The role of the impurity-potential range in disordered d-wave superconductors

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Abstract. We analyse how the range of disorder affects the localization properties of quasiparticles in a two-dimensional d-wave superconductor within the standard non-linear $\sigma$-model approach to disordered systems. We show that for purely long range disorder, which only induces intra-node scattering processes, the approach is free from the ambiguities which often beset the disordered Dirac-fermion theories, and gives rise to a Wess–Zumino–Novikov–Witten action leading to vanishing density of states and finite conductivities. We also study the crossover induced by inter-node scattering due to a short range component of the disorder, thus providing a coherent non-linear $\sigma$-model description in agreement with all the various findings of different approaches.

Keywords: disordered systems (theory), sigma models (theory)

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1. Introduction

The low-temperature quasiparticle transport in two-dimensional d-wave superconductors like cuprates is a fascinating issue due to the presence of four nodes in the energy spectrum of the Bogoliubov quasiparticles, around which the low-lying excitations have a Dirac-like dispersion. Within the self-consistent coherent phase approximation in the limit of weak disorder, spin and thermal conductivities are found to be related by a Wiedemann–Franz law and to acquire universal values which do not depend on the disorder strength [1].

Inclusion of quantum interference in the framework of the standard non-linear σ-model approach to disordered systems [2,3] leads to a variety of universality classes [4]–[11]. In the ‘generic’ case of short ranged non-magnetic impurities full localization of Bogoliubov quasiparticles is predicted [5].

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Nevertheless, with the only exceptions of those on YBCO(124) [12] and Pr$_{2-x}$Ce$_x$CuO$_4$ [13], experiments on cuprate materials like YBCO(123) [14–16], BSCCO(2212) [15] and LSCO [17] do not show any evidence of strong or even weak localization in the superconducting phase down to 0.1 K. Various physical effects may be invoked to explain the disagreement between theory and experiments.

For instance one may argue that the origin of the discrepancy is spin-flip scattering events, even though the systems are nominally free from magnetic impurities. Indeed, in the presence of spin flips, the non-linear $\sigma$-model predicts that quantum interference has a delocalizing effect [9]. Alternatively, or in addition, strong dephasing processes might set the temperature scale for the onset of localization effects below the experimentally accessed region. This question has not been settled yet [18]. Residual interactions among quasiparticles can also favour delocalization [9,11], even though, in the weak disorder limit, they are expected to be less effective since their coupling is proportional to the density of states which, in the Born approximation, is already very small.

Another possible explanation invokes the range of the impurity potential. In the case of purely long range disorder, forward processes dominate and scattering occurs mainly within each node. In the extreme case of intra-node scattering only, it has been shown [19,20] that the density of states behaves quite differently from the isotropic scattering case. In addition the eigenstates have been argued [20] to be delocalized, unlike for short range impurity potential. Even though real disorder will always have an isotropic component which provides scattering among all four nodes, and eventually drive the system to localization, one might argue that a large value of the intra-node with respect to the inter-node scattering could lower the crossover temperature for the appearance of localization precursor effects. A sizable amount of long range disorder has indeed been argued to be present in cuprates on the basis of STM and microwave conductivity experiments [21]. This is not surprising since superconducting cuprates are intrinsically disordered by the out-of-plane charge dopants which mainly provide a long range disordered potential. Further doping with iso-valent impurities which substitute in-plane Cu ions only adds a short range component on top of the pre-existing long range tail of the disordered potential.

The results in the presence of purely intra-node impurity scattering have been obtained [19,20] within an approach which is conceptually different from the standard non-linear $\sigma$-model approach to disordered systems. The latter starts from the Born approximation, i.e. from impurity-damped quasiparticles, and treats perturbatively what is beyond that, i.e. quantum interference effects on the diffusive motion. In contrast, the alternative methods used in [19,20] do not rely on the Born approximation but just map the action of ballistic nodal quasiparticles in the presence of disorder onto the action of one-dimensional (1D) fermions in the presence of an interaction, which is generated by the disorder average. Within the mapping, one of the two spatial dimensions transforms into the time coordinate of the 1D model while the other transforms into the 1D spatial coordinate. The final model is then analysed using Abelian and non-Abelian bosonization. The outcome of this analysis is that for purely intra-node or inter-opposite-node disorder, where essentially exact results can be obtained [19,20], the density of states is power-law vanishing at the chemical potential with an exponent which is disorder dependent in the former case and universal in the latter. In both cases the results suggest that a diffusive quasiparticle motion never sets in, namely quasiparticles move ballistically down to zero
energy. When the disorder also couples adjacent nodes, strong coupling arguments are invoked [20] which predict localization of quasiparticles and linearly vanishing density of states. Yet, even this case seems to suggest a scenario in which quasiparticle motion from ballistic motion turns directly into localized motion without crossing any intermediate diffusive regime.

The standard non-linear $\sigma$-model approach applies the replica trick to average over disorder from the outset. The resulting fermion interaction is then decoupled by introducing $Q$-matrix fields in terms of which an effective action is derived after integrating out the fermions. The saddle point solution is just the Born approximation, which provides, in the case of nodal Bogoliubov quasiparticles, a finite density of states and hence a finite damping. Finally, long wavelength transverse fluctuations are taken with respect to the saddle point leading to a non-linear $\sigma$-model action for the $Q$-matrix fields. However, unlike in conventional disordered systems, in this particular case an additional term may appear in the non-linear $\sigma$-model action, namely a so-called Wezz–Zumino–Novikov–Witten (WZW) term. It was actually argued [7, 20, 22] that two opposite nodes share the same WZW term, while two adjacent ones have opposite WZW terms. As a consequence, when the two pairs of opposite nodes are uncoupled by disorder the WZW term is effective and the $\beta$ function of the spin conductance flows towards an intermediate coupling fixed point, signalling a delocalized behaviour. In contrast, when disorder couples all nodes together, the two WZW terms cancel exactly and the non-linear $\sigma$-model no longer has protection against flowing towards a zero-conductance strong coupling regime characterized by a linearly vanishing density of states [6].

From the above discussion one might be led to conclude that the agreement between the conclusions drawn with either method is merely an accident which does not justify per se the conventional non-linear $\sigma$-model approach when dealing with Dirac fermions. The main objection against the conventional non-linear $\sigma$-model is that it is not appropriate to start from a symmetry breaking saddle point solution, associated with a mean field-like finite density of states, when the outcome of including fluctuations is a vanishing density of states. Put in a different language, it is hard to believe in a method which starts by assuming a diffusive behaviour if at the end it is discovered that a diffusive regime never appears. This criticism could invalidate also the results obtained with isotropic scattering, even though in this case the dimensionless coupling of the non-linear $\sigma$-model can be made small\(^5\) by assuming a large anisotropy of the Dirac dispersion (i.e. the velocities along and orthogonal to the Fermi surface). A related objection that can be raised is that the non-linear $\sigma$-model approach to disordered systems is commonly believed to be valid for length scales longer than the mean free path and energy scales smaller than the inverse relaxation time $1/\tau$. In contrast, the solution of the intra-node scattering problem demonstrates that disorder starts to affect for instance the density of states at energies of the order of the superconducting gap, and hence much bigger than $1/\tau$, namely in the regime when quasiparticle motion should be still ballistic.

It is the aim of the present work to clear up these inconsistencies of the non-linear $\sigma$-model approach to disordered d-wave superconductors. We will demonstrate that the above, apparently contradictory, approaches can be actually reconciled. This is of particular interest since it provides further support to the standard non-linear $\sigma$-

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\(^5\) For intra- and inter-opposite-node scattering, the non-linear $\sigma$-model coupling is instead of order 1 irrespectively of the velocity anisotropy; see the text below.
model approach based on the replica trick within the fermionic path integral formalism, which remains so far one of the few available tools for dealing with disorder in a generic situations with a Fermi sea of interacting quasiparticles. Let us briefly summarize our main results.

First we are going to present a simple and straightforward way to extract the WZW term. Indeed it is well known how one derives the WZW term within field theories defined on a continuous space with Dirac-like dispersing particles. However in disordered lattice models the existence of such a term is not at all a common situation. We will show that the WZW term emerges quite naturally within the conventional derivation of the non-linear $\sigma$-model for disordered systems as a consequence of the non-analytical properties of the spectrum within the Brillouin zone. More specifically, the spin current density in momentum space, $J_k$, in a d-wave superconductor is a $2 \times 2$ matrix in the Nambu spinor space. The WZW term arises just because the vector product $J_k \land J_k$ is finite and actually gives a measure of the vorticity around each node. In the presence of purely intra-node impurity scattering we obtain the same WZW action of the non-Abelian bosonization from the 1D mapping [19,20], with however the inverse mean free path as a momentum upper cut-off instead of the inverse lattice spacing which is usually the ultraviolet regularizer of the 1D Dirac theory. In this context we elucidate the role of the WZW term in providing the correct scaling behaviour of the density of states depending on the impurity-potential range.

Another issue we clarify is the relationship between the coupling constant of the non-linear $\sigma$-model and the actual spin conductance. The two quantities are known to coincide at the level of the Born approximation. However, rigorously speaking, the spin conductance has to be calculated through a Kubo formula which involves advanced and retarded Green’s functions. Since it makes a difference whether quasiparticles are right at the nodal points, with zero density of states, or slightly away from them, in which case the density of states is finite, one might wonder whether the two quantities, spin conductance and non-linear $\sigma$-model coupling, remain equal even beyond the Born approximation, especially in the case of intra-node and inter-opposite-node scattering. We will show that this is actually the case.

Finally we will show that the range of the impurity potential crucially affects the energy scale at which localization precursor effects starts to appear. In particular we will explicitly show that keeping fixed the inverse relaxation time within the Born approximation, $1/\tau_0$, and increasing the relative weight of the long range with respect to the short range components of the disorder leads to strong reduction below $1/\tau_0$ of the energy scale for the onset of localization. This may provide a natural explanation to the partial lack of experimental evidence of quasiparticle localization in superconducting cuprates.

The paper is organized as follows. In section 2 we introduce the model which we reformulate within the fermionic path integral formalism using the replica trick in section 3. The global symmetries of the action are discussed in section 4. In section 5 we start the derivation of the non-linear $\sigma$-model which includes two terms. The ‘conventional’ one is worked out in section 6, and its drawbacks discussed in section 7. The ‘unconventional’ WZW term is derived in section 8 and its consequences for intra-nodal disorder are discussed in section 9. In section 10 we analyse the role of inter-nodal scattering processes and finally section 11 is devoted to the concluding remarks.
2. The model

The model we study is described by an Hartree–Fock Hamiltonian for d-wave superconductors in the presence of disorder

\[ H = \sum_k \psi_k^\dagger [\epsilon_k \tau_3 + \Delta_k \tau_1] \psi_k + \sum_{kq} V(q) \psi_k^\dagger \tau_3 \psi_{k+q} \]

\[ \equiv \sum_k \psi_k^\dagger H_k^{(0)} \psi_k + \sum_{kq} V(q) \psi_k^\dagger \tau_3 \psi_{k+q}. \]  

(1)

Here \( V(q) \) is the impurity potential, \( \psi_k^\dagger = (c_{k\uparrow}^\dagger, c_{-k\downarrow}) \) the Nambu spinor, \( \Delta_k = \Delta (\cos k_x a - \cos k_y a)/2 \) the superconducting gap with d-wave symmetry and \( \epsilon_k \) the band dispersion measured with respect to the chemical potential. The Pauli matrices \( \tau_i, i = 1, 2, 3 \), act on the Nambu spinor components. The spectrum of the Bogoliubov quasiparticles is given by

\[ E_k = \sqrt{\epsilon_k^2 + \Delta_k^2}, \]  

(2)

and shows four nodal points at \( k_1 = k_F (1, 1)/\sqrt{2}, k_\bar{1} = -k_1, k_2 = k_F (-1, 1)/\sqrt{2} \) and \( k_\bar{2} = -k_2, k_F \) being the Fermi momentum along the diagonals of the Brillouin zone. Around the nodes it is actually more convenient to rotate the axes by \( 45^\circ \) through

\[ k_\xi = \frac{1}{\sqrt{2}} (k_x + k_y), \quad k_\eta = \frac{1}{\sqrt{2}} (k_y - k_x). \]  

(3)

In the new reference frame, \( k_1 = k_F (1, 0) \) and \( k_2 = k_F (0, 1) \). In what follows we define as

\[ \kappa \equiv k - k_a \]  

(4)

the momentum deviation from any of the four nodal points \( k_a, a = 1, 1, 2, 2 \). For small \( |\kappa| \ll k_F \) the spectra around nodes \( 1(\bar{1}) \) and \( 2(\bar{2}) \) are, respectively,

\[ E_{k_1 + \kappa} \equiv E_1 \kappa \simeq \sqrt{v_F^2 \kappa_\xi^2 + v_\Delta^2 \kappa_\eta^2}, \]  

(5)

\[ E_{k_2 + \kappa} \equiv E_2 \kappa \simeq \sqrt{v_F^2 \kappa_\eta^2 + v_\Delta^2 \kappa_\xi^2}, \]  

(6)

thus having a conical Dirac-like form. Here \( v_F = |\nabla \epsilon_{k_1}| \) and \( v_\Delta = |\nabla \Delta_{k_a}| \).

In the presence of disorder the motion of the gapless quasiparticles may become diffusive in the hydrodynamic regime. However diffusive propagators appear only in those channels which refer to conserved quantities. Hence, in superconductors, only thermal and spin density fluctuations might acquire a diffusive behaviour. Let us therefore briefly discuss some properties of the spin current operator which are going to play an important role in our analysis.

The \( z \)-component of the spin current operator in the Nambu representation satisfies

\[ q \cdot \sum_k \psi_k^\dagger J_k(q) \psi_{k+q} = \sum_k \psi_k^\dagger \left[ H_k^{(0)} - H_k^{(0)} \right] \psi_{k+q}. \]  

(7)

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and hence, for \( q \to 0 \),
\[
J_k(q) \to J_k \equiv \nabla \epsilon_k \tau_3 + \nabla \Delta_k \tau_1.
\]

Since the Pauli matrices anticommute, the following vector product turns out to be non-zero:
\[
J_k \wedge J_k = 2i \tau_2 \nabla \epsilon_k \wedge \nabla \Delta_k.
\]

The vector product (9) actually probes the vorticity of the spectrum in momentum space; nodes '1' and '1' have the same vorticity, \( J_k \wedge J_k \to 2i \tau_2 \nu \Delta \), although opposite to that for node '1', \( J_k \wedge J_k \to -2i \tau_2 \nu \Delta \). We are going to show that this property is crucial for uncovering the physical behaviour in the presence of disorder.

3. Path integral formulation

To analyse the effect of disorder in this model, we are going to use a replica trick method within the fermionic path integral approach. For that we associate with each fermionic operator Grassmann variables through
\[
c_{k\sigma}^\dagger \to \bar{c}_{k\sigma}, \quad c_{k\sigma} \to c_{k\sigma}.
\]

Notice that, unlike the original fermionic operators, \( c_{k\sigma}^\dagger \) and \( c_{k\sigma} \) are independent variables. After introducing the Nambu spinors
\[
\bar{\psi}_{nk} = (\bar{c}_{k\uparrow}(-i\omega_n), c_{k\downarrow}(i\omega_n)), \quad \psi_{nk} = \left( c_{k\uparrow}(i\omega_n), \bar{c}_{k\downarrow}(-i\omega_n) \right),
\]
where \( \bar{c}_{k\sigma}(i\omega_n) \) and \( c_{k\sigma}(i\omega_n) \) are the Grassmann variables in Matsubara frequencies, the path integral action without disorder reads
\[
S_0 = \sum_n \sum_k \bar{\psi}_{nk} \left[ \epsilon_k \tau_3 + \Delta_k \tau_1 - i\omega_n \right] \psi_{nk}.
\]

The disorder introduces the additional term
\[
S_{\text{imp}} = \sum_n \sum_{kq} V(q) \bar{\psi}_{nk} \tau_3 \psi_{nk+q}.
\]

Since \( \bar{\psi}_{nk} \) and \( \psi_{nk} \) are independent variables, the global transformation
\[
\psi_{nk} \to e^{i\theta \tau_2} \psi_{nk}, \quad \bar{\psi}_{nk} \to \bar{\psi}_{nk} e^{i\theta \tau_2}.
\]

becomes allowed within the path integral and is indeed a symmetry transformation of the full action \( S = S_0 + S_{\text{imp}} \) when \( \omega_n = 0 \). A finite frequency, \( \omega_n \neq 0 \), spoils this symmetry. If, in addition, the disorder is long range, namely it does not induce inter-node scatterings, it is possible to define four independent symmetry transformations of the above kind, one for each node. This type of chiral symmetry plays a crucial role for long range disorder, as was first emphasized in [19].

We notice that the disorder does not induce any mixing between different Matsubara frequencies, so that the action decouples into separate pieces, each one relating to a pair of opposite Matsubara frequencies, \( \pm \omega_n \), which are coupled together by the superconducting term. For this reason we will just consider one of these pairs, with frequencies which we denote by \( \pm \omega \), discarding all the others. This is enough to extract information about
the quasiparticle density of states (DOS) at given frequency as well as about transport coefficients.

In order to derive a long wavelength effective action, we resort to the replica trick technique; hence we first introduce $N$ replicas of the Grassmann variables, $\Psi_{ak\sigma}(\pm i\omega) \rightarrow \Psi_{ak\sigma}(\pm i\omega)$ and $\bar{c}_{k\sigma}(\pm i\omega) \rightarrow \bar{c}_{ak\sigma}(\pm i\omega)$, with $a = 1, \ldots, N$. At the end we shall send $N \rightarrow 0$. Next we define the column vectors $c_k$ and $\bar{c}_k$, with $4N$ elements ($N$ replicas, two spins and two frequencies) $c_{ak\sigma}(\pm i\omega)$ and $\bar{c}_{ak\sigma}(\pm i\omega)$, respectively. Finally we introduce new Nambu spinors through \[3\]

$$ \Psi_k = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{-k} \\ i\sigma_2 c_k \end{pmatrix}, $$

as well as

$$ \overline{\Psi}_k = \Psi^t_{-k} c', $$

with the charge conjugacy operator defined by \[3\]

$$ c = i \sigma_2 \tau_1. $$

Here the Pauli matrices $\sigma_1$, $\sigma_2$ and $\sigma_3$ act on the spin components of the column vectors $c_k$ and $\bar{c}_k$, while $\tau_1$, $\tau_2$ and $\tau_3$ act on the Nambu components (the diagonal elements relate to the particle–hole channels and the off-diagonal ones to the particle–particle channels). For later convenience, we also introduce the Pauli matrices $s_1$, $s_2$ and $s_3$, which act on the opposite frequency partners, as well as the identity matrices in the spin, Nambu and frequency subspaces, $\sigma_0$, $\tau_0$ and $s_0$, respectively.

By means of the above definitions, the clean action can be written as

$$ S_0 = \sum_k \overline{\Psi}_k [\epsilon_k + i \Delta_k \tau_2 s_1 - i \omega s_3] \Psi_k. \tag{13} $$

Since we are interested in the low-energy long wavelength behaviour, we shall focus our attention only in small areas around each of the four nodal points. In the rotated reference frame (3), the nodes lie at $k_1 = k_F (1, 0)$, $k_\tau = -k_1$, $k_2 = k_F (0, 1)$ and $k_\tau = -k_2$. Using the definition (4) for the small momentum deviations away from the nodal points, we introduce new Grassmann variables defined around each node through

$$ c_{a,k} \equiv c_{ka+k}, \quad \overline{c}_{a,k} \equiv \overline{c}_{ka+k}, $$

$a = 1, \overline{1}, 2, \overline{2}$, as well as the corresponding Nambu spinors

$$ \Psi_{1\kappa} = \begin{pmatrix} \tau_{1,-\kappa} \\ i\sigma_2 c_{1,\kappa} \end{pmatrix}, \quad \Psi_{\overline{1}\kappa} = \begin{pmatrix} \tau_{1,-\kappa} \\ i\sigma_2 c_{1,\kappa} \end{pmatrix}, $$

$$ \Psi_{2\kappa} = \begin{pmatrix} \tau_{2,-\kappa} \\ i\sigma_2 c_{2,\kappa} \end{pmatrix}, \quad \Psi_{\overline{2}\kappa} = \begin{pmatrix} \tau_{2,-\kappa} \\ i\sigma_2 c_{2,\kappa} \end{pmatrix}. \tag{14} $$

One notices that for $a = 1, 2$

$$ \overline{\Psi}_{ak\kappa} = \frac{1}{\sqrt{2}} (-c_{ak\kappa}, -i \overline{c}_{ak\kappa} \sigma_2) = \Psi^t_{ak\kappa} \sigma_2 c', $$

$$ \overline{\Psi}_{ak\kappa} = \frac{1}{\sqrt{2}} (-c_{ak\kappa}, -i \overline{c}_{ak\kappa} \sigma_2) = \Psi^t_{ak\kappa} \sigma_2 c'. $$

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The non-disordered action expanded around the nodes reads
\[ S_0 = \sum_{\kappa} \overline{\Psi}_{1\kappa} \left[ v_F \kappa_\xi + i v_\Delta \kappa_\eta \tau_2 s_1 \right] \Psi_{1\kappa} - \overline{\Psi}_{T\kappa} \left[ v_F \kappa_\xi + i v_\Delta \kappa_\eta \tau_2 s_1 \right] \Psi_{T\kappa} 
+ \overline{\Psi}_{2\kappa} \left[ v_F \kappa_\xi + i v_\Delta \kappa_\eta \tau_2 s_1 \right] \Psi_{2\kappa} - \overline{\Psi}_{T\kappa} \left[ v_F \kappa_\xi + i v_\Delta \kappa_\eta \tau_2 s_1 \right] \Psi_{T\kappa} 
- i\omega \sum_{i=1,2,7} \overline{\Psi}_{1\kappa} s_3 \Psi_{i\kappa}. \]

We find it useful to define new spinors with components in each of the nodes through
\[ \Psi_{\kappa} = \begin{pmatrix} \Psi_{1\kappa} \\ \Psi_{2\kappa} \\ \Psi_{T\kappa} \\ \overline{\Psi}_{T\kappa} \end{pmatrix}, \]
so that
\[ \overline{\Psi}_{\kappa} = \left( \Psi_{T-\kappa}, \overline{\Psi}_{2-\kappa}, \Psi_{1-\kappa}, \overline{\Psi}_{2-\kappa} \right) c' = \Psi_{-\kappa} \gamma_1 c', \]
where the Pauli matrices \( \gamma \) act on the ‘m’ and ‘m’ subspace, namely connecting two opposite nodes. This naturally leads to a new charge conjugacy defined through
\[ C = e^{\gamma_1} = i \sigma_2 \tau_1 \gamma_1. \tag{15} \]

In what follows we denote as pair 1 the two opposite nodes ‘1’ and \( T \), and as pair 2 the other two nodes, ‘2’ and \( \overline{T} \). Consequently we need to introduce also matrices in the two-pair subspace, 1 and 2, namely connecting adjacent nodes, which we will denote as \( \rho_0 \), the identity, \( \rho_1 \), \( \rho_2 \) and \( \rho_3 \), the three Pauli matrices.

The clean action reads now
\[ S_0 = \sum_{\kappa} \overline{\Psi}_{\kappa} \gamma_3 \frac{1}{2} \left( \kappa_\xi + \kappa_\eta \right) \left[ v_F + i v_\Delta \tau_2 s_1 \right] \Psi_{\kappa} 
+ \overline{\Psi}_{\kappa} \gamma_3 \rho_3 \frac{1}{2} \left( \kappa_\xi - \kappa_\eta \right) \left[ v_F - i v_\Delta \tau_2 s_1 \right] \Psi_{\kappa} 
- i\omega \sum_{\kappa} \overline{\Psi}_{\kappa} s_3 \Psi_{\kappa} \]
\[ \equiv \sum_{\kappa} \overline{\Psi}_{\kappa} \left[ \mathcal{T}_0(\kappa) - i\omega s_3 \right] \Psi_{\kappa}. \tag{16} \]

We notice that within this path integral formulation, the four independent chiral symmetry transformations of the form (12) translate into
\[ \Psi \rightarrow T \Psi, \quad \overline{\Psi} \rightarrow \overline{\Psi} C T^t c', \]
with \( T \) given by any of the following expressions:
\[ T = e^{i \theta \tau_2 s_1 \gamma_0 \rho_0 \sigma \cdot u}, \tag{18} \]
\[ T = e^{i \theta \tau_2 s_1 \gamma_3 \rho_0}, \tag{19} \]
\[ T = e^{i \theta \tau_2 s_1 \gamma_0 \rho_3 \sigma \cdot u}, \tag{20} \]
\[ T = e^{i \theta \tau_2 s_1 \gamma_3 \rho_3}, \tag{21} \]
where \( \theta \) is a phase factor and \( u \) an arbitrary unit vector.

We assume that the scattering potential \( V(q) = V(-q)^* \) induced by disorder has independent components which act inside each node, \( V(q) \equiv U_0(q) \), between opposite

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nodes, $V(q + 2k_1) \equiv U_1(q)$ and $V(q + 2k_2) \equiv U_2(q)$, and between adjacent nodes, $V(q + k_1 - k_2) \equiv U_{12}(q)$ and $V(q + k_1 + k_2) \equiv U_{11}(q)$, so that the impurity contribution to the action is

$$S_{\text{imp}} = \sum_{\kappa, q} U_0(q) \bar{\Psi}_\kappa \Psi_{\kappa + q}$$

$$+ \sum_{\kappa, q} U_1(q) \bar{\Psi}_{1\kappa} \Psi_{1\kappa + q} + U_1(q)^* \bar{\Psi}_{1\kappa + q} \Psi_{1\kappa}$$

$$+ \sum_{\kappa, q} U_2(q) \bar{\Psi}_{2\kappa} \Psi_{2\kappa + q} + U_2(q)^* \bar{\Psi}_{2\kappa + q} \Psi_{2\kappa}$$

$$+ \sum_{\kappa, q} U_{12}(q) \left[ \bar{\Psi}_{1\kappa} \Psi_{2\kappa + q} + \bar{\Psi}_{2\kappa} \Psi_{1\kappa + q} \right] + U_{12}(q)^* \left[ \bar{\Psi}_{2\kappa + q} \Psi_{1\kappa} + \bar{\Psi}_{1\kappa + q} \Psi_{2\kappa} \right]$$

$$+ \sum_{\kappa, q} U_{11}(q) \left[ \bar{\Psi}_{1\kappa} \Psi_{2\kappa + q} + \bar{\Psi}_{2\kappa} \Psi_{1\kappa + q} \right] + U_{11}(q)^* \left[ \bar{\Psi}_{2\kappa + q} \Psi_{1\kappa} + \bar{\Psi}_{1\kappa + q} \Psi_{2\kappa} \right].$$

Apart from the first term in the right-hand side of (22), we have for convenience kept the node labels. Since the following relations hold:

$$\sum_{\kappa} \bar{\Psi}_{2\kappa} \Psi_{1\kappa + q} = \sum_{\kappa} \Psi^t_{2\kappa} c^t \bar{\psi}^t_{1\kappa - q} = \sum_{\kappa} \bar{\Psi}_{1\kappa} \Psi_{2\kappa + q},$$

$$\sum_{\kappa} \bar{\Psi}_{2\kappa} \Psi_{1\kappa + q} = \sum_{\kappa} \Psi^t_{2\kappa} c^t \bar{\psi}^t_{1\kappa - q} = \sum_{\kappa} \bar{\Psi}_{1\kappa} \Psi_{2\kappa + q},$$

we can rewrite (22) in the following way:

$$S_{\text{imp}} = \sum_{\kappa, q} U_0(q) \bar{\Psi}_\kappa \Psi_{\kappa + q}$$

$$+ \sum_{\kappa, q} U_1(q) \bar{\Psi}_{1\kappa} \Psi_{1\kappa + q} + U_1(q)^* \bar{\Psi}_{1\kappa + q} \Psi_{1\kappa}$$

$$+ \sum_{\kappa, q} U_2(q) \bar{\Psi}_{2\kappa} \Psi_{2\kappa + q} + U_2(q)^* \bar{\Psi}_{2\kappa + q} \Psi_{2\kappa}$$

$$+ 2 \sum_{\kappa, q} U_{12}(q) \bar{\Psi}_{1\kappa} \Psi_{2\kappa + q} + U_{12}(q)^* \bar{\Psi}_{2\kappa + q} \Psi_{1\kappa}$$

$$+ 2 \sum_{\kappa, q} U_{11}(q) \bar{\Psi}_{1\kappa} \Psi_{2\kappa + q} + U_{11}(q)^* \bar{\Psi}_{2\kappa + q} \Psi_{1\kappa}. \quad (23)$$

Finally we go back in real space, which now corresponds to the low-energy continuum limit of the original lattice model, and obtain the clean action

$$S_0 = \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) \left[ \frac{i}{2} \gamma_3 \left( \gamma_3 \bar{\gamma}_3 \right) \right] \left[ v_F + i v_\Delta \gamma_2 s_1 \right] \Psi(\mathbf{r})$$

$$+ \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) \left[ \frac{i}{2} \gamma_3 \rho_3 \left( \gamma_3 \bar{\gamma}_3 \right) \right] \left[ v_F - i v_\Delta \gamma_2 s_1 \right] \Psi(\mathbf{r})$$

$$- i \omega \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) s_3 \Psi(\mathbf{r}) \equiv \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) \left[ \mathcal{H}_0(\mathbf{r}) - i \omega s_3 \right] \Psi(\mathbf{r}), \quad (24)$$

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We further assume that the disorder is intra-node disorder is present, namely $u$ $\delta$-like correlated with $v \neq 0$ only $T$ leave the action invariant; (ii) if $u \neq 0$ and $v \neq 0$ only (18) and (20) are allowed; (iii) finally if $u \neq 0$, $v \neq 0$ and $w \neq 0$ only (18) remains.

4. Symmetry properties

Since we have been obliged to introduce so many Pauli matrices, including the identity denoted as the zeroth Pauli matrix, for the sake of clarity we prefer to start this section by first summarizing in table 1 the subspaces in which any of them act.

| Pauli matrices | Subspace of action |
|----------------|-------------------|
| $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ | Spin components |
| $\tau_0, \tau_1, \tau_2, \tau_3$ | Nambu components |
| $s_0, s_1, s_2, s_3$ | Opposite frequency partners |
| $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ | Opposite nodal points |
| $\rho_0, \rho_1, \rho_2, \rho_3$ | The two pairs of opposite nodes |

and the impurity term

$$S_{\text{imp}} = \int dr \, U_0(r) \overline{\Psi}(r) \Psi(r)$$

$$+ U_1(r) \overline{\Psi}_1(r) \Psi_1^\ast(r) + U_1^\ast(r) \overline{\Psi}_1^\ast(r) \Psi_1(r)$$

$$+ U_2(r) \overline{\Psi}_2(r) \Psi_2(r) + U_2^\ast(r) \overline{\Psi}_2^\ast(r) \Psi_2(r)$$

$$+ 2 U_{12}(r) \overline{\Psi}_1(r) \Psi_2(r) + 2 U_{12}^\ast(r) \overline{\Psi}_2^\ast(r) \Psi_1(r)$$

$$+ 2 U_{12}^\ast(r) \overline{\Psi}_1^\ast(r) \Psi_2^\ast(r) + 2 U_{12}^\ast(r) \overline{\Psi}_2^\ast(r) \Psi_1^\ast(r).$$

(25)

We further assume that the disorder is $\delta$-like correlated with

$$\langle U_0(r) \rangle = 0, \qquad \langle U_0(r) U_0(r') \rangle = u^2 \delta (r - r'),$$

$$\langle U_1(r) \rangle = 0, \qquad \langle U_1(r) U_1(r')^\ast \rangle = 2 u^2 \delta (r - r'),$$

$$\langle U_2(r) \rangle = 0, \qquad \langle U_2(r) U_2(r')^\ast \rangle = 2 v^2 \delta (r - r'),$$

$$\langle U_{12}(r) \rangle = 0, \qquad \langle U_{12}(r) U_{12}(r')^\ast \rangle = 2 w^2 \delta (r - r'),$$

$$\langle U_{12}^\ast(r) \rangle = 0, \qquad \langle U_{12}^\ast(r) U_{12}^\ast(r') \rangle = 2 w^2 \delta (r - r').$$

(26)

We conclude by noticing that (i) if only $u \neq 0$ all transformations (18)–(21) leave the action invariant; (ii) if $u \neq 0$ and $v \neq 0$ only (18) and (20) are allowed; (iii) finally if $u \neq 0$, $v \neq 0$ and $w \neq 0$ only (18) remains.

4. Symmetry properties

Since we have been obliged to introduce so many Pauli matrices, including the identity denoted as the zeroth Pauli matrix, for the sake of clarity we prefer to start this section by first summarizing in table 1 the subspaces in which any of them act.

Now, let us uncover all global symmetry transformations. We start assuming that only intra-node disorder is present, namely $U_0 \neq 0$ but $U_1 = U_2 = U_{12} = U_{12}^\ast = 0$.

If the frequency is zero, the action $S_0 + S_{\text{imp}}$ is invariant under unitary global transformations $T$ such that

$$C_l T^d C \gamma_3 T = \gamma_3,$$

$$C_l T^d C \gamma_3 \tau_2 s_1 T = \gamma_3 \tau_2 s_1,$$

$$C_l T^d C \gamma_3 \rho_3 T = \gamma_3 \rho_3,$$

$$C_l T^d C \gamma_3 \rho_2 s_1 T = \gamma_3 \rho_3 \tau_2 s_1,$$

$$C_l T^d C T = 1.$$
They imply that
\[ C^t T^t C = T^{-1}, \]  
(27)
as well as that
\[ [T, \gamma_3] = 0, \quad [T, \rho_3] = 0, \quad [T, \tau_2 s_1] = 0. \]  
(28)
We parametrize \( T \) as
\[ T = \frac{1}{2} (T_1 + T_2) \rho_0 + \frac{1}{2} (T_1 - T_2) \rho_3, \]  
(29)
where the suffix ‘1’ refers to the pair 1 (opposite nodes 1 and \( \overline{1} \)) and ‘2’ to the pair 2 (opposite nodes 2 and \( \overline{2} \)). The symmetry modes for pair 1 and 2 can be parametrized by the \( 16N \times 16N \) unitary operators
\[ T_{1(2)} = \frac{1}{2} (\gamma_0 + \gamma_3) V_{1(2)} + \frac{1}{2} (\gamma_0 - \gamma_3) V_{\overline{1}(\overline{2})} \equiv \begin{pmatrix} V_{1(2)} & 0 \\ 0 & V_{\overline{1}(\overline{2})} \end{pmatrix}, \]
where the \( V \)’s are \( 8N \times 8N \) unitary matrices. Because of (27) we get
\[ C^t T_{1(2)}^t C = T_{1(2)}^{-1} = T_{1(2)}^t, \]
and \( V_{1(2)} \) and \( V_{\overline{1}(\overline{2})} \) are actually not independent as
\[ c^t V_{1(2)}^t c = V_{1(2)}^t. \]
(30)
Therefore
\[ T_{1(2)} = \begin{pmatrix} V_{1(2)} & 0 \\ 0 & c^t \left( V_{1(2)}^t \right)^t c \end{pmatrix}, \]
is indeed parametrized by a single \( 8N \times 8N \) unitary matrix \( V_{1(2)} \). Since the two pairs of nodes behave similarly, in what follows we drop the suffices 1 and 2. According to (28) we still need to impose that
\[ [V, \tau_2 s_1] = 0. \]  
(31)
To this end let us introduce the following unitary operation:
\[ U = \left[ \frac{1}{2} (\tau_0 + \tau_2) - \frac{1}{2} (\tau_0 - \tau_2) \right] s_3 e^{-i(\pi/4)s_2}, \]  
(32)
which transforms
\[ \Psi \rightarrow U \Psi, \quad \overline{\Psi} \rightarrow \Psi^t U^t c^t = \Psi^t c^t U^t = \overline{\Psi} U^t, \]
so that for any operator \( \mathcal{O} \)
\[ \overline{\Psi} O \Psi \rightarrow \overline{\Psi} U^t \mathcal{O} U \Psi. \]
One readily shows that
\[ U^t \tau_2 s_1 U = s_3, \quad U^t s_3 U = -s_1, \]  
(33)
and hence (31) transforms into
\[ [V, s_3] = 0, \]  
(34)
which is fulfilled by the general expression
\[ V = \frac{1}{2} (A + B) s_0 + \frac{1}{2} (A - B) s_3, \]  
(35)
with $A$ and $B$ being independent $4N \times 4N$ unitary matrices. Therefore the original symmetry turns out to be $U(4N) \times U(4N)$ for pair 1, and analogously for pair 2, in total $U(4N) \times U(4N) \times U(4N) \times U(4N)$.

We notice that, in the presence of a finite frequency, $\omega \neq 0$, we shall further impose through (33) that

$$[V, s_1] = 0,$$

which is satisfied by

$$V = Cs_0,$$

with $C$ belonging to $U(4N)$. Therefore the symmetry is lowered by the frequency down to $U(4N)$ for the pair 1 times $U(4N)$ for the pair 2.

Within the non-linear $\sigma$-model approach to disordered systems, the frequency actually plays the role of a symmetry breaking field. This leads to the identification of the transverse modes $V^\perp$ as those satisfying

$$s_1 V^\perp s_1 = (V^\perp)^{-1} = (V^\perp)^\dagger,$$

implying $A = B^\dagger$ in equation (35) so that we can write

$$V^\perp = \frac{1}{2} \left[ \sqrt{G} + \sqrt{G^\dagger} \right] s_0 + \frac{1}{2} \left[ \sqrt{G} - \sqrt{G^\dagger} \right] s_3,$$

(36)

where $G$ belongs to $U(4N)$. In other words the coset space spanned by the transverse modes is still a group, namely $U(4N)$. It is convenient to factorize out of $G$ the Abelian component. For that we write

$$G = \exp \left( i \left( \frac{\pi}{N} \phi \right) \right) g,$$

(37)

where $\phi$ is a scalar and $g$ belongs to $SU(4N)$. The forms (36) and (37) will be useful in the following for expressing the non-linear $\sigma$-model directly in terms of $g$ and $\phi$. After the transformation (32) from equation (30) we find that

$$V^\perp_{1(2)} = \left[ \tau_1 \sigma_2 \left( V^\perp_{1(2)} \right)^\dagger \tau_1 \sigma_2 \right]^t \equiv \left[ \tau_1 \sigma_2 s_1 V^\perp_{1(2)} s_1 \tau_1 \sigma_2 \right]^t,$$

(38)

which leads to

$$\phi_{1(2)} = -\phi_{1(2)}, \quad g_{1(2)} = \tau_1 \sigma_2 \left( g_{1(2)}^\dagger \right)^t \sigma_2 \tau_1.$$

(39)

Let us conclude by showing what would change for more general disorder potential. First let us consider the case in which the disorder also contains terms which couple opposite nodes, i.e. $U_1 \neq 0$ and $U_2 \neq 0$. In this case the $\gamma_{23}$-modes are no longer allowed by symmetry, so we have to impose that $V_{1(2)} = V^\perp_{1(2)}$, namely, through (39), that $\phi_{1(2)} = 0$ and

$$g_{1(2)}^\dagger = g_{1(2)}^{-1} = \tau_1 \sigma_2 g_{1(2)}^\dagger \sigma_2 \tau_1.$$

(40)

The coset becomes now the group $Sp(2N)$ of unitary–symplectic matrices (also called $USp(4N)$) for pair 1 and analogously for pair 2, i.e. $Sp(2N) \times Sp(2N)$.

Finally, if also scattering between adjacent nodes is allowed, $U_{12} \neq 0$ and $U_{11} \neq 0$, then also the $\rho_{23}$-modes do not leave the action invariant. In this case we have to impose that $V^\perp_1 = V^\perp_2$, so that the coset space is the group $Sp(2N)$.
5. The non-linear $\sigma$-model

In what follows we begin analysing just the case in which the disorder only induces intra-node scattering processes. At the end we will return to the most general case. Within the replica trick, we can average the action over the disorder probability distribution, after which (25) with $v = w = 0$ (see equation (26)) transforms into

$$S_{\text{imp}} = -\frac{u^2}{2} \int dr \left[ \overline{\Psi(r)} \Psi(r) \right]^2. \quad (41)$$

We define $32N \times 32N$ matrices $X(r)$ by

$$X(r) \equiv \Psi(r) \overline{\Psi(r)},$$

so that (41) can be also written as

$$S_{\text{imp}} = \frac{u^2}{2} \int dr \text{Tr} \left( X(r) X(r) \right). \quad (42)$$

By means of an Hubbard–Stratonovich transformation one can show that

$$\exp \left\{ -\frac{u^2}{2} \int dr \text{Tr} \left( X(r) X(r) \right) \right\} = \int \mathcal{D}Q(r) \exp \left\{ -\int dr \frac{1}{2u^2} \text{Tr} \left[ (Q(r))^2 \right] - i \text{Tr} \left[ Q(r) X(r) \right] \right\}, \quad (43)$$

with $Q(r)$ being Hermitian $32N \times 32N$ auxiliary matrix fields.

In conclusion the full action, (24) plus (43), becomes

$$S = \frac{1}{2u^2} \int dr \text{Tr} \left[ (Q(r))^2 \right] + \int dr \overline{\Psi(r)} \left[ \mathcal{H}_0(r) - i Q(r) - i\omega_3 \right] \Psi(r). \quad (44)$$

One obtains the effective action which describes the auxiliary field $Q(r)$ by integrating out the Grassmann variables, thus getting

$$S[Q] = \frac{1}{2u^2} \int dr \text{Tr} \left[ (Q(r))^2 \right] - \frac{1}{2} \text{Tr} \ln \left[ iQ + i\omega_3 - \mathcal{H}_0 \right]. \quad (45)$$

We further proceed in the derivation of a long wavelength effective action for $Q(r)$ by following the conventional approach. First of all we calculate the saddle point expression of $Q(r)$, which we denote by $Q_0 s_3$, assuming it is uniform, in the presence of an infinitesimal symmetry breaking term $\omega s_3$. Then we neglect longitudinal fluctuations and parametrize the actual $Q(r)$ by

$$Q(r) \simeq T(r)^{-1} Q_0 s_3 T(r) = \frac{1}{2} (\rho_0 + \rho_3) Q_1(r) + \frac{1}{2} (\rho_0 - \rho_3) Q_2(r), \quad (46)$$

where through (29) $Q_{ab}(r) = \delta_{ab} Q_a(r)$ with

$$Q_a(r) = T_a(r)^{-1} Q_0 s_3 T_a(r) = Q_0 s_3 T_a(r)^2 = T_a(r)^{-2} Q_0 s_3, \quad (47)$$

being the auxiliary field in the pair subspace $a, b = 1, 2$. Notice that, even though the most general $Q(r)$ would couple the nodes together, namely would include off-diagonal elements $Q_{12}(r)$, (46) does not contain any mixing term. This simply reflects that the off-diagonal components are not diffusive, and hence are massive.

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In order to derive the long wavelength action we find it convenient to decompose the action into the real part
\[ S_{NLσM} = \frac{1}{2u^2} \int \text{d}r \text{Tr} \left[ Q(r)^2 \right] \]
\[ - \frac{1}{4} \text{Tr} \ln \left[ iQ + iω s_3 - H_0 \right] - \frac{1}{4} \text{Tr} \ln \left[ -iQ - iω s_3 - H_0^\dagger \right], \]
which, as we shall see, gives the conventional non-linear σ-model, and the imaginary part
\[ S_Γ = - \frac{1}{4} \text{Tr} \ln \left[ iQ + iω s_3 - H_0 \right] + \frac{1}{4} \text{Tr} \ln \left[ -iQ - iω s_3 - H_0^\dagger \right], \]
which we will show gives rise to a WZW term.

6. Conventional σ-model

By means of (46), the real part of the action, (48), can be written as
\[ S_{NLσM} = \frac{V_{\text{eff}}}{2u^2} 32N Q_0^2 - \frac{1}{4} \text{Tr} \ln \left[ G^{-1} \right], \]
where \( V_{\text{eff}} \) is the effective volume corresponding to the long wavelength theory (roughly speaking \( V_{\text{eff}} = V/4 \), since we have implicitly folded the Brillouin zone into a single quadrant, in order to make all nodes coincide), and
\[ G^{-1} = \mathcal{H}_0 \mathcal{H}_0^\dagger + Q_0^2 + ω^2 + ω (Q s_3 + s_3 Q) + i\mathcal{H}_0 Q - iQ \mathcal{H}_0^\dagger \]
\[ ≡ G_0^{-1} - Σ, \]
where
\[ G_0^{-1} = \mathcal{H}_0 \mathcal{H}_0^\dagger + Q_0^2 + ω^2, \]
\[ Σ = -ω \{Q, s_3\} - i\mathcal{H}_0 Q + iQ \mathcal{H}_0^\dagger. \]
We notice that \( s_3 \mathcal{H}_0 s_3 = \mathcal{H}_0^\dagger \), and therefore
\[ i\mathcal{H}_0 Q - iQ \mathcal{H}_0^\dagger = i\mathcal{H}_0 Q s_3 s_3 - iQ s_3 \mathcal{H}_0 s_3 = i \{\mathcal{H}_0, Q s_3\} s_3 = J \cdot \nabla Q, \]
where we made use of the equivalence \( Q s_3 ≡ T^{-2} \) and of the expression of the spin current operator in the long wavelength limit
\[ J(r) = i \{\mathcal{H}_0(r), r\}. \]

Analogously one can show that
\[ i\mathcal{H}_0 Q - iQ \mathcal{H}_0^\dagger = i s_3 \mathcal{H}_0^\dagger s_3 Q - i s_3 s_3 Q \mathcal{H}_0^\dagger = -i s_3 \left[ s_3 Q, \mathcal{H}_0^\dagger \right] = \nabla Q \cdot J^\dagger, \]
so that the self-energy operator can be written as
\[ Σ = -ω \{Q, s_3\} - J \cdot \nabla Q = -ω \{Q, s_3\} - \nabla Q \cdot J^\dagger. \]
We then expand the action up to second order in $\Sigma$ and obtain

$$S_{\text{NL}:\mathbf{M}} = \frac{V_{\text{eff}}}{2u^2} 32N \left( Q_0^2 - \frac{1}{4} \right) \text{Tr} \ln [G^{-1}]$$

$$= \frac{V_{\text{eff}}}{2u^2} 32N \left( Q_0^2 + \frac{1}{4} \right) \text{Tr} \ln [G_0] - \frac{1}{4} \text{Tr} \ln [1 - G_0 \Sigma]$$

$$= \frac{V_{\text{eff}}}{2u^2} 32N \left( Q_0^2 + \frac{1}{4} \right) \text{Tr} \ln [G_0] + \frac{1}{4} \text{Tr} [G_0 \Sigma] + \frac{1}{8} \text{Tr} [G_0 \Sigma G_0 \Sigma]$$

$$= \frac{V_{\text{eff}}}{2u^2} 32N \left( Q_0^2 + \frac{1}{4} \right) \text{Tr} \ln [G_0] - \frac{\omega^2}{2} \text{Tr} [Q s_3 G_0]$$

$$+ \frac{\omega^2}{8} \text{Tr} [G_0 \{Q, s_3\} G_0 \{Q, s_3\}] + \frac{1}{8} \text{Tr} \left[ G_0 \nabla Q \cdot J^\dagger G_0 J \cdot \nabla Q \right].$$

Notice that we have arrived at $S_{\text{NL}:\mathbf{M}}$ in terms of $\nabla Q$ without passing through a gauge transformation to carry out the gradient expansion so avoiding any problem related to a proper treatment of the Jacobian determinant [20].

### 6.1. Saddle point equation

In momentum space one finds that

$$\mathcal{H}_{0\mathbf{k}} \mathcal{H}_{0\mathbf{k}}^\dagger = \frac{1}{2} \left( \kappa_\xi^2 + \kappa_\eta^2 \right) \left( \nu_\xi^2 + \nu_\eta^2 \right) + \rho_3 \frac{1}{2} \left( \kappa_\xi^2 - \kappa_\eta^2 \right) \left( \nu_\xi^2 - \nu_\eta^2 \right)$$

$$= \frac{1}{2} \left( E_{1\mathbf{k}}^2 + E_{2\mathbf{k}}^2 \right) + \rho_3 \frac{1}{2} \left( E_{1\mathbf{k}}^2 - E_{2\mathbf{k}}^2 \right),$$

where $E_{1\mathbf{k}}$ is the spectrum of quasiparticles around nodes ‘1’ and ‘T’, and analogously for $E_{2\mathbf{k}}$, see (5) and (6). Therefore

$$G_{0\mathbf{k}} = \frac{1}{2} [G_{1\mathbf{k}} + G_{2\mathbf{k}}] + \rho_3 \frac{1}{2} [G_{1\mathbf{k}} - G_{2\mathbf{k}}],$$

where

$$G_{1\mathbf{k}} = \frac{1}{E_{1\mathbf{k}}^2 + Q_0^2 + \omega^2}, \quad G_{2\mathbf{k}} = \frac{1}{E_{2\mathbf{k}}^2 + Q_0^2 + \omega^2},$$

so that

$$G_{0\mathbf{k}} = \frac{1}{2} \left[ G_{1\mathbf{k}} + G_{2\mathbf{k}} \right] + \rho_3 \frac{1}{2} \left[ G_{1\mathbf{k}}^2 - G_{2\mathbf{k}}^2 \right].$$

The saddle point equation is obtained by taking $Q(\mathbf{r}) = Q_0$ and minimizing the action, which leads to the self-consistency equation

$$\frac{Q_0}{Q_0 + \omega} = u^2 \frac{1}{V} \sum_k \frac{1}{E_k^2 + (Q_0 + \omega)^2},$$

originally derived in [1]. The solution of this equation, in the limit $\omega \to 0$, reads

$$Q_0 = \Lambda \exp \left( -\frac{\pi \nu_0 u \Delta}{2u^2} \right),$$

where $\Lambda$ is an ultraviolet cut-off which is roughly the energy scale above which the spectrum deviates appreciably from a linear one, in other words $\Lambda \simeq \Delta$. We notice that in the generic case with finite inter-nodal scattering, namely with non-zero $\nu^2$ and $w^2$, the saddle point equation remains the same apart form $u^2 \to u^2 + 2w^2 + v^2$.
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The quasiparticle density of states \( N(\epsilon) \), after the analytic continuation of \( i\omega \) to the positive real axis, \( i\omega \to \epsilon > 0 \), turns out to be, for small \( \epsilon \),

\[
N(\epsilon) = \frac{2}{\pi^2 v_F \nu} \left[ Q_0 \ln \frac{\Lambda}{\sqrt{\epsilon^2 + Q_0^2}} + \frac{\epsilon}{2} \left( \frac{\pi}{2} - \tan^{-1} \frac{Q_0^2 - \epsilon^2}{2\epsilon Q_0} \right) \right]. \tag{57}
\]

Therefore the density of states at the saddle point acquires a finite value \( N_0 \sim Q_0/v^2 \) at the chemical potential \( \epsilon = 0 \), while, for \( \epsilon \gg Q_0 \), it turns back to the linear dependence \( N(\epsilon) \sim \epsilon \) as in the absence of disorder.

### 6.2. Frequency dependent terms

The first-order term in the frequency is just

\[
-\frac{\omega}{2} \text{Tr} [Q_s G_0] = -\omega \frac{\pi}{2} \frac{N_0}{Q_0} \int \text{d}r \ \text{Tr} [Q(r) s_3]. \tag{58}
\]

The second-order term is readily found to be

\[
\frac{\omega^2}{8} \text{Tr} [G_0 \{Q, s_3\} G_0 \{Q, s_3\}] = \frac{1}{32 \pi^2 v_F \nu} \frac{1}{Q_0^2} \int \text{d}r \ \text{Tr} \left[ \{Q(r), s_3\}^2 \right]. \tag{59}
\]

This term is negligible as compared to (58) for frequencies \( \omega \) much smaller than \( v_F \nu \). The latter is therefore the energy scale below which diffusion sets in at the mean field level, namely the so-called inverse relaxation time in the Born approximation, \( 1/\tau_0 \).

### 6.3. Gradient expansion

The Fourier transform of the current operator is

\[
J(q) = \gamma_3 \frac{1}{2} [J_1(q) + J_2(q)] + \gamma_3 \rho_3 \frac{1}{2} [J_1(q) - J_2(q)],
\]

where, at long wavelengths \( q \to 0 \),

\[
J_1(q) \to J_1 = (v_F, i v_\Delta \tau_2 s_1), \\
J_2(q) \to J_2 = (i v_\Delta \tau_2 s_1, v_F).
\]

It is then easy to derive the following expression for the tensor product \( J \otimes J^\dagger \):

\[
J \otimes J^\dagger = \left( \begin{array}{cc} J_\xi J_\xi^\dagger & J_\eta J_\eta^\dagger \\ J_\eta J_\xi^\dagger & J_\eta J_\eta^\dagger \end{array} \right) = \frac{1}{2} \left( v_F^2 + v_\Delta^2 \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \\
+ \frac{1}{2} \rho_3 \left( v_F^2 - v_\Delta^2 \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) - i v_F v_\Delta \tau_2 s_1 \rho_3 \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).
\tag{61}
\]

To evaluate the second-order gradient correction in (54) we notice that

\[
\frac{1}{8} \text{Tr} \left[ G_0 \nabla Q \cdot J^\dagger G_0 J \cdot \nabla Q \right] = \frac{1}{8} \sum_{i,j} \text{Tr} \left[ \partial_i Q \partial_j Q \left( G_0 J_i^\dagger G_0 J_j + G_0 J_i^\dagger G_0 J_j \right) \right] \\
= \frac{1}{16} \sum_{i,j} \text{Tr} \left[ \partial_i Q \partial_j Q \left( G_0 J_i^\dagger G_0 J_j + G_0 J_i^\dagger G_0 J_j \right) \right].
\]

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In the long wavelength limit

\[ G_0 J^i_1 G_0 J_i + G_0 J^i_1 G_0 J_j = \frac{1}{V} \sum_\kappa G_{\kappa 1} J^i_1 G_{\kappa 1} J_i + G_{\kappa 1} J^i_1 G_{\kappa 1} J_j \]

\[ = \delta_{ij} \frac{1}{4V} \sum_\kappa [G^2_{1\kappa} + G^2_{2\kappa}] \left[ (v^2_F + v^2_\Delta) + \rho_3 \left( v^2_F - v^2_\Delta \right) (\delta_{i\kappa} - \delta_{j\kappa}) \right], \]

since \( \sum_\kappa G^2_{1\kappa} - G^2_{2\kappa} = 0 \). We find that

\[ \frac{1}{4V} \sum_\kappa G^2_{1\kappa} + G^2_{2\kappa} = \frac{1}{2V} \sum_\kappa G^2_{1\kappa} = \frac{1}{2} \int \frac{d\kappa \xi d\kappa \eta}{4\pi^2} \left( \frac{1}{v^2_F \kappa^2 + v^2_\Delta \kappa^2 + Q_0^2} \right)^2 \]

\[ = \frac{1}{8\pi v_F v_\Delta} \int_0^\infty dx^2 \left( \frac{1}{x^2 + Q_0^2} \right)^2 = \frac{1}{8\pi v_F v_\Delta} \frac{1}{Q_0^2}. \]

The Drude spin conductivity is defined by [1]

\[ \sigma = \frac{1}{4\pi^2} \frac{v^2_F + v^2_\Delta}{v_F v_\Delta}, \]

so that the second-order gradient correction can be written as

\[ \frac{\pi}{16 Q_0^2} \sigma \int dr Tr \left[ \nabla Q(r) \cdot \nabla Q(r) \right] \]

\[ + \frac{\pi}{16 Q_0^2} \sigma \int dr \frac{v^2_F - v^2_\Delta}{v^2_F + v^2_\Delta} Tr \left\{ \rho_3 \left[ \partial_\xi Q(r) \partial_\xi Q(r) - \partial_\eta Q(r) \partial_\eta Q(r) \right] \right\}. \]

6.4. The final form of \( S_{NL\sigma M} \)

Collecting all contributions which are second order in the gradient expansion and first order in the frequency we eventually obtain

\[ S_{NL\sigma M} = \frac{\pi}{16 Q_0^2} \sigma \int dr Tr \left[ \nabla Q(r) \cdot \nabla Q(r) \right] \]

\[ + \frac{\pi}{16 Q_0^2} \sigma \int dr \frac{v^2_F - v^2_\Delta}{v^2_F + v^2_\Delta} Tr \left\{ \rho_3 \left[ \partial_\xi Q(r) \partial_\xi Q(r) - \partial_\eta Q(r) \partial_\eta Q(r) \right] \right\} \]

\[ - \omega \frac{\pi N_0}{2 Q_0} \int dr Tr [Q(r) s_3] \]

\[ = \sum_{i=1,2} \int dr \frac{\pi}{16 Q_0^2} \sigma Tr \left[ \partial_\mu Q_i(r) \alpha^{(i)(i)}_{\mu\nu} \partial_\nu Q_i(r) \right] - \omega \frac{\pi N_0}{2 Q_0} Tr [Q_i(r) s_3]. \] (63)

Here we have defined a metric tensor for the pair 1 which includes nodes ‘1’ and ‘T’,

\[ \alpha^{(1)} = \frac{2}{v^2_F + v^2_\Delta} \left( \begin{array}{cc} v^2_F & 0 \\ 0 & v^2_\Delta \end{array} \right), \]

and for the pair 2, i.e. for nodes ‘2’ and ‘T’,

\[ \alpha^{(2)} = \frac{2}{v^2_F + v^2_\Delta} \left( \begin{array}{cc} v^2_\Delta & 0 \\ 0 & v^2_F \end{array} \right). \]

In reality the above action is not the most general one allowed by symmetry. As we discussed earlier, the theory possesses two chiral Abelian sectors, which actually occur in the singlet channels \( \gamma_3 \tau_2 s_1 \rho_0 \) and \( \gamma_3 \tau_2 s_1 \rho_3 \), see (19) and (21). From analogy with [23],

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we expect that upon integrating out longitudinal fluctuations the following term would appear:

$$\delta S_{NL\sigma M} = \frac{\pi}{16^2 Q_0^2} \sigma \Pi \sum_{i=1,2} \int dr \, \text{Tr} \left[ \gamma_3 \tau_2 s_1 Q_i(r) \partial_{\mu} Q_i(r) \right] \alpha_{\mu\nu}^{(i)} \text{Tr} \left[ \gamma_3 \tau_2 s_1 Q_i(r) \partial_{\nu} Q_i(r) \right],$$

with $\Pi \propto u^2$.

In conclusion the most general non-linear $\sigma$-model is given by

$$S_{NL\sigma M} = \sum_{i=1,2} \int dr \, \frac{\pi}{16^2 Q_0^2} \sigma \text{Tr} \left[ \partial_{\mu} Q_i(r) \alpha_{\mu\nu}^{(i)} \partial_{\nu} Q_i(r) \right] - \omega \frac{\pi N_0}{2 Q_0} \text{Tr} \left[ Q_i(r) s_3 \right]$$

$$+ \frac{\pi}{16^2 Q_0^2} \sigma \Pi \text{Tr} \left[ \gamma_3 \tau_2 s_1 Q_i(r) \partial_{\mu} Q_i(r) \right] \alpha_{\mu\nu}^{(i)} \text{Tr} \left[ \gamma_3 \tau_2 s_1 Q_i(r) \partial_{\nu} Q_i(r) \right]. \quad (65)$$

7. Failure of the conventional $\sigma$-model

The non-linear $\sigma$-model (65) belongs to one of the known chiral $\sigma$-models encountered when two-sublattice symmetry holds; see [9, 23, 24]. If we simply borrow known results (see e.g. table 1 in [9]), we should expect the $\beta$ function of the conductivity to vanish in the zero-replica limit, $N \to 0$. That would imply absence of localization and persistence of diffusive modes. Moreover we should predict a quasiparticle density of states $N(\epsilon)$ which diverges like [25]–[27]

$$N(\epsilon) \approx \frac{1}{\epsilon} \exp \left[ -A \frac{3}{\ln B} \frac{\sqrt{\ln B}}{\epsilon} \right],$$

with $A$ and $B$ some positive constants. This is clearly suspicious since in the absence of disorder the density of states actually vanishes, $N(\epsilon) \sim \epsilon$.

One is tempted to correlate the above suspicious result with what is found for the elementary loops of the Wilson–Polyakov renormalization group approach. Here one integrates out iteratively degrees of freedom in a momentum shell from the highest cut-off $\Lambda$ to $\Lambda/s$, with $s \geq 0$ eventually sent to infinity. In our case these fundamental loops for either nodes are given by

$$\frac{1}{\pi} \int_{\Lambda/s < |k| < \Lambda} \frac{d\kappa}{4\pi^2} \frac{1}{\kappa_{\mu\nu}^{(1,2)}} \frac{1}{\kappa_{\nu}^{(1,2)}} = \ln s \equiv g \ln s,$$

and provide the definition of the dimensionless coupling constant $g$, which should necessarily be small to justify a loop expansion in $g \ln s \ll 1$. In our case it turns out that $g = 1$, making any loop expansion meaningless.

This results is at odds with the standard non-linear $\sigma$-models for disordered systems where $g \sim 1/\sigma \ll 1$ for weak disorder. Here, however weak the disorder, $g = 1$. This peculiar fact was originally discussed in [19], where the authors identified correctly the complete failure of the non-crossing approximation as a starting point of perturbation theory due to the absence of small control parameters. This might lead to the conclusion that the non-linear $\sigma$-model we have so far derived is useless in this problem, since it heavily relies upon the assumption that quantum fluctuations around the saddle point solution can be controlled perturbatively.
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In the following section we are going to show that the above conclusion is not correct and that one only needs to be more careful in deriving the proper non-linear $\sigma$-model action.

8. Wess–Zumino–Novikov–Witten term

Indeed we have not yet accomplished all of our plan, as we still need to evaluate the imaginary part of the action (49). At leading order we can drop the frequency dependence of (49), and hence

$$S_\Gamma = -\frac{1}{4} \text{Tr} \ln [iQ - \mathcal{H}_0] + \frac{1}{4} \text{Tr} \ln [-iQ - \mathcal{H}_0^\dagger].$$

(67)

It is more convenient to evaluate the variation of $S_\Gamma$ along a massless path defined through

$$\delta Q(r) Q(r) + Q(r) \delta Q(r) = 0.$$

(68)

We find that

$$\delta S_\Gamma = -\frac{i}{4} \text{Tr} [G \delta Q] + \frac{i}{4} \text{Tr} [G^\dagger \delta Q],$$

(69)

where

$$G = (iQ - \mathcal{H}_0)^{-1}.$$

(70)

We notice that, in the long wavelength limit,

$$\left(-iQ - \mathcal{H}_0^\dagger\right) \left(iQ - \mathcal{H}_0\right) \simeq G_0^{-1} - \nabla Q \cdot J = G_0^{-1} - J^\dagger \cdot \nabla Q,$$

(71)

and

$$\left(iQ - \mathcal{H}_0\right) \left(-iQ - \mathcal{H}_0^\dagger\right) \simeq G_0^{-1} + \nabla Q \cdot J = G_0^{-1} + J^\dagger \cdot \nabla Q,$$

(72)

and hence

$$G = [1 + G_0 \nabla Q \cdot J]^{-1} G_0 \left(-iQ - \mathcal{H}_0^\dagger\right)$$

$$\simeq [1 + G_0 \nabla Q \cdot J + G_0 \nabla Q \cdot J G_0 \nabla Q \cdot J] G_0 \left(-iQ - \mathcal{H}_0^\dagger\right),$$

$$G^\dagger = [1 + G_0 \nabla Q \cdot J]^{-1} G_0 \left(iQ - \mathcal{H}_0\right)$$

$$\simeq [1 - G_0 \nabla Q \cdot J + G_0 \nabla Q \cdot J G_0 \nabla Q \cdot J] G_0 \left(iQ - \mathcal{H}_0\right).$$

The leading non-vanishing contribution to (69) derives from the second-order gradient correction to $G - G^\dagger$, which reads

$$G - G^\dagger \simeq -2i G_0 \nabla Q \cdot J G_0 \nabla Q \cdot J G_0 Q.$$

(73)

Therefore

$$\delta S_\Gamma = -\frac{1}{2} \text{Tr} \left[\delta Q G_0 \nabla Q \cdot J G_0 \nabla Q \cdot J G_0 Q \right]$$

$$= -\frac{1}{2} \text{Tr} \left[\delta Q G_0 \nabla Q \cdot J G_0 J^\dagger \cdot \nabla Q G_0 Q \right].$$

(74)
The only term which contributes comes from the anti-symmetric component of the tensor $\mathbf{J} \otimes \mathbf{J}^\dagger$:
\[ \frac{1}{2} \left( J_i J_j^\dagger - J_j J_i^\dagger \right) = -i \epsilon_{ij} v_F v_\Delta \tau_2 s_1 \rho_3, \]
(75)
$\epsilon_{ij}$ being the Levi-Civita tensor, thus leading to
\[ \delta S_{\Gamma} = i \Gamma \sum_{i,j} \epsilon_{i,j} \text{Tr} \left[ \partial_j Q \, \partial_i Q \, \tau_2 \, s_1 \, \rho_3 \right] \]
(76)
where
\[ \Gamma = \frac{1}{4V} \sum_{\kappa} v_F v_\Delta \left[ G_{1\kappa}^3 + G_{2\kappa}^3 \right] = \frac{1}{8\pi} \int_0^\infty dx^2 \left( \frac{1}{x^2 + Q_0^2} \right)^3 = \frac{1}{16\pi} \frac{1}{Q_0^4}. \]
(77)
Introducing a fictitious coordinate which parametrizes the massless path, we finally get
\[ S_{\Gamma} = i \frac{1}{12\pi} \frac{1}{4Q_0^6} \int d^3r \epsilon_{\mu\nu\eta} \text{Tr} \left[ \tau_2 s_1 \rho_3 Q(r) \partial_\mu Q(r) Q(r) \partial_\nu Q(r) Q(r) \partial_\eta Q(r) \right] \]
\[ = i \frac{1}{12\pi} \frac{1}{4Q_0^6} \int d^3r \epsilon_{\mu\nu\eta} \text{Tr} \left[ \tau_2 s_1 Q_1(r) \partial_\mu Q_1(r) Q_1(r) \partial_\nu Q_1(r) Q_1(r) \partial_\eta Q_1(r) \right] 
- i \frac{1}{12\pi} \frac{1}{4Q_0^6} \int d^3r \epsilon_{\mu\nu\eta} \text{Tr} \left[ \tau_2 s_1 Q_2(r) \partial_\mu Q_2(r) Q_2(r) \partial_\nu Q_2(r) Q_2(r) \partial_\eta Q_2(r) \right]. \]
(78)
This actually represents opposite WZW terms for each of the two pairs of nodes, namely for each of the two independent $U(1) \times SU(4N)$ $\sigma$-models. It is clear that if the disorder coupled the two pairs of nodes, we would be forced to identify $Q_1(r)$ with $Q_2(r)$, so that the WZW term would cancel in that case. To make (78) more explicit we call that, for $a = 1, 2$,
\[ Q_a(r) = T_a(r)^{-2} s_3 Q_0 = s_3 Q_0 T_a(r)^2, \]
(79)
where
\[ T_a^2(r) = \frac{1}{2} (\gamma_0 + \gamma_3) V_a^2(r) + \frac{1}{2} (\gamma_0 - \gamma_3) c^t [V_a^{-2}(r)]^t c, \]
(80)
and
\[ V_a^2(r) = U \left[ \frac{1}{2} (s_0 + s_3) c^t \sqrt{\pi/N} \phi_a(r) g_a(r) + \frac{1}{2} (s_0 - s_3) e^{-i \sqrt{\pi/N} \phi_a(r)} g_a(r)^\dagger \right] U^\dagger, \]
(81)
$U$ being defined in equation (32). By means of (79), (80) and (81), the expression (78) can be finally written as
\[ S_{\Gamma} = i \frac{1}{12\pi} \int d^3r \epsilon_{\mu\nu\eta} \text{Tr} \left[ g_1(r)^\dagger \partial_\mu g_1(r) g_1(r)^\dagger \partial_\nu g_1(r) g_1(r)^\dagger \partial_\eta g_1(r) \right] 
- i \frac{1}{12\pi} \int d^3r \epsilon_{\mu\nu\eta} \text{Tr} \left[ g_2(r)^\dagger \partial_\mu g_2(r) g_2(r)^\dagger \partial_\nu g_2(r) g_2(r)^\dagger \partial_\eta g_2(r) \right], \]
(82)
appropriate for two WZW models $SU(4N)_1$. 

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Let us express all other terms in the action by means of $\phi_a$ and $g_a$. First of all the density of states operator becomes

$$\frac{1}{Q_0} \text{Tr} \left[ s_3 Q(r) \right] = \text{Tr} \left[ T^2(r) \right] = 2 \sum_{a=1}^2 e^{i \sqrt{\pi/N} \phi_a(r)} \text{Tr} \left[ g_a(r) \right] + e^{-i \sqrt{\pi/N} \phi_a(r)} \text{Tr} \left[ g_a(r)^\dagger \right].$$

Moreover one readily finds that

$$\frac{1}{Q_0} \text{Tr} \left[ \partial_{\mu} Q_a(r) \alpha^{(a)}_{\mu \nu} \partial_{\nu} Q_a(r) \right] = 16 \pi \partial_{\mu} \phi_a(r) \alpha^{(a)}_{\mu \nu} \partial_{\nu} \phi_a(r) + 4 \text{Tr} \left[ \partial_{\mu} g_a(r) \alpha^{(a)}_{\mu \nu} \partial_{\nu} g_a(r)^\dagger \right],$$

and

$$\frac{1}{Q_0} \text{Tr} \left[ \gamma_3 \tau_2 s_1 Q_a(r) \partial_{\mu} Q_a(r) \right] \alpha^{(a)}_{\mu \nu} \text{Tr} \left[ \gamma_3 \tau_2 s_1 Q_a(r) \partial_{\nu} Q_a(r) \right] = \frac{\pi (16N)^2}{N} \partial_{\mu} \phi_a(r) \alpha_{\mu \nu} \partial_{\nu} \phi_a(r).$$

In conclusion the action expressed in terms of the fundamental fields and including the WZW term reads

$$S_{WZW} = \sum_{a=1,2} \int d^3r \frac{\pi}{4} \sigma \text{Tr} \left[ \partial_{\mu} g_a(r) \alpha^{(a)}_{\mu \nu} \partial_{\nu} g_a(r) \right] + \pi^2 \sigma (1 + N \Pi) \partial_{\mu} \phi_a(r) \alpha^{(a)}_{\mu \nu} \partial_{\nu} \phi_a(r)$$

$$- \omega \pi N_0 \left\{ e^{i \sqrt{\pi/N} \phi_a(r)} \text{Tr} \left[ g_a(r) \right] + e^{-i \sqrt{\pi/N} \phi_a(r)} \text{Tr} \left[ g_a(r)^\dagger \right] \right\} + S^{(a)}_{\Gamma},$$

with

$$S^{(a)}_{\Gamma} = \pm i \frac{1}{12\pi} \int d^3r \epsilon_{\mu \nu \eta} \text{Tr} \left[ g_a(r)^\dagger \partial_{\mu} g_a(r) g_a(r)^\dagger \partial_{\nu} g_a(r) g_a(r)^\dagger \partial_{\eta} g_a(r) \right],$$

where the plus refers to $a = 1$ and the minus to $a = 2$.

9. Consequences of the WZW term

We have just shown that the actual field theory which describes d-wave superconductors in the presence of a disorder which only permits intra-node scattering processes is not a conventional non-linear $\sigma$-model but instead it represents two decoupled $U(1) \times SU(4N)_1$ WZW models. Moreover if, for instance, we consider pair 1, then

$$\frac{\pi}{4} \sigma \frac{2}{v_F^2 + v_\Delta^2} \int d^3r \ v_F^2 \text{Tr} \left[ \partial_{\xi} g_a(r) \partial_{\xi} g_a(r)^\dagger \right] + v_\Delta^2 \text{Tr} \left[ \partial_{\eta} g_a(r) \partial_{\eta} g_a(r)^\dagger \right]$$

$$= \frac{\pi}{4} \frac{1}{4 \pi^2} \frac{v_F^2}{v_F v_\Delta} \frac{2}{v_F^2 + v_\Delta^2} \int d^3r \ v_F^2 \text{Tr} \left[ \partial_{\xi} g_a(r) \partial_{\xi} g_a(r)^\dagger \right]$$

$$+ v_\Delta^2 \text{Tr} \left[ \partial_{\eta} g_a(r) \partial_{\eta} g_a(r)^\dagger \right]$$

$$= \frac{1}{8\pi} \int d^3r \ v_F^2 \text{Tr} \left[ \partial_{\xi} g_a(r) \partial_{\xi} g_a(r)^\dagger \right] + v_\Delta^2 \text{Tr} \left[ \partial_{\eta} g_a(r) \partial_{\eta} g_a(r)^\dagger \right],$$

which, upon the change of variable $\xi \rightarrow v_F \xi$ and $\eta \rightarrow v_\Delta \eta$, shows that each WZW model is right at its fixed point. Hence there is no ambiguity in the zero-replica limit.

Now we can derive some consequences of what we have found. The first is that the average value of the density of states (DOS) at the chemical potential stays zero, as in the

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absence of disorder and contrary to the Born approximation. In particular the dimension of the density of states operator in the zero-replica limit $N \to 0$ is

$$\Delta Q = \frac{N-1}{N} + \frac{1}{N(1+N\Pi)} \to 1 - \Pi,$$

while the dimension of the frequency is $[\omega] = 2 - (1 - \Pi) = 1 + \Pi$. This implies that the DOS at finite frequency behaves as

$$N(\omega) \sim |\omega|^{(1-\Pi)/(1+\Pi)},$$

in agreement with [19].

Notice that (86) stems from the fact that the WZW terms modifies the non-linear $\sigma$-model results leading to equation (66) in two ways:

(i) it makes $\Pi$ unrenormalized as well as $\sigma$,

(ii) it adds the further contribution 1 to the dimension of the density of states operator.

The only differences between the 1D mapping and the non-linear $\sigma$-model results for the density of states are the constant factors fixed by the range of validity. 1D mapping: $N(\omega) \sim 1/(v_F v_A) \omega(\omega/\Delta)^{-2\Pi/(1+\Pi)}$ with $\omega < \Delta$; non-linear $\sigma$-model: $N(\omega) \sim N_0(\omega_0)^{(1-\Pi)/(1+\Pi)}$ with $\omega < 1/\tau_0$, $N_0$ being the saddle point value of the density of states.

At leading order in $u^2$ the matching of the two expressions is provided by $N_0 \tau_0 \sim 1$.

The second consequence concerns the transport properties. We have shown that within the Born approximation a finite spin conductivity arises. Is this result still true beyond that approximation? The renormalization group analysis which identifies the spin conductivity with the coupling of the non-linear $\sigma$-model would say that $\sigma_s$ stays unrenormalized to its fixed point value. Can we understand this in the 1D mapping language?

In order to answer this question, we introduce a uniform spin vector potential within the action, which in our path integral approach has the form

$$A = A_0 \sigma_3 + A_1 \sigma_3 s_1.$$  

(87)

The action becomes now a functional of $A$, i.e. $S \to S(A)$, and the spin conductivity turns out to be given by

$$\sigma_s = \frac{1}{2\pi} \left( \frac{\partial^2 \ln Z(A)}{\partial A_{0,\xi}^2} - \frac{\partial^2 \ln Z(A)}{\partial A_{1,\xi}^2} \right)_{A=0} = \frac{1}{2\pi} \left( \frac{\partial^2 \ln Z(A)}{\partial A_{0,\eta}^2} - \frac{\partial^2 \ln Z(A)}{\partial A_{1,\eta}^2} \right)_{A=0},$$

(88)

where

$$Z(A) = \int DQ(r)^{-S(A)}$$

is the generating functional in the presence of $A$. It is possible to show that at second order in $A$ one needs just to make the following substitution in the action:

$$\partial_\mu Q(r) \to \partial_\mu Q(r) + \frac{i}{2} [Q(r), A_\mu].$$

(89)

Upon the action of (32)

$$A \to U^\dagger A U = A_0 \sigma_3 + A_1 \sigma_3 \tau_2 s_3,$$

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and we can show that (89) implies the following transformation law of the matrix field $G(r)$ in the presence of $A$:

$$G(r) \to \exp \left( -\frac{i}{2} \int dr' \cdot (A_0(r') \sigma_3 - A_1(r') \tau_2 \sigma_3) \right) G(r)$$

$$\times \exp \left( \frac{i}{2} \int dr' \cdot (A_0(r') \sigma_3 + A_1(r') \tau_2 \sigma_3) \right).$$

(90)

In order to better understand the role of $A$ it is convenient to translate the WZW action into the language of the underlying free one-dimensional fermions. One may identify the component $G_{1,\alpha\beta}(r)$ of the matrix field for pair ‘1’, where $\alpha$ and $\beta$ run over $4N$ indices, with

$$G_{1,\alpha\beta}(r) \to i \Psi_{1;R\alpha}(r) \Psi_{1;L\beta}^\dagger(r),$$

(91)

$\Psi_{1;R\alpha}(r)$ and $\Psi_{1;L\alpha}(r)$ being right and left moving one-dimensional Fermi fields, respectively, and the two-component vector $r$ playing the role of space and time coordinates. Since pair ‘2’ has the opposite WZW term of pair ‘1’, it is more appropriate to define

$$G_{2,\alpha\beta}(r) \to -i \Psi_{2;L\alpha}(r) \Psi_{2;R\beta}^\dagger(r),$$

(92)

which formally yields equal WZW terms. Then (90) implies for the fermions the transformation laws

$$\Psi_{1;R}(r) \to \exp \left( -\frac{i}{2} \int dr' \cdot (A_0(r') \sigma_3 - A_1(r') \tau_2 \sigma_3) \right) \Psi_{1;R}(r),$$

$$\Psi_{1;L}(r) \to \exp \left( -\frac{i}{2} \int dr' \cdot (A_0(r') \sigma_3 + A_1(r') \tau_2 \sigma_3) \right) \Psi_{1;L}(r),$$

$$\Psi_{2;R}(r) \to \exp \left( -\frac{i}{2} \int dr' \cdot (A_0(r') \sigma_3 + A_1(r') \tau_2 \sigma_3) \right) \Psi_{2;R}(r),$$

$$\Psi_{2;L}(r) \to \exp \left( -\frac{i}{2} \int dr' \cdot (A_0(r') \sigma_3 - A_1(r') \tau_2 \sigma_3) \right) \Psi_{2;L}(r),$$

which we may interpret as if $A_0$ couples to the spin density operator

$$\sum_{i=1,2} \Psi_{i;R}^\dagger(r) \sigma_3 \Psi_{i;R}(r) + \Psi_{i;L}^\dagger(r) \sigma_3 \Psi_{i;L}(r),$$

(93)

while $A_1$ couples to the spin current operator

$$\Psi_{1;R}^\dagger(r) \sigma_3 \tau_2 \Psi_{1;R}(r) - \Psi_{1;L}^\dagger(r) \sigma_3 \tau_2 \Psi_{1;L}(r) - \Psi_{2;R}^\dagger(r) \sigma_3 \tau_2 \Psi_{2;R}(r) + \Psi_{2;L}^\dagger(r) \sigma_3 \tau_2 \Psi_{2;L}(r).$$

(94)

Since the fermions are free, apart from the Abelian sector which is not coupled to $A$, the susceptibility towards $A_0$ is the same as that towards $A_1$, and hence the spin conductivity would seem to vanish at $\omega$ strictly equal to zero, again unlike what we found within the Born approximation. Actually one has to be more careful in drawing such a conclusion. Let us suppose to do the same calculation at finite frequency $\omega$ and afterwards send $\omega \to 0$. A frequency $\omega \neq 0$ plays the role of an explicit dimerization in the one-dimensional

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fermionic problem:

\[-\pi \omega N_0 \sum_{i=1,2} \text{Tr} \left[ G_i(r) + G_i^\dagger(r) \right] = -i \pi \omega N_0 \sum_{i,\alpha} \Psi_{iR\alpha}^\dagger(r) \Psi_{iL\alpha}(r) - \Psi_{iL\alpha}(r) \Psi_{iR\alpha}(r).\]

(95)

It is straightforward to show that in the presence of a finite dimerization the current–
current susceptibility is finite and practically equal to that in the absence of dimerization,
while the density–density susceptibility is zero. This discontinuous behaviour at \( \omega = 0 \)
as opposed to \( \omega \rightarrow 0 \) is again a manifestation of the chiral anomaly which plays such an
important role in this problem [19]. Since it is physically more appropriate to identify the
spin conductance through the \( \omega \rightarrow 0 \) limit, we conclude that, in spite of the vanishing
DOS, the spin conductivity is finite. In other words, in spite of the fact that the DOS is
vanishing at the chemical potential, namely that quasiparticle motion is undamped and
hence remains ballistic, yet the spin conductivity acquires a finite value in agreement with
the Drude approximation.

10. Inter-node scattering processes

So far we have just considered the role of impurity scattering within each node. Now let
us extend our analysis by including also inter-node scattering processes. Upon integrating
out the most general disorder, we find two additional terms. The first describes opposite
node scattering processes, and it reads

\[ \delta S_{\text{imp}}^I = -v^2 \int dr \left[ \Psi_1(r) \Psi_{T1}(r) \right] \left[ \Psi_2(r) \Psi_{T1}(r) \right] \left[ \Psi_2(r) \Psi_{T2}(r) \right] \left[ \Psi_1(r) \Psi_{T2}(r) \right], \]

(96)

The second derives from the impurity scattering processes which couple the two pairs of
nodes, and it is given by

\[ \delta S_{\text{imp}}^II = -w^2 \int dr \left[ \Psi_1(r) \Psi_2(r) \right] \left[ \Psi_2(r) \Psi_1(r) \right] \left[ \Psi_2(r) \Psi_{T1}(r) \right] \left[ \Psi_1(r) \Psi_{T2}(r) \right] \left[ \Psi_2(r) \Psi_{T2}(r) \right] \left[ \Psi_1(r) \Psi_{T1}(r) \right] \]

\[ + \left[ \Psi_1(r) \Psi_{T1}(r) \right] \left[ \Psi_2(r) \Psi_{T1}(r) \right] \left[ \Psi_2(r) \Psi_{T2}(r) \right] \left[ \Psi_1(r) \Psi_{T2}(r) \right] \left[ \Psi_2(r) \Psi_{T2}(r) \right]. \]

(77)

As before we can decouple the four-fermion terms by introducing auxiliary Hubbard–
Stratonovich fields. Since these fields are expected to be massive, we can further expand
the action up to second order in those fields after integrating out fermions. We finally
integrate on the auxiliary massive fields. The net result is still an action for the \( Q \) matrices
only which includes now the additional terms:

\[ \delta S_{\text{imp}}^I = -\lambda^I \frac{N_0}{Q_0} \int dr \text{Tr} \left[ Q(r) \gamma_1 Q(r) \gamma_1 \right], \]

(98)

and

\[ \delta S_{\text{imp}}^II = -\lambda^II \frac{N_0}{Q_0} \int dr \sum_{i=0,1} \text{Tr} \left[ Q(r) \rho_i \gamma_i Q(r) \rho_i \gamma_i \right], \]

(99)

where

\[ \lambda^I \sim \frac{v^2}{u^2}, \quad \lambda^II \sim \frac{w^2}{u^2}. \]

The first term (98) tends to lock \( G_1 = G_T \) and \( G_2 = G_T^2 \), while (99) locks \( G_1 = G_2 \).
When opposite node scattering is added, still leaving pairs of opposite nodes uncoupled, only the symmetric $Q$-combinations of opposite nodes stay massless, the $\Pi$-term disappears and the coset for soft modes is $Sp(2N)$ for each pair of nodes. The $\beta$ function is vanishing only because of the contribution of the WZW term, and density of states vanishes with a universal exponent in agreement with the known results \[7,19,22\]. In the absence of isotropic scattering the vanishing of the $\beta$ function still indicates that the spin and heat conductivities have a metallic behaviour.

Finally, in the generic case in which all nodes are coupled only the four-node symmetric combination of the $Q$’s is required for the soft modes. The coset is again $Sp(2N)$, but now it represents degrees of freedom coming from all nodes. The two WZW terms are written in terms of the same $Q$ and they cancel since they have opposite sign. The action then reduces to $S_{NL\sigma M}$ as derived by \[5\].

### 10.1. Scaling analysis of the general model

In order to elucidate the role of inter-node scattering processes, it is convenient to transform the two-dimensional non-linear $\sigma$-model into a 1 + 1-dimensional model of interacting fermions.

We represent the matrix fields for pair ‘1’ according to

\[
G_{1;\tau\sigma a,\tau'\sigma' a'} = i \Psi_{1;R,\tau\sigma a} \Psi_{1;L,\tau'\sigma' a'}^\dagger
\]

\[
G_{1;\tau\sigma a,\tau'\sigma' a'} = -i \Psi_{1;L,\tau\sigma a} \Psi_{1;R,\tau'\sigma' a'}^\dagger
\]

\[
G_{1;\tau\sigma a,\tau'\sigma' a'} = \left( \tau_1 \sigma_2 G_{1;\tau\sigma a}^\dagger \sigma_2 \tau_1 \right)_{\tau'\sigma' a',\tau\sigma a}
\]

\[
G_{1;\tau\sigma a,\tau'\sigma' a'} = -i \sigma' \Psi_{1;L,-\tau'\sigma' a'} \Psi_{1;R,-\tau\sigma a}^\dagger
\]

\[
G_{1;\tau\sigma a,\tau'\sigma' a'} = i \sigma' \Psi_{1;R,-\tau'\sigma' a'} \Psi_{1;L,-\tau\sigma a}^\dagger
\]

As we mentioned, since pair ‘2’ has the opposite WZW term to pair ‘1’, it is convenient to define $G_{2;\tau\sigma a,\tau'\sigma' a'} = -i \Psi_{2;L,\tau\sigma a} \Psi_{2;R,\tau'\sigma' a'}^\dagger$. In such a way that the two WZW terms become equal leaving no ambiguity in mapping the non-linear $\sigma$-model onto a one-dimensional model of interacting electrons with the interaction vertices drawn in figure 1.

We notice that the coupling $g$ in figure 1 derives from the two terms in equation (64). The bare values of the coupling constants are approximately

\[
g^{(0)} \simeq u^2
\]

\[
g_3^{(0)} \simeq v^2
\]

\[
g_b = g_{3b} \simeq w^2
\]

Generally $u^2 \geq w^2 \geq v^2$, the equality holding only for short range impurity potential. An important observation is that two-loop corrections to the renormalization group (RG) equations vanish in the zero-replica limit, and hence the RG equations valid up to two-loop order are found to be

\[
\frac{dg}{d\ln s} = g_3^2 + g_b^2 + g_{3b}^2
\]

\[
\frac{dg_3}{d\ln s} = 4 g g_3 + 4 g_b g_{3b}
\]
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Figure 1. Interaction vertices for the most general disorder. Solid (dashed) lines refer to right (left) moving fermions. The label $i = 1, 2$ refers to the two pairs of opposite nodes, $\tau, \tau' = \pm 1$ to the two Nambu components, $\sigma, \sigma' = \uparrow, \downarrow$ to the spin and $a, a' = 1, \ldots, N$ to the replicas. The symbol $\sigma \sigma'$ in front of the $g_3$ and $g_{3b}$ coupling constants means $+1$ if $\sigma = \sigma'$ and $-1$ otherwise.

\[ \frac{dg_b}{d \ln s} = 2 g g_b + 2 g_3 g_{3b}, \]
\[ \frac{dg_{3b}}{d \ln s} = 2 g_3 g_b + 2 g_{3b} g, \]

where $s \to \infty$ is the scaling parameter. As discussed in [20], the velocity anisotropy does not enter the RG equations, which remains true at least up to two-loop order in our fermionic replica trick approach. It is convenient to define $g_{\pm} = g_{3b} \pm g_b$, so that

\[ \frac{dg}{d \ln s} = g_3^2 + \frac{1}{2} (g_+^2 + g_-^2), \]
\[ \frac{dg_3}{d \ln s} = 4 g g_3 + (g_+^2 - g_-^2), \]
\[ \frac{dg_+}{d \ln s} = 2 (g + g_3) g_+, \]
\[ \frac{dg_-}{d \ln s} = 2 (g - g_3) g_-. \]

Given the appropriate bare values of the amplitudes, one readily recognizes that the RG flow maintains the initial condition $g_- = 0$, and hence the scaling equations reduce to

\[ \frac{dg}{d \ln s} = g_3^2 + \frac{1}{2} g_+^2, \] (101)
\[ \frac{dg_3}{d \ln s} = 4 g g_3 + g_+^2, \] (102)
\[ \frac{dg_+}{d \ln s} = 2 (g + g_3) g_+. \] (103)

The RG equations with the appropriate initial conditions always flow to strong coupling with

\[ g_3 \sim 2 g \sim g_+ \to \frac{1}{3} \frac{1}{\ln s_c - \ln s}, \]

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where $s_c$ can be interpreted as the correlation length of the modes which acquire a mass gap $E_c$ by the interaction, with

$$E_c \simeq \frac{\sqrt{v_F v_D}}{s_c}.$$  

Therefore the two pairs of nodes get strongly coupled, in agreement with references [19, 20, 22].

### 10.2. Strong coupling analysis

In order to gain further insight into the strong coupling phase towards which RG flows, let us consider the case of a single replica $N = 1$. For further simplification it is convenient to adopt the same approach as in [5, 19, 20, 22] and neglect the role of the opposite frequencies, which amounts to dropping the $\tau$ label. The model thus reduces to two interacting chains of spinful fermions, each chain representing a pair of nodes.

The coupling $g$ of figure 1 only couples to the charge sector and makes the intra-chain Umklapp coupling, the coupling $g_3$, a relevant perturbation which opens a charge gap on each chain. Therefore the model is equivalent to two coupled chains of spin 1/2. If we denote by $\vec{n}_1(2)(x)$ and $\epsilon_1(2)(x)$ respectively the staggered magnetization and the dimerization of chain 1 (2), the coupling among the chains is ferromagnetic and given by

$$w^2 \int dx \epsilon_1(x) \epsilon_2(x) - 4 \vec{n}_1(x) \cdot \vec{n}_2(x).$$

As shown in [28], this model is equivalent to an $SO(4)$ Gross–Neveu one which turns out to be fully massive or, equivalently, to four two-dimensional off-critical classical Ising models, three ordered and one disordered, $\langle \sigma_1 \rangle = \langle \sigma_2 \rangle = \langle \sigma_3 \rangle = \langle \mu_4 \rangle = \sigma \neq 0$, where $\sigma_i$ and $\mu_i$, $i = 1, \ldots, 4$, are order and disorder parameters, respectively. The ground state is rigid to an external magnetic field and to a spin vector potential opposite for the two chains, and hence the conductivity is zero. As we discussed, a finite frequency amounts to adding a term

$$\omega \int dx \epsilon_1(x) + \epsilon_2(x) \propto \omega \int dx \sigma_1(x) \sigma_2(x) \sigma_3(x) \sigma_4(x) \simeq \omega \sigma^3 \int dx \sigma_4(x),$$

which actually plays the role of an external magnetic field acting on the fourth disordered Ising copy. The net result is that $\langle \sigma_4(x) \rangle_{\omega \neq 0} \sim \omega$, which in turn means that the DOS remains linear in frequency. Even in the presence of an explicit dimerization, the susceptibilities towards a magnetic field or towards a spin vector potential opposite for the two chains still vanish. Were these results valid for any $N$, even for $N \to 0$, we should conclude that (i) the model is indeed insulating; (ii) the DOS is linear in frequency, in full agreement with [6].

### 10.3. Identification of relevant energy scales

The previous analysis of the $N = 1$-model shows that the vanishing of thermal and spin conductivities in the model for a disordered d-wave superconductor translates in the language of the effective one-dimensional fermionic model into the existence of a finite spin gap. The correlation length associated with this gap should then represent the localization length of the Bogoliubov quasiparticles. We may estimate this correlation length as the
scale $s_c$ at which the RG equations (101)–(103) encounter a singularity. In addition we may introduce the mass gap $E_c$ through equation (104) which can be identified as the energy scale around which localization effects appear. It is worth noticing that $s_c$ underestimates the spin correlation length, and hence the actual localization length. The reason is that the RG equations blow up on a scale which is related to the largest gap in the excitation spectrum. Since the coupling $g$ only affects charge degrees of freedom, the largest gap is expected in the charge sector, the spin gap being smaller. Keeping this in mind, in what follows we shall discuss how $s_c$, or better $E_c$, depends on the range of the disorder potential.

First of all we need to identify some reference scale to compare with $E_c$. The natural candidate would be the inverse relaxation time $1/\tau_0$ in the Born approximation. In the generic case $u^2 \geq w^2 \geq v^2$, $1/\tau_0 = 2Q_0$ where $Q_0$ is obtained by equation (56) with $u^2$ substituted by $u^2 + 2w^2 + v^2$. We expect the actual $E_c$ to be always smaller then $1/\tau_0$, the two values being closest for extremely short range disorder, namely $u^2 = w^2 = v^2$. Since the derivation of the non-linear $\sigma$-model does not provide the precise dependence of the initial values of the coupling constants, $g^{(0)}$, $g_3^{(0)}$ and $g_+^{(0)}$, on the impurity potential, we will assume that the short range disorder corresponds to $2g^{(0)} = g_3^{(0)} = g_+^{(0)}$ and moreover that the mass gap $E_c$ in this case can be identified as $1/\tau_0$ in the Born approximation. Upon integrating the RG equations (101)–(103) with this initial condition, the value of $E_c \approx 1/\tau_0$ is found to be

$$
\frac{1}{\tau_0} \approx \Lambda \exp \left(-\frac{1}{2g^{(0)} + g_3^{(0)} + g_+^{(0)}}\right),
$$

where $\Lambda$ is the ultraviolet cut-off of the order of the gap $\Delta$. In order to appreciate the role of the range of the disorder potential, let us analyse the RG equations (101)–(103) keeping fixed the combination $g_0 = 2g^{(0)} + g_3^{(0)} + g_+^{(0)}$, i.e. at constant $1/\tau_0$, and increasing the value of $g^{(0)}$ at the expense of $g_3^{(0)} + g_+^{(0)} = g_0 - 2g^{(0)}$. One readily realizes that as $g^{(0)}$ increases, $E_c$ decreases from its short range value (105). For instance, if we assume $g_3^{(0)} = 0$ and $g_+^{(0)} = g_0 - 2g^{(0)} \ll g_0$, then

$$
E_c \approx \frac{1}{\tau_0} \left(\frac{e^{g^{(0)}}}{4g^{(0)}}\right)^{1/g_0} \ll \frac{1}{\tau_0},
$$

which explicitly shows that localization effects may show up at energies/temperatures much smaller than the $1/\tau_0$ in the Born approximation.

11. Conclusions

In this work we have presented a derivation of the non-linear $\sigma$-model for disordered d-wave superconductors able to deal with a finite range impurity potential, namely with an intra-node scattering potential generically different from the inter-nodal one. Within this derivation we have been able to clarify some controversial issues concerning the validity of a conventional non-linear $\sigma$-model approach when dealing with disordered Dirac fermions. We have indeed found that the non-linear $\sigma$-model approach is actually equivalent to the alternative method first introduced in [19] which consists in mapping the disordered model onto a one-dimensional model of interacting fermions. The energy upper cut-off is provided by the inverse Born relaxation time $1/\tau_0$ in the non-linear $\sigma$-model approach.
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and by the superconducting gap $\Delta$ in the 1D mapping. A closely related aspect which also emerges for d-wave superconductors is the existence of a Wess–Zumino–Novikov–Witten term related to the vorticity of the spectrum in momentum space. Both of these features are responsible for several interesting phenomena. For instance, unlike for conventional disordered systems, in this case disorder starts playing a role (particularly in the density of states) when the quasiparticle motion is still ballistic. In contrast, the onset of localization precursor effects is pushed towards energies/temperatures lower than the inverse relaxation time in the Drude approximation. More specifically, the longer the range of the impurity potential, the later the localization effects appear. This result may explain why experiments have so often failed to detect localization precursor effects in cuprate superconductors.

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