BOUNDEDNESS AND COMPACTNESS OF COMPOSITION OPERATORS ON SEGAL-BARGMANN SPACES

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Abstract. For $E$ a Hilbert space, let $\mathcal{H}(E)$ denote the Segal-Bargmann space (also known as the Fock space) over $E$, which is a reproducing kernel Hilbert space with kernel $K(x, y) = \exp(\langle x, y \rangle)$ for $x, y$ in $E$. If $\varphi$ is a mapping on $E$, the composition operator $C_\varphi$ is defined by $C_\varphi h = h \circ \varphi$ for $h \in \mathcal{H}(E)$ for which $h \circ \varphi$ also belongs to $\mathcal{H}(E)$. We determine necessary and sufficient conditions for the boundedness and compactness of $C_\varphi$. Our results generalize results obtained earlier by Carswell, MacCluer and Schuster for finite dimensional spaces $E$.

1. Introduction

Let $\mathcal{H}$ be a Banach space of functions on a set $X$ and $\varphi : X \to X$ be a map. We define the composition operator $C_\varphi$ by $C_\varphi h = h \circ \varphi$ for all functions $h \in \mathcal{H}$ for which the function $h \circ \varphi$ also belongs to $\mathcal{H}$. We are often interested in the problem of classifying the functions $\varphi$ which induce bounded or compact operators $C_\varphi$. There is a vast literature on this problem when $\mathcal{H}$ is the Hardy, Bergman or Bloch space over the unit disc on the plane or the unit ball in $\mathbb{C}^n$ (see, for example, [2, 6, 9, 11, 14, 15] and references therein). In [5], Carswell, MacCluer and Schuster studied composition operators on the Segal-Bargmann space (also known as the Fock space) over $\mathbb{C}^n$. They obtained necessary and sufficient conditions on the functions $\varphi$ that give rise to bounded or compact $C_\varphi$. They showed any such function $\varphi$ must be affine with an additional restriction. They also provided a formula for the norm of $C_\varphi$. This is an interesting result since the problem of computing the norm of a bounded composition operator on other classical function spaces is still an open problem.

Let $n \geq 1$ be an integer. We denote by $d\mu(z) = \pi^{-n} \exp(-|z|^2)dV(z)$ the Gaussian measure on $\mathbb{C}^n$, where $dV$ is the usual Lebesgue volume measure on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. The Segal-Bargmann (Fock) space $\mathcal{F}_n$ is the space of all entire functions on $\mathbb{C}^n$ that are square integrable with respect to $d\mu$. For $f, g \in \mathcal{F}_n$, the inner product $\langle f, g \rangle$ is given by

$$\langle f, g \rangle = \int_{\mathbb{C}^n} f(z)g(z) \ d\mu(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z)\overline{g(z)} \exp(-|z|^2) \ dV(z).$$

It is well known that $\mathcal{F}_n$ contains an orthonormal basis consisting of monomials. In fact, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, if

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we put $f_\alpha(z) = (\alpha!)^{-1/2}z^\alpha$, where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, then \{f_\alpha : \alpha \in \mathbb{Z}^n_{\geq 0}\} is an orthonormal basis for $F_n$. It is also well known that $F_n$ is a reproducing kernel Hilbert space of functions on $\mathbb{C}^n$ with kernel $K(z, w) = \exp((z, w))$. For more details on $F_n$, see, for example, Section 1.6 in [8]. We would like to alert the reader that other authors use slightly different versions of the Gaussian measure (for example, $d\mu(z) = (2\pi)^{-n}\exp(-|z|^2/2)dV(z)$) and hence the resulting reproducing kernels have different formulas (for example, $K(z, w) = \exp((z, w)/2)$). Our choice of the constant here is just for the simplicity of the formulas.

The following theorem [5, Theorem 1] characterizes bounded and compact composition operators on $F_n$.

**Theorem 1.1** (Carswell, MacCluer and Schuster). Suppose $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a holomorphic mapping.

(a) $C_\varphi$ is bounded on $F_n$ if and only if $\varphi(z) = Az + b$, where $A$ is an $n \times n$ matrix with $\|A\| \leq 1$ and $b$ is an $n \times 1$ vector such that $\langle A\zeta, b \rangle = 0$ whenever $|A\zeta| = |\zeta|$.

(b) $C_\varphi$ is compact on $F_n$ if and only if $\varphi(z) = Az + b$, where $\|A\| < 1$ and $b$ is any $n \times 1$ vector.

The norm of $C_\varphi$ is given by the next theorem, which is Theorem 4 in [5]. We alert the reader that the formula presented here is slightly different from the original formula given in [5] because our reproducing kernel is $K(z, w) = \exp((z, w))$ whereas theirs was $\exp(\frac{1}{4}(z, w))$.

**Theorem 1.2.** Suppose $\varphi(z) = Az + B$, where $\|A\| \leq 1$ and $\langle A\zeta, b \rangle = 0$ whenever $|A\zeta| = |\zeta|$. Then the norm of $C_\varphi$ on $F_n$ is given by

$$\|C_\varphi\| = \exp\left(\frac{1}{2}(\|w_0\|^2 - |Aw_0|^2 + |b|^2)\right),$$

where $w_0$ is any solution to $(I - A^*A)w_0 = A^*b$.

Motivated by the above results, we study in this paper composition operators on the Segal-Bargmann space $\mathcal{H}(E)$ over an arbitrary Hilbert space $E$. The proof of Theorem 1.1 in [5] makes use of the change of variables and the fact that any $n \times n$ matrix $A$ can be written in the form $A = UDV$ for unitary matrices $U, V$ and a diagonal matrix $D$. Since this approach relies heavily on the finiteness of the dimension of $E$, it does not seem to work when the dimension of $E$ is infinite. It turns out that there is an alternative approach, based on the theory of reproducing kernels. This idea appeared in E. Nordgren’s work [13] and it was employed in [10], where M. Jury proved the boundedness of composition operators on the Hardy and Bergman spaces of the unit disk without using Littlewood Subordination Principle. We will see that $C_\varphi$ is bounded if and only if $\varphi$ is an affine map as in Theorem 1.1 but we need a stronger condition on the vector $b$ when $E$ is an infinite dimensional Hilbert space (in the case $E$ has finite dimension, our condition on $b$ is equivalent to the condition in Theorem 1.1). In the course of proving
boundedness, we also obtain a formula for \( \|C\varphi\| \). Our formula is stated in a different way and it agrees with \( \|\varphi\| \) when \( E = \mathbb{C}^n \). For the compactness of \( C\varphi \), besides the condition that \( \varphi(z) = Az + b \) for some linear operator \( A \) on \( E \) with \( \|A\| < 1 \), it is also necessary that \( A \) be a compact operator (this condition is of course superfluous when the dimension of \( E \) is finite).

We now state our main results. The first result studies the boundedness and the norm formula for \( C\varphi \) on \( \mathcal{H}(E) \).

**Theorem 1.3.** Let \( \varphi \) be a mapping from \( E \) into itself. Then the composition operator \( C\varphi \) is bounded on \( \mathcal{H}(E) \) if and only if \( \varphi(z) = Az + b \) for \( z \in E \), where \( A \) is a linear operator on \( E \) with \( \|A\| \leq 1 \) and \( A^*b \) belongs to the range of \( (I - A^*A)^{1/2} \).

Furthermore, the norm of \( C\varphi \) is given by

\[
\|C\varphi\| = \exp\left(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|b\|^2\right),
\]

where \( v \) is the unique vector in \( E \) of minimum norm that satisfies the equation \( A^*b = (I - A^*A)^{1/2}v \).

The second main result characterizes compact operators \( C\varphi \).

**Theorem 1.4.** Let \( \varphi \) be a mapping from \( E \) into itself. Then the composition operator \( C\varphi \) is compact on \( \mathcal{H}(E) \) if and only if there is a compact linear operator \( A \) on \( E \) with \( \|A\| < 1 \) and a vector \( b \in E \) such that \( \varphi(z) = Az + b \) for all \( z \in E \).

2. Compositions operators on \( \mathcal{H}(E) \)

In the first part of this section we study the space \( \mathcal{H}(E) \), where \( E \) is an arbitrary Hilbert space. Since Gaussian measure is not available when \( E \) is of infinite dimension, our approach here follows the same lines as the construction of the Drury-Arveson space given in [4]. In the second part of the section, we consider composition operators on \( \mathcal{H}(E) \). Using kernel functions, we provide a criterion for the boundedness of these operators.

2.1. The construction of \( \mathcal{H}(E) \). For each integer \( m \geq 1 \), we write \( E^m \) for the symmetric tensor product of \( m \) copies of \( E \). We also define \( E^0 \) to be \( \mathbb{C} \) with its usual inner product. We have \( E^1 = E \) and for \( m \geq 2 \), \( E^m \) is a closed subspace of the full tensor product \( E \otimes^m \) consisting of all vectors that are invariant under the natural action of the symmetric group \( S_m \). The action of \( S_m \) on \( E \otimes^m \) is defined on elementary tensors by

\[
\pi \cdot (x_1 \otimes \cdots \otimes x_m) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(m)} \quad \text{for} \quad \pi \in S_m \quad \text{and} \quad x_1, \ldots, x_m \in E.
\]

For an element \( z \in E \), we write \( z^m = z \otimes \cdots \otimes z \in E^m \) for the \( m \)-fold tensor product of copies of \( z \) (here \( z^0 \) denotes the number 1 in \( E^0 = \mathbb{C} \)). Each space \( E^m \) is a Hilbert space with an inner product inherited from the inner product on \( E \). We will generally write \( \langle \cdot, \cdot \rangle \) for any inner product without referring to the space on which it is defined. The defining space will be clear from the context.
A continuous mapping \( p : E \to \mathbb{C} \) is called a continuous \( m \)-homogeneous polynomial on \( E \) if there exists an element \( \zeta \) in \( E^m \) such that \( p(z) = \langle z^m, \zeta \rangle \) for \( z \in E \). A continuous mapping \( f : E \to \mathbb{C} \) is called a polynomial if \( f \) can be written as a finite sum of continuous homogeneous polynomials. In other words, there is an integer \( m \geq 0 \) and there are vectors \( a_0 \in \mathbb{C}, a_1 \in E^1, \ldots, a_m \in E^m \) such that

\[
f(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle = a_0 + \langle z, a_1 \rangle + \cdots + \langle z^m, a_m \rangle.
\]  

(2.1)

When \( E = \mathbb{C}^n \) for some positive integer \( n \), the notion of polynomials that we have just given coincides with the usual definition of polynomials in \( n \) complex variables. In fact, each polynomial in \( z = (z_1, \ldots, z_n) \) is a linear combination of monomials of the form \( z_1^{j_1} \cdots z_n^{j_n} \) for non-negative integers \( j_1, \ldots, j_n \). Let \( \{e_1, \ldots, e_n\} \) denote the standard basis for \( \mathbb{C}^n \), where \( e_k = (0, \ldots, 0, 1, 0, \ldots) \) with the number 1 in the \( k \)th component. Then

\[
z_1^{j_1} \cdots z_n^{j_n} = \langle z, e_1\rangle^{j_1} \cdots \langle z, e_n\rangle^{j_n} = \langle z^l, a_l \rangle_E,
\]

where \( l = j_1 + \cdots + j_n \) and \( a_l \) is the orthogonal projection of \( e_1^{\otimes j_1} \cdots e_n^{\otimes j_n} \) on \( E^l \). This shows that any polynomial in the variables \( z_1, \ldots, z_n \) can be written in the form (2.1).

We denote by \( \mathcal{P}_n(E) \) the space of all continuous \( n \)-homogeneous polynomials and \( \mathcal{P}(E) \) the space of all continuous polynomials on \( E \). For more detailed discussions of polynomials between Banach spaces and locally convex spaces, see [7, 12].

For two continuous polynomials \( f, g \) in \( \mathcal{P}(E) \), we can find an integer \( m \geq 0 \) and vectors \( a_j, b_j \in E^j \) for \( 0 \leq j \leq m \) such that \( f(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle \) and \( g(z) = \sum_{j=0}^{m} \langle z^j, b_j \rangle \). We define

\[
\langle f, g \rangle = \sum_{j=0}^{m} j! \langle b_j, a_j \rangle.
\]  

(2.2)

It can be checked that (2.2) defines an inner product on \( \mathcal{P}(E) \). We denote by \( \mathcal{H}(E) \) the completion of \( \mathcal{P}(E) \) in the norm induced by this inner product.

There is a natural anti-unitary operator from \( \mathcal{H}(E) \) onto the symmetric (boson) Fock space \( \mathcal{F}(E) = E^0 \oplus E^1 \oplus E^2 \oplus \cdots \), where the sum denotes the infinite direct sum of Hilbert spaces. We skip the proof which is straightforward from the definition of \( \mathcal{H}(E) \) and \( \mathcal{F}(E) \).

**Proposition 2.1.** For each element \( f \in \mathcal{P}(E) \) given by formula (2.1), we define an element in \( \mathcal{F}(E) \) by

\[
Jf = (a_0, \sqrt{1!} a_1, \sqrt{2!} a_2, \sqrt{3!} a_3, \ldots)
\]
where \( a_j = 0 \) for \( j > m \). Then \( J \) is an anti-unitary from \( \mathcal{P}_m(E) \) onto \( E^m \) for each \( m \geq 0 \) and it extends uniquely to an anti-unitary operator from \( \mathcal{H}(E) \) onto \( \mathcal{F}(E) \).

As in the case of the Drury-Arveson space, we can realize the elements of \( \mathcal{H}(E) \) in more concrete terms, as entire functions on \( E \).

**Proposition 2.2.** Each element \( f \) in \( \mathcal{H}(E) \) can be identified as an entire function on \( E \) having a power expansion of the form

\[
f(z) = \sum_{j=0}^{\infty} \langle z^j, a_j \rangle \quad \text{for all } z \in E,
\]

where \( a_0 \in \mathbb{C} \), \( a_1 \in E \), \( a_2 \in E^2 \), \ldots. Furthermore, \( \|f\|^2 = \sum_{j=0}^{\infty} j!^2 \|a_j\|^2 \).

Conversely, if \( \sum_{j=0}^{\infty} j! \|a_j\|^2 < \infty \), then the power series \( \sum_{j=0}^{\infty} \langle z^j, a_j \rangle \) defines an element in \( \mathcal{H}(E) \).

**Proof.** By Proposition 2.1 each element \( f \) has a formal power series of the form

\[
f(z) = \sum_{j=0}^{\infty} \langle z^m, a_m \rangle, \quad (2.3)
\]

where \( a_j \) belongs to \( E^j \) for \( j \geq 0 \) and \( \sum_{j=0}^{\infty} j! \|a_j\|^2 = \|f\|^2 < \infty \). For any \( z \in E \), since \( \|z^m\| = \|z\|^m \), we have

\[
\sum_{j=0}^{\infty} |\langle z^j, a_j \rangle| \leq \sum_{j=0}^{\infty} \|z^j\| \|a_j\| = \sum_{j=0}^{\infty} \|z\|^j \|a_j\| = \sum_{j=0}^{\infty} \frac{\|z\|^j}{\sqrt{j!}} \sqrt{j!} \|a_j\|
\]

\[
\leq \left( \sum_{j=0}^{\infty} \frac{\|z\|^{2j}}{j!} \right)^{1/2} \left( \sum_{j=0}^{\infty} j! \|a_j\|^2 \right)^{1/2} = \exp(\|z\|^2/2)\|f\|.
\]

This shows that the power series \( (2.3) \) converges uniformly on any bounded ball in \( E \). It follows that \( f \) can be considered as an entire function on \( E \).

The converse follows from the fact that the sequence of polynomials \( \{p_m\}_{m=1}^{\infty} \) defined by \( p_m(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle \) for \( m = 1, 2, \ldots \), is a Cauchy sequence in \( \mathcal{H}(E) \). \( \square \)

For \( w \) in \( E \), put

\[
K_w(z) = \exp(\langle z, w \rangle) = \sum_{j=0}^{\infty} \frac{1}{j!} \langle z, w \rangle^j = \sum_{j=0}^{\infty} \langle z^n, \frac{w^j}{j!} \rangle \quad \text{for } z \in E.
\]

By Proposition 2.2 \( K_w \) belongs to \( \mathcal{H}(E) \). For any \( f \) given by (2.3), we have

\[
\langle f, K_w \rangle = \sum_{j=0}^{\infty} j! \langle \frac{w^j}{j!}, a_j \rangle = f(w) \quad \text{for } w \in E.
\]

Therefore, the function \( K(z, w) = K_w(z) \) for \( z, w \in E \) is the reproducing kernel function for \( \mathcal{H}(E) \). Furthermore, the linear span of the set \( \{K_w : w \in E\} \) is dense in \( \mathcal{H}(E) \). This shows that \( \mathcal{H}(E) \) is a reproducing kernel.
Hilbert space. For a general theory of these spaces, see, for example, [3] or [1, Chapter 2].

**Remark 2.3.** The space $\mathcal{H}(E)$ can be defined in an abstract way by the kernel function $K(z, w)$. However, it is not clear from the abstract definition why $\mathcal{H}(E)$ consists of the power series given in Proposition 2.2. We have chosen a more concrete construction to exhibit the decomposition

$$\mathcal{H}(E) = \bigoplus_{m \geq 0} P_m(E) = C \oplus P_1(E) \oplus P_2(E) \oplus \cdots,$$

which will be useful for us later.

When $E = \mathbb{C}^n$ for some positive integer $n$, the space $\mathcal{H}(\mathbb{C}^n)$ coincides with $F_n$, which we discuss in the Introduction.

The following facts are well known in the case $E = \mathbb{C}^n$ and they continue to hold for arbitrary Hilbert space $E$. We skip the proofs, which make use of the fact that a sequence in a reproducing kernel Hilbert space is weakly convergent if and only if it is bounded in norm and it converges point-wise.

**Lemma 2.4.** The following statements hold in $\mathcal{H}(E)$.

(a) $\lim_{\|z\| \to \infty} \|K_z\|^{-1} K_z = 0$ weakly in $\mathcal{H}(E)$.

(b) Let $\{u_m\}$ be a sequence converging weakly to 0 in $E$ (in particular, $\{u_m\}$ is bounded). For each $m$, put $f_m(z) = \langle z, u_m \rangle$ for $z \in E$. Then $\lim_{m \to \infty} f_m = 0$ weakly in $\mathcal{H}(E)$.

2.2. **Composition operators.** For any mapping $\varphi$ from $E$ into itself, we recall that the composition operator $C_\varphi$ is defined by $C_\varphi h = h \circ \varphi$ for all $h$ in $\mathcal{H}(E)$ for which $h \circ \varphi$ also belongs to $\mathcal{H}(E)$. Since $C_\varphi$ is a closed operator, it follows from the closed graph theorem that $C_\varphi$ is bounded if and only if $h \circ \varphi$ belongs to $\mathcal{H}(E)$ for all $h \in \mathcal{H}(E)$.

Now suppose that $C_\varphi$ is a bounded operator on $\mathcal{H}(E)$. A priori we do not impose any condition $\varphi$ but it follows from the boundedness of $C_\varphi$ that $\varphi$ must be an entire function (at least in the weak sense). In fact, for any $a \in E$, the function $\langle \varphi(\cdot), a \rangle = C_\varphi(\langle \cdot, a \rangle)$ belongs to $\mathcal{H}(E)$, hence it is entire on $E$ by Proposition 2.2.

For $z \in E$ and $h \in \mathcal{H}(E)$, since $\langle h, C_\varphi K_z \rangle = \langle C_\varphi h, K_z \rangle = h(\varphi(z)) = \langle h, K_{\varphi(z)} \rangle$, we obtain the well known formula

$$C_\varphi^* K_z = K_{\varphi(z)}.$$  

(2.5)

This formula was used in [5] for the proof of the necessity of Theorem 1.1. It turns out that the formula plays an important role in our proof of both the necessity and sufficiency on the boundedness of $C_\varphi$.

Let $\mathcal{M}$ denote the linear span of the kernel functions $\{K_z : z \in E\}$. We already know that $\mathcal{M}$ is dense in $\mathcal{H}(E)$. Motivated by (2.5), for any mapping $\varphi$ (even when $C_\varphi$ is not a bounded operator on $\mathcal{H}(E)$), we define a linear
operator $S_\varphi$ with domain $\mathcal{M}$ by the formula

$$S_\varphi\left(\sum_{j=1}^{m} c_j K_{x_j}\right) = \sum_{j=1}^{m} c_j K_{\varphi(x_j)},$$

for distinct elements $x_1, \ldots, x_m$ in $E$ and any complex numbers $c_1, \ldots, c_m$. The operator $S_\varphi$ is well defined since the kernel functions $K_{x_1}, \ldots, K_{x_m}$ are linearly independent. It follows from (2.5) that if $C_\varphi$ is bounded on $\mathcal{H}(E)$, then $S_\varphi = C_\varphi^*$ on $\mathcal{M}$ and hence $S_\varphi$ extends to a bounded operator on $\mathcal{H}(E)$. On the other hand, if $S_\varphi$ extends to a bounded operator on $\mathcal{H}(E)$, then since

$$(C_\varphi h)(z) = h(\varphi(z)) = \langle h, K_\varphi(z) \rangle = \langle h, S_\varphi K_z \rangle = (S_\varphi^* h)(z)$$

for all $h \in \mathcal{H}(E)$ and all $z \in E$, we conclude that $C_\varphi = S_\varphi^*$ and hence $C_\varphi$ is also a bounded operator. It turns out, with the help of kernel functions, that it is more convenient for us to work with $S_\varphi$ than with $C_\varphi$ directly.

For elements $x_1, \ldots, x_m$ in $E$ and complex numbers $c_1, \ldots, c_m$, since

$$\left\| S_\varphi\left(\sum_{j=1}^{m} c_j K_{x_j}\right) \right\|^2 = \sum_{j,l} \overline{c_j} c_l \langle K_{\varphi(x_j)}, K_{\varphi(x_l)} \rangle = \sum_{j,l} \overline{c_j} c_l K(\varphi(x_l), \varphi(x_l)),$$

and

$$\left\| \sum_{j=1}^{m} c_j K_{x_j} \right\|^2 = \sum_{j,l=1}^{m} \overline{c_j} c_l K(x_l, x_l),$$

we see that $S_\varphi$ is bounded with $\|S_\varphi\| \leq M$ if and only if

$$\sum_{j,l=1}^{m} c_j \overline{c_l} (M^2 K(x_l, x_l) - K(\varphi(x_l), \varphi(x_l))) \geq 0. \quad (2.6)$$

Put $\Phi_M(z, w) = M^2 K(z, w) - K(\varphi(z), \varphi(w))$ for $z, w \in E$. Since (2.6) holds for arbitrary $x_1, \ldots, x_m$ in $E$ and arbitrary complex numbers $c_1, \ldots, c_m$, we say that $\Phi_M$ is a positive semi-definite kernel on $E$ (in Section 3 we will discuss more about these kernels). Therefore, $S_\varphi$ (and hence, $C_\varphi$) is bounded with norm at most $M$ if and only if $\Phi_M$ is a positive semi-definite kernel. This criterion for boundedness of composition operators on general reproducing kernel Hilbert spaces was obtained by Nordgren in [13, Theorem 2].

Using the formula $K(z, w) = \exp(\langle z, w \rangle)$, we have

**Lemma 2.5.** For any mapping $\varphi$ from $E$ into itself, the composition operator $C_\varphi$ is bounded on $\mathcal{H}(E)$ with norm at most $M$ if and only if the function

$$\Phi_M(z, w) = M^2 \exp(\langle z, w \rangle) - \exp(\langle \varphi(z), \varphi(w) \rangle)$$

is positive semi-definite.

In particular, we have $\Phi_M(z, z) \geq 0$, which is equivalent to

$$M^2 \exp(\|z\|^2) \geq \exp(\|\varphi(z)\|^2) \iff 2 \ln M \geq \|\varphi(z)\|^2 - \|z\|^2,$$

for all $z \in E$. 
In Section 3, we discuss in more detail positive semi-definite kernels and find conditions on \( \phi \) under which the function \( \Phi_M \) above is positive semi-definite. Using these conditions we obtain a proof of Theorem 1.3.

3. Boundedness of composition operators on \( \mathcal{H}(E) \)

3.1. Positive semi-definite kernels. Let \( X \) be a set. A function \( F : X \times X \to \mathbb{C} \) is a positive semi-definite kernel if for any finite set \( \{ x_1, \ldots, x_m \} \) of points in \( X \), the matrix \( (F(x_l, x_j))_{1 \leq l,j \leq m} \) is positive semi-definite. That is, for any complex numbers \( c_1, \ldots, c_m \), we have

\[
\sum_{j,l=1}^{m} c_l c_j F(x_l, x_j) \geq 0.
\]

We list here a few immediate facts about positive semi-definite kernels.

(F1) Sums of positive semi-definite kernels are positive semi-definite. (A sum here may be an infinite sum provided that it converges pointwise.)

(F2) Since the Schur (entry-wise) product of two positive semi-definite square matrices is also positive semi-definite, the product of two positive semi-definite kernels is positive semi-definite.

(F3) Suppose \( F \) is a positive semi-definite kernel, then it follows from (F1) and (F2) that the function \( \tilde{F} = \exp(F) - 1 \) is also a positive semi-definite kernel.

(F4) If there is a vector space \( \mathcal{H} \) over \( \mathbb{C} \) with inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and norm \( \| \cdot \|_{\mathcal{H}} \) and there is a vector-valued function \( f : X \to \mathcal{H} \) such that \( F(x,y) = \langle f(y), f(x) \rangle_{\mathcal{H}} \) for \( x,y \) in \( X \), then for any \( x_1, \ldots, x_m \) in \( X \) and real numbers \( c_1, \ldots, c_m \), we have

\[
\sum_{j,l=1}^{m} c_l c_j F(x_l, x_j) = \sum_{j=1}^{m} \langle c_j f(x_j), c_l f(x_l) \rangle_{\mathcal{H}} = \| \sum_{j=1}^{m} c_j f(x_j) \|_{\mathcal{H}}^2 \geq 0.
\]

Therefore, \( F \) is a positive semi-definite kernel on \( X \). It turns out \([1]\) Theorem 2.53] that any positive semi-definite kernel can be represented in this form.

Now let \( E \) be a Hilbert space and \( T \) be a bounded linear operator on \( E \). Define \( F(z,w) = \langle Tz, w \rangle \) for \( z,w \in E \). It is clear that if \( F \) is positive semi-definite on \( E \), then \( F(z,z) \geq 0 \) for all \( z \in E \), which implies that \( T \) is a positive operator. Conversely, if \( T \) is positive, then since \( F(z,w) = \langle T^{1/2}z, T^{1/2}w \rangle \) (here \( T^{1/2} \) denotes the positive square root of \( T \)), it follows from fact (F4) above that \( F \) is positive semi-definite. The following proposition provides a generalization of this observation.

**Proposition 3.1.** Let \( u \) be a vector in \( E \), \( T \) be a self-adjoint operator on \( E \), and \( M \) be a real number. Define the function

\[
F(z,w) = \langle Tz, w \rangle - \langle z, u \rangle - \langle u, w \rangle + M^2 \text{ for } z, w \in E.
\] (3.1)
Then the followings are equivalent

(a) The function $F$ is a positive semi-definite kernel on $E$.
(b) $F(z, z) \geq 0$ for all $z \in E$.
(c) $T$ is a positive operator on $E$ and $u = T^{1/2} \hat{u}$ for some $\hat{u} \in E$ with $\|\hat{u}\| \leq M$.

Furthermore, if the conditions in (c) are satisfied and $v$ is the vector of smallest norm such that $u = T^{1/2}v$, then we have

$$ \inf \{ F(z, z) : z \in E \} = -\|v\|^2 + M^2. \quad (3.2) $$

The vector $v$ is characterized by two conditions: (i) $T^{1/2}v = u$ and (ii) $v$ belongs to $\text{ran}(T^{1/2})$.

**Proof.** It is obvious from the definition of positive semi-definite kernels that (a) implies (b). Now suppose (b) holds. Let $z$ be in $E$. Choose a complex number $\gamma$ such that $|\gamma| = 1$ and $\langle \gamma z, u \rangle = \gamma \langle z, u \rangle = |\langle z, u \rangle|$. For any real number $r$, since $F(r \gamma z, r \gamma z) \geq 0$, we obtain

$$ r^2(Tz, z) - 2r|\langle z, u \rangle| + M^2 \geq 0. $$

Because this inequality holds for all $r \in \mathbb{R}$ we see that $\langle Tz, z \rangle \geq 0$ and $|\langle z, u \rangle|^2 \leq M^2 \langle Tz, z \rangle$. Since $z$ was arbitrary, we conclude that $T$ is a positive operator and we have $|\langle z, u \rangle| \leq |M||T^{1/2}z||$ for $z \in E$. That this fact implies that $u$ belongs to the range of $T^{1/2}$ is well known but for completeness, we include here a proof. Define a linear functional on the range of $T^{1/2}$ by $\Lambda(T^{1/2}z) = \langle z, u \rangle$, for $z \in E$. By the inequality, $\Lambda$ is well defined and bounded on $T^{1/2}(E)$ with $\|\Lambda\| \leq |M|$. Extending $\Lambda$ to all $E$ by the Hahn-Banach theorem and using the Riesz’s representation theorem, we obtain an element $\hat{u}$ in $E$ with $\|\hat{u}\| = \|\Lambda\| \leq |M|$ such that $\Lambda(w) = \langle w, \hat{u} \rangle$ for all $w \in E$. We then have, for any $z \in E$,

$$ \langle z, u \rangle = \Lambda(T^{1/2}z) = \langle T^{1/2}z, \hat{u} \rangle = \langle z, T^{1/2} \hat{u} \rangle. $$

Thus $u = T^{1/2} \hat{u}$ and hence (c) follows.

Now assume that (c) holds. For any $z, w$ in $E$,

$$ F(z, w) = \langle T^{1/2}z, T^{1/2}w \rangle - \langle T^{1/2}z, \hat{u} \rangle - \langle \hat{u}, T^{1/2}w \rangle + M^2 $n
n $$

$$ = \langle T^{1/2}z - \hat{u}, T^{1/2}w - \hat{u} \rangle - \|\hat{u}\|^2 + M^2. $$

Since $-\|\hat{u}\|^2 + M^2 \geq 0$, we conclude that $F$ is positive semi-definite.

Now the preimage of $u$ under $T^{1/2}$ is the non-empty, closed, convex set $\hat{u} + \ker(T^{1/2})$. By a property of Hilbert spaces, there exists a unique vector $v$ of smallest norm in this set. In fact, $v$ is the orthogonal projection of $\hat{u}$ on $(\ker(T^{1/2}))^\perp$. Since $(\ker(T^{1/2}))^\perp = \text{ran}(T^{1/2})^\perp$, we conclude that $v = P_{\text{ran}(T^{1/2})} \hat{u}$, where $P_{\text{ran}(T^{1/2})}$ is the orthogonal projection from $E$ onto the closure of the range of $T^{1/2}$. Using the facts that $u = T^{1/2}v$ and that $v$
belongs to \( \text{ran}(T^{1/2}) \), we obtain
\[
\inf \{ F(z, z) : z \in E \} = \inf \{ \|T^{1/2}z - v\|^2 : z \in E \} - \|v\|^2 + M^2
\]
\[= -\|v\|^2 + M^2. \]

To prove the characterization of \( v \), let \( v' \) be a vector in \( \text{ran}(T^{1/2}) \) with \( u = T^{1/2}v' \). Then the difference \( v - v' \) belongs to both \( \ker(T^{1/2}) \) and \( \text{ran}(T^{1/2}) \). Since these subspaces are orthogonal complements of each other, we conclude that \( v = v' \).

**Remark 3.2.** In the case \( E = \mathbb{C}^n \) for some integer \( n \geq 1 \), since \( \text{ran}(T^{1/2}) = \text{ran}(T^{1/2}) \), the vector \( v \) in Proposition 3.1 is characterized by \( v = T^{1/2}\zeta \) for any \( \zeta \in E \) that satisfies \( T\zeta = u \).

### 3.2. Bounded composition operators

We are now ready for the proof of Theorem 1.3 on the boundedness of \( C_\varphi \).

**Proof of Theorem 1.3.** Suppose first that \( C_\varphi \) is bounded on \( \mathcal{H}(E) \). By Lemma 2.5, for any \( z \in E \), \( \|C_\varphi\|^2 \exp(\|z\|^2) - \exp(\|\varphi(z)\|^2) \geq 0 \). This implies
\[
\|\varphi(z)\|^2 - \|z\|^2 \leq 2\ln \|C_\varphi\|. \tag{3.3}
\]
For a fixed unit vector \( a \in E \), put \( f_a(z) = \langle z, a \rangle \) and \( F_a(z) = \langle \varphi(z), a \rangle \). Then \( F_a \), which equals to \( C_\varphi(f_a) \), belongs to \( \mathcal{H}(E) \). Therefore \( F_a \) can be represented as a power series
\[
F_a(z) = F_a(0) + \sum_{m=1}^\infty \langle z^m, \zeta_m \rangle \text{ for all } z \in E,
\]
where \( \zeta_1 \in E, \zeta_2 \in E^2, \ldots \). Now the inequality \( |F_a(z)| \leq \|\varphi(z)\| \) together with (3.3) gives \( |F_a(z)|^2 - \|z\|^2 \leq 2\ln(\|C_\varphi\|) \) for all \( z \in E \). This implies that \( \|\zeta_1\| \leq 1 \) and \( \zeta_m = 0 \) for all \( m \geq 2 \). In particular, \( z \mapsto F_a(z) - F_a(0) \) is linear functional with norm at most 1.

Since the map \( z \mapsto \langle \varphi(z) - \varphi(0), a \rangle = F_a(z) - F_a(0) \) is a linear functional with norm at most one for any unit vector \( a \in E \), we conclude that \( z \mapsto \varphi(z) - \varphi(0) \) is a linear operator with norm at most 1. Therefore, \( \varphi(z) = Az + b \) for some linear operator \( A \) on \( E \) with \( \|A\| \leq 1 \) and some vector \( b \) in \( E \).

Now (3.3) gives \( \|z\|^2 - \|Az + b\|^2 \geq 2\ln(\|C_\varphi\|) \geq 0 \) for all \( z \in E \), which is equivalent to
\[
\langle (I - A^*A)z, z \rangle - \langle z, A^*b \rangle - \langle A^*b, z \rangle - \|b\|^2 + 2\ln(\|C_\varphi\|) \geq 0. \tag{3.4}
\]
By Proposition 3.1, we conclude that \( A^*b \) belongs to the range of \( (I - A^*A)^{1/2} \). Choose \( v \in E \) of smallest norm such that \( A^*b = (I - A^*A)^{1/2}(v) \). Then by Proposition 3.1 again, the quantity
\[
2\ln(\|C_\varphi\|) - \|v\|^2 - \|b\|^2,
\]
being the infimum of the left hand side of (3.4), is non-negative. Thus we have

$$\|C_\varphi\| \geq \exp \left( \frac{1}{2}\|v\|^2 + \frac{1}{2}\|b\|^2 \right).$$

(3.5)

Conversely, suppose \( \varphi(z) = Az + b \) such that \( \|A\| \leq 1 \); \( A^*b \) belongs to the range of \((I - A^*A)^{1/2}\); and \( v \in E \) is of smallest norm satisfying \( A^*b = (I - A^*A)^{1/2}(v) \). We will show that the operator \( C_\varphi \) is bounded on \( \mathcal{H}(E) \) with norm at most the quantity on the right hand side of (3.5) (hence the inequality in (3.5) is in fact an equality).

We define for \( z, w \in E \),

$$F(z, w) = \langle z, w \rangle - \langle \varphi(z), \varphi(w) \rangle + \|b\|^2 + \|v\|^2$$

$$= \langle (I - A^*A)z, w \rangle - \langle z, A^*b \rangle - \langle A^*b, w \rangle + \|v\|^2.$$

By Proposition 3.1 \( F \) is a positive semi-definite kernel, which implies that \( \exp(F) - 1 \) is positive semi-definite. Now let \( G \) denote the positive semi-definite kernel defined by \( G(z, w) = \exp(\langle \varphi(z), \varphi(w) \rangle) \) for \( z, w \in E \). Then \( G \cdot (\exp(F) - 1) \) is also a positive semi-definite kernel. Since for \( z, w \in E \),

$$G(z, w)(\exp(F(z, w)) - 1) = \exp(\|b\|^2 + \|v\|^2)\exp(\langle z, w \rangle) - \exp(\langle \varphi(z), \varphi(w) \rangle),$$

we conclude, using Lemma 2.5 that \( C_\varphi \) is bounded on \( \mathcal{H}(E) \) and

$$\|C_\varphi\| \leq \exp \left( \frac{1}{2}\|b\|^2 + \frac{1}{2}\|v\|^2 \right).$$

(3.6)

This completes the proof of the theorem. \( \square \)

Remark 3.3. If \( \|A\| < 1 \), then the operator \( I - A^*A \) is invertible, hence \((I - A^*A)^{1/2}\) is also invertible and as a result, \( A^*b \) belongs to \((I - A^*A)^{1/2}(E)\) for any \( b \) in \( E \). Theorem 1.3 then shows that \( C_\varphi \) is bounded for any \( \varphi \) of the form \( \varphi(z) = Az + b \). It turns out (by Theorem 1.4) that \( C_\varphi \) is in fact compact.

3.3. The finite-dimensional case. We discuss here the case \( E = \mathbb{C}^n \) for some positive integer \( n \). Suppose \( A \) is a bounded operator on \( \mathbb{C}^n \) with \( \|A\| \leq 1 \) and \( b \) is a vector in \( \mathbb{C}^n \). We claim that \( A^*b \) belongs to the range of \((I - A^*A)^{1/2}\) if and only if \( \langle b, A\zeta \rangle = 0 \) whenever \( \|A\zeta\| = \|\zeta\| \). In fact, for \( \zeta \in \mathbb{C}^n \), we have

$$\|\zeta\|^2 - \|A\zeta\|^2 = \langle \zeta, \zeta \rangle - \langle A^*A\zeta, \zeta \rangle = \langle (I - A^*A)\zeta, \zeta \rangle = \|(I - A^*A)^{1/2}\zeta\|^2.$$ 

Therefore \( \|A\zeta\| = \|\zeta\| \) if and only if \( \zeta \) belongs to \( \ker(I - A^*A)^{1/2} \). This shows that \( \langle b, A\zeta \rangle = 0 \) for all such \( \zeta \) if and only if \( A^*b \) is in the orthogonal complement of \( \ker(I - A^*A)^{1/2} \), which is \( \overline{\text{ran}}(I - A^*A)^{1/2} \). On \( \mathbb{C}^n \), the identity \( \overline{\text{ran}}(I - A^*A)^{1/2} = \text{ran}(I - A^*A)^{1/2} \) holds, so the claim follows. We then recover Theorem 1.1.

Also, by Remark 3.2, the vector \( v \) in (1.2) is characterized by \( v = (I - A^*A)^{1/2}w_0 \) for any \( w_0 \in \mathbb{C}^n \) satisfying \( (I - A^*A)w_0 = A^*b \). It then follows
that
\[ \|v\|^2 + \|b\|^2 = \|(I - A^*A)^{1/2}w_0\|^2 + \|b\|^2 = \|w_0\|^2 - \|A w_0\|^2 + \|b\|^2. \]

Therefore, we recover the norm formula given in Theorem 1.2.

3.4. **The infinite-dimensional case.** As we have seen above, the requirement that \( \langle A \zeta, b \rangle = 0 \) whenever \( \|A \zeta\| = \|\zeta\| \) is equivalent to the requirement that \( A^*b \) belongs to the closure of the range of \( (I - A^*A)^{1/2} \). In the case \( E \) has infinite dimension, this certainly does not imply that \( A^*b \) belongs to the range of \( (I - A^*A)^{1/2} \) and hence, by Theorem 1.3, the composition operator \( C_\varphi \) (with \( \varphi(z) = Az + b \)) may not be bounded on \( \mathcal{H}(E) \).

We provide here a concrete example. Let \( E \) be a separable Hilbert space with an orthonormal basis \( \{v_m : m = 1, 2, \ldots\} \). Let \( A \) be the diagonal operator with \( Av_m = \alpha_m v_m \) where \( (1 - m^{-3})^{1/2} < \alpha_m < 1 \) for all integers \( m \geq 1 \). Put \( b = \sum_{m=1}^{\infty} m^{-1} v_m \), which belongs to \( E \). Since \( \|A \zeta\| = \|\zeta\| \) if and only if \( \zeta = 0 \), we see that the condition \( \langle A \zeta, b \rangle = 0 \) whenever \( \|A \zeta\| = \|\zeta\| \) holds trivially.

Define \( \varphi(z) = Az + b \) for \( z \in E \). We claim that
\[ \sup_{z \in E} (\|\varphi(z)\| - \|z\|^2) = \infty \quad (3.7) \]
and hence, by Lemma 2.5, the operator \( C_\varphi \) is not bounded on \( \mathcal{H}(E) \). For any integer \( m \geq 1 \), put \( t_m = \alpha_m (1 - |\alpha_m|^2)^{-1} m^{-1} \). A simple calculation gives
\[
\|\varphi(t_m v_m)\|^2 - \|t_m v_m\|^2 = \|t_m A v_m + b\|^2 - t_m^2 \geq (t_m \alpha_m + m^{-1})^2 - t_m^2
= -(1 - |\alpha_m|^2) t_m^2 + 2t_m m^{-1} \alpha_m + m^{-2}
= \alpha_m^2 (1 - m^{-2})^2 m^{-2}
= (1 - \alpha_m^2)^{-1} m^{-2}
> m \quad \text{since } (1 - \alpha_m^2) < m^{-3}.
\]
This then gives (3.7).

4. **Compactness of composition operators on \( \mathcal{H}(E) \)**

In this section we characterize mappings \( \varphi \) that induce compact operators \( C_\varphi \) on \( \mathcal{H}(E) \). Before discussing the general case, let us consider first the case \( \varphi(z) = Az \), where \( A \) is a linear operator on \( E \) with \( \|A\| \leq 1 \). In what follows, we will simply write \( C_A \) for \( C_\varphi \).

It turns out that via the anti-unitary \( J \) that we have seen in Proposition 2.1, the operator \( C_A \) has an easy description. Let \( f \) be a continuous \( m \)-homogeneous polynomial on \( E \). Then there is an element \( a_m \in E^m \) such that \( f(z) = \langle z^m, a_m \rangle \) for \( z \in E \). This gives
\[
(C_A f)(z) = \langle (Az)^m, a_m \rangle = \langle A \otimes^m (z^m), a_m \rangle = \langle z^m, (A^*) \otimes^m a_m \rangle,
\]
where \( A \otimes^m \) denotes the tensor product of \( m \) copies of \( A \). We conclude that \( C_A f \) is also a continuous \( m \)-homogeneous polynomial. Therefore, the space
\( \mathcal{P}_m(E) \) of continuous \( m \)-homogeneous polynomials is invariant under \( C_A \) and we have the identity \( C_A \mathcal{P}_m(E) = J^{-1}(A^*)^{\otimes m} J \). This, together with the decomposition (2.4), gives

\[
C_A = J^{-1}(1 \oplus A^* \oplus (A^*)^{\otimes 2} \oplus (A^*)^{\otimes 3} \oplus \cdots) J,
\]

where the sum is an infinite direct sum of operators. The identity (4.1) shows that \( C_A \) is compact if and only if \( (A^*)^{\otimes m} \) is compact for each \( m \geq 1 \) and \( \| (A^*)^{\otimes m} \| \to 0 \) as \( m \to \infty \). Using the fact that \( (A^*)^{\otimes m} \) is compact if and only if \( A^* \) (and hence \( A \)) is compact and the well known identity \( \| (A^*)^{\otimes m} \| = \| A^m \| = \| A \|^m \), we conclude that \( C_A \) is compact if and only if \( A \) is compact and \( \| A \| < 1 \). We have thus proved a special case of Theorem 1.4. A proof of the full version of Theorem 1.4 is given below.

**Proof of Theorem 1.4.** Assume first that \( C_\varphi \) is compact. By Theorem 1.3 there is a linear operator \( A \) on \( E \) with \( \| A \| \leq 1 \) and a vector \( b \in E \) such that \( A^*b \in (I - A^*A)^{1/2}(E) \) and \( \varphi(z) = Az + b \) for all \( z \in E \). We will show that \( A \) is compact and \( \| A \| < 1 \).

Let \( \{ u_m \}_{m=1}^\infty \) be a sequence in \( E \) that converges weakly to zero. For each \( m \), put \( f_m(z) = \langle z, u_m \rangle \) for \( z \in E \). Then \( f_m \to 0 \) weakly as \( m \to \infty \) by Lemma 2.4. This implies that \( \lim_{m \to \infty} \| C_\varphi f_m \| = 0 \). But

\[
(C_\varphi f_m)(z) = f_m(\varphi(z)) = \langle Az + b, u_m \rangle = \langle z, A^*u_m \rangle + \langle b, u_m \rangle,
\]

so \( \| C_\varphi f_m \|^2 = \| A^*u_m \|^2 + \| b, u_m \|^2 \). We then have \( \lim_{m \to \infty} \| A^*u_m \|^2 = 0 \). Therefore, \( A^* \) is a compact operator and hence, \( A \) is also compact.

Suppose it were true that \( \| A \| = 1 \). Then \( \| A^*A \| = 1 \). Since \( A^*A \) is a positive compact operator, 1 is its eigenvalue. So there is a vector \( w \neq 0 \) such that \( A^*Aw = w \), which is equivalent to \( (I - A^*A)^{1/2}w = 0 \). Since \( A^*b \) belongs to the range of \( (I - A^*A)^{1/2} \), we infer that \( \langle w, A^*b \rangle = 0 \), or equivalently, \( \langle Aw, b \rangle = 0 \). For any real number \( r \), the identity \( C_\varphi(K_{rw}) = K_{\varphi(rw)} \) together with a computation reveals

\[
\left\| C_\varphi \left( \frac{K_{rw}}{\| K_{rw} \|} \right) \right\|^2 = \left\| \frac{K_{\varphi(rw)}}{\| K_{rw} \|} \right\|^2 = \exp \left( \| \varphi(rw) \|^2 - \| rw \|^2 \right) = \exp \left( \| Aw + b \|^2 - r^2 \| w \|^2 \right) = \exp(\| b \|^2).
\]

Since \( K_{rw}/\| K_{rw} \| \to 0 \) weakly as \( r \to \infty \) by Lemma 2.4 again, it follows that \( C_\varphi \) is not a compact operator. Hence \( C_\varphi \) is not compact either. This gives a contradiction. Therefore, we have \( \| A \| < 1 \).

Conversely, suppose \( \varphi(z) = Az + b \), where \( A \) is a compact operator on \( E \) with \( \| A \| < 1 \) and \( b \) is an arbitrary vector in \( E \). Choose a positive number \( \alpha \) such that \( \| A \| < \alpha < 1 \). Put \( \varphi_1(z) = \alpha^{-1}Az \) and \( \varphi_2(z) = \alpha z + b \) for \( z \in E \). Then as we have shown above, \( C_{\varphi_1} \) is compact. By Theorem 1.3, \( C_{\varphi_2} \) is bounded. Since \( \varphi = \varphi_2 \circ \varphi_1 \), it follows that \( C_\varphi = C_{\varphi_1}C_{\varphi_2} \) and hence \( C_\varphi \) is a compact operator. \( \square \)
5. Normal, Isometric and Co-isometric Composition Operators

As consequences of Theorem 1.3 we determine in this section the mappings \( \varphi \) that give rise to normal, isometric or co-isometric operators \( C_\varphi \) (recall that an operator on the Hilbert space is called co-isometric if its adjoint is an isometric operator). We will make use of the formulas \( C_\varphi K_0 = K_\varphi(0) \) (by (2.5)) and \( C_\varphi^* C_\varphi K_0 = K_\varphi(0) \), where \( K_0 \equiv 1 \) is the reproducing kernel function of \( \mathcal{H}(E) \) at 0.

We first show that if \( C_\varphi \) is either a normal, isometric or co-isometric operator on \( \mathcal{H}(E) \), then \( \varphi(0) = 0 \). The argument is fairly standard. In fact, if \( C_\varphi \) is normal, then we have \( \| C_\varphi^* K_0 \| = \| C_\varphi K_0 \| \), which gives \( \| K_\varphi(0) \| = \| K_0 \| \). If \( C_\varphi \) is isometric, then \( C_\varphi^* C_\varphi K_0 = K_0 \), which gives \( K_\varphi(0) = K_0 \) and hence, in particular, \( \| K_\varphi(0) \| = \| K_0 \| \). If \( C_\varphi \) is co-isometric then we also have \( \| K_0 \| = \| C_\varphi^* K_0 \| = \| K_\varphi(0) \| \). Since \( \| K_\varphi(0) \|^2 = \exp(-\| \varphi(0) \|^2) \) and \( \| K_0 \|^2 = 1 \), we conclude that in each of the above cases, \( \varphi(0) = 0 \).

Now since \( \varphi(0) = 0 \), Theorem 1.3 shows that \( \varphi(z) = Az \) for some operator \( A \) on \( E \) with \( \| A \| \leq 1 \). Then \( C_\varphi = C_A \) and \( C_\varphi^* = C_A^* \) and hence

\[
C_\varphi^* C_\varphi = C_A A^* = C_{AA^*} \quad \text{and} \quad C_\varphi C_\varphi^* = C_A C_A^* = C_{A^*A}.
\]

We obtain

**Proposition 5.1.** Let \( \varphi \) be a mapping on \( E \) such that \( C_\varphi \) is a bounded operator on \( \mathcal{H}(E) \).

(a) \( C_\varphi \) is normal if and only if there exists a normal operator \( A \) on \( E \) with \( \| A \| \leq 1 \) such that \( \varphi(z) = Az \) for all \( z \in E \).

(b) \( C_\varphi \) is isometric if and only if there exists a co-isometric operator \( A \) on \( E \) such that \( \varphi(z) = Az \) for all \( z \in E \).

(c) \( C_\varphi \) is co-isometric if and only if there exists an isometric operator \( A \) on \( E \) such that \( \varphi(z) = Az \) for all \( z \in E \).

**Remark 5.2.** Statement (a) in Proposition 5.1 holds also for composition operators on the Hardy and Bergman spaces of the unit ball (see [6, Theorem 8.1]), where an analogous result to Theorem 1.3 is not available. (In fact, on the Hardy and Bergman spaces, mappings that are not affine can still give rise to bounded composition operators.) The proof of [6, Theorem 8.1] can be adapted to prove Proposition 5.1 (a) without appealing to Theorem 1.3 in the case \( E \) has finite dimension. On the other hand, since that proof relies on the finiteness of the dimension, it does not seem to work when \( E \) has infinite dimension.

**Remark 5.3.** In the case \( E = \mathbb{C}^n \) for some positive integer \( n \), isometric operators on \( E \) are also co-isometric and vice versa, and all these operators are unitary. Statements (b) and (c) in Proposition 5.1 then imply that \( C_\varphi \) is isometric on \( F_n \) if and only if it is co-isometric if and only if it is unitary.
COMPOSITION OPERATORS

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