Operator Product Expansion for Exclusive Decays: $B^+ \rightarrow D^{*+}_{s,d}e^+e^-$ and $B^+ \rightarrow D^{*+}_{s,d}e^+e^-$

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The decays $B^+ \rightarrow D^{*+}_{s,d}e^+e^-$ and $B^+ \rightarrow D^{*+}_{s,d}e^+e^-$ proceed through a weak and electromagnetic interaction. This is a typical “long distance” process, usually difficult to compute systematically. We propose that over a large fraction of phase space a combination of an operator product and heavy quark expansions effectively turns this process into one in which the weak and electromagnetic interactions occur through a local operator. Moreover, we use heavy quark spin symmetry to relate all the local operators that appear in leading order of the operator expansion to two basic ones. We use this operator expansion to estimate the decay rates for $B^+ \rightarrow D^{*+}_{s,d}e^+e^-$. The decays $B \rightarrow D^-\gamma$ [1] and $B \rightarrow D^0\bar{\nu}$ [2] have been measured directly, and the results are confirmed by hadronic analysis of $B \rightarrow D^*\pi$ [3]. How-ever, it was already realized in [2,3] that there were technical problems with the calculations. We demonstrate that over a large fraction of phase space a combination of an operator product and heavy quark expansions renders $B^+ \rightarrow D^{*+}_{s,d}e^+e^-$ computable. Our methods are fairly general and can be applied to a variety of other processes.

The Operator Product Expansion (OPE) is commonly used in the calculation of inclusive decay rates. One uses the optical theorem and the performs an OPE on forward decay amplitudes. There is a simple physical motivation for the use of the OPE in the decay amplitude of, for example, $B^+ \rightarrow D^{(*)+}e^+e^-$. Consider the hadronic part of the amplitude to lowest order in electromagnetic and weak interactions,

$$\langle D^{(*)+} | \int d^4x \ e^{iq\cdot x} \ T(j_{\mu}^{\text{em}}(x)\mathcal{O}(0))|B^+ \rangle. \quad (1)$$

Here $\mathcal{O} = \bar{b}\gamma^\mu(1-\gamma_5)u\bar{c}\gamma^\nu(1-\gamma_5)d$ is a four quark operator responsible for the weak transition, $j_{\mu}^{\text{em}}$ is the electromagnetic current and $T$ stands for time ordering of these operators. The energy denominators from intermediate states of energy $E$ are $M_B - E$; since the $B$ mass, $M_B$, is much larger than that of the $D$ meson and its excitations, many intermediate states contribute significantly to the amplitude. Since the available energy is much larger (in units of $\Lambda_{\text{QCD}}$) than the energy spacings between intermediate states, the time ordered product in (1) should be well approximated by an expansion in local operators (OPE).

This OPE is not a short distance expansion since the momentum transfer is not in the Euclidean domain. Hence its validity relies on quark-hadron duality. This is exactly analogous to the the use of an OPE in the computation of heavy hadron lifetimes. For large enough $M_B$ violations to duality can be described by (uncomputable) powered suppressed operators in the OPE [4]. Therefore, we can trust at least the leading term in our computation of the amplitude.

To use an OPE in Eq. (1) we take the heavy quark limit for the $b$ and $c$ quarks. Rather than the matrix element with external physical mesons of Eq. (1), we first consider a Green function with two external quarks and two external anti-quarks. For the momenta of the heavy quarks we write $m_b\nu + k_b$ and $m_c\nu' + k_c$, while for the light quarks we take $k_u$ and $k_d$. The residual momenta are small, $k_i \ll m_{b,c}$, provided $\nu \equiv \nu' \equiv \nu \cdot \nu'$ remains of order unity. Just as in the case of semileptonic inclusive decays [4] the combined heavy-quark and operator product expansions give an expansion in powers of $k_i/m_{b,c}$.

To see how this works explicitly, consider the first Feynman diagram of Fig. 1. To lowest order in $k_i$ one has

$$Q_c\gamma^\mu \frac{m_b\phi}{m_b^2 - m_c^2} + \frac{m_c}{m_b^2 - m_c^2} \gamma^\nu(1-\gamma_5) \otimes \gamma_\nu(1-\gamma_5), \quad (2)$$

where $Q_c = 2/3$ is the charge of the $c$ quark. This corresponds to a local operator which depends on the heavy masses and the velocity $v$, which is not a kinematic vari-
able but a parameter in the heavy quark effective theory. Corrections appear as higher dimension operators suppressed by powers of the large mass. For example, the leading correction is of the form

$$Q_c\gamma^\mu - \frac{2m_b v m_c}{m_b^2 - m_c^2} (m_b \gamma^\nu + m_c) \gamma^\nu(1 - \gamma_5) \otimes \gamma_\nu(1 - \gamma_5).$$

Of course, in the operator language, $k_c$ becomes $-i\partial$ acting on the $c$-quark field.

Consider next the second Feynman diagram of Fig. 1. Neglecting the light quark masses we have

$$- Q_d\gamma^\nu(1 - \gamma_5) \frac{m_b \gamma^\mu - m_c \gamma^\nu}{(m_b v - m_c v')^2} \gamma^\mu \otimes \gamma_\nu(1 - \gamma_5).$$

This term differs in an important way from (2). The large denominator is not uniformly large over the whole kinematic range. In fact the denominator is just the square of the leading part of the momentum out of the electromagnetic current, $q = m_b v - m_c v' + \sum k_i$. For the decay $B^+ \to D^{(*)}+e^+e^-$, the kinematic range is $0 \leq q^2 \leq (M_B - M_D)^2$. The approximation is valid provided $\Lambda_{QCD} \ll m_{c,b}$, e.g., the corrections are order $\Lambda_{QCD}/m_{c,b}$. There are also corrections of order $\Lambda_{QCD} m_{b,c}/q^2$. So our results are limited to the region where $q^2$ scales like $m_{c,b}^2$. The region where $q^2$ does not scale like $m_{c,b}^2$ is parametrically small, so the arguments we present are theoretically sound. We emphasize that our method cannot be applied to $B^+ \to D^+ \gamma$, for which $q^2 = 0$, identically.

The third and fourth diagrams in Fig. 1 are similarly computed. In an obvious notation, these are

$$- Q_b \gamma_\nu(1 - \gamma_5) \otimes \gamma^\mu \frac{m_b \gamma^\mu - m_c \gamma^\nu}{(m_b v - m_c v')^2} \gamma^\nu(1 - \gamma_5)$$

and

$$- Q_d \gamma_\nu(1 - \gamma_5) \otimes \gamma^\nu(1 - \gamma_5) \frac{m_b \gamma^\mu - m_c \gamma^\nu}{(m_b v - m_c v')^2} \gamma^\nu.\ (5)$$

Finally, for the calculation of the rate $B^+ \to D^{(*)}+e^+e^-$ one must compute the matrix elements of these four local operators.

The matrix elements of all of these local operators can be expressed, by use of heavy quark spin symmetry, in terms of only two invariant functions. Spin symmetry is best exploited using the Wigner-Eckart theorem. We represent the meson by

$$H^{(c)} = \left( \frac{1 + \gamma'}{2} \right) [D^\ast_{\nu} \gamma^\nu - D \gamma_5]$$

and the anti-meson by

$$H^{(b)} = [B^\ast_{\nu} \gamma^\nu - B \gamma_5] \left( \frac{1 - \gamma'}{2} \right).$$

We will also need the field conjugate,

$$\bar{H}^{(c)} = \gamma^0 H^{(c)\dagger} \gamma^0 = [D^\ast_{\nu} \gamma^\nu + D \gamma_5] \left( \frac{1 + \gamma'}{2} \right).$$

In terms of these we find

$$\langle H^{(c)} \bar{H}^{(c)\dagger} \Gamma_c d \bar{h}_v \Gamma_b u | H^{(b)} \rangle = \frac{\beta(w)}{4} \text{Tr}(\bar{H}^{(c)} \Gamma_c) \text{Tr}(H^{(b)} \Gamma_b) + \frac{\gamma(w)}{4} \text{Tr}(\bar{H}^{(c)} \Gamma_c H^{(b)} \Gamma_b),$$

where $\Gamma_{b,c}$ are arbitrary $4 \times 4$ matrices, $\bar{h}_v^{(c)}$ is the field that creates a heavy quark with velocity $v'$ and $\bar{h}_v^{(b)}$ annihilates a heavy anti-quark with velocity $v$. Invariance under spin symmetry of the heavy quarks readily implies

$$\langle H^{(c)} \bar{h}_v^{(c)\dagger} \Gamma_c d \bar{h}_v \Gamma_b u | H^{(b)} \rangle \propto \bar{H}^{(c)} \Gamma_c \otimes H^{(b)} \Gamma_b.$$
obtain rather than only the vacuum, between the currents. We assume:

\[ \langle B|\bar{b}\gamma^\mu(1-\gamma_5)d\bar{\nu}\gamma_\mu(1-\gamma_5)d|B\rangle = \frac{2}{3}B_B f_B^2 M_B^2, \]  
\[ \langle B|\bar{b}(1-\gamma_5)d\bar{\nu}(1-\gamma_5)d|B\rangle = -\frac{5}{6}B_B f_B^2 M_B^2. \]

Isospin relates these matrix elements to the matrix element in (3) symmetrized in \( u \leftrightarrow d \). This symmetrized operator can be related to (3) if we make the following assumption:

\[ \langle H'_0^{(c)}|\bar{h}_{c}^{(c)}\Gamma T^A d \bar{h}_{d}^{(c)}\Gamma T^A u|H'_0^{(b)}\rangle = 0. \]  

Here \( T^A \) is a generator of color-SU(3). This assumption is weaker than vacuum insertion because it holds true even if we insert a complete set of physical states, rather than only the vacuum, between the currents. We obtain \( \beta(1) = \frac{1}{6}(B_B + 5B_S)f_B f_D \sqrt{M_B M_D} \) and \( \gamma(1) = -\frac{2}{5}(B_B - B_S)f_B f_D \sqrt{M_B M_D} \) and \( \gamma(1) = 0 \) can be trivially modified to account for these lattice results, \( \beta(1) \approx 0.8f_B f_D \sqrt{M_B M_D} \) and \( \gamma(1) \approx 0 \). We do not use these lattice results below because \( B_B \) and \( B_S \) are computed for different heavy quarks and different renormalizations, and the result for \( B_B \) differs by 20\% from other lattice studies.

The effective Hamiltonian for the weak transition is

\[ H'_\text{eff} = \frac{G_F}{\sqrt{2}} V_{ub} V_{cd}^*(c(\mu/M_W)O + c_8(\mu/M_W)O_8), \]

where \( O_8 = \bar{b}\gamma^\nu(1-\gamma_5)T^u \bar{c}\gamma^\nu(1-\gamma_5)T^A d \). The dependence on the renormalization point \( \mu \) of the short distance coefficients \( c \) and \( c_8 \) cancels the \( \mu \)-dependence of operators, so matrix elements of the effective Hamiltonian are \( \mu \)-independent. Resuming the leading logs, \( c(\mu_0) = 4.4^2 + \frac{5}{6} x^{-1} \) and \( c_8(\mu_0) = x^{-1} - x^2 \), where \( x = (\alpha_s(\mu_0)/\alpha_s(M_W))^{6/23} \).

Defining

\[ h^{(*)}\mu = \langle D^{(\ast)+}|\int d^4x e^{iqx} T(j_{\text{em}}(x)H_{\text{eff}}(0))|B^+\rangle, \]

the decay rate for \( B^+ \to D^{(\ast)+}e^+\bar{\nu}e^- \) is given in terms of \( q^2 \) and \( t \equiv (p_D + p_{e^-})^2 = (p_B - p_{e^-})^2 \) by

\[ \frac{dt}{dq^2 dt} = \frac{1}{2\pi^3 M_B^2} \left| \int \frac{d^3p}{(2\pi)^3} \tilde{\ell}_{\mu} h^{(*)}\mu \right|^2 \]

where \( \tilde{\ell}_{\mu} = \bar{u}(p_{e^-})\gamma^\mu v(p_{e^-}) \) is the leptons’ electromagnetic current. A sum over final state lepton helicities, and polarizations in the \( D^* \) case, is implicit. Using the OPE and spin symmetry we obtain

\[ h^{\mu} = \frac{\kappa}{3} \left[ \frac{(4wm_b - 3m_c)v^\mu - (3m_b - 2wm_c)v^\mu}{m_B v^\mu + m_c v^\mu} \right] \left( \frac{m_B v^\mu + m_c v^\mu}{m_B^2 - m_c^2} \right) \]

and

\[ h^{(*)}\mu = \frac{\kappa}{3} \left[ \frac{(m_B v^\mu - m_c v^\mu)}{m_B^2 - m_c^2} \right] \left[ \frac{3im_c \epsilon_{\alpha\beta\gamma}}{(m_B v - m_c v)^2} \right] \left( \frac{3im_c \epsilon_{\alpha\beta\gamma}}{(m_B v - m_c v)^2} \right) \]

Here \( \kappa = G_F/\sqrt{2} V_{ub} V_{cq}^{\ast} [\beta + c_8 \beta_8] \) and we have defined the matrix element of \( O_8 \) as \( \beta_8 \), in analogy to Eq. (3). These expressions are our central results, demonstrating that the decay rates for \( B^+ \to D^{(+)}e^+\bar{\nu}e^- \) can be expressed in terms of the matrix elements \( \beta \) and \( \beta_8 \). Below we make an educated guess of these matrix elements, but for reliable results they should be determined from first principles, say, by monte carlo simulations of lattice QCD.

There are short distance QCD corrections to the coefficients (3)–(6) in the OPE. In leading-log order they are determined by the renormalization group. The matrix elements of the operators are all related by spin symmetry to the reduced matrix elements \( \beta, \beta_8, \gamma, \gamma_8 \). However, \( \beta \) and \( \beta_8 \) do not mix into \( \gamma \) and \( \gamma_8 \). Moreover, since we use \( \gamma = \gamma_8 = 0 \), we need only the \( 2 \times 2 \) matrix of anomalous dimensions of \( \beta \) and \( \beta_8 \). A straightforward one loop computation gives the anomalous dimension for \( \beta \) and \( \beta_8 \) as follows:

\[ \gamma = \frac{\alpha_s}{4\pi} \left( \frac{8}{9}(wr(w) - 1) \right), \]

where \( r(w) = \frac{1}{\sqrt{w^2 - 1}} \ln(w + \sqrt{w^2 - 1}) \). Note that the diagonal entry for \( \beta \) is precisely twice the anomalous dimension of the heavy-light current, as if the operator was factorized. This is in accordance with the results of calculations of \( \gamma \) of the operator for \( B \to \bar{B} \) mixing [9]. Moreover, at \( w = 1 \) the matrix simplifies, \( \gamma = \frac{2}{27} \).

Denote by \( \hat{c} \) and \( \hat{c}_8 \) the coefficients of \( \beta \) and \( \beta_8 \), respectively. They satisfy a renormalization group equation,

\[ \mu \frac{d}{d\mu} \hat{c} = -\gamma T \hat{c}. \]

Here “\( T \)” denotes transpose of a matrix and \( \hat{c} \) is a column vector, \( \hat{c}^T = (\hat{c}, \hat{c}_8) \). In leading-log order the solutions are

\[ \hat{c}(\mu) = z^\psi \left[ \frac{1}{3}(2z^\xi + z^{-\xi})\hat{c}(\mu_0) + \frac{2}{9} (z^\xi - z^{-\xi}) \hat{c}_8(\mu_0) \right], \]

\[ \hat{c}_8(\mu) = z^\psi \left[ (z^\xi - z^{-\xi}) \hat{c}(\mu_0) + \frac{1}{3} (z^\xi + z^{-\xi}) \hat{c}_8(\mu_0) \right], \]

where \( z = \alpha_s(\mu)/\alpha_s(\mu_0) \),

\[ \psi = \frac{13 - wr(w)}{3b_0}, \quad \xi = \frac{wr(w) - 1}{b_0}. \]

\( b_0 \) is the coefficient of the one loop beta function in QCD, \( b_0 = 11 - \frac{2}{3}n_f \), with \( n_f = 3 \) light flavors in our case.
For our numerical estimates below we match the coefficients $\tilde{c}$ and $c_s$ to $c$ and $c_s$ at the intermediate scale $\mu_0 = \sqrt{m_b m_c}$. Numerically, with $\alpha_s(\mu_0)/\alpha_s(M_W) \approx 2.24$ one has $c \approx 1.0$ and $c_s \approx -0.7$. We will use $\mu = 1$ GeV, with the implicit understanding that the matrix elements $\beta(w)$ and $\beta_s(w)$ are computed at that value of the renormalization point.

It is now a trivial exercise to compute the differential decay rate. We integrate in Eq. (14) over the variable $t$ and obtain, for $B^+ \to D_s^{(*)} e^+ e^-$,

$$\frac{d\Gamma}{dq^2} = \frac{\alpha^2 G_F^2}{288\pi M_B^3} |V_{ub} V_{cs}|^2 (c + c_s) F(q). \quad (22)$$

Here $F(q)$ is a dimensionless function of $q \equiv \sqrt{q^2/m_b^2}$ and $\tilde{m} \equiv M_D/\sqrt{M_B}$. For $B^+ \to D_s^+ e^+ e^-$ it is given by

$$F(q) = \frac{\sqrt{2(1 - (\tilde{q} + \tilde{m})^2)(1 - (\tilde{q} - \tilde{m})^2)}}{3\tilde{q}^2 \tilde{m}(1 - \tilde{m}^2)^2} \left( \begin{array}{c} -36\tilde{n}^8 + 8\tilde{q}^8 + 9\tilde{n}^{10} - 6\tilde{q}^6\tilde{m}^4 + 8\tilde{m}^6\tilde{q}^4 \\
-17\tilde{m}^8\tilde{q}^2 - 30\tilde{q}^4\tilde{m}^4 + 38\tilde{q}^2\tilde{m}^6 - 4\tilde{q}^6\tilde{m}^2 + 4\tilde{q}^6 \\
+4\tilde{q}^2 - 8\tilde{q}^4 - 36\tilde{m}^4 + 9\tilde{n}^2 - 4\tilde{q}^2\tilde{m}^2 \\
+30\tilde{q}^4\tilde{m}^2 - 21\tilde{q}^2\tilde{m}^4 + 54\tilde{n}^6 \end{array} \right). \quad (24)$$

Note that we have not distinguished between heavy quark and meson masses. The distinction enters at order $1/m_Q$ in the heavy mass expansion, and we have not considered such corrections in this work.

To estimate the branching fraction numerically, we use the vacuum insertion approximation, $\beta(w) = e^{-4/9}g_b f_{BD} \sqrt{M_B M_D}$ and $\beta_s(w) = 0$, and integrate the rate from $q^2 = 1$ GeV up to the kinematic limit $q^2 = (M_B - M_D)^2$. The lower limit is an estimate of how low $q^2$ may be before our operator expansion breaks down. We find

$$\text{Br}(B^+ \to D_s^{(*)} e^+ e^-)_{q^2 > 1 \text{ GeV}} = 1.8 \times 10^{-9} \quad (25)$$

$$\text{Br}(B^+ \to D_s^{(*)} e^+ e^-)_{q^2 > 1 \text{ GeV}} = 2.7 \times 10^{-10} \quad (26)$$

$$\text{Br}(B^+ \to D^+ e^+ e^-)_{q^2 > 1 \text{ GeV}} = 9.1 \times 10^{-11} \quad (27)$$

$$\text{Br}(B^+ \to D^+ e^+ e^-)_{q^2 > 1 \text{ GeV}} = 1.4 \times 10^{-11} \quad (28)$$

where we have used $|V_{ub} V_{cs}| = 0.004$, $|V_{ub} V_{cd}| = 8.8 \times 10^{-4}$, $f_B = 170$ MeV and $f_D = f_B \sqrt{M_B M_D}$. It is important to observe that the portion of phase space $q^2 > 1$ GeV is expected to give a small fraction of the total rate since the pole at $q^2 = 0$ dramatically amplifies the rate for small $q^2$.

In summary, we have applied the operator product and heavy quark expansions to the exclusive decay amplitude of heavy mesons. At zero hadronic recoil the matrix element of the leading operator in the OPE is related by heavy quark symmetries and octet suppression to the matrix element for $B\bar{B}$ mixing. Although the rates we compute are too small to be observable at $B$-factories, they may be accessible to experiments in hadronic colliders. We have demonstrated the method by calculating the rate for $B^+ \to D_{s,d}^{(*)} e^+ e^-$ to leading order in the operator expansion and to leading-log order in QCD. Systematic corrections to both of these expansions could and should be computed. Irreducible errors are expected from the application of the operator product expansion in the time-like regime, which is analogous to the assumption of local quark-hadron duality in calculations of heavy hadron lifetimes.

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