On the Frobenius number of certain numerical semigroups

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Abstract

Let $0 < \lambda \leq 1$, $\lambda \notin \{2/7, 2/11, 2/13, \ldots \}$, be a real and $p$ a prime number, with $[p, p + \lambda p]$ containing at least two primes. Denote by $f_\lambda(p)$ the largest integer which cannot be written as a sum of primes from $[p, p + \lambda p]$. Then

$$f_\lambda(p) \sim \left\lfloor \frac{2 + 2}{\lambda} \right\rfloor \cdot p,$$

as $p$ goes to infinity.

Further a question of Wilf about the ‘Money-Changing Problem’ has a positive answer for all semigroups of multiplicity $p$ containing the primes from $[p, 2p]$. In particular, this holds for the semigroup generated by all primes not less than $p$. The latter special case was already shown in a previous paper.

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1 Introduction

A numerical semigroup is an additively closed subset $S$ of $\mathbb{N}$ with $0 \in S$ and only finitely many positive integers outside from $S$, the so-called gaps of $S$. The genus $g$ of $S$ is the number of its gaps. The set $E = S^* \setminus (S^* + S^*)$, where $S^* = S \setminus \{0\}$, is the (unique) minimal system of generators of $S$. Its elements are called the atoms of $S$; their number $e$ is the embedding dimension of $S$. The multiplicity of $S$ is the smallest element $p$ of $S^*$.

From now on we assume that $S \neq \mathbb{N}$. Then the greatest gap $f$ is called the Frobenius number of $S$.

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For a certain class of numerical semigroups we shall study the relationship between the various invariants mentioned above.

In particular for some of these semigroups we will give an affirmative answer to

**Wilf’s question** [15]: Is it true that

\[
\frac{g}{1 + f} \leq \frac{e - 1}{e}
\]

We shall consider the following semigroups: Let \( p \) be a prime, \( \lambda \) a positive real number, \( I_\lambda(p) \) the interval \([p, p + \lambda p]\) and \( D_\lambda \) the set of all primes \( p \) such that \( I_\lambda(p) \) contains at least two primes. For such a \( p \) we denote by \( S_\lambda(p) \) the numerical semigroup generated by all primes from \( I_\lambda(p) \) and by \( f_\lambda(p) \) its Frobenius number. According to Bertrand’s postulate, \( D_1 \) is the set \( \mathbb{P} \) of all primes, further \( \mathbb{P} \setminus D_\lambda \) is finite for all \( \lambda > 0 \) by the prime number theorem.

Let \( p_1 = 2, p_2 = 3, p_3 = 5, \ldots \) be the sequence of prime numbers in natural order and let \( S_n \) be the semigroup generated by all primes not less than \( p_n \).

**Proposition 1.1.**

Let \( \lambda > 0, p \in D_\lambda \) and \( S \) any numerical semigroup of multiplicity \( p \) containing \( S_\lambda(p) \).

a) There is an integer \( C(\lambda) > 0 \) such that for \( p > C(\lambda) \), the semigroups \( S \) from above satisfy Wilf’s inequality (1).

b) In case \( \lambda = 1 \) formula (1) holds for all \( p \). In particular (1) is true for \( S = S_n \).

This will be seen in section 3.

In section 2 we shall show the following result.

**Theorem 1.2.**

Let \( 0 < \lambda \leq 1, \lambda \notin \{\frac{2}{7}, \frac{2}{10}, \frac{2}{13}, \ldots\} \), be a real. Then

\[
\lim_{p \to \infty} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor.
\]

For \( \lambda = \frac{2}{m} \) with an integer \( m \geq 2 \) and \( m \equiv 1 \mod 3 \), at least

\[
\limsup_{p \to \infty} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor.
\]

In particular, since \( f_\lambda(p) \) is decreasing as a function of \( \lambda \), for each \( \lambda > 0 \) there is a constant \( c_\lambda \), such that \( f_\lambda(p) \leq c_\lambda \cdot p \) for all primes \( p \in D_\lambda \), cf. [8, Remark 2c)].

According to our table t4_quotient_not_always_6.pdf from [19] possibly the series \( \frac{f_\lambda(p)}{p} \) may not converge as \( p \) goes to infinity. Hence we do not expect
that $\frac{f_{2}(p)}{p} \sim 2 + m$ holds in the exceptional cases $m \geq 4$ and $m \equiv 1 \mod 3$ from Theorem 1.2 as well. See figure 1.

By computational evidence (see [19]), we suspect that $\lim_{p \to \infty} \frac{f_{\lambda}(p)}{p} = 3$ for $\lambda > 1$. See the algorithm in [19] and figure 2 below.

In particular for the Frobenius number $f_{n}$ of $S_{n}$ we would have $\lim_{n \to \infty} \frac{f_{n}}{p_{n}} = $
3, hence large even numbers would be the sum of two primes (cf. [8, Proposition 2]).

2 Bounds for the Frobenius number of certain numerical semigroups generated by primes

In order to verify Theorem 1.2 for integers $m \geq 2$ consider the statement

\( A(m) \) For every $\delta > 0$ there is an $N(\delta, m) > 0$ such that all integers $N \geq N(\delta, m)$ of the same parity as $m$ can be written as a sum of primes

(2) \( N = q_1 + \cdots + q_m \) with the restriction \( \left| \frac{N}{m} - q_i \right| < \delta \cdot N \) for $i = 1, \ldots, m$.

For short: “Large $N$ of the same parity as $m$ are sums of $m$ almost equal primes.”

**Proposition 2.1.** Suppose $A(m)$ holds for some $m \geq 2$. Then for each pair $(\varepsilon, \lambda)$ of reals $\varepsilon > 0$ and $\lambda > \frac{2}{m}$,

(3) \( f_\lambda(p) < (m + 1 + \varepsilon)p \) for large $p \in D_\lambda$.

**Proof:** W.l.o.g. we may assume that $\varepsilon < \lambda - \frac{2}{m}$. Let $p \in D_\lambda$. Set $\delta := \frac{\varepsilon}{m(m + 2 + \varepsilon)}$. From $A(m)$ we get for large $p$:

Every integer $N \in [(m + \varepsilon)p, (m + 2 + \varepsilon)p]$ of the same parity as $m$ can be written as $N = q_1 + \cdots + q_m$ with primes $q_i$ such that

(4) \( \left| \frac{N}{m} - q_i \right| < \delta \cdot N \) for $i = 1, \ldots, m$.

We will see in a moment, that $q_i \in S_\lambda(p)$ for $i = 1, \ldots, m$, hence $N \in S_\lambda(p)$: Inequality (4) implies

\[
q_i > \frac{N}{m} - \delta \cdot N = \frac{m + 2 + 2\varepsilon}{m(m + 2 + \varepsilon)} \geq \frac{(m + 2)(m + \varepsilon)}{m(m + 2 + \varepsilon)}p > p
\]

and

\[
q_i < \frac{N}{m} + \delta \cdot N = \frac{m + 2 + 2\varepsilon}{m(m + 2 + \varepsilon)} \leq \frac{m + 2 + 2\varepsilon}{m}p \leq p + \left( \frac{2}{m} + \varepsilon \right)p < p + \lambda p,
\]

since $m \geq 2$ and $\varepsilon < \lambda - \frac{2}{m}$. Hence $N \in S_\lambda(p)$.

Considering $N$ and $N + p$ we see, that $\mathbb{Z} \cap [(m + 1 + \varepsilon)p, (m + 2 + \varepsilon)p]$ is contained in $S_\lambda(p)$ and is a set of at least $p$ consecutive integers. Hence $f_\lambda(p) < (m + 1 + \varepsilon)p$.

It will be immediate from the following improved version of [11, Theorem 1.1], that $A(m)$ holds for all integers $m \geq 3$. 

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Proposition 2.2. Let $\theta = \frac{11}{20} + \epsilon$, $\epsilon > 0$, and $m \geq 3$ an integer. Then every sufficiently large integer $N$ of the same parity as $m$ can be written as the sum $N = q_1 + \cdots + q_m$ of $m$ primes with

$$\left| \frac{N}{m} - q_i \right| \leq N^\theta$$

for $i = 1, \ldots, m$.

Proof. due to Kaisa Matomäki (private communication [10]).

Given $\epsilon > 0$ and $m \geq 3$ as above, let $N$ be a sufficiently large integer with $N - m$ even. By the existence of primes in short intervals, cf. [11, Theorem 1], there is a prime $p \in \left[ \frac{N}{m} - \frac{\theta}{4m}, \frac{N}{m} \right]$, hence

$$0 \leq \frac{N}{m} - p \leq \left( \frac{N}{m} \right) \frac{\theta}{4m} < N^{\frac{11}{50} + \frac{\epsilon}{2}}.$$ 

(5)

Now use [11, Theorem 1.1] for the odd integer $n := N - (m - 3)p = 3\frac{N}{m} + (m - 3)\left( \frac{N}{m} - p \right)$ and $\theta = \frac{11}{20} + \frac{\epsilon}{2}$. You get primes $q_1, q_2, q_3$ with $n = q_1 + q_2 + q_3$ and

$$\left| n - q_i \right| \leq n^{\frac{11}{50} + \frac{\epsilon}{2}} \leq N^{\frac{11}{50} + \frac{\epsilon}{2}}$$

for $i = 1, 2, 3$.

(6)

For $N \gg 0$ we also have

$$\frac{m}{3} \leq N^{\frac{\epsilon}{3}}.$$ 

(7)

Finally by (5), (6) and (7)

$$\left| \frac{N}{m} - q_i \right| = \left| \frac{n}{3} - q_i \right| = \left| \frac{N}{3} - q_i \right| - \frac{m - 3}{3} \frac{N}{m} - p \right|$$

$$\leq \frac{m}{3} N^{\frac{11}{50} + \frac{\epsilon}{2}} \leq N^{\frac{11}{50} + \frac{\epsilon}{2}}$$

$$= \frac{m}{3} N^{\frac{11}{50} + \frac{\epsilon}{2}} \leq N^{\frac{11}{50} + \frac{\epsilon}{2}}$$

for $i = 1, 2, 3$.

Hence $n = q_1 + q_2 + q_3 + (m - 3)p$ is the sum of $m$ primes of size as desired. □

Corollary 2.3. $A(m)$ is true for all integers $m \geq 3$.

Proof. Let $m \geq 3$ and $\delta > 0$. Apply 2.2 with $\epsilon = \frac{1}{20}$. Then $1 > \theta = \frac{3}{5} > \frac{11}{20}$ and $N^\theta < \delta \cdot N$ for large $N$. Hence $A(m)$ is true by 2.2. □

Proposition 2.1 and Corollary 2.3 together imply the following result.

Corollary 2.4. Let $\epsilon > 0$, $m \geq 3$ and $\lambda > \frac{2}{m}$. Then

$$f_\lambda(p) < (m + 1 + \epsilon)p$$

for large $p \in D_\lambda$. □
Remark. Let $p = p_n$ the $n$th prime in the natural order, $S_n$ the semigroup generated by all primes not less than $p$ and $f_n$ its Frobenius number. $S_{\lambda = 1}(p_n)$ is contained in $S_n$, hence $f_n$ is at most $f_{\lambda = 1}(p_n)$. Therefore, an application of Proposition \ref{2.4} with $m = 3$ gives $\limsup_{n \to \infty} \frac{f_n}{p_n} \leq 4$. This has been shown by a somewhat different way in our former paper \cite[Remark 2.a)]{8}.

On our way to Theorem \ref{1.2} Proposition \ref{2.6} below will give us lower bounds for $\frac{f_{\lambda}(p)}{p}$, $p \in D_{\lambda}$. Let $p(\lambda) := \max(I_{\lambda}(p) \cap \mathbb{P})$. From now on let $m \geq 2$. Set

$$T(m) := \{t \in \mathbb{N} \mid 1 + tm, 3 + tm, 1 + t(m + 2) \in \mathbb{P}\}.$$ 

Lemma 2.5. a) Let $p \in D_{\frac{m}{2}}$ and $p > m$. If $f_{\frac{m}{2}}(p) < (m + 2)p - 2$, then there is a $t \in T(m)$ such that $p = 1 + tm$, in particular $p + 2$ is a prime as well.

b) If $T(m)$ is finite, then

$$\frac{f_{\frac{m}{2}}(p)}{p} \geq m + 2 - \frac{2}{p}$$

for large $p \in D_{\frac{m}{2}}$.

c) $T(2) = \{1\}$, and $T(m)$ is empty if $m > 2$ and $m$ is incongruent to 1 modulo 3.

Proof. Let $p \in D_{\lambda}$.

a) By definition of $p(\frac{m}{2})$, $mp(\frac{m}{2}) \leq (m + 2)p$, hence $mp(\frac{m}{2}) < (m + 2)p$ since $p > m$. For reasons of parity we even have $mp(\frac{m}{2}) \leq (m + 2)p - 2$. 

Let $z := (m + 2)p - 2 > f_{\frac{m}{2}}(p)$, hence $z \in S_{\frac{m}{2}}(p)$. Since $mp(\frac{m}{2}) < z < (m + 3)p$, because of parity $z$ is the sum of exactly $m + 2$ atoms from $S_{\frac{m}{2}}(p)$, hence $p + 2$ must be a prime. Similarly $w := (m + 2)p - 2 > f_{\frac{m}{2}}(p)$ is in $S_{\frac{m}{2}}(p)$; hence $w = mp(\frac{m}{2})$ because of its parity and since $mp(\frac{m}{2}) \leq w < (m + 2)p$. For $t := \frac{p - 1}{m}$ we have primes $p = 1 + tm$, $p + 2 = 3 + tm$ and $p(\frac{m}{2}) = 1 + t(m + 2)$. Further $2t = p(\frac{m}{2}) - p$ is an even integer; hence $t \in T(m)$.

b) is immediate from a).

c) Let $m \geq 2$ and $t > 0$ be integers. Elementary calculations modulo 3 show:

If $m$ is incongruent to 1 modulo 3, then 3 divides $(1 + tm)(3 + tm)(1 + t(m + 2))$. Hence $1 + tm$, $3 + tm$ and $1 + t(m + 2)$ are primes if and only if $m = 2$ and $t = 1$. \hfill \Box

Proposition 2.6. a) $\frac{f_{\lambda}(p)}{p} \geq 3 - \frac{2}{p}$ for all $\lambda > 0$ and $p \in D_{\lambda}$ (cf. \cite[Proposition 1f]{8}).

b) If $m$ is incongruent to 1 modulo 3, then

$$(8) \quad \frac{f_{\frac{m}{2}}(p)}{p} \geq m + 2 - \frac{2}{p}$$

for large $p \in D_{\frac{m}{2}}$.

In case $m \equiv 1 \mod 3$, (8) at least holds for large isolated primes.
c) Let $m \geq 2$ be arbitrary and $0 < \lambda < \frac{2}{m}$. Then

$$\frac{f_\lambda(p)}{p} \geq m + 2 - \frac{2}{p} \text{ for } p > \frac{2}{2 - \lambda \cdot m}, \quad p \in D_\lambda.$$ 

**Proof:**

b) immediately follows from Lemma 2.5.

c) Elementary calculation shows that, since $p > \frac{2}{2 - \lambda \cdot m}$,

$$m(1 + \lambda)p < (m + 2)p - 2,$$

consequently

$$m \cdot p(\lambda) \leq m(1 + \lambda) \cdot p < (m + 2)p - 2 < (m + 2)p.$$

Hence $(m + 2)p - 2$ is a gap of $S_\lambda(p)$, for reasons of parity and magnitude. 

Now we are ready to restate and prove:

**Theorem 1.2.** Let $0 < \lambda \leq 1$, $\lambda \notin \{\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots\}$, be a real. Let $D_\lambda := \{p \in \mathbb{P} \mid [p, p + \lambda p] \text{ contains at least two primes}\}$ and let $f_\lambda(p)$ be the Frobenius number of the numerical semigroup generated by all primes from $[p, p + \lambda p]$. Then

$$\lim_{p \to \infty} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor.$$

For $\lambda = \frac{2}{m}$ with an integer $m \geq 2$ and $m \equiv 1 \mod 3$, at least

$$\limsup_{p \to \infty} \frac{f_\lambda(p)}{p} = \left\lfloor 2 + \frac{2}{\lambda} \right\rfloor.$$

**Proof.** Immediate from Corollary 2.4 and Proposition 2.6.

Notice, that in case $m \equiv 1 \mod 3$ Dickson’s conjecture implies, that $T(m)$ is infinite, so the above proof probably will not work. So in this case, Lemma 2.5 does not include formula (8) for large $p$.

### 3 The question of Wilf for certain numerical semigroups

This section is devoted to the proof of Proposition 1.1, restated below for the reader’s comfort.

Since Wilf’s inequality (1) holds by [2] and [6] if $p < 19$ or $f < 3p$, in what follows we may assume that $p \geq 19$ and $f > 3p$. Proposition 1.1a) will be an easy consequence of the following result.
Theorem 3.1. There is a constant $c > 0$ such that every numerical semigroup $S$ of multiplicity $p \geq c$ ($p$ not necessarily a prime number) and containing the primes from $J(p) := [p, p + p^{0.525}]$, satisfies Wilf’s inequality

\[
\frac{g}{1 + f} \leq 1 - \frac{1}{e}, \text{ equivalently } e(1 + f - g) \geq 1 + f.
\]

Proof. Let $\pi(x)$ be the number of primes less than or equal to $x$. For sufficiently large integers $p$, by [1, p. 562] we have

\[
e \geq |J(p) \cap p| \geq 0.09 \cdot p^{0.525} \log p =: e^*(p) \geq 2p^{0.5},
\]

the latter since $\lim_{x \to \infty} \frac{p^{0.525}}{\log x} = \infty$. Notice, that $1 + f - g$ is the number of elements of $S$ lying below $f$, sometimes called sporadic for $S$. Let $m$ be an integer such that $mp < f < (m + 1)p$. We have $m \geq 3$ since by assumption $f > 3p$. Hence the $(m - 1) \cdot |J(p) \cap p|$ many elements

\[s = ip + q, \ 0 \leq i \leq m - 2 \text{ and } q \in J(p) \text{ a prime,}
\]

are sporadic for $S$, since $s \leq (m - 2)p + q < mp < f$. Finally we get by (10) and since $m \geq 3$

\[e(1 + f - g) \geq e^*(p) \cdot (m - 1) \cdot e^*(p) \geq (m - 1)4p > (m + 1)p > f. \]

\[\Box
\]

Lemma 3.2. Let $S(p)$ be the semigroup generated by the primes from $I(p) = [p, 2p]$ and $f(p)$ its Frobenius number. Then

\[f(p) < 2(\pi(2p) - \pi(p))^2, \text{ if } n = \pi(p) > 674.
\]

Proof. Fundamental for this are the approximate formulas for the functions $p_n$ and $\pi(x)$ from the papers [12] and [13] by Rosser and Schoenfeld. According to [13] we have

\[\pi(2x) < 2\pi(x) \text{ for } x \geq 11.
\]

In [12] it is shown, that

\[p_n < n(\log n + \log \log n) \text{ for } n > 5,
\]

\[\pi(x) < \frac{x}{\log x - \frac{3}{2}} \text{ if } \log x > \frac{3}{2} \text{ and}
\]

\[\pi(x) > \frac{x}{\log x - \frac{1}{2}} \text{ for } x \geq 67.
\]

Since for the embedding dimension $e(p_n)$ of $S(p_n)$ we have

\[e(p_n) = \pi(2p_n) - n + 1 < n + 1 < p_n \text{ for } n > 674 \text{ by (12)},
\]
the approximation of the Frobenius number by [14] page 2, last line can be applied to \( S(p) \) if \( \pi(p) = n > 674 \). We get:

\[
(16) \quad f(p) \leq 2p \pi(2p) \cdot [p/(\pi(2p) - n + 1)] - p
\]
\[
< 2pp \pi(2p)/ (\pi(2p) - n + 1)
\]
\[
< 2p_n \cdot p_{2n}/(\pi(2p) - n).
\]

From (14) and (15) we get

\[
(17) \quad \pi(2x) > 2 \cdot \frac{x}{\log(2x) - \frac{1}{2}} > 2 \cdot \frac{\log x - \frac{3}{4}}{\log(2x) - \frac{1}{2}} \cdot \pi(x) =: l(x) \cdot \pi(x), \ x \geq 67.
\]

It is easily seen that the function \( l(x), x \geq 67 \), is strictly increasing. Together with (12) and (17) we get at the places \( x = p_n, n \geq 675 \), i.e. \( p_n \geq 5039 \)

\[
(18) \quad 2n > \pi(2p_n) > l(p_n) \cdot \pi(p_n) \geq l(5039) \cdot n,
\]

where \( l(5039) \) is approximately 1.61158.

The function

\[
l_2(x) := \frac{2 \cdot (\log x + \log \log x)(\log(2x) + \log \log(2x))}{x}, \ x \geq 675
\]

is strictly decreasing. As one can check,

\[
(19) \quad l_2(675) < (l(5039) - 1)^3.
\]

Applying (13) to the right hand side of formula (16), we get for \( n \geq 675 \)

\[
f(p_n) < 2p_n p_{2n}/(\pi(2p_n) - n)
\]
\[
\leq 2 \cdot l_2(n) \frac{n^3}{\pi(2p_n) - n}
\]
\[
\leq 2 \cdot l_2(675) \cdot \frac{n^3}{\pi(2p_n) - n}
\]
\[
< 2 \cdot (l(5039) - 1)^3 \cdot \frac{n^3}{\pi(2p_n) - n}
\]
\[
\leq 2 \cdot \frac{(l(p_n) - 1)n^3}{\pi(2p_n) - n}
\]
\[
< 2 \cdot (\pi(2p_n) - n)^2.
\]

For \( \lambda > 0 \), let \( D_{\lambda} := \{ p \in \mathbb{P} \mid [p, p + \lambda p] \text{ contains at least two primes} \} \).

**Proposition 1.1.** Let \( \lambda > 0, \ p \in D_{\lambda} \) and \( S \) any numerical semigroup of multiplicity \( p \) containing \([p, p + \lambda p] \cap \mathbb{P}\).

a) There is an integer \( C(\lambda) > 0 \) such that for \( p > C(\lambda) \), the semigroups \( S \) from above satisfy Wilf’s inequality (1).
b) In case \( \lambda = 1 \) formula (1) holds for all \( p \). In particular (1) is true for \( S = S_n := \text{numerical semigroup generated by all primes not less than } p_n \).

**Proof.** a) Let \( \lambda > 0 \), \( p \in D_\lambda \) and \( S \) as in the statement. Let \( c \) be the constant from Theorem 3.1 choose \( C(\lambda) \geq c \) such that \( C(\lambda)^{0.525} < \lambda \cdot C(\lambda) \). Then for every prime \( p \geq C(\lambda) \) we have \( p^{0.525} < \lambda p \) as well. Hence \( p \geq c \) and \( J(p) \cap P \subseteq [p, p + \lambda p) \cap \mathbb{P} \subseteq S \), as requested in Theorem 3.1 and \( S \) satisfies (9).

b) Let \( S \) be as in 1.1b), \( S(p) \) the semigroup generated by the primes from \( I(p) := [p, 2p] \) and \( f(p) \) its Frobenius number. Since \( 3p < f \), the primes from \( I(p) \subseteq S \) are atoms as well as sporadic elements for \( S \). The latter also holds for the even numbers \( p + q \), \( q \) a prime from \( I(p) \) as well as for \( 3p \). Hence \( 1 + f - g \geq 2(\pi(2p) - \pi(p) + 1) + 1 \), and all together

\[
(20) \quad e(1 + f - g) \geq 2(\pi(2p) - \pi(p) + 1)^2 + \pi(2p) - \pi(p) + 1.
\]

Since \( f \leq f(p) \), (20) together with (11) from Lemma 3.2 imply (9) for \( n > 674 \). Therefore, it remains only to prove Proposition 1.1b) in case \( 7 < n < 675 \).

According to the last column of table **wilf_for_p_to_2p.pdf** from [19] inequality (9) holds for \( S(p) \), if \( 8 \leq \pi(p) \leq 675 \). Hence we may assume that \( S \) is different from \( S(p), p = p_n \). Then \( e \geq \pi(2p_n) - n + 2 \), and (20) can be improved to

\[
e(1 + f - g) \geq (2 \cdot (\pi(2p_n) - n + 1) + 1)(\pi(2p_n) - n + 2).
\]

The second last column of table **wilf_for_p_to_2p.pdf** from [19] mentioned above shows, that \( f(p) \) (and, a fortiori, \( f \)) is less than the right hand side of this inequality, if \( 10 \leq \pi(p) \leq 675 \). The remaining cases are \( p = 19 \) and \( p = 23 \).

For \( p = 23 \), by assumption we have \( f > 69 = 3 \cdot p \), and \( S(23) \) contains 17 elements less than 70, which then are sporadic for \( S \). Since \( S \) is different from \( S(23), \) we have \( e \geq e(23) + 1 = 7 \); finally \( e(1 + f - g) \geq 7 \cdot 17 > 102 = f(23) \geq f \).

Analogously for \( p = 19 \) we have \( f > 57 = 3 \cdot p \) and \( e \geq 6 \). Further \( 58 = 29 + 29 \) is in \( S \), hence \( f \geq 59 \). Since 60, 61 and 62 are also in \( S \), either \( f = 59 \) or \( f \geq 63 \).

a) Case \( f \geq 63 \): \( S(19) \) contains 19 elements < 63. As above \( e(1 + f - g) \geq 6 \cdot 19 > 101 = f(19) \geq f \).

b) Case \( f = 59 \): \( S(19) \) contains 16 elements < 59. It follows \( e(1 + f - g) \geq 6 \cdot 16 > 59 = f \). \( \square \)

The fraction \( d = \frac{1 + f - g}{1 + f} \) describes the density of the sporadic elements of \( S \) in \([0, f] \cap \mathbb{Z} \). In terms of this density, Wilf’s conjecture says that \( d \) is at least \( \frac{1}{2} \).

We will see in a moment that for the semigroups \( S(p) \) generated by the primes from \([p, 2p]\) this bound becomes extremely weak, as \( p \) goes to infinity.

As above let \( 2 \geq \lambda > 0 \) be a real parameter, \( S_\lambda(p) \) the semigroup generated by the primes from \( I_\lambda(p) \) and \( e_\lambda(p) \) its embedding dimension. Here \( e_\lambda(p) \sim \lambda \cdot \pi(p) \), hence \( \frac{1}{e_\lambda(p)} \) is a null sequence. In contrast, the results of [3] and [11] imply the following result.
Proposition 3.3. If $S = S(p)$ is the semigroup generated by the primes from $[p, 2p]$ then

$$d \sim \frac{3}{8}.$$ 

Proof Let $\frac{s}{8} < t < 1$. By [3, Corollary], for all $2N \in [2p, 4p]$ but $O\left(\frac{2p}{\log(2p)}\right)$ exceptions, we have $2N = q_1 + q_2$ with

$$N - N^t \leq q_i \leq N + N^t, q_i \text{ a prime for } i = 1, 2.$$ 

If even

$$p + (2p)^t \leq N \leq 2p - (2p)^t$$

then it easily follows that $p < q_i < 2p$, hence $2N \in S(p)$. Since $\frac{t}{p}$ and $\frac{1}{\log(2p)}$ are null sequences, this shows that for large $p$, almost all even elements from $[2p, 4p]$ are in $S(p)$. 

By similar arguments we see from [11, Theorem 1.1], that for large primes $p$, every odd integer $N$ with

$$p + (6p)^t \leq N \leq 2p - (6p)^t$$

is contained in $S(p)$. Hence for large $p$, almost all odd elements from $[3p, 6p]$ are in $S(p)$. Further by Theorem [12], $f(p) \sim 4p$. 

4 Binary Goldbach for large numbers: Sufficient conditions

The Binary Goldbach conjecture for large numbers, which seems to be open, states that each large enough even integer can be written as a sum of two primes.

In this section we present some consequences which would follow if “Binary Goldbach for large numbers” should be false. We cannot disprove any of these consequences and we do not believe that our results mean some practical progress on a way to prove Binary Goldbach for large numbers.

Recall that by $S_n$ we denote the semigroup generated by all primes not less than $p_n$, and by $f_n$ we denote the Frobenius number of $S_n$. 

Obviously $3p_n \notin S_{n+1}$, hence $3p_n \leq f_{n+1}$ for all $n \geq 2$.

On the other hand, the table full_numerical_semigroups_created_by_primes_greater_nth.pdf from [19] shows that, for $2 \leq n \leq 10000$,

(i) $f_{n+1} < 3p_n + 2n$.

(ii) $f_{n+1}$ is odd, with the exception $f_4 = 16$.

(iii) $f_{n+1} = 3p_n + 2n - 2$ for $n = 4, 6, 7, 9$ or $15$.

Analogously, improving [8, Lemma 3] we get the following result.
Lemma 4.1. For large $n$, each odd integer $N \geq 3p_n + 2n$ is contained in $S_{n+1}$.
In particular,

\[ f_{n+1} < 3p_n + 2n \text{ if } f_{n+1} \text{ is odd, and} \]

\[ f_{n+1} < 3p_n + p_{n+1} + 2n \text{ if } f_{n+1} \text{ is even.} \]

Proof: By [11, Theorem 1.1] each odd number $N \geq U_n := 3p_n + 2n \gg 0$ is the sum

\[ N = q_1 + q_2 + q_3 \text{ of primes } q_i \text{ with } \left| \frac{N}{3} - q_i \right| \leq N^\frac{2}{5}, \ i = 1, 2, 3. \]

Since $F(x) := \frac{x}{3} - x^\frac{2}{5}$ is increasing for $x \geq 5$, and $p_n \sim n \log n$,

\[ q_i \geq \frac{N}{3} - N^\frac{2}{5} \]

\[ \geq \frac{U_n}{3} - U_n^\frac{2}{5} \]

\[ > p_n + \frac{2n}{3} - (6n \log n + 2n)^\frac{2}{5} \]

\[ > p_n \]

for large $n$. Hence $N = q_1 + q_2 + q_3$ is contained in $S_{n+1}$. \qed

This implies for each $1 \geq \varepsilon > 0$: Since $\frac{2n}{p_n}$ is a null sequence and $\lim_{n \to \infty} \frac{p_{n+1}}{p_n} = 1$, there is an $n(\varepsilon) > 0$ such that for $n \geq n(\varepsilon)$

\[ f_{n+1} \frac{p_{n+1}}{p_n} < 3 + \varepsilon \text{ if } f_{n+1} \text{ is odd, } f_{n+1} \frac{p_{n+1}}{p_n} < 4 + \varepsilon \text{ if } f_{n+1} \text{ is even and} \]

\[ \frac{p_{n+1}}{p_n} < 1 + \frac{\varepsilon}{5}. \]

Hence for all $n \geq n(\varepsilon)$

\[ \frac{f_{n+1}}{p_n} - \frac{f_{n+2}}{p_{n+1}} \leq \frac{f_{n+1}}{p_{n+1}} \frac{p_{n+1}}{p_n} - \frac{f_{n+1}}{p_{n+1}} \]

\[ = \frac{f_{n+1}}{p_{n+1}} \left( \frac{p_{n+1}}{p_n} - 1 \right) \]

\[ < 5 \cdot \frac{\varepsilon}{5} \]

\[ = \varepsilon \]

by (21) and (22).

Assumption The Binary Goldbach conjecture for large numbers is false.

Conclusions for the sequence $(f_n)$
a) For infinitely many $n > 0$, $\frac{f_{n+1}}{p_n} \geq 4$. See [8, Lemma 1].

b) For each integer $k > 0$ there is an integer $n \geq n\left(\frac{1}{k}\right)$ such that the $k$ Frobenius numbers $f_{n+1}, \ldots, f_{n+k}$ are even.

Proof: By a) we can find an $n \geq n\left(\frac{1}{k}\right)$ such that $\frac{f_{n+1}}{p_n} \geq 4$. By (23), for $1 \leq m \leq k$ we have

$$\frac{f_{n+m}}{p_{n+m-1}} = \frac{f_{n+1}}{p_n} - \left(\frac{f_{n+1}}{p_n} - \frac{f_{n+2}}{p_{n+1}}\right) - \cdots - \left(\frac{f_{n+m-1}}{p_{n+m-2}} - \frac{f_{n+m}}{p_{n+m-1}}\right)$$

$$\geq 4 - (m-1) \cdot \frac{1}{k}$$

$$\geq 3 + \frac{1}{k}$$

hence the integers $f_{n+1}, \ldots, f_{n+k}$ are even by (21).

Note on the even $f_n$:
1. We apply \cite{3, Corollary} to any \(0 < \varepsilon < \frac{3}{8}\) like e.g. \(\varepsilon = \frac{1}{8}\): \(U = m^{\frac{3}{8} + \varepsilon}\) is, because of \(\varepsilon < \frac{3}{8}\), smaller than \(\frac{9m}{10}\) for \(m\) large.

Hence apart from at most \(O\left(\frac{N}{\log N}\right)^A\) exceptions (\(A\) arbitrary) each even integer

\[
2m \in [N, 2N]
\]

is a sum

\[
2m = q_1 + q_2
\]

of two primes numbers

\[
\frac{9}{10} m \leq q_1, q_2 \leq \frac{11}{10} m.
\]

2. Let \(n\) be large enough and suppose \(f_n\) is even. By \cite{8} Proposition 1 and Lemma 3, \(N := 3p_n - 6 \leq f_n \leq 2N\); in particular, \(p_n \leq \frac{9m}{10} \cdot \frac{1}{2}\). Hence the gap \(f_n\) of \(S_n\) is always an exception in the sense of (1).

Note on coding Our numerical experiments may be reproduced by using the corresponding codes from the repository Bleiglanz/On_The_FrobeniusNumber in \cite{19}.

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