INJECTIVE HYPERBOLICITY FOR QUOTIENTS OF BALLS AND POLYDISKS

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ABSTRACT. In this article we study the injective Kobayashi metric on complex surfaces.

1. Introduction

For complex manifolds $Y$ and $X$ we let $\mathcal{O}(Y,X)$ denote the set of holomorphic maps $f : Y \to X$, and we let $\mathcal{O}_i(Y,X)$ be the set of elements $f \in \mathcal{O}(Y,X)$ such that $f$ is injective. We let $\Delta$ denote the unit disk in the complex plane.

Definition 1.1. Let $X$ be a complex manifold. For a point $(x,v) \in TX$ we set

$$
\omega^X_K(x,v) := \inf \left\{ \frac{1}{\lambda} : f \in \mathcal{O}(\Delta,X), f(0) = x, df(0)(1) = \lambda v \right\},
$$

and we set

$$
\omega^X_i(x,v) := \inf \left\{ \frac{1}{\lambda} : f \in \mathcal{O}_i(\Delta,X), f(0) = x, df(0)(1) = \lambda v \right\}.
$$

Then $\omega^X_K$ is the familiar (infinitesimal) Kobayashi metric, and we will call $\omega^X_i$ the injective Kobayashi metric.

Remark 1.2. Upon finishing a preprint of the current article we were made aware of the fact that the injective Kobayashi metric was already introduced in [3], and that it already appeared in the one-dimensional case in [11]. In [3] the corresponding object(s) are referred to as Hahn functions/metrics. Furthermore, these objects were studied in [5] and [6], and Theorem 1.5 below may be proved by the methods in [5] (where the corresponding result was proved for non-simply connected hyperbolic domains in $\mathbb{C}$ - see also [4] Chapter 8), or as an application of the result therein.

The main problem is the following.

Problem 1.3. For which 2-dimensional complex manifolds $X$ do we have that $\omega^X_i = \omega^X_K$?

In complex dimension one, i.e., in the the case that $X$ is a Riemann surface, the corresponding problem is quite simple. If $X$ is hyperbolic the metrics coincide if and only if $X$ is the unit disk (see [10] where the injective Kobayashi metric was introduced on foliations), if $X = \mathbb{C}$ or $X = \mathbb{P}^1$ both metrics vanish identically, and if $X = \mathbb{C}^*$ or $X$ is a torus, the metrics are different due to the Koebe $\frac{1}{4}$-theorem. Furthermore, in complex dimension larger than 2, the metrics always coincide, see [9].

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In this note, we will here give some initial results, focusing on quotients of balls and bi-disks.

**Theorem 1.4.** Let $\Gamma \subset \text{Aut}_{\text{hol}}(\mathbb{B}^2)$ be a Kleinian group, set $X := \mathbb{B}^2 / \Gamma$, and assume that $X$ is compact. Then $\omega_K^X \neq \omega_I^X$.

Theorem 1.4 will be proved in Section 4.

The following result is essentially due to Jarnicki [5].

**Theorem 1.5.** (Jarnicki [5]) Let $Y_1$ and $Y_2$ be compact hyperbolic Riemann surfaces, and set $X := Y_1 \times Y_2$. Then $\omega_K^X \neq \omega_I^X$.

The last theorem will be a consequence of a more general result proved in Section 5, where we will also construct non-trivial examples where the two metrics coincide.

## 2. Preliminaries

### 2.1. Definitions.

Throughout this article $\Delta$ will denote the unit disk in the complex plane $\mathbb{C}$, and $\Delta^2$ will denote the unit bidisk in $\mathbb{C}^2$. For any domain $\Omega \subset \mathbb{C}^n$ we let $b\Omega$ denote its topological boundary. For a point $p \in \mathbb{C}^2$ and $\delta > 0$, we let $B_\delta(p)$ denote the ball of radius $\delta$ centred at $p$.

Recall that for $X = \Delta$ we have that

$$\omega_\Delta^\Delta(z, v) = \omega_P(z, v) = \frac{|v|}{1 - |z|^2},$$

where $\omega_P$ denotes the Poincaré metric. Equipped with this metric, the holomorphic automorphism group $\text{Aut}_{\text{hol}}(\Delta)$ of the unit disk, is the group of orientation preserving isometries of $\Delta$, and any Riemann surface of hyperbolic type is the quotient of $\Delta$ by a Fuchsian sub-group $\Gamma$.

**Definition 2.1.** Let $\Gamma \subset \text{Aut}_{\text{hol}}(\Delta)$ be a sub-group. We will call $\Gamma$ a Fuchsian group if $\Gamma$ acts properly discontinuously on $\Delta$, i.e., if for every point $z \in \Delta$ there is an open set $U$ containing $z$ such that if $\phi \in \Gamma$ and if $\phi(U) \cap U \neq \emptyset$, then $\phi = id$. If $\Delta$ is replaced by $\mathbb{B}^2$ or $\Delta^2$ we will call such a group a Kleinian group.

Recall that any element $\phi \in \text{Aut}_{\text{hol}}(\Delta^2)$ is of the form $\phi = (\varphi_1, \varphi_2)$ with $\varphi_j \in \text{Aut}_{\text{hol}}(\Delta)$. The elements $\varphi \in \text{Aut}_{\text{hol}}(\Delta)$ are classified into one of three types:

1. **hyperbolic** if $\varphi$ has precisely two distinct fixed points on $\overline{\Delta}$, and both are contained in $b\Delta$,
2. **parabolic** if $\varphi$ has precisely one fixed points on $\overline{\Delta}$, and it is contained in $b\Delta$,
3. **elliptic** if $\varphi$ has precisely one fixed points on $\Delta$, and it is contained in $\Delta$.

Elements of $\text{Aut}_{\text{hol}}(\mathbb{B}^2)$ are classified correspondingly, were in (1)-(3) we replace $\Delta$ by $\mathbb{B}^2$.

Clearly, a Kleinian/Fuchsian group cannot contain any elements of elliptic type, so we will here consider hyperbolic and parabolic automorphisms. Recall that a hyperbolic automorphism of $\Delta$ is conjugate to an automorphism

$$\varphi(z) = \frac{z + r}{1 + rz} \quad (2.1)$$
with $0 < r < 1$. For parabolic automorphisms it is more convenient to identify $\triangle$ with the upper half plane $H$, and there any parabolic automorphism is conjugate to

$$\text{either } \varphi^+(z) = z + 1 \text{ or } \varphi^-(z) = z - 1.$$  \hfill (2.2)

In dimension two, any hyperbolic automorphism of $\mathbb{B}^2$ is conjugate to an automorphism

$$\phi(z, w) = \left( \frac{z + r}{1 + rz}, e^{i\theta} \frac{\sqrt{1 - r^2}}{1 + rz} w \right)$$  \hfill (2.3)

for $0 < r < 1$ and $\theta \in [0, 2\pi)$. For parabolic automorphisms it is more convenient to identify $\mathbb{B}^2$ with the Siegel upper half plane

$$H_2 = \{(z, w) \in \mathbb{C}^2 : \Im(w) > |z|^2 \}.$$ 

In this case, any parabolic automorphism of $H_2$ is conjugate (after possibly passing to inverses) to one of the following two types

$$\phi(z, w) = (e^{i\theta} z, w + 1)$$ \hfill (2.4)

or

$$\phi(z, w) = (z - i, w - 2z + i).$$ \hfill (2.5)

2.2. Extremal maps in $\mathbb{B}^2$. For a point $p \in \mathbb{B}^2$ and a tangent vector $v$, and extremal map $f : \triangle \to \mathbb{B}^2$ is a map such that $f(0) = p$, and $\omega_K(p, v) = \frac{1}{|f'(0)|}$. If we want to determine all extremal maps for a point $p$, since $\text{Aut}_{hol}(\mathbb{B}^2)$ acts transitively on $\mathbb{B}^2$, it suffices to consider the case $p = 0$. And since the isotropy group at the origin acts transitively on directions, it suffices to consider the case $v = (1, 0)$. Then using Schwarz Lemma, it follows that the map $f(z) = (z, 0)$ is extremal, and furthermore that $f$ is the unique extremal map for $\omega_K(0, v)$.

2.3. Extremal maps in $\triangle^2$. For any point $z \in \triangle$ and any vector $v \in \mathbb{C}^2$, it follows by Montel’s Theorem that there exists a map $f : \triangle \to \triangle^2$ such that $f(0) = z$ and $\omega_{\triangle^2}(z, v) = \frac{1}{|f'(0)|}$. Consider such extremal maps for $z = 0$ and $v = (1, \xi)$ with $|\xi| \leq 1$. Then a natural candidate for an extremal map is the map $f(z) = (z, z \cdot \xi)$. This map clearly has a left inverse $\psi : \triangle^2 \to \triangle$, namely $\psi(z_1, z_2) = z_1$, and using this it is clear from the Schwarz Lemma that $f$ indeed is an extremal map. Moreover, if $|\xi| = 1$ the map $f$ is the unique extremal map, which can be seen by applying Schwarz Lemma after projecting to the diagonal.

3. The general strategy

**Proposition 3.1.** Let $\Omega \subset \mathbb{C}^2$ be a bounded domain, let $X$ be a complex manifold, and suppose that $\pi : \Omega \to X$ is a holomorphic covering map. Assume that $f : \triangle \to \Omega$ is a proper holomorphic embedding such that $f$ is a unique extremal map for the Kobayashi metric through the point $p = f(0)$ with tangent vector $v = f'(0)$. Assume further that there is a point $q \in Z = f(\triangle)$ and a $\delta > 0$ such that

$$\pi(B_\delta(p) \cap Z) \cap \pi(B_\delta(q) \cap Z) = \pi(p) = \pi(q).$$

Then $\omega_X(p, v_*) > \omega_K(p, v_*)$ where $v_* = \pi_* v$. Consequently $\omega_X \neq \omega_K$. 

Proof. We will show first that if \( \tilde{f} : \triangle \to \Omega \) is sufficiently close to \( f \), then \( \pi \circ \tilde{f} : \triangle \to X \) is not injective. Set \( a = f^{-1}(q) \) and fix \( \epsilon > 0 \) such that \( \pi(f(B_\epsilon(q))) \cap \pi(f(B_\epsilon(a))) = \pi(p) \). Define \( G_j : \triangle \to X \) by \( G_1(z) = \pi(f(z)) \), and \( G_2(z) = \pi(f(z + a)) \). We will show that if \( G_j \) is sufficiently close to \( G_j \) for \( j = 1, 2 \), then \( G_1(\triangle) \cap G_2(\triangle) \neq \emptyset \). By passing to a local coordinate chart, and possibly having to decrease \( \epsilon \), we may assume that \( G_j(\triangle) \subset \mathbb{C}^2 \), \( G_j(0) = 0 \), and \( dG_1(0)(1) = (1,0) \), and further that \( G_1(z) = (g(z), h(g(z))) \). Setting \( H(z_1,z_2) = z_2 - h(z_1) \) we now have that the function \( H(G_2(z)) \) has an isolated zero at the origin on \( \Delta_{\epsilon/2} \).

Now suppose that \( G_k \to G_j \) uniformly as \( k \to \infty \) on \( \Delta_\epsilon \) for \( j = 1, 2 \). Then for sufficiently large \( k \) we may write \( G_k^j(z) = (g_k(z), h_k(g_k(z))) \), and we have that \( g_k \to g, h_k \to h \), and so setting \( H_k(z) = z_2 - h_k(z_1) \) we have that \( H_k \to H \) uniformly as \( k \to \infty \). Then \( H_k \circ G_k^j \to H \circ G_j \) uniformly as \( k \to \infty \), and so by Hurwitz’ Theorem \( H_k \circ G_k^j \) has a zero for sufficiently large \( k \). This means precisely that the images of the two discs intersect.

To finish the proof of the proposition, let \( \tilde{f}_i : \triangle \to X \) be a sequence of holomorphic maps with \( f(0) = \pi(p) \), \( df_i(0)(1) = \lambda_i^{-1} \cdot v_\alpha \), and such that \( \lambda_i \to \lambda = \omega_K^X(\pi(p), v_\alpha) \). Letting \( f_j : \triangle \to \Omega \) be liftings such that \( f_j = \pi \circ f_j \), we get by uniqueness of \( f \) that \( f_i \to f \), and by our previous conclusion we have that \( f_i \) is not injective for sufficiently large \( i \).

\[ \square \]

4. Quotients of the unit ball

The simplest situation where we in our context can find extremal maps to apply Proposition 3.1 is found where we consider a Kleinian subgroup \( \Gamma \subset \text{Aut}_{\text{hol}}(\mathbb{B}^2) \) which contains at least one hyperbolic element.

Theorem 4.1. Let \( \Gamma \subset \text{Aut}_{\text{hol}}(\mathbb{B}^2) \) be a Kleinian group, set \( X := \mathbb{B}^2 / \Gamma \), and assume that \( \Gamma \) contains at least one hyperbolic element. Then \( \omega_K^X \neq \omega_1^X \).

Proof. After conjugation we may achieve that a hyperbolic element is of the form

\[ \phi(z,w) = \left( \frac{z + r}{1 + rz}, e^{i\theta} \frac{\sqrt{1-r^2}w}{1 + rz} \right) \]

with \( 0 \leq \theta < 2\pi \). Note that, by replacing \( \phi \) with a high iterate, we may assume that \( r \) and \( e^{i\theta} \) are both arbitrarily close to 1. For \( \alpha \in (0,\sqrt{1-r^2}) \) to be determined further, consider the straight line \( L_\alpha := \{(z,\alpha) : |z|^2 < 1 - \alpha^2 \} \). Then \( \phi \) sends \( L_\alpha \) to the straight line

\[ L^\phi_\alpha = \{(z,w) : w = \frac{\alpha e^{i\theta}(1-rz)}{\sqrt{1-r^2}} \}. \]

The intersection point between \( L_\alpha \) and \( L^\phi_\alpha \) occurs for \( z_0 = \frac{1-\sqrt{1-r^2}e^{-i\theta}}{r} \). We have that

\[ |z_0|^2 - 1 < 0 \iff (1 - e^{-i\theta} \sqrt{1-r^2})(1 - e^{i\theta} \sqrt{1-r^2}) - r^2 < 0 \]
\[ \iff 1 - \sqrt{1-r^2} \cdot 2 \cos \theta + (1-r^2) - r^2 < 0 \]
\[ \iff 2(1-r^2) - 2 \sqrt{1-r^2} \cdot \cos \theta < 0 \]
\[ \iff 2 \sqrt{1-r^2} (\sqrt{1-r^2} - \cos \theta) < 0. \]
So if $r$ is close enough to 1 and if $\theta$ is close enough to 0 we have that $z_0$ is in the unit disk. Then $|z_0| < \sqrt{1-\alpha^2}$ if $\alpha$ is chosen small enough. The conditions in Proposition 3.1 are therefore fulfilled. □

Proof of Theorem 1.4: By Theorem 4.1 it suffices to prove that if $X$ is compact, then $\Gamma$ contains a hyperbolic element. In fact, if $X$ is compact, we have that $\Gamma$ contains only hyperbolic elements.

Lemma 4.2. Let $X = \mathbb{B}^2/\Gamma$ be a compact complex manifold. Then $\Gamma$ contains only hyperbolic elements.

Proof. Note that a compact hyperbolic manifold cannot have arbitrarily short non-trivial loops. So to prove the lemma it suffices to prove that any parabolic element $\phi \in \text{Aut}_{\text{hol}}(\mathbb{B}^2)$ identifies points with arbitrarily small Kobayashi distance between them. We demonstrate this in the Siegel upper half plane $H_2 = \{\text{Im}(w) > |z|^2\}$, and up to conjugation there are two cases to consider

$$\phi(z, w) = (e^{i\theta} z, w + 1) \text{ and } \phi(z, w) = (z - i, w - 2z + i).$$

In the first case we consider points $a_s = (0, i \cdot s)$ and $\phi(a_s) = (0, is + 1) \in \{0\} \times \{\text{Re}(w) > 0\} =: H_2^0$. In $H_2^0$ the Kobayashi metric is given by $\frac{|dw|^2}{\text{Im}(w)^2}$, and so it is clear that $\text{dist}_K(a, \phi(a_s)) \to 0$ as $s \to \infty$.

In the second case we consider points $a_s = (i, is), s > 1$, and we have that $\phi(a_s) = (0, i(s - 1))$. Then to estimate the distance between $a_s$ and $\phi(a_s)$ we connect the two points by joining two paths $\gamma_1^s$ and $\gamma_2^s$, the first one being the straight line segment between $i(s - 1)$ and $is$ in $H_2^0$, and the second being the straight line segment between $(0, is)$ and $(i, is)$ inside the complex disk $D_s = \{(z, w) \in H_2 : w = is\}$.

Then, by the the formula for the Kobayashi metric in $H_2^0$ above, it is clear that the length of $\gamma_1$ goes to zero as $s \to \infty$. So we consider $\gamma_2^s$. Then

$$D_s = \{(z, is) : |z|^2 < s\},$$

and so it is clear that the Kobayashi length of $\gamma_2^s$ in $D_s$ goes to zero as $s \to \infty$. □

Definition 4.3. We now extend our definition of a Fuchsian group to include certain sub-groups of $\text{Aut}_{\text{hol}}(\mathbb{B}^2)$. For a Fuchsian group $\Gamma \subset \text{Aut}_{\text{hol}}(\Delta)$ we may extend each element

$$\varphi(z) = e^{i\theta} \frac{z + \alpha}{1 - \alpha z}$$

to an element

$$\phi_\varphi(z, w) = (e^{i\theta} \frac{z + \alpha}{1 - \alpha z}, e^{i\psi_\varphi} \sqrt{1 - |\alpha|^2} \frac{1 - |\alpha|^2}{1 - \alpha z}(w))$$

of $\text{Aut}_{\text{hol}}(\mathbb{B}^2)$, and it is easy to see that the group $\tilde{\Gamma}$ generated by the $\phi_\varphi$s is a Kleinian group. We will refer to such special Kleinian groups as 2-dimensional Fuchsian groups.

Theorem 4.4. Let $\Gamma$ be a 2-dimensional Fuchsian group, and set $X = \mathbb{B}^2/\Gamma$. Then one of the two following cases can occur.
(1) $\Gamma$ is generated by a single parabolic $\phi_\varphi \in \text{Aut}_{\text{hol}}(\mathbb{B}^2)$, $\psi_\varphi = 0$, and $\omega_\tau = \omega_\tau^X$.

(2) $\omega_\tau^X \neq \omega_\tau^X$.

Proof. We consider first the case where $\Gamma$ is not generated by a single parabolic element. Then if $\Gamma$ contains a hyperbolic element we have that (2) follows from the previous theorem. Suppose then that $\Gamma$ contains extensions of two parabolic elements $\gamma$ and $\tilde{\gamma}$, representing two distinct generators of the fundamental group of the quotient. We will show that then $\langle \gamma, \tilde{\gamma} \rangle$ contains a hyperbolic element. For this it is more convenient to pass to the upper half plane, where first of all, up to conjugation, we may assume that $\gamma(z) = z + 1$. Furthermore, any automorphism of the upper half plane is of the form $a z + b \in \mathbb{R}, ad - bc = 1$.

and moreover, an automorphism of the form (4.1) is parabolic if and only if $a + d = \pm 2$. Now $Y = H/\langle \gamma, \tilde{\gamma} \rangle$ is an open Riemann surface, and so it is known that $\langle \gamma, \tilde{\gamma} \rangle$ is a free group, and so the two elements do not commute. So we may assume that $c \neq 0$.

Suppose now that $\gamma \circ \tilde{\gamma}$ is parabolic. We have that

$$
\tilde{\gamma}(z) = \frac{az + b}{cz + d}
$$

with $a, b, c, d \in \mathbb{R}, ad - bc = 1$, (4.1)

which is parabolic if and only if $a + c + d = \pm 2$. If $a + d = 2$ this would imply that $c = -4$. But then

$$
\gamma^{-1}(\gamma(z)) = \frac{(a - c)z + b - d}{cz + d}
$$

is not parabolic, since $a - c + d = 6$. The analogous argument applies if $a + d = -2$.

Suppose next that $\Gamma$ is generated by a single parabolic element $\gamma$. It is then more convenient to consider the Siegel upper half-plane model for the unit ball $H_2$ where we up to conjugation have that

$$
\gamma(z, w) = (e^{i\theta} z, w + 1).
$$

Suppose first that $\theta \neq 2k\pi, k \in \mathbb{Z}$. Setting

$$
L_{a,b} = \{(z, w) : z + aw + b = 0\}
$$

we have that

$$
G_{a,b} = L_{a,b} \cap H_2
$$

are unique geodesics in $H_2$. We are looking for $G_{a,b}$ that will allow us to apply Proposition 3.1, i.e., such that $G_{a,b} \cap \gamma(G_{a,b})$ is a single point. So we consider the set of equations

$$
z + aw + b = 0
$$

and

$$
e^{i\theta} z + a(w + 1) + b = 0
$$

We set $w = -\frac{b + z}{a}$ and solve

$$
e^{i\theta} z + a - b - z + b = 0,
$$

and we see that we may set $z_0 = \frac{a}{1-e^{i\theta}}$ and then $w_0 = -\frac{b}{a} + \frac{1}{1-e^{i\theta}}$ to get that $(z_0, w_0) = G_{a,b} \cap \gamma(G_{a,b})$. Finally, note that for any $a$ one may choose $b$ such that $(z_0, w_0) \in H_2$. Now Proposition 3.1 applies.
It remains to consider the case that $\theta = 0$. In that case, the above calculations show that the only possibility to achieve that $G_{a,b} \cap \gamma (G_{a,b}) \neq \emptyset$ is to set $a = 0$, i.e., to consider straight vertical lines $G_b = G_{0,b}$. In that case, we have that $G_b$ is invariant under $\gamma$, we have that $Z = G_b/\langle \gamma \rangle$ is conformally the punctured disk, and $Z \to X$ is a closed submanifold. Now, if we consider the covering map $\pi = H_2 \to X$ restricted to $G_b$, we have that $\pi : G_b \to Z$ is a universal covering map, so in this case it is not a priori clear if anything prevents $\pi|_{G_b}$ from being a uniform limit (on compacts) of injective holomorphic embeddings. In fact, we will show that it is.

Let $H_b = \{ \zeta \in \mathbb{C} : \text{Im}(\zeta) > |b| \}$. Set $g_b(\zeta) = (-b, \zeta)$ so that $g$ maps $H_b$ onto $G_b$, and set $f_b = \pi \circ g_b$. Then $f_b$ is an extremal map for any point in $H_b$, and we will pick an arbitrary point $\zeta_0 \in H_b$, and show that for any compact set $K \subset H_b$, we have that $f_b$ may be approximated arbitrarily well on $K$ by injective holomorphic embeddings $\tilde{f}_b : K \to X$, with the additional property that $d\tilde{f}_b(\zeta_0)(1) = df_b(\zeta_0)(1)$ are co-linear.

By Siu’s theorem we have that $f_b(H_b)$ has a Stein neighbourhood $\Omega \subset X$. Choose local coordinates near $f_b(\zeta_0)$ such that, in the local coordinates in $\mathbb{C}^2$, we have that $f_b(\zeta_0) = 0$ and $df_b(\zeta_0)(1) = (1,0)$. Since $\Omega$ is Stein there are holomorphic vector fields $V_1$ and $V_2$ on $\Omega$ such that in the local coordinates just chosen, we have that $V_1(0) = V_2(0) = 0$ and $V_1(\zeta_0) = V_2(\zeta_0) = 1$.

Now, for a compact set $K \subset H_b$ we let $W \subset X$ be a neighborhood of $f_b(K)$ such that the composition of flows $\psi_2^\delta \circ \psi_1^\delta$ exists on $W$ for $|t_1|, |t_2| < \epsilon$ for some $\epsilon > 0$. Now for $\delta > 0$ we set $g_b^\delta(\zeta) = (-b + \delta(\zeta - \zeta_0), \zeta)$, and further $f_b^\delta = \pi \circ g_b^\delta$. Then $f_b^\delta : K \to X$ is injective since $g_b^\delta$ maps $H_b$ onto a non-vertical line, and $g_b^\delta \to g_b$ uniformly on $K$ as $\delta \to 0$. Now in the local coordinates we have that $df_b^\delta(\zeta_0)(1) = (1 + \eta(\delta), \mu(\delta))$ where $\eta(\delta), \mu(\delta) \to 0$ as $\delta \to 0$. Now provided $\delta$ is sufficiently small we may set $s_1(\delta) = -\frac{\mu(\delta)}{1 + \eta(\delta)}$ and $s_2(\delta) = \log((1 + \eta(\delta))^{-1})$ and set

$$\tilde{f}_b^\delta = \psi_2^{s_2(\delta)} \circ \psi_1^{s_1(\delta)} \circ f_b^\delta,$$

and we get that $d\tilde{f}_b^\delta(\zeta_0)(1) = (1,0)$ for all $\delta$ (small) and $f_b^\delta \to f_b$ uniformly on $K$ as $\delta \to 0$.

5. Quotients of the bi-disk

In this section we will give two results on quotients of the bi-disk. A reason why the case of a bi-disk is more involved than the simple case of the unit ball, is that the extremal disks are not unique. Hence, Proposition 3.1 cannot be applied to any extremal holomorphic disk, and we have to work with the “diagonals” $D_\xi = \{(z, z \cdot \xi)\}$ with $|\xi| = 1$.

Theorem 1.5 is a consequence of the following.

**Theorem 5.1.** *(Jarnicki [5])* Let $\Gamma_j \subset \text{Aut}_{\ho}(\Delta)$ be Fuchsian groups for $j = 1, 2$, and set $X := \Delta^2/\Gamma$, with $\Gamma = \Gamma_1 \oplus \Gamma_2$. Suppose that there exists at least one element $\phi = (\varphi_1, \varphi_2) \in \Gamma$ such that $\varphi_j \neq \text{id}$ for $j = 1, 2$. Then $\omega_t^X \neq \omega^X$. 

The reason why Theorem 1.5 follows from this, is that if \( \Gamma \) only contains elements of the form either \((\varphi, \text{id})\) or \((\text{id}, \varphi)\), then \( X \) would not be compact.

Our next result is positive.

**Theorem 5.2.** Let \( \Gamma \subset \text{Aut}_{\text{hol}}(\triangle^2) \) be a Kleinian group, and set \( X := \triangle^2/\Gamma \). Suppose all elements \( \phi \in \Gamma \) are of the form \((\varphi, \text{id})\). Then if \( (\triangle \times \{0\})/\Gamma \) is an open Riemann surface we have that \( \omega_{K}^X = \omega^K \).

The main step in proving Theorem 5.1 is to prove it in the special case that \( \Gamma = \Gamma_1 \oplus \Gamma_2 \) where the \( \Gamma_j \)s are cyclic groups in \( \text{Aut}_{\text{hol}}(\triangle) \), i.e., when \( X = Y_1 \times Y_2 \) where the \( Y_j \) are hyperbolic Riemann surfaces with \( \pi_1(Y_j) = \mathbb{Z} \). We will consider three cases separately, and then we will explain the general case.

5.1. **The case where the \( Y_j \)'s are both annuli.** In this section we will prove the following:

**Theorem 5.3.** Let \( Y_1 \) and \( Y_2 \) be annuli, and set \( X := Y_1 \times Y_2 \). Then \( \omega_{K}^X \neq \omega_{K}^X \).

To prove this theorem we include a subsection where we introduce a way of measuring conformal moduli of annuli, and then we give the proof in the subsection following it.

5.2. **Conformal moduli of annuli.** Suppose \( X \) and \( Y \) are Riemann surfaces with a biholomorphism \( \psi : X \to Y \). Then a choice of lifting of \( \psi(\pi(0)) \) induces an automorphism \( g \in \text{Aut}_{\text{hol}}(\triangle) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\triangle & \xrightarrow{g} & \triangle \\
\downarrow{\pi} & & \downarrow{\pi'} \\
X & \xleftarrow{\psi} & Y
\end{array}
\]

Then if \( \Gamma \) denotes the Deck-group associated to \( \pi \) we have that \( \Gamma' = g\Gamma g^{-1} \) is the Deck group associated to \( \pi' \). On the other hand, if \( \Gamma \) is a Fuchsian group, if \( \pi : \triangle \to X = \triangle/\Gamma \) is the universal cover, and if \( g \in \text{Aut}_{\text{hol}}(\triangle) \), then \( g \) induces a biholomorphism \( \psi : X \to Y \), where \( Y = \triangle/\Gamma' \) with \( \Gamma' = g\Gamma g^{-1} \). Conjugating by such a \( g \) corresponds to a change of basepoint and direction for the universal covering map.

Now suppose \( X \) is an annulus, i.e., that \( X = \triangle/\langle \varphi \rangle \) where \( \varphi \) is hyperbolic. Then \( \varphi \) has precisely two fixpoints \( p^\alpha \) and \( p^\rho \) on \( b\triangle \), one attracting and one repelling, and we let \( \lambda^\alpha \) and \( \lambda^\rho \) denote their multipliers. Furthermore, \( p^\alpha \) and \( p^\rho \) are joined by the closure of a unique geodesic \( \gamma \subset \mathbb{D} \). After conjugation (change of base point) we may assume that \( p^\alpha = x \) and \( p^\rho = -1 \), in which case \( \gamma = \mathbb{R} \cap \triangle \), we have that \( \gamma \) is \( \varphi \)-invariant, and we have that \( \varphi \) is on the form

\[
\varphi(z) = \frac{z + r}{1 + rz}. \tag{5.1}
\]

Then 1 is an attracting fixed point for \( \varphi \), and \( \varphi^n \to 1 \) uniformly on compact subsets of \( \overline{\triangle} \setminus \{-1\} \) as \( n \to \infty \), so \( \gamma \) is the unique \( \varphi \)-invariant geodesic in \( \triangle \). Since the multipliers are invariant under conjugation we may compute them directly from the form (5.1) and we see that \( \lambda^\alpha = 1 - r \) and
\( \lambda^\rho = 1 + r \). It follows that \( r \) is completely determined by any one of the multipliers, that the multipliers are determined completely by \( r \), and we set

\[
\mathcal{M}(X) = \frac{1}{2} \log \frac{1+r}{1-r} = l_K(\gamma).
\]

**Proposition 5.4.** Let \( X \) and \( Y \) be two annuli. Then \( X \) is biholomorphic to \( Y \) if and only if \( \mathcal{M}(X) = \mathcal{M}(Y) \).

**Proof.** Suppose \( \mathcal{M}(X) = \mathcal{M}(Y) \). Then after conjugation we may assume that both \( X \) and \( Y \) are quotients of the disk by the group generated by the same map \((5.1)\), and so they are biholomorphic.

Next suppose \( X \) is biholomorphic to \( Y \). After conjugation we may assume that both \( X \) and \( Y \) are quotients of the disk by the groups generated by maps \( \varphi_{r_X} \) and \( \varphi_{r_Y} \) on the form \((5.1)\), but this time a priori with different dilations \( r_X \) and \( r_Y \). As noted above, the biholomorphism induces a conjugation \( g \), which necessarily has to fix the points \( \pm 1 \) individually, and so \( g \) is also on the form \((5.1)\). Consider what happens for the conjugation

\[
\varphi_{r_Y} = g \circ \varphi_{r_X} \circ g^{-1}
\]
at the fixed point 1. By the chain rule, the map \( g \circ \varphi_{r_X} \circ g^{-1} \) has the same multiplier as the map \( \varphi_{r_X} \), and so the map \( \varphi_{r_X} \) has the same multiplier as the map \( \varphi_{r_Y} \), from which it follows that \( r_X = r_Y \). \( \square \)

Next we would like to establish a growth of length description for certain families of non-trivial loops in \( X \), and that the loop \( \gamma \) in the definition of \( \mathcal{M}(X) \) is in fact the shortest non-trivial loop in \( X \).

**Proposition 5.5.** Let \( \phi(z) = \frac{z-r}{1+r(z)} \) and fix \( \theta \in (0, \pi) \cup (\pi, 2\pi) \). For \( s \in [0, 1] \) set

\[
\eta(s) := d_K(se^{i\theta}, \phi(se^{i\theta})).
\]

Then \( \eta(s) \) is strictly increasing in \( s \) and \( \lim_{s \to 1} \eta(s) = \infty \).

**Proof.** Letting \( d_M(\cdot, \cdot) \) denote the Möbius distance, we may prove that

\[
\lim_{s \to 1} = \tau(s) := d_M^2(se^{i\theta}, \phi(se^{i\theta})) = 1,
\]

and that \( \tau \) is an increasing function of \( s \). We have that

\[
\phi(se^{i\theta}) = \frac{se^{i\theta} + r}{1 + rs e^{i\theta}},
\]

and further we get that

\[
d_M(se^{i\theta}, \phi(se^{i\theta})) = \left| \frac{se^{i\theta} + r - se^{i\theta}}{1 - se^{-i\theta} \frac{se^{i\theta} + r}{1 + rs e^{i\theta}}} \right|
\]

\[
= \frac{se^{i\theta} + r - se^{i\theta}(1 + rs e^{i\theta})}{(1 + rs e^{i\theta}) - se^{-i\theta}(se^{i\theta} + r)}
\]

\[
= \frac{r(1 - s^2 e^{2i\theta})}{1 - s^2 + rs(e^{i\theta} - e^{-i\theta})},
\]
so we see that
\[ \lim_{s \to 1} d_M(se^{i\theta}, \phi(se^{i\theta})) = |\frac{(1 - e^{2i\theta})}{e^{i\theta} - e^{-i\theta}}| = 1. \]

Further we have that
\[ \tau(s) = r^2 \frac{1 - s^2 \cos(2\theta) + s^4 \sin^2(2\theta)}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)} \]
\[ = r^2 \frac{1 - 2s^2 \cos(2\theta) + s^4 \cos^2(2\theta) + s^4 \sin^2(2\theta)}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)} \]
\[ = r^2 \frac{1 - 2s^2(\cos^2(\theta) - \sin^2(\theta)) + s^4}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)} \]
\[ = r^2 \frac{1 - 2s^2(1 - 2\sin^2(\theta)) + s^4}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)} \]
\[ = r^2 \frac{1 - 2s^2 + 4s^2 \sin^2(\theta)) + s^4}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)} \]
\[ = r^2 \frac{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)} \]
\[ = r^2 [1 + \frac{4s^2 \sin^2 \theta - 4r^2 s^2 \sin^2 \theta}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)}] \]
\[ = r^2 [1 + \frac{s^2 \cdot (4(1 - r^2) \sin^2 \theta)}{(1 - s^2)^2 + 4r^2 s^2 \sin^2(\theta)}] \]

So \( \tau(s) \) is strictly increasing in \( s \) if the function
\[ f(x) = \frac{x}{(1 - x)^2 + x\alpha} \]
is strictly increasing for \( \alpha > 0 \). Computing the nominator \( N(f'(x)) \) we see that
\[ N(f'(x)) = (1 - x)^2 + x\alpha - x(2(1 - x)(-1) + \alpha) = (1 - x)^2 + 2x(1 - x) \]
which is strictly positive for \( 0 \leq x < 1 \). □

**Corollary 5.6.** For an annulus \( X \) we have that \( M(X) \) is the Kobayashi length of the shortest non-trivial loop in \( X \).

**Proof.** Choose the universal covering map \( \pi : \triangle \to X \) such that \( \text{Deck}(X) \) is generated by
\[ \varphi(z) = \frac{z + r}{1 + rz}. \]
Let \( \gamma : [0, 1] \to X \) be a continuous map with \( \gamma(0) = \gamma(1) = p \), and let \( \tilde{\gamma} : [0, 1] \to \triangle \) be a lifting of \( \gamma \). Assuming that \( \gamma \) is a candidate for a shortest non-trivial loop, we may assume that \( \tilde{\gamma} \) is a geodesic arc in \( \triangle \) connecting \( \tilde{\gamma}(0) \) and \( \tilde{\gamma}(1) \). Assume first that \( \tilde{\gamma}(0) \notin \mathbb{R} \) (in which case \( \tilde{\gamma}(1) \notin \mathbb{R} \)). Write \( se^{i\theta} \), and fix \( n \in \mathbb{Z} \) such that \( \tilde{\gamma}(1) = \varphi^n(se^{i\theta}) \). Then
\[ \varphi^n(z) = \frac{z + r'}{1 + r'z} \]
for some $r'$ with $|r'| \geq |r|$, and applying the proposition with $\varphi^n$, we see that the Kobayashi length of $\hat{\gamma}$ is strictly longer than the segment between 0 and $\varphi^n(0)$, which in turn has length $\mathcal{M}(X)$ if and only if $n = \pm 1$.

In the remaining case, if $\hat{\gamma}(0) \in \mathbb{R}$ then $\hat{\gamma} \subset \mathbb{R}$ and $\hat{\gamma}(1) = \varphi^n(\hat{\gamma}(0))$ for $n \in \mathbb{Z}$. By the minimality assumption we have that $n = \pm 1$, in which case $\hat{\gamma}$ has the same Kobayashi length as the line segment between 0 and $\varphi(0)$. 

\[\square\]

5.3. Proof of Theorem 5.3. We start by providing a lemma. Recall that $D \subset \Delta^2$ denotes the diagonal $\{(z, z)\}$. For any point $p \in D$ and $\delta > 0$ we let $D_\delta(p)$ denote the set $B_\delta(p) \cap D$, where $B_\delta(p)$ denotes the ball of radius $\delta$ centred at $p$ in $\Delta^2$.

Lemma 5.7. Let $\Gamma \subset \text{Aut}_\text{hol}(\Delta^2)$ be a Fuchsian group, assume that $\phi = (\varphi_1, \varphi_2) \in \Gamma$, with $\varphi_1(0) = \varphi_2(0)$, and $\varphi_1 \neq \varphi_2$, and consider the quotient $\pi : \Delta^2 \to X : = \Delta^2/\Gamma$. Then $\pi(D)$ is a singular curve in $X$, and there exists a $\delta > 0$ such that $\pi(D_\delta(0))$ and $\pi(D_\delta(\phi(0)))$ are distinct (locally) irreducible components of $\pi(D)$.

Proof. Set $q = \phi(0)$, and choose $\delta$ sufficiently small such that $\pi$ is injective on $D_\delta(0)$ and $D_\delta(q)$. Then $\pi(D_\delta(0))$ and $\pi(D_\delta(q))$ are smooth subsets of $\pi(D)$ and they intersect at the point $\pi(0) = \pi(q)$. Consider the set

$$Z = \{(z, z) \in D_\delta(0) : \pi(z, z) \subset \pi(D_\delta(q))\} = \{(z, z) \in D_\delta(0) : \phi(z, z) \subset D_\delta(q)\}.$$ 

Then $(z, z) \in Z$ if and only if the equation $\varphi_1(z) = \varphi_2(z)$ is satisfied, but since $\varphi_1 \neq \varphi_2$ we have that 0 is an isolated point satisfying this equation, and the conclusion of the lemma follows after possibly having to decrease $\delta$. 

Suppose first that $Y_1$ is not conformally equivalent to $Y_2$. According to Proposition 5.4 we have that $\mathcal{M}(Y_1) \neq \mathcal{M}(Y_2)$, and so without loss of generality we assume that $\mathcal{M}(Y_1) < \mathcal{M}(Y_2)$. Let $\pi_j : \Delta \to Y_j$ be universal covering maps for $j = 1, 2$, and let $\phi_j$ be generators for the corresponding Deck-groups. After conjugating $\phi_j$ for $j = 1, 2$, and possibly taking inverses, we may assume that

$$\phi_j(z) = \frac{z + r_j}{1 + r_j z}. \quad (5.2)$$

(Note that conjugating groups corresponds to changing the base points for $\pi_j$ and a choice of directional derivative for the universal covering map.) We then have that $r_1 < r_2$. By Lemma 5.5 there exists a point $z \in \Delta$ such that $d_K(z, \phi_1(z)) = \mathcal{M}(Y_2)$.

Now let $C_1$ denote the straight line segment between -1 and 1, and let $C_2$ be the geodesic in $\Delta$ that contains $z$ and $\phi_1(z)$. Then there is a (unique) Möbius transformation $\psi$ that maps $C_1$ onto $C_2$, and with $\psi(0) = z$, $\psi(r_2) = \phi_1(z)$. Set $\tilde{\phi}_1 = \psi^{-1} \circ \phi_1 \circ \psi$. Then $\tilde{\phi}_1(0) = r_2 = \phi_2(0)$. However, note that $\tilde{\phi}_1 \neq \phi_2$, since $C_2$ is not an invariant geodesic for $\phi_1$, which implies that $C_1$ is not an invariant geodesic for $\phi_1$.

We now consider the universal covering of $X$ given by

$$\pi : \Delta^2 \to \Delta^2/(\tilde{\phi}_1(z_1), \phi_2(z_2)).$$

Since $\tilde{\phi}_1(0) = \phi_2(0) = r_2$ it follows from Lemma 5.7 that the diagonal $D \subset \Delta \times \Delta$ is mapped onto a singular locally reducible curve in $X$. Moreover, setting $q = (\tilde{\phi}_1(0), \phi_2(0))$, near the point
π(0) = π(q) we have that π(D_δ(0)) and π(D_δ(q)) (see notation from Lemma [5.7]) are (local) irreducible components of π(D). This concludes the proof in the case that M(Y_1) ≠ M(Y_2) by an application of Proposition [3.1].

In the remaining case, after an initial conjugation, we may assume that φ_1 = φ_2, and that they are both on the form (5.2). We may then conjugate φ_1 as before to obtain an element ˜φ_1 such that ˜φ_1(0) = φ_2^2(0), while ˜φ_1 ≠ φ_2. Again, the conclusion is that the diagonal D descends to a singular curve in the quotient X.

5.4. The case where Y_1 is an annulus and Y_2 is the punctured disk. In this case we have generators φ_1 and φ_2 where φ_1 is hyperbolic and φ_2 is parabolic. Then for any 0 < r < ∞ we have that there exists a point z ∈ △ such that d_M(z, φ_2(z)) = r. Pick a point z_0 such that d_M(z_0, φ_2(z_0)) = d_M(0, φ_1(0)). Choose a map γ such that γ(0) = z_0 and set ˜φ_2(z) = γ^{-1}(φ_2(γ(z))). Then |φ_1(0)| = |φ_2(0)|, and so after another conjugation we may assume that φ_1(0) = ˜φ_2(0). Since φ_1 is hyperbolic and ˜φ_2 is parabolic, we have that φ_1 ≠ ˜φ_2, and so the proof is concluded as in the previous case.

5.5. The case where the Y_j’s are both the punctured disk. The punctured disk is the quotient of the upper half plane by a cyclic group generated by a parabolic element. Any such element is conjugate to an element z ↦ z ± 1, and so we may initially assume that φ_1(z) = z + 1 and φ_2(z) = z - 1 (if necessary we may also use inverses).

Lemma 5.8. Set

\[ \psi(z) = \frac{(7/5)z - (1/5)}{(4/5)z + (3/5)}. \]

Then ψ is conjugate to φ_2.

Proof. We have ψ(1/2) = 1/2. Set γ(z) = \( \frac{z+2}{2z} \) and γ\(^{-1}\)(z) = \( \frac{1}{z-1/2} \).

\[ \gamma^{-1}(\psi(\gamma(z))) = \frac{-1}{\frac{7(\frac{z+1}{2z}) - 1}{4(\frac{z+1}{2z}) + 3} - (1/2)} = \frac{-1}{\frac{(7z+14) - 2z}{(4z+3)+4z} - (1/2)} = \frac{-1}{\frac{10z+14}{10z+8} - (1/2)} = \frac{-1}{\frac{10z-8}{10z} - (1/2)} = \frac{-10z + 8}{5z - 14 - 5z + 4} = z - 4/5. \]

So (5/4)γ\(^{-1}\)(ψ(γ(4/5)z))) = z - 1. □
By the lemma, after conjugation we may assume that the two groups are generated by $\phi_1$ and $\psi$. Then $\phi_1(i) = \psi(i)$ while $\phi_1 \neq \psi$. So the proof is concluded as in the first case.

5.6. **Proof of Theorem 5.2.** We will consider how the extremal curves in $\Delta^2$ descend to $X$. So let $(\alpha, \beta) \in \Delta^2$ be an arbitrary point, and let $v \in \mathbb{C}^2 \setminus \{0\}$. Note first that if $v$ is vertical then the vertical line through $(\alpha, \beta)$ is mapped injectively into $X$, so we have an injective extremal curve. Next we assume that $v$ is not vertical and not horizontal. By conjugating $\Gamma$ we may assume that $\alpha = 0$. Now let $\psi$ be an automorphism of $\Delta$ such that $\psi(\beta) = 0$, set $F(z_1, z_2) := (z_1, \psi(z_2))$, and set $\tilde{v} := F_*(0, \beta)(v)$.

There is an extremal map $(\xi, \lambda \xi)$ through the origin in the direction $\tilde{v}$, and so the curve $(\xi, \psi^{-1}(\lambda \xi))$ is extremal in the direction $v$ at $(0, \beta)$. Now assume two points on this extremal curve are identified by $\Gamma$, i.e., there is a point $(\xi, \psi^{-1}(\lambda \xi))$ and an element $\varphi \in \Gamma$ such that $(\varphi(\xi), \psi^{-1}(\lambda \xi))$ equals $(\varphi(\xi), \psi^{-1}(\lambda \varphi(\xi)))$. Then $\lambda \xi = \lambda \varphi(\xi)$, but then $\varphi = \text{id}$ since $\Gamma$ is fixed point free.

It remains to consider horizontal directions, and this is done in the same way as the last part of the proof of Theorem 4.3.

6. **Examples**

6.1. **The case of dimension one.** In complex dimension one we have that the two metrics are the same on $\Delta, \mathbb{C}$ and $\mathbb{P}^1$. On $\mathbb{C}^*$ they are different because of the Koebe-$\frac{1}{4}$ theorem, which also gives that they are different on any torus $T$. On all other Riemann surfaces they are different.

6.2. **Some easy cases in dimension two.** In complex dimension two we have that the two metrics coincide on $\mathbb{C}^2, \mathbb{C}^* \times \mathbb{C}, \mathbb{C}^* \times \mathbb{C}^*, \mathbb{C}^* \times \mathbb{P}^1, \mathbb{C}^* \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$. It is unknown if they agree on $T \times \mathbb{P}^1$ where $T$ is a torus. The metrics also always coincide on any manifold $X$ with the Density Property, since for any point $x \in X$ there is a Fatou-Bieberbach domain $\Omega \subset X$ with $x \in \Omega$.

6.3. **The case of dimension greater than two.** If $X$ is a complex manifold of dimension $\dim(X) \geq 3$ we have that $\omega^X_K = \omega^X_\iota$ due to transversality.

6.4. **Convex domains.** It is a consequence of Lempert’s theory that the two metrics always coincide on a bounded strictly convex domain $\Omega \subset \mathbb{C}^2$ with boundary of class $C^3$ ([2], [3]).

6.5. **The symmetrized bi-disk.** The symmetrized bi-disk $\mathbb{S}$ gives an example of a non-convex domain for which the two metrics coincide. Agler and Young showed that every two points in $\mathbb{S}$ can be joined by a unique complex geodesic for $\omega^\mathbb{S}_K$ that has a left inverse ([41]). This example we might generalise as follows:

**Proposition 6.1.** Let $D \subset \mathbb{C}^n$ be a bounded taut domain such that for any two points $z_1, z_2 \in \Omega$ there exist a holomorphic map $\varphi : \Delta \to \Omega$ with $z_1, z_2 \in \varphi(\Delta)$, and a holomorphic map $\psi : D \to \Delta$ such that $\psi(\varphi(\zeta)) = \zeta$ for all $\zeta \in \Delta$. Then $\omega^D_\iota = \omega^K_\iota$.

**Proof.** Let $z \in D$ and let $v \in \mathbb{C}^n$. Let $z_j = z + (1/j)(v)$, $\varphi_j(0) = z$, $\varphi_j(\zeta_j) = z_j$, and $\psi_j(\varphi_j(\zeta)) = \zeta$. Without loss of generality we may assume that $\varphi_j \to \varphi : \Delta \to D, \psi_j \to \psi : D \to \Delta$ uniformly on compacta. Then $\psi(\varphi(\zeta)) = \zeta$ for all $\zeta \in \Delta$. So $\varphi$ is a holomorphic embedding, $\varphi'(0) = \lambda v$ for some $\lambda \neq 0$, and $\Omega^G_K(z, v) = 1/|\lambda|$. □
7. Open problems

Problem 7.1. Determine if the injective Kobayashi metric vanishes identically on $\mathbb{P}^1 \times T_1$ and $T_1 \times T_2$ (here $T_j$ are tori).

Problem 7.2. Let $R$ be a compact hyperbolic Riemann surface, and let $S$ denote either the unit disk $\Delta$, the complex plane $\mathbb{C}$, the Riemann sphere $\hat{\mathbb{C}}$, or a torus $T$. Set $X = R \times S$. Do we have $\omega^X_{i} = \omega^K_X$?

Problem 7.3. Let $X$ be an Oka manifold. Do we have $\omega^X_{i} = \omega^K_X$?

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