DETECTING BIFURCATION VALUES AT INFINITY OF REAL POLYNOMIALS

LUI S RENATO G. DIAS AND MIHAI TIBĂR

Abstract. We present a new approach for estimating the set of bifurcation values at infinity. This yields a significant shrinking of the number of coefficients in the recent algorithm introduced by Jelonek and Kurdyka for reaching critical values at infinity by rational arcs.

1. Introduction

The bifurcation locus of a polynomial mapping $f : \mathbb{K}^n \to \mathbb{K}^p$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $n \geq p$ is the smallest subset $B(f) \subset \mathbb{K}^p$ such that $f$ is a locally trivial $C^\infty$-fibration over $\mathbb{K}^p \setminus B(f)$. It is well known that $B(f) = f(Sing(f)) \cup B_\infty(f)$, where $B_\infty(f)$ denotes the set of bifurcation values at infinity (Definition 2.1). A simple example is $f(x, y) = x + x^2y$, where we have $Sing(f) = \emptyset$ and $B_\infty(f) = \{0\}$.

While the set of critical values $f(Sing(f))$ is relatively well understood, the other set $B_\infty(f)$ is still mysterious. In case $p = 1$ the bifurcation set $B(f)$ is finite, as proved by Thom [Th], see also [Ph, Ve]. However, one can precisely detect the bifurcation set $B(f)$ only in case $p = 1$ and $n = 2$ by using several types of (equivalent) tests, see [Su], [HL], [Ti2], [Dur], [Ti3] over $\mathbb{C}$ and [TZ], [CP] over $\mathbb{R}$.

For $p = 1$ and more than two variables one can only estimate $B(f)$ by “reasonably good” supersets $A(f) \supset B(f)$ of the form $A(f) = f(Sing(f)) \cup A_\infty(f)$, where $A_\infty(f)$ is a finite set which depends on the choice of some regularity condition at infinity: tameness [Br1], Malgrange regularity [Pa], $\rho$-regularity [NZ], $t$-regularity [ST1], [Ti2].

In case $p > 1$ it is known that $B_\infty(f)$ is contained in a one codimensional semi-algebraic subset of $\mathbb{R}^p$, or an algebraic subset of $\mathbb{C}^p$, respectively, as proved by Kurdyka, Orro and Simon [KOS]. Sharper such subsets have been obtained in [DRT].

This paper addresses the problem of estimating the bifurcation locus at infinity $B_\infty(f)$ and is motivated by the recent algorithm presented by Jelonek and Kurdyka [JK2] for finding the set of values $K_\infty(f)$ for which the Malgrange condition at infinity fails (Definition 2.7). Their algorithm applies to a space $AV(f)$ of rational arcs in $\mathbb{R}^n$, the coefficients of which are solutions of a certain set of equations.

We present here the construction of a different space of rational arcs $\text{Arc}(f)$ to which we attach the same set of equations for the coefficients, and thus the same algorithm can be run.

Date: March 27, 2014.
2010 Mathematics Subject Classification. 14D06, 58K05, 57R45, 14P10, 32S20, 58K15.
Key words and phrases. bifurcation locus of real polynomials, regularity at infinity, fibrations.
The authors acknowledge the support of the USP-COFECUB Uc Ma 133/12 project.

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Our main results, Theorem 3.4 together with Theorems 2.5 and 2.8 show that the resulting subset of asymptotic arcs \( \text{Arc}_\infty(f) \) detects a certain set of values that includes \( \mathcal{B}_\infty(f) \) and is included in \( K_\infty(f) \). Consequently, the Jelonek-Kurdyka algorithm applied to our \( \text{Arc}(f) \) will produce a (smaller) set of values including \( \mathcal{B}_\infty(f) \) in considerably shorter time since the number of coefficients of the rational arcs is drastically reduced, namely:

\[
\dim \text{Arc}(f) = n(1 + d^n) \quad \text{versus} \quad \dim \text{AV}(f) = n(2 + d(d + 1)^n(d^n + 2)^{n-1}).
\]

Our result is also relevant for optimisation and complexity problems since bifurcation values at infinity appear for instance in the optimization of real polynomials, e.g. [HP2], [Sa].

2. Regularity conditions at infinity

2.1. Bifurcation values at infinity. We start by recalling the basic definitions after [Ti2], [Ti3], [DRT], eventually adapting them to any \( n \geq p \geq 1 \).

Definition 2.1. Let \( f: \mathbb{R}^n \to \mathbb{R}^p \) be a polynomial mapping, where \( n \geq p \). We say that \( t_0 \in \mathbb{R}^p \) is a typical value of \( f \) if there exists a disk \( D \subset \mathbb{R}^p \) centered at \( t_0 \) such that the restriction \( f_0: f^{-1}(D) \to D \) is a locally trivial \( C^{\infty} \)-fibration. Otherwise we say that \( t_0 \) is a bifurcation value (or atypical value). We denote by \( B(f) \) the bifurcation locus, i.e. set of bifurcation values of \( f \).

We say that \( f \) is topologically trivial at infinity at \( t_0 \in \mathbb{R}^p \) if there exists a compact set \( K \subset \mathbb{R}^n \) and a disk \( D \subset \mathbb{R}^p \) centered at \( t_0 \) such that the restriction \( f_1: f^{-1}(D) \setminus K \to D \) is a locally trivial \( C^{\infty} \)-fibration. Otherwise we say that \( t_0 \) is a bifurcation value at infinity of \( f \). We denote by \( \mathcal{B}_\infty(f) \) the set of bifurcation values at infinity of \( f \).

As we have claimed in the Introduction, there is no general characterisation\(^1\) of the bifurcation locus besides the setting \( n = 2 \) and \( p = 1 \). In case \( n > 2 \) one uses regularity conditions at infinity in order to control the topological triviality. We work here with the \( \rho \)-regularity and with the Malgrange-Kuo-Rabier condition.

2.2. The \( \rho \)-regularity. We adapt the definition of \( \rho \)-regularity from [Ti2] to Euclidean spheres centered at any point of \( \mathbb{R}^n \). This allows us to define a new set, denoted here by \( S_\infty(f) \), which produces a sharper estimation of \( B(f) \).

Let \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) and let \( \rho_a: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) be the Euclidian distance function to \( a \), i.e. \( \rho_a(x) = (x_1 - a_1)^2 + \ldots + (x_n - a_n)^2 \).

Definition 2.2 (Milnor set at infinity). Let \( f: \mathbb{R}^n \to \mathbb{R}^p \) be a polynomial mapping, where \( n \geq p \). We fix \( a \in \mathbb{R}^n \). We call Milnor set of \((f, \rho_a)\) the critical set of the mapping \( (f, \rho_a): \mathbb{R}^n \to \mathbb{R}^{p+1} \) and denote it by \( M_a(f) \).

In case \( p = 1 \) we need the following statement, which was noticed in [HP1] (see also [Dut, Lemma 2.2]). We provide a proof and use some details of it in the proof of Theorem 3.4.

\(^1\)Still, one can treat particular cases when “singularities at infinity” are isolated in a certain sense, e.g. [ST], [Pa], [Ti3], [TT].
Lemma 2.3. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a polynomial mapping. There exists an open dense subset $\Omega_f \subset \mathbb{R}^n$ such that, for every $a \in \Omega_f$, $M_a(f) \setminus \text{Sing}f$ is either a non-singular curve or it is empty.

Proof. We claim that the following semi-algebraic set:

\[ Z := \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in M_a(f) \setminus \text{Sing}f\} \]

is a smooth $(n+1)$-dimensional manifold. For some fixed $(x_0, a_0) \in Z$, there is $1 \leq i \leq n$ such that $\frac{\partial f}{\partial x_i}(x_0) \neq 0$ and moreover, since $\text{Sing} f$ is closed, there exists an open set $U \subset \mathbb{R}^n$ such that $\frac{\partial f}{\partial x_i}(x) \neq 0, \forall x \in U$. For $1 \leq j \leq n, j \neq i$, we set:

\[ m_j(x, a) := \frac{\partial f}{\partial x_i}(x)(x_j - a_j) - \frac{\partial f}{\partial x_i}(x)(x_i - a_i). \]

We have $Z \cap (U \times \mathbb{R}^n) = \{(x, a) \in U \times \mathbb{R}^n \mid m_j(x, a) = 0; 1 \leq j \leq n, j \neq i \}$. Let $\varphi: U \times \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the mapping $\varphi = (m_1, \ldots, m_n)$, where $m_i$ is missing. By its definition we have $\varphi^{-1}(0) = Z \cap (U \times \mathbb{R}^n)$. Let us notice that the rank of the gradient matrix $D\varphi$ is $n-1$ at any point of $U \times \mathbb{R}^n$. Indeed, the minor $\frac{\partial m_l}{\partial x_i}(x, a)$ is a diagonal matrix of which the entries on the diagonal are all equal to $-\frac{\partial f}{\partial x_i}(x)$, hence non-zero. This shows that $Z$ is a manifold of dimension $n+1$.

We next consider the projection $\tau: Z \to \mathbb{R}^n, \tau(x, a) = a$. Thus, $\tau^{-1}(a) = (M_a(f) \setminus \text{Sing}f) \times \{a\}$. By Sard’s Theorem, we conclude that, for almost all $a \in \mathbb{R}^n$, $\tau^{-1}(a) = (M_a(f) \setminus \text{Sing}f) \times \{a\}$ is a smooth curve or that it is an empty set. 

\[ \square \]

Definition 2.4 ($\rho_a$-regularity at infinity). Let $f: \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial mapping, where $n \geq p$. Let $a \in \mathbb{R}^n$. We call:

\[ S_a(f) := \{t_0 \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset M_a(f), \lim_{j \to \infty} \|x_j\| = \infty \text{ and } \lim_{j \to \infty} f(x_j) = t_0\} \]

the set of asymptotic $\rho_a$-nonregular values. If $t_0 \notin S_a(f)$ we say that $t_0$ is $\rho_a$-regular at infinity. We set $S_\infty(f) := \bigcap_{a \in \mathbb{R}^n} S_a(f)$.

The above condition is a “Milnor type” condition that controls the transversality of the fibres of $f$ to the spheres centered at $a \in \mathbb{R}^n$.

We derive the following result about $S_\infty(f)$ from [DRT] Theorem 5.7, Proposition 6.4 where it was stated for $S_0(f)$:

Theorem 2.5. Let $f: \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial mapping, where $n \geq p \geq 1$. Then $B_\infty(f) \subset S_\infty(f)$.

Moreover:

(a) The sets $S_\infty(f)$ and $f(\text{Sing}f) \cup S_\infty(f)$ are closed sets.

(b) For any $a \in \mathbb{R}^n, S_a(f)$ and $f(\text{Sing}f) \cup S_a(f)$ are semi-algebraic sets of dimension $\leq p - 1$.

Footnote: For complex polynomial functions, transversality to large spheres was used in [Br1] p.229, in [NZ] where it is called $M$-tameness, later in [ST], [Dur] etc. In the real setting, it was used in [T1], [T2], later in [HP1] etc.
Proof. To prove the inclusion $B_0(\infty) \subset S_\infty(\infty)$, let $t_0 \in \mathbb{R}^p \setminus S_\infty(\infty)$. It follows that there exists $a \in \mathbb{R}^n$ such that $t_0 \notin S_a(f)$. By [DRT] Prop. 6.4 we obtain that $f$ is topologically trivial at infinity at $t_0 \in \mathbb{R}^p$, after Definition 2.1, where the later is an open set (as shown at (a) below). A completely similar reasoning shows the inclusion $B(\infty) \subset f(S_0(f)) \cup S_\infty(\infty)$, thus ending the proof of the first part of our statement.

(a). The proof of [DRT] Theorem 5.7(a), p. 337 yields that $S_0(f)$ and $S_0(f) \cup f(\text{Sing} f)$ are closed sets. The same proof holds true when the point 0 is replaced by any other point $a \in \mathbb{R}^n$. This implies that $S_\infty(f) = \bigcap_{a \in \mathbb{R}^n} S_a(f)$ and $\bigcap_{a \in \mathbb{R}^n} A_{\rho_a}(f)$ are closed sets.

The claim (b) was proved as [DRT] Theorem 5.7(b)] for $S_0(f)$ and the same proof holds for any $a \in \mathbb{R}^n$. \hfill $\Box$

Remark 2.6. Since $M_a(f)$ is semi-algebraic, for any value $c \in S_a(f)$ there exist paths $\phi : [0, \varepsilon] \to M_a(f) \subset \mathbb{R}^n$ such that $\lim_{t \to 0} \|\phi(t)\| = \infty$ and $\lim_{t \to 0} f(\phi(t)) = c$. This follows from the Curve Selection Lemma at infinity, as remarked in [DRT] and [CDTT].

2.3. Relation with the Malgrange-Kuo-Rabier condition. Jelonek an Kurdyka [JK1] gave an estimation for $B(f)$ by using the notion of asymptotic critical values of $f$. This is based on the Malgrange-Kuo-Rabier regularity at infinity. We have shown in [Ti1] and [DRT] that this condition implies the $\rho$-regularity at infinity, where $\rho$ denotes the Euclidean distance, but that they are not the same and therefore the associated sets of “critical values at infinity” may be different (see Example 2.9).

Definition 2.7. [Ra] Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial mapping, where $n \geq p$. Let $Df(x)$ be the Jacobian matrix of $f$ at $x$. Let

$$K_\infty(f) := \left\{ t \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n, \lim_{j \to \infty} \|x_j\| = \infty, \lim_{j \to \infty} f(x_j) = t, \lim_{j \to \infty} \|Df(x_j)\| = 0 \right\},$$

where $\nu(A) := \inf_{\varphi : \|\varphi\| = 1} \|A^*(\varphi)\|$, for a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^p$ and its adjoint $A^* : (\mathbb{R}^p)^* \to (\mathbb{R}^n)^*$. We call $K_\infty(f)$ the set of asymptotic critical values of $f$.

Note that in case $p = 1$ the last limit in (4) amounts to $\lim_{j \to \infty} \|x_j\| = 0$ thus the above definition recovers the definition of Malgrange non-regular values at infinity (cf [Pa], [ST], [Ti1] etc).

The next key result shows the inclusion $S_a(f) \subset K_\infty(f)$ for any $a \in \mathbb{R}^n$ (which may be strict, see Example 2.9). Moreover, it shows not only that all the values of $S_a(f)$ may be detected by paths $\phi$ like in the above Remark 2.6 but that the same paths $\phi$ verify the conditions (4) in the definition of $K_\infty(f)$. This will be a key ingredient in the proof of Theorem 3.4.

Theorem 2.8. Let $f = (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial mapping, where $n > p$. Let $\phi : [0, \varepsilon] \to M_a(f) \subset \mathbb{R}^n$ be an analytic path such that $\lim_{t \to 0} \|\phi(t)\| = \infty$ and $\lim_{t \to 0} f(\phi(t)) = c$. Then $\lim_{t \to 0} \|\phi(t)\| \nu(Df(\phi(t))) = 0$.

In particular $S_a(f) \subset K_\infty(f)$ and $S_\infty(f) \subset K_\infty(f)$.

The particular inclusion $S_0(f) \subset K_\infty(f)$ was proved in [CDTT] Proposition 2.4.
Proof. Let \( c = (c_1, \ldots, c_p) \in S_a(f) \). By the definition of \( M_a(f) \), we have \( \phi(t) \in M_a(f) \) if and only if \( \text{rank } D(f, \rho_a)(\phi(t)) < p + 1 \). It follows that there exist real coefficients \( \lambda(t), b_1(t), \ldots, b_p(t) \) which are actually analytic functions of \( t \) and not all equal to zero for the same \( t \), such that:

\[
(5) \quad \lambda(t) \cdot (\phi_1(t) - a_1, \ldots, \phi_n(t) - a_n) = b_1(t) \frac{\partial f_1}{\partial x}(\phi(t)) + \cdots + b_p(t) \frac{\partial f_p}{\partial x}(\phi(t)),
\]

where \( \frac{\partial f_i}{\partial x}(\phi(t)) := \left( \frac{\partial f_1}{\partial x}(\phi(t)), \ldots, \frac{\partial f_n}{\partial x}(\phi(t)) \right) \) for \( i = 1, \ldots, p \).

Let us denote \( \hat{\lambda}(t) := \frac{\lambda(t)}{\|\phi(t)\|} \) and \( \hat{b}(t) := \frac{b(t)}{\|\phi(t)\|} \), where \( b(t) = (b_1(t), \ldots, b_p(t)) \). Since \( b(t) \neq 0 \) \( \forall t \in [0, \varepsilon[ \) for small enough \( \varepsilon \), the equality \( (5) \) writes:

\[
(6) \quad \sum_{i=1}^{p} \hat{b}_i(t) \frac{\partial f_i}{\partial x}(\phi(t)) = \hat{\lambda}(t)(\phi(t) - a).
\]

From this we obtain:

\[
(7) \quad \sum_{i=1}^{p} \hat{b}_i(t) \frac{d}{dt} f_i(\phi(t)) = \left( \sum_{i=1}^{p} \hat{b}_i(t) \frac{\partial f_i}{\partial x}(\phi(t)), \phi'(t) \right) = \frac{1}{2} \hat{\lambda}(t) \frac{d}{dt} \|\phi(t) - a\|^2.
\]

We shall denote by \( \text{ord}_t \) the order at 0 of some analytic parametrisation. The condition \( \lim_{t \to 0} f_i(\phi(t)) = c_i \) implies that \( \text{ord}_t \left( \frac{d}{dt} f_i(\phi(t)) \right) \geq 0, i = 1, \ldots, p \), and the condition \( \|\hat{b}(t)\| = 1 \) implies that the order of the first sum in \( (7) \) is also non-negative. We may therefore derive from \( (7) \):

\[
(8) \quad 0 \leq \text{ord}_t \left( \hat{\lambda}(t) \frac{d}{dt} \|\phi(t) - a\|^2 \right) < \text{ord}_t \left( \hat{\lambda}(t) \|\phi(t) - a\|^2 \right).
\]

By taking norms in \( (5) \) and multiplying by \( \|\phi(t) - a\| \) we get:

\[
(9) \quad \text{ord}_t \left( \|\phi(t) - a\| \|\hat{b}_1(t) \frac{\partial f_1}{\partial x}(\phi(t)) + \cdots + \hat{b}_p(t) \frac{\partial f_p}{\partial x}(\phi(t))\| \right) = \text{ord}_t \left( \|\hat{\lambda}(t)\| \|\phi(t) - a\|^2 \right),
\]

which is positive by \( (8) \). This implies:

\[
\lim_{t \to 0} \|\phi(t) - a\| \|\hat{b}_1(t) \frac{\partial f_1}{\partial x}(\phi(t)) + \cdots + \hat{b}_p(t) \frac{\partial f_p}{\partial x}(\phi(t))\| = 0,
\]

which, in turn, implies \( \lim_{t \to 0} \|\phi(t) - a\| \nu(dF(\phi(t))) = 0 \). Since \( \text{ord}_t \|\phi(t) - a\| = \text{ord}_t \|\phi(t) - a\| < 0 \), we get \( \lim_{t \to 0} \|\phi(t)\| \nu(dF(\phi(t))) = 0 \), which shows that \( c \in K_\infty(F) \).

In order to show the claimed inclusions \( S_a(f) \subset K_\infty(f) \) and \( S_\infty(f) \subset K_\infty(f) \) we use Remark 2.6 and the above proof.

The inclusion \( S_\infty(f) \subset K_\infty(f) \) may be strict, as showed by the next example.

Example 2.9. [PZ] The polynomials \( f_{nq} : K^3 \to K, f_{nq}(x, y, z) := x - 3x^{2n+1}y^{2q} + 2x^{3n+1}y^{3q} + y^2z \), where \( n, q \in \mathbb{N} \setminus \{0\} \). We have \( S_0(f_{nq}) = \emptyset \). For \( K = \mathbb{C} \), it is shown in [PZ] that \( f_{nq} \) satisfies Malgrange’s condition for any \( t \in \mathbb{C} \) if and only if \( n \leq q \). One can check that the same holds for \( K = \mathbb{R} \). For \( n > q \) we therefore get \( S_0(f_{nq}) \subsetneq K_\infty(f_{nq}) \neq \emptyset \).
2.4. Questions and Conjecture. We have seen above that the inclusions $B(f) \subset S_a(f) \cup f(Sing f) \subset K_\infty(f) \cup f(Sing f)$ hold for any $a \in \mathbb{R}^n$. On other hand, one can have $S_a(f) \neq S_0(f)$, as shown in the following example for which we skip the computations:

Example 2.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y(x^2y^2 + 3xy + 3)$. Then $S_{(0,0)}(f) = \emptyset$ and $S_{(0,1)}(f) \neq \emptyset$.

In particular, for the polynomial $g(x, y) := f(x, y-1)$ we have that $\emptyset \neq S_0(g) \not\subseteq \mathcal{B}_\infty(g) = \emptyset$ which is a real counterexample to a conjecture in [NZ, p. 686, point 3]. A new conjecture can be stated as below.

These facts support the following natural questions:

(i) Is there a minimal set $S_a(f)$ in the collection $\{S_b(f), b \in \mathbb{R}^n \}$?

(ii) If (i) is true, does the minimality hold for some open dense subset $a \in \mathbb{R}^n$?

The following conjecture seems also natural:

Conjecture 2.11. $B_\infty(f) = S_\infty(f)$.

3. Detecting Bifurcation Values at Infinity by Parametrized Curves

3.1. Bounding the set of asymptotic critical values $S_\infty(f)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree $d \geq 2$. If $t_0 \in S_\infty(f)$ then, by Proposition 2.3 and Definition 2.4, there exists an open dense set of points $a \in \mathbb{R}^n$ such that $M_a(f)$ is a real affine algebraic set of dimension 1 and there exists a real asymptotic branch $\Gamma \subset M_a(f)$ such that $\lim_{z \to \Gamma_{\infty}} f(x) = t_0$. Therefore such branches $\Gamma$ detect all the values in $S_\infty(f)$.

Moreover, $\#S_a(f)$ is precisely the number of finite values taken by $f$ when restricted to branches at infinity of $M_a(f)$. It follows that $\#S_\infty(f)$ is majorated by the number of branches at infinity.

This interpretation yields the same bound for the number of bifurcation values at infinity $\#B_\infty(f)$ as the bound found by Jelonek and Kurdyka for $\#K_\infty(f)$.

Proposition 3.1. [JK1, Corollary 1.2]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree $d \geq 2$. The number $\#S_\infty(f)$ of asymptotic critical values is majorated by $d^{n-1} - 1$.

Proof. We may assume that $M_a(f)$ is not empty for $a \in \Omega_f$. By the proof of Lemma 2.3, $M_a(f)$ is a real curve which is a local complete intersection defined by $n - 1$ equations as in [2]. Let us consider the closure $\overline{M_a(f)}$ in $\mathbb{P}_\mathbb{R}^n$.

The complexification $M_a(f)^C$ of $M_a(f)$ is defined locally as the complex solutions of the same system of equations, hence it is a complex curve of degree equal to $d^{n-1}$. Let us denote by $\overline{M_a(f)^C}$ its closure in $\mathbb{P}_c^n$, and notice that this is a complex curve of the same degree $d^{n-1}$.

The hyperplane at infinity $H_\infty := \mathbb{P}^n_c \setminus C^n$ intersects $\overline{M_a(f)^C}$ at finitely many points and by Bézout’s theorem this number is bounded by the degree $d^{n-1}$. If $B(f) \setminus f(Sing f) \neq \emptyset$ then the restriction of $f_C$ to $M_a(f)^C$ cannot be bounded, since there is at least one branch on which the holomorphic function $f$ is non-constant. Hence we can get at most $d^{n-1} - 1$ finite values as limits of $f$. □
3.2. Branches at infinity of the Milnor set and asymptotic parametrisations. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree $d \geq 2$. If $t_0 \in S_{\infty}(f)$ then, by Proposition 2.3 and Definition 2.4, there exists an open dense set of points $a \in \mathbb{R}^n$ such that $M_a(f)$ is a real affine algebraic set of dimension 1 and moreover, there exists a real asymptotic branch $\Gamma \subset M_a(f)$ such that $\lim_{x \in \Gamma, ||x|| \to \infty} f(x) = t_0$.

Let us describe briefly following the ideas of [JK2] how to produce real parametrisations of the branches at infinity of $M_a(f)$. Let $M_a(f)^C$ denote its complexification defined by the same equations. We recall that this is a locally complete intersection and has degree $D := d^{n-1}$. One may construct Puiseux parametrisations for the branches at infinity $\Gamma_C$ of $M_a(f)^C$ like in [JK2]. Let $\gamma(t) = \sum_{-\infty \leq i \leq D} c_i t^i$, where $t \in \mathbb{C}$ and $|t| > R$ for some radius $R \gg 1$. The bound to the right is $D = d^{n-1}$.

Since each real branch at infinity $\Gamma$ of $M_a(f)$ is contained in some complex branch of $\Gamma_C$ of $M_a(f)^C$ then, following Milnor’s procedure [Mi] pag. 28-29, one may further construct a real parametrisation $\gamma(t) = \sum_{-\infty \leq i \leq D} a_i t^i$ of $\Gamma$ with the same bound $D = d^{n-1}$ to the right, where $t \in \mathbb{R}$ with $|t| > R$ and the same radius $R \gg 1$.

Next we truncate the parametrisation $\gamma(t)$ at the left to the bound $-D'$, where $D' := (d-1)D$ and $d$ is the degree of $f$. This truncation $\hat{\gamma}(t) = \sum_{-(d-1)D \leq i \leq D} a_i t^i$ does not verify anymore the equations of $M_a(f)$. Nevertheless we can show the following:

**Proposition 3.2.** The above defined truncation $\hat{\gamma}(t)$ has the following properties:
(a) $\lim_{t \to \infty} \|\hat{\gamma}(t)\| = \lim_{t \to \infty} \|\gamma(t)\| = \infty$.
(b) $\lim_{t \to \infty} f(\hat{\gamma}(t)) = \lim_{t \to \infty} f(\gamma(t)) = t_0$.
(c) $\lim_{t \to \infty} \frac{\partial f}{\partial x_i}(\hat{\gamma}(t)) = \lim_{t \to \infty} \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$, for any $i$.
(d) $\lim_{t \to \infty} x_j \frac{\partial f}{\partial x_i}(\hat{\gamma}(t)) = \lim_{t \to \infty} x_j \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$, for any $i, j$.

**Proof.** (a) is obvious from the definitions. (b) follows from [JK2] Lemma 3.3 since the degree of $f$ is $d$.
(c) and (d). We claim that the limits involving $\gamma(t)$ are zero. If this is true, then the same will follow for the truncation $\hat{\gamma}(t)$ by using [JK2] Lemma 3.3 since the degrees of $\frac{\partial f}{\partial x_i}$ and of $x_j \frac{\partial f}{\partial x_i}$ are both $\leq d$.

In order to prove our claim, we use our key result Theorem 2.8 for $p = 1$, as follows. Since in this case $\nu(Df(x)) = \|\text{grad}\, f(x)\|$, the equality $\lim_{t \to 0} \|\gamma(t)\| = \|\frac{\partial f}{\partial x_1}(\gamma(t)), \ldots, \frac{\partial f}{\partial x_n}(\gamma(t))\|$ is provided by Theorem 2.8 clearly implies $\lim_{t \to \infty} x_j \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$ and $\lim_{t \to \infty} \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$.

This completes our proof. \hfill $\Box$

3.3. The arc space. Jelonek and Kurdyka [JK2] found an algorithm for reaching the values of $K_{\infty}(f)$ by parametrized curves with bounded expansion. They construct a finitely dimensional space of such rational arcs (cf [JK2], Definition 6.9). Based on our framework we construct a new space of rational arcs of considerably lower dimension. We shall see that this detects at least the bifurcation set $S_{\infty}(f)$ without necessarily detecting all the values in $K_{\infty}(f)$.

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4this part of our construction is different from the procedure given in [JK2] Lemma 3.2.
We consider here the following space of arcs associated to the real polynomial \( f : \mathbb{R}^n \to \mathbb{R} \) of degree \( d \):

\[
\text{Arc}(f) = \left\{ \xi(t) = \sum_{-(d-1)d^{n-1} \leq k \leq d^{n-1}} a_k t^k, a_k \in \mathbb{R}^n \right\}
\]

which is isomorphic to \( \mathbb{R}^{n(1+d^n)} \).

**Remark 3.3.** The dimension of the arc space \( AV(f) \) constructed in [JK2] is \( n(2 + d(d + 1)^n)(d^n + 2)^{n-1} \), while ours is \( \dim \text{Arc}(f) = n(1 + d^n) \).

In a similar manner as in [JK2, Definition 6.10], we define the *asymptotic variety of arcs*, \( \text{Arc}_\infty(f) \subset \text{Arc}(f) \), as the algebraic subset of the rational arcs \( \xi(t) \in \text{Arc}(f) \) such that:

\[
\begin{align*}
(a') \quad & \sum_{k>0} \sum_{i=1}^n a_{kj}^2 = 1, & \text{where } a_k = (a_{k1}, \ldots, a_{kn}). \\
(b') \quad & f(\xi(t)) = b_0 + \sum_{k=1}^\infty b_k t^{-k}, & \text{where } b_0, b_k \in \mathbb{R}. \\
(c') \quad & \frac{\partial f}{\partial x_i}(\xi(t)) = \sum_{k=1}^\infty c_{ik} t^{-k}, & \text{for any } i, \text{ where } c_{ik} \in \mathbb{R}. \\
(d') \quad & x_j \frac{\partial f}{\partial x_i}(\xi(t)) = \sum_{k=1}^\infty d_{ijk} t^{-k}, & \text{for any } i, j, \text{ where } d_{ijk} \in \mathbb{R}.
\end{align*}
\]

The conditions (a’)–(d’) for \( \xi \) are equivalent with the corresponding properties (a)–(d) applied to \( \xi \) instead of \( \hat{\gamma} \). For instance the first equivalence follows by renormalising the coefficients.

Let \( b_0 : \text{Arc}_\infty(f) \to \mathbb{R}, \ b_0(\xi(t)) = \lim_{t \to \infty} f(\xi(t)). \)

**Theorem 3.4.** \( \mathcal{S}_\infty(f) \subset \text{Arc}_\infty(f) \subset K_\infty(f) \).

**Proof.** The inclusion \( b_0(\text{Arc}_\infty(f)) \subset K_\infty(f) \) is a direct consequence of the definitions of \( \text{Arc}_\infty(f) \) and \( K_\infty(f) \) since properties (a’), (b’) and (d’) characterise the values \( b_0 \in K_\infty(f) \).

Let us show the first inclusion. If \( b_0 \in \mathcal{S}_\infty(f) \) then there is some \( a \in \mathbb{R}^n \) and there exists a path \( \gamma(t) \in M_a(f) \), such that \( \lim_{t \to \infty} f(\gamma(t)) = b_0 \). Then Theorem 2.8 shows that \( \gamma \) verifies the conditions (a)–(d) of Proposition 3.2. Moreover, by the same Proposition 3.2, one has that the truncation \( \hat{\gamma} \) defined at §3.2 verifies the same properties, hence the equivalent conditions (a’)–(d’) too. This ends our proof. \( \square \)

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FACULDADE DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE UBERLÂNDIA, AV. JOÃO NAVES DE ÁVILA 2121, 1F-153 - CEP: 38408-100, UBERLÂNDIA, BRAZIL

E-mail address: lrgdias@famat.ufu.br

MATHEMÁTIQUES, LABORATOIRE PAUL PAINLEVÉ, UNIVERSITÉ LILLE 1, 59655 VILLENEUVE D’ASCQ, FRANCE.

E-mail address: tibar@math.univ-lille1.fr