CONTACT SPHERES AND HYPERKÄHLER GEOMETRY

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Abstract. A taut contact sphere on a 3-manifold is a linear 2-sphere of contact forms, all defining the same volume form. In the present paper we completely determine the moduli of taut contact spheres on compact left-quotients of SU(2) (the only closed manifolds admitting such structures). We also show that the moduli space of taut contact spheres embeds into the moduli space of taut contact circles.

This moduli problem leads to a new viewpoint on the Gibbons-Hawking ansatz in hyperkähler geometry. The classification of taut contact spheres on closed 3-manifolds includes the known classification of 3-Sasakian 3-manifolds, but the local Riemannian geometry of contact spheres is much richer. We construct two examples of taut contact spheres on open subsets of $\mathbb{R}^3$ with nontrivial local geometry; one from the Helmholtz equation on the 2-sphere, and one from the Gibbons-Hawking ansatz. We address the Bernstein problem whether such examples can give rise to complete metrics.

1. Introduction

We begin with the definition of our basic objects of interest. Recall that a contact form on a 3-manifold is a differential 1-form $\alpha$ such that $\alpha \wedge d\alpha \neq 0$.

Definition 1. A contact sphere is a triple of 1-forms $(\alpha_1, \alpha_2, \alpha_3)$ on a 3-manifold such that any non-trivial linear combination of these forms (with constant coefficients) is a contact form.

In other words, we require that the 3-form

$$(\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3) \wedge (\lambda_1 d\alpha_1 + \lambda_2 d\alpha_2 + \lambda_3 d\alpha_3)$$

be nowhere zero, i.e. a volume form, for any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \neq 0$. The name ‘contact sphere’ derives from the fact that it suffices to check this condition for points $(\lambda_1, \lambda_2, \lambda_3)$ on the unit sphere $S^2 \subset \mathbb{R}^3$.

Definition 2. A contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ is called taut if the contact form $\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3$ defines the same volume form for all $(\lambda_1, \lambda_2, \lambda_3) \in S^2$.

The requirement for a contact sphere to be taut is equivalent to the system of equations (for $i \neq j$)

$$\alpha_i \wedge d\alpha_i = \alpha_j \wedge d\alpha_j \neq 0,$$

$$\alpha_i \wedge d\alpha_j = -\alpha_j \wedge d\alpha_i.$$

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A straightforward calculation shows that one can then find a 1-form $\beta$ and a nowhere zero function $\Lambda$ such that
\begin{equation}
d\alpha_i = \beta \wedge \alpha_i + \Lambda \alpha_j \wedge \alpha_k,
\end{equation}
where $(i,j,k)$ runs over the cyclic permutations of $(1,2,3)$. Notice that $\Lambda$ is defined by
\begin{equation}
\alpha_i \wedge d\alpha_i = \Lambda \alpha_1 \wedge \alpha_2 \wedge \alpha_3.
\end{equation}
The analogous structure of a (taut) contact circle, defined in terms of two contact forms $(\alpha_1, \alpha_2)$, was studied in our previous papers \cite{15}, \cite{16}, \cite{17}. In \cite{15} we gave a complete classification of the closed, orientable 3-manifolds that admit a taut contact circle or a taut contact sphere:

**Theorem 3.** Let $M$ be a closed 3-manifold.

(a) $M$ admits a taut contact circle if and only if $M$ is diffeomorphic to a quotient of the Lie group $G$ under a discrete subgroup $\Gamma$ acting by left multiplication, where $G$ is one of the following.

(i) $S^3 = SU(2)$, the universal cover of $SO(3)$.
(ii) $\tilde{SL}_2$, the universal cover of $PSL_2 \mathbb{R}$.
(iii) $\tilde{E}_2$, the universal cover of the Euclidean group (that is, orientation preserving isometries of $\mathbb{R}^2$).

(b) $M$ admits a taut contact sphere if and only if it is diffeomorphic to a left-quotient of $SU(2)$.

In the course of this paper we shall present a new proof, more self-contained than the one given in \cite{15}, of the fact that the universal cover of a closed 3-manifold admitting a taut contact sphere is diffeomorphic to $S^3$.

In \cite{16} we showed that every closed, orientable 3-manifold admits a (non-taut) contact circle, and we gave examples of contact spheres. For instance, $S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$, described in terms of coordinates $(\theta,x,y,z)$, does not admit any taut contact circles by Theorem \cite{3} but it admits the contact sphere
\begin{align*}
\alpha_1 &= x \, d\theta + y \, dz - z \, dy, \\
\alpha_2 &= y \, d\theta + z \, dx - x \, dz, \\
\alpha_3 &= z \, d\theta + x \, dy - y \, dx.
\end{align*}

In \cite{17} we described deformation spaces for taut contact circles and gave a complete classification of taut contact circles. The present paper achieves the corresponding classification for taut contact spheres.

The investigation of taut contact spheres amounts to a systematic study of hyperkahler metrics with a homothety, in a sense made precise below. Constructions of complete hyperkahler metrics with translational invariance have played a prominent role in general relativity and supersymmetric field theories, beginning with the Gibbons-Hawking ansatz \cite{18}. This ansatz will be discussed in Section 5.2 in the context of explicit constructions of taut contact spheres with nontrivial local geometry. See \cite{2} for a fairly recent discussion of several constructions related to the Gibbons-Hawking ansatz (Eguchi-Hanson metric, Taub-NUT metric, Atiyah-Hitchin metric).

Hyperkahler metrics with a homothety are equally important in physical applications, see \cite{11} and \cite{19}. The latter pays particular attention to homotheties that are hypersurface orthogonal (such homotheties are called dilatations in \cite{19}). This

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1Our manifolds are always understood to be connected.
is equivalent to saying that the hyperkähler metric on $U \times \mathbb{R}$, where $U$ is some 3-manifold, is the cone metric over a 3-Sasakian metric on $U$, cf. [3]. For more general information on Sasakian and in particular 3-Sasakian geometry see the definitive survey [6] or the monograph [7]; some of the definitions will be recalled in Section 7. There we use our methods to recover the classification of the closed 3-manifolds admitting 3-Sasakian structures. In contrast with 3-Sasakian structures, taut contact spheres do not, in general, give rise to a cone metric on $U \times \mathbb{R}$ (in other words, the relevant homothety is not a dilatation; such general homotheties also appear in [11]). This implies that taut contact spheres are definitely more general than 3-Sasakian structures. We elaborate on this point in Section 7.

By comparison, Kähler metrics on $U \times \mathbb{R}$ admitting a dilatation correspond to a Sasakian structure on $U$. (For a classification of the closed 3-dimensional manifolds admitting Sasakian structures see [14].) In [20] it is shown that if the Sasakian analogue of the Kähler potential satisfies a Monge-Ampère equation, then the metric on $U$ is Sasakian-Einstein. This happens in particular if $U \times \mathbb{R}$ is Ricci-flat, cf. [19].

A taut contact sphere on a closed manifold $M$ always gives rise to a flat hyperkähler metric on $M \times \mathbb{R}$ (Theorem 10). This must be read as a global rigidity phenomenon, because the theorem fails for open 3-manifolds. In Section 5.1 we use a Monge-Ampère equation to construct a taut contact sphere on an open subset $U$ of $\mathbb{R}^3$ giving rise to a non-flat hyperkähler metric on $U \times \mathbb{R}$. In the appendix we use a contact transformation to relate this construction to the Helmholtz equation on the 2-sphere. In Section 5.2 we use the Gibbons-Hawking ansatz to give an even simpler construction of a non-flat example. In Section 6 we discuss the question, known as a Bernstein problem, whether such non-flat examples can give rise to complete metrics. The answer to this question depends on the choice of one of the two natural metrics associated with a taut contact sphere (see Definition 15, Theorem 16, and the comments following it).

This paper supersedes our preprint “Contact spheres and quaternionic structures”.

2. Statement of results

We now describe in outline some of the main results of the present paper. Our notational convention throughout will be that $M$ denotes a closed, orientable 3-manifold; $U$ will denote a 3-manifold (without boundary) that need not be compact.

The relation

$$(\alpha_1, \alpha_2, \alpha_3) \sim (v\alpha_1, v\alpha_2, v\alpha_3)$$

for some smooth function $v: U \to \mathbb{R}^+$ is easily seen to be an equivalence relation within the set of (taut) contact spheres.

**Definition 4.** Two (taut) contact spheres are **conformally equivalent** if one is obtained from the other by multiplying each contact form by the same positive function.

**Definition 5.** We call a contact sphere **naturally ordered** if $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$ is a positive multiple of $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$.

Throughout this paper we shall assume our contact spheres to satisfy this condition.
Since $\alpha_i \wedge d\alpha_i$ and $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$ scale with the second and third power of $v$, respectively, it is obvious that every conformal equivalence class of naturally ordered taut contact spheres contains, for any $c \in \mathbb{R}^+$, a unique representative satisfying

\[ \alpha_i \wedge d\alpha_i = c\alpha_1 \wedge \alpha_2 \wedge \alpha_3, \quad i = 1, 2, 3. \]

**Definition 6.** We call a taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ $c$-normalised if it satisfies equation (2).

**Remark 7.** This condition is equivalent to $\Lambda \equiv c$ in equation (1); beware that $\beta$ in that equation is not an invariant of the conformal equivalence class.

It is implicit in [13] and follows by a simple extension of the ideas from [15] that a taut contact sphere on $U$ gives rise to a hyperkähler structure on $U \times \mathbb{R}$. In Section 3 we analyse this situation a little more carefully. One of the results proved there is the following.

**Proposition 8.** A contact sphere on $U$ determines an oriented conformal structure on $U \times \mathbb{R}$. A naturally ordered taut contact sphere on $U$ determines a hyperkähler structure on $U \times \mathbb{R}$. Conformally equivalent (taut) contact spheres determine isomorphic conformal (resp. hyperkähler) structures.

As we shall see in Section 3 the hyperkähler structure $(g, J_1, J_2, J_3)$ on $U \times \mathbb{R}$ induced by a taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ on $U$ is given by the equations

\[ -g(\cdot, J_i \cdot) = d(e^t \alpha_i) =: \Omega_i, \quad i = 1, 2, 3, \]

where $t$ denotes the $\mathbb{R}$-coordinate, and the complex structures $J_i$ are $\partial_t$-invariant. Often we write the hyperkähler structure as the triple $(\Omega_1, \Omega_2, \Omega_3)$ of symplectic forms.

These symplectic forms are homogeneous of degree 1 with respect to the vector field $\partial_t$, that is, $L_{\partial_t} \Omega_i = \Omega_i$. The hyperkähler metric $g$ has the same homogeneity.

For reversing this construction, it is useful to make the following definition.

**Definition 9.** A vector field $Y$ on a hyperkähler manifold is called tri-Liouville if it is a Liouville vector field for every parallel self-dual 2-form $\Omega$, that is, $L_X \Omega = \Omega$.

Notice that a tri-Liouville vector field is automatically homothetic for the hyperkähler metric, but the converse is not true.

The construction of taut contact spheres in this paper uses two ingredients: a hyperkähler metric $g$ and a nowhere zero tri-Liouville vector field $Y$. For any conformal basis $(\Omega_1, \Omega_2, \Omega_3)$ of parallel self-dual 2-forms, the corresponding taut contact sphere is defined on any transversal to the flow of $Y$ by restricting the triple of 1-forms $(i_Y \Omega_1, i_Y \Omega_2, i_Y \Omega_3)$ to that transversal. For the formal statement see Proposition [21].

We also show that for a naturally ordered taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ on $U$ there is the following pointwise model on $U \times \mathbb{R}$ for the triple of symplectic forms $(\Omega_1, \Omega_2, \Omega_3)$, expressed in quaternionic notation: At any point $x \in U \times \mathbb{R}$, there is a quaternionic coordinate $dq_x$ for the tangent space $T_x(U \times \mathbb{R})$ such that

\[ d(e^t(i\alpha_1 + j\alpha_2 + k\alpha_3))_x = -dq_x \wedge \partial_t x. \]

The key to the classification of taut contact spheres is then the following statement.
Theorem 10 (Global rigidity). The hyperkähler metric on $M \times \mathbb{R}$ induced by a taut contact sphere on a closed 3-manifold $M$ is flat.

Proof. Hyperkähler metrics are always Ricci flat [4, 14.13], and any Ricci flat Kähler manifold of complex dimension 2 is anti-self-dual [1], that is, the self-dual part $W^+$ of the Weyl tensor of the metric $g$ vanishes. Since the Weyl tensor is an invariant of the conformal class of a metric [4, 1.159], $W^+$ also vanishes for the metric $g/g(\partial_t, \partial_t)$, which descends to the quotient $M \times S^1$ of $M \times \mathbb{R}$ under the map $(p, t) \mapsto (p, t+1)$, say. (In fact, the hyperhermitian structure $(g/g(\partial_t, \partial_t), J_1, J_2, J_3)$ descends to that quotient, and one may also appeal to a result of Boyer [5] saying that a hyperhermitian metric on a 4-manifold is anti-self-dual.)

Then the signature formula for $(M \times S^1, g/g(\partial_t, \partial_t))$ yields

$$\tau(M \times S^1) = \frac{1}{12\pi^2} \int_{M \times S^1} W^+ - W^-$$

But $\tau(M \times S^1) = 0$ for purely topological reasons. So the Weyl tensor $W = W^+ + W^-$ vanishes for $g/g(\partial_t, \partial_t)$. Again appealing to the conformal invariance of the Weyl tensor, we deduce that it also vanishes for $g$. For Ricci flat metrics (thus, in particular, for the hyperkähler metric $g$) the full curvature tensor equals its Weyl part [4, 1.116]. This proves the theorem. □

This theorem implies that on a closed manifold the pointwise model above for the triple of symplectic forms $d(e^t\alpha_i)$, $i = 1, 2, 3$, coming from a taut contact sphere, is actually a local model, since the three symplectic forms and the hypercomplex structure $(J_1, J_2, J_3)$ are all parallel with respect to the flat hyperkähler metric $g$.

It is then not very difficult, using properties of the $\partial_t$-flow, to derive the following classification statement. The proof will be given in Section 4.

Theorem 11. If $M$ is diffeomorphic to a lens space $L(m, m-1)$, including the 3-sphere $L(1,0)$, then the (naturally ordered) taut contact spheres on $M$, up to diffeomorphism and conformal equivalence, are given by the following family of $\mathbb{Z}_m$-invariant quaternionic 1-forms on $S^3 \subset \mathbb{H}$,

$$i\alpha_1 + j\alpha_2 + k\alpha_3 = \frac{1}{2}(dq \cdot \overline{q} - q \cdot d\overline{q}) - \nu d(g\overline{\mathbb{I}}), \ \nu \in \mathbb{R},$$

where $\nu$ and $-\nu$ determine equivalent structures.

If $M$ is diffeomorphic to $\Gamma \backslash SU(2)$ with $\Gamma \subset SU(2)$ a non-abelian group, there is a unique equivalence class of taut contact spheres on $M$, described by the formula above with $\nu = 0$.

All these taut contact spheres are homogeneous under a natural $SO(3)$-action. In particular, all great circles in a given taut contact sphere are isomorphic taut contact circles.

Note that the manifolds listed in this theorem exhaust all the possible left-quotients of $SU(2)$, cf. [15]. The $\mathbb{Z}_m$-action on $S^3$ that produces the quotient $L(m, m-1)$ is generated by right multiplication with $\cos(2\pi/m) + i\sin(2\pi/m)$.

Remark 12. As we shall see in the construction of the moduli space of taut contact spheres, all taut contact spheres on a given closed 3-dimensional manifold yield the same 4-dimensional metric $g$ up to global isometry. So it is not this induced metric
which determines the modulus, but in fact the different possible homothetic vector fields \( \partial_t \). In the cases where non-trivial moduli exist, neither the vector field \( \partial_t \) nor the function \( e^t \) (which turns out to equal \( g(\partial_t, \partial_t) \)) are unique for that \( g \).

With our coordinate conventions in Section 3 below, which seem natural in that context, left multiplication on \( \mathbb{C}^2 \) by

\[
\begin{pmatrix}
  a & -\overline{b} \\
  b & \overline{a}
\end{pmatrix} \in \text{SU}(2),
\]

where \( a = a_1 + ia_2, b = b_1 + ib_2, |a|^2 + |b|^2 = 1 \), corresponds to right multiplication on \( \mathbb{H} \) by the unit quaternion \( u = a_1 + ia_2 + jb_1 + kb_2 \).

The quaternionic 1-form \( dq \cdot \overline{u} - q \cdot d\overline{q} \) is invariant under this right multiplication \( q \mapsto qu \) by unit quaternions \( u \), and therefore descends to all left-quotients of \( \text{SU}(2) \).

The 1-form \( d(q\overline{q}) \) is invariant under right multiplication by unit quaternions of the special form \( a_1 + ia_2 \), hence it descends to all abelian quotients.

Here are some details about what we mean by ‘homogeneity under a natural \( \text{SO}(3) \)-action’ in Theorem 11. The action of \( \text{SO}(3) \) on \( i\alpha_1 + j\alpha_2 + k\alpha_3 \) rotates the contact forms and is given by conjugating the quaternionic 1-form by elements \( u \in S^3 \subset \mathbb{H} \). This action is induced from the \( S^3 \)-action \( q \mapsto uq \) on \( \mathbb{H} \), and that latter action cannot be replaced by an \( \text{SO}(3) \)-action. This amounts to a spinor phenomenon, see also the proof of Proposition 20. Since this left multiplication by unit quaternions commutes with all quaternionic right multiplications, it descends to all left-quotients of \( \text{SU}(2) \).

**Remark 13.** With \( q = x_0 + ix_1 + jx_2 + kx_3 \) we obtain the following real expression for the taut contact sphere in Theorem 11 corresponding to \( \nu = 0 \), where \((i,j,k)\) runs over the cyclic permutations of \((1,2,3)\):

\[
\alpha_i = x_0 dx_i - x_i dx_0 + x_j dx_k - x_k dx_j.
\]

Notice that it satisfies \((1)\) with \( \beta \equiv 0 \) and \( \Lambda \equiv 2 \). In particular, it is a 2-normalised taut contact sphere.

Restricting this to the hyperplane \( \{x_0 = 1\} \), we obtain a simple expression for a taut contact sphere on \( \mathbb{R}^3 \):

\[
\alpha_i = dx_i + x_j dx_k - x_k dx_j.
\]

See Proposition 21 for the principle behind this observation.

One might suspect that taut contact spheres constitute such a rigid structure that Theorem 10 would also hold locally and for open manifolds. However, this turns out to be false, even conformally.

**Theorem 14.** There are examples of taut contact spheres on open domains \( U \) in \( \mathbb{R}^3 \) inducing a metric on \( U \times \mathbb{R} \) that is not conformally flat.

This theorem will be proved in Section 5 where we present two methods for constructing such examples. Our first construction, in Section 5.1, starts from the observation that (locally) the conditions for a taut contact sphere lead to a complex Monge-Ampère equation. After imposing an additional homogeneity amounting to the existence of a tri-Hamiltonian vector field, we are able to find concrete solutions of that equation whose associated 4-dimensional metric is non-flat. In fact, subject to this extra homogeneity, the Monge-Ampère equation can be linearised to yield the Helmholtz equation \( \Delta u + 2u = 0 \) for the Laplacian \( \Delta \) on the 2-sphere.
Solutions of that Helmholtz equation then give rise to taut contact spheres with a tri-Hamiltonian symmetry.

Our second construction, in Section 5.2, starts from the Gibbons-Hawking ansatz, which is essentially a construction of an $\mathbb{R}$-invariant hyperkähler metric on $U_0 \times \mathbb{R}$, starting from a harmonic function on an open subset $U_0$ of $\mathbb{R}^3$. For an appropriate choice of such a harmonic function, one obtains a non-flat hyperkähler metric giving rise to a taut contact sphere on a suitable hypersurface $U \subset U_0 \times \mathbb{R}$. Beware that the $\mathbb{R}$-factor in this splitting $U_0 \times \mathbb{R}$ is not the one corresponding to the tri-Liouville vector field.

Taut contact spheres come associated with two natural metrics on the 3-manifold. In order to describe these, we observe that the 2-forms $\Omega_i = d(e^t \alpha_i)$ can be written with the help of the structure equation (1) as follows:

$$\Omega_i = e^t (\Lambda^{1/2} (dt + \beta) \wedge \Lambda^{1/2} \alpha_i + \Lambda^{1/2} \alpha_j \wedge \Lambda^{1/2} \alpha_k).$$

So the hyperkähler metric $g$ is given by

$$g = e^t (\Lambda^{-1} (dt + \beta)^2 + \Lambda (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)).$$

In particular, if the contact sphere is 1-normalised, we have

$$g = e^t ((dt + \beta)^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2).$$

This motivates the following definition.

**Definition 15.** The short metric associated with a 1-normalised taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ on $U$ is the metric

$$g_s = \alpha_1^2 + \alpha_2^2 + \alpha_3^2.$$  

The long metric associated with this contact sphere is

$$g_l = \beta^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2.$$  

Observe that $g_l$ is simply the restriction of $g$ to $U \equiv U \times \{0\}$, so from the viewpoint of hyperkähler geometry this is the more natural metric to consider.

In either of the above-mentioned constructions of taut contact spheres on $U$ giving rise to a non-flat hyperkähler metric on $U \times \mathbb{R}$, the induced long metric $g_l$ on $U$ is incomplete, and so is, a fortiori, the short metric $g_s$. In Section 6 we raise the question whether one can find such examples where $g_l$, at least, is complete. This type of question is known as a Bernstein problem [9]. Concerning $g_s$, we provide a partial answer to this problem for contact spheres with additional symmetries.

For $g_l$ there is a positive answer to the Bernstein problem, even subject to the additional symmetry requirement:

**Theorem 16.** Let $(\alpha_1, \alpha_2, \alpha_3)$ be a 1-normalised taut contact sphere on $U$ giving rise to the short metric $g_s$ and the long metric $g_l$ on $U$.

(a) If $g_s$ is complete and admits a non-trivial Killing vector field that preserves $(\alpha_1, \alpha_2, \alpha_3)$, then $U$ is necessarily compact, and hence $(\alpha_1, \alpha_2, \alpha_3)$ belongs to the family described in Theorem 11 (and in particular gives rise to a flat hyperkähler metric).

(b) There are examples of $S^1$-invariant taut contact spheres on $U = D \times S^1$, where $D$ is the open unit disc, for which $g_l$ is complete. The induced hyperkähler metrics on $U \times \mathbb{R}$ are not flat.
Part (a) will be proved in Section 6. Theorem 24 in that section is a more explicit reformulation of this part.

We reserve the proof of part (b) for a forthcoming paper. It turns out that in the $\mathbb{R}$- or $S^1$-invariant context one can use the Gibbons-Hawking ansatz in order to develop a complete theory of such contact circles. An infinite-dimensional family of examples giving rise to complete long metrics is then found with the help of Blaschke products on $\mathbb{D} \subset \mathbb{C}$.

3. Hyperkähler linear algebra

In this section we discuss the linear algebraic aspects of contact spheres, leading to a proof of Proposition 8. This prepares the ground for the proof of Theorem 11.

Let $(\alpha_1, \alpha_2, \alpha_3)$ be a contact sphere on a 3-manifold $U$. This gives rise to the symplectic forms $\Omega_i = d(\alpha_i)$, $i = 1, 2, 3$, on $U \times \mathbb{R}$. At any point $x$ of $U \times \mathbb{R}$, these symplectic forms span a definite 3-plane in the space $\bigwedge^2 T_x^* (U \times \mathbb{R})$ of skew-symmetric bilinear forms on the tangent space $T_x(U \times \mathbb{R})$. If the contact sphere is taut, we have in addition the identities (for $i \neq j$)

$$\Omega_i \wedge \Omega_i = \Omega_j \wedge \Omega_j \neq 0,$$

$$\Omega_i \wedge \Omega_j = 0.$$

First we are going to study the linear algebra of this situation. Thus, let $V_4$ be a 4-dimensional real vector space and write $V_6 = \bigwedge^2 V_4^*$. Consider the quadratic form $Q: V_6 \rightarrow \mathbb{R}$, $Q(A) = A \wedge A$,

of signature $(3, 3)$. We call a triple $(A_1, A_2, A_3)$ of elements of $V_6$ a symplectic triple on $V_4$ if it spans a definite 3-plane for $Q$ in $V_6$, and a conformal symplectic triple if the stronger condition

$$A_i \wedge A_i = A_j \wedge A_j \neq 0,$$

$$A_i \wedge A_j = 0,$$

is satisfied for $i \neq j$, cf. [13]. The same terminology will be used for triples of symplectic forms $(\Omega_1, \Omega_2, \Omega_3)$ on a 4-manifold as described above.

**Remark on Notation.** In the sequel, any equation (or other statement) involving the indices $i, j, k$ is meant to be read as three equations, with $(i, j, k)$ ranging over the cyclic permutations of $(1, 2, 3)$. We write bold face $\mathbf{i}$ for $\sqrt{-1} \in \mathbb{C}$, and bold face $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for the standard quaternionic units with $\mathbf{i} \mathbf{j} = \mathbf{k}$. The relation between real, complex, and quaternionic coordinates will be given by

$$z_1 = x_0 + \mathbf{i} x_1, \quad z_2 = x_2 + \mathbf{i} x_3;$$

$$q = x_0 + \mathbf{i} x_1 + \mathbf{j} x_2 + \mathbf{k} x_3 = z_1 + z_2 \mathbf{j}.$$

For $J$ a complex structure on $V_4$, we denote by $\bigwedge^{(p,q)}_J$ the space of exterior forms of type $(p, q)$ on $V_4$. As shown in [13], a conformal symplectic couple $(A_i, A_j)$ determines a unique complex structure $J_k$ on $V_4$ for which $A_i + \mathbf{i} A_j \in \bigwedge^{(2,0)}_J$. Thus, a conformal symplectic triple $(A_1, A_2, A_3)$ induces three complex structures $J_1$, $J_2$, $J_3$. In [13] it was shown that these complex structures satisfy the quaternionic identities as well as the relations

$$g(v, w) = A_i(v, J_i w), \quad i = 1, 2, 3, \text{ for all } v, w \in V_4,$$
for a unique definite symmetric bilinear form \( g \). We call a conformal symplectic triple \textbf{naturally ordered} if this \( g \) is \textit{positive} definite. Notice that the sign of \( g \) is well defined, since the \((-J_1, -J_2, -J_3)\) do not satisfy the quaternionic identities. (In particular, hyperkähler structures are always naturally ordered.)

An alternative way to define this definite bilinear form \( g \) is via the identity

\[
(v \cdot J_1 A_1) \wedge (v \cdot J_2 A_2) \wedge (v \cdot J_3 A_3) = \frac{1}{2} g(v, v) v \cdot (A_1^2) \quad \text{for all } v \in V_4.
\]

This is an obvious consequence of the \( \text{SO}(3) \)-homogeneous normal form for conformal symplectic triples discussed in the next two propositions.

**Proposition 17.** Let \((A_1, A_2, A_3)\) be a naturally ordered conformal symplectic triple on \(V_4\). Then there are real linear coordinates \(dx_0, dx_1, dx_2, dx_3\) on \(V_4\) such that

\[
A_i = dx_0 \wedge dx_i + dx_j \wedge dx_k.
\]

In terms of the corresponding complex and quaternionic coordinates we have

\[
A_1 = \frac{i}{2} (dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2),
\]

\[
A_2 + iA_3 = dz_1 \wedge d\overline{z}_2,
\]

and

\[
iA_1 + jA_2 + kA_3 = -\frac{1}{2} dq \wedge d\overline{q}.
\]

**Remark 18.** Because of the non-commutativity of \( \mathbb{H} \) some care is necessary in interpreting the wedge product of \( \mathbb{H} \)-valued 1-forms \( \alpha, \beta \) on a vector space \( V \). Our convention is to read it as

\[
(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v) \quad \text{for all } v, w \in V.
\]

This ensures \( \alpha \wedge q\beta = \alpha q \wedge \beta \), and that \( \alpha \wedge \overline{\alpha} \) is always purely imaginary.

**Proof of Proposition 17.** From \( 0 \neq A_2 + iA_3 \in \bigwedge^{(2, 0)} A_1 \) and \( A_1 \wedge (A_2 + iA_3) = 0 \) we conclude that the \((0, 2)\)-part of \( A_1 \) with respect to \( J_1 \) is zero. The form \( A_1 \) being real, its \((2, 0)\)-part must also vanish, hence \( A_1 \in \bigwedge^{(1, 1)} A_1 \).

Let \( \{\ell_1', \ell_2'\} \) be a basis for \( \bigwedge^{(1, 0)} A_1 \) and set \( \ell' = \left(\begin{array}{c} \ell_1' \\ \ell_2' \end{array}\right) \). Then we can write

\[
A_1 = i(\ell')^T \wedge A_1 \ell',
\]

with \( A_1 \) a hermitian \((2 \times 2)\)-matrix. By our assumption on \((A_1, A_2, A_3)\) being naturally ordered, the matrix \( A_1 \) is positive definite. Hence there is a matrix \( C \in \text{GL}_2(\mathbb{C}) \) such that

\[
C^T A_1 C = \left(\begin{array}{cc} 1/2 & 0 \\ 0 & 1/2 \end{array}\right),
\]

and so in terms of the basis \( \{\ell_1, \ell_2\} \) for \( \bigwedge^{(1, 0)} A_1 \) defined by \( \ell' = C\ell \) we have

\[
A_1 = \frac{i}{2} (\ell_1 \wedge \overline{\ell}_1 + \ell_2 \wedge \overline{\ell}_2)
\]

and

\[
A_2 + iA_3 = c \ell_1 \wedge \ell_2
\]
for some \( c \in \mathbb{C} \). We then find
\[
|c|^2 \ell_1 \wedge \ell_2 \wedge \overline{\ell}_1 \wedge \overline{\ell}_2 = (A_2 + iA_3) \wedge (A_2 - iA_3)
= A_2^2 + A_3^2 = 2A_1^2
= -\ell_1 \wedge \overline{\ell}_1 \wedge \ell_2 \wedge \overline{\ell}_2,
\]
from which we conclude \(|c| = 1\). The linear complex coordinates \( z_1, z_2 \) corresponding to \( \{ c \ell_1, \ell_2 \} \) then give the desired complex normal form. The real and quaternionic normal forms can be derived easily from the complex one. \( \square \)

**Remark 19.** (1) Notice that in terms of these pointwise coordinates we have \( g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \) and \((J_1, J_2, J_3) = (i, j, k)\). Moreover, we recognise the 3-plane in \( V_6 \) spanned by \( A_1, A_2, A_3 \) as the space of self-dual 2-forms for the metric \( g \) and the orientation of \( V_4 \) defined by \( A_i \wedge A_i \). The length of the \( A_i \) equals \( \sqrt{2} \).

(2) Here is another characterisation of taut contact spheres that can be read off from the preceding proposition: The purely imaginary 1-form \( \alpha = i\alpha_1 + j\alpha_2 + k\alpha_3 \) defines a taut contact sphere if and only if \( \alpha_1 \) is a contact form and at each point \( x \) of the manifold there is an \( \mathbb{H} \)-valued linear form \( \beta_x \) on the tangent space at \( x \) such that \( d(e^i\alpha)_x = \pm \beta_x \wedge \overline{\beta}_x \). The case \( d(e^i\alpha)_x = -\beta_x \wedge \overline{\beta}_x \) corresponds to \( (\alpha_1, \alpha_2, \alpha_3) \) being naturally ordered.

The following proposition will be an important ingredient in the proof of Proposition \( \mathbb{S} \) while the last part of its proof is used in the proof of Theorem \( \mathbb{T} \).

**Proposition 20** (The spinor equivariance). There is a one-to-one correspondence between oriented conformal structures on \( V_4 \) and definite 3-planes \( V_3 \) in \( V_6 \) (with respect to \( Q \)).

**Proof.** Given an oriented conformal structure on \( V_4 \), define \( V_3 \) as the corresponding space of self-dual 2-forms on \( V_4 \).

For the converse, we recall some quaternionic linear algebra. Under the identification of the purely imaginary quaternions with \( \mathbb{R}^3 \), any element \( \phi \) of \( \text{SO}(3) \) can be written as quaternionic conjugation \( \phi = \phi_u \),
\[
\mathbb{R}^3 \ni x \mapsto \phi_u(x) = uxu^\ast,
\]
with some unit quaternion \( u \in S^3 \subset \mathbb{H} \). The map \( u \mapsto \phi_u \) is the standard double covering \( S^3 \to \text{SO}(3) \).

Given a definite 3-plane in \( V_6 \), choose a naturally ordered conformal symplectic triple \( (A_1, A_2, A_3) \) spanning it. An orientation on \( V_4 \) is then given by \( A_i \wedge A_i \).

Let \( q \) be a quaternionic coordinate for \( V_4 \) as in Proposition \( \mathbb{L} \) and \( g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \) the inner product on \( V_4 \) determined by \( (A_1, A_2, A_3) \).

Suppose that \( A'_1, A'_2, A'_3 \) is another naturally ordered conformal symplectic triple spanning the same 3-plane \( V_3 \) and satisfying \( A'_i \wedge A'_i = A_i \wedge A_i \). Notice that \( Q \) defines an inner product on \( V_3 \) for which \( (A_1, A_2, A_3) \) and \( (A'_1, A'_2, A'_3) \) are orthogonal bases consisting of vectors of equal length, and defining the same orientation. Hence there is an element \( \phi_u \in \text{SO}(3) \) such that
\[
\begin{pmatrix}
A'_1 \\
A'_2 \\
A'_3
\end{pmatrix} = \phi_u
\begin{pmatrix}
A_1 \\
A_2 \\
A_3
\end{pmatrix}.
\]
By the preceding discussion this can be written as
\[ iA'_1 + jA'_2 + kA'_3 = u(iA_1 + jA_2 + kA_3) \]
\[ = -\frac{1}{2} u dq \wedge d\overline{q} \]
\[ = -\frac{1}{2} d(uq) \wedge d(\overline{uq}). \]

So a quaternionic coordinate corresponding to \((A'_1, A'_2, A'_3)\) is given by \(uq\), which gives rise to the same inner product \(g\) on \(V\), since left multiplication on \(\mathbb{H}\) by a unit quaternion is an isometry for \(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\).

If the conformal symplectic triple \((A_1, A_2, A_3)\) is replaced by \((vA_1, vA_2, vA_3)\), \(v \in \mathbb{R}^+\), then the induced inner product changes to \(vg\). This proves the proposition. \(\square\)

**Proof of Proposition** A (taut) contact sphere \((\alpha_1, \alpha_2, \alpha_3)\) on \(U\) gives rise to a (conformal) symplectic triple \((\Omega_1, \Omega_2, \Omega_3)\) on \(U \times \mathbb{R}\) as described at the beginning of this section. The complex structures \(J_1, J_2, J_3\) defined on each tangent space \(T_x(U \times \mathbb{R})\) depend smoothly on the point \(x\), and thus define almost complex structures on \(U \times \mathbb{R}\), which are integrable in the taut case, see [15]. So the statement concerning contact spheres and conformal structures is immediate from the preceding proposition. Notice that the orientation and conformal class of the metric on \(U \times \mathbb{R}\) are completely characterised as the unique ones for which the \(\Omega_i\) are self-dual.

The statement about taut contact spheres and hyperkähler structures follows similarly; see [13] for an explicit argument. A non-taut contact sphere determines at each point a linear 2-sphere worth of complex structures, since pointwise the symplectic triple \((\Omega_1, \Omega_2, \Omega_3)\) can be replaced by a conformal symplectic triple spanning the same 3-plane of skew-symmetric forms; this does not change the space of corresponding almost complex structures. Although this replacement can be done globally, leading again to triples \((J_1, J_2, J_3)\) satisfying the quaternionic identities, there does not seem to be a canonical choice for doing it.

If two contact spheres are related by multiplication by the function \(v: M \rightarrow \mathbb{R}^+\), the induced structures on \(U \times \mathbb{R}\) are related by the diffeomorphism given by \((p, t) \mapsto (p, t + \log v(p))\). \(\square\)

The following proposition explains how to go back from a hyperkähler structure and a tri-Liouville vector field to a taut contact sphere.

**Proposition 21.** Let \((\Omega_1, \Omega_2, \Omega_3)\) be a hyperkähler structure on a 4-manifold \(W\) with a nowhere zero tri-Liouville vector field \(Y\). Then the equations \(\alpha_i = Y \cdot \Omega_i\) define a naturally ordered taut contact sphere \((\alpha_1, \alpha_2, \alpha_3)\) on any transversal of \(Y\). Shifting the points of the transversal along the orbits of \(Y\) will change the taut contact sphere within its conformal class. \(\square\)

The proof is a straightforward computation, using \(\Omega_i \wedge \Omega_i = \Omega_j \wedge \Omega_j = 0\) and \(\Omega_i \wedge \Omega_j = 0\) for \(i \neq j\), cf. [13], as well as the identities \(Y \cdot \Omega_i^2 = 2\alpha_i \wedge d\alpha_i\), cf. identity [13]. If the hyperkähler structure comes from a taut contact sphere, then the above construction with \(Y = \partial_i\) recovers that contact sphere.

4. **Classification of taut contact spheres**

This section is largely devoted to the proof of Theorem [11]. As promised, this includes a new proof that the universal cover of a closed 3-manifold carrying a taut contact sphere is diffeomorphic to \(S^3\).
The strategy for the proof is as follows. From Theorem 10 we know that a taut contact sphere on a closed 3-manifold $M$ induces a flat metric on $M \times \mathbb{R}$. With the help of the developing map of this metric we are led to study hyperkähler structures on Euclidean 4-space $\mathbb{E}^4$, with the hyperkähler metric being the flat Euclidean metric. Parallel 2-forms for this metric have constant coefficients in Euclidean coordinates. The classification of taut contact spheres is thus reduced to a straightforward problem of determining the possible tri-Liouville vector fields.

Proof of Theorem 17 Let $M$ be a closed 3-manifold with a taut contact sphere, and let $(\alpha_1, \alpha_2, \alpha_3)$ be the lifted taut contact sphere on its universal cover $\tilde{M}$. Write $g$ for the induced metric on $\tilde{M} \times \mathbb{R}$. The fact that $g$ is flat and $\tilde{M} \times \mathbb{R}$ simply-connected implies that we have a developing map $\Phi: \tilde{M} \times \mathbb{R} \to \mathbb{E}^4$ for this metric which is a local isometry.

If $W \subset \tilde{M} \times \mathbb{R}$ is a domain on which $\Phi$ restricts to a diffeomorphism, then $(\Phi | W) \partial_t$ is a vector field $Y_W$ on the domain $\Phi(W) \subset \mathbb{E}^4$ generating a 1-parameter group of homotheties of the Euclidean metric (because $L_{\partial_t} g = g$ by the construction of $g$). Since homothetic transformations of a Riemannian manifold are affine transformations (this is easy to see for the Euclidean metric), $Y_W$ is the restriction $Y_{\Phi(W)}$ of a homothetic vector field $Y$ defined on all of $\mathbb{E}^4$. Then $\partial_t$ and $\Phi^* Y$ are homothetic vector fields for $g$ that coincide on the open set $W$, which forces $\partial_t = \Phi^* Y$ on all of $\tilde{M} \times \mathbb{R}$.

A homothetic vector field on $\mathbb{E}^4$ vanishes at a single point, and without loss of generality we may assume that $Y$ vanishes at 0. Let $\pi: \mathbb{E}^4 \setminus \{0\} \to S^3_{\mathbb{R}}$ be the projection onto the orbit space of $Y$ (which we can identify with the unit sphere $S^3_{\mathbb{R}} \subset \mathbb{E}^4$, for $Y$ is a genuinely expanding homothetic vector field and hence transverse to any sphere centred at 0). Then the composition $\pi$ is a local diffeomorphism and therefore, $\tilde{M}$ being simply-connected, a diffeomorphism. So we have proved $\tilde{M} \cong S^3$. Moreover, the property $\partial_t = \Phi^* Y$ implies that $\Phi$ is a diffeomorphism from $S^3 \times \mathbb{R}$ to $\mathbb{E}^4 \setminus \{0\}$, hence a global isometry.

From now on we write $S^3$ instead of $\tilde{M}$. Now $\Phi$ sends $S^3 \times \{0\}$ to some transversal of $Y$ in $\mathbb{E}^4$, and by Proposition 21 the original contact sphere is equivalent to the one induced on $S^3_{\mathbb{R}}$.

To simplify notation, we continue to write $e^t \alpha_i$, $\Omega_i$, $J_i$ for the push-forwards of these objects to $\mathbb{E}^4 \setminus \{0\}$, and we identify $\partial_t$ with $Y$. Thus $(J_1, J_2, J_3)$ defines a hyperkähler structure with respect to the Euclidean metric $g_{\mathbb{E}}$, and $(\Omega_1, \Omega_2, \Omega_3)$ are the corresponding Kähler forms. In particular, the $J_i$ and $\Omega_i$ are parallel with respect to $g_{\mathbb{E}}$, and thus have constant coefficients in any linear coordinate system for $\mathbb{E}^4$. As a consequence, there are linear coordinates $x_0, x_1, x_2, x_3$ on $\mathbb{E}^4$ with respect to which the formulae of Proposition 17 hold (with $A_i$ replaced by $\Omega_i$). Observe that this forces $x_0, x_1, x_2, x_3$ to be an orthonormal coordinate system with respect to $g_{\mathbb{E}}$.

Write $\psi_t$ for the flow of $Y$. This flow commutes with the $J_i$ and satisfies $\psi_t^* g_{\mathbb{E}} = e^{t} g_{\mathbb{E}}$ and $\psi_t^* \Omega_i = e^{t} \Omega_i$, in particular $\psi_t^* (dz_1 \wedge dz_2) = e^{t} dz_1 \wedge dz_2$.

Since the flow of $Y$ preserves $J_1$, we have a holomorphic vector field $Y_C$ on $\mathbb{E}^4 = \mathbb{C}^2$ with $Y = 2 \text{Re}(Y_C)$. As a homothetic vector field vanishing at zero, $Y$ can
We also observe and j (using spheres on This shows that, up to conformal equivalence and diffeomorphism, the taut contact formula in Theorem 11 is a universal local model for (naturally ordered) taut contact spheres inducing a flat hyperkähler metric. This analysis can be carried out locally. As a result, the quaternionic formula in Theorem \ref{thm:universal_model} is a universal local model for (naturally ordered) taut contact spheres inducing a flat hyperkähler metric.

To translate the preceding equations into quaternionic notation, we observe

\[ q \cdot \bar{q} = z_1 \overline{d\bar{z}_1} + z_2 \overline{d\bar{z}_2} - 2iz_1 \overline{dz}_2 + \overline{z}_2 dz_1 \]

and

\[ dq \cdot \bar{q} = \overline{z}_1 dz_1 + \overline{z}_2 dz_2 + z_1 dz_2 \overline{j} - z_2 dz_1 \overline{j} \]

(using \( j \overline{z} = \overline{jz} \) for \( z \in \mathbb{C} \) and \( z_1 + z_2 \overline{j} = \overline{z}_1 - z_2 \overline{j} \)), hence

\[ dq \cdot \bar{q} - q \cdot d\bar{q} = \overline{z}_1 dz_1 - z_1 \overline{d\bar{z}_1} + \overline{z}_2 dz_2 - z_2 d\bar{z}_2 + 2(z_1 d\bar{z}_2 - z_2 dz_1). \]

We also observe

\[ q\overline{q} = (|z_1|^2 - |z_2|^2) - 2i(z_1 \overline{z}_2). \]

Putting all this together, we find

\[ e^t(ia_1 + ja_2 + ka_3) = e^t(ia_1 + (a_2 + i \overline{a}_3)j) = \frac{1}{4}(dq \cdot \bar{q} - q \cdot d\bar{q}) - \frac{\nu}{2} d(q\overline{q}). \]

This shows that, up to conformal equivalence and diffeomorphism, the taut contact spheres on \( S^3 \) are as described in Theorem 11. The fact that different non-negative values of \( \nu \) give non-isomorphic contact spheres follows from the corresponding
classification of taut contact circles in \[15\], \[17\]. We shall be a bit more explicit about this point below, where we give a pictorial description of the moduli spaces in question. This will include a synthetic method for determining the modulus, independent of our previous papers.

Next we want to show that \(\nu\) and \(-\nu\) correspond to equivalent structures. The diffeomorphism of \(S^3\) given by \(q \mapsto qj\) is isotopic to the identity and pulls the quaternionic 1-form with parameter value \(\nu\) to that with value \(-\nu\), because \(j\) anti-commutes with \(i\).

After having determined the possible lifted taut contact spheres on \(\widetilde{M} \cong S^3\), we now consider the taut contact sphere on \(M\) itself. From Theorem 3 we know that \(M\) has to be a left-quotient of \(SU(2)\). The induced flat hyperkähler structure on \(M \times \mathbb{R}\) lifts to just such a structure on \(S^3 \times \mathbb{R}\), invariant under the deck transformation group \(\Gamma\). As was already argued in \[15\] for taut contact circles, this implies that in the complex coordinates \((z_1, z_2)\) which give a normal form as described above, one has \(\Gamma \subset SU(2)\). Moreover, again as in \[15\], the parameter \(\nu\) is forced to be zero for non-abelian \(\Gamma\), and it can take any value for the group \(\Gamma = \mathbb{Z}_m \subset SU(2)\) generated by \((\varepsilon^0 0 -1)\), where \(\varepsilon\) is some \(m\)th root of unity.

Now we address the homogeneity issue in the statement of Theorem 11. For any unit quaternion \(u\) we have
\[
u \left( \frac{1}{2} (dq \cdot q - q \cdot dq) - \nu d(qdq) \right) = \frac{1}{2} (d(uq) \cdot uq - uq \cdot d(uq)) - \nu d(uqduq).
\]
Therefore, if two taut contact spheres are related by an element \(\phi_u\) of \(SO(3)\) as in the proof of Proposition 20 then one is the pull-back of the other under a diffeomorphism of \(\Gamma \setminus SU(2)\) induced by the map \(q \mapsto uq\). In particular, this \(SO(3)\)-action on taut contact spheres shows that any taut contact sphere can be swept out by great circles, all defining isomorphic taut contact circles.

This concludes the proof of Theorem 11.

This theorem allows us to define a map from the space of taut contact spheres to that of taut contact circles: simply pick any great circle.

**Theorem 23.** Let \(M\) be any left-quotient of \(SU(2)\). The map sending a taut contact sphere on \(M\) to any of its great circles induces an embedding of the moduli space of taut contact spheres on \(M\) into the moduli space of taut contact circles.

**Proof.** Recall the classification (up to diffeomorphism and homothety, i.e. conformal equivalence and rotation) of taut contact circles \((\alpha_2, \alpha_3)\) on left-quotients of \(S^3\), see \[15\], \[17\]. On the lens spaces \(L(m, m-1)\) we have the continuous family of taut contact circles induced by the \(\mathbb{Z}_m\)-invariant complex 1-form
\[
\alpha_2 + i\alpha_3 = \left( \frac{1}{2} + \delta \right) z_1 dz_2 - \left( \frac{1}{2} - \delta \right) z_2 dz_1
\]
(restricted to \(S^3 \subset \mathbb{C}^2\)) with \(\delta \in \mathbb{C}, -1/2 < \text{Re}(\delta) < 1/2\), modulo replacing \(\delta\) by \(-\delta\), which corresponds to the diffeomorphism defined by \((z_1, z_2) \mapsto (z_2, z_1)\) and changing from \((\alpha_2, \alpha_3)\) to \((-\alpha_2, -\alpha_3)\). So from Theorem 11 we see that

- only the taut contact circles with \(\delta\) purely imaginary extend to taut contact spheres,
• this extension is unique up to automorphisms of $\alpha_2 + i\alpha_3$, and
• two taut contact circles giving rise to isomorphic taut contact spheres must be isomorphic.

This proves the theorem in the given case. For the abelian left-quotients the map on moduli spaces amounts to the inclusion

$$\mathbb{R}_0^+ \quad \nu \quad \mapsto \quad \{ \delta \in \mathbb{C} : \quad -1/2 < \text{Re}(\delta) < 1/2 \} / \sim \delta.$$ 

The moduli space of taut contact circles on $L(m,m-1)$ also includes a discrete family described by

$$\alpha_2 + i\alpha_3 = nz_1 dz_2 - z_2 dz_1 + z_n^2 dz_2,$$

where $n$ ranges over the natural numbers congruent $-1$ mod $m$. These contact circles, however, do not extend to any taut contact sphere, so they are not being considered here. \hfill \square

The moduli spaces described in the foregoing proof (without the discrete family) are illustrated on the left-hand side of Figure 1. The moduli space of taut contact circles is the orbifold

$$\{ \delta \in \mathbb{C} : \quad -1/2 < \text{Re}(\delta) < 1/2 \} / \sim \delta.$$

The half-line

$$\{ \delta \in i\mathbb{R} \} / \sim \delta$$

constituting the moduli space of taut contact spheres is shown as a dashed line. The real part

$$\{ \delta \in \mathbb{R} : \quad -1/2 < \delta < 1/2 \} / \sim \delta$$

of this moduli space, shown in bold, corresponds to so-called Cartan structures. The origin represents the unique 3-Sasakian structure. These structures will be discussed in Section 7.

Under the mapping $\delta \mapsto \delta^2$, the moduli space of taut contact circles is mapped bijectively to the interior of the parabola

$$\left\{ x + iy \in \mathbb{C} : \quad x = \frac{1}{4} - y^2 \right\}$$

shown on the right-hand side of Figure 1. So this singular mapping flattens the cone point and yields a representation of the moduli space as an open subset of $\mathbb{C}$.

Here is the promised synthetic characterisation of the modulus $\nu$ of a taut contact sphere on $S^3$. One can define a canonical slice inside the hyperkähler manifold $S^3 \times \mathbb{R}$ as the subset $\{ \| \partial_t \| = 1 \}$. A straightforward calculation shows that for the taut contact sphere of modulus $\pm \nu$ we have

$$\| \partial_t \|^2 = \left( \frac{1}{4} + \nu^2 \right) \cdot (|z_1|^2 + |z_2|^2).$$

So the canonical slice is isometric with the 3-sphere of radius $2/\sqrt{1 + 4\nu^2}$. The canonical slice of maximal radius corresponds to the unique 3-Sasakian structure.

The canonical slice has the property that the 1-forms $e^t \alpha_i$ induce on it a 1-normalised taut contact sphere. We conclude that the family in Theorem 11 induces on any Euclidean sphere a $c$-normalised taut contact sphere, for some constant $c$. 
Recall now Definition 15. Once the contact sphere is 1-normalised, the canonical slice is given by \( \{ t = 0 \} \) and we see that the hyperkähler metric induces the long metric \( g_l \) on it. Thus, on a closed 3-manifold the long metric is always spherical.

The short metric can be written as \( g_s = g_l - \beta^2 \). We claim that \( \beta \) is invariant under the maps \( q \mapsto uq \) with \( u \in S^3 \), which shows that \( g_s \) is a Berger metric. To see this invariance property, notice that the system of structure equations (1) is invariant under rotations of the triple \((\alpha_1, \alpha_2, \alpha_3)\), and we have seen that left multiplication by unit quaternions \( u \) induces such rotations.

The contact sphere with \( \nu = 0 \) in Theorem 11 is invariant under right multiplication by unit quaternions, which makes the corresponding \( \beta \) bi-invariant, but recall that in this case \( \beta \equiv 0 \) and so there is no contradiction here.

We briefly expand on the point in the proof of Theorem 23 concerning the automorphisms of \( \alpha_2 + i \alpha_3 \). From the arguments in [15, Section 5.2] it follows that automorphisms \( \phi \) of \( \alpha_2 + i \alpha_3 \) are \( \mathbb{C} \)-linear maps of \( \mathbb{C}^2 \), in fact, elements of \( \text{SL}_2 \mathbb{C} \). Given an \( \alpha_1 \) extending \((\alpha_2, \alpha_3)\) to a taut contact sphere, the other extensions are given by \( \phi^* \alpha_1 \).

For \( S^3 \) and \( \nu = 0 \), any element of \( \text{SL}_2 \mathbb{C} \) defines an automorphism of \( \alpha_2 + i \alpha_3 \), so the possible extensions are parametrised by \( \text{SL}_2 \mathbb{C}/\text{SU}(2) \).

For \( S^3 \) and \( \nu \neq 0 \), the condition that \( \phi \) preserve \( \alpha_2 + i \alpha_3 \) forces it to be a diagonal map \( \phi_c(z_1, z_2) = (cz_1, c^{-1}z_2) \). The same is true for the lens spaces \( \mathbb{Z}_m \setminus \text{SU}(2) \) other than \( S^3 \), even in the case \( \nu = 0 \); here the condition for \( \phi \) to be diagonal follows from the fact that it has to lie in the normaliser of \( \mathbb{Z}_m \). In these two cases, the possible extensions are parametrised by \( \mathbb{R}^+ \), since

\[
\phi_c^* \alpha_1 = \frac{1}{4} (|c|^2 (z_1 d\bar{z}_1 - z_1 dz_1) + |c|^{-2} (z_2 d\bar{z}_2 - z_2 dz_2)) - \frac{\nu}{2} d(|c|^2 |z_1|^2 - |c|^{-2} |z_2|^2).
\]

Finally, for the quotients of \( \Gamma \setminus \text{SU}(2) \) with \( \Gamma \) non-abelian, the fact that \( \phi \) lies in the normaliser of \( \Gamma \) forces it to be plus or minus the identity map. Thus the extension \( \alpha_1 \) is unique.

5. Non-flat metrics

In this section we describe two constructions of taut contact spheres giving rise to non-flat hyperkähler metrics. These examples constitute a proof of Theorem 14.
since a non-flat hyperkähler metric is not even conformally flat (cf. the proof of Theorem 10).

5.1. The Helmholtz equation on the 2-sphere. We have seen in Proposition 21 how to construct a taut contact sphere corresponding to a suitable hyperkähler structure \((\Omega_1, \Omega_2, \Omega_3)\). Notice that in terms of the holomorphic structure given by \(J_1\), an equivalent description of this hyperkähler structure is given by a holomorphic symplectic form \(\Omega = \Omega_2 + i\Omega_3\) and a closed real \((1, 1)\)-form \(\Omega_1\) with \(2\Omega_1^2 = \Omega \wedge \overline{\Omega}\).

We now make the following ansatz: Let \(z_1, z_2\) be complex coordinates on \(\mathbb{C}^2\) and identify \(\mathbb{C}^2\) with \(\mathbb{R}^3 \times \mathbb{R}\) by equating the \(\mathbb{R}\)-direction with the real part of \(2z_1\), so that \(\partial_t = \text{Re}(\partial_{z_1})\) and \(\psi_t(z_1, z_2) = (z_1 + t/2, z_2)\). Set

\[
\Omega = \Omega_2 + i\Omega_3 = \lambda(z_1, z_2)\, dz_1 \wedge dz_2
\]

with \(\lambda(z_1, z_2)\) a nowhere zero holomorphic function, and

\[
\Omega_1 = \frac{i}{2} \partial \overline{\partial} H(z_1, z_2)
\]

with \(H(z_1, z_2)\) a real-valued function.

To satisfy the conditions \(\psi_t^* \Omega_i = e^{t} \Omega_i\) it is sufficient to have

\[
\psi_t^* \lambda = e^{t} \lambda, \quad \psi_t^* H = e^{t} H.
\]

Further, the identity \(2\Omega_1^2 = \Omega \wedge \overline{\Omega}\) is equivalent to the complex Monge-Ampère equation

\[
\det \begin{pmatrix} H_{z_1 \overline{z}_1} & H_{z_2 \overline{z}_2} \\ H_{\overline{z}_1 z_1} & H_{\overline{z}_2 z_2} \end{pmatrix} = \lambda \overline{\lambda}.
\]

If all these conditions are met, Proposition 21 tells us how to recover the corresponding taut contact sphere.

We satisfy condition (6) by taking \(\lambda = e^{2z_1}\) and simplifying the ansatz further to

\[
H(z_1, z_2) = e^{2\text{Re}(z_1)} h(2 \text{Im}(z_1), 2 \text{Re}(z_2))
\]

\[
= e^{z_1 + \overline{z}_1} h(\overline{z}_1 - i z_1, z_2 + \overline{z}_2),
\]

with \(h(s_1, s_2)\) a function of two real variables. Then, writing \(h_i\) for the partial derivatives \(h_s = \partial h/\partial s_i\), etc., we have

\[
\partial \overline{\partial} H = e^{z_1 + \overline{z}_1} \left( \left( h + h_{11} \right) dz_1 \wedge d\overline{z}_1 + \left( h_2 - i h_{12} \right) dz_1 \wedge d\overline{z}_2 
\]

\[
+ \left( h_2 + i h_{12} \right) dz_2 \wedge d\overline{z}_1 + h_{22} dz_2 \wedge d\overline{z}_2 \right),
\]

and equation (7) becomes

\[
1 = \begin{vmatrix} h + h_{11} & h_2 - i h_{12} \\ h_2 + i h_{12} & h_{22} \end{vmatrix} = \begin{vmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{vmatrix} + \begin{vmatrix} h & h_2 \\ h_2 & h_{22} \end{vmatrix}.
\]

The reader may check directly that the function

\[
h(s_1, s_2) = \cos s_1 \int_1^{s_2 / \cos s_1} \sqrt{\xi^2 - 1} \, d\xi
\]

satisfies this equation. In the appendix we shall explain how to derive this sample solution, and verify that it leads to a non-flat hyperkähler metric. We arrive at this example after proving a general result that relates solutions of (8) to solutions of the Helmholtz equation on the 2-sphere.
5.2. The Gibbons-Hawking ansatz. The Gibbons-Hawking ansatz \cite{Hawking:1974rv} starts with a positive function \( V \) in three variables \( x_1, x_2, x_3 \) (locally on \( \mathbb{R}^3 \)) and a triple of functions \( b_1, b_2, b_3 \) in the same variables, satisfying the condition

\[
(\theta) \quad \nabla V = -\text{curl}(b_1, b_2, b_3),
\]

so that in particular

\[
\Delta V = \text{div}(\nabla V) = -\text{div}(\text{curl}(b_1, b_2, b_3)) = 0,
\]

i.e. \( V \) is harmonic. Set \( \beta = b_1 \, dx_1 + b_2 \, dx_2 + b_3 \, dx_3 \) and consider the triple of 2-forms on \( \mathbb{R}^3 \times \mathbb{R} \) (with \( \theta \) denoting the \( \mathbb{R} \)-coordinate) defined by

\[
\Omega_i = (d\theta + \beta) \wedge dx_i + V \, dx_j \wedge dx_k.
\]

The relation between \( V \) and \( \beta \) implies that these 2-forms are closed. Writing

\[
\theta_0 := V^{-1/2}(d\theta + \beta) \quad \text{and} \quad \theta_i := V^{1/2}dx_i, \quad i = 1, 2, 3,
\]

we have

\[
\Omega_i = \theta_0 \wedge \theta_i + \theta_j \wedge \theta_k,
\]

which shows that \((\Omega_1, \Omega_2, \Omega_3)\) is a conformal symplectic triple. The corresponding hyperkähler metric is

\[
\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 = V^{-1}(d\theta + \beta)^2 + V(dx_1^2 + dx_2^2 + dx_3^2).
\]

Observe the formal similarity with the metric in \( \Omega \).

The translational invariance of this metric in \( \theta \)-direction is obvious\(^2\) A homothety can be built into this ansatz by choosing \( \beta \) appropriately.

Here is an example. As domain \( U_0 \subset \mathbb{R}^3 \) we take the half-space given by the condition \( x_1 > 0 \), and \( V(x_1, x_2, x_3) := x_1 \) is our positive function on that domain. Set \( (b_1, b_2, b_3) = (0, x_3, 0) \), that is, \( \beta = x_3 \, dx_2 \). Then

\[
\nabla V = (1, 0, 0) = -\text{curl}(b_1, b_2, b_3),
\]

i.e. condition \( \Omega \) is satisfied. Then the Gibbons-Hawking ansatz yields the hyperkähler structure\(^3\)

\[
\begin{align*}
\Omega_1 &= (dx_0 + x_3 \, dx_2) \wedge dx_1 + x_1 \, dx_2 \wedge dx_3, \\
\Omega_2 &= dx_0 \wedge dx_2 + x_1 \, dx_3 \wedge dx_1, \\
\Omega_3 &= (dx_0 + x_3 \, dx_2) \wedge dx_3 + x_1 \, dx_1 \wedge dx_2,
\end{align*}
\]

with corresponding hyperkähler metric

\[
g = \frac{1}{x_1}(dx_0 + x_3 \, dx_2)^2 + x_1 (dx_1^2 + dx_2^2 + dx_3^2).
\]

A tri-Liouville vector field for this hyperkähler structure is

\[
Y := \frac{2}{3}x_0 \partial_{x_0} + \frac{1}{3}(x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3}).
\]

By Proposition \( \Omega \), the equations \( \alpha_i = Y \cdot \Omega_i, \quad i = 1, 2, 3 \), define a taut contact sphere on any transversal \( U \) to \( Y \). By construction, \( g \) is in turn the hyperkähler metric on \( U \times \mathbb{R} \) induced by this taut contact sphere, with the \( \mathbb{R} \)-factor now corresponding to the flow lines of \( Y \).

---

\(^2\)In particular, the metric descends to \( \mathbb{R}^3 \times S^1 \); this is the usual form of the Gibbons-Hawking ansatz.

\(^3\)The change in notation from \( \theta \) to \( x_0 \) is meant to emphasise that we want this to be an \( \mathbb{R} \)-, not an \( S^1 \)-coordinate.
The surface \( \Sigma := \{ x_2 = x_3 = 0 \} \subset U_0 \times \mathbb{R} \) is totally geodesic for the metric \( g \), since it is the fixed point set of the isometric involution \((x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, -x_2, -x_3)\). The metric on \( \Sigma \) induced by \( g \) is 
\[
\frac{1}{x_1} dx_0^2 + x_1 dx_1^2,
\]
and, from the well-known formula for computing the Gauß curvature of a metric in diagonal form, one obtains \( K_\Sigma = -1/x_1^3 \). This proves that \( g \) is non-flat.

5.3. Comparison of the two examples. We observe that our ansatz from Section 5.1 can also be related to a harmonic function on \( \mathbb{R}^3 \):

Write \( \Delta S^2 \) for the Laplacian on the unit 2-sphere in \( \mathbb{R}^3 \), and \( \Delta \mathbb{R}^3 \) for the Laplacian on \( \mathbb{R}^3 \). Moreover, let \( \rho \) be the radial coordinate on \( \mathbb{R}^3 \). Given a differentiable function \( \tilde{u} : \mathbb{R}^3 \to \mathbb{R} \), we have the relation 
\[
\Delta S^2 (\tilde{u}|_{S^2}) = (\Delta \mathbb{R}^3 \tilde{u} - \tilde{u} \rho^2 - 2\tilde{u}' \rho)|_{S^2};
\]
the last summand reflects the fact that both principal curvatures of \( S^2 \) are equal to 1.

If \( \tilde{u} \) is of the form 
\[
\tilde{u}(x_1, x_2, x_3) = \rho u(x_1/\rho, x_2/\rho, x_3/\rho),
\]
then this last equation simplifies to 
\[
\Delta S^2 u + 2u = (\Delta \mathbb{R}^3 u)|_{S^2}.
\]
In that particular case, \( \tilde{u} \) is homogeneous of degree 1 in \( \rho \), hence \( \Delta \mathbb{R}^3 \tilde{u} \) is homogeneous of degree -1. This implies that the vanishing of \( (\Delta \mathbb{R}^3 \tilde{u})|_{S^2} \) is sufficient for the vanishing of \( \Delta \mathbb{R}^3 \tilde{u} \) on all \( \mathbb{R}^3 \). In conclusion, solutions to our Monge-Ampère equation \( (5) \) are — by the preceding discussion and Proposition \( (36) \) in the appendix — in direct correspondence with harmonic functions on \( \mathbb{R}^3 \) that are homogeneous of degree 1 in \( \rho \).

Finally, notice that both examples admit a tri-Hamiltonian vector field: in the example of Section 5.1, this is the vector field \( \partial_{x_3} \), with \( x_3 := \text{Im}(z_2) \); in the example of Section 5.2, it is \( \partial_{x_2} \). As we plan to show in a forthcoming paper, all taut contact spheres with such a symmetry can be related to a Gibbons-Hawking ansatz, although this relation, in the Helmholtz case, is far from straightforward.

6. A Bernstein problem

Are there any 1-normalised taut contact spheres — on a suitable open domain \( U \) — inducing a non-flat metric \( g \) on \( U \times \mathbb{R} \) such that the long metric \( g_l \) (see Definition \( (15) \)) induced on \( U \equiv U \times \{0\} \) is complete? This kind of completeness question is known as a Bernstein problem, see \[9\].

In order to appreciate the difficulty of this question, it is helpful to consider how the ansatz in Section 5.1 has to be extended if it is to be of any use in providing an answer. First of all, we ensured the homogeneity of \( \partial \overline{\partial} H \) by taking \( H \) to be homogeneous. However, the example 
\[
H = z_1 e^{\overline{z}_1} + \overline{z}_1 e^{z_1},
\]
where 
\[
\overline{\partial} H = (z_1 e^{\overline{z}_1} + e^{z_1}) d\overline{z}_1 \text{ and } \partial \overline{\partial} H = (e^{\overline{z}_1} + e^{z_1}) dz_1 \wedge d\overline{z}_1,
\]

and, from the well-known formula for computing the Gauß curvature of a metric in diagonal form, one obtains \( K_\Sigma = -1/x_1^3 \). This proves that \( g \) is non-flat.
shows that it is perfectly possible for $\bar{\partial}H$ to be homogeneous in $e^{\text{Re}(z_1)}$ without either $H$ or $\partial H$ having this property. Secondly, we took $H$ to be independent of $\text{Im}(z_2)$, but to discuss the general case one needs to allow the auxiliary function $h$ to depend on all three variables $\text{Im}(z_1)$, $\text{Re}(z_2)$, and $\text{Im}(z_2)$. Thirdly, while a potential $H$ always exists locally, it need not exist globally.

The question we just raised has a venerable history. In [10] Calabi describes a construction of Kähler-Einstein metrics, and in particular Ricci-flat Kähler metrics, on complex tubular domains $D \times i\mathbb{R}^n \subset \mathbb{C}^n$, where $D$ is some connected, open subset of $\mathbb{R}^n$. The construction is based on a real Monge-Ampère equation (with a constant on the right-hand side), or equivalently, a complex Monge-Ampère equation for a function depending only on the real parts of $n$ complex variables. The resulting metrics are invariant under the group of translations along $i\mathbb{R}^n$. Earlier results of Calabi [8] allowed him to show that, in the Ricci-flat case, metrics obtained via this construction can never be complete, except for the trivial case with $D = \mathbb{R}^n$ and a flat metric.

Here is a partial answer to this Bernstein problem, dealing with contact spheres that possess additional symmetries. In place of the long metric $g_l$ we consider the short metric $g_s$. Completeness of $g_s$ is a stronger condition than completeness of $g_l$.

The following is part (a) of Theorem 16.

**Theorem 24.** Let $(\omega_1, \omega_2, \omega_3)$ be a $1$-normalised taut contact sphere on a $3$-manifold $U$ with the property that the short metric $g_s := \omega_2^2 + \omega_3^2 + \omega_3^2$ is complete and admits a Killing field $\mathbf{X} \neq 0$ that preserves each form, i.e.

$$L_\mathbf{X} \omega_1 = L_\mathbf{X} \omega_2 = L_\mathbf{X} \omega_3 = 0.$$  

Then $U$ is compact and hence a left-quotient of $\text{SU}(2)$, and $(\omega_1, \omega_2, \omega_3)$ is isomorphic to one of the taut contact spheres described in Theorem 11.

The proof of this theorem will take up the remainder of this section. We begin by observing that $X$ does not have any zeros, which can be seen as follows. Arguing by contradiction, assume that $p \in U$ is a point with $X(p) = 0$. Then the flow of $X$ preserves the distance spheres from $p$, and is then necessarily a rotation about an axis through $p$ (in geodesic normal coordinates). This is incompatible with the fact that this flow preserves the coframe $(\alpha_1, \alpha_2, \alpha_3)$.

The fact that $X$ does not have any zeros means that

$$\Lambda := (\omega_1(X)^2 + \omega_2(X)^2 + \omega_3(X)^2)^{1/2}$$

defines a function $\Lambda: U \rightarrow \mathbb{R}^+$. Set

$$\alpha_i := \omega_i/\Lambda, \quad i = 1, 2, 3,$$

so that

$$\alpha_1(X)^2 + \alpha_2(X)^2 + \alpha_3(X)^2 = 1.$$  

Notice that the $\alpha_i$ satisfy the equations (1) with that very $\Lambda$, and they are likewise invariant under the flow of $X$.

Consider the map

$$\Psi := (x_1, x_2, x_3) := (\alpha_1(X), \alpha_2(X), \alpha_3(X)) : U \rightarrow S^2.$$

It is clear that the differential of $\Psi$ satisfies $T \Psi(X) = 0$, i.e. each flow line of $X$ is mapped to a single point in $S^2$. Write $\|\cdot\|_s$ for the length of tangent vectors to $U$.
with respect to the short metric \(g_s\), and \(\|\cdot\|_{S^2}\) for the length of tangent vectors to \(S^2 \subset \mathbb{R}^3\) with respect to the standard metric \(dx_1^2 + dx_2^2 + dx_3^2\).

**Lemma 25.** If \(Z \in T_p U\) is a tangent vector \(g_s\)-orthogonal to \(X\), then

\[
\|T\Psi(Z)\|_{S^2} \geq \|Z\|_s.
\]

**Proof.** The \(X\)-invariance of the \(\alpha_i\) gives, with the Cartan formula for the Lie derivative,

\[
dx_i = d(\alpha_i(X)) = L_X \alpha_i - X \cdot d\alpha_i = -X \cdot d\alpha_i.
\]

All the following computations are made at the single point \(p\). Rotate the contact sphere so that at that point \(p\) we have \(\alpha_1(X) = 1\) and \(\alpha_2(X) = \alpha_3(X) = 0\), i.e. \(\Psi(p) = (1, 0, 0)\). Then, with \(\mathbf{1}\),

\[
dx_1 = -\beta(X)\alpha_1 + \beta, 
dx_2 = -\beta(X)\alpha_2 + \Lambda \alpha_3, 
dx_3 = -\beta(X)\alpha_3 - \Lambda \alpha_2.
\]

The condition that \(Z\) be \(g_s\)-orthogonal to \(X\) means that \(\alpha_1(Z) = 0\). Hence

\[
dx_1(Z) = \beta(Z), 
dx_2(Z) = -\beta(X)\alpha_2(Z) + \Lambda \alpha_3(Z), 
dx_3(Z) = -\beta(X)\alpha_3(Z) - \Lambda \alpha_2(Z).
\]

From \(\Psi(p) = (1, 0, 0)\) we have \(dx_1(Z) = 0\). Then

\[
\|T\Psi(Z)\|_{S^2} = dx_1(Z)^2 + dx_2(Z)^2 + dx_3(Z)^2
\]

\[
= dx_2(Z)^2 + dx_3(Z)^2
\]

\[
= (\beta(X)^2 + \Lambda^2) \cdot (\alpha_2(Z)^2 + \alpha_3(Z)^2)
\]

\[
\geq \Lambda^2 (\alpha_1(Z)^2 + \alpha_2(Z)^2 + \alpha_3(Z)^2)
\]

\[
= \|Z\|_s^2. \quad \square
\]

This lemma implies in particular that \(T\Psi\) has full rank at every point, so \(\Psi(U)\) is an open subset of \(S^2\).

**Lemma 26.** The map \(\Psi: U \rightarrow S^2\) is surjective.

**Proof.** Since \(\Psi(U) \subset S^2\) is open, it suffices to show that \(\Psi(U)\) is complete, i.e. that every path \(\gamma: [0,1] \rightarrow \Psi(U)\) of finite length has a limit point inside \(\Psi(U)\). Let \(\tilde{\gamma}: [0, t_0] \rightarrow U\) be a maximal lift of such a path, \(g_s\)-orthogonal to \(X\). By the previous lemma, this lift is non-empty and of finite length in the short metric. Since \(U\) is complete, we deduce that \(t_0 = 1\) and that the lift \(\tilde{\gamma}\) has a limit point in \(U\). Therefore \(\gamma\) has a limit point in \(\Psi(U)\). \(\square\)

**Lemma 27.** For each \(q \in S^2\), the preimage \(\Psi^{-1}(q) \subset U\) is a single orbit of \(X\).

**Proof.** Arguing by contradiction, we assume that \(q \in S^2\) is a point in the image of two distinct orbits \(O_0, O_1\) of \(X\). Let \(\tilde{\gamma}: [0, 1] \rightarrow U\) be a path joining these two orbits; \(\gamma := \Psi \circ \tilde{\gamma}: [0, 1] \rightarrow S^2\) is then a loop based at \(q\). By the argument in the proof of the preceding lemma we may assume that \(\tilde{\gamma}\) is orthogonal to \(X\). Let \(\gamma_s, s \in [0, 1]\), be a homotopy of \(\gamma = \gamma_0 \text{ rel } \{0, 1\}\) to the constant path \(\gamma_1\) at \(q\), and \(\tilde{\gamma}_s\)
the corresponding homotopy of lifts orthogonal to $X$ with initial point $\tilde{\gamma}_s(0) = \tilde{\gamma}(0)$ for all $s \in [0, 1]$. The endpoints $\tilde{\gamma}_s(1)$ form a smooth path in $\Psi^{-1}(q)$.

The completeness of $(U, g_s)$ entails that the flow of $X$ is complete, and this in turn ensures that $\tilde{\gamma}_s(1) \in O_1$ for all $s$. But the constant path $\gamma_1$ lifts to the constant path $\tilde{\gamma}_1$ at $\tilde{\gamma}(0) \in O_0$, hence $\tilde{\gamma}_1(1) \in O_0 \cap O_1$, which is impossible. □

As preimages of single points, the orbits of $X$ are closed subsets of $U$. This means that every orbit is either periodic or a proper embedding of $\mathbb{R}$ in $U$. If there is a periodic orbit $O_0$, all other orbits remain at a bounded distance from $O_0$, since the flow of $X$ is by isometries. This precludes proper embeddings of $\mathbb{R}$, i.e., in this case all orbits must be periodic, and $U$ is compact, as asserted in Theorem 24.

It remains to consider the complementary case, when all orbits are proper embeddings of $\mathbb{R}$. In this case, the time-1 map of the flow of $X$ will disjoin any sufficiently small compact set $K$ from itself, since each single orbit through $K$ is proper and the flow is by isometries. It follows that this time-1 map defines a free and properly discontinuous $\mathbb{Z}$-action on $U$.

The quotient manifold under this action is, by the first case, a compact manifold with a taut contact sphere, and hence a left-quotient of $SU(2)$. Such manifolds do not admit infinite covers. In other words, this second case cannot occur.

This completes the proof of Theorem 24.

7. 3-Sasakian structures

In this section we consider taut contact spheres $(\alpha_1, \alpha_2, \alpha_3)$ on a 3-dimensional domain $U$ that satisfy the stronger condition $\alpha_i \wedge d\alpha_j = 0$ for $i \neq j$. The conditions for a naturally ordered taut contact sphere then imply that $d\alpha_i = \Lambda \alpha_j \wedge \alpha_k$ for some $\Lambda: U \to \mathbb{R}^+$. Differentiation of this equation yields

$$0 = d(d\alpha_i) = d\Lambda \wedge \alpha_j \wedge \alpha_k + \Lambda d\alpha_j \wedge \alpha_k - \Lambda \alpha_j \wedge d\alpha_k = d\Lambda \wedge \alpha_j \wedge \alpha_k.$$ 

It follows that $d\Lambda \equiv 0$, so $\Lambda \equiv c$ for some constant $c$ and the contact sphere is $c$-normalised.

Recall that the Reeb vector field $R$ of a contact form $\alpha$ is defined by $\alpha(R) \equiv 1$ and $R \lrcorner \ d\alpha \equiv 0$. We now have the following simple lemma, where $R_i$ denotes the Reeb vector field of $\alpha_i$, and $\beta$ is the 1-form from equation (11). The proof is left to the reader.

**Lemma 28.** For a taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$, the following conditions are equivalent:

(i) $\alpha_i \wedge d\alpha_j = 0$ for $i \neq j$.

(ii) The Reeb vector fields $(R_1, R_2, R_3)$ constitute a frame dual to the coframe $(\alpha_1, \alpha_2, \alpha_3)$.

(iii) $\beta = 0$.

(iv) The short metric $g_s$ equals the long metric $g_l$.

If any of these conditions holds and $c$ is the normalisation constant, then one has $[R_i, R_j] = -cR_k$. This implies that the metric $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ has constant curvature equal to $c^2/4$. In particular, the metric $g_s = g_l$ has curvature identically equal to $1/4$. □

In analogy with our terminology in [15] we call a taut contact sphere satisfying any of the conditions in this lemma a **Cartan structure**.
Remark 29. Of the taut contact spheres in the statement of Theorem 11 exactly those with \( \nu = 0 \) are Cartan structures. Those with \( \nu \neq 0 \) are not even conformally equivalent to a Cartan structure; this follows for instance from [15, Prop. 6.1].

We now want to relate such Cartan structures to 3-Sasakian structures. Recall the definition of these structures.

Definition 30. A metric \( g \) on a 3-manifold \( U \) is called 3-Sasakian if the cone metric \( C = e^{2s}(\mathcal{g} + ds^2) \) on \( U \times \mathbb{R} \) is a hyperkähler metric.

Given such a cone metric \( C \), we observe that \( \partial_s \) is a homothetic vector field and \( \partial_s \downarrow C = e^{2s} \, ds \) is a closed 1-form.

Remark 31. Given a Riemannian metric \( g \) and a vector field \( Z \), the following are equivalent:

(i) \( \nabla Z \) is the identity on every tangent space.
(ii) \( L_Z g = g \) and \( Z \downarrow g \) is closed.
(iii) \( L_Z g = g \), and where \( Z \) is non-vanishing it is orthogonal to a codimension 1 foliation.
(iv) On the open set \( \{ Z \neq 0 \} \) we have local descriptions \( g = e^{2s}(\mathcal{g} + ds^2) \) with \( \partial_s = Z \) and \( \mathcal{g} \) being the metric induced by \( g \) in a transversal orthogonal to \( Z \).

Following [19], we call a vector field satisfying either of these conditions a dilatation. Thus, any particular description of \( g \) as a cone metric corresponds to a non-vanishing dilatation.

Proposition 32. A 1-normalised taut contact sphere on a 3-dimensional manifold \( U \) is a Cartan structure if and only if the induced hyperkähler metric \( g \) on \( U \times \mathbb{R} \) is a cone metric with the induced tri-Liouville vector field \( \partial_t \) as dilatation.

Proof. By formula (4) for the metric \( g \), the 1-form \( \partial_t \downarrow g \) equals \( e^t(dt + \beta) \), which is closed if and only if \( \beta \equiv 0 \). \( \square \)

Lemma 33. Given any metric \( \mathcal{g} \) on \( U \), any endomorphism field on \( U \times \mathbb{R} \) parallel with respect to the corresponding cone metric is invariant under the flow of \( \partial_s \).

Proof. Let \( (x_1, x_2, x_3) \) be local coordinates on \( U \). Computing the Levi-Civita connection of the cone metric in the coordinates \( (x_1, x_2, x_3, s) \), one finds that \( e^{-s}\partial_{x_1}, e^{-s}\partial_{x_2}, e^{-s}\partial_{x_3}, e^{-s}\partial_s \) are parallel along the radii. The coefficients of a parallel endomorphism field in this frame are constant along the radii. Those coefficients are the same in the frame \( \partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_s \). \( \square \)

It is a fact in Riemannian geometry that a family of metrics of the form \( \mathcal{g}_\lambda = \lambda \mathcal{g} \), \( \lambda \in \mathbb{R}^+ \), in general gives rise to a family of non-isometric cone metrics. We use the methods of this paper to give a simple proof of the following result, well-known in Sasakian geometry, which can be read as saying that for \( \mathcal{g} \) having constant positive curvature, only the \( \mathcal{g}_\lambda \) of curvature equal to 1 gives rise to a hyperkähler cone.

Proposition 34. A 3-Sasakian metric in dimension 3 has constant curvature equal to 1. Therefore, if a 4-dimensional hyperkähler metric admits a dilatation, it must be flat.
Proof. Let $\overline{g}$ be a Riemannian metric on a 3-manifold $U$ and assume that the cone metric $C = e^{2s}(ds^2 + \overline{g})$ is hyperkähler. By the preceding lemma the flow of the dilatation $(1/2)\partial_s$ is tri-holomorphic and also tri-Liouville. Introduce the new coordinate $t = 2s$, so that $\partial_t = (1/2)\partial_s$. By the theory we have developed, there is a 1-normalised taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ on $U$ such that

$$C = e^t(dt^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2).$$

Notice that $\overline{g}$ is recovered from the cone metric and the dilatation $\partial_t$ via the formula $g = \left. C \right|_{s=0} = \left. C \right|_{\|\partial_s\|=1}$. But $\|\partial_t\| = 2\|\partial_s\|$, so

$$\overline{g} = e^t(dt^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)|_{e^t=1/4} = (\alpha_1/2)^2 + (\alpha_2/2)^2 + (\alpha_3/2)^2.$$

We see that $\overline{g}$ has a 2-normalised Cartan structure as an orthonormal frame, making it a metric of constant curvature equal to 1.

Then $e^{2s}(\overline{g} + ds^2)$ describes the 4-dimensional Euclidean metric in spherical coordinates.

This result must be read as local rigidity of 3-Sasakian structures. As our constructions of taut contact spheres giving rise to non-flat hyperkähler metrics show, taut contact spheres have much richer local geometry. In particular, their associated hyperkähler metrics admit homotheties, but no dilatations. So they are not cone metrics in any way.

As we have mentioned in Section 4, the $c$-normalised taut contact spheres on closed 3-manifolds are orthonormal for Berger metrics, which are spherical only in the Cartan case. The long metrics, on the other hand, are always spherical.

By the discussion in this section, our Theorems 3 and 11 can be read as a classification of the closed 3-Sasakian 3-manifolds:

**Corollary 35.** The closed 3-Sasakian 3-manifolds are precisely the left-quotients of $SU(2)$.

For the parametric family of taut contact spheres in Theorem 11 the induced hyperkähler metric is always the standard Euclidean metric on $E^4 \setminus \{0\}$, so it is always a cone metric. But only for the parameter value $\nu = 0$ (corresponding to the unique class containing Cartan structures) does the tri-Liouville vector field $Y$ point in the radial direction (i.e. the direction of the only non-vanishing dilatation of $E^4 \setminus \{0\}$).

In fact, only the tri-Liouville vector field changes within this parametric family. The $d\alpha_i$ obviously do not depend on $\nu$, and a little computation shows that neither do the Reeb vector fields $R_i$ of the $\alpha_i$.

A proof of Corollary 35 previous to the one given above was indeed based on the observation that a 3-Sasakian 3-manifold is a space of constant curvature 1; the classification of 3-Sasakian manifolds among the 3-dimensional space forms was achieved by Sasaki [22] (who still spoke of normal contact metric 3-structures).

**Appendix: A contact transformation leading to Helmholtz’ equation**

The following intriguing proposition relates solutions of equation (8) from Section 5.1 to solutions of the Helmholtz equation (cf. [12] for this terminology) $\Delta u + 2u = 0$ on the 2-sphere $S^2$ with its metric of constant curvature 1. In geodesic polar coordinates this metric is given by

$$ds^2 = dr^2 + \sin^2 r \, d\theta^2.$$
Hence the gradient of a differentiable function \( f : S^2 \to \mathbb{R} \) is computed via

\[
\nabla f = f_r \partial_r + \frac{1}{\sin^2 r} f_\theta \partial_\theta.
\]

The area element is \( A = \sin r \, dr \wedge d\theta \), so from

\[
d(X \cdot \nabla A) = L_X A = \text{div}(X) \cdot A \quad \text{and} \quad \Delta f = \text{div}(\nabla f)
\]

we see that the **spherical** Laplacian \( \Delta = \Delta^{S^2} \) takes the form

\[
\Delta f = \frac{\cos r}{\sin r} f_r + f_{rr} + \frac{1}{\sin^2 r} f_{\theta\theta}.
\]

**Proposition 36.** Let \( h(s_1, s_2) \) be a solution of equation (8). Set

\[
(T) \begin{cases}
\theta &= s_1, \\
r &= \text{arccot}(h(s_2)) \in (0, \pi), \\
u &= -s_2 \cos r + h \sin r.
\end{cases}
\]

If \( h_{s_2 s_2} \neq 0 \), then \( u = u(r, \theta) \) is a function of the independent variables \( r \) and \( \theta \) that solves the spherical Helmholtz equation

\[
(10) \quad \Delta u + 2u = 0.
\]

Conversely, a solution \( u(r, \theta) \) of (10) satisfying \( u + u_{rr} \neq 0 \) gives rise to independent variables \( s_1, s_2 \) and a solution \( h(s_1, s_2) \) of (8) via the inverse transformation

\[
(T^{-1}) \begin{cases}
s_1 &= \theta, \\
s_2 &= u_r \sin r - u \cos r, \\
h &= u_r \cos r + u \sin r.
\end{cases}
\]

**Proof.** Given \( h(s_1, s_2) \), let \((r, \theta, u)\) be defined by \((T)\). The condition \( h_{s_2 s_2} \neq 0 \) is obviously necessary and sufficient for \((s_1, s_2) \mapsto (r, \theta)\) to be an invertible coordinate transformation. From

\[
\begin{align*}
du &= -\cos r \, ds_2 + s_2 \sin r \, dr + (h_{s_1} \, ds_1 + h_{s_2} \, ds_2) \sin r + h \cos r \, dr \\
&= (s_2 \sin r + h \cos r) \, dr + h_{s_1} \, \sin r \, d\theta
\end{align*}
\]

we see that \( u \) is a function of \( r \) and \( \theta \) with

\[
(T) \begin{cases}
u_r &= s_2 \sin r + h \cos r, \\
u_\theta &= h_{s_1} \, \sin r.
\end{cases}
\]

Conversely, given \( u(r, \theta) \), let \((s_1, s_2, h)\) be defined by \((T^{-1})\). Since

\[
d s_2 = (u + u_{rr}) \sin r \, dr + (u_r \sin r - u_\theta \cos r) \, d\theta,
\]

the condition \( u + u_{rr} \neq 0 \) is necessary and sufficient for \((r, \theta) \mapsto (s_1, s_2)\) to be a coordinate transformation, because \( r \in (0, \pi) \). From

\[
\begin{align*}
dh &= (u + u_{rr}) \cos r \, dr + (u_r \cos r + u_\theta \sin r) \, d\theta \\
&= \frac{u_\theta}{\sin r} \, ds_1 + \cot r \, ds_2
\end{align*}
\]

we infer that \( h \) is a function of \( s_1 \) and \( s_2 \) with

\[
(T^{-1}) \begin{cases}
h_{s_1} &= u_\theta \, \sin r, \\
h_{s_2} &= \cot r.
\end{cases}
\]
It is then a straightforward check that \((T)\) and \((T^{-1})\) are indeed inverse transformations of each other. In particular, one finds that

\[
h_{s_2s_2} = -\frac{1}{(u + u_{rt}) \sin^3 r},
\]

which shows that \((T)\) is defined on \(h\) if and only if \((T^{-1})\) is defined on \(u\).

In order to show that solutions of \((8)\) correspond to solutions of \((10)\) under the transformation \((T)\), it is convenient to rewrite \((8)\) as an exterior differential system \((11)\):

\[
dh = p_1 ds_1 + p_2 ds_2,
\]

\[
ds_1 \wedge ds_2 = dp_1 \wedge dp_2 + h ds_1 \wedge dp_2 - p_2^2 ds_1 \wedge ds_2. \tag{11}
\]

We then compute, using \((T^{-1})\) and \((T^{-1})'\),

\[
dp_1 \wedge dp_2 + h ds_1 \wedge dp_2 - (1 + p_2^2) ds_1 \wedge ds_2 =
\]

\[
= \frac{u_{\theta\theta}}{\sin^2 r} + \frac{1}{\sin^2 r} u_r \cos r \sin r + u \frac{1}{\sin r} + \frac{1}{\sin^2 r} (u + u_{rr}) \sin r \right) dr \wedge d\theta
\]

\[
= \frac{1}{\sin r} \left( \frac{1}{\sin^2 r} u_{\theta\theta} + \cos r \sin r u_r + 2u \right) dr \wedge d\theta
\]

\[
= \frac{1}{\sin r} (\Delta u + 2u) dr \wedge d\theta.
\]

This completes the proof of Proposition 36. \(\square\)

Since the reader is bound to wonder how we arrived at the transformation \((T)\), we present, in nuce, our chain of discovery: First, one can get rid of mixed derivatives in the exterior differential system \((11)\) by introducing \(p_2\) as an independent variable. The identity

\[
dh - p_1 ds_1 - p_2 ds_2 = d(h - p_2 s_2) - p_1 ds_1 + s_2 dp_2
\]

suggests the contact transformation \(h(s_1, s_2) \sim k(t_1, t_2)\) given by

\[
t_1 = s_1, \ t_2 = p_2 = h_{s_2}, \ k = h - s_2 h_{s_2},
\]

with inverse transformation

\[
s_1 = t_1, \ s_2 = -k_{t_2}, \ h = k - t_2 k_{t_2}.
\]

(Here one needs \(h_{s_2s_2} \neq 0\) or \(k_{t_2t_2} \neq 0\), respectively, for these to be honest, i.e. invertible, transformations.) We compute

\[
dp_1 \wedge dp_2 = dk_{t_1} \wedge dt_2 = k_{t_1 t_2} dt_1 \wedge dt_2,
\]

\[
h ds_1 \wedge dp_2 = (k - t_2 k_{t_2}) dt_1 \wedge dt_2,
\]

\[
-(1 + p_2^2) ds_1 \wedge ds_2 = (1 + t_2^2) dt_1 \wedge dk_{t_2} = (1 + t_2^2) k_{t_2 t_2} dt_1 \wedge dt_2.
\]

So the contact transformation takes equation \((8)\) to the following equation, where we now write \(k_i\) for \(k_{t_i}\), etc.: \(\tag{12}\)

\[
k_{i_1} + (1 + t_2^2) k_{i_2} - t_2 k_2 + k = 0.
\]
The first order term in this equation can be made to disappear by setting $x = t_1$, $\sinh y = t_2$ and $w(x, y) = k(x, \sinh y) / \cosh y$. This turns equation (12) into
\begin{equation}
(13) \quad w_{xx} + w_{yy} + \frac{2}{\cosh^2 y} w = 0,
\end{equation}
which is the Helmholtz equation for the metric
\[
\begin{pmatrix}
\frac{1}{\cosh^2 y} & 0 \\
0 & \frac{1}{\cosh^2 y}
\end{pmatrix}.
\]
The Gauß curvature of this metric turns out to be identically equal to 1. Indeed, it is the spherical metric in Mercator coordinates. Therefore, we pass from (13) to (10) by making the substitution
\[
cosh y = \frac{1}{\sin r}, \quad \sinh y = \cot r, \quad \theta = x, \quad u(r, \theta) = w(x, y).
\]
We now want to find an explicit solution of (10). The ansatz
\[
u(r, \theta) = \cos \theta \sin r g(r)
\]
leads to
\[
3 g_r \cos r + g_{rr} \sin r = 0,
\]
which has the solution $g_r = 1 / \sin^3 r$; that in turn integrates to
\[
g(r) = \int_{\cot r}^{0} \sqrt{1 + t^2} \, dt.
\]
The resulting $u$ corresponds under $(T^{-1})$ to the solution
\[
h(s_1, s_2) = \cos s_1 \int_{1}^{s_2 / \cos s_1} \sqrt{\xi^2 - 1} \, d\xi
\]
of equation (8). As domain of definition we may take
\[
U' = \{(s_1, s_2) \in \mathbb{R}^2 : |s_1| < \pi/2, \ s_2 > \cos s_1\}.
\]
The Kähler potential $H$ corresponding to this solution gives rise to a hyperkähler metric $g$ on $U \times \mathbb{R}$ inducing a taut contact sphere on $U$, with
\[
U = \{(s_1, s_2, s_3) \in \mathbb{R}^3 : (s_1, s_2) \in U'\}.
\]
The relation with the complex coordinates is given by $t + is_1 = 2z_1$ and $s_2 + is_3 = 2z_2$, say.

We claim that $g$ is non-flat. The coefficients of this metric are
\[
g_{\alpha \bar{\alpha}} = g(\partial_{z_{\alpha}}, \partial_{\bar{z}_{\alpha}}) = \Omega_1(\partial_{z_{\alpha}}, J_1 \partial_{\bar{z}_{\beta}}) = \frac{1}{2} H_{z_{\alpha} \bar{z}_{\beta}}.
\]
Write $G$ for the $(2 \times 2)$-matrix $(g_{\alpha \bar{\alpha}})$. Then the curvature tensor $K_{\alpha \bar{\alpha} \gamma \bar{\beta}}$, read for fixed $\gamma, \bar{\beta}$ as a $(2 \times 2)$-matrix indexed by $\alpha$ and $\bar{\alpha}$, is computed by
\[
(K_{\alpha \bar{\alpha} \gamma \bar{\beta}}) = G_{z_{\alpha} \bar{z}_{\alpha}} - G_{z_{\gamma} \bar{z}_{\gamma}} G^{-1} G_{\gamma \bar{\beta}},
\]
cf. [21] p. 159.
We now want to show that $K_{\mathcal{H}}^2$ is non-zero. In the following computations we write $\ast$ for any matrix entry that is irrelevant for the final result:

\[
G = \frac{1}{2} e^{z_1 + \overline{z}_1} \begin{pmatrix}
 h + h_{11} & h_2 - i h_{12} \\
 h_2 + i h_{12} & h_{22}
\end{pmatrix},
\]

\[
G_{z_2} = \frac{1}{2} e^{z_1 + \overline{z}_1} \begin{pmatrix}
 \ast & \ast \\
 h_{22} + i h_{122} & h_{222}
\end{pmatrix},
\]

\[
G_{\overline{z}_2} = \frac{1}{2} e^{z_1 + \overline{z}_1} \begin{pmatrix}
 \ast & h_{22} - i h_{122} \\
 \ast & h_{222}
\end{pmatrix},
\]

\[
G_{z_2 \overline{z}_2} = \frac{1}{2} e^{z_1 + \overline{z}_1} \begin{pmatrix}
 \ast & \ast \\
 \ast & h_{2222}
\end{pmatrix}.
\]

On the hyperplane $\{2 \text{Im}(z_1) = z_1 - \overline{z}_1 = 0\}$, corresponding to the line $\{s_1 = 0\}$, we have
\[ h_1 = 0 \text{ and } h = \int_1^{s_2} \sqrt{\xi^2 - 1} \, d\xi. \]

Writing $\sqrt{\xi^2 - 1} = \sigma$, for short, we have along that same line
\[
 h_2 = \sigma, \quad h_{22} = s_2/\sigma, \quad h_{222} = -1/\sigma^3, \quad h_{2222} = 3s_2/\sigma^5.
\]

At $(s_1, s_2) = (0, \sqrt{2})$ we thus find
\[ h_2 = 1, \quad h_{22} = \sqrt{2}, \quad h_{222} = -1, \quad h_{2222} = 3\sqrt{2}, \]
\[ h_{11} = h_{12} = 0, \text{ and } h + h_{11} = \sqrt{2}. \]

That last equality can be computed from (8). Hence, at $(z_1, z_2) = (0, \sqrt{2}/2)$ we obtain
\[
\left( \begin{array}{cc} \ast & \ast \\ \ast & K_{\mathcal{H}\overline{\mathcal{H}}} \end{array} \right) = \frac{1}{2} \left( \begin{array}{cc} \ast & \ast \\ \ast & 3\sqrt{2} \end{array} \right) - \frac{1}{2} \left( \begin{array}{cc} \ast & \ast \\ \ast & \sqrt{2} \end{array} \right) \left( \begin{array}{cc} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{array} \right) \left( \begin{array}{cc} \ast & \ast \\ \ast & -1 \end{array} \right),
\]
which yields $K_{\mathcal{H}\overline{\mathcal{H}}} = -\sqrt{2} \neq 0$. Thus $g$ is indeed non-flat.

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