Black di-ring and infinite nonuniqueness

Hideo Iguchi and Takashi Mishima

Laboratory of Physics, College of Science and Technology, Nihon University,
Narashinodai, Funabashi, Chiba 274-8501, Japan

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Abstract

We show that the $S^1$-rotating black rings can be superposed by the solution generating technique. We analyze the black di-ring solution for the simplest case of multiple rings. There exists an equilibrium black di-ring where the conical singularities are cured by the suitable choice of physical parameters. Also there are infinite numbers of black di-rings with the same mass and angular momentum. These di-rings can have two different continuous limits of single black rings. Therefore we can transform the fat black ring to the thin ring with the same mass and angular momentum by way of the di-ring solutions.

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I. INTRODUCTION

One of the most important recent findings of the higher-dimensional general relativity is a single-rotational black ring solution by Emparan and Reall [1]. (See also [2].) This solution is a vacuum, axially symmetric and asymptotically flat solution of the five-dimensional general relativity. The topology of the event horizon is $S^1 \times S^2$. The black ring rotates along the direction of $S^1$. The balanced black ring which has no conical singularity has a minimum of angular momentum for a fixed mass parameter. When the angular momentum is near this minimum, there are two different black rings with the same angular momentum. They are called fat and thin black rings according to their shapes. In addition we have a single-rotational spherical black hole [3] with the same asymptotic parameters. This finding entails the discrete nonuniqueness of the five dimensional vacuum solutions. It has been shown that the black rings can have dipole charges which are independent of all conserved charges [4]. Therefore the dipole rings imply the infinite violation of uniqueness by a continuous parameter.

Recently the present authors found a black ring solution with $S^2$ rotation by using a solitonic solution-generating technique [5, 6]. The seed solution of this ring is a simple Minkowski spacetime. Because the effect of rotation cannot compensate for the gravitational attractive force, the ring has a kind of strut structure. We have also generated the black ring with $S^1$ rotation by the same solitonic solution-generating technique [7]. The seed solution is not a Minkowski spacetime but an Euclidean C-metric solution. It has been shown that these two solutions can be obtained by the inverse scattering method [8, 9, 10, 11]. Rotating dipole black ring solutions have been systematically generated in five-dimensional Einstein-Maxwell-dilaton gravity [12, 13]. The relations between the seed and the solitonic solutions can be easily understood through the analysis of their rod structures [14, 15]. The seed of $S^1$-rotating black ring has been constructed by the help of the rod structure analysis. Thus the rod structure analysis is expected to be a useful guide to construct seed solutions for new solutions.

In this paper, we consider the multiplexed $S^1$-rotating black rings arranged in a concentric pattern. The seed solution can be constructed by the help of rod structure analysis as in the case of the $S^1$-rotating black ring [7]. The exact expressions of metric functions can be written down by the solitonic transformation, but in rather complicated forms. In this
paper we analyze di-ring solutions as the first step in the black ring multiplication. In the
supersymmetric system the solution of multiple black rings exist indeed \[16, 17\]. Also the
solution of concentric static extremal black rings has been considered \[18\].

The simplest multiple black rings solution is a black di-ring. This solution has two ring-
like event horizons of different radii with the same topology of \(S^1 \times S^2\). Both horizons can
rotate along the direction of \(S^1\). As similar as the single black ring solutions, this solutions
has conical singularities for general values of parameters. However these conical singularities
can be cured by an appropriate choice of the parameters as in the case of the \(S^1\)-rotating
black ring. The black di-rings can have the same mass and angular momentum for infinite
numbers of sets of parameters. Therefore the black di-rings realize an infinite nonuniqueness
without dipole charges. There can exist one Myers-Perry black hole, two \(S^1\)-rotating black
rings and infinite numbers of black di-rings for the same mass and angular momentum. Also
these single black rings are continuous limits in the black di-rings. Therefore these two
different black rings are connected by the black di-rings with the same mass and angular
momentum. When we shrinks the inner ring down to zero radius, we obtain a solution
describing a black hole sitting at the common center of the outer ring.

The plan of the paper is as follows. In Sec. II we briefly review the solution-generating
 technique used in the analysis. The rod structure analysis is explained in Sec. III We give
the seed solutions of black multi- and di-ring solutions and analyze the some features of
di-ring solution in Sec. IV In Sec. V we give a summary of this article.

II. BRIEF REVIEW OF SOLUTION GENERATING TECHNIQUE

At first we briefly explain the procedure to generate axisymmetric solutions in the five-
dimensional general relativity. The spacetimes which we considered satisfy the following
conditions: (c1) five dimensions, (c2) asymptotically flat spacetimes, (c3) the solutions of
vacuum Einstein equations, (c4) having three commuting Killing vectors including time
translational invariance and (c5) having a single non-zero angular momentum component.

Under the conditions (c1) – (c5), we can employ the following Weyl-Papapetrou metric form

\[
\begin{align*}
    ds^2 &= -e^{2U_0}(dx^0 - \omega d\phi)^2 + e^{2U_1}\rho^2(d\phi)^2 + e^{2U_2}(d\psi)^2 + e^{2(\gamma+U_1)}(d\rho^2 + dz^2),
\end{align*}
\]

(1)
where $U_0$, $U_1$, $U_2$, $\omega$ and $\gamma$ are functions of $\rho$ and $z$. Then we introduce new functions $S := 2U_0 + U_2$ and $T := U_2$ so that the metric form (1) is rewritten into

$$ds^2 = e^{-T} [-e^S(dx^0 - \omega d\phi)^2 + e^{T+2U_0} \rho^2 (d\phi)^2 + e^{2(\gamma+U_1)} (d\rho^2 + dz^2)] + e^{2T} (d\psi)^2. \quad (2)$$

Using this metric form the Einstein equations are reduced to the following set of equations,

(i) $\nabla^2 T = 0$,

(ii) $\begin{cases} \partial_\rho \gamma T = \frac{3}{4} \rho \left[ (\partial_\rho T)^2 - (\partial_z T)^2 \right] \\ \partial_z \gamma T = \frac{3}{2} \rho \left[ \partial_\rho T \partial_z T \right] \end{cases}$,

(iii) $\nabla^2 \mathcal{E}_S = \frac{2}{\mathcal{E}_S + \mathcal{E}_S} \nabla \mathcal{E}_S \cdot \nabla \mathcal{E}_S$,

(iv) $\begin{cases} \partial_\rho \gamma S = \frac{\rho}{2(\mathcal{E}_S + \mathcal{E}_S)} \left( \partial_\rho \mathcal{E}_S \partial_\rho \mathcal{E}_S - \partial_\rho \mathcal{E}_S \partial_z \mathcal{E}_S \right) \\ \partial_z \gamma S = \frac{\rho}{2(\mathcal{E}_S + \mathcal{E}_S)} \left( \partial_\rho \mathcal{E}_S \partial_z \mathcal{E}_S + \partial_\rho \mathcal{E}_S \partial_\rho \mathcal{E}_S \right) \end{cases}$,

(v) $(\partial_\rho \Phi, \partial_\Phi) = \rho^{-1} e^{2S} (-\partial_\rho \omega, \partial_\rho \omega)$,

(vi) $\gamma = \gamma_S + \gamma_T$,

(vii) $U_1 = -\frac{S + T}{2}$,

where $\Phi$ is defined through the equation (v) and the function $\mathcal{E}_S$ is defined by $\mathcal{E}_S := e^S + i \Phi$.

The most non-trivial task to obtain new metrics is to solve the equation (iii) because of its non-linearity. To overcome this difficulty here we use the method similar to the Neugebauer’s Bäcklund transformation [19] or the Hoenselaers-Kinnersley-Xanthopoulos transformation [20].

To write down the exact form of the metric functions, we follow the procedure given by Castejon-Amenedo and Manko [21]. In the five dimensional spacetime we start from the following form of a seed static metric

$$ds^2 = e^{-T(0)} \left[ -e^{S(0)} (dx^0)^2 + e^{-S(0)} \rho^2 (d\phi)^2 + e^{2S(0)} - S(0) (d\rho^2 + dz^2) \right] + e^{2T(0)} (d\psi)^2.$$ 

For this static seed solution, $e^{S(0)}$, of the Ernst equation (iii), a new Ernst potential can be written in the form

$$\mathcal{E}_S = e^{S(0)} \frac{x(1 + ab) + iy(b - a) - (1 - ia)(1 - ib)}{x(1 + ab) + iy(b - a) + (1 - ia)(1 - ib)}.$$
where \( x \) and \( y \) are the prolate spheroidal coordinates: 
\[ \rho = \sigma \sqrt{x^2 - 1} \sqrt{1 - y^2}, \quad z = \sigma xy, \]
with \( \sigma > 0 \). The ranges of these coordinates are \( 1 \leq x \) and \( -1 \leq y \leq 1 \). The functions \( a \) and \( b \) satisfy the following simple first-order differential equations

\[
(x - y)\partial_x a = a \left[ (xy - 1)\partial_x S^{(0)} + (1 - y^2)\partial_y S^{(0)} \right],
\]
\[
(x - y)\partial_y a = a \left[ -(x^2 - 1)\partial_x S^{(0)} + (xy - 1)\partial_y S^{(0)} \right],
\]
\[
(x + y)\partial_x b = -b \left[ (xy + 1)\partial_x S^{(0)} + (1 - y^2)\partial_y S^{(0)} \right],
\]
\[
(x + y)\partial_y b = -b \left[ -(x^2 - 1)\partial_x S^{(0)} + (xy + 1)\partial_y S^{(0)} \right].
\]

(3)

For the typical seed
\[ S^{(0)} = \frac{1}{2} \ln[R_d + (z - d)], \]
the following \( a \) and \( b \) satisfy the differential equations (3),

\[ a = l^{-1}_\sigma e^{2\phi_{d,c}}, \quad b = -l^{-1}_{-\sigma} e^{-2\phi_{d,c}}, \]

(5)

where

\[ \phi_{d,c} = \frac{1}{2} \ln \left[ e^{-\tilde{U}_d} \left( e^{2U_c} + e^{2\tilde{U}_d} \right) \right]. \]

(6)

Here the functions \( \tilde{U}_d \) and \( U_c \) are defined as \( \tilde{U}_d := \frac{1}{2} \ln [R_d + (z - d)] \) and \( U_c := \frac{1}{2} \ln [R_c - (z - c)] \). Because of the linearity of the differential equations (3) for \( S^{(0)} \), we can easily obtain \( a \) and \( b \) which correspond to a general seed function.

The metric functions for the five-dimensional metric (2) are obtained by using the formulas shown by [21],

\[ e^S = e^{S^{(0)}} \frac{A}{B}, \]
\[ \omega = 2\sigma e^{-S^{(0)}} \frac{C}{A} + C_1 \]
\[ e^{2\gamma} = C_2(x^2 - 1)^{-1} A e^{2\gamma'}, \]

(7) (8) (9)

where \( C_1 \) and \( C_2 \) are constants and \( A, B \) and \( C \) are given by

\[ A := (x^2 - 1)(1 + ab)^2 - (1 - y^2)(b - a)^2, \]
\[ B := [(x + 1) + (x - 1)ab]^2 + [(1 + y)a + (1 - y)b]^2, \]
\[ C := (x^2 - 1)(1 + ab)[(1 - y)b - (1 + y)a] \\
\] 
\[ + (1 - y^2)(b - a)[x + 1 - (x - 1)ab]. \]
In addition the $\gamma'$ in Eq. (9) is a $\gamma$ function corresponding to the static metric,

$$ds^2 = e^{-T(0)} \left[ -e^{2U_0(BH)} + S^{(0)} (dx^0)^2 + e^{-2U_0(BH) - S^{(0)}} \rho^2 (d\phi)^2 
+ e^{2(\gamma' - U_0(BH)) - S^{(0)}} (d\rho^2 + dz^2) \right] + e^{2T(0)} (d\psi)^2$$

where $U_0(BH) = \frac{1}{2} \ln \left( \frac{x - 1}{x + 1} \right)$. Therefore the function $\gamma'$ obeys the following equations,

$$\partial_\rho \gamma' = \frac{1}{4} \rho \left[ (\partial_\rho S')^2 - (\partial_z S')^2 \right] + \frac{3}{4} \rho \left[ (\partial_\rho T')^2 - (\partial_z T')^2 \right],$$

$$\partial_z \gamma' = \frac{1}{2} \rho \left[ (\partial_\rho S' \partial_z S') + \frac{3}{2} \rho [\partial_\rho T' \partial_z T'] \right],$$

where the first terms are contributions from Eq. (iv) and the second terms come from Eq. (ii). Here the functions $S'$ and $T'$ can be read out from Eq. (10) as

$$S' = 2U_0(BH) + S^{(0)},$$

$$T' = T^{(0)}.$$

To integrate these equations we can use the following fact that, the partial differential equations

$$\partial_\rho \gamma'_{cd} = \rho \left[ \partial_\rho \tilde{U}_c \partial_\rho \tilde{U}_d - \partial_z \tilde{U}_c \partial_z \tilde{U}_d \right],$$

$$\partial_z \gamma'_{cd} = \rho \left[ \partial_\rho \tilde{U}_c \partial_z \tilde{U}_d + \partial_\rho \tilde{U}_d \partial_z \tilde{U}_c \right],$$

have the following solution,

$$\gamma'_{cd} = \frac{1}{2} \tilde{U}_c + \frac{1}{2} \tilde{U}_d - \frac{1}{4} \ln Y_{cd},$$

where $Y_{cd} := R_c R_d + (z - c)(z - d) + \rho^2$. The general solution of $\gamma'$ is given by the linear combination of the functions $\gamma'_{cd}$. And then the function $T$ is equal to $T^{(0)}$ and $U_1$ is given by the Einstein equation (vii).

### III. ROD STRUCTURE ANALYSIS

We give a brief explanation of the rod structure analysis elaborated by Harmark [15]. See [15] for complete explanations.

Here we denote the D-dimensional axially symmetric stationary metric as

$$ds^2 = G_{ij} dx^i dy^j + e^\nu (d\rho^2 + dz^2)$$

(18)
where $G_{ij}$ and $\nu$ are functions only of $\rho$ and $z$ and $i, j = 0, 1, \ldots, D - 3$. The $D - 2$ by $D - 2$ matrix field $G$ satisfies the following constraint

$$\rho = \sqrt{|\det G|}. \quad (19)$$

The equations for the matrix field $G$ can be derived from the Einstein equation $R_{ij} = 0$ as

$$G^{-1} \nabla G = (G^{-1} \nabla G)^2, \quad (20)$$

where the differential operator $\nabla$ is the gradient in three-dimensional unphysical flat space with metric

$$d\rho^2 + \rho^2 d\omega^2 + dz^2. \quad (21)$$

Because of the constraint $\rho = \sqrt{|\det G|}$, at least one eigenvalue of $G(\rho, z)$ goes to zero for $\rho \to 0$. However it was shown that if more than one eigenvalue goes to zero as $\rho \to 0$, we have a curvature singularity there. Therefore we consider solutions which have only one eigenvalue goes to zero for $\rho \to 0$, except at isolated values of $z$. Denoting these isolated values of $z$ as $a_1, a_2, \ldots, a_N$, we can divide the $z$-axis into the $N + 1$ intervals $[-\infty, a_1], [a_1, a_2], \ldots, [a_N, \infty]$, which is called as rods. These rods correspond to the source added to the equation (21) at $\rho = 0$ to prevent the break down of the equation there.

The eigenvector for the zero eigenvalue of $G(0, z)$

$$v = v^i \frac{\partial}{\partial x^i}, \quad (22)$$

which satisfies

$$G_{ij}(0, z)v^i = 0, \quad (23)$$

determines the direction of the rod. If the value of $\frac{G_{ij}v^iv^j}{\rho^2}$ is negative (positive) for $\rho \to 0$ the rod is called timelike (spacelike). Each rod corresponds to the region of the translational or rotational invariance of its direction. The timelike rod corresponds to a horizon. The spacelike rod corresponds to a compact direction.

### IV. $S^1$-ROTATING BLACK MULTI-RING

The $n$-multiplexed $S^1$-rotating black ring can be obtained in the following manner. At first we prepare the seed solution of multi-ring as in Fig. [Fig. 1]. To assure the asymptotical
FIG. 1: Schematic pictures of rod structures of multi-ring and its seed. The left panel shows the rod structure of seed metric of $S^1$-rotating black multi-ring. The right panel shows the rod structure of $S^1$-rotating black multi-ring. The finite spacelike rod $[\eta_1^{\sigma}, \eta_2^{\sigma}]$ in the left panel is altered to the finite timelike rod by the solution-generating transformation. All static timelike rods may be transformed to stationary ones by the solitonic transformation. To denote the rotation of the event horizons, we put the finite timelike rods between the lines of $x^0$ and $\phi$.

Flatness, we need two semi-infinite spacelike rods in the different directions. There is a finite spacelike rod with the direction vector $\partial/\partial \phi$ around the $z = 0$. Between this finite rod and the semi-infinite spacelike rod of $\phi$-direction, we alternately arrange $n$ spacelike rods in $\psi$-direction and $(n - 1)$ static finite timelike rods. The finite spacelike rod of $\phi$-direction is changed to a finite timelike rod with $\phi$ rotation by the solitonic transformation [6, 7]. In addition the finite timelike rods of seed solution can get the $\phi$ components in their direction vectors through the transformation.

In the following we investigate the simplest multiple black rings, i.e., the black di-ring solution. The rod structure of the seed and the di-ring solution are given in Fig. 2. The rod structure of di-ring is determined by 4 length of finite rods and 2 angular velocities of timelike rods. We have five physical parameters $\eta_1, \eta_2, \delta_1, \delta_2$ and $\lambda$, except for the freedom of scaling. These parameters should satisfy the condition $-1 < \eta_1 < \eta_2 < 1 < \delta_1 < \delta_2 < \lambda$ for the di-ring solution. Note that when we set $\delta_2 = \lambda$, the inner ring shrinks to $S^3$ sphere. When $\delta_1 = \delta_2$, these structures are exactly the same as the case of single $S^1$-rotating black ring.
FIG. 2: Schematic pictures of rod structures of black di-ring and its seed. The left panel shows the rod structure of seed metric of $S^1$-rotating black di-ring. The right panel shows the rod structure of $S^1$-rotating black di-ring. The finite spacelike rod $[\eta_1 \sigma, \eta_2 \sigma]$ in the left panel is altered to the finite timelike rod by the solution-generating transformation.

The seed functions of black di-ring are given by the following functions,

$$T^{(0)} = \tilde{U}_{\lambda \sigma} + \tilde{U}_{\delta_1 \sigma} - \tilde{U}_{\delta_2 \sigma} + \tilde{U}_{\eta_1 \sigma} - \tilde{U}_{\eta_2 \sigma},$$  \hspace{1cm} (24)

$$S^{(0)} = \tilde{U}_{\lambda \sigma} - (\tilde{U}_{\delta_1 \sigma} - \tilde{U}_{\delta_2 \sigma}) + \tilde{U}_{\eta_1 \sigma} - \tilde{U}_{\eta_2 \sigma}.$$  \hspace{1cm} (25)

The corresponding auxiliary potentials of solitonic solutions are obtained as

$$a = \frac{\alpha}{2 \sigma^{1/2}} \left( \frac{e^{2U_\sigma} + e^{2\tilde{U}_{\lambda \sigma}}}{e^{2\tilde{U}_{\lambda \sigma}} + e^{2U_\sigma}} \frac{e^{2\tilde{U}_{\delta_1 \sigma}} + e^{2\tilde{U}_{\delta_2 \sigma}}}{e^{2\tilde{U}_{\delta_2 \sigma}} + e^{2\tilde{U}_{\delta_1 \sigma}}} \frac{e^{2\tilde{U}_{\eta_1 \sigma}} + e^{2\tilde{U}_{\eta_2 \sigma}}}{e^{2\tilde{U}_{\eta_2 \sigma}} + e^{2\tilde{U}_{\eta_1 \sigma}}} \right),$$  \hspace{1cm} (26)

$$b = 2\sigma^{1/2} \beta \left( \frac{e^{\tilde{U}_{\lambda \sigma}} + e^{2\tilde{U}_{\lambda \sigma}}}{e^{2U_\sigma} + e^{2\tilde{U}_{\lambda \sigma}}} \frac{e^{2\tilde{U}_{\delta_1 \sigma}} + e^{2\tilde{U}_{\delta_2 \sigma}}}{e^{2\tilde{U}_{\delta_1 \sigma}} + e^{2\tilde{U}_{\delta_2 \sigma}}} \frac{e^{2\tilde{U}_{\eta_1 \sigma}} + e^{2\tilde{U}_{\eta_2 \sigma}}}{e^{2\tilde{U}_{\eta_1 \sigma}} + e^{2\tilde{U}_{\eta_2 \sigma}}} \right),$$  \hspace{1cm} (27)

where $\alpha$ and $\beta$ are integration constants. The functions $S'$ and $T'$ in Eqs. (11) and (12) are obtained as

$$S' = 2 U_0^{(BH)} + S^{(0)}$$  

$$T' = T^{(0)} = \tilde{U}_{\lambda \sigma} + \tilde{U}_{\delta_1 \sigma} - \tilde{U}_{\delta_2 \sigma} + \tilde{U}_{\eta_1 \sigma} - \tilde{U}_{\eta_2 \sigma},$$  \hspace{1cm} (28)
therefore the function $\gamma'$ becomes the following sum of the functions $\gamma'_{cd}$:

$$\gamma' = \gamma_{\sigma} + \gamma'_{\sigma,-a} + \gamma'_{\lambda \sigma, \lambda a} + \gamma'_{\delta \sigma, \delta a} + \gamma'_{\eta \sigma, \eta a} + \gamma'_{n \sigma, n a} + \gamma'_{m \sigma, m a} + \gamma'_{\psi \sigma, \psi a}$$

$$-2\gamma_{\sigma,-a} + \gamma_{\lambda \sigma, \lambda a} - \gamma_{\delta \sigma, \delta a} + \gamma_{\eta \sigma, \eta a} - \gamma_{n \sigma, n a}$$

$$-\gamma_{\lambda \sigma, \lambda a} + \gamma_{\delta \sigma, \delta a} - \gamma_{\eta \sigma, \eta a} + \gamma_{n \sigma, n a}$$

$$+\gamma_{\lambda \sigma, \lambda a} - \gamma_{\delta \sigma, \delta a} + 2\gamma_{\lambda \sigma, \eta a} - 2\gamma'_{\lambda \sigma, n a} - 2\gamma_{\delta \sigma, \delta a}$$

$$+\gamma'_{\delta \sigma, \eta a} - \gamma'_{\delta \sigma, \eta a} - \gamma'_{\delta \sigma, \eta a} + \gamma'_{\delta \sigma, \eta a} - 2\gamma_{\eta \sigma, \eta a}.$$  

Using these functions we can write down the metric functions of black di-ring. The constants $C_1$ and $C_2$ of Eq. (30) and (31) are fixed as

$$C_1 = \frac{2\sigma^{1/2}}{1 + \alpha \beta}, \quad C_2 = \frac{1}{\sqrt{2(1 + \alpha \beta)^2}},$$

to assure that the spacetime does not have global rotation and that the periods of $\phi$ and $\psi$ become $2\pi$ at the infinity, respectively.

To make the metric component $g_{\phi \phi}$ be regular, we have to set the integration constants $\alpha$ and $\beta$ as

$$\alpha = \pm \sqrt{\frac{2(\delta_1 - 1)(1 - \eta_2)}{(\lambda - 1)(\delta_2 - 1)(1 - \eta_1)}}, \quad \beta = \pm \sqrt{\frac{(\lambda + 1)(\delta_2 + 1)(1 + \eta_1)}{2(\delta_1 + 1)(1 + \eta_2)}}. \tag{30}$$

These conditions also assure the non-existence of closed timelike curves around the event horizons.

To cure the conical singularities, we have to set the periods of angular coordinates appropriately. The periods of the coordinates $\phi$ and $\psi$ are defined as

$$\Delta \phi = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 g_{\rho \rho}}{g_{\phi \phi}}} \quad \text{and} \quad \Delta \psi = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 g_{\rho \rho}}{g_{\psi \psi}}} \tag{31}$$

We already set the periods of $\phi$ and the one of $\psi$ outside the ring to be $2\pi$. In addition the periods of $\psi$ can be obtained from

$$\Delta \psi = \frac{2\pi}{1 + \alpha \beta} \sqrt{\frac{(\lambda + 1)(\lambda - \delta_2)}{(\lambda - 1)(\lambda - \delta_1)} \left(\frac{\lambda - \eta_2}{\lambda - \eta_1}\right) \left(1 + \frac{\lambda - 1}{\lambda + 1} \alpha \beta\right)} \tag{32}$$

for $\delta_2 \sigma < z < \lambda \sigma$ and

$$\Delta \psi = \frac{2\pi}{1 + \alpha \beta} \sqrt{\frac{(\lambda + 1)(\delta_1 - 1)(\delta_2 + 1)(\delta_1 - \eta_2)(\delta_2 - \eta_1)}{(\lambda - 1)(\delta_1 + 1)(\delta_2 - 1)(\delta_1 - \eta_1)(\delta_2 - \eta_2)}}$$

$$\times \left[\frac{\lambda - \eta_2}{\lambda - \eta_1} \left(1 + \frac{(\lambda - 1)(\delta_1 + 1)(\delta_2 - 1)}{(\lambda + 1)(\delta_1 - 1)(\delta_2 + 1)} \alpha \beta\right)\right]. \tag{33}$$
for $\eta_2 \sigma < z < \delta_1 \sigma$. The parameters can be adjusted to make the both values of $\Delta \psi$ equal $2\pi$.

Asymptotic form of $\mathcal{E}_S$ near the infinity $\tilde{r} = \infty$ becomes

$$\mathcal{E}_S = \tilde{r} \cos \theta \left[ 1 - \frac{\sigma P(\alpha, \beta, \lambda)}{\tilde{r}^2 (1 + \alpha \beta)^2} + \cdots \right] + 2 i \sigma^{1/2} \left[ \frac{\alpha}{1 + \alpha \beta} - \frac{2 \sigma \cos^2 \theta \sin \theta}{\tilde{r}^2 (1 + \alpha \beta)^3} Q(\alpha, \beta, \lambda) + \cdots \right],$$

where we introduced the new coordinates $\tilde{r}$ and $\theta$ through the relations

$$x = \frac{\tilde{r}^2}{2 \sigma} + \lambda + (\eta_1 - \eta_2) + (\delta_1 - \delta_2), \quad y = \cos 2\theta,$$

and

$$P = 4(1 + \alpha^2 - \alpha^2 \beta^2) + 2(1 + \alpha \beta)^2(\delta_2 - \delta_1),$$

$$Q = \alpha(2\alpha^2 - \delta_1 + \delta_2 + \eta_1 - \eta_2 + \lambda + 3) - 2\alpha^2 \beta^3$$

$$- \beta \left[ 2(2\alpha\beta + 1)(\alpha^2 + 1) + (\delta_1 - \delta_2 - \eta_1 + \eta_2 - \lambda - 1)\alpha^2(\alpha\beta + 2) \right].$$

From the asymptotic behavior of the Ernst potential, we can compute the mass parameter $m^2$ and rotational parameter $m^2 a_0$ as

$$m^2 = \sigma \frac{P}{(1 + \alpha \beta)^2}, \quad m^2 a_0 = 4\sigma^{3/2} \frac{Q}{(1 + \alpha \beta)^3}.$$

The angular velocities of event horizons are obtained from the direction vectors of finite timelike rods. For the finite timelike rod of inner ring [$\delta_1 \sigma < z < \delta_2 \sigma$], the direction vector is calculated as

$$v = (1, \Omega_1, 0), \quad \Omega_1 = -\frac{2\beta(1 + \alpha \beta)}{\sqrt{\sigma}((\lambda - 1)\alpha \beta + \lambda + 1)((\delta_2 - 1)\alpha \beta + \delta_2 + 1)}.$$  (35)

The outer ring [$\eta_1 \sigma < z < \eta_2 \sigma$] has a direction vector

$$v = (1, \Omega_2, 0), \quad \Omega_2 = \frac{(1 + \alpha \beta)((2\beta(\delta_1 + 1)(1 + \eta_2) - \alpha(\lambda + 1)(\delta_2 + 1)(1 - \eta_1))}{2\sqrt{\sigma}(2\alpha \beta(\delta_1 + 1)(1 + \eta_2) - (\lambda + 1)(\delta_2 + 1)(\alpha^2(1 - \eta_1) + 2\alpha \beta + 2))}. \quad (36)$$

When $\eta_1 = -1$, the inner ring becomes static because of $\Omega_1 = 0$. In this case the rotation of the outer ring only can cause the absence of the conical singularity. When $\eta_2 = 1$, the both rings rotate along the same direction.
Analyzing the mass and angular momentum parameters, we can show the infinite nonuniqueness of black di-ring which means that the di-ring solution has a continuous parameter region to have the same mass and angular momentum. In addition there can be two different single ring limits, thin and fat black rings, of the black di-ring with the same mass and angular momentum. Therefore these two single rings can be transformed into each other through the black di-ring of the same mass and angular momentum.

To show this fact, we consider the black di-ring of $\eta_2 = 1$. In Fig. 3 we plot the variable

$$\frac{a_0^2}{m^2} = \frac{16Q^2}{P^3}$$

as a function of $\delta_1$ and $\eta_1$. At first we numerically decide the values of $\lambda$ and $\delta_2$ for the balanced black di-ring for which the right hand sides of Eqs. (32) and (33) become $2\pi$ with respect to given $\delta_1$ and $\eta_1$. Next we obtain the value of $\frac{a_0^2}{m^2}$ by substituting the parameters which satisfy the condition $\delta_1 < \delta_2 < \lambda$. The bold line of Fig. 3 is the single ring limit where $\delta_1 = \delta_2$. When $\delta_1 = \delta_2$ we can show that Eq. (37) of equilibrium ring is reduced to the following form,

$$\frac{a_0^2}{m^2} = \frac{(1 + \eta_1)^3}{8\eta_1},$$

which corresponds with the single $S^1$-rotating black ring. The Figure 4 is a plot of Eq. (38), where $0.5 < \eta_1 < 1$ corresponds to the fat ring and $\eta_1 < 0.5$ the thin ring. Along the bold line in Fig. 3 the value of $\frac{a_0^2}{m^2}$ has the same $\eta_1$ dependence of Eq. (38). Apparently there is a continuous path of di-ring between fat and thin black rings which have the same mass and angular momentum.

V. SUMMAY AND DISCUSSION

In this paper we have shown that the $S^1$-rotating black ring can be superposed concentrically. The solution of this multiple rings can be written down by the solitonic transformation for the appropriately arranged seed solution. We have obtained the functions needed to write down the metric of black di-ring which is the simplest multiple $S^1$-rotating black ring. To regularize the metric function, the integration constants $\alpha$ and $\beta$ should be set appropriately. For the equilibrium black di-ring we need the two additional conditions of parameters. We have analyzed the mass and angular momentum of black di-ring from the asymptotic form of Ernst potential.
FIG. 3: Plot of $\frac{a^2}{m^2}$ as a function of $\eta_1$ and $\delta_1$ where $\lambda$ and $\delta_2$ are determined by the equilibrium conditions and $\eta_2 = 1$. The bold line correspond to the single black ring of $\delta_1 = \delta_2$.

FIG. 4: Plot of $\frac{a^2}{m^2}$ of single black ring as a function of $\eta_1$. The region $0.5 < \eta_1 < 1$ correspond to fat rings and $\eta_1 < 0.5$ thin rings.

The most important feature of black di-rings is that they entail the infinite nonuniqueness of the vacuum neutral solutions of five dimensional general relativity. To show this, we have numerically plotted the spin parameter of the equilibrium black di-rings as a function of the two independent parameters. This plot shows that there are infinite numbers of black di-rings with the same mass and angular momentum. In addition we have shown that the black di-ring can be a pathway between the fat and thin $S^1$-rotating black rings.

The nonuniqueness we have shown is derived from the existence of one-parameter family of black di-rings with the same conserved parameters because we have fixed one parameter in the analysis. The parameters set for which the general black di-rings have the same conserved
parameters would be a two-dimensional surface in the three-dimensional parameters space. The physical features of black di-ring will be analyzed in detail.

The generalization of the solution to have two angular momenta would be important. Recently, the generalization of the single black ring solution to this direction has been considered by the inverse scattering method [22] and by the numerical study [23]. After this work was completed we noticed a preprint [24], which considers a black saturn: a spherical black hole surrounded by a black ring. It would be important to consider the relation between the black saturn and the black di-ring solutions.

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[25] In the previous version, one of these relations is inappropriate. As a result of this revision, the mass parameter is improved to be consistent with the ADM mass obtained by Yazadjiev, arXiv:0805.1600 [hep-th].