Weak and strong $L^p$-limits of vector fields with finitely many integer singularities in dimension $n$

Riccardo Caniato Filippo Gaia

November 23, 2022

Abstract

For every given $p \in [1, +\infty)$ and $n \in \mathbb{N}$ with $n \geq 1$, the authors identify the strong $L^p$-closure $L^p_\sharp(D)$ of the class of vector fields having finitely many integer topological singularities on a domain $D$ which is either bi-Lipschitz equivalent to the open unit $n$-dimensional cube or to the boundary of the unit $(n + 1)$-dimensional cube. Moreover, for any $n \in \mathbb{N}$ with $n \geq 2$ the authors prove that $L^p_\sharp(D)$ is weakly sequentially closed for every $p \in (1, +\infty)$ whenever $D$ is an open domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to the open unit cube. As a byproduct of the previous analysis, a useful characterisation of such class of objects is obtained in terms of existence of a (minimal) connection for their singular set.

MSC (2020): 46N20 (primary); 58E15 (secondary).

Contents

1. Introduction 3
   1.1. Statement of the results and motivation . . . . . . . . . . . . . . . . . . . . . 3
   1.2. Related literature and open problems . . . . . . . . . . . . . . . . . . . . . . 8
   1.3. Organization of the paper . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
   1.4. Acknowledgements . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
   1.5. Notation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

2. The strong $L^p$-approximation Theorem 10
   2.1. Choice of a suitable cubic decomposition . . . . . . . . . . . . . . . . . . . . . 11
   2.2. Smoothing on the $(n - 1)$-skeleton of the cubic decomposition . . . . . . . . 16
   2.3. Extensions on good and bad cubes . . . . . . . . . . . . . . . . . . . . . . . . 18
   2.4. Proof of Theorem 1.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21
   2.5. A characterization of $\Omega_{p,\sharp}^{n-1}$. . . . . . . . . . . . . . . . . . . . . . . . . 27
   2.6. The case of $\partial Q_{1}^{n+1}(0)$ . . . . . . . . . . . . . . . . . . . . . . . . . 31
   2.7. Corollaries of Theorem 1.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35
3. The weak $L^p$-closure of $\Omega^{n-1}_{p,\mathbb{Z}}(Q^n_1(0))$

3.1. Slice distance on $S^{n-1}(0)$ .................................................. 39
3.2. Slice distance on $\partial Q^n_1(0)$ ........................................... 43
3.3. Slice functions and their properties ........................................ 44
3.4. Proof of Theorem 1.3 for $Q^n_1(0)$ .......................................... 48

A. Minimal connections for forms with finitely many integer singularities 55

B. Laplace equation on spheres ....................................................... 58

C. Some technical lemmata ............................................................. 63
1. Introduction

Throughout the following paper we will always use the notations listed in subsection 1.4.

1.1. Statement of the results and motivation

Given any smooth map \( u : X \to Y \) between closed, oriented and connected \((n-1)\)-dimensional manifolds, the degree of \( u \) is a measure of how many times \( X \) wraps around \( Y \) under the action of \( u \). It can be defined as follows:\[ \text{deg}(u) = \int_X u^*\omega, \]
with \( \omega \in \Omega^n(Y) \) being a renormalized volume form on \( Y \), i.e. \( \omega \) is nowhere vanishing and \( \int_Y \omega = 1 \).

Now let \( D \subset \mathbb{R}^n \) be an open and bounded Lipschitz domain and consider a map \( u \in W^{1,p}(D,Y) \) for some \( p \geq 1 \) being smooth up to finitely many point singularities, which simply means that \( u \in C^\infty(D \setminus S_u, Y) \) for some finite set \( S_u \subset D \). In this case we write \( u \in R^{1,p}(D,Y) \). We define the degree of \( u \) at some singular point \( x \in S_u \) as
\[ \text{deg}(u,x) := \text{deg}(u|_{\partial D'}) = \int_{\partial D'} u^*\omega \in \mathbb{Z}, \tag{1.1} \]
where \( D' \subset \subset D \) is any open, piecewise smooth domain in \( D \) such that \( \overline{D'} \cap S_u = \{ x \} \). Notice that Definition (1.1) is independent from the choice of the set \( D' \).

If \( \text{deg}(u,x) \neq 0 \) for some \( x \in S_u \), then we say that \( x \) is a topological singularity of \( u \) and we refer to the subset of \( S_u \) made of the topological singularities of \( u \) as the topological singular set of \( u \), which we denote by \( S_{u}^{\text{top}} \).

Notice that if \( u \in R^{1,n-1}(D) \) then \( u^*\omega \in \Omega_1^{n-1}(D) \), moreover by (1.1) we see that the \( u^*\omega \) "detects" the topological singularities of \( u \), in the sense that
\[ \int_{\partial D'} u^*\omega = \sum_{x \in S_{u}^{\text{top}} \cap D'} \text{deg}(u,x) \tag{1.2} \]
for every open, piecewise smooth domain \( D' \subset \subset D \) such that \( \partial D' \cap S_u = \emptyset \). From (1.2) one can deduce that
\[ *d(u^*\omega) = \sum_{x \in S_{u}^{\text{top}}} \text{deg}(u,x)\delta_x \text{ in } \mathcal{D}'(D). \]

Remark 1.1. Sobolev maps smooth up to a finite set of topological singularities arise frequently as solutions of variational problems in critical or supercritical dimension. For example, this is the best regularity which is possible to guarantee for energy minimizing harmonic

---

\(^1\)An alternative definition can be given in terms of the orientations of the preimages of regular points of \( u \), see for instance [3, Chapter 7].
maps in $W^{1,2}(B^3, \mathbb{S}^2)$ (see [25, Theorem II]). Again, quite recently the second author and T. Rivière considered the following "weak $S^1$-harmonic map equation"
\[
\text{div}(u \wedge \nabla u) = 0 \quad \text{in } D'(B^2)
\]
and gave a completely variational characterization of the solutions in $R^{1,p}(B^2, S^1)$ with finite "renormalized Dirichlet energy" for $p > 1$ (see [12, Theorem I.3] for further details).

We also remark that the presence of topological singularities is deeply linked to fundamental questions concerning the strong $W^{1,p}$-approximability through smooth maps of elements in $W^{1,p}(D, Y)$ (see [3], [13]).

The previous discussion motivates the following general definition.

**Definition 1.1.** Let $p \in [1, \infty]$. Let $F \in \Omega^{n-1}_p(D)$. We say that $F$ has **finitely many integer singularities** if there exists a finite set of points $S \subset D$ such that $F \in \Omega^{n-1}(D \setminus S)$ and
\[
* dF = \sum_{x \in S} a_x \delta_x,
\]
where $a_x \in \mathbb{Z}$ for every $x \in S$. The class of $L^p$ integrable $(n-1)$-forms on $D$ having finitely many integer singularities will be denoted by $\Omega^{n-1}_{p,R}(D)$.

As we have seen above, $u^\ast \omega \in \Omega^{n-1}_{1,R}(D)$ for every $u \in R^{1,n-1}(D, Y)$, for any closed, oriented and connected $n - 1$-manifold $Y$. Other simple examples of elements of $\Omega^{n-1}_{p,R}(D)$ can be constructed as follows. Let $\sigma : D \to \mathbb{R}$ denote the fundamental solution of the Laplace equation, i.e.
\[
\sigma(x) = \begin{cases} 
-|x| & \text{if } n = 1, \\
-\frac{1}{2\pi} \log|x| & \text{if } n = 2 \\
\frac{1}{n(n-2)\alpha(n)|x|^{n-2}} & \text{if } n \geq 3,
\end{cases}
\]
where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Then $* d\sigma \in \Omega^{n-1}_{p,R}(D)$ for any $p \in [1, \frac{n}{n-1})$. In fact $* d(* d\sigma) = \Delta \sigma = \delta_0$.

Clearly any finite linear combination with integer coefficients of translations of $* d\sigma$ also belongs to $\Omega^{n-1}_{p,R}(D)$. In fact one can show that any element $F$ of $\Omega^{n-1}_{p,R}(D)$ can be decomposed as such a linear combination plus some $\tilde{F} \in \Omega^{n-1}_{p,R}(D)$ with $* d\tilde{F} = 0$. In particular, if $p \geq \frac{n}{n-1}$ then $* dF = 0$. Thus the class $\Omega^{n-1}_{p,R}(D)$ is relatively simple from an analytical point of view and so it is natural to ask which forms in $\Omega^{n-1}_{p,R}(D)$ can be approximated by elements in $\Omega^{n-1}_{p,R}(D)$. The main purpose of the present paper consists in giving a description of the strong and weak closure of the class $\Omega^{n-1}_{p,R}(D)$ for any open domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to the open unit $n$-cube $Q_1(0) \subset \mathbb{R}^n$.

First we will address the strong closure in the case of the open, unit $n$-dimensional cube $Q_1(0) \subset \mathbb{R}^n$. To this end we introduce the class of $(n-1)$-forms with integer-valued fluxes. For any $F \in \Omega^{n-1}_{p,loc}$ for any $x_0 \in Q_1(0)$ let $\tilde{F}_{x_0} \subset (0, r_0)$ be the set of radii $\rho \in (0, r_0)$ such that

1. the hypersurface $\partial Q_\rho(x_0)$ consists $H^{n-1}$-a.e. of Lebesgue points of $F$,
2. there holds $|F| \in L^p(\partial Q_\rho(x_0), H^{n-1})$. 


One can check that $\mathcal{L}^1((0, r_0) \setminus \tilde{R}_F, x_0) = 0$.

**Definition 1.2.** Let $p \in [1, \infty]$, let $F \in \Omega^{n-1}_{L,loc} \cap \Omega^{n-1}_p(Q_1(0), \mu)$ for some Radon measure $\mu$, for any $x_0 \in Q_1(0)$ let $\tilde{R}_{F, x_0}$ be defined as above. We say that $F$ has integer-valued fluxes if for any $x_0 \in Q_1(0)$, for $\mathcal{L}^1-$a.e. $\rho \in \tilde{R}_{F, x_0}$ there holds:

$$
\int_{\partial Q_\rho(x_0)} i^*_\rho F \in \mathbb{Z}.
$$

(1.3)

The space of $L^p(\mu)$-vector fields with integer valued fluxes will be denoted by $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0), \mu)$. The set of radii $\rho \in \tilde{R}_{F, x_0}$ for which (1.3) holds will be denoted by $R_{F, x_0}$

We will always write $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$ for $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0), \mathcal{L}^n)$, where $\mathcal{L}^n$ denotes the $n$-dimensional Lebesgue measure.

First of all we observe that (1.2) implies that $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0)) \subset \Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$. More general examples of forms in $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$ can be constructed as follows. Let again $Y$ be a smooth, closed, oriented and connected $n-1$-dimensional manifold. Let $u \in W^{1,n-1}(Q_1(0), Y)$. Then for any $x_0 \in Q_1(0)$, for a.e. $\rho \in (0, 2 \text{dist}_x(x_0, \partial Q_1(0)))$, $u\mid_{\partial Q_\rho(x_0)} \in W^{1,n-1}(\partial Q_\rho(0), Y)$. Therefore for any such $\rho$

$$
\int_{\partial Q_\rho(x_0)} i^*_\rho(u \mid_{\partial Q_\rho(x_0)}) \in \mathbb{Z}
$$

(1.4)

Notice that $\text{deg}(u\mid_{\partial Q_\rho(x_0)})$ is well defined (by means of approximation by functions in $W^{1,\infty}(\partial Q_\rho(x_0), Y)$, see [10], Section I.3).

We will show that in fact the closure of $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$ in $\Omega^{n-1}_p(Q_1(0))$ is exactly $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$. More precisely we have

**Theorem 1.1.** Let $n \in \mathbb{N}$ and let $p \in [1, \infty]$. Let $F \in \Omega^{n-1}_{p,\mathcal{L}}(Q_1(0), \mu)$. Then we have

1. if $q \in [0, 1]$ and $p \in \left[1, \frac{n}{n-1}\right)$, then there exists a sequence $\{F_k\}_{k \in \mathbb{N}}$ in $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$ such that $F_k \to F$ in $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$ as $k \to \infty$.

2. if $q \in (-\infty, 0]$ and $p \in \left[\frac{n}{n-1}, +\infty\right)$, then $*dF = 0$.

The reason why we have introduced the weighted measures $\mu = f \mathcal{L}^n$ for $q \neq 0$ is that forms belonging to $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0), \mu)$ appear naturally in the proof of Corollary 2.2. Nevertheless, we advise the reader to assume $q = 0$ (i.e. $\mu = \mathcal{L}^n$) throughout section 2 at a first reading of the present paper. This allows to skip many technicalities without losing formality, since all the results of this paper are independent on Corollary 2.2.

With the help of Theorem 1.1 we will get another characterization of the $L^p$-closure of $\Omega^{n-1}_{p,\mathcal{L}}(Q_1(0))$. For this we recall the following definition (compare with [8], Section II):

\[ \int_{\partial Q_\rho(x_0)} V \cdot \nu_{\partial Q_\rho(x_0)} d\mathcal{H}^{n-1} \in \mathbb{Z}. \]
Definition 1.3 (Connection and minimal connection). Let $M \subset \mathbb{R}^n$ be any embedded Lipschitz $m$-dimensional submanifold of $\mathbb{R}^n$ (with or without boundary) such that $\bar{M}$ is compact as a subset of $\mathbb{R}^n$.

A 1-dimensional current $I \in \mathcal{R}_1(M)$ is said to be a connection for (the singular set of) $F$ if $M(I) < +\infty$ and $\partial I = *dF$ in $(W_0^{1,\infty}(M))^*$. A 1-dimensional current $L \in \mathcal{R}_1(M)$ is said to be a minimal connection for (the singular set of) $F$ if it is a connection for $F$ and

$$M(L) = \inf_{T \in \mathcal{D}_1(M)} M(T).$$

We will see in Corollary 2.1 that $F$ admits a connection if and only if it admits a minimal connection.

Here is the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ in terms of minimal connections:

**Theorem 1.2.** Let $n \in \mathbb{N}_{>0}$, let $p \in [1, +\infty)$. Let $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. Then, the following are equivalent:

1. there exists $L \in \mathcal{R}_1(Q_1(0))$ such that $\partial L = *dF$ in $(W_0^{1,\infty}(Q_1(0)))^*$.
2. for every Lipschitz function $f : Q_1(0) \to [a, b] \subset \mathbb{R}$ such that $f|_{\partial Q_1(0)} \equiv b$, we have

$$\int_{f^{-1}(t)} i_{f^{-1}(t)} F \in \mathbb{Z}, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b];$$

3. $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$.

In other words, $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ if and only if $F$ admits a (minimal) connection. This characterization allows to generalize the definition of the class $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ to general Lipschitz domains:

Definition 1.4. Let $M \subset \mathbb{R}^n$ be any embedded Lipschitz $m$-dimensional submanifold of $\mathbb{R}^n$ (with or without boundary). We define

$$\Omega^{n-1}_{p,\mathbb{Z}}(M) := \{ F \in \Omega^{n-1}_{p}(M) \text{ s.t. } \exists L \in \mathcal{R}_1(M) \text{ connection for } F \}.$$

Notice that if $M = Q_1(0)$, Definition 1.2 and Definition 1.4 coincide by Theorem 1.2.

We will deduce from the previous results that the approximation result can be extended to any open domain which is bi-Lipschitz equivalent to $Q_1(0)$ of $\partial Q_1(0)$ (see Theorem 2.3).

We mention here two other corollaries of Theorem 1.1.

**Corollary 1.1.** Let $n \in \mathbb{N}$. Let $I \in \mathcal{R}_1(Q^n_1(0))$ be an integer rectifiable one-current. Then there exists an one form $\omega \in \Omega^{n-1}_{1,\mathbb{Z}}(Q^n_1(0))$ such that $*d\omega = \partial I$ and $\partial I$ can be approximated in $(W_0^{1,\infty}(Q^n_1(0)))^*$ by finite sums of Dirac-deltas with integer coefficients. More precisely, there exist sequences $(P_i)_{i \in \mathbb{N}}$ and $(N_i)_{i \in \mathbb{N}}$ of points in $Q^n_1(0)$ such that

$$\partial I = \sum_{i \in \mathbb{N}} (\delta_{P_i} - \delta_{N_i}) \text{ in } (W_0^{1,\infty}(Q^n_1(0)))^* \text{ and } \sum_{i \in \mathbb{N}} |P_i - N_i| < \infty.$$

Moreover if $I$ is supported on a Lipschitz submanifold $M$ of $\mathbb{R}^n$ compactly contained in $Q_1(0)$, the points in the sequences $(P_i)_{i \in \mathbb{N}}$ and $(N_i)_{i \in \mathbb{N}}$ can be chosen to belong to $M$. 

6
The next corollary was obtained first by R. Schoen and K. Uhlenbeck ([26, Section 4]) and F. Bethuel and X. Zheng ([3, Theorem 4]).

**Corollary 1.2.** Let \( Q_1(0) \subset \mathbb{R}^2 \) be the unit cube in \( \mathbb{R}^2 \). Let \( u \in W^{1,p}(Q_1(0), S^1) \) for some \( p \in (1, \infty) \).

If \( p \geq 2 \), then \( u \) can be approximated in \( W^{1,p} \) by a sequence of functions in \( C^\infty(Q_1(0), S^1) \).

If \( p < 2 \), then \( u \) can be approximated in \( W^{1,p} \) by a sequence of functions in

\[
\mathcal{R} := \{ v \in W^{1,p}(Q_1(0), S^1); v \in C^\infty(Q_1(0) \setminus A, S^1), \text{ where } A \text{ is some finite set} \}.
\]

In the second part of the paper we turn our attention to the weak closure of the space \( \Omega^{n-1}_{p,R}(D) \) for a domain \( D \subset \mathbb{R}^n \) which is bi-Lipschitz equivalent to \( Q_1(0) \) (or equivalently of \( \Omega^{n-1}_{p,\mathbb{Z}}(D) \)). We will show the following.

**Theorem 1.3** (Weak closure). Let \( n \in \mathbb{N}_{\geq 2}, p \in (1, +\infty) \) and \( D \subset \mathbb{R}^n \) be any open and bounded domain in \( \mathbb{R}^n \) which is bi-Lipschitz equivalent to the \( n \)-dimensional unit cube \( Q_1^n(0) \). Then, the space \( \Omega^{n-1}_{p,R}(D) \) is weakly sequentially closed in \( \Omega^{n-1}_{p}(D) \).

Notice that, by Theorem 1.2 (or more generally by Theorem 2.3), the statement of Theorem 1.3 is trivial for \( p \in [n/(n-1), +\infty) \). Thus, we just need to provide a proof in case \( p \in (1, n/(n-1)) \).

We first treat the case of the open unit \( n \)-cube \( Q_1(0) \subset \mathbb{R}^n \) by exploiting the characterization of \( \Omega^{n-1}_{p,-}(Q_1(0)) \) given by Theorem 1.2 and a suitable slice distance à la Ambrosio-Kirchheim (see [2] and [14]). We then address the general case by standard arguments (see Remark 3.5).

We remark that the case \( n = 1 \) is different. In fact for any interval \( I \subset \mathbb{R} \) there holds \( \Omega^{0}_{p,R}(I) = \Omega^{0}_{p}(I) \) (see Lemma 3.3).

Our main motivation to look at forms (instead of maps) with finitely many integer topological singularities is the need of developing geometric measure theory for principal bundles in order to face the still deeply open questions arising in the study of \( p \)-Yang-Mills lagrangians.

Let \( p \in [1, +\infty) \) and \( G \) be any compact matrix Lie group. Consider the trivial \( G \)-principal bundle on \( B^n \) given by \( \text{pr}_1 : P := B^n \times G \to B^n \), where \( \text{pr}_1 \) is the canonical projection on the first factor. The \( p \)-Yang-Mills lagrangian on \( P \) is given by

\[
\text{YM}_p(A) := \int_{B^n} |F_A|^p \, d\mathcal{L}^n = \int_{B^n} |dA + A \wedge A|^p \, d\mathcal{L}^n, \quad \forall A \in \Omega^1(B^n, \mathfrak{g}),
\]

where \( \mathfrak{g} \) denotes the Lie algebra of \( G \).

As it is described in [16], the reason why we aim to extend the set of the by now classical Sobolev connections is purely analytic and justified by issues arising in the application of the direct method of calculus of variations to \( p \)-Yang-Mills Lagrangians. On the other hand, the need to extend the notion of bundles in order to allow more and more singularities to appear has already been faced in many geometric applications, which brought to the introduction of coherent and reflexive sheaves in gauge theory (see [17, 18]).
Notice that all results mentioned above can be formulated in terms of vector fields: for any \( F \in \Omega^{n-1}_p(D) \) we can consider the associated vector field \( V_F := (\ast F)^b \). In fact for the proof of some of the results we preferred to work with vector fields instead of \( n-1 \)-forms.

1.2. Related literature and open problems

Theorem 1.1 was firstly announced to hold for a 3-dimensional domain in \([16]\) and a full proof of the 3-dimensional case was eventually given by the first author in \([11]\). Some form of the 2-dimensional case was treated in \([21]\), where M. Petrache proved that a strong approximability result holds for 1-forms admitting a connection both on the 2-dimensional disk and on the 2-sphere. In both cases, the proof that we give here is more general and simple. The 3-dimensional version of Theorem 1.3 was already treated by M. Petrache and T. Rivière in \([23]\). Nevertheless, here we took the opportunity to present the arguments in a more detailed and complete way. Both Theorem 1.1 and Theorem 1.3 in dimension \( n \neq 2,3 \) are instead completely new.

The first open problems that relate directly to our results are linked to the celebrated Yang-Mills Plateau problem. Indeed, the weak sequential closedness of the class \( \Omega^2_{p,2}(B^3) \) implies that such forms behave well-enough to be considered as suitable "very weak" curvatures for the resolution of the \( p \)-Yang-Mills Plateau problem for \( U(1) \)-bundles on \( B^3 \) (see the introduction of \([23]\) for further details). The question to address would be if and how we can exploit the same kind of techniques in order to face the existence and regularity issues linked to the so called "non abelian case" (i.e. the case of bundles having a non abelian structure group) in supercritical dimension. An interesting proposal in this sense is due to M. Petrache and T. Rivière and can be found in \([22]\), where a suitable class of weak connections in the supercritical dimension 5 is introduced and studied.

One could also hope that the technique presented in this paper could be adapted to show \( L^p \)-closedness (weak and strong) of classes of differential forms exhibiting "integer fluxes" properties similar to the one described in Definition 1.2. As an example we define here the class \( \Omega_{p,\text{Hopf}}^n(Q^n_1(0)) \) of differential forms with "Hopf singularities". Recall that for any \( n \in \mathbb{N}_1, \) for any smooth map \( f : S^{2n-1} \rightarrow S^n \) the Hopf invariant of \( f \) is defined as follows: let \( \omega \) be the standard volume form on \( S^n \). Let \( \alpha \in \Omega^{n-1}(S^{2n-1}) \) be such that \( f^*\omega = d\alpha \). Then the Hopf invariant of \( f \) is given by

\[
H(f) := \int_{S^{2n-1}} \alpha \wedge d\alpha.
\]

One can show that \( H(f) \in \mathbb{Z} \) and that it is independent of the choice of \( \alpha \) (see \([9]\), Proposition 17.22). In the spirit of Definition 1.2 we say that a form \( F \in \Omega^2_{p,\text{Hopf}}(Q^n_1(0)) \) belongs to \( \Omega_{p,\text{Hopf}}^n(Q^n_2(0)) \) for some \( p \geq 2 - \frac{1}{n} \) if there exists \( A \in \Omega^{n-1}_{W^{1,p}}(Q^n_2(0)) \) such that \( dA = F \) and if for every \( x_0 \in Q^n_2(0) \) there exists a set \( R_{F,x_0} \subset (0, r_{x_0}) \), with \( r_{x_0} := 2 \text{dist}_\infty(x_0, \partial Q^n_2(0)) \) such that:

1. \( \mathcal{L}^1((0, r_{x_0}) \setminus R_{F,x_0}) = 0; \)
2. for every \( \rho \in R_{F,x_0}, \) the hypersurface \( \partial Q^n_\rho(x_0) \) consists \( \mathcal{H}^{2n-1} \)-a.e. of Lebesgue points of \( F \), \( A \) and \( \nabla A \) (the matrix of all the partial derivatives of the components of \( A \)).
3. for every $\rho \in R_{F,x_0}$ we have $|F|, |A|, |\nabla A| \in L^p(\partial Q^2_{\rho}(x_0), \mathcal{H}^{n-1})$;
4. for every $\rho \in R_{F,x_0}$ it holds that
$$\int_{\partial Q^2_{\rho}(x_0)} i^*_\rho(\partial Q^2_{\rho}(x_0))(A \land F) \in \mathbb{Z}.$$  

Notice that if $u \in W^{1,2n-1}(Q^2_1(0), S^n)$, then $u^* \omega \in \Omega^{n}_{p,H}(Q^2_1(0))$.

1.3. Organization of the paper

The paper is organized as follows. Section 2 is dedicated to the strong $L^p$-closure of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0), \mu)$. First we present some preliminary and rather technical lemmata (Sections 2.1-2.3), then we give a proof of Theorem 1.1 (Section 2.4). In section 2.5 we show the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ in terms of (minimal) connections (Theorem 1.2). In Section 2.6 we exploit this result to extend the approximation result to other Lipschitz manifolds, and in particular to $\partial Q^n_1(0)$. Finally in Section 2.7 we prove Corollary 1.1 and Corollary 1.2.

In section 3 we discuss the weak $L^p$ closure of $\Omega^{n-1}_{p,\mathbb{R}}(Q_1(0))$. First we will introduce a slice distance à la Ambrosio-Kirchheim, first on spheres (Section 3.1) and then on cubes (Section 3.2). In Section 3.3 we discuss some of the properties of the slice distance and in Section 3.4 we use it to obtain a proof of Theorem 1.3. We will also discuss briefly the special case $n = 1$.

1.4. Acknowledgements

We are grateful to prof. Tristan Rivière for encouraging us to work on this subject, for his insights and for the helpful discussions. We would also like to thank Federico Franceschini for the interesting discussions.

This work has been supported by the Swiss National Science Foundation (SNF 200020_192062).

1.5. Notation

Let $M^m \subset \mathbb{R}^n$ be any $m$-dimensional, embedded Lipschitz submanifold of $\mathbb{R}^n$ (with or without boundary) such that $\overline{M}$ is compact as a subset of $\mathbb{R}^n$.

- We denote by $i_M : M \hookrightarrow \mathbb{R}^n$ the usual inclusion map.
- We always assume that $M$ is endowed with the $L^\infty$-Riemannian metric given by $g_M := i_M^* g_e$, where $g_e$ denotes the standard euclidean metric on $\mathbb{R}^n$.
- For every $k = 1, \ldots, m$, we define the following spaces of smooth $k$-forms on $M$:
  \begin{align*}
  \Omega^k(M) &:= \{i_M^* \omega \text{ s.t. } \omega \text{ is a smooth } k\text{-form on } \mathbb{R}^n\}, \\
  D^k(M) &:= \{\omega \in \Omega^k(M) \text{ s.t. } \text{spt}(\omega) \subset \subset M\}.
  \end{align*}

For every $p \in [1, +\infty]$, we denote by $\Omega^k_p(M)$ and $\Omega^k_{W^{1,p}}(M)$ the completions of $\Omega^k(M)$ with respect to the usual $L^p$-norm and $W^{1,p}$-norm respectively. We call $\Omega^k_p(M)$ the space of $L^p$ $k$-forms on $M$ and $\Omega^k_{W^{1,p}}(M)$ the space of $W^{1,p}$ $k$-forms on $M$. 

9
By the symbol "∗", we denote the Hodge star operator associated with the metric \( g_M \) on \( M \). By "♭" and "♯" we denote the usual musical isomorphisms associated with the metric \( g_M \). Recall that, under this notation, the map

\[
\Omega^{n-1}_p(M) \ni \omega \mapsto (\ast \omega)^\flat \in L^p(M, \mathbb{R}^n)
\]

(1.5)
gives an isomorphism onto its image. Exploiting this fact, we frequently identify \((n-1)\)-forms with vector fields on \( M \).

Let \( x_0 \in \mathbb{R}^n \) and \( \rho > 0 \).

- We denote by \( \| \cdot \|_\infty : \mathbb{R}^n \to [0, +\infty) \) the following norm on \( \mathbb{R}^n \):

\[
\|x\|_\infty := \max_{j=1, \ldots, n} |x_j|.
\]

We denote by \( d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) the distance associated to \( \| \cdot \|_\infty \).

- We let

\[
Q^n_\rho(x_0) := \left\{ x \in \mathbb{R}^n \text{ s.t. } \|x - x_0\|_\infty < \frac{\rho}{2} \right\}
\]

be the open cube in \( \mathbb{R}^n \) centered at \( x_0 \) and having edge-length \( \rho \). We will sometimes omit the \( n \) when the dimension is clear from the context.

We denote by \( B^n_\rho(x_0) \) the open ball in \( \mathbb{R}^n \) centered at \( x_0 \) with radius \( \rho \) (here again we will sometime omit the \( n \)).

- We define

\[
\mathcal{D}_k(M) := \{ k\text{-currents on } M \}, \quad \mathcal{M}_k(M) := \{ T \in \mathcal{D}_k(M) \text{ s.t. } M(T) < +\infty \}, \quad \mathcal{N}_k(M) := \{ T \in \mathcal{D}_k(M) \text{ s.t. } M(T), M(\partial T) < +\infty \}, \quad \mathcal{R}_k(M) := \{ T \in \mathcal{D}_k(M) \text{ s.t. } T \text{ is integer-multiplicity rectifiable} \},
\]

where \( M(\cdot) \) denotes the mass of a current (see [19] for further explanations). Moreover, given any \( F \in \Omega^k_1(M) \) we define the \((m-k)\)-current associated to \( F \) by

\[
\langle T_F, \omega \rangle := \int_M F \wedge \omega, \quad \forall \omega \in \mathcal{D}^{m-k}(M).
\]

2. The strong \( L^p \)-approximation Theorem

In this section we provide a proof of Theorem [11]. In an attempt to make the proof more accessible, we reformulate the Theorem in terms of vector fields. For any Radon measure \( \mu := f \mathcal{L}^n \) with \( f = (\frac{1}{2} - \| \cdot \|_\infty)^q \), with \( q \in (-\infty, 1] \) let

\[
L^p_{\mu}(Q_1(0)) := \{ V \in L^p(Q_1(0), \mu) \text{ vector field s.t. } \ast V^\flat \in \Omega^{n-1}_{p, \mu}(Q_1(0)) \}
\]

and let

\[
L^p_{Z}(Q_1(0), \mu) := \{ V \in L^p(Q_1(0), \mu) \text{ vector field s.t. } \ast V^\flat \in \Omega^{n-1}_{p, Z}(Q_1(0)) \}.
\]

We will sometime write \( L^p_{\mu}(Q_1(0)) \) for \( L^p_{\mu}(Q_1(0), \mathcal{L}^n) \), where \( \mathcal{L}^n \) denotes the \( n \)-dimensional Lebesgue measure.
Theorem 2.1. Let $V \in L^p_{Z}(Q^1_1(0), \mu)$. The following facts hold:

1. if $q \in [0, 1]$ and $p \in \left[1, \frac{n}{n-1}\right)$, then there exists a sequence $\{V_k\}_{k \in \mathbb{N}} \subset L^p_{R}(Q^1_1(0), \mu)$ such that $V_k \to V$ strongly in $L^p_{R}(Q^1_1(0), \mu)$;
2. if $q \in (-\infty, 0]$ and $p \in \left[\frac{n}{n-1}, +\infty\right)$, then $\text{div}(V) = 0$ distributionally on $Q^1_n(0)$.

The case $n = 1$ is particularly easy and is treated in Lemma 2.6. For the proof in the case $n \geq 2$ we follow the ideas of [21] and [11]. We present here a plan of the proof, reducing to the case $q = 0$ for simplicity and without losing generality.

First of all we show that for any $\varepsilon > 0$ it is possible to decompose $Q^1_1(0)$ into cubes $Q$ of size $\varepsilon$ (plus a negligible rest) so that

$$\int_{\partial Q} V \cdot \nu_{\partial Q} d\mathcal{H}^{n-1} \in \mathbb{Z}$$

and so that the number of cubes where the integral is different from zero is controlled (Section 2.1). We will then show that $V$ can be approximated on the boundaries of the small cubes $Q$ by smooth vector fields $(V_\varepsilon)_{\varepsilon > 0}$ with similar properties (Section 2.2). In Section 2.3 we show that the vector fields $V_\varepsilon$ can be extended inside the cubes $Q$ in such a way that the extension $\hat{V}_\varepsilon$ has a finite number of singularities in $Q$ (more precisely $\hat{V}_\varepsilon|Q \in L^p_{R}(Q)$) and is close to $V$ in $L^p(Q)$. In section 2.4 we will combine the previous elements to show that the approximating fields constructed above (up to some shifting and smoothing) satisfy the claim of the Theorem.

2.1. Choice of a suitable cubic decomposition

Fix any $\varepsilon \in (0, 1/4)$ and $a \in Q_1(0)$. Let

$$q_\varepsilon := \max\{q \in \mathbb{N} \text{ s.t. } 1 - \varepsilon \leq q\},$$
$$C_\varepsilon := \left\{\left(j + \frac{1}{2}\right)\varepsilon - \frac{1}{2} \text{ with } j = 1, \ldots, q_\varepsilon - 1\right\},$$
$$\mathcal{E}_{\varepsilon,a} := \left\{Q_\varepsilon(x) + a, \text{ with } x \in C_\varepsilon\right\}.$$

We say that $\mathcal{E}_{\varepsilon,a}$ is the cubic decomposition of $Q_1(0)$ with origin in $a$ and mesh thickness $\varepsilon$. Let

$$\mathcal{F}_{\varepsilon,a} := \left\{F | F \text{ is an (n-1)-dimensional face of } \partial Q, \text{ for some open cube } Q \in \mathcal{E}_{\varepsilon,a}\right\},$$
$$S_{\varepsilon,a} := \bigcup_{F \in \mathcal{F}_{\varepsilon,a}} F.$$

We say that $S_{\varepsilon,a}$ is the $(n-1)$-skeleton of the cubic decomposition $\mathcal{E}_{\varepsilon,a}$.

Lemma 2.1 (Choice of the cubic decomposition). Let $n \in \mathbb{N}_{>0}$. Let $V \in L^p_{Z}(Q_1(0), \mu)$ where $\mu := f\mathcal{L}^n$ with

$$f := \left(\frac{1}{2} - ||\cdot||_{\infty}\right)^q$$

for some $q \in (-\infty, 1]$. Then, there exists a subset $E_V \subset (0, 1)$ satisfying the following properties:
1. \( L^1((0,1) \setminus E_V) = 0; \)
2. for every \( \varepsilon \in E_V \), there exists \( a_\varepsilon \in Q_\varepsilon(0) \) such that \( \varepsilon \in R_{V,cQ} \) for every \( Q \in \mathcal{C}_\varepsilon,a_\varepsilon \) and

\[
\lim_{\varepsilon \in E_V} \varepsilon \left( \sum_{Q \in \mathcal{C}_\varepsilon,a_\varepsilon} \int_{\partial Q} |V - (V)_Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \right) = 0,
\]

where \( (V)_Q = \int_Q V \, d\mathcal{L}^n. \)

**Proof.** For any \( x \in Q_1(0) \) let \( R_{V,x} := R_{V^0,x} \) (see Definition 1.2). By assumption

\[
\int_0^1 \int_{Q_1(0)} \mathbb{1}_{\rho \in R_{V,x}} \, d\mathcal{L}^n \, d\rho = \int_{Q_1(0)} \mathcal{L}^1(R_{V,x}) \, d\mathcal{L}^n = \int_{Q_1(0)} 2 \text{dist}(x, \partial Q_1(0)) \, d\mathcal{L}^n
\]

(2.2)

For any \( \rho \in (0,1) \) let

\[
X_\rho = \{ x \in Q_1(0) : \rho \notin R_{V,x} \},
\]

then by (2.2) \( \mathcal{L}^n(X_\rho) = 0 \) for a.e. \( \rho \in (0,1) \). Now notice that for any \( \rho \in (0,1) \)

\[
\mathcal{L}^n(X_\rho) \supseteq \sum_{c \in C_\rho} \int_{Q_\rho(c)} \mathbb{1}_{X_\rho}(x + c) \, d\mathcal{L}^n = \int_{Q_\rho(0)} \sum_{c \in C_\rho} \mathbb{1}_{X_\rho}(x + c) \, d\mathcal{L}^n,
\]

thus for a.e. \( \rho \in (0,1) \) we have that for a.e. \( a_\rho \in Q_\rho(0) \) there holds \( \rho \in R_{V,cQ} \) for any \( Q \in \mathcal{C}_\rho,a_\rho \). Let \( E_V \) be the set of all such \( \rho \in (0,1) \).

Now let \( \varepsilon \in E_V \). We claim that

\[
I_\varepsilon := \int_{Q_\varepsilon(0)} \sum_{Q \in \mathcal{C}_\varepsilon,a_\varepsilon} \int_{\partial Q} |V(x) - (V)_Q|^p f(c_Q) \, d\mathcal{H}^{n-1}(x) \, d\mathcal{L}^n(a) = o(\varepsilon^{n-1}), \quad \text{as} \ \varepsilon \to 0^+ \text{ in } E_V.
\]

(2.3)

Indeed, let \( F \) be the set of the faces of the cube \( Q_1(0) \subset \mathbb{R}^n \) and notice that

\[
I_\varepsilon = \int_{Q_\varepsilon(0)} \sum_{F_0 \in F} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \left| V(x + c + a) - \int_{Q_\varepsilon(c+a)} V \right|^p f(c+a) \, d\mathcal{H}^{n-1}(x) \, d\mathcal{L}^n(a)
\]

\[
= \sum_{F_0 \in F} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \left| V(x + c + a) - \int_{Q_\varepsilon(c+a)} V \right|^p f(c+a) \, d\mathcal{L}^n(a) \, d\mathcal{H}^{n-1}(x)
\]

\[
= \sum_{F_0 \in F} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \left| V(x + y) - \int_{Q_\varepsilon(y)} V \right|^p f(y) \, d\mathcal{L}^n(y) \, d\mathcal{H}^{n-1}(x).
\]

Observe that for any \( c \in C_\varepsilon, x \in \partial Q_\varepsilon(0) \)

\[
\int_{Q_\varepsilon(c)} \left| V(x + y) - \int_{Q_\varepsilon(y)} V \right|^p f(y) \, d\mathcal{L}^n(y)
\]

12
As \( \tilde{V} \) is dense in \( L^p(Q_1(0), \mu) \) (see [20, Theorem 4.3]), given any \( \delta > 0 \) we can find \( \tilde{V} \in C_c^0(Q_1(0)) \) such that

\[
\int_{Q_1(0)} |V - \tilde{V}|^p d\mu \leq \delta.
\]

Notice that by Taylor’s Theorem

\[
\left| \frac{f(x + \alpha) - f(x)}{f(x)} \right| = \left( \frac{1}{2} - \frac{||x + \alpha||_\infty}{||x||_\infty} \right)^q \leq q \sup_{y \in Q_1(0)} \left( \frac{1}{2} - \frac{||x||_\infty}{||y||_\infty} \right)^q \leq q \varepsilon^{q-1} \left( \frac{1}{2} - ||x||_\infty \right)^{-1} \leq C
\]

for every \( x \in Q_{1-\varepsilon}(0), \alpha \in Q_\varepsilon(0) \) and for some constant \( C > 0 \) depending only on \( q \). Thus

\[
\|V - V(\cdot - \alpha)\|_{L^p(Q_{1-\varepsilon}(0), \mu)}^p \leq 4^{p-1} \left( \int_{Q_{1-\varepsilon}(0)} |V - \tilde{V}|^p d\mu + \int_{Q_{1-\varepsilon}(0)} |\tilde{V} - V(\cdot - \alpha)|^p d\mu \right)
\]

\[
+ \int_{Q_{1-\varepsilon}(0)} |\tilde{V}(\cdot - \alpha) - V(\cdot - \alpha)|^p d\mu
\]

\[
= 4^{p-1} \left( 2 \int_{Q_1(0)} |V - \tilde{V}|^p d\mu + \int_{Q_{1-\varepsilon}(0)} |\tilde{V} - V(\cdot - \alpha)|^p d\mu \right)
\]

\[
+ \int_{Q_{1-\varepsilon}(0)} |\tilde{V} - \tilde{V} f(\cdot + \alpha) - \tilde{V} f(\cdot - \alpha) f| d\mu
\]

\[
\leq 4^{p-1}(2 + C)\delta + 4^{p-1} \int_{Q_{1-\varepsilon}(0)} |\tilde{V} - \tilde{V}(\cdot - \alpha)|^p d\mu.
\]

As \( \tilde{V} \in C_c^0(Q_1(0)) \)

\[
\sup_{\alpha \in Q_\varepsilon(0)} \|\tilde{V} - \tilde{V}(\cdot - \alpha)\|_{L^p(Q_{1-\varepsilon}(0), \mu)}^p \to 0 \text{ as } \varepsilon \to 0^+,
\]

\[
\leq \int_{\mathcal{F}} \int_{Q_{\varepsilon}(0)} \int_{Q_{\varepsilon}(c)} |V(x + y) - V(z + y)|^p f(y)d\mathcal{L}^n(y)d\mathcal{L}^n(z).
\]

Thus for any \( F_0 \in \mathcal{F} \)

\[
\sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_{\varepsilon}(c)} |V(x + y) - V(z + y)|^p f(y)d\mathcal{L}^n(y)d\mathcal{L}^n(z) \]

\[
\leq \sum_{\varepsilon \in \mathcal{F}} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_{\varepsilon}(c)} |V(x + y) - V(z + y)|^p f(y)d\mathcal{L}^n(y)d\mathcal{L}^n(z) \]

\[
\leq 2^{p-1} \varepsilon^n \int_{Q_{\varepsilon}(0)} |V(x + y) - V(y)|^p f(y)d\mathcal{L}^n(y)d\mathcal{L}^n(z) \]

\[
\leq 2^p \varepsilon^n \sup_{\alpha \in Q_{\varepsilon}(0)} \|V - V(\cdot - \alpha)\|_{L^p(Q_{1-\varepsilon}, \mu)}^p.
\]
we have

\[ \limsup_{\varepsilon \to 0^+} \sup_{\alpha \in Q_\varepsilon(0)} \| V - V(\cdot - \alpha) \|_{L^p(Q_{1-\varepsilon}(0), \mu)}^p \leq 4^{p-1}(2 + C)\delta. \]

By letting \( \delta \to 0^+ \) in the previous inequality we get

\[ \sup_{\alpha \in Q_\varepsilon(0)} \| V - V(\cdot - \alpha) \|_{L^p(Q_{1-\varepsilon}(0), \mu)}^p \to 0 \text{ as } \varepsilon \to 0^+, \] (2.5)

and claim (2.3) follows.

By Fubini’s theorem, for every fixed \( \varepsilon \in E \), there exists some non-negligible subset \( T_\varepsilon \subset Q_\varepsilon(0) \) such that for any \( a \in T_\varepsilon \), \( \varepsilon \in R_{V,c,Q} \) for any \( Q \in C_{\varepsilon,a} \) and

\[
\sum_{Q \in C_{\varepsilon,a}} \int_{\partial Q} |V - (V)_{Q}|^p f(c_Q) \, d\mathcal{H}^{n-1} \leq \frac{1}{\varepsilon^n} \sum_{Q \in C_{\varepsilon,a}} \int_{\partial Q} |V - (V)_{Q}|^p f(c_Q) \, d\mathcal{H}^{n-1} \, d\mathcal{L}^n(a) = \frac{1}{\varepsilon^n} I_{\varepsilon}.
\]

By (2.3), for every \( a \in T_\varepsilon \) we have

\[
\sum_{Q \in C_{\varepsilon,a}} \int_{\partial Q} |V - (V)_{Q}|^p f(c_Q) \, d\mathcal{H}^{n-1} = o(\varepsilon^{-1})
\]
as \( \varepsilon \to 0^+ \) in \( E \). The statement follows.

Fix any \( V \in L^p_{ loc}(Q_1(0)) \) and \( \varepsilon \in E \). From now on, we will denote simply by \( C_{\varepsilon} \) the cubic decomposition \( C_{\varepsilon,a} \), provided by Lemma 2.1. Accordingly, the subscript "\( a \)" will be omitted in any writing referring to such a cubic decomposition.

Given any \( Q \in C_{\varepsilon} \), we say that \( Q \) is a good cube if

\[ \int_{\partial Q} V \cdot \nu_{\partial Q} = 0 \]

and that \( Q \) is a bad cube otherwise. We denote by \( C_{\varepsilon}^g \) the subfamily of \( C_{\varepsilon} \) made of all the good cubes and by \( C_{\varepsilon}^b \) the one made of all the bad cubes. Moreover, we let

\[
\Omega_{\varepsilon} := \bigcup_{Q \in C_{\varepsilon}} Q, \quad \Omega_{\varepsilon}^g := \bigcup_{Q \in C_{\varepsilon}^g} Q, \quad \Omega_{\varepsilon}^b := \bigcup_{Q \in C_{\varepsilon}^b} Q, \\
S_{\varepsilon} := \bigcup_{Q \in C_{\varepsilon}} \partial Q, \quad S_{\varepsilon}^g := \bigcup_{Q \in C_{\varepsilon}^g} \partial Q, \quad S_{\varepsilon}^b := \bigcup_{Q \in C_{\varepsilon}^b} \partial Q.
\]

**Lemma 2.2.** Assume that \( n \geq 2 \). Then, we have

\[
\lim_{\varepsilon \to 0^+} \varepsilon^n \sum_{Q \in \Omega_{\varepsilon}^g} f(c_Q) = 0.
\]

In particular if \( q = 0 \) (and thus \( \mu = \mathcal{L}^n \)) we have

\[
\lim_{\varepsilon \to 0^+} \mathcal{L}^n(\Omega_{\varepsilon}^b) = 0.
\]
Proof. Notice that by estimate (2.4)

\[ \frac{|f - f(c_Q)|}{f} \leq C \]  
for some universal constant $C > 0$. For every bad cube $Q \in \mathcal{C}_e$, it holds that

\[ 1 \leq \left| \int_{\partial Q} V \cdot v_{\partial Q} \, d\mathcal{H}^{n-1} \right| \leq \int_{\partial Q} |V| \, d\mathcal{H}^{n-1}. \]

By multiplying the previous inequality by $f(c_Q)$ and summing over all the bad cubes, we get

\[
\sum_{Q \in b \mathcal{C}_E} f(c_Q) \leq \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V| f(c_Q) \, d\mathcal{H}^{n-1}
\leq \left( \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V|^p f(c_Q) \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left( \sum_{Q \in b \mathcal{C}_E} f(c_Q) \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}}
= (2n)^{\frac{1}{p}} \varepsilon \frac{\alpha}{p} \left( \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V|^p f(c_Q) \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left( \sum_{Q \in b \mathcal{C}_E} f(c_Q) \right)^{\frac{1}{p}},
\]

which is equivalent to

\[
\sum_{Q \in b \mathcal{C}_E} f(c_Q) \leq (2n)^{p-1} \varepsilon (p-1)(n-1) \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V|^p f(c_Q) \, d\mathcal{H}^{n-1}.
\]

Hence, by the triangle inequality, we get

\[
\sum_{Q \in b \mathcal{C}_E} f(c_Q) \leq (4n)^{p-1} \varepsilon (p-1)(n-1)^{-1} \left( \varepsilon \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \right)
+ 2n \sum_{Q \in b \mathcal{C}_E} \int_{Q} |V|^p f(c_Q) \, d\mathcal{L}^n
\leq (4n)^{p-1} \varepsilon (p-1)(n-1)^{-1} \left( \varepsilon \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \right)
+ 2n \sum_{Q \in b \mathcal{C}_E} \int_{Q} |V|^p f(c_Q) \, d\mathcal{L}^n + 2n \sum_{Q \in b \mathcal{C}_E} \int_{Q} \left| |V|^p \frac{f(c_Q) - f}{f} \right| \, d\mathcal{L}^n
\leq (4n)^{p-1} \varepsilon (p-1)(n-1)^{-1} \left( \varepsilon \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \right)
+ 2n(1 + C) \int_{\Omega_e} |V|^p f \, d\mathcal{L}^n. \]

Therefore

\[
\varepsilon^n \sum_{Q \in b \mathcal{C}_E} f(c_Q) \leq (4n)^{p-1} \varepsilon (p-1)(n-1)^{-1} \left( \varepsilon \sum_{Q \in b \mathcal{C}_E} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \right)
\]

15
+ 2n(1 + C) \int_{Q_1(0)} \vert V \vert^p f \, d\mathcal{L}^n \right)

and the statement follows from (2.1) (here we need the assumption \( n > 1 \)). \qed

Remark 2.1. Assume that \( p \in \left[ n/(n - 1), +\infty \right) \). In this case \( \varepsilon^{(p-1)(n-1)-1} \) remains bounded as \( \varepsilon \to 0^+ \). Now by Lemma 2.1

\[ \varepsilon \sum_{Q \in \mathcal{C}_\varepsilon} \int_{\partial Q} \vert V - (V)_Q \vert^p f(c_Q) \, d\mathcal{H}^{n-1} \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V. \]

Moreover, by Lemma 2.2 we have

\[ \int_{Q_\varepsilon^b} f \, d\mathcal{L}^n = (1 + C)\varepsilon^n \sum_{Q \in \mathcal{C}_\varepsilon^b} f(c_Q) \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V. \]

This implies \( \mathcal{L}^n(Q_\varepsilon^b) \to 0 \) as \( \varepsilon \to 0^+ \) in \( E_V \), therefore

\[ \int_{Q_\varepsilon^b} \vert V \vert^p f \, d\mathcal{L}^n \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V \]

by absolute continuity of the integral. Thus it follows from (2.7) that

\[ \sum_{Q \in \mathcal{C}_\varepsilon^b} f(c_Q) \to 0^+ \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V. \]

Let \( N^b_\varepsilon \) be the number of bad cubes in \( \mathcal{C}_\varepsilon \). Notice that for \( q \leq 0 \), we have \( f \geq 2^{-q} \) on \( Q_1(0) \). This implies

\[ N^b_\varepsilon \leq 2^q \sum_{Q \in \mathcal{C}_\varepsilon^b} f(c_Q) \to 0^+ \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V. \]

Since \( N^b_\varepsilon \in \mathbb{Z} \) for any \( \varepsilon \in E_V \), \( N^b_\varepsilon = 0 \) for every \( \varepsilon \in E_V \) small enough. Hence, whenever \( p \in \left[ n/(n - 1), +\infty \right) \) and \( q \leq 0 \) we will assume, without losing generality, that there are no bad cubes in our chosen decomposition.

2.2. Smoothing on the \((n-1)\)-skeleton of the cubic decomposition

Lemma 2.3. Let \( Q \subset \mathbb{R}^n \) be an \( n \)-dimensional cube with side length \( R \). Let \( V \in L^p(Q, \mathbb{R}^m) \).

Let \( \varepsilon > 0 \). There exists \( V_\varepsilon \in C^\infty_c(Q, \mathbb{R}^m) \) such that

\[ \int_Q V_\varepsilon \, d\mathcal{L}^n = \int_Q V \, d\mathcal{L}^n \]

and

\[ \| V_\varepsilon - V \|_{L^p(Q)} < \varepsilon. \]
Proof. Without loss of generality, we will assume that $Q$ is centered in the origin of $\mathbb{R}^n$. Let $\psi \in C_c^\infty(\frac{1}{2}Q)$ and $r_0 \in (1/2, 1)$ such that
\[
\int_Q \psi \, d\mathcal{L}^n = 1 \quad \text{and} \quad R^n(1 - r_0^n) < \frac{\varepsilon}{\|\psi\|_{L^p(Q)}}.
\]

Let $r \in (r_0, 1)$ be such that
\[
\|V\|_{L^p(Q \setminus rQ)} \leq \min \left\{ \left( \frac{1}{\|\psi\|_{L^p(Q)}} \right)^{\frac{1}{p}}, \varepsilon \right\}.
\]

Set
\[
s := \int_{Q \setminus rQ} V \, d\mathcal{L}^n, \quad \tilde{V} := \chi_{rQ} V + s\psi \in L^p(Q, \mathbb{R}^m).
\]

Then
\[
\int_Q \tilde{V} \, d\mathcal{L}^n = \int_{Q \setminus rQ} V \, d\mathcal{L}^n + \left( \int_{Q \setminus rQ} V \, d\mathcal{L}^n \right) \int_Q \psi \, d\mathcal{L}^n = \int_Q V \, d\mathcal{L}^n.
\]

Moreover
\[
|s| = \left| \int_{Q \setminus rQ} V \, d\mathcal{L}^n \right| \leq |Q \setminus rQ| \left( \frac{1}{\|\psi\|_{L^p(Q)}} \right)^{\frac{1}{p}} \left\| V \right\|_{L^p(Q \setminus rQ)} \leq (R^n(1 - r^n))^{\frac{1}{p}} \left( \frac{\varepsilon}{\|\psi\|_{L^p(Q)}} \right)^{\frac{1}{p}} \leq \varepsilon.
\]

Therefore
\[
\|s\psi\|_{L^p(Q)} = \|\psi\|_{L^p(Q)} |s| \leq \varepsilon
\]
and, by choice of $\tilde{V}$,
\[
\|V - \tilde{V}\|_{L^p(Q)} \leq \|s\psi\|_{L^p(Q)} + \|V\|_{L^p(Q \setminus rQ)} \leq 2\varepsilon.
\]

Notice that $\tilde{V}|_{Q \setminus rQ} \equiv 0$.

Let $\eta \in C_c^\infty(B_1(0))$ with $\int_{B_1(0)} \eta \, d\mathcal{L}^n = 1$. For any $\delta > 0$ let
\[
\eta_\delta(x) := \frac{1}{\delta^n} \eta \left( \frac{x}{\delta} \right) \quad \forall \, x \in \mathbb{R}^n.
\]

Choose $\delta_0 > 0$ such that
\[
2\delta_0 < \text{dist} \left( \partial Q, \partial (rQ) \right) \quad \text{and} \quad \|\tilde{V} - \tilde{V} * \eta_{\delta_0}\|_{L^p(Q)} \leq \varepsilon.
\]

Set $V_\varepsilon := \tilde{V} * \eta_{\delta_0}$. Then $V_\varepsilon \in C_c^\infty(Q)$,
\[
\int_Q V_\varepsilon \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \eta_{\delta_0} \, d\mathcal{L}^n \int_Q \tilde{V} \, d\mathcal{L}^n = \int_Q \tilde{V} \, d\mathcal{L}^n = \int_Q V \, d\mathcal{L}^n
\]
and
\[
\|V - V_\varepsilon\|_{L^p(Q)} \leq \|V - \tilde{V}\|_{L^p(Q)} + \|\tilde{V} - V_\varepsilon\|_{L^p(Q)} \leq 3\varepsilon.
\]

$\blacksquare$
2.3. Extensions on good and bad cubes

**Lemma 2.4** (Extension on the good cubes). Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and $p \in [1, \infty)$. Let $f \in L^p(\partial \Omega)$ with
\[ \int_{\partial \Omega} f \, d\mathcal{H}^{n-1} = 0 \quad (2.8) \]
There exists a vector field $V \in L^p(\Omega)$ such that
\[ \int_{\Omega} V \cdot \nabla \varphi \, d\mathcal{L}^n = \int_{\partial \Omega} f \varphi \, d\mathcal{H}^{n-1} \quad \forall \, \varphi \in C^\infty(\mathbb{R}^n) \quad (2.9) \]
and
\[ \int_{\Omega} |V|^p \, d\mathcal{L}^n \leq C(p, \Omega) \int_{\partial \Omega} |f|^p \, d\mathcal{H}^{n-1} \quad (2.10) \]
for some constant $C(p, \Omega)$ depending only on $p$ and $\Omega$.
Moreover if $p = 1$, then $V \in L^q(\Omega)$ for any $q \in [1, \frac{n}{n-1})$.

**Remark 2.2.** Observe that (2.9) implies that $V$ is a distributional solution of the following Neumann problem
\[
\begin{align*}
\text{div}(V) &= 0 \quad \text{in} \ \Omega \\
V \cdot \nu_{\partial \Omega} &= f \quad \text{on} \ \partial \Omega.
\end{align*}
\]

**Proof.**

**Step 1:** First we consider the case $p \in (1, \infty)$.
Let $p' := \frac{p}{p-1}$. For any $u \in W^{1,p'}(\Omega)$ let
\[ E_p(u) = \frac{1}{p'} \int_{\Omega} |
abla u|^{p'} \, d\mathcal{L}^n - \int_{\partial \Omega} fu \, d\mathcal{H}^{n-1}. \]
Recall that any function $u \in W^{1,p'}(\Omega)$ has a trace in $L^{p'}(\partial \Omega)$, and that the trace operator is continuous. Thus for any $v \in W^{1,p'}(\Omega)$ with $\int_{\Omega} v = 0$ by Poincaré Lemma there holds
\[ \left| \int_{\partial \Omega} fv \, d\mathcal{H}^{n-1} \right| \leq \|f\|_{L^{p'}(\partial \Omega)} \|v\|_{L^{p'}(\partial \Omega)} \leq C(p, \Omega) \|f\|_{L^p(\partial \Omega)} \|\nabla v\|_{L^{p'}(\Omega)} \]
for some constant $C(p, \Omega)$ depending only on $p$ and $\Omega$. In particular the energy $E_p$ is well defined on $W^{1,p'}(\Omega)$.
Let
\[ W^{1,p'}(\Omega) := \left\{ v \in W^{1,p'}(\Omega), \int_{\Omega} v \, d\mathcal{L}^n = 0 \right\} \]
and observe that $E_p$ is strictly convex on $W^{1,p'}(\Omega)$. Let $u$ be the unique minimizer of $E_p$ in $W^{1,p'}(\Omega)$. Then
\[ \int_{\Omega} |\nabla u|^{p'-2} \nabla u \cdot \nabla \varphi \, d\mathcal{L}^n = \int_{\partial \Omega} f \varphi \, d\mathcal{H}^{n-1}, \quad \forall \varphi \in C^\infty(\mathbb{R}^n). \quad (2.11) \]
\[ \text{The argument above shows that (2.11) holds for any } \varphi \in C^\infty(\mathbb{R}^n) \text{ with } \int_{\Omega} \varphi \, d\mathcal{L}^n = 0, \text{ but assumption (2.8) implies that (2.11) remains valid for any } \varphi \in C^\infty(\mathbb{R}^n). \]
Moreover, as \( u \) is a minimizer of \( E_p \), \( E_p(u) \leq E_p(0) = 0 \). It follows that
\[
\frac{1}{p'} \int_\Omega |\nabla u|^{p'} d\mathcal{L}^n \leq \int_{\partial \Omega} f u d\mathcal{H}^{n-1} \leq \|f\|_{L^p(\partial \Omega)}\|u\|_{L^{p'}(\partial \Omega)} \leq C(p, \Omega)\|f\|_{L^p(\partial \Omega)}\|\nabla u\|_{L^{p'}(\Omega)}.
\]
Thus
\[
\int_\Omega |\nabla u|^{p'} d\mathcal{L}^n \leq (p' C(p, \Omega))^p \int_{\partial \Omega} |f|^p d\mathcal{H}^{n-1}.
\]
Set \( V := |\nabla u|^{p'-2} \nabla u \) in \( \Omega \). Then by (2.11)
\[
\hat{\Omega} V \cdot \nabla \varphi d\mathcal{L}^n = \hat{\partial \Omega} f \varphi d\mathcal{H}^{n-1} \quad \forall \varphi \in C^\infty(\mathbb{R}^n)
\]
and
\[
\int_\Omega |V|^p d\mathcal{L}^n = \int_\Omega |\nabla u|^{p'} d\mathcal{L}^n \leq (p' C(p, \Omega))^p \int_{\partial \Omega} |f|^p d\mathcal{H}^{n-1}.
\]

**Step 2:** Next we consider the case \( p = 1 \).
Let \( s > n \). For any \( u \in W^{1,s}(\Omega) \) let
\[
E_s(u) = \frac{1}{s} \int_\Omega |\nabla u|^s d\mathcal{L}^n - \int_{\partial \Omega} f u d\mathcal{H}^{n-1}.
\]
Notice that \( E_s \) is well defined and strictly convex in \( \dot{W}^{1,s}(\Omega) \).
Recall the Sobolev embedding
\[
W^{1,s}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})
\]
for \( \alpha = 1 - \frac{n}{s} \). Then for any \( u \in W^{1,s}(\Omega) \) the trace of \( u \) on \( \partial \Omega \) lies in \( C^{0,\alpha}(\partial \Omega) \) and if \( \int_\Omega u d\mathcal{L}^n = 0 \). Poincaré inequality implies
\[
\|u\|_{L^\infty(\partial \Omega)} \leq C(s, \Omega)\|\nabla u\|_{L^{s}(\Omega)}
\]
for some constant \( C(s, \Omega) \) depending only on \( s \) and \( \Omega \).
Let \( u \) be the unique minimizer of \( E_s \) in \( W^{1,s}(\Omega) \). Then since \( E_s(u) \leq E_s(0) = 0 \) there holds
\[
\frac{1}{s} \int_\Omega |\nabla u|^s d\mathcal{L}^n \leq \int_{\partial \Omega} f u d\mathcal{H}^{n-1} \leq \|f\|_{L^1(\partial \Omega)}\|u\|_{L^{\infty}(\partial \Omega)} \leq C(s, \Omega)\|f\|_{L^1(\partial \Omega)}\|\nabla u\|_{L^{s}(\Omega)}.
\]
Therefore
\[
\left( \int_\Omega |\nabla u|^s d\mathcal{L}^n \right)^{\frac{1}{s}} \leq s C(s, \Omega) \int_{\partial \Omega} |f|^s d\mathcal{H}^{n-1}.
\]
Moreover, since \( u \) is a minimizer of \( E_s \),
\[
\int_\Omega |\nabla u|^{s-2} \nabla u \cdot \nabla \varphi d\mathcal{L}^n = \int_{\partial \Omega} f \varphi d\mathcal{H}^{n-1} \quad \forall \varphi \in C^\infty(\mathbb{R}^n).
\]
Similarly as before set $V := |\nabla u|^{-2} \nabla u$ in $\Omega$. Then

$$\int_{\Omega} |V| \, d\mathcal{L}^n = \int_{\Omega} |\nabla u|^{-1} \, d\mathcal{L}^n \leq \mathcal{L}^n(\Omega) \frac{1}{2} \left( \int_{\Omega} |\nabla u|^s \, d\mathcal{L}^n \right)^{\frac{1}{s}} \leq sC(s, \Omega) \mathcal{L}^n(\Omega) \frac{1}{2} \int_{\partial \Omega} |f| \, d\mathcal{H}^{n-1}. $$

Moreover

$$\int_{\Omega} |V| \, d\mathcal{L}^n = \int_{\Omega} |\nabla u|^s \, d\mathcal{L}^n < \infty$$

Proof. Without losing generality, we assume that $\varepsilon > 0$ one can choose $\varepsilon C(p, Q) = \varepsilon C(p, Q)$.

Remark 2.3. Let $Q \subset \mathbb{R}^n$ be the unit cube and let $C(p, Q)$ be the corresponding constant in (2.10). By an easy scaling argument one sees that for any $\varepsilon > 0$ one can choose $C(p, \varepsilon Q) = \varepsilon C(p, Q)$.

Lemma 2.5 (Extension on the bad cubes). Let $Q := Q_\varepsilon(c_Q) \subset \mathbb{R}^n$ and $r(x) := \|x - c_Q\|_\infty$, for every $x \in \mathbb{R}^n$.

Consider any vector field $V : Q \to \mathbb{R}^n$ having the form

$$V(x) := \frac{1}{2^{n-1}} f \left( \frac{\varepsilon x - c_Q}{r(x)} + c_Q \right) \frac{x - c_Q}{r(x)^n} \quad \forall x \in Q,$$

for some $f \in L^\infty(\partial Q)$. Then, the following facts hold:

1. $V \in L^p(Q)$ for every $p \in [1, n/(n-1))$;
2. for some constant $C(n, p) > 0$ depending only on $n$ and $p$ we have

$$\int_Q |V|^p \, d\mathcal{L}^n \leq \varepsilon C(n, p) \int_{\partial Q} |f|^p \, d\mathcal{H}^{n-1},$$

(2.12)

3. for every $\varphi \in C^\infty(\mathbb{R}^n)$ we have

$$\int_Q V \cdot \nabla \varphi \, d\mathcal{L}^n = \int_{\partial Q} f \varphi \, d\mathcal{H}^{n-1} - \left( \int_{\partial Q} f \, d\mathcal{H}^{n-1} \right) \varphi(c_Q).$$

(2.13)

Proof. Without losing generality, we assume that $\varepsilon = 1$ and $c_Q = 0$. First, notice that $r : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz map such that $|\nabla r(x)| = 1$, for a.e. $x \in \mathbb{R}^n$. Moreover, since all the norms are equivalent on $\mathbb{R}^n$ there exists a constant $\tilde{C}(n) > 0$ depending only on $n$ such that $|x| \leq \tilde{C}r(x)$, for a.e. $x \in \mathbb{R}^n$. Now choose any $p \in (1, n/(n-1))$. By coarea formula we have

$$\int_Q |V|^p \, d\mathcal{L}^n \leq \frac{\tilde{C}^p}{2^{(n-1)p}} \int_0^\frac{1}{\rho^{n-1}} \int_{\partial Q_\varepsilon(0)} \left| f \left( \frac{x}{2\rho} \right) \right|^p \, d\mathcal{H}^{n-1}(x) \, d\rho
$$

$$= \frac{\tilde{C}^p}{2^{(n-1)p}} \left( \int_0^1 \frac{1}{\rho^{n-1}(p-1)} \, d\rho \right) \left( \int_{\partial Q} |f(y)|^p \, d\mathcal{H}^{n-1}(y) \right)
$$

$$= C \int_{\partial Q} |f|^p \, d\mathcal{H}^{n-1},$$

20
with

\[ C = C(n, p) := \frac{C_p}{2^{(n-1)(p-1)}} \int_0^{\frac{1}{2}} \frac{1}{\rho^{(n-1)(p-1)}} d\rho < +\infty. \]

Hence, 1. and 2. follow in once. We remark that the condition \( p \in \left[ 1, n/(n-1) \right) \) is needed in order to guarantee the convergence of the integral in \( \rho \).

We still need to prove 3. Pick any \( \varphi \in C^\infty(\mathbb{R}^n) \). By the coarea formula we have

\[
\int_Q V \cdot \nabla \varphi d\mathcal{L}^n = \frac{1}{2^{n-1}} \int_0^{\frac{1}{2}} \int_{\partial Q_{2\rho}(0)} f \left( \frac{x}{2\rho} \right) (x \cdot \nabla \varphi(x)) d\mathcal{H}^{n-1}(x) d\rho
\]

\[
= 2 \int_0^{\frac{1}{2}} \int_{\partial Q} f(y) (y \cdot \nabla \varphi(2\rho y)) d\mathcal{H}^{n-1}(y) d\rho
\]

\[
= \int_{\partial Q} f(y) \int_0^{\frac{1}{2}} \frac{d}{d\rho} (\varphi(2\rho y)) d\rho d\mathcal{H}^{n-1}(y)
\]

\[
= \int_{\partial Q} f \varphi d\mathcal{H}^{n-1} - \left( \int_{\partial Q} f d\mathcal{H}^{n-1} \right) \varphi(0)
\]

and 3. follows.

### 2.4. Proof of Theorem 1.1

We are finally ready to prove Theorem 1.1.

**Proof.** Let \( V \in L^p_\nu(Q_1(0)) \) and let \( \varepsilon \in E_V \) (constructed in Lemma 2.1). First, we notice that by using Lemma 2.3 separately on every face \( F \in \mathcal{F}_\varepsilon \) we can build a vector field \( V_\varepsilon \in C^\infty(S_\varepsilon) \) such that

\[
\int_{\partial Q} V_\varepsilon \cdot \nu_{\partial Q} d\mathcal{H}^{n-1} = \int_{\partial Q} V \cdot \nu_{\partial Q} d\mathcal{H}^{n-1} \quad \forall Q \in \mathcal{C}_\varepsilon
\]

and

\[
\sum_{Q \in \mathcal{C}_\varepsilon} \int_{\partial Q} |V_\varepsilon - V|^p f(c_Q) d\mathcal{H}^{n-1} < \varepsilon.
\]

Let \( \tilde{V}_\varepsilon \) be the vector field defined \( \mathcal{L}^n \)-a.e. on \( \Omega_\varepsilon \) as follows:

1. if \( Q \in \mathcal{C}_\varepsilon \) is a good cube, then we let \( \tilde{V}_\varepsilon := W_\varepsilon + (V)_Q \) on \( Q \), where \( W_\varepsilon \) is the extension of the datum \( f := (V_\varepsilon - (V)_Q) \cdot \nu_{\partial Q} \) given by Lemma 2.4 (notice that for any good cube condition (2.8) is satisfied by our choice of \( f \));

2. if \( Q \in \mathcal{C}_\varepsilon \) is a bad cube, then we let

\[
\tilde{V}_\varepsilon := \frac{1}{2^{n-1}} f \left( \frac{\varepsilon}{2} \frac{x - c_Q}{\|x - c_Q\|_\infty} + c_Q \right) \frac{x - c_Q}{\|x - c_Q\|_\infty}, \quad \forall x \in Q,
\]

with \( f := V_\varepsilon \mid_{\partial Q} \cdot \nu_{\partial Q} \in L^\infty(\partial Q) \).
We recall that no bad cubes will appear in the cubic decomposition in case \( p \in \left[ n/(n-1), +\infty \right) \) (see Remark 2.1).

**Claim 1.** We claim that
\[
\text{div}(\tilde{V}_\varepsilon) = \sum_{Q \in C^b} d_Q \delta_{c_Q} \quad \text{distributionally on } \Omega_\varepsilon,
\]
where
\[
d_Q := \int_{\partial Q} V_\varepsilon \cdot \nu_{\partial Q} \, dH^{n-1} = \int_{\partial Q} V \cdot \nu_{\partial Q} \, dH^{n-1} \in \mathbb{Z} \setminus \{0\}, \quad \forall Q \in C^b_\varepsilon.
\]
Indeed, pick any \( \varphi \in C^\infty_c(\Omega_\varepsilon) \). Let \( Q \in C^g_\varepsilon \) be a good cube. By the properties of the extension given by Lemma 2.4 and the divergence theorem we have
\[
\int_Q \tilde{V}_\varepsilon \cdot \nabla \varphi \, d\mathcal{L}^n = \int_{\partial Q} (V_\varepsilon - (V)_Q) \cdot \nu_{\partial Q} \varphi \, dH^{n-1} + \int_{\partial Q} ((V)_Q \cdot \nu_{\partial Q}) \varphi \, dH^{n-1}
\]
\[
= \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q}) \varphi \, dH^{n-1}.
\]
On the other hand, let \( Q \in C^b_\varepsilon \) be a bad cube. By (2.13), we have
\[
\int_Q \tilde{V}_\varepsilon \cdot \nabla \varphi \, d\mathcal{L}^n = \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q}) \varphi \, dH^{n-1} - d_Q \langle \delta_{c_Q}, \varphi \rangle.
\]
Hence we conclude that
\[
\int_{\Omega_\varepsilon} \tilde{V}_\varepsilon \cdot \nabla \varphi \, d\mathcal{L}^n = \sum_{Q \in C^g_\varepsilon} \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q}) \varphi \, dH^{n-1} - \sum_{Q \in C^b_\varepsilon} d_Q \langle \delta_{c_Q}, \varphi \rangle
\]
\[
= - \sum_{Q \in C^b_\varepsilon} d_Q \langle \delta_{c_Q}, \varphi \rangle.
\]
The claim follows.

**Claim 2.** We claim that \( \| \tilde{V}_\varepsilon - V \|_{L^n(\Omega_\varepsilon, \mu)} \to 0 \) as \( \varepsilon \to 0^+ \) in \( E_V \).

Recall estimate (2.6) and notice that
\[
\| \tilde{V}_\varepsilon - V \|_{L^n(\Omega_\varepsilon, \mu)} \leq (1 + C) (A_\varepsilon + B_\varepsilon),
\]
with
\[
A_\varepsilon := \sum_{Q \in C^g_\varepsilon} \int_Q |\tilde{V}_\varepsilon - V|^p f(c_Q) \, d\mathcal{L}^n,
\]
\[
B_\varepsilon := \sum_{Q \in C^b_\varepsilon} \int_Q |\tilde{V}_\varepsilon - V|^p f(c_Q) \, d\mathcal{L}^n.
\]
By triangle inequality and by the estimate in Lemma 2.4, we have that
\[
A_\varepsilon \leq 2^{n-1} \left( \sum_{Q \in C^g_\varepsilon} \int_Q |\tilde{V}_\varepsilon - (V)_Q|^p f(c_Q) \, d\mathcal{L}^n + \sum_{Q \in C^b_\varepsilon} \int_Q |V - (V)_Q|^p f(c_Q) \, d\mathcal{L}^n \right)
\]
\[
A_\varepsilon \leq 2^{n-1} \left( \sum_{Q \in C^g_\varepsilon} \int_Q |\tilde{V}_\varepsilon - (V)_Q|^p f(c_Q) \, d\mathcal{L}^n + \sum_{Q \in C^b_\varepsilon} \int_Q |V - (V)_Q|^p f(c_Q) \, d\mathcal{L}^n \right)
\]
22
We notice that

\[ \sum_{Q \in \mathscr{E}_q} \int_Q |W_\epsilon|^p f(c_Q) \, d\mathcal{L}^n + \sum_{Q \in \mathscr{E}_q} \int_Q |(V - (V)_Q)^p f(c_Q) | \, d\mathcal{L}^n \]

\[ \leq 2^{p-1} \left( \sum_{Q \in \mathscr{E}_q} \int_Q |(V - (V)Q)^p f(c_Q) | \, d\mathcal{L}^n + \sum_{Q \in \mathscr{E}_q} \int_Q |V - (V)Q|^p f(c_Q) | \, d\mathcal{L}^n \right), \]

where \( C_p := C(p, Q) \) (see Remark 2.3). Again by triangle inequality and because of our choice of \( \epsilon \), we have

\[ \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V_\epsilon - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \leq 2^{p-1} \left( \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V_\epsilon - V|^p f(c_Q) \, d\mathcal{H}^{n-1} \right) \]

\[ + \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \]

\[ \leq 2^{p-1} \left( 2\epsilon^2 + \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \right). \]

Thus by Lemma 2.4 it follows that

\[ \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V_\epsilon - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0^+ \text{ in } E_V. \]

Moreover, by (2.3) we have

\[ \sum_{Q \in \mathscr{E}_q} \int_Q |V - (V)Q|^p f(c_Q) \, d\mathcal{L}^n \leq 2^p (1 + C) \sum_{Q \in \mathscr{E}_q} \int_{Q_{(0)}} |V(x + y) - V(y)|^p f(y) \, d\mathcal{L}^n \]

\[ \leq 2^p (1 + C) \int_{Q_{(0)}} \|V(x + \cdot) - V\|_{L^p(\Omega_\epsilon, \mu)}^p \rightarrow 0 \]

as \( \epsilon \rightarrow 0^+ \) in \( E_V \). Hence, \( A_\epsilon \rightarrow 0 \) as \( \epsilon \rightarrow 0^+ \) in \( E_V \).

On the other hand, by (2.12) we have

\[ B_\epsilon \leq 2^{p-1} \left( \sum_{Q \in \mathscr{E}_q} \int_Q |\tilde{V}_\epsilon|^p f(c_Q) \, d\mathcal{L}^n + \sum_{Q \in \mathscr{E}_q} \int_Q |V|^p f(c_Q) \, d\mathcal{L}^n \right) \]

\[ \leq 2^{p-1} \left( C\epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V_\epsilon|^p f(c_Q) \, d\mathcal{H}^{n-1} + \int_{\Omega_\epsilon} |V|^p f(c_Q) \, d\mathcal{L}^n \right). \]

We notice that

\[ \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V_\epsilon|^p f(c_Q) \, d\mathcal{H}^{n-1} \leq 2^{p-1} \left( \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V_\epsilon - V|^p f(c_Q) \, d\mathcal{H}^{n-1} \right) \]

\[ + \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V|^p f(c_Q) \, d\mathcal{H}^{n-1} \]

\[ \leq 4^{p-1} \left( 2\epsilon^2 + \epsilon \sum_{Q \in \mathscr{E}_q} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \right). \]
Moreover, by (2.6) we have
\[
\varepsilon \sum_{Q \in \mathcal{E}_\varepsilon^p} \int_{\partial Q} |(V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} \leq \varepsilon \sum_{Q \in \mathcal{E}_\varepsilon^p} \int_{\partial Q} \left( \int_{Q} |V|^p f(c_Q) \, d\mathcal{L}^n \right) \, d\mathcal{H}^{n-1}
\]
\[
\leq \sum_{Q \in \mathcal{E}_\varepsilon^p} \varepsilon \left( \int_{Q} |V|^p f(c_Q) \, d\mathcal{L}^n \right) \int_{\partial Q} \, d\mathcal{H}^{n-1}
\]
\[
\leq 2n \sum_{Q \in \mathcal{E}_\varepsilon^p} \int_{Q} |V|^p f(c_Q) \, d\mathcal{L}^n = 2n \int_{\Omega^b_E} |V|^p f(c_Q) \, d\mathcal{L}^n
\]
\[
\leq (1 + C)2n \int_{\Omega^b_E} |V|^p f \, d\mathcal{L}^n.
\]

Thus, we have obtained
\[
B_\varepsilon \leq C \left( \varepsilon^2 + \sum_{Q \in \mathcal{E}_\varepsilon^p} \int_{\partial Q} |V - (V)Q|^p f(c_Q) \, d\mathcal{H}^{n-1} + \int_{\Omega^b_E} |V|^p f \, d\mathcal{L}^n \right),
\]
for some constant $C > 0$ which does not depend on $\varepsilon \in E_V$. By Lemma 2.4 and Remark 2.1 we get that $B_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in $E_V$. Hence, the claim follows.

Next we show that by rescaling $V_\varepsilon$ we obtain a vector field with similar properties defined on the whole $Q_1(0)$. Let
\[
\alpha_\varepsilon := \sup \{ \alpha \in [1/2, 1) \, \text{s.t.} \, Q_1(0) \subset \alpha^{-1} \Omega_\varepsilon \}, \quad \forall \varepsilon \in E_V.
\]
Notice that $\alpha_\varepsilon \to 1^-$ as $\varepsilon \to 0^+$ in $E_V$. Define the vector field $\nabla_\varepsilon := \alpha_\varepsilon^{-1} \nabla_\varepsilon(\alpha_\varepsilon \cdot) : Q_1(0) \to \mathbb{R}^n$. It’s straightforward that $\nabla_\varepsilon \in L^p(Q_1(0), \mu)$ in case $p > 1$ and $\nabla_\varepsilon \in L^s(Q_1(0), \mu)$ for some $s > 1$ in case $p = 1$, for every given $\varepsilon \in E_V$. A direct computation also shows that the distributional divergence of $\nabla_\varepsilon$ on $Q_1(0)$ is given by
\[
\text{div}(\nabla_\varepsilon) = \sum_{Q \in \mathcal{E}_\varepsilon^p} \delta_{Q} \delta_{\alpha_\varepsilon^{-1} c_Q},
\]
with
\[
\delta_Q = \begin{cases} 
dQ & \text{if } \alpha_\varepsilon^{-1} c_Q \in Q_1(0), \\
0 & \text{otherwise}.
\end{cases}
\]

We claim that $\nabla_\varepsilon \to V$ in $L^p(Q_1(0), \mu)$. Indeed, we have
\[
\int_{Q_1(0)} |\nabla_\varepsilon - V|^p f \, d\mathcal{L}^n = \int_{Q_1(0)} |\alpha_\varepsilon^{n-1} \nabla_\varepsilon - V(x)|^p f(x) \, d\mathcal{L}^n(x)
\]
\[
= \alpha_\varepsilon^{p(n-1)-n} \int_{\alpha_\varepsilon Q_1(0)} |\nabla_\varepsilon(y) - \alpha_\varepsilon^{-(n-1)} V(\alpha_\varepsilon^{-1} y)|^p f(\alpha_\varepsilon^{-1} y) \, d\mathcal{L}^n(y)
\]
\[
= \alpha_\varepsilon^{p(n-1)-n} \left( \int_{\Omega_\varepsilon} |\nabla_\varepsilon(y) - \alpha_\varepsilon^{-(n-1)} V(\alpha_\varepsilon^{-1} y)|^p f(y) \, d\mathcal{L}^n(y) \right)
\]
Now we are ready to reach the conclusions 1 and 2 of Theorem 2.1. (see Lemma C.2 for the definition of \(P_{\alpha^{-1}}V\)). By Lemma C.2 and since \(\tilde{V}_\varepsilon \to V\) in \(L^p(\Omega_\varepsilon, \mu)\), our claim follows.

Thus, we have built a vector field \(\nabla\varepsilon\) such that:

1. \(\nabla\varepsilon \in L^p(Q_1(0), \mu)\) and \(\nabla\varepsilon \in L^s(Q_1(0), L^n)\) for \(s = p\) if \(p > 1\) and \(s > 1\) if \(p = 1\)\(^4\)

2. the distributional divergence of \(\nabla\varepsilon\) on \(Q_1(0)\) is given by a finite sum of delta distributions supported on a finite set \(X_\varepsilon \subset Q_1(0)\) with integer weights \(\{d_x \text{ s.t. } x \in X_\varepsilon\}\);

3. \(|\nabla\varepsilon - V|_{L^p(Q_1(0), \mu)} \to 0\) as \(\varepsilon \to 0^+\) in \(E_V\).

Now we are ready to reach the conclusions 1 and 2 of Theorem 2.1.

1. If \(q \in [0, 1]\) and \(p \in \left[1, \frac{n}{n-1}\right]\) we possibly have \(X_\varepsilon \neq \emptyset\), since bad cubes can appear in the cubic decompositions. Since \(\nabla\varepsilon\) always belongs to \(L^s(Q_1(0))\) for some \(s > 1\) (with \(s = p\) if \(p\) itself is already greater than 1), we can Hodge-decompose \(\nabla\varepsilon\) as \(\nabla\varepsilon^0 = d\varphi + d^* A\) for some \(A \in \Omega^{2}_{\text{vert}}(Q_1(0))\) and some \(\varphi \in W^{1,s}(Q_1(0))\). Applying \(d^*\) to the previous decomposition we obtain

\[
\Delta \varphi = d^* (\nabla\varepsilon) = \text{div}(\nabla\varepsilon) = \sum_{x \in X_\varepsilon} d_x \delta_x.
\]

By standard elliptic regularity, \(\varphi \in C^\infty(Q_1(0) \setminus X_\varepsilon)\). Choose \(A_\varepsilon \in \Omega^2(Q_1(0))\) such that \(|A_\varepsilon - A|_{\Omega^{2}_{\text{vert}}(Q_1(0))} \leq \varepsilon\). Then \(|d^* A_\varepsilon - d^* A|_{\Omega^{1}_{\text{vert}}(Q_1(0))} \leq \varepsilon\). Let \(U_\varepsilon := v^\# + d^* (A_\varepsilon)\) and let \(U_\varepsilon \in L^p(Q_1(0), \mu)\) for every \(\varepsilon \in E_V\) and \(U_\varepsilon \to V\) in \(L^p(Q_1(0), \mu)\).

2. If \(q \in (-\infty, 0]\) and \(p \in \left[\frac{n}{n-1}, +\infty\right)\) no bad cubes are allowed in the cubic decomposition, thus \((\nabla\varepsilon)_{\varepsilon \in E_V}\) is a sequence of divergence-free vector fields converging to \(V\) in \(L^p(Q_1(0), \mu)\) as \(\varepsilon \to 0^+\) in \(E_V\). Hence \(V\) itself is divergence-free.

\[\square\]

**Remark 2.4.** Notice that if \(p = 1\), \(V_k \in L^s(Q^n_1(0))\) for any \(k \in \mathbb{N}\) and for any \(s \in \left[1, \frac{n}{n-1}\right]\).

**Remark 2.5.** Observe that this proof can be used to show that the analogous approximation result holds if we assume that \(V\) satisfies the first three conditions of Definition 1.2 and in addition we require that for every \(\rho \in R_{F,x_0}\) we have that

\[
\int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in S
\]

for a set \(S \subset \mathbb{R}\) such that \(0 \in S\) and 0 is an isolated point in \(S\). In this case the vector field \(V\) can be approximated in \(L^p\) by a sequence of vector fields \((V_n)_{n \in \mathbb{N}}\) smooth outside a finite

---

\(^4\)In fact even when \(\mu\) is different from \(\mathcal{L}^n\), \(\tilde{V}_\varepsilon\) is constructed through Lemmata 2.4 and 2.5 as extension of a smooth boundary datum, thus \(\tilde{V}_\varepsilon\) lies in \(L^r(\Omega)\) for any \(r \in [1, \frac{n}{n-1}]\) if \(p < n\) and in \(L^s(\Omega)\) for any \(r \in [1, \infty)\) if \(p \geq \frac{n}{n-1}\). It follows that \(\nabla\varepsilon \in L^r(\Omega)\) for any \(r \in [1, \frac{n}{n-1}]\) if \(p < n\) and \(\nabla\varepsilon \in L^s(\Omega)\) for any \(r \in [1, \infty)\) if \(p \geq \frac{n}{n-1}\).
In other words, $L^p_R(\mathbb{I})$ consists of all integer-valued step functions and their translations by a constant.

First we show the inclusion $\subset$: let $f = g + a$ with $g \in L^p(\mathbb{I}, \mathbb{Z})$ and $a \in [0, 1)$ and let $\varepsilon > 0$. For any $k \in \mathbb{N}$ set $g_k := 1_{|g| < k}$. Then there exists $K \in \mathbb{N}$ such that $\|g_K - g\|_{L^p(\mathbb{I})} < \frac{\varepsilon}{2}$. Now for any $j \in \{-K, ..., K\}$ $g_K^j - 1 = g^{-1}(j)$ is a measurable set, therefore there exists a finite collection of disjoint intervals $(I_j^i)_{i \in J}$ such that $\mathcal{L}\left(\bigcup_{i \in J} I_j^i \triangle g^{-1}(j)\right) \leq \frac{\varepsilon}{2(2K + 1)^2}$.

For any $j \in \{-K, ..., K\}$ set $A_j := \bigcup_{i \in J} I_j^i \setminus \left(\bigcup_{j' \neq j} \bigcup_{i \in J} I_j^i\right)$. Then $A_j$ is a finite union of intervals and

$$\mathcal{L}(A_j \triangle g^{-1}(j)) \leq \mathcal{L}(A_j \setminus g^{-1}(j)) + \sum_{j' \neq j} \mathcal{L}\left(\bigcup_{i \in J} I_j^i \cap g^{-1}(j)\right) \leq \frac{\varepsilon}{2(2K + 1)}.$$

Set $\tilde{g}_K = \begin{cases} j & \text{if } x \in A_j \\ 0 & \text{otherwise} \end{cases}$

Then by construction $\tilde{g}_K \in L^p_R(\mathbb{I})$ and $\|\tilde{g}_K - g\|_{L^p(\mathbb{I})} \leq \varepsilon$. We conclude that any $g \in L^p(\mathbb{I}, \mathbb{Z})$ lies in the closure of $L^p_R(\mathbb{I})$. This shows $\subset$. As $L^p(\mathbb{I}, \mathbb{Z}) + [0, 1)$ is closed in $L^p(\mathbb{I})$, the inclusion $\supset$ holds as well.

Next we show the second equality. Let's start with $\supset$. Let $g \in L^p(\mathbb{I}, \mathbb{Z})$, $a \in \mathbb{R}$ and $f = g + a$. Let $x_0 \in \mathbb{I}$, then for a.e. $r \in (0, \text{dist}(x_0, \partial \mathbb{I}))$ we have $f(x_0 + r) - f(x_0 - r) = g(x_0 + r) - g(x_0 - r) \in \mathbb{Z}$. Let $\tilde{R}_{F,x_0}$ denote the set of all such $r$. Set $R_{R,x_0}$ to be the intersection of $\tilde{R}_{F,x_0}$ with the set of Lebesgue points of $f$. Then $f$ satisfies Definition 1.2 and thus $f \in L^p_Z(\mathbb{I})$.

To show $\supset$ let $f \in L^p_Z(\mathbb{I})$. Set $F : I \to S^1$, $x \mapsto e^{i2\pi f(x)}$. Then $F$ is a measurable bounded function. Notice that for any $x_0 \in I$, for a.e. $r \in (0, \text{dist}(x_0, \partial \mathbb{I}))$ there holds $F(x_0 + r) = F(x_0 - r)$. This implies that $F$ is constant: this can be seen for instance approximating $F$ by smooth functions with the same symmetry properties away from $\partial \mathbb{I}$ (convolving with a symmetric mollifier with small support), which then have to be constant.

Choose $a \in \mathbb{R}$ such that $F \equiv e^{i2\pi a}$, then $f - a \in L^p(\mathbb{I}, \mathbb{Z})$. This completes the proof.

Remark 2.6. From Theorem 2.1 it follows immediately that

$$\overline{L^p_R(\mathbb{I})}^{L^p} = L_Z^p(\mathbb{I})^p.$$
To see this it is enough to check that $L^p_Z(Q_1(0))$ is closed in $L^p$, which can be shown by simple application of the coarea formula.

Notice that in case $p \in [n/(n-1), +\infty)$ we can approximate $V$ strongly in $L^p$ with smooth and divergence free vector fields. This is a straightforward consequence of Hodge decomposition.

### 2.5. A characterization of $\Omega^{n-1}_{p,Z}$

First of all we apply the Strong approximation Theorem to obtain a useful characterization of the class $F \in \Omega^{n-1}_{p,Z}(Q_1(0))$.

**Theorem 2.2.** Let $n \in \mathbb{N}_0$, $p \in [1, +\infty)$. Let $F \in \Omega^{n-1}_{p,Z}(Q_1(0))$. Then, the following are equivalent:

1. there exists $L \in \mathcal{R}_1(Q_1(0))$ such that $\partial L = *dF$ in $(W^{1,\infty}_0(Q_1(0)))^*$ and
   
   $$M(L) = \sup_{\varphi \in \mathcal{D}(Q_1(0)), \|d\varphi\|_{L^\infty} \leq 1} \int_{Q_1(0)} F \wedge d\varphi.$$

2. for every Lipschitz function $f : Q_1(0) \to [a, b] \subset \mathbb{R}$ such that $f|_{\partial Q_1(0)} \equiv b$, we have
   
   $$\int_{f^{-1}(t)} i^*_f -1(t) F \in \mathbb{Z}, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b];$$

3. $F \in \Omega^{n-1}_{p,Z}(Q_1(0))$.

**Proof.** We just need to show that $1 \Rightarrow 2, 2 \Rightarrow 3$ and $3 \Rightarrow 1$. We prove these implications separately.

1 $\Rightarrow$ 2. Assume 1. Let $L \in \mathcal{R}_1(Q_1(0))$ be given by

$$\langle L, \omega \rangle = \int_{\Gamma} \theta(\omega, \vec{L}) d\mathcal{H}^1, \quad \forall \omega \in \mathcal{D}^1(Q_1(0)),$$

where $\Gamma \subset Q_1(0)$ is a locally 1-rectifiable set, $\vec{L}$ is a Borel measurable unitary vector field on $\Gamma$ and $\theta \in L^1(\Gamma, \mathcal{H}^1)$ is a $\mathbb{Z}$-valued function. Pick any Lipschitz function $f : Q_1(0) \to \mathbb{R}$ and let $\varphi \in C^{\infty}_c((-\infty, b))$ be such that $\int_{\mathbb{R}} \varphi \ d\mathcal{L}^1 = 0$. By the coarea formula we have

$$\int_{Q_1(0)} F \wedge f^*(\varphi \text{vol}_\mathbb{R}) = \int_{-\infty}^{+\infty} \varphi(t) \left( \int_{f^{-1}(t)} i^*_f -1(t) F \right) dt.$$

At the same time, by the coarea formula for countably 1-rectifiable sets, we have

$$\langle L, f^*(\varphi \text{vol}_\mathbb{R}) \rangle = \int_{\Gamma} \theta(f^*(\varphi \text{vol}_\mathbb{R}), \vec{L}) d\mathcal{H}^1 = \int_{-\infty}^{+\infty} \varphi(t) \left( \int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} \right) dt,$$

where $\tilde{\theta} : \Gamma \to \mathbb{Z}$ is given by $\tilde{\theta} := \text{sgn}((f^* \text{vol}_\mathbb{R}, \vec{L}))\theta$. Let $\Phi \in C^{\infty}_c((-\infty, b))$ satisfy $d\Phi = \varphi \text{vol}_\mathbb{R}$. Notice that since $f$ is proper $f^*\Phi \in W^{1,\infty}_0(Q_1(0))$. Then, by hypothesis, we have

$$\int_{Q_1(0)} F \wedge f^*(\varphi \text{vol}_\mathbb{R}) = \int_{Q_1(0)} F \wedge df^*\Phi = \langle \star dF, f^*\Phi \rangle = \langle \partial L, f^*\Phi \rangle.$$
Therefore
\[
\int_{-\infty}^{\infty} \varphi(t) \left( \int_{f^{-1}(t)} i^*_0 F - \int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} \right) dt = 0, \quad \forall \varphi \in C_c^\infty((-\infty, b)) \text{ s.t. } \int_\mathbb{R} \varphi = 0.
\]
We conclude that
\[
\int_{f^{-1}(t)} i^*_0 F - \int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} = c, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b],
\]
for some constant \( c \in \mathbb{R} \). We claim that \( c = 0 \). In fact let \( m \in \mathbb{N} \setminus \{0\} \). Integrating both sides on \((-m, m)\) we get
\[
\int_{\{ |f| < m \}} F \wedge f^* \mathrm{vol}_\mathbb{R} - \int_{\Gamma \cap \{ |f| < m \}} \theta(f^* \mathrm{vol}_\mathbb{R}, \tilde{L}) = 2mc. \tag{2.14}
\]
Since \( f^* \mathrm{vol}_\mathbb{R} = df \), we have
\[
\int_{Q_1(0)} F \wedge f^* \mathrm{vol}_\mathbb{R} - \int_{\Gamma} \theta(f^* \mathrm{vol}_\mathbb{R}, \tilde{L}) = \int_{Q_1(0)} F \wedge df - \langle L, df \rangle = \langle \ast F - \partial L, f \rangle = 0.
\]
Thus, by letting \( m \to +\infty \) in (2.14), we get that the left-hand-side converges to 0 whilst the right-hand-side diverges to \(+\infty\), unless \( c = 0 \). Hence we conclude that \( c = 0 \), i.e.
\[
\int_{f^{-1}(t)} i^*_0 F = \int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} \in \mathbb{Z}, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b],
\]
since \( \Gamma \cap f^{-1}(t) \) consists of finitely many points for \( \mathcal{L}^1\text{-a.e. } t \in \mathbb{R} \).

2 \implies 3. Assume 2. Given \( x_0 \in Q_1(0) \), let \( f_{x_0} := \min \{ \| - x_0 \|, \frac{r_{x_0}}{2} \} \). We claim that we can find \( R_{f,x_0} \) as in Definition 1.2. Indeed, let \( L \subseteq Q_1(0) \) be the set of the Lebesgue points of \( F \). Let \( r_{x_0} := 2 \text{dist}_{\infty}(x_0, \partial Q_1(0)) \). Then, by the coarea formula, we have
\[
0 = \mathcal{L}^n(L_{r_{x_0}}(x_0) \setminus L) = \frac{1}{2^n} \int_{0}^{r_{x_0}} \mathcal{H}^{n-1}((Q_1(0) \setminus L) \cap \partial Q_\rho(x_0)) d\rho,
\]
which implies that there exists a set \( E_{x_0} \subset (0, r_{x_0}) \) such that
1. \( \mathcal{L}^1((0, r_{x_0}) \setminus E_{x_0}) = 0 \);
2. for every \( \rho \in E_{x_0}, \mathcal{H}^{n-1}\text{-a.e. } x \in \partial Q_\rho(x_0) \) is a Lebesgue point for \( F \).

Hence, for every \( \rho \in E_{x_0} \) it makes sense to consider the pointwise restriction of \( F \) to \( \partial Q_\rho(x_0) \). Notice that, by the coarea formula, we have
\[
\int_{E_{x_0}} \left( \int_{\partial Q_\rho(x_0)} |F|^p d\mathcal{H}^{n-1} \right) d\rho = 2^n \int_{Q_{r_{x_0}}(x_0)} |F|^p d\mathcal{L}^n < +\infty,
\]
which implies
\[
\int_{\partial Q_\rho(x_0)} |F|^p d\mathcal{H}^{n-1} < +\infty, \quad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0, E_{x_0}).
\]
Moreover, by Statement 2., we have
\[
\int_{f^{-1}(\rho)}^{i_{\rho}^{-1}(\rho)} F = \int_{\partial Q_{\rho}(x_0)} \nu^\rho F \in \mathbb{Z}, \quad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0, E_{x_0}).
\]

Our claim follows immediately.

3 \Rightarrow 1. Assume 3. By Theorem 1.1, we can find a sequence \( \{F_k\}_{k \in \mathbb{N}} \subset \Omega^\infty_{p,R}(Q_1(0)) \) such that \( F_k \to F \) strongly in \( L^p \). Since the estimate
\[
|\langle T_{F_k} - T_F, \omega \rangle| \leq C||F_k - F||_{L^p}
\]
holds for every \( \omega \in \mathcal{D}^1(Q_1(0)) \) such that \( ||\omega||_{L^\infty} \leq 1 \) and for every \( k \in \mathbb{N} \), we conclude that
\[
\sup_{\varphi \in W^1_0(\Omega, Q_1(0)), ||\varphi||_{L^\infty} \leq 1} \langle *dF_k - *dF, \varphi \rangle \leq C||F_k - F||_{L^p} \to 0 \quad \text{as } k \to \infty. \quad (2.15)
\]

Fix any \( \varepsilon \in (0, 1) \). By (2.15) we can find a subsequence \( \{F_{k_j}(\varepsilon)\}_{j \in \mathbb{N}} \subset \Omega^\infty_{p,R}(Q_1(0)) \) such that
\[
\|dF_{k_j}(\varepsilon) - *dF_{k_{j+1}}(\varepsilon)\|_{(W^1_0(\Omega, Q_1(0)))^*} \leq \frac{\varepsilon}{2^j}, \quad \text{for every } j \in \mathbb{N}.
\]

For every \( h \in \mathbb{N} \), let \( L^\varepsilon_h \) be a minimal connection for the singular set of \( F_{k_h}(\varepsilon) \) (the existence of such a minimal connection is proved in Proposition A.1). Analogously, for every \( j \in \mathbb{N} \), let \( L^\varepsilon_{j,j+1} \) be the minimal connection for the singular set of \( F_{k_j}(\varepsilon) - F_{k_{j+1}}(\varepsilon) \). Define the following sequence of integer 1-currents on \( Q_1(0) \):
\[
\tilde{L}^\varepsilon_h := \begin{cases} 
L^\varepsilon_0 & \text{if } h = 0, \\
L^\varepsilon_0 - \sum_{j=0}^{h-1} L^\varepsilon_{j,j+1} & \text{if } h > 0,
\end{cases}
\]
for every \( h \in \mathbb{N} \).

Clearly
\[
\partial L^\varepsilon_h = \partial L^\varepsilon_0 - \sum_{j=0}^{h-1} \partial L^\varepsilon_{j,j+1} = \partial L^\varepsilon_0 - \sum_{j=0}^{h-1} (\partial L^\varepsilon_j - \partial L^\varepsilon_{j+1}) = \partial L^\varepsilon_h = *dF_{k_h}(\varepsilon).
\]

Moreover, since \( L^\varepsilon_{j,j+1} \) is a minimal connection, it holds that
\[
\mathcal{M}(L^\varepsilon_{j,j+1}) = \| *dF_{k_j}(\varepsilon) - *dF_{k_{j+1}}(\varepsilon) \|_{(W^1_0(\Omega, Q_1(0)))^*} \leq \frac{\varepsilon}{2^j}, \quad \text{for every } j \in \mathbb{N}.
\]

Thus,
\[
\mathcal{M}(\tilde{L}^\varepsilon_{h+1} - \tilde{L}^\varepsilon_h) = \mathcal{M}(L^\varepsilon_{h,h+1}) \leq \frac{\varepsilon}{2^h}, \quad \text{for every } h \in \mathbb{N},
\]
which amounts to saying that the sequence \( \{\tilde{L}^\varepsilon_h\}_{h \in \mathbb{N}} \) is a Cauchy sequence in mass. Hence, by the closure of integer currents under mass convergence (see Lemma C.1), there exists an integer 1-current \( \tilde{L}^\varepsilon \in \mathcal{R}_1(Q_1(0)) \) such that
\[
\mathcal{M}(\tilde{L}^\varepsilon_h - \tilde{L}^\varepsilon) \to 0 \quad \text{as } h \to \infty,
\]
for every \( h \in \mathbb{N} \).
Notice that
\[
\partial \tilde{L}^\varepsilon = \lim_{h \to \infty} \partial \tilde{L}_h^\varepsilon = \lim_{h \to \infty} *dF_{k_h(\varepsilon)} = *dF \quad \text{in} \quad (W^{1,\infty}_0(Q_1(0)))^*.
\]

The family of integer 1-cycles \( \{\tilde{L}^\varepsilon - \tilde{L}^{1/2}\}_{0<\varepsilon<1} \subset \mathcal{R}_1(Q_1(0)) \) is uniformly bounded in mass. Indeed, first we notice that by (2.10) it holds that
\[
\mathcal{M}(\tilde{L}_h^\varepsilon) = \| *dF_{k_h(\varepsilon)} \|_{(W^{1,\infty}_0(Q_1(0)))^*} \leq C, \quad \forall h \in \mathbb{N}, \forall \varepsilon \in (0,1),
\]
where \( C > 0 \) is a constant independent on \( h \) and \( \varepsilon \). Thus, we have
\[
\mathcal{M}(\tilde{L}_h^\varepsilon) \leq \mathcal{M}(\tilde{L}_0^\varepsilon) + \sum_{j=0}^{h-1} \mathcal{M}(L_{j,j+1}^{\varepsilon}) \leq C + \sum_{j=0}^{+\infty} \frac{\varepsilon}{2^j} \leq C + \varepsilon \leq C + 1 \quad \forall h \in \mathbb{N}, \forall \varepsilon \in (0,1).
\]

Since \( \tilde{L}_h^\varepsilon \to \tilde{L}^\varepsilon \) in mass as \( h \to +\infty \) for every \( \varepsilon \in (0,1) \), we have
\[
\mathcal{M}(\tilde{L}^\varepsilon - \tilde{L}^{1/2}) \leq \mathcal{M}(\tilde{L}^\varepsilon) + \mathcal{M}(\tilde{L}^{1/2}) \leq 2(C+1).
\]

Hence, by standard compactness arguments for currents (see for instance [19], Theorem 7.5.2), we can find a sequence \( \varepsilon_k \to 0 \) and an integer 1-cycle \( \tilde{L} \in \mathcal{R}_1(Q_1(0)) \) with finite mass such that \( \tilde{L}^{\varepsilon_k} - \tilde{L}^{1/2} \to \tilde{L} \) weakly in \( D_1(Q_1(0)) \) as \( k \to +\infty \). If we let \( L := \tilde{L}^{1/2} + \tilde{L} \) then we get \( \tilde{L}^{\varepsilon_k} \to L \) weakly in \( D_1(Q_1(0)) \). By construction, \( L \) is again an integer 1-current with finite mass such that \( \partial L = \partial \tilde{L}^{1/2} = *dF \) in \( (W^{1,\infty}(Q_1(0)))^* \). We claim that
\[
\mathcal{M}(L) = \inf_{T \in \mathcal{M}_1(Q_1(0)), \partial T = *dF} \mathcal{M}(T).
\]

By contradiction, assume that we can find \( T \in \mathcal{M}_1(Q_1(0)) \) such that \( \partial T = *dF \) and
\[
\mathcal{M}(T) < \mathcal{M}(L) \leq \lim_{k \to \infty} \mathcal{M}(\tilde{L}^{\varepsilon_k}),
\]
where the last inequality follows by weak convergence and lower semicontinuity of the mass. Then, we can find some \( h \in \mathbb{N} \) such that
\[
\mathcal{M}(T) < \mathcal{M}(\tilde{L}^{\varepsilon_h}) - 2\varepsilon_h.
\]

Moreover, since \( \mathcal{M}(L_0^{\varepsilon} - \tilde{L}^\varepsilon) \leq 2\varepsilon \) for every \( 0 < \varepsilon < 1 \), it holds that
\[
\mathcal{M}(L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}) \leq 2\varepsilon_h.
\]

We define \( \tilde{T} := T + L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h} \) and we notice that \( \partial \tilde{T} = *dF_{k_h(\varepsilon_h)} \). Moreover, by the minimality of \( L_0^{\varepsilon_h} \), we conclude that
\[
\mathcal{M}(L_0^{\varepsilon_h}) \leq \mathcal{M}(\tilde{T}) \leq \mathcal{M}(T) + \mathcal{M}(L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}) < \mathcal{M}(L_0^{\varepsilon_h}),
\]
which is a contradiction. Thus, our claim follows.

Since \( L \in \mathcal{R}_1(Q) \), we get that
\[
\mathcal{M}(L) = \inf_{T \in \mathcal{R}_1(Q_1(0)), \partial T = *dF} \mathcal{M}(T) = \inf_{T \in \mathcal{M}_1(Q_1(0)), \partial T = *dF} \mathcal{M}(T).
\]
and, by Lemma 3.2, we have
\[
\inf_{T \in M_1(Q_1(0))} M(T) = \sup_{\varphi \in D(Q_1(0)), ||d\varphi||_{L^\infty} \leq 1} \int_M F \wedge d\varphi.
\]
Hence, 1. follows.

Remark 2.7. We notice that in the proof of Theorem 2.2 we have never used the minimality property of \( L \) while showing that \( 1 \Rightarrow 2 \). Hence whenever \( F \in \Omega^{n-1}_p(Q^n_1(0)) \) admits a connection we have \( F \in \Omega^{n-1}_{p,\infty}(Q^n_1(0)) \) in the sense of Definition 1.2 and the conclusions of Theorem 1.1 hold for \( F \).

By the previous remark we can deduce the following result from the proof of Theorem 2.2.

Corollary 2.1. Let \( F \in \Omega^{n-1}_p(Q^n_1(0)) \) and assume that there exists an integer 1-current of finite mass \( I \in R_1(Q_1(0)) \) such that \( \partial I = *dF \). Then there exists an integer 1-current \( L \in R_1(Q_1(0)) \) of finite mass such that \( \partial L = *dF \) and
\[
M(L) = \inf_{T \in R_1(Q_1(0))} M(T).
\]
In other words, whenever there exists a connection for \( F \), then there exists a minimal connection for \( F \).

2.6. The case of \( \partial Q^{n+1}_1(0) \)

In order to extend the previous result to more general manifolds we introduce the following definition.

Definition 2.1. Let \( M \) be a Lipschitz \( m \)-manifold embedded in \( \mathbb{R}^n \). Let \( p \in [1, \infty) \). Set
\[
\Omega^1_{p, R, \infty}(M) := \left\{ \alpha \in \Omega^1_p(M) \cap \Omega^1_{p, \infty}(M \setminus S) : *d\alpha = \sum_{i \in I} d_i \delta_{P_i} \right\},
\]
where \( I \) is a finite index set, \( d_i \in \mathbb{Z} \), \( P_i \in S \) for any \( i \in I \) and \( F := \{P_i\}_{i \in I} \).

The previous definition is motivated by the following observation: let \( M \) be a Lipschitz \( m \)-manifold, \( N \) a smooth \( m \)-manifold, \( \varphi : M \to N \) a bi-Lipschitz map. Let \( F \in \Omega^1_p(N) \). Then \( \varphi^*F \in \Omega^1_{p, R, \infty} \) (see Lemma 2.7).

Corollary 2.2. Let \( n \in \mathbb{N}_{\geq 2} \). Assume that \( F \in \Omega^{n-2}_{p, \infty}(\partial Q^n_1(0)) \) admits a connection. Then, the following facts hold:
1. if \( p \in \left[1, \frac{(n-1)(n-2)}{n(n-2)}\right) \), then there exists a sequence \( \{F_k\}_{k \in \mathbb{N}} \subset \Omega^{n-2}_{p, R, \infty}(\partial Q^n_1(0)) \) such that \( F_k \to F \) strongly in \( L^p \);
2. if \( p \in \left(\frac{(n-1)(n-2)}{n(n-2)}, +\infty\right) \), then \( *dF = 0 \) distributionally on \( \partial Q^n \).
Proof. Let \( N = (0, \ldots, 0, \frac{1}{2}) \) be the north pole in \( \partial Q^n_1(0) \subset \mathbb{R}^n \) and let
\[
U := \{(x_1, \ldots, x_{n-1}, x_n) \in \partial Q^n_1(0) \text{ s.t. } x_n = \frac{1}{2}\}
\]
be the upper face of \( \partial Q^n_1(0) \). Let \( q := (n - 1) - (n - 2)p \). For every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we let \( x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \). Define \( \Phi : \partial Q^n_1(0) \setminus N \subset \mathbb{R}^n \to Q^n_1(0) \) by
\[
\Phi(x) := \begin{cases} 
\left( \frac{1}{2} - \frac{\sqrt{2}}{4} \|x'\|_\infty \right) \frac{x'}{\|x'\|_\infty} & \text{on } U \setminus N, \\
g(x) & \text{on } \partial Q^n_1(0) \setminus U,
\end{cases}
\]
where the map \( g : \partial Q^n_1(0) \setminus U \to Q^n_{1/2}(0) \) is any bi-Lipschitz homeomorphism such that
\[
g = \left( \frac{1}{2} - \frac{\sqrt{2}}{4} \|x'\|_\infty \right) \frac{x'}{\|x'\|_\infty} \quad \text{on } \partial U.
\]
Notice that \( \Phi \) is an homeomorphism from \( \partial Q^n_1(0) \setminus N \) to \( Q^n_{1/2}(0) \). We denote its inverse map by \( \Psi \).

We have that \( \Psi \) is Lipschitz on \( Q^n_{1/2}(0) \) and \( \Phi \) is Lipschitz on every compact set \( K \subset \partial Q^n_1(0) \setminus N \), since there exists \( C > 0 \) such that
\[
|d\Phi(x)| \leq \frac{C}{\|x - N\|_\infty^2}, \quad \forall x \in \partial Q^n_1(0) \setminus N,
\]
\[
|d\Psi(y)| \leq C \left( \frac{1}{2} - \|y\|_\infty \right), \quad \forall y \in Q^n_{1/2}(0).
\]

Define \( \tilde{F} := \Psi^*F \) and fix any \( \varepsilon \in (0, 1/4) \). Notice that
\[
\int_{Q^n_{1/2}(0)} \left( \frac{1}{2} - \|\cdot\|_\infty \right)^q |\tilde{F}|^p d\mathcal{H}^{n-1} \leq C \int_{Q^n_{1/2}(0)} \left( \frac{1}{2} - \|\cdot\|_\infty \right)^{n-1} |F \circ \Psi|^p d\mathcal{H}^{n-1} \\
\quad \leq C \int_{\partial Q^n_1(0)} |F|^p d\mathcal{H}^{n-1} < +\infty.
\]

Moreover if \( I \in \mathcal{R}_1(\partial Q^n_1(0)) \) and \( *dF = \partial I \), then \( \Phi_*I \in \mathcal{R}_1(\partial Q^n_{1/2}(0)) \) and \( *d\tilde{F} = \partial \Phi_*I \) (see Lemma 2.7). This implies that \( \tilde{F} \in \Omega^p_{p,2}(Q^n_{1/2}(0), \mu) \) with \( \mu := \left( \frac{1}{2} - \|\cdot\|_\infty \right)^q L^n \) in the sense of Definition 2.2.

Let’s consider first the case \( p \in \left[1, (n-1)/(n-2)\right) \). Notice that in this case \( q > 0 \).

By Theorem 2.1 with \( f := (\frac{1}{2} - \|\cdot\|_\infty)^q \) there exists a \((n-2)\)-form \( \tilde{F}_\varepsilon \in \Omega^n_{p,R}(\Omega_{\varepsilon,a_\varepsilon}) \) such that
\[
\|\tilde{F}_\varepsilon - \tilde{F}\|_{L^p(\Omega_{\varepsilon,a_\varepsilon})} \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } E_{\tilde{F}}.
\]

Define \( F_\varepsilon := \Phi_*a_\varepsilon \tilde{F}_\varepsilon \) on \( \partial Q^n_1(0) \setminus U_\varepsilon \), with \( \Phi_{a_\varepsilon} := \Phi + a_\varepsilon \) and \( U_\varepsilon := \Phi_{a_\varepsilon}^{-1}(Q^n_{1}(0) \setminus \Omega_{\varepsilon,a_\varepsilon}) \).

Notice that
\[
\|F_\varepsilon - F\|_{L^p(\partial Q^n_1(0) \setminus U_\varepsilon)} \leq C \int_{\partial Q^n_1(0) \setminus U_\varepsilon} \frac{1}{N^{(n-2)p}} |\tilde{F}_\varepsilon \circ \Phi_{a_\varepsilon} - \tilde{F} \circ \Phi_{a_\varepsilon}|^p d\mathcal{H}^{n-1}
\]

32
\[ + C \int_{\partial Q^2_\varepsilon(0) \setminus U_\varepsilon} \frac{1}{|\cdot - N|^{(n-2)p}} |\tilde{F} \circ \phi_{a\varepsilon} - \tilde{F} \circ \phi|^p \, d\mathcal{H}^{n-1} \]
\[ \leq C \left( \int_{\Omega_{\varepsilon,a\varepsilon}} |\tilde{F}_\varepsilon - \tilde{F}|^p f \, d\mathcal{H}^{n-1} + \int_{Q^2_{\varepsilon,a\varepsilon}(0)} |\tilde{F}(-a\varepsilon) - \tilde{F}|^p f \, d\mathcal{H}^{n-1} \right). \]

The first term tends to zero as \( \varepsilon \to 0^+ \) in \( E_\varepsilon \) by (2.16), while the second tends to zero as \( \varepsilon \to 0^+ \) by (2.5). Therefore we have
\[ \|F_\varepsilon - F\|_{L^p(\partial Q^2_\varepsilon(0) \setminus U_\varepsilon)} \to 0 \]
as \( \varepsilon \to 0^+ \) in \( E_\varepsilon \).

Now we notice that
\[ \int_{\partial U_\varepsilon} \tilde{i}_\varepsilon^* F_\varepsilon = - \int_{\partial(\partial Q^2_\varepsilon(0) \setminus U_\varepsilon)} \tilde{i}_\varepsilon^*(F_\varepsilon) \]
\[ = \int_{\partial \Omega_{\varepsilon,a\varepsilon}} \tilde{i}_\varepsilon^* \tilde{F}_\varepsilon = \int_{\partial \Omega_{\varepsilon,a\varepsilon}} \tilde{i}_\varepsilon^* \tilde{F} = : b_\varepsilon \in \mathbb{Z}. \]

If \( b_\varepsilon = 0 \), then we use Lemma 2.4 to extend \( F_\varepsilon \) inside \( U_\varepsilon \). If \( b_\varepsilon \neq 0 \), then we use Lemma 2.5 to extend \( F_\varepsilon \) inside \( U_\varepsilon \) (notice that \( U_\varepsilon \) is an \((n-1)\)-cube of side-length \( 8\varepsilon^2 \) contained in \( U \) and centered at \( N \), for \( \varepsilon \) sufficiently small). In both cases, the following estimate holds:
\[ \int_{U_\varepsilon} |F_\varepsilon|^p \, d\mathcal{H}^{n-1} \leq C\varepsilon^2 \int_{\partial U_\varepsilon} |F_\varepsilon|^p \, d\mathcal{H}^{n-2} \leq C\varepsilon^2 \left( \int_{\partial U_\varepsilon} |F - F_\varepsilon|^p \, d\mathcal{H}^{n-2} + \int_{\partial U_\varepsilon} |F|^p \, d\mathcal{H}^{n-2} \right). \]

Notice that the first term on the right hand side tends to zero as \( \varepsilon \to 0^+ \) in \( E_\varepsilon \), since \( \|\tilde{F}_\varepsilon - \tilde{F}\|_{L^p(S_{\varepsilon,a\varepsilon})} \leq \varepsilon \) for any \( \varepsilon \in E_\varepsilon \). In order to control the second term, pick any \( \delta > 0 \) sufficiently small and notice that, by coarea formula, we have
\[ \int_0^\delta \varepsilon^2 \int_{\partial U_\varepsilon} |F|^p \, d\mathcal{H}^{n-2} \, d\mathcal{L}^1(\varepsilon) \leq \int_0^\delta \varepsilon \int_{Q^1_{8\varepsilon^2}(N)} |F|^p \, d\mathcal{H}^{n-2} \, d\mathcal{L}^1(\varepsilon) \]
\[ \leq \frac{1}{16} \int_0^{8\varepsilon^2} \int_{\partial Q^1_{\varepsilon}(N)} |F|^p \, d\mathcal{H}^{n-2} \, d\mathcal{L}^1(\varepsilon) \]
\[ \leq \frac{2n-1}{16} \int_{Q^1_{8\varepsilon^2}(N)} |F|^p \, d\mathcal{H}^{n-1} \to 0^+ \]
as \( \delta \to 0^+ \). This implies that we can pick a sequence \( \{\varepsilon_j\}_{j \in \mathbb{N}} \subset E_\varepsilon \) such that \( \varepsilon_j \to 0^+ \) and
\[ \varepsilon_j^2 \int_{\partial U_{\varepsilon_j}} |F|^p \, d\mathcal{H}^{n-1} \to 0^+ \]
as \( j \to +\infty \). Thus
\[ \|F_{\varepsilon_j} - F\|_{L^p(U_{\varepsilon_j})} \leq 2^{n-1} \left( \|F_{\varepsilon_j}\|_{L^p(U_{\varepsilon_j})} + \|F\|_{L^p(U_{\varepsilon_j})} \right) \to 0 \]
as \( j \to +\infty \), since \( \mathcal{H}^{n-1}(U_{\varepsilon_j}) \to 0^+ \) as \( \varepsilon_j \to 0^+ \). Hence, we conclude that
\[ \|F_{\varepsilon_j} - F\|_{L^p(\partial Q^2_\varepsilon(0))} \to 0 \]
as \( j \to +\infty \). Moreover, by construction we have that \(*dF_{\varepsilon_j}\) is a finite sum of Dirac-deltas with integer coefficients, for any \( j \in \mathbb{N} \). Thus arguing as in the final step of the proof of Theorem 2.1 (i.e. by Hodge decomposition), for any \( j \in \mathbb{N} \) we can find \( \tilde{F}_{\varepsilon_j} \in \Omega^1_{p,R,\infty}(\partial Q_1^n(0)) \) such that \( \|F_{\varepsilon_j} - \tilde{F}_{\varepsilon_j}\|_{\Omega^1_{p,R,\infty}(\partial Q_1^n(0))} < \varepsilon_j \). The sequence \( \{\tilde{F}_{\varepsilon_j}\}_{j \in \mathbb{N}} \) then has the desired properties. This concludes the proof in the case \( p \in [1, (n-1)/(n-2)] \).

If \( p \in [(n-1)/(n-2), +\infty) \) notice that \( q \leq 0 \). Therefore by Remark 2.1 we may assume, up to passing to a subsequence, that no bad cube appears in the construction of \( F_\varepsilon \). Hence repeating the first part of the proof as in the previous case, we get \( b_{\varepsilon_j} = 0 \) and \(*dF_{\varepsilon_j} = 0\) on \( \partial Q_1^n(0) \) for any \( j \in \mathbb{N} \). Thus we obtain that \( F \) can be approximated in \( \Omega^m_{p-1}(\partial Q_1^n(0)) \) by a sequence \( \{F_{\varepsilon_j}\}_{j \in \mathbb{N}} \) such that for any \( j \in \mathbb{N} \) there holds \(*dF_{\varepsilon_j} = 0\), and this property passes to the limit. \( \square \)

Remark 2.8. Notice that if \( p = 1 \), for any \( k \in \mathbb{N} \) we have that \( F_k \in \Omega^{m-1}_q(\partial Q_1^n(0)) \) for some \( q > 1 \) which does not depend on \( k \) (compare with Remark 2.4).

Lemma 2.7. Let \( M, N \subset \mathbb{R}^n \) be Lipschitz \( m \)-manifolds in \( \mathbb{R}^n \). Let \( \varphi : M \to N \) be a bi-Lipschitz map.

Let \( F \in \Omega^m_{p-1}(M) \) and assume that there exists a \( 1 \)-rectifiable current \( I \in \mathcal{R}_1(M) \) of finite mass such that \(*dF = \partial I \in (W^{1,\infty}_0(\partial I))^* \). Then \( \varphi_*, F \in \Omega^m_{p-1}(N) \) and \(*d(\varphi_*, F) = \partial \varphi_* I \).

If \( (F_k)_{k \in \mathbb{N}} \) is a sequence in \( \Omega^m_{p,R,\infty}(M) \) and \( F_k \to F \) in \( \Omega^m_{p-1}(M) \) as \( k \to \infty \), then \( (\varphi_*, F_k)_{k \in \mathbb{N}} \) is a sequence in \( \Omega^m_{p,R,\infty}(N) \) and \( \varphi_*, F_k \to \varphi_* F \) in \( \Omega^m_{p-1}(N) \) as \( k \to \infty \).

If in addition we assume that \( N \) is smooth and closed or a bounded simply connected Lipschitz domain and for any \( k \in \mathbb{N} \) we have \( F_k \in \Omega^{m-1}_q(N) \) for some \( q > 1 \) (possibly dependent on \( k \)), then \( \varphi_*, F \) can be approximated in \( \Omega^m_{p-1}(N) \) by \( (m-1) \)-forms in \( \Omega^{m-1}_p(N) \).

Proof. Assume that \(*dF = \partial I \in (W^{1,\infty}_0(\partial I))^* \). Then for any \( f \in W^{1,\infty}_0(N) \) \( \varphi^* f \in W^{1,\infty}_0(M) \) and thus

\[
(*d(\varphi_*, F), f) = \int_N \varphi_* F \wedge df = \int_M F \wedge d\varphi^* f = (\partial I, \varphi^* f) = (I, \varphi^* df) = (\partial \varphi_* I, f),
\]

therefore \(*d(\varphi_*, F) = \partial \varphi_* I \in (W^{1,\infty}_0(N))^* \).

Now assume that \( (F_k)_{k \in \mathbb{N}} \) is a sequence in \( \Omega^m_{p,R,\infty}(M) \) such that \( F_k \to F \) in \( \Omega^m_{p-1}(M) \) as \( k \to \infty \). Then for any \( k \in \mathbb{N} \) there exists \( I_k \in \mathcal{R}_1(M) \) of finite mass and so that \( \partial I_k \) supported in a finite subset of \( M \) such that \(*dF_k = \partial I_k \). As we saw above, \(*d(\varphi_*, F_k) = \partial \varphi_* I_k \), therefore \( \varphi_*, F_k \in \Omega^{m-1}_{p,R,\infty}(N) \). Moreover we have \( \varphi_*, F_k \to \varphi_*, F \) in \( \Omega^{m-1}_p(N) \) as \( k \to \infty \).

Finally if \( N \) is smooth and closed or a bounded Lipschitz domain and for any \( k \in \mathbb{N} \) we have that \( F_k \in \Omega^{m-1}_q(N) \) for some \( q > 1 \), we can improve the approximating sequence \( (\varphi^* F_k)_{k \in \mathbb{N}} \) as follows: for any \( k \in \mathbb{N} \), let \( \alpha_k \in \Omega^{m-2}_{W^{1,\infty}_p}(N) \), \( \beta_k \in \Omega^m_{W^{1,\infty}_p}(N) \) and \( h_k \in \Omega^m_{h_k}(N) \) (the space of harmonic \( (m-1) \)-forms on \( N \)) such that \( \varphi_*, F_k = d\alpha_k + \star \beta_k + h_k \). Then \(*\Delta \beta = *d\beta \) is supported in a finite set of points, thus \( \beta \) is smooth in \( N \) outside of a finite number of points. Now let \( \delta \in \Omega^{m-2}(N) \) such that \( \|\alpha_k - \delta_k\|_{W^{1,p}} \leq \frac{1}{k} \) and set \( \tilde{F}_k := d\delta_k + \beta_k + h_k \). Then by construction \( \tilde{F}_k \in \Omega^{m-1}_{R}(N) \) and \( \tilde{F}_k \to \varphi_*, F \) in \( \Omega^{m-1}_p(N) \) as \( k \to \infty \). \( \square \)

Theorem 1.1, Corollary 2.2, Lemma 2.7 and Remark 2.7 can be combined to obtain the following general statement.
Theorem 2.3. Let $M \subset \mathbb{R}^n$ be any embedded $m$-dimensional Lipschitz submanifold of $\mathbb{R}^n$ which is bi-Lipschitz equivalent either to $Q^m_1(0)$ or $\partial Q^m_{1+1}(0)$. Then:

$$\Omega^{m-1}_{p,R,\infty}(M)^{L^p} = \begin{cases} \Omega^{m-1}_{p,Z}(M) & \text{if } p \in [1, m/(m - 1)), \\ \{ F \in \Omega^{m-1}_{p}(M) \text{ s.t. } \ast dF = 0 \} & \text{if } p \in [m/(m - 1), +\infty). \end{cases}$$

Moreover, if $M$ is smooth we have

$$\Omega^{m-1}_{p,R}(M)^{L^p} = \Omega^{m-1}_{p,R,\infty}(M)^{L^p}.$$ 

2.7. Corollaries of Theorem 1.1

Finally we present a couple of Corollaries of Theorem 1.1. First we show that the boundary of any $I \in \mathcal{R}_1(D)$ having finite mass can be approximated strongly in $(W^{1,\infty}_0(D))^*$ by finite sums of deltas with integer coefficients.

Corollary 2.3. Let $D \subset \mathbb{R}^n$ be any open and bounded domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to $Q^1_1(0)$. Let $I \in \mathcal{R}_1(D)$ with finite mass. Then, there exists a vector field $V \in L^1(D)$ such that

$$\text{div}(V) = \partial I \quad \text{in} \quad (W^{1,\infty}_0(D))^*.$$ 

Thus $\partial I$ can be approximated strongly in $(W^{1,\infty}_0(D))^*$ by finite sums of deltas with integer coefficients. More precisely there exist sequences of points $(P_i)_{i \in \mathbb{N}}$ and $(N_i)_{i \in \mathbb{N}}$ in $D$ such that

$$\partial I = \sum_{i \in \mathbb{N}} (\delta_{P_i} - \delta_{N_i}) \quad \text{in} \quad (W^{1,\infty}_0(D))^* \quad \text{and} \quad \sum_{i \in \mathbb{N}} |P_i - N_i| < \infty. \quad (2.17)$$

Proof. By Lemma 2.7 it is enough to consider the case $D = Q_1(0)$. Let $I$ be as above. By [1, Theorem 5.6] there exists a map $u \in W^{1,n-1}(Q_1(0), \mathbb{S}^{n-1})$ such that

$$\ast d \left( \frac{1}{n} \sum_{i=1}^n (-1)^{i-1} u_i \bigwedge_{j \neq i} du_j \right) = \alpha_{n-1} \partial I,$$

where $\alpha_{n-1}$ denotes the volume of the $(n-1)$-dimensional ball. Set

$$\omega := \frac{1}{n \alpha_{n-1}} \sum_{i=1}^n (-1)^{i-1} u_i \bigwedge_{j \neq i} du_j$$

Notice that $\omega \in \Omega^{n-1}_1(Q_1(0))$ and $\ast d \omega = \partial I$. Now let $V := (\ast \omega)^\sharp$. Then $V \in L^1(Q_1(0))$ and

$$\text{div}(V) = \partial I.$$ 

By Theorem 1.1 there exists a sequence $(V_k)_{k \in \mathbb{N}}$ in $L^1_R(Q_1(0))$ such that $V_k \to V$ in $L^1$. Then

$$\text{div}(V_k) \to \text{div}(V) = \partial I \quad \text{in} \quad (W^{1,\infty}_0(Q_1(0)))^*.$$
As for any \( k \in \mathbb{N} \) we have that \( \text{div}(V_k) \) is a finite sum of deltas with integer coefficients, by Proposition A.1 of [24] holds.

**Corollary 2.4.** Let \( M \) be a complete Lipschitz \( m \)-manifold, with or without boundary, compactly contained in the open cube \( Q_2(0) \). Let \( I \in \mathcal{R}_1(Q_2(0)) \) be a rectifiable current of finite mass supported on \( M \). Then there exist two sequences of points \((p_i)_{i \in \mathbb{N}}\) and \((n_i)_{i \in \mathbb{N}}\) in \( M \) such that

\[
\partial I = \sum_{i \in \mathbb{N}} (\delta_{p_i} - \delta_{n_i}) \text{ in } (W^{1,\infty}(Q_2(0)))^* \text{ and } \sum_{i \in \mathbb{N}} |p_i - n_i| < \infty.
\]

**Proof.** Let \( I \in \mathcal{R}_1(M) \) be a rectifiable 1-current of finite mass supported in \( M \subset Q_2(0) \). By Corollary 2.3 there exists a vector field \( V \in L^1(Q_2(0)) \) such that \( \text{div}(V) = \partial I \). Thus we can apply the arguments of the proof of Theorem 2.1 to \( V \). For any \( \varepsilon \in E_V \) this yields a vector field \( \tilde{V}_\varepsilon \in L^1(Q_{a\varepsilon,\varepsilon}) \) with the following properties: all bad cubes \( Q \in \mathcal{C}_{a\varepsilon,\varepsilon} \) are such that \( Q \cap M \neq \emptyset \), therefore the topological singularities of \( \tilde{V}_\varepsilon \) lie at a distance of at most \( \sqrt{\varepsilon} \) from \( M \). Moreover notice that if \( \varepsilon \) is sufficiently small, \( \int_{Q_2(0)} \text{div}(\tilde{V}_\varepsilon) d\mathcal{L}^n = 0 \) (one can see this by testing \( \text{div}(\tilde{V}_\varepsilon) \) against a function \( \varphi \in C^\infty_c(Q_2(0)) \) such that \( \varphi \equiv 1 \) in a neighbourhood of \( M \)). Thus \( \text{div}(\tilde{V}_\varepsilon) \) can be represented by

\[
\text{div}(\tilde{V}_\varepsilon) = \sum_{i = 1}^{Q^\varepsilon} (\delta_{p_i^\varepsilon}^\varepsilon - \delta_{n_i^\varepsilon}^\varepsilon)
\]

for some \( Q^\varepsilon \in \mathbb{N} \) and points \( p_i^\varepsilon \) and \( n_i^\varepsilon \) (possibly repeated) in a \( \sqrt{\varepsilon} \)-neighbourhood of \( M \). By the argument of Lemma 2.2 (with \( Q_2(0) \) in place of \( Q_1(0) \)) we have \( \varepsilon Q^\varepsilon \to 0 \) as \( \varepsilon \to 0^+ \) in \( E_V \). Now for any \( i \in \{1, ..., Q^\varepsilon\} \) let \( \tilde{p}_i^\varepsilon \) and \( \tilde{n}_i^\varepsilon \) in \( M \) such that \( |p_i^\varepsilon - \tilde{p}_i^\varepsilon| < 2\sqrt{\varepsilon} \) and \( |n_i^\varepsilon - \tilde{n}_i^\varepsilon| < 2\sqrt{\varepsilon} \). Let \( I_{p_i^\varepsilon} \in \mathcal{R}_1(Q_2(0)) \) be the rectifiable current given by integration on the segment joining \( p_i^\varepsilon \) and \( \tilde{p}_i^\varepsilon \) oriented from \( p_i^\varepsilon \) to \( \tilde{p}_i^\varepsilon \) and let \( I_{n_i^\varepsilon} \in \mathcal{R}_1(Q_2(0)) \) be the rectifiable current given by integration on the segment joining \( n_i^\varepsilon \) and \( \tilde{n}_i^\varepsilon \) oriented from \( \tilde{n}_i^\varepsilon \) to \( n_i^\varepsilon \). Let \( I_\varepsilon \in \mathcal{R}_1(Q_2(0)) \) be a rectifiable 1-current of finite mass such that \( \text{div}(\tilde{V}_\varepsilon) = \partial I_\varepsilon \).

Set \( \tilde{I}_\varepsilon = I_\varepsilon + \sum_{i = 1}^{Q^\varepsilon} (I_{p_i^\varepsilon} + I_{n_i^\varepsilon}) \). Then

\[
\partial \tilde{I}_\varepsilon = \sum_{i = 1}^{Q^\varepsilon} (\delta_{\tilde{p}_i^\varepsilon} - \delta_{\tilde{n}_i^\varepsilon})
\]

is supported in \( M \). Moreover we have

\[
\|\partial I - \partial \tilde{I}_\varepsilon\|_{(W^{1,\infty}(M))^*} \leq \|\partial I - \partial I_\varepsilon\|_{(W^{1,\infty}(Q_2(0)))^*} + \|\partial I_\varepsilon - \partial \tilde{I}_\varepsilon\|_{(W^{1,\infty}(Q_2(0)))^*}
\]

(here \( M \) is endowed with the euclidean distance in \( Q_2(0) \); notice that we are making use of the fact that any Lipschitz function on \( M \) can be extended to a Lipschitz function on \( Q_2(0) \) with same Lipschitz constant). Now since \( \|\tilde{V}_\varepsilon - V\|_{L^p(\mathcal{C}_{a\varepsilon,\varepsilon})} \to 0 \) as \( \varepsilon \to 0 \) in \( E_V \) and \( \partial I, \partial I_\varepsilon \)

\footnote{Here Proposition A.1 in [24] is applied to the following metric space: for any \( x, y \in Q_1(0) \) let \( d(x, y) = \min\{d(x, y), \text{dist}(x, \partial Q_1(0)) + \text{dist}(y, \partial Q_1(0))\} \), where \( d \) denotes the Euclidean distance in \( Q_1(0) \). Let \((\Omega_1(0), d)\) denote the completion of \( Q_1(0) \) with respect to the distance \( d \). Then Lipschitz functions on \((\Omega_1(0), d)\) corresponds to functions in \( W^{1,\infty}_d(Q_1(0)) \) (with same Lipschitz constant) modulo additive constants.}
are supported in a compact subset of $Q_2(0)$, the first term on the right hand side tends to zero as $\varepsilon \to 0^+$ in $E_V$. Moreover the second term is bounded by $4\sqrt{n}\varepsilon^2 Q^\varepsilon$ (see for instance Lemma 2 in [21]) and thus tends to 0 as $\varepsilon \to 0^+$ in $E_V$. This shows that $\partial I$ belongs to the (strong) $(W^{1,\infty}(M))^*$ closure of the class of 0-currents $T$ on $M$ such that

$$T = \sum_{j \in J} (\delta_{p_j} - \delta_{n_j}) \in (W^{1,\infty}(M))^*$$

for a countable set $J$ and points $p_j, n_j$ in $M$. By [24, Proposition A.1] applied to the complete metric space $(M, d)$ (where $d$ denotes the Euclidean distance in $Q_2(0)$) this space is closed in $(W^{1,\infty}(M))^*$, therefore $\partial I$ is also of this form. Since any Lipschitz function $\varphi \in W^{1,\infty}(Q_2(0))$ has a Lipschitz trace $\varphi|_M$ on $M$ and $\langle \partial I, \varphi \rangle_{Q_2(0)} = \langle \partial I, \varphi|_M \rangle_M$, we conclude that $\partial I$ can be represented as in (2.18) also as an element of $(W^{1,\infty}(Q_2(0)))^*$.

Theorem 1.1 could also be useful to obtain approximation results for Sobolev maps with values into manifolds. For instance we can use it to recover the following result, due to R. Schoen and K. Uhlenbeck (for $p = 2$, see [21], Section 4) and F. Bethuel and X. Zheng (for $p > 2$, see [1], Theorem 4).

**Corollary 2.5.** Let $u \in W^{1,p}(Q_1(0), S^1)$ for some $p \in (1, \infty)$. If $p \geq 2$, then

$$\text{div}(u \wedge \nabla^\perp u) = 0 \quad \text{in} \mathcal{D}'(Q_1(0))$$

and $u$ can be approximated in $W^{1,p}$ by a sequence of functions in $C^\infty(Q_1(0), S^1)$. If $p < 2$, then

$$\frac{1}{2\pi} \text{div}(u \wedge \nabla^\perp u) = \partial I,$$

(2.19)

where $I \in \mathcal{R}_1(Q_1(0))$ is a 1-rectifiable current of finite mass, and $u$ can be approximated in $W^{1,p}$ by a sequence of functions in

$$\mathcal{R} := \{ v \in W^{1,p}(Q_1(0), S^1); v \in C^\infty(Q_1(0) \setminus A, S^1), \text{ where $A$ is some finite set} \}.$$  

**Proof.** First we claim that the vector field $u \wedge \nabla^\perp u$ belongs to $L_p^2(Q_1(0))$. In fact notice that for any $x_0 \in Q_1(0)$, for a.e. $\rho \in (0, 2 \text{dist}_{\infty}(x_0, \partial Q_1(0)))$ $\partial Q_\rho(x_0)$ consists $\mathcal{H}^{n-1}$-a.e. of Lebesgue points of $u \wedge \nabla^\perp u$. Moreover for almost any such $\rho$ we have

$$\frac{1}{2\pi} \int_{\partial Q_\rho(x_0)} (u \wedge \nabla^\perp u) \cdot \nu_{\partial Q_\rho(x_0)} = \text{deg} \left( u|_{\partial Q_\rho(x_0)} \right) \in \mathbb{Z}.$$

Hence the vector field $\frac{1}{2\pi} u \wedge \nabla^\perp u$ belongs to $L_p^2(Q_1(0))$. Thus if $p \geq 2$ by Theorem 1.1 there holds $\text{div}(u \wedge \nabla^\perp u) = 0$ in $\mathcal{D}'(Q_1(0))$, while if $p < 2$ there exists a sequence of vector fields $(V_n)_{n \in \mathbb{N}}$ in $L_p^2(Q_1(0))$ such that

$$V_n \to \frac{1}{2\pi} u \wedge \nabla^\perp u \quad \text{in} \ L^p(Q_1(0)) \text{ as } n \to \infty.$$

For any $n \in \mathbb{N}$ by Hodge decomposition there exist $a_n \in W^{1,p}(Q_1(0)), b_n \in W^{1,p}_0(Q_1(0))$ such that

$$2\pi V_n = \nabla a_n + \nabla b_n.$$
For any \( n \in \mathbb{N} \) let \( \tilde{a}_n \in C^\infty(Q_1(0)) \) be such that \( \| \tilde{a}_n - a_n \|_{L^p} \leq \frac{1}{n} \). Moreover notice that there exists \( d_n \in W^{1,p}(Q_1(0), S^1) \cap C^\infty(Q_1(0) \setminus A) \), where \( A \) is a finite set, such that
\[
\nabla b_n = d_n \wedge \nabla^\perp d_n.
\]

In fact
\[
\Delta b_n = 2\pi \text{div}(V_n) = 2\pi \sum_{i=1}^{Q^n} d^n_i \delta_i^n
\]
for some \( Q^n \in \mathbb{N} \), \( p_i^n \in Q_1(0) \) and \( d_i^n \in \mathbb{Z} \), thus \( b_n = -\sum_{i=1}^{Q^n} \log|x - p_i^n|d_i^n + h_n \) for an harmonic function \( h_n \). Then \( d_n \) can be chosen to be
\[
d_n(x) = e^{-i\tilde{h}_n} \prod_{i=1}^{Q^n} \left( \frac{x - p_i^n}{|x - p_i^n|} \right)^{d_i^n},
\]
where \( \tilde{h}_n \) is the harmonic conjugate of \( h_n \) (the product has to be understood as complex multiplication in \( \mathbb{C} \simeq \mathbb{R}^2 \)).

For any \( n \in \mathbb{N} \) set \( u_n := e^{i\tilde{a}_n} d_n \). Then by construction \( u_n \in \mathcal{R} \) and
\[
u_n \wedge \nabla^\perp u_n \to u \wedge \nabla^\perp u \quad \text{in} \ L^p(Q_1(0)) \text{ as } n \to \infty.
\]

Therefore there is \( c \in [0, 2\pi) \) so that up to a subsequence
\[
e^{ic} u_n \to u \quad \text{in} \ L^p(Q_1(0)) \text{ as } n \to \infty.
\]

\[
\square
\]

\textbf{Remark 2.9.} Equation (2.19) was obtained in [9, Theorem 3] with the help of the approximation result of F. Bethuel and X. Zheng.

3. The weak \( L^p \)-closure of \( \Omega_{p,\mathbb{Z}}^{n-1}(Q_1^n(0)) \)

In the present section we follow the ideas presented in [23] in order to prove that the space \( \Omega_{p,\mathbb{Z}}^{n-1}(Q_1^n(0)) \) is weakly sequentially closed for every \( n \geq 2 \) and \( p \in (1, \infty) \). The main reason why such techniques couldn’t be used before in this context for \( n \neq 3 \) was the lack of a strong approximation theorem like Theorem 1.1 for general dimension \( n \). Such result is needed in order to define a suitable notion of distance between the cubical slices of a form \( F \in \Omega_{p,\mathbb{Z}}^{n-1}(Q_1^n(0)) \), given by \( (x \mapsto x_0 + \rho x)^*i_{\partial Q_\rho(x_0)}^* F \) for \( L^1 \)-a.e. \( \rho \) (see subsections 3.1 and 3.2 for the precise definition). Once we have turned the space of the cubical slices of \( F \) into a metric space, we will show that the "slice function" associated to \( F \), given by \( \rho \mapsto (x \mapsto x_0 + \rho x)^*i_{\partial Q_\rho(x_0)}^* F \), is locally \( \frac{1}{p} \)-Hölder continuous (see subsection 3.3). Moreover we will see that if \( \{F_k\}_{k \in \mathbb{N}} \subset \Omega_{p,\mathbb{Z}}^{n-1}(Q_1^n(0)) \) converges weakly in \( L^p \), then the sequence of the slice functions associated to each \( F_k \) is locally uniformly \( \frac{1}{p} \)-Hölder continuous. Finally, we will use the previous facts together with some technical lemmata to conclude the proof of Theorem 1.3 for \( D = Q_1^n(0) \). Notice that by Theorem 1.1 the result is clear if \( p \in [\frac{n}{n/(n-1)}, \infty) \), here we will focus on the case \( p \in (1, \frac{n}{n/(n-1)}) \).
3.1. Slice distance on $\mathbb{S}^{n-1}$

Throughout the following section, we will assume that $p \in (1, n/(n-1))$. Moreover, we will denote by "*" the Hodge star operator associated with the standard round metric on $\mathbb{S}^{n-1}$. We will denote by $Z$ the linear subspace of $\Omega^1_p(\mathbb{S}^{n-1})$ given by

$$Z := \left\{ h \in \Omega^1_p(\mathbb{S}^{n-1}) \text{ s.t. } \int_{\mathbb{S}^{n-1}} h = 0 \right\}.$$ 

Remark 3.1. It's clear that $Z$ is weakly (and thus strongly) $L^p$ closed in $\Omega^1_p(\mathbb{S}^{n-1})$. Indeed, let $\{h_k\}_{k \in \mathbb{N}} \subset Z$ be any sequence such that $h_k \to h$ weakly in $\Omega^1_p(\mathbb{S}^{n-1})$, i.e.

$$\int_{\mathbb{S}^{n-1}} \varphi h_k \to \int_{\mathbb{S}^{n-1}} \varphi h, \quad \forall \varphi \in L^p(\mathbb{S}^{n-1}).$$

Then, the statement follows by picking $\varphi \equiv 1$ and noticing that a convergent sequence of integer numbers is definitively constant.

Fix any arbitrary point $q \in \mathbb{S}^{n-1}$. We define the functions $d, \tilde{d} : Z \times Z \to [0, +\infty]$ by

$$d(h_1, h_2) := \inf \left\{ ||\alpha||_{L^p} \text{ s.t. } * (h_1 - h_2) = d^* \alpha + \partial I + \left( \int_{\mathbb{S}^{n-1}} h_1 - h_2 \right) \delta_q \right\},$$

with $\alpha \in \Omega^1_p(\mathbb{S}^{n-1}), I \in \mathcal{R}_1(\mathbb{S}^{n-1})$, and

$$\tilde{d}(h_1, h_2) := \inf \left\{ ||\alpha||_{L^p} \text{ s.t. } * (h_1 - h_2) = d^* \alpha + \partial I + \left( \int_{\mathbb{S}^{n-1}} h_1 - h_2 \right) \delta_q \right\},$$

with $\alpha \in \Omega^1_p(\mathbb{S}^{n-1}), I \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$.

Remark 3.2 ($d$ and $\tilde{d}$ are always finite on $Z$). We claim that $d, \tilde{d} < +\infty$. Since obviously $d(h_1, h_2) \leq \tilde{d}(h_1, h_2)$, it is enough to show that $\tilde{d}(h_1, h_2) < +\infty$, for every $h_1, h_2 \in Z$. This just amounts to saying that given any $h_1, h_2 \in Z$ we can always find $\alpha \in \Omega^1_p(\mathbb{S}^{n-1}), I \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$ satisfying

$$*(h_1 - h_2) = d^* \alpha + \partial I + \left( \int_{\mathbb{S}^{n-1}} h_1 - h_2 \right) \delta_q.$$ 

Indeed, let

$$a := \int_{\mathbb{S}^{n-1}} h_1 - h_2 \in \mathbb{Z}.$$

Consider the following first order differential system on $\mathbb{S}^{n-1}$:

$$\begin{cases}
d^* \omega = *(h_1 - h_2) - a \delta_q =: F, \\
d\omega = 0.
\end{cases}$$

Since $p \in (1, n/(n-1)), F \in \mathcal{L}(W^{1,p}(\mathbb{S}^{n-1}))$. Moreover, $\langle F, 1 \rangle = 0$. Hence, by Lemma B.2 we know that the previous differential system has a solution $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ and the statement follows.
Remark 3.3. As observed above, it is clear that \( d(h_1, h_2) \leq \bar{d}(h_1, h_2) \), for every \( h_1, h_2 \in Z \). We claim that actually \( d(h_1, h_2) = \bar{d}(h_1, h_2) \), for every \( h_1, h_2 \in Z \). In order to prove the remaining inequality, fix any \( h_1, h_2 \in Z \) and let \( \{\alpha_k\}_{k \in \mathbb{N}} \subset \Omega^1_p(S^{n-1}) \), \( \{I_k\}_{k \in \mathbb{N}} \subset \mathcal{R}_1(S^{n-1}) \) be such that

\[
\begin{cases}
*(h_1 - h_2) = d^* \alpha_k + \partial I_k + a\delta_q, & \forall k \in \mathbb{N}, \\
||\alpha_k||_{L^p} \to d(h_1, h_2) \text{ as } k \to \infty,
\end{cases}
\]

with

\[
a := \int_{S^{n-1}} h_1 - h_2.
\]

By Corollary 3.1, the linear differential equation

\[
\Delta u = *(h_1 - h_2) - a\delta_q
\]

has a weak solution \( \psi \in \dot{W}^{1,p}(S^{n-1}) \). Let \( \omega_k := d\psi - \alpha_k \), for every \( k \in \mathbb{N} \). Notice that

\[
d^* \omega_k = \partial I_k, \quad \forall k \in \mathbb{N}.
\]

By Theorem 2.3, for every \( k \in \mathbb{N} \) there exists a sequence \( \{\omega^j_k\}_{j \in \mathbb{N}} \subset \Omega^1_p(R(S^{n-1})) \) such that

\[
r^j_k := \omega_k - \omega^j_k \to 0 \text{ strongly in } L^p \text{ as } j \to \infty.
\]

By construction, it follows that

\[
*(h_1 - h_2) = d^*(\alpha_k + r^j_k) + d^* \omega^j_k + a\delta_q, \quad \forall k, j \in \mathbb{N}.
\]

We observe that by Proposition A.1 for every \( k, j \in \mathbb{N} \) there exist \( I^j_k \in \mathcal{R}_1(S^{n-1}) \cap \mathcal{N}_1(S^{n-1}) \) such that \( d^* \omega^j_k = \partial I^j_k \). This implies that

\[
d(h_1, h_2) \leq ||\alpha_k + r^j_k||_{L^p} \leq ||\alpha_k||_{L^p} + ||r^j_k||_{L^p}, \quad \forall k, j \in \mathbb{N}.
\]

By letting first \( j \to \infty \) and then \( k \to \infty \) in the previous inequality, our claim follows.

**Proposition 3.1.** \((Z, d)\) is a metric space.

**Proof.** We need to check symmetry, triangular inequality and non-degeneracy.

**Symmetry.** This is clear since both the \( L^p \)-norm and the space \( \mathcal{R}_1(S^{n-1}) \) are invariant under sign change.

**Triangular inequality.** Let \( h_1, h_2, h_3 \in Z \). By definition of infimum, for every \( \varepsilon > 0 \) we can write

\[
\begin{cases}
*(h_1 - h_2) = d^* \alpha_\varepsilon + \partial I_\varepsilon + \left(\int_{S^{n-1}} h_1 - h_2\right)\delta_q, \\
*(h_2 - h_3) = d^* \alpha'_\varepsilon + \partial I'_\varepsilon + \left(\int_{S^{n-1}} h_2 - h_3\right)\delta_q,
\end{cases}
\]

with \( \alpha_\varepsilon, \alpha'_\varepsilon \in \Omega^1_p(S^{n-1}) \) and \( I_\varepsilon, I'_\varepsilon \in \mathcal{R}_1(S^{n-1}) \) satisfying

\[
\begin{cases}
||\alpha_\varepsilon||_{L^p} \leq d(h_1, h_2) + \varepsilon, \\
||\alpha'_\varepsilon||_{L^p} \leq d(h_2, h_3) + \varepsilon.
\end{cases}
\]

40
We notice that
\[ *(h_1 - h_3) = d^*(\alpha_\varepsilon + \alpha'_\varepsilon) + \partial(I_\varepsilon + I'_\varepsilon) + \left( \int_{S^{n-1}} h_1 - h_3 \right) \delta_q, \quad \forall \varepsilon > 0. \]

Then, by definition of \( d \), we have
\[ d(h_1, h_3) \leq \| \alpha_\varepsilon + \alpha'_\varepsilon \|_{L^p} \leq \| \alpha_\varepsilon \|_{L^p} + \| \alpha'_\varepsilon \|_{L^p} \leq d(h_1, h_2) + d(h_2, h_3) + 2\varepsilon, \quad \forall \varepsilon > 0. \]

By letting \( \varepsilon \to 0^+ \) in the previous inequality, we get our claim.

**Non-degeneracy.** Assume that \( d(h_1, h_2) = 0 \), for some \( h_1, h_2 \in Z \). Let
\[ a := \int_{S^{n-1}} h_1 - h_2 \in \mathbb{Z}. \]

Then, since \( d = \tilde{d} \) (see Remark 3.3) and by definition of \( \tilde{d} \), there exist \( \{ \alpha_k \}_{k \in \mathbb{N}} \subset \Omega^1_p(S^{n-1}) \) and \( \{ I_k \}_{k \in \mathbb{N}} \subset \mathcal{R}_1(S^{n-1}) \cap N_1(S^{n-1}) \) such that
\[ *(h_1 - h_2) = d^* \alpha_k + \partial I_k + a \delta_q \]
and \( \alpha_k \to 0 \) strongly in \( L^p \) as \( k \to \infty \). Observe that
\[ \partial I_k \to *(h_1 - h_2) - a \delta_q \quad \text{in} \quad (W^{1, \infty}(S^{n-1}))^*. \]

Now for any \( k \in \mathbb{N} \), \( \partial I_k \) can be represented as
\[ \partial I_k = \sum_{j=1}^{J_k} (\delta_{p_j} - \delta_{n_j}) \]
for some \( J_k \in \mathbb{N} \) and points \( p_j, n_j \in S^{n-1} \). But the space of distributions of the form
\[ \sum_{j \in J} (\delta_{p_j} - \delta_{n_j}) \quad \text{such that} \quad \sum_{j \in J} |p_j - n_j| < \infty \quad (3.1) \]
(for a countable set \( J \) and points \( p_j, n_j \in S^{n-1} \)) is closed with respect to the (strong) topology of \( (W^{1, \infty}(S^{n-1}))^* \) (see Proposition A.1 in [24]), thus there exists a distribution \( T \) as in \( (3.1) \) such that
\[ *(h_1 - h_2) = T - a \delta_q. \quad (3.2) \]
But this implies that \( *(h_1 - h_2) = 0 \), since the left-hand-side in \( (3.2) \) is in \( L^p \) whilst the right-hand-side is in \( L^p \) if and only if it is equal to zero.

**Remark 3.4.** Notice that the proof above relies on the fact that \( d = \tilde{d} \), which was proved using Corollary 2.2. As the proof of the Corollary 2.2 was rather cumbersome, we remark here that there is a way to skip that passage. Indeed, in the proof of the non-degeneracy of \( d \) it is not necessary to assume that \( \{ I_k \}_{k \in \mathbb{N}} \) lies in \( N_1(S^{n-1}) \). In fact, it follows from Corollary 2.4 that \( \partial I_k \) is of the form \( (3.1) \), and thus the limit of \( \{ I_k \}_{k \in \mathbb{N}} \) in \( (W^{1, \infty}(S^{n-1}))^* \) will also be of that form.
Proposition 3.2. Let \( \{h_k\}_{k \in \mathbb{N}} \subset Z \) and \( h \in Z \). Then the following are equivalent:

1. \( \{h_k\}_{k \in \mathbb{N}} \subset Z \) is uniformly bounded w.r.t. the \( L^p \)-norm and \( d(h_k, h) \to 0 \) as \( k \to \infty \);
2. \( h_k \rightharpoonup h \) weakly in \( L^p \) as \( k \to \infty \).

Proof. We prove separately the two implications.

2 \( \Rightarrow \) 1. Pick any subsequence of \( \{h_k\}_{k \in \mathbb{N}} \) (not relabelled). For every \( k \in \mathbb{N} \), let

\[
a_k := \langle h_k - h, 1 \rangle = \int_{S^{n-1}} h_k - h.
\]

Since \( h_k \rightharpoonup h \) weakly in \( L^p \) as \( k \to \infty \), it follows that \( a_k \to 0 \) as \( k \to \infty \). Since \( \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{Z} \), there exists \( K \in \mathbb{N} \) such that \( a_k = 0 \) for every \( k \geq K \). Fix any \( k \geq K \). By Lemma B.2, the linear differential system

\[
\begin{cases}
    d^* \omega = *(h_k - h) \\
    d\omega = 0,
\end{cases}
\]

respectively (if \( n = 2 \))

\[
\begin{cases}
    d^* \omega = *(h_k - h) \\
    d\omega = 0 \\
    \int_{S^1} \omega = 0,
\end{cases}
\]

has a unique weak solution \( \alpha_k \in \Omega^1_p(S^{n-1}) \). By Remark B.3 we have

\[
||\alpha_k||_{W^{1,p}} \leq C(||d\alpha_k||_{L^p} + ||d^*\alpha_k||_{L^p}) = C||h_k - h||_{L^p}.
\]

Since \( \{h_k\}_{k \in \mathbb{N}} \) is weakly convergent, we know that it is also uniformly bounded w.r.t. the \( L^p \)-norm. Then \( \{\alpha_k\}_{k \geq K} \) is uniformly bounded w.r.t. the \( W^{1,p} \)-norm. Hence, by weak compactness in \( W^{1,p} \), there exists a subsequence \( \{\alpha_{k_l}\}_{l \in \mathbb{N}} \subset \{\alpha_k\}_{k \geq K} \) and a one-form \( \alpha \in \Omega^1_{W^{1,p}}(S^{n-1}) \) such that \( \alpha_{k_l} \rightharpoonup \alpha \) weakly in \( W^{1,p} \). By Rellich-Kondrakov theorem, it follows that \( \alpha_{k_l} \to \alpha \) strongly in \( L^p \). We claim that \( \alpha = 0 \). Indeed,

\[
\langle \alpha, \omega \rangle_{L^p-L^p'} = \lim_{l \to \infty} \langle \alpha_{k_l}, d\varphi + d^*\beta \rangle_{L^p-L^p'}
\]

\[
= \lim_{l \to \infty} -\langle d^*\alpha_{k_l}, \varphi \rangle_{L^p-L^p'}
\]

\[
= \lim_{l \to \infty} -\int_{S^{n-1}} (h_{k_l} - h) \wedge \varphi = 0,
\]

\( \forall \omega = d\varphi + d^*\beta \in \Omega^1(S^{n-1}) \),

respectively (if \( n = 2 \))

\[
\langle \alpha, \omega \rangle_{L^p-L^p'} = \lim_{l \to \infty} \langle \alpha_{k_l}, d\varphi + d^*\beta + \eta \rangle_{L^p-L^p'}
\]

\[
= \lim_{l \to \infty} \langle \alpha_{k_l}, d\varphi + d^*\beta \rangle_{L^p-L^p'}
\]

\[
= \lim_{l \to \infty} -\langle d^*\alpha_{k_l}, \varphi \rangle_{L^p-L^p'}
\]

\[
= \lim_{l \to \infty} -\int_{S^1} (h_{k_l} - h) \wedge \varphi = 0,
\]

\( \forall \omega = d\varphi + d^*\beta + \eta \in \Omega^1(S^1) \),

42
where $\eta \in \Omega^1(S^1)$ is a harmonic 1-form on $S^1$ (hence a constant 1-form) and the second equality follows because $\alpha_{k_l}$ is distributionally closed.

Hence, we have shown that $\alpha_{k_l} \to 0$ strongly in $L^p$ as $l \to \infty$. As $*(h_{k_l} - h) = d^*\alpha_{k_l}$, for every $l \in \mathbb{N}$, we have

$$d(h_{k_l}, h) \leq ||\alpha_{k_l}||_{L^p} \to 0, \quad \text{as } l \to \infty.$$  

We have just proved that any subsequence of $\{h_k\}_{k \in \mathbb{N}}$ has a further subsequence converging to $h$ with respect to $d$, therefore 1. follows.

1 $\Rightarrow$ 2. Pick any subsequence of $\{h_k\}_{k \in \mathbb{N}}$ (not relabelled). Since $\{h_k\}_{k \in \mathbb{N}} \subset Z$ is uniformly bounded w.r.t. the $L^p$-norm, by weak $L^p$-compactness there exists a subsequence $\{h_{k_l}\}_{l \in \mathbb{N}}$ of $\{h_k\}_{k \in \mathbb{N}}$ and a $h \in Z$ such that $h_{k_l} \to h$ weakly in $L^p$. Since we have just shown that $2 \Rightarrow 1$, we know that $d(h_{k_l}, h) \to 0$ as $l \to \infty$. By uniqueness of the limit, we get $h_w = h$. We have just proved that any subsequence of $\{h_k\}_{k \in \mathbb{N}} \subset Z$ has a further subsequence converging to $h$ weakly in $L^p$, hence, 2. follows.

3.2. Slice distance on $\partial Q^*_1(0)$

Let $Q_1(0) \subset \mathbb{R}^n$ be the unit cube in $\mathbb{R}^n$ centered at the origin and let $\Psi : \mathbb{S}^{n-1} \to \partial Q_1(0)$ be a bi-Lipschitz homeomorphism. We let $Y$ be the linear subspace of $\Omega^{n-1}_{L^p}\partial Q_1(0)$ given by

$$Y := \left\{ h \in \Omega^{n-1}_{L^p}\partial Q_1(0) \mid \int_{\partial Q_1(0)} h = 0 \right\}.$$  

Remark 3.5. Notice that $h \in Y$ if and only if $\Psi^* h \in Z$. Indeed, given any $h \in Z$ we have

$$C_{\Psi}^{-1} \int_{\partial Q_1(0)} |h|^p \, d\mathcal{H}^{n-1} \leq \int_{\mathbb{S}^{n-1}} |\Psi^* h|^p \, d\mathcal{H}^{n-1} \leq C_{\Psi} \int_{\partial Q_1(0)} |h|^p \, d\mathcal{H}^{n-1},$$

with $C_{\Psi} := \left( \max\{||d\Psi||_{L^\infty}, ||d\Psi^{-1}||_{L^\infty} \} \right)^{(n-1)(p-1)}$, and

$$\int_{\mathbb{S}^{n-1}} \Psi^* h = \int_{\partial Q_1(0)} h.$$  

Thus, the functions $d_\Psi, \tilde{d}_\Psi : Y \times Y \to [0, +\infty)$ given by

$$d_\Psi(h_1, h_2) := d(\Psi^* h_1, \Psi^* h_2) \quad \forall h_1, h_2 \in Y, \\ \tilde{d}_\Psi(h_1, h_2) := \tilde{d}(\Psi^* h_1, \Psi^* h_2) \quad \forall h_1, h_2 \in Y,$$

are well-defined and coincide on $Y \times Y$ by Remarks 3.2 and 3.3. Moreover, $(Y, d_\Psi)$ is a metric space as a direct consequence of Proposition 3.1 and the following statement is a corollary of Proposition 3.2.

Corollary 3.1. Let $\{h_k\}_{k \in \mathbb{N}} \subset Y$ and $h \in Y$. Then, the following are equivalent:

1. $\{h_k\}_{k \in \mathbb{N}} \subset Y$ is uniformly bounded w.r.t. the $L^p$-norm and $d_\Psi(h_k, h) \to 0$ as $k \to \infty$;
2. $h_k \to h$ weakly in $L^p$ as $k \to \infty$.  

43
Remark 3.6. Let $\Psi_1, \Psi_2 : S^{n-1} \to \partial Q_1(0)$ be bi-Lipschitz homeomorphisms. We claim that the distances $d_{\Psi_1}$ and $d_{\Psi_2}$ induced on $Y$ by $\Psi_1$ and $\Psi_2$ respectively are equivalent. Indeed notice that given any bi-Lipschitz map $\Lambda : S^{n-1} \to S^{n-1}$ we have

$$d(\Lambda^* h_1, \Lambda^* h_2) \leq \|d\Lambda\|_{L^\infty}^{-2} \|d\Lambda^{-1}\|_{L^\infty}^{-1} d(h_1, h_2), \quad \forall h_1, h_2 \in \mathbb{R}.$$  \hspace{1cm} (3.3)

To see this notice that if $\alpha \in \Omega_p^1(S^{n-1})$ is a competitor in the definition of $d(h_1, h_2)$ then the form given by $(-1)^{n-2} \Lambda^* (\alpha^*) \in \Omega_p^1(S^{n-1})$ is a competitor in the definition of $d(\Lambda^* h_1, \Lambda^* h_2).$ Hence

$$d(\Lambda^* h_1, \Lambda^* h_2) \leq \|\Lambda^* (\alpha^*)\|_{L^p} \leq \|\Lambda\|_{L^\infty}^{-1} \|\Lambda^{-1}\|_{L^\infty}^{-1} \|\alpha\|_{L_p},$$

for every competitor $\alpha$ in the definition of $d(h_1, h_2).$ By taking the infimum on all the competitors in the previous inequality, (3.3) follows. By applying (3.3) we obtain

$$d_{\Psi_2}(h_1, h_2) = d(\Psi_2^* h_1, \Psi_2^* h_2) = d((\Psi_1^{-1} \circ \Psi_2)^* \Psi_1^* h_1, (\Psi_1^{-1} \circ \Psi_2)^* \Psi_1^* h_2) \leq C_{\Psi_1, \Psi_2} d(\Psi_1^* h_1, \Psi_1^* h_2) = C_{\Psi_1, \Psi_2} d_{\Psi_1}(h_1, h_2) \quad \forall h_1, h_2 \in Y,$$

Analogously, we get

$$d_{\Psi_1}(h_1, h_2) \leq C_{\Psi_1, \Psi_2} d_{\Psi_2}(h_1, h_2) \quad \forall h_1, h_2 \in Y,$$

with $C_{\Psi_1, \Psi_2} := \max \{ \|d(\Psi_2^{-1} \circ \Psi_1)\|_{L^\infty}, \|d(\Psi_1^{-1} \circ \Psi_2)\|_{L^\infty} \}^{n-2n\alpha}.$

3.3. Slice functions and their properties

Definition 3.1 (Slice functions). Let $F \in \Omega_p^{n-1}(Q_1(0)).$ Given any arbitrary $x_0 \in Q_1(0),$ we let $\rho_0 := 2 \text{dist}_\infty(x_0, \partial Q_1(0)).$

We call the slice function of $F$ at $x_0$ the map $s : \text{Dom}(s) \subset (0, \rho_0) \to Y$ given by

$$s(\rho) := (x \mapsto \rho x + x_0)^* i_{\partial Q_\rho(x_0)}^* F, \quad \forall \rho \in \text{Dom}(s),$$

where $\text{Dom}(s)$ is the subset of $(0, \rho_0)$ defined as follows: $\rho \in \text{Dom}(s)$ if and only if the following conditions hold:

1. $\mathcal{H}^{n-1}$-a.e. point in $\partial Q_\rho(x_0)$ is a Lebesgue point for $F,$
2. $|F| \in L^p(\partial Q_\rho(x_0), \mathcal{H}^{n-1}),$
3. $\rho$ is a Lebesgue point for the $L^p$-function

$$(0, \rho_0) \ni \rho \mapsto \int_{\partial Q_\rho(x_0)} i_{\partial Q_\rho(x_0)}^* F,$$

4. $(x \mapsto \rho x + x_0)^* i_{\partial Q_\rho(x_0)}^* F \in Y.$

Remark 3.7. Notice that $\text{Dom}(s)$ has $\mathcal{L}^1$ full measure in $(0, \rho_0).$ Moreover $s \in L^p((0, \rho_0); Y),$ in the following sense: letting $j_\rho : x \mapsto \rho x + x_0,$ we have

$$\|s(\rho)\|_{L^p}^p = \int_{\partial Q_1(0)} |j_\rho^* F| d\mathcal{H}^{n-1} = \int_{\partial Q_\rho(x_0)} |F|\rho^{(n-1)(p-1)} d\mathcal{H}^{n-1}$$

and thus

$$\int_0^{\rho_0} \|s(\rho)\|_{L^p}^p d\rho \leq \int_0^{\rho_0} \int_{\partial Q_\rho} |F|\rho^{(n-1)(p-1)} d\mathcal{H}^{n-1} d\rho \leq 2^n \int_{Q_{\rho_0}(x_0)} |F| d\mathcal{H}^n$$

44
Proposition 3.3. Let \( x_0 \in Q_1(0) \) and set \( \rho_0 := 2 \text{dist}(x_0, \partial Q_1(0)) \). Fix any \( F \in \Omega_{p,z}^{n-1}(Q_1(0)) \) and let \( s \in L^p((0, \rho_0), Y) \) be the slice function of \( F \) at \( x_0 \). Let \( K \subset (0, \rho_0) \) be compact. Then, there exists a subset \( E \subset K \) such that \( \mathcal{L}^1(K \setminus E) = 0 \) and a representative \( \tilde{s} \) of \( s \) defined pointwise on \( E \) such that
\[
d_{\Psi}(\tilde{s}(\rho_1), \tilde{s}(\rho_2)) \leq C_{p,K,\Psi} \|F\|_{L^p}|\rho_1 - \rho_2|^{\gamma}, \quad \forall \rho_1, \rho_2 \in E, \tag{3.4}
\]
where
\[
C_{p,K,\Psi} := C_{p,\Psi} \max_{\rho \in K} \rho^{1-n}
\]

Proof. Denote by \( T_F \in D_1(Q_1(0)) \) the 1-current on \( Q_1(0) \) given by
\[
\langle T_F, \omega \rangle = \int_{Q_1(0)} F \wedge \omega, \quad \forall \omega \in D^1(Q_1(0)).
\]

Since \( F \in \Omega_{p,z}^{n-1}(Q_1(0)) \), by Theorem 2.2 there exists \( I \in \mathcal{R}_1(Q_1(0)) \) such that \( M(I) < +\infty \) and \( *dF = \partial I \). By definition of integral 1-current, there exist a locally 1-rectifiable set \( \Gamma \subset Q_1(0) \), a Borel measurable unitary vector field \( \vec{I} \) on \( \Gamma \) and a positive \( \mathbb{Z} \)-valued \( \mathcal{H}^1 \Gamma \)-integrable function \( \theta \in L^1(\Gamma, \mathcal{H}^1) \) such that
\[
\langle I, \omega \rangle = \int_{\Gamma} \theta(\omega, \vec{I}) \, d\mathcal{H}^1, \quad \forall \omega \in D^1(Q_1(0)).
\]

By the coarea formula, there exists \( G \subset K \) such that \( \mathcal{L}^1(K \setminus G) = 0 \) and such that \( \Gamma \cap \partial Q_\rho(x_0) \) is a finite set for every \( \rho \in G \).

Let \( \Psi : \mathbb{S}^{n-1} \to \partial Q_1(0) \) be the bi-Lipschitz map given by
\[
\Psi(x) := \frac{x}{2\|x\|_\infty}, \quad \forall x \in \mathbb{S}^{n-1}.
\]

Consider the map \( \Phi : \mathbb{S}^{n-1} \times [0, \rho_0] \to \text{Im}(\Phi) = \overline{Q_{\rho_0}(x_0)} \subset \overline{Q_1(0)} \) given by
\[
\Phi(y, t) := x_0 + t\Psi(y), \quad \forall (y, t) \in \mathbb{S}^{n-1} \times [0, \rho_0].
\]

Notice that \( \Phi|_{\mathbb{S}^{n-1} \times [\rho_1, \rho_2]} \) is a bi-Lipschitz homeomorphism onto its image for every \( \rho_1, \rho_2 \in (0, 1) \).

We claim that estimate (3.4) holds on a full-measure subset of \( G \). Indeed, fix any \( \rho_1, \rho_2 \in G \). Without loss of generality, assume that \( \rho_2 > \rho_1 \). Let \( \hat{\Phi} := \Phi|_{\mathbb{S}^{n-1} \times [\rho_1, \rho_2]} \). Define \( \pi := \text{pr}_1 \circ \Phi^{-1} : Q_{\rho_0}(x_0) \to \mathbb{S}^{n-1} \), where \( \text{pr}_1 : \mathbb{S}^{n-1} \times [0, \rho_0] \to \mathbb{S}^{n-1} \) is the canonical projection on the first factor, and notice that \( \pi \) is a Lipschitz and proper map. Then, \( \pi_*\langle T_F, \text{Im}(\hat{\Phi}) \rangle \in D_1(\mathbb{S}^{n-1}) \) can be expressed as follows: for any \( \omega \in \Omega^1(\mathbb{S}^{n-1}) \)
\[
\langle \pi_*\langle T_F, \text{Im}(\hat{\Phi}) \rangle, \omega \rangle = \langle T_F, \text{Im}(\hat{\Phi}), \pi^*\omega \rangle = \int_{\text{Im}(\hat{\Phi})} F \wedge \pi^*\omega
\]
\[
= \int_{\text{Im}(\hat{\Phi})} (\Phi^{-1})^* (\Phi^* F \wedge \Phi^* \pi^*\omega)
\]
\[
= \int_{\mathbb{S}^{n-1} \times [\rho_1, \rho_2]} \Phi^* F \wedge \text{pr}_1^*\omega
\]
\[= \int_{\mathbb{S}^{n-1} \times [\rho_1, \rho_2]} \text{pr}_1^\ast \omega \wedge (*\Phi^* F)\]
\[= \int_{\mathbb{S}^{n-1} \times [\rho_1, \rho_2]} (\text{pr}_1^\ast \omega, *\Phi^* F) \, d\text{vol}_{\mathbb{S}^{n-1} \times [\rho_1, \rho_2]}(y, t)\]
\[= \int_{\rho_1}^{\rho_2} \left( \int_{\mathbb{S}^{n-1}} (\text{pr}_1^\ast \omega, *\Phi^* F) \, d\text{vol}_{\mathbb{S}^{n-1}}(y) \right) dt\]
\[= \int_{\rho_1}^{\rho_2} \left( \int_{\mathbb{S}^{n-1}} \langle \omega, i_{\mathbb{S}^{n-1} \times \{t\}}^\ast *\Phi^* F \rangle \, d\text{vol}_{\mathbb{S}^{n-1}}(y) \right) dt\]
\[= \int_{\mathbb{S}^{n-1}} \omega \wedge \left( \int_{\rho_1}^{\rho_2} (i_{\mathbb{S}^{n-1} \times \{t\}}^\ast *\Phi^* F) \right) dt\]
\[= (-1)^{n-1} \int_{\mathbb{S}^{n-1}} \omega \wedge \alpha,\]

where

\[\alpha := (-1)^{n-1} \int_{\rho_1}^{\rho_2} (i_{\mathbb{S}^{n-1} \times \{t\}}^\ast *\Phi^* F) \, dt \in \Omega_p^{n-2}(\mathbb{S}^{n-1}).\]

In particular,

\[
\langle \partial \pi_\ast (T_F \mathbf{L} \text{Im}(\hat{\Phi})), \varphi \rangle = \langle \pi_\ast (T_F \mathbf{L} \text{Im}(\hat{\Phi})), d\varphi \rangle \\
= (-1)^{n-1} \int_{\mathbb{S}^{n-1}} d\varphi \wedge \alpha = \langle d^\ast (*\alpha), \varphi \rangle, \ \ \ \forall \varphi \in C^\infty(\mathbb{S}^{n-1}).
\]

Recall that the restriction of an integral current to a measurable set is still an integral current. Moreover, the push-forward of an integral current through a Lipschitz and proper map remains an integral current (see \cite{19}, Chapter 7, §7.5). Then, \( \bar{I} := -\pi_\ast (I \mathbf{L} \text{Im}(\hat{\Phi})) \in \mathcal{R}_1(\mathbb{S}^{n-1}). \)

So far, we have shown that

\[
\partial \pi_\ast ((T_F - I) \mathbf{L} \text{Im}(\hat{\Phi})) = \partial \pi_\ast (T_F \mathbf{L} \text{Im}(\hat{\Phi})) - \partial \pi_\ast (I \mathbf{L} \text{Im}(\hat{\Phi})) = d^\ast (*\alpha) + \partial \bar{I}.
\]

Let \( \zeta \in C^\infty_c((-1, 1)) \) such that \( \int_{\mathbb{R}} \zeta = 1 \). For any \( \varepsilon \in (0, \min\{\rho_1, \rho_0 - \rho_2\}) \) set \( \zeta_\varepsilon = \frac{1}{\varepsilon} \zeta (\frac{\cdot}{\varepsilon}) \)
and let \( \chi_\varepsilon \) be the unique solution of

\[
\begin{cases}
\chi_\varepsilon'(x) = \zeta_\varepsilon(x - \rho_1) - \zeta_\varepsilon(x - \rho_2) \\
\chi_\varepsilon(0) = 0.
\end{cases}
\]

Let \( \psi \in C^\infty(\mathbb{S}^{n-1} \times [0, \rho_0]) \) and let \( \text{pr}_2 : \mathbb{S}^{n-1} \times [0, \rho_0] \to [0, \rho_0] \) be the projection on the second factor. We compute

\[
\langle (\Phi^{-1})_\ast I, \psi \rangle d(\chi_\varepsilon \circ \text{pr}_2) = \int_{\Phi^{-1}(\Gamma)} \theta(\Phi^{-1})_\ast I \psi(\chi_\varepsilon' \circ \text{pr}_2)(d\text{pr}_2, \bar{I}(\Phi^{-1})_\ast I) \, d\mathcal{H}^1
\]
\[= \int_{\rho_1 - \varepsilon}^{\rho_1 + \varepsilon} \zeta_\varepsilon(t) \left( \int_{\Phi^{-1}(\Gamma) \cap (\mathbb{S}^{n-1} \times \{t\})} \psi \, d\mathcal{H}^0 \right) \, d\mathcal{L}^1(t)
\]

46
\[- \int_{\rho_2 + \varepsilon}^{\rho_2 - \varepsilon} \zeta(t) \left( \int_{\Phi^{-1}(\Gamma) \cap (S^{n-1} \times \{t\})} \psi d\mathcal{H}^0 \right) d\mathcal{L}^1(t)\]

with \(\tilde{\theta} = \theta_{(\Phi^{-1})^*} \text{sgn}((d\pi_{\rho_2}, \tilde{\Pi}_{(\Phi^{-1})^*})) \in L^1(\Phi^{-1}(\Gamma), \mathbb{Z})\). Moreover

\[
\langle (\Phi^{-1})^* T_F, \psi d(\chi_\varepsilon \circ \text{pr}_2) \rangle = \int_{S^{n-1} \times [0, \rho_0]} \psi(\chi_\varepsilon' \circ \text{pr}_2) \Phi^* F \wedge d\pi_{\rho_2} \\
= \int_{\rho_1 + \varepsilon}^{\rho_1 - \varepsilon} \zeta(t) \left( \int_{S^{n-1} \times \{t\}} \psi(\Phi|_{S^{n-1} \times \{t\}})^* F \right) d\mathcal{L}^1(t) \\
- \int_{\rho_2 + \varepsilon}^{\rho_2 - \varepsilon} \zeta(t) \left( \int_{S^{n-1} \times \{t\}} \psi(\Phi|_{S^{n-1} \times \{t\}})^* F \right) d\mathcal{L}^1(t).
\]

Now observe that

\[
((\Phi^{-1})^* (T_F - I), (\chi_\varepsilon \circ \text{pr}_2) d\psi) \to (((\Phi^{-1})^* (T_F - I)) \mathbf{L}(S^{n-1} \times [\rho_1, \rho_2]), d\psi)
\]
as \(\varepsilon \to 0^+\), by dominated convergence. On the other hand, since \(\partial(T_F - I) = 0\), we have

\[
((\Phi^{-1})^* (T_F - I), (\chi_\varepsilon \circ \text{pr}_2) d\psi) = \langle (\Phi^{-1})^* I, \psi d(\chi_\varepsilon \circ \text{pr}_2) \rangle - \langle (\Phi^{-1})^* T_F, \psi d(\chi_\varepsilon \circ \text{pr}_2) \rangle.
\]

Therefore for almost every \(\rho_1, \rho_2 \in (0, \rho_0)\) (depending on \(\psi\)) we have

\[
\langle \partial((\Phi^{-1})^* (T_F - I)) \mathbf{L}(S^{n-1} \times [\rho_1, \rho_2]), \psi \rangle = \int_{\Phi^{-1}(\Gamma) \cap (S^{n-1} \times \{\rho_1\})} \psi \tilde{\theta} d\mathcal{H}^0 \\
- \int_{\Phi^{-1}(\Gamma) \cap (S^{n-1} \times \{\rho_2\})} \psi \tilde{\theta} d\mathcal{H}^0 \\
- \int_{S^{n-1} \times \{\rho_1\}} \psi(\Phi|_{S^{n-1} \times \{\rho_1\}})^* F \\
+ \int_{S^{n-1} \times \{\rho_2\}} \psi(\Phi|_{S^{n-1} \times \{\rho_2\}})^* F. \quad (3.5)
\]

Now let \(\{\psi_k\}_{k \in \mathbb{N}} \subset C^\infty(S^{n-1} \times [0, \rho_0])\) be a countable sequence dense in \(C^1(S^{n-1} \times [0, \rho_0])\). For every \(k \in \mathbb{N}\), let \(E_k \subset G\) be the set such that \((3.5)\) holds with \(\psi = \psi_k\) (i.e. the set of the \(\rho \in G\) which are "\(\zeta\)-Lebesgue points" of the integrands in \((3.5)\), with \(\psi = \psi_k\)) and define

\[
E := \bigcap_{k \in \mathbb{N}} E_k.
\]

Then \(\mathcal{L}^1(E) = \mathcal{L}^1(K)\) and for every \(\rho_1, \rho_2 \in E\) estimate \((3.5)\) holds with \(\psi = \psi_k\) for every \(k \in \mathbb{N}\). By density of \(\{\psi_k\}_{k \in \mathbb{N}}\) in \(C^1(S^{n-1} \times [0, \rho_0])\), we can pass to the limit in \((3.5)\) and get that for any given couple of parameters \(\rho_1, \rho_2 \in \bar{E}\) such estimate holds for every \(\psi \in C^\infty(S^{n-1} \times [0, \rho_0])\). In particular, for every \(\rho_1, \rho_2 \in E, \varphi \in C^\infty(S^{n-1})\) we have

\[
\langle \partial \pi_\nu ((T_F - I) \mathbf{L} \text{Im}(\Phi)), \varphi \rangle = \sum_{x \in \Gamma_{\rho_1}} \tilde{\theta}(x, \rho_1) \varphi(x) - \sum_{x \in \Gamma_{\rho_2}} \tilde{\theta}(x, \rho_2) \varphi(x) \\
- \int_{S^{n-1}} \varphi \Psi^* s(\rho_1) + \int_{S^{n-1}} \varphi^* \Psi^* s(\rho_2),
\]

47
where

\[
\Gamma_{\rho_1} := \text{pr}_1(\Phi^{-1}(\Gamma) \cap (\mathbb{S}^{n-1} \times \{\rho_1\})) \subset \mathbb{S}^{n-1},
\]
\[
\Gamma_{\rho_2} := \text{pr}_1(\Phi^{-1}(\Gamma) \cap (\mathbb{S}^{n-1} \times \{\rho_2\})) \subset \mathbb{S}^{n-1}
\]

are finite set for any \(\rho_1, \rho_2 \in G\).

Gathering together what we have proved so far, we have

\[
\ast(*) \Psi s(\rho_2) - \Psi s(\rho_1) = d'(*\alpha) + \partial I' + \left( \int_{\mathbb{S}^{n-1}} \Psi s(\rho_2) - \Psi s(\rho_1) \right) d_\mathbb{H}.
\]

where \(I' \in \mathcal{R}_1(\mathbb{R}^{n-1})\) is any rectifiable one-current of finite mass such that

\[
\partial I' = \sum_{x \in \Gamma_{\rho_1}} \hat{\theta}(x, \rho_1) \delta_x - \sum_{x \in \Gamma_{\rho_2}} \hat{\theta}(x, \rho_2) \delta_x + \partial I + \left( \sum_{x \in \Gamma_{\rho_1}} \hat{\theta}(x, \rho_1) - \sum_{x \in \Gamma_{\rho_2}} \hat{\theta}(x, \rho_2) \right) \delta_q,
\]

i.e. \(*\alpha\) is a competitor in the definition of \(d(\Psi s(\rho_2), \Psi s(\rho_1))\). Hence, in order to estimate \(d(\Psi s(\rho_2), \Psi s(\rho_1))\) we just need to find an upper bound for \(||*\alpha||_{L^p}\).

Notice that \(|d\Phi| \leq t|d\Psi| + \frac{\sqrt{n}}{2}\). Moreover since

\[
\Phi^{-1}(x) = \left( \Psi^{-1} \left( \frac{x - x_0}{2\|x - x_0\|_\infty} \right), 2\|x - x_0\|_\infty \right)
\]

we have \(|J\Phi^{-1}(x)| \leq 2^{-(n-2)} \left( \frac{\|d\Psi^{-1}\|_{L^\infty}}{\|x - x_0\|_\infty} \right)^{n-1}\). Therefore

\[
||*\alpha||_{L^p}^p \leq \int_{\mathbb{S}^{n-1}} \left| \int_{\rho_1}^{\rho_2} |\Phi^* F| dt \right|^p d\mathcal{H}^{n-1} \leq |\rho_1 - \rho_2|^p \int_{\mathbb{S}^{n-1} \times \{\rho_1, \rho_2\}} |\Phi^* F|^p d\mathcal{H}^{n-1} dt
\]

\[
\leq \left( 2^{-(n-2)} \left( \|d\Psi\|_{L^\infty} + \frac{\sqrt{n}}{2} \right)^{p(n-1)} \frac{\|d\Psi^{-1}\|_{L^\infty}^{-1}}{\rho_1} \right) |\rho_1 - \rho_2|^p \|F\|_{L^p(Q_1(0))}^p
\]

and our claim follows.

\[\square\]

3.4. Proof of Theorem 1.3 for \(Q^n_1(0)\)

For the proof of Theorem 1.3 we need two technical Lemmata.

**Lemma 3.1.** Let \(\{f_k\}_{k \in \mathbb{N}} \subset L^1(0, 1)\) be such that \(\|f_k\|_{L^1} \leq C\) for any \(k \in \mathbb{N}\). Then there exist a sequence of compact subsets \(\{W_h\}_{h \in \mathbb{N}_{>2}} \subset (0, 1)\) such that for every \(h \in \mathbb{N}_{>2}\) the following properties hold:

1. \(L^1(W_h) = 1 - \frac{C + 2}{h}\);
2. \(W_h \subset (1/h, 1)\);
3. for almost every \(\rho \in W_h\) and every \(k \in \mathbb{N}\) there exists \(k' > k\) such that \(|f_{k'}(\rho)| \leq h\).
Proof. Let $h \in \mathbb{N}_{\geq 2}$. For any $l \in \mathbb{N}$ let

$$A^h_l := \bigcap_{k=l}^{\infty} f_k^{-1}([-h, h]^c).$$

Notice that for any $l \in \mathbb{N}$ $A^h_l \subset A^h_{l+1}$ and set

$$I_h = \bigcup_{l=1}^{\infty} A^h_l.$$

Let $m \in \mathbb{N}$ and let $k \geq m$. Notice that

$$C \geq \int_0^1 |f_k(\rho)|\,d\rho \geq \int_{A^h_m} |f_k(\rho)|\,d\rho > h\mathcal{L}^1(A^h_m).$$

By letting $m \to \infty$ in the previous inequality, be obtain

$$\mathcal{L}^1(I_h) \leq \frac{C}{h}.$$

Then, by defining $E_h := I_h^c \cap (1/h, 1)$, we clearly get

$$\mathcal{L}^1(E_h) = \mathcal{L}^1((I_h \cup (0, 1/h))^c) = 1 - \mathcal{L}^1(I_h \cup (0, 1/h)) \geq 1 - \frac{C + 1}{h}.$$

Moreover, $E_h$ and any of its subsets satisfy the properties 2. and 3.. Finally, since $E_h$ is measurable, we can find a compact set $W_h \subset E_h$ such that

$$\mathcal{L}^1(W_h) = 1 - \frac{C + 2}{h}.$$

By construction, $W_h$ satisfies 1, 2 and 3.

Since we do not know if the space $(Y, d)$ is complete, we will also need the following lemma.

**Lemma 3.2.** Let $K \subset [0, 1]$ be compact and let $S \subset K$ be dense and countable. Let $\{f_k\}_{k \in \mathbb{N}} \subset C^0(K, Y)$ be such that

1. $\{f_k\}_{k \in \mathbb{N}}$ is uniformly Cauchy from $K$ to $(Y, d)$ (i.e. it is a Cauchy sequence w.r.t. uniform convergence);
2. for some $C > 0$,

$$\sup_{\rho \in S} \sup_{k \in \mathbb{N}} ||f_k(\rho)||_{L^p} \leq C;$$

3. for some $A > 0$ and some $\alpha \in (0, 1]$, we have

$$d(f_k(\rho), f_k(\rho')) \leq A|\rho - \rho'|^{\alpha}, \quad \forall \rho, \rho' \in S.$$

Then, there exists $f \in C^0(K, Y)$ such that $f_k \to f$ uniformly.
Proof. Fix any $\rho \in S$. By hypothesis 2, $\{f_k(\rho)\}_{k \in \mathbb{N}}$ is bounded in $L^p$ and therefore it has a subsequence converging weakly in $L^p$ to a limit $f(\rho) \in Y$ (recall that $Y$ is closed with respect to the weak $L^p$ convergence). By Corollary 3.1 such a subsequence converges in $(Y,d)$ to the same limit $f(\rho)$. Since $\{f_k\}_{k \in \mathbb{N}}$ is uniformly Cauchy from $K$ to $(Y,d)$, we have that $\{f_k(\rho)\}_{k \in \mathbb{N}}$ is Cauchy in $(Y,d)$, therefore $f_k(\rho) \overset{d}{\to} f(\rho)$ and, by Corollary 3.1 $f_k(\rho) \to (\rho)$ weakly in $L^p$.

Fix any $\rho \in K$ and let $\{\rho_i\}_{i \in \mathbb{N}} \subset S$ be such that $\rho_i \to \rho$. We claim that there exists $f_\rho \in Y$ such that $f(\rho_i) \to f_\rho$ w.r.t. $d$ and $f_\rho$ doesn’t depend on the choice of the sequence $\{\rho_i\}_{i \in \mathbb{N}}$. Indeed by lower semicontinuity of the $L^p$ norm w.r.t. the weak $L^p$ convergence, we have

$$||f(\rho_i)||_{L^p} \leq \liminf_{k \to \infty} ||f_k(\rho_i)||_{L^p} \leq C,$$

for every $i \in \mathbb{N}$. Then there exists a subsequence $\{\rho_{i_j}\}_{j \in \mathbb{N}}$ and a $f_\rho \in L^p$ such that $f(\rho_{i_j}) \to f_\rho$ weakly in $L^p$. Since $Y$ is weakly closed in $L^p$ we have $f_\rho \in Y$. By Corollary 3.1 we also have $f(\rho_{i_j}) \to f_\rho$ w.r.t. $d$.

Relabel the subsequence as $\{\rho_i\}_{i \in \mathbb{N}}$. To see that $f_\rho$ doesn’t depend on the subsequence (and thus it doesn’t depend on $\{\rho_i\}_{i \in \mathbb{N}}$), assume that $\{\tilde{\rho}_i\}_{i \in \mathbb{N}}$ is another sequence in $S$ with $\tilde{\rho}_i \to \rho$ and $f(\tilde{\rho}_i) \to \tilde{f}_\rho$ w.r.t. $d$. To see that $f_\rho = \tilde{f}_\rho$, first notice that by hypothesis 3. and by triangle inequality we have

$$d(f(\rho_i), f(\tilde{\rho}_i)) \leq d(f(\rho_i), f_k(\rho_i)) + d(f_k(\rho_i), f(\tilde{\rho}_i)) + d(f_k(\tilde{\rho}_i), f(\tilde{\rho}_i)),$$

for every $i \in \mathbb{N}$. Hence, passing to the limit as $k \to \infty$ in the previous inequality, we get

$$d(f(\rho_i), f(\tilde{\rho}_i)) \leq A|\rho_i - \tilde{\rho}_i|^\alpha.$$

Thus we finally obtain

$$d(f_\rho, \tilde{f}_\rho) \leq d(f_\rho, f(\rho_i)) + d(f(\rho_i), f(\tilde{\rho}_i)) + d(f(\tilde{\rho}_i), \tilde{f}_\rho) \leq d(f_\rho, f(\rho_i)) + A|\rho_i - \tilde{\rho}_i|^\alpha + d(f(\tilde{\rho}_i), \tilde{f}_\rho),$$

and, passing to the limit as $i \to \infty$, we get $d(f_\rho, \tilde{f}_\rho) = 0$, i.e. $f_\rho = \tilde{f}_\rho$.

For any $\rho \in K$ let

$$f(\rho) := \lim_{i \to \infty} f(\rho_i),$$

where $\{\rho_i\}_{i \in \mathbb{N}} \subset S$ is any sequence such that $\rho_i \to \rho$ and the limit is understood w.r.t. $d$. Then $f$ is a well-defined function on $K$.

To see that $f_k \to f$ uniformly, let $\varepsilon > 0$ and let $K \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}_{\geq K}$

$$\sup_{\rho \in S} d(f_m(\rho), f_n(\rho)) < \varepsilon.$$

Let $\rho \in K$ and let $\{\rho_i\}_{i \in \mathbb{N}}$ be a sequence in $S$ such that $\rho_i \to \rho$. Then for any $k \in \mathbb{N}_{\geq K}$ we have

$$d(f_k(\rho), f(\rho)) = \lim_{i \to \infty} d(f_k(\rho_i), f(\rho_i)) = \lim_{i \to \infty} \lim_{m \to \infty} d(f_k(\rho_i), f_m(\rho_i)) \leq \varepsilon.$$

This shows that $f$ is the uniform limit of $\{f_k\}_{k \in \mathbb{N}}$ in $K$, with respect to $d$. As $f_k \in C^0(K,Y)$ for any $k \in \mathbb{N}$, we have that $f \in C^0(K,Y)$ (this also follows directly from the construction of $f$).
Theorem 3.1 (Weak closure for $Q_1^2(0)$). Fix any $n \in \mathbb{N}_{\geq 2}$ and assume that $p \in (1, n/(n - 1))$. Then $\Omega_p^{n-1}(Q_1^2(0))$ is weakly sequentially closed.

Proof. Assume that $F \in \Omega_p^{n-1}(Q_1^2(0))$ belongs to the weak $L^p$-closure of $\Omega_p^{n-1}(Q_1^2(0))$, i.e. there exists $\{F_k\}_{k \in \mathbb{N}} \subset \Omega_p^{n-1}(Q_1^2(0))$ such that $F_k \xrightarrow{L^p} F$. What we need to show is that $F \in \Omega_p^{n-1}(Q_1^2(0))$, which amounts to saying that

$$
\int_{\partial Q^1(x_0)} i^*_\partial Q^1(x_0) F \in \mathbb{Z}, \tag{3.6}
$$

for every $x_0 \in Q_1^2(0)$ and for a.e. $\rho \in (0, 2 \text{dist}_e(x_0, \partial Q_1^2(0)))$. Without losing generality, we will just show (3.6) for $x_0 = 0$.

Step 1. For any $k \in \mathbb{N}$ let $s_k$ be the slice function of $F_k$ at 0. Fix any $h \in \mathbb{N}_{\geq 2}$ and let $W_h \subset (1/h, 1)$ be the compact set given by applying Lemma 3.1 with $f_k = \|s_k\|_{L^p}$ and $C = 27 \sup_{k \in \mathbb{N}} \|F_k\|_{L^p}$ (see Remark 3.7). Let $F_h^k \subset W_h$ denote the subset associated to $W_h$ and $s_k$ by Proposition 3.3 let $E_h = \bigcap_{k \in \mathbb{N}} E_h^k$ and let $s_k$ denote its $1/p$-Hölder representative on $E_h$, for any $k \in \mathbb{N}$. By property 3. in Lemma 3.1, for almost every $\rho \in E_h$ we can find a subsequence $\{s_{k_j}(\rho)\}_{j \in \mathbb{N}} \subset \{s_k(\rho)\}_{k \in \mathbb{N}}$ such that $s_{k_j}(\rho)$ is uniformly bounded in $L^p$ by $h$. Denote by $E_h$ the set of all such $\rho$ and observe that $L^1(E_h) = L^1(W_h) = 1 - C_{h, p}^2$. Then for any $\rho \in E_h$, $\{s_k(\rho)\}_{k \in \mathbb{N}}$ has a subsequence that converges weakly in $L^p$ or, equivalently, with respect to $d_{\Psi}$ (see Corollary 3.1).

Let $S_h \subset E_h$ be a countable dense subset. By a diagonal extraction argument, we find a subsequence $\{s_{k_l}\}_{l \in \mathbb{N}}$ such that $\{s_{k_l}(\rho)\}_{l \in \mathbb{N}}$ is convergent with respect to $d_{\Psi}$ and weakly in $L^p$, and is uniformly bounded in $L^p$ by $h$, for every $\rho \in S_h$.

Step 2. Next we claim that for any $l \in \mathbb{N}$, $s_{k_l}$ can be extended to a $1/p$-Hölder continuous function on $E_h$ (with the same Hölder constant, which is bounded uniformly in $l$). In fact let $f \in \{s_{k_l}\}_{l \in \mathbb{N}}$, let $\rho_0 \in E_h \setminus E_h$, let $\{\rho_i\}_{i \in \mathbb{N}}$ be a sequence in $S_h$ such that $\rho_i \to \rho_0$ as $i \to \infty$ (observe that such a sequence exists, since $S_h$ is dense in $E_h$, which in turn is dense in $E_h$). Since $\{f(\rho_i)\}_{i \in \mathbb{N}} \subset Y$ is uniformly bounded in $L^p$, by weak $L^p$-compactness there exists a subsequence $\{f(\rho_{j_l})\}_{l \in \mathbb{N}}$ of $\{f(\rho_i)\}_{i \in \mathbb{N}}$ such that $f(\rho_{j_l}) \to f_\rho$ weakly in $L^p$ for some $f_\rho \in Y$. By Corollary 3.1 we know that $d_{\Psi}(f(\rho_{j_l}), f_\rho) \to 0$. Since $f$ is $1/p$-Hölder continuous on $E_h$, $f_\rho$ does not depend on the sequence $\{\rho_i\}_{i \in \mathbb{N}}$. Hence, the function

$$
\tilde{f}(\rho) := \begin{cases} 
 f(\rho) & \text{if } \rho \in E_h \\
 f_\rho & \text{if } \rho \in E_h \setminus E_h
 \end{cases}
$$

is well-defined on $E_h$ and satisfies (3.4) on $E_h$.

In the following, in order to simplify the notation, we will denote again by $s_{k_l}$ the $1/p$-Hölder extension of $s_{k_l}$ to $E_h$, for any $l \in \mathbb{N}$.

Step 3. We show that $\{s_{k_l}\}_{l \in \mathbb{N}}$ converges uniformly on $E_h$ to some $s \in C^0(E_h, Y)$. Fix any $\varepsilon > 0$. By Step 2, we know that the sequence $\{s_{k_l}\}_{l \in \mathbb{N}}$ is equicontinuous from $E_h$ to $(Y, d_{\Psi})$. Therefore we can choose $\delta > 0$ such that

$$
d_{\Psi}(s_{k_l}(\rho), s_{k_l}(\rho')) < \varepsilon, \quad \forall \rho, \rho' \in E_h \text{ s.t. } |\rho - \rho'| < \delta \text{ and } \forall l \in \mathbb{N}.
$$
Notice that \( \{(\rho - \delta, \rho + \delta)\}_{\rho \in S_h} \) is an open cover of \( \overline{E_h} \). Since \( \overline{E_h} \) is compact, we can find a finite set \( \{\rho_1, \ldots, \rho_m\} \subset S_h \) such that \( \{(\rho_j - \delta, \rho_j + \delta)\}_{j=1,\ldots,m} \) is a finite open cover of \( \overline{E_h} \).

Now let \( \rho \in \overline{E_h} \). Observe that there exists a point \( \rho_j \in \{\rho_1, \ldots, \rho_m\} \) such that \( \rho \in (\rho_j - \delta, \rho_j + \delta) \), i.e., \( |\rho - \rho_j| < \delta \). By our choice of \( \delta \), this implies \( d(s_{k_l}(\rho), s_{k_l}(\rho_j)) < \varepsilon \), for every \( l \in \mathbb{N} \). By triangle inequality, we have

\[
d\phi(s_{k_l}(\rho), s_{k_m}(\rho)) \leq d\phi(s_{k_l}(\rho), s_{k_l}(\rho_j)) + d\phi(s_{k_l}(\rho_j), s_{k_m}(\rho)) + d\phi(s_{k_m}(\rho_j), s_{k_m}(\rho)) < 2\varepsilon + d\phi(s_{k_l}(\rho_j), s_{k_m}(\rho_j)).
\]

But since \( \rho_j \in S_h \), we know that there exists \( L_j > 0 \) such that

\[
d\phi(s_{k_l}(\rho_j), s_{k_m}(\rho_j)) < \varepsilon, \quad \forall l, m \geq L_j.
\]

Hence, by letting \( L := \max_{j=1,\ldots,m} L_j \), we have that

\[
d\phi(s_{k_l}(\rho), s_{k_m}(\rho)) < 3\varepsilon, \quad \forall l, m \geq L, \forall \rho \in \overline{E_h}.
\]

Here we have just proved that the sequence \( \{s_{k_l}\}_{l \in \mathbb{N}} \) is uniformly Cauchy on \( \overline{E_h} \). Since \( \{s_{k_l}\}_{l \in \mathbb{N}} \) satisfies all the hypotheses of Lemma 3.2, we get that there exists \( s \in C^0(\overline{E_h}, Y) \) such that \( s_{k_l} \to s \) uniformly on \( \overline{E_h} \) w.r.t. \( d\phi \).

Notice that since \( \|s_{k_l}(\rho)\|_{L^p} \leq h \) for any \( l \in \mathbb{N} \), for any \( \rho \in E_h \), and since \( s_{k_l}(\rho) \to s(\rho) \) w.r.t. \( d\phi \) for any \( \rho \in \overline{E_h} \), by Corollary 3.3 there holds \( s_{k_l}(\rho) \to s(\rho) \) in \( L^p \) for any \( \rho \in E_h \).

**Step 4.** Let \( s_0 : E_h \to Y \) be the restriction to \( E_h \) of the slice function of \( F \) at 0. We claim that \( s = s_0 \) a.e. in \( E_h \). To show this we will prove that

\[
\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi(s_{k_l}(\rho) - s_0(\rho)) \, d\rho \to 0, \quad \text{as } l \to \infty,
\]

for every \( \varphi \in C_c^\infty((0, 1)) \) and for every \( \psi \in \text{Lip}(\partial Q_1(0)) \). Indeed, an explicit computation gives

\[
\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi(s_{k_l}(\rho) - s_0(\rho)) \, d\rho
\]

\[
= \int_{E_h} \varphi(\rho) \left( \int_{\partial Q_r(0)} \psi\left(\frac{\cdot}{\rho}\right) i_{\partial Q_r(0)}^* F_{k_l} - \int_{\partial Q_r} \psi\left(\frac{\cdot}{\rho}\right) i_{\partial Q_r(0)}^{*} F \right) \, d\rho
\]

\[
= 2^n \int_{Q_1(0)} 1_{E_h}(2r) \varphi(2|\cdot|) \psi\left(\frac{\cdot}{|\cdot|}\right) dr \wedge (F_{k_l} - F) \to 0 \quad \text{as } l \to \infty,
\]

since

\[
1_{E_h}(2r) \varphi(2|\cdot|) \psi\left(\frac{\cdot}{|\cdot|}\right) dr \in \Omega^1_{\rho}(Q_1(0))
\]

and \( F_{k_l} \to F \) in \( L^p \). On the other hand, since \( s_{k_l}(\rho) \to s(\rho) \) in \( L^p \) for any \( \rho \in E_h \),

\[
\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi(s_{k_l}(\rho) - s(\rho)) \, d\rho \to 0, \quad \text{as } l \to \infty
\]
for every $\varphi \in C_c^\infty((0,1))$ and for every $\psi \in \text{Lip}(\partial Q_1(0))$, we obtain
\[
\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi(s(\rho) - s_0(\rho)) \, d\rho = 0, \quad \forall \varphi \in C_c^\infty((0,1)), \forall \psi \in \text{Lip}(\partial Q_1(0)).
\]
This means that $s_0(\rho) = s(\rho) \in Y$ for a.e. $\rho \in E_h$.

**Step 5.** Finally we show that \((3.6)\) holds for almost any $\rho \in (0,1)$.
In fact for any $\rho \in E_h$ such that $s_0(\rho) = s(\rho)$ we have
\[
\int_{\partial Q_1(0)} i_{\partial Q_1}^* F = \int_{\partial Q_1(0)} s_0(\rho) = \int_{\partial Q_1(0)} s(\rho) = \lim_{k_i \to \infty} \int_{\partial Q_1(0)} s_{k_i}(\rho) \in \mathbb{Z},
\]
since $s_{k_i}(\rho) \to s(\rho)$ in $L^p$. Thus \((3.6)\) holds for $L^1$-a.e. $\rho \in E_h$. Since the previous step can be repeated for any $h \in \mathbb{N}_{\geq 2}$, and since
\[
\lim_{h \to +\infty} L^1(E_h) = \lim_{h \to +\infty} 1 - \frac{C + 1}{h} = 1,
\]
we conclude that \((3.6)\) holds for $L^1$-a.e. $\rho \in (0,1)$. $\blacksquare$

**Remark 3.8.** Let $D \subset \mathbb{R}^n$ be any open and bounded domain which is bi-Lipschitz equivalent to $Q_1(0)$. From Theorem 3.1 follows that $\Omega_{p,\mathbb{R}}^{n-1}(D)$ (see Definition 1.4) is a weakly sequentially closed subspace of $\Omega_{p,\mathbb{R}}^{n-1}(D)$. Indeed, let $\varphi : Q_1(0) \to D$ be any bi-Lipschitz homeomorphism and let $\{F_k\}_{k \in \mathbb{N}} \subset \Omega_{p,\mathbb{R}}^{n-1}(D)$ be such that $F_k \overset{L^p}{\to} F$ on $D$. Then, by Lemma 2.7 we have $\{\varphi F_k\}_{k \in \mathbb{N}} \subset \Omega_{p,\mathbb{R}}^{n-1}(Q_1(0))$ and as $\varphi$ is bi-Lipschitz we have $\varphi F_k \overset{L^p}{\to} \varphi F$ on $Q_1(0)$. By the Weak Closure Theorem (Theorem 3.1), $\varphi F \in \Omega_{p,\mathbb{R}}^{n-1}(Q_1(0))$. Thus $F \in \Omega_{p,\mathbb{R}}^{n-1}(D)$ (again by Lemma 2.7). This shows that Theorem 3.3 holds true.

Observe that Theorem 3.3 does not hold if $n = 1$. In fact in this case the following holds.

**Lemma 3.3.** Let $I$ be a bounded connected interval in $\mathbb{R}$. Let $p \in [1, \infty)$.

The weak closure of $L^p_{\mathbb{R}}(I)$ in $L^p(I)$ is $L^p(I)$.

**Proof.** Since $C^0(\overline{I})$ is dense in $L^p(I)$, it is enough to show that any function $f \in C^0(\overline{I})$ can be approximated weakly in $L^p(I)$ by functions in $L^p_{\mathbb{R}}(I)$. Without loss of generality we can assume that $I = [0,1)$. For any $n \in \mathbb{N}$ let’s define $f_n : I \to \mathbb{R}$ as follows:
for any $k \in \{1,...,2^n\}$ let $I_k := [-\frac{k-1}{2^n}, \frac{k}{2^n})$, for any $k \in \{1,...,2^n\}$ let $c_k := \int_{I_k} f(x) \, dx$ and for any $x \in I_k$ set
\[
f_n(x) := \begin{cases} 
[c_k] & \text{if } x - \frac{k-1}{2^n} \leq \frac{c_k}{2^n|c_k|} \\
0 & \text{otherwise}. 
\end{cases}
\]
Then $f_n \in L^p(I,\mathbb{R})$ and $\int_{I_k} f_n(x) \, dx = \int_{I_k} f(x) \, dx$ for any $k \in \{1,...,2^n\}$, for any $n \in \mathbb{N}$.
Moreover notice that since $f$ is bounded, the sequence $(f_n)_{n \in \mathbb{N}}$ is bounded in $L^p(I)$. Therefore if $p > 1$ $(f_n)_{n \in \mathbb{N}}$ converges weakly in $L^p(I)$, up to a subsequence, to a function $\tilde{f} \in L^p(I)$. Testing against continuous functions on $\overline{I}$ it is easy to check that $f = \tilde{f}$.
If $p = 1$ we have to check that, up to a subsequence,
\[
\lim_{n \to \infty} \int_I f_n g = \int_I f g \quad \forall g \in L^\infty(I). \tag{3.7}
\]
Since $L^\infty(I) \subset L^q(I)$ for any $q > 1$, \((3.7)\) follows from the case $p > 1$ (with $p = q'$). $\blacksquare$
Remark 3.9. Let $n \geq 2$ and let $D \subset \mathbb{R}^n$ be any open, bounded and Lipschitz domain in $\mathbb{R}^n$. It is still unknown if the space $L^1_2(D)$ is weakly sequentially closed. Surely it is not weakly-$*$ closed, a proof this fact can be easily achieved by generalising the arguments in [22, Section 8].
A. Minimal connections for forms with finitely many integer singularities

Throughout Appendix A, we will denote by \( M^m \subset \mathbb{R}^n \) some arbitrary embedded Lipschitz and connected \( m \)-dimensional submanifold of \( \mathbb{R}^n \). Let \( p \in [1, \infty] \). We will denote by \( \Omega^1_{p,R,\infty}(M) \) the space introduced in Definition 2.1.

**Lemma A.1.** Let \( F \in \Omega^{n-1}_{p,R,\infty}(M) \). Then, there exists a connection for \( F \).

**Proof.** Throughout the following proof, given any couple of points \( x, y \in M \) we will denote by \( (x, y) \) an arbitrarily chosen oriented Lipschitz curve with finite length joining \( x \) and \( y \). By assumption, it holds that

\[
*df = \sum_{j=1}^{N} d_j \delta_{x_j}, \quad \text{for some } d_1, ..., d_N \in \mathbb{Z} \setminus \{0\} \text{ and } x_1, ..., x_N \in M.
\]

We define

\[
\{i_1, ..., i_p\} := \{ j \in \{1, ..., N\} \text{ s.t. } d_j > 0\},
\]

\[
\{j_1, ..., j_q\} := \{ j \in \{1, ..., N\} \text{ s.t. } d_j < 0\},
\]

\[
d := \sum_{j=1}^{N} d_j \in \mathbb{Z}.
\]

We build a family \( \mathcal{F} = \{I_\alpha\}_{\alpha \in A} \) of oriented Lipschitz curves in \( M \) as follows. If there is no point \( x_j \) such that \( d_j < 0 \), then we set \( \mathcal{F} = \emptyset \). Else, we start from \( x_{i_1} \) and we add to the family \( \mathcal{F} \) the curves \((x_{j_1}, x_{i_1}), ..., (x_{j_{k_1}}, x_{i_1})\), until we reach the condition \( k_1 = q \) or the condition

\[
r_1 := d_{i_1} + \sum_{l=1}^{k_1} d_{j_l} \leq 0.
\]

If \( k_1 = q \), then we stop. Else, we move to the point \( x_{i_2} \).

If \( r_1 = 0 \), then we add to \( \mathcal{F} \) the segments \((x_{j_{k_1+1}}, x_{i_2}), ..., (x_{j_{k_2}}, x_{i_2})\), where \( k_2 \in \{1, ..., q\} \) is the smallest value such that

\[
r_2 := d_{i_2} + \sum_{l=k_1+1}^{k_2} d_{j_l} \leq 0.
\]

If there is no \( k \in \{1, ..., q\} \) such that

\[
d_{i_2} + \sum_{l=k_1+1}^{k} d_{j_l} \leq 0,
\]

then we add to \( \mathcal{F} \) the segments \((x_{j_{k_1+1}}, x_{i_2}), ..., (x_{j_{q}}, x_{i_2})\) and we set \( k_2 = q \).
If \( r_1 < 0 \), then we add to \( \mathcal{F} \) the segments \((x_{j_{k_1}}, x_{i_2})\) and \((x_{j_{k_1+1}}, x_{i_2}), \ldots, (x_{j_{k_2}}, x_{i_2})\), where \( k_2 \in \{1, \ldots, q\} \) is the smallest value such that

\[
r_2 := d_{i_2} + r_1 + \sum_{l=k_1+1}^{k_2} d_{j_l} \leq 0.
\]

If there is no \( k \in \{1, \ldots, q\} \) such that

\[
d_{i_2} + r_1 + \sum_{l=k_1+1}^{k} d_{j_l} \leq 0,
\]

then we add to \( \mathcal{F} \) the segments \((x_{j_{k_1}}, x_{i_2})\) and \((x_{j_{k_1+1}}, x_{i_2}), \ldots, (x_{j_{k}}, x_{i_2})\) and we set \( k_2 = q \).

We proceed iteratively in this way, moving on to the subsequent points \( x_{i_s} \) until \( k_s = q \) or \( s = p \). Then, the construction of the family \( \mathcal{F} \) is complete. We let \( x_{i_h} \) be the last node that is visited before the iteration stops and, for every \( I_\alpha = (x_j, x_i) \in \mathcal{F} \), we define its multiplicity \( m_\alpha \) as

\[
m_\alpha := \begin{cases} |d_j| - |r_1| & \text{if } i = i_l \text{ and } j = i_{k_1}, \\
\min\{|d_i|, |r_{l-1}|\} & \text{if } i = i_l \text{ and } j = i_{k_{l-1}}, \\
\min\{|d_j|, |d_i|\} & \text{else}.
\end{cases}
\]

Finally, we divide three cases:

1. **Case \( d = 0 \)**. Notice that this is always the case if \( M \) has no boundary. We define the integer 1-current \( I \in \mathcal{R}_1(M) \) given by

\[
\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega, \quad \text{for every } \omega \in \mathcal{D}^1(M).
\]

2. **Case \( d > 0 \)**. We fix a point \( x_0 \in \partial M \) and we let \( I_s^h := (x_0, x_{i_s}) \), for every \( s = h, \ldots, p \).

We define the integer 1-current \( I \in \mathcal{R}_1(M) \) given by

\[
\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega + r_h \int_{I_h^h} \omega + \sum_{s=h+1}^{p} d_{i_s} \int_{I_s^h} \omega, \quad \text{for every } \omega \in \mathcal{D}^1(M).
\]

3. **Case \( d < 0 \)**. We fix a point \( x_0 \in \partial M \) and we let \( I_s^n := (x_j, x_0) \), for every \( s = k_h, \ldots, q \).

We define the integer 1-current \( I \in \mathcal{R}_1(M) \) given by

\[
\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega + |r_h| \int_{I_h^n} \omega + \sum_{s=k_h+1}^{q} |d_{i_s}| \int_{I_s^n} \omega, \quad \text{for every } \omega \in \mathcal{D}^1(M).
\]

By direct computation, we verify that \( I \) has the desired properties and the statement follows.

\[\blacksquare\]

**Lemma A.2.** Let \( F \in \Omega_p^{m-1}(M) \) (see Definition 1.4). Then,

\[
\inf_{T \in \mathcal{D}_1(M), \partial T = s \ast dF} \mathcal{M}(T) = \inf_{T \in \mathcal{M}_1(M), \partial T = s \ast dF} \mathcal{M}(T) = \sup_{\varphi \in W^{1,\infty}_0(M), ||d\varphi||_{L^\infty} \leq 1, \int_M F \wedge d\varphi < +\infty} \int_M F \wedge d\varphi < +\infty, \quad (A.1)
\]

where \( \mathcal{M}_1(M) \) denotes the set of all the 1-currents with finite mass on \( M \). Moreover, the infimum on the left-hand-side of the previous chain of equalities is achieved.
Proof. By definition, there exists an integer 1-current \( I \in \mathcal{R}_1(M) \) with finite mass such that \( \partial I = *dF \). Hence
\[
\inf_{T \in D_1(M), \partial T = *dF} \mathcal{M}(T) \leq \mathcal{M}(I) < +\infty.
\]
The first equality in (A.1) is clear.

Notice that for every \( T \in \mathcal{M}_1(M) \) such that \( \partial T = *dF \) it holds that
\[
\int_M F \wedge d\varphi = \langle *dF, \varphi \rangle = \langle \partial T, \varphi \rangle = \langle T, d\varphi \rangle \leq \mathcal{M}(T) \|d\varphi\|_{L^\infty(M)}, \quad \forall \varphi \in W_0^{1,\infty}(M).
\]
Hence,
\[
\inf_{T \in \mathcal{M}_1(M), \partial T = *dF} \mathcal{M}(T) \geq \sup_{\varphi \in W_0^{1,\infty}(M), \|d\varphi\|_{L^\infty} \leq 1} \int_M F \wedge d\varphi. \quad (A.2)
\]
To prove that the former inequality is actually an equality, it suffices to show that the supremum on its right-hand-side is greater than the mass of some 1-current with finite mass \( T \) on \( M \) such that \( \partial T = *dF \). Define the vector subspace \( X \subset \Omega^1_\infty(M) \) given by
\[
X := \{ \omega \in \Omega^1_\infty(M) \text{ s.t. } \omega = d\varphi, \text{ for some } \varphi \in W_0^{1,\infty}(M) \}.
\]
Consider the linear functional \( \phi : X \subset (\Omega^1(M), \| \cdot \|_{L^\infty}) \to \mathbb{R} \) given by
\[
\langle \phi, \omega \rangle = \int_M F \wedge \omega, \quad \forall \omega \in X.
\]
By (A.2) we get that \( \phi \) is continuous on \( X \), i.e.
\[
\| \phi \|_{\mathcal{L}(X)} = \sup_{\omega \in X, \| \omega \|_{L^\infty} \leq 1} \int_M F \wedge \omega = \sup_{\varphi \in W_0^{1,\infty}(M), \|d\varphi\|_{L^\infty} \leq 1} \int_M F \wedge d\varphi \leq \inf_{T \in \mathcal{M}_1(M), \partial T = *dF} \mathcal{M}(T) < +\infty.
\]
By Hahn-Banach theorem, we can extend \( \phi \) to a linear functional \( T : \Omega^1(M) \to \mathbb{R} \) such that
\[
\|T\|_{\mathcal{L}(\Omega^1_\infty(M))} = \| \phi \|_{\mathcal{L}(X)} = \sup_{\varphi \in W_0^{1,\infty}(M), \|d\varphi\|_{L^\infty} \leq 1} \int_M F \wedge d\varphi.
\]
But then, \( T \) is a 1-current on \( M \) having finite mass and such that
\[
\mathcal{M}(T) \leq \|T\|_{\mathcal{L}(\Omega^1_\infty(M))} = \sup_{\varphi \in W_0^{1,\infty}(M), \|d\varphi\|_{L^\infty} \leq 1} \int_M F \wedge d\varphi.
\]
Moreover,
\[
\langle \partial T, \varphi \rangle = \langle T, d\varphi \rangle = \int_M F \wedge d\varphi = \langle *dF, \varphi \rangle, \quad \forall \varphi \in W_0^{1,\infty}(M).
\]
Hence,
\[
\mathcal{M}(T) = \inf_{T \in D_1(M), \partial T = *dF} \mathcal{M}(T) = \inf_{T \in \mathcal{M}_1(M), \partial T = *dF} \mathcal{M}(T) = \sup_{\varphi \in W_0^{1,\infty}(M), \|d\varphi\|_{L^\infty} \leq 1} \int_M F \wedge d\varphi < +\infty
\]
and the statement follows. \( \square \)
Proposition A.1. Let $F \in \Omega^{n-1}_{p,R}(M)$. Then, there exists an integer 1-current $L \in \mathcal{R}_1(M)$ such that $\partial L = *dF$ and

$$
M(L) = \inf_{T \in \mathcal{R}_1(M), \partial T = *dF} M(T) = \sup_{\varphi \in W^{1,\infty}_0(M), ||d\varphi||_{L^\infty} \leq 1} \int_M F \wedge d\varphi.
$$

In particular,

$$
M(L) \leq C ||F||_{L^p}.
$$

Proof. Notice that by [13, Chapter 1, Section 3.4, Theorem 8], we have

$$
\inf_{T \in \mathcal{R}_1(M), \partial T = *dF} M(T) = \inf_{T \in \mathcal{D}_1(M), \partial T = *dF} M(T).
$$

Since the mass $M(\cdot)$ is lower semicontinuous with respect to the weak convergence in $\mathcal{D}_1(M)$ and since $M$-bounded subsets of the competition class $\mathcal{R}_1(M) \cap \{T \in \mathcal{D}_1(M) \text{ s.t. } \partial T = *dF\}$ are weakly sequentially compact (for a reference, see e.g. [19, Equation (7.5), Theorem 7.5.2]), by the direct method of calculus of variations we conclude that there exists an integer 1-current $L \in \mathcal{R}_1(M)$ such that $\partial L = *dF$ and

$$
M(L) = \inf_{T \in \mathcal{R}_1(M), \partial T = *dF} M(T) = \inf_{T \in \mathcal{D}_1(M), \partial T = *dF} M(T) = \sup_{\varphi \in W^{1,\infty}_0(M), ||d\varphi||_{L^\infty} \leq 1} \int_M F \wedge d\varphi,
$$

where the last equality follows from Lemma A.2. The statement follows.

B. Laplace equation on spheres

Let $n \in \mathbb{N}$ be such that $n \geq 2$ and fix any $p \in (1, +\infty)$. We let

$$
\dot{W}^{1,p}(S^{n-1}) := \left\{ u \in W^{1,p}(S^{n-1}) \text{ s.t. } \bar{u} := \int_{S^{n-1}} u \, d\text{vol}_{S^{n-1}} = 0 \right\}.
$$

We can endow the space $\dot{W}^{1,p}(S^{n-1})$ with the usual $W^{1,p}$-norm induced by $W^{1,p}(S^{n-1})$, given by

$$
||u||_{W^{1,p}} := ||u||_{L^p} + ||du||_{L^p}, \quad \forall u \in \dot{W}^{1,p}(S^{n-1}).
$$

Lemma B.1 (Poincaré inequality on $\dot{W}^{1,p}$). There exists a constant $C > 0$ such that

$$
\int_{S^{n-1}} |u|^p \, d\text{vol}_{S^{n-1}} \leq C \int_{S^{n-1}} |du|^p \, d\text{vol}_{S^{n-1}}, \quad \forall u \in \dot{W}^{1,p}(S^{n-1}).
$$

Proof. By contradiction, assume that for every $k > 0$ there exists $u_k \in \dot{W}^{1,p}(S^{n-1})$ such that $||u_k||_{L^p} = 1$ and

$$
1 > k \int_{S^{n-1}} |du_k|^p \, d\text{vol}_{S^{n-1}}.
$$
This implies immediately that $\|du_k\|_{L^p} \to 0$ as $k \to \infty$. In particular, the sequence $\{u_k\}_{k \in \mathbb{N}}$ is bounded with respect to the $W^{1,p}$-norm. Hence, by weak compactness of $W^{1,p}(S^{n-1})$, there exists a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ of $\{u_k\}_{k \in \mathbb{N}}$ such that $u_{k_j} \rightharpoonup u$ in $W^{1,p}(S^{n-1})$. Moreover, by Rellich-Kondrakov theorem, we have $u_{k_j} \to u$ strongly in $L^p(S^{n-1})$. Since $du_{k_j} \to 0$ strongly in $L^p$ we get $du = 0$. Then, $u$ is constant on $S^{n-1}$. Since $u_{k_j} \rightharpoonup u$ in $L^p(S^{n-1})$, it follows that

$$0 = \lim_{j \to \infty} \int_{S^{n-1}} u_{k_j} \, d\vol_{S^{n-1}} = \int_{S^{n-1}} u \, d\vol_{S^{n-1}},$$

and this leads to $u = 0$. But this is absurd, since by strong $L^p$-convergence of $\{u_{k_j}\}_{j \in \mathbb{N}}$ to $u$ we obtain $\|u\|_{L^p} = 1$.

\[\square\]

**Remark B.1.** By Lemma B.1 we conclude that we can endow $\dot{W}^{1,p}$ with the following much more convenient norm:

$$\|u\|_{\dot{W}^{1,p}} := \|du\|_{L^p}, \quad \forall u \in \dot{W}^{1,p}(S^{n-1}).$$

Moreover, such a norm is equivalent to $W^{1,p}$-norm.

**Remark B.2.** Notice that a linear functional on $W^{1,p}(S^{n-1})$ restricts to an element of $(\dot{W}^{1,p}(S^{n-1}))^*$ if and only if it is $W^{1,p}$-continuous and $\langle F, 1 \rangle = 0$.

**Lemma B.2.** Let $F \in (W^{1,p}(S^{n-1}))^*$ be such that $\langle F, 1 \rangle = 0$. Then, the following facts hold.

1. If $n \geq 3$ the linear differential system

   $$\begin{cases}
d^* \omega = F \\
d\omega = 0
\end{cases}$$

   has a unique weak solution $\alpha \in \Omega^1_p(S^{n-1})$.

2. If $n = 2$ the linear differential system

   $$\begin{cases}
d^* \omega = F \\
d\omega = 0 \\
\int_{S^1} \omega = 0
\end{cases}$$

   has a unique weak solution $\alpha \in \Omega^1_p(S^1)$.

In both cases, $\alpha$ satisfies the following estimate:

$$\|\alpha\|_{L^p} \leq C \|F\|_{\mathcal{L}(W^{1,p'}(S^{n-1}))},$$

for some constant $C > 0$ depending only $n$.

**Proof.** Observe that, by Remark B.2 $F$ restricts to an element of $(\dot{W}^{1,p'}(S^{n-1}))^*$. Consider the linear functional $\phi : \Omega^1(S^{n-1}) \to \mathbb{R}$ given by

$$\langle \phi, \omega \rangle = \langle F, u \rangle, \quad \forall \omega = du + d^* \beta + \eta \in \Omega^1(S^{n-1}),$$

59
where \( \eta \) is a harmonic 1-form on \( S^{n-1} \). Since \( \phi \) is continuous and linear on \( \Omega^1(S^{n-1}) \) w.r.t. the \( L^{p'} \)-norm, by Hahn-Banach theorem there exists a unique (recall that \( L^p \)-spaces are strictly convex) extension \( \Phi \in (\Omega^1_p(S^{n-1}))^* \) of \( \phi \) such that

\[
\|\Phi\|_{(\Omega^1_p(S^{n-1}))^*} = \|\phi\|_{(\Omega^1(S^{n-1}))^*} \leq C\|F\|_{\hat{L}^1(W^{1,p'}(S^{n-1}))^*}.
\]

By Riesz representation theorem, there exists a unique \( \alpha \in \Omega^1_p(S^{n-1}) \) such that

\[
\langle \alpha, \omega \rangle_{L^p-L^{p'}} := \int_{S^{n-1}} \alpha \wedge *\omega = \langle \Phi, \omega \rangle, \quad \forall \omega \in \Omega^1_p(S^{n-1}) \quad (B.1)
\]

and

\[
\|\alpha\|_{L^p} = \|\Phi\|_{(\Omega^1_p(S^{n-1}))^*} \leq C\|F\|_{(\hat{L}^1(W^{1,p'}(S^{n-1}))^*)}.
\]

Finally, by applying equation (B.1) with get

\[
\langle \alpha, du \rangle_{L^p-L^{p'}} = \langle \Phi, du \rangle = \langle \phi, du \rangle = \langle F, u \rangle, \quad \forall u \in C^\infty(S^{n-1}),
\]

and

\[
\langle \alpha, d^* \beta \rangle_{L^p-L^{p'}} = \langle \Phi, d^* \beta \rangle = \langle \phi, d^* \beta \rangle = 0, \quad \forall \beta \in \Omega^2(S^{n-1}).
\]

The two previous equations are exactly the weak forms of the equations \( d^* \alpha = F \) and \( d\alpha = 0 \) respectively. Moreover, in case \( n = 2 \), we have

\[
\int_{S^1} \alpha = \int_{S^1} \alpha \wedge *1 = \langle \alpha, *1 \rangle_{L^p-L^{p'}} = \langle \Phi, *1 \rangle = \langle F, 1 \rangle = 0.
\]

This concludes about the existence of a solution to the differential systems given in points 1 and 2. For what concerns uniqueness, assume that \( \alpha \) and \( \alpha' \) are two solutions of the differential system given in point 1 (resp. 2) and define \( \beta = \alpha - \alpha' \). Then, we distinguish the two cases:

**Case** \( n \geq 3 \). In this case, \( \beta \) satisfies

\[
\begin{cases}
  d^* \beta = 0 \\
  d\beta = 0.
\end{cases}
\]

Hence, \( \beta \) is a harmonic 1-forms on \( S^{n-1} \) for \( n \geq 1 \), which implies \( \beta = 0 \).

**Case** \( n = 2 \). In this case, \( \beta \) satisfies

\[
\begin{cases}
  d^* \beta = 0 \\
  d\beta = 0 \\
  \int_{S^1} \beta = 0.
\end{cases}
\]

Hence, \( \beta \) is a harmonic 1-forms on \( S^1 \), which implies \( \beta = c \text{vol}_{S^1} \) for some \( c \in \mathbb{R} \). But since \( \beta \) has vanishing integral on \( S^1 \), we get \( \beta = 0 \). \( \square \)
**Definition B.1** (Sobolev spaces of differential forms). Fix any $k \in \mathbb{N} \setminus \{0\}$. We define the Sobolev space of $W^{1,p}$-regular differential $k$-forms on $\mathbb{S}^{n-1}$ by

$$\Omega^k_{W^{1,p}}(\mathbb{S}^{n-1}) := \{ \omega \in \Omega^k_p(\mathbb{S}^{n-1}) \text{ s.t. } d\omega, d^*\omega \in L^p \}.$$ 

We endow such space with the norm

$$||\omega||_{W^{1,p}} := ||\omega||_{L^p} + ||d\omega||_{L^p} + ||d^*\omega||_{L^p}, \quad \forall \omega \in \Omega^k_{W^{1,p}}(\mathbb{S}^{n-1}).$$

**Remark B.3.** It can be shown (see [27, §3 and §4]) that such Sobolev spaces are completely equivalent to the usual ones, namely the space of $k$-forms having local coefficients in $W^{1,p}$. Moreover, in case $n \geq 3$ there exists $C > 0$ such that

$$||\omega||_{W^{1,p}} \leq C(||d\omega||_{L^p} + ||d^*\omega||_{L^p}), \quad \forall \omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}). \quad (B.2)$$

Indeed, let

$$X := \{ d\alpha \text{ s.t. } \alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}) \},$$

$$Y := \{ d^*\beta \text{ s.t. } \beta \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}) \}.$$ 

By [27, Proposition 7.1], both $X$ and $Y$ are closed linear subspaces respectively of $\Omega^1_p(\mathbb{S}^{n-1})$ and $\dot{W}^{1,p}(\mathbb{S}^{n-1})$. Then, $X \oplus Y$ is a Banach space with respect to the standard norm on the direct sum of two Banach spaces. We claim that the linear operator $T : \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}) \rightarrow X \oplus Y$ given by $T\omega = (d\omega, d^*\omega)$, for every $\omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ is a continuous linear bijection between Banach spaces. Indeed, the fact that $T$ is injective follows form the fact that there no non-zero harmonic forms on $\mathbb{S}^{n-1}$ for $n \geq 3$. Hence, we just need to show that $T$ is surjective. Pick any $(d\alpha, d^*\beta) \in X \oplus Y$. By Lemma [3,2], the linear differential system

$$\begin{cases}
    d^*\omega = d^*(\beta - \alpha) \\
    d\omega = 0
\end{cases}$$

has a unique weak solution $\tilde{\omega} \in \Omega^1_p(\mathbb{S}^{n-1})$. Since by construction we have $d\tilde{\omega}, d^*\tilde{\omega} \in L^p$, we conclude that $\tilde{\omega} \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$. Then, by letting $\omega := \tilde{\omega} + \alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ we see that

$$T\omega = (d\omega, d^*\omega) = (d\tilde{\omega} + d\alpha, d^*\tilde{\omega} + d^*\alpha) = (d\alpha, d^*\beta)$$

and we have proved our claim. This proves that $T$ has a continuous inverse and the statement follows with $C = ||T^{-1}||_{\mathcal{L}(X \oplus Y, \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}))}$. 

In case $n = 2$, the estimate (B.2) still holds for every $\omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^1)$ such that

$$\int_{\mathbb{S}^1} \omega = 0.$$ 

The proof is completely analogous.

**Remark B.4** ($L^p$-Hodge decomposition). Let

$$X := \{ d\varphi \text{ s.t. } \varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1}) \},$$

$$Y := \{ d^*\beta \text{ s.t. } \beta \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}) \},$$

61
Lemma B.3 (A weak version of Poincaré lemma)

We follow the notation of Remark B.4 and we notice that, since

\[ d\alpha \]

is a continuous and linear isomorphism between Banach spaces. Hence, \( T \) has a continuous inverse. We let

\[ C_H := \|T^{-1}\|_{L(\Omega^1_p(S^{n-1}), X \oplus Y)}. \]

We conclude that for every \( \omega \in \Omega^1_p(S^{n-1}) \) there exist \( \varphi \in \dot{W}^{1,p}(S^{n-1}) \), \( \beta \in \Omega^2_{W^{1,p}}(S^{n-1}) \) and \( \eta \in Z \) such that \( \omega = d\varphi + d^*\beta + \eta \) and

\[ \|d\varphi\|_{L^p} + \|d^*\beta\|_{L^p} + \|\eta\|_{L^p} \leq C_H \|\omega\|_{L^p}. \]  

(B.3)

Lemma B.3 (A weak version of Poincaré lemma). Let \( n \geq 3 \) and let \( \alpha \in \Omega^1_p(S^{n-1}) \) be such that \( d\alpha = 0 \) weakly on \( S^{n-1} \). Then, there exists a Sobolev function \( \varphi \in \dot{W}^{1,p}(S^{n-1}) \) such that \( d\varphi = \alpha \) weakly on \( S^{n-1} \).

Proof. We follow the notation of Remark B.4 and we notice that, since \( n \geq 3 \) we have \( Z = \{0\} \). Hence, we write \( \alpha = d\varphi + d^*\beta \), for \( \varphi \in \dot{W}^{1,p}(S^{n-1}) \) and \( \beta \in \Omega^2_{W^{1,p}}(S^{n-1}) \). We observe that

\[ \langle d^*\beta, \omega \rangle_{L^p - L^{p'}} = \langle d^*\beta, d\psi + d^*\gamma \rangle_{L^p - L^{p'}} \]

\[ = \langle d^*\beta, d^*\gamma \rangle_{L^p - L^{p'}} \]

\[ = \langle \alpha - d\varphi, d^*\gamma \rangle_{L^p - L^{p'}} \]

\[ = \langle \alpha, d^*\gamma \rangle_{L^p - L^{p'}} = 0, \quad \forall \omega = d\psi + d^*\gamma \in \Omega^1(S^{n-1}). \]

This implies \( d^*\beta = 0 \) and the statement follows. \( \square \)

Corollary B.1 (Laplace equation on spheres). Let \( F \in \mathcal{L}(W^{1,p'}(S^{n-1})) \) such that \( \langle F, 1 \rangle = 0 \). Then, the linear differential equation

\[ \Delta u = F \]

has a unique weak solution \( \varphi \in \dot{W}^{1,p}(S^{n-1}) \) satisfying

\[ ||\varphi||_{W^{1,p}} \leq C ||F||_{(W^{1,p'}(S^{n-1}))^*}. \]

Proof. First, we face the case \( n \geq 3 \). By Lemma B.2 we can find \( \alpha \in \Omega^1_p(S^{n-1}) \) satisfying

\[ \begin{cases} 
  d^*\alpha = F \\
  d\alpha = 0 
\end{cases} \]

and

\[ ||\alpha||_{L^p} \leq C ||F||_{(W^{1,p'}(S^{n-1}))^*}. \]
Since $d\alpha = 0$, by Lemma B.3 there exists $\varphi \in \dot{W}^{1,p}(S^{n-1})$ such that $\alpha = d\varphi$. Hence, we get
$$\Delta \varphi = d^*d\varphi = d^*\alpha = F.$$ Moreover, by Lemma B.1 we have
$$||\varphi||_{\dot{W}^{1,p}} \leq C||d\varphi||_{L^p} = C||\alpha||_{L^p} \leq C||F||_{(\dot{W}^{1,p'(S^{n-1})})^*}.$$ This concludes the proof in case $n \geq 3$.

If $n = 2$, then by Lemma B.2 we can find $\alpha \in \Omega^1(S^{n-1})$ satisfying
$$\begin{cases}
d^*\alpha = F \\
d\alpha = 0 \\
\int_{S^1} \alpha = 0
\end{cases}$$
and
$$||\alpha||_{L^p} \leq C||F||_{(\dot{W}^{1,p'(S^{n-1})})^*}.$$ By setting $\varphi := \ast \alpha$, the statement follows.

C. Some technical lemmata

In this section we will make use of the following notation: let $T$ be an $m$-rectifiable current in $\mathbb{R}^n$, then $T$ can be represented as follows:
$$\langle T, \omega \rangle = \int_{\mathbb{R}^n} \theta(\omega, \xi) d\mathcal{H}^m \big|_\Sigma \quad \forall \omega \in \Omega^m(\mathbb{R}^n),$$
where $\Sigma$ is a locally $m$-rectifiable set,
$$\theta : \Sigma \to \mathbb{Z}$$
is a locally $\mathcal{H}^m$-integrable, non-negative function and
$$\xi : \Sigma \to \Lambda_m \mathbb{R}^n$$is an $\mathcal{H}^m$-measurable function such that for $\mathcal{H}^m$-almost every point $x \in \Sigma$, $\xi(x)$ is a simple unit $m$-vector in $T_x \Sigma$.
In this case we write
$$T = \tau(\Sigma, \theta, \xi).$$

Lemma C.1. For any $k \in \mathbb{N}$ let
$$T_k = \tau(\Sigma_k, \theta_k, \xi_k)$$be an $m$-rectifiable current on $\mathbb{R}^n$ of finite mass. Assume that $(T_k)_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to the convergence in mass.
Then there exists an $m$-rectifiable current
$$T = \tau(\Sigma, \theta, \xi)$$such that
$$T_k \to T \quad (k \to \infty) \quad \text{in mass.}$$
Proof. Replacing the original sequence by a subsequence if necessary, we may assume that for any \( k \in \mathbb{N} \)
\[
\mathbb{M}(T_k - T_{k+1}) \leq 2^{-k}.
\]
Now for any \( k \in \mathbb{N} \) let
\[
\tilde{T}_k := \begin{cases} 
T_1 & \text{if } k = 1 \\
T_k - T_{k-1} & \text{if } k > 1.
\end{cases}
\]
Then for any \( k \in \mathbb{N} \) we have
\[
T_k = \sum_{i=1}^{k} \tilde{T}_i.
\]
For any \( k \in \mathbb{N} \) write
\[
\tilde{T}_k = \langle \tilde{\Sigma}_k, \tilde{\theta}_k, \tilde{\xi}_k \rangle.
\]
Notice that
\[
\sum_{i=1}^{k} \tilde{\theta}_i \tilde{\xi}_i = \theta_k \xi_k \quad \mathcal{H}^m \text{- a.e. on } \Sigma,
\]
for every \( k \in \mathbb{N} \). Set
\[
\Sigma := \bigcup_{k \in \mathbb{N}} (\tilde{\Sigma}_k \setminus \tilde{\theta}_k^{-1}(0)).
\]
Then \( \Sigma \) is \( m \)-rectifiable as countable union of \( m \)-rectifiable sets. Moreover \( \mathcal{H}^m(\Sigma) < \infty \). In fact
\[
\mathcal{H}^m(\Sigma) \leq \sum_{k \in \mathbb{N}} \mathcal{H}^m(\tilde{\Sigma}_k \setminus \tilde{\theta}_k^{-1}(0)) \leq \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^n} |\tilde{\theta}_k| d\mathcal{H}^m \tilde{\Sigma}_k = \sum_{k \in \mathbb{N}} \mathbb{M}(\tilde{T}_k) < \infty.
\]
Next let
\[
\theta = \sum_{k \in \mathbb{N}} \tilde{\theta}_k,
\]
where \( \tilde{\theta}_k \) is extended by zero on \( \Sigma \setminus \tilde{\Sigma}_k \) for any \( k \in \mathbb{N} \).
By Beppo-Levi Theorem
\[
\int_{\mathbb{R}^n} |\theta| d\mathcal{H}^m \Sigma = \int_{\mathbb{R}^n} |\tilde{\theta}_k| d\mathcal{H}^m \tilde{\Sigma}_k = \sum_{k \in \mathbb{N}} \mathbb{M}(\tilde{T}_k) < \infty. \quad \text{(C.1)}
\]
Therefore \( \theta \) is finite \( \mathcal{H}^m \)-a.e. in \( \Sigma \), i.e. for \( \mathcal{H}^m \)-a.e. \( x \in \Sigma \) there are only finitely many \( k \in \mathbb{N} \) so that \( \theta_k(x) \neq 0 \). In particular the sum
\[
\sum_{k \in \mathbb{N}} \tilde{\theta}_k(x) \tilde{\xi}_k(x)
\]
64
is well defined and finite for $\mathcal{H}^m$-a.e. $x \in \Sigma$ (again $\xi_k$ is extended by zero on $\Sigma \setminus \tilde{\Sigma}_k$ for any $k \in \mathbb{N}$) and we can write
\[
\theta(x)\xi(x) = \sum_{k \in \mathbb{N}} \tilde{\theta}_k(x)\xi_k(x)
\]
for some $\theta(x) \in \mathbb{Z}_{\geq 0}$ and for some simple unit $m$-vector $\xi(x)$ in $T_x \Sigma$, for $\mathcal{H}^m$-a.e. $x \in \Sigma$.

Observe that $\theta$ is an $\mathcal{H}^m$-measurable function on $\Sigma$ as the absolute value of the a.e.-limit of $\mathcal{H}^m$-measurable functions. Analogously, $\xi$ is an $\mathcal{H}^m$-measurable map on $\Sigma$ as the a.e.-limit of $\mathcal{H}^m$-measurable maps. We set
\[
T := \tau(\Sigma, \theta, \xi)
\]
and we claim that
\[
T_k \to T \quad (k \to \infty) \quad \text{in mass.}
\]

In fact we know that since the space of $m$-currents is complete under the convergence in mass (as a dual space), there exists an $m$-current $T'$ such that
\[
T_k \to T' \quad (k \to \infty) \quad \text{in mass.}
\]

To see that $T = T'$ observe that for any $\omega \in \mathcal{D}^m(\mathbb{R}^n)$
\[
\theta_k(\omega, \xi_k) = \sum_{i=1}^k \tilde{\theta}_i(\omega, \tilde{\xi}_i) \to \theta(\omega, \xi) \quad \mathcal{H}^m$-a.e. in $\Sigma,
\]
thus by (C.1) and Dominated Convergence Theorem we conclude that
\[
\langle T_k, \omega \rangle \to \langle T, \omega \rangle \quad (k \to \infty).
\]

In particular $T = T'$.

\[\square\]

**Lemma C.2.** Let $\alpha \in (1, +\infty)$, $q \in (-\infty, 1]$, $\varepsilon \in (0, 1)$ and let $\Omega \subset Q^1_{1-\varepsilon}(0)$ be open, Lipschitz and bounded. For any $p \in [1, +\infty)$ and $\mu := f \mathcal{L}^n$ with $f = \left(\frac{1}{2} - \|\cdot\|_{\infty}\right)^q$, consider the continuous linear operator $P_\alpha : L^p(\Omega, \mu; \mathbb{R}^n) \to L^p(\Omega, \mu; \mathbb{R}^n)$ given by
\[
(P_\alpha V)(x) := \begin{cases} 
\alpha^{n-1}V(\alpha x) & \text{if } x \in \alpha^{-1}\Omega, \\
0 & \text{on } \Omega \setminus \alpha^{-1}\Omega.
\end{cases}
\]

Then:

1. For every $\alpha \in (1, +\infty)$ such that $|1 - \alpha^{-1}| \leq \varepsilon$ holds that
\[
\|P_\alpha V\|_{L^p(\mu)} \leq C\alpha^{-1-\frac{m}{n}}\|V\|_{L^p(\mu)},
\]
for some constant $C > 0$ depending only on $q$ and $p$.

2. For every $V \in L^p(\Omega, \mu; \mathbb{R}^n)$ we have that $P_\alpha V \to V$ in $L^p(\Omega, \mu; \mathbb{R}^n)$ as $\alpha \to 1^+$. 

65
Proof. First, we prove 1. Fix any \( V \in L^p(\Omega, \mu; \mathbb{R}^n) \) and compute

\[
\int |P_\alpha V|^p \, d\mu = \alpha^{p(n-1)} \int |V(\alpha x)|^p \, d\mu(x)
\]

\[
\leq \alpha^{p(n-1)-n} \left( \int |V(y)|^p \, d\mu(y) + \int |V(y)|^p \frac{f(y)}{f(y)} \, d\mu(y) \right).
\]

As in (2.4) we can estimate

\[
\left| \frac{f(\alpha^{-1} y) - f(y)}{f(y)} \right| \leq q \left( \frac{1}{2} - \|y\|_\infty \right)^{-1} \|y\|_\infty (1 - \alpha^{-1}) \leq C
\]

for any \( y \in Q_{1-\varepsilon}(0) \) and any \( \alpha \geq 1 \) such that \( |1 - \alpha^{-1}| \leq \varepsilon \), for some constant \( C \) depending only on \( q \). Therefore

\[
\int |P_\alpha V|^p \, d\mu \leq (C + 1)\alpha^{p(n-1)-n} \int |V(y)|^p \, d\mu(y).
\]

Hence 1. follows. We are left to prove 2. Fix any \( \delta > 0 \) and let \( V_\delta \in C_0^\infty(\Omega; \mathbb{R}^n) \) be such that

\[\|V_\delta - V\|_{L^p(\mu)} \leq \delta.\]

By 1, we have

\[
\|P_\alpha V - V\|_{L^p(\mu)} \leq \|P_\alpha (V - V_\delta)\|_{L^p(\mu)} + \|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)} + \|V_\delta - V\|_{L^p(\mu)}
\]

\[
\leq (C\alpha^{n-1-\frac{n}{p}} + 1)\delta + \|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)}.
\]

for every \( \alpha \in (1, +\infty) \) such that \( |1 - \alpha^{-1}| \leq \varepsilon \). Since \( V_\delta \) is continuous and compactly supported, it follows from dominated convergence that \( \|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)} \to 0 \) as \( \alpha \to 1^+ \). Hence, by letting \( \alpha \to 1^+ \) in the previous inequality we get

\[
\limsup_{\alpha \to 1^+} \|P_\alpha V - V\|_{L^p(\mu)} \leq (C + 1)\delta.
\]

As \( \delta > 0 \) was arbitrary, 2 follows. \( \square \)

References

[1] G. Alberti, S. Baldo, and G. Orlandi. “Functions with prescribed singularities”. In: *J. Eur. Math. Soc. (JEMS)* 5.3 (2003), pp. 275–311.

[2] L. Ambrosio and B. Kirchheim. “Currents in metric spaces”. In: *Acta Math.* 185.1 (2000), pp. 1–80.

[3] M. Berger and B. Gostiaux. *Degree Theory*. Springer New York, 1988, 244–276. ISBN: 978-1-4612-1033-7. DOI: 10.1007/978-1-4612-1033-7. URL: https://doi.org/10.1007/978-1-4612-1033-7.

[4] F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein. “A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds”. In: *Nematics (Orsay, 1990)*. Vol. 332. NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, 1991, pp. 15–23.
[5] F. Bethuel and X. M. Zheng. “Density of smooth functions between two manifolds in Sobolev spaces”. In: *J. Funct. Anal.* 80.1 (1988), pp. 60–75.

[6] R. Bott and L. W. Tu. *Differential forms in algebraic topology*. Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. xiv+331.

[7] H. Brezis. “Liquid crystals and energy estimates for $S^2$-valued map”. In: *Theory and Applications of Liquid Crystals*. Springer, 1987, pp. 31–52.

[8] H. Brezis, J.-M. Coron, and E. H. Lieb. “Harmonic maps with defects”. In: *Comm. Math. Phys.* 107.4 (1986), pp. 649–705.

[9] H. Brezis, P. Mironescu, and A. C. Ponce. “$W^{1,1}$-maps with values into $S^1$”. In: *Geometric analysis of PDE and several complex variables*. Vol. 368. Contemp. Math. Amer. Math. Soc., Providence, RI, 2005, pp. 69–100.

[10] H. Brezis and L. Nirenberg. “Degree theory and BMO; part I: Compact manifolds without boundaries”. In: *Selecta Mathematica* 1 (1995), pp. 197–263.

[11] R. Caniato. “The strong $L^p$-closure of vector fields with finitely many integer singularities on $B^3$”. In: *J. Funct. Anal.* 281.6 (2021), Paper No. 109095, 50.

[12] F. Gaia and T. Rivière. “A variational approach to $S^1$-harmonic maps and applications”. In: (2021). URL: https://arxiv.org/abs/2111.14769

[13] M. Giaquinta, G. Modica, and J. Souček. *Cartesian currents in the calculus of variations. II*. Vol. 38. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Variational integrals. Springer-Verlag, Berlin, 1998, pp. xxiv+697.

[14] R. Hardt and T. Rivière. “Connecting topological Hopf singularities”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2.2* (2003), pp. 287–344.

[15] R. Hardt and T. Rivière. “Ensembles singuliers topologiques dans les espaces fonctionnels entre variétés”. In: Séminaire: Équations aux Dérivées Partielles, 2000–2001. Sémin. Équat. Dériv. Partielles. École Polytech., Palaiseau, 2001, Exp. No. VII, 14.

[16] T. Kessel and T. Rivière. “Singular bundles with bounded $L^2$-curvatures”. In: *Boll. Unione Mat. Ital.* (9) 1.3 (2008), pp. 881–901.

[17] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. Reprint of the 1963 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996, pp. xii+329.

[18] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. Reprint of the 1969 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996, pp. xvi+468.

[19] S. G. Krantz and H. R. Parks. *Geometric integration theory*. Cornerstones. Birkhäuser Boston, Boston, MA, 2008, pp. xvi+339.

[20] F. Maggi. *Sets of finite perimeter and geometric variational problems*. Vol. 135. Cambridge Studies in Advanced Mathematics. An introduction to geometric measure theory. Cambridge University Press, Cambridge, 2012, pp. xx+454.
[21] M. Petrache. “An integrability result for $L^p$-vector fields in the plane”. In: Adv. Calc. Var. 6.3 (2013), pp. 299–319.

[22] M. Petrache and T. Rivière. “The resolution of the Yang-Mills Plateau problem in super-critical dimensions”. In: Adv. Math. 316 (2017), pp. 469–540.

[23] M. Petrache and T. Rivière. “Weak closure of singular abelian $L^p$-bundles in 3 dimensions”. In: Geom. Funct. Anal. 21.6 (2011), pp. 1419–1442.

[24] A. C. Ponce. “On the distributions of the form $\sum_i(\delta_{p_i} - \delta_{n_i})$”. In: Journal of Functional Analysis 210 (2004), pp. 391–435.

[25] R. Schoen and K. Uhlenbeck. “A regularity theory for harmonic maps”. In: J. Differential Geometry 17.2 (1982), pp. 307–335.

[26] R. Schoen and K. Uhlenbeck. “Boundary regularity and the Dirichlet problem for harmonic maps”. In: J. Differential Geom. 18.2 (1983), pp. 253–268.

[27] C. Scott. “$L^p$ theory of differential forms on manifolds”. In: Trans. Amer. Math. Soc. 347.6 (1995), pp. 2075–2096.

R. Caniato, Department of Mathematics, ETH, Rämistrasse 101, CH-8093 Zürich, Switzerland
riccardo.caniato@math.ethz.ch

F. Gaia, Department of Mathematics, ETH, Rämistrasse 101, CH-8093 Zürich, Switzerland
filippo.gaia@math.ethz.ch

68