UNIQUENESS AND EXPONENTIAL MIXING FOR THE MEASURE OF MAXIMAL ENTROPY FOR PIECEWISE HYPERBOLIC MAPS

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Abstract. For a class of piecewise hyperbolic maps in two dimensions, we propose a combinatorial definition of topological entropy by counting the maximal, open, connected components of the phase space on which iterates of the map are smooth. We prove that this quantity dominates the measure theoretic entropies of all invariant probability measures of the system, and then construct an invariant measure whose entropy equals the proposed topological entropy. We prove that our measure is the unique measure of maximal entropy, that it is ergodic, gives positive measure to every open set, and has exponential decay of correlations against Hölder continuous functions. As a consequence, we also prove a lower bound on the rate of growth of periodic orbits. The main tool used in the paper is the construction of anisotropic Banach spaces of distributions on which the relevant weighted transfer operator has a spectral gap. We then construct our measure of maximal entropy by taking a product of left and right maximal eigenvectors of this operator.

1. Introduction

There has been a flurry of recent activity in establishing the existence and uniqueness of equilibrium states for broad classes of potentials and systems outside the uniformly hyperbolic setting. This topic traces back to the work of Margulis [Ma1], who proved that the number of periodic orbits of length at most \( L \) for the geodesic flow on a compact manifold of strictly negative curvature grow at an exponential rate determined by the topological entropy \( h_{\text{top}} \) of the flow. To prove this result, Margulis constructed an invariant measure \( \mu_{\text{top}} \) via conditional measures on the local stable and unstable manifolds of the flow which scaled by \( e^{\pm th_{\text{top}}} \). An important feature of the measure \( \mu_{\text{top}} \) is that it is the unique measure of maximal entropy for the flow: its measure-theoretic entropy equals the topological entropy \( h_{\text{top}} \).

These results were generalized and further developed for broader classes of Anosov and Axiom A flows and diffeomorphisms through the work of Sinai, Bowen, Ruelle and many others using thermodynamic formalism [S, BR, Ru1], topological techniques [Bo1, Bo2, Bo3, Bo4], and dynamical zeta functions [Pa1, Pa2]. Later, Dolgopyat’s proof of exponential decay of correlations for some geodesic flows [Do] led to more precise asymptotics for counting periodic orbits [PS].

Recent attempts to extend proofs of existence and uniqueness of equilibrium states in general, and measures of maximal entropy in particular, to the nonuniformly hyperbolic setting have employed symbolic dynamics [Sa1, Sa2, LiM, BS], as well as adapting the approach of Bowen via a notion of non-uniform specification [BCFT, CFT, CKW, CPZ]. These works have greatly broadened the classes of systems for which one can prove the existence and uniqueness of equilibrium states, yet they do not usually provide rates of mixing for these measures.

Simultaneously, there have been advances made in the study of the transfer operator associated with hyperbolic systems with singularities, first to piecewise hyperbolic maps (with bounded derivatives) [DL, BG1, BG2], and then to dispersing and other hyperbolic billiards [DZ1, DZ2, DZ3]. This approach, which avoids the coding associated with Markov partitions or extensions, exploits the hyperbolicity of the system to prove that the action of the transfer operator on appropriately defined Banach spaces has good spectral properties. It was used recently to prove exponential decay
of correlations for the finite horizon Sinai billiard flow \[BDL\], adapting ideas of Dolgopyat \[Do\] and Liverani \[L2\]. It was then applied to prove the existence and uniqueness of a measure of maximal entropy for finite horizon Sinai billiard maps \[BD\], establishing a variational principle for this class of billiards.

For hyperbolic systems with discontinuities, a priori results that guarantee the existence of an invariant measure maximizing the entropy, or even a simple definition of topological entropy, are not available as they are for continuous maps and flows. Indeed, in order to overcome this shortcoming, one approach is to redefine the map as a continuous map on a noncompact space, and then apply generalized definitions of topological entropy in this setting. Yet such definitions can be cumbersome to work with, and the resulting entropy can depend on the choice of metric in the reduced space.

To simplify matters, the first step in \[BD\] is to define an intuitive notion of growth in complexity given by the number of domains of continuity \(\mathcal{M}_0^n\) for the map \(T^n\). This leads to an asymptotic quantity \(h_*\), which plays the role of topological entropy \[BD, Definition 2.1\] (see also Definition 2.5 below). This quantity is proved to equal the supremum of the measure-theoretic entropies of the invariant measures for the billiard map, and the unique measure \(\mu_*\) whose entropy achieves this maximum is constructed by taking a product of left and right maximal eigenvectors of an associated weighted transfer operator \(L\), following the methods in \[GL\] which generalize the classical Parry construction.

Despite this success, the weight in the relevant transfer operator in \[BD\] is unbounded due to the unbounded expansion and contraction that occur near grazing collisions in dispersing billiards. The presence of this weight forced significant changes in the Banach spaces from \[DZ1\] on which the operator acted, and it was not possible to establish a spectral gap in this context. Indeed, the rate of mixing for the measure of maximal entropy is an open question for billiards.

The purpose of the present paper is to demonstrate that under the additional assumption that the derivative of the map is bounded, the techniques employed in \[BD\] are sufficient to prove the existence and uniqueness of a measure of maximal entropy that is exponentially mixing. To this end, we study a class of piecewise hyperbolic maps, defined in Section 2. The existence and statistical properties of Sinai-Ruelle-Bowen (SRB) measures\[1\] for this class of maps has been studied via a variety of techniques \[P, L1, Y, DL, BG2\]. Transfer operators with more general potentials were studied in \[BG2\] and a bound on the essential spectral radius was obtained; however, lower bounds on the spectral radius of the transfer operator were not obtained, so that no spectral gap was established and the related invariant measures were not constructed. Currently there are no results regarding measures of maximal entropy, nor more general equilibrium states for this class of maps.

In structure, this paper mainly follows the approach in \[BD\]. Yet there are several key differences between the class of piecewise hyperbolic maps studied here and dispersing billiards. The primary simplification is that our maps have bounded derivatives, as mentioned above, and this fact permits us to prove a spectral gap for the relevant transfer operator, which leads to exponential decay of correlations for the measure of maximal entropy \(\mu_*\). However, there are two additional difficulties in the current setting that are not present in Sinai billiards.

(i) We do not assume that the singularity curves for our map \(T\) satisfy the continuation of singularities property enjoyed by billiards.

(ii) We do not assume the map is associated with a continuous flow.

Point (i) creates significant complications in the study of the rate of growth of \(\#\mathcal{M}_0^n\), the number of maximal, connected domains of continuity of \(T^n\). In particular, the submultiplicative property of \(\#\mathcal{M}_0^n\) proved in \[BD\, Lemma 3.3\], and often exploited in that work, may fail in the present context due to the fact that dynamical refinements of \(\#\mathcal{M}_0^n\) may have elements that are not simply connected. Indeed, the uniform exponential upper and lower bounds on \(\#\mathcal{M}_0^n\) proved in

\[1\]Recall that an SRB measure for a hyperbolic system is an invariant probability measure whose conditional measures on local unstable manifolds are absolutely continuous with respect to the Riemannian volume.
Proposition 2.12 are completed only after the spectral gap for the operator $L$ is established. Point (ii) has several consequences. The first is that the continuous flow provides a linear bound on the growth in complexity, which is exploited in [BD]. In the present work, this property is replaced by the complexity assumption (P1) introduced in Section 2.1 while the growth in complexity may be exponential for our class of maps, it is slow relative to the minimum hyperbolicity constant for the map (and therefore also relative to $h_*$ by Lemma 3.6(d)). The second consequence of (ii) is that in [BD], there is a positive minimum distance between orbits that belong to different elements of $M_0$. In the present context this may fail, so in Section 2.1 we define an adapted metric which we use to define the dynamical Bowen balls instrumental in the estimation of the entropy of $\mu_*$ in Section 5.

The structure of the paper is as follows. We begin by defining in Definition 2.5 the exponential rate of growth in complexity, $h_*$, which counts the number of domains of continuity $M_0$ of $T^n$. This quantity dominates the measure-theoretic entropies of the invariant measures (Theorem 2.8). We then proceed to study the action of a weighted transfer operator, defined in Section 3.2. The Banach spaces we use are similar to those defined in [DL] (not [BD]) for this class of maps, yet the operator has significant differences from the transfer operator with respect to the SRB measure studied in [DL]. By proving a series of growth and fragmentation lemmas in Sections 3.5 and 3.6 that control the prevalence of short and long connected components of $T^{-n}W$ for local stable manifolds $W$, we are able to establish that the operator has a spectral gap in Section 4. Finally, in Section 5 we construct a measure $\mu_*$ out of the left and right eigenvectors of the transfer operator and show that it has exponential decay of correlations and that it is the unique invariant measure with entropy equal to $h_*$. The properties of the measure $\mu_*$ are summarized in Theorem 2.9. In Corollary 2.11 we derive our asymptotic bound on the growth rate of periodic orbits, applying results of [LiM] and [Bu]. Finally, as a byproduct of this approach, uniform growth rates are established for $\#M_0^n$ and the length $|T^{-n}W|$ of stable manifolds $W$; these are stated in Proposition 2.12 and Corollary 2.13 respectively.

2. Setting, Definitions and Results

In this section, we introduce a set of formal assumptions on our class of piecewise hyperbolic maps and state the principal results of the paper.

2.1. Piecewise Hyperbolic Maps. Let $M$ be a compact two-dimensional Riemannian manifold, possibly with boundary and not necessarily connected, and let $T : M \subset \subset$ be a piecewise uniformly hyperbolic map in the sense described below. There exist a finite number of pairwise disjoint open, simply connected regions \{$M^+_i\}_{i=1}^d$ such that $\cup \overline{M^+_i} = M$ and $\partial M^+_i$ comprises finitely many $C^1$ curves of finite length. We will refer to $S^+ = M \setminus \cup_i M^+_i$ as the singularity set for $T$.

Define $M^-_i = T(M^+_i)$. We assume that $\cup_i \overline{M^-_i} = M$ and refer to the set $S^- = M \setminus \cup_i M^-_i$ as the singularity set for $T^{-1}$. We require that $T \in \text{Diff}^2(M \setminus S^+, M \setminus S^-)$ and that on each $M^+_i$, $T$ has a $C^2$ extension\footnote{This implies in particular that $||DT||$ is bounded on each $M^+_i$, so that this class of maps does not include dispersing billiards.} to $M^+_i$. Since the extension of $T$ is defined on $\partial M^+_i$, we will write $T(S^+)$ to denote the set of images of these boundary curves (on which the extension of $T$ may be multi-valued). In this notation, $T(S^+) = S^-$ and $T^{-1}(S^-) = S^+$.

On each $M_i$, $T$ is uniformly hyperbolic: i.e., there exist constants $\Lambda > 1$, $\kappa \in (0, 1)$ and two $DT$-strictly-invariant families of cones $C^a$ and $C^s$, continuous in each $\overline{M^+_i}$ which satisfy,
There exist each $M$ overlap. Following convention 2.1, the extended space is compact in the metric $d$. Denote by $\bar{M}_i^+$ the set of singularity curves for $T^n$ and by $\bar{S}_n^+ = \cup_{i=0}^{n-1} T^n s^+$ the set of singularity curves in $S^+$. Let $K(n)$ denote the maximum number of singularity curves in $S_n^+$ or in $S_n^+$ which intersect at a single point. We make the following assumption regarding the complexity of $T$.

\begin{equation}
(P1) \quad \text{There exist } \alpha_0 > 0 \text{ and an integer } n_0 > 0, \text{ such that } \Lambda \kappa^{\alpha_0} > 1 \text{ and } (\Lambda \kappa^{\alpha_0})^{n_0} > K(n_0).
\end{equation}

Condition (P1) can always be satisfied if $K(n)$ has polynomial growth (as is the case with a Sinai billiard on a torus); however, since (P1) is required only for some fixed $n_0$, it is not necessary to control $K(n)$ for all $n$ in order to verify the condition.

\textbf{Remark 2.2.} If property (P1) holds for $\alpha_0$, then it holds for all $0 < \alpha < \alpha_0$ with the same $n_0$. Notice also that $K(kn_0) \leq K(n_0)^k$ which implies that the inequality in (P1) can be iterated to make $(\Lambda \kappa^{\alpha_0})^{-k n_0} K(kn_0)$ arbitrarily small once (P1) is satisfied for some $n_0$.

In Section 3.1 we will define a set of admissible stable curves $\hat{W}^s$, with tangent vectors belonging to the stable cone, which we will use to define our norms. For $W \in \hat{W}^s$, let $\hat{K}_n$ denote the number of smooth connected components of $T^{-n} W$. For a fixed $N$, by shrinking the maximum length $\delta_0$ of leaves in $\hat{W}^s$, we can require that $K_N \leq K(N) + 1$. This implies that choosing $N = k n_0$, we can make $(\Lambda \kappa^{\alpha_0})^{-N} K_N$ arbitrarily small.
We use the embedding lemma, Lemma 3.3, connecting our Banach spaces to the standard spaces of distributions.

Thus, if one wants to apply the present results to a smooth map, for example a toral automorphism, one should first partition the torus into a finite number of simply connected ‘rectangles’ with boundaries transverse to $C^u$ and $C^s$. Then Theorem 2.9 implies that the rate of growth in cardinality of dynamical refinements of this partition, $h_s$, will equal the topological entropy of the automorphism.

Remark 2.4. Property (P1) is standard for piecewise hyperbolic maps, and a variant of it has been used in [11, 12, 24, 15]. The most common form is only to require the complexity bound in one direction, for example on $S^n_-$ in [11, 12, 15]. Here, we assume the symmetric version on both $S^-_n$ and $S^+_n$ in order to prove the super-multiplicativity property for $\#M^n_0$, Proposition 3.12.

It follows from the piecewise hyperbolicity of $T$ and (P1) that $T$ admits an SRB measure [P, Theorem 1]. The requirement that $\mu_{SRB}$ be smooth in Property (P2) is less essential to our argument. We use $\mu_{SRB}$ as our reference measure rather than the Riemannian volume $\mu$ in order to simplify the estimates involving the transfer operator. Assuming that $\mu_{SRB}$ is smooth allows us to prove the embedding lemma, Lemma 3.3, connecting our Banach spaces to the standard spaces of distributions.

Our assumptions on the hyperbolicity of $T$ imply the following uniform expansion and bounded distortion properties along stable curves, which we record for future use. There exists $C_e > 0$ such that for any $W \in \hat{W}^s$ and $n \geq 0$,

$$|T^{-n}W| \geq C_e \Lambda^n |W|,$$

where $|W|$ denotes the arc length of $W$ in the metric induced by the Riemannian metric on $M$.

Suppose $W \in \hat{W}^s$ is such that $T^n$ is smooth on $W$ and $T^n W \in \hat{W}^s$, for $i = 0, \ldots, n$. We denote by $J_W T^n$ the Jacobian of $T^n$ along $W$ with respect to arc length. There exists $C_d > 0$, independent of $W$, such that for all $x, y \in W$ and all $n \geq 0$,

$$\left| \frac{J_W T^n(x)}{J_W T^n(y)} - 1 \right| \leq C_d d_W(x, y),$$

where $d_W(\cdot, \cdot)$ denotes arc length distance along $W$.

2.2. A Definition of Topological Entropy. Following [BD], for $k, n \geq 0$, let $\mathcal{M}^n_{\pm k}$ denote the set of maximal connected components of $M \setminus (S^n_+ \cup S^n_-)$, where we define $S^n_0 = \emptyset$. Note that by definition, elements of $\mathcal{M}^n_{\pm k}$ are open in $M$. With this notation, $\mathcal{M}^n_0$ denotes the set of maximal, open, simply connected components of $M$ on which $T^n$ is continuous, while $\mathcal{M}^n_{\pm}$ has the analogous property for $T^{-n}$. We remark also that the requirement that each $M^n_i$ be open and simply connected prevents the partition $\mathcal{M}^n_0$ from being trivial, and implies in particular that the diameter of elements of $\mathcal{M}^n_{\pm}$ tends to 0 as $n$ gets large.

Similarly, the transversality assumptions coupled with the finiteness requirement on the number of smooth curves in $S^+$ guarantee that $\#\mathcal{M}^n_{\pm}$ is finite for each $k$ and $n$.

Definition 2.5. (Topological entropy of $T$.) Define $h_s(T) = \lim_{n \to \infty} \frac{1}{n} \log (\# \mathcal{M}^n_0)$. [3]
By definition, if $A \in \mathcal{M}_0^n$, then $T^n A \in \mathcal{M}_0^{-n}$, so that $\# \mathcal{M}_0^n = \# \mathcal{M}_0^{-n}$. Thus $h_+(T) = h_+(T^{-1})$, i.e. this definition is symmetric in time. Indeed, the limsup in the definition is in fact a limit, which follows from Proposition 2.12.

We begin by establishing that the quantity $h_+$ is finite.

**Lemma 2.6.** For a piecewise hyperbolic map $T$ as defined in Section 2.1 but not necessarily satisfying conditions (P1) and (P2), the quantity $h_+ < \infty$.

**Proof.** The elements of $\mathcal{M}_i^1$ are simply the domains $M_i^+$. For any $n \geq 1$, elements of $\mathcal{M}_0^{n+1}$ are created by (the image under $T^{-1}$ of) the connected components of the intersection of an element of $\mathcal{M}_0^n$ with one of the domains $M_i^{-}$. By assumption, $S_i^+$ and $S_i^-$ comprise finitely many $C^1$ curves which either coincide or are uniformly transverse. Since $T$ is $C^2$ on the closure of each $M_i^+$, the same is true of the sets $S_n^+$ and $S_n^-$. Moreover, elements of $S_n^+$ have a uniform bound (in $n$) on their derivative.

Consider the intersection $A \cap M_i^-$ for $A \in \mathcal{M}_0^n$. Connected components of this set are created by intersections of $\partial A$ with elements of $S_i^-$. Since $\partial A \subset S_i^+$, by the compactness of $M$ and uniform transversality, $\partial A$ can intersect each smooth curve in $S_i^-$ a finite number of times, with uniform upper bound $B > 0$ independent of $n$. Thus the number of connected components of $A \cap M_i^-$ is bounded by $B(\#S_i^-)$. Since this bound holds for each $A \in \mathcal{M}_0^n$, we have

$$\# \mathcal{M}_0^{n+1} \leq (\# \mathcal{M}_0^n)B(\#S_i^-) \leq dB^n(\#S_i^-)^n,$$

where $d$ is the number of domains $M_i^+$.

In order to connect $h_+ = h_+(T)$ to the dynamical refinements of a fixed partition, for each $k \in \mathbb{N}$, define $\mathcal{P}_k$ to be the maximal connected components of $M$ on which $T^k$ and $T^{-k}$ are continuous. That is, $\mathcal{P}_k$ is the partition of $M$ defined by $M \setminus (S_k^+ \cup S_k^-)$, together with the boundary curves associated to each element, according to Convention 2.1. If we let $\mathring{\mathcal{P}}_k$ denote the collection of interiors of elements of $\mathcal{P}_k$, then we have $\mathring{\mathcal{P}}_k = \mathcal{M}_0^{k-1}$.

For $n \geq 1$, define $\mathcal{P}_k^n = \bigcup_{i=0}^n T^{-i}\mathcal{P}_k$. $\mathcal{P}_k^n$ is still a pointwise partition of $M$, yet its elements may not be open sets, and it may occur that $\mathcal{P}_k^n$ contains isolated points due to multiple boundary curves intersecting at one point. Furthermore, we do not assume that the elements of $\mathcal{P}_k^n$ are connected sets. Thus, although the collection of interiors $\mathring{\mathcal{P}}_k^n$ is a partition of $M \setminus (S_{k+n}^+ \cup S_{k+n}^-)$, it may be that $\mathring{\mathcal{P}}_k^n \neq \mathcal{M}_0^{k+n}$.

Our next lemma provides a rough upper bound on the number of isolated points that can be created by refinements of $\mathcal{P}_k$. Let $\#S_{\pm}$ denote the number of smooth components of $S_{\pm}$.

**Lemma 2.7.** For each $k, n \geq 1$, the number of isolated points in $\mathcal{P}_k^n$ is at most

$$2(\#S^- + \#S^+) \sum_{j=1}^{k+n} \#M_0^j.$$

**Proof.** By Convention 2.1 there are no isolated points in $\mathcal{P}_1$. Next, for each $n \geq 1$, at time $n$, isolated points in $\mathcal{P}_1^n$ can be produced by intersections of corner points in the boundary of $\mathcal{P}_1^{n-1}$ with elements of $S^-$. Moreover, each pair of smooth curves $S \in S^+_n$ and $S' \in S^-$ intersect at most twice per element of $\mathcal{M}_0^n$. Thus the number of new isolated points created at time $n$ is at most $2\#S^- \#M_0^n$. Applying this estimate inductively, we have

$$\text{number of isolated points in } \mathcal{P}_1^n \leq 2\#S^- \sum_{j=1}^n \#M_0^j.$$
Next, for each \( k \), applying a similar inductive argument to \( T^{-1} \), we have
\[
\text{number of isolated points in } P_k \leq 2^k \sum_{j=1}^{k} \#M_{-j}^0 + 2^k \sum_{j=1}^{k} \#M_{j}^0 \\
\leq 2(\#S^+ + \#S^-) \sum_{j=1}^{k} \#M_{j}^0,
\]
where we have used the fact that \( \#M_{j}^0 = \#M_{-j}^0 \). Finally, refining \( P_k \), we create at most \( 2^{j+1} \) new isolated points in \( P_k^j \) at time \( j \). Summing over \( j \leq n \), we complete the proof of the lemma.

\[\square\]

2.3. Statement of Main Results. Our first result establishes a connection between the rates of growth of \( \#P_k^n \) and \( \#M_k^n \), and uses this to prove that \( h_* \) dominates the measure-theoretic entropies of the invariant measures of \( T \).

**Theorem 2.8.** Let \( T \) be a piecewise hyperbolic map as defined in Section 2.1, but not necessarily satisfying conditions (P1) and (P2).

a) For each \( k,n \geq 1 \), \( \#P_k^n \leq \#M_{-k}^{k+n} \) and \( \#P_k^n \leq C(k+n)\#M_{-k}^{k+n} \), for some \( C > 0 \) depending only on \( T \).

b) For all \( k \geq 1 \), \( \lim_{n \to \infty} \frac{1}{n} \log(\#M_{-k}^n) = h_* \).

c) \( \sup_{k} \lim_{n \to \infty} \frac{1}{n} \log \#P_k^n = \sup_{k} \lim_{n \to \infty} \frac{1}{n} \log \#P_k^n \leq h_* \).

d) \( h_* \geq \sup \{ h_\mu(T) : \mu \text{ is an invariant probability measure for } T \} \).

**Proof.** a) The first inequality is straightforward since by definition, both \( \hat{P}_k^n \) and \( M_{-k}^{k+n} \) are partitions of \( M \setminus (S_+^{k+n} \cup S_-^k) \), yet \( \hat{P}_k^n \) may have disconnected components. Thus \( M_{-k}^{k+n} \) is a refinement of \( \hat{P}_k^n \). The second inequality follows by noting that \( \#P_k^n \) equals \( \#P_k^n \) plus isolated points, and then applying Lemma 2.7.

b) The value of the limsup is the same for each \( k \) since by definition, \( A \in M_k^n \) if and only if \( T^k A \in M_{0+k}^{n+k} \). Thus \( \#M_k^n = \#M_{0+k}^{n+k} \).

c) We first remark that \( \#P_{k+m}^n \leq \#P_k^n \#P_m^n \), and also \( \#\hat{P}_{k+m}^n \leq \#\hat{P}_k^n \#\hat{P}_m^n \) (which can be proved as in [BD, Lemma 3.3]), thus the two limits in part (c) exist by subadditivity. The fact that both limits are bounded by \( h_* \) follows from parts (a) and (b) of the theorem.

d) Let \( \mu \) denote a \( T \)-invariant probability measure. The assumptions of uniform hyperbolicity imply that both \( T \) and \( T^{-1} \) are expansive with respect to the metric \( d \) defined in (2.2).

\[
(2.5) \quad \text{There exists } \varepsilon_0 > 0 \text{ such that if } d(T^j x, T^j y) < \varepsilon_0 \text{ for all } j \in \mathbb{Z}, \text{ then } x = y.
\]

By (2.1), the uniform transversality of stable and unstable cones, and the assumption that each \( M_+^k \) is simply connected, the maximum diameter of elements of \( M_{-k}^k \) and hence of \( P_k \) is bounded by \( C \Lambda^{-k} \). Choosing \( k \) large enough that \( C \Lambda^{-k} \leq \varepsilon_0 \), we conclude that \( P_k \) is a generator for \( T \) [W, Theorem 5.23]. Then by [W, Theorem 4.22],
\[
h_\mu(T) = h_\mu(T, P_k) = \lim_{n \to \infty} \frac{1}{n} H_\mu(P_k^n) \leq \lim_{n \to \infty} \frac{1}{n} \log(\#P_k^n) \leq h_* ,
\]
applying part (c) of the present theorem. Thus \( h_\mu(T) \leq h_* \).

\[\square\]

Next we state our main theorem, which requires the additional hypotheses (P1) and (P2).
Theorem 2.9. Let $T$ be a piecewise hyperbolic map as defined in Section 2.1, satisfying conditions (P1) and (P2).

There exists a $T$-invariant probability measure $\mu_*$ with the following properties.

a) The measure $\mu_*$ has no atoms, and there exists $C > 0$ such that for any $\varepsilon > 0$,

$$
\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}^\pm)) \leq C\varepsilon^{1/p},
$$

where $p > 1$ is from (3.4) and $\mathcal{N}_\varepsilon(\cdot)$ denotes the $\varepsilon$-neighborhood of a set in the Riemannian metric on $M$. This implies in particular, that $\mu_*$-a.e. $x \in M$ has a stable and unstable manifold of positive length, and that $x$ approaches $\mathcal{S}^\pm$ at a subexponential rate.

b) $\mu_*(O) > 0$ for any open set $O \subset M$.

c) $(T^n, \mu_*)$ is ergodic for all $n \in \mathbb{Z}^+$.

d) $\mu_*$ has exponential decay of correlations against Hölder continuous functions.

e) The measure $\mu_*$ is the unique $T$-invariant probability measure satisfying $h_{\mu_*}(T) = h_*$.

Theorem 2.9 will be proved in Section 5.1. In particular, items (a)-(c) are proved in Section 5.1 item (d) is proved in Proposition 5.3, and item (e) is proved in Sections 5.2 and 5.3.

Corollary 2.10. Let $T$ be a piecewise hyperbolic map as defined in Section 2.1, satisfying conditions (P1) and (P2).

$T$ satisfies the following variational principle: For all $k \geq 0$,

$$
\lim_{n \to \infty} \frac{1}{n} \log (\#M^0_{-k}) = h_* = \sup\{h_\mu(T) : \mu \text{ is an invariant probability measure for } T\}.
$$

Proof. The fact that the limit defining $h_*$ exists (rather than simply the $\lim\sup$ from Definition 2.3) follows from Proposition 2.12, and the independence from $k$ follows from Theorem 2.8(b). The second equality follows from Theorem 2.8(d) together with Theorem 2.9(e). □

Theorem 2.9(a) implies that $\int_M |\log d(x, \mathcal{S}^\pm)|\,d\mu_* < \infty$ (see Corollary 5.5(c)), so that $\mu_*$ is $T$-adapted in the language of LiM. This allows us to make the following connection to the growth of periodic orbits of $T$. Let $P_n(T) = \{x \in M : \#(T^kx : k \in \mathbb{Z}) = n\}$ denote the set of points of prime period $n$ for $T$.

Corollary 2.11. Under the assumptions of Theorem 2.9, $\liminf_{n \to \infty} \frac{1}{n} \log \#P_n(T)e^{-nh_*} = 1$.

Proof. The proof relies on the construction of a countable Markov partition for hyperbolic maps with singularities carried out in LiM. The class of maps in the present paper satisfy conditions (A1)-(A6) in LiM, which are general enough to admit dispersing billiards. Since $\mu_*$ is $T$-adapted and hyperbolic (see Corollary 5.5), we may apply LiM Corollary 1.2 to conclude that there exist $p \geq 1$ and $C > 0$ such that the number of points of period $np$ for $T$ is at least $Ce^{nh_*}$ for all $n$ sufficiently large.

Next, applying Bu Main Theorem as in Bu Theorem 1.5, we conclude that we may take $p = 1$ and asymptotically, $C = 1$ for large $n$. □

In the course of proving the growth lemmas in Section 3, we establish the following uniform bounds on the growth of $\#M'_0$, which may be of independent interest, and are needed for the proof of uniqueness in Section 5.3.

Proposition 2.12. There exists a constant $C_\# > 0$ such that for all $n \geq 1$,

$$
C_\# e^{nh_*} \leq \#M'_0 = \#M^0_n \leq C_\#^{-1} e^{nh_*}.
$$

Proof. The upper bound is Corollary 3.13 while the lower bound is Lemma 5.1. □

Corollary 2.13. There exists $\bar{C} > 0$ such that for all stable curves $W \in \hat{W}^s$ with $|W| \geq \delta_1/3$ and all $n \geq n_1$, where both $\delta_1 > 0$ and $n_1$ are from (3.9), we have

$$
\bar{C} e^{nh_*} \leq |T^{-n}W| \leq \bar{C}^{-1} e^{nh_*}.
$$
Proof. Let $W \in \hat{W}^s$ with $|W| \geq \delta_1/3$. We use the notation of Section 3.5 regarding the connected components $\mathcal{G}_n(W)$ of $T^{-n}W$. Lemma 3.6(b), Lemma 3.11 and Proposition 2.12 together yield,
\[ c_0 C_{\rho} e^{nh_*} \leq c_0 \# \mathcal{M}^0_n \leq \# \mathcal{G}_n(W) \leq C \delta_0^{-1} \# \mathcal{M}^0_n \leq C \delta_0^{-1} C_{\rho}^{-1} e^{nh_*}. \]
Then on the one hand,
\[ |T^{-n}W| = \sum_{W_i \in \mathcal{B}_n(W)} |W_i| \leq \delta_0 \# \mathcal{G}_n(W), \]
since each element of $\mathcal{G}_n(W)$ has length at most $\delta_0$, completing the upper bound of the corollary. On the other hand, by (4.12),
\[ |T^{-n}W| = \sum_{W_i \in \mathcal{G}^+_n(W)} |W_i| \geq \frac{2\delta_1}{9} \# \mathcal{G}_n(W), \]
proving the lower bound. \qed

3. Banach Spaces and Growth Lemmas

In this section we define the Banach spaces we will use in the analysis of the transfer operator and prove several key lemmas controlling the growth in complexity of $T^n$.

3.1. Stable Curves. We begin with a definition of stable curves as graphs of functions in local charts, following [DL]. We will use the fact that the uniform hyperbolicity of $T$ guarantees the existence of stable $E^s(x)$ and unstable $E^u(x)$ directions in the tangent space $T_x M$ at Lebesgue-almost every $x \in M$.

For $\tau$ sufficiently small, we define the stable cone at $x \in M$ by
\[ \hat{C}^s(x) = \{ u + v \in T_x M : u \in E^s(x), v \perp E^s(x), \|v\| \leq \tau \|u\| \}. \]
Define $\hat{C}^u(x)$ analogously. These families of cones are strictly invariant, $DT^{-1}(x) \hat{C}^s(x) \subset \hat{C}^s(T^{-1}x)$ and $DT(x) \hat{C}^u(x) \subset \hat{C}^u(Tx)$.

For each $i$, we choose a finite number of coordinate charts $\{\chi_j\}_{j=1}^L$, whose domains $R_j$ are either $(-r_j, r_j)^2$ if $\chi_j$ maps only to the interior of $M^+_i$, or $(-r_j, r_j)$ restricted to one side of a piecewise $C^1$ curve (the preimage of a piece of $\partial M^+_i$) which we place so that it passes through the origin. For each $j$, $R_j$ has a centroid $x_j$, and $\chi_j$ satisfies,
(a) $D\chi_j(x_j)$ is an isometry;
(b) $D\chi_j(x_j) : (\mathbb{R} \times 0) \simeq E^s(\chi_j(x_j));$
(c) The $C^2$-norm of $\chi_j$ and its inverse are bounded by $1 + \tau$;
(d) There exists $c_j \in (\tau, 2\tau)$ such that the cone $C_j = \{ u + v \in \mathbb{R}^2 : u \in \mathbb{R} \times \{0\}, v \in \{0\} \times \mathbb{R}, \|v\| \leq c_j \|u\| \}$ satisfies: For each $y \in R_j$ such that $\chi_j(y) \notin S^-$, $D\chi_j(y) C_j \subset C^s(\chi_j(y))$, and $DT^{-1}(D\chi_j(y) C_j) \subset \hat{C}^s(T^{-1}(\chi_j(y)))$;
(e) $M^+_i \subset \bigcup_{j=1}^L \chi_j(R_j \cap (-\frac{r_j}{2}, \frac{r_j}{2})^2)$.

Choose $r_0 \leq \frac{\tau}{2} \min_j r_j; r_0$ may be further reduced later, depending on $\delta$. Fix $B < \infty$ and consider the set of functions
\[ \Xi := \{ F \in C^2([{-r}, {r}], \mathbb{R}) : r \in (0, r_0], F(0) = 0, |F|_{C^1} \leq \tau, |F|_{C^2} \leq B \}. \]
Define $I_r = (-r, r)$. For $x \in R_j \cap (-r_j/2, r_j/2)^2$ such that $x + (t, F(t)) \in R_j$ for $t \in I_r$, define $G(x, r, F)(t) := \chi_j(x + (t, F(t)))$ for $t \in I_r$, i.e. $G(x, r, F)$ is the lift of the graph of $F$ to $M$.

To abbreviate notation, we will refer to $G(x, r, F)$ as $G_F$. It follows from the construction that $|G_F|_{C^1} \leq (1 + \tau)^2$ and $G_F^{-1} \leq 1 + \tau$.

Our set of admissible stable curves is defined by,
\[ \hat{W}^s := \{ W = G(x, r, F)(I_r) : x \in R_j \cap (r_j/2, r_j/2)^2, r \leq r_0, F \in \Xi \}. \]
If necessary, we reduce \( r_0 \) so that \( \sup_{W \in \hat{W}^s} \| W \| \leq \delta_0 \), where \( \delta_0 \) is the length scale chosen in Conven-
tion\(^5\). Due to the uniform hyperbolicity of \( T \), if \( T^{-n} \hat{W}^s \) represents the connected components of \( T^{-n} W \) for \( W \in \hat{W}^s \), then choosing \( B \) large enough, it follows that \( T^{-n} \hat{W}^s \subset \hat{W}^s \), up to subdivision of long curves. With this choice of \( B \), the set of real local stable manifolds of length at most \( \delta_0 \), which we denote by \( \mathcal{V}^s \), satisfies \( \mathcal{V}^s \subset \hat{W}^s \).

Next, we define two notions of distance\(^6\), which are used in the definition of our norms, namely the strong unstable norm. For two curves \( W_1(x_{i_1}, x_{r_1}, F_1) \) and \( W_2(x_{i_2}, x_{r_2}, F_2) \), we define the distance between them to be

\[
d_{\hat{W}^s}(W_1, W_2) = \eta(i_1, i_2) + |x_1 - x_2| + |r_1 - r_2| + |F_1 - F_2|_{C^1(I_r_1 \cap I_r_2)},
\]

where \( \eta(i_1, i_2) = 0 \) if \( i_1 = i_2 \) and \( \eta(i_1, i_2) = \infty \) otherwise, i.e. we only compare curves in the same chart.

Given \( W_1, W_2 \) with \( d_{\hat{W}^s}(W_1, W_2) < \infty \) and two functions \( \psi_1 \in C^0(W_1) \), we define the distance between them to be

\[
d_0(\psi_1, \psi_2) = |\psi_1 \circ G_{F_1} - \psi_2 \circ G_{F_2}|_{C^0(I_r_1 \cap I_r_2)}.\]

### 3.2. Transfer operator

The main tool we will use to construct the measure of maximal entropy is a weighted transfer operator, \( \mathcal{L} \). Because we do not have a conformal measure at our disposal a priori, we will define the transfer operator acting on distributions defined via local stable manifolds. Let \( \hat{W}^s \) denote the set of maximal connected local stable manifolds of \( T \) restricted to each \( \hat{M}^+ \). Note that such manifolds have uniformly bounded length due to the finite diameter of \( M \) and the assumption that \( \hat{M}^+ \) is simply connected. Due to the uniform hyperbolicity of \( T \), \( \mu_{\text{SRB}} \)-almost every point in \( M \) has a stable manifold of positive length.

For any local stable manifold \( W \), and \( \alpha \in (0, 1] \), define the \( \alpha \)-Hölder norm of a test function \( \psi : M \to \mathbb{C} \) by

\[
|\psi|_{C^\alpha(W)} = |\psi|_{C^0(W)} + H^\alpha_W(\psi) := \sup_W |\psi| + \sup_{x \neq y \in W} \frac{|\psi(x) - \psi(y)|}{d_W(x, y)\alpha},
\]

where \( d_W(\cdot, \cdot) \) denotes distance induced by the Riemannian metric restricted to \( W \). Let \( \hat{C}^\alpha(W) \) denote the set of functions in \( C^0(W) \) with finite \( | \cdot |_{C^\alpha(W)} \) norm. With this notation, \( \hat{C}^1(W) \) denotes the set of Lipschitz functions on \( W \).

Analogously, for each \( n \geq 0 \), define \( H^\alpha_{\hat{W}^s}(\psi) = \sup_{W \in \hat{W}^s} H^\alpha_W(\psi) \), and

\[
\hat{C}^\alpha(\hat{W}^s) = \{ \psi : M \to \mathbb{C} \mid |\psi|_{C^\alpha(W)} + H^\alpha_{\hat{W}^s}(\psi) < \infty \}.
\]

The set \( \hat{C}^\alpha(\hat{W}^s) \) together with the norm \( |\psi|_{C^\alpha(\hat{W}^s)} := |\psi|_\infty + H^\alpha_{\hat{W}^s}(\psi) \) is a Banach space.

Since stable manifolds cannot be cut under \( T^n \), if \( W \in \hat{W}^s \), then \( T^n W \subset V \in \hat{W}^s \) for each \( n \geq 0 \). This together with the uniform hyperbolicity of \( T \) and (2.1) implies that if \( \psi \in C^\alpha(\hat{W}^s) \), then \( \psi \circ T \in C^\alpha(\hat{W}^s) \) (see also (4.4)).

Then if \( f \in (\hat{C}^\alpha(\hat{W}^s))^* \) belongs to the dual of \( C^\alpha(\hat{W}^s) \), the operator \( \mathcal{L} : (\hat{C}^\alpha(\hat{W}^s))^* \to (\hat{C}^\alpha(\hat{W}^s))^* \) is defined by

\[
\mathcal{L} f(\psi) = f \left( \frac{\psi \circ T}{J^sT} \right) \quad \forall \psi \in C^\alpha(\hat{W}^n),
\]

where \( J^sT \) denotes the stable Jacobian of \( T \). By (2.4), it follows that \( J^sT^n \in \hat{C}^1(\hat{W}^s) \) for each \( n \geq 1 \).

---

\(^5\)Neither of these distances will satisfy the triangle inequality, but that is irrelevant for our purposes.

\(^6\)For \( x \in W \in \hat{W}^s \), \( J^sT(x) = J_W T(x) \).
If \( f \in \mathcal{C}^0(M) \), then we identify \( f \) with a signed measure absolutely continuous with respect to \( \mu_{\text{SRB}} \). We denote this integration by,

\[
f(\psi) = \int_M \psi \ f \ d\mu_{\text{SRB}},
\]

for \( \psi \in \mathcal{C}^0(M) \). With this identification, we consider \( \mathcal{C}^0(M) \subset (\tilde{\mathcal{C}}^\alpha(\tilde{W}^s))^* \). Then also by (3.2), for any \( n \geq 1 \), \( \mathcal{L}^n f \) is absolutely continuous with respect to \( \mu_{\text{SRB}} \) with density,

\[
\mathcal{L}^n f = \frac{f \circ T^{-n}}{J^n}.
\]

### 3.3. Definition of Norms

Let \( \mathcal{W}^s \) denote those local stable manifolds having length at most \( \delta_0 \), where \( \delta_0 \) is from Convention 2.3. Note that \( \mathcal{W}^s \subset \hat{\mathcal{W}}^s \), yet \( \mathcal{W}^s \not\subset \mathcal{W}^s \) since \( \mathcal{W}^s \) contains only maximal local stable manifolds (which are necessarily disjoint), while \( \hat{\mathcal{W}}^s \) contains stable manifolds of any length less than \( \delta_0 \), many of which may overlap. We will define our norms by integrating on elements of \( \mathcal{W}^s \) against Hölder continuous test functions.

For \( W \in \mathcal{W}^s \) and \( \alpha > 0 \), let \( \mathcal{C}^\alpha(W) \) denote the closure of \( \tilde{\mathcal{C}}^1(W) \) in the \( \mathcal{C}^\alpha \) norm, defined in (3.1). In this notation, then \( \mathcal{C}^1(W) = \mathcal{C}^1(W) \).

Now given a function \( f \in \mathcal{C}^1(M) \), define the strong norm of \( f \) by

\[
|f|_s = \sup_{W \in \mathcal{W}^s} \sup_{\psi \in \mathcal{C}^1(W)} \int_W f \, \psi \, dm_W,
\]

where \( m_W \) denotes arc length along \( W \). Let \( ||W|| = m_W(W) \).

Next, choose \( \alpha, \beta < 1 \) and \( p > 1 \) such that

\[
0 < 2\beta \leq 1/p \leq 1 - \alpha \leq \alpha_0, \quad \text{and} \quad 1/p < \alpha.
\]

Define the strong stable norm of \( f \) by

\[
\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\psi \in \mathcal{C}^\alpha(W)} \int_W f \, \psi \, dm_W.
\]

Recalling the notion of distance \( d_{\mathcal{W}^s}(\cdot, \cdot) \) between curves \( W \in \mathcal{W}^s \) and the distance \( d_0(\cdot, \cdot) \) between test functions on nearby curves defined in Section 3.1 and fixing \( \varepsilon_0 \leq r_0 \), we define the strong unstable norm of \( f \) by

\[
\|f\|_u = \sup_{\varepsilon \leq \varepsilon_0} \sup_{W_1, W_2 \in \mathcal{W}^s} \sup_{\psi \in \mathcal{C}^1(W)} \varepsilon^{-\beta} \left| \int_{W_1} f \, \psi_1 \, dm_{W_1} - \int_{W_2} f \, \psi_2 \, dm_{W_2} \right|.
\]

Define the strong norm of \( f \) by \( \|f\| = \|f\|_s + c_u \|f\|_u \), where \( c_u > 0 \) is a constant to be chosen in the proof of Lemma 4.3.

Finally, our weak space \( \mathcal{B}_w \) is defined to be the completion of \( \mathcal{C}^1(M) \) in the weak norm, \( |\cdot|_w \), while our strong space \( \mathcal{B} \) is defined to be the completion of \( \mathcal{C}^1(M) \) in the strong norm \( \|\cdot\|_B \).

### Remark 3.1

The definition of our spaces \( \mathcal{B} \) and \( \mathcal{B}_w \) is nearly the same as that in [DL, Section 2.2], the key difference being that the norms in [DL] integrate along cone-stable curves \( \hat{\mathcal{W}}^s \), while our norms here integrate on local stable manifolds \( \mathcal{W}^s \). This change is necessary since the potential for our weighted transfer operator, \( 1/J^n T \), is Hölder continuous along real stable manifolds, yet may only be measurable along arbitrary stable curves. By restricting our norms to this smaller set of curves, we are able to prove the essential Lasota-Yorke inequalities, Proposition 4.2.

---

\(^7\)This space is strictly smaller than the set of \( \mathcal{C}^{\alpha} \) functions, yet contains \( \mathcal{C}^{\alpha'} \) for each \( \alpha' > \alpha \). We adopt this usage in order that the embedding of our strong space in our weak space is injective (Lemma 3.4).
3.4. Preliminary facts about the Banach spaces.

**Lemma 3.2.** Let \( Q \) be a \((\text{mod } 0 \text{ w.r.t. } \mu_{\text{SRB}})\) finite partition of \( M \) into open, simply connected sets such that there exist constants \( K, C_Q > 0 \) such that for each \( Q \in \mathcal{Q}, \) and \( W \in \mathcal{W}^s, Q \cap W \) comprises at most \( K \) connected components and for any \( \varepsilon > 0, \) \( m_W(N_\varepsilon(\partial Q) \cap W) \leq C_Q \varepsilon^{1/2}. \)

- Let \( \gamma > \beta/(1 - \beta) \) and suppose \( \varphi \) is a function on \( M \) such that \( \sup_{Q \in \mathcal{Q}} |\varphi| c^1(Q) < \infty. \) Then \( \varphi \in B. \)

- There exists \( C > 0 \) such that if \( \varphi \) is such that \( \sup_{Q \in \mathcal{Q}} |\varphi| c^1(Q) < \infty \) and \( f \in B, \) then \( \varphi f \in B \) and \( \|\varphi f\|_B \leq C\|f\|_B \sup_{Q \in \mathcal{Q}} |\varphi| c^1(Q). \)

**Proof.** To prove (a), a function \( \varphi \) as in the statement of the lemma can be approximated by \( C^1 \) functions using mollification precisely as in [DZ3, Lemma 3.5]. Part (b) follows along similar lines using [DZ3, Lemma 5.3]. Both proofs use the restrictions in (3.4) we have assumed for the parameters appearing in the norms. In particular, we need \( \beta \leq 1/(2p), \) rather than simply \( \beta \leq 1/p, \) due to the weak transversality condition assumed on \( Q. \)

**Lemma 3.3.** Let \( f \in C^1(M) \) and \( \psi \in \tilde{C}^1(\tilde{\mathcal{W}}^s). \) Then,

\[
|f(\psi)| = \left| \int_M f \psi \, d\mu_{\text{SRB}} \right| \leq C|f|_w|\psi|_\infty + H_{W^s}(\psi).
\]

**Proof.** Let \( f \in C^1(M) \) and \( \psi \in \tilde{C}^1(\tilde{\mathcal{W}}^s). \) We will estimate

\[
f(\psi) = \int_M f \psi \, d\mu_{\text{SRB}}.
\]

To this end, we choose a foliation \( \mathcal{F} = \{W_\xi\}_{\xi \in \Xi} \subset \mathcal{W}^s \) of maximal local stable manifolds subdivided according to the length scale \( \delta_0. \) We then disintegrate the measure \( \mu_{\text{SRB}} \) on \( W_\xi \in \mathcal{F} \) and a factor measure \( \tilde{\mu}_{\text{SRB}}(\xi) \) on the index set \( \Xi \) of stable manifolds. Since \( \mu_{\text{SRB}} \) is smooth by assumption \((P2), \) it follows from [P, Proposition 6] (see also [CZ, eq. (3.7)]) that the conditional measures \( \tilde{\mu}_{\text{SRB}}(\xi) \) are absolutely continuous with respect to arc length, \( d\tilde{\mu}_{\text{SRB}} = |W_\xi|^{-1} g_\xi \, dm_{W_\xi}, \) where \( g_\xi \) is given by\(^8\)

\[
\frac{g_\xi(x)}{g_\xi(y)} = \lim_{n \to \infty} \frac{J_{W_\xi} T^n(x)}{J_{W_\xi} T^n(y)} \quad \text{for all } x, y \in W_\xi.
\]

This characterization, plus the normalization \( \tilde{\mu}_{\text{SRB}}(W_\xi) = 1, \) uniquely determines \( g_\xi. \) It follows from a standard estimate\(^9\) and [2.4] that \( g_\xi \) is uniformly log-Lipschitz continuous on \( W_\xi, \) i.e. there exists \( C_g \geq 1 \) such that

\[
0 < C_g^{-1} \leq \inf_{\xi \in \Xi} W_\xi \sup_{\xi \in \Xi} |g_\xi| c^1(W_\xi) \leq C_g < \infty.
\]

\(^8\)Both [P] and [CZ] give the analogous formula for the conditional measures of \( \mu_{\text{SRB}} \) on unstable manifolds. Yet, due to our assumption \((P2), \) \( \mu_{\text{SRB}} \) is an SRB measure for \( T^{-1} \) as well, and so enjoys the analogous properties on stable manifolds of \( T. \)

\(^9\)Note \( J_{W_\xi} T^n(z) = \prod_{j=0}^{n-1} J_{T_j W_\xi} T(T_j z) \) and for brevity let \( g_n = J_{W_\xi} T^n. \) The limit of \( g_n(x)/g_n(y) \) exists if the limit of \( \log(g_n(x)/g_n(y)) \) exists. Now for \( n, k \geq 1, \) we may estimate using [2.3] and [2.4],

\[
\log \frac{g_n(x)}{g_n(y)} - \log \frac{g_{n+k}(x)}{g_{n+k}(y)} = \frac{J_{T^n W_\xi} T^k(T^n x)}{J_{T^n W_\xi} T^k(T^n y)} \leq C_d d_{T^n W_\xi}(T^n x, T^n y) \leq C_d C_{\varepsilon}^{-1} A^{-n} d_{W_\xi}(x, y),
\]

so that the sequence \( \log(g_n(x)/g_n(y)) \) is Cauchy and therefore converges. Thus the limit defining \( g_\xi \) exists. A similar estimate shows that \( g_\xi \) is log-Lipschitz with Lipschitz constant at most \( C_d, \) bounded independently of \( n, \) and so this bound carries over to \( g_\xi. \)
Using this disintegration, we write,

$$|f(\psi)| = \left| \int_{\xi \in \Xi} \int_{W_\xi} f \psi g_\xi |W_\xi|^{-1} d\mu_{\text{SRB}}(\xi) \right|$$

(3.6)

$$\leq \int_{\xi \in \Xi} |f|_\infty |\psi|_{C^1(W_\xi)} |g_\xi|_{C^1(W_\xi)} |W_\xi|^{-1} d\mu_{\text{SRB}}(\xi)$$

$$\leq C_g |f|_\infty \left( |\psi|_\infty + H_{\psi}(\psi) \right) \int_{\xi \in \Xi} |W_\xi|^{-1} d\mu_{\text{SRB}}(\xi).$$

To bound this last integral, we will apply some results of [CZ], which studies hyperbolic maps with singularities in an axiomatic context (Assumptions (H.1)-(H.5) in that paper), which include the class of maps in the present paper, in addition to many dispersing and semi-dispersing billiards.

Indeed, the final integral in (3.6) is precisely the Z-function, $Z_1(\mathcal{F})$, defined in [CZ] eq. (4.7) which governs the average length of stable manifolds in the family $\mathcal{F}$. (See also [CM], Exercise 7.15 and Proposition 7.17 for a similar application of these ideas.) The parameters $p$ and $q$ in [CZ] are both equal to 1 in our context, due to our property (P1) and Convention 2.3 which imply that $T$ satisfies the one-step expansion condition, [CZ] Condition (H.5)] with parameter $q = 1$,

$$\sup_{W \in \mathcal{W}_s} \sum_{V_i \subset T^{-1}W \text{ connected}} \left( \left| W_i \right| \right)^q \left| \frac{TV_i}{W_i} \right| \leq K_1 \Lambda^{-1} \leq \rho < 1,$$

where $V_i$ are the maximal, connected components of $T^{-1}W$. The required bound on $Z_1(\mathcal{F})$ follows from [CZ] Lemma 4 (again with $q = 1$) since $\mu_{\text{SRB}}$ is obtained as the limit of standard pairs with finite valued Z-function.

**Lemma 3.4.** There is a sequence of continuous inclusions,

$$\mathcal{C}^1(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^0(\mathcal{W}_s))^*.$$

The first two inclusions are injective.

**Proof.** The continuity of the first inclusion follows from Lemma 3.2 and its injectivity is obvious. The continuity of the second inclusion follows from $| \cdot |_w \leq || \cdot ||_s$. Its injectivity is a result of the fact that we have defined $|| \cdot ||_s$ with respect to $\mathcal{C}^0(W)$ rather than $\mathcal{C}^0(W)$, and $\mathcal{C}^1(W)$ is dense in $\mathcal{C}^0(W)$. Finally, the continuity of the third inclusion follows from Lemma 3.3.

By adding an additional weight to the weak norm, one can make the third inclusion in Lemma 3.4 injective as well (see for example [DZ3], Lemma 3.8), but we will not need this property here. Our final lemma in this section is essential for proving the quasi-compactness of $\mathcal{L}$ on $\mathcal{B}$.

**Lemma 3.5.** The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.

**Proof.** The lemma follows from [DL], Lemma 3.5. The fact that [DL] Lemma 3.5] uses the family of admissible curves $\tilde{\mathcal{W}}^s$ while we use the smaller set $W^s \subset \tilde{\mathcal{W}}^s$ does not affect the argument since the family of functions defining $\mathcal{W}^s$ in each chart is still compact in the $\mathcal{C}^1$-metric.

3.5. **Growth Lemmas.** In this section, we prove several growth lemmas which will be instrumental in establishing precise upper and lower bounds on the spectral radius of our transfer operator. Many of the results in this subsection and the next parallel those of [BD, Section 5].

Given a curve $W \in \tilde{\mathcal{W}}^s$, let $G_1(W)$ denote the maximal connected components of $T^{-1}W$ on which $T$ is smooth, with long pieces subdivided so that they have length between $\delta_0/2$ and $\delta_0$. In particular, elements of $G_1(W)$ must belong to a single element of $\mathcal{M}^+_0$, i.e. to a single component $M^+_i$ of $M$. Inductively, define $G_n(W)$ to denote the collection of maximal connected components of $T^{-1}V$, where $V \in G_{n-1}(W)$, again subdividing long pieces into curves of length between $\delta_0/2$ and $\delta_0$. We call $G_n(W)$, the $n$th generation of $W$. 
For each \( n \), let \( L_n(W) \) denote those elements of \( G_n(W) \) having length at least \( \delta_0/3 \). Let \( I_n(W) \) denote those elements \( W_i \in G_n(W) \) such that for each \( 0 \leq k \leq n - 1 \), \( T^k W_i \subset V \in G_{n-k}(W) \) and \( |V| < \delta_0/3 \), i.e. \( I_n(W) \) represents those elements in \( G_n(W) \) that have always been contained in a short element of \( G_{n-k}(W) \) from time 1 to time \( n \).

**Lemma 3.6.** There exists \( C > 0 \) such that for all \( W \in \hat{W}^s \), and all \( n \geq 0 \),

a) \( \# I_n(W) \leq K_1^n \leq \rho^n \kappa^{n} \Lambda^n \);  
b) \( \# G_n(W) \leq C \delta_0^{-1} \# \mathcal{M}_0^n \);  
c) \( \sum_{W_i \in \hat{G}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} \leq C \delta_0^{-1+1/p} \kappa^{-n/p} (\# \mathcal{M}_0^n)^{1-1/p} \);  
d) \( \# \mathcal{M}_0^n \geq C \delta_0 \Lambda^n \).

**Proof.** (a) This estimate follows from the fact that curves \( W_i \in I_n(W) \) have always been contained in a short element of \( G_{n-k}(W) \) for each \( k \) between 0 and \( n - 1 \). Thus property (P1) (recalling also Convention [2.3]) can be applied inductively in \( k \) to each element of \( I_{n-k}(W) \), yielding the claimed bound on the cardinality of these elements.

(b) The bound is trivial since each element of \( G_n(W) \) belongs by definition to one element of \( \mathcal{M}_0^n \). Since the stable diameter of each component of \( \mathcal{M}_0^n \) is uniformly bounded in \( n \), the connected components of \( T^{-n} W \) are subdivided into at most \( C \delta_0^{-1} \) curves to form the elements of \( G_n(W) \), for some uniform \( C > 0 \).

(c) Note that for \( W_i \in \hat{G}_n(W) \), using [2.1],

\[
|T^n W_i| = \int_{W_i} J_{W_i} T^n \, dm_{W_i} \geq |W_i| \kappa^n.
\]

Thus,

\[
\sum_{W_i \in \hat{G}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} \leq \kappa^{-n/p} \sum_{W_i \in \hat{G}_n(W)} \frac{|T^n W_i|^{1/p}}{|W|^{1/p}} \leq \kappa^{-n/p} \left( \sum_{W_i \in \hat{G}_n(W)} 1 \right)^{1-1/p} \leq C \kappa^{-n/p} \delta_0^{-1+1/p} (\# \mathcal{M}_0^n)^{1-1/p},
\]

where we have used the Hölder inequality and part (b) of the lemma.

(d) Applying part (b) of the lemma, we have

\[
|T^{-n} W| = \sum_{W_i \in \hat{G}_n(W)} |W_i| \leq \delta_0 \# G_n(W) \leq C \# \mathcal{M}_0^n.
\]

Then recalling [2.3] and applying this to \( W \in \mathcal{W}^s \) with \( |W| = \delta_0 \) completes the proof of the lemma.

Next we proceed to show that most elements of \( G_n(W) \) are long, if the length scale is chosen appropriately. For \( \delta \in (0, \delta_0) \) and \( W \in \hat{W}^s \), define \( \mathcal{G}_n^\delta(W) \) to be the smooth components of \( T^{-n} W \), with pieces longer than \( \delta \) subdivided to have length between \( \delta/2 \) and \( \delta \), i.e. \( \mathcal{G}_n^\delta(W) \) is defined precisely like \( \hat{G}_n(W) \), but with \( \delta_0 \) replaced by \( \delta \). Define \( L_n^\delta(W) \) to be the set of curves in \( \mathcal{G}_n^\delta(W) \) having length at least \( \delta/3 \), and let \( S_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W) \). Similarly, let \( I_n^\delta(W) \) denote those elements of \( \hat{S}_n^\delta(W) \) that have no ancestors of length at least \( \delta/3 \).

**Lemma 3.7.** For all \( \varepsilon > 0 \), there exist \( \delta \in (0, \delta_0) \) and \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1 \),

\[
\# L_n^\delta(W) \geq (1 - \varepsilon) \# \mathcal{G}_n^\delta(W), \quad \text{for all } W \in \hat{W}^s \text{ with } |W| \geq \delta/3.
\]
Proof. Fix $\varepsilon \in (0,1)$ and by Property (P1) and Remark 2.2, choose $n_1$ sufficiently large that $3C_e^{-1}(K(n_1 + \ell) + 1)\Lambda^{-n_1-\ell} < \varepsilon/2$ for all $0 \leq \ell \leq n_1 - 1$, where $C_e \leq 1$ is from (2.3). Next, choose $\delta > 0$ sufficiently small that if $W \in \hat{\mathcal{W}}^s$ with $|W| \leq \delta$, then $T^{-n}W$ comprises at most $K(n) + 1$ smooth components of length at most $\delta_0$ for all $n \leq 2n_1$.

Now let $W \in \hat{\mathcal{W}}^s$ with $|W| \geq \delta/3$. We shall prove that for $n \geq n_1$, $$\#S_n^\delta(W) \leq \#G_n^\delta(W).$$

For $n \geq n_1$, write $n = kn_1 + \ell$ for some $0 \leq \ell < n_1$. If $k = 1$, the above inequality follows immediately since there are at most $K(n_1 + \ell) + 1$ elements of $S_{n_1+\ell}(W)$ by choice of $\delta$, while by (2.3), $|T^{-n_1-\ell}W| \geq C_e\Lambda^{n_1+\ell}|W| \geq C_e\Lambda^{n_1+\ell}\delta/3$. Thus $G_n^\delta(W)$ must contain at least $C_e\Lambda^{n_1+\ell}/3$ curves since each has length at most $\delta_0$. Thus, $$\#S_{n_1+\ell}^\delta(W) \leq 3C_e^{-1}K(n_1 + \ell) + 1/\Lambda^{n_1+\ell} < \varepsilon/2,$$ by assumption on $n_1$.

On the other hand, if $k > 1$ then we split $n$ into $k - 1$ blocks of length $n_1$ and one block of length $n_1 + \ell$. We group elements $W_i \in S_{n_1+\ell}(W)$ by most recent long ancestor $V_j \in L_{n_1}(W)$: $t$ is the greatest index $\leq k - 1$ such that $T^{(k-t)n_1+\ell}W_i \in V_j$ and $V_j \in L_{n_1}(W)$. Note that we only consider ancestors occurring in blocks of length $n_1$. It is irrelevant for our estimate whether $W_i$ has a long ancestor at an intermediate time.

Since each $|V_j| \geq \delta/3$, it follows that $G_{n_1+\ell}^\delta(V_j)$ must contain at least $C_e\Lambda^{(k-t)n_1}/3$ curves of length at most $\delta_0$. Thus using Lemma 3.6(a), we have

$$\frac{\#S_{kn_1+\ell}^\delta(W)}{\#G_{kn_1+\ell}^\delta(W)} = \frac{\#I_{kn_1+\ell}^\delta(W)}{\#G_{kn_1+\ell}^\delta(W)} + \frac{\sum_{t=1}^{k-1} \sum_{V_j \in L_{n_1}(W)} \#I_{kn_1+\ell}^\delta(V_j)}{\#G_{kn_1+\ell}^\delta(W)} \leq \frac{(K(n_1) + 1)^k}{C_e\Lambda^{kn_1}/3} + \sum_{t=1}^{k-1} \sum_{V_j \in L_{n_1}(W)} \frac{K(n_1) + 1)^k - t}{C_e\Lambda^{(k-t)n_1}/3} \leq 3C_e^{-1} \sum_{t=1}^{k} (K(n_1)) + 1)\Lambda^{-tn_1} \leq \sum_{t=1}^{k} \left(\frac{\varepsilon}{2}\right)^t < \varepsilon.$$ 

□

The following corollary extends Lemma 3.7 to arbitrarily short curves, and is used in Lemma 4.5 to prove the positivity of our maximal eigenvector on all elements of $\mathcal{W}^s$.

**Corollary 3.8.** There exists $C_2 > 0$ such that for any $\varepsilon, \delta$ and $n_1$ as in Lemma 3.7,

$$\#I_n^\delta(W) \geq (1 - 2\varepsilon)\#G_n^\delta(W), \quad \forall W \in \hat{\mathcal{W}}^s, \forall n \geq C_2n_1 \frac{|\log(|W|/\delta)|}{|\log \varepsilon|}.$$ 

Proof. Fix $\varepsilon, \delta$ and $n_1$ from Lemma 3.7. Suppose $W \in \hat{\mathcal{W}}^s$ has $|W| < \delta/3$, and let $n > n_1$. We decompose $G_n^\delta(W)$ as in Lemma 3.7 and estimate the second sum in (3.8) precisely as before.

The first term on the right hand side of (3.8), $\#I_n^\delta(W)/\#G_n^\delta(W)$, is handled differently. Let $n_2$ denote the least integer $\ell$ such that $G_\ell^\delta(W)$ contains at least one element of length $\delta/3$. Since $|T^{-\ell}W| \geq C_e\Lambda^{\ell}|W|$ by (2.3), and $G_\ell^\delta(W) \leq K_1^\ell$ by (P1) and Convention 2.3, as long as $|T^{-\ell}W| \leq \delta_0$, at least one element of $G_\ell^\delta(W)$ must have length at least $C_e\Lambda^\ell|W|/K_1^\ell \geq C_e\rho^{-\ell}|W|$. Thus

$$n_2 \leq \frac{|\log(3C_e|W|\delta^{-1})|}{|\log \rho|}.$$
Then calling $V$ the element of $\mathcal{G}_{n_2}^{\delta}(W)$ having length at least $\delta/3$, we have
\[ \#\mathcal{G}_{n}^{\delta}(W) \geq \#\mathcal{G}_{n-n_2}^{\delta}(V) \geq C_{c}\Lambda^{n-n_2}/3. \]

Thus
\[ \frac{\#I^{\delta}_{n}(W)}{\#G_{n}^{\delta}(W)} \leq \frac{3(K(n_1) + 1)^{[n/n_1]}}{C_{c}\Lambda^{n}} \Lambda^{n_2} \leq \left( \frac{3}{2} \right)^{[n/n_1]} \Lambda^{n_2}. \]

Finally, since $n_2 = O(|\log(|W|/\delta)|)$, we may choose $C_2$ sufficiently large, that if $n \geq C_2n_1 \frac{|\log(|W|/\delta)|}{|\log \varepsilon|}$, then the quantity on the right is at most $\varepsilon$, completing the proof of the corollary.

Choosing $\varepsilon = 1/3$, we let $\delta_1 > 0$ and $n_1$ be the corresponding quantities from Lemma 3.7. Fixing this choice of $\delta_1$ and $n_1$, we have
\[ (3.9) \quad \#L^\delta_{n}(W) \geq 2^n \#G^\delta_{n}(W), \quad \text{for all } W \in \widehat{\mathcal{W}}^n \text{ with } |W| \geq \delta_1/3 \text{ and all } n \geq n_1. \]

Our next lemma shows that a positive fraction of elements of $\mathcal{M}_{0}^-_n$ and $\mathcal{M}_0^-_n$ have length at least $\delta_1$ in some direction. This will be essential to establishing the lower bounds of Section 3.6. For $A \subset M$, let $\text{diam}^s(A)$ denote the stable diameter of $A$, i.e. the length of the longest stable curve in $A$. Similarly, define the unstable diameter $\text{diam}^u(A)$ to be the length of the longest unstable curve in $A$.

The boundary of the partition defined by $\mathcal{M}_{0}^-_n$ is comprised of unstable curves belonging to $S_n^- = \cup_{i=0}^{n-1} T^{-i}(S^+)$. Similarly, $\partial \mathcal{M}_0^-_n$ is comprised of the stable curves, $S_n^+ = \cup_{i=0}^{n-1} T^{-i}(S^-)$. In what follows, we will find it convenient to invoke Convention 2.1 regarding the definition of $T^{\pm 1}$ on each smooth component of $S^\pm$. Let $L_u(\mathcal{M}_{0}^-_n)$ denote those elements of $\mathcal{M}_{0}^-_n$ whose unstable diameter is at least $\delta_1/3$, and let $L_s(\mathcal{M}_{0}^+_n)$ denote those elements of $\mathcal{M}_{0}^+_n$ whose stable diameter is at least $\delta_1/3$. The following lemma is the analogue of Lemma 3.7 for these dynamically defined partitions.

**Lemma 3.9.** There exist $C_{n_1} > 0$ and $n_3 \geq n_1$ such that for all $n \geq n_3$,
\[ \#L_s(\mathcal{M}_{0}^+_n) \geq C_{n_1} \delta_1 \#\mathcal{M}_{0}^+_n \quad \text{and} \quad \#L_u(\mathcal{M}_{0}^-_n) \geq C_{n_1} \delta_1 \#\mathcal{M}_{0}^-_n. \]

**Proof.** We prove the bound for $L_s(\mathcal{M}_{0}^+_n)$. In order to prove the lemma, we will use the fact that the boundary of $\mathcal{M}_{0}^+_n$ is the set $\cup_{j=0}^{n-1} T^{-j}S^+$. Let $S_s(\mathcal{M}_{0}^+_n)$ denote the elements of $\mathcal{M}_{0}^+_n$ whose stable diameter is less than $\delta_1/3$. We have $\mathcal{M}_{0}^+_n = L_s(\mathcal{M}_{0}^+_n) \cup S_s(\mathcal{M}_{0}^+_n)$. Similarly, let $S_s(T^{-j}S^+)$ denote the set of stable curves in $T^{-j}S^+$ whose length is less than $\delta_1/3$.

The following sublemma will prove useful for establishing key claim in the proof.

**Sublemma 3.10.** If a smooth stable curve $V_i \subset T^{-i}S^+$ intersects a smooth curve $V_j \subset T^{-j}S^+$ for $i < j$, then $V_j$ must terminate on $V_i$.

**Proof of Sublemma 3.10.** Suppose such an intersection occurs for $j > i$. Then $T^{i+1}(V_i) \subset S^-$ is an unstable curve, while $T^{i+1}(V_j) \subset S^+_i$ is a stable curve. Thus $T^{i+1}(V_j)$ must cross $S^-$ transversally, and so $T^{i}(V_j)$ will be split into at least two smooth components since $S^-$ is the singularity set for $T^{-1}$. This implies that $V_j$ cannot be a single smooth curve.

Using the sublemma, we establish the following claim:
\[ (3.10) \quad \#S_s(\mathcal{M}_{0}^+_n) \leq 2 \sum_{j=0}^{n-1} \#S_s(T^{-j}S^+) + B_1n, \]
for some $B_1 > 0$. According to the sublemma, if $A \in S_s(\mathcal{M}_{0}^+_n)$, then either $\partial A$ contains a short curve in $T^{-j}S^+$ or $\partial A$ contains an intersection point of two curves in $T^{-j}S^+$, for some $0 \leq j \leq n - 1$. But intersections of curves within $T^{-j}S^+$ are images of intersections of curves within $S^+$, and the cardinality of cells created by such intersections is bounded by some uniform constant $B_1 > 0$. 
depending only on \( S^+ \). Since each short curve in \( T^{-j} S^+ \) belongs to the boundary of at most two elements of \( \mathcal{S}_L(M_0^n) \), the claim follows.

Now, we subdivide \( S^+ \) into \( \ell_0 \) smooth curves \( V_i \) of length between \( \delta_1/3 \) and \( \delta_1 \). For \( j \geq n_1 \), recalling the notation \( \mathcal{S}_j^{\delta_1}(V_i) \) for the short elements of the \( j \)th generation \( \mathcal{G}_j^{\delta_1}(V_i) \) of subcurves in \( T^{-j} V_i \), we have by (3.9),

\[
\#S_s(T^{-j} S^+) = \sum_{i=1}^{\ell_0} \#S_j^{\delta_1}(V_i) \leq \frac{1}{3} \sum_{i=1}^{\ell_0} \#L_j^{\delta_1}(V_i).
\]

Next, using (3.11), we estimate the sum over \( j \) in (3.10) by splitting it over two parts,

\[
\#S_s(M_0^n) \leq B_1 n + 2 \sum_{j=0}^{n_1-1} \#S_s(T^{-j} S^+) + \frac{2}{3} \sum_{j=n_1}^{n-1} \sum_{i=1}^{\ell_0} \#L_j^{\delta_1}(V_i).
\]

The cardinality of the first sum up to \( n_1 - 1 \) is bounded by some constant \( \bar{C} n_1 \), depending only on the map \( T \) and \( n_1 \), but independent of \( n \).

Next, we wish to relate \( \#L_j^{\delta_1}(V_i) \) to \( \#L_s(M_0^n) \) for \( j \geq n_1 \). Note that if \( V' \in L_j^{\delta_1}(V_i) \), then \( |T^{-n-j} V'| \geq CA^{-n-j} \delta_1/3 \), so that \( \#G_{n-j}(V') \geq CA^{-n-j}/3 \).

Now, for each \( j \) such that \( n_1 \leq j \leq n - 1 - n_1 \), and \( V' \in L_j^{\delta_1}(V_i) \), we may apply (3.9), so that

\[
\#L_{n-1}^{\delta_1}(V_i) \geq \sum_{V' \in L_j^{\delta_1}(V_i)} \#L_{n-1-j}^{\delta_1}(V') \geq C' A^{-n-1-j} \#L_j^{\delta_1}(V_i).
\]

For \( j > n - n_1 \), we compare \( L_j^{\delta_1}(V_i) \) with \( L_{n-1}^{\delta_1}(V_i) \). Since \( K_1 < \Lambda \), there is at least one element of \( L_{j+1}^{\delta_1}(V_i) \) for each element of \( L_j^{\delta_1}(V_i) \). Applying this inductively to \( j \), we conclude,

\[
\#L_{n-1}^{\delta_1}(V_i) \geq \#L_j^{\delta_1}(V_i).
\]

Putting together this estimate with (3.13) in (3.12), we estimate,

\[
\#S_s(M_0^n) \leq B_1 n + \bar{C} n_1 + \sum_{j=n_1}^{n-1-n_1} CA^{j+1-n} \#L_s(T^{-n+1} S^+) + \sum_{j=n-n_1}^{n-1} \#L_s(T^{-n+1} S^+),
\]

\[
\leq B_1 n + \bar{C} n_1 + C\delta_1^{-1} \#L_s(M_0^n) + n_1 C \delta_1^{-1} \#L_s(M_0^n),
\]

where in the second line we have used the fact that \( \#L_s(T^{-n+1} S^+) \leq C \delta_1^{-1} \#L_s(M_0^n) \), which follows from Sublemma 3.10.

Finally, since \( \#M_0^n = \#L_s(M_0^n) + \#S_s(M_0^n) \), we estimate,

\[
\#L_s(M_0^n) \geq \#M_0^n - \bar{C} n_1 - B_1 n \frac{n}{1 + C \delta_1^{-1}(1 + n_1)}.
\]

Since \( \#M_0^n \geq C \delta_0 \Lambda^n \) by Lemma 3.6(d) and \( n_1 \) is fixed, we may choose \( n_2 \in \mathbb{N} \) such that \( \#M_0^n - \bar{C} n_1 - B_1 n \geq \frac{1}{2} \#M_0^n \), for all \( n \geq n_2 \). We conclude that there exists \( C n_1 > 0 \) such that for \( n \geq n_2 \),

\[
\#L_s(M_0^n) \geq \bar{C} n_1 \delta_1 \#M_0^n,
\]

completing the proof of the lemma for \( L_s(M_0^n) \).

The lower bound for \( \#L_u(M_0^n) \) follows similarly, using the fact that (P1) also allows us to control the evolution of unstable curves under \( T^n \) by controlling the complexity of \( S^+_n \). Note that the analogue of Lemma 3.7 holds for forward iterates of unstable curves using precisely the same proof. The constant \( \kappa \) does not appear in this argument, i.e. the fact that the rate of expansion has a maximum is not needed for the proof. \( \square \)
3.6. Lower bounds on growth. The prevalence of long pieces established in Lemmas 3.7 and 3.9 have the following important consequences.

**Lemma 3.11.** Let \( \delta_1 \) be the length scale from (3.9). There exists \( c_0 > 0 \), depending on \( \delta_1 \), such that for all \( W \in \mathcal{W}^s \) with \( |W| \geq \delta_1/3 \) and \( n \geq 1 \), we have \( \#G_n(W) \geq c_0\#M_0^0 \).

This lemma, in turn, implies the supermultiplicativity property for \( \#M_0^0 \).

**Proposition 3.12.** There exists \( c_1 > 0 \) such that for all \( j, n \in \mathbb{N} \) with \( j \leq n \), it holds,

\[
\#M_0^0 \geq c_1\#M_{n-j}^0 \#M_0^j.
\]

In order to establish Lemma 3.11, we recall the construction of Cantor rectangles. For \( x \in M \), let \( W^s(x) \) and \( W^u(x) \) denote the maximal smooth components of the local stable and unstable manifolds of \( x \) (which, by definition, belong to a single domain \( M^+_\mu \)).

We begin by defining a solid rectangle \( D \subset M \) to be a closed region whose boundary comprises exactly two stable manifolds and two unstable manifolds of positive length. Given such a region \( D \), define the **locally maximal Cantor rectangle** \( R \) in \( D \) to be the union of all points in \( D \) whose local stable and unstable manifolds completely cross \( D \). Locally maximal Cantor rectangles are endowed with a natural product structure: for any \( x, y \in R \), \( W^u(x) \cap W^s(y) \) belongs to \( R \). Such rectangles are closed, so their boundary coincides with the boundary of \( D \). In this case, we write \( D = D(R) \) to denote the fact that \( D \) is the smallest solid rectangle containing \( R \).

Following [L1], for a Cantor rectangle \( R \), we call the core of \( R \) to be \( R \cap D_{1/4} \), where \( D_{1/4} \) is an approximately concentric rectangle in \( D(R) \) with side lengths \( 1/4 \) the side lengths of \( D \).

For a locally maximal Cantor rectangle \( R \), we say that a stable (respectively unstable) curve \( W \) **properly crosses** \( R \) if \( W \) intersects the rectangle \( D_{1/4}(R) \), but does not terminate in \( D(R) \), and \( W \) does not cross either of the stable (resp. unstable) boundaries of both \( D(R) \) and \( D_{1/4}(R) \).

**Proof of Lemma 3.11.** Applying [L1] Theorem 4.10], we may choose locally maximal Cantor rectangles \( R_{\delta_1} = \{R_1, \ldots, R_k\} \), with \( \mu_{\text{SRB}}(R_i) > 0 \), whose stable and unstable boundaries have length at most \( \frac{1}{10} \delta_1 \) such that any stable or unstable curve of length at least \( \delta_1/3 \) properly crosses at least one of them. Furthermore, we may choose the rectangles sufficiently small that both \( R_i \) and \( R_i \cap D_{1/4}(R_i) \) have positive \( \mu_{\text{SRB}} \)-measure for each \( i \). The number of rectangles \( k \) depends on \( \delta_1 \).

For brevity, denote by \( R^*_i = R_i \cap D_{1/4}(R_i) \), the core of \( R_i \). Due to the mixing property of \( (T, \mu_{\text{SRB}}) \), there exist \( \varepsilon > 0 \) and \( n_4 \in \mathbb{N} \) such that for all \( n \geq n_4 \), and all \( 1 \leq i, j, k \), \( \mu_{\text{SRB}}(R^*_i \cap T^{-n}R^*_j) \geq \varepsilon \).

We claim that for each \( n \), at least one Cantor rectangle \( R_i \in R_{\delta_1} \) is fully crossed in the unstable direction by at least \( \frac{1}{k} \#L_u(M^0_{n-1}) \) elements of \( M^0_{n} \). This is because if \( A \in M^0_{n} \), then \( \partial A \) is comprised of unstable curves belonging to \( S^n_{\mu} \). Since unstable manifolds cannot be cut under iteration by \( T^{-n} \), \( S^n_{\mu} \) cannot intersect the unstable boundaries of \( R_i \). Thus if \( A \cap R_i \neq \emptyset \), then either \( \partial A \) terminates inside \( R_i \) or \( A \) fully crosses \( R_i \). This implies that elements of \( L_u(M^0_{n}) \) fully cross at least one \( R_i \), and so at least one \( R_i \) must be fully crossed by at least \( \frac{1}{k} \) such elements.

With the claim established, for each \( n \), let \( R_n \) denote a Cantor rectangle that is fully crossed by at least \( \frac{1}{k} \#L_u(M^0_{n-1}) \) elements of \( M^0_{n} \).

Now take \( W \in \mathcal{W}^s \) with \( |W| \geq \delta_1/3 \). By construction, there exists \( R_j \in R_{\delta_1} \) such that \( W \) properly crosses \( R_j \) in the stable direction. For each \( n \in \mathbb{N} \), using mixing, we have \( \mu_{\text{SRB}}(R^*_n \cap T^{-n}R^*_j) \geq \varepsilon \).

By [L1] Lemma 4.13, there is a curve \( V \in \mathcal{G}^1_{n_4}(W) \) that properly crosses \( R_n \) in the stable direction. By choice of \( R_n \), this implies that \( \#G_n(V) \geq \frac{1}{k} \#L_u(M^0_{n}) \). Thus,

\[
\#G_{n+n_4}(W) \geq \frac{1}{k} \#L_u(M^0_{n}) \implies \#G_n(W) \geq \frac{c}{k} \#L_u(M^0_{n}),
\]

\(^{10}\)Once a Cantor rectangle of some size is constructed around \( \mu_{\text{SRB}} \)-almost-every \( x \in M \), the existence of such a finite family for any fixed length scale \( \delta_1 \) follows from the compactness of the set of stable (and also unstable) curves of length \( \geq \delta_1/3 \) in the Hausdorff metric, as in [CM] Lemma 7.87.
where \((C')^{-1} = C\delta_0^{-1} \#\mathcal{M}_0^{n_4}\) since \(\#\mathcal{G}_{n+n_4}(W) \leq C\delta_0^{-1} \#\mathcal{M}_0^{n_4} \#\mathcal{G}_a(W)\) by Lemma 3.6b.

Finally, by Lemma 3.9 \(\|L_u(\mathcal{M}_{0-n})\| \geq C_{n_1} \delta_1 \#\mathcal{M}_0^n\), which proves the lemma for \(n \geq \max\{n_3, n_4\}\) since \(\#\mathcal{M}_{0-n} = \#\mathcal{M}_0^n\). The lemma extends to all \(n \in \mathbb{N}\) by possibly reducing the constant \(c_0\) since there are only finitely many values to correct for.

**Proof of Proposition 3.12.** Recall that since \(T^{-j}(S_j^- \cup S_{n-j}^+) = S_{n-j}^+\), there is a one-to-one correspondence between elements of \(\mathcal{M}_{0-j}^-\) and \(\mathcal{M}_0^n\) for each \(j < n\). Thus \(\#\mathcal{M}_0^n = \#\mathcal{M}_{0-j}^-\), and this latter partition is obtained by taking the maximal connected components of \(\mathcal{M}_0^n \setminus \mathcal{M}_{0-j}^-\).

To prove the lemma, we will show that a positive fraction, independent of \(j\) and \(n\), of elements of \(\mathcal{M}_{0-j}^-\) intersect a positive fraction of elements of \(\mathcal{M}_0^n\). Recall that \(L_u(\mathcal{M}_{0-j}^-)\) denotes those elements of \(\mathcal{M}_{0-j}^-\) with unstable diameter of length at least \(\delta_1/3\) while \(L_s(\mathcal{M}_{0-j}^-)\) denotes those elements of \(\mathcal{M}_{0-j}^-\) with stable diameter of length at least \(\delta_1/3\).

If \(A \in L_s(\mathcal{M}_{0-j}^-)\) and \(V \subset A\) is a stable curve with \(|V| \geq \delta_1/3\), then \(\#\mathcal{G}_j(V) \geq c_0 \#\mathcal{M}_0^n\) by Lemma 3.11. Remark that up to subdivision of long pieces, each component of \(\mathcal{G}_j(V)\) corresponds to one component of \(V \setminus S_j^-\). Thus \(V\) intersects at least \(c_0 \#\mathcal{M}_0^n = c_0 \#\mathcal{M}_{0-j}^-\) elements of \(\mathcal{M}_{0-j}^\prime\).

Applying this estimate to each \(A \in L_s(\mathcal{M}_{0-j}^-)\), we obtain

\[
\#\mathcal{M}_0^n \geq \#L_s(\mathcal{M}_{0-j}^-) \cdot c_0 \#\mathcal{M}_0^n \geq C_{n_1} \delta_1 c_0 \#\mathcal{M}_{0-j}^- \#\mathcal{M}_0^n,
\]

where we have applied Lemma 3.9 in the second inequality. This proves the lemma when \(n-j \geq n_3\). For \(n-j \leq n_3\), since \(\#\mathcal{M}_{0-j}^- \leq \#\mathcal{M}_0^n\), we obtain the lemma by possibly decreasing the value of \(c_1\) since there are only finitely many values to correct for.

**Corollary 3.13.** For all \(n \in \mathbb{N}\), \(\#\mathcal{M}_0^n \leq 2c_1^{-1} e^{nh^*}\), where \(c_1 > 0\) is from Proposition 3.12.

**Proof.** The proof follows using Proposition 3.12 precisely as in [BD, Proposition 4.6].

### 4. Spectral Properties of \(L\)

In this section, we prove the following theorem.

**Theorem 4.1.** The operator \(L\) acting on \(\mathcal{B}\) is quasi-compact, with spectral radius equal to \(e^{h^*}\) and essential spectral radius bounded by \(\max\{\Lambda^{-\beta}, \rho\} e^{h^*}\).

Since \(T\) is topologically mixing, \(L\) has a spectral gap: \(e^{h^*}\) is a simple eigenvalue (multiplicity 1 and no Jordan blocks) and the rest of the spectrum of \(L\) is contained in a disk of radius strictly smaller than \(e^{h^*}\).

Let \(\nu_0 \in \mathcal{B}\) be an eigenfunction for eigenvalue \(e^{h^*}\) defined by

\[
\nu_0 := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh^*} L^k 1.
\]

Then \(\nu_0 \neq 0\) is a non-negative Radon measure on \(M\).

The quasi-compactness of \(L\) is proved in Lemma 4.3 following the Lasota-Yorke inequalities of Proposition 4.2. The fact that \(L\) has a spectral gap is proved in Lemma 4.6 while the characterization of \(\nu_0\) is proved in Lemma 4.4.

#### 4.1. Lasota-Yorke Inequalities

The following proposition is the key component in establishing the quasi-compactness of \(L\).

**Proposition 4.2.** There exists \(C > 0\) such that for all \(n \geq 0\) and \(f \in \mathcal{B}\),

\[
\begin{align*}
|L^n f|_w &\leq C\delta_0^{-1} \#\mathcal{M}_0^n |f|_w, \\
\|L^n f\|_s &\leq C\delta_0^{-2} \#\mathcal{M}_0^n ((\Lambda^{-\alpha n} + \rho^n) \|f\|_s + \kappa^{-n/p} |f|_w), \\
\|L^n f\|_u &\leq C\delta_0^{-1} \#\mathcal{M}_0^n (\Lambda^{-\beta n} \|f\|_u + \kappa^{-n/p} |f|_s).
\end{align*}
\]
By density, it suffices to prove the proposition for \( f \in C^1(M) \).

4.1.1. Weak norm bound. Take \( f \in C^1(M) \), \( W \in \mathcal{W}^s \) and \( \psi \in C^1(W) \), \( |\psi|_{C^1(W)} \leq 1 \). Recalling that \( \mathcal{G}_n(W) \) denotes the decomposition of \( T^{-n}W \) into elements of \( \mathcal{W}^s \), we estimate for \( n \geq 1 \),

\[
\int_W \mathcal{L}^n f \psi \, dm_W = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f \psi \circ T^n \, dm_{W_i} \leq |f|_W \sum_{W_i \in \mathcal{G}_n(W)} |\psi \circ T^n|_{C^1(W_i)},
\]

where we have applied the weak norm of \( f \) to the integral on each \( W_i \). Next, using the uniform contraction of \( T \) along stable curves, we have

\[
(4.4) \quad \frac{|\psi \circ T^n(x) - \psi \circ T^n(y)|}{d_W(x, y)} = \frac{|\psi \circ T^n(x) - \psi \circ T^n(y)|}{d_W(T^n(x), T^n(y))} \leq C |J^s T^n|_{C^0(W_i)} H^1_W(\psi),
\]

for some uniform constant \( C > 0 \), using (2.1). Then since \( |\psi \circ T^n|_{C^0(W_i)} \leq |\psi|_{C^0(W)} \), we have \( |\psi \circ T^n|_{C^1(W_i)} \leq C |\psi|_{C^1(W)} \leq C \). Finally, applying Lemma 3.6(b) to the sum over \( \mathcal{G}_n(W) \) and taking the supremum over \( \psi \in C^1(W) \) and \( W \in \mathcal{W}^s \) completes the proof of (4.1).

4.1.2. Strong stable norm bound. Let \( f \in C^1(M) \), \( W \in \mathcal{W}^s \) and \( \psi \in C^\alpha(W) \) with \( |\psi|_{C^\alpha(W)} \leq |W|^{-1/p} \). Let \( n \geq 1 \). For each \( W_i \in \mathcal{G}_n(W) \), define \( \overline{\psi}_i = |W_i|^{-1} f_{W_i} \psi \circ T^n \, dm_{W_i} \). Proceeding as before, we estimate

\[
(4.5) \quad \int_W \mathcal{L}^n f \psi \, dm_W = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f (\psi \circ T^n - \overline{\psi}_i) \, dm_{W_i} + \sum_{W_i \in \mathcal{G}_n(W)} \overline{\psi}_i \int_{W_i} f \, dm_{W_i}.
\]

To each term in the first sum on the right hand side, we apply the strong stable norm,

\[
\int_{W_i} f (\psi \circ T^n - \overline{\psi}_i) \leq \|f\|_{s} |W_i|^{1/p} |\psi \circ T^n - \overline{\psi}_i|_{C^\alpha(W_i)} \leq C \|f\|_{s} \left( \frac{|W_i|}{|W|} \right)^{1/p} |J^s T^n|_{C^\alpha(W_i)},
\]

where we have applied the analogous estimate to (4.4) to the difference \( \psi \circ T^n - \overline{\psi}_i \) with the exponent \( \alpha \). Since \( \alpha > 1/p \), using bounded distortion (2.4), we estimate

\[
|W_i|^{1/p} |J^s T^n|_{C^\alpha(W_i)} \leq C |T^n W_i|^{1/p} \Lambda^{-n(\alpha - 1/p)}.
\]

Finally, summing over \( W_i \), we obtain

\[
(4.6) \quad \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f (\psi \circ T^n - \overline{\psi}_i) \leq C \|f\|_{s} \Lambda^{-n(\alpha - 1/p)} \sum_{W_i \in \mathcal{G}_n(W)} \frac{|T^n W_i|^{1/p}}{|W|^{1/p}}

\leq C \|f\|_{s} \Lambda^{-n(\alpha - 1/p)} \left( \sum_{i} \frac{|T^n W_i|}{|W|} \right)^{1/p} \Lambda^{-1/p} \# \mathcal{G}_n(W_i)

\leq C \delta_0^{-1+1/p} \|f\|_{s} \Lambda^{-n(\alpha - 1/p)} \left( \# \mathcal{M}_0 \right)^{1-1/p} \leq C \delta_0^{-1} \Lambda^{-\alpha} \|f\|_{s} \# \mathcal{M}_0^n,
\]

where in the second line we have used the Hölder inequality and in the third we have used Lemma 3.6(b) and (d).

Next, we estimate the second sum in (4.5). For this estimate, we group \( W_i \in \mathcal{G}_n(W) \) by most recent long ancestor as follows. Recall that \( \mathcal{I}_k(W) \) denotes those elements of \( \mathcal{G}_k(W) \) whose length is at least \( \delta_0/3 \). If \( V_j \in \mathcal{L}_k(W) \) is such that \( T^{n-k}(W_i) \subset V_j \) and \( k \leq n \) is the largest such index with this property, then we say that \( V_j \) is the most recent long ancestor of \( W_i \). Let \( \mathcal{I}_{n-k}(V_j) \) denote those elements of \( \mathcal{G}_n(W) \) whose most recent long ancestor is \( V_j \). If no such ancestor exists, then \( W_i \in \mathcal{I}_n(W) \). Thus,

\[
\sum_{W_i \in \mathcal{G}_n(W)} \overline{\psi}_i \int_{W_i} f \, dm_{W_i} = \sum_{k=1}^{n} \sum_{V_j \in \mathcal{L}_k(W)} \sum_{W_i \in \mathcal{I}_{n-k}(V_j)} \overline{\psi}_i \int_{W_i} f \, dm_{W_i} + \sum_{W_i \in \mathcal{I}_n(W)} \overline{\psi}_i \int_{W_i} f \, dm_{W_i}.
\]
We use the strong stable norm to estimate the terms in $\mathcal{I}_n(W)$,
\begin{equation}
\sum_{W_i \in \mathcal{I}_n(W)} \overline{\psi}_i \int_{W_i} f \, dm_{W_i} \leq \|f\|_s \sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} \sum_{W_i \in \mathcal{I}_n(W)} \frac{|T^n W_i|^{1/p}}{|W|^{1/p}} \leq \|f\|_{sK^{-n/p}} \sum_{W_i \in \mathcal{I}_n(W)} \frac{|T^n W_i|^{1/p}}{|W|^{1/p}} \leq \|f\|_{sK^{-n/p}K_1^{n(1-1/p)}} \leq \|f\|_{sK^{-n/p}\rho^n \kappa^\alpha \Lambda^n} \leq \|f\|_{s\rho^n C\delta_0^{-1} \# \mathcal{M}_0^n},
\end{equation}
where we have used (2.1) for the second inequality, the Hölder inequality and Lemma 3.6(a) for the third and fourth inequalities, and the fact that $\alpha_0 \geq 1/p$ (from (3.4)) and Lemma 3.6(d) for the last inequality.

For the remainder of the terms, we use the weak norm of $f$, and sum using Lemma 3.6(a) from time $k$ to time $n$,
\begin{equation}
\sum_{k=1}^{n} \sum_{V_j \in \mathcal{L}_k(W)} \overline{\psi}_i \int_{W_i} f \, dm_{W_i} \leq \sum_{k=1}^{n} \sum_{V_j \in \mathcal{L}_k(W)} \sum_{W_i \in \mathcal{I}_{n-k}(V_j)} |W|^{-1/p} |f|_w \leq \sum_{k=1}^{n} 3\delta_0^{-1} K_1^{n-k} |V_j|^{1/p} |f|_w \leq \sum_{k=1}^{n} C\delta_0^{-1} K_1^{n-k} \kappa^{-k/p} (#\mathcal{M}_0^k)^{-1-1/p} |f|_w \leq C\delta_0^{-1} |f||_{w} \rho^{-k} \kappa^{\alpha (n-k)} \Lambda^{n-k} # \mathcal{M}_0^k \leq C\delta_0^{-2} \epsilon_1^{-1} \kappa^{-n/p} # \mathcal{M}_0^n |f|_w,
\end{equation}
where we have used Lemma 3.6(c) to sum over $V_j \in \mathcal{L}_k(W)$, as well as the fact that
\[ \Lambda^{n-k} # \mathcal{M}_0^k \leq C\delta_0^{-1} # \mathcal{M}_0^{n-k} # \mathcal{M}_0^k \leq C\delta_0^{-1} \epsilon_1^{-1} # \mathcal{M}_0^n, \]
by Proposition 3.12. Putting this estimate together with (4.7) and (4.6) in (4.5) yields
\[ \int_W \mathcal{L}_n f \psi \, dm_W \leq C\delta_0^{-2} \left( ((\Lambda^{-\alpha n} + \rho^n) \|f\|_s + \kappa^{-n/p} |f|_w) \# \mathcal{M}_0^n \right), \]
and taking the appropriate suprema over $W$ and $\psi$ completes the proof of (4.2).

4.1.3. Strong unstable norm bound. Let $f \in C^1(M)$ and $\varepsilon \in (0, \varepsilon_0)$. Take $W^1, W^2 \in \mathcal{W}_s$ with $d_{W^1}(W^1, W^2) \leq \varepsilon$, and $\psi_k \in C^1(W^k)$ such that $|\psi_k|_{C^1(W^k)} \leq 1$ and $d_0(\psi_1, \psi_2) = 0$. For $n \geq 1$, we subdivide $\mathcal{G}_n(W^k)$ into matched and unmatched pieces as follows.

To each $W_i \in \mathcal{G}_n(W^k)$, we associate a family of vertical (in the chart) segments $\{\gamma_x\}_{x \in W^i}$ of length at most $C \Lambda^{-n} \varepsilon$ such that if $\gamma_x$ is not cut by an element of $\mathcal{S}_+^+$, its image $T^n \gamma_x$ will have length $C \varepsilon$ and will intersect $W^2$. Due to the uniform transversality of stable and unstable cones, such a segment $T^n \gamma_x$ will belong to the unstable cone for each $i = 0, \ldots, n$, and so undergo the uniform expansion due to (2.1).

In this way, we obtain a partition of $W^1$ into intervals for which $T^n \gamma_x$ is not cut and intersects $W^2$ and subintervals for which this is not the case. This defines an analogous partition of $T^{-n} W^1$ and $T^{-n} W^2$. We call two curves $U^1_j \subset T^{-n} W^1$ and $U^2_j \subset T^{-n} W^2$ matched if they are connected by the foliation $\gamma_x$ and their images under $T^n$ are connected by $T^n \gamma_x$. We call the remaining components of $T^{-n} W^k$ unmatched and denote them by $V^k_i$. With this decomposition, there is at most one matched piece and two unmatched pieces for each $W^k_i \in \mathcal{G}_n(W^k)$, and we may write $T^{-n} W^k = (\cup_j U^1_j) \cup (\cup_i V^k_i)$.

We proceed to estimate,
\begin{equation}
\left| \int_{W^1} \mathcal{L}_n f \psi_1 - \int_{W^2} \mathcal{L}_n f \psi_2 \right| \leq \sum_j \left| \int_{U^1_j} f \psi_1 \circ T^n - \int_{U^2_j} f \psi_2 \circ T^n \right| + \sum_{k,i} \left| \int_{V^k_i} f \psi_k \circ T^n \right|
\end{equation}

We begin by estimating the contribution from unmatched pieces. We say a curve $V^1_i$ is created at time $j$, $1 \leq j \leq n$, if $j$ is the first time that $T^{-j} V^1_i$ is not part of a matched curve in $T^{-j} W^1$. \[ \mathcal{M}_0^n = \{ \mathcal{M}_0^n, \mathcal{M}_0^n, \ldots, \mathcal{M}_0^n \}. \]
Define,

\[ V_{j,t} = \{ i : V_i^{1} \text{ is created at time } j \text{ and } T^{n-j}V_i^{1} \subset W^{1}_{t} \in G_j(W^{1}) \} . \]

Note that \( \cup_{i \in V_{j,t}} V_i^{1} = W^{1}_{t} \). Due to the expansion of \( T \) in the unstable cone and the uniform transversality of \( S^{1}_{j} \) with the stable cone, it follows that \( |W^{1}_{t}| \leq CA^{-j}\varepsilon \). Now applying the strong stable norm to each such curve at the time it is created,

\[(4.9) \quad \sum_{i} \int_{V_{j}} f \psi_{1} \circ T^{n} \leq \sum_{j=1}^{n} \sum_{W_{t}^{1} \in G_j(W^{1})} \int_{W_{t}^{1}} L^{n-j} f \psi_{1} \circ T^{n-j} \]

\[ \leq \sum_{j=1}^{n} \sum_{W_{t}^{1} \in G_j(W^{1})} |W_{t}^{1}|^{1/p} \| L^{n-j} f \psi_{1} \|_{s} T^{n-j} |C^{0}(W_{t}^{1})| \]

\[ \leq \sum_{j=1}^{n} \sum_{W_{t}^{1} \in G_j(W^{1})} C A^{-j/p} \varepsilon^{1/p} \delta_{0}^{-1} \kappa^{-2(n-j)/p} \| \mathcal{M}_{0}^{-n-j} \| \| f \|_{s} \]

\[ \leq C \delta_{0}^{-1} \varepsilon^{1/p} \| f \|_{s} \kappa^{-n/p} \sum_{j=1}^{n} \Delta^{-j/p} \# \mathcal{M}_{0}^{j} \mathcal{M}_{0}^{-n-j} \]

where we have applied \((4.2)\) in the second inequality (actually, a simpler version suffices with no need to subtract the average of the test function on each \( W_{t}^{1} \), and Proposition \( 3.12 \) in the fourth. A similar estimate holds over the curves \( V_{t}^{2} \).

Next, we estimate the matched pieces. Recall that according to our notation in Section \( 3.1 \) the curve \( U_{j}^{1} \) is associated with the quadruple \( (i,j,x_{j},r_{j},F_{j}^{1}) \) so that \( F_{j}^{1} \) is defined in the chart \( \chi_{i,j} \) and \( U_{j}^{1} = G(x_{j},r_{j},F_{j}^{1})(I_{r_{j}}) \). By definition of our matching process, it follows that \( U_{j}^{2} = G(x_{j},r_{j},F_{j}^{2})(I_{r_{j}}) \) for some uniform \( F_{j}^{2} \) defined in the same chart, so that the point \( x_{j} + (t,F_{j}^{1}(t)) \) is associated with the point \( x_{j} + (t,F_{j}^{2}(t)) \) by the vertical line \( (0,s)_{s \in \mathbb{R}} \) in the chart.

Recall that \( G_{F_{j}^{k}} = \chi_{i,j}(x_{j} + (t,F_{j}^{k}(t)), \text{ for } t \in I_{r_{j}} \). Define

\[ \tilde{\psi}_{j} = \psi_{1} \circ T^{n} \circ G_{F_{j}^{1}} \circ G_{F_{j}^{2}}^{-1} . \]

The function \( \tilde{\psi}_{j} \) is well-defined on \( U_{j}^{2} \) and \( d_{0}(\tilde{\psi}_{j}, \psi_{1} \circ T^{n}) = 0 \). We can then estimate,

\[(4.10) \quad \sum_{j} \left| \int_{U_{j}^{1}} f \psi_{1} \circ T^{n} - \int_{U_{j}^{2}} f \psi_{2} \circ T^{n} \right| \leq \sum_{j} \left| \int_{U_{j}^{1}} f \psi_{1} \circ T^{n} - \int_{U_{j}^{2}} f \tilde{\psi}_{j} \right| + \left| \int_{U_{j}^{2}} f \tilde{\psi}_{j} - \psi_{2} \circ T^{n} \right| . \]

We estimate the first term on the right side of \((4.10)\) using the strong unstable norm. It follows from the uniform hyperbolicity of \( T \) and the usual graph transform arguments (see \[ DL \] Section 4.3)), that

\[ d_{W^{1}}(U_{j}^{1},U_{j}^{2}) \leq CA^{-n}\varepsilon . \]

Moreover, by definition \( G_{F_{j}^{1}},G_{F_{j}^{2}}^{-1} \in \mathcal{C}^{1} \) so that by \((4.4)\), \( |\tilde{\psi}_{j}|_{\mathcal{C}^{1}(U_{j}^{2})} \leq C |\psi_{1}|_{\mathcal{C}^{1}(W^{1})} \) for some uniform constant \( C \). Thus,

\[(4.11) \quad \sum_{j} \left| \int_{U_{j}^{1}} f \psi_{1} \circ T^{n} - \int_{U_{j}^{2}} f \tilde{\psi}_{j} \right| \leq C \varepsilon^{\beta} A^{-n}\delta_{0}^{-1} \| f \|_{u} \# \mathcal{M}_{0}^{n} , \]

where we have used Lemma \( 3.6(b) \) to sum over the matched pieces since there is at most one matched piece per element of \( G_{n}(W^{1}) \).
We estimate the second term on the right side of (4.10) using the strong stable norm,

\[ \sum_j \left| \int_{U_j^n} f (\tilde{\psi}_j - \psi_2 \circ T^n) \right| \leq \sum_j \|f\|_s \|U_j^n\|^{1/p} \left| \tilde{\psi}_j - \psi_2 \circ T^n \right|_{C^0(U_j^n)}. \]

It follows from [DL, Lemma 4.2 and eq. (4.20)] that,

\[ \left| \tilde{\psi}_j - \psi_2 \circ T^n \right|_{C^0(U_j^n)} \leq C \epsilon^{1-\alpha}. \]

Putting this together with the above estimate and summing over \( j \) yields,

\[ \sum_j \left| \int_{U_j^n} f (\tilde{\psi}_j - \psi_2 \circ T^n) \right| \leq C \epsilon^{1-\alpha} \|f\|_s \delta_0^{-1} \# M_0^n. \]

Finally, collecting the above estimate with (4.11) in (4.10) and adding the estimate over unmatched pieces from (4.9), yields by (4.8),

\[ \left| \int_{W^1} L^n f \psi_1 - \int_{W^2} L^n f \psi_2 \right| \leq C \delta_0^{-1} (\epsilon \Lambda^{-\beta n} \|f\|_u + \epsilon^{1-\alpha} \|f\|_s + \epsilon^{1/p} \kappa^{-n/p} \|f\|_s) \# M_0^n. \]

Then, since \( \beta \leq \min\{1 - \alpha, 1/p\} \) according to (3.4), we may divide through by \( \epsilon^\beta \), and take the appropriate suprema to complete the proof of (4.3).

### 4.2. A spectral gap for \( L \)

We prove that \( L \) has a spectral gap in a series of lemmas, first establishing its quasi-compactness, Lemma 4.3, then characterizing elements of its peripheral spectrum, Lemmas 4.4 and 4.5, and finally concluding the existence of a spectral gap, Lemma 4.6. These are all the items of Theorem 4.1.

**Lemma 4.3.** The spectral radius of \( L \) on \( B \) is \( e^{h^*} \), while its essential spectral radius is at most \( \sigma e^{h^*} \) for any \( \sigma > \max\{\Lambda^{-\beta}, \rho\} \). Thus \( L \) is quasi-compact on \( B \). Moreover, the peripheral spectrum of \( L \) contains no Jordan blocks.

**Proof.** First we establish the upper bound on the spectral radius of \( L \) using Proposition 4.2 and Corollary 3.13. Fix \( \sigma < 1 \) such that \( \sigma > \max\{\Lambda^{-\beta}, \rho\} \). Next, choose \( N > 0 \) such that \( C \delta_0^{-2} e_1^{-1} \max\{\Lambda^{-\beta N}, \rho^N\} \leq \frac{1}{2} \sigma^N \). Finally, choose \( c_u > 0 \) such that \( c_u C \delta_0^{-2} e_1^{-1} \kappa^{-N/p} \leq \frac{1}{2} \sigma^N \). Then,

\[
\|L^N f\|_B = \|L^N f\|_s + c_u \|L^N f\|_u
\leq \left( \frac{1}{2} \sigma^N \|f\|_s + C \delta_0^{-2} e_1^{-1} \kappa^{-N/p} \|f\|_w + c_u \frac{1}{2} \sigma^N \|f\|_u + c_u C \delta_0^{-2} e_1^{-1} \kappa^{-N/p} \|f\|_s \right) e^{nh^*}
\leq \left( \sigma^N \|f\|_B + C \delta_0^{-2} \kappa^{-N/p} \|f\|_w \right) e^{nh^*}.
\]

This is the standard Lasota-Yorke inequality for \( L \), which, coupled with the compactness of the unit ball of \( B \) in \( B_w \) (Lemma 3.5), is sufficient to conclude that the essential spectral radius of \( L \) is at most \( \sigma e^{h^*} \), and its spectral radius is at most \( e^{h^*} \).

To prove the lower bound on the spectral radius, we estimate using (3.9) and Lemma 3.11. Take \( W \in W^s \) with \( |W| \geq \delta_1/3 \). Then for \( n \geq n_1 \) we have,

\[
\|L^n 1\|_B \geq \int_W L^n 1 \ dm_W = \sum_{W_i \in G_0^n} |W_i| \geq \sum_{W_i \in L^N_1(W)} \delta_1/3
\geq \frac{2 \delta_1}{9} \|G_n(W) \geq \frac{2 \delta_1}{9} e_0 \# M_0^n.
\]

Then taking the limit as \( n \to \infty \) and using the definition of \( h^* \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|L^n 1\|_B \geq \limsup_{n \to \infty} \frac{1}{n} \log \left( \|L^n 1\|_B / \|1\|_B \right) \geq \limsup_{n \to \infty} \frac{1}{n} \log (\# M_0^n) = h^*.
\]
which proves that the spectral radius of \( \mathcal{L} \) is at least \( e^{h_*} \). We conclude that the spectral radius of \( \mathcal{L} \) is in fact \( e^{h_*} \) and so \( \mathcal{L} \) is quasi-compact since its essential spectral radius is bounded by \( \sigma e^{h_*} \).

Finally, the lack of Jordan blocks stems from Corollary 3.13 and Proposition 4.2, which together imply \( \|\mathcal{L}^n\|_B \leq C e^{nh_*} \) for all \( n \geq 0 \).

Let \( V_\theta \) denote the eigenspace associated to the eigenvalue \( e^{h_*+2\pi i \theta} \). Due to the quasi-compactness of \( \mathcal{L} \) and the absence of Jordan blocks, the spectral projector \( \Pi_\theta : \mathcal{B} \to V_\theta \) is well-defined in the uniform topology of \( L(\mathcal{B}, \mathcal{B}) \) and can be realized as,

\[
(4.13) \quad \Pi_\theta = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} e^{-2\pi i \theta k} \mathcal{L}^k.
\]

Let \( V = \bigoplus_\theta V_\theta \), where the sum is taken over \( \theta \) corresponding to eigenvalues of \( \mathcal{L} \). Note that \( V \) is finite dimensional by the quasi-compactness of \( \mathcal{L} \). Analogously, and as in the statement of Theorem 4.1, define

\[
(4.14) \quad \nu_0 = \Pi_0 1 := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \mathcal{L}^k 1.
\]

Since we have proved uniform bounds of the form \( \|\mathcal{L}^k\|_B \leq C e^{kh_*} \), the limit above exists and satisfies \( \mathcal{L} \nu_0 = e^{h_*} \nu_0 \). A priori, however, \( \nu_0 \) may be 0 (if \( e^{h_*} \) is not in the spectrum of \( \mathcal{L} \)). The following lemma shows this is not the case, and provides an important characterization of the peripheral spectrum of \( \mathcal{L} \).

**Lemma 4.4.** (Peripheral spectrum of \( \mathcal{L} \))

a) The distribution \( \nu_0 = \Pi_0 1 \neq 0 \) is a non-negative Radon measure and \( e^{h_*} \) is in the spectrum of \( \mathcal{L} \).

b) All elements of \( V \) are signed measures, absolutely continuous with respect to \( \nu_0 \).

c) The spectrum of \( e^{-h_*} \mathcal{L} \) consists of a finite number of cyclic groups; in particular, each \( \theta \) is rational.

**Proof.** (a) By density of \( C^1(M) \) in \( \mathcal{B} \), since \( V_\theta \) is finite-dimensional, it follows that \( \Pi_\theta C^1(M) = V_\theta \). Thus for each \( \nu \in V \), \( \nu \neq 0 \), there exists \( f \in C^1(M) \) such that \( \Pi_\theta f = \nu \). Moreover, for every \( \psi \in C^1(M) \), we have

\[
(4.15) \quad |\nu(\psi)| = |\Pi_\theta f(\psi)| \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-h_* k} |\mathcal{L}^k f(\psi)| \leq |f|_{\infty} |\Pi_0 1(\psi)|,
\]

so that \( \Pi_0 1 \neq 0 \) since \( \nu \neq 0 \). In particular, \( e^{h_*} \) is an eigenvalue of \( \mathcal{L} \). Moreover, since \( \Pi_0 1 \) is positive as an element of \( (C^1(M))^* \), it follows from [Sch] Sect. I.4] that \( \nu_0 = \Pi_0 1 \) is a non-negative Radon measure on \( M \).

(b) Applying (4.15) again to \( \nu \in V_\theta \), we conclude that every element of \( V_\theta \) is a signed measure, absolutely continuous with respect to \( \nu_0 \). Moreover, setting \( f_\nu = \frac{\nu}{d(\nu_0)} \), it follows that \( f_\nu \in L^\infty(M, \nu_0) \).

(c) Suppose \( \nu \in V_\theta \). Then using part (b), for any \( \psi \in C^1(M) \),

\[
(4.16) \quad \int_M \psi f_\nu \, d\nu_0 = \nu(\psi) = e^{-h_*} e^{-2\pi i \theta} \mathcal{L} \nu(\psi) = e^{-h_*} e^{-2\pi i \theta} \nu(\psi \circ T^{-1})
\]

\[
= e^{-h_*} e^{-2\pi i \theta} \nu_0(f_\nu \psi \circ T^{-1}) = e^{-h_*} e^{-2\pi i \theta} \mathcal{L} \nu_0(\psi f_\nu \circ T^{-1})
\]

\[
= e^{-2\pi i \theta} \int_M \psi f_\nu \circ T^{-1} \, d\nu_0.
\]
Thus \( f_\nu \circ T^{-1} = e^{2\pi i \theta} f_\nu \), \( \nu_0 \)-a.e. Define \( f_{\nu,k} = (f_\nu)^k \in L^\infty(\nu_0) \). It follows as in [BD, Lemma 5.5], that \( d\nu_k := f_{\nu,k} d\nu_0 \in \mathcal{B} \) for each \( k \in \mathbb{N} \). Then since \( \mathcal{L}\nu_k = e^{2\pi i k\theta} \nu_k \), it follows that \( e^{2\pi i k \theta} \) is in the peripheral spectrum of \( \mathcal{L} \) for each \( k \). By the quasi-compactness of \( \mathcal{L} \), this set must be finite, and so \( \theta \) must be rational. □

We remark that elements of \( \mathcal{B}_w \) can be viewed as both distributions on \( M \), as well as families of leafwise distributions on stable manifolds as follows (cf. [BD, Definition 7.5]). For \( f \in C^1(M) \), the map defined by

\[
\mathcal{K}_{(W,f)}(\psi) = \int_W f \psi \, dm_W, \quad \psi \in C^1(W),
\]

can be viewed as a distribution of order 1 on \( W \). Since \( \mathcal{K}_{(W,f)}(\psi) \leq |f|_w |\psi|_{C^1(W)} \), \( \mathcal{K}_{(W,-)} \) can be extended to \( f \in \mathcal{B}_w \). We denote this extension by \( \int_W f \psi \), and we call the associated family of distributions the leafwise distribution \((f,W)_{W \in W^s} \) corresponding to \( f \). If, in addition, \( f \in \mathcal{B}_w \) satisfies \( \int_W \psi f \geq 0 \) for all \( \psi \geq 0 \), then by [Sch, Section I.4], the leafwise distribution is in fact a leafwise measure.

Recall the disintegration of \( \mu_{SRB} \) used in the proof of Lemma 3.3 into conditional measures \( \mu_{SRB}^\xi \) on the family of stable manifolds \( F = \{W_\xi\}_{\xi \in \Xi} \), and a factor measure \( \hat{\mu}_{SRB} \) on the index set \( \Xi \). We have \( d\mu_{SRB}^\xi = |W_\xi|^{-1} g_\xi \, d\nu_k \), where \( g_\xi \) is uniformly log-Hölder continuous by (3.5).

**Lemma 4.5.** Let \( \nu_0^\xi \) and \( \hat{\nu}_0 \) denote the conditional measures on \( W_\xi \) and factor measure on \( \Xi \), respectively, obtained by disintegrating \( \nu_0 \) on the family of stable manifolds \( F \). For all \( \psi \in C^1(M) \),

\[
\int_{W_\xi} \psi \, d\nu_0^\xi = \int_{W_\xi} \psi g_\xi \, \nu_0 \quad \text{for all } \xi \in \Xi, \quad \text{and} \quad \hat{\nu}_0(\xi) = |W_\xi|^{-1} \left( \int_{W_\xi} g_\xi \, \nu_0 \right) \, d\hat{\mu}_{SRB}(\xi).
\]

Moreover, viewed as a leafwise measure, \( \nu_0(W) > 0 \) for all \( W \in W^s \).

**Proof.** We prove the last claim first. For \( W \in W^s \), let \( n_2 \leq \tilde{C}_2 \log(|W|/\delta_1) \) be the constant from the proof of Corollary 3.8 applied in the case \( \varepsilon = 1/3 \) and \( \delta_1 \) as chosen in (3.9). Let \( V \in \mathcal{G}_{n_2}(W) \) have \( |V| \geq \delta_1/3 \). Then using (3.9) and Lemma 3.11,

\[
\int_W \nu_0 = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \int_W \mathcal{L}^k 1 \, dm_W \geq \lim_{n \to \infty} \frac{1}{n} \sum_{k=n_1+n_2}^{n-1} e^{-kh_*} \sum_{W_i \in \mathcal{G}_{n_1+n_2}(V)} |W_i| \\
\geq \lim_{n \to \infty} \frac{1}{n} \sum_{k=n_1+n_2}^{n-1} e^{-kh_*} \frac{\delta_1}{9} c_0 \lambda_0^{k-n_2} = \frac{2\delta_1}{9} e^{-(n_1+n_2)h_*} \lim_{n \to \infty} \sum_{k=0}^{\infty} e^{-kh_*} \# \mathcal{M}_0^k.
\]

We claim that the last limit cannot be 0. For suppose it were 0. Then for any \( W \in W^s \), \( \psi \in C^1(W) \), we would have by Lemma 3.6(b),

\[
\int_W \psi \cdot \nu_0 = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \int_W \psi \mathcal{L}^k 1 \, dm_W \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \sum_{W_i \in \mathcal{G}_k(W)} |\psi|_\infty |W_i| \\
\leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} C \# \mathcal{M}_0^k = 0,
\]

which would imply \( \nu_0 = 0 \), a contradiction. This proves the claim, and recalling the definition of \( n_2 \), we conclude that

\[
(4.17) \quad \nu_0(W) \geq C |W|^h \tilde{C}_2 \quad \text{for all } W \in W^s.
\]

With (4.17) established, the remainder of the proof follows from the definition of convergence in the weak norm, precisely as in [BD, Lemma 7.7]. □
We are finally ready to prove the final point of our characterization of the peripheral spectrum of \( \mathcal{L} \).

**Lemma 4.6.** \( \mathcal{L} \) has a spectral gap on \( B \).

**Proof.** Recalling Lemma 4.4(c), suppose \( \nu_q \in \mathcal{V}_{p/q} \). Then \( \mathcal{L}^q \nu_q = e^{qh_*} \nu_q \) and \( \mathcal{L}^q \nu_0 = e^{qh_*} \nu_0 \). Since \( T^q \) is also mixing and the spectral radius of \( \mathcal{L}^q \) is \( e^{qh_*} \), it suffices to prove that mixing implies the eigenspace corresponding to \( e^{h_*} \) is simple in order to conclude that \( \mathcal{L} \) can have no other eigenvalues of modulus \( e^{h_*} \), i.e. \( \mathcal{L} \) has a spectral gap. We proceed to prove this claim.

Suppose \( \nu_1 \in \mathcal{V}_0 \). We will show that \( \nu_1 = c \nu_0 \) for some constant \( c > 0 \). By [4.16], there exists \( f_1 \in L^\infty(\nu_0) \) such that \( f_1 \nu_1 = \nu_0 \) and \( f_1 \circ T = f_1, \nu_0 \)-a.e. Letting

\[
S_n f_1(x) = \sum_{k=0}^{n-1} f_1 \circ T^k(x),
\]

it follows that the ergodic average \( \frac{1}{n} S_n f_1 = f_1 \) for all \( n \geq 0 \). This implies that \( f_1 \) is constant on stable manifolds. In addition, since by Lemma 4.5 and (4.17), the factor measure \( \hat{\nu}_0 \) is equivalent to \( \hat{\mu}_{SRB} \) on the index set \( \Xi \), we have that \( f_1 = f_1 \circ T \) on \( \hat{\mu}_{SRB} \) a.e. \( W_\xi \in \mathcal{F} \), i.e. \( f_1 = f_1 \circ T \), \( \hat{\mu}_{SRB} \)-a.e. By the ergodicity of \( \mu_{SRB} \), \( f_1 = \text{constant} \mu_{SRB} \)-a.e. But since this constant value holds on each stable manifold \( W_\xi \in \mathcal{F} \), using again the equivalence of \( \hat{\nu}_0 \) and \( \hat{\mu}_{SRB} \), we conclude that \( f_1 \) is constant \( \nu_0 \)-a.e. \( \square \)

5. **Construction and Properties of the Measure of Maximal Entropy**

Since \( \mathcal{L} : B \to B \) has a spectral gap, we may decompose \( \mathcal{L} \) as

\[
\mathcal{L}^n f = e^{nh_*} \Pi_0 f + R^n f \quad \text{for any } n \geq 1, f \in B,
\]

where \( \Pi_0 = \Pi_0 R = R \Pi_0 = 0 \) and there exists \( \bar{\sigma} < 1 \) and \( C > 0 \) such that \( \|e^{-nh_*} R^n\|_B \leq C \bar{\sigma}^n \).

Indeed, we may recharacterize the definition of the spectral projector \( \Pi_0 \) in (4.13) as,

\[
\Pi_0 f = \lim_{n \to \infty} e^{-nh_*} \mathcal{L}^n f,
\]

where convergence is in the \( B \) norm. Indeed, letting \( W \in \mathcal{W}^s \) with \( |W| \geq \delta_1/3 \), we have by Lemma 3.6(b) and (4.17),

\[
0 < \nu_0(W) = \lim_{n \to \infty} e^{-nh_*} \int_W \mathcal{L}^n 1 \, dm_W = \lim_{n \to \infty} e^{-nh_*} \sum_{W_i \in \mathcal{G}_s(W)} |W_i| \\
\leq \liminf_{n \to \infty} C e^{-nh_*} \# \mathcal{M}_s^n.
\]

This implies the final limit cannot be \( 0 \). We have proved the following.

**Lemma 5.1.** There exists \( \bar{c}_1 > 0 \) such that \( \# \mathcal{M}_s^n \geq \bar{c}_1 e^{nh_*} \) for all \( n \geq 1 \).

Next, consider the dual operator, \( \mathcal{L}^* : B^* \to B^* \), which also has a spectral gap. Recalling our identification of \( f \in C^1(M) \) with the measure \( f d\mu_{SRB} \) from Section 3.2, define

\[
\tilde{\nu}_0 := \lim_{n \to \infty} e^{-nh_*} (\mathcal{L}^*)^n d\mu_{SRB},
\]

where convergence is in the dual norm, \( \| \cdot \|_{B^*} \). Clearly, \( \tilde{\nu}_0 \in \mathcal{B}^* \), and \( \mathcal{L}^* \tilde{\nu}_0 = e^{h_*} \tilde{\nu}_0 \). By the positivity of the operator \( \mathcal{L}^* \), we have \( \tilde{\nu}_0(f) \geq 0 \) for each \( f \in C^1(M) \) with \( f \geq 0 \) (recalling \( C^1(M) \subset \mathcal{B} \)). Thus again applying [Sch. Section I.4], we conclude that \( \tilde{\nu}_0 \) is a Radon measure on \( M \).

Next, defining \( f_n = e^{-nh_*} \mathcal{L}^n 1 \in \mathcal{B} \) for \( n \geq 1 \), we have,

\[
\tilde{\nu}_0(f_n) = \lim_{k \to \infty} e^{-kh_*} \langle f_n, (\mathcal{L}^*)^k d\mu_{SRB} \rangle = \lim_{k \to \infty} e^{-kh_*} \langle \mathcal{L}^k f_n, d\mu_{SRB} \rangle.
\]
The expression on the right is bounded by where we have used Lemma 3.2(b) for the first inequality and (5.1) for the second, since in particular,

We prove the proposition for Proposition 5.3. The result for $q > 0$, there exist constants $C = C(q)$ and $\gamma = \gamma(q) > 0$ such that for all $\varphi, \psi \in C^0(M)$,

$$\left| \int_M \varphi \psi \circ T^n d\mu_* - \int_M \varphi d\mu_* \int_M \psi d\mu_* \right| \leq C|\varphi|_{C^0(M)}|\psi|_{C^0(M)}e^{-\gamma n} \quad \text{for all } n \geq 0.$$  

Proof. We prove the proposition for $\varphi, \psi \in C^1(M)$. The result for $q \in (0,1)$ then follows by a standard approximation argument.

First we verify that $\psi \circ T^n \tilde{\nu}_0$ is an element of $\mathcal{B}^*$ for $\psi \in C^1(M)$ and $n \geq 1$. We do this by noting that for any $\psi \in C^1(M)$, $\psi \tilde{\nu}_0 \in \mathcal{B}^*$ by simply defining,

$$\langle f, \psi \tilde{\nu}_0 \rangle := \langle \psi f, \tilde{\nu}_0 \rangle \quad \text{for any } f \in \mathcal{B},$$

and the expression on the right is bounded by $|\psi|_{C^1} \|f\|_{\mathcal{B}} \|\tilde{\nu}_0\|_{\mathcal{B}^*}$ by Lemma 3.2(b), and so the pairing defines a bounded, linear functional on $\mathcal{B}$, with norm at most $|\psi|_{C^1} \|\tilde{\nu}_0\|_{\mathcal{B}^*}$. Next, define for $n \geq 1$,

$$\langle f, \psi \circ T^n \tilde{\nu}_0 \rangle := \langle e^{-nh} \mathcal{L}^n f, \psi \tilde{\nu}_0 \rangle = \langle \psi e^{-nh} \mathcal{L}^n f, \nu_0 \rangle.$$  

The expression on the right is bounded by

$$\|\psi e^{-nh} \mathcal{L}^n f\|_{\mathcal{B}} \|\tilde{\nu}_0\|_{\mathcal{B}^*} \leq |\psi|_{C^1(M)} e^{-nh} \|\mathcal{L}^n f\|_{\mathcal{B}} \|\tilde{\nu}_0\|_{\mathcal{B}^*} \leq C|\psi|_{C^1(M)} \|f\|_{\mathcal{B}} \|\tilde{\nu}_0\|_{\mathcal{B}^*},$$

where we have used Lemma 3.2(b) for the first inequality and (5.1) for the second, since in particular, $e^{-nh} \|\mathcal{L}^n f\|_{\mathcal{B}} \leq C$. Thus (5.4) defines a bounded, linear functional on $\mathcal{B}$, so $\psi \circ T^n \tilde{\nu}_0 \in \mathcal{B}^*$.
Finally, using Definition 5.2 and (5.4), noting that \( \phi\nu_0 \in \mathcal{B} \) by Lemma 3.2(b), and recalling again (5.1), we write

\[
\int_M \varphi \psi \circ T^n d\mu_* = \langle \varphi \nu_0, \psi \circ T^n \nu_0 \rangle = \frac{\langle e^{-nh_*} L^n(\varphi \nu_0), \psi \nu_0 \rangle}{\langle \nu_0, \nu_0 \rangle} = \frac{\langle \nu_0, \nu_0 \rangle \langle \varphi \nu_0, \psi \nu_0 \rangle + \langle e^{-nh_*} R^n(\varphi \nu_0), \psi \nu_0 \rangle}{\langle \nu_0, \nu_0 \rangle},
\]

where we have used (5.3). The first term on the right is simply \( \int_M \varphi \psi \mu_* \). The second term is bounded by,

\[
C e^{-nh_*} R^n(\varphi \nu_0) \leq C' e^{-\sigma n} \|\nu_0\| \|\psi\|_{C^1} \|\nu_0\| \|\psi\|_{C^0} \leq C' e^{-\sigma n} |\phi|_{C^1} |\psi|_{C^0},
\]

where we have used Lemma 3.2(b) and \( C' \) depends on \( \|\nu_0\|, \|\psi\|_{C^0}, \text{ and } \langle \nu_0, \nu_0 \rangle \).

\[\square\]

5.1. Hyperbolicity and Ergodicity of \( \mu_* \). We begin by showing that \( \mu_* \) gives small measure to \( \varepsilon \)-neighborhoods of the singularity sets \( S_k^\pm \).

**Lemma 5.4.** For any \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that

\[
\mu_*(\mathcal{N}_\varepsilon(S_k^\pm)) \leq C_k \varepsilon^{1/p}.
\]

In particular, for any \( \gamma > p \) and \( k \in \mathbb{N} \), for \( \mu_* \)-a.e. \( x \in M \), there exists \( C > 0 \) such that

\[
d(T^n x, S_k^\pm) \geq C n^{-\gamma}, \quad \text{for all } n \geq 0.
\]

**Proof.** First we prove the claimed bounds with respect to \( \nu_0 \) for each \( S_k^\pm \), \( k \geq 1 \). Let \( 1_{k,\varepsilon} \) denote the indicator function of the set \( \mathcal{N}_\varepsilon(S_k^\pm) \). Since \( S_k^- \) comprises finitely many smooth curves, all uniformly transverse to the stable cone, by Lemma 3.2(b), \( 1_{k,\varepsilon} \nu_0 \in \mathcal{B} \), and as a consequence, \( 1_{k,\varepsilon} \nu_0 \in \mathcal{B}_w \). We claim that,

\[
\nu_0(\mathcal{N}_\varepsilon(S_k^-)) \leq C |1_{k,\varepsilon} \nu_0|_w \leq C_k \varepsilon^{1/p}.
\]

Indeed, the first inequality follows from Lemma 3.3. To prove the second inequality, let \( W \in \mathcal{W}_w \) and \( \psi \in C^1(W) \) with \( |\psi|_{C^1(W)} \leq 1 \). Due to the transversality of \( S_k^- \) with the stable cone, \( W \cap \mathcal{N}_\varepsilon(S_k^-) \) comprises at most a finite number \( N_k \) of curves, depending only on \( S_k^- \) and \( \delta_0 \), and not on \( W \), each having length at most \( C \varepsilon \). Thus,

\[
\int_W 1_{k,\varepsilon} \psi \nu_0 = \sum_i \int_{W_i} \psi \nu_0 \leq \sum_i \|\nu_0\|_B \|W_i\|^{1/p} |\psi|_{C^0(W_i)} \leq C N_\varepsilon(\psi) \leq C N\varepsilon(\psi),
\]

and taking the supremum over \( \psi \) and \( W \), the second inequality in (5.6) follows. Next, it follows from (5.2) and Lemma 3.3 that

\[
|\tilde{\nu}_0(f)| \leq C |f|_w, \quad \text{for all } f \in \mathcal{B}_w,
\]

so that in fact \( \tilde{\nu}_0 \in \mathcal{B}_w^\star \subset \mathcal{B}_w^\star \). Thus for each \( k \geq 1 \), by (5.6),

\[
\mu_*(\mathcal{N}_\varepsilon(S_k^-)) = \frac{|\tilde{\nu}_0(1_{k,\varepsilon} \nu_0)|}{|\tilde{\nu}_0(\nu_0)|} \leq C |1_{k,\varepsilon} \nu_0|_w \leq C C_k \varepsilon^{1/p}.
\]

To prove the bound for \( S_k^+ \), we use the invariance of \( \mu_* \) together with the fact that \( T^{-k} S_k^- = S_k^+ \). Moreover, we have \( T^k(\mathcal{N}_\varepsilon(S_k^+)) \subset \mathcal{N}_{C_k \varepsilon}(\mathcal{N}_\varepsilon(S_k^+)) \), where \( \kappa_+ \) is the maximum rate of expansion in the unstable cone.

Finally, to prove (5.5), we fix \( \gamma > p \) and estimate for each \( k \in \mathbb{N} \),

\[
\sum_{n \geq 1} \mu_*(\mathcal{N}_{n^{-\gamma}}(S_k^\pm)) \leq C_k \sum_{n \geq 1} n^{-\gamma/p} < \infty.
\]

Thus by the Borel-Cantelli Lemma, \( \mu_* \)-a.e. \( x \in M \) visits \( \mathcal{N}_{n^{-\gamma}}(S_k^\pm) \) only finitely many times along its orbit, completing the proof of the lemma. \[\square\]
Lemma 5.4 immediately implies the following corollary.

**Corollary 5.5.** The following items establish the hyperbolicity of the measure $\mu_*$.

- a) For any $C^1$ curve $V$ uniformly transverse to the stable cone, there exists $C > 0$ such that $\nu_0(N_\epsilon(V)) \leq C\epsilon$ for all $\epsilon > 0$.
- b) The measures $\nu_0$ and $\mu_*$ have no atoms, and $\mu_*(W) = 0$ for all local stable and unstable manifolds, $W$.
- c) $\int_M |\log d(x, S^u_x)| d\mu_* < \infty$.
- d) $\mu_*$-a.e. $x \in M$ has a stable and an unstable manifold of positive length.

**Proof.** The proof follows directly from the control established on the measures of the neighborhoods of the singularity sets in Lemma 5.4. The argument follows exactly as in [BD, Corollary 7.4]. □

With the control established in Lemma 5.4, we may follow the same arguments as in [BD, Section 7.3] to establish the ergodicity of the measure $\mu_*$. Indeed, our control is stronger than the bounds $\mu_*(N_\epsilon(S^u_x)) \leq C_\gamma |\log \epsilon|^\gamma$ for some $\gamma > 1$ available in [BD], and the Hölder continuity of our strong norm $\| \cdot \|_u$ is stronger than the logarithmic modulus of continuity available in [BD]. The key result is establishing the absolute continuity of the unstable foliation with respect to $\mu_*$. Given a locally maximal Cantor rectangle $R$, let $W^u(R)$ be the set of stable/unstable manifolds that cross $D(R)$ completely (see Section 3.6).

**Proposition 5.6.** Let $R$ be a locally maximal Cantor rectangle with $\mu_*(R) > 0$. Fix $W^0 \in W^s(R)$, and for $W \in W^s(R)$, let $\Theta_W : W^0 \cap R \to W \cap R$ denote the holonomy map sliding along unstable manifolds in $W^u(R)$. Then $\Theta_W$ is absolutely continuous with respect to $\mu_*$.  

**Proof.** This is [BD, Corollary 7.9]. Its proof relies on the analogous property of absolute continuity for $\nu_0$, which in turn follows from the control established by the strong norm and Lemma 5.4. The final step in the proof is to show that on each $W \in W^u(R)$, the conditional measure $\mu_*^W$ of $\mu_*$ is equivalent to the leafwise measure $\nu_0$ restricted to $W$, i.e. there exists $C_W > 0$ such that

$$C_W \mu_*(W) \leq \nu_0|W | \leq C_W^{-1} \mu_*(W). \tag{5.8}$$

This equivalence of the measures follows from the representation of $\nu_0$ as a family of leafwise measures given by Lemma 4.5 as well as the characterization of $\mu_*$ via the limit,

$$\mu_*(\psi) = \tilde{\nu}_0(\nu)^{-1} \tilde{\nu}_0(\psi \nu) = \tilde{\nu}_0(\nu)^{-1} \lim_{n \to \infty} e^{-nh_*(L^*)^n} d\mu_{SRB}(\psi \nu),$$

from [5.2]. □

**Corollary 5.7.** The absolute continuity of the unstable holonomy with respect to $\mu_*$ implies the following additional properties.

- a) $(T^n, \mu_*)$ is ergodic for all $n \geq 1$.
- b) For any open set $O \subset M$, we have $\mu_*(O) > 0$.

**Proof.** a) Using absolute continuity, one establishes that each Cantor rectangle belongs to a single ergodic component following the usual Hopf argument [BD, Lemma 7.15]. Then the ergodicity of $T^n$ follows from the assumption that $T$ is topologically mixing [BD, Proposition 7.16].

b) The proof is identical to the proof of [BD, Proposition 7.11]. □

5.2. **Entropy of $\mu_*$.** In this section, we prove that the measure-theoretic entropy of $\mu_*$ is $h_*$, by estimating the measure of dynamically defined Bowen balls for $T^{-1}$. Recall the metric $\bar{d}$ defined in (2.2). For $n \geq 0$ and $\epsilon > 0$ and $x \in M$, define

$$B_n(x, \epsilon) = \{ y \in M : \bar{d}(T^{-j}y, T^{-j}x) \leq \epsilon, \forall 0 \leq j \leq n \}.$$
Lemma 5.8. There exists $C > 0$ such that for all $\varepsilon > 0$ sufficiently small and all $n \geq 0$, we have\footnote{The extra factor of $n$ in this estimate is due to the fact that we do not assume the dynamical refinements of $\mathcal{M}_0^1$ are simply connected. Such an assumption would allow us to eliminate this factor, as in [BD] Proposition 7.12.}
\[
\mu_*(B_n(x, \varepsilon)) \leq C n e^{-nh_*}.
\]

Proof. Fix $x \in M$, $\varepsilon > 0$ and $n \geq 0$, and let $1_{B,\varepsilon}$ denote the indicator function of the Bowen ball $B_n(x, \varepsilon)$. We shall prove
\[
\mu_*(B_n(x, \varepsilon)) = \frac{\bar{v}_0(1_{B,\varepsilon} \nu_0)}{\nu_0(v_0)} \leq C |1_B \nu_0|_w \leq C n e^{-nh_*},
\]
where $C > 0$ can be chosen independent of $\varepsilon$. The first inequality follows from (5.7), once we show that $1_{B,\varepsilon} \nu_0 \in B_w$. To see this, write
\[
1_{B,\varepsilon} = \prod_{j=0}^{n-1} \mathcal{L}_{\text{SRB}}(1_{N_j(T^{-j}x)}),
\]
where $\mathcal{L}_{\text{SRB}}$ denotes the transfer operator with respect to $\mu_{\text{SRB}}$. Since $\mathcal{L}_{\text{SRB}}$ preserves $B_w$ (and also $B$) by [BD], the claim follows since $1_{N_j(T^{-j}x)}$ satisfies the assumptions of Lemma 3.2. Completeness of this transversality condition of that lemma for $\varepsilon$ sufficiently small. Applying Lemma 3.2(b) inductively in $\mathcal{L}$ completes the proof of the claim, and of the first inequality in (5.9).

Next, since $\nu_0$ is a non-negative leafwise measure by Lemma 4.5, we have $\int_W \psi \nu_0 \geq 0$ for all $W \in W^s$ and $\psi \geq 0$. Then since $|\int_W \psi \nu_0| \leq \int_W \psi | \nu_0|$, we can achieve the supremum in the weak norm of $\nu_0$ by restricting to test functions $\psi \geq 0$.

Now take $W \in W^s$, $\psi \in C^1(W)$ with $\psi \geq 0$ and $|\psi|_{C^1(W)} \leq 1$, and suppose that $W \cap B_n(x, \varepsilon) \neq \emptyset$. Then using that $\nu_0$ is an eigenfunction of $\mathcal{L}$,
\[
\int_W \psi 1_{B,\varepsilon} \nu_0 = \int_W \psi 1_{B,\varepsilon} e^{-n \cdot h_*} \mathcal{L}^n \nu_0 = e^{-n \cdot h_*} \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} \psi \circ T^n 1_{B,\varepsilon} \circ T^n \nu_0.
\]

Observe that $1_{B,\varepsilon} \circ T^n = 1_{T^{-n}(B_n(x,\varepsilon))}$, and that
\[
T^{-n}(B_n(x, \varepsilon)) = \{y \in M : \bar{d}(T^{j-n}x, T^j y) \leq \varepsilon, \forall 0 \leq j \leq n\}.
\]
Thus on each $W_i \in \mathcal{G}_n(W)$ such that $W_i \cap T^{-n}(B_n(x, \varepsilon)) \neq \emptyset$, the positivity of $\nu_0$ implies,
\[
\int_{W_i} \psi \circ T^n 1_{B,\varepsilon} \circ T^n \nu_0 \leq \nu_0(W_i) \leq |\nu_0|_w,
\]

It remains to estimate the cardinality of such $W_i$. Recalling (2.2), if $\varepsilon < 10 \text{diam}(M)$, and $\bar{d}(T^{-j-n}x, T^j y) \leq \varepsilon$, then $T^{-j-n}x$ and $T^j y$ belong to the same set $\bar{M}^+_j$ for each $j$. We would like to conclude that then $T^{-n}(B_n(x, \varepsilon))$ belongs to a single element of $\mathcal{M}_0^1$, yet this may fail since both the dynamical refinements of $\mathcal{M}_0^1$ and the local components of $T^{-j}W \subset \bar{M}^+_{ij}$ may not be connected. Figure 1 shows an example of how these multiple components may arise due to intersections of $S^+_j$ with $S$.

Yet suppose $V \subset V' \in \mathcal{G}_j(W)$, $|V| < \varepsilon$. Since $S^-$ comprises a finite number of smooth curves uniformly transverse to the stable cone, for $\varepsilon$ sufficiently small there can be at most two connected components of $V$ that lie in the same $\bar{M}^-_{ij}$; these will be mapped to the same $\bar{M}^+_ij$ under $T^{-1}$. Since this subdivision of a set of radius $\varepsilon$ can occur at most once per iterate, we have at most $n$ elements $W_i \in \mathcal{G}_n(W)$ such that $W_i \cap T^{-n}(B_n(x, \varepsilon)) \neq 0$ for $\varepsilon$ sufficiently small. Putting these estimates together yields,
\[
\int_W \psi 1_{B,\varepsilon} \nu_0 \leq e^{-n \cdot h_* n}|\nu_0|_w.
\]
and taking the supremum over $\psi$ and $W$ yields the final inequality in (5.9). \hfill \Box

**Proposition 5.9.** For $\mu_*$ defined by Definition 5.2 we have $h_{\mu_*}(T) = h_*$. 

*Proof.* Recall that $\int_M |\log d(x, S_j^+)| \, d\mu_* < \infty$ by Corollary 5.5(c), and that $\mu_*$ is ergodic by Corollary 5.7. Thus applying [Dwy] Proposition 3.1, we conclude that for $\mu_*$-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_*(B_n(x, \varepsilon)) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_*(B_n(x, \varepsilon)) = h_{\mu_*}(T^{-1}) = h_{\mu_*}(T).$$

On the other hand, Lemma 5.8 implies that for all $\varepsilon > 0$ sufficiently small,

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_*(B_n(x, \varepsilon)) \geq h_*.$$

Thus $h_{\mu_*}(T) \geq h_*$. But $h_{\mu_*}(T) \leq h_*$ by Theorem 2.8(d), so equality follows. \hfill \Box

### 5.3. Uniqueness of $\mu_*$. In this section we prove that $\mu_*$ is the unique invariant probability measure with $h_{\mu_*}(T) = h_*$. 

The proof of uniqueness follows very closely the proof of uniqueness in [BD; Section 7.7]. We include the proof to point out several differences in the initial estimates on elements of $\mathcal{M}_{-n}^0$, and for completeness. The idea of the proof is to adapt Bowen’s proof of the uniqueness of equilibrium states to the setting of maps with discontinuities. The key estimates will be to show that while not all elements of $\mathcal{M}_{-n}^0$ satisfy good lower bounds on their measure, most elements (in the sense of Lemma 5.10) have satisfied good lower bounds at some point in the recent past (in the sense of Lemma 5.11). Recall that $\mathcal{M}_0^n$ denotes the set of maximal, open connected components on which $T^n$ is smooth, while $\mathcal{M}_{-n}^0$ denotes the analogous set for $T^{-n}$.

Choose $\delta_2 > 0$ sufficiently small that for all $n, k \in \mathbb{N}$, if $A \in \mathcal{M}_k^n$ is such that $\text{diam}^u(A) \leq \delta_2$ and $\text{diam}^s(A) \leq \delta_2$, then $A \setminus S^\pm$ consists of no more that $K_1$ connected components. Such a choice of $\delta_2$ is possible by property (P1) and Convention 2.3.

For $n \geq 1$, define

$$B_{2n}^0 = \{ A \in \mathcal{M}_{-2n}^0 : \forall j, 0 \leq j \leq n/2, T^{-j}A \subset E \in \mathcal{M}_{-n+j}^0 \text{ such that } \text{diam}^u(E) \leq \delta_2 \}.$$

Define $B_{2n}^0 \subset \mathcal{M}_{-2n}^0$ analogously with $\text{diam}^u(E)$ replaced by $\text{diam}^s(E)$. Next, let

$$B_{2n} = \{ A \in \mathcal{M}_{-2n}^0 : \text{ either } A \in B_{2n}^0 \text{ or } T^{-2n}A \in B_{2n}^0 \}.$$

\hfill \footnotesize
\begin{itemize}
  \item Which is a slight modification of the Brin-Katok local entropy theorem [BK], applying [M; Lemma 2]. See also [BD; Corollary 7.17].
\end{itemize}
and $G_{2n} = M_{2n}^0 \setminus B_{2n}$. We think of $B_{2n}$ as the set of ‘bad’ elements and $G_{2n}$ as the set of ‘good’ elements.

Note that for any $n \geq 1$, each $A \in M_{0}^0$ satisfies $\text{diam}^s(A) \leq C \Lambda^{-n}$. We choose $\tilde{n} \in \mathbb{N}$ such that $C \Lambda^{-\tilde{n}} \leq \delta_2$. Our first lemma shows that the cardinality of $B_{2n}$ is small relative to $e^{2nh^*}$ for large $n$.

**Lemma 5.10.** There exists $C > 0$ such that for all $n \geq \tilde{n}$,

$$\# B_{2n} \leq C e^{3 nh^*/2} K_{1}^{n/2} \leq C \rho^{n/2} e^{2 nh^*}.$$  

*Proof.* For $n \geq \tilde{n}$, suppose $A \in B_{2n}^0 \subset M_{0}^0$. For simplicity assume $n$ is even; otherwise, we may use $\lfloor n/2 \rfloor$ in place of $n/2$. For $0 \leq j \leq n/2$, let $A_j$ denote the element of $M_{-3n/2-j}^0$ containing $T^{-(n/2-j)} A \in M_{-3n/2-j}^0$.

Since $A \in B_{2n}^0$ and by choice of $\tilde{n}$, it follows that $\max\{\text{diam}^s(A_j), \text{diam}^u(A_j)\} \leq \delta_2$ for each $0 \leq j \leq n/2$. By choice of $\delta_2$, the number of connected components of $M_{-3n/2-j}^0$ in each $A_j$ is at most $K_1$. Fixing $A_0 \in M_{0}^{0} \cap M_{-3n/2}^0$ and applying this estimate inductively in $j$, we conclude that

$$\# \{A' \in B_{2n}^0 : T^{-n/2} A' \subset A_0\} \leq K_{1}^{n/2}.$$  

Summing over the possible $A_0 \in M_{0}^{0}$ yields,

$$\# B_{2n}^0 \leq \# M_{-3n/2}^0 K_{1}^{n/2} \leq C e^{3 nh^*/2} \rho^{n/2} K_{1}^{n/2} \leq C \rho^{n/2} e^{2 nh^*},$$

where we have used Proposition 2.12 and Convention 2.3 for the second inequality, and Lemma 3.6(d) for the third.

Next, if $A \in M_{0}^0$, then $\text{diam}^u(A) \leq C \Lambda^{-n}$ as well, so the same choice of $\tilde{n}$ permits the analogous estimate to hold for $\# B_{2n}^0$ for $n \geq \tilde{n}$. Finally, since there is a one-to-one correspondence between elements of $M_{0}^0$ and $M_{0}^{0}$, we have $\# B_{2n} \leq \# B_{2n}^0 + \# B_{2n}^0$, completing the proof of the lemma. □

Our next lemma shows that long elements of $M_{0}^{-j}$ enjoy good lower bounds on their $\mu_s$-measure. These lower bounds will eventually be linked to elements of $G_{2n}$.

**Lemma 5.11.** There exists a constant $C_{\delta_2} > 0$ such that for all $j \geq 1$ and $A \in M_{0}^{-j}$ such that $\min\{\text{diam}^u(A), \text{diam}^s(T^{-j} A)\} \geq \delta_2$, it follows that,

$$\mu_s(A) \geq C_{\delta_2} e^{-j h^*}.$$  

*Proof.* As in the proof of Lemma 3.11, we choose a finite set $\mathcal{R}_{\delta_2} = \{R_1, \ldots, R_T\}$ of locally maximal Cantor rectangles with $\mu_s(R_i) > 0$, such that every stable curve of length $\delta_2$ properly crosses at least one $R_i$ in the stable direction, and every unstable curve of length $\delta_2$ properly crosses at least one $R_i$ in the unstable direction.

Now let $j \geq 1$ and $A \in M_{0}^{-j}$ be as in the statement of the lemma. By choice of $\mathcal{R}_{\delta_2}$, an unstable curve in $A$ properly crosses at least one $R_i \in \mathcal{R}_{\delta_2}$. Since $\partial A \subset S^{-}$, $\partial A$ cannot intersect any unstable manifolds in $R_i$ since unstable manifolds cannot be cut under $T^{-n}$. Thus $A$ must fully cross $R_i$ in the unstable direction. Similarly, $T^{-j} A \in M_{0}^j$ must fully cross at least one rectangle $R_k \in \mathcal{R}_{\delta_2}$ in the stable direction.

Let $\Xi_i$ denote the index set of the family of stable manifolds comprising $R_i$. If $\xi \in \Xi$, set $W_{\xi, A} = W_{\xi} \cap A$. Since $T^{-j}$ is smooth on $A$ and $T^{-j} A$ fully crosses $R_k$ in the stable direction, it must be that $T^{-j}(W_{\xi, A})$ is a single curve that properly crosses $R_k$, and so contains a stable manifold in the family corresponding to $R_k$.

Let $s > 0$ denote the length of the shortest stable manifold in the rectangles belonging to $\mathcal{R}_{\delta_2}$. Applying (4.17), we estimate for $\xi \in \Xi_i$,

$$\int_{W_{\xi, A}} v_0 = e^{-j h^*} \int_{W_{\xi, A}} L_j v_0 = e^{-j h^*} \int_{T^{-j}(W_{\xi, A})} v_0 \geq e^{-j h^*} C^s s^h C_2.$$  

Next, we let $D(R_i)$ denote the smallest solid rectangle containing $R_i$, and disintegrate $\mu_s$ on $\{W_{\xi, A} \}_{\xi \in \Xi_i}$ into conditional measures $\mu^\xi_s$ and a factor measure $\hat{\mu}_s$ on $\Xi_i$. Then using the equivalence
of the conditional measure $\mu^\xi_s$ with $\nu_0$ on $\mu_s$-a.e. $\xi \in \Xi_i$ from (5.8), we have

$$\mu_s(A) \geq \mu_s(A \cap D(R_i)) \geq \int_{\Xi_i} \mu^\xi_s(A) d\hat{\mu}_s(\xi) \geq \int_{\Xi_i} C^{-1}_\xi \nu_0(W_{\xi,A}) d\hat{\mu}_s(\xi) \geq C'h^\xi e^{-\int_{\Xi_i} C^{-1}_\xi d\hat{\mu}_s(\xi)},$$

which completes the proof of the lemma due to the finiteness of $\mathcal{R}_\delta_2$. □

Our main proposition of the section is the following.

**Proposition 5.12.** The measure $\mu_s$ is the unique measure of maximal entropy.

**Proof.** Since $\mu_s$ is ergodic, it suffices to prove that if $\mu$ is an invariant probability measure that is singular with respect to $\mu_s$, then $h_\mu(T) < h_{\mu_s}(T)$.

Recall from (2.5) that with respect to the metric $d$ defined in (2.2), $T$ and $T^{-1}$ are expansive: there exists $\varepsilon_0 > 0$ such that if $d(T^j x, T^j y) < \varepsilon_0$ for all $j \in \mathbb{Z}$, then $x = y$.

For $n \geq 1$, define $Q_n$ to be the partition of maximal, connected components of $M$ with boundary points doubled according to Convention 2.1 on which $T^{-n}$ is continuous. By the discussion of Section 2.2, $Q_n$ consists of elements with non-empty interior which correspond to elements of $\mathcal{M}_0^{\Xi_n}$, plus isolated points. Since the entropy of an atomic measure is 0, we may assume that $\mu$ gives 0 mass to the isolated points, and it follows from Lemma 5.4 that $\mu_s$ does as well. Thus the only elements of $Q_n$ with positive measure correspond to elements of $\mathcal{M}_0^{\Xi_n} = B_n \cup G_n$. Accordingly, we throw out the atoms in $Q_n$ and continue to call this collection of sets by the same name.

Since $\mu$ is singular with respect to $\mu_s$, there exists a Borel set $F \subset M$ with $T^{-1}F = F$, $\mu_s(F) = 0$, and $\mu(F) = 1$. Our first step is to approximate $F$ by elements of $Q_n$.

**Sublemma 5.13.** For each $n \geq \bar{n}$, there exists a finite union $C_n$ of elements of $Q_n$ such that

$$\lim_{n \to \infty} (\mu + \mu_s)((T^{-n/2}C_n) \triangle F) = 0.$$

This is [BD, Sublemma 7.24], and its proof relies on the fact that the diameters of elements of $T^{-n/2}(Q_n)$ tend to 0 as $n$ increases due to the uniform hyperbolicity of $T$. The invariance of $F$ implies in addition that

$$\lim_{n \to \infty} (\mu + \mu_s)(C_n \triangle F) = \lim_{n \to \infty} (\mu + \mu_s)((T^{n/2}C_n) \triangle F) = 0.$$

By the proof of [BD, Sublemma 7.24], for each $n$, there exists a compact set $K(n)$ that defines the approximating collection $C_n = T^{-n/2}C_n \subset \mathcal{M}_{-n/2}^{0}$, and satisfying $K(n) \not\cup F$ as $n \to \infty$. To exploit this approximation, we group elements $Q \subset Q_{2n}$ according to whether $T^{-n}Q \subset \cup C_n$ or $T^{-n}Q \cap (\cup C_n) = \emptyset$, where $\cup C_n$ denotes the union of elements of $C_n$ in $M$. Since we have eliminated isolated points, if $T^{-n}Q \cap (\cup C_n) \neq \emptyset$, then $T^{-n}Q \in \mathcal{M}_{-n/2}$ is contained in an element of $\mathcal{M}_{-n/2}$ that intersects $K(n)$. Thus $Q \subset \cup T^n C_n = \cup T^n C_n$.

As noted above, the diameters of $T^{-n}Q_{2n}$ tend to 0 as $n \to \infty$, so by the expansive property of $T$, since the image under $T^{2n}$ of each element of $Q_{2n}$ is simply connected, $Q_{2n}$ is a generating partition for $T^{2n}$ for $n$ large enough. Thus,

$$h_\mu(T^{2n}) = h_\mu(T^{2n}, Q_{2n}) \leq H_\mu(Q_{2n}) = - \sum_{Q \in Q_{2n}} \mu(Q) \log \mu(Q).$$
And so,
\[ 2nh_{\mu}(T) = h_{\mu}(T^{2n}) \leq - \sum_{Q \in \mathcal{Q}_{2n}} \mu(Q) \log \mu(Q) \]
\[ \leq - \sum_{Q \subset \bigcup T^n \mathcal{C}_n} \mu(Q) \log \mu(Q) - \sum_{Q \cap (\bigcup T^n \mathcal{C}_n) \neq \emptyset} \mu(Q) \log \mu(Q) \]
\[ \leq \frac{2}{e} + \mu(\bigcup T^n \mathcal{C}_n) \log \#(Q_{2n} \cap T^n \mathcal{C}_n) + \mu(M \setminus (\bigcup T^n \mathcal{C}_n)) \log \#(Q_{2n} \setminus (T^n \mathcal{C}_n)), \]
where in the last line we have used that for \( p_j > 0 \), \( \sum_{j=1}^{N} p_j \leq 1 \), it holds that
\[ - \sum_{j=1}^{N} p_j \log p_j \leq \frac{1}{e} + (\log N) \sum_{j=1}^{N} p_j; \]
see for example [KH, eq. (20.3.5)]. We have applied this fact with \( p_j = \mu(Q) \) to both sums separately.

Next, since \(-h_{\mu_*}(T) = (\mu(\bigcup T^n \mathcal{C}_n) + \mu(M \setminus (\bigcup T^n \mathcal{C}_n))) \log e^{-h_*}\), we estimate for \( n \geq \bar{n} \),
\[ 2n(h_{\mu}(T) - h_{\mu_*}(T)) - \frac{2}{e} \leq \mu(\bigcup \mathcal{C}_n) \log \sum_{Q \subset \bigcup T^n \mathcal{C}_n} e^{-2nh_{\mu}} + \mu(M \setminus (\bigcup T^n \mathcal{C}_n)) \log \sum_{Q \in Q_{2n} \setminus (T^n \mathcal{C}_n)} e^{-2nh_*} \]
\[ \leq \mu(\bigcup \mathcal{C}_n) \log \left( \sum_{Q \in G_{2n} \cap T^n \mathcal{C}_n} e^{-2nh_{\mu}} + \sum_{Q \in B_{2n} \cap T^n \mathcal{C}_n} e^{-2nh_{\mu}} \right) \]
\[ + \mu(M \setminus (\bigcup \mathcal{C}_n)) \log \left( \sum_{Q \in G_{2n} \setminus (T^n \mathcal{C}_n)} e^{-2nh_*} + \sum_{Q \in B_{2n} \setminus (T^n \mathcal{C}_n)} e^{-2nh_*} \right), \]
where for the last inequality, we have used the invariance of \( \mu \). By Lemma 5.10, the sums over the two subsets of \( B_{2n} \) are bounded by \( C \rho^{n/2} \). We focus on estimating the sums over the two subsets of \( G_{2n} \).

The following is proved in [BD, Section 7.7]: For each \( Q \in G_{2n} \subset \mathcal{M}^0_{-2n} \), there exists \( j, k \in \mathbb{N} \), \( 0 \leq j, k \leq n/2 \) and \( \bar{E} \in \mathcal{M}^0_{-2n+j+k} \) such that \( T^{-j}Q \subset \bar{E} \) and \( \min\{\text{diam}^s(\bar{E}), \text{diam}^s(T^{-2n+j+k})\} \geq \delta_2 \). We call such a triple \((\bar{E}, j, k)\) an admissible triple for \( Q \in G_{2n} \), and note that by Lemma 5.11,
\[ \mu_*\left( \bar{E} \right) \geq C \delta_2 e^{(-2n+j+k)h_*}. \]

There may be many admissible triples for a fixed \( Q \in G_{2n} \). Define the unique maximal triple for \( Q \) by taking first the maximum \( j \), then the maximum \( k \) over all admissible triples for \( Q \).

Denote by \( \mathcal{E}_{2n} \) the set of maximal triples corresponding to elements of \( G_{2n} \), and for \((\bar{E}, j, k) \in \mathcal{E}_{2n}\), set
\[ \mathcal{A}_{M}(\bar{E}, j, k) = \{ Q \in G_{2n} : (\bar{E}, j, k) \text{ is the maximal triple for } Q \} \].

Since \( \bar{E} \in \mathcal{M}^0_{-2n+j+k} \) and \( G_{2n} \subset \mathcal{M}^0_{-2n} \), it follows from Proposition 2.12 that \#\( \mathcal{A}_{M}(\bar{E}, j, k) \leq C e^{(j+k)h_*} \) for some \( C \) independent of \( (\bar{E}, j, k) \) and \( n \).

The following sublemma is [BD, Sublemma 7.25], which implies that if we organize our counting according to maximal triples, we avoid unwanted redundancies.

Sublemma 5.14. If \((\bar{E}_1, j_1, k_1)\) and \((\bar{E}_2, j_2, k_2)\) are distinct elements of \( \mathcal{E}_{2n} \) with \( j_2 \geq j_1 \), then \( T^{-(j_2-j_1)} \bar{E}_1 \cap \bar{E}_2 = \emptyset \).

If \( Q \in T^n \mathcal{C}_n \cap \mathcal{A}_{M}(\bar{E}, j, k) \), then by definition of maximal triple, \( T^{-n+j} \bar{E} \in \mathcal{M}^{n-j}_{-n+k} \) contains \( T^{-n}Q \). Since \( j, k \leq n/2 \), \( T^{-n+j} \bar{E} \) is contained in an element of \( \mathcal{M}^{n/2}_{-n/2} \) that also contains \( T^{-n}Q \) and
intersects $\mathcal{K}(n)$. Thus $T^{-n+j} \bar{E} \subset \bar{C}_n$ whenever $T^n \bar{C}_n \cap A_M(\bar{E}, j, k) \neq \emptyset$, and so $A_n(\bar{E}, j, k) \subset T^n \bar{C}_n$ whenever $T^n \bar{C}_n \cap A_M(\bar{E}, j, k) \neq \emptyset$.

Using these observations together with (5.12), we estimate

\[
\sum_{Q \in G_{2n} \cap T^n \bar{C}_n} e^{-2nh_*} \leq \sum_{(\bar{E}, j, k) \in E_{2n} \cap E \cap T^n \bar{C}_n} \sum_{Q \in A_M(\bar{E}, j, k)} e^{-2nh_*} \\
\leq \sum_{(\bar{E}, j, k) \in E_{2n} \cap E \cap T^n \bar{C}_n} C e^{-2n+2j} \leq \sum_{(\bar{E}, j, k) \in E_{2n} \cap E \cap T^n \bar{C}_n} C' \mu_* (\bar{E}) \\
\leq \sum_{(\bar{E}, j, k) \in E_{2n} \cap E \cap T^n \bar{C}_n} C' \mu_* (T^{-n+j} \bar{E}) \leq C' \mu_* (\cup \bar{C}_n) = C' \mu_* (\cup C_n),
\]

where we have used the invariance of $\mu_*$ and the constant $C'$ is independent of $n$. In the last line we have used Sublemma 5.14 in order to sum over the elements of $E_{2n}$ without double counting.

Similarly, since $T^{-n+j} \bar{E} \subset M \setminus \bar{C}_n$ whenever $T^n \bar{C}_n \cap A_M(\bar{E}, j, k) = \emptyset$, the sum over $Q \in G_{2n} \setminus T^n \bar{C}_n$ in (5.11) is bounded by $C' \mu_* (M \setminus (\cup C_n))$.

Putting these estimates together with (5.11) allows us to conclude the argument,

\[
2n(h_\mu (T) - h_{\mu_*} (T)) - \frac{2}{e} \leq \mu (\cup C_n) \log (C' \mu_* (\cup C_n) + C \rho^{n/2}) \\
+ \mu (M \setminus (\cup C_n)) \log (C' \mu_* (M \setminus (\cup C_n)) + C \rho^{n/2}).
\]

Then since $\mu (\cup C_n) \rightarrow 1$ and $\mu_* (\cup C_n) \rightarrow 0$ as $n \rightarrow \infty$, the quantity on the right side of the inequality tends to $-\infty$. This forces $h_\mu (T) < h_{\mu_*} (T)$ to permit the left side to tend to $-\infty$ as well. \qed

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