Stability of radially symmetric, monotone vorticities of 2D Euler equations

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Abstract
We consider the incompressible Euler equations in $\mathbb{R}^2$ when the initial vorticity is bounded, radially symmetric and non-increasing in the radial direction. Such a radial distribution is stationary, and we show that the monotonicity produces stability in some weighted norm related to the angular impulse. For instance, it covers the cases of circular vortex patches and Gaussian distributions. Our stability does not depend on $L^\infty$-bound or support size of perturbations. The proof is based on the fact that such a radial monotone distribution minimizes the impulse of functions having the same level set measure.

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1 Introduction

Stability for special solutions of the incompressible Euler equations has been an important topic in hydrodynamics since Kelvin’s work [38]. In this paper, we are interested in stability of circular flows generated by radial, monotone vorticities.

We consider the incompressible Euler equations in $\mathbb{R}^2$ in vorticity form:

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^2,$$

$$\omega|_{t=0} = \omega_0 \quad \text{for} \quad x \in \mathbb{R}^2,$$

(1.1)

where the velocity $u$ is determined from the vorticity $\omega$ by the Biot-Savart law given as

$$u(x) = \int_{\mathbb{R}^2} K(x - y)\omega(y)dy, \quad K(x) = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right).$$

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Yudovich’s theory [40] says that if the initial vorticity \( \omega_0 \) lies on \( (L^1 \cap L^\infty)(\mathbb{R}^2) \), then there exists a unique global-in-time weak solution \( \omega(t) \) of (1.1). For modern treatments of the theory, we refer to Majda-Bertozzi [31, Ch. 8], Marchioro-Pulvirenti [33, Ch. 2], Chemin [15, Ch. 5]. We denote the angular impulse of a function \( f \) by

\[
J(f) := \int_{\mathbb{R}^2} |x|^2 f(x) dx.
\]

For \( p \in [1, \infty) \), we define the norm \( \|\cdot\|_{J^p} \) by

\[
\|g\|_{J^p} := \|g\|_{L^p(\mathbb{R}^2)} + J(|g|) = \left( \int_{\mathbb{R}^2} |g(x)|^p dx \right)^{1/p} + \int_{\mathbb{R}^2} |x|^2 |g(x)| dx,
\]

for \( g \in L^p(\mathbb{R}^2) \) that satisfies \( J(|g|) < \infty \). We remind readers that for any nonnegative initial data \( \omega_0 \in L^\infty(\mathbb{R}^2) \) with \( J(\omega_0) < \infty \), the corresponding solution \( \omega(t) \) of (1.1) is nonnegative and has conservations of any \( L^p \)-norm \( \|\omega(t)\|_{L^p} \) for \( p \in [1, \infty) \) and the angular impulse \( J(\omega(t)) \). In particular, the Lebesgue measure of each level set

\[
|\{x \in \mathbb{R}^2 : \omega(t, x) > \alpha\}|, \quad \alpha > 0
\]

(1.2)
is preserved in time.

Throughout this paper, we denote \( B_r \) as the disk centered at the origin with radius \( r > 0 \);

\[
B_r := \{x \in \mathbb{R}^2 : |x| < r\}.
\]

For convention, we also denote

\[
B_0 := \text{the empty set}, \quad D := \text{the unit disk } B_1.
\]

1.1 Main results

It is a well-known fact that the characteristic function \( 1_D \) of the unit disk is a stationary solution of (1.1). As a pioneer work, Wan-Pulvirenti [39] obtained \( L^1 \)-stability of \( 1_D \) in patch-type perturbations for circular bounded domains \( B_R, \ R > 1 \). For the whole space \( \mathbb{R}^2 \), Sideris-Vega [35] proved \( L^1 \)-stability with the explicit estimate

\[
\sup_{t \geq 0} |\Omega_t \triangle D|^2 \leq 4\pi \sup_{\Omega_0 \triangle D} \left| |x|^2 - 1 \right| \cdot |\Omega_0 \triangle D|,
\]

where \( 1_{\Omega_0} \) is the solution for patch-type initial data \( 1_{\Omega_0} \), and the symbol \( \triangle \) means the symmetric difference. We also refer to the result of Dritschel [23]. Classical numerical results around \( 1_D \) can be found in Deem-Zabusky [21] and Buttke [13]. We also see [18], [16], [19], Elgindi-Jeong [24] for some applications about growth in perimeter of patch boundary and winding number of particles.

First, we obtain \( J_2 \)-stability of \( 1_D \) allowing nonpatch-type perturbations which are not necessarily compactly supported.

**Theorem 1.1** For \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if a nonnegative \( \omega_0 \in L^\infty(\mathbb{R}^2) \) with \( J(\omega_0) < \infty \) satisfies

\[
\|\omega_0 - 1_D\|_{J_2} \leq \delta,
\]

then the solution \( \omega(t) \) of (1.1) satisfies

\[
\sup_{t \geq 0} \|\omega(t) - 1_D\|_{J_2} \leq \varepsilon.
\]
Such stability lies on the fact that $1_D$ uniquely minimizes the angular impulse $J$ of non-negative functions $\xi$ bounded by 1 and having the same $L^1$-norm as $1_D$ (see Proposition 2.7);

$$0 \leq \xi \leq 1, \quad \|\xi\|_{L^1(\mathbb{R}^2)} = \|1_D\|_{L^1(\mathbb{R}^2)}.$$  

During the proof, $\delta$ is implicitly obtained by a contradiction argument.

Now we consider a generalization of circular patches. We first note that every radially symmetric vorticity is also a stationary solution of (1.1) because it makes the stream function $\phi = \Delta^{-1}\omega$ be radially symmetric as well, which gives us

$$u \cdot \nabla \omega = \nabla \perp \phi \cdot \nabla \omega = 0$$

(cf. see Gómez Serrano-Park-Shi-Yao [25] for a related converse statement). Let’s take any function $\zeta \in L^\infty(\mathbb{R}^2)$ which is nonnegative, radially symmetric, and non-increasing; there exists a function $f \in L^\infty([0, \infty))$ such that

$$\zeta(x) = f(|x|) \quad \text{for} \quad x \in \mathbb{R}^2, \quad f \geq 0, \quad f(r_1) \geq f(r_2) \quad \text{for} \quad r_1 \leq r_2.$$  

Theorem 1.2 below says that $\zeta$ is $J_p$-stable for $p \in [1, \infty)$ with an explicit estimate (1.4) whenever $\zeta$ is compactly supported in $\mathbb{R}^2$. For instance, Theorem 1.1 is just a particular case ($\zeta = 1_D$ and $p = 2$) of Theorem 1.2.

**Theorem 1.2 (supp $(\zeta)$ : compact)** For any constants $R, M > 0$, there exists a constant $C = C(R, M) > 0$ such that if $\zeta \in L^\infty(\mathbb{R}^2)$ is nonnegative, radially symmetric, non-increasing, and compactly supported with

$$\text{supp} \ (\zeta) \subset B_R, \quad \|\zeta\|_{L^\infty} \leq M,$$

then for any nonnegative $\omega_0 \in L^\infty(\mathbb{R}^2)$ with $J(\omega_0) < \infty$, the solution $\omega(t)$ of (1.1) satisfies

$$\sup_{t \geq 0} \|\omega(t) - \zeta\|_{J_p} \leq C \left( \|\omega_0 - \zeta\|_{J_p}^{1/p} + \|\omega_0 - \zeta\|_{J_p} \right) \quad \text{for any} \quad p \in [1, \infty).$$ (1.4)

As in the case $1_D$, the key idea is that such a profile $\zeta$ minimizes the angular impulse $J$ of nonnegative functions $\xi$ having the same level set measure as $\zeta$;

$$\|\{\xi > \alpha\}\| = \|\{\zeta > \alpha\}\|, \quad \alpha > 0.$$  

We note that Marchioro-Pulvirenti [32] already showed $L^1$—stability of such monotone profiles for any circular bounded domains $B_R, R > 0$ (also see Burton [10, Theorem 3]). The paper [32] also showed $L^1$-stability of any monotone function $\zeta(x, y) = \zeta(y)$ for bounded channels $\mathbb{T} \times [0, R], R > 0$.

Lastly, we consider the case when a monotone profile $\zeta$ is not necessarily compactly supported. In this case, we request, instead, the profile $\zeta$ to have finite momentum of some higher order (1.5). For instance, we obtain stability of a vorticity with the Gaussian distribution $e^{-|x|^2}$. Bassom-Gilbert [4, Section 4] examined the Gaussian profile numerically, which was motivated by Lamb [29, 334a]. We also see Schecter-Dubin-Cass-Driscoll-Lansky-O’Neil [34] and references therein for linear analysis and experiments on radial profiles.

**Theorem 1.3 (supp $(\zeta)$ : not necessarily compact)** Let $\zeta \in L^\infty(\mathbb{R}^2)$ be nonnegative, radially symmetric, and non-increasing with

$$\int_{\mathbb{R}^2} |x|^6 \zeta \, dx < \infty,$$ (1.5)
and let $p \in [1, \infty)$. Then for $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \zeta, p) > 0$ such that if a nonnegative $\omega_0 \in L^\infty(\mathbb{R}^2)$ with $J(\omega_0) < \infty$ satisfies

$$\|\omega_0 - \zeta\|_{L^p} \leq \delta,$$

then the solution $\omega(t)$ of (1.1) satisfies

$$\sup_{t \geq 0} \|\omega(t) - \zeta\|_{L^p} \leq \varepsilon.$$

**Remark 1.4** All our Theorems 1.1, 1.2, 1.3 ask initial vorticity $\omega_0$ to be nonnegative. Removing the sign assumption looks highly non-trivial as long as we consider fluids in the whole space $\mathbb{R}^2$ (cf. no sign condition results [32], [10] for circular bounded domains). This is because our stability is essentially due to the conservation of the fluid impulse $\int_{\mathbb{R}^2} |x|^2 \omega(t, x) \, dx$. Indeed, when allowing negative part of $\omega$, it might be possible that some negative region might fly away (toward infinity) in time (cf. see the example in Ifrim-Sideris-Gambili [26, Sect. 3]). In other words, we do not know a global-in-time bound on $\int_{\{\omega < 0\}} |x|^2 \omega(t, x) \, dx$.

Technically speaking, for a function on the whole space $\mathbb{R}^2$, the nonnegativity is needed to define its symmetric-decreasing rearrangement (see Definition 2.1).

For strictly monotone radial profiles in $\mathbb{R}^2$, Bedrossian-Coti Zelati-Vicol [5] obtained the linear inviscid damping. We also mention asymptotic results for a point vortex by Coti Zelati-Zillinger [20] and by Ionescu-Jia [27]. For the infinite channel $\mathbb{T} \times \mathbb{R}$, Beichman-Denisov [7] obtained stability for long rectangular patches. It is interesting that for Couette flows in the same channel, Bedrossian-Masmoudi [6] proved even the inviscid damping with nonlinear asymptotic stability. For a bounded channel, we refer to Ionescu-Jia [28].

Orbital stability of other various vorticities can be found e.g. in Tang [36] on Kirchhoff’s ellipse, Burton-Nussenzveig Lopes-Lopes Filho [12] on various dipoles, Cao-Wan-Wang [14] on patches for bounded domains, [1] on Lamb (Lamb-Chaplygin) dipole, and [17] on Hill’s spherical vortex.

### 1.2 Key ideas

We follow the strategy via the variational method based on vorticity due to the classical papers by Kelvin [37] and Arnold [2] (also see the book of Arnold-Khesin [3]). To go into detail and explain, the stability in this paper is closely related to the following properties of the symmetric-decreasing rearrangement of functions. The first is that the angular impulse of a nonnegative function $f$ is greater than or equal to that of the rearrangement $f^*$ (see Definition 2.1) of $f$;

$$J(f^*) \leq J(f). \quad (1.6)$$

It is simply because the weight $|x|^2$ is radially symmetric and monotonically increasing. On the other hand, any two functions having the same measure for each level set have the same rearrangement. By this, the rearrangement of the solution $\omega(t)$ of the Euler flow stays the same as that of $\omega_0$ throughout all time;

$$\left(\omega(t)\right)^* = (\omega_0)^* \quad \text{for all} \quad t \geq 0.$$
The second is the nonexpansivity (Lemma 3.1) of rearrangements for nonnegative functions;
\[ \| g^* - h^*\|_{L^1} \leq \| g - h\|_{L^1}. \]

It says that \( L^1 \)-difference between two functions is non-increasing by replacing them with their rearrangements. It is one of well-known properties of rearrangements. For instance, a proof can be found in Lieb-Loss [30, Sect. 3.5].

The third is the following estimate (Lemma 3.2) which is a sharper version of (1.6):
\[ \| f - f^*\|_{L^1}^2 \leq 4\pi \| f\|_{L^\infty} \left[ J(f) - J(f^*) \right]. \]

It means that \( L^1 \)-difference between a function and its rearrangement can be controlled by the difference between their angular impulses. Such a rearrangement estimate appeared in the paper [32, Lemma 1] for bounded channel domains \( \mathbb{T} \times [0, R], \ R > 0 \). For other fine properties of rearrangements which are applicable to the Euler equations, we refer to a series of works from Burton [8], [9], Burton-McLeod [11] for bounded domains and Douglas [22] for unbounded domains.

We combine the above properties of rearrangements with the idea of cutting off a function to avoid using its supremum. It makes us to remove the dependence on \( \|\omega_0\|_{L^\infty} \) in \( L^1 \)-estimate (Lemma 3.3). In addition, we use the fact that the rearrangement operation and the cut-off operation commute with each other; the rearrangement of a cut-off of a function is the same as the cut-off of the rearrangement.

For the organization of the rest, in Sect. 2, we prove Theorem 1.1 via a contradiction argument. It is done by studying

1. Existence: Lemma 2.6, 2. Uniqueness: Proposition 2.7, 3. Compactness: Proposition 2.8

of the variational problem (2.5). In Sect. 3, we first show \( L^1 \)-estimate (3.7) in Lemma 3.3, which gives \( J_p \)-estimate (3.17) in Lemma 3.4. Then we have Theorems 1.2 and 1.3.

Lastly, we recall again that Theorem 1.1 is just a particular case of Theorem 1.2. Nevertheless, we decided to keep the proof of Theorem 1.1 in Sect. 2 because the variational method and the contradiction argument are more robust when applying to more complicated settings such as Lamb (Lamb-Chaplygin) dipole [1] or Hill’s spherical vortex [17]. However, the constructive computations done in Sect. 3 seem not applicable to those complicated cases since they heavily rely on the explicit rearrangement estimate (3.2) in Lemma 3.2. If one wants to cut to the chase toward Theorems 1.2 and 1.3, we recommend to skip Sect. 2 and to read Sect. 3.

We finish the introduction section by mentioning the easiest stability (1.7) of a circular patch for patch-type perturbations since it does not seem to be well-known in the community. The current form (1.7) is due to [35] while [23] also stated it in a convenient integral form. For any patch-type solution \( 1_{\Omega_1} \) of (1.1), the nonnegative quantity
\[ \int_{\Omega_t \Delta D} |x|^2 - 1 \, dx \]
is conserved in time by the direct computation:
\[
\int_{\Omega_t \triangle D} |x|^2 - 1 \, dx = \int_{\Omega_t \setminus D} (|x|^2 - 1) \, dx + \int_{D \setminus \Omega_t} (1 - |x|^2) \, dx
\]
\[
= \int_{\Omega_t} (|x|^2 - 1) \, dx + \int_{D} (1 - |x|^2) \, dx
\]
\[
= \int_{\Omega_0} (|x|^2 - 1) \, dx + \int_{D} (1 - |x|^2) \, dx
\]
\[
= \int_{\Omega_0 \triangle D} |x|^2 - 1 \, dx, \quad t \geq 0. \tag{1.7}
\]

2 Variational method

In this section, we use the notion of the symmetric-decreasing rearrangement of a function. To define this rearrangement rigorously, we first need the notion of the symmetric rearrangement of a measurable set with finite measure. Below we follow the definition in the textbook [30, Sec. 3.3].

**Definition 2.1**

(i) Let \( A \subset \mathbb{R}^2 \) be a measurable set with \( |A| < \infty \). Then the symmetric rearrangement \( A^* \) of \( A \) is defined as \( B_r \) for some \( r \geq 0 \) having the same measure as \( A \);

\[ A^* := B_r, \quad \pi r^2 = |A|. \]

(ii) Let \( f \in L^p(\mathbb{R}^2) \) for some \( p \in [1, \infty) \). Then the symmetric-decreasing rearrangement \( f^* \) of \( f \) is defined as the following;

\[ f^*(x) := \int_0^\infty 1_{\{y \in \mathbb{R}^2 : |f(y)| > \xi\}}^*(x) \, d\xi, \quad x \in \mathbb{R}^2. \]

**Remark 2.2** It is easy to see that \( f^* \) is nonnegative, radially symmetric, non-increasing, and that each level set of \( f^* \) is the symmetric rearrangement of the level set of \( |f| \);

\[ \{ x \in \mathbb{R}^2 : f^*(x) > \alpha \} = \{ x \in \mathbb{R}^2 : |f(x)| > \alpha \}^* \quad \text{for all } \alpha > 0. \tag{2.1} \]

So, we have

\[ |\{ x \in \mathbb{R}^2 : f^*(x) > \alpha \}| = |\{ x \in \mathbb{R}^2 : |f(x)| > \alpha \}| \quad \text{for all } \alpha > 0. \tag{2.2} \]

Thus, the \( L^p \)-norm is preserved;

\[ \| f^* \|_{L^p} = \| f \|_{L^p}. \tag{2.3} \]

From now on, we simply call \( A^* \) and \( f^* \) the rearrangement of \( A \) and \( f \), respectively.

**Remark 2.3** If a nonnegative \( f \in L^\infty(\mathbb{R}^2) \) satisfies \( J(f) < \infty \), as in the setting of our theorems, then we have \( f \in L^p(\mathbb{R}^2) \) for every \( p \in [1, \infty] \) because of

\[ \| f \|_{L^1} = \int_D f \, dx + \int_{\mathbb{R}^2 \setminus D} f \, dx \leq \pi \| f \|_{L^\infty} + J(f), \]

and

\[ \| f \|_{L^p}^p \leq \| f \|_{L^\infty}^{p-1} \| f \|_{L^1}, \quad p \in [1, \infty). \]

So the rearrangement \( f^* \) is well-defined.

In this section, the basic property (1.6) of \( f^* \) is mainly used.
2.1 Existence and uniqueness

We introduce two admissible classes and the corresponding variational problems.

**Definition 2.4** We define admissible classes of functions by

\[ P = \{ f \in L^1(\mathbb{R}^2) : f = 1_{\Omega} \text{ for some measurable } \Omega \subset \mathbb{R}^2, \ J(f) < \infty, \ \| f \|_{L^1} = \pi \} , \]

\[ P' = \{ f \in L^1(\mathbb{R}^2) : 0 \leq f \leq 1, \ J(f) < \infty, \ \| f \|_{L^1} = \pi \} . \]

We also set variational problems

\[ I = \inf_{f \in P} J(f), \quad (2.4) \]

\[ I' = \inf_{f \in P'} J(f), \quad (2.5) \]

and denote sets of minimizers of the above problems by

\[ S = \{ f \in P : J(f) \leq J(g) \text{ for all } g \in P \} , \]

\[ S' = \{ f \in P' : J(f) \leq J(g) \text{ for all } g \in P' \} . \]

**Remark 2.5** Since the weight \(|x|^2\) of \(J\) is radially symmetric and monotonically increasing, it is obvious that the variational problem (2.4) has the unique minimizer \(1_D\), that is,

\[ S = \{ 1_D \} \quad \text{and} \quad I = J(1_D) = \frac{\pi}{2}. \quad (2.6) \]

The following lemma says that the characteristic function \(1_D\) of the unit disk is a minimizer even in the larger class \(P'\).

**Lemma 2.6** *(Existence)*

\[ I = I' \quad \text{and} \quad 1_D \in S'. \]

This lemma implies that \(S\) is contained in \(S'\).

**Proof** By (2.6) and by \(P \subset P'\), it’s enough to show

\[ J(1_D) \leq J(f) \text{ for all } f \in P'. \quad (2.7) \]

Without loss of generality, we may assume \(f = f^*\), thanks to the basic property (1.6).

To show (2.7), for each \(n \in \mathbb{N}\), we define level sets \(A_k^{(n)}\) of \(f\) for \(k = 1, \ldots, n\) by

\[ A_k^{(n)} = \left\{ x \in \mathbb{R}^2 : f(x) > \frac{k}{n} \right\} , \quad (2.8) \]

and set the simple function \(g^{(n)}\) by

\[ g^{(n)} = \sum_{k=1}^{n-1} \frac{1}{n} 1_{A_k^{(n)}}. \quad (2.9) \]

Then it forms a sequence of simple functions \(\{g^{(n)}\}_{n=1}^{\infty}\) that is dominated by \(f\) and converges to \(f\) pointwise. Thus, \(g^{(n)}\) satisfies

\[ \| g^{(n)} \|_{L^1} \to \| f \|_{L^1} = \pi \quad \text{as} \quad n \to \infty. \quad (2.10) \]
For each $n \in \mathbb{N}$, we set $\bar{r}^{(n)} \geq 0$ by
\[ |B_{\bar{r}^{(n)}}| = \|g^{(n)}\|_{L^1}. \] (2.11)

Then by (2.10), we have
\[ \bar{r}^{(n)} \to 1 \quad \text{as} \quad n \to \infty. \]

We claim for each $n$, we have
\[ \mathcal{J}(1_{B_{\bar{r}^{(n)}}}) \leq \mathcal{J}(g^{(n)}). \] (2.12)

Once this claim is shown, then taking $n \to \infty$ on both sides of (2.12) gives us (2.7).

Let’s prove the above claim. We fix $n \in \mathbb{N}$ and for simplicity, we drop the parameter $n$;
\[ g = g^{(n)}, \quad A_k = A_k^{(n)}, \quad \bar{r} = \bar{r}^{(n)}. \]

Because of the property (2.1) and $f = f^*$ being radially symmetric and non-increasing, we have
\[ A_k \supset A_{k+1}, \quad k = 1, \ldots, n - 1, \]
\[ A_k = B_{s_k} \quad \text{for some} \quad s_k \geq 0 \quad \text{satisfying} \quad \pi s_k^2 = |A_k|, \quad k = 1, \ldots, n. \]

In particular, we have $s_n = 0$ because $A_n$ is the empty set. Furthermore, $g$ can be rewritten as
\[ g = \sum_{k=1}^{n-1} h_k, \] (2.13)

where $h_k$ for each $k = 1, \ldots, n - 1$ is given as
\[ h_k = \frac{k}{n} 1_{A_k \setminus A_{k+1}} = \frac{k}{n} 1_{\{y \in \mathbb{R}^2 : s_k \leq |y| < s_{k+1}\}}. \]

Then $h_k$ has magnitude $\frac{k}{n} \leq 1$ and satisfies
\[ \|h_k\|_{L^1} = \frac{k}{n} \int_{A_k \setminus A_{k+1}} 1 \, dx = \frac{\pi k}{n} (s_k^2 - s_{k+1}^2), \]
\[ J(h_k) = \frac{k}{n} \int_{A_k \setminus A_{k+1}} |x|^2 \, dx = \frac{2\pi k}{n} \int_{s_k}^{s_{k+1}} r^3 \, dr = \frac{\pi k}{2n} (s_k^4 - s_{k+1}^4). \]

Additionally, we define a function $h'_k$ by
\[ h'_k = 1_{\{y \in \mathbb{R}^2 : s_k \leq |y| < c_k\}}, \]
where $c_k \geq 0$ is chosen to satisfy
\[ \|h_k\|_{L^1} = \|h'_k\|_{L^1}, \] (2.14)

that is,
\[ \frac{k}{n} \left| \{y \in \mathbb{R}^2 : s_{k+1} \leq |y| < s_k\} \right| = \left| \{y \in \mathbb{R}^2 : s_{k+1} \leq |y| < c_k\} \right|. \]

More specifically, we have
\[ \frac{\pi k}{n} (s_k^2 - s_{k+1}^2) = \pi (c_k^2 - s_{k+1}^2), \] (2.15)
which gives us

\[ c_k = \sqrt{\left(1 - \frac{k}{n}\right) s_{k+1}^2 + \frac{k}{n} s_k^2}. \]  

(2.16)

We observe that \( h_k' \) is the function which has greater magnitude compared to \( h_k \), the annulus with smaller outer radius, yet has the same inner radius and \( L^1 \)-norm. Then we have

\[ J(h_k) \geq J(h_k'). \]  

(2.17)

Indeed, this holds because using (2.15) and (2.16), we have

\[
J(h_k) - J(h_k') = \frac{\pi k}{2n} (s_k^4 - s_{k+1}^4) - \frac{\pi}{2} (c_k^4 - s_{k+1}^4)
\]

\[
= \frac{\pi k}{2n} (s_k^2 + s_{k+1}^2) (s_k^2 - s_{k+1}^2) - \frac{1}{2} \left[ \left(1 - \frac{k}{n}\right) s_{k+1}^2 + \frac{k}{n} s_k^2 + s_{k+1}^2 \right] \frac{\pi k}{n} (s_k^2 - s_{k+1}^2)
\]

\[
= \frac{\pi k}{2n} \left(1 - \frac{k}{n}\right) (s_k^2 - s_{k+1}^2)^2 \geq 0.
\]

Since \( h_k' \) is a characteristic function of an annulus for each \( k = 1, \cdots, n-1 \), the rearrangement of the summation \( \sum_{k=1}^{n-1} h_k' \) is the characteristic function of some disk; there exists \( \widehat{\tau} = \widehat{\tau}(n) \geq 0 \) such that

\[ 1_{B_{\widehat{\tau}}} = \left( \sum_{k=1}^{n-1} h_k' \right)^* \]

Since the annuli \( \{ s_{k+1} \leq |x| < s_k \} \) are pairwise disjoint and the annuli \( \{ s_{k+1} \leq |x| < c_k \} \) satisfy the same as well, we get \( \varphi = \widehat{\varphi} \), where \( \varphi \) is from the equation (2.11). Indeed, together with the preservation property (2.3) and the equation (2.14), we have

\[ |B_{\widehat{\tau}}| = \|g\|_{L^1} = \left| \sum_{k=1}^{n-1} h_k \right|_{L^1} = \left| \sum_{k=1}^{n-1} h_k' \right|_{L^1} = \left| \sum_{k=1}^{n-1} h_k' \right|_{L^1} = \left| \sum_{k=1}^{n-1} h_k' \right|_{L^1} = |B_{\widehat{\tau}}|. \]

Finally by the basic property (1.6), the equation (2.13), (2.17), and the linearity of \( J \), we obtain (2.12);

\[ J(g) = J \left( \sum_{k=1}^{n-1} h_k \right) = \sum_{k=1}^{n-1} J(h_k) \geq \sum_{k=1}^{n-1} J(h_k') = J \left( \sum_{k=1}^{n-1} h_k' \right) \geq J \left( \left( \sum_{k=1}^{n-1} h_k' \right)^* \right) = J(1_{B_{\widehat{\tau}}}). \]

Next, we show that \( 1_D \) is the unique minimizer of the variational problem (2.5).

Proposition 2.7 (Uniqueness)

\[ S' = \{1_D\}. \]

Proof The equation (2.6) and Lemma 2.6 show \( S = \{1_D\} \subset S' \), so it suffices to show \( S' \subset S \).

We let \( \phi \in S' \). Due to \( I' = I \) by Lemma 2.6, we want

\[ \phi \in P. \]
To prove this, we need to show
\[ \left| \{ x \in \mathbb{R}^2 : 0 < \phi(x) < 1 \} \right| = 0. \]
For a contradiction, we suppose \[ \left| \{ x \in \mathbb{R}^2 : 0 < \phi(x) < 1 \} \right| > 0. \] Then there exists \( \delta > 0 \) such that
\[ \left| \{ x \in \mathbb{R}^2 : \delta \leq \phi(x) \leq 1 - \delta \} \right| > 0. \] (2.18)
We let
\[ A = \{ x \in \mathbb{R}^2 : \delta \leq \phi(x) \leq 1 - \delta \}. \]
Then we have \( 0 < |A| < \infty \) and \( 0 < J(1_A) < \infty \) because \( \phi \) lies on \( P' \). To begin with, we fix a function \( h \in L^\infty(\mathbb{R}^2) \) that satisfies
\[ \text{supp} \ (h) \subset A, \quad \int_{\mathbb{R}^2} h dx = 1, \quad J(h) = 0. \] (2.19)
For instance, we can construct such \( h \) explicitly in the following way: First, take positive real numbers \( 0 < r_1 < r_2 < \infty \) such that \( 0 < |A \cap B_{r_1}| < \infty, \ 0 < |A \setminus B_{r_2}| < \infty. \)
Second, take \( c_1, c_2 \in \mathbb{R} \) that satisfy
\[ c_1 |A \cap B_{r_1}| + c_2 |A \setminus B_{r_2}| = 1, \quad c_1 J(1_{A \cap B_{r_1}}) + c_2 J(1_{A \setminus B_{r_2}}) = 0. \] (2.20)
Such \( c_1, c_2 \) always exist because the linear system (2.20) has a non-singular matrix;
\[
\begin{bmatrix}
|A \cap B_{r_1}| & |A \setminus B_{r_2}|
\end{bmatrix}
\begin{bmatrix}
J(1_{A \cap B_{r_1}}) \\
J(1_{A \setminus B_{r_2}})
\end{bmatrix}
= |A \cap B_{r_1}| \int_{A \setminus B_{r_2}} |x|^2 dx - |A \setminus B_{r_2}| \int_{A \cap B_{r_1}} |x|^2 dx
\geq (r_2^2 - r_1^2) |A \cap B_{r_1}| |A \setminus B_{r_2}| > 0.
\]
Finally, define \( h \) as
\[ h(x) = \begin{cases} 
  c_1 & \text{if } x \in A \cap B_{r_1}, \\
  c_2 & \text{if } x \in A \setminus B_{r_2}, \\
  0 & \text{otherwise.}
\end{cases} \]
Then we see such \( h \) satisfies the condition (2.19).
Now we define \( \eta \in L^\infty(\mathbb{R}^2) \) by
\[ \eta = 1_A - |A|h. \]
Then \( \eta \) is supported on \( A \) and it satisfies
\[ \int_{\mathbb{R}^2} \eta dx = |A| - |A| \int_{\mathbb{R}^2} h dx = 0, \]
\[ J(\eta) = J(1_A) - |A| J(h) = J(1_A) \in (0, \infty). \] (2.21)
Now we take \( \varepsilon_0 > 0 \) small enough such that \( \varepsilon_0 \|\eta\|_{L^\infty} \leq \delta \). Then we have
\[ 0 \leq (\phi - \varepsilon_0 \eta) \leq 1. \]
In addition, we also have
\[ J(\phi - \varepsilon_0 \eta) = J(\phi) - \varepsilon_0 J(\eta) < \infty, \]
\[ \int_{\mathbb{R}^2} (\phi - \varepsilon_0 \eta) \, dx = \int_{\mathbb{R}^2} \phi \, dx = \pi, \]
which shows \((\phi - \varepsilon_0 \eta) \in P^\prime\). Then due to (2.21) and the initial assumption \(\phi \in S^\prime\), we have
\[ J(\phi) \leq J(\phi - \varepsilon_0 \eta) = J(\phi) - \varepsilon_0 J(\eta) = J(\phi) - \varepsilon_0 J(1_A). \]
This implies
\[ -\varepsilon_0 J(1_A) \geq 0, \]
which gives us \(J(1_A) = 0\). Thus, we have \(|A| = 0\), which is a contradiction to (2.18). \(\square\)

2.2 Compactness

In this subsection, we prove that a sequence of functions which has norm convergences with
the convergence
\[ J(f_n) \longrightarrow J(1_D) \]
should have strong convergences.

Proposition 2.8 Let \(\{f_n\}_{n=1}^\infty \subset L^\infty(\mathbb{R}^2)\) be a sequence of nonnegative functions such that
\[ \int_{\{x \in \mathbb{R}^2 : f_n(x) \geq 1 + a_n\}} f_n \, dx \longrightarrow 0 \quad \text{for some sequence} \quad \{a_n\}_{n=1}^\infty \subset \mathbb{R}_{>0} \text{ satisfying} \ a_n \searrow 0, \tag{2.22} \]
\[ \|f_n\|_{L^1} \longrightarrow \|1_D\|_{L^1}, \tag{2.23} \]
\[ \|f_n\|_{L^2} \longrightarrow \|1_D\|_{L^2}, \tag{2.24} \]
\[ J(f_n) \longrightarrow J(1_D) \quad \text{as} \quad n \longrightarrow \infty. \tag{2.25} \]

Then \(\{f_n\}\) satisfies
\[ \|f_n - 1_D\|_{L^1} + \|f_n - 1_D\|_{L^2} + J(|f_n - 1_D|) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \]

Proof Suppose that the conclusion is false, that is, there exists \(\varepsilon_0 > 0\) and a subsequence \(\{f_{n_j}\}_{j=1}^\infty\) of \(\{f_n\}_{n=1}^\infty\) such that
\[ \|f_{n_j} - 1_D\|_{L^1} + \|f_{n_j} - 1_D\|_{L^2} + J(|f_{n_j} - 1_D|) \geq \varepsilon_0 \quad \text{for all} \quad j \in \mathbb{N}. \tag{2.26} \]
Now we show that there exists a subsequence \(\{f_{n_{jk}}\}_{k=1}^\infty\) of \(\{f_{n_j}\}_{j=1}^\infty\) such that
\[ \|f_{n_{jk}} - 1_D\|_{L^1} + \|f_{n_{jk}} - 1_D\|_{L^2} + J(|f_{n_{jk}} - 1_D|) \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \]
Once this is shown, then it contradicts (2.26), completing the proof.

Due to
\[ \sup_{j \in \mathbb{N}} \|f_{n_j}\|_{L^2} < \infty, \]

\(\square\)
which comes from the $L^2$-norm convergence (2.24), there exists $f \in L^2(\mathbb{R}^2)$ and a subsequence $\{f_{n_j}\}_{j=1}^\infty$ of $\{f_{n_j}\}_{j=1}^\infty$ such that

$$f_{n_j} \rightharpoonup f \quad \text{in} \quad L^2(\mathbb{R}^2) \quad \text{as} \quad k \longrightarrow \infty. \quad (2.27)$$

For notational convenience, we simply denote the subsequence $\{f_{n_j}\}_{j=1}^\infty$ by $\{f_n\}$. The proof is done in 8 steps:

**Step 1.** $0 \leq f \leq 1$.

**Step 2.** $\|f\|_{L^1} \leq \|1_D\|_{L^1}$ and $J(f) \leq J(1_D)$.

**Step 3.** $\|f\|_{L^1} = \|1_D\|_{L^1}$.

**Step 4.** $J(f) = J(1_D)$.

**Step 5.** $f = 1_D$.

**Step 6.** $f_n \longrightarrow 1_D$ in $L^2$ as $n \longrightarrow \infty$.

**Step 7.** $J(|f_n - 1_D|) \longrightarrow 0$ as $n \longrightarrow \infty$.

**Step 8.** $f_n \longrightarrow 1_D$ in $L^1$ as $n \longrightarrow \infty$.

Step 1, 2, and 3 are to show $f \in P'$. Then Step 4 and Proposition 2.7 show Step 5. This step confirms that the $L^2$-weak limit $f$ is indeed the unique minimizer $1_D$ in $P'$. The remaining steps finish our proof.

**Step 1.** $0 \leq f \leq 1$.

We trivially have $f \geq 0$ due to $f_n \geq 0$ for each $n \in \mathbb{N}$. To prove $f \leq 1$, we show

$$|\{x \in \mathbb{R}^2 : f(x) > 1\}| = 0.$$

For a contradiction, we suppose

$$|\{x \in \mathbb{R}^2 : f(x) > 1\}| > 0.$$

Then there exists $\delta > 0$ such that

$$0 < |\{x \in \mathbb{R}^2 : f(x) \geq 1 + \delta\}| < \infty, \quad (2.28)$$

where the measure is finite because of $f \in L^2(\mathbb{R}^2)$. We denote

$$C = \{x \in \mathbb{R}^2 : f(x) \geq 1 + \delta\}, \quad E_n = \{x \in \mathbb{R}^2 : f_n(x) \geq 1 + a_n\}.$$

Then we have

$$\int_C f\,dx \geq \int_C (1 + \delta)\,dx = (1 + \delta)|C|,$$

and

$$\int_C f_n\,dx = \int_{C\cap E_n} f_n\,dx + \int_{C\setminus E_n} f_n\,dx \leq \int_{E_n} f_n\,dx + \int_{C\setminus E_n} (1 + a_n)\,dx \leq \int_{E_n} f_n\,dx + (1 + a_n)|C|.$$

Then we obtain

$$\int_C (f_n - f)\,dx \leq \int_{E_n} f_n\,dx + (a_n - \delta)|C|. \quad (2.29)$$

Then the left-hand side of (2.29) goes to 0 because of the $L^2$-weak convergence (2.27) and the finite-measured set (2.28), and the right-hand side of (2.29) converges to $-\delta|C|$ due to (2.22) as $n \longrightarrow \infty$. This gives a contradiction to (2.28).

**Step 2.** $\|f\|_{L^1} \leq \|1_D\|_{L^1}$, $J(f) \leq J(1_D)$.
We observe
\[
\int_{\mathbb{R}^2} f_n 1_{B_r} \, dx = \int_{B_r} f_n \, dx \leq \|f_n\|_{L^1}, \quad r > 0.
\]  
(2.30)

Then because of the $L^1$-norm convergence (2.23) and the $L^2$-weak convergence (2.27), we have by taking $n \to \infty$ on both sides of (2.30),
\[
\int_{B_r} f \, dx \leq \|1_D\|_{L^1}, \quad r > 0.
\]
Then we have
\[
\|f\|_{L^1} = \lim_{r \to \infty} \int_{\mathbb{R}^2} f 1_{B_r} \, dx = \lim_{r \to \infty} \int_{B_r} f \, dx \leq \|1_D\|_{L^1}.
\]

Similarly, by replacing $1_{B_r}$ with $|x|^2 1_{B_r} \in L^2(\mathbb{R}^2)$, the convergence (2.25) of angular impulse and the $L^2$-weak convergence (2.27) produce
\[
J(f) \leq J(1_D).
\]
(2.31)

**Step 3.** $\|f\|_{L^1} = \|1_D\|_{L^1}$.  
To show this, we fix $\varepsilon > 0$. Then because of $f \in L^1(\mathbb{R}^2)$, there exists $R_1 > 0$ such that
\[
\int_{\mathbb{R}^2 \setminus B_R} f \, dx \leq \varepsilon \quad \text{for all} \quad R \geq R_1.
\]
(2.32)
Furthermore, by the convergence (2.25) of angular impulse, we have
\[
\sup_{n \in \mathbb{N}} J(f_n) < \infty.
\]
Let’s denote $M = \sup_{n \in \mathbb{N}} J(f_n)$. Then due to
\[
M \geq J(f_n) \geq \int_{\mathbb{R}^2 \setminus B_r} |x|^2 f_n \, dx \geq r^2 \int_{\mathbb{R}^2 \setminus B_r} f_n \, dx \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad r > 0,
\]
we get
\[
\frac{M}{r^2} \geq \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2 \setminus B_r} f_n \, dx \quad \text{for all} \quad r > 0.
\]
So choosing $R_2 \geq \sqrt{\frac{M}{\varepsilon}}$, we have
\[
\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2 \setminus B_R} f_n \, dx \leq \varepsilon \quad \text{for all} \quad R \geq R_2.
\]
(2.33)

We take $R = \max \{R_1, R_2\}$. Then because of $1_{B_R} \in L^2(\mathbb{R}^2)$ and the $L^2$-weak convergence (2.27), there exists $N \in \mathbb{N}$ such that
\[
\left| \int_{B_R} f_n \, dx - \int_{B_R} f \, dx \right| \leq \varepsilon \quad \text{for all} \quad n \geq N.
\]
(2.34)

By collecting all the estimates (2.32), (2.33), and (2.34), we have, for all $n \geq N$,
\[
\left| \|f_n\|_{L^1} - \|f\|_{L^1} \right| \leq \left| \int_{B_R} f_n \, dx - \int_{B_R} f \, dx \right| + \left| \int_{\mathbb{R}^2 \setminus B_R} f_n \, dx - \int_{\mathbb{R}^2 \setminus B_R} f \, dx \right|
\]
\[
\left| \int_{B_R} f_n \, dx - \int_{B_R} f \, dx \right| + \int_{\mathbb{R}^2 \setminus B_R} f_n \, dx + \int_{\mathbb{R}^2 \setminus B_R} f \, dx
\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.
\]

Hence by the \(L^1\)-norm convergence (2.23), we have
\[
\| f \|_{L^1} = \lim_{n \to \infty} \| f_n \|_{L^1} = \| 1_D \|_{L^1}.
\]

**Step 4.** \(J(f) = J(1_D)\).

Step 1, 2, and 3 show
\[
f \in P'.
\]
(2.35)

Then by Proposition 2.7, we have
\[
J(f) \geq I' = J(1_D).
\]

Then by the inequality (2.31) in Step 2, we obtain
\[
J(f) = J(1_D).
\]
(2.36)

**Step 5.** \(f = 1_D\).

By (2.35) and by (2.36), Proposition 2.7 implies
\[
f \in S' = \{ 1_D \}.
\]
(2.37)

Up to now, we have shown that the \(L^2\)-weak limit \(f\) is exactly \(1_D\). From now on, we show that the weak limit is, in fact, the strong limit.

**Step 6.** \(f_n \rightharpoonup 1_D\) in \(L^2\) as \(n \to \infty\).

From the \(L^2\)-weak convergence (2.27) and (2.37) in Step 5, we have
\[
f_n \rightharpoonup 1_D \quad \text{in} \quad L^2 \quad \text{as} \quad n \to \infty.
\]

Then this together with the \(L^2\)-norm convergence (2.24) gives us
\[
f_n \to 1_D \quad \text{in} \quad L^2 \quad \text{as} \quad n \to \infty.
\]
(2.38)

**Step 7.** \(J(|f_n - 1_D|) \to 0\) as \(n \to \infty\).

To show this, we split the range of the integral of \(J(|f_n - 1_D|)\) into \(D\) and \(\mathbb{R}^2 \setminus D\):
\[
J(|f_n - 1_D|) = \int_D |x|^2 f_n - 1_D |dx + \int_{\mathbb{R}^2 \setminus D} |x|^2 f_n dx.
\]

For \((I)\), using the Hölder’s inequality, by the strong \(L^2\)-convergence (2.38) in Step 6, we have
\[
(I) \leq \left( \int_{\mathbb{R}^2} |f_n - 1_D|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |x|^4 |1_D|^2 \, dx \right)^{\frac{1}{2}} \leq C \| f_n - 1_D \|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.
\]

To show the same holds for \((II)\), we use the following decomposition;
\[
\int_{\mathbb{R}^2 \setminus D} |x|^2 f_n \, dx = \left( J(f_n) - J(1_D) \right) - \left( \int_D |x|^2 f_n \, dx - \int_D |x|^2 1_D \, dx \right).
\]
Again using the Hölder’s inequality, by the convergence (2.25) of angular impulse and by (2.38), we have

\[(II) \leq |J(f_n) - J(1_D)| + \left| \int_D |x|^2 f_n dx - \int_D |x|^2 1_D dx \right| \]

\[\leq |J(f_n) - J(1_D)| + C \|f_n - 1_D\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.\]

Therefore, we have \(J(|f_n - 1_D|) \to 0\) as \(n \to \infty\).

**Step 8.** \(f_n \to 1_D\) in \(L^1\) as \(n \to \infty\).

By the Hölder’s inequality, the \(L^1\)-norm can be controlled by the \(J_2\)-norm as the following:

\[\|g\|_{L^1} = \int_D |g| dx + \int_{\mathbb{R}^2 \setminus D} |g| dx \leq \sqrt{\pi} \|g\|_{L^2} + J(|g|) \leq \pi \|g\|_{J_2}, \quad g \in L^2, \quad J(|g|) < \infty.\]

(2.39)

Then by Step 6, 7, and the above with \(g = f_n - 1_D\), we have \(\|f_n - 1_D\|_{L^1} \to 0\) as \(n \to \infty\). \(\square\)

### 2.3 Contradiction argument

**Proof of Theorem 1.1** We suppose that the conclusion is false. Then there exists \(\varepsilon_0 > 0\) such that there exists a sequence of nonnegative initial data \(\{\omega_{n,0}\}_{n=1}^{\infty} \subset L^\infty(\mathbb{R}^2)\) with \(J(\omega_{n,0}) < \infty\) and a sequence \(\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}_{\geq 0}\) such that for each \(n \in \mathbb{N}\), we have

\[\|\omega_{n,0} - 1_D\|_{J_2} \leq \frac{1}{n},\]  

(2.40)

but

\[\|\omega_{n}(t_n) - 1_D\|_{J_2} \geq \varepsilon_0,\]  

(2.41)

where \(\omega_n(t)\) is the solution of (1.1) for \(\omega_{n,0}\). First, we recall that for each \(n\), the \(L^p\)-norm for \(p \in [1, \infty]\) and \(J(\omega_{n}(t))\) are preserved in time. This gives us

\[\|\omega_{n}(t_n)\|_{L^1} = \|\omega_{n,0}\|_{L^1}, \quad \|\omega_{n}(t_n)\|_{L^2} = \|\omega_{n,0}\|_{L^2}, \quad J(\omega_{n}(t_n)) = J(\omega_{n,0}).\]  

(2.42)

Also we recall that for each \(n\), the corresponding flow for \(\omega_{n}(t)\) preserves the measure of each level set in time. Thus, \(\omega_{n}(t_n)\) is nonnegative and we get

\[\int_{\{\omega_{n}(t_n) \geq 1 + \frac{\varepsilon_0}{\sqrt{\pi}}\}} \omega_{n}(t_n) dx = \int_{\{\omega_{n,0} \geq 1 + \frac{\varepsilon_0}{\sqrt{\pi}}\}} \omega_{n,0} dx.\]  

(2.43)

For notational convenience, we denote \(\omega_n = \omega_{n}(t_n)\). By the estimate (2.39) and (2.40) above, we have

\[\|\omega_{n,0} - 1_D\|_{L^1} \leq \pi \\|\omega_{n,0} - 1_D\|_{J_2} \leq \frac{\pi}{n}.\]  

(2.44)

Then by (2.40), (2.42), and (2.44), we get

\[\|\omega_n\|_{L^1} = \|\omega_{n,0}\|_{L^1} \to \|1_D\|_{L^1},\]

\[\|\omega_n\|_{L^2} = \|\omega_{n,0}\|_{L^2} \to \|1_D\|_{L^2},\]  

(2.45)

\[J(\omega_n) = J(\omega_{n,0}) \to J(1_D) \quad \text{as} \quad n \to \infty.\]
We claim
\[ \int_{\{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\}} \omega_n,0 \, dx \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \] (2.46)

Indeed, we split the range of the above integral;
\[
\int_{\{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\}} \omega_n,0 \, dx = \int_{\{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \cap D} \omega_n,0 \, dx + \int_{\{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \setminus D} \omega_n,0 \, dx.
\]

For (II), by the \(L^1\)-convergence (2.44), we have
\[
(II) \leq \int_{\mathbb{R}^2 \setminus D} \omega_n,0 \, dx \leq \|\omega_n,0 - 1_D\|_{L^1} \leq \frac{\pi}{n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

To prove the similar holds for (I), we use the \(L^1\)-convergence (2.44) once more;
\[
\frac{\pi}{n} \geq \|\omega_n,0 - 1_D\|_{L^1} \geq \int_D |\omega_n,0 - 1| \, dx \geq \int_{\{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \cap D} |\omega_n,0 - 1| \, dx \geq \frac{1}{\sqrt{n}} \left| \{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \cap D \right|,
\]
which gives us
\[
\left| \{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \cap D \right| \leq \frac{\pi}{\sqrt{n}}.
\]

Then we have
\[
(I) = \int_{\{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \cap D} (\omega_n,0 - 1_D) \, dx + \int_{\{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \setminus D} 1_D \, dx \leq \|\omega_n,0 - 1_D\|_{L^1} + \left| \{\omega_n,0 \geq 1 + \frac{1}{\sqrt{n}}\} \cap D \right| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Hence, (2.46) holds. Then by the equation (2.43), we have
\[ \int_{\{\omega_n \geq 1 + \frac{1}{\sqrt{n}}\}} \omega_n \, dx \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \] (2.47)

In sum, we obtained all the assumptions of Proposition 2.8 from (2.45) and (2.47). Therefore by the proposition, the sequence \(\{\omega_n\}_{n=1}^{\infty}\) satisfies
\[ \|\omega_n - 1_D\|_{J^2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \]
which is a contradiction to (2.41).

\[\square\]

3 Stability estimate

3.1 Rearrangement estimates

Throughout this section, we use the following fine properties.
Lemma 3.1 (Nonexpansivity of rearrangement) For nonnegative functions \( g, h \in L^1(\mathbb{R}^2) \), we have
\[
\|g^* - h^*\|_{L^1} \leq \|g - h\|_{L^1}.
\] (3.1)

The result follows from the convexity of \( |\cdot| \). For a proof, one may refer [30, Sect. 3.5] for details. The reason why we need \( \zeta \) to be radially symmetric and non-increasing in order to get stability is because it satisfies \( \zeta^* = \zeta \). Additionally, we shall use the rearrangement estimate (3.2) below, which is a sharper version of the basic property (1.6). We prove the estimate by following the spirit of [32, Lemma 1]. As our setting is the whole space \( \mathbb{R}^2 \) while the lemma of [32] was stated for Euler flows in bounded channel \( \mathbb{T} \times [0, R], R > 0 \), we present the proof below for completeness.

Lemma 3.2 Let a nonnegative function \( f \in L^\infty(\mathbb{R}^2) \) satisfy \( J(f) < \infty \) and \( f^* \) be its rearrangement. Then they satisfy
\[
\|f - f^*\|_{L^1}^2 \leq 4\pi \|f\|_{L^\infty} \left[ J(f) - J(f^*) \right].
\] (3.2)

Proof Due to \( J(af) = aJ(f) \) and \( (af)^* = af^*, a > 0 \), it’s enough to show only for the case \( \|f\|_{L^\infty} = 1 \). Then for each \( n \in \mathbb{N} \), we use similar level sets and simple functions from (2.8) and (2.9) in the proof of Lemma 2.6;
\[
A_k^{(n)} = \left\{ x \in \mathbb{R}^2 : f(x) > \frac{k}{n} \right\}, \quad C_k^{(n)} = \left\{ x \in \mathbb{R}^2 : f^*(x) > \frac{k}{n} \right\}, \quad k = 1, \ldots, n - 1,
\]
\[
\xi^{(n)} = \sum_{k=1}^{n-1} \frac{1}{n} A_k^{(n)}, \quad \eta^{(n)} = \sum_{k=1}^{n-1} \frac{1}{n} C_k^{(n)}.
\]
in which \( \xi^{(n)} \) and \( \eta^{(n)} \) are dominated by and converge to \( f \) and \( f^* \) pointwise, respectively. We claim
\[
\frac{4\pi(n - 1)}{n} \left[ J(\xi^{(n)}) - J(\eta^{(n)}) \right] \geq \left\| \xi^{(n)} - \eta^{(n)} \right\|_{L^1}^2, \quad n \in \mathbb{N}.
\] (3.3)

Once we have this, by the dominated convergence theorem, taking \( n \rightarrow \infty \) on both sides of the above claim gives us (3.2).

Let’s prove the above claim. We fix \( n \in \mathbb{N} \) and for the convenience of notation, we drop the parameter \( n \);
\[
A_k = A_k^{(n)}, \quad C_k = C_k^{(n)}, \quad \xi = \xi^{(n)}, \quad \eta = \eta^{(n)}.
\]
Additionally, we take \( s_k, \beta_k \geq 0 \) satisfying
\[
C_k = B_{s_k}, \quad \beta_k = \frac{1}{2} \left\| 1_{A_k} - 1_{C_k} \right\|_{L^1} = \frac{1}{2} |A_k \triangle C_k|, \quad k = 1, \ldots, n - 1.
\]

We recall \( |A_k| = |C_k| \) for each \( k \), due to the property (2.2). This gives us
\[
|A_k \setminus C_k| = |C_k \setminus A_k| = \frac{1}{2} |A_k \triangle C_k| = \beta_k, \quad k = 1, \ldots, n - 1.
\] (3.4)
Now we compute

\[ J(\xi) - J(\eta) = \frac{1}{n} \sum_{k=1}^{n-1} \left( \int_{A_k} |x|^2 \, dx - \int_{C_k} |x|^2 \, dx \right) = \frac{1}{n} \sum_{k=1}^{n-1} \left( \int_{A_k \setminus C_k} |x|^2 \, dx - \int_{C_k \setminus A_k} |x|^2 \, dx \right) \]

\[ \geq \frac{1}{n} \sum_{k=1}^{n-1} \left( \int_{\Sigma_{k,1}} |x|^2 \, dx - \int_{\Sigma_{k,2}} |x|^2 \, dx \right), \]

where for each \( k \), \( \Sigma_{k,1} \subset \mathbb{R}^2 \setminus B_{s_k} \) is the annulus that has the same measure as \( A_k \setminus C_k \) with the inner radius \( s_k \), and \( \Sigma_{k,2} \subset B_{s_k} \) is the annulus having the same measure as \( C_k \setminus A_k \) with the outer radius \( s_k \):

\[ \Sigma_{k,1} = \left\{ x \in \mathbb{R}^2 : s_k \leq |x| \leq r_{k,1} \right\}, \quad \Sigma_{k,2} = \left\{ x \in \mathbb{R}^2 : r_{k,2} \leq |x| \leq s_k \right\} . \]

with \( r_{k,1} \geq s_k \geq r_{k,2} \geq 0 \) satisfying

\[ |\Sigma_{k,1}| = |A_k \setminus C_k|, \quad |\Sigma_{k,2}| = |C_k \setminus A_k|. \]

Thus, by (3.4), we have

\[ |\Sigma_{k,1}| = |\Sigma_{k,2}| = \beta_k, \]

in which a further calculation gives us

\[ r_{k,1} = \sqrt{s_k^2 + \frac{\beta_k}{\pi}}, \quad r_{k,2} = \sqrt{s_k^2 - \frac{\beta_k}{\pi}}. \]

Then we have

\[ J(\xi) - J(\eta) \geq \frac{1}{n} \sum_{k=1}^{n-1} \left( \int_{\Sigma_{k,1}} |x|^2 \, dx - \int_{\Sigma_{k,2}} |x|^2 \, dx \right) = \frac{2\pi}{n} \sum_{k=1}^{n-1} \left( \int_{s_k}^{r_{k,1}} \rho^2 \, d\rho - \int_{r_{k,2}}^{s_k} \rho^2 \, d\rho \right) \]

\[ = \frac{\pi}{2n} \sum_{k=1}^{n-1} (r_{k,1}^4 - r_{k,2}^4 - 2s_k^4) = \frac{\pi}{2n} \sum_{k=1}^{n-1} \left( s_k^4 + \frac{\beta_k}{\pi} \right) \left( s_k^4 - \frac{\beta_k}{\pi} \right) \]

\[ = \frac{1}{4\pi n} \sum_{k=1}^{n-1} \beta_k^2 = \frac{1}{4\pi n} \sum_{k=1}^{n-1} \left\| 1_{A_k} - 1_{C_k} \right\|_{L^1}^2. \]  \hspace{1cm} (3.5)

On the other hand, using the Cauchy-Schwartz inequality, we have

\[ \|\xi - \eta\|_{L^1} = \int_{\mathbb{R}^2} \left\| \frac{1}{n} \sum_{k=1}^{n-1} (1_{A_k} - 1_{C_k}) \right\|_{L^1} \, dx \]

\[ \leq \frac{1}{n} \sum_{k=1}^{n-1} \left\| 1_{A_k} - 1_{C_k} \right\|_{L^1} \leq \sqrt{n-1} \left( \frac{1}{n} \sum_{k=1}^{n-1} \left\| 1_{A_k} - 1_{C_k} \right\|_{L^1}^2 \right)^{\frac{1}{2}}, \]

which gives

\[ \sum_{k=1}^{n-1} \left\| 1_{A_k} - 1_{C_k} \right\|_{L^1}^2 \geq \frac{n^2}{n-1} \|\xi - \eta\|_{L^1}^2. \]  \hspace{1cm} (3.6)

Collecting (3.5) and (3.6), we have (3.3).

\[ \square \]
3.2 Stability in $L^1$

We first prove $L^1$-stability. We note that the estimate below does not depend on $\|\omega_0\|_{L^\infty}$.

**Lemma 3.3** Let $R, M, \alpha \in (0, \infty)$. Then there exist constants $C_1 = C_1(R, M, \alpha) > 0$ and $C_2 = C_2(M) > 0$ such that if $\xi \in L^\infty(\mathbb{R}^2)$ is nonnegative, radially symmetric, non-increasing with $J(\xi) < \infty$, and satisfies

$$\|\xi\|_{L^\infty} \leq M, \quad |\xi|_{L^1} \leq \alpha,$$

then for any nonnegative $\omega_0 \in L^\infty(\mathbb{R}^2)$ with $J(\omega_0) < \infty$, the corresponding solution $\omega(t)$ of (1.1) satisfies

$$\sup_{t \geq 0} \|\omega(t) - \xi\|_{L^1} \leq C_1 \left[ \|\omega_0 - \xi\|_{L^1}^2 + \|\omega_0 - \xi\|_{L^1} + J(|\omega_0 - \xi|) \right] + C_2 \left( \int_{\mathbb{R}^2 \setminus B_R} |\xi|^2 \right)^{\frac{1}{2}}.$$

(3.7)

**Proof** During the proof, for any nonnegative function $f \in L^\infty(\mathbb{R}^2)$, we define a set $A_f \subset \mathbb{R}^2$ by

$$A_f := \{x \in \mathbb{R}^2 : f(x) \leq M + 1\},$$

and a function $\tilde{f} \in L^\infty(\mathbb{R}^2)$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A_f \\ M + 1 & \text{if } x \in \mathbb{R}^2 \setminus A_f. \end{cases}$$

(3.9)

Then we have

$$\int_{\mathbb{R}^2 \setminus A_{\omega_0}} |f(x) - \xi(x)| dx \leq \int_{\mathbb{R}^2 \setminus A_{\omega_0}} |f(x) - \tilde{f}(x)| dx + \int_{\mathbb{R}^2 \setminus A_{\omega_0}} \|f(x) - \tilde{f}(x)\|_{L^1} dx.$$

(3.10)

where the first equality holds due to the conservation of level set measure (1.2). Let’s fix $t \geq 0$ and for simplicity, we drop the parameter $t$;

$$\omega = \omega(t), \quad A_\omega = A_{\omega(t)}, \quad \tilde{\omega} = \tilde{\omega}(t).$$

Then we have

$$\|\omega - \xi\|_{L^1} \leq \int_{A_\omega} |\tilde{\omega} - \xi| dx + \int_{\mathbb{R}^2 \setminus A_\omega} |\omega - \xi| dx \leq \|\tilde{\omega} - \xi\|_{L^1} + \int_{\mathbb{R}^2 \setminus A_\omega} \omega dx + \int_{\mathbb{R}^2 \setminus A_\omega} \xi dx \leq \|\tilde{\omega} - (\tilde{\omega})^*\|_{L^1} + \|\tilde{\omega} - \xi\|_{L^1} + \int_{\mathbb{R}^2 \setminus A_{\omega_0}} \omega dx + \int_{\mathbb{R}^2 \setminus A_{\omega_0}} \xi dx.$$

(III) $\leq \int_{\mathbb{R}^2 \setminus A_{\omega_0}} |\omega_0 - \xi| dx + \int_{\mathbb{R}^2 \setminus A_{\omega_0}} \xi dx$

To estimate (III) and (IV), using the estimate (3.10) and the Hölder’s inequality, we have

(3.10)
Then we split

\[(IV) \leq \|\xi\|_{L^2}^{2} A_{\omega} \|\xi\|_{L^2}^{2} \leq \|\omega_{0} - \xi\|_{L^1} + \|\xi\|_{L^2} \|\omega_{0} - \xi\|_{L^1}^{1}\]

Thus, we get

\[(III) + (IV) \leq \|\omega_{0} - \xi\|_{L^1} + 2 \|\xi\|_{L^2} \|\omega_{0} - \xi\|_{L^1} \leq \|\omega_{0} - \xi\|_{L^1} + 2 \sqrt{MAN} \|\omega_{0} - \xi\|_{L^1}^{1}\]

To estimate (II), we observe the fact

\[\omega^{*} = (\omega_{0})^{*},\]

because both \(\omega\) and \(\omega_{0}\) have the same measure for each level set. In other words, functions with same level set measure have the same rearrangement. Similarly, we have

\[(\tilde{\omega})^{*} = (\tilde{\omega}_{0})^{*},\]

because the level set of both \(\tilde{\omega}\) and \(\tilde{\omega}_{0}\) is the empty set for \(a \geq M + 1\) while for the case \(a < M + 1\), we get

\[\left|\left\{x \in \mathbb{R}^2 : \tilde{\omega}(x) > a\right\}\right| = \left|\left\{x \in \mathbb{R}^2 : \omega(x) > a\right\}\right| = \left|\left\{x \in \mathbb{R}^2 : \omega_{0}(x) > a\right\}\right|.

Then the nonexpansivity estimate (3.1) in Lemma 3.1 and (3.13) gives

\[(II) = \left\|\tilde{\omega}_{0} - \xi\right\|_{L^1} \leq \left\|\omega_{0} - \xi\right\|_{L^1} = \int_{A_{\omega_{0}}} |\omega_{0} - \xi| dx + \int_{A_{\omega_{0}}} |(M + 1) - \xi| dx \leq \int_{A_{\omega_{0}}} |\omega_{0} - \xi| dx + \int_{A_{\omega_{0}}} |\omega_{0} - \xi| dx = \|\omega_{0} - \xi\|_{L^1}.

(3.14)

This is the first time we use the fact that \(\xi\) is radially symmetric and non-increasing so that we have \(\xi^{*} = \xi\).

To estimate (I), we recall \(\|\tilde{\omega}\|_{L^{\infty}} \leq M + 1\). Thus, the rearrangement estimate (3.2) in Lemma 3.2 says

\[(I) \leq \frac{\sqrt{4\pi (M + 1)}}{2} \left[ J(\tilde{\omega}) - J((\tilde{\omega})^{*}) \right]^{1/2}.

Then we split \(J(\tilde{\omega}) - J((\tilde{\omega})^{*})\) into 4 terms by adding and subtracting suitable terms:

\[J(\tilde{\omega}) - J((\tilde{\omega})^{*}) = \left[ J(\tilde{\omega}) - J(\omega) \right] + \left[ J(\omega) - J(\xi) \right] + \left[ J(\xi) - J((\omega_{0})^{*}) \right] + \left[ J((\omega_{0})^{*}) - J((\tilde{\omega})^{*}) \right].

The term \((I_{a})\) is nonpositive, due to \(\tilde{\omega} \leq \omega_{0}\), so it can be dropped. The estimate of \((I_{b})\) follows from the conservation of the angular impulse of \(\omega\):

\[(I_{b}) = J(\omega_{0}) - J(\xi) \leq J(|\omega_{0} - \xi|).

The estimate of \((I_{c})\) becomes, using the equation (3.12),

\[(I_{c}) = J(\xi) - J((\omega_{0})^{*}) \leq \int_{B_{R}} |x|^2 \left| \xi - (\omega_{0})^{*} \right| dx + \int_{R^{2} \setminus B_{R}} |x|^2 \left| \xi - (\omega_{0})^{*} \right| dx\]
because both functions \((\omega_0)^*\) and \((\omega_0^\bullet)^*\) are radially symmetric and non-increasing with the same measure for each level set. Indeed, if we have \(a \geq M + 1\), then each level set of both functions is the empty set and when \(a < M + 1\), we have, from the conservation of level set measure \((2.2)\),

\[
\left| \{ x \in \mathbb{R}^2 : (\tilde{\omega}_0)^* (x) > a \} \right| = \left| \{ x \in \mathbb{R}^2 : \tilde{\omega}_0 (x) > a \} \right| = \left| \{ x \in \mathbb{R}^2 : \omega_0 (x) > a \} \right| = \left| \{ x \in \mathbb{R}^2 : (\omega_0)^* (x) > a \} \right|.
\]

Now using equations \((3.12)\), \((3.13)\), and \((3.15)\), we have

\[
(I_d) = J \left( (\omega_0)^* \right) - J \left( (\omega_0^\bullet)^* \right) = J \left( (\omega_0)^* \right) - J \left( (\omega_0^\bullet)^* \right).
\]

Then splitting the integral range into \(A_{(\omega_0)^*}^1 \) and \(\mathbb{R}^2 \setminus A_{(\omega_0)^*}^1 \) for each of the last terms above, the integral terms on \(A_{(\omega_0)^*}^1 \) get cancelled out;

\[
(I_d) = \int_{A_{(\omega_0)^*}^1} |x|^2 (\omega_0)^* dx + \int_{\mathbb{R}^2 \setminus A_{(\omega_0)^*}^1} |x|^2 (\omega_0)^* dx

- \left( \int_{A_{(\omega_0^\bullet)^*}^1} |x|^2 (\omega_0^\bullet)^* dx + \int_{\mathbb{R}^2 \setminus A_{(\omega_0^\bullet)^*}^1} |x|^2 (\omega_0^\bullet)^* dx \right) \leq \int_{\mathbb{R}^2 \setminus A_{(\omega_0^\bullet)^*}^1} |x|^2 (\omega_0^\bullet)^* dx.
\]

Then using \(\mathbb{R}^2 \setminus A_{(\omega_0)^*}^1 = B_\tau \) for some \(\tau \geq 0\), the conservation of level set measure \((2.2)\), and the estimate \((3.10)\), we have

\[
\pi \tau^2 = |\mathbb{R}^2 \setminus A_{(\omega_0)^*}^1| = \left| \{ y \in \mathbb{R}^2 : (\omega_0)^* > M + 1 \} \right| = \left| \{ y \in \mathbb{R}^2 : \omega_0 > M + 1 \} \right|.
\]
Using this and the conservation of $L^1$-norm (2.3), we have

\[
(I_a) \leq \frac{72}{\pi} \int_{B_R} (|\omega_0|^\alpha \, dx) \leq \frac{1}{\pi} || \omega_0 - \xi ||_{L^1} || (\omega_0)^\alpha ||_{L^1} = \frac{1}{\pi} || \omega_0 - \xi ||_{L^1} || \omega_0 ||_{L^1}
\]

\[
\leq \frac{1}{\pi} || \omega_0 - \xi ||_{L^1}^2 + \frac{1}{\pi} || \xi ||_{L^1} || \omega_0 - \xi ||_{L^1} \leq \frac{1}{\pi} || \omega_0 - \xi ||_{L^1}^2 + \frac{\alpha}{\pi} || \omega_0 - \xi ||_{L^1}.
\]

Now gathering all the estimates of $(I_a), (I_b), (I_c), \text{ and } (I_d)$, we have the following estimate for \((I)\);

\[
(I) \leq \sqrt{4\pi (M + 1)} \left[ (I_a) + (I_b) + (I_c) + (I_d) \right]^{\frac{1}{2}} \leq C_{R, M, \alpha} \left[ || \omega_0 - \xi ||_{L^1}^2 + || \omega_0 - \xi ||_{L^1} + J(|| \omega_0 - \xi ||_{L^1}) \right]^{\frac{1}{2}} + C_M \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^{-2} \xi \, dx \right)^{\frac{1}{2}},
\]

where $C_{R, M, \alpha} > 0$ is a constant depending only on $R, M, \alpha$ and $C_M > 0$ is a constant depending only on $M$. Finally, we have the estimate (3.7) from estimates (3.11) of \((III) + (IV), (3.14)\) of \((II), \text{ and } (3.16)\) of \((I)\).

### 3.3 Stability in $J_p$

Before proving our main theorems, we first obtain $J_p$-stability.

**Lemma 3.4** Let $R, M, \alpha \in (0, \infty)$. Then there exist constants $C_3 = C_3(R, M, \alpha) > 0$ and $C_4 = C_4(M) > 0$ such that if a function $\xi \in L^\infty(\mathbb{R}^2)$ with $J(\xi) < \infty$ is nonnegative, radially symmetric, non-increasing, and satisfies

\[
|| \xi ||_{L^\infty} \leq M, \quad || \xi ||_{L^1} \leq \alpha,
\]

then for any nonnegative $\omega_0 \in L^\infty(\mathbb{R}^2)$ with $J(\omega_0) < \infty$, the solution $\omega(t)$ of \((1.1)\) satisfies

\[
\sup_{t \geq 0} || \omega(t) - \xi ||_{J^p} \leq C_3 \left[ || \omega_0 - \xi ||_{J^p} \right] + C_4 \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \xi \, dx \right)^{\frac{1}{p}} + C_4 \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \xi \, dx \right), \quad \text{for any } p \in [1, \infty)
\]

**Proof** As in the proof of the previous lemma, we fix $t \geq 0$ and drop the parameter $t$ for simplicity;

\[
\omega = \omega(t).
\]

To begin with, we use the decomposition

\[
\int_{\mathbb{R}^2 \setminus B_R} |x|^2 \, dx = \left( J(\omega) - J(\xi) \right) - \int_{B_R} |x|^2 (\omega - \xi) \, dx + \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \xi \, dx,
\]

to get

\[
J(|\omega - \xi|) = \int_{B_R} |x|^2 |\omega - \xi| \, dx + \int_{\mathbb{R}^2 \setminus B_R} |x|^2 |\omega - \xi| \, dx \leq R^2 \int_{B_R} |\omega - \xi| \, dx + \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \omega \, dx + \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \xi \, dx
\]
\[ \leq R^2 \int_{B_R} |\omega - \zeta| \, dx + \left( J(\omega) - J(\zeta) \right) + \int_{B_R} |x|^2 |\omega - \zeta| \, dx + 2 \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx, \]

which gives us

\[ J(|\omega - \zeta|) \leq 2R^2 \int_{B_R} |\omega - \zeta| \, dx + \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx, \]

\[ \leq 2R^2 \|\omega - \zeta\|_{L^1} + J(|\omega_0 - \zeta|) + 2 \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx. \]  

(3.18)

Then combined with the \( L^1 \)-estimate (3.7) in Lemma 3.3, we have

\[ J(|\omega - \zeta|) \leq C \left[ \|\omega_0 - \zeta\|_{L^1}^{\frac{1}{2}} + \|\omega_0 - \zeta\|_{L^p} + J(|\omega_0 - \zeta|) \right] \]

\[ + R^2 C'' \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2}} + C' \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx, \]  

(3.19)

where \( C' = C'(R, M, \alpha) > 0 \) and \( C'' = C''(M) > 0 \) are constants.

Then adding (3.7) from Lemma 3.3 and (3.19) gives us (3.17) when \( p = 1 \).

Now we assume \( p \in (1, \infty) \). We may change constants \( C' = C'(R, M, \alpha) > 0 \) and \( C'' = C''(M) > 0 \) line by line, but they are remained independent of the choice of \( p \). Then applying the estimate

\[ \|g\|_{L^1} \leq \pi^{\frac{p-1}{p}} \|g\|_{L^p} + J(|g|) \leq \pi \|g\|_{L^p} + J(|g|), \quad g \in L^p, \quad J(|g|) < \infty, \]  

(3.20)

on (3.7), we get

\[ \|\omega - \zeta\|_{L^1} \leq C' \left[ \|\omega_0 - \zeta\|_{L^1}^{\frac{1}{2}} + \|\omega_0 - \zeta\|_{L^p} + J(|\omega_0 - \zeta|) \right] \]

\[ + C'' \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2}}. \]  

(3.21)

Then using (3.21) on (3.18), we have

\[ J(|\omega - \zeta|) \leq C' \left[ \|\omega_0 - \zeta\|_{L^1}^{\frac{1}{2}} + \|\omega_0 - \zeta\|_{L^p} + J(|\omega_0 - \zeta|) \right] \]

\[ + R^2 C'' \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2}} + C' \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx. \]  

(3.22)

To get the \( L^p \) estimate, we use the set \( A_\omega \) and the function \( \tilde{\omega} \) defined by (3.8) and (3.9) with \( f = \omega \) from the proof of Lemma 3.3 once more. First, we have

\[ \|\omega - \zeta\|_{L^p} \leq \|\tilde{\omega} - \zeta\|_{L^p} + \int_{\mathbb{R}^2 \setminus A_\omega} |\omega - \zeta| \, dx \]

\[ \leq \|\tilde{\omega} - \zeta\|_{L^p} + \int_{A_\omega} |\omega - \zeta| \, dx + \int_{\mathbb{R}^2 \setminus A_\omega} |\omega - \zeta| \, dx \]

\[ \leq (M + 1)^{p-1} \|\omega - \zeta\|_{L^1} + 2^p \left( \int_{\mathbb{R}^2 \setminus A_\omega} |\omega| \, dx + \int_{\mathbb{R}^2 \setminus A_\omega} |\zeta| \, dx \right) \]

(11)
We also borrow the estimate (3.10); 
\[ \| \mathbb{R}^2 \setminus A_{\omega_0} \| \leq \| \omega_0 - \zeta \|_{L^1}. \]

Then we get
\[ (I) \leq 2^p \left( \int_{\mathbb{R}^2 \setminus A_{\omega_0}} |\omega_0 - \zeta|^p \, dx + \int_{\mathbb{R}^2 \setminus A_{\omega_0}} |\zeta|^p \, dx \right) \leq 2^p \left( \| \omega_0 - \zeta \|_{L^p}^p + M^P \| \mathbb{R}^2 \setminus A_{\omega_0} \| \right) \leq 2^p \left( \| \omega_0 - \zeta \|_{L^p}^p + M^P \| \omega_0 - \zeta \|_{L^1} \right). \]

The estimate of (II) follows in the same way;
\[ (II) \leq M^P \| \mathbb{R}^2 \setminus A_{\omega} \| = M^P \| \mathbb{R}^2 \setminus A_{\omega_0} \| \leq M^P \| \omega_0 - \zeta \|_{L^1}. \]

Then we have
\[ \| \omega - \zeta \|_{L^p}^p \leq (M + 1)^{p-1} \| \omega - \zeta \|_{L^1} + 2^p (2^p + 1) M^P \| \omega_0 - \zeta \|_{L^1} + 2^{2p} M^P \| \omega_0 - \zeta \|_{L^p}^p \]
\[ \leq (M + 1)^p \| \omega - \zeta \|_{L^1} + 2^p M^P \| \omega_0 - \zeta \|_{L^1} + 2^{2p} M^P \| \omega_0 - \zeta \|_{L^p}. \]

Thus, by estimates (3.20) and (3.21), we have
\[ \| \omega - \zeta \|_{L^p} \leq C' \left[ \| \omega_0 - \zeta \|_{L^p}^p + \| \omega_0 - \zeta \|_{L^p} + J(|\omega_0 - \zeta|)^{\frac{1}{p}} + J(|\omega_0 - \zeta|)^{\frac{1}{2p}} \right] + C'' \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2p}}. \] (3.23)

Finally, adding (3.22) and (3.23), we obtain (3.17) for \( p \in (1, \infty) \). \[ \square \]

Now we are ready to prove our main theorems.

**Proof of Theorem 1.2** We let \( R, M \in (0, \infty) \), \( p \in [1, \infty) \), and set \( \alpha := \pi R^2 M \). Also, we let \( \zeta \in L^\infty(\mathbb{R}^2) \) be nonnegative, radially symmetric, non-increasing, and compactly supported with (1.3). Additionally, we let \( \omega_0 \in L^\infty(\mathbb{R}^2) \) be nonnegative with \( J(\omega_0) < \infty \) and \( \omega(t) \) be the solution of (1.1) for \( \omega_0 \). Then we have \( \| \zeta \|_{L^1} = \int_{B_R} \zeta \, dx \leq \pi R^2 M \leq \alpha \), so we can use the conclusion (3.17) in Lemma 3.4. Furthermore, we have

\[ \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx = 0, \]

which makes all the terms containing \( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \) on the right-hand side of (3.17) zero. In sum, we get

\[ \sup_{t \geq 0} \| \omega(t) - \zeta \|_{J_p} \leq C_3 \left[ \| \omega_0 - \zeta \|_{J_p}^{\frac{1}{p}} + \| \omega_0 - \zeta \|_{J_p} \right], \]

with \( C_3 = C_3(R, M, \alpha) = C_3(R, M, \pi R^2 M) \). \[ \square \]

**Proof of Theorem 1.3** We let \( \varepsilon > 0, p \in [1, \infty) \), and \( \zeta \in L^\infty(\mathbb{R}^2) \) be nonnegative, radially symmetric, non-increasing with

\[ \int_{\mathbb{R}^2} |x|^6 \zeta \, dx < \infty. \] (3.24)

We set \( M, \alpha \in (0, \infty) \) by \( \| \zeta \|_{L^\infty} = M, \| \zeta \|_{L^1} = \alpha \). (If we have \( \zeta \equiv 0 \), then the conclusion is trivial.) Then we borrow the constant \( C_4 = C_4(M) \) from Lemma 3.4. We also let \( \omega_0 \in \)
\( L^\infty(\mathbb{R}^2) \) be nonnegative with \( J(\omega_0) < \infty \) and \( \omega(t) \) be the solution of (1.1) for \( \omega_0 \). Then due to \( J(\zeta) < \infty \) by (3.24), there exists (large) \( R_1 > 0 \) such that

\[
C_4 \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{p}} + \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \leq \epsilon \quad \text{for all} \quad R \geq R_1.
\]

In addition, there exists (large) \( R_2 > 0 \) such that

\[
R^2 \cdot C_4 \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2}} \leq \epsilon \quad \text{for all} \quad R \geq R_2.
\]

Indeed, this holds due to (3.24);

\[
R^2 \cdot \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^6 \zeta \, dx \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad R \to \infty.
\]

We take \( R = \max \{ R_1, R_2 \} \). Now we take the constant \( C_3 = C_3(R, M, \alpha) \) from Lemma 3.4. Finally, by taking (small) \( \delta > 0 \) that satisfies \( C_3(\delta^{\frac{1}{2p}} + \delta) \leq \epsilon \), if we have

\[
\| \omega_0 - \zeta \|_{J^p} \leq \delta,
\]

then the estimate (3.17) in Lemma 3.4 says

\[
\sup_{t \geq 0} \| \omega(t) - \zeta \|_{J^p} \leq C_3 \left[ \| \omega_0 - \zeta \|_{J^p}^{\frac{1}{2p}} + \| \omega_0 - \zeta \|_{J^p} \right] + R^2 \cdot C_4 \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2}}
\]

\[
+ C_4 \left( \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \right)^{\frac{1}{2p}} + \int_{\mathbb{R}^2 \setminus B_R} |x|^2 \zeta \, dx \leq \epsilon + \epsilon + \epsilon = 3\epsilon.
\]

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