Can rigidly rotating polytropes be sources of the Kerr metric?

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Abstract
We use a recent result by Cabezas et al (2007 Gen. Rel. Grav. 39 707) to build up an approximate solution to the gravitational field created by a rigidly rotating polytrope. We solve the linearized Einstein equations inside and outside the surface of zero pressure including second-order corrections due to rotational motion to get an asymptotically flat metric in a global harmonic coordinate system. We prove that if the metric and their first derivatives are continuous on the matching surface up to this order of approximation, the multipole moments of this metric cannot be fitted to those of the Kerr metric.

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1. Introduction
In this paper we explore the possibilities arising from the method introduced by Cabezas et al [1] in a recent paper (hereinafter CMMR) for computing approximate solutions of Einstein’s equations which can describe the stationary axisymmetric gravitational field of a rigidly rotating perfect fluid. In their paper, Cabezas et al study a fluid with a very simple equation of state; the mass–energy density is constant. However, they suggest that their approximation scheme may also be implemented with more realistic—and more complex—equations of state. Here we use their approach with some minor changes to deal with a polytropic fluid. However, we do not carry the approximation as far as Cabezas et al did. The main reason for this is that, in our problem, the metric cannot be written using elementary functions; it involves a few functions defined by non-trivial differential equations. Therefore, we stop at the level of the linearized Einstein theory but include quadratic effects in the rotational motion of the fluid. We hope, however, to work out true nonlinear terms for this metric in the future.

We aim, first, to show how a polytropic fluid can be incorporated into the CMMR framework and set up the mechanism to run. Secondly, we use the results the approach...
provides (which do not differ much from those of Newtonian theory) to question the Kerr metric as a ‘good’ metric to describe the gravitational field outside a rigidly rotating polytrope.

The existence of a suitable source for the Kerr metric which be a reasonable perfect fluid has been discussed at some length. This problem is different to that solved by W Israel in his famous paper ‘Source of the Kerr metric’ [2], since one looks for finite balls of matter instead of studying the singularities of the Kerr metric; nevertheless, many of the former works on this topic can be found among the references included in this paper. An important result is due to Roos [3], who points out that a perfect fluid cannot be ruled out as an interior of the Kerr metric by local arguments based on the constraints that matching imposes on the interior metric: there always exists an interior solution in a neighborhood of the surface of the fluid. However, this argument is insufficient for a problem which involves elliptic differential equations. Moreover, the proof that Mars and Senovilla [4] and Vera [5] give for the uniqueness of the exterior metric surrounding a stationary axisymmetric perfect fluid requires not only the Darmois matching conditions but also an asymptotically flat exterior metric. All this strengthens the case for the solution to this problem being global, since it implies that there are conditions on the metric in domains of the spacetime manifold which are not close to one another: regularity conditions on the symmetry axis, matching on the fluid surface and asymptotic flatness at infinity. Even though we do not actually obtain an exact solution to all these problems, we deal with all of them and solve them in a coherent way since they are all worked out up to the order of approximation we are considering. In this sense, the CMMR approach that we follow in this paper casts a reasonable doubt on the existence of a normal source for the Kerr metric.

There is a rather heuristic argument due to Wolf and Neugebauer [6] which also suggests that the Kerr metric is not suitable to describe the exterior gravitational field. Although it does not follow the scheme we have sketched above, it too questions perfect fluids as sources of the Kerr metric. For this reason we feel it is worth mentioning. As we have already mentioned, one of the most important aspects of the CMMR approach is its global character. This arises from several assumptions, one of the most important of which concerns the coordinates. They are harmonics and cover the entire spacetime manifold in such a way that the metric is of class $C^1$ on the matching surface (Lichnerowicz matching conditions [7]). Another important assumption concerns the metric itself. It is assumed that the metric can be expanded in a double power series of two dimensionless parameters, $\lambda$ and $\Omega$, the first of which takes into account the weakness of the field and the second is related to the rotation of the fluid (which is taken to be in rigid motion). This general framework is implemented by making other complementary assumptions concerning the dependence of the metric on the spherical coordinates, the form of the matching surface and the expansion of the metric in terms of the parameters $\lambda$ and $\Omega$. We aim to provide just enough information about all these assumptions to allow the reader to follow our paper easily. However, for a more detailed explanation we direct the reader to the original CMMR article and to a previous paper by Cabezas and Ruiz [8].

In the first three sections of the paper we introduce the polytropic problem guided by the CMMR approach. Our aim is to set up the notation and almost everything that is necessary to understand the approximation scheme. We devote section 4 to solving the interior problem and section 5 to matching it to the CMMR exterior solution. The relationship between the global approximate metric we build up and the Kerr metric is discussed in section 6. We prove a theorem that excludes the Kerr metric from the set of admissible exterior metrics. In section 7 we make some comments and develop an argument to extend our result to other barotropic equations of state.
2. Density and pressure

We consider a Papapetrou-type stationary and axisymmetric metric. We can therefore choose coordinates \( \{t, r, \theta, \phi\} \) adapted to the two commuting Killing fields defining the symmetry, and to the two-dimensional surfaces orthogonal to their orbits. We denote by \( \xi = \partial_t \), the timelike Killing field, and by \( \zeta = \partial_\phi \), the Azimuthal spacelike field, so the metric can be written as follows:

\[
g = \gamma_{tt} \omega_t \otimes \omega_t + \gamma_{t\phi} \omega_t \otimes \omega_\phi + \gamma_{\phi\phi} \omega_\phi \otimes \omega_\phi,
\]

where \( \gamma \)'s are functions of \( r \) and \( \theta \) alone, and \( \omega_t, \omega_\phi = \partial_t, \partial_\phi \). We want to link this metric to a rigidly rotating perfect fluid. To do so we consider an energy–momentum tensor,

\[
T = (\mu + p) u \otimes u + pg,
\]

invariant under the two Killing fields, so the functions \( \mu \) and \( p \) (the density and pressure of the fluid) are also functions of \( r \) and \( \theta \) alone, and \( u \), the velocity of the fluid, is a linear combination of the two Killing fields,

\[
u = \psi (\xi + \omega \zeta),
\]

where \( \omega \) is a constant and \( \psi \),

\[
\psi \equiv \left[ -(\gamma_{tt} + 2\omega \gamma_{t\phi} r \sin \theta + \omega^2 \gamma_{\phi\phi} r^2 \sin^2 \theta) \right]^{-\frac{1}{2}}
\]

is a normalization factor such that \( g(u, u) = -1 \).

In the framework defined by equations (1)–(3), the energy–momentum conservation law reduces to a couple of first-order linear differential equations [9],

\[
\partial_a p - (\mu + p) \partial_a \ln \psi = 0 \quad (a, b, \ldots = r, \theta).
\]

These can be integrated if the density and pressure of the fluid are related by a barotropic equation of state, \( p = p(\mu) \). For a polytropic fluid,

\[
p = k \mu^{1+\frac{2}{n}} \quad (n > 0),
\]

and a simple calculation leads to the following expressions for \( \mu \) and \( p \) as functions of the potential \( \psi \):

\[
\mu = \frac{1}{k^n} \left[ \left( \frac{\psi}{\psi_\Sigma} \right)^{\frac{1}{n}} - 1 \right]^n, \quad p = \frac{1}{k^n} \left[ \left( \frac{\psi}{\psi_\Sigma} \right)^{\frac{1}{n}} - 1 \right]^{n+1}.
\]

We have chosen the integration constant \( \psi_\Sigma \) in such a way that the pressure and density of the fluid vanish on the surface

\[
\Sigma : \psi(r, \theta) = \psi_\Sigma,
\]

thus defining the boundary of the fluid.

Equations (7) and (8) play an important part in our approach. Both results are exact for a rigidly rotating perfect fluid.

3. Weak field approximation

We look for a solution to the problem set up in the preceding section. We assume it takes the form of a metric \( g(\lambda, \Omega) \) depending on two dimensionless parameters \( \lambda \) and \( \Omega \) having the
following properties:

(1) \( g(\lambda, \Omega) \) tends to the Minkowski metric if \( \lambda \) goes to 0, that is,
\[
g(0, \Omega) = \eta = -\omega^t \otimes \omega^t + \omega^r \otimes \omega^r + \omega^\theta \otimes \omega^\theta + \omega^\phi \otimes \omega^\phi;
\]
we use this limit to identify coordinates \( \{t, r, \theta, \phi\} \) as the standard polar coordinates of flat spacetime;

(2) \( g(\lambda, \Omega) \) tends to a static spherically symmetric metric when \( \Omega \) goes to 0, which is a solution of Einstein’s equations for a fluid with the same equation of state, say a polytrope, as that of the stationary axisymmetric solution.

The parameter \( \Omega \) must be related to the rotational motion of the fluid. Following CMMR, we assume \( \omega = \lambda^{1/2} \Omega_0^{-1} \), where \( r_0 \) is a constant with the dimensions of length that may be identified as the radius of the fluid ball in the static limit. The other parameter \( \lambda \) obviously accounts for the gravitational field strength. Though we cannot yet identify it with any definite quantity (but we suppose that it may be proportional to the quotient of the ‘mass’ of the fluid and its ‘radius’) we are sure about the role it plays in the approximation scheme: we expect the metric to behave as follows for small values of \( \lambda \):
\[
\begin{align*}
\gamma_{tt} &\approx -1 + \lambda f_{tt}, & \gamma_{t\phi} &\approx \lambda^{3/2} \Omega f_{t\phi}, & \gamma_{\phi\phi} &\approx 1 + \lambda f_{\phi\phi}, \\
\gamma_{rr} &\approx 1 + \lambda f_{rr}, & \gamma_{r\theta} &\approx \lambda f_{r\theta}, & \gamma_{\theta\theta} &\approx 1 + \lambda f_{\theta\theta}.
\end{align*}
\]
These expressions agree with the two conditions we impose on the metric and they also give a more precise meaning to the kind of approximation we are proposing.

The above expansion for the metric in \( \lambda \) leads to a similar expansion of the energy–momentum tensor. Taking into account (10), it can easily be checked that all the quantities entering in \( \mathcal{T} \) except \( \mu \) and \( p \) have non-zero values at \( \lambda = 0 \): \( g \approx \eta, \psi \approx 1 \) and \( u \approx -\omega^t \) (we use the same symbol to denote the vector field and the 1-form). However, a coherent perturbation scheme based on the parameter \( \lambda \) needs an energy–momentum tensor which tends to zero with the first power of \( \lambda \). This can only be achieved if the density and the pressure are at least first-order quantities in \( \lambda \). This does not seem to be evident.

The normalization factor up to first order in \( \lambda \) reads:
\[
\psi \approx 1 + \frac{\lambda}{2} (f_{tt} + \Omega^2 \eta^2 \sin^2 \theta),
\]
where \( \eta = r/r_0 \). Since the constant \( \psi_{\Sigma} \) is equal to the value of \( \psi \) on the zero pressure surface, we can assume a similar expansion for it in \( \lambda \) and \( \Omega \), so we write
\[
\psi_{\Sigma} \approx 1 + \lambda \left[ M_0 + \Omega^2 \left( \frac{1}{3} - M_0 \kappa \right) \right],
\]
where \( M_0 \) and \( \kappa \) are constants. (This is just a way to parametrize the two arbitrary constants entering into the expression of \( \psi_{\Sigma} \) which does not impose any constraint on it; that choice is related to the assumptions we make on the matching, as will be seen in section 5.) Substituting expressions (11) and (12) into equation (7) and expanding in \( \lambda \), we get
\[
\begin{align*}
\mu &\approx \frac{\lambda^n}{k^n (n+1)^n} \left[ \frac{1}{2} (f_{tt} - 2M_0) + \Omega^2 \left( \frac{1}{2} \eta^2 \sin^2 \theta - \frac{1}{3} + M_0 \kappa \right) \right]^n, \\
&\equiv \frac{\lambda^n}{k^n (n+1)^n} q^n, \\
p &\approx \frac{\lambda^{n+1}}{k^n (n+1)^{n+1}} q^{n+1}.
\end{align*}
\]
This shows that we can make \( \mu \) a quantity of first order in \( \lambda \) if we assume \( k \propto \lambda^{1/2} \); a suitable choice that accounts for physical density units and other technical facts related to the
matching is
\[
\frac{1}{k^n} = \frac{(n + 1)^n}{4\pi r_s^2} \lambda^{1-n},
\]  
(15)

where \( r_s \) is a new length constant. This equation can actually be a definition of the parameter \( \lambda \) whenever we can establish a relationship between the length scales, \( r_0 \) and \( r_s \). We shall do this in section 5.

We now come to the field equations. We do not aim to work out an exact solution but rather an approximate one. Following CMMR once more, we use the so-called post-Minkowskian approximation scheme (see, for instance, [10] for a general description of the approach). It requires the introduction of new coordinates \( \{t, x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = \cos \theta\} \); standard Cartesian coordinates associated with the spherical-type coordinates \( \{t, r, \theta, \phi\} \), and the quantity
\[
h_{\alpha\beta} \equiv g_{\alpha\beta} - \eta_{\alpha\beta},
\]  
(16)

which is assumed to be of at least first order in \( \lambda \). Here the indices \( \alpha, \beta, \ldots \) stand for the new coordinates, and expression (16) relates the components of the spacetime metric and the Minkowski metric in this system of coordinates. That is, \( (\eta_{\alpha\beta}) = \text{diag}(-1, 1, 1, 1) \) and \( g_{\alpha\beta} \) are some combinations of \( \gamma \)'s introduced in (1). These assumptions are consistent with those made before in terms of the old coordinates (equation (10)). Moreover, we require the new Cartesian-like coordinates to be harmonic coordinates.

Under the assumptions we have been making concerning the metric and the coordinates up to now, the linearized Einstein equations imply that \( h_{\alpha\beta} \) must be a solution of the differential system
\[
\Delta h_{00} = -2 \frac{\lambda}{r_s^2} q^n,
\]  
(17)

\[
\Delta h_{0i} = 4 \frac{\lambda^{3/2} \Omega}{r_s^2} q^n \eta \sin \theta m_i,
\]  
(18)

\[
\Delta h_{ij} = -2 \frac{\lambda}{r_s^2} q^n \delta_{ij},
\]  
(19)

\[
\partial^k \left( h_{k\mu} - \frac{1}{2} h_{\mu\nu} \right) = 0,
\]  
(20)

where \( \Delta \equiv \delta^{ij} \partial_i \partial_j \) stands for the standard Laplacian in Cartesian coordinates, \( h \equiv \eta^{\nu\mu} h_{\nu\mu} \), and \( (m_i) \equiv (\sin \phi, \cos \phi, 0) \). As defined in (13), \( q \) depends on \( f_t \), which is not a great problem since \( h_{00} \approx \lambda f_t \). Interestingly the pressure does not contribute to the right-hand side of equations (17), (18), or (19). It can easily be checked from definition (15) and formula (14) that \( p \) is a second-order quantity in \( \lambda \).

4. Slow rotation solution

In this section we work out an approximate metric which describes the geometry of the spacetime we are interested in, which involves the rotation parameter \( \Omega \). We do not intend to find an exact solution to the linear problem defined by equations (17)–(20), but rather an approximate solution up to order \( \Omega^2 \). The differential equations used to describe this system include one that should verify the component \( h_{00} \), which seems to be the most difficult to
solve, since $h_{00}$ appears on the right-hand side of the equation in a non-trivial way. So, first we deal with this equation and later we try to find a solution for the whole system.

Let us set $\Omega = 0$ in (13). Substituting the result of (17) and replacing $h_{00}$ by $\lambda f_{tt}$ (so we eliminate any dependence on $\lambda$) gives the following equation for the time–time component:

$$\Delta f_{tt} = \frac{2}{r^2} \left( \frac{1}{2} f_{tt} - M_0^n \right) .$$

We can assume spherical symmetry in this limit; that is, $f_{tt}(r)$. Then, by introducing a new function,

$$f_{tt} = 2 \left[ \Phi \left( \frac{r}{r_s} \right) + M_0 \right] ,$$

and changing the independent variable, $r = r_s s$, we arrive at the Lane–Emden equation of Newtonian theory [11] (see also [12, 13]),

$$\Phi'' + \frac{2}{s} \Phi' + \Phi^n = 0$$

(23)

We need a solution of this equation which is regular at the origin of the coordinates, $s = 0$. We can select it by looking at well-known results of classical polytrope theory: the solution of the Lane–Emden equation that satisfies the initial data $\Phi(0) = 1$ and $\Phi'(0) = 0$ is chosen (see [11]). Hereafter we shall identify the symbol $\Phi$ with that particular solution of the Lane–Emden equation. It is important to remember that there are only analytic expressions for $\Phi$ for a few polytropic indices, $n = 0$ (constant mass density), $n = 1$ (the Lane–Emden equation is linear) and $n = 5$ [11]. In all other cases $\Phi$ has to be calculated using numerical integration methods.

We now have an approximation of $h_{00}$ which is of zeroth order in $\Omega$. To obtain a second-order correction, we replace $h_{00}$ by

$$h_{00}(r, \theta) \approx \lambda f_{tt}(r, \theta) \rightarrow 2 \lambda \left[ \Phi(s) + M_0 \right] + \lambda \Omega^2 f_{tt}(r, \theta) .$$

Substituting this expression into (17), expanding this equation to include all the quadratic terms in $\Omega$, and taking into account that $\Phi$ verifies (23), we get the following linear equation for the new $f_{tt}$:

$$\Delta f_{tt} + \frac{n}{r_s^2} \Phi^{n-1} f_{tt} = -\frac{2n}{r_s^2} \Phi^{n-1} \left( \frac{1}{2} \eta^2 \sin^2 \theta - \frac{1}{3} + M_0 \kappa \right) .$$

(25)

This does not appear easy to solve. However, since $\Phi$ does not depend on the $\theta$ coordinate, we can look for a solution of the homogeneous equation associated with (25) as a function of $r$ times a Legendre polynomial. Following the CMMR approach, we expect that any term of order $\Omega^2$ in the interior metric component $g_{00}$ cannot depend on a Legendre polynomial of order higher than 2. Let us note this is the dependence on the $\theta$ coordinate we get for such terms in the $g_{00}$ component of the exterior metric assuming that Geroch–Hansen multipole moments of this asymptotically flat vacuum metric depend on the rotational velocity in a similar way to that exhibited by the mass multipole moments of MacLaurin ellipsoids (see [1, 8] for more details). If we accept this rule, the following function may be enough for our purposes:

$$f_{tt}(r, \theta) = \frac{4r^2}{r_0^2} \phi_0(s) + \frac{2}{3} (1 - \eta^2) - 2 M_0 \kappa + \left[ \phi_2(s) + \frac{2}{3} \eta^2 \right] P_2(\cos \theta) ,$$

(26)

where $\phi_0(s)$ and $\phi_2(s)$ are, respectively, solutions of the following two linear differential equations,

$$\phi_0'' + \frac{2}{s} \phi_0' + n \Phi^{n-1} \phi_0 = 1, \quad \phi_2'' + \frac{2}{s} \phi_2' + \left( n \Phi^{n-1} - \frac{6}{s^2} \right) \phi_2 = 0 .$$

(27)
Both equations admit a 1-parameter family of solutions which are regular at \( s = 0 \). We refer to these regular solution when we write the symbols \( \phi_0 \) and \( \phi_2 \).

We now use the approximate solution for \( h_{00} \) given by (24) and (26) to expand the right-hand side of (18). Eliminating terms in \( \Omega/3 \) amounts to substituting \( q \) by \( \Phi(s) \), and gives us:

\[
\Delta h_{00} = 4 \frac{\lambda^3/2}{r_0^2} \Phi^\eta \eta \sin \theta m_i.
\] (28)

According to the CMMR prescription, the solution of this equation that we need has two parts: a specific solution of the full equation and a solution of the homogeneous equation that depends on the angular coordinates through a spherical harmonic vector, that is:

\[
h_{00}(r, \theta, \varphi) = -\frac{\lambda^3/2}{2} \frac{\Omega}{r_0^2} \left[ j_1 + 4 \left( \Phi(s) - \frac{1}{3} \frac{2r_0^3}{r_0^3 \eta^3} I(s) \right) \right] \eta \sin \theta m_i,
\] (29)

where \( j_1 \) is a constant and

\[
I(s) \equiv \int_0^s r^2 \Phi(r) \, dr.
\] (30)

This solution is regular at \( s = 0 \) and it also satisfies the harmonic condition (20).

The equations involving the components \( h_{ij} \) are easier to solve. Comparing (19) with (17), it is obvious that \( h_{00} h_{ij} \) is a specific solution of the first equation if \( h_{00} \) is a solution of the second one. Moreover, the couple \( h_{00} \) and \( h_{ij} \) so defined always satisfies the harmonic condition (20). We may add to \( h_{ij} \) any solution of the homogeneous equation and the harmonic condition; that is, the terms associated with the constants \( a_0, b_0, a_2 \) and \( b_2 \), but not \( m_2 \), in equation (33) of CMMR. This part of the solution will be set equal to zero once the matching conditions are imposed, so we prefer not to make it explicit in order to avoid a lot of definitions. There is another reason to ignore these terms: they all can be eliminated up to this order of approximation by a simple change of harmonic coordinates, so to include them or not can be seen as a gauge choice.

Finally, bringing together the results obtained in this section and joining the components of the approximate interior metric in a tensor expression we have

\[
g_{\text{int}} \approx (-1 + 2\lambda \Phi + 2\lambda M_0) T_0 + (1 + 2\lambda \Phi + 2\lambda M_0) D_0
\]

\[
+ \lambda \Omega^2 \left[ \frac{4r_0^2}{r_0^2} \Phi_0 + \frac{2}{3} (1 - \eta^2) - 2M_0 \kappa \right] (T_0 + D_0)
\]

\[
+ \lambda \Omega^2 \left( \Phi_0 + \frac{2}{3} \eta^2 \right) (T_2 + D_2)
\]

\[
+ \lambda^{3/2} \Omega \eta \left[ j_1 + 4 \left( \Phi - \frac{1}{3} \frac{2r_0^3}{r_0^3 \eta^3} I \right) \right] Z_1,
\] (31)

where we have introduced the CMMR notations to denote spherical harmonic tensors, \( T_0 \equiv \omega^i \otimes \omega^i \), \( D_0 \equiv \delta_{ij} \, dx^i \otimes dx^j \), \( T_2 \equiv P_2(\cos \theta) \omega^i \otimes \omega^i \), \( D_2 \equiv P_2(\cos \theta) \delta_{ij} \, dx^i \otimes dx^j \) and \( Z_1 \equiv P_1(\cos \theta) (\omega^\varphi \otimes \omega^\varphi + \omega^\psi \otimes \omega^\psi) \).

5. Global metric

The next step is to connect the interior metric (31), through the zero pressure surface defined by (8), with an asymptotically flat vacuum metric. To accomplish this we use the CMMR
The constant labeled $J$ must be viewed as a choice of gauge. Anyhow, the matching conditions impose that they all must vanish. The constant we omit in the interior metric, so to set them equal to zero can be viewed as a choice of gauge. Anyhow, the matching conditions impose that they all must vanish. The constant labeled $J_3$ in CMMR has not been included above because it is related to terms of order $\Omega^3$, and we stop our approximation at quadratic level in the $\Omega$ parameter.

We have an approximate expression for the interior metric so we can use it to get a parametric equation for the matching surface (8). However, it is preferable to do this using the exterior metric, since we assume that the metric is continuous over the zero pressure surface, as in CMMR. This leads to

$$
\Sigma : r \approx r_0 \left[ 1 + \Omega^2 \kappa + \frac{\Omega^2}{M_0} \left( M_2 - \frac{1}{3} \right) P_2(\cos \theta) \right],
$$

and also to the expression for $\psi$, which we have been using (12). We have included a new constant, $\kappa$, which was absent in the CMMR expression for the parametric equation of the matching surface. Even though it was superfluous there, we need it to solve the matching problem here. This is rather a technical fact linked to the lack of freedom we have by using the standard solution of the Lane–Emden equation, since it does not include any arbitrary constant. Nevertheless, we want to remark that these procedures applied to the constant density problem leads to the CMMR metric, and it also gives the same matching surface, that is $\kappa$ must be equal to zero.

We understand the matching of interior and exterior metrics in the same way as Lichnerowicz does: the metric and its derivatives must be continuous on the matching surface [7]. However, since we do not have an exact solution, we require these conditions to be fulfilled up to the same order of approximation as the metric in a solution of the field equations. This means that we evaluate both metrics on the surface defined by (33), then we expand the result in $\Omega$ neglecting terms of a higher order than $\Omega^2$, then we develop the matching conditions in spherical harmonic tensors, and finally we equate all the coefficients of the expansion to zero to get a set of algebraic equations. This procedure leads to the following six constraints,

$$
\Phi \left( \frac{r_0}{r_s} \right) + \Omega^2 \left[ 2r_0^2 \phi_0 - \frac{r_0}{r_s} \Phi + \frac{r_0}{r_s} \Phi' \left( \frac{r_0}{r_s} \right) \right] \approx 0,
$$

$$
M_0 + \frac{r_0}{r_s} \Phi' \left( \frac{r_0}{r_s} \right) + \Omega^2 \left[ 2r_0^2 \phi_0 + \frac{r_0^2}{r_s^2} \kappa \Phi'' \left( \frac{r_0}{r_s} \right) - \frac{2}{3} - 2M_0 \kappa \right] \approx 0,
$$

$$
\phi_2 \left( \frac{r_0}{r_s} \right) + \frac{2r_0}{r_s M_0} \left( M_2 - \frac{1}{3} \right) \Phi' \left( \frac{r_0}{r_s} \right) \approx 0,
$$

$$
M_2 + \frac{4}{3} + \frac{r_0}{2r_s} \phi_2' \left( \frac{r_0}{r_s} \right) + \frac{r_0^2}{r_s^2} M_0 \left( M_2 - \frac{1}{3} \right) \Phi'' \left( \frac{r_0}{r_s} \right) \approx 0,
$$

$$
J_1 - 2J_1 = -\frac{4}{3} - \frac{8r_0^3}{r_0^3} I \left( \frac{r_0}{r_s} \right) + 4 \Phi \left( \frac{r_0}{r_s} \right) \approx 0,
$$

$$
J_1 + 4J_1 = -\frac{4}{3} + \frac{16r_0^3}{r_0^3} I \left( \frac{r_0}{r_s} \right) - 4 \Phi \left( \frac{r_0}{r_s} \right) + 4 \frac{r_0}{r_s} \Phi' \left( \frac{r_0}{r_s} \right) \approx 0.
$$
One may expect a large number of constraints, but as \( h_{ij} \) is essentially equal to \( h_{00} \), all the constraints we can derive from the matching of these components are already included in the matching of \( h_{00} \).

There are two types of constant in system (34). Two of which are of the first type, \( r_0/r_s \) and \( M_0 \), come from the static limit; the others, of the second type, appear at a level where rotation is taken into account. This last class can be taken as pure numbers when we try to solve the matching conditions but this is not the case for the first class, which may be linear functions of \( \lambda \Ω^2 \) (constants are seen as functions of \( \lambda \) and \( \Ω \) in the CMMR approach). However, in our problem we mean that it is better to consider \( r_s \) as a true constant and to assume that there is no \( \Ω^2 \) term in the expansion of \( M_0 \). This leads to

\[
\Phi \left( \frac{r_0}{r_s} \right) = 0, \quad M_0 \approx -\frac{r_0}{r_s} \Phi' \left( \frac{r_0}{r_s} \right),
\]

(35)

two well-known predictions of Newtonian theory [11]. In order to justify our choice: first, recall expression (15), which becomes an interesting definition of the parameter \( \lambda \) if \( r_s \) is not a free constant. The first equation in (35) ensures this since it permits \( r_0/r_s \) to be determined in terms of the zeroes of \( \Phi(s) \). So we can write, \( r_s = r_0/s_0 \), where \( s_0 \) is a number3. Another technical reason is based on a careful inspection of the first matching constraint in (34). It seems that any dependence on \( \Omega^2 \) we assign to \( r_s \) may be absorbed into the extra constant \( \kappa \). Second, we do not expect any correction to the mass at the Newtonian level; this is the meaning of our assumption concerning \( M_0 \), the monopole moment of the exterior gravitational field.

Taking into account (35), we can solve the matching conditions (34) to get approximate expressions for the first multipole moments of the metric,

\[
M_2 \approx \frac{1}{3} + \frac{1}{2} \Phi(s_0), \quad J_1 \approx \frac{2}{3} M_0 - \frac{4}{s_0} I(s_0),
\]

(36)

the constants of the interior metric,

\[
\kappa \approx 2 \frac{\Phi(s_0)}{s_0^2 M_0}, \quad j_1 \approx \frac{4}{3} (1 + M_0),
\]

(37)

and the initial data needed to pick out regular solutions of the differential equations (27),

\[
\Phi_0(s_0) \approx \frac{s_0}{3}, \quad \frac{1}{2} s_0 \Phi_2(s_0) + \frac{3}{2} \Phi_2(s_0) + \frac{5}{3} \approx 0.
\]

(38)

Substituting (36) and (37) into expressions (31) and (32), we get an approximate solution of Einstein’s equations inside and outside the polytropic fluid up to order \( \lambda^3/2 \) and \( \Omega^2 \), which is of class \( C^1 \) on the surface of the fluid up to the same order of approximation if functions \( \Phi_0(s) \) and \( \Phi_2(s) \) satisfy conditions (38).

6. Kerr metric

The vacuum metric (32) can be used to describe the Kerr metric near infinity by choosing suitable values for the constants \( M_0 \), \( J_1 \) and \( M_2 \). This is a straightforward consequence of multipole moment theory in harmonic coordinates [14]. So, to the extent that a stationary axisymmetric vacuum metric can be identified by its multipole moments [15, 16] we can say that metric (32) coincides with the Kerr metric if4:

\[
m = \lambda r_0 M_0, \quad ma = \lambda^{3/2} r_0^2 J_1, \quad -ma^2 = \lambda \Omega^2 r_0^3 M_2,
\]

(39)

3 A classical theorem proves that the solution of the Lane–Emden equation corresponding to the initial data \( \Phi(0) = 1 \) and \( \Phi(0) = 0 \) has a first zero in the interval \( 0 < s < \infty \) if \( 0 \leq n < 5 \) [11].

4 For a development of Kerr multipole moments that is closer to our point of view, see [17].
where \( m \) and \( a \) are the standard parameters of the Kerr metric in Boyer–Lindquist coordinates. However, if we consider metric (32) to be the exterior gravitational field of a polytrope, the matching constraints restrict the values that \( M_0, J_1 \) and \( M_2 \) can take. Therefore, we may ask if we still have enough freedom to make the exterior metric into the Kerr metric.

Solving the first two expressions in (39) for \( m \) and \( a \), and substituting the results into the third one, we find that the Kerr quadrupole moment, \(-ma^2\), is a quantity of order \( \lambda^2 \) not of order \( \lambda \) as it appears on the right-hand side of the equation. This condition cannot be fulfilled unless \( M_2 \) is a quantity of order \( \lambda \). Equations (36) and (38) transform this condition on \( M_2 \) into two initial data for the function \( \phi_2(s) \)

\[
\phi_2(s_0) \approx -\frac{2}{3}, \quad \phi_2'(s_0) \approx -\frac{4}{3s_0}.
\]

We must add to them the regularity condition \( s = 0 \). This means there are too many conditions for a function defined by a linear second-order differential equation. Let us prove that it is not possible.

**Theorem.** Let \( \Phi(s) \) be the solution of the Lane–Emden equation \((0 \leq n < 5)\) defined by the initial data \( \Phi(0) = 1, \Phi'(0) = 0 \), and let \( s_0 \) be the first zero of \( \Phi(s) \) in the interval \((0, +\infty)\), that is \( \Phi(s_0) = 0 \). The differential equation

\[
(s^2\phi_1')' + (ns^2\Phi^{n-1} - 6)\phi_2 = 0,
\]

has no smooth solution in the interval \((0, s_0)\) such that

\[
\phi_2(0) = 0, \quad \phi_2(s_0) = -\frac{2}{3}, \quad \phi_2'(s_0) = -\frac{4}{3s_0}.
\]

Regular solutions of equation (41) vanish at \( s = 0 \) as \( \phi_2(s) \sim c_2s^2 \) \((c_2 \neq 0)\). Then, to prove the theorem we have to demonstrate that any solution of this kind cannot take the values (42) at \( s_0 \). Let us first prove the following.

**Lemma.** If a solution \( \phi_2(s) \neq 0 \) of the differential equation (41) is regular at \( s = 0 \), then it does not vanish in the interval \((0, s_0)\).

Let us consider the differential equation,

\[
(s^2\phi_1')' + (ns^2\Phi^{n-1} - 2)\phi_1 = 0.
\]

Since it is just the derivative with respect to \( s \) of the Lane–Emden equation (23), its regular solution at \( s = 0 \) is \( \phi_1(s) = c_1\Phi'(s) \) \((c_1 \neq 0)\). If \( c_1 > 0 \), \( \phi_1(s) \) is negative in \((0, s_0)\) because \( \Phi(s) \) is a decreasing function of \( s \) in that domain. Let us set \( c_1 = 1 \).

Multiplying (41) by \( \Phi'(s) \) and (43) by \( \phi_2(s) \), then subtracting them, and integrating the result over \((0, s)\), we get

\[
W(s) = \Phi'(s)\phi_2'(s) - \Phi''(s)\phi_2(s) = \frac{4}{s^2} \int_0^s \Phi'(\tau)\phi_2(\tau) \, d\tau.
\]

The integral on the right-hand side is negative near \( s = 0 \) if \( c_2 > 0 \) and positive if \( c_2 < 0 \). Let us take \( c_2 > 0 \) (the argument runs the same for \( c_2 < 0 \)). Since \( \Phi'(s) \) is negative, the product under the integral symbol is negative in \((0, s_0)\) unless \( \phi_2(s) \) vanishes. Let us assume that \( \phi_2(s_0) = 0, 0 < s_0 < s_0 \), and \( \phi_2(s) > 0 \) if \( 0 < s < s_0 \). Then, since \( \phi_2(s_0) \leq 0 \), we have \( W(s) \geq 0 \), but the right-hand side of (44) is still negative at \( s_0 \). Therefore, we conclude that all the zeroes of any regular \( \phi_2(s) \) should be outside the interval \((0, s_0)\).

Let us return to the proof of the theorem. We introduce a new function,

\[
y(s) \equiv s^n \left[ \frac{\phi_2(s)}{\phi_2'(s)} - \frac{2}{3} \right].
\]
If \( \varphi_2(s) \) is a solution of (41), \( y(s) \) is a solution of the first-order differential equation

\[
y' + \frac{y^2}{s^6} + ns^6\Phi^{-1} = 0.
\]

(46)

Clearly \( y(s) \) is the same function for all regular solutions of (41), particularly the solution of (46) defined by the initial data \( y(0) = 0 \). The other two conditions in (42) lead to a further one on \( y(s) \): \( y(s_0) = 0 \). The second and third terms in equation (46) are positive in \((0, s_0)\), so \( y(s) \) is a decreasing function there. Also \( y(s) \) is negative near \( s = 0 \) since \( y(0) = 0 \), and it is negative in \((0, s_0)\) if it does not run away to minus infinity for an intermediate value of \( s \). From the definition of \( y(s) \) and the preceding lemma, we know that this does not happen. Therefore \( y(s) < 0 \) at \( s_0 \) so \( y(s_0) \neq 0 \) and (42) is not fulfilled.

7. Comments

We have given an argument based on a perturbation approach which points to rejecting the Kerr metric as the exterior metric of a polytropic fluid. Though we do not go further than the linear approximation, but considering quadratic terms in the rotation, the approximation surprisingly seems to be enough to arrive at that conclusion by analyzing the constraints that the Kerr metric imposes on the quadrupole moment of the exterior field.

One relevant fact of our approach is that the global metric can be written in a global harmonic coordinate system. This is a straightforward consequence of the Lichnerowicz matching condition we impose on the metric, which requires the metric and their first derivatives to be continuous on the limiting surface. It works, but this is not the standard way to set the matching problem. Therefore one may question if there is another member of the family of approximate polytropic metrics we get which can be matched properly to a vacuum asymptotically flat exterior metric but it actually does not admit a global harmonic coordinate system. We mean that it is not true, but we have not yet proved it. The answer to that problem surely needs more elaborated methods than those we have used in this paper in order to be solved (see [18]).

However, let us point out that our global metric does contain the desired number of free constants: \( \omega \), which accounts for the rotation; and \( r_0 \), which may be related to the central density by means of equations (13), (15) and (37),

\[
\mu(0) \approx \frac{\lambda}{4\pi r_0^2} \left[ s_0^2 + 2n\Omega^2\varphi(0) \right].
\]

(47)

And it is widely thought that such a 2-parameter metric is the general solution to this problem in Einstein’s theory as well as in Newtonian gravity.

Let us give an argument which may extend our results to perfect fluids with other barotropic equations of state which are not of the polytropic kind. The integrability condition of Boyer’s equations (5) can be fulfilled by setting the density \( \mu \) to be a function of the normalization factor \( \psi \). Introducing the new variable \( X = \psi/\psi_s \), we can write \( \mu = f(X) \), and the pressure is given by

\[
p(X) = X \int_1^X \chi^{-2} f(\chi) d\chi.
\]

(48)

Then \( X = 1 \) defines the fluid boundary, \( p(1) = 0 \). It is clear that \( p \) is a function of \( \mu \), at least in a local sense, so the fluid admits a barotropic equation of state. We have seen (equations (11) and (12)) that to expand the metric in \( \lambda \) implies \( X \approx 1 + \lambda q \). Therefore, what is relevant to our approach is the form that the function \( f(X) \) takes near \( X = 1 \). For instance, let \( f(X) = \mu X g(X)(X-1)^n (n \geq 0) \), where \( g(X) \) is a well-behaved function at \( X = 1 \), \( g(1) = 1 \),
and $\mu_s$ is a constant we introduce for convenience to make $g(X)$ dimensionless. Then we have $\mu \approx \mu_s (X - 1)^n [1 + o(X - 1)]$, for small values of $X - 1$, and $\mu \approx \mu_s \lambda^n q^n$, for the first term in the $\lambda$ expansion of $\mu$. This expression is similar to that we obtain for the density of a polytropic fluid (13), though it will lead to a different value of the parameter $\lambda$. Furthermore, we can integrate (48) by using the form of $f(X)$ for small values of $X - 1$, and by expanding the result in $\lambda$ to check that the pressure is at least one order higher in $\lambda$ than the density. All this suggests the existence of a common metric up to first order in $\lambda$ for all these barotropic fluids.

Finally, let us note that if the density is not zero at $X = 1$ ($n = 0$), the resultant first-order metric must coincide with the CMMR metric. It was shown in the CMMR article that the Kerr metric does not fit the exterior metric of a constant density fluid, and neither does it fit the exterior field of this other kind of barotropic fluid.

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