Explosive higher-order Kuramoto dynamics on simplicial complexes

Ana P. Millán and Joaquín J. Torres
Departamento de Electromagnetismo y Física de la Materia and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, 18071 Granada, Spain

Ginestra Bianconi
School of Mathematical Sciences, Queen Mary University of London, E1 4NS London, United Kingdom
Alan Turing Institute, The British Library, London, United Kingdom

The higher-order interactions of complex systems, such as the brain are captured by their simplicial complex structure and have a significant effect on dynamics. However the existing dynamical models defined on simplicial complexes make the strong assumption that the dynamics resides exclusively on the nodes. Here we formulate the higher-order Kuramoto model which describes the interactions between oscillators placed not only on nodes but also on links, triangles, and so on. We show that higher-order Kuramoto dynamics can lead to explosive synchronization transition by using an adaptive coupling dependent on the solenoidal and the irrotational component of the dynamics.

PACS numbers:

From the brain [1–3] to social interactions [4–6] and complex materials [7–8], a vast number of complex systems have the underlying topology of simplicial complexes [9–10]. Simplicial complexes are topological structures formed by simplices of different dimension such as nodes, links, triangles, tetrahedra and so on, and capture the many body interactions between the elements of an interacting complex systems. In the last years simplicial complexes modelling has attracted significant attention, and there has been significant progress in extending random graphs [11], the configuration model [12], growing network models [13] and activity-driven models [14] to simplicial complexes, revealing very important and non-trivial effects of network topology and geometry [15]. Modelling complex systems using simplicial complexes opens the very fertile perspective to consider the role that higher-order interactions have on dynamical processes. For instance, recent works have characterized topological percolation [16], epidemic spreading [4–6] and synchronization [17–20] on simplicial complexes finding very interesting consequences of the topology and geometry of the simplicial complexes and the cooperative phenomena due to the many-body interactions present in these structures.

In the last years explosive synchronization [21–22] is attracting increasing scientific interest. Different pathways to explosive synchronization have been explored in the framework of the Kuramoto dynamics of single and multilayer networks. These include notably correlating the intrinsic frequency of the nodes to their degree [23] or modulating the coupling between different oscillators adaptively using the local order parameter in single networks and in multiplex networks [24–25]. An outstanding open question is to establish the conditions that allow explosive synchronization on simplicial complexes.

A recent work [18] has proposed a many-body Kuramoto model where the phases associated with the nodes of the network can be coupled in triplets or quadruplets if the corresponding nodes share a triangle or a tetrahedron. Interestingly in this context it has been shown that the many-body Kuramoto dynamics can lead to explosive, i.e. discontinuous phase transitions. However this work, and in general in the vast majority of works that address the study of dynamics in simplicial complexes, have the strong limitation that they associate a dynamic variable exclusively with nodes of a network. Here we are interested in a much more general scenario where the dynamics can be associated with the faces of dimension $n \geq 0$ of a simplicial complex. Indeed dynamical processes might not just reside on nodes, instead they might be related directly to dynamics defined on higher dimensional simplices. For instance we might desire to associate with each link or with each triangle a flow or a phase. This more general scenario leads to the definition of topological dynamical signals [26].

In this Letter we formulate a higher-order Kuramoto dynamics where the dynamical variables are coupled oscillators associated with higher dimensional simplices such as nodes, links, triangles and so on. By using Hodge decomposition we show that the dynamics defined on an $n$-dimensional simplex can be projected on the dynamics defined on $n+1$ and $n-1$ dimensional simplices. We propose a simple higher-order Kuramoto dynamics in which these two projected dynamics are decoupled and display a continuous phase transition. We then formulate the explosive higher-order Kuramoto dynamics which adaptively couples the two projected dynamics with a mechanism inspired by Ref. [24], showing that in this case the explosive higher-order Kuramoto dynamics leads to a discontinuous synchronization transition. This implies for instance that a dynamics defined on links can induce a simultaneous explosive synchronization on the dynamics projected on nodes and triangles. Therefore our work elucidates an important mechanism leading to higher-order explosive Kuramoto dynamics.

Definition of simplicial complexes- Simplicial complexes represent higher-order networks, which include interactions between two or more nodes, described by sim-
A simplicial complex $K$ is formed by a set of simplices that satisfy the condition of closure (given a simplex belonging to the simplicial complex all its faces also belong to the simplicial complex). Among the increasing number of available models of large random simplicial complexes [11–15], in this work we will use the configuration model [12] of simplicial complexes, which naturally generalizes the configuration model of networks. In particular, the $d$-dimensional configuration model generates simplicial complexes formed by gluing $d$-dimensional simplices such that every node is incident to a given number of $d$-dimensional simplices called its generalized degree. In topology simplices have also an orientation. A $n$-dimensional oriented simplex $\alpha$ is a set of ordered $n+1$ nodes

$$\alpha = [i_0, i_1, \ldots, i_n].$$

(1)

For instance, a link $\alpha = [i, j]$ has opposite sign of the link $[j, i]$, i.e.

$$[i, j] = -[j, i].$$

(2)

Similarly, we associate an orientation to higher-order simplices satisfying

$$[i_0, i_1, \ldots, i_n] = (-1)^{\sigma(\pi)}[i_{\pi(0)}, i_{\pi(1)}, \ldots, i_{\pi(n)}],$$

(3)

where $\sigma(\pi)$ indicates the parity of the permutation $\pi$. Here we consider the orientation induced by the labelling of its nodes, i.e. for every simplex in a simplicial complex we give positive orientation as the one provided by the increasing list of node labels (see Figure 1).

In topology [26–29], the boundary map is defined as the map that associates every $n$-dimensional simplex $\alpha = [i_0, i_2, \ldots, i_n]$ with a linear combination of the $(n-1)$-dimensional oriented faces at its boundary, given by

$$\partial_n[i_0, i_1, \ldots, i_n] = \sum_{p=0}^{n} (-1)^p [i_0, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_n].$$

(4)

The boundary map satisfies the important property that

$$\partial_{n-1} \partial_n = 0,$$

(5)

that is usually expressed by saying that the boundary of a boundary is null. Given a simplicial complex with $N_{[n]}$ $n$-dimensional simplices, the boundary map $\partial_n$ can be described using the $N_{n-1} \times N_{[n]}$ incidence matrix $B_{[n]}$. For instance, in Figure 1 we show an example of a simplicial complex formed by the set of nodes $\{[1], [2], [3], [4]\}$, the set of links $\{[1, 2], [1, 3], [2, 3], [3, 4]\}$, and the set of triangles $\{[1, 2, 3]\}$. The incidence matrices [26][29] of this simplicial complex are given by

$$B_{[1]} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{[2]} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$  (6)

FIG. 1: An example of a small simplicial complex with the orientation of the simplices induced by the labelling of the nodes.

Higher-order Laplacians - The graph Laplacian is widely used to study dynamical processes defined on the nodes of a network. It can be expressed in terms of the boundary matrix $B_{[1]}$ as

$$L_{[0]} = B_{[1]}B_{[1]}^\top.$$  (7)

The higher-order Laplacian $L_{[n]}$ [26][29], with $n > 0$, generalizes the graph Laplacian by describing diffusion taking place on $n$ dimensional faces. The $n$-th Laplacian $L_{[n]}$ is an $N_{[n]} \times N_{[n]}$ matrix given by

$$L_{[n]} = B_{[n]}^\top B_{[n]} + B_{[n+1]}^\top B_{[n+1]}.$$  (8)

The spectral properties of the higher-order Laplacian can be proven to be independent of the orientation of the simplices as long as the orientation is induced by a labelling of the nodes. The main property of the higher-order Laplacian is that the degeneracy of the zero eigenvalue of the $n$ Laplacian $L_{[n]}$ is equal to the Betti number $\beta_n$, and that their corresponding eigenvectors localize around the corresponding $n$-dimensional cavities of the simplicial complex.

The higher-order Laplacians have notable spectral properties induced by the topological properties of the boundary map [26]. In fact, given Eq. 5 we have $B_{[n-1]}B_{[n]} = 0$ and, similarly, $B_{[n]}^\top B_{[n-1]} = 0$. It follows that the eigenvectors associated with the non-null eigenvalues of $L_{[n]}^{\text{up}} = B_{[n+1]}^\top B_{[n+1]}$ are orthogonal to the eigenvectors associated with the non-null eigenvalues of $L_{[n]}^{\text{down}} = B_{[n]}^\top B_{[n]}$. It follows that the non-null eigenvalues of $L_{[n]}$ are either the non-null eigenvalues of $L_{[n]}^{\text{up}}$ or the non-null eigenvalues of $L_{[n]}^{\text{down}}$. This property of the higher-order Laplacian can be exploited to prove that
every vector $x_{[n]}$ defined on $n$-dimensional simplices can be decomposed according to the Hodge decomposition as

$$x_{[n]} = x^H_{[n]} + B^T_{[n]} z_{[n-1]} + B_{[n+1]} z_{[n+1]}$$

(9)

where $x^H_{[n]}$ is the harmonic component that satisfies $B^T_{[n+1]} x^H_{[n]} = B_{[n]} x^H_{[n]} = 0$, the term $B^T_{[n]} z_{[n-1]}$ is the irrotational component as we have $B^T_{[n+1]} B_{[n]} z_{[n-1]} = 0$ and the third term $B_{[n+1]} z_{[n+1]}$ is the solenoidal component as we have $B_{[n]} B_{[n+1]} z_{[n+1]} = 0$.

FIG. 2: The projection of the higher-order $(n = 1)$ Kuramoto dynamics on $(n-1)$-dimensional faces and $(n+1)$-dimensional faces is investigated by plotting the order parameters $R^{[1]}$ (left panel) and $R^{[2]}$ (right panel), both for the simple (blue circles) and explosive (red squares) dynamics. Here both the simple and the explosive higher-order Kuramoto model have $\Omega = 2$ and are defined on a configuration model of $N_{[0]} = 1000$ nodes, $N_{[1]} = 5299$ links and $N_{[2]} = 4147$ triangles with generalized degree of the nodes that is power-law distributed with power-law exponent $\gamma = 2.8$.

Higher-order Kuramoto dynamics - The Kuramoto dynamics [30] defined on nodes, i.e. simplices of dimension $n = 0$, indicated with label $i = 1, 2, \ldots, N_{[0]}$, can be expressed in terms of the incidence matrix $B_{[0]}$

$$\dot{\theta}_i = \omega_i - \sigma \sum_{\ell \in S_{d,n}} [B_{[0]}]_{i\ell} \sin \left( \sum_{\beta \in S_{d,n}} [B^T_{[0]}]_{\ell,\beta} \theta_\beta \right),$$

(10)

where here and in the following we indicate with $S_{d,n}$ the set of all simplices of dimension $n$ (of cardinality $|S_{d,n}| = N_{[n]}$) present in the simplicial complex under consideration. The internal frequency of each oscillator $i$ is indicated with $\omega_i$ and the intensity of the coupling between oscillators is indicated with the parameter $\sigma$. Here we define the simple higher-order Kuramoto dynamics defined for phases $\theta_\alpha$ associated to simplices $\alpha$ of dimension $n > 0$ as the natural extension of the above definition as

$$\dot{\theta}_\alpha = \omega_\alpha - \sigma \sum_{\beta \in S_{d,n+1}} [B_{[n]}]_{\alpha\beta} \sin \left( \sum_{\alpha' \in S_{d,n}} [B^T_{[n]}]_{\beta,\alpha} \theta_{\alpha'} \right) - \sigma \sum_{\beta \in S_{d,n-1}} [B^T_{[n-1]}]_{\alpha\beta} \sin \left( \sum_{\alpha' \in S_{d,n}} [B_{[n-1]}]_{\beta,\alpha'} \theta_{\alpha'} \right),$$

where we draw the intrinsic frequencies $\omega_\alpha$ from a normal distribution with mean $\Omega$ and variance 1, i.e. $\omega \sim N(\Omega, 1)$.

The higher-order Kuramoto dynamics describes a dynamics of phases associated to simplices of dimension $n$ as links ($n = 1$) or triangles ($n = 2$) etc. An important question to ask is whether the dynamics associated to $n$-dimensional simplices induces a dynamics on lower or higher dimensional simplices. For instance, if we have a Kuramoto dynamics defined on links, an important question is: what is the effect of this dynamics on nodes and triangles? It turns out that there is a natural way to project the dynamics defined on links into dynamics defined on nodes and triangles suggested by topology. More in general, we can project the dynamics defined on $n$ simplices to the dynamics defined on simplices of dimension $n-1$ and $n+1$ by using the higher-order incidence matrices. To this end, let us indicate with $\theta^{[+]}$ the vector of $N_{[n+1]}$ phases associated to each $n+1$ simplex of the simplicial complex. This vector describes the projection of the dynamics on simplices of dimension $n+1$. Similarly, let us indicate with $\theta^{-[1]}$ the vector of $N_{[n-1]}$ phases associated with each $n-1$ simplex of the simplicial complex. This vector represents the projection of the dynamics on simplices of dimension $n-1$. Topological considerations naturally suggest to define $\theta^{[+]}$ and $\theta^{-[-]}$ respectively as the "discrete curl" and "discrete divergence" of $\theta$ i.e.

$$\theta^{[+]} = B^T_{[n+1]} \theta,$$

$$\theta^{-[1]} = B_{[n]} \theta,$$

(11)

Using the Hodge decomposition it is easy to show that $\theta^{[+]}$ depends only on the solenoidal component of the dynamics defined on $n$-dimensional phases, whereas $\theta^{-[-]}$ depends only on the irrotational component. Since we have that $B_{[n]} B_{[n-1]} = 0$ and $B_{[n-1]} B_{[n]} = 0$, if $\theta$ obeys the higher-Kuramoto dynamics, then the projected dynamical variables $\theta^{[+]}$ and $\theta^{-[-]}$ evolve independently ac-
According to
\[
\frac{d\theta^{[+]}}{dt} = B_{[n]}^T \omega - \sigma L_{[n+1]}^{[down]} \sin(\theta^{[+]}),
\]
\[
\frac{d\theta^{-}}{dt} = B_{[n-1]} \omega - \sigma L_{[n-1]}^{[up]} \sin(\theta^{-}).
\]
(12)

Therefore, the dynamics defined on \( n \)-dimensional simplices can naturally be decoupled into two non-interacting dynamics acting on \( n - 1 \) and on \( n + 1 \) dimensional simplices. The order parameter for each of these two independent dynamics are \( R^{[+]} \) and \( R^{-} \).

In order to investigate the properties of the dynamics defined on \( n \)-dimensional simplices, we can consider the standard order parameter \( R \) given by \( R = \left( \sum_{\alpha=1}^{N_{d,n}} e^{i\theta_{\alpha}} \right) / N_{[n+1]} \) and two additional order parameters \( R^{[1]} = \left( \sum_{\alpha=1}^{N_{d,n}} e^{i\theta_{\alpha}^{[1]}} \right) / N_{[n]}, \) and \( R^{[2]} = \left( \sum_{\alpha=1}^{N_{d,n}} e^{i\theta_{\alpha}^{[2]}} \right) / N_{[n]}, \) where \( y^{[1]} = L_{[n]}^{[up]} \theta = L_{[n]}^{[up]} B_{[n+1]} z_{[n+1]} \) depends only on the solenoidal component of the dynamics on \( n \)-dimensional simplices, and \( y^{[2]} = L_{[n]}^{[down]} \theta = L_{[n]}^{[down]} B_{[n-1]} z_{[n-1]} \) depends only on the irrotational component of the dynamics on \( n \)-dimensional simplices.

We have simulated the higher-order \((n = 1)\) Kuramoto dynamics on the 3-dimensional simplicial complexes produced by the configuration model with power-law generalized degree distribution of the nodes. These simplicial complexes have Betti numbers \( \beta_1 > 0, \beta_2 = 0 \). We observe that the projected dynamics on the 2-dimensional simplices and the 0-dimensional simplices display a continuous synchronization transition (see Figure 2). When we investigate the three order parameters for the dynamics defined on \( n \)-dimensional simplices we observe that \( R \) does not capture the collective behavior of the phases due to the fact that the harmonic component of their dynamics is not coupled by the higher-order Kuramoto dynamics, however the order parameters \( R^{[1]} \) and \( R^{[2]} \) are sensible to the synchronization of the solenoidal and irrotational component of the dynamics of the phases (see Fig. 3).

**Explosive higher-order Kuramoto dynamics**—In order to explore whether it is possible to enforce an explosive phase transition we include a coupling between the equations determining the dynamics of \( \theta^{[+]i} \) and \( \theta^{-} \). The way we coupled these two independent dynamics is inspired by the coupling of the dynamics of multiplex Kuramoto dynamics in Ref. [24]. However while in the explosive multiplex Kuramoto dynamics the coupling between the phases in one layer is modulated by the local order parameter of each node in the other layer, here we consider a modulation of the coupling between the phases \( \theta^{[+]i} \) and \( \theta^{-} \) given respectively by the global order parameters \( R^{[+]i} \) and \( R^{-} \). This choice is clearly driven by the fact that the \((n+1)\)-dimensional faces are not in a one-to-one relation with the \((n-1)\)-dimensional faces. Given these considerations we propose the following explosive higher-order Kuramoto dynamics:

\[
\dot{\theta}_\alpha = \omega_\alpha - \sigma R^{-} \sum_{\beta \in S_{d,n+1}} [B_{[n]}]_{\alpha \beta} \sin \left( \sum_{\alpha' \in S_{d,n}} [B_{[n]}]_{\beta \alpha'} \theta_{\alpha'} \right)
\]
\[-\sigma R^{[+]} \sum_{\beta \in S_{d,n-1}} [B_{[n-1]}]_{\alpha \beta} \sin \left( \sum_{\alpha' \in S_{d,n}} [B_{[n-1]}]_{\beta \alpha'} \theta_{\alpha'} \right),
\]

which can be projected on the dynamics of \( n + 1 \) and \( n - 1 \) dimensional simplices producing now two equations coupled by the global order parameters \( R^{[+]i} \) and \( R^{-} \):

\[
\frac{d\theta^{[+]}}{dt} = B_{[n]}^T \omega - \sigma R^{-} L_{[n+1]}^{[down]} \sin(\theta^{[+]i}),
\]
\[
\frac{d\theta^{-}}{dt} = B_{[n-1]} \omega - \sigma R^{[+]} L_{[n-1]}^{[up]} \sin(\theta^{-}).
\]
(13)

We have simulated the explosive higher-order Kuramoto dynamics on simplices of dimension \( n = 1 \) on the configuration model of simplicial complexes with power-law distribution of generalized degrees. We clearly observe a discontinuous phase transition in \( R^{[+]} \) and \( R^{-} \) (see Fig. 2). This transition is reflected also on the irrotational and solenoidal components of the dynamics on the \( n \)-dimensional faces captured by the order parameters \( R^{[1]} \) and \( R^{[2]} \), while due to the presence of the uncoupled harmonic component \( R \) remains close to zero (see Fig. 3).

**Conclusions**—We have introduced the higher-order Kuramoto dynamics designed to characterize the coupling between phases associated with higher-dimensional simplices, such as links, triangles and so on. The higher-order Kuramoto dynamics defined on \( n \)-dimensional simplices allows us to define a topologically projected dynamics on faces of dimension \( n - 1 \) and \( n + 1 \), which obey a dynamics of coupled oscillators. We have considered two versions of the higher-order Kuramoto dynamics, the simple and the explosive higher-order Kuramoto dynamics, and we have simulated them on the simplicial complex configuration model. We have found that the simple higher-order Kuramoto dynamics displays continuous phase transitions for the projected dynamics defined on \( n + 1 \) and \( n - 1 \) faces. Interestingly, however, when we introduce a coupling between the dynamics projected on the \( n + 1 \) and \( n - 1 \) dynamical phases, as it is done in the definition of the explosive higher-order Kuramoto dynamics, the system displays an explosive synchronization transition. This work opens innovative perspectives in characterizing the Kuramoto dynamics on higher-dimensional simplices, and it shows that a higher-order
synchronization dynamics defined on \( n \)-dimensional simplices (as for example links) can induce a simultaneous discontinuous transition on its projected dynamics defined on \((n-1)\) and \((n+1)\)-dimensional simplices (i.e., nodes and triangles).

This research utilized Queen Mary’s Apocrita HPC facility, supported by QMUL Research-IT.

[1] C. Giusti, R. Ghrist and D. S. Bassett, J. Comp. Neuro., **41**, 1 (2016).
[2] G. Petri, et. al., Jour. Roy. Soc. Interface, **11**, 20140873 (2014).
[3] M. W. Reimann, M. Nolte, M. Scolamiero, et al., Front. Comp. Neuro. **11**, 48 (2017).
[4] I. Iacopini, G. Petri, A. Barrat and V. Latora, Nature Comm., **10**, 2485 (2019).
[5] B. Jhun, M. Jo, and B. Kahng, arXiv preprint arXiv:1910.00375 (2019).
[6] J. T. Matamalas, S. Gómez and A. Arenas, arXiv preprint arXiv:1910.03069 (2019).
[7] D.S. Bassett, E.T. Owens, K. E. Daniels and M. A. Porter, Phys. Rev. E, **86**, 041306 (2012).
[8] M. Šuvakov, M. Andjelković, and B. Tadić, Sci. Rep. **8**, 1987 (2018).
[9] G. Bianconi, EPL (Europhysics Letters) **111**, 56001 (2015).
[10] V. Salnikov, D. Cassese, and R. Lambiotte, Eur. Jour. Phys. **40**, 014001 (2018).
[11] A. Costa and M. Farber, In Configuration spaces (pp. 129-153). Springer, Cham (2016).
[12] O. T. Courtney and G. Bianconi, Phys. Rev. E, **93**, 062311 (2016).
[13] G. Bianconi and C. Rahmede, Sci. Rep. **7** 41974 (2017).
[14] G. Petri and A. Barrat, Phys. Rev. Lett. **121**, 228301 (2018).
[15] Z. Wu, G. Menichetti, C. Rahmede and G. Bianconi, Sci. Rep. **5**, 10073 (2014).
[16] G. Bianconi and R. M. Ziff, Phys. Rev. E, **98**, 052308 (2018).
[17] P.S. Skardal and A. Arenas, Phys. Rev. Lett., **122**, 248301 (2019).
[18] P.S. Skardal and A. Arenas, arXiv preprint arXiv:1909.08057 (2019).
[19] A. P. Millán, J. J. Torres and G. Bianconi, Sci. Rep. **8**, 9910 (2018).
[20] A.P. Millán, J. J. Torres, and G. Bianconi. Phys. Rev. E **99**, 022307 (2019).
[21] R. M. D’Souza, J. Gómez-Gardeñes, J. Nagler and A. Arenas, Adv. Phys., **68**, 123 (2019).
[22] S. Boccaletti, et.al., Phys. Rep. **660**, 1 (2016).
[23] J. Gómez-Gardeñes, S., Gómez, A. Arenas and Y. Moreno, Phys. Rev. Lett., **106**,128701 (2011).
[24] X. Zhang, S. Boccaletti, S. Guan and Z.Liu, Z., Phys. Rev. Lett., **114**, 038701 (2015).
[25] M. M. Danziger, I. Bonamassa, S. Boccaletti and S. Havlin, Nature Phys., **15**, 178 (2019).
[26] S. Barbarossa and S. Sardellitti, arXiv preprint arXiv:1907.11577 (2019).
[27] T. E. Goldberg, Senior Thesis, Bard College (2002).
[28] A. Muhammad and M. Egerstedt. In Proc. of 17th International Symposium on Mathematical Theory of Networks and Systems, 1024 (2006).
[29] D. Horak and J. Jost, Adv. in Math. **244**, 303 (2013).
[30] Y. Kuramoto, Lect. Notes Phys. **39**, 420 (1975).