Factorization of Shapovalov elements

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Abstract

Shapovalov elements $\theta_{\beta,m}$ are special elements in a Borel subalgebra of a classical or quantum universal enveloping algebra parameterized by a positive root $\beta$ and a positive integer $m$. They relate the canonical generator of a reducible Verma module with highest vectors of its Verma submodules. For $m = 1$, they can be explicitly obtained as matrix elements of the inverse Shapovalov form. We extend this approach to $m > 1$ for all $\beta$ but three roots in $g_2$, $f_4$, and $e_8$, presenting $\theta_{\beta,m}$ as a product of matrix elements of weight $\beta$.

Key words: Shapovalov elements, Shapovalov form, R-matrix, Verma modules

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1 Introduction

The Berstein-Gelfand-Gelfand representation category $\mathcal{O}$ of semi-simple Lie algebras and quantum groups [1] is one of the fundamental concepts appearing in various fields of mathematics and mathematical physics. In particular, it accommodates finite-dimensional and numerous important infinite dimensional representations like parabolic modules and their generalizations [2]. There are distinguished objects in $\mathcal{O}$ called Verma modules that feature a universality property: all irreducible modules in $\mathcal{O}$ are their quotients. The maximal proper submodule in a Verma module is generated by extremal vectors [3], which are invariants of the positive triangular subalgebra. This makes extremal vectors critically important in representation theory.

Extremal vectors in a Verma module are related with a vacuum vector of highest weight via special elements $\theta_{\beta,m}$ of the negative Borel subalgebra that are called Shapovalov elements [4, 5]. They are parameterized with a positive root $\beta$ and an integer $m \in \mathbb{N}$ validating, in the classical version, the Kac-Kazhdan condition $2(\lambda + \rho, \beta) - m(\beta, \beta) = 0$ on the highest weight $\lambda$ (with $\rho$ being the half-sum of positive roots). This condition guarantees that the Verma module is reducible. In the special case when the root $\beta$ is simple, $\theta_{\beta,m} = f_{\beta}^m$, where $f_{\beta}$ is the corresponding Chevalley generator. For compound $\beta$, the Shapovalov element is a polynomial in the simple root generators with coefficients in the Cartan subalgebra.

Factorization of $\theta_{\beta,m}$ to a product of polynomials of lower degree is convenient both for their explicit construction and for analysis of their properties. For example it is good for the study of their classical limit in the case of quantum groups, which is crucial for quantization of conjugacy classes and their equivariant vector bundles [22].

A description of extremal vectors in Verma modules over Kac-Moody algebras is available in [7] via a special calculus of polynomials with complex exponents. An inductive construction of extremal vectors in the case of quantum groups was suggested in [8]. In a factorized form $\theta_{\beta,m} = \theta_{\beta,1}^m$, Shapovalov elements for classical Lie algebras are presented in [9], with the help of extremal projectors [10]. While Zhelobenko’s construction gives an answer in the case of classical simple Lie algebras, there remains a problem of explicit description of the structure of factors.

We suggest an alternative approach to the problem based on a contravariant bilinear form on Verma modules. It is not as universal as Zhelobenko’s, but has its own advantages as it gives explicit expressions for Shapovalov elements in a factorized form in almost all cases. There are only three roots in the exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, and $\mathfrak{e}_8$ that are not covered by our method in its current version.

Extremal vectors generate the kernel of a canonical contravariant form on a Verma module,
which is a specialization of the "universal" Shapovalov form on the Borel subalgebra [4]. This form itself is extremely important and has numerous applications, see e.g. [11, 12, 13, 14]. For a generic weight, the Verma module is irreducible and the form is non-degenerate. The inverse form gives rise to an element $S$ of extended tensor product of positive and negative subalgebras of the (quantized) universal enveloping algebra. Sending the positive leg of $S$ to a representation yields a matrix with entries in the negative subalgebra which we call Shapovalov matrix.

Our approach consists in relating $\theta_{\beta,m}$ with entries of the Shapovalov matrix, which is explicitly known for all classical and quantum groups. It was obtained in [6] in generalization of Nagel-Moshinski expression for raising and lowering operators of $\mathfrak{sl}(n)$ [15]. It can also be derived (in the quantum setting) from the ABRR equation on the dynamical twist [16, 11]. Our method provides not only factorization of $\theta_{\beta,m}$ to a product of (possibly shifted) $\theta_{\beta,1}$ but also an efficient recipe for description of $\theta_{\beta,1}$ in a very elementary way, by a generalized Nagel-Moshinsky rule.

Our approach is absolutely parallel for a classical semi-simple Lie algebra $\mathfrak{g}$ and its Drindel-Djimbo quantum group. The classical case can be done directly or obtained as the limit case $q \to 1$ of the deformation parameter. Let us describe the method in more detail.

With a finite dimensional module $V$ and a pair of non-zero vectors $v, f_\beta v \in V$ we associate a Shapovalov matrix element which belongs to the negative Borel subalgebra rationally extended over the Cartan subalgebra. Under certain assumptions on $V$ and $v$, such matrix elements deliver factors in $\theta_{\beta,m}$. These factors normalize positive root vectors of the semi-simple subalgebra $\mathfrak{l} \subset \mathfrak{g}$ whose negative counterparts annihilate $v$. This way they become lowering operators in the Mickelsson algebras of the pair $(\mathfrak{g}, \mathfrak{l})$, [17].

The vector $f_\beta v$ determines a homomorphisms $V_{\lambda_2} \to V \otimes V_{\lambda_1}$, where $V_{\lambda_i}$ are irreducible Verma modules of highest weights $\lambda_i$ and $\lambda_2 - \lambda_1$ is the weight of $f_\beta v$. Factorization of $\theta_{\beta,m}$ follows from factorization of the matrix element of the pair $v^{\otimes m}, (f_\beta v)^{\otimes m}$, and from the chain of homomorphisms

$$V_{\lambda_m} \to V \otimes V_{\lambda_{m-1}} \to \ldots \to V^{\otimes m} \otimes V_{\lambda_0}.$$ 

The vector $v$ should be highest for the support of $\beta$ (the minimal simple subalgebra in $\mathfrak{g}$ that accommodates $\beta$) and generate a 2-dimensional submodule of the subalgebra generated by the root spaces $\mathfrak{g}_{\pm \beta}$. These conditions are feasible for all $\beta$ but the three exceptional roots mentioned above.

As a result, we obtain $\theta_{\beta,m}(\lambda)$ as a product $\prod_{i=0}^{m-1} \theta_{\beta,1}(\lambda_i)$ with $\lambda_0 = \lambda$. The factors $\theta_{\beta,1}$ can be calculated by the generalized Nagel-Moshinsky rule (3.4); that is done in the last section of the paper. Viewed as an element of the Borel subalgebra, $\theta_{\beta,m}$ becomes a product of shifted $\theta_{\beta,1}$, by the weight of $v$. This shift degenerates to trivial if $\beta$ contains a simple root $\alpha$ of the
same length with multiplicity 1. In this case $v$ can be chosen of weight $\omega_\alpha$, where $\omega_\alpha$ is the corresponding fundamental generator of the weight lattice. In that case, $\theta_{\beta,m}$ becomes a power of $\theta_{\beta,1}$. The element $\theta_{\beta,1}$ is a Mickelsson generator for the pair $(g, l)$, where simple roots of $l$ are complementary to $\alpha$. Different $\alpha$ results in different presentations.

Except for the last section, we present only the $q$-version of the theory. The classical case can be obtained by sending $q$ to 1. The expression for $\theta_{\beta,1}$ is greatly simplified for $q = 1$, so we give a special consideration to this case in the last section.

## 2 Preliminaries

Let $g$ be a simple complex Lie algebra and $h \subset g$ its Cartan subalgebra. Fix a triangular decomposition $g = g_- \oplus h \oplus g_+$. Denote by $R \subset h^*$ the root system of $g$, and by $R^+$ the subset of positive roots with basis $\Pi$ of simple roots. The basis $\Pi$ generates a root lattice $\Gamma \subset h^*$ with the positive semigroup $\Gamma^+ = \mathbb{Z}_+\Pi \subset \Gamma$.

Choose an ad-invariant form $(\cdot, \cdot)$ on $g$, restrict it to $h$, and transfer to $h^*$ by duality. For every $\lambda \in h^*$ there is a unique element $h_\lambda \in h$ such that $\mu(h_\lambda) = (\mu, \lambda)$, for all $\mu \in h^*$. For a non-zero $\mu \in h^*$ set $\mu^\vee = \frac{2}{(\mu, \mu)}\mu$ and $h_\mu^\vee = \frac{2}{(\mu, \mu)}h_\mu$.

Let $\omega_\alpha, \alpha \in \Pi$ denote fundamental weights determined by equations $(\omega_\alpha, \beta^\vee) = \delta_{\alpha, \beta}$, for all $\alpha, \beta \in \Pi$.

Fix a non-zero complex number $q$ that is not a root of unity and set $[z]_q = \frac{z - q^{-z}}{q - q^{-1}}$ for $z \in h + \mathbb{C}$. The standard Drinfeld-Jimbo quantum group $U_q(g)$ was introduced in [18, 19]. It is a complex Hopf algebra with the set of generators $e_\alpha, f_\alpha$, and $q^{\pm h_\alpha}, \alpha \in \Pi$, satisfying relations

$$q^{h_\alpha}e_\beta = q^{(\alpha, \beta)}e_\beta q^{h_\alpha}, \quad [e_\alpha, f_\beta] = \delta_{\alpha, \beta}[h_\alpha]_q, \quad q^{h_\alpha}f_\beta = q^{-(\alpha, \beta)}f_\beta q^{h_\alpha}, \quad \alpha, \beta \in \Pi.$$  

The elements $q^{h_\alpha}$ are invertible, with $q^{h_\alpha}q^{-h_\alpha} = 1$, while $\{e_\alpha\}_{\alpha \in \Pi}$ and $\{f_\alpha\}_{\alpha \in \Pi}$ also satisfy quantized Serre relations, see [20] for details.

A Hopf algebra structure on $U_q(g)$ is introduced by the comultiplication

$$\Delta(f_\alpha) = f_\alpha \otimes 1 + q^{-h_\alpha} \otimes f_\alpha, \quad \Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha}, \quad \Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha$$

set up on the generators and extended as an algebra homomorphism $U_q(g) \rightarrow U_q(g) \otimes U_q(g)$. The antipode is an algebra anti-automorphism of $U_q(g)$ that acts on the generators by the assignment

$$\gamma(f_\alpha) = -q^{h_\alpha}f_\alpha, \quad \gamma(q^{\pm h_\alpha}) = q^{\mp h_\alpha}, \quad \gamma(e_\alpha) = -e_\alpha q^{-h_\alpha}.$$  

The counit homomorphism $\epsilon: U_q(g) \rightarrow \mathbb{C}$ returns

$$\epsilon(e_\alpha) = 0, \quad \epsilon(f_\alpha) = 0, \quad \epsilon(q^{h_\alpha}) = 1.$$
Denote by $U_q(h), U_q(g_+), \text{and } U_q(g_-)$ subalgebras in $U_q(g)$ generated by \( \{ q^{\pm h_a} \}_{a \in \Pi}, \{ e_a \}_{a \in \Pi}, \text{and } \{ f_a \}_{a \in \Pi} \), respectively. The quantum Borel subgroups are defined as $U_q(b_\pm) = U_q(g_\pm)U_q(h)$; they are Hopf subalgebras in $U_q(g)$. We will also need their extended version $\hat{U}_q(b_\pm) = U_q(g_\pm)\hat{U}_q(h)$, where $\hat{U}_q(h)$ is the ring of fractions of $U_q(h)$ over the multiplicative system generated by $[h_\alpha - c]_q$ with $\alpha \in \Gamma_+$ and $c \in \mathbb{Q}$.

We extend the notation $f_\alpha, e_\alpha$ to all $\alpha \in \mathbb{R}^+$ meaning the Lusztig root vectors with respect to some normal ordering of $\mathbb{R}^+$, [20]. They are known to generate a Poincare-Birkhoff-Witt (PBW) basis in $U_q(g_\pm)$.

Given a $U_q(g)$-module $V$, a non-zero vector $v$ is said to be of weight $\mu$ if $q^{h_a}v = q^{(\mu, \alpha)}v$ for all $\alpha \in \Pi$. The linear span of such vectors is denoted by $V[\mu]$. A module $V$ is said to be of highest weight $\lambda$ if it is generated by vector $v \in V[\lambda]$ that is killed by all $e_\alpha$. The vector $v$ is called highest; it is defined up to a non-zero scalar multiplier.

We consider an involutive coalgebra anti-automorphism and algebra automorphism $\sigma : U_q(g) \rightarrow U_q(g)$ setting it on the generators by the assignment

$$
\sigma : e_\alpha \mapsto f_\alpha, \quad \sigma : f_\alpha \mapsto e_\alpha, \quad \sigma : q^{h_a} \mapsto q^{\sigma(h_a)}.
$$

The involution $\omega = \gamma^{-1} \circ \sigma = \sigma \circ \gamma$ is an algebra anti-automorphism and preserves comultiplication.

A symmetric bilinear form $(\_, \_)$ on a $g$-module $V$ is called contravariant if $(\omega(x), y) = (x, \omega(y))$ for all $x, y \in U_q(g)$. A module of highest weight has a unique $\mathbb{C}$-valued contravariant form such that the highest vector has squared norm 1. We call this form canonical and apply this term to the form on tensor products that is the product of canonical forms on tensor factors. This is consistent because $\omega$ is a coalgebra map.

Let us recall the definition of the Shapovalov of $U_q(h)$-valued Shapovalov form on the Borel subalgebra $U_q(b_-)$, [4]. Regard $U_q(b_-)$ as a free right $U_q(h)$-module generated by $U_q(g_-)$. The triangular decomposition $U_q(g) = U_q(g_-)U_q(h)U_q(g_+)$ facilitates projection $\varphi : U_q(g) \rightarrow U_q(h)$ along the sum $g_-U_q(g) + U_q(g)g_+$, where $g_-U_q(g)$ and $U_q(g)g_+$ are right and left ideals generated by the negative and positive generators, respectively. Set

$$(x, y) = \varphi(\omega(x)y), \quad x, y \in U_q(g).$$

This form is $U_q(h)$-linear and contravariant. It follows that the left ideal $U_q(g)g_+$ is in the kernel, so the form descends to a form on the quotient $U_q(g)/U_q(g)g_+ \simeq U_q(b_-)$.

A Verma module $V_\lambda$ is an induced module $U_q(g) \otimes_{U_q(b_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the 1-dimensional $U_q(b_+)$-module that is trivial on $U_q(g_+)$ and returns weight $\lambda$ on $U_q(h)$. Its highest vector of weight $\lambda$ is denoted by $v_\lambda$, which is also called vacuum vector. It freely generates $V_\lambda$ as a module over $U_q(g_-)$.
Specialization of the Shapovalov form at $\lambda \in \mathfrak{h}^*$ gives the canonical contravariant $C$-valued form $(x, y)_\lambda = \lambda \left( \varphi(\omega(x)y) \right)$ on $V_\lambda$, upon a natural isomorphism $U_q(\mathfrak{g}_-) \simeq V_\lambda$ of $U_q(\mathfrak{g}_-)$-modules generated by the assignment $1 \mapsto v_\lambda$. Conversely, the canonical contravariant form on $V_\lambda$ regarded as a function of $\lambda$ descends to the Shapovalov form if one views $U_q(\mathfrak{h})$ as the algebra of polynomial functions on $\mathfrak{h}^*$. By an abuse of terminology, we also mean by Shapovalov form the canonical contravariant form on $V_\lambda$.

It is known that the contravariant form on a Verma module goes degenerate if and only if its highest weight is in the union of $H_{\beta, m} = \{ \lambda \mid q^{2(\lambda + \rho, \beta) - m(\beta, \beta)} = 1 \}$ over $\beta \in \mathbb{R}^+$ and $m \in \mathbb{N}$, where $\rho$ is the half-sum of positive roots, [21]. In the classical case $q = 1$, $H_{\beta, m}$ becomes a Kac-Kazhdan hyperplane of weights satisfying $2(\lambda + \rho, \beta) = m(\beta, \beta)$.

Recall that a vector $v \in V_\lambda$ of weight $\lambda - \mu$ with $\mu \in \Gamma_+, \mu \neq 0$, is called extremal if $e_\alpha v = 0$ for all $\alpha \in \Pi$. We call its image under the isomorphism $V_\lambda \to U_q(\mathfrak{g}_-)$ a Shapovalov element. Extremal vectors are in the kernel of the contravariant form and generate submodules of the corresponding highest weights. We will be interested in the special case when $\mu = m\beta$ with $\beta \in \mathbb{R}^+$ and $m \in \mathbb{N}$. Then the highest weight $\lambda$ has to be in $H_{\beta, m}$.

For simple $\beta$ the Shapovalov element $\theta_{\beta, m}$ is just the $m$-th power of the root vector, $\theta_{\beta, m} = f_{\beta}^m$. That is not the case for compound $\beta$. The goal of this work is to find explicit expressions for $\theta_{\beta, m}$ when $\beta$ is compound.

3 Shapovalov inverse and its matrix elements

Define the opposite $U_q(\mathfrak{g})$-module $V'_{\lambda}$ of lowest weight $-\lambda$ as follows. The underlying vector space of $V'_{\lambda}$ is taken to be $V_\lambda$, while the representation homomorphism $\pi'_{\lambda}$ is twisted by $\sigma$, that is $\pi'_{\lambda} = \pi_{\lambda} \circ \sigma$. The module $V'_{\lambda}$ is freely generated over $U_q(\mathfrak{g}_+)$ by its lowest vector $v'_{\lambda}$.

Let $\sigma_{\lambda} : V_\lambda \to V'_{\lambda}$ denote the isomorphism of vector spaces, $xv_{\lambda} \mapsto \sigma(x)v'_{\lambda}$, $x \in U_q(\mathfrak{g}_-)$. It intertwines the representations homomorphism, $\pi'_{\lambda} \circ \sigma = \sigma_{\lambda} \circ \pi_{\lambda}$. The map $\sigma_{\lambda}$ relates the contravariant form on $V_\lambda$ with an invariant pairing $V_\lambda \otimes V'_{\lambda} \to V_\lambda \otimes V_\lambda \to C$.

Suppose that the module $V_\lambda$ is irreducible. Then its invariant pairing is non-generate (as well as the contravariant form on $V_\lambda$). The inverse form is an element of a completed tensor product $V'_{\lambda} \hat{\otimes} V_\lambda$. Under the isomorphisms $V_\lambda \to U_q(\mathfrak{g}_-)$, $V'_{\lambda} \to U_q(\mathfrak{g}_+)$, it goes to an element that we denote by $S \in U_q(\mathfrak{g}_+) \hat{\otimes} U_q(\mathfrak{g}_-)$ and call universal Shapovalov matrix. Varying the highest weight $\lambda$ we get a rational trigonometric dependance of $S$. As a function of $\lambda$, $S$ is regarded as an element of $U_q(\mathfrak{g}_+) \hat{\otimes} U_q(\mathfrak{b}_-)$. In other words, the weight dependance is accommodated by the right tensor leg of $S$. 


Given a finite dimensional $U_q(\mathfrak{g})$-module with representation homomorphism $\pi: U_q(\mathfrak{g}) \to \text{End}(V)$ the image $S = (\pi \otimes \text{id})(\mathcal{S})$ is a matrix with entries in $U_q(\mathfrak{g}_-)$. One can work directly with a rational trigonometric operator function $S$ and forget that it came from $\mathcal{S}$.

An explicit expression of $S$ in a weight basis $\{v_i\}_{i \in I} \subset V$ can be given in terms of Hasse diagram $\mathfrak{H}(V)$. Such a diagram is associated with any partially ordered sets. Arrows are simple root vectors $e_\alpha$ connecting basis elements $v_j \leftarrow v_i$ whose weight difference is $\nu_j - \nu_i = \alpha$. We introduce a partial order on $\{v_i\}_{i \in I}$ by writing $v_i \succ v_j$ if the inclusion $\nu_i - \nu_j \in \Gamma_+ \setminus \{0\}$ holds for their weights. The matrix $S$ is triangular: $s_{ii} = 1$ and $s_{ij} = 0$ if $v_i$ is not succeeding $v_j$. The entry $s_{ij}$ is a rational trigonometric function $\mathfrak{h}^* \to U_q(\mathfrak{g}_-)$. Its value carries weight $\nu_j - \nu_i \in -\Gamma_+$.

The matrix $S$ depends only on the $U_q(\mathfrak{h}_+)$-module structure on $V$. Therefore, to calculate a matrix element $s_{ij}$, one can choose a weight basis that extends a weight basis in the submodule $U_q(\mathfrak{g}_+)v_j$. Then, in particular, $s_{ij} = 0$ if $v_i \not\in U_q(\mathfrak{g}_+)v_j$.

We recall a construction of $S$ following [6]. Let $\{h_{i\alpha}\}_{i=1}^{1kq} \in \mathfrak{h}$ be an orthonormal basis. The element $q \sum h_{i\alpha} \otimes h_i$ belongs to a completion of $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{h})$ in the $h$-adic topology, where $h = \ln q$. Choose an R-matrix of $U_q(\mathfrak{g})$ such that $\mathcal{R} = q^{-\frac{1}{2}} \sum h_{i\alpha} \otimes h_i \mathcal{R} \in U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{g}_-)$ and set $C = \frac{1}{q^{-q^{-1}}} (\mathcal{R} - 1 \otimes 1)$. The key identity on $C$ that facilitates the $q$-version of the theory is [6]

$$[1 \otimes e_\alpha, C] + (e_\alpha \otimes q^{-h_\alpha})C - C(e_\alpha \otimes q^{h_\alpha}) = e_\alpha \otimes [h_\alpha]_q, \quad \forall \alpha \in \Pi^+. \tag{3.1}$$

In the classical limit, $C = \sum_{\alpha \in R^+} e_\alpha \otimes f_\alpha$ is the polarized split Casimir without its Cartan part. One then recovers

$$[1 \otimes e_\alpha, C] + [e_\alpha \otimes 1, C] = e_\alpha \otimes h_\alpha \tag{3.2}$$

for each simple root $\alpha$.

We rectify the Hasse diagram and the partial ordering by removing arrows $v_i \leftarrow v_j$ if $C_{ij} = 0$. This will not affect the formula (3.4) for matrix elements.

For each weight $\mu \in \Gamma_+$ put

$$\eta_\mu = h_\mu + (\mu, \rho) - \frac{1}{2}(\mu, \mu) \in \mathfrak{h} \oplus \mathbb{C}. \tag{3.3}$$

Regard it as an affine function on $\mathfrak{h}^*$ by the assignment $\eta_\mu : \zeta \mapsto (\mu, \zeta + \rho) - \frac{1}{2}(\mu, \mu), \zeta \in \mathfrak{h}^*$. Observe that $\eta_{m\beta} = m(h_\beta + (\beta, \rho) - \frac{m}{2}(\beta, \beta))$. That is, $[\eta_{m\beta}(\lambda)]_q$ vanishes on $\mathcal{H}_{\beta, m}$ (and only on the Kac-Kazhdan hyperplane in the classical case).

For a pair of non-zero vectors $v, w \in V$ define a matrix element $\langle w|v \rangle = (w, S_1v)S_2 \in U_q(\mathfrak{g}_-)$, where $S_1 \otimes S_2$ stands for a Sweedler-like notation for $\mathcal{S}$ and the pairing is with respect to the canonical contravariant form on $V$. Its specialization at a weight $\lambda$ is denoted by $\langle w|v \rangle_\lambda$. For each
w, the map \( V \to V_\lambda \), \( v \mapsto \langle v|w\rangle v_\lambda = \langle w|v\rangle_\lambda v_\lambda \) satisfies: \( e_\alpha \langle v|w\rangle v_\lambda = \langle \sigma(f_\alpha)v|w\rangle v_\lambda \) for all \( \alpha \in \Pi \). This is a consequence of \( U_q(\mathfrak{g}_+)-\)invariance of the element \( S(1 \otimes v_\lambda) \in U_q(\mathfrak{g}_+) \otimes V_\lambda \).

Let \( c_{ij} \) denote the entries of the matrix \((\pi \otimes \text{id})(C)\) in an orthonormal weight basis \( \{v_i\}_{i \in I} \in V \). The entries \( s_{ij} \) of the matrix \( S \) are the matrix elements \( \langle v_i|v_j\rangle \). Fix a "start" node \( v_a \) and an "end" node \( v_b \) such that \( v_b \succ v_a \). Then the re-scaled matrix element \( \hat{s}_{ab} = -s_{ba}[\eta_{\nu_b-v_a}]q^{-\nu_b-v_a} \) can be calculated by the formula
\[
\hat{s}_{ba} = c_{ba} + \sum_{k \geq 1} \sum_{v_{i_k} \succ v_{i_{k-1}} \succ \ldots \succ v_{i_1} \succ v_a} c_{bk} \ldots c_{1a} \frac{(-1)^k q^{\nu_{i_k}} \cdots q^{\nu_{i_1}}}{[\eta_{\nu_b}]_q \cdots [\eta_{\nu_1}]_q} \in \hat{U}(\mathfrak{b}_-),
\]
where \( \mu_l = \nu_l - \nu_a \in \Gamma_+ \), \( l = 1, \ldots, k \). Here the summation is performed over all possible routes from \( v_a \) to \( v_b \), see [6] for details.

It is straightforward that \( U_q(\mathfrak{g}_+)-\)invariance of the tensor \( S(v_a \otimes v_\lambda) \) implies
\[
e_\alpha \hat{s}_{ba}(\lambda)v_\lambda \propto [\eta_{\nu_b-v_a}(\lambda)]_q \sum_k \pi(f_\alpha)_{kb} s_{ka}(\lambda)v_\lambda.
\]

(3.5)

It follows that \( \hat{s}_{ba}(\lambda)v_\lambda \) is an extremal vector in \( V_\lambda \) for \( \lambda \) satisfying \( [\eta_{\nu_b-v_a}(\lambda)]_q = 0 \) provided

1. \( \hat{s}_{ba}(\lambda) \neq 0 \),

2. \( \lambda \) is a regular point for all \( s_{ka}(\lambda) \) and all \( \alpha \).

We aim to find an appropriate matrix element for \( \theta_{\beta,m} \) that satisfies these conditions.

Let \( V \) be a finite dimensional \( U_q(\mathfrak{g}) \)-module with a pair of weight vectors \( v_a, v_b \in V \) such that \( v_a = f_\beta v_b \) for \( \beta \in R^+ \). We call the triple \((V, v_a, v_b)\) a \( \beta \)-representation.

**Proposition 3.1.** Let \((V, v_a, v_b)\) be a \( \beta \)-representation for \( \beta \in R^+ \). Then for generic \( \lambda \in H_{\beta,1} \) the vector \( \hat{s}_{ba}(\lambda)v_\lambda \in V_\lambda \) is extremal.

**Proof.** The factors \( q^{\nu_{i_k}}_{[\nu_{i_k}]_q} \) in (3.4) go singular on the union of a finite number of the null-sets \( \{\lambda \mid [\eta_\mu(\lambda)]_q = 0\} \). None of them coincides with the \( H_{\beta,1} \), hence \( \hat{s}_{ba}(\lambda) \) is regular at generic \( \lambda \in H_{\beta,1} \). By the same reasoning, all \( s_{ka}(\lambda) \) are regular at such \( \lambda \). Finally, the first term \( c_{ba} \) (and only this one) involves the Lusztig root vector \( f_\beta \), a generator of a PBW basis in \( U_q(\mathfrak{g}_-) \). It is therefore independent of the other terms, and \( \hat{s}_{ba}(\lambda) \neq 0 \). \( \square \)

Upon identification of rational \( U_q(\mathfrak{g}_-) \)-valued functions on \( \mathfrak{h}^* \) with \( \hat{U}(\mathfrak{b}_-) \) we conclude that \( \hat{s}_{ba} \) is a Shapovalov element \( \theta_{\beta,1} \) and denote it by \( \theta_\beta \). Uniqueness of extremal vector of given weight implies that all matrix elements \( \hat{s}_{ba} \) with \( v_a = f_\beta v_\beta \) deliver the same \( \theta_\beta \), up to a scalar factor. However, when we aim at \( \theta_{\beta,m} \) with \( m > 1 \), we have to choose matrix elements for \( \theta_\beta \) more carefully.
It was relatively easy to secure the above two conditions in the case of \( m = 1 \). For higher \( m \) we will choose a different strategy: we will satisfy the first condition by the very construction and bypass a proof of the second with different arguments.

## 4 Factorization of Shapovalov elements

For a positive root \( \beta \in \Pi \) denote by \( \Pi_\beta \subset \Pi \) the set of simple roots entering the expansion of \( \beta \) over the basis \( \Pi \) with positive coefficients. A simple Lie subalgebra \( \mathfrak{g}(\beta) \subset \mathfrak{g} \) generated by \( e_\alpha, f_\alpha \) with \( \alpha \in \Pi_\beta \) is called support of \( \beta \). Its universal enveloping algebra is quantized as a Hopf subalgebra in \( U_q(\mathfrak{g}) \). Since \([e_\mu, f_\nu] = 0\) for \( \nu \in \Pi_\beta \) and \( \nu \in \Pi \setminus \Pi_\beta \), we can restrict to \( \mathfrak{g} = \mathfrak{g}(\beta) \) without loss of generality.

### Definition 4.1.

Let \( \beta \in \mathbb{R}^+ \) be a positive root and \((V, v_a, v_b)\) a \( \beta \)-representation such that \( e_\alpha v_b = 0 \) for all \( \alpha \in \Pi_\beta \) and \( (v_b, \beta^\vee) = 1 \). We call such \( \beta \)-representation admissible.

In other words, a triple \((V, v_a, v_b)\) is admissible if \( v_b \) is the highest vector of a \( U_q(\mathfrak{g}(\beta)) \)-submodule in \( V \) that generates a 2-dimensional submodule of the \( U_q(\mathfrak{sl}(2)) \)-subalgebra generated by \( f_\beta, e_\beta \). It is clear that if a root has an admissible representation, then one of the fundamental representations is admissible. It is also clear that \( v_a \) and \( v_b \) can be included in an orthonormal weight basis in \( V \).

Assuming the triple \((V, v_a, v_b)\) admissible, denote by \( V^{(m)} \) the irreducible (finite-dimensional) \( U_q(\mathfrak{g}) \)-module with highest weight \( m \nu_a \) and highest vector \( v^m_b \). The weight \( m \nu_a \) is related with \( m \nu_b \) by the reflection \( \sigma_\beta : \beta \rightarrow -\beta \) from the Weyl group, hence \( \dim V^{(m)}[m \nu_a] = 1 \). Set \( v^m_a = f^m_{\beta} v^m_b \in V^{(m)}[m \nu_a] \). This is a non-zero vector.

A factorization for \( \theta_{\beta, m} \) we are seeking for is a consequence of the following factorization of Shapovalov matrix elements in tensor product modules.

### Lemma 4.2.

Let \((V, v_a, v_b)\) be an admissible \( \beta \)-representation. Then for all \( m \in \mathbb{N} \),

\[
\langle v^m_b | v^m_a \rangle_{\lambda_0} = c \langle v_b | v_a \rangle_{\lambda_{m-1}} \cdots \langle v_b | v_a \rangle_{\lambda_0}
\]

where \( \lambda_k = \lambda + k \nu_a \in \mathfrak{h}^* \) for \( k \in \mathbb{Z}_+ \) and \( c \) is a non-zero scalar.

**Proof.** We realize \( V^{(m)} \) as a submodule in the tensor product \( V^\otimes m \) generated by the highest vector \( v^m_b = v^\otimes m_b \). Let \( \lambda \) be such that all Verma modules \( V_{\lambda k} \) of highest weights \( \lambda_k, k = 0, \ldots, m - 1 \), are irreducible and consider a chain of module homomorphisms

\[
V_{\lambda_m} \rightarrow V \otimes V_{\lambda_{m-1}} \rightarrow \cdots \rightarrow V^\otimes m \otimes V_{\lambda_0}.
\]
They send the highest vectors \( v_{\lambda k} \in V_{\lambda k} \) to the extremal vectors \( S(v_{a} \otimes v_{\lambda k-1}) \in V \otimes V_{\lambda k-1} \). The highest vector \( v_{\lambda m} \) goes over to \( w_{m} \otimes v_{\lambda 0} \), where \( w_{m} \in V^{\otimes m} \) is of weight \( mv_{a} \). It is related with \( v_{a}^{\otimes m} \) by an invertible operator from \( \text{End}(V^{\otimes m}) \), which is \( m-1 \)-fold dynamical twist \([11]\).

Pair \( S(w_{m} \otimes v_{a}) \) with \( v_{b}^{\otimes m} \) and calculate \( \langle v_{b}^{\otimes m}|w_{m}\rangle_{\lambda 0} \):

\[
\left( v_{b}^{\otimes(m-1)}, \langle v_{b}|v_{a}\rangle_{\lambda_{m-1}}^{(1)} S \right) w_{m-1}^{(1)} \left( \langle v_{b}|v_{a}\rangle_{\lambda_{m-1}}^{(2)} S_{2} \right) (\lambda_{0}) = \langle v_{b}|v_{a}\rangle_{\lambda_{m-1}}^{(2)} \left( \omega \left( \langle v_{b}|v_{a}\rangle_{\lambda_{m-1}}^{(1)} v_{b}^{\otimes(m-1)}|w_{m-1}\right)_{\lambda_{0}},
\]

where we use the Sweedler notation \( \Delta(x) = x^{(1)} \otimes x^{(2)} \in U_q(b_-) \otimes U_q(g_-) \) for the coproduct of \( x \in U_q(g_-) \). Since \( yq^{h_q}v_{b} = \epsilon(y)q^{(\alpha,\beta)}v_{b} \) for all \( y \in U_q(g_+) \) and \( \alpha \in \Gamma_+ \), we arrive at

\[
\langle v_{b}^{\otimes m}|w_{m}\rangle_{\lambda 0} = q^{-\beta_v} \langle v_{b}|v_{a}\rangle_{\lambda_{m-1}}^{(2)} \langle v_{b}^{\otimes(m-1)}|w_{m-1}\rangle_{\lambda_{0}}.
\]

Proceeding by induction on \( m \) we conclude that \( \langle v_{b}^{\otimes m}|w_{m}\rangle_{\lambda 0} \) equals the right-hand side of (4.6), up to the factor \( q^{-m(\beta,\nu_b)} \). Finally, we replace \( w_{m} \) with its orthogonal projection to \( V^{(m)} \), which is proportional to \( v_{a}^{m} \) because \( \dim V^{(m)}[mv_{a}] = 1 \). This proves the lemma for generic and hence for all \( \lambda \) where the right-hand side of (4.6) makes sense.

It follows from the above factorization that the least common denominator of the extremal vector \( u = S(v_{a}^{m} \otimes v_{\lambda}) \in V^{(m)} \otimes V_{\lambda} \) contains \( d(\lambda) = [\eta_{\beta}(\lambda + (m-1)v_{a})]_{q} = [(\lambda + \rho) - \frac{m}{2}(\beta,\beta)]_{q}. \)

It comes from the leftmost factor \( \langle v_{b}|v_{a}\rangle_{\lambda_{m-1}} \) in the right-hand side of (4.6). Denote by \( s_{v_{b}^{m},v_{a}^{m}}(\lambda) \) the matrix element \( \langle v_{b}^{m}|v_{a}^{m}\rangle_{\lambda} \). Since \( d \) divides \( [\eta_{m\beta}]_{q} \), the re-scaled matrix element

\[
\tilde{s}_{v_{b}^{m},v_{a}^{m}}(\lambda) = c(\lambda) d(\lambda) s_{v_{b}^{m},v_{a}^{m}}(\lambda) \propto \prod_{k=0}^{m-1} \theta(\lambda_k),
\]

where \( c(\lambda) = -q^{-\eta_{m\beta}(\lambda_{m-1})} \frac{[m_{\lambda_{m}}(\lambda)]_{q}}{d(\lambda)} \), is regular and does not vanish at generic \( \lambda \in H_{\beta,m} \) because \( d(\lambda) \) cancels the pole in \( \langle v_{b}|v_{a}\rangle_{\lambda_{m-1}} \). Put \( \tilde{u} = d^{k}(\lambda)u \), where \( k \geq 1 \) is the maximal degree of this pole in \( u \). It is an extremal vector in \( V^{(m)} \otimes V_{\lambda} \) that is regular at generic \( \lambda \in H_{\beta,m} \).

**Theorem 4.3.** For generic \( \lambda \in H_{\beta,m} \), \( \theta_{m,\beta}(\lambda) \propto \tilde{s}_{v_{b}^{m},v_{a}^{m}}(\lambda) \).

**Proof.** The vector \( \tilde{u} \) is presentable as

\[
\tilde{u} = v_{b}^{m} \otimes d^{k}(\lambda)v_{\lambda} + \ldots + v_{a}^{m} \otimes d^{k-1}(\lambda)c(\lambda)\tilde{s}_{v_{b}^{m},v_{a}^{m}}(\lambda)v_{\lambda}.
\]

We argue that \( \tilde{u} = v_{a}^{m} \otimes c(\lambda)\tilde{s}_{v_{b}^{m},v_{a}^{m}}(\lambda)v_{\lambda} \) for generic \( \lambda \) in \( H_{\beta,m} \), where \( d(\lambda) = 0 \). Indeed, the \( V_{\lambda} \)-components of \( \tilde{u} \) span a \( U_q(g_{+}) \)-submodule in \( V_{\lambda} \) isomorphic to a quotient of the \( V^{(m)} \)-dual. A vector of maximal weight in this span is extremal and distinct from \( v_{\lambda} \). But \( \theta_{\beta,m}(\lambda)v_{\lambda} \) is the only, up to a factor, extremal vector in \( V_{\lambda} \), for generic \( \lambda \). Therefore \( k = 1 \) and \( \theta_{\beta,m} \propto \tilde{s}_{v_{b}^{m},v_{a}^{m}}. \)
One can pass to the "universal form" of $\theta_\beta$ regarding it as an element of $\hat{U}_q(b_-)$. Then

$$\theta_{\beta,m} = (\tau_{v_0}^{m-1}\theta_\beta) \ldots (\tau_{v_0}\theta_\beta) \theta_\beta,$$

(4.7)

where $\tau_\nu$ is an automorphism of $\hat{U}_q(\mathfrak h)$ generated by the affine shift of the weight $\nu$, that is, $(\tau_\nu \varphi)(\mu) = \varphi(\mu + \nu)$, $\varphi \in \hat{U}_q(\mathfrak h)$, $\mu \in \mathfrak h^*$. One may ask when the shift is trivial, $\tau_{v_0}\theta_\beta = \theta_\beta$, and $\theta_{\beta,m}$ is just the $m$-th power of $\theta_\beta$.

**Proposition 4.4.** Let $\beta$ be a positive root. Suppose that there is $\alpha \in \Pi$ of the same length as $\beta$ that enters the expansion of $\beta$ over the basis $\Pi$ with multiplicity 1. Then $\theta_{\beta,m} = \theta_\beta^m \in U_q(b_-)$.

**Proof.** Let $I \subset g(\beta)$ be the semi-simple subalgebra generated by simple root vectors $f_\mu, e_\mu$ with $\mu \neq \alpha$. Take for $V$ the fundamental module of highest weight $\omega_\alpha$. Put $v_b$ to be the highest vector and $v_a = f_\beta v_b$. Then $(V,v_a,v_b)$ is an admissible $\beta$-representation because $(\nu_b,\beta) = (\omega_\alpha,\alpha) = \frac{(\alpha,\alpha)}{2} = \frac{(\beta,\beta)}{2}$. We write (3.4) as

$$\theta_\beta(\lambda) = c_{ba} + \sum_{v_b \succ v_i \succ v_a} c_{bi}s_{ia}(\lambda).$$

The highest vector $v_b$ is killed by $I_-$, therefore the Hasse diagram between $v_a$ and $v_b$ looks

$$v_b \xleftarrow{e_\alpha} f_\alpha v_b \ldots v_a,$$

where arrows in the suppressed part are simple root vectors from $U_q(I_+)$. But then the only copy of $f_\alpha$ is in $c_{ia}$ while all $s_{ia}$ belong to $U_q(I_-\hat{U}_q(\mathfrak h_I))$, the extended Borel subalgebra of $U_q(I)$.

Finally, since $\Pi_I$ is orthogonal to $v_b$, we have $(\mu,\nu_a) = - (\mu,\beta)$ for all $\mu \in R^+_I$. Therefore

$$\theta_\beta(\lambda_k) = \theta_\beta(\lambda - k\beta), \quad \theta_{\beta,m}(\lambda) = \prod_{k=0}^{m-1} \theta_\beta(\lambda - k\beta),$$

where the product is taken in the descending order from left to right. This proves the plain power factorization because each $\theta_\beta$ carries weight $-\beta$.

**Remark 4.5.** Here are a few comments on the conditions of Lemma 4.2 and Proposition 4.4.

1. In the case when $g$ is one of the four classical types, there is an admissible representation for each compound root $\beta \in R^+_g$. It can be realized in the minimal fundamental module for all $\beta$ except for a short root of $\mathfrak{so}(2n+1)$, in which case the fundamental spin module does the job, [22]. The conditions of Proposition (4.4) hold in all these cases except for a long root of $\mathfrak{sp}(2n)$. In the latter case, one should take for $V$ the fundamental module of highest weight $\omega_\alpha$, where $\alpha$ is the simple long root.
2. If \( g \) is simply laced and \( \beta \) contains a simple root \( \alpha \) with multiplicity 1, the fundamental module with highest weight \( \omega_\alpha \) satisfies the conditions of Proposition 4.4. That covers all roots of \( \mathfrak{c}_6 \) and \( \mathfrak{c}_7 \) and all but the maximal root of \( \mathfrak{c}_8 \). If all multiplicities of simple roots in \( \beta \) are 2 or higher (as in the maximal root in \( \mathfrak{c}_8 \)), there are not admissible \( \beta \)-representations.

3. There are three roots that have no admissible representations:
   - a short root in \( \mathfrak{g}_2 \) that is a sum of one long and two short simple roots,
   - the root \( \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \) in \( \mathfrak{f}_4 \) with respect to the enumeration \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \),
   - the maximal root of \( \mathfrak{e}_8 \).

In the \( \mathfrak{g}_2 \)- and \( \mathfrak{f}_4 \)-cases, the only simple root with multiplicity 1 in \( \beta \) is longer than \( \beta \). In the simply laced case of \( \mathfrak{c}_8 \), all simple roots enter maximal \( \beta \) with multiplicities \( \geq 2 \).

4. It follows that a root may have a few plain power factorizations of its Shapovalov elements. Say, if \( \beta \) is a root of height \( k \) for \( \mathfrak{g} = \mathfrak{sl}(\mathfrak{n}) \), then \( \theta_{\beta,m} \) admits \( k \) (apparently different) presentations. This is in accordance with the results of Zhelobenko whose presentations are parameterised by normal orderings of positive roots.

5 **Shapovalov elements of degree 1**

In this section we describe the factor \( \theta_\beta \) for a particular admissible \( \beta \)-representation \((V, v_a, v_b)\). To a large extent it reduces to description of the relevant part of the Hasse diagram \( \mathcal{H}(V) \). We give a complete explicit solution in the classical case. In the case of \( q \neq 1 \), we do it up to calculation of the entries of the matrix \( C \).

To that end, we need to figure out the Hasse sub-diagram \( \mathcal{H}(v_b, v_a) \subset \mathcal{H}(V) \) including all possible routes from \( v_a \) to \( v_b \). We argue that \( \mathcal{H}(v_b, v_a) \) can be extracted from a diagram \( \mathcal{H}(\mathfrak{b}_-) \) we associate with the adjoint representation \( \simeq \mathfrak{g} \) as follows. The nodes of \( \mathcal{H}(\mathfrak{b}_-) \) are elements of the Cartan-Weyl basis in \( \mathfrak{b}_- \): \( h^\vee_\alpha, \alpha \in \Pi, \) and \( f_\mu, \mu \in \mathbb{R}^+ \). Here \( h^\vee_\alpha = \frac{2}{(\alpha,\alpha)} h_\alpha \) so that \( \alpha(h^\vee_\alpha) = 2 \).

Arrows are \( f_\mu \overset{e_\alpha}{\leftarrow} f_\nu \) if \([e_\alpha, f_\nu] \propto f_\mu \) and \( h^\vee_\alpha \overset{e_\alpha}{\leftarrow} f_\alpha, \alpha \in \Pi \). For example, in the case of \( \mathfrak{g} = \mathfrak{g}_2 \) we have

\[
\begin{array}{c}
\circ \quad e_{\alpha_1} \quad e_{\alpha_1} \\
\circ \quad f_{\alpha_1} \quad e_{\alpha_1} \\
\circ \quad e_{\alpha_2} \quad f_{\alpha_1} \quad e_{\alpha_2} \\
\circ \quad h^\vee_{\alpha_2} \quad f_{\alpha_2} \quad e_{\alpha_2} \\
\circ \quad h^\vee_{\alpha_1} \quad f_{\alpha_1} \quad e_{\alpha_1} \\
\end{array}
\]

\[
\begin{array}{c}
\circ \quad e_{\alpha_2} \quad e_{\alpha_2} \\
\circ \quad f_{\alpha_1+\alpha_2} \quad f_{\alpha_1+2\alpha_2} \quad f_{\alpha_1+3\alpha_2} \quad f_{2\alpha_1+3\alpha_2} \\
\end{array}
\]

\[
\begin{array}{c}
\mathfrak{b}_- \\
\end{array}
\]
where $\alpha_1$ is the long simple root and $\alpha_2$ is short. This is a part of $\mathfrak{h}(\mathfrak{g})$ without arrows $b^\gamma_\alpha \leftarrow f_\mu$, $\alpha \neq \mu$. For each $\alpha \in \Pi$ and $\beta \in \mathbb{R}^+$, the vector space underlying $\mathfrak{h}(b^\gamma_\alpha, f_\beta)$ is a $\mathfrak{g}_+$-module (a subquotient of $\mathfrak{g}$ by submodules generated by all $e_\mu$ and simple $f_\mu$ with $\mu \neq \alpha$).

**Proposition 5.1.** Let $V$ be a fundamental $\mathfrak{g}$-module with highest weight $\omega_\alpha$ and highest vector $v$. Suppose that $(V, f_\beta v, v)$ is an admissible representation for $\beta \in \mathbb{R}^+$. Then $\dim V[\mu] = 1$ for each weight $\mu$ such that $\omega_\alpha \geq \mu \geq \omega_\alpha - \beta$. Furthermore, the assignment $h^\gamma_\alpha \mapsto v, f_\gamma \mapsto f_\gamma v, \gamma = \omega_\alpha - \mu$, induces an isomorphism of Hasse sub-diagram $\mathfrak{h}(h^\gamma_\alpha, f_\beta) \subset \mathfrak{h}(\mathfrak{b}_-) \subset \mathfrak{h}(V)$. 

**Proof.** It is sufficient to prove it for the case $q = 1$ because the weight structure is independent of $q$. First of all, observe that $f_\gamma v \neq 0$ once $\gamma \leq \beta$, because $(\gamma, \omega_\alpha) \neq 0$. We are left to prove that $\psi_\gamma v \propto f_\gamma v$ for each $\psi \in U(\mathfrak{g}_-)$ of weight $\gamma \in \mathbb{R}^+$. Let $\ell \in \mathbb{N}$ be the multiplicity of $\alpha$ in $\beta$, then $\ell(\alpha, \alpha) = (\beta, \beta)$. Suppose that $\ell = 1$. The weight subspace $V[\omega_\alpha - \gamma]$ with $\omega_\alpha \succ \omega_\alpha - \gamma \succ \omega_\alpha - \beta$ is constructed as follows. For $\gamma = \alpha$ it is $f_\alpha v$. Let $\psi \in U(\mathfrak{g}_-)$ be of weight $\gamma$ which contains $\alpha$ with multiplicity 1. Then $\psi v \propto f_\gamma v$ if $\gamma \in \mathbb{R}^+$ and $\psi v = 0$ otherwise, because all simple root vectors other than $f_\alpha$ annihilate $v$ and every monomial in $\psi$ can be replaced with a composition of commutators. The same is true in the case $\ell = 2$. Indeed, it is sufficient to check it for $f_\alpha \psi_\gamma f_\sigma v$ where $\psi_\gamma$ a monomial in $f_\sigma, \sigma \neq \alpha$ of weight $\gamma \in \mathbb{R}^+$. As we already proved, $\psi_\gamma f_\alpha v$ is proportional to $[f_\gamma, f_\alpha]v$. But then $f_\alpha \psi_\gamma f_\sigma v \propto f_\alpha [f_\gamma, f_\alpha]v = 2f_\alpha f_\gamma f_\alpha v = [f_\alpha, [f_\gamma, f_\alpha]]v$ because $f_\gamma v = 0 = f_\alpha^2 v$. Finally, the case $\ell = 3$ occurs only in $\mathfrak{g} = \mathfrak{g}_3$ (in the roots $\beta = \alpha_1 + 3\alpha_2$ and $2 = 2\alpha_1 + 3\alpha_2$). Then $\alpha = \alpha_2, \omega_\alpha = \alpha_1 + 2\alpha_2, V = \mathbb{C}^7$, and the statement can be checked directly.

Thus there is a linear bijection between $\mathbb{C} h^{\gamma}_\alpha + \sum_{\alpha \leq \gamma \leq \beta} \mathfrak{g} - \gamma \subset \mathfrak{g}$ and $\mathbb{C} v + \sum_{\alpha \leq \gamma \leq \beta} \mathfrak{g} - \gamma v \subset V$ that shifts weights by $\omega_\alpha$. It is determined by the assignment on the basis as stated. It is easy to see that it is an isomorphism of $\mathfrak{g}_-$-modules $\mathfrak{g}(h^\gamma_\alpha, f_\beta)$ and $V(v, f_\beta v)$ which induces an isomorphism of the corresponding sub-diagrams. \hfill $\square$

Specialization of the formula (3.4) for $\theta_\beta$ in the light of Proposition 5.1 requires the knowledge of matrix $(\pi \otimes \text{id})(C) \in \text{End}(V) \otimes U_q(\mathfrak{g}_-)$, we do it for the case of $q = 1$. Let $\beta \in \mathbb{R}^+$ and $\alpha \in \Pi$ be such as in the above proposition. For $\nu, \gamma \in \mathbb{R}^+$, denote by $C_{\nu, \gamma} \in \mathbb{C}$ the scalars such that $[e_\nu, f_\gamma] = C_{\nu, \gamma} f_{\gamma - \nu}$, if $\gamma - \nu \in \mathbb{R}^+$, $C_{\nu, \gamma} = \frac{(\beta, \beta) f_{\nu, \gamma}}{2 f_{\alpha, \beta}}$, and $C_{\nu, \gamma} = 0$ otherwise. Then

$$(\pi \otimes \text{id})(C)(f_\gamma v_b \otimes 1) = C_{\gamma, \gamma} v_b \otimes f_\gamma + \sum_{\nu < \gamma} C_{\nu, \gamma} f_{\gamma - \nu} v_b \otimes f_\nu,$$

for all $\gamma$ satisfying $\alpha \leq \gamma \leq \beta$. The formula (3.4) becomes

$$\theta_\beta = C_{\beta, \beta} f_\beta + \sum_{k \geq 1} \sum_{\nu_1 + \ldots + \nu_k = \beta} (C_{\nu_{k+1}, \gamma_k} \ldots C_{\nu_1, \gamma_0})(f_{\nu_{k+1}} \ldots f_{\nu_1}) \frac{(-1)^k}{\eta_{\nu_k} \ldots \eta_{\nu_1}}. \tag{5.8}$$
The internal summation is performed over all partitions of $\beta$ to a sum of positive roots $\nu_i$ subject to the following: all $\gamma_i = \gamma_{i-1} - \nu_i$ for $i = 1, \ldots, k$ with $\gamma_0 = \beta$ are positive roots and $\gamma_i \succeq \alpha$. The weights $\mu_i$ are defined as $\mu_i = \gamma_0 - \gamma_i + \nu_1 + \ldots + \nu_i$. Note that in the $q \neq 1$ case this sum may involve terms with matrix entries of $C$ whose weights are not roots.

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