VARIATIONAL DESTRUCTION OF INVARIANT CIRCLES

Lin Wang

Abstract. We construct a sequence of generating functions \((h_n)_{n \in \mathbb{N}}\), arbitrarily close to an integrable system in the \(C^r\) topology with \(r < 4\) for \(n\) large enough. With the variational method, we prove that for a given rotation number \(\omega\) and \(n\) large enough, the exact monotone area-preserving twist maps generated by \((h_n)_{n \in \mathbb{N}}\) admit no invariant circles with rotation number \(\omega\).

Key words. invariant circles, minimal configuration, Peierls’s barrier

AMS subject classifications (2000). 58F27, 58F05, 58F30, 58F11

1. Introduction

M. Herman constructed an example in [H2]. With geometrical method he proved the example has the property that the invariant circle (i.e. a homotopically non-trivial invariant curve) with a given rotation number can be destructed by an arbitrarily small perturbation in the \(C^{3-\epsilon}\) topology for the exact monotone area-preserving twist map. The KAM theory (see [S]) implies the persistence of invariant circle with a Diophantine rotation number under the small perturbation in the \(C^{3+\epsilon}\) topology. Hence, Herman’s result is optimal. However, the perturbation in the example in [H2] is too artificial. In this paper, following the ideas and techniques developed by J.N. Mather in the series of papers [M1], [M2], [M3] and [M4], we construct an example with a more natural perturbation to achieve the same goal. More precisely, our example has the property as follow:

Property: For a given rotation number \(\omega\) and \(n\) large enough, there exists a sequence of generating functions \((h_n)_{n \in \mathbb{N}}\), arbitrarily close to an integrable system in the \(C^r\) topology with \(r < 4\) such that the exact monotone area-preserving twist maps generated by \((h_n)_{n \in \mathbb{N}}\) admit no invariant circles with rotation number \(\omega\).

In [M1], [M2], [M3] and [M4], Mather introduced a notion called Peierls’s barrier as follows

\[ P^h_\omega(\xi) = \min_{x_0=\xi} \sum_I (h(x_i, x_{i+1}) - h(x_i^-, x_{i+1}^-)), \]

where \(I = \mathbb{Z}\), if \(\omega\) is not a rational number, \(I = \{0, ..., q-1\}\), if \(\omega = p/q\), and \((x_i)_{i \in I} \in \prod_{i \in I} [x_i^-, x_i^+]\) satisfying \(x_0 = \xi\). Moreover, he proved that \(P^h_\omega(\xi)\) is a non-negative Lipschitz function with respect to the variable \(\xi \in \mathbb{R}\) with the modulus of continuity with respect to \(\omega\) and found a criterion of the existence of invariant circles. Namely, the exact area-preserving monotone twist map generated by \(h\) admits an invariant circle if and only if \(P^h_\omega(\xi) \equiv 0\) for every \(\xi \in \mathbb{R}\).

For our example, the modulus of continuity can be improved due to the hyperbolicity of the perturbation, which follows from similar ideas of [F]. More precisely, (Lemma 5.1 below) if \(\omega\) is suitable small, then

\[ |P^h_{\omega_n}(\xi) - P^h_{\omega_n}(\xi)| \leq C \exp(-n^\delta), \]
where $\delta$ is a small positive constant independent of $n$. Based on the improvement, we obtain that there exists $\xi \in \mathbb{R}$ such that $P^{h_n}_\omega(\xi) \neq 0$ for $n$ large enough. It follows that $h_n$ doesn’t admit any invariant circles for $n$ large enough.

2. Preliminaries

2.1. Minimal configuration

Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) be an exact area-preserving monotone twist map and $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a generating function for the lift $F$ of $f$ to $\mathbb{R}^2$, namely $F$ is generated by the following equations

$$\begin{align*}
y &= -\partial_1 h(x, x'), \\
y' &= \partial_2 h(x, x'),
\end{align*}$$

where $F(x, y) = (x', y')$. The lift $F$ gives rise to a dynamical system whose orbits are given by the images of points of $\mathbb{R}^2$ under the successive iterates of $F$. The orbit of the point $(x_0, y_0)$ is the bi-infinite sequence

$$\ldots, (x_{-k}, y_{-k}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k), \ldots,$$

where $(x_k, y_k) = F(x_{k-1}, y_{k-1})$. The sequence

$$\ldots, x_{-k}, \ldots, x_{-1}, x_0, x_1, \ldots, x_k, \ldots$$

denoted by $(x_i)_{i \in \mathbb{Z}}$ is called stationary configuration which stratifies the identity

$$\partial_1 h(x_i, x_{i+1}) + \partial_2 h(x_{i-1}, x_i) = 0, \text{ for every } i \in \mathbb{Z}.$$ 

Given a sequence of points $(z_i, \ldots, z_j)$, we can associate its action

$$h(z_i, \ldots, z_j) = \sum_{i \leq s < j} h(z_s, z_{s+1}).$$

A configuration $(x_i)_{i \in \mathbb{Z}}$ is called minimal if for any $i < j \in \mathbb{Z}$, the segment of $(x_i, \ldots, x_j)$ minimizes $h(z_i, \ldots, z_j)$ among all segments $(z_i, \ldots, z_j)$ of the configuration satisfying $z_i = x_i$ and $z_j = x_j$. It is easy to see that every minimal configuration is a stationary configuration. By [B], minimal configurations satisfy a group of remarkable properties as follows:

- Two distinct minimal configurations cross at most once, which is so called Aubry’s crossing lemma.

- For every minimal configuration $x = (x_i)_{i \in \mathbb{Z}}$, the limit

$$\rho(x) = \lim_{n \to \infty} \frac{x_{i+n} - x_i}{n}$$

exists and doesn’t depend on $i \in \mathbb{Z}$. $\rho(x)$ is called the rotation number of $x$.

- For every $\omega \in \mathbb{R}$, there exists a minimal configuration with rotation number $\omega$. Following the notations of [B], the set of all minimal configurations with rotation number $\omega$ is denoted by $M^h_{\omega}$, which can be endowed with the topology induced from the product topology on $\mathbb{R}^\mathbb{Z}$. If $x = (x_i)_{i \in \mathbb{Z}}$ is a minimal configuration, considering the projection $pr : M^h_{\omega} \to \mathbb{R}$ defined by $pr(x) = x_0$, we set $A^h_{\omega} = pr(M^h_{\omega})$. 


By [M3], modulus of continuity with respect to \( \omega \) is defined as follows:

\[
\omega \text{ to } t \text{urbation implies the exponential approximation from } P \text{ will be provided in Section 5.}
\]

if the Peierls’s barrier \( P \) map generated by \( h \) of existence of invariant circle. Namely, the exact area-preserving monotone twist follows

\[
\text{• A} \cdot h \text{• } \xi \text{• A} \cdot h \text{• A} \cdot h \text{• A} \cdot h \text{• A} \cdot h \text{• A} \cdot h \text{• A} \cdot h \text{• A} \cdot h \text{• A} \cdot h
\]

If \( \omega \in \mathbb{Q} \), say \( \omega = p/q \) (in lowest terms), then it is convenient to define the rotation symbol to detect the structure of \( M^h_{p/q} \). If \( x \) is a minimal configuration with rotation number \( p/q \), then the rotation symbol \( \sigma(x) \) of \( x \) is defined as follows

\[
\sigma(x) = \begin{cases} 
    p/q+, & \text{if } x_{i+q} > x_i + p \text{ for all } i, \\
    p/q, & \text{if } x_{i+q} = x_i + p \text{ for all } i, \\
    p/q-, & \text{if } x_{i+q} < x_i + p \text{ for all } i.
\end{cases}
\]

Moreover, we set

\[
M^h_{p/q+} = \{ x \ a \text{ is minimal configuration with rotation symbol } p/q \text{ or } p/q+ \},
\]

\[
M^h_{p/q-} = \{ x \ a \text{ is minimal configuration with rotation symbol } p/q \text{ or } p/q- \},
\]

then both \( M^h_{p/q+} \) and \( M^h_{p/q-} \) are totally ordered. Namely, every two configurations in each of them do not cross. We denote \( pr(M^h_{p/q+}) \) and \( pr(M^h_{p/q-}) \) by \( A^h_{p/q+} \) and \( A^h_{p/q-} \) respectively.

If \( \omega \in \mathbb{R}\setminus\mathbb{Q} \) and \( x \) is a minimal configuration with rotation number \( \omega \), then \( \sigma(x) = \omega \) and \( M^h_\omega \) is totally ordered.

\( A_\omega^h \) is a closed subset of \( \mathbb{R} \) for every rotation symbol \( \omega \).

### 2.2. Peierls’s barrier

In [M3], Mather introduced the notion of Peierls’s barrier and gave a criterion of existence of invariant circle. Namely, the exact area-preserving monotone twist map generated by \( h \) admits an invariant circle with rotation number \( \omega \) if and only if the Peierls’s barrier \( P^h_\omega(\xi) \) vanishes identically for all \( \xi \in \mathbb{R} \). The Peierls’s barrier is defined as follows:

- If \( \xi \in A^h_\omega \), we set \( P^h_\omega(\xi) = 0 \).
- If \( \xi \in A^h_\omega \), since \( A^h_\omega \) is a closed set in \( \mathbb{R} \), then \( \xi \) is contained in some complementary interval \((\xi^-, \xi^+)\) of \( A^h_\omega \) in \( \mathbb{R} \). By the definition of \( A^h_\omega \), there exist minimal configurations with rotation symbol \( \omega \), \( x^- = (x_i^-)_{i \in \mathbb{Z}} \) and \( x^+ = (x_i^+)_{i \in \mathbb{Z}} \) satisfying \( x_0^- = \xi^- \) and \( x_0^+ = \xi^+ \). For every configuration \( x = (x_i)_{i \in \mathbb{Z}} \) satisfying \( x_i^- \leq x_i \leq x_i^+ \), we set

\[
G_\omega(x) = \sum_{i \in I} (h(x_i, x_{i+1}) - h(x_i^-, x_{i+1}^-)),
\]

where \( I = \mathbb{Z} \), if \( \omega \) is not a rational number, and \( I = \{0, ..., q-1\} \), if \( \omega = p/q \).

\( P^h_\omega(\xi) \) is defined as the minimum of \( G_\omega(x) \) over the configurations \( x \in \Pi = \prod_{i \in I} [x_i^-, x_i^+] \) satisfying \( x_0 = \xi \). Namely

\[
P^h_\omega(\xi) = \min\{G_\omega(x) \mid x \in \Pi \text{ and } x_0 = \xi \}.
\]

By [M3], \( P^h_\omega(\xi) \) is a non-negative periodic function of the variable \( \xi \in \mathbb{R} \) with the modulus of continuity with respect to \( \omega \).

To our example (see Section 3), the modulus of continuity of \( P^h_\omega(\xi) \) with respect to \( \omega \) can be improved significantly. Loosely speaking, the hyperbolicity of the perturbation implies the exponential approximation from \( P^h_\omega(\xi) \) to \( P^h_0(\xi) \). The details will be provided in Section 5.
3. Construction of the generating functions

Consider a completely integrable system with the generating function
\[ h_0(x, x') = \frac{1}{2} (x - x')^2 \quad x, x' \in \mathbb{R}. \]

We construct the perturbation consisting of two parts. The first one is
\[ u_n(x) = \frac{1}{na} (1 - \cos(2\pi x)) \quad x \in \mathbb{R}, \]
where \( n \in \mathbb{N} \) and \( a \) is a positive constant independent of \( n \). The second one is a non-negative function \( v_n(x) \) satisfying
\[
\begin{align*}
&v_n(x + 1) = v_n(x), \\
&\text{supp} v_n \cap [0, 1] \subset \left[ \frac{1}{n} - \frac{1}{n^a}, \frac{1}{n} + \frac{1}{n^a} \right], \\
&\max v_n = n^{-s}, \\
&||v_n||_{C^k} = O(n^{-s'}),
\end{align*}
\]
where we require \( s' > a \). It is enough to take \( s = (k + 2)a \) for achieving that. The generating function of the nearly integrable system is constructed as follow:
\[ h_n(x, x') = h_0(x, x') + u_n(x') + v_n(x'), \]
where \( n \in \mathbb{N} \). Moreover, we have the following theorem.

**Theorem 3.1** For \( \omega \in \mathbb{R} \setminus \mathbb{Q} \) and \( n \) large enough, the exact area-preserving monotone twist map generated by \( h_n \) does not admit any invariant circles with the rotation number satisfying
\[ |\omega| < n^{-\frac{2}{a} - \delta}, \]
where \( \delta \) is a small positive constant independent of \( n \).

We will prove Theorem 3.1 in the following sections. First of all, based on the theorem, we verify that our example has the property aforementioned in Section 1.

If \( \omega \in \mathbb{Q} \), then the invariant circles with rotation number \( \omega \) could be easily destructed even though the perturbation is \( C^\infty \) close to 0. Therefore it suffices to consider the irrational \( \omega \). The case with a given irrational rotation number can be easily reduced to the one with a small enough rotation number. More precisely,

**Lemma 3.2** Let \( h_P \) be a generating function as follow
\[ h_P(x, x') = h_0(x, x') + P(x'), \]
where \( P \) is a periodic function of periodic 1. Let \( Q(x) = q^{-2} P(qx), q \in \mathbb{N} \), then the exact area-preserving monotone twist map generated by \( h_Q(x, x') = h_0(x, x') + Q(x') \) admits an invariant circle with rotation number \( \omega \in \mathbb{R} \setminus \mathbb{Q} \) if and only if the exact area-preserving monotone twist map generated by \( h_P \) admits an invariant circle with rotation number \( q\omega - p, p \in \mathbb{Z} \).
We omit the proof and for more details, see [H2]. For the sake of simplicity of notations, we denote \( Q_n \) by \( Q_n \) and the same to \( u_{q_n}, v_{q_n} \) and \( h_{q_n} \). Let

\[
Q_n(x) = q_n^{-2}(u_n(q_n x) + v_n(q_n x)),
\]

where \((q_n)_{n \in \mathbb{N}}\) is a sequence satisfying Dirichlet approximation

\[
|q_n \omega - p_n| < \frac{1}{q_n},
\]

where \( p_n \in \mathbb{Z} \) and \( q_n \in \mathbb{N} \). Since \( \omega \in \mathbb{R} \setminus \mathbb{Q} \), we say \( q_n \to \infty \) as \( n \to \infty \). Let \( \tilde{h}_n(x, x') = h_0(x, x') + Q_n(x') \), we have

**Corollary 3.3** For a given rotation number \( \omega \in \mathbb{R} \setminus \mathbb{Q} \) and every \( \varepsilon \), there exists \( N \) such that for \( n > N \), the exact area-preserving monotone map generated by \( \tilde{h}_n \) admits no invariant circle with rotation number \( \omega \) and

\[
||\tilde{h}_n - h_0||_{C^{4-\delta'}} < \varepsilon,
\]

where \( \delta' \) is a small positive constant independent of \( n \).

**Proof** Based on Theorem 3.1 and Dirichlet approximation (3.4), it suffices to take

\[
\frac{1}{q_n} \leq \frac{1}{q_n^{4+\delta}},
\]

which implies

\[
a \leq 2 - 2\delta.
\]

From (3.1), (3.2) and (3.3), it follows that

\[
||\tilde{h}_n(x, x') - h_0(x, x')||_{C^r} = ||Q_n(x')||_{C^r},
\]

\[
\leq q_n^{-2}(||u_n(q_n x')||_{C^r} + ||v_n(q_n x')||_{C^r}),
\]

\[
\leq q_n^{-2}(q_n^{-a}(2\pi)^r q_n r + C_1 q_n^{-s'} q_n r),
\]

\[
\leq C_2 q_n^{r-a-2},
\]

where \( C_1, C_2 \) are positive constants only depending on \( r \).

To complete the proof, it is enough to make \( r - a - 2 < 0 \), which together with (3.5) implies

\[
r < a + 2 \leq 4 - 2\delta.
\]

We set \( \delta' = 2\delta \), then the proof of Corollary 3.3 is completed. \( \square \)

The following sections are devoted to prove Theorem 3.1. For simplicity, we don’t distinguish the constant \( C \) in following different estimate formulas.

4. **Estimate of lower bound of** \( P_{h_n}^{h_0} \)

In this section, we will estimate the lower bound of \( P_{h_n}^{h_0} \) at the given point. To achieve that, we need to estimate the distances of pairwise adjacent elements of the minimal configuration.
4.1. A spacing lemma

**Lemma 4.1** Let \((x_i)_{i \in \mathbb{Z}}\) be a minimal configuration of \(\bar{h}_n\) with rotation symbol \(0^+\), then
\[
x_{i+1} - x_i = O(n^{-\frac{a}{2}}), \quad \text{for} \quad x_i \in \left[\frac{1}{4}, \frac{3}{4}\right],
\]
where \(\bar{h}_n(x_i, x_{i+1}) = h_0(x_i, x_{i+1}) + u_n(x_{i+1})\).

**Proof** Without loss of generality, we assume \(x_i \in [0, 1]\) for all \(i \in \mathbb{Z}\). By Aubry’s crossing lemma, we have
\[
0 < ... < x_{i-1} < x_i < x_{i+1} < ... < 1.
\]
We consider the configuration \((\xi_i)_{i \in \mathbb{Z}}\) defined by
\[
\xi_j = \begin{cases} x_j, & j < i, \\ x_{j+1}, & j \geq i. \end{cases}
\]
Since \((x_i)_{i \in \mathbb{Z}}\) is minimal, we have
\[
\sum_{i \in \mathbb{Z}} \bar{h}_n(\xi_i, \xi_{i+1}) - \sum_{i \in \mathbb{Z}} \bar{h}_n(x_i, x_{i+1}) \geq 0.
\]
By the definitions of \(\bar{h}_n\) and \((\xi_i)_{i \in \mathbb{Z}}\), we have
\[
0 \leq \sum_{i \in \mathbb{Z}} \bar{h}_n(\xi_i, \xi_{i+1}) - \sum_{i \in \mathbb{Z}} \bar{h}_n(x_i, x_{i+1})
= \bar{h}_n(x_{i-1}, x_{i+1}) - \bar{h}_n(x_i, x_i) - \bar{h}_n(x_i, x_{i+1})
= (x_{i+1} - x_i)(x_i - x_{i-1}) - u_n(x_i).
\]
Moreover,
\[
u_n(x_i) \leq (x_{i+1} - x_i)(x_i - x_{i-1}) \leq \frac{1}{4}(x_{i+1} - x_{i-1})^2.
\]
Therefore,
\[
x_{i+1} - x_{i-1} \geq 2\sqrt{u_n(x_i)}.
\]
For \(x_i \in \left[\frac{1}{4}, \frac{3}{4}\right]\), \(u_n(x_i) \geq n^{-a}\), hence,
\[
x_{i+1} - x_{i-1} \geq 2n^{-\frac{a}{2}}.
\]
(4.1)

On the other hand, we consider another configuration \((\eta_i)_{i \in \mathbb{Z}}\) defined by
\[
\eta_j = \begin{cases} x_{j+1}, & j < i, \\ \frac{1}{2}(x_j + x_{j+1}), & j = i, \\ x_j, & j > i. \end{cases}
\]
Based on the minimality of \((x_i)_{i \in \mathbb{Z}}\), following the deduction as similar as that above, we have
\[
x_{i+1} - x_i \leq 2\sqrt{u_n(\eta_i)} \leq 2\sqrt{2}n^{-\frac{a}{2}}.
\]
Since \((x_i)_{i \in \mathbb{Z}}\) is a stationary configuration, we have
\[
x_{i+1} - x_i = -\partial_1 \bar{h}_n(x_i, x_{i+1}),
= \partial_2 \bar{h}_n(x_i, x_i),
= x_i - x_{i-1} + u'_n(x_i).
\]
Since \( u_n(x) = \frac{2\pi}{n} \sin(2\pi x) \), it follows from (3.1) that
\[
x_{i+1} - x_i \geq C n^{-\frac{3}{2}}.
\]
Therefore, we have
\[
x_{i+1} - x_i = O(n^{-\frac{3}{2}}), \quad x_i \in \left[ \frac{1}{4}, \frac{3}{4} \right].
\]
The proof of Lemma 4.1 is completed.

4.2. The lower bound of \( P_{0^+}^{h_n} \)

By the definition of \( v_n \), \( \text{supp} \ v_n \cap [0, 1] \subset \left[ \frac{1}{2} - \frac{1}{n^2}, \frac{1}{2} + \frac{1}{n^2} \right] \) and \( v_n(x + 1) = v_n(x) \).
Let \( (x_i)_{i \in \mathbb{Z}} \) be the minimal configuration of \( h_n(x_i, x_{i+1}) = h_0(x_i, x_{i+1}) + u_n(x_{i+1}) \) with rotation symbol \( 0^+ \) satisfying \( x_0 = \frac{1}{2} - \frac{1}{n^2} \), then
\[
(x_i)_{i \in \mathbb{Z}} \cap \text{supp} v_n = \emptyset.
\]
Moreover, for all \( i \in \mathbb{Z} \),
\[
v_n(x_i) = 0.
\]
Let \( (\xi_i)_{i \in \mathbb{Z}} \) be a minimal configuration of \( h_n \) defined by (3.3) with rotation symbol \( 0^+ \) satisfying \( \xi_0 = \eta \), where \( \eta \) satisfies \( v_n(\eta) = \max v_n(x) = n^{-s} \), then
\[
\sum_{i \in \mathbb{Z}} (h_n(\xi_i, \xi_{i+1}) - h_n(\xi_{i}^+, \xi_{i+1}^-))
\geq v_n(\eta) + \sum_{i \in \mathbb{Z}} h_0(x_i, x_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1})
\geq v_n(\eta) + \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1})
= v_n(\eta) - \sum_{i \in \mathbb{Z}} v_n(x_{i+1})
= v_n(\eta).
\]
Therefore,
\[
P_{0^+}^{h_n}(\eta) = \min_{x_0 = \eta} \sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(x_i^-, x_{i+1}^-)) \geq v_n(\eta) = n^{-s}.
\]
We conclude that there exists a point \( \xi \in \left[ \frac{1}{2} - \frac{1}{n^2}, \frac{1}{2} + \frac{1}{n^2} \right] \) such that
\[
(4.2)
P_{0^+}^{h_n}(\xi) \geq n^{-s}.
\]

5. The approximation from \( P_{0^+}^{h_n} \) to \( P_{\omega}^{h_n} \)

In this section, we will prove the improvement of modulus of continuity of Peierls’s barrier based on the hyperbolicity of \( h_n \). Namely

**Lemma 5.1** For every irrational rotation symbol \( \omega \) satisfying \( 0 < \omega < n^{-\frac{3}{2} - \delta} \), we have
\[
|P_{\omega}^{h_n}(\xi) - P_{0^+}^{h_n}(\xi)| \leq C \exp(-n^\delta).
\]
where \( \xi \in \left[ \frac{1}{2} - \frac{1}{n^2}, \frac{1}{2} + \frac{1}{n^2} \right] \) and \( \delta \) is a small positive constant independent of \( n \).
5.1. Some counting lemmas

To prove the lemma, we need to do some preliminary work. First of all, we count the number of the elements of a minimal configuration \((x_i)_{i \in \mathbb{Z}}\) with arbitrary rotation symbol \(\omega\) in a given interval. With the method of [F], we can conclude the following lemma.

**Lemma 5.2** Let \((x_i)_{i \in \mathbb{Z}}\) be a minimal configuration of \(h_n\) with rotation symbol \(\omega\),
\[
J_n = \left[ \exp \left( -n^\frac{\delta}{2} \right), \frac{1}{2} \right], \quad \Lambda_n = \{i \in \mathbb{Z} | x_i \in J_n\},
\]
then
\[
\sharp \Lambda_n \leq C n^\frac{\alpha}{2} + \frac{\delta}{2},
\]
where \(\sharp \Lambda_n\) denotes the number of elements in \(\Lambda_n\) and \(\delta\) is a small positive constant independent of \(n\).

**Proof** Let \(x^+ = 1 - \exp \left( -n^\frac{\delta}{2} \right)\), \(x^- = \frac{1}{2}\) and \(\sigma = \left( \frac{x^+}{x^+ - x^-} \right)^\frac{1}{N}\), hence,
\[
\ln \sigma = \frac{\ln(x^+) - \ln(x^-)}{N}.
\]

We choose \(N \in \mathbb{N}\) such that \(1 \leq \ln \sigma \leq 2\), then \(N = O \left( n^\frac{\delta}{4} \right)\).

We consider the partition of the interval \(J_n = [x^-, x^+]\) into the subintervals \(J_n^k = [\sigma^k x^-, \sigma^{k+1} x^-]\) where \(0 \leq k < N\). Hence, \(J_n = \bigcup_{k=0}^{N-1} J_n^k\). We set \(S_k = \{i \in \Lambda_n | (x_{i-1}, x_{i+1}) \subset J_n^k\}\) and \(m_k = \sharp S_k\).

By the similar deduction as the one in Lemma 4.1, we have
\[
x_{i+1} - x_{i-1} \geq 2 \sqrt{u_n(x_i) + v_n(x_i)} \geq C n^{-\frac{\delta}{4}} x_i, \quad \text{for } x_i \in \left[ 0, \frac{1}{2} \right].
\]

For simplicity of notation, we write \(C n^{-\frac{\delta}{4}}\) by \(\alpha_n\).

If there exists \(k\) such that \(i \in S_k\) for \((x_i)_{i \in \mathbb{Z}}\), then \(x_{i+1} - x_{i-1} \geq \alpha_n \sigma^k x^-\), moreover,
\[
m_k \alpha_n \sigma^k x^- \leq 2 \mathcal{L}(J_n^k) = 2(\sigma - 1) \sigma^k x^-,
\]
where \(\mathcal{L}(J_n^k)\) denotes the length of the interval of \(J_n^k\). Hence \(m_k \leq 2(\sigma - 1) \alpha_n^{-1}\).

On the other hand, if \(i \notin S_k\) for any \(k\), then there exists \(l\) satisfying \(1 \leq l < N\) such that
\[
x_{i-1} < \sigma^l x^- < x_{i+1}.
\]
Hence,
\[
\sharp \{i \in \Lambda_n | i \notin S_k \text{ for any } k\} \leq 2N.
\]

Therefore,
\[
\sharp(\Lambda_n) \leq 2N(\sigma - 1) \alpha_n^{-1} + 2N.
\]

Since \(1 \leq \ln \sigma \leq 2\) and \(N = O \left( n^\frac{\delta}{4} \right)\), then we have
\[
\sharp \Lambda_n \leq C n^\frac{\alpha}{2} + \frac{\delta}{2}.
\]

The proof of Lemma 5.2 is completed.
Remark 5.3 Let \((x_i)_{i \in \mathbb{Z}}\) be a minimal configuration of \(h_n\) defined by (5.3) with rotation symbol \(\omega\). An argument as similar as the one in Lemma 5.2 implies that
\[
\| \begin{bmatrix} i \in \mathbb{Z} \mid x_i \in \left[\exp\left(-n\frac{\omega}{2}\right), 1 - \exp\left(-n\frac{\omega}{2}\right)\right] \end{bmatrix} \| \leq Cn^\frac{\omega}{2} + \frac{\omega}{4}.
\]

Second, it is easy to count the number of the elements of a minimal configuration with irrational rotation symbol. More precisely, we have the following lemma.

Lemma 5.4 Let \((x_i)_{i \in \mathbb{Z}}\) be a minimal configuration with rotation number \(\omega \in \mathbb{R}\setminus \mathbb{Q}\). Then for every interval \(I_k\) of length \(k\), \(k \in \mathbb{N}\),
\[
\frac{k}{\omega} - 1 \leq \| \{ i \in \mathbb{Z} \mid x_i \in I_k \} \| \leq \frac{k}{\omega} + 1.
\]

Proof For every minimal configuration \((x_i)_{i \in \mathbb{Z}}\) with rotation number \(\omega\), there exists an orientation-preserving lift \(\phi \) such that \(\rho(\Phi) = \omega\), where \(\Phi : \mathbb{R} \to \mathbb{R}\) denotes the lift of \(\phi\). Since \(\omega \in \mathbb{R}\setminus \mathbb{Q}\), thanks to [H1], \(\phi\) has a unique invariant probability measure \(\mu\) on \(\mathbb{T}\) such that \(\mu[x, \Phi(x)] = \omega\) for every \(x \in \mathbb{R}\). In particular,
\[
\mu[x, x+1] = \omega, \quad \text{for every } i \in \mathbb{Z}.
\]
From \(\mu(I_k) = k\), it follows that
\[
\omega(\| \{ i \in \mathbb{Z} \mid x_i \in I_k \} \| - 1) \leq k,
\omega(\| \{ i \in \mathbb{Z} \mid x_i \in I_k \} \| + 1) \geq k,
\]
which completes the proof of Lemma 5.4.

Based on Lemma 5.2 and Lemma 5.4, if \(0 < \omega < \frac{n}{2} - \delta\) and \(\omega\) is irrational, then
\[
\| \{ i \in \mathbb{Z} \mid x_i \in [0, 1] \} \| \geq \frac{1}{\omega} - 1 \geq C_1n^\frac{\omega}{2} + \delta > C_2n^\frac{\omega}{2} + \frac{\omega}{4}.
\]
So far, we have proved the following conclusion.

Lemma 5.5 Let \((x_i)_{i \in \mathbb{Z}}\) be a minimal configuration of \(h_n\) defined by (5.3) with rotation symbol \(0 < \omega < \frac{n}{2} - \delta\), then there exists \(j^-, j^+ \in \mathbb{Z}\) such that
\[
0 < x_{j^-} - 1 < x_{j^-} < x_{j^+} < x_{j^+ + 1} < 1.
\]
Without loss of generality, we assume that
\[
j^+ - j^- \geq Cn^\frac{\omega}{2} + \frac{\omega}{4}.
\]

If \(\xi \in \mathcal{A}_{\omega}^{h_n}\), then \(P_{\omega}^{h_n}(\xi) = 0\). Hence, it suffices to consider the case with \(\xi \notin \mathcal{A}_{\omega}^{h_n}\) for destruction of invariant circles. Let \((\xi^-, \xi^+)\) be the complementary interval of \(\mathcal{A}_{\omega}^{h_n}\) in \(\mathbb{R}\) and contains \(\xi\). Let \(\xi^\pm = (\xi^\pm_i)_{i \in \mathbb{Z}}\) be the minimal configurations with rotation symbol \(\omega\) satisfying \(\xi^\pm_0 = \xi^\pm\) and let \((\xi_i)_{i \in \mathbb{Z}}\) be a minimal configuration of \(h_n\) with rotation symbol \(\omega\) satisfying \(\xi_0 = \xi\) and \(\xi^-_i \leq \xi_i \leq \xi^+_i\). By the definition of Peierls barrier, we have
\[
P_{\omega}^{h_n}(\xi) = \sum_{i \in \mathbb{Z}} (h_n(\xi_i, \xi_{i+1}) - h_n(\xi^-_i, \xi^+_i)).
Since $P^h_n(\xi)$ is 1-periodic with respect to $\xi$, without loss of generality, we assume that $\xi \in [0, 1]$. We set $d(x) = \min\{|x|, |x - 1|\}$ and write $\exp(-n^{\frac{1}{2}})$ by $\epsilon(n)$. By Lemma 5.5 there exist $i^-, i^+$ such that

\begin{equation}
    d(\xi^-) < \epsilon(n) \quad \text{and} \quad \xi^-_{i+1} - \xi^-_{i-1} \leq \epsilon(n) \quad \text{for} \quad i = i^-, i^+.
\end{equation}

Thanks to Aubry’s crossing lemma, we have $\xi^-_i \leq \xi_i \leq \xi^+_i \leq \xi^-_{i+1}$. Hence,

\begin{equation}
    \xi_i - \xi^-_i \leq \epsilon(n) \quad \text{for} \quad i = i^-, i^+.
\end{equation}

5.2. Proof of Lemma 5.1

In the following, we will prove Lemma 5.1 with the method similar to the one developed by Mather in [M3]. The proof can be proceeded in the following two steps.

5.2.1. Step 1

We consider the number of the elements in a segment of the configuration as the length of the segment. In the first step, we approximate $P^h_n(\xi)$ for $\xi \in \left[\frac{1}{2} - \frac{1}{\pi r}, \frac{1}{2} + \frac{1}{\pi r}\right]$ by the difference of the actions of the segments of length $i^+ - i^- + 1$. To achieve that, we define the following configurations

\begin{align*}
    x_i &= \begin{cases}
    \xi_i, & i \neq i^-, i^+; \\
    \xi^-_i, & i = i^-; i^+,
    \end{cases}
    \quad \text{and} \quad
    y_i &= \begin{cases}
    \xi_i, & i^- < i < i^+; \\
    \xi^-_i, & i \leq i^-; i \geq i^+.
    \end{cases}
\end{align*}

It is easy to see that $\xi_0 = \xi$ is contained both of $(x_i)_{i \in \mathbb{Z}}$ and $(y_i)_{i \in \mathbb{Z}}$ up to the rearrangement of the index $i$ since $\xi \in \left[\frac{1}{2} - \frac{1}{\pi r}, \frac{1}{2} + \frac{1}{\pi r}\right]$. Hence, by the minimality of $(\xi_i)_{i \in \mathbb{Z}}$ satisfying $\xi_0 = \xi$, we have

\begin{equation}
    P^h_n(\xi) \leq \sum_{i \in \mathbb{Z}} (h_n(y_i, y_{i+1}) - h_n(\xi^-_i, \xi^-_{i+1})).
\end{equation}

Since $\omega$ is irrational, then $(x_i)_{i \in \mathbb{Z}}$ is asymptotic to $(\xi^-_i)_{i \in \mathbb{Z}}$, which together with the minimality of $(\xi^-_i)_{i \in \mathbb{Z}}$ yields

\begin{equation}
    \sum_{i \in \mathbb{Z}} (h_n(y_i, y_{i+1}) - h_n(\xi^-_i, \xi^-_{i+1})) \leq \sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(\xi^-_i, \xi^-_{i+1})).
\end{equation}

We set

\[ h(x_1, ..., x_j) = \sum_{i \leq s < j} h(x_s, x_{s+1}), \]

then

\[ \sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(\xi_i, \xi_{i+1})) = \sum_{i = i^-, i^+} (h_n(\xi_{i-1}, \xi^-_i, \xi^+_{i+1}) - h_n(\xi_{i-1}, \xi_i, \xi_{i+1})). \]
By the construction of \(v_n\) and Lemma 5.5 we have \(v_n(\xi_i^-) = 0\). It follows that

\[
\begin{align*}
   h_n(\xi_i^-, \xi_i^+ - \xi_i^-) - h_n(\xi_i^-, \xi_i^- - \xi_i^-) &= h_n(\xi_i^-, \xi_i^- - \xi_i^-) - h_n(\xi_i^-, \xi_i^- - \xi_i^-), \\
   &= h_n(\xi_i^-, \xi_i^- - \xi_i^-) - h_n(\xi_i^-, \xi_i^- - \xi_i^-), \\
   &= 4(\xi_i^- - \xi_i^-)(\xi_i^- - \xi_i^- + \xi_i^- - \xi_i^-) + u_n(\xi_i^-) - u_n(\xi_i^-), \\
   &\leq 4\epsilon(\xi_i^- - \xi_i^-)^2 + 2\pi \epsilon(n) + u_n(\eta)(\xi_i^- - \xi_i^-), \\
   &\leq C\epsilon(n)^2,
\end{align*}
\]

where \(\eta \in (\xi_i^-, \xi_i^-)\). It is similar to obtain

\[
\begin{align*}
   h_n(\xi_i^+, \xi_i^+ + \xi_i^+ - \xi_i^+) - h_n(\xi_i^+, \xi_i^+ + \xi_i^+ - \xi_i^+) &\leq C\epsilon(n)^2.
\end{align*}
\]

Hence,

\[
\sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(\xi_i, \xi_{i+1})) \leq C\epsilon(n)^2.
\]

Moreover,

\[
\sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(\xi_i^-, \xi_{i+1}^-)) \\
= \sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(\xi_i^-, \xi_{i+1}^-) + h_n(\xi_i, \xi_{i+1}) - h_n(\xi_i^-, \xi_{i+1}^-)), \\
= \sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(\xi_i, \xi_{i+1})) + P^{h_n}_\omega(\xi), \\
\leq P^{h_n}_\omega(\xi) + C\epsilon(n)^2.
\]

Therefore, it follows from (5.3) and (5.4) that

\[
P^{h_n}_\omega(\xi) \leq \sum_{i \in \mathbb{Z}} (h_n(y_i, y_{i+1}) - h_n(\xi_i^-, \xi_i^+)) \leq P^{h_n}_\omega(\xi) + C\epsilon(n)^2,
\]

where

\[
\sum_{i \in \mathbb{Z}} (h_n(y_i, y_{i+1}) - h_n(\xi_i^-, \xi_i^+)) = h_n(y_i^-, y_{i+1}^-) - h_n(\xi_i^-, y_{i+1}^-).
\]

5.2.2. Step 2

It follows from [M4] that the Peierls’s barrier \(P^{h_n}_{0+}(\xi)\) could be defined as follows

\[
P^{h_n}_{0+}(\xi) = \min_{\eta_i \in \mathbb{Z}} h_n(\eta_i, \eta_{i+1}) - \min_{\eta_i \in \mathbb{Z}} h_n(\eta_i, \eta_{i+1}),
\]

where \((\eta_i)_{i \in \mathbb{Z}}\) and \((z_i)_{i \in \mathbb{Z}}\) are monotone increasing configurations limiting on 0, 1. We set

\[
\begin{align*}
   K(\xi) &= \min_{\eta_i \in \mathbb{Z}} h_n(\eta_i, \eta_{i+1}), \\
   K &= \min_{\eta_i \in \mathbb{Z}} h_n(\eta_i, \eta_{i+1}).
\end{align*}
\]
First of all, it is easy to see that $K(\xi)$ and $K$ are bounded. Second, $P_{0^+}^{h_n}(\xi) = 0$ for $\xi = 0$ or 1. Hence, we only need to consider the case with $\xi \in (0, 1)$. Following the ideas of [M6], let $\xi^-$ and $\xi^+$ be minimal configurations of rotation symbol $0^+$ and let $(\xi^-_n, \xi^+_n)$ be the complementary interval of $A_{0^+}^{h_n}$ and contains $\xi$. Based on the definition

$$P_{0^+}^{h_n}(\xi) = \min_{x_0 = \xi} \{G_{0^+}(x)|\xi^-_i \leq x_i \leq \xi^+_i\},$$

where

$$G_{0^+}(x) = \sum_{i \in \mathbb{Z}} (h_n(x_i, x_{i+1}) - h_n(\xi^-_i, \xi^+_{i+1})) = -K + \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1}),$$

the proof of [M7] will be completed when we verify that the configuration $(x_i)_{i \in \mathbb{Z}}$ achieving the minimum in the definition of $K(\xi)$ satisfies $\xi^-_i \leq x_i \leq \xi^+_i$. It can be easily obtained by Aubry’s crossing lemma. In fact, since $(\xi^-_i)_{i \in \mathbb{Z}}$ and $(x_i)_{i \in \mathbb{Z}}$ are minimal and both are $\alpha$-asymptotic to 0 as well as $\omega$-asymptotic to 1, by Aubry’s crossing lemma, $(\xi^-_i)_{i \in \mathbb{Z}}$ and $(x_i)_{i \in \mathbb{Z}}$ do not cross. Similarly $(\xi^+_i)_{i \in \mathbb{Z}}$ and $(x_i)_{i \in \mathbb{Z}}$ do not cross. It follows from $x_0 \in (\xi^-_0, \xi^+_0)$ that $(x_i)_{i \in \mathbb{Z}}$ achieving the minimum in the definition of $K(\xi)$ satisfies $\xi^-_i \leq x_i \leq \xi^+_i$.

In the following, we will compare $K$, $K(\xi)$ with $h_n(\xi^-_i, ..., \xi^+_i)$, $h_n(y_i, ..., y_i)$ respectively.

First, we consider $K$ and $h_n(\xi^-_i, ..., \xi^+_i)$. Let $(z_i)_{i \in \mathbb{Z}}$ be a monotone increasing configuration limiting on 0, 1 such that $K = \sum_{i \in \mathbb{Z}} h_n(z_i, z_{i+1})$. By Lemma 5.2

$$\sharp \{i \in \mathbb{Z}|(z_i)_{i \in \mathbb{Z}} \cap [\epsilon(n), 1 - \epsilon(n)]\} \leq Cn^{\frac{3}{2}+\frac{3}{2}}.$$

On the other hand, since $(z_i)_{i \in \mathbb{Z}}$ has the rotation number $0^+$, then from [M1], it follows that up to the rearrangement of the index $i$, there exists a subset of length $i^+ - i^-$ of $(z_i)_{i \in \mathbb{Z}}$, denoted by $\{z_i, z_{i+1}, ..., z_{i^-}, z_{i^+}\}$ such that

$$z_{i^-} \leq \epsilon(n), \quad z_{i^+} \geq \epsilon(n).$$

We consider the configuration $(\bar{x}_i)_{i \in \mathbb{Z}}$ defined by

$$\begin{cases}
\bar{x}_i = \xi^-_i, & i^- < i < i^+, \\
\bar{x}_i = 0, & i \leq i^-, \\
\bar{x}_i = 1, & i \geq i^+.
\end{cases}$$

By the definition of $h_n$, $h_n(\bar{x}_i, \bar{x}_{i+1}) = 0$ for $i < i^-$ or $i \geq i^+$, then

$$\sum_{i \in \mathbb{Z}} h_n(\bar{x}_i, \bar{x}_{i+1}) = h_n(\bar{x}_{i^-}, ..., \bar{x}_{i^+}).$$

Moreover, by the minimality of $(z_i)_{i \in \mathbb{Z}}$, we have

$$K \leq \sum_{i \in \mathbb{Z}} h_n(\bar{x}_i, \bar{x}_{i+1}) = h_n(\bar{x}_{i^-}, ..., \bar{x}_{i^+}).$$
By the construction of $h_n$, we have $v_n(x_{i-1}^+) = v_n(x_{i+1}^-) = 0$. Hence,

$$h_n(x_{i-1}, x_{i+1}) - h_n(x_{i-1}, x_{i+1}) = \frac{1}{2}(x_{i-1} - x_{i+1})^2 + u_n(x_{i-1}) - \frac{1}{2}(x_{i-1} - x_{i+1})^2 - u_n(x_{i-1}),$$

(5.9)

$$= \frac{1}{2}(x_{i-1}^+)^2 - \frac{1}{2}(x_{i-1}^+ - x_{i-1})^2,$$

$$= \frac{1}{2}x_{i-1}^+ (2x_{i-1}^+ - x_{i-1})^2,$$

$$\leq C\varepsilon(n)^2.$$

It is similar to obtain

(5.10)  

$$h_n(x_{i+1}, x_{i+1}) - h_n(x_{i+1}, x_{i+1}) \leq C\varepsilon(n)^2.$$

Since

$$h_n(x_{i-1}, ..., x_{i+1}) - h_n(x_{i-1}, ..., x_{i+1}) = h_n(x_{i-1}, x_{i+1}) + h_n(x_{i+1}, x_{i}) - h_n(x_{i-1}, x_{i}),$$

then

(5.11)  

$$h_n(x_{i-1}, ..., x_{i+1}) - h_n(x_{i-1}, ..., x_{i+1}) \leq C\varepsilon(n)^2.$$

From (5.8) and (5.11) we have

(5.12)  

$$K \leq h_n(x_{i-1}, ..., x_{i+1}) + C\varepsilon(n)^2.$$

To obtain the reverse inequality of (5.12), we consider the configuration as follows

$$\begin{cases}
\tilde{x}_i = z_i, & i^- < i < i^+, \\
\tilde{x}_i = 0, & i^- \leq i^-, \\
\tilde{x}_i = 1, & i^+ \geq i^+.
\end{cases}$$

From the definition of $h_n$, it follows that $v_n(z_{i-1}^+) = 0$ and $h_n(z_i, z_{i+1}) \geq 0$ for all $i \in \mathbb{Z}$. Moreover, we have

$$h_n(x_{i-1}, ..., x_{i+1}) - K = h_n(x_{i-1}, x_{i+1}) + h_n(x_{i+1}, x_{i}) - \sum_{i < i^- \leq i^+} h_n(z_i, z_{i+1}),$$

$$\leq \frac{1}{2}(z_{i-1})^2 + u_n(z_{i-1}) + \frac{1}{2}(z_{i+1} - 1)^2,$$

$$\leq u_n'(\eta) z_{i-1} + C_1\varepsilon(n)^2,$$

$$\leq 2\pi n^{-a} \sin(2\pi\eta) z_{i-1} + C_1\varepsilon(n)^2,$$

$$\leq C_2 n^{-a} (z_{i-1})^2 + C_1\varepsilon(n)^2,$$

$$\leq C\varepsilon(n)^2.$$

where $\eta \in (0, z_{i-1}^+)$. Namely

(5.13)  

$$h_n(x_{i-1}, ..., x_{i+1}) \leq K + C\varepsilon(n)^2.$$
Furthermore, we consider the finite segment of the configuration defined by
\[
\begin{cases}
\eta_i = x_i, & i^- < i < i^+ , \\
\eta_i = \xi_i^-, & i = i^- , \\
\eta_i = \xi_i^+, & i = i^+ .
\end{cases}
\]

Then, the minimality of \((\xi_i^-)_{i \in \mathbb{Z}}\) implies \(h_n(\xi_i^-, ..., \xi_i^+) \leq h_n(\eta_i^-, ..., \eta_i^+)\). Hence, by (5.13), we have
\[
h_n(\xi_i^-, ..., \xi_i^+) \leq K + C\epsilon(n)^2 + h_n(\eta_i^-, ..., \eta_i^+) - h_n(x_i^-, ..., x_i^+),
\]
where
\[
h_n(\eta_i^-, ..., \eta_i^+) - h_n(x_i^-, ..., x_i^+)
= h_n(\eta_i^-, \eta_i^{-+1}) - h_n(x_i^-, x_i^{-+1}) + h_n(\eta_i^{+-1}, \eta_i^+) - h_n(x_i^{+-1}, x_i^+).
\]

By the deduction as similar as (5.9), we have
\[
h_n(\eta_i^-, \eta_i^{-+1}) - h_n(x_i^-, x_i^{-+1}) \leq C\epsilon(n)^2,
\]
\[
h_n(\eta_i^{+-1}, \eta_i^+) - h_n(x_i^{+-1}, x_i^+) \leq C\epsilon(n)^2.
\]

Moreover,
\[
h_n(\eta_i^-, ..., \eta_i^+) - h_n(x_i^-, ..., x_i^+) \leq C\epsilon(n)^2.
\]

Hence, from (5.14) and (5.15), it follows that
\[
h_n(\xi_i^-, ..., \xi_i^+) \leq K + C\epsilon(n)^2,
\]
which together with (5.12) and (5.16) implies
\[
|h_n(\xi_i^-, ..., \xi_i^+) - K| \leq C\epsilon(n)^2.
\]

Next, we compare \(h_n(y_i^-, ..., y_i^+)\) with \(K(\xi)\).

Since \((\xi_i)_{i \in \mathbb{Z}}\) is minimal among all configurations with rotation symbol \(\omega\) satisfying \(\xi_0 = \xi\). By (5.2) and Aubry’s crossing lemma, we have
\[
d(\xi_i) \leq \epsilon(n), \quad \text{for } i = i^-, i^+ ,
\]
where \(d(\xi_i) = \min\{| \xi_i |, | \xi_i - 1 | \}\). By an argument as similar as the one in the comparison between \(K\) and \(h_n(\xi_i^-, ..., \xi_i^+)\), we have
\[
|h_n(y_i^-, ..., y_i^+) - K(\xi)| \leq C\epsilon(n)^2.
\]

By the construction of \((y_i)_{i \in \mathbb{Z}}\), namely
\[
y_i = \begin{cases}
\xi_i, & i^- < i < i^+ , \\
\xi_i^-, & i \leq i^- , i \geq i^+ .
\end{cases}
\]

we have
\[
\begin{align*}
h_n(y_i^-, ..., y_i^+) - h_n(\xi_i^-, ..., \xi_i^+)
&= h_n(\xi_i^-, \xi_i^{-+1}) - h_n(\xi_i^-, \xi_i^{-+1}) + h_n(\xi_i^{+-1}, \xi_i^+) - h_n(\xi_i^{+-1}, \xi_i^+).
\end{align*}
\]
By the deduction as similar as (5.11), we have

\[ |h_n(y_{i-}, \ldots, y_{i+}) - h_n(\xi_{i-}, \ldots, \xi_{i+})| \leq C \epsilon(n)^2. \]  

Finally, from (5.6), (5.17), (5.18) and (5.19) we obtain

\[
|P h_n \omega(\xi) - P h_n(0) + (\xi)| \leq |h_n(y_{i-}, \ldots, y_{i+}) - h_n(\xi_{i-}, \ldots, \xi_{i+}) + K - K(\xi)| + C_1 \epsilon(n)^2,
\]

\[
\leq |h_n(\xi_{i-}, \ldots, \xi_{i+}) - K(\xi)| + |h_n(\xi_{i-}, \ldots, \xi_{i+}) - K|
\]

\[
+ |h_n(y_{i-}, \ldots, y_{i+}) - h_n(\xi_{i-}, \ldots, \xi_{i+})| + C_1 \epsilon(n)^2,
\]

\[
\leq C \epsilon(n)^2,
\]

\[
= C \exp(-n^\delta),
\]

which completes the proof of Lemma 5.1. \(\square\)

6. Proof of Theorem 3.1

Based on the preparation above, it is easy to prove Theorem 3.1. We assume that there exists an invariant circle with rotation number \(0 < \omega < n^{-\frac{a}{2} - \delta}\) for \(h_n\), then \(P _{\omega} h_n(\xi) \equiv 0\) for every \(\xi \in \mathbb{R}\). By Lemma 5.1 we have

\[
|P _{0+} h_n(\xi)| \leq C \exp(-n^\delta), \quad \text{for} \quad \xi \in \left[\frac{1}{2} - \frac{1}{n^a}, \frac{1}{2} + \frac{1}{n^a}\right].
\]

On the other hand, (4.2) implies that there exists a point \(\tilde{\xi} \in \left[\frac{1}{2} - \frac{1}{n^a}, \frac{1}{2} + \frac{1}{n^a}\right]\) such that

\[ P _{0+} h_n(\tilde{\xi}) \geq n^{-s}. \]

Hence, we have

\[ n^{-s} \leq C \exp(-n^\delta). \]

It is an obvious contradiction for \(n\) large enough. Therefore, there exists no invariant circle with rotation number \(0 < \omega < n^{-\frac{a}{2} - \delta}\).

For \(-n^{-\frac{a}{2} - \delta} < \omega < 0\), by comparing \(P _{\omega} h_n(\xi)\) with \(P _{0} h_n(\xi)\), the proof is similar. We omit the details. Therefore, the proof of Theorem 3.1 is completed. \(\square\)

Acknowledgement The author would like to thank Prof. C-Q.Cheng, W.Cheng for many helpful discussions, and X-J. Cui for his careful proofreading. This work is under the support of the National Basic Research Programme of China (973 Programme, 2007CB814800) and Basic Research Programme of Jiangsu Province, China (BK2008013).

References

[AD] S.Aubry and P.Y.Le Daeron. The discrete Frenkel-Kontorova model and its extensions I: exact results for the ground states. Phys. Rev. D 8 (1983), 381-422.

[B] V.Bangert. Mather sets for twist maps and geodesics on tori. Dynamics Reported 1 (1988), 1-45.

[F] G.Forni. Analytic destruction of invariant circles. Ergod. Th. & Dynam. Sys. 14 (1994), 267-298.
[H1] M.R.Herman. *Sur la conjugation différentiable des difféomorphismes du cercle à des rotations*. Publ. Math. IHES 49 (1979), 5-233.

[H2] M.R.Herman. *Sur les courbes invariantes par les difféomorphismes de l’anneau*. Astérisque 103-104 (1983), 1-221.

[M1] J.N.Mather. *Existence of quasi periodic orbits for twist homeomorphisms of the annulus*. Topology 21 (1982), 457-467.

[M2] J.N.Mather. *A criterion for the non-existence of invariant circle*. Publ. Math. IHES 63 (1986), 301-309.

[M3] J.N.Mather. *Modulus of continuity for Peierls’s barrier*. Periodic Solutions of Hamiltonian Systems and Related Topics. ed. P.H.Rabinowitz et al. NATO ASI Series C 209. Reidel: Dordrecht, (1987), 177-202.

[M4] J.N.Mather. *Destruction of invariant circles*. Ergod. Th. & Dynam. Sys. 8 (1988), 199-214.

[S] D.Salamon. *The Kolmogorov-Arnold-Moser theorem*. Math. Phys. Eletron. J. 10 (2004), 3-37.

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, CHINA.

E-mail address: linwang.math@gmail.com