A reformulation of an ordinary differential equation

Oscar A. Barraza

Departamento de Ciencias Complementarias
Facultad de Ciencias Económicas, UNLP

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Abstract

The purpose of this note is to present a formulation of a given non-linear ordinary differential equation into an equivalent system of linear ordinary differential equations. It is evident that the easiness of such a procedure would be able to open a new way in order to calculate or approximate the solution of an ordinary differential equation. Some examples are presented.
1 Introduction

Let us consider an initial value problem (IVP) of first order like

\[
\begin{cases}
y' = \frac{dy}{dt} = \varphi(y) \\
y(t_0) = y_0,
\end{cases}
\]

(1)

or

\[
\begin{cases}
y' = \frac{dy}{dt} = \varphi(y) + g(t) \\
y(t_0) = y_0,
\end{cases}
\]

(2)

where the function \( \varphi(y) \) is supposed to be sufficiently smooth on an open interval \((t_0 - \delta, t_0 + \delta)\) for some \( \delta > 0 \).

Usually, problems like (1) or (2) are solved by means of known special methods according to the particular type of function \( \varphi \), for example, separation of variables, homogeneous equations, linear equations, Bernoulli equations, etc. However, the difficulty to find out an exact solution is related to some particular facts associated to the particular method of resolution. For instance, when the method of separation of variables is attempted to be applied the existence of a general primitive function of \( \frac{1}{\varphi(y)} \) is required; but the fact of computing such a primitive is not a trivial challenge.

The goal in this work consists to transform problems like (1) or (2) into a following system of infinite linear ordinary differential equations

\[
X' = AX(t), \quad \text{or} \quad X' = AX(t) + b(t)
\]

(3)

such that \( X(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \end{pmatrix} \) in an infinite dimensional function vector,

\[
A = \begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is a real constant entries infinite dimensional matrix and
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\[ b(t) = \begin{pmatrix} g(t) \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \] is an other infinite dimensional function vector.

The organization of this work is as follows. The aim of Section 2 consists to display some particular transforms through some simple examples. Section 3 is devoted to expose the procedure to transform an initial value problem into a system of linear ordinary differential equations. In this section the equivalence between the solutions of the IVP and the system is shown. In Section 4 some suggestions about possible generalizations are presented.

2 Some examples

Before developing the details of the procedure, two examples are presented in order to show how this technique works.

Example 2.1. Let us present an easy example because it is an initial value problem where the ordinary differential equation is linear. That is,

\[ \begin{align*}
    y' &= \frac{dy}{dt} = y - t \\
    y(0) &= y_0,
\end{align*} \]

for some real number \( y_0 \).

- It is straightforward to compute the exact solution because the ODE is linear. Then the general solution is

\[ y(t) = K e^t + t + 1, \]

for an arbitrary real constant \( K \). Using the initial condition the constant is \( K = y_0 - 1 \). So, the solution of the IVP is

\[ y = (y_0 - 1) e^t + t + 1. \]

- Let us present the transformation putting

\[ x_0(t) = y(t) \]
then
\[ x'_0(t) = y'(t) = y(t) - t = x_1(t) \quad \therefore x'_0 = x_1. \]

So,
\[ x'_1(t) = (y(t) - t)' = y'(t) - 1 = y(t) - t - 1 = x_2(t) \quad \therefore x'_1 = x_2. \]

One more step
\[ x'_2(t) = (y(t) - t - 1)' = y'(t) - 1 = y(t) - t - 1 = x_3(t) \quad \therefore x'_2 = x_3. \]

Then, in general for all integer \( j \geq 0 \) it is deduced that
\[ x'_j = x_{j+1}. \]

In this way the matrix system is
\[
X'(t) = \begin{pmatrix}
  0 & 1 & 0 & 0 & \cdots \\
  0 & 0 & 1 & 0 & \cdots \\
  0 & 0 & 0 & 1 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} X(t) + \begin{pmatrix}
  -t \\
  0 \\
  0 \\
  \vdots
\end{pmatrix}
\]

= \[ A \]

= \[ b(t) \]

It is recalled that the exponential matrix of a matrix \( B \) belonging to a Banach algebra \( \mathcal{A} \) is defined by
\[ e^B = I + B + \frac{1}{1!}B^2 + \frac{1}{2!}B^3 + \cdots = \]
\[ = I + \sum_{j=1}^{+\infty} \frac{1}{j!}B^j = \lim_{n \to +\infty} S_n \quad (5) \]

where \( I \) is the identity matrix and \( S_n = I + \sum_{j=1}^{n-1} \frac{1}{j!}B^j \) is the \( n \)th-partial sum. This series is absolutely convergent in the Banach algebra \( \mathcal{A} \) because on one side
\[ \|B^k\| \leq \|B\|^k \]
for all integer \( k \geq 1 \), and then
\[ \|e^B\| \leq 1 + \sum_{j=1}^{+\infty} \frac{1}{j!}\|B\|^j = e^{\|B\|}, \]
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showing that the exponential matrix $e^B$ is bounded in the norm of the Banach algebra $\mathcal{A}$ for all matrix $B \in \mathcal{A}$. On the other side the sequence of the partial sums $\{S_n\}_n$ is Cauchy convergent. For the details the reader is referred, for instance, to ([5]).

Taking into account the variation of constants formula, the solution of the previous linear system is

$$X(t) = e^{A.t}.X(0) + \int_0^t e^{A(t-s)}.b(s) \, ds$$

and, as a consequence, the solution of the given IPV is then

$$y(t) = x_0(t) = (1\text{st. row of } X(t)) =$$

$$= (1\text{st. row of } e^{A.t}), \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} + \int_0^t (1\text{st. row of } e^{A(t-s)}), \begin{pmatrix} -s \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \, ds$$

$$= c_0 + c_1 t + \frac{1}{2!} c_2 t^2 + \frac{1}{3!} c_3 t^3 + \cdots + \int_0^t (-s) \, ds =$$

$$= c_0 + c_1 t + \frac{1}{2!} c_2 t^2 + \frac{1}{3!} c_3 t^3 + \cdots - \frac{s^2}{2} \bigg|_0^t.$$  

$$= c_0 + c_1 t + \frac{1}{2!} c_2 t^2 + \frac{1}{3!} c_3 t^3 + \cdots - \frac{t^2}{2}.$$  

This implies that

$$x_0(0) = c_0 = y_0$$

and that from the equations of the system

$$x_j(0) = \frac{d^j x_0}{dt^j}(0) = \begin{cases} c_j, & \text{for } j \geq 1 \text{ and } j \neq 2, \\ c_2 - 1, & \text{for } j = 2, \end{cases}$$

so, the solution is written as

$$y(t) = y_0 + y_0 t + \left(\frac{1}{2!} y_0 - \frac{1}{2}\right) t^2 + \frac{1}{3!} (y_0 - 1) t^3 + \cdots + \frac{1}{j!} (y_0 - 1) t^j + \cdots$$

$$= (y_0 - 1).e^t + t + 1.$$  

- The reader can observe that through both procedures the same solution is reached for all real $t$.  

Example 2.2.

\[
\begin{aligned}
y' &= \frac{dy}{dt} = y^2, \\
y(0) &= y_0.
\end{aligned}
\]  

(6)

• First of all, the general solution is presented. This solution can be easily computed by means of the separation of variables method. The initial value \(y_0\) must be different to zero in order to guarantee the nontrivial solution. Then, under the assumption \(y_0 \neq 0\), the exact solution of problem (6) defined for \(t \neq \frac{1}{y_0}\) is given by

\[
y(t) = \frac{y_0}{1 - y_0(t - t_0)} = \frac{y_0}{1 - y_0 t},
\]  

(7)

since \(t_0 = 0\). It is mandatory to keep the domain

\[
\left(-\frac{1}{y_0}, +\infty\right) \quad \text{if} \quad y_0 > 0
\]

or

\[
\left(-\infty, \frac{1}{|y_0|}\right) \quad \text{if} \quad y_0 < 0.
\]

For \(y_0 = 0\) the solution obtained is the trivial one \(y(t) = 0\), for all \(t\).

• Now, we show how to transform problem (6) into a system of infinite linear ordinary equations.

Let us define

\[
x_0(t) = y(t)
\]

with \(x_0(0) = y_0\). Then,

\[
x_1(t) := x_0'(t) = y'(t) = y^2(t),
\]

so

\[
x_1'(t) = 2y(t)y'(t) = 2y^3(t).
\]

The next variable \(x_2\) is defined by

\[
x_2(t) = y^3(t),
\]

and then

\[
x_2'(t) = 3y^2(t)y'(t) = 3y^4(t).
\]
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Continuing with the precedent scheme let us define

\[ x_3(t) = y'^4(t), \]

and then

\[ x'_3(t) = 4y'^3(t).y'(t) = 4y^5(t), \]

and so on.

The general expression in the present example is

\[ x_j(t) = (y(t))^{j+1}, \]

for all integer \( j \geq 0. \)

Thus the set of the infinite linear differential equations can be summarized by

\[ x'_{j-1}(t) = j.x_j(t), \quad \text{for all integer } j \geq 1, \]

with the initial condition \( x_0(t_0) = y_0. \) So, this system is expressed in its matrix form as

\[
\begin{pmatrix}
  x_1(t) \\
  2.x_2(t) \\
  3.x_3(t) \\
  4.x_4(t) \\
  \vdots
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 2 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 3 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 4 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
  x_0(t) \\
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  \vdots
\end{pmatrix}
= A \begin{pmatrix}
  x_0(t) \\
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  \vdots
\end{pmatrix}
\]

where

\[
X(t) = \begin{pmatrix}
  x_0(t) \\
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  \vdots
\end{pmatrix}
\]

and

\[
A = \begin{pmatrix}
  0 & 1 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 2 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 3 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 4 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]
The general solution of system (8) has the form

$$X(t) = e^{At},$$

where

$$
\begin{pmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  \vdots
\end{pmatrix}
$$

is the vector of arbitrary constants.

Finally, taking into account (9), (5) and each one of the powers of the matrix $A$, that is

$$A^2 = \begin{pmatrix}
  0 & 0 & 2 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 6 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 12 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 0 & 20 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

$$A^3 = \begin{pmatrix}
  0 & 0 & 0 & 6 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 24 & 0 & \cdots \\
  0 & 0 & 0 & 3 & 0 & 60 & \cdots \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

and so on, the general solution of problem (9) is given by

$$y(t) = x_0(t) = (1\text{st. row of } e^{At}) \cdot \begin{pmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  \vdots
\end{pmatrix} =$$

$$= \left( \text{1st. row of } \left( I + \sum_{j=1}^{+\infty} \frac{1}{j!}(At)^j \right) \right) \cdot \begin{pmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  \vdots
\end{pmatrix} =$$
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\[ \begin{pmatrix} \text{1st. row of } \mathbf{I} + \sum_{j=1}^{+\infty} \frac{1}{j!} \text{(1st. row of } (\mathbf{A})^j_i) i^j \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = \]

\[ = c_0 + c_1 t + \frac{1}{2!} c_2 t^2 + \frac{1}{3!} c_3 t^3 + \cdots + \frac{1}{j!} c_j t^j + \cdots = \]

\[ = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_j t^j + \cdots \]

Since the initial condition satisfies \( y_0 = y(0) = x_0(0) = c_0 \) and for all integer \( j \geq 0 \) it is clear that \( x_j(t) = (y(t))^{j+1} \), it is deduced that \( c_j = x_j(0) = (y(0))^{j+1} = c_0^{j+1} \) for all \( j \geq 1 \). As a consequence the solution calculated above takes the form

\[ y(t) = c_0 + c_0^2 t + c_0^3 t^2 + c_0^4 t^3 + \cdots + c_0^{j+1} t^j + \cdots = \]

\[ = c_0 \left[ 1 + c_0 t + (c_0 t)^2 + (c_0 t)^3 + \cdots + (c_0 t)^j + \cdots \right] = \]

\[ = c_0 \cdot \frac{1}{1 - c_0 t} = \frac{y_0}{1 - y_0 t} \tag{10} \]

defined under the assumption \( |y_0 t| < 1 \). Note that in the open interval \( \left( -\frac{1}{|y_0|}, \frac{1}{|y_0|} \right) \) solutions \( (7) \) and \( (11) \) agree each to other. According to the sign of \( y_0 \), the domain of this solution could be extended to \(+\infty\) or to \(-\infty\).

For the special case where \( y_0 = 0 \), the solution by the series expansion is the zero solution as before.

Example 2.3.

\[ \left\{ \begin{array}{c}
 y' = \frac{dy}{dt} = e^y \\
 y(0) = y_0,
\end{array} \right. \tag{11} \]

As in the previous example the solution of problem \( (10) \) can be rapidly computed by means of the separation of variables method to obtain for all initial value \( y_0 \) that

\[ y(t) = -\ln(e^{-y_0} - t), \tag{12} \]
Its domain, for \( y_0 \) given, is defined by the condition \( e^{-y_0} - t > 0 \) which is equivalent to \( t < e^{-y_0} \), that is \( e^{y_0} \cdot t < 1 \).

- Let us pass to the transformation of problem (XII) into a system of infinite linear ordinary equations.

Let us define

\[ x_0(t) = y(t) \]

with \( x_0(0) = y_0 \). So

\[ x_1(t) := x'_0(t) = y'(t) = e^{y(t)}, \]

and

\[ x'_1(t) = e^{y(t)} \cdot y'(t) = e^{2y(t)}. \]

Next the variable \( x_2 \) is defined by

\[ x_2(t) = e^{2y(t)}, \]

and then

\[ x'_2(t) = 2 \cdot e^{2y(t)} \cdot y'(t) = 2e^{3y(t)}. \]

Let us continue defining

\[ x_3(t) = e^{3y(t)}, \]

thus

\[ x'_3(t) = 3 \cdot e^{3y(t)} \cdot y'(t) = 3e^{4y(t)}, \]

and so on.

The general expression in this third example is

\[ x_j(t) = e^{jy(t)}, \]

for all integer \( j \geq 1 \).

Therefore the set of the infinite linear differential equations can be briefly expressed by

\[ x'_{j-1}(t) = (j - 1) \cdot x_j(t), \quad \text{for all integer} \ j \geq 1, \]
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with the initial condition \( x_0(t_0) = y_0 \). Consequently, this system is expressed in its matrix form as

\[
X'(t) = \begin{pmatrix}
x_1(t) \\
x_2(t) \\
2.x_3(t) \\
3.x_4(t) \\
\vdots
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 2 & \cdots \\
0 & 0 & 0 & 0 & 3 & \cdots \\
\vdots
\end{pmatrix}
\begin{pmatrix}
x_0(t) \\
x_1(t) \\
x_2(t) \\
x_3(t) \\
\vdots
\end{pmatrix}
= A
\]

where

\[
X(t) = \begin{pmatrix}
x_0(t) \\
x_1(t) \\
x_2(t) \\
x_3(t) \\
\vdots
\end{pmatrix}
\]

and

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 2 & \cdots \\
0 & 0 & 0 & 0 & 3 & \cdots \\
\vdots
\end{pmatrix}
\]

The general solution of system (13) has the form

\[
X(t) = e^{At} \begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
\vdots
\end{pmatrix},
\]

(14)

where \( \begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
\vdots
\end{pmatrix} \) is the vector of arbitrary constants.
Finally, considering (14), (15) and each one of the powers of the matrix $A$, that is

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 6 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 6 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 24 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and so on, the general solution of problem (11) is given by

$$y(t) = x_0(t) = \left(1\text{st. row of } e^{At}\right) \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = \left(1\text{st. row of } I + \sum_{j=1}^{\infty} \frac{1}{j!} (At)^j \right) \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = \left(1\text{st. row of } I + \sum_{j=1}^{\infty} \frac{1}{j!} \left(1\text{st. row of } (A)^j \right) t^j \right) \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = c_0 + c_1 t + \frac{1}{2!} c_2 t^2 + \frac{1}{3!} 2!c_3 t^3 + \cdots + \frac{1}{j!} (j-1)! c_j t^j + \cdots = c_0 + c_1 t + \frac{c_2}{2} t^2 + \frac{c_3}{3} t^3 + \cdots + \frac{c_j}{j} t^j + \cdots.$$
Taking into account that the initial condition is \( y_0 = y(0) = x_0(0) = c_0 \) and that \( x_j(t) = e^{jyt(t)} \) for all integer \( j \geq 1 \), it is deduced that \( c_j = x_j(0) = e^{jy_0} = e^{jy_0} \) for all \( j \geq 1 \). Therefore the solution computed above takes the form

\[
y(t) = y_0 + e^{y_0}t + \frac{1}{2}e^{2y_0}t^2 + \frac{1}{3}e^{3y_0}t^3 + \cdots + \frac{1}{j}e^{jy_0}t^j + \cdots = \\
= y_0 + (e^{y_0}t) + \frac{1}{2}(e^{y_0}t)^2 + \frac{1}{3}(e^{y_0}t)^3 + \cdots + \frac{1}{j}(e^{y_0}t)^j + \cdots = \\
= y_0 - \ln (1 - e^{y_0}t) = y_0 - \ln [e^{y_0} \cdot (e^{-y_0} - t)] = \\
= y_0 - y_0 - \ln (e^{-y_0} - t) = -\ln (e^{-y_0} - t),
\]

(15)

provided that \( |e^{y_0}t| < 1 \). In the open interval \((-e^{-y_0}, e^{-y_0})\) solutions (12) and (15) are equal and the last one (15) can be extended to the interval \((-\infty, e^{-y_0})\), that is \( e^{y_0}t < 1 \) as in (12).

3 Transforming an initial value problem into a system of linear ordinary differential equations

The aim of this section is to establish the relationship between the solution of an initial value problem of kind (1) or (2) and the solution of a system of infinite linear differential equations of kind (3).

Let us begin with a necessary condition to be satisfied by the solution \( X(t) \) of a system of infinite linear differential equations (3)

\[
X' = AX(t) + b(t)
\]

such that \( X(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \end{pmatrix} \) is an infinite dimensional real function vector,

\[
A = \begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is a real constant entries infinite
dimensional matrix and \( b(t) = \begin{pmatrix} g(t) \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \) is an other infinite dimensional real function vector as it was presented in the introduction. Let us suppose that \( a_{ij} = 0 \) for \( 0 \leq i \leq j < +\infty \) and for \( j + 1 < i < +\infty \), and that \( a_{j,j+1} \neq 0 \), for all integer \( j \geq 0 \). In this way the matrix \( A \) has the explicit form

\[
A = \begin{pmatrix}
0 & a_{01} & 0 & 0 & 0 & \cdots \\
0 & 0 & a_{12} & 0 & 0 & \cdots \\
0 & 0 & 0 & a_{23} & 0 & \cdots \\
0 & 0 & 0 & 0 & a_{34} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

and the equations of the system are clearly given by

\[
x_0' = a_{01} x_1 + g(t) \quad \text{(16)}
\]
\[
x_1' = a_{12} x_2 \quad \text{(17)}
\]
\[
x_2' = a_{23} x_3 \quad \text{(18)}
\]
\[
\ldots
\]
\[
x_j' = a_{j,j+1} x_{j+1} \quad \text{for } j \geq 0 \quad \text{(19)}
\]

Let us also assume that the function \( \varphi(y) \) is real analytic at the value \( y = y_0 \) and the function \( g(t) \) is real analytic at the value \( t = t_0 \). Therefore, there exists a unique solution \( y(t) \) in some open interval centered at the value \( t_0 \) for the IVP (2), which includes the IVP (1),

\[
\left\{ \begin{array}{l}
y' = \frac{dy}{dt} = \varphi(y) + g(t) \\
y(t_0) = y_0,
\end{array} \right.
\]

Extending the variation of constants formula for the finite dimensional case ( (2) or (3)), it is deduced in a straightforward way that the solution of the system is

\[
X(t) = e^{A(t-t_0)} C + \int_{t_0}^{t} e^{A(t-s)} b(s) \, ds \quad \text{(20)}
\]
where the infinite dimensional vector of the arbitrary real constants $\mathbf{C}$ coincides to the vector of the initial values $\mathbf{X}(t_0)$, that is

$$\mathbf{C} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = \mathbf{X}(t_0).$$

**Remark 3.1.** Minimal hypotheses on smoothness of the vector function operator $\mathbf{T}^1$ on a Banach space of infinite dimensional vector functions $\mathbf{X}(t)$, where $\mathbf{T}(\mathbf{X})(t) = \mathbf{A} \mathbf{X}(t) + \mathbf{b}(t)$ for the case in which $\mathbf{A}$ is a constant matrix, or $\mathbf{T}(\mathbf{X})(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{b}(t)$ for the case in which $\mathbf{A}(t)$ is a function matrix, guarantee a unique local fixed point of operator $\mathbf{T}$, and then a unique local solution of the former system of infinite ODE with the initial vector value $\mathbf{C}$. Besides, as is shown in ([5]), the exponential of an operator on a Banach algebra can be defined by symbolic calculus, and then it is differentiable satisfying the standard rule

$$\frac{d}{dt} \left( e^{\int_{t_0}^{t} \mathbf{A} \, ds} \right) = \mathbf{A}.e^{\int_{t_0}^{t} \mathbf{A} \, ds}.$$

As a direct consequence, equality (20) provides the solution for the linear system (3).

To reach the goal of this article which is the resolution of problem (2), the following equality has to be imposed

---

1It is usually known as the Picard fixed point theorem for abstract Banach spaces; see, for instance, ([5]). Alternatively, it is called the Contraction Mapping Principle on complete metric spaces; see, ([4]) or ([1]).
\[ \varphi(y(t)) + g(t) = y'(t) = x'_0(t) = \sum_{j=1}^{\infty} \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] c_j \frac{(t-t_0)^{j-1}}{(j-1)!} + g(t), \]
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so

\[ x_j(t) = \frac{1}{a_{j-1,j}^{-1}} x_{j-1}'(t) = \cdots = \frac{1}{\prod_{h=j}^{j-1} a_{k,k+1}} \sum_{h=j}^{+\infty} \prod_{k=0}^{h-1} a_{k,k+1} \right] c_h \cdot (t - t_0)^{h-j} = \frac{1}{\prod_{k=0}^{j-1} a_{k,k+1}} \]

And finally, by evaluating at \( t = t_0 \) it is obtained from (21) that

\[ \bullet c_0 = y(t_0) = y_0, \tag{23} \]

and that

\[ \bullet c_j = \frac{1}{\prod_{k=0}^{j-1} a_{k,k+1}} \frac{d^j}{dy} \varphi(y(t)) \bigg|_{t=t_0}, \tag{24} \]

for all \( j \geq 1 \), in coincidence with the construction of the coefficients of Taylor series (22) where

\[ \frac{d^j}{dy} \varphi(y(t)) \bigg|_{t=t_0} = \prod_{k=0}^{j-1} a_{k,k+1} \cdot c_j. \]

The previous set of infinite equalities allows to compute the constants \( c_j \) for all integer \( j \geq 0 \) in terms of the function \( \varphi(y) \), the derivative of \( y(t) = x_0(t) \). In fact, the following statement has been proved.

**Theorem 3.2.** Let us suppose an infinitely derivable function \( g(t) \) at \( t = t_0 \) and the existence of the function \( y(t) \) solution of problem (2)

\[ \begin{cases} y'(t) = \frac{dy}{dt} & = \varphi(y(t)) + g(t) \\ y(t_0) = y_0, \end{cases} \]

for a function \( \varphi(y(t)) \) which is represented by its Taylor series about \( t = t_0 \)

\[ \varphi(y(t)) = \sum_{j \geq 1} \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] \cdot c_j \cdot \frac{(t - t_0)^{j-1}}{(j - 1)!}. \]
on the open interval \( |t - t_0| < \delta \) defined for some real number \( \delta > 0 \).

Then, the family of infinite functions given by the following relations for \( x_0(t) = y(t) \)

\[
x_0'(t) = \varphi(y(t)) + g(t) = a_{01}.x_1(t),
\]
\[
x_1'(t) = a_{12}.x_2(t),
\]
\[
x_2'(t) = a_{23}.x_3(t),
\]
\[
\vdots
\]
\[
x_j'(t) = a_{j,j+1}.x_{j+1}(t), \quad \text{for all integer } j \geq 1
\]
defines an infinite dimensional function vector

\[
\mathbf{X}(t) = \\
\begin{pmatrix}
  x_0(t) \\
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  \vdots
\end{pmatrix}
\]

which is the solution of the system of infinite linear ordinary differential equations

\[
\mathbf{X}'(t) = \mathbf{A}.\mathbf{X}(t) + \mathbf{b}(t)
\]

with the initial vector value

\[
\mathbf{X}(t_0) = \\
\begin{pmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  \vdots
\end{pmatrix}
\]

Proof. The assumptions about functions \( g(t) \) and \( \varphi(y(t)) \) allows to establish that the (unique) solution \( y(t) \) of the initial value problem on an open interval centered at the value \( t = t_0 \) is infinitely differentiable there thanks to the application of the chain rule. In this way the existence of the constants \( c_j \), for all integer \( j \geq 0 \), is guaranteed according to the family of formulas preceding this theorem. Taking into account these computations, it is straightforward to conclude that the function vector \( \mathbf{X}(t) \) is the solution
of system (3) satisfying the initial conditions \( \mathbf{X}(t_0) = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \), as it was the purpose of this result.

\[ \square \]

**Theorem 3.3.** Conversely, let \( \mathbf{C} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \) be an infinite dimensional real vector and if

1. the function \( g(t) \) is infinitely differentiable on \( t = t_0 \),

2. the function \( \varphi(y) \) is infinitely differentiable for all real \( y \) such that \( |y - c_0| < \beta \) for some real number \( \beta > 0 \),

3. the power series

\[
\sum_{j \geq 1} \left[ \frac{1}{\prod_{k=0}^{j-1} a_{k,k+1}} \right] . c_j . \frac{(t - t_0)^{j-1}}{(j-1)!}
\]

converges on the open interval \( |t - t_0| < \alpha \) for some real number \( \alpha > 0 \), and

4. if, besides, system (3)

\[
\mathbf{X}'(t) = \mathbf{A} \cdot \mathbf{X}(t) + \mathbf{b}(t)
\]

with initial vector value \( \mathbf{X}(t_0) = \mathbf{C} \) admits as solution the infinite dimensional function vector

\[
\mathbf{X}(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \end{pmatrix}
\]
then, its first component \( y(t) = x_0(t) \) is the solution of the initial value problem (3) on an appropriated open interval \( |t - t_0| < \delta \) for some real number \( \delta > 0 \), where \( y_0 = c_0 \).

**Proof.** In order to verify this theorem, let us define \( y(t) = x_0(t) \), the first component of the vector \( X(t) \) solution of system (3)

\[
X'(t) = A.X(t) + b(t)
\]

with initial vector value \( X(t_0) = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} := C \).

As it was mentioned above, thanks to the variation of constants method, the solution of this system has the form

\[
X(t) = e^{A.(t-t_0)} . C + \int_{t_0}^{t} e^{A.(t-s)}.b(s) \, ds,
\]

since \( A \) is a constant matrix and then it commutes with the exponential one \( e^{A.t} \).

Therefore, the first component \( y(t) = x_0(t) \) of vector \( X(t) \), is given by the first row rule; that is,

\[
y(t) = x_0(t) = c_0 + a_{01}.c_1(t - t_0) + a_{01}.a_{12}.c_2 \frac{(t - t_0)^2}{2!} + a_{01}.a_{12}.a_{23}.c_3 \frac{(t - t_0)^3}{3!} + \cdots +
\]

\[
+ \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] c_j \frac{(t - t_0)^j}{j!} + \cdots + \int_{t_0}^{t} g(s) \, ds.
\]

Or more briefly

\[
y(t) = c_0 + \sum_{j=1}^{\infty} \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] c_j \frac{(t - t_0)^j}{j!} + \int_{t_0}^{t} g(s) \, ds,
\]
which has sense for all real $t$ in the interval of convergence $|t - t_0| < \alpha$ of the last series, for some real number $\alpha > 0$.

First, by evaluating on $t_0$ it is clear that $y(t_0) = c_0 = y_0$.

Next, by differentiating term to term the series with respect to the variable $t$ in the convergence interval $|t - t_0| < \alpha$ and applying the fundamental calculus theorem it is evident that

$$y'(t) = \sum_{j=1}^{+\infty} \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] c_j \frac{(t - t_0)^{j-1}}{(j-1)!} + g(t).$$

By hypothesis, function $g(t)$ is infinitely differentiable on $t = t_0$ and also the last power series on the open interval $|t - t_0| < \alpha$. Thus, the function $y(t)$ is infinitely differentiable on $t = t_0$ as well, so by applying the chain rule indefinitely to the function composition $\varphi(y(t))$ on $t_0$ the identities \[24\] are rapidly recovered. As a consequence the series

$$\sum_{j=1}^{+\infty} \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] c_j \frac{(t - t_0)^{j-1}}{(j-1)!}$$

becomes the Taylor series of the function $\varphi(y(t))$ on an appropriated interval $|t - t_0| < \delta$, for some real number $\delta$ depending on $t_0$, on $\alpha$, on $\beta$ and on the values of $g(t)$ on a neighborhood of $t_0$. This fact shows that $y'(t) = \varphi(y(t)) + g(t)$ proving that $y(t)$ is the solution of the initial value problem \[24\], as desired.

\[\square\]

**Remark 3.4.** According to a particular situation, the domain of the definition of the solution concerning the power series could be extended to a larger interval. In fact, this was observed in the two previous examples.

### 3.1 An special extension of the method

Let us now continue with the IVP

$$\begin{cases}
y'(t) = \frac{dy}{dt} = \Phi(y(t),t) = \varphi(y(t)).f(t) + g(t) \\
y(t_0) = y_0,
\end{cases}$$

which its corresponding system of infinite linear differential equations is supposed to be

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{b}(t)$$

\[25\]
such that as before \( X(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \end{pmatrix} \) is an infinite dimensional real function vector,

\[
A(t) = \begin{pmatrix} 0 & a_{01} f(t) & 0 & 0 & \cdots \\ 0 & 0 & a_{12} f(t) & 0 & \cdots \\ 0 & 0 & 0 & a_{23} f(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = A.f(t) \tag{26}
\]

is an infinite dimensional matrix with real function entries, where the matrix

\[
A = \begin{pmatrix} 0 & a_{01} & 0 & 0 & \cdots \\ 0 & 0 & a_{12} & 0 & \cdots \\ 0 & 0 & 0 & a_{23} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]

has constant entries and \( b(t) = \begin{pmatrix} g(t) \\ 0 \\ 0 \\ \vdots \end{pmatrix} \)

is an other infinite dimensional real function vector. Thus, the system is explicitly given by

\[
x'_0 = a_{01}. f(t) x_1 + g(t) \tag{27}
\]

\[
x'_1 = a_{12}. f(t) x_2 \tag{28}
\]

\[
x'_2 = a_{23}. f(t) x_3 \tag{29}
\]

\[
x'_j = a_{j,j+1}. f(t) x_{j+1} \quad \text{for } j \geq 1
\]

Since the matrix \( A(t) = A.f(t) \) and the integral \( \int_{t_0}^t A(s) \, ds = A \int_{t_0}^t f(s) \, ds \) commute, then \( A(t) \) and the exponential matrix \( e^{\int_{t_0}^t A(s) \, ds} = e^{A \int_{t_0}^t f(s) \, ds} \)
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commute as well, and thanks to the variation of constants formula the solution of the previous linear system can be expressed as

\[
X(t) = e^{\int_{t_0}^t A(s) ds} X(t_0) + \int_{t_0}^t e^{\int_{s}^t A(r) dr} b(r) dr = e^{A \cdot \int_{t_0}^t f(s) ds} X(t_0) + \int_{t_0}^t e^{A \cdot \int_{s}^t f(r) dr} b(r) dr.
\]

(31)

See, for instance, (2) or (3).

Thus, under little adaptations to this case, it is not difficult to deduced the corresponding extensions of the previous two theorems.

**Theorem 3.5.** Let us suppose an infinitely derivable function \( g(t) \) at \( t = t_0 \), the existence of the function \( y(t) \) solution of problem

\[
\begin{align*}
\frac{dy}{dt} &= \Phi(y(t), t) = \varphi(y(t)).f(t) + g(t) \\
y(t_0) &= y_0,
\end{align*}
\]

(32)

for a function \( \varphi(y(t)) \) which is represented by its Taylor series about \( t = t_0 \)

\[
\varphi(y(t)) = \sum_{j \geq 1} \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] \cdot c_j \cdot \frac{(t - t_0)^j}{(j - 1)!}
\]

on an open interval \( |t - t_0| < \delta \) defined for some real number \( \delta > 0 \), and an integrable function \( f(t) \) on the same interval.

Then, the family of infinite functions defined by the following relations

\[
x_0(t) = y(t)
\]

\[
x'_0(t) = \varphi(y(t)).f(t) + g(t) = a_{01}.f(t) x_1(t),
\]

\[
x'_1(t) = a_{12}.f(t) x_2(t),
\]

\[
x'_2(t) = a_{23}.f(t) x_3(t),
\]

\[
\ldots \ldots
\]

\[
x'_j(t) = a_{j,j+1}.f(t) x_{j+1}(t), \quad \text{for all integer } j \geq 1
\]
defines an infinite dimensional function vector

\[
X(t) = \begin{pmatrix}
  x_0(t) \\
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  \vdots
\end{pmatrix}
\]

which is the local solution of the system of infinite linear ordinary differential equations (25)

\[
X'(t) = A(t)X(t) + b(t)
\]

with the initial vector value

\[
X(t_0) = \begin{pmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  \vdots
\end{pmatrix},
\]

and where \( A(t) \) is given by (26).

**Theorem 3.6.** Conversely, let \( C = \begin{pmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  \vdots
\end{pmatrix} \) be an infinite dimensional real vector and if

1. the function \( g(t) \) is infinitely differentiable on \( t = t_0 \),
2. the function \( \varphi(y) \) is infinitely differentiable for all real \( y \) such that \( |y - c_0| < \beta \) for some real number \( \beta > 0 \),
3. the power series

\[
\sum_{j \geq 1} \left[ \prod_{k=0}^{j-1} a_{k,k+1} \right] .c_j .(t-t_0)^{j-1} \frac{1}{(j-1)!}
\]
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converges on the open interval \( |t - t_0| < \alpha \) for some real number \( \alpha > 0 \), and

4. if, besides, system (3)

\[
X'(t) = A.X(t) + b(t)
\]

with initial vector value \( X(t_0) = C \) admits as solution the infinite dimensional function vector

\[
X(t) = \begin{pmatrix}
x_0(t) \\
x_1(t) \\
x_2(t) \\
x_3(t) \\ \\
\vdots
\end{pmatrix},
\]

then, its first component \( y(t) = x_0(t) \) is the solution of the initial value problem (2) on an appropriated open interval \( |t - t_0| < \delta \) for some real number \( \delta > 0 \), where \( y_0 = c_0 \).

In order to illustrate this extension, let us present the following example.

Example 3.7.

\[
\begin{align*}
y' &= \frac{dy}{dt} = y^2.t \\
y(0) &= y_0,
\end{align*}
\]

- The general solution is easily computed by means of the separation of variables method. If the initial value \( y_0 \neq 0 \) the solution is a nontrivial one; then, the exact solution of problem (33) is given by

\[
y(t) = \frac{2y_0}{2 - y_0.t^2}.
\]

Its domain is given by the condition

\[y_0.t^2 \neq 2\]

For \( y_0 = 0 \) the solution obtained is the trivial one \( y(t) = 0 \), for all \( t \), which can be included in the previous formula.
Let us pass to the transformed of problem (33) into a system of infinite linear ordinary equations.

Let us define

\[ x_0(t) = y(t) \]

with \( y_0 = x_0(0) = c_0 \). Then,

\[ x'_0(t) = y'(t) = y^2(t).t = t.x_1(t), \]

where

\[ x_1(t) = y^2(t). \]

So

\[ x'_1(t) = 2.y(t).y'(t) = 2t.y^3(t) = 2t.x_2(t), \]

where

\[ x_2(t) = y^3(t). \]

Then

\[ x'_2(t) = 3.y^2(t).y'(t) = 3t.y^4(t) = 3t.x_3(t), \]

and so on. In this way the general expression is

\[ x_j(t) = y^{j+1}(t) = x_0^{j+1}(t), \]

and

\[ x'_j(t) = (j + 1).y^j(t).y'(t) = (j + 1).t.y^{j+2}(t) = (j + 1).t.x_{j+1}(t), \]

for all integer \( j \geq 0 \).

Thus, the system is expressed in its matrix form as

\[
X'(t) = \begin{pmatrix}
t.x_1(t) \\
2t.x_2(t) \\
3t.x_3(t) \\
4t.x_4(t) \\
\vdots
\end{pmatrix} = \begin{pmatrix}
0 & t & 0 & 0 & 0 & \cdots \\
0 & 0 & 2t & 0 & 0 & \cdots \\
0 & 0 & 0 & 3t & 0 & \cdots \\
0 & 0 & 0 & 0 & 4t & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \begin{pmatrix}
x_0(t) \\
x_1(t) \\
x_2(t) \\
x_3(t) \\
\vdots
\end{pmatrix} = \mathbf{A}(t) \cdot X(t)
\]

\[ = \mathbf{A}(t).X(t) \]
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where

\[
X(t) = \begin{pmatrix}
x_0(t) \\
x_1(t) \\
x_2(t) \\
x_3(t) \\
\vdots
\end{pmatrix}
\]

and

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 3 & 0 & \cdots \\
0 & 0 & 0 & 0 & 4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The general solution of system (35) has the form

\[
X(t) = e^{\int_0^t A \cdot s \, ds} \begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
\vdots
\end{pmatrix},
\]

where \(\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
\vdots
\end{pmatrix}\) is the vector of arbitrary constants.

Finally, taking into account (36), (5), the matrix \(\int_0^t A \cdot s \, ds = A \cdot \int_0^t s \, ds = A \cdot \frac{t^2}{2}\) and each one of the powers of the matrix \(A\), that is

\[
A^2 = \begin{pmatrix}
0 & 0 & 2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 6 & 0 & \cdots \\
0 & 0 & 0 & 0 & 12 & \cdots \\
0 & 0 & 0 & 0 & 0 & 20 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
\[ A^3 = \begin{pmatrix} 0 & 0 & 0 & 6 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 24 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & 60 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \]

and so on, the general solution of problem (33) is given by

\[ y(t) = x_0(t) = \left( \begin{array}{c} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{array} \right) = \left( \begin{array}{c} 0 \\ 3 \\ 60 \\ 0 \end{array} \right) = \left( \begin{array}{c} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{array} \right) = \left( \begin{array}{c} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{array} \right) = \left( \begin{array}{c} 0 \\ 3 \\ 60 \\ 0 \end{array} \right) \]

Since the initial condition satisfies \( y_0 = y(0) = x_0(0) = c_0 \) and for all integer \( j \geq 0 \) it is clear that \( x_j(t) = (y(t))^{j+1} \), it is deduced that \( c_j = x_j(0) = (y(0))^{j+1} = c_0^{j+1} \) for all \( j \geq 1 \). As a consequence the
solution calculated above takes the form

\[ y(t) = y_0 + y_0^2 \left( \frac{t^2}{2} \right) + y_0^3 \left( \frac{t^2}{2} \right)^2 + y_0^4 \left( \frac{t^2}{2} \right)^3 + \cdots + y_0^{j+1} \left( \frac{t^2}{2} \right)^j + \cdots = \]

\[ = y_0 \left[ 1 + y_0 \left( \frac{t^2}{2} \right) + \left( \frac{y_0 t^2}{2} \right)^2 + \left( \frac{y_0 t^2}{2} \right)^3 + \cdots + \left( \frac{y_0 t^2}{2} \right)^j + \cdots \right] = \]

\[ = y_0 \cdot \frac{1}{1 - \left( \frac{y_0 t^2}{2} \right)} = \frac{2y_0}{2 - y_0 t^2}. \]  

(37)

defined under the assumption \( |y_0 t^2| < 2 \), that is \( t^2 < \frac{2}{|y_0|} \), if \( y_0 \neq 0 \).

Note that in the open interval \( -\sqrt{\frac{2}{|y_0|}} < t < \sqrt{\frac{2}{|y_0|}} \), solutions (34) and (37) agree each to other and solution (37) accepts an extension to all real number \( t \neq \pm \sqrt{\frac{2}{|y_0|}} \).

Besides, for \( y_0 = 0 \) the solution (37) is the zero series.

### 4 Some conclusions

It is impossible to deny the importance of the role of the differential equations in all the processes which involve a dynamical component of a model for continuous variables. These processes are illustrated in a large variety of domains related, for instance, to applications to Astronomy, Biology, Chemistry, Economy, Engineering and Physics.

The necessity to describe the behavior of the exact or approximated solutions of those dynamical models provides an understanding of the underlying processes. The fact of computing the exact solutions of a differential equation is a desirable purpose but in some cases it is a hard task to accomplish, in most of them it is impossible. So, finding alternative ways to solve ordinary differential equations are always good news or, at least, it could open some paths to obtain approximated solutions. For example, keeping in mind that the solution of the system

\[ X'(t) = A(t)X(t) + b(t) \]
with initial vector value \( X(t_0) = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = C \) is given by formula (31)

\[
X(t) = e^{M(t)}X(t_0) + \int_{t_0}^{t} e^{[M(t)−M(r)]}b(r)dr,
\]

where the matrix \( M(t) = \int_{t_0}^{t} A(s) ds \), one possibility consists in replacing the exponential matrix in that formula by its \( n \)-th partial sum. In this way the sequence of approximations to the desired solution is defined by

\[
X_0(t) = X(t_0) = C
\]

and for each integer \( n \geq 1 \)

\[
X_n(t) = \left( I + \sum_{j=1}^{n-1} \frac{[M(t)]^j}{j!} \right)X(t_0) + \int_{t_0}^{t} \left( I + \sum_{j=1}^{n-1} \frac{[M(t)−M(s)]^j}{j!} \right)b(s)ds.
\]

For the special case of a constant infinite dimensional matrix \( A(t) = A \), the matrix \( M(t) \) takes the form \( M(t) = \int_{t_0}^{t} A ds = (t − t_0).A \) and then the last sequence is reduced to

\[
X_n(t) = \left( I + \sum_{j=1}^{n-1} \frac{(t − t_0)^j}{j!}A^j \right)X(t_0) + \int_{t_0}^{t} \left( I + \sum_{j=1}^{n-1} \frac{(t − s)^j}{j!}A^j \right)b(s)ds,
\]

for each integer \( n \geq 1 \).

In particular, the method presented in this note which consists to transform a nonlinear differential equation into a system of infinite linear ordinary differential equations has the advantage to translate the difficulty to treat directly with nonlinearity into the problem of the treatment of the infinite quantity of linear ODE with constant coefficients or simple variable ones. The methodology of reducing the difficult of nonlinearity keeps a certain analogy between the methodology of reducing a \( n \)th-order ODE to a first-order system of \( n \) ODE because, in general, solving an isolated linear ODE with constant coefficients is easier than solving a nonlinear one.
However, the transformation of a nonlinear ODE to a system of infinite linear ODE’s is only developed here for the case of first-order differential equations; this is the first restriction of our method. A second limitation is due to the hypotheses about the smoothness of functions \( \varphi \) and \( g \) in problems (1) and (2) and the existence of the series expansion detailed in the theorem. The third constraint of this method is the form of the derivative \( y'(t) \) in the first two problems. A more general situation to be studied with this procedure in a next work could concern an ordinary differential equation like \( y'(t) = \varphi(y, t) \), for more general function \( \varphi(y, t) \).

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