Unimodality of generalized Gaussian coefficients.

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January 1991

Abstract

A combinatorial proof of the unimodality of the generalized $q$-Gaussian coefficients $\left[ \frac{N}{\lambda} \right]_q$ based on the explicit formula for Kostka-Foulkes polynomials is given.

10. Let us mention that the proof of the unimodality of the generalized Gaussian coefficients based on theoretic-representation considerations was given by E.B. Dynkin [1] (see also [4], [10], [11]). Recently K.O’Hara [6] gave a constructive proof of the unimodality of the Gaussian coefficient $\left[ \frac{n + k}{n} \right]_q = s(k)(1, \cdots, q^k)$, and D. Zeilberger [12] derived some identity which may be considered as an “algebraization” of O’Hara’s construction. By induction this identity immediately implies the unimodality of $\left[ \frac{n + k}{n} \right]_q$.

Using the observation (see Lemma 1) that the generalized Gaussian coefficient $\left[ \frac{n}{\lambda'} \right]_q$ may be identified (up to degree $q$) with the Kostka-Foulkes polynomial $K_{\lambda', \mu}(q)$ (see Lemma 1), the proof of the unimodality of $\left[ \frac{n}{\lambda'} \right]_q$ is a simple consequence of the exact formula for Kostka-Foulkes polynomials contained in [4]. Furthermore the expression for $K_{\lambda', \mu}(q)$ in the case $\lambda = (k)$ coincides with identity (KOH) from [8]. So we obtain a generalization and a combinatorial proof of (KOH) for arbitrary partition $\lambda$. 

1
2. Let us recall some well known facts which will be used later. We base ourselves \cite{9} and \cite{5}. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition, $|\lambda|$ be the sum of its parts $\lambda_i$, $n(\lambda) = \sum_i (i-1)\lambda_i$ and $\left[ \begin{array}{c} n \\ \lambda \end{array} \right]_q$ be the generalized Gaussian coefficient.

Recall that

$$s_\lambda(1, \cdots, q^n) = q^{n(\lambda)} \left[ \begin{array}{c} n \\ \lambda' \end{array} \right]_q = \prod_{x \in \lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}}. \quad (1)$$

Here $c(x)$ is the content and $h(x)$ is the hook length corresponding to the box $x \in \lambda$, \cite{5}.

**Lemma 1** Let $\lambda$ be a partition and fix a positive integer $n$. Consider new partitions $\tilde{\lambda} = (n \cdot |\lambda|, \lambda)$ and $\mu = (|\lambda|^{n+1})$. Then

$$q^{\frac{n(n-1)}{2} - |\lambda| + n(\lambda)} \left[ \begin{array}{c} n \\ \lambda' \end{array} \right]_q = K_{\tilde{\lambda},\mu}^\prime(q). \quad (2)$$

Proof. We use the description of the polynomial $q^{\lambda} \cdot \left[ \begin{array}{c} n \\ \lambda' \end{array} \right]_q$ as a generating function for the standard Young tableaux of the shape $\lambda$ filled with numbers from the interval $[1, \cdots, n]$. Let us denote this set of Young tableaux by $STY(\lambda, \leq n)$. Then

$$q^{\lambda} \cdot \left[ \begin{array}{c} n \\ \lambda' \end{array} \right]_q = \sum_{T \in STY(\lambda, \leq n)} q^{|T|}. \quad (3)$$

Here $|T|$ is the sum of all numbers filling the boxes of $T$. For any tableau $T$ (or diagram $\lambda$) let us denote by $T[k]$ (or $\lambda[k]$) the part of $T$ (or $\lambda$) consisting of rows starting from the $(k+1)$-st one. Given tableau $T \in STY(\lambda, \leq n)$, then consider tableau $\tilde{T} \in STY(\tilde{\lambda}, \mu)$ such that $\tilde{T}[1] = T + supp \lambda[1]$, and we fill the first row of $\tilde{T}$ with all remaining numbers in increasing order from left to right. Here for any diagram $\lambda$ we denote by $supp\lambda$ the plane partition of the shape $\lambda$ and content $(1^{\lambda})$. It is easy to see that

$$c(\tilde{T}) = |T| + \frac{(n+1)(n-2)}{2} \cdot |\lambda|,$$

so we obtain the identity (2).
Let us consider an explanatory example. Assume $\lambda = (2, 1)$, $n = 3$. Then $\bar{\lambda} = (9, 2, 1)$, $\mu = (3^4)$. It is easy to see that $|STY(\lambda, \leq 3)| = 8$.

| $T$  | $|T|$ | $\bar{T}$   | $c(\bar{T})$ |
|------|------|-------------|--------------|
| 1 1  | 4    | 1 1 1 1 2 3 3 4 4 4 | 10           |
| 2    |      | 2 2          |              |
| 3    |      | 3            |              |
| 1 1  | 5    | 1 1 1 1 2 3 3 4 4 4 | 11           |
| 3    |      | 2 2          |              |
| 4    |      | 4            |              |
| 1 2  | 5    | 1 1 1 1 2 2 3 4 4 4 | 11           |
| 2    |      | 2 3          |              |
| 3    |      | 3            |              |
| 1 2  | 6    | 1 1 1 1 2 2 3 3 4 4 4 | 12           |
| 3    |      | 2 3          |              |
| 4    |      | 4            |              |
| 1 3  | 6    | 1 1 1 1 2 2 3 3 4 4 4 | 12           |
| 2    |      | 2 4          |              |
| 3    |      | 3            |              |
| 1 3  | 7    | 1 1 1 1 2 2 3 3 3 4 4 | 13           |
| 3    |      | 2 4          |              |
| 4    |      | 4            |              |
| 2 2  | 7    | 1 1 1 1 2 2 2 3 4 4 4 | 13           |
| 3    |      | 3 3          |              |
| 4    |      | 4            |              |
| 2 3  | 8    | 1 1 1 1 2 2 2 3 3 4 4 | 14           |
| 3    |      | 3 4          |              |
| 4    |      | 4            |              |

Now we would like to use the formula for Kostka-Foulkes polynomials, obtained in [4].
First let us recall some definitions from [4]. Given a partition $\lambda$ and composition $\mu$, a configuration $\{\nu\}$ of the type $(\lambda, \mu)$ is, by definition, a collection of partitions $\nu^{(1)}, \nu^{(2)}, \ldots$ such that

1) $|\nu^{(k)}| = \sum_{j \geq k+1} \lambda_j$;
2) $P_n^{(k)}(\lambda, \mu) := Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}) \geq 0$ for all $k, n \geq 1$, where $Q_n(\lambda) := \sum_{j \leq n} \lambda_j'$, and $\nu^{(0)} = \mu$.

**Proposition 1** [4] Let $\lambda$ be a partition and $\mu$ be a composition, then

$$K_{\lambda, \mu}(q) = \sum_{\{\nu\}} q^{c(\nu)} \prod_{k,n} \left[ P_n^{(k)}(\lambda, \mu) + m_n(\nu^{(k)}) \right]_q,$$

where the summation in (4) is taken over all configurations of $\{\nu\}$ of the type $(\lambda, \mu)$, $m_n(\nu^{(k)}) = (\nu^{(k)})'_n - (\nu^{(k)})'_n+1$.

From Lemma 1 and Proposition 1 we deduce

**Theorem 1** Let $\lambda$ be a partition. Then

$$\left[ \begin{array}{c} N \\ \lambda' \end{array} \right]_q = \sum_{\{\nu\}} q^{c(\nu)} \prod_{k,n} \left[ P_n^{(k)}(\lambda, N) + m_n(\nu^{(k)}) \right]_q,$$

where the summation in (5) is taken over all collections $\{\nu\}$ of partitions $\{\nu\} = \{\nu^{(1)}, \nu^{(2)}, \ldots\}$ such that

1) $|\nu^{(k)}| = \sum_{j \geq k} \lambda_j$, $k \geq 1$, $|\nu^{(0)}| = 0$,

2) $P_n^{(k)}(\nu, N) := n(N+1) \cdot \delta_{k,1} + Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}) \geq 0$, for all $k, n \geq 1$. Here

$$c_0(\nu) = n(\nu^{(1)}) - n(\lambda) + \sum_{k,n \geq 1} \left( \frac{\alpha_n^{(k)} - \alpha_n^{(k+1)}}{2} \right), \quad \alpha_n^{(k)} := (\nu^{(k)})'_n$$

and by definition $\left( \begin{array}{c} \alpha \\ 2 \end{array} \right) := \frac{\alpha(\alpha-1)}{2}$ for any $\alpha \in \mathbb{R}$.

The identity (5) may be considered as a generalization of the (KOH) identity (see [8]) for arbitrary partition $\lambda$.

**Corollary 1** The generalized $q$-Gaussian coefficient $\left[ \begin{array}{c} n \\ \lambda' \end{array} \right]_q$ is a symmetric and unimodal polynomial of degree $(N-1)|\lambda| - n(\lambda)$.
Proof. First, it is well known (e.g. [10],[11]) that the product of symmetric and unimodal polynomials is again symmetric and unimodal. Secondly, we use a well known fact (e.g.[10]), that the ordinary $q$-Gaussian coefficient \[ \binom{m+n}{n}_q \] is a symmetric and unimodal polynomial of degree $mn$. So in order to prove Corollary 1, it is sufficient to show that the sum

\[ 2c_0(\nu) + \sum_{k,n} m_n(\nu(k))P_n^{(k)}(\nu, N) \tag{7} \]

is the same for all collections of partitions $\{\nu\}$ which satisfy the conditions 1) and 2) of the Theorem 1. In order to compute the sum (7), we use the following result (see [4]):

**Lemma 2** Assume $\{\nu\}$ to be a configuration of the type $(\lambda, \nu)$. Then

\[ \sum_{k,n} m_n(\nu(k))P_n^{(k)}(\nu, \mu) = 2n(\mu) - 2c(\nu) - \sum_{n \geq 1} \mu'_n \cdot \alpha_n^{(1)}. \tag{8} \]

Using Lemma 2, it is easy to see that the sum (7) is equal to $(N - 1)|\lambda| - n(\lambda)$. This concludes the proof.

Note that in the proof of Corollary 1 we use symmetry and unimodality of the ordinary $q$-Gaussian coefficient \[ \binom{m+n}{n}_q \]. However, we may prove the unimodality of \[ \binom{m+n}{n}_q \] by induction using the identity (5) for the case $\lambda = (1^n)$, $N = m$.

**Remark 1.** The unimodality of generalized $q$-Gaussian coefficients was also proved in the recent preprint [7]. The proof in [7] uses the result from [4]. However [7] does not contain the identity (5).

**Remark 2.** The proof of the identity (4) given in [4] is based on the construction and properties of the bijection (see [4])

\[ STY(\lambda, \mu) \Leftrightarrow QM(\lambda, \mu). \]

It is an interesting task to obtain an analytical proof of (5). In the case $q = 1$ such a proof was obtained in [3].

**Acknowledgements.** The final version of this paper was written during the author’s stay at RIMS. The author expresses his deep gratitude to RIMS for its hospitality.
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