On existence and regularity of a terminal value problem for the time fractional diffusion equation

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Abstract
In this paper we consider a final value problem for a diffusion equation with time-space fractional differentiation on a bounded domain \(D\) of \(\mathbb{R}^k\), \(k \geq 1\), which includes the fractional power \(L^\beta\), \(0 < \beta \leq 1\), of a symmetric uniformly elliptic operator \(L\) defined on \(L^2(D)\). A representation of solutions is given by using the Laplace transform and the spectrum of \(L^\beta\). We establish some existence and regularity results for our problem in both the linear and nonlinear case.

Keywords: existence and regularity, final value problem, time fractional derivative, uniqueness

1. Introduction
Nonlinear diffusion equations, an important class of parabolic equations, come from many diffuse phenomena that appear widely in nature. They are proposed as mathematical models of...
physical problems in many areas, such as filtering, phase transition, biochemistry and dynamics of biological groups. Many new ideas and methods have been developed to consider some various kinds of nonlinear diffusion equations. We list some selected works in recent time, for example Caffarelli et al [1], Duzaar et al [2–4], Vazquez et al [5–8] and the references therein.

We present existence and regularity estimates for the solution to a final boundary value problem for a space-time fractional diffusion equation. Let $D$ be an open and bounded domain in $\mathbb{R}^k$, ($k \geq 1$) with boundary $\partial D$. Given $0 < \alpha < 1$ and $0 < \beta \leq 1$, a forcing (or source) function $F$, we consider the final value problem for the time fractional diffusion equation

$$^\alpha D^\alpha_t u(t,x) = -\mathcal{L}^\beta u(t,x) + F(t,x,u(t,x)), \quad (t,x) \in J \times D, \quad (1.1)$$

with the boundary condition

$$\mathcal{H}u(t,x) = 0, \quad (t,x) \in J \times \partial D, \quad (1.2)$$

and the final condition

$$u(x,T) = \varphi(x), \quad x \in D, \quad (1.3)$$

where $\varphi$ is a given function. Here $J$ is the interval $(0,T)$, the notation $^\alpha D^\alpha_t$ for $0 < \alpha < 1$ represents the left Caputo fractional derivative of order $\alpha$ which is defined by

$$^\alpha D^\alpha_t v(t) := \int_0^t \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} v(t-s)^{\alpha-1}, \quad t \geq 0, \quad (1.1)$$

provided that $I_1^\alpha v(t) := g_\alpha(t) \ast v(t)$, here $g_\alpha(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, $t > 0$, $\ast$ denotes the convolution.

For $\alpha = 1$, we consider the usual time derivative $\frac{d}{dt}$. The fractional power $\mathcal{L}^\beta 0 < \beta \leq 1$ of the Laplacian operator $\mathcal{L}$ on $D$ is defined by its spectrum. The symmetric uniformly elliptic operator is defined on the space $L^2(D)$ by

$$\mathcal{L}u(x) = \sum_{i=1}^k \frac{\partial}{\partial x_i} \left( \sum_{j=1}^k L_{ij} \frac{\partial}{\partial x_j} u(x) \right) + b(x)u(x),$$

provided that $L_{ij} \in C^1(\overline{D})$, $b \in C(\overline{D})$, $b(x) \geq 0$ for all $x \in \overline{D}$, $L_{ij} = L_{ji}$, $1 \leq i, j \leq k$, and $\xi^T \left[ L_{ij}(x) \right] \xi \geq L_0 |\xi|^2$ for some $L_0 > 0$, $x \in \overline{D}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_k) \in \mathbb{R}^k$. The equation (1.1) is equipped with $\mathcal{H}v = v$ or $\mathcal{H}v = (\mathcal{L}v_n)v$, where $\mathcal{L} = \left[ L_{ij}(x) \right]_{ij=1}^k$ is a $k \times k$ matrix and $n$ is the outer normal vector of $\partial D$. Then the operator $\mathcal{L}$ is self-adjoint under this impedance boundary condition.

The time fractional reaction diffusion equation arises in describing ‘memory’ occurring in physics such as plasma turbulence [9] and it was introduced by Nigmatullin [10] to describe diffusion in media with fractal geometry, which is a special type of porous media and is applied in the flow in highly heterogeneous aquifer [11] and single-molecular protein dynamics [12]. In a physical model presented in [13], the fractional diffusion corresponds to a diverging jump length variance in the random walk, and a fractional time derivative arises when the characteristic waiting time diverges.

If the final condition (1.3) is replaced by the initial condition

$$u(x,0) = u_0(x), \quad x \in D \quad (1.4)$$

then problem (1.1), (1.2), (1.4) is called a forward problem (or an initial value problem) for time-space fractional diffusion equations; for applications of this type of equation see [14] and for the abstract form of (1.1)–(1.4) see [15]. Carvalho et al [16] established a local theory of mild solutions for problem (1.1)–(1.4) where $\mathcal{L}^\beta$ is a sectorial (nonpositive) operator.
Guswanto [17] studied the existence and uniqueness of a local mild solution for a class of initial value problems for nonlinear fractional evolution equations and the study of existence of initial value problems was considered by Warma et al [14]. A significant number of papers were devoted to extend properties holding in the standard setting to the fractional one (see for example [18, 19, 20–22]).

Numerical approximation for solutions for problem (1.1)–(1.4) was studied by Jin et al [23, 24] and for other works on fractional diffusion see [25–28]. However, the literature on regularity of the initial value problem for fractional diffusion-wave equations is scarce; for the linear case see [26, 29–31], and for the nonlinear case see [14, 32–34]. Although there are many works on the direct problem, but the results on inverse problem for fractional diffusion are scarce. We can list some papers of Yamamoto and his group see [35–37, 38, 39, 40], of Kaltenbacher et al [41, 42], of Rundell et al [43, 44], of Janno see [45, 46], etc.

In practice, initial data of some problems may not be known since many phenomena cannot be measured at the initial time. Phenomena can be observed at a final time \( t = T \), such as, in the image processing area. A picture is not processed at the capturing time \( t = 0 \). Instead, one wishes to recover the original information of the picture from its blurry form. Hence, inverse problems or terminal value problems or final value problems (IPs/FVPs), i.e., the fractional differential equations (FDEs) equipped with final value data, have been considered. IPs/FVPs are important in engineering in detecting the previous status of physical fields from its present information. If \( F = 0 \) in (1.1), Yamamoto et al [31] showed that problem (1.1)–(1.3) has a unique weak solution when \( \varphi \in H^2(\Omega) \). If \( b = 0 \) and \( F(u(x, t)) = F(x, t) \), Tuan et al [47] showed that problem (1.1)–(1.3) has a unique weak solution when \( \varphi \in H^2(\Omega) \) and \( F \in L^\infty(0, T : H^2(\Omega)) \), and other works on the homogeneous case for problem (1.1)–(1.3) can be found in [25, 47–49]. Wei and her group [62–52] studied some regularization methods for homogeneous backward problem and Yamamoto et al [53] considered a backward problem in time for a time-fractional partial differential equation in the one-dimensional case. When \( \alpha = 1 \), systems (1.1)–(1.3) are reduced to the backward problem for classical reaction diffusion equations, and were studied in [54–56].

To the best of the authors’ knowledge this is the first paper that analyzes problem (1.1)–(1.3). We present existence and uniqueness results and derive regularity estimates both in time and space. In what follows, we analyze the difficulties of this problem. By letting \( u(t, \cdot) = O(t)(f, \varphi) \), the solution operator \( O(0) \) is really not bounded in \( L^2(D) \). Hence continuity of mild solutions does not hold at the initial time \( t = 0 \). In addition, since the fractional derivative is non-locally defined, if we put \( v(t) = u(T - t) \) then \( D^\alpha_t v(t) \big|_{t=0} \) does not equal \( D^\alpha T v(t) \), so the problem cannot be changed to an initial value problem. As a result we need new techniques to deal with the FVP (1.1)–(1.3). To the best of our knowledge, the work on the final value problem is still limited.

Our main results in this paper can be split into two parts, linear and nonlinear source functions. Linear models are sometimes good approximations of the real problems under consideration and provides mathematical tools needed to study nonlinear phenomena, especially for semi-linear and quasi-linear equations. In part 1, we consider the regularity property of the solution in the linear case \( F \). We seek to address the following question: if the data is regular, how regular is the solution? Our task in this part is to find a suitable Banach space for the given data \((\varphi, F)\) in order to obtain regularity results for the corresponding solution. In part 2, we discuss existence, uniqueness and regularity for the solutions to (1.1)–(1.3) for the nonlinear problem. Our main motivation for deriving regularity results is that one needs it for a rigorous study of a numerical scheme to approximate the solution. To the best of our knowledge, regularity results on inverse initial value problems (final value problems) for fractional
diffusion is still unavailable in the literature. For initial value problems, McLean et al [30], Jin et al [24] and Ahmad et al [34] considered existence and regularity results of the solution in $C([0, T] ; L^2(D))$. However it seems the techniques in [24, 34] cannot be applied for our problems (it is impossible to apply some well-known fixed point theorems with some spaces in [24] for establishing unique solutions). To overcome this we need data $\varphi$ in a suitable space and we will use Picard iteration argument and then develop some new techniques to obtain existence and regularity of the solution.

The rest of this paper is organized as follows. In section 2, we give basic notations and preliminaries, and we propose a mild solution of our problem. In section 3, we give some regularity results of the linear inhomogeneous problem. Section 4 is devoted to existence and regularity for nonlinear problems and section 5 considers global existence.

2. Notations and preliminaries

2.1. Functional space

In this subsection, we introduce some functional spaces for solutions of FVP (1.1)–(1.3). By $\{m_j\}_{j \geq 1}$ and $\{e_j(x)\}_{j \geq 1}$, we denote the spectrum and sequence of eigenfunctions of $L$ which satisfy $e_j \in \{v \in H^2(D) : Hv = 0\}$, $L e_j(x) = m_j e_j(x)$, $0 < m_1 \leq m_2 \leq \ldots \leq m_j \leq \ldots$, and $\lim_{j \to \infty} m_j = \infty$. The sequence $\{e_j(x)\}_{j \geq 1}$ forms an orthonormal basis of the space $L^2(D)$. For a given real number $p \geq 0$, the Hilbert scale space $H^p(D)$ is defined by

$$
\left\{ v \in L^2(D) \quad \text{such that} \quad \|v\|_{H^p(D)}^2 := \sum_{j=1}^{\infty} (v, e_j^2 m_j^p)^{\frac{2}{p}} < \infty \right\},
$$

where $(\cdot, \cdot)$ is the usual inner product of $L^2(D)$. The fractional power $L^{\beta}$, $\beta \geq 0$, of the Laplacian operator $L$ on $D$ is defined by

$$
L^{\beta} v(x) := \sum_{j=1}^{\infty} (v, e_j^2 m_j^\beta) e_j(x). \tag{2.1}
$$

Then, $\{m_j^\beta\}_{j \geq 1}$ is the spectrum of the operator $L^{\beta}$. We denote by $V_\beta$ the domain of $L^{\beta}$, and then

$$
V_\beta = \{ v \in L^2(D) : \|L^{\beta} v\| < \infty \}
$$

where $\| \cdot \|$ is the usual norm of $L^2(D)$, and $V_\beta$ is a Banach space with respect to the norm $\|v\|_{V_\beta} = \|L^{\beta} v\|$. Moreover, the inclusion $V_\beta \subset H^{2\beta}(D)$ holds for $\beta > 0$. We identify the dual space $(L^2(D))^\prime = L^2(D)$ and define the domain $V_{-\beta} := D(L^{-\beta})$ by the dual space of $V_\beta$, i.e., $V_{-\beta} = \{ v \in V_\beta \}$. Then, $V_{-\beta}$ is a Hilbert space endowed with the norm

$$
\|v\|_{V_{-\beta}} := \left\{ \sum_{j=1}^{\infty} (v, e_j^2 m_j^{-2\beta}) \right\}^{1/2},
$$
where $\langle \cdot , \cdot \rangle_{-\beta,\beta}$ denotes the dual inner product between $V_{-\beta}$ and $V_{\beta}$. We note that the Sobolev embedding $V_\beta \hookrightarrow L^2(D) \hookrightarrow V_{-\beta}$ holds for $0 < \beta < 1$, and $\langle \tilde{v}, v \rangle_{-\beta,\beta} = \langle \tilde{v}, v \rangle$, for $\tilde{v} \in L^2(D)$, $v \in V_\beta$. Hence, we have

$$
\langle e_i, e_j \rangle_{-\beta,\beta} = (e_i, e_j) = \delta_{ij}
$$

(2.2)

where $\delta_{ij}$ is the Kronecker delta for $i, j \in \mathbb{N}, i, j \geq 1$. Moreover, for given $p_1 \geq 1$ and $0 < \eta < 1$, we denote by $\mathcal{X}_{p_1,\eta}(J \times D)$ the set of all functions $f$ from $J$ to $L^{p_1}(D)$ such that

$$
|||f|||_{p_1,\eta} := \sup_{0 \leq t \leq T} \left( \int_0^t ||f(t, \cdot)||_{p_1}^{p_1} \, dt \right)^{\frac{1}{p_1}} < \infty,
$$

(2.3)

where $||.||_{p_1}$ is the norm of $L^{p_1}(D)$. Note that, for fixed $t > 0$, the Hölder’s inequality shows that

$$
\int_0^t ||f(t, \cdot)||_{p_1}^{p_1} \, dt \leq \left( \int_0^t ||f(t, \cdot)||_{p_1}^{p_1} \, dt \right)^{\frac{1}{p_1}} \left( \int_0^t (t - \tau)^{p_1 - 1} \, d\tau \right)^{\frac{1}{p_1 - 1}}.
$$

In the above inequality, we note that the function $\tau \to (t - \tau)^{p_1 - 1}$ is integrable for $p_2 > \frac{1}{2}$.

Therefore, if we let $L^{p_2}(0, T; L^{p_1}(D))$, $p_1, p_2 \geq 1$, be the space of all Bochner’s measurable functions $f$ from $J$ to $L^{p_1}(D)$ such that

$$
||f||_{L^{p_2}(0, T; L^{p_1}(D))} := \left( \int_0^t ||f(t, \cdot)||_{p_1}^{p_2} \, dt \right)^{\frac{1}{p_2}} < \infty,
$$

then the following inclusion holds

$$
L^{p_2}(0, T; L^{p_1}(D)) \subset \mathcal{X}_{p_1,\eta}(J \times D), \quad \text{for } p_2 > \frac{1}{\eta},
$$

(2.4)

and there exists a positive constant $C > 0$ such that

$$
|||f|||_{p_1,\eta} \leq C||f||_{L^{p_2}(0, T; L^{p_1}(D))},
$$

(2.5)

here, $C$ depends only on $p_2$, $\eta$, and $T$. Moreover, for a given number $s$ such that $0 < s < \eta$, we have $\mathcal{X}_{p_1,\eta-s}(J \times D) \subset \mathcal{X}_{p_1,\eta}(J \times D)$ since $|||f|||_{p_1,\eta-s} \leq T^s|||f|||_{p_1,\eta}$. Let $B$ be a Banach space, and we denote by $C((0, T], B)$ the space of all continuous functions from $(0, T]$ to $B$ endowed with the norm $||v||_{C((0, T], B)} := \sup_{0 \leq t \leq T} ||v(t)||_B$, and by $C^\theta((0, T], B)$, $0 < \theta \leq 1$, the subspace of $C((0, T]; B)$ which includes all Hölder-continuous functions, and is equipped with the norm

$$
||v||_{C^\theta((0, T], B)} := \sup_{0 \leq t_1 < t_2 \leq T} \frac{||v(t_2) - v(t_1)||_B}{|t_2 - t_1|^\theta}.
$$

In some cases, a given function might not be continuous at $t = 0$. Hence, it is useful to consider the set $C((0, T]; B)$ which consists of all continuous functions from $(0, T]$ to $B$. We define by $C^\theta_0((0, T]; B)$ the weighted Banach space of all functions $v$ in $C((0, T]; B)$ such that

$$
||v||_{C^\theta_0((0, T]; B)} := \sup_{0 \leq t \leq T} t^\theta ||v(t)||_B < \infty.
$$
Now, we discuss solutions of the FVP for the fractional ordinary equation
\[ ^c D_t^\alpha v(t) = g(t, v(t)) - mv(t), \quad t \in J, \quad \text{and} \quad v(T) = v_T, \quad (2.6) \]
where \( m, v_T \) are given real numbers. Here, we wish to find a representation formula for \( v \) in terms of the given function \( g \) and the final value data \( v_T \). By writing \( ^c D_t^\alpha = I_t^{1-\alpha} D_t \), and applying the fractional integral \( I_t^\alpha \) on both sides of equation (2.6), we obtain
\[ v(t) = v(0) + \int_0^t [g(t, v(t)) - mv(t)] \, dt. \]
The Laplace transform yields that
\[ \hat{v} = \frac{\lambda^{\alpha-1}}{\lambda^{\alpha} + m} v(0) + \frac{1}{\lambda^{\alpha} + m} \hat{g}(v), \]
where \( \hat{v} \) is the Laplace transform of \( v \). Hence, the inverse Laplace transform implies
\[ v(t) = v(0) E_{\alpha,1}(-mt^{\alpha}) + g(t, v(t)) \ast [t^{\alpha-1} E_{\alpha,1}(-mt^{\alpha})], \quad (2.7) \]
Here, \( E_{\alpha,1} \) and \( E_{\alpha,\alpha} \) are the Mittag-Leffler functions which are generally defined by \( E_{\alpha,b}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k + b)} \), for \( \alpha > 0, b \in \mathbb{R}, z \in \mathbb{C} \). Now, a representation of the solution of FVP (2.6) can be obtained by substituting \( t = T \) into (2.7), and using the final value data \( v(T) = v_T \), i.e.,
\[ v(t) = g(t, v(t)) \ast \tilde{E}_{\alpha,\alpha}(-mt^{\alpha}) + \left[ v_T - \left( g(r, v(r)) \ast \tilde{E}_{\alpha,\alpha}(-mt^{\alpha}) \right) \right]_{r=T} \frac{E_{\alpha,1}(-mt^{\alpha})}{E_{\alpha,1}(-mT^{\alpha})}, \quad (2.8) \]

2.2. Mild solutions of FVP (1.1)–(1.3) and unboundedness of solution operators

A representation of solutions and the definition of mild solutions are given in this subsection, and then we analyze the unboundedness of solution operators. By the definition (2.1) of \( \mathcal{L}^\beta \), the identity \( \mathcal{L}^\beta e_j(x) = m_j^\beta e_j(x) \) holds. Hence, in view of the Fourier expansion \( u(t, x) = \sum_{j=1}^{\infty} u_j(t) e_j(x) \), where \( u_j(t) = (u(t, \cdot), e_j) \), equation (1.1) can be rewritten as
\[ ^c D_t^\alpha \sum_{j=1}^{\infty} u_j(t) e_j = - \left( \mathcal{L}^\beta \sum_{j=1}^{\infty} u_j(t) e_j \right) + \left( F(t, x, u(t, x)), e_j \right), \quad t \in J. \]
This is equivalent to the equation
\[ ^c D_t^\alpha u_j(t) = F_j(t, u(t)) - m_j^\beta u_j(t), \quad F_j(t, u(t)) = (F(t, x, u(t, x)), e_j). \]
By applying the method of solutions of FVPs for fractional ordinary equations in subsection 2.1, and using the final value data (1.3), we derive
\[ u_j(t) = F_j(t, u(t)) \ast \tilde{E}_{\alpha,\alpha}(-m_j^\beta t^{\alpha}) + \left[ \varphi_j - \left( F_j(r, u(r)) \ast \tilde{E}_{\alpha,\alpha}(-m_j^\beta r^{\alpha}) \right) \right]_{r=T} \frac{E_{\alpha,1}(-m_j^\beta T^{\alpha})}{E_{\alpha,1}(-m_j^\beta r^{\alpha}), \quad (2.9) \]
where \( \varphi_j = (\varphi, e_j) \). Therefore, we obtain a spectral representation for \( u \) as follows:

\[
\begin{align*}
\varphi_j &= \langle \varphi, e_j \rangle. \\
&\text{Therefore, we obtain a spectral representation for } u \text{ as follows:} \\
u(t, x) &= \sum_{j=1}^{\infty} F_j(t, u(t)) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta T^\alpha) e_j(x) \\
&\quad + \sum_{j=1}^{\infty} \left[ \varphi_j = \langle F_j(r, u(r)) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta T^\alpha) \rangle \right]_{r=T} E_{\alpha, 1}(-m_j^\beta T^\alpha) e_j(x).
\end{align*}
\]

For \( g \in L^2(0, T; L^2(D)) \) and \( v \in L^2(D) \), let us denote by \( O_n, 1 \leq n \leq 3 \), the following operators

\[
(\mathcal{O}_1 g)(t, x) := \sum_{j=1}^{\infty} g_j(t) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta T^\alpha) e_j(x),
\]

\( (\mathcal{O}_2(t)v)(x) := \sum_{j=1}^{\infty} v_j E_{\alpha, 1}(-m_j^\beta T^\alpha) e_j(x) \),

and \( (\mathcal{O}_3 g)(t) := -\mathcal{O}_2(t)(\mathcal{O}_1 g)(T) \) on \( L^2(D) \), for \( t \in J \). Then, the solution \( u \) can be represented as

\[
\begin{align*}
\varphi_j &= \langle \varphi, e_j \rangle. \\
&\text{Therefore, we obtain a spectral representation for } u \text{ as follows:} \\
u(t, x) &= (\mathcal{O}_1 F)(t, x) + (\mathcal{O}_2(t)\varphi)(x) + (\mathcal{O}_3 F)(t, x),
\end{align*}
\]

for \( (t, x) \in J \times D \).

One of the most important things, when we consider the well-posedness of a PDE, is the boundedness of solution operators. Corresponding to the initial value problem (1.1), (1.2), (1.4), the solution operators are usually bounded in \( L^2(D) \); see e.g., [27, 30, 33, 34, 57]. Unfortunately, some solution operators of FVP (1.1)–(1.3) are not bounded on \( L^2(D) \) at \( t = 0 \). For this purpose, we recall that, for \( 0 < \alpha < 1 \) and \( \varepsilon < 0 \), there exist positive constants \( c_{\alpha}, \tilde{c}_{\alpha} \) such that

\[
\begin{align*}
c_{\alpha} \leq |E_{\alpha, 1}(z)| \leq \tilde{c}_{\alpha}, & \quad |E_{\alpha, \alpha}(z)| \leq \min \left\{ \frac{\tilde{c}_{\alpha}}{1 + |z|}, \frac{\tilde{c}_{\alpha}}{1 + |z|^2} \right\},
\end{align*}
\]

see, for example [58, 59, 60]. Now, let \( u_0 \) be defined by \( u_0 = (v_0, e_j) = j^{1/2} m_j^\beta, j \geq 1 \). Then, it is easy to see that \( u_0 \) belongs to \( V_{\beta, \gamma} \), for \( 0 \leq \gamma < 1 \), and does not for \( \gamma \geq 1 \). Using the inequalities (2.11), we have

\[
\begin{align*}
\|O_2(0)u_0\|^2 &= \sum_{j=1}^{\infty} \frac{v_{0,j}^2}{E_{\alpha, 1}(-m_j^\beta T^\alpha)} \geq \sum_{j=1}^{\infty} \frac{v_{0,j}^2}{1 + m_j^\beta T^\alpha} \geq \frac{c_{\alpha}^2 T^2 \alpha \sum_{j=1}^{\infty} v_{0,j}^2}{1 + m_j^\beta T^\alpha} \geq \frac{c_{\alpha}^2 T^2 \alpha \sum_{j=1}^{\infty} 1}{1 + m_j^\beta T^\alpha} = \infty,
\end{align*}
\]

which shows the unboundedness of \( O_2(0) \) on \( L^2(D) \). Similarly, the unboundedness of \( O_3(0) \) on \( L^2(D) \) can be shown.

### 3. FVP with a linear source

In this section, we study the regularity of mild solutions of FVP (1.1)–(1.3) corresponding to the linear source function \( F \), i.e., \( F(t, x, u(t, x)) = F(t, x) \) which does not include \( u \). We will investigate the regularity of the following FVP

\[
\begin{align*}
\mathcal{D}^\alpha u(t, x) &= -L^\alpha u(t, x) + F(t, x), & (t, x) &\in J \times D, \\
\mathcal{H}u(t, x) &= 0, & (t, x) &\in J \times \partial D, \\
u(x, T) &= \varphi(x), & x &\in D,
\end{align*}
\]

where \( \varphi_j = (\varphi, e_j) \). Therefore, we obtain a spectral representation for \( u \) as follows:

\[
\begin{align*}
\varphi_j &= \langle \varphi, e_j \rangle. \\
&\text{Therefore, we obtain a spectral representation for } u \text{ as follows:} \\
u(t, x) &= \sum_{j=1}^{\infty} F_j(t, u(t)) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta T^\alpha) e_j(x) \\
&\quad + \sum_{j=1}^{\infty} \left[ \varphi_j = \langle F_j(r, u(r)) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta T^\alpha) \rangle \right]_{r=T} E_{\alpha, 1}(-m_j^\beta T^\alpha) e_j(x).
\end{align*}
\]
where $\varphi$, $F$ will be specified later. In order to consider this problem, it is necessary to give a definition of mild solutions based on (2.10) as follows.

**Definition 3.1.** If a function $u$ belongs to $L^p(0,T;L^q(D))$, for some $p, q \geq 1$, and satisfies the equation

$$u(t, x) = (\mathcal{O}_1 F)(t, x) + (\mathcal{O}_2(t)\varphi)(x) + (\mathcal{O}_3 F)(t, x),$$

(3.2)

then $u$ is said to be a mild solution of FVP (3.1).

In what follows, we introduce some assumptions on the final value data $\varphi$ and the linear source function $F$.

- (R1) $0 < p, q < 1$ such that $p + q = 1$;
- (R2) $0 < r \leq \frac{1-q}{q}$;
- (R3) $0 < s < \min (\alpha q, 1 - \alpha q)$;
- (R4) $0 < p' \leq p$, $q' = 1 - p'$, $0 < r \leq \frac{1-q}{q'}$;
- (R5) $0 \leq \hat{q} \leq \min (p, q, \frac{s}{p})$, $\hat{p} = 1 - \hat{q}$, $0 < \hat{r} \leq \frac{1-q}{\hat{q}}$.

In the following lemma, we will show that solutions of FVP (3.1) must be bounded by a power function $t^{-\alpha q}$, for some appropriate number $q$, i.e.,

$$\|u(t, \cdot)\| \lesssim t^{-\alpha q},$$

for all $0 < t \leq T$.

**Lemma 3.2.** Let $p, q$ be defined by (R1), and $u$ satisfies (3.2). If $\varphi \in \mathcal{V}_{\beta p}$, and $F \in \mathcal{K}_{2,\alpha q}(J \times D)$, then there exists a constant $C_0 > 0$ such that

$$\|u(t, \cdot)\| \leq C_0 \left( \|\varphi\|_{\mathcal{V}_{\beta p}} + \|F\|_{\mathcal{K}_{2,\alpha q}} \right) t^{-\alpha q}.$$  

(3.3)

**Proof.** The inequalities (2.11) shows that

$$E_{\alpha, \alpha}(-m_j^\beta (t - \tau)\alpha) \leq \tilde{c}_\alpha \left[ 1 + m_j^\beta (t - \tau)\alpha \right]^{-p} \leq \tilde{c}_\alpha m_j^{-\beta p}(t - \tau)^{-\alpha p}. $$

(3.4)

Combined with the definition $(\mathcal{O}_1 F)(t, x) = \sum_{j=1}^\infty F_j(t) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta t^\alpha) \xi_j(x)$, we have that

$$\|(\mathcal{O}_1 F)(t, \cdot)\| \leq \int_0^t \left\| \sum_{j=1}^\infty F_j(\tau) \tilde{E}_{\alpha, \alpha}(-m_j^\beta (t - \tau)\alpha) \xi_j \right\| d\tau$$

$$= \int_0^t \left\{ \sum_{j=1}^\infty F_j^2(\tau) \tilde{E}_{\alpha, \alpha}^2(-m_j^\beta (t - \tau)\alpha) (t - \tau)^{2\alpha - 2} \right\}^{1/2} d\tau$$

$$\leq \tilde{c}_\alpha \int_0^t \left\{ \sum_{j=1}^\infty F_j^2(\tau)m_j^{-2\beta p}(t - \tau)^{-2\alpha p} (t - \tau)^{2\alpha - 2} \right\}^{1/2} d\tau.$$  

(3.5)
Hence, we obtain the following estimate

$$
\|(O_1 F) (t, \cdot)\| \leq \tilde{c}_\alpha m_1^{-\beta_p} \int_0^T \|F(\tau, \cdot)\|(t-\tau)^{\alpha q-1} d\tau \leq M_1 t^{-\alpha q} \| |F||_{\lambda_2^{\alpha q}}, \quad (3.6)
$$

by noting (2.3) and letting $M_1 = \tilde{c}_\alpha m_1^{-\beta_p} T^{\alpha q}$. In addition, the norm $\|O_2(t)\phi\|$ can be estimated as

$$
\|O_2(t)\phi\| = \left\{ \sum_{j=1}^{\infty} \psi_j t \left( \frac{E_{\alpha,1}(-m_1^\beta T^\alpha)}{E_{\alpha,1}(-m_1^\beta T^\alpha)} \right) \right\}^{1/2} \leq \tilde{c}_\alpha c_\alpha \left\{ \sum_{j=1}^{\infty} \psi_j^2 \left( \frac{1 + m_j^\beta T^\alpha}{1 + m_j^\beta T^\alpha} \right)^2 \right\}^{1/2}.
$$

The ratio $\frac{1 + m_j^\beta T^\alpha}{1 + m_j^\beta T^\alpha}$ is clearly bounded by both $1 + m_j^\beta T^\alpha$ and $\frac{T^\alpha}{m_j^\beta}$. Moreover, the increasing property of the sequence $(m_j)_{j \geq 1}$ shows $1 \leq m_1^\beta m_j^\beta$. Thus, we have $1 + m_j^\beta T^\alpha \leq (m_j^\beta + T^\alpha) m_j^\beta$. By noting $p + q = 1$, one can deduce that the ratio is bounded by the product of $T^{\alpha q} t^{-\alpha q}$ and $(1 + m_j^\beta T^\alpha)^p$. Bring the above arguments together, and this leads to

$$
\|O_2(t)\phi\| \leq \tilde{c}_\alpha c_\alpha \left\{ \sum_{j=1}^{\infty} \psi_j t \left( \frac{E_{\alpha,1}(-m_1^\beta T^\alpha)}{E_{\alpha,1}(-m_1^\beta T^\alpha)} \right) \right\}^{1/2} \leq M_2 t^{-\alpha q} \| \phi \|_{V^{1,p}}, \quad (3.7)
$$

where

$$
M_2 = \tilde{c}_\alpha c_\alpha \left\{ \sum_{j=1}^{\infty} \psi_j t \left( \frac{E_{\alpha,1}(-m_1^\beta T^\alpha)}{E_{\alpha,1}(-m_1^\beta T^\alpha)} \right) \right\}^{1/2}.
$$

Now, we proceed to estimate $\|(O_3 F) (t, \cdot)\|$ by using the same techniques as in (3.5) and (3.7). As a consequence of

$$(O_3 F)(t, x) = -O_2(t)(O_1 F)(T)(x) = -\sum_{j=1}^{\infty} \left( F_j(r) \ast \tilde{E}_{\alpha,1}(-m_1^\beta T^\alpha) \right) \frac{E_{\alpha,1}(-m_j^\beta T^\alpha)}{E_{\alpha,1}(-m_1^\beta T^\alpha)} \phi_j(x),$$

we can obtain the following estimates

$$
\| (O_3 F)(t, \cdot) \| \leq \int_0^T \left\{ \sum_{j=1}^{\infty} \frac{F_j(\tau) E_{\alpha,1}(-m_j^\beta (T-\tau)^{\alpha}) (T-\tau)^{\alpha q - 1} E_{\alpha,1}(-m_1^\beta T^\alpha)}{E_{\alpha,1}(-m_1^\beta T^\alpha)} \right\} d\tau^{1/2} = \int_0^T \left\{ \sum_{j=1}^{\infty} \frac{F_j^2(\tau) E_{\alpha,1}^2(-m_j^\beta (T-\tau)^{\alpha}) (T-\tau)^{2\alpha q - 1} E_{\alpha,1}^2(-m_1^\beta T^\alpha)}{E_{\alpha,1}^2(-m_1^\beta T^\alpha)} \right\} d\tau^{1/2} \leq \tilde{c}_\alpha \left\{ \sum_{j=1}^{\infty} \frac{F_j^2(\tau) m_j^{-2\beta_p} (T-\tau)^{2 \alpha q - 2} T^{2\alpha q} (1 + m_j^\beta T^\alpha)^{2p}}{E_{\alpha,1}^2(1 + m_j^\beta T^\alpha)} \right\} d\tau^{1/2}.
$$

A simple computation shows that

$$
\| (O_3 F)(t, \cdot) \| \leq M_3 t^{-\alpha q} \int_0^T \|F(\tau, \cdot)\|(T-\tau)^{\alpha q - 1} d\tau \leq M_3 t^{-\alpha q} \| |F||_{\lambda_2^{\alpha q}}, \quad (3.8)
$$

where we let $M_3 = \tilde{c}_\alpha M_2$. Finally, it follows from (3.6)–(3.8), and the identity (3.2) that
\|u(t, \cdot)\| \leq \|(O_2 F)(t, \cdot)\| + \|(O_2 F)(t, \cdot)\| + \|(O_2 F)(t, \cdot)\| \\
\leq \left( \sum_{1 \leq n \leq 3} M_n \right) \left( \|\varphi\|_{V_{\beta q}} + \|F\|_{L_{x,\alpha q}} \right) t^{-\alpha q}.

The inequality (3.3) is proved by letting $C_0 = \sum_{1 \leq n \leq 3} M_n$. \qed

Based on lemma 3.2, we consider existence, uniqueness, and regularity of solutions in the following theorem which is divided into two parts. In the first part, we obtain the existence and uniqueness of a mild solution in the space $L^{\frac{1}{\alpha q}}(0, T; L^2(D))$ for some suitable numbers $q, r$ and for the given assumptions on $\varphi$ and $F$ as in lemma 3.2. In the second part, we improve the smoothness of the mild solution by considering the spatial-fractional derivative $L^{\beta(p-r')}$. It is very important to investigate the continuity of the mild solution. We first show that the mild solution is continuous from $(0, T)$ to $L^2(D)$. Moreover, we establish the continuity on the closed interval $[0, T]$ which corresponds to lower spatial-smoothness, $V_{-\beta q'}$, for a relevant number $q'$. Theorem 3.3.

(a) Let $p, q, r$ be defined by (R1), (R2). If $\varphi \in V_{\beta p}$ and $F \in X_{2,\alpha q}(J \times D)$, then FVP (3.1) has a unique solution $u$ in $L^{\frac{1}{\alpha q}}(0, T; L^2(D))$. Moreover, there exists a positive constant $C_1$ such that

$$\|u\|_{L^{\frac{1}{\alpha q}}(0, T; L^2(D))} \leq C_1 \|\varphi\|_{V_{\beta p}} + C_1 \|F\|_{X_{2,\alpha q}}. \tag{3.9}$$

(b) Let $p, q, r, p', q'$ be defined by (R1), (R5), (R4). If $\varphi \in V_{\beta p}$, and $F \in X_{2,\alpha q}(J \times D)$, then FVP (3.1) has a unique solution $u$ such that

$$u \in L^{\frac{1}{\alpha q}}(0, T; V_{\beta(p-r')} \cap C_{\alpha q}'((0, T); L^2(D)) \cap C'((0, T); V_{-\beta q'}).$$

Moreover, there exists a positive constant $C_2$ such that

$$\|u\|_{L^{\frac{1}{\alpha q}}(0, T; V_{\beta(p-r')} \cap C_{\alpha q}'((0, T); L^2(D)) \cap C'((0, T); V_{-\beta q')} \leq C_2 \|\varphi\|_{V_{\beta p}} + C_2 \|F\|_{X_{2,\alpha q}}. \tag{3.10}$$

Proof. The proof of part (a) can be easily obtained from lemma 3.2. Indeed, the inequality (3.3) leads to

$$\|u\|_{L^{\frac{1}{\alpha q}}(0, T; L^2(D))} \leq C_0 \left( \|\varphi\|_{V_{\beta p}} + \|F\|_{X_{2,\alpha q}} \right) \left\{ \int_0^T t^{-\alpha q} \left( \frac{1}{\alpha q} - r \right) \right\}^{\frac{1}{\alpha q}}.$$

Since $-\alpha q \left( \frac{1}{\alpha q} - r \right) > -1$, the integral in the above inequality exists, i.e., $t^{-\alpha q}$ belongs to $L^{\frac{1}{\alpha q}}(0, T; R)$. Hence, FVP (3.1) has a solution $u$ in $L^{\frac{1}{\alpha q}}(0, T; L^2(D))$. The uniqueness of $u$ depends on the uniqueness of the ODE (2.6). For the uniqueness of this ODE, see [61] (theorem 3.25). Moreover, the inequality (3.9) is derived by letting $C_1 = C_0 \|t^{-\alpha q}\|_{L^{\frac{1}{\alpha q}}(0, T; R)}$. Now, we proceed to prove part (b) which will be presented in the following steps.
Step 1: We prove \( u \in L^\infty_\text{loc}(0, T; \mathbf{V}_{\beta_0}) \). Firstly, by the same argument as in the proof of (3.5), we derive the following chain of inequalities

\[
\| (O_1 F)(t, \cdot) \|_{\mathbf{V}_{\beta_0}} \leq \int_0^T \left\| L^{(p, p')} \sum_{j=1}^{\infty} F_1(t) \mathcal{E}_{a, \alpha}(-m_j^\beta(T - \tau)\alpha) e_j \right\| \, d\tau,
\]

\[
\leq \tilde{c}_\alpha \int_0^T \left\{ \sum_{j=1}^{\infty} F_j^2(t) m_j^{-2\beta}(T - \tau)^{-2\alpha q}(T - \tau)^{2n-2} m_j^{2\beta(p-p')} \right\} \frac{1}{2} \, d\tau.
\]

\[
\leq \tilde{c}_\alpha M_4 \int_0^T |F(t, \cdot)| \| (t - \tau)^{\alpha q-1} \| \, d\tau \leq M_4 T^{-\alpha q-1} \| F \|_{\mathbf{V}_{\gamma_{\alpha q-\gamma}}}, \quad (3.11)
\]

for \( M_4 = \tilde{c}_\alpha m_1^{-\beta \gamma} T^\alpha \gamma \), where the inequality \( \| F \|_{\mathbf{V}_{\gamma_{\alpha q-\gamma}}} \leq T \| F \|_{\mathbf{V}_{\gamma_{\alpha q-\gamma}}} \) holds. Similarly, from \( \| O_2(t) \phi \|_{\mathbf{V}_{\beta_0}} = \| L^{(p, p')} O_2(t) \phi \| \) and the same way as in the proof of (3.7), we have

\[
\| O_2(t) \phi \|_{\mathbf{V}_{\beta_0}} \leq \tilde{c}_\alpha c_\alpha^{-1} \left( \sum_{j=1}^{\infty} \frac{\mathcal{E}_j^2(t) (1 + m_j^\beta T^\alpha \gamma)^{2\beta p'} m_j^{2\beta(p-p')}}{m_j^{2\beta(p-p')}} \right)^{1/2} \leq M_5 T^{-\alpha q} \| \phi \|_{\mathbf{V}_{\beta_0}}, \quad (3.12)
\]

where we let \( M_5 = \tilde{c}_\alpha c_\alpha^{-1} T^\alpha \gamma (m_1^{-\beta \gamma} + T^\alpha \gamma \gamma) \). Now, we will estimate the norm \( \| (O_3 F)(t, \cdot) \|_{\mathbf{V}_{\beta_0}} \), which will use the same estimate for the fraction \( \frac{E_{a, 1}(-m_j^\beta T^\alpha \gamma)}{E_{a, 1}(-m_j^\beta T^\alpha)} \) as in (3.12). Indeed, noting that \((1 + m_j^\beta T^\alpha \gamma)^{\gamma} \leq (m_1^{-\beta \gamma} + T^\alpha \gamma \gamma)^{\gamma} \), by using (2.11), we see that

\[
\| (O_3 F)(t, \cdot) \|_{\mathbf{V}_{\beta_0}} \leq \int_0^T \left\| L^{(p, p')} \sum_{j=1}^{\infty} F_j(t) E_{a, \alpha}(-m_j^\beta(T - \tau)\alpha) T^{-\alpha q-1} \frac{E_{a, 1}(-m_j^\beta T^\alpha \gamma)}{E_{a, 1}(-m_j^\beta T^\alpha)} e_j \right\| \, d\tau
\]

\[
\leq \int_0^T \left\{ \sum_{j=1}^{\infty} \frac{F_j^2(t) E_{a, \alpha}^2(-m_j^\beta(T - \tau)\alpha) T^{-\alpha q-1} E_{a, 1}(-m_j^\beta T^\alpha \gamma)}{E_{a, 1}(-m_j^\beta T^\alpha)} m_j^{2\beta(p-p')} \right\} \frac{1}{2} \, d\tau.
\]

\[
\leq \tilde{c}_\alpha M_5 \int_0^T \left\{ \sum_{j=1}^{\infty} F_j^2(t) m_j^{-2\beta}(T - \tau)^{-2\alpha q}(T - \tau)^{2n-2} m_j^{2\beta(p-p')} \right\} \frac{1}{2} \, d\tau.
\]

Thus, by some simple computations, one can get

\[
\| (O_3 F)(t, \cdot) \|_{\mathbf{V}_{\beta_0}} \leq \tilde{c}_\alpha M_5 T^{-\alpha q-1} \int_0^T |F(t, \cdot)| \| (T - \tau)^{\alpha q-1} \| \, d\tau \leq M_6 T^{-\alpha q-1} \| F \|_{\mathbf{V}_{\gamma_{\alpha q-\gamma}}}, \quad (3.13)
\]
with \( M_6 = \tilde{e}_a M_3 T' \), where the inequality \( \| F \|_{X_2, eq} \leq T' \| F \|_{X_2, eq - r} \) has been used. Bring (3.11)--(3.13) and (3.2) together, we have

\[
\| u(t, \cdot) \|_{V_{(p-r')}^{\alpha, \psi}} \leq M_7 \left( \| \varphi \|_{V_{p}^{\alpha, \psi}} + \| F \|_{X_2, eq - r} \right) r^{-\alpha q'},
\]

where \( M_7 = \sum_{4 \leq p \leq 6} M_6 \). Since the function \( t \rightarrow t^{-\alpha q'} \) is clearly contained in the space \( L^{\alpha q'}(0, T; \mathbb{R}) \), we can take the \( L^{\alpha q'}(0, T; \mathbb{R}) \)-norm on both sides of the above inequality, namely

\[
\| u \|_{L^{\alpha q'}(0, T; V_{(p-r')}^{\alpha, \psi})} \leq M_8 \| \varphi \|_{V_{p}^{\alpha, \psi}} + M_8 \| F \|_{X_2, eq - r}, \tag{3.14}
\]

where \( M_8 = M_7 \| r^{-\alpha q'} \|_{L^{\alpha q'}(0, T; \mathbb{R})} \).

**Step 2:** We prove \( u \in \mathcal{C}^0_w((0, T); L^2(D)) \). Let us consider \( 0 < t_1 < t_2 \leq T \). By (3.2), the difference \( u(t_2, \cdot) - u(t_1, \cdot) \) can be calculated as

\[
u(t_2, x) - u(t_1, x) = \sum_{j=1}^{\infty} F_j(t) \ast \tilde{E}_{\alpha, \gamma}(-m_j^2 \omega^\gamma) \bigg|_{t=t_1}^{t=t_2} e_j(x) + \sum_{j=1}^{\infty} \varphi \frac{E_{\alpha, \gamma}(-m_j^2 \omega^\gamma)}{E_{\alpha, \gamma}(-m_j^2 T)} \bigg|_{t=t_1}^{t=t_2} e_j(x)
\]

\[-\sum_{j=1}^{\infty} \left( F_j(t) \ast \tilde{E}_{\alpha, \gamma}(-m_j^2 \omega^\gamma) \right) \bigg|_{t=t_1}^{t=t_2} \frac{E_{\alpha, \gamma}(-m_j^2 \omega^\gamma)}{E_{\alpha, \gamma}(-m_j^2 T)} e_j(x).
\]

By using the differentiation identities

\[ D_\omega E_{\alpha, \gamma}(-m_j^2 \omega^\gamma) = -m_j^2 \tilde{E}_{\alpha, \gamma}(-m_j^2 \omega^\gamma) \text{ and } D_\omega \left( \tilde{E}_{\alpha, \gamma}(-m_j^2 \omega^\gamma) \right) = \omega^{\alpha-2} E_{\alpha, \gamma-1}(-m_j^2 \omega^\gamma), \]

see, for example [58, 59, 60], we have

\[
F_j(t) \ast \tilde{E}_{\alpha, \gamma}(-m_j^2 \omega^\gamma) \bigg|_{t=t_1}^{t=t_2}
= \int_0^{t_1} F_j(\tau) \tilde{E}_{\alpha, \gamma}(-m_j^2 \omega^\gamma) \bigg|_{\omega=\tau^{-1}}^{\omega=\tau^{-2}} d\tau + \int_{t_1}^{t_2} F_j(\tau) \tilde{E}_{\alpha, \gamma}(-m_j^2 (t_2 - \tau)^\gamma) d\tau
= \int_0^{t_1} \int_{t_1}^{t_2} F_j(\tau) \omega^{\alpha-2} E_{\alpha, \gamma-1}(-m_j^2 \omega^\gamma) d\omega d\tau + \int_{t_1}^{t_2} F_j(\tau) \tilde{E}_{\alpha, \gamma}(-m_j^2 (t_2 - \tau)^\gamma) d\tau,
\]

and

\[ E_{\alpha, \gamma}(-m_j^2 \omega^\gamma) \bigg|_{t=t_1}^{t=t_2} = -m_j^2 \int_{t_1}^{t_2} \tilde{E}_{\alpha, \gamma}(-m_j^2 \omega^\gamma) d\omega.
\]

Combining the above arguments gives
\[ u(t_2, x) - u(t_1, x) \]
\[ = \sum_{j=1}^{\infty} \int_0^{t_1} \int_{t_1-\tau}^{t_2} F_j(\tau) F_{a,p-2} E_{a,q-1}(-m_j^2 \omega^2) d\omega d\tau e_j(x) \]
\[ + \sum_{j=1}^{\infty} \int_{t_1}^{t_2} F_j(\tau) E_{a,q-1}(-m_j^2 (t_2 - \tau) \omega^2) d\omega d\tau e_j(x) - L^2 \sum_{j=1}^{\infty} \int_{t_1}^{t_2} F_j(\tau) E_{a,q-1}(-m_j^2 \omega^2) d\omega d\tau e_j(x) \]
\[ + L^2 \sum_{j=1}^{\infty} \int_0^{t_1} \int_{t_1-\tau}^{t_2} F_j(\tau) E_{a,q-1}(-m_j^2 (T - \tau) \omega^2) \tilde{E}_{a,q-1}(-m_j^2 \omega^2) d\omega d\tau e_j(x) \]
\[ := I_1 + I_2 + I_3 + I_4. \tag{3.15} \]

Now, we will establish estimates for \( I_j, j = 1, 2, 3, 4 \), and show that \( I_j \) tends to 0 as \( t_2 - t_1 \to 0 \).

Firstly, by the inequality (2.11), we see that the absolute value of \( E_{a,q-1}(-m_j^2 \omega^2) \) is bounded by \( \tilde{c}_m m_j^{-\beta/\omega} \). This implies
\[ \omega^2 \left| E_{a,q-1}(-m_j^2 \omega^2) \right| \leq \tilde{c}_a \omega^2 m_j^{-\beta/\omega}. \]

Moreover, for \( 0 < \tau < t_1 \), we have
\[ \int_{t_1-\tau}^{t_2} \omega^2 \omega^2 d\omega = \frac{1}{1 - \alpha q} \left( t_2 - \tau \right)^{1-\alpha q} \left( t_1 - \tau \right)^{1-\alpha q} \left( t_2 - t_1 \right)^{1-\alpha q}, \]
where we note that the estimates \( (t_2 - \tau)^{1-\alpha q} \leq (t_2 - t_1)^{1-\alpha q} \) and \( (t_2 - \tau)^{1-\alpha q} \geq (t_1 - \tau)^{1-\alpha q} \) can be showed easily from \( 0 < \alpha q < 1 \), and \( 1 - \alpha q - s > 0 \). Hence, we deduce
\[ \|I_1\| \leq \int_0^{t_1} \left\| \sum_{j=1}^{\infty} \int_{t_1-\tau}^{t_2} F_j(\tau) E_{a,p-2} E_{a,q-1}(-m_j^2 \omega^2) d\omega d\tau \right\| d\tau \]
\[ \leq \tilde{c}_a m_1^{-\beta/\omega} \int_0^{t_1} \int_{t_1-\tau}^{t_2} \omega^2 \omega^2 d\omega \left\| F(\tau, \omega) \right\| d\tau \]
\[ \leq \tilde{c}_a m_1^{-\beta/\omega} \int_0^{t_1} \|F(\tau, \omega)\| (t_1 - \tau)^{1-\alpha q-1} d\tau (t_2 - t_1)^{1-\alpha q}. \]

This leads to
\[ \|I_1\| \leq M_3 \|F\|_{X_{a,q-1}} (t_2 - t_1)^{1-\alpha q}, \tag{3.16} \]
where we let \( M_3 = \frac{\tilde{c}_a m_1^{-\beta/\omega}}{1-\alpha q} \). Secondly, an estimate for the term \( I_2 \) can be shown by using (2.11) as follows.
\[ \|I_2\| \leq \int_0^{t_1} \sum_{j=1}^{\infty} F_j(\tau) E_{a,q-1}(-m_j^2 (t_2 - \tau)^2) e_j \left( t_2 - \tau \right)^{1-\alpha q} d\tau \]
\[ \leq \tilde{c}_a \int_0^{t_1} \|F(\tau, \omega)\| (t_2 - \tau)^{1-\alpha q-1} (t_2 - t_1)^{1-\alpha q} d\tau \]
\[ \leq M_{10} \|F\|_{X_{a,q-1}} (t_2 - t_1)^{1-\alpha q}. \tag{3.17} \]
where we let $M_{10} = \tilde{c}_0T^{\alpha}$. Thirdly, we will estimate the term $I_3$. We have

$$
\|I_3\| = \left\|L^j \sum_{j=1}^{\infty} \varphi_j \int_{\hat{t}_1}^{\hat{t}_2} \frac{\tilde{E}_{\alpha,\alpha}(-m_j^\beta \omega^{\alpha})}{E_{\alpha,1}(-m_j^\beta T^\alpha)} \omega \, d\omega\right\|.
$$

Here, the fraction can be estimated as follows

$$
\frac{\tilde{E}_{\alpha,\alpha}(-m_j^\beta \omega^{\alpha})}{E_{\alpha,1}(-m_j^\beta T^\alpha)} \leq \tilde{c}_0 c_1^{-1} \left[1 + m_j^\beta T^\alpha\right]^p \left[1 + m_j^\beta \omega^{2\alpha}\right] \omega^{\alpha-1},
$$

by using (2.11) and $\tilde{E}_{\alpha,\alpha}(-m_j^\beta \omega^{\alpha}) = E_{\alpha,\alpha}(-m_j^\beta \omega^{\alpha})\omega^{\alpha-1}$. Moreover, we can see that

$$
\frac{1 + m_j^\beta T^\alpha}{1 + m_j^\beta \omega^{2\alpha}} \leq (m_1^{-\beta} + T^\alpha)m_j^{-1}m_j^{-2\alpha} \omega^{-2\alpha} \leq (m_1^{-\beta} + T^\alpha)m_j^{-\beta} \omega^{-2\alpha}.
$$

Taking these estimates together, we thus obtain the following chain of the inequalities

$$
\|I_3\| \leq \left\{ \sum_{j=1}^{\infty} \varphi_j^2 m_j^{2\beta} \left[ \int_{\hat{t}_1}^{\hat{t}_2} \frac{\tilde{E}_{\alpha,\alpha}(-m_j^\beta \omega^{\alpha})}{E_{\alpha,1}(-m_j^\beta T^\alpha)} \omega \, d\omega \right]^2 \right\}^{1/2}
$$

$$
\leq M_{11} \left\{ \sum_{j=1}^{\infty} \varphi_j^2 m_j^{2\beta} \left[ \int_{\hat{t}_1}^{\hat{t}_2} \omega^{-\alpha q} m_j^{-\beta q \omega^{-2\alpha q} \omega^{\alpha-1}} \omega \, d\omega \right]^2 \right\}^{1/2}
$$

$$
\leq M_{11} \left\{ \sum_{j=1}^{\infty} \varphi_j^2 m_j^{2\beta} \left[ \int_{\hat{t}_1}^{\hat{t}_2} \omega^{-\alpha q - 1} \omega \, d\omega \right]^2 \right\}^{1/2},
$$

which implies that

$$
\|I_3\| \leq M_{12} t_1^{-\alpha q} (t_2 - t_1)^{\alpha q} \|\varphi\|_{V_{\beta q}},
$$

(3.19)

where $M_{11} = \tilde{c}_0 c_1^{-1} T^{\alpha} (m_1^{-\beta} + T^\alpha)^q$, and $M_{12} = M_{11} |\alpha q|^{-1}$. Fourthly, we proceed to estimate $I_4$. According to (3.18), we have $\frac{\tilde{E}_{\alpha,\alpha}(-m_j^\beta \omega^{\alpha})}{E_{\alpha,1}(-m_j^\beta T^\alpha)} \leq M_{11} m_j^{-\beta q} \omega^{-\alpha q - 1}$. Moreover, $\tilde{E}_{\alpha,\alpha}(-m_j^\beta (T - \tau)^\alpha) \leq \tilde{c}_0 m_j^{-\beta q} (T - \tau)^{\alpha q - 1}$ can be established by using the inequalities (2.11).

Hence, we obtain

$$
\|I_4\| \leq \int_0^T \left\|L^j \sum_{j=1}^{\infty} \int_{\hat{t}_1}^{\hat{t}_2} F_j(\tau) \tilde{E}_{\alpha,\alpha}(-m_j^\beta (T - \tau)^\alpha) \frac{\tilde{E}_{\alpha,\alpha}(-m_j^\beta \omega^{\alpha})}{E_{\alpha,1}(-m_j^\beta T^\alpha)} \omega \, d\omega \right\| \, d\tau
$$

$$
\leq \int_0^T \left\{ \sum_{j=1}^{\infty} m_j^{2\beta} F_j^2(\tau) \left[ \int_{\hat{t}_1}^{\hat{t}_2} \frac{\tilde{E}_{\alpha,\alpha}(-m_j^\beta (T - \tau)^\alpha)}{E_{\alpha,1}(-m_j^\beta T^\alpha)} \omega \, d\omega \right]^2 \right\}^{1/2} \, d\tau
$$

$$
\leq \tilde{c}_0 M_{11} \int_0^T \left\{ \sum_{j=1}^{\infty} m_j^{2\beta} F_j^2(\tau) \left[ \int_{\hat{t}_1}^{\hat{t}_2} m_j^{-\beta q} (T - \tau)^{\alpha q - 1} m_j^{-\beta q \omega^{-\alpha q - 1}} \omega \, d\omega \right]^2 \right\}^{1/2} \, d\tau
$$

$$
\leq \tilde{c}_0 M_{11} \int_0^T \int_0^{\hat{t}_2} \frac{(T - \tau)^{\alpha q - 1}}{\hat{t}_1^{\alpha q - 1}} \left[ \int_{\hat{t}_1}^{\hat{t}_2} m_j^{-\beta q} (T - \tau)^{\alpha q - 1} m_j^{-\beta q \omega^{-\alpha q - 1}} \omega \, d\omega \right] \, d\tau,
$$

and we arrive at
\[ \|I_4\| \leq \tilde{c}_0 M_1 \frac{t_2 - t_1}{t_2 - t_1} \|F\|_{\mathcal{L}^2_{\alpha q-s}}, \]  

(3.20)

where \( M_1 = M_1 T \). We deduce from (3.16), (3.17), (3.19), (3.20) that \( \|\sum_{j \leq 1} I_j\| \) tends to 0 as \( t_2 - t_1 \) tends 0 for \( 0 < t_1 < t_2 \leq T \). Thus, \( u \) belongs to the set \( C(0, T) : L^2(D) \). On the other hand, by assumption (R3), we have \( 0 < \alpha q-s < \alpha q \) and \( \mathcal{X}_{\alpha q-s}(J \times D) \subset \mathcal{X}_{\alpha q-s}(J \times D) \). Therefore, the assumptions on \( \varphi \) and \( F \) in this theorem also fulfill lemma 3.2. Hence, the inequality (3.3) holds, i.e.,

\[ t^q \|u(t, \cdot)\| \leq M_4 \left( \|\varphi\|_{\mathcal{Y}} + \|F\|_{\mathcal{X}_{\alpha q-s}} \right), \quad t > 0. \]  

(3.21)

This implies \( u \) belongs to \( C^\alpha_0((0, T) : L^2(D)) \). Moreover, by taking the supremum on both sides of (3.21) on \( (0, T] \), we obtain

\[ \|u\|_{C^\alpha_0((0, T) : L^2(D))} \leq M_4 \|\varphi\|_{\mathcal{Y}} + T^q M_4 \|F\|_{\mathcal{X}_{\alpha q-s}}, \]  

(3.22)

**Step 3:** We prove \( u \in C^\alpha((0, T) : \mathcal{Y}^{-\beta q'}) \). In this step, we establish the continuity of the solution on the closed interval \([0, T]\). Now, we consider \( 0 \leq t_1 < t_2 \leq T \). If \( t_1 = 0 \), then \( I_1 = 0 \). If \( t_1 > 0 \), then combining (2.2) in the same way as in (3.16) gives

\[ \|I_1\|_{\mathcal{Y}^{-\beta q'}} \leq \int_0^{t_1} \left\{ \sum_{j=1}^\infty \int_{t_1 - \tau}^{t_2 - \tau} F_j(\tau) \omega^{\alpha-2} E_{\alpha,\alpha-1}(-m_j^\alpha \omega^\alpha) d\omega e_j \right\} \|F_j\|_{\mathcal{Y}^{-\beta q'}} d\tau \]

\[ \leq \int_0^{t_1} \left\{ \sum_{j=1}^\infty \int_{t_1 - \tau}^{t_2 - \tau} F_j(\tau) \omega^{\alpha-2} E_{\alpha,\alpha-1}(-m_j^\alpha \omega^\alpha) d\omega e_j \right\} \|F_j\|_{\mathcal{Y}^{-\beta q'}} d\tau \]

and so

\[ \|I_1\|_{\mathcal{Y}^{-\beta q'}} \leq \tilde{c}_0 m_1^{-\beta q'-\beta p} \frac{1}{t_2 - t_1} \|F\|_{\mathcal{X}_{\alpha q-s}}. \]  

(3.23)

On the other hand, the inequality (3.17) also holds for all \( 0 \leq t_1 < t_2 \leq T \). Hence, the same way as in the proof (3.17) shows that

\[ \|I_2\|_{\mathcal{Y}^{-\beta q'}} \leq \int_0^{t_2} \left\{ \sum_{j=1}^\infty \int_{t_1 - \tau}^{t_2 - \tau} F_j(\tau) E_{\alpha,\alpha}(-m_j^\alpha (t_2 - \tau)\omega^\alpha) e_j \right\} \|F_j\|_{\mathcal{Y}^{-\beta q'}} d\tau \]

\[ \leq m_1^{-\beta q'} \tilde{c}_0 (t_2 - t_1) \|\varphi\|_{\mathcal{Y}} + \|F\|_{\mathcal{X}_{\alpha q-s}}, \]  

(3.24)

Now, we will establish estimates for \( \|I_3\|_{\mathcal{Y}^{-\beta q'}} \) and \( \|I_4\|_{\mathcal{Y}^{-\beta q'}} \). Indeed, we have

\[ \frac{\tilde{E}_{\alpha,\alpha}(-m_j^\alpha \omega^\alpha)}{E_{\alpha,1}(-m_j^\alpha \omega^\alpha)} \leq \tilde{c}_0 c_0^{-1} \frac{1 + m_j^\alpha T^\alpha}{1 + m_j^\alpha \omega^\alpha} \left( \frac{1 + m_j^\alpha T^\alpha}{1 + m_j^\alpha \omega^\alpha} \right)^{(p-p')/\omega^\alpha-1}, \]

\[ \leq \tilde{c}_0 c_0^{-1} (m_1^{-\beta q'} + T^{q+p} m_j^{\beta q'(p-p')/\omega^\alpha-1}(q+p)\omega^{-\alpha(q+p)} \omega^{-\alpha(q+p)}). \]
Thus, we can derive

\[
\left| \frac{\tilde{E}_{n,n}(m_j^{\beta-\beta'})}{E_{n,1}(m_j^{\beta-\beta'})} \right| \leq M_{14} m_j^{\beta(p-q')} \omega^{\alpha(p-q')-1}, \tag{3.24}
\]

where we let

\[
M_{14} = \tilde{c}_n^{-1}(m_1^{-\beta} + T^{\alpha(p-q')}T^{\alpha(p-q')}).
\]

This leads to

\[
\|I_3\|_{V_{\beta q'}} \leq \left\{ \sum_{j=1}^{\infty} \varphi_j^2 m_j^{2\beta} \left[ \int_{T_j}^{T_2} \left| \tilde{E}_{n,n}(m_j^{\beta-\beta'}) \frac{E_{n,1}(m_j^{\beta-\beta'})}{E_{n,1}(m_j^{\beta-\beta'})} \right|^2 \, d\omega \right]^{1/2} \middle/ \sum_{j=1}^{\infty} \varphi_j^2 m_j^{2\beta} \left[ \int_{T_j}^{T_2} \omega^{\alpha(p-q')-1} \right]^{1/2} \right\}.
\]

Thus, by letting \(M_{15} = \frac{M_{14}}{\alpha(p-q')}T^{\alpha(p-q')-1}\), we obtain the estimate

\[
\|I_3\|_{V_{\beta q'}} \leq \frac{M_{14}}{\alpha(p-q')} \|\varphi\|_{V_{\beta q'}} \left( t_2^{\alpha(p-q')} - t_1^{\alpha(p-q')} \right),
\tag{3.25}
\]

where we have used that

\[
t_2^{\alpha(p-q')} - t_1^{\alpha(p-q')} \leq (t_2 - t_1)^{\alpha(p-q')} \leq T^{\alpha(p-q')} - (t_2 - t_1)^{\alpha(p-q')},
\]

for \(p' \leq p - \frac{2\alpha}{\beta}\). By employing (3.24) and following the same way as in (3.25), we have

\[
\|I_4\|_{V_{\beta q'}} \leq \int_0^T \left\{ \sum_{j=1}^{\infty} \varphi_j^2 F_j^2(\tau) \left[ \int_{T_j}^{T_2} \tilde{E}_{n,n}(m_j^{\beta-\beta'}) \frac{E_{n,1}(m_j^{\beta-\beta'})}{E_{n,1}(m_j^{\beta-\beta'})} \, d\omega \right]^{1/2} \right\} \, d\tau
\]

\[
\leq \sum_{j=1}^{\infty} \varphi_j^2 F_j^2(\tau) \left[ \int_{T_j}^{T_2} \tilde{E}_{n,n}(m_j^{\beta-\beta'}) \frac{E_{n,1}(m_j^{\beta-\beta'})}{E_{n,1}(m_j^{\beta-\beta'})} \, d\omega \right]^{1/2} \, d\tau
\]

\[
\leq \tilde{c}_n M_{14} \sum_{j=1}^{\infty} \varphi_j^2 F_j^2(\tau) \left[ \int_{T_j}^{T_2} (T - \tau)^{\alpha(p-q')-1} m_j^{\beta(p-q')} \omega^{\alpha(p-q')-1} \, d\omega \right]^{1/2} \, d\tau
\]

\[
\leq \tilde{c}_n M_{14} \sum_{j=1}^{\infty} \varphi_j^2 F_j^2(\tau) \left[ \int_{T_j}^{T_2} (T - \tau)^{\alpha(p-q')-1} m_j^{\beta(p-q')} \omega^{\alpha(p-q')-1} \, d\omega \right]^{1/2} \, d\tau,
\]

where

\[
|\tilde{E}_{n,n}(m_j^{\beta}(T - \tau)^\alpha)| \leq m_j^{-\beta(p-q)}(T - \tau)^{\alpha(p-q')-1}.
\]

This implies that

\[
\|I_4\|_{V_{\beta q'}} \leq M_{14} (t_2 - t_1)^{\alpha(p-q')} \|F\|_{L_{2(q-\alpha)}},
\tag{3.26}
\]

where
where $M_{16} = \hat{c}_u T^\gamma M_{15}$. Combining the above arguments guarantees that $u$ belongs to $C^\alpha([0, T] : V_{\beta_\gamma q'})$. Moreover, there also exists a positive constant $M_{17}$ such that
\[
\|u\|_{C^\alpha([0, T] : V_{\beta_\gamma q'})} \leq M_{17}\|\varphi\|_{V_{\beta p}} + M_{17}\|\beta\|_{L^\infty \cap \frac{1}{L^\infty \cap J}}.
\] (3.27)

Finally, the inequality (3.10) is obtained by taking the inequality (3.14), (3.22) and (3.27) together. The proof is complete. \hfill \Box

In the next theorem, we will investigate the time-space fractional derivative of the mild solution $u$. More specifically, we investigate $\mathcal{D}_t^{\gamma \alpha \beta} L^{-\beta q} u$, for a suitably chosen number $\hat{q} \leq q$. We also establish the continuity of $\mathcal{D}_t^{\gamma \alpha \beta} L^{-\beta q} u$ on the interval $[0, T]$ without establishing it at $t = 0$ since this requires a strong assumption of $F$, for example, $F$ must be continuous on whole interval $[0, T]$.

**Theorem 3.4.** Let $p, q, s, p', q', \hat{p}, \hat{q}, r, \hat{r}$ be defined by (R1), (R3), (R4), (R5).

(a) If $\varphi \in V_{\beta p+\hat{q}}$, and $F \in L^{-\hat{r}}(0, T; L^2(D))$, then FVP (3.1) has a unique solution $u$ such that
\[
u \in L^{\hat{r}}(0, T; V_{\beta_{p'} q}) \cap C_w((0, T] : L^2(D)) \cap C^\alpha([0, T] : V_{\beta_\gamma q'}),
\]
\[
\mathcal{D}_t^{\gamma \alpha \beta} u \in L^{\hat{r}}(0, T; \mathcal{V}_{-\beta q}).
\]
Moreover, there exists a positive constant $C_3$ such that
\[
\|\mathcal{D}_t^{\gamma \alpha \beta} u\|_{L^{\hat{r}}(0, T; \mathcal{V}_{-\beta q})} \leq C_3\|\varphi\|_{V_{\beta p+\hat{q}}} + C_3\|F\|_{L^{-\hat{r}}(0, T; L^2(D))}.
\] (3.28)

(b) If $\varphi \in V_{\beta p+\hat{q}}$, and $F \in L^{-\hat{r}}(0, T; L^2(D)) \cap C_w((0, T] : V_{\beta_\gamma q})$, then FVP (3.1) has a unique solution $u$ such that
\[
u \in L^{\hat{r}}(0, T; V_{\beta_{p'} q}) \cap C_w((0, T] : L^2(D)) \cap C^\alpha([0, T] : V_{\beta_\gamma q'}),
\]
\[
\mathcal{D}_t^{\gamma \alpha \beta} u \in L^{\hat{r}}(0, T; \mathcal{V}_{-\beta q}) \cap C_w((0, T] : V_{\beta_\gamma q}).
\]
Moreover, there exists a positive constant $C_4$ such that
\[
\|\mathcal{D}_t^{\gamma \alpha \beta} u\|_{L^{\hat{r}}(0, T; \mathcal{V}_{-\beta q})} + \|\mathcal{D}_t^{\gamma \alpha \beta} u\|_{C_w((0, T] : V_{\beta_\gamma q})} \leq C_4\|\varphi\|_{V_{\beta p+\hat{q}}} + C_4\|F\|_{L^{-\hat{r}}(0, T; L^2(D))} + C_4\|F\|_{C_w((0, T] : V_{\beta_\gamma q})}.
\] (3.29)

**Proof.** (a) By (R5), we have $0 < \alpha q - s < 1$, and $\frac{1}{\alpha q - s} + \frac{1}{\alpha q - s}$, and $\frac{1}{\alpha q - s} > 1$. Thus, one can deduce from (2.5) that
\[
L^{\frac{1}{\alpha q - s}}(0, T ; L^2(D)) \subset \mathcal{X}_{\alpha q - s}(F \times D),
\] (3.30)
where we have used the inclusion (2.4). Moreover, the Sobolev embedding $V_{\beta p+\hat{q}} \hookrightarrow V_{\beta p}$ holds. Therefore, the assumptions of this theorem also fulfills part (b) of theorem 3.3. Hence, FVP (3.1) has a unique solution
\[
u \in L^{\frac{1}{\alpha q - s}}(0, T ; V_{\beta_{p'} q}) \cap C_w((0, T] : L^2(D)) \cap C^\alpha([0, T] : V_{\beta_\gamma q'}).\]

Now, we prove $\mathcal{D}_t^{\gamma \alpha \beta} u$ exists and belongs to $L^{\hat{r}}(0, T ; \mathcal{V}_{-\beta q}) \cap C_w((0, T] : V_{\beta_\gamma q})$. It follows from the identities
see, for example [58, 59, 60], and equation (2.9) that
\[
\begin{align*}
\mathcal{D}^c_\alpha u_j(t) &= \mathcal{D}^c_\alpha \left[ F_j(t) \ast \hat{E}_{\alpha,\alpha}(-m_j^\beta \tau, \nu) \right] + \left[ \varphi_j - \left( F_j(r) \ast \hat{E}_{\alpha,\alpha}(-m_j^\beta \tau, \nu) \right) \right]_{r=T} \mathcal{D}^c_\alpha E_{\alpha,1}(-m_j^\beta \tau, \nu, \nu) \\
&= F_j(t) - m_j^\beta F_j(t) \ast \hat{E}_{\alpha,\alpha}(-m_j^\beta \tau, \nu) - \varphi_j \mathcal{D}^c_\alpha E_{\alpha,1}(-m_j^\beta \tau, \nu) \%
\end{align*}
\] 
\[
\mathcal{D}^c_\alpha \hat{E}_{\alpha,\alpha}(-m_j^\beta \tau, \nu) = \mathcal{D}^c_\alpha E_{\alpha,1}(-m_j^\beta \tau, \nu, \nu),
\]

for all \( j \in \mathbb{N} \). Firstly, let us consider the sum \( \sum_{n_1 \leq j \leq n_2} \psi_j^{(1)}(t)e_j \), for \( n_1, n_2 \in \mathbb{N}, 1 \leq n_1 < n_2 \). By the definition of the dual space \( V_{-\beta q - \tilde{q}} \) of \( V_{\beta q - \tilde{q}} \), and the identity (2.2) of their dual inner product, we have
\[
\begin{align*}
\left\| \sum_{n_1 \leq j \leq n_2} \psi_j^{(1)}(t)e_j \right\|_{V_{-\beta q - \tilde{q}}} &\leq \int_0^t \left\| \sum_{n_1 \leq j \leq n_2} m_j^\beta F_j(\tau) \hat{E}_{\alpha,\alpha}(-m_j^\beta (t - \tau), \nu)e_j \right\|_{V_{-\beta q - \tilde{q}}} \, d\tau \\
&\leq \int_0^t \left\{ \sum_{n_1 \leq j \leq n_2} m_j^{2\beta(p+\tilde{q})} \left( \sum_{n_1 \leq j \leq n_2} F_j(\tau) \hat{E}_{\alpha,\alpha}(-m_j^\beta (t - \tau), \nu)e_j, e_j \right) \right\}^{1/2} \, d\tau \\
&\leq \int_0^t \left\{ \sum_{n_1 \leq j \leq n_2} m_j^{2\beta(p+\tilde{q})} F_j^2(\tau) \hat{E}_{\alpha,\alpha}^2(-m_j^\beta (t - \tau), \nu) \right\}^{1/2} \, d\tau.
\end{align*}
\] 
Assumption (R5) shows that \( 0 < p + \tilde{q} < 1 \). Hence, by using the inequalities (2.11), we have \( \hat{E}_{\alpha,\alpha}(-m_j^\beta (t - \tau), \nu) \leq \hat{c}_\alpha m_j^{3\beta(p+\tilde{q})} (t - \tau)^{-\alpha(p+\tilde{q})(t - \tau)\alpha^{-1}} \). This together with the above argument gives
\[
\left\| \sum_{n_1 \leq j \leq n_2} \psi_j^{(1)}(t)e_j \right\|_{V_{-\beta q - \tilde{q}}} \leq M_{18} t^{-\alpha} \int_0^t (t - \tau)^{\alpha(q - \tilde{q}) - 1} \left\{ \sum_{n_1 \leq j \leq n_2} F_j^2(\tau) \right\}^{1/2} \, d\tau, \quad (3.31)
\]
where \( M_{18} = \hat{c}_\alpha T^n \). Secondly, we proceed to establish an estimate for the sum \( \sum_{n_1 \leq j \leq n_2} \psi_j^{(2)}(t)e_j \). Using the inequality (2.11), the absolute value of \( \frac{E_{\alpha,1}(-m_j^\beta \tau, \nu, \nu)}{E_{\alpha,1}(-m_j^\beta \tau, \nu, \nu)} \) is bounded by \( \hat{c}_\alpha c^{-1} \). Therefore, we derive
\[
\begin{align*}
\left\| \sum_{n_1 \leq j \leq n_2} \psi_j^{(2)}(t)e_j \right\|_{V_{-\beta q - \tilde{q}}} &= \left\{ \sum_{n_1 \leq j \leq n_2} m_j^{2\beta(p+\tilde{q})} \psi_j^2 E_{\alpha,1}(-m_j^\beta \tau, \nu, \nu) \right\}^{1/2} \, d\tau \\
&\leq \hat{c}_\alpha c^{-1} \left\{ \sum_{n_1 \leq j \leq n_2} m_j^{2\beta(p+\tilde{q})} \psi_j^2 t^{-2\alpha} \right\}^{1/2}.
\end{align*}
\]
which shows that
\[
\left\| \sum_{n_1 \leq j \leq n_2} \psi_j^{(2)}(\tau)e_j \right\|_{V_{-\beta(q-\bar{q})}} \leq M_{19} t^{-\alpha} \left\{ \sum_{n_1 \leq j \leq n_2} m_j^{2\beta(p+\bar{q})} \varphi_j^2 \right\}^{1/2},
\]  
(3.32)
where \( M_{19} := \overline{c}_\alpha c_\alpha^{-1} T^{-\alpha} \). Thirdly, we proceed to establish an estimate for the sum \( \sum_{n_1 \leq j \leq n_2} \psi_j^{(3)}(\tau)e_j \). By a similar argument as in (3.32), we have
\[
\left\| \sum_{n_1 \leq j \leq n_2} \psi_j^{(3)}(\tau)e_j \right\|_{V_{-\beta(q-\bar{q})}} \leq \int_0^T \left\| \sum_{n_1 \leq j \leq n_2} F_j(\tau) \tilde{E}_{\alpha,\alpha}(-m_j^\beta(T - \tau)^\alpha) m_j^\beta E_{\alpha,1}(-m_j^\beta T^\alpha) \right\| \, d\tau \leq \int_0^T \left\{ \sum_{n_1 \leq j \leq n_2} m_j^{2\beta(p+\bar{q})} F_j^2(\tau) \tilde{E}_{\alpha,\alpha}(-m_j^\beta(T - \tau)^\alpha) E_{\alpha,1}^2(-m_j^\beta T^\alpha) \right\}^{1/2} \, d\tau.
\]
By a similar argument as in (3.30), we have that \( F(\tau, \cdot) \) belongs to \( L^2(D) \). This implies \( \sum_{1 \leq j \leq N} F_j(\tau)e_j \) is a Cauchy sequence in \( L^2(D) \). This together with the embedding
\[
L^2(D) \hookrightarrow V_{-\beta(q-\bar{q})}
\]
implies that \( \sum_{1 \leq j \leq N} F_j(\tau)e_j \) is also a Cauchy sequence in \( V_{-\beta(q-\bar{q})} \). On the other hand, it follows from \( \varphi \in V_{\beta(p+\bar{q})} \) that
\[
\lim_{n_1, n_2 \to \infty} \sum_{n_1 \leq j \leq n_2} \varphi_j^2 m_j^{2\beta(p+\bar{q})} = 0.
\]
By (R5), \( 0 \leq \bar{q} \leq \frac{\alpha}{\alpha} \), and we obtain the inclusion \( \mathcal{X}_{2,\alpha q - \bar{q}}(J \times D) \subset \mathcal{X}_{2,\alpha \alpha q - \bar{q}}(J \times D) \). We deduce that \( F \in \mathcal{X}_{2,\alpha \alpha q - \bar{q}}(J \times D) \), and
\[
\lim_{n_1, n_2 \to \infty} \int_0^T (T - \tau)^{\alpha(q-\bar{q})-1} \left\{ \sum_{n_1 \leq j \leq n_2} F_j^2(\tau) \right\}^{1/2} \, d\tau = 0,
\]  
(3.34)
by the dominated convergence theorem. Combining these with the estimates (3.31)–(3.33), we have that
\[
\lim_{n_1, n_2 \to \infty} \left\| \sum_{n_1 \leq j \leq n_2} \hat{c} D_j^\alpha u_j(t)e_j \right\|_{V_{-\beta(q-\bar{q})}} = 0.
\]
Hence $\sum_{i=1}^{n} C_i^i u_i(t)e_i$ is a Cauchy sequence and a convergent sequence in $V_{-\beta(q-\hat{q})}$. We then conclude that $\sum_{i=1}^{n} c_i^i u_i(t)e_i$ finitely exists in the space $V_{-\beta(q-\hat{q})}$. Moreover, by taking the inequalities (3.31)–(3.33), there exists a constant $M_{20} > 0$ such that

$$
\|\sum_{i=1}^{n} C_i^i u_i(t)e_i\|_{L^{\frac{2}{\beta(q-\hat{q})}}(0,T;V_{-\beta(q-\hat{q})})} \leq M_{20} \| F(t,\cdot)\|_{L^{\frac{2}{\beta(q-\hat{q})}}(0,T;V_{-\beta(q-\hat{q})})} + M_{20} \| \varphi \|_{L^{\frac{2}{\beta(q-\hat{q})}}(0,T;V_{-\beta(q-\hat{q})})} \tau^{-\alpha}.
$$

Now, it follows from $0 < \hat{r} \leq \frac{1}{\alpha q}$ and $0 < \alpha q - s < \alpha$ that $1 \leq \frac{1}{\alpha} - \hat{r} < \frac{1}{\alpha q - \hat{r}}$ and $\hat{r}$. This implies the following Sobolev embedding

$$
L^{\frac{2}{\beta(q-\hat{q})}}(0,T;V_{-\beta(q-\hat{q})}) \hookrightarrow L^{\frac{2}{\alpha q - \hat{r}}} (0,T;V_{-\beta(q-\hat{q})}).
$$

Moreover, by the assumption (R5), $\hat{r} < \frac{1}{\alpha}$, we have $\frac{1}{\alpha q - \hat{r}} > \frac{1}{\alpha q - \hat{r}}$. This implies that there exists a constant $C_3 > 0$ such that

$$
\| F \|_{L^{\frac{2}{\alpha q - \hat{r}}} (0,T;V_{-\beta(q-\hat{q})})} \leq C_3 \| F \|_{L^{\frac{2}{\beta(q-\hat{q})}}(0,T;V_{-\beta(q-\hat{q})})}.
$$

Hence, we deduce from (3.35) that there exists a constant $M_{21} > 0$ satisfying

$$
\|\sum_{i=1}^{n} C_i^i u_i(t)\|_{L^{\frac{2}{\alpha q - \hat{r}}} (0,T;V_{-\beta(q-\hat{q})})} \leq M_{21} \| F \|_{L^{\frac{2}{\beta(q-\hat{q})}}(0,T;V_{-\beta(q-\hat{q})})} + M_{21} \| \varphi \|_{L^{\frac{2}{\beta(q-\hat{q})}}(0,T;V_{-\beta(q-\hat{q})})} \tau^{-\alpha},
$$

where we note that $\| \tau^{-\alpha} \|_{L^{\frac{2}{\alpha q - \hat{r}}} (0,T;\mathbb{R})} < \infty$. The inequality (3.28) is proved by letting $C_3 = M_{21}$.

(b) It is clear that the assumptions of this part also satisfy part (a). Therefore, by part (a), it is necessary to prove $u \in C_{a0}([0,T];V_{-\beta q})$, i.e.,

$$
\lim_{t_2-t_1 \to 0} \|\sum_{i=1}^{n} C_i^i u_i(t_2,\cdot) - \sum_{i=1}^{n} C_i^i u_i(t_1,\cdot)\|_{V_{-\beta q}} = 0,
$$

where we note $0 < t_1 < t_2 \leq T$. After some simple computations we find that

$$
\sum_{i=1}^{n} C_i^i u_i(t_2,\cdot) = \sum_{i=1}^{n} C_i^i u_i(t_1,\cdot),
$$

where $\mathcal{J}_n = -\mathcal{L}^2 \mathcal{J}_n$, and $\mathcal{J}_n$ is defined by (3.15). Since $\mathcal{F}$ is in $C_{a0}([0,T];V_{-\beta q})$, we have just to prove $\| \mathcal{J}_n \|_{V_{-\beta q}}$ approaches 0 as $t_2 - t_1$ approaches 0. Let us first consider $\| \mathcal{J}_1 \|_{V_{-\beta q}}$. The inequalities (2.11) yields that

$$
\| \mathcal{J}_1 \|_{V_{-\beta q}} \leq \int_0^{t_1} \left\| \sum_{j=1}^{J_1} \mathcal{J}_1^\partial \left( \int_{t_1}^{t_1+\tau} \sum_{j=1}^{J_1} \frac{m_j^2 \beta^2}{m_j^{2/3} \beta^2} \mathcal{F}_j^2 (\tau) \mathcal{F}_j (\tau) \right) \right\|_{V_{-\beta q}} d\tau.
$$

$$
\leq \int_0^{t_1} \left\{ \sum_{j=1}^{J_1} \frac{m_j^2 \beta^2}{m_j^{2/3} \beta^2} \right\} \int_{t_1}^{t_1+\tau} \sum_{j=1}^{J_1} \frac{m_j^2 \beta^2}{m_j^{2/3} \beta^2} \mathcal{F}_j^2 (\tau) \left( \int_{t_1}^{t_1+\tau} \mathcal{F}_j^2 (\tau) \right)^{1/2} d\tau.
$$

Hence, we deduce from (3.35) that there exists a constant $M_{21} > 0$ satisfying

$$
\|\sum_{i=1}^{n} C_i^i u_i(t_2,\cdot) - \sum_{i=1}^{n} C_i^i u_i(t_1,\cdot)\|_{V_{-\beta q}} = 0,
$$

where we note $0 < t_1 < t_2 \leq T$. After some simple computations we find that

$$
\sum_{i=1}^{n} C_i^i u_i(t_2,\cdot) = \sum_{i=1}^{n} C_i^i u_i(t_1,\cdot),
$$

where $\mathcal{J}_n = -\mathcal{L}^2 \mathcal{J}_n$, and $\mathcal{J}_n$ is defined by (3.15). Since $\mathcal{F}$ is in $C_{a0}([0,T];V_{-\beta q})$, we have just to prove $\| \mathcal{J}_n \|_{V_{-\beta q}}$ approaches 0 as $t_2 - t_1$ approaches 0. Let us first consider $\| \mathcal{J}_1 \|_{V_{-\beta q}}$. The inequalities (2.11) yields that

$$
\| \mathcal{J}_1 \|_{V_{-\beta q}} \leq \int_0^{t_1} \left\| \sum_{j=1}^{J_1} \frac{m_j^2 \beta^2}{m_j^{2/3} \beta^2} \mathcal{F}_j^2 (\tau) \right\|_{V_{-\beta q}} d\tau.
$$

$$
\leq \int_0^{t_1} \left\{ \sum_{j=1}^{J_1} \frac{m_j^2 \beta^2}{m_j^{2/3} \beta^2} \right\} \int_{t_1}^{t_1+\tau} \sum_{j=1}^{J_1} \frac{m_j^2 \beta^2}{m_j^{2/3} \beta^2} \mathcal{F}_j^2 (\tau) \left( \int_{t_1}^{t_1+\tau} \mathcal{F}_j^2 (\tau) \right)^{1/2} d\tau.
$$

20
We recall that, by (3.30), $F$ belongs to $X_{2,\omega_q}(J \times D)$. Thus, we can deduce from the above inequality that

$$\|J_1\|_{\mathcal{V}_{-\alpha q}} \leq \tilde{c}_\alpha \int_0^{t_1} \left\| F(\cdot, \cdot) \right\|_{L^2} \omega^{\alpha q} \omega d\omega \leq \frac{\tilde{c}_\alpha}{1-\alpha q} \|F\|_{X_{2,\omega_q}(t_2-t_1)^{\alpha}}, \quad (3.40)$$

where we have used the same argument as in the estimate (3.16). Let us secondly consider $\|J_2\|_{\mathcal{V}_{-\alpha q}}$. We have

$$\|J_2\|_{\mathcal{V}_{-\alpha q}} \leq \int_{t_1}^{t_2} \left\| \mathcal{L}^{3} \sum_{j=1}^{\infty} F_j(\tau) E_{\alpha,\alpha}(-m_j^{\beta}(t_2-\tau)^{\alpha}) e_j \right\|_{\mathcal{V}_{-\alpha q}} \|t_2-\tau\|^{\alpha-1} d\tau$$

Further, we have

$$\leq \tilde{c}_\alpha \int_{t_1}^{t_2} \left\{ \sum_{j=1}^{\infty} m_j^{3 \beta} \omega^{2 \alpha q} m_j^{2 \beta} \omega^{2 \alpha q} (t_2-\tau)^{-2 \alpha q} \right\} \|t_2-\tau\|^{\alpha-1} d\tau$$

$$\leq \tilde{c}_\alpha \int_{t_1}^{t_2} \|F(\cdot, \cdot)\| \|t_2-\tau\|^{\alpha-1} \|t_2-\tau\|^{\alpha} d\tau \leq \tilde{c}_\alpha \|F\|_{X_{2,\omega_q}(t_2-t_1)^{\alpha}}, \quad (3.41)$$

where (2.11) has been used. Thirdly, we consider the norm $\|J_3\|_{\mathcal{V}_{-\alpha q}}$. It is clear that

$$J_3 = \mathcal{L}^{3} \mathcal{I}_3 = \mathcal{L}^{2} \sum_{j=1}^{\infty} g_j \int_{t_1}^{t_2} \frac{E_{\alpha,\alpha}(-m_j^{\beta} \omega^{\alpha})}{E_{\alpha,1}(-m_j^{\beta} T^\alpha)} \omega d\omega e_j.$$

Hence, we deduce

$$\|J_3\|_{\mathcal{V}_{-\alpha q}} = \left\{ \sum_{j=1}^{\infty} m_j^{2 \beta} \omega^{-2 \alpha q} \left\| \int_{t_1}^{t_2} \frac{E_{\alpha,\alpha}(-m_j^{\beta} \omega^{\alpha})}{E_{\alpha,1}(-m_j^{\beta} T^\alpha)} \omega d\omega \right\|_{\mathcal{V}_{-\alpha q}}^2 \right\}^{1/2}.$$

By applying (2.11), the absolute value of $\frac{E_{\alpha,\alpha}(-m_j^{\beta} \omega^{\alpha})}{E_{\alpha,1}(-m_j^{\beta} T^\alpha)}$ is bounded by $\tilde{c}_\alpha c_\alpha^{-1} \frac{1+\Omega}{1+(m_j^{\beta} \omega^{\alpha})}$. This is associated with $1 + m_j^{\beta} T^\alpha \leq (m_j^{\beta} + T^\alpha) m_j^{\beta}$ that $\frac{1+\Omega}{1+(m_j^{\beta} \omega^{\alpha})} \leq (m_j^{\beta} + T^\alpha) m_j^{\beta} \omega^{-2 \alpha q}$. Thus, we obtain

$$\|J_3\|_{\mathcal{V}_{-\alpha q}} \leq \tilde{c}_\alpha c_\alpha^{-1} \left( m_j^{\beta} + T^\alpha \right) \left\{ \sum_{j=1}^{\infty} m_j^{4 \beta} \omega^{-2 \alpha q} \int_{t_1}^{t_2} m_j^{\beta} \omega^{-2 \alpha q} \omega d\omega \right\}^{1/2}$$

$$\leq \frac{M_{23}}{c_\alpha} \|t_2-t_1\|^{2 \alpha q} \|\varphi\|_{\mathcal{V}_{\beta q}} (t_2-t_1), \quad (3.42)$$

where $M_{23} = \tilde{c}_\alpha c_\alpha^{-1} (m_1^{\beta} + T^\alpha)$. Finally, we can look at $\|J_4\|_{\mathcal{V}_{-\alpha q}}$ as follows:
where the fraction \( \frac{\tilde{E}_{\alpha,\beta}(m^2_\omega \omega^{\alpha})}{E_{\alpha,1}(m^2 T^\alpha)} \) can be estimated in the same way as in the proof of (3.42), and \( M_{24} = \tilde{c}_a M_{23} \). This leads to

\[
\| \mathcal{J}_4 \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} \leq \frac{M_{24}}{\alpha} \epsilon_1^{2\alpha} (t'_2 - t'_1) \int_0^T \| F(\tau, \cdot) \| (T - \tau)^{\alpha - 1} d\tau,
\]

which shows that

\[
\| \mathcal{J}_4 \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} \leq \frac{M_{24}}{\alpha} \epsilon_1^{2\alpha} \| \| F(\cdot, \cdot) \| \| \varphi \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} (t'_2 - t'_1).
\]

(4.34)

This implies (3.38) by taking (3.39)–(3.43) together. Thus, \( \mathcal{D}_u \) is contained in \( C((0, T], \mathcal{V}^{\infty}_{-\beta \lambda q}) \).

On the other hand, it is easy to see that the estimates (3.31)–(3.33) also hold for \( \tilde{q} = 0 \). Hence, we deduce from (3.35) and (3.36) that

\[
\epsilon_1 \| \mathcal{D}_u(t, \cdot) \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} \leq M_{20} \epsilon_1^\alpha \| F(t, \cdot) \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} + M_{20} \left( \| \varphi \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} + M_{20} \| \varphi \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} + M_{20} \| \varphi \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} \right) \int_0^T (T - \tau)^{\alpha - 1} d\tau.
\]

(4.44)

Now \( \mathcal{D}_u \in C_0^\infty((0, T]; \mathcal{V}^{\infty}_{-\beta \lambda q}) \). In addition, there exists a positive constant \( C > 0 \) such that

\[
\| \mathcal{D}_u \|_{C_0^\infty((0, T]; \mathcal{V}^{\infty}_{-\beta \lambda q})} \leq M_{20} \| F \|_{C_0^\infty((0, T]; \mathcal{V}^{\infty}_{-\beta \lambda q})} + C M_{20} \| \varphi \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} + C M_{20} \| \varphi \|_{\mathcal{V}^{\infty}_{-\beta \lambda q}} \int_0^T (T - \tau)^{\alpha - 1} d\tau.
\]

We can complete the proof by taking (3.28) and the above inequality together.

\[
\square
\]

### 4. FVP with a nonlinear source

In this section, we study the existence, uniqueness, and regularity of mild solutions of FVP (1.1)–(1.3) corresponding to the nonlinear source function \( F(t, x, u(t, x)) \). It is suitable considering assumptions that \( u(t, \cdot) \) and \( F(t, \cdot, u(t, \cdot)) \) belong to the same spatial space \( H \). In view of most considerations of PDEs, we let \( H = L^2(D) \).

We introduce the following assumptions on the numbers \( p, q, p', q', \tilde{p}, \tilde{q}, r, \tilde{r} \).

- (R1b) \( 0 < q < p < 1 \) such that \( p + q = 1 \);
- (R4b) \( 0 < p' < p \), \( q' = 1 - p' \), \( 0 < r \leq \frac{1 - q' d}{r q' d} \);
- (R4c) \( 0 < p' < p - q \), \( q' = 1 - p' \), \( 0 < r \leq \frac{1 - q' d}{r q' d} \);
- (R5b) \( 0 < \tilde{q} < q \), \( \tilde{p} = 1 - \tilde{q} \), \( 0 < \tilde{r} \leq \frac{1}{\alpha \tilde{q}} \).
In our work, we will assume on $F(t, \cdot, u(t, \cdot))$ the following assumptions

- (A1) $F(t, \cdot, 0) = 0$, and there exists a constant $K > 0$ such that, for all $v_1, v_2 \in L^2(D)$ and $t \in J$,
  $$
  \|F(t, \cdot, v_1) - F(t, \cdot, v_2)\| \leq K \|v_1 - v_2\|.
  $$

- (A2) $F(t, \cdot, 0) = 0$, and there exists a constant $K > 0$ such that, for all $v_1, v_2 \in L^2(D)$ and $t_1, t_2 \in J$,
  $$
  \|F(t_1, \cdot, v_1) - F(t_2, \cdot, v_2)\| \leq K_s (|t_1 - t_2| + \|v_1 - v_2\|).
  $$

Note that the assumption (A1), (A2) imply that, for $v \in L^2(D)$,
  $$
  \|F(t, \cdot, v)\| \leq K \|v\|. 
  \tag{4.1}
  $$

We try to develop the ideas of the linear FVP (3.1) to deal with the nonlinear FVP (1.1)–(1.3). In section 3, for the linear function $F(t, x)$ we assume that
  $$
  F \in \mathcal{A}_2(J \times D), \quad \text{or} \quad F \in \mathcal{A}_2(J \times D), \quad \text{or} \quad F \in L^{1+s-\gamma}(0, T; L^2(D)), \tag{4.2}
  $$
where $p, q, s, \gamma$ are defined by (R1), (R3), (R5). However, we cannot suppose that the nonlinear source function $F(t, x, u(t, x))$ satisfies the same assumptions as in (4.2), and then find the solution $u$. A natural idea might be to combine the idea of lemma 3.2 with the inequality (4.1), i.e., we predict the solution $u$ may be contained in the set
  $$
  W^0_{p,q}(J \times D) := \{w \in \mathcal{A}_2(J \times D) : \|w(t, \cdot)\| \leq \rho T^{-\gamma}, \quad \text{for } 0 < t \leq T \},
  $$
for $\rho > 0, 0 < \gamma \leq \eta < 1$.

The prediction will be proved in the next lemma. However, it is necessary to give some useful notes on $W^0_{p,q}(J \times D)$ as follows. For $w \in W^0_{p,q}(J \times D)$, we see
  $$
  \text{ess sup}_{0 \leq t \leq T} \int_0^t \|w(\tau, \cdot)\|(t - \tau)^{\gamma-1} d\tau \leq \rho \text{ ess sup}_{0 \leq t \leq T} \int_0^t \tau^{-\gamma}(t - \tau)^{\gamma-1} d\tau.
  $$

The function $\tau \mapsto \tau^{-\gamma}(t - \tau)^{\gamma-1}$ is integrable on $(0, t)$ since both numbers $-\gamma$ and $\eta - 1$ are greater than $-1$. In addition, we have $\int_0^t \tau^{-\gamma}(t - \tau)^{\gamma-1} d\tau = T^{-\gamma} B(\eta, 1 - \gamma)$, where $B(\cdot, \cdot)$ is the beta function see, for example, [58, 59, 60]. Hence, we have
  $$
  |||w|||_{\mathcal{A}_2} \leq \rho T^{-\gamma} B(\eta, 1 - \gamma). \tag{4.3}
  $$
Moreover, if $\gamma < \eta$, then there always exists a real number $p$ such that $1 < \frac{1}{\eta} < p < \frac{1}{\gamma}$. This implies that the function $r^{-\gamma}$ belongs to $L^p(0, T; L^2(D))$. Therefore, we can obtain the following inclusions
  $$
  W^0_{p,q}(J \times D) \subset L^p(0, T; L^2(D)) \subset \mathcal{A}_2(J \times D).
  $$

In the following lemma, we will consider the case $\gamma = \eta$, which we will denote by $W^0_{p}(J \times D) := W^0_{p,q}(J \times D)$.

Now, the Sobolev embedding $V_{bp} \hookrightarrow L^2(D)$ shows that there exists a positive constant $C_D$ depending on $D, \beta, q$ such that $\|v\| \leq C_D \|v\|_{V_{bp}}$ for all $v \in V_{bp}$. In this section, we let
  $$
  k_0(T) = KB(\alpha q, 1 - \alpha q)T^\alpha \left[\tilde{c}_q m_1^{-\beta p} + \tilde{c}_q^2 c_\alpha^{-1}(m_1)^\beta + T^{\alpha p}\right],
  $$
and

\[ M_0 = C_0 T^{\varphi} + \tilde{c}_\alpha c^{-1}_\alpha T^{\varphi}(m_1^{-\beta} + T^{\psi})^\rho . \]

**Lemma 4.1.** Let \( p, q \) be defined by (R1) and \( \phi \) be a function on \( D \). Let \( \{w_{(0)}\} \) be defined by \( w_{(0)} = \phi \),

\[ w_{(0)}(t, x) = (Ow_{(0-1)})(t, x) \]

\[ = (O_1 F(w_{(0-1)}))(t, x) + O_2(t) \phi(x) + (O_3 F(w_{(0-1)}))(t, x), \quad n \in \mathbb{N}, n \geq 1. \] (4.4)

If \( \phi \) belongs to \( V_{\beta p}, F \) satisfies (A1), and \( k_0(T) < 1 \), then

\[ \{w_{(0)}\}_{t \geq 0} \subset W_{\alpha q}^{\tilde{C}_0}(J \times D), \] (4.5)

where \( \tilde{C}_0 := \tilde{C}_0 ||| \phi |||_{V_{\beta q}} \) and \( \tilde{C}_0 = \frac{M_0}{1 - k_0(T)} \).

**Proof.** First, we have

\[ |||w_{(0)}||| = |||\phi||| \leq K C_D |||\phi|||_{V_{\beta p}} \leq M_0 |||\phi|||_{V_{\beta p}} T^{-\alpha q} \leq \tilde{C}_0 T^{-\alpha q}. \]

Hence, inequality (4.3) and \( \alpha q < 1 \), imply \( w_{(0)} \in W^{\tilde{C}_0}_{\alpha q}(J \times D) \). Now, we assume that \( w_{(n-1)} \) belongs to \( W^{\tilde{C}_0}_{\alpha q}(J \times D) \) for some \( n \geq 1 \). Then, by using (4.3), we have

\[ |||w_{(n-1)}|||_{X_{\alpha q}} \leq \tilde{C}_0 B(\alpha q, 1 - \alpha q). \] (4.6)

By induction, the inclusion (4.5) will be proved by showing that \( w_{(n)} \) belongs to \( W^{\tilde{C}_0}_{\alpha q}(J \times D) \). Indeed, by using the same arguments as in the proof of (3.6), we have

\[ \|(O_1 F(w_{(n-1)}))(t, \cdot)\| \leq \tilde{c}_\alpha m_1^{-\beta} \int_0^T |F(\tau, \cdot, w_{(n-1)}(\tau, \cdot))|(t - \tau)^{\alpha q - 1} d\tau, \] (4.7)

\[ \leq K \tilde{c}_\alpha m_1^{-\beta} |||w_{(n-1)}|||_{X_{\alpha q}} \leq M_2 \tilde{C}_0 T^{-\alpha q}, \]

where we have used (4.1), (4.6) and let \( M_{26} = K \tilde{c}_\alpha m_1^{-\beta} B(\alpha q, 1 - \alpha q)T^{\psi} \). On the other hand, the norm \( \|O_2(t)\phi\| \) is estimated by (3.7), i.e.,

\[ \|O_2(t)\phi\| \leq M_0 \|\phi\|_{V_{\beta p}} T^{-\alpha q}, \]

where we note that

\[ M_2 = \tilde{c}_\alpha c^{-1}_\alpha T^{\varphi}(m_1^{-\beta} + T^{\psi})^\rho \leq M_0. \]

The norm \( \|(O_3 F(w_{(n-1)}))(t, \cdot)\| \) can be estimated in the same way as in the proof of (3.8). That is,

\[ \|(O_3 F(w_{(n-1)}))(t, \cdot)\| \leq M_3 T^{-\alpha q} \int_0^T |F(\tau, \cdot, w_{(n-1)}(\tau, \cdot))|(T - \tau)^{\alpha q - 1} d\tau \] (4.8)

\[ \leq K M_3 T^{-\alpha q} |||w_{(n-1)}|||_{X_{\alpha q}} \leq M_{27} \tilde{C}_0 T^{-\alpha q}, \]

where (4.1), (4.6) have been used. Here, \( M_{27} = K M_3 B(\alpha q, 1 - \alpha q), \) \( M_3 = \tilde{c}_\alpha c^{-1}_\alpha T^{\varphi} (m_1^{-\beta} + T^{\psi})^\rho \).
We deduce from the above arguments and \( w_{(n)}(t, \cdot) = \mathcal{O}(t, \cdot)w_{(n-1)} \) that
\[
\|w_{(n)}(t, \cdot)\| \leq \|(\mathcal{O}_1 F(w_{(n-1)}))(t, \cdot)\| + \|\mathcal{O}_2(t)\phi\| + \|(\mathcal{O}_3 F(w_{(n-1)}))(t, \cdot)\|
\leq k_0(T)\hat{C}_0 t^{-\alpha q} + M_0\|\phi\|_{\mathcal{V}_{\beta q}} t^{-\alpha q},
\]
(4.9)
by noting that \( k_0(T) = M_{26} + M_{27} \). Since \( \hat{C}_0 = \frac{M_0}{1-k_0(T)}\|\phi\|_{\mathcal{V}_{\beta q}} \), the above inequality implies that \( \|w_{(n)}(t, \cdot)\| \leq \hat{C}_0 t^{-\alpha q} \). Therefore, from \( \alpha q < 1 \), we obtain the inclusion (4.5).

Next, it is necessary to give a definition of mild solutions of FVP (1.1)–(1.3).

**Definition 4.2.** Let \( F \) be defined by (A1) or (A2). If a function \( u \) belongs to \( L^p(0, T; L^p(D)) \), for some \( p, p' \geq 1 \), and satisfies equation (2.10) where \( F \) stands for the nonlinear source function \( F(t, u, u(t, x)) \), then \( u \) is said to be a mild solution of FVP (1.1)–(1.3).

The following theorems presents existence, uniqueness, and regularity of a mild solution of FVP (1.1)–(1.3).

**Theorem 4.3.**

(a) Let \( p, q, r, p', q' \) be defined by (R1), (R4b). If \( \varphi \) belongs to \( \mathcal{V}_{\beta q} \), \( F \) satisfies (A1), and \( k_0(T) < 1 \), then FVP (1.1)–(1.3) has a unique solution
\[
u \in L^\frac{1}{\alpha q - r}(0, T; \mathcal{V}_{\beta q - p'}) \cap \mathcal{C}_{\alpha q}((0, T]; L^2(D)),
\]
and there exists a positive constant \( \mathcal{C}_5 \) such that
\[
\|\nu\|_{L^\frac{1}{\alpha q - r}(0, T; \mathcal{V}_{\beta q - p'})} + \|\nu\|_{\mathcal{C}_{\alpha q}((0, T]; L^2(D))} \leq \mathcal{C}_5\|\varphi\|_{\mathcal{V}_{\beta q}}.
\]

(b) Let \( p, q, r, p', q' \) be defined by (R1b), (R4c). If \( \varphi \) belongs to \( \mathcal{V}_{\beta q} \), \( F \) satisfies (A1), and \( k_0(T) < 1 \), then FVP (1.1)–(1.3) has a unique solution
\[
u \in L^\frac{1}{\alpha q - r}(0, T; \mathcal{V}_{\beta q - p'}) \cap \mathcal{C}_{\alpha q}((0, T]; L^2(D)) \cap \mathcal{C}_{\alpha q}([0, T]; V_{-\beta q'}),
\]
and there exists a positive constant \( \mathcal{C}_6 \) such that
\[
\|\nu\|_{L^\frac{1}{\alpha q - r}(0, T; \mathcal{V}_{\beta q - p'})} + \|\nu\|_{\mathcal{C}_{\alpha q}((0, T]; L^2(D))} + \|\nu\|_{\mathcal{C}_{\alpha q}([0, T]; V_{-\beta q'})} \leq \mathcal{C}_6\|\varphi\|_{\mathcal{V}_{\beta q}}.
\]

**Proof.** (a) We divide the proof of this part into the following steps.

**Step 1:** We prove the existence and uniqueness of a mild solution. In order to prove the existence of a mild solution of FVP (1.1)–(1.3), we will construct a convergent sequence in \( L^\frac{1}{\alpha q - r}(0, T; L^2(D)) \) whose limit will be a mild solution of the problem. Here, \( r \) is defined by (R2). Let \( \{\nu(n)\}_{n \geq 0} \) be a sequence defined by lemma 4.1 with respect to \( \varphi = \varphi \in \mathcal{V}_{\beta q} \), then
\[
\{\nu(n)\}_{n \geq 0} \subset W^{\alpha q}_{\alpha q}(J \times D)
\]
where \( \hat{C}_0 = \frac{M_0}{1-k_0(T)}\|\varphi\|_{\mathcal{V}_{\beta q}} \). Therefore,
\[
\|\nu(n)(t, \cdot)\| \leq \hat{C}_0 t^{-\alpha q}, \quad 0 < t \leq T,
\]
for all \( n \geq 1 \). This together with \( t^{-\alpha q} \) belonging to \( L^\frac{1}{\alpha q - r}(0, T; \mathcal{R}) \) implies that \( \{\nu(n)\}_{n \geq 0} \) is a bounded sequence in \( L^{\frac{1}{\alpha q - r}}(0, T; L^2(D)) \). Now, we will show that \( \{\nu(n)\}_{n \geq 0} \) is convergent by
proving that it is also a Cauchy sequence. For fixed \( n \geq 1 \) and \( k \geq 1 \), the definition (4.4) of \( \{w_{n(\alpha)}\}_{n \geq 0} \) yields that
\[
w_{(n+k)}(t, x) - w_{(n)} = O_1 \left[ F(w_{(n-1+k)}) - F(w_{(n-1)}) \right] + O_3 \left[ F(w_{(n-1+k)}) - F(w_{(n-1)}) \right].
\]

Since \( F \) satisfies lemma 4.1, the latter equation shows that we can apply the same arguments as in lemma 4.1 with \( \phi = 0 \). Hence, one can deduce
\[
\begin{align*}
\|w_{(n+k)}(t, \cdot) - w_{(n)}(t, \cdot)\| & \leq \|O_1 \left[ F(w_{(n-1+k)}) - F(w_{(n-1)}) \right](t, \cdot)\| + \|O_3 \left[ F(w_{(n-1+k)}) - F(w_{(n-1)}) \right](t, \cdot)\| \\
& \leq \tilde{c}_n m^\beta K \int_0^T \|F(t, \cdot) - w_{(n)}(t, \cdot)\| (t - \tau)^{\alpha q - 1} d\tau \\
& \quad + M_3 K T^{-\alpha q} \int_0^T \|F(t, \cdot) - w_{(n)}(t, \cdot)\| (T - \tau)^{\alpha q - 1} d\tau \\
& \leq \tilde{c}_n m^\beta K (2\tilde{C}_0) \int_0^T \tau^{-\alpha q} (t - \tau)^{\alpha q - 1} d\tau + M_3 K (2\tilde{C}_0) T^{-\alpha q} \int_0^T \tau^{-\alpha q} (T - \tau)^{\alpha q - 1} d\tau \\
& \leq \left(\tilde{c}_n m^\beta K T^{-\alpha q} B(\alpha q, 1 - \alpha q) + M_3 K B(\alpha q, 1 - \alpha q)\right) (2\tilde{C}_0) T^{-\alpha q}.
\end{align*}
\]
From the definition of \( k_0(T) \), we conclude that
\[
\|w_{(n+k)}(t, \cdot) - w_{(n)}(t, \cdot)\| \leq k_0(T) (2\tilde{C}_0) T^{-\alpha q}.
\]
Iterating this method \( n \)-times shows
\[
\|w_{(n+k)}(t, \cdot) - w_{(n)}(t, \cdot)\| \leq k_0^n(T) (2\tilde{C}_0) T^{-\alpha q}.
\]
Taking the \( L^{\alpha q}(0, T; \mathbb{R}) \)-norm of both sides of the above inequality directly implies
\[
\|w_{(n+k)} - w_{(n)}\|_{L^{\alpha q}(0, T; L^2(D))} \leq k_0^n(T) (2\tilde{C}_0) T^{-\alpha q} \|
\]
(4.10).

Here, we emphasise that the constants in (4.10) also do not depend on \( (n, k) \). Therefore, by letting \( n \) go to infinity, we obtain
\[
\lim_{n, k \to \infty} \|w_{(n+k)} - w_{(n)}\|_{L^{\alpha q}(0, T; L^2(D))} = 0,
\]

i.e., \( \{ u_n \}_{n \geq 0} \) is a bounded Cauchy sequence in \( L^{\frac{1}{\beta}}(0, T; L^2(D)) \). Hence, there exists a function \( u \) in \( L^{\frac{1}{\beta}}-r(0, T; L^2(D)) \) such that

\[
u = \lim_{n \to \infty} u_n, \quad \text{in} \quad L^{\frac{1}{\beta}}-r(0, T; L^2(D)),
\]

and \( u \) satisfies equation (2.10), i.e., \( u \) is a mild solution of FVP problem (1.1)–(1.3). Moreover, the boundedness (4.9) of \( \{ u_n \}_{n \geq 0} \) gives

\[
\| u(t, \cdot) \| \leq \tilde{C}_0 \| \varphi \|_{V_{\beta q}},
\]

and so that

\[
\| u \|_{L^{\frac{1}{\beta}}-r(0, T; L^2(D))} \leq M_{28} \| \varphi \|_{V_{\beta q}}
\]

where \( M_{28} = \tilde{C}_0 \| \cdot \|_{L^{\frac{1}{\beta}}-r(0, T; \mathbb{R})} \). Now, we show the uniqueness of the solution \( u \). Assume that \( \tilde{u} \) is another solution of FVP (1.1)–(1.3). Then, by applying the same arguments as in (4.10), we also have

\[
\| u - \tilde{u} \|_{L^{\frac{1}{\beta}}-r(0, T; L^2(D))} \leq k'_0(T) \| \cdot \|_{L^{\frac{1}{\beta}}-r(0, T; \mathbb{R})}
\]

for all \( n \in \mathbb{N}, n \geq 1 \). Thus \( \| u - \tilde{u} \|_{L^{\frac{1}{\beta}}-r(0, T; L^2(D))} = 0 \) by letting \( n \) go to infinity. Hence, \( u = \tilde{u} \in L^{\frac{1}{\beta}}-r(0, T; L^2(D)) \).

**Step 2:** We prove that \( u \in L^{\frac{1}{\beta}}-r(0, T; V_{\beta q}) \). This will be proved by using the inequality (4.11). We now apply the same arguments as in the proofs of (3.11), and (3.13) to estimate \( \| u(t, \cdot) \|_{V_{\beta q}} \) as follows. First, we have

\[
\| (C_2F(u))(t, \cdot) \|_{V_{\beta q}} \leq \tilde{C}_0 \int_0^T \left\{ \sum_{j=1}^{\infty} F_j^2(\tau, u(\tau)) m_j^{2b_1(\beta q - \rho')} m_j^{-2b_1(\beta q - \rho')} (t - \tau)^{2q - 2} (t - \tau)^{2w - 2} \right\}^{1/2} \, d\tau
\]

\[
\leq \tilde{c}_m m_1^{-\beta q} K \int_0^T \| F(\tau, u(\tau), \cdot) \| (t - \tau)^{q - 1} \, d\tau
\]

\[
\leq \tilde{c}_m m_1^{-\beta q} K \tilde{C}_0 \int_0^T \| u(\tau, \cdot) \| (t - \tau)^{q - 1} \, d\tau
\]

\[
\leq \tilde{c}_m m_1^{-\beta q} K \tilde{C}_0 \int_0^T \tau^{-\alpha q} (t - \tau)^{q - 1} \, d\tau \leq M_{29} \| \varphi \|_{V_{\beta q}} \tau^{-\alpha q'},
\]

where we let

\[
M_{29} = \tilde{c}_m m_1^{-\beta q} K \tilde{C}_0 B(\alpha q, 1 - \alpha q) T^{\alpha q'}.
\]
Secondly,
\[
\begin{align*}
\|(\partial_y^2 F(u))(t, \cdot)\|_{V^s(\partial_y^p - p')} & \leq M_6 t^{-\alpha q} \int_0^T \|F(\tau, \cdot, u(\tau, \cdot))\|(T - \tau)^{\alpha q - 1} d\tau \\
& \leq M_6 t^{-\alpha q} K \int_0^T \|u(\tau, \cdot)\|(T - \tau)^{\alpha q - 1} d\tau \\
& \leq M_6 t^{-\alpha q} K \tilde{C}_0 \int_0^t \tau^{-\alpha q}(t - \tau)^{\alpha q - 1} d\tau \\
& \leq M_{30} \|\varphi\|_{V^s(\partial_y^p - \alpha q)},
\end{align*}
\]
where we let \(M_{30} = M_6 K \tilde{C}_0 B(\alpha q, 1 - \alpha q).\) We recall that \(\|\partial_y^2(t)\varphi\|_{V^s(\partial_y^p - p')}\) have been estimated by (3.12). According to the above arguments, we arrive at the estimate
\[
\begin{align*}
\|u(t, \cdot)\|_{V^s(\partial_y^p - p')} & \leq \|(\partial_y^2 F(u))(t, \cdot)\|_{V^s(\partial_y^p - p')} + \|\partial_y^2(t)\varphi\|_{V^s(\partial_y^p - p')} + \|(\partial_y^2 F(u))(t, \cdot)\|_{V^s(\partial_y^p - p')} \\
& \leq M_{31} \|\varphi\|_{V^s(\partial_y^p - \alpha q)},
\end{align*}
\]
for \(M_{31} = M_{29} + M_5 + M_{30}.\) By taking the \(L^{\infty, \alpha q}((0, T) ; \mathbb{R})\)-norm, then the latter inequalities can be transformed into the following estimate
\[
\|u\|_{L^{\infty, \alpha q}((0, T) ; V^s(\partial_y^p - p'))} \leq M_{32} \|\varphi\|_{V^s(\partial_y^p - \alpha q)},
\]
where \(M_{32} = M_{31} \|\varphi\|_{L^{\infty, \alpha q}((0, T) ; \mathbb{R})}.
\]

**Step 3:** We prove that \(u \in C^0_{\alpha}(0, T) ; L^2(D))\). Let us consider \(0 < t_1 < t_2 \leq T.\) By the same arguments as in (3.15), we have
\[
\begin{align*}
\|u(t_2, x) - u(t_1, x)\| &= \sum_{j=1}^\infty \int_{t_1}^{t_2} F_j(\tau, u(\tau))\omega^\alpha E_{\alpha, \alpha - 1}(-m_j^2 \omega^2) d\omega d\tau e_j(x) \\
& \quad + \sum_{j=1}^\infty \int_{t_1}^{t_2} F_j(\tau, u(\tau))\tilde{E}_{\alpha, \alpha}(-m_j^2 (t_2 - \tau)^\alpha) d\tau e_j(x) \\
& \quad - L^3 \sum_{j=1}^\infty \int_{t_1}^{t_2} \tilde{E}_{\alpha, \alpha}(-m_j^2 \omega^2) E_{\alpha, \alpha}(\omega^2, 0) d\omega e_j(x) \\
& \quad + L^3 \sum_{j=1}^\infty \int_{t_1}^{t_2} F_j(\tau, u(\tau))\tilde{E}_{\alpha, \alpha}(-m_j^2 (T - \tau)^\alpha) E_{\alpha, \alpha}(\omega^2, T^\alpha) d\omega d\tau e_j(x) \\
& := I_1 + I_2 + I_3 + I_4.
\end{align*}
\]
Here, by (3.19), \(\|I_3\|\) tends to 0 as \(t_2 - t_1\) tends to 0. In what follows, we will establish the convergence for \(\|I_n\|, n = 1, 2, 4\) which can be treated similarly as in (3.16), (3.17), (3.20) based on the Lipschitzian assumption (A1). We first see that
Thus, due to the substitution \( \tau = t_2 \mu \), we have

\[
\int_1^{t_2} \tau^{\alpha q - 1} \left[ (t_1 - \tau)^{\alpha q - 1} - (t_2 - \tau)^{\alpha q - 1} \right] \, d\tau = B(\alpha q, 1 - \alpha q) - B(\alpha q, 1 - \alpha q) - \int_1^{t_2} \mu^{\alpha q - 1} \, d\mu 
\]

As a consequence, \( \lim_{t_2 \to 1} \int_1^{t_2} \tau^{\alpha q - 1} \left[ (t_1 - \tau)^{\alpha q - 1} - (t_2 - \tau)^{\alpha q - 1} \right] \, d\tau = 0 \), and so \( \| \mathcal{I}_1^N \| = 0 \). Secondly we proceed to deal with \( \mathcal{I}_2^N \). Now \( 0 < p_0 < p \), by (2.11), we have

\[
|E_{\alpha,o}(m^j(t_2 - \tau)^\alpha)| \leq \tilde{c}_0 m^{1 - \beta(p - p_0)}(t_2 - t)^{-\alpha(p - p_0)}. \tag{4.18}
\]

We deduce the following chain of estimates

\[
\| \mathcal{I}_2^N \| \leq \int_1^{t_2} \left\| \sum_{j=0}^{\infty} F_j(\tau, u(\tau))E_{\alpha,o}(m^j(t_2 - \tau)^\alpha) \right\| (t_2 - \tau)^{\alpha q - 1} \, d\tau 
\]

\[
\leq \tilde{c}_0 m^{1 - \beta(p - p_0)} \int_1^{t_2} \left\| F(\tau, u(\tau)) \right\| (t_2 - \tau)^{\alpha q - 1} \, d\tau 
\]

\[
\leq \tilde{c}_0 m^{1 - \beta(p - p_0)} \int_1^{t_2} \left\| u(\tau) \right\| (t_2 - \tau)^{\alpha q - 1} \, d\tau 
\]

\[
\leq \tilde{c}_0 m^{1 - \beta(p - p_0)} \tilde{K}_0 \| \varphi \| \| \mathcal{V} \|_p \int_0^{t_2} \tau^{\alpha q - 1} (t_2 - \tau)^{\alpha q - 1} \, d\tau 
\]

\[
\leq \tilde{c}_0 m^{1 - \beta(p - p_0)} \tilde{K}_0 \| \varphi \| \| \mathcal{V} \|_p B(\alpha q, 1 - \alpha q)(t_2 - t)^{\alpha(p - p_0)}. \tag{4.19}
\]
This implies $\lim_{t_2 \to 0} \| I_N^N \| = 0$. Next, we thirdly proceed to consider $I_N^N$. The same argument as in (3.20) gives

$$
\| I_N^N \| \leq \tilde{c}_0 M_{11} \frac{t_2^q - t_1^q}{t_1^q} \int_0^T \| F(\tau, u(\tau, \cdot))\| (T - \tau)^{\alpha q - 1} d\tau \\
\leq \tilde{c}_0 M_{11} \tilde{C}_0 \| \varphi \| \frac{t_2^q - t_1^q}{t_1^q} \int_0^T \tau^{- \alpha q} (T - \tau)^{\alpha q - 1} d\tau \\
\leq \tilde{c}_0 M_{11} \tilde{C}_0 (1 - \alpha q) \| \varphi \| \frac{t_2^q - t_1^q}{t_1^q},
$$

(4.20)

and we arrive at $\lim_{t_2 \to 0} \| I_N^N \| = 0$. The above arguments prove $u \in C((0, T] : L^2(D))$. This combines with (4.11) so that $u \in C^v_0((0, T] : V_{-\beta q})$. In this part, we consider $0 \leq t_1 < t_2 \leq T$. Let us first show

$$
\| I_N^N \|_{L^{\infty}(0, T] : L^2(D)} \leq \tilde{C}_0 \| \varphi \| v_{t_1} (t_2 - t_1)^\alpha q,
$$

(4.21)

for some positive constant $M_{33}$, where the case $t_1 = 0$ is trivial. It is necessary to prove (4.22) for $t_1 > 0$. From the proof of the estimate (3.23), we have

$$
\| I_N^N \|_{v_{-\beta q}} \leq \int_0^{t_1} \left\{ \sum_{j=1}^{\infty} m_j \frac{2^{2j/3} F^2_j(\tau, u(\tau))}{(T - \tau)^{2}} \int_{T - \tau}^{\tau} \omega^{-2} E_{\alpha q - 2}(m_j^\beta \omega) d\omega \right\}^{1/2} d\tau.
$$

In addition, the inequalities (2.11) yield that $|E_{\alpha q - 2}(m_j^\beta \omega)| \leq \tilde{c}_0 m_j^{\beta q} \omega^{-\alpha q'}$. This associates with $\alpha q' = 2 = \alpha q - 2 + \alpha (p - p')$ so that

$$
\| I_N^N \|_{v_{-\beta q}} \leq \tilde{c}_0 m_j^{\beta q} \int_0^{t_1} \| F(\tau, u(\tau, \cdot))\| \int_{T - \tau}^{\tau} \omega^{\alpha q - 2 + \alpha (p - p')} d\omega d\tau \\
\leq \tilde{c}_0 m_j^{\beta q} \tilde{C}_0 \left( t_2 - t_1 \right)^{\alpha q - 2 + \alpha (p - p')} d\tau \\
\leq \tilde{c}_0 m_j^{\beta q} \tilde{C}_0 \left( t_2 - t_1 \right)^{\alpha q - 2 + \alpha (p - p')} \left( \frac{t_2}{t_1} - 1 \right)^{\alpha q} \\
\leq M_{33} \| \varphi \| v_{t_1} (t_2 - t_1)^\alpha q,
$$

where we let

$$
M_{33} = \frac{\tilde{c}_0 m_j^{\beta q}}{(1 - \alpha q) \alpha q} \tilde{C}_0 T^{\alpha (p - p') - \alpha q}.
$$
Proof. Taking the above estimates for solution $u$ satisfying that Theorem 4.4. Here, we have used (4.19) with respect to $0 < p_0 = p - p' < p$. Thirdly, we now consider $I_N^2$. By applying the same arguments as in (3.25), one can get

$$
\|I_N^2\|_{V_{-\beta q'}} \leq \frac{M_{14}}{\alpha(p - p')} \|\varphi\|_{V_{q'}} \left( t_2^{\alpha(p-p')} - t_1^{\alpha(p-p')} \right) \leq \frac{M_{14}}{\alpha(p - p')} T^{\alpha(p-p') - \alpha q} \|\varphi\|_{V_{q'}} (t_2 - t_1)^{\alpha q}. \quad (4.23)
$$

Finally, the arguments proving (3.26) give

$$
\|I_4\|_{V_{-\beta q'}} \leq \tilde{c}_\alpha \frac{M_{14}}{\alpha(p - p')} \left( \int_{t_2}^{t_1} (T - \tau)^{\alpha q - 1} d\tau \right) \|F(\tau, \cdot, u(\tau, \cdot))\|_{L^2(\tau)} \leq \tilde{c}_\alpha \frac{M_{14}}{\alpha(p - p')} (t_2 - t_1)^{\alpha q} K \int_0^T \|u(\tau, \cdot)\|_{L^2(\tau)} d\tau \leq \tilde{c}_\alpha \frac{M_{14}}{\alpha(p - p')} T^{\alpha(p-p') - \alpha q} \|\varphi\|_{V_{q'}} (t_2 - t_1)^{\alpha q}.
$$

Taking the above estimates for $I_N^2$, $1 < n \leq 4$ together, we conclude that $u$ belongs to $C^{\alpha q}([0, T]; V_{-\beta q'})$. Moreover, there exists a positive constant $M_3$ such that

$$
\|u\|_{C^{\alpha q}([0, T]; V_{-\beta q'})} \leq M_3 \|\varphi\|_{V_{q'}}.
$$

By combining this inequality with (4.14), (4.21), we complete the proof. \hfill \Box

**Theorem 4.4.** Let $p, q, p', q', \tilde{p}, \tilde{q}, r, \tilde{r}$ be defined by (R1), (R4b), (R5b). If $\varphi$ belongs to $V_{\beta p + \tilde{q}'}$, $F$ satisfies the assumptions (A2), and $k_0(T) < 1$, then FVP (1.1)–(1.3) has a unique solution $u$ satisfying that

$$
u \in L^\infty_\infty((0, T]; V_{-\beta q'}). \cap C^\alpha_w((0, T]; L^2(D)), \quad \tilde{D}_r u \in L^\infty_\infty((0, T]; V_{-\beta q'}) \cap C^\alpha_w((0, T]; L^2(D)).
$$

Moreover, there exists a constant $C_7 > 0$ such that

$$
\|\tilde{D}_r u\|_{L^\infty_\infty((0, T]; V_{-\beta q'})} + \|\tilde{D}_r^t u(t, \cdot)\|_{C^\alpha_w((0, T]; V_{-\beta q'})} \leq C_7 \|\varphi\|_{V_{\beta p + \tilde{q}'}}. \quad (4.24)
$$

**Proof.** Since $F$ satisfies (A2), $F$ also satisfies (A1) with respect to the Lipschitz constant $K$. In addition, the Sobolev imbedding $V_{\beta p + \tilde{q}'} \hookrightarrow V_{\tilde{q}'}$ shows that $\varphi$ belongs to $V_{\tilde{q}'}$. Hence, by theorem 4.3, FVP (1.1)–(1.3) has a unique solution

$$
u \in L^\infty_\infty((0, T]; V_{-\beta q'}) \cap C^\alpha_w((0, T]; L^2(D)).
$$

Moreover, the inequality (4.11) also holds. We deduce that, for $0 < t \leq T$,

$$
\|F(t, \cdot; u(t, \cdot))\| \leq K_4 \|\tilde{C}_0\| \|\varphi\|_{V_{\tilde{q}'}} T^{-\alpha q} \leq M_{34} K_4 \|\tilde{C}_0\| \|\varphi\|_{V_{\tilde{q}'} + \beta p} T^{-\alpha q}. \quad (4.25)
$$
The remainder of this proof falls naturally into two steps as follows.

**Step 1:** We prove \( D_j^\gamma u \) finitely exists and belongs to \( L^{2+\gamma}(0, T; \mathbb{V}_{\beta(\alpha, \gamma)}) \). By the same way as in part (a) of theorem 3.4, we have

\[
D_j^\gamma u(t) = F_j(t, u(t)) - m_j^\beta F_j(t, u(t)) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta \tau) - \tilde{\varphi} E_{\alpha, 1}(-m_j^\alpha T) + \left( F_j(t, u(r)) \ast \tilde{E}_{\alpha, \alpha}(-m_j^\beta \tau) \right) - m_j^\beta E_{\alpha, 1}(-m_j^\beta T) =: F_j(t, u(t)) + \psi_t^N_1(t) + \psi_t^N_2(t) + \psi_t^N_3(t),
\]

for all \( j \in \mathbb{N}, j \geq 1 \). In view of (4.25), \( F(t, \cdot, u(t, \cdot)) \) is contained in \( L^2(D) \) for \( 0 < t \leq T \). This associates with the Sobolev embedding \( L^2(D) \hookrightarrow \mathbb{V}_{\beta(\alpha, \gamma)} \) that \( F(t, \cdot, u(t, \cdot)) \) is contained in \( \mathbb{V}_{\beta(q, \gamma)} \), namely \( \sum_{j=1}^{\infty} F_j(t, u(t))e_j \) is contained in \( \mathbb{V}_{\beta(q, \gamma)} \). On the other hand, \( \psi_t^N_i = \psi_t^N_1 \), and the norm \( \| \sum_{j=1}^{\infty} \psi_t^N_i(t)e_j \|_{\mathbb{V}_{\beta(q, \gamma)}} \) exists finitely by (3.32). Now, we consider \( \| \sum_{j=1}^{\infty} \psi_t^N_i(t)e_j \|_{\mathbb{V}_{\beta(q, \gamma)}} \), \( n = 1, 3 \). According to the estimates (3.31) and (3.33), the following ones hold:

\[
\begin{align*}
\left\| \sum_{n_1 \leq j \leq n_2} \psi_t^N_1(t)e_j \right\|_{\mathbb{V}_{\beta(q, \gamma)}} & \leq \tilde{C}_\alpha T^{\alpha-t} \int_0^t (t - \tau)^{\alpha(q - \gamma) - 1} \left\{ \sum_{n_1 \leq j \leq n_2} F_j^2(t, u(\tau)) \right\}^{1/2} d\tau, \\
\left\| \sum_{n_1 \leq j \leq n_2} \psi_t^N_3(t)e_j \right\|_{\mathbb{V}_{\beta(q, \gamma)}} & \leq \tilde{C}_\alpha M_{10} t^{\alpha-t} \int_0^t (T - \tau)^{\alpha(q - \gamma) - 1} \left\{ \sum_{n_1 \leq j \leq n_2} F_j^2(t, u(\tau)) \right\}^{1/2} d\tau.
\end{align*}
\]

For \( 0 < \tau < T \), we have \( F(\tau, \cdot, u(\tau, \cdot)) \) belonging to \( L^2(D) \). This follows that the sequence \( \{ G_n(\tau) \} \), which is defined by \( G_n(\tau) = \left\{ \sum_{j=1}^{\infty} F_j^2(t, u(\tau)) \right\}^{1/2} \), converges pointwise to 0 as \( n \) goes to infinity. Moreover, by (4.25), we have

\[
(t - \tau)^{\alpha(q - \gamma) - 1} G_n(\tau) \leq M_{36} K_0 \| \varphi \|_{\mathbb{V}_{\beta(p + q)}} (t - \tau)^{\alpha(q - \gamma) - 1} t^{-\alpha q} \| \psi \|_{\mathbb{V}_{\beta(q, \gamma)}}.
\]

The function \( \tau \rightarrow (t - \tau)^{\alpha(q - \gamma) - 1} t^{-\alpha q} \) is integrable on the open interval \( (0, t) \), \( t > 0 \), since

\[
\int_0^t (t - \tau)^{\alpha(q - \gamma) - 1} t^{-\alpha q} d\tau = t^{1 - \alpha q} B(\alpha(q - \gamma), 1 - \alpha q).
\]

Therefore, the dominated convergence theorem yields that

\[
\lim_{n \to \infty} \int_0^t (t - \tau)^{\alpha(q - \gamma) - 1} G_n(\tau) d\tau = 0.
\]

This together with \( \left\{ \sum_{n_1 \leq j \leq n_2} F_j^2(\tau, u(\tau)) \right\}^{1/2} \leq G_n(\tau) \) gives

\[
\lim_{n_1, n_2 \to \infty} \int_0^t (t - \tau)^{\alpha(q - \gamma) - 1} \left\{ \sum_{n_1 \leq j \leq n_2} F_j^2(\tau, u(\tau)) \right\}^{1/2} d\tau = 0.
\]

Similarly, we also have

\[
\lim_{n_1, n_2 \to \infty} \int_0^T (T - \tau)^{\alpha(q - \gamma) - 1} \left\{ \sum_{n_1 \leq j \leq n_2} F_j^2(\tau, u(\tau)) \right\}^{1/2} d\tau = 0.
\]
We deduce \( \| \sum_{j=1}^{\infty} \psi_j(t) e_j \|_{V^{\beta} - \bar{q}} \), \( n = 1, 3 \) exist finitely. Taking all the above arguments together, we conclude that \( \| \sum_{j=1}^{\infty} D^j u(t)e_j \|_{V^{\beta} - \bar{q}} \) finitely exists. In addition, the Sobolev embedding \( L^2(D) \hookrightarrow V^{\beta - \bar{q}} \) yields that there exists a positive constant \( M_{37} \) such that

\[
\| F(t, \cdot, u(t, \cdot)) \|_{V^{\beta - \bar{q}}} \leq M_{37} \| F(t, \cdot, u(t, \cdot)) \|.
\]

Hence,

\[
\| D^j u(t, \cdot) \|_{V^{\beta - \bar{q}}} \leq \| F(t, \cdot, u(t, \cdot)) \|_{V^{\beta - \bar{q}}} + \sum_{1 \leq p \leq 3} \| \sum_{j=1}^{\infty} \psi_j(t) e_j \|_{V^{\beta - \bar{q}}}
\]

\[
\leq M_{37} \| F(t, \cdot, u(t, \cdot)) \| + \tilde{C}_\alpha \int_0^t (t - \tau)^{\alpha(q - \bar{q}) - 1} \| F(\tau, \cdot, u(\tau, \cdot)) \| d\tau
\]

\[
+ M_{38} t^{\alpha - \alpha q} \| \varphi \|_{V^{\beta + \bar{q}}} + \tilde{C}_\alpha M_{38} t^{\alpha - \alpha q} \int_0^T (T - \tau)^{\alpha(q - \bar{q}) - 1} \| F(\tau, \cdot, u(\tau, \cdot)) \| d\tau.
\]

We now note that

\[
\int_0^t (t - \tau)^{\alpha(q - \bar{q}) - 1} \tau^{-\alpha q} d\tau \leq T^{\alpha - \alpha q} B(\alpha(q - \bar{q}), 1 - \alpha q) t^{-\alpha q},
\]

and

\[
\int_0^T (T - \tau)^{\alpha(q - \bar{q}) - 1} \tau^{-\alpha q} d\tau = T^{\alpha - \alpha q} B(\alpha(q - \bar{q}), 1 - \alpha q).
\]

This combines with (4.25) and there exists a constant \( M_{38} > 0 \) such that

\[
\| D^j u(t, \cdot) \|_{V^{\beta - \bar{q}}} \leq M_{38} \| \varphi \|_{V^{\beta + \bar{q}}} t^{\alpha - \alpha q},
\]

(4.26)

which leads to

\[
\| D^j u \|_{L^{\infty}(0, T; V^{\beta - \bar{q}})} \leq M_{38} \| \tau^{-\alpha q} \|_{L^{\infty}(0, T; \mathbb{R}^d)} \| \varphi \|_{V^{\beta + \bar{q}}}.
\]

(4.27)

**Step 2:** We prove \( D^j u \in C^0_0((0, T) ; V^{\beta - \bar{q}}) \). We consider \( 0 < t_1 < t_2 \leq T \). A similar argument as in (3.39) yields

\[
D^j u(t_2, x) - D^j u(t_1, x) = F(t_2, x, u(t_2, x)) - F(t_1, x, u(t_1, x)) + \sum_{1 \leq p \leq 4} J^N_n,
\]

where \( J^N_n = -L^\beta T^N_n \) and \( T^N_n \) is defined by (4.15). By applying the Sobolev embedding \( L^2(D) \hookrightarrow V^{\beta - \bar{q}} \), there exists a positive constant \( M_{39} \) such that

\[
\lim_{t_2 \to t_1^-} \| F(t_2, \cdot, u(t_2, \cdot) - F(t_1, \cdot, u(t_1, \cdot)) \|_{V^{\beta - \bar{q}}}
\]

\[
\leq \lim_{t_2 \to t_1^-} M_{39} \| F(t_2, \cdot, u(t_2, \cdot)) - F(t_1, \cdot, u(t_1, \cdot)) \|
\]

\[
\leq \lim_{t_2 \to t_1^-} M_{39} K_n \left( \| t_2 - t_1 \| + \| u(t_2, \cdot) - u(t_1, \cdot) \| \right) = 0.
\]
where we note that $u \in C^α_w((0, T); L^2(D))$. From (3.40) and (4.17), we have

$$
\|\mathcal{J}_1^N\|_{V_{\alpha \beta q}} \leq \tilde{c}_0 \int_0^{t_1} \|F(\tau, \cdot, u(\tau, \cdot))\| \int_{t_1 - \tau}^{t_1 - \tau} \omega^{\alpha q - 2} d\omega \ d\tau \\
\leq \frac{\tilde{c}_0 M_3 K_c \tilde{C}_0}{1 - \alpha q} \|\varphi\|_{V_{\beta p + \tilde{q}}} \int_0^{t_1} \tau^{-\alpha q} [(t_1 - \tau)^{\alpha q - 1} - (t_2 - \tau)^{\alpha q - 1}] d\tau \\
\leq \frac{\tilde{c}_0 M_3 K_c \tilde{C}_0}{(1 - \alpha q)^{\alpha q} t_1} \|\varphi\|_{V_{\beta p + \tilde{q}}} \left(\frac{t_2}{t_1} - 1\right)^{\alpha q}.
$$

In addition, by

$$
\int_{t_1}^{t_2} \tau^{-\alpha q} (t_2 - \tau)^{\alpha q - 1} \ d\tau = \int_{t_1/t_2}^1 \mu^{-\alpha q} (1 - \mu)^{\alpha q - 1} \ d\mu \leq \frac{1}{\alpha q} \left(\frac{t_2}{t_1} - 1\right)^{\alpha q}
$$

and (4.25), we can obtain the following chain of the inequalities

$$
\|\mathcal{J}_2^N\|_{V_{\alpha \beta q}} \leq \tilde{c}_0 \int_0^{t_2} \|F(\tau, \cdot, u(\tau, \cdot))\| (t_2 - \tau)^{\alpha q - 1} d\tau \\
\leq \frac{\tilde{c}_0 M_3 K_c \tilde{C}_0}{1 - \alpha q} \|\varphi\|_{V_{\beta p + \tilde{q}}} \int_0^{t_2} \tau^{-\alpha q} (t_2 - \tau)^{\alpha q - 1} d\tau \\
\leq \frac{\tilde{c}_0 M_3 K_c \tilde{C}_0}{1 - \alpha q} \|\varphi\|_{V_{\beta p + \tilde{q}}} \left(\frac{1}{\alpha q} \int_{t_2}^{t_1} \tau^{-\alpha q} (t_2 - t_1)^{\alpha q}\right).
$$

Finally, the norm $\|\mathcal{J}_3^N\|_{V_{\alpha \beta q}}$ has been estimated by (3.42), and the norm $\|\mathcal{J}_4^N\|_{V_{\alpha \beta q}}$ can be estimated as follows:

$$
\|\mathcal{J}_3^N\|_{V_{\alpha \beta q}} \leq \frac{M_3}{\alpha} \int_0^{t_1} \int_0^T \|F(\tau, \cdot, u(\tau, \cdot))\| (T - \tau)^{\alpha q - 1} d\tau \\
\leq \frac{M_3}{\alpha} \int_0^{t_1} \int_0^T \tau^{-\alpha q} (T - \tau)^{\alpha q - 1} d\tau \\
\leq \frac{M_3}{\alpha} \int_0^{t_1} \tau^{-\alpha q} (T - t_1)^{\alpha q - 1} d\tau.
$$

It follows from the above arguments that $\partial_t^\alpha D^\beta u$ belongs to $C((0, T); V_{\gamma p})$. On the other hand, the estimate (4.26) also holds for $\tilde{p} = 0$ and $\tilde{q} = 1$, i.e., we have

$$
\|\partial_t^\alpha D^\beta u(t, \cdot)\|_{V_{\gamma p}} \leq M_3 \|\varphi\|_{V_{\beta p}},
$$

for $0 < t < T$. Therefore, $\partial_t^\alpha D^\beta u \in C^\alpha_w((0, T); V_{\gamma p})$ there exists a constant $M_{40} > 0$ such that

$$
\|\partial_t^\alpha D^\beta u(t, \cdot)\|_{C^\alpha_w((0, T); V_{\gamma p})} \leq M_{40} \|\varphi\|_{V_{\beta p + \tilde{q}}}. \tag{4.29}
$$

by the Sobolev embedding $V_{\beta p + \tilde{q}} \hookrightarrow V_{\beta p}$. The inequality (4.24) is derived by taking the inequality (4.27) and (4.29) together. We finally complete the proof. \square
Remark 4.1. At the beginning part of step 1 of the above proof, we recall that \( \psi_j^{N,2} = \psi_j^{(2)} \). This means that \( \psi_j^{N,2} \) (with respect to the nonlinear case) can be similarly estimated in the same way as in the linear case. More precisely, this term can be estimated as (3.31). Here, the formula of \( \psi_j^{(2)} \) is given by

\[
\psi_j^{(2)} = -\psi_j \frac{m_j^2 E_{\alpha,1}(-m_j^2 T)}{E_{\alpha,1}(-m_j^2 T)},
\]

see the proof of theorem 3.4. The appearance of the factor \( m_j^2 \) in (4.30) tells that we need \( \varphi \in V_{\beta(p+q)} \) to obtain (3.31). In summary, in order to bound the Caputo fractional derivative \( ^CD_t^\alpha u \), we need the stronger assumption \( \varphi \in V_{\beta(p+q)} \) rather than \( \varphi \in V_\beta \).

5. Discussion on global existence of solutions

In the previous section, we found a solution \( u \) of FVP (1.1)–(1.3) in the set \( W^{C,q}_0(J \times D) \). This allows that \( \|u(t, \cdot)\| \leq \tilde{C} \alpha t^{-\alpha q} \) for all \( 0 < t \leq T \). Then, we obtain \( u \in C^{\alpha q}_0((0, T]; L^2(D)) \) by establishing the time continuity of \( u \), which corresponds to the boundedness

\[
\sup_{0 < t \leq T} t^{\alpha q} \|u(t, \cdot)\| < +\infty.
\]

(5.1)

However, the existence given in theorem 4.3 requires the assumption \( k_0(T) < 1 \), which is equivalent to \( KT^{\alpha q} < M_4 \), where \( M_4 \) is a constant. This can occur if \( K \) or \( T \) is small enough.

‘Under what conditions is the contractivity condition \( k_0(T) < 1 \) satisfied?’ This motivates the result in this section.

The purpose of this section is to discuss global existence of solutions, namely, existence of solutions without any assumptions on \( K \) and \( T \). To overcome the difficulties of finding solutions in \( C^{\alpha q}_0((0, T]; L^2(D)) \), we shall seek solutions in a wider/weaker space than \( C^{\alpha q}_0((0, T]; L^2(D)) \).

The alternative solution space we are going to find is to take inspiration from replacing the supremum (5.1) by the following integral

\[
\int_0^T \left( \int_0^t r^\beta e^{-\beta t} \|u(t, \cdot)\| \right)^\mu dt < +\infty,
\]

(5.2)

with suitable parameter \( \beta, \rho, \) and \( \mu \). We expect that the mapping \( \mathcal{O} \) (formulated by lemma 4.1) on the alternative solution space is contracted as \( \rho \) tends to positive infinity.

The above arguments motivate us to denote by \( L^{\mu,\beta}_\rho(0, T; L^2(D)) \), \( \mu \geq 1, \rho > 0, b > 0 \), the weighted Lebesgue space of all functions \( v : (0, T) \to L^2(D) \) such that

\[
\|v\|_{L^{\mu,\beta}_\rho(0, T; L^2(D))} := \left( \int_0^T \left( \int_0^t r^\beta e^{-\beta t} \|v(t, \cdot)\| \right)^\mu dt \right)^{1/\mu} < \infty.
\]

In the next theorem, we present global existence for FVP (1.1)–(1.3) in \( L^{\mu,\beta}_\rho(0, T; L^2(D)) \). It is helpful to introduce the following special function

\[
\mathcal{F}_1(a, b, z) := \frac{\Gamma(b)}{\Gamma(b-a)} \int_0^1 (1 - \tau)^{b-a-1} \tau^{a-1} e^{-\tau} d\tau,
\]

which is called the Kummer function or hypergeometric function. We recall the following asymptotic behavior of this function

\[
\mathcal{F}_1(a, b, z) := \Gamma(b)(\Gamma(a))^{-1} e^z z^{-(b-a)} \left( 1 + O(|z|^{-1}) \right),
\]

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Due to a simple integration by substitution, for all $t \in (0, T)$ we observe that

$$
\int_0^t (t - \tau)^{a-1} e^{\tau} \, d\tau = \int_0^t (1 - \tau)^{a-1} e^{\tau} \, d\tau
$$

$$
= \frac{\Gamma(a)}{\Gamma(a + b)} \mathcal{F}_1(b, a + b, z)
$$

$$
= \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} \mathcal{F}_1(b, a + b, z) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} \mathcal{F}_1(b, a + b, z).
$$

(5.3)

**Theorem 5.1.** Assume that $\frac{1}{\alpha} < \alpha < 1$. Let $p, q$ be defined by (R1) such that $\frac{1}{2n} < q < 1$. Let $\mu$ and $b$ be such that $\frac{1}{\alpha q} < \mu < 2$, $\alpha q - \frac{1}{\mu} < b < 1 - \frac{1}{\mu}$. If $\varphi$ belongs to $V_{\beta p}$, $F$ satisfies (A1), then there exists $\hat{\rho} > 0$ such that FVP (1.1)–(1.3) has a unique solution $u_\Omega \in L^\mu_{\rho, q}(0, T; L^2(D))$ with $\rho \geq \hat{\rho}$, and furthermore

$$
\int_0^T (\hat{b} e^{-\rho t} \| u_\Omega(t, \cdot) \|^2) \, dt < +\infty.
$$

**Proof.** For $v_1, v_2 \in L^\mu_{\rho, q}(0, T; L^2(D))$, we first estimate the $L^\mu_{\rho, q}(0, T; L^2(D))$-norm of $\mathcal{O}_1 v_1 - \mathcal{O}_1 v_2$. The main idea is splitting the quantity $\| F(\tau, \cdot, v_1(\tau, \cdot)) - F(\tau, \cdot, v_2(\tau, \cdot)) \| (t - \tau)^{\alpha q - 1}$ into the product of $(t - \tau)^{\alpha q - 1} e^{\tau} \cdot e^{-\rho t}$ and $\hat{b} e^{-\rho t} \| F(\tau, \cdot, v_1(\tau, \cdot)) - F(\tau, \cdot, v_2(\tau, \cdot)) \|$. Then the $L^\mu_{\rho, q}(0, T; L^2(D))$-norm of $\| v_1(\tau, \cdot) - v_2(\tau, \cdot) \|$ can be obtained by applying the Hölder inequality and the Lipschitz assumption (A1). Indeed, one can see that

$$
\int_0^T \left( \hat{b} e^{-\rho t} \int_0^t \| v_1(\tau, \cdot) - v_2(\tau, \cdot) \| (t - \tau)^{\alpha q - 1} \, d\tau \right) \, dt
$$

$$
\leq \int_0^T (\hat{b} e^{-\rho t})^\mu \left( \int_0^t (t - \tau)^{\alpha q - 1} \, d\tau \right)^{\mu - 1} \int_0^T \left( \hat{b} e^{-\rho t} \| v_1(\tau, \cdot) - v_2(\tau, \cdot) \| \right)^\mu \, d\tau \, dt
$$

$$
\leq C_1^\mu \| v_1 - v_2 \|^\mu_{L^\mu_{\rho, q}(0, T; L^2(D))} \int_0^T (\hat{b} e^{-\rho t})^\mu \left( \int_0^t \int_0^\infty \frac{e^{-\sigma t}}{\sigma^{1-\omega}} (\sigma t)^{\frac{1-\omega}{\omega}} \left( 1 + t^{-1} O(\rho^{-1}) \right) \right) \, d\sigma \, dt
$$

$$
\leq C_1^\mu \| v_1 - v_2 \|^\mu_{L^\mu_{\rho, q}(0, T; L^2(D))} \left( T + O(\rho^{-1}) \right)^{\mu - 1} \rho^{1-\omega q} \int_0^T t^{1-\mu} \, dt,
$$

where we denote $C_1 := \Gamma \left( (\alpha q - 1)/(\mu - 1) \right) (\mu - 1)/(\alpha q - 1)/(\mu - 1)$. Here, the asymptotic behavior (5.3) had been used in the second estimate, where we note that

$$
\frac{\alpha q - 1}{\mu - 1} + 1 = \frac{\alpha q - 1}{\mu - 1} + \frac{\alpha q (1/(\alpha q) - 1)}{\mu - 1} = 0,
$$

as $\mu > 1/(\alpha q)$, and $1 - \beta \mu/(\mu - 1) > 0$ as $b < (\mu - 1)/\mu$. Moreover, by taking $\rho$ large enough we can bound $(T + O(\rho^{-1}))^{\mu - 1}$ by a constant independently of $x$, $t$. Since $\mu > 1/(\alpha q)$, the factor $\rho^{1-\omega q \mu}$ obviously tends to zero as $\rho$ tends to infinity. Furthermore, the latter improper integral is convergent as $\mu < 2$. Summarizing, we can find a constant $\rho_1 > 0$ such that
\[
\|O_1 v_1 - O_1 v_2\|_{L^p_{b,T}(0,T; L^2(D))}^2 \\
\leq (K_1 m_1)^{\beta p} \int_0^T \left( \int_0^t e^{-\rho\tau} \int_0^\tau \|F(\tau,\cdot, v_1(\tau,\cdot)) - F(\tau,\cdot, v_2(\tau,\cdot))\|(T - \tau)^{\alpha q - 1} \, d\tau \right) \, dt \\
\leq (K_1 m_1)^{\beta p} \int_0^T \left( \int_0^t e^{-\rho\tau} \int_0^\tau \|v_1(\tau,\cdot) - v_2(\tau,\cdot)\|(T - \tau)^{\alpha q - 1} \, d\tau \right) \, dt 
\]

so we arrive at the following estimate

\[
\|O_1 v_1 - O_1 v_2\|_{L^p_{b,T}(0,T; L^2(D))} \leq \frac{1}{4} \|v_1 - v_2\|_{L^p_{b,T}(0,T; L^2(D))},
\]

for all \( \rho \geq \rho_1 \), where we have used (4.7) in the first estimate and the Lipschitz assumption (A1) in the second estimate.

Second, we will estimate the \( L^p_{b,T}(0,T; L^2(D)) \)-norm of the difference \( O_3 v_1 - O_3 v_2 \). Based on estimating the \( L^p_{b,T}(0,T; L^2(D)) \)-norm as above, this can be treated by combining the Hölder inequality, the Lipschitz assumption (A1), and the inequality (4.8). Indeed, we see that

\[
\|O_3(t,\cdot)v_1 - O_3(t,\cdot)v_2\|_{L^p_{b,T}(0,T; L^2(D))}^2 \\
\leq M_3^\mu \int_0^T \left( \int_0^t e^{-\rho\tau} \int_0^\tau \|F(\tau,\cdot, v_1(\tau,\cdot)) - F(\tau,\cdot, v_2(\tau,\cdot))\|(T - \tau)^{\alpha q - 1} \, d\tau \right) \, dt \\
\leq (K M_3)\mu \int_0^T \left( \int_0^t e^{-\rho\tau} \int_0^\tau \|v_1(\tau,\cdot) - v_2(\tau,\cdot)\|(T - \tau)^{\alpha q - 1} \, d\tau \right) \, dt \\
\leq (K M_3)\mu \|v_1 - v_2\|_{L^p_{b,T}(0,T; L^2(D))}^2 \\
\int_0^T \left( \int_0^t \int_0^\tau e^{-\rho\tau} \int_0^\tau e^{-\rho\tau} \, d\tau \right) \, dt \\
\leq \sigma_{b,\rho}(T) \int_0^T \mu b^{\alpha - \mu} e^{-\mu T} \, dt.
\]

where the asymptotic behavior (5.3) was employed. Let us denote by \( T_{b,\rho} \), the latter integral, then

\[
\sigma_{b,\rho}(T) \int_0^T \mu b^{\alpha - \mu} e^{-\mu T} \, dt.
\]

Note that \((b - \alpha q)\mu > ((\alpha q - 1/\mu) - \alpha q)\mu = -1\) as \( b > \alpha q - 1/\mu \). Hence, using the asymptotic behavior (5.3), we have

\[
\sigma_{b,\rho}(T) \int_0^T b^{\alpha - \mu} e^{-\mu T} \, dt \\
\leq \sigma_{b,\rho}(T) T^{b - \alpha q} \mu^{-1} e^{-\mu T} (1 + O((\rho\mu T)^{-1})) \\
= (1 + T^{-1} O(\rho^{-1}))^{\mu^{-1}} (1 + O((\rho\mu T)^{-1})) \frac{\mu T^{\alpha q - \mu}}{\rho^\mu},
\]

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where the latter right-hand side tends to zero as \( \rho \) tends to infinity. Thus, there exists a \( \rho_2 > 0 \) such that
\[
\| O_3 v_1 - O_3 v_2 \|_{L^p_{\rho_2}(0,T;L^2(D))} \leq \frac{1}{4} \| v_1 - v_2 \|_{L^p_{\rho_2}(0,T;L^2(D))},
\] (5.5)
for all \( \rho \geq \rho_2 \). Taking the estimates (5.4) and (5.5) together gives
\[
\| O v_1 - O v_2 \|_{L^p_{\rho_2}(0,T;L^2(D))} \leq \sum_{j \in \{1,3\}} \| O j v_1 - O j v_2 \|_{L^p_{\rho_2}(0,T;L^2(D))}
\leq \frac{1}{2} \| v_1 - v_2 \|_{L^p_{\rho_2}(0,T;L^2(D))},
\]
for all \( \rho \geq \max\{\rho_1;\rho_2\} \), namely, \( O : L^p_{\rho_2}(0,T;L^2(D)) \to L^p_{\rho_2}(0,T;L^2(D)) \) is a contraction mapping. This means that \( O \) has only one fixed point in \( L^p_{\rho_2}(0,T;L^2(D)) \), and so FVP (1.1)–(1.3) has a unique solution \( u_0 \) in the weighted Lebesgue space \( L^p_{\rho_2}(0,T;L^2(D)) \). The desired inequality is obvious. \( \square \)

**Remark 5.1.** In fact, we tried to find solutions in the space
\[
C^{\phi,p}_w((0,T];L^2(D)) := \left\{ v \in C((0,T];L^2(D)) \mid \left\| v \right\|_{C^{\phi,p}_w((0,T];L^2(D))} = \sup_{0 < t \leq T} \rho^p e^{-\rho t} \| v(t, \cdot) \| < +\infty \right\}.
\]
Note that the following inclusion holds
\[
C^{\phi,p}_w((0,T];L^2(D)) \subset L^p_{\rho_2}(0,T;L^2(D)).
\]
Indeed, if \( v \in C^{\phi,p}_w((0,T];L^2(D)) \) then
\[
\left\| v \right\|_{L^p_{\rho_2}(0,T;L^2(D))} \leq \left( \int_0^T dt \right)^{1\over p} \left( \sup_{0 < t \leq T} \rho^p e^{-\rho t} \| v(t, \cdot) \| \right) = T^{1\over p} \left\| v \right\|_{C^{\phi,p}_w((0,T];L^2(D))} < +\infty.
\]
In order to find solutions in this space, for all \( v_1, v_2 \in C^{\phi,p}_w((0,T];L^2(D)) \), it requires to bound the following quantities
\[
Q_1(b,\rho) := \sup_{0 < t \leq T} \rho^p e^{-\rho t} \int_0^T \| v_1(\tau, \cdot) - v_2(\tau, \cdot) \|(T - \tau)^{\alpha q - 1} d\tau,
\]
\[
Q_2(b,\rho) := \sup_{0 < t \leq T} \rho^p e^{-\rho t} \int_0^T \| v_1(\tau, \cdot) - v_2(\tau, \cdot) \|(T - \tau)^{\alpha q - 1} d\tau,
\]
by \( k_j(\rho)\| v_1 - v_2 \|_{C^{\phi,p}_w((0,T];L^2(D))} \), with \( k_j(\rho), j = 1,2 \), tend to zero as \( \rho \) tends to infinity. Unfortunately, it does not occur with the term \( Q_2(b,\rho) \). Indeed, since \( v_1, v_2 \in C^{\phi,p}_w((0,T];L^2(D)) \) we have
\[
\| v_1(t, \cdot) - v_2(t, \cdot) \| \leq T^{\alpha q} \| v_1 - v_2 \|_{C^{\phi,p}_w((0,T];L^2(D))},
\]
which gives
\[
Q_2(b,\rho) \leq \left( \sup_{0 < t \leq T} \rho^p e^{-\rho t \alpha q} \int_0^T \tau^{-\alpha q} e^{\alpha q} (T - \tau)^{\alpha q - 1} d\tau \right) \| v_1 - v_2 \|_{C^{\phi,p}_w((0,T];L^2(D))}.
\]
The following conclusions are obvious:

- Due to the asymptotic behavior (5.3), the supremum of $\hat{\sigma}_{b,\rho}(t, T)$ on $(0, T]$ tends to infinity as $\rho$ tends to infinity. Hence, $\mathcal{O}: L^\mu_{b,\rho}(0, T; L^2(D)) \rightarrow L^\mu_{b,\rho}(0, T; L^2(D))$ cannot be a contraction mapping for arbitrary $T$. This is the main reason why we did not find solutions in $C_{\text{loc}}^0((0, T]; L^2(D))$.

- The idea of using the space $L^\mu_{b,\rho}(0, T; L^2(D))$ fortuitously came when we realized that

$$\int_0^T (\hat{\sigma}_{b,\rho}(t, T))^\mu \ dt \xrightarrow{\rho \to \infty} 0$$

with a suitable number $\mu \geq 1$. Here, we replaced the supremum (5.1) by the integral (5.2).

- One can show that $Q_1(b, \rho)$ is bounded by $k_1(\rho)\|v_1 - v_2\|_{C_{\text{loc}}^0((0, T]; L^2(D))}$, where $k_1(\rho)$ tends to zero as $\rho$ tends to infinity. This means that we can establish the existence of a mild solution to the initial value problem (1.1), (1.2), (1.4) in $C_{\text{loc}}^0((0, T]; L^2(D))$ without any assumptions on $K$ and $T$.

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References

[1] Allen M, Caffarelli L and Vasseur A 2016 A parabolic problem with a fractional time derivative Arch. Ration. Mech. Anal. 221 603–30

[2] Bögelein V, Duzaar F, Marcellini P and Scheven C 2018 Doubly nonlinear equations of porous medium type Arch. Ration. Mech. Anal. 229 503–45

[3] Bögelein V, Duzaar F, Marcellini P and Signoriello S 2015 Nonlocal diffusion equations J. Math. Anal. Appl. 432 398–428

[4] Duzaar F and Habermann J 2012 Partial regularity for parabolic systems with non-standard growth J. Evol. Equ. 12 203–44

[5] Bonforte M, Sire Y and Vazquez J L 2017 Optimal existence and uniqueness theory for the fractional heat equation Nonlinear Anal. 153 142–68

[6] Bonforte M and Vazquez J L 2014 Quantitative local and global a priori estimates for fractional nonlinear diffusion equations Adv. Math. 250 242–84

[7] Hung N Q and Vazquez J L 2018 Porous medium equation with nonlocal pressure in a bounded domain Commun. PDE 43 1502–39

[8] Stan D, del Teso F and Vázquez J L 2019 Existence of weak solutions for a general porous medium equation with nonlocal pressure Arch. Ration. Mech. Anal. 233 451–96

[9] del-Castillo-Negrete D, Carreras B A and Lynch V E 2005 Nondiffusive transport in plasma turbulence: a fractional diffusion approach Phys. Rev. Lett. 94 065003

[10] Nimatullin R R 1986 The realization of the generalized transfer equation in a medium with fractal geometry Phys. Status Solidi b 133 425–30
[11] Berkowitz B, Klafker J, Metzler R and Scher H 2002 Physical pictures of transport in heterogeneous media: advection-dispersion, random-walk, and fractional derivative formulations Water Resour. Res. 38 9
[12] Kou S 2008 Stochastic modeling in nanoscale biophysics: subdiffusion within proteins Ann. Appl. Stat. 2 501–35
[13] Zaslavsky G M 2002 Chaos, fractional kinetics, and anomalous transport Phys. Rep. 371 461–580
[14] Gal C G and Warma M 2017 Fractional in Time Semilinear Parabolic Equations and Applications, HAL Id: hal-01578788
[15] Clement P, Londen S-O and Simonett G 2004 Quasilinear evolutionary equations and continuous interpolation spaces J. Differ. Equ. 196 418–47
[16] de Andrade B, Carvalho A N, Carvalho-Neto P M and Marin-Rubio P 2015 Semilinear fractional differential equations: global solutions, critical nonlinearities and comparison results Topol. Methods Nonlinear Anal. 45 439–67
[17] Guswanto B H and Suzuki T 2015 Existence and uniqueness of mild solutions for fractional semilinear differential equations Electron. J. Differ. Equ. 2015 16
[18] Dong H and Kim D 2019 $L_p$-estimates for time fractional parabolic equations with coefficients measurable in time Adv. Math. 345 289–345
[19] Giga Y and Namba T 2017 Well-posedness of Hamilton-Jacobi equations with Caputo’s time fractional derivative Commun. PDE 42 1088–120
[20] Kim I, Kim H K and Lim S 2017 An $L_2$($L_p$)-theory for the time fractional evolution equations with variable coefficients Adv. Math. 306 123–76
[21] Li L and Liu G L 2018 Some compactness criteria for weak solutions of time fractional PDEs SIAM J. Math. Anal. 50 3963–95
[22] Taylor M Remarks on Fractional Diffusion Equations www.unc.edu/math/Faculty/met/fdif.pdf
[23] Jin B, Lazarov R, Liu Y and Zhou Z 2015 The Galerkin finite element method for a multi-term time-fractional diffusion equation J. Comput. Phys. 281 825–43
[24] Jin B, Li B and Zhou Z 2018 Numerical analysis of nonlinear subdiffusion equations SIAM J. Numer. Anal. 56 1–23
[25] Li Z, Liu Y and Yamamoto M 2015 Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients Appl. Math. Comput. 257 381–97
[26] Nochetto R H, Otárola E and Salgado A J 2016 A PDE approach to space-time fractional wave problems SIAM J. Numer. Anal. 54 848–73
[27] Otárola E and Salgado A J Regularity of solutions to space-time fractional wave equations: Fractional Calculus Appl. Anal. 21 1262–93
[28] Zhou G H and Guo Z B 2018 Boundary feedback stabilization for an unstable time fractional reaction diffusion equation SIAM J. Control Optim. 56 75–10
[29] Dang D T, Nane E, Nguyen D M and Tuan N H 2018 Continuity of solutions of a class of fractional equations Potential Anal. 49 423–78
[30] McLean W 2010 Regularity of solutions to a time-fractional diffusion equation ANZIAM J. 52 123–38
[31] Sakamoto K and Yamamoto M 2011 Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems J. Math. Anal. Appl. 382 426–47
[32] Alvarez E, Gal C G, Keyantuo V and Warma M 2019 Well-posedness results for a class of semilinear super-diffusive equations Nonlinear Anal. 181 24–61
[33] Kian Y and Yamamoto M 2017 On existence and uniqueness of solutions for semilinear fractional wave equations Fractional Calculus Appl. Anal. 20 117–38
[34] Mu J, Ahmad B and Huang S 2017 Existence and regularity of solutions to time-fractional diffusion equations Comput. Math. Appl. 73 985–96
[35] Jiang D, Li Z, Liu Y and Yamamoto M 2017 Weak unique continuation property and a related inverse source problem for time-fractional diffusion-advection equations Inverse Problems 33 21
[36] Li Z, Imanuvilov O Y and Yamamoto M 2016 Uniqueness in inverse boundary value problems for fractional diffusion equations Inverse Problems 32 16
[37] Li G, Zhang D, Jia X and Yamamoto M 2013 Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation Inverse Problems 29 36
[38] Luchko Y, Rundell W, Yamamoto M and Zuo L 2013 Uniqueness and reconstruction of an unknown semilinear term in a time-fractional reaction-diffusion equation Inverse Problems 29 16
[39] Miller L and Yamamoto M 2013 Coefficient inverse problem for a fractional diffusion equation *Inverse Problems* **29** 8

[40] Wang W, Yamamoto M and Han B 2013 Numerical method in reproducing kernel space for an inverse source problem for the fractional diffusion equation *Inverse Problems* **29** 15

[41] Kaltenbacher B and Rundell W 2019 On an inverse potential problem for a fractional reaction–diffusion equation *Inverse Problems* **35** 065004

[42] Kaltenbacher B and Rundell W 2019 Regularization of a backward parabolic equation by fractional operators *Inverse Problems Imaging* **13** 401–30

[43] Rundell W and Zhang Z 2018 Recovering an unknown source in a fractional diffusion problem *J. Comput. Phys.* **368** 299–314

[44] Rundell W and Zhang Z 2017 Fractional diffusion: recovering the distributed fractional derivative from overposed data *Inverse Problems* **33** 035008

[45] Janno J and Kinash N 2018 Reconstruction of an order of derivative and a source term in a fractional diffusion equation from final measurements *Inverse Problems* **34** 19

[46] Janno J and Kasemets K 2017 Uniqueness for an inverse problem for a semilinear time-fractional diffusion equation *Inverse Problems Imaging* **11** 125–49

[47] Tuan N H, Long L D, Thinh N V and Tran T 2017 On a final value problem for the time-fractional diffusion equation with inhomogeneous source *Inverse Problem Sci. Eng.* **25** 1367–95

[48] Jia J, Peng J, Gao J and Li Y 2018 Backward problem for a time-space fractional diffusion equation *Inverse Problems Imaging* **12** 773–99

[49] Wei T and Zhang Y 2018 The backward problem for a time-fractional diffusion–wave equation in a bounded domain *Comput. Math. Appl.* **75** 3632–48

[50] Wei T and Xian J 2019 Variational method for a backward problem for a time-fractional diffusion equation *ESAIM: Math. Modelling Numer. Anal.* **53** 1223–44

[51] Xian J and Wei T 2019 Determination of the initial data in a time-fractional diffusion-wave problem by a final time data *Comput. Math. Appl.* **78** 2525–40

[52] Wei T and Wang J G 2014 A modified quasi-boundary value method for the backward time-fractional diffusion problem *ESAIM: Math. Modelling Numer. Anal.* **48** 603–21

[53] Liu J J and Yamamoto M 2010 A backward problem for the time-fractional diffusion equation *Appl. Anal.* **89** 1769–88

[54] Au V V, Kirane M and Tuan N H 2019 Determination of initial data for a reaction-diffusion system with variable coefficients *Discrete Continuous Dyn Syst - Ser A* **39** 771–801

[55] Hao D N, Duc N V and Thang N V 2018 Backward semi-linear parabolic equations with time-dependent coefficients and local Lipschitz source *Inverse Problems* **34** 33

[56] Tuan N H, Khoa V A and Au V V 2019 Analysis of a quasi-reversibility method for a terminal value quasi-linear parabolic problem with measurements *SIAM J. Math. Anal.* **51** 60–85

[57] Kian Y, Oksanen L, Soccorsi E and Yamamoto M 2018 Global uniqueness in an inverse problem for time fractional diffusion equations *J. Differ. Equ.* **264** 1146–70

[58] Diethelm K 2010 *The Analysis of Fractional Differential Equations* (Berlin: Springer)

[59] Podlubny I 1999 *Fractional Differential Equations* (London: Academic)

[60] Samko S G, Kilbas A A and Marichev O I 1987 *Fractional Integrals and Derivatives, Theory and Applications* (Minsk: Gordon and Breach Science, Nauka Tekhnika)

[61] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* (volume 204 of North-Holland Mathematics Studies) (Amsterdam: Elsevier)

[62] Chen Y, Gao H, Garrido-Atienza M and Schmalfuß B 2014 Pathwise solutions of SPDEs driven by Hölöder-continuous integrators with exponent larger than 1/2 and random dynamical systems *Discrete Continuous Dyn Syst - Ser A* **34** 79–98

[63] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (New York: Dover)