ABSTRACT. The associator of a non-associative algebra is the curvature of the Hochschild quasi-complex. The relationship “curvature-associator” is investigated. Based on this generic example, we extend the geometric language of vector fields to a purely algebraic setting, similar to the context of Gerstenhaber algebras. We interpret the elements of a non-associative algebra with a Lie bracket as “vector fields” and the multiplication as a connection. Conditions for the existence of an “algebra of functions” having as algebra of derivations the original non-associative algebra, are investigated.

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Date: October 1999.

1991 Mathematics Subject Classification. Primary: 58A12, 14A22; Secondary: 17A75.

Key words and phrases. Hochschild DGLA, Gerstenhaber algebra, curvature, associator, connection.

I thank Jim Stasheff for helpful remarks, and for pointing to the related work of [DMM, DM].
1. Introduction

The associator $\alpha$ of a non-associative algebra $(A, \mu)$ is the curvature of the Hochschild quasi-complex. It has properties analogous to the properties of the curvature of a linear connection, suggesting the interpretation of its elements as formal vector fields. Gerstenhaber algebras motivate this interpretation, as a non-commutative analog of Poisson algebras.

In this article we investigate the possibility of "reconstructing" an algebra of "functions" representing a "non-commutative space", such that the original algebra is the algebra of its derivations ("vector fields"). A differential calculus based on derivations is only sketched. Such a calculus was developed in detail in [DMM, DM].

Classical differential geometry is built on the notion of space: differential manifolds. A rough "hierarchy" is: space, functions, vector fields and differential forms, connections etc.

Algebraic geometry starts at the “second level” (functions) by considering an arbitrary commutative algebra and then constructing the “first level”, the substitute for a space: its spectrum. A space (affine variety) is roughly a pair consisting of a topological space and its algebra of functions.

A natural question arises: “What can be derived starting from the “third level” - vector fields - and to what extent is it profitable?”

We will be interested in geometry with connections, so we will investigate the interpretation of the associator of a non-associative algebra as a curvature (section 1.1). We recall in section 2 some well known facts about the Lie algebra structure on the Hochschild quasi-complex ($d^2 \neq 0$) of a non-associative algebra. Motivated by this generic example, we consider a Lie algebra, with elements thought of as vector fields, with a multiplication (not necessarily associative) thought of as a linear connection, and we define in section 3 the algebra of functions (2nd level). Conditions are derived for the existence of a natural “algebra of functions”, having as derivations the original non-associative algebra (theorem 3.1). Section 4 considers examples of torsion algebras. The main example is the pre-Lie algebra of Hochschild cochains with Gerstenhaber composition (theorem 4.1).

As motivation for the emphasis on vector fields we mention two sources. Physical understanding evolved from considering phase spaces (Poisson manifolds) rather than configuration spaces. Moreover, the actual goal is to model the space of evolutions of a system.

The second motivation is the correspondence between Poisson-Lie group structures and Lie bialgebras $g$. After quantization of $Ug$ one has deformed differential operators and, in some sense deformed vector fields. It would be convenient to have a procedure to recover an algebra of “functions” (possibly non-commutative).

In deformation quantization of Poisson manifolds, one keeps the classical observables and deforms the “laws of mechanics” to account for the Heisenberg bracket. We consider this approach as slightly conservative and consider that the basic level for quantum physics: states - vectors, evolution - operators, does not need an actual (configuration) space. Our “functions” are naturally operators on the given algebra of “vector fields”.

2
1.1. **Associator - curvature / monoidal structure.** Another argument for a geometric interpretation of an algebra \((A, \mu)\) is as follows. The associator \(\alpha(x, y, z) = (xy)z - x(yz)\) of the nonassociative algebra is formally a “curvature” of the left regular quasi-representation \(L : (A, \mu) \to (End_k(A), \circ)\):

\[
L(xy) \xrightarrow{\sigma_{x,y}} L(x)L(y)
\]

\[
\sigma(x, y)(z) = (L(x)L(y) - L(xy))(z) = -\alpha(x, y, z)
\]

A quasi-representation is a morphism in the underlying category, not necessarily preserving the additional structure (e.g. commuting with the group operation). In the situation at hand, the above map \(L\) is assumed to be only \(k\)-linear. At the infinitesimal (Lie algebra) level, the same map \(L\), interpreted as a “quasi-Lie” representation \(L : (A, [\cdot, \cdot]) \to (End_k(A), [\cdot, \cdot])\), defines a formal curvature \(K\):

\[
L([x, y]) \xrightarrow{K_{x,y}} [L(x), L(y)], \quad K(x, y) = [L(x), L(y)] - L([x, y])
\]

Moreover, in the Hochschild quasi-complex \((C^\bullet(A), d_\mu)\), the differential \(d_\mu\) has properties analog to a covariant derivative, e.g. \(d_\mu^2 f = [\alpha, f]\). Also \(\alpha = \frac{1}{2} [\mu, \mu]\) (the “curvature”) is closed, i.e. a “Bianchi identity” holds. (see \(\text{2.1}\)).

This approach is only a tentative to a partial model of a “local” differential geometry.

The “global” point of view should consider the non-linear (multiplicative) interpretation of the failure of a quasi-representation (or underlying morphism) to preserve the structure as a non-strict monoidal structure. This “monoidal” interpretation of the associator comes from looking at quasi-representations, and more general at functions defined on groups \(s : G \to N\) as defining a 2-cocycle \(f : G \times G \to N, f(a \otimes b) = s(a)s(b)s(ab)^{-1}\) (“factor set” corresponding to a given action of \(G\) on \(N\)). If interpreting \(G\) and \(N\) as monoidal groupoids \([\mathbb{L}]\), then \(f\) is a monoidal structure \(f_{a,b} : s(a \otimes b) \to s(a) \otimes s(b)\) of the monoidal functor \(s\).

2. **Hochschild Quasi-complex**

We begin by recalling the Hochschild cohomology \([\mathbb{G}e, \mathbb{G}S]\) in the more general case of a possibly non-associative algebra.

The obstruction to the cohomological study of a non-associative algebras, \(d^2 \neq 0\), is interpreted as a curvature. The use of the language of differential geometry is considered.

Throughout this section, \(A\) will denote a module over a commutative ring \(R\). We will assume that 2 and 3 do not annihilate non-zero elements in \(A\).

2.1. **Pre-Lie Algebra of a Module.** Consider

\[
C^{p,q}(A, A) = \{f : A^\otimes_p \to A^\otimes_q | f \text{ R-linear} \} \quad p, q \geq 0
\]

and \(C^\bullet(A, A) = \bigoplus_{p,q \geq 0} C^{p,q}\) with total degree \(\partial \text{deg}(f^{p,q}) = p - q\). \(C^{0,q}\) is identified with \(A^\otimes_q\).

We will be interested in the first column \(C^\bullet(A) = \bigoplus_{p \in \mathbb{N}} C^{p,1}\). The grading induced by the total degree is \(C^{p-1}(A) = C^{p,1}(A, A)\) with \(p \geq 0\).
We recall briefly the comp operation and the Lie algebra structure it defines on the graded module of Hochschild cochains \([\mathcal{C}, \mathcal{KPS}]\).

If \(f^p \in C^p(A)\) and \(g^q \in C^q(A)\), define the composition into the \(i^{th}\) place, where \(i = 1, \ldots, p + 1\):

\[
f^p \circ_i g^q(a_1, \ldots, a_{p+q-1}) = f^p(a_1, \ldots, a_{i-1}, g^q(a_i, \ldots, a_{i+q-1}), a_{i+q}, \ldots, a_{p+q-1})
\]

and the \(\text{comp}\) operation:

\[
f^p \circ g^q = \sum_{i=1}^{p+1} (-1)^{(i-1)q} f^p \circ_i g^q \in C^{p+q}
\]

It is assumed that the composition is zero whenever \(p = -1\). Note that the (non-associative) composition respects the grading. Denote by \(\alpha(f, g, h) = (f \circ g) \circ h - f \circ (g \circ h)\) the associator of \(\circ\), as a “measure” of non-associativity of the comp operation.

The graded commutator is defined by:

\[
[f^p, g^q] = f^p \circ g^q - (-1)^{pq} g^q \circ f^p
\]

It is graded commutative:

\[
[f^p, g^q] = -(-1)^{pq}[g^q, f^p]
\]

and the graded Jacobi identity holds:

\[
(-1)^{FH}[f, [g, h]] + (-1)^{GF}[g, [h, f]] + (-1)^{HG}[h, [f, g]] = 0
\]  

(2.1)

where \(F, G, H\) denote the degrees of \(f, g, h\) respectively. It is equivalent to \(ad\) being a representation of graded Lie algebras:

\[
[f, [g, h]] = [[f, g], h] + (-1)^{pq}[g, [f, h]]
\]

\[
ad_f([g, h]) = [ad_f(g), h] + (-1)^{pq}[g, ad_f(h)]
\]

(2.2)

Notation 2.1. We denote by \(f \circ_{(i, j)} (g, h)\) the simultaneous insertion of two functions \(g\) and \(h\) in the \(i^{th}\) and \(j^{th}\) arguments of \(f\) respectively.

Lemma 2.1. If \(f, g, h\) have degrees \(p, q, r\), then:

(i) \(f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i-1+j} h\), \(1 \leq i \leq p + 1, 1 \leq j \leq q + 1\)

(ii) \((f \circ g) \circ h = f \circ (g \circ h) = \sum_{i \neq j} \epsilon(i, j)(-1)^{(i-1)(q+j-i-r)} f \circ_{(i, j)} (g, h)\), where \(\epsilon(i, j) = 1\) if 

\[
1 \leq j < i \leq p + 1
\]

and equals \((-1)^{pr}\) if \(1 \leq i < j \leq p + 1\).

(iii) \(\alpha(f, g, h) = (-1)^{qr}\alpha(f, h, g)\)

Proof. (i) and (ii) follow from a straightforward inspection of trees and signs. The key is that the only trees which survive, build out of \(f, g, h\) in the associator \(\alpha\), are of the type \(f \circ_{(i, j)} (g, h)\). The “supercommutativity” sign \((-1)^{qr}\) appears when \(i\) passes over \(j\) and the order of insertion (\(g\) before \(h\)) changes. \(\square\)
Notation 2.2. Let $\mu \in C^1(A)$ and $\mu = \mu_- + \mu_+$ the natural decomposition, with $\mu_-(a,b) = \mu(a,b) - (-1)^{pq}\mu(b,a)$ and $\mu_+(a,b) = \mu(a,b) + (-1)^{pq}\mu(b,a)$ the graded skew and symmetric part of $\mu$. Alternatively $\mu_-$ will be denoted as $[\cdot]_\mu$ or just $[\cdot]$ if no confusion is expected. The corresponding associator will be denoted as $\alpha_\mu$.

**Definition 2.1.** A (possibly non-associative) algebra $(A,\mu)$ is called a *pre-Lie algebra* if $\mu_-$ is a Lie bracket.

**Lemma 2.2.** Let $(A,\mu)$ be an algebra and $\alpha$ its associator.
(i) $\text{Alt}(\alpha_{\mu_+}) = 0$
(ii) $\text{Alt}(\alpha_{\mu_-}) = 4\text{Alt}(\alpha)$
(iii) $(A,\mu)$ is a pre-Lie algebra iff $\text{Alt}(\alpha) = 0$.

If $A$ is graded, then a graded alternation $\text{Alt}$ is assumed.

**Proof.** A direct computation.

From lemma 2.2 and the above lemma it immediately follows the well-known fact that the comp operation on Hochschild cochains defines a Lie bracket.

**Corollary 2.1.** $(C^\bullet,\circ)$ is a (graded) pre-Lie algebra.

**Proof.** Since $\alpha(f,g,h) = (-1)^{qr}\alpha(f,h,g)$, we have:
$$\text{Alt}(\alpha)(f,g,h) = \sum_{cyc} \epsilon(f,g,h)(\alpha(f,g,h) - (-1)^{qr}\alpha(f,h,g)) = 0$$

where $\epsilon(f,g,h)$, in the graded case, is not necessarily 1. For example $\epsilon(g,h,f) = (-1)^{(q+r)p}$.

2.2. **Quasi-complex of a Non-Associative Algebra.**

**Definition 2.2.** A *quasi-complex* is a sequence of objects and morphisms in a category $\mathcal{A}$
$$C^\bullet = \{ \cdots \to C^{-1} \to C^0 \to C^1 \to \cdots \}.$$ 

The family of morphisms $d^\bullet$ is called a *quasi-differential*, for which $d^2$ may be non-zero.

Now assume that an element $\mu : A \otimes A \to A$ of degree one is fixed. Thus $(A,\mu)$ is a (non-associative) $R$-algebra. Define the quasi-differential as the adjoint $\mu$ action:
$$d_\mu(f^p) = [\mu,f]$$

Then $(C^\bullet(A),d_\mu)$ is a quasi-differential graded Lie algebra, called the Hochschild quasi-complex corresponding to the algebra $(A,\mu)$.

Note the difference of sign when compared with [Ge, GS]:
$$d_{Ge} = -[f,\mu] = (-1)^p[\mu,f] = (-1)^pd_\mu f$$
As an example, for $p = 1$ (with $d = d_{\mu}$ and $\mu(x, y) = xy$ if no confusion is possible):

$$
\begin{align*}
\text{df}(x, y, z) &= \mu \circ f(x, y, z) + f \circ \mu(x, y, z) \\
&= \mu(f(x, y), z) - \mu(x, f(y, z)) + f(\mu(x, y), z) - f(x, \mu(y, z)) \\
&= -\{xf(y, z) - f(xy, z) + f(x, yz) - f(x, y)z\}
\end{align*}
$$

which is the usual Hochschild differential modulo the sign:

$$
d_{\text{Hoch}} = (-1)^{\partial_{\text{deg}}} \cdot d_{\mu} = d_{G_e}
$$

Note that $d_{\mu} : C^p \to C^{p+1}$ has degree one, $[,] : C^p \otimes C^q \to C^{p+q}$ is of degree zero and $d_{\text{Hoch}}$ is not a graded derivation, since it does not satisfy the “Leibniz identity” 2.2. We state the following fact about graded Lie algebras, which is an immediate consequence of the graded Jacobi identity.

**Lemma 2.3.** Let $(g, [,])$ a graded Lie algebra over a commutative ring $R$. Then, if $x$ is an even degree element $[x, x] = 0$. If $x$ is odd, $[x, [x, x]] = 0$ and $ad_{[x, x]} = 2(ad_x)^2$.

If the multiplication $\mu$ is associative, then (2.4):

$$
d\mu(x, y, z) = 2\{(xy)z - x(yz)\} = 0
$$

and $[\mu, \mu] = 0$. By the previous lemma $2[\mu, [\mu, f]] = d_{[\mu, \mu]}(f) = 0$, and thus $(C^\bullet(A), d_{\mu})$ is a complex of $R$-modules.

We introduce the following definition:

**Definition 2.3.** A quasi-differential graded Lie algebra $(C^\bullet, d)$ is called coboundary if there is a degree zero element $I$ such that the multiplication (also called torsion) $\mu = dI$ verifies the relation $d = ad_{\mu}$.

Then $2\alpha = d\mu$ defines the associator $\alpha$ (or curvature).

Since $d\mu = [\mu, \mu] = 2\mu \circ \mu$ and 2 was assumed not to have right divisors, the associator is well-defined by the above equation.

The motivation for the geometric terminology comes from a formal analogy in the context of a derivation law in an $A$-module, where $A$ is an $R$-algebra (see [Ko]) or a linear connection $D$ on a vector bundle, where the torsion $T$ and curvature $F$ of the total covariant derivative $d$ are defined as:

$$
T = dI, \quad dI(X, Y) = D_X I(Y) - D_Y I(X) - I([X, Y])
$$

$$
F(X, Y) = [D_X, D_Y] - D_{[X, Y]}, \quad d^2 s = [F, s]
$$

Here $I$ is the identity tensor.

The geometric interpretation will be considered in section 3.

**Proposition 2.1.** Let $(A, \mu)$ be an $R$-algebra, possibly non-associative. Then $(C^\bullet(A), d_{\mu})$ is a coboundary quasi-differential algebra. In the adjoint representation the zero-degree element $I$ corresponds to the grading character:

$$
ad_I(f) = -\partial_{\text{deg}}(f)f
$$
and Bianchi’s identity holds:
\[ d\alpha_\mu = 0, \quad \alpha_\mu(x, y, z) = (xy)z - x(yz) \]
Moreover
\[ d^2 s = [\alpha, s] \]

**Proof.** Note first that the identity map \( I : A \to A \) has degree zero. If \( f \in C^p(A) \)
\[ ad_I(f) = I \circ f - (-1)^{p-1} f \circ I \]
\[ = f - (p + 1)f \]
\[ = -\partial eg(f)f \quad (2.6) \]
Since \([I, f] = -[f, I]\), the right adjoint action of the unit is scalar multiplication by the degree map.

Obviously \( d_\mu I = [\mu, I] = \partial eg(\mu)\mu = \mu \), thus \( I \) is a unit. Now, by the Jacobi identity, the
assocciator is a cocycle \( d_\mu \alpha = 0 \) (the curvature is closed):
\[ d_\mu[\mu, \mu] = [\mu, [\mu, \mu]] = 0 \]
where the assumption that 2 and 3 do not annihilate non-zero elements in \( A \) was used.

The second equation follows from lemma 2.3.

In the context of the previous proposition, for any element \( x \) of even degree, we have
\([x, x] = 0\), since \( 0 = [I, [x, x]] = -2\partial eg(x)[x, x] \) (see lemma 2.3).

To define the cohomology of certain classes of non-associative algebras, we give the following definition:

**Definition 2.4.** An algebra \((A, \mu)\) is called \( N \)-coherent if \( d^N = 0 \).

Note that an algebra is associative iff it is a 2-coherent algebra, and a 1-coherent algebra
is just the trivial one: \( \mu = 0 \).

Of course an algebra \((A, \mu)\) is \( N \)-coherent iff \( ad_\mu \) is a nilpotent element of order \( N \).

For an \( N \)-coherent algebra the quasi-differential graded Lie algebra \((C^\bullet, d_\mu)\) is an \( N \)-
complex as defined in \( [K] \).

Next we will consider examples of non-associative algebras.

**Example 2.1.** If \((A, \mu = [\cdot, \cdot])\) is a Lie algebra, then its associator is:
\[ \alpha(a, b, c) = ([a, b], c) - [a, [b, c]] \]
\[ = [b, [c, a]] \quad (2.7) \]
and \( Alt(\alpha) = 0 \) as expected.

**Example 2.2.** If \((A, m)\) is an associative algebra, we can associate to it the Jordan algebra
\((A, \mu_+)\), with \( \mu_+(a, b) = \{a, b\} = ab + ba \), and its Lie algebra \((A, \mu_-)\), with \( \mu_-(a, b) = [a, b] = ab - ba \) . The corresponding associators are
\[ \alpha_+(a, b, c) = \{cab + bac - acb - bca\} = -[b, [c, a]] \]
and
\[ \alpha_-(a, b, c) = \{ -bac - cab + acb + bca \} = [b, [c, a]]. \]

Thus they have the same even quasi-differential Lie algebra \((C^+, d_{\alpha_\pm})\) (modulo a sign).

Note also that \(\text{Alt}(\alpha_\pm) = 0\).

3. **Torsion Algebras**

Hochschild differential complex is defined for an associative algebra with coefficients in a symmetric \(A\)-bimodule \(M\). When relaxing both conditions, associativity and action requirement, one obtains formulas which are familiar in differential geometry, and correspond to a non-flat connection. It may be thought of as a geometry of “vector fields” without starting from a function algebra.

**Definition 3.1.** A torsion algebra \(\mathcal{M} = (C, \mu, [\, , ]_C)\) is a (non-associative) \(k\)-algebra \((C, \mu)\) together with a Lie bracket \([\, , ]_C\). Its torsion is \(T = \mu_- - [\, , ]_C\), where \(\mu = \mu_+ + \mu_-\) is the decomposition into its symmetric \((\text{quasi-Jordan})\) and skewsymmetric \((\text{quasi-Lie})\) part:

\[ T(X, Y) = \mu(X, Y) - \mu(Y, X) - [X, Y]_C, \quad X, Y \in C. \]

It is a generalization of the most important classes of algebras. Associative algebras, with the usual Lie bracket \([x, y] = xy - yx\) are torsion algebras, with \(T = 0\). Lie algebras \((C, [\, , ])_C\), with \(\mu = \frac{1}{2}[\, , ]\) are again torsion algebras with zero torsion. Poisson algebras (compatibility between \(\mu\) and \([\, , ]\)) and Gerstenhaber algebras (non-commutative Poisson algebras) can be interpreted as torsion algebras in several ways. Pre-Lie algebras (non-associative algebras such that the skew part of the multiplication is a Lie bracket) with \([\, , ] = \mu_-\) are torsion algebras with zero torsion \((T = 0)\).

We think of \((C, [\, , ])\) as a Lie algebra of vector fields and the multiplication \(\mu\) as a generalized connection.

**Example 3.1.** Obviously any manifold \(V\) with a connection \(D\) defines a torsion algebra. Take \(C\) as the Lie algebra of vector fields on \(V\) and interprete the connection as a non-associative multiplication \(\mu(X, Y) = D_X Y\). Then the torsion is \(T = D_- - [\, , ]\), i.e.

\[ T(X, Y) = D_X Y - D_Y X - [X, Y]. \]

In this geometric example, the torsion tensor coincides with the torsion in the sense of definition 3.1.

If \(V\) is the real line, then the Lie algebra of vector fields \(X_f = f \partial_t\) can be identified with \((C^\infty(V), [\, , ])\), where \([f, g] = fg' - gf'\). Also any connection \(D\) has a canonical Christoffel symbol \(\Gamma\) and

\[ D f g = f(g' + g\Gamma) \quad (D = d + \Gamma). \]
3.1. **Notation and background.** \((C, \mu)\) will denote a possibly non-associative \(k\)-algebra, where \(k\) is a ring. We will write \(D_X Y = \mu(X, Y)\), in order to emphasize the geometric interpretation. Basic definitions for the usual algebraic model are assumed, following [Ko]. The prefix \(M\) will be used with notions referring to the formal context ("non-commutative" space), and the prefix \(A\) to refer to the usual notions in the context of a geometric example, e.g. on a manifold \(V\).

In the "geometric world", functions can be identified as \(k\)-endomorphisms (multiplication of vector fields by functions) for which the connection is linear in the first argument.

3.2. **The Algebra of Functions.** We define as "functions" the annihilator of the left commutator of the multiplication \(D\):

**Definition 3.2.** Let \((C, D, [,]_C)\) a torsion algebra. Its elements are called \(M\)-vector fields. The set of \(M\)-functions is:

\[
A = \{ \phi \in \text{End}_k(C) | D_{\phi(x)} Y = \phi(D_X Y) \}
\]

The multiplication of \(M\)-functions is the natural composition of \(k\)-endomorphisms in \(\text{End}_k(C)\).

Note that the multiplication of \(M\)-functions is an internal operation:

\[
D_{(\phi \circ \psi)x} = D_{\phi(\psi x)} = \phi D_{\psi x} = \phi \psi D_x
\]

and that the set of \(M\)-vector fields \(C\) is a left \(A\)-module.

The multiplication \(D\) defines a \(k\)-linear map:

\[
\tilde{D} : C \rightarrow \text{End}_k(C)
\]

called the **left regular quasi-representation** of \((C, D)\) as a non-associative algebra.

We will test the notions introduced against the simple geometric example of the real line.

**Example 3.2.** In the context of example [3.1], multiplication of vector fields \(C \cong C^\infty(V)\) by functions is just the regular left representation \(L : C^\infty(V) \rightarrow \text{End}(C^\infty(V))\) of \(C^\infty(V)\) (in the usual sense):

\[
(fX_g) = f(g \partial_t) = (fg) \partial_t = X_{fg}
\]

Moreover the \(M\)-functions \(A\) are naturally identified as \(A\)-functions \(C^\infty(V)\). Indeed, if \(\phi \in \text{End}_k(C)\) “left commutes” with \(D\):

\[
D_{\phi(f)g} = \phi(D_{fg})
\]

then (see equation [3.1]):

\[
\phi(f)(g' + g\Gamma) = \phi(f(g' + g\Gamma)).
\]

But it is clear that \(g' + g\Gamma = h\) has a solution for any \(h \in C^\infty(V)\). Thus \(\phi(fh) = \phi(f)h\), so \(\phi(h) = \phi(1)h\) and \(\phi\) corresponds to left multiplication by \(\phi(1)\).

We note that \(\phi\) is a function iff \(\tilde{D} \circ \phi = L_{\phi} \circ \tilde{D}\), where:

\[
L : \text{End}(C) \rightarrow \text{End}(\text{End}(C))
\]
is the regular representation of the associative algebra \((\text{End}(C), \circ)\). In other words, \(\tilde{D}\) intertwines \(\phi\) and \(L_\phi\):
\[
\tilde{D} \circ \phi = L_\phi \circ \tilde{D}
\]

We will interpret the \(\mathcal{M}\)-vector fields as derivations on \(A\). Let \(X \in C\) and \(\phi \in A\).

**Lemma 3.1.** Any two of the following conditions imply the third:

(i) The action of \(C\) on functions is defined by:
\[
(X \cdot \phi)(Y) = [X, \phi(Y)]_C - \phi([X, Y]_C), \quad Y \in C
\]
\[(3.2)\]

(ii) \(D\) is a derivation law:
\[
D_X(\phi Y) = (X \cdot \phi)Y + \phi D_X Y, \quad (X \cdot \phi = [D_X, \phi])
\]
\[(3.3)\]

(iii) The torsion is \(A\)-bilinear.

**Proof.** Note that the torsion \(T\) is skewsymmetric and:
\[
T(X, \phi Y) = D_X(\phi Y) - D_\phi Y X - [X, \phi Y]
\]
\[
= \{D_X(\phi Y) - \phi D_X Y - (X \cdot \phi) Y\} + \phi T(X, Y)
\]
\[
+ \{(X \cdot \phi) Y + \phi[X, Y] - [X, \phi Y]\}
\]

Now it is clear that any two conditions imply the third:
\[
T(X, \phi Y) - \phi T(X, Y) = \{D_X(\phi Y) - \phi D_X Y - (X \cdot \phi) Y\}
\]
\[
+ \{(X \cdot \phi) Y + \phi[X, Y] - [X, \phi Y]\}
\]

We will adopt the second condition in lemma 3.1 as a definition for the action of a vector field on a function.

**Definition 3.3.** A vector field \(X \in C\) acts on a function \(\phi \in A\) by:
\[
X \cdot \phi = [D_X, \phi]
\]
\[(3.4)\]

Note that it measures the failure of \(D\) to be right \(A\)-linear.

**Proposition 3.1.** The \(\mathcal{M}\)-vector fields act as (external) derivations on \(A\).

**Proof.** If \(\phi\) and \(\psi\) are \(\mathcal{M}\)-functions, then:
\[
(X \cdot (\phi \circ \psi))Y = D_X(\phi(\psi(Y))) - (\phi \circ \psi)D_X Y
\]
and
\[
(X \cdot \phi) \circ \psi(Y) + \phi \circ (X \cdot \psi)(Y) = D_X(\phi(\psi(Y))) - \phi(\psi(D_X Y))
\]

For \(X \cdot \phi\) to be again a function, so that elements of \(C\) act as derivations, we note the following:
Lemma 3.2. The following conditions are equivalent:

(i) For any \( X \in C \) and \( \phi \in A \), \( X \cdot \phi \) is an \( M \)-function.

(ii) The associator \( \alpha \) of \( D \) is \( A \)-linear in the first two variables.

Proof. Recall that the associator is:

\[
\alpha(X, Y, Z) = (XY)Z - X(YZ) = D_{D_X Y}Z - D_X D_Y Z
\]

where multiplicative notation was alternatively used. The following are equivalent:

\[
D_{(X \cdot \phi)Y}Z = (X \cdot \phi)D_Y Z
\]

\[
D_{D_X \phi Y}Z - \phi D_{D_X Y}Z = D_X(\phi D_Y Z) - \phi D_X D_Y Z
\]

Since:

\[
D_{D_X Y}Z = D_{\phi D_X Y}Z = \phi(D_{D_X Y}Z)
\]

the linearity of the associator in the first variable is clear:

\[
\alpha(\phi X, Y, Z) = D_{\phi X Y}Z - \phi(D_X D_Y Z)
\]

\[
= \phi\alpha(X, Y, Z).
\]

A direct computation proves the \( A \)-linearity in the second variable:

\[
\alpha(X, \phi Y, Z) = D_{D_X \phi Y}Z - D_X D_{\phi Y}Z = D_{D_X \phi Y}Z - D_X(\phi D_Y Z)
\]

\[
= D_{(X \cdot \phi)Y + \phi D_X Y}Z - ((X \cdot \phi)D_Y Z + \phi D_X D_Y Z)
\]

\[
= \phi(D_{D_X Y}Z - D_X D_Y Z) = \phi\alpha(X, Y, Z).
\]

Definition 3.4. A torsion algebra \( \mathcal{M} = (C, D, [\cdot, \cdot]_C) \) is called regular if the torsion and the associator are \( A \)-bilinear in the first two variables.

Since condition (ii) of lemma 3.1 holds by definition in a torsion algebra, any of the other two imply the third. Now we can easily prove the following:

Theorem 3.1. Let \( \mathcal{M} = (C, D, [\cdot, \cdot]_C) \) be a regular torsion algebra. Then, for all \( X, Y \in C \) and \( \phi \in \text{End}_k(C) \) an \( \mathcal{M} \)-function:

1. \( X \cdot \phi \) is an \( \mathcal{M} \)-function.
2. \( X \) acts as a derivation on \( \mathcal{M} \)-functions.
3. The associator of \( D \) is \( A \)-linear in the first two variables.
4. \( [X, \phi Y] = \phi[X, Y] + (X \cdot \phi)Y \)
5. \( D \) is a connection on \( \mathcal{M} : D_X(\phi Y) = (X \cdot \phi)Y + \phi D_X Y \).
6. The torsion \( T \) is \( A \)-bilinear.
Proof. (1) holds by assumption.
Since by definition the conditions (ii) and (iii) from lemma 3.1 hold, (2) follows.
The other statements are clear from lemma 3.1 and 3.2.

In what follows we will omit the $\mathcal{A}/\mathcal{M}$ prefix.

3.3. Differential Forms. The exterior derivative will be defined as the differential of the Chevalley-Eilenberg quasi-complex.

In the context of non-associativity, it is natural to relax the action requirement as well. The associator (algebraic point of view) may be interpreted as a curvature (geometric point of view) or as a monoidal structure through categorification (failure to be a morphism) (see []). Now, an action of $\mathcal{A}$ on $\mathcal{M}$ is a morphism $\rho : \mathcal{A} \to \text{End}(\mathcal{M})$ and an associative multiplication is an action $\mathcal{A} \to \text{End}(\mathcal{A})$.

Definition 3.5. A quasi-action of $\mathcal{A}$ on $\mathcal{M}$, in the category of $k$-modules, is a $k$-linear map $L : \mathcal{A} \to \text{End}_k(\mathcal{M})$.

Let $\mathcal{M}$ be a left $\mathcal{A}$-module with a derivation law $D^\mathcal{M} : \mathcal{C} \to \text{End}_k(\mathcal{M})$.

Definition 3.6. The $\mathcal{M}$-valued $\mathcal{M}$-differential forms are defined as usual:

$$\Omega^p(\mathcal{M}, \mathcal{M}) = \{\omega : \mathcal{C} \times \ldots \times \mathcal{C} \to \mathcal{M} | \omega \text{ alternating and } \mathcal{A} - \text{multilinear}\}$$

Then $\Omega(\mathcal{M}, \mathcal{M})$ is just the alternate part of the Hochschild cochains $\mathcal{C}^\bullet(\mathcal{C}; \mathcal{M})$ with coefficients in $\mathcal{M}$ (Chevalley cochains).

To define first the Hochschild quasi-complex ($d^2$ not necessarily zero), consider the following $\mathcal{C}$-quasi-bimodule structure on $\mathcal{M}$:

$$\lambda : \mathcal{C} \times \mathcal{M} \to \mathcal{M}, \quad \lambda(X, u) = D^\mathcal{M}_X u$$

$$\rho : \mathcal{M} \times \mathcal{C} \to \mathcal{M}, \quad \rho(u, X) = -D^\mathcal{M}_X u$$

where $\lambda^{op}$ is the opposite quasi-action using the signed braiding. In the associative case with $\mathcal{M} = \mathcal{A}$, the use of the signed braiding gives $\mathcal{M}$ a structure of $(\mathcal{A}, \mathcal{A}^{op})$ supersymmetric bimodule structure: $am = -ma$.

Instead of the Hochschild quasi-complex derived from the associated graded Lie algebra $(\mathcal{C}^\bullet(\mathcal{C}), [, ])$, with $d\omega = [\mu, \omega] = \mu \circ \omega - (-1)^p \omega \circ \mu$, consider the Hochschild quasi-complex $\mathcal{C}^p(\mathcal{C}; \mathcal{M}) = \text{Hom}_\mathcal{A}(\mathcal{C}^p, \mathcal{M})$, $p \geq -1$ of the Lie algebra $(\mathcal{C}, [, ]_\mathcal{C})$ as a non-associative algebra, with coefficients the $\mathcal{C}$-quasi-bimodule $\mathcal{M}$:

$$d\omega = (-1)^p((\lambda, \rho) \circ \omega - (-1)^p \omega \circ [, ]_\mathcal{C})$$

$$d\omega(a_1, ..., a_{p+2}) = \lambda(a_1, \omega(a_2, ..., a_{p+2})) - \omega([a_1, a_2]_\mathcal{C}, ..., a_{p+2}) + ... + (-1)^p \rho(\omega(a_1, ..., a_{p+1}), a_{p+2})$$

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Then for $u \in C^{-1} = M$ and $\omega \in C^0$:

\[
\begin{align*}
  du(X) &= \lambda(X, u) - \rho(u, X) = 2D_X u \\
  d\omega(X, Y) &= D_X \omega(Y) - D_Y \omega(X) - \omega([X, Y]_C) \\
  ddu(X, Y) &= 2(D_X D_Y u - D_Y D_X u - D_{[X,Y]} u) = K(X,Y) u
\end{align*}
\]

To obtain the usual formulas in geometry, consider the alternating part $\Lambda^\bullet(A; M)$ of the above complex, and project the differential $d_{Ch} = Alt \circ d$.

**Definition 3.7.** $(\Lambda^\bullet(A; M), d_{Ch})$ is called the associated Chevalley-Eilenberg quasi-complex of $C$ with coefficients in $M$.

Then, for example:

\[
  d_{Ch}\omega(X, Y, Z) = \sum_{cycl} D_X \omega(Y, Z) - \omega([X, Y], Z)
\]

### 3.4. The Lie Derivative

Let $M = (C, D, [\cdot, \cdot]_C)$ be a regular torsion algebra. Consider the $A$-module $M = A$ and the corresponding differential forms $\Omega(M)$. The **canonical derivation law** on $A$ is:

\[
  D_X \phi = X \cdot \phi.
\]

Extend as usual the Lie derivative defined on functions and vector fields as a derivation on the tensor algebra commuting with contractions. It is easy to see that it is an internal operation. For example, if $\omega : C \to A$ is a 1-form, then $(L_X \omega)(Z) = D_X \omega(Z) - \omega([X, Z])$ is $A$-linear.

An exterior differential on forms $\Omega(A; F)$ is defined by the homotopy formula: $L_X = di_X + i_X d$. The usual explicit formula holds for $d$. It coincides with $d_{Ch}$ defined above.

### 4. Examples

We will start by investigating torsion algebras for which $T = 0$.

Let $(C, D)$ be a unital associative algebra. Consider the corresponding Lie algebra structure: $[X, Y]_C = D_X Y - D_Y X$. Then the torsion is $T = 0$. The associator is zero and $(C, D, [\cdot, \cdot])$ is a regular torsion algebra. If $\phi \in End_k(C)$ is a function, then $D_{\phi(X)} Y = \phi(D_X Y)$ in multiplicative notation is just $\phi(X) Y = \phi(X Y)$. Thus $\mathcal{M}$-functions are left multiplication by elements of $C$ and the algebra of functions is isomorphic to the initial algebra. The morphism $C \to Der(A)$, realizing $C$ as derivations of $A$, is the usual Lie algebra representation.

Thus we have the following:

**Proposition 4.1.** Any associative algebra $(C, \mu)$ has a natural structure of a regular torsion algebra. The algebra $C$ is isomorphic with the algebra of $\mathcal{M}$-functions.
4.1. **Hochschild Pre-Lie Algebra.** Let $V$ be a $k$-module and $C = (C^\bullet(V), \varpi)$ the corresponding graded pre-Lie algebra, where $\varpi$ denotes the Gerstenhaber composition operation \cite{Ger}. Recall that $[x, y] = x \varpi y - (-1)^{pq} y \varpi x$, defines a Lie bracket, where $x \in C^p(V) = \text{Hom}_k(V^{p+1}, V)$ and $y \in C^q(V) = \text{Hom}_k(V^{q+1}, V)$ are two Hochschild cochains.

Then $C$ is a torsion algebra with $D = \varpi$ and $T = D - [\, , ] = 0$.

A $k$-endomorphism $\phi \in \text{End}_k(C)$ is an $\mathcal{M}$-function iff:

$$D\phi xy = \phi(Dxy) \quad (4.1)$$

and an argument similar to the case of associative algebras gives $\phi = L_{\phi(1)}$, where $1 = \text{id}_V$ and $L : C \to (\text{End}_k(C), \circ)$ is the regular quasi-representation. Note that $\varpi$ is not associative and $L_{\text{id}_V}$ is only a projector on the even part of $C$.

Denote $\phi(1)$ by $f$. Then equation (4.1) holds iff:

$$D_f \varpi xy = f\varpi(Dxy)$$

i.e. $(f\varpi)x\varpi y = f\varpi(x\varpi y)$ for any $x, y \in C^\bullet$. It is easy to see that this is true iff $f \in C^0(V)$ and thus the set of functions is $A = C^0(V)$.

The composition of functions is composition of $k$-endomorphisms:

$$L_f \circ L_g = L_{f \varpi g}, \quad f, g \in A.$$ 

since $\varpi$ reduces to the usual composition $\circ$ of $k$-endomorphisms of $C^0(V) = \text{End}_k(V)$.

Thus we have:

**Theorem 4.1.** Let $V$ be a $k$-module and $(C^\bullet(V), \varpi)$ the corresponding pre-Lie algebra. Then:

(i) $C = (C^\bullet(V), D, [\, , ])$ is a zero torsion algebra where $D = \varpi$ is called the canonical connection.

(ii) The algebra of functions of $C$ is $A = (C^0(V), \varpi)$, i.e. $(\text{End}_k(V), \circ)$.

(iii) $C$ acts through exterior derivations on $A$:

$$(x \cdot L_f)(y) = D_x(L_f(y)) - L_f(D_xy), \quad x, y, f \in C$$

where $L : C \to \text{End}_k(C)$ is the regular left quasi-representation of $C$.

We note that the failure to be a regular torsion algebra comes from the $A$ non-linearity of the associator. Recall that the associator is graded skew-symmetric in the last two variables (2.1). Thus being a regular torsion algebra would be equivalent to $\alpha$ being an $A$-multilinear form.

4.2. **Poisson Algebras.** Let $(C, \cdot, \{ , \})$ be a Poisson algebra, with $D = \cdot$ commutative and associative and Lie bracket $[\, , ]$ being the Poisson bracket $\{ , \}$. Then $(C, D, [\, , ])$ is a torsion algebra with torsion $T = -[\, , ]$. Since $D$ is associative, its algebra of $\mathcal{M}$-functions $A$ is isomorphic to $C$, in a manner similar to the associative algebra case. In this way, a Poisson algebra is not a regular torsion algebra.

If $D = [\, , ] = \{ , \}$ then it becomes a zero torsion algebra, but it is not clear what the algebra $(A, \circ)$ of $\mathcal{M}$-functions is and what is the relation with the multiplication of functions.
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