Controlled $K$-Fusion Frame for Hilbert Spaces

Nadia ASSILA$^1$, Samir KABBAJ$^2$, and Brahim MOALIGE$^3$

Abstract. $K$-fusion frames are a generalization of fusion frames in frame theory. In this paper, we extend the concept of controlled fusion frames to controlled $K$-fusion frames, and we develop some results on the controlled $K$-fusion frames for Hilbert spaces, which generalize some well known results of controlled fusion frame case. Also we discuss some characterizations of controlled Bessel $K$-fusion sequences and of controlled $K$-fusion frames. Further, we analyze stability conditions of controlled $K$-fusion frames under perturbation.

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1. Introduction

Frames are more flexible than bases to solve some problems in Hilbert spaces. They were firstly introduced by Duffin and Schaeffer [11] to study nonharmonic Fourier series in 1952, and widely studied by Daubechies, Grossman and Meyer [10] in 1986. More results of frames are in [7].
Fusion frames as a generalisation of frames were introduced by Casazza and Kutyniok in [6] and further there were developed in their joint paper [9] with Li. The theory for fusion frames...
is available in arbitrary separable Hilbert spaces (finite-dimensional or not). The motivation behind fusion frames comes from signal processing, more precisely, the desire to process and analyze large data sets efficiently. A natural idea is to split such data sets into suitable smaller “blocks” which can be treated independently. From a pure mathematical point of view, fusion frames are special cases of the $g$-frames [22]. However, the connection to concrete applications is less apparent from the more abstract definition of $g$-frames. In 2012, L. Gavruta [15] introduced the notions of $K$-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator $K$. Controlled frames in Hilbert spaces have been introduced by P. Balaz [1] to improve the numerical efficiency of iterative algorithms for inventing the frame operator. Further A. Khosravi [17] generalized this concept to the case of fusion frames. He has showed that controlled fusion frame as a generalization of fusion frames give a generalized way to obtain numerical advantage in the sense of preconditioning to check the fusion frame condition. In 2015 Rahimi [20] defined the concept of controlled $K$-frames in Hilbert spaces and showed that controlled $K$-frames are equivalent to $K$-frames.

Motivated by the above literature, we introduce and investigate some properties of controlled $K$-fusion frames, we also generalize some known results for controlled fusion frames to controlled $K$-fusion frames. Finally, we present perturbation result for controlled $K$-fusion frames. This paper is organized as follows. In Section 2, we recall several definitions about fusion frames, $K$-fusion frames and controlled fusion frames. Then, we give a basic properties about a bounded linear operator. In Section 3, we introduce the concept of controlled $K$-fusion frames and discuss their properties. In section 4, we analyze stability conditions of controlled $K$-fusion frames under perturbation.

2. Preliminaries and Notations

Throughout this paper, we will adopt the following notations. $\mathcal{H}$ is a separable Hilbert space, $\{W_i\}_{i \in I}$ is a sequence of closed subspaces of $\mathcal{H}$, where $I$ is a countable index set. the family of all bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$. We denote $R_T$, $N_T$, range and null space of a bounded linear operator $T$, respectively. $GL(\mathcal{H})$ is the set of all bounded invertible operators on $\mathcal{H}$ with bounded inverse, and $GL(\mathcal{H})^+$ denotes the set of all positive operators in $GL(\mathcal{H})$. $\pi_{W_i}$ is the orthogonal projection from $\mathcal{H}$ into $W_i$, and $\{w_i\}_{i \in I}$ is a family of weights, i.e. $w_i > 0$, for any $i \in I$.

The space $(\oplus_{i \in I} \mathcal{H})_2$ which is defined by

$$(\oplus_{i \in I} \mathcal{H})_2 = \{\{f_i\}_{i \in I} : f_i \in \mathcal{H}, i \in I, \sum_{i \in I} \|f_i\|^2 < \infty\},$$

with the inner product as

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle_{\mathcal{H}},$$

is a separable Hilbert space [17].

2.1. Fusion frames.
If only the right inequality of (2.1) holds, we call the family

\[ K \]

Definition 2.2. [3] Let \( K \)

Where \( K \)

\[ K \]

Frame for \( H \)

Where \( K \)

Its adjoint operator, which is called the analysis operator \( K \)

2.2. \( K \)-fusion frames.

Definition 2.2. [3] Let \( K \in B(H) \), let \( \{ W_i \}_{i \in I} \) be a family of closed subspaces of a Hilbert space \( H \), and let \( \{ w_i \}_{i \in I} \) be a family of weights. Then the family \( W = \{ W_i, w_i \}_{i \in I} \) is called a \( K \)-fusion frame for \( H \), if there exist positive constants \( A \leq B < \infty \) such that

\[ A \| f \|^2 \leq \sum_{i \in I} w_i^2 \| \pi_{W_i} f \|^2 \leq B \| f \|^2, f \in H. \tag{2.1} \]

\( A \) and \( B \) are called the lower and upper bounds of fusion frame, respectively. If only the right inequality of (2.1) holds, we call the family \( \{ W_i, w_i \}_{i \in I} \) a fusion Bessel sequence.

2.2. \( K \)-fusion frames.

Definition 2.2. [3] Let \( K \in B(H) \), let \( \{ W_i \}_{i \in I} \) be a family of closed subspaces of a Hilbert space \( H \), and let \( \{ w_i \}_{i \in I} \) be a family of weights. Then the family \( W = \{ W_i, w_i \}_{i \in I} \) is called a \( K \)-fusion frame for \( H \), if there exist positive constants \( A \leq B < \infty \) such that

\[ A \| K^* f \|^2 \leq \sum_{i \in I} w_i^2 \| \pi_{W_i} f \|^2 \leq B \| f \|^2, f \in H. \tag{2.2} \]

Where \( K^* \) is the adjoint operator of \( K \).

\( A \) and \( B \) are called the lower and upper bounds of \( K \)-fusion frame, respectively.

suppose that \( \{ W_i, w_i \}_{i \in I} \) is a fusion Bessel sequence for \( H \), then the synthesis operator of \( \{ W_i, w_i \}_{i \in I} \) is defined by \( T_W : (\sum_{i \in I} \oplus W_i)_{l^2} \rightarrow H \),

\[ T_W(\{ f_i \}_{i \in I}) = \sum_{i \in I} w_i f_i, \quad \{ f_i \}_{i \in I} \in (\sum_{i \in I} \oplus W_i)_{l^2}. \]

Where

\[ (\sum_{i \in I} \oplus W_i)_{l^2} = \{ \{ f_i \}_{i \in I} : f_i \in W_i, i \in I, \sum_{i \in I} \| f_i \|^2 < \infty \}. \]

Its adjoint operator, which is called the analysis operator \( T^*_W : H \rightarrow (\sum_{i \in I} \oplus W_i)_{l^2} \), is defined by

\[ T^*_W(f) = \{ w_i \pi_{W_i} f \}_{i \in I}, \quad f \in H. \]

And the \( K \)-fusion frame operator associated is \( S_W : H \rightarrow H \).

\[ S_W(f) = \sum_{i \in I} w_i^2 \pi_{W_i} f, \quad f \in H. \tag{2.3} \]

2.3. Controlled fusion frame.

Definition 2.3. [17] Let \( \{ W_i \}_{i \in I} \) be a family of closed subspaces of a Hilbert space \( H \), let \( \{ w_i \}_{i \in I} \) be a family of weights, and let \( T, U \in GL(H) \). Then the family \( W = \{ W_i, w_i \}_{i \in I} \) is called a \( (T, U) \)-controlled fusion frame for \( H \), if there exist positive constants \( A \leq B < \infty \) such that

\[ A \| f \|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W} T f, \pi_{W} U f \rangle \leq B \| f \|^2, f \in H. \tag{2.4} \]

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A and B are called the lower and upper bounds of \((T, U)\)-controlled fusion frame, respectively. For further information in \(K\)-fusion frame and controlled fusion frame theory we refer the reader to [3], [17] [18] and [9].

In theory of frames, often use the following theorem, which describes some properties of the adjoint operator.

**Theorem 1.** [7] Let \(H_1, H_2\) be Hilbert spaces, and suppose that \(U \in B(H_1, H_2)\). Then,

i) \(U^* \in B(H_2, H_1)\) and \(\|U^*\| = \|U\|\).

ii) \(U\) is surjective if and only if \(\exists A > 0\) such that \(\|U^*h\|_{H_2} \geq A\|h\|_{H_1}\).

It is well-known that not all bounded operator \(U\) on a Hilbert space \(H\) is invertible: an operator \(U\) needs to be injective and surjective in order to be invertible. For doing this, one can use right-inverse operator. The following lemma shows that if an operator \(U\) has closed range, there exists a "right-inverse operator" \(U^\dagger\) in the following sense:

**Lemma 2.** [7] Let \(H_1, H_2\) be Hilbert spaces, and \(U \in B(H_1, H_2)\) be a closed range \(R_U\). Then there exists a bounded operator \(U^\dagger : H_2 \rightarrow H_1\) for which

\[
UU^\dagger x = x, \quad x \in \mathcal{R}_U,
\]

and

\[
(U^*)^\dagger = (U^\dagger)^*.
\]

The operator \(U^\dagger\) is called the Pseudo-inverse of \(U\).

In the literature, one will often see the pseudo-inverse of an operator \(U\) with closed range defined as the unique operator \(U^\dagger\) satisfying that

\[
\mathcal{N}_{U^\dagger} = \mathcal{R}_{U}^\perp, \quad UU^\dagger x = x, \quad x \in \mathcal{R}_U.
\]

The following lemma is necessary for our results.

**Lemma 3.** [14] Let \(V \subseteq H\) be a closed subspace, and \(T\) be a linear bounded operator on \(H\). Then

\[
\pi_V T^* = \pi_V T^* \pi_{TV}.
\]

If \(T\) is a unitary (i.e. \(T^* T = T T^* = \text{Id}_H\)), then

\[
\pi_{TV} T = T \pi_V.
\]

**Proposition 2.1.** [21] Let \(T : H \rightarrow H\) be a linear operator. Then the following condition are equivalent:

1. There exist \(m > 0\) and \(M < \infty\), such that \(m I \leq T \leq M I\);
2. \(T\) is positive and there exist \(m > 0\) and \(M < \infty\), such that \(m \|f\|^2 \leq \|T^{1/2} f\|^2 \leq M\|f\|^2\) for all \(f \in H\);
3. \(T\) is positive and \(T^{1/2} \in \text{GL}(H)\);
4. There exists a self-adjoint operator \(A \in \text{GL}(H)\), such that \(A^2 = T\);
5. \(T \in \text{GL}^+(H)\).

The following lemma will be used in the sequel.
Lemma 4. [12] Let $F$, $G$, $H$ be Hilbert spaces. Let $T \in B(F, G)$ and $T' \in B(H, G)$ with $\overline{R}_{T'}$ be orthogonally complemented. Then the following statements are equivalent:

i) $T' T'^* \leq \lambda T T^*$ for some $\lambda > 0$.

ii) There exists $\mu > 0$ such that $\|T^* z\| \leq \mu \|T^* z\|$ for all $z \in G$.

Theorem 5. [13] Let $T$ be a positive linear bounded operator on $H$. $T$ possesses a unique positive bounded square root which commutes with every bounded operator that commutes with $T$.

3. Controlled $K$-fusion frame

In this section, we introduce the notion of controlled $K$-fusion frames in Hilbert spaces and we discuss some their properties.

Definition 3.1. Let $K \in B(H)$, and $\{W_i\}_{i \in I}$ be a family of closed subspaces of a Hilbert space $H$. Also, let $\{w_i\}_{i \in I}$ be a family of weights, and let $C, C' \in GL(H)$. Then $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is called a $K$-fusion frame controlled by $C$ and $C'$ or $(C, C')$-controlled $K$-fusion frame if there exist two constants $0 < A_{CC'} \leq B_{CC'} < \infty$ such that

$$A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 < \pi_{W_i} C f, \pi_{W_i} C' f > \leq B_{CC'} \|f\|^2, f \in H. \quad (3.1)$$

Where $K^*$ is the adjoint operator of $K$.

$A_{CC'}$ and $B_{CC'}$ are called lower and upper bounds of a $(C, C')$-controlled $K$-fusion frame respectively.

1. We call $\mathcal{W}$ a $(C, C')$-controlled Parsval $K$-fusion frame if $A_{CC'} = B_{CC'} = 1$.

2. If only the second inequality (3.1) is required, we call $\mathcal{W}$ a $(C, C')$-controlled Bessel $K$-fusion sequence with Bessel bound $B$.

Remark 6. i) If $K = I$ (where is the identity operator), then every $(C, C')$-controlled $K$-fusion frame is a $(C, C')$-controlled fusion frame.

ii) If $C = C' = I$, then every $(C, C')$-controlled $K$-fusion frame is a $K$-fusion frame.

iii) Every $(C, C')$-controlled fusion frame is a $(C, C')$-controlled $K$-fusion frame. Indeed, by definition (3.1) there exist constants $0 < A_{CC'} \leq B_{CC'}$, such that for all $f \in H$, we have $A_{CC'} \|f\|^2 \leq \sum_{i \in I} w_i^2 < \pi_{W_i} C f, \pi_{W_i} C' f > \leq B_{CC'} \|f\|^2$.

Therefore, for $\|K\| > 0$, one has $A_{CC'} \|K^* f\|^2 \leq B_{CC'} \|f\|^2 \leq A_{CC'} \|K\|^2 \|f\|^2$, that is,

$$\frac{A_{CC'}}{\|K\|^2} \|K^* f\|^2 \leq A_{CC'} \|f\|^2,$$
it follows that,
\[
\frac{A_{CC'}}{\|K\|^2} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 < \pi_{W_i} C f, \pi_{W_i} C' f > \leq B_{CC'} \|f\|^2.
\]

Hence, the family \( W \) is a \((C, C')\)-controlled \( K \)-fusion frame for \( \mathcal{H} \).

The next example shows that in general, frames may be controlled \( K \)-fusion frame without being a controlled fusion frame.

**Example 3.1.** Let \( \mathcal{H} = l_2(\mathbb{C}) = \{\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \mid \sum_{n=0}^{+\infty} |a_n|^2 < \infty\} \) be a Hilbert space, with respect to the inner product
\[
\langle \{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \rangle = \sum_{n \in \mathbb{N}} a_n \overline{b_n},
\]
equipped with the norm
\[
\|\{a_n\}_{n \in \mathbb{N}}\|_{l_2(\mathbb{C})} = \left( \sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.
\]
Consider two operators \( C \) and \( C' \) defined by
\[
C : \mathcal{H} \rightarrow \mathcal{H}, \quad \{a_n\}_{n \in \mathbb{N}} \mapsto \{\alpha a_n\}_{n \in \mathbb{N}}
\]
and
\[
C' : \mathcal{H} \rightarrow \mathcal{H}, \quad \{a_n\}_{n \in \mathbb{N}} \mapsto \{\beta a_n\}_{n \in \mathbb{N}}
\]
where \( \alpha, \beta \in \mathbb{R}^*_+ \).

It is easy to see that:
- \( C \) and \( C' \) are positives.
- \( C \) and \( C' \) are invertibles.

Then, invertible operators are given respectively by:
\[
C^{-1} : \mathcal{H} \rightarrow \mathcal{H}, \quad \{a_n\}_{n \in \mathbb{N}} \mapsto \{\alpha^{-1} a_n\}_{n \in \mathbb{N}},
\]
and
\[
C'^{-1} : \mathcal{H} \rightarrow \mathcal{H}, \quad \{a_n\}_{n \in \mathbb{N}} \mapsto \{\beta^{-1} a_n\}_{n \in \mathbb{N}}.
\]
Let \( E_i = \{a_j\}_{j \in \mathbb{N}}, \) where \( a_j = \{\delta^i_j\}_{j \in \mathbb{N}} \) (where \( \delta^i_j \) is the Kronecker symbol). Let \( \{W_i\}_{i \in \mathbb{N}} \) be a closed subspaces of \( \mathcal{H} \) such that \( W_i = CE_i \), and let \( w_i = \frac{1}{\sqrt{i+1}} \), for all \( i \in \mathbb{N} \).

The family \( W = \{W_i, w_i\}_{i \in \mathbb{N}} \) is a \((C, C')\)-controlled Bessel fusion sequence. Indeed for each \( \{a_n\}_{n \in \mathbb{N}} \in \mathcal{H} \), we have
\[
\sum_{i \in \mathbb{N}} w_i^2 \langle \pi_{W_i} C(\{a_n\}_{n \in \mathbb{N}}), \pi_{W_i} C'(\{a_n\}_{n \in \mathbb{N}}) \rangle = \alpha \beta \sum_{i \in \mathbb{N}} \frac{1}{i+1} |a_i|^2
\]
\begin{align*}
\leq \alpha\beta \sum_{i \in \mathbb{N}} |a_i|^2 \\
= \alpha\beta \|\{a_n\}_{n \in \mathbb{N}}\|_H^2.
\end{align*}

But is not \((C,C')\)-controlled fusion frame, For this, assume the contrary that exists \(A_{CC'} > 0\) such that:

\begin{equation}
A_{CC'} \sum_{i \in \mathbb{N}} |a_i|^2 \leq \sum_{i \in \mathbb{N}} \frac{\alpha\beta}{i+1} |a_i|^2.
\end{equation}

Hence

\[
\sum_{i \in \mathbb{N}} |a_i|^2 < \infty \implies \lim_{i \to +\infty} a_i = 0
\]

So, we have

\[
|a_j| \to 0 \quad \text{as} \quad j \to \infty;
\]

\[
\frac{\alpha\beta}{i+1} \to 0 \quad \text{as} \quad i \to \infty.
\]

\[
\implies \left\{ \begin{array}{ll}
\forall \varepsilon \geq 0 & \exists N \in \mathbb{N} : \ j \geq N \implies |a_j| < \varepsilon,
\forall \gamma \geq 0 & \exists N \in \mathbb{N} : \ j \geq M \implies \frac{\alpha\beta}{j+1} < \gamma.
\end{array} \right.
\]

\[
\implies \sum_{i \in \mathbb{N}} \frac{\alpha\beta - (j+1)A_{CC'}}{j+1} |a_j|^2 \geq 0.
\]

By fixing \(\varepsilon = \gamma\), there exist \(N, M \in \mathbb{N}^*\), such that

\[
\left\{ \begin{array}{ll}
j \geq N & \implies |a_j| < \varepsilon,
\end{array} \right.
\]

Now, let \(N_1 = \max(N, M)\), then \(\forall j \geq N_1\), \(|a_j| < \varepsilon\) and \(\frac{\alpha\beta}{j+1} < \varepsilon\). Hence

\[
\sum_{i=0}^{N_1-1} \frac{\alpha\beta - (i+1)A_{CC'}}{i+1} |a_i|^2 + \varepsilon^2 \sum_{i=N_1}^{\infty} (\varepsilon - A_{CC'}) \geq 0.
\]

Now, for \(\varepsilon = \frac{A_{CC'}}{2}\), we obtain

\[
\sum_{i=0}^{N_1-1} \frac{\alpha\beta - (i+1)A_{CC'}}{i+1} |a_i|^2 + \left(\frac{A_{CC'}}{2}\right)^2 \sum_{i=N_1}^{\infty} \left(-\frac{A_{CC'}}{2}\right) \geq 0.
\]

absurde.

Now if we considere the operator

\[
K : \mathcal{H} \to \mathcal{H} \quad \{a_n\}_{n \in \mathbb{N}} \mapsto \left\{ \frac{a_n}{\sqrt{n+1}} \right\}_{n \in \mathbb{N}}.
\]
Then, $K$ is a bounded linear operator on $\mathcal{H}$. Furthermore, for each $\{a_n\}_{n \in \mathbb{N}} \in \mathcal{H}$, we have

$$\langle K^*(\{a_n\}_{n \in \mathbb{N}}), K^*(\{a_n\}_{n \in \mathbb{N}}) \rangle = \sum_{i=0}^{\infty} \frac{|a_n|^2}{n+1}. $$

So,

$$\sum_{i=0}^{\infty} \frac{\alpha \beta}{n+1} |a_n|^2 \leq \sum_{i=0}^{\infty} \frac{\alpha \beta}{n+1} |a_n|^2 \leq \alpha \beta \sum_{i=0}^{\infty} |a_n|^2,$$

The following proposition provides a relation between controlled $K$-fusion frames and controlled fusion frames.

**Proposition 7.** Let $K \in \mathcal{B}(\mathcal{H})$ be a closed range operator. Then, every $(C, C')$-controlled $K$-fusion frame is a $(C, C')$-controlled fusion frame for $\mathcal{R}_K$.

**Proof.** Let $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ be a $(C, C')$-controlled $K$-fusion frame with frame bounds $A_{CC'}$ and $B_{CC'}$. Then for all $f \in \mathcal{R}_K$, we have

$$A_{CC'}\|K^*f\|^2 \leq \sum_{i \in I} w_i^2 < \pi_{W_i}Cf, \pi_{W_i}C'f > \leq B_{CC'}\|f\|^2.$$

Therefore, via lemma 2, we have

$$A_{CC'}\|f\|^2 \leq A_{CC'}\|(K^*)^\dagger f\|^2\|K^*f\|^2.$$

Hence,

$$\frac{A_{CC'}}{\|(K^*)^\dagger\|^2}\|f\|^2 \leq A_{CC'}\|K^*f\|^2.$$

Thus,

$$\frac{A_{CC'}}{\|(K^*)^\dagger\|^2}\|f\|^2 \leq \sum_{i \in I} w_i^2 < \pi_{W_i}Cf, \pi_{W_i}C'f > \leq B_{CC'}\|f\|^2.$$

So, we have the result.

If $\mathcal{W}$ is a $(C, C')$-controlled $K$-fusion frame and $C'^*\pi_{W_i}C$ is a positive operator for each $i \in I$, then $C'^*\pi_{W_i}C = C^*\pi_{W_i}C'$ and we have

$$A_{CC'}\|K^*f\|^2 \leq \sum_{i \in I} w_i^2 \|(C'^*\pi_{W_i}C)^{\frac{1}{2}}f\|^2 \leq B_{CC'}\|f\|^2, f \in \mathcal{H}.$$

Indeed,

$$\sum_{i \in I} w_i^2 \langle \pi_{W_i}Cf, \pi_{W_i}C'f \rangle = \sum_{i \in I} w_i^2 \langle (C'^*\pi_{W_i}C)^{\frac{1}{2}}f, (C'^*\pi_{W_i}C)^{\frac{1}{2}}f \rangle = \sum_{i \in I} w_i^2 \|(C'^*\pi_{W_i}C)^{\frac{1}{2}}f\|^2.$$
\[ f \mapsto T_{CC'}(f) := (w_i(C'^* \pi_{W_i} C)^{1/2} f)_{i \in I}, \]

Where

\[ K = \{ (w_i(C'^* \pi_{W_i} C)^{1/2} f)_{i \in I} | f \in \mathcal{H} \} \subseteq (\oplus_{i \in I} \mathcal{H})_2. \]

\( K \) is closed [17] and \( T_{CC'} \) is well defined. Moreover \( T_{CC'} \) is a bounded linear operator. Its adjoint operator is given by

\[ T^*_{CC'} : K \rightarrow \mathcal{H} \]

\[ (w_i(C'^* \pi_{W_i} C)^{1/2} f)_{i \in I} \mapsto T^*_{CC'}((w_i(C'^* \pi_{W_i} C)^{1/2} f)_{i \in I}) := \sum_{i \in I} w_i^2 C'^* \pi_{W_i} C f, \]

and is called the controlled synthesis operator.

Therefore, we define the controlled \( K \)-fusion frame operator \( S_{CC'} \) on \( \mathcal{H} \) by

\[ S_{CC'} = T^*_{CC'} T_{CC'}(f) = \sum_{i \in I} w_i^2 C'^* \pi_{W_i} C f, \quad f \in \mathcal{H}. \] (3.3)

In fact, many of the properties of the ordinary \( K \)-fusion frames are valid in this case.

**Lemma 8.** Let \( W = \{ W_i, w_i \}_{i \in I} \) be a \((C, C')\)-controlled \( K \)-fusion frame with bounds \( A_{CC'} \) and \( B_{CC'} \). Then the operator \( S_{CC'} \) (3.3) is a well-defined, linear, positive, bounded and self-adjoint operator. Furthermore, we have

\[ A_{CC'} K K^* \leq S_{CC'} \leq B_{CC'} \text{Id}_H. \] (3.4)

**Proof.**
- By definition, \( S_{CC'} \) is a linear bounded and well-defined operator, and it is clear to see that \( S_{CC'} \) is a positive and self-adjoint operator.
- The family \( W = \{ W_i, w_i \}_{i \in I} \) is a \((C, C')\)-controlled \( K \)-fusion frame for \( \mathcal{H} \) with bounds \( A_{CC'} \) and \( B_{CC'} \) if and only if

\[ A_{CC'} \| K^* f \|^2 \leq \langle S_{CC'} f, f \rangle = \langle \sum_{i \in I} w_i^2 C'^* \pi_{W_i} C f, f \rangle \leq B_{CC'} \| f \|^2, \quad f \in \mathcal{H}, \]

that is,

\[ A_{CC'} \langle K^* f, f \rangle \leq \langle S_{CC'} f, f \rangle \leq B_{CC'} \langle f, f \rangle, \quad f \in \mathcal{H}. \]

Hence,

\[ A_{CC'} K K^* \leq S_{CC'} \leq B_{CC'} \text{Id}_H, \]

so the conclusion holds.

The next theorem generalizes the situation of controlled Bessel \( K \)-fusion sequence. Since it has similar procedure, the proof is omitted.

**Theorem 9.** \( W \) is a \((C, C')\)-controlled Bessel \( K \)-fusion sequence with bound \( B_{CC'} \) if and only if \( T^*_{CC'} \) is well-defined bounded operator and \( \| T_{CC'} \| \leq \sqrt{B} \).
Controlled K-fusion frame operator of \((C, C')\)-controlled K-fusion frame is not invertible in general, but we can show that it is invertible on the subspace \(R_K \subset \mathcal{H}\). In fact, since \(R_K\) is closed
\[
KK^\dagger |_{R_K} = id_{R_K},
\]
so we have
\[
id^*_R = (K^\dagger |_{R_K})^* K^*.
\]
Hence for any \(f \in R_K\)
\[
\|f\| = \|(K^\dagger |_{R_K})^* K^* f\| \leq \|K^\dagger\| \|K^* f\|,
\]
that is,
\[
\|f\|^2 \leq \|K^\dagger\|^2 \|K^* f\|^2.
\] (3.5)
Combined with (3.1) we have
\[
\langle S_{CC'}f, f \rangle \geq A_{CC'} \|K^* f\|^2 \geq A_{CC'} \|K^\dagger\|^2 \|f\|^2, \quad \forall f \in R_K.
\]
So from the definition of \((C, C')\)-controlled K-fusion frame, one implies that \(S : \mathcal{R}_K \rightarrow S(\mathcal{R}_K)\) is an isomorphism, furthermore we have
\[
B_{CC'}^{-1} \|f\| \leq \|S^{-1} f\| \leq A_{CC'}^{-1} \|K^\dagger\|^2 \|f\|, \quad \forall f \in (S(\mathcal{R}_K)).
\]

**Theorem 10.** Let \(K \in B(\mathcal{H})\) be a closed range operator, then \(W\) is a \((C, C')\)-controlled K-fusion frame with bounds \(A_{CC'}\) and \(B_{CC'}\) if and only if \(T_{CC'}^*\) is well-defined and surjective.

**Proof.** Let the sequence \(W\) be a \((C, C')\)-controlled K-fusion frame for \(\mathcal{H}\), and let \(S_{CC'}\) be its controlled K-fusion frame operator. Then, it is a \((C, C')\)-controlled Bessel K-fusion sequence and therefore, by Theorem 9, the bounded operator \(T_{CC'}^*\) is well-defined. It remains to show that By definition, for each \(f \in \mathcal{H}\), we have
\[
A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 < \pi_{W_i} C f, \pi_{W_i} C' f > \leq B_{CC'} \|f\|^2.
\]
In particular, we have
\[
A_{CC'} \|K^* f\|^2 \leq \langle S_{CC'}f, f \rangle \leq \|S_{CC'}f\| \|f\|.
\]
Since, \(S_{CC'} = T^* T\), then
\[
\|S_{CC'}f\| \|f\| \leq \|T\| \|Tf\| \|f\|.
\]
Hence,
\[
A_{CC'} \|T\|^{-1} \|K^* f\|^2 \leq \|Tf\| \|f\|.
\]
Since \(R_K\) is closed, and via lemma we have
\[
KK^\dagger |_{R_K} = id_{R_K},
\]
so we have
\[
id^*_R = (K^\dagger |_{R_K})^* K^*.
\]
Hence for any \( f \in \mathcal{R}_K \)
\[
\|f\| = \|(K^*|_{\mathcal{R}_K})^*K^*f\|,
\]
that is,
\[
\|f\|^2 \leq \|(K^*)^t\|^2\|K^*f\|^2. \tag{3.6}
\]
Therefore, we have
\[
A_{CC'}\|f\|^2 \leq A_{CC'}\|(K^*)^t\|^2\|K^*f\|^2.
\]
Hence,
\[
\frac{A_{CC'}}{\|(K^*)^t\|^2}\|f\|^2 \leq A_{CC'}\|K^*f\|^2.
\]
\[
\|Tf\| \geq \frac{A_{CC'}}{\|T\|(K^*)^t\|^2}\|f\|.
\]
Thus, \( T_{CC'}^* \) is a surjective operator.
Conversely, let \( T_{CC'}^* \) be a well-defined, bounded and surjective, then theorem 9 shows that \( \mathcal{W} \) is a \((C, C')\)-controlled Bessel \( K \)-fusion sequence for \( \mathcal{H} \). Therefore, for each \( f \in \mathcal{H} \), since \( T_{CC'}^* \) is surjective, then, by Lemma 2, there exists an operator \((T_{CC'}^*)^t : \mathcal{H} \rightarrow \mathcal{K}, \) such that
\[
T_{CC'}^*(T_{CC'}^*)^t = id.
\]
Hence,
\[
T_{CC'}^tT_{CC'} = id.
\]
So, for each \( f \in \mathcal{H} \), we have
\[
\|K^*f\|^2 \leq \|K\|^2\|T_{CC'}^t\|^2\|T_{CC'}f\|^2
\]
\[
= \|T_{CC'}^t\|^2\|K\|^2\sum_{i \in I} w_i^2 < \pi_{W_i}Cf, \pi_{W_i}C'f >.
\]
Therefore, \( \mathcal{W} \) is a \((C, C')\)-controlled \( K \)-fusion frame for \( \mathcal{H} \).

**Proposition 11.** Let \( K \in B(\mathcal{H}) \), \( C, C' \in GL^+(\mathcal{H}) \) and let \( \mathcal{W} \) be a \((C, C')\)-controlled \( K \)-fusion frame for \( \mathcal{H} \) with bounds \( A_{CC'} \) and \( B_{CC'} \) with \( \mathcal{R}_T \) is orthogonally complemented. If \( T \in B(\mathcal{H}) \) with \( \mathcal{R}_T \subset \mathcal{R}_K \). Then \( \mathcal{W} \) is a \((C, C')\)-controlled \( T \)-fusion frame for \( \mathcal{H} \).

**Proof.** Assume that \( \mathcal{W} \) be a \((C, C')\)-controlled \( K \)-fusion frame for \( \mathcal{H} \) with bounds \( A_{CC'} \) and \( B_{CC'} \). Then for each \( f \in \mathcal{H} \), we have
\[
A_{CC'}\langle K^*f, K^*f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i}Cf, \pi_{W_i}C'f \rangle \leq B_{CC'}\langle f, f \rangle.
\]
Since \( \mathcal{R}_T \subset \mathcal{R}_K \), so by using lemma 4, there exists some \( \lambda > 0 \) such that
\[
TT^* \leq \lambda KK^*.
\]
This implies that for all \( f \in \mathcal{H} \), we have
\[
A_{CC'}\langle T^*f, T^*f \rangle \leq A_{CC'}\lambda\langle K^*f, K^*f \rangle.
\]
Therefore,
\[
\frac{A_{CC}^i}{\lambda} \langle T^* f, T^* f \rangle \leq A_{CC}^i \langle K^* f, K^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'}^i \langle f, f \rangle.
\]

Then, \( W \) is a \((C, C')\)-controlled \(T\)-fusion frame for \( \mathcal{H} \) with bounds \( \frac{A_{CC}^i}{\lambda} \) and \( B_{CC'}^i \).

**Theorem 12.** Let \( K_1, K_2 \in B(\mathcal{H}) \) such that \( \mathcal{R}_{K_1^*} \perp \mathcal{R}_{K_2^*} \). If \( \mathcal{W} \) is a \((C, C')\)-controlled \( K_i \)-fusion frame for \( \mathcal{H} \) \((i = 1, 2)\). Then \( \mathcal{W} \) is a \((C, C')\)-controlled \((aK_1 + \beta K_2)^*\)-fusion frame for \( \mathcal{H} \), where \( a, \beta \in \mathbb{C} \).

**Proof.** Since \( \mathcal{W} \) is a \((C, C')\)-controlled \( K_i \)-fusion frame for \( \mathcal{H} \) \((i = 1, 2)\), there exist \( A_{CC'}^i, B_{CC'}^i > 0 \), such that for all \( f \in \mathcal{H}, j = 1, 2 \), we have
\[
A_{CC'}^i \langle K_j^* f, K_j^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'}^i \langle f, f \rangle.
\]

Then for any \( f \in \mathcal{H} \), we have
\[
\langle (aK_1 + \beta K_2)^* f, (aK_1 + \beta K_2)^* f \rangle = \langle \pi K_1^* f + \overline{\beta} K_2^* f, \pi K_1^* f + \overline{\beta} K_2^* f \rangle = |a|^2 \langle K_1^* f, K_1^* f \rangle + |\beta|^2 \langle K_2^* f, K_2^* f \rangle + \overline{\beta} \pi \langle K_1^* f, K_1^* f \rangle + \beta \pi \langle K_2^* f, K_2^* f \rangle.
\]

Since \( \mathcal{R}_{K_1} \perp \mathcal{R}_{K_2} \), then, for any \( f \in \mathcal{H} \), we have
\[
\langle K_1^* f, K_2^* f \rangle = 0,
\]
\[
\langle K_2^* f, K_1^* f \rangle = 0.
\]

Thus,
\[
\langle (aK_1 + \beta K_2)^* f, (aK_1 + \beta K_2)^* f \rangle = |a|^2 \langle K_1^* f, K_1^* f \rangle + |\beta|^2 \langle K_2^* f, K_2^* f \rangle.
\]

Therefore, for any \( f \in \mathcal{H} \), we have
\[
|a|^2 \langle K_1^* f, K_1^* f \rangle + |\beta|^2 \langle K_2^* f, K_2^* f \rangle \leq \frac{|a|^2}{A_{CC'}^1} + \frac{|\beta|^2}{A_{CC'}^2} \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq \frac{A_{CC'}^1 |a|^2 + A_{CC'}^2 |\beta|^2 + 1}{A_{CC'}^1 A_{CC'}^2} \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle.
\]

Then,
\[
0 < \frac{A_{CC'}^1 A_{CC'}^2}{(A_{CC'}^2 |a|^2 + A_{CC'}^1 |\beta|^2 + 1)} \langle (aK_1 + \beta K_2)^* f, (aK_1 + \beta K_2)^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle.
\]

Then, we have
\[
\frac{A_{CC'}^1 A_{CC'}^2}{(A_{CC'}^2 |a|^2 + A_{CC'}^1 |\beta|^2 + 1)} \langle (aK_1 + \beta K_2)^* f, (aK_1 + \beta K_2)^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle.
\]
then,
\[
\sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle = \frac{1}{2} \left( \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C f \rangle + \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \right)
\leq \frac{B_{1CC'}^1 + B_{1CC'}^2}{2} \langle f, f \rangle.
\]

Thus, \( W \) is a \((C, C')\)-controlled \( \alpha K_1 + \beta K_2 \)-fusion frame with bounds \( A_{CC'}^1 \) and \( B_{CC'}^1 \)\( A_{CC'}^2 \) and \( B_{CC'}^1 + B_{CC'}^2 \).

**Lemma 13.** Let \( K \in \mathcal{B}(\mathcal{H}) \), and \( C, C' \in \mathcal{GL}^+(\mathcal{H}) \) such that \( CC' = C'C \). Assume that \( CK = KC \), \( C'K = KC' \); \( SC = CS \), \( SC' = C'S \) and \( \pi_{W_i} C = \pi_{W_i} C' \). Then, \( W \) is a \((C, C')\)-controlled \( K \)-fusion frame for \( \mathcal{H} \) if and only if \( W \) is a \( K \)-fusion frame for \( \mathcal{H} \).

Where \( S \) is the \( K \)-fusion frame operator (2.3), defined by
\[
S f = \sum_{i \in I} w_i^2 \pi_{W_i} f, \quad f \in \mathcal{H}.
\]

**Proof.** Assume that \( W \) is a \( K \)-fusion frame with bounds \( A \) and \( B \). Then for each \( f \in \mathcal{H} \), we have
\[
A \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2 \leq B \|f\|^2.
\]

Since, \( C \) and \( C' \) are linear bounded operators, applying 2.1, there exist constants \( m, m', M \) and \( M' > 0 \) such that
\[
\begin{cases}
ml \leq C \leq Ml, \\
m'l \leq C' \leq M'l.
\end{cases}
\]

\[\langle SC f, f \rangle = \langle f, CS f \rangle.\]

Then,
\[
mKK^* \leq CS \leq MS \leq MBl.
\]

We deduce that
\[
mm'KK^* \leq C'SC \leq MM'B1.
\]

Therefore, for each \( f \in \mathcal{H} \), we have
\[
mm' A \langle K^* f, K^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq MM'B \|f\|^2.
\]

Thus, \( W \) is a \((C, C')\)-controlled \( K \)-fusion frame.

Conversely, Assume that \( W \) is a \((C, C')\)-controlled \( K \)-fusion frame with bounds \( A \) and \( B \). Then for each \( f \in \mathcal{H} \), we have
\[
A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \|f\|^2.
\]
By assumption, $C, C' \in GL^+(\mathcal{H})$, $CC' = C'C$ and via theorem 5, we have

$$((C)^{\frac{1}{2}}(C')^{-\frac{1}{2}})^* = (C'^*)^{-\frac{1}{2}}(C^*)^{\frac{1}{2}}$$
$$= (C')^{-\frac{1}{2}}(C)^{\frac{1}{2}}$$
$$= (C)^{\frac{1}{2}}(C')^{-\frac{1}{2}}.$$

Then, for each $f \in \mathcal{H}$, we have

$$A_{CC'}(K^*f, K^*f) = A_{CC'}((CC')^{\frac{1}{2}}(CC')^{-\frac{1}{2}}K^*f, (CC')^{\frac{1}{2}}(CC')^{-\frac{1}{2}}K^*f)$$
$$\leq \|(CC')^{\frac{1}{2}}\|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i} C(CC')^{-\frac{1}{2}}f, \pi_{W_i} C'(CC')^{-\frac{1}{2}}f \rangle$$
$$= \|(CC')^{\frac{1}{2}}\|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i}(C)^{\frac{1}{2}}(C')^{-\frac{1}{2}}f, \pi_{W_i}(C')^{\frac{1}{2}}(C')^{-\frac{1}{2}}f \rangle$$
$$= \|(CC')^{\frac{1}{2}}\|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i}(C)^{\frac{1}{2}}(C')^{-\frac{1}{2}}f, \pi_{W_i}(C')^{\frac{1}{2}}(C')^{-\frac{1}{2}}f \rangle$$
$$= \|(CC')^{\frac{1}{2}}\|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i}(C)(C')^{-\frac{1}{2}}f, \pi_{W_i}(C')^{\frac{1}{2}}(C')^{-\frac{1}{2}}f \rangle$$
$$= \|(CC')^{\frac{1}{2}}\|^2 \sum_{i \in I} w_i^2 \pi_{W_i} f, f \rangle.$$

$$\implies A_{CC'}\|CC'\|^2 \|K^*f, K^*f\| \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} f, \pi_{W_i} f \rangle.$$

In the other hand

$$\sum_{i \in I} w_i^2 \langle \pi_{W_i} f, \pi_{W_i} f \rangle = \langle Sf, f \rangle,$$

where $Sf = \sum_{i \in I} w_i^2 \pi_{W_i} f$.

$$\langle Sf, f \rangle = \langle (CC')^{-\frac{1}{2}}(CC')^{\frac{1}{2}}Sf, f \rangle$$
$$= \langle (CC')^{\frac{1}{2}}Sf, (CC')^{-\frac{1}{2}}f \rangle$$
$$= \langle (CC')^{-\frac{1}{2}}Sf, (CC')^{\frac{1}{2}}f \rangle$$
$$= \langle CC'(CC')^{-\frac{1}{2}}Sf, (CC')^{-\frac{1}{2}}f \rangle$$
$$= \langle CC' S, (CC')^{-\frac{1}{2}}f \rangle$$
$$\leq B_{CC'}\|CC'\|^2 \|f\|^2.$$

Thus, $W$ is a $K$-fusion frame with bounds $A_{CC'}\|CC'\|^2 \|K^*f, K^*f\|^{-2}$ and $B_{CC'}\|CC'\|^{-2}$. 


**Theorem 14.** Let $K \in B(\mathcal{H})$, let $\mathcal{W}$ be a $(C, C)$-controlled $K$-fusion frame with bounds $A_{CC}$ and $B_{CC}$. If $U \in B(\mathcal{H})$ is an invertible operator such that $U^*C = CU^*$ and $K(U^*)^{-1} = (U^*)^{-1}K^*$, then $(UW_i, w_i)_{i \in I}$ is a $(C, C)$-controlled $K$-fusion frame for $\mathcal{H}$.

**Proof.** Assume that $\mathcal{W}$ is a $(C, C)$-controlled $K$-fusion frame with bounds $A_{CC}$ and $B_{CC}$. By definition, for each $f \in \mathcal{H}$, we have

$$A_{CC}\|K^*f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i}Cf\|^2 \leq B_{CC}\|f\|^2.$$ 

Now, let $f \in \mathcal{H}$. Via lemma 3 and since $UW_i$ is closed, we have

$$\|\pi_{W_i}CU^*f\| = \|\pi_{W_i}U^*Cf\| = \|\pi_{W_i}U^*\pi_{UW_i}Cf\| \leq \|U\|\|\pi_{UW_i}Cf\|.$$ 

Therefore,

$$A_{CC}\|K^*U^*f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i}CU^*f\|^2 \leq \|U\|^2 \sum_{i \in I} w_i^2 \|\pi_{UW_i}Cf\|^2.$$ 

$$\|K^*f\|^2 = \|K^*(U^*)^{-1}U^*f\|^2 = \|(U^*)^{-1}K^*U^*f\|^2 \leq \|U^{-1}\|^2\|K^*U^*f\|^2.$$ 

Then, we have

$$\frac{A_{CC}}{\|U^{-1}\|^2\|U\|^2} \|K^*f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{UW_i}Cf\|^2.$$ 

On the other hand, via lemma 3, we obtain with $U^{-1}$ instead of $T$:

$$\pi_{UW_i} = \pi_{UW_i}(U^*)^{-1}\pi_{W_i}U^*.$$ 

Thus,

$$\|\pi_{UW_i}Cf\|^2 = \|\pi_{UW_i}(U^*)^{-1}\pi_{W_i}U^*Cf\|^2 \leq \|U^{-1}\|^2\|\pi_{W_i}U^*Cf\|^2,$$

and it follows

$$\sum_{i \in I} w_i^2 \|\pi_{UW_i}Cf\|^2 \leq \|U^{-1}\|^2 \sum_{i \in I} w_i^2 \|\pi_{W_i}U^*Cf\|^2.$$ 

hence,

$$\sum_{i \in I} w_i^2 \|\pi_{UW_i}Cf\|^2 \leq B_{CC}\|U^{-1}\|^2\|U\|^2\|f\|^2.$$ 

Thus, $\mathcal{W}$ is a $(C, C)$-controlled $K$-fusion frame with bounds $A_{CC}\|U^{-1}\|^2\|U\|^2$ and $B_{CC}\|U^{-1}\|^2\|U\|^2$. 


Corollary 15. Let $K \in B(\mathcal{H})$, let $\mathcal{W}$ be a $(C, C)$-controlled $K$-fusion frame with bounds $A_{CC}$ and $B_{CC}$. If $U \in B(\mathcal{H})$ is a unitary operator such that $U^{-1}C = CU^{-1}$ and $K^*U = UK^*$, then $(UW_i, w_i)_{i \in I}$ is a $(C, C)$-controlled $K$-fusion frame for $\mathcal{H}$.

Proof. The result follows from Theorem 14.

4. Perturbation on Controlled $K$-fusion frame

The following result provides a sufficient condition on a family of closed subspaces of $\mathcal{H}$ to be a controlled $K$-fusion frame, in the presence of another controlled $K$-fusion frame. In fact it is a generalisation of Proposition 2.4 in [2], Proposition 4.6 in [8] and Proposition 2.6 in [17].

Proposition 16. Let $K \in B(\mathcal{H})$ be a closed range operator $\mathcal{R}_K$, let $T, U \in GL(\mathcal{H})$ and let $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ be a $(C, C')$-controlled $K$-fusion frame for $\mathcal{H}$ with lower and upper bounds $A_{CC'}$ and $B_{CC'}$, respectively. Let $\{V_i\}_{i \in I}$ be a family of closed subspaces of $\mathcal{H}$. If there exists a number $0 < R < A_{CC'}$ such that

$$0 < \sum_{i \in I} w_i^2 \langle C' (\pi_V - \pi_W) C f, f \rangle \leq R \|f\|^2, \forall f \in \mathcal{H},$$

(4.1)

then $\mathcal{V} = \{V_i, w_i\}_{i \in I}$ is a $(C, C')$-controlled Bessel $K$-fusion sequence for $\mathcal{H}$ and a $(C, C')$-controlled $K$-fusion frame for $\mathcal{R}_K$.

Proof. Let $f \in \mathcal{H}$. Considering that the family $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is a $(C, C')$-controlled $K$-fusion frame for $\mathcal{H}$, we have

$$A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \langle C' \pi_W C f, f \rangle \leq B_{CC'} \|f\|^2.$$

Firstly, let us prove that $\{V_i, w_i\}_{i \in I}$ is a $(C, C')$-controlled Bessel $K$-fusion sequence for $\mathcal{H}$. We have

$$\sum_{i \in I} w_i^2 \langle C' \pi_V C f, f \rangle = \sum_{i \in I} w_i^2 \langle C' (\pi_V - \pi_W) C f, f \rangle + \sum_{i \in I} w_i^2 \langle C' \pi_W C f, f \rangle \leq R \|f\|^2 + B_{CC'} \|f\|^2,$$

consequently,

$$\sum_{i \in I} w_i^2 \langle C' \pi_V C f, f \rangle \leq (R + B_{CC'}) \|f\|^2.$$

Now, let us establish for $\{V_i, w_i\}_{i \in I}$ the left-hand side. We obtain

$$\sum_{i \in I} w_i^2 \langle C' \pi_V C f, f \rangle = \sum_{i \in I} w_i^2 \langle C' \pi_W C f, f \rangle + \sum_{i \in I} w_i^2 \langle C' (\pi_V - \pi_W) C f, f \rangle \geq \sum_{i \in I} w_i^2 \langle C' \pi_W C f, f \rangle - \sum_{i \in I} w_i^2 \langle C' (\pi_V - \pi_W) C f, f \rangle \geq A_{CC'} \|K^* f\|^2 - R \|f\|^2.$$  

(4.2)
Therefore, for any \( f \in \mathcal{R}_K \), we have
\[
\|f\| = \|(K^+ |_{\mathcal{R}_K})^* K^* f\| \leq \|K^*\| \|K^* f\|,
\]
that is, \( \|K^* f\|^2 \geq \|K^+\|^{-2} \|f\|^2 \).

Then, according to (4.2) and (4.3), we have
\[
\sum_{i \in I} w_i^2 \langle C_i \pi_i V_i C f, f \rangle \geq (A_{CC} - R \|K^+\|^{-2}) \|K^* f\|^2.
\]
Which completes the proof.

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