ON RECTIFIABLE SPACES AND PARATOPOLOGICAL GROUPS

FUCAI LIN* AND RONGXIN SHEN

Abstract. We mainly discuss the cardinal invariants and generalized metric properties on paratopological groups or rectifiable spaces, and show that: (1) If $A$ and $B$ are $\omega$-narrow subsets of a paratopological group $G$, then $AB$ is $\omega$-narrow in $G$, which give an affirmative answer for [7, Open problem 5.1.9]; (2) Every bisequential or weakly first-countable rectifiable space is metrizable; (3) The properties of Fréchet-Urysohn and strongly Fréchet-Urysohn are coincide in rectifiable spaces; (4) Every rectifiable space $G$ contains a (closed) copy of $S_\omega$ if and only if $G$ has a (closed) copy of $S_2$; (5) If a rectifiable space $G$ has a $\sigma$-point-discrete closed $k$-network, then $G$ contains no closed copy of $S_{\omega_1}$; (6) If a rectifiable space $G$ is pointwise canonically weakly pseudocompact, then $G$ is a Moscow space. Also, we consider the remainders of paratopological groups or rectifiable spaces, and give a partial answer to questions posed by C. Liu in [20] and C. Liu, S. Lin in [21], respectively.

1. Introduction

Recall that a topological group $G$ is a group $G$ with a (Hausdorff) topology such that the product maps of $G \times G$ into $G$ is jointly continuous and the inverse map of $G$ onto itself associating $x^{-1}$ with arbitrary $x \in G$ is continuous. A paratopological group $G$ is a group $G$ with a topology such that the product maps of $G \times G$ into $G$ is jointly continuous. A topological space $G$ is said to be a rectifiable space provided there are a surjective homeomorphism $\varphi : G \times G \rightarrow G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \rightarrow G$ is the projection to the first coordinate. If $G$ is a rectifiable space, then $\varphi$ is called a rectification on $G$. It is well known that rectifiable spaces and paratopological groups are all good generalizations of topological groups. In fact, for a topological group with the neutral element $e$, then it is easy to see that the map $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on $G$. However, there exists a paratopological group which is not a rectifiable space; Sorgenfrey line ([11, Example 1.2.2]) is such an example. Also, the 7-dimensional sphere $S_7$ is rectifiable but not a topological group [25 § 3]. Further, it is easy to see that paratopological groups and rectifiable spaces are all homogeneous.

By a remainder of a space $X$ we understand the subspace $bX \setminus X$ of a Hausdorff compactification $bX$ of $X$.

In this article, we mainly consider the cardinal invariants and generalized metric properties of paratopological groups or rectifiable spaces. We also consider the

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question: When does a Tychonoff paratopological group or rectifiable space $X$ have a Hausdorff compactification $bX$ with a remainder belonging to a given class of spaces?

2. Preliminaries

Let $X$ be a space. A collection of nonempty open sets $\mathcal{U}$ of $X$ is called a $\pi$-base at point $x$ if for every nonempty open set $O$ with $x \in O$, there exists an $U \in \mathcal{U}$ such that $x \in O \subset U$. The $\pi$-character of $x$ in $X$ is defined by $\pi(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a $\pi$-base at point } x \in X\}$. The $\pi$-character of $X$ is defined by $\pi(X) = \sup\{\pi(x, X) : x \in X\}$.

Definition 2.1. $[7]$ Let $\zeta$ and $\eta$ be any family of non-empty subsets of $X$.

1. The family $\zeta$ is called a prefilter on a space $X$ if, whenever $P_1$ and $P_2$ are in $\zeta$, there is a $P \in \zeta$ such that $P \subseteq P_1 \cap P_2$.
2. A prefilter $\zeta$ on a space $X$ is said to converge to a point $x \in X$ if every open neighborhood of $x$ contains an element of $\zeta$.
3. If $x \in X$ belongs to the closure of every element of a prefilter $\zeta$ on $X$, we say that $\zeta$ accumulates to $x$ or a cluster point for $x$.
4. Two prefilters $\zeta$ and $\eta$ are called to be synchronous if, for any $P \in \zeta$ and any $Q \in \eta$, $P \cap Q \neq \emptyset$.
5. A space $X$ is called bisequential $[7]$ if, for every prefilter $\zeta$ on $X$ accumulating to a point $x \in X$, there exists a countable prefilter $\xi$ on $X$ converging to the same point $x$ such that $\zeta$ and $\xi$ are synchronous.

Definition 2.2. Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space $X$ such that for each $x \in X$, (a) if $U, V \in \mathcal{P}_x$, then $W \cup U \cap V$ for some $W \in \mathcal{P}_x$; (b) the family $\mathcal{P}_x$ is a network of $x$ in $X$, i.e., $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with $U$ open in $X$, then $P \subseteq U$ for some $P \in \mathcal{P}_x$.

The family $\mathcal{P}$ is called a weak base for $X$ $[2]$ if, for every $G \subseteq X$, the set $G$ must be open in $X$ whenever for each $x \in G$ there exists $P \in \mathcal{P}_x$ such that $P \subseteq G$. The space $X$ is weakly first-countable if $\mathcal{P}_x$ is countable for each $x \in X$.

Definition 2.3. Let $\kappa$ be an infinite cardinal.

1. A space $X$ is called an $S_\kappa$-space if $X$ is obtained by identifying all the limit points of $\kappa$ many convergent sequences;
2. A space $X$ is called an $S_2$-space (Arens’ space) if $X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ and the topology is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of $x_n$ is $\{x_n\} \cup \{x_n(m) : m > k, \text{for some } k \in \mathbb{N}\}$; a basic neighborhood of $\infty$ is $\{\infty\} \cup \{V_n : n > k, \text{for some } k \in \mathbb{N}\}$, where $V_n$ is a neighborhood of $x_n$.

Theorem 2.4. $[10] [14] [24]$ A topological space $G$ is rectifiable if and only if there exists $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x \in G, y \in G$ the next identities hold:

$$p(x, q(x, y)) = q(x, p(x, y)) = y, q(x, x) = e.$$ 

In fact, we can assume that $p = \pi_2 \circ \phi^{-1}$ and $q = \pi_2 \circ \phi$ in Theorem 2.4.

Fixed a point $x \in G$, then $f_x \circ g_x : G \to G$ defined with $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$, for each $y \in G$, are homeomorphism, respectively. We denote $f_x, g_x$ with $p(x, G), q(x, G)$, respectively.
Let $G$ be a rectifiable space, and let $p$ be the multiplication on $G$. Further, we sometime write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subseteq G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$; since $x \cdot e = x \cdot q(x, y) = x$ and $x \cdot q(x, e) = e$, it follows that $e$ is a right neutral element for $G$ and $q(x, e)$ is a right inverse for $x$. Hence a rectifiable space $G$ is a topological algebraic system with operation $p, q$, 0-ary operation $e$ and identities as above. It is easy to see that this algebraic system need not to satisfy the associative law about the multiplication operation $p$. Clearly, every topological loop is rectifiable.

All spaces are $T_1$ and regular unless stated otherwise. The notation $\mathbb{N}$ denotes the set of all positive natural numbers. The letter $e$ denotes the neutral element of a group and the right neutral element of a rectifiable space, respectively. Readers may refer to [7, 11, 12] for notations and terminology not explicitly given here.

### 3. Cardinal invariants in paratopological groups or rectifiable spaces

A subset $B$ of a paratopological group $G$ is called $\omega$-narrow in $G$ if, for each neighborhood $U$ of the identity in $G$, there is a countable subset $F$ of $G$ such that $B \subseteq FU \cap UF$.

**Question 3.1.** [7, Open problem 5.1.9] Let $A$ and $B$ be $\omega$-narrow subsets of a paratopological group $G$. Is the set $AB$ necessarily $\omega$-narrow in $G$?

Now we give an affirmative answer to this question.

**Theorem 3.2.** Let $A$ and $B$ be $\omega$-narrow subsets of a paratopological group $G$. Then $AB$ is $\omega$-narrow in $G$.

**Proof.** Let $U$ be a neighborhood of the neutral element $e$ in $G$. Since $G$ is a paratopological group, we can choose an open neighborhood $V$ of $e$ in $G$ with $V^2 \subseteq U$. Since $B$ is $\omega$-narrow, there is a subset $C$ of $G$ with $|C| \leq \omega$ satisfying $B \subseteq CV$. For every $y \in C$, we can choose a neighborhood $W_y$ of $e$ in $G$ such that $y^{-1}W_y \subseteq V$. Since $A$ is $\omega$-narrow in $G$, for every $y \in C$, there is a subset $K_y$ of $G$ with $|K_y| \leq \omega$ and $A \subseteq K_yW_y$. Let $K = \bigcup_{y \in C} K_y$ and $M = KC$. Obviously, $|M| \leq \omega$. Now we show that $AB \subseteq MU$. Assume that $a \in A$ and $b \in B$. Then there is a point $y \in C$ such that $b \in yV$. Therefore, there exists a point $x \in K_y$ with $a \in xW_y$. Hence we have

$$ab \in xW_y yV = xy(y^{-1}W_y y)V \subseteq xyVV \subseteq xyU,$$

that is, $ab \in MU$. Thus we proved that $AB \subseteq MU$. Similarly, we can prove that there exists a subset $P$ of $G$ with $|P| \leq \omega$ and $AB \subseteq UP$. Put $D = M \cup P$. Obviously, we have $AB \subseteq DU \cap UD$ and $|D| \leq \omega$. Therefore, the product $AB$ is $\omega$-narrow in $G$. \hfill $\Box$

It is well known that a bisequential or weakly first-countable topological group is metrizable. Now, we show that a bisequential or weakly first-countable rectifiable space is also metrizable.

**Theorem 3.3.** Every bisequential rectifiable space $G$ is metrizable.
Proof. Since $G$ is a bisequential space, there exists a countable open prefilter $\gamma$ on $G$ converging to some point $g \in G$ [7, Lemma 4.7.11]. Since $G$ is homogeneous, without loss of generality, we can assume that $g = e$. Then the family $\{q(B, B) : B \in \gamma\}$ is a base at $e$. Indeed, for any open neighborhood $U$ of $e$, we can find an open neighborhood $V$ of $e$ and $B \in \gamma$ such that $q(V, V) \subset U$ and $B \subset V$. So we have

$$e \in q(B, B) \subset q(V, V) \subset U.$$ 

Therefore, the space $G$ is first-countable at point $e$. Since $G$ is homogeneous, the space $G$ is first-countable. Thus $G$ is metrizable by [14, Theorem 3.2].

\[\Box\]

Lemma 3.4. Suppose that $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ is a weak base in a rectifiable space. For each $x \in G$ and each $n \in \mathbb{N}$, put $W_n(x) = x \cdot V_n(e)$. Then $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is a weak base in $G$.

Proof. For each $x \in G$, let $f_x : G \to G$ with $f_x(y) = p(x, y)$ for each $y \in G$. Therefore, the map $f_x$ is a homeomorphism of $G$ onto $G$ such that $f_x(e) = p(x, e) = p(x, q(x, x)) = x$. Put $W_n(x) = f_x(V_n(e)) = x \cdot V_n(e)$. It follows from [7, Proposition 4.7.2] that $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is a weak base in $G$.

\[\Box\]

Lemma 3.5. Suppose that $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ is a weak base in a rectifiable space. For each $x \in G$ and each $n \in \mathbb{N}$, put $W_n(x) = (x \cdot V_n(e)) \cdot V_n(e)$. Then $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is a weak base in $G$.

Proof. By Lemma 3.4, we can assume that $V_n(x) = x \cdot V_n(e)$, for each $x \in G$ and each $n \in \mathbb{N}$. Since $p$ is a continuous map from $G \times G$ onto $G$, it is easy to see that $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is a weak base in $G$.

\[\Box\]

Lemma 3.6. [7] Suppose that $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ and $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ are two weak bases in a space $G$. Then, for each $g \in G$ and every $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $W_k(g) \subset V_n(g)$.

Theorem 3.7. Every weakly first-countable rectifiable space $G$ is metrizable.

Proof. Let $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ be a weak base in $G$. For each $x \in G$ and $n \in \mathbb{N}$, we have $V_n(x) = x \cdot V_n(e)$ by Lemma 3.4. Let $U_n = \{x \in V_n(e) : x \cdot V_k(e) \subset V_n(e)\}$ for some $k \in \mathbb{N}$. Obviously, we have $e \in U_n \subset V_n(e)$ by Lemma 3.4 and 3.6. Next we show that $U_n$ is open in $G$. Indeed, take any $y \in U_n$. Then $y \cdot V_k(e) \subset V_n(e)$ for some $k \in \mathbb{N}$. By Lemma 3.4, 3.5 and 3.6, it is easy to see that there exists an $m \in \mathbb{N}$ such that $(y \cdot V_m(e)) \cdot V_n(e) \subset y \cdot V_k(e)$. Hence $(y \cdot V_m(e)) \cdot V_n(e) \subset V_n(e)$, which implies that $V_m(y) = y \cdot V_m(e) \subset U_n$. Therefore, the set $U_n$ is open in $G$. Thus $\{U_m : m \in \mathbb{N}\}$ is a countable base of $G$ at $e$. Then $G$ is metrizable by [14, Theorem 3.2].

A space $X$ is said to be Fréchet-Urysohn if, for each $x \in \overline{A} \subset X$, there exists a sequence $\{x_n\}$ such that $\{x_n\}$ converges to $x$ and $\{x_n : n \in \mathbb{N}\} \subset A$. A space $X$ is said to be strongly Fréchet-Urysohn if the following condition is satisfied

(SFU) For every $x \in X$ and each sequence $\eta = \{A_n : n \in \mathbb{N}\}$ of subsets of $X$ such that $x \in \bigcap_{n \in \mathbb{N}} A_n$, there is a sequence $\zeta = \{a_n : n \in \mathbb{N}\}$ in $X$ converging to $x$ and intersecting infinitely many members of $\eta$.

Obviously, a strongly Fréchet-Urysohn space is Fréchet-Urysohn. However, the space $S_\alpha$ is Fréchet-Urysohn and non-strongly Fréchet-Urysohn.

Next, we show that the properties of Fréchet-Urysohn and strongly Fréchet-Urysohn coincide in rectifiable spaces.

\[\Box\]
Theorem 3.8. If a rectifiable space $G$ is Fréchet-Urysohn, then it is strongly Fréchet-Urysohn.

Proof. We can assume that $G$ is non-discrete. It is enough to verify the condition (SFU) for $x = e$ since $G$ is homogeneous. Suppose $e \in \bigcap_{n \in \mathbb{N}} A_n$, where each $A_n$ is a subset of $G$. Fix a sequence $\{a_n : n \in \mathbb{N}\} \subset G \setminus \{e\}$ converging to $e$. For each $n \in \mathbb{N}$, since $p(e, e) = e$, we can fix an open neighborhood $V_n$ of $e$ such that $a_n \notin p(V_n, V_n)$. Since $q(a_n, a_n) = e$, we can fix an open neighborhood $W_n$ of $e$ such that $q(a_n \cdot W_n, a_n) \subset V_n$ and $W_n \subset V_n$. Moreover, for each $n \in \mathbb{N}$, since $e \in A_n$, we may assume that $A_n \subset W_n$. Put $C_n = a_n \cdot A_n$, for each $n \in \mathbb{N}$.

Claim 1: We have $e \notin C_n$ and $a_n \in C_n$ for each $n \in \mathbb{N}$.

Indeed, if $e \in C_n$, then $W_n \cap a_n \cdot A_n \neq \emptyset$. Then we can choose $w_n \in W_n$ and $w'_n \in A_n$ such that $w_n = a_n \cdot w'_n$. Therefore, we have $a_n = (a_n \cdot w'_n) \cdot q(a_n \cdot w'_n, a_n) = p(a_n \cdot w'_n, q(a_n \cdot w'_n, a_n)) \subset p(W_n, V_n) \cap p(V_n, V_n)$, which is a contradiction. Hence $e \notin C_n$ for each $n \in \mathbb{N}$. For each $g \in G$ and $A \subset G$, since $p(g, A) = p(g, A)$, it is easy to see that $a_n \in C_n$ for each $n \in \mathbb{N}$.

Let $C = \bigcup_{n \in \mathbb{N}} C_n$, and hence $e \in \overline{C}$. Since $G$ is Fréchet-Urysohn, there exists a sequence $\zeta = \{c_n : n \in \mathbb{N}\}$ in $C$ converging to $e$. By the Claim 1, we have $e \notin C_n$, and hence the sequence $\zeta$ must intersects $C_n$ for infinitely many values of $n$. For every $n \in \mathbb{N}$, choose a $k_n \in \mathbb{N}$ such that $c_n \in C_{k_n}$. Then $c_n = a_{k_n} \cdot a'_{k_n}$ for each $n \in \mathbb{N}$, where $a'_{k_n} \in A_{k_n}$. Put $b_n = a'_{k_n} = q(a_{k_n}, p(a_{k_n}, a'_{k_n})) = q(a_{k_n}, a_{k_n} \cdot a'_{k_n}) = q(a_{k_n}, c_n) \to e$ as $n \to \infty$.

Therefore, the sequence $\{b_n : n \in \mathbb{N}\}$ converges to $e$ and intersects infinitely many $A_n$’s. Thus, the condition (SFU) is satisfied, and hence $G$ is strongly Fréchet-Urysohn.

Theorem 3.9. The product of a Fréchet-Urysohn rectifiable space $G$ with a first-countable space $M$ is Fréchet-Urysohn.

Proof. Take any subset $A$ of $G \times M$ and any point $(x, y) \in \overline{A}$. Fix a decreasing countable base $\{U_n : n \in \mathbb{N}\}$ of $M$ at the point $y$. Put $B_n = \pi_1((G \times U_n) \cap A)$. Clearly, we have $x \in B_n$ for each $n \in \mathbb{N}$. We also have $B_{n+1} \subset B_n$, since $U_{n+1} \subset U_n$. It follows from Theorem 3.8 that there exists a sequence $\{b_n : n \in \mathbb{N}\} \subset G$ converging to $x$ and intersecting $B_n$ for infinitely many $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, there are $b_k \in B_{n_k}$ and $c_k \in U_{n_k}$ such that $(b_k, c_k) \in A$ and $n_k > k$. Then, clearly, the sequence $\{(b_k, c_k) : k \in \mathbb{N}\}$ converges to the point $(x, y)$.

Theorem 3.10. The following conditions are equivalent for a rectifiable space $G$:

1. every compact subspace of $G$ is first-countable;
2. every compact subspace of $G$ is metrizable.

Proof. Obviously, we have (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Let $F$ be a non-empty compact subset of $G$. Consider the map $g : G \times G \to G$ defined by $g(x, y) = q(x, y)$, for all $x, y \in G$. Clearly, the map $g$ is continuous, and the image $F_1 = g(F \times F)$ is a compact subset of $G$ which contains $e$ of $G$. Since every compact subspace of $G$ is first-countable, the compact subspace $F_1$ is first-countable, thus $\chi(e, F_1) \leq \mathbb{N}$. Denote by $f$ the restriction of $g$ to $F \times F$. For each $x \in G$, since $q(x, G)$ are homeomorphism and $q(x, x) = e$, it is easy to see that $\Delta_F = f^{-1}(e)$, where $\Delta_F$ is the diagonal in $F \times F$. Since $f$ is a closed map, it
follows that \( \chi(\Delta_F, F \times F) = \chi(e, F_1) \leq \mathbb{N} \). Therefore, the subspace \( F \) is metrizable by [12].

4. Generalized metrizable properties on rectifiable spaces

In this section, we discuss the generalized metrizable properties on rectifiable spaces.

**Theorem 4.1.** Let \( G \) be a rectifiable space. Then \( G \) contains a (closed) copy of \( S_{\omega} \) if and only if \( G \) has a (closed) copy of \( S_2 \).

**Proof.** Sufficiency. Since \( G \) is homogeneous, without loss of generality, we can assume that \( A = \{ e \} \cup \{ x_n : n \in \mathbb{N} \} \cup \{ x_n(m) : m, n \in \mathbb{N} \} \) is a closed copy of \( S_2 \), where, \( x_n \to e \) as \( n \to \infty \) and \( x_n(m) \to x_n \) as \( m \to \infty \) for each \( n \in \mathbb{N} \). For each \( m, n \in \mathbb{N} \), put \( y_n(m) = q(x_n, x_n(m)) \). Then, for each \( n \in \mathbb{N} \), we have \( y_n(m) = q(x_n, x_n(m)) \to q(x_n, x_n) = e \) as \( m \to \infty \). For every \( n \in \mathbb{N} \), let \( A_n = \{ y_n(m) : m \in \mathbb{N} \} \).

Claim 2: For each \( m \in \mathbb{N} \), the set \( F = \{ n : A_m \cap A_n \text{ is infinite} \} \) is finite. Indeed, if not, then there exists an \( m \in \mathbb{N} \) such that \( F = \{ n : A_m \cap A_n \text{ is infinite} \} \) is infinite. Take distinct \( q(x_n, x_n(m_i)) \in A_m \cap A_n \), for each \( i \in \mathbb{N} \). Thus \( q(x_n, x_n(m_i)) \to e \) as \( i \to \infty \) since \( q(x_n, x_n(m_i)) \in A_m \) for each \( i \in \mathbb{N} \). Since \( x_n \to e \) as \( n \to \infty \), we can assume that \( x_n(m_i) \to e \) as \( i \to \infty \), which is a contradiction.

By the Claim 2, without loss of generality, we can assume that \( A_i \cap A_j = \emptyset \) for distinct \( i, j \in \mathbb{N} \). Put \( B = \{ e \} \cup \{ y_n(m) : m, n \in \mathbb{N} \} \).

Claim 3: The subspace \( B \) of \( G \) is a closed copy of \( S_{\omega} \).

First, the set \( B \) is closed in \( G \). Indeed, if not, then there exists a point \( x \in G \setminus B \) such that \( x \in B \). Obviously, we have \( x \neq e \), and hence \( p(e, x) \neq e \) since \( p(e, e) = e \) and \( p(e, G) \) is a homeomorphism. Since \( A \) is closed, there is an open neighborhood \( V \) of \( e \) such that \( |p(V, x \cdot V) \cap A| \leq 1 \) or \( p(V, x \cdot V) \cap A \subset \{ x_k \} \cup \{ x_k(m) : m \in \mathbb{N} \} \) for some \( k \in \mathbb{N} \). Let \( U \) be an open neighborhood of \( e \) with \( U \cdot (x \cdot U) \subset V \cdot (x \cdot V) \) and \( e \notin x \cdot U \). Then \( x \cdot U \) contains an infinite subset \( \{ y_n(m_i) : i \in \mathbb{N} \} \subset B \), where \( n_i \neq n_j \) for distinct \( i, j \in \mathbb{N} \). Since \( x_n \to e \) as \( n \to \infty \), we can assume that \( \{ x_n : i \in \mathbb{N} \} \subset U \). Therefore, for each \( i \in \mathbb{N} \), we have

\[
x_n(m_i) = p(x_n, q(x_n, x_n(m_i))) \subset p(U, x \cdot U) \subset p(V, x \cdot V),
\]

which implies that \( \{ x_n(m_i) : i \in \mathbb{N} \} \subset p(V, x \cdot V) \). This is a contradiction.

Let \( f : \mathbb{N} \to \mathbb{N} \). Then \( C = \{ y_n(m) : m \leq f(n), n \in \mathbb{N} \} \) does not have any cluster point. Indeed, if not, then there exists a point \( x \in C \setminus \{ x \} \). Suppose that \( V_1 \) is an open neighborhood of \( e \) with \( |p(V_1, x \cdot V_1) \cap \{ x_n(m) : m \leq f(n), n \in \mathbb{N} \}| \leq 1 \), and that \( U_1 \) is an open neighborhood of \( e \) with \( U_1 \cdot (x \cdot U_1) \subset V_1 \cdot (x \cdot V_1) \). Then \( x \cdot U_1 \) contains an infinite subset \( \{ y_k(l_i) : i \in \mathbb{N} \} \subset C \). Since \( x_n \to e \) as \( n \to \infty \), we can assume that \( \{ x_k : i \in \mathbb{N} \} \subset U_1 \). Therefore, for each \( i \in \mathbb{N} \), we have

\[
x_k(l_i) = p(x_k, q(x_k, x_k(l_i))) \subset p(U_1, x \cdot U_1) \subset p(V_1, x \cdot V_1),
\]

which implies that \( \{ x_k(l_i) : i \in \mathbb{N} \} \subset p(V_1, x \cdot V_1) \). This is a contradiction.

Necessity. Let \( A = \{ e \} \cup \{ y_n(m) : m, n \in \mathbb{N} \} \) be a closed copy of \( S_{\omega} \), where for each \( n \in \mathbb{N} \), \( y_n(m) \to e \) as \( m \to \infty \). It is obvious that there is a non-trivial sequence \( \{ x_n \} \) converging to \( e \) as \( n \to \infty \), where \( x_n \neq e \) for each \( n \in \mathbb{N} \). For each
Let $U_n$ be an open neighborhood of $x_n$ with $\overline{U_i} \cap \overline{U_j} = \emptyset$ for distinct $i, j \in \mathbb{N}$.

Let $x_n(m) = x_n \cdot y_n(m)$, for each $n, m \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have $x_n(m) \to x_n$ as $m \to \infty$. Without loss of generality, we assume that $\{x_n(m) : m \in \mathbb{N}\} \subset U_n$. Put $B = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : n, m \in \mathbb{N}\}$.

Claim 4: The subspace $B$ of $G$ is a closed copy of $S_2$.

First, we show that $B$ is closed in $G$. Suppose not, there is a point $x \in G \setminus B$ such that $x \in \overline{B}$. It is easy to see that $q(e, x) \neq e$. Since $A$ is closed, there is an open neighborhood $V$ of $e$ such that $q(V, x \cdot V) \cap (A \setminus \{q(e, x)\}) = \emptyset$. Let $U$ be an open neighborhood of $e$ with $q(U, x \cdot U) \subset q(V, x \cdot V)$ and $x \cdot U \cap \{x_n : n \in \mathbb{N}\} = \emptyset$. Clearly, for each $n \in \mathbb{N}$, the set $x \cdot U \cap \{x_n(m) : m \in \mathbb{N}\}$ is finite. Moreover, the set $x \cdot U$ contains infinitely many elements of $\{x_n(m) : n, m \in \mathbb{N}\}$, and we denote them by $\{x_n(m_i) : i \in \mathbb{N}\}$. Since $x_n \to e$ as $n \to \infty$, without loss of generality, we assume that $\{x_n : n \in \mathbb{N}\} \subset U$. Therefore, for each $i \in \mathbb{N}$, we have

$$y_n(m_i) = q(x_n, p(x_n, y_n(m_i))) \subset q(U, x \cdot U) \subset q(V, x \cdot V),$$

which implies that $q(V, x \cdot V)$ contains infinitely many $y_n(m_i)$’s. This is a contradiction.

If $f : \mathbb{N} \to \mathbb{N}$, similarly as in the proof of Claim 3, then $\bigcup \{x_n(m) : m \leq f(n), n \geq k \text{ for some } k \in \mathbb{N}\} \subset U$. Hence $B$ is a closed copy of $S_2$. \hfill $\square$

**Definition 4.2.** Let $\mathcal{P}$ be a family of subsets of a space $X$.

1. The family $\mathcal{P}$ is called a *wcs*-network [19] of $X$, if whenever sequence $\{x_n\}$ converges to $x \in U \in \tau(X)$, there is a $P \in \mathcal{P}$ such that $P \subset U$ and for each $n \in \mathbb{N}$, $x_m \in P$ for some $m_n > n$.

2. The family $\mathcal{P}$ is called a *k-network* [22] if whenever $K$ is a compact subset of $X$ and $K \subset U \in \tau(X)$, there is a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{P}' \subset U$.

**Corollary 4.3.** Let $G$ be a sequential rectifiable space. If $G$ has a point-countable *wcs*-network, then $G$ is metrizable if and only if $G$ contains no closed copy of $S_2$.

**Proof.** Obviously, it is sufficient to show the sufficiency. If $G$ contains no closed copy of $S_2$, then $G$ contains no closed copy of $S_\omega$ by Theorem [11] and hence $G$ is first-countable space by [18, Corollary 2.1.11]. Therefore, it follows that $G$ is metrizable by [14]. \hfill $\square$

Let $\mathcal{B} = \{B_\alpha : \alpha \in H\}$ be a family of subsets of a space $X$. The family $\mathcal{B}$ is *point-discrete* if $\{x_\alpha : \alpha \in H\}$ is closed and discrete in $X$, whenever $x_\alpha \in B_\alpha$ for each $\alpha \in H$. The family $\mathcal{B}$ is hereditarily closure-preserving (abbrev. HCP) if, whenever a subset $S(P) \subset P$ is chosen for each $P \in \mathcal{B}$, the family $\{S(P) : P \in \mathcal{B}\}$ is closure-preserving.

**Theorem 4.4.** Let $G$ be a rectifiable space. If $G$ has a $\sigma$-point-discrete closed *k-network*, then $G$ contains no closed copy of $S_{\omega_1}$.

**Proof.** Suppose that $G$ contains a closed copy of $S_{\omega_1} = \{e\} \cup \{x_n(\alpha) : \alpha < \omega_1, n \in \mathbb{N}\}$, where, for each $\alpha < \omega_1$, $x_n(\alpha) \to e$ as $n \to \infty$. Obviously, there exists a non-trivial sequence $\{x_n\}$ such that $x_n$ converges to $e$. By the regularity of $G$, we can take an open subset $U_n$ of $G$ such that $x_n \in U_n$, $\overline{U_i} \cap \overline{U_j} = \emptyset$ for distinct $i, j \in \mathbb{N}$ and $\overline{U_n} \cap \{x_i : i \in \mathbb{N}\} = \{x_n\}$. For each $m \in \mathbb{N}$ and $\alpha < \omega_1$, it is easy to see that $x_m \cdot x_n(\alpha) \to x_n$ as $n \to \infty$ and $\{x_m \cdot x_n(\alpha) : n \in \mathbb{N}\}$ is eventually in $U_m$. Without loss of generality, we assume that $\{x_m \cdot x_n(\alpha) : n \in \mathbb{N}\} \subset U_m$. 

Claim 5: The subspace $B = \{x_{n(\alpha)} \cdot x_{m(\alpha)}(\alpha) : \alpha < \omega_1\}$ of $G$ is discrete for $n(\alpha), m(\alpha) \in \mathbb{N}$.

Case 1: The set $\{n(\alpha) : \alpha < \omega_1\}$ is finite.

We denote $\{n(\alpha) : \alpha < \omega_1\}$ by $\{l_1, \cdots, l_k\}$. Since $\{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}$ is discrete for each $g : \omega_1 \to \mathbb{N}$ and, for each $x \in G$, the map $p(x, G)$ is a homeomorphism, the set $\{x_{i} \cdot x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}$ is discrete for each $1 \leq i \leq k$. Therefore, the subspace $B$ is discrete.

Case 2: The set $\{n(\alpha) : \alpha < \omega_1\}$ is infinite.

Suppose that $B$ is non-discrete, and that $x$ is a cluster point of $B$. For each $g : \omega_1 \to \mathbb{N}$, there exist an open neighborhood $V$ of $x$ with

$$|q(V, (x \cdot V)) \cap \{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}| \leq 1.$$  

Let $U$ be an open neighborhood of $e$ with $q(U, (x \cdot U)) \subset q(V, (x \cdot V))$. Obviously, the set $C = x \cdot U \cap \{x_{n(\alpha)} \cdot x_{m(\alpha)}(\alpha) : \alpha < \omega_1\} \neq \emptyset$ for infinitely many $n(\alpha)$, and we denote it by $\{n_i : i \in \mathbb{N}\}$. Since $x_n \to e$ as $n \to \infty$, without loss of generality, we assume that $\{x_n : n \in \mathbb{N}\} \subset U$. Obviously, for each $i \in \mathbb{N}$, we have

$$x_{m(\alpha)}(\alpha) = q(x_{n_i(\alpha)}, p(x_{n_i(\alpha)}, x_{m(\alpha)}(\alpha))) \subset q(U, x \cdot U) \subset q(V, x \cdot V).$$

Then $|q(V, x \cdot V) \cap \{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}| \geq \omega$, which is a contradiction.

For $\alpha < \omega_1$, let $C_\alpha = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n_i} \cdot x_i(\alpha) : n \in \mathbb{N}, i \geq f_n(\alpha)\}$. Obviously, we have $x_n \cdot x_{n_i}(\alpha) \to e$ as $n \to \infty$, where $j_n > f_n(\alpha)$. Since every infinitely subset of $C_\alpha$ has a cluster point in $C_\alpha$, it follows that $C_\alpha$ is countably compact. It is well known that every countably compact space with a $\sigma$-point-discrete network has a countable network, and hence $C_\alpha$ is compact.

Let $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a $\sigma$-point-discrete $k$-network consisting of closed subsets of $G$. Then there exists a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $C_0 \subset \mathcal{P}'$. Pick a $P_0 \in \mathcal{P}'$ such that $P_0$ contains a point $a_0 = x_{n(0)} \cdot x_{m(0)}(0)$ and infinitely many $x_n$’s. Then we have $C_\beta \subset G \setminus \{a_0 : \alpha < \beta\}$, which is open in $G$ by the Claim 5. Therefore, there exists a finite subfamily $\mathcal{P}'' \subset \mathcal{P}$ such that $C_\beta \subset \mathcal{P}' \subset G \setminus \{a_0 : \alpha < \beta\}$. Pick a $P_\beta \in \mathcal{P}$ such that $P_\beta$ contains a point $a_\beta = x_{n(\beta)} \cdot x_{m(\beta)}(\beta)$ and infinitely many $x_n$’s. By induction, we can pick that $\{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$ such that $P_\alpha \neq P_\beta$ whenever $\alpha \neq \beta$ and each $P_\alpha$ contains infinitely many $x_n$’s. Therefore, there are uncountably many $P_\alpha \in \mathcal{P}_n$ for some $n \in \mathbb{N}$. Since $\mathcal{P}_n$ is point-discrete, there is a subsequence $L$ of $\{x_n : n \in \mathbb{N}\}$ such that $L$ is discrete, which is a contradiction. \qed

In [16], H.J. Junnila and Y.Z. Qiu have proved that a space $X$ is an $\aleph$-space\(^1\) if and only if $X$ has a $\sigma$-HCP $k$-network and contains no closed copy of $S_{\omega_1}$. Therefore, we have the following corollary by Theorem 4.3.

**Corollary 4.5.** A rectifiable space $G$ is an $\aleph$-space if and only if $G$ has a $\sigma$-HCP $k$-network.

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\(^1\)A space $X$ is called an $\aleph$-space if $X$ has a $\sigma$-locally finite $k$-network.
5. When is a rectifiable space a Moscow space?

A space $X$ is called Moscow [7] if for each open subset $U$ of $X$, the set $\overline{U}$ is the union of a family of $G_δ$-sets in $X$, that is, for every $x \in \overline{U}$, there exists a $G_δ$-set $P$ in $X$ with $x \in P \subset \overline{U}$.

A point $x \in X$ is said to be a point of canonical weak pseudocompactness [7] or a cwp-point of $X$, for brevity, if the following condition is satisfied:

(CWP) For every regular open subset $U$ of $X$ such that $x \in \overline{U}$, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of $U$ such that $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, and for every indexed family $η = \{O_n : n \in \mathbb{N}\}$ of open subsets of $X$ satisfying $O_n \cap A_n \neq \emptyset$ for any $n \in \mathbb{N}$, the family $η$ has an accumulation point in $X$.

A space $X$ is called pointwise canonically weakly pseudocompact [7] if each point of $X$ is a cwp-point.

**Theorem 5.1.** If a rectifiable space $G$ is pointwise canonically weakly pseudocompact, then $G$ is a Moscow space.

**Proof.** Let $U$ be a regular open subset of $G$. Obviously, it suffices to show that if $e \in \overline{U}$, then there is a $G_δ$-set $P \subset G$ such that $e \in P \subset \overline{U}$. Thus let us assume that $e \in \overline{U}$ and fix subsets $A_n \subset U$ such as in condition (CWP), where $x = e$.

Next we are going to define a sequence $\{V_n : n \in \mathbb{N}\}$ of open neighborhood $e$, and a sequence $\{a_n : n \in \mathbb{N}\}$ with $a_n \in A_n$, for each $n \in \mathbb{N}$. Firstly, take an $a_1 \in A_0$, and let $V_1$ be an open neighborhood of $e$ with $a_1 \cdot V_1 \subset U$. Assume that an open neighborhood $V_k$ of $e$ is already defined, for some $k \in \mathbb{N}$. Choose a point $a_{k+1} \in A_{k+1} \cap V_k$. Let $V_{k+1}$ be an open neighborhood of $e$ such that $\overline{V_{k+1}} \cdot V_{k+1} \subset V_k$, $q(V_{k+1}, V_{k+1}) \subset V_k$ and $a_{k+1} \cdot V_{k+1} \subset U$. The recursive definition is complete. Let $ζ = \{a_n \cdot V_{n+1} : n \in \mathbb{N}\}$ and $H$ be the set of all accumulation point of $ζ$ in $G$. Since $G$ is pointwise canonically weakly pseudocompact, it follows that $H \neq \emptyset$. Put $B = \bigcap_{n \in \mathbb{N}} V_n$. Obviously, we have $B = \bigcap_{n \in \mathbb{N}} \overline{V_n}$, and hence $B$ is also closed in $G$.

Claim 6: The set $H$ contains $B$.

It follows from $a_n \in V_{n-1}$ that we have $a_n \cdot V_{n+1} \subset V_{n-1} \cdot V_{n+1} \subset V_{n-1} \cdot V_{n-1} \subset V_{n-2}$, for each $n \geq 3$. Therefore, we have $H \subset \bigcup \{a_n \cdot V_{n+1} : n \in \mathbb{N}, n \geq k + 1\} \subset V_{k-1}$ for each $k \geq 2$. Thus $H \subset B$, whence Claim 6 follows.

Claim 7: We have $a \cdot B = B$, for every $a \in H$.

In fact, fix a point $a \in H$, we have $a \cdot B = B$ by Claim 6, and hence, the point $a \in V_n$, for each $n \in \mathbb{N}$. For each $b \in B$ and $k \in \mathbb{N}$, since $p(a, b) \in V_k$, we have $p(a, b) \in B$. Then $a \cdot B \subset B$. Take any point $b \in B$. For each $n \in \mathbb{N}$, since $q(a, b) \in V_n$, it follows that $q(a, b) \in B$. Therefore, we have $b = p(a, q(a, b)) = a \cdot q(a, b) \in a \cdot B$. Thus $B \subset a \cdot B$, and hence $a \cdot B = B$.

Fix a point $a \in H$. Then, by Claim 7, we have

$$B = a \cdot B \subset \bigcup_{n \in \mathbb{N}} (a_n \cdot V_{n+1}) \cdot B \subset \bigcup_{n \in \mathbb{N}} (a_n \cdot V_{n}) \cdot V_n \subset \overline{U}.$$  

Since $e \in B$, it follows that $e \in B \subset \bigcup_{n \in \mathbb{N}} (a_n \cdot V_{n}) \cdot V_n \subset \overline{U}$. Since $B$ is a $G_δ$-set, the space $G$ is Moscow. □
Lemma 5.2. [25] Let $X$ be a Moscow space, and suppose that $Y$ is a $G_δ$-dense subset of $X$. Then $Y$ is $C$-embedded in $X$.

It follows from Theorem 5.1 and Lemma 5.2 that we have the following corollary.

Corollary 5.3. Let $G$ be a pointwise canonically weakly pseudocompact rectifiable space, and $Y$ a $G_δ$-dense subspace of $G$. Then $Y$ is $C$-embedded in $G$.

Let $G$ be a rectifiable space, and $U \subseteq G$. A subset $A$ of $G$ is called $\omega$-deep in $U$ if there is a $G_δ$-set $B$ in $G$ with $e \in B$ and $A \cdot B \subseteq U$. We say that the $g$-tightness $t_g(G)$ of $G$ is countable if, for each regular open subset $U$ of $G$ and each $x \in U$, there is an $\omega$-deep subset $A$ of $U$ such that $x \in \overline{A}$.

Theorem 5.4. Every rectifiable space of countable $g$-tightness is a Moscow space.

Proof. Take any regular open subset $U$ of $G$, and any point $x \in U$. Since $t_g(G) \leq \omega$, there is an $\omega$-deep subset $A$ of $U$ with $x \in \overline{A}$. Then we can fix a $G_δ$-subset $B$ of $G$ with $e \in B$ and $A \cdot B \subseteq U$. For each $a \in G$, since $p(a, G)$ is a homeomorphism map, we have $x \in x \cdot B \subseteq \overline{A} \cdot B \subseteq U$, and $x \cdot B$ is a $G_δ$-subset of $G$. Thus, the space $G$ is Moscow.

The following lemma is an easy exercise.

Lemma 5.5. The union of any countable family of $\omega$-deep subsets of $U$ is an $\omega$-deep subset of $U$, for any set $U$ of a rectifiable space $G$.

Lemma 5.6. If $G$ is a rectifiable space of countable tightness, then the $g$-tightness of $G$ is countable.

Proof. It is easy to see by Lemma 5.5. \qed

Lemma 5.7. If $G$ is a rectifiable space of countable $o$-tightness, then the $g$-tightness of $G$ is countable. In particular, if $G$ has a countable cellularity, then $g$-tightness of $G$ is also countable.

Proof. Let $U$ be a regular open subset of $G$, and assume that $x \in U$. Denote by $ζ$ the family of all non-empty $\omega$-deep open subsets of $U$. We have $U = \cup ζ$, since $G$ is a rectifiable space. Since the $o$-tightness of $G$ is countable, there is a countable subfamily $γ \subseteq ζ$ with $x \in \cup γ$. Then the family $η$ is countable, and hence, the set $V = \cup γ$ is an $\omega$-deep subset of $U$ by Lemma 5.5. Hence, we have $t_g(G) \leq \omega$. Since the $o$-tightness of a space $X$ is less than or equal to the cellularity of $X$, whence the proof is complete. \qed

The next two lemmas are obvious.

Lemma 5.8. If $G$ is an extremally disconnected rectifiable space, then the $g$-tightness of $G$ is countable.

\(^2\) A subset $Y$ of $X$ is called $G_δ$-dense if every $G_δ$ of $X$ meets $Y$.
\(^3\) Remember that $Y$ is $C$-embedded in $X$ if every continuous real-valued function on $Y$ extends continuously over $X$.
\(^4\) A space $X$ is called has countable $o$-tightness if whenever a point $a \in X$ belongs to the closure of $\cup η$, where $η$ is any family of open subsets in $X$, there exists a countable subfamily $ζ$ of $η$ such that $a \in \overline{ζ}$.
\(^5\) We recall that a space $X$ is extremally disconnected if the closure of any open subset of $X$ is open.
Lemma 5.9. If $G$ is rectifiable space of countable pseudocharacter, then the $g$-tightness of $G$ is countable.

Theorem 5.10. Every dense subspace of a rectifiable space of countable $g$-tightness is a Moscow space. In particular, if a rectifiable space $G$ satisfies at least one of the following conditions, then it is Moscow.

1. the space $G$ is a dense subspace of a $\kappa$-Fréchet-Urysohn rectifiable space;
2. the tightness of $G$ is countable;
3. the $o$-tightness of $G$ is countable;
4. the cellularity of $G$ is countable;
5. the pseudocharacter of $G$ is countable;
6. the space $G$ is extremely disconnected.

Proof. It follows from Lemma 5.9 and 5.10 that (1) holds. It is easy to see that (3), (4), (5) and (6) follow from Theorem 5.13 and Lemma 5.7, 5.8 and 5.9. □

Lemma 5.11. Let $G$ be a rectifiable space, $U$ a subset of $G$, and $b$ an element of $G$. If there exists a countable $\omega$-deep subsets $\gamma$ of $U$ such that $b \in \overline{\gamma}$, then there is a closed $G_{\delta}$-subset $P$ of $G$ such that $b \in P \subset U$.

Proof. Without loss of generality, we may assume that $b = e$. Also, we denote $\gamma$ by $\{F_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, it is easy to see that we can choose a closed $G_{\delta}$-subset $V_n$ such that $e \in V_n$, $F_n \cdot V_n \subset U$, $V_{n+1} \cdot V_{n+1} \subset V_n$ and $q(V_{n+1}, V_{n+1}) \subset V_n$. Now let $P = \cap \{V_n : n \in \mathbb{N}\}$. Obviously, the set $P$ is a closed $G_{\delta}$-subset of $G$. By the proof of Claim 7 in Theorem 5.4, we also have $P \in \overline{\gamma}$. Since $e \in \overline{\gamma}$, it follows that

$$P = e \cdot P \subset \bigcup \{F_n \cdot P : n \in \mathbb{N}\} \subset \bigcup \{F_n \cdot V_n : n \in \mathbb{N}\} \subset U.$$ □

By Lemma 5.11, we have the following theorem.

Theorem 5.12. Let $G$ be a rectifiable space. For each open subset $U$ and each point $b \in U$, if there exists a countable $\omega$-deep subsets $\gamma$ of $U$ such that $b \in \overline{\gamma}$, then $G$ is a Moscow space.

Theorem 5.13. Let $G$ be a sequential rectifiable space such that $G \setminus \{e\}$ is normal. Then $G$ has countable pseudocharacter.

Proof. We may assume that $e$ is a non-isolated point in $G$. Since $G$ is homogeneous, we only need to show that $e$ is a $G_{\delta}$-point. Suppose that $e$ is a non-$G_{\delta}$-point in $G$. Obviously, we have $G \setminus \{e\}$ is a non-sequentially closed subset, and hence there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset G \setminus \{e\}$ converging to $e$. Put $A = \{x_{2n-1} : n \in \mathbb{N}\}$ and $B = \{x_{2n} : n \in \mathbb{N}\}$. Clearly, we may assume that $A$ and $B$ are disjoint. Obviously, the sets $A$ and $B$ are closed in $G \setminus \{e\}$, and since $G \setminus \{e\}$ is normal, there exists a continuous function $f$ on $G \setminus \{e\}$ such that $f(x) = 1$, for each $x \in A$, and $f(x) = 0$, for each $x \in B$. Therefore, it is impossible to extend this function continuously to the point $e$. However, since $G$ is sequential, the space $G$ is Moscow by (2) in Theorem 5.10 and hence $G \setminus \{e\}$ is $C$-embedded in $G$ by Lemma 5.2, which is a contradiction. □
6. Remains of $k$-gentle paratopological groups or rectifiable spaces

In this section, we assume that all spaces are Tychonoff.

Let $f : X \to Y$ be a map. The map $f$ is called $k$-gentle if for each compact subset $F$ of $X$ the image $f(F)$ is also compact. A paratopological group $G$ is called $k$-gentle \[5\] if the inverse map $x \mapsto x^{-1}$ is $k$-gentle.

Lemma 6.1. \[5\] Suppose that $G$ is a $k$-gentle paratopological group. Then any remainder of $G$ in a compactification $bG$ of $G$ is either pseudocompact or Lindelöf.

Lemma 6.2. \[5\] Let $G$ be a $k$-gentle paratopological group such that some remainder of $G$ is Lindelöf. Then $G$ is a topological group.

Theorem 6.3. (Henriksen and Isbell \[15\]) A space $X$ is of countable type if and only if its remainder in any (in some) compactification of $X$ is Lindelöf.

Theorem 6.4. Suppose that $G$ is a paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has countable pseudocharacter, then at least one of the following conditions is satisfied

1. the space $G$ is of countable type\[4\],
2. the space $Y$ is first countable.

Proof. Let $Y$ be a non-first countable space. Then there exists a point $y_0 \in Y$ such that $Y$ is not first countable at point $y_0$. Since $Y$ has countable pseudocharacter, the point $y_0$ is a $G_\delta$-point in $Y$, and hence there exists a compact subset $F \subset bG$ such that $F$ has a countable open neighborhood base in $bG$ and $F \cap (bG \setminus G) = \{y_0\}$. Then $F \setminus \{y_0\} \neq \emptyset$ because $Y$ is not first countable at point $y_0$. Therefore, there is a non-empty compact subset $B \subset F$ such that $B$ has a countable neighborhood base in $bG$ and $y_0 \notin B$. It is obvious that $B \subset G$. It follows from \[5\] Proposition 4.1] that $G$ is of countable type. \hfill $\square$

Theorem 6.5. Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has locally a regular $G_\delta$-diagonal, then $G$ is a topological group, and hence, $bG$ and $Y$ are separable and metrizable spaces.

Proof. Clearly, the space $Y$ is nowhere locally compact since $G$ is non-locally compact. It follows from Lemma \[6.1\] that $Y$ is pseudocompact or Lindelöf.

Claim 8: The space $Y$ is Lindelöf.

Suppose not, we assume that $Y$ is pseudocompact. Since $Y$ has locally a regular $G_\delta$-diagonal, for each $y \in Y$, there is an open neighborhood $U_y$ of $y$ in $Y$ such that $\overline{U_y}$ has a regular $G_\delta$-diagonal. Obviously, the set $\overline{U_y}$ is pseudocompact because $Y$ is pseudocompact. For each $y \in Y$, the set $\overline{U_y}$ is metrizable, and hence $\overline{U_y}$ is compact. Then $Y$ is locally compact, which is a contradiction.

It follows from Lemma \[6.2\] and Claim 8 that $G$ is a topological group. Therefore, it follows that $G$, $bG$ and $Y$ are separable and metrizable by \[3\]. \hfill $\square$

\[7\] Recall that a space $X$ is of countable type if every compact subspace $F$ of $X$ is contained in a compact subspace $K \subset X$ with a countable base of open neighborhoods in $X$.

\[8\] A space $X$ is said to have a regular $G_\delta$-diagonal if the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of $\Delta$ in $X \times X$. 
Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has a regular $G_δ$-diagonal, then $G, bG$ and $Y$ are separable and metrizable spaces.

**Theorem 6.7.** Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has locally a $\sigma$-point-finite base, then $G$ is a topological group, and hence $G, bG$ and $Y$ are separable and metrizable spaces.

**Proof.** It follows from Lemma 6.1 that $Y$ is pseudocompact or Lindelöf.

Suppose that $Y$ is pseudocompact. Since $Y$ has locally a $\sigma$-point-finite base, for each $y \in Y$, there is an open neighborhood $U_y$ of $y$ in $Y$ such that $U_y$ has a $\sigma$-point-finite base. Clearly, the set $U_y$ is pseudocompact because $Y$ is pseudocompact. For each $y \in Y$, the set $U_y$ is metrizable \[23\], and hence $U_y$ is compact. Then $Y$ is locally compact which is a contradiction. Therefore, the space $Y$ is Lindelöf.

It follows from Lemma 6.2 that $G$ is a topological group. Therefore, we have $G, bG$ and $Y$ are separable and metrizable by [17].

**Corollary 6.8.** Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has locally an uniform base, then $G, bG$ and $Y$ are separable and metrizable spaces.

**Question 6.9.** Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has a point-countable base, are $G, bG$ and $Y$ separable and metrizable spaces?

**Question 6.10.** Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has a $G_δ$-diagonal, are $G, bG$ and $Y$ separable and metrizable spaces?

Next, we give some partial answers to Questions 6.9 and 6.10.

**Theorem 6.11.** Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has a point-countable base and a $G_δ$-diagonal, then $G, bG$ and $Y$ are separable and metrizable spaces.

**Proof.** Since $Y$ has a $G_δ$-diagonal, the space $G$ is $\sigma$-compact or is of countable type [3].

Case 1: The space $G$ is of countable type.

Then $Y$ is Lindelöf, and hence $G$ is a topological group by Lemma 6.2 Therefore, it follows that $G, bG$ and $Y$ are separable and metrizable spaces by [3].

Case 2: The space $G$ is $\sigma$-compact.

It is obvious that $c(G) \leq \omega$, and hence $c(bG) \leq \omega$. Since $Y$ is dense in $bG$, we have $c(Y) \leq \omega$. There exists a paracompact Čech-complete dense subset $Z \subseteq Y$ because $Y$ is Čech-complete. Since $Z$ has a point-countable base, the subspace $Z$ is metrizable by [12 Corollary 7.10]. We have $c(Z) \leq \omega$ because $Z$ is dense in $Y$. Hence $Z$ is separable, and $Y$ is a separable space. Therefore, the space $Y$ has a countable base since $Y$ is separable space with a point-countable base. Thus $Y$ is Lindelöf, and hence $G$ is a topological group by Lemma 6.2 Therefore, it follows that $G, bG$ and $Y$ are separable and metrizable spaces by [3].

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9Let the family $\mathcal{P}$ be a base of a space $X$. The family $\mathcal{P}$ is an *uniform base* [11] for $X$ if for each point $x \in X$ and $\mathcal{P}$, is a countably infinite subset of $(\mathcal{P})_x$, the family $\mathcal{P}$ is a neighborhood base at $x$. 

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Theorem 6.12. Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = \partial G \setminus G$ is a remainder of $G$. If $Y$ is locally normal with a $G_δ$-diagonal, then $G, bG$ and $Y$ are separable and metrizable spaces.

Proof. It follows from Lemma 6.1 that $Y$ is pseudocompact or Lindelöf.

Case 1: The space $Y$ is pseudocompact.

For every $y \in Y$, since $Y$ is locally normal with a $G_δ$-diagonal, there exists an open neighborhood $U$ of $y$ such that $\overline{U}$ is normal subspace with a $G_δ$-diagonal. Then $\overline{U}$ is pseudocompact, and hence $\overline{U}$ is countably compact and metrizable. It follows that $Y$ is locally compact. Therefore, it follows that $G, bG$ and $Y$ are separable and metrizable spaces by Theorem 6.11.

Case 2: The space $Y$ is Lindelöf.

It follows from Lemma 6.2 that $G$ is a topological group. Since $Y$ has locally a $G_δ$-diagonal, we have $G, bG$ and $Y$ are separable and metrizable spaces.

□

Theorem 6.13. Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = \partial G \setminus G$ is a remainder of $G$. If $Y$ is a c.c.c space with a sharp base $\mathcal{P}$, then $G, bG$ and $Y$ are separable and metrizable spaces.

Proof. It follows from Lemma 6.1 that $Y$ is pseudocompact or Lindelöf.

Case 1: The space $Y$ is pseudocompact.

Since a pseudocompact and c.c.c space with a sharp base is metrizable [6], it follows that $G, bG$ and $Y$ are separable and metrizable spaces by Theorem 6.11.

Case 2: The space $Y$ is Lindelöf.

It follows from Lemma 6.2 that $G$ is a topological group. Since $Y$ has a sharp base, the space $Y$ has a point-countable base [6]. Therefore, we have $G, bG$ and $Y$ are separable and metrizable spaces [3].

□

Next, we consider two questions posed in [20] and [21], respectively.

Question 6.14. [20] Question 6] Let $G$ be a non-locally compact topological group, if the remainder $Y = \partial G \setminus G$ is a quotient s-image of a metric space, are $G$ and $bG$ separable and metrizable?

Question 6.15. [21] Question 5.2] Let $G$ be a non-locally compact topological group, if the remainder $Y = \partial G \setminus G$ of a Hausdorff compactification of $G$ has a point-countable weak base, are $G$ and $bG$ separable and metrizable?

Now we give a partial answers to for Questions 6.14 and 6.15.

Lemma 6.16. Let $G$ be a $k$-gentle paratopological group. Then $G$ is Lindelöf if and only if there exists a compactification $bG$ such that, for any compact subset $F \subset Y = \partial G \setminus G$, there is a compact subset $L \subset Y$ which contains $F$ and is a $G_δ$-subset in $Y$.

Proof. If $G$ is Lindelöf, then $Y$ is of countable type by Theorem 6.3. Therefore, we only need to show the sufficiency. It follows from Lemma 6.1 that $Y$ is pseudocompact or Lindelöf.

Case 1: The space $Y$ is Lindelöf.

10Let the family $\mathcal{P}$ be a base of a space $X$. The family $\mathcal{P}$ is a sharp base [11] for $X$ if for each point $x \in X$ and $\{P_n : n \in \mathbb{N}\} \subset (\mathcal{P})_x$, where $P_n \neq P_m$ whenever $n \neq m$, the family $\{\cap_{n \leq i} P_n : i \in \mathbb{N}\}$ is a neighborhood base at $x$. 
It follows from Lemma 6.16 that $G$ is a topological group, and hence $G$ is a paracompact $p$-space. Therefore, the space $G$ is Lindelöf by [8, Lemma 2.3].

Case 2: The space $Y$ is pseudocompact.

Let $F \subset Y$ be a compact subset. Then there is a compact subset $L \subset Y$ which contains $F$ and is a $G_\delta$-subset in $Y$. Since compact subset $L$ of $Y$ is a $G_\delta$-set and $Y$ is pseudocompact, it is well known that compact subset $L \subset Y$ has a countably open neighborhood base, and hence $Y$ is of countable type. Therefore $G$ is Lindelöf by Theorem 6.3.

**Theorem 6.17.** Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. Then the following conditions are equivalent:

1. the space $Y$ is of subcountable type\(^{11}\).
2. the space $Y$ is of countable type.

**Proof.** It is easy to see by Lemma 6.16 and Theorem 6.3.

**Theorem 6.18.** Suppose that $G$ is a non-locally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ is $\kappa$-perfect\(^{12}\), then $Y$ is first countable.

**Proof.** It follows from Theorem 6.17 that $Y$ is of countable type. Since every point of $Y$ is a $G_\delta$-point, the space $Y$ is first countable.

Let $\mathcal{A}$ be a collection of subsets of $X$. The collection $\mathcal{A}$ is a $p$-metabase \([9]\) for $X$ if for distinct points $x, y \in X$, there exists an $\mathcal{F} \in \mathcal{A}^{\omega^\omega}$ such that $x \in (\cup \mathcal{F})^\circ \subset \cup \mathcal{F} \subset X - \{y\}$.

**Lemma 6.19.** Suppose that $X$ has a point-countable $p$-metabase. Then each countably compact subset of $X$ is a compact, metrizable $G_\delta$-subset of $X$.

**Proof.** Suppose that $\mathcal{V}$ is a point-countable $p$-metabase of $X$, and that $K$ is a compact subset of $X$. Then $K$ is compact by [9]. It follows from [13] that for distinct $x, y \in X$, there exists a finite subfamily $\mathcal{F} \subset \mathcal{V}$ such that $x \in (\cup \mathcal{F})^\circ \subset \cup \mathcal{F} \subset X - \{y\}$. According to a generalized Miščenko’s Lemma in [26, Lemma 6], there are only countably many minimal neighborhood-covers\(^{13}\) of $K$ by finite elements of $\mathcal{V}$, say $\{\mathcal{V}(n) : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $V(n) = \cup \mathcal{V}(n)$. Then $K \subset \cap \{V(n) : n \in \mathbb{N}\}$. Suppose that $x \in X \setminus K$. For each point $y \in K$, there is an $\mathcal{F}_y \in \mathcal{V}^{\omega^\omega}$ with $y \in (\cup \mathcal{F}_y)^\circ \subset \cup \mathcal{F}_y \subset X - \{x\}$. Then there is some sub-collection of $\cup \{\mathcal{F}_y : y \in K\}$ which is a minimal finite neighborhood-covers of $K$, since $K$ is compact. Therefore, we obtain one of the collections $\mathcal{V}(n)$ with $K \subset V(n) = \cup \mathcal{V}(n) \subset X - \{x\}$.

**Theorem 6.20.** \(^{17}\) Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ has a locally point-countable $p$-metabase. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.

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\(^{11}\)Recall that a space $X$ is of subcountable type \([8]\) if every compact subspace $F$ of $X$ is contained in a compact $G_\delta$ subspace $K$ of $X$.

\(^{12}\)Recall that a space $X$ is of $\kappa$-perfect \([8]\) if every compact subspace $F$ is a $G_\delta$ subspace of $X$.

\(^{13}\)Let $\mathcal{P}$ be a collection of subsets of $X$ and $A \subset X$. The collection $\mathcal{P}$ is a neighborhood-cover of $A$ if $A \subset (\cup \mathcal{P})^\circ$. A neighborhood-cover $\mathcal{P}$ of $A$ is a minimal neighborhood-cover if for each $P \in \mathcal{P}$, the family $\mathcal{P} \setminus \{P\}$ is not a neighborhood-cover of $A$. 

\(^{17}\)Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ has a locally point-countable $p$-metabase. Then $G$ and $bG$ are separable and metrizable if $\pi$-character of $Y$ is countable.
**Theorem 6.21.** Suppose that $G$ is a non-locally compact topological group, and that $Y = bG \setminus G$ has a locally point-countable $p$-metabase. Then $G$ and $bG$ are separable and metrizable spaces.

**Proof.** Claim 9: The space $Y$ is $\kappa$-perfect.

Let $F$ be any compact subset in $Y$. For each $y \in Y$, there exists an open subset $U_y$ such that $U_y$ has a point-countable $p$-metabase. Since $F$ is compact, there exists a finite subset $A \subset F$ such that $F \subset \bigcup_{y \in A} U_y$. Without loss of generality, we denote \{U_y : y \in A\} by \{U_{y_1}, \cdots, U_{y_m}\}. Then \{U_{y_i} \cap F : i = 1, \cdots, m\} is a relatively open cover of $F$. For each $1 \leq i \leq m$, there is a closed subset $F_i$ such that $F_i \subset U_{y_i}$ and $F = \bigcup_{i=1}^{i=m} F_i$. For each $1 \leq i \leq m$, the set $F_i$ is a $G_\delta$-set by Lemma 6.19 and hence there exists a sequence of open subsets \{V_{in}\} of $U_{y_i}$ such that $F_i = \bigcap_{n=1}^{\infty} V_{in}$. Put $W_n = \bigcup_{i=1}^{i=m} V_{in}$. Then $F = \bigcap_{n=1}^{\infty} W_n$. In fact, it is obvious that $F \subset \bigcap_{n=1}^{\infty} W_n$. We only need to show that $\bigcap_{n=1}^{\infty} W_n \subset F$. Suppose not, let $x \in \bigcap_{n=1}^{\infty} W_n \setminus F$. Since $x \notin F_i$ for each $1 \leq i \leq m$, there exists a $k_i \in \mathbb{N}$ such that $x \notin V_{ik_i}$. Let $l = \max\{k_i : 1 \leq i \leq m\}$. Then $x \notin \bigcup_{i=1}^{m} V_{il} = W_l$, which is a contradiction with $x \in W_l$.

By the Claim 9 and Theorem 6.18 $Y$ is first countable. Therefore, $G$ and $bG$ are separable and metrizable spaces by Theorem 6.20. \qed

Finally, we discuss the remainders of rectifiable spaces.

**Lemma 6.22.** [5] Suppose that $G$ is a paracompact rectifiable space, and $Y = bG \setminus G$ has a $G_\delta$-diagonal if and only if $Y$, $G$ and $bG$ are separable and metrizable spaces.

**Theorem 6.23.** Suppose that $G$ is a paracompact rectifiable space, and $Y = bG \setminus G$ has locally a $G_\delta$-diagonal if and only if $Y$, $G$ and $bG$ are separable and metrizable spaces.

**Proof.** Claim 10: The space $Y$ is Lindelöf.

We can assume that $G$ is non-locally compact, and otherwise, the space $Y$ is compact.

Case 1: The space $Y$ is countably compact.

Since $Y$ has locally a $G_\delta$-diagonal, the space $Y$ is locally metrizable, and hence $Y$ is locally compact, which is a contradiction with $Y$ is nowhere locally compact.

Case 2: The space $Y$ is non-countably compact.

By [5] Theorem 3.1, the space $Y$ is Lindelöf or pseudocompact. So we assume that $Y$ is pseudocompact. Since each point in $Y$ is a $G_\delta$-point, the space $Y$ is first countable. Since $Y$ is non-countably compact, it follows, by a standard argument, that $G$ has a countable $\pi$-base at some point which is an accumulation point of some countable subset of $Y$. Hence, the space $G$ is metrizable [14]. Therefore, the space $Y$ is Lindelöf, since $G$ is of countable type.

It follows from [20] Lemma 11) and the Claim 10 that $Y$ has a $G_\delta$-diagonal, and therefore, it is easy to see that the theorem is verified by Lemma 6.22. \qed

**Theorem 6.24.** Suppose that $G$ is a rectifiable space, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has countable pseudocharacter, then at least one of the following conditions satisfies

1. the space $G$ is a strong $p$-space;
2. the space $Y$ is first countable.
Proof. One can prove in a similar way in Theorem 6.4 that $G$ is of countable type or $Y$ is first countable. By [5, Corollary 2.8], it follows that $G$ is a strong $p$-space or $Y$ is a first countable space. □

The following question is still open.

**Question 6.25.** Let $G$ be a topological group. If $G$ has a first countable remainder, is $G$ metrizable?

It is well known that there exists a non-metrizable paratopological group with a first countable remainder. In fact, Alexandorff’s double-arrow space is a Hausdorff compactification of Sorgenfrey line, its reminder is still a copy of Sorgenfrey line. However, we have the following question.

**Question 6.26.** Let $G$ be a rectifiable space. If $G$ has a first countable remainder, is $G$ metrizable?

Next we give some partial answers to this question.

The proofs of the following two theorem are identical to the proofs of Theorem 2.1 and 2.2 in [4], respectively.

**Theorem 6.27.** Let $G$ be a non-compact rectifiable space such that $G^\omega$ has a first countable remainder. Then $G$ is metrizable.

**Theorem 6.28.** A rectifiable space $G$ is metrizable if there is a nowhere locally compact metrizable space (or first countable space) $M$ such that the product space $G \times M$ has a first countable remainder.

7. Open problems

Here, we list some open problems about rectifiable spaces, which mainly appear in [5] and [7].

**Question 7.1.** [7, Open problem 5.7.6] Suppose that $G$ is a (regular, Tychonoff) paratopological group which is also a rectifiable space. Is $G$ homeomorphic to a topological group?

**Question 7.2.** [7, Open problem 5.7.7] Is every regular rectifiable space Tychonoff?

**Question 7.3.** [7, Open problem 5.7.8] Is every regular rectifiable space of countable pseudocharacter submetrizable? Is it Tychonoff?

**Question 7.4.** [5, Problem 5.9] Is every rectifiable $p$-space paracompact? What if the space is locally compact?

**Question 7.5.** [5, Problem 5.10] Is every rectifiable $p$-space with a countable Souslin number Lindelöf? What if we assume the space to be separable? Separable and locally compact?

**Question 7.6.** [5, Problem 5.11] Is every rectifiable $p$-space a $D$-space?

**Question 7.7.** Is every sequential rectifiable space with a point-countable $k$-network a paracompact space?

It is easy to see that Corollary 4.3 gives a partial answer to this question.

**Question 7.8.** Let $G$ be a rectifiable. If $F, P$ are compact and closed subsets of $G$ respectively, is $P \cdot F$ or $F \cdot P$ closed in $G$?
Obviously, if both $F, P$ are compact subset of $G$, then $P \cdot F$ and $F \cdot P$ are compact in $G$ since $p$ is a continuous map from $G \times G$ onto $G$.

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Fucai Lin (corresponding author): Department of Mathematics and Information Science, Zhangzhou Normal University, Zhangzhou 363000, P. R. China
E-mail address: linfucai2008@yahoo.com.cn

Rongxin Shen: Department of Mathematics, Taizhou Teacher’s College, Taizhou 225300, P. R. China
E-mail address: srx20212021@163.com