A second order differential equation for a point charged particle

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Abstract

A model for the dynamics of a classical point charged particle interacting with higher order jet electromagnetic fields is described by an implicit ordinary second order differential equation. We show that such equation is free of run-away and pre-accelerated solutions of Dirac’s type. The theory is Lorentz invariant and is compatible with Newton’s first law and Larmor’s power radiation law.

1 Introduction

The existence of some fundamental problems in the electrodynamics theory of point charged particles interacting with its own radiation reaction field is one of the oldest open problems in field theory (see for instance [14, 20, 30, 34] and references therein). The investigation of the radiation reaction of point charged particles led to the Abraham-Lorentz-Dirac equation [12] (in short, the ALD equation). However, a main conceptual difficulty is that, although the derivation of the ALD is based on general principles, the equation has un-physical solutions.

Such problems on the standard ALD equation suggest that classical electrodynamics of point charged particles should not be based on the ALD equation together with Maxwell’s theory. Ideas to overcome the difficulties range from reduction of order schemes for the ALD equation [22], re-normalization group schemes [33, 34], extended models of the charge particle [7, 30], higher order derivative field theories [4, 24, 11], the Feynman-Wheeler’s absorber theory of electrodynamics [15], models where the observable mass is variable on time [3, 19, 23], non-linear electrodynamic theories [5], dissipative force models [27, 19] among other proposals (for an account of some of the above approaches, see [14] and [31]).
Some of the above mentioned proposals contain serious deficiencies that prevent them to be considered as consistent solutions of the radiation reaction problem. The imposition of the Dirac’s asymptotic condition seems an ad hoc procedure to eliminate the undesired solutions. Such point of view puts in disadvantage those approaches based on the Lorentz-Dirac equation as fundamental equation, in particular the models of reduction of order. On the other hand, simplicity in the assumptions makes natural to consider point charged particle as a model for the probe particle (this excludes extended models of particles). Otherwise, if we adopt more sophisticated models for the probe charged particle, one will need to explain the stability of the extended structure of the probe charged particle and prove the mathematical consistency of the theory. Therefore, in order to avoid some of such complications we will concentrate on the case that probe particles are described by point charged particles. For a probe point charged particles, the simplest dynamical equation that is Lorentz invariant, compatible with conservation of relativistic four momentum and with the hypothesis that the electromagnetic field are solutions of the Maxwell equations is the ALD equation [12]. Thus, the deep paradox is that the ALD equation is properly derived from first principles but is physically untenable.

One wonders whether the problems of the ALD is inherent to the classical description of point charged particles, since at a more fundamental level, the description of a charged particle should be in the framework of non-relativistic quantum electrodynamics. However, we are not pursuing here a quantum description of the problem. This is because our question is a classical issue, posed at the classical level and therefore it should be solvable (in positive or negative way) at the same level. Also, it is not clear that a quantum formulation could provide a consistent description of the radiation reaction phenomena. Even if there are some arguments favoring a semi-classical quantum treatment of the problem (see for instance [28] and references therein), there is no a complete proof of the resolution in the framework of the quantum theory. Thus, in the absence of a rigorous result showing the impossibility of solving the above problems in a classical framework or showing that they can be completely solved in the quantum mechanical framework, we find justified to formulate the problem and try to solve it in the simpler classical framework.

The fact that the ALD equation is properly derived from fundamental principles of classical field theory suggests that the way to obtain a consistent classical description of radiation reaction of point charged particles requires to relax some of the fundamental assumptions of classical electrodynamics. One of such new ideas was depicted by R. Feynman and Wheeler. In Wheeler-Feynman’s theory, the electromagnetic field in vacuum is identically zero, there is a complete time reversal symmetry not only in the equations for the potential. In their theory, the electromagnetic field appears only as a secondary, mathematically convenient object. That is, in Wheeler-Feynman proposal there are only particles and not free external fields.

In the companion paper [18], a generalization of the classical Maxwell-Lorentz electrodynamics in the form of a theory of abelian generalized higher order jets elec-
tromagnetic fields was developed\(^2\). It was shown that such notion of generalized higher order field is more flexible than the usual notion of classical field. Thus, in our proposal apart from probe particles, there is a smooth generalized electromagnetic field, whose dynamics and value is not independent of the dynamics of the probe particle used to measure it. It is in this sense that our theory resembles the Wheeler-Feynman’s theory, since the probe particle affects directly the field.

In the case of electromagnetic phenomena, such weaker notion of field can be partially justified as follows. The standard notion of classical fields as sections of a vector bundle over manifold \(M\) presupposes the independence of the field from the dynamical state of the probe particles used to measure the fields. However, we have learned from the quantum theory that the action of a measurement on a physical can disturb in a fundamental way the system being measured, such that after the measurement, the system has changed drastically the state. In a similar way, one can imagine that the notion of local classical electromagnetic field (that is, a closed, differential 2-form living on the spacetime manifold \(M\) and independent of the motion of the probe particle) fails to accommodate the effects of the dynamics of the probe charged particle, due to radiation reaction effects, that distort the original electromagnetic field. The relevant point is how this is accomplished. For instance, in the Wheeler-Feynman theory, the fields are still living on the manifold \(M\) and the particle-field interaction is formulated in a time-symmetric way. As a result, the radiation damping term is the same than in the standard Abraham-Lorentz theory. Such damping term it is the problematic one. Thus, Wheeler-Feynman’s appears more as a justification for the theory of Abraham-Lorentz-Dirac than a solution to the problem. One needs to modify the way the probe particle changes the field.

A notion of classical field that can be adapted to the dynamics of the probe charged point particle is of fundamental importance for a better understanding of feedback phenomena and in particular the radiation reaction in classical electrodynamics. We find that in this case, a field that depends on the details of the trajectory of the probe particle is the adequate notion. This lead us to a convenient notion of physical fields as a generalized differential form with values in higher order jet bundles along curves \(^{18}\). Such fields are non-local in the sense that they do not live on the spacetime manifold \(M\) (the generalized fields does not depend only on where they are measured, but also on how they are measured). As a consequence, the description of electrodynamics using generalized electromagnetic fields will not be local, since the fields depend not only on the position with time \(x(t)\) but also on the derivatives of the world-line particle probe.

In this work we adopt the mathematical framework introduced in \(^{18}\) to obtain in detail an implicit second order differential equation for point charged particles that is free of the run away and pre-accelerated problems of the ALD equation. We found that the new implicit differential equation is compatible with Larmor’s

\(^{2}\)The mathematical foundations for generalized higher order tensors and differential forms, the fundamental elements of the geometry of maximal acceleration, as well as the fundamental mathematical aspects of generalized fields and generalized higher order electrodynamics are developed in more detail in the companion paper \(^{18}\).
power radiation law, a fundamental constraint in searching for models of radiation reaction. The derivation of the new equation of motion for probe point charged particles is based on two new assumptions. First, the notion of \textit{generalized higher order fields} is applied to the case of electromagnetic fields. Second, the hypothesis that proper acceleration (measured with the Lorentzian metric \( \eta \)) of a point charged particle is bounded, an idea that convey us to the use of a \textit{maximal acceleration geometry} \cite{18}. This is a natural requirement to avoid run-away solutions. It can also be seen that such metrics appears in a natural way if one requires to preserve the first and second laws of dynamics (see \cite{18}, section 3). With the notion of generalized higher order fields and maximal acceleration on hand and demanding that the equation of motion of a probe point charged particle is of second order, it is possible to eliminate the \textit{Schott term} in the ALD equation consistently. The strategy of eliminating the Schott term in the equation of motion of a point particle was first brought to light by J. Larmor and later by W. Bonnor. In their theory, the observable rest mass of the point charged particle could vary on time, being this the source for the radiation. On the other hand, the bare mass was constant. In contrast, in our proposal the bare mass can vary during the evolution but the observable rest mass is constant. However, the variable bare mass should be thought as a consistent requirement, and not as a main principle (as it was in the variable mass hypothesis in Bonnor’s theory). Thus, in our theory the higher order electromagnetic fields contain additional degrees of freedom corresponding to the higher order corrections, that originate the change in the bare mass (see \cite{18}, section 7) and allow also for the radiation of energy-momentum. In addition, we keep the minimal higher order corrections necessary to obtain an implicit second order differential equation.

The two new assumptions (generalized fields and maximal acceleration geometry) are necessary to solve the problem of the motion of a point charge in a consistent way. We will show that the notion of generalized higher order electromagnetic field is not sufficient to provide an appropriate dynamical description of a point charged particle. Similarly, the hypothesis of maximal acceleration geometry is also not enough (see Section 4). However, in combination with the hypothesis of \textit{maximal proper acceleration} one can obtain a model compatible with the power radiation law of a point charged particle, obtaining equation (5.10). Thus, maximal acceleration is useful for controlling the grow of the acceleration and the order of the differential equation.

The new differential equation (5.10) is compatible with the first Newton’s law, it does not have un-physical solutions as the ALD equation and it is compatible with energy-momentum conservation. We show that the equation (5.10) can be approximated by an ordinary differential equation where the second time derivative is explicitly isolated. We argue that in the theoretical domain of applicability of the theory, the approximate equation is equivalent to the original new equation of motion. This approximated differential equation opens the possibility to use standard ODE theory to prove existence and uniqueness, bounds and regular properties of the solutions.
The derivation of equation (5.10) breaks down in two dynamical regimes. The first corresponds to curves of maximal acceleration. Thus the domain of applicability of the theory corresponds to world-lines far from the maximal acceleration regime. The second corresponds to covariant uniform acceleration. In the case of covariant uniform acceleration, a similar analysis as in the non covariant uniform acceleration case leads us to a differential equation which is a Lorentz force equation with a constant total electromagnetic field \[16\]. However, covariant uniform motion of point charged particles is unstable: in order to compensate the radiation reaction field and provide the total constant force of the uniform motion, an infinite precision in the calibration of the external fields is required, which seems to be an unstable regime.

The structure of this paper is the following. In section 2, the notion of generalized higher order electromagnetic field is motivated and briefly discussed. We have used local coordinates, since it is enough for our objective of finding an ODE for point charged particles. The necessary notation on jet bundle theory is put in a form of Appendix A in a way that the reader un-familiar with that theory can follow the local description. In section 2 we also provided the fundamental equations of the generalized Maxwell’s theory. In section 3, the basic notions of the geometry of metrics of maximal acceleration is presented. An heuristic argument in faubour of a principle of maximal acceleration is presented. The perturbation scheme of our theory is also introduced. Note that we will not develop in full the perturbation scheme, since we do not provide any definite value for the maximal proper acceleration. However, to consider the proper acceleration very large compared with the value of the maximal acceleration it will be enough to provide an effective model. In addition, to show the consistency of the notion of maximal acceleration geometry, we provide Appendix B, where it is proved that the proper-time of a metric of maximal acceleration is invariant under reparameterizations. Section 4 is devoted to explain a geometric method to obtain the ALD equation. This is based on the use of adapted frames and the imposition of consistency with Larmor’s radiation law of a point particle. In section 5, the method is used in the framework of generalized fields and maximal acceleration, obtaining the equation (5.10) (or the covariant version (5.11)). A short discussion of the theory is presented in section 6.

**Notation.** \(M\) will be a four dimensional spacetime manifold. \(TM\) is the tangent bundle of \(M\) and \(\eta\) is the Minkowski metric with signature \((-1, 1, 1, 1)\). The null cone bundle of \(\eta\) is \(\hat{\pi} : NC \to M\). Local coordinates on \(M\) are denoted by \((x^0, x^1, x^2, x^3)\) or simply by \(x^\mu\). We will find useful to identify points by its coordinate, since we are working on single homeomorphic open domains to \(R^4\). Natural coordinates on \(TM\) and \(NC\) are denoted as \((x, y)\). Further notation will be introduced in later sections, Appendix A and Appendix B.

**2 Notion of generalized higher order fields**

The notion of generalized field adopted in this paper is conveyed by the idea of linking the mathematical model of probe particle with an economical in postulates
description of fields capable to support a consistent dynamics with probe particles.

For classical fields, the simplest model of probe particle is the point charged particle,
described by a one dimensional world-line curve \( x: I \rightarrow M \). By considering the
departure of the world-line from being a geodesic, one can obtain information on
the value of the electromagnetic field. However, this inverse process is not enough
to completely determine the electromagnetic field and in general, it is necessary to
consider more than one probe particle to completely fix the field variables. Therefore,
a classical electromagnetic field could be thought as a functional (with co-domain
in a convenient space) defined over a quotient space of the path space of world-lines
of physical probe particles. Examples of such quotient spaces are the spacetime
manifold \( M \) itself and the corresponding jet bundles over the spacetime manifold.

From the mathematical side, the main problem for obtaining a consistent theory
is to determine functionals and quotient spaces compatible with experiment and
that admit a consistent mathematical formulation; from the physical side, the main
concern is that the logical system should be predictive and falsifiable.

Such functionals could be difficult to analyze. In this context, jet bundle theory
is quite adequate, since it deals systematically with Taylor expansions of functions
and sections with arbitrary order. Moreover, jet bundle formalism is useful when
dealing with geometric objects that are defined along maps, independently of their
geometric character. For instance, jet theory is extensively used in problems of
calculus of variations (see Appendix A and references therein), in the formulation of
generalized field theories like Bopp-Podolsky theory [11] and more generally, in the
formulation of problems involving partial differential equations.

Jet fields have a definite transformation rule under local coordinate transformations
and in some circumstances, a section of a vector bundle can be approximated locally
by jet sections. This is as a consequence of Peetre’s type theorems (see [21], pg.
176). Thus, for non-increasing operators acting on smooth sections, it is equivalent
to work with operators acting on convenient higher order jet fields.

An additional motivation for considering jet fields comes from the fact that, at
the quantum level description, the relevant dynamical variables in gauge theories
are holonomy variables or loop variables (see for instance the discussion in Chapter
1 of [9]). Thus, one could expect some degree of non-locality in an intermediate
description of physical systems between quantum and classical systems. Jet fields are
a natural mathematical tool accomplishing such properties, since they are classical
they live on the world-line curve \( x: I \rightarrow M \), which corresponds to a location of the
particle in spacetime) and at the same time they depend not only on the spacetime
point manifold \( x(I) \leftrightarrow M \) but also on the higher derivatives associated with the
time-like curve \( x: I \rightarrow M \). Note that although the comparison between the non-
locality of the holonomy variables in one hand and the corresponding for jet fields is
merely formal, it suggests a description of the holonomy variables in terms of infinite
higher order jets fields.
2.1 Notions of generalized electromagnetic field and current

Although we will use mainly the coordinate formulation of the generalized electromagnetic field, there is a coordinate-free formulation for such objects these objects. In particular, the Hodge star operator $\star$ and the nilpotent exterior derivative $d_4$ are well defined geometric objects acting on generalized forms. For the notation and basic definitions of jet bundle theory we refer to Appendix A.

Here we will adopt the following generalization of the Faraday and excitation tensor and electromagnetic density current,

**Definition 2.1** The electromagnetic field $\bar{F}$ along the lift $k^x : I \to J^k_0(M)$ is a 2-form that in local natural coordinates can be written as

$$\bar{F}(k^x) = \bar{F}(x, \dot{x}, \ddot{x}, ..., x^{(k)}) = \left(F_{\mu\nu}(x) + \Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, ..., x^{(k)})\right) d_4 x^\mu \wedge d_4 x^\nu,$$  \hspace{1cm} (2.1)

with $F(x) \in \Gamma \Lambda^2 M$. The excitation tensor along the lift $k^x : I \to J^k_0(M)$ is the 2-form

$$\bar{G}(k^x) = \bar{G}(x, \dot{x}, \ddot{x}, ..., x^{(k)}) = \left(G_{\mu\nu}(x) + \Xi_{\mu\nu}(x, \dot{x}, \ddot{x}, ..., x^{(k)})\right) d_4 x^\mu \wedge d_4 x^\nu.$$

The density current in electrodynamics is represented by a 3-form

$$\bar{J}(x, \dot{x}, \ddot{x}, ..., x^{(k)}) = \left(J_{\mu\nu\rho}(x) + \Phi_{\mu\nu\rho}(x, \dot{x}, \ddot{x}, ..., x^{(k)})\right) d_4 x^\mu \wedge d_4 x^\nu \wedge d_4 x^\rho.$$  \hspace{1cm} (2.2)

These generalized fields are localization of sections of $\Lambda^p(M, F(J^k_0(M)))$ (see [18] or the Appendix A).

A generalized field associates to each pair of tangent vectors $e_\mu$, $e_\nu$ and curve $x : I \to M$ an element of the $k$-jet along $x$, $(e_\mu(x(\tau)), e_\nu(x(\tau))) \mapsto \bar{F}(k^x(x(\tau)))$. That $F$ is defined on the Grassmannian $G(2, M)$ allows to define fluxes and to associate with them the outcome of macroscopic measurements. The fact that the fields take values on higher jet bundles over the spacetime manifold is the mathematical implementation of our idea that fields depend on the state of motion of the probe particle in a fundamental way. At his stage we did not fix the value of the order $k$. We will keep it free until later sections, where it will be fixed to be $k = 3$ by physical constraints.

With the aid of the nilpotent exterior derivative $d_4$ and the Hodge star operator $\star$ defined by $\eta$, one can write in complete analogy with Maxwell’s equations the following homogeneous and non-homogeneous equations [18]:

- The generalized homogeneous equations are
  \[ d_4 \bar{F} = 0. \]  \hspace{1cm} (2.4)

- Let us assume the simplest constitutive relation $\bar{G} = \star \bar{F}$. Then the generalized inhomogeneous Maxwell’s equations are
  \[ d_4 \star \bar{F} = J + d_4 \star \Upsilon. \]  \hspace{1cm} (2.5)
Using some constrains on $\Upsilon$ coming from compatibility with the equation of motion of point charged particles, one obtains an effective theory which is equivalent to the standard Maxwell’s theory. Thus the homogeneous equations are

$$dF = 0.$$  \hfill (2.6)

Similarly, the non-homogeneous equations are

$$d \star F = J,$$  \hfill (2.7)

and the conservation of the current density are

$$dJ = 0.$$  \hfill (2.8)

Since the fields are originally in higher order spaces, there are additional degrees of freedom to define a consistent particle dynamics. This also shows that the effective theory of higher order fields, in the sector of Maxwell’s fields, is reduced to the standard Maxwell’s theory. This will be different for the particle law of motion, as we will see in this work.

### 3 Maximal acceleration geometry

#### 3.1 Why maximal acceleration?

The notion of maximal acceleration is not new. It was developed first in the work of E. Caianiello and co-workers, that provided heuristic motivation for such notion in their framework of quantum geometry (see [6] and references there in). We give here an heuristic argument for the existence of a maximal acceleration based on the assumption that there is a minimal length $L_{\text{min}}$ and a maximal speed. The minimal length is the assumed scale of the spacetime region that can produce an effect on the system in the shortest period of time. This idea is not necessarily related with a quantification of spacetime, but requires a notion of extended local domain where cause-effect relations are originated. Therefore, the maximal acceleration could be relational, depending on the physical system. This is in contrast with universal maximal acceleration. However, we will require that whatever the maximal acceleration is, they will be very large compare with the acceleration of the probe particle. In this way, our perturbative scheme will be perfectly applicable.

By adopting the above hypotheses, the action on a particle done by its surrounding (assuming a certain form of locality) and one obtains for a maximal work as a result of such action to be

$$L_{\text{min}} m a \sim \delta m v_{\text{max}}^2,$$

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\[^3\text{A detailed derivation of the equations (2.4) to (2.8) from the generalized Maxwell’s equations is contained in ref. [18], Section 7.}\]
where $a$ is the value of the acceleration in the direction of the total exterior effort is done. Then one associates this value to the work over any fundamental degree of freedom evolving in $M$, caused by rest of the system. Since the speed must be bounded, $v_{\text{max}} \leq c$. Also, the maximal work produced by the system on a point particle is $\delta m = -m$. Thus, there is a bound for the value of the acceleration,

$$a_{\text{max}} \simeq \frac{c^2}{L_{\text{min}}}.$$  \hspace{1cm} (3.1)

Maximal acceleration in electrodynamics has been studied before, in particular in the theory of extended charged particles. The first instance on the limitation of acceleration is found in the Lorentz’s theory of the electron, where it appears as a causal condition for the evolution of the charge elements defining the electron (for a modern discussion of the theory see [34]). Another instance where maximal acceleration appears is in Caldirola’s extended model of charge particles [8], where the existence of a maximal speed and minimal elapsed time (called chronon) implies the maximal acceleration. In contrast, our probe particle is a point particle and the maximal acceleration is a postulate to allow us to define a perturbative model, with phenomenological consequences.

Universal maximal acceleration has appeared recently as a direct consequence of covariant loop gravity, in a way compatible with local Lorentzian geometry [32]. Before, it was argued that some models of string theory contains maximal acceleration too [29]. Thus, the notion of proper bound for maximal acceleration must naturally contained in the physical geometry of the spacetime.

### 3.2 Elements of covariant maximal acceleration geometry

In the same way that the electromagnetic field is described by a generalized higher order field, it is natural to describe the spacetime structures by generalized higher order metrics. The motivation for this is two-fold. First, when coupling gravity with generalized higher order fields, it will be natural to consider the gravitational field described by a generalized higher order field. Second, when considering maximal acceleration kinematics it turns out that the corresponding path structure is a generalized metric. In this paper we will consider the second point, leaving the relation between generalized fields and gravity for future work.

Let $D_\xi$ be the covariant derivative in the direction $\dot{x}$ associated with the Levi-Civita connection of the Minkowski metric $\eta$. It induces a Levi-Civita connection $\gamma^\mu_{\nu\rho}$ on $M$ and a non-linear connection on the bundle $\pi : TM \setminus NC \rightarrow M$ whose coefficients are given by

$$N^\mu_{\nu}(x,y) = \gamma^\mu_{\nu\rho} y^\rho, \quad \mu, \nu = 0, 1, 2, 3$$  \hspace{1cm} (3.2)

with $\gamma^\mu_{\nu\rho}$ the Christoffel symbols. The induced vertical forms are

$$\{\delta y^\mu = dy^\mu + N^\mu_{\nu} dx^\nu, \mu, \nu = 0, 1, 2, 3\}.$$
The set of local sections \( \{ \frac{\delta}{\delta x^i}|_u \}, \frac{\delta}{\delta x^i}|_u, u \in \pi^{-1}(x), x \in U \) generates the local horizontal distribution \( \mathcal{H}_U \) over \( TU \), with \( U \) an open set of \( M \).

The covariant acceleration vector field of the curve \( x : I \rightarrow M \) is the vector \( D_x \dot{x} \in T_x M \) that satisfies the differential equation in local coordinates

\[
(D_x \dot{x})^\mu := \ddot{x}^\mu + \gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho. \tag{3.3}
\]

The covariant formalism for geometries of maximal acceleration that we use was developed in [17] as a geometric formulation of Caianiello’s quantum geometry. In the way related with generalized tensors was developed in [18]. The Sasaki-type metric on the bundle \( TM \setminus NC \) is the pseudo-Riemannian metric

\[
g_S = \eta_{\mu\nu} dx^\mu \otimes dx^\nu + \frac{1}{A_{\text{max}}^2} \eta_{\mu\nu} \left( \delta y^\mu \otimes \delta y^\nu \right). \tag{3.4}
\]

Let us consider a timelike curve \( x : I \rightarrow M \), parameterized such that for the lifted vectors \( g_S(\dot{x}, 1\dot{x}) = -1 \). Then the metric \( (3.4) \) induces a bilinear, non-degenerate, symmetric form \( g \) along the lift \( 1x : I \rightarrow TM \). [18].

**Theorem 3.1** There is a bilinear, non-degenerate, symmetric form \( g \) along the lifted timelike curve \( 1x : I \rightarrow TM \), such for the corresponding horizontal lifted vector \( h \dot{x} \) conformaly equivalent to \( \eta \) such that the following relation holds,

\[
g_{\mu\nu}(x(\tau)) = \left( 1 - \frac{\eta_{\sigma\lambda} D_x \dot{x}^\sigma(\tau) D_x \dot{x}^\lambda(\tau)}{A_{\text{max}}^2} \right) \eta_{\mu\nu} + \text{higher order terms}. \tag{3.5}
\]

**Proof.** The tangent vector of the lift \( (x(\tau), \dot{x}(\tau)) \) is \( (\dot{x}, \ddot{x}) \). Using the non-linear connection, one considers the vector field \( T = \dot{x}^\mu \frac{\delta}{\delta x^\mu} + \ddot{x}^\mu \frac{\partial}{\partial y^\mu} \in T^1 J^1 M \). Then the metric \( g_S \) acting on the vector field \( T \) at the point \( (x(\tau), \dot{x}(\tau)) \) has the value

\[
g_S(T, T) = \left( \eta_{\mu\nu} dx^\mu \otimes dx^\nu + \frac{1}{A_{\text{max}}^2} \eta_{\mu\nu} \left( \delta y^\mu \otimes \delta y^\nu \right) \right)(T, T) = \eta(\dot{x}, \ddot{x}) \left( 1 + \frac{1}{A_{\text{max}}^2} \eta_{\mu\nu} D_x \dot{x}^\mu D_x \dot{x}^\nu \right) \eta(\dot{x}, \ddot{x}) \approx -\left( 1 - \frac{1}{A_{\text{max}}^2} \eta_{\mu\nu} D_x \dot{x}^\mu D_x \dot{x}^\nu \right),
\]

which coincides with the value \( (g_{\mu\nu} dx^\mu \otimes dx^\nu)(\dot{x}, \ddot{x}) \), with \( g_{\mu\nu} \) given by \( (3.5) \) at leading order.

The *generalized metric* \( \eta \) is called metric of maximal acceleration, because of the following property,

**Proposition 3.2** If \( g(\dot{x}, \ddot{x}) < 0 \), then \( g(D_x \dot{x}, D_x \ddot{x}) < A_{\text{max}}^2 \).

**Proof.** From the expression \( (3.3) \) and since \( g \) is non-degenerate, the factor

\[
(1 - \frac{\eta_{\mu\lambda} D_x \dot{x}^\sigma(\tau) D_x \dot{x}^\lambda(\tau)}{A_{\text{max}}^2})
\]
cannot be zero. Thus, if \( g(\dot{x}, \dot{x}) < 0 \), then it follows that \( \eta(D_2 \dot{x}, D_2 \dot{x}) < A_{\text{max}}^2 \), from which follows directly the thesis.

Since the metric \( \eta \) is flat, there is a local coordinate system where the connection coefficients \( \gamma^\mu_{\nu\rho} \) are zero (normal coordinate system). In such coordinate system we also has that \( N^\mu_{\nu}(x, y) = 0 \) and the bilinear form \( (3.5) \) reduces to

\[
g(\tau) = \left(1 - \frac{\eta_{\lambda\sigma} \ddot{x}^\sigma(\tau) \ddot{x}^\lambda(\tau)}{A_{\text{max}}^2}\right) \eta_{\mu\nu} \, dx^\mu \otimes dx^\nu.
\]

(3.6)

This non-covariant version was first discussed by Caianiello’s and co-workers [6].

For timelike curves, the domain \( \mathcal{D} \) of the metric of maximal acceleration is the intersection of the open domains \( \mathcal{D}_1 = \{ 2x \in J^0_0(M) \text{ s.t.} \eta(\dot{x}, \dot{x}) < 0 \} \) and \( \mathcal{D}_2 = \{ 2z \in J^0_0(M) \text{ s.t.} \eta(\ddot{x}, \ddot{x}) < A_{\text{max}}^2 \} \). Therefore, the domain of definition of \( g \) as an open domain \( \mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 \subset J^0_0(M) \). This is in concordance with the definition of \( g \) as a generalized metric (see Appendix B and for more details [18]). For lightlike vectors \( \eta(\dot{z}, \dot{z}) = 0 \), one also has \( g(\dot{z}, \dot{z}) = 0 \). Indeed, using the expression \( (3.4) \), we have for a lightlike vector of \( \eta \),

\[
g_s(\dot{x}, \dot{x}) = \left( \eta(\dot{x}, \dot{x}) + \frac{1}{A_{\text{max}}^2} \eta(\ddot{x}, \ddot{x}) \right) \eta_{\mu\nu} \, dx^\mu \otimes dx^\nu.
\]

On the other hand

\[
g(\dot{x}, \dot{x}) = g_s(\dot{x}, \dot{x}) + \mathcal{O}(\epsilon_0^2)
\]

\[
= \left(1 - \frac{\eta_{\lambda\sigma} \ddot{x}^\sigma(\tau) \ddot{x}^\lambda(\tau)}{A_{\text{max}}^2}\right) \eta(\dot{x}, \dot{x}) + \mathcal{O}(\epsilon_0^2)
\]

\[
= \mathcal{O}(\epsilon_0^2).
\]

Thus, for lightlike curves respect to \( \eta \), the two expressions are compatible iff \( \eta(\ddot{x}, \ddot{x}) = 0 \). In the regime \( \eta(\dot{x}, \dot{x}) << A_{\text{max}}^2 \) and \( g(\dot{x}, \dot{x}) = 0 \), this condition is equivalent to \( \ddot{x} = 0 \), or in covariant form, \( D_2 \dot{x} = 0 \). Therefore, for lightlike trajectories, the domain of the maximal acceleration \( g \) are the lightlike geodesics.

### 3.3 On the validity of the clock hypothesis and geometry of maximal acceleration

It is well known that the existence of ideal clocks and rods associated with accelerated particles is an assumption, which is commonly named the clock hypothesis. It is logically independent of the principle of relativity and the principle of constance of the speed of light in vacuum. Indeed, Einstein’s words, a standard clock and rod is such assumed that its behavior depends only upon velocities, and not upon accelerations, or at least, that the influence of acceleration does not counteract that of velocity (see [13], footnote in page 64).
Let us consider a normal coordinate system associated with the Minkowski metric \( \eta \) and let us denote by \( \tau \) the proper-time parameter along a given curve respect to \( g \).

\[
\tau[x] = \int_{t_0}^{t} \sqrt{-g(x'(t), x'(t))} \, dt.
\]  

(3.7)

Clearly, the clocks whose proper-time are calculated by using a generalized metric (3.5). Therefore, the justification of the use of proper-time (3.7) requires first to justify why the clock hypothesis is not universally valid. Such step was recognized by B. Mashhoon long time ago in [25]. After analyzing the equivalent hypothesis of locality, it was shown that such hypothesis is not valid when the so called acceleration time and length (a short of measure of the tidal effects on measurement devices) are not much more large compared with the intrinsic length and time of the physical system acting as a measurement device. However, this is exactly the situation for a radiating electron, that is the case of investigation in this paper: for a radiating electron the acceleration length and time are of the same order than the intrinsic length and time [25]. Therefore, for the situations that we are interesting in, the hypothesis of locality (or equivalently the ideal clock hypothesis) will not work. Another relevant instance where the hypothesis will not work is on the singularity points. That is, when the accelerations become extremely high or un-bounded.

In this situation, the proper-time given by equation (3.7) is compatible with Lorentz invariance, diffeomorphism reparameterization invariance and with maximal proper acceleration. Therefore, we adopt as the physical proper-time parameter. Note that doing this, we need to disregard the Lorentzian proper-time as the physical time. However, because the above properties of \( \tau \), this is indeed a moderate change. Moreover, due to the high value of the maximal acceleration, in usual situations the difference \( \tau[\gamma] - s[\gamma] \) is small, being increasing as the proper acceleration \( \eta(\ddot{x}, \ddot{x}) \) increases. Moreover, since the existence of maximal acceleration, it is prevented the existence of singularities, allowing to define \( \tau \) in the full domain of the curves \( \gamma \).

3.4 Perturbation scheme

We can parameterize each world-line by the corresponding proper time \( \tau \) associated to \( g \) and this should be understood below in this work if anything else is not stated. The acceleration square function is defined by the expression

\[
a^2(\tau) := \eta_{\mu\rho} \ddot{x}^\mu \ddot{x}^\rho. \tag{3.8}
\]

Then the function \( \epsilon \) is defined in a normal coordinate system of \( \eta \) by the relation

\[
\epsilon(\tau) = \frac{\eta_{\mu\lambda} \ddot{x}^\sigma(\tau) \ddot{x}^\lambda(\tau)}{A_{\text{max}}^2}. \tag{3.9}
\]

In an arbitrary coordinate system the parameter \( \epsilon(\tau) \) is given by the relation

\[
\epsilon(\tau) := \frac{\eta(D_{\dot{x}} \dot{x}, D_{\dot{x}} \dot{x})}{A_{\text{max}}^2}. \tag{3.10}
\]
The relation between $g$ and $\eta$ along $x : I \to M$ is a conformal factor,

$$g(\tau) = (1 - \epsilon(\tau))\eta.$$  
(3.11)

$\epsilon(\tau)$ determines a bookkeeping parameter $\epsilon_0$ by the relation $\epsilon(\tau) = \epsilon_0 h(\tau)$, where

$$\epsilon_0 = \max\{\epsilon(\tau), \tau \in I\}. \tag{3.12}$$

Also, the relation between the proper parameter of $\tau$ and the proper parameter of $g$ is determined by the relation

$$ds = (1 - \dot{\epsilon})^{-1} d\tau.$$  
(3.13)

For compact curves this definition always makes sense. However, we will need to bound higher order derivatives in order to keep such parameter finite for non compact curves. Then one can speak of asymptotic expansions on powers of $O(\epsilon_0^l)$, with the basis for asymptotic expansions being $\{\epsilon_0^l, l = -\infty, -1, 0, 1, ..., +\infty\}$.

In order to define the perturbative scheme, let us consider a normal coordinate system for $\eta$. We assume that the dynamics happens in a regime such that

$$a^2(\tau) \ll A^2_{\text{max}}.$$  
(3.14)

The curves $X : I \to M$ with $a^2(\tau) = A^2_{\text{max}}$ are curves of maximal acceleration. Since we assume that all the derivatives $(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, ...)$ are small, the dynamics of point charged particles will be away from the sector of maximal acceleration.

The metric $g$ determines different kinematical relations than $\eta$. Let us assume that the parametrization of the world-line $x : I \to M$ is such that $g(\dot{x}, \dot{x}) = -1$ and the monomials in powers of the derivatives of $\epsilon$ define a complete generator set for asymptotic expansions. In a geometry of maximal acceleration, the kinematical constrains are

$$g(\dot{x}, \dot{x}) = -1,$$  
(3.15)

$$g(\ddot{x}, \dot{x}) \simeq \frac{\dot{\epsilon}}{2} \eta(\dot{x}, \dot{x}),$$  
(3.16)

$$g(\dddot{x}, \dot{x}) + g(\ddot{x}, \ddot{x}) \simeq \frac{d}{d\tau} \left( \frac{\dot{\epsilon}\eta(\dot{x}, \dot{x})}{2} \right) + \dot{\epsilon}\eta(\ddot{x}, \ddot{x})$$  
(3.17)

and similar conditions hold for higher derivatives obtained by derivation of the previous ones. Therefore, in a geometry of maximal acceleration $(M, g)$, given the normalization $g(\dot{x}, \dot{x}) = -1$, the following approximate expressions hold:

$$\dddot{x}^\rho \dot{x}_\rho := g_{\mu\rho} \dot{x}^\mu \ddot{x}^\rho = -1,$$  
(3.18)

$$\ddot{x}^\rho \dot{x}_\rho := g_{\mu\rho} \ddot{x}^\mu \dot{x}^\rho = \frac{\dot{\epsilon}}{2} + O(\epsilon_0^2),$$  
(3.19)

$$\dddot{x}^\rho \dot{x}_\rho + \dddot{x}^\rho \ddot{x}_\rho = g_{\mu\rho} \dddot{x}^\mu \dot{x}^\rho + g_{\mu\rho} \dddot{x}^\mu \ddot{x}^\rho = \frac{d}{d\tau} \left( \frac{\dot{\epsilon}\eta(\dot{x}, \dot{x})}{2} \right) + \dot{\epsilon} + O(\epsilon_0^2).$$  
(3.20)
4 A simple derivation of the ALD force equation

4.1 Rohrlich’s derivation of the ALD equation

Let us assume that the spacetime is the Minkowski spacetime \((M, \eta)\). In a normal coordinate system of \(\eta\), the ALD equation is the third order differential equation

\[
m \ddot{x}^\mu = e F^\mu_\nu \dot{x}^\nu + 2 \frac{e^2}{3} \left( \dddot{x}^\mu - (\dddot{x}^\rho \dddot{x}_\rho) \dot{x}^\mu \right), \quad \dddot{x}_\mu := \dddot{x}^\sigma \eta_{\mu \sigma}. \tag{4.1}
\]

with \(e\) being the charge of the electron, \(m\) is the experimental inertial mass and the time parameter \(t\) is the proper time respect to the metric \(\eta\). The ALD has run-away and pre-accelerated solutions \([12, 20]\), both against what is observed in everyday experience and in contradiction with the first Newton’s law of classical dynamics.

We present an elementary derivation of the ALD equation \((4.1)\). This derivation is what we have called Rohrlich’s argument \([31]\). It illustrates an application of Cartan’s adapted frame method \([35]\) that we will use later in the contest of generalized higher order fields in combination with maximal acceleration geometry. In addition, it does not require the formal introduction of the energy-momentum tensor for the generalized fields. Indeed, one deals formally with the geometry of point particles and the covariant Larmor’s law.

One starts with the Lorentz force equation for a point charge particle interacting with an electromagnetic field \(F^\mu_\nu\),

\[
m_b \dddot{x}^\mu = e F^\mu_\nu \dot{x}^\nu, \tag{4.2}
\]

where \(m_b\) is the bare mass and \(e\) the electric charge of the particle. Note that both sides are orthogonal to \(\dot{x}\) by using the Minkowski spacetime metric \(\eta\). Also, in this sub-section, the parameter \(t\) is the proper parameter associated with \(\eta\). In order to generalize the equation \((4.2)\) to take into account the radiation reaction, one can add to the right hand side of \((4.2)\) a vector field \(Z\) along the curve \(x : I \rightarrow M\),

\[
m_b \dddot{x}^\mu = e F^\mu_\nu \dot{x}^\nu + Z(3x(t))
\]

The orthogonality condition \(\eta(Z(t), \dot{z}(t)) = 0\) implies the following general expression for \(Z\),

\[
Z^\mu(t) = P^\mu_\nu(t)(a_1 \dddot{x}^\nu(t) + a_2 \ddot{x}^\nu(t) + a_3 \dddot{x}^\nu), \quad P^\mu_\nu = \eta_{\mu \nu} + \dot{x}_\mu(t) \dot{x}_\nu(t), \quad \dot{x}_\mu = \eta_{\mu \nu} \dddot{x}^\nu. \tag{4.3}
\]

We can prescribe \(a_1 = 0\). Then using the kinematical relation \(\ddot{x}^\rho \dddot{x}_\sigma \eta_{\rho \sigma} = -\dddot{x}^\rho \ddot{x}_\sigma \eta_{\sigma \rho}\), one obtains the relation

\[
Z^\mu(t) = a_2 \dddot{x}^\mu(t) + a_3 (\dddot{x}^\mu - (\dddot{x}^\rho \dddot{x}_\rho) \dot{x}^\mu)(t).
\]

The term \(a_2 \dddot{x}\) combines with the left hand side to renormalize the bare mass,

\[
(m_b - a_2) \dddot{x}^\mu = m \dddot{x}^\mu. \tag{4.4}
\]
This procedure will be assumed to be valid independently of the values of $m_b$ and $a_2$ (that both could have infinite values but the different be finite). The argument from Rohrlich follows by requiring that the right hand side is compatible with the relativistic power radiation formula \[20, 31,\]

$$P^\mu_{\text{rad}}(t) = -\frac{2}{3} e^2 (\ddot{x}^\rho \dddot{x}^\sigma \eta_{\rho\sigma})(t) \dot{x}^\mu(t). \tag{4.5}$$

This condition is satisfied if

$$a_3 = \frac{2}{3} e^2.$$

In order to recover this relation, the minimal piece required in the equation of motion of a charged particle is $-2/3e^2(\dddot{x}^\rho \dddot{x}^\sigma \eta_{\rho\sigma})\dot{x}^\mu$. The Schott term $\frac{2}{3} e^2 \dot{x}$ is a total derivative. It does not contribute to the averaged power emission of energy-momentum. However, in the above argument, the radiation reaction term and the Schott term are necessary, due to the kinematical constraints of the metric $\eta$.

### 4.2 Rohrlich's derivation of the Abraham-Lorentz-Dirac equation with maximal acceleration

Using the kinematical constraints for metrics of maximal acceleration, we can repeat Rohrlich’s argument. However, there is a slightly modification caused by the bound in the acceleration and because now the kinematic relations are the corresponding to a geometry of maximal acceleration. If one writes know

$$Z^\mu(\tau) = P^\mu(\tau)(\lambda_1 \dot{x}^\nu(\tau) + \lambda_2 \ddot{x}^\nu(\tau) + \lambda_3 \dddot{x}^\nu),$$

and by an analogous procedure as before, $\lambda_1$ is arbitrary and we can prescribe $\lambda_1 = 0$. The equation is consistent with Larmor’s law if

$$\dot{\lambda}_2 + \lambda_3 \frac{d}{d\tau} (g(\dot{x}, \ddot{x})) = 0, \quad \lambda_3 = \frac{2}{3} e^2, \quad \lambda_k = 0, \quad \forall k \geq 4. \tag{4.6}$$

The corresponding modified ALD equation is

$$m \dddot{x}^\mu = e F_{\mu \nu} \dot{x}^\nu + \frac{2}{3} e^2 (\dddot{x}^\mu - (\dddot{x}^\rho \dddot{x}^\sigma \eta_{\rho\sigma})\dot{x}^\mu) + O(\epsilon_0^2), \tag{4.7}$$

with $F_{\mu \nu} := \eta^{\rho\sigma} F_{\rho\sigma}$. This equation is formally identical to the ALD equation. This fact implies that only maximal acceleration hypothesis is not enough to solve the problem of the Schott term in the ALD equation.

Similar considerations holds if we repeat the calculation in the framework of generalized higher order fields. In that case one can see that without the requirement of bounded acceleration one does not obtain a second order differential equation for the motion of the point charged probe. In that case, one obtains again a differential equation that is formally the same than the ALD equation. However, we will see in next section that if one combines maximal acceleration geometry with generalized higher order fields, it is possible to obtain an implicit second order differential equation for point charged particles.
5  A second order differential equation for point charged particles

Let us assume that the physical trajectory of a point charged particle is a smooth curve of class $C^k$ such that $g(\dot{x}, \ddot{x}) = -1$, with $\dot{x}^0 < 0$ and such that the square of the acceleration vector field $a^2$ is bounded from above. The motivation for this assertion is the hypothesis that the physical measurable metric structure is the corresponding to the maximal acceleration metric. We will follow closely an analogous argument to Rohrlich’s argument for generalized higher order fields in a maximal geometry back-ground. Thus one has the following general expression,

$$\Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots) = B_\mu \dot{x}_\nu - B_\nu \dot{x}_\mu + C_\mu \ddot{x}_\nu - C_\nu \ddot{x}_\mu + D_\mu \dddot{x}_\nu - D_\nu \dddot{x}_\mu + \ldots,$$

with $\dot{x}_\mu = g_{\mu\nu} \dddot{x}^\nu$. This implies that we will have the expression

$$m_b \dddot{x}^\mu = e F^\mu_{\ \nu} + \left( B^\mu \dddot{x}_\nu - \dot{x}_\nu B_\mu \right) \dddot{x}^\nu + \left( C^\mu \dddot{x}_\nu - \dot{x}_\nu C_\mu \right) \dddot{x}^\nu + \left( D^\mu \dddot{x}_\nu - \dot{x}_\nu D_\mu \right) \dddot{x}^\nu + \ldots,$$

and with $F^\mu_{\ \nu} = g^{\mu\rho} F_{\rho\sigma} = \eta^{\mu\rho} F_{\rho\sigma}$. On the right hand side of the above expression all the contractions that appear in expressions like $(B^\mu \dddot{x}_\nu - \dot{x}_\nu B_\mu) \dddot{x}^\nu$, etc, are performed with the metric $g$ instead of the Minkowski metric $\eta$. The other terms come from the higher order terms of the expression (5.1) of the electromagnetic field.

The general form of the $k$-jet field along a smooth curve $x : R \to M$, $B(\tau), C(\tau), D(\tau)$ are

$$B^\mu(\tau) = \beta_1 \dot{x}^\mu(\tau) + \beta_2 \ddot{x}^\mu(\tau) + \beta_3 \dddot{x}^\mu(\tau) + \beta_4 \dddot{x}^\mu(\tau) + \ldots,$$

$$C^\mu(\tau) = \gamma_1 \dot{x}^\mu(\tau) + \gamma_2 \ddot{x}^\mu(\tau) + \gamma_3 \dddot{x}^\mu(\tau) + \gamma_4 \dddot{x}^\mu(\tau) + \ldots,$$

$$D^\mu(\tau) = \delta_1 \dot{x}^\mu(\tau) + \delta_2 \ddot{x}^\mu(\tau) + \delta_3 \dddot{x}^\mu(\tau) + \delta_4 \dddot{x}^\mu(\tau) + \ldots.$$

Let us assume that there are not derivatives higher than 2 in the differential equation of a point charged particle. One way to achieve this is to impose that all the coefficients for higher derivations are equal to zero,

$$\gamma_k = \delta_k = 0, \ k \geq 0, \ \beta_k = 0, k > 3. \quad (5.1)$$

With this choice and using the kinetic relations for $g$, one obtains the expression

$$m_b \dddot{x}^\mu = e F^\mu_{\ \nu} - \beta_2 \dddot{x}^\mu - \beta_3 \dddot{x}^\mu - \frac{1}{2} \beta_2 \dot{\epsilon} \dot{x}^\mu - \beta_3 (-a^2(\tau) + \dot{\epsilon}) \dot{x}^\mu. \quad (5.2)$$

Under the assumption that the equation of motion of a point charged particle must be of second order and compatible with the power radiation formula (4.5) and for $\dot{\epsilon}(\tau) \neq 0$, one obtains the relations

$$\beta_2 = \frac{4}{3} e^2 a^2(\tau) \frac{1}{\dot{\epsilon}}, \quad (5.3)$$

$$\beta_k = 0, \ \forall k \geq 3. \quad (5.4)$$

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At leading order in $\epsilon_0$, we obtain the following differential constraint:

$$m_b(\tau) \dddot{x}^\mu = e F^{\mu \nu} \dot{x}^\nu - \frac{2}{3} e^2 a^2(\tau) \dot{x}^\mu - \frac{2}{3} e^2 a^2(\tau) \frac{1}{\epsilon} \dddot{x}^\mu. \quad (5.5)$$

There is a re-normalization of the bare mass,

$$m_b(\tau) + \frac{2}{3} e^2 a^2(\tau) \frac{1}{\epsilon} = m, \quad \epsilon \neq 0. \quad (5.6)$$

In this expression, the term $m_b$ contains already the infinite term from the Coulomb field, that is the infinite electrostatic mass has been already renormalized (for instance the term $\dddot{x}^\mu C_\nu \dot{x}^\nu$ is a term that can renormalize the Coulomb self-energy).

The constraint (5.6) implies differential constraints on the derivatives, obtaining by taking the derivative respect to $\tau$ in both sides. Thus, in order to keep consistency, the bare mass $m_b$ will be defined such that the derivatives $\dot{m}_b$, $\ddot{m}_b$, ..., following the corresponding constraint. Using that $\dot{m} = 0$ and the relation (3.9). For the first derivative of the bare mass $\dot{m}_b$, one has the constrain on the first derivative,

$$\frac{d^k}{d\tau^k} m_b + \frac{2}{3} e^2 A^2_{\max} \frac{d^k}{d\tau^k} \left( \frac{\epsilon}{\epsilon} \right) = 0, \quad k = 1, 2, 3, ... \quad (5.7)$$

All such initial conditions are satisfied if one imposes the condition

$$\epsilon = 0 \Rightarrow m_b(\tau) = m = constant. \quad (5.8)$$

With this assumption, the renormalization of mass relations (5.6) coincides with solution of the constraint (5.7).

The first derivative of the bare mass must be of the form,

$$\frac{d m_b}{d\tau} \bigg|_{\tau=0} = -\frac{2}{3} e^2 A^2_{\max} \left( \frac{\epsilon}{\epsilon} \right) \bigg|_{\tau=0}. \quad (5.9)$$

In a similar way, one can compute the defining properties for higher order derivatives of the bare mass $m_b$, that by definition will have the formal expression

$$m_b(\tau) = \sum_{k=0}^{\infty} \frac{m^{(k)}(\tau=0)}{k!} \tau^k. \quad (5.10)$$

Since $m_b$ is not observable, each coefficient $m^{(k)}(\tau=0)$ can be adjusted to be compatible with the derivative constraints like (5.7). Thus the bare mass $m_b$ is an element of the infinite jet bundle $m_b(\tau) \in J_0^\infty(M)$.

Therefore, for $\dot{\epsilon} \neq 0$ the following implicit differential equation should hold,

$$m \dddot{x}^\mu = e F^{\mu \nu} \dot{x}^\nu - \frac{2}{3} e^2 \eta_{\rho\sigma} \dddot{x}^\rho \dddot{x}^\sigma \dot{x}^\mu, \quad F^{\mu \nu} = g^{\mu \rho} F_{\rho \nu}. \quad (5.10)$$

The derivation of this implicit second order differential equation breaks down for $\dot{\epsilon} = 0$ and we will consider this case separately.
The covariant form of equation (5.10) in any coordinate system is

\[ m D_\tau \ddot{x} = e \tilde{\iota}_\tau F(x(\tau)) - \frac{2}{3} e^2 \eta(D_\tau \dot{x}, D_\tau \dot{x}), \]  

(5.11)

with \( \tilde{\iota}_\tau F(x(\tau)) \) the dual of the contraction \( \iota_\tau F \in \Gamma \Lambda^1 M \).

Note that since we are considering metrics of maximal acceleration avoids the orthogonality problem of taking away the Schott term in the ALD equation.

The derivatives in equations (5.10) and (5.11) are taken with respect to the proper-time determined by the metric of maximal acceleration. Thus, we need to prove that such parameter exists. This is done in Appendix B. In particular the proof applies to the reparameterization invariance between proper time of \( g \) and time coordinate. Moreover, it is important to remark that in our framework the physical metric, the time elapsed by an observer at rest with the particle, is measured by hypothesis by a metric of maximal acceleration, that depends explicitly of the acceleration of the particle measured by an inertial system. Thus, the clock hypothesis is not maintain in our theory. This corresponds with profound departure from Special and General Relativity, where the metric of the spacetime is Lorentzian. In contrast, in our framework, the Lorentzian metric \( \eta \) plays only a secondary role, and it is not associated with the physical proper-time (18, section 3).

It is in this context that one can ask for the transformation rule of equation (5.10) under the parameter change from the proper parameter \( s \) determined by \( \eta \) and the proper-time \( \tau \) determined by \( g \). Such transformation rule must be determined by the fundamental relation (3.13). In the transformation rule of the equation (5.10) under such parameter change, since the dependence on higher derivative, they will appear higher derivatives in the transformed equation of motion. However, the transformation \( \varphi(\tau) = s \) is not a diffeomorphism of the form \( \psi : I \to \tilde{I}, I, \tilde{I} \subset \mathbb{R} \). That is, \( s = \varphi(\tau) \) is not a reparameterization of the parameter \( \tau \) as usually is understood, in accordance with the identification of \( g \) as the physical measurable metric and \( \eta \) as only a convenient geometric device.

5.1 Properties of the equation (5.10)

Let us consider a normal coordinate system for \( \eta \). Let us multiply equation (5.10) by itself and contract with the metric \( g \). Using the kinetic relations of proposition

---

*Note that any second differential equation respect to a parameter \( r \) can be changed to an arbitrary order differential equation where the derivatives are taken respect a second parameter \( l \), if the relation \( l(s) \) contains higher order than zero derivatives. Thus it is not a surprise that this happens in the case of equation (5.10) when passing from \( \tau \) to \( s \).*
One obtains
\[ m^2 a^2 (1 - \epsilon) = e^2 F^{\mu}_\rho \dot{x}^\rho F^{\nu}_\lambda \dot{x}^\lambda (1 - \epsilon) \eta_{\mu\nu} + \left( \frac{2}{3} e^2 \right)^2 (a^2)^2 \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} \]
- \[ 2e \frac{2}{3} e^2 F^{\rho}_\mu \dot{x}^\rho \dot{x}^\nu (1 - \epsilon) \eta_{\mu\nu} \]
= \[ (1 - \epsilon) e^2 F^{\mu}_\rho \dot{x}^\rho F^{\nu}_\lambda \dot{x}^\lambda \eta_{\mu\nu} - \left( \frac{2}{3} e^2 \right)^2 (a^2)^2 \]
- \[ 2e \frac{2}{3} e^2 F^{\rho}_\mu \dot{x}^\rho \dot{x}^\nu (1 - \epsilon) \eta_{\mu\nu} \]
= \[ (1 - \epsilon) \left( F_L^2 - \frac{1}{1 - \epsilon} \left( \frac{2}{3} e^2 \right)^2 (a^2)^2 \right). \]

with the magnitude of the Lorentz force \( F_L \) given by
\[ F_L^2 = e^2 F^{\nu}_\mu F^{\rho}_\lambda \dot{x}^\nu \dot{x}^\lambda \eta_{\mu\rho}. \]

**Proposition 5.1** For any curve solution of equation (5.10) one has the following consequences,

1. The Lorentz force is always spacelike or zero,
   \[ F_L^2 \geq 0. \quad (5.12) \]

2. In the case the Lorentz force is zero, the magnitude of the acceleration is zero,
   \[ F_L^2 = 0 \iff a^2 = 0. \quad (5.13) \]

3. If there is an external electromagnetic field, the acceleration is bounded by the strength of the corresponding Lorentz force.

**Proof.** For \( \epsilon \neq 0 \), one can re-write the expression
\[ m^2 a^2 (1 - \epsilon) = (1 - \epsilon) \left( F_L^2 - \frac{1}{1 - \epsilon} \left( \frac{2}{3} e^2 \right)^2 (a^2)^2 \right) \]
as the following
\[ F_L^2 = \frac{1}{1 - \epsilon} \left( \frac{2}{3} e^2 \right)^2 (a^2)^2 + m^2 a^2, \quad (5.14) \]
from which follows the three consequences. \( \square \)

Contracting both sides of the differential equation with \( \dot{x} \)
\[ m \ddot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} = F^{\mu}_\rho \dot{x}^\rho \dot{x}^\nu \eta_{\mu\nu} - \frac{2}{3} e^2 a^2 g(\dot{x}, x^\prime) = \frac{2}{3} e^2 a^2. \quad (5.15) \]
Using the kinetic relations from \( g \), the relation (5.15) reduces to

\[
m g(\ddot{x}, \dot{x}) = -\frac{2}{3} e^2 a^2(\tau) + \mathcal{O}(\epsilon^2).
\]

Defining the characteristic time

\[
\tau_0 := \frac{2}{3m} e^2,
\]

the relation can be re-written as

\[
g(\ddot{x}, \dot{x}) \simeq -\tau_0 a^2(\tau).
\] (5.16)

The constraint (5.10) is an implicit differential equation. Locally, one can solve \( \ddot{x} \) in the left side under some approximations in the following way. First, note that equation (5.10) can be read as

\[
\ddot{x} \rho (m \delta_{\mu \rho} + \frac{2}{3} e^2 a^2 \dot{x} \rho) = e F_{\mu \nu} \dot{x} \nu.
\]

If the radiation reaction term \( \frac{2}{3} e^2 a^2 \dot{x} \rho \) is small compared with the Lorentz force term one can treat it as a perturbation. As a first approximation one can consider that the Lorentz equation holds,

\[
\ddot{x} = e \dot{x} F_{\rho \nu} \nu
\]

and substitute in the above expression. The inversion of the operator

\[
M_{\mu \rho} = (m \delta_{\mu \rho} + \frac{2}{3m} \dot{x} \mu \dot{x} \rho)
\] (5.17)

is given by an expression in local coordinates as

\[
O_{\mu \nu} = \frac{1}{m} \delta_{\mu \nu} - \frac{2}{3m^2} \dot{x} \mu F_{\nu \lambda} \dot{x} \lambda.
\] (5.18)

Thus, the second derivative \( \ddot{x} \) can be isolated as

\[
\ddot{x} = e \dot{x} F_{\rho \nu} \nu \left( \frac{1}{m} \delta_{\mu \rho} - \frac{2}{3m^2} \dot{x} \mu F_{\rho \sigma} \dot{x} \sigma \right).
\] (5.19)

This equation can be seen as an approximation for (5.10) at leading order in \( \epsilon_0 \). It is useful, since for this approximated equation (5.19) one can use existence and uniqueness theorem of ODE theory [11] to state the following result,

**Proposition 5.2** Let \( z_1 \) and \( z_2 \) be two solutions of the equation of (5.10) with the same initial conditions. Then they differ by a smooth function on the radiation reaction term \( \frac{2}{3} e^2 a^2 \).
Proof. Each of the solutions of equation (5.10) can be approximated by a solution of equation of the equation (5.19) and the difference is given locally as a power on \( \epsilon = \frac{2}{3} e^2 a^2 \), starting at least at power 1 in \( \epsilon \). By uniqueness of solutions of ODE [10], given a fixed initial conditions \((x^\mu(\tau_0), \dot{x}^\mu(\tau_0))\) the solution exits and is unique (in a finite interval of time \( \tau \)). In such interval the two solutions differ by a polynomial on \( \frac{2}{3} e^2 a^2 \) in a short time interval. \( \Box \)

Equation (5.19) can be written as a geodesic equation for a suitable connection. Let us consider a connection \( ^2\nabla \) on \( T.J^2_0(M) \) that in the local holonomic frame \( \{ \partial_\mu, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial \dot{x}^\mu} \} \) of \( T.J^2_0(M) \) has connection coefficients

\[
^2\Gamma^\mu_{\nu\sigma} := (^2\nabla_{\partial\nu} \partial\sigma)^\mu = \gamma^\mu_{\nu\sigma} + K^\mu_{\nu\sigma} + L^\mu_{\nu\sigma},
\]

(5.20)

with the tensors \( K \) and \( L \) defined by the expressions

\[
K^\mu_{\nu\sigma} = -\frac{1}{2m} \left( F^\mu_{\nu\sigma\lambda} \dot{x}^\lambda + F^\mu_{\nu\sigma\lambda} \dot{x}^\lambda \right) \frac{1}{1 - \frac{\eta_{\kappa\xi\dot{x}^\kappa\dot{x}^\xi}}{A_{max}}},
\]

(5.21)

\[
L^\mu_{\nu\sigma} = -\frac{2e^3}{3m^2} F_{\mu\rho\nu} F_{\rho\nu} \dot{x}^\mu.
\]

(5.22)

All the rest of connection coefficients of \( ^2\nabla \) are put equal to zero. Therefore,

**Proposition 5.3** The coefficients \( ^2\nabla \) define a covariant derivative on \( J^2_0(M) \) and the geodesics

\[
^2\nabla (\dot{z}^{\dot{x}}) \cdot ^2\dot{x} = 0
\]

(5.23)

coincide with the solutions of the equation (5.19).

The geometric interpretation of equation (5.19) opens the possibility to investigate approximations to (5.10) using geometric and kinetic methods.

### 5.2 Covariant uniform acceleration

The case of world-lines with \( \dot{\epsilon} = 0 \) requires special care. Let us consider the following definition of **covariant uniform acceleration**,

**Definition 5.4** A **covariant uniform acceleration curve** is a map \( x : I \rightarrow M \) such that along it, the following constrain holds:

\[
\dot{\epsilon} = \frac{d}{d\tau} \frac{\eta(D\dot{x}, D\dot{x})}{A_{max}^2} = 0.
\]

(5.24)
This notion of covariant uniform motion is general covariant. Thus, in order to make some computations we can adopt normal coordinates associated to \( \eta \). Then the kinetic relation (3.19) holds. Although the four-acceleration is in general not orthogonal by \( g \) to the four velocity vector field, in the case of covariant uniform accelerated motion one has that \( g(\ddot{x}, \dot{x}) = 0 \). By the relation (3.20),

\[
g(\ddot{x}, \ddot{x}) = -g(\dot{x}, \dot{x}).
\]

Since the right hand side is necessarily zero,

\[
g(\ddot{x}, \dot{x}) = (1 - \epsilon) \eta(\ddot{x}, \dot{x}) = (1 - \epsilon) \dot{\epsilon} = 0,
\]

\( \ddot{x} \) is also lightlike vector with the metric \( g \). The implication of this fact is that the four-acceleration cannot be constant in a normal coordinate system, except for \( a^2 = 0 \).

For maximal accelerations such that \( A_{\text{max}}^2 \leq \infty \) our definition coincides with the standard definition of covariant uniform acceleration appearing in standard references (see for instance [22]). It also implies the same notion of covariant uniform motion as a solution of an homogeneous differential equation [16].

Equation (5.10) fails to describe covariant uniform motion. The reason for this is that in order to apply the mass renormalization procedure (5.6) and the perturbation scheme the requirement \( \dot{\epsilon} \neq 0 \) is need. Therefore, one needs to consider separately the case \( \dot{\epsilon} = 0 \). Let us consider the following consequence of equations (5.6) and (5.5),

\[
m_b \ddot{x}^\mu = -e F^\mu_{\nu} \dot{x}^\nu + \tilde{\beta}_2 \ddot{x}^\mu.
\]

Thus one has the following differential equation for covariant uniform motion,

\[
m \ddot{x}^\mu = e F^\mu_{\nu} \dot{x}^\nu, \quad m = m_b(\tau) - \tilde{\beta}_2(\tau).
\]

For covariant uniform acceleration one can see easily that

\[
\frac{d}{d\tau} \left( g_{\mu \rho} \right) = 0 \quad (5.26)
\]

and this relation implies

\[
0 = \frac{d}{d\tau} a^2 = \frac{d}{d\tau} \left( \frac{F^2}{m} \right). \quad (5.27)
\]

We will discuss below that this notion of covariant uniform motion is highly unstable under small perturbations. Note that if the point \( \tau_0 \) with \( \dot{\epsilon}(\tau_0) = 0 \), \( \tau_0 \) is an isolated critical point of \( a^2 \), one can extend by continuity the solutions of (5.10) to the full world-line by continuity arguments.
5.3 Absence of pre-acceleration of Dirac’s type for the equation (5.10)

Let us adopt a normal coordinate system associated to $\eta$. We discuss the absence of non-physical solutions on (5.10). Run away solutions are solutions that have the following peculiar behavior: even if the external forces have a compact domain in the spacetime, the charged particle follows accelerating indefinitely. This is a pathological behavior of the ALD equation. However, since the condition (5.13) holds for equation (5.10), we show that our model is free of such problems for asymptotic conditions where the exterior field is bounded.

In order to investigate the existence of pre-acceleration in some of the solutions for the equation (5.10), let us consider the example of a pulsed electric field $\vec{E} = (\kappa \delta(\tau), 0, 0)$. For an external electric pulse $\vec{E} = (\kappa \delta(\tau), 0, 0)$, the equation (5.10) in the non-relativistic limit reduces to

$$a \ddot{x} = \kappa \delta(\tau), \quad a = \frac{3m}{2e^2}.$$  (5.28)

The solution of this equation is the Heaviside’s function,

$$a \dot{x} = \kappa, \quad \tau \geq 0,$$

$$0, \quad \tau < 0,$$

that do not exhibit pre-acceleration behavior. In order to prove uniqueness on the space of smooth functions, let us consider two solutions to the equation (5.28). Clearly, the two solutions must differ by an affine function and the requirement of having the same initial conditions, implies that affine function must be trivial. This shows that in the non-relativistic limit the equation (5.10) does not have pre-accelerated solutions of Dirac’s type, but since the theory is Lorentz covariant, the equation (5.10) does not have pre-accelerate solutions of Dirac’s type in any coordinate system. It is open the question if (5.10) is free of any other type of pre-accelerated solutions.

6 Discussion

Combining the notion of generalized higher order electromagnetic field with maximal acceleration geometry, we have obtained the implicit differential equation (5.10) as a description of the dynamics of a point charged particle that takes into account the radiation reaction. Equation (5.10) is free of run-away solutions and pre-accelerate solutions of Dirac’s type. The hypothesis of maximal acceleration is necessary in order to keep under control the value of the acceleration. It also provides a book-keeping parameter that allows to construct our perturbation method. The hypothesis of generalized higher order fields is fundamental too, since it provides us with the degree of freedom that we need to eliminate the Schott term in the ALD equation. Moreover, the physical interpretation of such fields is clear: fields should not be defined independently of the way they are measured or detected.
The hypotheses of generalized higher order fields for \( k = 3 \) and maximal acceleration do not fix completely the dynamics of a point charged particle. However, the requirements that

- The dynamics is compatible with the covariant power radiation law (4.5),
- The differential equation is second order and
- The notion of field is extended minimally to keep the lowest extension on the derivatives possible

are enough to fix the dynamics of point charged particles, except for covariant uniform motion (which is a degenerate case in our framework). This different behavior of the uniform motion is because the method that we follow breaks down when \( \dot{\epsilon} = 0 \).

For the case of covariant uniform acceleration one needs to have a special treatment. Indeed, for a covariant uniform motion of a point charged particle, one needs to provide not only the initial conditions \((x^\mu(\tau_0), \dot{x}^\mu(\tau_0))\), but also a mechanism to fix in anticipation the external field, such that along the world-line of the charged particle the total field is constant. This requires of infinite precision on the determination of the external electromagnetic field \( F_L \) in such a way that it compensates the radiation reaction field. This fine tuning is highly unstable, and the point charged particle will not behave uniformly accelerated under small fluctuations in the external field. Thus, a model of motion is not structurally stable. On the other hand, a stable description for the dynamics of a point charged particle is given by equation (5.10) only. This argument and the assumption of requiring an stable model\(^5\) force us to abandon the notion of covariant uniform acceleration as a possible physical dynamics for point charged particles.

It is natural to consider the differential equation (5.19) as an approximation to the motion of a point charge when the radiation reaction is negligible compared with the external field. The difference on the solutions of (5.19) and (5.10) is given by the tensor \( L^\mu_{\nu\sigma} \), which is by hypothesis, a small perturbation. Thus, for short time evolution and for initial conditions such that the radiation reaction is small compared with the total field Lorentz force, the difference on the solutions of the two equations is small. Also, the difference between the solutions of (5.10) and (5.25) is small.

The generalized fields are assumed to be well defined on the trajectory of the probe particle. This is contradictory with the fact that electromagnetic fields are infinite at the localization in spacetime of the point charged particle world-line, due to the Coulomb singularity. In order to treat this problem we have adopted the method of mass renormalization, with a time variable bare mass \( m_b(\tau) \). Although we do not think that is the last word on this, the method allows us to consider well defined fields over the world-line of charged particle probes, without the singularity of Coulomb type, which is renormalized to give the observable constant mass \( m \). Therefore, the

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\(^5\)By a stable model we mean a weakly structurally stable [1].
considerations in this model concern finite fields in the whole spacetime manifold \( M \).

A theory which in some sense resembles the one proposed in this lines, is Bonnor’s theory \[3\]. In such theory, the observable mass is time variable, and the constraint on the variable mass \( m_0 \) is given by

\[
\frac{d}{ds} m_0 = -\frac{2}{3} c^2 A_{\text{max}}^2 \epsilon.
\]  

(6.1)

However, this differential equation is completely different form the equation for the bare mass \( m_0 \), equation (5.7), that is trivially satisfied if the condition (5.8) holds.

Apart from the structural differences between the two models, there are at least two important formal properties in Bonnor’s theory that differ from our model. The first one is that in our model is the bare mass changes with time, while the observable mass is constant in time. This is in contrast with Bonnor’s model, where de observable mass changes with time. The second relevant formal difference is on the spacetime metric used and on the orthogonal relations used: in Bonnor’s model, the metric is the usual Minkowski metric and the orthogonality condition are

\[
\eta(\dot{x}, \ddot{x}) = 0,
\]  

(6.2)

such constraint is not true for metrics of maximal acceleration.

Finally, let us remark that the notion of generalized higher order electromagnetic field can be extended to non-abelian Yang-Mill fields, following a similar scheme. Also, the type of metric structure \( g \) is a generalized tensor. This is consistent with a general picture of generalized metrics and fields, coupled by a generalized version of Einstein’s field equations, still to be developed.

### Appendix A: Jet bundles and generalized tensors

In this appendix we collect some of the notions and definitions that we need to define generalized higher order fields. An extended version of the theory was developed in \[18\]. Given a smooth curve \( x : I \rightarrow M \), the set of derivatives \( (p = x(0, \frac{dx}{d\sigma}, \frac{d^2x}{d\sigma^2}, ..., \frac{d^kx}{d\sigma^k}) \) determines a point (jet) in the space of jets \( J^k_0(p) \) over the point \( p \in M \),

\[
J^k_0(p) := \left\{ (x(0), \frac{dx}{d\sigma}|_0, ..., \frac{d^kx}{d\sigma^k}|_0), \forall C^k\text{-curve } x : I \rightarrow M, x(0) = p \in M, 0 \in I \right\}.
\]

Thus the set of higher derivatives \( (p = x(0, \frac{dx}{d\sigma}, \frac{d^2x}{d\sigma^2}, ..., \frac{d^kx}{d\sigma^k}) \) determines a jet at the point \( p = x(0) \). The jet bundle \( J^k_0(M) \) over \( M \) is the disjoint union

\[
J^k_0(M) := \bigsqcup_{x \in M} J^k_0(x).
\]
The projection map is

\[ k\pi : J^k_0(M) \to M, \quad (x(0), \frac{dx}{d\sigma}|_0, \frac{d^2x}{d\sigma^2}|_0, \ldots, \frac{d^kx}{d\sigma^k}|_0) \mapsto x(0). \]

**Example A.1** The simplest example where jets appear is when we use Taylor’s expansions of smooth functions. This kind of approximation is an example of the type of approximation.

**Example A.2** Some of the applications of jet bundle formalism requires the previous introduction of more sophisticated notation and notion. However, there is an example that being natural and of fundamental relevance, we can mention in this Appendix as an example of use of jets in the investigation of ordinary differential equations. It is the case of the perturbative study of geodesic deviation equations of a connection (not necessarily affine) on the manifold \( M \). In that case, the condition for the difference on the coordinates \( \xi^\mu = x^\mu - X^\mu \) is of the geodesic is written as

\[ \ddot{\xi}^\mu + \Gamma^\mu_{\nu\sigma}(X + \xi, \dot{X} + \dot{\xi}) \left( \dot{X}^\nu + \dot{\xi}^\nu \right) \left( \dot{X}^\sigma + \dot{\xi}^\sigma \right) - \Gamma^\mu_{\sigma\nu}(X) \dot{X}^\sigma \dot{X}^\nu = 0. \]

Considering a bookkeeping parameter \( \epsilon \) on the functions \( \{\xi^\mu, \dot{\xi}^\mu, \ddot{\xi}^\mu, \ldots\} \). Then developing by Taylor’s expansion all the functions in the above expression, one obtains a series in terms of \( \epsilon \).

\[ \sum_{k=1}^{\infty} \epsilon^k G_k(\Xi^\mu, \dot{\Xi}^\mu, \ddot{\Xi}^\mu) = 0. \]

Equating to zero each term, a hierarchy of ordinary differential equations is obtained,

\[ G_k(\Xi^\mu, \dot{\Xi}^\mu, \ddot{\Xi}^\mu) = 0, \quad k = 1, 2, 3, \ldots \quad (A.1) \]

The equation obtained from the first order \( G_1(\Xi^\mu, \dot{\Xi}^\mu, \ddot{\Xi}^\mu) = 0 \) is the Jacobi equation of the connection. Higher order deviation equations are obtained by equating to zero the expressions \( G_k(\Xi^\mu, \dot{\Xi}^\mu, \ddot{\Xi}^\mu) = 0 \) for \( k = 2, 3, \ldots \). This method was pioneered by S.L. Bażański [2].

We observe clearly that this is a recursive system of ODE’s, and that to each \( r \)-order in \( \epsilon \) corresponds to an equation for an \( r \)-jet order fields approximations of \( \xi^\mu \).

Given a curve \( x : I \to M \), the \( k \)-lift is the curve \( k^x : I \to J^k_0(M) \) such that the following diagram commutes,

\[ \begin{array}{ccc}
J^k_0(M) & \xrightarrow{k\pi} & M \\
\downarrow{k\pi} \quad \downarrow{k\pi} \\
I & \xrightarrow{k^x} & M.
\end{array} \]

There are also the notions of lift of tangent vectors and smooth functions,
• Let \( X \in T_xM \) be a tangent vector at \( x \in M \). A lift \( k\pi_!(X) \) at \( u \in \pi^{-1}(x) \) is a tangent vector at \( u \) such that
\[
k\pi_!(k\pi_!(X)) = X.
\]

• Let us denote by \( \mathcal{F}(J^0(M)) \) the algebra of real smooth functions over \( J^0(M) \). Then there is defined the lift of a function \( f \in \mathcal{F}(M) \) to \( \mathcal{F}(J^0(M)) \) by
\[
k\pi_!(f)(u) = f(x), \quad \forall u \in \pi^{-1}(x).
\]

Each point on the lifted curve \( k\sigma : I \to J^0(M) \) has local coordinates given by \( (x(\sigma), \frac{dx(\sigma)}{d\sigma}, ..., \frac{d^kx(\sigma)}{d\sigma^k}) \), where \( \sigma \) is the parameter of the curves \( x : I \to M \) and \( k\sigma : I \to J^0(M) \). Also, the linear map \( k\pi : J^0(M) \to M \) is differentiable and the differential of the projection \( k\pi_! \) at \( u \in \pi^{-1}(x) \) is the linear map
\[
k\pi_!(u) : T_uJ^0(M) \to T_xM.
\]
We will denote by \( k\pi_! \) the projection \( (k\pi_! : TJ^0(M) \to TM \) such that at each \( u \) it is the linear map \( (k\pi_!)_u \).

The kernel of \( (k\pi_!)_x \) at \( x \in M \) is the vector space
\[
kV_x := (k\pi_!)^{-1}(0_x),
\]
where \( 0_x \) is the zero vector in \( T_xM \). Then vertical bundle over \( M \) is
\[
kV := \bigsqcup_x kV_x
\]
and the vertical bundle over \( J^0(M) \) is determined by the surjection
\[
k\pi_! : kV \to J^0(M), \quad (k\pi_!^{-1}(0_x)) \ni u \mapsto (\pi^{-1}(x)).
\]
\( kV \) is a real vector bundle over \( J^0(M) \), since it is the kernel of \( k\pi_! \). The composition of \( k\pi_! \) with \( k\pi \) determines also a real vector bundle over \( M \),
\[
k\pi \circ k\pi_! : kV \to M.
\]
One can introduce the notation \( k\pi_! = k\pi \circ k\pi_! \).

Note that we are interested in the case when the probe particles are described by world-lines. In the case that the probe particles have spacetime extension, the maps describing the particles must be other type of sub-manifolds \( \Theta : R^d \to M \) and the jets bundles that must be considered are \( J^0(R^d, M) \). Thus, we have considered in this paper the paper the case \( d = 1 \) only.

A relevant property of jet bundles is Peetre’s theorem. Recall that the support \( \text{supp} \) of a section \( S : M \to \mathcal{E} \) of a vector bundle is the closure of the sets \( \{ x \in M : S(x) \neq 0 \} \). An operator \( D \) between the bundles \( \pi_1 : \mathcal{E}_1 \to M \) and \( \pi_2 : \mathcal{E}_2 \to M \)
\( D : C^\infty(\mathcal{E}_1) \to C^\infty(\mathcal{E}_2) \) is said to be support non-increasing if \( \text{supp}(DS) \subset \text{supp}(S) \) for every section \( S \in \Gamma S \).

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Theorem A.3 (Peetre, 1960) Consider two vector bundles \( \pi_1 : E_1 \to M \) and \( \pi_2 : E_2 \to M \) and a non-increasing operator \( r \). Then \( D : C^\infty(E_1) \to C^\infty(E_2) \). For every compact \( K \in \text{supp} D \), there is an integer \( r(K) \) such that if the jets \( j^r(S_1) = j^r(S_2) \), then \( DS_1 = DS_2 \).

A tensor field of \((p,q)\)-type \( T \) is a special type of section of the bundle \( T^{(p,q)}J^k_0(M) \). To define it properly, it is necessary to introduce a connection \( \hat{\mathcal{H}} \) on \( J^k_0(M) \) (see for instance [21] and [26]). Given a connection \( \hat{\mathcal{H}} \) on \( J^k_0(M) \), one can define the notions of horizontal and vertical tensors of any order. For instance, a tensor field \( T \) of type \((2,0)\) is a section of the bundle \( T^{(2,0)}J^k_0(M) \) and is defined independently of the connection: in a local neighborhood, the sections of \( T^{(2,0)}J^k_0(M) \) are spanned (with coefficients on \( \mathcal{F}(J^k_0(M)) \)) by the tensor product of elements of the frame \( \{e_1(x), \ldots, e_n(x)\} \)

\[
T^{(2,0)}J^k_0(M) = \text{span} \left\{ e_i(x)x^i \otimes e_j(x)x^j, i, j = 1, \ldots, n(k+1) \right\}.
\]

On the other hand, a \((0,2)\)-horizontal tensor fields are locally spanned as

\[
T^{(0,2)}_h J^k_0(M) = \text{span} \left\{ \frac{\delta}{\delta x^\mu}|_{kx} \otimes \frac{\delta}{\delta x^\nu}|_{kx}, \mu, \nu = 1, \ldots, n \right\},
\]

where the horizontal local sections \( \{\frac{\delta}{\delta x^\mu}, \ldots, \frac{\delta}{\delta x^\nu}\} \) are defined by the expressions

\[
\frac{\delta}{\delta x^\mu}|_{(x,y)}(x,y) = \frac{\partial}{\partial x^\mu}|_{(x,y)} - N^A{}_{\nu}(x,y) \frac{\partial}{\partial y^A}|_{(x,y)}. \tag{A.2}
\]

The coefficients \( N^A{}_{\nu}(x,y) \) are the non-linear connection coefficients [26].

Similarly, the tensor bundle of \((2,0)\)-vertical tensors is locally spanned by the local frame

\[
T^{(2,0)}_v J^k_0(M) = \text{span} \left\{ \frac{\partial}{\partial y^A}|_{kx} \otimes \frac{\partial}{\partial y^B}|_{kx}, A, B = 1, \ldots, nk \right\}.
\]

The bundle over \( J^k_0(M) \) of \( hv \) tensors of type \((1,1)\) is locally spanned by the local frame

\[
T^{(1,1)} J^k_0(M) = \text{span} \left\{ \frac{\delta}{\delta x^\mu}|_{kx} \otimes \frac{\partial}{\partial y^A}|_{kx}, \mu = 1, \ldots, n, A = 1, \ldots, nk \right\}.
\]

One can consider an horizontal \((1,1)\) tensor, that generically will have the following expression in local coordinates

\[
T(kx) = T^i{}_j(kx) \frac{\delta}{\delta x^i}|_{kx} \otimes \delta x^j|_{kx}, \quad T^i{}_j(kx) \in \mathcal{F}(J^k_0(M)).
\]

Horizontal \( p \)-forms can be spanned locally in a similar way. For instance, a 2-form horizontal can be expressed in local coordinates as

\[
\omega(kx) = \omega_{ij}(kx) dx^i|_{kx} \wedge dx^j|_{kx}.
\]
Note that horizontal forms does not depend on the specific connection $k\mathcal{H}$ that we can choose.

The tensor product of horizontal tensors is an horizontal tensor. In a similar way, the exterior product of horizontal forms is an horizontal form. There is a notion of horizontal exterior derivative, for which definition we need a connection on $J^k_0(M)$ [18].

The following definition provides the fundamental notion of generalized higher order field,

**Definition A.4** A generalized tensor $T$ of type $(p,q)$ with values on $\mathcal{F}(J^k_0(M))$ is a smooth section of the bundle of $\mathcal{F}(M)$-linear homomorphisms

$T^{(p,q)}(M, \mathcal{F}(J^k_0(M))) := \text{Hom}(T^p M \times ...p... \times T^q M \times TM \times ...q... \times TM, \mathcal{F}(J^k_0(M))).$

A $p$-form $\omega$ with values on $\mathcal{F}(J^k_0(M))$ is a smooth section of the bundle of $\mathcal{F}(M)$-linear completely alternate homomorphisms

$\Lambda^p(M, \mathcal{F}(J^k_0(M))) := \text{Alt}(TM \times ...p... \times TM, \mathcal{F}(J^k_0(M))).$

The space of 0-forms is $\Gamma \Lambda^0(M, \mathcal{F}(J^k_0(M))) := \mathcal{F}(J^k_0(M)).$

It can be shown that this notion is equivalent to the notion of generalized higher order field as horizontal fields.

**Example A.5** A generalized electromagnetic field is a section of $\Lambda^2(M, \mathcal{F}(J^k_0(M))).$

There is a well defines Cartan’s calculus for generalized forms in a similar way as the standard Cartan calculus of smooth differential forms. The usual formulation of the inner derivation, exterior product and exterior derivative can be introduced in a coordinate free form and are independent of the connection used (for the construction and general properties of Cartan’s calculus of generalized forms, the reader can see [18]). For instance, there is a coordinate free definition of the operator $d_4$. The realization in local coordinates of the operator $d_4$ is straightforward: if $\phi$ is a generalized $k$-form, its exterior derivative is

\begin{align*}
\ d_4\phi &= \ d_4(\phi_i(x(s), \dot{x}, \ddot{x}, ..., x^{(k)})d_4x^i) \\
&= d(\phi_i(x(s), \dot{x}, \ddot{x}, ..., x^{(k)})) \wedge d_4x^i \\
&= \partial_j \phi_i(x(s), \dot{x}, \ddot{x}, ..., x^{(k)})d_4x^j \wedge d_4x^i.
\end{align*}

This operator is nil-potent, $(d_4)^2 = 0$. There is a definition of integration of differential forms which is diffeomorphic invariant and a corresponding Stoke’s theorem.

### B Re-parameterization invariance of the proper-time associated with the metric of maximal acceleration

The general theory of generalized metrics is considered in [18]. In this paper, we consider the particular case of a metric of maximal acceleration [18], showing that
the associated proper-time is re-parameterization invariant. The proper time of the
form \(3.5\) is
\[
\tau^{(2)}_x[t] = \int_{t_0}^{t} \sqrt{-g(x'(t), x'(t))} \, dt.
\]
The parameter \(t \in I \subset \mathbb{R}\) is arbitrary. Thus the natural question is if the proper-
time \(\tau^{(2)}_x[t]\) is invariant under an arbitrary re-parameterization
\[\phi : I \to \tilde{I}, \, t \mapsto \phi(t) = s.\] (B.1)
The natural way to check this is through the definition of gene-
eralized metric and some of its fundamental properties. A convenient definition of generalized metric is
the following (see [18], in particular subsection (2.6)):

**Definition B.1** A generalized metric is a section \(\bar{g} \in \Gamma T^{(0,2)}(M, \mathcal{F}(J^k_0(M)))\) such
that the following condition holds:

1. \(\bar{g}\) is smooth in the sense that for all \(X_1, X_2\) smooth vector fields along the
curve \(x : I \to M\), the function \(\bar{g}(X_1, X_2)\) is smooth except when it takes the
zero value.

2. It is homogeneous of degree zero: if \(kx : I \to M\) has local coordinates \((x^\mu(s), \dot{x}^\mu(s), \ddot{x}^\mu(s)...)\),
then
\[
\bar{g}(x^\mu(s), \lambda_1 \dot{x}^\mu(s), \lambda_1^2 \ddot{x}^\mu(s) + \lambda_2 \dddot{x}^\mu(s) + 3\lambda_2 \lambda_1 \dddot{x} + \lambda_1^3 \dddot{x}^i,...)(X, X) =
\bar{g}(x^\mu(s), \dot{x}^\mu(s), \dddot{x}^\mu(s), ... x^{(k)}(s))(X, X)
\]
for all \(X \in T_x M\) and \(\lambda_i > 0, i = 1,..., k\).

3. \(\bar{g}\) is symmetric, in the sense that
\[
\bar{g}(X_1, X_2) = \bar{g}(X_2, X_1)
\]
for all \(X_1, X_2\) smooth vector fields along the curve \(x : I \to M\).

4. It is bilinear,
\[
\bar{g}(X_1 + f(kx)X_2, X_3) = \bar{g}(X_1, X_3) + f(kx)\bar{g}(X_2, X_3),
\]
for all \(X_1, X_2, X_3\) arbitrary smooth vector fields along \(x : I \to M\) and \(f \in \mathcal{F}(M)\).

5. It is non-degenerate, in the sense that if \(\bar{g}(X, Z) = 0\) for all \(Z\) smooth along
\(x : I \to M\), then \(X = 0\).
The metric of maximal acceleration $g$ determines a section of $\Gamma T^{(0,2)}(M, \mathcal{F}(J^2_0(M)))$, with connection components

$$g_{\mu\nu}^{(2)}(x) = \left(1 - \frac{\eta(\hat{x}, \hat{x})}{A_{\text{max}}^2}\right) \eta_{\mu\nu}(x), \quad \mu, \nu = 1, \ldots, n.$$ 

The homogeneity property of the Definition 3.1 when applied to the metric of the maximal acceleration reads,

$$\hat{g}(x^\mu(s), \hat{x}^\mu(s), \hat{x}^\mu(s))(X, X) = g(x^\mu(s), \lambda_1 \hat{x}^\mu(s), \lambda_{21} \hat{x}^\mu(s) + \lambda_2 \hat{x}^\mu(s))(X, X).$$

where the $^{(2)}\hat{x}$ is the second jet of the map $\hat{x} : \tilde{I} \to M, \ s \to \hat{x}(s), \ X \in T_{\hat{x}(s)}M$ and $\lambda_1, \lambda_{21}, \lambda_2$ are arbitrary real functions along $\hat{x} : \tilde{I} \to M$. In particular we have that by the bilinearity property on $X$,

$$g(\hat{x}^\mu(s), \hat{x}^\mu(s), \hat{x}^\mu(s))(\lambda_1 X, \lambda_1 X) = (\lambda_1)^2 g(\hat{x}^\mu(s), \lambda_1 \hat{x}^\mu(s), \lambda_{21} \hat{x}^\mu(s) + \lambda_2 \hat{x}^\mu(s))(X, X).$$

In short,

$$g(\hat{x}^\mu(s), \hat{x}^\mu(s), \hat{x}^\mu(s))(\lambda_1 X, \lambda_1 X) = (\lambda_1)^2 g(\hat{x}^\mu(s), \lambda_1 \hat{x}^\mu(s), \lambda_{21} \hat{x}^\mu(s) + \lambda_2 \hat{x}^\mu(s))(X, X).$$

(B.2)

Now the property of re-parameterization invariance of the proper-time of the metric of maximal acceleration $g \in \Gamma T^{(0,2)}(M, \mathcal{F}(J^2_0(M)))$ follows directly,

**Theorem B.2** The proper-time (3.7) is invariant under re-parameterizations.

**Proof.** Let us consider two different curves $x : I \to M$ and $\tilde{x} : \tilde{I} \to M$, related by re-parameterization $\phi : I \to \tilde{I}$. Then we have that

$$\tau^{(2)}[x] = \int_{t_0}^t \sqrt{-g(\hat{x}(t), \hat{x}(t))} \, dt$$

$$= \int_{t_0}^t \sqrt{-\left(1 - \frac{\eta(\hat{x}, \hat{x})}{A_{\text{max}}^2}\right) \eta(\hat{x}(t), \hat{x}(t))} \, dt$$

$$= \int_{s_0}^s \sqrt{-\left(1 - \frac{\eta(\hat{x}, \hat{x})}{A_{\text{max}}^2}\right) \eta(\lambda x^\mu(s), \lambda x^\mu(s)) |\lambda|^{-1}} \, ds = *,$$

where $^{(2)}\hat{x} = (\hat{x}^\mu(s), \lambda(\hat{x})^\mu(s), \lambda^2(\hat{x}')^\mu(s) + \lambda \lambda'(\hat{x}'')^\mu(s))$, $\hat{x}'' = \lambda^2(\hat{x}'')^\mu(s) + \lambda \lambda'(\hat{x}'')^\mu(s)$ and $\lambda = \frac{d\phi}{dt}$. Using the homogeneity condition for the generalized metric $g$,

$$* = \int_{s_0}^s \sqrt{-\left(1 - \frac{\eta(\hat{x}, \hat{x})}{A_{\text{max}}^2}\right) \eta(\lambda x^\mu(s), \lambda x^\mu(s)) |\lambda|^{-1}} \, ds$$

$$= \int_{s_0}^s \sqrt{-\left(1 - \frac{\eta(\hat{x}, \hat{x})}{A_{\text{max}}^2}\right) \eta(x^\mu(s), x^\mu(s))} \, ds$$

$$= \tau^{(2)}[\hat{x}].$$

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