The amplitude of lee waves on the boundary-layer inversion

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This study presents an analytical model for the amplitude of lee waves on the boundary-layer inversion in two-dimensional flow. Previous linear lee wave models, in which the amplitude depends on the power spectrum of topography, can be inaccurate if the amplitude is large. Our model incorporates nonlinear effects by assuming that lee waves originate at a region of transition between super- and subcritical flow (internal jump) downstream of topography. Energy flux convergence at this location is compensated by the radiation of laminar lee waves. The available energy is estimated using a hydraulic jump model and the resulting wave amplitude is determined from linear theory. According to this model, the amplitude of lee waves depends essentially on their wavelength and on the inversion height difference across the jump. The new amplitude model is verified against numerical simulations and water tank experiments. The agreement between the model and the numerical results is excellent, while the verification with water tank experiments reveals that the accuracy of the model is comparable to that of numerical simulations. Finally, we derive a nonlinearity parameter for interfacial lee waves and discuss the regime transition from lee waves to hydraulic jumps in terms of the Froude number and the non-dimensional mountain and inversion heights.

Key Words: trapped lee waves; surface water waves; wave breaking; hydraulic jump; water tank experiments; numerical simulations

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1. Introduction

Lee waves are a special type of internal gravity waves, generated when stratified airflow is lifted over a mountain. Instead of propagating vertically, lee waves are trapped in a layer of finite depth and propagate horizontally over large distances, up to a few hundred kilometres. When condensation occurs in the wave crests, trapped lee waves can be observed in satellite images as cloud stripe patterns in the lee of mountains, with a typical wavelength of 3–15 km. Since lee waves can cause strong vertical motion in their up- and downdraughts, accurate forecasts of this phenomenon are important for aircraft pilots. On the one hand, the strong updraughts allow long and smooth glider flights to high altitudes. On the other hand, they can lead to severe low-level turbulence in atmospheric rotors (Doyle and Durran, 2007; Strauss et al., 2015) and pose a potential hazard for aviation.

Dynamically, atmospheric trapped lee waves are analogous to waves on a free water surface (Scorer, 1949). However, unlike free waves on a stream of water, atmospheric lee waves form in the interior of the fluid. Scorer found that they do not only develop along layer interfaces such as temperature inversions, but may also occur within continuously stratified layers, if vertically propagating waves are reflected downwards (resonant waves). This happens when the parameter

\[ F^2 \approx \frac{N^2}{U^2} - \frac{1}{U} \frac{d^2 U}{dz^2} \]

(where \( N \) is the buoyancy frequency and \( U \) the wind speed) decreases with height and the layer in which the wave is reflected is sufficiently deep. Resonant lee waves have been studied extensively in the literature (e.g. Durran, 1986b; Wurtele et al., 1996; Teixeira, 2014), while interfacial lee waves have been long overlooked and re-examined only recently (Vosper, 2004; Teixeira et al., 2013; Sachsperger et al., 2015, 2016). Linear theory shows that the wavelength of resonant and interfacial waves depends on the...
flow profile far upstream of the mountain (Scorer, 1949; Vosper, 2004) and is independent of the mountain half-width and height (in contrast to vertically propagating internal gravity waves). While linear theory explains well the wavelength, it gives only limited insight into the behaviour of the wave amplitude (Corby and Wallington, 1956; Smith, 1976; Nance and Durran, 1997), because nonlinear effects may be important in that respect.

The development of large-amplitude lee waves can be understood also by analogy to shallow-water flow over an obstacle. If the obstacle is sufficiently high, the flow is partially blocked upstream of it and a hydraulic jump forms in the lee (Houghton and Kasahara, 1968). Dissipation in the jump implies that the flow loses energy in the form of turbulence as it transitions from super- to subcritical. Rayleigh (1914) showed that the amount of energy loss depends on the jump size (the altitude difference of the interface across the jump). Observations in laboratory experiments by Favre (1935) revealed that, if the jump size is small, the transition to subcritical flow does not lead to energy dissipation, but rather to energy radiation through laminar waves. Motivated by this result, Lemoine (1948) successfully combined the energy source model of Rayleigh with a linear wave model that relates the available energy to the wave amplitude, as described by Benjamin and Lighthill (1954).

Since under certain conditions the behaviour of atmospheric flows is similar to that of hydraulic flow (e.g. Durran, 1986a; Vosper, 2004), we extend Lemoine’s approach in order to determine the amplitude of atmospheric lee waves. To the knowledge of the authors, this has not been done thus far. In this article, we only focus on lee waves forming on a density discontinuity (e.g. on the boundary-layer inversion), leaving the case of resonant lee waves for future research.

In section 2, we extend Lemoine’s model to atmospheric flows and discuss briefly the hydraulics of two-layer flow. Theoretical results are verified against numerical simulations and water tank experiments, as described in sections 3 and 4. Furthermore, in section 5 we discuss the large-amplitude limit of lee waves showing that it corresponds to a regime transition to hydraulic jumps. Conclusions are drawn in section 6.

2. Theoretical considerations

In this section, we develop a simple analytical model for the amplitude of lee waves forming on the boundary-layer inversion. The model consists of two components that describe the wave propagation and the energy source separately. The wave-propagation component is based on linear gravity wave theory and relates the wave amplitude to the wave energy flux, while the wave-energy-source component is based on hydraulic theory and gives an estimation of the available energy for wave propagation. We consider two-layer flow with a neutrally stratified lower layer representing a well-mixed boundary layer, a continuously stratified upper layer representing the free atmosphere, and a temperature inversion at the layer interface.

2.1. Wave energy propagation

Whenever waves are excited, they radiate energy away from the energy source. It can be shown that, for small-amplitude (linear) waves, the rate of energy radiation, i.e. the wave energy flux, is proportional to the wave amplitude. We use this property to determine the wave amplitude for a given rate of energy propagation.

The linear wave-energy-propagation model considers a two-layer Boussinesq fluid (Figure 1). The two fluid layers are continuously stratified with \( N_i \geq 0 \) and are of depth \( h_i \), where the subscript \( i \in \{1, 2\} \) denotes the lower and upper layer respectively. In what follows we use \( N_1 = 0 \), since we consider a well-mixed boundary layer, as described above. For the moment, we assume that the fluid is at rest \((U = 0)\). The case of a moving fluid is discussed further below.

The wave disturbance between the two layers is assumed to be of form \( \eta_0 = A \cos(k(x - C t)) \), where \( A \) is the displacement amplitude, \( k = 2\pi / \lambda \) is the wavenumber and \( C \) is the intrinsic phase speed. The displacement field \( \eta(x, z) \) in each fluid layer depends on the imposed wave mode \( \eta_0 \) and can be determined using generic solutions to the Taylor–Goldstein equation and zero-displacement boundary conditions at the top and bottom of the fluid (the Appendix gives details).

The total horizontal and vertical wave energy fluxes are the sum of the respective fluxes in each fluid layer, defined as (Nappo, 2012)

\[
EF_x = \int_{-h_1}^{h_1} p_i w_i' dz' = \int_{-h_1}^{0} \hat{p}_i \hat{w}_i' dz' + \int_{0}^{h_1} p_i \hat{w}_i' dz' \quad (1)
\]

and

\[
EF_z = \int_{-h_1}^{h_1} p_i w_i' dz' = \int_{-h_1}^{0} \hat{p}_i \hat{w}_i' dz' + \int_{0}^{h_1} p_i \hat{w}_i' dz', \quad (2)
\]

where \( z' = z - h_1 \) is the geometric height with respect to the interface, \( p_i' \) is the pressure perturbation and \( w_i' \) are the horizontal and vertical wind speed perturbations. The overbar in the correlation terms represents averaging over one wavelength. Since the wave disturbance is either trapped on the interface or reflected from the rigid boundary, its phase lines are vertical and the energy flux \( EF_z \) is zero in both layers. The perturbations \( \hat{p}_i' \) and \( \hat{w}_i' \) can be derived from \( \eta_0 \) using the phase relationships for internal gravity waves (the Appendix gives details). Inserting \( \hat{p}_i' \) and \( \hat{w}_i' \) into Eq. (1) gives the total vertically integrated horizontal wave energy flux across the fluid (in \( \text{J m}^{-1} \text{s}^{-1} \))

\[
EF_x = \xi A^2 C^3 \left| \begin{array}{l}
N_1 \left( e^{2n_1 h_1 - e^{-2n_1 h_1} + 4n_1 h_1} \right) \\
+ N_2 \left( e^{2n_2 h_2 - e^{-2n_2 h_2} + 4n_2 h_2} \right)
\end{array} \right|. \quad (3)
\]

Here, \( n_i = -(k^2 - N_i^2 / C^2)^{1/2} \) is the decay rate of the displacement \( \eta \) with distance from the interface in each layer and \( \xi \) is the density at the interface. Solving Eq. (3) for the amplitude yields

\[
A = \sqrt{\frac{4EF_x}{\rho C^3}} \left| \begin{array}{l}
N_1 \left( e^{2n_1 h_1 - e^{-2n_1 h_1} + 4n_1 h_1} \right) \\
+ N_2 \left( e^{2n_2 h_2 - e^{-2n_2 h_2} + 4n_2 h_2} \right)
\end{array} \right|^{1/2}. \quad (4)
\]

So far, we have assumed that the fluid is at rest. However, in the atmosphere lee waves are excited when flow with speed \( U (> 0) \) moves over an obstacle. Since lee waves are typically stationary with respect to a fixed location at the ground \((C = C + U = 0)\), the intrinsic phase speed of the wave disturbance \( \eta_0 \) must oppose
The vertically integrated energy flux is taken constant with $EF \propto C$.

First, in order to maintain a real root, the energy flux has to be negative because $C = -U$ is negative, similar to non-hydrostatic internal gravity waves (Nappo, 2012). This implies that energy is transported against the wind in a reference frame moving with the mean flow.

Second, the amplitude depends significantly (through $N_1$ and $N_2$) on the wavelength of the disturbance, as is evident in Figure 2(a). For large wavelengths, a given wave energy flux leads to larger amplitudes than it does for smaller wavelengths. Furthermore, if the upper layer is stratified, a cut-off wavelength ($k^2 = N_2^2/U^2$) exists above which wave energy propagates vertically without being trapped at the interface (e.g., Sachseperger et al., 2015). Consequently, for the atmosphere (infinitely deep) the solid curve (black) ends at the cut-off wavelength in Figure 2(a), because longer lee waves are not possible. However, when a zero-displacement condition is enforced at the top boundary (as assumed in Eq. (4)) vertically propagating waves are reflected from it and longer wave modes are supported (grey line in Figure 2(a)), similar to the case $N_2 = 0$.

Third, the amplitude becomes independent of the layer depths beyond a certain threshold (Figure 2(b, c)). This happens when the wave disturbance decays in the vertical before it can be reflected from the boundaries, i.e. when $h_i/h_o$ is large.

The only a priori unknown parameter in Eq. (4) is the wave energy flux $EF$, which needs to be parametrized. This can be done using a hydraulic jump model for the amount of energy that becomes available at an internal jump, as is now explained.

### 2.2. Wave energy source

It is instructive to consider first the relatively simple case of an external jump on the free surface of a single shallow-fluid layer. The fluid has constant density and expands rapidly in depth at the jump. Assuming mass and momentum conservation in the layer, it can be shown that the energy flux converges at the jump (Baines, 1995, p. 37). As mentioned above, the energy accumulated by energy flux convergence can either be dissipated in a jump or radiated in waves.

Similarly, the energy flux converges at an internal jump between two fluid layers of different densities (more generally, at the transition region from super- to subcritical flow). In this case, however, momentum (in contrast to mass) is not conserved in the individual fluid layers, but only in the entire column. Since there is only one momentum conservation condition for two fluid layers, the system is underdetermined and requires a closure (Yih and Guha, 1955). One possibility to close the system is to assume energy conservation in one of the layers, while the energy flux convergence occurs in the other (Wood and Simpson, 1984).

The comparison with laboratory experiments and numerical simulations suggests that assuming energy conservation in the lower (expanding) layers and energy accumulation in the upper (contracting) layer is more realistic, as shown by Klemp et al. (1997, hereafter KRS). In this work, we use the KRS model to estimate the available energy at an internal jump. While this model assumes hydrostatic flow far up- and downstream of the flow transition, it poses no restriction on the nature of the expansion region. If the flow is hydrostatic and long-wave modes dominate, a proper internal jump occurs. If, instead, non-hydrostatic effects are locally important, the horizontal dispersion of short-wave modes generates lee waves. However, in both cases, the energy available at the flow transition is controlled by the flow profiles far up- and downstream of it.

For the flow sketched in Figure 3, the energy flux convergence at the internal jump is (according to KRS)

$$\Delta EF = \frac{\hat{\rho} U^3 (d - h_o)(d - h_t)(h_t - h_o)^3}{2h_i^2 (d - h_i)^2 (2d - h_t - h_o)}$$  (5)

(units J m⁻¹ s⁻¹). Here, $d$ is the fluid depth, while $h_t$ and $h_o$ denote the interface height upstream and downstream of the jump respectively. The horizontal wind speed $U$ is vertically uniform and $\hat{\rho} = \rho_o \equiv \rho_1$ (KRS) is the fluid density at the interface. The densities in both fluid layers are assumed to be constant. In the case of stratified fluid layers (e.g. in the atmosphere) the constant density approximation is reasonable provided that the layers are shallow compared to their scale depth. Equation (5) differs from the KRS model by a minus sign. This is required, because the jump moves in the opposite direction in our case. A detailed derivation of Eq. (5) is out of the scope of this article and can be found in KRS and Li and Cummins (1998). The expression for the energy flux convergence in Eq. (5) can be cast in non-dimensional form using the parameters $\Delta EF/\hat{\rho} U^3 h_o$, $\alpha = h_o/d$ and $\beta = h_t/h_o$:

$$\Delta EF/\hat{\rho} U^3 h_o = -\frac{(1 - \alpha)(1 - 2\alpha \beta)(\alpha \beta - \alpha)^3}{\alpha^2 \beta^2(1 - \alpha \beta)^2(2 - \alpha \beta - \alpha)}$$  (6)
The dependence of the non-dimensional energy flux convergence on total fluid depth and jump size is shown in Figure 4 for the most relevant parameter space in the context of lee waves. Clearly, the influence of the jump size dominates over that of the fluid depth. It is also apparent that the impact of $d$ can be neglected for an infinitely deep layer, because $\Delta EF / (\tilde{\rho} U^3 h_a)$ becomes independent of $h_f / d$ as $h_f / d \to 0$. Therefore, the KRS model, which is developed for two-layer flow with rigid lid conditions at the boundaries, can be applied to atmospheric flows.

The wave-propagation model (Eq. (4)) can be coupled with the energy-source model (Eq. (5)) assuming that $EF_q = \Delta EF$, i.e. no dissipation occurs at the jump. This is explained more in detail in section 4 below.

3. Data and experimental design

In order to verify the analytical amplitude model presented in the previous section, we use numerical simulations and data from observations in water tank experiments. A description of the two data sources is presented in what follows.

3.1. Numerical model

Numerical simulations of two-layer atmospheric flow with a temperature inversion at the layer interface are carried out with the Bryan cloud model (CM1; Bryan and Fritsch, 2002; Doyle et al., 2011). CM1 is a fully nonlinear and non-hydrostatic large-eddy-simulation model. The model equations are integrated using a third-order Runge–Kutta scheme in time and a fifth-order finite-difference scheme in space. Subgrid-scale turbulent fluxes are parametrized after Deardoff (1980). The computational domain is two-dimensional and spans $n_x \times n_z = 2880 \times 145$ grid points in the horizontal and vertical directions with an isotropic grid spacing of $\Delta x = \Delta z = 50$ m in the layer where lee waves are expected to form (below $z = 2$ km). Above $z = 2$ km, the vertical grid spacing increases linearly to $\Delta z = 200$ m at $z = 7$ km and remains constant until the model top. At $x_0 = 55$ km a Gaussian-shaped mountain of the form

$$h(x) = H \exp \left\{ -\left( x - x_0 \right)^2 / 2\sigma^2 \right\}$$

is placed. Using $\sigma = 1.744$ km results in an approximate mountain half-width of $L = 5$ km at the surface. In the experiments described below, two different mountain heights $H = 200$ m and $H = 1000$ m are used.

The boundary conditions are open-radiative at the inflow and outflow boundaries, while they are periodic in the cross-stream direction. To minimize spurious wave reflections from the domain boundaries, Rayleigh damping is applied in a 40 km-wide zone at each of the streamwise boundaries and above $z = 10$ km. The undamped part of the domain, i.e. the physical domain, spans $40 < x < 104$ km and $0 < z < 10$ km, which is the region where lee wave formation is expected. In addition to minimizing reflections, we also reduce artifacts arising from the abrupt model start by accelerating the flow from rest to the designated flow speed ($U = 10 \text{ m s}^{-1}$) in the first simulation hour (the total integration time is 200 min). Furthermore, since we use the simulations to verify the frictionless analytical amplitude model, free-slip lower boundary conditions are applied. The details of the two-layer flow configuration (e.g. stratification, inversion height and strength) depend on the experiment and are described in section 4.

3.2. Water tank experiments

In addition to numerical simulations, we use water tank measurements of density-stratified two-layer flow over two-dimensional topography, taken during the HYIV-CNRS-SECORO experiments (Stiperski et al., 2016; personal communication). Laboratory experiments are useful to study interfacial waves, because the flow can be controlled and environmental influences are minimized, in contrast to measurements of atmospheric flows in the field. Therefore, laboratory flows are expected to compare well with theory. HYIV-CNRS-SECORO was carried out to acquire observational evidence of lee wave interference over double-mountain orography (Grušišić and Stiperski, 2009; Stiperski and Grušišić, 2011). In total, 395 experiments were carried out using both single- and double-mountain obstacles. The density profiles of the fluid represent typical boundary-layer flow with two layers, the lower one neutrally stratified ($N_1 = 0$) and the upper one continuously stratified ($N_2 > 0$), with a density jump $\Delta \rho$ between the layers. A Gaussian-shaped obstacle (Eq. (7)) was towed at various speeds along the bottom boundary. Depending on the Froude number $Fr = U / (g' h_1)^{1/2}$ and the non-dimensional mountain height $H / h_1$, different flow responses have been observed to occur in the lee; $g' = g \Delta \rho / \rho_0$ and $H$ are reduced gravity and mountain height respectively. The flow responses for the single mountain experiments were subjectively categorized into two types of response (lee waves and hydraulic jumps). The lee wave category is defined as the class of experiments where laminar undulations are present on the lee side of the obstacle with no evidence of significant mixing along the interface (Figure 5(a)). The hydraulic jump category includes the experiments where a sudden jump in the interface height is evident and mixing visibly occurs in its vicinity (Figure 5(b)). Ambiguous flow responses are not categorized. The dependence of the flow types on $Fr$ and $H / h_1$ is shown in Figure 6. Lee waves are not expected to occur for large values of $Fr$ in flows with an infinite upper layer (Vosper, 2004). They are instead observed in the HYIV-CNRS-SECORO experiments, in which they are likely caused by the reflection of vertically propagating long-wave modes at the free water surface. In this work, we focus on a subset of the single-mountain lee wave cases only. Further details on the HYIV-CNRS-SECORO experiments can be found in Stiperski et al. (2016; personal communication).

3.3. Scaling between numerical simulations and water tank experiments

Since the atmospheric numerical simulations ($\rho \sim 1 \text{ kg m}^{-3}$, horizontal scale $\sim 10$ km) are on a different scale than the laboratory experiments ($\rho \sim 1000 \text{ kg m}^{-3}$, horizontal scale $\sim 1$ m), we need to determine the simulation set-up in order to

\[ \Delta EF / (\tilde{\rho} U^3 h_a) \]

Figure 4. Impact of non-dimensional fluid depth ($x$ axis) and non-dimensional jump size ($y$ axis) on the non-dimensional energy flux convergence $\Delta EF / (\tilde{\rho} U^3 h_a)$ (shades and contours). Contours are drawn every 0.02.
achieve comparable results. Scaling arguments suggest that flow responses at different scales will be similar when fundamental non-dimensional numbers, depending on the physical parameters of the system, are constant. In our case, keeping the vertical aspect ratio $H/L$ of the mountain and the nonlinearity parameter $NH/U$ constant allows us to retain the wave structure in the continuously stratified upper layer (Baines, 1995, p. 237). Likewise, the interface response in the lee of the mountain is similar when the non-dimensional mountain height $H/h_1$ and Froude number $Fr$ are constant (Houghton and Kasahara, 1968). In addition, keeping these dimensionless parameters constant also retains the non-dimensional wavenumber $Lk$ across scales, as becomes apparent in the expression

$$Lk = \frac{1}{2} \frac{H}{h_1} Fr^{-2}.$$  

Here, we have used

$$k = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \frac{g}{U^2},$$

where the approximation $k \approx g'/(2U^2)$ is valid for internal interfacial waves in deep-water flow (i.e. when $\rho_2 \approx \rho_1$ and $kh_1 \gg 1$; Turner, 1973). Equation (8) is an important result because it proves that the applied scaling parameters will also keep the lee wavelength in scale with the obstacle. This becomes evident also in the flow comparison in Figure 7. The flow response in the water tank experiment and the corresponding numerical simulation is similar, although the spatial scale is different by a factor 1000. Details on the scaling of the wave amplitude are provided in section 5.

3.4. Experimental design

Using the data sources described above, two sets of experiments are carried out. First, idealized numerical simulations are made in order to verify the relationship between the amplitude and the wave energy flux (Eq. (4)). Second, observed amplitudes in laboratory flows and corresponding numerical simulations are compared with estimations of the analytical amplitude model with parametrized energy source (Eqs (4) and (5)).

In the first experiment, which we refer to as the energy-flux experiment, we compare the wave amplitudes measured in numerical simulations with those estimated using the linear model in Eq. (4). In particular, we use Eq. (4) to obtain an amplitude estimate from the wave energy flux $EF_x$ computed from the numerical simulations. The simulations are initialized with two-layer flow and different inversion strengths (Table 1). The mountain height is set to 200 m, in order to generate wave perturbations with relatively small amplitude, which should be
well reproduced by a linear wave model. In other words, these simulations serve as a check for the model described in section 2.1.

From the simulations we determine $\Delta F_x$ according to Eq. (1). The covariance $\rho^2$ is computed from the simulated pressure and horizontal wind speed field, averaged over one wavelength and integrated vertically over a layer that is twice as deep as the interface height ($0 \leq z \leq 2h_1$). Averaging over more than one wave crest, as well as averaging over a deeper layer did not show a significant effect on the results. The amplitude and wavelength of the simulated lee wave are determined from the displacement of the isentrope in the centre of the inversion layer, $\theta_{inv} = \theta_0 + \Delta \theta/2$. In the second experiment, which we refer to as the amplitude experiment, we compare the measured amplitudes of the simulations $A_{sim}$ and water tank experiments $A_{obs}$ with the amplitude $A_0$ estimated from Eqs (4) and (5). The numerical simulations in the amplitude experiment are similar to those in the energy-flux experiment, except for the flow set-up. The upstream flow conditions ($N_2, h_1$ and $\Delta \theta$) correspond to those of the water tank experiments and are determined from scaling arguments, that is, preserving $H/h_1$, $NH/U$ and $Fr$. Also, a higher mountain ($H = 1000\text{ m}$) is used in order to achieve typical values for the atmospheric boundary-layer-inversion height $h_1$ (determined from $H/h_1$). The details are shown in Table 2.

In order to determine $A_{obs}$, we begin with estimating the available energy flux $\Delta F_x$ using Eq. (5). To this end, we need to know the upstream flow parameters ($N_2, U, \Delta \theta, h_1$ and $d$) and the downstream jump parameters ($h_3, h_4$ and $\lambda$). The former are known a priori, while the latter are obtained from the interface displacement in the simulations and water tank experiments. We determine $h_3$ as the minimum interface height in the lee of the mountain and $h_4$ as the mean interface height in the wave period with the largest amplitude (Figure 3). Once $h_3$ and $h_4$ are known, $\Delta F_x$ can be calculated with Eq. (5). Assuming that $\Delta F_x = \Delta E_F$, i.e. the wave energy flux equals the energy accumulation at the hydraulic transition, we can finally estimate the amplitude $A_{th}$ using Eq. (4). The results from the energy-flux and amplitude experiments are presented below.

### 4. Results

#### 4.1. Energy-flux experiment

We begin by examining the wavelengths and wave energy fluxes obtained from numerical simulations. In the corresponding model runs, the inversion strength has been varied ($\Delta \theta = 6$ to $20\text{ K}$). According to interfacial wave theory (e.g. Vosper, 2004; Sachsperger et al., 2015), an increasing inversion strength leads to a shorter wavelength $\lambda$. Respectively, $\lambda$ varies between 2.3 and 5.7 km, as shown in Figure 8(a) (black dots). In addition to the wavelength, the inversion strength also impacts the wave energy flux $\Delta F_x$ through the lee wave train, which reaches a maximum for $\Delta \theta = 11\text{ K}$ ($Fr = 0.52$). This value of $Fr$ is approximately the same as that for which hydraulic theory predicts the maximum size of a hydraulic jump, $h_a = h_4$ (Houghton and Kasahara, 1968). This finding lends further support to the idea that energy convergence at the transition to subcritical flow provides the energy source for the non-hydrostatic lee wave train.

Both $\lambda$ and $\Delta F_x$ impact the amplitude $A = \max(\theta_0)$ of the lee wave (Eq. (4)). The values of $A$ predicted by Eq. (4) in the parameter space spanned by the simulations are shown as grey shades in Figure 8(a). It appears, that simulations with different values of $\Delta F_x$ can have similar amplitude. For example, this is the case for $\Delta \theta = 14\text{ K}$ and $\Delta \theta = 8\text{ K}$ where $\Delta F_x$ differs by about 40%, but the maximum displacement is $A \approx 200\text{ m}$ in both cases.

| No. | $Fr$ | $H/h_1$ | $h_1 (\text{m})$ | $N_1 (\text{s}^{-1})$ | $\Delta \theta (\text{kg m}^{-3})$ | $U (\text{m s}^{-1})$ | $h_1 (\text{km})$ | $N_2 (\text{s}^{-1})$ | $\Delta \theta (\text{K})$ |
|-----|-----|--------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 188 | 0.89 | 0.89   | 0.15            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 189 | 0.96 | 0.89   | 0.15            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 195 | 0.78 | 0.70   | 0.19            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 196 | 0.97 | 0.69   | 0.19            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 197 | 1.07 | 0.69   | 0.19            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 203 | 0.67 | 0.50   | 0.26            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 204 | 0.76 | 0.50   | 0.26            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 205 | 0.86 | 0.50   | 0.26            | 0.83            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 213 | 0.80 | 0.11   | 0.12            | 0.85            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 214 | 0.90 | 0.11   | 0.12            | 0.84            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 215 | 1.01 | 0.10   | 0.12            | 0.85            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 324 | 0.78 | 0.90   | 0.15            | 0.49            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 325 | 0.85 | 0.90   | 0.15            | 0.49            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 326 | 0.98 | 0.90   | 0.15            | 0.49            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 330 | 0.78 | 0.51   | 0.26            | 0.62            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 333 | 0.88 | 0.50   | 0.26            | 0.62            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 334 | 0.65 | 0.50   | 0.26            | 0.62            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 336 | 0.83 | 0.50   | 0.26            | 0.62            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 340 | 0.77 | 0.70   | 0.19            | 0.59            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 341 | 0.88 | 0.70   | 0.19            | 0.59            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 342 | 0.98 | 0.70   | 0.19            | 0.59            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 373 | 0.89 | 0.90   | 0.15            | 0.52            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |
| 382 | 0.88 | 0.70   | 0.19            | 0.58            | 0.31            | 26.1            | 1.4             | 0.92            | 3.5             |

The parameters $U, N_1$ and $\theta_0$ in the simulations are the same as in Table 1.
The comparison of the linear amplitude estimates (grey shades in Figure 8(a)) with the amplitudes obtained from the simulations is presented in Figure 8(b). The agreement between theory and simulations is excellent, even when amplitudes are large. In the simulation with the largest wave-energy flux ($\Delta \theta = 11$ K) the amplitude is almost 300 m, which is approximately one third of the lower-layer height ($h_1 = 1000$ m).

### 4.2. Amplitude experiment

In this experiment we carry out an intercomparison of the wave amplitudes obtained from the water tank experiments, the numerical simulations and the analytical amplitude model with parametrized energy source (Eqs (4) and (5)). We begin with the comparison of the water tank experiments with the numerical simulations, which are expected to be most accurate in reproducing the laboratory flows. Then, the numerical simulations are compared with the analytical model. Finally, the analytical model is compared with the laboratory experiments.

In both the water tank experiments and the simulations, we determine the amplitude from the largest lee wave crest and non-dimensionalize it with the mountain height in order to make the results comparable. Figure 9(a) shows that simulations and observations are in good agreement. There is significant scatter with a linear correlation coefficient (Pearson) of $R = 0.75$ (a similar value results for the wavelength, not shown). A large part of the scatter likely results from the different boundary condition at the top of the fluid. While the laboratory flows are of finite depth and allow reflections of vertically propagating internal waves from the free water surface, those disturbances are absorbed before reaching the upper boundary in the numerical simulations and do not impact the low-level flow. Scatter may also arise from friction at the bottom and the side walls of the tank (whereas the simulations are carried out with a free-slip lower boundary condition), and from irregularities in the shape of the bottom boundary in the experiments, not reproduced in the model runs (e.g. obstacle mounts and towing wires).

The results of the comparison between theory and numerical simulations is shown in Figure 9(b). The agreement between the two is very good, despite the fact that the theoretical model systematically overestimates the wave amplitude. Since the amplitude–energy-flux relationship (Eq. (4)) is very accurate (Figure 8(b)), the systematic error has to arise mainly from the energy source model (Eq. (5)). It is reasonable to expect that Eq. (5) overestimates the available energy of the simulated flows. In the numerical simulations, the upper layer is stratified and supports vertically propagating wave disturbances which transport energy upward. Since stratification effects are neglected in the KRS model (Eq. (5)), vertical wave propagation is not supported and no energy leakage occurs, causing the overestimation of available energy. Another process that might play a role is topographic blocking of the upstream flow, which reduces the mass and energy flux over the obstacle. Since Eq. (5) neglects blocking effects, fluxes upstream of the jump, as well as amplitudes, are overestimated. A detailed analysis on the impact of these two processes on the energy budget of the flow is out of the scope of the article and left for future research. In any case, the influence of these processes on the present results is marginal.

In Figure 9(c) the comparison of theory with water tank experiments is presented. Again, the agreement is reasonably good, considering that the comparison of the laboratory flows with numerical simulations showed large scatter as well (Figure 9(a)).

In summary, we showed that wave amplitudes are proportional to the wave energy flux even when amplitudes are large (Figure 8(b)). Furthermore, the verification of the analytical

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**Figure 8.** Verification of the amplitude–energy-flux relation, Eq. (4). Panel (a) shows amplitudes $A$ (grey shading) calculated from Eq. (4) as a function of $EF$, and $\lambda$. Black dots indicate the measured values of $EF$, and $\lambda$ in numerical simulations ($\Delta \theta = 6$ to 20 K) of the energy-flux experiment at $t = 200$ min. Panel (b) shows a scatter plot of the measured amplitude in the simulations and the respective linear theory values. The labels of the dots refer to the inversion strength $\Delta \theta$ of the simulation while $R$ represents the Pearson correlation coefficient.

**Figure 9.** Intercomparison of the non-dimensional wave amplitudes obtained from water tank experiments (subscript obs), numerical simulations (subscript sim) and the analytical model (subscript th) assuming $EF_0 = \Delta EF$. The panels show the comparison of (a) simulations and water tank experiments, (b) analytical model and simulations and (c) analytical model and water tank experiments. $R$ represents the Pearson correlation coefficient.
amplitude model (Eqs (4) and (5)) showed very good agreement with the numerical simulations (Figure 9(b)). In a third test, we found that the performance of the analytical model is comparable to that of numerical simulations in predicting the wave amplitude in water tank experiments (Figure 9(a) and (c)).

5. Discussion

The amplitude model described in sections 2 and 4 can be used to understand the conditions under which an interfacial lee wave reaches critical amplitude and breaks. In general, wave breaking occurs when the horizontal wind speed perturbation opposes the mean flow causing stagnation, i.e. when $|u'/U| \geq 1$ (Baines, 1995). For vertically propagating hydrostatic waves in a continuously stratified atmosphere, this happens when the nonlinearity parameter $NH/U \geq 1$. In analogy to $NH/U$, a nonlinearity parameter can be formulated also for interfacial waves.

The relevance of $NH/U$ as a nonlinearity parameter for internal gravity waves is described by Baines (1995, p. 239). The scale for $u'$ is derived from the scale for $w'$ using the incompressible mass continuity constraint $\partial u'/\partial x = |\partial w'/\partial z|$. For flow over a mountain with scales $H$ and $L$, the vertical wind speed perturbation and its vertical gradient are obtained from the parallel flow condition as $w' = U\partial h/\partial x \sim UH/L$ and $|\partial w'/\partial z| = NH/L; \partial \omega/\partial z \sim N/U$ since $U/N$ is the maximum vertical displacement in stratified flow (Smith, 1990). It follows that the horizontal wind speed perturbation scales as $NH$, and consequently that $|u'/U| \sim NH/U$.

A similar approach can be used for interfacial waves in two-layer flow. The parallel flow condition along the wave disturbance $h_0$, gives an expression for the vertical wind speed perturbation $w' = U\partial h_0/\partial x \sim UA/D$, where $A$ is the amplitude and $D$ is the characteristic length-scale of the interfacial wave. Since the trapped wave is evanescent in the upper layer, its amplitude decays exponentially with height according to $w'e^{-z/z}$. Thus, the vertical gradient of $w'$ at the interface ($z' = 0$) scales approximately as $\partial w'/\partial z \sim AN_0D/U$. Using again the continuity equation for the horizontal wind speed perturbation yields $u' \sim AN_0U$ and

$$|u'/U| \sim |AN_0| = \left\{A^2k^2 - (NA/U)^2\right\}^{1/2}. \quad (10)$$

Alternatively, this nonlinearity parameter can be obtained from the linear wave solution of $u'_H$, assuming that $h_2 \to \infty$ (cf. Eqs (10) and (A7) in the Appendix). Equation (10) reveals that for small $NA/U$ the nonlinearity of interfacial waves is related to the steepness of the wave crest ($Ak$).

In order to determine the nonlinearity parameter (Eq. (10)) entirely from the upstream flow profile, we need to scale the horizontal wavenumber $k$ and the amplitude $A$. For $k$, we use the frequency dispersion relationship of lee waves on the boundary-layer inversion $k = g^2/(2U^2) + N^2/(2g')$, which is accurate for waves in deep-water flow (when $kh_1 \gg 1$; Sachsperger et al., 2015). For the amplitude, we use the mountain height $H$, which is on the same order of magnitude of the maximum deflection of the interface (i.e. $H$ is approximately an upper bound for $A$, as visible in Figure 9, where generally $A/H < 1$). Consequently, the resulting nonlinearity parameter predicts $|u'/U|$ for the case when the wave disturbance reaches maximum amplitude. With these two scales for $k$ and $A$, Eq. (10) can be rearranged to

$$|u'/U| \sim \left(\frac{1}{2} \frac{H}{h_1} Fr^{-2} \left[1 - \frac{1 - (NH)^2}{U^2} \right]^{1/2}\right) = \gamma, \quad (11)$$

where $\sigma = (UN/g')^2$ is the ratio between the perturbation pressure associated with waves above the inversion ($\propto NU$) and the density jump across the inversion ($\propto g$) (Jiang, 2014). In the case of non-hydrostatic lee waves, $\sigma$ is small (Sachsperger et al., 2015) and therefore neglected. Since $\gamma$ depends only on the characteristic non-dimensional numbers of the system, it becomes clear that the ratio $|u'/U|$, i.e. the nonlinearity of the wave, is constant across scales as well.

According to the breaking criterion $|u'/U| \geq 1$, lee waves must reach critical amplitude if the parameter $\gamma$ in Eq. (11) exceeds unity. The contour of $\gamma = 1$ (with $NH/U = 0.5$; $NH/U$ ranges in the experiments between 0.3 and 0.6) is overlaid (dashed line) on the flow regimes of the water tank experiments in Figure 6. This line represents the critical steepness of the wave (Eq. (10)) and accurately reproduces the regime transition between lee waves and hydraulic jumps. When lee waves reach critical amplitude, the flow regime changes from lee waves to a hydraulic jump, since dissipation consumes most of the available energy. A similar result can be obtained by overlaying this contour (black dashed) over the regime diagram by Vosper (2004), Figure 10.

An alternative explanation for the regime transition in Figure 10 has been provided by Teixeira et al. (2013), who found that lee waves occur only when the non-dimensional lee wavelength $\lambda_{2}/(2\pi) > 0.3$ to 0.4; $\gamma$ being the Scorer parameter in the upper layer. This result can be interpreted also in terms of the wave steepness which, for a given wave amplitude, is larger for short lee waves. It appears that the critical wave steepness ($\gamma = 1$, indicating the onset of wave breaking) approximately coincides with the range of critical wavelengths ($\lambda_{2}/(2\pi) = 0.3-0.4$) suggested by Teixeira et al. (2013).

Furthermore, Figure 10 reveals that free-atmospheric stratification has relatively weak influence on $\gamma$. Although $NH/U = 0, 0.5, 1, 1.5$ (black dashed and grey lines) is varied substantially, the difference between the curves is relatively small. Hence, the first term in the square brackets in Eq. (11) is dominant and stratification only has a secondary effect on the breaking of lee waves.

The small discrepancy between the theoretical curve ($\gamma = 1$ with $NH/U = 0.5$) and the regime transition in the no-slip simulations in Figure 10 likely results from surface friction, which is neglected in $\gamma$. The viscous boundary layer in the simulations absorbs energy and, thus, reduces the wave amplitude. Consequently, theory predicts the critical wave slope to be reached at larger Froude numbers (longer wavelengths) than in the simulations by Vosper (2004).

Figure 10. Regime diagram (adapted from Vosper (2004)) for different flow responses (lee wave, lee wave rotors and hydraulic jumps) determined from two-dimensional numerical simulations. The dashed line shows the contour $\gamma = 1$ (Eq. (11) with $NH/U = 0.5$) and indicates the transition from lee waves $\gamma < 1$ to hydraulic jumps $\gamma > 1$. Grey solid lines represent the influence of stratification showing $\gamma = 1$ for $NH/U = 0, 1, 1.5$; the lowest line represents $NH/U = 1.5$. The solid black line represents the critical Froude number for which lee waves are possible.
6. Summary and conclusions

We have extended Lemoine’s (1948) theoretical model for the amplitude of surface water waves to atmospheric flows, in order to determine the amplitude of interfacial lee waves. The model consists of two parts that describe the wave propagation and the energy source separately. The accuracy of the theoretical model in predicting measured wave amplitudes in water tank experiments is comparable to that of high-resolution numerical simulations. Moreover, the comparison of the theoretical results with the simulations is excellent, even for large-amplitude lee waves. In analogy to surface water waves, interfacial lee waves are similar to those of surface waves. Given the good agreement between theory, simulations and experiments, we may draw further conclusions on the dynamics of atmospheric lee waves on the boundary-layer inversion.

Our investigations confirm Scorer’s (1949) statement that the dynamics of interfacial lee waves are similar to those of surface water waves. This becomes apparent as the extended Lemoine model can explain well the amplitude behaviour of atmospheric interfacial lee waves. In analogy to surface water waves, interfacial lee waves develop when the flow over a mountain transitions from subcritical to supercritical at the crest. According to single-layer hydraulic theory, this happens when

\[
\frac{H}{h_1} - \left( \frac{3}{2} \frac{F_r^{3/2}}{h_1} + \frac{1}{2} \frac{F_r}{h_1} \right) \geq 1
\]

(Baines, 1995, p. 39) and leads to the formation of a downstream hydraulic jump at which energy accumulates. We showed, using the hydraulic model by Klemp et al. (1997), that the amount of available energy, and thus, the lee wave amplitude, depends on the size of the jump, i.e., the height difference of the interface across the jump.

Modelling the energy source of the wave appropriately is crucial for accurate wave amplitude predictions. This is likely the reason why traditional linear models are highly inaccurate in predicting the amplitude of lee waves (an issue that was raised by Corby and Wallington, 1956, and Vosper, 2004). Those models link the energy source to the power spectrum of the topography. This is relatively accurate for vertically propagating mountain waves (Smith, 1989), which are generated at the ground, but inaccurate for interfacial lee waves, which originate at the density interface further aloft in the interior of the fluid. Consequently, their characteristics are not directly determined by topography, rather they depend on the dynamics of two-layer flow over a mountain in which nonlinear effects may be important. Therefore, hydraulic theory is more appropriate to describe the energy source, while linear theory is still accurate enough to describe the propagation of the wave, as becomes apparent in Figure 8(b).

Lastly, we showed that hydraulic jumps correspond to the limiting case of breaking lee waves. In analogy to NH/\(U\), which describes the nonlinearity \(|u'/U|\) of internal gravity waves, we derived a non-linearity parameter \(\gamma \sim |u'/U|\) (Eq. (11)) for interfacial waves in deep-water flow (\(kh_1 \gg 1\)). We showed that the wave breaking criterion \((\gamma \geq 1)\) accurately predicts the regime transition between lee waves and hydraulic jumps in water tank experiments and simulations (Vosper, 2004). When \(\gamma \geq 1\), energy accumulation at the expansion region is too large to be radiated by lee waves, and the available energy is converted to turbulent kinetic energy in a hydraulic jump; in other words, the first lee wave crest reaches critical amplitude and breaks.

The amplitude model developed in this study is semi-prognostic, because the jump size and the wavelength are determined diagnostically from observed and simulated flows. Potential for further improvement therefore lies in the parametrization of the two parameters in order to make the model entirely prognostic. Furthermore, the inclusion of a boundary-layer representation in the amplitude model may be useful to predict the onset of rotor formation (e.g., similar to Vosper et al., 2006, and Teixeira, 2016; personal communication).

Finally, in this work we focused on interfacial lee waves on the boundary-layer inversion. However, it is reasonable to expect that the dynamics of resonant lee waves are similar, since the interface behaviour is analogous to hydraulic flow (Durrant, 1986a). Whether Lemoine’s approach can be successfully applied to determine the amplitude of resonant waves remains to be clarified in future research.

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Appendix

Displacement field

The displacement field \(\eta(x, z)\) in the stratified fluid layers \(i \in \{1, 2\}\) can be determined from generic solutions to the Taylor–Goldstein equation in each layer

\[
\eta(x, z') = \cos(k(x - C_0 t)) \left[ a_1 e^{\eta_1 x} + b_1 e^{-\eta_1 x} \right]. \tag{A1}
\]

The coefficients \(a_i\) and \(b_i\) need to be determined from the boundary conditions. Zero-displacement conditions at the walls of the fluid \((\eta = 0\) at \(z' = -h_1\) and \(z' = h_2\)) give

\[
\eta_1(x, z') = 0 = a_1 \left[ 1 - e^{-2\eta_1 h_1} \right] \cos(k(x - C_0 t)), \tag{A2}
\]

and the interface condition \(\eta_2(x, z' = 0) = \eta_0 = A \cos(k(x - C_0 t))\) yields the wave displacements

\[
\eta_1(x, z') = \eta_0 \left[ \frac{e^{\eta_1 (z' + h_1)} - e^{-\eta_1 (z' + h_1)}}{e^{\eta_1 h_1} - e^{-\eta_1 h_1}} \right], \tag{A4}
\]

\[
\eta_2(x, z') = \eta_0 \left[ \frac{e^{\eta_2 (z' - h_2)} - e^{-\eta_2 (z' - h_2)}}{e^{-\eta_1 h_2} - e^{-\eta_2 h_2}} \right]. \tag{A5}
\]

Using \(\eta' = \partial \eta/\partial t\), the wave perturbations \(\eta'\) and \(p'\) can be derived using the polarization relations of internal gravity waves (Nappo, 2012) and become

\[
u'_1 = n_1 C_0 \eta_0 \left[ \frac{e^{\eta_1 (z' + h_1)} + e^{-\eta_1 (z' + h_1)}}{e^{\eta_1 h_1} - e^{-\eta_1 h_1}} \right], \tag{A6}
\]

\[
\nu'_2 = n_2 C_0 \eta_0 \left[ \frac{e^{\eta_1 (z' - h_2)} + e^{-\eta_1 (z' - h_2)}}{e^{-\eta_1 h_2} - e^{-\eta_2 h_2}} \right], \tag{A7}
\]

\[
p'_1 = n_1 C_2 \rho \eta_0 \left[ \frac{e^{\eta_1 (z' + h_1)} + e^{-\eta_1 (z' + h_1)}}{e^{\eta_1 h_1} - e^{-\eta_1 h_1}} \right], \tag{A8}
\]

\[
p'_2 = n_2 C_2 \rho \eta_0 \left[ \frac{e^{\eta_1 (z' - h_2)} + e^{-\eta_1 (z' - h_2)}}{e^{-\eta_1 h_2} - e^{-\eta_2 h_2}} \right]. \tag{A9}
\]

For \(h_2 \to \infty\), and taking \(C = -U\) and \(\eta_0 = A\), Eq. (A7) reduces to Eq. (10).

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