Fourth Painlevé and Ermakov equations: quantum invariants and new exactly-solvable time-dependent Hamiltonians

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Abstract
In this work, we introduce a new realization of exactly-solvable time-dependent Hamiltonians based on the solutions of the fourth Painlevé and the Ermakov equations. The latter is achieved by introducing a shape-invariant condition between an unknown quantum invariant and a set of third-order intertwining operators with time-dependent coefficients. New quantum invariants are constructed after adding a deformation term to the well-known quantum invariant of the parametric oscillator. Such a deformation depends explicitly on time through solutions of the Ermakov equation, a property that simultaneously ensures the regularity of the new time-dependent potentials at each time. The fourth Painlevé equation appears after introducing an appropriate reparametrization of the spatial coordinate and the time parameter, where the parameters of the fourth Painlevé equation dictate the spectral information of the quantum invariant. In this form, the eigenfunctions of the third-order ladder operators lead to several sequences of solutions to the Schrödinger equation, which are determined in terms of the solutions of the Riccati equation, Okamoto polynomials, and nonlinear bound states of the derivative nonlinear Schrödinger equation. Remarkably, it is noticed that the solutions in terms of the nonlinear bound states lead to a quantum invariant with equidistant eigenvalues, which contains both a finite-dimensional and an infinite-dimensional

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sequences of eigenfunctions. The resulting family of time-dependent Hamiltonians is such that, to the authors’ knowledge, have been unnoticed in the literature of stationary and nonstationary systems.

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(Some figures may appear in colour only in the online journal)

1. Introduction

During the last several decades, physicists have realized the importance of nonlinear equations in the study of physical systems, even in cases where the dynamical laws are governed by linear equations. In quantum mechanics, the nonlinear Riccati equation [1, 2] has played a fundamental role in the construction and study of new exactly-solvable models, for it relates the factorization method [3–8] with the Darboux transformation [9, 10]. The latter can be formulated in the so-called supersymmetric quantum mechanics (SUSYQM) [11–14] after noticing that the factorization operators together with the former and new Hamiltonians define a set of intertwining relationships. These factorization operators define the so-called super charges in the supersymmetric formulation of quantum mechanics of Witten [15–18] for models in potential theory. In this regard, systems with spectrum on-demand are obtained either by adding energy level not present in the original model, or by removing levels through the Darboux–Crum transformation [19]. This formalism has led to outstanding progress in the study of quantum systems; it has allowed extending the families of exactly-solvable models [20, 21] beyond the most well-known ones such as the harmonic oscillator, hydrogen atom, interaction between diatomic molecules, and some few others. Furthermore, quantum mechanics in the non-Hermitian regime has been explored by generalizing the factorization method, where the solutions to the related Riccati equation are now allowed to be complex-valued functions [22]. Recently, it was found that such a complex-valued Riccati equation is intrinsically related to the Ermakov equation through a proper reparametrization [23], where the reality of the spectrum is preserved in systems with either broken and unbroken parity-time (PT) symmetry [24–26], extending the conventional systems with PT-symmetry and real spectrum [27–29]. From the latter, the nonlinear Ermakov equation [30–32] emerges naturally from the complexified Riccati one, the solutions of which ensures the regularity of the new complex-valued potentials. For more details on the applications of the Riccati and Ermakov equations in physics, see [2] and references therein.

In addition, the Painlevé transcendents $P_I$–$P_{VI}$ define a family of six nonlinear second-order differential equations determined in terms of some complex parameters, where, for arbitrary values of those parameters, the solutions cannot be expressed in terms of elementary functions $[33, 34]$. Nevertheless, for some specific values of the parameters, a seed function can be used to generate complete hierarchies of solutions through the Bäcklund transformation [35], which can be thought as a nonlinear counterpart of recurrence relations in linear systems. In particular, the fourth Painlevé equation can be taken into a Riccati equation with the appropriate choice of the parameters. We thus solve a ‘simpler’ nonlinear equation instead. Moreover, the fourth Painlevé equation has brought new results in the trend of exceptional orthogonal polynomials, where new families were discovered through the hierarchies of rational solutions constructed in terms of the generalized Okamoto [36], generalized Hermite and Yablonskii–Vorob’ev polynomials [37, 38]. The Painlevé transcendents have also found interesting applications in the study of physical models in nonlinear optics [39], quantum gravity [40], and SUSYQM [41–43], to
mention some. Interestingly, the fourth Painlevé equation arises quite naturally in third-order shape-invariant SUSYQM [42], where the parameters of the Painlevé transcendent define the eigenvalues of the new Hamiltonians. The respective intertwining operators serve at the same time as ladder operators, from where the eigenfunctions are determined. In general, the so-constructed intertwining operators are not factorizable in terms of first-order operators. Thus, the results obtained in [42] generalize those presented in [44]. It is worth to notice that higher-order ladder operators have been also studied in the context of supersymmetric (SUSY) partners for the stationary oscillator in both the Hermitian [21, 45] and non-Hermitian regimes [24].

Although a vast literature on families of solvable stationary systems is available, the time-dependent counterparts have not been widely explored. The difficulty lies in the dynamical law, the Schrödinger equation, which is defined in terms of a partial differential equation that, in general, cannot be reduced to an ordinary differential equation. Under some circumstances, we can extract information of the system through approximation techniques such as the sudden and the adiabatic approximations [46]. The latter restricts the range of applicability of the so-obtained solutions. Despite all these difficulties, time-dependent phenomena find exciting applications in physical systems such as electromagnetic traps of charged particles [47–50], plasma physics [51], and in optical-analogs under the paraxial approximation [52–54].

For time-dependent (nonstationary) models, some generalizations of the conventional Darboux transformation have been developed. For instance, Bargov and Samsonov introduced in [55, 56] a first-order differential operator that intertwines two different Schrödinger equations, one already known and the other to be constructed from a seed solution of the former one. Later, a further generalization was introduced by Cannata et al [57], where the authors consider intertwining operators of first-order in the spatial coordinate and first-order in the time parameter. In this formalism, orthogonality is no longer a property that can be taken for granted (see for instance [58]), since the method by itself does not provide essential information about the system such as the constants of motion, which have to be determined separately. Despite such a difficulty, several new families of exactly-solvable time-dependent potentials have been reported in the literature [58–60].

Among the nonstationary quantum systems, the parametric oscillator [61–64] is perhaps the most well-known model that admits a set of exact solutions. Lewis and Riesenfeld [61] addressed the problem by noticing the existence of a nonstationary eigenvalue equation associated with the appropriate constant of motion (quantum invariant) of the system in which the time dependence appears in the coefficients of the related ordinary differential equation. Such an eigenvalue equation can indeed be factorized so that the conventional Darboux transformation applies straightforwardly [65, 66], resulting in a new quantum invariant rather than a Hamiltonian. Then, the appropriate ansatz allows to determine the respective Hamiltonian and time-dependent potentials with ease [65]. The solutions, and the complex-phases introduced by Lewis–Riesenfeld, are inherited from the former system, ensuring an orthogonal set of solutions for the new system.

In this work, we exploit the quantum invariants and introduce a new approach to construct time-dependent and exactly-solvable models by adapting the shape-invariant structure to those invariants instead of the Hamiltonians (as it is usually done in conventional approaches). Remarkably, the latter allows us introducing the time dependence into the quantum invariant while preserving the eigenvalue equation structure so that the orthogonality is guaranteed by construction. It is worth to recall that, for time-dependent Hamiltonians, the Schrödinger equation reduces to an eigenvalue equations for the Hamiltonian itself only under the adiabatic approximation, as shown in [67]. Such a case will be discarded throughout this work since we look for exact solutions.
From the quantum invariants, we determine the respective time-dependent Hamiltonians, where the nonstationary eigenfunctions lead to exact solutions to the Schrödinger equation. To this end, we combine the solutions of the Ermakov and fourth Painlevé equations. This is achieved by considering a set of third-order intertwining operators in the spatial variable with time-dependent coefficients, together with an unknown quantum invariant to be determined such that a third-order shape-invariant condition is fulfilled. The invariant is constructed as a deformation of the one associated with the parametric oscillator. In this form, we generalize the construction presented in [42, 43]. The time dependence is introduced into the intertwining operators through the solutions of the Ermakov equation, which ensure the regularity of the nonstationary eigenfunctions at each time. On the other hand, with the aid of the appropriate reparametrization, the fourth Painlevé equation appears, the parameters and solutions of which determine the spectral information and the exact form of the quantum invariant. Finally, we modify the transitionless tracking algorithm [68] to determine the time-dependent Hamiltonians, and the respective solutions to the Schrödinger equation are determined with the addition of a time-dependent complex-phase.

The text is structured as follows. In section 2, we introduce the basic notions of shape-invariance for time-dependent Hamiltonians and their respective quantum invariants. Then, a couple of differential ladder operators of third-order are introduced such that an initial, and unknown, quantum invariant satisfies a higher-order shape-invariant relationship. From the latter, the explicit form of the ladder operators and the quantum invariants are determined. In section 3, with the aid of the third-order ladder operators, we determine the respective spectral information of the quantum invariant. In section 4, the time-dependent Hamiltonians associated with the quantum invariants are identified, together with the solutions to the Schrödinger equation. In section 5, we discuss the solutions of the Ermakov equation for some specific time-dependent frequency profiles. In particular, it is shown that the constant-frequency case leads to periodic potentials, whose solutions are in agreement with the Floquet theorem. Also, the appropriate limit to recover the well-known stationary results is presented. In turn, in section 6, we consider some particular solutions of the Painlevé equation obtained through solutions of the Riccati equation, in the form of rational solutions, or by solutions of another nonlinear models such as the derivative nonlinear Schrödinger equation. For completeness, in appendix A, we briefly revisit the parametric oscillator and its solutions through the approach of Lewis–Riesenfeld.

2. Time-dependent quantum invariants, third-order ladder operators and the fourth Painlevé equation

In quantum mechanics, the quantum invariants play a fundamental role in determining the exact solutions of the quantum models. For stationary systems (time-independent Hamiltonians), the Hamiltonian is one constant of motion and it leads to an eigenvalue equation in the form of a Sturm–Liouville problem. For time-dependent Hamiltonians, the Hamiltonian is no longer a constant of motion, and determining any of the invariants becomes a challenging task. A prime example is given by the parametric oscillator, where the respective quantum invariants are determined with relative ease. As pointed out in [61], the appropriate invariant operator admits a nonstationary eigenvalue equation so that its spectral information leads to solutions of the Schrödinger equation. For self-consistency, we provide a brief discussion on that matter in appendix A. It is worth to mention that, even for stationary systems, there are constants of motion other than the Hamiltonian, some of which may depend on time explicitly. The latter was noticed and exploited for the stationary singular oscillator [66] to construct new time-dependent potentials.
Before proceeding, we would like to stress the meaning of shape-invariance in SUSYQM for both stationary and nonstationary systems. It is said that two time-independent Hamiltonians $\hat{H}_{\pm}$ are shape-invariant [14, 17] if their respective potentials $V_{\pm}(x; \{c_a\})$ and $V_{\pm}(x; \{d_a\})$, with $\{c_a\}$ and $\{d_a\}$ two sets of constant parameters, are related by the condition [17] $V_{\pm}(x; \{c_a\}) = V_{\pm}(\{d_a\}) + S(\{c_a\})$, where $S(\{c_a\})$ is a function of the set of parameters $\{c_a\}$ and independent of $x$. In turn, for nonstationary systems, we define the shape-invariance considering two quantum invariants $I_{\pm}(t)$ by means of the relationship $I_{\pm}(t) = I_{\pm}(0) + f(t)$, with $f(t)$ a real constant. The latter implies that the respective time-dependent Hamiltonians $\hat{H}_{\pm}(t)$ are related as $\hat{H}_{\pm}(t) = \hat{H}_{\pm}(0) + f(t)$, with $f(t)$ an arbitrary real-valued function of time. As a consequence, the solutions of the respective Schrödinger equations differ only from a global time-dependent complex-phase of the form $e^{i\int dt'/f(t')}$, see [65] for details.

In this section, and throughout the manuscript, we construct a new family of time-dependent Hamiltonians $\hat{H}(t)$ such that the respective quantum invariants $\hat{I}(t)$ fulfill a third-order shape-invariant condition. To this end, let us first introduce $\hat{H}(t)$ as an unknown time-dependent Hamiltonian and $\hat{H}_{\pm}(t)$ as an auxiliary Hamiltonian so that both define the respective Schrödinger equations

$$i\frac{\partial}{\partial t}\psi^{(j)}(t) = \hat{H}(t)\psi^{(j)}(t), \quad \hat{H}(t) := \hat{p}^2 + V_j(\hat{x}, t), \quad j = 1, 2,$$

where, without loss of generality, the reduced Planck’s constant $\hbar$, together with the factor 1/2 and the mass constant in the kinetic energy term, have been properly reparametrized to get dimensionless quantities. The states $|\psi^{(j)}(t)\rangle$ denote the solutions to the Schrödinger equation, $V_j(\hat{x}, t)$ the potential energy operators, while $\hat{x}$ and $\hat{p}$ the canonical position and momentum operators, respectively. Equation (1) can be cast into a partial differential equation through the coordinate representation $\hat{x} \equiv x$ and $\hat{p} \equiv -i\frac{\partial}{\partial x}$ as

$$i\frac{\partial}{\partial t}\psi^{(j)}(x, t) = -\frac{\partial^2}{\partial x^2}\psi^{(j)}(x, t) + V_j(x, t)\psi^{(j)}(x, t), \quad \psi^{(j)}(x, t) := \langle x|\psi^{(j)}(t)\rangle,$$

with $j = 1, 2$, and $\psi^{(j)}(x, t)$ the wave functions. In this form, the physical system under consideration is described either by the Hamiltonian $\hat{H}(t)$ or the potential $V_j(x)$. Now, let $\hat{I}_1(t)$ and $\hat{I}_2(t)$ be the quantum invariants related to $\hat{H}_1(t)$ and $\hat{H}_2(t)$, respectively, in the following sense [61, 69]:

$$i\left[\hat{H}(t), \hat{I}_j(t)\right] + \frac{\partial \hat{I}_j(t)}{\partial t} = 0, \quad j = 1, 2.$$

The latter implies that the expectation value $\langle \psi^{(j)}(t)|\hat{I}_j(t)|\psi^{(j)}(t)\rangle$, for any state solution $|\psi^{(j)}(t)\rangle$ of the Schrödinger equation (1), is constant in time. The corresponding results hold for $\hat{I}_2(t)$. The invariant $\hat{I}_3(t)$ is an auxiliary operator, as well as $\hat{H}_2(t)$, which is useful to determine the spectral properties of $\hat{I}_1(t)$ and $\hat{H}_1(t)$, as will be discussed in section 3. Therefore, the spectral properties of the auxiliary invariant $\hat{I}_2(t)$ and the solutions related to the Hamiltonian $\hat{H}_2(t)$ are not relevant to our discussion.

From the Lewis–Riesenfeld approach [61], we conclude that equation (3) leads to the nonstationary eigenvalue equation

$$\hat{I}_j(t)\phi^{(j)}_n(x, t) = \Lambda^{(j)}_n\phi^{(j)}_n(x, t), \quad j = 1, 2,$$

with real and time-independent eigenvalues $\Lambda^{(j)}_n$, where the nonstationary eigenfunctions $\phi^{(j)}_n(x, t) := \langle x|\phi^{(j)}_n(t)\rangle$ satisfy the finite-norm condition $|\langle x|\phi^{(j)}_n(t)\phi^{(j)}_n(t)\rangle| < \infty$ to ensure the
construct the vector spaces. In general, the time-dependent Hamiltonian is already studied for the parametric oscillator \[63, 65\] and the second-order one for the nonstationary oscillator. The latter is achieved after introducing the real-valued functions such that, for a fixed order \(N\), the explicit form of \(I_j(t)\) can be determined. Some particular cases have been considered previously in the literature. For instance, the first-order case has been studied for the parametric oscillator \[63, 65\] and the second-order one for the nonstationary singular oscillator \[66, 70\]. In all those cases, the time-dependent Hamiltonians are already known and the ladder operators and quantum invariant are determined from the underlying group structure.

In this manuscript, we focus on the construction of the unknown invariant \(I_j(t)\) such that \(\{\hat{A}(t), \hat{A}^\dag(t)\}\) is a set of third-order ladder operators. In this form, the shape-invariant condition translates into the intertwining relationships

\[
I_1(t)\hat{A}^\dag(t) = \hat{A}^\dag(t) [I_1(t) + 2\lambda], \quad I_1(t)\hat{A}(t) = \hat{A}(t) [I_1(t) - 2\lambda],
\]

(6)

with \(\lambda > 0\) independent of time. In the latter, and from now on, we omit the identity operator \(1\) each time it multiplies a constant. From (6), it is clear that \(\hat{A}(t)\) and \(\hat{A}^\dag(t)\) are indeed the annihilation and creation operators, respectively, for the elements of the set \(\{\phi_n^{(1)}(x,t)\}_{n=0}^\infty\). These operators are also known as intertwining operators. Also, it follows that the action of \(\hat{A}^\dag(t)\) (\(\hat{A}(t)\)) on \(\phi_n^{(1)}(x,t)\) increases (decreases) the eigenvalue \(\lambda_n^{(1)}\) by 2\(\lambda\) units.

Furthermore, to give an explicit form to the unknown quantum invariants, we consider \(\hat{I}_j(t)\) \((j = 1, 2)\) as deformations of the quantum invariant \(\hat{I}_0(t)\) associated with the parametric oscillator. The latter is achieved after introducing the real-valued functions \(R_1(x,t)\) and \(R_2(x,t)\), henceforth called the deformation terms, such that

\[
\hat{I}_j(t) = \hat{I}_0(t) + R_j(x,t) \equiv -\sigma^2 \frac{\partial^2}{\partial x^2} + ix\sigma \frac{\partial}{\partial x} + R(x,t) + R_j(x,t), \quad j = 1, 2,
\]

(7)

with

\[
R(x,t) = \left(\frac{\sigma^2}{4} + \frac{1}{\sigma^2}\right)x^2 + \frac{i\sigma}{2}, \quad \dot{\sigma} \equiv \frac{d\sigma(t)}{dt},
\]

(8)

where \(\sigma(t)\) is solution to the Ermakov equation \(\ddot{\sigma} + 4\Omega^2(t)\sigma = 4\sigma^{-3}\), with \(\Omega^2(t)\) a time-dependent frequency term (see appendix A for a detailed discussion). The deformation terms \(R_1(x,t)\) and \(R_2(x,t)\) are determined from the shape-invariant condition (6). The existence of the term \(R_1(x,t)\) implies that \(\hat{I}_1(t)\) is different from \(\hat{I}_0(t)\), and thus the resulting Hamiltonian \(\hat{H}_1(t)\) describes a physical system other than the parametric oscillator. In particular, if \(R_1(x,t)\) either vanishes or reduces to a constant (independent of both \(x\) and \(t\)), \(\hat{I}_1(t)\) becomes shape-invariant.
with respect to \( \hat{I}_0(t) \) and no new potentials are obtained. See the definition of shape-invariance at the beginning of this section. Therefore, from now on, we discard those two cases.

For the stationary case, as discussed in [42], the third-order intertwining operators can be factorized as products of first-order operators for a reducible factorization, or as combination of first and second-order operators for irreducible factorizations [71]. Throughout the manuscript, we focus on the reducible factorization of the intertwining relationships (6). However, for the sake of completeness, in this section we discuss the more general irreducible factorization. We thus decompose the set of intertwining operators \( \{ \hat{A}(t), \hat{A}^\dagger(t) \} \) as the product of first and second-order differential operators \( \{ \hat{Q}^\dagger(t), \hat{Q}(t) \} \) and \( \{ \hat{M}(t), \hat{M}^\dagger(t) \} \), respectively, as follows:

\[
\hat{A}^\dagger(t) = \hat{Q}^\dagger(t)\hat{M}(t), \quad \hat{A} = \hat{M}^\dagger(t)\hat{Q}(t). \tag{9}
\]

The new operators give rise the additional set of intertwining relationships of the form

\[
\hat{I}_1(t)\hat{Q}^\dagger(t) = \hat{Q}^\dagger(t) \left[ \hat{I}_2(t) + 2\lambda \right], \quad \hat{I}_2(t)\hat{Q}(t) = \hat{Q}(t) \left[ \hat{I}_1(t) - 2\lambda \right], \tag{10}
\]

\[
\hat{I}_2(t)\hat{M}(t) = \hat{M}(t)\hat{I}_1(t), \quad \hat{I}_1(t)\hat{M}^\dagger(t) = \hat{M}^\dagger(t)\hat{I}_2(t), \tag{11}
\]

where in the latter we have introduced the auxiliary quantum invariant \( \hat{I}_2(t) \) as an intermediate. In contradistinction to (6), the equations (10) and (11) by themselves do not define a shape-invariant relation. Nevertheless, their combined action take us back to the shape-invariant condition (6), see figure 1 for details.

Now, given that \( \hat{Q}(t) \) and \( \hat{Q}^\dagger(t) \) are considered as first-order differential operators, we use the general form introduced in [66], that is,

\[
\hat{Q}^\dagger = \sigma \frac{\partial}{\partial x} + w(x, t), \quad \hat{Q} = -\sigma \frac{\partial}{\partial x} + w^*(x, t), \tag{12}
\]

with \( w(x, t) \) a complex-valued function, \( \sigma \equiv \sigma(t) \) given in (A-7), and \( f^* \) stands for the complex-conjugate of \( f \). In turn, the second-order differential operators \( \hat{M}(t) \) and \( \hat{M}^\dagger(t) \) are constructed as a generalization of those reported in [71] by introducing the time-dependent coefficients of
the form

\[
\dot{M} = \sigma^2 \frac{\partial^2}{\partial x^2} - 2g(x, t) \frac{\partial}{\partial x} + b(x, t),
\]

\[
\dot{M} = \sigma^2 \frac{\partial^2}{\partial x^2} + 2g^*(x, t) \frac{\partial}{\partial x} + b^*(x, t) - 2[g'(x, t)]^*,
\]

(13)

where \( b(x, t) \) and \( g(x, t) \) are complex-valued functions, and \( g'(x, t) \) stands for the partial derivative of \( g(x, t) \) with respect to \( x \).

The complex-valued functions \( w(x, t) \), \( g(x, t) \) and \( b(x, t) \) are determined from the respective intertwining relationships. For instance, \( w(x, t) \) is obtained after substituting (12) in (10), leading to

\[
w(x, t) = -i \frac{\dot{\sigma}}{2} x + W(z(x, t)), \quad z(x, t) := \frac{x}{\sigma}.
\]

(14)

Given that the solution to the Ermakov equation \( \sigma(t) \) is a nodeless function, we can guarantee that the reparametrized variable \( z(x, t) \) is non-singular for \( t \in \mathbb{R} \). In turn, \( W(z(x, t)) \) is a real-valued function that solves the Riccati equations

\[
z^2 + R_1(z) = \partial_t W + W^2, \quad z^2 + R_2(z) = -\partial_t W + W^2 - 2\lambda, \quad \partial_t \equiv \frac{\partial}{\partial t}.
\]

(15)

From (15) we also get \( R_2(z) - R_1(z) = -2\partial_t W - 2\lambda \), which resembles the conventional relationship between the potential and the super-potential of the conventional stationary SUSYQM construction [42, 43].

On the other hand, the functions \( g(x, t) \) and \( b(x, t) \) are computed after inserting (13) in (11), leading to

\[
g(x, t) = i \frac{\sigma \dot{\sigma}}{2} x + \sigma G(z(x, t)),
\]

(16)

\[
b(x, t) = i \dot{\sigma} x G(z(x, t)) - i \frac{\sigma \dot{\sigma}}{2} - \frac{\dot{\sigma}^2}{4} x^2 + B(z(x, t)),
\]

with the real-valued functions \( G(z(x, t)) \) and \( B(z(x, t)) \) determined from the nonlinear relationships

\[
B = 2G^2 + \partial_t G - (z^2 + R_2(z)) + \gamma,
\]

\[
z^2 + R_1(z) = -2\partial_t G + G^2 + \frac{(\partial_t G)^2}{2G} - \frac{(\partial_t G)^2}{4G^2} - \frac{d}{4G^2} + \gamma,
\]

(17)

\[
z^2 + R_2(z) = 2\partial_t G + G^2 - \frac{(\partial_t G)^2}{2G} - \frac{(\partial_t G)^2}{4G^2} - \frac{d}{4G^2} + \gamma,
\]

with \( \gamma \) and \( d \) constants of integration with respect to \( z(x, t) \), that is, those constants do not depend on \( x \) or \( t \). From (17) we obtain a complementary relationship of the form \( R_2(z) - R_1(z) = 4\partial_t G \) that, together with the one obtained from (15), gives

\[
W(z) = -2G(z) - \lambda z.
\]

(18)

A differential equation for \( G(z) \) can be found after substituting (18) into any of the Riccati equations in (15) and comparing with (17). The straightforward calculation leads to

\[
\partial_t^2 G = \frac{(\partial_t G)^2}{2G} + 6G^3 + 8\lambda \dot{z} G^2 + 2 \left[ \dot{z}^2 - (\gamma + \lambda) \right] G + \frac{d}{2G}.
\]

(19)
where the following reparametrizations:

\[ y = \sqrt{\lambda} z, \quad G = \frac{\sqrt{\lambda}}{2} w(y), \quad \alpha = \frac{\gamma}{\lambda} + 1, \quad \beta = \frac{2d}{\lambda^2}, \]  

allow us to rewrite (19) as the Painlevé-IV differential equation [33, 37, 38]

\[ w_{yy} = \frac{(w_y)^2}{2w} + \frac{3}{2} w^3 + 4yw^2 + 2(y^3 - \alpha)w + \frac{\beta}{w}. \]  

Solutions for the Painlevé-IV equation have been extensively studied in the literature, in particular it is known that \( w(y) \) can be determined in terms of elementary functions [33, 34].

Before finishing this section, we would like to recall that the factorization (9) allowed us to find the functions \( R_{1,2}(x, t) \), which define uniquely the respective quantum invariants \( I_{1,2}(t) \) in terms of the solutions of the fourth Painlevé equation (21). A summary of the steps followed so far is presented in the diagram of figure 1, where the time-dependent Hamiltonians \( H_{1,2}(t) \) are discussed in section 4.

3. Spectral information of \( I_1(t) \)

As pointed out in the previous section, the shape-invariant condition (6) implies that \( \hat{A}(t) \) and \( \hat{A}^\dagger(t) \) are the ladder operators for the nonstationary eigenfunctions \( \phi_{n,t}^{(1)}(x) \) of \( I_1(t) \). The latter indeed allows determining the spectral information (4), for \( j = 1 \). To this end, we first determine the zero-mode eigenfunction, which is an element in the kernel of the annihilation operator \( \mathcal{K}_A \equiv \text{Ker}(\hat{A}(t)) = \{ \phi^{(1)} \} \), with \( \hat{A}(t)\phi^{(1)} = 0 \). However, in our case, the annihilation operator under consideration is a differential third-order one, and thus \( \mathcal{K}_A \) is composed of three linearly independent zero-mode solutions, \( \mathcal{K}_A \equiv \{ \phi_{0,1}^{(1)}, \phi_{0,2}^{(1)}, \phi_{0,3}^{(1)} \} \). Nevertheless, we must verify whether the elements in \( \mathcal{K}_A \) fulfill the finite-norm condition. With the zero-modes already identified, the remaining eigenfunctions are computed from the iterated action of the creation operator \( \hat{A}^\dagger(t) \) on the zero-mode eigenfunctions, and the respective eigenvalues increase by 2\( \lambda \) at each iteration.

For convenience, in this section we consider the case for which \( \hat{M}(t)^\dagger \) factorizes as the product of two first-order operators, that is, a reducible case. Let us consider the factorization

\[ \hat{M}(t) \equiv \hat{M}_1(t)\hat{M}_2(t), \quad \hat{M}(t) = \hat{M}_2(t)\hat{M}_1(t), \]  

where \( \hat{M}_{1,2}(t) \) are first-order operators constructed in analogy to (12) as

\[ \hat{M}_1(t) := \left( \sigma \frac{\partial}{\partial x} - i \frac{\hat{y}}{2} x + W_1(z) \right), \quad \hat{M}_2(t) := \left( \sigma \frac{\partial}{\partial x} - i \frac{\hat{y}}{2} x + W_2(z) \right). \]  

The straightforward calculations show that the real-valued functions \( W_1(z) \) and \( W_2(z) \) are given by

\[ W_1 = -G + \left( \frac{G_z - \sqrt{-d}}{2G} \right), \quad W_2 = -G - \left( \frac{G_z - \sqrt{-d}}{2G} \right). \]  

From the latter result, it is clear that the factorization of \( \hat{M}(t) \) requires \( d < 0 \). Recall that \( \hat{M}^\dagger(t) \) intertwines the quantum invariant \( I_1(t) \) with \( I_2(t) \). Thus, to inspect the respective intertwining relationships fulfilled by \( M_{1,2}(t) \) we introduce a new auxiliary quantum invariant

\[ \tilde{\lambda}_n(x, t) = \lambda_n \tilde{\phi}_n(x, t), \]  

10
with $\mathcal{H} = \text{Span}\{\hat{\phi}_n\}_{n=0}^{\infty}$ the respective vector space composed with the finite-norm solutions. The spectral information of $\mathcal{H}$ is not relevant, for it just serves as an aid to solve the eigenvalue problem associated with $\hat{I}_1(t)$. For this reason, the respective Hamiltonian associated with $\mathcal{H}$ is not considered throughout the rest of the text.

The new auxiliary invariant satisfies the intertwining relationships

\[ \hat{I}_1(t)\hat{M}_1(t) = \hat{M}_1(t)\hat{I}_1(t), \quad \hat{I}_2(t)\hat{M}_2(t) = \hat{M}_2(t)\hat{I}_2(t). \]  

(26)

Given that both operators $\hat{M}_{1,2}(t)$ are of first-order, the relationships (26) are then equivalent to

\[ \hat{I}_1(t) = \hat{M}_1(t)\hat{I}_1(t) + \epsilon_1, \quad \hat{I}_2(t) = \hat{M}_2(t)\hat{I}_2(t) + \epsilon_2, \]  

(27)

where the substitution of (23) and (24) into (27) leads to

\[ \epsilon_1 = \gamma - \sqrt{-d}, \quad \epsilon_2 = \gamma + \sqrt{-d}. \]  

(28)

From (10) and (26), we can see that the first-order operators define mappings among the vector spaces $\mathcal{H}_1(t)$, $\mathcal{H}_2(t)$ and $\mathcal{H}(t)$ in the following form:

\[ \hat{M}_2(t) : \mathcal{H}_2(t) \rightarrow \mathcal{H}_1(t), \quad \hat{M}_1(t) : \mathcal{H}_1(t) \rightarrow \mathcal{H}_2(t), \]  

\[ \hat{M}_1(t) : \mathcal{H}_1(t) \rightarrow \mathcal{H}(t), \quad \hat{M}_2(t) : \mathcal{H}_2(t) \rightarrow \mathcal{H}(t). \]  

(29)

From the intertwining relationships (6) and (11) we get the respective mappings

\[ \hat{A}(t) : \mathcal{H}_1(t) \rightarrow \mathcal{H}_1(t), \quad \hat{A}(t) : \mathcal{H}_2(t) \rightarrow \mathcal{H}_2(t), \]  

\[ M(t) : \mathcal{H}_1(t) \rightarrow \mathcal{H}_2(t), \quad M(t) : \mathcal{H}_2(t) \rightarrow \mathcal{H}_1(t). \]  

(30)

Notice that $\hat{A}(t)$ and $\hat{A}(t)$ are endomorphisms in $\mathcal{H}_1(t)$, as expected since they are ladder operators.

From these mappings, we construct the elements of $\mathcal{K}_A$ (zero-modes), that is, the eigenfunctions annihilated by $\hat{A}(t)$. We thus have

\[ \hat{A}(t)\phi_{0,k}^{(1)} = \hat{M}_1(t)\hat{Q}(t)\phi_{0,k}^{(1)} = \hat{M}_1(t)\hat{M}_2(t)\hat{Q}(t)\phi_{0,k}^{(1)} = 0, \quad k = 1, 2, 3, \]  

(31)

where it is worth discussing three different cases.

- $\hat{Q}(t)\phi_{0,1}^{(1)} = 0$. Here, a first solution is determined, up to a normalization constant, by solving a trivial first-order differential equation.
- $\hat{M}_2(t)\hat{Q}(t)\phi_{0,2}^{(1)} = 0$ with $\hat{Q}(t)\phi_{0,2}^{(1)} \neq 0$. From the mappings defined by the intertwining relationship (10), it is clear that $\hat{Q}(t)\phi_{0,2}^{(1)} = \phi_{0,2}^{(2)} \in \mathcal{H}_2(t)$, with the latter being annihilated by $\hat{M}_2(t)$. Thus, in order to determine the zero-mode $\phi_{0,2}^{(1)}$, we should solve the first-order differential equation $\hat{M}_2(t)\phi_{0,2}^{(1)} = 0$, and from it we determine $\phi_{0,2}^{(1)}$, together with the respective eigenvalue, after mapping $\phi_{0,2}^{(1)}$ through $\hat{Q}(t)$ (see figure 2).
- $\hat{M}_1(t)\hat{M}_2(t)\hat{Q}(t)\phi_{0,3}^{(1)} = 0$ with $\hat{M}_2(t)\hat{Q}(t)\phi_{0,3}^{(1)} \neq 0$. In this case, the non-null element $\hat{M}_2(t)\hat{Q}(t)\phi_{0,3}^{(1)} = \phi_0 \in \mathcal{H}_1(t)$ is annihilated by $\hat{M}_1(t)$. To extract the zero-mode $\phi_{0,3}^{(1)}$, we solve the first-order differential equation $\hat{M}_1(t)\phi_{0} = 0$. Then, we take $\phi_0$ to $\mathcal{H}_1(t)$ by consecutively performing the mappings $\hat{M}_2(t)$ and $\hat{Q}(t)$ (see figure 2).
The complete procedure is summarized in the scheme depicted in figure 2. The straightforward calculations lead to

$$\phi^{(1)}_{0;1}(x,t) := N^{(1)}_{0;1} \frac{e^{i\hat{A}^{\dagger} x^2}}{\sqrt{\sigma}} e^{\int f'd' W(z')},$$

$$\phi^{(1)}_{0;2}(x,t) = N^{(1)}_{0;2} [W(z) - W_2(z)] \frac{e^{i\hat{A}^{\dagger} x^2}}{\sqrt{\sigma}} e^{-\int f'd' W_2(z')},$$

$$\phi^{(1)}_{0;3}(x,t) = N^{(1)}_{0;3} [-2\sqrt{-d} + (W(z) - W_2(z))(W_1(z) + W_2(z))] \frac{e^{i\hat{A}^{\dagger} x^2}}{\sqrt{\sigma}} e^{-\int f'd' W_1(z')},$$

with the respective eigenvalues $\Lambda^{(1)}_{0;1} = 0$, $\Lambda^{(1)}_{0;2} = \epsilon_2 + 2\lambda = \gamma + \sqrt{-d} + 2\lambda$ and $\Lambda^{(1)}_{0;3} = \epsilon_1 + 2\lambda = \gamma - \sqrt{-d} + 2\lambda$. The terms $N^{(1)}_{0;j}$ stand for the normalization factors that might depend on time. From (32), the rest of the eigenfunctions are determined from the action $[\hat{A}^{\dagger}(t)]^n$, for $n = 0, 1, \ldots$, on each element $\phi^{(1)}_{0;j}$. By doing so, we generate at most three sequences of eigenfunctions, where the eigenvalues $\Lambda^{(1)}_{0;j}$, for $j = 1, 2, 3$, increase by $2n\lambda$.

For the conventional stationary oscillator, it is well-known that the creation operator does not lead to finite-norm eigenfunctions. On the other hand, the one-step SUSY partner Hamiltonians admit a creation operator for which a finite-norm eigenfunction is achieved. In the context of SUSYQM, such an eigenfunction is the so-called missing state [21]. Thus, it is natural to look for the solutions that are annihilated by the creation operator. In the case under consideration, we have constructed $\hat{A}^{\dagger}(t)$ as a third-order differential operator, which admits three linearly independent eigenfunctions, and at least one finite-norm solution is possible. The existence of the latter implies a truncation of the sequences generated from the zero-modes $\phi^{(1)}_{0;j}$. We thus define $K_{A^{\dagger}} := \text{Ker}(\hat{A}^{\dagger}(t)) = \{\Phi^{(1)}_{0;1}, \Phi^{(1)}_{0;2}, \Phi^{(1)}_{0;3}\}$ as the set containing the finite-norm eigenfunctions of $\hat{A}^{\dagger}(t)$. If the set $K_{A^{\dagger}}$ is empty, three infinite sequences are generated (see section 6.2). In turn, if $K_{A^{\dagger}}$ contains one single element, we generate at most two infinite sequences, together with one finite-dimensional sequence (see section 6.3), which in particular could be a singlet (see section 6.1).
Following the same steps as in (32), it is straightforward to show that the eigenfunctions of $A'$ are

$$
\Phi_{0,1}(x, t) = N_{0,1} \frac{e^{i \frac{\sigma}{\sqrt{\sigma}} x^2}}{\sqrt{\sigma}} e^{i \int c' W_1(c')},
$$

$$
\Phi_{0,2}(x, t) = N_{0,2} [W_1(z) + W_2(z)] \frac{e^{i \frac{\sigma}{\sqrt{\sigma}} x^2}}{\sqrt{\sigma}} e^{i \int c' W_2(c')},
$$

$$
\Phi_{0,3}(x, t) = N_{0,3} [\epsilon_2 + 2 \lambda + (W_1(z) + W_2(z))(W_2(z) - W(z))] \frac{e^{i \frac{\sigma}{\sqrt{\sigma}} x^2}}{\sqrt{\sigma}} e^{i \int c' W_3(c')},
$$

where the respective eigenvalues are given by $N_{0,1} = \epsilon_1 = \gamma - \sqrt{-d}$, $N_{0,2} = \epsilon_2 = \gamma + \sqrt{-d}$ and $N_{0,3} = -2 \lambda$.

In general, we cannot say which solutions in (32) and (33) fulfill the finite-norm condition, since it depends on the specific solutions of the fourth Painlevé equation. However, we may get more insight by considering the possible behavior of the asymptotics. To this end, let us suppose that the real-valued functions $W_1(z)$, $W_2(z)$ and $W(z)$ are smooth, and such that they converge to a finite value for $|z| \to \infty$. Then, finite-norm solutions are achieved depending on the convergence of the exponential functions in (32) and (33). For instance, if $e^{i \int c' W_1(c')} \to 0$ for $|z| \to \infty$, then $\phi_{0,1}(x, t)$ becomes a finite-norm solution, whereas $\phi_{0,3}(x, t)$ do not. The same analysis can be extended to the rest of solutions in (32) and (33). In such a case, we may conclude that at most three out of the six solutions have a finite-norm. This indeed corresponds to the solutions discussed in sections 6.1 and 6.2.

4. New families of time-dependent Hamiltonians

So far, we have determined the family of exactly-solvable quantum invariants $I_1(t)$ related to the fourth Painlevé equation. Now, to completely identify the physical system related to $I_1(t)$, it is necessary to determine explicitly the corresponding time-dependent Hamiltonian $H(t)$. To this end, we consider an approach based on the transitionless tracking algorithm [68, 69] that relies on the nonstationary eigenfunctions of the quantum invariant, which have been already identified in section 3. Although we are only interested in $H_1(t)$, the procedure applies to $H_2(t)$ as well, but the latter will not be explored in this work.

Let us consider the Schrödinger equation in coordinate-free representation

$$
i \partial_t |\psi^{(j)}_n(t)\rangle = \hat{H}(t) |\psi^{(j)}_n(t)\rangle, \quad |\psi^{(j)}_n(t)\rangle = e^{i H |\phi^{(j)}_n(t)\rangle}, \quad j = 1, 2,$$

where the relationship between the wavefunctions $\psi^{(j)}_n(x, t) = \langle x |\psi^{(j)}_n(t)\rangle$ and the nonstationary eigenfunctions $\phi^{(j)}_n(x, t) = \langle x |\phi^{(j)}_n(t)\rangle$ is defined through a time-dependent complex-phase $\theta^{(j)}_n(t)$, as stated by Lewis and Riesenfeld$^4$ [61]. The straightforward calculation shows that $\theta^{(j)}_n(t)$ is computed from (34) through the expectation value

$$
\frac{d}{dt} \theta^{(j)}_n(t) = \langle \phi^{(j)}_n(t) | [i \partial_t - \hat{H}(t)] |\phi^{(j)}_n(t)\rangle, \quad j = 1, 2,
$$

provided that the Hamiltonian is already known. However, in our case, both the Hamiltonian and the complex-phase are unknown, and a workaround should be implemented. To this

$^4$ Although Lewis and Riesenfeld focused explicitly on the parametric oscillator, their results hold for any time-dependent Hamiltonian.
end, we consider the time-evolution operator $\hat{U}(t; t_0)$, that is, an operator that maps a solution defined at a time $t_0$ into one defined at a time $t$, $|\psi_n^{(b)}(t)\rangle = \hat{U}(t; t_0)|\psi_n^{(b)}(t_0)\rangle$. Given that the Hamiltonians $\hat{H}(t)$ and the quantum invariants $\hat{I}(t)$ are self-adjoint, it follows that the time-evolution operator is unitary and it takes the diagonal form

$$
\hat{U}(t; t_0) := \sum_{n=0}^{\infty} |\psi_n^{(b)}(t)\rangle \langle \psi_n^{(b)}(t_0)| = e^{\sum_{n=0}^{\infty} \mathbb{I}[\hat{\theta}_n^{(b)}(t) - \hat{\theta}_n^{(b)}(t_0)]} |\psi_n^{(b)}(t)\rangle \langle \psi_n^{(b)}(t_0)|,
$$

where it is worth to recall that $\langle \phi_m^{(b)}(t')|\phi_n^{(b)}(t)\rangle \neq \delta_{m,n}$, for $t' \neq t$.

Now, substituting (36) and $|\psi_n^{(b)}(t)\rangle = \hat{U}(t; t_0)|\psi_n^{(b)}(t_0)\rangle$ into (34) lead us to an expression for the Hamiltonian $\hat{H}(t)$ in terms of $\hat{U}(t; t_0)$ as

$$
\hat{H}(t) = \left[\mathbb{I}\partial_t \hat{U}(t; t_0)\right] \hat{U}^\dagger(t; t_0) = -\sum_{n=0}^{\infty} \hat{\theta}_n^{(b)}(t)|\phi_n^{(b)}(t)\rangle \langle \phi_n^{(b)}|
$$

$$
+ i \sum_{n=0}^{\infty} \left[\partial_t |\phi_n^{(b)}(t)\rangle\right] \langle \phi_n^{(b)}(t)|.
$$

From (37), the Hamiltonian $\hat{H}(t)$ is determined once the complex-phase $\theta_n^{(b)}(t)$ has been specified. Straightforward calculations show that the Hamiltonian obtained from (37) is such that $\hat{I}(t)$ is its respective quantum invariant. Such a conclusion holds true regardless of the choice of $\theta_n^{(b)}(t)$. From (37), we have to point out that $\hat{H}(t)$ is composed by the sum of a diagonal and a non-diagonal operator. Thus, in general, $\hat{H}(t)$ is not diagonalizable in $\mathcal{H}_j(t)$. Moreover, since $\hat{H}(t)$ and $\hat{I}(t)$ do not commute, a common basis that simultaneously diagonalizes both operators does not exist.

Given that $\theta_n^{(b)}(t)$ is arbitrary, we introduce it in the following convenient form:

$$
\dot{\theta}_n^{(b)}(t) \equiv \frac{d}{dt} \theta_n^{(b)}(t) = \frac{\Lambda_n^{(b)}}{\sigma^2(t)},
$$

with $\Lambda_n^{(b)}$ the eigenvalues in (4), so that the Hamiltonian takes the form

$$
\hat{H}(t) = \frac{1}{\sigma^2} \hat{I}(t) + \hat{F}(t), \quad \hat{F}(t) := \sum_{n=0}^{\infty} \left[\partial_t |\phi_n^{(b)}(t)\rangle\right] \langle \phi_n^{(b)}(t)|,
$$

where the first part of the Hamiltonian becomes proportional to the invariant operator $\hat{I}(t)$, and the factor $\sigma^{-2}(t)$ in (38) has been introduced to recover the kinetic energy term $\hat{p}^2$. The operator $\hat{F}(t)$ can be simplified by using the coordinate representation $\langle x|\partial_t |\phi_n^{(b)}(t)\rangle$, where the nonstationary eigenfunctions obtained in section 3 are all of the form $\phi_n^{(b)}(x,t) = e^{i\omega x^2/4\sigma} \sigma^{-1/2} K(z(x,t))$, with $z(x,t) = x/\sigma$, and $K(z(x,t))$ a function that depends explicitly on $z$, and implicitly on $x$ and $t$. After some calculations we obtain

$$
\partial_t |\phi_n^{(b)}(t)\rangle = \frac{i}{4} \left( \frac{\dot{\sigma}}{\sigma} + \frac{\dot{\sigma}^2}{\sigma^2} \right) \hat{x}^2 - \frac{i}{2} \frac{\dot{\sigma}}{\sigma} [\hat{x}, \hat{p}] |\phi_n^{(b)}\rangle.
$$

From the latter, together with the Ermakov equation (A-6), the time-dependent Hamiltonians take the form

$$
\hat{H}_j(t) = \hat{p}^2 + V_j(\hat{x},t), \quad V_j(x,t) = \Omega^2(t)x^2 + \frac{1}{\sigma^2} R_j(x,t), \quad j = 1, 2,
$$

13
where $\hat{x} \equiv x$ and $\hat{p} \equiv -i\partial_x$ stand for the position and momentum operators, respectively, and the time-dependent potentials $V_j(x, t)$ are written in terms of $R_j(x, t)$ given in (15). Thus, it is clear that the new time-dependent potential $V_1(x, t)$ in (41) is in general different to that of the parametric oscillator, except in the case $R_1(x, t) = c$, with $c$ a real constant. As mentioned in section 2, that case is excluded from the present discussion, since no new potential is obtained.

It is worth to remark that a different choice of $\theta_n^{(j)}(t)$ leads to a Hamiltonian that, in general, cannot be written in terms of the position and momentum operators. The physical meaning of such Hamiltonians is not clear, and that case will not be considered in the rest of this work.

Finally, the solutions to the Schrödinger equation $\psi_n^{(j)}(x, t)$, given in (34), are simplified by using the solutions to the Ermakov equation [25, 63, 65], leading to

$$\psi_n^{(j)}(x, t) = e^{i\theta_n^{(j)}(t)} \phi_n^{(j)}(x, t),$$

and for $n > 0$,

$$\theta_n^{(j)}(t) = -\Lambda_n^{(j)} \int_0^t \frac{dt'}{\sigma^2(t')} = -\Lambda_n^{(j)} \text{arctan} \left[ \frac{W_0}{2} \left( \sqrt{ac - \frac{4}{W_0^2}} + c \frac{q_1(t)}{q_2(t)} \right) \right],$$

with $a, c$ some arbitrary positive constants given in (A-7), and $W_0$ the Wronskian of two linearly independent solutions $q_{1,2}(t)$ of the linear equation (A-8).

5. Frequency profiles and solutions of the Ermakov equation

In this section, we discuss the specific form of $\sigma(t)$ by considering some particular forms of the time-dependent frequency term $\Omega(t)$ that appears in the new time-dependent potentials $V_j(x, t)$ given in (41), and in the parametric oscillator Hamiltonian (A-1). With the solutions to the Ermakov equation properly identified, we determine the reparametrization $z(x, t) = x/\sigma(t)$, and equivalently $y(x, t) = \sqrt{a}z(x, t)$, which are singular-free at each time (see section A). We have to remark that the form of $V_j(x, t)$ depend on the solutions of the Ermakov and fourth Painlevé equations. However, the latter are independent one of the other, and thus the solutions constructed in this section are valid for any solution of the fourth Painlevé equation discussed in section 6.

To exemplify our results, we consider two different time-dependent frequency profiles. First, we consider the simplest constant frequency case, for which time-dependent potentials are achieved in the most general case, and the stationary results are determined as a particular limit. On the other hand, we consider a frequency profile that changes smoothly from a constant value at $t \to -\infty$ to another different constant at $t \to \infty$. Such a profile can be seen as a regularization of the Heaviside distribution [72].

Frequency $\Omega^2(t) = 1$

In this case, two linearly independent solutions of (A-8) are given by $q_1(t) = \cos 2(t - t_0)$ and $q_2(t) = \sin 2(t - t_0)$, with $t_0 \in \mathbb{R}$ an arbitrary initial time and the Wronskian $W_0 = 2$. After some calculations we obtain

$$\sigma(t) = \left[ \frac{a + c}{2} + \sqrt{ac - 1} \sin 4(t - t_0) + \frac{a - c}{2} \cos 4(t - t_0) \right]^{1/2}, \quad (43)$$

with $a, c > 0$ such that $ac \geq 1$. Notice that, even if the frequency $\Omega(t)$ is a constant, the resulting potentials $V_{1,2}(x, t)$ are in general time-dependent, and periodic functions in time. This class of systems are usually studied under the Floquet theory, and already discussed for the parametric oscillator in [49].
Figure 3. Functions $\sigma(t)$ (solid-blue), $q_1(t)$ (dashed-red), and $q_2(t)$ (dotted-green) for the frequency profile $4\Omega^2(t) = \Omega_1 + \Omega_2 \tanh(kt)$. The parameters have been fixed to $k = 1/2$, $\Omega_1 = 15$, $\Omega_2 = 10$ and $a = 1/2$.

For $a = c = 1$, it follows that $\sigma(t) = 1$ and $z(x, t) = x$. Thus, the conventional stationary results reported in [42, 43] are recovered.

**Frequency $4\Omega^2(t) = \Omega_1 + \Omega_2 \tanh(kt)$**

In this case, we introduce the constraint $\Omega_1 > \Omega_2$ to ensure that $\Omega(t)$ is a positive function at each time. Exact solutions can be determined for any value of the parameters $k$, $\Omega_1$ and $\Omega_2$ by taking the linear differential equation (A-8) into the hypergeometric form [73]. After some calculations we obtain two linearly independent solutions

$$q_1(t) = (1 - \Xi(t))^{-\frac{i\mu}{2}} (1 + \Xi(t))^{\frac{1 - i\mu}{2}} F_1 \left( -i\mu, 1 - i\mu; \frac{1 - \Xi(t)}{2} \right),$$

$$q_2(t) = [q_1(t)]^*, \quad \mu = \frac{1}{k} \sqrt{\frac{\Omega_1 + \sqrt{\Omega_1^2 - \Omega_2^2}}{2}}, \quad r_\pm = \mu \pm \frac{\Omega_2}{2k^2\mu}, \quad \Xi(t) = \tanh(kt),$$

with $F_1(a, b; c; z)$ the hypergeometric function [72]. Given that $q_2(t) = [q_1(t)]^*$, it is trivial to realize that the respective Wronskian becomes $W_0 = q_1q_2 - \dot{q_1}q_2 = -2i kr_+$, that is, a pure imaginary constant. Thus, a real-valued solution $\sigma(t)$ is determined if $a = c$ in (A-7), leading to

$$\sigma^2(t) = 2a \Re[q_1^2(t)] + 2 \sqrt{a^2 + \frac{1}{k^2 r_+^2}} |q_1(t)|^2.$$

The behavior of $\sigma(t)$ and $q_{1,2}(t)$ is depicted in figure 3. For asymptotic times $|t| \gg 1$, the frequency function $\Omega(t)$ converges to a constant value, and the respective linear solutions $q_{1,2}(t)$ approximate to sinusoidal functions. Thus, the resulting time-dependent potential $V_j(x, t)$ behaves as a periodic function in the asymptotic limit.

**6. Solutions of the Painlevé equation**

As discussed in section 2, the solutions to the fourth Painlevé equation $w(y)$ allow us to construct the functions $R_j(x, t)$ required to determine the time-dependent potentials $V_j(x, t)$ given in (41). Also, the finite-norm condition of the zero-mode solutions discussed section 3 depends strongly on the asymptotic behavior and regularity of $w(y)$. The fourth Painlevé equation has been widely studied in the literature [33, 34, 74], and in this section we discuss some hierarchies of solutions that can be implemented in the construction of time-dependent systems. To
this end, let us recall the fourth Painlevé equation,
\[ w_{yy} = \frac{(w_y)^2}{2w} + \frac{3}{2}w^3 + 4yw^2 + 2(y^2 - \alpha)w + \frac{\beta}{w}, \]  
(46)
whose solutions \( w \equiv w(y; \alpha, \beta) \) are determined according to the values of the parameters \( \alpha \) and \( \beta \). From the latter, the time-dependent Hamiltonian \( \hat{H}_1(t) \) given in (15) is defined in terms of the time-dependent potential
\[ V_1(x, t) = \left[ \Omega^2(t) + \frac{\lambda^2 - 1}{\sigma^4} \right] x^2 - \frac{\lambda}{\sigma^2} [ \beta_x - w^2 - 2yw + 1 ] \quad \text{and} \quad y = \sqrt{\lambda z} = \sqrt{\lambda} x, \]  
(47)
and the respective zero-modes are given in section 3. Throughout the rest of this section, we consider three different hierarchies of solutions \( w(y; \alpha, \beta) \), namely the Riccati-like, rational, nonlinear solutions. Those hierarchies have been considered because the spectral information obtained in each case reveals the existence of sequences of solutions, which can be either truncated, infinite, or a combination of both. In addition, we obtain eigenvalues that are equidistant, or equidistant with intermediate gaps.

6.1. Solutions in terms of the Riccati equation

It is well-known that the solutions of the fourth Painlevé equation can be determined through a Riccati equation [33] of the form
\[ w_y = \mu w^2 + 2\mu y w - 2(1 + \alpha \mu), \quad \mu^2 = 1, \]  
(48)
provided that \( \beta = -2(1 + \alpha \mu)^2 \), with \( \alpha \in \mathbb{C} \). In this form, we have to solve (48), which can be linearized with ease [1] through the use of a logarithmic derivative as
\[ w = -\frac{1}{\mu} u, \quad u_y - 2\mu y u_y - 2\mu(1 + \alpha \mu)u = 0. \]  
(49)
For the physical case under consideration, the set of Painlevé parameters \( \{\alpha, \beta\} \) are related to the set of physical parameters \( \{\lambda, \gamma, d\} \) through the relationships given in (20). In this form, for the Riccati-like solutions of (46), we obtain the constraint \( d = -[\lambda(1 + \mu) + \mu \gamma]^2 \), where the reducible condition \( d \leq 0 \) of section 3 is automatically fulfilled. In general, the linearized equation (49) has two linearly independent solutions of the form
\[ u_1(y) = _1F_1 \left( \frac{1}{2} + \frac{\lambda + \gamma}{2\lambda}, \frac{1}{2}; \mu y^2 \right), \quad u_2(y) = \sqrt{\mu y^2} _1F_1 \left( \frac{1}{2} + \frac{\lambda + \gamma}{2\lambda}, \frac{3}{2}; \mu y^2 \right), \]  
(50)
with \(_1F_1(a, b; z)\) the confluent hypergeometric function [72].

Interestingly, from (48), one realizes that the time-dependent potential (47) reduces, for \( \mu = 1 \), to
\[ V_1(x, t) = \left[ \Omega^2(t) + \frac{\lambda^2 - 1}{\sigma^4(t)} \right] x^2 + \frac{\lambda}{\sigma^2(t)} \left( \frac{3 + \gamma}{\lambda} \right), \]  
(51)
which in the context of time-dependent systems corresponds to a shape-invariant potential of the parametric oscillator (see discussion in section 2). The latter holds for any linear combination of the solutions (50). We thus discard the case $\mu = 1$ for the rest of the text.

In turn, for $\mu = -1$, the new time-dependent potential takes the form

$$V_1(x, t) = \left[ \Omega^2(t) + \frac{\lambda^2 - 1}{\sigma^2(t)} \right] x^2 - \frac{\lambda}{\sigma^2(t)} \left[ 2\partial_x - 2\frac{\gamma}{\lambda} + 1 \right],$$

(52)

where now we have, in general, a potential different from the class of shape-invariants. From (50), we can either choose

$$u_1(y) = iF_1 \left( -\frac{\gamma}{2\lambda}, \frac{1}{2}; -y^2 \right), \quad u_2(y) = iyF_1 \left( \frac{\lambda - \gamma}{2\lambda}, \frac{3}{2}; y^2 \right),$$

(53)

or the equivalent Kummer transformations

$$u_1(y) = e^{-y^2} iF_1 \left( \frac{\gamma + \lambda}{2\lambda}, \frac{1}{2}; y^2 \right), \quad u_2(y) = ye^{-y^2} iF_1 \left( \frac{2\lambda + \gamma}{2\lambda}, \frac{3}{2}; y^2 \right),$$

(54)

as the set of linearly independent solutions. For (53), the general solution is constructed as the linear combination

$$u(y) = k_1u_1(y) + k_2u_2(y), \quad \left| \frac{k_1}{k_2} \right| > \Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{\lambda + \gamma}{2\lambda} \right) \Gamma \left( \frac{\lambda + \gamma}{2\lambda} \right),$$

(55)

where the imaginary number i has been absorbed in the constant $k_1$, and the constraint between the real constants $k_1$ and $k_2$ is determined from the asymptotic behavior of the confluent hypergeometric function to ensure the existence of a nodeless solution $u(y)$ for $y \in \mathbb{R}$, see also [75]. The latter is required to avoid singularities in the solution of the fourth Painlevé equation given in (49), and consequently the potential $V_1(x, t)$ in (52). Similar results are obtained by using the solutions (54) instead.

Additionally, the asymptotic behavior of the confluent hypergeometric function reveals that $u(y) \to \infty$ and $u(y) \to 0$ at $y \to \pm \infty$. We thus determine the finite-norm elements in $\mathcal{K}_\Lambda$ and $\mathcal{K}_{\Lambda_1}$ from (32) and (33), respectively, leading to the following spectral information

$$\phi^{(1)}_{0}(x, t) \equiv \phi^{(1)}_{0,1}(x, t) = \Phi^{(1)}_{0,1}(x, t) = \Lambda^{(1)}_{0} \frac{e^{\frac{\gamma + \lambda}{2\lambda}x^2}}{\sqrt{\sigma}} u(y), \quad \Lambda^{(1)}_{0} = 0,$$

$$\phi^{(1)}_{1}(x, t) \equiv \phi^{(1)}_{0,2}(x, t) = \Lambda^{(1)}_{1} \frac{e^{\frac{\gamma + \lambda}{2\lambda}x^2}}{\sqrt{\sigma}} \left( \frac{\mu}{u} + 2y \right) e^{-y^2/2}. \quad \Lambda^{(1)}_{1} = 2(\gamma + \lambda),$$

(56)

with $y(x, t) = \sqrt{\lambda x}/\sigma(t)$. From the latter, the zero-mode solution $\phi^{(1)}_{0}$ belongs to both $\mathcal{K}_\Lambda$ and $\mathcal{K}_{\Lambda_1}$, that is, $\phi^{(1)}_{0}$ is annihilated by both the creation and annihilation operators. In turn, $\phi^{(1)}_{1}$ is annihilated only by $\Lambda(t)$, and from it we generate a single sequence of states $\{ \phi^{(n+1)}_{1} \}_{n=0}^{\infty}$ through the iterated operation $\phi^{(n+1)}_{1} \propto [\hat{A}^{\dagger}(t)]^{n} \phi^{(1)}_{1}$, up to a normalization constant, with $n = 0, 1, \ldots$. On the other hand, the respective eigenvalues are determined by increasing $\Lambda^{(1)}_{n+1}$ in 2 units for each iteration, leading to $\Lambda^{(1)}_{n+1} = 2[\gamma + \lambda(n + 1)]$. For a summary of the spectral information of the quantum invariant, see figure 4.

Several special cases can be discussed from the general solution (55), leading to specific hierarchies of solutions of the Painlevé equation.
For $\gamma = 0$ together with $2\sqrt{w}k_a = 1 - \sqrt{2\pi}k^2$ and $k_b = k^2$, we obtain the set of parameters $\{\alpha, \beta\} = \{1, 0\}$. In such a case we recover the complementary error function hierarchy solutions of the form

$$w(y; 1, 0) = \frac{2\sqrt{2}k^2 e^{-y^2}}{1 - \sqrt{2\pi}k^2 \text{Erf}c(y)},$$

leading to the equidistant eigenvalues $\Lambda_{\alpha}^{(1)} = 2n\lambda$. The respective potential $V_1(x, t)$, determined from (52), reduces to a time-dependent variation of the stationary deformed oscillator potentials reported [8]. Such a time-dependent potential has been obtained previously in [58] through the Bagrov–Samsonov approach [55, 56].

Another interesting case is recovered for $k_b = 0$ and $\gamma = 2N\lambda$, with $N = 0, 1, \ldots$, where we obtain the rational solutions

$$w(y) = H_{2N}(y), \quad w(y; 2N + 1, -2(2N)^2) = 2N \frac{H_{2N-1}(y)}{H_{2N}(y)},$$

with $H_N(y) = (-i)^N H_N(iy)$ and $H_n(z)$ the pseudo-Hermite and Hermite polynomials [72], respectively. Contrary to the previous case, the eigenvalues are non-equidistant and given by $\Lambda_{\alpha}^{(1)} = 0$ and $\Lambda_{\alpha+1}^{(1)} = 2\lambda(2N + n + 1)$. It is worth to remark that the even pseudo-Hermite polynomials are nodeless, whereas the odd ones have one zero at the origin. Thus, the Painlevé solution in (58) is well defined for every $y \in \mathbb{R}$. Such a property is essential since it leads to a rational, nonsingular, and time-dependent potential $V_1(x, t)$, where $y \equiv y(x, t) = \sqrt{\lambda x}/\sigma(t)$. This particular case leads to eigenfunctions written in terms of the exceptional Hermite polynomials, previously discussed for stationary [76, 77] and time-dependent systems [65].

6.2. Hierarchies of rational solutions

In general, it is well-known that the fourth Painlevé equation admits hierarchies of rational solutions if and only if [33] the set of parameters $\{\alpha, \beta\}$ take either the values $\{m, -2(2n - m + 1)^2\}$ or $\{m, -2(2n - m + 1/3)^2\}$. The class of all the rational solutions are classified as
hierarchies, whereas the nonlinear recurrence relationship $M$ calculations, let us consider '...

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generalized Hermite polynomials $H_n$ with $Q$ together with the finite-norm zero-modes $Q_{M}$ where $Q_M$ stands for the respective normalization constants. From (62) one can see that

\[
\phi_{0}(1)(x, t) \equiv \phi_{0:1} = N_{0}^{(1)} \frac{e^{\frac{t}{\sigma}}}{\sqrt{\sigma}} e^{-\frac{1}{\sigma}} \frac{(2y^2 + 3)}{8y^6 + 60y^4 + 90y^2 + 135},
\]

\[
\phi_{1}(1)(x, t) \equiv \phi_{0:2} = N_{1}^{(1)} \frac{e^{\frac{t}{\sigma}}}{\sqrt{\sigma}} e^{-\frac{1}{\sigma}} y \frac{(8y^4(2y^4 + 24y^2 + 63) - 2835)}{(8y^6 + 60y^4 + 90y^2 + 135)},
\]

\[
\phi_{2}(1)(x, t) \equiv \phi_{0:3} = N_{2}^{(1)} \frac{e^{\frac{t}{\sigma}}}{\sqrt{\sigma}} e^{-\frac{1}{\sigma}} \frac{(2(4y^2(2y^2 + 15)(2y^4 - 45) - 6075)y^2 + 6075)}{(8y^6 + 60y^4 + 90y^2 + 135)},
\]

\[
\phi_{3}(1)(x, t) \equiv \phi_{0:4} = N_{3}^{(1)} \frac{e^{\frac{t}{\sigma}}}{\sqrt{\sigma}}
\]

where $N_{n}^{(1)}$ stands for the respective normalization constants. From (62) one can see that
(a) Potential $V_1(x,t)$ computed through the hierarchy of rational solutions $w_2$ (61) defined in terms of the Okamoto polynomials, together with the frequency $\Omega_2(t) = 1$ and parameters \( \{ \lambda = 2.5, a = 2, c = 1 \} \). (b) Potential $V_1(x,t)$ computed through the hierarchy of nonlinear bound states $w = 2\sqrt{2}\eta^2(\chi,N)$ of section 6.3 with the frequency $\Omega_2(t) = 1$ and parameters \( \{ N = 3, k = 0.44/\sqrt{3}, \lambda = 1, a = 2, c = 1 \} \). (Second row) Probability densities $|\phi_0^{(1)}|^2$ (c), $|\phi_3^{(1)}|^2$ (d), and $|\phi_4^{(1)}|^2$ (e) of the zero-modes (62) related to the potential (a). (Third row) Probability densities $|\phi_0^{(1)}|^2$ (f), $|\phi_N^{(1)}|^2$ (g), and $|\phi_{N+1}^{(1)}|^2$ (h) of the zero-modes (68) related to the potential (b).

$\phi_0^{(1)}$, $\phi_3^{(1)}$, and $\phi_4^{(1)}$ have exactly zero, three, and four real nodes, respectively. Moreover, the associated eigenvalues are $\Lambda_0^{(1)} = 0$, $\Lambda_3^{(1)} = 14\lambda/3$, and $\Lambda_4^{(1)} = 16\lambda/3$. Every zero-mode in (62) is an element of $\mathcal{K}_\lambda$, and consequently, each mode generates an infinite sequence of solutions, that is, we have three infinite sequences (see figure 5). The behavior of the respective potential and the probability densities associated with the zero-modes is depicted in figures 6(a)–(e),
where the frequency profile has been fixed as a constant, $\Omega^2(t) = 1$, with $\sigma(t)$ given in (43). Notice that, even in such a case, the resulting potential depends explicitly on time.

### 6.3. Solutions in terms of nonlinear bound states

Another special class of solutions to (46) is determined by considering the set of parameters $\{\alpha, \beta\} = \{2\nu + 1, 0\}$, together with the reparametrization

$$w(y; 2\nu + 1, 0) = 2\sqrt{2}\eta_\nu^2(\xi; \nu), \quad y = \frac{\xi}{\sqrt{2}}. \quad (63)$$

The latter leads to a nonlinear differential equation for $\eta_\nu(\xi; \nu)$ of the form [74]

$$\frac{d^2\eta_\nu}{d\xi^2} = 3\eta_\nu^5 + 2\xi\eta_\nu^3 + \left(\frac{1}{4}\xi^2 - \nu - \frac{1}{2}\right)\eta_\nu, \quad \eta \equiv \eta_\nu(\xi; \nu), \quad \nu \in \mathbb{R}, \quad (64)$$

which arises in the study of the derivative nonlinear Schrödinger equation [79]. A striking feature of $\eta_\nu$ is provided by the asymptotic behavior $\eta_\nu(\xi; \nu) \sim kD_\nu(\xi)$ as $\xi \rightarrow +\infty$ for $\nu \in \mathbb{R}$ and $D_\nu(\xi)$ the parabolic cylinder functions [72]. In turn, determining the asymptotic behavior for $\xi \rightarrow -\infty$ becomes a challenging task, where the asymptotic value depends on $\nu$ and it is computed from a connection formula, see [80] for details. In this section, we restrict ourselves to the special case $\nu = N$, with $N = 0, 1, \ldots$. In such a case, there are solutions $\eta_\nu(\xi; N)$ for $\xi \in \mathbb{R}$ with asymptotic behavior

$$\eta_\nu(\xi; N) \sim \begin{cases} k\xi^N e^{-\xi^2/4} & x \rightarrow +\infty \\ k\xi^N e^{-\xi^2/4} \sqrt{1 - 2\sqrt{2}\pi N!k^2} & x \rightarrow -\infty \end{cases}, \quad k^2 < \frac{1}{2\sqrt{2}\pi n!}. \quad (65)$$

That is, the solutions decay exponentially to zero at both $\xi \rightarrow \pm \infty$. The exact form of $\eta_\nu(\xi; N)$ is determined in a recursive way through the combination of several Bäcklund transformations such that $\beta = 0$ is preserved in each iteration [74]. We thus have

$$\eta_{\nu + 1}(\xi; N + 1) = \frac{\xi\eta_\nu(\xi; N) + 2\eta_\nu^2(\xi; N) - 2\eta_\nu^3(\xi; N)}{2\left[N + 1 + 2\eta_\nu(\xi; N)\eta_\nu'(\xi; N) - \xi\eta_\nu(\xi; N) - 2\eta_\nu(\xi; N)\right]^{1/2}}. \quad (66)$$

with $\eta_\nu'$ the partial derivative of $\eta_\nu$ with respect to $\xi$. Thus, the $N$th solution is determined by iterating $N$ times the solution associated with $N = 0$ in the recursion formula (66). The $N = 0$ solution is related to the complementary error function hierarchy (57) as

$$\eta_\nu(\xi; 0) \equiv \frac{1}{2\sqrt{2}} \left[w(\xi/\sqrt{2}; 1, 0)\right]^{1/2} = \frac{k e^{-\xi^2/4}}{\left[1 - \sqrt{2\pi k^2}\text{Erfc}(\xi/\sqrt{2})\right]^{1/2}}. \quad (67)$$

The behavior of the solutions $\eta_\nu(\xi; N)$ are depicted in figure 7(a) for several values of $N$. In such a figure it can be seen that, indeed, the solutions contain exactly $N$ zeroes while they converge to zero at the boundary points of the domain.
determine the set of finite-norm zero-mode eigenfunctions. From (32) and (33) we obtain

\[
\phi_n^{(1)}(x,t) = \phi_0^{(1)} = N_0 \frac{e^{x^2/4}}{\sqrt{\sigma}} e^{-\lambda t^2/2} e^{-2\int dz' G(z')}.
\]

with the respective eigenvalues \( \Lambda_0^{(1)} = 0 \), \( \Lambda_N^{(1)} = 2\lambda N \), and \( \Lambda_{N+1}^{(1)} = 2\lambda(N+1) \). From the asymptotic behavior (65), one realizes that the term \( \exp \left( \int dz' G(z') \right) \) converges to a finite value for \( z \to \pm \infty \), since the integral approximates to the error function at the asymptotic value. Thus, every zero-mode eigenfunction in (68) converges to zero at \( z \to \pm \infty \) and, indeed, we have finite-norm solutions. The remaining elements of the spectrum are determined from the action of the creation operator \( \hat{A}^\dagger(t) \) on the zero modes (68), as usual. Notice that \( \phi_0^{(1)} \in \mathcal{K}_{\hat{A}^\dagger} \), and thus \( \phi_N^{(1)} \) is annihilated by the creation operator \( \hat{A}^\dagger(t) \). Therefore, the creation operator generates a \( (N+1) \)-dimensional sequence of eigenfunctions \( \{ \phi_n^{(1)} \}_{n=0}^N \) through the iteration

\[
\hat{A}^\dagger(t) \phi_n^{(1)}, \text{ for } n = 0, 1, \ldots, N.
\]

In turn, an additional infinite sequence \( \{ \phi_n^{(1)} \}_{n=N+1}^\infty \) is generated from the operation \( \hat{A}^\dagger(t) \phi_N^{(1)} \), for \( n = 0, 1, \ldots \). In this form, the case discussed in this section generalizes the complementary error function hierarchy, where the latter is obtained.
as the special case $N = 0$. This spectral information is summarized in the diagram depicted in figure 7(b).

Finally, from the zero modes (68), together with the properties of the nonlinear bound states $\eta_k(\xi; N)$, it follows that $\phi^{(1)}_0$, $\phi^{(1)}_N$, and $\phi^{(1)}_{N+1}$ are solutions with exactly zero, $N$, and $N + 1$ nodes, respectively. Therefore, the oscillation theorem for the Sturm–Liouville associated with the quantum invariant $I_1(t)$ is verified. Additionally, given that the action of the creation operator increases the eigenvalue by $2\lambda$ units, we determine that in general the eigenvalues of $I_1(t)$ are equidistant, $\Lambda^{(1)}_n = 2\lambda n$, for $n = 0, 1, \ldots$. The behavior of the respective time-dependent potential $V_1(x, t)$ and the probability densities of the zero-modes is depicted in figures 6(b)–(h), where we have chosen $\Omega^2(t) = 1$, with $\sigma(t)$ given in (43).

7. Conclusions

The results of this manuscript can be seen from two different perspectives. On the one hand our approach represents a time-dependent generalization of the families of potentials reported previously in the stationary regime [42], on the other hand we also introduce some new quantum potentials unnoticed in the literature of stationary models. To this end, it was essential to address the shape-invariant problem from the more general perspective of the quantum invariants rather than the Hamiltonians. In this form, the time-dependence is introduced to both quantities, where the conventional spectral analysis is now carried on for the quantum invariant. Regardless of its time-dependence, the eigenvalues associated with the quantum invariant are time-independent, as it was first proved by Lewis–Reisenfeld [61]. Interestingly, after introducing the time parameter in the construction, a second nonlinear equation appears, namely the Ermakov equation, in such a way that the resulting time-dependent potentials and solutions to the Schrödinger equation are free of singularities at each time. In turn, the fourth Painlevé equation emerges after using a convenient reparametrization, where the parameters of the Painlevé equation dictate the distribution of eigenvalues of the quantum invariant, provided that the respective zero-modes are physically acceptable. It is worth to mention that both nonlinear equations, Ermakov and Painlevé, are not interlaced to each other; that is, the solutions of one equation do not modify the outcome of the solutions of the other equation. We can thus study each equation independently.

Regarding the fourth Painlevé equation, a first family of solutions is determined through the related Riccati equation. This does not only allows us to recover the one-step rational extensions of the parametric oscillator, reported previously in [65], but also leads to a family of one-parameter solutions in terms of the error function. The respective potential corresponds to a time-dependent generalization of the deformed oscillator reported by Mielnik [8]. On the other hand, the hierarchy of rational solutions ‘$-2y/3$’ in terms of the Okamoto polynomials allows constructing a quantum invariant with several gaps in its spectrum, which is generated by three infinite sequences of independent eigenfunctions, that is, the respective eigenvalues in each sequence do not overlap. As a particular example, we have shown that the Okamoto polynomial $Q_2(y)$ generates two gaps. Nevertheless, our results can be separated for an arbitrary polynomial $Q_N(y)$, with $N = 0, 1, \ldots$, where the spectrum acquires precisely $N$ gaps. The latter is indeed a property that could reveal an intrinsic structure in terms of exceptional polynomials. Further analysis is required, and results on the matter will be reported elsewhere.

Although a closed expression for the $N$th nonlinear bound state is not available, a nonlinear recurrence relation in the form of a Bäcklund transformation allows computing any solution by iterative means from the seed solution given by the error function. Remarkably, the Bäcklund transformation [74] is such that preserves $\beta = 0$ for any $N$th nonlinear bound state.
The latter implies that the spectrum of the quantum invariant is equidistant, for any $N$. Furthermore, the eigenfunctions are classified by two sequences, one that is $(N + 1)$-dimensional and one infinite-dimensional. The finite-dimensional sequence has two zero-modes, constructed as eigenfunctions of the annihilation (nodeless function) and creation ($N$ nodes function) operators. The zero-mode related to the infinite sequence is also an eigenfunction of the annihilation operator such that it has exactly $N + 1$ nodes. Interestingly, this case also brings new results in the stationary regime, for it generalizes the singlet and doublet structure introduced in [42]. Thus, it is clear that the families of nonlinear bound states can be explored even further in the context of stationary Hamiltonians, and a detailed analysis will be discussed in an upcoming contribution.

The construction of time-dependent systems can also be achieved using canonical point transformations, where the appropriate deformation of the spatial coordinate and the time parameter modifies the Schrödinger equation of the stationary oscillator into the one of the parametric oscillator [63, 81], which have been also exploited to construct oscillators with time-dependent mass [82], as well as the Caldirola–Kanai oscillator through the Arnold transformation [83]. Similarly, coordinates transformations to solve time-dependent systems have been studied in the context of conformal groups [84, 85] and further considered for quadratic Hamiltonians [86]. The relationship between these methods and the quantum invariants presented in current manuscript is a task that deserves special attention by itself. Results in this regard are in progress and will be reported elsewhere.

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Appendix A. Parametric oscillator

In this appendix, we briefly introduce the basic notions of the parametric oscillator, also known as nonstationary oscillator. The latter system is characterized by a time-dependent quadratic Hamiltonian of the form

$$\hat{H}_0(t) := \hat{p}^2 + \Omega^2(t)\hat{x}^2 \equiv -\frac{\partial^2}{\partial x^2} + \Omega^2(t)x^2,$$

where $\hat{x} \equiv x$ and $\hat{p} \equiv -i\frac{\partial}{\partial x}$ the canonical position and momentum operators, respectively, and $\Omega(t) > 0$ the time-dependent frequency of oscillation. In contradistinction to the stationary oscillator, the Hamiltonian $\hat{H}_0(t)$ does not admit an eigenvalue equation. However, from the approach of Lewis–Riesenfeld [61], it is known that solutions of the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi_n^{(0)}(x,t) = \hat{H}_0(t)\psi_n^{(0)}(x,t),$$

are determined from the eigenvalue problem

$$\hat{I}_0(t)\phi_n^{(0)}(x,t) = \Lambda_n^{(0)}\phi_n^{(0)}(x,t),$$
with \( \Lambda_n^{(0)} \) the time-independent eigenvalues and \( \phi_n^{(0)}(x, t) \) the nonstationary eigenfunctions of the quantum invariant \( I_0(t) \) of the system. Such an invariant is computed from the invariance condition

\[
i [\hat{H}_0, \hat{J}_0(t)] + \frac{\partial \hat{J}_0(t)}{\partial t} = 0,
\]

and it takes the form \([61, 63]\)

\[
\hat{J}_0(t) = \sigma^2 \hat{p}^2 + \left( \frac{\dot{\sigma}^2}{4} + \frac{1}{\sigma^2} \right) \hat{x}^2 - \frac{\sigma \dot{\sigma}}{2} [\hat{x}, \hat{p}], \quad \dot{\sigma} = \frac{d\sigma(t)}{dt},
\]

with \( \sigma = \sigma(t) \) a solution of the nonlinear equation

\[
\dot{\sigma} + 4\Omega^2(t)\sigma = \frac{4}{\sigma^2}.
\]

The latter equation is known as the Ermakov equation \([30–32]\), and a solution is found through the nonlinear combination \([23, 24]\)

\[
\sigma(t) = [aq^2(t) + bq_1(t)q_2(t) + cq^2(t)]^{1/2}, \quad b^2 - 4ac = -\frac{16}{W_0^2},
\]

with \( W_0 = W(q_1, q_2) \) the Wronskian of two linearly independent solutions \( q_{1,2}(t) \) of the linear homogeneous equation

\[
\ddot{q}_{1,2} + 4\Omega^2(t)q_{1,2} = 0.
\]

From the form of the differential equation (A-8), it is straightforward to realize that the Wronskian \( W_0 \) is time-independent, regardless of the structure of \( \Omega^2(t) \). The constraint in the constants given in (A-7) guarantees that \( \sigma(t) \) is a nodeless function at any time. Such a feature is essential to construct regular solutions \( \psi_n^{(0)}(x, t) \), and also in determining new nonsingular time-dependent potentials, as discussed in section 2.

The spectral problem (A-3) has been already determined in the literature through several techniques, such as solving the differential equation directly [61], using a particular complex reparametrization [49], with the aid of the Fourier transform [64], and performing geometrical transformations [63]. Thus, the spectral information of \( I_0(t) \) is given by

\[
\phi_n^{(0)}(x, t) = e^{\frac{i}{2} \dot{\sigma}^2} e^{\frac{1}{2} \sigma^2 H_n \left( \frac{x}{\sigma} \right)}, \quad \Lambda_n^{(0)} = 2n + 1, \quad n = 0, 1, \ldots,
\]

with \( H_n(x) \) the Hermite polynomials [72]. Clearly, the elements of the set \( \{\phi_n^{(0)}\}_{n=0}^{\infty} \) do not fulfill (A-2), but it can be easily shown that

\[
\psi_n^{(0)}(x, t) = e^{i\theta_n^{(0)}(t)} \phi_n^{(0)}(x, t),
\]

is indeed a solution, where \( \theta_n^{(0)}(t) \) is determined after substituting (A-10) in (A-2). It takes the following expression \([61, 63]\)

\[
\theta_n^{(0)}(x, t) = -(2n + 1) \int_t^\infty \frac{d\tau'}{\sigma^2(\tau')}
\]

\[
= - \left( n + \frac{1}{2} \right) \arctan \left[ \frac{W_0}{2} \left( \sqrt{ac - \frac{4}{W_0^2}} + \frac{q_2(t)}{q_1(t)} \right) \right].
\]
Contrary to the stationary case, the time-dependent complex phase does not represent the time-evolution of the system.

In summary, to completely determine the solutions of the parametric oscillator, we only need to find two linearly independent solutions of \( (A-8) \), provided that \( \Omega(t) \) has been already specified.

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