Efficient Online Estimation of Causal Effects by Deciding What to Observe

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Abstract

Researchers often face data fusion problems, where multiple data sources are available, each capturing a distinct subset of variables. While problem formulations typically take the data as given, in practice, data acquisition can be an ongoing process. In this paper, we aim to estimate any functional of a probabilistic model (e.g., a causal effect) as efficiently as possible, by deciding, at each time, which data source to query. We propose online moment selection (OMS), a framework in which structural assumptions are encoded as moment conditions. The optimal action at each step depends, in part, on the very moments that identify the functional of interest. Our algorithms balance exploration with choosing the best action as suggested by current estimates of the moments. We propose two selection strategies: (1) explore-then-commit (OMS-ETC) and (2) explore-then-greedy (OMS-ETG), proving that both achieve zero asymptotic regret as assessed by MSE. We instantiate our setup for average treatment effect estimation, where structural assumptions are given by a causal graph and data sources may include subsets of mediators, confounders, and instrumental variables.

1 Introduction

Statistical and causal modeling typically proceed from the assumption that we already know which variables are (and are not) observed. However, this perspective fails to address the difficult data collection decisions that precede such modeling efforts. Doctors must select a set of tests to run. Survey designers must select a slate of questions to ask. Companies must select which datasets to purchase. Whether or not we model these decisions, they pervade the practice of data science, influencing both what questions we can ask and how accurately we can answer them.

One might ask, why not collect everything? The answers are two-fold: First, data acquisition can be expensive. In a medical setting, blood tests can cost anywhere from tens to thousands of dollars. Running every test for every patient is infeasible. Likewise, space on surveys is limited, and asking every conceivable question of every respondent is infeasible. Second, in many settings, we lack complete control over the set of variables observed. Instead, we might have access to multiple data sources, each capturing a different subset of variables. Such data fusion problems pervade economic modeling and public health, and present interesting challenges: (i) efficiently estimating (or even identifying) a population parameter of interest often requires intelligently combining data from multiple sources; (ii) data collection is often iterative, with tentative conclusions at each stage informing choices about what data to collect next.

In this paper, we formalize the sequential problem of deciding, at each time, which data source to query (i.e., what to observe) in order to efficiently estimate a target parameter. We propose online moment selection (OMS), a framework that applies the generalized method of moments (GMM) to both estimate the parameter and to decide which data sources to query. This framework can be
applied to estimate any statistical parameter that can be identified by a set of moment conditions. For example, OMS can address (i) any (regular) maximum likelihood estimation problem [13, Page 109]; and (ii) estimating average treatment effects (ATEs) using instrumental variables (IVs), backdoor adjustment sets, mediators, and/or other identification strategies.

Our strategy requires only that the agent has sufficient structural knowledge to formulate the set of moment conditions and that each moment can be estimated using the variables returned by at least one of the data sources. Interestingly, the optimal decisions which lead to estimates with the lowest mean squared error (MSE) depend on the (unknown) model parameters. This motivates our adaptive strategy: as we collect more data, we better estimate the underlying parameters, improving our strategy for allocating our remaining budget among the available data sources.

We first address the setting where the cost per instance is equal across data sources (Section 4). First, we show that any fixed policy that differs from the oracle suffers constant asymptotic regret, as assessed by MSE. We then overcome this limitation by proposing two adaptive strategies—explore-then-commit (OMS-ETC) and explore-then-greedy (OMS-ETG)—both of which choose data sources based on the estimated asymptotic variance of the target parameter.

Under OMS-ETC, we use some fraction of the sample budget to explore randomly, using the collected data to estimate the model parameters. We then exploit the current estimated model, collecting the remaining samples according to the fraction expected to minimize our estimator’s asymptotic variance. In OMS-ETG, we continue to update our parameter estimates after every step as we collect new data. We prove that both policies achieve zero asymptotic regret. To overcome the non-i.i.d. nature of the sample moments, we draw upon martingale theory. To derive zero asymptotic regret, we show uniform concentration of sample moments and a finite-sample concentration inequality for the GMM estimator with dependent data. Next, we adapt OMS-ETC and OMS-ETG to handle heterogeneous costs over the data sources (Section 5) and prove that they still have zero asymptotic regret.

Finally, we validate our findings experimentally (Section 6). Motivated by ATE estimation in causal models encoded as directed acyclic graphs, we generate synthetic data from a variety of causal graphs and show that the regret of our proposed methods converges to zero. Furthermore, we see that despite being asymptotically equivalent, OMS-ETC outperforms OMS-ETG in finite samples. Finally, we demonstrate the effectiveness of our methods on two semi-synthetic datasets: the Infant Health Development Program (IHDP) dataset [19] and a Vietnam era draft lottery dataset [2].

1 Related Work

Many works attempt to identify and estimate causal effects from multiple datasets. [4, 21] study the problem of combining multiple heterogeneous datasets and propose methods for dealing with various biases. Other works study causal identification when observational and interventional distributions involving different sets of variables are available [28, 40]. [12] introduce estimators of the ATE that efficiently combine two datasets, one where confounders are observed (enabling the backdoor adjustment) and another where mediators are observed (enabling the frontdoor adjustment). [29] derive an estimator for the ATE in linear causal models with multiple confounders, where the confounders are observed in different datasets.

Another related line of work addresses finding optimal adjustment sets for covariate adjustment [18, 53, 45, 50]. While these works take for granted the collection of available datasets, we focus on the problem of deciding which data to collect. Our work shares motivation with active learning, where the learner strategically chooses which (unlabeled) samples to label in order to learn most efficiently [35, 26]. [8] design algorithms for actively collecting samples in a manner that minimizes the learner’s variance. In settings where there is a cost associated with collecting each feature, active feature acquisition methods incrementally query feature values to improve a predictive model [22, 34, 20]. [47] propose an active learning criterion to find the most informative questions to ask each respondent in a survey. In the context of causal inference, [37] study active structure learning of causal DAGs by finding cost-optimal interventions. [11] demonstrate that for ATE estimation, actively deconfounding data can improve sample complexity. [44] propose strategies for acquiring missing confounders to efficiently estimate the ATE.

The code and data are available at https://www.github.com/acmi-lab/online-moment-selection
Others have studied moment selection and IV selection from batch data. [1] introduce consistent moment selection procedures for the GMM setting with some incorrect moments. [6] propose an information-based lasso method for excluding invalid or redundant moment conditions. [41] propose a variable selection framework that uses lasso regression to decide which covariates to include. [14] propose four statistical criteria—including estimation efficiency and non-redundancy—for selecting among a set of candidate IVs. [9] develop an IV selection criteria based on asymptotic MSE and develop it for the GMM and generalized empirical likelihood estimators. [17] develop conservative confidence intervals for structural parameters when the off-diagonal entries of the covariance matrix of the empirical moments are unknown (e.g., when the moments are obtained from different datasets). By contrast, we are interested in selecting the data sources for the moments in an online setting.

Previous works address learning from adaptively collected data. [25] propose methods for adaptive experimental design, where at each step, the experimenter must decide the treatment probability using past data in order to efficiently estimate the ATE. [23, 24] propose a doubly-robust estimator of the empirical moments are unknown (e.g., when the moments are obtained from different datasets). Our methods are applicable to any setup, the moment conditions can be written as the Hadamard product, \(m \otimes g\), where \(m\) is some (possibly data dependent) positive definite matrix. In this work, we use the two-step GMM estimator, where the one-step estimator \(\tilde{\theta}_T\) is computed with \(\tilde{W} := I\) (identity) and the two-step estimator with \(\hat{W} := [\Omega_T(\tilde{\theta}_T)]^{-1}\), where \(\Omega_T(\tilde{\theta}_T) = \frac{1}{T} \sum_{t=1}^{T} g_t(\tilde{\theta}) g_t(\tilde{\theta})^\top\).

Let \(V\) be the set of variables of interest and \(\psi\) a collection of subsets of \(V\), each corresponding to the specific variables observable via one of the available data sources. Our methods are applicable whenever the target parameter can be identified by a set of moment conditions such that each moment depends on variables simultaneously observable in at least one data source. The selection vector, denoted by \(s_t \in \{0, 1\}^{|\psi|}\), is the binary vector indicating the data source selected at time \(t\).

**Assumption 1.** The agent queries one data source at each step: \(\sum_{i=1}^{|\psi|} s_{t,i} = 1\), i.e., \(s_t\) is one-hot.

We can handle the querying of multiple sources by adding the union of their variables to \(\psi\). In our setup, the moment conditions can be written as \(g_t(\theta) = m(s_t) \otimes \tilde{g}_t(\theta) \in \mathbb{R}^M\), where \(\otimes\) is the Hadamard product, \(m : \{0, 1\}^{|\psi|} \to \{0, 1\}^M\) is a fixed known function such that \(m(s_t)\) determines which moments get selected, and \(\tilde{g}_t(\theta)\) are i.i.d. across \(t\). For concreteness, we instantiate our setup with a simple example:

**Example 1** (Instrumental Variable (IV) graph). Consider a linear IV causal model (Figure 2a) with instrument \(Z\), treatment \(X\), outcome \(Y\), and the following data-generating process:

\[X := \alpha Z + \eta, \quad Y := \beta X + \epsilon, \quad \epsilon \perp \perp \eta, \quad \epsilon \perp \perp Z, \quad \eta \perp \perp Z.\]

The target parameter is the ATE \(\beta\). For \(\psi = \{\{Z, X\}, \{Z, Y\}\}\), the moment conditions are

\[g_t(\theta) = \begin{bmatrix} s_{t,1} Z_t(X_t - \alpha Z_t) \\ s_{t,2} Z_t(Y_t - \alpha Z_t) \end{bmatrix} = \begin{bmatrix} s_{t,1} 1_{Z_t(X_t - \alpha Z_t)} \\ (1 - s_{t,1}) 1_{Z_t(Y_t - \alpha Z_t)} \end{bmatrix},\]

where \(\theta = [\beta, \alpha]^\top\) and \(\{Z_t, X_t, Y_t\}\) are i.i.d.
For some known function $f_{\text{tar}}: \Theta \to \mathbb{R}$, let $\beta^* := f_{\text{tar}}(\theta^*)$ be the target parameter (e.g., the ATE). In practice, we estimate the target parameter by plugging-in the GMM estimate: $\hat{\beta} = f_{\text{tar}}(\hat{\theta})$. Let $H_t$ represent the history or the data collected until time $t$ with $H_0 = \emptyset$ and space $\mathcal{H}_t$. A data collection policy $\pi$ consists of a sequence of functions $\pi_t: \mathcal{H}_{t-1} \to \{0, 1\}^{|\psi|}$ with $s_t = \pi_t(H_{t-1})$. Thus $s_t$ can be dependent on data collected until time $(t-1)$ and so the sample moments $g_i(\theta)$ are not i.i.d.

**Definition 1** (Selection ratio). The selection ratio, denoted by $\kappa_T^{(\pi)}$, encodes the fraction of samples collected from each data source until time $t$: $\kappa_t^{(\pi)} = \frac{1}{t} \sum_{i=1}^t s_t \in \Delta_1^{\psi-1}$ (standard simplex).

We use $\hat{\theta}_t^{(\pi)}$ and $\tilde{\theta}_t^{(\text{os})}$ to denote the two-step and one-step GMM estimators, respectively, that use the data $H_t$. To reduce clutter, we use $\Delta_\psi := \Delta_1^{\psi-1}$, $\text{ctr} \(\Delta_\psi\) = \left[\frac{1}{|\psi|}, \frac{1}{|\psi|}, \ldots, \frac{1}{|\psi|}\right]$ (center of the simplex), and might drop the superscript $\pi$ from $\kappa_t$, $\hat{\theta}_t$, and $\tilde{\theta}_t$. $\|\cdot\|$ denotes the spectral and $l_2$ norms for matrices and vectors, respectively, and $N_\epsilon(\theta) := \{\theta' : \|\theta' - \theta\| \leq \epsilon\}$ ($\epsilon$-ball around $\theta$).

4 Adaptive Data Collection

The central challenge in this work is to make strategic decisions about which data to observe so that we most efficiently estimate the target functional. In this section, we present three policies: (i) fixed: query the data sources according to a pre-specified ratio; (ii) OMS-ETC: query uniformly for a specified exploration period, estimate the optimal ratio based on the inferred parameters and thereafter continue with the oracle ratio; and (iii) OMS-ETG: same as OMS-ETC but continue to update parameter (and thus oracle ratio) estimates after the exploration period. For now, we analyze these policies for the case where the cost to query is identical across data sources. For $T \in \mathbb{N}$, we denote the (known) horizon, which can be thought of as the agent’s data acquisition budget. We defer all proofs to Appendix [A].

We now present sufficient conditions for consistency and asymptotic normality of the GMM estimator under adaptively collected data. We later use these results to derive the regret of our policies.

**Assumption 2.** (a) (Identification) $\forall \theta \neq \theta^*, \mathbb{P} \left(\lim inf_{T \to \infty} Q(\theta) > 0\right) = 1$, where $Q(\theta) = g_T^*(\theta)\tilde{W}g_T^*(\theta)^\top$ and $g_T^*(\theta) = \left[\frac{1}{T} \sum_{t=1}^T m(s_t) \otimes \mathbb{E} \left[\tilde{g}_t(\theta)\right]\right]$; (b) $\Theta \subset \mathbb{R}^D$ is compact; (c) $\forall \theta, \mathbb{E} \left[\tilde{g}_t(\theta)\right]$ is twice continuously differentiable (c.d.); (d) $\forall \theta, \mathbb{E} \left[\tilde{g}_t(\theta)\right]$ is continuous; and (e) $f_{\text{tar}}$ is c.d. at $\theta^*$.

By Assumption 2(a), the GMM objective is uniquely minimized at $\theta^*$. Informally, this means that each moment is (asymptotically) collected enough times to allow identification. If $M = D$ (just-identified case), this holds when (i) an asymptotically non-negligible fraction of every moment is collected: $\forall j \in [M], \mathbb{P} \left(\lim inf_{T \to \infty} \frac{1}{T} \sum_{t=1}^T m_j(s_t) \neq 0\right) = 1$; and (ii) $\forall \theta \neq \theta^*, \mathbb{E} \left[\tilde{g}_t(\theta)\right] \neq 0$.

**Property 1** (ULLN). Let $a_i(\theta) := a(X_i; \theta) \in \mathbb{R}$ be a continuous function with $X_i$ sampled i.i.d. We say that $a_i(\theta)$ satisfies the ULLN property if (i) $\forall \theta$, $\mathbb{E} \left[a_i(\theta)^2\right] < \infty$; (ii) $a_i(\theta)$ is dominated by a function $A(X_i)$: $\forall \theta, |a_i(\theta)| \leq A(X_i)$; and (iii) $\mathbb{E}[A(X_i)] < \infty$.

**Proposition 1** (Consistency). Suppose that (i) Assumption 2 holds, (ii) $\forall j \in [M], \tilde{g}_t(\theta)\tilde{g}_t(\theta)^\top$ satisfies Property 1, and (iii) $\forall (i, j) \in [M]^2$, $[\tilde{g}_t(\theta)\tilde{g}_t(\theta)^\top]_{i,j} \text{ satisfies Property 1}$. Then, $\hat{\theta}_T^{(\pi)} \xrightarrow{T \to \infty} \theta^*$.

**Proposition 2** (Asymptotic normality). Suppose that (i) $\tilde{\theta}_T^{(\pi)} \xrightarrow{T \to \infty} \theta^*$; (ii) $\forall (i, j) \in [M] \times [D], \left[\frac{\partial^2 \mathbb{E}}{\partial \theta^2}(\tilde{\theta}_T^{(\pi)})\right]_{i,j}$ satisfies Property 1; (iii) $\exists \delta > 0$ such that $\mathbb{E} \left[\|\tilde{\theta}_T(\theta^*)\|^2 + \delta\right] < \infty$, and (iv) (Selection ratio convergence) $\kappa_T^{(\pi)} \xrightarrow{T \to \infty} k$ for some constant $k \in \Delta_\psi$. Then $\hat{\theta}_T$ is asymptotically normal:

$$\sqrt{T}(\hat{\theta}_T^{(\pi)} - \theta^*) \overset{d}{\to} \mathcal{N}(0, \Sigma(\theta^*, k)),$$

where $\Sigma(\theta^*, k)$ is a constant matrix that depends only on $\theta^*$ and $k$ (see Appendix [A] for the complete expression). By Assumption 2(e) and the Delta method, $\hat{\beta}_r$ is asymptotically normal:

$$\sqrt{T}(\hat{\beta}_T - \beta^*) \overset{d}{\to} \mathcal{N}(0, V(\theta^*, k)),$$

where $V(\theta^*, k) = \nabla_\theta f_{\text{tar}}(\theta^*)^\top \left[\Sigma(\theta^*, k)\right] \nabla_\theta f_{\text{tar}}(\theta^*)$.

Proposition 2 shows that for a policy under which the selection ratio $\kappa_T$ converges in probability to a constant, the GMM estimator $\hat{\theta}_T$ can be asymptotically normal. The specific order in which the data sources are queried does not affect asymptotic normality as long the selection ratio $\kappa_T^{(\pi)}$ converges.
Assumption 3 holds. Case (a): For a fixed policy \( \pi \), we have \( \kappa^\pi_T = \kappa^\pi \) as \( T \to \infty \) for some constant \( \kappa^\pi \). Here, the collection decisions do not depend on the data and each data source is queried a fixed fraction of the time. By Proposition 2, \( \hat{\kappa}_T(\pi) \) is asymptotically normal. The oracle policy, denoted by \( \pi^\ast \), is the fixed policy with the lowest asymptotic variance. Thus for \( \pi^\ast \), we have \( \kappa^\pi^\ast_T = \kappa^\ast \), where \( \kappa^\ast = \arg\min_{\kappa} V(\theta^\ast, \kappa) \). We call \( \kappa^\ast \) the oracle selection ratio. For the oracle policy, we have \( \sqrt{T}(\hat{\beta}^{(\pi^\ast)}_T - \beta^\ast) \overset{d}{\to} \mathcal{N}(0, V(\theta^\ast, \kappa^\ast)) \). The following assumption ensures that \( \kappa^\ast \) is unique and consequently the data collection decisions are meaningful.

**Assumption 3.** \( \kappa^\ast \) uniquely minimizes \( V(\theta^\ast, \kappa) \): \( \forall \kappa \in \Delta_{\psi} \) s.t. \( \kappa \neq \kappa^\ast \), \( V(\theta^\ast, \kappa) > V(\theta^\ast, \kappa^\ast) \).

**Definition 2 (Asymptotic regret).** The asymptotic regret captures how close the scaled asymptotic error of a given policy is to the oracle policy. We define the asymptotic regret of a policy \( \pi \) as

\[
R_{\infty}(\pi) = \text{AMSE} \left( \sqrt{T} (\hat{\beta}^{(\pi)}_T - \beta^\ast) \right) - V(\theta^\ast, \kappa^\ast),
\]

where \( \text{AMSE} \) is the asymptotic MSE (i.e., the MSE of the limiting distribution).

For any fixed policy \( \pi_k \) such that \( \kappa_T(\pi_k) = k \) for some constant \( k \neq \kappa^\ast \), we have \( R_{\infty}(\pi_k) = |V(\theta^\ast, k) - V(\theta^\ast, \kappa^\ast)| > 0 \) (by Assumption 3). This shows that a fixed policy suffers constant regret. This motivates the design of adaptive policies, where the regret asymptotically converges to zero.

### 4.1 Online Moment Selection via Explore-then-Commit (OMS-ETC)

OMS-ETC is inspired by the ETC strategy for multi-armed bandits (MABs) [27, Chapter 6]. Under OMS-ETC, we first explore by collecting a fixed number of samples for each choice. Then, we use these samples to estimate the oracle selection ratio \( \kappa^\ast \). Finally, we commit to this ratio for the remaining time steps. We denote the OMS-ETC policy by \( \pi_{\text{ETC}} \) (Figure 1a).

The policy \( \pi_{\text{ETC}} \) is characterized by an exploration fraction \( e \in (0, 1) \). We first collect \( Te \) samples by querying each data source equally so that \( \kappa_T e = \text{ctr}(\Delta_{\psi}) \). We then estimate \( \hat{\theta}_e \) and obtain the plugin estimate of \( \kappa^\ast \) as \( \hat{k} = \arg\min_{\kappa \in \Delta_{\psi}} V(\hat{\theta}_e, \kappa) \). The feasible region for \( \kappa_T \) is defined as the set of values that \( \kappa_T \) can take after we have devoted \( Te \) samples to exploration and is given by \( \hat{\Delta} = \{ e\kappa_T e + (1-e)\kappa : \kappa \in \Delta_{\psi} \} \) (proof in Appendix C). We collect the remaining \( T(1-e) \) samples such that \( \kappa_T \) is as close to \( \hat{k} \) as possible: \( \kappa_T = \text{proj}(\hat{k}, \hat{\Delta}) \), where \( \text{proj}(\hat{k}, \hat{\Delta}) = \arg\min_{\kappa \in \hat{\Delta}} \| \hat{k} - \kappa \| \).

**Remark.** The feasible region shrinks as \( e \) increases because, as \( e \) increases, the \( T(1-e) \) samples that remain after exploration decrease thereby shrinking the possible values that \( \kappa_T \) can take.

**Theorem 1 (Regret of OMS-ETC).** Suppose that (i) Conditions (i)-(iii) of Proposition 2 hold and (ii) Assumption 2 holds. Case (a): For a fixed \( e \in (0, 1) \), if \( \kappa^\ast \in \hat{\Delta} \), then the regret converges to zero.
The theorem provides sufficient conditions for when the regret converges to zero. Case (a) of the theorem shows that if we explore for a fixed fraction of the horizon $T$, the regret will only converge to zero if $\kappa^*$ is inside the feasible region. Thus the regret will not converge to zero over the entire parameter space $\Theta$ as there would be certain parameter values for which $\kappa^*$ would be outside the feasible region. Case (b) shows that we can achieve zero asymptotic regret for every $\theta^* \in \Theta$ by setting $e$ such that it becomes asymptotically negligible ($e \in o(1)$). The main idea in the proof (see Appendix A.3) is to show that $\kappa_T \not \rightarrow \kappa^*$ and apply Proposition 2. In Case (b), the feasible region $\hat{\Delta}$ asymptotically covers the entire simplex $\Delta \not \rightarrow \Delta_0$ and this is sufficient to show that $\kappa_T \not \rightarrow \kappa^*$.

### 4.2 Online Moment Selection via Explore-then-Greedy (OMS-ETG)

We extend OMS-ETC by periodically updating our estimate of $\kappa^*$ as we collect additional samples instead of committing to a value after exploration. The data is collected in batches. OMS-ETG (Figure 1b) is characterized by a batch fraction $s \in (0, 1)$. The algorithm runs for $J = \bar{J}$ rounds and we collect $b = Ts$ samples in each round. In the first round, we explore and thus $k_0 = c\tau(\Delta_0)$. After every round $j \in [J - 1]$, we estimate $\hat{\theta}_b$ and the oracle selection ratio: $\hat{k}_b = \arg\min_{\kappa \in \Delta_0} V(\hat{\theta}_b, \kappa)$. The feasible region for round $j + 1$ (the set of values of $\kappa_{b(j+1)}$ can take) is $\hat{\Delta}_{j+1}(\kappa_{b_j}) = \{\frac{\kappa_{b-j}+\kappa}{2} : \kappa \in \Delta_0\}$ (proof in Appendix C). In round $(j + 1)$, we (greedily) collect samples such that $\kappa_{b(j+1)}$ is as close to $\hat{k}_j$ as possible: $\kappa_{b(j+1)} = \text{proj}\left(\hat{k}_j, \hat{\Delta}_{j+1}(\kappa_{b_j})\right)$.

Theorem 2 states sufficient conditions for when OMS-ETG has zero regret. We first state a finite-sample tail bound for the two-step GMM estimator under adaptively collected (non-i.i.d.) data in Lemma 1 which might be of independent interest and use it to prove the theorem.

**Property 2 (Concentration).** Let $\hat{a}_i(\theta) := \hat{a}(X_i, \theta) \in \mathbb{R}$ with $X_i$ sampled i.i.d., $a_*(\theta) = \mathbb{E} [\hat{a}(X_i, \theta)]$, $A(X_i, \theta) = \frac{\partial \hat{a}(X_i, \theta)}{\partial \theta}$, $u_i(\eta) = \sup_{\theta, \theta' \in \Theta, \|\theta - \theta'\| \leq \eta} |\hat{a}(\theta) - \hat{a}(\theta')|$, and $u_*(\eta) = \mathbb{E} [u_i(\eta)]$. Let $L_{1, \eta_0}$ and $A_0$ be some positive constants. We say that $\hat{a}_i(\theta)$ satisfies the Concentration property if (i) $\hat{a}_i(\theta)$ is $L_1$-Lipschitz, (ii) $\forall \theta \in \Theta$, $|\hat{a}_i(\theta) - \hat{a}_i(\theta')| \leq u_*(\eta)$, (iii) $\mathbb{E} [\|A(X_i, \theta)\|] < A_0 < \infty$, and (iv) one of the following two conditions hold: (a) $\forall \eta \in (0, \eta_0)$, $\|u_i(\eta) - u_*(\eta)\|$ is sub-Exponential, or (b) $\sup_{\theta \in \Theta} \|A(X_i, \theta)\|$ is sub-Exponential.

**Remark.** Property 2 is used to derive a uniform law (see Lemma 4) that is used to prove Lemma 2. Property 2(iv) might be hard to check but (iv)(a) is satisfied for bounded function classes, i.e., when $\|\hat{a}_i\|_\infty < A < \infty$ (see Proposition 5) and (iv)(b) for linear function classes with sub-Exponential data (see Proposition 10). For the linear IV model in Example 2, Property 2(iv) would hold if $Z_i$ is sub-Exponential (e.g., when $Z_i$ is sub-Gaussian).

**Lemma 1 (GMM concentration inequality).** Let $\lambda_*, C_0, \eta_1, \eta_2$, and $\delta_0$ be some positive constants. Suppose that (i) Assumption 2 holds; (ii) $\forall j$, $\hat{g}_i(\theta)$ satisfies Property 2; (iii) The spectral norm of the GMM weight matrix $\hat{W}$ is upper bounded with high probability: $\forall \delta \in (0, C_0), P \left(\|\hat{W}\| \leq \Lambda_\delta \right) \geq 1 - \frac{1}{\delta^2} \exp \{ - \Omega (T \delta^2) \}$ (see Remark 1); (iv) (Local strict convexity) $\forall \theta \in N_{\eta_1}(\theta^*)$, $P \left(\|\hat{W}(\hat{\theta}(RT))^{-1} \| \leq \delta \right) = 1$ ($\hat{Q}(\theta)$ is defined in Assumption 2); (v) (Strict minimization) $\forall \theta \in N_{\eta_2}(\theta^*)$, there is a unique minimizer $\kappa(\theta) = \arg\min_{\kappa} V(\theta, \kappa)$ s.t. $V(\theta, \kappa) - V(\theta, \kappa(\theta)) \leq \delta^2 \Rightarrow \|\kappa - \kappa(\theta)\| \leq \delta$; and (vi) $\sup_{\theta \in \Theta} \|V(\theta, \kappa) - V(\theta', \kappa)\| \leq L \|\theta - \theta'\|$. Then, for $\hat{k}_T = \arg\min_{\kappa \in \Delta_0} V(\hat{\theta}_T, \kappa)$, any policy $\pi$, and $\forall \delta \in (0, \delta_0)$,

$$P \left(\|\hat{a}_i(\theta) - \hat{a}(\theta)\| > \delta \right) < \frac{1}{\delta^2} \exp \{ - \Omega (T \delta^4) \} \quad \text{and} \quad P \left(\|\hat{k}_T - \kappa^*\| > \delta \right) < \frac{1}{\delta^2} \exp \{ - \Omega (T \delta^4) \}.$$

Better rates for $\hat{k}_T$ are applicable under additional restrictions on $\theta^*$ (see Lemma 6).

To derive the tail bound for $\hat{\theta}_T$, we first prove that the minimized empirical GMM objective is close to $\hat{Q}(\theta^*)$ with high probability (w.h.p.) (using Conditions (i)-(iii)). Then we show that $\hat{\theta}_T$ is close to
\( \theta^* \) w.h.p. (using Condition (iv)). Next, we use the inequality for \( \hat{\theta} T \) to derive the tail bound for \( \hat{k} \). For this, we show that \( V(\theta^*, \hat{k}) \) is close to \( V(\theta^*, k^*) \) (using Condition (vi)) and then show that \( \hat{k} \) is close to \( k^* \) w.h.p. (using Condition (v)).

**Theorem 2** (Regret of OMS-ETG). Suppose that Conditions (i)-(iv) of Proposition 2 hold. Let 
\[ \Delta(s) = \{ \kappa \in \Delta : \kappa \in \Delta \} \]. Case (a): For a fixed \( s \in (0, 1) \), if the oracle selection ratio \( \kappa^* \in \Delta(s) \), then the regret converges to zero: \( R_\infty(\pi_{ETG}) = 0 \). If \( \kappa^* \in \Delta(s) \), then \( R_\infty(\pi_{ETG}) > 0 \) (non-zero regret). Case (b): Now also suppose that the conditions for Lemma 7 hold. If \( s = CT^{0.1} \) for some constant \( C \) and any \( \eta \in (0, 1) \), then \( \forall \hat{\theta}^* \in \Theta, R_\infty(\pi_{ETG}) = 0 \).

Similar to OMS-ETC, Case (a) of the theorem shows that if the batch size is a constant fixed fraction, some values of \( \kappa^* \) will be outside the feasible region and thus we do not get zero regret over the entire parameter space. Case (b) shows that to get zero asymptotic regret for every \( \theta^* \in \Theta \), \( s \) must depend on \( T \) and be asymptotically negligible. But unlike OMS-ETC, the estimate of \( \kappa^* \) is updated after every round. The batch fraction \( s \) can be as small as \( CT^{-1} \) allowing the agent to collect a constant number of samples in each batch.

We prove the theorem by showing that \( \kappa_T \xrightarrow{P} \kappa^* \) and applying Proposition 2. To do so for Case (b), we show that the estimated oracle ratio after every round is close to \( \kappa^* \), i.e., \( \forall \epsilon > 0, P \left( \| \kappa_T - \kappa^* \| < \epsilon \right) \rightarrow 1 \) (by using Lemma 1). Since we move as close as possible to \( \kappa_T \) after every round, this ensures that \( \forall \epsilon > 0, P \left( \| \kappa_T - \kappa^* \| < \epsilon \right) \rightarrow 1 \) (and thus \( \kappa_T \xrightarrow{P} \kappa^* \)).

Both OMS-ETC and OMS-ETG are asymptotically equivalent as they can achieve zero regret for every \( \theta^* \in \Theta \). But our experiments show that OMS-ETG outperforms OMS-ETC (in terms of regret) for small sample sizes. This may be because with OMS-ETG, the estimate of \( \kappa^* \) keeps improving as more samples are collected instead of being fixed after exploration. This suggests that better estimates of \( \kappa^* \) may lead to higher-order reductions in MSE.

**Remark 1** (Weight matrix \( \hat{W} \)). For OMS-ETG, the GMM weight matrix \( \hat{W} \) needs to satisfy Condition (iii) of Lemma 1 till round \((J - 1)\). We denote this matrix by \( \hat{W}_{valg} \) in Figure 1b. For the final step of OMS-ETG, we use the standard efficient two-step GMM weight matrix (denoted by \( \hat{W}_{efficient} \) in Figure 1b) and thus the final estimate of \( \theta^* \) is still asymptotically efficient. Condition (iii) of Lemma 1 would hold for the efficient two-step GMM weight matrix (i.e., for \( \hat{W} := \left[ \Omega_T(\tilde{\theta}_T^{*})^{-1} \right] \) if \( \forall(j,k), [g_{i,j}(\theta)]_{i,j} \) satisfies Property 2 (see Lemma 5 for proof). This would hold if the moments \( \tilde{g}_{i,j}(\theta) \) are uniformly bounded. Condition (iii) can also be satisfied with a regularized weight matrix:
\( \hat{W} := \left[ \Omega_T(\tilde{\theta}_T^{*})^{-1} \right] + \lambda_W I \) for some \( \lambda_W > 0 \) as this ensures that \( \| \hat{W} \| \leq \lambda_W^{-1} \).

**Additional exploration.** MAB algorithms usually require additional exploration to perform well (e.g., \( \epsilon \)-greedy and upper confidence bound strategies [27, Chapter 7]). However, we empirically noticed that additional exploration hurts performance in our setup. This might be because unlike typical bandit setups where pulling one arm does not improve the estimates of another arm, querying any data source can improve the estimates of the model parameter \( \theta^* \) in our setup.

## 5 Incorporating a Cost Structure

In many real-world settings, the agent has a budget constraint and must pay a different cost to query each data source. We adapt OMS-ETC and OMS-ETG to this setting where a cost structure is associated with the data sources in \( \psi \). We prove that these policies still have zero asymptotic regret for every \( \theta^* \in \Theta \). Let \( \{\psi_i\}_{i \in [\psi]} \) be an indexed family. We denote the (known) budget by \( B \in \mathbb{N} \) and by \( c \in \mathbb{R}^{[\psi]} \), a cost vector such that \( c_i \) is the cost of querying data source \( \psi_i \). Due to the cost structure, the horizon \( T \) is a random variable dependent on \( \pi \) with \( T = \left[ \frac{B}{\kappa^*c} \right] \). The setting in Section 4 is a special case of this formulation when \( \forall i, c_i = 1 \) and \( B = T \). We defer proofs to Appendix B.

For a fixed policy \( \pi_k \), we have \( \kappa_T(\pi_k) = k \), for some constant \( k \in \Delta \). By Proposition 2 as \( B \to \infty \), we have \( \sqrt{B} \left( \bar{\beta}_T(\pi_k) - \beta^* \right) \xrightarrow{d} N \left( 0, V(\theta^*, k)(k^T c) \right) \). Here we scale by \( \sqrt{B} \) instead of
\( \sqrt{T} \) to make comparisons across policies meaningful. The oracle selection ratio is now defined as 
\( \kappa^* = \arg \min_{\kappa \in \Delta_\psi} \left[ V(\theta^*, \kappa) \left( \kappa^\top c \right) \right] \) and the asymptotic regret of policy \( \pi \) now is 
\[
R_\infty(\pi) = \text{AMSE} \left( \sqrt{B} \left( \beta^*(\theta) - \beta^* \right) \right) - V(\theta^*, \kappa^*) \left( \kappa^* \right)^\top c.
\]

OMS-ETC-CS (OMS-ETC with cost structure) is an adaptation of OMS-ETC for this setting. We use 
\( Be \) budget to explore and estimate \( \kappa^* \) by 
\( \hat{k} = \arg \min_{\kappa \in \Delta_\psi} \left[ V(\hat{\theta}_{T_e}, \kappa) \left( \kappa^\top c \right) \right], \) where 
\( T_e = \left( \frac{eB}{\kappa^\top c} \right) \) and \( \kappa_{T_e} = \text{crt} (\Delta_\psi). \) Exploration strategies that utilize the cost structure can also be used (e.g., 
evenly dividing the budget across data sources while exploring). With the remaining budget, we 
collect samples such that \( \kappa_T = \text{proj} (\hat{k}, \Delta) \), where \( \Delta \) is the feasibility region (expression given in Appendix C). The following proposition shows that OMS-ETC-CS can achieve zero regret.

**Proposition 3** (Regret of OMS-ETC-CS). Suppose that the conditions of Theorem 1 hold. If \( e = \Theta(1) \) such that \( Be \to \infty \) as \( B \to \infty \), then \( \forall \theta^* \in \Theta, R_\infty(\pi_{ETC-CS}) = 0. \)

We propose two ways of adapting OMS-ETG to this setting: (i) OMS-ETG-FS (OMS-ETG with 
fixed samples) where we collect a fixed number of samples in every round and (ii) OMS-ETG-FB 
(OMS-ETG with fixed budget) where we spend a fixed fraction of the budget in every round.

Let 
\( c_{\text{max}} = \max_{i \in |\psi|} c_i \) and 
\( c_{\text{min}} = \min_{i \in |\psi|} c_i. \) In OMS-ETG-FS (Figure 6a), we collect 
\( b = \left( \frac{B}{c_{\text{max}}} \right) \) samples in every round except the last one (the last batch can be smaller as we may not 
have enough budget left for a full batch). The number of rounds \( J \) is random since, in every round, 
depending on what we collect, we use up a different amount of the budget. Like OMS-ETG, after 
each round, we estimate \( \kappa^* \) and greedily collect samples to get as close to it as possible.

In OMS-ETG-FB (Figure 6b), we spend \( Bs \) budget in each round. Thus the number of rounds \( J \) is 
fixed with \( J = \frac{1}{e} \) but the number of samples collected per round is now random. Like OMS-ETG, 
after each round, we estimate \( \kappa^* \) and collect samples to get as close to the estimate as possible. The 
next two Propositions show that both OMS-ETG-FS and OMS-ETG-FB have zero asymptotic regret.

**Proposition 4** (Regret of OMS-ETG-FS). Suppose that the conditions of Theorem 2b hold. If 
\( s = B^{\eta - 1} \) for any \( \eta \in [0, 1], \) then \( \forall \theta^* \in \Theta, R_\infty(\pi_{ETG-FS}) = 0. \)

**Proposition 5** (Regret of OMS-ETG-FB). Suppose that the conditions of Theorem 2b hold. If 
\( s = B^{\eta - 1} \) for any \( \eta \in [0, 1], \) then \( \forall \theta^* \in \Theta, R_\infty(\pi_{ETG-FB}) = 0. \)

### 6 Experiments

#### 6.1 Synthetic data

We validate our methods on synthetic data generated from known causal graphs (see Appendix D) for 
parameter values and moment conditions used. In our experiments (including the ones in Section 6.2, 
for OMS-ETG, we use the regularized GMM weight matrix with \( \lambda_W := 0.01 \) (see Remark 1). The 
regret is only minimally affected with an unregularized matrix \( \lambda_W := 0 \) despite theoretical 
guarantees only holding for the regularized case (the maximum change in regret was 1.98\%) and thus 
our conclusions do not change. We first simulate data from a linear IV graph (Figure 2a) and compare 
the performance of different polices (Figure 3a). We use \( \psi = \{\{Z, X\}, \{Z, Y\}\} \) and assume that 
both sources cost the same. We set parameter values such that \( \kappa^* \approx [0.36, 0.64]^\top. \) We compare 
policies based on relative regret (RR) with respect to the oracle policy:

\[
\text{Relative regret } = \text{RR}(\pi) = \frac{\text{MSE}(\pi) - \text{MSE}(\text{oracle})}{\text{MSE}(\text{oracle})} \times 100%.
\]

The MSE values are computed across 12,000 runs. The label \( \text{etc}_x \) in the plot refers to OMS-ETC 
with exploration fraction \( e = x \) (e.g., \( \text{etc}_0.1 \) means \( e = 10\%). Similarly, \( \text{etg}_x \) refers to 
OMS-ETG with \( s = x \). We see that as the horizon increases, the RR of all policies converges to zero. 
This supports the claim that both OMS-ETC and OMS-ETG have zero asymptotic regret. However, 
when the horizon is small (\( T = 300 \)), both \( \text{etg}_0.1 \) and \( \text{etg}_0.2 \) perform poorly due to insufficient 
exploration. In contrast, OMS-ETG has close to zero RR even for small horizons which shows that 
repeatedly improving the parameter estimates can lead to faster convergence of regret.
(b) Two IVs model.
(c) Confounder-mediator model.

Figure 2: Examples of causal models—with treatment \( X \) and outcome \( Y \)—where the ATE can be identified by different data sources returning different subsets of variables.

![Graphs](image)

(a) Instrumental variable graph. (b) Two IVs graph. (c) Confounder-mediator graph.

Next, we simulate data from a linear graph with two IVs (Figure 2b). Here \( \psi = \{(Z, X), (Z, Y)\} \) and both choices cost the same. We set the parameters such that \( \kappa^* = [0, 1] \) (\( \kappa^* \) is on the corner of the simplex). We compare the RR across various horizons (Figure 3b). We see the OMS-ETC performs worse than OMS-ETG for small horizons. One difference from the previous case (Figure 3a) is that OMS-ETC performs poorly even for large horizons. This is because after using 40% of the samples for exploration, the feasibility region is not large enough to get close to the corner of the simplex. This demonstrates another benefit of OMS-ETG over OMS-ETC: OMS-ETG can achieve close to zero regret in finite samples when the oracle ratio \( \kappa^* \) is either on the boundary or in the interior of the simplex.

Finally, we simulate data from a linear confounder-mediator graph (Figure 2c). Here, both the backdoor (using \( \{X, Y, W\} \)) and frontdoor (using \( \{X, Y, M\} \)) adjustments are applicable [31] Section 3.3. We use \( \psi = \{(X, Y, W), \{X, Y, M\}\} \) with cost structure \( c = [1.8, 1] \) (confounders \( W \) cost more than the mediators \( M \)). We set the parameters such that \( \kappa^* \approx [0.15, 0.85] \). We see similar conclusions as the previous cases. OMS-ETC with low exploration performs poorly but converges for large horizons. Both OMS-ETG variants—OMS-ETG-FS and OMS-ETG-FB—have close to zero RR for all horizons. We see no significant difference between the regret of OMS-ETG-FS and OMS-ETG-FB. The policy \( \text{fixed}_\text{equal} \) is a fixed policy that collects an equal fraction of both subsets. Its RR does not converge and is substantially higher than the oracle (\( \approx 20\% \)). This demonstrates that adaptive policies can lead to significant gains in MSE over fixed policies and that our methods remain applicable even with an associated cost structure on the data sources.

6.2 Semi-synthetic data

IHDP. Hill [19] constructed a dataset based on the Infant Health and Development Program (IHDP). The data [10] is from a randomized experiment studying the effect of home visits by a trained provider on future cognitive test scores of children. Following Hill [19], we create an observational dataset by removing a non-random subset of the data. The treatment \( X \) is binary. The dataset contains pre-treatment covariates which are measurements on the mother and the child. For simplicity, we only use two covariates: birth weight (continuous) \((W_1)\) and whether the mother smoked (binary) \((W_2)\). For each sample of the generated semi-synthetic data, \((X, W_1, W_2)\) are sampled uniformly at random from the real data. The outcome \( Y \) (continuous) is simulated:

\[
Y := \alpha_1 W_1 + \alpha_2 W_2 + \beta X + \epsilon_y,
\]
Figure 4: Relative regret (RR) on semi-synthetic data (error bars denote 95% CIs). For both IHDP (b) and Earnings data (c), adaptive policies converge to zero RR but fixed policies (collect_all, fixed_equal) suffer constant regret. OMS-ETG outperforms OMS-ETC for small horizons.

where $\epsilon_y \sim \mathcal{N}(0, \sigma^2_y)$, $\alpha_1, \alpha_2, \beta \in \mathbb{R}$, and $\sigma^2_y \in \mathbb{R}^+$ (see Figure 4a). Here $\alpha_1, \alpha_2, \beta$, and $\sigma_y$ are model parameters with $\beta$ being the ATE (target parameter).

For this experiment, we use $\psi = \{\{X, Y, W_1\}, \{X, Y, W_2\}, \{X, Y, W_1, W_2\}\}$ with cost structure $c = [1, 3, 3.5]^\top$. Thus, at each step, the agent can collect either one of the covariates or both of them, and each choice has a distinct cost. Setting model parameters such that $\kappa^* = [0.59, 0.0, 0.41]^\top$, we compare performance across policies for various total budgets (Figure 4b). The policy collect_all is a fixed policy that collects $\{X, Y, W_1, W_2\}$ at every step. This policy has higher RR ($\approx 50\%$ higher MSE than the oracle) for all budgets demonstrating that collecting all covariates for every sample can be sub-optimal. The policy etc_0.2 does poorly with a small budget whereas etc_0.4 has close to zero RR. Both OMS-ETG-FB and OMS-ETG-FS have close to zero RR for all horizons.

The Vietnam draft and future earnings. Angrist [2] computed the effect of veteran status on future earnings from the Vietnam draft lottery data [3] using an IV (Figure 2a). The IV Z (binary) indicates whether an individual was eligible for the draft based on a random lottery. The treatment X (binary) indicates whether they actually served. The outcome Y (continuous) represents their future earnings. The IV removes bias caused by certain types of men being more likely to serve. In this dataset, $\{Z, X\}$ and $\{Z, Y\}$ were collected using different data sources (thus $\{Z, X, Y\}$ are not observed simultaneously), which suits our framework. We construct a semi-synthetic dataset that closely matches the real data so that we know the ground-truth causal effect (needed to compute the MSE). For each instance, we sample Z uniformly at random from the empirical distribution. X is sampled from the Bernoulli distribution $\hat{P}(X|Z)$ with conditional probabilities given by the empirical distribution (values taken from [2] Table 2). We generate the outcome as $Y := \beta X + \gamma + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2_\epsilon)$ and $\epsilon \perp \perp X$. The parameters $\beta, \gamma$, and $\sigma^2_\epsilon$ are set such that the distribution of $(Z, Y)$ is close to the real data (see Appendix D.5 for details). We compare the RR of our policies on this dataset (Figure 4d). Most adaptive policies converge to near zero RR as the horizon gets large. OMS-ETC does poorly with low exploration while OMS-ETG policies have significantly lower RR for smaller horizons. By contrast, fixed_equal (the fixed policy that queries both data sources equally) suffers constant regret and has $\approx 25\%$ higher regret than the oracle even for large horizons. This demonstrates that adaptive policies can lead to substantial MSE gains in a real-world setting.

7 Conclusion

This paper takes some initial strides towards endogenizing decisions about which variables to solicit into the modeling process. Addressing the problem of deciding, sequentially, which data sources to query in order to efficiently estimate a parameter, we developed the online moment selection (OMS) framework and two instantiations: OMS-ETC and OMS-ETG. We prove that over the entire parameter space, adaptive data collection with either method can provide substantial MSE gains. While our work focuses on ATE estimation, our framework is more broadly applicable to any parameter identified by moment conditions. In future work, we hope to apply our framework to more general prediction problems, addressing practical considerations including high-dimensional data and complex model classes (e.g., neural networks). Moreover, in real-world settings, common assumptions like ignorability rarely hold. We hope to extend our framework to overcome issues such as model misspecification, or to overcome biases present in some, but not all data sources.
References

[1] D. W. Andrews. Consistent moment selection procedures for generalized method of moments estimation. *Econometrica*, 1999.

[2] J. D. Angrist. Lifetime earnings and the vietnam era draft lottery: evidence from social security administrative records. *The American Economic Review*, 1990.

[3] J. D. Angrist. Replication data for: Lifetime Earnings and the Vietnam Era Draft Lottery: Evidence from Social Security Administrative Records, 2009. URL https://doi.org/10.7910/DVN/PLF0YL.

[4] E. Bareinboim and J. Pearl. Causal inference and the data-fusion problem. *Proceedings of the National Academy of Sciences*, 2016.

[5] J. Bell. The lindeberg central limit theorem, 2015.

[6] X. Cheng and Z. Liao. Select the valid and relevant moments: An information-based lasso for gmm with many moments. *Journal of Econometrics*, 2015.

[7] M. D. Cocci and M. Plagborg-Møller. Standard errors for calibrated parameters. *Princeton University*, 2019.

[8] D. A. Cohn, Z. Ghahramani, and M. I. Jordan. Active learning with statistical models. *Journal of artificial intelligence research*, 1996.

[9] S. G. Donald, G. W. Imbens, and W. K. Newey. Choosing instrumental variables in conditional moment restriction models. *Journal of Econometrics*, 2009.

[10] V. Dorie. Non-parametrics for causal inference. https://github.com/vdorie/npci 2016.

[11] K. Gan, A. A. Li, Z. C. Lipton, and S. Tayur. Causal inference with selectively-deconfounded data. *arXiv preprint arXiv:2002.11096*, 2020.

[12] S. Gupta, Z. C. Lipton, and D. Childers. Estimating treatment effects with observed confounders and mediators. *arXiv preprint arXiv:2003.11991*, 2020.

[13] A. R. Hall. *Generalized method of moments*. Oxford university press, 2005.

[14] A. R. Hall and F. P. Peixe. A consistent method for the selection of relevant instruments. *Econometric reviews*, 2003.

[15] P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic press, 1980.

[16] J. D. Hamilton. *Time series analysis*. Princeton university press, 1994.

[17] L. P. Hansen. Large sample properties of generalized method of moments estimators. *Econometrica: Journal of the Econometric Society*, 1982.

[18] L. Henckel, E. Perković, and M. H. Maathuis. Graphical criteria for efficient total effect estimation via adjustment in causal linear models. *arXiv preprint arXiv:1907.02435*, 2019.

[19] J. L. Hill. Bayesian nonparametric modeling for causal inference. *Journal of Computational and Graphical Statistics*, 2011.

[20] S.-J. Huang, M. Xu, M.-K. Xie, M. Sugiyama, G. Niu, and S. Chen. Active feature acquisition with supervised matrix completion. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery &amp; Data Mining*, 2018.

[21] P. Hünermund and E. Bareinboim. Causal inference and data-fusion in econometrics. *arXiv preprint arXiv:1912.09104*, 2019.

[22] S. Ji and L. Carin. Cost-sensitive feature acquisition and classification. *Pattern Recognition*, 2007.
[23] M. Kato. Adaptive doubly robust estimator and paradox concerning logging policy: Off-policy evaluation from dependent samples, 2021.

[24] M. Kato. Adaptive doubly robust estimator from non-stationary logging policy under a convergence of average probability, 2021.

[25] M. Kato, T. Ishihara, J. Honda, and Y. Narita. Adaptive experimental design for efficient treatment effect estimation, 2020.

[26] P. Kumar and A. Gupta. Active learning query strategies for classification, regression, and clustering: A survey. *Journal of Computer Science and Technology*, 2020.

[27] T. Lattimore and C. Szepesvári. *Bandit algorithms*. Cambridge University Press, 2020.

[28] S. Lee and E. Bareinboim. Causal effect identifiability under partial-observability. In *International Conference on Machine Learning*. PMLR, 2020.

[29] H. Li, W. Miao, Z. Cai, X. Liu, T. Zhang, F. Xue, and Z. Geng. Causal data fusion methods using summary-level statistics for a continuous outcome. *Statistics in Medicine*, 2020.

[30] W. K. Newey and D. McFadden. Large sample estimation and hypothesis testing. *Handbook of econometrics*, 1994.

[31] J. Pearl. *Causality*. Cambridge university press, 2009.

[32] A. Rakhlin, K. Sridharan, and A. Tewari. Sequential complexities and uniform martingale laws of large numbers. *Probability Theory and Related Fields*, 2015.

[33] A. Rotnitzky and E. Smucler. Efficient adjustment sets for population average treatment effect estimation in non-parametric causal graphical models. *arXiv preprint arXiv:1912.00306*, 2019.

[34] M. Saar-Tsechansky, P. Melville, and F. Provost. Active feature-value acquisition. *Management Science*, 2009.

[35] B. Settles. Active learning literature survey. *Computer Science Technical Report 1648*, 2009.

[36] E. Smucler, F. Sapienza, and A. Rotnitzky. Efficient adjustment sets in causal graphical models with hidden variables. *arXiv preprint arXiv:2004.10521*, 2020.

[37] C. Squires, S. Magliacane, K. Greenewald, D. Katz, M. Kocaoglu, and K. Shanmugam. Active structure learning of causal dags via directed clique trees. In *Advances in Neural Information Processing Systems*, 2020.

[38] T. Tao. *Topics in random matrix theory*. American Mathematical Soc., 2012.

[39] G. Tauchen. Diagnostic testing and evaluation of maximum likelihood models. *Journal of Econometrics*, 1985.

[40] S. Tikka, A. Hyttinen, and J. Karvanen. Causal effect identification from multiple incomplete data sources: A general search-based approach. *arXiv preprint arXiv:1902.01073*, 2019.

[41] O. Urminsky, C. Hansen, and V. Chernozhukov. Using double-lasso regression for principled variable selection. *Available at SSRN 2733374*, 2016.

[42] R. Vershynin. *High-dimensional probability: An introduction with applications in data science*. Cambridge university press, 2018.

[43] M. J. Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*. Cambridge University Press, 2019.

[44] S. Wang, S. E. Yi, S. Joshi, and M. Ghassemi. Confounding feature acquisition for causal effect estimation. In *Machine Learning for Health*. PMLR, 2020.

[45] J. Witte, L. Henckel, M. H. Maathuis, and V. Didelez. On efficient adjustment in causal graphs. *Journal of Machine Learning Research*, 2020.
[46] R. Zhan, Z. Ren, S. Athey, and Z. Zhou. Policy learning with adaptively collected data. *arXiv preprint arXiv:2105.02344*, 2021.

[47] C. Zhang, S. J. Taylor, C. Cobb, J. Sekhon, et al. Active matrix factorization for surveys. *Annals of Applied Statistics*, 2020.

[48] K. W. Zhang, L. Janson, and S. A. Murphy. Inference for batched bandits, 2021.

[49] K. W. Zhang, L. Janson, and S. A. Murphy. Statistical inference with m-estimators on bandit data, 2021.
A  Omitted Proofs for Section 4

A.1 Proof of Proposition 1 (Consistency)

Proposition 6 (MDS LLN [15, Example 7.11]). Let $\bar{Y}_T$ be the sample mean from a martingale difference sequence (MDS), $\bar{Y}_T = \frac{1}{T} \sum_{i=1}^{T} Y_i$, with $\mathbb{E}[|Y_i|] < \infty$ for some $T > 1$. Then $\bar{Y}_T \xrightarrow{p} 0$.

Lemma 2 (Uniform convergence). Let $a_i(\theta) := S_i \bar{a}(\theta, X_i)$ be a real-valued function where $S_i \in \{0, 1\}$ is $H_{i-1}$-measurable and $X_i$ are i.i.d. Suppose that (i) $\Theta$ is compact and (ii) $\bar{a}(\theta, X_i)$ satisfies Property 1. Then

$$\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - S_i a_*(\theta)] \right| \xrightarrow{p} 0,$$

where $a_*(\theta) = \mathbb{E}[\bar{a}(\theta, X_i)]$.

Proof. We follow a standard uniform law of large numbers proof (e.g. Tauchen [39, Lemma 1]) and modify it to work for dependent data. The key modification is replacing the law of large numbers (LLN) in that proof with a MDS LLN.

Let $(\theta_1, \theta_2, \ldots, \theta_K)$ be a minimal $\delta$-cover of $\Theta$ and $N_\delta(\theta_k)$ denote the $\delta$-ball around $\theta_k$. By compactness of $\Theta$, $K$ is finite. For $k \in [K]$ and $\theta \in N_\delta(\theta_k)$, we have

$$\frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - S_i a_*(\theta)] = \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - a_i(\theta_k) + a_i(\theta_k) - S_i a_*(\theta_k) + S_i a_*(\theta_k) - S_i a_*(\theta)] \leq \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - a_i(\theta_k)] + \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_k) - S_i a_*(\theta_k)] + \frac{1}{T} \sum_{i=1}^{T} [S_i a_*(\theta_k) - S_i a_*(\theta)]$$

$$= \frac{1}{T} \sum_{i=1}^{T} [\bar{a}(\theta; X_i) - a(\theta; X_i)] + \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_k) - S_i a_*(\theta_k)] + \frac{1}{T} \sum_{i=1}^{T} [S_i a_*(\theta_k) - a_*(\theta)] \

\leq \frac{1}{T} \sum_{i=1}^{T} [\bar{a}(\theta; X_i) - \bar{a}(\theta_k; X_i)] + \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_k) - S_i a_*(\theta_k)] + |a_*(\theta_k) - a_*(\theta)|.$$

We now show that each of the three terms on the RHS above is small. In the third term, by continuity of $a_*(\theta), \forall \epsilon > 0, \exists \delta > 0$ s.t. $|a_*(\theta_k) - a_*(\theta)| < \epsilon$.

In the second term, $[a_i(\theta_k) - S_i a_*(\theta_k; S_i)]$ is a MDS. By Property 1(i) and Proposition 6, we have

$$\frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_k) - S_i a_*(\theta_k)] \xrightarrow{p} 0.$$

Next, we examine first term on the RHS. Let $u_i(\delta) = \sup_{\theta, \theta' \in \Theta, \|\theta - \theta'\| \leq \delta} |\bar{a}(\theta, X_i) - \bar{a}(\theta', X_i)|$. By continuity of $\bar{a}(\theta, X_i)$, compactness of $\Theta$, and the Heine-Cantor theorem, $\bar{a}(\theta, X_i)$ is uniformly continuous in $\theta$. This ensures that $u_i(\delta)$ is continuous in $\delta$ and thus $u_i(\delta) \xrightarrow{\delta} 0$ as $\delta \downarrow 0$. Since $u_i(\delta) \leq 2A(X_i)$ (by Property 1(iii)), using dominated convergence, we have $\mathbb{E}[u_i(\delta)] \downarrow 0$ as $\delta \downarrow 0$. Therefore, $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\mathbb{E}[u_i(\delta)] < \epsilon$. Thus we can write the first term as

$$\frac{1}{T} \sum_{i=1}^{T} [\bar{a}(\theta; X_i) - \bar{a}(\theta_k; X_i)] \leq \frac{1}{T} \sum_{i=1}^{T} u_i(\delta)$$

$$= \frac{1}{T} \sum_{i=1}^{T} u_i(\delta) - \mathbb{E}[u_i(\delta)] + \mathbb{E}[u_i(\delta)] \\

\leq \frac{1}{T} \sum_{i=1}^{T} u_i(\delta) - \mathbb{E}[u_i(\delta)] + \epsilon$$

$\overset{(a)}{=} o_p(1) + \epsilon$,
where (a) follows by the weak law of large numbers which applies because \( E[u_i(\delta)] \leq E[A(X_i)] < \infty \) (by Property \( \mathbb{P} \text{iii} \)).

**Proposition (Consistency).** Suppose that (i) Assumption 2 holds, (ii) \( \forall j \in [M] \), \( \tilde{g}_{i,j}(\theta) \) satisfies Property 1 and (iii) \( \forall (i, j) \in [M]^2 \), \( [\tilde{g}_{i,j}(\theta)\tilde{g}_{i,j}(\theta)^\top]_{i,j} \) satisfies Property 1. Then, for any policy \( \pi \), \( \tilde{\theta}_T \xrightarrow{p}{\to} \theta^* \).

**Proof.** We begin by defining the empirical and population analogues of the two-step GMM objective for a given policy \( \pi \):

Empirical objective: \( \hat{Q}_T^{(\pi)}(\theta) = \left[ \frac{1}{T} \sum_{t=1}^{T} g_t(\theta) \right]^\top \hat{W} \left[ \frac{1}{T} \sum_{t=1}^{T} g_t(\theta) \right] \),

Population objective: \( \bar{Q}_T^{(\pi)}(\theta) = \left[ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[g_t(\theta)|H_{t-1}] \right]^\top \bar{W} \left[ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[g_t(\theta)|H_{t-1}] \right] = \left[ \left( \frac{1}{T} \sum_{t=1}^{T} m(s_t) \right) \otimes g_*(\theta) \right]^\top \bar{W} \left[ \left( \frac{1}{T} \sum_{t=1}^{T} m(s_t) \right) \otimes g_*(\theta) \right] = [m_T \otimes g_*(\theta)]^\top \bar{W} [m_T \otimes g_*(\theta)] \),

where \( g_*(\theta) = \mathbb{E} [\tilde{g}_t(\theta)] \) and \( m_T = \frac{1}{T} \sum_{t=1}^{T} m(s_t) \). We have \( \hat{W} = \left[ \tilde{\Omega}_T(\tilde{\theta}_T^{(\pi)}) \right]^{-1} \), where \( \tilde{\theta}_T^{(\pi)} \) is the one-step GMM estimate and

\[
\tilde{\Omega}_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} \left[ g_t(\theta)g_t(\theta)^\top \right] = \frac{1}{T} \sum_{t=1}^{T} \left( [m(s_t)m(s_t)^\top] \otimes [\tilde{g}_t(\theta)\tilde{g}_t(\theta)^\top] \right).
\]

Furthermore, we have \( W = [m_{\Omega}(\kappa_T) \otimes \Omega(\theta^*)]^{-1} \), where

\[
m_{\Omega}(\kappa_T) = \sum_{t=1}^{T} (m(s_t)m(s_t)^\top) \),
\]
\[
\Omega(\theta) = \mathbb{E} \left[ \tilde{g}_t(\theta)\tilde{g}_t(\theta)^\top \right] .
\]

The two-step GMM estimator is obtained by minimizing the empirical objective: \( \tilde{\theta}_T = \arg \min_{\theta \in \Theta} \hat{Q}_T^{(\pi)}(\theta) \). At the true parameter \( \theta^* \), \( \bar{Q}_T^{(\pi)}(\theta^*) = 0 \) and by Assumption 2(a), \( \theta^* \) uniquely minimizes \( \bar{Q}_T^{(\pi)}(\theta) \). By Newey and McFadden [30 Theorem 2.1], \( \sup_{\theta \in \Theta} \left| \hat{Q}_T^{(\pi)}(\theta) - \bar{Q}_T^{(\pi)}(\theta^*) \right| \xrightarrow{p} 0 \Rightarrow \tilde{\theta}_T \xrightarrow{P} \theta^* \).

**Uniform convergence of \( \hat{Q}_T^{(\pi)}(\theta) \).** We now prove that \( \sup_{\theta \in \Theta} \left| \hat{Q}_T^{(\pi)}(\theta) - \bar{Q}_T^{(\pi)}(\theta^*) \right| \xrightarrow{p} 0 \). Following the proof of Newey and McFadden [30 Theorem 2.6], we have

\[
\left| \hat{Q}_T^{(\pi)}(\theta) - \bar{Q}_T^{(\pi)}(\theta^*) \right| \leq \left\| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m(s_t) \otimes g_*(\theta)] \right\|^2 \|\hat{W}\|^2 + 2 \|g_*(\theta)\| \left\| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m(s_t) \otimes g_*(\theta)] \right\| \|\tilde{W}\| + \|g_*(\theta)\|^2 \|\tilde{W} - W\| .
\]

(2)
We first prove that \( \| \hat{W} - W \| \overset{p}{\to} 0 \). Due to Condition (iii) of the theorem, we can apply Lemma 2 to get

\[
\forall (i, j) \in [M]^2, \forall \epsilon > 0, \quad P \left( \frac{\sum_{\theta \in \Theta} \left| \hat{\Omega}(\theta)_{i,j} - (\kappa_T \otimes \Omega(\theta)) \right|}{\epsilon} \right) \to 0,
\]

\[
\therefore \forall (i, j) \in [M]^2, \forall \epsilon > 0, \quad P \left( \frac{\left| \hat{\Omega}(\theta)_{i,j} - (\kappa_T \otimes \Omega(\theta)) \right|}{\epsilon} \right) \to 0,
\]

\[
\therefore \forall (i, j) \in [M]^2, \forall \epsilon > 0, \quad P \left( \frac{\left| \hat{\Omega}(\theta)_{i,j} - (\kappa_T \otimes \Omega(\theta)) \right|}{\epsilon} \right) \overset{(a)}{\to} 0,
\]

\[
\therefore \forall (i, j) \in [M]^2, \forall \epsilon > 0, \quad P \left( \left\| \hat{\Omega}(\theta)_{i,j} - (\kappa_T \otimes \Omega(\theta)) \right\| > \epsilon \right) \overset{(b)}{\to} 0,
\]

where (a) follows because \( \hat{\Omega}(\theta)_{i,j} \to \kappa_T \otimes \Omega(\theta) \) (by Proposition 1) and (b) by the continuous mapping theorem. Therefore, we have

\[
\left\| \hat{W} - W \right\| \leq \| W \| + o_p(1)
\]

Substituting these results in Eq. 2, we get

\[
\left| \hat{\bar{Q}}_{T_n}^{(\pi)}(\theta) - \bar{Q}_{T_n}^{(\pi)}(\theta) \right| \leq \left| \frac{1}{T} \sum_{i=1}^{T} [g_i(\theta) - m_i(\theta)] \right| \lambda_0 + \| g_i(\theta) \| \left\| \frac{1}{T} \sum_{i=1}^{T} [g_i(\theta) - m_i(\theta)] \right\| \lambda_0 + o_p(1).
\]

Thus, to show uniform convergence of \( \hat{Q}_{T_n}^{(\pi)}(\theta) \), we need to show that

\[
\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [g_i(\theta) - m_i(\theta)] \right| \overset{P}{\to} 0.
\]

For any \( \epsilon > 0 \), we have

\[
P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [g_i(\theta) - m_i(\theta)] \right| < \epsilon \right) \geq P \left( \left\| \frac{1}{M} \sum_{j=1}^{M} \left( \frac{1}{T} \sum_{i=1}^{T} [g_i(\theta) - m_i(\theta) g_j(\theta)] \right) \right\| < \epsilon \right)
\]

\[
\overset{(a)}{\geq} 1 - \frac{M}{\epsilon} P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [g_i(\theta) - m_i(s_i) g_j(\theta)] \right| \right) \geq 1 - o_p(1),
\]

\[
\overset{(b)}{\geq} 1 - o_p(1),
\]

where (a) follows by the union bound and (b) by applying Lemma 2 for every \( j \in [M] \) (using Condition (ii)).

A.2 Proof of Proposition 2 (Asymptotic normality)

**Proposition 7** (Martingale CLT [15] Corollary 3.1). Let \( M_i \) with \( 1 \leq i \leq n \) be a martingale adapted to the filtration \( F_i \) with differences \( X_i = M_i - M_{i-1} \) and \( M_0 = 0 \). Suppose that the following two conditions hold: (i) (Conditional Lindeberg) \( \forall \epsilon > 0, \sum_{i=1}^{n} E \left[ X_i^2 I \left( |X_i| > \epsilon \right) | F_{i-1} \right] \overset{p}{\to} 0 \), and (ii) (Convergence of conditional variance) For some constant \( \tau > 0, \sum_{i=1}^{n} E \left[ X_i^2 | F_{i-1} \right] \overset{p}{\to} \sigma^2 \).

Then \( \sum_{i=1}^{n} X_i \overset{d}{\to} N(0, \sigma^2) \).

**Proposition (Asymptotic normality).** Suppose that (i) \( \hat{\theta}_{T_n}^{(\pi)} \overset{p}{\to} \theta^* \); (ii) \( \forall (i, j) \in [M] \times [D], \frac{\partial g_i}{\partial \theta_j}(\theta) \) satisfies Property \( \mathbb{P} \); (iii) \( \exists \delta > 0 \) such that \( E \left[ \| \tilde{g}_i(\theta^*) \|^2 + \delta \right] < \infty \), and (iv) (Selection ratio convergence) \( \kappa_{T_n}^{(\pi)} \overset{p}{\to} k \) for some constant \( k \in \Delta_\psi \). Then \( \hat{\theta}_T \) is asymptotically normal: \( \sqrt{T} (\hat{\theta}_{T_n}^{(\pi)} - \theta^*) \overset{d}{\to} N(0, \Sigma(\theta^*, k)) \),
where $\Sigma(\theta^*, k)$ is a constant matrix that depends only on $\theta^*$ and $k$. By Assumption 2(e) and the Delta method, $\hat{\beta}_T$ is asymptotically normal:
\[
\sqrt{T}(\hat{\beta}_T - \beta^*) \xrightarrow{d} N(0, V(\theta^*, k)), \quad \text{where} \quad V(\theta^*, k) = \nabla_{\theta} f_{\text{tar}}(\theta^*)^\top \Sigma(\theta^*, k) \nabla_{\theta} f_{\text{tar}}(\theta^*). \]

**Proof.** We follow a standard GMM asymptotic normality proof (e.g. Newey and McFadden [30, Theorem 3.4]) and modify it to work for dependent data. Applying the GMM first-order condition to the two-step GMM estimator, we get
\[
\sqrt{T}(\hat{\theta}_T - \theta^*) = \left[ \hat{G}^\top(\hat{\theta}_T) \hat{\Omega}(\hat{\theta}_T) \hat{G}(\hat{\theta}) \right]^{-1} \hat{G}^\top(\hat{\theta}_T) \hat{\Omega}(\hat{\theta}_T) \hat{G}(\hat{\theta}) - \theta^* + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} g_t(\theta^*),
\]
where $\hat{\theta}_T$ is the one-step GMM estimator, $\hat{\theta}$ is a point on the line-segment joining $\hat{\theta}_T$ and $\theta^*$,

\[
\hat{G}(\theta) = \frac{1}{T} \sum_{t=1}^{T} \frac{\partial g_t(\theta)}{\partial \theta},
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{\partial m(s_t) \otimes \hat{g}_t(\theta)}{\partial \theta},
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left( m(s_t) \otimes \left[ \nabla_{\theta} \hat{g}_t(\theta) \right] \right),
\]

\[
\hat{\Omega}(\theta) = \frac{1}{T} \sum_{t=1}^{T} \left[ g_t(\theta) g_t(\theta) \right],
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left( m(s_t) m(s_t) \otimes \left[ \nabla_{\theta} \hat{g}_t(\theta) \nabla_{\theta} \hat{g}_t(\theta) \right] \right),
\]

where $m_G(s_t) = [m(s_1), m(s_2), \ldots, m(s_T)]$ is a $M \times D$ matrix and $m_{\Omega}(s_t) = m(s_t) m(s_t)^\top$.

**Convergence of $\hat{G}(\hat{\theta}_T)$.** Let $G(\theta) = E \left[ \frac{\partial g_t(\theta)}{\partial \theta} \right]$. Applying Lemma 2 to every element of $\hat{G}$ (using Condition (ii)) and using the union bound, we get
\[
\sup_{\theta \in \Theta} \left\| \hat{G}(\theta) - \left( \frac{1}{T} \sum_{t=1}^{T} m_G(s_t) \right) \otimes G(\theta) \right\| \xrightarrow{P} 0,
\]
\[
\therefore \forall \epsilon > 0, \quad P \left( \left\| \hat{G}(\hat{\theta}_T) - \left( \frac{1}{T} \sum_{t=1}^{T} m_G(s_t) \right) \otimes G(\hat{\theta}_T) \right\| > \epsilon \right) \to 0. \quad (3)
\]

Since $\kappa_T \xrightarrow{P} k$ for some constant $k$ (by Condition (iv)), $\left( \frac{1}{T} \sum_{t=1}^{T} m_G(s_t) \right) \xrightarrow{P} m_G^*(k)$ also converges in probability to a constant matrix. That is, $\frac{1}{T} \sum_{t=1}^{T} m_G(s_t) \xrightarrow{P} m_G^*(k)$ for some constant matrix $m_G^*(k)$ that only depends on $k$. By the continuity of $G$ and the fact that $\hat{\theta}_T \xrightarrow{P} \theta^*$ (by Condition (i)), we have $G(\hat{\theta}_T) \xrightarrow{P} G(\theta^*)$. Using these results with Eq. (3) we get
\[
\hat{G}(\hat{\theta}_T) \xrightarrow{P} m_G^*(k) \otimes G(\theta^*),
\]
\[
G(\theta), \quad (4)
\]

Similarly, $\hat{G}(\theta) \xrightarrow{P} G_*(\theta^*, k)$,
\[
\quad (5)
\]

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where $G_*(\theta^*, k) = m_G^*(k) \otimes G(\theta^*)$ and (a) follows because $\tilde{\theta} \xrightarrow{p} \theta^*$.

**Convergence of the weight matrix $\tilde{W}$**. Let $\Omega(\theta) = \mathbb{E} [\tilde{g}_t(\theta) \tilde{g}_t(\theta)^\top]$. By applying Lemma 3 to every element of $\tilde{\Omega}$ (using Condition (iii)) and the union bound, we get

$$
\sup_{\theta \in \Theta} \left\| \tilde{\Omega}(\theta) - \left( \frac{1}{T} \sum_{t=1}^T m_{\Omega}(s_t) \right) \otimes \Omega(\theta) \right\| \xrightarrow{p} 0,
$$

$$
\therefore \forall \epsilon > 0, P \left( \left\| \tilde{\Omega}(\tilde{\theta}_T^{(o)}) - \left( \frac{1}{T} \sum_{t=1}^T m_{\Omega}(s_t) \right) \otimes \Omega(\tilde{\theta}_T^{(o)}) \right\| > \epsilon \right) \rightarrow 0. \tag{6}
$$

Since $\kappa T \xrightarrow{p} k$ for some constant $k$ (by Condition (iv)), $\left( \frac{1}{T} \sum_{t=1}^T m_{\Omega}(s_t) \right) \xrightarrow{p} m_{\Omega}^*(k)$ for some constant matrix $m_{\Omega}^*(k)$ that only depends on $k$. By continuity of $\tilde{\Omega}$ and the fact that $\tilde{\theta}_T^{(o)} \xrightarrow{p} \theta^*$ (which follows by Proposition 1), we have $\tilde{\Omega}(\tilde{\theta}_T^{(o)}) \xrightarrow{p} \Omega(\theta^*)$. Using these results with Eq. 6, we get

$$
\tilde{\Omega}(\tilde{\theta}_T^{(o)}) \xrightarrow{p} m_{\Omega}^*(k) \otimes \Omega(\theta^*) = \Omega_*(\theta^*, k),
$$

$$
\therefore \tilde{W} = \tilde{\Omega}(\tilde{\theta}_T^{(o)})^{-1} \xrightarrow{p} \Omega_*(\theta^*, k)^{-1}, \tag{7}
$$

where $\Omega_*(\theta^*, k) = m_{\Omega}^*(k) \otimes \Omega(\theta^*)$.

**Asymptotic normality of $\frac{1}{\sqrt{T}} \sum_{t=1}^T v^\top g_t(\theta^*)$**. For this part, we use the Cramer-Wold theorem and the martingale CLT in Proposition 7. For any $v \in \mathbb{R}^M$ s.t. $\|v\| = 1$,$ \frac{v^\top g_t(\theta^*)}{\sqrt{T}}$ is a MDS because $\mathbb{E} [v^\top g_t(\theta^*)|H_{t-1}] = v^\top \mathbb{E} [g_t(\theta^*)|H_{t-1}] = 0$. We now show that the two conditions of Proposition 7 apply to this MDS.

(i) **Conditional Lindeberg**: The Lyapunov condition implies the Lindeberg condition [5] pg. 6]. In our case, the Lyapunov condition is easier to check and we show that it holds. For some $\delta > 0$, we have

$$
\frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \left\| v^\top g_t(\theta^*) \right\|^{2+\delta} \xrightarrow{(a)} \frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \left\| v \right\|^{2+\delta} \left\| g_t(\theta^*) \right\|^{2+\delta} \leq \frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \left\| g_t(\theta^*) \right\|^{2+\delta} \xrightarrow{(b)} \frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \left\| g_t(\theta^*) \right\|^{2+\delta},
$$

$$
\therefore \frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \mathbb{E} \left[ \left\| v^\top g_t(\theta^*) \right\|^{2+\delta} \mid H_{t-1} \right] \leq \frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \mathbb{E} \left[ \left\| g_t(\theta^*) \right\|^{2+\delta} \mid H_{t-1} \right] = \frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \mathbb{E} \left[ \left\| m(s_t) \otimes \tilde{g}_t(\theta^*) \right\|^{2+\delta} \right] \xrightarrow{(c)} \frac{1}{T^{1+\delta/2}} \sum_{t=1}^T \mathbb{E} \left[ \left\| \tilde{g}_t(\theta^*) \right\|^{2+\delta} \right] \xrightarrow{(d)} 0,
$$

where (a) follows by Cauchy-Schwarz, (b) because $\|v\| = 1$, (c) because $m(s_t)$ is a binary vector, and (d) because $\mathbb{E} \left[ \left\| \tilde{g}_t(\theta^*) \right\|^{2+\delta} \right] < \infty$ (by Condition (iii)).
(ii) Convergence of conditional variance: The conditional variance can be written as

\[
\frac{1}{T} \sum_{t=1}^{T} E \left[ v^T g_t(\theta^*) g_t(\theta^*)^T v | H_{t-1} \right] = \frac{1}{T} \sum_{t=1}^{T} v^T E \left[ g_t(\theta^*) g_t(\theta^*)^T | H_{t-1} \right] v
\]

Thus, by the Cramer-Wold theorem, we get

\[
\sqrt{T} \left( \theta_T - \theta^* \right) \overset{d}{\rightarrow} \mathcal{N} \left( 0, \Omega_*(\theta^*, k) \right),
\]

where \( (a) \) holds because \( \kappa_T \overset{p}{\rightarrow} k \) (by Condition (iv)). Thus, using Proposition 7 \( \forall v \in \mathbb{R}^M \) s.t. \( \|v\| = 1 \), we have

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} v^T g_t(\theta^*) v \overset{d}{\rightarrow} \mathcal{N} \left( 0, \Omega_*(\theta^*, k) v \right).
\]

Thus, by the Cramer-Wold theorem, we get

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} g_t(\theta^*) v \overset{d}{\rightarrow} \mathcal{N} \left( 0, \Omega_*(\theta^*, k) \right).
\]

A.3 Proof of Theorem 1 (Regret of OMS-ETC)

Lemma 3 (Consistency of \( \hat{k}_t \)). Suppose that Assumption 3 holds. If \( \hat{\theta}_t \overset{p}{\rightarrow} \theta^* \), then \( \hat{k}_t \overset{p}{\rightarrow} \kappa^* \) where

\[
\hat{k}_t = \arg\min_{k \in \Delta_v} V(\hat{\theta}_t, k).
\]

Proof. By continuity of \( V \), compactness of \( \Delta_v \), and Assumption 3 \( \hat{k}_t \overset{p}{\rightarrow} \arg\min_{k \in \Delta_v} V(\theta^*, k) = \kappa^* \).

Theorem (Regret of OMS-ETC). Suppose that (i) Conditions (i)-(iii) of Proposition 2 hold and (ii) Assumption 3 holds. Case (a): For a fixed \( e \in (0, 1) \), if \( \kappa^* \in \Delta \), then the regret converges to zero:

\[
R_\infty(\pi_{ETC}) = 0.
\]

Case (b): If \( e \) depends on \( T \) such that \( e = o(1) \) and \( Te \rightarrow \infty \) as \( T \rightarrow \infty \) (e.g. \( e = \frac{1}{\sqrt{T}} \)), then

\[
R_\infty(\pi_{ETC}) = 0.
\]

Proof. We first analyze Case (a) of the theorem where \( e \) is fixed. By Condition (i), \( \hat{\theta}_{Te} \overset{p}{\rightarrow} \theta^* \). We have \( \hat{k} = \arg\min_{k \in \Delta_v} V(\hat{\theta}_{Te}, k) \) and therefore \( \hat{k} \overset{p}{\rightarrow} \kappa^* \) (by Lemma 3). Thus, if \( \kappa^* \in \Delta \), then \( \kappa_T \overset{p}{\rightarrow} \hat{k} \) and therefore \( \kappa_T \overset{p}{\rightarrow} \kappa^* \). Using Proposition 2 we get

\[
\sqrt{T} \left( \hat{\beta}_T - \beta^* \right) \overset{d}{\rightarrow} \mathcal{N} \left( 0, V(\theta^*, \kappa^*) \right)
\]

Thus, if \( \kappa^* \in \Delta \), then

\[
R_\infty(\pi_{ETC}) = V(\theta^*, \kappa^*) - V(\theta^*, \kappa^*) = 0.
\]

If \( \kappa^* \notin \Delta \), then \( \kappa_T \overset{p}{\rightarrow} \kappa \neq \kappa^* \), where \( \kappa = \arg\min_{k \in \Delta} V(\theta^*, k) \). Using Proposition 2 we have

\[
\sqrt{T} \left( \hat{\beta}_T - \beta^* \right) \overset{d}{\rightarrow} \mathcal{N} \left( 0, V(\theta^*, \kappa) \right)
\]

Then, if \( \kappa^* \notin \Delta \), then

\[
R_\infty(\pi_{ETC}) = V(\theta^*, \kappa) - V(\theta^*, \kappa^*) \overset{(a)}{=} 0,
\]

\[
\begin{aligned}
1 &\sum_{t=1}^{T} E \left[ v^T g_t(\theta^*) g_t(\theta^*)^T v | H_{t-1} \right] = 1 \sum_{t=1}^{T} v^T E \left[ g_t(\theta^*) g_t(\theta^*)^T | H_{t-1} \right] v \\
&= v^T \left[ \left( 1 \sum_{t=1}^{T} m(s_t)m(s_t)^T \right) \otimes \Omega(\theta^*) \right] v \\
&= \frac{(a)}{p} v^T \left[ m_\delta^T(k) \otimes \Omega(\theta^*) \right] v \\
&= v^T \left[ \Omega_*(\theta^*, k) \right] v,
\end{aligned}
\]
where (a) follows by Condition (ii).

Now we analyze part (b) of the theorem. When \( e \) depends on \( T \) such that \( e = o(1) \), the feasible region converges to the entire simplex: \( \Delta \to \Delta_{\phi} \) as \( T \to \infty \). Thus \( \kappa_T \to \kappa^* \) and \( \Delta_{\phi} \to \Delta^* \). Using Proposition \[2\] we get the desired result. \[ \square \]

### A.4 Proof of Lemma 1 (GMM concentration inequality)

**Proposition 8** (MDS concentration inequality [43, Theorem 2.19]). Let \( \{ (D_k, F_k) \} \to \infty \) be a MDS, and suppose that \( \mathbb{E} [\exp \{ \lambda D_k \} | F_{k-1}] \leq \exp \left\{ \frac{\lambda^2 \nu^2}{2} \right\} \) almost surely for any \( \lambda < \frac{1}{\alpha} \). Then the sum satisfies the concentration inequality

\[
P \left( \left| \frac{1}{n} \sum_{k=1}^{n} D_k \right| > \eta \right) \leq 2 \exp \left\{ -\frac{\eta^2 \nu^2}{2 \alpha} \right\} \text{ if } 0 \leq \eta < \frac{\nu^2}{\alpha}.
\]

**Lemma 4** (Uniform law for dependent data). Let \( a_i(\theta) := S_i \tilde{a} (\theta; X_i) \), where \( a_i \) is a real-valued function, \( S_i \in \{0,1\} \) is \( H_{i-1} \)-measurable, and \( X_i \sim \mathbb{P}_{\theta} \). Let \( \tilde{a}_* (\theta) = \mathbb{E}[\tilde{a}(\theta; X_i)] \). Suppose that \( \tilde{a}(\theta) \) satisfies Property 2. Note that \( \mathbb{E}[a_i(\theta)|H_{i-1}] = S_i \tilde{a}_*(\theta) \). Then, for some constant \( \delta_0 > 0 \) and \( \forall \delta \in (0, \delta_0) \),

\[
P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - S_i \tilde{a}_*(\theta)] \right| > \delta \right) < \frac{1}{\delta^2} \exp \left\{ -\mathcal{O} \left( T \delta^2 \right) \right\}.
\]

**Proof.** Let \( U = \{ \theta_1, \theta_2, \ldots, \theta_N \} \) be a minimal \( \delta \)-cover of \( \Theta \). We have \( N \leq \frac{C}{\delta^2} \) for some constant \( C \).

Let \( q : \Theta \to U \) be a function that returns the closest point from the cover: \( q(\theta) = \arg \min_{\theta' \in U} \| \theta - \theta' \| \). We have

\[
\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - S_i \tilde{a}_*(\theta)] \right|
= \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - a_i(q(\theta))] + a_i(q(\theta)) - S_i \tilde{a}_*(q(\theta)) + S_i \tilde{a}_*(q(\theta)) - S_i \tilde{a}_*(\theta)] \right|
\leq \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - a_i(q(\theta))] + \max_{n \in [N]} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_n) - S_i \tilde{a}_*(\theta_n)] \right| + \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} S_i [\tilde{a}_*(q(\theta)) - \tilde{a}_*(\theta)] \right| \right|
\leq \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [\tilde{a}(\theta, X_i) - \tilde{a}_*(q(\theta), X_i)] + \max_{n \in [N]} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_n) - S_i \tilde{a}_*(\theta_n)] \right| + \sup_{\theta \in \Theta} \left| \tilde{a}_*(q(\theta)) - \tilde{a}_*(\theta) \right| \right|.
\]

We now examine the three terms on the RHS one at a time.

**Third term.** By Lipschitzness of \( \tilde{a}_* \) (Property 2(i)), we have:

\[
\sup_{\theta \in \Theta} \left| \tilde{a}_*(q(\theta)) - \tilde{a}_*(\theta) \right| \leq L_1 \sup_{\theta \in \Theta} \| q(\theta) - \theta \| \leq L_1 \delta.
\]

20
\textbf{Second term.} We note that it is a sum of a MDS. By Property \textsuperscript{2}(ii) and Proposition \textsuperscript{8} there exists a constant $C_1 > 0$ such that for $\delta \in (0, C_1)$, we have

$$\forall n \in [N], \; P \left( \frac{1}{T} \sum_{i=1}^{T} |\tilde{a}_i(\theta_n) - S_i\tilde{a}_*(\theta_n)| < \delta \right) > 1 - \exp \left\{ -\mathcal{O}(T^2\delta^2) \right\}$$

$$\therefore \; P \left( \max_{n \in [N]} \frac{1}{T} \sum_{i=1}^{T} |\tilde{a}_i(\theta_n) - S_i\tilde{a}_*(\theta_n)| < \delta \right) > 1 - P \left( \bigcup_{n \in [N]} \frac{1}{T} \sum_{i=1}^{T} |a_i(\theta_n) - S_i\tilde{a}_*(\theta_n)| > \delta \right)$$

$$> 1 - N \exp \left\{ -\mathcal{O}(T^2\delta^2) \right\}$$

$$> 1 - \frac{1}{\delta^D} \exp \left\{ -\mathcal{O}(T^2\delta^2) \right\}.$$

\textbf{First term.} We have

$$u_*(\eta) = E \left[ \sup_{\theta, \theta' \in \Theta; \|\theta - \theta'\| \leq \eta} |\tilde{a}_i(\theta, X_i) - \tilde{a}_i(\theta', X_i)| \right]$$

$$\leq E \left[ \sup_{\theta \in \Theta} \|A(X_i, \theta)\| \sup_{\theta, \theta' \in \Theta; \|\theta - \theta'\| \leq \eta} \|\theta - \theta'\| \right]$$

$$\leq \eta \sup_{\theta \in \Theta} \|A(X_i, \theta)\|$$

$$\leq A_0\eta, \quad (9)$$

where (a) follows by Property \textsuperscript{2}(iii).

Suppose that Property \textsuperscript{2}(iv)(a) holds. Then

$$\sup_{\theta \in \Theta} \frac{1}{T} \sum_{i=1}^{T} |\tilde{a}_i(\theta, X_i) - \tilde{a}_i(q(\theta), X_i)| \leq \frac{1}{T} \sum_{i=1}^{T} u_i(\delta)$$

$$\leq \frac{1}{T} \sum_{i=1}^{T} u_i(\delta) - u_*(\delta) + u_*(\delta)$$

$$\leq \frac{1}{T} \sum_{i=1}^{T} u_i(\delta) - u_*(\delta) + A_0\delta,$$

where (a) follows by Eq. \textsuperscript{9}. By Property \textsuperscript{2}(iv)(a), $(u_i(\delta) - u_*(\delta))$ is sub-Exponential. By the sub-exponential tail bound \textsuperscript{43} Proposition 2.9], for some constant $C_2 > 0$ and $\delta \in (0, C_2)$, we have

$$P \left( \left| \frac{1}{T} \sum_{i=1}^{T} u_i(\delta) - u_*(\delta) \right| < \delta \right) > 1 - \exp \left\{ -\mathcal{O}(T^2\delta^2) \right\}$$

$$\therefore \; P \left( \sup_{\theta \in \Theta} \frac{1}{T} \sum_{i=1}^{T} |\tilde{a}_i(\theta, X_i) - \tilde{a}_i(q(\theta), X_i)| < (A_0 + 1)\delta \right) > 1 - \exp \left\{ -\mathcal{O}(T^2\delta^2) \right\}$$

$$\therefore \; P \left( \sup_{\theta \in \Theta} \frac{1}{T} \sum_{i=1}^{T} |\tilde{a}_i(\theta, X_i) - \tilde{a}_i(q(\theta), X_i)| < \delta \right) > 1 - \exp \left\{ -\mathcal{O}(T^2\delta^2) \right\}.$$

Now suppose that Property \textsuperscript{2}(iv)(b) holds instead. Then

$$\sup_{\theta \in \Theta} \frac{1}{T} \sum_{i=1}^{T} |\tilde{a}_i(\theta, X_i) - \tilde{a}_i(q(\theta), X_i)| \leq \frac{1}{T} \sum_{i=1}^{T} \sup_{\theta \in \Theta} \|A(X_i, \theta)\| \sup_{\theta \in \Theta} \|\theta - q(\theta)\|$$

$$\leq \frac{\delta}{T} \sum_{i=1}^{T} \sup_{\theta \in \Theta} \|A(X_i, \theta)\|.$$
Thus \( u \) is sub-Exponential. Then, i.e.,

\[
\| \tilde{a}_i(q(\theta), X_i) \|_\infty < A < \infty.
\]

Combining these results together using the union bound, we get

\[
P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - S_i \tilde{a}_*(\theta; k)] \right| > (L_1 + L_2 + 2)\delta \right) < (L_1 + L_2 + 2)\delta
\]

\[
> P \left( \max_{n \in [N]} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_n) - S_i \tilde{a}_*(\theta_n)] \right| < \delta, \quad \left| \frac{1}{T} \sum_{i=1}^{T} u_i(\delta) - u_*(\delta) \right| < \delta \right)
\]

\[
> 1 - \sum_{n=1}^{N} P \left( \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta_n) - S_i \tilde{a}_*(\theta_n)] \right| > \delta \right) - P \left( \left| \frac{1}{T} \sum_{i=1}^{T} u_i(\delta) - u_*(\delta) \right| > \delta \right)
\]

\[
> 1 - \frac{1}{\delta^2} \exp \left\{ -\mathcal{O}(T\delta^2) \right\}
\]

\[
\therefore P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{i=1}^{T} [a_i(\theta) - S_i \tilde{a}_*(\theta; k)] \right| < \delta \right) > 1 - \frac{1}{\delta^2} \exp \left\{ -\mathcal{O}(T\delta^2) \right\}.
\]

\[\square\]

**Proposition 9** (Boundedness and Property [2](iv)(a)). Property [2](iv)(a) is satisfied for bounded function classes, i.e., when \( \| \tilde{a}_i \|_\infty < A < \infty \).

**Proof.** We have:

\[
u_i(\eta) = \sup_{\theta, \theta' \in \Theta, \| \theta - \theta' \| \leq \eta} |\tilde{a}(\theta, X_i) - \tilde{a}(\theta', X_i)|
\]

\[
\leq 2 \sup_{\theta \in \Theta} |\tilde{a}_i|
\]

\[
\leq 2A.
\]

Thus \( u_i(\eta) \) is bounded and therefore sub-Gaussian for every \( \eta \). \[\square\]

**Proposition 10** (Linearity and Property [2](iv)(b)). Suppose that (i) \( \tilde{a}(\theta, X_i) \) is a linear function of \( \theta \), i.e., \( \tilde{a}(\theta, X_i) = \theta^T \phi(X_i) + \rho(X_i) \), where \( \phi \) and \( \rho \) are arbitrary functions; and (ii) \( \forall d \in [D], \phi(X_i)_d \) is sub-Exponential. Then \( \tilde{a}(\theta, X_i) \) satisfies Property [2](iv)(b).
Proof. We have that $A(X_i, \theta) = \frac{\partial \tilde{g}(X_i, \theta)}{\partial \theta} = \phi(X_i)$ and thus $\sup_{\theta \in \Theta} \|A(X_i, \theta)\| = \|\phi(X_i)\| \leq \sum_{d=1}^{D} |\phi(X_i)_d|$. Therefore, for any $\eta > 0$, we have

$$
P \left( \sup_{\theta \in \Theta} \|A(X_i, \theta)\| < \eta \right) = P \left( \|\phi(X_i)\| < \eta \right) \geq P \left( \sum_{d=1}^{D} |\phi(X_i)_d| < \eta \right) \geq P \left( \forall d \in [D], |\phi(X_i)_d| < \frac{\eta}{D} \right) \geq 1 - \sum_{d=1}^{D} P \left( |\phi(X_i)_d| > \frac{\eta}{D} \right) \geq 1 - \sum_{d=1}^{D} \exp \{-O(\eta)\} \geq 1 - \exp \{-O(\eta)\},$$

where (a) follows by the union bound and (b) because $\phi(X_i)_d$ is sub-Exponential. This shows that $\sup_{\theta \in \Theta} \|A(X_i, \theta)\|$ is also sub-Exponential (see Vershynin [42, Definition 2.7.5]). \qed

Remark. Rakhlin et al. [32] derive a uniform martingale LLN and develop sequential analogues of classical complexity measures used in empirical process theory. These techniques are a potential alternative for deriving the tail bound in Lemma 4. However, the conditions required for these techniques are difficult to check. In our case, the dependent and i.i.d. components can be separated more easily. Thus we opted for deriving a uniform concentration bound by modifying the classical uniform LLN proof. Zhan et al. [46] also derive a uniform LLN without requiring boundedness of the martingale difference terms, but with structural assumptions on the summands related to their specific application.

Lemma (GMM concentration inequality). Let $\lambda_*, C_0, \eta_1, \eta_2$, and $\delta_0$ be some positive constants. Suppose that (i) Assumption 2 holds; (ii) $\forall j, \tilde{g}_{i,j}(\theta)$ satisfies Property 2; (iii) The spectral norm of the GMM weight matrix $\tilde{W}$ is upper bounded with high probability; $\forall \delta \in (0, C_0), P \left( \|\tilde{W}\| \leq \lambda_* \right) \geq 1 - \frac{1}{D^2} \exp \{-O(T \delta^2)\}$ (see Remark 1); (iv) (Local strict convexity) $\forall \theta \in N_{\eta_1}(\theta*)$, $P \left( \|\tilde{g}_{i}(\theta)\|^{-1} \leq h \right) = 1$ ($\tilde{g}(\theta)$ is defined in Assumption 2(a)); (v) (Strict minimization) $\forall \theta \in N_{\eta_2}(\theta*)$, there is a unique minimizer $\kappa(\theta) = \arg \min_{\kappa} V(\theta, \kappa)$ s.t. $V(\theta, \kappa) - V(\theta, \kappa(\theta)) \leq \delta^2 \implies \|\kappa - \kappa(\theta)\| \leq \delta$; and (vi) $\sup_{\kappa} |V(\theta, \kappa) - V(\theta', \kappa)| \leq L \|\theta - \theta'\|$. Then, for $\tilde{\kappa}_T = \arg \min_{\kappa \in \Delta_\epsilon} V(\tilde{\theta}_T, \kappa)$, any policy $\pi$, and $\forall \delta \in (0, \delta_0)$,

$$
P \left( \|\tilde{\theta}_T - \theta^*\| > \delta \right) < \frac{1}{\delta^2D} \exp \{-O(T \delta^4)\} \quad \text{and} \quad P \left( \|\tilde{\kappa}_T - \kappa^*\| > \delta \right) < \frac{1}{\delta^2D} \exp \{-O(T \delta^4)\}.$$

Proof. Below we give the empirical and population analogues of the GMM objective for a given policy $\pi$:

Empirical objective: $\tilde{Q}_T^{(\pi)}(\theta) = \left[ \frac{1}{T} \sum_{t=1}^{T} g_t(\theta) \right]^\top \tilde{W} \left[ \frac{1}{T} \sum_{t=1}^{T} g_t(\theta) \right],$

Population objective: $\tilde{Q}_T^{(\pi)}(\theta) = g_T^*(\theta) \tilde{W} g_T^*(\theta)^\top$

$$= \left[ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[g_t(\theta)|H_{t-1}] \right]^\top \tilde{W} \left[ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[g_t(\theta)|H_{t-1}] \right]$$

$$= \left[ \left( \frac{1}{T} \sum_{t=1}^{T} m(s_t) \right) \otimes \tilde{g}_*(\theta) \right]^\top \tilde{W} \left[ \left( \frac{1}{T} \sum_{t=1}^{T} m(s_t) \right) \otimes \tilde{g}_*(\theta) \right],$$

where $\tilde{g}_*(\theta) = \mathbb{E}[\tilde{g}_t(\theta)].$
To simplify notation, let \( m_t = m(s_t) \). By the triangle and Cauchy-Schwartz inequalities (see Newey and McFadden [30] Theorem 2.6),

\[
\left| \hat{Q}_T^{(\pi)}(\theta) - \tilde{Q}_T^{(\pi)}(\theta) \right| \\
\leq \left| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)] \right| \| W \|^2 + 2 \| \hat{g}_s(\theta) \| \left| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)] \right| \| W \|
\]

\[
\leq \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)] \| W \|^2 + 2 \left[ C \left( \frac{1}{T} \sum_{t=1}^{T} |g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)| \right) \| W \| \right],
\]

where \( C = \sup_{\theta \in \Theta} \| \hat{g}_s(\theta) \| \). By applying Lemma 4 to each element of the vector \( g_t(\theta) \) and using the union bound, we get:

\[
P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)] \right| < \delta \right) \geq P \left( \bigcap_{j=1}^{M} \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} [g_{t,j}(\theta) - m_{t,j} \otimes \tilde{g}_{s,j}(\theta)] \right| < \frac{\delta}{M} \right)
\]

\[
\geq 1 - \sum_{j=1}^{M} P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} [g_{t,j}(\theta) - m_{t,j} \otimes \tilde{g}_{s,j}(\theta)] \right| > \frac{\delta}{M} \right)
\]

\[
\geq 1 - \frac{1}{\delta^{2D}} \exp \left\{ -\Theta(T \delta^2) \right\}.
\]

This means that, for \( 0 < \delta < 1, \)

\[
\left| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)] \right| \leq \delta, \| W \| \leq \lambda \Rightarrow \left| \hat{Q}_T^{(\pi)}(\theta) - \tilde{Q}_T^{(\pi)}(\theta) \right| \leq \lambda^2 \delta^2 + 2\lambda \cdot C \delta
\]

\[
= (2C + \lambda \cdot \delta) \lambda \delta
\]

\[
< (2C + \lambda \cdot \delta) \lambda \delta.
\]

\[
\therefore P \left( \sup_{\theta \in \Theta} \left| \hat{Q}_T^{(\pi)}(\theta) - \tilde{Q}_T^{(\pi)}(\theta) \right| < (2C + \lambda \cdot \delta) \lambda \delta \right) \geq P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)] \right| \leq \delta, \| W \| \leq \lambda \right)
\]

\[
\geq \left( a \right) 1 - P \left( \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} [g_t(\theta) - m_t \otimes \tilde{g}_s(\theta)] \right| > \delta \right) - P \left( \| W \| > \lambda \right)
\]

\[
\geq \left( b \right) 1 - \frac{1}{\delta^{2D}} \exp \left\{ -\Theta(T \delta^2) \right\}
\]

\[
\therefore P \left( \sup_{\theta \in \Theta} \left| \hat{Q}_T^{(\pi)}(\theta) - \tilde{Q}_T^{(\pi)}(\theta) \right| < \delta \right) \geq 1 - \frac{1}{\delta^{2D}} \exp \left\{ -\Theta(T \delta^2) \right\},
\]

where (a) follows by the union bound and (b) follows by Eq. 10 and Condition (iii). Using this uniform concentration bound, we get

\[
P \left( \hat{Q}_T^{(\pi)}(\hat{\theta}_T) < \tilde{Q}_T^{(\pi)}(\hat{\theta}_T) + \frac{\delta}{2} \right) \geq 1 - \frac{1}{\delta^{2D}} \exp \left\{ -\Theta(T \delta^2) \right\},
\]

\[
P \left( \hat{Q}_T^{(\pi)}(\hat{\theta}^*) < \tilde{Q}_T^{(\pi)}(\hat{\theta}^*) + \frac{\delta}{2} \right) \geq 1 - \frac{1}{\delta^{2D}} \exp \left\{ -\Theta(T \delta^2) \right\}.
\]

Since \( \hat{\theta}_T \) minimizes \( \hat{Q}_T^{(\pi)} \) almost surely, we have \( P \left( \hat{Q}_T^{(\pi)}(\hat{\theta}_T) \leq \hat{Q}_T^{(\pi)}(\hat{\theta}^*) \right) = 1 \). Combining these inequalities using the union bound, we get

\[
P \left( \hat{Q}_T^{(\pi)}(\hat{\theta}_T) < \tilde{Q}_T^{(\pi)}(\hat{\theta}^*) + \delta \right) \geq 1 - \frac{1}{\delta^{2D}} \exp \left\{ -\Theta(T \delta^2) \right\}
\]

\[
\therefore P \left( \hat{Q}_T^{(\pi)}(\hat{\theta}) < \delta \right) \geq 1 - \frac{1}{\delta^{2D}} \exp \left\{ -\Theta(T \delta^2) \right\},
\]

where (a) follows because \( \hat{Q}_T^{(\pi)}(\hat{\theta}^*) = 0 \).
Intuitively, if \( \tilde{Q}_T^{(\pi)}(\hat{\theta}_T) \) is small, then we would expect \( \hat{\theta}_T \) to be close to \( \theta^* \). To formally show this, we use the local curvature of \( \tilde{Q}_T^{(\pi)} \). By Condition (iv), \( \tilde{Q}_T^{(\pi)} \) is locally strictly convex in the \( \eta_1 \)-ball \( N_{\eta_1}(\theta^*) \). Therefore, there exists a closed \( \gamma \)-ball \( N_\gamma(\theta^*) \) such that

\[
\forall \theta \notin N_\gamma(\theta^*), \quad \tilde{Q}_T^{(\pi)}(\theta) > Q_N, \quad \text{where} \quad Q_N = \sup_{\theta \in N_\gamma(\theta^*)} \tilde{Q}_T^{(\pi)}(\theta).
\]

This is analogous to an identification condition and ensures that \( \tilde{Q}_T^{(\pi)}(\theta) \leq Q_N \implies \theta \in N_\gamma(\theta^*) \).

Let \( H(\theta) = \frac{\partial^2 \tilde{Q}_T^{(\pi)}}{\partial \theta \theta}(\theta) \). Then, by twice continuous differentiability of \( g \), for \( \theta \in N_\gamma(\theta^*) \), we have

\[
\tilde{Q}_T^{(\pi)}(\theta) = (a) \tilde{Q}_T^{(\pi)}(\theta^*) + (\theta - \theta^*) [H(\theta')](\theta - \theta^*)^\top
\]

\[
(b) (\theta - \theta^*) [H(\theta')](\theta - \theta^*)^\top,
\]

\[
\therefore \|\theta - \theta^*\|^2 \leq \tilde{Q}_T^{(\pi)}(\theta) \|H^{-1}(\theta')\|
\]

\[
(c) \left[ \tilde{Q}_T^{(\pi)}(\theta) \right] h,
\]

where in (a), \( \theta' \) is a point on the line segment joining \( \theta \); (b) follows because \( \tilde{Q}_T^{(\pi)}(\theta^*) = 0 \); and (c) follows by Condition (iv). Thus, for \( \delta < Q_N \), we have

\[
\tilde{Q}_T^{(\pi)}(\hat{\theta}_T) < \delta \implies \|\hat{\theta}_T - \theta^*\| < \sqrt{\delta h}
\]

\[
\therefore P\left(\|\hat{\theta}_T - \theta^*\| < \delta\right) \geq P\left(\tilde{Q}_T^{(\pi)}(\hat{\theta}_T) < \delta^2 \right)
\]

\[
\geq 1 - \frac{1}{\delta^{2d}} \exp \left\{-O\left(T\delta^4\right)\right\}.
\]

**Concentration inequality for \( \hat{k}_T \)**

By Condition (vi), \( \sup_{\kappa \in \Delta_\psi} |V(\hat{\theta}_T, \kappa) - V(\theta^*, \kappa)| \leq L\|\hat{\theta}_T - \theta^*\| \). Therefore,

\[
\|\hat{\theta}_T - \theta^*\| < \delta \implies \sup_{\kappa \in \Delta_\psi} |V(\hat{\theta}_T, \kappa) - V(\theta^*, \kappa)| \leq L\delta.
\]

Furthermore, we have

\[
\sup_{\kappa \in \Delta_\psi} |V(\hat{\theta}_T, \kappa) - V(\theta^*, \kappa)| \leq L\delta \implies V(\theta^*, \hat{k}_T) < V(\theta^*, \hat{k}_T) + L\delta,
\]

\[
V(\theta^*, \kappa^*) < V(\theta^*, \kappa^*) + L\delta.
\]

Since \( \hat{k}_T \) is the minimizer, we have \( V(\hat{\theta}_T, \hat{k}_T) \leq V(\hat{\theta}_T, \kappa^*) \). Combining these inequalities, we get

\[
\|\hat{\theta}_T - \theta^*\| < \delta \implies V(\theta^*, \hat{k}_T) - V(\theta^*, \kappa^*) < 2L\delta.
\]

Due to Condition (v), we have

\[
V(\theta^*, \hat{k}_T) - V(\theta^*, \kappa^*) < 2L\delta \implies \|\hat{k}_T - \kappa^*\| < \sqrt{\frac{2L\delta}{c}},
\]

\[
\therefore \|\hat{\theta}_T - \theta^*\| < \delta \implies \|\hat{k}_T - \kappa^*\| < \sqrt{\frac{2L\delta}{c}}
\]

\[
\therefore P(\|\hat{k}_T - \kappa^*\| < \delta) > 1 - P\left(\|\hat{\theta}_T - \theta^*\| < O\left(\delta^2\right)\right)
\]

\[
> 1 - \frac{1}{\delta^{2d}} \exp \left\{-O\left(T\delta^8\right)\right\}.
\]

\[\square\]

**Lemma 5 (Sufficient condition for \( \hat{W} \)).** Suppose that \( \forall (j, k), \ [\hat{g}_{i,j}(\theta)\hat{g}_{i,k}(\theta)] \) satisfies Property 2. Let \( \hat{W}_T(\hat{\theta}_T^{(\alpha)}) = \Omega_T(\hat{\theta}_T^{(\alpha)})^{-1} = \left[ \frac{1}{T} \sum_{t=1}^T \hat{g}_t(\hat{\theta}_T^{(\alpha)})\hat{g}_t^\top(\hat{\theta}_T^{(\alpha)}) \right]^{-1} \), where \( \hat{\theta}_T^{(\alpha)} \) is the one-step GMM estimator (that uses \( \hat{W} = I \)). Then \( \hat{W}_T(\hat{\theta}_T^{(\alpha)}) \) satisfies Condition (iii) of Lemma 7.
Proof. We define $W_T(\theta^*)$ as

$$W_T(\theta^*) = \Omega_T(\theta^*)^{-1} = \left[ \frac{1}{T} \sum_{t=1}^{T} E \left[ g_t(\theta^*) g_t^T(\theta^*) \mid H_{t-1} \right] \right]^{-1}$$

$$= \left[ \left( \frac{1}{T} \sum_{t=1}^{T} m(s_t) m^T(s_t) \right) \otimes E \left[ \tilde{g}_t(\theta^*) \tilde{g}_t^T(\theta^*) \right] \right]^{-1}.$$

Let $\Delta = \hat{\Omega}_T(\hat{\theta}^{(m)}) - \Omega_T(\hat{\theta}^{(m)})$ and $\lambda_{\min}$ denote smallest eigenvalue. Using the eigenvalue stability inequality [38, Section 1.3.3], we get:

$$\lambda_{\min} \left( \hat{\Omega}_T(\hat{\theta}^{(m)}) \right) - \lambda_{\min} \left( \Omega_T(\hat{\theta}^{(m)}) \right) \leq \|\Delta\|,$$

$$\therefore \|\hat{W}_T(\hat{\theta}^{(m)})\| = \|\hat{\Omega}_T(\hat{\theta}^{(m)})^{-1}\| \leq \frac{1}{\lambda_{\min} \left( \Omega_T(\hat{\theta}^{(m)}) \right) - \|\Delta\|} \leq \frac{1}{\lambda_{\min} \left( \hat{\Omega}_T(\hat{\theta}^{(m)}) \right) - \|\Delta\|}.$$ (11)

By applying Lemma 4 to each term of the matrix and using the union bound, we have

$$P \left( \sup_{\theta \in \Theta} \|\hat{\Omega}_T(\theta) - \Omega_T(\theta)\| \leq \delta \right) \geq P \left( \sup_{\theta \in \Theta} \|\hat{\Omega}_T(\theta) - \Omega_T(\theta)\|_F \leq \delta \right)$$

$$\geq P \left( \sup_{\theta \in \Theta} \sum_{i,j} \hat{\Omega}_{T,i,j}(\theta) - \Omega_{T,i,j}(\theta) \leq \delta \right)$$

$$\geq 1 - \sum_{i,j} P \left( \sup_{\theta \in \Theta} \hat{\Omega}_{T,i,j}(\theta) - \Omega_{T,i,j}(\theta) > \frac{\delta}{M^2} \right)$$

$$= 1 - \frac{1}{\delta^2} \exp \left\{ -O \left( T \delta^2 \right) \right\}.$$ (12)

where in (a) $\|\cdot\|_F$ denotes the Frobenius norm.

For some $\delta_0 > 0$, let $\lambda = \inf_{\theta \in \Theta} \lambda_{\min}(\Omega_T(\theta))$. For $\delta \leq \min \left\{ \delta_0, \frac{\lambda}{2} \right\}$, we have

$$\|\Delta\| \leq \delta \implies \|\hat{W}_T(\hat{\theta})\| \leq \frac{2}{\lambda}.$$ (13)

$$\therefore \ P \left( \left\| \hat{W}_T(\hat{\theta}) \right\| \leq \frac{2}{\lambda} \right) \geq P \left( \|\Delta\| \leq \delta \right)$$

$$\geq 1 - \frac{1}{\delta^2} \exp \left\{ -O \left( T \delta^2 \right) \right\},$$ (b)

where (a) follows by Eq. [13] and (b) by Eq. [12].

In the next lemma, we present a concentration inequality for $\hat{k}_T$ with better rates under additional restrictions on $\theta^*$. We do not require these better rates for proving zero regret for OMS-ETG. We present this lemma for the sake of completeness.

**Lemma 6** (Another concentration inequality for $\hat{k}_T$). Let $\kappa(\theta) = \arg \min_{\kappa} V(\theta, \kappa)$, $\Theta_{\text{boundary}} = \{ \theta \in \Theta : \kappa(\theta) \in \text{boundary} (\Delta_{\psi}) \}$, where boundary $(\Delta_{\psi}) = \{ \kappa \in \Delta_{\psi} : \exists i, \text{ s.t. } \kappa_i = 0 \}$, $\Theta_{\text{minima}} = \{ \theta \in \Theta : \frac{\partial V}{\partial \kappa}(\theta, \kappa) = 0 \}$, and $\Theta_{\text{restricted}} = \Theta \setminus (\Theta_{\text{boundary}} \cap \Theta_{\text{minima}})$ Suppose that (i) the conditions of Lemma 4 hold, and (ii) $\theta \in \Theta_{\text{restricted}}$. Then

$$P \left( \left\| \hat{k}_T - \kappa \right\| > \delta \right) < \frac{1}{\delta^2 D} \exp \left\{ -O \left( T \delta^4 \right) \right\}.$$

This means that if $\theta^*$ is not such that the minimizer $\kappa(\theta) = \arg \min_{\kappa} V(\theta, \kappa)$ is on the boundary of the simplex and is also a local minimum of $V(\theta, \kappa)$ (informally, $\kappa(\theta)$ is not “just” on the boundary), we can get better rates.
Figure 5: Illustration of the proof of OMS-ETG algorithm. When the event \( \mathcal{I}(\varepsilon) \) occurs, (a) if the selection ratio \( \kappa_{b_j} \) is outside \( N_2(\kappa^*) \), then selection ratio in the next round \( \kappa_{b_{j+1}} \) will move closer to \( N_2(\kappa^*) \), and (b) if \( \kappa_{b_j} \) is inside \( N_2(\kappa^*) \), it remains inside for all future rounds.

**Proof.** Now we use the tail bound for \( \hat{\theta}_T \) to derive a concentration inequality for \( \hat{k}_T \) when \( \theta \in \Theta_{\text{restricted}} \). \( \hat{k}_T \) is the solution to the following constrained optimization problem:

\[
\min_{\kappa \in \mathbb{R}^{|\psi|}} V(\hat{\theta}_T, \kappa) \quad \text{subject to} \quad \sum_{i=1}^{|\psi|} \kappa_i = 1.
\]

The Lagrangian function is

\[
\mathcal{L}(\theta, \kappa, \lambda) = V(\theta, \kappa) + \lambda \left( \sum_{i=1}^{|\psi|} \kappa_i - 1 \right).
\]

Let \( f(\theta, \kappa, \lambda) = \frac{\partial \mathcal{L}}{\partial \kappa}(\theta, \kappa, \lambda) = \frac{\partial V}{\partial \kappa}(\theta, \kappa) + \lambda |1, 1, \ldots, 1|^{\top} \). Since \( \lambda |1, 1, \ldots, 1|^{\top} \neq 0 \), there exists a Lagrange multiplier \( \lambda^* \in \mathbb{R} \) such that \( f(\theta^*, \kappa^*, \lambda^*) = 0 \).

Condition (ii) is required to ensure that \( f(\theta, \kappa, \lambda^*) \) is continuously differentiable in \((\theta, \kappa)\) which allows us to use the implicit function theorem. To show this, we divide the space \( \Theta_{\text{restricted}} \) into two disjoint sets: (i) \( \Theta_{\text{interior}} = \Theta \setminus \Theta_{\text{boundary}} \), and (ii) \( \Theta_{\text{strict-boundary}} = \Theta_{\text{boundary}} \cap \Theta_{\text{minima}} \). When \( \theta \in \Theta_{\text{interior}} \), the constraint will be active and thus \( \lambda^* = 0 \). When \( \theta \in \Theta_{\text{strict-boundary}} \), the constraint will be active and thus \( \lambda^* > 0 \). In both cases, \( f(\theta, \kappa, \lambda^*) \) will be continuously differentiable in \((\theta, \kappa)\).

Note that if \( \theta \in \Theta \setminus \Theta_{\text{restricted}} \), then \( \lambda^* = 0 \) but \( f \) is not differentiable because the constraint is “just” inactive.

Let \( Y(\theta, \kappa) = \frac{\partial f}{\partial \kappa}(\theta, \kappa) = \frac{\partial^2 V}{\partial \kappa^2}(\theta, \kappa) \), and \( X(\theta, \kappa) = \frac{\partial f}{\partial \theta}(\theta, \kappa) = \frac{\partial^2 V}{\partial \theta \partial \kappa}(\theta, \kappa) \). By the implicit function theorem, since \( Y(\theta^*, \kappa^*) \) is invertible (by Condition (v)), there exist neighbourhoods \( N(\theta^*) \) and \( N(\kappa^*) \) and a function \( \phi : N(\theta^*) \to N(\kappa^*) \) such that \( \hat{k}_T = \phi(\hat{\theta}_T) \) and \( \frac{\partial \phi}{\partial \theta}(\theta) = -[Y(\theta, \phi(\theta))^{-1}X(\theta, \phi(\theta))] \). By a Taylor expansion, we get

\[
\hat{k}_T = \phi(\hat{\theta}_T) \overset{(a)}{=} \phi(\theta^*) + \frac{\partial \phi}{\partial \theta}(\hat{\theta}) (\hat{\theta}_T - \theta^*) \\
= \kappa^* + \frac{\partial \phi}{\partial \theta}(\hat{\theta}) (\hat{\theta}_T - \theta^*) \\

\therefore \|\hat{k}_T - \kappa^*\| \leq \left\| \frac{\partial \phi}{\partial \theta}(\hat{\theta}) \right\| \|\hat{\theta}_T - \theta^*\| \\
\leq C \left\| \hat{\theta}_T - \theta^*\right\|,
\]

where in (a) \( \hat{\theta} \) is a point on the line segment joining \( \hat{\theta}_T \) and \( \theta^* \), and \( C = \sup_{\theta \in N(\theta^*)} \left\| \frac{\partial \phi}{\partial \theta}(\theta) \right\| \).

Therefore, we have

\[
P\left(\|\hat{k}_T - \kappa^*\| \leq \delta\right) \geq P\left(\|\hat{\theta}_T - \theta^*\| \leq \frac{\delta}{C}\right) \\
\geq 1 - \frac{1}{\delta exp \{ -O(t\delta^4) \}}.
\]

\[\square\]

A.5 Proof of Theorem (Regret of OMS-ETG)

**Theorem** (Regret of OMS-ETG), Suppose that Conditions (i)-(iv) of Proposition hold. Let \( \Delta(s) = \{sk_0 + (1-s)k : k \in \Delta_v \} \). Case (a): For a fixed \( s \in (0, 1) \), if the oracle selection ratio
\[ \kappa^* \in \hat{\Delta}(s), \text{ then the regret converges to zero: } R_\infty(\pi_{ETG}) = 0. \] If \( \kappa^* \notin \hat{\Delta}(s) \), then \( R_\infty(\pi_{ETG}) = r \) for some constant \( r > 0 \). Case (b): Now also suppose that the conditions for Lemma \[7] hold. If \( s = CT^{\eta-1} \) for some constant \( C \) and any \( \eta \in [0, 1) \), then \( \forall \theta^* \in \Theta \), the regret converges to zero: \( R_\infty(\pi_{ETG}) = 0. \)

**Proof.** We prove this theorem by first showing that \( \kappa_T \xrightarrow{p} \kappa^* \). Then we can apply Proposition \[2] to get the desired result. Recall that \( b = Ts \) is the batch size.

**Case 1:** when \( s \in (0, 1) \) is a fixed constant and \( \kappa^* \in \hat{\Delta}(s) \).

Let \( \mathcal{I}(\epsilon) \) be the event that \( \hat{k}_{bj} \) remains inside an \( \epsilon \)-ball of \( \kappa^* \) (denoted by \( N_\epsilon(\kappa^*) \)) for all rounds \( j \in [J] \). That is, \( \mathcal{I}(\epsilon) = \left\{ \forall j \in [J], \hat{k}_{bj} \in N_\epsilon(\kappa^*) \right\} \). If \( \kappa^* \in \hat{\Delta}(s) \), then to prove that \( \kappa_T \xrightarrow{p} \kappa^* \), it is sufficient to show that \( \forall \epsilon > 0 \), \( \mathcal{I}(\epsilon) \) occurs w.p.a. \( 1 \).

This is because in OMS-ETG, after every round, we move as close to \( \hat{k}_{bj} \) as possible. This is illustrated in Figure \[5\] for the case when \( \Delta_\psi \) is a 1-simplex. When \( \mathcal{I}(\epsilon) \) occurs, if the selection ratio \( \kappa_{bj} \) after round \( j \) is outside \( N_\epsilon(\kappa^*) \), we move towards it in the subsequent round and thus \( \kappa_{b(j+1)} \) will be closer to \( N_\epsilon(\kappa^*) \). Once the selection ratio enters \( N_\epsilon(\kappa^*) \) (which it is guaranteed to if \( \kappa^* \in \hat{\Delta}(s) \)), it will remain inside \( N_\epsilon(\kappa^*) \) for every round after that. Thus \( \mathcal{I}(\epsilon) \implies \kappa_T \in N_\epsilon(\kappa^*) \). Therefore, we have

\[
\forall \epsilon > 0, \quad P(\kappa_T \in N_\epsilon(\kappa^*)) \geq P(\mathcal{I}(\epsilon)) \\
= P(\forall j \in [J], \hat{k}_{bj} \in N_\epsilon(\kappa^*)) \\
= 1 - P\left( \bigcup_{j=1}^{J} \left\| \hat{k}_{bj} - \kappa^* \right\| > \epsilon \right) \\
\geq 1 - \sum_{j=1}^{J} P\left( \left\| \hat{k}_{bj} - \kappa^* \right\| > \epsilon \right) \\
\xrightarrow{(a)} 1, \\
\xrightarrow{(b)} 1, \\
\therefore \kappa_T \xrightarrow{p} \kappa^*,
\]

where (a) follows by the union bound and (b) follows because \( J \) is finite and \( \forall j, \hat{k}_{bj} \xrightarrow{p} \kappa^* \) (by Lemma \[3\]).

**Case 2:** when \( s \) depends on the horizon \( T \).

**Case 2(a):** when \( s \in \Omega(T^{\eta-1}) \) for any \( \eta \in (0, 1) \).

Similar to Case 1, it is sufficient to show that the event \( \mathcal{I}(\epsilon) = \left\{ \forall j \in [J], \hat{k}_{bj} \in N_\epsilon(\kappa^*) \right\} \) occurs w.p.a. \( 1 \) for every \( \epsilon > 0 \). However, since \( J \to \infty \), consistency of \( \hat{k}_{bj} \) is no longer sufficient to prove
this. Instead, we use the concentration inequality in Lemma 1:

\[ \forall \epsilon > 0, \ P(\kappa_T \in N_\epsilon(\kappa^*)) \geq P(\mathcal{J}(\epsilon)) \]

\[ = P\left( \forall j \in [J], \ \tilde{k}_{bj} \in N_\epsilon(\kappa^*) \right) \]

\[ = P\left( \forall j \in [J], \ \|\tilde{k}_{bj} - \kappa^*\| \leq \epsilon \right) \]

\[ = 1 - P\left( \bigcup_{j=1}^{J} \|\tilde{k}_{bj} - \kappa^*\| > \epsilon \right) \]

\[ \geq 1 - \sum_{j=1}^{J} P\left( \|\tilde{k}_{bj} - \kappa^*\| > \epsilon \right) \]

\[ \geq 1 - \sum_{j=1}^{J} \frac{1}{\epsilon^{4D}} \exp \left\{ -\mathcal{O} \left( -Tsj\epsilon^8 \right) \right\} \]

\[ \geq 1 - \sum_{j=1}^{J} \frac{1}{\epsilon^{4D}} \exp \left\{ -\mathcal{O} \left( -Ts\epsilon^8 \right) \right\} \]

\[ = 1 - \frac{J}{\epsilon^{4D}} \exp \left\{ -\mathcal{O} \left( -Ts\epsilon^8 \right) \right\} \]

\[ = 1 - \frac{1}{s\epsilon^{4D}} \exp \left\{ -\mathcal{O} \left( -Ts\epsilon^8 \right) \right\} \]

\[ \rightarrow 1 \text{ if } s = CT\eta^{-1} \]

for any \( \eta \in (0, 1) \) and some constant \( C \). Here (a) follows by the union bound, (b) by Lemma 1, and (c) because \( j \geq 1 \).

**Case 2(b): when \( s = \frac{C}{T} \) for some constant \( C > 0 \).**

We prove this similarly to Case 2(a). However, in this case, the number of rounds \( J = \frac{1}{s} \in \mathcal{O}(T) \).

Let \( f = T^{-\gamma-1} \) for some \( \gamma \in (0, 1) \) and \( \mathcal{J}(f, \epsilon) = \left\{ \forall j \in [Jf + 1, \ldots, J], \ \tilde{k}_{bj} \in N_\epsilon(\kappa^*) \right\} \) be the event that \( \tilde{k}_{bj} \) remains inside \( N_\epsilon(\kappa^*) \) after the first \( Jf \) rounds.
Since $f \in o(1)$, we have $\mathcal{I}(f, \epsilon) \implies \kappa_T \in N_\epsilon(\kappa^*)$ for every $\epsilon > 0$. This is because the fraction $f$ is asymptotically negligible and thus we can effectively ignore the first $Jf$ rounds. Therefore we have

$$\forall \epsilon > 0, \ P(\kappa_T \in N_\epsilon(\kappa^*)) \geq P(\mathcal{I}(f, \epsilon))$$

$$= P \left( \forall j \in [Jf + 1, Jf + 2, \ldots, J], \left\| \hat{k}_{bj} - \kappa^* \right\| \leq \epsilon \right)$$

$$= 1 - P \left( \bigcup_{j=Jf + 1}^{J} \left\| \hat{k}_{bj} - \kappa^* \right\| > \epsilon \right)$$

$$\geq 1 - \sum_{j=Jf}^{J} P \left( \left\| \hat{k}_{bj} - \kappa^* \right\| > \epsilon \right)$$

$$\geq 1 - \sum_{j=Jf + 1}^{J} \frac{1}{e^{4D}} \exp \left\{ -\mathcal{O}(Ts\epsilon^8) \right\}$$

(a) $$\geq 1 - \sum_{j=Jf + 1}^{J} \frac{1}{e^{4D}} \exp \left\{ -\mathcal{O}(j\epsilon^8) \right\}$$

(b) $$\geq 1 - \sum_{j=Jf + 1}^{J} \frac{1}{e^{4D}} \exp \left\{ -\mathcal{O}(Jf\epsilon^8) \right\}$$

(c) $$\geq 1 - \sum_{j=Jf + 1}^{J} \frac{1}{e^{4D}} \exp \left\{ -\mathcal{O}(T\gamma \epsilon^8) \right\}$$

$$\geq 1 - \frac{T}{e^{4D}} \exp \left\{ -\mathcal{O}(T\gamma \epsilon^8) \right\}$$

$$\to 1,$$

where (a) follows because $Ts = C$, (b) because $j \geq Jf$, and (c) because $Jf = O(T\gamma)$. We note that it is possible to unify the analysis of Case 2(a) and Case 2(b) by ignoring the first $Jf$ rounds in Case 2(a) as well. We prove the two cases separately for the sake of clarity.

\[ \square \]

### B Incorporate Cost Structure

#### B.1 Proof of Proposition 3 (Regret of OMS-ETC-CS)

**Proposition (Regret of OMS-ETC-CS).** Suppose that the conditions of Theorem 1 hold. If $e = o(1)$ such that $Be \to \infty$ as $B \to \infty$, then $\forall \theta^* \in \Theta$, $R_\infty(\pi_{ETC-CS}) = 0$.

**Proof.** The proof is almost exactly like that of Theorem 1. We prove that $\kappa_T \to \kappa^*$ and then apply Proposition 2. Let the number of samples used for exploration be $T_e$. Since $\kappa_{T_e} = \left[ \frac{1}{|\psi|}, \frac{1}{|\psi|}, \ldots, \frac{1}{|\psi|} \right]$, we have

$$T_e = \frac{Be}{\kappa_T^*}.$$

$T_e$ is not a random variable because $\kappa_{T_e}$ is fixed. By Lemma 3, we have $\hat{k}_{T_e} \to \kappa^*$.

When $e \in o(1)$, the feasible region converges to the entire simplex, i.e., $\Delta \to \Delta_\psi$. Thus $\kappa_T \to \hat{k}_{T_e} \to 0$. \[ \square \]
**Figure 6:** Algorithms for OMS-ETG-FS and OMS-ETG-FB.

### B.2 Proof of Proposition 4 (Regret of OMS-ETG-FS)

**Proposition (Regret of OMS-ETG-FS).** Suppose that the conditions of Theorem 2 hold. If \( s = B^{\gamma - 1} \) and any \( \eta \in [0, 1) \), then \( \forall \theta^* \in \Theta, R_\infty(\pi_{ETG-FS}) = 0 \).

**Proof.** We can prove this similarly to Theorem 2. The key difference is that the number of rounds \( J \) is now a random variable. But we can use the fact that \( J \) is bounded:

\[
\frac{1}{s} \leq J \leq \frac{c_{\text{max}}}{s_{\text{min}}},
\]

\[
\therefore J \in \mathcal{O}(\frac{1}{s}).
\]

Now we can proceed like Case 2 in the proof of Theorem 2. 

### B.3 Proof of Proposition 5 (Regret of OMS-ETG-FB)

**Proposition (Regret of OMS-ETG-FB).** Suppose that the conditions of Theorem 2 hold. If \( s = B^{\gamma - 1} \) and any \( \eta \in [0, 1) \), then \( \forall \theta^* \in \Theta, R_\infty(\pi_{ETG-FB}) = 0 \).

**Proof.** We show this similarly to Theorem 2. In this case, the size of each batch is random but the numbers of rounds \( J = \frac{1}{s} \) is not random. Thus we can’t use the concentration inequality in Lemma 1 directly since that only holds for a fixed time step \( t \). We get around this by showing that the estimated selection ratio \( \tilde{k}_t \) will remain in an \( \epsilon \)-ball around \( \kappa^* \) uniformly over all time steps after some asymptotically negligible fraction of the horizon \( T \).

Let \( T_j \) be the number of samples collected after round \( j \), i.e., \( T_j = \frac{B_{sj}}{\kappa_j c} \). Let \( f = B^\gamma - 1 \) for some \( \gamma \in (0, 1) \). Like the proof of Theorem 2, we can ignore the first \( J f \) rounds since they are \( f \in o(1) \) is
an asymptotically negligible fraction. And similarly to the proof of Theorem 2, in order to show that \( \kappa_T \xrightarrow{P} \kappa^* \), it is sufficient to show that \( \mathbf{P} \left( \forall j \in [J_f + 1, J_f + 2, \ldots, J], \left\| \hat{k}_{T_j} - \kappa^* \right\| \leq \epsilon \right) \xrightarrow{B \to \infty} 1 \). We can show this as follows:

\[
\mathbf{P} \left( \forall j \in [J_f + 1, J_f + 2, \ldots, J], \left\| \hat{k}_{T_j} - \kappa^* \right\| \leq \epsilon \right) \geq \mathbf{P} \left( \forall t \in [T_{J_f + 1}, \ldots, T_J], \left\| \hat{k}_t - \kappa^* \right\| \leq \epsilon \right).
\]

The minimum and maximum batch sizes are \( b_{\text{min}} = \frac{B s}{c_{\text{max}}} \) and \( b_{\text{max}} = \frac{B s}{c_{\text{min}}} \), respectively. Therefore,

\[
T_{J_f + 1} \geq J f b_{\text{min}} = J f \frac{B s}{c_{\text{max}}}, \\
T_J \leq J b_{\text{max}} = J \frac{B s}{c_{\text{min}}}.
\]

Using these facts and continuing Eq. 13, we get:

\[
\mathbf{P} \left( \forall j \in [J_f + 1, J_f + 2, \ldots, J], \left\| \hat{k}_{T_j} - \kappa^* \right\| \leq \epsilon \right) \geq \mathbf{P} \left( \forall t \in [J f b_{\text{min}}, \ldots, J b_{\text{max}}], \left\| \hat{k}_t - \kappa^* \right\| \leq \epsilon \right)
\]

\[
\geq 1 - \sum_{t = J f b_{\text{min}}}^{J b_{\text{max}}} \frac{1}{e^{4D}} \exp \left\{ -O \left( t \epsilon^8 \right) \right\}
\]

\[
\geq 1 - \sum_{t = J f b_{\text{min}}}^{J b_{\text{max}}} \frac{1}{e^{4D}} \exp \left\{ -O \left( J f b_{\text{min}} \epsilon^8 \right) \right\}
\]

\[
\geq 1 - \sum_{t = J f b_{\text{min}}}^{J b_{\text{max}}} \frac{1}{e^{4D}} \exp \left\{ -O \left( B f \epsilon^8 \right) \right\}
\]

\[
\geq 1 - \frac{J b_{\text{max}}}{e^{4D}} \exp \left\{ -O \left( B f \epsilon^8 \right) \right\}
\]

\[
\geq 1 - \frac{1}{e^{4D}} \exp \left\{ -O \left( B \gamma \epsilon^8 \right) \right\}
\]

where (a) follows by the union bound and (b) because \( t \geq J f b_{\text{min}} \).

\[\square\]

C Feasible regions

In this section, we derive the feasibility regions for the various policies.

OMS-ETC

Recall that in OMS-ETC, we first collect \( T e \) samples such that \( \kappa_T e = \text{ctr} (\Delta_\psi) \). For the remaining \( T(1 - e) \) samples, the agent can query the data sources with any fraction \( \kappa \in \Delta_\psi \). Therefore, the feasible values of \( \kappa_T \) are

\[
\tilde{\Delta} = \left\{ \frac{T e \kappa_T e + T(1 - e)\kappa}{T} : \kappa \in \Delta_\psi \right\}
\]

\[
= \left\{ e \kappa_T e + (1 - e) \kappa : \kappa \in \Delta_\psi \right\}.
\]

OMS-ETG

After \( j \) rounds, the selection ratio is denoted by \( \kappa_{b_j} \). In every round, we collect \( b = T s \) samples. For the batch collected in round \( j + 1 \), the agent can query the data sources with any fraction \( \kappa \in \Delta_\psi \).
Therefore, the feasible values of $\kappa_{b(j+1)}$ are
\[
\tilde{\Delta}_{j+1}(\kappa_{b}) = \left\{ \frac{bj\kappa_{b} + b\kappa}{b(j+1)} : \kappa \in \Delta_{\psi} \right\} \\
= \left\{ \frac{Tsj\kappa_{b} + Ts\kappa}{Ts(j+1)} : \kappa \in \Delta_{\psi} \right\} \\
= \left\{ \frac{j\kappa_{b} + \kappa}{(j+1)} : \kappa \in \Delta_{\psi} \right\}.
\]

**OMS-ETC-CS**

The agent uses $Be$ budget to uniformly query the available data sources. Let $T_{e}$ denote the number of samples collected after exploration. We have
\[
T_{e} = \frac{Be}{\kappa_{T}^\top c},
\]
where $\kappa_{T} = \text{ctr}(\Delta_{\psi})$ and $c$ is the cost vector. With the remaining $B(1 - e)$ budget, the agent can collect samples with any fraction $\kappa \in \Delta_{\psi}$. However, since the data sources can have different costs, the total number of samples $T$ depends on the choice of $\kappa$:
\[
T = T_{e} + \frac{B(1 - e)}{\kappa^\top c},
\]
for $\kappa \in \Delta_{\psi}$. Therefore the feasible values of $\kappa_{T}$ are
\[
\tilde{\Delta} = \left\{ \frac{T_{e}\kappa_{T_{e}} + (T - T_{e})\kappa}{T} : \kappa \in \Delta_{\psi} \right\} \\
= \left\{ \frac{Be}{\kappa_{T_{e}}^\top c} \kappa_{T_{e}} + \frac{B(1 - e)}{\kappa^\top c} \kappa}{Be} + \frac{B(1 - e)}{\kappa^\top c} : \kappa \in \Delta_{\psi} \right\} \\
= \left\{ \frac{e (\kappa^\top c) \kappa_{T_{e}} + (1 - e) (\kappa_{T_{e}}^\top c) \kappa}{e (\kappa^\top c) + (1 - e) (\kappa_{T_{e}}^\top c)} : \kappa \in \Delta_{\psi} \right\}.
\]

**OMS-ETG-FS**

Since we collect a fixed number of samples in each round, the feasibility region for OMS-ETG-FS is that same as OMS-ETG:
\[
\tilde{\Delta}_{j+1}(\kappa_{b}) = \left\{ \frac{j\kappa_{b} + \kappa}{(j+1)} : \kappa \in \Delta_{\psi} \right\}.
\]

**OMS-ETG-FB**

Let the selection ratio after $j$ rounds be $\kappa_{T_{j}}$, where $T_{j}$ number of samples collected after round $j$:
\[
T_{j} = \frac{Bs_{j}}{\kappa_{b_{j}}^\top c}.\]
For the batch collected in round $j + 1$, the agent can query the data sources with any fraction $\kappa \in \Delta_{\psi}$. However, the number of samples collected in round $j + 1$ would depend on the choice $\kappa$ due to the cost structure. Therefore the number of samples collected after round $j + 1$ is
\[
T_{j+1} = T_{j} + \frac{Bs_{j}}{\kappa_{T_{j}}^\top c},
\]
for $\kappa \in \Delta_{\psi}$. Hence, the feasible values of $\kappa_{T_{j+1}}$ are
\[
\tilde{\Delta}_{j+1}(\kappa_{T_{j}}) = \left\{ \frac{T_{j}\kappa_{T_{j}} + (T_{j+1} - T_{j})\kappa}{T_{j+1}} : \kappa \in \Delta_{\psi} \right\} \\
= \left\{ \frac{Bs_{j}}{\kappa_{b_{j}}^\top c} \kappa_{T_{j}} + \frac{Bs_{j}}{\kappa_{b_{j}}^\top c} \kappa}{Bs_{j}} + \frac{Bs_{j}}{\kappa_{b_{j}}^\top c} : \kappa \in \Delta_{\psi} \right\} \\
= \left\{ \frac{j (\kappa^\top c) \kappa_{T_{j}} + (\kappa_{T_{j}}^\top c) \kappa}{j (\kappa^\top c) + (\kappa_{T_{j}}^\top c)} : \kappa \in \Delta_{\psi} \right\}.
\]
D Experiments

D.1 Linear IV graph

Data from the linear IV graph (Figure 2a) is simulated as follows:

\[ Z \sim \mathcal{N}(0, \sigma_Z^2), \]
\[ U \sim \mathcal{N}(0, \sigma_U^2), \]
\[ X := \alpha Z + \gamma U + \epsilon_x, \quad \epsilon_x \sim \mathcal{N}(0, \sigma_x^2), \]
\[ Y := \beta X + \phi U + \epsilon_y, \quad \epsilon_y \sim \mathcal{N}(0, \sigma_y^2), \]

where \( \epsilon_x \) and \( \epsilon_y \) are exogenous noise terms independent of other variables and each other and \( U \) is an unobserved confounder. Here \( \{\beta, \alpha, \gamma, \phi, \sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\} \) are parameters that we set for simulating the data. For the experiment in Section 6.1, we used \( \beta = 1, \alpha = 1, \gamma = 1, \phi = 1, \sigma_z = 1, \sigma_u = 1, \sigma_x = 1, \sigma_y = 1 \).

The moment conditions used for estimation are

\[ g_t(\theta) = \frac{s_{t,1}}{s_{t,2}} \otimes \begin{bmatrix} Z_t(X_t - \alpha Z_t) \\ Z_t(Y_t - \beta Z_t) \end{bmatrix}, \]

The parameter we estimate is \( \theta = [\beta, \alpha]^\top \) and \( \beta = f_{\text{law}}(\theta) = \theta_0 \).

D.2 Two IVs graph

Data from the two IVs graph (Figure 2b) is simulated as follows:

\[ Z_1 \sim \mathcal{N}(0, \sigma_Z^2), \]
\[ Z_2 \sim \mathcal{N}(0, \sigma_Z^2), \]
\[ U \sim \mathcal{N}(0, \sigma_U^2), \]
\[ X := \alpha_1 Z_1 + \alpha_2 Z_2 + \gamma U + \epsilon_x, \quad \epsilon_x \sim \mathcal{N}(0, \sigma_x^2), \]
\[ Y := \beta X + \phi U + \epsilon_y, \quad \epsilon_y \sim \mathcal{N}(0, \sigma_y^2), \]

where \( \epsilon_x \) and \( \epsilon_y \) are exogenous noise terms independent of other variables and each other and \( U \) is an unobserved confounder. For the experiment in Section 6.1, we used \( \beta = 1, \alpha = 1, \gamma = 1, \phi = 1, \sigma_z = 1, \sigma_u = 1, \sigma_x = 1, \sigma_y = 1 \).

The moment conditions used for estimation are

\[ g_t(\theta) = \frac{s_{t,1}}{s_{t,2}} \otimes \begin{bmatrix} (Z_1)(Y_t - \alpha Z_1) \\ (Z_2)(Y_t - \beta Z_t) \end{bmatrix}, \]

The parameter we estimate is \( \theta = [\beta] \) and \( \beta = f_{\text{law}}(\theta) = \theta_0 \).

D.3 Confounder-mediator graph

Data from the confounder-mediator graph (Figure 2b) is simulated as follows:

\[ W \sim \mathcal{N}(0, \sigma_W^2), \]
\[ X := dW + \epsilon_x, \quad \epsilon_x \sim \mathcal{N}(0, \sigma_x^2), \]
\[ M := \frac{\beta}{\alpha} X + \epsilon_m, \quad \epsilon_m \sim \mathcal{N}(0, \sigma_m^2), \]
\[ Y := aM + bW + \epsilon_y, \quad \epsilon_y \sim \mathcal{N}(0, \sigma_y^2), \]

where \( \epsilon_x, \epsilon_m, \) and \( \epsilon_y \) are exogenous noise terms independent of other variables and each other. For the experiment in Section 6.1, we used \( \beta = -0.32, \alpha = 0.33, \beta = -0.34, d = 0.45, \sigma_w = 1, \sigma_x = 1, \sigma_m = 1, \sigma_y = 1 \).
The moment conditions used for estimation are

\[ g_t(\theta) = \begin{bmatrix} s_{t,1} \\ s_{t,1} \\ s_{t,2} \\ s_{t,2} \\ s_{t,1} \\ s_{t,1} \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} X_t(Y_t - bW_t - \beta X_t) \\ W_t(Y_t - bW_t - \beta X_t) \\ X_t(M_t - \frac{\beta}{\alpha} X_t) \\ M_t(\gamma_t - aM_t - \frac{b\delta x^2}{\sigma^2 u + \sigma^2_y}) X_t \\ X_t(Y_t - aM_t - \frac{b\delta x^2}{\sigma^2 u + \sigma^2_y}) X_t \\ W_2^2 - \sigma^2_y \\ W_t(\tilde{X}_t - dW) \\ X_t^2 - (d^2 \sigma^2 u + \sigma^2_y) \end{bmatrix} = \tilde{g}_t(\theta) \]  

The parameter we estimate is \( \theta = [\beta, \alpha, b, d, \sigma^2 u, \sigma^2 y] \) and \( \beta = f_{\text{true}}(\theta) = \theta_0. \)

### D4 IHDP dataset

To generate semi-synthetic IHDP dataset, we use two covariates: birth weight (denoted by \( W_1 \)) and whether the mother smoked (denoted by \( W_2 \)). The binary treatment is denoted by \( X \) and the outcome is denoted by \( Y \). The corresponding causal graph is shown in Figure 4a. For every sample of the semi-synthetic dataset, \( W_1, W_2, \) and \( X \) are sampled uniformly at random from the real data. The outcome \( Y \) is simulated as follows:

\[ Y := \beta X + \alpha_1 W_1 + \alpha_2 W_2 + \epsilon_y, \quad \epsilon_y \sim \mathcal{N}(0, \sigma_y^2), \]

where \( \epsilon_y \) is an independent exogenous noise term. For the experiment in Section 6.2, we used \( \beta = 1, \alpha_1 = 1, \alpha_2 = 0.1, \sigma_y = 1. \)

The moment conditions used for estimation are

\[ g_t(\theta) = \begin{bmatrix} 1 - s_{t,2} \\ 1 - s_{t,2} \\ 1 - s_{t,1} \\ 1 - s_{t,1} \\ s_{t,3} \\ 1 - s_{t,2} \\ 1 - s_{t,1} \\ 1 - s_{t,1} \\ 1 - s_{t,2} \\ 1 - s_{t,1} \end{bmatrix} \otimes \begin{bmatrix} (W_1)_t ((Y_t - \alpha_1(W_1)_t - \beta X_t) - \alpha_2 d) \\ X_t ((Y_t - \alpha_1(W_1)_t - \beta X_t) - \alpha_2 \tau_2) \\ (W_2)_t ((Y_t - \alpha_2(W_2)_t - \beta X_t) - \alpha_1 d) \\ X_t ((Y_t - \alpha_2(W_2)_t - \beta X_t) - \alpha_1 \tau_1) \\ (W_1)_t(W_2)_t - d \\ X(W_1)_t - \tau_1 \\ X(W_2)_t - \tau_2 \\ (W_1)^2 - \sigma^2 u_1 \\ (W_2)^2 - \sigma^2 u_2 \\ (Y_t - \alpha_1(W_1)_t - \beta X)_t^2 - \alpha_2^2 \sigma^2 u_1 - \sigma^2_y \\ (Y_t - \alpha_2(W_2)_t - \beta X)_t^2 - \alpha_1^2 \sigma^2 u_2 - \sigma^2_y \end{bmatrix} = \tilde{g}_t(\theta) \]  

The parameter we estimate is \( \theta = [\beta, \alpha_1, \alpha_2, d, \tau_1, \tau_2, \sigma^2 u_1, \sigma^2 u_2, \sigma^2 y] \) and \( \beta = f_{\text{true}}(\theta) = \theta_0. \)

### D5 The Vietnam draft and future earnings dataset

The causal graph for this dataset corresponds to Figure 2a with a binary IV \( Z \), binary treatment \( X \) and continuous outcome \( Y \). In this dataset, \( \{Z, X\} \) and \( \{Z, Y\} \) are collected from different data sources and thus \( \{Z, X, Y\} \) are not observed simultaneously. For our experiment, we only use data from the 1951 cohort.

In the semi-synthetic dataset, we sample \( Z \) uniformly at random from the real dataset. The treatment \( X \) is generated similarly to a probit model. We first generate an intermediate variable \( X^* \) and then use that to generate \( X \) as follows:

\[ X^* := \alpha Z + c^* + \epsilon_x, \quad \epsilon_x \sim \mathcal{N}(0, 1), \]

\[ X := 1(X^* \geq 0), \]

where \( 1 \) is the indicator function. To reduce clutter, let \( \mu_x = \tilde{P}(Z = 1) = 0.3425, \mu_x^{(1)} = P(X = 1|Z = 1) \) and \( \mu_x^{(0)} = P(X = 1|Z = 0). \) We set the parameters \( \alpha \) and \( c^* \) such that \( \mu_x^{(1)} = 0.2831 \) and
\[ \mu_x^{(0)} = 0.1468 \] (these values have been taken from [2, Table 2] to match the empirical distribution):

\[
\begin{align*}
\mu_x^{(0)} &= P(1(X^* > 0)|Z = 0) \\
&= P(c^* + \epsilon_x > 0) \\
&= P(\epsilon_x > -c^*) \\
&= P(\epsilon_x < c^*) \\
&= \Phi(c^*),
\end{align*}
\]

\[
\therefore \ c^* = \Phi^{-1}(\mu_x^{(0)})
\]

\[
\begin{align*}
\mu_x^{(1)} &= P(1(X^* > 0)|Z = 1) \\
&= P(\alpha + c^* + \epsilon_x > 0)
\end{align*}
\]

\[
\therefore \ \alpha = \Phi^{-1}(\mu_x^{(1)}) - c^*
\]

\[
\begin{align*}
&= \Phi^{-1}(\mu_x^{(1)}) - \Phi^{-1}(\mu_x^{(0)}) \\
&= \Phi^{-1}(0.2831) - \Phi^{-1}(0.1468) \\
&= 0.4766,
\end{align*}
\]

where \( \Phi \) is the cumulative distribution function of the standard normal distribution.

In the real data, we standardize the outcome \( Y \) by subtracting its mean and dividing by its standard deviation and thus \( \bar{E}[Y] = 0 \) and \( \bar{\text{Var}}(Y) = 1 \). To generate the simulated outcome \( Y \), we use \( Y := \beta X + \gamma + c_0 \epsilon_x + \epsilon_y \), where \( \epsilon_y \sim N(0, \sigma_{\epsilon_y}^2) \). When \( c_0 \neq 0 \), the noise term \( (c_0 \epsilon_x + \epsilon_y) \) determines the extent of the confounding between \( X \) and \( Y \).

Thus \( c_0 \) determines the extent of the confounding between \( X \) and \( Y \).

We now describe how we set \( \beta \) and \( \gamma \). Since \( E[Y] = 0 \), we have

\[
\begin{align*}
\gamma &= -\beta E[X] \\
&= -\beta \left( \mu_x^{(0)}(1 - \mu_z) + \mu_x^{(1)}\mu_z \right) \\
&= -0.1934\beta.
\end{align*}
\]

Using the covariance of \( Y \) and \( Z \), we have

\[
\begin{align*}
\text{Cov}(Y, Z) &= E[YZ] \\
&= \beta E[ZX] + \gamma E[Z] \\
&= \beta (E[ZX] - E[X]E[Z]) \\
&= \beta (E[Z1(\alpha Z + c^* + \epsilon_x > 0)] - E[X]E[Z]) \\
&= \beta (E[ZE1(\alpha Z + c^* + \epsilon_x > 0)|Z] - E[X]E[Z]) \\
&= \beta (E[ZE1(\epsilon_x > -(\alpha Z + c^*))|Z] - E[X]E[Z]) \\
&= \beta (E[Z\Phi(\alpha Z + c^*)] - E[X]E[Z]) \\
&= \beta (\Phi(\alpha Z + c^*)\mu_z - E[X]E[Z]) \\
&= \beta \mu_z \left( \mu_x^{(1)} - E[X] \right).
\end{align*}
\]

Therefore, we set \( \beta \) and \( \gamma \) as

\[
\begin{align*}
\beta &= \frac{\bar{E}[YZ]}{\mu_z \left( \mu_x^{(1)} - E[X] \right)} = -0.4313, \\
\gamma &= -0.1934\beta = 0.0834.
\end{align*}
\]

Now we describe how we set \( c_0 \) and \( \sigma_{\epsilon_y}^2 \). For this, we use the variance of \( Y \):

\[
\text{Var}(Y) = 1 = \beta^2 \text{Var}(X) + c_0^2 \sigma_{\epsilon_y}^2 + 2\beta c_0 E[X\epsilon_x].
\]  \hspace{1cm} (14)
We have
\[ \text{Var}(X) = \text{Var}[E(X|Z)] + E[\text{Var}(X|Z)] \]
\[ = \text{Var} \left( Z \mu_x^{(1)} + (1 - Z)\mu_x^{(0)} \right) + \mu_x \mu_x^{(1)} (1 - \mu_x^{(1)}) + (1 - \mu_x^{(0)}) (1 - \mu_x^{(0)}) \]
\[ = \mu_x (1 - \mu_x) (\mu_x^{(1)} - \mu_x^{(0)})^2 + \mu_x \mu_x^{(1)} (1 - \mu_x^{(1)}) + (1 - \mu_x^{(0)}) (1 - \mu_x^{(0)}) \]
\[ = 0.1560, \]
\[ E[X\epsilon_x] = E[\text{Var}(X|Z)] \]
\[ = E[\text{Var}(X|Z)] \]
\[ = \left[ \int_{-(Z\alpha + c^*)}^{\infty} xf(x)dx \right] \]
\[ = E \left[ \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{-(Z\alpha + c^*)^2}{2} \right\} \right] \]
\[ = E \left[ \frac{1}{\sqrt{2\pi}} \right] \left\{ \exp \left\{ \frac{-(c^*)^2}{2} (1 - \mu_x) + \exp \left\{ \frac{-(\alpha + c^*)^2}{2} \right\} \mu_z \right\} \right\] \]
\[ = 0.2670, \]
where in (a), \( f(x) \) is the probability density function of the standard normal distribution. We set \( c_0 = 0.5 \) and using Eq. [14] we get \( \sigma_{\epsilon_y}^2 = 0.6058 \).

To summarize, the data is generated as follows:
\[ Z \sim \text{Bernoulli}(\mu_z), \]
\[ X^* := \alpha Z + c^* + \epsilon_x, \epsilon_x \sim \mathcal{N}(0, 1), \]
\[ X := 1(X^* > 0), \]
\[ Y := \beta X + \gamma + c_0 \epsilon_x + \epsilon_y, \epsilon_y \sim \mathcal{N}(0, \sigma_{\epsilon_y}^2), \]
where \( \mu_z = 0.3424, \alpha = 0.4766, c^* = -1.0502, \beta = -0.4313, \gamma = 0.0834, \) and \( \sigma_{\epsilon_y}^2 = 0.6058 \).

The moment conditions used for estimation are
\[ g_t(\theta) = \left[ \begin{array}{c} Z_t(Y_t - \mu_1) \\ s_{t,1} \\ s_{t,1} \\ s_{t,2} \\ s_{t,2} \end{array} \right] \otimes \left[ \begin{array}{c} (1 - Z_t)(Y_t - \mu_0) \\ Z_t(X_t - \tau_1) \\ Z_t(X_t - \tau_1) \end{array} \right] = \tilde{m}_t(\theta) \]

The parameter we estimate is \( \theta = [\mu_1, \mu_0, \tau_1, \tau_0] \) and the target parameter is \( \beta = \hat{f}_{\text{tar}}(\theta) = \frac{\mu_1 - \mu_0}{\tau_1 - \tau_0}. \)