On Continuous Weighted Finite Automata

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Abstract

We investigate the continuity of the $\omega$-functions and real functions defined by weighted finite automata (WFA). We concentrate on the case of average preserving WFA. We show that every continuous $\omega$-function definable by some WFA can be defined by an average preserving WFA and then characterize minimal average preserving WFA whose $\omega$-function or $\omega$-function and real function are continuous.

We obtain several algorithmic reductions for WFA-related decision problems. In particular, we show that deciding whether the $\omega$-function and real function of an average preserving WFA are both continuous is computationally equivalent to deciding stability of a set of matrices.

We also present a method for constructing WFA that compute continuous real functions.

Key words: continuity, decidability, matrix semigroup, stability, weighted finite automaton

2010 MSC: 68Q17, 26A15, 20M35, 15A99

1. Introduction

Weighted finite automata (WFA) over $\mathbb{R}$ are finite automata with transitions labelled by real numbers. They can be viewed as devices to compute functions from words to real numbers, or even as a way to define real functions. Weighted finite automata and transducers have many nice applications in natural language processing, image manipulation etc, see \cite{1, 2, 3, 4, 5, 6, 7} and references therein. On the other hand, weighted automata over more general semi-rings have been extensively studied as a natural extension of ordinary automata. A good source for more background information on various aspects of weighted automata is a forthcoming handbook on the field \cite{1}.

\textsuperscript{*}Abbreviations used: ap (average preserving), RCP (right-convergent product), WFA (weighted finite automata)

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Preprint submitted to Linear Algebra and its Applications August 26, 2009
WFA provide a natural and intrinsic description for some self-similar real functions. Smooth real functions defined by WFA are limited to polynomials [8, 9]. However, many more functions in lower differentiability classes can be generated. In this paper, we study those WFA functions that are continuous. We have two concepts of continuity: the continuity of the function $f_A$ that assigns real numbers to infinite words, and the continuity of the corresponding function $\hat{f}_A$ that assigns values to points of the unit interval.

Culik and Karhumäki have stated various results about real functions defined by the so-called level automata in [10]. In this work, we generalize many of these results to apply to the general setting. We also use the closely related theory of right-convergent product (RCP) sets of matrices as developed in [11, 12] and [13].

The paper is organized as follows: In Section 2 we give basic definitions, introduce the concepts of stable sets and RCP sets of matrices, provide some key results on RCP sets from [11, 12], define weighted finite automata and discuss the important concepts of minimality and average preservation (ap).

In Section 3 we study WFA as devices that assign real numbers to infinite words. We prove that any continuous function that can be defined by a WFA can, in fact, be defined using average preserving WFA, so restricting the attention to ap WFA is well motivated. We establish a canonical form for the average preserving WFA whose $\omega$-function $f_A$ is continuous (Corollary 37). We obtain algorithmic reductions between the decision problems of determining convergence and continuity of $f_A$, and the stability, product convergence and continuous product convergence of matrix sets. If stability of finite sets of square matrices is undecidable (which is not presently known) then all questions considered are undecidable as well.

In Section 4 we consider the real functions defined by WFA. Connections between the continuity of the $\omega$-function and the corresponding real function are formulated. We specifically look into those ap WFA whose $\omega$- and real functions are both continuous. If the $\omega$-function is continuous then there is a simple and effectively testable additional condition for the continuity of the corresponding real function. Again, we see that the stability of matrix products plays an important role in algorithmic questions. Finally, we provide a method to generate continuous ap WFA when a stable pair of matrices is given.

2. Preliminaries

Let $\Sigma$ be a non-empty finite set. In this context, we call $\Sigma$ an alphabet and its elements letters. With concatenation as the binary operation and the empty word $\varepsilon$ as the unit element, $\Sigma$ generates the monoid $\Sigma^*$, the elements of which are called words.

We denote by $|v|$ the length of the word $v \in \Sigma^*$. Denote the $i$-th letter of the word $v$ by $v_i$ and the factor $v_i v_{i+1} \cdots v_j$ by $v_{[i,j]}$. By $\text{pref}_k(v)$ we denote the prefix of length $k$ of the word $v$. An infinite word $w$ is formally a mapping $w : \mathbb{N} \rightarrow \Sigma$. Denote the set of all infinite words by $\Sigma^\omega$.

The set $\Sigma^\omega$ is a metric space with the Cantor metric (or prefix metric) defined as follows:

$$d_C(w, w') = \begin{cases} 0 & \text{if } w = w', \\ \frac{1}{2^k} & \text{otherwise}, \end{cases}$$
where $k$ is the length of the longest common prefix of $w$ and $w'$. The space $\Sigma^\omega$ is a product of the compact spaces $\Sigma$, therefore $\Sigma^\omega$ itself is compact.

The set of reals $\mathbb{R}$ is a complete metric space with the usual Euclidean metric

$$d_E(x, y) = |x - y| \text{ for all } x, y \in \mathbb{R}.$$  

We denote by $E$ the unit matrix (of appropriate size). We use the same notation $\|A\|$ both for the usual $l^2$ vector norm, if $A$ is a vector, and for the corresponding matrix norm, if $A$ is a matrix.

Assume that for each letter $a \in \Sigma$ we have an $n \times n$ square matrix $A_a$. Then for $v \in \Sigma^*$ let $A_v$ denote the matrix product $A_{v_1}A_{v_2} \ldots A_{v_k}$. If $v$ is empty, let $A_v = E$. If $w \in \Sigma^\omega$, we let $A_w = \lim_{k \to \infty} A_{\text{pref}_k(w)}$ if the limit exists.

In this paper, we assume that the elements of all matrices are defined in such a way that we can algorithmically perform precise operations of addition, multiplication and division as well as decide equality of two numbers. We can obtain such effective arithmetics by limiting ourselves, for example, to matrices and vectors with rational elements.

**Definition 1.** Let $B = \{A_a \mid a \in \Sigma\}$ be a nonempty set of $n \times n$ matrices such that $A_w = 0$ for every $w \in \Sigma^\omega$. Then we call $B$ a stable set.

Given a finite set $B$ of matrices, we will call the algorithmic question “Is $B$ stable?” the Matrix Product Stability problem. For $|B| = 1$, Matrix Product Stability is easy to solve using eigenvalues and Lyapunov equation (see [14, page 169]). Moreover, there is a semi-algorithm that halts iff $B$ is a stable set (idea of this algorithm is to check whether the joint spectral radius of $B$ is less than 1, see [11]). However, it is not known whether there exists an algorithm deciding Matrix Product Stability; even the binary ($|B| = 2$) case is as hard as the general stability problem, see [15] and [16] (we also prove this in Lemma 46).

The following Lemma is stated as Corollary 4.1a in [11]. For the sake of completeness we offer a short proof here.

**Lemma 2.** Let $\{A_a \mid a \in \Sigma\}$ be a stable set of $n \times n$ matrices. Then the convergence of $A_{\text{pref}_k(w)}$ to zero is uniform. That is, for every $\varepsilon > 0$ there exists $k_0$ such that for any $v \in \Sigma^*$ with $|v| > k_0$ we have $\|A_v\| < \varepsilon$.

**Proof.** Assume that the statement is not true. Then there is an $\varepsilon > 0$ such that there exist arbitrarily long $v \in \Sigma^*$ such that $\|A_v\| \geq \varepsilon$. From compactness of $\Sigma^\omega$ we obtain that there exists an infinite word $u \in \Sigma^\omega$ with the property that for each $l$ there exists a $v_l \in \Sigma^*$ such that $\|A_{u[l, l]}v_l\| \geq \varepsilon$. But then

$$\|A_u \| \|v_l\| \geq \|A_{u[l, l]}v_l\| \geq \varepsilon,$$

so

$$\|v_l\| \geq \frac{\varepsilon}{\|A_u \|}.$$  

Because $A_u = 0$, we have that the set $\{\|A_v\| \mid v \in \Sigma^*\}$ is unbounded. In the rest of the proof, we use the reasoning from [17] (proof of Lemma 1.1).
For each $k$, let $v^{(k)}$ be a word of length at most $k$ such that $\|A_{v^{(k)}}\|$ is maximal (note that $v^{(k)}$ might be empty). Denote $l = |v^{(k)}|$. First, we show that $\|A_{v^{(k)}}\| \geq 1$ for all $1 \leq i \leq l$. If for some such $i$ we would have $\|A_{v^{(k)}}\| < 1$, then

$$
\|A_{v^{(k)}}\| \|A_{v^{(k)}}\| \geq \|A_{v^{(k)}}\|, \text{ so}
$$

$$
\|A_{v^{(k)}}\| > \|A_{v^{(k)}}\|,
$$

contradicting the maximality of $\|A_{v^{(k)}}\|$. We conclude that $\|A_{v^{(k)}}\| \geq 1$ for all $i$.

As the norm of matrices $A_v$ is unbounded, the length of $v^{(k)}$ goes to infinity. Then we obtain from the compactness of $\Sigma^\omega$ that there exists a word $w \in \Sigma^\omega$ such that for each $i$ we can find $k_i$ such that $w_{[1,i]} = v^{(k_i)}$. But this means that $\|A_{w_{[1,i]}}\| \geq 1$ for each $i$, a contradiction with the stability of $\{A_a \mid a \in \Sigma\}$. □

**Definition 3.** A set of matrices $\{A_a \mid a \in \Sigma\}$ is called right-convergent product set or RCP set if the function $A_w : w \mapsto \lim_{k \to \infty} A_{\text{pref}_k(w)}$ is defined on the whole set $\Sigma^\omega$. If $A_w$ is continuous on $\Sigma^\omega$, we say that the set is continuous RCP.

Clearly every stable set is a continuous RCP set. In [1], the authors prove several results about RCP sets of matrices. Most importantly, Theorem 4.2 from [11] (with errata from [12]) gives us a characterization of continuous RCP sets of matrices. For $V, E_1$ subspaces of $\mathbb{R}^n$ such that $R^n = V \oplus E_1$ denote by $P_V : \mathbb{R}^n \to \mathbb{R}^n$ the projection to $V$ along $E_1$.

**Theorem 4** (Theorem 4.2 in [11]). Let $B = \{A_a \mid a \in \Sigma\}$ be a finite set of $n \times n$ matrices. Then the following conditions are equivalent:

1. The set $B$ is a continuous RCP set.

2. All matrices $A_a$ in $B$ have the same left 1-eigenspace $E_1 = E_1(A_a)$, and this eigenspace is simple for all $A_a$. There exists a vector space $V$ with $\mathbb{R}^n = E_1 \oplus V$, having the property that $P_V B P_V$ is a stable set.

3. The same as (2), except that $P_V B P_V$ is a stable set for all vector spaces $V$ such that $\mathbb{R}^n = E_1 \oplus V$.

From Theorem 4 and Lemma 2 follows a corollary (stated as Corollary 4.2a in [11]), which generalizes Lemma 2.

**Corollary 5.** If $B = \{A_a \mid a \in \Sigma\}$ is a continuous RCP set then all the products $A_{\text{pref}_k(w)}$ for $w \in \Sigma^\omega$ converge uniformly at a geometric rate.

Another important result that we will need is part (a) of Theorem I in [13], stated below (slightly modified) as Theorem 6. A set of matrices $B$ is called product-bounded if there exists a constant $K$ such that the norms of all finite products of matrices from $B$ are less than $K$. Notice that as all matrix norms on $n \times n$ matrices are equivalent, being product-bounded does not depend on our choice of matrix norm.
Theorem 6. Let $B$ be an RCP set of matrices. Then $B$ is product-bounded.

So we have the following sequence of implications:

$$B \text{ stable } \implies B \text{ continuous RCP } \implies B \text{ RCP } \implies B \text{ product bounded}$$

The problem of determining whether a given finite $B$ is product bounded is undecidable [16], while it is not known whether it is decidable if $B$ is stable, RCP or continuous RCP. We will explore the relationship between RCP sets and WFA later in the paper.

Weighted Finite Automata

A weighted finite automaton (WFA) $A$ is a quintuple $(Q, \Sigma, I, F, \delta)$. Here $Q$ is the state set, $\Sigma$ a finite alphabet, $I : Q \to \mathbb{R}$ and $F : Q \to \mathbb{R}$ are the initial and final distributions, respectively, and $\delta : Q \times \Sigma \times Q \to \mathbb{R}$ is the weight function. If $\delta(p, a, q) \neq 0$ for $a \in \Sigma$, $p, q \in Q$, we say that there is a transition from $p$ to $q$ labelled by $a$ of weight $\delta(p, a, q)$. We denote the cardinality of the state set by $|Q| = n$. Note that we allow $Q$ to be empty.

A more convenient representation of $A$ is by vectors $I \in \mathbb{R}^{1 \times n}$, $F \in \mathbb{R}^{n \times 1}$ and a collection of weight matrices $A_a \in M_{n \times n}(\mathbb{R})$ defined by

$$\forall a \in \Sigma, \forall i, j \in Q : (A_a)_{ij} = \delta(i, a, j).$$

A WFA $A$ defines the word function $F_A : \Sigma^* \to \mathbb{R}$ by

$$F_A(v) = IA_vF.$$

We denote by $A_v$ the automaton which we get from the WFA $A$ by substituting $IA_v$ for the original initial distribution $I$. Obviously, $F_{A_v}(w) = F_A(vw)$ for all $w \in \Sigma^*$.

Remark 7. Notice that for any $n \times n$ regular matrix $M$, we can take a new automaton with distributions $IM, M^{-1}F$ and the set of weight matrices $\{M^{-1}A_aM \mid a \in \Sigma\}$ without affecting the computed word function. We will call this operation changing the basis.

This means that whenever $I, F \neq 0$, we can change either $I$ or $F$ to any nonzero vector of our choice by switching to a different basis.

Given a word function $F$, we can define $\omega$-function $f$ on infinite words. For $w \in \Sigma^\omega$, we let

$$f(w) = \lim_{k \to \infty} F(pref_k(w)),$$

if the limit exists. If the limit does not exist then $f(w)$ remains undefined. In the following, we will use this construction to define $\omega$-function $f_A$ using $F_A$ for some $A$ weighted finite automaton.

As usual, the $\omega$-function $f : \Sigma^\omega \to \mathbb{R}$ is continuous at $w \in \Sigma^\omega$ if for every positive real number $\varepsilon$ there exists a positive real number $\delta$ such that all $w'$ in $\Sigma^\omega$ such that $d_C(w, w') < \delta$ satisfy $d_E(f(w), f(w')) < \varepsilon$. In particular, if $f$ is continuous at $w$ then $f$ must be defined in some neighborhood of $w$. We say that $f$ is continuous if it is continuous at every $w \in \Sigma^\omega$.

Throughout the paper, we will be mostly talking about the case when the convergence in the limit (1) is uniform.
**Definition 8.** We say that a word function $F$ is uniformly convergent if for every $\varepsilon > 0$, there exists a $k_0$ such that for all $w \in \Sigma^\omega$ and all $k > k_0$ we have

$$|F(\text{pref}_k(w)) - f(w)| < \varepsilon.$$ 

A WFA $\mathcal{A}$ is uniformly convergent if $F_{\mathcal{A}}$ is uniformly convergent.

**Lemma 9.** If a word function $F$ is uniformly convergent then the corresponding $\omega$-function $f$ is defined and continuous in the whole $\Sigma^\omega$.

**Proof.** From the definition of uniform convergence we obtain that $f(w)$ must exist for every $w \in \Sigma^\omega$. Continuity follows from the fact that $f$ is the uniform limit of continuous functions $f_k$ defined as $f_k(w) = F(\text{pref}_k(w))$ for all $w \in \Sigma^\omega$. \(\square\)

The following Lemma gives another formulation of the uniform convergence.

**Lemma 10.** The function $F$ is uniformly convergent iff for each $\varepsilon > 0$ there exists $m$ such that for all $w \in \Sigma^\omega$ and $v \in \Sigma^*$ such that $\text{pref}_m(v) = \text{pref}_m(w)$ we have

$$|F(v) - f(w)| < \varepsilon.$$ 

**Proof.** Obviously, if $F$ satisfies the condition on the right then letting $k_0 = m$ and $v = \text{pref}_k(w)$ for $k > k_0$ yields that $F$ is uniformly convergent.

For the converse, assume $\varepsilon > 0$ is given. We need to find $m$ with the required properties.

If $F$ is uniformly convergent then $f$ is continuous by Lemma 9 and from the compactness of $\Sigma^\omega$ we obtain uniform continuity of $f$. So there exists $l$ such that $\text{pref}_l(w) = \text{pref}_l(z)$ implies $|f(w) - f(z)| < \varepsilon/2$ for $w, z \in \Sigma^\omega$. Let now $k_0$ be such that $|F(\text{pref}_k(z)) - f(z)| < \varepsilon/2$ for all $z$ and all $k > k_0$. Choose $m > k_0, l$. Given $v \in \Sigma^*, w \in \Sigma^\omega$ with $\text{pref}_m(v) = \text{pref}_m(w)$, choose $z \in \Sigma^\omega$ such that $v$ is a prefix of $z$. Then we can write:

$$|F(v) - f(w)| \leq |F(v) - f(z)| + |f(z) - f(w)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

concluding the proof. \(\square\)

In contrast to Lemma 2, the following example shows that convergence to zero everywhere does not guarantee that a WFA converges uniformly.

**Example 11.** Consider the automaton $\mathcal{A}$ on the alphabet $\Sigma = \{0, 1\}$ described by Figure 1.

In the figure, the two numbers inside each state denote the initial and final distribution, respectively, while the numbers next to the arrow express the label and weight of the transition (weight is in parentheses). The matrix presentation of this automaton is

$$I = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

6
This automaton computes the word function

\[ F_A(v) = \begin{cases} (-1)^n & v \in 0^n1 \\ 0 & \text{otherwise} \end{cases} \]

so it defines the zero \( \omega \)-function. However, the convergence is not uniform at the point \( w = 0^\omega \).

Definition 12. A WFA \( \mathcal{A} \) with \( n \) states is said to be left minimal if

\[ \dim(IA_u, u \in \Sigma^*) = n. \]  \hfill (2)

Similarly, \( \mathcal{A} \) is called right minimal if

\[ \dim(A_uF, u \in \Sigma^*) = n. \]  \hfill (3)

If \( \mathcal{A} \) is both left and right minimal, we call it minimal.

In other words, \( \mathcal{A} \) is minimal when each of its distributions generates the space \( \mathbb{R}^n \). Moreover, \( \mathcal{A} \) is minimal according to our definition if it is also minimal in the sense that no other WFA with fewer states than \( n \) can compute the same word function \( F_A \) (see \([9]\), Proposition 3.1). Observe that minimality is clearly invariant under the change of basis.

Lemma 13. Given a WFA \( \mathcal{A} \), we can effectively find WFA \( \mathcal{A}' \) such that \( \mathcal{A}' \) is minimal and \( F_{\mathcal{A}'} = F_{\mathcal{A}} \).

For proof, see \([9]\), Proposition 3.1. Also, if the transition matrices of \( \mathcal{A} \) had rational entries then we can choose \( \mathcal{A}' \) so that its transition matrices have rational entries. In the following, we will often assume that \( \mathcal{A} \) is minimal.

Definition 14. A function \( F : \Sigma^* \rightarrow \mathbb{R} \) is average preserving \((\text{ap})\), if for all \( v \in \Sigma^* \),

\[ \sum_{a \in \Sigma} F(va) = kF(v), \quad \text{where } k = |\Sigma|. \]

The WFA \( \mathcal{A} \) with the final distribution \( F \) and weight matrices \( A_a \) is called average preserving \((\text{ap})\), if

\[ \sum_{a \in \Sigma} A_a F = kF, \quad \text{where } k = |\Sigma|. \]
Every ap WFA defines an average preserving word function and every average preserving word function definable by some WFA can be defined by an ap WFA (see [3, pages 306, 310]). In fact, any minimal WFA computing an ap word function must be ap. Notice also that neither a change of basis nor minimizing (as in Lemma 13) destroys the ap property of an ap automaton.

Lemma 15. The only ap word function defining the zero \( \omega \)-function is the zero function.

Proof. Assume that for every \( w \in \Sigma^\omega \) we have \( f(w) = 0 \), yet (without loss of generality) \( F(v) = s > 0 \) for some word \( v \in \Sigma^* \). By the ap property of \( F \), we have:

\[
\frac{1}{|\Sigma|} \sum_{a \in \Sigma} F(va) = F(v).
\]

This means that \( \max\{F(va) \mid a \in \Sigma\} \geq F(v) \) and so \( F(va) \geq s \) for some \( a \). Repeating this argument, we obtain \( w \in \Sigma^\omega \) such that \( f_A(vw) \geq s > 0 \), a contradiction. \( \square \)

Corollary 16. Let \( F,G \) be two ap functions defining the same \( \omega \)-function \( f \) and suppose that \( f(w) \) is defined for every \( w \in \Sigma^\omega \). Then \( F = G \).

Proof. As the function \( F - G \) is ap and defines the zero \( \omega \)-function, Lemma 15 gives us that \( F - G = 0 \) and so \( F = G \). \( \square \)

Example 17. As the only ap word function defining \( f_A = 0 \) is the zero function (Lemma 15), it is easy to decide if a given minimal ap WFA \( A \) computes \( f_A = 0 \). Such automaton is the unique zero state WFA (whose \( I,F \) are zero vectors from the space \( \mathbb{R}^0 \)). However, in the non-ap case, we might encounter automata such as in Figure 2. The automaton \( A \) has \( I = F = (1) \) and \( A_0 = A_1 = (\frac{1}{2}) \). Obviously, \( A \) is minimal (but not ap) and computes the word function \( F_A(v) = 1/2^{|v|} \) and the \( \omega \)-function \( f_A = 0 \).

3. Properties of \( \omega \)-functions

In this section, we will study the \( \omega \)-function \( f_A \), where \( A \) is an automaton operating on the alphabet \( \Sigma \). We will put the emphasis on \( \omega \)-functions defined using ap word functions, as these have numerous useful traits.
3.1. Average preserving word functions

We begin by showing that any automaton computing a continuous $\omega$-function can be modified to be $\text{ap}$ and still compute the same $\omega$-function. Hence we do not miss any WFA definable continuous functions if we restrict the attention to $\text{ap}$ WFA.

Actually, the following theorem is even more general and allows for some "defects" of continuity: we only expect $f_A$ to be uniformly continuous on a certain dense set $\Delta \subseteq \Sigma^\omega$ (as usual, this means that the function $f_A$ may even be undefined outside $\Delta$).

We will need the more general formulation later in Section 4 for Corollary 58.

**Theorem 18.** Let $A$ be a WFA and $w \in \Sigma^\omega$. Suppose that $f_A$ is continuous on the set $\Delta = \Sigma^*w$, and suppose that there exists a continuous $g : \Sigma^\omega \to \mathbb{R}$ such that $f_A|_\Delta = g|_\Delta$. Then there is an average preserving WFA $B$ such that $f_B = g$. Moreover, if $A$ is left-minimal then $B$ can be obtained from $A$ by changing the final distribution.

**Remark 19.** The condition that "There exists a continuous $g : \Sigma^\omega \to \mathbb{R}$ such that $f_A|_\Delta = g|_\Delta." is equivalent with demanding that $f_A|_\Delta$ be uniformly continuous. Moreover, if such a $g$ exists then it is unique because $\Delta$ is dense in $\Sigma^\omega$.

**Proof.** Using Lemma 13 we can assume that the automaton $A = (I, \{A_a | a \in A\}, F)$ is a left-minimal WFA. Denote by $A_i$ the WFA obtained from $A$ by replacing $I$ by $I_i$, the $i$-th element of the canonical basis of $\mathbb{R}^n$. Let us first prove that each $f_{A_i}$ is uniformly continuous on $\Delta$. From left-minimality of $A$ we obtain that there are words $u_1, \ldots, u_n \in \Sigma^*$ and coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $F_{A_i}(v) = \sum_{j=1}^n \alpha_j f_{A_i}(u_j v)$ for all $v \in \Sigma^*$. This implies that $f_{A_i}(v) = \sum_{j=1}^n \alpha_j f_A(u_j v)$ for all $v \in \Delta$. But then $f_{A_i}$ is uniformly continuous on $\Delta$ as a linear combination of uniformly continuous functions $v \mapsto f_A(u_j v)$.

Recall that $w$ is the infinite word such that $\Delta = \Sigma^*w$. Observe that the limit $G = \lim_{k \to \infty} (A_{\text{pref}_k(w)} F)$ exists, as we have a simple formula for the $i$-th component of $G$:

$$G_i = \lim_{k \to \infty} (I_i A_{\text{pref}_k(w)} F) = f_{A_i}(w).$$

Denote $L = |\Sigma|$ and let $B = (I, \{A_a\}, F')$ be the WFA with the modified final distribution

$$F' = \lim_{i \to \infty} \left( \frac{\sum_{a \in \Sigma} A_a}{L} \right)^i G = \lim_{i \to \infty} \frac{1}{L^i} \sum_{|u| = i} \lim_{k \to \infty} A_{\text{pref}_k(uw)} F. \quad (4)$$

First we show that the limit (4) exists. The $j$-th coordinate of the $i$-th vector in the sequence has the following presentation:

$$\phi_{i}^{(j)} = \frac{1}{L^i} \sum_{|u| = i} \lim_{k \to \infty} (I_j A_{\text{pref}_k(uw)} F) = \frac{1}{L^i} \sum_{|u| = i} f_{A_j}(uw).$$

To show that $\lim_{i \to \infty} \phi_{i}^{(j)}$ exists, it suffices to show that $\{\phi_{i}^{(j)}\}_{i=1}^{\infty}$ is a Cauchy sequence as $\mathbb{R}$ is a complete metric space.
Let $\varepsilon > 0$. As $f_{A_j}$ is uniformly continuous on $\Delta$, there is a $k_{\varepsilon}$ such that $\text{pref}_{k_{\varepsilon}}(z) = \text{pref}_{k_{\varepsilon}}(z')$ implies $|f_{A_j}(z) - f_{A_j}(z')| < \varepsilon$ for any $z, z' \in \Delta$. Let $\phi^{(j)}_s$ and $\phi^{(j)}_{s+t}$ be two elements of the sequence with $s \geq k_{\varepsilon}, t \in \mathbb{N}$. Then

$$
|\phi^{(j)}_{s+t} - \phi^{(j)}_s| = \frac{1}{L^{s+t}} \left| \sum_{|u|=s+t} f_{A_j}(uw) - \frac{1}{L^s} \sum_{|u|=s} f_{A_j}(uw) \right|
$$

$$
= \frac{1}{L^{s+t}} \left| \sum_{|u|=s} \left( \sum_{|v|=t} f_{A_j}(uw) - L^t f_{A_j}(uw) \right) \right|
$$

$$
\leq \frac{1}{L^{s+t}} \sum_{|u|=s} \sum_{|v|=t} |f_{A_j}(uw) - f_{A_j}(uw)|
$$

$$
< \frac{1}{L^{s+t}} L^{s+t} \varepsilon = \varepsilon.
$$

We see that the vector sequence $\{\phi_i\}_{i=1}^{\infty}$ converges element-wise, and hence the limit (4) exists.

It remains to show that $B$ is average preserving and verify the equality $f_B = g$. To prove the ap property of $B$, we compute the product

$$
\left( \sum_{a \in \Sigma} A_a \right) F' = \left( \sum_{a \in \Sigma} A_a \right) \lim_{i \to \infty} \left( \frac{\sum_{a \in \Sigma} A_a}{L} \right)^i G
$$

$$
= L \lim_{i \to \infty} \sum_{a \in \Sigma} A_a \left( \frac{\sum_{a \in \Sigma} A_a}{L} \right)^i G
$$

$$
= L \lim_{i \to \infty} \left( \frac{\sum_{a \in \Sigma} A_a}{L} \right)^{i+1} G
$$

$$
= LF'.
$$

To show that $f_B = g$, let $v \in \Sigma^\omega$ be an arbitrary word. Then

$$
f_B(v) = \lim_{j \to \infty} IA_{\text{pref}_j(v)} F'
$$

$$
= \lim_{j \to \infty} IA_{\text{pref}_j(v)} \lim_{i \to \infty} \frac{1}{L^i} \sum_{|u|=i} \lim_{k \to \infty} A_{f_{A_j}(uw)} F
$$

$$
= \lim_{j \to \infty} \lim_{i \to \infty} \sum_{|u|=i} \frac{1}{L^i} \lim_{k \to \infty} IA_{\text{pref}_j(v)\text{pref}_k(uw)} F
$$

$$
= \lim_{j \to \infty} \lim_{i \to \infty} \sum_{|u|=i} \frac{1}{L^i} f_A(\text{pref}_j(v)uw)
$$

$$
= \lim_{j \to \infty} \lim_{i \to \infty} \sum_{|u|=i} \frac{1}{L^i} g(\text{pref}_j(v)uw)
$$

$$
= g(v),
$$
where the last equality follows from the uniform continuity of $g$. 

**Corollary 20.** Every continuous function $\Sigma^\omega \to \mathbb{R}$ that can be computed by a weighted finite automaton can be computed by some average preserving weighted finite automaton.

### 3.2. Continuity of $\omega$-functions

We now prove several results about WFA with continuous $\omega$-functions. While the behavior of general WFA with continuous $\omega$-functions can be complicated, we can obtain useful results for uniformly convergent WFA and the uniform convergence assumption is well justified: As we show in Lemma 21 all ap WFA with continuous $\omega$-function are uniformly convergent. Together with Theorem 18 we then have that uniformly convergent WFA compute all WFA-computable continuous functions.

**Lemma 21.** Let $F$ be an ap word function. Let its $\omega$-function $f$ be continuous. Then $F$ is uniformly convergent.

**Proof.** Let $\varepsilon > 0$. By continuity of $f$ and compactness of $\Sigma^\omega$, there exists an index $k$ such that $|f(w) - f(w')| < \varepsilon$ for every $w, w'$ for which $\text{pref}_k(w') = \text{pref}_k(w)$. Fix any $w \in \Sigma^\omega$ and let $v \in \Sigma^*$ be its prefix whose length is at least $k$.

By the ap property of $F$, we obtain:

$$\frac{1}{|\Sigma|} \sum_{a \in \Sigma} F(va) = F(v).$$

This means that

$$\max\{F(va) \mid a \in \Sigma\} \geq F(v) \geq \min\{F(va) \mid a \in \Sigma\}.$$

So for some letters $a, b \in \Sigma$, we have $F(va) \geq F(v) \geq F(vb)$. We can now continue in this manner, obtaining words $w_1, w_2 \in \Sigma^\omega$ such that $f(vw_1) \geq F(v) \geq f(vw_2)$. By the choice of $k$, we have $f(w) + \varepsilon > f(vw_1)$ and $f(vw_2) > f(w) - \varepsilon$. Therefore, $|f(w) - F(v)| < \varepsilon$ and the claim follows.

**Remark 22.** While most of the theorems in this section deal with uniformly convergent functions and automata, uniform convergence is difficult to verify. The ap property of minimal automata, on the other hand, is easy to check. Thanks to Lemma 21 we can rewrite all following theorems by replacing the uniform convergence assumption on $F$ by the demand that $F$ be ap and $f$ be continuous. This is how we will mostly use the results of this section, as ap WFA are often used in applications. However, it turns out that uniform convergence is the essential feature that makes the following theorems valid, so we present the proofs in this more general setup.

We now state a simple but important property of uniformly convergent word functions.
Lemma 23. Let $F$ be a word function defining the $\omega$-function $f$. If $F$ is uniformly convergent, then the following equality holds for all $w \in \Sigma^\omega$ and $u \in \Sigma^*$:

$$\lim_{k \to \infty} F(\text{pref}_k(w)u) = f(w).$$

Proof. By Lemma 10, for any $\varepsilon > 0$ there exists $k_0$ such that for all $k > k_0$ we have $|F(\text{pref}_k(w)u) - f(w)| < \varepsilon$. \qed

Example 24. Example 11 shows that average preservation is a necessary condition in Lemmas 21 and 23. The WFA $\mathcal{A}$ in Figure 1 is minimal but not average preserving. It defines the (continuous) zero $\omega$-function, but the convergence is not uniform. Likewise, the conclusion of Lemma 23 also does not hold for $F_\mathcal{A}$: Choose $w = 0^\omega$ and $u = 1$. Then $f(0^\omega) = 0$ while $\lim_{k \to \infty} F(0^k1) = 1$.

Next, we show that changing the initial distribution of a left minimal WFA does not alter convergence and continuity properties:

Lemma 25. Let $\mathcal{A}$ be a left minimal WFA and let $B$ be a WFA obtained from $\mathcal{A}$ by changing the initial distribution. Then the following holds:

1. If $f_\mathcal{A}$ is defined on the whole $\Sigma^\omega$ then so is $f_B$.
2. If $f_\mathcal{A}$ is continuous then so is $f_B$.
3. If $\mathcal{A}$ is uniformly convergent then so is $B$.

Proof. Let us obtain $B$ from $\mathcal{A}$ by changing the initial distribution to $I'$. By the left minimality of $\mathcal{A}$, there are words $u_i \in \Sigma^*$ and coefficients $\alpha_i \in \mathbb{R}$ such that $I' = \alpha_1 IA_{u_1} + \ldots + \alpha_n IA_{u_n}$. Then $B$ computes a function which is a linear combination of the functions $F_{\mathcal{A}_{u_i}}$:

$$F_B(v) = \alpha_1 IA_{u_1} A_v F + \ldots + \alpha_n IA_{u_n} A_v F = \alpha_1 f_\mathcal{A}(u_1 v) + \ldots + \alpha_n f_\mathcal{A}(u_n v)$$

So, assuming that $f_\mathcal{A}$ is defined everywhere, we obtain:

$$f_B(w) = \alpha_1 f_\mathcal{A}(u_1 w) + \ldots + \alpha_n f_\mathcal{A}(u_n w),$$

proving (1). Moreover, it is easy to observe (2) and (3) from these equalities. \qed

It follows from Lemma 25 that if $f_\mathcal{A}$ is a left minimal continuous WFA then the sequence $\{A_{\text{pref}_k(w)}F\}_{k=1}^\infty$ of vectors converges element-wise as $k$ tends to infinity: To see that $\{A_{\text{pref}_k(w)}F\}_{k=1}^\infty$ converges, we change the initial distribution to the $i$-th element of the canonical basis $I_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Denote the resulting automaton by $\mathcal{A}_i$. Then $f_{\mathcal{A}_i}(w)$ is continuous and

$$f_{\mathcal{A}_i}(w) = \lim_{k \to \infty} (I_i A_{\text{pref}_k(w)}F) = \lim_{k \to \infty} (A_{\text{pref}_k(w)}F)_i.$$
We see that $\lim_{k \to \infty} A_{\text{pref}_k(w)} F$ is the vector with $i$-th component equal to $f_A(w)$ for $i = 1, \ldots, n$.

We now look into the effect of changing the final distribution of a right minimal WFA. If the WFA is uniformly convergent then the outcome is the same as multiplying the $\omega$-function by a constant.

**Lemma 26.** Let $A$ be right minimal and uniformly convergent. Then changing the final distribution of $A$ keeps uniform convergence and affects $f_A$ by a multiplicative constant only.

**Proof.** Let $F'$ be any final distribution, and let $u_1, \ldots, u_n \in \Sigma^*$ be words such that $F' = \alpha_1 A_{u_1} F + \ldots + \alpha_n A_{u_n} F$ for some $\alpha_1, \ldots, \alpha_n$. Such words exist by the right minimality of $A$. Denote by $B$ the WFA $A$ with the final distribution $F'$. Then

$$f_B(w) = \lim_{k \to \infty} (I A_{\text{pref}_k(w)} F')$$

$$= \lim_{k \to \infty} (I A_{\text{pref}_k(w)} (\alpha_1 A_{u_1} F + \ldots + \alpha_n A_{u_n} F))$$

$$= \alpha_1 \lim_{k \to \infty} (I A_{\text{pref}_k(w)} A_{u_1} F) + \ldots + \alpha_n \lim_{k \to \infty} (I A_{\text{pref}_k(w)} A_{u_n} F)$$

$$= (\alpha_1 + \ldots + \alpha_n) f_A(w)$$

where we have used Lemma 23 in the last equality.

Uniform convergence of $B$ easily follows, as the functions $F_i(v) = F_A(v u_i)$ are all uniformly convergent and $F_B = F_1 + F_2 + \cdots + F_n$. $\square$

Putting Lemmas 25 and 26 together, we obtain a theorem about the continuity of $\omega$-functions.

**Theorem 27.** Let $A$ be a minimal uniformly convergent WFA. Then any automaton $B$ obtained from $A$ by changing $I$ and $F$ is also uniformly convergent (and therefore continuous).

**Proof.** To prove the theorem, we change first $I$ and then $F$.

Lemma 25 tells us that changing $I$ does not break uniform convergence of $A$. Also, it is easy to observe that changing $I$ does not affect right-minimality of $A$, so the conditions of Lemma 26 are satisfied even after a change of initial distribution. Recall that uniform convergence implies continuity by Lemma 9. $\square$

Recall that for $w \in \Sigma^\omega$ we define $A_w = \lim_{k \to \infty} A_{\text{pref}_k(w)}$ if the limit exists. We are now prepared to prove that the weight matrices of a minimal uniformly convergent WFA form a continuous RCP set.

**Corollary 28.** Let $A$ be a minimal uniformly convergent WFA. Then the limit

$$A_w = \lim_{k \to \infty} A_{\text{pref}_k(w)}$$

exists for all $w \in \Sigma^\omega$, the elements of the matrix $A_w$ are continuous functions of $w$ and we have $f_A(w) = IA_w F$. 13
Proof. Taking $I_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $F_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ with one on the $i$-th and $j$-th place, we obtain the automaton $A_{ij}$ computing $(A_w)_{ij}$. From Lemma 27, we see that $f_{A_{ij}}$ is continuous on the whole $\Sigma$, so elements of $A_w$ are continuous functions of $\Sigma$.

Multiplications by constant vectors $I$ and $F$ are continuous operations so we can write

$$f_A(w) = \lim_{k \to \infty} I A_{pref_k(w)} F = I \left( \lim_{k \to \infty} A_{pref_k(w)} \right) F = I A_w F,$$

concluding the proof. □

Next we look into the matrices $A_w$ and prove that they have some very particular properties.

**Lemma 29.** Let $A$ be minimal and uniformly convergent, $w \in \Sigma^\omega$ and $u \in \Sigma^*$. Then

$$A_w A_u = A_w.$$

**Proof.** By Theorem 27 we can change the initial and final distributions of $A$ to any $I$ and $F$ without affecting uniform convergence. Then Lemma 23 gives us that

$$IA_w A_u F = I \left( \lim_{k \to \infty} A_{pref_k(w)} A_u F \right) = \lim_{k \to \infty} (IA_{pref_k(w)} A_u F) = \lim_{k \to \infty} (IA_{pref_k(w)} F) = IA_w F.$$

As the above equality holds for all $I$ and $F$, we have

$$A_w A_u = A_w. \square$$

**Corollary 30.** Let $A$ be minimal and uniformly convergent. If $f_A$ is a non-zero function, then we can effectively find a vector $I_c \neq 0$ such $I_c A_a = I_c$ for all $a \in \Sigma$.

**Proof.** Suppose $f_A \neq 0$. Then $IA_w F \neq 0$ for some $w \in \Sigma^\omega$. Let $I_c = IA_w$. Consider now the WFA $B$ obtained from $A$ by replacing the initial distribution $I$ with $I_c$. Then, by Lemma 29 we have for all $u \in \Sigma^*$:

$$I_c A_u F = IA_w A_u F = IA_w F = I_c F \neq 0.$$

Thus $B$ computes a non-zero constant function.

Next we notice that for all $u \in \Sigma^*$ and all $a \in \Sigma$ we have the equality $I_c A_a A_u F = I_c F = I_c A_u F$. This together with the right minimality of $A$ gives us that $I_c A_a = I_c$ for all $a \in \Sigma$.

We have shown that the matrices $A_a$, $a \in \Sigma$ always have a common left eigenvector belonging to the eigenvalue 1. We can find such common left eigenvector $I_c$ effectively by solving the set of linear equations $\{ I_c (A_a - E) = 0, a \in \Sigma \}$. □

**Remark 31.** It is easy to see from Corollary 30 that any minimal and uniformly convergent WFA that computes a non-zero function can be made to compute a non-zero constant function just by changing its initial distribution to $I_c$.

If $A$ is uniformly convergent minimal WFA, then the rows of all limit matrices $A_w$ are multiples of the same vector.
Lemma 32. Let $\mathcal{A}$ be minimal uniformly convergent WFA and let $f_\mathcal{A} \neq 0$. Then for all $w \in \Sigma^\omega$ the row space $V(A_w)$ of $A_w$ is one-dimensional. Moreover, $V(A_w)$ is the same for all $w \in \Sigma^\omega$.

Proof. By Lemma 29, $A_w A_u F = A_u F$ and thus $A_w (A_u F - F) = 0$ for all $u \in \Sigma^*$. We see that vector $A_u F - F$ is orthogonal to $V(A_w)$ irrespective of the choice of $u$.

Denote $W = \langle A_u F - F \mid F \in \Sigma^* \rangle$. Now the minimality of $\mathcal{A}$ implies $\dim W \geq n - 1$, because $\dim(W + \langle F \rangle) = n$. On the other hand, for all $w \in \Sigma^\omega$ we have $V(A_w) \subseteq W^\perp$ so $\dim V(A_w) \leq \dim W^\perp \leq 1$. If $A_w = 0$, for some $w$ then $f_\mathcal{A}(uw) = I A_u A_w F = 0$ for all $u \in \Sigma^*$ and so, by continuity of $f_\mathcal{A}$, we would have $f_\mathcal{A} = 0$. This means that $\dim V(A_w) = 1$ and $V(A_w) = W^\perp$ for all $w$.

Remark 33. From Lemma 32 it follows that the vector $I_c$ from Corollary 30 belongs to $V(A_w)$ and is therefore unique up to multiplication by a scalar.

Remark 34. Let $\mathcal{A}$ be minimal and uniformly convergent. If $F$ is an eigenvector belonging to the eigenvalue $\lambda$ of some $A_u$, then for all $w \in \Sigma^\omega$

$$A_w F = A_w A_u F = A_w \lambda F = \lambda A_w F,$$

and thus either $\lambda = 1$ or $A_w F = 0$ for all $w \in \Sigma^\omega$. In the latter case $f_\mathcal{A} = 0$.

Using Corollary 30 and Lemma 32 we can transform all minimal uniformly convergent automata to a “canonical form”. This transformation is a simple change of basis, so it preserves minimality as well as the the ap property:

Lemma 35. Let $\mathcal{A}$ be a minimal uniformly convergent automaton such that $f_\mathcal{A} \neq 0$. Then we can algorithmically find a basis of $\mathbb{R}^n$ such that the matrices $A_a$ are all of the form

$$A_a = \begin{pmatrix} \vspace{1mm} B_a \\ 0 \end{pmatrix},$$

(5)

where $\{B_a \mid a \in \Sigma\}$ is a stable set of matrices.

Proof. Suppose that $\mathcal{A}$ is minimal and uniformly convergent. Using Corollary 30 we can algorithmically find a vector $I_c$ such that $I_c A_a = I_c$ for all $a \in \Sigma$. Let us change the basis of the original automaton so that $I_c = (0, \ldots, 0, 1)$ (this does not affect uniform convergence or minimality of $\mathcal{A}$).

As we have $I_c A_a = I_c$, the lowest row of every weight matrix $A_a$ must be equal to $(0, \ldots, 0, 1)$. In other words, we have shown that for every $a \in \Sigma$, matrix $A_a$ has the form

$$A_a = \begin{pmatrix} \vspace{1mm} B_a \\ 0 \end{pmatrix},$$

where $0$ and $b_a$ are row and column vectors, respectively. For $v$ word (finite or infinite), denote

$$A_v = \begin{pmatrix} \vspace{1mm} B_v \\ 0 \\ \vdots \end{pmatrix}.$$
From the formula for matrix multiplication, we obtain

\[ A_{uv} = A_u \cdot A_v = \begin{pmatrix} B_u & b_u \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} B_v & b_v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B_u B_v & B_u b_v + b_u \\ 0 & 1 \end{pmatrix}, \]

in particular \( B_{uv} = B_u B_v \) and so \( B_v \) is simply a product \( B_{v_1} B_{v_2} \cdots B_{v_n} \).

For all \( w \in \Sigma^\omega \), we have:

\[ A_w = \begin{pmatrix} B_w & b_w \\ 0 & 1 \end{pmatrix}. \]

By Lemma 32, we know that if \( A \) defines a continuous function, then the rows 1, \ldots, \( n-1 \) in \( A_w \) are multiples of row \( n \). This means that \( B_w = 0 \) for all \( w \in \Sigma^\omega \) and so \( \{ B_a \mid a \in \Sigma \} \) is a stable set.

One might ask if all automata with matrices of the form (5) and \( \{ B_a \mid a \in \Sigma \} \) stable are uniformly convergent. We show that the answer is yes and prove an even more general statement along the way (we are going to need this more general form later when proving Theorem 61).

Lemma 36. Let \( \{ A_a \mid a \in \Sigma \} \) be a finite set of matrices of the form

\[ A_a = \begin{pmatrix} B_a & C_a \\ 0 & D_a \end{pmatrix}, \]

where \( B_a \) and \( D_a \) are square matrices and the set \( \{ B_a \mid a \in \Sigma \} \) is stable. Then the following holds:

1. If \( \{ D_a \mid a \in \Sigma \} \) is product-bounded then \( \{ A_a \mid a \in \Sigma \} \) is product-bounded.
2. If \( \{ D_a \mid a \in \Sigma \} \) is RCP then \( \{ A_a \mid a \in \Sigma \} \) is RCP.
3. If \( \{ D_a \mid a \in \Sigma \} \) is continuous RCP then \( \{ A_a \mid a \in \Sigma \} \) is continuous RCP.
4. If \( \{ D_a \mid a \in \Sigma \} \) is stable then \( \{ A_a \mid a \in \Sigma \} \) is stable.

Proof. As before, denote

\[ A_v = \begin{pmatrix} B_v & C_v \\ 0 & D_v \end{pmatrix}. \]

It is easy to see that \( B_v \) and \( D_v \) are equal to the matrix products \( B_{v_1} B_{v_2} \cdots B_{v_n} \) and \( D_{v_1} D_{v_2} \cdots D_{v_n} \), respectively, while for \( C_v \) the equality \( C_{uv} = B_u C_v + C_u D_v \) holds.

1. Let \( K \) be a constant such that \( \| D_u \| < K \) for all \( u \in \Sigma^* \). We need to prove that there exists a constant \( L \) such that \( \| C_u \| < L \) for all \( u \in \Sigma^* \).

   By Lemma 2 there exists a \( k \) such that for all words \( u \) of length at least \( k \) we have \( \| B_u \| < 1/2 \). Denote \( M = \max \{ \| B_u \| \mid u \in \Sigma^*, |u| < k \} \) and \( N = \max \{ \| C_u \| \mid a \in \Sigma \} \).
Let $m = |u|$. It is easy to see that when $m \geq k$, the inequality $\|B_u\| < M \cdot 2^{-\lfloor m/k \rfloor}$ holds. Moreover, a quick proof by induction yields that:

$$C_u = \sum_{j=0}^{m} B_{u_1 \ldots u_{j-1}} C_{u_j} D_{u_{j+1} \ldots u_m}.$$ 

Hence, we can write (for $m > k$):

$$\|C_u\| \leq \sum_{j=0}^{k-1} \|B_{u_1 \ldots u_{j-1}}\| \|C_{u_j}\| \|D_{u_{j+1} \ldots u_m}\| + \sum_{j=k}^{m} \|B_{u_1 \ldots u_{j-1}}\| \|C_{u_j}\| \|D_{u_{j+1} \ldots u_m}\|$$

$$\leq \sum_{j=0}^{k-1} M N K + \sum_{j=k}^{m} M \cdot 2^{-\lfloor j/k \rfloor} \cdot N K$$

The first sum is exactly $k M N K$ while the second one can bounded from the above by $k M N K$. All in all, we obtain that $\|C_u\| \leq 2k M N K$ and so $\{A_a | a \in \Sigma\}$ is product-bounded.

(2) Using Theorem 6, we obtain that the set $\{D_a | a \in \Sigma\}$ is product-bounded. Therefore, using the part (1) of this Lemma, we see that $\{A_a | a \in \Sigma\}$ is product-bounded and so there exists some $L > 0$ such that $\|C_u\| < L$ for all $u \in \Sigma^\omega$.

We only need to show that for every $w \in \Sigma^\omega$, the sequence $\{C_{\text{pref}_k(w)}\}_{k=1}^\infty$ satisfies the Bolzano-Cauchy condition.

Assume $\varepsilon > 0$ is given. Because $\{B_a | a \in \Sigma\}$ is stable, there exists a $k$ such that $\|B_{\text{pref}_k(w)}\| < \varepsilon/(4L)$. Denote $u = \text{pref}_k(w)$ and let $x \in \Sigma^\omega$ be such that $w = ux$. The sequence $\{D_{\text{pref}_j(x)}\}_{j=1}^\infty$ converges so there exists a number $j$ such that for all positive $i$ we have $\|D_{\text{pref}_j(x)} - D_{\text{pref}_{j+i}(x)}\| < \varepsilon/(2L)$. Let $v = \text{pref}_j(x)$ and write $w = uvy$ where $y$ is an appropriate infinite suffix.

We will now prove that for all prefixes $uv$ of $w$ we have $\|C_{uv} - C_{uv}\| < \varepsilon$. Using the equalities

$$C_{uv} = B_u C_v + C_u D_v,$$

$$C_{uv} = B_u C_v + C_u D_v,$$

we obtain

$$\|C_{uv} - C_{uv}\| \leq \|B_u\| \|C_v - C_{uv}\| + \|C_u\| \|D_v - D_{uv}\| < \frac{\varepsilon}{4L} \cdot 2L + L \cdot \frac{\varepsilon}{2L} = \varepsilon,$$

this means that the sequence $\{C_{\text{pref}_k(w)}\}_{k=1}^\infty$ is Cauchy and so the proof is finished.
(3) By case (2) we have that $A_w$ exists for all $w \in \Sigma^\omega$. As $B_w$, $D_w$ depend continuously on $w$, all we need to show is that the map $w \mapsto C_w$ is also continuous.

As before, by Theorem 6 the set $\{C_a \mid a \in \Sigma\}$ is product-bounded. By passing to limits, we see that there exists $L$ such that $\|C_w\| < L$ for all infinite $w \in \Sigma^\omega$.

The function $w \mapsto D_w$ is continuous on a compact space and so it is uniformly continuous. Given $\varepsilon > 0$, we find $k$ such that for all $u, v$ of length $k$ and all $w, z \in \Sigma^\omega$ we have:

$$\|B_u\| < \frac{\varepsilon}{4L},$$
$$\|D_{uv} - D_{vz}\| < \frac{\varepsilon}{2L}.$$  

We can now, similarly to case (2), write:

$$\|C_{uz} - C_{uvw}\| \leq \|B_u\|\|C_z - C_{vw}\| + \|C_u\|\|D_{vz} - D_{vw}\|$$
$$< \frac{\varepsilon}{4L} \cdot 2L + L \cdot \frac{\varepsilon}{2L} = \varepsilon,$$

proving continuity.

(4) Using case (2), we obtain that $C_z$ exists for all $z \in \Sigma^\omega$ and moreover, by Theorem 6, there exists $L > 0$ such that $\|C_z\| < L$ for all $z \in \Sigma^\omega$.

Let $w \in \Sigma^\omega$ and $\varepsilon > 0$. If we prove that $\|C_w\| < \varepsilon$, we are done. There is a finite prefix $u$ of $w$ such that $\|B_u\| < \varepsilon/L$. Let $w = uz$, where word $z \in \Sigma^\omega$ is the remaining infinite suffix of $w$. We now have:

$$\|C_w\| = \|C_{uz}\| = \|B_uC_z + C_uD_z\| = \|B_uC_z\| \leq \|B_u\||C_z| < \frac{\varepsilon}{L} = \varepsilon,$$

where we have used the equality $D_z = 0$. This means that $\|C_w\| = 0$ and we are done. \qed

Observe that by letting $D_a = 1$ for all $a \in \Sigma$, we obtain from case (3) of Lemma 36 and Corollary 5 a partial converse to Lemma 35. All automata of the form (5) where $\{B_a \mid a \in \Sigma\}$ stable are uniformly convergent.

Therefore, putting Lemmas 35 and 36 together, we obtain (under the assumption that $A$ is minimal and $f_A \neq 0$) a characterization of uniformly convergent automata.

**Corollary 37.** Let $A$ be a minimal automaton such that $f_A \neq 0$. Then $A$ is uniformly convergent iff there exists a basis of $\mathbb{R}^n$ in which all the transition matrices $A_a$ have the form (5) where $\{B_a \mid a \in \Sigma\}$ is a stable set.

Note that we could have relied on Theorem 4 here: Together with Lemma 40, it directly gives us Corollary 37. (We only need to realize that the dimension of the space $E_1$ is one in this case, which follows from Lemma 32.) However, we wanted to show how to algorithmically obtain the form (5) and we will also need Lemma 36 later on.
Remark 38. If $\mathcal{A}$ is a minimal ap automaton then it is easy to verify algorithmically whether $f_{\mathcal{A}} = 0$, because $f_{\mathcal{A}} = 0$ iff $F_{\mathcal{A}} = 0$.

Remark 39. In [10], the authors define level automata as automata satisfying the following conditions:

1. Only loops of length one (i.e. $q \rightarrow q$) are allowed.
2. The transition matrices and distribution vectors are non-negative.
3. For every state $p$, if there exist a state $q \neq p$ and a letter $a$ such that $(A_a)_{p,q} \neq 0$ then $(A_a)_{p,q} < 1$ for all $q$ and $a$. If there are no such $q$ and $a$ then $(A_a)_{p,p} = 1$ for all letters $a$.
4. The automaton is reduced; it does not have useless states.

As all level automata have (after proper ordering of states) matrices of the form

$$A_a = \begin{pmatrix} B_a & C_a \\ 0 & E \end{pmatrix},$$

where $E$ is the unit matrix and $B_a$ are upper triangular matrices with entries in the interval $[0, 1)$, the automata from case (3) of Lemma 36 are actually a generalization of level automata.

3.3. WFA and RCP sets

In this part we explicitly connect the notions of RCP sets and functions computed by WFA.

Theorem 40. Let $\mathcal{A}$ be a WFA and let $B = \{A_a \mid a \in \Sigma\}$ be its set of transition matrices. Then the following holds:

1. If $B$ is an RCP set then $f_{\mathcal{A}}$ is defined everywhere.
2. If $B$ is a continuous RCP set then $\mathcal{A}$ is uniformly convergent (and therefore $f_{\mathcal{A}}$ is continuous).

For the converse, we need to assume minimality:

3. If $\mathcal{A}$ is uniformly convergent and minimal then $B$ is a continuous RCP set.

Proof. (1) If the limit $A_w = \lim_{k \to \infty} A_{\text{pref}_k(w)}$ exists then

$$f_{\mathcal{A}}(w) = \lim_{k \to \infty} F_{\mathcal{A}}(\text{pref}_k(w)) = IA_w F.$$

As $A_w$ is defined everywhere, so is $f_{\mathcal{A}}$. 

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Similarly to the first proof, we have \( f_A(w) = IA_wF \) where \( w \mapsto A_w \) is a continuous function, so \( w \mapsto f_A(w) \) is continuous. Uniform convergence follows from Corollary 5, continuity from Lemma 9.

This is precisely Corollary 28.

The uniform convergence and minimality conditions in the third statement are both necessary, as we can see from the following two examples where \( f_A \) is continuous but \( A \) is not even RCP:

**Example 41.** We construct a counterexample that is ap (and thus uniformly convergent by Lemma 21) but not minimal. Let \( I = (0,1), F = (0,1)^T \) and

\[
A_0 = A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

This automaton is ap and computes the constant function \( f_A(w) = 1 \), yet the set \( \{A_0, A_1\} \) is not RCP.

**Example 42.** To obtain a minimal automaton that computes a continuous function, but does not have RCP set of transition matrices, take the automaton \( \mathcal{A} \) from Example 11. This automation computes the zero \( \omega \)-function and has transition matrices:

\[
A_0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

Now observe that

\[
A_0^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 0 \end{pmatrix},
\]

so \( \mathcal{A} \) is not RCP.

The next example shows that even if \( \mathcal{A} \) is minimal and ap, and if \( f_A \) is everywhere defined and continuous everywhere except at one point, we can not infer that \( A \) is RCP.

**Example 43.** Let \( I = (1,0), F = (0,1)^T \) and

\[
A_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}
\]

It is easy to see that \( \mathcal{A} \) is both ap and minimal. Moreover, we have

\[
F(0^n) = (-1)^n IF = 0 \\
F(0^n1w) = (-1)^n \\
F(0^n2w) = (-1)^{n+1}
\]

for every \( w \in \Sigma^* \). This means that \( f_A \) is defined everywhere. However, \( f_A \) is not continuous at \( 0^\omega \). The set \( A \) is not RCP, because we have

\[
A_0^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}.
\]
3.4. Decision problems for $\omega$-functions

In this part, we present several results about decidability of various properties of the $\omega$-function $f_A$ in the case of ap automata. In particular, we are interested to know how to determine if the $\omega$-function $f_A$ is everywhere defined, or everywhere continuous. It turns out that the questions are closely related to the decidability status of the matrix stability problem: If it is undecidable whether a given finite set of matrices is stable then it is also undecidable for a given ap WFA $A$ whether $f_A$ is everywhere defined, or whether $f_A$ is continuous. We also show that in this case it is undecidable if a given finite matrix set is RCP, or if it is continuous RCP. Conversely, if it were the case that stability is decidable then continuity of $f_A$ is decidable, as is the question of whether a given matrix set is continuous RCP. The central algorithmic problem is therefore the following:

**Matrix Product Stability:**

**Input:** A finite set $\{A_a \mid a \in \Sigma\}$ of $n \times n$ matrices.

**Question:** Is $\{A_a \mid a \in \Sigma\}$ stable?

We begin with the equivalence problem of two ap WFA.

**Theorem 44.** Given two ap WFA $A$ and $B$ such that at least one of the $\omega$-functions $f_A$ and $f_B$ is everywhere defined, one can algorithmically decide whether $f_A = f_B$.

**Proof.** To decide $f_A = f_B$, we construct ap automaton $C$ computing the difference $f_A - f_B$ and then minimize $C$, obtaining some automaton $D$. Minimization is effective by Lemma 13. Now from Lemma 15 we get that $f_A - f_B = 0$ iff $D$ is the trivial automaton. Note that $f_A - f_B$ is not defined on those $w \in \Sigma^\omega$ for which exactly one of the functions $f_A$ and $f_B$ is undefined. Hence $f_A - f_B = 0$ is equivalent to $f_A = f_B$.

Note that the process in the previous proof fails if $f_A = f_B$ is not everywhere defined: in this case also $f_A - f_B$ will be undefined for some $w \in \Sigma^\omega$, yielding (wrongly) a negative answer.

In contrast to Theorem 44, if **Matrix Product Stability** is undecidable then the analogous question is undecidable without the ap assumption. In this case one cannot even determine if a given non-ap WFA defines the zero-function.

**Theorem 45.** **Matrix Product Stability** is algorithmically reducible to the problem of determining if $f_A = 0$ for a given WFA $A$.

**Proof.** Given a set of matrices $B = \{A_a \mid a \in \Sigma\}$, we construct automata $A_{ij}$ with transition matrices $A_a$, initial distribution $I_i$, and final distribution $I_j^F$ (where $I_1, \ldots, I_n$ is a basis of $\mathbb{R}^n$). Obviously, $B$ is stable iff all the $\omega$-functions computed by $A_{ij}$ are zero.

We conjecture that Theorem 45 holds even under the additional assumption that $f_A$ is known to be everywhere defined and continuous, but we can not offer a proof.

Recall that Theorem 18 tells us that for every WFA computing a continuous function there is an ap WFA that computes the same function. It would be interesting to know
whether this conversion can be done effectively. One consequence of Theorems \[44\] and \[45\] is that, assuming Matrix Product Stability is undecidable, we cannot effectively convert a non-ap WFA into an ap WFA with the same \(\omega\)-function.

In the following we reduce Matrix Product Stability to the following decision problems:

**AP-WFA Convergence:**

*Input:* An average preserving WFA \(A\).
*Question:* Is \(f_A\) everywhere defined?

**AP-WFA Continuity:**

*Input:* An average preserving WFA \(A\).
*Question:* Is \(f_A\) everywhere continuous?

**Matrix Product Convergence:**

*Input:* A finite set \(\{A_a \mid a \in \Sigma\}\) of \(n \times n\) matrices.
*Question:* Is \(\{A_a \mid a \in \Sigma\}\) an RCP set?

**Matrix Product Continuity:**

*Input:* A finite set \(\{A_a \mid a \in \Sigma\}\) of \(n \times n\) matrices.
*Question:* Is \(\{A_a \mid a \in \Sigma\}\) a continuous RCP set?

To simplify our constructions, we use the fact that the problems Matrix Product Stability, Matrix Product Convergence and Matrix Product Continuity are as hard for a pair of matrices as they are for any finite number of matrices, see \[15\]. The elementary proof for Matrix Product Stability we present is based on \[16\].

**Lemma 46.** The Matrix Product Stability problem for a set \(\{A_1, A_2, \ldots, A_m\}\) of matrices, is algorithmically reducible to Matrix Product Stability for a pair of matrices \(\{B_0, B_1\}\).

*Proof.* For given \(m\) matrices \(A_1, A_2, \ldots, A_m\) of size \(n \times n\) we construct two matrices of size \(mn \times mn\) that in the block form are

\[
B_0 = \begin{pmatrix}
0 & E_{m(n-1)} \\
0 & 0
\end{pmatrix} \quad B_1 = \begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
A_m & 0 & \ldots & 0
\end{pmatrix}.
\]

Here \(E_{m(n-1)}\) is the \(m(n-1) \times m(n-1)\) identity matrix, and \(0\) indicates the zero matrix of appropriate size.

In the same way that we produce graphs of WFA, we construct the graph in Figure 3 (We are actually constructing a WFA over the ring of \(n \times n\) matrices.)

Consider now the matrix \(B_v\) where \(v \in \{0, 1\}^\ast\). This matrix can be divided into \(m \times m\) blocks of size \(n \times n\). To calculate the value of the block at the position \(i, j\), we add up all the products along all paths labeled by \(v\) from vertex \(i\) to vertex \(j\). Due to the shape of the
Figure 3: Directed graph whose paths correspond to blocks in the products of $B_0$ and $B_1$ in the proof of Lemma 46.

In the graph in Figure 3, there will be always at most one such path for each $i, j, v$ and each $B_v$ will have at most $m$ nonzero $n \times n$ blocks.

Moreover, it is easy to see that the blocks in infinite products are exactly all the infinite products of matrices $A_i$ (or zero matrices), so it is clear that $\{B_0, B_1\}$ is stable if and only if $\{A_1, A_2, \ldots, A_m\}$ is stable.

**Theorem 47.** Matrix Product Stability is algorithmically reducible to problems AP-WFA convergence, AP-WFA continuity, Matrix Product Convergence and Matrix Product Continuity.

**Proof.** Let $B = \{B_\alpha \mid \alpha \in \Sigma\}$ be a set of matrices whose stability we want to decide. Thanks to Lemma 46, we can assume $\Sigma = \{0, 1\}$.

We create several ap-automata $A_{ij}$ such that:

- if $B$ is stable then the function $f_{A_{ij}}$ is continuous and the matrices of $A_{ij}$ form a continuous RCP set for each $i, j$, while

- if $B$ is not stable then for some $i, j$ the function $f_{A_{ij}}$ is not everywhere defined and the transition matrices of $A_{ij}$ are not an RCP set.

The result then follows directly.

We choose the transition matrices for $A_{ij}$ as follows:

$$A_0 = \begin{pmatrix} B_0 & b_{0,j} \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} B_1 & b_{1,j} \\ 0 & 1 \end{pmatrix}.\]
Here the column vectors $b_{0,j}$ and $b_{1,j}$ have all entries zero except for the $j$-th. The $j$-th entry of $b_{0,j}$ is 1 while the $j$-th entry of $b_{1,j}$ is $-1$.

The initial distribution of $A_{ij}$ is $I_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with one on the $i$-th place. The final distribution is the same for all automata: $F = (0, \ldots, 0, 1)^T$.

First observe that
\[
(A_0 + A_1)F = \begin{pmatrix}
    B_0 + B_1 & 0 \\
    0 & 2
\end{pmatrix} \cdot \begin{pmatrix}
    0 \\
    \vdots \\
    0 \\
    1
\end{pmatrix} = \begin{pmatrix}
    0 \\
    \vdots \\
    0 \\
    2
\end{pmatrix} = \Sigma \cdot F,
\]
so each $A_{ij}$ is ap. If $B$ is a stable set then from the case (3) of Lemma 36 we obtain that for all $i, j$ the set $\{a \mid a \in \Sigma\}$ is a continuous RCP set, and therefore all $f_{A_{ij}}$ are continuous.

Assume then that $B$ is not stable, so for some $w$ the limit $\lim_{k \to \infty} B_{\text{pref}_k(w)}$ is not zero or does not exist. Then there exists a pair $(i, j)$ such that the sequence $\{(B_{\text{pref}_k(w)})_{ij}\}_{k=1}^{\infty}$ does not converge to zero. Consider the value of $F_{A_{ij}}(\text{pref}_k(w))$. The product of transition matrices will be
\[
A_{\text{pref}_k(w)} = \begin{pmatrix}
    B_{\text{pref}_k(w)} & b_k \\
    0 & 1
\end{pmatrix},
\]
where $b_k$ is some column vector. The value of $F_{A_{ij}}(\text{pref}_k(w))$ is equal to $I_i A_{\text{pref}_k(w)} F$, which, after a short calculation, turns out to be the $i$-th element of $b_k$.

Moreover, it is straightforward to verify that the vectors $b_k$ satisfy the equation $b_{k+1} = b_k + B_{\text{pref}_k(w)} b_{w_{k+1,j}}$. Taking the $i$-th element of $b_{k+1}$, we get the equation for $F_{A_{ij}}(\text{pref}_k(w))$:
\[
F_{A_{ij}}(\text{pref}_{k+1}(w)) = F_{A_{ij}}(\text{pref}_k(w)) + c_{w_{k+1}} (B_{\text{pref}_k(w)})_{ij},
\]
where $c_{w_{k+1}}$ is the $j$-th element of $b_{w_{k+1,j}}$, i.e. either 1 or $-1$. We obtain
\[
|F_{A_{ij}}(\text{pref}_{k+1}(w)) - F_{A_{ij}}(\text{pref}_k(w))| = \left|(B_{\text{pref}_k(w)})_{ij}\right|.
\]
Because the sequence $\{(B_{\text{pref}_k(w)})_{ij}\}_{k=1}^{\infty}$ does not tend to zero, neither does the difference $|F_{A_{ij}}(\text{pref}_{k+1}(w)) - F_{A_{ij}}(\text{pref}_k(w))|$. But then the sequence of values $\{F_{A_{ij}}(\text{pref}_k(w))\}_{k=1}^{\infty}$ does not satisfy the Bolziano-Cauchy condition and can not converge. Therefore, $f_{A_{ij}}(w)$ remains undefined.

By Theorem 40 the matrices of $A_{ij}$ are then not an RCP set, which concludes the proof.

\[\square\]

**Remark 48.** Note that the proof showed, in fact, more: if Matrix Product Stability is undecidable then the continuous ap WFA are recursively inseparable from the ap WFA that are not everywhere defined. Recall that two disjoint sets $A, B$ are called recursively inseparable if there does not exist an algorithm that on input $x$ returns value 0 if $x \in A$, value 1 if $x \in B$ and may return either value if $x \not\in A \cup B$. If membership in either $A$
or $B$ is decidable then clearly $A$ and $B$ are not recursively inseparable, but the converse is not true. The reduction in the previous proof always produced ap WFA whose $\omega$-function is either everywhere continuous, or not everywhere defined, so the recursive inseparability follows directly.

Analogously, the proof shows that if Matrix Product Stability is undecidable then one cannot recursively separate those finite matrix sets that are continuously RCP from those that are not RCP.

Next we consider the implications if Matrix Product Stability turns out to be decidable.

**Theorem 49.** Problems AP-WFA continuity and Matrix Product Continuity are algorithmically reducible to Matrix Product Stability.

*Proof.* The reduction from Matrix Product Continuity to Matrix Product Stability was proved in [11]. Let us prove the reduction from AP-WFA continuity, so let $A$ be a given ap automaton whose continuity we want to determine.

We begin by minimizing $A$. If the resulting automaton computes the zero function, we are done. Otherwise, we run the procedure from Lemma 35 to obtain the form (5) of transition matrices. If any step of the algorithm fails (that is, nontrivial $I_c$ does not exist), $A$ can not define a continuous function. Otherwise, $f_A$ is continuous iff $\{B_a | a \in \Sigma\}$ in (5) is stable.

From Theorems 47 and 49 we conclude that decision problems Matrix Product Stability, AP-WFA continuity and Matrix Product Continuity are computationally equivalent.

If we drop the requirement that the WFA is ap, we can make the following observation:

**Theorem 50.** Matrix Product Convergence is algorithmically reducible to the problem of determining if a given WFA is everywhere defined.

*Proof.* Use the same reduction as in the proof of Theorem 45.

4. Real functions defined by WFA

Let $\Sigma = \{0, 1\}$ be the binary alphabet and $A$ a WFA over $\Sigma$. Then we can use $f_A$ to define the real function $\hat{f}_A : [0, 1) \rightarrow \mathbb{R}$ via the binary addressing scheme on the half-open interval $[0, 1)$. For $w \in \Sigma^\omega$ denote by $\text{num}(w)$ the real number

$$\text{num}(w) = \sum_{i=1}^{\infty} w_i 2^{-i}.$$ 

Let $\Omega = \Sigma^\omega \setminus \Sigma^*1^\omega$. It is easy to see that by taking $\text{num}_{|\Omega}$, we obtain a one-to-one correspondence between words of $\Omega$ and numbers in the interval $[0, 1)$. Denote $\text{bin}$ the inverse mapping to $\text{num}_{|\Omega}$, i.e.

$$\forall w \in \Omega, \text{bin}(x) = w \iff \text{num}(w) = x.$$
We emphasize that the correspondence is between sets $[0, 1)$ and $\Omega$, not $[0, 1)$ and $\Sigma^\omega$. A point $x \in [0, 1)$ with a word presentation of the form $bin(x) = v0^\omega$ for some $v \in \Sigma^*$ is called dyadic. Points without such a presentation are non-dyadic.

Let $f$ be a (partial) function from $\Sigma^\omega$ to $\mathbb{R}$. Then we define the corresponding (partial) real function $\hat{f} : [0, 1) \to \mathbb{R}$ by:

$$\hat{f}(x) = f(bin(x)).$$

As usual, if $f(bin(x))$ is not defined then $\hat{f}(x)$ remains undefined.

4.1. Continuity of real functions defined by WFA

We will call the real function $\hat{f}$ continuous resp. uniformly continuous if it is continuous resp. uniformly continuous in the whole $[0, 1)$. Note that $\hat{f}$ is uniformly continuous iff it can be extended to a continuous function on the whole closed interval $[0, 1]$.

The following two examples show that the function $\hat{f}_A$ can be continuous without being uniformly continuous: in these examples the left limit $\lim_{x \to 1^-} \hat{f}_A(x)$ does not exist.

**Example 51.** The ap WFA in Figure 4 computes a piecewise linear function $\hat{f}_A : [0, 1) \to \mathbb{R}$ that does not have the left limit at point 1 (see its graph in Figure 5). The $\omega$-function $f_A$ is everywhere defined, but the convergence at point $1^\omega$ is not uniform. Note that $f_A(1^\omega) = 1/2$. Function $f_A$ is continuous at all points except $1^\omega$.

**Example 52.** The ap WFA in Figure 6 computes a piecewise linear function that maps $1 - 1/2^n \mapsto 2^n - 1$ for $n \in \mathbb{N}$. See the graph in Figure 7. Obviously, $\lim_{x \to 1^-} f(x) = \infty$. The $\omega$-function $f_A$ is not defined at point $1^\omega$.

The following Lemma establishes correspondence between the continuity of the real function $\hat{f}$ and the corresponding $\omega$-function $f$ in its relevant domain $\Omega$. Continuity of $f$ in $\Omega$ corresponds to the continuity of $\hat{f}$ at all non-dyadic points together with continuity of $\hat{f}$ from the right at all dyadic points.
Figure 5: Graph from Example 51

Figure 6: Automaton from Example 52
Lemma 53. Let $f$ be any $\omega$-function, and let $\hat{f}$ be the corresponding real function. Let $x \in [0,1)$ and denote $w = \text{bin}(x)$. Function $f$ is continuous at $w$ as a function $\Omega \to \mathbb{R}$ if and only if $\hat{f}$ is continuous (continuous from the right) at the point $x$, provided $x$ is non-dyadic (dyadic, respectively).

Proof. Let us show first that for $u, v \in \Sigma^\omega$, we have the inequality between the Euclidean and Cantor metrics

$$d_E(\text{num}(u), \text{num}(v)) \leq d_C(u, v).$$

Let $d_C(u, v) = 2^{-j}$. Then $u_i = v_i$ for all $1 \leq i \leq j$. Therefore

$$d_E(\text{num}(u), \text{num}(v)) = \left| \sum_{i=1}^{\infty} (u_i - v_i)2^{-i} \right| = \left| \sum_{i=j+1}^{\infty} (u_i - v_i)2^{-i} \right|$$

$$\leq \sum_{i=j+1}^{\infty} |(u_i - v_i)|2^{-i} \leq 2^{-j} = d_C(u, v).$$

We have obtained for all $u, v \in \Sigma^\omega$ the implication

$$d_C(u, v) < \delta \implies d_E(\text{num}(u), \text{num}(v)) < \delta,$$

so it follows directly that the continuity of $\hat{f}$ at $\text{num}(w)$ implies the continuity of $f$ at $w \in \Omega$. Suppose now that $\text{num}(w)$ is dyadic. Then $w = v0^\omega$ for some finite word $v$ of length $k$. We
have
\[ d_C(u, w) \leq 2^{-k} \implies \text{num}(u) \geq \text{num}(w), \]
so in this case continuity of \( \hat{f} \) at \( \text{num}(w) \) from the right is enough to obtain the continuity of \( f \) at \( w \).

Let us prove the converse direction. Suppose that \( f \) is continuous at \( w \in \Omega \). For every \( k \) there exists \( \delta > 0 \) such that whenever \( \text{num}(w) \leq \text{num}(v) < \text{num}(w) + \delta \), then \( \text{pref}_k(w) = \text{pref}_k(v) \). We can accomplish this by choosing \( \delta = \text{num}(\text{pref}_k(w)1^\omega) - \text{num}(w) \).

Similarly, if \( w \) does not end in \( 0^\omega \) (i.e. \( \text{num}(w) \) is not dyadic), we can choose \( \delta = \text{num}(w) - \text{num}(\text{pref}_k(w)0^\omega) \) and see that \( \text{num}(w) - \delta < \text{num}(v) \leq \text{num}(w) \) implies \( \text{pref}_k(w) = \text{pref}_k(v) \).

This means that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ d_E(\text{num}(u), \text{num}(w)) < \delta \text{ (and } \text{num}(u) > \text{num}(w) \text{ if } \text{num}(w) \text{ is dyadic) } \implies d_C(u, w) < \varepsilon. \]
This is enough to see that \( \hat{f} \) is continuous at \( x = \text{num}(w) \) if \( x \) is not dyadic, and continuous from right if \( x \) is dyadic.

The following example shows that Lemma 53 cannot be extended to continuity from the left at dyadic points.

**Example 54.** Let \( \mathcal{A} \) be a WFA with
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.
\]
It is easy to see that \( f_A(1v) = 1 \) and \( f_A(0v) = 0 \) for all \( v \in \Sigma^\omega \). Clearly, \( f_A \) is continuous: For each \( w, w' \in \Sigma^\omega \), \( d_C(w, w') < 1 \) implies \( d_E(f_A(w), f_A(w')) = 0 \). However, \( \hat{f}_A \) is not continuous at the point \( x = 1/2 \), as \( \hat{f}_A(1/2) = 1 \), but \( \hat{f}_A(y) = 0 \) for any \( y < 1/2 \).

Based on Lemma 53 we can now characterize those real functions \( \hat{f} \) whose corresponding \( \omega \)-function \( f \) is continuous or uniformly continuous in \( \Omega \).

**Corollary 55.** Let \( f \) be an \( \omega \)-function and let \( \hat{f} \) be the corresponding real function. Then:

1. Function \( f \) is continuous in the set \( \Omega \) if and only if \( \hat{f} \) is continuous at every non-dyadic point and continuous from the right at every dyadic point.
2. Function \( f \) is uniformly continuous in the set \( \Omega \) if and only if \( \hat{f} \) is continuous at every non-dyadic point, continuous from the right at every dyadic point, and has a limit from the left at all nonzero dyadic points as well as at the point \( x = 1 \).

Note that \( f \) might not even be defined at points in \( \Sigma^*1^\omega \).
Proof. Part (1) follows directly from Lemma 53, so we focus on part (2).

Suppose that \( f \) is uniformly continuous in \( \Omega \). By part (1) it is sufficient to show that \( \hat{f} \) has a limit from the left at each point \( \text{num}(v1^\omega) \) for \( v \in \Sigma^* \). As \( \Omega \) is dense in \( \Sigma^\omega \), there exists a (unique) continuous \( g : \Sigma^\omega \to \mathbb{R} \) such that \( g|_\Omega = f|_\Omega \). Then

\[
\lim_{x \to \text{num}(v1^\omega)_-} \hat{f}(x) = \lim_{w \to v1^\omega} f(w) = \lim_{w \to v1^\omega} g(w) = g(v1^\omega),
\]

so the limit exists.

For the other direction of (2), assume that \( \hat{f} \) has a limit from the left at all dyadic points, including 1. By (1) we have that \( f \) is continuous in \( \Omega \). We need to prove that \( f \) is uniformly continuous in \( \Omega \). We show this by constructing a continuous \( g : \Sigma^\omega \to \mathbb{R} \) such that \( g|_\Omega = f|_\Omega \). Uniform continuity of \( f \) then follows from the compactness of \( \Sigma^\omega \). For every \( v1^\omega \), set

\[
g(v1^\omega) = \lim_{x \to \text{num}(v1^\omega)_-} \hat{f}(x) = \lim_{w \to v1^\omega} f(w),
\]

while for \( w \in \Omega \) we let \( g(w) = f(w) \). Because the limit from the left exists at every \( \text{num}(v1^\omega) \), the function \( g \) is everywhere defined. It remains to verify that \( g \) is continuous in \( \Sigma^\omega \).

Let \( v \in \Sigma^\omega \) and \( \varepsilon > 0 \). From the definition of \( g \) and properties of \( \hat{f} \) we obtain that there exists \( \delta > 0 \) such that

\[
\forall u \in \Omega, d_C(v, u) < \delta \Rightarrow |g(v) - g(u)| < \frac{1}{2}\varepsilon.
\]

Now whenever \( u = z1^\omega \) and \( d_C(v, u) < \delta \), the value \( g(u) \) is the limit of the sequence \( \{g(z1^n0^\omega)\}_{n=1}^\infty \) whose elements belong to \( \Omega \). Observe that for all \( n \) large enough we have \( d_C(v, z1^n0^\omega) < \delta \) and so \( |g(v) - g(z1^n0^\omega)| < \varepsilon/2 \). Therefore, \( |g(v) - g(u)| < \varepsilon \).

We have shown for all \( u \) that if \( d_C(v, u) < \delta \) then \( |g(v) - g(u)| < \varepsilon \), proving continuity. \( \square \)

Remark 56. By (2) of Corollary 55, uniform continuity of \( f \) in \( \Omega \) implies the existence of \( \lim_{x \to 1^-} \hat{f}(x) \). So in this case, if \( \hat{f} \) is continuous it is uniformly continuous. In particular, continuity of \( f \) in \( \Sigma^\omega \) and \( \hat{f} \) in \([0, 1) \) imply uniform continuity of \( \hat{f} \).

Uniform continuity of \( \hat{f} \) is stronger than uniform continuity of \( f \). The additional requirement is the continuity of \( \hat{f} \) from the left at all dyadic points:

Corollary 57. The function \( \hat{f} : [0, 1) \to \mathbb{R} \) obtained from the \( \omega \)-function \( f \) is uniformly continuous if and only if:

(1) Function \( f \) is uniformly continuous in \( \Omega \), and

(2) for all finite words \( v \), the equality \( g(v10^\omega) = g(v01^\omega) \) holds, where \( g \) is the (unique) continuous function \( g : \Sigma^\omega \to \mathbb{R} \) such that \( f|_\Omega = g|_\Omega \).
Proof. If \( \hat{f} \) is uniformly continuous in \([0, 1)\) then it has a right limit at \( x = 1 \), so \( \hat{f} \) satisfies the conditions in part (2) of Corollary 55. Therefore, \( f \) is uniformly continuous in \( \Omega \). Let \( g \) be the continuous extension of \( f \) to \( \Sigma^\omega \). Because \( \hat{g} \) is continuous at dyadic points, we have

\[
g(v10^\omega) = \lim_{w \to v01^\omega} g(w) = g(v01^\omega).
\]

Assume now that conditions (1) and (2) hold. Using Lemma 53, we obtain continuity of \( \hat{f} \) at non-dyadic points and continuity from the right at dyadic points. Now continuity of \( \hat{f} \) from the left at dyadic points follows from (2) and the continuity of \( g \).

We also have \( \lim_{x \to 1^-} \hat{f}(x) = g(1^\omega) \), so we can continuously extend \( \hat{f} \) to the whole interval \([0, 1]\), proving uniform continuity of \( \hat{f} \).

If \( \hat{f}_A \) is uniformly continuous then we know that \( f \) is uniformly continuous in \( \Omega \). Because \( \Sigma^*0^\omega \subseteq \Omega \) we can choose \( w = 0^\omega \) and \( \Delta = \Sigma^*0^\omega \) in Theorem 18 and obtain an average preserving WFA computing \( f \).

**Corollary 58.** If a uniformly continuous function \( \hat{f}_A \) is computed by some WFA \( A \), then there is an average preserving WFA \( B \) such that \( \hat{f}_A = \hat{f}_B \) and \( f_B \) is continuous in \( \Sigma^\omega \). Automaton \( B \) can be produced from \( A \) by first minimizing \( A \) and then changing the final distribution.

Note that \( B \) itself need not be right-minimal but we can minimize it. Putting together the Corollary 58 and Lemma 55, we obtain the main result of this section:

**Corollary 59.** If a nonzero uniformly continuous function \( \hat{f} \) is computed by some WFA \( A \) then \( \hat{f} \) is also computed by a minimal, average preserving WFA with transition matrices of the form

\[
A_i = \begin{pmatrix} B_i & b_i \\ 0 & 1 \end{pmatrix},
\]

where \( i = 0, 1 \) and \( \{B_0, B_1\} \) is a stable set of matrices.

4.2. Decision problems concerning the real function continuity

In this section we study how does the decision problem \textsc{Matrix Product Stability} relate to the problem of deciding the uniform continuity of the real function determined by a WFA.

Note that we do not address non-uniform continuity of \( \hat{f}_A \) for which Corollary 59 fails. On the other hand, by Corollary 58 any uniformly continuous \( \hat{f}_A \) is generated by an ap WFA with continuous \( f_A \), so we restrict the attention to such WFA. The decision problem of interest is then the following:

**AP-WFA uniform continuity:**

**Input:** An average preserving WFA \( A \) over the binary alphabet \( \Sigma = \{0, 1\} \).

**Question:** Are both \( f_A \) and \( \hat{f}_A \) everywhere continuous?
Note that the question is equivalent to asking about the uniform continuity of \( f_A \) and \( \hat{f}_A \) (see Remark 56).

To decide AP-WFA \textsc{uniform continuity} we need to verify that \( f_A \) is continuous and then check the condition (2) of Corollary 57. It turns out that, if \( A \) is ap and \( f_A \) continuous, condition (2) is easy to test.

**Lemma 60.** Let \( A \) be an average preserving WFA such that \( f_A \) is continuous on \( \Sigma^\omega \). Then condition (2) of Corollary 57 is decidable for the function \( f_A \).

**Proof.** As minimization is effective we can assume that the input automaton \( A \) is minimal and average preserving. First we can effectively check whether \( f_A = 0 \), in which case condition (2) of Corollary 57 is satisfied. Suppose than that \( f_A \neq 0 \). By Lemma 35 we can effectively transform the automaton to the form with transition matrices

\[
A_0 = \begin{pmatrix} B_0 & b_0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} B_1 & b_1 \\ 0 & 1 \end{pmatrix},
\]

where \( \{B_0, B_1\} \) is a stable set. Because \( f_A \) is continuous on \( \Sigma^\omega \) the condition (2) says that for all \( v \in \Sigma^* \)

\[
f_A(v10^\omega) = f_A(v01^\omega).
\]

From minimality we obtain that the sufficient and necessary condition for this to hold is that \( A_{01^\omega} = A_{10^\omega} \).

Consider matrices \( A_{01^k} \) and \( A_{10^k} \). They are of the following forms:

\[
A_{01^k} = \begin{pmatrix} B_0 B_1^k & b_0 + B_0 (\sum_{i=0}^{k-1} B_1^i) b_1 \\ 0 & 1 \end{pmatrix}, \quad A_{10^k} = \begin{pmatrix} B_1 B_0^k & b_1 + B_1 (\sum_{i=0}^{k-1} B_0^i) b_0 \\ 0 & 1 \end{pmatrix}.
\]

Observe that

\[
\sum_{i=0}^{k-1} B_0^i (E - B_0) = E - B_0^k \quad \text{and} \quad \sum_{i=0}^{k-1} B_1^i (E - B_1) = E - B_1^k.
\]

As the set \( \{B_0, B_1\} \) is stable, we must have \( B_0^n, B_1^n \to 0 \) and so all eigenvalues of both \( B_0 \) and \( B_1 \) must lie inside the unit disc. Thus the sums \( \sum_{i=0}^{\infty} B_0^i \) and \( \sum_{i=0}^{\infty} B_1^i \) converge. It follows that

\[
\sum_{i=0}^{\infty} B_0^i = (E - B_0)^{-1} \quad \text{and} \quad \sum_{i=0}^{\infty} B_1^i = (E - B_1)^{-1}.
\]

This means that we have the limits:

\[
A_{01^\omega} = \begin{pmatrix} 0 & b_0 + B_0 (E - B_1)^{-1} b_1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad A_{10^\omega} = \begin{pmatrix} 0 & b_1 + B_1 (E - B_0)^{-1} b_0 \\ 0 & 1 \end{pmatrix}.
\]
So we are left with the simple task of checking the equality
\[ b_0 + B_0(E - B_1)^{-1}b_1 = b_1 + B_1(E - B_0)^{-1}b_0. \]

We are ready to prove the main result of this section. Recall that for \( \mathcal{A} \) average preserving, continuity of \( f_\mathcal{A} \) is computationally as hard as stability. We show that also simultaneous continuity of \( f_\mathcal{A} \) and \( \hat{f}_\mathcal{A} \) is as hard.

**Theorem 61.** Decision problems Matrix Product Stability and Ap-WFA uniform continuity can be algorithmically reduced to each other.

**Proof.** Suppose first that Matrix Product Stability is decidable, and let \( \mathcal{A} \) be a given ap WFA over the binary alphabet. By Theorem 49 we can effectively determine if \( f_\mathcal{A} \) is continuous in \( \Sigma^\omega \). If the answer is positive then – according to Lemma 60 – we can effectively check whether the function \( f_\mathcal{A} \) satisfies the condition (2) in Lemma 57. By Lemma 57 this is enough to determine whether \( \hat{f}_\mathcal{A} \) is uniformly continuous, so we get the answer to Ap-WFA uniform continuity.

For the converse direction, let us assume that Ap-WFA uniform continuity is decidable. By Lemma 46 it is enough to show how we can determine if a given pair \( \{B_0, B_1\} \) of \( n \times n \) matrices is stable. Because we can check whether \( \lim_{n \to \infty} B_i^n = 0 \) for \( i = 0, 1 \) (using the Lyapunov equation method as in [14, page 169]), we can assume that \( \{B_0\} \) and \( \{B_1\} \) are stable sets.

In the following we effectively construct several ap WFA \( \mathcal{A}_{ij} \) over the binary alphabet such that

- if \( \{B_0, B_1\} \) is stable then the functions \( f_{\mathcal{A}_{ij}} \) and \( \hat{f}_{\mathcal{A}_{ij}} \) are continuous for each \( i, j \), while
- if \( \{B_0, B_1\} \) is not stable then for some \( i, j \) the function \( f_{\mathcal{A}_{ij}} \) is not continuous.

The result then follows directly. The construction of \( \mathcal{A}_{ij} \) is similar to the proof of Theorem 47. Again, we write down the transition matrices in the the block form
\[
A_0 = \begin{pmatrix} B_0 & C_0 \\ 0 & D_0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} B_1 & C_1 \\ 0 & D_1 \end{pmatrix},
\]
only this time, instead of constant \( D_0 = D_1 = 1 \), we use the \( 3 \times 3 \) matrices
\[
D_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.
\]

These matrices (with initial and final distributions \( I = (1, 0, 0) \) and \( F = (1/2, 1/2, 1)^T \)) form a minimal ap WFA \( \mathcal{D} \) that computes the continuous real function shown in Figure 8. An important feature of this function, implicit in the proof below, is the fact that it has
value zero at both endpoints of the domain interval. Also, by Theorem \[40\] \( \{D_0, D_1\} \) is a continuous RCP set.

Let \( i, j \in \{1, \ldots n\} \). Denote by \( C_0 \) the following \( n \times 3 \) matrix:

\[
C_0 = \begin{pmatrix}
0 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 0
\end{pmatrix},
\]

where the single 1 is in the \( j \)-th row. Let \( C_1 = -C_0 \).

We now construct the ap WFA \( A_{ij} \) with transition matrices

\[
A_0 = \begin{pmatrix}
B_0 & C_0 \\
0 & D_0
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
B_1 & C_1 \\
0 & D_1
\end{pmatrix},
\]

initial distribution \( I_i \) (\( i \)-th element of the canonical basis) and final distribution \( F = (0, \ldots, 0, 1/2, 1/2, 1)^T \).

Assume for a moment that \( f_{A_{ij}} \) is continuous. We show that then

\[
G = \lim_{n \to \infty} A_0^n F = (0, \ldots, 0, 0, 1)^T, \quad H = \lim_{n \to \infty} A_1^n F = (0, \ldots, 0, 1, 1)^T.
\]
Consider only $G$; the case of $H$ is similar. As $\{B_0\}$ is stable and $\{D_0\}$ is RCP, an application of Lemma 36 on the singleton set $\{A_0\}$ shows that the limit $G$ exists. The vector $G$ is a 1-eigenvector of $A_0$ and by direct computation we obtain that the last three elements of $G$ are $0, 0, 1$.

Notice now that the vector $G' = (0, \ldots, 0, 1)^T$ is a 1-eigenvector of $A_0$. Were $G \neq G'$, we would have the 1-eigenvector $G - G'$ whose last three elements are zero. But then the first $n$ elements of $G - G'$ form a 1-eigenvector of $B_0$ and so $\{B_0\}$ is not stable, a contradiction. Thus $G = G'$. The proof that $H = (0, \ldots, 0, 1, 1)^T$ is analogous.

We are now ready to finish the proof. Assume first that $\{B_0, B_1\}$ is a stable set. We claim that then $f_{A_{ij}}, \hat{f}_{A_{ij}}$ are both continuous. Now the general form of Lemma 36 comes into play: According to part (3) of that Lemma, the set $\{A_0, A_1\}$ is a continuous RCP set and so, by Theorem 10, the function $f_{A_{ij}}$ is continuous.

By Corollary 57 we only need to show that condition (2) of that Corollary is satisfied. We can compute the limits

$$\lim_{n \to \infty} A_1 A_0^n F = A_1 G = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix} = A_0 H = \lim_{n \to \infty} A_0 A_1^n F.$$ 

This implies that $f_{A_{ij}}(v01^\omega) = f_{A_{ij}}(v10^\omega)$ for all $v \in \Sigma^*$, so by Corollary 57 the function $\hat{f}_{A_{ij}}$ is continuous.

Suppose then that the set $\{B_0, B_1\}$ is not stable. Then there exist $i, j$ and $w \in \Sigma^*$ such that for some $\varepsilon > 0$ there are infinitely many $n$ such that $|((B_{pref_n(w)})_{i,j})| > \varepsilon$. Consider the automaton $A_{ij}$ for these $i, j$. We want to prove that $f_{A_{ij}}$ is not continuous in this case.

We will proceed by contradiction, assuming that $f_{A_{ij}}$ is continuous. Then $A_{ij}$ is uniformly convergent by Lemma 21. Then from Lemma 23 and continuity of $f_{A_{ij}}$ we obtain that

$$\lim_{n \to \infty} \left[ 2F_{A_{ij}}(pref_n(w)0) - f_{A_{ij}}(pref_n(w)10^\omega) - f_{A_{ij}}(pref_n(w)0^\omega) \right] = 0.$$ 

This means that $\lim_{n \to \infty} IA_{pref_n(w)}(2A_0 F - A_1 G - G) = 0$. However, a straightforward calculation shows that the vector $2A_0 F - A_1 G - G$ is the $j$-th element of the canonical basis and so $IA_{pref_n(w)}(2A_0 F - A_1 G - G) = (A_{pref_n(w)})_{i,j}$ which does not converge to zero. Therefore, $f_{A_{ij}}$ can not be continuous.

\textbf{Remark 62.} Analogously to Remark 18 we can note that in the case that \textit{Matrix Product Stability} is undecidable we have in fact showed the recursive inseparability of ap WFA whose $f_A$ and $\hat{f}_A$ are both continuous from those ap WFA whose $f_A$ is not continuous.

4.3. \textit{Constructing WFA defining continuous real functions}

We end our paper by giving a few notes on how to construct nontrivial ap WFA with continuous $f_A$ and $\hat{f}_A$, for all initial distributions.
Lemma 63. Let $\mathcal{A}$ be a left-minimal ap automaton. Then the following statements are equivalent:

1. $f_{\mathcal{A}}$ is constant
2. $F_{\mathcal{A}}$ is constant
3. $A_a F = F$ for all $a \in \Sigma$.

Proof. Implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious. We prove (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

Assume (1): $f_{\mathcal{A}}(w) = c$ for all $w \in \Sigma^\omega$. Then $f_{\mathcal{A}}$ is the $\omega$-function corresponding to both $F_{\mathcal{A}}$ and the constant word function $G(u) = c$. Then $F_{\mathcal{A}} - G$ is an average preserving word function whose $\omega$-function is zero, so by Lemma 15 we have $F_{\mathcal{A}} - G = 0$, i.e. condition (2) holds.

To prove (3), assuming (2), we note that for all $u \in \Sigma^*$ and $a \in \Sigma$ the equality

$$ I A_u A_a F = F_{\mathcal{A}}(u a) = F_{\mathcal{A}}(u) = I A_u F $$

holds. By left minimality this implies $A_a F = F$.

Notice that even without left-minimality we have the following: if $\mathcal{A}$ is an ap WFA such that $A_a F \neq F$ for some $a \in \Sigma$ then there exists a choice for the initial distribution $I$ such that $f_{\mathcal{A}}$ is not constant.

Lemma 64. Let $\{B_0, B_1\}$ be a stable set of matrices. Then $\det(B_0 + B_1 - 2E) \neq 0$.

Proof. Were it not the case, there would exist a vector $v \neq 0$ such that for each $n$ we would have

$$ \left( \frac{B_0 + B_1}{2} \right)^n v = v. $$

Then we can write:

$$ \|v\| = \left\| \left( \frac{B_0 + B_1}{2} \right)^n v \right\| = \left\| \sum_{w \in \Sigma^n} B_w v \right\| \leq \sum_{w \in \Sigma^n} \|B_w\| \|v\|. $$

However, by Lemma 2 there exists $n$ such that $\|B_w\| < 1$ for each $w$ of length $n$. For such $n$,

$$ \|v\| < \sum_{w \in \Sigma^n} \frac{1}{2^n} \|v\| = \|v\|, $$

a contradiction.

The following theorem (and its proof) gives us tools to generate ap WFA with non-constant, continuous $f_{\mathcal{A}}$ and $\hat{f}_{\mathcal{A}}$. We know from Corollary 59 that we can limit the search to ap WFA with transition matrices in the form 59, for stable $\{B_0, B_1\}$. The minimality condition can be replaced by the weaker concept that all initial distribution yield a continuous WFA.

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Theorem 65. Let \( \{B_0, B_1\} \) be a stable set of matrices. Consider the problem of finding vectors \( b_0, b_1 \) and a final distribution \( F \) so that, for any choice of the initial distribution \( I \), the transition matrices

\[
A_i = \begin{pmatrix} B_i & b_i \\ 0 & 1 \end{pmatrix}, \quad i = 0, 1,
\]

describe an ap WFA \( A \) with continuous \( f_A \) and \( \hat{f}_A \). We also want \( A_0 F \neq F \), so that for some initial distribution \( A \) does not define the constant function.

1. If \( \det(B_0 + B_1 - E) = 0 \) then we can algorithmically find such vectors \( b_0, b_1 \) and \( F \).
2. If \( \det(B_0 + B_1 - E) \neq 0 \) then such choices do not exist: only the constant function \( f_A \) can be obtained.

Proof. We are going to obtain sufficient and necessary conditions for the vectors \( b_0, b_1 \) and \( F \).

By definition, the ap condition is \( (A_0 + A_1)F = 2F \). Let \( F' \) be the vector obtained from \( F \) by removing the last element. Note that the last element of \( F \) cannot be zero, because then the ap condition would require \( (B_0 + B_1)F' = 2F' \), which only has the solution \( F' = 0 \) by Lemma 64. Without loss of generality, we fix the last element of \( F \) to be 1. (We can do this because multiplication of the final distribution by any non-zero constant \( c \) has only the effect of multiplying \( f_A \) and \( \hat{f}_A \) by \( c \).)

The ap condition becomes

\[
\begin{pmatrix} B_0 + B_1 & b_0 + b_1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} F' \\ 1 \end{pmatrix} = 2 \begin{pmatrix} F' \\ 1 \end{pmatrix},
\]

that is,

\[
(B_0 + B_1 - 2E)F' + b_0 + b_1 = 0. \tag{6}
\]

From Lemma 64, we have that \( B_0 + B_1 - 2E \) is regular. This means that for any choice of vectors \( b_0, b_1 \) there is a unique \( F' \), given by (6), that makes the WFA average preserving.

The requirement that \( f_A \) is continuous is automatically satisfied as \( \{B_0, B_1\} \) is stable (the case (3) of Lemma 36 and the case (2) of Theorem 40). By Corollary 57 continuity of \( \hat{f}_A \) is then equivalent to the condition \( f_A(v10^\omega) = f_A(v01^\omega) \) for all \( v \in \Sigma^* \). Since we require \( \hat{f}_A \) to be continuous for all initial distributions, we have the equivalent condition that

\[
\lim_{k \to \infty} (A_0 A_1^k) F = \lim_{k \to \infty} (A_1 A_0^k) F.
\]

As in the proof of Lemma 60, we obtain

\[
A_{01^\omega} = \begin{pmatrix} 0 & b_0 + B_0(E - B_1)^{-1}b_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{10^\omega} = \begin{pmatrix} 0 & b_1 + B_1(E - B_0)^{-1}b_0 \\ 0 & 1 \end{pmatrix}.
\]
Therefore, we can rewrite $A_{01}F = A_{10}F$ as an equation for vectors $b_0$ and $b_1$:

$$b_0 + B_0(E - B_1)^{-1}b_1 = b_1 + B_1(E - B_0)^{-1}b_0.$$  

This can be written equivalently as

$$(B_0 + B_1 - E)[(E - B_1)^{-1}b_1 - (E - B_0)^{-1}b_0] = 0$$  

(7)

So choices of $b_0$, $b_1$ and $F$ that satisfy the requirements of the theorem (except for $A_0F \neq F$) are exactly the ones that satisfy (6) and (7).

Consider now the final requirement $A_0F \neq F$. This is equivalent to

$$(B_0 - E)F' + b_0 \neq 0,$$

and further to $F' \neq -(B_0 - E)^{-1}b_0$. Substituting for $F'$ in the ap condition (6), and recalling that matrix $B_0 + B_1 - 2E$ is regular, we obtain the equivalent condition

$$-b_0 - (B_1 - E)(B_0 - E)^{-1}b_0 + b_0 + b_1 \neq 0,$$

which can be rewritten as

$$(E - B_1)^{-1}b_1 - (E - B_0)^{-1}b_0 \neq 0.$$  

(8)

We have obtained sufficient and necessary conditions (6), (7) and (8).

Now we can prove parts (1) and (2) of the theorem. If $\det(B_0 + B_1 - E) \neq 0$ then (7) and (8) are contradictory, so no choice of $b_0$, $b_1$ and $F$ can satisfy all the requirements. On the other hand, if $\det(B_0 + B_1 - E) = 0$ we can choose $b_0$, $b_1$ so that $(E - B_1)^{-1}b_1 - (E - B_0)^{-1}b_0$ is a nonzero element of the kernel of matrix $B_0 + B_1 - E$. This can be easily done by, for example, choosing any nonzero $k \in ker(B_0 + B_1 - E)$ and an arbitrary vector $b_0$, and setting

$$b_1 = (E - B_1) \left[ k + (E - B_0)^{-1}b_0 \right].$$

These choices of $b_0$ and $b_1$ satisfy (7) and (8). We can then calculate the unique $F'$ that satisfies (6).

We see that in order to generate non-constant functions we need a stable pair of matrices $\{B_0, B_1\}$ such that $\det(B_0 + B_1 - E) = 0$.

The following numerical example illustrates the previous proof.

**Example 66.** Let

$$B_0 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{2}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$  

It is easy to see that $\{B_0, B_1\}$ is stable and $\det(B_0 + B_1 - E) = 0$. The kernel of $B_0 + B_1 - E$ is generated by $(1, 0)^T$. If we (arbitrarily) choose $k = (9, 0)^T$ and $b_0 = (3, 0)^T$ we can solve

$$b_1 = (E - B_1) \left[ k + (E - B_0)^{-1}b_0 \right] = (5, 6)^T.$$
From (6) we get
\[ F' = -(B_0 + B_1 - 2E)^{-1}(b_0 + b_1) = (10, 6)^T. \]
So we have the ap WFA
\[
A_0 = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & 3 \\
\frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
A_1 = \begin{pmatrix}
\frac{2}{3} & 0 & 5 \\
-\frac{1}{3} & \frac{2}{3} & 6 \\
0 & 0 & 1
\end{pmatrix}, \quad
F = \begin{pmatrix}
10 \\
6 \\
1
\end{pmatrix}
\]
which with the initial distribution \((1, 0, 0)\) defines the real function \(\hat{f}_A\) whose graph is shown in Figure 9.

**Example 67.** It is easy to see that one-state continuous ap WFA can compute precisely all constant functions.

Let us find all two-state ap WFA with continuous \(f_A\) and \(\hat{f}_A\). Now \(B_0\) and \(B_1\) are numbers, and the condition that \(\text{det}(B_0 + B_1 - E) = 0\) forces them to add up to one. Stability requires both numbers to be less than 1 in absolute value, so we have \(B_0 = a\) and \(B_1 = 1 - a\) for some \(0 < a < 1\). We can choose \(b_0\) and \(b_1\) arbitrarily, and calculate \(F' = b_0 + b_1\). We get the continuous ap WFA with
\[
A_0 = \begin{pmatrix}
a & b_0 \\
0 & 1
\end{pmatrix}, \quad
A_1 = \begin{pmatrix}
1 - a & b_1 \\
0 & 39
\end{pmatrix}, \quad
F = \begin{pmatrix}
b_0 + b_1 \\
1
\end{pmatrix},
\]
for $0 < a < 1$ and $b_0, b_1 \in \mathbb{R}$. Note that we did not require (3) to hold, which means that we also get the constant functions when

$$\frac{b_1}{a} = \frac{b_0}{1 - a}.$$ 

5. Conclusions

We have investigated the relationship between continuity of WFA and properties of its transition matrices. We have obtained a “canonical form” for ap WFA computing continuous functions (the form (5) from Lemma 35). These results generalize some of the theorems in [10] and are similar to those obtained in a slightly different setting in the article [11]. Moreover, we present a method of constructing continuous WFA.

We have also asked questions about decidability of various incarnations of the continuity problem. Mostly, these problems turn out to be equivalent to the Matrix Product Stability problem. This is why we believe that any interesting question about continuity of functions computed by WFA is at least as hard as Matrix Product Stability.

There are numerous open questions in this area. Most obviously, settling the decidability of the Matrix Product Stability problem would be a great step forward. However, as this problem has resisted efforts of mathematicians so far, we offer a few other open problems:

**Open Question 68.** Given an automaton computing a continuous $\omega$-function, can we algorithmically find the ap automaton computing the same function?

**Open Question 69.** Given ap automaton computing $\omega$-function which is uniformly continuous on $\Omega$, can we algorithmically find automaton computing the function $g$ from Theorem 18?

**Open Question 70.** Is deciding the continuity of $\hat{f}_A$ for ap automata computationally equivalent with deciding Matrix Product Stability?

Other interesting questions that can be posed on WFA are whether a given $\hat{f}_A$ converges everywhere, and whether it is bounded. We know that all level WFA (as described in [10]) are both everywhere convergent and bounded but both properties remain to be characterized in the general case. We also point out that similar results on higher differentiability classes (e.g. continuously differentiable WFA functions) are likely to exist and should be investigated.

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