Uniform Distribution of Sequences and its interplay with Functional Analysis

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Abstract

In this paper we apply ideas from the theory of Uniform Distribution of sequences to Functional Analysis and then drawing inspiration from the consequent results, we study concepts and results in Uniform Distribution itself. So let $E$ be a Banach space. Then we prove:

(a) If $F$ is a bounded subset of $E$ and $x \in \overline{\text{co}}(F)$ (= the closed convex hull of $F$), then there is a sequence $(x_n) \subseteq F$ which is Cesàro summable to $x$.

(b) If $E$ is separable, $F \subseteq E^*$ bounded and $f \in \overline{\text{co}}w^*(F)$, then there is a sequence $(f_n) \subseteq F$ whose sequence of arithmetic means $\frac{f_1 + \cdots + f_N}{N}$, $N \geq 1$ weak$^*$-converges to $f$.

By the aid of Krein-Milman’s theorem, both (a) and (b) have interesting implications for closed, convex and bounded subsets $\Omega$ of $E$ such that $\Omega = \overline{\text{co}}\Omega$ and for weak$^*$ compact and convex subsets of $E^*$. For instance, if we apply (b) to $\Omega = B_{M(K)}$, where $K$ is a compact metric space, we get that for every signed Borel measure $\mu$ on $K$ with $\|\mu\| \leq 1$, there are sequences $(x_n) \subseteq K$ and $(\varepsilon_n) \subseteq \{\pm 1\}$ such that the sequence

$$\frac{\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n}{N} \xrightarrow{\text{weak}^*} \mu.$$

Assuming that $\|\mu\| = 1$ we can prove in addition that $(x_n)$ can be chosen to be $|\mu|$-u.d., i.e. $\frac{x_1 + \cdots + x_N}{N} \xrightarrow{\text{weak}^*} |\mu|$.

By further expanding the previous ideas and results, we are able to generalize a classical theorem of Uniform Distribution which is valid for increasing functions $\phi : I = [0,1] \to \mathbb{R}$ with $\phi(0) = 0$ and $\phi(1) = 1$, for functions $\phi$ of bounded variation on $I$ with $\phi(0) = 0$ and total variation $V^I_0 \phi = 1$. So for every such $\phi$, there are sequences $(x_n) \subseteq I$ and $(\varepsilon_n) \subseteq \{\pm 1\}$ such that for each $x \in I$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{[0,x]}(x_k) = \nu(x) \quad \text{and} \quad \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k \chi_{[0,x]}(x_k) = \phi(x),$$

where $\nu$ is the function of total variation of $\phi$ on $I$.

Introduction

Our aim in this paper is twofold. We first study consequences of ideas coming from the theory of Uniform Distribution of sequences [15] in Functional Analysis (section 1) and
then we investigate concepts and results of the theory of Uniform Distribution itself, setting them in a more general frame (section 2).

In the first section we generalize and improve an important result of Niederreiter [19], which we state as Theorem 1. The "translation" of this theorem into the language of Functional Analysis is Theorem 2, which is the main result of this section. The first assertion of this theorem is strongly related to a celebrated Lemma due to Maurey (see Lemma D of [5]) and roughly says that given a point \( x \) in the convex hull \( co(F) \) of a bounded subset of some Banach space, then for every \( N \in \mathbb{N} \), \( x \) can be approximated (an estimation of the approximation error is also given) by the arithmetic mean of \( N \) points of \( F \). The second assertion says that, if \( x \) belongs to \( \overline{co}(F) \), then there is a sequence of points of \( F \) which is Cesàro summable to \( x \). A number of easy consequences of Theorem 2 for a Banach space \( E \) are the following:

A) If \((y_n) \subseteq E \) is any weakly null sequence, then there is a function \( \varphi : \mathbb{N} \to \mathbb{N} \), such that the sequence \( x_n = y_{\varphi(n)} \), \( n \geq 1 \) is Cesàro summable to zero (Prop.2).

B) If \( \Omega \) is a closed convex bounded subset of \( E \) equal to the closed convex hull of its extreme points \( ez\Omega \), then for every \( x \in \Omega \) there is a sequence \((x_n) \subseteq ez\Omega \) which is Cesàro summable to \( x \) (Prop.3). This result has, by the aid of Krein-Milman’s theorem, obvious implications for the unit ball of a Banach space which is reflexive or of the form \( C(K) \) (=the space of real continuous functions on \( K \), where \( K \) is any compact totally disconnected space (Corollaries 1 and 2).

C) A result analogous to Theorem 2(2) (with analogous proof) is valid for the dual \((E^*, weak^*) \) of a separable Banach space \( E \). When \( F \) is a bounded subset of \( E^* \) and \( f \) belongs to \( \overline{co}^{weak^*}(F) \), there is a sequence \((f_n) \subseteq F \) whose arithmetic means weak*-converge to \( f \) (Prop.4).

This result (again using Krein-Milman’s theorem) has obvious implications for a weak*?-compact and convex subset \( \Omega \) of \( E^* \), which generalize classical results (see Prop.5, Cor.3, Th.3 and Th.4). So Theorem 3 is the well known result stating that every \( \mu \in \Omega = P(K) \) (\( K \) is any compact metric space) admits a u.d. sequence, but Theorem 4 says something that seems to be new: for every signed measure \( \mu \in \Omega = B_M(K) \), there is a sequence \((x_n) \subseteq K \) and a sequence of signs \((\varepsilon_n) \subseteq \{ \pm 1 \} \), so that the sequence \( \mu_N = \frac{\varepsilon_1 \delta_{x_1} + \cdots + \varepsilon_N \delta_{x_N}}{N}, N \geq 1 \), weak*-converges to \( \mu \).

The last result is our motivation for the second section of this paper. Drawing inspiration from Theorem 4, we extend the classical concept of uniformly distributed sequence defined for measures \( \mu \in P(K) \), where \( K \) is a compact space (see Def. 1.1 of [15]), to every signed measure \( \mu \in M(K) \) with \( \|\mu\| = 1 \). Thus we will say that a sequence \((x_n) \subseteq K \) is \( \mu \)-u.d. iff \((x_n) \) is \( |\mu|-u.d. \) (in the classical sense) and if also there is a sequence of signs \((\varepsilon_n) \subseteq \{ \pm 1 \} \), so that the sequence \( \mu_N = \frac{\varepsilon_1 \delta_{x_1} + \cdots + \varepsilon_N \delta_{x_N}}{N}, N \geq 1 \), weak*-converges to \( \mu \) (Def.1).

Then (generalizing Theorem 3) we prove Theorem 5, which states that given a compact metric space \( K \) and \( \mu \in M(K) \) with \( \|\mu\| = 1 \), then \( \mu \) admits a u.d. sequence \((x_n) \subseteq K \) (in the sense of the aforementioned definition). So if \( f \in C(K) \), we have

\[
\lim_{N \to \infty} \frac{f(x_1) + \cdots + f(x_N)}{N} = \int_K f \, d\mu \quad \text{and} \quad \lim_{N \to \infty} \frac{\varepsilon_1 f(x_1) + \cdots + \varepsilon_N f(x_N)}{N} = \int_K f \, d\mu.
\]
A further generalization of the last theorem is Theorem 6, which says that both of the above equalities are also valid for $\mu$-Riemann integrable functions. Now let $K$ be a compact interval of the real line, say for simplicity $K = [0, 1]$. Taking into account the standard identification of signed Borel measures on $I$ with (proper) functions of bounded variation (BV) on $I$, Theorem 6 yields Theorem 7: Let $\varphi : I \to \mathbb{R}$ be a BV function with $\varphi(0) = 0$, $V_0^1 \varphi = 1$ and $\varphi$ is right continuous on $I$. Then there are sequences $(x_n) \subseteq I$ and $(\varepsilon_n) \subseteq \{\pm 1\}$, such that for every point of continuity $x$ of $\varphi$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{[0,x]}(x_k) = v(x) \quad \text{and} \quad \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k \chi_{[0,x]}(x_k) = \varphi(x),$$

where $v$ is the function of total variation of $\varphi$ on $I$ and $v(1) = V_0^1 \varphi$. The last theorem partially generalizes a classical result from [15] (Theorem 8 in our treatment) which says that the equalities of Theorem 7 are valid for every $\varphi(0) = 0$ and $\varphi(1) = 1$ (of course, since $\varphi$ is increasing, we have $v = \varphi$ and $\varepsilon_k = 1$ for all $k \geq 1$).

The rest of this section is devoted to the proof of Theorem 9, that is, of the fact that Theorem 7 holds true for every BV function $\varphi$ on $I$ with $\varphi(0) = 0$, $V_0^1 \varphi = 1$ and for each point $x \in I$. This result is a common generalization of Theorems 7,8 and is the main result of the second section. The proof of Theorem 9 is rather elaborate and is presented in several steps (Lemmas 5,6,7 etc.). We also note that the notion of discrepancy of a sequence in $I$ is crucial in its proof.

**Preliminaries**

If $E$ is any Banach space, then $B_E$ denotes its closed unit ball. A subset $L$ of $E$ is said to be total in $E$, if its linear span $\langle L \rangle$ of $L$ is dense in $E$. A sequence $(x_n) \subseteq E$ is said to be Cesàro summable, if the corresponding sequence $\frac{x_1 + \cdots + x_n}{n}$, $n \geq 1$ of arithmetic means of $(x_n)$ converges in norm. Let $A \subseteq E$, then $\co(A)$ is the convex hull of $A$ and $\overline{\co}(A)$ the norm closure of $\co(A)$, which by a classical theorem of Mazur coincides with the weak closure of $\co(A)$. If $A \subseteq E^*$, then $\overline{\co}^\ast(A)$ denotes the closure of $\co(A)$ in the weak$^\ast$ topology of $E^*$. Let $x \in \co(A)$ with $x = \sum_{k=1}^{n} \lambda_k x_k$, where $x_1, \ldots, x_n$ are distinct points of $A$, $\lambda_k > 0$ for $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} \lambda_k = 1$, then we set $\text{supp}x = \{x_1, \ldots, x_n\}$.

Let $K$ be a compact Hausdorff space, then $C(K)$ is the Banach space with sup-norm (denoted by $\| \cdot \|_\infty$) of all continuous real valued functions on $K$. The dual $C(K)^\ast$ of $C(K)$ is isometrically identified via the classical Riesz representation theorem with the space $\mathcal{M}(K)$ of all finite regular signed Borel measures on $K$, with norm $\| \mu \| = |\mu|(K)$. By $\mathcal{M}^+(K)$ (resp. $\mathcal{P}(K)$) we denote the positive (resp. probability) measures on $K$. When $\mu \in \mathcal{M}^+(K)$, a bounded function $f : X \to \mathbb{R}$ is said to be $\mu$-Riemann integrable, if the set of discontinuity points of $f$ has $\mu$-measure zero. It is an easy consequence of Lusin’s theorem that each $\mu$-Riemann integrable function is $\mu$-measurable and hence $\mu$-integrable.

Let $X$ be a (nonempty) set; then $|X|$ denotes the cardinality of $X$ and $\ell_\infty(X)$ the Banach space (with sup-norm) of all bounded real valued functions on $X$. It is well
known that $\ell_\infty(X)$ is linearly isometric to the space $C(\beta X)$, where $\beta X$ is the Stone-
\v{C}ech compactification of the discrete set $X$. If $x \in X$, then $\delta_x$ denotes the point mass at $x$, that is, the Dirac measure $\delta_x : \ell_\infty(X) \to \mathbb{R}$ such that $\delta_x(f) = f(x)$, for $f \in \ell_\infty(X)$. We denote by $\mathcal{F}(X)$ the set of probability measures of finite support on $X$, thus $\mathcal{F}(X) = co(\{\delta_x : x \in X\})$; if $\mu \in \mathcal{F}(X)$, $x_1, \ldots, x_n$ are distinct points of $X$ so that $\mu(\{x\}) = \lambda_k > 0$, for $k = 1, 2, \ldots, n$ and $\sum_{k=1}^n \lambda_k = 1$, then $\mu = \sum_{k=1}^n \lambda_k \delta_{x_k}$ and thus $supp \mu = \{x_1, \ldots, x_n\}$. We note that if $X$ is compact Hausdorff, then $\mathcal{P}(X)$ is weak$^*$ compact and convex subset of $\mathcal{M}(X) = C(X)^*$ and the set of its extreme points $ex\mathcal{P}(X)$ coincides with the set of Dirac measures on $X$; therefore by Krein-Milman’s theorem $\mathcal{P}(X) = co_{w^*}(\{\delta_x : x \in X\}) = \mathcal{F}(X)^{w^*}$.

Whilst most of our results remain valid in the complex case, we assume for simplicity that all Banach spaces (and functions) are real and in certain cases we indicate what happens in the complex case.

1.

We begin by generalizing and improving an important result of Niederreiter, essentially following the proof of the original result (see Th.1 of [19]).

**Theorem 1.** Let $X$ be a nonempty set, $L$ a subset of the closed unit ball $B$ of $\ell_\infty(X)$ and $(\mu_j) \subseteq \mathcal{F}(X)$. Assume that the sequence $(\mu_j)$ converges pointwise on $L$, that is, there exists a function $\mu : L \to \mathbb{R}$ such that

$$
\mu_j(f) \xrightarrow[j \to \infty]{} \mu(f) \quad \forall f \in L.
$$

Then there is a sequence $\omega = (x_n) \subseteq \bigcup_{j=1}^\infty supp \mu_j$ such that

1. The sequence $\nu_N = \frac{\delta_{x_1} + \cdots + \delta_{x_N}}{N} \xrightarrow[N \to \infty]{} \mu$ pointwise on $L$.

2. Moreover, if the sequence $(\mu_j)$ converges to $\mu$ uniformly on $L$, then $(\nu_N)$ converges to $\mu$ uniformly on $L$.

The main tool for proving the above theorem is the following Lemma (see Lemma 1 of [19]).

**Lemma 1.** Let $\mu \in \mathcal{F}(X)$; then there exists a positive constant $C(\mu)$ and a sequence $\omega = (y_n)$ in $X$, such that

$$
\left| \frac{1}{N} \sum_{k=1}^N \chi_M(y_k) - \mu(M) \right| \leq \frac{C(\mu)}{N} \quad (1)
$$

for all $N \in \mathbb{N}$ and for all subsets $M \subseteq X$. In particular

$$
C(\mu) = (m - 1) \left\lfloor \frac{m}{2} \right\rfloor
$$

will do, where $m = |supp \mu|$. 4
It is necessary for our purposes to prove that a modification of the above Lemma holds, not only for characteristic functions, but also for every bounded function \( f : X \to \mathbb{R} \). We recall that the set of extreme points \( \text{exB} \) of the unit ball \( B \) of \( l_\infty(X) \) consists of all functions \( f : X \to \mathbb{R} \) such that \( |f(x)| = 1 \), for all \( x \in X \) and the well known fact that \( B = \text{co}(\text{exB}) \).

**Proposition 1.** Let \( f : X \to \mathbb{R} \) be any bounded function. Then inequality (1) of Lemma 1 holds in the following modified form

(a) If \( f \in \text{co}(\text{exB}) \) then we have

\[
\left| \frac{1}{N} \sum_{k=1}^{N} f(y_k) - \int_X f \, d\mu \right| \leq 2 \frac{C(\mu)}{N} \tag{2}
\]

for all \( N \geq 1 \).

(b) If \( f \) is any bounded function, then we have

\[
\left| \frac{1}{N} \sum_{k=1}^{N} f(y_k) - \int_X f \, d\mu \right| \leq \frac{2\|f\|_\infty}{N} (1 + C(\mu)) \tag{3}
\]

for all \( N \geq 1 \).

**Proof.** (a) Assume first that \( f \in \text{exB} \). Set \( V = \{x \in X : f(x) = 1\} \), then the complement of \( V \) is the set \( V^c = \{x \in X : f(x) = -1\} \). Therefore \( f = \chi_V - \chi_{V^c} \). So we get for \( N \in \mathbb{N} \) that

\[
\sum_{k=1}^{N} f(y_k) = \sum_{k=1}^{N} \chi_V(y_k) - \sum_{k=1}^{N} \chi_{V^c}(y_k) \quad \text{and} \quad \int_X f \, d\mu = \mu(V) - \mu(V^c).
\]

Now from Lemma 1 we have,

\[
\left| \frac{1}{N} \sum_{k=1}^{N} f(y_k) - \int_X f \, d\mu \right| \leq \left| \frac{1}{N} \sum_{k=1}^{N} \chi_V(y_k) - \mu(V) \right| + \left| \frac{1}{N} \sum_{k=1}^{N} \chi_{V^c}(y_k) - \mu(V^c) \right|
\]

\[
\leq \frac{C(\mu)}{N} + \frac{C(\mu)}{N} = 2 \frac{C(\mu)}{N}.
\]

It now follows easily from the last inequality that (2) remains valid, for all \( f \in \text{co}(\text{exB}) \).

(b) It is clear that it suffices to prove (3) for \( f \in B \). Since \( B = \text{co}(\text{exB}) \), there is a sequence \( (f_n) \subseteq \text{co}(\text{exB}) \) such that \( f_n \to f \) uniformly on \( X \). Given \( N \in \mathbb{N} \), consider \( n_0 \in \mathbb{N} \) such that

\[
\|f - f_{n_0}\|_\infty < \frac{1}{N} \tag{4}.
\]

Then from assertion (a) and (4) we get that

\[
\left| \frac{1}{N} \sum_{k=1}^{N} f(y_k) - \int_X f \, d\mu \right| =
\]

\[
\left| \frac{1}{N} \sum_{k=1}^{N} (f(y_k) - f_{n_0}(y_k)) + \int_X f_{n_0} \, d\mu - \int_X f \, d\mu \right| + \left( \frac{1}{N} \sum_{k=1}^{N} f_{n_0}(y_k) - \int_X f_{n_0} \, d\mu \right) \leq
\]

\[
\frac{2\|f\|_\infty}{N} (1 + C(\mu)) + \frac{1}{N} \sum_{k=1}^{N} |f_{n_0}(y_k) - f(y_k)| + \frac{1}{N} \sum_{k=1}^{N} |f(y_k) - f_{n_0}(y_k)| \leq
\]

\[
\frac{2\|f\|_\infty}{N} (1 + C(\mu)) + \frac{1}{N} \sum_{k=1}^{N} |f_{n_0}(y_k) - f(y_k)| + \frac{1}{N} \sum_{k=1}^{N} |f(y_k) - f_{n_0}(y_k)| \leq
\]

\[
\frac{2\|f\|_\infty}{N} (1 + C(\mu)) + \frac{1}{N} \sum_{k=1}^{N} |f_{n_0}(y_k) - f(y_k)| + \frac{1}{N} \sum_{k=1}^{N} |f(y_k) - f_{n_0}(y_k)| \leq
\]
\[
\frac{1}{N} \sum_{k=1}^{N} |f(y_k) - f_{n_0}(y_k)| + \int_X |f - f_{n_0}| \, d\mu + \frac{1}{N} \sum_{k=1}^{N} f_{n_0}(y_k) - \int_X f_{n_0} \, d\mu \leq \frac{1}{N} \cdot N \cdot \frac{1}{N} + \frac{1}{N} \mu(X) + 2 \frac{C(\mu)}{N} = \frac{2}{N}(1 + C(\mu)).
\]

\[\blacksquare\]

**Remark 1.** Let \( f = Re f + iIm f \) be any complex function so that 
\[ |f(x)| = \sqrt{(Re f(x))^2 + (Im f(x))^2} \leq 1 \text{ for all } x \in X. \] 
Then it is easy to prove that
\[ \left| \frac{1}{N} \sum_{k=1}^{N} f(y_k) - \int_X f \, d\mu \right| \leq \frac{2\sqrt{2}}{N}(1 + C(\mu)) \quad (5) \]
for all \( N \geq 1. \)

In particular (5) is valid for any extreme point \( f \) of the unit ball \( B \) of the complex Banach space \( \ell_\infty(X) \) (recall that the extreme points of \( B \) are the functions of the form \( f : X \to \mathbb{C} \), such that \( |f(x)| = 1 \) for all \( x \in X \)).

We now proceed with the proof of Theorem 1. Note that assertion (a) is slightly more general than Theorem 1 of [19]; assertion (b) is new.

**Proof.** (of Theorem 1) (1) Assume that \( (\mu_j) \) converges pointwise on \( L \) to \( \mu \). By Lemma 1 there exist positive constants \( C_j = C(\mu_j) \) and sequences \( \omega_j = (x_{j,n})_{n \geq 1}, \ j \in \mathbb{N} \) such that relation (1) of Lemma 1 holds. For each \( j \in \mathbb{N} \), choose a positive integer \( r_j \) so that \( r_j \geq \max\{j^2, j(C_1 + \cdots + C_{j+1})\} \). Put \( r_0 = 0; \) we define a sequence \( \omega = (x_n) \) as follows.

Every positive integer \( n \) has a unique representation of the form \( n = r_0 + r_1 + \cdots + r_{j-1} + s \) with \( j \geq 1 \) and \( 0 < s \leq r_j \); we set \( x_n = x_{j,s} \). Take an integer \( N > r_1; \) \( N \) can be written in the form \( N = r_1 + \cdots + r_k + s \) with \( 0 < s \leq r_{k+1} \). For any function \( f \in L \) we get
\[
\sum_{n=1}^{N} f(x_n) = \sum_{j=1}^{k} \left( \sum_{\lambda=1}^{r_j} f(x_{j,\lambda}) \right) + \sum_{\lambda=1}^{s} f(x_{k+1,\lambda}).
\]

Therefore \( |\upsilon_N(f) - \mu(f)| = \left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \mu(f) \right| =
\]
\[
\left| \sum_{j=1}^{k} \frac{r_j}{N} \left( \frac{1}{r_j} \sum_{\lambda=1}^{r_j} f(x_{j,\lambda}) - \mu_j(f) \right) + \frac{s}{N} \left( \frac{1}{s} \sum_{\lambda=1}^{s} f(x_{k+1,\lambda}) - \mu_{k+1}(f) \right) \right| + \sum_{j=1}^{k} \frac{r_j}{N} \mu_j(f) + \frac{s}{N} \mu_{k+1}(f) - \mu(f) \right|
\]
(\text{using Proposition 1})
\[
\leq \sum_{j=1}^{k} \frac{r_j}{N} \left( \frac{2}{r_j} (1 + C_j) \right) + \frac{s}{N} \left( \frac{2}{s} (1 + C_{k+1}) + \frac{1}{N} \sum_{j=1}^{k} r_j \mu_j(f) + s \mu_{k+1}(f) \right) - \mu(f) \right|
\]
(by letting \( K(N, f) = \frac{1}{N} \sum_{j=1}^{k} r_j \mu_j(f) + s \mu_{k+1}(f) \) - \mu(f))
\[ \leq \frac{2}{r_k} \sum_{j=1}^{k+1} (1 + C_j) + |K(N, f)| = \frac{2}{r_k} (k + 1) + \frac{2}{r_k} \sum_{j=1}^{k+1} C_j + |K(N, f)| \leq \]

(since \( r_k \geq \max \{k^2, k(C_1 + \cdots + C_{k+1})\} \))

\[ \leq \frac{2(k + 1)}{k^2} + \frac{2}{k} + |K(N, f)|. \]

If \( N \to \infty \) then \( k \to \infty \) and the sum of the first two terms tends to zero. In order to prove that the third term tends to zero, we set for every \( N > r_1 \)

\[ A_N = \left( \frac{r_1}{N}, \frac{r_2}{N}, \ldots, \frac{r_k}{N}, \frac{s}{N}, 0, \ldots \right), \]

where \( N = r_1 + r_2 + \cdots + r_k + s, \; 0 < s \leq r_{k+1} \). Then \( A = (A_N) \) defines an infinite real matrix that is a regular method of summability. If we set \( H_f = (\mu_j(f))_{j \geq 1} \), where \( f \in L \), then we have

\[ A_N \cdot H_f = \frac{1}{N} \left[ \sum_{j=1}^{k} r_j \mu_j(f) + s \mu_{k+1}(f) \right], \quad N \geq 1. \]

Since \( \mu_j(f) \underset{j \to \infty}{\longrightarrow} \mu(f) \) for \( f \in L \) and \( A \) is a regular method of summability, we get that

\[ |A_N \cdot H_f - \mu(f)| = |K(N, f)| \underset{N \to \infty}{\longrightarrow} 0 \; \forall f \in L. \]

So we are done.

(2) We assume now that \((\mu_j)\) converges to \( \mu \) uniformly on \( L \). Since \( A \) is a regular method of summability, we get that

\[ A_N \cdot H_f \underset{N \to \infty}{\longrightarrow} \mu(f) \; \text{uniformly on} \; L. \]

Therefore given \( \varepsilon > 0 \), there is \( N_0 = N_0(\varepsilon) \) such that

\[ N \geq N_0 \Rightarrow |A_n \cdot H_f - \mu(f)| = |K(N, f)| \leq \frac{\varepsilon}{2} \; \forall f \in L \]

and of course \( 2 \left( \frac{k+1}{k^2} + \frac{1}{k} \right) \leq \frac{\varepsilon}{2} \), if \( N_0 \) is quite large. It then follows from the above that

\[ N \geq N_0 \Rightarrow \left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \mu(f) \right| \leq \varepsilon \; \forall f \in L, \]

which means that

\[ v_N = \frac{\delta x_1 + \cdots + \delta x_N}{N} \underset{N \to \infty}{\longrightarrow} \mu \]

uniformly on \( L \).
Remark 2. We notice that using inequality (5) of Remark 1 in the proof of Th. 1 instead of inequality (3) of Prop. 1, we can easily prove that Th. 1 is also valid assuming that \( L \) consists of complex functions. Therefore Th. 2, which we are going to prove, and everything depending on this theorem is also valid in the complex case.

The rest of this section is devoted to some applications of the previous results (Prop. 1 and Th. 1) in Banach space theory. We first prove the following

**Theorem 2.** Let \( E \) be a Banach space, \( F \) a bounded subset of \( E \) so that \( F \subseteq B(0, R) \) and \( x \in E \). Then we have:

1. Assume that \( x \in \text{co}(F) \) and let \( F_0 \subseteq F \) be any finite set so that \( x \in \text{co}(F_0) \). Then there is a sequence \( (x_n) \subseteq F_0 \) and a positive constant \( C = C(|F_0|) \) such that

\[
\left\| \frac{1}{N} \sum_{k=1}^{N} x_k - x \right\| \leq \frac{2R}{N} \left( 1 + C \right) \text{ for all } N \geq 1;
\]

in particular

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k \|\| = x.
\]

2. Assume that \( x \in \overline{\text{co}}(F) \). Then there is a sequence \( (x_n) \subseteq F \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k \|\| = x.
\]

**Proof.** Assume without loss of generality that \( R = 1 \), that is \( F \subseteq B_E \); otherwise we replace \( F \) by \( \frac{R}{2} F \) and \( x \) by \( \frac{R}{2} x \). We set \( X = B_E \) and notice that each \( f \in E^* \) can be identified with a bounded (continuous) function on \( X \) through the isometry operator \( T : f \in E^* \mapsto T(f) = f/X \in \ell_\infty(X) \).

(1) Let \( x \in \text{co}(F) \), then \( x = \alpha_1 y_1 + \cdots + \alpha_m y_m \), where \( y_1, \ldots, y_m \) are distinct points of \( F \), \( \alpha_k > 0 \) for \( k = 1, 2, \ldots, m \) and \( \sum_{k=1}^{m} \alpha_k = 1 \). We can consider \( x \) as a finitely supported measure \( \mu \) on \( X \), by letting \( \mu = \sum_{k=1}^{m} \alpha_k \delta y_k \); clearly \( \text{supp} \mu = \{y_1, \ldots, y_m\} \).

Then \( \mu \) represents \( x \), that is, for every \( f \in B_{E^*} \)

\[
\int_X f d\mu = \sum_{k=1}^{m} \alpha_k f(y_k) = f \left( \sum_{k=1}^{m} \alpha_k y_k \right) = f(x).
\]

It then follows from Prop. 1 (see also Remark 1) that there is \( (x_n) \subseteq F_0 \), where \( F_0 = \text{supp} (\mu) \), such that for every \( f \in \ell_\infty(X) \) with \( \|f\| \leq 1 \) we have

\[
\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \int_X f d\mu \right| \leq \frac{2}{N} \left( 1 + C(\mu) \right) \text{ for all } N \geq 1.
\]
In particular, if \( f \in E^* \) with \( \|f\| \leq 1 \), then we have

\[
\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \int_X f d\mu \right| = \left| f \left( \frac{1}{N} \sum_{k=1}^{N} x_k - x \right) \right| \leq \frac{2}{N}(1 + C(\mu)) \text{ for all } N \geq 1,
\]

which implies that for all \( N \geq 1 \),

\[
\left| \frac{1}{N} \sum_{k=1}^{N} x_k - x \right| = \sup \left\{ \left| f \left( \frac{1}{N} \sum_{k=1}^{N} x_k - x \right) \right| : f \in B_{E^*} \right\} \leq \frac{2}{N}(1 + C(\mu)).
\]

We set \( C = C(\mu) \), hence \( C \) depends on \( m = |F_0| \) and obtain the desired result.

(2) Let \( x \in co(F) \); then there is a sequence \((\mu_j)\) of convex combinations of elements of \( F \) such that \( \mu_j \to x \) uniformly on \( L \).

We consider each \( \mu_j \) as a finitely supported probability measure on \( F \subseteq X = B_{E^*} \). So if we set \( L = T(B_{E^*}) \subseteq B_{\ell_\infty}(X) \), then (6) means that \( \mu_j \to x \) uniformly on \( L \). It then follows from Theorem 1(2) that there is a sequence \((x_n) \subseteq \bigcup_{j=1}^{\infty} \text{supp}\mu_j \subseteq F \) such that the sequence of arithmetic means

\[
\frac{\delta x_1 + \cdots + \delta x_N}{N} \to x \text{ uniformly on } L,
\]
equivalently \( \frac{x_1 + \cdots + x_N}{N} \to x \). The proof of the theorem is complete.

Assertion (1) of the above theorem has a strong relationship with a Lemma due to Maurey (see Lemma D of [5]) which states that

**Lemma 2.** Let \( E \) be a Banach space of type \( p \) for some \( p > 1 \), \( F \subseteq E \) and \( x \in co(F) \). Set \( q = \frac{p}{p-1} \). Then for every \( N \in \mathbb{N} \) there exist \( x_1, \ldots, x_N \in F \) such that

\[
\left\| \frac{1}{N} \sum_{k=1}^{N} x_k - x \right\| \leq \text{diam}(F) \frac{T_p(E)}{N^{\frac{1}{q}}},
\]

\((T_p(E) \text{ is the type } p \text{ constant of } E, \text{ see pp. 137-8 of [1]}).\)

In our case the constant \( C \) depends on the cardinality of the finite subset of \( F \) that supports \( x \); in Maurey’s Lemma it depends only on the space \( E \), which must be of type \( p \). If we assume, as we may, that \( F \subseteq B_E \) (thus \( \text{diam}(F) \leq 2 \)) and \( N \) is quite large, then clearly \( \frac{1+C}{N} \leq \frac{T_p(E)}{N^{\frac{1}{q}}} \), hence we get Maurey’s Lemma. In some way assertion (1) of Th.2 is the "pointwise" version of Maurey’s Lemma.

From assertion (2) of Th.2 together with Mazur’s classical result, stating that the weak and the norm closure of any convex subset of a Banach space coincide, we obtain the following interesting consequence.
Proposition 2. Let $E$ be a Banach space and $(y_n) \subseteq E$ be any weakly convergent sequence, so that $y_n \xrightarrow{w} y$. Then there is a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that the sequence $x_n = y_{\varphi(n)}$, $n \geq 1$ is Cesàro summable to $y$, that is \[
abla_{N \to \infty} \left( \frac{y_1 + \ldots + y_N}{N} \right) \to y.\]

Proof. The set $F = \{ y_n : n \geq 1 \}$ is bounded. Since $y \in F^w$, we get that $y \in \text{co}^w(F)$. By Mazur’s Theorem we have that $\text{co}^w(F) = \text{co}(F)$. So $y \in \text{co}(F)$ and then assertion (2) of Th.2 can be applied.

Remark 3. It is well known that the function $\varphi$ of Prop.2 cannot in general be chosen strictly increasing (neither 1-1). In fact, it is possible to find a weakly null sequence $(y_n)$, such that for every subsequence $(y'_n)$ of $(y_n)$ the sequence of arithmetic means $\frac{y'_1 + \ldots + y'_N}{N}$, $N \geq 1$ is not norm convergent. The first such example was constructed by J. Schreier (see [17] and [3]). We note in this connection that in [3] is given a complete classification of the complexity of weakly null sequences, by the aid of a hierarchy of summability methods introduced there.

Still another immediate but useful consequence of assertion (2) of Th.2 is the following

Proposition 3. Let $E$ be a Banach space and $\Omega$ be a closed convex bounded subset of $E$, so that $\Omega$ is equal to the closed convex hull of its extreme points $ex\Omega$, that is $\Omega = \text{co}(ex\Omega)$. Then for every $x \in \Omega$, there is a sequence $(x_n) \subseteq ex\Omega$ which is Cesàro summable to $x$.

We now present some applications of Prop.3.

Corollary 1. Let $\Omega$ be a weakly compact and convex subset of a Banach space $E$ (in particular $\Omega = B_E$ and $E$ is reflexive). Then for every $x \in \Omega$ there is a sequence $(x_n)$ of extreme points of $\Omega$ which is Cesàro summable to $x$.

Proof. Since $\Omega$ is weakly compact and convex, by Krein-Milman’s Theorem we have that $\Omega = \text{co}(ex\Omega)$. Hence the result is an immediate consequence of Prop.3.

The next result concerns Banach spaces of the form $C(K)$, where $K$ is a compact Hausdorff space. We recall that $K$ is called totally disconnected, if it has a base for its topology consisting of open and closed (clopen) sets.

Corollary 2. Let $K$ be a compact Hausdorff space. We assume that either (a) $C(K)$ is the space of continuous complex functions on $K$, or (b) $K$ is totally disconnected and $C(K)$ is the space of continuous real functions on $K$. Then for every $f \in B = B_{C(K)}$, there exists a sequence $(f_n)$ of extreme points of $B$ such that
\[
\left( \frac{f_1 + \ldots + f_N}{N} \right) \xrightarrow{N \to \infty} f.
\]
Proof. In either case we have that $B = \overline{\text{co}}(\text{ex}B)$ (see Ths. 1.6 and 1.8 of [3]). Hence Prop. 3 can be applied.

Remark 4. (a) Recall that if $K$ is a compact Hausdorff space, then $f \in \text{ex}B$ iff $|f(x)| = 1$, for all $x \in K$ (see Th. 1.3 of [3]). If $K$ is in addition totally disconnected, $C(K)$ is the space of real continuous functions on $K$ and $f \in \text{ex}B$, then the sets $V = \{x \in K : f(x) = 1\}$ and $V^c = \{x \in K : f(x) = -1\}$ constitute a partition of $K$ in two clopen sets. Thus the extreme points of $B$ are completely determined by the clopen nonempty subsets of $K$.

(b) We have already used the above remark in the special case of the Banach space $\ell_\infty(X)$ (cf. the proof of Prop. 1). Indeed $\ell_\infty(X)$ is isometric to $C(\beta X)$, where $\beta X$ is the Stone-Čech compactification of the discrete set $X$, which is a compact extremally disconnected space.

We continue our investigation, applying Th. 1 to the weak* topology of the dual $E^*$ of a separable Banach space $E$. As we shall see, results similar to Th. 2 (2) and Prop. 3 are valid. Moreover, our approach has interesting applications for the dual $M(K) = C(K)^*$ of $C(K)$, where $K$ is any compact metric space.

Proposition 4. Let $E$ be a separable Banach space, $F$ a bounded subset of its dual $E^*$ and $f \in \overline{\text{co}}^*(F)$. Then there is a sequence $(f_n) \subseteq F$ such that

$$\frac{f_1 + \cdots + f_N}{N} \overset{w^*}{\longrightarrow} f.$$ \(\square\)

Proof. The proof is similar to the proof of Th. 2 (2). We set $X = B_{E^*}$ and assume without loss of generality that $F \subseteq X$. Note that, since $E$ is separable, $X$ is weak* compact and metrizable and also that each $x \in E$ can be identified with a continuous function on $X$ through the linear isometry 

$$T : x \in E \mapsto T(x) = x/X \in C(X) \subseteq \ell_\infty(X).$$

Since the weak* closed convex hull $\overline{\text{co}}^*(F) \subseteq X$ is a weak* compact and metrizable set, given any $f \in \overline{\text{co}}^*(F)$ there is a sequence $(\mu_j)$ of convex combinations of elements of $F$ such that

$$\mu_j \overset{w^*}{\longrightarrow} f \quad (7).$$

We consider each $\mu_j$ as a finitely supported probability measure on $F \subseteq X$. So if we set $L = T(B_E) \subseteq B_{\ell_\infty(X)}$ then (7) means that $\mu_j \rightarrow f$ pointwise on $L$.

It then follows from Th. 1 (1) that there is a sequence $(f_n) \subseteq \bigcup_{j=1}^\infty \text{supp}\mu_j \subseteq F$ such that

$$\frac{\delta f_1 + \cdots + \delta f_N}{N} \overset{N \rightarrow \infty}{\longrightarrow} f \text{ pointwise on } L;$$

equivalently $$\frac{f_1 + \cdots + f_N}{N} \overset{N \rightarrow \infty}{\longrightarrow} f. \quad \square$$
**Proposition 5.** Let $E$ be a separable Banach space and $\Omega$ be a weak* compact and convex subset of $E^*$. Then for every $f \in \Omega$ there is a sequence $(f_n) \subseteq \text{ex} \Omega$ such that

$$\frac{f_1 + \cdots + f_N}{N} \xrightarrow{w^*} f_{\infty}.$$  

**Proof.** It follows immediately from Krein-Milman’s theorem and Prop.4. \qed

Since the dual unit ball $B_{E^*}$ of any Banach space $E$ is a weak* compact and convex set, we immediately obtain the following

**Corollary 3.** Let $E$ be a separable Banach space. Then for every $f \in B_{E^*}$, there is a sequence $(f_n)$ of extreme points of $B_{E^*}$ such that

$$\frac{f_1 + \cdots + f_N}{N} \xrightarrow{w^*} f_{\infty}.$$ 

An immediate consequence of Prop.5 is the following well known result.

**Theorem 3.** Let $K$ be any compact metric space. Then every Borel probability measure $\mu$ on $K$ (i.e. $\mu \in P(K)$) admits a u.d. sequence.

**Proof.** Since $K$ is compact Hausdorff, we have that $P(K)$ is a weak* compact and convex subset of $C(K)^* = M(K)$ with $exP(K) = \{\delta_x : x \in K\}$, thus $P(K) = \overline{w^*}\{\delta_x : x \in K\}$. $K$ is a metrizable space, hence $C(K)$ is separable and so Prop.5 can be applied. \qed

Applying corollary 3 to $C(K)^*$, where $K$ is a compact metric space, gives the following result that seems to be new and will be our motivation for the next section.

**Theorem 4.** Let $K$ be any compact metric space. Then for every $\mu \in B = B_{C(K)}$, there are sequences $(x_n) \subseteq K$ and $(\varepsilon_n) \subseteq \{\pm 1\}$ such that

$$\frac{\varepsilon_1\delta_{x_1} + \cdots + \varepsilon_N\delta_{x_N}}{N} \xrightarrow{w^*} \mu_{\infty}.$$  

(In case when $C(K)$ is the space of continuous complex functions, $(\varepsilon_n) \subseteq T = \{z \in \mathbb{C} : |z| = 1\}$).

**Proof.** We first assume that $C(K)$ is the space of continuous real functions. We then have $exB = \{\pm \delta_x : x \in K\}$ and hence $B = \overline{w^*}\{\pm \delta_x : x \in K\}$. In the complex case we have that $exB = \{\alpha \delta_x : x \in K$ and $\alpha \in \mathbb{C}, |\alpha| = 1\}$ and hence $B = \overline{w^*}\{\alpha \delta_x : x \in K \text{ and } \alpha \in \mathbb{C}, |\alpha| = 1\}\}$ (see Th.1.9 of [4]). So the result follows immediately from Corollary 3. \qed

**Remark 5.** (a) Theorem 3 is of course a direct consequence of Niederreiter’s main result (see Th.2 of [19]). We state Theorem 3 here, because it shows that Prop.5 can be considered as a generalization of such a well known result to the much wider class of separable Banach spaces.

(b) A compact Hausdorff space $K$ is said to be angelic iff for every $A \subseteq K$ and each $x \in \overline{A}$ there is a sequence $(x_n) \subseteq A$ such that $x_n \to x$. It is clear that, if the
dual unit ball \((B_{E^*}, w^*)\) of a Banach space \(E\) is an angelic space, then Prop.4 and its consequences (Prop.5, Cor.3 and Ths 3,4) remain valid. Well known classes of (not necessarily separable) Banach spaces with angelic dual balls are weakly compactly generated (WCG) and their generalizations, like weakly countably determined (WCD) Banach spaces, etc. (see [3], [10] and [14]).

We also note that if \(K\) is any compact Hausdorff space so that the convex hull \(\text{co}\{\delta_x : x \in K\}\) is weak* sequentially dense in \(P(K)\), then it is easy to see that every \(\mu \in M(K)\) with \(\|\mu\| = 1\) satisfies the conclusion of Theorem 4 (in particular, by a result of Niederreiter mentioned in (a), every \(\mu \in P(K)\) admits a u.d. sequence).

Finally, assuming Martin’s axiom plus the negation of Continuum Hypothesis (MA+¬CH), Theorem 4 remains valid for every compact separable space \(K\) of topological weight \(w(K) < c\), where \(c\) is the cardinality of the continuum (the proof is essentially the same as the proof of Prop. 2.21 of [18]).

(c) Let \(E\) be a Banach space not containing an isomorphic copy of \(\ell_1\). Then by a result of Haydon, every weak* compact and convex subset \(\Omega\) of \(E^*\) is the norm closed convex hull of its extreme points (see [13]). It then follows from this result and the aforementioned considerations that for every \(f \in \Omega\) there exists a sequence \((f_n) \subseteq \text{ex}\Omega\) norm Cesàro summable to \(f\). It follows in particular that if \(E\) is of the form \(C(K)\), where \(K\) is compact and Hausdorff (since \(\ell_1 \not\subseteq C(K)\), every \(\mu \in P(K)\) is purely atomic), then for every \(\mu \in P(K)\) there is a \(\mu\)-u.d. sequence \((x_n)\) in \(K\) with the stronger property

\[
\frac{\delta_{x_1} + \cdots + \delta_{x_N}}{N} \xrightarrow{\|\cdot\|_\infty} \mu.
\]

Note that the last result can also be proved by a direct method. We also note that a class of (not necessarily separable) Banach spaces not containing \(\ell_1\) is that of Asplund spaces; a Banach space \(E\) is called Asplund, if every separable subspace of \(E\) has separable dual (see [10]).

2.

In this section we shall concentrate on duals \(C(K)^* = M(K)\) of Banach spaces of the form \(C(K)\) with \(K\) compact (metrizable) space and shall further investigate the effect of the results of the previous section to the uniform distribution of sequences in \(K\).

Theorem 4 inspires the following generalization of the classical concept of uniformly distributed sequences defined for regular Borel probability measures on compact spaces (see Def.1.1 of [13]).

**Definition 1.** Let \(K\) be a compact Hausdorff space and \(\mu \in M(K)\) with total variation \(|\mu|(K) = 1\) (i.e. \(\mu\) is a regular Borel signed measure with \(\|\mu\| = 1\)). We say that a sequence \((x_n) \subseteq K\) is \(\mu\)-u.d. if both of the following conditions are satisfied

1. \((x_n)\) is \(|\mu|-\text{u.d. (in the classical sense)}\) and
2. there is a sequence of signs \((\varepsilon_n) \subseteq \{\pm 1\}\) such that

\[
\frac{\varepsilon_1 \delta_{x_1} + \cdots + \varepsilon_N \delta_{x_N}}{N} \xrightarrow{\|\cdot\|_\infty} \mu
\]
(that is, \( \lim_{N \to \infty} \frac{\varepsilon_1 f(x_1) + \cdots + \varepsilon_N f(x_N)}{N} = \int_K f \, d\mu \), for all \( f \in C(K) \)).

We note that:
(a) By applying the above equality for the constant function \( f = 1 \), we get that \( \mu(K) = \int_K d\mu = \lim_{N \to \infty} \varepsilon_1 + \cdots + \varepsilon_N \)
and
(b) if \( \mu \in P(K) \) admits a u.d. sequence \((x_n)\), then we can take \( \varepsilon_n = 1 \) for all \( n \geq 1 \).

Let \( \mu \in M(K) \) with \( \|\mu\| = 1 \) (\( = |\mu|(K) \)). It then follows from Radon-Nikodym theorem that there is a Borel function \( h : K \to \mathbb{R} \) with \( |h(x)| = 1 \), for all \( x \in K \) such that
\[ d\mu = hd|\mu| \iff h = \frac{d\mu}{d|\mu|}. \]

The function \( h \) is the so-called Radon-Nikodym derivative of \( \mu \) with respect to its total variation \( |\mu| \). So we have that
\[ \int_K f \, d\mu = \int_K fhd|\mu|, \]
for all bounded Borel measurable functions \( f : K \to \mathbb{R} \).

With the above notation we have the following

**Proposition 6.** Assume that \( |\mu| \) admits a u.d. sequence \((x_n)\) (in the classical sense) and also that the function \( h \) is \(|\mu|-\text{Riemann integrable} \). Then the sequence \((x_n)\) is \( \mu \)-u.d. (in the sense of Def.1).

**Proof.** Let \( f \in C(K) \); then the function \( fh \) is \(|\mu|-\text{Riemann integrable} \) and since \((x_n)\) is \(|\mu|-\text{u.d.} \), we get that
\[ \int_K f \, d\mu = \int_K fhd|\mu| = \lim_{N \to \infty} \frac{(fh)(x_1) + \cdots + (fh)(x_N)}{N}. \]
So the desired sequence of signs is the sequence \( \varepsilon_n = h(x_n), \ n \geq 1 \). \( \square \)

**Remark 6.** Later in this section, we shall present a class of measures \( \mu \in P(K) \), where \( K = [a, b] \) is a compact interval of the real line, so that the function \( h = \frac{d\mu}{d|\mu|} \) is \(|\mu|-\text{Riemann integrable} \).

It follows in particular from Prop.6 and standard results (cf. Th.2.2 of \cite{15}) that, whenever \( K \) is compact metric, \( \mu \in M(K) \) with \( \|\mu\| = 1 \) so that the function \( h = \frac{d\mu}{d|\mu|} \) is \(|\mu|-\text{Riemann integrable} \), then \( \mu \) admits a u.d. sequence in the sense of Def.1. In the sequel we are going to show, essentially by the method of proof of Th.2.2 of \cite{15}, that every measure \( \mu \in M(K) \) with \( \|\mu\| = 1 \) admits a u.d. sequence.

We first cite some preliminaries. Let \( K \) be a compact Hausdorff space; we denote by \( K^\infty \) the cartesian product of countably many copies of \( K \). Then \( K^\infty \) is a compact Hausdorff space endowed with the product topology. If \( \mu \in P(K) \), then \( \mu \) induces the product measure \( \mu^\infty \) in \( K^\infty \), which we may assume to be complete. We also denote by \( B(K) \) the Banach space of bounded Borel functions on \( K \) endowed with supremum norm.
Theorem 5. Let $K$ be a compact metric space and $\mu \in M(K)$ with $\|\mu\| = 1$; also let $S$ be the set of all sequences in $K$, which are $\mu$-u.d. considered as a subset of $K^\infty$. Then $\mu^\infty(S) = 1$.

Proof. We consider a countable total subset $L = \{f_n : n \geq 1\}$ of $C(K)$ with $f_1 \equiv 1$. Also let $h = \frac{d\mu}{d|\mu|}$ (=the Radon-Nikodym derivative of $\mu$ with respect to $|\mu|$). Set $M = L \cup hL = \{f_n : n \geq 1\} \cup \{f_nh : n \geq 1\}$. As the members of the set $M$ are bounded Borel functions and $|\mu| \in P(K)$, for each $g \in M$ there is a $|\mu|^\infty$-measurable subset $B_g$ of $K^\infty$ with $|\mu|^\infty(B_g) = 1$ such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} g(x_k) = \int_{K} g \, |\mu| \quad \forall (x_1, \ldots, x_k, \ldots) \in B_g \quad (8)$$

(see Lemma 2.1, p.182 of [15]).

Set $B = \cap_{g \in M} B_g$; as the set $M$ is countable, we get that $|\mu|^\infty(B) = 1$. Let $(x_1, \ldots, x_k, \ldots) \in B$. Since the set $L$ is total in $C(K)$, we get that this sequence is $|\mu|$-u.d. in $K$, that is (8) is valid for every $f \in C(K)$. Note that the operator $T : f \in C(K) \mapsto hf \in B(K)$ is a linear isometry, thus the set $hL$ is total in the closed subspace $T(C(K))$ of the Banach space $B(K)$. So we get that equation (8) also holds for each member of the space $T(C(K))$, that is

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k f(x_k) = \int_{K} fhd|\mu| = \int_{K} fd\mu \quad \forall f \in C(K) \quad (9)$$

where $\varepsilon_k = h(x_k), k = 1, 2, \ldots$. So we are done. \hfill \square

Remark 7. Note that equalities (1) and (2) of Definition 1 (for $f \in C(K)$) are equivalent to the following

$$\lim_{N \to \infty} \frac{(1 + \varepsilon_1)f(x_1) + \cdots + (1 + \varepsilon_N)f(x_N)}{2N} = \int_{K} f \, d\mu^+ \quad (10)$$

and

$$\lim_{N \to \infty} \frac{(1 - \varepsilon_1)f(x_1) + \cdots + (1 - \varepsilon_N)f(x_N)}{2N} = \int_{K} f \, d\mu^- \quad (11),$$

where $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$ are the positive and negative variations of the measure $\mu$.

Indeed, assuming that (1) and (2) of Def.1 are valid, for $f \in C(K)$ we have

$$\int_{K} f \, d\mu^+ = \frac{1}{2} \left[ \int_{K} f \, d(|\mu| + \mu) \right] = \frac{1}{2} \left[ \int_{K} f \, d|\mu| + \int_{K} f \, d\mu \right] = \lim_{N \to \infty} \frac{(1 + \varepsilon_1)f(x_1) + \cdots + (1 + \varepsilon_N)f(x_N)}{2N}$$

so (10) holds. In a similar way we get equality (11).

In the converse direction, by adding and subtracting (10) and (11) we get (1) and (2) of Def.1 respectively.
Concerning Remark 7, it should be noticed that equalities (10) and (11) (and hence (1) and (2) of Def.1) are also valid for every $\mu$-Riemann integrable function (i.e. a bounded function $f : K \to \mathbb{R}$ that is $|\mu|$-Riemann integrable). In order to prove this, we shall use the following well known facts:

**Fact I** Let $\nu \in M^+(K)$ (with $\nu(K) > 0$), then a bounded function $f : K \to \mathbb{R}$ is $\nu$-Riemann integrable iff for every $\varepsilon > 0$ there are $f_1, f_2 \in C(K)$ such that

$$f_1 \leq f \leq f_2 \text{ and } \int_K (f_2 - f_1) d\nu \leq \varepsilon.$$  

(see p.90 of [13])

**Fact II** Let $\nu \in P(K)$ and $(x_n) \subseteq K$ be a $\nu$-u.d. sequence. Then for every $\nu$-Riemann integrable function $f : K \to \mathbb{R}$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_K f d\nu.$$  

The proofs of Facts I and II are essentially contained in the proofs of Th.1, ch.1 and Th.2, ch.3 (see also ex.1.12, p.179) of [15].

Let us prove, for instance, equality (10). So let $f : K \to \mathbb{R}$ be any $\mu$-Riemann integrable function and $\varepsilon > 0$. Then by Fact I there are $f_1 \leq f \leq f_2$ continuous functions, such that

$$0 \leq \int_K (f_2 - f_1) d|\mu| \leq \varepsilon.$$ 

Since $0 \leq \frac{1 + h}{2} \leq 1 \left(h = \frac{d\mu}{d|\mu|}\right)$ we get that

$$F_1 = f_1 \left(\frac{1 + h}{2}\right) \leq F = f \left(\frac{1 + h}{2}\right) \leq F_2 = f_2 \left(\frac{1 + h}{2}\right) \text{ and}$$

$$0 \leq \int_K (F_2 - F_1) d|\mu| = \int_K (f_2 - f_1) \left(\frac{1 + h}{2}\right) d|\mu| \leq \int_K (f_2 - f_1) d|\mu| \leq \varepsilon.$$ 

Set $I = \int_K F d|\mu| \left(= \int_K f \left(\frac{1 + h}{2}\right) d|\mu| = \int_K f d\mu^+\right)$. Then we have that

$$I - \varepsilon = \int_K F d|\mu| - \varepsilon \leq \int_K F_1 d|\mu| = \int_K f_1 d\mu^+ = \lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} f_1(x_k) \leq \liminf_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} f(x_k) \leq$$

$$\limsup_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} (1 + \varepsilon_k) f(x_k) \leq \lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} (1 + \varepsilon_k) f_2(x_k) =$$

$$\int_K f_2 d\mu^+ = \int_K F_2 d|\mu| \leq \int_K F d|\mu| + \varepsilon = I + \varepsilon.$$ 

Since $\varepsilon$ is arbitrarily small, we get (10).

Taking into account the above remarks and Th.5, we obtain the following result.
Theorem 6. Let $K$ be a compact metric space and $\mu \in M(K)$ with $\|\mu\| = 1$. Then there is a sequence $(x_n) \subseteq K$ and a sequence of signs $(\varepsilon_n) \subseteq \{\pm 1\}$ (where $\varepsilon_n = h(x_n)$ and $h = \frac{du}{d|\mu|}$) such that, for every $\mu$-Riemann integrable function $f : K \to \mathbb{R}$ we have

(a) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_K f \, d|\mu|$; in particular $(x_n)$ is $|\mu|$-u.d.

(b) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k f(x_k) = \int_K f \, d\mu$,

(c) $\lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} (1 + \varepsilon_k) f(x_k) = \int_K f \, d\mu^+$ and

(d) $\lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} (1 - \varepsilon_k) f(x_k) = \int_K f \, d\mu^-$.

In the sequel, we focus on the special case of Theorem 6 when $K$ is a compact interval of the real line, say $K = [a, b]$. Let $\varphi : [a, b] \to \mathbb{R}$ be a function of bounded variation. We denote by $v, p$ and $n$ the (increasing) functions of total, positive and negative variation of $\varphi$ $(v(x) = V_a^x \varphi, x \in [a, b])$.

We note that these functions are connected as follows:

$$p(x) = \frac{1}{2}(v(x) + \varphi(x) - \varphi(a))$$

and

$$n(x) = \frac{1}{2}(v(x) - \varphi(x) + \varphi(a)),$$

(see p.208 of [7]).

We recall that the space $M(K)$ of signed Borel measures on $K$ is in one-to-one correspondence with the space of functions of bounded variation on $K$ which are right continuous on $(a, b)$ with $\varphi(a) = 0$, in the sense that each $\mu \in M(K)$ is uniquely defined by such a $\varphi$ by the rule

$$\mu((y, x]) = \varphi(x) - \varphi(y), \text{ for } a \leq y < x \leq b$$

(see Th. 3.29 of [9] and Th. 14.26 of [7]).

With the above notation and terminology, Theorem 6 yields the following

Theorem 7. Let $\varphi : [a, b] \to \mathbb{R}$ be a right continuous function (of bounded variation) with total variation $v(b) = V_a^b \varphi = 1$ and $\varphi(a) = 0$. Then there are sequences $(x_n) \subseteq [a, b]$ and $(\varepsilon_n) \subseteq \{\pm 1\}$, such that for every point of continuity $x \in [a, b]$ of $\varphi$ we have

(a) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{[a,x]}(x_k) = v(x)$

(b) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k \chi_{[a,x]}(x_k) = \varphi(x)$,

(c) $\lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} (1 + \varepsilon_k) \chi_{[a,x]}(x_k) = p(x)$ and

(d) $\lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^{N} (1 - \varepsilon_k) \chi_{[a,x]}(x_k) = n(x).$
Proof. Let \( \mu = \mu_\varphi \) be the signed (Lebesgue-Stieljes) measure defined by \( \varphi \) on \([a, b]\) by the rule \( \mu((a, x]) = \varphi(x)(= \mu([a, x])) \) for \( x \in (a, b] \). As is well known, the Jordan decomposition and the total variation of \( \mu \) are given by

\[
\mu = \mu^+ - \mu^- , \quad |\mu| = \mu^+ + \mu^- ,
\]

where \( |\mu| = \mu_\varphi \), \( \mu^+ = \mu_p \) and \( \mu^- = \mu_n \), thus in particular \( ||\mu|| = \nu(b) = 1 \) (see Th. 3.29 and exs. 28, 29, p.107 of [9]).

Let \( D_\varphi \) be the (countable) set of discontinuity points of \( \varphi \); then every interval \( I \subseteq [a, b] \) whose both endpoints do not belong to \( D_\varphi \) has characteristic function which is \( \mu \)-Riemann integrable. Then by applying Theorem 6 to intervals of the form \((a, x)\), with \( x \notin D_\varphi \) we get the conclusion (of course \( \varepsilon_n = h(x_n) , n \geq 1 \) where \( h = \frac{d\mu}{d|\mu|} \)).

The last theorem partially generalizes an important result from [15] (Th. 4.3, p.138) stating that:

**Theorem 8.** Let \( \varphi : I = [0, 1] \to \mathbb{R} \) be an increasing function, with \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \). Then there is a sequence \((x_n) \subseteq I\), such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{[0, x)}(x_k) = \varphi(x) , \quad \text{for } 0 \leq x \leq 1.
\]

We then say that \((x_n)\) has \( \varphi \) as the asymptotic distribution function mod 1 (abbreviated a.d.f.(mod 1) \( \varphi(x) \)).

The proof of this result is given in two steps. First, the continuous case is proved (Lemma 4.2, p.137 of [15]) and then the general case follows, using a result from real analysis (Lemma 4.3, p.138 of [15]). We state both of these results for the reader’s convenience.

**Lemma 3.** Let \( \varphi : I \to \mathbb{R} \) be a continuous increasing function, with \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \). Then there is a sequence \((x_n) \subseteq I\), such that

\[
\left| \frac{1}{N} \sum_{k=1}^{N} \chi_{[0, x]}(x_k) - \varphi(x) \right| \leq \frac{\log(N + 1)}{N \log 2}
\]

for all \( N \geq 1 \) and \( 0 \leq x \leq 1 \).

Before we state the real analysis Lemma, we recall that a continuous function \( \varphi : [a, b] \to \mathbb{R} \), \((a, b \in \mathbb{R}, a < b)\) is said to be polygonal (or piecewise linear), if its graph consists of finitely many straight line segments.

**Lemma 4.** Let \( \varphi : [a, b] \to \mathbb{R} \) be an increasing function. Then there is a sequence \((\varphi_k)\) of polygonal increasing functions defined on \([a, b]\), satisfying \( \varphi_k(a) = \varphi(a) \) and \( \varphi_k(b) = \varphi(b) \) for \( k \geq 1 \), which converges pointwise to \( \varphi \), that is, \( \lim_{k \to \infty} \varphi_k(x) = \varphi(x) \), for all \( x \in [a, b] \).
Our aim is to give a full generalization of both Theorems 7 and 8, in the sense that assertions (a) to (d) of Theorem 7 are valid for every function $\varphi : [a, b] \to \mathbb{R}$ of bounded variation (with $\varphi(a) = 0$ and $V_a^b\varphi = 1$) and for each point $x \in [a, b]$. We start by generalizing Lemma 4.

**Lemma 5.** Let $\varphi : [a, b] \to \mathbb{R}$ be a function of bounded variation (with $V_a^b\varphi > 0$). Also, let $\nu, p$ and $n$ be the functions of total, positive and negative variation of $\varphi$. Then there are sequences of increasing polygonal functions $(g_n)$ and $(h_n)$ defined on $[a, b]$, such that if we let $\varphi_k = g_k - h_k$, for $k \geq 1$, then we have that $(\varphi_k)$ is polygonal and

(a) $g_k \to p$, $h_k \to n$ and (thus) $\varphi_k \to \varphi$ pointwise on $[a, b]$; moreover $g_k(a) = p(a)$, $g_k(b) = p(b)$ and $n_k(a) = n(a)$, $n_k(b) = n(b)$, for $k \geq 1$.

(b) If $\nu_k$ denotes the function of total variation of $\varphi_k$, then $\nu_k$ is polygonal and $\nu_k \to \nu$ pointwise on $[a, b]$.

**Proof.** For each $k \geq 1$ we choose a partition $\mathcal{P}_k = \{t_0^k = a < t_1^k < \cdots < t_m^k = b\}$ of $[a, b]$ with $t_i^k - t_{i-1}^k < \frac{1}{k}$, for $0 \leq i < m_k$, that contains all points $x \in (a, b)$ with $v(x + 0) - v(x - 0) > \frac{1}{k}$ (since $v$ is increasing, there can only be finitely many such $x$). We define the functions $g_k$ and $h_k$ as follows: Set $g_k(t_i^k) = p(t_i^k)$ and $h_k(t_i^k) = n(t_i^k)$, for $0 \leq i \leq m_k$ and then extend $g_k$ and $h_k$ on $[a, b]$ so as to be linear on the intervals $[t_i^k, t_{i+1}^k]$, $0 \leq i < m_k$. Then clearly $g_k$ and $h_k$ are polygonal and increasing on $[a, b]$ and (hence) $\varphi_k$ is polygonal on $[a, b]$.

(a) We shall prove that $(g_k)$ converges pointwise to $p$ (the proof for $(h_k)$ is analogous). This is trivial for the endpoints $a$ and $b$. Let $x \in (a, b)$; assume first that $x$ is a discontinuity point of $v$. Then $v(x + 0) - v(x - 0) > 0$ and so, from some $k_0$ on, we will have that $x = t_i^k$, with $0 < i < m_k$. Therefore $g_k(x) = p(x)$ for sufficiently large $k$.

Now let $v$ be continuous at $x$, then $\varphi$, $p$, $n$ are also continuous at $x$. So let $\varepsilon > 0$ be given. Then, for all sufficiently large $k$ (say $k \geq k_0$) we will have

$$y \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right) \Rightarrow p(y) \in (p(x) - \varepsilon, p(x) + \varepsilon).$$

Yet for each $k$ we have $t_i^k \leq x \leq t_{i+1}^k$, for some $i = i(k)$ with $0 \leq i < m_k$. Since $0 < t_{i+1}^k - t_i^k < \frac{1}{k}$, both $t_i^k$, $t_{i+1}^k$ lie in $(x - \frac{1}{k}, x + \frac{1}{k})$. Hence, for $k \geq k_0$ we obtain

$$g_k(t_i^k) = p(t_i^k) = p(x) - \varepsilon \quad \text{and} \quad g_k(t_{i+1}^k) = p(t_{i+1}^k) < p(x) + \varepsilon;$$

since $g_k$ is increasing, we get that

$$g_k(t_i^k) \leq g_k(x) \leq g_k(t_{i+1}^k)$$

and so $p(x) - \varepsilon < g_k(x) < p(x) + \varepsilon$ for $k \geq k_0$, which shows that $g_k(x) \to p(x)$.

(b) Let $x \in (a, b)$ (clearly $v(a) = v_k(a) = 0$ for $k \geq 1$); since $\varphi_k$ is a polygonal and hence piecewise $C^1$ function, we get that

$$v_k(x) = \int_a^x |\varphi_k'(t)| \, dt = \sum_{\lambda=0}^{i(k)-1} |\varphi_k(t_{\lambda+1}^k) - \varphi_k(t_\lambda^k)| + |\varphi_k(t_{i(k)}^k) - \varphi_k(t_{i(k)}^k) - \varphi_k(x)| =$$
(where \( x \in (t^k_{i(k)}, t^k_{i(k)+1}] \) and \( 0 \leq i(k) < m_k \))

\[
\sum_{\lambda=0}^{i(k)-1} |\varphi(t^k_{\lambda+1}) - \varphi(t^k_{\lambda})| + |\varphi(t^k_{i(k)}) - \varphi_k(x)| \leq V^r_{a} \varphi + |\varphi(t^k_{i(k)}) - \varphi_k(x)| \tag{12}.
\]

It is clear that

\[
V^r_{a} \varphi + |\varphi(t^k_{i(k)}) - \varphi(x)| \leq V^z_{a} \varphi; \tag{13}
\]

since \( \varphi_k(x) \to \varphi(x) \), we get from (12) and (13) that

\[
\limsup_{k \to \infty} \upsilon_k(x) = \limsup_{k \to \infty} (V^r_{a} \varphi + |\varphi(t^k_{i(k)}) - \varphi_k(x)|) = \limsup_{k \to \infty} (V^r_{a} \varphi + |\varphi(t^k_{i(k)}) - \varphi(x)|) \leq V^z_{a} \varphi. \tag{14}
\]

But since \( \varphi_k \to \varphi \) pointwise on \([a, b]\), we have that

\[
v(x) = V^z_{a} \varphi \leq \liminf_{k \to \infty} \upsilon_k(x) \tag{15}
\]

(see ex.12, p.205 of [7]).

It then follows from (14) and (15) that

\[
v(x) = \lim_{k \to \infty} \upsilon_k(x), \text{ for } x \in (a, b).
\]

We finally note that it is easy to verify that the function of total variation of a polygonal function is also polygonal. Therefore each \( \upsilon_k \) is a polygonal (and increasing) function.

\[\square\]

We note that the proof of claim (a) of Lemma 5 is similar to the proof of Lemma 4.

**Remark 8.** (1) Regarding the previous Lemma, we set

\[
p_k = \frac{1}{2}(v_k + \varphi_k - \varphi_k(a)) \text{ and } n_k = \frac{1}{2}(v_k - \varphi_k + \varphi_k(a)),
\]

where \( \upsilon_k \) is the function of total variation of \( \varphi_k \). Then we have that:

(a) \( p_k \) and \( n_k \) are the positive and negative variations of \( \varphi_k \).

(b) the function \( \varphi_k \) is polygonal, the functions \( v_k, p_k, n_k \) are polygonal and increasing and

(c) \( \varphi_k \to \varphi, v_k \to v \) pointwise on \([a, b]\) and hence \( p_k \to p, n_k \to n \) pointwise on \([a, b]\).

(2) Assume now that \( \varphi(a) = 0 \) and \( V^b_{a} \varphi = 1 \). We then have that \( \varphi_k(a) = 0 \) for \( k \geq 1 \) and \( V^b_{a} \varphi_k = v_k(b) \to v(b) = V^b_{a} \varphi = 1 \). Now we define

\[
\Phi_k = \frac{\varphi_k}{v_k(b)}, \ \Upsilon_k = \frac{v_k}{v_k(b)}, \ \Pi_k = \frac{p_k}{v_k(b)}, \ \Nu_k = \frac{n_k}{v_k(b)}
\]

and notice the following:

(a) \( \Upsilon_k, \Pi_k, \Nu_k \) are the functions of total, positive and negative variation of \( \Phi_k \), so that
\begin{align*}
\Phi_k(a) = 0 \text{ and } \Upsilon_k(b) = V^b_k \Phi_k = 1, \text{ for } k \geq 1. \\
(b) \Phi_k \text{ is polygonal and } \Upsilon_k, P_k, N_k \text{ are polygonal and increasing.} \\
(c) \text{ For every } x \in [a, b] \text{ we have that } \Phi_k(x) \to \varphi(x), \ U_k(x) \to v(x) \text{ and hence } P_k(x) \to p(x) \text{ and } N_k(x) \to n(x). \\
\end{align*}

It follows from the aforementioned remark that Lemma 5 can be stated as follows:

**Proposition 7.** Let \( \varphi : [a, b] \to \mathbb{R} \) be a function of bounded variation with \( \varphi(a) = 0 \) and \( V^b_a \varphi = 1 \). Also, let \( v, p \) and \( n \) be the functions of total, positive and negative variation of \( \varphi \). Then there is a sequence \( \varphi_k : [a, b] \to \mathbb{R}, \ k \geq 1 \) of polygonal functions, such that if \( v_k, p_k \) and \( n_k \) are the total, positive and negative variations of \( \varphi_k \), then (these functions are polygonal and)

\begin{align*}
(a) & \quad \varphi_k \to \varphi, \ v_k \to v, \ p_k \to p \text{ and } n_k \to n \text{ pointwise on } [a, b]. \\
(b) & \quad \varphi_k(a) = 0 \text{ and } V^b_a \varphi_k = 1, \text{ for } k \geq 1. \\
\end{align*}

Let \( \varphi : I = [0, 1] \to \mathbb{R} \) be a continuous function of bounded variation with \( \varphi(0) = 0 \) and \( V^1_0 \varphi = 1 \). Denote, as usual, the function of total variation of \( \varphi \) by \( v \). Then by Theorem 7, there are sequences \( \omega = (x_n) \subseteq I \) and \( \varepsilon = (\varepsilon_n) \subseteq \{ \pm 1 \} \) such that

\begin{align*}
(a) & \quad \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{[0,x)}(x_k) = v(x) \text{ and} \\
(b) & \quad \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k \chi_{[0,x)}(x_k) = \varphi(x), \\
\end{align*}

for \( 0 \leq x \leq 1 \).

We can now define the discrepancy \( D_N(\omega; v) \) of \( \omega = (x_n) \) with respect to the (continuous) function \( v \) by the rule

\[ D_N(\omega; v) = \sup_{0 \leq a < b \leq 1} \frac{1}{N} \sum_{k=1}^{N} \left| \chi_{[a,b)}(x_k) - (v(b) - v(a)) \right|, \]

and one can prove that \( \lim_{N \to \infty} D_N(\omega; v) = 0 \) (see Th. 1.1, p.89 and the remarks before Th. 1.2, p.90 of [15]).

Similarly, we define the discrepancy of \( \omega \) with respect to \( \varphi \) as follows

\[ D_N(\omega; \varphi) = \sup_{0 \leq a < b \leq 1} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k \chi_{[a,b)}(x_k) - (\varphi(b) - \varphi(a)). \]

Then an analogous result can be shown, the proof of which is similar to the proof of Th. 1.1, p.89 of [15]. So, with the above assumptions and notation for \( \varphi \) we have the following

**Lemma 6.** \( \lim_{N \to \infty} D_N(\omega; \varphi) = 0 \)

**Proof.** We first define the discrepancies of the functions \( p \) and \( n \) in the obvious way. It is easy to prove that

\[ D_N(\omega; \varphi) \leq D_N(\omega; p) + D_N(\omega; n) \quad (16). \]
Since \( p \) and \( n \) are continuous and hence uniformly continuous on the compact interval \( I \), given any \( \varepsilon > 0 \) there is \( m \in \mathbb{N} \) such that

\[
x, y \in I \text{ and } |x - y| < \frac{1}{m} \Rightarrow |p(x) - p(y)| < \frac{\varepsilon}{2} \text{ and } |n(x) - n(y)| < \frac{\varepsilon}{2}. \tag{17}
\]

We may pick \( m \) quite large, so that \( \frac{1}{m} < \varepsilon \).

For such an integer \( m \), set \( I_k = \left( \frac{k}{m}, \frac{k+1}{m} \right) \), \( 0 \leq k \leq m - 1 \). Using the equalities (c) and (d) of Th.7, we can get a \( N_0 = N_0(m) \in \mathbb{N} \), such that for every \( N \geq N_0 \) and each \( k = 0, 1, \ldots, m \) we have

\[
\mu^+(I_k) - \frac{1}{m^2} \leq \frac{1}{N} \sum_{\lambda=1}^{N} \left( \frac{1 + \varepsilon_{\lambda}}{2} \right) \chi_{I_k}(x_{\lambda}) \leq \mu^+(I_k) + \frac{1}{m^2} \tag{18}
\]

and

\[
\mu^-(I_k) - \frac{1}{m^2} \leq \frac{1}{N} \sum_{\lambda=1}^{N} \left( \frac{1 - \varepsilon_{\lambda}}{2} \right) \chi_{I_k}(x_{\lambda}) \leq \mu^-(I_k) + \frac{1}{m^2} \tag{19}
\]

where \( \mu^+(I_k) = p\left( \frac{k+1}{m} \right) - p\left( \frac{k}{m} \right), \mu^-(I_k) = n\left( \frac{k+1}{m} \right) - n\left( \frac{k}{m} \right) \) and \( \mu^+= \mu_p, \mu^- = \mu_n \) are the positive and negative variations of \( \mu = \mu_{\omega} \) (cf. the proof of Th.7).

Now consider an arbitrary interval \( J = [a, b] \subseteq I \), then there are subintervals \( J_1, J_2 \) of \( J \) each one of them being a finite union of successive intervals \( I_k \), such that \( J_1 \subseteq J \subseteq J_2 \) and

\[
\mu^+(J) - \mu^+(J_1) < \varepsilon, \mu^+(J_2) - \mu^+(J) < \varepsilon \text{ and } \mu^-(J) - \mu^-(J_1) < \varepsilon, \mu^-(J_2) - \mu^-(J) < \varepsilon. \tag{20}
\]

These inequalities are easy consequence of (17), i.e. of the uniform continuity of \( p \) and \( n \). By adding at most \( m \) inequalities of the form (18), we get that

\[
\mu^+(J_1) - \frac{1}{m} \leq \frac{1}{N} \sum_{\lambda=1}^{N} \left( \frac{1 + \varepsilon_{\lambda}}{2} \right) \chi_{J_1}(x_{\lambda}) \leq \frac{1}{N} \sum_{\lambda=1}^{N} \left( \frac{1 + \varepsilon_{\lambda}}{2} \right) \chi_{J}(x_{\lambda}) \leq \frac{1}{N} \sum_{\lambda=1}^{N} \left( \frac{1 + \varepsilon_{\lambda}}{2} \right) \chi_{J_2}(x_{\lambda}) \leq \mu^+(J_2) + \frac{1}{m};
\]

then using (20) we conclude that

\[
\mu^+(J) - 2\varepsilon < \mu^+(J) - \frac{1}{m} - \varepsilon \leq \frac{1}{N} \sum_{\lambda=1}^{N} \left( \frac{1 + \varepsilon_{\lambda}}{2} \right) \chi_{J}(x_{\lambda}) \leq \mu^+(J) + \frac{1}{m} + \varepsilon < \mu^+(J) + 2\varepsilon. \tag{21}
\]

In a similar way we get that

\[
\mu^-(J) - 2\varepsilon < \mu^-(J) - \frac{1}{m} - \varepsilon \leq \frac{1}{N} \sum_{\lambda=1}^{N} \left( \frac{1 - \varepsilon_{\lambda}}{2} \right) \chi_{J}(x_{\lambda}) \leq \mu^-(J) + \frac{1}{m} + \varepsilon < \mu^-(J) + 2\varepsilon. \tag{22}
\]

Since (21) and (22) are independent of \( J \), we conclude that \( \lim_{N \to \infty} D_N(\omega; p) = 0 \), \( \lim_{N \to \infty} D_N(\omega; n) = 0 \) and thus by (16) \( \lim_{N \to \infty} D_N(\omega; \varphi) = 0 \).  \( \Box \)
Remark 9. It is also possible (and useful) to define the concept of $D_N^*$ discrepancy for the functions $\varphi$, $\upsilon$, $p$ and $n$ (cf. Def. 1.2, p.90 of [15]). For instance we may define

$$D_N^*(\omega; \varphi) = \sup_{0 \leq x \leq 1} \left| \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k \chi_{[0,x)}(x_k) - \varphi(x) \right|.$$ 

It is then easy to see that

$$D_N^*(\omega; \varphi) \leq D_N(\omega; \varphi) \leq 2D_N^*(\omega; \varphi)$$

(cf. Th. 1.3, p.91 of [15]). So we get that

$$\lim_{N \to \infty} D_N(\omega; \varphi) = 0 \iff \lim_{N \to \infty} D_N^*(\omega; \varphi) = 0.$$ 

Let $(f_n)$ be a sequence of scalar valued functions defined on a set $X$. Given a strictly increasing sequence of positive integers $1 \leq N_1 < N_2 < \cdots < N_k < \cdots$, we arrange the terms of $(f_n)$ setting

$$g_N = f_1, \text{ for } 1 \leq N < N_1 \text{ and } g_N = f_k, \text{ for } N_{k-1} \leq N < N_k, \text{ } k \geq 2.$$ 

Then the following Lemma has an easy proof, which we omit.

**Lemma 7.** If $f_n \to f$ pointwise on $X$, then $g_N \to f$ pointwise on $X$.

Let now $(f_n) : \mathbb{N} \to \mathbb{R}$, $n \geq 1$ be a sequence of functions, so that for every $n \geq 1$, $\lim_{m \to \infty} f_n(m) = 0$. We consider a strictly increasing sequence of positive integers $(N_k)_{k \geq 1}$ such that

$$m \geq N_k \Rightarrow |f_k(m)| \leq \frac{1}{k}, \text{ for } k \geq 1.$$ 

(Since each $f_n$ is a null sequence of scalars, such a sequence exists). If we apply the above arrangement to $(f_n)$ defined by $(N_k)$, we get the following

**Lemma 8.** \(\lim_{N \to \infty} g_N(N) = 0\).

**Proof.** Let $N \geq N_1$, then there is $k \geq 2 : N_{k-1} \leq N < N_k$, hence $g_N(N) = f_k(N)$. But since $N \geq N_k$, we get (from the definition of $(N_k)$) that $|f_k(N)| \leq \frac{1}{k}$ and so

$$|g_N(N)| = |f_k(N)| \leq \frac{1}{k}.$$ 

As $N \to \infty$ implies $k \to \infty$, we obtain the desired result. 

We are now in a position to prove the desired generalization of Theorems 7 and 8.

**Theorem 9.** Let $\varphi : [a, b] \to \mathbb{R}$ be a function of bounded variation with $\varphi(a) = 0$ and $V_{[a,b]}^1 \varphi = 1$. Also let $\upsilon$, $p$ and $n$ be the functions of total, positive and negative variations of $\varphi$. Then there are sequences $\tau = (x_n) \subseteq [a, b]$ and $\varepsilon = (\varepsilon_n) \subseteq \{\pm 1\}$, such that for every $x \in [a, b]$ we have
(a) \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \chi_{[a,x]}(x_k) = v(x) \)

(b) \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \varepsilon_k \chi_{[a,x]}(x_k) = \varphi(x) \),

(c) \( \lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^N (1 + \varepsilon_k) \chi_{[a,x]}(x_k) = p(x) \) and

(d) \( \lim_{N \to \infty} \frac{1}{2N} \sum_{k=1}^N (1 - \varepsilon_k) \chi_{[a,x]}(x_k) = n(x) \).

**Proof.** We first reduce the theorem to the case when \([a,b]\) is the unit interval \(I = [0,1] \).

Consider the affine continuous function \(g(x) = (b-a)x + a, \ x \in I\); clearly \(g\) is strictly increasing, with \(g(0) = a\) and \(g(1) = b\). Set \(\Phi = \varphi \circ g\) and notice that

(i) \(\Phi\) is of bounded variation on \(I\) with \(\Phi(0) = 0\) and \(V_0^1 \Phi = V_0^1 \varphi = 1\) and

(ii) \(v_\Phi(x) = v_\varphi(g(x)), \ p_\Phi(x) = p_\varphi(g(x))\) and \(n_\Phi(x) = n_\varphi(g(x))\), for \(x \in I\). Now let \((z_n) \subseteq I\) and \((\varepsilon_n) \subseteq \{ \pm 1 \}\) satisfying conditions (a) to (d) for the function \(\Phi\). Then the sequences \(x_n = g(z_n), n \geq 1\) and \((\varepsilon_n)\) satisfy the same conditions for \(\varphi\).

In order to prove the theorem (with \([a,b] = I\)) we will follow the method of proof of Theorem 8 (Th. 4.3, p.138 of [15]) and use Prop. 7. So let \(\varphi_k, v_k\) (and \(p_k, n_k\)) be as in Prop. 7. Since each \(v_k\) is continuous and increasing with \(v_k(0) = 0\) and \(v_k(1) = 1\), by Lemma 3 (Lemma 4.2, p.137 of [15]) there is a sequence \(\tau_k = (x_1^k, x_2^k, \ldots, x_n^k, \ldots)\) satisfying

\[
\left| \frac{1}{N} \sum_{n=1}^N \chi_{[0,x]}(x_n^k) - v_k(x) \right| \leq \frac{\log(N + 1)}{N \cdot \log 2} \quad (23)
\]

for all \(N \geq 1\) and \(x \in I\).

It is clear that if we fix some \(k \in \mathbb{N}\), then letting \(N \to \infty\) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \chi_{[0,x]}(x_n^k) = v_k(x) \quad \text{for} \ x \in I. \quad (24)
\]

Let \(\mu_k = \mu_{\varphi_k}\) be the Lebesgue-Stieltjes measure that \(\varphi_k\) defines on \(I\) and \(h_k = |d\mu_k|/|d[p_k]|\) be the corresponding Radon-Nikodym derivative. Since each \(\varphi_k\) is polygonal, its derivative is a step function, hence \(h_k\) is \(|\mu_k|\)-Riemann integrable, which implies by Prop.6 that the sequence \(\varepsilon_k = (\varepsilon_n^k)_{n \geq 1}\), where \(\varepsilon_n^k = h_k(x_n^k), n \geq 1\) has the property

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \varepsilon_n^k \chi_{[0,x]}(x_n^k) = \varphi_k(x) \quad \text{for} \ x \in I. \quad (25)
\]

It follows from Lemma 6 and Remark 9 that, if we set

\[
D_N^*(\tau_k; \varphi_k) = \sup_{0 \leq x \leq 1} \left| \frac{1}{N} \sum_{n=1}^N \varepsilon_n^k \chi_{[0,x]}(x_n^k) - \varphi_k(x) \right|
\]

then we have

\[
D_N^*(\tau_k; \varphi_k) \to 0, \quad \text{for every} \ k \geq 1. \quad (26)
\]
We notice that we may furthermore assume that
\[
D_N^*(\tau_N; \varphi_N) \xrightarrow{N \to \infty} 0. \tag{27}
\]
In order to attain (27), we consider a strictly increasing sequence of positive integers \((N_k)_{k \geq 1}\) such that
\[
m \geq N_k \Rightarrow D_m^*(\tau_k; \varphi_k) \leq \frac{1}{k} \quad \text{for } k \geq 1.
\]
Then we arrange the sequence of functions \((\varphi_k)_{k \geq 1}\) as in Lemma 7, that is we set
\[
\Phi_N = \varphi_1, \quad \text{for } 1 \leq N < N_1 \quad \text{and} \quad \Phi_N = \varphi_k, \quad \text{for } N_{k-1} \leq N < N_k, \quad k \geq 2.
\]
It then follows from Lemmas 7 and 8 that \(\Phi_k \to \varphi, (Y_k \to v, \text{ etc.})\) pointwise on \(I\) and that (23) to (27) remain valid for the sequence \((\Phi_k)_{k \geq 1}\). So we may (and will) assume without loss of generality that \(\Phi_k = \varphi_k, \quad \text{for } k \geq 1\).

Now we construct the sequence \(\tau = (x_n) \subseteq I\) by listing successively the first term of \(\tau_1\), the first two terms of \(\tau_2\), . . . , the first \(k\) terms of \(\tau_k\), that is,
\[
\tau = (x_1^1, x_1^2, x_2^2, \ldots, x_k^k) = (x_1^1, x_2^2, \ldots, x_k^k, \ldots).
\]
The sequence of signs \(\varepsilon = (\varepsilon_n)\) is constructed similarly; so we set
\[
\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_2^2, \ldots, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots).
\]
We are going to prove that \(\tau\) and \(\varepsilon\) are the desired sequences. Assertion (a) of this theorem is proved the same way as assertion (a) of Theorem 8.

Indeed, by Lemma 4.1, p.136 of [15], it suffices to prove that
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \chi_{[0,x]}(x_i^k) = v(x), \quad \text{for } x \in I.
\]
By using (23) and as (by Prop.7) \(v_k(x) \to v(x)\), for \(x \in I\), we get that
\[
\left| \frac{1}{k} \sum_{i=1}^{k} \chi_{[0,x]}(x_i^k) - v(x) \right| \leq \left| \frac{1}{k} \sum_{i=1}^{k} \chi_{[0,x]}(x_i^k) - v_k(x) \right| + |v_k(x) - v(x)| \leq \frac{\log(k+1)}{k \log 2} + |v_k(x) - v(x)| \xrightarrow{k \to \infty} 0, \quad \text{for } x \in I.
\]
This way assertion (a) is proved.

Now, to prove assertion (b), using again Lemma 4.1 of [15] as above it suffices to show that
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \varepsilon_i^k \chi_{[0,x]}(x_i^k) = \varphi(x), \quad \text{for } x \in I.
\]
We now use (27) and the fact that $\varphi_k(x) \to \varphi(x)$, for $x \in I$ (see Prop.7), so we get

$$\left| \frac{1}{k} \sum_{i=1}^{k} \varepsilon_i x(x_i^k) - \varphi(x) \right| \leq \left| \frac{1}{k} \sum_{i=1}^{k} \varepsilon_i x(x_i^k) - \varphi_k(x) \right| + |\varphi_k(x) - \varphi(x)| \leq$$

$$\leq D_k^* (\tau_k; \varphi_k) + |\varphi_k(x) - \varphi(x)| \xrightarrow[k \to \infty]{} 0.$$

Assertions (c) and (d) follow easily from (a) and (b). The proof of the theorem is now complete.

\[ \square \]

**Concluding remarks**

1. Let $\varphi : [a, b] \to \mathbb{R}$ be a differentiable function with ($\varphi(a) = 0$ and) bounded derivative. Then it is Lipschitz continuous and (hence) of bounded variation. Let $\mu = \mu_\varphi$ be the Lebesgue-Stieljes measure defined by $\varphi$ on $[a, b]$ and $h = \frac{d\mu}{d|\mu|}$. Assuming that $\varphi'$ is Riemann integrable, it is not difficult to show that $h$ is $|\mu|$-Riemann integrable. It then easily follows that if $\varphi$ is piecewise $C^1$ (for instance a polygonal function), then $h$ has the desired property.

On the other hand if $\varphi'$ is not Riemann integrable, that is, $\varphi$ is a Volterra type function, then the function $h$ may or may not be $|\mu|$-Riemann integrable. For examples (and the properties) of functions of Volterra type, we refer the reader to the books [11], pp.35-36 and [6], pp.22-25 and 33-35.

2. Concerning future work, we note the following:

(a) It would be interesting to have a generalization of Theorems 7 and 8 for functions of “bounded variation” of several variables, that is, for functions $f$ defined on the cube $I^n$ for $n \geq 2$, which satisfy a proper notion of bounded variation (see for instance Def. 5.2, p.147 of [15]).

(b) Besides compact metric spaces (see Th.3), there are several classes of compact non-metrizable spaces $K$ with the property that every measure $\mu \in P(K)$ admits a u.d. sequence (see [16] and [18]). For such spaces it would be interesting to know if every signed measure $\mu$ with $\|\mu\| = 1$ admits a u.d. sequence in the sense of Def.1 (cf. also Remark 5(b)). In our opinion the most interesting case is that of compact separable groups $G$: since we know that under special set-theoretic assumptions, i.e. Continuum Hypothesis (CH) (see [16]) or Martin’s axiom plus the negation of Continuum Hypothesis (MA+$\neg$ CH), (see [3]), every measure $\mu \in P(G)$ admits a u.d. sequence. We note that it is enough to consider the compact group $\{0,1\}^c$, where $c = \text{the cardinality of the continuum}$; this is so because, as is well known, every compact separable group is a dyadic space, i.e. a continuous image of $\{0,1\}^c$ (see [12] and [18]).

(c) Regarding our consideration, the case of locally compact and separable metrizable spaces is also of interest, see [15] Notes, pp.177-178.
References

[1] F. Albiac and N.J. Kalton, Topics in Banach Space Theory, Grad. Texts in Math., vol.233, Springer, New York (2006).

[2] S. Argyros and S. Mercourakis, On Weakly Lindelöf Banach Spaces, Rocky Mountain J. Math. 23 (1993), no. 2, 395–446.

[3] S. Argyros, S. Mercourakis and A. Tsarpalias, Convex unconditionality and summability of weakly null sequences, Israel J. Math. 107 (1998), 157—193.

[4] W.G. Bade, The Banach Space C(S), Lecture Notes Series no. 26, Matematisk Institut, Aarhus Universitet, Aarhus, 1971.

[5] J. Bourgain, A. Pajor, S.J. Szarek and N. Tomczak-Jaegermann, On the duality problem for entropy numbers of operators, In Geometric Aspects of Functional Analysis, Lecture Notes in Math., 1376, Springer, Berlin (1989), 50–63.

[6] Andrew Bruckner, Differentiation of Real functions, CRM Monograph Series, Amer. Math. Soc., 1994.

[7] N.L. Carothers, Real Analysis, Cambridge Univ. Press, 2000.

[8] R. Frankiewicz, G. Plebanek. On asymptotic density and uniformly distributed sequences, Studia Mathematica 119 (1996), no. 1, 17–26.

[9] Gerald B. Folland, Real Analysis: Modern Techniques and Their Applications, second Edition, J. Wiley, 1999.

[10] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, Banach Space Theory, The Basis for Linear and Nonlinear Analysis, CMS Books in Mathematics, Canadian Mathematical Society, Springer, 2011.

[11] Russel A. Gordon, The Integrals of Lebesgue, Denjoy, Perron, and Henstock, Grad. Studies in Math., Amer. Math. Soc., 1994.

[12] S. Grekas and S. Mercourakis, On the measure theoretic structure of compact groups, Trans. Amer. Math. Soc. 350 (1998), 2779–2796.

[13] Richard Haydon, Some more characterizations of Banach spaces containing $\ell_1$, Math. Proc. Cambridge Philos. Soc. 80 (1976), no. 2, 269–276.

[14] P. Hájek, V. Montesinos Santalucia, J. Vanderwerff and V. Zizler, Biorthogonal systems in Banach spaces, CMS Books in Mathematics/ Ouvrages de Mathématiques de la SMC, 26, Springer, New York, 2008.

[15] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, J. Wiley, New York, 1974.

[16] V. Losert, On the existence of uniformly distributed sequences in compact topological spaces I, Trans. Amer. Math. Soc. 246 (1978), 463–471.

[17] S. Mercourakis, On Cesaro summable sequences of continuous functions, Mathematika 42 (1995), 87–104.

[18] S. Mercourakis, Some remarks on countably determined measures and uniform distribution of sequences, Mh. Math. 121 (1996), 79–111.

[19] H. Niederreiter, On the existence of uniformly distributed sequences in compact spaces, Compositio Mathematica 25 (1972), no. 1, 93–99.

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