Nonparaxial Cartesian and azimuthally symmetric waves with concentrated wavevector and frequency spectra

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Abstract
In this paper, we develop a theoretical analysis to efficiently handle superpositions of waves with concentrated wavevector and frequency spectra, allowing an easy analytical description of fields with interesting transverse profiles. First, we analyze an extension of the paraxial formalism that is more suitable for superposing these types of waves, as it does not rely on the use of coordinate rotations combined with paraxial assumptions. Second, and most importantly, we leverage the obtained results to describe azimuthally symmetric waves composed of superpositions of zero-order Bessel beams with close cone angles that can be as large as desired, unlike in the paraxial formalism. Throughout the paper, examples are presented, such as Airy beams with enhanced curvatures, nonparaxial Bessel–Gauss beams and circular parabolic-Gaussian beams (which are based on the Cartesian parabolic-Gaussian beams), and experimental data illustrates interesting transverse patterns achieved by superpositions of beams propagating in different directions.

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1. Introduction
Appropriate superpositions of waves propagating in different directions may create interesting and useful interference patterns in the transverse plane that keep their shape up to a certain longitudinal distance, determined by the interval in which the waves interfere significantly. Consequently, this range is higher when the waves’ wavevector spectra are concentrated around their propagation directions, implying larger spot sizes and slower diffraction rates. Since these directions can make large angles with respect to the longitudinal axis, the paraxial wave equation may not describe the propagation of the superposition properly.

In principle, each wave could be analyzed separately using the paraxial formalism in a system of coordinates whose z-axis coincides with its propagating direction. Applying a coordinate rotation would then express the wave as a function of the desired coordinates. However, this procedure is not straightforward and results in complicated expressions, as the rotations mix the transverse and longitudinal nonrotated coordinates.

To handle superpositions of waves with concentrated spectra in an analytically simpler way, we analyze and apply an extension of the paraxial formalism that deals with combinations of plane waves whose wavevectors are close to any desired direction without relying on coordinate rotations. The resulting expressions do not mix transverse and longitudinal coordinates while still incorporate the inclination of the plane waves very clearly through parameters. If desired, the method can also be straightforwardly extended to include higher-order corrections, so it is not limited to the paraxial approximation’s accuracy.

As the main contribution of this work, we then leverage these results to develop a novel analytical treatment for...
azimuthally symmetric waves with concentrated spectra, that is, waves composed of superpositions of zero-order Bessel beams with close cone angles. However, in contrast to the paraxial formalism, this analysis encompasses cone angles as large as desired and provides simple expressions for this class of nonparaxial waves. In particular, similarities with the uni-dimensional\(^2\) Cartesian waves allow us to build azimuthally symmetric versions of them, which can be used to derive new types of waves. As examples, we provide analytical expressions for nonparaxial Bessel–Gauss (BG) beams and for azimuthally symmetric versions of parabolic-Gaussian beams (which we will refer to as circular parabolic-Gaussian beams).

Throughout this work, we also present experimental demonstrations and comparisons to numerically calculated Rayleigh–Sommerfeld diffraction integrals to corroborate the validity and accuracy of the theoretical results.

This paper is organized as follows: from section 2 to section 5, we present the fundamental analysis of single Cartesian waves with concentrated spectra, which includes the integral formulation and the wave equation (section 2), the diffraction integral (section 3), the extension of the Helmholtz–Gauss beams expressions (section 4) and the basic vectorial expressions for linearly polarized waves (section 5); then, in section 6, we present some examples of solutions; in section 7, we exemplify how the superpositions of these waves can generate more complex patterns and, in section 8, we experimentally demonstrate some profiles; then, in section 9, we analyze the properties of azimuthally symmetric waves with concentrated spectra and, in section 10, we show some examples of solutions and compare the precision of the analytical expressions with numerical calculations; finally, in section 11, we present our conclusions.

2. Integral formulation and wave equation

2.1. Bidimensional case

Let \(\Psi(x, y, z, t)\) be an arbitrary solution of the homogeneous scalar wave equation \((\nabla^2 - \frac{1}{c^2} \partial^2_t)\Psi = 0\). It can be represented by a superposition of plane waves with different complex amplitudes, wavevectors \(k = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}\) and angular frequencies \(\omega\), given that \([k] = k(\omega) = \omega/c\), where \(c\) is the speed of light in the medium. If we consider only waves that propagate in the +\(\hat{z}\) direction, \(k_z = \sqrt{k^2(\omega) - (k_x^2 + k_y^2)}\) and \(\Psi(x, y, z, t)\) can be written as

\[
\Psi(x, y, z, t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \hat{S}(k_x, k_y, \omega) e^{-i\omega t} e^{i(k_x x + k_y y + \sqrt{k^2(\omega) - (k_x^2 + k_y^2)} z)},
\]

where \(\hat{S}(k_x, k_y, \omega)\) is the spectrum. If it is concentrated around \((k_x, k_y, \omega) = (k_x^0, k_y^0, \omega_0)\), the plane waves have wavevectors that vary only slightly around \(k_0^0 = k_{x0} \hat{x} + k_{y0} \hat{y} + k_{z0} \hat{z}\), where \(k_{0\omega} = \frac{k(\omega_0)}{c} = \omega_0/c\). Mathematically, this means that \([\hat{S}(k_x, k_y, \omega)]\) is significant only in the region where \(|k_x^0/k_{x0}| \ll 1, |k_y^0/k_{y0}| \ll 1\) and \(|u/\omega_0| \ll 1\) with \(k^0_x = k_x - k_{x0}, k^0_y = k_y - k_{y0},\) and \(u = \omega - \omega_0\) being defined as the deviation variables.\(^3\) In this region, \(k_z\) can be well approximated by two consecutive Taylor expansions: first, we use a second-order expansion in \(k^0_x\) and \(k^0_y\), resulting in coefficients that still depend on \(\omega\) due to \(k(\omega)\) appearances; second, we perform a first order approximation in \(\omega\) only in the most significant term, which is \(\sqrt{k^2(\omega) - k^2_{z0}},\) where \(k^2_{z0} = k_x^2 + k_y^2\). In all the other terms, we take \(k(\omega) \approx k_0^\omega\). Then, performing the integrals of equation (1) on the deviation variables results in

\[
\Psi(x, y, z, t) \approx e^{-i\omega_0 t} e^{ik_{x0} x + ik_{y0} y + \sqrt{k^2(\omega_0) - k^2_{z0}} z} A(x, y, z, t),
\]

\[
A(x, y, z, t) \equiv \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y S(k'_x, k'_y, u) e^{-i\omega_0 t} e^{ik_{x0} x + ik_{y0} y + \sqrt{k^2(\omega_0) - k^2_{z0}} z},
\]

where \(S(k'_x, k'_y, u) = \hat{S}(k'_x + k_x^0, k'_y + k_y^0, u + \omega_0)\), the coefficients of the expansion are

\[
a_x \equiv \frac{k_{x0}}{\sqrt{k^2_{00} - k^2_{z0}}}, \quad a_y \equiv \frac{k_{y0}}{\sqrt{k^2_{00} - k^2_{z0}}},
\]

\[
b \equiv \frac{1}{2} \frac{k^2_{00}}{(k^2_{00} - k^2_{z0})^2}, \quad d \equiv \frac{k_{x0} k_{y0}}{(k^2_{00} - k^2_{z0})},
\]

\[
e \equiv \frac{k_0}{\sqrt{k^2_{00} - k^2_{z0}}},
\]

and the deviation of \(k_z\) with respect to \(k_{z0} \equiv \sqrt{k^2_{00} - k^2_{z0}}\) is

\[
k'_z = -a_x k'_x - a_y k'_y - d(k'_x k'_y) - b(k'_x^2 + k'_y^2) + \frac{e}{c} u. \quad (5)
\]

Equation (2) shows that \(\Psi(x, y, z, t)\) is approximately a plane wave with wavevector \(k_0\) and frequency \(\omega_0\) modulated by an envelope \(A(x, y, z, t)\) which, according to equation (3), is composed of a superposition of plane waves with wavevectors \(k'_x \hat{x} + k'_y \hat{y} + k'_z \hat{z}\) and angular frequencies \(u\), with the relation among \(k'_x, k'_y, k'_z, u\) and \(\omega\) given by equation (5). Since \([k'_x], [k'_y], [k'_z]\) and \(|u|\) are small compared to \([k_{x0}], [k_{y0}], [k_{z0}]\) and \(\omega_0\), respectively, \(A(x, y, z, t)\) has slow variations in \(x, y, z, t\). Mathematically, this means

\(^3\) Additionally, we may assume that \(k_{x0}^2 + k_{y0}^2 < k_{z0}^2\), so that all the waves are propagating. In this case, even though we keep the limits of the integrals in \(k_x\) and \(k_y\) from \(-\infty\) to \(\infty\), the concentration of the spectrum implies that there are no significant evanescence waves in the superposition.

\(^4\) If desired, higher-order corrections can be easily included to improve accuracy, even though the second-order expansion in \(k'_x\) and \(k'_y\) and the first order approximation in \(\omega\) are usually enough.
with respect to the problem at hand. Further, we find it convenient to introduce the notation
\[
\frac{\partial A}{\partial x_i^2} = \kappa_{x_i} A, \quad \frac{\partial^2 A}{\partial x_i^2} = \kappa_{x_i} \frac{\partial A}{\partial x_i}.
\]
(6a)

\[
\frac{\partial A}{\partial t} = \left| \omega_0 A \right|, \quad \frac{\partial^2 A}{\partial t^2} = \omega_0 \frac{\partial A}{\partial t}.
\]
(6b)

where \(x_i\) stands for any spatial coordinate.

The differential equation satisfied by \(A(x, y, z, t)\) is uniquely defined by the relation among the frequency and wavevector components of its composing plane waves, given by equation (5). Each differentiation of \(A(x, y, z, t)\) with respect to a variable implies a multiplication of its spectrum by a factor proportional to the corresponding reciprocal variable, resulting in the correspondences:

\[
\frac{1}{i} \frac{\partial}{\partial x_i} \leftrightarrow k_i', \quad -\frac{\partial^2}{\partial x_i^2} \leftrightarrow k_i'^2, \quad -\frac{1}{i} \frac{\partial}{\partial t} \leftrightarrow u
\]

where \(x_i\) stands again for any spatial coordinate. Therefore, equation (5) is equivalent to

\[
\frac{\partial A}{\partial z} = \frac{\partial A}{\partial t} - \frac{\partial}{\partial x} \psi_1 A + \frac{\partial}{\partial y} \psi_2 A + \frac{dA}{dx} \frac{\partial^2 A}{\partial x^2}.
\]
(8)

where \(\psi_1 \equiv a_x \xi + a_y \eta\) and \(\psi_2 \equiv \xi \theta x + \eta \theta y\). If both \(k_{y0}\) and \(k_{x0}\) are zero, equation (8) reduces to the well-known paraxial wave equation \(\frac{\partial A}{\partial z} = -\frac{i}{c} \frac{\partial A}{\partial t} + \frac{1}{2c} \frac{\partial^2 A}{\partial x^2} A\), as in this case the superposed plane waves have wavevectors almost parallel to the z-direction.

2.2. Unidimensional case

If \(\Psi\) is a function of only one transverse coordinate, say \(\Psi(x, z, t)\), a similar analysis provides:

\[
\Psi(x, z, t) \approx e^{-i\omega_0' t} e^{i\theta' x} e^{i\xi' z} \Psi(x, z, t),
\]
(9)

\[
A(x, z, t) = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dk'' \mathcal{S}(k'', u) e^{-i\omega_0' t} e^{i\theta' x} e^{i\xi' z},
\]
(10)

\[
k'' = -a_x k_x - b k_y + \gamma u,
\]
(11)

\[
\frac{\partial A}{\partial z} = -\frac{\partial A}{c \partial t} - a_x \frac{\partial A}{\partial x} + i b \frac{\partial^2 A}{\partial x^2},
\]
(12)

with \(k_{y0} = k_{x0}\). It is also possible to write equation (12) in a dimensionless form by defining dimensionless constants and variables:

\[
\gamma \equiv \frac{k_{x0}}{k_{y0}}, \quad \alpha \equiv k_{y0}, \quad \beta \equiv \frac{k_{x0}}{k_{y0}},
\]
(13)

\[
s \equiv \frac{x}{x_0}, \quad \xi \equiv \frac{z}{k_{y0}} \left(1 - \gamma^2\right)^{1/2}, \quad \tau \equiv \frac{c}{k_{x0}} t,
\]
(14)

where \(x_0\) is a constant. The differential equation for \(A(x, \xi, \tau)\) then becomes

\[
\frac{\partial^2 A}{\partial \xi^2} + 2i \frac{\partial A}{\partial \xi} + \frac{2i\alpha(1 - \gamma^2)}{\partial \xi} + \frac{2i(1 - \gamma^2)}{\partial \tau} = 0.
\]
(15)

In addition, if we define the dimensionless wavevector \(k'_{y0} = k'_{x0} x_0\), equation (9) becomes

\[
\Psi(s, \xi, \tau) \approx e^{-i\omega_0' t} e^{i\theta' x} e^{i\xi' z} \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d(k''_x, u) \mathcal{S}(k''_x, u) e^{-i\omega_0' t} e^{i\theta' x} e^{i\xi' z}\right],
\]
(16)

where \(\mathcal{S}(k''_x, u) \equiv S(k''_x / x_0, u / x_0)\) is a concentrated spectrum around \((k''_x, u) = (0, 0)\).

2.3. Reduction to the paraxial wave equation

By changing variables, it is possible to reduce the general equation (8) to the paraxial wave equation, so we can leverage all we know about this equation to equation (8). The first order derivatives are eliminated if we use the shifted variables \(z' = x - a_x z, \xi = y - a_y z, \tau = t - \varepsilon z/c\), and \(\tau = \xi \frac{c}{a_x} \). Note that the shifts in the transverse coordinates express the inclination of the wavevector with respect to the z-direction and the shift in the temporal variable represents the wave’s travel time to arrive at the longitudinal position \(z\). In fact, this shows that any solution of equation (8) must be of the form \(A(x - a_x z, y - a_y z, t - \varepsilon z/c)\).

To eliminate the remaining mixed-derivative term \(\partial_{x\tau} A\), we can use one of the following sets of transformations:

\[
\zeta = \frac{1}{\sqrt{4b^2 - d^2}} (2b x - d y), \quad \eta = \varepsilon \zeta.
\]
(17a)

\[
\eta = \frac{1}{\sqrt{2(4b^2 - d^2)}} \left[2b \tau - (d + \sqrt{4b^2 - d^2}) \zeta\right],
\]
(18a)

so that equation (8) becomes

\[
\frac{\partial A}{\partial \zeta} = i b \left[\frac{\partial^2 A}{\partial \zeta^2} + \frac{\partial^2 A}{\partial \eta^2}\right]
\]
(19)

which is like a paraxial wave equation in the variables \(\zeta\) and \(\eta\). An additional scaling in \(\zeta\) and \(\eta\) reduces equation (19) precisely to the paraxial wave equation in the new scaled variables, which are \(\tilde{\zeta} = \zeta \sqrt{2b k_0}\) and \(\tilde{\eta} = \eta \sqrt{2b k_0}\). Note that a separable solution for \(A\) in the variables \(\tilde{\zeta}\) and \(\tilde{\eta}\) exists only when the ‘coupling factor’ \(d\) is zero. In the unidimensional case, equation (12) can be reduced to \(\partial_{x\tau} A = i b \partial_{x\tau} A\) or to the paraxial wave equation in the scaled variable \(\chi = \tilde{\xi} \sqrt{2b k_0}\).
Therefore, any solution of the paraxial wave equation can be extended to the case being considered, which is natural, since this approach should be analogous to a rotation of coordinates. Note, however, that the former is simpler, as it does not mix the longitudinal and transverse variables. Also, note that if a beam with envelope $A(x, y, z)$ satisfies equation (8), the pulse with envelope $\mathcal{A}(x, y, z) = A(x, y, z)B(t - \epsilon z/c)$ is also a solution of that equation for an arbitrary differentiable function $B(t - \epsilon z/c)$.

3. Diffraction integral

For the bidimensional case, it is evident from equation (3) that $A(x, y, 0, t)$ is a Fourier transform of $S(k_x', k_y', u)$. If we invert the transform and apply the result to equation (3) to eliminate $S(k_x', k_y', u)$, we get, after a few calculations, the following diffraction integral:

$$A(x, y, z, t) = \frac{1}{4\pi i \nu} \int_{-\infty}^{\infty} \frac{d\nu'}{1 - \frac{d^2}{4\nu^2}} \left[ i k_x' \frac{A(x', 0, t - \frac{\epsilon}{c} z)}{4b(z(1 - \frac{d^2}{4\nu^2})} \right. $$

$$\left. \times \exp \left( \frac{i}{4b(z(1 - \frac{d^2}{4\nu^2}))} \left( x - x' - a_x z \right)^2 \right. \right. $$

$$\left. \left. \times \left( y - y' - a_y z \right)^2 \frac{d}{b} (x - x' - a_x z)(y - y' - a_y z) \right) \right] \right)$$

(20)

For beams, that is, when $A = A(x, y, z)$, equation (20) is still valid if we remove the $t$ dependencies. As it should be, for $k_x = k_y = 0$, it reduces to the Fresnel diffraction integral. An alternative procedure to derive equation (20) is to write the Fresnel diffraction integral for the transformed variables $\zeta$ and $\eta$ and arrange the terms in $x - a_x z$ and $y - a_y z$.

For the unidimensional case, an analogous procedure starting with equation (10) results in

$$A(x, z, t) = \frac{1 - i}{2\sqrt{2\pi b z}} \int_{-\infty}^{\infty} \frac{d\nu}{1 - \frac{d^2}{4\nu^2}} \left[ i k_x' \frac{A(x', 0, t - \frac{\epsilon}{c} z)}{4b(z(1 - \frac{d^2}{4\nu^2})} \right. $$

$$\left. \times \exp \left( \frac{i}{4b(z(1 - \frac{d^2}{4\nu^2}))} \left( x - x' - a_x z \right)^2 \right. \right. $$

$$\left. \left. \times \left( y - y' - a_y z \right)^2 \frac{d}{b} (x - x' - a_x z)(y - y' - a_y z) \right) \right] \right)$$

(21)

4. Helmholtz–Gauss beams

For every monochromatic solution of the paraxial wave equation, there exists another solution that describes the propagation of the same beam apodized at $z = 0$ by a Gaussian function. These waves are generically called Helmholtz–Gauss beams and are particularly valuable to provide finite-energy versions of beams with infinite power flux, as is the case of Bessel beams (then resulting in BG beams) and other ideal nondiffractive beams.

Since $A(\zeta, \eta, z)$ satisfies equation (19), the paraxial wave equation with $(2k_0)^{-1}$ replaced by $b$, there exists a corresponding Helmholtz–Gauss beam with envelope $A_{HG}(\zeta, \eta, z)$ given by

$$A_{HG}(\zeta, \eta, z) = \frac{1}{\mu} \exp \left[ \frac{-q^2(\zeta^2 + \eta^2)}{\mu} \right] A \left( \frac{\zeta}{\mu}, \frac{\eta}{\mu}, \frac{z}{\mu} \right)$$

(22)

where $\mu = 1 + i4q^2b_z$.5

Using the transformation of equation (17), the envelope is expressed in the original Cartesian coordinates as

$$A_{HG}(x, y, z) = \frac{1}{\mu} \exp \left[ -\frac{q^2}{\mu \left( 1 - \frac{d^2}{4\nu^2} \right)} \right] \left[ (x - a_x z)^2 \right. $$

$$\left. + (y - a_y z)^2 \frac{d}{b} (x - a_x z)(y - a_y z) \right] \right) \times A \left( \frac{x}{\mu}, \frac{y}{\mu}, \frac{z}{\mu} \right)$$

(23)

which is therefore the generalized expression of Helmholtz–Gauss beams for the class nonparaxial beams we are considering. At $z = 0$, we have

$$A_{HG}(x, y, 0) = A(x, y, 0) \exp \left[ -\frac{q^2}{\left( 1 - \frac{d^2}{4\nu^2} \right)} \right] \times \left( x^2 + y^2 \frac{d}{b} xy \right)$$

(24)

which is a Gaussian apodization with an extra mixed-product term proportional to $d$.

5 Note that $\mu = 1 + 2iq^2/\nu_b$ for the paraxial wave equation and we have to replace $(2k_0)^{-1}$ with $b$ to be consistent with equation (19).

5. Nonparaxial linearly polarized electromagnetic field with concentrated spectra

Since any Cartesian component of an electric field $\tilde{E}$ obeys the scalar wave equation, a scalar wave with concentrated spectra can be assigned to one of them and the other field components can be obtained via Maxwell’s equations. In the calculations, we use integration by parts and apply equation (6) to write

$$\int (\partial A/\partial x) e^{i k_x z^2} dz \approx (\partial A/\partial x) \int e^{i k_x z^2} dz (with x_i being any spatial coordinate) and$$

$$\int (\partial A/\partial t) e^{-i \omega t} dt \approx (\partial A/\partial t) \int e^{-i \omega t} dt. Due to equation (6), we may also neglect second-order derivatives.
If we choose a linearly polarized electric field in the $x$-direction $\vec{E} = E_x \hat{x} + E_z \hat{z}$ with

$$E_x(x, y, z, t) = A(x, y, z, t)e^{-i\omega_0 t}e^{ik_y y}e^{ik_z z}e^{ik_0 z}$$

the $E_z$ component may be obtained from Gauss’s law ($\vec{\nabla} \cdot \vec{E} = 0$):

$$E_z(x, y, z, t) = A(x, y, z, t)e^{-i\omega_0 t}e^{ik_y y}e^{ik_z z} \left[-a_y A + \frac{i}{k_z} \left( \frac{\partial A}{\partial x} - a_x \frac{\partial A}{\partial z} \right) \right].$$

(25)

From Faraday’s law ($\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$), $\vec{B}$ can be calculated:

$$B_x = -\frac{i}{\omega_0} e^{-i\omega_0 t}e^{i(k_y y+k_z z)} \left[ -dk_z \left( \frac{1}{\omega_0} \frac{\partial A}{\partial t} + a_x \frac{\partial A}{\partial y} - a_y \frac{\partial A}{\partial z} + ik_z \frac{\partial A}{\partial z} \right) \right].$$

(26a)

$$B_y = -\frac{i}{\omega_0} e^{-i\omega_0 t}e^{i(k_y y+k_z z)} \left[ -dk_z \left( \frac{1}{\omega_0} \frac{\partial A}{\partial t} + a_x \frac{\partial A}{\partial y} + a_y \frac{\partial A}{\partial z} \right) \right].$$

(26b)

$$B_z = \frac{i}{\omega_0} e^{-i\omega_0 t}e^{i(k_y y+k_z z)} \left[ \frac{\partial A}{\partial y} + ik_y \left( \frac{1}{\omega_0} \frac{\partial A}{\partial t} \right) \right].$$

(26c)

6. Examples of Cartesian solutions

The first three solutions presented below are just tilted versions of their paraxial counterparts, that is, they can be regarded as paraxial waves propagating in the direction defined by $k_0$. Therefore, their elementary properties are not different from those of the paraxial solutions and hence are not explored in depth. Nevertheless, these waves are useful for building other solutions, as pointed out after their expressions are shown. On the other hand, the fourth wave presented has fundamental distinctions from its paraxial counterpart and thus is analyzed in more detail.

6.1. Gaussian pulse with Gaussian bidimensional spatial profile

If we choose a separable Gaussian spectrum in $k_x'$ and $k_y'$ and $u$: $S(k_x', k_y', u) = C \frac{r_x}{2\pi} e^{-\frac{k_x^2}{2r_x^2}} e^{-\frac{k_y'^2}{2r_y'^2}}$, where $C$ is an arbitrary constant and $r_x$, $r_y$, and $r_t$ are related to the spot sizes in $x$, $y$, and $t$, respectively, and use the integral result $\int_{-\infty}^{\infty} \exp(-px^2 \pm qx)dx = \frac{\sqrt{\pi}}{p} \exp(\frac{q^2}{4p})$, valid for $\text{Re}(p^2) > 0$. equations (2) and (3) give

$$\Psi_{\text{GB}}(x, y, z, t) = C e^{-i\omega_0 t}e^{ik_y y}e^{ik_z z} \sqrt{1 + \frac{i2\pi}{r_t^2}} \exp \left[ \frac{-i(x - a_x z)^2 - i(y - a_y z)^2 + i dz (x - a_x z)(y - a_y z)}{r_t^2 \left( \frac{1}{4} + i\frac{\pi}{r_t^2} \right)} \right].$$

(29)

If we want a monochromatic wave (i.e. a beam) with a spatial Gaussian profile and frequency $\omega_0$ instead of a pulse, we may just replace the factor $r_t \exp(-r_t^2 u^2/4)/(2\sqrt{\pi})$ in equation (29) by $\delta(u)$, where $\delta(\cdot)$ is the Dirac delta function. The resulting wave has the same expression of equation (29) but without the factor $\exp[-(t - iz/c)^2/r_t^2]$. A superposition of Gaussian beams of this type is used in section 8 to obtain waves with structured transverse intensity patterns.

6.2. Unidimensional Gaussian beam (GB)

For a single frequency ($\omega_0$) and a Gaussian spectrum in $k_z'$:

$$S(k_z', u) = C \delta(u) \frac{r_t}{2\sqrt{\pi}} e^{-\frac{k_z'^2}{2r_t^2}}$$

(30)

the resultant wave is a unidimensional Gaussian beam (GB):

$$\Psi_{\text{GB}}(x, y, z, t) = C e^{-i\omega_0 t}e^{ik_y y}e^{ik_z z} \sqrt{1 + \frac{i4\pi}{r_t^2}} \exp \left[ \frac{-i(x - a_x z)^2}{r_t^2 \left( 1 + i\frac{4\pi}{r_t^2} \right)} \right].$$

(31)

This solution is important as a basis for building a non-paraxial version of a BG beam, as shown in section 10.1.

6.3. Nonparaxial Cartesian parabolic-Gaussian (PG) beams

The solutions of the paraxial wave equation known as Cartesian parabolic-Gaussian (PG) beams were introduced by Bandres et al in [3] and further analyzed in [4]. Their
envelope \( \psi_A(x, z) \) is described by

\[
\psi_A(x, z) = \psi_0(z) \exp \left[ \frac{ik}{4} \left( \frac{1}{p} - \frac{1}{q} \right) x^2 \right] \\
\times P_i \left( \sqrt{\frac{k}{p - q}} x \right)
\]

with

\[
p = p_0 + z, \quad p_0 = \frac{zR}{n + i}.
\]

\[
q = q_0 + z, \quad q_0 = -p_0^*.
\]

\[
\psi_0(z) = 2^{1/4 - n/2} \left( \frac{q/q_0} {p/p_0} \right)^{i\nu/2 - 1/4} (p/p_0)^{-\nu/2 + 1/4},
\]

\[
P_i(x) = \left( x e^{i\nu/4} e^{-x^2/4} \right) \exp \left[ -\frac{i\nu^2}{4} + \frac{ix^2}{2} \right].
\]

where \( z_R = kn_0^2/2 \) is the Rayleigh distance, \( r_0 \) is the waist of the Gaussian envelope, \( n \) is 1/2 for even beams and 3/2 for odd beams, \( \nu \) is the order of the parabolic cylinder function. \( P_i(x) \) is the confluent hypergeometric function and \( n \in \mathbb{R} \) is a real parameter. At \( z = 0 \), it becomes

\[
\psi_A(x, z = 0) = \frac{1}{2^{\nu/2 - 1/2}} \exp \left[ -\frac{x^2}{r_0^2} \right] P_i \left( \frac{2\sqrt{\nu}}{r_0} x \right).
\]

To obtain the nonparaxial version of this envelope, it is only necessary to replace \( x \) by \( (x - ac)/\sqrt{2\hbar k_0} \), as explained in section 2.3. This solution is important as a basis for building a nonparaxial and azimuthally symmetric version of the PG beam, as shown in section 10.2.

Although the previous nonparaxial examples are just tilted versions of the usual paraxial beams, they are useful for describing superpositions of beams (as commented in section 7) and for generating new expressions for azimuthally symmetric waves (as shown in section 9 and exemplified in section 10). However, it is also possible to use the presented formalism to obtain waves with enhanced characteristics with respect to their paraxial counterparts, as illustrated in the next example.

### 6.4. Nonparaxial Airy beam

The spectrum

\[
\mathcal{S}(k_x' u) = \frac{\delta(u)}{2\pi} \exp \left[ i \left( \frac{k_x'^3}{3} + (\beta + i\sigma)k_x'^2 + \partial k_x'^2 \right) \right]
\]

with free-to-choose real parameters \( \beta, \partial \) and \( \sigma \) results in a nonparaxial Airy beam. \( \sigma \) causes a Gaussian apodization of the spectrum, which is manifest in space as an exponential apodization of the Airy spatial profile at \( z = 0 \). As in the paraxial case, the spectrum \( \mathcal{S}(k_x' u) \) is not concentrated unless the beam is apodized.

Using the integral result \( \int_{-\infty}^{\infty} \exp \left[ i \left( \frac{t^3}{3} + vt^2 + wt \right) \right] dt = 2\pi \exp \left[ i \left( \frac{v^3}{3} - vw \right) \right] A_i(w - v^2) \) [5], the expression of the beam becomes

\[
\Psi_A(s, \xi, t) = e^{-i\nu} e^{\frac{ik}{2} \left( 1 - \gamma^2 \right)} A \left( s + \vartheta \right) \\
- \beta^2 + \sigma^2 - 2i\beta\sigma + \xi (\beta + i\sigma - \alpha(1 - \gamma^2)) + \frac{\xi^2}{4}
\]

\[
\times \exp \left[ i \left( \frac{2}{3} (\beta + i\sigma - \xi \gamma^2) - (\beta + i\sigma - \xi \gamma^2) \right) \right] (s - \alpha(1 - \gamma^2) \xi + \vartheta).\]

Denoting \( \delta \approx 1.01879 \), the parabolic trajectory of the beam’s position of maximum amplitude is

\[
s + \vartheta = -\beta^2 + \sigma^2 - 2i\beta\sigma + \xi (\beta + i\sigma - \alpha(1 - \gamma^2)) + \frac{\xi^2}{4} = -\delta
\]

since \( x = \delta \) is where \( \text{Ai}(\delta) \) has its maximum value. Therefore, \( \beta \) controls the linear coefficient and \( \vartheta \), the independent coefficient. If we choose \( \sigma = 0 \), \( \beta = \alpha(1 - \gamma^2) \) and \( \vartheta = -\delta + \beta^2 \), the trajectory equation becomes simply

\[
s = \frac{\xi^2}{4} \Rightarrow x = \frac{1}{4k_x'n_0^3} \left( \frac{z^2}{1 - \gamma^2} \right)^3.
\]

While \( x_0 \) controls the spot size of the beam, \( k_{s_0} \) can be used to choose a desired curvature for its trajectory. In this way, we can obtain Airy beams with much steeper curvatures than it would be possible in the usual paraxial regime, in which \( k_{s_0} = 0 \). To illustrate this, figure 1 shows the changes in the beam’s parabolic trajectory for \( x_0 = 50 \mu m \) when \( k_{s_0} \) is varied. As we can see, the higher \( k_{s_0} \) is, the faster the beam curves.

6. To interpret the trajectory this way, we have to ignore the imaginary terms, although they are kept in equation (40).
7. Symmetric superposition of Cartesian waves

As explained in the introduction, the basic goal of the analysis so far is to provide a simple analytic way to describe the propagation of a superposition of waves traveling in different directions, thus composing interesting interference patterns in the transverse plane. Although these waves obey different approximate equations, as their dissimilar wavevector spectrum has peaks concentrated around both $+k_{x_0}$ and $-k_{x_0}$, they are all approximate solutions of the homogeneous scalar wave equation, which implies that their expressions can be legitimately combined.

Let us take as an example the superposition of two equal unidimensional beams with the same initial envelope $A(x, z = 0)$ traveling in opposite directions so that the resulting wavevector spectrum has peaks concentrated around both $+k_{x_0}$ and $-k_{x_0}$. Applying the diffraction integral (equation (21)) for each beam and combining the results, we have

$$
\Psi(x, z, t) = e^{-i \omega t} e^{ik_z z} \left( \frac{1}{2\pi b z} \right) \left( 1 - i \right) \frac{a_z^2}{4b^2} \int_{-\infty}^{+\infty} dx' A(x', 0) \cos \left( k_{x_0} x - \frac{a_z (x - x')}{2b} \right),
$$

(42)

where $a_z$ is related to $+k_{x_0}$. At the initial plane $z = 0$, the superposition of the two patterns gives

$$
\Psi(x, z = 0, t) = 2e^{-i \omega t} A(x, 0) \cos(k_{x_0} x)
$$

(43)

showing that the slowly varying envelope is modulated by the fast-oscillating function $\cos(k_{x_0} x)$. Therefore, the spot radius $\Delta x_0$ of the resulting beam is determined by a quarter of this cosine period, that is, $\Delta x_0 = \pi/(2k_{x_0})$.

In any superposition, the resulting field depth $Z$ can be estimated by the distance within which the waves interfere significantly. In this simple case, we can apply a reasoning analogous to that of the upcoming section 9.4 to obtain

$$
Z \approx w_0 k_0/(|k_{x_0}| \sqrt{2b k_0}),
$$

where $w_0$ is a characteristic width of the envelope $A(x, z = 0)$.

8. Experimental results

To demonstrate that the proposed theoretical expressions can be used to easily describe a superposition of beams with concentrated spectra propagating in different directions, we experimentally generated two superpositions of Gaussian beams that result in illustrative interference patterns. The experimental setup is shown in figure 2 and used a reflective spatial light modulator (SLM) to generate the desired patterns.

A He–Ne laser ($\lambda \approx 632.8$ nm) is expanded and collimated by a system of lenses ($L_1$ and $L_2$) and then sent to the SLM, which is a LC-R1080 model from HOLOEYE Photonics and possesses a display matrix of $1920 \times 1200$ with 8.1 $\mu$m pixel pitch. The device was used in amplitude modulation mode, with the analyzer and analyzer angles (0° and 90°, respectively) measured with respect to the SLM axis. The reflected beam is then subject to a 4f spatial filtering system. The SLM is positioned at the input plane (at a focal distance $f$ behind the lens $L_3$) and the spatial filtering mask (SF, a bandpass circular pupil) is placed at the Fourier plane to select the first diffraction order containing the information of the desired complex field pattern. After another Fourier transform, the field is obtained at the Fourier plane of lens $L_4$ and its intensity profile for different propagation distances $z$ is measured by a CCD camera (DMK 41BU02.H from The Imaging Source, with a 1280 $\times$ 960 display and 4.65 $\mu$m pixel size).

Due to the size of the SLM’s pixel, it cannot resolve well the fast amplitude and phase variations of beams with $k_{x_0}/k_0$ and/or $k_{y_0}/k_0$ ratios that are not small, so we had to choose close-to-zero values for them. Even though these beams are not significantly nonparaxial, the results illustrate how interesting interference patterns can be produced. With suitable equipment, it would be straightforward to generate the same kinds of beams in a nonparaxial regime.

8.1. Superposition of two Gaussian beams

The first example consists in a superposition of two Gaussian beams with same amplitudes and spot sizes $r_x = r_y = 300/\sqrt{2} \, \mu$m, but with different values of $k_{x_0}$ and $k_{y_0}$. One of them has $(k_{x_0}, k_{y_0}) = (+0.003k_0, 0k_0)$ and the other has $(k_{x_0}, k_{y_0}) = (-0.003k_0, 0k_0)$, so that they propagate in opposite directions in the $x$-axis but do not shift in the $y$-direction. Figure 3 shows what we should expect for a $y = 0$ cut of the beam’s intensity profile: a sinusoidal pattern enveloped by a Gaussian function. Therefore, it can be viewed as a kind of ‘cartesian analog’ of a BG beam, which consists in a superposition of GBs over a cone and has a Bessel function profile enveloped by a Gaussian function.

The propagation distance $Z$ for which the sinusoidal pattern is kept can be estimated by the expression in section 7, resulting in

$$
Z \approx r_x k_0/(|k_{x_0}| \sqrt{2b k_0}) \approx 70.71 \, \text{mm},
$$

which agrees with figure 3. Figure 4 shows the good agreement between the theoretical and the measured intensity patterns for $y = 0$ cuts at different propagation distances and also presents the measured patterns at each of these positions.

---

Note that, according to section 2.3, $A$ is a function of $x$ and $z$ and, at $z = 0$,

$$
\chi = x/\sqrt{2bk_0}.
$$

Since $b$ is the same for $+k_{x_0}$ and $-k_{x_0}$, the initial envelope is equal for the two beams.

---

Figure 2. Experimental setup for generating beams in free-space. Laser: He–Ne laser; $L_1$ to $L_4$: lenses; PH: pinhole; $P_1$: polarizer; BS: beam splitter; SLM: reflective Spatial Light Modulator, model LC-R1080 from HOLOEYE Photonics; $P_2$: analyzer; SF: circular pupil; CCD: CCD camera model DMK 41BU02.H from The Imaging Source.
Slight deviations between the predicted and obtained results may be due to small imprecisions in the generation of the beam’s initial pattern (in particular its phase), whether it is related to the resolution of the SLM or not.

8.2. Superposition of four Gaussian beams

The second example consists in a superposition of four Gaussian beams with same amplitudes and spot sizes \( r_x = r_y = 300 / \sqrt{2} \) μm, but with different values of \( k_{x_0} \) and \( k_{y_0} \). Two of them are the same as the ones in section 8.1 and the other two have \( (k_{x_1}, k_{y_1}) = (0k, +0.003k) \) and \( (k_{x_2}, k_{y_2}) = (0k, -0.003k) \), so that, in addition to the two beams being shifted in the \( x \)-direction, we now have two more beams that are shifted along the \( y \)-axis in opposite directions. Therefore, the interference pattern is a little more sophisticated, with more maxima and minima.

Figure 5 shows the good agreement between the theoretical and measured intensity patterns for \( y = 0 \) cuts at different propagation distances and captured patterns at each of them. See visualization 2 in the supplementary material for measurements and theoretically expected results for \( z \) ranging from 0 to 120 mm in 2 mm increments.

Slight deviations between the predicted and obtained results may be due to small imprecisions in the generation of the beam’s initial pattern (in particular its phase), whether it is related to the resolution of the SLM or not.

9. Nonparaxial azimuthally symmetric waves with concentrated spectra

Following a procedure similar to the one in section 2, we now present how the previous unidimensional results can be leveraged to describe nonparaxial azimuthally symmetric waves with concentrated spectra.

9.1. Superposition of Bessel beams with close cone angles

Any azimuthally symmetric wave \( \Psi(\rho, z, t) \) can be expressed as a superposition of Bessel beams of order zero, that is

\[
\Psi(\rho, z, t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int_{0}^{\infty} dk_\rho \times \hat{S}(k_\rho, \omega) J_0(\rho k_\rho) e^{ik^2 z} \tag{44}
\]

with the spectrum \( \hat{S}(k_\rho, \omega) \) defining the amplitudes of each beam. Note that the upper limit of \( k_\rho \) is not limited to \( \omega/c \), so that evanescent waves are allowed.

For an arbitrary spectrum, the integrals in equation (44) generally do not have a closed analytical representation, but when \( \hat{S}(k_\rho, \omega) \) is concentrated around \( (k_\rho, \omega) = (k_{\rho_0}, \omega_0) \) some simplifications can be applied using an approximate expression for \( J_0(\cdot) \). As in section 2.1, a concentrated spectrum means that \( |\hat{S}(k_\rho, \omega)| \) is significant only inside the region \( |k_\rho - k_{\rho_0}| \ll 1 \) and \( |\omega - \omega_0| \ll 1 \), where \( k_{\rho_0} \) and \( \omega_0 \). In other words, all the superposed Bessel
beams have almost the same cone angle. Note, however, that it can be as large as desired, unlike in the paraxial approximation, which restricts the angles to small values.

Similarly to what was done in section 2, the hypothesis of concentrated spectrum allows us to accurately approximate \( k_z = \sqrt{k^2 - k^2_p} \) to second order in \( k^2_p \) and to first order in \( u \):

\[
\sqrt{k^2 - k^2_p} \approx \sqrt{k_0^2 - k^2_p} - ak^2_p - bk^{22}_p + \frac{a}{c}u
\]

with

\[
a \equiv \frac{k_{0}}{\sqrt{k_{0}^{2} - k^{2}_{p}}}, \quad b \equiv \frac{1}{2} \frac{k_{0}^{2}}{(k_{0}^{2} - k_{p}^{2})^{2}}, \quad c \equiv \frac{k_{0}}{\sqrt{k_{0}^{2} - k^{2}_{p}}},
\]

Changing the integration variable in equation (44) from \( k^2_p \) to \( k^2_{0p} \) results in

\[
\Psi(\rho, z, t) \approx e^{-i\omega_{0}t}e^{i k_{0} z - \frac{k_{0}^{2}}{2} z} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk^2_p S(k^2_p, u) J_0[(k^2_p + k^2_{0p})\rho]e^{ik^2_p z} \]

with \( S(k^2_p, u) \) defined as \( \tilde{S}(k^2_p + k^2_{0p}, u + \omega_{0}) \) and

\[
k'^{2}_z = -ak'^{2}_p - bk'^{22}_p + \frac{c}{u}.
\]

For \( k_{0p} = 0 \), the spectrum is concentrated around \( k^2_p = 0 \) and its amplitude is negligible in the region \( k^2_p \geq k^2_{0p} \). Therefore, the lower limit of the integral in \( k^2_p \) in equation (47) can be extended to \(-\infty\) without loss of accuracy.

### 9.2. An approximate expression for \( J_0(x) \)

Now, we will make use of an unusual, although very accurate, approximate expression for \( J_0(x) \). It was first introduced in

[6], but the reasoning behind its derivation was not presented. Therefore, we show here how this approximation is obtained.

The asymptotic approximation for \( J_0(x) \) is

\[
J_0(x) \approx \frac{2}{\sqrt{\pi x}} \cos \left( x - \frac{\pi}{4} \right).
\]

Although it is rigorously valid only when the argument is large, figure 6 shows that this asymptotic approximation is precise even for relatively small values of \( x \), failing only due to the divergence when \( x \to 0 \). Therefore, one could attempt to improve the accuracy of equation (49) for small arguments by introducing a function in the denominator of the square root to eliminate the divergence, that is,

\[
F(x) \approx \frac{2}{\sqrt{\pi x + g(x)}} \cos \left( x - \frac{\pi}{4} \right)
\]

where \( g(x) \) is chosen in such a way that \( F(x) \approx J_0(x) \) for all \( x \). To this end, \( g(x) \) must have the following properties:

- \( g(0) = 1 \), so that \( F(0) = J_0(0) = 1 \).
- \( g'(0) = - (\pi - 2) \), so that \( F'(0) = J'_0(0) = 0 \).
- \( g(x) \to 0 \) sufficiently fast when \( x \) increases, so that \( F(x) \approx J_0(x) \) for \( x \gg 0 \).

The simplest function that satisfies all these conditions is \( g(x) = e^{-x-2x^2} \) and we can hence accurately approximate \( J_0(x) \) with the expression

\[
J_0(x) \approx \frac{2}{\sqrt{\pi x + e^{-x-2x^2}}} \cos \left( x - \frac{\pi}{4} \right).
\]

for any \( x \geq 0 \), even small. To allow negative arguments, \( x \) in the right-hand side of equation (51) should be replaced by \( |x| \), so that the expression is even, as is \( J_0(x) \). Figure 6 displays a comparison between the exact \( J_0(x) \), the asymptotic approximation and the unusual approximation, showing that the latter represents \( J_0(x) \) very accurately.

It is worth noting that the approximation in equation (51) for \( J_0(x) \) separates the slowly decaying envelope \( \sqrt{\frac{2}{\pi x + e^{-x-2x^2}}} \) from the fast oscillations expressed by \( \cos \left( x - \frac{\pi}{4} \right) \).

### 9.3. Similarities with the unidimensional Cartesian case

Extending the lower limit of the integral in equation (47) to \(-\infty\) and using the approximation for the Bessel function, the expression becomes

\[
\Psi(\rho, z, t) \approx e^{-i\omega_{0}t}e^{i k_{0} z - \frac{k_{0}^{2}}{2} z} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk^2_p S(k^2_p, u) J_0[(k^2_p + k^2_{0p})\rho]e^{ik^2_p z} \]

\[
\times \sqrt{\frac{2}{\pi (k^2_p + k^2_{0p})\rho + e^{-x-2x^2}}}
\times \cos \left[ (k^2_p + k^2_{0p})\rho - \frac{\pi}{4} \right] e^{ik^2_p z}.
\]
Since by hypothesis \( |k_0^\prime| \ll k_{01} \), where \(|S(k_{01}, u)|\) is significant, the argument \((k_0^\prime + k_{01})_0 \approx k_{01}\rho \) of the Bessel function is always positive in this region, so there is no need to include the absolute value for the approximate expression to be valid. The effect of the small variation \( k_0^\prime \) is important in the fast oscillations of the cosine, but can be neglected in the envelope, which remains always approximately equal to \( 2 \sqrt{\frac{2k_{01}\rho + e^{-(\pi - 2\pi)}}} \) and can be taken outside the integral. Expanding the cosine into complex exponentials, we finally get

\[
\Psi(\rho, z, t) \approx \frac{e^{-i\omega t}e^{-i\frac{\rho^2}{2k_{01}}}Z}{\sqrt{2(\pi k_{01}\rho + e^{-(\pi - 2\pi)})}} \times \left[ e^{ik_{01}\rho - \frac{\pi}{4}}A(\rho, z, t) + e^{-ik_{01}\rho - \frac{\pi}{4}}A(-\rho, z, t) \right],
\]

(53)

where

\[
A(\rho, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_{01} S(k_{01}, u)e^{-i\omega t}e^{ik_{01}\rho + ik_{01}z}.
\]

(54)

has the same functional form of a unidimensional nonparaxial Cartesian envelope (equation (10)) with \( x \) replaced by \( \rho \). The differential equation satisfied by \( A(\rho, z, t) \) is then

\[
\frac{\partial A}{\partial z} = \frac{e}{c} \frac{\partial A}{\partial t} - \frac{\partial A}{\partial \rho} + i\hbar \frac{\partial^2 A}{\partial \rho^2}
\]

(55)

and the unidimensional transformations of section 2.3 are applicable if \( x \) is replaced by \( \rho \).

Therefore, the azimuthally symmetric version of a nonparaxial Cartesian wave can be obtained by simply applying the known solution \( A(x, z, t) \) to equation (53) with \( x \) replaced by \( \rho \) and \( k_{01} \) by \( k_{01} \). The minus sign in \( A(-\rho, z, t) \) can be viewed alternatively as a change in the sign of \( a \), implying that in equation (53) there are waves traveling in opposite directions.

Over the axis (\( \rho = 0 \)), the wave is

\[
\Psi(0, z, t) \approx e^{-i\omega t}e^{-i\frac{\rho^2}{2k_{01}}}A(0, z, t)
\]

(56)

that is, \(|\Psi(0, z, t)| = |A(0, z, t)|\), thus allowing the easy prediction of the on-axis intensity. On the other hand, at \( z = 0 \), the result is

\[
\Psi(\rho, 0, t) \approx e^{-i\omega t}e^{i(\frac{\rho^2}{2k_{01}} - \frac{\pi}{4})}A(\rho, 0, 0) + e^{-i(\frac{\rho^2}{2k_{01}} + \frac{\pi}{4})}A(-\rho, 0, 0) \frac{\sqrt{2(\pi k_{01}\rho + e^{-(\pi - 2\pi)})}}{\sqrt{2(\pi k_{01}\rho + e^{-(\pi - 2\pi)})}} \frac{C}{\sqrt{1 + \frac{1}{4\omega z}}}
\]

(57)

If \( A(\rho, 0, t) \) has a definite parity in \( \rho \), the numerator of equation (57) becomes \( 2\cos(k_{01}\rho - \pi/4)A(\rho, 0, t) \) if \( A \) is even and \( 2i\sin(k_{01}\rho - \pi/4)A(\rho, 0, t) \) if \( A \) is odd, showing that the slowly varying envelope is modulated by a fast-oscillating sine or cosine factor with period \( 2\pi/k_{01} \), as in section 7. In these cases, we may expect to have waves with spot radii of the order of \( \Delta \rho \sim \pi/(2k_{01}) \).

### 9.4. Approximate field depth

Since the wave in equation (53) can be qualitatively viewed as a superposition of waves with envelope \( A(\rho, z, t) \) over a cone of half-angle \( \theta = \arcsin(k_{01}/\rho) \), we can estimate its field depth \( Z \) as the distance within which their interference is significant. From [8], we know that this interpretation, besides being qualitatively appealing, is also quantitatively accurate for determining the field depth of paraxial waves, so we can use the unidimensional transformations of section 2.3 to generalize this result to the nonparaxial case considered here.

Let \( w_0 \) be a characteristic width of \( A(\rho, z, t) \) at \( z = 0 \) which embraces its main features, such as its spot radius, and let us assume its change is negligible when propagating the distance \( Z \). This is reasonable because \( A(\rho, z, t) \) is a slowly varying envelope by hypothesis. In the paraxial case, \( Z \) can be estimated by \( Z \approx w_0 / \sin \theta = w_0 k_{01}/k_{01}^{\prime} \) [8]. Since the expression for the nonparaxial envelope \( A(\rho, z, t) \) is obtained from an initial paraxial expression by replacing \( \rho \) by \( (\rho - az)/\sqrt{2bk_0} \) (see section 2.3), if the characteristic width \( w_0 \) of \( A(\rho, z, t) \) is converted back to a paraxial counterpart of this envelope the result is a characteristic width of \( w_0 / \sqrt{2bk_0} \). Therefore, combining this converted length with the paraxial approximation for \( Z \), we get

\[
Z \approx \frac{w_0}{\sqrt{2bk_0}} \frac{k_{01}}{k_{01}^\prime} = \frac{w_0}{\sqrt{2bk_0}} \frac{k_{01}}{k_{01}^\prime}
\]

(58)

which is an accurate estimate for the field depth of any azimuthally symmetric wave with concentrated spectra, as illustrated in the examples of section 10.

### 10. Examples of nonparaxial azimuthally symmetric beams

#### 10.1. BG beam

A nonparaxial BG beam is obtained by choosing a Gaussian spectrum concentrated around \( k_{01}^\prime \), as the one in equation (30) with \( k_0^\prime \) replaced by \( k_{01}^\prime \) and \( r_0 \) by \( r_0^\prime \). Using the Cartesian expression of equation (31) in equation (53), the result is

\[
\Psi_{BG}(\rho, z, t) = \frac{e^{-i\omega t}e^{i(k_0^\prime)^2 - k_{01}^\prime \rho^2}}{\sqrt{2(\pi k_{01}^\prime \rho + e^{-(\pi - 2\pi)})}} \frac{C}{\sqrt{1 + \frac{1}{4\omega z}}} \times \begin{cases}
    e^{i(k_0^\prime)^2} \exp\left[\frac{-(\rho - az)^2}{r_0^\prime \left(1 + \frac{1}{4\omega z}\right)}\right] \\
    \left[\frac{-(\rho + az)^2}{r_0^\prime \left(1 + \frac{1}{4\omega z}\right)}\right]
\end{cases}
\]

(59)
Figure 7. Comparison between the paraxial (equation (60)) and generalized nonparaxial (equation (59)) expressions for a BG beam with $r_0 = 1500 \mu m$ under paraxial ($k_{r_0} = 0.001k_0$) and nonparaxial ($k_{r_0} = 0.8k_0$) regimes. In both cases, $\lambda = 632.8$ nm. The first line shows the spectra, the following two lines depict the normalized $|\psi(\rho, z, t)|^2$ profiles according to the expressions and the last line compares the predicted on-axis intensities with the correct results calculated via Rayleigh–Sommerfeld diffraction integral.
At $z = 0$, the beam’s pattern is a Bessel function $J_0(k_{\rho_0}\rho)$ (in the form of equation (51)) with spot size $\Delta \rho_0 \approx 2.4/k_{\rho_0}$ apodized by a Gaussian function of waist $r_0$. For practical situations, it is desired to have $r_0 \gg \Delta \rho_0$, so that a large number of the Bessel’s rings are unattenuated and the BG beam behaves like a Bessel beam for longer distances, resulting in a higher field depth, which can be estimated by making $w_0 = r_0$ in equation (58). Note that a large $r_0$ implies a concentrated spectrum, which is consistent with our assumptions.

For comparison, the usual expression of a BG beam, valid for paraxial regime, is [8]

$$\psi_{BG}^{\text{paraxial}}(\rho, z, t) = -e^{-ik_{\rho_0}C} \exp \left( \frac{ik_{\rho_0}(z + \rho^2/2z)}{2z} \right)$$

$$\times J_0 \left( \frac{ik_{\rho_0}k_{\rho_0}\rho}{2z} \right) \exp \left[ -\frac{1}{4Q} \left( k_{\rho_0}^2 + k_{\rho_0}^2 \rho^2/z^2 \right) \right] \tag{60}$$

with $Q = 1/r_0^2 - i/k_{\rho_0}/(2z)$.

Figure 7 contrasts equations (60) and (59) for two situations: paraxial ($k_{\rho_0} = 0.001k_0$) and highly nonparaxial ($k_{\rho_0} = 0.8k_0$). In both cases, the medium is air, $\lambda = 632.8$ nm and $r_0 = 1500 \mu$m. The first line of figure 7 depicts the Gaussian spectrum, showing that they are concentrated. The following two lines show the intensities $|\psi(\rho, z, t)|^2$ predicted by the paraxial expression (equation 60), second line of figure) and the generalized nonparaxial expression (equation 59, third line of figure) normalized by their peak at $z = 0$. It is clear that the two expressions agree in the paraxial regime, but in the nonparaxial case equation (60) overestimates the field depth predicted by equation (59). The last line shows the intensities at $\rho = 0$ predicted by both expressions and compares them with the exact intensity calculated using the Rayleigh–Sommerfeld diffraction integral for the initial pattern $\Psi(\rho, 0, t) = e^{-i\omega t}J_0(k_{\rho_0}\rho)\exp(-\rho^2/r_0^2)$. The three results coincide for the paraxial regime, but only equation (59) agrees with the diffraction integral for the nonparaxial situation. According to equation (58), the estimated field depths are $Z \approx 1.5$ m and $Z \approx 0.87$ mm for the paraxial and nonparaxial examples, respectively, which accurately match the diffraction integral results.

It is worth mentioning that equation (59), while valid for the cases of practical interest in which $r_0 \gg \Delta \rho_0$, is much simpler than other nonparaxial extensions of the BG beams found in the literature, which rely on nonparaxial correction terms [9], series of functions [10] or integral representations [11].

### 10.2. Circular parabolic-Gaussian (CPG) beam

If we apply the nonparaxial version of the envelope in equation (32) to equation (53), we get an azimuthally symmetric version of the PG beams, which we will refer to as circular Parabolic-Gaussian (CPG) beams.

To illustrate, we will consider two examples in air. The first is an odd beam ($n = 3/2$) with $\nu = h = 1$ and $r_0 = 200 \mu$m while the second is an even beam ($n = 1/2$) with $\nu = -3/2$, $h = 0.5$ and $r_0 = 200 \mu$m. Both are analyzed in two regimes: paraxial ($k_{\rho_0} = 0.01k_0$) and nonparaxial ($k_{\rho_0} = 0.5k_0$), with $\lambda = 632.8$ nm.

Since the envelopes of PG beams have definite parity in $\rho$ at $z = 0$, the initial patterns of CPG beams are Gaussian-apodized parabolic cylinder functions, with an additional $1/\sqrt{2[\pi k_{\rho_0}^2 + e^{-(\nu-2)k_{\rho_0}^2\rho^2}]}$ decay factor, modulated by a fast oscillating sinusoidal pattern with period $2\pi/k_{\rho_0}$, as explained in section 9.3. Figure 8 shows the initial patterns of the CPG beams for the paraxial case, in which $\Delta \rho_0 \sim 16 \mu$m. In the nonparaxial regime, the patterns are similar, but with much faster oscillations ($\Delta \rho_0 \sim 0.32 \mu$m).

Figure 9 presents the resulting spectra and the intensity profiles of the odd beam. The first line depicts the spectra, calculated from the inverse Fourier transform of the initial profile (analogously to section 3). Since the PG beams are solutions of the paraxial wave equation, their spectra are indeed concentrated and, in the case of their CPG counterparts, they are centered at $k_{\rho} = k_{\rho_0}$. The second line shows the resulting intensity profiles, normalized by their peak.
intensities at $z = 0$. Although the odd symmetry makes the on-axis amplitude zero at $z = 0$, the sidelobes shown in figure 8 interfere constructively after some distance and create a very high-intensity peak over the axis, due to the concentration of the energy they carry in a small area. The effect is even more intense in the nonparaxial regime, in which the spot size is much smaller. The last line compares the on-axis intensities predicted by equation (56) with the numerical calculations of Rayleigh–Sommerfeld diffraction integral with the initial patterns given by equation (57). It is clear that in both paraxial and nonparaxial regimes the analytical expression is accurate.

Figure 9 presents the same kinds of results of figure 9 but for the even beam. We see again that the spectra are concentrated and that the on-axis intensities predicted by the analytical expressions are accurate. Note that the even symmetry makes the peak intensities appear at $(\rho, z) = (0, 0)$, so, due to the normalization, we do not see higher-intensity on-axis peaks as in figure 9.

The field depths of the examples presented can be estimated by equation (58) without recurring to the plots of intensity profiles. We could use $w_0 = r_0$ for a reasonable estimate, but better results are obtained by choosing $w_0$ based on the envelopes of figure 8. For the odd beam, $w_0 = 0.35 \text{ mm}$ embraces all its significant lobes, while the same happens for the even beam for $w_0 = 0.3 \text{ mm}$. Using these values in equation (58), we get $Z \approx 35 \text{ mm}$ and $Z \approx 0.56 \text{ mm}$ for the odd beam under paraxial and nonparaxial regimes, respectively, while $Z \approx 30 \text{ mm}$ and $Z \approx 0.48 \text{ mm}$ for the even beam under paraxial and
nonparaxial regimes, respectively. According to figures 9 and 10, these estimates are reasonable.

11. Conclusions

In this work, we developed a theoretical analysis to efficiently describe the propagation of superpositions of waves with concentrated wavevector and frequency spectra, thus allowing a simple analytical description of fields with interesting transverse profiles. Starting with an extension of the paraxial formalism, we show how it can be applied to easily handle combinations of waves traveling in different directions without recurring to coordinate rotations. Moreover, and more importantly, we used a similar procedure to create a novel analytical description of azimuthally symmetric waves with concentrated spectra that can leverage all the previously presented results for unidimensional Cartesian waves, allowing us to build azimuthally symmetric versions of them and, therefore, to possibly derive new types of waves. Since these waves are composed of superpositions of zero-order Bessel beams with close cone angles that can be as large as desired, unlike in the paraxial formalism, we showed that it can also provide simple expressions for nonparaxial versions of known beams, such as the BG beam. Finally, the validity and accuracy of the theory was

Figure 10. Characteristics of a CPG beam with $n = 1/2$, $\nu = -3.5$, $h = 0.5$ and $r_0 = 200 \mu m$ in both paraxial ($k_{p_0} = 0.01k_0$) and nonparaxial ($k_{p_0} = 0.5k_0$) regimes for the wavelength $\lambda = 632.8$ nm. The first line shows the spectra, the second depicts the $|\Psi(\rho, z, t)|^2$ profiles according to the analytical expressions and the third compares the predicted on-axis intensities with the correct results calculated via Rayleigh–Sommerfeld diffraction integral.
corroborated by experimental demonstrations and comparisons to numerically calculated Rayleigh–Sommerfeld diffraction integrals.

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