Deformations of Adjoint orbits for semisimple Lie algebras and Lagrangian submanifolds

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Abstract

We give a coadjoint orbit’s diffeomorphic deformation between the classical semisimple case and the semi-direct product given by a Cartan decomposition. The two structures admit the Hermitian symplectic form defined in a semisimple complex Lie algebra. We provide some applications such as the constructions of Lagrangian submanifolds.

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1 Introduction

Let $\mathfrak{g}$ be a non-compact semisimple Lie group with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{a} \subset \mathfrak{s}$ maximal abelian. In the vector space underlying $\mathfrak{g}$ there is another Lie algebra structure $\mathfrak{k}_{ad} = \mathfrak{k} \times_{ad} \mathfrak{s}$ given by the semi-direct product defined by the adjoint representation of $\mathfrak{k}$ in $\mathfrak{s}$, which is viewed as an abelian Lie algebra.

Let $G = \text{Aut}\mathfrak{g}$ be the adjoint group of $\mathfrak{g}$ (identity component of the automorphism group) and put $K = \exp \mathfrak{k} \subset G$. The semi-direct product $K_{ad} = K \times_{Ad} \mathfrak{s}$ obtained by the adjoint representation of $K$ in $\mathfrak{s}$ has Lie algebra $\mathfrak{k}_{ad} = \mathfrak{k} \times_{ad} \mathfrak{s}$ (see Subsection 3.1).

In this paper we consider coadjoint orbits for both Lie algebras $\mathfrak{g}$ and $\mathfrak{k}_{ad}$. These orbits are submanifolds of $\mathfrak{g}^{*}$ that we identify with $\mathfrak{g}$ via the Cartan-Killing form of $\mathfrak{g}$, so that the orbits are seen as submanifolds of $\mathfrak{g}$. These are just the adjoint orbits for the Lie algebra $\mathfrak{g}$ while for $\mathfrak{k}_{ad}$ they are the orbits in $\mathfrak{g}$ of

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the representation of $K_{\text{ad}}$ obtained by transposing its coadjoint representation. The orbits through $H \in \mathfrak{g}$ are denoted by $\text{Ad}(G) \cdot H$ and $K_{\text{ad}} \cdot H$, respectively.

We consider the orbits through $H \in \mathfrak{a} \subset \mathfrak{s}$. In this case the compact orbit $\text{Ad}(K) \cdot H$ (contained in $\mathfrak{s}$) is a flag manifold of $\mathfrak{g}$, say $\mathbb{F}_H$. In Gasparim-Grama-San Martin [2] it was proved that $\text{Ad}(G) \cdot H$ is diffeomorphic to the cotangent space $T^*\mathbb{F}_H$ of $\mathbb{F}_H = \text{Ad}(K) \cdot H$. We prove here that the same happens to the semi-direct product orbit $K_{\text{ad}} \cdot H$ (as foreseen by Jurdjevic [3]). So that $\text{Ad}(G) \cdot H$ and $K_{\text{ad}} \cdot H$ diffeomorphic to each other.

In this paper we define a deformation $\mathfrak{g}_r$ of the original Lie algebra $\mathfrak{g}$ (see Section 3.1). The deformation is parameterized by $r > 0$ and satisfies $\mathfrak{g}_1 = \mathfrak{g}$. For each $r$ the Lie algebra $\mathfrak{g}_r$ is isomorphic to $\mathfrak{g}$ (hence semisimple) and $\mathfrak{g}_r = \mathfrak{k} \oplus \mathfrak{s}$ is a Cartan decomposition as well with $\mathfrak{k}$ a subalgebra of $\mathfrak{g}_r$. Furthermore as $r \to \infty$ the Lie algebra $\mathfrak{k}_{\text{ad}}$ is recovered. (The deformation amounts essentially to change the brackets $[X,Y]$, $X,Y \in \mathfrak{s}$, by $(1/r) [X,Y]$ and keeping the other brackets unchanged.) A Lie algebra $\mathfrak{g}_r = \mathfrak{k} \oplus \mathfrak{s}$, $r > 0$, has its own automorphism group whose identity component is denoted by $G_r$. Thus the adjoint orbits in $\mathfrak{g}_r$ are $\text{Ad}(G_r) \cdot H$ and by the isomorphism $\mathfrak{g}_r \approx \mathfrak{g}$ it follows that $\text{Ad}(G_r) \cdot H$ is diffeomorphic to $\text{Ad}(G) \cdot H$ and hence to the cotangent space $T^*\mathbb{F}_H$. Thus the Lie algebra deformation yields a continuous one parameter family of embeddings of $T^*\mathbb{F}_H$ into the vector space underlying $\mathfrak{g}$. The family is parameterized in $(0, +\infty]$ where $+\infty$ is the embedding given by the the semi-direct product orbit $K_{\text{ad}} \cdot H$.

The example with $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, presented in Subsection 3.1 is elucidative of this deformation. In $\mathfrak{sl}(2, \mathbb{R}) \approx \mathbb{R}^3$ the semi-direct product orbit is the cylinder $x^2 + y^2 = 1$ while the adjoint orbit is the one-sheet hyperboloid $x^2 + y^2 - z^2 = 1$. In the deformation the adjoint orbit in $\mathfrak{g}_r$ is the paraboloid $x^2 + y^2 - z^2/r = 1$ that converges to the cylinder as $r \to +\infty$. The hyperboloids as well as the cylinder are unions of straight lines in $\mathbb{R}^3$ crossing the circle $x^2 + y^2 = 1$ with $z = 0$. As is well known the hyperboloids are obtained by twisting the generatrices of the cylinder.

This picture of “twisting generatrices” holds in a general Lie algebra $\mathfrak{g}$. The semi-direct product orbit $K_{\text{ad}} \cdot H$ has the cylindrical shape

$$K_{\text{ad}} \cdot H = \bigcup_{X \in \text{Ad}(K)H} (X + \text{ad}(X)\mathfrak{s})$$

where $\text{ad}(X)\mathfrak{s}$ is a subspace of $\mathfrak{k}$. While the adjoint orbit $\text{Ad}(G) \cdot H$ has the hyperboloid shape

$$\text{Ad}(G) \cdot H = \bigcup_{k \in K} \text{Ad}(k)(H + \mathfrak{n}^+_H)$$

where $\mathfrak{n}^+_H$ is the nilpotent subalgebra which is the sum of the eigenspaces of $\text{ad}(H)$ associated to positive eigenvalues. The deformation of $\mathfrak{g}$ into $\mathfrak{g}_r$ has the effect of twisting the generatrix $H + \text{ad}(H)\mathfrak{s} \subset \mathfrak{k}$ into $H + \mathfrak{n}^+_H$ where $\mathfrak{n}^+_H$ the nilpotent Lie subalgebra of $\mathfrak{g}_r$ defined the same way as $\mathfrak{n}$ from the adjoint $\text{ad}_r(H)$ of $H$ in $\mathfrak{g}_r$. The deformation of orbits allows to transfer geometric
properties from semi-direct product orbit $K_{\text{ad}} \cdot H$ to the adjoint orbit $\text{Ad}(G) \cdot H$. This can be useful since in several aspects the geometry of $K_{\text{ad}} \cdot H$ is more manageable than that of $\text{Ad}(G) \cdot H$.

In this paper we apply this transfer approach to adjoint orbits in a complex semisimple Lie algebra $\mathfrak{g}$ (see Section 5). In the complex case $\mathfrak{g}$ is endowed with a Hermitian metric

$$\mathcal{H}_\tau (X,Y) = \langle X,Y \rangle + i\Omega (X,Y)$$

where $\langle \cdot,\cdot \rangle$ is an inner product and $\Omega$ is a symplectic form. The form $\Omega$ restricts to symplectic forms on the orbit $\text{Ad}(G) \cdot H$ since this is a complex submanifold. Although not as immediate we prove that the restriction of $\Omega$ to the semi-direct product orbit $K_{\text{ad}} \cdot H$ is also a symplectic form. By construction the diffeomorphisms between the adjoint orbits (including $K_{\text{ad}} \cdot H$) are symplectomorphisms.

Based on these facts in Section 5.1 we construct Lagrangian submanifolds in $K_{\text{ad}} \cdot H$ (w.r.t. $\Omega$) and then transport them to $\text{Ad}(G) \cdot H$ through the deformation. To conclude in section 5.2, we describe the Lagrangian orbits given by the adjoint action of $U$ and the Hermitian form $\Omega$ on $\text{Ad}(G) \cdot H$.

2 Semi-direct products

Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and take a representation $\rho : G \rightarrow \text{Gl}(V)$ on a vector space $V$ (with $\dim V < \infty$). The infinitesimal representation of $\mathfrak{g}$ on $\text{gl}(V)$ is also going to be denoted by $\rho$.

In particular, the vector space $V$ can be seen as an abelian Lie group (or abelian Lie algebra). In this way, we can take the semi-direct product $G \times_\rho V$ which is a Lie group whose underlying manifold is the Cartesian product $G \times V$. This group is going to be denoted by $G_\rho$ and its Lie algebra $\mathfrak{g}_\rho$ is the semi-direct product

$$\mathfrak{g}_\rho = \mathfrak{g} \times_\rho V.$$ 

The vector space of $\mathfrak{g}_\rho$ is $\mathfrak{g} \times V$ with bracket

$$[(X,v),(Y,w)] = ([X,Y],\rho(X)w - \rho(Y)v).$$

Our purpose is to describe the coadjoint orbit on the dual $\mathfrak{g}_\rho^*$ of $\mathfrak{g}_\rho$. To begin, let’s see how to determine the $\rho$-adjoint representation $\text{ad}_\rho(X,v)$, with $(X,v) \in \mathfrak{g} \times_\rho V$. Thus, take a basis of $\mathfrak{g} \times V$ denoted by $\mathcal{B} = \mathcal{B}_\mathfrak{g} \cup \mathcal{B}_V$ with $\mathcal{B}_\mathfrak{g} = \{X_1, \ldots, X_n\}$ and $\mathcal{B}_V = \{v_1, \ldots, v_d\}$ basis of $\mathfrak{g}$ and $V$, respectively. On this basis the matrix of $\text{ad}_\rho(X,v)$ is given by

$$\text{ad}_\rho(X,v) = \begin{pmatrix} \text{ad}(X) & 0 \\ A(v) & \rho(X) \end{pmatrix},$$

where $\text{ad}(X)$ is the adjoint representation of $\mathfrak{g}$ while for each $v \in V$, $A(v)$ is the linear map $\mathfrak{g} \rightarrow V$ defined by

$$A(v)(X) = \rho(X)(v).$$
Since $G$ is compact, then $V$ admits a $G$-invariant inner product $\langle \cdot, \cdot \rangle$ (Hermitian, in the complex case). Define a map $\mu : V \otimes V \to g^*$ (called moment map of $\rho$) given by
\[
\mu(v \otimes w)(X) = \langle \rho(X)v, w \rangle,
\]
where $\rho(g)$ is an isometry and $\rho(X)$ is an anti-symmetric linear application of $\langle \cdot, \cdot \rangle$ for $g \in G$ and $X \in g$, then $\mu$ is anti-symmetric. Hence the moment map $\mu$ is defined in the exterior product $\wedge^2 V = V \wedge V$. Furthermore, a compact Lie algebra $g$ admits an ad-invariant inner product such that we can identify $g^*$ with $g$, then
\[
\mu : V \wedge V \to g.
\]

Similarly, the dual $g^* \times V^*$ of $g_\rho = g \times V$ are identified by its inner product. This means that the coadjoint representation of $g \times V$ is written in $g \times V$ as type matrices (on orthonormal bases):
\[
\operatorname{ad}_{\rho}^*(X, v) = \begin{pmatrix} \operatorname{ad}(X) & -A(v) \\ 0 & \rho(X) \end{pmatrix} \quad X \in g, \ v \in V
\]
where for each $v \in V$, $A(v) : V \to g$ can be identified by $A(v)(w) = \mu(v \wedge w)$. Then the representations $\operatorname{Ad}_\rho$ and $\operatorname{Ad}^\rho_{\rho}$ of $G_\rho$ are obtained by exponentials of representations in $g_\rho$. In particular, the following matrices are obtained (on the basis given above):
\[
e^{t \operatorname{ad}_{\rho}(0,v)} = \begin{pmatrix} 1 & 0 \\ tA(v) & 1 \end{pmatrix}, \quad e^{t \operatorname{ad}_{\rho}^*(0,v)} = \begin{pmatrix} 1 & -tA(v) \\ 0 & 1 \end{pmatrix}. \quad (1)
\]
On the other hand, for $g \in G$ the restriction of $\operatorname{Ad}_{\rho}(g)$ to $V$ coincides with $\rho(g)$. Thus the coadjoint orbit can be described by the moment map. Additionally, the moment map $\mu$ is bilinear in $V \times V$, then setting $w \in V$ implies that $\mu_w : V \to g$ is a linear map and its image $\mu_w(V) = A(w)(V)$ is a subspace of $g$. The following proposition shows that the coadjoint orbit for $v \in V$ is the union of subspaces $A(w)(V)$.

**Proposition 1.** The coadjoint orbit $\operatorname{Ad}_{\rho}^*(G_\rho)v$ of $v \in V \subset g \times V$ is given by
\[
\operatorname{Ad}_{\rho}^*(G_\rho) \cdot v = \bigcup_{w \in \rho(G)v} \mu_w(V) \times \{w\} \subset g \times V,
\]
and writing $g \times V$ as $g \oplus V$ (with the proper identifications)
\[
\operatorname{Ad}_{\rho}^*(G_\rho) \cdot v = \bigcup_{w \in \rho(G)v} w + A(w)(V).
\]

**Proof.** If $g \in G$, we can identify $\operatorname{Ad}_{\rho}(g)$ with $\rho(g)$ on $V \subset g \times V$. Therefore $\rho(g) \cdot v \subset \operatorname{Ad}_{\rho}(G_\rho) \cdot v$ and by (1) given an element $w \in V \subset g \oplus V = g \times V$
\[
e^{t \operatorname{ad}_{\rho}^*(0,v)} \cdot w = w - t \cdot A(v)(w) \quad \text{with} \quad A(v)(w) = \mu_w(v).
\]
Varying \( v \in V \), shows that the affine subspace \( w + \mu_v(V) \) is contained in the coadjoint orbit of \( v \). Next to the fact that \( \rho(G) \cdot v \subset \text{Ad}_\rho^*(G_\rho) \cdot v \), we conclude that
\[
\bigcup_{w \in \rho(G) \cdot v} w + A(w)(V) \subset \text{Ad}_\rho^*(G_\rho) \cdot v.
\]

Conversely, if \( g \in G \) and \( w \in V \)
\[
\text{Ad}_\rho^*(g) (w + A(w)(V)) = \rho(g) \cdot w + \text{Ad}_\rho^*(g) \mu_w(V)
\]
\[
= \rho(g) \cdot w + \mu_{\rho(g) \cdot w}(V).
\]

For \( h \in G_\rho \) there are \( g \in G \) and \( \tilde{v} \in V \) such that
\[
\text{Ad}_\rho^*(h) \cdot v = \text{Ad}_\rho^*(g) \cdot \text{Ad}_\rho^*(e^{i(0, \tilde{v})}) \cdot v,
\]
as \( \text{Ad}_\rho^*(e^{i(0, \tilde{v})}) \cdot v \in v + \mu_v(V) \) implies \( \text{Ad}_\rho^*(h) \cdot v \in \rho(g) \cdot v + \mu_{\rho(g) \cdot v}(V) \).

By Proposition 1, the coadjoint orbit \( \text{Ad}_\rho^*(G_\rho) \cdot x \), for \( x \in V \) is the union of vector spaces and a fiber over \( \rho(G)x \). This union is disjoint because given \( Z \in (w + \mu_w(V)) \cap (v + \mu_v(V)) \), then
\[
Z = w + X = v + Y \quad X = \mu_w(x), \quad Y = \mu_v(w)
\]
with \( X, Y \in g \). Since the sum \( g \oplus V \) is direct, it follows that \( w = v \) and \( X = Y \). Therefore there is a fibration \( \text{Ad}_\rho^*(G_\rho) \cdot x \to \rho(G)x \) such that an element \( Z = w + X \in w + \mu_w(V) \) associates \( w \in \rho(G)x \), and whose fibers are vector spaces. The following proposition shows that this fibration is the cotangent space of \( \rho(G)x \).

**Proposition 2.** \( \text{Ad}_\rho^*(G_\rho) \cdot x \) is diffeomorphic to the cotangent bundle \( T^* (\rho(G)x) \) of \( \rho(G)x \), by the diffeomorphism
\[
\phi : \text{Ad}_\rho^*(G_\rho) \cdot x \to T^* (\rho(G)x),
\]
that satisfies
\[
\phi (w + \mu_w(V)) = T^*_w (\rho(G)x), \quad w \in \rho(G)x.
\]
The restriction of \( \phi \) to a fiber \( w + \mu_w(V) \) is given by a linear isomorphism
\[
\mu_w(V) \to T^*_w (\rho(G)x).
\]

**Proof.** Take \( X \in \text{Ad}_\rho^*(G_\rho) \cdot x \), from the above observation there is a unique \( w \in \rho(G)x \), such that \( X \in \mu_w(V) \), then there is \( v \in V \) with \( X = \mu(v \wedge w) \). The vector \( v \in V \) defines a linear functional \( f_v \) on \( T_w (\rho(G)x) \) and therefore an element of \( T^*_w (\rho(G)x) \). Set
\[
\phi (X) = f_v \in T^*_w (\rho(G)x) \quad \text{with} \quad X = w + \mu (v \wedge w).
\]

An application \( \phi \) is a linear injective application and the linear application \( \mu (v \wedge w) \mapsto f_v \) is surjective. Furthermore, the restriction of \( \phi \) to a fiber \( w +
\( \mu_w (V) \) is given by the isomorphism: \( \mu_w (V) \rightarrow T^*_w (\rho (G) x) \). It follows that \( \phi \) is a bijection. Finally \( \phi \) is diffeomorphism because both \( \phi \) and \( \phi^{-1} \) are differentiable as follows by construction: \( \phi \) is the identity application at the base of the bundles of \( \rho (G) x \) and \( \phi \) is linear on the fibers.

**Example 3** \((G = SO(n))\). Take the canonical representation of \( g = so (n) \) in \( \mathbb{R}^n \). The moment map with values in \( g \) is given by

\[ \mu (v \wedge w) (B) = \langle Bv, w \rangle \quad B \in so (n), \]

where taking \( v \) and \( w \) as a column vectors \( n \times 1 \), we have

\[ v \wedge w = vw^T - wv^T \]

which is a \( n \times n \) matrix. Then

\[ \text{ad}^* (B, v) = \begin{pmatrix} \text{ad} (B) & -A (v) \\ 0 & B \end{pmatrix} \quad A \in so (n), \ v \in \mathbb{R}^n \]

where for each \( v \in \mathbb{R}^n \), \( A (v) : \mathbb{R}^n \rightarrow so (n) \) is the application

\[ A (v) (w) = v \wedge w = vw^T - wv^T. \]

The representation \( so (n) \times \mathbb{R}^n \) defines a representation of the semi-direct product \( G \rho = SO (n) \times \mathbb{R}^n \) on \( so (n) \times \mathbb{R}^n \) by exponentials. As discussed earlier a \( G \rho \)-orbit of \( v \in \mathbb{R}^n \subset so (n) \times \mathbb{R}^n \) is given by

\[ \bigcup_{w \in \mathcal{O}} w + A (w) (\mathbb{R}^n) \quad \mathcal{O} = SO (n) \cdot v. \]

In this case, the orbits of \( SO (n) \) in \( \mathbb{R}^n \) are the \( (n-1) \)-dimensional spheres centered at the origin. In particular, for \( n = 2 \), \( so (2) \times \mathbb{R}^2 \approx \mathbb{R}^3 \) and for all \( w \) the image \( A (w) (\mathbb{R}^2) = so (2) \), therefore the coadjoint orbits of the semi-direct product are the circular cylinders with axis on line determined by \( so (2) \) in \( so (2) \times \mathbb{R}^2 \approx \mathbb{R}^3 \).

In the coadjoint orbit \( \text{Ad}^*_\rho (G \rho) \cdot x \) we can define the Kostant-Kirilov-Souriaux (KKS) symplectic form, denoted by \( \Omega \), in the same way for the cotangent bundle \( T^* (\rho (G) x) \) we can define the canonical symplectic form \( \omega \). The following proposition shows that these symplectic forms are related by the diffeomorphism \( \phi \) of the Proposition 2. The best way to relate these symplectic forms is through the action of the semi-direct product \( G \rho = G \times V \) on the cotangent bundle of \( \rho (G) x \). This action is described in Proposition 25 (in a general case), the action of \( G \rho \) on \( T^* (\rho (G) x) \) is Hamiltonian and therefore defines a moment

\[ m : T^* (\rho (G) x) \rightarrow \mathfrak{g}_\rho. \]

The construction of \( m \) shows that it is the inverse of the diffeomorphism \( \phi \) of the Proposition 2. Moreover, \( m \) is equivariant, that is, it exchanges the actions on \( T^* (\rho (G) x) \) and the adjoint orbit, which implies that \( m \) is a symplectic morphism. Then we conclude:
Proposition 4. Let $\Omega$ be the symplectic form $KKS$ in $\text{Ad}_\rho^*(G_\rho) x$ and $\omega$ the canonical symplectic form in $T^*(\rho(G)x)$. If $\phi$ is the diffeomorphism of the Proposition 3 then $\phi^*\omega = \Omega$. In other words, the diffeomorphism $\phi$ is symplectic.

3 Adjoint orbits in semisimple Lie algebras

Let $\mathfrak{g}$ be a non-compact semisimple (real or complex) Lie algebra and let $G$ be a connected Lie group with finite centre and Lie algebra $\mathfrak{g}$ (for example $G = \text{Aut}_0 \mathfrak{g}$). The usual notation is:

1. The Cartan decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, with global decomposition $G = KS$.

2. The Iwasawa decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, with global decomposition $G = KAN$.

3. $\Pi$ is the set of roots of $\mathfrak{a}$, with a choice of a set of positive roots $\Pi^+$ and simple roots $\Sigma \subset \Pi^+$ such that $n^+ = \sum_{\alpha > 0} \mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha$ is the root space of the root $\alpha$. The corresponding positive Weyl chamber is $a^+$.

4. A subset $\Theta \subset \Sigma$ defines a parabolic subalgebra $\mathfrak{p}_\Theta$ with parabolic subgroup $P_\Theta$ and a flag $F_\Theta = G/P_\Theta$. The flag is also $F_\Theta = K/K_\Theta$, where $K_\Theta = K \cap P_\Theta$. The Lie algebra of $K_\Theta$ is denoted by $K_\Theta$.

5. Given an element $H \in \text{cl}(a^+)$ it determines $\Theta_H \subset \Sigma$ such that $\Theta_H = \{ \alpha \in \Sigma : \alpha(H) = 0 \}$. Then $\text{Ad}(K) \cdot H = G/P_H = K/K_H$ is the flag denoted by $F_H$ (where $P_H$ and $K_H$ denotes the subgroups $P_{\Theta_H}$ and $K_{\Theta_H}$, respectively).

6. $b_H = 1 \cdot K_H = 1 \cdot P_H$ denotes the origin of the flag $F_H$.

7. We write

$$n_H^+ = \sum_{\alpha(H) > 0} \mathfrak{g}_\alpha, \quad n_H^- = \sum_{\alpha(H) < 0} \mathfrak{g}_\alpha$$

so that $\mathfrak{g} = n_H^- \oplus N_H \oplus n_H^+$, where $N_H$ is the centralizer of $H$ in $\mathfrak{g}$.

8. $T_{b_H}F_H \simeq \sum_{\alpha(H) < 0} \mathfrak{g}_\alpha = n_H^-$.

9. $Z_H = \{ g \in G : \text{Ad}(g) \cdot H = H \}$ is the centralizer in $G$ of $H$. Its Lie algebra is $z_H$. Moreover, $K_H$ is the centralizer of $H$ in $K$:

$$K_H = Z_K(H) = Z_H \cap K = \{ k \in K : \text{Ad}(k) \cdot H = H \}.$$
3.1 Semi-direct product of Cartan decomposition

Let \( g \) be a non-compact semisimple Lie algebra with Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{s} \). As \([ \mathfrak{k}, \mathfrak{s} ] \subset \mathfrak{s}\), the subalgebra \( \mathfrak{k} \) can be represented on \( \mathfrak{s} \) by the adjoint representation. Then, we can define the semi-direct product \( \mathfrak{k}_{\text{ad}} = \mathfrak{k} \times \mathfrak{s} \), where \( \mathfrak{s} \) can be seen as an abelian algebra. This is a new Lie algebra structure on the same vector space \( g \) where the brackets \( [X,Y] \) are the same when \( X \) or \( Y \) are in \( \mathfrak{k} \), but the bracket changes when \( X, Y \in \mathfrak{s} \). The identification between \( \mathfrak{k}_{\text{ad}} = \mathfrak{k} \times \mathfrak{s} \) and its dual \( \mathfrak{k}^*_{\text{ad}} = \mathfrak{k}^* \times \mathfrak{s}^* \) is given by the inner product

\[
\langle \cdot, \cdot \rangle = \langle \cdot, \theta \cdot \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the Cartan-Killing form of \( g \) and \( \theta \) is a Cartan involution. If \( A \in \mathfrak{k} \), then \( \text{ad} (A) \) is anti-symmetric with respect to \( \langle \cdot, \cdot \rangle \), while \( \text{ad} (X) \) is symmetric for \( X \in \mathfrak{s} \). The moment map is given by

\[
\mu (X \wedge Y) (A) = B_\theta (\text{ad} (A) X, Y) \quad A \in \mathfrak{k}; \ X, Y \in \mathfrak{s},
\]

the second part of that equality is

\[
B_\theta ([A, X], Y) = -B_\theta ([X, A], Y) = -B_\theta (A, [X, Y]) = -\langle A, [X, Y] \rangle
\]

because \( [X, Y] \in \mathfrak{k} \). Therefore the moment map of the adjoint representation of \( \mathfrak{k} \) on \( \mathfrak{s} \) is

\[
\mu (X \wedge Y) = [X, Y] \in \mathfrak{k} \quad X, Y \in \mathfrak{s},
\]

where \( [\cdot, \cdot] \) is the usual bracket of \( g \). Therefore, the coadjoint representation of the semi-direct product \( \mathfrak{k} \times \mathfrak{s} \) is given by (in an orthonormal basis)

\[
\text{ad}^* (X,Y) = \begin{pmatrix}
\text{ad} (X) & -A (Y) \\
0 & \text{ad} (X)
\end{pmatrix} \quad X \in \mathfrak{k}, \ Y \in \mathfrak{s}
\]

where for each \( Y \in \mathfrak{s}, \ A (Y) : \mathfrak{s} \to \mathfrak{k} \) is the map \( A (Y) (Z) = [Y, Z] \).

Let \( K \subset G \) the subgroup given by \( K = \langle \exp \mathfrak{k} \rangle \). The semi-direct product of \( K \) in \( g \) will be denoted by \( K_{\text{ad}} = K \times \mathfrak{s} \). The coadjoint orbit of \( \tilde{X} \in \mathfrak{s} \subset \mathfrak{k} \times \mathfrak{s} \) is the union of the fibers \( A (Y) (\mathfrak{s}) \) with \( Y \) passing through the \( K \)-adjoint orbit of \( \tilde{X} \) in \( \mathfrak{s} \). As \( A (Y) (Z) = [Y, Z] \), then \( A (Y) (\mathfrak{s}) = \text{ad} (Y) (\mathfrak{s}) \) where \( \text{ad} \) is the adjoint representation in \( g \).

To detail the coadjoint orbits of the semi-direct product, take a maximal abelian subalgebra \( \mathfrak{a} \subset \mathfrak{s} \). The \( \text{Ad} (K) \)-orbits in \( \mathfrak{s} \) passing through \( \mathfrak{a} \), thus are the flags on \( g \). Take a positive Weyl chamber \( \mathfrak{a}^+ \subset \mathfrak{a} \), if \( H \in \text{cl} (\mathfrak{a}^+) \) then the orbit \( \text{Ad} (K) H \) is the flag manifold \( \mathbb{F}_H \). By Proposition 2 the \( K_{\text{ad}} \)-orbit in \( H \in \text{cl} (\mathfrak{a}^+) \) is diffeomorphic to the cotangent bundle of \( \mathbb{F}_H \), thus the \( K_{\text{ad}} \)-orbit itself is the union of the fibers \( \text{ad} (Y) (\mathfrak{s}) \), with \( Y \in \mathbb{F}_H \). In this union the fiber over \( H \) is \( H + \text{ad} (H) (\mathfrak{s}) \) with \( \text{ad} (H) (\mathfrak{s}) \subset \mathfrak{k} \). With the notations above this subspace of \( \mathfrak{k} \) is given by

\[
\text{ad} (H) (\mathfrak{s}) = \sum_{\alpha (H) > 0} \mathfrak{t}_\alpha.
\]
Example 5. Take \( \mathfrak{sl}(2, \mathbb{R}) \) with basis \( \{ H, S, A \} \) given by
\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and coordinates \((x, y, z) = xH + yS + zA\). The Cartan decomposition \( \mathfrak{k} \oplus \mathfrak{s} \) is given by \( \mathfrak{k} = \mathfrak{so}(2) = \langle A \rangle \) and \( \mathfrak{s} = \langle H, S \rangle \). The adjoint representation of \( \mathfrak{so}(2) \) in \( \mathfrak{s} \) coincides with its canonical representation in \( \mathbb{R}^2 \). Hence, the coadjoint orbits of the semi-direct product are the cylinders \( x^2 + y^2 = r \), with \( r > 0 \) and on the \( z \)-axis (generated by \( A \)) the orbits degenerate into points.

4 Deformations of Lie algebras

Let \( \mathfrak{g} \) be a non-compact semisimple Lie algebra, we will provide a compatible structure that allows us to deform the adjoint orbit of \( G \) in the coadjoint orbit of \( K_{\text{ad}} \). As previously stated, the idea is based on case \( \mathfrak{sl}(2, \mathbb{R}) \) (Example 5), where we obtain a cylinder as an orbit, while in the usual case the result is an hyperboloid.

Fixing \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \) a Cartan decomposition, with \( \theta \) its Cartan involution, the two structures of Lie algebras for \( \mathfrak{g} \) (semisimple and semi-direct product) give rise to different coadjoint orbits. In both cases, the orbits that pass through any element of \( \mathfrak{s} \) are diffeomorphic to the cotangent bundles of the flags of \( \mathfrak{g} \), and therefore diffeomorphic to each other.

For this, consider \( r > 0 \) and define the linear map
\[
T_r : \mathfrak{g} \to \mathfrak{g} \quad \text{such that} \quad T_r(X) = rX \quad \forall X \in \mathfrak{k} \quad \text{and} \quad T_r(Y) = Y \quad \forall Y \in \mathfrak{s},
\]
and induce the Lie bracket
\[
[T_rX, T_rY]_r = T_r[T_r^{-1}X, T_r^{-1}Y],
\]
such that \( (\mathfrak{g}, [], \cdot , \cdot , r) \) is a Lie algebra. In general we have:

Lemma 6. For \( r > 0 \), denote by \( \mathfrak{g}_r \) the Lie algebra \( (\mathfrak{g}, [], \cdot , \cdot , r) \) and by \( \langle \cdot , \cdot , r \rangle \) its Cartan-Killing form. Then

1. \( T_r : \mathfrak{g} \to \mathfrak{g}_r \) is an isomorphism of Lie algebras.
2. \( \langle X, Y \rangle_r = \langle T_r^{-1}X, T_r^{-1}Y \rangle \) for all \( X, Y \in \mathfrak{g} \).
3. \( \mathfrak{g}_r \) supports the same Cartan decomposition of \( \mathfrak{g} \).

Proof. The isomorphism between that Lie algebras is also immediate because
\[
[T_rX, T_rY]_r = T_r[T_r^{-1}T_rX, T_r^{-1}T_rY] = T_r[X, Y],
\]
that is, \( T_r \) is a homomorphism of Lie algebras. In relation to the Cartan-Killing form, as \( T_r \) is an isomorphism, then
\[
\text{ad}_r(X) = T_r \circ \text{ad}(T_r^{-1}X) \circ T_r^{-1}.
\]
Hence
\[
\langle X, X \rangle_r = \text{tr} \left( \text{ad}_r \left( X \right) \right)^2 = \text{tr} \left( \text{ad} \left( T_r^{-1}X \right) \right)^2 = \langle T_r^{-1}X, T_r^{-1}X \rangle
\]
showing the second statement. For the last statement, if \( \theta \) is a Cartan involution corresponding to the decomposition \( g = \mathfrak{k} \oplus \mathfrak{s} \), then \( \tilde{\theta} = T_r \circ \theta \circ T_r^{-1} \) is a Cartan involution that satisfies
\[
\mathfrak{k} = \{ X \in g_r : \tilde{\theta}X = X \} \quad \text{and} \quad \mathfrak{s} = \{ Y \in g_r : \tilde{\theta}Y = -Y \},
\]
that is, \( g_r \) has the same Cartan decomposition. \( \square \)

**Remark.** As a consequence of the Lemma 6, in \( g_r \) we can choose the same maximal abelian algebra \( \mathfrak{a} \subset \mathfrak{s} \) and the same simple root system \( \Sigma \) (consequently the same root system \( \Pi \) and positive Weyl chamber \( \mathfrak{a}^+ \)), but the root spaces are going to change.

Denote by \( \text{ad}_r : g_r \to g_r \) the \( r \)-adjoint representation, given by
\[
\text{ad}_r(X)(Y) = [X, Y]_r \quad X, Y \in g_r.
\]
Then for \( \alpha \in \Pi \) the corresponding \( r \)-root space is defined by
\[
g^r_\alpha = \{ X \in g_r : \text{ad}_r(H)X = \alpha(H)X \quad \forall H \in \mathfrak{a} \}.
\]
If \( g_\alpha \) is the usual root space (of \( g \)). Then we can define the application \( \psi_r : g \to g \) given by
\[
\psi_r(Z) = Z + \frac{r - 1}{r + 1} \theta Z,
\]
such that \( g^r_\alpha = \psi_r \left( g_\alpha \right) \) for all \( \alpha \in \Pi \). So if \( n^+ = \sum_{\alpha > 0} g_\alpha \), we have:
\[
n^+_r = \sum_{\alpha > 0} g^r_\alpha = \sum_{\alpha > 0} \psi_r \left( g_\alpha \right) = \psi_r \left( n^+ \right),
\]
thus given \( H \in \text{cl}(\mathfrak{a}^+) \)
\[
n^+_r H = \sum_{\alpha(H) > 0} g_\alpha \quad \text{and} \quad n^+_r H = \psi_r \left( n^+_r H \right).
\]

Let be \( G_r = \text{Aut}_0 g_r \), that is, \( G_r \) is semisimple and diffeomorphic to \( G \), whose adjoint orbits are manifolds in \( g \). The \( r \)-adjoint representation of \( G_r \) is going to be defined (identified) by:
\[
\text{Ad}_r(G) \cdot H = \text{Ad}_r(\mathfrak{g}) \left( H + \psi_r \left( n^+_r H \right) \right).
\]
Now, let’s see some results that will allow to describe the \( r \)-adjoint orbit:

**Lemma 7.** For \( r > 0 \), \( X \in \mathfrak{n} \) and \( Y \in g \), then
\[
\text{Ad}_r \left( e^{rX} \right) \cdot Y = \text{Ad} \left( e^{\frac{r}{2}X} \right) \cdot Y.
\]
Proof. For \( r > 0 \), \( X \in \mathfrak{k} \) and \( Y \in \mathfrak{g} \), then
\[
\text{ad}_r(X) \cdot Y = T_r \left[ \frac{1}{r} X, \frac{1}{r} \kappa(Y) + \sigma(Y) \right] = \frac{1}{r} \text{ad}(X) \cdot Y.
\]
Inductively \( \text{ad}_k^r(X) \cdot Y = \frac{1}{k} \text{ad}^k(X) \cdot Y \), then
\[
\sum_{k>0} \frac{t^k \text{ad}_k^r(X)}{k!} Y = \sum_{k>0} \frac{(\frac{t}{r})^k \text{ad}^k(X)}{k!} Y = \text{Ad} \left( e^{\frac{t}{r}X} \right) \cdot Y.
\]
Therefore, given \( H \in \text{cl}(\mathfrak{a}^+) \) and \( X \in \mathfrak{k} \):
\[
\text{Ad}_r \left( e^{tX} \right) \cdot H = \text{Ad} \left( e^{\frac{t}{r}X} \right) \cdot H,
\]
that is, they determine the same flag manifold. In addition, given \( \alpha \in \Pi \)
\[
\text{Ad}_r \left( e^{tX} \right) : X_\alpha^r = e^{\frac{t}{r} \text{ad}(X) (\psi_r(X_\alpha))}
\]
\[
= \sum_{k>0} \frac{(\frac{t}{r})^k \text{ad}^k(X)(\psi_r(X_\alpha))}{k!}
\]
\[
= \psi_r \left( \sum_{k>0} \frac{(\frac{t}{r})^k \text{ad}^k(X)}{k!} (X_\alpha) \right) = \psi_r \left( \text{Ad} \left( e^{\frac{t}{r}X} \right) \cdot X_\alpha \right).
\]
Thus for all \( k \in K \)
\[
\text{Ad}_r(k) \left( n_{r,H}^+ \right) = \text{Ad}_r(k) \cdot \psi_r \left( n_H^+ \right) = \psi_r \left( \text{Ad}(k) \left( n_H^+ \right) \right)
\]
Hence, we conclude:

**Proposition 8.** For \( r > 0 \) and \( H \in \text{cl}(\mathfrak{a}^+) \)
\[
\text{Ad}_r(G) \cdot H = \bigcup_{k \in K} \text{Ad}(k)(H) + \psi_r \left( \text{Ad}(k) \left( n_H^+ \right) \right),
\]
that is, the \( r \)-adjoint orbit of \( G \) is a \( r \)-deformation of the adjoint orbit of \( G \).

**Remark.** The construction of \( \psi_r \) is given by the fact that there are \( X_\alpha \in \mathfrak{g}_\alpha \)
such that \( \theta X_\alpha = -X_{-\alpha} \in \mathfrak{g}_{-\alpha} \) for all \( \alpha \in \Pi \). Then given \( H \in \mathfrak{a} \subset \mathfrak{s} \):
\[
\text{ad}_r(H)(\psi_r X_\alpha) = \text{ad}_r(H)(X_\alpha) + \frac{r-1}{r+1} \text{ad}_r(H)(X_{-\alpha}),
\]
and
\[
\text{ad}_r(H) X_\alpha = \alpha(H) \left( \frac{1+r^2}{2r} X_\alpha - \frac{1-r^2}{2r} X_{-\alpha} \right).
\]
Therefore \( \text{ad}_r(H) (X_\alpha^r) = \alpha(H) \left( X_\alpha + \frac{1-r^2}{2r+1} X_{-\alpha} \right) = \alpha(H) (X_\alpha^r) \).
In addition, the representation of $K_H$ in $\mathfrak{g}_r$ makes invariant the subspace $n^+_{r,H}$, because if $k \in K$, then $\text{Ad}_r(k)$ commutes with $\text{ad}_r(H)$. Therefore $\text{Ad}_r(k)$ takes eigenspaces of $\text{ad}_r(H)$ in eigenspaces. Thus we can induce the representation $\rho_r$ of $K_H$ in $n^+_{r,H}$, and by [2], we have

$$
\psi_r(\rho(k) \cdot X) = \rho_r(k) \cdot \psi_r(X) \quad k \in K_H, \ X \in n^+_H,
$$

where $\rho$ is the representation in the case $r = 1$ (that is, the usual representation $\text{Ad}$). Therefore

$$
K \times_{\rho_r} n^+_{r,H} = K \times_{\rho} \psi_r(n^+_H).
$$

So let’s induce a diffeomorphism between $\text{Ad}(G) \cdot H$ and $\text{Ad}_r(G) \cdot H$ using the following map (this construction was proved in [2, Proposition 2.4])

$$
\gamma_r : \text{Ad}_r(G) \cdot H \to K \times_{\rho} \psi_r(n^+_H),
$$

such that

$$
Y = \text{Ad}_r(k)(H + X) \mapsto (k, X) \in K \times_{\rho} \psi_r(n^+_H)
$$

is a diffeomorphism that satisfies:

1. $\gamma_r$ is equivariant with respect to the action of $K$.
2. $\gamma_r$ leads fibers into fibers.
3. $\gamma_r$ leads the orbit $\text{Ad}_r(K) \cdot H$ in the null section of $K \times_{\rho} \psi_r(n^+_H)$.

It’s easy to see that $(d\gamma_r)_x = \text{id}$, for $x = \text{Ad}_r(k)(H + Y) \in \text{Ad}_r(G) \cdot H$.

Furthermore, diffeomorphism $\gamma_r$ is defined by the vector bundle $K \times_{\rho} \psi_r(n^+_H)$ associated with the main bundle $K \to K/H$ of $\text{Ad}_r(G) \cdot H$, which is a homogeneous space. Using this diffeomorphism for $r > 0$ we define the map $\tilde{\psi}_r$ as follows:

$$
\begin{array}{cccc}
\text{Ad}(G) \cdot H & \xrightarrow{\tilde{\psi}_r} & \text{Ad}_r(G) \cdot H \\
\gamma & \downarrow & \gamma_r \\
K \times_{\rho} n^+_H & \xrightarrow{\psi_r} & K \times_{\rho} \psi_r(n^+_H)
\end{array}
$$

which is a diffeomorphism, because $\psi_r$ is linear (in the complex is the sum of linear and anti-linear applications) and $\gamma_r$ is a diffeomorphism, as seen above. We conclude

$$
(d\tilde{\psi})_x = \psi_r, \quad x \in \text{Ad}_r(G) \cdot H.
$$

Hence joining these constructions, we conclude that

**Theorem 9.** Let $H \in \text{cl}(\mathfrak{a}^+)$ and $r > 0$, then the manifolds $\text{Ad}(G) \cdot H$ and $\text{Ad}_r(G) \cdot H$ are diffeomorphic by $\tilde{\psi}_r$. 

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In addition, our intention is to take the above diffeomorphism to a diffeomorphism between $\text{Ad}_r(G) \cdot H$ and $K_{\text{ad}} \cdot H$, for this define the application

$$\psi(X) = X + \theta X,$$

and notice that when $r \to \infty$

$$\psi_r \to \psi.$$

**Lemma 10.** The application $\psi$ defined above satisfies:

1. The image of $\psi$ is in $\mathfrak{k}$ and the kernel in $\mathfrak{s}$.
2. Let $X \in \mathfrak{k}$, we have $\psi \circ \text{ad}(X) = \text{ad}(X) \circ \psi$.
3. If $X \in \mathfrak{k}$ and $\alpha \in \Pi$, then

$$\text{Ad} \left( e^{tX} \right) \psi X_\alpha = \psi \left( \text{Ad} \left( e^{tX} \right) X_\alpha \right).$$

**Proof.** Item 1 is immediate from the definition of $\kappa$.

2. Let $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$

$$\psi[X,Y] = [X,Y] + \theta[X,Y] = [X,Y] + [X,\theta Y] = [X,\psi Y].$$

3. Let $X \in \mathfrak{k}$ and $\alpha \in \Pi$, note that for $Y \in \mathfrak{g}$

$$\text{ad}(Y) \cdot \theta X_\alpha = [Y,\theta X_\alpha] = \theta[Y,X_\alpha] = \theta \text{ad}(Y)X_\alpha,$$

inductively, we have $\text{ad}^k Y \cdot \theta X_\alpha = \theta \text{ad}^k (\theta Y) \cdot X_\alpha$, then

$$e^{t \text{ad}(X)} \left( \theta X_\alpha \right) = \sum_{k>0} \frac{t^k \text{ad}^k(X)}{k!} \cdot \theta X_\alpha = \sum_{k>0} \frac{t^k \theta \text{ad}^k (\theta X)}{k!} \cdot X_\alpha = \theta \cdot e^{t \text{ad}(X)} (X_\alpha),$$

because $\theta X = X$ and we have

$$\text{Ad} \left( e^{tX} \right) \cdot \psi X_\alpha = e^{t \text{ad}(X)} (X_\alpha) + \theta \cdot e^{t \text{ad}(X)} (X_\alpha) = \psi \left( e^{t \text{ad}(X)} \cdot X_\alpha \right) = \psi \left( \text{Ad} \left( e^{tX} \right) X_\alpha \right).$$

Hence define

$$\text{Ad}_\infty(G) \cdot H := \bigcup_{k \in K} \text{Ad}(k) \left( H + \psi \left( n_H^* \right) \right),$$

(3)

when $r \to \infty$ we have

$$\text{Ad}_r(G) \cdot H \to \text{Ad}_\infty(G) \cdot H.$$
It is convenient to define the $\infty$-root spaces by $g_\infty^\alpha = \psi(g_\alpha)$, and consequently
\[
\mathfrak{n}^+_{\infty,H} = \sum_{\alpha(H) > 0} g_\infty^\alpha = \sum_{\alpha(H) > 0} \psi(g_\alpha) = \psi\left(\mathfrak{n}^+_H\right).
\]

Therefore, analogous to the above, define
\[
\gamma_\infty : \text{Ad}_\infty(G) \cdot H \to K \times_\rho \psi\left(\mathfrak{n}^+_H\right),
\]
such that
\[
Y = \text{Ad}(k)(H + X) \mapsto (k, X) \in K \times_\rho \psi(\mathfrak{n}^+_H).
\]

So we have that the application $\gamma_\infty$ is a diffeomorphism, seen as a vector bundle. The map $\gamma_\infty$ is well defined as a consequence of $\psi$, the bijectivity is a consequence of the way in which the manifold $\text{Ad}(G)\cdot H$ was defined, and the differentiability is given by the idea of making $r \to \infty$, in the diffeomorphism $\gamma_r$.

Then we can define the diffeomorphism $\tilde{\psi}$, given by:
\[
\begin{array}{ccc}
\text{Ad}(G) \cdot H & \xrightarrow{\tilde{\psi}} & \text{Ad}_\infty(G) \cdot H \\
\gamma & \downarrow & \gamma_\infty \\
K \times_\rho \mathfrak{n}^+_H & \xrightarrow{\psi} & K \times_\rho \psi(\mathfrak{n}^+_H)
\end{array}
\]
in the same way as for $\tilde{\psi}_r$. So using $r \to \infty$
\[
(dx\gamma_\infty)_x = \text{id} \quad \text{and} \quad \left(dx\tilde{\psi}\right)_x = \psi.
\]

Soon, we can conclude that

**Theorem 11.** The manifolds $\text{Ad}_\infty(G)H$ and $\text{Ad}_r(G) \cdot H$ are diffeomorphic for $r > 0$, the diffeomorphisms are given by $\tilde{\psi}$ and $\tilde{\psi}_r$ defined above.

Then
\[
K_{ad} \cdot H = \bigcup_{k \in K} \text{Ad}(k)(H) + [\text{Ad}(k) \cdot H, s].
\]
The fiber in $H$ is $H + [H, s]$, but $s = a \oplus \sigma(n)$, then $[H, s] = [H, \sigma(n)]$ because $[H, a] = 0$. In addition, for $\alpha \in \Pi^+$
\[
[H, X_\alpha] = \alpha(H)X_\alpha,
\]
such that
\begin{itemize}
  \item $\alpha(H) = 0$ if $\alpha \notin (\Theta_H)^+$, then $\alpha(H)g_\alpha = 0$ for $\alpha \notin (\Theta_H)^+$.
  \item $\alpha(H) > 0$ if $\alpha \in (\Theta_H)^+$, then $\alpha(H)g_\alpha = g_\alpha$ for $\alpha \in (\Theta_H)^+$.
\end{itemize}
Thus \([H, \sigma(n)] = \frac{1}{2} ([H, X_\alpha] - [H, \theta X_\alpha]),\) but
\[
[H, \theta X_\alpha] = \theta [\theta H, X_\alpha] = \theta [-H, X_\alpha] = -\theta [H, X_\alpha],
\]
because \(H \in a \subset s,\) then
\[
[H, \sigma(X_\alpha)] = \frac{1}{2} ([H, X_\alpha] + \theta [H, X_\alpha]) = \frac{1}{2} \psi (\alpha(H) \cdot X_\alpha),
\]
and as \([H, X_\alpha] \neq 0\) if and only if \(\alpha \in \langle \Theta_H \rangle^+,\) we have
\[
[H, s] = \psi \left( n_H^+ \right).
\]
So the fibers in \(H\) of \(K_{ad} \cdot H\) and \(Ad_\infty(G) \cdot H\) coincide. In an equivalent way, we can identify the other fibers of these spaces for each \(k \in K,\)
\[
Ad(k) \cdot (H) + [Ad(k) \cdot H, s] \mapsto Ad(k) \cdot (H) + \psi \left( Ad(k) \cdot n_H^+ \right).
\]
So we can identify the manifolds \(K_{ad} \cdot H\) and \(Ad_\infty(G) \cdot H\) that are diffeomorphic from the bundle \(T^*F_H.\) We can conclude:

**Corollary 12.** The adjoint orbit \(Ad(G) \cdot H\) deforms in \(K_{ad} \cdot H,\) by \(\psi_r.\)

### 5 Hermitian symplectic form

In this section we will take advantage of the construction given in Section 5 for complex semisimple algebras. Let \(g\) be a complex semisimple algebra and \(u\) the compact real form of \(g,\) such that \(g = u \oplus iu\) is a Cartan decomposition of \(g,\) with Cartan involution \(\tau.\) Let \(G = \text{Aut}_0 g,\) then for \(X, Y \in g\)
\[
\mathcal{H}_\tau(X, Y) = -\langle X, \tau Y \rangle
\]
is a Hermitian form of \(g,\) where \(\langle \cdot , \cdot \rangle\) is the complex Cartan-Killing form of \(g\) (see [6 Lema 12.17]). The imaginary part of \(\mathcal{H}_\tau\) will be denoted by \(\Omega_\tau,\) and \(\Omega_\tau\) is a symplectic form on \(g.\) In addition, for \(H \in g\) the restriction of \(\Omega_\tau\) in the adjoint orbit \(Ad(G) \cdot H\) is a symplectic form. Furthermore, the restriction of \(\Omega_\tau\) in \(Ad_r(G) \cdot H\) is a symplectic form for \(r > 0,\) because \(\mathcal{H}_\tau\) is given by the Cartan-Killing form of \(u\) which is \(Ad(U)-\)invariant.

#### 5.1 Lagrangian sections

As \(g = u \oplus iu\) is a Cartan decomposition with Cartan involution \(\tau,\) for \(g\) semisimple complex Lie algebra. If \(U \subset G\) is the compact subgroup with Lie algebra \(u.\) Then, the representation of the semi-direct product (described above in the general case) is \(U_{ad}.\) If \(H \in s = iu,\) its semi-direct orbit is denoted by \(U_{ad} \cdot H.\) To begin with, let’s see that the restriction of \(\Omega_\tau(\cdot, \cdot) = \text{im} \left( \mathcal{H}_\tau(\cdot, \cdot) \right)\) is a symplectic form in \(U_{ad} \cdot H."
Proposition 13. The form $\Omega_\tau$ of $\mathfrak{g}$ restricted to $U_{ad} \cdot H$ is a symplectic form, for $H \in \text{cl}(a^+)$. 

Proof. The restriction is a closed 2-form because it is the pull-back of the imaginary part of $\mathcal{H}_\tau$ by inclusion. Hence, it remains to be seen that the restriction is a non-degenerate 2-form. Take a semi-direct coadjoint orbit 

$$
\mathcal{O} = \bigcup_{Y \in \text{Ad}(U)H} \left( Y + \text{ad}(Y)(\text{iu}) \right), \quad H \in \text{cl}(a^+) .
$$

The tangent space to a fiber $Y + \text{ad}(Y)(\text{iu})$ is a subspace of $\mathfrak{u}$, and a Lagrangian subspace of $\mathfrak{g}$. Hence the tangent spaces to the fibers are isotropic subspaces for the restriction of $\Omega_\tau$. The dimension of a fiber is half the dimension of the total orbit. Therefore, by Proposition 26 to prove that the restriction of $\Omega_\tau$ is non-degenerate, it is enough to show that the tangent spaces to fibers are maximal isotropic. Take an element $\xi = H + X$ in the fiber over the origin $H$ with $X \in \text{ad}(H)(\text{iu})$. In terms of root spaces 

$$
\text{ad}(H)(\text{iu}) = \sum_{\alpha \in \Pi} u_\alpha,
$$

where $u_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{u}$. The tangent space $T_\xi \mathcal{O}$ of the orbit $\mathcal{O}$ in $\xi = H + X$ is generated by this vertical space $\text{ad}(H)(\text{iu})$ and by the vectors $\text{ad}(A)\xi$, with $A \in \mathfrak{u}$, such that 

$$
[A, H + X] = [A, H] + [A, X] \quad X \in \text{ad}(H)(\text{iu}) \subset \mathfrak{u}.
$$

The component $[A, X] \in \mathfrak{u}$, so if $v \in \text{ad}(H)(\text{iu})$ is a vector of the vertical tangent space then 

$$
\Omega_\tau(v, [A, H] + [A, X]) = \Omega_\tau(v, [A, H])
$$

since the Hermitian form $\mathcal{H}_\tau$ is real in $\mathfrak{u}$, that is, $\mathfrak{u}$ is a Lagrangian subspace for $\Omega_\tau$. Then, to show that the tangent space to the fiber is maximal isotropic it must be shown that given $[A, H]$ with $A \in \mathfrak{u}$, there is an element $v$ of the tangent space to the fiber such that $\Omega_\tau(v, [A, H]) \neq 0$. Now, the subspace 

$$
\{ [A, H] : A \in \mathfrak{u} \}
$$

is nothing less than the tangent space to the orbit $\text{Ad}(U) \cdot H$ and is given by $\text{ad}(H)(\text{iu}) = i\text{ad}(H)(\text{iu})$. Therefore, it all comes down to verify that given $Z \in \text{ad}(H)(\text{iu}) \subset \mathfrak{s} = \mathfrak{iu}$, $Z \neq 0$, there is $v \in \text{ad}(H)(\text{iu})$ such that $\Omega_\tau(v, Z) \neq 0$. But this is immediate as $\mathcal{H}_\tau(Z, Z) > 0$ since $\mathcal{H}_\tau$ is positively defined in $\mathfrak{s}$. Hence if $v = iZ \in \text{ad}(H)(\text{iu})$ then 

$$
\mathcal{H}_\tau(v, Z) = \mathcal{H}_\tau(iZ, Z) = i\mathcal{H}_\tau(Z, Z)
$$

is imaginary and $\neq 0$ which means that $\Omega_\tau(v, Z) \neq 0$. 

In short, it was shown that (the restriction of) $\Omega_\tau$ is a symplectic form along the fiber $H + \text{ad}(H)(\text{iu})$ over the origin $H$. In the other fibers the result is obtained by using the fact that $\Omega_\tau$ is invariant by $U$ and taking into account that the fiber over $Y = \text{Ad}(u) \cdot H$, $u \in U$, is given by $\text{Ad}(u)(H + \text{ad}(H)(\text{iu}))$. □
Then the restriction of $\Omega_\tau$ in the coadjoint orbit will also be denoted by $\Omega_\tau$. In addition, the restriction of $\mathcal{H}_\tau$ to $\mathfrak{s} = i\mathfrak{u}$ is the Cartan-Killing form $\langle \cdot, \cdot \rangle$, which is an inner product in $\mathfrak{s}$ and induces an $U$-invariant Riemannian metric in an orbit $\text{Ad}(U) \cdot H$.

**Proposition 14.** Given $Y \in \mathfrak{s}$ and $Z \in \text{Ad}(U) \cdot H$, suppose that $Y \in T_Z \text{Ad}(U) \cdot H$. Then $iY \in \text{ad}(Z)(i\mathfrak{u})$, that is, $Z + iY$ is in the fiber over $Z$ of the semi-direct coadjoint orbit.

**Proof.** Take first $Z = H \in \mathfrak{a} \subset \mathfrak{s}$. Then,
\[
T_H \text{Ad}(U) \cdot H = \sum_{\alpha(H) > 0} s_\alpha,
\]
while
\[
iT_H \text{Ad}(U) \cdot H = \sum_{\alpha(H) > 0} u_\alpha = \text{ad}(Z)(i\mathfrak{u}).
\]
By these expressions, it is immediate that $iY$ is tangent to the fiber if $Y$ is tangent to the orbit $\text{Ad}(U) \cdot H$. For $Z = \text{Ad}(u) \cdot H, u \in U$, the same result applying $\text{Ad}(u)$.

A vector field in the orbit $\text{Ad}(U) \cdot H$ is an application $x \mapsto Y(x) \in \mathfrak{s}$ that assumes values in the tangent space to $x$. By the Proposition above, $iY(x) \in \mathfrak{u}$ is tangent to the fiber over the semi-direct orbit. Thus, given a vector field $Y$ in $\text{Ad}(U) \cdot H$, the vector field $iY(x)$ is defined in the semi-direct orbit, such that in the fiber $x + \text{ad}(x)(\mathfrak{s})$ is a constant field.

**Proposition 15.** Let $Y = \text{grad} f$ be a gradient field in $\text{Ad}(U) \cdot H$. Thus $iY$ is the Hamiltonian vector field of the function $\tilde{f} = f \circ \pi$ with respect to the symplectic form $\Omega_\tau$.

**Proof.** If $W$ is a vertical vector then $d\tilde{f}(W) = 0$ and $\Omega_\tau(W, iY(x)) = 0$, because both $W$ and $iY(x)$ are in $\mathfrak{u}$. On the other hand, take a vector of type $[A, x + X] = [A, x] + [A, X]$ with $X \in \text{ad}(x)(i\mathfrak{u}) \subset \mathfrak{u}$ (these vectors, together with the vertical space, generate the tangent space as in Proposition 3). The component $[A, X] \in \mathfrak{u}$, so that $d\tilde{f}([A, X]) = 0$ and $\Omega_\tau([A, X], iY(x)) = 0$. Since the component $v = [A, x]$ is the tangent space to $x$, hence
\[
d\tilde{f}(v) = df(v) = \langle Y(x), v \rangle,
\]
because $Y = \text{grad} f$. But,
\[
\Omega_\tau(iY(x), v) = \mathcal{H}_\tau(iY(x), v) = i\langle Y(x), v \rangle,
\]
because in this sequence of equality all terms are purely imaginary. Consequently, for vectors of type $w = [A, x + X] = [A, x] + [A, X]$, holds $d\tilde{f}(w) = \Omega_\tau(iY(x), v)$, as this equality is also true for vertical vectors, it is shown that $iY(x)$ is the Hamiltonian vector field of $\tilde{f}$. \hfill \Box
**Corollary 16.** Let $Y$ be a gradient field on the flag manifold $F_H = \text{Ad} (U) \cdot H$, and for $t \in \mathbb{R}$ we define the application

$$\sigma_t Y (x) = x + ti Y (x).$$

This application is a section of $U_{\text{ad}} \cdot H$. Then, the image of $\sigma_t Y$ is a Lagrangian submanifold of $U_{\text{ad}} \cdot H$ with respect to the symplectic form $\Omega_\tau$.

**Proof.** By Proposition 15, $i Y (x)$ is a Hamiltonian vector field that is constant in each fiber, which means that if $\sigma_t Y$ is its flow, then

$$\sigma_t Y (x + X) = x + X + ti Y (x)$$

to $x + X$ in the fiber over $x$. In particular, the image of $\sigma_t Y$ on $F_H$ (0-section) is a Lagrangian submanifold because the 0-section is Lagrangian, which concludes the demonstration. 

Denote by $L_{\sigma_t Y}$ the image of section $\sigma_t Y$, which is Lagrangian submanifold of $U_{\text{ad}} \cdot H$. The next step is to find the tangent space to the section $x \mapsto x + i Y (x)$. If $i Y (x)$ is a section of $U_{\text{ad}} \cdot H \to F_H$, then the tangent space $T_x F_H$ is generated by the vectors $\tilde{A} (x) = [A, x]$ with $A \in \mathfrak{u}$. Therefore, to determine the space tangent to the section we have to compute the differential of $Y$ in the direction of $A (x) = [A, x]$. By the formula of the Lie bracket of vector fields we have

$$dY_x \left( \tilde{A} (x) \right) = \left[ Y, \tilde{A} \right] (x) + d\tilde{A}_x (Y (x))$$

and since $\tilde{A} (x) = [A, x]$ is a linear field it follows that

$$dY_x \left( \tilde{A} (x) \right) = \left[ Y, \tilde{A} \right] (x) + [A, Y (x)].$$

Multiplying this differential by $i$ and adding the base vector, we get a vector tangent to the image of the section as

$$[A, x] + i \left[ Y, \tilde{A} \right] (x) + i [A, Y (x)] \quad A \in \mathfrak{u}.$$

These vectors are in fact tangent to the orbit $U_{\text{ad}} \cdot H$ because $\left[ Y, \tilde{A} \right] (x) \in T_x F_H$ and therefore $i \left[ Y, \tilde{A} \right] (x)$ is tangent to the fiber over $x$. The sum

$$[A, x] + i [A, Y (x)] = [A, x + i Y (x)]$$

is tangent to the orbit because $A \in \mathfrak{u}$. This last equality is written as $\text{ad} (A) (\sigma_Y (x))$ where $\sigma_Y (x) = x + i Y (x)$ is the section defined by the field $Y$. Thus the tangent vectors to the section $\sigma_Y$ are

$$\text{ad} (A) (\sigma_Y (x)) + i \left[ Y, \tilde{A} \right] (x) \quad A \in \mathfrak{u}.$$

This proves the following characterization of the tangent spaces to the sections.
Proposition 17. The tangent space to $L_{tY}$ on the section $\sigma_{tY}(x) = x + itY(x)$ of $\text{U}_{\text{ad}} \cdot H \rightarrow \mathbb{P}_H$ is generated by

$$[A, x] + ti \left[ Y, \tilde{A} \right](x) + ti [A, Y(x)] = \text{ad}(A)(\sigma_{tY}(x)) + ti \left[ Y, \tilde{A} \right](x),$$

with $A \in \mathfrak{u}$.

Now, we are interested in using these Lagrangian submanifolds to transport them by a symplectomorphism between $\text{U}_{\text{ad}} \cdot H$ and $\text{Ad}(G) \cdot H$, with respect to $\Omega_\tau$. For this, consider the following proposition

Proposition 18. For $r > 0$ we have

$$\tilde{\psi}_r^*(\Omega_\tau) = \Omega_\tau,$$

that is $\tilde{\psi}_r$ is symplectomorphism for $r > 0$.

Proof. If $x \in \text{Ad}(G) \cdot H$ and $\tilde{x} = \tilde{\psi}_r(x)$, then

$$\left( \tilde{\psi}_r^* \Omega_\tau \right)_x (X, Y) = \left( \Omega_\tau \right)_x \left( (d\tilde{\psi}_r)_x X, (d\tilde{\psi}_r)_x Y \right)$$

$$= \left( \Omega \right)_x (\psi_r X, \psi_r Y),$$

for $X, Y \in \mathfrak{n}_H \simeq T_x \text{Ad}(G) \cdot H$ and their corresponding $\psi_r(X), \psi_r(Y)$ in $\mathfrak{n}_{r, H} \simeq T_{\tilde{x}} \text{Ad}_r(G) \cdot H$, as seen above $\psi_r(X_\alpha) = X'_\alpha$ are the generators for $\alpha \in \langle \Theta_H \rangle$.

Similarly, taking $r \to \infty$, in the manifold $\text{Ad}_\infty(G) \cdot H$ we have

$$\tilde{\psi}^* (\Omega_\tau) = \Omega_\tau,$$

where $\Omega_\tau$ is a symplectic form of $\text{Ad}_\infty(G) \cdot H$, because as seen above and by Proposition 17 it coincides with $\text{U}_{\text{ad}} \cdot H$. Then we conclude that

Theorem 19. The manifolds $K_{\text{ad}} \cdot H$ and $\text{Ad}(G) \cdot H$ are symplectomorphic, with the symplectic form $\Omega_\tau$.

and consequently

Corollary 20. The manifolds $\tilde{\psi}^{-1}(L_{tY})$ are Lagrangian submanifolds of $\text{Ad}(G) \cdot H$ with the symplectic form $\Omega_\tau$.

5.2 Lagrangian submanifolds in the adjoint action

We are interested in finding the isotropic or Lagrangian orbits given by the action of $U$ and its subgroups on $\mathfrak{g}$, or more specifically for $\text{Ad}(G) \cdot H$, where we use some techniques of [1] and [3]. Therefore, the action of $U$ is symplectic in relation to $\Omega_\tau$. We can describe the action in terms of the moment map in $\mathfrak{g}$, and we can specify it in an adjoint orbit. Let $B_r(X, Y) = -\langle X, \tau Y \rangle_\mathbb{R}$ a inner product, where $\langle \cdot, \cdot \rangle_\mathbb{R}$ is the Cartan-Killing (real) form of $\mathfrak{g}_\mathbb{R}$, that satisfies

$$B_r(X, Y) = 2\text{Re} \left( \mathcal{H}_r(X, Y) \right) \quad \text{and} \quad B_r(iX, iY) = B_r(X, Y),$$
then unless multiplying by $\frac{1}{2}$:

$$
\Omega_r(X,Y) = B_r(iX,Y) = -(iX,\tau Y)_R.
$$

For this, we will describe the action in terms of the moment map in $\mathfrak{g}$ and then specifying for the adjoint orbits. So for $A \in \mathfrak{u}$, define the bilinear form:

$$
\beta_A(X,Y) = \Omega_r(\text{ad}(A) \cdot X,Y) = B_r(i \text{ad}(A) \cdot X,Y),
$$

such that is symmetric:

$$
\beta_A(Y,X) = B_r(\text{ad}(A) \cdot Y,X) = B_r(\text{ad}(A) \cdot iX,Y) = \beta_A(X,Y),
$$

because $\text{ad}(A)$ is anti-symmetric in relation to $B_r$. Then define the quadratic form $Q(X) = \beta_A(X,X) = \Omega_r(\text{ad}(A)X,X)$.

**Proposition 21.** If $A \in \mathfrak{u}$ then $\text{ad}(A)$ is a Hamiltonian field with Hamiltonian function $\frac{1}{2}Q(x)$.

**Proof.** Let $\alpha(t)$ be any curve, then

$$
\frac{d}{dt} \left( \frac{1}{2}Q(\alpha(t)) \right) = \frac{d}{dt} \left( \frac{1}{2} \beta_A(\alpha(t),\alpha(t)) \right) = \beta_A(\alpha'(t),\alpha(t)) = \Omega_r(\text{ad}(A) \cdot \alpha'(t),\alpha(t)).
$$

therefore a vector field $x \mapsto \text{ad}(A) \cdot x$ is Hamiltonian with function $\frac{1}{2}Q(x)$. 

From this Hamiltonian function we can write the moment map $\mu : \mathfrak{g} \rightarrow \mathfrak{u}$, for $A \in \mathfrak{u}$:

$$
\langle \mu(x),A \rangle_{\mathfrak{u}} = \frac{1}{2}Q(x),
$$

where $\langle \cdot,\cdot \rangle_{\mathfrak{u}}$ is the Cartan-Killing form of $\mathfrak{u}$. Therefore we have

$$
\langle \mu(x),A \rangle_{\mathfrak{u}} = \frac{1}{2} \Omega_r(\text{ad}(A) \cdot x,x)
$$

$$
= -\frac{1}{2} \langle i \text{ad}(A) \cdot x,\tau x \rangle_R = \frac{1}{2} \langle A,[\tau ix,x] \rangle_R.
$$

Hence $\mu(x)$ is the orthogonal projection on $\mathfrak{u}$ of $[\tau ix,x]$, that is

$$
\mu(x) = \frac{1}{2} ([\tau ix,x] + \tau [\tau ix,x]) = [\tau ix,x] \in \mathfrak{u}.
$$

**Corollary 22.** The moment map $\mu$ for the adjoint action of $\mathfrak{g}$ in $\mathfrak{g}$ (and thus for the action in each orbit $\text{Ad}(G) \cdot H$) is given to $A \in \mathfrak{u}$ by

$$
\mu(x) = [\tau ix,x] = -i[\tau x,x] \in \mathfrak{u} \quad x \in \mathfrak{g}.
$$
From this expression for $\mu$ and [3, Prop. 4], it follows that the orbit $\text{Ad}(U) \cdot x$ is isotropic for $\Omega_\tau$ if and only if $[\tau x, x] = 0$, since $u$ is semisimple. Put another way, $\text{Ad}(U) \cdot x$ is isotropic if and only if $x$ commutes with $\tau x$.

Example 23. Let $g = \mathfrak{sl}(n, \mathbb{C})$, we have

$$\tau x = -x^* = x^T,$$

therefore the isotropic orbits are the orbits of normal transformations.

One case where the adjoint orbit $\text{Ad}(U) \cdot H$ is isotropic is when $H \in \mathfrak{s} = iu$. In this case, $\text{Ad}(U) \cdot H = \mathbb{F}_H$ is a flag manifold of $g$. Moreover, we have $\dim (\text{Ad}(G) \cdot H) = 2 \dim \mathbb{F}_H$, hence $\mathbb{F}_H$ is Lagrangian submanifold of $\text{Ad}(G) \cdot H$ with respect to $\Omega_\tau$. Then

Theorem 24. The only isotropic $\text{Ad}(U)$-orbit in $\text{Ad}(G) \cdot H$ is the flag manifold $\mathbb{F}_H$, since it is the only orbit with dimension less or equal to $\frac{1}{2} \dim (\text{Ad}(G) \cdot H)$.

Proof. It should be proved that if $0 \neq X \in \mathfrak{n}_H^+$, then the isotropy subgroup $U_{H+X}$ on $H + X$ has a strictly smaller dimension than the dimension of $U_H$ on $H$, as this shows that

$$\dim \text{Ad}(U)(H + X) > \dim \mathbb{F}_H = \frac{1}{2} \dim (\text{Ad}(G) \cdot H).$$

For this it is observed that if

$$\text{Ad}(u)(H + X) = \text{Ad}(u) \cdot H + \text{Ad}(u) \cdot X = H + X$$

then $\text{Ad}(u) \cdot H = H$ and $\text{Ad}(u) \cdot X = X$. The first equality means that $U_{H+X} \subset U_H$. Take the torus $T_H = \text{cl}\{e^{itH} : t \in \mathbb{R}\}$ which has dimension greater than 0, then $T_H \subset U_H$ but $\text{Ad}(v) \cdot X \neq X$ for some $v \in T_H$ since $\text{Ad}(T_H)$ has no fixed points in $\mathfrak{n}_H^+$, because the eigenvalues of $\text{ad}(H)$ in $\mathfrak{n}_H^+$ are strictly positive.

This shows that $\text{ad}(iH)$ is not in the isotropy algebra $H + X$ and therefore $\dim U_{H+X} < \dim U_H$. \qed

Appendix

Representations and symplectic geometry

Let $M \subset W$ be a immersed submanifold of the vector space $W$ (real, that is, $W = \mathbb{R}^N$). The cotangent bundle $\pi : T^*M \to M$ is provided with the canonical symplectic form $\omega$. Given a function $f : TM \to \mathbb{R}$ denote by $X_f$ the corresponding Hamiltonian field, such that $df(\cdot) = \omega(X_f, \cdot)$. If $\alpha \in W^*$, the height function $f_\alpha : M \to \mathbb{R}$ is given by

$$f_\alpha(x) = \alpha(x)$$

and also denote by $f_\alpha$ its lifting $f_\alpha \circ \pi$ which is constant on the fibers of $\pi$. Denote by $X_\alpha$ the Hamiltonian field of this function. Since $f_\alpha$ is constant in the
fibers, the field $X_\alpha$ is vertical and the restriction to the fiber $T_x^*M$ is constant in the direction of the vector $(df_\alpha)_x \in T_x^*M$. Furthermore, if $\alpha, \beta \in W^*$, the vector fields $X_\alpha$ and $X_\beta$ commutes. In terms of the action of Lie groups and algebras, the commutativity $[X_\alpha, X_\beta] = 0$ means that the application $\alpha \mapsto X_\alpha$ is an infinitesimal action of $W^*$, seen as an abelian Lie algebra. This infinitesimal action can be extended to an action of $W^*$ (seen as an abelian Lie group because the fields $X_\alpha$ are complete).

Now, let $R : L \to \text{Gl}(W)$ be a representation of the Lie group $L$ on $W$ and take a $L$-orbit given by $M = \{R(g)x : g \in L\}$. The action of $G$ on $M$ lifts to an action in the cotangent bundle $T^*M$ for linearity. If $l$ is the Lie algebra of $L$, then the infinitesimal action of $l$ in the orbit $M$ is given by the fields $y \in M \mapsto R(X)y$, where $X \in l$ and $R(X)$ also denotes the infinitesimal representation associated to $R$. The infinitesimal action of the lifting in $T^*M$ is given by $X \in l \mapsto H_X$, where $H_X$ is the Hamiltonian field on $T^*M$, such that the Hamiltonian function is $F_X : T^*M \to \mathbb{R}$ given by

$$F_X(\alpha) = \alpha(R(X)y) \quad \alpha \in T^*_yM.$$  

The actions of $L$ and $W^*$ in $T^*M$ are going to define an action of the semi-direct product $L \times W^*$, defined by the dual representation $R^*$. The action of $L \times W^*$ on $T^*M$ is Hamiltonian in the sense that the corresponding infinitesimal action of $l \times W^*$ is formed by Hamiltonian fields. When we have a Hamiltonian action we can define its moment application (See [7, Section 14.4]). In this case, an application

$$m : T^*M \to (l \times W^*)^* = l^* \times W.$$  

In the action on $T^*M$, the field induced by $X \in l$ is the Hamiltonian field $H_X$ of the function $F_X(\alpha) = \alpha(R(X)y)$, while the field induced by $\alpha \in W^*$ is the Hamiltonian field of the function $f_\alpha$. So if $\gamma \in T^*_yM, y \in M \subset W$ then for $X \in l$ and $\alpha \in W^*$

$$m(\gamma)(X) = \gamma(R(X)y) \quad \text{and} \quad m(\gamma)(\alpha) = \alpha(y).$$  

The first term coincides with the moment $\mu : W \otimes W^* \to l^*$ of the representation $R$, that is, $m(\gamma) = \mu(y \otimes \gamma)$ such that the restriction of $\gamma \in W^*$ to the tangent space $T_yM$ is equal to $\gamma$. The second term shows that the linear functional $m(\gamma)$ restricted to $W^*$ is exactly $y$. Consequently,

**Proposition 25.** The moment application $m : T^*M \to l^* \times W$ is given by

$$m(\gamma_y) = \mu(y \otimes \gamma) + y,$$  

where $\gamma_y \in T^*_yM$ and $\gamma \in W^*$, such that its restriction to $T_yM = \{R(X)y : X \in l^*\}$ is equal to $\gamma$.

**Skew-symmetric bilinear form**

Let $V$ be a vector space (over $\mathbb{R}$ and $\dim V < \infty$) and $\omega$ a skew-symmetric bilinear form in $V$. The radical $R^\omega$ of $\omega$ is given by

$$R^\omega = \{v \in V : \forall w \in V, \omega(v, w) = 0\}.$$  

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By definition, \( \omega \) is non-degenerate if and only if \( R^{\omega} = \{0\} \). In this case \( \dim V \) is even and \( \omega \) is called a linear symplectic form.

**Proposition 26.** \( \omega \) is non-degenerate if and only if there is a maximal isotropic subspace \( W \), with \( 2 \dim W = \dim V \).

**Proof.** As it is well known, if \( \omega \) is a symplectic form then dimension of the maximal isotropic subspaces (Lagrangian subspaces) is half the dimension of \( V \). Furthermore, every isotropic subspace is contained in some Lagrangian subspace. For the reciprocal, take the quotient space \( V/R^{\omega} \) and define the form \( \overline{\omega} \) in \( V/R^{\omega} \) by \( \overline{\omega}(\overline{v},\overline{w}) = \omega(v,w) \) which is a skew-symmetric bilinear form in \( V/R^{\omega} \). The radical \( R^{\overline{\omega}} \) of \( \overline{\omega} \) vanishes, because if \( \overline{v} \in R^{\overline{\omega}} \) then \( \omega(v,w) = \overline{\omega}(\overline{v},\overline{w}) = 0 \), for all \( w \in V \). Hence if \( \omega \) is not identically null, then \( \overline{\omega} \) is a symplectic form.

Now let \( W \subset V \) be an isotropic subspace. So, the projection \( \overline{W} \subset V/R^{\omega} \) is isotropic subspace for \( \overline{\omega} \). If \( W \) is maximal isotropic then \( R^{\omega} \subset W \) and as follows from the definition, \( \overline{W} \) is maximal isotropic and therefore \( \dim V/R^{\omega} = 2 \dim \overline{W} \). In this case \( \dim W = \dim \overline{W} + \dim R^{\omega} \), then

\[
2 \dim W = 2 \dim \overline{W} + 2 \dim R^{\omega} = \dim V - \dim R^{\omega} + 2 \dim R^{\omega} = \dim V + \dim R^{\omega}.
\]

Hence, if \( \omega \) is degenerate then \( \dim R^{\omega} > 0 \) and therefore \( 2 \dim W > \dim V \), concluding the demonstration. \( \Box \)

**References**

[1] Bedulli, L. and Gori, A.: Homogeneous Lagrangian submanifolds, *Communications in Analysis and Geometry*, 16, 591–615, 2008.

[2] Gasparim, E.; Grama, L.; and San Martin, L. A. B.: Adjoint orbits of semisimple Lie groups and Lagrangean submanifolds. *Proceedings Edinburgh Mathematical Society*, 60(2), 361-385, 2017.

[3] Gasparim, E.; San Martin, L. and Valencia, F.: Infinitesimally Tight Lagrangian Orbits, *arXiv:1903.03717*, 2019.

[4] Helgason, S.: Differential geometry, Lie groups, and symmetric spaces, *Academic press*, 80, 1979.

[5] Jurdjevic, V.: Affine-Quadratic Problems on Lie Groups: Tops and Integrable Systems. *Journal of Lie Theory* 30 (2020), 425-444.

[6] San Martin, L.: Álgebras de Lie, *Editora Unicamp*, 2010.

[7] San Martin, L.: Grupos de Lie, *Editora Unicamp*, 2016.