Ramanujan Graphs and the Solution of the
Kadison–Singer Problem

Adam W. Marcus∗ Daniel A. Spielman† Nikhil Srivastava

Abstract. We survey the techniques used in our recent resolution of the Kadison–
Singer problem and proof of existence of Ramanujan Graphs of every degree: mixed
characteristic polynomials and the method of interlacing families of polynomials. To
demonstrate the method of interlacing families of polynomials, we give a simple proof of
Bourgain and Tzafriri’s restricted invertibility principle in the isotropic case.

Mathematics Subject Classification (2010). Primary, 05C50, 46L05; Secondary,
26C10.

Keywords. Interlacing polynomials, Kadison–Singer, mixed characteristic polynomials,
Ramanujan graphs, mixed discriminants, restricted invertibility.

1. Introduction

In a recent pair of papers [30, 31], we prove the existence of infinite families of bipar-
tite Ramanujan graphs of every degree and we affirmatively resolve the Kadison–
Singer Problem. The techniques that we use in the papers are closely related. In
both we must show that certain families of matrices contain particular matrices of
small norm. In both cases, we prove this through a new technique that we call the
method of interlacing families of polynomials. In the present survey, we review this
technique and the polynomials that we analyze with it, the mixed characteristic
polynomials.

We begin by defining Ramanujan Graphs, explaining the Kadison–Singer Prob-
lem, and explaining how these problems are related. In particular, we connect the
two by demonstrating how they are both related to the problem of sparsifying
graphs.

1.1. Ramanujan Graphs. Let \( G \) be an undirected graph with vertex set
\( V \) and edge set \( E \). The adjacency matrix of \( G \) is the symmetric matrix \( A \) whose

∗Research partially supported by an NSF Mathematical Sciences Postdoctoral Research Fel-
lowship, Grant No. DMS-0902962.
†Research partially supported by NSF grants CCF-0915487 and CCF-1111257, a Simons In-
vestigator Award, and a MacArthur Fellowship.
rows and columns are indexed by vertices in \( V \) with entries

\[
A(a, b) = \begin{cases} 
1 & \text{if } (a, b) \in E \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( A \) is symmetric it has \(|V|\) real eigenvalues, which we will also refer to as the eigenvalues of \( G \).

Consider a function \( f : V \rightarrow \mathbb{R} \). Multiplication by \( A \) corresponds to the operator that replaces the value of \( f \) at a given vertex with the sum of the values at its neighbors in \( G \). In this way, \( A \) is related to random walks and diffusion on \( G \).

It is well known that the speed of the convergence of these processes is determined by the eigenvalues of \( A \) and related matrices.

We will restrict our attention to graphs that are connected and \( d \)-regular. When \(|V|\) is finite, it is easy to check that every such graph has an eigenvalue of \( d \) corresponding to the eigenvector of all 1’s. Furthermore, in the case that \( G \) is bipartite, one can check that the eigenvalues of \( A \) are symmetric about the origin. Thus every finite bipartite \( d \)-regular graph must also have an eigenvalue of \(-d\). Because these eigenvalues are unavoidable (they are an artifact of being finite), they are often referred to as the trivial eigenvalues.

The graphs on which random walks mix the fastest are those whose non-trivial eigenvalues are as small as possible. An infinite family of connected \( d \)-regular graphs all of whose non-trivial eigenvalues are at most \( \alpha \) for some constant \( \alpha < d \) is called a family of expander graphs. Constructing \( d \)-regular expanders with a small number of vertices (relative to \( d \)) is easy: for example, the complete graph on \( d + 1 \) vertices has all non-trivial eigenvalues \(-1\) and the complete bipartite graph with \( 2d \) vertices has all non-trivial eigenvalues \( 0 \). The interesting problem is to construct \( d \)-regular expanders with an arbitrarily large number of vertices. Margulis \[32\] was the first to find an explicit construction of such an infinite family.

Expander graphs have proved to be incredibly useful in a variety of contexts. We refer the reader who is interested in learning more about expander graphs, with a focus on their applications in computer science, to the survey of Hoory, Linial, and Wigderson \[25\]. Many applications of expanders depend upon the magnitudes of their non-trivial eigenvalues. A theorem of Alon and Boppana provides a bound on how small the non-trivial eigenvalues can be.

**Theorem 1.1** (\[3, 35\]). For every integer \( d \geq 3 \) and every \( \epsilon > 0 \), there exists an \( n_0 \) so that every \( d \)-regular graph \( G \) with more than \( n_0 \) vertices has a non-trivial eigenvalue that is greater than \( 2\sqrt{d - 1} - \epsilon \).

The number \( 2\sqrt{d - 1} \) in Theorem 1.1 has a meaning: it is the spectral radius of the infinite \( d \)-regular tree, whose spectrum is the closed interval \([-2\sqrt{d - 1}, 2\sqrt{d - 1}]\) (it has no trivial eigenvalues because it is not finite) \[25\]. Since Theorem 1.1 says that no infinite family of \( d \)-regular graphs can have eigenvalues that are asymptotically smaller than \( 2\sqrt{d - 1} \), we may view this infinite tree as being the “ideal” expander. A natural question is whether there exist infinite families of finite \( d \)-regular graphs whose eigenvalues are actually as small as those of the tree.
Lubotzky, Phillips and Sarnak [29] and Margulis [33] were the first to construct infinite families of such graphs. Their constructions were Cayley graphs, and they exploited the algebraic properties of the underlying groups to prove that all of the nontrivial eigenvalues of their graphs have absolute value at most \(2\sqrt{d-1}\). Their proofs required the proof of the Ramanujan Conjecture, and so they named the graphs they obtained *Ramanujan graphs*. As of 2013, all known infinite families of Ramanujan graphs were obtained via constructions similar to [29, 33]. As a result, all known families of Ramanujan graphs had degree \(p^k+1\) for \(p\) a prime and \(k\) a positive integer.

The main theorem of [30] is that there exist infinite families of \(d\)-regular bipartite Ramanujan graphs for every integer \(d \geq 3\). This is achieved by proving a variant of a conjecture of Bilu and Linial [9], which implies that every \(d\)-regular Ramanujan graph has a \(2\)–cover which is also Ramanujan, immediately establishing the existence of an infinite sequence. In contrast to previous results, the proof is completely elementary, and we will sketch most of it in this survey.

Bilu and Linial’s conjecture is a purely linear algebraic statement about *signings* of adjacency matrices. To define a signing, recall that we can write the adjacency matrix of any graph \(G = (V, E)\) as

\[
A = \sum_{(a,b) \in E} A_{(a,b)},
\]

where \(A_{(a,b)}\) is the adjacency matrix of a single edge \((a,b)\). Then, a signing is any matrix of the form

\[
\sum_{(a,b) \in E} s_{(a,b)}A_{(a,b)},
\]

where \(s_{(a,b)} \in \{-1, +1\}\) are signs. A graph with \(m\) edges has exactly \(2^m\) signings.

Bilu and Linial conjectured that every \(d\)-regular adjacency matrix \(A\) has a signing \(A_s\) with \(\|A_s\| \leq 2\sqrt{d-1}\). We prove the following weaker statement, which is equivalent to their conjecture in the bipartite case, as in this case the eigenvalues are symmetric about zero.

**Theorem 1.2.** Every \(d\)-regular adjacency matrix \(A\) has a signing \(A_s\) with

\[
\lambda_{\text{max}}(A_s) \leq 2\sqrt{d-1}.
\]

This is a statement about the existence of a certain sum of rank two matrices of type \(s_{(a,b)}A_{(a,b)}\), but it is useful to rewrite it as a statement about a sum of rank one matrices by making the substitution

\[
s_{(a,b)}A_{(a,b)} = (e_a + s_{(a,b)}e_b)(e_a + s_{(a,b)}e_b)^T - e_a e_a^T - e_b e_b^T,
\]

where \(e_a\) is the standard basis vector with a 1 in position \(a\). For a \(d\)-regular graph, we now have

\[
A_s = \sum_{(a,b) \in E} s_{(a,b)}A_{(a,b)} = \sum_{(a,b) \in E} (e_a + s_{(a,b)}e_b)(e_a + s_{(a,b)}e_b)^T - dI. \tag{1}
\]
So, Theorem 1.2 is equivalent to the statement that there is a choice of \( s(a,b) \) for which
\[
\lambda_{\text{max}} \left( \sum_{(a,b) \in E} (e_a + s(a,b)e_b)(e_a + s(a,b)e_b)^T \right) \leq d + 2\sqrt{d-1}.
\]
The existence of such a choice can be written in probabilistic terms by defining for each \((a, b) \in E\) a random vector
\[
r_{(a,b)} := \begin{cases} (e_a + e_b) & \text{with probability } 1/2 \\ (e_a - e_b) & \text{with probability } 1/2 \end{cases}.
\]
Notice that
\[
E \sum_{(a,b) \in E} r_{(a,b)}r_{(a,b)}^T = dI.
\]
Thus, Theorem 1.2 is equivalent to the statement that for every \(d\)-regular \(G = (V, E)\),
\[
\lambda_{\text{max}} \left( \sum_{(a,b) \in E} r_{(a,b)}r_{(a,b)}^T \right) \leq \lambda_{\text{max}} \left( E \sum_{(a,b) \in E} r_{(a,b)}r_{(a,b)}^T \right) + 2\sqrt{d-1}
\]
with positive probability.

Such a sum may be analyzed using tools of random matrix theory, but this approach does not give the sharp bound we require, and it is known that it cannot in general as there are graphs for which the desired signing is exponentially rare (consider a union of disjoint cliques on \(d\) vertices).

The main subject of this survey is an approach that succeeds in proving exactly. The methodology also succeeds in resolving several other important questions about sums of independent random rank one matrices, including Weaver’s conjecture and thereby the Kadison–Singer problem. We review these first and describe their connection to Ramanujan graphs before proceeding to describe the actual technique. The proof of and Theorem 1.2 will be sketched in Section 5.1.

1.2. Sparse Approximations of Graphs. Spielman and Teng observed that one can view an expander graph as an approximation of a complete graph, and asked if one could find analogous approximations of arbitrary graphs. In this context, it is more natural to consider the class of general weighted graphs rather than just unweighted \(d\)-regular graphs, and to study the Laplacian matrix instead of the adjacency matrix. Recall that the Laplacian of a weighted graph \(G = (V, E, w)\) may be defined as the following sum of rank one matrices over the edges:
\[
L_G = \sum_{(a,b) \in E} w_{(a,b)}(e_a - e_b)(e_a - e_b)^T.
\]
In the unweighted \(d\)-regular case, it is easy to see that \(L = dI - A\), so the eigenvalues of the Laplacian are just \(d\) minus the eigenvalues of the adjacency...
matrix. The Laplacian matrix of a graph always has an eigenvalue of 0; this is a trivial eigenvalue, and the corresponding eigenvectors are the constant vectors.

Following Spielman and Teng, we say that two graphs $G$ and $H$ on the same vertex set $V$ are spectral approximations of each other if their Laplacian quadratic forms multiplicatively approximate each other:

$$\kappa_1 \cdot x^T L_H x \leq x^T L_G x \leq \kappa_2 \cdot x^T L_H x \quad \forall x \in \mathbb{R}^V,$$

for some approximation factors $\kappa_1, \kappa_2 > 0$. We will write this as

$$\kappa_1 \cdot L_H \preceq L_G \preceq \kappa_2 \cdot L_H,$$

where $A \preceq B$ means that $B - A$ is positive semidefinite, i.e., $x^T (B - A) x \geq 0$ for every $x$.

The complete graph on $n$ vertices, $K_n$, is the graph with an edge of weight 1 between every pair of vertices. All of the eigenvalues of $L_{K_n}$ other than 0 are equal to $n$. If $G$ is a $d$-regular non-bipartite Ramanujan graph, then 0 is the trivial eigenvalue of its Laplacian matrix, $L_G$, and all of the other eigenvalues of $L_G$ are between $d - 2\sqrt{d} - 1$ and $d + 2\sqrt{d} - 1$. After a simple rescaling, this allows us to conclude that

$$(1 - 2\sqrt{d} - 1/d) L_K \preceq (n/d) L_G \preceq (1 + 2\sqrt{d} - 1/d) L_K.$$

So, $(n/d)L_G$ is a good approximation of $L_{K_n}$.

Batson, Spielman and Srivastava proved that every weighted graph has an approximation that is almost this good.

**Theorem 1.3 ([7]).** For every $d > 1$ and every weighted graph $G = (V, E, w)$ on $n$ vertices, there exists a weighted graph $H = (V, F, \tilde{w})$ with $\lceil d(n - 1) \rceil$ edges that satisfies:

$$\left(1 - \frac{1}{\sqrt{d}}\right)^2 L_G \preceq L_H \preceq \left(1 + \frac{1}{\sqrt{d}}\right)^2 L_G.$$

However, their proof had very little to do with graphs. In fact, they derived their result from the following theorem about sparse weighted approximations of sums of rank one matrices.

**Theorem 1.4 ([7]).** Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$ with

$$\sum_i v_i v_i^T = V.$$

For every $\epsilon \in (0, 1)$, there exist non-negative real numbers $s_i$ with

$$|\{i : s_i \neq 0\}| \leq \lceil n/\epsilon^2 \rceil$$

so that

$$(1 - \epsilon)^2 V \preceq \sum_i s_i v_i v_i^T \preceq (1 + \epsilon)^2 V.$$
Taking $V$ to be a Laplacian matrix written as a sum of outer products and setting $\epsilon = 1/\sqrt{d}$ immediately yields Theorem 1.3.

Theorem 1.4 is very general and turned out to be useful in a variety of areas including graph theory, numerical linear algebra, and metric geometry (see, for instance, the survey of Naor [34]). One of its limitations is that it provides no guarantees on the weights $s_i$ that it produces, which can vary wildly. So it is natural to ask: is there a version of Theorem 1.4 in which all the weights are the same?

This may seem like a minor technical point, but it is actually a fundamental difference. In particular, Gil Kalai observed that the statement of Theorem 1.4 with $V = I$ is similar to Weaver’s Conjecture, which was known to imply a positive solution to the Kadison–Singer Problem. It turns out that the natural unweighted variant of it is essentially the same as Weaver’s conjecture. We discuss the Kadison–Singer problem and this connection in the next section.

1.3. The Kadison-Singer Problem and Weaver’s Conjecture. In 1959, Kadison and Singer [26] asked the following fundamental question: does every pure state on the abelian von Neumann algebra $D(\ell_2)$ of diagonal operators on $\ell_2$ have a unique extension to a pure state on $B(\ell_2)$, the von Neumann algebra of all bounded operators on $\ell_2$? In their original paper, they suggested an approach to resolving this question: they showed that the answer is yes if every operator in $B(\ell_2)$ can be ‘paved’ by a constant number of operators which are strictly smaller in the operator norm. Beginning with the work of Anderson [4, 5, 6], this was shown to be equivalent to several combinatorial questions about decomposing finite matrices into a small number of strictly smaller pieces.

Among these questions is Akemann and Anderson’s “projection paving conjecture” [2], which Nik Weaver [44] later showed was equivalent to the following discrepancy-theoretic conjecture that he called $KS_2$.

Conjecture 1.5. There exist positive constants $\alpha$ and $\epsilon$ so that for every $n$ and $d$ and every set of vectors $v_1, \ldots, v_n \in \mathbb{C}^d$ such that $\|v_i\| \leq \alpha$ for all $i$ and

$$\sum_i v_i v_i^* = I,$$

there exists a partition of $\{1, \ldots, n\}$ into two sets $S_1$ and $S_2$ so that for each $j \in \{1, 2\}$

$$\left\| \sum_{i \in S_j} v_i v_i^* \right\| < 1 - \epsilon. \quad (7)$$

To see the similarity between this conjecture and Theorem 1.4 observe that for any partition $S_1 \cup S_2$:

$$\sum_{i \in S_1} v_i v_i^* + \sum_{i \in S_2} v_i v_i^* = I,$$

so that condition (7) is equivalent to

$$\epsilon I \preceq \sum_{i \in S_1} v_i v_i^* \preceq (1 - \epsilon)I.$$
Thus, choosing a subset of the weights $s_i$ to be non-zero in Theorem 1.4 is similar to choosing the set $S_1$. The difference is that Conjecture 1.5 assumes a bound on the lengths of the vectors $v_i$ and in return requires the stronger conclusion that all of the $s_i$ are either 0 or 1. It is easy to see that long vectors are an obstacle to the existence of a good partition; an extreme example is provided by considering an orthonormal basis $e_1, \ldots, e_n$. Weaver’s conjecture asserts that this is the only obstacle.

Overcoming this seemingly small difference turns out to require substantial new machinery beyond the techniques used in the proof of Theorem 1.4. However, much of this machinery is built on two key ideas which are contained in [7]. The first is the use of “barrier functions” to bound the roots of polynomials, which is discussed in Section 3.2. The second, which was presented purely for motivational purposes in [7], is the examination of expected characteristic polynomials.

As in the case of Ramanujan graphs, Weaver’s conjecture can be written in terms of sums of independent random rank one matrices. Given vectors $v_1, \ldots, v_m \in \mathbb{C}^d$, define for each $i$ the random vector $r_i \in \mathbb{C}^{2d}$

$$r_i = \begin{pmatrix} v_i \\ 0_d \end{pmatrix} \text{ with probability } 1/2 \quad \text{and} \quad \begin{pmatrix} 0_d \\ v_i \end{pmatrix} \text{ with probability } 1/2,$$

where $0_d \in \mathbb{C}^d$ is the zero vector. Then it is easy to see that every realization of $r_1, \ldots, r_m$ corresponds to a partition $S_1 \cup S_2 = [m]$ in the natural way, and that

$$\sum_i r_i r_i^* = \begin{pmatrix} \sum_{i \in S_1} v_i v_i^* \\ 0 \\ \sum_{i \in S_2} v_i v_i^* \end{pmatrix}. $$

Moreover, the norm of this matrix is the maximum of the norms of the matrices in the upper-left and lower-right blocks. Thus, Weaver’s conjecture is equivalent to the statement that when the $\|v_i\| \leq \alpha$, the following holds with positive probability:

$$\lambda_{\text{max}} \left( \sum_{i=1}^m r_i r_i^* \right) \leq 1 - \epsilon \quad (9)$$

Once again, it is possible to apply tools of random matrix theory to analyze this sum. This gives a proof of the conjecture with $\alpha = 1/\log n$, essentially recovering a result of Bourgain and Tzafriri [14], which was essentially the best partial solution to Kadison–Singer until recently.

The main result of [31] is the following strong form of Weaver’s conjecture.

**Theorem 1.6.** Let $v_1, \ldots, v_m \in \mathbb{C}^d$ satisfy $\sum_i v_i v_i^* = I$ and $\|v_i\|^2 \leq \alpha$ for all $i$. Then, there exists a partition of $\{1, \ldots, m\}$ into sets $S_1$ and $S_2$ so that for $j \in \{1, 2\}$,

$$\left\| \sum_{i \in S_j} v_i v_i^* \right\| \leq \frac{(1 + \sqrt{2\alpha})^2}{2}. \quad (10)$$

We will sketch the proof of Theorem 1.6 which is closely related to the proof of Theorem 1.2 in Sections 4 and 5.
1.4. Sums of Independent Rank One Random Matrices. As witnessed by equations (1) and (9), the common thread in the problems described above is that they can all be resolved by showing that a certain sum of independent random rank one matrices has small eigenvalues with nonzero probability. Prior to this line of work, there were already well-developed tools in random matrix theory for reasoning about such sums, generally called Matrix Chernoff Bounds [1, 37, 42]. As mentioned earlier, these provide bounds that are worse than those we require by a factor that is logarithmic in the dimension. However, they hold with high probability rather than the merely positive probability that we obtain.

Our approach to analyzing the eigenvalues of sums of independent rank one random matrices rests on the following connection between possible values of any particular eigenvalue, and the corresponding root of its expected characteristic polynomial. We will use $\lambda_1 \geq \lambda_2, \ldots, \geq \lambda_n \in \mathbb{R}$ to denote the eigenvalues of a Hermitian matrix as well as the roots of a real-rooted polynomial.

**Theorem 1.7 (Comparison with Expected Polynomial).** Suppose $r_1, \ldots, r_m \in \mathbb{C}^n$ are independent random vectors. Then, for every $k$,

$$
\lambda_k \left( \sum_{i=1}^{m} r_i r_i^* \right) \leq \lambda_k \left( \mathbb{E} \left[ \sum_{i=1}^{m} r_i r_i^* \right] (x) \right),
$$

with positive probability, and the same is true with $\geq$ instead of $\leq$.

In the special case when the $r_i$ are identically distributed with $\mathbb{E} r_i r_i^* = I$, there is short proof of Theorem 1.7 that only requires univariate interlacing. We present this proof as Lemma 3.2 and use it to establish a variant of Bourgain and Tzafriri’s restricted invertibility theorem. In Section 4 we prove the theorem in full generality using tools from the theory of real stable polynomials. This yields mixed characteristic polynomials, which are then analyzed in Sections 5.1 and 5.2 to prove the existence of infinite families of Bipartite Ramanujan Graphs as well as Weaver’s Conjecture.

2. Interlacing Polynomials

A defining characteristic of the proofs in [30] and [31] is that they analyze matrices solely through their characteristic polynomials. This is perhaps a counterintuitive way to proceed; on the surface, we are losing information by considering characteristic polynomials, which only know about eigenvalues and not eigenvectors. However, the structure we gain far outweighs the losses in two ways: the characteristic polynomials satisfy a number of algebraic identities which make calculating their averages tractable, and they are amenable to a set of analytic tools that do not naturally apply to matrices.

As hinted at earlier, we study the roots of *averages of* polynomials. In general, averaging polynomials coefficient-wise can do unpredictable things to the roots. For instance, the average of $(x - 1)(x - 2)$ and $(x - 3)(x - 4)$, which are both
real-rooted quadratics, is $x^2 - 5x + 7$, which has complex roots $2.5 \pm \sqrt{3}i$. Even when the roots of the average are real, there is in general no simple relationship between the roots of two polynomials and the roots of their average.

The main insight is that there are nonetheless many situations where averaging the coefficients of polynomials also has the effect of averaging each of the roots individually, and that it is possible to identify and exploit these situations. The key to doing this systematically is the classical notion of interlacing.

**Definition 2.1 (Interlacing).** Let $f$ be a degree $n$ polynomial with real roots $\{\alpha_i\}$, and let $g$ be degree $n$ or $n-1$ with real roots $\{\beta_i\}$ (ignoring $\beta_n$ in the degree $n-1$ case). We say that $g$ interlaces $f$ if their roots alternate, i.e.,

$$\beta_n \leq \alpha_n \leq \beta_{n-1} \leq \ldots \beta_1 \leq \alpha_1,$$

and the largest root belongs to $f$.

If there is a single $g$ which interlaces a family of polynomials $f_1, \ldots, f_m$, we say that they have a common interlacing.

It is an easy exercise to show that $f_1, \ldots, f_m$ of degree $n$ have a common interlacing iff there are closed intervals $I_n \leq I_{n-1} \leq \ldots I_1$ (where $\leq$ means to the left of) such that the $i$th roots of all the $f_j$ are contained in $I_i$. It is also easy to see that a set of polynomials has a common interlacing if every pair of them has a common interlacing (this may be viewed as Helly’s theorem on the real line).

We now state our main theorem about averages of polynomials with common interlacings.

**Theorem 2.2 (Lemma 4.1 in [K]).** Suppose $f_1, \ldots, f_m$ are real-rooted of degree $n$ with positive leading coefficients. Let $\lambda_k(f_j)$ denote the $k$th largest root of $f_j$ and let $\mu$ be any distribution on $[m]$. If $f_1, \ldots, f_m$ have a common interlacing, then for all $k = 1, \ldots, n$

$$\min_j \lambda_k(f_j) \leq \lambda_k(\mathbb{E}_{j \sim \mu} f_j) \leq \max_j \lambda_k(f_j).$$

The proof of this theorem is a three line exercise, which essentially amounts to applying the intermediate value theorem inside each interval $I_i$.

An important feature of common interlacings is that their existence is equivalent to certain real-rootedness statements. Often, this characterization gives us a systematic way to argue that common interlacings exist. The following seems to have been discovered a number of times. It appears as Theorem 2.1 of Dedieu [16], (essentially) as Theorem 2’ of Fell [17], and as (a special case of) Theorem 3.6 of Chudnovsky and Seymour [15]. The proof of it included below assumes that the roots of a polynomial are continuous functions of its coefficients (which may be shown using elementary complex analysis).

**Theorem 2.3.** If $f_1, \ldots, f_m$ are degree $n$ polynomials and all of their convex combinations $\sum_{i=1}^m \mu_i f_i$ have real roots, then they have a common interlacing.

**Proof.** Since common interlacing is a pairwise condition, it suffices to handle the case of two polynomials $f_0$ and $f_1$. Let

$$f_t := (1-t)f_0 + tf_1$$
with $t \in [0, 1]$. Assume without loss of generality that $f_0$ and $f_1$ have no common roots (if they do, divide them out and put them back in at the end). As $t$ varies from 0 to 1, the roots of $f_t$ define $n$ continuous curves in the complex plane $C_1, \ldots, C_n$, each beginning at a root of $f_0$ and ending at a root of $f_1$. By our assumption the curves must all lie in the real line. Observe that no curve can cross a root of either $f_0$ or $f_1$ in the middle: if $f_t(r) = 0$ for some $t \in (0, 1)$ and $f_0(r) = 0$, then immediately we also have $f_t(r) = tf_1(r) = 0$, contradicting the no common roots assumption. Thus, each curve defines a closed interval containing exactly one root of $f_0$ and one root of $f_1$, and these intervals do not overlap except possibly at their endpoints, establishing the existence of a common interlacing.

It is worth mentioning that the converse of Theorem 2.3 is true as well, but we will not use this fact.

While interlacing and real-rootedness are entirely univariate notions as discussed above, the most powerful ways to apply them arise by viewing them as restrictions of multivariate phenomena. There are two important generalizations of real-rootedness to more than one variable: real stability and hyperbolicity.

We were inspired by the development of the theory of real stability in the works of Borcea and Brändén, including \cite{Borcea, Borcea2, Borcea3}. Their results center primarily around characterizations of stable polynomials, including closure properties (that is, operations that preserve real stability of polynomials) and showing that properties of various mathematical structures an be related to the stability of some “generating polynomial” of that structure.

There is an isomorphism between real stable polynomials and hyperbolic polynomials, a concept that originated in a series of papers by Gårding \cite{Garding} in his investigation of partial differential equations. The theory of hyperbolic polynomials was developed further in the optimization community (see the survey of Renegar \cite{Renegar}). However, it was not until Gurvits’s use of hyperbolic polynomials in his proof of the van der Waerden conjecture \cite{Gurvits}, that their combinatorial power was revealed.

While it is well known that the concepts of real stability and hyperbolicity are essentially equivalent (one can translate easily between the two), various features of the way each property is defined have led to a natural separation of results: algebraic closure properties and characterization in real stability and analytic properties such as convexity in hyperbolicity. The “method of interlacing polynomials” discussed in this survey, is in many ways a recipe for mixing the ideas from these two communities into a single proof technique.

The method of interlacing polynomials consists of two somewhat distinct parts. The first is to show that a given collection of polynomials forms what we call an interlacing family, which is broadly speaking any class of polynomials for which the roots of its average can be related to those of the individual polynomials. This falls naturally into the realm of results regarding real stable polynomials as it often reduces to that showing various linear combinations of polynomials are real-rooted. The second part is to bound one of the roots of the expected polynomial under some distribution. This is more of an analytic task, for which the convexity properties studied in the context of hyperbolicity are relevant. For instance, in
Ramanujan Graphs and the Solution of the Kadison–Singer Problem

[31], the analysis of the largest root is based on understanding the evolution of the root surfaces defined by a multivariate polynomial as certain differential operators are applied to it, and draws on the same convexity properties that are at the core of hyperbolic polynomials.

3. Restricted Invertibility

The purpose of this section is to give the simplest possible demonstration of the method of interlacing families of polynomials. It will be completely elementary and self-contained, relying only on classical facts about univariate polynomials, and should be accessible to an undergraduate. Nonetheless, it is structurally almost identical to the proof of Weaver’s conjecture and contains most of the same conceptual components in a primitive form.

Bourgain and Tzafriri’s restricted invertibility theorem [13] states that any square matrix $B$ with unit length columns and small operator norm contains a large column submatrix $B_S$ which is well-invertible on its span. That is, the least singular value of the submatrix, $\sigma_{|S|}(B_S)$, is large. This may be seen as a robust, quantitative version of the fact that any matrix contains an invertible submatrix of size equal to its rank. The theorem was generalized to arbitrary rectangular $B$ by Vershynin [43], and further sharpened in [38, 45]. We will give a proof of the following theorem from [38], which corresponds to the important case $BB^T = I$, when the columns of $B$ are isotropic.

**Theorem 3.1.** Suppose $v_1, \ldots, v_m \in \mathbb{C}^n$ are vectors with $\sum_{i=1}^m v_i v_i^T = I_n$. Then for every $k < n$ there is a subset $S \subset [m]$ of size $k$ with

$$\lambda_k \left( \sum_{i \in S} v_i v_i^T \right) \geq \left( 1 - \sqrt{\frac{k}{n}} \right)^2 \frac{n}{m}.$$

The proof of this theorem has two parts. The first part is the special case of Theorem 3.7 in which $r_1, \ldots, r_n$ are independent and identically distributed (i.i.d.) and $\mathbb{E} r_i r_i^* = c I$. It reduces the problem of showing the existence of a good subset to that of analyzing the roots of the expected characteristic polynomial.

**Lemma 3.2.** Suppose $r_1, \ldots, r_k$ are i.i.d. copies of a finitely supported random vector $r$ with $\mathbb{E} r r^* = c I$. Then, with positive probability,

$$\lambda_k \left( \sum_{i=1}^k r_i r_i^* \right) \geq \lambda_k \left( \mathbb{E} \chi \left[ \sum_{i=1}^k r_i r_i^* \right] \right).$$

The second part is the calculation of the expected polynomial and the derivation of a bound on its roots.

**Lemma 3.3.** Suppose $r_1, \ldots, r_k$ are i.i.d. copies of a random vector $r$ with $\mathbb{E} r r^* = I$. Then,

$$\mathbb{E} \chi \left[ \sum_{i=1}^k r_i r_i^* \right] (x) = (1 - D)^k x^n = x^{n-k} (1 - D)^n x^k.$$
Moreover,
\[ \lambda_k ((1 - D)^n x^k) \geq \left( 1 - \frac{\sqrt{k}}{n} \right)^2 n. \]

### 3.1. Interlacing and \((1 - D)\) operators.

Let us begin with the first part. To relate the expected characteristic polynomial to its summands, we will inductively apply Theorem 2.2, which requires the existence of certain common interlacings. These will be established by a combination of two ingredients. The first is the following classical fact, which says that rank-one updates naturally cause interlacing.

**Lemma 3.4** (Cauchy’s Interlacing Theorem). If \( A \) is a symmetric matrix and \( v \) is a vector then \( \chi[A](x) \) interlaces \( \chi[A + vv^*](x) \).

One can easily derive this from the matrix determinant lemma:

**Lemma 3.5.** If \( A \) is an invertible matrix and \( u, v \) are vectors, then
\[ \det(A + uv^*) = \det(A)(1 + v^* A^{-1} u) \]

The second ingredient is the following correspondence between isotropic random rank one updates and differential operators.

**Lemma 3.6.** Suppose \( r \) is a random vector with \( E rr^* = cI \) for some constant \( c \geq 0 \). Then for every matrix \( A \), we have
\[ E \chi[A + rr^*](x) = (I - cD)\chi[A](x), \]
where \( D \) denotes differentiation with respect to \( x \).

**Proof.** Using Lemma 3.5, we obtain
\[
E \det(xI - A - rr^*) = E \det(xI - A)(1 - r^*(xI - A)^{-1} r) = \det(xI - A)(1 - Tr[(Err^*)(xI - A)^{-1}]) = \det(xI - A)(1 - cTr(xI - A)^{-1})
\]

Letting \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of \( A \), this quantity becomes
\[
\prod_{i=1}^{n} (x - \lambda_i) \left( 1 - c \sum_{i=1}^{n} \frac{1}{x - \lambda_i} \right) = \chi(A)(x) - c \sum_{i=1}^{n} \prod_{j \neq i} (x - \lambda_j) = (1 - cD)\chi(A)(x),
\]
as desired.  

The purpose of Lemma 3.6 is twofold. First, it allows us to easily calculate expected characteristic polynomials, which a priori could be intractably complicated sums. Second, the operators \((1 - cD)\) have other nice properties which witness that the expected polynomials we generate have real roots and common interlacings.
Lemma 3.7 (Properties of Differential Operators).

(1) If $f$ has real roots then so does $(I-cD)f$.

(2) If $f_1,\ldots,f_m$ have a common interlacing, then so do $(I-cD)f_1,\ldots,(1-cD)f_m$.

Proof. For part (1), assume that $f$ and $f'$ have no common roots (otherwise, these are also common roots of $f$ and $f-cf'$ which are clearly real). Consider the rational function

$$
\frac{f(x)-cf'(x)}{f(x)} = 1 - c \frac{f'(x)}{f(x)} = 1 - c \sum_{i=1}^{n} \frac{1}{x - \lambda_i}
$$

where $\lambda_i$ are the roots of $f$. Inspecting the poles of this function and applying the intermediate value theorem shows that $f-cf'$ has the same number of zeros as $f$, all distinct from those of $f$.

For part (2), Theorem 2.2 tells us that all convex combinations $\sum_{i=1}^{m} \mu_i f_i$ have real roots. By part (1) it follows that all

$$(1-cD)\sum_{i=1}^{m} \mu_i f_i = \sum_{i=1}^{m} \mu_i (1-cD)f_i
$$

also have real roots. By Theorem 2.3, this means that the $(1-cD)f_i$ must have a common interlacing.

With these facts in hand, we can easily complete the proof of Lemma 3.2.

Proof. Assume $r$ is uniformly distributed on some set $v_1,\ldots,v_m \in \mathbb{C}^n$. We need to show that there is a choice of indices $j_1,\ldots,j_k \in [m]$ for which

$$
\lambda_k \left( \sum_{i=1}^{k} v_{j_i}^* v_{j_i} \right) \geq \lambda_k \left( \mathbb{E} \chi \left[ \sum_{i=1}^{k} r_i r_i^* \right] \right).
$$

For any partial assignment $j_1,\ldots,j_\ell$ of the indices, consider the “conditional expectation” polynomial:

$$
q_{j_1,\ldots,j_\ell}(x) := \mathbb{E}_{r_{\ell+1},\ldots,r_k} \chi \left[ \sum_{i=1}^{\ell} v_{j_i}^* v_{j_i} + \sum_{i=\ell+1}^{k} r_i r_i^* \right].
$$

Since the $r_i$ are independent, and $\mathbb{E}r_i = (1/m)I$, applying Lemma 3.6 $k-\ell$ times reveals that:

$$
q_{j_1,\ldots,j_\ell}(x) = (1-(1/m)D)^{k-\ell} \chi \left[ \sum_{i=1}^{\ell} v_{j_i}^* v_{j_i} \right](x).
$$

We will show that there exists a $j_{\ell+1} \in [m]$ such that

$$
\lambda_k(q_{j_1,\ldots,j_{\ell+1}}) \geq \lambda_k(q_{j_1,\ldots,j_{\ell}}),
$$

(11)
which by induction will complete the proof. Consider the matrix

$$A = \sum_{i=1}^{\ell} v_j v_j^*,$$

By Lemma 3.4, $\chi[A]$ interlaces $\chi[A + v_{j\ell+1} v_{j\ell+1}^*]$ for every $j\ell+1 \in [m]$. Lemma 3.7 tells us $(1 - (1/m)D)$ operators preserve common interlacing, so the polynomials

$$(1 - (1/m)D)^{k-(\ell+1)}\chi(A + v_{j\ell+1} v_{j\ell+1}^*) = q_{j_1,\ldots,j_{\ell},j_{\ell+1}}(x)$$

must also have a common interlacing. Thus, some $j_{\ell+1} \in [m]$ must satisfy (11), as desired. 

3.2. Laguerre Polynomials and the Univariate Barrier Argument. We now move on to the second part, Lemma 3.3, in which we prove a bound on the $k$th root of the expected polynomial, which after rescaling by a factor of $m$ is just:

$$E\chi\left[m \cdot \sum_{i=1}^{k} r_i r_i^*\right](x) = (1 - D)^k x^n.$$

We begin by observing that $(1 - D)^k x^n = x^{n-k}(1 - D)^n x^k$. This may be verified by term-by-term calculation, or by appealing to the correspondence between $(1 - D)$ operators and random isotropic rank one updates established in Lemma 3.6 as follows. Let $G$ be an $n$-by-$k$ matrix of random, independently distributed, $N(0,1)$ entries. The covariance matrix of each column is the $n$-dimensional identity matrix, and the covariance of each row is the $k$-dimensional identity. So,

$$(1 - D)^k x^n = E_G \chi(GG^*)(x) = E_G x^{n-k} \chi(G^*G)(x) = x^{n-k}(1 - D)^n x^k.$$

Thus, we would like to lower bound the least root of $(1 - D)^n x^k$. The easiest way to do this is to observe that it is a constant multiple of a known polynomial, namely an associated Laguerre polynomial $L_k^{(n-k)}(x)$. These are classical orthogonal polynomials and a lot is known about the locations of their roots; in particular, they are known to be contained in the interval $[n(1 - \sqrt{k/n})^2, n(1 + \sqrt{k/n})^2]$ (see, for instance, [27]).

In order to keep the presentation self-contained, and also because it is a key tool in the proof of Kadison–Singer and more generally in the analysis of expected characteristic polynomials, we now give a direct proof of Lemma 3.3 based on the “barrier method” introduced in [7]. The basic idea is to study the effect of each $(1 - D)$ operator on the roots of a polynomial $f$ via the associated rational function

$$\Phi_f(b) := \frac{f'(b)}{f(b)} = -\frac{\partial \log f(b)}{\partial b} = \sum_{i=1}^{n} \frac{1}{\lambda_i - b}.$$
which we will refer to as the lower barrier function. The poles of this function are the roots $\lambda_1, \ldots, \lambda_n$ of $f$, and we remark that it is the same up to a multiplicative factor of $(-1/n)$ as the Stieltjes transform of the discrete measure supported on these roots. It is immediate from the above expression that $\Phi_f(b)$ is positive, monotone increasing, and convex for $b$ is strictly less than the roots of $f$, and that it tends to infinity as $b$ approaches the smallest root of $f$ from below.

We now use the inverse of $\Phi_f$ to define a robust lower bound for the roots of a polynomial $f$:

$$s_{\min}^{\varphi}(f) := \min\{x \in \mathbb{R} : \Phi_f(x) = \varphi\},$$

where $\varphi > 0$ is a sensitivity parameter. Since $\Phi_f(b) \to 0$ as $b \to -\infty$, it is immediate that we always have $s_{\min}^{\varphi}(f) \leq \lambda_{\min}(f)$. The number $\varphi$ controls the tradeoff between how accurate a lower bound $s_{\min}^{\varphi}$ is an how smoothly it varies — in particular the extreme cases are $s_{\min}^{\infty}(f) = \lambda_{\min}(f)$, which is not always well-behaved, and $s_{\min}^{0}(f) = -\infty$, which doesn’t even depend on $f$. This quantity was implicitly introduced and used in [7] and explicitly defined in [41], where it was called the ‘soft spectral edge’; for an intuitive discussion of its behavior in terms of an electrical repulsion model, we refer the reader to the latter paper.

We also remark that the inverse Stieltjes transform was used by Voiculescu in his development of Free Probability theory to study the limiting spectral distributions of certain random matrix ensembles as the dimension tends to infinity. We view the use of $s_{\min}$ as a non-asymptotic analogue of that idea, except that we use it to reason about the edge of the spectrum rather than the bulk.

The following lemma tells us that $s_{\min}^{\varphi}(f)$ grows in a smooth and predictable way when we apply a $(1 - D)$ operator to $f$.

**Lemma 3.8.** If $f$ has real roots and $\varphi > 0$, then

$$s_{\min}^{\varphi}((1 - D)f) \geq s_{\min}^{\varphi}(f) + \frac{1}{1 + \varphi}.$$

**Proof.** Let $b = s_{\min}^{\varphi}(f)$. To prove the claim it suffices to find a $\delta \geq (1 + \varphi)^{-1}$ such that $b + \delta$ is below the roots of $f$ and $\Phi_{(1 - D)f}(b + \delta) \leq \varphi$. We begin by writing the barrier function of $(1 - D)$ in terms of the barrier function of $f$:

$$\Phi_{(1 - D)f} = - \frac{(f - f')'}{f''} = - \frac{(f(1 + \Phi_f))'}{f(1 + \Phi_f)} = - \frac{f'}{f} - \frac{\Phi_f'}{1 + \Phi_f} = \Phi_f - \frac{\Phi_f'}{1 + \Phi_f}. \tag{13}$$

This identity tells us that for any $\delta \geq 0$:

$$\Phi_{(1 - D)f}(b + \delta) = \Phi_f(b + \delta) - \frac{\Phi_f'(b + \delta)}{1 + \Phi_f(b + \delta)},$$

which is at most $\varphi = \Phi_f(b)$ whenever

$$\frac{\Phi_f'(b + \delta)}{1 + \Phi_f(b + \delta)} \geq \Phi_f(b + \delta) - \Phi_f(b).$$
This is in turn equivalent to

\[
\frac{\Phi'(b + \delta)}{\Phi_f(b + \delta) - \Phi_f(b)} - \Phi_f(b + \delta) \geq 1.
\]

Expanding each \(\Phi_f\) as a sum of terms as in (12) and applying Cauchy-Schwartz appropriately reveals \(1/\delta - \Phi_f(b)\) that the left-hand side of this inequality it at least \(1\) for all \(\delta \leq (1 + \varphi)^{-1}\).

We conclude that \(\Phi(1 - D) \Phi_f(b + \delta)\) is bounded by \(\varphi\) for all \(\delta \in [0, (1 + \varphi)^{-1}]\), which implies in particular that \(b + \delta\) is below the roots of \((1 - D)f\).

Applying the lemma \(n\) times immediately yields the following bound on our polynomial of interest:

\[
\lambda_k((1 - D)^n x^k) \geq \text{smin}_\varphi((1 - D)^n x^k) \geq \text{smin}_\varphi(x^k) + \frac{n}{1 + \varphi} = -\frac{k}{\varphi} + \frac{n}{1 + \varphi} \quad \text{since } \Phi_{x^k}(b) = -k/b.
\]

Setting \(\varphi = \frac{\sqrt{n}}{\sqrt{1 - \sqrt{k}}}\) yields Lemma 3.3, completing the proof of Theorem 3.1.

We remark that we have, as a byproduct, derived a sharp bound on the least root of an associated Laguerre polynomial.

In Lemma 5.2 we use a multivariate version of the analogous bound for the largest root of the associated Laguerre polynomial. A crucial aspect of the proof of the upper bound on the largest root is that it essentially depends only on the convexity and monotonicity of the barrier function. For a real-rooted polynomial \(f\), we define the upper barrier function as \(\Phi^f(b) = f'(b)/f(b)\) and

\[
\text{smax}_\varphi(f) := \max\{x \in \mathbb{R} : \Phi^f(x) = \varphi\}.
\]

**Lemma 3.9.** If \(f\) has real roots and \(\varphi > 0\), then

\[
\text{smax}_\varphi((1 - D)f) \leq \text{smax}_\varphi(f) + \frac{1}{1 - \varphi}.
\]

**Proof.** Let \(b = \text{smax}_\varphi(f)\). As before, we may derive

\[
\Phi^{1 - D}f = \Phi^f - (D\Phi^f)/(1 - \Phi^f).
\]

So, to show that

\[
\text{smax}_\varphi((1 - D)f) \leq b + \delta,
\]

\footnote{The simple but slightly cumbersome calculation appears as Claim 3.6 of [7]; we have chosen to omit it here for the sake of brevity.}
it suffices to prove that

\[ \Phi_f(b) - \Phi_f(b + \delta) \geq \frac{-D\Phi_f(b + \delta)}{1 - \Phi_f(b + d)}. \]

As \( \Phi_f(b) \) is monotone decreasing for \( b \) above the roots of \( f \), \( D\Phi_f(b + \delta) \) is negative. As \( \Phi_f(b) \) is convex for the same \( b \),

\[ \Phi_f(b) - \Phi_f(b + \delta) \geq \delta(-D\Phi_f(b + \delta)). \]

Thus, we only require

\[ \delta \geq \frac{1}{1 - \Phi_f(b + d)}. \]

As \( \Phi_f(b) \) is monotone decreasing, this is satisfied for \( \delta = 1/(1 - \varphi) \).

Setting \( \varphi = \frac{\sqrt{k}}{\sqrt{n} + \sqrt{k}} \), we obtain our upper bound the largest root of an associated Laguerre polynomial.

**Lemma 3.10.** The largest root of \((1 - D)^n x^k\) is at most \( n(1 + \sqrt{k/n})^2 \).

4. Mixed Characteristic Polynomials

The argument given in the previous section is a special case of a more general principle: that the expected characteristic polynomials of certain random matrices can be expressed in terms of differential operators, which can then be used to establish the existence of common interlacings as well as to analyze the roots of the expected polynomials themselves. In the isotropic case of Bourgain–Tzafriri, this entire chain of reasoning can be carried out by considering univariate polynomials only. Morally, this is because the covariance matrices of all of the random vectors involved are multiples of the identity (which trivially commute with each other), and all of the characteristic polynomials involved are simple univariate linear transformations of each other (of type \((I - cD)\)).

On the other hand, the proofs of Kadison-Singer and existence of Ramanujan graphs involve analyzing sums of independent rank one matrices which come from *non-identically distributed* distributions whose covariance matrices do not commute. This leads to a much more general family of expected polynomials which we call *mixed characteristic polynomials*. The special structure of these polynomials is revealed crisply when we view them as restrictions of certain multivariate polynomials. Their qualitative and quantitative properties are, correspondingly, established using multivariate differential operators and barrier functions, which are analyzed using tools from the theory of real stable polynomials.

In the remainder of this section we will sketch a proof of Theorem 1.7. The proof hinges on the following central identity, which describes the general correspondence between sums of independent random rank one matrices and (multivariate) differential operators.
Theorem 4.1. Let $r_1, \ldots, r_m$ be independent random column vectors in $\mathbb{C}^d$. For each $i$, let $A_i = E r_i r_i^*$. Then,

$$
E \chi \left[ \sum_{i=1}^m r_i r_i^* \right] (x) = \left( \prod_{i=1}^m 1 - \partial z_i \right) \det \left( xI + \sum_{i=1}^m z_i A_i \right) \bigg|_{z_1=\cdots=z_m=0}.
$$

(14)

In particular, the expected characteristic polynomial of a sum of independent rank one Hermitian random matrices is a function of the covariance matrices $A_i$. We call this polynomial the mixed characteristic polynomial of $A_1, \ldots, A_m$, and denote it by $\mu [A_1, \ldots, A_m] (x)$. The name mixed characteristic polynomial is inspired by the fact that the expected determinant of this matrix is called the mixed discriminant. Notice that when $A_1 = A_2 = \ldots = A_m = I$, it is just a multiple of an associated Laguerre polynomial as in Section 3.

Theorem 4.1 may be proved fairly easily by inductively applying an identity similar to Lemma 3.6 or by appealing to the Cauchy-Binet formula; we refer the reader to [31] for a short proof. We remark that it and all of the other results in this section depend crucially on the fact that the $r_i r_i^*$ are rank one, and fail rather spectacularly for rank 2 or higher matrices.

The most important consequence of Theorem 4.1 is that mixed characteristic polynomials always have real roots. To prove this, we will need to consider a multivariate generalization of real-rootedness called real stability.

Definition 4.2. A multivariate polynomial $f \in \mathbb{R}[z_1, \ldots, z_m]$ is real stable if it has no roots with all coordinates strictly in the upper half plane, i.e., if

$$
\text{Im}(z_i) > 0 \quad \forall i \quad \Rightarrow \quad f(z_1, \ldots, z_m) \neq 0.
$$

Notice that stability is the same thing as real rootedness in the univariate case, since complex roots occur in conjugate pairs.

A natural and relevant example of real stable polynomials is the following:

Lemma 4.3 ([10]). If $A_1, \ldots, A_m$ are positive semidefinite matrices, then

$$
f(z_1, \ldots, z_m) = \det \left( \sum_{i=1}^m z_i A_i \right)
$$

is real stable.

One reason real stability is such a useful notion for us is that it has remarkable closure properties which are extremely well-understood. In particular, Borcea and Brändén have completely characterized the linear operators preserving real stability [12]. What this means heuristically is that proofs of stability can often be reduced to a formal exercise: to prove that a particular polynomial is stable, one must simply write it as a composition of known stability-preserving operations.

To prove that mixed characteristic polynomials are real stable, we will only require the following elementary closure properties.
Lemma 4.4 (Closure Properties). If $f(z_1, \ldots, z_m)$ is real stable, then so are

$$(1 - \partial_{z_i})f$$

for every $i$ and

$$f(\alpha, z_2, \ldots, z_m)$$

for every $\alpha \in \mathbb{R}$.

The first part was essentially established by Lieb and Sokal in [28]. It follows easily by considering a univariate restriction to $z_i$ and studying the associated rational function, as in the (entirely univariate) proof of Lemma 3.7. The second part is trivial for $\alpha$ strictly in the upper half plane, and may be extended to the real line by appealing to Hurwitz’s theorem.

Combining these properties with Theorem 4.1 instantly establishes the following important fact.

Theorem 4.5. If $A_1, \ldots, A_m$ are positive semidefinite, then $\mu[A_1, \ldots, A_m](x)$ is real-rooted.

We are now in a position to prove Theorem 1.7. As in Lemma 3.2, we will do this inductively by showing that the relevant “conditional expectation” polynomials have common interlacings. However, instead of explicitly finding these common interlacings using Cauchy’s theorem, we will guarantee their existence implicitly using Theorem 4.5.

Proof of Theorem 1.7. For any partial assignment $v_1, \ldots, v_\ell$ of $r_1, \ldots, r_\ell$, consider the conditional expected polynomial

$$q_{v_1, \ldots, v_\ell}(x) := \mathbb{E}_{\chi} \left[ \sum_{i=1}^{\ell} v_i v_i^* + \sum_{i=\ell+1}^{m} r_i r_i^* \right](x).$$

Suppose $r_{\ell+1}$ is supported on $w_1, \ldots, w_N$. Then, for all convex coefficients $\sum_{i=1}^{N} \mu_i = 1, \mu_i \geq 0$, the convex combination

$$\sum_{i=1}^{N} \mu_i q_{v_1, \ldots, v_\ell, w_i}(x)$$

is itself a mixed characteristic polynomial, namely

$$\mu \left[ v_1 v_1^*, \ldots, v_\ell v_\ell^*, \sum_{i=1}^{N} \mu_i w_i w_i^*, \mathbb{E} r_{\ell+1} r_{\ell+1}^*, \ldots, \mathbb{E} r_m r_m^* \right](x),$$

which has real roots by Theorem 4.5. This establishes that the $q_{v_1, \ldots, v_\ell, w_i}(x)$ have a common interlacing, which by Theorem 2.2 implies that for every $k$ there exists an $i \in [N]$ for which

$$\lambda_k(q_{v_1, \ldots, v_m, w_i}(x)) \leq \lambda_k(q_{v_1, \ldots, v_m}(x)),$$

completing the induction. \hfill \square
The above proof highlights the added flexibility of allowing the $r_i$ to have different distributions: by taking some of these distributions to be deterministic, we can encode any conditioning and more generally any addition of a positive semidefinite matrix while remaining in the class of mixed characteristic polynomials.

5. Analysis of Expected Polynomials

In this section, we describe two situations in which we are able to bound the largest roots of mixed characteristic polynomials. The first is very specific: we observe that the expected characteristic polynomial of a random signing of an adjacency matrix of a graph is equal, up to a shift, to the matching polynomial of the graph. The zeros of this polynomial have been studied for decades and elementary combinatorial arguments due to Heilmann and Lieb [23] can be used to give a sharp bound on its largest root. The main consequence of this bound is the existence of infinite families of bipartite Ramanujan graphs of every degree.

The second situation is almost completely general. We show that given any collection of matrices satisfying $\sum_{i=1}^m A_i = I$, the mixed characteristic polynomial $\mu[A_1, \ldots, A_m](x)$ has roots bounded by $1 + \sqrt{\max_{i} \text{Tr}(A_i)}^2$. This is achieved by a direct multivariate generalization of the barrier function argument that we used in Section 3 to upper bound the roots of associated Laguerre polynomials. The main consequence of this bound is a proof of Weaver’s conjecture and thereby a positive solution to the Kadison–Singer problem.

5.1. Matching Polynomials. We are now ready to prove the bound (4) and thereby Theorem 1.2. For any $d$–regular graph $G = (V, E)$, let the random vectors \( \{r(a,b)\}_{(a,b) \in E} \) be defined as in (2). Applying Theorem 1.7 with $k = 1$ and subtracting $d$ from both sides, we find that:

$$
\lambda_{\max} \left( \sum_{(a,b) \in E} r(a,b) r^*(a,b) - dI \right) = \lambda_{\max} \left( \sum_{(a,b) \in E} r(a,b) r^*(a,b) \right) - d
$$

$$
\leq \lambda_{\max} \left( \mathbb{E} \chi \left[ \sum_{(a,b) \in E} r(a,b) r^*(a,b) \right](x) \right) - d
$$

$$
= \lambda_{\max} \left( \mathbb{E} \chi \left[ \sum_{(a,b) \in E} r(a,b) r^*(a,b) - dI \right](x) \right),
$$

with positive probability. Switching back to signed adjacency matrices by applying (1), we conclude that

$$
\lambda_{\max}(A_s) \leq \lambda_{\max} (\mathbb{E} \chi [A_s](x))
$$

(15)

with positive probability for a uniformly random signing $A_s$. 
We now observe that this expected characteristic polynomial is equal to the matching polynomial of the graph. A matching is a graph in which every vertex has degree at most one. The matching polynomial is a generating function which counts the number of matchings that are subgraphs of a graph; for a graph on \( n \) vertices, it is defined as

\[
\mu_G(x) := \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i x^{n-2i} m_i,
\]

where \( m_i \) is the number of subgraphs of \( G \) with \( i \) edges that are matchings.

Godsil and Gutman [20] showed that the matching polynomial of a graph is equal to the expected characteristic polynomial of a random signing of its adjacency matrix:

\[
E \chi[As](x) = \mu_G(x).
\]

This identity may be proved easily by expanding \( \chi[As](x) = \det(xI - As) \) as a sum of permutations and observing that the only terms that do not vanish are the permutations with all orbits of size two, which correspond to the matchings.

About a decade before this, Heilmann and Lieb [23] studied the matching polynomial in the context of monomer-dimer systems in statistical physics. In that paper, they showed that \( \mu_G(x) \) always has all real roots (a fact which we have also just proved by writing it as a shift of a mixed characteristic polynomial), and that

\[
\lambda_{\text{max}}(\mu_G(x)) \leq 2\sqrt{d-1}
\]

for a graph with maximum degree \( d \). They proved this bound by finding certain simple combinatorial recurrences satisfied by \( \mu_G(x) \), induced by edge and vertex deletions. The appearance of the number \( 2\sqrt{d-1} \) is not a coincidence; Godsil [19] later showed using similar recurrences that \( \mu_G(x) \) divides the characteristic polynomial of a certain tree associated with \( G \), which is an induced subgraph of the infinite \( d \)--regular tree.

Combining (15), (16), and (17) yields Theorem 1.2. There is also a generalization of this theorem which proves the existence of “irregular” Ramanujan graphs, which were not previously known to exist; we refer the interested reader to [30] for details.

5.2. The Multivariate Barrier Argument. The tight bound of \( 2\sqrt{d-1} \) obtained above relies heavily on the fact that the random vectors \( r_{(a,b)} \) come from a graph and have combinatorial structure. Remarkably, it turns out that we can prove a bound that is almost as sharp by completely ignoring this structure and relying only on the much weaker property that the \( rr^\ast \) are rank one matrices of bounded trace. This type of generic bound is precisely what one needs to control the roots of the quite general mixed characteristic polynomials which arise in the proof of Weaver’s conjecture, and thereby prove Kadison–Singer.

**Theorem 5.1.** Suppose \( A_1, \ldots, A_m \) are positive semidefinite matrices with \( \sum_{i=1}^{m} A_i = I \) and \( \text{Tr}(A_i) \leq \epsilon \). Then,

\[
\lambda_{\text{max}}(\mu[\ldots, A_m]\,(x)) \leq (1 + \sqrt{\epsilon})^2.
\]
At a high level, the proof of this theorem is very similar to that of Lemma 3.10: we express \( \mu [A_1, \ldots, A_m](x) \) as a product of differential operators applied to some nice initial polynomial, and show that each differential operator perturbs the roots in a predictable way. The difference is that the differential operators and roots are now multivariate rather than univariate.

To deal with this issue, we begin by defining a notion of multivariate upper bound: we say that \( b \in \mathbb{R}^m \) is above the roots of a real stable polynomial \( f(z_1, \ldots, z_m) \) if \( f(z) > 0 \) for all \( z \geq b \) coordinate-wise. It is best to think of an “upper bound” for the roots of \( f \) as a set rather than as a single point — the set of all points above the roots of \( f \).

As we did in the univariate case, we soften this notion by studying certain rational functions associated with \( f \) which interact naturally with the \((1 - \partial_{z_j})\) operators we are interested in. For each coordinate \( j \), define the multivariate barrier function
\[
\Phi^j_f(z_1, \ldots, z_m) = \frac{\partial f}{\partial z_j} \frac{f(z_1, \ldots, z_m)}{f(z_1, \ldots, z_m)},
\]
and notice that
\[
\Phi^j_f(z_1, \ldots, z_m) = \sum_{i=1}^d \frac{1}{z_j - \lambda_i},
\]
where \( \lambda_1, \ldots, \lambda_d \) are the roots of the univariate restriction obtained by fixing all the coordinates other than \( z_j \).

For a sensitivity parameter \( \varphi < 1 \), we define a \( \varphi \)-robust upper bound on \( f(z_1, \ldots, z_m) \) to be any point \( b \) above the roots of \( f \) with \( \Phi^j_f(b) \leq \varphi \) for all \( j \). We denote the set of all such robust upper bounds by \( \overrightarrow{s\text{ma}x}_\varphi(f) \). The following multivariate analogue of Lemma 3.9 holds for \( \overrightarrow{s\text{ma}x}_\varphi \). It says that applying an \((1 - \partial_{z_j})\) operator simply moves the set of robust upper bounds in direction \( j \) by a small amount.

**Lemma 5.2.** If \( f(z_1, \ldots, z_m) \) is real stable and \( \varphi < 1 \), then
\[
\overrightarrow{s\text{ma}x}_\varphi ((1 - \partial_{z_j})f) \supseteq \overrightarrow{s\text{ma}x}_\varphi(f) + \frac{1}{1 - \varphi} e_j,
\]
where \( e_j \) is the elementary basis vector in direction \( j \).

The proof of this lemma is syntactically almost identical to that of Lemma 3.9 except that it is less obvious that the barrier functions \( \Phi^j_f \) are monotone and convex in the coordinate directions. In [31] we prove this by appealing to a powerful representation theorem of Helton and Vinnikov [24], which says that bivariate restrictions of real stable polynomials can always be written as determinants of positive semidefinite matrices, which are easy to analyze. Later, elementary proofs of this fact were given by James Renegar (using tools from the theory of hyperbolic polynomials [8]) and Terence Tao (using a combination of elementary calculus and complex analysis, along with Bezout’s theorem).

With Lemma 5.2 in hand, one can prove Theorem 5.1 by an induction similar to the one we used in Lemma 3.3. We refer the reader to [31] for details.

Applying Theorems 1.7 and 5.1 to the random vectors defined in [8] immediately yields Theorem 1.6.
6. Ramanujan Graphs and Weaver’s Conjecture

We conclude by showing how the generic bound derived above may be used to analyze the random signings that occur in the proof of Theorem 1.2. This turns out to be very instructive and is quite natural, since when $G = (V, E)$ is $d$-regular, (3) tells us that

$$
E \sum_{(a,b) \in E} \frac{r(a,b) r^*(a,b)}{d} = I.
$$

Thus, each vector has the same norm $\|r(a,b)\|^2 = 2/d$, and applying Theorems 1.7 and 5.1 shows that

$$
\sum_{(a,b) \in E} r(a,b) r^*(a,b) \leq d \left( 1 + \sqrt{2/d} \right)^2 = d + 2 + 2 \sqrt{2d}
$$

with positive probability. This bound has asymptotically the same dependence on $d$ as the correct bound established using matching polynomials. Moreover, it immediately proves that the dependence on $\epsilon$ in Theorem 5.1 cannot be improved: if it could, the above argument would imply the existence of signings with largest eigenvalue $o(\sqrt{d})$, contradicting the Alon–Boppana bound. Thus, the matrices arising in the study of Ramanujan graphs witness the sharpness of our bounds on mixed characteristic polynomials.

References

[1] R. Ahlswede and A. Winter. Strong converse for identification via quantum channels. Information Theory, IEEE Transactions on, 48(3):569–579, 2002.
[2] C. A. Akemann and J. Anderson. Lyapunov theorems for operator algebras. Number 458. American Mathematical Soc., 1991.
[3] N. Alon. Eigenvalues and expanders. Combinatorica, 6(2):83–96, 1986.
[4] J. Anderson. Extensions, restrictions, and representations of states on $C^*$-algebras. Transactions of the American Mathematical Society, 249(2):303–329, 1979.
[5] J. Anderson. Extreme points in sets of positive linear maps on $B(H)$. Journal of Functional Analysis, 31(2):195–217, 1979.
[6] J. Anderson. A conjecture concerning the pure states of $B(H)$ and related theorem. Topics in modern operator theory (Timisoara/Herculane, 1980), Birkhäuser, Basel-Boston, Mass, pages 27–43, 1981.
[7] J. Batson, D. A. Spielman, and N. Srivastava. Twice-Ramanujan sparsifiers. SIAM Journal on Computing, 41(6):1704–1721, 2012.
[8] H. H. Bauschke, O. Güler, A. S. Lewis, and H. S. Sendov. Hyperbolic polynomials and convex analysis. Canadian Journal of Mathematics, 53(3):470–488, 2001.

\[2\] We remark that the Alon–Boppana bound can also be used to show that the dependence on $\alpha$ in Theorem 1.6 itself is tight by recursively applying it to the Laplacian of the complete graph. We point the reader to [40] or [22] for a complete argument.
[9] Y. Bilu and N. Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26(5):495–519, 2006.

[10] J. Borcea and P. Brändén. Applications of stable polynomials to mixed determinants: Johnson’s conjectures, unimodality, and symmetrized fischer products. *Duke Mathematical Journal*, 143:205–223, 2008.

[11] J. Borcea and P. Brändén. The Lee–Yang and Pólya–Schur programs, I. Linear operators preserving stability. *Invent. Math.*, 177(3):541–569, 2009.

[12] J. Borcea and P. Brändén. Multivariate Polya-Schur classification problems in the Weyl algebra. *Proc. London Mathematical Society*, (3) 101(1):73–104, 2010.

[13] J. Bourgain and L. Tzafriri. Invertibility of large submatrices with applications to the geometry of Banach, spaces and harmonic analysis. *Israel journal of mathematics*, 57(2):137–224, 1987.

[14] J. Bourgain and L. Tzafriri. On a problem of Kadison and Singer. *J. reine angew. Math.*, 420:1–43, 1991.

[15] M. Chudnovsky and P. Seymour. The roots of the independence polynomial of a clawfree graph. *Journal of Combinatorial Theory, Series B*, 97(3):350–357, 2007.

[16] J. P. Dedieu. Obreschkoff’s theorem revisited: what convex sets are contained in the set of hyperbolic polynomials? *J. Pure and Applied Algebra*, 81(3):269–278, 1992.

[17] H. J. Fell. Zeros of convex combinations of polynomials. *Pacific J. Math.*, 89(1):43–50, 1980.

[18] L. Gårding. Linear hyperbolic partial differential equations with constant coefficients. *Acta Mathematica*, 85(1):1–62, 1951.

[19] C. D. Godsil. Matchings and walks in graphs. *J. Graph Theory*, 5(3):285–297, 1981.

[20] C. D. Godsil and I. Gutman. On the matching polynomial of a graph. In L. Lovász and V. T. Sós, editors, *Algebraic Methods in graph theory*, volume I of *Colloquia Mathematica Societatis János Bolyai*, 25, pages 241–249. 1981.

[21] L. Gurvits. Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all. *the electronic journal of combinatorics*, 15(R66):1, 2008.

[22] N. J. A. Harvey and N. Olver. Pipage rounding, pessimistic estimators and matrix concentration. In *ACM-SIAM Symposium on Discrete Algorithms*, 2014.

[23] O. J. Heilmann and E. H. Lieb. Theory of monomer-dimer systems. *Communications in Mathematical Physics*, 25(3):190–232, 1972.

[24] J. W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Communications on pure and applied mathematics*, 60(5):654–674, 2007.

[25] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439–561, 2006.

[26] R. V. Kadison and I. M. Singer. Extensions of pure states. *American Jour. Math.*, 81:383–400, 1959.

[27] I. Krasikov. On extreme zeros of classical orthogonal polynomials. *Journal of computational and applied mathematics*, 193(1):168–182, 2006.

[28] E. H. Lieb and A. D. Sokal. A general Lee-Yang theorem for one-component and multicomponent ferromagnets. *Communications in Mathematical Physics*, 80(2):153–179, 1981.
Ramanujan Graphs and the Solution of the Kadison–Singer Problem

[29] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. Combinatorica, 8(3):261–277, 1988.
[30] A. W. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families I: Bipartite Ramanujan graphs of all degrees. In Proceedings of the 54th IEEE Symposium on Foundations of Computer Science, pages 529–537, 2013.
[31] A. W. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem. arXiv preprint arXiv:1306.3969, 2013.
[32] G. A. Margulis. Explicit constructions of concentrators. Problemy Peredachi Informatsii, 9(4):71–80, October-December 1973.
[33] G. A. Margulis. Explicit group theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators. Problems of Information Transmission, 24(1):39–46, July 1988.
[34] A. Naor. Sparse quadratic forms and their geometric applications (after batson, spielman and srivastava). arXiv preprint arXiv:1101.4324, 2011.
[35] A. Nilli. On the second eigenvalue of a graph. Discrete Math, 91:207–210, 1991.
[36] J. Renegar. Hyperbolic programs, and their derivative relaxations. Foundations of Computational Mathematics, 6(1):59–79, 2006.
[37] M. Rudelson and R. Vershynin. Sampling from large matrices: An approach through geometric functional analysis. Journal of the ACM, 54(4):21, 2007.
[38] D. Spielman and N. Srivastava. Graph sparsification by effective resistances. SIAM Journal on Computing, 40(6):1913–1926, 2011.
[39] D. A. Spielman and S.-H. Teng. Spectral sparsification of graphs. SIAM Journal on Computing, 40(4):981–1025, 2011.
[40] N. Srivastava. Windows on theory: Discrepancy, graphs, and the kadison-singer problem, 2013. from http://windowsontheory.org/2013/07/11/discrepancy-graphs-and-the-kadison-singer-conjecture-2/.
[41] N. Srivastava and R. Vershynin. Covariance estimation for distributions with $2 + \varepsilon$ moments. The Annals of Probability, 41(5):3081–3111, 2013.
[42] J. A. Tropp. User-friendly tail bounds for sums of random matrices. Foundations of Computational Mathematics, 12(4):389–434, 2012.
[43] R. Vershynin. John’s decompositions: Selecting a large part. Israel Journal of Mathematics, 122(1):253–277, 2001.
[44] N. Weaver. The Kadison–Singer problem in discrepancy theory. Discrete Mathematics, 278(1–3):227–239, 2004.
[45] P. Youssef. Restricted invertibility and the Banach–Mazur distance to the cube. Mathematika, pages 1–18, 2012.

Yale University and Crisply, Inc
E-mail: adam.marcus@yale.edu
2. Yale University
E-mail: spielman@cs.yale.edu
3. Microsoft Research, India
E-mail: niksri@microsoft.com