ABSTRACT: The Dirac equation is solved in the Einstein-Yang-Mills background found by Bartnik and McKinnon. We find a normalizable zero-energy fermion mode in the s-wave sector. As shown recently, their solution corresponds to a gravitational sphaleron which mediates transitions between topologically distinct vacua. Since the Bartnik-McKinnon solution is unstable, it will either collapse to form a black hole or radiate away its energy. In either case, as the Chern-Simons number of the configuration changes, there will be an accompanying anomalous change in fermion number.
1. Introduction

Neither the Yang-Mills nor Einstein field equations admit static finite-energy, non-singular solutions. However, such particle-like solutions have been found by Bartnik and McKinnon \[1\] in the combined Einstein-Yang-Mills theory. These solutions were later interpreted as sphalerons, that is, static saddlepoint solutions lying at the top of an energy barrier in field configuration space separating vacua with different Chern-Simons number \[2\] [3]. This, in particular, accounts for their instability. If the sphaleron is perturbed, it will either radiate its energy to infinity or collapse to form a black hole. Since either process involves a change in Chern-Simons number, one expects an equal anomalous change in chiral fermion number. In this paper we initiate a study of this problem. In Section 2, the Bartnik-McKinnon solution and its interpretation as a sphaleron is reviewed. In Sections 3-5, the Dirac equation in the background of the static Bartnik-McKinnon sphaleron is analyzed. We find a zero energy bound state in the $s$-wave sector. In Section 6, a Higgs field is included. In Section 7, the conformal invariance of the Dirac equation is exploited to prove a general no-hair theorem for fermions.

2. Bartnik-McKinnon Solutions

In \[1\] a class of spherically symmetric particle-like solutions to the Einstein-Yang-Mills equations with $SU(2)$ gauge group were found numerically. It was shown that there are an infinite sequence of solutions $(A_n(r), B_n(r), K_n(r))$ for the metric and gauge field

$$ds^2 = -A^2dt^2 + B^2dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$A_i = \frac{1 - K}{2gr} \epsilon_{ijk} n^j \tau^k \quad *$$

where $g$ is the gauge coupling constant, $\tau^i$ are the generators of $SU(2)$, and $\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. $K_n$ has $n$ nodes and satisfies the boundary condition $K_n(0) = 1$ and $K_n(\infty) = (-1)^n$. At large distances, the metric approaches Schwarzschild and the gauge field strength decays as $1/r^3$ with $A_i$ approaching the pure gauge $A_i = -\frac{i}{g} \partial_i U U^{-1}$ where $U = 1$ for $n$ even and $U = -i\vec{n} \cdot \vec{\tau}$ for $n$ odd. Near a node $r = r_0$ ($K(r_0) = 0$), the gauge field corresponds to a $P = 1/g$ Dirac monopole, and the metric to extremal Reissner-Nordstrom. The masses of the Bartnik-McKinnon

* This form of the gauge field is gauge equivalent to that used in \[1\].
solutions are proportional to the only mass scale in the problem, \( g^{-1} G^{-1/2} \), where \( G \) is the gravitational constant. The masses increase with \( n \) approaching the mass of the extremal Reissner-Nordstrom black hole with magnetic charge, \( P = 1/g \) as \( n \to \infty \). Heuristically, in this limit, \( K_n \) fluctuates rapidly approaching its mean zero value corresponding to the Dirac monopole. The existence of the Bartnik-McKinnon solutions was subsequently established rigourously \[4\], and generalized to include horizons \[3\].

Soon after their discovery, it was shown that both particle-like and black hole solutions are unstable \[3\]. In fact, the Bartnik-McKinnon solutions correspond to a gravitational analog of the sphaleron \[2\][3]. We recall that sphalerons \[7\] are static saddle-point solutions that lie at the top of an energy barrier separating topologically distinct vacua. For asymptotically flat spacetimes, the ADM energy provides a positive definite mass functional on field configuration space. The zero energy “vacua” are flat spacetime metrics and pure gauge Yang-Mills configurations \( A_i = -\frac{i}{g} \partial_i U U^{-1} \). Each pure gauge yields a map from space into the group manifold of \( SU(2) \). Demanding \( U \to 1 \), this becomes a map \( U : S^3 \to S^3 \). Since maps with different winding, or Chern-Simons, numbers cannot be continuously deformed into one another, paths in configuration space connecting these vacua must enter non-vacuum regions, and therefore by the positive definiteness of the mass functional must traverse an energy barrier. * Each such path has a maximal energy configuration. Among these configurations the one with minimum energy will be an extremum of the mass functional and therefore correspond to an unstable saddlepoint solution. By considering paths connecting the vacuum sector with Chern-Simons number zero and the sector with Chern-Simons number one, one obtains the Bartnik-McKinnon solution \( K_1 \). One can see by symmetry that its Chern-Simons number is \( 1/2 \). Repeating this procedure, with paths connecting \( K_1 \) and its gauge equivalent solution with Chern-Simons number \( -1/2 \), the \( K_2 \) solution with zero Chern-Simons number is obtained. Continuing in this way, the entire Bartnik-McKinnon sequence is produced. This procedure also yields the black hole solutions if one considers asymptotically flat metrics with fixed horizon area.

As the fields evolve dynamically along a path in configuration space connecting vacua with different Chern-Simons numbers and passing through the sphaleron, anomalous fermion production will occur \[8\]. The total number of chiral fermions created equals

\* We shall regard vacua which are related by so called large gauge transformations as distinct. One can, if one wishes, identify them, in which case the paths we refer to below become closed non-contractible loops in this identified configuration space.
the change in the Chern-Simons number of the field configuration. As one passes between
topologically distinct vacua, a fermion state will emerge from the negative energy Dirac
sea, enter the discrete spectrum, and finally merge with the positive energy continuum.
By symmetry, therefore, there should be a zero energy bound state at the midpoint of this
path where the sphaleron configuration is located. If perturbed, the sphaleron will fall
in one of two directions. The Yang-Mills field will dominate, in which case the sphaleron
will radiate away its energy to infinity. The fermion zero mode would then also escape
to infinity perhaps after being reflected at the origin. Alternatively, gravity may induce
the object to collapse to a black hole swallowing the Yang-Mills field, and presumably the
fermion mode as well.

3. Dirac Equation

We now proceed to test this picture by solving the Dirac equation in the sphaleron
background. As expected from the above discussion, we find a zero-energy bound state
in the $s$-wave sector. We also derive the full set of time dependent radial equations. To
determine more precisely the fate of the zero mode if the sphaleron collapses to a black
hole would require numerically solving the time dependent equations. Such an analysis
was done for the Yang-Mills-Higgs sphaleron in [9][10].

The massless Dirac equation in an Einstein-Yang-Mills background for an isodoublet
fermion is given by

$$i\gamma^\mu(\nabla_\mu - i g A_\mu)\Psi = 0$$

(3.1) where $\nabla_\mu$ is the covariant derivative $\nabla_\mu = \partial_\mu - \frac{1}{4} \omega^{ab}_{\mu} \gamma_a \gamma_b$. $\mu$ and $a$ are tangent and
spacetime indices respectively and are related by $e^a_\mu \equiv e^a$, a basis of orthonormal one-
forms. $\omega^{ab}_\mu \equiv \omega^{ab}$ are the associated connection one-forms obeying $de^a + \omega^a_b \wedge e^b = 0$, and
$\gamma^a$ are Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$ with $\eta^{00} = -1$. The metric and gauge
field are given in (2.1) and (2.2) where $\tau^i$ are now the Pauli spin matrices. In terms of the
orthonormal basis

$$e^0 = A dt \quad e^r = B dr \quad e^\theta = r d\theta \quad e^\phi = r \sin \theta d\phi$$

(3.2)

the connection one-forms are

$$\omega^{0r} = B^{-1} A' dt \quad \omega^{\theta r} = B^{-1} d\theta \quad \omega^{\phi r} = B^{-1} \sin \theta d\phi \quad \omega^{\phi \theta} = \cos \theta d\phi$$

(3.3)
where $t \equiv \frac{\partial}{\partial r}$. In the chiral representation, the gamma matrices are

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

(3.4)

where $\sigma^i$ are Pauli spin matrices. $\Psi$ can be decomposed into its left and right chiral components $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ where $\psi_L(R)$ carry two-component Lorentz and isospin indices. Since the Dirac equation is massless, the two chiralities decouple. In the following, we consider just the right-handed component $\psi \equiv \psi_R$. The Dirac equation then becomes

$$
\frac{\partial \psi}{\partial t} + \frac{\vec{\sigma} \cdot \vec{n}}{r} A^{1/2} \frac{\partial}{\partial r} (r A^{1/2} \psi) + \frac{A}{r} D_T \psi - ig A \vec{\sigma} \cdot \vec{A} \psi = 0
$$

(3.5)

where $D_T$ is the Dirac operator on the unit two sphere and $\vec{a} \cdot \vec{b} \equiv a^i b^i$. Eqn. (2.2) implies $\vec{\sigma} \cdot \vec{A} = \frac{(K-1)}{2gr} (\vec{n} \cdot \vec{\sigma} \times \vec{\tau})$. The conserved inner product is given by

$$
\langle \psi_1 | \psi_2 \rangle = \int \psi_1^\dagger \psi_2 Br^2 dr d\Omega.
$$

(3.6)

We now discuss the effect of charge conjugation, $C$, and parity, $P$ on the sphaleron. The action for a Dirac fermion coupled to a Yang-Mills field with arbitrary gauge group is invariant under $C$ and $P$ separately. Under $C$, the fields transform as

$$
\Psi \rightarrow \Psi^C = \gamma^2 \Psi^* \\
A_\mu \rightarrow A^C_\mu = -A^*_\mu
$$

(3.7)

and under $P$ as

$$
\Psi(x^i, t) \rightarrow \Psi^P = \gamma^0 \Psi(-x^i, t) \\
A_i(x^i, t) \rightarrow A^P_i = -A_i(-x^i, t) \\
A_0(x^i, t) \rightarrow A^P_0 = A_0(-x^i, t).
$$

(3.8)

The metric being real and spherically symmetric is invariant under $C$ and $P$. Because $C$ and $P$ interchange chirality, the action for chiral fermions is only invariant under the combined $CP$ transformation. Using (3.7), (3.8), and $-\tau^{i*} = \tau^2 \tau^i \tau^2$, the gauge field $A_i$ transforms as $A_i \rightarrow A^CP_i = \tau^2 A_i \tau^2$. The $CP$ invariance of the theory then implies that given a solution $\Psi$ to the Dirac equation in the sphaleron background with energy $E$, $\tau^2 \Psi^{CP}$ is a solution in the same background, but with energy $-E$. 

4
4. S-wave Sector and Zero Mode

We now solve the Dirac equation (3.5) by separating variables. Since the total angular momentum $\vec{K} = \vec{L} + \vec{S} + \vec{T}$ commutes with the Hamiltonian, $\psi$ can be expanded in eigenstates of $K^2$ and $K^z$ with eigenvalues $k$ and $m$. $\vec{L}$, $\vec{S} = \vec{\sigma}/2$, and $\vec{T} = \vec{\tau}/2$ are the orbital angular momentum, spin, and isospin. The $s$-wave sector corresponding to $k = m = 0$ is spanned by the two states $\chi_1$ and $\chi_2 \equiv \vec{\sigma} \cdot \vec{n} \chi_1$ where

$$\chi_1 = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_S \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_T - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_S \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_T \right]$$

is the hedgehog spinor satisfying $(\vec{\sigma} + \vec{\tau}) \chi_1 = 0$. The action of the various operators in the Dirac equation on the two states can be determined from the hedgehog property and the spin commutation relations. The transverse Dirac operator can be written as

$$D_T = (2\vec{S} \cdot \vec{\sigma} + 1)(\vec{\sigma} \cdot \vec{n}) = (J^2 - \vec{L}^2 - S^2 + 1)(\vec{\sigma} \cdot \vec{n})$$

where $\vec{J} \equiv \vec{L} + \vec{S}$. Since $[J^2, \vec{\sigma} \cdot \vec{n}] = 0$ and $\vec{\sigma} \cdot \vec{n}$ changes the value of the orbital angular momentum by one, we find that $D_T \chi_1 = -\chi_2$ and $D_T \chi_2 = \chi_1$. The operator $\vec{n} \cdot \vec{\sigma} \times \vec{\tau}$ appearing in the gauge field term in the Dirac equation also interchanges $\chi_1$ and $\chi_2$: $(\vec{n} \cdot \vec{\sigma} \times \vec{\tau}) \chi_1 = -2i \chi_2$ and $(\vec{n} \cdot \vec{\sigma} \times \vec{\tau}) \chi_2 = 2i \chi_1$.

Thus, (3.5) becomes

$$\begin{align*}
\frac{\partial f}{\partial t} + \frac{\partial g}{\partial r^*} + \frac{AK}{r} g &= 0 \\
\frac{\partial g}{\partial t} + \frac{\partial f}{\partial r^*} - \frac{AK}{r} f &= 0
\end{align*}$$

(4.2)

where $\psi = r^{-1} A^{-1/2} f \chi_1 + r^{-1} A^{-1/2} g \chi_2$ and $r^*$ is the “tortoise” coordinate satisfying $\frac{dr^*}{dr} = A^{-1} B$.

Eqn. (4.2) admits a zero energy bound state of the form

$$f = \exp \int_{r_0}^{r} B \frac{K}{r} dr, \quad g = 0$$

(4.3)

for $K = K_n$ with $n$ odd. For $n$ even, $K \rightarrow 1$ asymptotically, and therefore $f$ diverges. The zero mode (4.3) vanishes at $r = 0$ and falls off as $1/r$ asymptotically. The charge density of the wave function from (3.6) is given by $4\pi A^{-1} B |f|^2$. The density of the zero mode (4.3) is peaked in the monopole region, $K = 0$, which is effectively acting as a potential well. For a spacetime with horizon at $r = r_H$, the metric functions behave as $A \sim (r - r_H)^{1/2}(1 + O(r - r_H))$ and $B \sim (r - r_H)^{-1/2}(1 + O(r - r_H))$. From (4.3) we observe that the wavefunction diverges as $(r - r_H)^{-1/4}$, and its norm logarithmically. This is perhaps to be anticipated in light of the no-hair theorem for fermions. The effective potential within
the s-wave sector is found by rewriting the two coupled first order equations as decoupled second order equations

\[ \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r^2} + V(r) f = 0, \quad \dot{g} = -\frac{\partial f}{\partial r^*} + Hf \]  

(4.4)

where

\[ V(r) = \frac{\partial H}{\partial r^*} + H^2, \quad H \equiv A \frac{K}{r}. \]  

(4.5)

Near the horizon, the effective potential vanishes as \( V \sim (r - r_H)^{1/2} \).

5. \( k > 0 \) Modes

The eigenspaces with eigenvalues \( k \) and \( m \) are four-dimensional for \( k \geq 1 \). Since \( L^2 \) and \( K^i \) commute, they can be simultaneously diagonalized. Within the four dimensional eigenspace, there are two states denoted \( |k, m, \pm > \) with \( L^2 \) eigenvalues \( l = k \pm 1 \), and an orthogonal two-dimensional space with eigenvalue \( l = k \). In addition, either \( J^2 = (L + S)^2 \) or \( R^2 \equiv (S + T)^2 \) can be diagonalized, but not simultaneously since they do not commute. The eigenstates of \( J^2 \) are \( |k, m, \pm > \) with eigenvalues \( j = k \pm 1/2 \) and \( |k, m, j_\pm > \), lying in the \( l = k \) subspace, with eigenvalues \( j = k \pm 1/2 \). The eigenstates of \( R^2 \) are \( |k, m, \pm > \) both with eigenvalues \( r = 1 \) and \( |k, m, 0 > \) and \( |k, m, 1 > \), lying in the \( l = k \) subspace, with eigenvalues \( r = 0 \) and 1. The \( J^2 \) and \( R^2 \) eigenstates in the \( l = k \) subspace are related by a rotation matrix of angle \( \xi \) with \( \tan \xi = \sqrt{\frac{k}{k+1}} : \)

\[
|k, m, j_+ > = \frac{1}{\sqrt{2k+1}} \left( (\sqrt{k+1}|k, m, 0 > + \sqrt{k}|k, m, 1 >) \right) \\
|k, m, j_- > = \frac{1}{\sqrt{2k+1}} \left( -\sqrt{k}|k, m, 0 > + \sqrt{k+1}|k, m, 1 >) \right). 
\]  

(5.1)
The $R^2$ eigenstates when expressed in terms of spherical harmonics take the form:

$$ |k, m, +> = \frac{1}{\sqrt{(2k + 2)(2k + 3)}} \left[ \sqrt{(k + m + 1)(k + m + 2)} Y_{k+1}^{m+1} |1, -1 > - \sqrt{2(k + m + 1)(k - m + 1)} Y_{k}^{m} |1, 0 > + \sqrt{(k - m + 1)(k + m + 2)} Y_{k}^{m-1} |1, 1 > \right] $$

$$ |k, m, -> = \frac{1}{\sqrt{2k(2k - 1)}} \left[ \sqrt{(k - m)(k - m - 1)} Y_{k-1}^{m+1} |1, -1 > + \sqrt{2(k + m)(k - m)} Y_{k-1}^{m} |1, 0 > + \sqrt{(k + m - 1)(k + m)} Y_{k-1}^{m-1} |1, 1 > \right] $$

$$ |k, m, 0> = Y_k^m |0, 0 > $$

$$ |k, m, 1> = \frac{1}{\sqrt{2k(k + 1)}} \left[ \sqrt{(k - m)(k + m + 1)} Y_{k}^{m+1} |1, -1 > + m\sqrt{2} Y_{k}^{m} |1, 0 > - \sqrt{(k + m)(k - m + 1)} Y_{k-1}^{m-1} |1, 1 > \right] $$

(5.2)

where $|1, 1 >$, $|1, 0 >$, $|1, -1 >$ are the spin-one triplet of $R^2$.

We now proceed to express the various operators appearing in the Dirac equation (3.5) as matrices in the basis ($|k, m, +>$, $|k, m, j_+>$, $|k, m, j_->$, $|k, m, ->$) in which $K^2$, $K_z$, $L^2$, and $J^2$ are diagonalized. Since $\vec{\sigma} \cdot \vec{n}$ commutes with $J^2$ and squares to unity, after appropriate normalization of states, one finds that $\vec{\sigma} \cdot \vec{n} |k, m, \pm> = \pm |k, m, j_\pm>$. Using this result, one can show that the transverse Dirac operator $\mathcal{D}_T = (J^2 - L^2 - S^2 + 1)\vec{\sigma} \cdot \vec{n}$ takes the form

$$ \mathcal{D}_T = \begin{pmatrix}
0 & -k - 1 & 0 & 0 \\
-k - 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -k \\
0 & 0 & k & 0 \\
\end{pmatrix}. $$

(5.3)

Finally, one can determine the matrix form of the operator $\vec{n} \cdot \vec{\sigma} \times \vec{\tau}$ appearing in the gauge field term in the Dirac equation (3.5). Since it is antisymmetric in $\vec{\sigma}$ and $\vec{\tau}$, it must interchange the $r = 0, 1$ states, and because of the factor $\vec{n}$, it changes the orbital angular momentum by one. Further calculation shows

$$ \vec{n} \cdot \vec{\sigma} \times \vec{\tau} = \frac{2i}{2k + 1} \begin{pmatrix}
0 & k + 1 & -\sqrt{k(k + 1)} & 0 \\
-k - 1 & 0 & 0 & \sqrt{k(k + 1)} \\
\sqrt{k(k + 1)} & 0 & 0 & k \\
0 & -\sqrt{k(k + 1)} & k & 0 \\
\end{pmatrix}. $$

(5.4)

Substituting (5.3) and (5.4) in (3.5), the Dirac equation reduces to four coupled linear first-order equations. It appears that higher mode zero energy bound states are forbidden.
since the wave function diverges at \( r = 0 \). Near the horizon, \( r = r_H \), \( A = \alpha(r - r_H)^{1/2}(1 + O(r - r_H)) \) and \( B = \beta(r - r_H)^{-1/2}(1 + O(r - r_H)) \) while \( K \) is constant. Therefore, to leading order in \( r - r_H \), the Dirac equation becomes

\[
\dot{\psi} + \frac{\alpha}{\beta} \vec{\sigma} \cdot \vec{n} [(r - r_H)\psi' + \frac{1}{4} \psi] = 0
\]

(5.5)

with solutions \( \psi \sim e^{-iEt}(r - r_h)^{(\pm iE\beta/\alpha - 1/4)} \) which as before diverge at the horizon.

### 6. Higgs Field

In this section, we consider the effect of a Higgs field. The Dirac equation then becomes

\[
i\gamma^\mu (\nabla_\mu - ig A_\mu) \Psi - \Phi \Psi = 0
\]

(6.1)

where \( \Phi = \phi^i \tau^i \) is the Higgs field in the adjoint representation. We have absorbed the Yukawa coupling constant in \( \Phi \). Consider the following spherically symmetric ansatz for \( \Phi \)

\[
\phi^i = F(r) n^i.
\]

(6.2)

Since the two chiralities of the fermion, \( \psi_L \) and \( \psi_R \), no longer decouple, the Dirac equation now reduces to four coupled first order equations in the s-wave sector and to eight in the higher mode sector. Using the fact that \( \vec{\tau} \cdot \vec{n} \chi_1 = -\chi_2 \) and \( \vec{\tau} \cdot \vec{n} \chi_2 = -\chi_1 \), we find that the s-wave equations become

\[
\begin{align*}
\frac{\partial f_R(L)}{\partial t} &\pm \left[ \frac{\partial g_R(L)}{\partial r^*} + \frac{\partial g_R(L)}{\partial r} \right] - iAFg_R(L) = 0 \\
\frac{\partial f_R(L)}{\partial t} &\pm \left[ \frac{\partial f_R(L)}{\partial r^*} - \frac{\partial f_R(L)}{\partial r} \right] - iAFf_R(L) = 0
\end{align*}
\]

(6.3)

where \( \Psi_R(L) = r^{-1/2}f_R(L)\chi_1 + r^{-1/2}g_R(L)\chi_2 \). There is in fact still a zero-energy bound state solution to these equations:

\[
f_R = -i f_L = \exp \int_{r_0}^{r} B(\frac{K}{r} - F) dr, \quad g_R = g_L = 0.
\]

(6.4)

Moreover, since the Higgs field causes the wave-function to decay exponentially at infinity, \( \Psi \sim e^{-F(\infty)r} \), there is a bound state for both odd and even \( n \). As before, for a space-time with horizon the zero mode diverges there. The s-wave equations can be written as decoupled second order equations

\[
\begin{align*}
\frac{\partial^2 f_R}{\partial t^2} - \frac{\partial^2 f_R}{\partial r^{*2}} + V(r)f_R &= 0, \quad f_R = i f_R, \quad \dot{g}_R = i g_R = -\frac{\partial f_R}{\partial r^*} + H f_R
\end{align*}
\]

(6.5)
where
\[ V(r) = \frac{\partial H}{\partial r^*} + H^2, \quad H \equiv A\left(\frac{K}{r} - F\right). \quad (6.6) \]

The equations for the higher modes can be found as well. The matrix associated with the operator \( \vec{\tau} \cdot \vec{n} \) appearing in the Higgs field term is obtained from \( \vec{\sigma} \cdot \vec{n} \) by exchanging \( \sigma \) and \( \tau \) and using the fact that the \( r = 0, 1 \) states are antisymmetric and symmetric in \( \sigma \) and \( \tau \) respectively:

\[
\vec{\tau} \cdot \vec{n} = \begin{pmatrix}
0 & -\cos 2\xi & \sin 2\xi & 0 \\
-\cos 2\xi & 0 & 0 & \sin 2\xi \\
\sin 2\xi & 0 & 0 & \cos 2\xi \\
0 & \sin 2\xi & \cos 2\xi & 0
\end{pmatrix} \tag{6.7}
\]

where \( \tan \xi = \sqrt{\frac{k}{k+1}} \).

7. Fermion No-Hair Theorem

Assuming that the endpoint of gravitational collapse is a stationary black hole, then the uniqueness theorems imply it must be Kerr-Newman. In general, the no-hair theorems assert that the only external fields that a black hole can generate are those yielding conserved charges in the form of a surface integral at infinity. If one attempts to find a neighbouring solution with hair, the solution for the perturbation will necessarily diverge at the horizon of the black hole (assuming it vanishes at spatial infinity.) We should point out that a Yang-Mills black hole such as the one discussed earlier does not unambiguously violate the no-hair theorems since the non-linear nature of the field allows one to view it and not the black hole as being the source of the external Yang-Mills field. * After all, one would certainly ascribe the complicated gravitational field of an accretion disk surrounding a black hole to the disk and not to the black hole. The higher multipoles moments of the gravitational field can be unambiguously identified with the higher moments of the matter density distribution of the disk. The Yang-Mills field, however, does not have well defined multipole moments. In particular, attempts to construct a total charge by integrating the Yang-Mills magnetic field does not yield a gauge invariant quantity since there is a free uncontracted group index.

The fact that the zero mode (4.3) (6.4) diverges on the horizon is expected from these no-hair theorems. In this section, we prove a general no-hair theorem for fermions.

* In fact, Einstein-Skyrme black holes with Yang-Mills hair have been constructed recently[11], and are stable under linear perturbations[12].
This has already been done for Schwarzschild in [13] and [14] by employing the conformal invariance of the Dirac equation, we show that all static solutions in a spherically symmetric spacetime diverge at the horizon. Any static, spherically symmetric spacetime may be written in isotropic coordinates

$$ds^2 = -V^2 dt^2 + W^2 dx \cdot dx$$ (7.1)

where $V$ and $W$ are functions of $|x|$. (This ansatz is, in fact, more general than being spherically symmetric applying to multi-black hole metrics as well.) Consider the gauged Dirac equation

$$i\gamma^\mu(\nabla_\mu - igA_\mu)\Psi - m\Psi = 0$$ (7.2)

where $m$ might depend on a Higgs field and hence on position. In $d$ spacetime dimensions we have the following result: if $(\Psi, g_{\mu\nu}, A_\mu, m)$ is a solution of (7.2), then $(\Omega^{d-1\over 2}\Psi, \Omega^{-2}g_{\mu\nu}, A_\mu, \Omega m)$ is also a solution. For the metric (7.1) with $A_0 = 0$, this implies that $V^{3/2}\Psi$ solves (7.2) with mass term $Vm$ in the ultrastatic optical metric

$$ds^2 = -dt^2 + W^2 V^2 dx \cdot dx.$$ (7.3)

As is the case for no-hair theorems, we are interested in time-independent solutions to the Dirac equation. Applying conformal invariance again, but now to the spatial part of the metric implies that $\chi \equiv V^{1/2}W\Psi$ solves the flat three-dimensional gauged Dirac equation with mass term $mW$.

For $m = A_i = 0$ the flat space Dirac equation may be solved in a variety of ways, but perhaps the most illuminating from the present conformal viewpoint is to note that in flat Euclidean $n$-space,

$$\chi^\alpha = T^\alpha_{i_1i_2...i_l}x^{i_1}x^{i_2}...x^{i_l}$$ (7.4)

clearly solves the flat space Dirac equation with $m = A_i = 0$ provided $T^\alpha_{i_1...k}$ satisfies

$$\gamma^{i_\alpha}T^\beta_{i_1...k} = 0.$$ (7.5)

Solutions which vanish at spatial infinity can be obtained from (7.4) by an inversion. To each static solution $\chi(x^i)$ to the flat space Dirac equation in $n$ spatial dimensions, there is an inverted solution $\gamma^{i_\alpha}x^{i_\alpha} \chi(x^i/n^2)$. Applying this to (7.4), we obtain the general multipole solution

$$\chi = \gamma_i n^i T_{i_1i_2...i_l}n^{i_1}n^{i_2}...n^{i_l}$$ (7.6)
where $n^i = x^i / r$. Thus, if $m = A_i = 0$, the general solution in the spacetime (7.1) which decays at infinity is given by

$$\Psi = V^{-1/2} W^{-1} \frac{\gamma_i n^i}{r^{l+2}} T_1^{i_1} \ldots i_{i_l} n^{i_{l+1}} \ldots n^{i_l}.$$  \hfill (7.7)

It is now clear that $\Psi$ will blow up at a non-extreme horizon for which $V = 0$ for $r > 0$. In the extreme case for a regular horizon where $W \to r^{-1}$ and $V \to r$, we see that $\Psi$ still blows up for any permitted value of $l$.

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