Almost-Sure Reachability in Stochastic Multi-Mode System

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Abstract

A constant-rate multi-mode system is a hybrid system that can switch freely among a finite set of modes, and whose dynamics is specified by a finite number of real-valued variables with mode-dependent constant rates. We introduce and study a stochastic extension of a constant-rate multi-mode system where the dynamics is specified by mode-dependent compactly supported probability distributions over a set of constant rate vectors. Given a tolerance ε > 0, the almost-sure reachability problem for stochastic multi-mode systems is to decide the existence of a control strategy that steers the system almost-surely from an arbitrary start state to an ε-neighborhood of an arbitrary target state while staying inside a pre-specified safety set. We prove a necessary and sufficient condition to decide almost-sure reachability and, using this condition, we show that almost-sure reachability can be decided in polynomial time. Our algorithm can be used as a path-following algorithm in combination with any off-the-shelf path-planning algorithm to make a robot or an autonomous vehicle with noisy low-level controllers follow a given path with arbitrary precision.

1 Introduction

Planning and control of autonomous vehicles (or robots) are increasingly hierarchical in nature [14, 18] as this provides abstraction to dissociate the complications involved in lower-level hardware control from higher level planning decisions. This naturally gives rise to compositional design frameworks where the central problem is to design control so as to provide performance guarantees for planners at higher levels by assuming performance guarantees from the controllers at lower levels. Le Ny and Pappas [24] recently presented a general notion of robust motion specification at a lower level and a mechanism to sequentially compose them to satisfy a higher-level control objective. In this paper lower-level controllers are abstracted as modes having constant-rate dynamics with stochastic noise; the control objective is to almost surely follow an arbitrary path with an arbitrary precision. We prove a necessary and sufficient condition ensuring the existence of such control.

In order to restrict ourselves to decidable models, we extend the constant-rate multi-mode system framework of Alur et al. [6] by allowing bounded stochastic uncertainties with various modes. These systems, that we call stochastic multi-mode systems or SMMS, consist of a finite set of continuous variables, whose dynamics is given by mode-dependent constant-rates that can vary within given bounded sets according to given probability distributions. This dynamics gives rise to a one-and-half player game between a controller and the environment, where at each step the controller chooses a mode and time duration and the environment chooses a rate vector for that mode from the given bounded set following its distribution. The system evolves with that rate for the chosen time and the game continues in this fashion from the resulting state. A key problem for these systems is almost-sure reachability, which is
Almost-sure reachability is a concern when solving path-planning problem for autonomous vehicles with finitely many modes associated with noisy dynamics. For instance, consider the problem of navigating a robot with a set of three motion directions along with bounded uncertainty distributions, shown as $\mu_1$, $\mu_2$, and $\mu_3$ in Figure 1. An important problem for such systems is to decide so-called $\varepsilon$-reachability property that asks whether it is possible for the given robot to almost surely follow a given trajectory, with arbitrary precision, as shown by the open tube in Figure 1 and if so, to compute the controller strategy.

Our key result is that given a set of stochastic modes and a path-connected and bounded safety set, there is a strategy to reach an arbitrary neighborhood of an arbitrary target state from any given state, if and only if for every direction (vector) $\vec{v}$ there is a stochastic mode such that its expected direction has a positive projection along the direction $\vec{v}$. (For every mode, the expected rate is depicted as a thick arrow in Figure 1.) It is a straightforward consequence of this result that for probability distributions permitting an efficient computation of their expected values, this property can be checked in polynomial time. Our results can be combined with paths returned by off-the-shelf path-planning algorithms, such as rapidly exploring random trees [23] or Canny’s algorithm [12], to accomplish motion planning in the presence of stochastic uncertainties.

For a detailed survey of well-known motion planning algorithms we refer the reader to excellent expositions by Latombe [21], LaValle [22] and by de Berg et al. [9]. For path-following and trajectory tracking of autonomous robots under uncertainty we refer the reader to [1]. Planning using composition of lower-level motion primitives has been studied, among others, by [24, 17, 16, 8]. A general modeling framework for specifying hybrid systems is provided by hybrid automata [2, 3]. Given the expressiveness of hybrid automata, it is not surprising that simple verification questions like reachability are undecidable [19] for the general class of hybrid automata. Given this result, there has been a growing body of work on decidable subclasses of hybrid automata [3, 11]. Most notable among these classes are initialized rectangular hybrid automata [19], piecewise-constant derivative systems [7], timed automata [4], and multi-mode systems.

As mentioned earlier, stochastic multi-mode systems are a generalization of constant-rate multi-mode systems. Alur, Trivedi, and Wojtczak [6] considered constant-rate multi-mode systems and showed that the reachability problem—deciding the reachability of a specified state while staying in a given safety set—and the schedulability problem—deciding the existence of a non-Zeno control so that the system always stays in a given bounded and convex safety set—for this class of systems can be solved in polynomial time.

Alur et al. [5] introduced bounded-rate multi-mode systems where the rate in each mode is a constant that is picked from a given bounded set. These systems can be considered as constant-rate multi-mode
systems with uncertainties. It is known [5, 10] that the schedulability and reachability problems for bounded-rate multi-mode systems are, although intractable (co-NP-complete), decidable. To the best of our knowledge, there is no known result on stochastic extensions of multi-mode systems.

The paper is organized as follows. We begin by reviewing necessary background on probability theory in the next section, followed by the problem formulation in Section 3. In Section 4 we prove our key theorem for a simpler setting of one-dimensional stochastic multi-mode systems. We treat the case of general multi-dimensional systems in Section 5. Although, the proof for one-dimensional system follows from the proof for the general case, the proof for one-dimensional case is different and much simpler. We provide the algorithms based on our theorem to solve motion planning problem in Section 6, before concluding in Section 7 by discussing potential future directions.

2 Preliminaries

Let \( \mathbb{R} \) be the set of real numbers. We write \([m]\) for the set \{1, \ldots, m\}. For vectors \(u, v \in \mathbb{R}^n\), we write \(u \cdot v\) for the inner product of \(u\) and \(v\), i.e., \(u \cdot v := \sum_{i=1}^{n} u_i v_i\). We use \(\| \cdot \|\) to denote the standard Euclidean-norm in \(\mathbb{R}^n\), i.e., \(\|u\| := \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}\). We say that a set \(S \subseteq \mathbb{R}^n\) is bounded, if there exists a \(\rho \geq 0\) such that \(\|x\| \leq \rho\) for all \(x \in S\). For a vector \(x \in \mathbb{R}^n\) and \(d > 0\), we define the ball \(B(x, d)\) of radius \(d\) around \(x\) as

\[
B(x, d) := \{ y \in \mathbb{R}^n \mid \|x - y\| < d \}.
\]

We also let the \(\ell_1\) norm of \(u\) be \(\|u\|_1 := \sum_{i=1}^{n} |u_i|\).

2.1 Probability Space

Here, we present background material and the main results in probability theory that will be needed later. Readers are referred to standard references such as [13] for more details. Let \((\Omega, \mathcal{F}, P(\cdot))\) be a probability space where \(\Omega\) is a sample space, \(\mathcal{F}\) is a \(\sigma\)-algebra containing all the events of interest in the probability space, and \(P : \mathcal{F} \to [0,1]\) is a probability measure on \((\Omega, \mathcal{F})\). For a given probability space, we say that a property \(p\) holds with probability 1 or almost surely (a.s.) if

\[
P(\{\omega \in \Omega \mid \omega \text{ satisfies } p\}) = 1.
\]

We assume all the probability measures in \(\mathbb{R}^n\) discussed in this paper to be Borel measures.

We say that a distribution (probability measure) \(\mu\) over \(\mathbb{R}^n\) is compactly supported if \(\mu(\{x \in \mathbb{R}^n \mid \|x\| \geq \rho\}) = 0\) for some \(\rho > 0\). For a distribution \(\mu\) over \(\mathbb{R}^n\), we write \(\overline{\mu}\) to denote its expected vector \(E(\mu)\), which is the expected vector of a random vector \(Z\) whose distribution is \(\mu\). Moreover, for an event \(A \in \mathcal{F}\), we write \(1_A(\omega)\) for its characteristic function, defined as

\[
1_A(\omega) := \begin{cases} 
1 & \text{if } \omega \in A \\
0 & \text{otherwise} 
\end{cases}.
\]

For two random variables \(X\) and \(Y\), we use the wedge-notation to denote their minimum, i.e., \(X \wedge Y := \min(X,Y)\).

For an ensemble of random variables \(\{\xi_k\}_{k \in I}\) over \((\Omega, \mathcal{F})\), we define \(\sigma(\{\xi_k\}_{k \in I})\) to be the smallest \(\sigma\)-algebra such that \(\xi_k\)'s are measurable with respect to it. We also use the notation \(\{y(k, \omega)\}_{k \geq 0}\) to denote a fixed sample path \(\omega \in \Omega\) of a (discrete-time) random processes \(\{y(k)\}\).

2.2 Martingales

A filtration \(\{\mathcal{F}_k\}\) in a probability space \((\Omega, \mathcal{F}, P(\cdot))\) is a sequence of sub-\(\sigma\)-algebras (of \(\mathcal{F}\)) such that \(\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots\). Let \(\{\alpha(k)\}\) be a random process with \(E(|\alpha(k)|) < \infty\). We say that \(\{\alpha(k)\}\) is adapted to \(\{\mathcal{F}_k\}\), if \(\alpha(k)\) is measurable with respect to \(\mathcal{F}_k\) for all \(k \geq 0\).
Definition 1 (Martingales) We say that a random process \( \{\alpha(k)\} \) adapted to a filtration \( \{\mathcal{F}_k\} \) is a martingale with respect to a filtration \( \{\mathcal{F}(k)\} \) if

\[
E(\alpha(k+1) \mid \mathcal{F}_k) = \alpha(k),
\]

It is a submartingale if

\[
E(\alpha(k+1) \mid \mathcal{F}_k) \geq \alpha(k),
\]

and a supermartingale if

\[
E(\alpha(k+1) \mid \mathcal{F}_k) \leq \alpha(k).
\]

We will make use of the following important result in martingale theory (cf. Theorem 5.2.8 in [13]).

Theorem 1 (Martingale convergence theorem) Let \( \{\alpha(k)\} \) be a martingale such that \( E(|\alpha(k)|) < B \) for all \( k \geq 0 \) and some bound \( B \in \mathbb{R} \). Then,

\[
\alpha = \lim_{k \to \infty} \alpha(k)
\]

exists almost surely and \( E(|\alpha|) < \infty \).

Throughout this work, we assume that \( \{\mathcal{F}_k\} \) is the natural filtration for the underlying process \( \{\alpha(k)\} \) and hence,

\[
E(\alpha(k+1) \mid \mathcal{F}_k) = E(\alpha(k+1) \mid y(0), y(1), \ldots, y(k)).
\]

One of the immediate consequences of the martingale convergence theorem is the following result that follows from the Robbins-Siegmund Theorem (cf. Theorem 7.11 in [26]).

Corollary 1 If \( \{\alpha(k)\} \) is a submartingale such that for all \( k \geq 0 \) we have \( E(|\alpha(k)|) < B \) and

\[
E(\alpha(k+1) \mid \mathcal{F}_k) \geq \alpha(k) + \xi(k),
\]

(1)

where \( \xi(k) \geq 0 \) almost surely, then

\[
\sum_{k=0}^{\infty} \xi(k) < \infty \quad \text{almost surely.}
\]

In addition to the above theorem, we make use of the following result, which follows immediately from the definition of martingale and the dominated convergence theorem (cf. Theorem 2.24 in [15]).

Lemma 1 Let \( \{\alpha(k)\} \) be a uniformly bounded supermartingale, i.e., \(|\alpha(k)| \leq B\) almost surely for some real number \( B > 0 \) and all \( k \geq 0 \). Then, if

\[
\alpha = \lim_{k \to \infty} \alpha(k),
\]

then we have

\[
E(\alpha) \leq \lim_{k \to \infty} E(\alpha(k)).
\]

The same result holds for a uniformly bounded submartingale with the direction of the inequality reversed, i.e.,

\[
E(\alpha) \geq \lim_{k \to \infty} E(\alpha(k)).
\]

In particular, if \( \{\alpha(k)\} \) is a uniformly bounded martingale, then \( \lim_{k \to \infty} E(\alpha(k)) = E(\alpha) \).
3 Problem Formulation

An $n$-dimensional stochastic multi-mode system (SMMS) $\mathcal{M}$ is a plant that is governed by a set of stochastic modes in $\mathbb{R}^n$, i.e., a set of distributions on $\mathbb{R}^n$, $\{\mu_1, \ldots, \mu_\gamma\}$ for $\gamma \geq 1$. The plant dynamics starts at a point $x(0) \in \mathbb{R}^n$ at the time $t_0 = 0$. At each discrete iteration $k = 0, 1, \ldots$, the controller chooses a mode $m(k) \in [\gamma]$ and dwelling time $d(k) > 0$ for that mode and then, the plant’s dynamics follows

$$\dot{x}(t) = \eta_{k+1}, \text{ for } t \in [t_k, t_k + d(k)),$$

where $\eta_{k+1} \in \mathbb{R}^n$ is a sample point from the chosen distribution $\mu_{m(k)}$ and $t_k$ is recursively defined by

$$t_k = \begin{cases} 0 & \text{if } k = 0, \\ t_{k-1} + d(k-1) & \text{for } k > 0. \end{cases}$$

In other words, $x(t) = x(t_k) + (t - t_k)\eta_{k+1}$ for $t \in [t_k, t_{k+1})$. When the dimension $n$ is clear from the context, we simply refer to a stochastic multi-mode system as a set $\mathcal{M} = \{\mu_1, \ldots, \mu_\gamma\}$ of distributions. An SMMS is deterministic if for each mode $\mu \in \mathcal{M}$, we have $P(\mu = v) = 1$ for some (deterministic) vector $v \in \mathbb{R}^n$. We denote the state of the plant at the decision times $t_0, t_1, \ldots$ by

$$y(k) := x(t_k), \text{ for } k = 0, 1, \ldots.$$

For the control of an SMMS, our focus is to determine the mode $m(k) \in [\gamma]$ and the time $d(k)$ based on the past observation of the system’s behavior.

**Definition 2 (Control policy)** A control policy (or control strategy) is a sequence $\{m(k), d(k)\}_{k \geq 0}$, where $m(k)$ is a random variable supported in $[\gamma]$ and $d(k)$ is a non-negative random variable for all $k \in \mathbb{Z}^+$. A control policy is causal if $m(k)$ and $d(k)$ are measurable with respect to $\mathcal{F}_k = \sigma(\eta_1, \ldots, \eta_k)$ ($\mathcal{F}_0 = \{\emptyset, \Omega\}$) for all $k \geq 0$.

Throughout this work, all control policies of interest are assumed to be causal. The main property that we investigate is the following reachability property.

**Definition 3 ($\epsilon$-Reachability with safety set $S$)** For a SMMS $\mathcal{M}$ and a safety set $S \subseteq \mathbb{R}^n$, we say that $\mathcal{M}$ satisfies the $\epsilon$-reachability property with safety set $S$, or simply almost-sure reachability, if for any starting point $x_0 \in S$, any terminal point $x_T \in S$, and any $\epsilon$-neighborhood of $x_T$, there exists a causal policy (controller) that steers $\mathcal{M}$ from the initial point $x(0) = x_0$ to a target point $x(T) = y(k) \in B(x_T, \epsilon)$ in a finite time $T = t_k < \infty$ such that $x(t) \in S$ for all $t \in [0, T]$ almost surely.

Our approach to solve the $\epsilon$-reachability problem is to characterize necessary and sufficient conditions that guarantee $\epsilon$-reachability as described below.

**Problem 1** Given an open set $S \subseteq \mathbb{R}^n$, under what conditions on $S$ and the stochastic modes can one guarantee $\epsilon$-reachability with safety set $S$?

For a bounded safety set $S$, one can only use compactly supported measures $\mu$ to maintain safety, as otherwise there is a non-zero probability that the dynamics (2) does not satisfy safety. Therefore we henceforth assume that all the distributions in $\mathcal{M}$ are compactly supported and we let

$$L_\mathcal{M} = \max_i \inf \{ \rho \mid \mu_i(\{x \in \mathbb{R}^n \mid \|x\| \geq \rho\}) = 0 \}.$$

Our key observation is that the $\epsilon$-reachability property of a stochastic multi-mode system is closely related to the $\epsilon$-reachability property of the associated (deterministic) expected multi-mode system defined as follows.
Definition 4 (Expected Multi-mode System) The expected multi-mode system $\overline{M}$ of a stochastic multi-mode system $M = \{\mu_1, \ldots, \mu_\gamma\}$ is the deterministic multi-mode system whose dynamics in each mode is given by the expected direction of the corresponding mode in $M$, i.e., $\overline{M}$ is the deterministic multi-mode system with modes $\overline{M} = \{\overline{\mu}_1, \ldots, \overline{\mu}_\gamma\}$.

The main contribution of this work is the following result.

**Theorem** Let $M = \{\mu_1, \ldots, \mu_\gamma\}$ be a stochastic multi-mode system with a finite set of compactly supported distributions and let $S \subset \mathbb{R}^n$ be a path-connected and bounded open safety set. The following statements are equivalent:

a. $M$ satisfies the $\epsilon$-reachability property with safety set $S$.

b. The expected multi-mode system $\overline{M}$ of $M$ satisfies the $\epsilon$-reachability property with safety set $S$.

c. For every non-zero vector $v \in \mathbb{R}^n$, there exists a mode $\mu \in M$ such that $\overline{\mu} \cdot v > 0$.

In the next section, we visit this theorem in the context of 1-dimensional stochastic multi-mode systems and give a simpler proof for this result than that of an $n$-dimensional SMMS. Also, the statement and the proof of the main result for the 1-dimensional dynamics sheds light on the statement and the proof of the main theorem in its complete generality.

### 4 One-Dimensional Dynamics

For the rest of this section we assume that the given stochastic multi-mode system $M = \{\mu_1, \mu_2, \ldots, \mu_\gamma\}$ is a 1-dimensional system. Before we characterize the necessary and sufficient condition for the $\epsilon$-reachability, we establish the following result.

**Lemma 2** Let $X$ be a random variable with $E(X) > 0$ and $|X| \leq 1$ almost surely. Then we have that

$$E(\log(1 + \delta X)) > 0,$$

for every $\delta < E(X)/2$.

**Proof:** Let $\delta < E(X)/2 \leq 1/2$. For $z \in [-\delta, \delta]$, the Taylor expansion of $f(z) = \log(1 + z)$ around $z = 0$ implies:

$$\log(1 + z) = z - \frac{1}{(1 + \tilde{z})^2} \frac{z^2}{2} \geq z - \frac{1}{(1 - \delta)^2} \frac{z^2}{2} \geq z - 2z^2,$$

for some $\tilde{z} \in [-\delta, \delta]$, where the first inequality follows from the fact that $1/(1 + \tilde{z})^2$ is a decreasing function of $\tilde{z}$ and the second inequality follows from the fact that

$$\delta \leq E(X)/2 \leq \frac{1}{2}.$$ 

Therefore, for a random variable $X$ whose support is in $[-1, 1]$, we have:

$$E(\log(1 + \delta X)) \geq \delta E(X) - 2\delta^2 E(X^2) \geq 2\delta (E(X)/2 - \delta) > 0,$$

where the last inequality follows from $\delta \in (0, E(X)/2)$.

Next, we show that almost sure $\epsilon$-reachability with safety set $S = (a, b)$ is achievable if and only if there exist two modes $\mu_+$ and $\mu_-$ such that $\overline{\mu}_+ > 0$ and $\overline{\mu}_- < 0$. Note that for a deterministic system this is indeed necessary and sufficient: it is necessary because otherwise one cannot move the deterministic
system from \(x_s = \frac{3a+b}{4}\) to \(x_t = \frac{a+3b}{4}\) or vice versa. It is sufficient because, once we have a positive and a negative control direction, then one can steer the system towards left and right to the desired position without violating safety. However, to show such a result for stochastic systems one must account for the possibly adversarial effects of the noise in the control vectors. Through a proper choice of control policy we want to make sure that the system will reach the target while ensuring safety.

\[\text{Theorem 2}\]

Let \(\mathcal{M} = \{\mu_1, \ldots, \mu_\gamma\}\) be a SMMS with a finite set of compactly supported distributions and, without loss of generality, assume that each \(\mu_i \in \mathcal{M}\) is distributed over the unit interval. Further assume that the safety set is \(S = (a, b)\) for some \(a < b\).

Then, \(\mathcal{M}\) satisfies the \(\epsilon\)-reachability property with safety set \(S\) if and only if there exist modes \(\mu_+ \in \mathcal{M}\) and \(\mu_- \in \mathcal{M}\) such that \(E(\mu_+) > 0\) and \(E(\mu-) < 0\).

In other words, \(\mathcal{M}\) satisfies the \(\epsilon\)-reachability property with safety set \(S\) if and only if the expected deterministic system \(\overline{\mathcal{M}}\) satisfies the same property.

\[\text{Proof:}\] Without loss of generality, we assume that \(a = 0\).

(Necessity.) Suppose that \(\mathcal{M}\) does not have a mode \(\mu_-\) with \(\overline{\mu_-} < 0\). We prove a stronger statement: we show that there does not exists any causal policy that can almost surely reach the \(\epsilon\)-neighborhood of \(x_t\) for any starting point \(x_s \in (a, b)\), any target point \(x_t \in (a, b)\) with \(x_t < x_s\) and any \(\epsilon < (x_s - x_t)\).

To show this consider any causal policy \(\{m(k), d(k)\}_{k \geq 0}\) that guarantees safety almost surely. Consider the random process \(\{y(k)\}\) defined by (3). Note that \(y(k)\) is adapted to \(\mathcal{F}_k\) (the natural filtration for the \(\eta_k\)). It follows that:

\[E(y(k+1) \mid \mathcal{F}_k) = y(k) + d(k+1)E(\mu_{m(k)}) \geq y(k).\]

Therefore, \(\{y(k)\}\) is a submartingale w.r.t. the filtration \(\{\mathcal{F}_k\}\). Also, since the policy guarantees almost sure safety and \(S\) is bounded, \(\{y(k)\}\) is a bounded submartingale and hence by Theorem 1, it is convergent almost surely. Let us define the stopping time \(T\) as follows:

\[T := \inf\{k \geq 0 \mid y(k) \in B(x_t, \epsilon)\}.\]

Since \(\{y(k)\}\) is a bounded submartingale, \(\{y(k) \wedge T\}\) would be a bounded submartingale and it is convergent almost surely, i.e.,

\[y = \lim_{k \to \infty} y(k \wedge T)\]  \hspace{1cm} (4)

exists almost surely. Hence by Lemma 1 we get that

\[E(y) = \lim_{k \to \infty} E(y(k \wedge T)) \geq E(y(0)) = x_s.\]

In particular, if we let \(p = P(y \in B(x_t, \epsilon))\), then

\[x_s \leq E(y) \leq (x_t + \epsilon)p + (1 - p)b,\]

and hence

\[p \leq \frac{b - x_s}{b - (x_t + \epsilon)} < 1.\]

Therefore almost-sure convergence is impossible.

The impossibility of almost sure \(\epsilon\)-reachability for the case that SMMS does not contain a mode \(\mu_+\) with \(E(\mu_+) > 0\) follows from the same argument presented above.

(Sufficiency.) To show the sufficiency part of the theorem, let \(x_s \in (a, b)\) be any given initial condition and \(x_t \in (a, b)\) be any given target point and let \(\epsilon > 0\). We assume that \(x_t + \epsilon < b\). If this condition is not met, we replace \(\epsilon\) with \(\tilde{\epsilon} = (b - x_t)\) in the following argument (note that if \(x(t)\) is in \(B(x_t, \tilde{\epsilon})\) it will also belong to \(B(x_t, \epsilon)\) as \(B(x_t, \tilde{\epsilon}) \subseteq B(x_t, \epsilon)\)).
Recall that without loss of generality we may assume that $a = 0$ (so that the length of $S$ is $b$) and $x_s < x_t$. Consider the constant control policy for mode selection $\{\mu_+\}$ and the dwelling time sequence $\{d(k)\}$ be as follows:

$$d(k) = \delta y(k - 1), \quad (5)$$

for a constant $\delta < \min(\mu_+/2, \epsilon/(b - a))$. We show that $y(k)$ enters $B(x_t, \epsilon)$ with probability 1 while maintaining safety. One can show inductively that almost surely the policy (5) guarantees $x(t) > 0$ for all $t \geq 0$. Also, note that

$$y(k) = y(k - 1) + d(k)Z(k) \leq y(k - 1) + \frac{\epsilon}{b - a}Z(k) \leq y(k - 1) + \epsilon,$$

where $Z(k)$ is sampled from $\mu_+$. Note that

$$y(k) = y(k - 1) + \delta y(k - 1)Z(k) = y(k - 1)(1 + \delta Z(k)) = y(0) \prod_{i=1}^{k}(1 + \delta Z(i)),$$

and therefore,

$$\log(y(k)) = \log(y(0)) + \sum_{i=1}^{k} \log(1 + \delta Z(i)). \quad (6)$$

Note that $Z(i)$-s are i.i.d. random variables and hence $\log(1 + \delta Z(i))$-s are i.i.d. random variables. By Lemma 2 and the choice of $\delta$, it follows that

$$E(\log(1 + \delta Z(i))) > 0.$$

Therefore, invoking the strong law of large numbers (cf. Theorem 2.4.1. [13]), it follows that

$$\lim_{k \to \infty} \log(y(k)) = \infty$$

and we have $y(k) > x_t$ with probability one for some $k$ (depending on $\omega \in \Omega$). Finally, since

$$y(k) \leq y(k - 1) + \epsilon,$$

therefore, almost surely $y(k) \in B(x_t, \epsilon)$ for some $k(\omega)$. The proof is now complete. □

## 5 Higher-Dimensional Dynamics

In this section, we prove the extension of Theorem 2 for an arbitrary open, bounded, and path-connected safety set in $\mathbb{R}^n$. Namely, we show that in order to drive a system from any starting point to any target point, for any direction $v$ there must exist a stochastic mode $\mu$ such that we can positively move along $v$ in expectation using $\mu$, i.e., $v \cdot \bar{\mu} > 0$. So, the main result of this section is as follows.

**Theorem 3** Let $\mathcal{M} = \{\mu_1, \ldots, \mu_\gamma\}$ be a stochastic multi-mode system with a finite set of compactly supported distributions and let $S \subset \mathbb{R}^n$ be a path-connected and bounded open safety set. The following statements are equivalent:

a. $\mathcal{M}$ satisfies the $\epsilon$-reachability property with safety set $S$. 


b. The expected multi-mode system $\overline{\mathcal{M}}$ of $\mathcal{M}$ satisfies the $\epsilon$-reachability property with safety set $S$.

c. For every non-zero vector $v \in \mathbb{R}^n$, there exists a mode $\mu \in \mathcal{M}$ such that $\overline{\mu} \cdot v > 0$.

As for 1-dimensional dynamics, we prove $a \Rightarrow c$ by a martingale argument. For the converse, we break the problem into sub-problems: we show that if condition $c$ holds, then

i. If the safety set $S$ is a ball, the controller can reach the $\epsilon$-neighborhood of the center of the ball from any starting point $x_s \in S$ (Lemma 3).

ii. If the safety set $S$ is a ball, the controller can reach the $\epsilon$-neighborhood of any point $x_t \in S$ from any starting point $x_s \in S$ (Lemma 4).

iii. For any path-connected open set $S$, the controller can traverse from any starting point $x_s \in S$ to any $\epsilon$-neighborhood of any target point $x_t \in S$ by moving inside a sequence of balls that are strictly within the safety set.

We proceed by formulating and proving the above intermediate steps.

**Lemma 3** Let $\mathcal{M} = \{\mu_1, \ldots, \mu_\gamma\}$ be a stochastic multi-mode system with a finite set of compactly supported distributions and let the safety set $S$ be a ball $B(x_0, r) \subset \mathbb{R}^n$. If condition $c$ of Theorem 3 holds then, for any starting point $x_s \in S$ and any $\epsilon > 0$, the system can reach the $\epsilon$-neighborhood of the center $x_t = x_0$ almost surely.

**Proof:** Let $\mathcal{M} = \{\mu_1, \ldots, \mu_\gamma\}$ be a stochastic multi-mode system with a finite set of compactly supported distributions. Suppose that the safety set $S$ is $B(x_0, r) \subset \mathbb{R}^n$ and w.l.o.g. assume that $x_0 = 0$ (see, Figure 2a). Assume that $x_t = x_0 = 0$ (i.e., $x_t$ is the center of the safety ball) and assume condition $c$ of Theorem 3 holds. We show that for any starting point $x_s \in S$ and any $\epsilon > 0$, there is a control policy to steer the system to the $\epsilon$-neighborhood of $x_0$ almost surely while staying within the safety set $S$.

Without loss of generality assume that $\epsilon \leq r$. Let $L_M$ be $L_M$, the maximum support of the modes in $\mathcal{M}$. We use the following controller policy for the $\epsilon$-reachability of $x_t$:

At iteration $k$, if $y(k) \notin B(0, \epsilon)$, we let:

$$m(k) \in \operatorname{argmax}_{i \in [m]} \overline{\mu}_i \cdot (-y(k)),$$

$$d(k) = \delta(r^2 - \|y(k)\|^2),$$

for a sufficiently small positive constant $\delta$ satisfying

$$\delta < \min\left(\frac{1}{L}, \frac{1}{2Lr}\right).$$

Figure 2: Stochastic multi-mode system: proof sketch.
We require $\delta$ to satisfy further inequalities that will be discussed later. If $y(k) \in B(0, \epsilon)$, we simply let $d(k) = 0$ (or in other words, we stop the process). Let us denote the event $y(k) \notin B(0, \epsilon)$ by $A_k$.

If $\|y(k)\| < r$, then for any $\delta \in (0, 1/(2Lr))$ and on $A_k$, almost surely we have:

$$\|y(k + 1)\| = \|y(k) + d(k)Z(k)\|$$

$$\leq \|y(k)\| + d(k)\|Z(k)\|$$

$$< \|y(k)\| + \delta(r^2 - \|y(k)\|^2)L$$

$$= \|y(k)\| + \delta(r - \|y(k)\|)(r + \|y(k)\|)L$$

$$< \|y(k)\| + (r - \|y(k)\|)\delta 2r L$$

$$< \|y(k)\| + (r - \|y(k)\|) = r,$$

where the last inequality follows from $\delta < \frac{1}{2L}$ and $\|y(k)\| < r$. Hence, inductively, policy (7) satisfies safety almost surely.

By expanding $\|y(k + 1)\|^2 = (y(k) + d(k)Z(k)) \cdot (y(k) + d(k)Z(k))$, we get:

$$\|y(k + 1)\|^2 = \|y(k)\|^2 + 1_{A_k} \left( \delta^2(r^2 - \|y(k)\|^2)\|Z(k)\|^2 + 2\delta(r^2 - \|y(k)\|^2)y(k) \cdot Z(k) \right).$$

Subtracting both sides of the above equality from $r^2$, we get:

$$r^2 - \|y(k + 1)\|^2 = (r^2 - \|y(k)\|^2)$$

$$- 1_{A_k} (r^2 - \|y(k)\|^2) \times$$

$$(\delta^2(r^2 - \|y(k)\|^2)\|Z(k)\|^2 + 2\delta y(k) \cdot Z(k)).$$

Letting $\alpha(k) := (r^2 - \|y(k)\|^2)$ (see, Figure 2a), this equality simplifies to:

$$\alpha(k + 1)$$

$$= \alpha(k)(1 - 1_{A_k}(\delta^2(r^2 - \|y(k)\|^2)\|Z(k)\|^2 + 2\delta y(k) \cdot Z(k)))$$

$$\geq \alpha(k)(1 - 1_{A_k}(\delta^2 r^2 L^2 + 2\delta y(k) \cdot Z(k))). \tag{9}$$

Note that $g(v) = v \cdot \bar{n}_i$ is a continuous functional on $\mathbb{R}^n$ therefore, the function $v \rightarrow \max_{i \in M} v \cdot \bar{n}_i$ is a continuous function. Since, the set $\{v \in \mathbb{R}^n \mid \|v\| = 1\}$ is a compact set in $\mathbb{R}^n$, we get

$$\lambda := \inf_{v \in \mathbb{R}^n : \|v\| = 1} \max_{i \in [\gamma]} v \cdot \bar{n}_i > 0.$$

Let $p(k) := 2y(k) \cdot Z(k) + \epsilon \lambda$. Then, (9) simplifies to:

$$\alpha(k + 1) \geq \alpha(k)(1 + 1_{A_k}(-\delta^2 r^2 L^2 + \delta \epsilon \lambda - \delta p(k))). \tag{10}$$

Note that for

$$\delta \leq \frac{\epsilon \lambda}{r^2 L^2}, \tag{11}$$

we have $-\delta^2 r^2 L^2 + \delta \epsilon \lambda \geq 0$. Therefore, for a $\delta$ satisfying (8) and (11), we have:

$$\alpha(k + 1) \geq \alpha(k)(1 - \delta 1_{A_k}p(k)).$$

Applying $\log(\cdot)$ on both sides of the above inequality, we get:

$$\log(\alpha(k + 1)) \geq \log(\alpha(k)) + 1_{A_k} \log(1 - \delta p(k)).$$
Note that on $A_k$,
\[ |p(k)| \leq 2\|y(k)\|\|Z(k)\| + \epsilon\lambda \leq 2rL + \epsilon\lambda. \] (12)
Also,
\[ E(-p(k) \mid \mathcal{F}_k) = (E(2(-y(k)) : Z(k) - \epsilon\lambda \mid \mathcal{F}_k)) \geq (2\|y(k)\|\lambda - \epsilon\lambda) \geq (2\epsilon\lambda - \epsilon\lambda) = \epsilon\lambda > 0, \]
where the first inequality follows from the choice of $m(k)$ in (7) and the definition of $\lambda$. Since $E(p(k) \mid \mathcal{F}_k) > 0$ and $|p(k)| \leq 2rL + \epsilon\lambda$, by Lemma 2, for
\[ \delta \leq \frac{\epsilon\lambda}{4rL + 2\epsilon\lambda} \] (13)
that also satisfies (8) and (11), we have:
\[ E(\log(1 + 1A_k\delta p(k)) \mid \mathcal{F}_k) = 1A_kE(\log(1 + \delta p(k)) \geq 1A_k\xi, \] (14)
for some $\xi > 0$. Therefore, for such a small $\delta$, we have:
\[ E(\log(\alpha(k + 1)) \mid \mathcal{F}_k) \geq \log(\alpha(k)) + 1A_k\xi. \]
Since, $\alpha(k + 1)$ is bounded, by Corollary 1, it follows that
\[ \sum_{k=0}^{\infty} 1A_k < \infty, \]
almost surely. Therefore, almost surely, the trajectories of the dynamics will enter $B(x_t, \epsilon) = B(0, \epsilon)$. \qed

Using this result, the next step is to show that the controller can achieve almost sure $\epsilon$-reachability for the safety set being a ball (assuming the conditions of Lemma 3).

**Lemma 4** Consider a stochastic multi-mode system with a finite set $\mathcal{M} = \{\mu_1, \ldots, \mu_\gamma\}$ of compactly supported stochastic modes satisfying (e). If the safety set $S$ is a ball $B(x_0, r) \subset \mathbb{R}^n$, then the SMMS satisfies the almost sure $\epsilon$-reachability property with the safety set $S = B(x_0, r)$.

**Proof:** Let $x_s$ and $x_t$ be arbitrary starting and target points in the safety set $S = B(x_0, r)$ for some $r \geq \epsilon > 0$. Let
\[ \tilde{r} = \frac{1}{2} \min(r - \|x_s - x_0\|, r - \|x_t - x_0\|). \]
Note that for any point $v \in \{\beta x_s + (1 - \beta)x_t \mid \beta \in [0, 1]\} \subset S$, the segment connecting $x_s$ and $x_t$ in $S$, $B(v, 2\tilde{r}) \subset S$ holds.

Let $N = \lceil(\|x_t - x_s\|)/\tilde{r} \rceil + 1$ and let $u_i = x_s + (i/N)(x_t - x_s)$ for any $i = 0, \ldots, N$, where $z = \lfloor z \rfloor$ is the largest integer that satisfies $z \leq \beta$. Note that $u_0 = x_s$ and $u_N = x_t$. These intermediate points are illustrated in Figure 2b.

Let $\tilde{\epsilon} < \min(\epsilon, \tilde{r})$. Then, for any $u \in B(u_i, \tilde{\epsilon})$, we have:
\[ \|u_{i+1} - u\| = \|u_{i+1} - u_i + u_i - u\| \leq \|u_{i+1} - u_i\| + \|u_i - u\| \leq \tilde{r} + \tilde{\epsilon} < 2\tilde{r}, \]
for $i = 0, \ldots, N$. By Lemma 3, we can move from any point in the $\tilde{\epsilon}$-neighborhood of $u_i$ to a point in $\tilde{\epsilon}$-neighborhood of $u_{i+1}$ satisfying the safety $B(u_{i+1}, 2\tilde{r}) \subset S$. Therefore, by induction, almost surely, we can traverse from $u_0 = x_s$ to $B(u_N = x_t, \tilde{\epsilon}) \subset B(x_t, \epsilon)$ while satisfying safety $S$ almost surely. \qed

Finally, we are in a position to complete the proof of the main result.
Proof: [of Theorem 3] Let $\mathcal{M} = \{\mu_1, \ldots, \mu_n\}$ be a stochastic multi-mode system with finite set of compactly supported distributions and let $S \subset \mathbb{R}^n$ be a path-connected and bounded open safety set. It suffices to show the equivalence of $a$ and $c$ as the same equivalence holds for the deterministic system and the directions of its modes. In particular we show that $\mathcal{M}$ satisfies almost sure $\epsilon$-reachability property with the safety set $S$ if and only if for any $v \in \mathbb{R}^n$ there exists a mode $\mu \in \mathcal{M}$ such that $\overline{v} \cdot v > 0$.

(a $\Rightarrow$ c) Suppose that $a$ holds but $c$ does not. Let $v \neq 0$ be a vector such that for every $\mu \in \mathcal{M}$, $\overline{v} \cdot v \leq 0$. If a vector $v$ satisfies such a property, then the unit-length vector $\frac{v}{\|v\|}$ also does. So, without loss of generality we assume that $\|v\| = 1$. Let $x_s \in S$ be an arbitrary starting point. Since $S$ is an open set, there exists a $\delta > 0$ such that $B(x_s, \delta) \subset S$. Let $x_t = x_s + \frac{\delta}{4}v$ and let $\epsilon = \frac{\delta}{4}$ (see Figure 2c). Now, consider an arbitrary causal control policy $(m(k), d(k))$ and let $\{y(k)\}$ be defined as in (3) and $\{\mathcal{F}_k\}$ be the $\sigma$-algebra that is adapted to $\{y(k)\}$. Define $\alpha(k) := v \cdot y(k)$ for all $k \in \mathbb{Z}^+$. Note that $\{\alpha(k)\}$ is adapted to $\{\mathcal{F}_k\}$ and also,

$$E(\alpha(k+1) \mid \mathcal{F}_k) = E(v \cdot (y(k) + d(k)Z(k)) \mid \mathcal{F}_k) = v \cdot y(k) + d(k)E(v \cdot Z(k)) \leq v \cdot y(k) = \alpha(k),$$

where $Z(k)$ is the random vector whose distribution is $\mu_{m(k)}$. Therefore, the sequence $\{\alpha(k)\}$ would be a supermartingale. Note that $S$ is bounded and hence, if the policy guarantees almost sure safety, this supermartingale is convergent to a random variable $\alpha$. Therefore,

$$E(\alpha) \leq E(\alpha(0)) = v \cdot x_s. \quad (15)$$

On the other hand, for any $u \in B(x_t, \frac{\delta}{4})$, we have:

$$v \cdot u = v \cdot (u - x_t + x_t) = v \cdot (u - x_t) + v \cdot x_t \geq -\frac{\delta}{4}\|v\|^2 + v \cdot (x_s + \frac{\delta}{2}v) = v \cdot x_s + \frac{\delta}{4}\|v\|^2 > v \cdot x_s = E(\alpha(0)),$$

where the inequality follows from the Cauchy-Schwartz inequality and the fact that $u - x_t \in B(0, \frac{\delta}{4})$. Therefore, if the sample paths almost surely reach $B(x_t, \frac{\delta}{4})$, we have $E(\alpha) > E(\alpha(0))$ which contradicts (15).

(c $\Rightarrow$ a) Indeed this part of the Theorem applies for any path-connected open set $S$ (that is not necessarily bounded). Let $S$ be an arbitrary path-connected and open set and let $x_s, x_t \in S$. Since $S$ is path-connected, there exists a continuous path $\nu : [0, 1] \to S$ such that $\nu(0) = x_s$ and $\nu(1) = x_t$. For any $\theta \in [0, 1]$, let $\rho_\theta > 0$ be such that $B_\theta := B(\theta, \rho_\theta) \subset S$. Such $\rho_\theta$ exists because $S$ is an open set. Also, the image of $[0, 1]$ under $\nu$, i.e., the set

$$\mathcal{C} := \{\nu(\theta) \mid \theta \in [0, 1]\}$$

is a compact subset of $S$ as it is the image of a compact interval $[0, 1]$ under the continuous map $\nu$. Finally, the collection $I = \{B_\theta\}_{\theta \in [0, 1]}$ is an open cover for $\mathcal{C}$, i.e.

$$\mathcal{C} \subset \bigcup_{\theta \in [0, 1]} B_\theta.$$

By compactness of $\mathcal{C}$, there exists a finite open sub-cover of $I$ that covers $\mathcal{C}$. In other words, there exists $\theta_1, \ldots, \theta_q \in [0, 1]$ such that

$$\mathcal{C} \subset B_{\theta_1} \cup B_{\theta_2} \cup \cdots \cup B_{\theta_q},$$
for some finite number $q \in \mathbb{Z}^+$. Construct the undirected intersection graph $G = ([q], E)$ where $[q] := \{1, \ldots, q\}$ and

$$E = \{\{i, j\} \mid i, j \in [q], B_{\theta_i} \cap B_{\theta_j} \neq \emptyset\}.$$ 

One can verify that the intersection graph $G$ should be connected, as otherwise, the set $C$ would be a (path) disconnected set.

Without loss of generality assume that $x_s \in B_{\theta_1}$ and $x_t \in B_{\theta_q}$. Let $u_1 = 1 \to u_2 \to \cdots \to u_k = q$ be a directed path in $G$ that connects vertex 1, which is associated with $B_{\theta_1}$, which contains $x_s$, to the vertex $q$, which is associated with $B_{\theta_q}$, which contains $x_t$. Now let $x_1 = x_s$, $x_k = x_t$ and choose the points $x_i \in B_{\theta_{u_{i+1}}} \cap B_{\theta_{u_i+1}}$ for $i \in \{2, \ldots, k-1\}$. By Lemma 4, there exists a control policy that starting from any starting point in $B(x_i, \epsilon) \subset B_{\theta_{u_i}}$ the controller can move to some point in $B(x_{i+1}, \epsilon) \subset B_{\theta_{u_{i+1}}}$ while maintaining safety $B_{\theta_{u_{i+1}}} \subset S$ almost surely. Therefore, the controller can traverse from $x_s$ to $x_t$ by concatenating these control policies while maintaining safety $S$ almost surely. The proof is now complete.

Note that the characterization in Theorem 3 is independent of the safety set as long as it is open, bounded, and path-connected. Therefore, one may regard $\epsilon$-reachability to be a property of the SMMS independent of the safety set (as long as the latter satisfies those conditions).

Another observation about Theorem 3 is that the bounded condition on the safety set $S$ is absolutely required to prove that $c$ is necessary for $\epsilon$-reachability. To show the importance of this condition let us discuss a simple example.

**Example 1** Consider the safety set $S = \mathbb{R}$ and the SMMS $\mathcal{M} = \{\mu_1\}$ where

$$P(\mu_1 = +1) = P(\mu_1 = -1) = \frac{1}{2}.$$ 

Notice that this is the case of a simple random walk on $\mathbb{Z}$. Consider the simple control policy $d(k) = \epsilon$ and $m(k) = 1$ for all $k \geq 0$. It can be shown that (cf. Theorem 4.1.2 in [13]) for any initial condition $y(0) = x(0) = x_s \in \mathbb{R}$:

$$\liminf_{k \to \infty} y(k) = -\infty \text{ and } \limsup_{k \to \infty} y(k) = +\infty.$$ 

Therefore, starting from any starting point $x_s$, this controller will almost surely visit the $\epsilon$-neighborhood of any target point $x_t \in \mathbb{R}$. Therefore, this SMMS $\mathcal{M}$ satisfies almost reachability in $S = \mathbb{R}$.

However, the expected value of each mode is $\bar{\mu}_1 = 0$ which clearly does not satisfy $c$. The reason that Theorem 2 and Theorem 3 fail in this case is that the safety set is no longer a bounded set.

Although the boundedness of the safety set $S$ is necessary to prove that $a$ implies $c$, the proof of the reverse implication does not rely on the boundedness of the safety set.

**Corollary 2** Let $\mathcal{M} = \{\mu_1, \ldots, \mu_\gamma\}$ be a stochastic multi-mode system with a finite set of compactly supported distributions. Also, let $S \subset \mathbb{R}^n$ be a path-connected open set. Then, if for any non-zero $v \in \mathbb{R}^n$, there exists a mode $\mu \in \mathcal{M}$ such that $\bar{\mu} \cdot v > 0$, the $\epsilon$-reachability property with safety set $S$ holds almost surely.

### 6 Algorithms

Given a stochastic multi-mode systems $\mathcal{M}$, an arbitrary high-dimensional open-connected safety set $S$, starting point $x_s \in S$, and target point $x_t \in S$, a typical hierarchical motion planning procedure for stochastic multi-mode systems include the following steps:

1. (path-finding) find a path from $x_s$ to $x_t$,

2. (error-margin estimation) find a finite open cover for the path connecting $x_s$ to $x_t$ in $S$, and
Algorithm 1: REACH-IN_ARB_PC_SET(ℳ, ε, xₛ, xₜ, S)

Input: An n-dimensional SMMS ℳ = {μ₁, μ₂, . . . , μₙ}, starting point xₛ, target point xₜ, open and path-connected safety set S, and precision ε > 0
Output: Dynamic reachability algorithm to reach ε neighborhood of xₜ using SMMS ℳ.

1 if IS_ALMOST-SURE_REACHABLE(ℳ) = No then
2 return Can not guarantee almost-sure reachability
3 else
4 Compute continuous path ν : [0, 1] → S from xₛ to xₜ using RRT or Canny’s algorithm
5 Let B₁, B₂, . . . , B_q be a finite set of open balls that covers the path ν and stays inside the safety set S, Bᵢ ∩ Bᵢ₊₁ ≠ ∅, xₛ ∈ B₁, and xₜ ∈ B_q
6 Let x₀ = xₛ, x₁, . . . , x_q₋₁, x_q = xₜ be set of points such that xᵢ ∈ Bᵢ ∩ Bᵢ₊₁ for all 1 ≤ i < q and xₛ ∈ B₁ and xₜ ∈ B_q
7 Set y(0) = xₛ
8 Set k = 0
9 while ∥y(k) − xₜ∥ > ε do
10 k := k + 1
11 y(k) := REACH_IN_A BALL(ℳ, ε, y(k), x_k, B_k)
12 return y(k)

3. (path-following) compute the control policy to steer the system from xₛ to an arbitrary neighborhood of xₜ while ensuring safety.

Algorithm 1 provides pseudocode for the this motion planning problem that invokes Algorithms 2, 3, and 4, and a call to an off-the-shelf path-finding algorithm. There are well-established algorithms to explore non-convex, high-dimensional spaces including the rapidly exploring random tree (RRT) algorithm [23].

Intuitively, the RRT algorithm can return a path from the source to the destination by random exploration of the state space. This path can be robustly followed by repeated applications of our algorithm in the context of systems modeled as SMMSs by exploiting the fact that S is an open set and the image C of the path ν is compact, and hence find:

\[ r* = \min_{x \in C, y \in S} \|x - y\| > 0, \]  \hspace{1cm} (16)

which exists due to the compactness of C and closedness of S = \( \mathbb{R}^n \) \( S \). Also, note that if the discovered path is a piece-wise linear path and S is defined by a set of linear inequalities, then \( r* \) in (16) can be lower-bounded by the smallest of the minimum distances of vertices of C from the faces of the hyperplanes that define S.

Once \( r* \) is found, let \( x₁, \ldots, x_k \in S \) be such that \( x₁ = xₛ \) and \( x_k = xₜ \) and

\[ \|x_i - x_{i+1}\| \leq r*/2 \text{ for } i = 1, \ldots, k - 1. \]

Then, \{B(xᵢ, 3r* / 4)\}_{i∈[k]} would be a cover satisfying

\[ B(xᵢ, 3r* / 4) \cap B(x_{i+1}, 3r* / 4) \neq \emptyset \]

and hence, the proof technique of Theorem 3 can be applied to establish a safe routing from xₛ to xₜ.

The steps 5 and 6 of Algorithm 1 assume the existence of such sequence of balls and repeatedly invoke Algorithm 3 to accomplish reachability within a ball given the stochastic multi-mode system satisfies ε-reachability property (Algorithm 4).

Now we turn our focus to analyze computational complexity of deciding ε-reachability property for path-connected and bounded open safety sets. For the sake of algorithmic analysis of the problem we
assume that, for a given SMMS, the distributions in all of the modes are computationally tractable, i.e., the expected vector for each stochastic mode is rational and can be computed in polynomial time. The following complexity result follows from the necessary and sufficient condition developed in the previous sections.

Lemma 5 The decision version of the ε-reachability problem for stochastic multi-mode systems is in PTIME.

Proof: Let $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_{\gamma}\}$ be an SMMS with a finite set of compactly-supported modes. According to Theorem 3, deciding ε-reachability is equivalent to deciding whether for all $x \in \mathbb{R}^n$ we have that $x \cdot \overline{\mu}_i > 0$ for some $i$.

This later fact is equivalent to showing that any vector $x \in \mathbb{R}^n$ can be written as a non-negative linear combination of vectors in $\{\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_{\gamma}\}$. It is well known [27] that this property holds if and only if the following conditions hold:

a. The set $\{\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_{\gamma}\}$ spans $\mathbb{R}^n$, and

b. $\sum_{i=1}^{\gamma} \alpha_i \overline{\mu}_i = 0$ for some real numbers $\alpha_i \geq 1$.

This observation implies the correctness of Algorithm 4 in deciding the almost-sure ε-reachability problem for stochastic multi-mode systems.

Note that the key computation effort in the algorithm is to check the solution of a linear program and to verify the full-rank condition of a matrix. Since both linear programming [25] and matrix rank computation [20] can be solved in polynomial time, it follows that almost-sure ε-reachability problem can be decided in polynomial time.

Figure 3 shows a simulation of a two-dimensional SMMS that models a self driving car with stochastic control directions. This model consists of modes $\mathcal{M} = \{(0, 1) + U, (1, 0) + U, (0, -1) + U, (-1, 0) + U\}$, where $U$ is a uniform distribution over the $[-7.5, 7.5] \times [-7.5, 7.5]$ box in $\mathbb{R}^2$. Note that the variance of the noise is much higher than the length of the expected control directions, i.e., the plant has a very low signal to noise ratio. The control objective is to take the (red) car out of the parking lot while maintaining
Algorithm 3: \textsc{Reach\_In\_A\_Ball}(\mathcal{M}, \varepsilon, x_s, x_t, B(x_0, r))

\begin{algorithm}[H]
\begin{algorithmic}[1]
\State \textbf{Input}: An \textit{n}-dimensional SMMS $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_\gamma\}$, starting point $x_s$, target point $x_t$, safety set $B(x_0, r)$, and precision $\varepsilon > 0$
\State \textbf{Output}: Dynamic reachability algorithm to reach $\varepsilon$ neighborhood of $x_t$ using SMMS $\mathcal{M}$.
\State Set $\tilde{r} = \frac{1}{2} \min(r - \|x_s - x_0\|, r - \|x_t - x_0\|)$
\State Set $N = \left\lceil \frac{\|x_t - x_s\|}{\tilde{r}} \right\rceil + 1$
\State Set $y(0) = x_s$
\State Set $k = 0$
\State \While{$\|y(k) - x_t\| > \varepsilon$}
\State $k := k + 1$
\State $u(k) := x_s + \frac{k}{N}(x_t - x_s)$
\State $y(k) := \text{Reach\_Center}(\mathcal{M}, \min(\varepsilon, \tilde{r}), y(k - 1), u(k), 2\tilde{r})$
\EndWhile
\State \Return $y(k)$
\end{algorithmic}
\end{algorithm}

Algorithm 4: \textsc{Is\_Almost\_Sure\_Reachable}(\mathcal{M})

\begin{algorithm}[H]
\begin{algorithmic}[1]
\State \textbf{Input}: An \textit{n}-dimensional SMMS $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_\gamma\}$.
\State \textbf{Output}: Yes, if the $\varepsilon$-reachability property holds for path-connected and bounded safety sets, and No otherwise.
\State Compute the expected multimode system
\State $\mathcal{M} = \{\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_\gamma\}$
\State Compute
\State $P := \min_{i=1}^{\gamma} \|\alpha_i \overline{\mu}_i\|_1$
\State subject to: $\alpha_i \geq 1$ for all $1 \leq i \leq \gamma$.
\State \If{$(\text{Rank}[\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_\gamma] = n)$ and $(P = 0)$}
\State \Return Yes
\EndIf
\State \Else \Return No
\end{algorithmic}
\end{algorithm}

Safety. Figure 3a shows a sample path generated by the proposed algorithm. The circles mark the width of the safe region. As it can be seen in that picture, safety is maintained while taking the car out of the parking lot. Figure 3b shows a zoomed-in view of the same sample path.

7 Conclusion

We introduced and studied stochastic multi-mode systems, which are a natural formalism in motion planning when hierarchical control is combined with noisy sensors or actuators. A key result of this paper is that it is possible to efficiently decide the $\varepsilon$-reachability problem, that is, whether a control strategy exists that steers a stochastic multi-mode system from any point in an open, path-connected set to an arbitrary neighborhood of any other point in that set with probability one. We have shown in particular that a stochastic system enjoys the $\varepsilon$-reachability property if and only if the associated deterministic system does. This condition implies that for almost-sure $\varepsilon$-reachability of stochastic multi-mode systems can be checked efficiently.
As a natural next step we are investigating the possibility of employing decision procedures for stochastic multi-mode systems in a counterexample-guided abstraction-refinement framework in order to develop motion planning algorithms for systems with richer dynamics (linear-hybrid systems) in the presence of stochastic uncertainties.

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