Velocity dependence of point masses, moving on timelike geodesics, in weak gravitational fields.

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I. INTRODUCTION

General relativity provides the standard description of generalized motion and gravity. Geodesics of free fall particles follow space-time curvature in inhomogeneous gravitational fields. These are characterized by tidal forces described by the geodesic deviation equations of a pseudo-Riemannian metric. The principle of equivalence although not playing a highly prominent role in the modern version of the theory, is still useful if the limits of its applicability are clearly delineated. That is, the principle is believed to hold in the limit of small local space-time regions of the gravitational field. We begin by using the principle of equivalence to simple cases of acceleration from a special relativistic viewpoint, in an attempt to elucidate the qualitative result of acceleration dependence on velocity. We find this translates into velocity dependence for timelike geodesics in gravitational fields.

Einstein reasoned that light would bend under gravity based on acceleration arguments and applying the principle of equivalence. He then calculated the more precise result using general relativity, taking into account the effect of curved spacetime. Using a similar approach we deduce that acceleration under gravity is velocity dependent using accelerating frames under special relativity, which we then confirm with the Schwarzschild solution of general relativity, incidentally producing a result that coincides with one by Hilbert. We find a discrepancy of a factor of two between the result using flat space accelerations and the Schwarzschild analysis, the reason for which we further explore.

We can begin by defining a spacetime coordinate differential with a four-vector

\[ dx^\mu = (cdt, dx, dy, dz), \]

with contribution from three spatial dimensions and \( t \) is the time in a particular reference frame and \( c \) is the invariant speed of light. In this paper we are able to focus exclusively on one-dimensional motion and so we can suppress two of the space dimensions writing a space-time vector \( dx^\mu = (cdt, dx) \). We have the metric tensor \( g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) that defines the covariant vector \( dx_\mu = g_{\mu\nu} dx^\nu = (cdt, -dx) \). In the co-moving frame we have \( dx = 0 \) and so \( dx^\mu = (cdt, 0) \), which defines \( \tau \) the local proper time. We define the four-velocity

\[ v^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau} \frac{dx^\mu}{dt} = (\gamma c, \gamma v), \]

where \( v = dx/dt \) and

\[ \gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \]

We then have the magnitude of the velocity four-vector

\[ \sqrt{v^\mu v_\mu} = \sqrt{\gamma^2 c^2 - \gamma^2 v^2} = c \]

that is a Lorentz invariant, where we have used the Einstein summation convention. We also have the four-acceleration

\[ a^\mu = \frac{d^2 x^\mu}{d\tau^2} = (\gamma^4 v a/c, \gamma^4 a). \]

As we are describing one-dimensional motion, as opposed to motion in three-dimensions, so that we are implicitly assuming the special case in which \( v \) is parallel to \( a = d^2x/d\tau^2 \). We then find the magnitude of the four-acceleration

\[ \sqrt{a^\mu a_\mu} = \sqrt{\gamma^8 c^2 a^2/c^2 - \gamma^8 a^2} = \gamma^3 a. \]

Now, in the momentarily co-moving inertial frame (MCIF) we have \( v = 0 \) giving the acceleration four-vector \( a^\mu = (0, a) \) and the four-velocity \( v^\mu = (c, 0) \),
which gives $\sqrt{\alpha^\mu a_{\mu}} = \alpha$ and the expected orthogonality $v^\mu a_{\mu} = 0$. Hence, comparing the magnitudes of the four-acceleration in Eq. (6) with the magnitude in the MCIF we find $\alpha = \gamma^3 a$ so that in an alternate inertial frame we calculate

$$a = \alpha/\gamma^3.$$  \hfill (7)

To confirm Eq. (5) we can apply a Lorentz boost to the MCIF four-acceleration $a^\mu = (0, \alpha)$, with the transformation $t = \gamma(t' + vx'/c^2)$ and $x = \gamma(x' + vt')$. This produces $a^\mu = (\gamma v \alpha/c^2, \gamma \alpha)$ and so comparing this with Eq. (5) we have $\gamma \alpha = \gamma^3 a$ or $\alpha = \gamma^3 a$, confirming Eq. (7).

A. Thought experiment

Consider a rocket out in space far from the effects of any gravitational influences. Within this, effectively flat region of space, we place small frames of reference that individually can measure the acceleration of passing objects. We will call these types of frames PG1 for particle group 1. The PG1 frames are currently at rest relative to the rocket and also with respect to each other and they are spread throughout the space surrounding the rocket. The rocket also has a hole at the top and bottom so that the PG1 can pass straight through allowing them to measure the acceleration of the rocket. The rocket also has an inbuilt mechanism so that, when the rocket is accelerating, it will drop a second group of particles, labeled PG2, from the top of the rocket, at predetermined fixed time intervals as measured by the rocket. Thus, PG2 can also measure the rocket’s acceleration.

Now, for the sake of argument, let the rocket be accelerated at 9.8 ms$^{-2}$ and as specified, PG2 will be dropping from the top of the rocket. The rocket now accelerates away from the PG2 frames with acceleration $\alpha = T/m = 9.8$ ms$^{-2}$, where $m$ is the mass of the rocket and assuming $T$ is an applied thrust in order to maintain a constant proper acceleration. The PG2 particles, once released, comprise inertial objects not partaking in the rocket’s acceleration. Additionally, as the rocket continues its acceleration it will encounter PG1 lying in its path that will enter the hole at the top of the rocket and while passing through measure the acceleration of the rocket. Now, as the rocket is maintaining a steady acceleration, clearly the velocity of the rocket will be steadily increasing. Hence the rocket will be encountering the PG1 at higher and higher relative velocities.

The question we now wish to consider is: Will PG1 and PG2 measure the same acceleration for the rocket?

Based on standard theory, we expect the answer to be in the negative. This is because special relativity asserts that, as viewed by PG1, the rocket’s velocity will converge to the light speed upper bound, and so the acceleration will appear to decrease. Since, this physical setting is described by Eq. (4), the one-dimensional relativistic equation for acceleration $a$, as measured in the PG1 frames, can be written as

$$a = \frac{\alpha}{\gamma^3} = \frac{T}{m} \left(1 - \frac{v^2}{c^2}\right)^{3/2},$$  \hfill (8)

where $\alpha$ is the acceleration measured in the co-moving inertial frames PG2, $v$ is the velocity of the rocket relative to PG1. Eq. (8) may have been qualitatively anticipated from special relativity, as it asserts that as viewed by PG1, the rocket’s velocity will appear to converge to the light speed upper bound, and hence the acceleration will appear to decrease, as shown by $\gamma^3$ in the denominator.

This shows that the PG2 accelerations can be reduced, or effectively transformed away for $v \to c$ with a Lorentz boost to the non-accelerating frame of PG1. Interestingly if PG1 were to observe a light beam fired horizontally across the rocket, this would also transform away the bending of the light as seen inside the rocket frame. That is, as PG1 are not accelerating with the rocket, PG1 will see light travel in a straight line. This means both cases of light bending and acceleration dependence, can be transformed away and are therefore coordinate dependent phenomena. Of further interest is that due to the low acceleration of the rocket, we can ignore the effect of Einstein’s time dilation occurring as a function of ‘vertical’ position in the rocket, and hence the rocket can consider itself a satisfactory candidate for viewing the accelerations of PG1. We will find the PG1 acceleration to be less than PG2.

Now, given these results, we can ask a pivotal question: *Given the principle of equivalence will the above mentioned results for accelerating observers, be replicated under a gravitational field?*

We presume for appropriately local regions of a homogeneous gravitational field, the answer must be in the affirmative, provided we now replace the words ‘acceleration of the rocket with respect to both PG1 and PG2’ with the words ‘fixed observer with respect to the source of gravitational field’ and we replace the words ‘acceleration of PG1 and PG2 with respect to the rocket’ with ‘PG1 and PG2 being freely falling particles on timelike geodesics’.

B. Gravitational fields

The central role played by the equivalence principle in the general theory was stated by Einstein in 1907: *we [...] assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system.*

Einstein’s equivalence principle was initially based on the well established equivalence of gravitational and inertial mass, also called the weak equivalence principle, which has now been confirmed by experiment\textsuperscript{2} to an accuracy better than $1 \times 10^{-15}$. It is now recognized that the full Einstein equivalence principle requires a curved spacetime metric theory of gravity in which particles follow geodesics, in accordance with the general theory\textsuperscript{2}.
Hence, incorporating the equivalence principle, our current proposition is that since Eq. (8) pertains to a reference frame described above with an accelerating rocket then when using the equivalence principle this will apply to an homogeneous gravitational field. That is, we write

\[ a = g \left( 1 - \frac{v^2}{c^2} \right)^{3/2}, \] (9)

where \( g \) represents the free fall of an object in a gravitational field. Under general relativity, for objects ‘stationary’ in gravity it describes a proper acceleration analogous to \( \alpha \). The point here is not to claim that this actually represents the true description in gravity in the sense of GR—clearly this is not a GR equation—the point is that the principle of equivalence used here tells us that we can expect some kind of dependence of the rate of free fall geodesics on initial velocities. See appendix A for further clarification on this point. Hence, using an approach originally proposed by Einstein in his 1907 formulation of the equivalence principle, our result obviously requires a more precise formulation in GR—clearly this is not a GR equation—the point is that the principle of equivalence used here tells us that we can expect some kind of dependence of the rate of free fall geodesics on initial velocities. See appendix A for further clarification on this point. Hence, using an approach originally proposed by Einstein in his 1907 formulation of the equivalence principle, that is from a Newtonian view point, before the full spacetime curvature formulation, we produce a result which predicts that a free-falling object (equivalent to PG1) in a static homogeneous gravitational field, is velocity dependent. This approximate result obviously requires a more precise formulation in GR terms. Hence in the section following, we attempt to formulate a more accurate description of this by deriving a result using the Schwarzschild solution of general relativity.

II. SCHWARZSCHILD SOLUTION

For a static, non-rotating, spherical mass the field equations of general relativity give the Schwarzschild solution

\[ c^2 d\tau^2 = \left( 1 - \frac{2\mu}{r} \right) c^2 dt^2 - \left( 1 - \frac{2\mu}{r} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \cos^2 \theta d\phi^2, \] (10)

where \( \mu = GM/c^2 \) and \( r \) is measured from the center and outside the mass. As geodesics follow paths of maximal proper time, we can divide through by \( d\tau^2 \) to define a Lagrangian

\[ \mathcal{L} = \left( 1 - \frac{2\mu}{r} \right) \dot{t}^2 - \frac{1}{c^2} \left( 1 - \frac{2\mu}{r} \right)^{-1} \dot{r}^2 = 1, \] (11)

where \( \ddot{t} = \frac{d}{d\tau} \) and \( \ddot{r} = \frac{d}{d\tau} \) and for purely radial motion we have assumed that the angular terms are zero. As we are assuming we are dealing with particles with mass we can parametrize their motion using the proper time \( \tau \).

Lagrange’s equations for \( t \), namely \( \frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{t}} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0 \), gives

\[ \frac{d}{d\tau} \left( \left( 1 - \frac{2\mu}{r} \right) c^2 \dot{t} \right) = \frac{d\mathcal{L}}{dt} = 0. \] (12)

Hence we have a constant of the motion

\[ \left( 1 - \frac{2\mu}{r} \right) \dot{t} = \frac{E_0}{mc^2}, \] (13)

where \( E_0 \) represents the conserved total energy for its motion. Substituting Eq. (13) into Eq. (11) we find

\[ \frac{dr}{d\tau} = \pm c \sqrt{\frac{E_0^2}{m^2c^4} - \left( 1 - \frac{2\mu}{r} \right)}. \] (14)

Differentiating with respect to proper time \( \tau \) gives

\[ a^\tau = \frac{d^2 r}{d\tau^2} = \frac{dr}{d\tau} \frac{d}{dr} \left( \frac{dr}{d\tau} \right) = -\frac{\mu c^2}{r^2}, \] (15)

as measured by a co-moving observer. This implies the magnitude of the four-acceleration, for an observer at fixed coordinates, is

\[ \sqrt{g_{\mu\nu} a^\mu a^\nu} = \sqrt{g_{rr}} \frac{GM}{r^2} = \frac{1}{\sqrt{1 - 2\mu/r}} \frac{GM}{r^2}, \] (16)

which is the acceleration felt by a ‘stationary’ observer. Now, using Eq. (13) we find

\[ \frac{dr}{dt} = \frac{d\tau}{dt} \frac{dr}{d\tau} = \pm c \left( 1 - \frac{2\mu}{r} \right) \sqrt{1 - \frac{m^2c^4}{E_0^2} \left( 1 - \frac{2\mu}{r} \right) - 2}. \] (17)

For particles entering the gravitational field from infinity with an initial velocity \( v \), we have \( E_0 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \), where \( v = \frac{dr}{dt} \). For the alternate case of bound particles we have \( E_0 < mc^2 \), also calculated from Eq. (17). Now, using the chain rule \( \frac{d^2 r}{d\tau^2} = \frac{d}{dr} \frac{d^2 r}{dt^2} \frac{dr}{d\tau} \), we find

\[ \frac{d^2 r}{dt^2} = -\frac{\mu c^2}{r^2} \left( 1 - \frac{2\mu}{r} \right) \left( 1 - \frac{3}{c^2} \left( \frac{dr}{dt} \right)^2 \left( 1 - \frac{2\mu}{r} \right)^{-2} \right). \] (18)

Finally, using the relation between \( E_0 \) and \( \frac{dr}{dt} \) in Eq. (17) we find

\[ \frac{d^2 r}{dt^2} = -\frac{\mu c^2}{r^2} \left( 1 - \frac{2\mu}{r} \right) \left( 1 - \frac{3}{c^2} \left( \frac{dr}{dt} \right)^2 \left( 1 - \frac{2\mu}{r} \right)^{-2} \right). \] (19)

This shows the velocity dependence \( \frac{d^2 r}{dt^2} \) based on the Schwarzschild coordinates, for an observer at infinity. In the weak field, low velocity limit, \( \frac{d^2 r}{dt^2} = -\frac{\mu c^2}{r^2} = -\frac{GM}{r^2} \) and represents the acceleration required to remain at rest at a fixed radius \( r \) in Schwarzschild coordinates and corresponds to the acceleration \( \alpha \) calculated earlier. This shows a constant acceleration as assumed for the rocket frame as measured by PG2, referred to earlier as proper acceleration. This thus corresponds with Eq. (9) when \( v = 0 \). For the case of the weak field only we have \( \frac{d^2 r}{dt^2} \to 0 \) and so

\[ \frac{d^2 r}{dt^2} = -\frac{GM}{r^2} \left( 1 - \frac{3v^2}{c^2} \right), \] (20)
a result first derived by Hilbert in 1917, for particles moving radially in the Schwarzschild metric. In Appendix C we also show how this same result is obtained for a choice of other common coordinate systems, such as isotropic coordinates, Brillouin coordinates or indeed Schwarzschild’s original metric, showing that it is not simply an artifact of the coordinates. Indeed, any variation of the coordinates subject to the condition that they approach flat space for \( r \to \infty \), will produce this velocity dependence, as shown in Appendix C. This shows the surprising result that gravity appears to become repulsive for high input velocities to the field. However we note that the radial velocity, in Eq. (17), does not change sign and so the apparent repulsive force only acts to slow down the particle rather than eject it from the field.

Now, for a shell observer located at a fixed \( r \) in Schwarzschild coordinates, we can write the line element as

\[
c^2 dt^2 = c^2 dT^2 - dR^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,
\]

where the conversion between the observer at a coordinate \( r \) and the observer at infinity is now \( dt^2 = (1 - \frac{2\mu}{r}) dt^2 \) and \( dR^2 = (1 - \frac{2\mu}{r})^{-1} dr^2 \). Therefore the radial velocity is

\[
d\tau = \left(1 - \frac{2\mu}{r}\right)^{-\frac{1}{2}} \frac{dr}{dt} = \frac{mc^2}{E_0} \frac{dr}{d\tau}.
\]

Therefore

\[
\frac{d^2 R}{dt^2} = \frac{dR}{dt} \frac{d}{dR} \left( \frac{dR}{dt} \right) = \frac{m^2 c^4}{E_0^2} \frac{d}{dr} \left( \frac{dr}{d\tau} \right) \frac{dr}{d\tau} = \frac{m^2 c^4}{E_0^2} \left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}} \frac{d^2 r}{dr^2} = \frac{\mu c^2 m^2 c^4}{r^2 E_0^2} \left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}} = -\frac{GM}{r^2} \left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}} \frac{dR}{c dt} \left(1 - \frac{2\mu}{r}\right)^{-\frac{1}{2}}.
\]

This equation shows a tradeoff between the space components of the metric that increases the rate of free fall and the velocity dependence that decreases the rate of free fall, for geodesics. Eq. (23) is relevant for a terrestrial experiment located in the gravity field of the Earth measuring radially moving particles.

Therefore we can see that the Schwarzschild solution gives velocity dependent geodesics for all observers at rest with respect to the gravitational field coordinates. This could be interpreted as an apparent weakening of the field strength in gravity, for radially moving objects.

Now, to a first approximation, we have a velocity dependence for acceleration in special relativity given in Eq. (8) of \( 1 - \frac{3v^2}{c^2} \) compared with the dependence found from the Schwarzschild solution, shown in Eq. (20) of \( 1 - \frac{2\mu}{r} \). We can immediately see that we have a factor of two discrepancy between these two results. One obvious difference between the two calculations is that in special relativity we assume a flat space. Indeed, if we set the spatial coefficient to one in the Schwarzschild metric, as shown in Appendix B and Eq. (B13), we find the velocity dependence becomes \( 1 - \frac{2\mu}{r} \), in the weak field, and so much closer to our result. This final discrepancy is effectively a \( \gamma \) factor difference. In fact, we note that \( \alpha = \frac{d}{dt} \left( \frac{dx}{d\tau} \right) \) and hence, as \( dt = \gamma d\tau \), we have identified the missing \( \gamma \). That is, \( \alpha = \frac{d}{dt} \left( \frac{dx}{d\tau} \right) = \frac{1}{\gamma} \frac{d^2 x}{d\tau^2} \) so that \( a = \frac{1}{\gamma} \frac{d^2 x}{d\tau^2} \). Hence, we need to select Eq. (5) and Eq. (20) in order to form a correct comparison. In the weak field, because \( -\frac{GM}{r^2} = \frac{d^2 x}{d\tau^2} \), both equations effectively relate acceleration with respect to proper time with coordinate acceleration. Hence from Eq. (5) we have \( \frac{d^2 x}{d\tau^2} = \gamma^{-4} \frac{d^2 x}{d\tau^2} \approx \left(1 - \frac{v^2}{c^2}\right)^2 \frac{d^2 x}{d\tau^2} \approx \left(1 - \frac{2\mu}{r}\right) \frac{d^2 x}{d\tau^2} \)

where the Schwarzschild solution with spatial curvature set to flat space gives \( 1 - \frac{2\mu}{r} \), shown in Appendix B and Eq. (B13). That is, the two results now agree to first order, once the spatial curvature is removed from the Schwarzschild metric.

This then perfectly reconciles the results and hence our approach potentially provides a natural pathway from accelerations in special relativity to the behavior of radial geodesics in gravitational fields.

### III. EXPERIMENTAL TESTS

In order to experimentally test this predicted behavior of radial geodesics in an earth bound frame, we can possibly use accurate clocks to measure such deviations from expected accelerations, comparing an acceleration \( g = -\frac{GM}{r^2} \) with the acceleration predicted in Eq. (23). Note that, due to the rocket’s mild acceleration rate, then inside the rocket frame itself, there will be extremely minor time dilation effects. This allows the stationary frame in gravity, to be the frame of reference to measure fairly accurately the rates of acceleration of PG1 and PG2. It is therefore proposed that this should be the reference frame for an experimental test of the principle. The maximum effect predicted in Eq. (8) will obviously be for particles falling in the Earth’s gravitational field at velocities approaching the speed of light.

### IV. DISCUSSION

We show in this paper that by considering accelerating objects within the context of special relativity and using the equivalence principle, their behavior can be predicted in weak approximately uniform homogeneous gravitational fields. Specifically, we have shown that the behavior of timelike geodesics in gravity, are a function of the radial particle velocity as shown in Eq. (5). We note the similarity to a little-known result first derived by Hilbert in 1917, shown in Eq. (20). From the viewpoint
of an observer fixed with respect to fixed Schwarzschild coordinates this might also be interpreted as a weakening of the field.

We note that although the velocity dependence effect is real, it is as real in the same sense as the bending of light is real. The bending of light as seen by an observer which is a geodesic falling with the light, will not see the same bending of light as an observer outside the field, or accelerating with fixed Schwarzschild coordinates in the field. Likewise the acceleration dependence can also be transformed away by a boost from PG1 to PG2. Having said this the result can be shown in Eq. (20) and Eq. (10). We have shown there is no violation of the principle of equivalence, since velocity dependence holds under both flat space accelerations and general relativity. Indeed, we redo our calculations for the Schwarzschild solution but set the spatial curvature to zero, then we find that we obtain a factor of two as opposed to three, which is much closer to our result in Eq. (20) and so appears to explain this discrepancy. Refer to Appendix B, which shows the precise result with the coefficient of two now on the velocity dependence, in Eq. (B13). Hence the velocity dependent effect still occurs when we eliminate the radial space curvature and consider time dilation alone, although it is reduced. This treats the field rather unrealistically as homogeneous, yet surprisingly isolates the cause of the rate of decrease of geodesic motion primarily to that of velocity dependence on the seemingly innocuous Eq. (9), derived based purely on simple arguments of the equivalence principle. We consider this an important insight of the paper. Having said this we must qualify the equivalence principle. We consider this an important point here is not to claim that this actually represents the true description of gravity in the sense of GR—clearly this is not a GR equation—the point is that the principle of equivalence used here tells us that in GR we can expect some kind of dependence of the rate of free fall geodesics on velocities. This is the key point we wish to make at this stage of the paper.

Appendix A: Simplified Newtonian version of equivalence principle

We can revert back to a Newtonian perspective as a pedagogical aid to make a qualitative point. In Eq. (8), \( a \) is the acceleration as viewed by the inertial PG1 frame. However \( a \) is the acceleration as seen by PG2. Hence PG2 views this as proper acceleration, which is thus acceleration relative to a free-fall, or inertial, observer who is momentarily at rest relative to the object being measured. Hence it is also equivalent to Newton’s acceleration of \( \alpha = F/m \). However if the rocket was windowless then inside this rocket by the principle of equivalence we have \( \alpha = \frac{GM}{r^2} \). So in this limited sense we have that \( a = \frac{\alpha}{c^2} = \frac{GM}{r^2} \) and hence

\[
a = \frac{\alpha}{c^2} = \frac{GM}{r^2} \left(1 - \frac{v^2}{c^2}\right)^{3/2}.
\]

As stated the point here is not to claim that this actually represents the true description of gravity in the sense of GR but set the spatial curvature to zero, then we find that for the bending of starlight account is removed from the Schwarzschild metric.

Appendix B: Geodesic equation

We have the geodesic equation

\[
a^\alpha = \frac{dv^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu}v^\mu v^\nu.
\]

From the metric we have \( g_{rr} = -(1 - \frac{2\mu}{r})^{-1} \) and \( g_{tt} = (1 - \frac{2\mu}{r}) \). If we select purely radial motion, then we have the non-zero Christoffel symbols

\[
\Gamma^t_{rr} = -\Gamma^r_{rr} = \frac{1}{2}g^{rr}\partial_r g_{rr} = \frac{\mu}{r^2} \left(1 - \frac{2\mu}{r}\right)^{-1},
\]

\[
\Gamma^r_{tt} = -\frac{1}{2}g^{rr}\partial_t g_{tt} = \frac{\mu}{r^2} \left(1 - \frac{2\mu}{r}\right).
\]

The radial coordinate acceleration is

\[
\frac{d^2r}{d\tau^2} = -\Gamma^r_{rr} \left(\frac{dr}{d\tau}\right)^2 - \Gamma^r_{tt} \left(\frac{dt}{d\tau}\right)^2 = \frac{\mu}{r^2} \left(1 - \frac{2\mu}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - \frac{\mu^2}{r^2} \left(1 - \frac{2\mu}{r}\right) \left(\frac{dt}{d\tau}\right)^2.
\]
The last term in brackets is simply the metric in Eq. (10) and so equal to one, and so
\[ a^r = \frac{d^2 r}{d \tau^2} = -\frac{\mu c^2}{r^2} = -\frac{GM}{r^2}. \]  
(B4)

Hence, for the observer at fixed spatial coordinates we have the four-velocity \( u^\beta = \left( c \left( 1 - \frac{2\mu}{r} \right)^{-1/2}, 0 \right) \) and the four-acceleration \( a^\beta = \left( 0, -\frac{GM}{r^2} \right) \), where we have suppressed the two angular coordinates. This observer then sees the four-velocity of the radial infalling particle \( u^\beta = \gamma \left( c \left( 1 - \frac{2\mu}{r} \right)^{-1/2}, v \left( 1 - \frac{2\mu}{r} \right)^{1/2} \right) \).

Now, we can write Eq. (11) as
\[
\frac{d}{d\tau} \left( \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 - \frac{GM}{r} \right) = 0 \tag{B5}
\]
and so
\[
\frac{1}{c^2} \left( \frac{dr}{d\tau} \right)^2 - 2 \frac{GM}{r^2} = \text{constant} = \frac{E_0^2}{m^2 c^4} - 1, \tag{B6}
\]
where \( E_0 \) can be shown to be the conserved total energy of the particle. Hence
\[
\frac{dr}{d\tau} = \pm c \sqrt{\frac{2\mu}{r} + \frac{E_0^2}{m^2 c^4} - 1}, \tag{B7}
\]
as measured by a co-moving observer.

Now \( \frac{dr}{dt} = \frac{dr}{d\tau} \frac{d\tau}{dt} \) and so using Eq. (11) we determine
\[
\frac{dt}{d\tau} = \frac{E_0}{mc^2 \left( 1 - \frac{2\mu}{r} \right)} \tag{B8}
\]
and so we find
\[
\frac{dr}{dt} = \pm c \left( 1 - \frac{2\mu}{r} \right) \sqrt{1 - \frac{m^2 c^4}{E_0^2} \left( 1 - \frac{2\mu}{r} \right)}. \tag{B9}
\]

Using the chain rule, \( \frac{d^2 r}{d\tau^2} = \frac{d(dr/d\tau)}{dt} \frac{d\tau}{dt} \), we find
\[
\frac{d^2 r}{d\tau^2} = -\frac{\mu c^2}{r^2} \left( 1 - \frac{2\mu}{r} \right) \left( 1 - 3 \left( \frac{dr}{cdt} \right)^2 \left( 1 - \frac{2\mu}{r} \right)^{-2} \right). \tag{B10}
\]
Then, using the relation between \( E_0 \) and \( \frac{dr}{dT} \) we find
\[
\frac{d^2 r}{dT^2} = \frac{\mu c^2}{r^2} \left( 1 - \frac{2\mu}{r} \right) \left( 1 - 3 \left( \frac{dr}{cdt} \right)^2 \left( 1 - \frac{2\mu}{r} \right)^{-2} \right), \tag{B11}
\]
in agreement with Eq. (11).

If we enforce a spatially flat space, with \( g_{rr} = -1 \) and repeat the derivation above, then we find \( \frac{dr}{d\tau} \) is unaffected but we now have
\[
\frac{dr}{d\tau} = \pm c \left( 1 - \frac{2\mu}{r} \right)^{-1/2} \sqrt{\frac{E_0^2}{m^2 c^4} - \left( 1 - \frac{2\mu}{r} \right)}, \tag{B12}
\]
\[
\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} \frac{E_0^2}{m^2 c^4} \left( 1 - \frac{2\mu}{r} \right)^{-2},
\]
and
\[
\frac{d^2 r}{dT^2} = -\frac{\mu c^2}{r^2} \left( 1 - 2 \left( \frac{dr}{cdt} \right)^2 \left( 1 - \frac{2\mu}{r} \right)^{-1} \right). \tag{B13}
\]

This shows that in the weak field limit, the coefficient on the velocity dependence reduces from 3 to 2 in flat space.

Now, for a shell observer located at a fixed \( r \) in Schwarzschild coordinates, where we have the line element as in Eq. (21), but assuming a spatial flat Schwarzschild metric, we have the conversion between the observer at a coordinate \( r \) and the observer at infinity is now \( dT^2 = \left( 1 - \frac{2\mu}{r} \right) dt^2 \) and \( dR^2 = d\tau^2 \). Therefore the radial velocity is now given by
\[
\left( 1 - \frac{2\mu}{r} \right)^{-1/2} \frac{dR}{dT} = \left( 1 - \frac{2\mu}{r} \right)^{-1} \frac{dr}{dt} = \frac{mc^2}{E_0} \frac{dr}{d\tau}. \tag{B14}
\]
Therefore
\[
\frac{d^2 R}{dT^2} = \frac{dR}{dT} \frac{d}{dt} \frac{dR}{dT} = \frac{dR}{dT} \
\frac{d}{dT} \frac{dR}{dT} \tag{B15}
\]
using Eq. (13). This essentially coincides with the shell observer found earlier in Eq. (22), but with a modified radial scaling factor.

\section*{Appendix C: Common forms of the Schwarzschild metric}

A general form of the Schwarzschild metric can be written as
\[
c^2 d\tau^2 = \left( 1 - \frac{2\mu}{D} \right) c^2 dt^2 - (D')^2 \left( 1 - \frac{2\mu}{D} \right)^{-1} dr^2 \tag{C1}
\]
\[-D^2 d\theta^2 - D^2 \cos^2 \theta d\phi^2.
\]
The four common variants, which are time independent, are: Schwarzschild’s original metric with \( D = (r^2 + 8\mu^3)^{1/3} \), isotropic coordinates with \( D = r(1 + \frac{\mu}{2r})^2 \), Brillouin coordinates with \( D = r + 2\mu \) and the more common form of the metric with \( D = r \). We notice that all these choices enforce a flat space condition, such that for large \( r, D \rightarrow r \). This also implies that \( D' \rightarrow 1 \) and \( D'' \rightarrow 0 \) as \( r \rightarrow \infty \).
Now, given new spatial coordinates \( \rho = D(r) \), giving \( d\rho = D'dr \), where \( D' = \frac{dD}{dr} = \frac{d\rho}{dr} \), then

\[
\frac{d^2\rho}{dt^2} = \frac{d}{dt} \left( \frac{dr}{dt} \frac{d\rho}{dr} \right) = \frac{d\rho}{dr} \left( \frac{d}{dt} \frac{dr}{dt} \right)^2 + \left( \frac{dr}{dt} \right)^2 \frac{d}{dr} \left( \frac{d\rho}{dr} \right). \tag{C2}
\]

Hence, if \( \frac{d\rho}{dr} = D' = \text{constant} \) the second term is zero and so we recover the same velocity dependence in the weak field, as found in Eq. (20). This implies we limit new coordinate systems, in the weak field regime far from the source, to \( \rho = a + br \), where \( a, b \) are constants, giving \( \frac{d\rho}{dr} = b = \text{constant} \), as required.

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