Uncertainty Relations and Wave Packets on the Quantum Plane

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Abstract

(2+2)-dimensional quantum mechanical q-phase space which is the semi-direct product of the quantum plane $E_q(2)/U(1)$ and its dual algebra $e_q(2)/u(1)$ is constructed. Commutation and the resulting uncertainty relations are studied. “Quantum mechanical q-Hamiltonian” of the motion over the quantum plane is derived and the solution of the Schrödinger equation for the q-semiclassical motion governed by the expectation value of that Hamiltonian is solved.

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1 Introduction

How the quantum mechanical effects are altered if we replace the space–time continuum by a non–commutative geometry is an interesting question. Especially the phenomena which are directly related to the geometry of the environment, such as the Casimir effect, Aharonov–Bohm effect and the cosmological pair production may exhibit peculiarities. To obtain the required mathematical tools to study the above mentioned physical effect a program dealing with the Green functions for q–group spaces has already started [1], [2].

Another consequence of the non–commutative nature of the geometry is the changes occurring in the structure of the quantum mechanical phase space: What are the new commutation relations and the uncertainty relations? There exists an extensive literature on the commutation relations between the position and momentum operators resulting from the quantum group related associative algebras and the corresponding new uncertainty relations [3]. Since most of the above mentioned literatures employ 1–dimensional position operators the corresponding configuration spaces are commutative. The operators act on the Hilbert space of the square integrable functions of the commutative coordinates or momenta.

In this paper we study the case of a 2–dimensional non–commutative configuration space $E_q^2$ which is obtained from the space of q–Euclidean group $E_q(2)$. We have mutually non–commutative position operators $\hat{x}$ and $\hat{y}$ (and thus mutually non–commutative momentum operators $\hat{p}_x$, $\hat{p}_y$). They act on the Hilbert space $H(E_q^2)$ of square integrable functions of non–commutative coordinates $x$ and $y$. The latter can be realized in the Hilbert space $H(S)$ of square integrable functions on the circle $S$.

In the following section we present our $(2+2)$–dimensional quantum mechanical q–phase space.

In Section III the wave packets are obtained and the uncertainty relations are studied.

In Section IV the free motion Hamiltonian $\hat{H}$ which commutes with $\hat{p}_x$, $\hat{p}_y$ and the angular momentum like operator $\hat{l}$ is introduced. We then write the q–expectation value of this Hamiltonian, and then considering the expectation value of $\hat{p}_x$, $\hat{p}_y$ and $\hat{l}$ as continuously varying numbers obtain a q–semiclassical Hamiltonian $H$. Schrödinger equation corresponding to this q–semiclassical Hamiltonian is also solved.
All the required background on the quantum groups and algebras are presented in the Appendix.

2 (2+2)–Dimensional Quantum Mechanical q–Phase Space

We start with defining the basic concepts that we employ. All the known definitions related to the quantum group $E_q(2)$ and its dual $e_q(2)$ are given in the Appendix.

(i) q–configuration space $E^2_q$ is the subspace of the quantum group $E_q(2)$ defined as

$$E^2_q = \sum_{j \in \mathbb{Z}} \oplus E[j, 0],$$

(II.1)

where $E[i, j]$ is given by (A.8). In other words q–configuration space is the left sided coset space $E_q(2)/U(1)$. One can show that $E^2_q$ is generated by $z_{\pm}$, which satisfy the commutation relation

$$[z_+, z_-] = (1 - q^2)z_+ z_-.$$  

(II.2)

We can also define the cartesian $x, y$ and the polar $r, o$ coordinates as

$$x = \frac{1}{\sqrt{2}}(z_+ + z_-), \quad y = \frac{1}{\sqrt{2i}}(z_+ - z_-)$$

(II.3)

and

$$z_+ = re^{io}, \quad z_- = e^{-io}r$$

(II.4)

respectively. By the virtue of (II.2) we get the commutation relations

$$[x, y] = i \tanh \Lambda (x^2 + y^2),$$

(II.5)

$$[r, o] = i\Lambda r,$$

(II.6)

where $\Lambda = \log q^{-1}, \quad q < 1$. Involution are

$$x^* = x, \quad y^* = y.$$  

(II.7)
Inspecting the commutation relations (II.5) and (II.6) we conclude that for larger radial distance one observes larger non–commutativity. The operators \( z_\pm \) and consequently \( x, y \) and \( r, o \) are defined in the Hilbert space \( H(S) \) (B.1). In a fashion parallel to the quantum mechanics we can interpret \( x, y \) as observables on the states in \( H(S) \). The commutative coordinates \( x \) and \( y \) are the expectation values of the self–adjoint operators \( x \) and \( y \):

\[
x = (u, xu), \quad y = (u, yu),
\]

where \( u \in H(S) \).

(ii) q–momentum space \( e^2_q \) is the quotient algebra \( e_q(2)/u(1) \). It is generated by \( p_\pm \), which by the virtue of (A.10) satisfy the commutation relation

\[
[p_+, p_-] = (1 - q^2)p_+ p_-.
\]

q–momentum space \( e^2_q \) and q–configuration space \( E^2_q \) are in duality. We define the momenta \( p_x, p_y \) in cartesian coordinates as

\[
p_x = \frac{1}{\sqrt{2}}(p_+ + p_-), \quad p_y = \frac{i}{\sqrt{2}}(p_+ - p_-).
\]

By the virtue of (II.10) and (A.11) we get the commutation relation

\[
[p_x, p_y] = i \tanh \Lambda (p_x^2 + p_y^2)
\]

and the involution

\[
p_x^* = p_x, \quad p_y^* = p_y.
\]

We observe that \( p_x \) and \( p_y \) satisfy the same commutation relations as \( x \) and \( y \). Therefore the elements of the q–momentum space are operators in the Hilbert space \( H(S) \). In the orthonormal basis (B.2) we have

\[
p_+ e_j = p_0 q^{-j+\frac{1}{2}} e_{j-1}, \quad p_- e_j = p_0 q^{-j-\frac{1}{2}} e_{j+1},
\]

where \( p_0 \) is a constant of dimension of momentum. The commutative momenta \( p_x, p_y \) are the expectation values of the operators \( p_x \) and \( p_y \) in \( H(S) \).
q–phase space $E^2_q \otimes e^2_q$ is the direct product of q–configuration and q–momentum spaces. It is realized in the Hilbert space $H(S) \otimes H(S)$.

(iii) “quantum mechanical q–phase space” $E^2_q \ast e^2_q$ is the semi–direct product of the q–configuration and q–momentum spaces. The position $\hat{z}_\pm$ and momentum $\hat{p}_\pm$ operators are defined in the Hilbert space $H(E^2_q)$ as

$$\hat{z}_\pm f = f z_\pm, \quad \hat{p}_\pm f = \hbar \mathcal{R}(p_\pm) f,$$

where $f \in \Phi(E^2_q) \subset H(E^2_q)$ and $\mathcal{R}$ is the right representation (B.9) of the quantum algebra $e_q(2)$ corresponding to the left quasi–regular representation (B.8) of the quantum group $E_q(2)$. Here $\Phi(E^2_q)$ is dense subset in $H(E^2_q)$ defined in (B.3). The position $\hat{x}$, $\hat{y}$ and momentum $\hat{p}_x$, $\hat{p}_y$ operators in cartesian coordinates are given by

$$\hat{x} f = f x, \quad \hat{y} f = f y, \quad \hat{p}_x f = \hbar \mathcal{R}(p_x) f, \quad \hat{p}_y f = \hbar \mathcal{R}(p_y) f,$$

where $f \in \Phi(E^2_q)$ and $\hbar$ is the Planck constant. The quasi–regular representation (B.8) is unitary with respect to the scalar product (B.7). Unitarity implies that the operators $\hat{p}_x$ and $\hat{p}_y$ are at least symmetric in the Hilbert space $H(E^2_q)$. We assume that they have self–adjoint extensions. Using (B.15) we have the following realization of the self–adjoint operators $\hat{p}_x$ and $\hat{p}_y$

$$\hat{p}_x f(z_+, z_-) = \frac{i \hbar}{\sqrt{2}} (D^{z_+}_{q^{-2}} f(z_+, z_-) + D^{z_-}_{q^2} f(q^{-2} z_+, z_-))$$

$$\hat{p}_y f(z_+, z_-) = \frac{\hbar}{\sqrt{2}} (D^{z_+}_{q^{-2}} f(z_+, z_-) - D^{z_-}_{q^2} f(q^{-2} z_+, z_-)),$$

where $f \in \Phi(E^2_q)$ is regular function of $z_+$ and $z_-$ with the ordering (B.19). The commutation relations satisfied by the elements of the “quantum mechanical q–phase” space are

$$[\hat{x}, \hat{y}] = -i \tanh \Lambda (\hat{x}^2 + \hat{y}^2), \quad [\hat{p}_x, \hat{p}_y] = -i \tanh \Lambda (\hat{p}_x^2 + \hat{p}_y^2)$$

$$[\hat{p}_x, \hat{x}] = [\hat{p}_y, \hat{y}] = i \hbar e^{\frac{A \Lambda i}{\hbar}}, \quad [\hat{p}_x, \hat{y}] = [\hat{p}_y, \hat{x}] = 0,$$
where $\hat{l}$ is a function of $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ which acts in $\Phi(E^2_q)$ as
\[
\hat{l}(z_+, z_-) = i\hbar [z_+ \partial_{z_+} f(z_+, z_-) - z_- \partial_{z_-} f(q^{-2}z_+, z_-)].
\] (II.21)

It satisfies the commutation relations
\[
[\hat{l}, \hat{y}] = -i\hbar \hat{x}, \quad [\hat{l}, \hat{x}] = i\hbar \hat{y},
\] (II.22)
\[
[\hat{l}, \hat{p}_x] = i\hbar \hat{p}_y, \quad [\hat{l}, \hat{p}_y] = -i\hbar \hat{p}_x.
\] (II.23)

From the second commutation relations of (II.19) we see that for higher energies the non–commutativity between the momentum operators is bigger. Note that the above commutation relations (II.22) and (II.23) are the familiar quantum mechanical relations if $\hat{l}$ is interpreted as the angular momentum operator $\hat{p}_x \hat{y} - \hat{p}_y \hat{x}$ in $q \to 1$ limit. Inspection of the commutation relations (II.20) reveals that only the section of the Hilbert space on which the expectation value of $\hat{l}$ is non–negative is acceptable. Otherwise we may have smaller (even zero ) than the usual non–commutativity between the momenta and position operators which is unacceptable. This problem however is specific to the two dimensional nature of the space. In three dimensions we would not have such a unphysical situation.

Using (II.4) and (B.19) we can represent an arbitrary element $f \in \Phi(E^2_q)$ as the infinite series
\[
f(r, o) = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} c_{nj} r^n e^{ij o}
\] (II.24)
subject to the square integrability condition (B.3). The position $\hat{r}$ and $\hat{o}$ operators in the polar coordinate system are given by
\[
\hat{r} f = f r, \quad \hat{o} f = f o, \quad f \in \Phi(E^2_q).
\] (II.25)

The radial momentum operator $\hat{p}$ is expressed as
\[
\hat{p} = (\hat{p}_+ e^{i o} + e^{-i o} \hat{p}_-).
\] (II.26)

By the virtue of (B.13) we get the following realization of $\hat{p}$
\[
\hat{p} f(r, o) = \frac{i\hbar}{1 + q} (qD_q^r + q^{-1} D_{q^{-1}}^r) f(r, o) + \frac{i\hbar}{(1 + q) r^2} f(r, o)
\] (II.27)
in the Hilbert space $H(E_q^2)$. Putting
\[ f(r, o) = r^{-1/2}\Psi(r, o) \] (II.28)
in (II.27) we get the following realization of the radial momentum operator $\hat{p}$
\[ \hat{p}\Psi(r, o) = i\hbar\hat{D}_q^r\Psi(r, o) \] (II.29)
in the Hilbert space $H'(E_q^2)$. The scalar product in $H'(E_q^2)$ is
\[ (\Psi, \Psi') = \psi(r^{-1}\Psi\Psi'^*) \] (III.30)
where $\psi(\cdot)$ is the invariant integral on $E_q(2)$ given by (B.7). In (II.29) $\hat{D}_q^r$ is
$q$–derivative defined as
\[ \hat{D}_q^r\Psi(r) = \frac{\Psi(qr) - \Psi(q^{-1}r)}{(q - q^{-1})r}. \] (II.31)
For the radial momentum operator we have the following commutation relations
\[ [\hat{p}, \hat{r}] = i\hbar\hat{a}, \quad [\hat{p}, \hat{o}] = 0, \] (III.32)
where $\hat{a}$ is the self–adjoint operator in $H'(E_q^2)$ defined as
\[ \hat{a}\Psi(r, o) = \frac{q^{1/2}\Psi(qr, o) + q^{-1/2}\Psi(q^{-1}r, o)}{(q^{1/2} + q^{-1/2})}. \] (II.33)

3 Uncertainty Relations and Wave Packets

Let $X_1, X_2$ be the self–adjoint operators satisfying the commutation relation
\[ [X_1, X_2] = iX_3. \] (III.1)
The uncertainty relation for operators $X_1, X_2$ in the state $\psi$ is given by
\[ (\Delta X_1)(\Delta X_2) \geq \frac{1}{2} | (\psi, \{X_1', X_2'\}\psi) + i(\psi, X_3\psi) |, \] (III.2)
where
\[ X_j' = X_j - (\psi, X_j\psi); \quad j = 1, 2; \] (III.3)
and \(\{\cdot, \cdot\}\) is the anticommutator. The equality sign in \((\text{III.}2)\) holds if the condition
\[
X_1'\psi = cX_2'\psi
\]
(\text{III.4})
is fulfilled. Here \(c\) is some complex number. If \(c\) is pure imaginary, that is if \(c^* = -c\) the first term of the right hand side of \((\text{III.}2)\) vanishes. We then have
\[
(\Delta X_1)(\Delta X_2) \geq \frac{1}{2} | (\psi, X_3\psi) |; \quad c^* = -c.
\]
(\text{III.5})
The corresponding state \(\psi\) is called the wave packet.

(i) **uncertainty relation for position operators**
Consider the position operators \((\text{II.}25)\) in the polar coordinates. One can not have the regular solutions of \((\text{III.}4)\). It is not difficult to check that the expectation value of the anticommutator \(\{\hat{r}', \hat{o}'\}\) is zero on the vectors \(f(r) \in \Phi(E_q^2)\) depending only of \(r\), that is
\[
(\Delta \hat{r})(\Delta \hat{o}) \geq \frac{\Lambda}{2} | (f, \hat{r}f) |,
\]
(\text{III.6})
where the scalar product \((\cdot, \cdot)\) is given by \((\text{B.}7)\). We can minimize the right hand side of \((\text{III.}6)\) on the gaussian
\[
f(r) = \sqrt{\frac{2\log q^2}{q^2 - 1}} e^{-\frac{r^2}{\varepsilon}}
\]
(\text{III.7})
as
\[
(\Delta \hat{r})(\Delta \hat{o}) \geq \frac{\Lambda}{2} \sqrt{\frac{\pi}{2}} \varepsilon.
\]
(\text{III.8})
Making \(\varepsilon\) very small we can decrease the uncertainty between the position operators \(\hat{r}\) and \(\hat{o}\). Note that we cannot put \(\varepsilon = 0\) since otherwise the wave packet \((\text{III.}7)\) becomes a non–regular function (precisely generalized function).

(ii) **Uncertainty relation for momentum operators**
In this case the condition \((\text{III.}4)\) reads
\[
(\hat{p}_x - c\hat{p}_y)f = (\langle p_x \rangle - c\langle p_y \rangle)f.
\]
(\text{III.9})
For \( c = i \) and \( c = -i \) the above equation becomes

\[
\hat{p}_+ f_{\hat{p}_+} = \langle \hat{p}_+ \rangle f_{\hat{p}_+} \tag{III.10}
\]

and

\[
\hat{p}_- f_{\hat{p}_-} = \langle \hat{p}_- \rangle f_{\hat{p}_-} \tag{III.11}
\]

respectively. By making use of (B.15) we get the solutions

\[
f_{\hat{p}_+} = N_+ e^{\frac{i}{\hbar}(\hat{p}_+ z + \hat{p}_- z)} = N_+ e^{\frac{i}{\hbar}(\hat{p}_+ + \hat{p}_- + \hat{p}_- \hat{p}_+)}; \quad c = i \tag{III.12}
\]

and

\[
f_{\hat{p}_-} = N_- e^{\frac{i}{\hbar}(\hat{p}_- z - \hat{p}_+ z)} = N_- e^{\frac{i}{\hbar}(\hat{p}_- + \hat{p}_+ \hat{p}_-)}; \quad c = -i. \tag{III.13}
\]

Here \( N_\pm \) are normalization constants. The above wave functions are the \( q \)-analogues of the plane waves. They are not square integrable and therefore do not belong to the Hilbert space \( H(E^2_q) \). The wave packets are given by

\[
f = \int db \ c(b) f_b, \tag{III.14}
\]

where \( c(b) \) is regular function of \( b \) with the main value in the neighborhood of \( \langle \hat{p}_+ \rangle \) or \( \langle \hat{p}_- \rangle \) according to the choice (III.12) and (III.13). We have the following uncertainty relations

\[
(\Delta \hat{p}_x)(\Delta \hat{p}_y) \geq \frac{\text{tanh} \Lambda}{2} |(f, (\hat{p}_x^2 + \hat{p}_y^2)f)| \sim \Lambda \langle \hat{p}_+ \hat{p}_- \rangle. \tag{III.15}
\]

(iii) \( (\Delta \hat{p})(\Delta \hat{r}) \) uncertainty

It is very difficult to solve (II.4) for \( \hat{p}_x, \hat{x} \) and \( \hat{p}_y, \hat{y} \) pairs. On the other hand the commutation relations (II.19), (II.20) display a circular symmetry. We then consider it is more meaningful to study the uncertainty relation between the radial position \( \hat{r} \) and momenta \( \hat{p} \) defined in (II.27) and (II.28). For these operators the condition (II.4) reads

\[
(\hat{p} - c\hat{r})\Psi(r, o) = (\langle \hat{p} \rangle - c\langle \hat{r} \rangle) \Psi(r, o) = d\Psi(r, o), \tag{III.16}
\]
where \( \Psi(\mathbf{r}, \mathbf{o}) \) is the vector from the Hilbert space \( H'(E_q^2) \) with scalar product (II.30). By the virtue of (II.29) we have

\[
i\hbar \hat{D}_q^r \Psi(\mathbf{r}, \mathbf{o}) - c \mathbf{r} \Psi(\mathbf{r}, q^{-1} \mathbf{o}) = d\Psi(\mathbf{r}, \mathbf{o}),
\]

(III.17)

where \( \hat{D}_q^r \) is the q–derivative defined in (II.31). For the sake of simplicity let us consider the case \( d = 0 \). The square integrable solution of the equation (III.17) exists for \( c = -i |c| q^{-j} \) which is given by

\[
\Psi_j(\mathbf{r}, \mathbf{o}) = NE_q^2 (-\frac{|c| q^{-j}}{(1+q)\hbar} \mathbf{r}^2) e^{i j \phi}, \quad j = 0, \ 1, \ 2, \ldots,
\]

(III.18)

where \( N \) is the normalization constant and \( E_q^2(x) \) is the q–exponential defined as

\[
E_q^2(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q^2}!}.
\]

(III.19)

The q–factorial \([n]_{q^2}!\) is

\[
[n]_{q^2}! = [1]_{q^2}[2]_{q^2} \cdots [n]_{q^2}, \quad [n]_{q^2} = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}.
\]

(III.20)

Note that the solution (III.18) is the eigenfunction of the angular momentum operator \( \hat{l} \) corresponding to the e–value \( j\hbar \):

\[
\hat{l}\Psi_j(\mathbf{r}, \mathbf{o}) = \hbar j \Psi_j(\mathbf{r}, \mathbf{o}).
\]

(III.21)

Note that as we already discussed in the paragraph following (II.22) physically only the chiral states with positive \( j \) values are acceptable. The minimal uncertainty relation between the radial momentum and position is

\[
(\Delta \hat{p})(\Delta \mathbf{r}) \geq \frac{\hbar}{2} |(\Psi_j, \hat{a} e^{\frac{j}{\hbar} \hat{\mathbf{r}}})|,
\]

(III.22)

where \( \hat{a} \) is the self–adjoint operator defined in (II.33). One can verify that in \( q \to 1 \) limit the first contribution from the operator \( \hat{a} \) to the usual Heisenberg uncertainty relation is of order \( \Lambda^2 \). Therefore we can put \( \hat{a} = 1 \) in the right hand side of the above inequality to obtain the correction of the order \( \Lambda \). We then have

\[
(\Delta \hat{p})(\Delta \mathbf{r}) \geq \frac{\hbar}{2} |1 + \Lambda j|.
\]

(III.23)
4 Free q–Semi–Classical Motion over $E_q^2$

The dynamical model we have on the q–configuration space $E_q^2$ employs classical time $t$. We now define the "quantum mechanical q–Hamiltonian" which commutes with $\hat{p}_x$, $\hat{p}_y$ and $\hat{l}$:

$$[\hat{H}, \hat{p}_x] = [\hat{H}, \hat{p}_y] = [\hat{H}, \hat{l}] = 0. \quad (IV.1)$$

It is easy to check that the following operator

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2)e^{-\frac{2\Lambda}{\hbar}} \quad (IV.2)$$

satisfies the desired property (IV.1), where $m$ is the mass of the particle moving on $E_q^2$. We further assume that the commutator of any operator with $\hat{H}$ gives the time evolution of that operator. Thus the time evolution of the position operators $\hat{x}$, $\hat{p}$ are

$$\frac{d}{dt}\hat{x} = \frac{1}{2m}\left[(q + q^{-1})^2\hat{p}_x + \frac{q^2 - q^{-2}}{2}\hat{p}_y\right] + \frac{i\hbar}{2}(q - q^{-1})^2 \hat{x} + \frac{q^2 - q^{-2}}{2}\hat{y} \quad (IV.3)$$

$$\frac{d}{dt}\hat{y} = \frac{1}{2m}\left[(q + q^{-1})^2\hat{p}_y - \frac{q^2 - q^{-2}}{2}\hat{p}_x\right] + \frac{i\hbar}{2}(q - q^{-1})^2 \hat{y} - \frac{q^2 - q^{-2}}{2}\hat{x}. \quad (IV.4)$$

q–Hamiltonian $H$ corresponding to (IV.2) is given by

$$H = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2)e^{-\frac{2\Lambda}{\hbar}}. \quad (IV.5)$$

$H$ is the element of the q–phase space $E_q^2 \otimes \mathbb{C}_q^2$ defined in the subsection (ii) of Sec. II. It acts in the Hilbert space $H(S \times S)$.

We define the q–semiclassical Hamiltonian $H$ as the expectation value of $H$ in the Hilbert space $H(S \times S)$

$$H = (u \otimes v, H u \otimes v), \quad (IV.6)$$

where $u$ is the state from $H(S)$ on which the uncertainty between $x$ and $y$ is minimized and $v$ is the state on which the uncertainty between $p_x$ and $p_y$ is minimized. They are given by

$$u(\phi) = Ne^{-(\theta - \phi)^2/2\Lambda - i\hbar\Lambda/\alpha(\phi - \theta)} \quad (IV.7)$$
and
\[ v(\phi) = Ne^{-(\phi-\phi')^2 + \log \frac{p}{\Lambda}(\phi-\phi')}, \]  
(respectively. On this states we have
\[ x = (u, xu) = r \cos \theta, \quad y = (u, xu) = r \sin \theta \]  
and
\[ p_x = (v, pxv) = p \cos \theta', \quad p_y = (u, pyu) = p \sin \theta'. \]  
It may be suggestive to sketch the quantum mechanical motion governed by
the q–semiclassical Hamiltonian
\[ H = \frac{1}{2m}(p_x^2 + p_y^2)e^{-\frac{2\Lambda t}{\hbar}}, \]  
where \( p_x, p_y \) and \( l = xp_y - yp_x \) are the classical variables. The Schrödinger
equation corresponding to the above q–semiclassical Hamiltonian, when it is
written in plane–polar coordinates is
\[ -\hbar^2 \frac{1}{2m}(\frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_\theta^2)e^{-i\Lambda \theta_0} \psi(r, \theta, t) = \frac{i}{\hbar} \psi(r, \theta, t) \]  
solved by
\[ \psi_j(r, \theta, t) = e^{-i\epsilon t} \frac{1}{\sqrt{2\pi}} e^{i j \theta} j_\frac{2m \epsilon}{\hbar} e^{-\Lambda j/2r}. \]  
where \( \epsilon = \frac{1}{2m}(p_x^2 + p_y^2). \) The above Schrödinger equation and its solution
exhibit the peculiarity which is due to the non–commutative nature of plane
where semi q–classical motion we are considering is originated. For states
with \( j = 0 \) the motion is exactly same as usual one with \( q = 1. \) However
for particles with angular momentum \( j \) the radial distance is shorter for the
chiral states corresponding to \( j > 0 \)
\[ r \to e^{-\Lambda j/2r} = q^{j/2}r, \quad \text{for} \quad q < 1 \]  
5 Discussions

Non–commutativity of the coordinates of the quantum plane is stronger for
larger radial distances (see (II.5) and (II.6)). Since the value of \( q \) is supposed
to be very close to 1, the non–commutativity of $\hat{x}$, $\hat{y}$ coordinate operators is negligible if the expectation value of $(\hat{x}^2 + \hat{y}^2)$ is small. Thus we can say that in the vicinity of the point where the observer is located the plane is almost classical. For every observer the non–commutativity is same for the points of plane corresponding to the same expectation value of $(\hat{x}^2 + \hat{y}^2)$. In other words quantum plane is isotropic.

Non–commutativity of $\hat{p}_x$ and $\hat{p}_y$ on the other hand can be appreciable at extremely high energies. We can then say that this aspect of q–quantization should also manifest itself at very small distances especially of the order of Planck scale.

Inspecting (III.23) we conclude that non–commutativity of $\hat{p}_x$ and $\hat{x}$ (and similarly $\hat{p}_y$ and $\hat{y}$) differs from the usual one only for very large values of angular momentum quantum number $j$.

**Appendix [4]**

A. Quantum Group $E_q(2)$ and it’s Dual $e_q(2)$

The Hopf algebra generated by $z_\pm$ and $n_{\mp}^\pm$ with relations

$$ z_+ z_- = q^{-2} z_- z_+, \quad z_\pm n = q^2 n z_\pm, \quad (A.1) $$

involutions

$$ n^* = n^{-1}, \quad z_{\mp}^* = z_\mp \quad (A.2) $$

and group operations

$$ \Delta(z_\pm) = z_\pm \otimes 1 + n_{\mp}^\pm \otimes z_\pm, \quad \Delta(n_{\mp}^\pm) = n_{\mp}^\pm \otimes n_{\mp}^\pm, \quad (A.3) $$

$$ \varepsilon(n_{\mp}^\pm) = 1, \quad \varepsilon(z_\pm) = 0, \quad (A.4) $$

$$ S(n_{\mp}^\pm) = n_{\mp 1}^\pm, \quad S(z_\pm) = -n_{\mp 1} z_\pm. \quad (A.5) $$

is called the quantum Euclidean group $E_q(2)$. The homomorphism

$$ \phi_K(z_\pm) = 0, \quad \phi_K(n_{\mp}^\pm) = t^{\pm 1} \quad (A.6) $$

defines the quantum subgroup $U(1)$ of $E_q(2)$. We have the decomposition

$$ E_q(2) = \sum_{i,j \in \mathbb{Z}} \oplus E[i,j] \quad (A.7) $$
of $E_+(2)$ with respect to its subgroup $U(1)$. Here $E[i,j]$ are the subspaces of $E_+(2)$ defined as

$$E[i,j] = \{ f \in E_+(2) : L_K(f) = t^i \otimes f; \ R_K(f) = f \otimes t^j \}, \quad (A.8)$$

where

$$L_K = (\phi_K \otimes id) \circ \Delta, \quad R_K = (id \otimes \phi_K) \circ \Delta. \quad (A.9)$$

The quantum algebra $e_+(2)$ is the Hopf algebra generated by $p_\pm$ and $k^{\pm 1}$ satisfying the relations

$$p_+ p_- = q^{-2} p_- p_+, \quad p_\pm k = q^{\pm 2} k p_\pm, \quad (A.10)$$

involutions

$$p_\pm^* = p_{\mp}, \quad k^* = k \quad (A.11)$$

and co–algebra operations

$$\Delta(p_\pm) = p_\pm \otimes 1 + k \otimes p_\pm, \quad \Delta(k^\pm) = k^\pm \otimes k^\pm, \quad (A.12)$$

$$S(p_\pm) = -k^{-1} p_\pm, \quad S(k^\pm) = k^{\mp 1}, \quad (A.13)$$

$$\varepsilon(p_\pm) = 0, \quad \varepsilon(k) = 1. \quad (A.14)$$

The quantum algebra $e_+(2)$ and the quantum group $E_+(2)$ are in duality. The duality bracket is given by

$$\langle p_+^{n'} p_-^{k'} k^{j'} | z_+^n z_-^k n^j \rangle = \frac{i^{m+k}(1-q^2)^n(1-q^{-2})^k}{(q^2; q^2)_n(q^{-2}; q^{-2})_k q^{-2j'k' - m - n}} \delta_{nn'} \delta_{kk'} \quad (A.15)$$

with $j', j \in \mathbb{Z}$ and $n, n', k$ and $k'$ being positive integers. Quantum subalgebra $u(1)$ of $e_+(2)$ corresponding to the quantum subgroup $U(1)$ of $E_+(2)$ is generated by $k^{\pm 1}$.

**B. Representation of $E_+(2)$ and it’s Dual $e_+(2)$**

We have the following realization of commutation relations (A.1)

$$z_+ = r_0 e^{-i \Lambda \frac{d}{d \phi} - i \phi}, \quad z_- = r_0 e^{-i \Lambda \frac{d}{d \phi} + i \phi}, \quad n = e^{-2i \phi} \quad (B.1)$$
(with $q = e^{-\Lambda}$) in the Hilbert space $H(S)$ of the square integrable functions on the circle $S^1$. In the above equation $r_0$ is a constant of dimension of length. In the orthonormal basis $e_j = \frac{1}{\sqrt{2\pi}} e^{ij\phi}$, $-\infty < j < \infty$ we have

$$z_+ e_j = r_0 q^{-j+\frac{1}{2}} e_{j-1}, \quad z_- e_j = r_0 q^{-j-\frac{1}{2}} e_{j+1}, \quad n^\pm e_j = e_{j\pm 2}. \quad (B.2)$$

Denote by $\Phi(E_q(2))$ the space of regular functions of $z_\pm$ and $n_\pm$ with finite norm

$$\| f \| < \infty, \quad f \in \Phi(E_q(2)), \quad (B.3)$$

where

$$\| f \| = \sqrt{\psi(ff^*)} \quad (B.4)$$

with $\psi$ being the invariant integral on $E_q(2)$ defined as

$$\psi(f) = (1 - q^2) \sum_{j=-\infty}^{\infty} (e_j, \xi f e_j), \quad \xi = z_+ z_- \quad (B.5)$$

If $f$ is only the function of $\xi$ the above expression can be rewritten by means of the $q$–integral as

$$\psi(f) = \int_0^{\infty} f(\xi)d_q \xi. \quad (B.6)$$

The space $\Phi(E_q(2))$ can be equipped with the scalar product

$$(f', f) = \psi(f'f^*). \quad (B.7)$$

In a similar manner we define the space $\Phi(E_q^2)$. It consists of the regular functions of coordinates $z_\pm$ satisfying the square integrability condition (B.3). Completing $\Phi(E_q^2)$ in the norm (B.4) we arrive at the Hilbert space $H(E_q^2)$.

The comultiplication

$$\Delta : \Phi(E_q^2) \to \Phi(E_q(2)) \otimes \Phi(E_q^2) \quad (B.8)$$

defines a left quasi–regular representation of the quantum group $E_q(2)$ in $\Phi(E_q^2)$. Since the scalar product in $\Phi(E_q^2)$ is defined by means of invariant integral this representation is unitary.

The right representation $\mathcal{R}$ of the quantum algebra $e_q(2)$ corresponding to the left quasi–regular representation (B.8) of $E_q(2)$ is given by

$$\mathcal{R}(\phi)f = (\phi \otimes id) \circ \Delta(f), \quad f \in \Phi(E_q^2), \quad (B.9)$$

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where $\phi \in e_q(2)$ and

$$
(\phi \otimes \text{id}) \circ \Delta(f) = \langle \phi | f_j \rangle f'_j, \quad \Delta(f) = f_j \otimes f'_j.
$$

(B.10)

Here $\langle \cdot | \cdot \rangle$ is duality bracket given in (A.13). The linear space $\Phi(E^2_q)$ is common invariant dense domain for the set of linear operators $\mathcal{R}(\phi)$, $\phi \in e_q(2)$. Since the representation (B.8) of $E_q(2)$ is unitary we have

$$
(\mathcal{R}(\phi)f, f') = (f, \mathcal{R}(\phi^*)f'), \quad f \in \Phi(E^2_q).
$$

(B.11)

Some of the relations satisfied by the right representation $\mathcal{R}$ are

$$
\mathcal{R}(\phi \phi') = \mathcal{R}(\phi') \mathcal{R}(\phi)
$$

(B.12)

and

$$
\mathcal{R}(p_{\pm})(f f') = \mathcal{R}(p_{\pm}) ff' + \mathcal{R}(k)f \mathcal{R}(p_{\pm})f',
$$

(B.13)

$$
\mathcal{R}(k)(f f') = \mathcal{R}(k)f \mathcal{R}(k)f'.
$$

(B.14)

We further have

$$
\mathcal{R}(p_+)(z_+, z_-) = iD_{q^{2}} f(z_+, z_-), \quad \mathcal{R}(p_-)(z_+, z_-) = iD_{q^{-2}} f(q^{-2}z_+, z_-),
$$

(B.15)

$$
\mathcal{R}(k)(z_+, z_-) = f(q^{-2}z_+, q^2z_-)
$$

(B.16)

and

$$
\mathcal{R}(p_+)(\xi) = iD_{q^{2}} f(\xi)z_-, \quad \mathcal{R}(p_-)(\xi) = iq^{-2}D_{q^{-2}} f(\xi)z_+.
$$

(B.17)

The q–derivative employed in the above equation is defined as

$$
D_q^x f(x) = \frac{f(x) - f(qx)}{(1 - q)x}.
$$

(B.18)

Note that ordering convention we employ is such that $z_+$’s are always on the left of $z_-$’s, that is

$$
f(z_+, z_-) = \sum_{n,m=0}^\infty c_{nm} z_+^n z_-^m,
$$

(B.19)

where $c_{nm}$ are arbitrary complex coefficients chosen to satisfy the condition (B.3).
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