T_i \), for \( i \leq k \). By induction let \( f_i : \Sigma_i^* \rightarrow \Gamma^* \) denote the transduction realized by \( T_i \), for \( i \in \{1, \ldots, k\} \). Let us define the image of a word \( w \) of \( \Sigma^* \) by the transduction realized by \( T \):

- Let \( r = c_1, \ldots, c_{n+1} \) be the accepting run of \( T_{k+1} \) over \( w \) and \( y_1, \ldots, y_n \) be the outputs of the corresponding configurations.
- For every \( j \in \{1, \ldots, n\} \), let \( u_j \) be the word obtained from \( y_j \) by replacing each occurrence of a letter \( c_{i'} \in \{ \text{call}_{i'}, \ldots, \text{call}_{k}\} \) by \( f_i(c_{i'}) \) (where \( c_{i'} \) is \( c_j \) without end-markers).

The image of \( w \) by \( T \) is the word \( u_1 \cdots u_n \).

**Example 5.** Let us consider the 2-nested transducer \( T = \langle T_1, T_2 \rangle \), where \( T_2 \) realizes a function \( f_{\text{pref}} \) similar to the one defined in Example 2: it makes a number of calls to \( T_1 \) corresponding the length of the \( a \)-prefix of the input (separated by \( \# \) symbols). \( T_1 \) realizes the transduction \( \tilde{f}_1 : \{ (a, b) \times 2^{Q_1} \}^* \rightarrow \{ (a, b) \}^* \) that copies a word, but erases the state predicates. Then the transduction realized by \( T \) is the function \( f : w \mapsto (w\#)^{f_{\text{pref}}(w)} \). The function \( f \) copies an input word as many times as the length of the prefix of the word with only \( a \)'s.

The picture from Figure 4 illustrates the behavior of \( T \): first \( T_2 \) runs over the input word, and each time \( T_2 \) produces a call \( l_1 \), this is interpreted as a call to \( T_1 \) to run over the current configuration.

\[
\begin{array}{c}
\xrightarrow{} aaba_1 \\
\xrightarrow{} aaba_2 \\
\xrightarrow{} aaba_3 \\
\xrightarrow{} aaba_4 \\
\xrightarrow{} aaba_5 \\
\xrightarrow{} aaba_6 \\
\end{array}
\]

**Figure 4.** Run of a 2-pebble transducer.

By definition \( f_1((a)q_1)ab) = f_1((a)\{aq_1\})ba) = aaba \), hence we obtain \( f(aaba) = aaba\#aaba\# \).

**Remark 6.** We don’t give the definition of a \( k \)-pebble transducer and refer the reader to [Bojańczyk 2018, Definition 2.5] for a definition of the model. The fact that \( k \)-pebble transducers and \( k \)-nested transducers are equivalent is trivial and
We start by stating our results. Note that we state the re-
kpebbles.

Terminology 8. Let $f : \Sigma^* \to \Gamma^*$ be a word to word function. We say that $f$ has degree $k$ growth (or, abusing notations, that it is in $O(n^k)$) if the size of the image of a word of size $n$ is in $O(n^k)$. For degrees 0, 1 we shall use respectively the terms bounded and linear growth.

The number of pebbles bounds in an obvious way the degree of a polyregular function:

Proposition 9. A function realized by a transducer with $k$ pebbles is in $O(n^k)$.

Proof. We prove the result for $k$-nested transducers. This is easily shown by observing that the number of configurations of a 1-pebble transducer over a word of size $n$ is in $O(n)$, hence a regular function is linear. Thus we get that a $k+1$-nested transducer is linear in the number of calls of a $k$-nested transducer which has growth in $O(n^k)$, by induction.

\qed

2 Results

We start by stating our results. Note that we state the results in terms of pebble transducers, as opposed to nested transducers, because they are the more commonly known model. First, the growth degree characterizes the number of necessary pebbles:

Theorem 10 (Characterization). A polyregular function is in $O(n^k)$ if and only if it can be realized by a transducer with $k$ pebbles.

Moreover, the characterization above is decidable and one can actually minimize the number of pebbles:

Theorem 11 (Minimization). Given a polyregular function $f$, one can compute an equivalent pebble transducer with the minimal number of pebbles. In particular, one can decide if a polyregular function is regular.

Finally as a consequence (using the result from [Bojańczyk et al. 2019, Theorem 7]) we obtain a correspondence between the dimension of mso interpretations and the number of pebbles of pebble transducers:

Theorem 12 (mso-dimension). A word-to-word function can be defined by an mso interpretation of dimension $k$ if and only if it can be realized by a 1-pebble transducer.

We now spend the rest of the article showing the above results. We start by introducing our main tool, the notion of transition morphism of a 1-pebble transducer. We first characterize the regular functions, then we tackle the general case of degree $k$ growth.

We show in Section 3 how to decide if a regular function is bounded. In Section 4 we prove the dichotomy Lemma which tells, given a $k$-pebble transducer, if one can construct an equivalent $k-1$-pebble transducer. Then, using the dichotomy Lemma, we prove the theorems above as corollaries.

3 Deciding if a regular function is bounded

We start by showing how to decide if a regular function is bounded or not. This not very deep result will serve as a stepping stone (as well as a warm-up) for the main contribution of the article. To characterize bounded regular function, our main tool will be the usual notion of transition morphism of a 1-pebble transducer.

3.1 Transition morphism of 1-pebble transducers

We present here the tool used to summarize the behavior of a 1-pebble transducer, called its transition monoid (resp. morphism). We map each word $w$ to an element of the monoid which gives the following kind of information e.g.: if the automaton enters the word from the left in state $q$, then it exits to the left in state $q'$, etc. Moreover, we will sometimes need to record information about the output produced in such a pass of the transducer; this information can for instance the whole output (in which case the monoid is infinite) or information of the kind “a letter a has been produced at least once” (here we recover finiteness).

Definitions 13 (Transition monoid/morphism). Let $T$ be a 1-pebble transducer with set of states $Q$ and output alphabet $\Gamma$.

We define the (infinite) transition monoid $M$ of $T$ as follows:

- its elements are functions of the form $f : Q \times \{\emptyset, \_\} \to Q \times \{\emptyset, \_\} \times \Gamma^*$;
- the composition $\cdot$ is defined as follows. Let $f, g$ be two elements of $M$, $q \in Q$ and $d \in \{\emptyset, \_\}$. We define the transition sequence between $f$ and $g$ starting from $(q, d)$ and its output sequence to be respectively the sequences $(q_i, d_i)_{i \in [0, n]}$ and $(w_i)_{i \in [1, n]}$ satisfying the following conditions:
  - $(q_0, d_0) = (q, d)$;
  - two cases:
    * either $n = 1$ and $d_1 \neq d_0$
    * or, $d_0 = d_1, d_{n-1} = d_n$ and $d_i \neq d_{i+1}$ for every $i \in [1, n-2]$;
  - if $d_0 = \emptyset$ then for every even $i$, $f(q_i, d_i) = (q_{i+1}, d_{i+1}, w_{i+1})$ and for every odd $i$, $g(q_i, d_i) = (q_{i+1}, d_{i+1}, w_{i+1})$;
then for every even \( i \), \( g(q_i, d_i) = (q_{i+1}, d_{i+1}, w_{i+1}) \)
and for every odd \( i \), \( f(q_i, d_i) = (q_{i+1}, d_{i+1}, w_{i+1}) \).
We set \((f \cdot g)(q, d)\) to be \((q_n, d_n, w_1 \cdots w_n)\).

We give in Fig 5 an illustration of the transition sequence of \( f, g \) starting in \((q, \rightarrow)\).

![Transition sequence of f, g starting in (q, \rightarrow).](image)

**Figure 5.** Transition sequence of \( f, g \) starting in \((q, \rightarrow)\).

We now define the transition morphism associated with transducer \( T \). Let \( \mu : (\Sigma_\ast, \ast) \rightarrow M \) be defined as follows:

For every \( d \in \{\rightarrow, \ast\} \), \( \mu(a)(q, d) = \delta(a, q) \).

Finally, let \( a \in \Gamma \), we consider the morphism \( \chi_a : \Gamma^+ \rightarrow \{0, 1\} \) defined by \( \chi_a(a) = 1 \), and \( \chi_a(b) = 0 \) if \( b \neq a \), which says if a word contains at least one letter \( a \). We can naturally extend this to a morphism \( \chi_a : M \rightarrow M_{\{0, 1\}} \), with \( M_{\{0, 1\}} = Q \times \{\rightarrow, \ast\} \rightarrow Q \times \{\rightarrow, \ast\} \times \{0, 1\} \). In explicit terms, for \( f \in M \), if \( f(p, d) = (q, e, w) \) we have \( \chi_a(f)(p, d) = (q, e, \chi_a(w)) \). We denote by \( \mu_a \) the composition \( \chi_a \circ \mu \), and we call this the \( a \)-transition morphism of \( T \).

**Example 14.** Let us consider the transducer given in Example 2. Its transition function is:

\[
\begin{align*}
(\ast, q_1) &\mapsto (q_1, \rightarrow, \ast) \\
(a, q_1) &\mapsto (q_1, \ast, \ast) \\
(b, q_1) &\mapsto (q_1, a, \ast) \\
(\ast, q_1) &\mapsto (q_1, b, \ast)
\end{align*}
\]

\( \delta : \)

As an example let us consider \( f = \mu(\ast ab) = \mu(aaababa) \), then \( f : (q_1, \rightarrow) \mapsto (q_1, \rightarrow, \ast) \). This means that the word \( ab \) (as well as the word \( aaababa \)) goes from \( q_1 \) to \( q_2 \) from left to right, producing at least one symbol \( \circ \).

3.2 Producing triples and bounded regular functions

Now that we have defined the transition morphism of a 1-pebble transducer, we can introduce the notion of producing triple which characterizes non-boundedness. Intuitively, a producing triple means a loop in the run of the transducer that produces a non-empty output and thus can be pumped to produce arbitrarily large outputs.

**Definition 15** (Producing triple). Let \( T = (\Sigma, \Gamma, Q, q_1, q_F, \delta) \) be a 1-pebble transducer, and let \( a \in \Gamma \). Let \((x, e, y) \in \mu(a)(\Sigma^+) \times \mu(a)(\Sigma^+) \times \mu(a)(\Sigma^+) \).

We say that the triple \((x, e, y)\) is a-producing if the transition sequence of \((xe, ey)\) starting from \((q_1, \rightarrow)\), \((q_i, d_i)\) satisfies the following conditions:

* \((q_n, d_n) = (q_F, \rightarrow)\);
* \(e\) is idempotent i.e. \( e \cdot e = e \);
* there exists \( i \in [1, n - 1] \) such that \( e(q_i, d_i) \) is of the form \((q, d, 1)\).

**Example 16.** Using the same transducer as in Example 14, we have that \((\mu(a)(\ast a)), \mu(a)(\ast a), \mu(a)(\ast a))\), for instance, is a \( \circ \)-producing triple.

**Definition 17.** Let \( f : \Sigma^+ \rightarrow \Gamma^+ \) be a function and let \( a \in \Gamma \). We say that \( f \) is bounded (resp. linear, etc) in \( a \) if \( \pi_a \circ f : \Sigma^+ \rightarrow a^+ \) is bounded (resp. linear, etc), where \( \pi_a : \Gamma^+ \rightarrow a^+ \) is the morphism erasing non \( a \) letters:

\[
\pi_a(b) = a \text{ if } b = a \\
= \epsilon \text{ otherwise.}
\]

The following lemma states that having a producing triple characterizes the functions that are unbounded.

**Lemma 18.** A 1-pebble transducer is bounded in \( a \) if and only if it has no \( a \)-producing triples.

For the proof of the lemma, we will use a notion of factorization in a morphism, which will also be used in Section 4.

**Definitions 19.** Let \( \mu : \Sigma^+ \rightarrow M \) be a monoid morphism and let \( w \in \Sigma^+ \). An (idempotent) \( k \)-factorization of \( w \) in the morphism \( \mu \) is given as a tuple of non-empty words \((w_0, x_{1_1}, w_1, \ldots, w_k, x_{k_k}) \) in \( \Sigma^+ \) verifying:

* \( w = w_0x_1w_1 \cdots x_kw_k \)
* for all \( i \in [1, k] \), \( \mu(x_i) = \mu(x_j) \)

We also generalize this definition to a \( k, r \)-factorization, which is a \( k \)-factorization so that each idempotent factor \( x_i \) is itself the product of \( r \) identical non-empty idempotent factors, i.e. \( x_i = x_{i,1} \cdots x_{i,r} \) with \( x_{i,1} \neq \epsilon \) and \( \mu(x_{i,j}) = \mu(x_{i,j}) \) for all \( i \in \{1, \ldots, k\} \), \( j \in \{1, \ldots, r\} \).

We say that such a factorization is **according to** the tuple \((m_0, e_1, m_1, \ldots, e_k, m_k) \) if for all \( i \in [0,k] \), \( \mu(w_i) = m_i \) and for all \( i \in [1, k] \), \( \mu(x_i) = e_i \).

Given a morphism \( \mu : \Sigma^+ \rightarrow M \), we will denote in the following by \( P_k \) the set of tuples of \( M^{2k+1} \) such that some word in \( \Sigma^+ \) has a factorization according to it (the morphism \( \mu \) being clear from context).

The next claim is a Ramsey-type argument which says that the set of words which do not admit a factorization is finite.

**Claim 20.** Let \( \mu : \Sigma^+ \rightarrow M \) be a morphism with \( M \) finite and let \( k, r \geq 1 \). The set of words without any \( k, r \)-factorization is finite.

**Proof.** Let \( R(c, r) \) be the number such that, according to Ramsey’s theorem, an \( R(c, r) \)-clique with edges colored using \( c \) distinct colors contains a monochromatic clique of size \( r \).
Let $\mu : \Sigma^* \rightarrow M$ be a morphism with $M$ finite. To a word $w \in \Sigma$, we associate the complete graph over $|w|$ vertices. Given $1 \leq i < j \leq |w|$ the edge $(i, j)$ is colored with $\mu[w[i, j]]$. Let us consider a monochromatic clique of size $r > 3$, $i_1 < i_2 < \ldots < i_r$ in this graph. Let $w_1 = w[i_1, i_2], w_2 = w[i_2, i_3], \ldots, w_{r-1} = w[i_{r-1}, i_r]$, we have that $\mu(w_1) = \mu(w_2) = \ldots = \mu(w_{r-1}) = \mu(w_1w_2)$ hence we have found $r-1$ consecutive identical non-empty idempotent factors in $w$.

Hence, any word of length greater than $kR(|M|, r + 1)$ must have a $k, r$-factorization. \hfill \square

Proof of Lemma 18. We know that a 1-pebble transducer realizes a linear function, from Proposition 9. Let $T$ be a 1-pebble transducer realizing a function $f : \Sigma^* \rightarrow \Gamma^*$, and without loss of generality, we can assume that $f = \{a\}$ since we only care about the size of outputs. Let $\mu_a$ be the $a$-transition monoid morphism of $T$.

Let us first assume that there exists an $a$-producing triple $(m_0, e_1, m_1) \in P_1$, and let $w$ be a word such that $\ell w$ has a 1, $3$-factorization $(w_0, x_1, y_1, z_1, w_1)$ according to this triple. Then we show that since $(m_0, e_1, m_1)$ is $a$-producing, $|f(w_0x_1y_1^jz_1^jw_1)|\Theta(n)$. By definition of $a$-producing triple, the output while reading a $y_1$ factor is non-empty, hence $f$ is not bounded.

Let us now assume that there are no $a$-producing triples in $P_1$. Using Claim 20, there exists an integer $d$ such that any word of length greater than $d$ has a 1, 3-factorization. Let $w$ be a word with a 1, 3-factorization $(w_0, x_1, y_1, z_1, w_1)$. Since there are no $a$-producing triples, $(\mu_a(w_0), \mu_a(y_1), \mu_a(w_1))$ is not $a$-producing. This means that the outputs corresponding to the factor $y_1$ in the run over $w$ are all empty, and thus we have $|f(w_0x_1y_1z_1w_1)| = |f(w_0x_1z_1w_1)|$. Hence we have $\{|f(w)| \mid w \in \Sigma^*\} = \{|f(w)| \mid w \in \Sigma^{\leq d}\}$, and $f$ is bounded. \hfill \square

4 Deciding if a polyregular function is in $O(n^k)$

Now that we have solved the case of 1-pebble transducers, we move on to general case: deciding if a $k + 1$-pebble transduction can be realized by a $k$-pebble transducer. The main idea is, given $(T_1, \ldots, T_{k+1})$, to modify $T_{k+1}$ so that it calls $T_k$ only when “necessary”. Then, if this modified $T_{k+1}$ is bounded in $(\text{call}_k)$, it means that the function can actually be realized by a transducer with $k$ pebbles.

In order to obtain the dichotomy Lemma, which is the main lemma of the section (actually of the article), we need several tools which we present below.

One first useful tool we will be using is that of mso-labelling of words, i.e. labelling each letter with some regular information. More formally an mso-labelling is a function of type $\ell : \Sigma^* \rightarrow (\Sigma \times L)^*$, which does not change the $\Sigma$ component. It is given by some unary mso-formulas $\phi_1(x), \cdots, \phi_p(x)$ and a function $g : 2^{\{1, \ldots, p\}} \rightarrow L$. Given a word $u \in \Sigma^*$ we define $v = \ell(u)$ by $v[i] = (u[i], l)$, with $l = g(I)$ such that $u \models \bigwedge_{j \in I} \phi_j(i) \land \bigwedge_{j \notin I} \neg \phi_j(i)$. We show that pre-composition with mso-labelling does not change the number of needed pebbles to realize a function.

Proposition 21. Transductions realized by $k$-nested transducers are closed under pre-composition with mso-labelling.

Proof. We show the result by induction on $k$. For $k = 1$, we use that mso-labelling are a particular case of regular functions, and that regular functions are closed under composition.

We assume that the proposition holds for $k$. Let $T = (T_1, \ldots, T_k, T_{k+1})$ be a $k + 1$-nested transducer realizing a function $f : (\Sigma \times L)^* \rightarrow \Gamma^*$. To simplify the proof and without loss of generality we assume that $T_{k+1}$ only makes calls to $T_k$, i.e. does not make calls to transducers with smaller indices and does not output anything in $\Gamma$. Let $f_k : (\Sigma \times L)^2 \rightarrow \Gamma^*$ be the function realized by $T_k$, with $Q$ the state space of $T_{k+1}$. Let $\ell : \Sigma^* \rightarrow (\Sigma \times L)^*$ be an mso-labelling, our goal is to show that $f \circ \ell$ can be realized by a $k + 1$-nested transducer. We extend $\ell$ naturally to $\ell : (\Sigma \times 2^Q)^* \rightarrow (\Sigma \times L \times 2^Q)^*$, just by ignoring the $2^Q$ component. Using the induction assumption, we can obtain $T_k'$ a $k$-nested transducer realizing $f_k \circ \ell$.

To obtain the result, we use the construction from [Engelfriet and Hoogeboom 2001, Lemma 6] which shows that 1-pebble transducers with mso look-around are as expressive as 1-pebble transducers. Thus we can define a transducer $T_{k+1}'$ which simulates $T_{k+1}$ over words in $\Sigma^*$, using the mso look-around. This transducer can call $T_k'$ at the right moments which itself simulates $T_k$ over words in $\Sigma^*$, and thus $f \circ \ell$ can be realized by a $k + 1$-pebble transducer. \hfill \square

The next claim says that if a $k + 1$-nested transducer $\langle T_1, \ldots, T_{k+1}\rangle$ only makes a bounded number of calls to $T_k$, then one nesting is superfluous. Intuitively, instead of calling transducer $T_k$, transducer $T_{k+1}$ can simulate it since it only needs to do it a bounded number of times.

Claim 22. Let $\langle T_1, \ldots, T_{k+1}\rangle$ be a nested transducer realizing a function $f$, such that $T_{k+1}$ is bounded in call$_k$. Then $f$ can be realized by a $k$-nested transducer.

Proof. Let $T = (T_1, \ldots, T_{k+1})$ be a nested transducer realizing a function $f$, such that $T_{k+1}$ is bounded in call$_k$. Let $N$ be such that the number of call$_k$ output over any word is $\leq N$. For all $i \in \{1, \ldots, N\}$ there is a formula $\phi_i(x)$ such that for any word $w$, $w \models \phi_i(j)$ if and only if the $i^{th}$ call$_k$ in the run of $T_{k+1}$ over $w$ is output at position $j$. We thus define the associated mso-labelling $\ell$ which tells at each position of $w$ the subset of $\{1, \ldots, N\}$ of call$_k$ output by $T_{k+1}$ at this position. Let $g = f \circ \ell^{-1}$ denote the function $f$ extended to labelled words just by ignoring the labelling. We define a $k$-nested transducer $\langle T_1', \ldots, T_k' \rangle$ realizing $g$, which simulates $T$ using the extra labelling information.

Let us describe the behavior of transducer $T_k'$; it simulates $T_{k+1}$ and has an additional counter, initialized at 0, which
counts how many call_k have been output. Instead of outputting call_k, it increments the counter value from let us say i to i + 1, keeps in memory the current state q of Tk+i. Then, it simulates Tk using the fact that some position is labelled by i + 1 and that it has q in memory. Once the runs of Tk is done, it resumes the run of Tk+i in state q at the position labelled with i + 1.

We have provided a k-nested transducer realizing g, and from Proposition 21 the function g ◦ ℓ = f ◦ ℓ−1 ◦ ℓ = f can also be realized by a k-pebble transducer.

\[\text{Claim 23. Let } (M, \cdot) \text{ be a monoid and } \mu : \Sigma^* \to M \text{ be a morphism. Let } w_1, w_2, w_3 \in \Sigma^* \text{ such that there exists } x, y, z, t, e, f \in M \text{ satisfying:}
\]

- \[\mu(w_1 w_2) = x \cdot e \text{ and } \mu(w_3) = e \cdot y\]
- \[\mu(w_1) = z \cdot f \text{ and } \mu(w_2 w_3) = f \cdot t\]
- e and f are idempotent.

For every u, v ∈ \Sigma^* such that \[\mu(u) = e \text{ and } \mu(v) = f\] we have that:

- \[\mu(w_1 u w_2) = x \cdot e\]
- \[\mu(w_2 u w_3) = f \cdot t\]

\[\text{Proof. We have that } \mu(w_1 v) = z \cdot f \cdot f = z \cdot f = \mu(w_1). \text{ Thus } \mu(w_1 u w_2) = \mu(w_1 v) \cdot \mu(w_3) = \mu(w_1) \cdot \mu(w_3) = \mu(w_1 w_3) = x \cdot e. \text{ We proceed in the same way for the other equality.} \]

\[\square\]

The following lemma is the technical core of this article. It basically says that the domain of a k-pebble transduction can be decomposed into two parts: one part where any word can be pumped in such a way that a cause, and this leads to a growth in \(\Theta(n^k)\), and another part where the transduction can be realized with only \(k-1\) pebbles. The result given in the lemma actually needs to be a bit stronger than that, in order to make the induction work.

\[\text{Lemma 24 (Dichotomy Lemma). Let } T = \langle T_1, \ldots, T_k \rangle \text{ be a nested transducer over input alphabet } \Sigma \text{ realizing a function } f. \text{ There exists a morphism in a finite monoid } \mu : (\Sigma^+) \to M \text{ and a set } \mathcal{P} \subseteq P_k \text{ such that:}
\]

- For any \(w \in \Sigma^* \) with \((w_0, x_1, w_1, \ldots, x_k, w_k)\) a \(k\)-factorization according to an element of \(P\), we have \(f(w_0 x_1^n w_1 \cdots x_k^n w_k) = \Theta(n^k)\)
- \(f\) restricted to words without \(k\), \(r\)-factorization according to any element of \(P\) can be realized by a \(k-1\)-nested transducer.

\[\text{Proof. This is shown by induction on } k. \text{ For } k = 1, \text{ it is a consequence of the proof of Lemma 18, with the convention that a bounded regular function is a } 0 \text{-nested transduction. Indeed any factorization according to a producing triple yields a linear growth. Conversely let } L_r \text{ be the language of words without any } 1, r\text{-factorization according to any producing tuple, then in any } 1, r\text{-factorization one can remove one of the idempotent factors without changing the output size. According to Claim 20, there is an integer } d \text{ such that any word larger than } d \text{ has a } 1, r\text{-factorization. Thus } |f(w)|, w \in L_r = |\{f(w) | w \in \Sigma^{<d}\}|, \text{ which means that } f \text{ restricted to } L_r \text{ is bounded.}
\]

We assume that the lemma holds for \(k\), let us show that it holds for \(k + 1\). Let \(T = \langle T_1, \ldots, T_k, T_{k+1} \rangle \) be a nested transducer realizing a function \(f : \Sigma^* \to \Gamma\), and let \(r > 0\). Let \(T_k = \langle T_1, \ldots, T_k \rangle \) and let \(g_k : \Sigma_k^\ast \to \Gamma^\ast\) be the function realized by \(T_k\). We consider \(\mu_{call_k} : \Sigma_k^\ast \to N\) the call_k-transition morphism of \(T_{k+1}\).

Let us apply the induction assumption to \(T_k\) and let \(\mu : (\Sigma_{k+1})^\ast \to M \) be given as in the lemma. For any \(s > 0\) let \(S_s\) be a \(k-1\)-nested transducer realizing the function \(f_k\) restricted to words without any \(k, s\)-factorization according to elements of \(P\), and let \(\sigma_s\) denote the function it realizes. We choose \(s\) to be large enough, namely larger than \(\max(R(|M|, r + 1), R(|N|, r + 1))\).

The main idea of the proof is to modify the transducer \(T_{k+1}\) into a new transducer which only outputs call_k when it is absolutely necessary, i.e. when the word can be factorized in such a way that, by pumping idempotents, one can obtain an output in \(\Theta(n^k)\). Otherwise, we have according to the lemma that we can outsource the computation to a transducer with only \(k-1\) nesting.

Let us define a new transducer \(R_{k+1}\) which behaves as \(T_{k+1}\), except that at each step where it should output the letter call_k, it checks, using some regular look-around if the word has a \(k, s\)-factorization according to an element of \(P\). If yes then it outputs call_k normally, calling \(T_k\), otherwise it calls \(S_s\) instead. Note that the head movement of \(R_{k+1}\) is the same as the head movement of \(T_{k+1}\).

The look-around is implemented by an \(\text{MSO}\) labelling \(\ell\) which labels each position by additional information. Let \(L = \mathcal{Q}_{k+1} \to \{S_s, T_k\}\) be the labelling alphabet, then \(\ell : (\Sigma^*)^\ast \to (\Sigma \times L)^\ast\) is defined as follows: Let \(w \in (\Sigma^*)^\ast\), the word \(z = \ell(w)\) has the same size as \(w\) and \(z[i] = (w[i], h)\) with \(h(q) = T_k\) if and only if the word obtained by replacing \(w[i]\) with \(\frac{[w[i]]}{[q]}\) has a \(k, s\)-factorization according to an element of \(P\). Let \(\Lambda = \Sigma \times L\) in the following. Transducer \(R_{k+1}\) thus reads words over alphabet \(\Lambda\).

\[\text{Claim 25. Let } u = \ell(v) \text{ be in } \Lambda^\ast. \text{ Let us consider } (w_0, x_1, \ldots, x_s, w_1) \text{ a } 1, s\text{-factorization of } v \text{ in } \mu_{call_k}. \text{ There exists } i, j \in \{1, \ldots, s\} \text{ such that the following holds. Let } y = x_1 \cdots x_{i-1}, z = x_i \cdots x_j, t = x_{j+1} \cdots x_s, \text{ let } u_n = w_0 y^n z w_1, \text{ then there exist } \alpha, \beta, \gamma \in \Lambda^\ast \text{ such that } u_n = \ell(u_n) = \alpha \beta \gamma, \text{ for all } n \in \mathbb{N}.
\]

Claim 25. Since \(s\) has been chosen large enough, we can choose \(i, j\) so that \(x_1 \cdots x_s\) can be decomposed into \(r\) consecutive identical non-empty idempotent factors according to \(\mu\). Using Claim 23, we see that pumping a factor of \(u\) which is idempotent for \(\mu\), we do not affect the existence of a \(k, s\)-factorization of a word. \(\square\)
From the above claim, we have that one can pump an idempotent of \( \mathcal{N} \) as without affecting the labelling, as long as this idempotent appears at least \( s \) times.

We now consider the producing triples of \( R_{k+1} \). If a word can be factorized into a producing triple of \( R_{k+1} \) this means it can also be factorized according to a tuple of \( P \). However the two factorizations need not be compatible as in the following picture.

Here the red factor represents the idempotent in the producing triple of \( R_{k+1} \). The blue factors correspond to the idempotents of the factorization according to \( \mu \). The idea is that if we ask the red factor to repeat at least three times, then we can be sure that the middle factor does not intersect with the blue factors, since the blue factors cannot completely cover the red factor (because only one position can have a state label).

Let \( N_{k+1} \) denote the transition monoid of \( R_k \). We consider a slightly different monoid which gives the following information about an idempotent word: for any context the subset of tuples \( t \in P \) so that a call \( k \) is output using a factorization according to \( t \). Let us consider the monoid \( M_{k+1} \), which is equal to the product of \( M \) and \( N_{k+1} \). We define the set \( P_{k+1} \) as the tuples of \( M_{k+1} \) that correspond to triples of \( N_{k+1} \) that are producing triples and tuples \( t \) of \( P \), so that the producing triple outputs a call \( k \) on a position such that the correspond configuration has a \( k \), \( s \)-factorization according to \( t \). By construction and using Claim 25 we have that iterating the corresponding idempotents must result in a growth in \( \Theta(n^{k+1}) \).

The only thing remaining is to show that the function restricted to words without any \( k + 1 \), \( s' \)-factorization according to an element of \( P_{k+1} \) can be realized by a transducer with only \( k \) pebbles, for any \( s' \). Let \( s'' \) be chosen large enough, we define \( R_{k+1}^{s''} \) which behaves just as \( R_{k+1} \) except that it asks for \( k \), \( s'' \)-factorizations according to \( P \). We want to show that a word without any \( k + 1 \), \( s' \)-factorization according to \( P_{k+1} \) does not have any \( 1 \), \( r'' \)-factorization according to a producing tuple of \( R_{k+1}^{s''} \), which will conclude the proof from Claim 22. The construction of \( R_{k+1}^{s''} \) ensures that a call \( k \) is produced only when a \( k \), \( s'' \)-factorization is present, and if \( s'' \) is large enough, we can assume that it is a \( k \), \( s' \)-factorization according to a tuple of \( M_{k+1} \). Taking \( r'' \) large enough similarly ensures that the idempotent of the producing triple appears at least \( s' + 2 \) times which means that, using the same technique as above, the idempotents don’t overlap. In the end, we have shown that the language of words without any \( k + 1 \), \( s' \)-factorization according to a tuple of \( M_{k+1} \) is in particular included in the language of words without \( 1 \), \( r'' \)-factorization according to a producing triple of \( R_{k+1}^{s''} \), which means that the transduction restricted to this language can be realized with only \( k \) pebbles.

Finally we show that we can remove the look-ahead, using Proposition 21, concluding the proof. \( \Box \)

5 Proofs of the main theorems

In this section we use the dichotomy Lemma to show the results given in Section 2.

Theorem 10 (Characterization). A polyregular function is in \( O(n^k) \) if and only if it can be realized by a transducer with \( k \) pebbles.

Proof. From Proposition 9, we already have that \( k \)-pebble transductions are in \( O(n^k) \). To show the converse we use the dichotomy Lemma. Let \( T \) be a \( j \)-pebble transducer realizing a function \( f \in O(n^k) \). If \( j \leq k \) then \( f \) can be realized by a \( k \)-pebble transducer. If \( j > k \), then we only need to show that we can obtain a \( j - 1 \)-pebble transducer realizing \( f \). Using the dichotomy Lemma, we know there is a morphism \( \mu : (\Sigma_+,)^* \to M \) and a set \( P \) such that any word with a \( j \)-factorization according to an element of \( P \) can be pumped to obtain growth in \( \Theta(n^k) \). The transduction over all other words can be realized by a transducer with \( j - 1 \) pebbles. Since \( f \) is in \( O(n^k) \), \( P \) has to be empty. Thus \( f \) restricted to words without \( j \)-factorization according to any element of \( P \) is just \( f \). Hence \( f \) can be realized by a \( j - 1 \)-pebble transducer. Repeating this until \( j = k \) we show that \( f \) can be realized by a \( k \)-pebble transducer. \( \Box \)

Theorem 11 (Minimization). Given a polyregular function \( f \), one can compute an equivalent pebble transducer with the minimal number of pebbles. In particular, one can decide if a polyregular function is regular.

Proof. We only need to show given a \( k \)-pebble transducer realizing a function how to obtain, if possible, an equivalent transducer with \( k - 1 \) pebbles.

Let \( T = \langle T_1, \ldots, T_k \rangle \) be a pebble transducer over input alphabet \( \Sigma \) realizing a function \( f \).

Using the dichotomy Lemma, we know there is a morphism \( \mu : (\Sigma_+,)^* \to M \) and a set \( P \) such that any word with a \( k \)-factorization according to an element of \( P \) can be pumped to obtain growth in \( \Theta(n^k) \). The transduction over all other words can be realized by a transducer with \( k - 1 \) pebbles. Here we use the same kind of argument as in the proof of Theorem 10. Either \( P \) is non-empty and \( f \) is in \( \Theta(n^k) \) and thus cannot be realized by a \( k - 1 \)-pebble transducer, or \( P \) is empty and \( f \) can be realized by a \( k - 1 \)-pebble transducer, which concludes the proof. \( \Box \)

Theorem 12 (mso-dimension). A word-to-word function can be defined by an mso interpretation of dimension \( k \) if and only if it can be realized by a \( k \)-pebble transducer.

Proof. An mso interpretation \( T \) from \( \Sigma^* \) to \( \Gamma^* \) of dimension \( k \) is given by the following:
We have shown that the number of pebbles of a pebble transducer exactly coincides with the growth degree of polyregular functions. As a corollary, using [Bojańczyk et al. 2019], we obtain that mso interpretations of dimension \(k\) compute the same functions as \(k\)-pebble transducers. Moreover, we have shown how to minimize the number of pebbles of pebble transducers. The two results put together entail that we can decide if a polyregular function is regular. Overall we have obtained a quite satisfying understanding of the growth of polyregular functions, at least in two of the five different models presented in [Bojańczyk 2018; Bojańczyk et al. 2019].

One natural extension of this work would be to study the growth of polyregular functions in terms of other models. A first observation in that direction is that any function realized by a \(k\)-pebble transducer can be obtained as the composition of two functions: first the \(\text{power}_k\) function, generalizing the square function in [Bojańczyk 2018], which produces \(n^{k-1}\) copies of an input of size \(n\), each with \(k\) underlined positions, concatenated in lexicographic order; and second a regular function. This means that up to extending the basis of atomic functions with these \(\text{power}_k\) functions (on top of the square) the atomic functions, as well as the polyregular list functions, enjoy a characterization in terms of growth degree, like pebble transducers and mso interpretations. Only one model seems not to fit that pattern: for programs. The nesting depth of for loops in a for program gives an upper bound on the growth degree of the function. However regular functions require an unbounded nesting depth of for loops, since for programs are inherently one-way and they need nesting to simulate head-reversal. This means that the nesting depth of for programs yields a different hierarchy of polyregular functions, which may be worth investigating further.

A minimization procedure for pebble transducers, not in terms of pebbles but in terms of state space, is another interesting research question. However, it seems out of reach for now for at least two reasons: 1) equivalence of pebble transducers is not known to be decidable, and minimization procedures often come from canonical models, which would give an algorithm for testing equivalence 2) already for regular functions no canonical model procedure is known, while equivalence is decidable.

\footnote{The definition of mso interpretation used in [Bojańczyk et al. 2019] is slightly different: they do not make use of copying. This difference does not change expressiveness, up to increasing the dimension by 1.}

\section*{Conclusion}
We have shown that the number of pebbles of pebble transducers exactly coincides with the growth degree of polyregular functions. As a corollary, using [Bojańczyk et al. 2019], we obtain that mso interpretations of dimension \(k\) compute the same functions as \(k\)-pebble transducers. Moreover, we have shown how to minimize the number of pebbles of pebble transducers. The two results put together entail that we can decide if a polyregular function is regular. Overall we have obtained a quite satisfying understanding of the growth of polyregular functions, at least in two of the five different models presented in [Bojańczyk 2018; Bojańczyk et al. 2019].

One natural extension of this work would be to study the growth of polyregular functions in terms of the other
A transduction is in power_k if and only if it is definable by a transducer with k pebbles.