Becchi-Rouet-Stora-Tyutin quantization of a soliton model in 2+1 dimensions

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Abstract

The Becchi-Rouet-Stora-Tyutin (BRST) method is applied to the quantization of the solitons of the non-linear $O(3)$ model in 2+1 dimensions. We show that this method allows for a simple and systematic treatment of zero-modes with a non-commuting algebra. We obtain the expression of the BRST Hamiltonian and show that the residual interaction can be perturbatively treated in an IR-divergence-free way. As an application of the formalism we explicitly evaluate the two-loop correction to the soliton mass.
I. INTRODUCTION

During the last few years the issue of soliton quantization has received renewed attention. One of the main reasons for this was the revival of soliton models as the low energy limit of QCD\(^1\). In this kind of models, baryons are described as solitonic excitations of a chiral lagrangian in 3 + 1 dimensions\(^2\). Although many aspects of the soliton quantization problem have been studied in the mid-seventies\(^3\) the methods developed at that time become quite cumbersome when applied to systems in more than one spatial dimension\(^4\). As an alternative to these canonical quantization methods one can apply the Becchi-Rouet-Stora-Tyutin (BRST) quantization formalism\(^8\). This application\(^1\) is based on the observation that there appears a fundamental gauge symmetry when collective coordinates are introduced to perform the quantization of the zero modes around the soliton. Such gauge symmetry consists of the group of all time-dependent transformations which simultaneously move the intrinsic frame and the soliton so as to reproduce the same physical situation. The overcompleteness associated with the introduction of collective coordinates has to be compensated by constraints and gauge conditions. In the BRST treatment both the collective coordinates and the Lagrange multipliers associated with the constraints are considered as dynamical variables on the same footing as the original coordinates. Ghosts (i.e., fermions carrying zero spin and having no direct physical meaning) are introduced from the beginning. As an illustration of this method, the authors of Ref.\(^10\) have applied it to the quantization of soliton models in 1 + 1 dimensions, where comparison with conventional methods can be easily performed.

However, it is not in such kind of problems where the advantage of using the BRST scheme is more clearly seen (except perhaps for the fact that solitons with arbitrary large momentum can be easily quantized\(^11\)). In the present paper we deal with the non-

\(^1\)Recently the problem of soliton quantization has been addressed in Ref.\(^7\) within the framework of the Kerman-Klein method.
linear $O(3)$ model in $2 + 1$ dimensions \[12\]. This model has already all the complexity associated with the presence of several zero modes with a non-commutative algebra, with the interplay between internal (isospin) and external transformations and with the survival of some unbroken symmetries. At the same time, its rather simple form allows for explicit expressions for the soliton configurations and zero-modes.

It should be stressed that this model has relevant applications on its own. For example, in solid state physics it has been used as a model for the continuum limit of a two-dimensional isotropic ferromagnet. In that context the soliton solutions represent the metastable ‘pseudo-particles’ which are responsible for the destruction of long-range order at any temperature \[13\]. More recently, the non-linear $O(3)$ model has also been studied in connection to high-$T_c$ superconductivity \[14\].

This paper is organized as follows. In Sec. II we review the non-linear $O(3)$ model in $2 + 1$ dimensions and present its soliton solutions. In Sec. III we introduce the collective coordinates and impose the corresponding constraints. In Sec. IV the quadratic hamiltonian is obtained and diagonalized. In Sec. V we describe the BRST quantization of the model. In Sec. VI we show explicitly that the two loop correction to the soliton mass is independent of the spurious parameters introduced by the BRST quantization. Finally, in Sec. VII our conclusions are given.

II. THE MODEL AND ITS SOLITON SOLUTIONS

The non-linear $O(3)$ model consists of three real scalar fields $\phi_a$ ($a = 1, 2, 3$) subject to the constraint

$$\phi_a \phi_a = 1. \quad (1)$$

The dynamics is determined by the lagrangian density ($\mu = 0, 1, 2$)

$$\mathcal{L} = \frac{f^2}{2} \partial_\mu \phi_a \partial^\mu \phi_a. \quad (2)$$
which is invariant under spatial translations and rotations and under internal $O(3)$ transformations. As usual we take $\hbar = c = 1$ and work in terms of adimensional quantities. The constant $f$ is taken as the large parameter of the model. In the case of the antiferromagnetism, $f$ corresponds to the spin of the ions which is assumed to be large in the usual quasi-classical expansion \[14\]. In the standard Skyrme model the role of $f$ is played by the pion decay constant $f_\pi$ which is of $O(N^{1/2})$ in the large-$N_c$ expansion, and therefore a large number.

As it is well-known, the $O(3)$ symmetry is spontaneously broken by the vacuum which we choose to be $\vec{\phi} = (0, 0, 1)$. Besides this solution the equations of motion that result from the minimization of the lagrangian (2) in the presence of the constraint (1) admit non-trivial, finite-energy, static solutions satisfying $\lim_{|\vec{x}| \to \infty} \vec{\phi} = (0, 0, 1)$ \[13\]. They represent mappings of $S_2^{(phy)}$ into $S_2^{(int)}$ and can be characterized by the winding number $W$ \[3\]

$W = \frac{1}{8\pi} \int \epsilon^{0ij} \epsilon^{abc} \phi_a \partial_j \phi_b \partial_i \phi_c.$

In order to obtain the explicit form of these soliton solutions it is convenient to introduce independent fields. As shown in Ref. \[12\] a useful choice is obtained by stereographically projecting $S_2^{(int)}$ onto a plane parallel to the $\{\phi_1, \phi_2\}$ plane which contains the south pole. The (internal) cartesian coordinates thus obtained are used to construct the complex fields

$\theta = \frac{\phi_1 + i\phi_2}{1 - \phi_3}; \quad \bar{\theta} = \frac{\phi_1 - i\phi_2}{1 - \phi_3}.$ \[4\]

In the same fashion, the coordinates of the two dimensional physical space $x^1, x^2$ are combined into independent complex coordinates

$z = x^1 + i x^2; \quad \bar{z} = x^1 - i x^2.$ \[5\]

In terms of the new fields the lagrangian density reads

$L = \frac{2f^2}{(1 + \theta \bar{\theta})^2} \left[ \dot{\theta} \dot{\bar{\theta}} - 2(\partial_1 \theta \partial_2 \bar{\theta} + \partial_2 \theta \partial_1 \bar{\theta}) \right].$ \[6\]

\[2\]Here and in what follows the surface element is omitted in the space integrals.
where

\[ \partial_z = \frac{1}{2}(\partial_1 - i\partial_2); \quad \bar{\partial}_z = \frac{1}{2}(\partial_1 + i\partial_2). \]  

(7)

The equations of motion can be now easily obtained by applying the Euler-Lagrange variational principle. For time-independent configurations we obtain

\[ \frac{2\bar{\theta}}{1 + \theta \bar{\theta}} \partial_z \theta \bar{\partial}_z \theta - \partial_{zz} \theta = 0, \]

and similarly for \( \bar{\theta} \). It easy to see that any static analytic \( \theta(z) \) (or antianalytic \( \bar{\theta}(\bar{z}) \)) function automatically solves Eq. (8). Since cuts are prohibited because of the single-valuedness of \( \phi_a(x) \), the only allowed singularities of \( \theta(z) \) are poles. As shown in Ref. [13], the general form of the soliton solution is therefore a quotient of polynomials. The corresponding winding number is given by the maximum between the degrees of the numerator and denominator. For winding number \( W = 1 \) the general solution satisfying \( \theta \to \infty \) (\(|x| \to \infty\)) is therefore the four-parameter family given by

\[ \theta(z) = az + b, \]  

(9)

where \( a, b \) are arbitrary complex constants. As a particular \( W = 1 \) soliton solution, we choose \( \theta = z \) (\( a = 1, b = 0 \)). Since the soliton mass is independent of the values of the parameters \( a, b \), there will be four zero-modes around this solution. They correspond to the transformations that connect our particular solution with any other given by Eq. (9). As it will become clear below two of these zero-modes are associated with spatial translations, one with spatial and internal (isospin) rotations\(^3\) and the remaining one with spatial dilatation.

\(^3\)There are three independent isospin rotations in the \( O(3) \) model. In fact, the rotations around the 1 and 2 axes would yield zero frequency (Goldstone) bosons already for the \( W = 0 \) sector, due to the chosen asymptotic vacuum \((0,0,1)\). In the \( W = 1 \) sector the existence of such modes becomes apparent if we use instead of (9) the more general six-parameter family \( \theta(z) = (az + b)/(cz + d) \)
III. COLLECTIVE COORDINATES AND CONSTRAINTS

In general, all the zero-modes are associated with symmetries of the action which are broken by the classical solution. An exception to this are the dilatations. In fact, the action is not invariant under dilatations (although the classical energy is). In addition, in more realistic models, where the effective lagrangian includes terms with higher powers in the field derivatives, the dilatation does not correspond to a zero-mode at all [15]. Considering this situation and since we are mainly interested in the treatment of collective variables associated with the breakdown of symmetries, we will ignore the motion associated with dilatations altogether.

In order to treat the remaining zero-modes of the system we include collective coordinates and describe the system from a rotated and translated spatial frame. The position of the moving frame with respect to the fixed (laboratory) frame is determined by the collective variables: the displacements \( Z \) and \( \bar{Z} \) for the translation and the angle \( \Phi \) for the rotation.

If \( z', \bar{z}' \) stand for the coordinates in the laboratory frame, and \( z, \bar{z} \) for the coordinates in the moving frame, we have

\[
\begin{align*}
    z' & \rightarrow z = e^{-i\Phi}(z' + Z) ; \\
    \bar{z}' & \rightarrow \bar{z} = e^{i\Phi}(\bar{z}' + \bar{Z}).
\end{align*}
\]  

(10)

The lagrangian density can be written as \( (\mu = t, z, \bar{z}) \)

\[
\mathcal{L} = \frac{2f^2}{(1 + \theta\bar{\theta})^2} g^{\mu\nu} \partial_\mu \theta \partial_\nu \bar{\theta},
\]

(11)

where the metric tensor \( g^{\mu\nu} \) is no longer constant in the moving frame, but instead reads

\[
\begin{pmatrix}
    1 & \dot{z} & \dot{\bar{z}} \\
    \dot{z} & \dot{z}^2 & \dot{z}\dot{\bar{z}} - 2 \\
    \dot{\bar{z}} & \dot{z}\dot{\bar{z}} - 2 & \bar{z}^2
\end{pmatrix},
\]

(12)

subject to the condition \( ad - bc = 1 \). For the two remaining isospin modes we assume that they can be IR-regularized including a (small) mass term for the Goldstone bosons, in both the \( W = 0,1 \) sectors.
\[ \dot{z} = v - i\dot{\Phi} z; \quad v = e^{-i\Phi} \dot{Z}, \]
\[ \ddot{z} = \ddot{v} + i\dot{\Phi} \ddot{z}; \quad \ddot{v} = e^{i\Phi} \ddot{Z}. \quad (13) \]

In order to include internal rotations we measure the fields from a moving internal frame related to the fixed internal frame by an \( U(1) \) transformation, i.e.,
\[ \theta' \rightarrow \theta = e^{i\alpha} \theta'. \quad (14) \]

The lagrangian density reads (\( k = z, \bar{z} \))
\[ L = \frac{2f^2}{(1 + \theta\bar{\theta})^2} \left[ (\dot{\theta} + i\dot{\alpha}\theta)(\dot{\bar{\theta}} - i\dot{\alpha}\bar{\theta}) + g^{0k}\partial_k\bar{\theta}(\dot{\theta} + i\dot{\alpha}\theta) + g^{0k}\partial_k\theta(\dot{\bar{\theta}} - i\dot{\alpha}\bar{\theta}) \right]. \quad (15) \]

Since we are considering the collective coordinates as true variables of the problem, the independent degrees of freedom are the independent fields \( \theta \) and \( \bar{\theta} \) plus the collective coordinates. To compute the canonical hamiltonian we must first find the conjugate momenta to these independent degrees of freedom. For the case of the field \( \theta \) we have
\[ \pi = \frac{\partial L}{\partial \dot{\theta}} = \frac{2f^2}{(1 + \theta\bar{\theta})^2} \left( \dot{\theta} - i\dot{\alpha}\bar{\theta} + g^{0k}\partial_k\bar{\theta} \right). \quad (16) \]

In a similar way \( \bar{\pi} \) can be calculated by replacing \( \theta \) by \( \bar{\theta} \) in Eq. (16). The Eqs. defining the conjugate momenta to the collective variables yield the primary constraints
\[ T_3 = \frac{\partial L}{\partial \dot{\alpha}} = i \int \left( \pi \theta - \bar{\pi} \bar{\theta} \right) \equiv t_3, \]
\[ J = \frac{\partial L}{\partial \Phi} = i \int \left[ \pi(\bar{z}\partial_z - z\partial_{\bar{z}})\theta + \bar{\pi}(\bar{z}\partial_z - z\partial_{\bar{z}})\bar{\theta} \right] \equiv j, \]
\[ P = \frac{\partial L}{\partial v} = \int (\pi\partial_z \theta + \bar{\pi}\partial_z \bar{\theta}) \equiv p, \]
\[ \bar{P} = \frac{\partial L}{\partial \bar{v}} = \int (\pi\partial_{\bar{z}} \theta + \bar{\pi}\partial_{\bar{z}} \bar{\theta}) \equiv \bar{p}, \quad (17) \]

\( T_3 \) is the generator of collective internal rotations around the 3-axis and \( J, P \) and \( \bar{P} \) are the generators of collective spatial rotations and translations. The operators \( t_3, j, p \) and \( \bar{p} \) transform correspondingly the fields. The external rotation and the translations are associated with the Euclidean E(2) group
\[ [j,p] = -p; \quad [j,\bar{p}] = \bar{p}; \quad [p,\bar{p}] = 0. \]  

(18)

For convenience we have defined \( P \) and \( \bar{P} \), which are the generators of collective translations in a frame which is rotated in an angle \( \Phi \) with respect to the lab. frame (cf. Eq. (13)). They do not commute with \( J \). The corresponding generators of translations parallel to the laboratory axes,

\[ P_L = \frac{\partial L}{\partial \dot{Z}} = e^{-i\Phi}P; \quad \bar{P}_L = \frac{\partial L}{\partial \dot{\bar{Z}}} = e^{i\Phi}\bar{P}, \]  

(19)

are, of course, the ones that commute with \( J \).

It must be noted that the classical solution \( \theta = z \) only partially breaks spatial and isospin rotational symmetry. It is invariant under the transformation generated by \( t_3 + j \). Therefore, quantum excitations can be classified by the eigenvalues of this operator. The collective variable conjugate to \( T_3 + J \) is redundant. The constraint \( T_3 + J = t_3 + j \) insures that the operations associated to \( T_3 + J \) are determined by the intrinsic structure.

IV. THE HAMILTONIAN

As it is well known, the collective coordinates and momenta do not appear in the expression of the canonical hamiltonian, namely

\[ H = \int \mathcal{H} = T_3\dot{\alpha} + J\dot{\Phi} + P\dot{v} + \bar{P}\bar{v} + \int \left( \pi\dot{\theta} + \bar{\pi}\dot{\bar{\theta}} \right) - \int \mathcal{L} \]

\[ = \frac{1}{2f^2} \int (1 + \theta\bar{\theta})^2\pi\bar{\pi} + \int \frac{4f^2}{(1 + \theta\bar{\theta})^2} \left( \partial_z\theta\partial_z\bar{\theta} + \partial\theta\partial\bar{\theta} \right). \]  

(20)

It follows immediately that the constraints are first-class.

In order to get the explicit form of the quadratic hamiltonian we expand the fields and their conjugate momenta around the classical solution as follows

\[ \theta = z + \frac{(1 + r^2)}{f}\dot{\theta}; \quad \pi = \frac{f}{1 + r^2}\bar{\pi}. \]  

(21)

The quadratic hamiltonian density reads
\[ \mathcal{H}^{(2)} = \frac{1}{2} \hat{p}^2 - \frac{8(1 - r^2)}{(1 + r^2)^2} \hat{\dot{\theta}} + 2\partial_r \hat{\theta} \partial_r \hat{\dot{\theta}} + \frac{2}{r^2} \partial_\varphi \hat{\theta} \partial_\varphi \hat{\dot{\theta}} + \frac{8i}{1 + r^2} \hat{\dot{\theta}} \partial_\varphi \hat{\dot{\theta}}. \]  

(22)

In order to diagonalize this Hamiltonian it is convenient to use the partial wave decomposition

\[ \hat{\theta} = \sum_{nm} \frac{a_{nm}}{\sqrt{2\pi}} R_{nm}(r) \exp(im\varphi), \]  

(23)

where \( a_{nm} \) are complex numbers that satisfy \( |a_{nm}|^2 = 1 \), and the real functions \( R_{nm} \) are conveniently normalized. The resulting eigenvalue equation reads

\[ R''_{nm}(r) + \frac{1}{r} R'_{nm}(r) - \left( \frac{m^2}{r^2} - \frac{4m}{1 + r^2} - \frac{4(1 - r^2)}{(1 + r^2)^2} \right) R_{nm}(r) = -\varepsilon_{nm}^2 R_{nm}(r). \]  

(24)

As expected, Eq. (24) has zero-energy solutions. In general, they have the form

\[ R_{0m} \propto \frac{r^m}{1 + r^2}. \]  

(25)

It should be noticed, however, that those with \( m > 1 \) do not correspond to zero-energy modes around the \( W = 1 \) soliton and therefore have to be dismissed. Combinations of the remaining four zero-energy wavefunctions (two independent choices for each value of \( a_{0m} \) with \( m = 0, 1 \)) describe the zero-modes mentioned in Sec. II. The explicit form of these linear combinations will be given below.

In addition to the zero-energy solutions, Eq. (24) has a continuum of finite energy solutions. The asymptotic forms of these solutions read

\[ R_{nm} \approx r^{|m|} \quad \text{for} \quad r \to 0, \]

\[ R_{nm} \approx r^{-\frac{1}{2}} \sin(\varepsilon_{nm} r + \delta_{nm}) \quad \text{for} \quad r \to \infty. \]  

(26)

Using these boundary conditions, the eigenvalue Eqs. can be numerically solved. In this way the phase shifts \( \delta_{nm} \) are obtained as a function of the energies \( \varepsilon_{nm} \).

The quadratic Hamiltonian (22) commutes with the linear expressions for the generators defined in (17)

\[ t_3^{(1)} = -j^{(1)} = if \int \frac{z \hat{\pi} - \bar{z} \hat{\pi}}{1 + r^2}, \]

\[ p^{(1)} = f \int \frac{\hat{\pi}}{1 + r^2}, \]

\[ \bar{p}^{(1)} = f \int \frac{\bar{\pi}}{1 + r^2}. \]  

(27)
The operator $t_3^{(1)} + j^{(1)}$ vanishes due to the fact that the symmetry is only partially broken. For this reason it is convenient heron to use the following linear combinations of generators:

$$v_{s=0'} = \frac{1}{2}(t_3 + j),$$
$$v_{s=0} = \frac{1}{2}(t_3 - j),$$
$$v_{s=+1} = i\sqrt{2\rho},$$
$$v_{s=-1} = -i\sqrt{2\rho}. \quad (28)$$

We may determine both the value of the inertia parameters $\Im s'$ associated with the collective motion and the expression for $G_{-s'}$, the variables conjugate to the $v^{(1)}_{s'}$, through the well known RPA Eqs. [16]

$$[H^{(2)}, G_{s'}] = -\frac{i}{\Im s'} v^{(1)}_{s'}; \quad [G_{s'}, v^{(1)}_{-s}] = i\delta_{s's'}, \quad (29)$$

which yield

$$G_0 = -\frac{2if}{\Im} \int \frac{z\hat{\theta} - \hat{z}\tilde{\theta}}{1 + r^2},$$
$$G_{-1} = -\frac{2\sqrt{2}if}{M} \int \frac{\hat{\theta}}{1 + r^2},$$
$$G_{+1} = \frac{2\sqrt{2}if}{M} \int \frac{\hat{\theta}}{1 + r^2},$$
$$\Im_0 = \Im = 4f^2 \int \frac{r^2}{(1 + r^2)^2},$$
$$\Im_{\pm1} = M = 4f^2 \int \frac{1}{(1 + r^2)^2},$$

where $G_{s'} = O(f^{-1})$ and $\Im_{s'} = O(f^2)$.

The integral defining $\Im$ diverges logarithmically. Therefore, we will consider all space integrals up to an upper radius $R$. The limit $R \to \infty$ can be taken safely at the end since

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4A primed $s$ will exclude $s = 0'$. Similar combinations and conventions hold for the collective generators, which are denoted by $V_s$. 10
the corrections to the classical results are expressed as ratios \((\Im \varepsilon_{nm})^{-1}\) between collective and intrinsic energies. In (more realistic) models displaying stability against dilatations, the rotational parameter would not vanish.

In terms of the creation and annihilation operators for the finite frequency modes (see Eq. (41)) and of the linear generators, the quadratic Hamiltonian reads

\[
H^{(2)}(\varepsilon_{nm}(c_{nm}^\dagger c_{nm} + d_{nm}^\dagger d_{nm} + 1) + \frac{1}{23}v_0^{(1)2} + \frac{1}{M}v_{+1}^{(1)}v_{-1}^{(1)}, \tag{30}
\]

where the sum over subindices \(n\) include only finite frequency modes.

The remaining terms in the Hamiltonian are at most of \(O(f^{-1})\). Therefore one might be tempted to use perturbation theory to evaluate the corrections to the quadratic contributions. However, due to the presence of the zero-modes such perturbative expansion would be plagued with infrared divergences. As it will be shown in the next sections the BRST quantization scheme provides a very convenient method to eliminate these divergences in a consistent way.

V. BRST QUANTIZATION

The constraining operators \(F_s \equiv v_s - V_s\) generate gauge transformations, which are a manifestation of the fact that transforming the fields and correspondingly moving the frame of reference must result into two completely equivalent physical descriptions.

For each collective degree of freedom we introduce a Lagrange multiplier \(\Omega_s\) and two ghost operators \(\eta_s, \bar{\eta}_s\), together with the corresponding conjugate operators \(\pi_s, \bar{\pi}_s\), which satisfy the non-vanishing commutation and anti-commutation relations

\[
[\Omega_s, B_{-l}] = i\delta_{sl}; \quad \{\eta_s, \pi_{-l}\} = \{\bar{\eta}_s, \bar{\pi}_{-l}\} = \delta_{sl}. \tag{31}
\]

The quantal constraints

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\(^5\text{As explained at the beginning of Sec. III, the dilatations have been ignored.}\)
on physical states are replaced by the requirement that physical states should be annihilated by the BRST charge $Q$ \[ Q \equiv B_s \bar{\pi}_{-s} - F_s \eta_{-s} + \frac{i}{2} C_{st}^u \eta_{-s} \eta_{-t} \pi_u, \] (33)

where $[v_s, v_t] = i C_{st}^u v_u$ and $[V_s, V_t] = -i C_{st}^u V_u$. The non-vanishing terms of $C_{st}^u = -C_{ts}^u$ are

\[ C_{0' + 1}^{+1} = -C_{0' - 1}^{-1} = C_{0' - 1}^{-1} = -C_{0 + 1}^{+1} = \frac{1}{2}, \] (34)

As a consequence of the fact that the charge $Q$ is a nilpotent and hermitian operator, we may add to the hamiltonian any term of the form $\{ \rho, Q \}$, without altering the overlaps of the original hamiltonian within the subspace annihilated by $Q$. We choose \[ \rho = \Omega_s' \pi_{-s'} + \omega^2_{s'} \left( G_{s'} - \frac{1}{23 s'} B_{s'} \right) \bar{\eta}_{-s'}, \] (35)

which yields

\[ \{ \rho, Q \} = -\Omega_s' F_{-s'} + \omega^2_{s'} \left( G_{s'} B_{-s'} - \frac{1}{23 s'} B_{s'} B_{-s'} \right) + i \pi_{s'} \pi_{-s'} \]

\[ + \omega^2_{s'} \eta_{-t} \bar{\eta}_{-s'} [G_{s'}, v_t] + i C_{s't}^u \Omega_{-s} \eta_{-t} \pi_u. \] (36)

The (spurious) frequencies $\omega_{s'}$ are arbitrary and should disappear from any physical result. Since the collective variables are to be considered as genuine variables of the problem, a tradeoff should take place and some original degrees of freedom must become spurious. At the quadratic level this is accomplished by including the zero modes of the hamiltonian (30) in the spurious sector, together with the quadratic terms of $\{ \rho, Q \}$. Moreover, at this level there is a separation of variables corresponding to each of the three spurious sectors

\[ H^{(2)}_{sp} = \frac{1}{23 s'} \left( v_{s'}^{(1)} v_{-s'}^{(1)} - \Omega_{s'} v_{-s'}^{(1)} + \omega^2_{s'} G_{s'} B_{-s'} - \frac{\omega^2_{s'}}{23 s'} B_{s'} B_{-s'} + i \pi_{s'} \pi_{-s'} + i \omega^2_{s'} \eta_{s'} \bar{\eta}_{-s'} \right) \]

\[ = \omega_{s'} \left( a_{s'1}^\dagger a_{s'1} - a_{s'0}^\dagger a_{s'0} + \bar{a}_{s'} a_{s'} + \bar{b}_{s'} b_{s'} \right). \] (37)

The normal boson and ghost modes may be obtained by applying the transformations
\[ a_{s'1} = \sqrt{\frac{1}{2s'\omega_{s'}}} v_{s'}^{(1)} - \sqrt{\frac{3s'}{2s'}} \Omega_{s'} - i\sqrt{\frac{3s'\omega_{s'}}{2}} G_{s'}, \]
\[ a_{s'0} = \sqrt{\frac{3s'}{2s'}} \Omega_{s'} - i\sqrt{\frac{\omega_{s'}}{2s'}} B_{s'} + i\sqrt{\frac{3s'\omega_{s'}}{2}} G_{s'}, \]
\[ a_{s'} = -i\bar{b}_{-s'} \sqrt{\frac{1}{2\omega_{s'}}} \bar{\pi}_{s'} - i\sqrt{\frac{\omega_{s'}}{2}} \bar{\eta}_{s'}, \]
\[ b_{s'} = i\bar{a}_{-s'} \sqrt{\frac{1}{2\omega_{s'}}} \bar{\pi}_{s'} + i\sqrt{\frac{\omega_{s'}}{2}} \bar{\eta}_{s'}. \]

(38)

Note the manifest supersymmetry of the spurious sector and the relations\(^6\)

\[ [a_{s'1}, \bar{a}_{-s'}] = -[a_{s'0}, a_{s'}^\dagger] = \{a_{s'}, \bar{a}_{s'}\} = \{b_{s'}, \bar{b}_{s'}\} = \delta_{s's'}. \]

(39)

The vacuum state of the spurious sector satisfies

\[ a_{s'1}|\rangle = a_{s'0}|\rangle = a_{s'}|\rangle = b_{s'}|\rangle = 0, \]

(40)

it is also annihilated (to leading order) by the BRST charge, and it is the only normalizable state of the spurious sector satisfying this condition. It represents the unperturbed spurious sector for any physical state.

We express the fields and their conjugate momenta in terms of the corresponding eigen-modes and of the creation (annihilation) operators for the finite frequency modes \(c_{nm}^\dagger, d_{nm}^\dagger\) \((c_{nm}, d_{nm})\) and for the spurious modes \(a_{s'1}^\dagger, a_{s'0}^\dagger\) \((a_{s'1}, a_{s'0})\). In the Schroedinger representation,

\[ \hat{\theta} = -\frac{1}{2} \sqrt{\frac{1}{2\omega_0}} \Psi_0^{ZM}(\vec{r}) \left(a_{0,1} - a_{0,1}^\dagger\right) - \frac{1}{2} \sqrt{\frac{1}{\omega_1}} \Psi_1^{ZM}(\vec{r}) \left(a_{1,1} - a_{-1,1}^\dagger\right) \]
\[ + \frac{1}{2} \sum_{n,m} \varepsilon_{nm}^{-1/2} \Psi_{nm}(\vec{r}) \left(c_{nm} + d_{nm}^\dagger\right), \]
\[ \hat{\pi} = -i\sqrt{\frac{\omega_0}{2}} \left[\Psi_0^{ZM}(\vec{r})\right]^* \left(a_{0,1} + a_{0,0} + a_{0,1}^\dagger + a_{0,0}^\dagger\right) \]

\^6The minus sign in the second commutation relation is a consequence of the fact that we use the existing freedom to describe the spurious sector by demanding that the vacuum state of the \(\Omega_s\)-oscillators should be annihilated by the operator \(-\frac{\partial}{\partial \Omega_s} + \Omega_s\).
\[-i\sqrt{\frac{\omega_1}{2}} \left[ \Psi^{ZM}(\vec{r}) \right]^* \left( a_{-1,1} + a_{-1,0} + a_{1,1}^+ + a_{1,0}^+ \right) \]
\[+i \sum_{n,m} \varepsilon^{l/2}_{nm} \Psi_{nm}^* (\vec{r}) \left( c_{nm}^+ - d_{nm} \right), \tag{41}\]

and the corresponding expressions for \( \hat{\theta} \) and \( \hat{\pi} \). The normalized eigenmodes are defined

\[\Psi_{nm}(\vec{r}) = \sqrt{\frac{1}{2\pi}} R_{nm}(r)e^{im\varphi}, \tag{42}\]

for the finite modes, and

\[\Psi_{0}^{ZM}(\vec{r}) = \frac{2fr e^{i\varphi}}{\sqrt{3(1+r^2)}},\]
\[\Psi_{1}^{ZM}(\vec{r}) = -\frac{2f}{\sqrt{M(1+r^2)}}, \tag{43}\]

for the zero modes.

The residual BRST terms are

\[H_{\text{BRST}}^{(\text{res})} = H^{(\text{res})} + \{\rho, Q\}^{(3)}, \tag{44}\]

where

\[\{\rho, Q\}^{(3)} = \Omega_{s'} V_{-s'} - \Omega_{s'} v_{s'}^{(2)} + \omega_{s'}^2 \eta_{-s'} - i C_{s'} v_{s'}^{(2)} + i C_{s'}^{-1} \Omega_{s'} \eta_{-s'} \pi_u, \tag{45}\]

and we have used the fact that \( v_{s'} \) is at most quadratic in the fluctuations. Thus \( \{\rho, Q\}^{(3)} \) is of \( \mathcal{O}(f^{-1}) \). As already mentioned the terms in \( H^{(\text{res})} \) are also of \( \mathcal{O}(f^{-1}) \) and higher. In Eq. (45) there appear the ghost conjugate operators \( \eta_{0'}, \pi_{0'} \) for which there is no corresponding term in the quadratic hamiltonian (37). The Hilbert space is thus divided into two degenerate subspaces by this ghost degree of freedom. However, there are no IR-divergences, since the ghost excitations always appear pairwise and the ghost accompanying the \( \eta_{0'}, \pi_{0'} \) operators has a frequency \( \omega_{s'} \). Therefore, the full residual hamiltonian may be treated in perturbation theory. However arbitrary, the spurious frequencies \( \omega_{s'} \) are finite and therefore the problem of infrared divergences has disappeared. Counterterms should still be introduced in order to solve the difficulties at the ultraviolet.
It should be noticed that, to $O(f^{-1})$, the residual BRST Hamiltonian has only off-diagonal terms (cf. Eq. (54)). Therefore, the lowest order corrections to diagonal quantities (like e.g., the mass of a soliton-meson state) turn out to be of $O(f^{-2})$.

The operator $L_0'$ defined as

$$L_0' = -\{Q_0, \pi_0' + iC_0^{\dagger}_s \Omega_{-s} \eta_i\}$$

$$= v_0' - V_0' - iC_0^{\dagger}_s \Omega_{-s} B_t - C_0^{\dagger}_s \eta_{-s} \pi_t - C_0^{\dagger}_s \bar{\eta}_{-s} \bar{\eta}_t,$$  \hspace{1cm} (46)

is the extension of $F_0'$ to include the transformation of the Lagrange multipliers and ghosts. It commutes with $H_{BRST}$ and is diagonalized by the transformation $\Lambda$.

$$L_0' = \Lambda - \frac{1}{2}(T_3 + J),$$  \hspace{1cm} (47)

where

$$\Lambda = \sum_{nm} \frac{1}{2}(m - 1)(c_{nm}^{\dagger}c_{nm} - d_{nm}^{\dagger}d_{nm})$$

$$+ \sum_{s'} C_{s's'}^{\dagger}(a_{s'0}^{\dagger}a_{s'0} - a_{s'1}^{\dagger}a_{s'1} + \bar{a}_{s'}a_{s'} - \bar{b}_{s'}b_{s'}).$$  \hspace{1cm} (48)

Let us consider as a basis of the Hilbert space the product form $|\text{intr}\rangle \otimes |\text{coll}\rangle$. The intrinsic subspace is characterized by the occupation numbers $n_{nm}^{c}, n_{nm}^{d} (=0,1,2,\ldots)$ of the real phonons and by the occupation numbers $n_{s'0}, n_{s'1} (=0,1,2,\ldots)$ and $n_{s'a}, n_{s'b} (=0,1)$ of the spurious phonons and ghosts. It carries the quantum number $\Lambda$.

A complete set of states for the collective sector depends on the angle $\alpha$ corresponding to internal rotation, the angle $\Phi$ determining the orientation of the spatial frame, and the coordinates $Z, \bar{Z}$ associated with the translational motion in the plane. It is convenient to use the complete set given by

$$|T_3, J, P_L, \bar{P}_L\rangle = e^{iT_3\alpha} e^{i\Phi} e^{i(P_LZ + \bar{P}_L\bar{Z})}.$$  \hspace{1cm} (49)

The operator $L_0'$ annihilates physical states since it is a “null” operator $[9]$. Therefore, for such states $T_3 = 2\Lambda - J$ and the collective subspace of interest is of the form (up to a trivial phase)
\[ |\Lambda; J, P_L, \bar{P}_L\rangle = e^{iJ(\Phi - \alpha)} e^{i(P_L Z + \bar{P}_L \bar{Z})}, \]  

(50)

where \( \Lambda \) is determined from the intrinsic structure.

The collective-intrinsic coupling \( \mathcal{O}(f^{-1}) \) is given by the first term in Eq. (45),

\[ \Omega_s V_{-s'} = \Omega_0(\Lambda - J) + i\sqrt{2} \left(e^{-i\Phi} \Omega_1 \bar{P}_L - e^{i\Phi} \Omega_{-1} P_L\right). \]  

(51)

The operator \( \Omega_0 \) does not change the value of \( \Lambda \), while \( \Omega_1 (\Omega_{-1}) \) decreases (increases) the value of \( \Lambda \) by one unit (cf. Eqs. (34), (38) and (48)). The conservation of \( L_0' \) is insured by the terms \( e^{\mp i\Phi} \) changing the value of \( J \) by the necessary amount.

The collective parameters may be calculated in perturbation theory using the fact that the intrinsic energies are of \( \mathcal{O}(1) \) and thus much larger than the collective energies \( \mathcal{O}(f^{-2}) \), cf. Eq. (52)). In second order of perturbation theory such term yields the well-known collective energies

\[ H_{coll} = \frac{1}{2\mathcal{S}_{s'V_{-s'}}} \left( J - \Lambda \right)^2 + \frac{2}{M} P_L \bar{P}_L, \]  

(52)

where Eqs. (19) and (30) have been used. The collective energies are given by the expectation values of (52) within the subset of states (50) associated with given values of \( \Lambda, J, P_L \) and \( \bar{P}_L \).

Higher orders of perturbation theory yield corrections to the parameters \( \mathcal{S} \) and \( M \), as well as higher than quadratic terms in the collective operators.

VI. CORRECTION TO THE SOLITON MASS

As an application of the formalism developed above in this section we evaluate explicitly the correction of \( \mathcal{O}(f^{-2}) \) to the soliton mass. For convenience we work with the real fields \( \theta_a (a = 1, 2) \) which are related to the fields \( \theta \) and \( \bar{\theta} \) of the previous sections by

\[ \theta = \theta_1 + i\theta_2; \quad \bar{\theta} = \theta_1 - i\theta_2 \]  

(53)

The correction of \( \mathcal{O}(f^{-2}) \) to the soliton mass is given by two-loops vacuum diagrams (Fig. 1) obtained from \( H_{BRST}^{(3)} \) and \( H_{BRST}^{(4)} \).
\[ H_{\text{BRST}}^{(3)} = \frac{1}{6f} \int \left\{ 3x_a \hat{\theta}_a \hat{\pi}_b \hat{\pi}_b + 4(1 + r^2)^3 G_{ab} \hat{\theta}_a \hat{\theta}_b + 12(1 + r^2)^2 G_{bc} \hat{\theta}_a \left[(1 + r^2) \hat{\theta}_a \hat{\theta}_b \hat{\theta}_c \right. \right. \]
\[ \left. + 12 G_b \nabla \left[(1 + r^2) \hat{\theta}_a \right] \cdot \nabla \left[(1 + r^2) \hat{\theta}_a \hat{\theta}_b \hat{\theta}_c \right] \right\} + \{ \rho, \mathcal{Q} \}^{(3)} \]  
\[ H_{\text{BRST}}^{(4)} = \frac{1}{24 f^2} \int \left\{ 12 x_a x_b \hat{\theta}_a \hat{\pi}_b \hat{\pi}_b + 6(1 + r^2) \hat{\theta}_a \hat{\pi}_b \hat{\pi}_b + 4(1 + r^2)^4 G_{abcd} \hat{\theta}_a \hat{\theta}_b \hat{\theta}_c \hat{\theta}_d \right. \]
\[ \left. + 16(1 + r^2)^3 G_{bc} \hat{\theta}_a \left[(1 + r^2) \hat{\theta}_a \hat{\theta}_b \hat{\theta}_c \hat{\theta}_d \right] \right. \]
\[ \left. + 24(1 + r^2)^2 G_{bc} \nabla \left[(1 + r^2) \hat{\theta}_a \right] \cdot \nabla \left[(1 + r^2) \hat{\theta}_a \hat{\theta}_b \hat{\theta}_c \right] \right\}, \]  
where
\[ G_{a_1 \ldots a_k} = \frac{\partial^k}{\partial a_1 \ldots \partial a_k} (1 + \vartheta^2)^{-2} \bigg|_{\vartheta = x_a} \]  
and \{ \rho, \mathcal{Q} \}^{(3)} is given in (55).

Each diagram of Fig. 4 has terms which depend on the spurious frequencies. They are
\[ (a) = -\frac{1}{6f} \sum_{n_1,n_2,s',t'} \frac{1}{\omega_{n_1} \omega_{n_2} \omega_{s'}} A_{s's'n_2} \left( A_{n_1 n_2 n_2} - 2\omega_{n_1}^2 B_{n_1 n_2 n_2} \right) \]
\[ -\frac{1}{128} \sum_{n,s',t'} \frac{1}{\omega_s \omega_{s'} \omega_{t'}} A_{s's'n_1} A_{t't'n_1}, \]  
\[ (b) = \frac{1}{8} \sum_{n_1,n_2,s',t'} \left[ \omega_{n_1} - \omega_{n_2} \right] \left[ 2\sqrt{3} \omega_s \omega_s' \omega_{s'} E_{s's'n_1 n_2} B_{n_1 s'n_2} - (\omega_{n_1} + \omega_{n_2} - \omega_{s'}) E_{s's'n_1 n_2}^2 \right] \]
\[ + \frac{1}{16} \sum_{n,s',t'} \left[ E_{s't'n_1} \left( \frac{\omega_{n_1}^2}{\omega_{s'} \omega_{t'}} - 2\omega_{n_1} \omega_{s'} + 2 \right) - 2\sqrt{3} \omega_s E_{s't'n_1} B_{s't'n_1} \right] \]
\[ -\frac{1}{256} \sum_{s',t',u'} \left( \frac{3u'}{3s'} C_{s't'u'} C_{s't'u'} + \frac{2}{3s'} C_{s't'u'} C_{s't'u'} \right), \]  
\[ (c) = \frac{1}{32} \sum_{n,s'} \frac{1}{\omega_{n} \omega_{s'}} \left( F_{n s's'} - 2\omega_n^2 D_{n s's'} - 8D_{n s's'} \right) \]
\[ + \frac{1}{64} \sum_{s,t'} \frac{1}{\omega_{s} \omega_{t'}} \left( F_{s't't'} - 8D_{s't't'} \right). \]  
The integrals \( A, B, D, E \) and \( F \) are defined in the Appendix. The indices \( n_k \) run only over finite frequency modes. Indices \( s',t',u' \) label the zero-modes.

Summing up the expressions \((a), (b) \) and \((c) \) and including the real sector contributions we obtain the total correction of order \( \mathcal{O}(f^{-2}) \) to the soliton mass. It reads:
\[ \Delta M^{(2)} = -\frac{1}{192} \sum_{n_1,n_2,n_3} \frac{\left( A_{n_1 n_2 n_3} + 2\omega_{n_1} \omega_{n_2} B_{n_1 n_2 n_3} + 2\omega_{n_2} \omega_{n_1} B_{n_2 n_1 n_3} + 2\omega_{n_3} \omega_{n_1} B_{n_3 n_2 n_1} \right)^2}{\omega_{n_1} \omega_{n_2} \omega_{n_3} (\omega_{n_1} + \omega_{n_2} + \omega_{n_3})}. \]
\[
-\frac{1}{128} \sum_{n_1,n_2,n_3} \frac{1}{\omega_{n_1} \omega_{n_2} \omega_{n_3}^2} \left( A_{n_1 n_2 n_3} - 2 \omega_{n_1}^2 B_{n_1 n_1 n_3} \right) \left( A_{n_2 n_2 n_3} - 2 \omega_{n_2}^2 B_{n_1 n_1 n_3} \right)
+ \frac{1}{8} \sum_{n_1,n_2,n_3} \frac{\omega_{n_1}}{\omega_{n_2}} B_{n_1 n_3 n_2}^2 + \frac{1}{64} \sum_{n_1,n_2} \left( \frac{1}{\omega_{n_1} \omega_{n_2}} F_{n_1 n_1 n_2} - \frac{4 \omega_{n_1}}{\omega_{n_2}} D_{n_1 n_1 n_2} \right)
+ \frac{1}{8} \sum_{n_1,n_2,n_3} \frac{\omega_{n_1} - \omega_{n_2}}{3 \omega_{n_2}} E_{n_1 n_1 n_2} \left( E_{n_1 n_1 n_2} - 2 \sqrt{3 \omega_{n_2}} B_{n_1 n_1 n_2} \right)
+ \frac{1}{8} \sum_{n_1,n_2,n_3} \frac{1}{3 \omega_{n_2}} E_{s',t',n} \left( E_{s',t',n} - \sqrt{3 \omega_{n_2}} B_{s',t',n} \right)
- \frac{1}{288} \sum_{s',t',u'} \left( \frac{3 \omega_{u'}}{3 \omega_{s'} \omega_{t'}} C_{s't'u'} C_{s't'u'} + \frac{2}{3 \omega_{s'} \omega_{t'}} C_{s't'u'} C_{s't'u'} \right). \] 

(60)

Using the identities (A10) listed in the Appendix it can be explicitly checked that \( \Delta M^{(2)} \) is independent of the arbitrary parameters \( \omega_{s'} \), as it should.

VII. CONCLUSIONS

We have carried out a quantum mechanical treatment of the \( O(3) \) model in (2+1) dimensions, both for the collective and the intrinsic excitations. We have derived the explicit expression of the BRST hamiltonian and shown that the corrections to the quadratic expressions may be perturbatively performed in terms of an expansion in the inverse of the large parameter of the model \( f \). Such expansion is free of IR-divergences related to translations and rotations of the classical solution. There still remains the IR-divergences due to massless mesons which are already present in the vacuum sector and can be regulated including a mass term.

The lagrangian we have studied is not UV-renormalizable. It must be considered as the low energy limit of a lagrangian with an infinite number of higher order derivative terms which can absorb the divergences [17].

Higher order terms must also be taken into account to avoid the dilatational instability. However, in the present work we have preferred not to include them and ignore the dilatation problem altogether. In this way we have the simplicity associated with the existence of analytical solutions and keep most of the features inherent to soliton models. Namely, there is a set of several zero-modes associated with broken symmetries, a non-commutative algebra
associated such symmetries, the interplay between external and internal transformations and the survival of some unbroken symmetries. We have shown that such problems may be successfully treated by the application of methods based on the BRST invariance. Moreover, such procedure may be carried out to other more realistic models in a fairly straightforward way, but for the fact that the equations should be numerically solved.

In our formalism we have considered that the collective velocities are small, i.e., $|V_{s'}|/\Im_{s'} \ll 1$. This allows to treat perturbatively the intrinsic-collective coupling $\Omega_{s'} V_{s'}$. In principle, the BRST procedure allows also to treat cases in which some expectation values of the Lagrange multipliers $\langle \Omega_{s'} \rangle = \langle V_{s'} \rangle / \Im_{s'}$ are of $\mathcal{O}(1)$. Such treatment has been applied, for instance, to the translational motion in [11] and corresponds to the classical solution used in [18] for the internal motion (in both cases (1+1) dimensions is assumed). An obvious and interesting extension of the present work would be the treatment of models with non-commutative zero-modes in the large collective momentum limit. This would allow, among other things, to study the stability of the fast rotating solitons. Another point that could be studied is the appearance of the long-time-seek Yukawa coupling in such a limit as it has been recently suggested in Ref. [18]. We hope to be able to report on these topics in the near future.

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APPENDIX:

In Sec. [VI] we made use of the following integrals which appear in the vertices of $H_{\text{BRST}}^{(3)}$ and $H_{\text{BRST}}^{(4)}$. 

where $G_{a_1...a_k}$ is defined in Eq. (56), and $\psi_{a,n}$ are the eigenmodes of the fluctuation (see Eqs. (12, 13)), which read

$$\psi_{1,nm}(\vec{r}) = \frac{1}{2\sqrt{\pi}} R_{nm}(r) \cos m \varphi; \quad \psi_{2,nm}(\vec{r}) = \frac{1}{2\sqrt{\pi}} R_{nm}(r) \sin m \varphi,$$

for the finite frequency modes, and

$$\psi_{1,0}^{ZM}(\vec{r}) = \sqrt{\frac{2}{3}} \frac{f x_1}{(1 + r^2)}; \quad \psi_{2,0}^{ZM}(\vec{r}) = \sqrt{\frac{2}{3}} \frac{f x_2}{(1 + r^2)},$$

$$\psi_{1,\pm 1}^{ZM}(\vec{r}) = \mp \sqrt{\frac{2}{M}} \frac{f}{(1 + r^2)}; \quad \psi_{2,\pm 1}^{ZM}(\vec{r}) = 0,$$

for the zero-modes.

We list below the relations that the integrals $A$, $B$, $E$ and $F$ satisfy when some of the indices $\bar{n}$ correspond to zero-modes:

$$A_{\bar{n}_1 \bar{n}_2 \bar{n}_3} = \frac{1}{6f} \int \left\{ (1 + r^2)^3 G_{abc} \psi_{a,\bar{n}_1} \psi_{b,\bar{n}_2} \psi_{c,\bar{n}_3} \right.$$  

$$+ 3(1 + r^2)^2 G_{bc} \partial_a \left[ (1 + r^2) \psi_{a,\bar{n}_1} \right] \psi_{b,\bar{n}_2} \psi_{c,\bar{n}_3}$$  

$$+ 3(1 + r^2)^2 G_{b} \nabla \left[ (1 + r^2) \psi_{a,\bar{n}_1} \right] \cdot \nabla \left[ (1 + r^2) \psi_{a,\bar{n}_2} \right] \psi_{b,\bar{n}_3} \right\}$$

$$+ \text{permutations of } (\bar{n}_1, \bar{n}_2, \bar{n}_3), \quad (A1)$$

$$B_{\bar{n}_1 \bar{n}_2 \bar{n}_3} = \frac{1}{2f} \int (1 + r^2)^3 G_{b} \psi_{a,\bar{n}_1} \psi_{a,\bar{n}_2} \psi_{b,\bar{n}_3},$$

$$D_{\bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4} = \frac{1}{4f^2} \int (1 + r^2)^4 G_{bc} \psi_{a,\bar{n}_1} \psi_{a,\bar{n}_2} \psi_{b,\bar{n}_3} \psi_{c,\bar{n}_4},$$

$$E_{0;\bar{n}_1 \bar{n}_2} = 0,$$

$$E_{\pm 1;\bar{n}_1 \bar{n}_2} = \pm \frac{i}{4\sqrt{2}f^2} \int (1 + r^2) \psi_{a,\bar{n}_1} \partial_{\bar{1}} \mp i \partial_{\bar{2}} \left[ (1 + r^2) \psi_{a,\bar{n}_2} \right],$$

$$F_{\bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4} = \frac{1}{48f^2} \int \left\{ (1 + r^2)^2 G_{abcd} \psi_{a,\bar{n}_1} \psi_{b,\bar{n}_2} \psi_{c,\bar{n}_3} \psi_{d,\bar{n}_4} ight.$$  

$$+ 4(1 + r^2)^2 G_{bcd} \partial_a \left[ (1 + r^2) \psi_{a,\bar{n}_1} \right] \psi_{b,\bar{n}_2} \psi_{c,\bar{n}_3} \psi_{d,\bar{n}_4}$$  

$$+ 6(1 + r^2)^2 G_{bc} \nabla \left[ (1 + r^2) \psi_{a,\bar{n}_1} \right] \cdot \nabla \left[ (1 + r^2) \psi_{a,\bar{n}_2} \right] \psi_{b,\bar{n}_3} \psi_{c,\bar{n}_4} \right\}$$

$$+ \text{permutations of } (\bar{n}_1, \bar{n}_2, \bar{n}_3, \bar{n}_4), \quad (A6)$$

for the finite frequency modes, and

$$\psi_{1,0}^{ZM}(\vec{r}) = \sqrt{\frac{2}{3}} \frac{f x_1}{(1 + r^2)}; \quad \psi_{2,0}^{ZM}(\vec{r}) = \sqrt{\frac{2}{3}} \frac{f x_2}{(1 + r^2)},$$

$$\psi_{1,\pm 1}^{ZM}(\vec{r}) = \mp \sqrt{\frac{2}{M}} \frac{f}{(1 + r^2)}; \quad \psi_{2,\pm 1}^{ZM}(\vec{r}) = 0,$$

for the zero-modes.
\[ A_{s't'n} = \frac{2}{\sqrt{3 s'}} \omega_n^2 E_{s't';n}, \]

\[ A_{s't'u'} = 0, \]

\[ B_{n_1 n_2; s'} = 0 \quad (A10) \]

\[ F_{s's'n_1 n_1} = \frac{1}{\sqrt{3 s'}} \sum_{n_2} E_{s's'n_2} \left( A_{n_1 n_1 n_2} - 2 \omega_{n_1}^2 B_{n_1 n_1 n_2} \right) + \frac{4}{3 s'} \sum_{n_2} \left( \omega_{n_1}^2 - \omega_{n_2}^2 \right) E_{s'n_1 n_1 n_2}^2 - \frac{4}{3 s'} \sum_{n'} \omega_{n_1}^2 E_{s'n_1 n_1 n_1}^2, \]

\[ F_{s's't't'} = \frac{2}{\sqrt{3 s't'}} \sum_{n_1} E_{s's';n_1} E_{t't';n_1} + \frac{4}{3 s'} \sum_{n_1} \omega_{n_1}^2 E_{s't';n_1}^2. \]

\( n_k \) denote finite frequency modes, and \( s', t' \) and \( u' \) zero-modes. The above Eqs. relate the spurious vertices from \( H^{(3)} \) and \( H^{(4)} \) to the vertices from \( \{ \rho, Q \}^{(i)} \), and are needed to show that the total contribution from diagrams of \( \mathcal{O}(f^{-2}) \) is independent of the spurious frequencies.
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FIG. 1. Diagrammatic corrections of $\mathcal{O}(f^{-2})$ to the soliton mass.