The Dirac-Hestenes Equation and its Relation with the Relativistic de Broglie-Bohm Theory

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Abstract

In this paper we provide using the Clifford and spin-Clifford formalism and some few results of the extensor calculus a derivation of the conservation laws that follow directly from the Dirac-Hestenes equation (DHE) describing a Dirac-Hestenes spinor field (DHSF) in interaction with an external electromagnetic field without using the Lagrangian formalism. In particular, we show that the energy-momentum and total angular momentum extensors of a DHSF is not conserved in spacetime regions permitting the existence of a null electromagnetic field \(F\) but a non null electromagnetic potential \(A\). These results have been used together with some others recently obtained (e.g., that the classical relativistic Hamilton-Jacobi equation is equivalent to a DHE satisfied by a particular class of DHSF) to obtain the correct relativistic quantum potential when the Dirac theory is interpreted as a de Broglie-Bohm theory. Some results appearing in the literature on this issue are criticized and the origin of some misconceptions is detailed with a rigorous mathematical analysis.

1 Introduction

Using the Clifford and spin Clifford bundle formalisms and some few results of the theory of extensor calculus we derive the conservation laws for the probability current, energy-momentum and total angular momentum \textit{extensor fields} resulting from the structure of Dirac-Hestenes equation (DHE) satisfied by a Dirac-Hestenes spinor field (DHSF) in interaction with an electromagnetic field
without using the Lagrangian formalism. We show in a quite simple way that the energy-momentum extensor field of the DHSF is not symmetric and that its antisymmetric part is the source of the spin extensor field of the free DHSF. Moreover we show that the conservation laws for the energy-momentum and total angular momentum extensor for the coupled system of the DHSF and the electromagnetic field is very different from what is expected from a classical point of view. In particular these laws show in an elegant way that in space-time regions with topology permitting the existence of an electromagnetic field potential $A \neq 0$ but with $F = 0$ the energy-momentum and total angular momentum of the sole DHSF are not conserved. This is clearly according to our view the origin of the Bohm-Aharonov effect.

In Section 3 we briefly recall results obtained in [26] where it was shown that (a): that the classical relativistic Hamilton-Jacobi equation (HJE) for a charged electrical particle in interaction with the electromagnetic field is equivalent to a DHE satisfied by a special class of DHSF that we call classical DHSF which are characterized by having the Takabayashi angle equal to 0 or $\pi$ and moreover that the DHE satisfied by a classical DHSF is equivalent to the HJE; (b) The identification of the correct relativistic quantum potential resulting from the DHE in a de Broglie-Bohm approach leading to a HJE like equation for the motion of (spinning) charged particle.

Equipped with these results and the ones of Section 2 we analyze in Section 3 results of [13, 14] were authors though that they have disclosed the relativistic quantum potential for the Dirac particle. We prove that the results of those papers are equivocated by explicitly showing with detailed calculations were authors get mislead.

In Section 4 we present our conclusions and in the Appendix we recall the notation and some results of the Clifford and spin-Clifford bundles formalism used in the paper.

2 Dirac-Hestenes Equation (DHE) and Conserved Currents

With the notations introduced in the Appendix the DHE for $\phi_L \in \text{sec} \mathcal{C}^{01}(M, \eta)$ and $\phi_R \in \text{sec} \mathcal{C}^{00}(M, \eta)$ in the presence of an electromagnetic potential $A \in \wedge^1 T^* M \hookrightarrow \text{sec} \mathcal{C}^{01}(M, \eta)$ are

$$\partial \phi_L \gamma_{21} - eA\phi_L - m\phi_L \gamma_0 = 0, \quad (1)$$

$$\gamma_{12} \phi_R \vec{\partial} - e\phi_R A - m\gamma_0 \phi_R = 0. \quad (2)$$

1 We recall that the concepts of multiform functions and extensor fields permit us to give a nice Lagrangian (and Hamiltonian) formalism for all relativistic fields and in particular to derive the conserved energy-momentum, spin and angular momentum extensor fields for interacting relativistic fields. Preliminary details of our formalism has been first presented in [21, 20, 22] and further elaborated in [25].

2 Related to the Tetrode tensor of the Dirac spinor field.
Multiplying Eq. (1) on the right by $\gamma_0 \phi_R$ we get

$$\pounds \phi_L \gamma_{012} \phi_R - eA \phi_L \gamma_0 \phi_R - m \phi_L \phi_R = 0$$  \hspace{1cm} (3)

Now, taking notice that for any odd $O \in \sec (\Lambda^1 T^* M + \Lambda^3 T^* M) \mapsto \sec \mathcal{C}(M, \eta)$ it is

$$\langle O \rangle_1 = \frac{1}{2} (O + \tilde{O})$$  \hspace{1cm} (4)

and that

$$\partial_\mu (\phi_L \gamma_{012} \phi_R) = \partial_\mu \phi_L \gamma_{012} \phi_R + \phi_L \gamma_{012} \partial_\mu \phi_R,$$  \hspace{1cm} (5)

we have

$$\langle \partial_\mu \phi_L \gamma_{012} \phi_R \rangle_1 = \frac{1}{2} (\partial_\mu \phi_L \gamma_{012} \phi_R - \phi_L \gamma_{012} \partial_\mu \phi_R).$$  \hspace{1cm} (6)

Summing Eq. (5) (multiplied by 1/2) with Eq. (6) and multiplying on the left by $\gamma^\mu$ we get

$$\frac{1}{2} \partial (\phi_L \gamma_{012} \phi_R) + \gamma^\mu \langle \partial_\mu \phi_L \gamma_{012} \phi_R \rangle_1 = \partial \phi_L \gamma_{012} \phi_R.$$  \hspace{1cm} (7)

Next, using Eq. (1) on the right side of Eq. (4) we get

$$\frac{1}{2} \partial (\gamma_5 s) + \gamma^\mu \langle \partial_\mu \phi_L \gamma_{012} \phi_R \rangle_1 = eA J + m \phi_L \phi_R$$  \hspace{1cm} (8)

where

$$s = \phi_L \gamma^3 \phi_R \text{ and } J = \phi_L \gamma^0 \phi_R$$  \hspace{1cm} (9)

are bilinear invariants of Dirac theory called respectively the spin 1-form field and the probability current 1-form field.

### 2.1 The Conserved Probability Current

From Eq. (1) we also have

$$\pounds \phi_L \gamma_0 \phi_R = eA \phi_L \gamma_{012} \phi_R + m \phi_L \gamma_{21}$$  \hspace{1cm} (10)

Now, taking also into account that for $O \in \sec (\Lambda^1 T^* M + \Lambda^3 T^* M) \mapsto \sec \mathcal{C}(M, \eta)$ it is

$$\langle O \rangle_3 = \frac{1}{2} (O - \tilde{O})$$  \hspace{1cm} (11)

we also have the identities

$$\partial_\mu (\phi_L \gamma_0 \phi_R) = \partial_\mu \phi_L \gamma_0 \phi_R + \phi_L \gamma_0 \partial_\mu \phi_R,$$  \hspace{1cm} (12)

$$\langle \partial_\mu \phi_L \gamma_0 \phi_R \rangle_3 = \frac{1}{2} (\partial_\mu \phi_L \gamma_0 \phi_R - \phi_L \gamma_0 \partial_\mu \phi_R)$$  \hspace{1cm} (13)

and summing Eqs. (12) and (13) and multiplying on the left by $\gamma^\mu$ we get

$$\frac{1}{2} \partial (\phi_L \gamma_0 \phi_R) + \gamma^\mu \langle \partial_\mu \phi_L \gamma_0 \phi_R \rangle_3 = \partial \phi_L \gamma_0 \phi_R.$$  \hspace{1cm} (14)

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3Details in [25].
Using Eq. (10) in Eq. (14) we get

$$\nabla J + \gamma^\mu (\partial_\mu \phi_L \gamma_0 \phi_R)_3 = -eA_5s. \quad (15)$$

Now, from Eq. (15) we get

$$\langle \nabla J + \gamma^\mu (\partial_\mu \phi_L \gamma_0 \phi_R)_3 \rangle_0 = \langle \nabla J \rangle_0 = \partial_\gamma J,$$

$$\langle -eA_5s \rangle_0 = -e\langle (A\gamma_3s)_2 + (A\gamma_5s)_2 \rangle_0 = 0 \quad (16)$$

and finally we have the conservation law for the probability current, i.e.,

$$\partial_\gamma J = -\delta J = 0. \quad (17)$$

3 The Energy-Momentum Extensor of the Dirac-Hestenes Field

Recalling that for $v \in \sec \bigwedge^1 T^* M \leftrightarrow \sec C\ell(M, \eta)$ and $C \in \sec C\ell(M, \eta)$ the following identity holds \[25\]

$$\nabla (vC) = \nabla (vC) - v(\nabla C) + 2(v:\nabla)C \quad (18)$$

we immediately get from Eq. (1) after some trivial algebra that

$$\nabla^2 \phi_L = (e^2 A^2 - m^2) \phi_L - e[(\nabla A) \phi_L + 2(A \nabla) \phi_L] \gamma_{21}. \quad (19)$$

Next, recalling again the identity given by Eq. (4) we have

$$\langle \nabla^2 \phi_L \gamma_{012} \phi_R \rangle_1 = \frac{1}{2} \left( \nabla^2 \phi_L \gamma_{012} \phi_R - \phi_L \gamma_{012} \nabla^2 \phi_R \right). \quad (20)$$

Next, using Eq. (20) on the right side of Eq. (19) multiplied by $\gamma_{012} \phi_R$ we get

$$\langle \nabla^2 \phi_L \gamma_{012} \phi_R \rangle_1 = eF \cdot J + e[(\nabla A) J + (A \nabla) J] \quad (21)$$

with $F = \nabla \wedge A = dA \in \sec \bigwedge^2 T^* M \leftrightarrow \sec C\ell(M, \eta)$ is the Faraday field.

We now write Eq. (21) in a convenient form which permit us to introduce the energy-momentum $(1,1)$-extensor field for the Dirac-Hestenes field, i.e., the object

$$T : \sec \bigwedge^1 T^* M \leftrightarrow \sec C\ell(M, \eta) \rightarrow \sec \bigwedge^1 T^* M \leftrightarrow \sec C\ell(M, \eta),$$

$$n \mapsto T(n) \quad (22)$$

such that

$$T(\gamma^\mu) \cdot \gamma^\nu = T^{\mu\nu} \quad (23)$$

\[\text{Recall that } \nabla = d - \delta, \text{ where } d = \nabla \wedge \text{ is the differential operator and } \delta = -\nabla \gamma \text{ is Hodge codifferential operator} \]
are the components of the energy-momentum tensor of the Dirac field. In order to proceed we introduce the differential operators

\[ \partial_n := \gamma^\mu \frac{\partial}{\partial n^\mu} \quad \text{and} \quad \partial_n \cdot \partial = \eta^{\mu\nu} \frac{\partial}{\partial n^\mu} \frac{\partial}{\partial n^\nu} \]  

(24)

acting on the bundle of the (1,1)-extensor fields. Recalling moreover that for any \( C \in \sec \bigwedge^1 T^* M \) it is

\[ \partial_n \cdot \partial (n \cdot \partial C) = \partial^2 C \]  

(25)

and

\[ \partial^\mu C = \gamma^\mu \cdot \partial_n (n \cdot \partial C) \]  

(26)

the left side of Eq. (21) can be written as

\[ \langle \partial^2 \phi_L \gamma_{012} \phi_R \rangle_1 = \partial_n \cdot \partial (n \cdot \partial C) = e F \cdot J. \]  

(27)

Now, observe that it is

\[ (\partial \cdot A)J + (A \cdot \partial)J = \gamma^\mu \cdot (\partial \mu A)J + \gamma^\mu \cdot A \partial^\mu J \]

\[ = \gamma^\mu \cdot \partial_n [(\partial^\mu n) \cdot AJ + n \cdot (\partial^\mu A)J + n \cdot A \partial^\mu J] \]

\[ = \gamma^\mu \cdot \partial_n \partial_m (n \cdot AJ) \]

\[ = \partial_n \cdot \partial (n \cdot AJ). \]  

(28)

Then, using Eqs. (24) and (25) we can write Eq. (21) as

\[ \partial_n \cdot \partial [\langle (n \cdot \partial) \phi_L \gamma_{012} \phi_R \rangle_1 - en \cdot AJ] = e F \cdot J. \]  

(29)

We now write

\[ T^\dagger (n) := \langle (n \cdot \partial) \phi_L \gamma_{012} \phi_R \rangle_1 - en \cdot AJ \]  

(30)

where \( T^\dagger \) denotes the adjoint of a (1,1)-extensor field \( T \), i.e., for \( n, m \in \sec \bigwedge^1 T^* M \) it is

\[ T(n) \cdot m = n \cdot T^\dagger (m) \]  

(31)

and write Eq. (29) as

\[ \partial_n \cdot \partial T^\dagger (n) = e F \cdot J \]  

(32)

which can also be written as

\[ \partial \cdot T^\dagger (\gamma_{\mu}) = e (F_{\mu}J) \cdot \gamma_{\mu} \]

When \( A = 0 \) we write \( T^\dagger (n) = T^\dagger_D (n) \)

\[ T^\dagger_D (n) = \langle (n \cdot \partial) \phi_L \gamma_{012} \phi_R \rangle_1 \]  

(33)

5 Recall that classically \( e F \cdot J \) represents the Lorentz force law.
which is the energy-momentum extensor field of the Dirac-Hestenes field. Obviously, in a region where \( F = 0 \) and \( A = 0 \), \( T_D^\dagger(n) \) is conserved, i.e.,

\[
\partial_n \cdot \partial T_D^\dagger(n) = 0.
\]

We write

\[
T^\mu = T(\gamma^\mu) := T^{\mu\nu} \gamma_\nu
\]

and of course

\[
T^\dagger_{\mu} = T^\dagger(\gamma^\mu) = T^{\nu\mu} \gamma_\nu.
\]

The objects \( T^\mu \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} \mathcal{C}(M, \eta) \) are said to be the energy-momentum 1-form fields of the Dirac field in interaction with an electromagnetic field. The \( T^\mu_{\nu} \) are the components of the so-called Tetrode energy-momentum tensor of the Dirac field and as it is well known and will be shown below it is

\[
T^\mu_{\nu} \neq T^\nu_{\mu},
\]

i.e., the Tetrode tensor is not symmetric, something that has a nontrivial implication shown in subsection 3.2, namely that the antisymmetric part of the energy-momentum tensor is the source of the spin field of the Dirac field.

### 3.1 The Trace and the Bif of the Extensor field \( T \)

To proceed, we need some results, in particular the trace and the bif of the extensor field \( T \) defined respectively by

\[
\text{tr}(T) = \text{tr}(T^\dagger) := \gamma^\mu \cdot T^\dagger(\gamma^\mu) = T^\dagger(\gamma^\mu) \cdot \gamma_\mu.
\]

\[
\text{bif}(T) = -\text{bif}(T^\dagger) = -T^\dagger(\gamma_\mu) \wedge \gamma^\mu = \gamma_\mu \wedge T^\dagger(\gamma_\mu).
\]

Now, the \( \langle \rangle_0 \) part of Eq.(8) is taking into account that

\[
\gamma^\mu \cdot \langle \partial_\mu \phi_L \gamma_{012} \phi_R \rangle_1 = eA \cdot J + m\phi_L \cdot \phi_R.
\]

Then,

\[
\text{tr}(T) = m\phi_L \cdot \phi_R
\]

Now, taking the \( \langle \rangle_0 \) part of Eq.(8) we get

\[
\frac{1}{2} \partial_\mu (\gamma_{5}s) + \gamma^\mu \wedge (\partial_\mu \phi_L \gamma_{012} \phi_R)_1 = eA \wedge J
\]

and then recalling also that \( \partial_\mu (\gamma_{5}s) = -\gamma_5 \partial_\mu s \)

\[
\text{bif}(T) = \frac{1}{2} \gamma_5 \partial \wedge s.
\]

Also, since

\[
T^\dagger(n) - T(n) = n \cdot \text{bif}(T)
\]

\[\text{Note that } \phi_L \cdot \phi_R = \phi_L \cdot \phi_L = \phi_R \cdot \phi_R.\]
we have
\[\partial_n \cdot \partial T^\dagger(n) - \partial_n \cdot \partial T(n) = \partial_n \cdot \partial(n \cdot \text{bif}(T))\]
\[= \gamma^\mu \cdot \partial_n [\partial_\mu n \cdot \text{bif}(T)] + n \cdot \partial^\mu \text{bif}(T)]\]
\[= \partial_n \text{bif}(T) = -\partial_\mu (\partial_\mu (\gamma_5 s)) = -\delta_\mu (\partial_\mu (\gamma_5 s)) = 0 \quad (45)\]
which means that
\[\partial_n \cdot \partial T^\dagger(n) = \partial_n \cdot \partial T(n). \quad (46)\]

### 3.2 The Source of Spin Extensor of the Dirac-Hestenes Field

Taking advantage that Minkowski spacetime is parallelizable we introduce the 1-form
\[x := x^\mu \gamma^\mu \quad (47)\]
which will be called the position 1-form of a point of \(x \in M\) relative to the point \(x_0 \in M\) which is the one mapped to the origin of the coordinates in \(\mathbb{R}^4\). If we make the exterior product of both members of Eq. (32) by \(x\) we get
\[|\partial_n \cdot \partial T(n) \wedge x = e(F \downarrow J) \wedge x \quad (48)\]
using the identity
\[\partial_n \cdot \partial T(n) \wedge x = [\partial_n \cdot \partial T(n)] \wedge x + \text{bif}(T) \quad (49)\]
in Eq. (48) we get
\[\partial_n \cdot \partial [T(n) \wedge x] = -\text{bif}(T) = e(F \downarrow J) \wedge x. \quad (50)\]

Now, we have that
\[\text{bif}(T) = \frac{1}{2} \gamma_5 \partial \wedge s = -\frac{1}{2} \gamma_5 \partial^\mu (s \wedge \gamma^\mu)\]
\[= -\frac{1}{2} \gamma^\mu \cdot \partial_n \gamma_5 [\partial_\mu s \wedge n + s \wedge \partial_\mu x]\]
\[-\gamma_5 \cdot \partial_n \partial_\mu \frac{1}{2} \gamma_5 (s \wedge n) = -\partial_n \cdot \partial \gamma_5 (s \wedge n)\]
and introducing the (1,2)-extensor field
\[S : \text{sec} \bigwedge^{1} T^* M \rightarrow \text{sec} \bigwedge^{2} T^* M \rightarrow \text{sec} \bigwedge C^\ell (M, \eta) \rightarrow \text{sec} \bigwedge C^\ell (M, \eta),\]
\[S(n) := \frac{1}{2} \gamma_5 (s \wedge n) \quad (51)\]
called the spin extensor field of the Dirac field we have that
\[\text{bif}(T) = -\partial_n \cdot \partial S(n) \quad (52)\]
If we contract Eq.\((52)\) on the left taking account of Eq.\((44)\) we get the fundamental result
\[
T(n) - T^\dagger(n) = n_\mu (\partial_n \cdot \partial S(n))
\]
(53)
which says that the source of the spin extensor is the antisymmetric part of the energy-momentum extensor. From Eq.\((53)\) we can immediately get that
\[
T^\mu\nu - T^\nu\mu = -\gamma^\nu \cdot \partial_\mu \{ \partial_\kappa S(\gamma^\kappa) \} = -\{ \gamma^\nu \wedge \gamma^\mu \} \cdot \partial_\kappa S(\gamma^\kappa).
\]
(54)
Finally calling
\[
J(n) := T(n) \wedge x + S(n)
\]
(55)
t we get using Eq.\((52)\) on the left side of Eq.\((50)\)
\[
\partial_n \cdot \partial J(n) = e(F_L J) \wedge x
\]
(56)
Recalling that \((F_L J) \wedge x\) is the torque produced by the Lorentz force \(e(F_L J)\) we see that Eq.\((56)\) says explicitly that in the presence of an external electromagnetic field the total angular momentum of the Dirac field is not conserved.

### 3.3 Origin of the Bohm-Aharonov Effect
\[
\partial_n \cdot \partial T^\dagger(n) = eF_L J
\]
(57)
In our opinion Eq.\((32)\) and Eq.\((56)\) encode the origin of the Bohm-Aharonov effect. Indeed, recalling Eq.\((30)\) and Eq.\((55)\) we have
\[
T(n) = T_D(n) - en \cdot AJ,
J(n) := T_D(n) \wedge x + S(n) - en \cdot AJ \wedge x = J_D(n) - en \cdot AJ \wedge x
\]
So, it follows that even in spacetime regions with topology permitting that \(F = 0\) but \(A \neq 0\) we must have (with \(A = A^\mu \gamma_\mu\)) from Eqs. \((32)\) and \((50)\)
\[
\partial_n \cdot \partial T_D(n) = \partial_\mu T_D(\gamma^\mu) = e\partial_n \cdot \partial(n \cdot AJ) = \partial_\mu (A^\mu J)
\]
(58)
\[
\partial_n \cdot \partial J_D(n) = \partial_\mu J_D(\gamma^\mu) = \partial_n \cdot \partial[n \cdot AJ \wedge x] = \partial_\mu (A^\mu J \wedge x)
\]
(59)
which says that in regions where \(F = 0\) but \(A \neq 0\) the energy-momentum \(T_D(n)\) and the total angular-momentum extensor \(J_D(n)\) of the Dirac field are not conserved. This certainly means that a particle described by a Dirac field entering a region with topology permitting that \(F = 0\) but \(A \neq 0\) will follow paths distinct from the ones where it enter a region where \(F = 0\) and \(A = 0\). This is exactly what is predicted in the Bohm-Aharonov effect and the experimental study of the possible trajectories (as done, e.g., in) in the de Broglie-Bohm interpretation of the Dirac equation seems a good test for such interpretation of quantum mechanics.
4 The Correct Relativistic Quantum Potential

In this section we analyze in details papers [13, 14] showing that it contains erroneous results and thus an erroneous identification of the quantum potential. Moreover, we recall form our recent paper [26] where we derive the correct generalized Hamilton-Jacobi equation which follows from the DHE satisfied by a DHSF and identifies the correct relativistic quantum potential.

So, to start our analysis, recall that from Eq. (33) we can write for the energy-momentum 1-form fields $T^\mu_D$ that

$$T^\mu_D = (\partial^\mu \phi_L \gamma_{012} \phi_R) = \frac{1}{2} \langle \partial^\mu \phi_L \gamma_{012} \phi_R - \phi_L \gamma_{012} \partial^\mu \phi_R \rangle$$  

(60)

In [13, 14] authors define the objects

$$\rho P^\mu = \frac{1}{2} (\partial^\mu \phi_L \gamma_{012} \phi_R - \phi_L \gamma_{012} \partial^\mu \phi_R)$$  

(61)

$$\rho W^\mu = -\frac{1}{2} \partial^\mu (\phi_L \gamma_{012} \phi_R) = -\frac{1}{2} \partial^\mu (\phi_L \gamma_{012} \phi_R)$$  

(62)

which taking into account Eqs. (4) and (11) permit us to write

$$\rho P^\mu = T^\mu_D = (\partial^\mu \phi_L \gamma_{012} \phi_R)_1 \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, \eta)$$  

(63)

$$\rho W^\mu = -(\partial^\mu \phi_L \gamma_{012} \phi_R)_3 \in \text{sec} \bigwedge^3 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, \eta)$$  

(64)

Authors of [13, 14] also introduce the object $J^7$

$$\rho J := \phi_L \gamma_{012} \phi_R \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, \eta)$$  

(65)

Now, recall that for a Dirac particle moving in a region where $A = 0$, Eq. (19) becomes

$$\partial^2 \phi_L + m^2 \phi_L = 0$$  

(66)

and of course we have also that

$$\partial^2 \phi_R + m^2 \phi_R = 0.$$  

(67)

Multiplying Eq. (66) on the right by $\phi_R$ and Eq. (67) on the right by $\phi_L$ and taking first the sum of these equations and second their difference we get $\partial^\mu \partial_\mu$

$$\partial^\mu (\partial^\mu \phi_L) \phi_R + \phi_L (\partial_\mu \partial^\mu \phi_R) + 2m^2 \phi_L \phi_R = 0$$  

(68)

\(^7\)Take notice that the object $J$ is not the same object we call $J$ (the probability current) given by Eq. (9) and is also different from the total angular momentum extensor which we denoted by $J(n)$. 

9
and
\[ \phi_L(\partial_\mu \partial^\mu \phi_R) - \partial_\mu (\partial^\mu \phi_L) \phi_R = 0 \quad (69) \]

Now, suppose that \( \phi_L \) is such that
\[ \phi_L \tilde{\phi}_L = \phi_L \phi_R \neq 0. \quad (70) \]
Then \( \phi_L \) is invertible and can be written
\[ \phi_L = R e^{\frac{1}{2} \gamma_5 \beta} U \quad (71) \]

where \( R, \beta \) are scalar functions (0-form fields) and for each \( x \in M, \ U \in \text{Spin}^0,3(\simeq \text{Sl}(2, \mathbb{C})) \subset \mathbb{R}^4,3 \) and so \( UU = UU = 1 \). Putting \( \rho = R^2 \) from Eqs. (61) and (62) we immediately get
\[ -\partial_\mu \phi_L = [P_\mu - W_\mu]\phi_L e^{-\gamma_5 \beta} \gamma_{012}, \quad (72) \]
\[ \partial_\mu \phi_R = \gamma_{012} \phi_R e^{-\gamma_5 \beta} [P^\mu + W^\mu]. \quad (73) \]

**Remark 1** Now, comes an important and crucial observation. Authors of [13, 14] write Eqs. (72) and (73) without the term \( e^{-\gamma_5 \beta} \). This means that they took the Takabayashi angle as null. In [26] we called invertible spinor fields for which \( \beta = 0 \) or \( \pi \) classical spinor fields. The reason for that name was that in [26] we prove that the classical Hamilton-Jacobi equation for a classical Dirac-Hestenes spinor field is equivalent to the classical Hamilton-Jacobi equation. This classical Dirac-Hestenes spinor field has the form
\[ \phi_L = R \Re(\Pi) e^{S_{\gamma}^2}, \]
\[ \Re(\Pi) = \frac{m + (\Pi + eA)^0 \gamma^0}{[2 (m + \Pi_0 + A_0)]^{1/2}}, \quad (74) \]

with
\[ \Pi : -\partial S = mV + eA \quad (75) \]
being the canonical momentum and
\[ V = \phi_L \gamma^0 \phi_L^{-1} \quad (76) \]
being the (1-form) velocity field such that for each possible world line of the classical particle (parametrized with the proper time) \( \sigma : \mathbb{R} \rightarrow M \) we have
\[ \eta(V, \cdot)|_\sigma = \sigma_* = v \quad (77) \]
where \( v \) is the velocity of the particle.

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8The function \( \beta \) is called the Takabayshi angle.
To end this remark we present [26] the correct generalized Hamilton-Jacobi equation (GHJE) that follows almost trivially in our formalism for the DHE satisfied by a general free DHSF that we write as
\[ \psi = \rho^{1/2} R(\Pi) e^{\frac{\beta \gamma_5}{2}} e^{S_{\gamma_2}} = \psi_0 e^{\frac{\beta \gamma_5}{2}} e^{S_{\gamma_2}} = e^{\frac{\beta \gamma_5}{2}} \psi_0 e^{S_{\gamma_2}}. \] (78)

in interaction with an external electromagnetic field. It is, with \( \Pi = -\partial S \) and \( \psi_0 \psi^{-1} = e^{\beta \gamma_5 V} \)
\[ -\partial S = m \cos \beta V + e A + (m \sin \beta \gamma_5 V + (\partial \ln \psi_0) \psi \gamma_2 \psi^{-1} + \frac{1}{2} \gamma_5 \partial (\ln \beta) \psi_0 \) \] (79)
where the following constraint must hold:
\[ \langle e \sin \beta \gamma_5 V + (\partial \ln \psi_0) \psi \gamma_2 \psi^{-1} + \frac{1}{2} \gamma_5 \partial (\ln \beta) \psi_0 \rangle_3 = 0 \] (80)

Thus the true ”quantum potential” is
\[ Q = \langle m \sin \beta \gamma_5 V + (\partial \ln \psi_0) \psi \gamma_2 \psi^{-1} + \frac{1}{2} \gamma_5 \partial (\ln \beta) \psi_0 \rangle_1 \] (81)
which differs considerably form the usual Bohm quantum potential. Moreover and contrary to the usual presentations of the de Broglie-Bohm theory the mass parameter of the particle in the GHJE (Eq.(79)) is not a constant, instead it is
\[ m' = m \cos \beta. \] (82)

Some analogous results using classical like equations of motion instead of the Hamilton Jacobi like equation as above has been obtained in a series of remarkable papers by Hestenes [10, 11, 12].

Now, we returning to the analysis of papers [13, 14] where as shown above used \( \beta = 0 \). Under these conditions if we substitute Eqs. (72) and (73) in Eq. (68) we get
\[ \rho(P^\mu P_\mu + W^\mu W_\mu) + \rho(\partial_\mu P^\mu - J_\mu P^\mu) - \rho(\partial_\mu W^\mu J + J_\mu W^\mu) - m^2 \rho = 0 \] (83)
Now, since \( \langle W^\mu W_\mu \rangle_2 = -\langle W^\mu W_\mu \rangle_2 = 0 \) we have that
\[ \langle P^\mu P_\mu + W^\mu W_\mu \rangle = P^\mu P_\mu + \langle W^\mu W_\mu \rangle_0 + \langle W^\mu W_\mu \rangle_2 = P^\mu P_\mu + W^\mu W_\mu \] (84)
\[ (\partial_\mu P^\mu J - J_\mu P^\mu) = 2 \partial_\mu P^\mu \wedge J. \] (85)
Since $\langle \partial_\mu W^\mu J \rangle_4 = 0$ and $\langle \partial_\mu W^\mu J \rangle_2 = - \langle \dot{J} \partial_\mu \dot{W}^\mu \rangle_2 = - \langle J \partial_\mu W^\mu \rangle_2$ it is

$$-(\partial_\mu W^\mu J + J \partial_\mu W^\mu) = -2 \partial_\mu W^\mu \cdot J.$$ (86)

Then, Eq.(83) becomes

$$P^\mu \cdot P_\mu + W^\mu \cdot W_\mu + 2 \partial_\mu P^\mu \wedge J - 2 \langle \partial_\mu W^\mu J \rangle_0 - m^2 = 0$$ (87)

which implies that:

$$P^\mu \cdot P_\mu + W^\mu \cdot W_\mu - 2 \langle \partial_\mu W^\mu J \rangle_0 - m^2 = 0,$$ (88)

$$2 \partial_\mu P^\mu \wedge J = 0.$$ (89)

Now (with $\beta = 0$), substituting Eqs.(72) and (73) in Eq.(69) we get

$$2 \rho \partial_\mu P^\mu \cdot J + 2 \rho \langle J \partial_\mu W^\mu \rangle_0 + 2 \rho \partial_\mu P^\mu - 2 \rho \partial_\mu W^\mu J$$

$$X + \phi \gamma_{012} \partial_\mu \phi_R P^\mu + P_\mu \partial_\mu \phi_R \gamma_{012} \phi_R$$

$$Y + \phi \gamma_{012} \partial_\mu \phi_R W^\mu + W_\mu \partial_\mu \phi_R \gamma_{012} \phi_R = 0$$ (90)

Calling the contents of the three lines of Eq.(90) respectively $X, Y, Z$ we have:

$$X = 4 \rho \partial_\mu P^\mu \cdot J + 2 \rho \langle J \partial_\mu W^\mu \rangle_0 + 2 \rho \langle J \partial_\mu W^\mu \rangle_2 - 2 \rho \langle J \partial_\mu W^\mu J \rangle_0 - 2 \rho \langle J \partial_\mu W^\mu J \rangle_2$$

$$4 \rho \partial_\mu P^\mu \cdot J + 4 \rho \langle J \partial_\mu W^\mu \rangle_2,$$ (91)

$$Y = - \rho (P^\mu \cdot W^\mu + W^\mu \cdot P^\mu) =$$

$$= - \rho [\langle P^\mu W^\mu \rangle_2 + \langle P^\mu W^\mu \rangle_4 + \langle W^\mu P_\mu \rangle_2 + \langle P_\mu W^\mu \rangle_4]$$

$$= -2 \rho \langle P^\mu W^\mu \rangle_2 = -2 \rho P^\mu \cdot W^\mu,$$ (92)

$$Z = - \rho (P^\mu \cdot W^\mu + W^\mu \cdot P^\mu);$$ (93)

So, Eq.(90) becomes

$$\partial_\mu P^\mu \cdot J + \langle J \partial_\mu W^\mu \rangle_0 - P^\mu \cdot W^\mu = 0.$$ (94)

**Remark 2** Authors of [13, 14] did not obtain Eqs.(88), (89) and (94) and moreover they interpreted the scalar part in Eq.(87) as being the GHJE in Bohm formalism. If this was the case we would identify $(W^\mu \cdot W_\mu - 2 \langle \partial_\mu W^\mu J \rangle_0)$ as the quantum potential.

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9Note that for any $A, B \in \wedge^3 T^* M \mapsto \text{sec} \mathcal{C}(M, \eta)$ always $(AB)_4 = 0$. 

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However we claim that such an identification is misleading in view of the results presented in [20] and also from the fact that authors of [13, 14] wrongly identified $2\rho P^\mu$ with the components $T^\mu_0$ of the energy-momentum of the Dirac field, which is a nonsequitur since the objects $2\rho P^\mu$ are 1-forms and objects $T^\mu_0$ are scalars. Moreover, for Dirac theory as Eq.(76ad) shows the quantum potential is a 1-form field, not a scalar function.

Remark 3 As a final remark, we emphasize that Eq.(94) does not describe the development of the spin. The equation of motion of the spin is the one given by Eq.(12).

5 Conclusions

Using the Clifford and spin-Clifford bundle formalisms and some few results of the theory of extensor calculus we derived the conservation laws for the probability current, energy-momentum and total angular momentum extensors resulting from the structure of DHE satisfied by a Dirac-Hestenes spinor field (DHSF) in interaction with the electromagnetic field. It was shown in a quite simple way that the energy-momentum extensor $T^\mu_0$ of the DHSF is not symmetric and that its antisymmetric part is the source of the spin extensor field of the free DHSF. In particular it was shown that these conservation laws implies that in spacetime regions with topology permitting the existence of an electromagnetic field potential $A \neq 0$ but with $F = 0$ the energy-momentum and total angular momentum of the sole DHSF are not conserved and this fact clearly is the origin of the Bohm-Aharonov effect. And here it must be emphasize that since this result can be directly derived for the DHE satisfied by a classical DHSF equivalent to the classical relativistic HJE satisfied by a (spinning) charged particle it is not of primely of quantum nature, more precisely, one can predict that the motion of spinning charged particles in a region with topology permitting $A \neq 0$ but with $F = 0$ is different from the motion of spinning charged particles in a region $A = 0$ but with $F = 0$.

We also briefly recalled results obtained in [20] where it was shown that (a): that the classical relativistic Hamilton-Jacobi equation (HJE) for a charged electrical particle in interaction with the electromagnetic field is equivalent to a DHE satisfied by a special class of DHSF that we call classical DHSF which are characterized by having the Takabayashi angle equal to 0 or $\pi$ and moreover that the DHE satisfied by a classical DHSF is equivalent to the HJE; (b) The identification of the correct relativistic quantum potential resulting from the DHE in a de Broglie-Bohm approach [11] leading to a HJE like equation for the motion of (spinning) charged particle. And equipped with these results and the ones of Section 2 we analyzed results of [13, 14] were authors though that they have disclosed the relativistic quantum potential for the Dirac particle. We show that the results of those papers are equivocated by explicitly showing with detailed calculations were authors get mislead.

\[10\] Related to the Tetrode tensor of the Dirac spinor field.
In view of the recent interest on experiments to verify if the trajectories of particles predicted by de Broglie-Bohm theory (see, e.g., [5, 19, 27]) and pertinent criticisms, like, e.g., in [2, 3, 16] we think that our results are worth to be appreciated. In particular, it clear for us that any prediction of trajectories even in the nonrelativistic limit must be done using the non relativistic approximation to the DHE, for as shown long ago in an important paper by Guther and Hestenes [9] the Schrödinger equation which may be derived for the DHE in fact describes not a spinless particle, but a particle in a spin auto state.

Concerning also the trajectories problem it is important to discover if particles follows the integral lines of the velocity field as defined in [26] or the integral lines of the vector field $\eta(T^{a}_{0}, \ldots) = T^{a}_{\mu} \frac{\partial}{\partial x^{\mu}}$ (which are the ones appearing in the conservation laws of energy-momentum and total angular momentum).

A Notation

In this paper the arena where physical phenomena is supposed to take place is the Minkowski spacetime structure $(M, \eta, D, \tau_{\eta}, \uparrow)$ where the manifold $M \simeq \mathbb{R}^{4}$, $\eta \in \sec T_{0}^{2}M$ is Minkowski metric, $D$ is the Levi-Civita connection of $\eta$. Moreover $M$ is oriented by $\tau_{\eta} \in \sec \Lambda^{4}T^{*}M$ and time oriented by $\uparrow$. We introduce in $M$ global coordinates $\{x^{b}\}$ in Einstein-Lorentz-Poincaré gauge. We put

$e_{\mu} := \frac{\partial}{\partial x^{\mu}} = \partial_{\mu}$, \hspace{1cm} $\gamma^{\mu} := dx^{\mu}$ \hspace{1cm} (95)

which are respectively global basis for $TM$ and $T^{*}M$. Then we can write $\eta = \eta_{\mu\nu} \gamma^{\mu} \otimes \gamma^{\nu}$. We also introduce the metric of the cotangent bundle, i.e., $\eta = \eta_{\mu\nu} e_{\mu} \otimes e_{\nu}$ and define the reciprocal basis of the basis $\{\gamma^{\mu}\}$ as being $\{\gamma_{\mu}\}$, with $\eta(\gamma_{\mu}, \gamma^{\nu}) = \delta^{\nu}_{\mu}$. We denote the Clifford bundle of differential forms by $\mathcal{Cl}(M, \eta)$. In what follows the Clifford product of sections of $\mathcal{Cl}(M, \eta)$ is denoted by juxtaposition of symbols. In particular we have the fundamental relation

$\gamma^{\mu} \gamma^{\nu} + \gamma^{\nu} \gamma^{\mu} = 2\eta^{\mu\nu}$. \hspace{1cm} (96)$

We use the notation $C \in \sec \mathcal{Cl}(M, \eta)$ to denote a general section of the Clifford bundle and since the bundle of differential forms $\Lambda T^{*}M$ shares with $\mathcal{Cl}(M, \eta)$ for each $x \in M$ the same vector space we can write that

$C = \sum_{i=0}^{4} C_{i} \hspace{1cm} (97)$

where $C_{i} \in \sec \Lambda^{i}T^{*}M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$. We also use the projection operators

$\langle \rangle_{i} : C \in \sec \mathcal{Cl}(M, \eta) \mapsto C_{i} \in \sec \Lambda^{i}T^{*}M \hookrightarrow \sec \mathcal{Cl}(M, \eta). \hspace{1cm} (98)$

We recall that if $A_{r} \in \sec \Lambda^{r}T^{*}M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$, $B_{s} \in \sec \Lambda^{s}T^{*}M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$

\[11\] See details in [25].
\( A_rB_s = (A_rB_s)_{|r-s|} + (A_rB_s)_{|r-s|+2} + \cdots + (A_rB_s)_{r+s} = \sum_{k=0}^{m} (A_rB_s)_{|r-s|+2k}, \)

Moreover, for \( A, C \in \sec \mathcal{C}(M, \eta) \) the following identity that will be used several times in the calculations of main text holds

\[ (AC)_r = (-1)^{\frac{r(r-1)}{2}}(\tilde{B}C)_r \]  

where

\[ \cdot : \sec \Lambda^r T^*M \ni \sec \mathcal{C}(M, \eta) \ni \sec \Lambda^r T^*M \ni \sec \mathcal{C}(M, \eta), \]

\[ C_r \mapsto \tilde{C}_r = (-1)^{\frac{r(r-1)}{2}}. \]  

We have also used the fact that \( B_r \in \sec \Lambda^r T^*M \ni \sec \mathcal{C}(M, \eta), \ B_s \in \sec \Lambda^s T^*M \ni \sec \mathcal{C}(M, \eta) \) with \( r \leq s \) the right and left contractions are related by

\[ B_r \cdot C_s = (B_rC_s)_{|r-s|} = (-1)^{\frac{r(r-1)}{2}}(C_sB_r)_{|r-s|} = C_s \cdot B_r. \]

Also,

\[ B_r \wedge C_s = (B_rC_s)_{|r-s|} = (-1)^{r}C_s \wedge B_r. \]

**Remark 4** We remark that with our conventions the scalar product (denoted by the symbol \( \cdot \)) of multiforms \( B_r, C_s \in \sec \Lambda^r T^*M \ni \sec \mathcal{C}(M, \eta) \) is null for \( r \neq s \) and for \( r = s \) it is related to the contractions by the following relation

\[ B_r \cdot C_r = B_r \cdot C_r = \tilde{B}_r \cdot C_r = B_r \cdot \tilde{C}_r. \]

**Remark 5** Moreover, we wrote the volume element \( \tau_\eta \in \sec \Lambda^5 T^*M \ni \sec \mathcal{C}(M, \eta) \) as

\[ \tau_\eta = \gamma^0 \wedge \gamma^1 \wedge \gamma^2 \wedge \gamma^3 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 =: \gamma^5. \]  

Besides \( \mathcal{C}(M, \eta) \) we need also the so-called left and right spin-Clifford bundles denoted \( \mathcal{C}_L^{Spin_{1,3}}(M, \eta) \) and \( \mathcal{C}_R^{Spin_{1,3}}(M, \eta) \) and the complexified left and right spin-Clifford bundles denoted \( \mathcal{C}_L^{Spin_{1,3}^c}(M, \eta) \) and \( \mathcal{C}_R^{Spin_{1,3}^c}(M, \eta) \). These bundles are all trivial\(^{22}\) and have the structures

\[ \mathcal{C}(M, \eta) = P_{Spin_{1,3}}(M) \times_{Ad} \mathbb{R}_{1,3}, \]

\[ \mathcal{C}_L^{Spin_{1,3}^c}(M, \eta) = P_{Spin_{1,3}^c}(M) \times_{\mathfrak{l}} \mathbb{R}_{1,3}, \quad \mathcal{C}_R^{Spin_{1,3}^c}(M, \eta) = P_{Spin_{1,3}^c}(M) \times_{\mathfrak{r}} \mathbb{R}_{1,3}, \]

\[ \mathcal{C}_L^{Spin_{1,3}}(M, \eta) = P_{Spin_{1,3}}(M) \times_{\mathfrak{l}} \mathbb{R} \otimes \mathbb{R}_{1,3}, \quad \mathcal{C}_R^{Spin_{1,3}}(M, \eta) = P_{Spin_{1,3}}(M) \times_{\mathfrak{r}} \mathbb{R} \otimes \mathbb{R}_{1,3}. \]

\(^{22}\)Indeed, all spinor bundles over a 4-dimensional Lorentzian spacetime are trivial. See (HEIS)
where $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$ is the spacetime algebra, $P_{\text{Spin}_{1,3}}(M)$ is a principal bundle called spinor structure bundle. Moreover, Ad: Spin$_{1,3} \rightarrow \text{Aut}(\mathbb{R}_{1,3})$ with $\text{Ad}_uX = uXu^{-1}$ and $l$ and $r$ denotes respectively the representations of Spin$_{1,3}(\simeq \text{Sl}(2, \mathbb{C}))$ on $\mathbb{R}_{1,3}$ given respectively by $l(u)X = uX$ and $r(u)X = Xu$. Left and right Dirac-Hestenes spinor fields (DHSF) are respectively sections of $\mathcal{CL}_{\text{Spin}_{1,3}}^L(M, \eta)$ and $\mathcal{CR}_{\text{Spin}_{1,3}}^R(M, \eta)$, the even subbundles of $\mathcal{CL}_{\text{Spin}_{1,3}}^L(M, \eta)$ and $\mathcal{CR}_{\text{Spin}_{1,3}}^R(M, \eta)$. We define also the bundles $S^L(M)$ and $S^R(M)$ such that

$$S^L(M) = P_{\text{Spin}_{1,3}}(M) \times_l I^L, \quad S^R(M) = P_{\text{Spin}_{1,3}}(M) \times_r I^R$$

with the minimal ideals $I^L$ and $I^R$ in $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$ being respectively

$$I^L = (\mathbb{R} \otimes \mathbb{R}_{1,3})f, \quad I^R = \tilde{f}(\mathbb{R} \otimes \mathbb{R}_{1,3}).$$

and generated by the idempotents $f$ and $\tilde{f}$ where

$$f = \frac{1}{2}(1 + \gamma_0)\frac{1}{2}(1 + i\gamma_1\gamma_2) \in I \subset \mathbb{C} \otimes \mathbb{R}_{1,3}, \quad i = \sqrt{-1}. \quad (108)$$

The sections of $S^L(M)$ can be represented in $\mathbb{C}(4)$ and there is a $1 - 1$ correspondence with ideal sections of the bundle $S^L(M) = P_{\text{Spin}_{1,3}}(M) \times_l \mathbb{C}(4)f$ where $f$, the representation of $f$ in $\mathbb{C}(4)$ is given by

$$f = \frac{1}{2}(1 + \gamma_0)\frac{1}{2}(1 + i\gamma_1\gamma_2), \quad i = \sqrt{-1}. \quad (109)$$

and where

$$\gamma_0 \mapsto \gamma_0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}; \quad \gamma_i \mapsto \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix},$$

where $1_2$ is the unit $2 \times 2$ matrix and $\sigma_i, (i = 1, 2, 3)$ are the standard Pauli matrices.

Finally, we recall that:

(i) The ideal sections of $S^L(M)$ are 1-1 correspondence with the sections of the bundle the Dirac spinor fields represented as columns complex vectors in the standard formalism used in Physics textbooks and which are sections of the bundle

$$S^L(M) = P_{\text{Spin}_{1,3}}(M) \times_l \mathbb{C}^4.$$  

(ii) Sections of the bundles $\mathcal{CL}_{\text{Spin}_{1,3}}^L(M, \eta)$ and $\mathcal{CR}_{\text{Spin}_{1,3}}^R(M, \eta)$ are represented by some well defined equivalence classes of sections of the even subbundle $\mathcal{CL}^L(M, \eta)$ of $\mathcal{CL}(M, \eta)$. A representative is given once we fix a spinorial frame.

In what follows we suppose that a spinorial frame has been fixed in order to simplify our notation. Details are given in [24, 23, 25]. Here we recall that the matrix representation in $\mathbb{C}(4)$ of $\Phi_L \in I^R$ will be denoted by the same letter in
boldface, i.e., \( \Phi_L \mapsto \Phi_L \in I = \mathbb{C}(4)f \). We have (see, e.g., [?], 18, 28)

\[
\Phi_L = \begin{pmatrix}
\psi_1 & 0 & 0 & 0 \\
\psi_2 & 0 & 0 & 0 \\
\psi_3 & 0 & 0 & 0 \\
\psi_4 & 0 & 0 & 0
\end{pmatrix}, \quad \psi_i \in \mathbb{C}.
\] (111)

Any element \( \Phi_L \in I \) can be written

\[
\Phi_L = \phi_L f, \quad \phi_L \in \mathbb{R}^{0, 1, 3} \subset \mathbb{C} \otimes \mathbb{R}_{1,3}
\] (112)

The matrix representation of \( \phi_L \) (a representative of a \textit{Dirac-Hestenes spinor} in a given spin frame) in \( \mathbb{C}(4) \) will be denoted by the same letter in boldsymbol, i.e.:

\[
\phi_L = \begin{pmatrix}
\psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\
\psi_2^* & \psi_1 & \psi_4 & -\psi_3^* \\
\psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\
\psi_4 & -\psi_3^* & \psi_2 & \psi_1^*
\end{pmatrix}
\] (113)

Remark 6 When (global) coordinates in Einstein-Lorentz-Poincaré gauge are introduced in \( M \) the spin-Dirac operator \( \partial^\ast \) acting on sections of the left spinor bundle will be denoted simply by \( \partial = \gamma^\mu \partial_\mu \). In this case, taking moreover into account that all spinor bundles in Lorentzian spacetime structures are trivial \([7, 8]\) we use in what follows a convenient obvious notation writing the Dirac equation satisfied by a spinor field \( \Phi_L : M \simeq \mathbb{R}^4 \to I = \mathbb{C}(4)f \) as

\[
i \partial \Phi_L - m \Phi_L = 0
\] (114)

In this paper we denote the Dirac conjugate\(^{13}\) of a spinor field \( \Phi_L \) as

\[
\Phi_R := \Phi_L \gamma^\mu \gamma_0
\] (115)

we can easily show (as it is well known) that it satisfies the following Dirac conjugate equation

\[
i \Phi_R \overset{\leftarrow}{\partial} + m \Phi_R = 0, \quad \Phi_R \overset{\leftarrow}{\partial} := \partial_\mu \Phi_R \gamma^\mu.
\] (116)

A.1 The Dirac-Hestenes Equation

Remark 7 Now, we recall that the field \( \phi_L : M \simeq \mathbb{R}^4 \to \mathbb{R}^{0,1,3} \subset \mathbb{C} \otimes \mathbb{R}_{1,3} \) which as already said above is a representative of a \textit{Dirac-Hestenes spinor field} (an object which is a section of a Spin-Clifford bundle) in the Clifford bundle once a spin frame is fixed. To have in mind that \( \phi_L \) is a representative of a DHSF is extremely important in order to avoid misconceptions. So, take notice that

\(^{13}\)Note that the Dirac conjugate spinor of \( \Phi_L \) is usually denote by \( \bar{\Phi}_L := \Phi_L^\dagger \gamma_0 \). We did not use this notation here because as already recalled we use in [25] then the symbol to denote the main involution operator.
(using coordinates in Einstein-Lorentz-Poincaré gauge for $M$) the representative in the Clifford bundle of the spin-Dirac operator $\partial^s$ acting on the bundle of Dirac-Hestenes spinor fields will be simply denoted by

$$\partial = \gamma^\mu \partial_\mu.$$  

(117)

Under these conditions one can easily shown that the Dirac equation is equivalent to the following equation which is known as the Dirac-Hestenes equation

$$\partial \phi_L \gamma_{21} - m \phi_L \gamma_0 = 0,$$

(118)

To continue we observe that if we multiply this equation on the left by the idempotent $f$ (Eq.(108)) we immediately get recalling Eq.(112) satisfies the following differential equation.

$$i \partial \Phi_L - m \Phi_L = 0,$$

(119)

whose matrix representation in $\mathbb{C}^4$ is of course given by Eq.(114).

The object representing $\Phi_R$ in the Clifford bundle formalism is

$$\Phi_R := \tilde{\Phi}_L^*: M \simeq \mathbb{R}^4 \rightarrow I_R = \tilde{f}^*(\mathbb{C} \otimes \mathbb{R}_{1,3})$$

where the idempotent

$$\tilde{f}^* = \frac{1}{2}(1 - i \gamma_{21})\frac{1}{2}(1 + \gamma_0)$$

generates (for each $x \in M$) the ideal $I_R = \tilde{f}^*(\mathbb{C} \otimes \mathbb{R}_{1,3})$. Of course,

$$\Phi_R := \tilde{\Phi}_L^* = \frac{1}{2}(1 - i \gamma_{21})\frac{1}{2}(1 + \gamma_0)\tilde{f} \tilde{\phi}_L, \quad \phi_R := \tilde{\phi}_L$$

(120)

and we immediately get

$$i \Phi_R \partial + m \Phi_R = 0.$$  

(121)

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\[14\] See details in Chapter 7 of [25].
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