Reified unit resolution and the failed literal rule

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Abstract

Unit resolution can simplify a CNF formula or detect an inconsistency by repeatedly assign the variables occurring in unit clauses. Given any CNF formula $\sigma$, we show that there exists a satisfiable CNF formula $\psi$ with size polynomially related to the size of $\sigma$ such that applying unit resolution to $\psi$ simulates all the effects of applying it to $\sigma$. The formula $\psi$ is said to be the reified counterpart of $\sigma$. This approach can be used to prove that the failed literal rule, which is an inference rule used by some SAT solvers, can be entirely simulated by unit resolution. More generally, it sheds new light on the expressive power of unit resolution.

1 Introduction

1.1 Unit resolution

A unit clause is a logical clause with only one literal, like $(a)$ or $(\overline{b})$. Unit resolution (also called unit propagation) consists to repeatedly fix the variables occurring into unit clauses in such a way to satisfy these clauses. For example, if there is a clause $(b)$ in the formula, then the variable $b$ is set to false, and the formula is simplified by removing all the clauses containing $b$ as well as all the occurrences of $b$ in the other clauses.

Sometimes, unit propagation produces the empty clause, meaning that the formula is not satisfiable. Because unit resolution in not a complete proof system, all unsatisfiable formulae cannot be solved in this way. In SAT solvers, unit propagation is used to fixe some variables in order to reduce the number of branches in the search tree.

1.2 The failed literal rule

This is an inference rule allowing SAT solvers to fix some variables which cannot be fixed by using only unit propagation. As an example, let us consider the following CNF formula:

$$\sigma = (a \lor b) \land (\overline{b} \lor c) \land (\overline{b} \lor \overline{c})$$

Because there is no unit clause, applying unit resolution to this formula does not fix any variable. Applying the failed literal rule to the literal $\overline{c}$ consists in trying to fix this variable to false and then to apply unit propagation. Because the empty clause is produced, $\sigma \land \overline{c}$ is not satisfiable. Then the variable $a$ must be set to true. The failed literal rule can be also applied to the literal $b$, with the result that the variable $b$ must be set to false.

1.3 Contribution

We will show that applying unit propagation to a CNF formula $(\overline{\sigma} \lor \overline{\overline{\sigma}}) \land \text{reif}(\sigma \land w, l)$ has the same effect as applying the failed literal rule to a formula $\sigma$ with the literal $w$. $\sigma' = \text{reif}(\sigma \land w, l)$ is said to be the reified counterpart of $\sigma \land (w)$ in the sense that applying unit propagation to $\sigma'$ cannot produce the empty clause, but fixes $l$ to true if and only if applying unit propagation to $\sigma \land (w)$ would produce the empty clause.

Although the size of the reified counterpart $\text{reif}(\psi, l)$ of a formula $\psi$ is polynomially related to the size of $\psi$, the interest of the concept is rather theoretical. It sheds new light on the expressive power of unit resolution.
2 Reified unit resolution

The unit propagation process can be decomposed into several steps, where each step fixes the variables which occur in unit clauses after the step $i - 1$ (if applicable) is completed. Because each step fixes at least one variable, and because the empty clause is produced when the same variable is fixed both to $true$ and $false$, the number of steps cannot exceed $n + 1$, where $n$ is the number of variables in the formula. Let $\sigma$ be a CNF formula with $n$ variables, and $\psi$ its reified counterpart. The formula $\psi$ can be decomposed in $n + 1$ sub-formulae $\psi_1, \ldots, \psi_{n+1}$, where each $\psi_i$ simulates the effect of the step $i$ of unit propagation on $\sigma$. For each variable $v$ of $\sigma$, there are $2(n + 1)$ variables, namely $v_1^+, v_1^-, \ldots, v_{n+1}^+, v_{n+1}^-$, in $\psi$. The formula $\psi$ is designed so that if $v$ is fixed to $true$ ($false$, respectively) after $i$ propagation steps on $\sigma$, then $v_i^+$ ($v_i^-$, respectively) is fixed to $true$ after $i$ propagation steps on $\psi$. As a manner of speaking, the assignments $v = true$ and $v = false$ are decoupled in $v_i^+ = true$ and $v_i^- = true$ in $\psi$, and no variable of $\psi$ can be set to $false$ by unit propagation.

Let us present the construction of $\psi$ form the formula

$$\sigma = (\pi) \land (a \lor b) \land (\overline{b} \lor c) \land (\overline{c} \lor \pi) \quad (2)$$

The sub-formula $\psi_1$ must allow unit propagation to fix $a_1^-$ to $true$ because at the first step of unit propagation on $\sigma$, the variable $a$ is fixed to $false$. Then

$$\psi_1 = (a_1^-) \quad (3)$$

The sub-formula $\psi_2$ must allow unit propagation to fix $a_2^-$ to $true$ because $a$ remains to $true$ at the second step of unit propagation on $\sigma$. This can be obtained thanks to the clause $(a_1^- \lor a_2^-)$, which will be called a propagation clause. It must also allow unit propagation on $\psi$ to simulate the effect of unit propagation on $\sigma$ regarding the clause $(a \lor b)$, given that $a$ is set to $false$. This can be obtained thanks to the clause $(a_1^- \lor b_2^+)$, which will be called a deduction clause.

Because the goal is to build the formula $\psi$ without knowing in advance which variables of $\sigma$ will be fixed by each unit resolution step, all the possible propagation and deduction clauses are added to each sub-formula $\psi_i, i > 1$.

$$\psi_i = \begin{cases} \text{propagation clauses} \\
(a_i^- \lor a_{i+1}^-) \land (a_i^+ \lor a_{i+1}^+) \land (b_i^- \lor b_{i+1}^-) \land (b_i^+ \lor b_{i+1}^+) \land (c_i^- \lor c_{i+1}^-) \land (c_i^+ \lor c_{i+1}^+) \\
\text{deduction clauses} \\
(a_i^- \lor b_{i+1}^+) \land (b_i^- \lor a_{i+1}^+) \land (b_i^+ \lor c_{i+1}^-) \land (c_i^- \lor b_{i+1}^-) \land (b_i^+ \lor c_{i+1}^+) \land (c_i^- \lor b_{i+1}^-) \end{cases} \quad (4)$$

For example, the third propagation clause $(b_i^- \lor b_{i+1}^+)$ says "if $b_i^- = true$ at the step $i$ of unit propagation on $\psi$, meaning that $b = false$ at the step $i$ of unit propagation on $\sigma$ then $b_{i+1}^-$ must be set to $true$ at the step $i + 1$ of unit propagation on $\psi$, meaning that $b = false$ at the step $i + 1$ of unit propagation on $\sigma$".

As another example, the third deduction clause $(b_i^- \lor c_{i+1}^+)$ says "According to the clause $(\overline{b} \lor c)$ of $\sigma$, if $b_i^+ = true$ at the step $i$ of unit propagation on $\psi$, meaning that $b = true$ at the step $i$ of unit propagation on $\sigma$, then $c_{i+1}^+$ must be set to $true$ at the step $i + 1$ of unit propagation on $\psi$, meaning that $c = true$ at the step $i + 1$ of unit propagation on $\sigma$".

The production of the empty clause by unit propagation on $\sigma$ (if applicable) can be reified by adding a new variable $s$ and the following clauses to $\psi$

$$((a_i^+ \lor a_i^- \lor s) \land (b_i^+ \lor b_i^- \lor s) \land (b_i^+ \lor b_i^- \lor s)) \quad (5)$$

Clearly, unit propagation on $\psi$ will fix $s$ to $true$ if and only if unit propagation on $\sigma$ produces the empty clause, i.e. implicitly fixes the same variable both to $true$ and $false$.

As it stands, the formula $\psi$ is of little interest because it can only allow to simulate one "scénario" of unit
propagation on $\sigma$. It is much more useful to simulate the effects of unit propagation when some variables of $\sigma$ have been previously fixed (for example by other inference rules or by branching rules in the context of the running of a SAT solver).

To this end, some of (or all) the variables of $\sigma$ can be injected into $\psi$ with the following clauses:

\[
(\overline{a} \lor a^+_1) \land (a \lor \overline{a}^-_1) \land (\overline{b} \lor b^+_1) \land (b \lor \overline{b}^-_1) \land (\overline{c} \lor c^+_1) \land (c \lor \overline{c}^-_1)
\]  

(6)

Thanks to these additional clauses, unit propagation on $\psi$ can simulate the effect of unit propagation on $\sigma$ under any given partial truth assignment of the variables of $\sigma$.

If $\sigma$ includes $n$ variables and $m$ clauses with at most $k$ literals per clause, then each sub-formula $\psi_i$ contains $2n$ binary propagation clauses and at most $km k$-ary deduction clauses. It follows that $\psi$ contains $O(n^2 + nkm)$ clauses.

### 3 Concluding remarks

We shown that for any formula $\sigma$, there exists a satisfiable formula $\psi$ such that unit propagation on $\psi$ can simulate the behavior of unit propagation on $\sigma$, even when the empty clause is produced. What this tells about the expressive power of unit propagation? Unit propagation can be seen as a way to compute functions mapping partial truth assignments to \{yes, no\} with two different approaches. In the first approach, the result yes corresponds to the assignment of a particular variable. In the second one, it corresponds to the production of the empty clause. The results presented in this report show that the same functions can be computed using these two approaches, and that the required numbers of clauses are polynomially related.