TWO NEW DEFINITIONS ON CONVEXITY AND RELATED INEQUALITIES

MEVLÜT TUNÇ

Abstract. We have made some new definitions using the inequalities of Young’ and Nesbitt’. And we have given some features of these new definitions. After, we established new Hadamard type inequalities for convex functions in the Young and Nesbitt sense.

1. Introduction

The following definition is well known in the literature: A function \( f : I \rightarrow \mathbb{R} \), \( \emptyset \neq I \subseteq \mathbb{R} \), is said to be convex on \( I \) if inequality

\[
(1.1) \quad f((1-t)x + ty) \leq tf(x) + (1-t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \). Geometrically, this means that if \( P, Q \) and \( R \) are three distinct points on the graph of \( f \) with \( Q \) between \( P \) and \( R \), then \( Q \) is on or below chord \( PR \).

Let \( I \) denote a suitable interval of the real line \( \mathbb{R} \). A function \( f : I \rightarrow \mathbb{R} \) is called convex in the Jensen sense or \( J \)-convex or midconvex if

\[
(1.2) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}
\]

for all \( x, y \in I \).

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function and \( a, b \in I \) with \( a < b \). The following double inequality:

\[
(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

is known in the literature as Hadamard inequality for convex function. Keep in mind that some of the classical inequalities for means can come from (1.3) for convenient particular selections of the function \( f \). If \( f \) is concave, this double inequality hold in the inversed way.

In [1], Pachpatte established two Hadamard-type inequalities for product of convex functions.

**Theorem 1.** Let \( f, g : [a,b] \subseteq \mathbb{R} \rightarrow [0,\infty) \) be convex functions on \( [a,b] \), \( a < b \). Then

\[
(1.4) \quad \frac{1}{b-a} \int_a^b f(x)g(x) \, dx \leq \frac{1}{3} M(a,b) + \frac{1}{6} N(a,b)
\]

2000 Mathematics Subject Classification. 26D15.

Key words and phrases. Hadamard’s, Young’s, Nesbitt’s, Pachpatte’s inequality.
and
\[ 2f \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b) \]

where \( M(a, b) = f(a)g(a) + f(b)g(b) \) and \( N(a, b) = f(a)g(b) + f(b)g(a) \).

We recall the well-known Young’s inequality which can be stated as follows.

**Theorem 2. (Young’s inequality, see [2], p. 117)** If \( a, b > 0 \) and \( p, q > 1 \) satisfy

\[ \frac{1}{p} + \frac{1}{q} = 1, \]

then

\[ \frac{1}{p} a^p + \frac{1}{q} b^q \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b) \]

Equality holds if and only if \( a^p = b^q \).

**Example 1. (Nesbitt’s inequality, see [3], p. 37)** For \( a, b, c \in \mathbb{R}^+ \), we have

\[ \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}. \]

In the following sections our main results are given. We establish two new class of convex functions and then we obtain new Hadamard type inequalities.

### 2. Young-Convexity and Related Inequalities

**Remark 1.** If we take \( a = t^{\frac{1}{p}} \) and \( b = t^{\frac{1}{q}} \) in (1.5), we have

\[ 1 \leq \frac{1}{p} t^{\frac{1}{p}-1} + \left( 1 - \frac{1}{p} \right) t^{\frac{1}{q}} \]

for all \( t \in [0, 1] \).

We now give a new definition of the following using the above remark.

**Definition 1.** Let \( f : I \to [0, \infty) \), \( \emptyset \neq I \subseteq \mathbb{R} \), \( x, y \geq 0 \) and \( p > 1 \). We say that \( f : I \subseteq \mathbb{R} \to [0, \infty) \) is Young-convex function, if

\[ f \left( tx + (1-t)y \right) \leq \left( \frac{1}{p} t^{\frac{1}{p}} + \frac{p-1}{p} t^{\frac{1}{p}+1} \right) f(x) \]

\[ + \left( \frac{p-1}{p} (1-t) t^{\frac{1}{p}} + \frac{1}{p} t^{\frac{1}{p}-1} (1-t) \right) f(y) \]

for all \( x, y \in I \) and \( t \in (0, 1) \). We denote this by \( f \in Yng(I) \). If the inequality (2.2) is reversed, then \( f \) is said to be Young-concave function. Obviously, if we take \( p \to 1 \) in (2.2), we obtain ordinary convex function in (1.1).

**Proposition 1.** If \( f \in Yng(I) \), then \( f \) is non-negative on \( [0, \infty) \).
Proof. We have, for $\kappa \in \mathbb{R}_+$

$$f(\kappa) = f\left(\frac{\kappa}{2} + \frac{\kappa}{2}\right)$$

$$\leq \left(\frac{1}{p} \left(\frac{1}{2}\right)^{\frac{1}{p}} + \frac{p - 1}{p} \left(\frac{1}{2}\right)^{1 + \frac{1}{p}}\right) f(\kappa)$$

$$+ \left(\frac{p - 1}{p} \left(\frac{1}{2}\right)^{\frac{p}{p+1}} + \frac{1}{p} \left(\frac{1}{2}\right)^{\frac{1}{p}}\right) f(\kappa)$$

$$= 2 \left(\frac{1}{p} \left(\frac{1}{2}\right)^{\frac{1}{p}} + \frac{p - 1}{p} \left(\frac{1}{2}\right)^{1 + \frac{1}{p}}\right) f(\kappa)$$

$$= \frac{p + 1}{p} \left(\frac{1}{2}\right)^{\frac{1}{p}} f(\kappa)$$

Thus, \( \left(\frac{p + 1}{p} \left(\frac{1}{2}\right)^{\frac{1}{p}} - 1\right) f(\kappa) \geq 0 \) and so \( f(\kappa) \geq 0 \).

\( \square \)

Theorem 3. Let \( f : I \subset \mathbb{R} \to [0, \infty) \) be a Young-convex function and \( p > 1 \). Then the following inequality holds:

(2.3) \[ \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{p^2 + 2p}{(p+1)(2p+1)} f(a) + \frac{3p^2}{(p+1)(2p+1)} f(b) \]

Proof. By the Young-convexity of \( f \), we have that

$$f(tx + (1-t)y) \leq \left(\frac{1}{p} t^{\frac{1}{p}} + \frac{p - 1}{p} t^{1+\frac{1}{p}}\right) f(x)$$

$$+ \left(\frac{p - 1}{p} (1-t) t^{\frac{1}{p}} + \frac{1}{p} t^{\frac{1}{p}-1} (1-t)\right) f(y)$$

Integrating both sides of the inequality above with respect to \( t \) on \([0,1]\), we have

$$\int_0^1 f(ta + (1-t)b) \, dt = \frac{1}{b-a} \int_a^b f(x) \, dx$$

$$\leq f(a) \frac{1}{p} \int_0^1 t^{\frac{1}{p}} \, dt + f(a) \frac{p - 1}{p} \int_0^1 t^{1+\frac{1}{p}} \, dt$$

$$+ f(b) \frac{p - 1}{p} \int_0^1 (1-t) t^{\frac{1}{p}} \, dt + f(b) \frac{1}{p} \int_0^1 t^{\frac{1}{p}-1} (1-t) \, dt$$

By computing the above integrals, we obtain

$$\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{p^2 + 2p}{(p+1)(2p+1)} f(a) + \frac{3p^2}{(p+1)(2p+1)} f(b)$$

which is the inequality in (2.3).

\( \square \)

Remark 2. If we choose \( p \to 1 \), in (2.3), we obtain right hand side of (1.3).
Theorem 4. Let $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ be a Young-convex function and $p > 1$. Then the following inequality holds:

\begin{equation}
2^{\frac{p}{p+1}} \frac{p}{p+1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \left(\frac{p(p+2)}{(p+1)(1+2p)} + \frac{p-1}{p} \beta \left(\frac{1+p}{p}, 2\right) + \frac{1}{p} \beta \left(\frac{1}{p}, 2\right)\right) \left(\frac{f(a)+f(b)}{2}\right)
\end{equation}

Proof. By the Young-convexity of $f$, we have that

\[ f\left(\frac{x+y}{2}\right) \leq \left(\frac{1}{p} + 1\right) \left(\frac{1}{2}\right)^{\frac{1}{p}+1} (f(x) + f(y)). \]

If we choose $x = ta + (1-t)b$, $y = tb + (1-t)a$, we get

\begin{equation}
\label{eq:2.5}
f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{p} + 1\right) \left(\frac{1}{2}\right)^{\frac{1}{p}+1} \int_0^1 (f(ta + (1-t)b) + f(tb + (1-t)a)) \, dt.
\end{equation}

for all $t \in [0,1]$. Then, integrating both side of (2.5) with respect to $t$ on $[0,1]$, we have

\[ f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{p} + 1\right) \left(\frac{1}{2}\right)^{\frac{1}{p}+1} \int_0^1 (f(ta + (1-t)b) + f(tb + (1-t)a)) \, dt. \]

Use of the changing of variable, we have

\[ 2^{\frac{p}{p+1}} \frac{p}{p+1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx, \]

which is the first inequality in (2.4).

To prove the second inequality in (2.4), we use the right side of (2.5) and using Young-convexity of $f$, we have

\[ \left(\frac{1}{p} + 1\right) \left(\frac{1}{2}\right)^{\frac{1}{p}+1} (f(ta + (1-t)b) + f(tb + (1-t)a)) \leq \left(\frac{1}{p} + 1\right) \left(\frac{1}{2}\right)^{\frac{1}{p}+1} (f(a) + f(b)) \left(\frac{1}{p} t^{\frac{1}{p}} + \frac{p-1}{p} t^{1+\frac{1}{p}} + \frac{p-1}{p} t^{\frac{1}{p}+1} + 1 - t\right). \]

Integrating the both side of the above inequality with respect to $t$ on $[0,1]$, we have

\[ \frac{1}{b-a} \int_a^b f(x) \, dx \leq \left(\frac{p(p+2)}{(p+1)(1+2p)} + \frac{p-1}{p} \beta \left(\frac{1+p}{p}, 2\right) + \frac{1}{p} \beta \left(\frac{1}{p}, 2\right)\right) \left(\frac{f(a)+f(b)}{2}\right) \]

Which is the second inequality in (2.4).

\[ \square \]

Remark 3. If we choose $p \rightarrow 1$, in (2.4), we obtain the inequality of (1.3).
Theorem 5. Let \( f, g : I \subset \mathbb{R} \to [0, \infty) \) be Young-convex functions and \( p > 1 \). Then the following inequality holds:

\[
\frac{1}{b - a} \int_a^b f(x)g(x) \, dx \\ \leq \left( \frac{1}{p(2 + p)} + \frac{p - 1}{p(1 + p)} + \frac{(p - 1)^2}{p(2 + 3p)} \right) f(a)g(a) \\ + \left( \left( \frac{p - 1}{p} \right)^2 \beta \left( \frac{2}{p} + 1, 3 \right) + \frac{2(p - 1)}{p^2} \beta \left( \frac{2}{p}, 3 \right) + \frac{1}{p} \beta \left( \frac{2}{p - 1}, 3 \right) \right) f(b)g(b) \\ + \left( \frac{p - 1}{p^2} \beta \left( \frac{2}{p}, 2 \right) + \left( \frac{p - 1}{p} \right)^2 \beta \left( \frac{2}{p} + 2, 2 \right) + \frac{1}{p} \beta \left( \frac{2}{p + 1}, 2 \right) \right) \left( f(a)g(b) + f(b)g(a) \right).
\]

Proof. Since \( f, g \) are Young-convex functions on \( I \), we have

\[
f(ta + (1 - t)b) \leq \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) f(a) + \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) f(b) \\
g(ta + (1 - t)b) \leq \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) g(a) + \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) g(b)
\]

for all \( t \in [0, 1] \). Since \( f \) and \( g \) are non-negative,

\[
\begin{align*}
& f(ta + (1 - t)b)g(ta + (1 - t)b) \\
& \leq \left[ \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) f(a) + \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) f(b) \right] \\
& \times \left[ \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) g(a) + \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) g(b) \right] \\
& = \left[ \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) \right] f(a)g(a) \\
& + \left[ \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) \right] f(b)g(b) \\
& + \left[ \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) \right] f(a)g(b) \\
& + \left[ \left( \frac{1}{p} t^\frac{p}{p-1} + \frac{p-1}{p} t^{1+\frac{1}{p-1}} \right) \left( \frac{p-1}{p} (1-t) t^\frac{p}{p-1} + \frac{1}{p} t^{1+\frac{1}{p-1}} (1-t) \right) \right] f(b)g(a)
\end{align*}
\]
Then if we integrate the both side of the inequality above with respect to $t$ on $[0, 1]$, we have

$$f (ta + (1 - t)b)g (ta + (1 - t)b) \leq \left( \frac{1}{p(2 + p)} + \frac{p - 1}{p(1 + p)} + \frac{(p - 1)^2}{p(2 + 3p)} \right) f (a)g (a)$$

$$+ \left( \left( \frac{p - 1}{p} \right)^2 \beta \left( \frac{2}{p} + 1, 1 \right) + \frac{2(p - 1)}{p^2} \beta \left( \frac{2}{p}, 3 \right) + \frac{1}{p^2} \beta \left( \frac{2}{p}, 3 \right) \right) f (b)g (b)$$

$$+ \left( \left( \frac{2(p - 1)}{p^2} \beta \left( \frac{2}{p} + 1, 1 \right) + \left( \frac{p - 1}{p} \right)^2 \beta \left( \frac{2}{p} + 2, 2 \right) + \frac{1}{p^2} \beta \left( \frac{2}{p}, 2 \right) \right) (f (a)g (b) + f (b)g (a)).$$

By changing of the variables, we obtain the desired result. $\square$

**Remark 4.** If we choose $p \to 1$, in (2.6), we obtain the inequality of (1.4).

### 3. Nesbitt-Convexity and Related Inequalities

**Remark 5.** If we take $a = t$, $b = \frac{1}{2}$ and $c = 1 - t$ in (1.6), we have

$$(3.1) \quad 1 \leq \frac{2t}{3 - 2t} + \frac{2(1 - t)}{1 + 2t}$$

for all $t \in [0, 1]$.

We now give a new definition of the following using the above remark.

**Definition 2.** Let $f : I \to [0, \infty)$, $\emptyset \neq I \subseteq \mathbb{R}$, $x, y > 0$. We say that $f : I \subset \mathbb{R} \to [0, \infty)$ is Nesbitt-convex function, if

$$(3.2) \quad f (tx + (1 - t)y) \leq \left( \frac{2t^2}{3 - 2t} + \frac{2t(1 - t)}{1 + 2t} \right) f (x) + \left( \frac{2t(1 - t)}{3 - 2t} + \frac{2(1 - t)^2}{1 + 2t} \right) f (y)$$

for all $x, y \in I$ and $t \in (0, 1]$. We denote this by $f \in Nsb (I)$. If the inequality (3.2) is reversed, then $f$ is said to be Nesbitt-concave function.

**Remark 6.** If we choose $t = \frac{1}{2}$, in (3.2), we obtain the inequality of (1.2).

**Theorem 6.** Let $f : I \subset \mathbb{R} \to [0, \infty)$ be a Nesbitt-convex function. Then the following inequality holds:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f (x)dx \leq \ln \frac{3\sqrt{3}}{e} (f (a) + f (b))$$

**Proof.** By the Nesbitt-convexity of $f$, we have that

$$f \left( \frac{x + y}{2} \right) \leq \frac{f (x) + f (y)}{2}.$$

If we choose $x = ta + (1 - t)b$, $y = tb + (1 - t)a$, we get

$$f \left( \frac{a + b}{2} \right) \leq \frac{f (ta + (1 - t)b) + f (tb + (1 - t)a)}{2}.$$
for all $t \in [0, 1]$. Then, integrating both side of the resulting inequality with respect to $t$ on $[0, 1]$, we have

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{2} \int_0^1 (f(ta + (1 - t)b) + f(tb + (1 - t)a)) \, dt.$$  

Use of the changing of variable, we have

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx,$$

which is the first inequality in (3.5).

To prove the second inequality in (3.5), we use the right side of (3.4) and by using Nesbitt-convexity of $f$, we have

$$\left( f(ta + (1 - t)b) + f(tb + (1 - t)a) \right) \leq \frac{2}{f(a) + f(b)} \left( \frac{2t^2}{3 - 2t} + \frac{2(1 - t)}{1 + 2t} \right) + \left( \frac{2t(1 - t)}{3 - 2t} + \frac{2(1 - t)^2}{1 + 2t} \right).$$

Integrating the both side of the above inequality with respect to $t$ on $[0, 1]$, we have

$$\frac{1}{b - a} \int_a^b f(x) \, dx \leq (3 \ln 3 - 2) \left[ \frac{f(a) + f(b)}{2} \right]$$

Which is the second inequality in (3.5).

**Theorem 7.** Let $f, g : I \subset \mathbb{R} \to [0, \infty)$ be Nesbitt-convex functions. Then the following inequality holds:

$$\frac{1}{b - a} \int_a^b f(x) g(x) \, dx \leq \left( \frac{125}{6} - \frac{147}{8} \ln 3 \right) (f(a) g(a) + f(b) g(b)) + \left( \frac{117}{8} \ln 3 - \frac{95}{6} \right) (f(a) g(b) + f(b) g(a)) .$$

**Proof.** Since $f, g$ are Nesbitt’s-convex functions on $I$, we have

$$f(ta + (1 - t)b) \leq \left( \frac{2t^2}{3 - 2t} + \frac{2(1 - t)}{1 + 2t} \right) f(a) + \left( \frac{2t(1 - t)}{3 - 2t} + \frac{2(1 - t)^2}{1 + 2t} \right) f(b)$$

and

$$g(ta + (1 - t)b) \leq \left( \frac{2t^2}{3 - 2t} + \frac{2(1 - t)}{1 + 2t} \right) g(a) + \left( \frac{2t(1 - t)}{3 - 2t} + \frac{2(1 - t)^2}{1 + 2t} \right) g(b)$$

for all $t \in (0, 1]$. Since $f$ and $g$ are non-negative,

$$f(ta + (1 - t)b) g(ta + (1 - t)b) \leq \left[ \left( \frac{2t^2}{3 - 2t} + \frac{2(1 - t)}{1 + 2t} \right) f(a) + \left( \frac{2t(1 - t)}{3 - 2t} + \frac{2(1 - t)^2}{1 + 2t} \right) f(b) \right] \times \left[ \left( \frac{2t^2}{3 - 2t} + \frac{2(1 - t)}{1 + 2t} \right) g(a) + \left( \frac{2t(1 - t)}{3 - 2t} + \frac{2(1 - t)^2}{1 + 2t} \right) g(b) \right]$$
Then if we integrate the both side of the resulting inequality with respect to \( t \) on \([0,1]\), we have

\[
\int_0^1 f (ta + (1-t)b) g (ta + (1-t)b) \, dt 
\leq \left[ \frac{125}{6} - \frac{147}{8} \ln 3 \right] [f(a) g(a) + f(b) g(b)] \\
+ \left[ \frac{117}{8} \ln 3 - \frac{95}{6} \right] [f(a) g(b) + f(b) g(a)].
\]

By changing of the variables, we obtain the desired result. \( \square \)

**Theorem 8.** Let \( f, g : I \subset \mathbb{R} \to [0, \infty) \) be Nesbitt-convex and similarly ordered functions. Then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x) g(x) \, dx 
\leq \left( 5 - \frac{30}{8} \ln 3 \right) (f(a) g(a) + f(b) g(b)) \\
= 0.8802 \times M(a,b).
\]

**Proof.** Using similar arguments as in the proof of Theorem \( \square \) from \( 3.6 \), we can write

\[
\int_0^1 f (ta + (1-t)b) g (ta + (1-t)b) \, dt 
\leq \left[ \frac{125}{6} - \frac{147}{8} \ln 3 \right] [f(a) g(a) + f(b) g(b)] \\
+ \left[ \frac{117}{8} \ln 3 - \frac{95}{6} \right] [f(a) g(b) + f(b) g(a)] \\
\leq \left[ \frac{125}{6} - \frac{147}{8} \ln 3 \right] [f(a) g(a) + f(b) g(b)] \\
+ \left[ \frac{117}{8} \ln 3 - \frac{95}{6} \right] [f(a) g(a) + f(b) g(b)]
\]

By changing of the variables, we obtain the desired result. \( \square \)

**References**

[1] B.G. Pachpatte, On some inequalities for convex functions, RGMIA Res. Rep. Coll., 6 (E), 2003.

[2] S.S. Dragomir, R.P. Agarwal and N.S. Barnett, Inequalities for beta and gamma functions via some classical and new integral inequalities, J. Inequal. & Appl., 5 (2000), 103-165.

[3] R.B. Manfrino, J.A.G. Ortega and R.V. Delgado, Inequalities A Mathematical Olympiad Approach, Birkhäuser, Basel-Boston-Berlin, 2009.

Department of Mathematics, Faculty of Art and Sciences, University of Kilis 7 Aralik, 79000, Turkey

E-mail address: mevluttunc@kilis.edu.tr