GENERALIZED BERNSTEIN OPERATORS ON THE CLASSICAL POLYNOMIAL SPACES

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Abstract. We study generalizations of the classical Bernstein operators on the polynomial spaces $\mathbb{P}_n[a,b]$, where instead of fixing 1 and $x$, we reproduce exactly 1 and a polynomial $f_1$, strictly increasing on $[a,b]$. We prove that for sufficiently large $n$, there always exist generalized Bernstein operators fixing 1 and $f_1$. These operators are defined by non-decreasing sequences of nodes precisely when $f_1' > 0$ on $(a, b)$, but even if $f_1'$ vanishes somewhere inside $(a, b)$, they converge to the identity.

1. Introduction

Let $\mathbb{P}_n[a,b]$ denote the space of polynomials of degree bounded by $n$, over the interval $[a,b]$. The classical Bernstein operator $B_n : C[a,b] \to \mathbb{P}_n[a,b]$, defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(a + \frac{k}{n} (b-a)\right) \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n},$$

reproduces exactly (or fixes) the affine functions, but from the design viewpoint, one might be interested in the precise reproduction of other functions. So a natural idea is to search for analogous operators that fix all functions in a given two-dimensional space, possibly different from the affine functions, and still converge to the identity.

Here we explore such generalized Bernstein operators $B_{f_1}^n$ on polynomial spaces, where fixing the constant function 1 and an injective polynomial $f_1$ is achieved, when possible, by modifying the location of the nodes $t_{n,k}$ (instead of having $t_{n,k} = a + \frac{k}{n} (b-a)$, as in (1)). A motivation for this approach is that it allows us to keep the Bernstein bases unchanged, a desirable feature given their several optimality properties, cf. for instance [10]. Multiplying by $-1$ if needed, we may assume that $f_1$ is increasing.

We shall show (cf. Theorem 5.1) that given any polynomial $f_1(x)$, strictly increasing on $[a,b]$, and of degree $m$, it is always possible to find a generalized Bernstein operator fixing 1 and $f_1$, on the space $\mathbb{P}_n[a,b]$ with the standard Bernstein basis, provided that $n \geq m$ and that $n$ is “sufficiently large”. The special case $f_1(x) = x^j$, $[a,b] = [0,1]$, had been previously solved in [4] Proposition 11; on the other hand, it is known that such operators do not exist if we are required to fix $f_0(x) = x^i$ and $f_1(x) = x^j$ on $[0,1]$, $1 \leq i < j$, cf. [11] Theorem 2010 Mathematics Subject Classification: Primary: 41A10.

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Regarding the meaning of “sufficiently large”, in general $n$ and the degree $m$ are not comparable, i.e., there is no constant $C > 0$ such that $B_n^f$ is well defined for all $n \geq Cm$.

We shall see that for the family $f_{1,t}(x) = (x-t)^3$ on $[0,1]$, where $t \in (0,1/2)$, we must have $n > 1/t$ (cf. Theorem 5.3).

Denote by $\gamma_{n,k}$, $k = 0, \ldots, n$ the coordinates of $f_1$ with respect to the Bernstein bases. Since we want to fix $f_1$, the nodes must be given by $t_{n,k} = f_1^{-1}(\gamma_{n,k})$, the injectivity of $f_1$ being used at this point: uniqueness of the coordinates entails that the nodes are also uniquely determined. However, it need not be true that all $f_1^{-1}(\gamma_{n,k}) \in [a,b]$, (cf. Theorem 5.3). We prove that if $n \geq m$ is sufficiently large then indeed all the coordinates $\gamma_{n,k}$ belong to $[f_1(a), f_1(b)]$. The proof proceeds by showing that we have $|\gamma_{n,k} - f_1(a + \frac{k}{m} (b-a))| = O(1/n)$, where the implicit constant in the big $O$ notation depends only on $f_1$ over the interval $[a,b]$, but not on $k$ nor $n$ (cf. Theorem 5.4).

A considerable difference arises between the cases where $f'_1$ vanishes at some point inside $(a,b)$, and where $f'_1 > 0$ on $(a,b)$: under the latter condition, the nodes form an increasing sequence and the separation between consecutive nodes is bounded by $O(1/n)$, while if $f'_1(x_0) = 0$ for some $x_0 \in (a,b)$, then there are reversals of the nodes no matter how big $n$ is, and the difference between consecutive nodes can be distinctly larger than $O(1/n)$ (cf. Theorem 3.2, Theorem 6.2, and Example 6.1). When $f'_1$ vanishes somewhere inside $(a,b)$, letting $s-1$ be the largest order of all such zeros, we are able to prove that the separation between consecutive nodes is bounded by $O(1/n^{1/s})$ (cf. Theorem 6.7), so despite the order reversal of some nodes, we still have convergence to the identity.

Let $\omega(f,t)$ stand for the modulus of continuity of a uniformly continuous function $f$. A consequence of the preceding results is that when $f'_1 > 0$,

$$|B^f_n f(x) - B_n f(x)| \leq \omega(f,Kn^{-1})$$

for some constant $K > 0$, while if $s-1$ is the largest order of all the zeros of $f'_1$ inside $(a,b)$, then

$$|B^f_n f(x) - B_n f(x)| \leq \omega(f,K'n^{-1/s})$$

for some $K' > 0$. Recalling that there is a $C > 0$ (which depends on $[a,b]$ only) such that

$$|f(x) - B_n f(x)| \leq C \omega(f,n^{-1/2})$$

we see that up to some constant and for $s \leq 2$, the rate of approximation of the generalized Bernstein operators in terms of the modulus of continuity, is no significantly worse than that for the classical operator:

$$|f(x) - B^f_n f(x)| \leq (K' + C) \omega(f,n^{-1/2}).$$

Now let us place the preceding results in context. Several variations of the Bernstein operators have been considered in the literature to address the problem of fixing functions other than $1$ and $e_1(x) = x$, sometimes modifying the Bernstein bases functions (consider, for instance, the nowadays called King’s operators, after [12]). Also, similar questions have been asked about related positive operators, (cf. for instance [1]).
Within the line of research followed here, previous work has focused on spaces different or more general than spaces of polynomials (cf. [15], [3], [4], [5], [13], [6], [14]), but convergence has also been studied in Müntz spaces, cf. [2], and for rational Bernstein operators, see [18].

Generally speaking, the situation regarding existence is well understood in the context of extended Chebyshev spaces (which generalize the space of polynomials of degree at most \( n \), by retaining the bound on the number of zeros) cf. [4]: One considers a two dimensional extended Chebyshev space \( U_1 \), for which a generalized Bernstein operator fixing it can always be defined, and inductively, this definition is extended to \( U_1 \subset U_2 \subset \cdots \subset U_n \), where each \( U_k \) is a \( k + 1 \)-dimensional extended Chebyshev space.

While the present paper returns to the classical polynomial spaces, its results go beyond the setting of chains of extended Chebyshev spaces (starting with dimension two) for oftentimes such chains, fixing \( 1 \) and \( f_1 \), are impossible to generate.

Understanding the polynomial case is a natural starting point towards possible generalizations to other spaces of functions, and it sheds light on the usefulness of different notions regarding generalized Bernstein operators. For instance, if one requires in the definition that the sequence of nodes be non-decreasing (as done in [14]) this will lead to better properties of the operators from the viewpoint of shape preservation (see the example at the end of the paper) but existence will never be obtained if \( f'_1 \) has a zero in \((a, b)\).

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2. Definitions and background.

Definition 2.1. Let \( U_n \) be an \( n + 1 \) dimensional subspace of \( C^n ([a, b], \mathbb{R}) \) (in this paper we consider real valued functions only). A Bernstein basis \( \{p_{n,k} : k = 0, \ldots, n\} \) of \( U_n \) is a basis with the property that each \( p_{n,k} \) has a zero of order \( k \) at \( a \), and a zero of order \( n - k \) at \( b \). The function \( p_{n,k} \) might have additional zeros inside \((a, b)\); this is not excluded by the preceding definition. A Bernstein basis is non-negative if for all \( k = 0, \ldots, n \), \( p_{n,k} \geq 0 \) on \([a, b]\), and positive if \( p_{n,k} > 0 \) on \((a, b)\). Finally, a non-negative Bernstein basis is normalized if \( \sum_{k=0}^{n} p_{n,k} \equiv 1 \).

It is easy to check that non-negative Bernstein bases are unique up to multiplication by a positive scalar, and that normalized Bernstein bases are unique.

Definition 2.2. If \( U_n \) has a non-negative Bernstein basis \( \{p_{n,k} : k = 0, \ldots, n\} \), we define a generalized Bernstein operator \( B_n : C[a, b] \to U_n \) by setting

\[
B_n(f) = \sum_{k=0}^{n} f(t_{n,k}) \alpha_{n,k} p_{n,k},
\]

where the nodes \( t_{n,0}, \ldots, t_{n,n} \) belong to the interval \([a, b]\), and the weights \( \alpha_{n,0}, \ldots, \alpha_{n,n} \) are positive.
We briefly comment on the rather weak assumptions made in the preceding definition. Non-negativity of the functions $p_{n,k}$ and positivity of the weights $\alpha_{n,0}, \ldots, \alpha_{n,n}$ are required so that the resulting operator is positive, a natural property from the viewpoint of shape preservation. Strict positivity of the weights entails that all the basis functions are used in the definition of the operator. Additionally, the nodes must belong to $[a, b]$. This is a natural condition, since in principle the domain of definition of the functions being considered is $[a, b]$. Note that no requirement is made in the preceding definition about the ordering of the nodes, and in particular, we do not ask that they be strictly increasing, i.e., that $t_{n,0} < t_{n,1} < \cdots < t_{n,n}$.

When we only have $t_{n,0} \leq t_{n,1} \leq \cdots \leq t_{n,n}$ we say that the sequence of nodes is increasing, or equivalently, non-decreasing.

The problem of existence, as studied in [4] and [5], arises when we choose two functions $f_0, f_1 \in U_n$, such that $f_0 > 0$, $f_1/f_0$ is strictly increasing, and we require that

$$B_n(f_0) = f_0 \text{ and } B_n(f_1) = f_1.$$  

If these equalities can be satisfied, they uniquely determine the location of the nodes and the values of the coefficients, cf. [4, Lemma 5]: in other words, there is at most one Bernstein operator $B_n$ of the form (2) satisfying (3). We will consistently use the following notation. Assume that $p_{n,k}$, $k = 0, \ldots, n$, is a Bernstein basis of the space $U_n$. Given $f_0, f_1 \in U_n$, there exist coefficients $\beta_{n,0}, \ldots, \beta_{n,n}$ and $\gamma_{n,0}, \ldots, \gamma_{n,n}$ such that

$$f_0(x) = \sum_{k=0}^{n} \beta_{n,k} p_{n,k}(x) \text{ and } f_1(x) = \sum_{k=0}^{n} \gamma_{n,k} p_{n,k}(x).$$

The following elementary fact regarding bases will be used throughout (cf. [4, Lemma 5]): suppose there exists a generalized Bernstein operator $B_n$ of the form (2), fixing $f_0$ and $f_1$; then it must be the case that for each $k = 0, \ldots, n$,

$$\beta_{n,k} = f_0(t_{n,k}) \alpha_{n,k} \text{ and } \gamma_{n,k} = f_1(t_{n,k}) \alpha_{n,k}.$$  

If $f_0 = 1$, using $\tilde{p}_{n,k} := \alpha_{n,k} p_{n,k}$ instead of $p_{n,k}$, we may assume that the Bernstein basis is normalized, and then we can take the coordinates of $1$ and the weights to be $1$. Thus, we have that

$$1 = \alpha_{n,k} = \beta_{n,k} \text{ and } t_{n,k} = f_1^{-1}(\gamma_{n,k}).$$

In this case, we denote the generalized Bernstein operator by $B_n^{f_1}$.

3. Characterizing when nodes increase.

For the remainder of the paper, $p_{n,k}(x)$ will denote the usual Bernstein basis function

$$p_{n,k}(x) = \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n}$$

on $\mathbb{P}_n[a,b]$, the space of polynomials on $[a,b]$, of degree bounded by $n$. 

While Theorem 3.2 and its corollary can be presented in greater generality, in order to minimize technicalities we shall restrict ourselves to the polynomial spaces \( \mathbb{P}_n[a, b] \) with their standard Bernstein bases, which is all we shall need in this paper.

The following lemma is well known.

**Lemma 3.1.** Let \( f_1 \) be a polynomial on \([a, b]\), of degree bounded by \( n \geq 1 \), with coordinates given by \( f_1(x) = \sum_{k=0}^{n} \gamma_{n,k} p_{n,k}(x) \), and let \( \frac{d}{dx} f_1(x) = \sum_{k=0}^{n-1} w_k p_{n-1,k}(x) \). For \( k = 1, \ldots, n \), we have

\[
\gamma_{n,k} - \gamma_{n,k-1} = \frac{(b-a)w_{k-1}}{n}.
\]

**Proof.** Taking the derivative of the Bernstein bases functions, we get, for \( k = 0 \),

\[
\frac{d}{dx} p_{n,0}(x) = -\frac{n}{b-a} p_{n-1,0}(x),
\]

for \( 0 < k < n \),

\[
\frac{d}{dx} p_{n,k}(x) = \frac{n}{b-a} \left( p_{n-1,k-1}(x) - p_{n-1,k}(x) \right),
\]

and for \( k = n \),

\[
\frac{d}{dx} p_{n,n}(x) = \frac{n}{b-a} p_{n-1,n-1}(x).
\]

Now using the preceding expressions and rearranging terms we get

\[
\sum_{k=0}^{n-1} w_k p_{n-1,k}(x) = \frac{d}{dx} f_1(x) = \sum_{k=0}^{n} \gamma_{n,k} \frac{d}{dx} p_{n,k}(x)
\]

\[
= \frac{n}{b-a} \left( \sum_{k=1}^{n} (\gamma_{n,k} - \gamma_{n,k-1}) p_{n-1,k-1}(x) \right),
\]

so \( w_{k-1} = \frac{n}{b-a} \left( \gamma_{n,k} - \gamma_{n,k-1} \right) \).

**Theorem 3.2.** Let \( f_1 \) be a strictly increasing polynomial on \([a, b]\), of degree bounded by \( n \geq 1 \), with coordinates given by \( f_1(x) = \sum_{k=0}^{n} \gamma_{n,k} p_{n,k}(x) \). Then the following are equivalent:

a) There exists a generalized Bernstein operator \( B_n^{f_1} : C[a, b] \to \mathbb{P}_n[a, b] \), fixing 1 and \( f_1 \), and defined by

\[
B_n^{f_1} f(x) = \sum_{k=0}^{n} f(t_{n,k}) p_{n,k}(x),
\]

where \( a = t_{n,0} \leq \cdots \leq t_{n,n} = b \) (resp. \( a = t_{n,0} < \cdots < t_{n,n} = b \)).

b) For \( k = 0, \ldots, n \), the coefficients \( \gamma_{n,k} \) are increasing (resp. strictly increasing).

c) For \( k = 0, \ldots, n-1 \), the coefficients \( w_k \) defined by \( \frac{d}{dx} f_1 = \sum_{k=0}^{n-1} w_k p_{n-1,k} \), are non-negative (resp. strictly positive).
Fix Corollary 3.4.

Proof. The equivalence between a) and b) is immediate from (6), where it is noted that if \( f_0 = 1 \), then \( \alpha_{n,k} = \beta_{n,k} = 1 \) for \( k = 0, \ldots, n \). Thus, if \( B_{f_1}^f \) fixes \( f_1 \) and the nodes are increasing with \( k \), since \( f_1(t_{n,k}) = \gamma_{n,k} \), the coordinates \( \gamma_{n,k} \) also increase, while if the coordinates increase, so do the nodes given by \( t_{n,k} = f_1^{-1}(\gamma_{n,k}) \), and, provided that the nodes belong to \([a, b] \), the operator defined by (13) clearly fixes \( f_1 \) and 1. So it is enough to check that \( t_{n,0} = a \) and \( t_{n,n} = b \). But this is obvious, since \( f_1(a) = \gamma_{n,0} \) \( p_{n,0}(a) = \gamma_{n,0} \), and \( f_1(b) = \gamma_{n,n} p_{n,n}(b) = \gamma_{n,n} \). The strictly increasing case is identical.

Regarding the equivalence of b) and c), by the preceding Lemma, \( \gamma_{n,k} - \gamma_{n,k-1} = \frac{(b-a)w_{k-1}}{n} \), so for \( k = 1, \ldots, n \), \( w_{k-1} \geq 0 \) (resp. \( w_{k-1} > 0 \)) if and only if \( \gamma_{n,k} \geq \gamma_{n,k-1} \) (resp. \( \gamma_{n,k} > \gamma_{n,k-1} \)).

Thus, we obtain the following necessary condition for the existence of a generalized Bernstein operator defined via an increasing sequence of nodes, with respect to the standard Bernstein basis on \( \mathbb{P}_n[a, b] \).

Corollary 3.3. Suppose that there exists a generalized Bernstein operator \( B_{f_1}^f : C[a, b] \rightarrow \mathbb{P}_n[a, b] \) with increasing nodes, fixing the constant function 1 and a strictly increasing polynomial \( f_1 \). Then \( f_1'(x) > 0 \) for all \( x \) in the open interval \( (a, b) \). If the sequence of nodes is strictly increasing, then \( f_1' > 0 \) on the closed interval \([a, b]\).

Proof. If the sequence of nodes is increasing, by Theorem 3.2, there are non-negative coefficients \( w_k, k = 0, \ldots, n-1 \), such that \( f_1(x) = \sum_{k=0}^{n-1} w_k p_{n-1,k}(x) \). Since \( f_1 \) is not identically zero, at least one of the coefficients is strictly positive, say \( w_j > 0 \). And since \( p_{n-1,j} > 0 \) on \((a, b)\), we also have \( f_1' > 0 \) on \((a, b)\).

If the sequence of nodes is strictly increasing, then all the coefficients \( w_k \) are strictly positive. Since the standard Bernstein basis on \( \mathbb{P}_{n-1}[a, b] \) forms a partition of unity, for all \( x \in [a, b] \),

\[
f_1'(x) = \sum_{k=0}^{n-1} w_k p_{n-1,k} \geq \min_{k=0,\ldots,n-1} w_k \mathbf{1}(x) > 0.
\]

Corollary 3.4. Fix \( k \geq 1 \) and \( a < 0 < b \). There is no generalized Bernstein operator \( B_{f_1}^f : C[a, b] \rightarrow \mathbb{P}_n[a, b] \) fixing 1 and \( f_1 := x^{2k+1} \), and defined by a non-decreasing sequence of nodes.

4. Strictly positive polynomials have positive Bernstein coordinates, eventually.

The title of this section recalls an old result of S. Bernstein, cf. [7], [8], according to which a polynomial \( g > 0 \) on \([a, b] \) has positive Bernstein coordinates in \( \mathbb{P}_N[a, b] \), provided \( N \geq \deg(g) \) is large enough. The case \( g > 0 \) on \((a, b)\) reduces to the previous one by factoring the zeros (if any) at the endpoints, and then we conclude that the Bernstein coordinates are non-negative. Note however that if \( g \geq 0 \) on \([a, b] \) and \( g(c) = 0 \) for some \( c \in (a, b) \), then some coordinate of \( g \) must be negative, since the Bernstein bases functions are positive on \((a, b)\).

Two additional proofs of Bernstein’s result can be found in the answers to [17] Problem 49 of Part VI]. It also follows from Theorem 3.4 below, which gives a proof analogous to Bernstein’s original, the difference being that we are interested in uniform estimates.
Let us remind the reader of the fact that strictly positive polynomials can have some negative Bernstein coordinates, so the “sufficiently large” clause is needed. The next example (essentially, the Example from [16, Pg. 4684]) illustrates the fact that the necessary condition “$f'_1 > 0$ on $(a, b)$” from Corollary 3.3 to ensure that nodes are non-decreasing, is not sufficient. In fact, even the assumption $f'_1 > 0$ on $[a, b]$ is not sufficient.

**Example 4.1.** For $n = 2$ and $[a, b] = [0, 1]$, the positive function $f'_1(x) := (x - 1/2)^2 + 1/8$ satisfies $f'_1(x) = 3p_{2,0}(x)/8 - p_{2,1}(x)/8 + 3p_{2,2}(x)/8$. Consider the primitive with constant term zero, given by $f_1(x) = 3x/8 - x^2/2 + x^3/3$. By Theorem 3.2, if a generalized Bernstein operator fixing 1 and $f_1(x)$ exists, then it is not defined via an increasing sequence of nodes. Actually, in this case the operator $B_3^{f_1}$ does exist: by computing coordinates we find that $f_1(x) = p_{3,1}(x)/8 + p_{3,2}(x)/12 + 5p_{3,3}(x)/24$, and since $f_1(0) = 0$ and $f_1(1) = 5/24$, all the nodes belong to $[0, 1]$; we have $0 < t_{3,0} < t_{3,2} = f_1^{-1}(1/12) < t_{3,1} = f_1^{-1}(1/8) < t_{3,3} = 1$. We shall see that for some $N > 3$ the nodes become disentangled, forming an increasing sequence. And once they become increasing, they stay that way, by Theorem 3.2 and the next result, which is a direct consequence of degree elevation.

**Proposition 4.2.** If $g(x) = \sum_{k=0}^{n} w_{n,k}p_{n,k}(x)$ on $[a, b]$ has non-negative coefficients $w_{n,k}$, then the Bernstein coefficients $w_{N,k}$ of $g(x)$ with respect to $p_{N,k}$, $k = 0, \ldots, N$, for any $N \geq n$, also satisfy $w_{N,k} \geq 0$.

**Proof.** By an induction argument it suffices to prove it for $N = n + 1$. Since

\[
p_{n,k} = a_{n+1,k}p_{n+1,k} + a_{n+1,k+1}p_{n+1,k+1},
\]

with constants $a_{n+1,k} > 0$ and $a_{n+1,k+1} > 0$,

\[
g(x) = \sum_{k=0}^{n} w_{n,k}p_{n,k}(x) = \sum_{k=0}^{n} w_{n,k}a_{n+1,k}p_{n+1,k} + \sum_{k=0}^{n} w_{n,k}a_{n+1,k+1}p_{n+1,k+1} = w_{n,0}a_{n+1,0}p_{n+1,0} + \sum_{k=1}^{n} (w_{n,k} + w_{n,k-1}) a_{n+1,k}p_{n+1,k} + w_{n,n}a_{n+1,n+1}p_{n+1,n+1}.
\]

\[\square\]

5. **Generalized Bernstein operators fixing 1 and a strictly increasing polynomial.**

In this section we prove that for $n$ sufficiently large, there is a generalized Bernstein operator fixing 1 and a nonconstant polynomial $f_1$ under the assumption $f'_1 \geq 0$ on $[a, b]$. More precisely, we prove the following theorem, presenting some preliminary results before the proof.

**Theorem 5.1.** Given any non constant polynomial $f_1$ with $f'_1 \geq 0$ on $[a, b]$, for every $N$ sufficiently large there is a generalized Bernstein operator $B_N^{f_1}$ based on $P_N[a, b]$, which fixes both 1 and $f_1$. If there exists an $x_0 \in (a, b)$ such that $f'_1(x_0) = 0$, the sequence of nodes will fail to be non-decreasing for all $N$ such that $B_N^{f_1}$ exists. Fix $N$ large enough so that $B_N^{f_1}$ is
well defined. If $f'_1$ has a zero of order $s_1$ at $a$, the first $s_1 + 1$ nodes are equal to $a$, and if $f'_1$ has a zero of order $s_2$ at $b$, the last $s_2 + 1$ nodes are equal to $b$. If $f'_1 > 0$ on $(a, b)$, then the nodes in $(a, b)$ form a strictly increasing sequence, while if $f'_1 > 0$ on $[a, b]$, then all nodes form a strictly increasing sequence.

Let us recall the well known relationship between the Bernstein bases and the monomial or power bases.

**Proposition 5.2.** Let $g : [a, b] \rightarrow \mathbb{R}$ be a polynomial of degree $m$ and let $n \geq m$ be a natural number. Write

$$g(x) = \sum_{l=0}^{m} c_l (x-a)^l = \sum_{k=0}^{n} w_{n,k} p_{n,k}(x).$$

Then the Bernstein coefficients $w_{n,k}$ of $g$ are given by

$$w_{n,k} = \sum_{l=0}^{\min(k,m)} \frac{k!}{n!(k-l)!} (b-a)^l c_l.$$

**Proof.** Since $1 = \sum_{k=0}^{n-l} p_{n-l,k}(x)$,

$$g(x) \cdot 1 = \sum_{l=0}^{m} c_l (x-a)^l \sum_{k=0}^{n-l} p_{n-l,k}(x) = \sum_{l=0}^{m} \sum_{k=0}^{n} c_l (x-a)^l p_{n-l,k-l}(x).$$

Now

$$(x-a)^l p_{n-l,k-l}(x) = \binom{n-l}{k-l} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^{n-l}} = \frac{k! (n-l)!}{(k-l)! n!} (b-a)^l p_{n,k}(x),$$

so

$$g(x) = \sum_{l=0}^{m} \sum_{k=0}^{n} c_l \frac{k! (n-l)!}{(k-l)! n!} (b-a)^l p_{n,k}(x)$$

$$= \sum_{k=0}^{n} p_{n,k}(x) \sum_{l=0}^{\min(k,m)} \frac{k! (n-l)!}{(k-l)! n!} (b-a)^l.$$

$\square$

How large must $N$ be so that $B^l_N f_1$ is well defined cannot be determined from the degree of $f_1$ alone, it also depends on the coefficients $c_l$.

**Theorem 5.3.** Set $f_1(x) = (x-N^{-1})^3$ on $[0, 1]$. Then the node $t_{N,2} < 0$, so $B^l_N f_1$ is not well defined.

**Proof.** Since

$$f_1(x) = \left(x - \frac{1}{N}\right)^3 = \frac{1}{N^3} + \frac{3x}{N^2} - \frac{3x^2}{N} + x^3$$
on $[0, 1]$, from Proposition 5.2 below we know that the Bernstein coordinate $\gamma_{N, 2}$ is given by

$$
(16) \quad \gamma_{N, 2} = \sum_{l=0}^{2} c_l \frac{2! (N - l)!}{N! (2 - l)!} = -\frac{1}{N^3} + \frac{6}{N^3} - \frac{6}{N^2 (N - 1)} = \frac{5}{N^3} - \frac{6}{N^3 - N^2}.
$$

Now $f_1^{-1}(y) = 1/N + y^{1/3}$ is increasing, so

$$
t_{N, 2} = f_1^{-1}\left(\frac{5}{N^3} - \frac{6}{N^3 - N^2}\right) < f_1^{-1}\left(\frac{5}{N^3} - \frac{6}{N^3}\right) = 0.
$$

Next we show that for an arbitrary polynomial $g$, restricted to the interval $[a, b]$, we have $|g(a + k/n (b - a)) - w_{n,k}| = O(1/n)$, where the constant implicit in the order notation depends only on $g$ and on $[a, b]$, but not on $n$ or $k$. In particular, the following proof entails that if $g$ is affine, then $g(a + k/n (b - a)) = w_{n,k}$, that is, we obtain the well known fact that the standard Bernstein operators fix the affine functions. So the result says something new only when $m = \deg(g) \geq 2$.

Write $e_j(x) := x^j$, and let $n \gg 1$ be even, say $n = 2k$. We know from Proposition 5.2 that for $e_2(x)$ on $[0, 1]$, $w_{n,k} = \frac{k(k-1)}{n(n-1)}$, so $|e_2(k/n) - w_{n,k}| = (4n - 4)^{-1}$. Thus, the estimate $|g(k/n) - w_{n,k}| = O(1/n)$ cannot in general be improved.

We use the standard conventions whereby the value of an empty sum is 0, and the value of an empty product, 1.

**Theorem 5.4.** Let $g(x) = \sum_{l=0}^{m} c_l (x - a)^l$ be a polynomial of degree $m$, restricted to the interval $[a, b]$. Define $c_{\max} := \max_{l=2, \ldots, m} |c_l|$. For $n > m$ and $k = 0, \ldots, n$, let $w_{n,k}$ be the Bernstein coefficients of $g$ in $P_n[a, b]$, so

$$
g(x) = \sum_{k=0}^{n} w_{n,k} \binom{n}{k} \frac{(x - a)^k (b - x)^{n-k}}{(b - a)^n}.
$$

Then

$$
|g\left(a + \frac{k}{n} (b - a)\right) - w_{n,k}| \leq m^3 c_{\max} \max\left\{\left(\frac{b - a}{n}\right)^2, \left(\frac{b - a}{n}\right)^m\right\}.
$$

**Proof.** By (14)

$$
g\left(a + \frac{k}{n} (b - a)\right) - w_{n,k} = \sum_{l=0}^{\min(k, m)} c_l \frac{(k/l) l! (n-l)!}{n! (k-l)!} (b - a)^l.
$$

$$
\sum_{l=k+1}^{m} c_l \frac{(k/l) l! (n-l)!}{n! (k-l)!} (b - a)^l.
$$
Since for \( l = 0,1, \frac{k}{n} l - \frac{k(l-1)!}{n(l-1)!} = 0 \), we can start the sums with \( 2 = l \leq m \) (in particular, if \( m < 2 \), then \( g(a + \frac{k}{n}(b-a)) = w_{n,k} \)). Thus

\[
\left| g\left(a + \frac{k}{n}(b-a)\right) - w_{n,k}\right| = \left| \sum_{l=2}^{\min(k,m)} c_l \left(\frac{k}{n} l - \frac{k(k-1)\ldots(k-(l-1))}{n(n-1)\ldots(n-(l-1))}\right)(b-a)^l + \sum_{l=k+1}^{m} c_l \left(\frac{k}{n} l\right)^l (b-a)^l \right|
\]

Note that for every \( 0 \leq s < n, \frac{k}{n} \geq \frac{k}{n-s} \), as can be checked just by simplifying. Furthermore, if \( b - a \geq 1 \), then for \( 2 \leq l \leq m \) we have \( (b-a)^l \leq (b-a)^m \), while if \( b - a \leq 1 \), then for \( 2 \leq l \leq m \) we have \( (b-a)^l \leq (b-a)^2 \). Write \( M := \max\{(b-a)^2, (b-a)^m\} \). Let us consider first the case \( k \leq m \). Then

\[ \leq \sum_{l=2}^{m} |c_l| \left(\frac{k}{n} l\right)^l (b-a)^l \]

Next, suppose \( k \geq m \). In this case,

\[ \leq M c_{\max} \sum_{l=2}^{m} \left(\frac{k}{n} l - \frac{k-m}{n-m}\right)^l \]

Using the mean value theorem for the function \( h_l(x) = x^l \) and the interval \([\frac{k-m}{n-m}, \frac{k}{n}]\), we have

\[ \left(\frac{k}{n}\right)^l - \left(\frac{k-m}{n-m}\right)^l \leq l \max_{\xi \in \left[\frac{k-m}{n-m}, \frac{k}{n}\right]} \xi^{l-1} \left(\frac{k}{n} - \frac{k-m}{n-m}\right) \]

\[ \leq m \left(\frac{k}{n}\right)^{l-1} \frac{m(n-k)}{n(n-m)} \leq \frac{m^2}{n} \]

It follows that

\[ \left| g\left(a + \frac{k}{n}(b-a)\right) - w_{n,k}\right| \leq M c_{\max} \sum_{l=2}^{m} \frac{m^2}{n} \leq M c_{\max} \frac{m^3}{n}. \]

Since the estimate from (26) is always larger than the estimate from (20) - (21), the result follows. \[ \square \]
Remark 5.5. The uniform (in \( k \)) approximation obtained in the preceding Theorem depends on the degree of the polynomial, its coefficients, and the length of the interval, cf. (17). The dependency on the last factor is easily explained: the smaller \( b-a \) is, the larger the “sampling rate” of the polynomial, and vice versa.

Also, from the preceding proof we see that near 0 (say, for \( 0 \leq k \leq m \)) we have
\[
\left| g \left( a + \frac{k}{n} (b-a) \right) - w_{n,k} \right| = O \left( \frac{1}{n^2} \right),
\]
which of course is better than the general \( \left| g \left( a + \frac{k}{n} (b-a) \right) - w_{n,k} \right| = O \left( \frac{1}{n} \right) \). We can also get \( \left| g \left( a + \frac{k}{n} (b-a) \right) - w_{n,k} \right| = O \left( \frac{1}{n^2} \right) \) at the other endpoint, for \( n-m \leq k \leq n \), just by using \( \frac{m(n-k)}{n(n-m)} \leq \frac{m^2}{n(n-m)} \) in (25).

The basic idea of the following proof is that the reversal of the nodes can only happen “far away” from the endpoints, and then by Theorem 5.3, for sufficiently large \( N \), if \( k/N \) is far away from the endpoints of \([0,1]\), we must have \( f_1(a) < \gamma_{N,k} < f_1(b) \).

Proof of Theorem 5.7. We assume, for simplicity in the expressions, that \([a,b] = [0,1]\).

There is no loss of generality in doing so, since the increasing affine change of variables that maps 0 to \( a \) and 1 to \( b \) preserves the non-negativity (resp. positivity) of \( f_1' \) on the closed interval (resp. on the open or on the closed interval).

Suppose that \( f_1' \) has at least one zero in \((0,1)\). The case where \( f_1' > 0 \) on \((0,1)\) is similar but simpler, since there are no reversals in the ordering of the nodes (cf. Corollary 3.3).

Let \( m \geq 2 \) be the degree of \( f_1 \). We need to show that for \( N \) sufficiently large, all nodes \( t_{N,k} \) belong to \([0,1]\).

Suppose that \( f_1' \) has a zero of order \( s_1 \geq 0 \) at 0, and a zero of order \( s_2 \geq 0 \) at 1; let us write \( f_1'(x) = x^{s_1} g(x) (1-x)^{s_2} \), with \( g = \sum_{k=0}^{n} \tilde{w}_{n,k} p_{n,k} \) for \( n \geq m-1 \). Thus
\[
f_1'(x) = \sum_{k=0}^{n} \tilde{w}_{n,k} x^{s_1} p_{n,k}(x) (1-x)^{s_2} = \sum_{k=0}^{n} \tilde{w}_{n,k} \left( \frac{n}{k+s_1} \right) p_{n+s_1+s_2,k+s_1}(x) = \sum_{k=0}^{n+s_1+s_2} w_{n+s_1+s_2,k} p_{n+s_1+s_2,k}(x).
\]

Equating coefficients, we find that for \( 0 \leq k \leq s_1-1 \) and for \( n+s_1+1 \leq k \leq n+s_1+s_2 \), \( w_{n+s_1+s_2,k} = 0 \), while for \( s_1 \leq k \leq n+s_1 \), \( \tilde{w}_{n,k-s_1} \) and \( w_{n+s_1+s_2,k} \) have the same sign.

Let \( x_0, x_1 \in (0,1) \) be the first and the last zeros of \( f_1' \) in \((0,1)\). Since \( f_1 \) is increasing, we can select a \( \delta > 0 \) such that \( f_1(0) < f_1(x_0/2) - \delta < \frac{f_1((x_1+1)/2) + \delta}{f_1(1)} \). We choose \( n_1 \) such that \( m^2 \gamma_{n_1} < \delta \). By Theorem 5.4, applied to \( f_1 \), whenever \( N \geq n_1 \) and \( k/N \leq (1 + x_1)/2 \), we have \( \gamma_{N,k} < f_1(1) \) and hence \( t_{N,k} < 1 \). Likewise, whenever \( k/N \geq x_0/2 \), we have \( \gamma_{N,k} > f_1(0) \) and \( t_{N,k} > 0 \). Thus, if \( x_0/2 \leq k/N \leq (x_1 + 1)/2 \), then \( t_{N,k} \in (0,1) \).

Using Theorem 5.3 again, this time applied to \( g \), we can select an even larger \( N \) so that for every \( k \) with either \( 0 \leq k/N \leq 3x_0/4 \) or \( (3x_1 + 1)/4 \leq k/N \leq 1 \), the coordinates \( \tilde{w}_{N-s_1-s_2,k} \) of \( g \) in dimension \( N-s_1-s_2 \) are strictly positive, and thus, the coordinates of
$f_1'$ in dimension $N$ satisfy $w_{N,k} \geq 0$. From Lemma 3.1 it follows that $\gamma_{N,k} \leq \gamma_{N,k+1}$ for every such $k$. Since $f_1'(0) = \gamma_{N,0} p_{N,0}(0) = \gamma_{N,0}$, we have, for $0 \leq k/N \leq 3x_0/4$, that the nodes $t_{N,k} = f_1^{-1}(\gamma_{N,k}) \geq t_{N,0} = 0$. Likewise, for $(3x_i + 1)/4 \leq k/N \leq 1$ we have $t_{N,k} \leq t_{N,N} = 1$.

Regarding the statements about the nodes, if $f_1' > 0$ on $[a, b]$, then for $N$ sufficiently large and $k = 0, \ldots, N$, the Bernstein coordinates $w_{N,k}$ of $f_1'$ satisfy $w_{N,k} > 0$, so by Lemma 3.1 or Theorem 3.2, the coordinates $\gamma_{N,k}$ of $f_1$ form a strictly increasing sequence, and hence, so do the nodes (their inverse image under $f_1$). Likewise, by Lemma 3.1 $\gamma_{N,0} = \cdots = \gamma_{N,s_1}$ when $f_1'$ has a zero of order $s_1$ at 0. The statements for the cases where $f_1'$ has a zero of order $s_2$ at 1, and where $f_1' > 0$ on $(a, b)$, follow in the same manner. Finally, the fact that nodes must decrease at some point if $f_1'$ vanishes somewhere inside $(0, 1)$, is immediate from Corollary 3.3.

6. Convergence to the identity.

Next we show that the generalized Bernstein operators fixing constants and an increasing polynomial $f_1$, denoted in this section by $B_n^{f_1}$, converge to the identity in the strong operator topology, as $n \to \infty$.

The proof proceeds as follows: We deduce the convergence of the operators $B_n^{f_1}$ to the identity from the convergence of the standard Bernstein operators $B_n$, by showing that the nodes $t_{n,k}$ of $B_n$ are “close” to the corresponding nodes $a + \frac{k}{n} (b - a)$ of $B_n$, which follows from the fact that for $0 \leq k \leq n$, $f_1(t_{n,k})$ is “close” to $f_1(a + \frac{k}{n} (b - a))$ (Theorem 5.4).

We consider first the case $f_1' > 0$ inside $(a, b)$, where estimates are better for the following reason: the zeros of $f_1'$ inside $(a, b)$ generate reversals of the order of the nodes. This back and forth movement entails that the average distance between consecutive nodes will be larger than $(b - a)/n$. And the distance between a concrete pair of nodes can be much larger than $(b - a)/n$, as the next example shows. However, when $f_1' > 0$ inside $(a, b)$ there are no order reversals; thus, the average distance between consecutive nodes is still $(b - a)/n$. We show that the distance between every pair of consecutive nodes is bounded by $O(1/n)$, so in fact it is never much larger than its average value. This has the obvious consequence that $B_n^{f_1} f(x)$ and $B_n f(x)$ are always at distance $O(\omega(f, 1/n))$.

**Example 6.1.** We use the same example $(x - t)^3$ as in Theorem 5.3 but taking $t = 1/2$ instead of $t = 1/N$. Consider

$$f_1(x) = \left(x - \frac{1}{2}\right)^3 = x^3 - \frac{3}{2} x^2 + \frac{3}{4} x - \frac{1}{8}$$

on $[0, 1]$. Since the largest order $s - 1$ of a zero of $f_1'$ is 2, we have $s = m = 3$.

From Proposition 5.2 we know that for $n \geq k \geq m$, the Bernstein coefficients of a polynomial $g$ of degree $m$, $g : [0, 1] \to \mathbb{R}$,

$$g(x) = \sum_{l=0}^m c_l x^l = \sum_{k=0}^n w_{n,k} p_{n,k}(x),$$

where
are given by

\[
(27) \quad w_{n,k} = \sum_{l=0}^{m} c_l \frac{k! (n-l)!}{n! (k-l)!}.
\]

In the specific instance \( f_1(x) = (x - \frac{1}{2})^3 \), for \( k \geq 3 \) the coordinates \( \gamma_{n,k} \) are

\[
\gamma_{n,k} = -\frac{1}{8} + \frac{3k}{4n} - \frac{3k (k-1)}{2n (n-1)} + \frac{k (k-1) (k-2)}{n (n-1) (n-2)}.
\]

Let \( K > 0 \) be any fixed constant, and let \( N \gg 16^2 K^3 \). When \( n = 2N \) and \( k = N \), we have \( \gamma_{2N,N} = 0 \), while if \( n = 2N \) and \( k = N + 1 \), a computation shows that \( \gamma_{2N,N+1} = -\frac{3}{16N^2 - 8N} < 0 \). Since \( t_{n,k} = f_1^{-1}(\gamma_{n,k}) \) and \( f_1^{-1}(t) = 1/2 + t^{1/3} \), we see that \( t_{2N,N} = 1/2 \), \( t_{2N,N+1} = 1/2 - (\frac{3}{16N^2 - 8N})^{1/3} \) and

\[
|t_{2N,N+1} - t_{2N,N}| = \left( \frac{3}{16N^2 - 8N} \right)^{1/3} \geq (16N)^{-2/3} \gg KN^{-1}.
\]

Also, the distance between \( k/n \) and \( t_{n,k} \) can be much larger than \( Kn^{-1} \). Choose \( N \gg 3K^2 \) so that both \( N/2 \) and \( (N/3)^{1/2} \) are integers. A computation shows that \( \gamma_{N,N/2+(N/3)^{1/2}} < 0 \), so \( t_{N,N/2+(N/3)^{1/2}} < 1/2 \) and

\[
|t_{N,N/2+(N/3)^{1/2}} - \frac{N + 2(N/3)^{1/2}}{2N}| \geq \frac{(N/3)^{1/2}}{N} = \frac{1}{(3N)^{1/2}} \gg \frac{K}{N}.
\]

**Theorem 6.2.** Let \( f_1 \) be a strictly increasing polynomial on \( [a,b] \), of degree \( m \), such that \( f_1'(x) > 0 \) for all \( x \) in the open interval \( (a,b) \). Then there exist an integer \( n_0 \) and a constant \( K > 0 \) such that for all \( n \geq n_0 \), and all \( 0 \leq k \leq n \), we have

\[
(28) \quad \left| a + \frac{k}{n} (b - a) - t_{n,k} \right| \leq \frac{K}{n},
\]

where \( t_{n,k} = f_1^{-1}(\gamma_{n,k}) \), and the \( \gamma_{n,k} \)'s are the Bernstein coordinates of \( f_1 \) in \( \mathbb{P}_n[a,b] \).

**Proof.** By a change of variables we assume that \( a = 0 \) and \( b = 1 \). This may alter the concrete value of \( K \) in inequality (28), since \( b - a \) appears in (17), but it will not change the bound \( |a + \frac{k}{n} (b - a) - t_{n,k}| = O \left( \frac{1}{n} \right) \). Choose \( n \geq n_0 \gg m \) so large that all the Bernstein coordinates of \( f_1' \) are non-negative in all dimensions \( n \geq n_0 - 1 \). Additional conditions will be imposed on \( n_0 \) later on. By Lemma 3.1 the coordinates of \( f_1 \), and hence the nodes, are non-decreasing for all \( n \geq n_0 \). We consider the case where \( f_1' \) vanishes both at 0 and at 1. The other cases are simpler and can be handled in the same way. Suppose that \( f_1' \) has a zero of positive order \( s - 1 \) at 0, so \( s - 1 > 0 \) and \( f_1(x) - f_1(0) = \sum_{l=s}^{m} c_l x^l \). Then \( c_s > 0 \) (since \( f_1 \) is increasing), and \( f_1'(x) = \sum_{l=s}^{m} c_l x^{l-1} = sc_s x^{s-1} + O(x^s) \). Thus, there exists a \( \delta_0 \in (0,1/2) \) such that for all \( y \in [0,\delta_0) \),

\[
(29) \quad \frac{c_s}{2} y^s \leq f_1(y) - f_1(0) \quad \text{and} \quad \frac{sc_s}{2} y^{s-1} \leq f_1'(y).
\]
Suppose next that \( f'_1 \) has a zero of order \( r - 1 > 0 \) at 1. Recalling that \( f \) is strictly increasing, we conclude that for \( y < 1 \), \( f_1(y) - f_1(1) < 0 \) and \( f_1(y) > 0 \). Thus, there exists a \( \delta_1 \in (0, 1/2) \) such that for all \( y \in (1 - \delta_1, 1) \),

\[
(30) \quad f_1(y) - f_1(1) = \sum_{l=r}^{m} c_l(y - 1)^l = c_r(y - 1)^r + O((y - 1)^{r+1}) \leq \frac{c_r}{2} (y - 1)^r \leq 0
\]

and

\[
(31) \quad f'_1(y) = \sum_{l=r}^{m} c_l(y - 1)^{l-1} = r c_r(y - 1)^{r-1} + O((y - 1)^r) \geq \frac{r c_r}{2} (y - 1)^{r-1} \geq 0.
\]

We write \([0, 1]\) as the union of the three subintervals \([0, \delta_0/2)\), \([\delta_0/4, 1 - \delta_1/4]\) and \((1 - \delta_1/2, 1]\); let

\[
0 < c = \min_{x \in [\delta_0/4, 1 - \delta_1/4]} f'_1(x).
\]

If both \( k/n \) and \( t_{n,k} \) belong to \([\delta_0/4, 1 - \delta_1/4]\), we use the fact that \( f_1^{-1} \) is Lipschitz on the interval \([\delta_0/4, 1 - \delta_1/4]\), with constant \( 1/c \), to conclude that \(|k/n - t_{n,k}| = O(n^{-1})\).

Let \( n_0 \gg 2/(c \min \{\delta_1, \delta_1')\} \). Suppose either \( k/n \) or \( t_{n,k} \) belong to \([0, \delta_0/2)\); if either \( k/n \) or \( t_{n,k} \) belong to \((1 - \delta_1/2, 1]\), the argument is entirely analogous. We must have that both \( k/n \) and \( t_{n,k} \) belong to \([0, \delta_0]\), for otherwise

\[
c \delta_0/2 \leq f_1(\delta_0) - f_1(\delta_0/2) \leq |f_1(k/n) - \gamma_{n,k}| = O(1/n)
\]

by Theorem 5.4 and a contradiction is obtained since \( n \geq n_0 \).

Let \( k_0 \) and \( j \) be the smallest integers such that

\[
(32) \quad n \delta_0 \leq 2k_0 \quad \text{and} \quad j \geq 2^s (sc_s)^{-1} c_{\max} m^3
\]

respectively, and assume that \( k < k_0 \), so \( k/n < \delta_0/2 \). Note that \( j > m \). First, since the nodes are increasing, if \( k \leq 2j \), then

\[
(33) \quad |t_{n,k} - k/n| \leq |t_{n,k} + k/n| \leq |t_{n,2j} + 2j/n| \leq |t_{n,2j} - 2j/n| + 4j/n.
\]

Thus, it is enough to prove that for all \( 2j \leq k < k_0 \),

\[
(34) \quad \frac{k-j}{n} \leq t_{n,k} \leq \frac{k+j}{n},
\]

since this implies, whenever \( 0 \leq k < k_0 \), that

\[
(35) \quad \left| t_{n,k} - \frac{k}{n} \right| \leq \max \left\{ \frac{j}{n}, |t_{n,2j} - 2j/n| + \frac{4j}{n} \right\} \leq \frac{5j}{n}.
\]

So suppose \( k \geq 2j > m \). Going back to formulas (22) - (25), the only difference here is that we have \( s \leq l \leq m \) instead of \( 2 \leq l \leq m \), so using the bound \((k/n)^{l-1} \leq (k/n)^{s-1}\) instead of \((k/n)^{l-1} \leq 1\) in (25), we obtain the following refinement of (26):

\[
(36) \quad \left| f_1 \left( \frac{k}{n} \right) - \gamma_{n,k} \right| \leq \left( \frac{k}{n} \right)^{s-1} c_{\max} \sum_{l=s}^{m} \frac{m^2}{n} \leq \left( \frac{k}{n} \right)^{s-1} c_{\max} \frac{m^3}{n},
\]
or equivalently,

\[(37) \quad f_1 \left( \frac{k}{n} \right) - \left( \frac{k}{n} \right)^{s-1} c_{\text{max}} \frac{m^3}{n} \leq \gamma_{n,k} \leq f_1 \left( \frac{k}{n} \right) + \left( \frac{k}{n} \right)^{s-1} c_{\text{max}} \frac{m^3}{n}. \]

Choose \(n_0\) so large that \(j/n_0 < \delta_0/8\). Since \(k/n \leq \delta_0/2\), we have that \((k+j)/n < \delta_0\), so by (29), on the interval \([k/n, (k+j)/n]\), \(f_1'(\xi) \geq sc_s \xi^{s-1}/2 \geq sc_s (k/n)^{s-1}/2\). By the Mean Value Theorem,

\[(38) \quad f_1 \left( \frac{k+j}{n} \right) - f_1 \left( \frac{k}{n} \right) \geq \frac{sc_s}{2} \left( \frac{k}{n} \right)^{s-1} \left( \frac{k+j}{n} - \frac{k}{n} \right) \geq c_{\text{max}} \frac{m^3}{n} \left( \frac{k}{n} \right)^{s-1}. \]

From (37) and (38) we obtain

\[(39) \quad \gamma_{n,k} \leq f_1 \left( \frac{k+j}{n} \right), \]

and since \(f_1\) is increasing,

\[(40) \quad t_{n,k} \leq \frac{k+j}{n}. \]

Using the fact that \(k \geq 2j\), an entirely analogous argument shows that

\[f_1 \left( \frac{k}{n} \right) - f_1 \left( \frac{k-j}{n} \right) \geq \frac{sc_s}{2} \left( \frac{k}{n} \right)^{s-1} \left( \frac{j}{n} \right) \geq \frac{c_{\text{max}} m^3}{n} \left( \frac{k}{n} \right)^{s-1}, \]

so \(\gamma_{n,k} \geq f_1 \left( \frac{k-j}{n} \right)\), and \(t_{n,k} \geq \frac{k-j}{n}\).

Finally, suppose that \(t_{n,k} < \delta_0/2 < k/n\). Recalling that \(j/n_0 < \delta_0/8\), that \(n \geq n_0\), and that the nodes are nondecreasing, by (32) and (34) we have

\[\delta_0/4 \leq \delta_0/2 - \frac{1}{n} - \frac{j}{n} \leq \frac{k_0-1}{n} - \frac{j}{n} \leq t_{n,k_0-1} \leq t_{n,k}. \]

Thus, the case \(t_{n,k} < \delta_0/2 \leq k/n\) has already been considered, since then both \(k/n, t_{n,k} \in [\delta_0/4, \delta_0) \subset [\delta_0/4, 1 - \delta_1/4]\).

**Definition 6.3.** The modulus of continuity of a function \(f \in C[a,b]\) is

\[\omega(f, \delta) := \sup \{ |f(x) - f(y)| : x, y \in [a,b], |x - y| \leq \delta \}. \]

**Remark 6.4.** Consider the classical Bernstein operator \(B_n\) over \([0,1]\). It is well known that for all \(f \in C[0,1]\), all \(x \in [0,1]\) and all \(n \geq 1\),

\[|f(x) - B_n f(x)| \leq c \omega \left( f, n^{-\frac{1}{2}} \right), \]

where \(c = \frac{4306 + 837\sqrt{6}}{5832} \approx 1.08988\) (cf. [19], [20]).
Theorem 6.5. Let \( f_1 : [a, b] \to \mathbb{R} \) be a polynomial of degree \( m \geq 1 \), such that \( f_1' > 0 \) on \( (a, b) \), and let \( B_n^{f_1} \) be the Bernstein operator over \([a, b]\), fixing \( f_1 \) and the constant function 1. Denote by \( B_n \) the classical Bernstein operator. Then there exist a constant \( K > 0 \) and a natural number \( n_0 \) such that for all \( f \in C[a, b] \), all \( x \in [a, b] \) and all \( n \geq n_0 \),
\[
|B_n^{f_1} f(x) - B_n f(x)| \leq \omega (f, Kn^{-1}).
\]

Proof. By Theorem 6.2, \( \frac{B_n^{f_1} f(x) - f(x)}{B_n f(x)} \) which is comparable to the rate of convergence of \( \frac{\alpha}{\beta} \) \( \leq 0 \)

Hence, \( B_n^{f_1} f(x) - B_n f(x) \) \( \leq \sum \left( f(t_{n,k}) - f(k/n) \right) p_{n,k}(x) \) and hence, \( \omega (f, K^{-1}) p_{n,k}(x) = \omega (f, K^{-1}). \)

which is comparable to the rate of convergence of \( B_n \).

Definition 6.6. A function \( f : [a, b] \to \mathbb{R} \) is Lipschitz if there exists a constant \( K > 0 \) such that for all \( x, y \in [a, b] \), \( |f(x) - f(y)| \leq K|x - y| \). If for some \( 0 < \alpha \leq 1 \) and \( K > 0 \) we have \( |f(x) - f(y)| \leq K|x - y|^\alpha \), then we say that \( f \) is Hölder continuous of order \( \alpha \).

We will often use the shorter expression “\( \alpha \)-Hölder”. Thus, 1-Hölder is the same as Lipschitz. Furthermore, if \( 0 < \alpha < \beta \leq 1 \) and \( f \) is \( \beta \)-Hölder, then it is also \( \alpha \)-Hölder, since \( |f(x) - f(y)| \leq K|x - y|^\beta = K|x - y|^\alpha \). Hence, \( |f(x) - f(y)| \leq K|x - y|^\alpha \).

The next theorem may be well known, but an internet search has yielded no results.

Theorem 6.7. Let \( p : [a, b] \to \mathbb{R} \) be an increasing polynomial of degree \( m \geq 1 \), and let \( 0 \leq s - 1 \) be the largest order of the zeros that \( p' \) may have in \([a, b]\). Then there exist a \( \delta > 0 \) and a \( K > 0 \) such that for all \( u, v \in [p(a), p(b)] \) with \(|u - v| \leq \delta\), \(|p^{-1}(u) - p^{-1}(v)| \leq K|u - v|^{1/s} \).

Hence, \( |p^{-1}(u) - p^{-1}(v)| \leq K'|u - v|^{1/m} \) for some \( K' > 0 \).

Proof. If \( p' > 0 \) on \([a, b]\), then \( p^{-1} \) is Lipschitz on \([p(a), p(b)]\) (with constant \( K = \|p^{-1}'\|_\infty \)) and hence, Hölder of order \( 1/s = 1 \). So suppose \( p' \) vanishes somewhere in \([a, b]\), say, at the distinct points \( a \leq x_0 < \cdots < x_j \leq b \) (if \( p' \) has only one zero, at \( x_0 \), then \( a \leq x_0 \leq b \), and the preceding notation is not intended to imply that \( x_0 < b \)).

For \( 0 \leq i \leq j \), let \( 1 \leq s_i - 1 \leq m - 1 \) be the order of the zero of \( p' \) at \( x_i \). Then
\[
p(x) - p(x_i) = \sum_{l = s_i}^m c_l (x - x_i)^l = c_{s_i}(x - x_i)^{s_i} + O(|x - x_i|^{s_i+1}).
\]
Since $p$ is increasing on $[a, b]$, for each $i$ there exists an $\varepsilon_i > 0$ such that if $x_0 = a$, then $p$ is convex on $[a, a + \varepsilon_0]$, if $x_i \in (a, b)$, then $p$ is concave on $[x_i - \varepsilon_i, x_i]$ and convex on $[x_i, x_i + \varepsilon_i]$, and if $x_j = b$, then $p$ is concave on $[b - \varepsilon_j, b]$.

Note that for $x_i \in [a, b)$, we have $c_{s_i} > 0$, since $p$ is increasing on $[x_i, b]$ (but if $x_i = b$ and $s_i$ is even, then $c_{s_i} < 0$). Suppose first that $x_i \in (a, b)$. Select $\delta_i \in (0, \varepsilon_i]$ so that for all $y \in [x_i, x_i + \delta_i]$,

$$\frac{c_{s_i}}{2} (y - x_i)^{s_i} \leq p(y) - p(x_i),$$

with the inequality reversed when $y \in [x_i - \delta_i, x_i]$:

$$\frac{c_{s_i}}{2} (y - x_i)^{s_i} \geq p(y) - p(x_i).$$

The one sided cases $x_0 = a$ and $x_j = b$ are simpler and handled in the same way: with the same notation as above, we also have

$$\frac{c_{s_0}}{2} (y - a)^{s_0} \leq p(y) - p(a) \quad \text{and} \quad \frac{c_{s_j}}{2} (y - b)^{s_j} \geq p(y) - p(b)$$

on $[a, a + \delta_0]$ and $[b - \delta_j, b]$ respectively. Next, choose $\delta > 0$ satisfying the following two conditions:

i) For $i = 0, \ldots, j$,

$$[a, b] \cap p^{-1}([p(x_i) - 2\delta, p(x_i) + 2\delta]) \subset [a, b] \cap [x_i - \delta_i, x_i + \delta_i].$$

ii) For $i = 0, \ldots, j$, the intervals $[p(x_i) - 2\delta, p(x_i) + 2\delta]$ are disjoint.

Now let $u, v \in [p(a), p(b)]$ satisfy $|u - v| \leq \delta$. On $[p(a), p(b)] \setminus \bigcup_{i=0}^{j} (p(x_i) - \delta, p(x_i) + \delta)$, $\|p^{-1}\|_\infty < \infty$, and hence $p^{-1}$ is Lipschitz there. Next, assume that for some $i$, either $u \in (p(x_i) - \delta, p(x_i) + \delta)$ or $v \in (p(x_i) - \delta, p(x_i) + \delta)$. Then both $u, v \in (p(x_i) - 2\delta, p(x_i) + 2\delta)$.

By a translation and by substracting a constant if needed, we may, without loss of generality, suppose that $0 = x_i = p(x_i)$, so $u, v \in (-2\delta, 2\delta)$.

Assume first that $x_i < b$. Since

$$\frac{c_{s_i} y^{s_i}}{2} \leq p(y) \quad \text{on} \quad [0, \delta_i], \quad \text{it follows that} \quad \left(\frac{2u}{c_{s_i}}\right)^{1/s_i} \geq p^{-1}(u) \quad \text{on} \quad [0, 2\delta].$$

By the concavity of $p^{-1}$ on $[0, 2\delta]$, if $0 \leq u, v$, then

$$|p^{-1}(u) - p^{-1}(v)| \leq p^{-1}(|u - v|) \leq \left(\frac{2|u - v|}{c_{s_i}}\right)^{1/s_i}.$$ 

If $0 = x_i = a$ we do not need to do anything else. If $0 = x_i \in (a, b)$, and $0 \geq u, v$, say, with $u < v \leq 0$, we argue in the same way, but paying attention to the negative signs. The interior zeros of $p'$ have even order, since $p' \geq 0$ on $[a, b]$, so $s_i - 1$ is even, and $s_i$, odd. By (43), for all $y \in [-\delta_i, 0]$,

$$\frac{c_{s_i} y^{s_i}}{2} \geq p(y).$$
Writing \( w = p(y) \) for \( w \in (-2\delta, 0] \), we have
\[
\frac{c_{s_i}}{2} \left( p^{-1}(w) \right)^{s_i} \geq w,
\]
or equivalently,
\[
\frac{c_{s_i}}{2} \left| p^{-1}(w) \right|^{s_i} \leq |w|;
\]
hence,
\[
|p^{-1}(w)| \leq \left( \frac{2}{c_{s_i}} |w| \right)^{1/s_i}.
\]
(45)

Now by the convexity of \( p^{-1} \) on \((-2\delta, 0]\),
\[
0 > p^{-1}(u) - p^{-1}(v) \geq p^{-1}(u-v),
\]
so
\[
|p^{-1}(u) - p^{-1}(v)| \leq p^{-1}(-v) - p^{-1}(v) \leq 2 \left( \frac{2|v|}{c_{s_i}} \right)^{1/s_i} \leq 2 \left( \frac{2|u-v|}{c_{s_i}} \right)^{1/s_i}.
\]
(46)

Lastly, if, say \( u < 0 < v \), and for instance, \( v \geq |u| \), then
\[
|p^{-1}(u) - p^{-1}(v)| \leq |p^{-1}(-v) - p^{-1}(v)| \leq 2 \left( \frac{2|v|}{c_{s_i}} \right)^{1/s_i} \leq 2 \left( \frac{2|u-v|}{c_{s_i}} \right)^{1/s_i}.
\]

To finish, if \( 0 = x_i = b \) and \( s_i \) is odd, the argument is exactly as in the case \( u, v \in (-2\delta, 0] \) seen above, while if \( s_i \) is even, then \( c_{s_i} < 0 \) in (41), since \( p \) is increasing on \([a, b]\); from (44) we conclude that for \( w = p(y) \in (-2\delta, 0] \),
\[
\frac{|c_{s_i}|}{2} \left( \left| p^{-1}(w) \right| \right)^{s_i} \leq |w|,
\]
and taking \(-2\delta < u < v \leq 0 \), we have
\[
|p^{-1}(u) - p^{-1}(v)| \leq |p^{-1}(u-v)| \leq 2 \left( \frac{2}{|c_{s_i}|} |u-v| \right)^{1/s_i}.
\]

Now \( i \) is arbitrary and \( s_i \leq s \leq m \), so the result follows. \( \square \)

It is well known and easy to prove, that if \( g : [a, b] \to \mathbb{R} \) is continuous, and there exist a \( \delta > 0 \) and a \( K > 0 \) such that for all \( x, y \in [a, b] \) with \( |x-y| \leq \delta \), we have \( |g(x) - g(y)| \leq K|x-y|^\alpha \), then \( g \) is Hölder continuous of order \( \alpha \). Though Theorem 6.7 is sufficient for our purposes, the next result is interesting in itself.

**Corollary 6.8.** Let \( p : [a, b] \to \mathbb{R} \) be an increasing polynomial of degree \( m \geq 1 \). Then \( p^{-1} : [p(a), p(b)] \to [a, b] \) is Hölder continuous of order \( 1/s \), where \( s-1 \geq 0 \) denotes the largest order of any zero of \( p' \) in \([a, b]\).
**Theorem 6.9.** Let $f_1$ be a strictly increasing polynomial on $[a, b]$, of degree $m$, and let $s - 1 \geq 0$ denote the largest order of the zeros that $f_1'$ may have in $[a, b]$. Then there exist constants $n_0 \geq m$ and $K > 0$ such that $B_{n}^{f_1}$ is well defined whenever $n \geq n_0$, and for all $0 \leq k \leq n$, we have
\[
\left| t_{n,k} - \left( a + \frac{k}{n} (b - a) \right) \right| \leq \frac{K}{n^{1/s}},
\]
where the points $t_{n,k} = f_1^{-1}(\gamma_{n,k})$ are the nodes of $B_{n}^{f_1}$.

**Proof.** The argument to handle the possible zeros of $f_1'$ at the endpoints is essentially identical to the one used in the proof of Theorem 6.2, keeping the analogous choices made there for $\delta_0$ and $\delta_1$, on $[a, a + \delta_0/2]$ and $(b - \delta_1/2, b]$ we have that the coordinates $\gamma_{n,k}$ of $f_1$ are non-decreasing when $k/n \in [a, a + \delta_0/2] \cup (b - \delta_1/2, b]$, and since $f_1$ is increasing, the nodes $t_{n,k}$ are also non-decreasing.

As for the central region $[a + \delta_0/2, b - \delta_1/2]$, we use Theorem 6.7 for $f_1([a + \delta_0/2, b - \delta_1/2])$; since by Theorem 6.2 we have
\[
\left| f_1 \left( a + \frac{k}{n} (b - a) \right) - \gamma_{n,k} \right| = O(1/n)
\]
(with constant depending only on $f_1$ and $[a, b]$) and $f_1^{-1}$ is $1/s$-Hölder over $f_1([a + \delta_0/2, b - \delta_1/2])$ (again with constant depending only on $f_1$ and $[a, b]$) we conclude that
\[
\left| a + \frac{k}{n} (b - a) - t_{n,k} \right| = \left| f_1^{-1} \left( f_1 \left( a + \frac{k}{n} (b - a) \right) \right) - f_1^{-1}(\gamma_{n,k}) \right|
\]
\[
= O \left( \left| f_1 \left( a + \frac{k}{n} (b - a) \right) - \gamma_{n,k} \right|^{1/s} \right) = O(n^{-1/s}).\]

Thus, the following variant of Theorem 6.5 with $1/n^{1/s}$ in the modulus of continuity instead of $1/n$, is obtained.

**Theorem 6.10.** Let $f_1 : [a, b] \to \mathbb{R}$ be an increasing polynomial of degree $m \geq 1$, and let $B_{n}^{f_1}$ be the Bernstein operator over $[a, b]$, fixing $f_1$ and the constant function 1. Denote by $B_{n}$ the classical Bernstein operator, and denote by $s - 1 \geq 0$ the largest order of the zeros of $f_1'$ in $[a, b]$. Then there exist a constant $K = K(f_1|_{[a,b]}) > 0$ and a natural number $n_0$ such that for all $f \in C[a, b]$, all $x \in [a, b]$ and all $n \geq n_0$,
\[
\|B_{n}^{f_1} f (x) - B_{n} f (x)\| \leq \omega \left( f, Kn^{-1/s} \right).
\]

**Theorem 6.11.** With the same notation as in the preceding result, for all $f \in C[a, b]$, $\lim_{n \to \infty} \|B_{n}^{f_1} f - f\|_{\infty} = 0$.

**Proof.** Since $f$ is continuous on $[a, b]$, $\lim_{n \to 0} \omega \left( f, Kn^{-1/s} \right) = 0$, so by Theorem 6.10 and the convergence of the standard Bernstein operator,
\[
\lim_{n \to \infty} \|B_{n}^{f_1} f - f\|_{\infty} \leq \lim_{n \to \infty} \|B_{n}^{f_1} f - B_{n} f\|_{\infty} + \lim_{n \to \infty} \|B_{n} f - f\|_{\infty} = 0.
\]
We finish with some remarks on shape preservation. Regarding the convexity preserving properties of the generalized Bernstein operator $B_n^{f_1}$ when $f_1' > 0$ on $(a,b)$, by [4] they are analogous to the ones of the standard Bernstein operator, but understood with $(1, f_1)$-convexity replacing ordinary convexity.

**Definition 6.12.** Let $E \subseteq \mathbb{R}$. A function $f : E \to \mathbb{R}$ is called $(f_0, f_1)$-convex on $E$ if for all $x_0, x_1, x_2$ in $E$ with $x_0 < x_1 < x_2$, the determinant

\[
\det_{x_0,x_1,x_2}(f) := \det \begin{pmatrix}
  f_0(x_0) & f_0(x_1) & f_0(x_2) \\
  f_1(x_0) & f_1(x_1) & f_1(x_2) \\
  f(x_0) & f(x_1) & f(x_2)
\end{pmatrix}
\]

is non-negative.

Note in particular that convexity is the same as $(1, x)$-convexity. In [4] Theorem 25 it is shown that when the nodes fail to be non-decreasing, $(1, f_1)$-convexity may be lost, but the example given there is not a polynomial space. The same phenomenon can occur in the polynomial context, as we prove next.

**Example 6.13.** Consider $\mathbb{P}_4[-1,1] = \text{Span}\{1, x, x^2, x^3, x^4\}$, with the standard Bernstein bases over $[-1,1]$. It is easy to check that the coordinates of $f_1(x) := x^3$ are $\gamma_{4,0} = -1$, $\gamma_{4,1} = 1/2$, $\gamma_{4,2} = 0$, $\gamma_{4,3} = -1/2$ and $\gamma_{4,4} = 1$ (just plug in, and simplify; alternatively, these coordinates can be obtained from formula (14)). The nodes are the cube roots of the corresponding coordinates, so

\[
B_4^{f_1}e_4(x) = \frac{(1-x)^4}{16} + \frac{(1+1)^4}{2} \frac{1}{4} (1-x)^3(1+x) + \frac{(1+1)^4}{2} \frac{1}{4} (1-x)(1+x)^3 + \frac{(1+1)^4}{16}.
\]

By [9] Theorem 5], $f$ is $(1, f_1)$-convex if and only if $f \circ f_1^{-1}$ is convex in the ordinary sense. Note that $e_4(x) = x^4$ is $(1, x^3)$-convex, since $e_4(x^{1/3}) = (x^{1/3})^4$ is convex. But $B_4^{f_1}e_4(x)$ is not $(1, x^3)$-convex: expanding, simplifying, and replacing $x$ with $x^{1/3}$, we get

\[
B_4^{f_1}e_4(x^{1/3}) = \frac{2^{2/3} + 1}{8} + \frac{3}{4} (x^{1/3})^2 - \left(\frac{2^{2/3} - 1}{8}\right) (x^{1/3})^4.
\]

Since both $x^{2/3}$ and $-x^{4/3}$ are concave on $[0,1]$, so is $B_4^{f_1}e_4(x^{1/3})$.

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