Semi-classical behavior of Pöschl-Teller coherent states

H. Bergeron\textsuperscript{1(a)}, J.-P. Gazeau\textsuperscript{2(b)}, P. Siegl\textsuperscript{2,3,4(c)} and A. Youssef\textsuperscript{2(d)}

\textsuperscript{1}Université Paris-Sud, ISMO, FRE 3363 - Bât. 351, F-91405 Orsay, France, EU
\textsuperscript{2}Université Paris Diderot Paris 7, Laboratoire APC - Case 7020, F-75205 Paris Cedex 13, France, EU
\textsuperscript{3}Czech Technical University in Prague, FNSPE - Břehová 7, 11519 Prague, Czech Republic, EU
\textsuperscript{4}Nuclear Physics Institute ASCR - 25068 Řež, Czech Republic, EU

received 21 July 2010; accepted in final form 28 November 2010
published online 11 January 2011

PACS 03.65.-w – Quantum mechanics

Abstract – We present a construction of semi-classical states for Pöschl-Teller potentials based on a supersymmetric quantum mechanics approach. The parameters of these “coherent” states are points in the classical phase space of these systems. They minimize a special uncertainty relation. Like standard coherent states they resolve the identity with a uniform measure. They permit to establish the correspondence (quantization) between classical and quantum quantities. Finally, their time evolution is localized on the classical phase space trajectory.

Introduction. – The search of quantum states that exhibit a semi-classical time behavior was initiated in Schrödinger’s pioneering work on “packets of eigenmodes” of the harmonic oscillator [1]: “I show that the packet of eigenmodes with high quantum number $n$ and with relatively small difference in quantum numbers may represent the mass point that moves according to the usual classical mechanics, i.e. it oscillates with [classical] frequency $\nu_0$.” These Schrödinger states have been given the name of coherent states by Glauber within the context of quantum optics. With regard to the huge amount of recent works on quantum dots and quantum wells in nanophysics it has become challenging to construct quantum states for infinite wells which display localization properties comparable to those nicely displayed by the Schrödinger states. Infinite wells are often modeled by Pöschl-Teller (also known as trigonometric Rosen-Morse) confining potentials [2,3] used, e.g., in quantum optics [4,5]. The infinite square well is a limit case of this family referred to in what follows as $\mathcal{T}$-potentials. The question is to find a family of normalized states: (a) phase space labelled, (b1) yielding a resolution of the identity, (b2) the latter holding with respect to the uniform measure on phase space, (c) allowing a reasonable classical-quantum correspondence (“CS” quantization) and (d) exhibiting semi-classical phase space properties with respect to $\mathcal{T}$-Hamiltonian time evolution. We refer to these states as coherent states (CS) as they share many striking properties with Schrödinger’s original semi-classical states.

Most of the CS encountered in the literature are built through a group-theoretical or algebraic approach. (Early examples of other constructions were given by Robertson, the so-called “intelligent” states, [6–8,]). Regarding $\mathcal{T}$-potentials, they belong to the class of shape-invariant potentials [9] that have been intensively studied either specifically within the framework of supersymmetric quantum mechanics (SUSYQM) [10] or using a pure algebraic approach [11,12]. Then various semi-classical states adapted to supersymmetric systems in general [12–14] or to $\mathcal{T}$-potentials in particular have been proposed in previous works (see [15–18] and references therein). Whereas most of them verify (b1) and (d), they do not really “live” on the genuine classical phase space of the system. Hence, a classical-quantum correspondence (property (d)) often lacks unambiguous interpretation. Moreover, the correspondence between classical and quantum momenta for a particle moving on an interval requires a thorough analysis; as a matter of fact, there exists a well-known ambiguity in the definition of the quantum momentum operator [17,19]. This is due to the nature of the boundary conditions imposed to the system, unlike the harmonic-oscillator case.

In this letter, we present a construction of coherent states for $\mathcal{T}$-potentials based on a general approach given by one of us in [20], and we display their remarkable qualities as classical-quantum “conveyers”. These
states can be defined in a systematic way for any one-dimensional Hamiltonian (as eigenstates of some Hamiltonian-dependent lowering operator obtained from a Darboux factorization), and they are not associated in general with some closed dynamical algebra. Nevertheless, their properties are especially interesting if the ground-state wave function $\phi_0(x)$ of the Hamiltonian possesses a specific analytical property, namely if $\phi_0(x)^{-2}$ is a Laplace transform of some positive measure, a condition which actually holds true in many cases. The harmonic oscillator is the simplest illustration of such a feature; in this case our CS are identical to the usual harmonic ones. Under the above unique assumption on the ground state, properties (a), (b1) and (c) are always satisfied, whilst (c) remains essentially fulfilled at a formal level. However, the proof of the property (b2) is not known in a general situation (although it could be conjectured as true) and the property (d) has not yet been fully analyzed. Most of the shape-invariant Hamiltonians verify the ground-state condition and therefore they can be treated within our formalism. Investigating the case of $\mathcal{T}$-potentials: i) we can prove explicitly that the condition (b2) is satisfied, ii) we have a large set of observables for which the classical-quantum correspondence can be studied (condition (c)), and iii) the condition (d) can be evaluated in a precise way. The validity of the condition (b2) is very interesting because it means that our CS do not favor any part of the classical phase space. This is unexpected because the classical phase space for $\mathcal{T}$-potentials is a strip, and so it is a manifold with boundaries, topologically very different from the whole plane of the usual (harmonic) CS. In the following we examine in detail the classical-quantum correspondence based on these states (“CS quantization”) and we eventually show that our states stand comparison with the Schrödinger CS in terms of semi-classical time behavior.

**Definition of SUSYQM coherent states.**—Let us consider the motion of a particle confined to the interval $[0, L]$ and submitted to the repulsive symmetric $\mathcal{T}$-potential

$$V_\nu(x) = \frac{\hbar^2 \pi^2}{2mL^2} \nu(n + \nu + 1) \sin^2 \frac{\pi}{L} x,$$  

(1)

where $\nu \geq 0$ is a dimensionless parameter. The limit $\nu \to 0$ corresponds to the infinite square well. The factor $E_0 = \hbar^2 \pi^2 (2mL^2)^{-1} \geq 0$ is chosen as the ground-state energy of the infinite square well. On the quantum level, the Hamiltonian acts in the Hilbert space $\mathcal{H} = L^2([0, L], dx)$ as

$$H_\nu = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_\nu(x).$$  

(2)

The eigenvalues $E_{n,\nu}$ and corresponding eigenstates $|\phi_{n,\nu}\rangle$ of $H_\nu$ read, respectively,

$$E_n = E_0 (n + \nu + 1)^2, \quad n = 0, 1, 2, \ldots, \quad (3)$$

$$\phi_{n,\nu}(x) = Z_n \sin^{\nu+1} \left( \frac{\pi}{L} x \right) \cos^{\nu+1} \left( \frac{\pi}{L} x \right), \quad (4)$$

where $C_{n+1}^{\nu+1}$ is a Gegenbauer polynomial and

$$Z_n = \frac{\Gamma(\nu + 1)^{\nu+1/2}}{\sqrt{\Gamma(\nu + 2)}} \sqrt{n!(n + \nu + 1)} \Gamma(n + 2\nu + 2)$$  

(5)

is the normalization constant. Eigenfunctions $\phi_n$ obey the Dirichlet boundary conditions $\phi_n(0) = \phi_n(L) = 0$. A detailed mathematical discussion on the boundary conditions and self-adjoint extensions for the $\mathcal{T}$-Hamiltonian can be found in [17,21].

In particular, the ground-state eigenfunction $\phi_0$ is $Z_0 \sin^{\nu+1} \frac{\pi}{L} x$ and the eigenfunctions for the infinite square well ($\nu = 0$) reduce to $\sqrt{\frac{\pi}{L}} \sin^2 \frac{n+1}{L} x$.

We define the superpotential $W_\nu(x)$ as

$$W_\nu(x) = -\hbar \frac{\phi_0'(x)}{\phi_0(x)} = -\hbar \frac{\pi}{L} (\nu + 1) \cot \frac{\pi}{L} x,$$

(6)

and the lowering and raising operators $A_\nu$ and $A_\nu^\dagger$ as

$$A_\nu \equiv W_\nu(x) + \beta \frac{d}{dx} \quad \text{and} \quad A_\nu^\dagger \equiv W_\nu(x) - \beta \frac{d}{dx}. \quad (7)$$

Thus, the $\mathcal{T}$-Hamiltonian $H_\nu$ can be rewritten in terms of these operators as

$$H_\nu = \frac{1}{2m} A_\nu A_\nu^\dagger + E_0. \quad (8)$$

As expected [10], the supersymmetric partner $H^{(S)}_\nu$, 

$$H^{(S)}_\nu = \frac{1}{2m} A_\nu A_\nu^\dagger + E_0, \quad (9)$$

coincides with the original Hamiltonian with increased $\nu$: $H^{(S)}_\nu = H_{\nu+1}$.

The classical phase space for the motion in a $\mathcal{T}$-potential is defined as the infinite band in the plane: $\mathcal{K} = \{(q, p) | q \in [0, L] \text{ and } p \in \mathbb{R} \}$. Let us introduce the operators

$$Q: \psi(x) \mapsto x \psi(x), \quad \text{and} \quad P: \psi(x) \mapsto -i \hbar \frac{d}{dx} \psi(x). \quad (10)$$

We then build our coherent states $|\eta_{q,p}\rangle$ as normalized eigenvectors of $A_\nu = W_\nu(Q) + iP$ with eigenvalue $W_\nu(q) + ip$, the latter being the classical counterpart of $A_\nu$ as shown below (see table 1),

$$|\eta_{q,p}\rangle = N_\nu(q) \left( \delta_{W(q),q+ip} \right), \quad (q, p) \in \mathcal{K}, \quad (11)$$

where $\xi_q(x) = e^{ax/b} \sin^{\nu+1} \left( \frac{\pi}{L} x \right)$ for $x \in [0, L]$. The normalization coefficient $N_\nu(q)$ is given by

$$N_\nu(q) = 2^{\nu+1} \sqrt{\Gamma(\nu + 2) - i(\nu + 1) \cot \frac{\pi}{L} q}$$

$$\times \exp \left[ \frac{\pi}{2} (\nu + 1) \cot \frac{\pi}{L} q \right]. \quad (12)$$

For the sake of simplicity we drop off systematically in the sequel the $\nu$ dependence of various used symbols when no confusion is possible. It is possible to show that the function $x \mapsto |\eta_{q,p}(x)|$ reaches its maximal value for $x = q$ and $(P)_{p,q} = p$. Finally, the uncertainty relation
\[ \Delta W_\nu(Q) \Delta P \geq \frac{\hbar}{2} \langle W'_\nu(Q) \rangle \] is minimized by our CS as proved in [20].

**CS quantization and expected values.** – As is proved in [22], the CS family (11) resolves the unity with respect to the uniform measure on the phase space \( \mathcal{K} \):

\[
\int_{\mathcal{K}} \frac{dq \, dp}{2\pi \hbar} |\eta_{q,p}\rangle \langle \eta_{q,p}| = \mathbb{I}. \quad (13)
\]

As an immediate consequence we proceed with the CS quantization of “classical observables” \( f(q,p) \) through the correspondence [23,24]

\[
f(q,p) \rightarrow F = \int_{\mathcal{K}} \frac{dq \, dp}{2\pi \hbar} f(q,p) |\eta_{q,p}\rangle \langle \eta_{q,p}|. \quad (14)
\]

This operator-valued integral is understood as the sesquilinear form,

\[
B_f(\psi_1, \psi_2) = \int_{\mathcal{K}} \frac{dq \, dp}{2\pi \hbar} f(q,p) (|\psi_1\rangle |\eta_{q,p}\rangle \langle \eta_{q,p}| \langle \psi_2|).
\]

The form \( B_f \) is assumed to be defined on a dense subspace of the Hilbert space. If \( f \) is real and at least semi-bounded, Friedrich’s extension ([25], Thm. X.23) of \( B_f \) univocally defines a self-adjoint operator. However, if \( f \) is not semi-bounded, there is no natural choice of a self-adjoint operator associated with \( B_f \). In this case, we can consider directly the symmetric operator \( F \) given by eq. (14) enabling us to obtain a self-adjoint extension (unique for particular operators). The question of what is the class of operators that may be so represented is a subtle one [23,24]. In table 1, we give a list of operators obtained through the CS quantization of basic functions \( f \).

One can also compute the so-called “lower” or “covariant” symbols [23,24] of operators defined as the expectation values of the latter in the CS. In table 2 we give a list of functions of the most important quantum operators.

**Semi-classical behavior.** – For any normalized state \( \phi \in \mathcal{H} = L^2([0,L], dx) \), the resolution of unity (13) allows us to get its phase space representation \( \Phi(q,p) \equiv \langle \eta_{q,p}| \phi \rangle / \sqrt{2\pi \hbar} \) and the resulting probability distribution on the phase space \( \mathcal{K} \):

\[
\mathcal{K} \ni (q,p) \mapsto \frac{1}{2\pi \hbar} |\langle \eta_{q,p}| \phi \rangle|^2 = \rho_\phi(q,p). \quad (16)
\]
H. Bergeron et al.

H. Bergeron et al.

Fig. 1: (Colour on-line) Phase space distribution (16) for \( \nu = 0 \) of the state \( \eta_{q_0, p_0} \) with \( q_0 = L/5 \), \( p_0 = 4\pi\hbar/L \) and \( L = 20 \AA \). The thick curve is the expected phase trajectory in the infinite square well, deduced from the semi-classical Hamiltonian in eq. (18). The particle is an electron, its mean energy deduced from eq. (18) is \( E = 1.6 \text{ eV} \). Increasing values of the function are encoded by the colors from blue to red. Note that for Schrödinger CS the corresponding distribution is a Gaussian localized on points of a circular trajectory in the complex plane. See attached multimedia file Animation.gif for an animated time evolution.

The phase space distribution \( \rho_{\eta_{q_0, p_0}}(q, p) \) for a particular state \( \phi = |\eta_{q_0, p_0}\rangle \) is shown in fig. 1 for \( \nu = 0 \) (infinite square well).

Let us now examine the time behavior \( t \mapsto \rho_{\phi(t)}(q, p) \) for a state \( \phi(t) \) evolving under the action of the infinite square-well Hamiltonian \( H_0 \) (there is no significant difference from a generic \( \nu \neq 0 \) case):

\[
|\phi(t)\rangle = e^{-iH_0 t/\hbar}|\phi\rangle = \sum_{n=0}^{\infty} e^{-i\epsilon_0(n+1)^2 t/\hbar}|\phi_{n, 0}\rangle|\phi\rangle|\phi_{n, 0}\rangle,
\]

where \( \phi_{n, 0} = \sqrt{2} \sin \frac{(n+1)\pi x}{L} \).

With \( \phi = \eta_{q_0, p_0} \) as an initial state, we have for a given \( \nu \) (see table 2)

\[
|\eta_{q_0, p_0}(t)\rangle|H_0|\eta_{q_0, p_0}(t)\rangle = \frac{p_0^2}{2m} + \frac{1}{2\nu + 1} \frac{\epsilon_0(\nu + 1)^2}{\sin^2 \frac{T q_0}{2}}.
\]

Since the lower symbols of \( W_\nu(Q) \) and \( P \) correspond to their classical original functions \( W_\nu(q) \) and \( p \), respectively (see table 2), one can expect that the time average of the probability law \( \rho_{\eta_{q_0, p_0}(t)}(p, q) \) corresponds to some fuzzy extension in phase space of the classical trajectory corresponding to the time-independent Hamiltonian in the r.h.s. of (18). This key result is illustrated in fig. 2 where we have represented the time average distribution \( \bar{\rho} \) defined as

\[
\bar{\rho}(q, p) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_{\eta_{q_0, p_0}(t)}(q, p) dt,
\]

for the same values of the parameters as in fig. 1. The time average distribution \( \bar{\rho} \) allows us to compare the quantum behavior with the classical trajectory, but the expression (19) hides the complex details of the wave packet dynamics. The latter exhibits a splitting of the initial wave packet into secondary ones during the sharp reflection phase, each of them following the classical trajectory, before they amalgamate to reconstitute a unique packet (revival time). This important point makes the difference with the time behavior of the Schrödinger states for the harmonic oscillator.

Conclusion. – We have presented a family of CS for the \( T \)-potentials that sets a sort of natural bridge between the phase space and its quantum counterpart. These CS share with the Schrödinger ones some of their most striking properties, e.g. resolution of identity with uniform measure and saturation of uncertainty inequalities. They also possess remarkable evolution stability.
Semi-classical behavior of Pöschl-Teller coherent states

features (not to be confused with CS temporal stability in the sense of [17] corresponding to the time parametric evolution): their time evolution generated by \( H_\nu \) is localized on the classical phase space trajectory. The approach developed in this paper can be easily extended to higher-dimensional bounded domains, provided that the latter be symmetric enough (e.g. square, equilateral triangle, etc.) to allow shape invariance integrability. Furthermore it is a challenging question to specify the range of validity of the properties (b2) and (d) obtained for our CS with the considered class of \( T \)-potentials (e.g. are they valid for all shape-invariant potentials?), and, if it is not, under which condition(s) they remain valid.

***

PS appreciates the support by the Grant Agency of the Czech Republic project No. 202/08/H072 and the Czech Ministry of Education, Youth and Sports within the project LC06002. We wish to thank A. Comtet, J. Dittrich, P. Exner, J. R. Klauder, T. Paul and J. Tolar for fruitful discussions and comments.

REFERENCES

[1] **SCHRÖDINGER E.**, Naturwissenschaften, 14 (1926) 664.
[2] **PÖSCHL G.** and **TELLER E.**, Z. Phys., 83 (1933) 143.
[3] **ROSEN N.** and **MORSE P. M.**, Phys. Rev., 42 (1932) 210.
[4] **YILDIRIM H.** and **TOMAK M.**, Phys. Rev. B, 72 (2005) 115340.
[5] **WANG G., GUO Q., WU L.** and **YANG X.**, Phys. Rev. B, 75 (2007) 205337.
[6] **ROBERTSON H. P.**, Phys. Rev., 35 (1930) 667.
[7] **ROBERTSON H. P.**, Phys. Rev., 46 (1934) 794.
[8] **DODONOV V. V., KURMYSHEV E. V.** and **MAN’KO V. I.**, Phys. Lett. A, 79 (1980) 150.
[9] **GENDENSHTEIN L.**, JETP Lett., 38 (1983) 356.
[10] **COOPER F., KHARE A.** and **SUKHATME U. P.**, Supersymmetry in Quantum Mechanics (World Scientific Publishing Company, Singapore) 2002.
[11] **BALANTEKIN A. B.**, Phys. Rev. A, 57 (1998) 4188.
[12] **FUJII T.** and **AIZAWA N.**, Phys. Lett. A, 180 (1993) 308.
[13] **FATYGA B. W., KOSTELECKÝ V. A., NIETO M. M.** and **TRUAX D. R.**, Phys. Rev. D, 43 (1991) 1403.
[14] **SHREECHARAN T., PANIGRAHI P. K.** and **BANERJI J.**, Phys. Rev. A, 69 (2004) 012102.
[15] **CRAWFORD M. G. A.** and **VRSCAY E. R.**, Phys. Rev. A, 57 (1998) 106.
[16] **ALEXIO A.** and **BALANTEKIN A. B.**, J. Phys. A, 40 (2007) 3463.
[17] **ANTOINE J.-P., GAZEAU J.-P., MONCEAU P.**, **KLAUDER J. R.** and **PENSON K. A.**, J. Math. Phys., 42 (2001) 2349.
[18] **EL. KINANI A. H.** and **DAOUD M.**, Phys. Lett. A, 283 (2001) 291.
[19] **REED M.** and **SIMON B.**, Methods of Modern Mathematical Physics I. Functional Analysis, Vol. 1 (Academic Press, New York) 1972.
[20] **BERGERON H.** and **VALANCE A.**, J. Math. Phys., 36 (1995) 1572.
[21] **GESZTESY F.** and **KIRSCH W.**, J. Reine Angew. Math., 362 (1985) 28.
[22] **BERGERON H., GAZEAU J.-P., SIEGL P.** and **YOUSSEF A.**, in preparation.
[23] **KLAUDER J. R.** and **SKAGERSTAM B. S.**, Coherent States, Applications in Physics and Mathematical Physics (World scientific, Singapore) 1985.
[24] **GAZEAU J.-P.**, Coherent States in Quantum Physics (Wiley-VCH, Berlin) 2009.
[25] **REED M.** and **SIMON B.**, Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness, Vol. 2 (Academic Press, New York) 1975.