Two dimensional RG flows and Yang-Mills instantons

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Abstract: We study RG flow solutions in (1,0) six dimensional supergravity coupled to an anti-symmetric tensor and Yang-Mills multiplets corresponding to a semisimple group $G$. We turn on $G$ instanton gauge fields, with instanton number $N$, in the conformally flat part of the 6D metric. The solution interpolates between two $(4,0)$ supersymmetric $AdS_3 \times S^3$ backgrounds with two different values of $AdS_3$ and $S^3$ radii and describes an RG flow in the dual 2D SCFT. For the single instanton case and $G = SU(2)$, there exist a consistent reduction ansatz to three dimensions, and the solution in this case can be interpreted as an uplifted 3D solution. Correspondingly, we present the solution in the framework of $N = 4 \ (SU(2) \ltimes \mathbb{R}^3)^2$ three dimensional gauged supergravity. The flows studied here are of v.e.v. type, driven by a vacuum expectation value of a (not exactly) marginal operator of dimension two in the UV. We give an interpretation of the supergravity solution in terms of the D1/D5 system in type I string theory on K3, whose effective field theory is expected to flow to a $(4,0)$ SCFT in the infrared.

Keywords: AdS/CFT correspondence, Gauge/Gravity correspondence, Supergravity Models.
1. Introduction

Renormalization Group flows have been a subject of much study soon after the proposal of the AdS/CFT correspondence [1]. For supersymmetric flows, solutions interpolating between two AdS critical points of the dual supergravity theories can be obtained by looking at BPS equations. The flows describe typically deformations of the CFT at the UV fixed point by relevant operators or, less typically, by vacuum expectation values of relevant operators and, like in the present case, by vevs of marginal operators. The theory is then driven to another fixed point in the IR. These flow solutions have been originally studied in the context of five dimensional or three dimensional gauged supergravities to describe RG flows in the dual four dimensional or two dimensional SCFT’s. Examples of these can be found in [2, 3, 4] for the 4D case and in [5, 6, 7] for the 2D case. In some cases, these solutions can be uplifted to higher dimensions by using consistent truncation formulae, and this will also be the case for a class of solutions of the present paper.

Here we study some RG flow solutions in the context of AdS$_3$/CFT$_2$. Our starting point is (1,0) six dimensional supergravity coupled to a tensor multiplet and to $SU(2)$ Yang-Mills multiplets. In an earlier paper [8], we have shown that the $SU(2)$ reduction of this theory gives rise to $N = 4$ three dimensional gauged supergravity with scalar manifold $SO(4,4)/SO(4) \times SO(4)$. We first discuss a 6D flow solution which, in fact, is the lift to 6D of an RG flow of the 3D $N = 4$ gauged supergravity and preserves half of the supersymmetries. This solution involves an $SU(2)$ instanton on $\mathbb{R} \times S^3$, $\mathbb{R}$ being the radial coordinate, with topological charge equal to 1, which in the 3D setting is seen as a scalar’s background. The instanton interpolates between the $|0\rangle$ Yang-Mills vacuum with winding number 0 in the IR and $|1\rangle$ vacuum with winding number 1 in the UV. We then move to study solutions involving multi-instanton gauge fields of an arbitrary semisimple gauge group $G$. In this case, the solution is genuinely six dimensional in the sense that it cannot be obtained as an uplifted solution of a three dimensional theory, roughly, because it involves higher modes on $S^3$. The instanton interpolates between $|N\rangle$ vacuum in the UV and $|0\rangle$ in the IR. The solution has been studied long ago in [9, 10] but in different contexts. In this paper, we will look at it from another point of view by regarding it as an RG flow solution interpolating between the UV and IR CFT’s corresponding to two AdS$_3$ limits. The central charge at the two fixed points of course respects the c-theorem and admits an interpretation in terms of the dynamics of the $D1/D5$ dual system giving rise to a $(4,0)$ SCFT in the decoupling limit[11, 12, 13]. What emerges from our solutions is that the flow from UV to IR is essentially a manifestation of the Coulomb branch, where $N$ D5-branes, if the instanton number is $N$, decouple from the D1D5 system as we move towards the
asymptotic infrared region. Therefore in the IR, the central charge, which is linear in
the number of D5 branes, decreases accordingly. We also discuss the contribution of
the gauge Chern-Simons terms to the central charge of $SU(2)$ left moving R-symmetry
current algebra. In doing this, we will give a different derivation of the result obtained
in [14]: our derivation involves the computation of chiral correlators of the stress energy
tensors or of the currents, respectively, in the spirit of [15]. As a byproduct, we will
obtain a holographic derivation of Virasoro’s and current algebra Ward identities.

The paper is organized as follows. We start by finding an RG flow solution in
the $(1,0)$ six dimensional supergravity coupled to an anti-symmetric tensor and $SU(2)$
Yang-Mills multiplets in section 2. As we mentioned, there exist a consistent reduction
to three dimensional gauged supergravity for the $SU(2)$ gauge fields describing a single
instanton. In appendix A, we also give the same solution in the framework of $N = 4$
three dimensional gauged supergravity. In section 3, we generalize the solution found
in section 2 to the case in which the gauge fields of an arbitrary semisimple gauge group
describe a configuration of $N$ instantons. We also give an interpretation of the solution
in term of D1/D5 brane system in type I string theory. We then discuss the central
charge of the dual CFT. As mentioned above, we rederive Virasoro and current algebra
central charges, and give some details of the derivation in appendix B. We end this
paper by making some conclusions and comments in section 4.

2. An RG flow solution from six dimensional supergravity on
$SU(2)$ group manifold

In this section, we study an RG flow solution of the $(1,0)$ six dimensional supergravity
coupled to an anti-symmetric tensor and $SU(2)$ Yang-Mills multiplets. This is a special
case, where $G = SU(2)$, of the theory studied in [8] in the context of $SU(2)$ reduction.
This theory is an ungauged version of the $(1,0)$ six dimensional gauged supergravity
constructed in [10, 11]. Here, we will find a flow solution in the six dimensional framework. At the fixed points, the solution can be interpreted as supersymmetric $AdS_3 \times S^3$
backgrounds.

We start by giving some necessary formulae we will use in this and the following
sections. Six dimensional supergravity coupled to an antisymmetric tensor multiplet
admits a Lagrangian formulation. According to the formulation in [17], the Bianchi
identity for the three form field $\hat{G}_3$ is modified in the presence of gauge fields $\hat{A}$ by

$$\hat{d}\hat{G}_3 = v^2 \text{Tr}(\hat{F} \wedge \hat{F}).$$  \hspace{1cm} (2.1)

The equation of motion for $\hat{G}_3$ is also modified to

$$\hat{d}(e^{2\Phi} \hat{G}_3) = -v^2 \text{Tr}(\hat{F} \wedge \hat{F}).$$ \hspace{1cm} (2.2)
where $\hat{\theta}$ is the scalar field in the tensor multiplet. It has been shown in [17] that when one of the parameters $v^z$ and $\tilde{v}^z$ vanishes, an invariant Lagrangian can be written down. In [8], we have chosen $v^z = 1$ and $\tilde{v}^z = 0$ and shown that there is a consistent $SU(2)$ reduction to $N = 4$ three dimensional gauged supergravity. Throughout this section, we will work with this choice so that the solution in this section can be considered as a flow solution of the $SU(2) \times SU(2) \sim SO(3) \times SO(3)$ three dimensional gauged supergravity. Furthermore, we will use $SU(2)$ and $SO(3)$ interchangeably in this paper.

In order to find supersymmetric solutions, we need supersymmetry transformations for fermionic fields. We only give here the expressions and refer the readers to [16, 17] for more details. We distinguish six and three dimensional fields by putting a hat on all six dimensional fields. The supersymmetry transformations of the gravitino $\psi_\mu$, gauginos $\lambda^I$ and the fermion in the tensor multiplet $\chi$ are given by [17]

$$\delta \psi_M = \hat{D}_M \epsilon + \frac{1}{24} e^{\hat{\theta}} \Gamma^{NPQ} \Gamma_M \hat{G}_{3NPQ} \epsilon,$$  

$$\delta \lambda^I = \frac{1}{4} \Gamma^{MN} \hat{F}^I_{MN} \epsilon,$$  

$$\delta \chi = \frac{1}{2} \Gamma^M \partial_M \hat{\theta} \epsilon - \frac{1}{12} e^{\hat{\theta}} \Gamma^{MN} \hat{G}_{3MN} \epsilon$$

where we have given only the bosonic contributions. Note that we have not put a hat on fermions because we will not consider three dimensional fermions here. This is all we need for finding BPS solutions.

We now review the reduction ansatz used in [8]

$$ds^2 = e^{2f} ds^2 + e^{2g} h_{\alpha\beta} \nu^\alpha \nu^\beta,$$

$$\hat{A}^I = A^I + A_\alpha^I \nu^\alpha, \quad \nu^\alpha = \sigma^\alpha - g_1 A^\alpha,$$

$$\hat{F}^I = d\hat{A}^I + \frac{1}{2} g_2 f_{IJK} \hat{A}^I \wedge \hat{A}^K$$

$$= F^I - g_1 A_\alpha^I F^\alpha + \mathcal{D} A^I_\alpha \wedge \nu^\alpha + \frac{1}{2} (g_2 A_\alpha^I A_\beta^K f_{IJK} - \epsilon_{\alpha\beta\gamma} A_\gamma^I) \nu^\alpha \wedge \nu^\beta,$$

$$\hat{G}_3 = \tilde{h} \epsilon_3 + F^\alpha \wedge \nu^\alpha + \frac{1}{2} K_{\alpha\beta} \wedge \nu^\alpha \wedge \nu^\beta + \frac{1}{6} \tilde{a} \epsilon_{\alpha\beta\gamma} \nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma$$

where

$$\tilde{h} = h \epsilon_3 + \tilde{F}^I \wedge A^I - \frac{1}{6} g_2 A^I \wedge A^J \wedge A^K f_{IJK},$$

$$\tilde{F}^\alpha = A_\alpha^I (2 F^I - g_1 A_\alpha^I F^\alpha) - 6 a g_1 F^\alpha,$$

$$\tilde{a} = 6 a - A_\alpha^I A_\alpha^I + \frac{1}{3} g_2 A_\alpha^I A_\beta^I A_\gamma^K f_{IJK} \epsilon_{\alpha\beta\gamma},$$

and

$$K_{\alpha\beta} = A_\beta^I \mathcal{D} A_\alpha^I - A_\alpha^I \mathcal{D} A_\beta^I.$$
In the present case, indices $I, J, K = 1, 2, 3$ are $SU(2)$ adjoint indices. The structure constant $f_{IJK}$ will be replaced by $SU(2)$ structure constant $\epsilon_{IJK}$. $\mathcal{D}$ is the $SU(2) \times SU(2)$ covariant derivative with coupling constants $g_1$ and $g_2$, respectively,

$$\mathcal{D}A^I_\alpha = dA^I_\alpha + g_1 \epsilon_{\alpha\beta\gamma} A^\beta A^I_\gamma + g_2 \epsilon_{IJK} A^J A^K_\alpha.$$  

(2.8)

In order to obtain the vielbein on the $S^3$ part, we introduce the scalar matrix $L_\alpha^i$ defined by

$$h_{\alpha\beta} = L_\alpha^i L_\beta^i.$$  

(2.9)

The left invariant $SU(2)$ one forms satisfy

$$d\sigma^\alpha = -\frac{1}{2} \epsilon_{\alpha\beta\gamma} \sigma^\beta \wedge \sigma^\gamma.$$  

(2.10)

Finally, there is a truncation in both bosonic and fermionic fields in the reduction of $S^3$. The relevant relation for our discussion is the bosonic one given by

$$h_{\alpha\beta} = e^{\theta - 2g} (12a \delta_{\alpha\beta} - 2A^I_\alpha A^I_\beta)$$  

(2.11)

where $a$ is a constant. This relation is a consequence of truncating out massive vector fields in three dimensions.

In the ansatz for the flow solution, we set $A^I_\mu = A^\alpha_\mu = 0$, $A^I_\alpha = \delta^I_\alpha A$ and $L_\alpha^i = \delta_\alpha^i$. After various scalings discussed in [8] together with the standard domain wall ansatz in three dimensions, we find

$$ds^2 = e^{2w(r)} (-dx_0^2 + dx_1^2) + dr^2 + \frac{e^{2q(r)}}{4g_1^2} \delta_{\alpha\beta} \sigma^\alpha \sigma^\beta,$$

$$\hat{A}^I = A^I_\alpha \sigma^\alpha = A \delta^I_\alpha \sigma^\alpha = A \sigma^I,$$

$$\hat{F}^I = \frac{1}{g_1} dA \wedge \sigma^I + \frac{1}{2g_1^2} (g_2 A^3 - g_1 A) \epsilon_{IJK} \sigma^J \wedge \sigma^K,$$

$$\hat{G}_3 = h \varepsilon + \frac{1}{6g_1^3} \left(g_1 + 2g_2 A^3 - 3g_1 A^2\right) \epsilon_{IJK} \sigma^I \wedge \sigma^J \wedge \sigma^K.$$  

(2.12)

The scalings have been performed to restore the $g_1$ and $g_2$ in the appropriate positions in the solution. This makes the comparison with the three dimensional solution given in appendix A clearer. Notice the particular ansatz for $A^I_\alpha$ which gives $K_{\alpha\beta} = 0$. This is the reason for the consistency of the truncation of the three dimensional gauge fields, $A^I_\mu = 0$ and $A^\alpha_\mu = 0$. It can be easily checked that all the three dimensional field equations given in [8] are satisfied by our ansatz. The $S^3$ part of the metric and that in (2.6) are related by

$$\frac{e^{2q}}{4g_1^2} \delta_{\alpha\beta} = e^{2g} h_{\alpha\beta} = 2e^g (1 - A^2) \delta_{\alpha\beta}$$  

(2.13)
where we have used (2.11) after scalings. We will see later that our solution satisfies this relation and is indeed a solution of the theory obtained in [8]. The supersymmetric flow solution can be found by considering the Killing spinor equations coming from the supersymmetry transformation of fermions. From the metric, we can read off the vielbeins

\[ \hat{e}^a = e^w dx^a, \quad \hat{e}^r = dr, \quad \text{and} \quad \hat{e}^i = \frac{e^q}{2g_1} \sigma^i \] (2.14)

We can compute the following spin connections

\[ \hat{\omega}^{a \hat{r}} = w' \hat{e}^a, \]
\[ \hat{\omega}^{\hat{r} i} = -q' \hat{e}^i, \]
\[ \hat{\omega}^{ij} = -g_1 e^{-q} \epsilon_{ijk} \hat{e}^k. \] (2.15)

The index \( a \) is the tangent space index for \( \mu = 0, 1 \), and \( ' \) means \( \frac{d}{dr} \). For conveniences, we also repeat here the decompositions of the six dimensional gamma matrices from [18]

\[ \Gamma^A = (\Gamma^a, \Gamma^i), \quad \Gamma^a = \gamma^a \otimes I_2 \otimes \sigma_1, \]
\[ \Gamma^i = I_2 \otimes \gamma^i \otimes \sigma_2, \quad \Gamma_7 = I_2 \otimes I_2 \otimes \sigma_3, \]
\[ \gamma^{abc} = \epsilon^{abc}, \quad \gamma^{ijk} = i \epsilon^{ijk}, \quad \{ \gamma_a, \gamma_b \} = 2\eta_{ab}, \quad \{ \gamma_i, \gamma_j \} = 2\delta_{ij} \] (2.16)

with the same conventions given in [18]. We further specify the three dimensional gamma matrices by the following choice

\[ \gamma^0 = i\tilde{\sigma}_2, \quad \gamma_1 = \tilde{\sigma}_1, \quad \gamma^2 = \tilde{\sigma}_3. \] (2.17)

Using (2.3), (2.4) and (2.3), we find

\[ \delta \lambda^I = 0 : \quad A' = -2(g_2 A^2 - g_1 A) e^{-q}, \]
\[ \delta \chi = 0 : \quad \theta' = e^{\theta} [h - 8 e^{-3q} (g_1 + 2g_2 A^3 - 3g_1 A^2)], \]
\[ \delta \psi_i = 0 : \quad q' = -g_1 e^{-q} + \frac{1}{2} [e^{\theta} [h + 8 e^{-3q} (g_1 + 2g_2 A^3 - 3g_1 A^2)]], \]
\[ \delta \psi_a = 0 : \quad w' = -\frac{1}{2} e^{\theta} [h + 8 e^{-3q} (g_1 + 2g_2 A^3 - 3g_1 A^2)] \] (2.18)

where we have used \( \tilde{\sigma}_3 \otimes I_2 \otimes I_2 \epsilon = \epsilon \). So, the solution preserves half of the (1,0) supersymmetry in six dimensions. As in [8], we fix \( h \) by using the equation of motion for \( \hat{G}_3 \)

\[ \hat{D}(e^{2\theta} \hat{e} \hat{G}_3) = 0. \] (2.19)
This gives $he^{3q+2\theta} = c_1$ with a constant $c_1$. Using this result and changing the coordinate $r$ to $\tilde{r}$ given by $\frac{dr}{d\tilde{r}} = e^{-q}$, we find that the above equations can be rewritten as

\begin{align*}
\theta' &= e^{\theta-2q}(c_1 e^{-2q} - 8\tilde{a}) , \\
q' &= -g_1 + \frac{1}{2} e^{\theta-2q}(e^{-2q} c_1 + 8\tilde{a}) , \\
w' &= -\frac{1}{2} e^{\theta-2q}(c_1 e^{-2q} + 8\tilde{a}) , \\
A' &= -2(g_2 A^2 - g_1 A),
\end{align*}

(2.20)

(2.21)

(2.22)

(2.23)

where $\tilde{a} = g_1 + 2g_2 A^3 - 3g_1 A^2$. The $'$ is now $\frac{d}{d\tilde{r}}$. Before solving these equations, let us look at the fixed points given by the conditions $\theta' = q' = A' = 0$. There are two fixed points:

- I:

\begin{align*}
A &= 0, \quad \theta = \frac{1}{2} \ln \frac{c_1}{8g_1}, \\
q &= \frac{1}{4} \ln \frac{8c_1}{g_1},
\end{align*}

(2.24)

- II:

\begin{align*}
A &= \frac{g_1}{g_2}, \quad \theta = \frac{1}{2} \ln \frac{c_1 g_2^2}{8g_1(g_2^2 - g_1^2)}, \\
q &= \frac{1}{4} \ln \frac{8c_1(g_2^2 - g_1^2)}{g_1 g_2^2}.
\end{align*}

(2.25)

Equation (2.23) can be solved and gives

\begin{align*}
A &= \frac{g_1}{g_2} e^{g_1 C_2 - 2g_1 \tilde{r}}.
\end{align*}

(2.26)

Taking the combination (2.20) + 2 (2.21), we find

\begin{align*}
z' &= 2e^{-z} c_1 - 2g_1
\end{align*}

(2.27)

where $z = \theta + 2q$. From (2.27), we find the solution for $z$ is

\begin{align*}
z &= \ln \frac{c_1 - e^{-2g_1 \tilde{r} + C_3}}{g_1}.
\end{align*}

(2.28)

From (2.20), we see that the fixed point I is at $\tilde{r} \to -\infty$ while the II point is at $\tilde{r} \to \infty$. Regularity of $A$ requires that $-e^{g_1 C_2}$ must have the same sign as $g_2$. For convenience, we choose

\begin{align*}
C_2 &= \frac{1}{g_1} \ln(-g_2).
\end{align*}
From (2.28), \( z \) blows up as \( \tilde{r} \to -\infty \), so the solution breaks down at the I point. To overcome this problem, we choose \( z \) to be constant in such a way that (2.27) is satisfied identically. This can be achieved by setting

\[
\begin{align*}
z &= \ln \frac{c_1}{g_1}. \tag{2.29}
\end{align*}
\]

This means \( \theta = \ln \frac{c_1}{g_1} - 2q \). We can see that this condition is satisfied at both fixed points, and equations (2.21) and (2.20) collapse to a single equation namely

\[
q' = 4e^{-4q} \frac{c_1}{g_1} \left( g_1 - \frac{e^{4g_1\tilde{r}}(3 + e^{2g_1\tilde{r}})g_1^3}{g_2^2(1 + e^{2g_1\tilde{r}})^2} \right) - \frac{g_1}{2}. \tag{2.30}
\]

This equation can be solved, and we find

\[
q = \frac{1}{4} \ln \left[ 8e^{-2g_1\tilde{r}} \left( c_1 e^{2g_1\tilde{r}} \left( \frac{1}{g_1} - \frac{g_1}{g_2^1} \right) - \frac{c_1 g_1(2 + 3e^{2g_1\tilde{r}})}{g_2^2(1 + e^{2g_1\tilde{r}})^2} + \frac{54C_4}{g_2^2} \right) \right]. \tag{2.31}
\]

In order to make the solution for \( q \) interpolates between the two values at both fixed points, we need to choose

\[
C_4 = \frac{c_1 g_1}{27}. \tag{2.32}
\]

We finally find

\[
\begin{align*}
A &= \frac{g_1}{g_2(1 + e^{-2g_1\tilde{r}})} \tag{2.33} \\
q &= \frac{1}{4} \ln \frac{8c_1(g_2^2 + 2g_2^2 e^{2g_1\tilde{r}} + (g_2^2 - g_1^2)e^{4g_1\tilde{r}})}{g_1 g_2^2(1 + e^{2g_1\tilde{r}})^2} \tag{2.34} \\
w &= -q - g_1\tilde{r} \tag{2.35}
\end{align*}
\]

We neglect all additive constants to \( w \) because they can be absorbed in the rescaling of \( x_0 \) and \( x_1 \). The solution for \( q \) approaches the fixed point I and II as \( \tilde{r} \to \mp \infty \), respectively.

At the fixed points, the six dimensional metric is given by

\[
\begin{align*}
ds^2 &= e^{-2q_0 - 2g_1\tilde{r}} dx_{1,1}^2 + e^{2q_0} d\tilde{r}^2 + \frac{e^{2q_0}}{4g_1^2} \delta_{\alpha\beta} \sigma^\alpha \sigma^\beta \tag{2.36}
\end{align*}
\]

where \( q_0 \) is the value of \( q \) at the fixed points. By rescaling the \( x^\mu \) and \( \tilde{r} \) by a factor of \( e^{-q_0} \) and \( -e^{q_0} \), respectively, we can write (2.36) as

\[
\begin{align*}
ds^2 &= e^{2\tilde{r}} dx_{1,1}^2 + d\tilde{r}^2 + \frac{R^2}{4} \delta_{\alpha\beta} \sigma^\alpha \sigma^\beta \tag{2.37}
\end{align*}
\]
which is the AdS$_3 \times S^3$ metric. The radii of AdS$_3$ and $S^3$ are given by $L = \frac{e^{q_0}}{g_1}$ and $R = \frac{e^{q_0}}{g_1}$, respectively. The central charge in the dual CFT is given by [14]

$$c = \frac{3L}{2G_N^{(3)}} \sim e^{4q_0}$$

where we have used the relation between Newton constants in three and six dimensions $G_N^{(3)} = \frac{G_N^{(6)}}{\text{Vol}(S^3)}$. We find the ratio of the central charges

$$\frac{c_I}{c_{II}} = \frac{e^{4q_0}|I|}{e^{4q_0}|II|} = \frac{1}{1 - \frac{g_2}{g_1^2}} > 1. \quad (2.38)$$

From this equation, we find that the flow respects the c-theorem as it should, and point I is the UV point while point II is the IR point. Note that $d\tilde{r} = -e^{q_1}d\tilde{r}$, so the UV and IR points correspond to $\tilde{r} \to \pm \infty$. We can interpret $\tilde{r}$ as an RG scale in the dual two dimensional field theory. From the solutions for $q, \theta$ and $A$, we can check that the relation (2.13) is satisfied. So, the solution is indeed a solution of the theory considered in [8] and can be obtained from three dimensional gauged supergravity. We also give this solution in the three dimensional framework in appendix A.

We briefly look at the behavior of the scalar fields near the UV point I. From (2.33) and (2.34), we find that

$$A \sim e^{2g_1\tilde{r}} \sim e^{-\frac{2g_1}{\tilde{r}}} \quad \text{and} \quad e^q \sim e^{-\frac{2g_1}{\tilde{r}}} \quad (2.39)$$

We can see that the flow is driven by a vacuum expectation value of a marginal operator of dimension two. Although this is not expected, we will confirm this fact in appendix A in which we will reobtain this solution in the three dimensional gauged supergravity. So, this flow is a vev. flow driven by a vacuum expectation value of a marginal operator. Notice that gauge-field background of (2.12) that we have found corresponds to a single SU(2) instanton on the four-space ($r, S^3$), interpolating between winding number 0 for $\tilde{r} \to -\infty$ and winding number 1 for $\tilde{r} \to +\infty$. In the next section we will generalize this result to a multi-instanton configuration for semisimple $G$ gauge fields, which therefore will not admit a three dimensional interpretation.

Throughout this section, we have mainly studied the flow solution in the context of the SU(2) reduction to three dimensions. This leads to the form of the solution given above. Before discussing the multi-instanton case, we would like to change the form of the solution to make contact with what we will find in the next section. First of all, we can change the coordinates in (2.12) to $R$ given by

$$\frac{dR}{dr} = -g_1 R e^{-q_1}. \quad (2.40)$$
We have put a minus sign in order to identify the UV point with $R \to \infty$ and the IR with $R \to 0$. We then find that the metric is given by

$$ds^2 = e^{2w}(-dx_0^2 + dx_1^2) + \frac{e^{2q}}{g_1 R^2} dy^i dy^i$$

(2.41)

where $dy^i dy^i = dR^2 + \frac{R^2}{4} \sigma^\alpha \sigma^\alpha$ is the flat metric of the four dimensional space. This is the form of the metric we will see in the next section in which the 4-dimensional part is conformally flat. The second point is the solution for $A$ in (2.26). Recall that the relation between $R$ and $\tilde{r}$ is $\frac{dR}{d\tilde{r}} = -g_1 R$, we can write

$$A = \frac{\chi^2}{g_2 R(\chi^2 + R^2)}$$

(2.42)

where we have chosen $C_2 = -\frac{1}{g_1} \ln \left( -\frac{g_2}{\chi^2} \right)$. This is a single instanton solution at the origin $R = 0$ in the polar coordinates. Notice that this is the instanton solution in the singular gauge in which the winding number come from the contribution near $R = 0$. In the next section, we will study a flow solution with $N$ instantons but in the regular gauge.

3. RG flow solutions and multi-instantons

In this section, we generalize the solution obtained in the previous section by considering the gauge field configuration describing $N$ instantons. We will further make an extension to gauge fields of an arbitrary gauge group $G$. The solution we will study is very similar to the solution given in [9] and further studied in [10]. In this paper, we give an interpretation of this solution in the context of an RG flow in the dual two dimensional field theory. We start by reobtaining this solution and then discuss its implication in term of the RG flow.

3.1 Flow solutions

Since we are going to use the full six-dimensional theory, we will now turn on both $v^z$ and $\tilde{v}^z$. Throughout this section, we also assume that both $v^z$ and $\tilde{v}^z$ are positive. If this is not the case, the phase transition discussed in [3] is unavoidable. The supersymmetry transformations of fermions are the same to leading order in fermionic fields and given by (2.3), (2.4) and (2.5). On the other hand, the bosonic field equations are

$$\hat{D}(e^{2\hat{\theta}} \hat{\ast} \hat{G}_3) + \tilde{v}^z \hat{F}^I \wedge \hat{F}^I = 0,$$

(3.1)

$$\hat{D}[(v^z e^{\hat{\theta}} + \tilde{v}^z e^{-\hat{\theta}}) \hat{\ast} \hat{F}^I] - 2v^z e^{2\hat{\theta}} \hat{G}_3 \wedge \hat{F}^I + 2\tilde{v}^z \hat{\ast} \hat{G}_3 \wedge \hat{F}^I = 0,$$

(3.2)

$$\hat{d} \hat{\ast} \hat{d} \hat{\theta} + (v^z e^{\hat{\theta}} + \tilde{v}^z e^{-\hat{\theta}}) \hat{\ast} \hat{F}^I \wedge \hat{F}^I + 2e^{2\hat{\theta}} \hat{\ast} \hat{G}_3 \wedge \hat{G}_3 = 0.$$
It is easy to see that if we set \( v^z = 1, \tilde{v}^z = 0 \) and take a spherically symmetric single instanton configuration (i.e. an instanton at the origin of \( \mathbb{R}^4 \)) for the gauge field \( A^I \), then the above equations reduce to the ones discussed in the previous section. The Bianchi identity is

\[
\hat{D}\hat{G}_3 = v^z \hat{F}^I \wedge \hat{F}^I. \tag{3.4}
\]

We take an ansatz for the metric as

\[
ds_6^2 = e^{2f}(-dx_0^2 + dx_1^2) + ds_4^2 \tag{3.5}
\]

where \( f \) only depends on the coordinates \( z^\alpha, \alpha = 2, \ldots, 5 \) of the four dimensional metric \( ds_4^2 = g_{\alpha\beta}dz^\alpha dz^\beta \). We first look at the \( \delta\lambda^I = 0 \) equation. We can satisfy this condition by choosing \( F^I_{\alpha\beta} \) to be self dual because of the anti-self duality of the \( \Gamma_{\alpha\beta}, \alpha, \beta = 2, \ldots, 5 \). The anti-self duality of \( \Gamma_{\alpha\beta} \) is implied by the condition \( \Gamma_{7\epsilon} = \epsilon \) and the two-dimensional chirality \( \Gamma_{01\epsilon} = \epsilon \) chosen in \( \delta\psi_\mu = 0 \) below. The indices \( I, J, \ldots = 1, 2, \ldots, \text{dim} \ G \) are now \( G \) adjoint indices. The gauge fields and three form field strength are

\[
\hat{A}^I = A^I, \quad \hat{F}^I = F^I, \\
\hat{G}_3 = G + dx_0 \wedge dx_1 \wedge d\Lambda. \tag{3.6}
\]

The hatted fields are six dimensional ones while the unhatted fields represented by differential forms without indices have only components along \( ds_4^2 \). The \( x_0 \) and \( x_1 \) components will be shown explicitly. The three form field satisfies the Bianchi identity (3.4) which gives \( DG = v^z F^I \wedge F^I \). The dual of \( \hat{G}_3 \) is

\[
\hat{*}\hat{G}_3 = e^{-2f} * d\Lambda - e^{2f} dx_0 \wedge dx_1 \wedge *G \tag{3.7}
\]

where \( \hat{*} \) and \( * \) are Hodge duals in six and four dimensions, respectively. We have used the same convention as [17] namely \( \epsilon^{012345} = 1 \). Using equation (3.1), we find

\[
D(e^{2\theta-2f} * d\Lambda) = v^z F^I \wedge F^I, \tag{3.8}
\]

\[
D(e^{2\theta+2f} * G) = 0 \Rightarrow \ast G = e^{-2\theta-2f} \tilde{d}\Lambda. \tag{3.9}
\]

We take \( F^I \) to be self dual with respect to the four dimensional \( * \). This corresponds to an instanton configuration. The dual of \( \hat{F}^I \) is given by

\[
\hat{*}\hat{F}^I = -e^{2f} dx_0 \wedge dx_1 \wedge *F^I. \tag{3.10}
\]
We now come to supersymmetry transformations. Using our ansatz and the results given above, we find the Killing spinor equations

\[ \delta \chi = \frac{1}{2} \partial \theta \epsilon - \frac{1}{12} e^\theta \hat{G}_3 \epsilon \]
\[ = \frac{1}{2} \partial \theta + \frac{1}{2} e^{\theta - 2f} \partial \Lambda - \frac{1}{2} e^{-\theta - 2f} \tilde{\partial} \tilde{\Lambda} = 0 \]  

(3.11)

\[ \delta \psi_\mu = D_\mu \epsilon + \frac{1}{24} e^\theta \hat{G}_3 \Gamma_\mu \epsilon, \mu = 0, 1 \]
\[ = \frac{1}{2} \Gamma_\mu \partial f - \frac{1}{4} e^{\theta - 2f} \Gamma_\mu \partial \Lambda - \frac{1}{4} e^{-\theta - 2f} \Gamma_\mu \tilde{\partial} \tilde{\Lambda} = 0. \]  

(3.12)

We have used the notation \( \hat{G}_3 \equiv G^{MNP} \hat{G}_{3MNP} \) and a projector \( \Gamma^{2345} \epsilon = \epsilon \) which is also equivalent to \( \Gamma_{01} \epsilon = \epsilon \). This implies that the solution preserves half of the six dimensional supersymmetry. Taking the combination \((3.11) - 2(3.12)\), we find

\[ 2d\Lambda = -e^{-\theta + 2f} d(\theta - 2f). \]  

(3.13)

The solution is easily found to be

\[ \Lambda = \frac{1}{2} e^{-\theta + 2f} + C_1 \]  

(3.14)

with a constant \( C_1 \). Similarly, the combination \((3.11) + 2(3.12)\) gives

\[ de^{\theta + 2f} - 2d\tilde{\Lambda} = 0 \Rightarrow \tilde{\Lambda} = \frac{1}{2} e^{\theta + 2f} + C_2 \]  

(3.15)

with a constant \( C_2 \). The equation from \( \delta \psi_\alpha = 0 \) gives

\[ D_\alpha \epsilon - \frac{1}{2} \partial f \Gamma_\alpha \epsilon = 0. \]  

(3.16)

We now make the following ansatz for the 4-dimensional metric

\[ ds_4^2 = e^{2g} dy^i dy^i. \]  

(3.17)

With the supersymmetry transformation parameter of the form \( \epsilon = e^\frac{f}{4} \tilde{\epsilon} \), we can write equation (3.10) as

\[ \partial \xi \tilde{\epsilon} - \frac{1}{2} \Gamma_{ji} \partial j (f + g) \tilde{\epsilon} = 0. \]  

(3.18)

To satisfy this equation, we simply choose \( g = -f \) and find that \( \tilde{\epsilon} \) is a constant spinor. So, we have solved all the Killing spinor equations.

We can easily check that equation (3.2) is identically satisfied with our explicit
forms of $\Lambda$ and $\tilde{\Lambda}$. We now solve equations (3.8) and the Bianchi identity (3.4). We start with the $SU(2)$ instanton configuration from [19]

$$A_i = \frac{i}{2} \bar{\sigma}_{ij} \partial_j \ln \rho, \quad \rho = 1 + \sum_{a=1}^{n} \frac{\lambda_a^2}{(y - y_a)^2}. \quad (3.19)$$

Notice that, we have rescaled the $A_i$ form [19] by a factor of $\frac{1}{2}$. This can be done without losing generalities, see for example, [21] for a discussion. The $\bar{\sigma}_{ij}$ matrices are anti-self dual and can be written in terms of Pauli matrices $\sigma_x$ as [19]

$$\bar{\sigma}_{xy} = \frac{1}{4i} [\sigma_x, \sigma_y], \quad \bar{\sigma}_{x4} = -\frac{1}{2} \sigma_x, \quad x, y = 1, 2, 3. \quad (3.20)$$

This solution can be generalized to any semi-simple group $G$ to obtain the solution given in [20]. We can write the above $\bar{\sigma}_{ij}$ as $\bar{\eta}^I_{ij} t^I$ where $\bar{\eta}^I_{ij}$ and $t^I$ are 't Hooft’s tensor generating $SU(2)$ subgroup of $SO(4)$ and $SU(2)$ generators, respectively. For any group $G$, the solution is [20]

$$A^I_a = G^I_a \bar{\eta}^I_{ij} \partial_j \ln \rho, \quad a = 1, 2, 3. \quad (3.21)$$

For $G = SU(2)$, we simply have $G^I_a = \delta^I_a$. The solution for any group $G$ can be obtained by embedding $SU(2)$ in $G$ [20]. In order to solve (3.8) and (3.4), we need to compute $F^I_{ij} F^{ij}$. For $SU(2)$ instanton, we have [19]

$$F^I_{ij} F^{ij} = -\Box \ln \rho. \quad (3.22)$$

For $G$ instanton, the result is the same up to some numerical factors, from [20],

$$F^I_{ij} F^{ij} = -\frac{2}{3} c(G) d(G) \Box \ln \rho. \quad (3.23)$$

We now come to the Bianchi identity for $G$, $DG = v^z F^I \wedge F^I$, which gives

$$\Box e^{-(\theta + 2f)} = -v^z F^I_{ij} F^{ij} = v^z \frac{2}{3} c(G) d(G) \Box \ln \rho \quad (3.24)$$

where we have used [19]

$$*(F^I \wedge F^I) = *(F^I \wedge F^I) = \frac{1}{2} F^I_{ij} F^{ij} = -\frac{1}{3} c(G) d(G) \Box \ln \rho. \quad (3.25)$$

We can solve (3.24) and obtain

$$e^{-(\theta + 2f)} = \frac{d}{r^2} + v^z \left( \frac{2}{3} c(G) d(G) \Box \ln \rho + \sum_{a=1}^{n} \frac{4}{(y - y_a)^2} \right)$$

$$\equiv \frac{d}{r^2} + v^z \frac{2}{3} c(G) d(G) \Box \ln \tilde{\rho}. \quad (3.26)$$
We have removed the singularities in the solution by defining \( \bar{\rho} = \rho \prod_{a=1}^{n} (y - y_a)^2 \). Inserting \( \Lambda \) from (3.14) in (3.8), with \( *d\Lambda \) replaced by \( e^{-2f} *d\Lambda \), gives

\[
\Box e^{-2f} = -\bar{v}^2 F_{ij} F^{ij} = \frac{2}{3} \bar{v}^2 c(G) d(G) \Box \ln \bar{\rho}.
\] (3.27)

The solution is similar to the previous equation

\[
e^{\theta-2f} = \frac{c}{r^2} + \bar{v}^2 \frac{2}{3} c(G) d(G) \Box \ln \bar{\rho}
\] (3.28)

where \( c \) is an integration constant. The two integration constants \( c \) and \( d \) are proportional, respectively, to the fluxes of \( \hat{\ast} \hat{G}_3 \) and \( \hat{G}_3 \) through the \( S^3 \). Therefore, they represent, respectively, the number of \( D1 \) and \( D5 \) branes. We can directly see this by considering for example, the \( \hat{G}_3 \) flux near \( r = 0 \)

\[
Q_1 = \frac{1}{8\pi^2} \int_{S^3} e^{2\theta} \hat{\ast} \hat{G}_3 = \frac{c}{4}
\] (3.29)
\[
Q_5 = \frac{1}{8\pi^2} \int_{S^3} \hat{G}_3 = \frac{d}{4}.
\] (3.30)

We have used the same normalization of \( Q_1 \) and \( Q_5 \) as in [10]. Indeed, we can regard our six dimensional theory as a subsector of type I theory compactified on \( K3 \). In this solution, we have \( D5 \) branes wrapped on \( K3 \) and \( D1 \) branes transverse to it. The solution we give here is the same as the gauge dyonic string studied in [9] and [10].

The behaviors of \( e^{-4f} \) near \( r \to \infty \) and \( r \to 0 \) are given by

\[
r \to \infty: \quad e^{-4f} \sim \frac{(c + 4\bar{v}^2 N)(d + 4\bar{v}^2 N)}{r^4} = \frac{16(Q_1 + \bar{v}^2 N)(Q_5 + \bar{v}^2 N)}{r^4},
\] (3.31)
\[
r \to 0: \quad e^{-4f} \sim \frac{cd}{r^4} = \frac{16Q_1 Q_5}{r^4}.
\] (3.32)

We have introduced the instanton number \( N \) given by

\[
N = \frac{1}{32\pi^2} \int d^4 y (\ast F)_{ij} F^{ij} = -\frac{1}{48\pi^2} c(G) d(G) \int d^4 y \Box \ln \bar{\rho}.
\] (3.33)

At the fixed points, the metric is given by

\[
ds_6^2 = \frac{r^2}{L^2} (-dx_0^2 + dx_1^2) + \frac{L^2}{r^2} dr^2 + L^2 ds^2(S^3).
\] (3.34)

where we have rewritten the four dimensional flat metric in the polar coordinates

\[
dy^i dy^i = dr^2 + r^2 ds^2(S^3).
\] (3.35)
The metric (3.34) is readily seen to be $AdS_3 \times S^3$ metric with the $AdS_3$ and $S^3$ having the same radius $L$. The AdS radii at the fixed points near $r \sim \infty$ and $r \sim 0$ are $L = [(c + 4\tilde v^z N)(d + 4v^z N)]^{1/4}$ and $L = (cd)^{1/4}$, respectively. In the dual two dimensional conformal field theory, this solutions describes an RG flow from the CFT UV to the CFT IR with the ratio of the central charges

$$\frac{c|_0}{c|_\infty} = \frac{e^{-4f}|_0}{e^{-4f}|_\infty} = \frac{cd}{(c + 4\tilde v^z N)(d + 4v^z N)} < 1 \quad (3.36)$$

where we have used the relation between the central charge and AdS radius $c \sim \frac{L_{\text{AdS}}}{{G_N}^{(3)}} \sim \frac{L_{\text{Vol}(S^3)}}{{G_N}^{(6)}} \sim e^{-4f}$. The UV point corresponds to $r = \infty$ while the IR point is at $r = 0$.

To extract the dimension of the operator driving the flow, we need to consider the behavior of the fluctuation of the metric around $AdS_3 \times S^3$ near the UV point. To simplify the manipulation, we first consider here a single instanton at the origin $y_i^a = 0$. With this simplification, the solution, up to group theory factors which are not relevant for this discussion, is given by

$$e^{-4f} = \left(\frac{c}{r^2} + \tilde v^z \Box \ln(r^2 + \lambda^2)\right)\left(\frac{d}{r^2} + v^z \Box \ln(r^2 + \lambda^2)\right). \quad (3.37)$$

As $r \to \infty$, the solution behaves

$$e^{-4f} \sim \frac{(c + 4\tilde v^z)(d + 4v^z)}{r^4} \left(1 + \frac{8\lambda^2[c + d + 4(\tilde v^z + v^z)]}{r^2(c + 4\tilde v^z)(d + 4v^z)} + \ldots\right)$$

or

$$e^{2g} = e^{-2f} \sim \frac{\sqrt{(c + 4\tilde v^z)(d + 4v^z)}}{r^2} \left(1 + \frac{4\lambda^2[c + d + 4(\tilde v^z + v^z)]}{r^2(c + 4\tilde v^z)(d + 4v^z)} + \ldots\right). \quad (3.38)$$

From this equation, we find the fluctuation

$$\delta e^g \sim \frac{2\lambda^2[c + d + 4(\tilde v^z + v^z)]}{r^2(c + 4\tilde v^z)(d + 4v^z)} \quad (3.39)$$

which give $\Delta = 2$ in agreement with the result of the previous section. We can also see this in the coordinate $\tilde r = [(c + 4\tilde v^z)(d + 4v^z)]^{1/4} \ln r$ in which

$$ds_6^2 = e^{2\tilde r}(-dx_0^2 + dx_1^2) + d\tilde r^2 + R^2 ds(S^3)^2 \quad (3.40)$$

and $\delta e^g \sim e^{-2\tilde r}$. We have identified the $AdS_3$ and $S^3$ radii $L = R = [(c + 4\tilde v^z)(d + 4v^z)]^{1/4}$. In the general case with $N$ instantons, it can be checked through a more complicated algebra that the fluctuation of the metric behaves as $\sim r^{-2}$ near the UV point. This
can be seen as follows. $\tilde{\rho}$ have an expansion in powers of $r^{2n} + r^{2n-2}$ with $r^2 = y_i y_i$. From this, we find that $\Box \ln \tilde{\rho} \sim \frac{1}{r^4}(r^2 + r^4 + \ldots + r^{4n-2})$ from which we see that $r^{-2}$ is the leading term we have found in (3.31) while the subleading $r^{-4}$ gives $\Delta = 2$ as in the single instanton case. So, our flow is a vev. flow driven by a vacuum expectation value of a marginal operator.

We end this subsection by giving a comment on the anti-instanton gauge field configuration. We need to choose the three dimensional chirality $\Gamma_{01} \epsilon = -\epsilon$ which implies the self dual $\Gamma_{\alpha\beta}$ from $\Gamma_7 \epsilon = -\Gamma_{2345} \epsilon = \epsilon$. So, the condition $\delta \lambda^I = 0$ is still satisfied. The BPS equations (3.11) and (3.12) are modified by some sign changes. We find the following equations

\begin{align}
\frac{1}{2} \partial / \theta - \frac{1}{2} e^{\theta - 2f} \partial \Lambda + \frac{1}{2} e^{-\theta - 2f} \tilde{\partial} \tilde{\Lambda} &= 0, \\
\frac{1}{2} \partial / f + \frac{1}{4} e^{\theta - 2f} \partial \Lambda + \frac{1}{4} e^{-\theta - 2f} \tilde{\partial} \tilde{\Lambda} &= 0. 
\end{align}

(3.41) (3.42)

This change results in an extra minus sign in $\Lambda = -\frac{1}{2} e^{\theta + 2f} + C_1$. The field strength $*F^I = -F^I$ gives an extra minus sign in equation $D G = v^{z} F^I \wedge F^I$. The final result is

\begin{align}
&\frac{d}{r^2} - v^{z} \frac{2}{3} c(G)d(G) \Box \ln \tilde{\rho} \\
&\frac{c}{r^2} - v^{z} \frac{2}{3} c(G)d(G) \Box \ln \tilde{\rho}
\end{align}

with the behavior near the fixed points

\begin{align}
\frac{d}{r^2} &\sim \frac{(c - 4 \tilde{v}^{z} N)(d - 4 v^{z} N)}{r^4}, \\
\frac{c}{r^2} &\sim \frac{cd}{r^4}.
\end{align}

(3.43) (3.44)

(3.45) (3.46)

In this case, $N$ is now negative.

### 3.2 Central charges of the dual CFT

We now give some comments on the central charge of the dual (4,0) CFT. We have mentioned that solutions to the six-dimensional supergravity given in the previous sections can be interpreted as a D1/D5 brane system in type I string theory on K3. As type I and heterotic theories are S-dual to each other [22], this D1/D5 system is dual to the F1/NS5 brane system in the heterotic theory. We choose to work with heterotic string theory on K3 with the string frame effective action given by [23]

\begin{align}
I_6 &= \frac{(2\pi)^3}{\alpha'^2} \int d^6x \sqrt{-g} e^{2\theta} \left[ R_6 + 4 \partial M \theta \partial M \theta - \frac{1}{12} G_{MNP} G^{MNP} \right] \\
+ \int_{M_6} \frac{1}{4(2\pi)^3 \alpha'} B \wedge \sum_{\alpha} \tilde{v}^{*} \text{tr} F_\alpha \wedge F_\alpha
\end{align}

(3.47)
where we have given only the relevant terms for our discussion. All the notations are the same as those in [23] including the modified three-form field strength

\[ G = dB - \frac{\alpha'}{4} \sum_\alpha \epsilon^{\alpha} \Omega(F^\alpha) \]  

(3.48)

where \( \Omega(F^\alpha) \) is the Chern-Simons term of the gauge field \( A^\alpha \).

To compute the central charge, we need to know the coefficient of the Einstein-Hilbert term. The central charge is then given by [14]

\[ c = \frac{3\ell}{2G^{(3)}_N} \]  

(3.49)

where \( \ell \) is the \( AdS_3 \) radius. Note that the central charge can be written as \( c = 24\pi\alpha \) with \( \alpha \) being the coefficient of the Einstein-Hilbert term. In appendix B, we give a derivation of this result by using the computation of chiral correlators involving \( T_{zz} \) (\( T_{zz} \)) in the spirit of [13].

Our ansatz for the metric is

\[ ds^2_6 = \ell^2 \left( \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} dx^2_{1,1} \right) + \ell^2 ds^2(S^3). \]  

(3.50)

We have used the \( AdS_3 \) metric in the form given in [24]. The three-form field is

\[ G = \ell^2 h\epsilon_3 + \ell^2 a\omega_3 \]  

(3.51)

where \( \epsilon_3 \) and \( \omega_3 \) are the volume form on the unit \( AdS_3 \) and \( S^3 \), respectively. The radii of \( AdS_3 \) and \( S^3 \) are the same by the equation of motion for \( \theta \) with constant \( \theta \). The \( G \) equations of motion and Bianchi identity require \( h \) and \( a \) to be constant. Einstein equation determines the value of \( h = a = 2 \). The fluxes of \( G \) and \( *G \) are given by

\[ Q_1 = \frac{(2\pi)^3}{\alpha'^2} \int_{S^3} e^{2\theta} *G = \frac{(2\pi)^5 \ell^2 e^{2\theta}}{\alpha'^2}, \]  

(3.52)

\[ Q_5 = \frac{(2\pi)^3}{\alpha'^2} \int_{S^3} G = \frac{(2\pi)^5 \ell^2}{\alpha'^2}. \]  

(3.53)

In order to relate \( Q_1 \) and \( Q_5 \) to the number of F1 strings and NS5 branes, \( N_1 \) and \( N_5 \), we match our ansatz to the F1/NS5 solution in six dimensions given in [25]. The three-form field strength is

\[ G = 2\alpha' N_5 (\text{Vol}_{AdS_3} + \text{Vol}_{S^3}). \]  

(3.54)
We find that, after matching the flux of this solution with that of our ansatz,
\[ N_1 = 2\pi\alpha'Q_1, \quad N_5 = \frac{\alpha'Q_5}{(2\pi)^5}. \] (3.55)

We now make a reduction of (3.47) on \( S^3 \) and obtain
\[
I_3 = \frac{\alpha'^2Q_1Q_5}{2(2\pi)^5} \int d^3x \sqrt{g}R + \frac{\alpha'Q_1}{4} \int_{M_3} v\Omega(F^\alpha) + \frac{\alpha'Q_5}{4(2\pi)^6} \int_{M_3} \tilde{v}\Omega(F^\alpha)
- \frac{\alpha'Q_1Q_5}{8(2\pi)^5} \int_{M_3} \Omega(F^I)
= \frac{N_1N_5}{4\pi} \int d^3x \sqrt{g}R + \frac{N_1}{8\pi} \int_{M_3} v\Omega(F^\alpha) + \frac{N_5}{8\pi} \int_{M_3} \tilde{v}\Omega(F^\alpha)
- \frac{N_1N_5}{2(8\pi)} \int_{M_3} \Omega(F^I),
\] (3.56)

where we have given only the Einstein-Hilbert and Chern-Simons terms which are relevant for the present discussion. The \( SU(2) \) Chern-Simons term \( \Omega(F^I) \) cannot be determined by the dimensional reduction of the action (3.47). As in our previous work [8], its presence in the effective action is implied by the equation of motion for \( F^I \).

We can now use (3.56), together with the results in [26, 32], which we will rederive in a different way in the Appendix B, to compute the central charges
\[
c_L = 6N_1N_5, \quad c_R = 6N_1N_5.
\] (3.57)

The Kac-Moody levels of the \( SU(2) \) and gauge group \( G_\alpha \) can be computed from the Chern-Simons terms of the \( SU(2) \) and \( G^\alpha \) gauge fields. Using the result in appendix B, we find
\[
SU(2) \text{ level : } k_{SU(2)} = 8\pi\beta = N_1N_5, \quad (3.58)
G_\alpha \text{ level : } k_\alpha = 2(v^\alpha N_1 + \tilde{v}^\alpha N_5). \quad (3.59)
\]

### 4. Conclusions

In this paper, we have found three analytic RG flow solutions in six and three dimensional supergravities. In six dimensional supergravity, we have found the solution in which the internal components, outside the \( AdS_3 \) part, of the gauge fields, describe a configuration with \( N \) instantons. We have discussed separately the case \( N = 1 \). This is interesting in the sense that the solution can be obtained from uplifting the three dimensional solution. We have also given the corresponding solution in the Chern-Simons
gauged supergravity. With the reduction given in [8], the solution can be lifted to six dimensions and easily seen that it is indeed the same as the six dimensional solution.

The flows describe a deformation of the UV CFT by a vacuum expectation value of a (not exactly) marginal operator. Interestingly, these RG flows have an interpretation in terms of Yang-Mills instantons tunnelling between $|N\rangle$ Yang-Mills vacuum in the UV and $|0\rangle$ in the IR, and this fact is in turn related to the different values of the central charge at the two fixed points. In the general $N$ instanton solution, there is a subtlety of phase transitions occurring whenever $v$ and $\tilde{v}$ change sign. We have avoided this issue by assuming the positivity of both $v$ and $\tilde{v}$. We do not have a clear interpretation of this phase transition in the dual CFT, so it would be interesting to study this issue in more detail. We finally give some comments on the central charge of the dual CFT along with the derivation of the central charge of the boundary CFT from the gravity theory in the bulk.

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**A. Flow solution from $N = 4$ three dimensional gauged supergravity**

In this appendix, we study a flow solution in $N = 4$ three dimensional $(SO(3) \ltimes \mathbb{R}^3) \times (G \ltimes \mathbb{R}^{\dim G})$ Chern-Simons gauged supergravity. It has been shown in [8] that this theory is equivalent to $SO(3) \times G$ Yang-Mills gauged theory obtained from $SU(2)$ reduction of six dimensional supergravity whose flow solution has been studied in section 2. We are interested in the case $G = SO(3)$. We will see that the solution we are going to find is the same as that in section 2 but now in another framework. As in our previous papers, we use the formulation of [27]. See [28, 29] for more details of various gaugings.

**A.1 $(SO(3) \ltimes \mathbb{R}^3) \times (SO(3) \ltimes \mathbb{R}^3)$ gauged supergravity**

We now construct three dimensional gauged supergravity with gauge groups $(SO(3) \ltimes \mathbb{R}^3) \times (SO(3) \ltimes \mathbb{R}^3)$. This can be obtained from the theory constructed in [8] by setting $G = SO(3)$. The scalar fields are described by $\frac{SO(4)}{SO(4) \times SO(4)}$ coset manifold. We
parametrize the coset by

\[
L = \begin{pmatrix}
\frac{1}{1 - A(r)^2} I_{3 \times 3} & 0 & \frac{A(r)}{1 - A(r)^2} I_{3 \times 3} & 0 \\
0 & \cosh h(r) & 0 & \sinh h(r) \\
\frac{A(r)}{1 - A(r)^2} I_{3 \times 3} & 0 & \frac{1}{1 - A(r)^2} I_{3 \times 3} & 0 \\
0 & \sinh h(r) & 0 & \cosh h(r)
\end{pmatrix}.
\]

(A.1)

The \(SO(3)\) generators are given by

\[
J_{a_1}^A = \epsilon_{ABC} e_{BC}, \quad J_{a_2}^A = \epsilon_{ABC} e_{B+4,C+4}, \quad A, B, C = 1, 2, 3
\]

(A.2)

where \((e_{AB})_{CD} = \delta_{AC}\delta_{BD}\) are \(8 \times 8\) matrices. The translational symmetries are generated by

\[
J_{b_1}^A = -e_{A4} + e_{A8} + e_{4A} + e_{8A},
J_{b_2}^A = -e_{4,4+A} + e_{A+4,4} - e_{A+4,8} + e_{8,4+4}, \quad A = 1, 2, 3.
\]

(A.3)

\(SO(4)\) R-symmetry generators are given by

\[
J_{IJ} = J_{IJ}^I = J_{IJ}^J = e_{IJ} - e_{JI}, \quad I, J, \ldots = 1, \ldots 4.
\]

(A.4)

In the \(N = 4\) theory, the \(SO(4)\) R-symmetry decomposes to \(SO(3)_{+}\) and \(SO(3)_{-}\), and each factors acts separately on the two scalar target spaces. In our case called the degenerate case, there is only one target space, so we have only one \(SO(3)\) which we will denote by \(SO(3)_{+}\). Non compact generators of \(SO(4, 4)\) are

\[
Y_{ab} = e_{a,b+4} + e_{b+4,a}, \quad a, b = 1, 2, 3.
\]

(A.5)

We now proceed as in \cite{[3]} using the formulation of \cite{[27]}. The embedding tensor gives the following \(T\)-tensors, \(T_{\alpha\beta} = \Theta_{\alpha\mu\nu} Y_{\alpha\beta} Y_{\beta\mu}^\nu\),

\[
T_{IJ, KL} = g_1(\nu_{a_1}^{A I J} \nu_{b_1}^{A K L} + \nu_{b_1}^{A I J} \nu_{a_1}^{A K L}) + g_2(\nu_{a_2}^{A I J} \nu_{b_2}^{A K L} + \nu_{b_2}^{A I J} \nu_{a_2}^{A K L}) + h_1 \nu_{b_1}^{A I J} \nu_{a_1}^{A K L} + h_2 \nu_{b_2}^{A I J} \nu_{a_2}^{A K L},
\]

(A.6)

\[
T_{ab}^{IJ} = g_1(\nu_{a_1}^{A I J} \nu_{b_1}^{A} + \nu_{b_1}^{A I J} \nu_{a_1}^{A}) + g_2(\nu_{a_2}^{A I J} \nu_{b_2}^{A} + \nu_{b_2}^{A I J} \nu_{a_2}^{A}) + h_1 \nu_{b_1}^{A I J} \nu_{b_1}^{A} + h_2 \nu_{b_2}^{A I J} \nu_{b_2}^{A}.
\]

(A.7)

With the coset representative \(L\), we can compute all the needed quantities using

\[
L^{-1} D_\mu L = \frac{1}{2} Q_\mu^{IJ} X^{IJ} + Q_\mu^a X^a + \epsilon_\mu^A Y^A,
L^{-1} t^\mu L = \frac{1}{2} \nu^\mu_{\alpha\beta} X^{IJ} + \nu^\mu_{\alpha} X^a + \nu^\mu_A Y^A.
\]

(A.8)
The consistency condition from supersymmetry, $\mathbb{P}_\theta T^{IJKL} = 0$, \cite{27} requires that $h_2 = -h_1$. The resulting $V^M_A$ are given by

$$V_{a_1,2}^{AKL} = -\frac{1}{2} \text{Tr}[L^{-1} J_{a_1,2}^A L J_{+}^{KL}],$$
$$V_{b_1,2}^{AKL} = -\frac{1}{2} \text{Tr}[L^{-1} J_{b_1,2}^A L J_{+}^{KL}],$$
$$V_{a_1,2}^{Aab} = \frac{1}{2} \text{Tr}[L^{-1} J_{a_1,2}^A L(e_{a,b+4} + e_{b+4,a})],$$
$$V_{b_1,2}^{Aab} = \frac{1}{2} \text{Tr}[L^{-1} J_{b_1,2}^A L(e_{a,b+4} + e_{b+4,a})]$$

(A.9)

where $A, B, \ldots = 1, 2, 3$ label $SO(3)$ gauge generators, and a pair of indices $a, b, \ldots = 1, 2, 3$ labels target space coordinates. With an appropriate normalization, the tensor $f^{IJ}$ is given by

$$f_{ab,cd}^{IJ} = 2 \text{Tr}(e_{ba} J_{IJ}^{+} e_{cd}).$$

(A.10)

With all these ingredients, we can now compute $A_1$ and $A_2$ tensors which give the scalar potential via

$$A_1^{IJ} = -\frac{4}{N-2} T^{IM,JM} + \frac{2}{(N-1)(N-2)} \delta^{IJ} T^{MN,MN},$$
$$A_2^{IJ} = \frac{2}{N} T^{IJ} j + \frac{4}{N(N-2)} f_{M}^{IJ} T_{j}^{JM} + \frac{2}{N(N-1)(N-2)} \delta^{IJ} f_{KL}^{m} T_{m}^{KL},$$
$$V = \frac{4}{N}(A_1^{IJ} A_1^{IJ} - \frac{1}{2} N g^{ij} A_{2i}^{IJ} A_{2j}^{IJ}).$$

(A.11)

The potential for these two scalars is given by

$$V = e^{-4h} \left[ h_1^2 + \frac{2e^{2h}}{(A^2 - 1)^2} (g_1^2 + A^2(g_2 + g_2 A(A^2 - 3)) - 3g_1^2) \right].$$

(A.12)

This simple looking potential admits five different critical points. We can identify supersymmetric critical points by using the procedures explained in \cite{2}. All non trivial critical points are given in table I.

**Table I**: Critical points of the potential (A.12). $A_0$ and $h_0$ are vacuum expectation values at the critical point of $A$ and $h$, respectively.
| Critical points | $A_0$ | $h_0$ | $V_0$ | Preserved supersymmetries |
|-----------------|------|------|------|--------------------------|
| I               | 0    | $\ln\left(-\frac{h_1}{g_1}\right)$ | $-\frac{g_1}{h_1^2}$ | non supersymmetric |
| II              | 0    | $\ln\left(\frac{h_1}{g_1}\right)$ | $-\frac{g_1}{h_1^2}$ | (4,0) |
| III             | $\frac{g_1}{g_2}$ | $\ln\left(\sqrt{-h_1^2g_1^2-2h_1g_1g_2}+g_2\right)$ | $-\frac{g_1^2g_2^2}{(g_1^2-g_2^2)^2h_1^4}$ | (4,0) |
| IV              | $\frac{g_1}{g_2}$ | $\ln\left(-\sqrt{-h_1^2g_1^2-2h_1g_1g_2}+g_2\right)$ | $-\frac{g_1^2g_2^4}{(g_1^2-g_2^2)^2h_1^4}$ | non supersymmetric |
| V               | $\frac{g_2}{g_1}$ | $\ln\frac{h_1\sqrt{g_1^2-2g_1^2g_2}}{g_1^2-g_1^2g_2+g_2^2}$ | $\frac{g_1^2g_2^4}{(g_1^2-g_2^2)^2h_1^2}+g_2^2$ | non supersymmetric |

Supersymmetric flow equations can be obtained from supersymmetry transformations of fermions, $\delta\psi^I_\mu = 0$ and $\delta\chi^{ij} = 0$. The metric ansatz is

$$ds^2 = e^{2f}dx_{A,1}^2 + dr^2. \quad (A.13)$$

Recall that

$$\delta\psi^I_\mu = D_\mu \epsilon^I + A^{I J}_1 \gamma_\mu \epsilon^J,$$
$$\delta\chi^{ij} = \frac{1}{2}(\delta^{IJ} \mathbf{1} - f^{IJ})^i \mathcal{D}^j \epsilon^J - NA^{JIi}_2 \epsilon^J, \quad (A.14)$$

together with the formulae in equation (A.11), we find, from $\delta\chi^{ji}$,

$$\frac{dA}{dr} = \frac{2e^{-A}(g_2A - g_1)}{\sqrt{1 - A^2}} \quad (A.15)$$
$$\frac{dh}{dr} = \frac{2e^{-2h}}{(1 - A^2)^{3/2}}[e^h(g_1 - g_2A^3) + h_1(A^2 - 1)\sqrt{1 - A^2}] \quad (A.16)$$

We can easily check that (A.15) and (A.16) have two critical points which are exactly the same as II and III points in table I. With a new function $g$ and new coordinate $\bar{r}$ given by

$$g = h + \ln \sqrt{1 - A^2} \quad \text{and} \quad d\bar{r} = e^{-g}dr, \quad (A.17)$$

we can write (A.15) as

$$A' = \frac{dA}{d\bar{r}} = 2A(g_2A - g_1). \quad (A.18)$$

The solution for $A$ is

$$A = -\frac{g_1}{e^{2g_1\bar{r} + g_1C_1} - g_2}. \quad (A.19)$$

As in section 2, we choose $C_1 = \frac{1}{g_1} \ln (-g_2)$ and end up with

$$A = \frac{g_1}{(e^{2g_1\bar{r} + 1})g_2}. \quad (A.20)$$
With (A.17) and (A.20), equation (A.16) can be rewritten as
\[ g' = \frac{dg}{d\tilde{r}} = -2 \left[ g_1 \left( -\frac{g_1 h_1 e^{-g}}{g_2^2 (e^{2g_1 \tilde{r}} + 1)^2} + \frac{g_1 + g_2}{g_2 e^{2g_1 \tilde{r}} + g_1 + g_2} + \frac{1}{g_2 e^{2g_1 \tilde{r}} + 1} - 2 \right) \right. \\
\left. + h_1 e^{-g} + g_1 \tanh(g_1 \tilde{r}) \right]. \]  
(A.21)

This can be solved, and the solution is
\[ g = \ln \left[ \frac{(h_1 + 16 C_2 g_1 e^{2g_1 \tilde{r}})(g_2^2 e^{4g_1 \tilde{r}} + 2g_2^2 e^{2g_1 \tilde{r}} - g_1^2 + g_2^2)}{g_1 g_2^2 (1 + e^{g_1 \tilde{r}})^2} \right]. \]  
(A.22)

This solution interpolates between II and III fixed points provided that we choose \( C_2 = 0 \). We now move to \( \delta \psi \). With the solutions for \( A \) and \( g \), the gravitino variation gives
\[ \frac{df}{d\tilde{r}} = -\frac{g_1^2 g_2^2 (e^{2g_1 \tilde{r}} + 1)(g_2^2 (e^{2g_1 \tilde{r}} - 1) + g_1^2 (e^{2g_1 \tilde{r}} + 1)^3)}{h_1 (g_2 e^{2g_1 \tilde{r}} - g_1 + g_2)^2 (g_2 e^{2g_1 \tilde{r}} + g_1 + g_2)^2}. \]  
(A.23)

After going to \( \tilde{r} \) coordinate, we find the solution
\[ f = g_1 \tilde{r} - \ln[2(1 + e^{2g_1 \tilde{r}})] + \frac{1}{2} \ln[2(g_1^2 - g_2^2(1 + e^{2g_1 \tilde{r}})^2)] \]  
(A.24)

where, as usual, we have ignored all additive constants because they can be absorbed by rescaling \( x^0 \) and \( x^1 \). The AdS\(_3\) radius is given by
\[ L = \frac{e^{g_0}}{g_1} = \frac{1}{\sqrt{V_0}}. \]  
(A.25)

We can compute the ratio of the central charges between the two fixed points
\[ \frac{c_{\text{II}}}{c_{\text{III}}} = \frac{1}{1 - \frac{g_1^2}{g_2^2}} > 1. \]  
(A.26)

By the c-theorem, we see that point II and III correspond to the UV and IR CFTs, respectively. This is in agreement with the solution found in section \[2\]. So, the solutions from both theories are the same. This is the result, at the level of solutions, of the fact that the two theories are equivalent as shown in \[8\]. Near the UV point, the scalars behave as
\[ \delta A \sim e^{-2r/L}, \quad \delta h \sim e^{-4r/L}, \quad L = \frac{h_1}{g_1^2}. \]  
(A.27)

We see that the flow is driven by a marginal operator dual to \( A \) of dimension 2. \( h \) is dual to an irrelevant operator of dimension 4. Up to quadratic order in the scalars, the potential (A.12) at the UV point is given by
\[ V = -\frac{g_1^4}{h_1^2} + \frac{4g_1^4}{h_1^2} h^2. \]  
(A.28)
We find that the scalar $A$ is massless at this point and dual to a marginal operator. The scalar kinetic terms are

$$\mathcal{L}_{\text{scalar kinetic}} = \frac{1}{2} \left( \frac{3A^2}{(A^2 - 1)^2} + h^2 \right). \quad (A.29)$$

At the UV point, $A = 0$, all the kinetic terms are canonically normalized, and we can read off the values of mass squared directly from the potential. In the unit of $\frac{1}{L^2}$, $h$ has $m^2 L^2 = 8$ which gives exactly $\Delta = 4$ in agreement with the asymptotic behavior. At the IR point, $A$ becomes massive with positive mass squared as can be seen from the expansion of the potential

$$V = -\frac{g_1^4}{h_1^2(1 - \frac{g_1^2}{g_2^2})} + \frac{12g_1^4g_2^8}{(g_1^2 - g_2^2)^4h_1^2}A^2 + \frac{4g_1^4g_2^4}{(g_1^2 - g_2^2)^2h_1^2}h^2. \quad (A.30)$$

The positive mass squared means that the potential has a minimum at the IR point, and the dual operator is irrelevant as it should. To compute the mass squared at the IR point, we redefine $A$ to $A = \tanh \phi$. Then, (A.29) becomes

$$\frac{1}{2}(3\phi'^2 + h'^2). \quad (A.31)$$

The potential near the IR point is, to quadratic terms in scalars, given by

$$\frac{V}{V_{0\text{IR}}} = 1 - 4h^2 - \frac{12g_2^4}{(g_1^2 - g_2^2)^2}g^2. \quad (A.32)$$

At this fixed point, $h$ and $\phi$ have $m^2 L^2 = 8$ and $m^2 L^2 = \frac{24}{1 - \frac{g_1^2}{g_2^2}}$, respectively. We find that

$$\Delta_h = 4 \quad \text{and} \quad \Delta_\phi = 1 + \sqrt{1 + \frac{24g_2^4}{(g_1^2 - g_2^2)^2}} > 2. \quad (A.33)$$

As expected, the operator dual to $\phi$ is irrelevant with dimension greater than 2.

**B. Derivation of the central charges**

In this Appendix we present a holographic derivation of the left- and right- central charges in a 2D CFT which, to our knowledge, has not appeared in the literature. We follow [15] and find the correlation functions of the stress-energy tensor. The two point function will directly give the value of the central charge. We first start with the gravitational action in three dimensions of the form

$$I = \alpha \left[ \int_M d^3x (\sqrt{G}R_G + 2\Lambda) - \int_{\partial M} d^2x \sqrt{\partial 2K} \right]. \quad (B.1)$$
The second term is the Gibbons-Hawking term with the induce metric \( g \) and extrinsic curvature of the boundary \( \partial M, K \). The coefficient \( \alpha \) is dimensionless provided that we use the unit \( AdS_3 \) in the measure \( d^3x\sqrt{g} \). We then adopt Euclidean signature and take the metric \[ ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \ell^2 \left( \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(x, \rho)}{\rho} dx^i dx^j \right) \, . \] (B.2)

In these coordinates, the extrinsic curvature is given by \( K_{ij} = -\frac{2\rho}{\ell} \partial_\rho g_{ij} \), and the boundary is at \( \rho = 0 \). We look for the solution of \( g_{ij} \) of the form \[ g(x, \rho) = g(0) + \rho g(2) + h(2) \rho \ln \rho + \ldots \, . \] (B.3)

The Einstein equations give \[ \rho[2g'' - 2g'g^{-1}g' + \text{Tr}(g^{-1}g')g'] + R_g - \text{Tr}(g^{-1}g')g = 0 \] (B.4)

\[ \nabla_i \text{Tr}(g^{-1}g') - \nabla_j g'_{ij} = 0 \] (B.5)

\[ \text{Tr}(g^{-1}g'') - \frac{1}{2} \text{Tr}(g^{-1}g'g^{-1}g') = 0 \, . \] (B.6)

Using the expansion (B.3), we find the relevant equations

\[ \nabla^i g_{ij}^{(2)} = \frac{1}{2} \nabla_i R_g \] (B.7)

\[ \text{Tr}g^{(2)} = = \frac{1}{2} R_g \, . \] (B.8)

We now expand the background metric \( g(0) \) about the flat metric. We use the complex coordinates with the convention of \[ and write the metric \( g(0) \) as

\[ g(0) = \left( \begin{array}{cc} h_{zz} & \frac{1}{2} + h_{zz} \\ \frac{1}{2} + h_{zz} & h_{zz} \end{array} \right) \, . \] (B.9)

To simplify the equations, we will keep only the \( h_{zz} \) component non zero. This is enough to find the \( T_{zz}T_{zz} \) correlation functions. In the complex coordinates, equation (B.7) takes the form

\[ \nabla_z g_{zz}^{(2)} + \nabla_z g_{zz}^{(2)} - 2h_{zz} \nabla_z g_{zz}^{(2)} = \frac{1}{4} \partial_z R_g \, . \] (B.10)

Equation (B.8) gives

\[ g_{zz}^{(2)} - h_{zz}g_{zz}^{(2)} = \frac{1}{8} R_g \, . \] (B.11)
This can be used to eliminate $g^{(2)}_{zz}$ in (B.10). We finally find
\[
\partial_z g^{(2)}_{zz} - h_{zz} \partial_z g^{(2)}_{zz} - 2 \partial_z h_{zz} g^{(2)}_{zz} = \frac{1}{8} \partial_z R_g .
\] (B.12)

This equation has precisely the structure of Virasoro’s Ward identity for the generating function of connected correlators of $T_{zz}$. A different holographic derivation of it has been discussed in [31].

The Ricci scalar $R_g$ is given by
\[
R_g = 2 g^{\bar{z}z}_{(0)} R_{\bar{z}z} = 4 \partial^2 h_{\bar{z}z} .
\] (B.13)

We can now solve (B.12) for $g^{(2)}$ order by order. The first order equation is simply given by
\[
\partial_z g^{(2)}_{zz} = \frac{1}{2} \partial^2 h_{\bar{z}z} .
\] (B.14)

This is easily solved by recalling $\partial_z \frac{1}{z} = 2\pi \delta(z)$ and taking
\[
g^{(2)}_{zz} = -\frac{3}{2\pi} \int d^2 w \frac{1}{(z - w)^4} h_{\bar{w}w}(w) .
\] (B.15)

Back to our action (B.1), we can evaluate this action on the solution (B.2) with the expansion (B.3). This gives [32]
\[
\delta I = \alpha \int d^2 x \sqrt{g} (g^{(2)ij} - g^{(0)}_{ij}) \delta g^{(0)}_{ij} .
\] (B.16)

Although, in our coordinates, the boundary is at the lower limit of the $\rho$ integration, $\rho = 0$, in contrast to [32] in which the boundary is at the upper limit of the $\eta$ integration, $\eta = \infty$, we find $\delta I$ with the same sign as that in [32]. This is because of the extra minus sign in the extrinsic curvature $K_{ij}$.

In the complex coordinates and with only $h_{\bar{z}z} \neq 0$, we find
\[
\delta I = 2\alpha i \int d^2 z g^{(2)zz} \delta g^{(0)}_{\bar{z}z} .
\] (B.17)

This gives the one point function for the stress-energy tensor. Using the solution (B.15), we can find the two point function by differentiate one more. The result is
\[
\langle T(z) T(w) \rangle = (-2\pi)^2 \frac{\delta^2}{\delta h_{\bar{z}z}(z) \delta h_{\bar{w}w}(w)} e^{iS} |_{h_{\bar{z}z}=0} = \frac{12\pi \alpha}{(z - w)^4} .
\] (B.18)

We have used our normalization factor of $-2\pi$ in the definition of the stress-energy tensor. This normalization has been determined by computing the three point function.
\langle T(z_1)T(z_2)T(z_3) \rangle which in turn can be obtained by solving (B.12) to the second order. After matching this three point function with the CFT \langle T(z_1)T(z_2)T(z_3) \rangle, we find the normalization factor. We then compare (B.18) with the OPE \( T(z)T(w) \sim \frac{c_L}{2(z-w)^4} + \ldots \), we obtain
\[ c_L = 24\pi \alpha. \] (B.19)

A similar analysis can be done for the \( \langle \bar{T} \bar{T} \rangle \) with non zero \( h_{zz} \). The right moving central charge is then given by
\[ c_R = 24\pi \alpha. \] (B.20)

In principle, we can use (B.12) to find any \( n \) point function of the CFT’s stress-energy tensor. However, the above analysis only involves ether \( h_{zz} \) or \( h_{z\bar{z}} \). With all \( h_{zz}, h_{z\bar{z}} \) and \( h_{z\bar{z}} \) non zero, we have also checked, to leading order, that there is no \( T \bar{T} \) correlation function, but there is a coupling between \( h_{zz} \) and \( h_{zz} \) and between \( h_{z\bar{z}} \) and \( h_{z\bar{z}} \). These couplings can be removed by adding some local counter terms to the two dimensional action. Beyond leading order, it is not clear what we can learn from a very complicated equation coming from (B.12).

We end this section by briefly discussing the contribution to the Kac-Moody level from the gauge Chern-Simons term. Following [33], the gauge field can be expanded as
\[ A = A^{(0)} + \rho A^{(1)} + \ldots. \] (B.21)

The Lagrangian for the gauge field including the Chern-Simons term is
\[ I = -\frac{1}{2} \int \ast F^a \wedge F^a - \frac{\beta}{2} \int \left( A^a \wedge dA^a + \frac{2}{3} f_{abc} A^a \wedge A^b \wedge A^c \right). \] (B.22)

We will suppress the gauge group index on \( A \) from now on to make the expression compact. From this action, it is straightforward to find the equation of motion and find, in the \( A_\rho = 0 \) gauge,
\[ \partial_i A^{(0)}_j - \partial_j A^{(0)}_i = 0. \] (B.23)

As discussed in [33], it is necessary to add a boundary term in order to obtain only the left moving current. This boundary term is given by
\[ I_b = \frac{\beta}{2} \int d^2x \sqrt{g} g^{ij} A^{(0)}_i A^{(0)}_j. \] (B.24)

Notice the sign change as oppose to the result in [33]. We can solve (B.23) in complex coordinates by taking
\[ A_z(z) = -\frac{1}{2\pi} \int d^2w \frac{1}{(z-w)^2} A_w(w). \] (B.25)
Inserting this into $I_b$, we obtain

$$I_b = -i \frac{\beta}{2\pi} \int d^2 z d^2 w A_{\bar{w}}(w) \frac{1}{(z - w)^2} A_{\bar{z}}(z). \quad (B.26)$$

We can now find

$$\langle J(z)J(w) \rangle = (-2\pi)^2 \frac{\delta^2}{\delta A_{\bar{z}}(z)\delta A_{\bar{w}}(w)} e^{i I_b} \bigg|_{A_{\bar{z}} = 0} = \frac{4\pi \beta}{(z - w)^2} \quad (B.27)$$

which gives $k = 8\pi \beta$ by using the OPE $J(z)J(w) \sim \frac{k}{2(z-w)^2} + \ldots$. The factor $-2\pi$ is again due to our normalization of the current. The central charge is $c = 6k = 48\pi \beta$.

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