Convergence of a Distributed Kiefer-Wolfowitz Algorithm

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Abstract
This paper proposes a proof of the convergence of a distributed and asynchronous version of the Kiefer-Wolfowitz algorithm.

Keywords
Optimization, Distributed, Asynchronous

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1. Introduction
The goal is to maximize a concave function of $K > 1$ variables. There are $K$ agents and each agent observes the values of the function, corrupted by observation noise, and adjusts his own variable without knowing the values of the other variables. The agents do not communicate their variables. This formulation is motivated by many applications where the agents do not know each other or are not able to communicate directly with one another. Moreover, the agents are not synchronized, so that they update their variable either at the same or different times.

Each agent experiments by perturbing his variable by a zero-mean change in order to estimate the partial derivative of the function with respect to that variable. He then updates his variable in proportion to the estimate of the partial derivative.

This algorithm is an extension of [2] and [5]. In [2], the authors introduce a gradient descent algorithm where the gradient is estimated by observing the function at perturbed values of its variable and they prove the convergence of the algorithm to the minimum of the function. [5] proposes a variation of the algorithm in the multivariate case where the partial derivatives with respect to the different variables are estimated by simultaneously perturbing each variable by an independent and zero-mean amount, instead of perturbing the variables one at a time. The author proves the convergence to the minimum of the function under the assumption that the variables return infinitely often to a compact set. In this paper, we extend the algorithm to the case where the different variables get updated asynchronously. Also, the proof does not require assuming returns to a compact set.

An agent corrupts the estimate of the partial derivative of another agent either when he experiments or updates his variable while the other agent calculates his estimate. Technically, the difficult aspect of our version is that the corruption of the estimate by the updates of other agents is not zero-mean, in contrast with the corruption by their experiments which is zero-mean. Proving the convergence of this asynchronous version requires careful bounds on the size of the corruptions. This is the technical contribution of the paper.

Some papers propose mechanisms where agents exchange the value of their variables, possibly with some delays, and they may know the function they want to maximize (e.g., [1], [3], [4], [6]). The key contribution of this paper is to show that such communication is not necessary for convergence. Also, the agents observe the values of the function with some observation noise but need not know its functional form. That is, the agents can observe the effect of their choice of value for their variable, but they could not calculate it. The algorithm is similar in spirit to tâtonnement (groping) in economics (see [7]).

2. Algorithm and Result
Let $f: \mathbb{R}^K \to \mathbb{R}$ be a concave function, strictly concave in a neighborhood of its maximizer $x^*$. Assume that the function is globally Lipschitz with constant $L$. Assume also that the second and third derivatives of $f(\cdot)$ are bounded. Let $\tau \geq 2$ be an integer and $p_k \in \{0, \ldots, \tau-1\}$ for $k = 1, \ldots, K$. Let also $T_k(n) = n\tau + p_k - \tau \{p_k = \tau - 1\}$ for $k = 1, \ldots, K$ and $n \geq 0$. Note that $T_k(n) + 1 \in \{n\tau, \ldots, n\tau + \tau - 1\}$. For $k = 1, \ldots, K$, agent $k$ experiments at times $\{T_k(n) + 1, n \geq 0\}$ and updates at times $\{T_k(n) + 2, n \geq 0\}$. Thus, the agents experiment and update every $\tau$ steps and they may be out of phase with one another. The case of a single agent (i.e., $K = 1$) is the same as in [2] while that of simultaneous updates (i.e., $p_1 = \cdots = p_K$) corresponds to [5].

The experiments and updates are defined as follows. For $k = 1, \ldots, K$ and $n \geq 0$, let $x_k(n)$ be the value of the variable of agent $k$ at step $n$. Let also $x(n)$ be the vector with components $x_k(n)$, for $n \geq 0$.

The algorithm is as follows. For $k = 1, \ldots, K$ and $n \geq 0$,
one has, for $m = T_k(n)$,

\[ x_k(m + 1) = x_k(m) + a_k(n)\varepsilon(n) \quad \text{(experiment)} \tag{1} \]
\[ x_k(m + 2) = x_k(m) + g_k(n)\gamma(n) \quad \text{(update)} \tag{2} \]

where

\[ g_k(n) = \frac{f(x(m + 1)) - f(x(m)) + \eta_k(n)}{x_k(m + 1) - x_k(m)}; \tag{3} \]

\[ a_k(n) \text{ are independent with}\]
\[ P(a_k(n) = -1) = P(a_k(n) = 1) = 0.5; \tag{4} \]

\[ \eta_k(n) \text{ independent, zero-mean, bounded}; \tag{5} \]

\[ \varepsilon(n), \gamma(n) \in (0, 1), \sum_{n=1}^{\infty} \frac{\gamma^2(n)}{\varepsilon^2(n)} < \infty, \quad \sum_{n=1}^{\infty} \gamma(n)\varepsilon^2(n) < \infty, \tag{6} \]

and $\gamma(n)$ is bounded. (For instance, $\gamma(n) = n^{-0.75}, \varepsilon(n) = n^{-0.2}$.)

Our objective is to prove the following theorem.

**Theorem 1.** One has

\[ x_n \to x^*, \text{ almost surely as } n \to \infty \]

where $x^*$ is the maximizer of $f(\cdot)$ on $\mathbb{R}^K$.

### 3. Proof Outline

Let $z(m)$ be the vector with components $z_k(m) = x_k(m) - a_k(n)\varepsilon(n)$, for $m = T_k(n)$, $1 \leq n \leq n_0$. That is, $z_k(m)$ is the latest updated value of $x_k$ by time $m$. Note that $z_k(m)$ does not change when user $k$ performs an experiment, only when he updates his variable. Of course $x$ changes during experiments and updates, and the gradient is estimated by observing $f(x)$, not $f(z)$.

Fix any $\delta > 0$. It is shown in the Appendix that $u(n) = \| z(n\tau) - x^* \|$ satisfies the following two inequalities:

\[ u(n + 1) \leq u(n) - [\gamma(n)\beta - \alpha(n)], \text{ whenever } u(n) > \delta \tag{7} \]

and

\[ u^2(n + 1) \leq u^2(n) + c(n), \text{ whenever } u(n) \leq \delta \tag{8} \]

In these expressions, $\beta > 0, c(n) \to 0$, and $\Sigma \alpha(n)$ converges to a finite random variable.

The claim is that these inequalities imply that $u(n) \leq 3\delta$ for all $n \geq n_0$ for some finite $n_0$. To see this, choose $n_0$ so that $c(n) \leq 3\delta^2$ for $n \geq n_0 - 1$ and $\sum_{n=n_0}^{\infty} \alpha(n) \leq \delta$ for all $m \geq 0$. Let $n_1$ be the first time after $n_0$ that $u(n) > \delta$. If there is no such time, we are done. Else, let $m_1$ be the first time after $n_1$ that $u(n) \leq \delta$. Such a time must exist because of (7), for otherwise $u(n) \to -\infty$ since $\Sigma \gamma(n) = \infty$ and $\Sigma \alpha(n) \to \infty$. Let then $n_2$ be the first time after $m_1$ that $u(n) > \delta$, then $m_2$ the first time after $n_2$ that $u(n) \leq \delta$, and so on. Finally, let $v(j)$ be the maximum value of $u(n)$ for $n \in \{n_j, \ldots, m_j - 1\}$. Because of (8), $u^2(n_j) \leq u^2(n_j - 1) + c(n_j - 1) \leq \delta^2 + 3\delta^2$, so that $u(n_j) \leq 2\delta$. Also, because of (7), $v(j) - u(n_j) \leq \max_{n \geq n_0} \alpha(n) \leq \delta$. Hence $v(j) \leq 3\delta$ for all $j$, so that $u(n) \leq 3\delta$ for all $n \geq n_0$.

Since $\delta > 0$ is arbitrary, it follows that $u(n) \to 0$. Since $\|z(n\tau) - x(\tau)\| \leq \varepsilon(n)$, this implies that $x(n) \to x^*$.

### 4. Conclusions

This paper proves the convergence of a distributed version of the Kiefer-Wolfowitz algorithm under some strong assumptions. The function is assumed to be strictly concave in a neighborhood of its maximizer and with bounded derivatives up to the third order. The observation noise is assumed to be bounded. The agents update periodically, with the same period, but possibly with different phases. The proof is self-contained and does not require assuming that the variables visit a compact set infinitely often. Instead, it shows that the updates prevent the variables from drifting away.

Many of these assumptions are stronger than necessary. For instance, the periods of the different agents could be different. This assumption can probably be relaxed further by assuming only that the rates of update converge. Convergence in probability should occur if only moments of the noise are bounded. Relaxing the assumptions and a projection version of the algorithm are left for further study.

### 5. Acknowledgements

This work was motivated by an application to wireless networks studied with Piotr Gawlowicz and Adam Wolisz. They identified the importance of asynchronous distributed updates in that application. I am grateful for their suggestions for this paper.

### 6. Appendix: Proof of (7)-(8)

We first give the main steps that lead to the inequalities. The rest of the appendix provides the details of the calculations.

#### Main Steps

Inequality (7) says that when $z$ is away from the maximizer $x^*$ the gradient updates bring it closer. This is intuitive since the gradient is then large. Inequality (8) says that when $z$ is close to the maximizer, the updates do not make it move far away. This happens because the gradient is then small.

Every $\tau$ steps, each variable $x_k$ gets updated roughly in the direction of the partial derivative of $f(\cdot)$ with respect to that variable. Thus, $z$ gets updated roughly in the direction of the gradient $\nabla f(x)$. Errors occur because of corruptions of the gradient estimate due to observation noise and the changes of the other variables by other agents. More precisely, using (3) one finds (see Lemma 4)

\[ w(n) := z(n\tau + \tau) - z(n\tau) = \gamma(n)\nabla f(z(n\tau)) + \mu(n)\gamma(n)/\varepsilon(n) + O(\rho(n)) \tag{9} \]
where \( \rho(n) = \max \{ \gamma^2(n)/\epsilon(n), \gamma(n)\epsilon^2(n) \} \) and \( \mu(n) \) is a bounded random vector that is zero-mean given \( \mathcal{F}_{n-1} \) where
\[
\mathcal{F}_n := \{ a_k(m), \eta_k(m), m \leq n; k = 1, \ldots, K \}.
\]
Also, in (9), \( O(\rho(n)) \) is a random vector whose components are bounded in absolute value by a constant times \( \rho(n) \).

Identity (9) implies (see Lemma 5),
\[
\|w(n)\|^2 = O(\gamma^2(n)/\epsilon^2(n)). \tag{10}
\]
Hence,
\[
u^2(n+1) = \|z(n\tau + \tau) - \bar{\mathbf{x}}\|^2 = \|z(n\tau) - \bar{\mathbf{x}} + w(n)\|^2
\]
\[
= \nu^2(n) + 2(z(n\tau) - \bar{\mathbf{x}})^\top w(n) + \|w(n)\|^2
\]
\[
= \nu^2(n) + 2\gamma(n/(\epsilon(n))(z(n\tau) - \bar{\mathbf{x}})^\top \mu(n)
\]
\[
+ 2\gamma(n)/(\epsilon(n))(z(n\tau) - \bar{\mathbf{x}})\n\]
\[
\mu(n)
\]
\[
+ 2(\gamma(n)/(\epsilon(n))(z(n\tau) - \bar{\mathbf{x}})^\top O(\rho(n)) + O(\gamma^2(n)/\epsilon^2(n)). \tag{12}
\]
Now,
\[
(z(n\tau) - \bar{\mathbf{x}})^\top \n\]
\[
\mu(n)
\]
\[
= u(n) \sum_k h_k(n) \mu_k(n)
\]
with
\[
h_k(n) = \frac{z_k(n\tau)}{u(n)} - \bar{x}_k^*.
\]
Hence, when \( u(n) > \delta \),
\[
u^2(n+1) \leq \nu^2(n) - 2\gamma(n)Bu(n)
\]
\[
+ 2u(n)((\gamma(n)/(\epsilon(n)) \sum_k h_k(n) \mu_k(n)
\]
\[
+ 2u(n)O(\rho(n)) + O(\gamma^2(n)/\epsilon^2(n))
\]
\[
\leq \nu^2(n) - 2\gamma(n)Bu(n)
\]
\[
+ 2u(n)((\gamma(n)/(\epsilon(n)) \sum_k h_k(n) \mu_k(n)
\]
\[
+ 2u(n)O(\rho(n)) + O(\gamma^2(n)/\epsilon^2(n))(2\delta)
\]
\[
= \nu^2(n) - 2\gamma(n)Bu(n)
\]
\[
+ 2u(n)((\gamma(n)/(\epsilon(n)) \sum_k h_k(n) \mu_k(n)
\]
\[
+ 2u(n)O(\rho(n)) + O(\gamma^2(n)/\epsilon^2(n))(2\delta)]
\]
\[
\leq \nu^2(n) - 2\gamma(n)Bu(n)
\]
\[
+ 2u(n)((\gamma(n)/(\epsilon(n)) \sum_k h_k(n) \mu_k(n) + O(\kappa(n)))
\]

where
\[
\kappa(n) = \max \{ \gamma^2(n)/\epsilon^2(n), \rho(n) \}
\]
\[
= \max \{ \gamma^2(n)/\epsilon^2(n), \gamma(n)\epsilon^2(n) \}.
\]

Hence,
\[
u^2(n+1) \leq \nu^2(n) - 2u(n)[\beta \gamma(n) - \alpha(n)] \tag{14}
\]

where
\[
\alpha(n) = (\gamma(n)/(\epsilon(n)) \sum_k h_k(n) \mu_k(n) + O(\kappa(n)).
\]

Now, (14) implies implies (7), i.e.,
\[
u(n+1) \leq u(n) - [\beta \gamma(n) - \alpha(n)].
\]

Indeed, if this last inequality were violated, one would have
\[
u^2(n+1) > \{u(n) - [\beta \gamma(n) - \alpha(n)]\}^2
\]
\[
= \nu^2(n) - 2u(n)[\beta \gamma(n) - \alpha(n)] + [\beta \gamma(n) - \alpha(n)]^2
\]
\[
\geq \nu^2(n) - 2u(n)[\beta \gamma(n) - \alpha(n)].
\]

and this would contradict (14).

To show that \( \sum \alpha(n) \) converges to a finite random variable in Lemma 8, one uses the martingale convergence theorem for the first term and the fact that \( \sum \kappa(n) < \infty \) by (6). For the first term, the key observation is that \( h_k^2(n) \leq 1 \). (See Lemma 8.)

When \( u(n) \leq \delta \), (12)
\[
u^2(n+1) \leq \nu^2(n) + 2(\gamma(n)/(\epsilon(n))(z(n\tau) - \bar{\mathbf{x}})^\top \mu(n) + O(\kappa(n))
\]
\[
= \nu^2(n) + c(n)
\]
with
\[
c(n) = 2(\gamma(n)/(\epsilon(n))(z(n\tau) - \bar{\mathbf{x}})^\top \mu(n) + O(\kappa(n)).
\]

The martingale convergence theorem implies that the first term goes to zero, because \( \|z(n\tau) - \bar{\mathbf{x}}\|^2 \leq \delta \) and \( \sum \gamma^2(n)/\epsilon^2(n) < \infty \). The last term also goes to zero. (See Lemma 9.)

The next section develops some estimates.

**Preliminary Calculations**

We recall the following notation that avoids having to keep track of explicit constants.

**Definition 1.** Let \( \{h(n), n \geq 0\} \) be a sequence of positive numbers. By definition, \( \{O(h(n)), n \geq 0\} \) designates a sequence of random variables such that
\[
|O(h(n))| \leq Ch(n), \forall n
\]
for some constant \( C \).

The same notation is used when the variables \( O(h(n)) \) are deterministic and in the vector case when the inequality holds componentwise.
An important observation is that the gradient estimates calculations of random variables time of agent \((\gamma(n))\) is bounded, by (6).

**Lemma 1.** One has
\[
\begin{align*}
\mathbb{E}[\mathcal{O}(h(n))] &= \mathbb{E}[\mathcal{O}(n^\alpha)], \forall \alpha > 0 \quad (15) \\
\mathbb{E}[\mathcal{O}(h_1(n)) \times \mathcal{O}(h_2(n))] &= \mathbb{E}[\mathcal{O}(h_1(n)h_2(n))],\quad \mathbb{E}[\mathcal{O}(h_1(n)) + \mathcal{O}(h_2(n))] = \mathbb{E}[\max\{h_1(n),h_2(n)\}] \\
\text{If } h_1(n) \leq C_1h_2(n) \leq C_2h_1(n), n \geq n_0, \text{ then } \mathbb{E}[\mathcal{O}(h_1(n))] = \mathbb{E}[\mathcal{O}(h_2(n))]
\end{align*}
\]
(17)

\[\max\{\mathbb{E}(n),\gamma(n)(A+B/\mathbb{E}(n))\} = \mathbb{E}(n).\]  
(19)

**Lemma 2.** Let \(m = T_k(n)\). We claim that
\[
\begin{align*}
f(x(m+1)) - f(x(m)) &= a_k(n)\mathbb{E}(n)f_k(z(n\tau)) \\
&+ V\mathbb{E}(n) + a_k(n)U\mathbb{E}(n) + a_k(n)V'\gamma(n) + O(\mathbb{E}^3(n))
\end{align*}
\]
(20)

where \(U,V,V'\) are bounded and independent of \(a_k(n)\) and \(\mathcal{F}_{n-1}\), and \(U\) is zero-mean. Also, \(f_k(z(n\tau))\) is the partial derivative of \(f(\cdot)\) with respect to \(z_k\) evaluated at \(z(n\tau)\).

**Proof.** Proof of Lemma 2

Let \(m = T_k(n)\). Recall that \(T_k(n) + 1\) is the experiment time of agent \(l\) during \(\{n\tau,\ldots,n\tau + \tau - 1\}\), so that \(T_k(n) + 2\) is his update time. The update equations (1) and (2) imply that, for \(m \in \{n\tau,\ldots,n\tau + \tau - 1\}\),
\[
q_l := x_l(m) - z_l(n\tau) = \begin{cases} 
  a_l(n)\mathbb{E}(n), & \text{if } T_l(n) = T_k(n) - 1 \\
  g_l(n)\gamma(n), & \text{if } T_l(n) \leq T_k(n) - 2 \\
  0, & \text{otherwise}
\end{cases}
\]
and
\[
r_l := x_l(m+1) - z_l(n\tau) = \begin{cases} 
  a_l(n)\mathbb{E}(n), & \text{if } T_l(n) = T_k(n) \\
  g_l(n)\gamma(n), & \text{if } T_l(n) \leq T_k(n) - 1 \\
  0, & \text{otherwise}
\end{cases}
\]

An important observation is that the gradient estimates \(g_l(n)\) for \(l \neq k\) are only affected by \(a_k(n)\) at and after time \(m + 1\) and then used to update \(x_l\) at or after time \(m + 2\). Thus, the random variables \(q_l\) and \(r_l\) for \(l \neq k\) that enter in the calculations of \(x(m)\) and \(x(m+1)\) are independent of \(a_k(n)\). Moreover, \(a_k(n)\) is independent of \(z(n\tau)\).

Definition (3) shows that, for all \(l = 1,\ldots,K\),
\[
|g_l(n)| \leq L + G/\mathbb{E}(n) = O(1/\mathbb{E}(n))
\]
where \(L\) is the Lipschitz constant and \(G\) is the bound on \(\eta_k(n)\).

The identities above show that \(|r| = O(\mathbb{E}(n))\) and \(|q| = O(\mathbb{E}(n))\) because \(g_l(n)\gamma(n) = O(\mathbb{E}(n))\) by (19). Taylor’s theorem implies the following identity:
\[
\begin{align*}
f(x(m+1)) - f(x(n\tau)) &= f(x(n\tau) + r) - f(x(n\tau)) \\
&= r'\nabla f(x(n\tau)) + \frac{1}{2}r'HR + O(\mathbb{E}^3(n))
\end{align*}
\]
where \(H = Hf(z(n\tau))\) is the Hessian of \(f(\cdot)\) evaluated at \(z(n\tau)\).

Similarly,
\[
\begin{align*}
f(x(m)) - f(x(n\tau)) &= f(x(n\tau) + q) - f(x(n\tau)) \\
&= q'\nabla f(x(n\tau)) + \frac{1}{2}q'HR + O(\mathbb{E}^3(n)).
\end{align*}
\]

Subtracting these two expressions, we find
\[
\begin{align*}
\mathbb{E}(m+1) - f(x(m)) &= (r - q)'HR + O(\mathbb{E}^3(n)) \\
&+ \frac{1}{2}(r - q)'HR + O(\mathbb{E}^3(n)).
\end{align*}
\]

Now,
\[
r_l - q_l = \begin{cases} 
  a_l(n)\mathbb{E}(n), & \text{if } T_l(n) = T_k(n) \\
  g_l(n)\gamma(n) - a_l(n)\mathbb{E}(n), & \text{if } T_l(n) = T_k(n) - 1 \\
  0, & \text{otherwise}
\end{cases}
\]
and
\[
r_l + q_l = \begin{cases} 
  a_l(n)\mathbb{E}(n), & \text{if } T_l(n) = T_k(n) \\
  g_l(n)\gamma(n) + a_l(n)\mathbb{E}(n), & \text{if } T_l(n) = T_k(n) - 1 \\
  2g_l(n)\gamma(n), & \text{if } T_l(n) \leq T_k(n) - 2 \\
  0, & \text{otherwise}
\end{cases}
\]

In the rest of this proof, \(U,U_1,U_2,U_3\) designate random variables that are bounded, zero-mean and independent of \(a_k(n)\) and \(V,V',V_1,V_2,V_3,V_4\) designate random variables that are bounded and independent of \(a_k(n)\).

By examining the terms in \(r - q\), we find that
\[
(r - q)'\nabla f(x(n\tau)) = a_k(n)\mathbb{E}(n)f_k(z(n\tau)) + W
\]
where \(W\) is a sum of terms of the forms
\[
a_l(n)\mathbb{E}(n)f_l(z(n\tau)) \quad \text{and} \quad g_l(n)\gamma(n)f_l(z(n\tau)).
\]

Thus, the terms of the above two types are either of the form
\[
U_1\mathbb{E}(n) \quad \text{or} \quad V_1\mathbb{E}(n).
\]

We conclude that
\[
(r - q)'\nabla f(x(n\tau)) = a_k(n)\mathbb{E}(n)f_k(z(n\tau)) + U_1\mathbb{E}(n) + V_1\mathbb{E}(n).
\]

The sum \((r - q)'H(r + q)\) is composed of terms that are multiples of one of the following three expressions:
\[
a_k(n)a_j(n)H_{ij}\mathbb{E}^2(n), a_k(n)g_j(n)H_{ij}\mathbb{E}(n)\mathbb{E}(n), g_k(n)g_j(n)H_{ij}\mathbb{E}^2(n).
\]

Terms of first type yield a sum \(a_k(n)U_2\mathbb{E}^2(n) + V_2\mathbb{E}^2(n)\) where \(a_k(n)U_2\mathbb{E}^2(n) = 2\sum_{l \neq k} a_k(n)a_j(n)H_{ij}\) and
\[
V_2\mathbb{E}^2(n) = H_{k,k}\mathbb{E}^2(n) + \sum_{l \neq k} a_l(n)a_j(n)\mathbb{E}(n)\mathbb{E}(n)H_{ij}.
\]

Terms of the second or third type yield a sum \(U_3\mathbb{E}(n) + a_k(n)V_3\mathbb{E}(n) + V_4\mathbb{E}^2(n)/\mathbb{E}^2(n).\)
Combining the observations above, we conclude that
\[
f(x(m + 1)) - f(x(m)) = a_k(n)\varepsilon(n)f_k(z(n\tau)) + U_1\varepsilon(n) + V_1\gamma(n)/\varepsilon(n) \\
+ a_k(n)U_2\varepsilon^2(n) + V_2\varepsilon^2(n) \\
+ U_3\gamma(n) + a_k(n)V_3\gamma(n) + V_4\gamma^2(n)/\varepsilon^2(n) + O(\varepsilon^3(n)) \\
= a_k(n)\varepsilon(n)f_k(z(n\tau)) + V\varepsilon(n) + a_k(n)U\varepsilon^2(n) \\
+ a_k(n)V'\gamma(n) + O(\varepsilon^2(n))
\]

where \(U, V, V'\) are defined as
\[
V\varepsilon(n) = U_1\varepsilon(n) + V_2\varepsilon^2(n) + U_3\gamma(n) \\
+ a_k(n)U_2\varepsilon^2(n) \\
a_k(n)V'\gamma(n) = a_k(n)V_3\gamma(n).
\]

This is (20).

You will note that in this derivation, all the terms involving \(g_k(n)\) are due to the asynchronous updates where some agents update while others are estimating the partial derivatives.

**Lemma 3.** Let \(m = T_k(n)\). We claim that
\[
g_k(n) = f_k(x(n\tau)) + \mu_k(n)/\varepsilon(n) + O(\gamma(n)/\varepsilon(n)) + O(\varepsilon^2(n))
\]
(21)
where \(\mu_k(n)\) is a bounded random variable that is zero-mean given \(\mathcal{F}_{n-1}\).

**Proof.** Proof of Lemma 3

Since \(a_k(n) = 1/\delta_k(n)\) (because \(a_k(n) \in \{1, 1\}\) one has, using Lemma 2,
\[
g_k(n) = \frac{f(x(m + 1)) - f(x(m)) + \eta_k(n)}{a_k(n)\varepsilon(n)} \\
= \frac{f_k(x(n\tau)) + a_k(n)\varepsilon(n)f_k(z(n\tau)) + V\varepsilon(n) + a_k(n)U\varepsilon^2(n) + V_3\gamma(n) + a_k(n)V'\gamma(n)/\varepsilon(n) + O(\varepsilon^2(n))}{\delta_k(n)}
\]
This expression is of the form (21), with
\[
\mu_k(n)/\varepsilon(n) = a_k(n)V + U\varepsilon(n) + a_k(n)\eta_k(n)/\varepsilon(n)
\]
and
\[
O(\gamma(n)/\varepsilon(n)) + O(\varepsilon^2(n)) = V'\gamma(n)/\varepsilon(n) + a_k(n)O(\varepsilon^2(n)).
\]

**Proofs of the Main Steps**

The following Lemma shows that (9) holds.

**Lemma 4.** Let \(w(n) = z(n\tau + \tau) - z(n\tau)\). One has
\[
w(n) = \gamma(n)\nabla f(z(n\tau)) + (\gamma(n)/\varepsilon(n))\mu(n) + O(\rho(n))
\]
(22)
where \(\mu(n)\) is a bounded random vector that is zero-mean given \(\mathcal{F}_{n-1}\) and \(\rho(n) = \max\{\gamma^2(n)/\varepsilon(n), \gamma(n)\varepsilon^2(n)\}\).

**Proof.** Proof of Lemma 4

One has
\[
z_k(n\tau + \tau) = z_k(n\tau) + \gamma(n)\mu_k(n),
\]
(23)
so that Lemma 3 implies that
\[
w(n) = \gamma(n)\nabla f(z(n\tau)) + (\gamma(n)/\varepsilon(n))\mu(n) \\
+ \gamma(n)[O(\gamma(n)/\varepsilon(n)) + O(\varepsilon^2(n))]
\]
\[
= \gamma(n)\nabla f(z(n\tau)) + (\gamma(n)/\varepsilon(n))\mu(n) + O(\rho(n)).
\]
Hence, (22) holds.

The following Lemma shows that (10) holds.

**Lemma 5.** Let \(w(n) = z(n\tau + \tau) - z(n\tau)\). One has
\[
||w(n)||^2 = O(\gamma^2(n)/\varepsilon^2(n)).
\]
(24)

**Proof.** Proof of Lemma 5

In (9), which is also (22), the gradient of \(f(\cdot)\) is bounded and so is the vector \(\mu(n)\). Hence,
\[
w(n) = O(\gamma(n)) + O(\gamma(n)/\varepsilon(n)) + O(\rho(n)) = O(\gamma(n)/\varepsilon(n)).
\]
Thus, (24) holds.

The following Lemma proves (11)

**Lemma 6.** One has
\[
(z(n\tau) - x^*)'\nabla f(z(n\tau)) \leq f(z(n\tau)) - f(x^*).
\]
(25)

**Proof.** Proof of Lemma 6

Let \(z = z(n\tau)\). For \(\rho \in [0, 1]\), one has
\[
(1 - \rho)f(z) + \rho f(x^*) \leq f(\rho x^* + (1 - \rho)z),
\]
by concavity of \(f(\cdot)\). By Taylor's theorem,
\[
f(\rho x^* + (1 - \rho)z) = f(z) + \rho(x^* - z)'\rho f(z) + O(\rho^2).
\]
Hence,
\[
f(z) + \rho(f(x^*) - f(z)) \leq f(z) + \rho(x^* - z)'\rho f(z) + O(\rho^2),
\]
so that
\[
\rho(f(x^*) - f(z)) \leq \rho(x^* - z)'\nabla f(z) + O(\rho^2),
\]
Dividing by \(\rho\), we get
\[
f(x^*) - f(z) \leq (x^* - z)'\nabla f(z) + O(\rho).
\]
Letting \(\rho \to 0\) yields (25).
The following Lemma proves (13).

**Lemma 7.** For any $\delta > 0$, there is some $\beta > 0$ such that

$$f(z) \leq f(x^\star) - \beta \|z - x^\star\|, \text{ if } \|z - x^\star\| \geq \delta.$$  \hfill (26)

**Proof.** Proof of Lemma 7 By continuity and strict concavity in a neighborhood of $x^\star$,

$$-\alpha := \max \{ f(z) - f(x^\star) \mid \|z - x^\star\| \geq \delta \} < 0.$$  

Let $\beta = \alpha / \delta$. Assume $\|z - x^\star\| > \delta$. Define $v$ as follows:

$$v = \rho z + (1 - \rho)x^\star$$

Then,

$$\|v - x^\star\| = \|(1 - \rho)z - (1 - \rho)x^\star\| = (1 - \rho)\|z - x^\star\| = \delta.$$

Consequently,

$$f(v) - f(x^\star) \leq -\alpha.$$  

Also, by concavity,

$$f(v) \geq \rho f(x^\star) + (1 - \rho)f(z).$$

Hence,

$$\theta f(x^\star) + (1 - \rho)f(z) \leq f(x^\star) - \alpha,$$

so that

$$f(z) \leq f(x^\star) - \frac{\alpha}{1 - \rho} = f(x^\star) - \beta \|z - x^\star\|,$$

as claimed.  \hfill $\square$

The following lemma shows that the sequence $a(n)$ in (7) sums to a finite random variable.

**Lemma 8.** Let

$$\alpha(n) = (\gamma(n) / \epsilon(n)) \sum_k h_k(n) \mu_k(n) + O(\kappa(n))$$

where $\kappa(n) = \max \{ \gamma^2(n) / \epsilon^2(n), \gamma(n) \epsilon^2(n) \}$.  

Then the sum of $\alpha(n)$ converges to a finite random variable.

**Proof.** Proof of Lemma 8 First consider

$$(\gamma(n) / \epsilon(n)) h_k(n) \eta_k(n).$$

Recall that $|h_k(n)| \leq 1$ and that the random variables $\eta_k(n)$ are bounded and zero-mean given $\mathcal{F}_{n-1}$. Thus, the sum

$$\sum_{k=0}^{m} (\gamma(n) / \epsilon(n)) h_k(n) \mu_k(n)$$

is a martingale with respect to that filtration $\mathcal{F}_m$. Moreover,

$$\sum_{n} (\gamma(n) / \epsilon(n))^2 < \infty$$

by assumption. Consequently, by the martingale convergence theorem, this sum converges to a finite random variable.

Also, the terms $O(\kappa(n))$ sum to finite numbers, by (6).  \hfill $\square$

The following lemma shows that the $c(n)$ in (8) converge to zero.

**Lemma 9.** Let

$$c(n) = 2(\gamma(n) / \epsilon(n)) (z(n) - x^\star) \mu(n) + O(\kappa(n))$$

for $n$ such that $\|z(n) - x^\star\| \leq \delta$ and $c(n) = 0$ otherwise. Then $c(n) \to 0$.

**Proof.** Proof of Lemma 9 Consider the term

$$(\gamma(n) / \epsilon(n)) (z_k(n) - x_k^\star) \mu_k(n)$$

Note that

$$| (\gamma(n) / \epsilon(n)) (z_k(n) - x_k^\star) |^2 \leq (\gamma^2(n) / \epsilon^2(n)) |z_k(n) - x_k^\star|^2$$

$$\leq (\gamma^2(n) / \epsilon^2(n)) \|z(n) - x^\star\|^2 \leq (\gamma^2(n) / \epsilon^2(n)) \delta^2.$$

Consequently, as in Lemma 8, these terms sum to a finite random variable. Hence, the terms converge to zero.

The terms $O(\kappa(n))$ also converge to zero.  \hfill $\square$

**References**

[1] Robert K. L. Kennedy, Taghi M. Khoshgoftaar, Flavio Villanustre, Timothy Humphrey. A parallel and distributed stochastic gradient descent implementation using commodity clusters. *J Big Data*, 6, 16 (2019)

[2] Kiefer, J. and Wolfowitz, J. Stochastic Estimation of the Maximum of a Regression Function. *Ann. Math. Statist.*, **Vol. 23**, No. 3, 462-466, 1952.

[3] Angelia Nedic and Asuman Ozdaglar. Distributed subgradient methods for multi-agent optimization. *IEEE Transactions on Automatic Control*, **54**(1):48, 2009.

[4] Arunselvan Ramaswamy. DSPG: Decentralized Simultaneous Perturbations Gradient Descent Scheme. *arXiv:1903.07050v2 [math.OC]*, 27 Aug 2019

[5] Spall, James. Multivariate Stochastic Approximation Using a Simultaneous Perturbation Gradient Approximation. *IEEE Transactions on Automatic Control*, **vol. 37**, No.3, March 1992.

[6] Brian Swenson, Ryan Murray, Soummya Kar, H. Vincent Poor. Distributed Stochastic Gradient Descent: Nonconvexity, Nonsmoothness, and Convergence to Local Minima. *arXiv:2003.02818v4 [math.OC]*, 19 August 2020.

[7] Walras, Léon. Principe d’une théorie mathématique de l’échange, *Journal des économistes*, **34**, April 1874.