Abstract

We have obtained the correct expression for the centrifugal force acting on a particle at the equatorial circumference of a rotating body in the locally non-rotating frame of the Kerr geometry. Using this expression for the equilibrium of an element on the surface of a slowly rotating Maclaurin spheroid, we obtain the expression for the ellipticity (as discussed earlier by Abramowicz and Miller) and determine the radius at which the ellipticity is maximum.

1 Introduction

The reversal of centrifugal force at the photon circular orbit of the Schwarzschild geometry as shown by Abramowicz and Prasanna [1] (AP) has had several interesting consequences and could in principle account for the maximum of the ellipticity of a collapsing Maclaurin spheroid preserving mass and angular momentum as demonstrated by Abramowicz and Miller [2] (AM).

In order to discuss the case of a rotating body, it is indeed necessary to use the Kerr geometry and consider the 3+1 splitting and identify the forces. Prasanna and Chakrabarti [3] had considered the study of angular momentum coupling using the optical reference geometry [4] in Kerr spacetime and had obtained the separation of the total four-force acting on a test particle in terms of gravitational, centrifugal and Coriolis forces. However, while considering the conformal splitting, they had used the Boyer-Lindquist form of the Kerr metric which has the restriction of ergosurface being the static limit surface. Hence, in that splitting, the discussion of the behaviour of forces can be made only beyond the ergosurface ($r = 2$ in the equatorial plane) and not from the event horizon onwards as in the Schwarzschild case. This indeed can be rectified by using the locally non-rotating frame (LNRF), which, in fact, is the most suitable one for discussing the dynamics of rotating configurations in general relativity.

In the following we start with the Kerr geometry in LNRF and then, using the conformal splitting, obtain the expression for the centrifugal force. This is then used in the case of a slowly rotating Maclaurin spheroid to obtain the expression for its ellipticity. We follow the same notation as in the earlier papers referred to.


2 Formalism

The Kerr metric as expressed in LNRF is given by

\[ ds^2 = -\frac{\Sigma \Delta}{B} dt^2 + \Sigma dr^2 + \Sigma d\theta^2 + \frac{B}{\Sigma} \sin^2 \theta d\hat{\phi}^2, \] (2.1)

where

\[ \Delta = r^2 - 2mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad B = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \] (2.2)

Considering the 3+1 conformal splitting as introduced by Abramowicz, Carter and Lasota [4],

\[ ds^2 = \frac{1}{\Phi} \left[ -dt^2 + \tilde{g}^{-1} \right], \]

\[ \frac{1}{\tilde{g}_{rr}} = \frac{B}{\Delta}, \quad \frac{1}{\tilde{g}_{\theta\theta}} = \frac{B}{\Delta}, \quad \frac{1}{\tilde{g}_{\phi\phi}} = \frac{B^2}{(\Delta \Sigma^2)}. \] (2.3)

we have

\[ \Phi = \frac{\Sigma \Delta}{B}, \]

\[ \tilde{g}_{rr} = \frac{B}{\Delta^2}, \quad \tilde{g}_{\theta\theta} = \frac{B}{\Delta}, \quad \tilde{g}_{\phi\phi} = \frac{B^2}{(\Delta \Sigma^2)}. \] (2.4)

Restricting the functions to the \( \theta = \pi/2 \) hyperplane we have

\[ \Phi = \frac{r \left( r^2 - 2mr + a^2 \right)}{\left( r^2 + a^2 r + 2ma^2 \right)}, \]

\[ \frac{1}{\tilde{g}_{rr}} = \frac{B}{\Delta} = \frac{r^4 + a^2 r^2 + 2ma^2 r}{(r^2 - 2mr + a^2)^2}, \]

\[ \frac{1}{\tilde{g}_{\phi\phi}} = \frac{\left( r^3 + a^2 r + 2ma^2 \right)^2}{\left[ r^2 \left( r^2 - 2mr + a^2 \right) \right]^2}. \] (2.5)

Using the same notation as in AP, it can be easily verified that the spatial 3-momentum \( p^i \) defined through \( p^i = \Phi P^i \), where \( P^i \) is the 3-momentum in the four-space, may be used to evaluate the centrifugal acceleration. For a particle in a circular orbit the invariant speed \( \tilde{v} \) in the projected manifold is given by

\[ \tilde{v}^2 = \Phi g_{ij} u^i u^j, \] (2.6)

where \( u^i \) is the four velocity of the particle. Using the usual definition of the four velocity in terms of Killing vectors, \( u^\alpha = A \left( \eta^\alpha + \Omega \xi^\alpha \right) \), where \( A \) is the redshift factor, one gets \( u^i = A \Omega \xi^i \) and then

\[ \tilde{v}^2 = \frac{\Omega^2 r^2}{1 - \Omega^2 \tau^2} = \frac{\Phi \Omega^2 \tau^2}{1 - \Omega^2 \tau^2} = \frac{\Omega^2 \tilde{g}_{\phi\phi} \tilde{g}_{tt}}{1 - \Omega^2 \tilde{g}_{\phi\phi}} \] (2.7)

The geodesic curvature radius \( R \) defined through [11]

\[ R = \tilde{r} / \left| \tilde{g}^{ij} \left( \tilde{\nabla}_i \tilde{r} \right) \left( \tilde{\nabla}_j \tilde{r} \right) \right|^{1/2} \] (2.8)

projected onto the instantaneously corotating frame with the spatial triad

\[ e^r_{(r)} = \left( \tilde{g}^{rr} \right)^{1/2}, \quad e^\theta_{(\theta)} = \left( \tilde{g}^{\theta\theta} \right)^{1/2}, \quad e^{\hat{\phi}}_{(\hat{\phi})} = \left( \tilde{g}^{\phi\phi} \right)^{1/2} \] (2.9)
is given by

\[ R = \tilde{r}/(\partial_r \tilde{r}) = 2\tilde{g}_{\tilde{\phi}\tilde{\phi}}/\left(\partial_r \tilde{g}_{\tilde{\phi}\tilde{\phi}}\right) = \frac{r\Delta (r^3 + a^2r + 2ma^2)}{(r^5 - 3mr^4 + a^2r^3 - 3ma^2r^2 + 6m^2a^2r - 2ma^4)}. \] (2.10)

If \( L \) denotes the angular momentum as measured by the stationary observer (instantaneously at rest), one has

\[ L = \xi^a u_a = A\Omega(\xi \xi) = A\tilde{\Omega} \tilde{g}_{\tilde{\phi}\tilde{\phi}} \] (2.11)

and thus from equation (2.6) and the definition of \( u^i \),

\[ L^2 = \tilde{v}^2 \tilde{g}_{\tilde{\phi}\tilde{\phi}}. \] (2.12)

Hence if we now consider the definition of centrifugal force as given in AP, \( C_f = m_0\tilde{v}^2/R \), we get

\[ C_f = \frac{m_0L^2 \partial_r \tilde{g}_{\tilde{\phi}\tilde{\phi}}}{2\tilde{g}_{\tilde{\phi}\tilde{\phi}}} = \frac{m_0L^2 \partial_r \tilde{g}_{\tilde{\phi}\tilde{\phi}}}{2(\tilde{g}_{\tilde{\phi}\tilde{\phi}})^2}, \] (2.13)

\[ C_f = \frac{L^2r (r^5 - 3mr^4 + a^2r^3 - 3ma^2r^2 + 6m^2a^2r - 2ma^4)}{(r^3 + a^2r + 2ma^2)^3}. \] (2.14)

It is interesting to note that the final expression for the centrifugal force (eq. 2.13) just turns out to be exactly

\[ \left(p^\tilde{\phi}\right)^2 \partial_r \tilde{g}_{\tilde{\phi}\tilde{\phi}}/2, \] (2.15)

as was obtained for the static spacetime. The parameters \( L \) and \( \tilde{\Omega} \), the angular velocity measured in the global inertial frame, are related through the expression

\[ L = \frac{\tilde{\Omega} (r^2 + a^2 + 2ma^2r)^{3/2}}{\left[(r^2 - 2mr + a^2) - \tilde{\Omega}^2 (r^2 + a^2 + 2ma^2/r)^2\right]^{1/2}}. \] (2.16)

It is indeed clear from the expression (2.14) that for all real values of \( L \), the centrifugal force reverses sign at the zeros of the function

\[ f(r) = r^5 - 3mr^4 + a^2r^3 - 3ma^2r^2 + 6m^2a^2r - 2ma^4. \] (2.17)

Using Sturm’s theorem one can ascertain that there are three real roots of the equation \( f(r) = 0 \) for \( 0 \leq r < \infty \). Of these three real roots, two lie between \( r = 0 \) and the event horizon and are thus of no consequence to any outside observer. The third root lies between \( r = 2m \) and \( r = 3m \), i.e., the ergosurface and the surface where centrifugal reversal occurs for non-rotating objects (\( a = 0 \)). Table (1) enlists the values of \( R (= r/m) \) for which \( f(R) = 0 \) for different values of \( \alpha (= a/m) \). As \( \alpha \) increases from 0 to 1 the surface of reversal moves inwards from \( R = 3 \). Fig. (1) shows the behaviour of the centrifugal force for non-zero values of \( \alpha \), depicting the inward movement of the zero and the maximum as well as the flattening of the arm beyond the maximum.
Table 1: Location of the zero of $C_f$ for different $\alpha$.

| $\alpha$ | R   | $\alpha$ | R   |
|----------|-----|----------|-----|
| 0.0      | 3.0000 | 0.6      | 2.9202 |
| 0.1      | 2.9978 | 0.7      | 2.8916 |
| 0.2      | 2.9911 | 0.8      | 2.8590 |
| 0.3      | 2.9800 | 0.9      | 2.8226 |
| 0.4      | 2.9645 | 1.0      | 2.7830 |
| 0.5      | 2.9445 |

3 Slowly rotating configurations

For slowly rotating Maclaurin spheroids the balance of forces on a surface element in the equatorial plane is expressed through the equation \[2, 5\]

\[(\text{Centrifugal force}) = (\text{Gravitational force}) f(e),\] \(\text{(3.1)}\)

with $e$ denoting the eccentricity of the spheroid, expressed through the standard relation, and $f(e)$ is given by

\[f(e) = \frac{9(1 - e^2)}{2e^2} \left[ \sin^{-1} e \sqrt{1 - e^2} \left(1 - \frac{2e^2}{3}\right) - e \right].\] \(\text{(3.2)}\)

The ellipticity $\epsilon$ is expressed in terms of $e$ by the relation (AM)

\[\epsilon = \left[1 - (1 - e^2)^{1/2}\right] / (1 - e^2)^{1/6}.\] \(\text{(3.3)}\)

Following the same approximation as in \[2\], we take for slow rotation

\[\epsilon = e^2/2, \quad f(\epsilon) = 4\epsilon/5.\] \(\text{(3.4)}\)

Further the angular momentum $J$ turns out to be

\[J = \frac{2}{5}(1 - e^2)^{-1/3}L \approx 2L/5.\] \(\text{(3.5)}\)

Using equations \(3.3\) and \(2.14\) in equation \(3.1\) we get, without any approximation in terms of $\bar{\Omega}$ or $a$,

\[\epsilon = \frac{125}{16} J^2 R^3 (R^5 - 3R^4 + \alpha^2 R^3 - 3\alpha^2 R^2 + 6\alpha^2 R - 2\alpha^4) / (R^3 + \alpha^2 R + 2\alpha^2)^3,\] \(\text{(3.6)}\)

with $R = r/m$ and $\alpha = a/m$. Thus $\bar{\epsilon} = \epsilon / J^2$ expressed in terms of $R$ is

\[\bar{\epsilon}(R) = \frac{125}{16} R^3 (R^5 - 3R^4 + \alpha^2 R^3 - 3\alpha^2 R^2 + 6\alpha^2 R - 2\alpha^4) / (R^3 + \alpha^2 R + 2\alpha^2)^3.\] \(\text{(3.7)}\)
Table 2: Location of the maximum of $\bar{\epsilon}$ for different $\alpha$ and the values of $\bar{\epsilon}$ and $C_f$ there.

| $\alpha$ | $R$     | $\epsilon_{\text{max}}$ | $C_f$  |
|---------|---------|--------------------------|--------|
| 0.0     | 6.0000  | 0.6510                   | 0.0145 |
| 0.1     | 6.0044  | 0.6506                   | 0.0144 |
| 0.2     | 6.0178  | 0.6491                   | 0.0143 |
| 0.3     | 6.0401  | 0.6467                   | 0.0142 |
| 0.4     | 6.0713  | 0.6435                   | 0.0140 |
| 0.5     | 6.1116  | 0.6393                   | 0.0137 |

Fig. (2) shows the behaviour of $\bar{\epsilon}$ as a function of $R$ and it may be seen that the maximum in $\bar{\epsilon}$, which appears at $R = 6$ for $\alpha = 0$, now moves slightly outwards for $\alpha \neq 0$. Thus one finds that taking into account the potential of the Kerr metric for the surface element of the rotating fluid configuration, the maximum of ellipticity occurs earlier during the collapse (i.e., for larger radii) than what was found using the Schwarzschild potential. It may also be seen directly that it is not necessary to make any approximation in the expression with respect to $\hat{\Omega}$ or $a$ to find the ellipticity in units of $J^2$. This is, in fact, true for the Schwarzschild case also. (In the paper of AM they had neglected the terms in $\Omega^2$ for slow rotation, which in fact is not necessary).

Table (2) lists the location of the occurrence of $\bar{\epsilon}_{\text{max}}$ for different $\alpha$ and also the values of $\bar{\epsilon}_{\text{max}}$ and the centrifugal force at that location. As $\bar{\epsilon}$ expresses the ratio of $C_f$ to the gravitational force and since $C_f$ flattens after the maximum for increasing $\alpha$ (see Fig. 1), the location of $\bar{\epsilon}_{\text{max}}$ shifts outward for increasing $\alpha$.

4 Concluding remarks

The study of equilibrium configurations for rotating fluid masses, a topic which has been analysed thoroughly in the Newtonian framework, is still to find a completely satisfactory analysis in general relativity, due to the lack of exact solutions. However, there have been a number of investigations to study the effects of general relativity on the dynamics of rotating fluids bearing out the necessity to improve the Newtonian results by including GR effects. The Kerr geometry, which is an exact representation of the field outside the rotating body, should be the correct framework to be used for studying the effects of rotation, particularly through the Locally Non-Rotating Frames introduced by Bardeen [3]. In most of the studies carried out earlier, when one used the Boyer-Lindquist coordinates one could not separate out the inertial frame dragging effects, leading to ambiguous interpretations. Thus in the analysis made above using LNRF we have first obtained the correct expression for the centrifugal force acting on the surface element (in the equatorial plane) of a rotating mass and then studied the behaviour of the ellipticity as was discussed by Chandrasekhar, Miller and Abramowicz [2, 4, 8].

The main result that the inclusion of the Kerr potential brings is that the ellipticity maximum occurs at a radius slightly larger than in the case of the Schwarzschild potential.
This can certainly have some consequence in the dynamics of rotating compact stars. In fact, it is known that most of the calculations of neutron star models assume that there is a centrifugal barrier and thus confine the rotation speed to certain limits, whereas if one considers the effect of general relativity through centrifugal force reversal there can be an effect both on the shape and size of the star different from what is assumed. As pointed out by AM, it is still necessary to include the effect of pressure gradient forces in a compatible way along with the correct form of centrifugal force as given in equation (2.14) to get a complete picture of the shape and size of a rotating star.

References

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