Asymptotics of non-minimizing stationary points of the Ohta-Kawasaki energy and its sharp interface version

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Abstract

We study a non-local Cahn-Hilliard energy arising in the study of di-block copolymer melts, often referred to as the Ohta-Kawasaki energy in that context. In this model, two phases appear, which interact via a Coulombic energy. As in [23]–[24], we focus on the regime where one of the phases has a very small volume fraction, thus creating “droplets” of the minority phase in a “sea” of the majority phase. In this paper, we address the asymptotic behavior of non-minimizing stationary points in dimensions \( n \geq 2 \) left open by the study of the \( \Gamma \)-convergence of the energy established in [23]–[24], which provides information only for almost minimizing sequences when \( n = 2 \). In particular, we prove that (asymptotically) stationary points satisfy a force balance condition which implies that the minority phase distributes itself uniformly in the background majority phase. Our proof uses and generalizes the framework of Sandier-Serfaty [37, 36], used in the context of stationary points of the Ginzburg-Landau model, to higher dimensions. When \( n = 2 \), using the regularity results obtained in [25], we also are able to conclude that the droplets in the sharp interface energy become asymptotically round when the number of droplets is constrained to be finite and have bounded isoperimetric deficit.

1 Introduction

This paper is devoted to the convergence of stationary points of the Ohta-Kawasaki energy functional [34] in the small volume regime. The energy functional has the following form:

\[
\mathcal{E}[u] = \int_{\Omega} \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right) dx + \frac{1}{2} \int_{\Omega} \int_{\Omega} (u(x) - \bar{u}) G_0(x, y)(u(y) - \bar{u}) \, dx \, dy,
\]

where \( \Omega \) is the domain occupied by the material, \( u : \Omega \to \mathbb{R} \) is the scalar order parameter, \( V(u) \) is a symmetric double-well potential with minima at \( u = \pm 1 \), such as the usual Ginzburg-Landau potential \( V(u) = \frac{1}{4}(1 - u^2)^2 \), \( \varepsilon > 0 \) is a parameter characterizing interfacial thickness, \( \bar{u} \in (-1, 1) \) is the background charge density, and \( G_0 \) is the Neumann Green’s function of the Laplacian, i.e., \( G_0 \) solves

\[
-\Delta G_0(x, y) = \delta(x - y) - \frac{1}{|\Omega|}, \quad \int_{\Omega} G_0(x, y) \, dx = 0,
\]

where \( \Delta \) is the Laplacian in \( x \) and \( \delta(x) \) is the Dirac delta-function, with Neumann boundary conditions. Note that \( u \) is also assumed to satisfy the “charge neutrality” condition

\[
\frac{1}{|\Omega|} \int_{\Omega} u \, dx = \bar{u}.
\]
For a discussion of the motivation and the main quantitative features of this model, see [23], as well as [32, 31]. For specific applications to physical systems, we refer the reader to [18, 42, 34, 33, 21, 27, 31, 30]. For the remainder of this paper we focus on the case where $\Omega$ is the flat $n$-dimensional torus $\mathbb{T}^n = [0,1]^n$ with periodic boundary conditions, unless otherwise specified.

We focus most of our attention on the following “sharp interface” version of (1):

$$E^\varepsilon[u] = \frac{\varepsilon}{2} \int_{\mathbb{T}^n} |\nabla u| + \frac{1}{2} \int_{\mathbb{T}^n \times \mathbb{T}^n} (u(x) - \bar{u})(x-y)(u(y) - \bar{u}) dxdy,$$

(reserving most of our analysis of the diffuse interface energy (1) to the final section of this paper (Section 6). In (4), $G$ is the screened Poisson kernel solving

$$-\Delta G + \kappa^2 G = \delta(x-y) \text{ in } \mathbb{T}^n,$$

for $\kappa = 1/\sqrt{V''(1)} > 0$ and $u \in A$ where

$$A := \{u \in BV(\mathbb{T}^n; \{-1,1\})\}. \quad (6)$$

The charge neutrality condition (cf. equation (3)) is no longer imposed, i.e. $\int_{\mathbb{T}^n} u \neq \bar{u}$. This is related to the fact that the charge of the minority phase is expected to partially redistribute itself into the majority phase to ensure screening of the induced non-local field (see [23] for a more detailed discussion). The energy (4) was first studied in [32] where the connection between (1) and (4) is made for exact minimizers. Moreover, when $n = 2$, it is shown that when $\bar{u}$ is close to $-1$, minimizers of (1) form almost spherical “droplets” of the minority phase $\{u = +1\}$ with the same radius, distributed uniformly throughout the domain. In [23, 24] the full $\Gamma$-limit of (4) was computed to first and second order near the onset of non-trivial minimizers (see [8] for an introduction to $\Gamma$-convergence), with [23] addressing the $\Gamma$ limit of (1) as well. There it is shown that, in addition, almost minimizers form (on average) almost spherical droplets of the phase $\{u = +1\}$, with almost the same radius and which are once again distributed uniformly throughout the domain. An important observation in these works is that, as $\varepsilon \to 0$, the number of disjoint connected components of $\{u^\varepsilon = +1\}$ may be unbounded [23, 24, 32], and the results can thus be seen as generalizations of the work of Choksi and Peletier who study a suitably rescaled version of (1) and (4) in the absence of screening (i.e. $\kappa = 0$) and when the number of droplets is constrained to be finite [10, 11]. More precisely, they compute the $\Gamma$-limit in this setting of (1) and (4) in [11] and [10] respectively, showing, in particular, that the droplets of the minority phase $\{u = +1\}$ shrink to points whose magnitudes and locations are determined via a limiting Coulombic interaction energy. A related result concerning minimizers is the work of Alberti-Choksi-Otto and Spadaro [1, 40], wherein it is shown that the energy of minimizers of (1) and (4) respectively is uniformly distributed throughout the domain.

All of the above results are concerned with minimizing stationary points of the energies (1) and (4). Moreover, all of the results regarding the asymptotics of minimizers when the number of droplets is unbounded work only in dimension $n = 2$. In this paper we address a question left open in the above analysis which is that of the asymptotic behavior of a priori non-minimizing stationary points of the energies (1) and (4) which, moreover, applies to any dimension $n \geq 2$. There has been some work in this context by Röger and Tonegawa [35]. They show that when the number of droplets is constrained to be finite in a bounded domain $\Omega$ with a fixed volume fraction, that any sequence of critical points $(u^\varepsilon)_{\varepsilon}$ of (1), i.e. solutions to

$$-\varepsilon^2 \Delta u^\varepsilon + V'(u^\varepsilon) + \phi^\varepsilon = \lambda^\varepsilon,$$
where \( \phi_\epsilon(x) = (G_0(x - \cdot) \ast (u_\epsilon - \bar{u}_\epsilon))(x) \) and \( \lambda_\epsilon \) is a Lagrange multiplier arising from \([3]\), satisfying mild bounds on the energy, converge in an appropriate sense to the Gibbs-Thompson law:

\[
\sigma H = \begin{cases} 
-\phi + \lambda & \text{for } x \in \partial^\text{\textcircled{a}} \{ u = +1 \} \\
0 & \text{for } x \in \partial \{ u = +1 \} \setminus \partial^\text{\textcircled{a}} \{ u = +1 \}. \end{cases} \tag{7}
\]

Here \( H \) is the mean curvature of \( \{ u = +1 \} \) where \( u \in BV(\Omega; \{-1, +1\}) \) and \( \phi \) are both appropriately rescaled limits of \( u_\epsilon^\cdot \) and \( \phi_\epsilon \) respectively, \( \sigma \) is an integer which arises from the ‘folding’ of the interfaces, \( \partial^\text{\textcircled{a}} \{ u = +1 \} \) denotes the reduced boundary of \( \{ u = +1 \} \) (see Section \([3]\) and \( \lambda \) is the limiting Lagrange multiplier constant. This establishes the connection between critical points of the diffuse interface energy \([1]\) and its sharp interface analogue (replacing the first Cahn-Hilliard term with perimeter). Our goal differs from that of \([35]\), as we wish to establish the distribution of the small droplets in the regime where the volume of the minority phase vanishes, and the number of droplets is not constrained to be finite a priori for \([1]\) and \([4]\).

To understand our goal more precisely, we recall some of the main results of \([23]\) for almost minimizers of \([1]\). We begin by setting

\[ \bar{u}_\epsilon = -1 + \delta(\epsilon), \]

in \([1]\) and show for almost minimizers of \([1]\), when \( \delta(\epsilon) = \epsilon^{2/3} \ln \epsilon^{1/3} \delta \) with \( \delta > 0 \), that the number of droplets of \( \{ u_\epsilon^\cdot = +1 \} \) is \( O(|\ln \epsilon|) \) as \( \epsilon \to 0 \) and, moreover, that

\[
\omega_\epsilon := \bar{\delta}(\epsilon)^{-1}(1 + \bar{u}_\epsilon) \to \bar{\omega} \text{ in } C(\mathbb{T}^2)^*, \tag{8}
\]

where \( \omega_\epsilon \) is the “normalized droplet density” of the phase \( \{ u_\epsilon^\cdot = +1 \} \) and where \( \bar{\omega} \) is the unique constant density minimizer to

\[
E^0[\omega] = \frac{\bar{\delta}^2}{2\kappa^2} + \left( \frac{2}{3} - \frac{2\bar{\delta}}{\kappa^2} \right) \int_{\mathbb{T}^2} d\omega + 2 \int_{\mathbb{T}^2 \times \mathbb{T}^2} G(x - y) d\omega(x) d\omega(y), \tag{9}
\]

over all Radon measures \( \omega \in H^{-1}(\mathbb{T}^2) \). Moreover, \( \bar{\omega} \) is given explicitly by

\[
\bar{\omega} = \max \left( \frac{1}{2} (\bar{\delta} - \bar{\delta}_c), 0 \right) \quad \text{with} \quad E^0[\bar{\omega}] = \frac{\bar{\delta}}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c), \tag{10}
\]

where \( \bar{\delta}_c > 0 \) is the critical volume fraction for the onset of non-trivial minimizers (ie. \( \bar{\omega} \neq 0 \)). In addition, setting \( v_\epsilon \) to be the solution to

\[
-\Delta v_\epsilon + \kappa^2 v_\epsilon = \omega_\epsilon,
\]

we conclude that

\[
\nabla v_\epsilon \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{T}^2). \tag{11}
\]

The convergence in equations \((8)\) and \((11)\) show that \( v_\epsilon \) and \( \omega_\epsilon \) are asymptotically constant in an averaged sense as \( \epsilon \to 0 \), which physically suggests the droplets are uniformly distributed throughout the domain. One way of phrasing the goal of this paper, is to ask the following question: Do the normalized droplet densities still converge weakly to a constant when we drop the assumption of minimality? Moreover, does this fact continue to hold in higher dimensions? We answer these questions under the single assumption that the perimeter of the set \( \{ u^\cdot = +1 \} \) vanishes as \( \epsilon \to 0 \). We make similar conclusions for the diffuse interface energy \([1]\), but reserve this discussion for a separate section (Section \(7\)).

Before we proceed, we give a precise definition of a stationary point of \([1]\). In addition to the class \( \mathcal{A} \) defined above, we occasionally consider stationary points in \( \mathcal{A} \) with mass constraint \( m \):

\[
\mathcal{A}_m := \left\{ u \in \mathcal{A} : \int_{\mathbb{T}^n} u = m \right\}. \tag{12}
\]
Definition 1. A function $u \in A$ is said to be a stationary point of (4) in $A$ if for any $C^1$ vector field $X : \mathbb{T}^n \to \mathbb{R}^n$ we have, setting $\phi_t(x) = x + tX(x)$, that
\[
\left. \frac{d}{dt} \right|_{t=0} E^\varepsilon(u \circ \phi_t) = 0.
\] (13)

If (13) holds only for all $\phi_t$ such that $u \circ \phi_t \in A_m$ for all $t$ sufficiently small, then we call $u$ a stationary point of (4) in $A_m$.

We proceed by showing that, away from a very small set on which the droplets are concentrated, we obtain a limiting condition on the measure $\omega_\varepsilon$ which takes the form
\[
\omega_\varepsilon \nabla v^\varepsilon \to \omega \nabla v = 0,
\] (14)
in a suitably weak sense. The convergence above clearly does not follow from the weak convergence of $\omega_\varepsilon$ (cf. equation (8)) and the weak convergence of the potential (cf. equation (11)). This is similar to the problem which arises when studying weak limits of solutions to the Euler equations in vorticity form as in [15, 16, 9, 14, 45] in dimension $n = 2$. It was originally Delort [14] who first recognized the phenomenon of “vorticity concentration cancellation”, which allows one to nonetheless pass to the limit in (14) in a distributional sense when $\omega_\varepsilon$ has a distinguished sign. Similar analysis was done by DiPerna and Majda [15, 16, 7] which allows for $\omega$ to have mixed signs under additional assumptions. There are, of course, natural regularity issues with the above equation, and we will see in Theorem 1 that the regularity we assume on $\omega$ allows us to obtain more precise information from (14). When $\omega$ is a smooth density for instance, it is easy to see that (14) implies that $\omega$ is constant on $\mathbb{T}^n$, so that the normalized droplet densities converge weakly to a constant. We obtain two characterizations of this condition, both of which imply that the droplets of the minority phase satisfy a kind of “force balance” condition, where the overall force on each limiting droplet is zero. Moreover, unlike the analysis of the Euler equations, our approach applies to all dimensions $n \geq 2$.

We have the additional difficulty, however, that we have contributions from the local terms in (4) and (1) which measure the perimeter of the level sets of $u$ when we take variations of the energy. Here we adopt, and generalize, the techniques in [37] and [36, Chapter 13] which were used to prove similar results in the context of Ginzburg-Landau. There it is shown that it suffices to establish (14) away from a very small set where the contributions of the surface terms are concentrated. Thus this framework can be seen as a generalization of the method of vorticity concentration cancellation introduced by Delort [14] for measures with distinguished sign, which allows for additional contributions to (14) that are concentrated on small sets, and which also allows for the measures to take on mixed signs, making it somewhat more similar to the work of DiPerna and Majda [15, 16, 7].

In order to make sense of (4) and its first variation, we must use extensively the theory of sets of finite perimeter (see [28, 17] for nice expositions, or [39] for a more general treatment which includes varifolds, which may have higher co-dimension). Unlike the analysis of the corresponding Euler-Lagrange equation which corresponds to minimizers in [32], here we will assume no minimality, and thus cannot expect global smoothness of the boundary. While it is known that local minimizers have boundaries which are of class $C^{3,\alpha}$ for some $\alpha > 0$ [4, 29, 43, 32, 41], the question of regularity of the reduced boundary of stationary points of (4) has only recently been addressed in [23]. More precisely, in [23], we provide a simple proof that the reduced boundary of any stationary point of (4) is of class $C^{3,\alpha}$, utilizing Allard’s regularity theorem [2], and present a rigorous derivation of the Euler-Lagrange equation satisfied by stationary points of (4). The additional regularity obtained therein allows us to make
stronger statements concerning the limiting behavior of critical points in dimension \( n = 2 \); in particular, we show that, in the case of a bounded number of droplets which have bounded isoperimetric deficit, the generalized mean curvature of each connected component of \( \{ u^\varepsilon = +1 \} \) (appropriately normalized) is asymptotically constant.

Our paper is organized as follows. In Section 2 we set up certain notation which will be used throughout the paper, and present our three main results in Sections 2.1, 2.3 and 2.4 respectively. In Section 3 we provide a brief introduction to the theory of sets of finite perimeter and weak mean curvature. In Section 4 we prove the main result of Section 2.1 for stationary points of the sharp interface energy (4). We then address the case of the diffuse interface energy (cf. equation (1)) in Section 6, where we prove the main result of Section 2.4.

**Notation:** We will denote \( D'(\Omega) \) as the space of distributions on \( \Omega \) and \( H^k(\Omega) \) and \( W^{k,p}(\Omega) \) will, as usual, denote the standard Sobolev spaces. We denote as \( H^k \) the standard \( k \)-dimensional Hausdorff measure. For a measurable set \( E \subset \Omega \), \( P(E \cap \Omega) \) will denote its relative perimeter (see Section 3 for definitions), and \( |E| \) will denote its standard \( n \)-dimensional Lebesgue measure. We write as \( T_n = [0,1)^n \), the standard flat \( n \)-dimensional torus. With some abuse of notation, we will sometimes say \( E \subset A \) (or \( A_m \)) when we mean \( \chi_E \), the indicator function of \( E \), belongs to \( A \) (respectively \( A_m \)). Finally we denote \( \alpha_n - 1 \) as the volume of the unit ball in \( \mathbb{R}^{n-1} \).

## 2 Problem formulation and main results

In this section, we first rewrite the energy (11) in a way which is more convenient for the subsequent presentation and analysis. We begin with the result of Ambrosio et al. [3] which allows us to decompose (up to \( \mathcal{H}^{n-1} \) negligible sets) \( \{ u = +1 \} \) into a countable collection of connected components \( \{ \Omega_i \} \) contained in a single cell of \( \mathbb{T}^n \) when \( \mathcal{H}^{n-1}(\{ u = +1 \}) \) is sufficiently small:

\[
 u(x) = -1 + 2 \sum_i \chi_{\Omega_i}(x),
\]

and we set

\[
 \tilde{u} = -1 + \delta(\varepsilon),
\]

where we assume \( \delta(\varepsilon) \) is bounded as \( \varepsilon \to 0 \). We define the “normalized droplet density”

\[
 \omega_\varepsilon := \sum_i \frac{\chi_{\Omega_i}}{|\Omega_i|},
\]

so that \( \omega_\varepsilon \) is a probability measure on \( \mathbb{T}^n \) for all \( \varepsilon > 0 \). If we insert (15) and (16) into the sharp interface energy (11) we obtain

\[
 E[ u^\varepsilon] = \varepsilon \sum_i P(\Omega_i) - \frac{2\delta(\varepsilon)}{\kappa^2} \sum_i \sum_i |\Omega_i| + 2 \int_{\mathbb{T}^n \times \mathbb{T}^n} G(x-y) \sum_i \chi_{\Omega_i}(x) \sum_i \chi_{\Omega_i}(y) dx \, dy + \frac{\delta(\varepsilon)^2}{2\kappa^2},
\]

where we set

\[
 -\Delta v_\varepsilon + \kappa^2 v_\varepsilon = \sum_i \frac{\chi_{\Omega_i}}{|\Omega_i|} =: \omega_\varepsilon.
\]

The rewriting of (11) expressed by (17) will turn out to be more convenient for our purposes, as it allows us to focus on a non-local energy which depends only on the normalized droplet
density $\omega_\varepsilon$ (and not $\bar{u}_\varepsilon$). Our goal is to derive a suitably weak form of (14). We proceed by computing the Euler-Lagrange equation of (14) and show that this is equivalent to a certain 2 tensor $\{S_{ij}\} = S^\varepsilon$ having zero divergence. The idea is then to pass to the limit in the condition
\[
\text{div} S^\varepsilon = 0
\]
as $\varepsilon \to 0$, and obtain a weak form of (14) as the limiting condition. This may at first appear surprising, as there will be contributions (in the form of curvature) from the perimeter term in (17), and (14) seems to depend only on the non-local terms. As alluded to before, we show that the contributions from these local terms occur in a very small set so that we are still able to conclude (14) in an appropriately weak sense outside of this set, and this turns out to be enough to make our main conclusions. More precisely, we show that the set where the local terms are concentrated in the Euler-Lagrange equations have arbitrarily small 1-capacity.

We recall from Evans-Gariepy [17] the definition of $p$-capacity of a set $E \subset \mathbb{R}^n$:
\[
\text{Cap}_p(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla \varphi|^p; \varphi \in L^{p^*}(\mathbb{R}^n), \nabla \varphi \in L^p(\mathbb{R}^n), E \subset \text{int}(\varphi \geq 1) \right\},
\]
where int($A$) denotes the interior of $A$ and $p^* = 2p/(2 - p)$. We will show that up to a set of very small 1-capacity, the tensor $S^\varepsilon$ is close to the tensor $T^\varepsilon$ in $L^1(T^n)$ defined by
\[
T^\varepsilon_{ij} = -\partial_i v_\varepsilon \partial_j v_\varepsilon + \frac{1}{2} \delta_{ij} (|\nabla v_\varepsilon|^2 + \kappa^2 v_\varepsilon^2),
\]
where the condition
\[
\text{div} T^\varepsilon = 0
\]
implies that
\[
\omega_\varepsilon \nabla v_\varepsilon = 0 \text{ in } L^1_{\text{loc}}.
\]
Our goal is to pass to the limit in this condition and obtain the weak form of (14):
\[
\text{div} T = 0,
\]
up to a set of arbitrarily small 1-capacity, where $T$ is the 2-tensor with components $T_{ij}$ given by
\[
T_{ij} = -\partial_i v \partial_j v + \frac{1}{2} \delta_{ij} (|\nabla v|^2 + \kappa^2 v^2),
\]
and $v$ is the distributional limit of $v^\varepsilon$ (cf. equation (18)) obtained from the weak convergence of $\omega_\varepsilon$ to $\omega$. The condition (19) is in fact obtained by taking variations of the non-local term in (17) of the form $v_t(x) = v(x + tX(x))$, often called “inner variations”. More precisely, condition (19) arises from the vanishing of
\[
\frac{d}{dt} \bigg|_{t=0} \int_{T^n} |\nabla v_t|^2 + \kappa^2 v_t^2 \, dx.
\]
The vanishing of the divergence of this tensor (cf. equation (20)) implies, in particular, that $v$ is constant on the support of $\omega$ if $\omega \in L^p(T^n)$ for large enough $p$ and a “vanishing gradient property”, first established in [6] in the context of Ginzburg-Landau, if $\omega = \sum_{i=1}^d b_i \delta_{a_i}$ (see Theorem 1), which formally states that the force on each particle is balanced by the others. We now make some of these notions precise in order to state our main result, and begin with the following definition, taken from [39].
Definition 2. (Divergence-free in finite part) Assume $X$ is a vector field in $\mathbb{T}^n$. We say $X$ is divergence-free in finite part if there exists a family of sets $\{E_\delta\}_{\delta > 0}$ such that

1. We have $\lim_{\delta \to 0} \text{Cap}_1(E_\delta) = 0$.
2. For every $\delta > 0$, $X \in L^1(\mathbb{T}^n \setminus E_\delta)$.
3. For every $\zeta \in C^\infty_c(\mathbb{T}^n)$, 
   \[ \int_{\mathbb{T}^n \setminus F_\delta} X \cdot \nabla \zeta = 0, \]
   where $F_\delta = \zeta^{-1}(\zeta(E_\delta))$.

If $T$ is a 2-tensor with coefficients $\{T_{ij}\}_{1 \leq i,j \leq n}$ we say $T$ is divergence-free in finite part if the vectors $T_i = (T_{i1}, T_{i2}, \cdots, T_{in})$ are, for $i = 1, 2, \cdots, n$.

To see that the above definition is consistent with the ordinary notion of divergence free, we borrow the following proposition from [36].

**Proposition 1.** Assume that $X$ is divergence free in finite part in $\mathbb{T}^n$ and that $X \in L^1(\mathbb{T}^n \setminus E)$. Then for every $\zeta \in C^\infty_c(\mathbb{T}^n)$ we have 
\[ \int_{\mathbb{T}^n \setminus F} X \cdot \nabla \zeta = 0, \]
where $F = \zeta^{-1}(\zeta(E))$. In particular, if $X$ is in $L^1(\mathbb{T}^n)$, then $F = \emptyset$ in the above and therefore $\text{div}X = 0$ in $\mathcal{D}'(\mathbb{T}^n)$.

### 2.1 Main result I: The sharp interface energy (4)

Our first main result concerning stationary points of (4) is the following.

**Theorem 1. (Equidistribution of droplets)** Let $u^\varepsilon \in A$ be a sequence of stationary points of (4) in $A$ in the sense of Definition 1 and assume 
\[ \limsup_{\varepsilon \to 0} \int_{\mathbb{T}^n} |\nabla u^\varepsilon| = 0. \quad (22) \]

Then for any $p \in (1, n/(n - 1))$, $\omega^\varepsilon$ converges in $W^{-1,p}$ to a probability measure $\omega$ and $v^\varepsilon$ converges in $W^{1,p}$ to $v$, where $v$ and $\omega$ are related via 
\[ -\Delta v + \kappa^2 v = \omega. \quad (23) \]

Moreover, the symmetric 2-tensor $T_\omega$ with coefficients $T_{ij}$ given by (21) is divergence free in finite part. In addition, we have the following characterizations of the divergence free condition on $T_\omega$.

0. If $\int d\omega^\varepsilon = 0$ for all $\varepsilon > 0$ sufficiently small, then 
   \[ \omega \equiv 0. \quad (24) \]

1. If $\omega \in H^{-1}(\mathbb{T}^n)$ then 
   \[ \text{div}T = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n). \quad (25) \]
2. If $\omega \in L^p$ for $p > 1$ when $n = 2$ and $p \geq 2n/(n+1)$ otherwise and $\omega \neq 0$, then in fact
\[ \omega = 1 \, dx, \]
the uniform Lebesgue measure on $\mathbb{T}^n$. 

3. If $\omega = \sum_{i=1}^d b_i \delta_{a_i}$, then setting $v(x) = \Phi(|x - a_i|) + H_i(x)$ where $\Phi$ is the fundamental solution to the Laplace equation in $\mathbb{R}^n$ and $H_i$ is smooth in a neighborhood of $a_i$, we have
\[ \nabla H_i(a_i) = 0, \quad (26) \]
for $i = 1, \ldots, d$.

Theorem 1 is analogous to the results obtained for Ginzburg-Landau \cite{37, 36}, with the droplets playing the role of the vortices in the magnetic Ginzburg-Landau model. The main difference in our case is that we are dealing with sharp interface version of (1) so that $u^\varepsilon$ takes on only the values $+1$ and $-1$. We must therefore be careful concerning regularity issues on the boundary of the set $\{u^\varepsilon = +1\}$, and consequently use the theory of finite perimeter sets (Section 3). Our proof, however, is in some ways simpler as we will have no contributions from the local terms outside the support of $\omega^\varepsilon$. This is no longer true for (1) in Section 6, and some additional analysis is needed. In addition, the vortices in the Ginzburg-Landau model are quantized, and we do not a priori know the shape or volume of the droplets in this model. Theorem 3 in Section 2.3 provides some information about the shape of these droplets; in particular, they are asymptotically round as $\varepsilon \to 0$ when $n = 2$ under assumptions on the number of droplets and their isoperimetric deficit ratio. We will see later that this is easily seen to be false for dimensions $n \geq 3$.

2.2 Interpretation of Theorem 1

The hypothesis (22) is essential to our proofs, as it will be seen to imply that $\text{Cap}_1(\{u^\varepsilon = +1\}) = o(1)$ as $\varepsilon \to 0$. This allows us to show that $\text{div}^\varepsilon T^\varepsilon$ converges, in a distributional sense, outside of the set $\{u^\varepsilon = +1\}$ to $\text{div}T^\omega$. The smallness of the set $\{u^\varepsilon = +1\}$ allows us to demonstrate that the limiting tensor $T^\omega$ is divergence free in finite parts.

The conditions of Cases 2 and 3 are simply consequences of the divergence free condition on $T^\omega$ (see Section 4). The condition (26) is called the ‘vanishing gradient property’, first established in the context of Ginzburg-Landau in \cite{6} where $\{(a_i, b_i)\}$ is a critical point of the “renormalized energy” associated to the problem. The condition (26) can be interpreted as saying the sum of the Coulombic forces from the neighboring droplets balance each other.

When $\omega$ is regular enough (Case 2) and non-zero, then in fact it is equal to the uniform Lebesgue measure on $\mathbb{T}^n$, meaning the droplets are uniformly distributed throughout the domain. When we only know that $\omega \in H^{-1}(\mathbb{T}^n)$ as in Case 1 above, the measure $\omega$ can be concentrated on lower dimensional hypersurfaces \cite{3, 26, 37}. This concentration phenomenon also occurs in the two-dimensional magnetic Ginzburg-Landau model where the limiting vortices of solutions, which bear much resemblance with the droplets in our case, can concentrate on lines \cite{3} \cite{30}. Analysis concerning the existence of solutions to (23) in a bounded domain $\Omega$ with $\omega$ concentrated on a smooth, closed curve $\Sigma \subset \subset \Omega$, and absolutely continuous with respect to the arc-length measure on $\Sigma$, is studied in \cite{26}. In all cases, the above analysis shows that we can have $\omega \in H^{-1}(\Omega)$, while it is not in general true that $\omega << dx$. Here we demonstrate a simple example on $\mathbb{T}^n$ for the screened Poisson kernel (an example for the non-screened kernel can be similarly constructed).
Example 1. Let $w(s)$ be the Green’s function of the operator $-\Delta + \kappa^2 I$ on $[-1, 1]$ with periodic boundary conditions. Then $w$ is the unique periodic solution to

$$-w''(s) + \kappa^2 w(s) = \delta(s) \text{ on } [-1, 1],$$

where $\delta(s)$ is the Dirac delta function at $s = 0$. Set $v(x_1, \ldots, x_n) = w(x_1)$ and we have

$$-\Delta v(x_1) + \kappa^2 v(x_1) = \delta(x_1) \text{ on } [-1, 1]^n.$$

In this case the divergence free condition $\text{div} T_\omega = 0$ is equivalent to requiring that

$$\int_{-1}^1 (-v^2_{x_1} + \kappa^2 v^2) \phi'(s)ds = 0 \text{ for all } \phi \in C^1([-1, 1]) \text{ periodic.}$$

It is easy to see from standard elliptic methods that $w \in W^{1,\infty}([-1, 1])$ and $w \in C^\infty(B_\rho(0)^c)$ for any fixed $\rho > 0$. Thus

$$\int_{-1}^1 (-v^2_{x_1} + \kappa^2 v^2) \phi'(s)ds = \int_{-1}^{-\rho} (-v^2_{x_1} + \kappa^2 v^2) \phi'(s)ds + \int_{\rho}^1 (-v^2_{x_1} + \kappa^2 v^2) \phi'(s)ds + o_\rho(1),$$

as $\rho \to 0$. Observing that if $w(s)$ solves (27), then so does $w(-s)$, we conclude from uniqueness of solutions to (27) that $w$ is an even function and therefore that $w'$ is odd. Thus integrating by parts and using periodicity of $v$ we obtain

$$\int_{-1}^{-\rho} (-v^2_{x_1} + \kappa^2 v^2) \phi'(s)ds + \int_{\rho}^1 (-v^2_{x_1} + \kappa^2 v^2) \phi'(s)ds = (-v^2_{x_1} + \kappa^2 v^2)\phi\big|_{x_1=+\rho} - (-v^2_{x_1} + \kappa^2 v^2)\phi\big|_{x_1=-\rho} = 0. \quad (29)$$

Combining (28) and (29) and then sending $\rho \to 0$, we conclude $\text{div} T_\omega = 0$ in $\mathcal{D}'([-1, 1]^n)$.

In the following section we recall that we say $E \subset A$ (respectively $E \subset A_m$) if the characteristic function of $E$, $\chi_E$, belongs to $A$ (respectively $A_m$).

2.3 Main Result II: Asymptotic roundness of droplets

We begin by recalling the main result of [25], applied specifically to the torus. For $\gamma \in \mathbb{R}$ we consider the more general functional $I_\gamma : A \to \mathbb{R}$ given by

$$I_\gamma(E) := P(E) + \gamma \int_E \int_E G(x, y) dy dx + \int_E f(x) dx, \quad (30)$$

where $f \in C^2(T^n)$, $\gamma \in \mathbb{R}$ is a constant parameter, $P(E)$ is the perimeter of $E$ (see Section 3) and $G \in L^1(T^n \times T^n)$ is the kernel of the Laplacian on $T^n$.

The reduced boundary of a set $E$ is said to be of class $C^{k,\alpha}$ if each point in $\partial^* E$ is locally contained in the graph of a function which is $C^{k,\alpha}$. Our main result in [25] for the regularity of the reduced boundary is the following.

Theorem 2. Let $E$ be a stationary point of the functional (30) in $A$ or $A_m$. Then the reduced boundary $\partial^* E$ belongs to the class $C^{3.1-n/p}$. In particular, the equation

$$H(x) + 2\gamma v_E + f(x) = \lambda,$$

holds strongly on $\partial^* E$ where $H$ is the mean curvature of $\partial^* E$, and $\lambda$ is a Lagrange multiplier. When $E$ is a stationary point in the class $A$, then $\lambda = 0$. Moreover, $\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$.  

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The proof of Theorem\textsuperscript{2} follows essentially from Allard’s regularity theorem and De Giorgi’s structure theorem. Theorem\textsuperscript{2} applied to (17) with \( n = 2, E = \{ u^\varepsilon = +1 \} \) and \( f = -\frac{2\delta(\varepsilon)}{\kappa^2} \) yields the equation
\begin{equation}
\varepsilon H_\varepsilon - \frac{2\delta(\varepsilon)}{\kappa^2} + v_\varepsilon \sum_j |\Omega_j| = 0 \quad \text{on } \partial^* \{ u^\varepsilon = +1 \}. \tag{31}
\end{equation}
We will use (31) to show that when the number of droplets is finite and they have bounded isoperimetric deficit, they become asymptotically round as \( \varepsilon \to 0 \) in \( n = 2 \).

We recall that the Green’s function on \( T^2 \) can be written as
\begin{equation}
G(x - y) = -\frac{1}{2\pi} \log |x - y| + S(x - y) \quad \text{for } x, y \in T^2, \tag{32}
\end{equation}
where \( S \) is a continuous function. If we consider a single round droplet so that \( u^\varepsilon = -1 + \chi_{B(x_\varepsilon,r_\varepsilon)} \), then formally we expect from (31) and (32) that
\begin{equation}
H_\varepsilon \simeq \frac{\log r_\varepsilon}{\varepsilon} r_\varepsilon^2 + \varepsilon^{-1} \delta(\varepsilon). \tag{33}
\end{equation}
When \( \varepsilon^{-1} \delta(\varepsilon) = O(r_\varepsilon^2 \log r_\varepsilon) \), as is the case for minimizers \textsuperscript{32} \textsuperscript{23} \textsuperscript{24}, then we have
\begin{equation}
H_\varepsilon = O \left( \frac{\log r_\varepsilon}{\varepsilon} r_\varepsilon^2 \right) \quad \text{as } \varepsilon \to 0. \tag{34}
\end{equation}
Equation (34) provides us with a hint of what the correct scaling of \( H_\varepsilon \) should be as the droplets shrink to points.

We now make the assumption that \( u^\varepsilon = -1 + \sum_{j=1}^{N(\varepsilon)} \chi_{\Omega_j} \) for \( N(\varepsilon) = O(1) \) as \( \varepsilon \to 0 \) so that the number of droplets is constrained to be finite. In the case that \( u^\varepsilon \) is minimizing, it is shown in \textsuperscript{32} \textsuperscript{23} \textsuperscript{24} that any two droplets stay sufficiently far apart, and this is due to the the Coulombic repulsion between droplets arising in the non-local term when bounds on the energy are assumed. This is no longer true in our case, and we must account for the situation where multiple droplets converge to the same point in \( T^2 \), while still finding an appropriate normalization of \( H_\varepsilon \) as the droplets shrink to points. Motivated from the above discussion, we define
\begin{equation}
\rho_\varepsilon := \frac{-\varepsilon}{\sum_{j=1}^{N(\varepsilon)} \log P(\Omega_j) \sum_{j=1}^{N(\varepsilon)} |\Omega_j|}, \tag{35}
\end{equation}
to be the “normalized radius” and
\begin{equation}
\tilde{\delta} := \liminf_{\varepsilon \to 0} \frac{-\delta(\varepsilon)}{\sum_{j=1}^{N(\varepsilon)} \log P(\Omega_j) \sum_{j=1}^{N(\varepsilon)} |\Omega_j|}, \tag{36}
\end{equation}
to be the “normalized volume fraction”. When we work in the scaling regime of minimizers as in \textsuperscript{32} \textsuperscript{23} \textsuperscript{24} then it is shown that there exists a \( \tilde{\delta}_{cr} > 0 \) such that whenever \( \tilde{\delta} > \tilde{\delta}_{cr} \) we have \( P(\Omega_j) = O(\varepsilon^{1/3} \ln \varepsilon^{-1/3}), |\Omega_j| = O(\varepsilon^{2/3} \ln \varepsilon^{-2/3}) \) and thus, when \( N_\varepsilon = O(1) \) as \( \varepsilon \to 0 \),
\begin{equation}
\rho_\varepsilon = O(\varepsilon^{1/3} \ln \varepsilon^{-1/3}) = O(r_\varepsilon) \quad \text{as } \varepsilon \to 0,
\end{equation}
where \( r_\varepsilon = 3^{1/3} \varepsilon^{1/3} \ln \varepsilon^{-1/3} \) is the energetically preferred radius of a single droplet as shown in \textsuperscript{32} \textsuperscript{23} \textsuperscript{24}. We have the following Theorem concerning the asymptotic roundness of droplets when \( n = 2 \) as \( \varepsilon \to 0 \).
Theorem 3. (Asymptotic roundness of droplets when $n = 2$) Assume the hypotheses of Theorem 1 and, in addition, that $u^{\varepsilon} = -1 + 2\sum_{i=1}^{N(\varepsilon)} \chi_{\Omega_i}$ for $N(\varepsilon) = O(1)$ as $\varepsilon \to 0$ with bounded isoperimetric deficit:

$$\limsup_{\varepsilon \to 0} \frac{\sum_{j=1}^{N(\varepsilon)} P(\Omega_j)^2}{\sum_{j=1}^{N(\varepsilon)} |\Omega_j|} < +\infty,$$

and $\delta \in (0, +\infty)$. Then there exists a $\delta_{cr}$ such that for $\delta > \delta_{cr}$ the following holds. Let $\Omega_{j_i}$ have center of mass converging (subsequentially) to $a_i$ for $j_i = 1, \cdots, d_i$. Then there exists a constant $c_i > 0$ such that such that

$$\|\rho^{\varepsilon} H - c_i\|_{L^\infty(\bigcup_{j_i=1}^{d_i} \partial \Omega_{j_i})} \to 0 \text{ as } \varepsilon \to 0,$$

up to subsequences, where $H^{\varepsilon}$ is the mean curvature of $\{u^{\varepsilon} = +1\}$ and $\rho^{\varepsilon}$ is given by (35).

Remark 1. The assumption (37) is required in order to ensure the next order term in the expansion of the potential $v^{\varepsilon}$ is controlled. In the case of minimizers as in [32, 23, 24], bounds on the energy imply the condition (37). It is easy to see in dimensions $n \geq 3$ that the above statement is false, by taking any solution in $n = 2$ and extending uniformly in the third direction we also obtain a solution which is composed of tubes (and not spherical droplets). The proof works in dimension $n = 2$ due to the specific scaling of the logarithmic potential, as can be seen by (32). Indeed, for very small droplets, the leading order contribution from the potential $v^{\varepsilon}$ is independent of the shape of the droplet.

2.4 Main result III: The diffuse interface energy equation (1)

For the diffuse interface energy (1), the analysis is very similar to that of the sharp interface energy (4), however we must use the unscreened kernel for the Laplace operator and thus define

$$\tilde{T}_{ij} = -\partial_i v^{\varepsilon}\partial_j v^{\varepsilon} + \frac{1}{2} \delta_{ij} |\nabla v^{\varepsilon}|^2,$$

where

$$v^{\varepsilon}(x) = \int_{\mathbb{T}^n} G(x - y) \frac{1}{1 + u^{\varepsilon}(y)} dy,$$

and we make the particular choice $V(u) = \frac{1}{4}(1 - u^2)^2$. We must now work in the class $\mathcal{A}_{\bar{u}}$ given by

$$\mathcal{A}_{\bar{u}} := \left\{ u \in H^1(\mathbb{T}^n) : \int_{\mathbb{T}^n} u = \bar{u} \right\},$$

due to (3). For the energy (1), we define a critical point as follows.

Definition 3. A function $u \in \mathcal{A}_{\bar{u}}$ is said to be a critical point of (1) if for any $v \in H^1(\mathbb{T}^n)$ satisfying $\int_{\mathbb{T}^n} v = 0$ we have

$$\frac{d}{dt} \bigg|_{t=0} \mathcal{E}^{\varepsilon}(u + tv) = 0.$$

A simple calculation along with standard elliptic theory reveals that $u^{\varepsilon}$ is $C^{3,\alpha}$ and solves the elliptic equation

$$-\varepsilon^2 \Delta u^{\varepsilon} + u^{\varepsilon}(1 - (u^{\varepsilon})^2) + \delta(\varepsilon)v^{\varepsilon} = \lambda^{\varepsilon} \text{ in } \mathbb{T}^n,$$

where $\lambda^{\varepsilon}$ is the Lagrange multiplier corresponding to the volume constraint when taking variations in Definition 3.
We show that if \( u^\varepsilon \in A_{\bar{u}^\varepsilon} \), with \( \bar{u}^\varepsilon = -1 + \delta(\varepsilon) \), is a sequence of critical points of \( E^\varepsilon \) with the perimeter of the minority phase vanishing, then \( \tilde{T}^\varepsilon \) converges up to a small set to the tensor \( \tilde{T}_\omega \) with coefficients defined by

\[
\tilde{T}_{ij} = -\partial_i v \partial_j v + \frac{1}{2} \delta_{ij} |\nabla v|^2,
\]

where now

\[-\Delta v = \omega - 1 \text{ on } \mathbb{T}^n,
\]

and \( \omega \) is a probability measure on \( \mathbb{T}^n \). More precisely, we prove the following.

**Theorem 4. (Diffuse interface energy)** Let \( u^\varepsilon \in A_{\bar{u}^\varepsilon} \) be a sequence of critical points of (1) in the sense of Definition 3 which satisfy

\[
\limsup_{\varepsilon \to 0} |\lambda_{E^\varepsilon}| < +\infty \text{ and }
\]

\[
\limsup_{\varepsilon \to 0} \mathcal{H}^{n-1}(\{u^\varepsilon \geq -1 + \delta(\varepsilon)^{1+\alpha}\}) = 0 \text{ for } \alpha > 0,
\]

with

\[
\bar{u}^\varepsilon = -1 + \delta(\varepsilon) \text{ and } \delta(\varepsilon) = o_\varepsilon(1) \text{ as } \varepsilon \to 0.
\]

Then for any \( p \in (1, n/(n-1)) \), \( \omega^\varepsilon := \frac{1 + u^\varepsilon}{\delta(\varepsilon)} \) converges in \( W^{-1,p} \) to a probability measure \( \omega \) and \( v^\varepsilon \) converges in \( W^{1,p} \) to \( v \) where

\[-\Delta v = \omega - 1 \text{ on } \mathbb{T}^n.
\]

Moreover, the symmetric 2-tensor \( T_\omega \) with coefficients \( T_{ij} \) given by (40) is divergence free in finite part. In particular, cases 0., 1., 2. and 3. of Theorem 1 continue to hold for \( \omega \).

**Remark 2.** The specific choice of \( \delta(\varepsilon)^{1+\alpha} \) in (41) is a technical limitation which is required in the proofs.

### 3 Mathematical preliminaries: Sets of finite perimeter

Here we introduce the basic notions of sets of finite perimeter. A detailed exposition on these topics can be found in [28]. For a more general treatment of varifolds, we refer the reader to [39]. Let \( E \in \mathbb{R}^n \) be a Lebesgue measurable set. We say that \( E \) has finite perimeter if

\[
\sup_{\varphi \in C_c^1(\mathbb{R}^n)} \int_E \text{div } \varphi < +\infty.
\]

By the Riesz-Representation theorem, the above implies the existence of a vector valued Radon measure \( \mu_E \) such that generalized Gauss-Green formula holds true

\[
\int_E \nabla \varphi = \int_{\mathbb{R}^n} \varphi d\mu_E \text{ for all } \varphi \in C_c^1(\mathbb{R}^n).
\]

The measure \( \mu_E \) is referred to as the Gauss-Green measure of \( E \) and the total perimeter of the set \( E \) is defined as

\[
P(E) = |\mu_E|(\mathbb{R}^n).
\]

In the case that \( E \) has a \( C^1 \) boundary, then we have

\[
\mu_E = \nu_E \mathcal{H}^{n-1} \partial E
\]

and

\[
P(E) = \mathcal{H}^{n-1}(\partial E).
\]
and, in particular, we have
\[ \nu_E(x) = \lim_{r \to 0^+} \int_{B(x,r) \cap \partial E} \nu_E dH^{n-1} = \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{\mu_E(B(x,r))}. \]

For a generic set \( E \) of finite perimeter, we therefore define the reduced boundary, denoted \( \partial^* E \), as those \( x \in \partial E \) such that the above limit exists and belongs to \( S^{n-1} \). The Borel vector field \( \nu_E : \partial^* E \to S^{n-1} \) is called the measure theoretic unit normal of \( E \). When \( \partial E \) is \( C^1 \), then the measure-theoretic outer unit normal agrees with the classical definition.

3.1 The first variation of perimeter

We wish to define a one-parameter family of diffeomorphisms with initial velocity \( X \in C^1_c(\Omega; \mathbb{R}^n) \) which is a collection \( \{ \phi_t \}_{t \in (-\epsilon, \epsilon)} \) for \( \epsilon > 0 \) defined as
\[ \phi_t(x) = x + tX(x), \quad x \in \Omega. \] (44)

We call \( \{ \phi_t \}_{-\epsilon < t < \epsilon} \) a local variation in \( \Omega \) associated with \( X \) if in addition
\[ \phi_t(\Omega) \subset \subset \Omega. \] (45)

The first variation of perimeter is then easily computed as (see [28, 17, 39])
\[ \frac{d}{dt} \bigg|_{t=0} P(\phi_t(E)) = \int \operatorname{div}_E X dH^{n-1}, \quad X \in C^1_c(\Omega; \mathbb{R}^n), \] (46)
where \( \operatorname{div}_E X \) is the tangential divergence of the vector field \( X \) with respect to \( E \):
\[ \operatorname{div}_E X = \operatorname{div} X - \nu_E(x) \cdot \nabla X(x) \nu_E(x). \]

Observe that the first variation is a linear functional on \( C^1_c(\Omega; \mathbb{R}^n) \). In the special case that it has a continuous extension to \( C_0^0(\Omega; \mathbb{R}^n) \) it can be represented by a vector valued Radon measure, which has a singular part with respect to \( \mu_E \) and a non-singular part, using the Radon-Nikodym theorem.

We thus have
\[ \int \operatorname{div}_E X dH^{n-1} = - \int X \cdot \vec{H} dH^{n-1} - \int X \cdot \nu_E d\sigma_E, \] (47)
where \( |\vec{H}| \in L^p_{\text{loc}}(\partial^* E) \) and \( \sigma_E \) denotes the singular part of the measure. We call \( \vec{H} \) the vector valued generalized mean curvature. When we can write \( \vec{H} = H\nu_E \), we call \( H \) the generalized mean curvature.

4 Proof of Theorem [1]

As seen previously in Section 2 (cf. equation [19]), a direct computation yields
\[ \operatorname{div} T^\epsilon = \nabla v_\epsilon \omega_\epsilon \text{ in } L^1_{\text{loc}}(T^n), \] (48)
where \( T^\epsilon \) is the 2-tensor with coefficients \( T_{ij} \) given by
\[ T_{ij} = -\partial_i v_\epsilon \partial_j v_\epsilon + \frac{1}{2} \left( |\nabla v_\epsilon|^2 + \kappa_\epsilon^2 v_\epsilon^2 \right) \delta_{ij}. \] (49)

As discussed in the beginning of Section [2], we proceed by showing that the Euler-Lagrange equation obtained in Theorem [2] is equivalent to the vanishing of a certain 2-tensor \( S^\epsilon \). The
part of $S^e$ which does not include $T^e$ will be shown to be concentrated on $\{w^e = +1\}$, which will be shown to have vanishing 1-capacity as $\varepsilon \to 0$, as a result of our assumption that $\mathcal{H}^{n-1}(\{w^e = +1\})$ vanishes as $\varepsilon \to 0$. The first step is the following proposition, which has been adapted from [36] and generalized to dimensions $n \geq 2$. The purpose of it will become clear in the proof of Theorem 1, where we will cover the set $\{w^e = +1\}$ by small balls and use the fact that the 1-capacity of a ball $B(x,r)$ is $\alpha_{n-1}r^{n-1}$ [17].

**Proposition 2.** Assume $K$ is a compact subset of $\mathbb{R}^n$. Then there exists a finite covering of $K$ by closed balls $B_1, \ldots, B_k$ such that

$$\sum_k r(B_k)^{n-1} \leq C \mathcal{H}^{n-1}(\partial K).$$

**Proof.** Since $\partial K$ is compact it suffices to work with a finite covering, and then taking closures and using Lemma 4.1 of [36], we may assume the balls are closed and disjoint, by possibly increasing the constant $C$ in the proposition. Indeed if $B_1$ and $B_2$ are two balls which intersect, then there exists a ball $B$ containing $B_1 \cup B_2$ such that $r(B) \leq r(B_1) + r(B_2)$ and thus $r(B)_{n-1} \leq C(r(B_1)^{n-1} + r(B_2)^{n-1})$.

In particular $A = \mathbb{R}^n \setminus \bigcup_{i=1}^k B_i$ is connected. Now if $B_1, \ldots, B_k$ cover $\partial K$, we claim they cover $K$. The claim follows by noting that $A$, which is connected, intersects the complement of $K$ since $K$ is bounded. Thus if $A$ intersected $K$ it would also intersect $\partial K$, which is impossible from the definition of $A$. Thus $K \subset \mathbb{R}^n \setminus A = \bigcup_{i=1}^k B_i$. The result then follows by the definition of $n-1$ dimensional Hausdorff measure. \hfill \Box

We now finally define precisely what we mean by $L^1$ convergence ‘up to a small set’. This definition is taken from [36].

**Definition 4.** We say a sequence $\{X_k\}_k$ in $L^1(\Omega)$ converges in $L^1_{\text{loc}}(\Omega)$ to $X$ if $X_k \to X$ in $L^1_{\text{loc}}(\Omega)$ except on a set of arbitrarily small 1-capacity, or precisely if there exists a family of sets $\{E_\delta\}_{\delta > 0}$ such that for any compact $K \subset \Omega$,

$$\lim_{\delta \to 0} \text{Cap}_1(K \cap E_\delta) = 0, \quad \forall \delta > 0 \quad \lim_{k \to +\infty} \int_{K \setminus E_\delta} |X_k - X| = 0.$$

We define similarly the convergence in $L^2_{\text{loc}}$ by replacing $L^1$ by $L^2$ in the above.

It is clear that $\nabla v^e$ cannot converge to $\nabla v$ strongly in $L^2$ in general, even if we have a uniform bound in $H^1(\mathbb{T}^n)$. However the fundamental observation is that away from a set of very small 1-capacity, we do in fact have strong $L^2$ convergence as long as the measures converge weakly in $(C(\mathbb{T}^n))^*$. The following result is adapted from [36] to work in higher dimensions.

**Proposition 3.** Assume $\{\alpha_k\}_k$ is a sequence of measures such that for some $p \in (1, n/(n-1))$

$$\lim_{k \to +\infty} \|\alpha_k\|_{W^{-1,p}(\Omega)}\|\alpha_n\|_{C^0(\Omega)^*} = 0,$$

for $\Omega \subset \mathbb{R}^n$ bounded and open where $\|\alpha_k\|_{C^0(\Omega)^*}$ denotes the total variation of $\alpha_k$, $\int_{\Omega} |\alpha_k|$. Then letting $h_k$ be the solution of

$$-\Delta h_k + \kappa^2 h_k = \alpha_k \text{ in } \Omega,$$

it holds that $h_k$ and $\nabla h_k$ converge to 0 in $L^2_{\text{loc}}(\Omega)$.\hfill 14
Proof. We begin by noticing that \( W^{1,q} \) embeds into \( C^0 \) for \( q > n \), and thus the \( (C^0)^* \) norm dominates the \( W^{-1,p} \) norm for \( p \in (1, n/(n - 1)) \). Thus the hypothesis implies that \( \|\alpha_k\|_{W^{-1,p}} \) tends to zero as \( k \to +\infty \). We let

\[
\delta_k = \left( \frac{\|\alpha_k\|_{W^{-1,p}}}{\|\alpha_k\|(C^0)^* + 1} \right)^{1/2}, \quad F_k = \{ x \in \Omega | |h_k| \geq \delta_k \}. \tag{50}
\]

Then we use the well known bound on p-capacity of \( F_k \) (see [17, Lemma 1])

\[
\text{Cap}_p(F_k) \leq C\frac{\|h_k\|^p_{W^{1,p}}}{\delta_k^p}. \tag{51}
\]

Then by elliptic regularity we have \( \|h_k\|_{W^{1,p}} \leq C\|\alpha_k\|_{W^{-1,p}} \) and so from (50)–(51) we have

\[
\text{Cap}_p(F_k) \leq C\|\alpha_k\|^p_{W^{-1,p}}(\|\alpha_k\|_{C^0(\Omega)^*} + 1)^{p/2},
\]

which therefore tends to 0 as \( k \to +\infty \). This implies that \( \text{Cap}_1(F_k) \to 0 \) as \( n \to +\infty \). From a well known property of Sobolev functions, the truncated function \( \bar{h}_k = \max(-\delta_k, \min(h_k, \delta_k)) \) satisfies \( \nabla \bar{h}_k = 0 \) a.e in \( F_k \), hence

\[
\int_{\Omega \setminus F_k} |\nabla h_k|^2 = \int_{\Omega} \nabla \bar{h}_k \cdot \nabla h_k.
\]

It follows that

\[
\int_{\Omega \setminus F_k} |\nabla h_k|^2 + \kappa^2 h_k^2 \leq \int_{\Omega} \nabla h_k \cdot \nabla \bar{h}_k + \kappa^2 h_k \bar{h}_k = \int_{\Omega} \bar{h}_k d\alpha_k,
\]

where the last equality follows from \( -\Delta h + \kappa^2 h = \alpha_k \). The right hand side is bounded above by \( \delta_k \|\alpha_k\|_{C^0(\Omega)^*} \), hence by \( \left( \|\alpha_k\|_{W^{-1,p}}\|\alpha_k\|_{C^0(\Omega)^*} \right)^{1/2} \) and therefore tends to zero as \( k \to +\infty \). Thus

\[
\lim_{k \to +\infty} \|h_k\|_{L^2(\Omega \setminus F_k)} = \lim_{k \to +\infty} \|\nabla h_k\|_{L^2(\Omega \setminus F_k)} = 0. \tag{52}
\]

To conclude, since \( \lim_{k \to +\infty} \text{Cap}_1(F_k) = 0 \) there is a subsequence still denoted by \( \{k\} \) so that \( \sum_k \text{Cap}_1(F_k) < +\infty \). We define

\[
E_\delta = \bigcup_{k>\delta} F_k.
\]

Then \( \text{Cap}_1(E_\delta) \) tends to zero as \( \delta \to 0 \) since it is bounded above by the tail of a convergent series. Moreover, for any \( \delta > 0 \) we have \( F_k \subset E_\delta \) when \( k \) is large enough and therefore (52) implies that \( \lim_{k \to +\infty} \|h_k\|_{L^2(\Omega \setminus E_\delta)} = \lim_{k \to +\infty} \|\nabla h_k\|_{L^2(\Omega \setminus E_\delta)} = 0. \)

We will see in the proof of Theorem 1 that Proposition 3 implies that \( T^\varepsilon \) converges to \( T \) in \( L^1_\varepsilon(\mathbb{T}^n) \). The proof of Theorem 1 then follows after applying the following proposition contained in [36].

**Proposition 4.** Assume \( \{T_k\}_{k \in \mathbb{N}} \) is a sequence of divergence-free vector fields which converge to \( T \) in \( L^1_\varepsilon(\mathbb{T}^n) \). Then \( T \) is divergence-free in finite part.

We are now ready to present the proof of Theorem 1. The characterizations of \( \omega \) in items 0,1,2,3 will be contained in Propositions 5 and 6 below.
 Proof of first part of Theorem 1: We begin by observing that if we define $J^\varepsilon$ to be the 2-tensor with coefficients $J_{ij} = (\delta_{ij} - \nu_i \nu_j) |\mu_\varepsilon|$, where $\mu_\varepsilon$ is the Gauss-Green measure of $\{u^\varepsilon = +1\}$ as in Section 3 we have
\[ \int_{\partial \{u^\varepsilon = +1\}} \text{div}_E X dH^{n-1} = \int_{\mathbb{T}^n} (\text{div}_E X - \partial_i X^j \nu_j) d|\mu_\varepsilon| = \int_{\mathbb{T}^n} J_{ij} \partial_i X^j. \]  
(53)

By Theorem 2 applied to (17), (48) and (53), we claim the criticality condition for $E^\varepsilon$ can be written as
\[ \text{div} S^\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n), \]
where $S^\varepsilon$ is the 2-tensor given by
\[ S^\varepsilon_{ij} = T^\varepsilon_{ij} - \frac{\varepsilon}{b_\varepsilon^2} (\delta_{ij} - \nu_i \nu_j) |\mu_\varepsilon| + \frac{1}{b_\varepsilon^2} \delta_{ij} \frac{(u_\varepsilon + 1) \delta(\varepsilon)}{\kappa^2} - \delta_{ij} v_\varepsilon \omega, \]  
(54)
where we’ve set $b_\varepsilon = \sum_j |\Omega_j|$. Indeed, applying Theorem 2 to (17) with $E = \{u^\varepsilon = +1\}$, $f = -\frac{\delta_\varepsilon}{\kappa^2}$, $\Omega = \mathbb{T}^n$ with Green’s potential $G$ of $\mathbb{T}^n$ we have
\[ \frac{\varepsilon}{b_\varepsilon^2} H_\varepsilon \mu_\varepsilon - \frac{\delta(\varepsilon)}{b_\varepsilon^2 \kappa^2} \mu_\varepsilon + \frac{1}{b_\varepsilon} v_\varepsilon \mu_\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n). \]
Using (48) and (53), a direct computation yields
\[ \text{div} S^\varepsilon = \nabla v_\varepsilon \omega - \frac{\varepsilon}{b_\varepsilon^2} H_\varepsilon \mu_\varepsilon + \frac{\delta(\varepsilon)}{b_\varepsilon^2 \kappa^2} \mu_\varepsilon - \nabla v_\varepsilon \omega - \frac{1}{b_\varepsilon} v_\varepsilon \mu_\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n). \]  
(55)

From Proposition 2 there exists a collection of balls $B_1, \ldots, B_k$ which cover $\{u^\varepsilon = +1\}$ with $\sum_{i=1}^k r(B_i)^{n-1} \leq C \mathcal{H}^{n-1}(\{u^\varepsilon = 1\})$. Define $Z_\varepsilon$ to be the union of these balls. Then we have
\[ S^\varepsilon = T_\varepsilon \text{ in } Z^\varepsilon_\varepsilon. \]

By subadditivity of the 1-capacity and the fact that the 1-capacity of a ball $B(x, r)$ is $\alpha_{n-1} r^{n-1}$ we have via the vanishing of $P(\{u^\varepsilon = +1\})$ (cf. equation (22)) that
\[ \text{Cap}_1(Z_\varepsilon) \to 0, \int_{\mathbb{T}^n \setminus Z_\varepsilon} |S_\varepsilon - T_\varepsilon| = 0. \]

Now choose a decreasing subsequence $\{\varepsilon_k\}$ tending to zero such that $\sum_k \text{Cap}_1(Z_\varepsilon_k) < +\infty$ and let
\[ E_\delta = \bigcup_{k \geq \frac{1}{\delta}} Z_\varepsilon_k. \]

Finally we define
\[ F_\delta := E_\delta \cup \tilde{E}_\delta, \]  
(56)
where, in view of the definition of $L^2_\delta$ convergence (cf. Definition 3), $\tilde{E}_\delta$ are the sets given by Proposition 3. Then once again by subadditivity of capacity we have
\[ \lim_{\delta \to 0} \text{Cap}_1(F_\delta) = 0. \]

Since $\omega_\varepsilon$ is a family of probability measures on $\mathbb{T}^n$, we have $\omega_\varepsilon \to \omega$ weakly in $(C^0(\mathbb{T}^n))^*$ up to a subsequence, and thus $\omega_\varepsilon \to \omega$ strongly in $W^{-1,p}$ for $p \in (1, n/(n-1))$ via the compact
embedding \((C^0(T^n))^* \subset \subset W^{-1,p}\) which follows from the compact embedding \(W^{1,q}(\mathbb{T}^n) \subset \subset C^0(\mathbb{T}^n)\) for \(q > n\). From Proposition 4 we therefore conclude

\[
\nabla v_\epsilon \to \nabla v \text{ in } L^2_\delta(\mathbb{T}^n).
\]

Thus, recalling \(S_\epsilon = T_\epsilon + F_\delta\) we have

\[
S_\epsilon - T_\omega \text{ converges to } 0 \text{ in } L^1_\delta(\mathbb{T}^n),
\]

where the sets \(F_\delta\) in Definition 2 are given by (56). Thus \(T_\omega\) is divergence free in finite part from Proposition 7. \(\square\)

It now remains to prove the characterizations of Theorem 1, i.e. items 0, 1, 2 and 3, which we divide into Propositions 5 and 6 below.

**Proposition 5.** Let \(-\Delta v + v = \omega \in H^{-1}(\mathbb{T}^n)\) and that \(T_\omega\) is divergence free in finite parts. Then it holds distributionally that

\[
\text{div}T_\omega = 0.
\]

Moreover we have the following

- If \(n = 2\) then \(v \in W^{1,\infty}\).
- If, in addition, \(\omega \in L^p\) for \(p \geq \frac{2n}{n+1}\) when \(n > 2\) and \(p \geq 1\) for \(n = 2\), then

\[
\omega = 1 dx.
\]

**Proof.** When \(n = 2\) for both cases, see [36]. The proofs are very similar to [36] but we generalize them for arbitrary dimension. First observe that \(\text{div}T_\omega = 0\) is an immediate consequence of Proposition 1. When \(n > 2\), if \(\omega \in L^p\) for \(p \geq \frac{2n}{n+1}\) then \(\nabla v \in L^q\) for \(q \leq \frac{p}{p-1}\) by standard elliptic theory. Let \(\omega_n = \omega * \rho_n\) where \(\{\rho_n\}_n\) is a regularizing kernel and define \(v_k = v * \rho_k\) and let \(T_k\) be the tensor with coefficients \(-\partial_i v_k \partial_j v_k + \frac{1}{2}(\nabla v_k^2 + v_k^2)\delta_{ij}\). Then \(\omega_k\) tends to \(\omega\) in \(L^p\) and since \(\nabla v \in L^{p/(p-1)}\), \(\nabla v_n\) tends to \(\nabla v\) in \(L^{p/(p-1)}\). By Hölder’s inequality we obtain

\[
\omega_k \nabla v_k \to \omega \nabla v, \quad T_n \to T_\omega \text{ in } L^1_{loc}(\mathbb{T}^n).
\]

It follows that \(\text{div}T_k \to \text{div}T_\omega = 0\) and that \(\omega_k \nabla v_k \to \omega \nabla v\) in \(D'(\mathbb{T}^n)\). Since \(\text{div}T_k = \omega_k \nabla v_k\) we conclude \(\omega \nabla v = \lim_k \text{div}T_k = 0\) in \(L^1_{loc}(\mathbb{T}^n)\) and thus a.e. Then since \(\Delta v = 0\) a.e on the set \(F = \{\nabla v = 0\}\), we have \(\omega = \kappa^2 v\) a.e on the set \(F\), and \(\omega = 0\) a.e on the complement of \(F\) from \(\omega \nabla v = 0\). Thus we obtain

\[
\omega = \kappa^2 v 1_{|\nabla v| = 0}.
\]

Multiply by \(v\) and integrating by parts, using the periodic boundary conditions on \(\mathbb{T}^n\) we obtain that \(\nabla v = 0\) a.e and thus \(v\), and therefore \(\omega\) is constant. Since \(\int \omega = 1\) it follows that \(\omega = 1 dx\). \(\square\)

We have the following interpretation of the divergence free condition when \(\omega\) is a finite linear combination of Dirac masses.

**Proposition 6.** Let \(-\Delta v + \kappa^2 v = \omega = \sum_{i=1}^d b_i \delta_{a_i}\) and assume that \(T_\omega\) is divergence free in finite parts. Then setting \(v(x) = \Phi(|x-a_i|) + H_i(x)\) it holds that

\[
\nabla H_i(a_i) = 0.
\]

Before we continue with the proof of Proposition 6 we need the following Proposition which follows almost immediately from Proposition 1 The proof is simple and contained in [36].
Proposition 7. If \( X \) is divergence free in finite parts and is continuous in a neighborhood \( U \) of the boundary of a smooth, compact set \( K \) in \( \Omega \), then

\[
\int_{\partial K} X \cdot \nu_K dS = 0.
\]

Proof of Proposition 6: We present the proof for \( n = 3 \); the general case is similar. Assume that \( \omega \) is a single Dirac mass at the origin with mass \( 4\pi \) without loss of generality. Then in spherical coordinates we have

\[
\nu = \frac{\partial}{\partial r} \tau = \frac{1}{r} \frac{\partial}{\partial \theta} \eta = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}.
\] (57)

We compute \( T_\omega \cdot \nu \) in the \((\nu, \tau, \eta)\) basis to find

\[
T_\omega \cdot \nu = \frac{1}{2} \left( \left( \frac{\partial \tau}{r^2} + 2 \frac{\partial \nu H}{r^2} \right) \nu + \left( \frac{\partial \eta H}{r^2} \right) \tau + \left( \frac{\partial \nu H}{r^2} \right) \eta \right).
\] (58)

Then we write \( h = \Phi + H \) where \( \Phi \) is the positive solution to \(-\Delta \Phi = 4\pi \delta(x)\) and \( H \) is smooth in a neighborhood of 0. Then we have \( \partial_r \Phi = -\frac{1}{r} + o_r(1) \) as \( r \to 0 \) and \( \partial_r \Phi = \partial_r \eta \Phi = 0 \). Thus as \( r \to 0 \) we have

\[
T_\omega \cdot \nu = \frac{1}{2} \left( -\frac{1}{r^4} + 2 \frac{\partial \nu H}{r^2} \right) \nu + \left( \frac{\partial \eta H}{r^2} \right) \tau + \left( \frac{\partial \nu H}{r^2} \right) \eta.
\] (59)

Now using the fact that the integral of \( \bar{I}(r) \) of \( T_\omega \cdot \nu \) over \( \partial B(0, r) \) is zero by Proposition 7, we have as \( r \to 0 \) that

\[
0 = \nabla H(0) \cdot \bar{I}(r) = 4\pi |\nabla H(0)|^2 + o_r(1).
\]

This implies \( \nabla H(0) = 0 \). \( \square \)

5 Proof of Theorem 3

We are now ready to prove Theorem 3. The main idea of the proof is simple. We use Theorem 2 to write down the Euler-Lagrange equation satisfied on the reduced boundary of \( \{u^\varepsilon = +1\} \). To leading order, the potential \( v_\varepsilon \) is constant on the boundary of an isolated droplet \( \Omega_j \) whose center of mass converges to \( a_i \) (up to a subsequence), due to the logarithmic scaling of \( G \) on \( T^2 \) (cf. equation (32)). The control of the isoperimetric deficit (37) controls the size of the error in making this approximation, and allows us to conclude the curvature is asymptotically constant on the reduced boundary of droplets converging to \( a_i \).

Proof of Theorem 3: By assumption, we have

\[
u(x) = -1 + 2 \sum_{J=1}^{N(\varepsilon)} \chi_{\Omega_j},
\] (60)

where \( N(\varepsilon) = O(1) \) as \( \varepsilon \to 0 \). We then apply Theorem 2 to (14) to conclude that

\[
\frac{\varepsilon}{\sum_j |\Omega_j|} H_\varepsilon - \frac{2\delta(\varepsilon)}{\kappa^2 \sum_j |\Omega_j|} + v_\varepsilon = 0 \text{ on } \partial^* \{u^\varepsilon = +1\},
\] (61)

holds for all \( \varepsilon > 0 \). Since \( N(\varepsilon) = O(1) \) as \( \varepsilon \to 0 \) and \( P(\Omega_i) \to 0 \) for each \( i \), we conclude from compactness of \( T^2 \) that the center of mass of each \( \Omega_i \) converges up to a subsequence to some \( a_i \).
Let $J_i$ be the set of indices so that the center of mass of $\Omega_j$ converges to $a_i$. We now expand the potential near $a_i$, first recalling that

$$v_\varepsilon(x) = \int_{T^2} G(x-y) \frac{\sum_j \chi_{\Omega_j}(y)}{\sum_j |\Omega_j|} dy. \tag{62}$$

Then, using (32), we have for $\varepsilon$ sufficiently small, in a neighborhood of $a_i$

$$v_\varepsilon(x) = \int_{T^2} -\frac{1}{2\pi} \log |x-y| \frac{\sum_{j \in J_i} \chi_{\Omega_j}(y)}{\sum_j |\Omega_j|} dy + S_\varepsilon^i(x), \tag{63}$$

where $S_\varepsilon^i$ is uniformly bounded in $\varepsilon$ in a neighborhood of $a_i$. Then letting $\bar{x}_j = P(\Omega_j)^{-1} x$, $\bar{y}_j = P(\Omega_j)^{-1} y$, $v_\varepsilon$ in these variables becomes

$$v_\varepsilon(x) = \int_{T^2} -\frac{1}{2\pi} \log |x-y| \frac{\sum_{j \in J_i} \chi_{\Omega_j}(y)}{\sum_j |\Omega_j|} dy + S_\varepsilon^i(x) \tag{64}$$

$$= \sum_{j \in J_i} \frac{-\frac{1}{2\pi} \log P(\Omega_j)|\Omega_j|}{\sum_j |\Omega_j|} + \sum_j P(\Omega_j)^2 \int_{\Omega_j} -\frac{1}{2\pi} \log |\bar{x}_j - \bar{y}_j| d\bar{y}_j + S_\varepsilon^i(x). \tag{65}$$

We then use the inequality

$$\text{essdiam}(\bar{\Omega}_j) \leq \frac{1}{2} P(\bar{\Omega}_j) = \frac{1}{2}, \tag{66}$$

which follows (for instance) from [3, Theorem 7 and Lemma 4] noting that in view of [3, Proposition 6(ii)] it suffices to consider only simple sets [3, Definition 3]. Thus we have from (66) and the definition of $\bar{x}_j$, $\bar{y}_j$

$$\left| \int_{\Omega_j} -\frac{1}{2\pi} \log |\bar{x}_j - \bar{y}_j| d\bar{y}_j \right| \leq \frac{1}{2\pi} \int_{B(0,2)} |\log|x||dx \leq C. \tag{67}$$

where $C > 0$ is independent of $\varepsilon$. Inserting (67) into (65) and using the bound on the isoperimetric deficit (37) we have, using the fact that $P(\{u^\varepsilon = +1\}) \to 0$ as $\varepsilon \to 0$ (cf. equation (22)), that for any $k \in J_i$ and $x^\varepsilon_j \in \bigcup_{j \in J_i} \partial^* \Omega_j$

$$\frac{v_\varepsilon(x^\varepsilon_j)}{\sum_j \log P(\Omega_j)} = -\frac{1}{2\pi} \sum_{j \in J_i} \frac{\log P(\Omega_j)|\Omega_j|}{\sum_j |\Omega_j|} + o_\varepsilon(1).$$

Rewriting the Euler-Lagrange equation (61) we have

$$\left\| \frac{-\varepsilon H_\varepsilon}{\sum_j \log P(\Omega_j)} \frac{1}{\sum_j |\Omega_j|} + c_i^\varepsilon \right\|_{L^\infty(U_{j \in J_i} \partial^* \Omega_j)} \to 0 \text{ as } \varepsilon \to 0, \tag{68}$$

where

$$c_i^\varepsilon = -\frac{1}{2\pi} \sum_{j \in J_i} \log P(\Omega_j)|\Omega_j| \sum_j |\Omega_j| + \frac{2}{\kappa^2} \sum_j \log P(\Omega_j) \sum_j |\Omega_j|^2 \delta(\varepsilon). \tag{69}$$

Now choose a subsequence $\varepsilon_k$ so that the liminf in the definition of $\delta$ (cf. (36)) is achieved as $\varepsilon_k \to 0$. It is clear that the first term in the definition of $c_i^\varepsilon$ is bounded uniformly and positive as $\varepsilon \to 0$, and therefore converges subsequentially to some $c_i^0 \geq 0$. Therefore we have (possibly taking a further subsequence) that

$$c_i^{\varepsilon_k} \to c_i^0 - \frac{2}{\kappa^2} \delta,$$

as $\varepsilon_k \to 0$. Choosing $\delta = \frac{\kappa^2 c_i^0}{2}$, we obtain the result. □
6 The diffuse interface energy

In this section we study

\[ E[u] = \int_{\Omega} \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right) \, dx + \frac{1}{2} \int_{\Omega} \int_{\Omega} (u(x) - \bar{u})G_0(x,y)(u(y) - \bar{u}) \, dx \, dy. \]  

We make the particular choice of \( V(u) = \frac{1}{4}(1 - u^2)^2 \), but our results will hold, with minor adjustments to the proofs, under general assumptions on \( V \). Recalling the discussion in Section 2.4 we know that any stationary point \( u^\varepsilon \) of (70) in the sense of Definition 1 is a critical point in the sense of Definition 3 (??), which is easily seen to be a solution to

\[ -\varepsilon^2 \frac{\lambda}{\delta(\varepsilon)} \Delta u^\varepsilon - \frac{1}{\delta(\varepsilon)} u^\varepsilon(1 - (u^\varepsilon)^2) + v^\varepsilon = \lambda^\varepsilon, \]  

where \( \lambda^\varepsilon \) is the Lagrange multiplier arising from the volume constraint and

\[ v^\varepsilon(x) = \int_{\Gamma^n} G(x - y) \frac{1 + u^\varepsilon(y)}{\delta(\varepsilon)} \, dy. \]

We recall our main assumption that

\[ \limsup_{\varepsilon \to 0} \mathcal{H}^{n-1}(\{u^\varepsilon \geq -1 + \delta(\varepsilon)^{1+\alpha}\}) = 0 \]  

for some \( \alpha > 0 \).

Our methods will be very similar to those of the sharp interface energy (4), and follow closely the methods of [37, 36] for Ginzburg-Landau. In particular, we first show that (71) is equivalent to a certain 2-tensor \( S^\varepsilon = \{S_{ij}\} \) having zero divergence (cf. Proposition 8 below). We then use (72) to cover the set where \( u^\varepsilon \) is close to +1 by balls whose boundaries have very small \( \mathcal{H}^{n-1} \) measure (cf. Proposition 2). Finally we show that away from the set where \( u^\varepsilon \) is close to +1, \( S^\varepsilon \) is close in \( L^1 \) to the tensor \( T^\varepsilon = \{T_{ij}\} \) defined by

\[ T_{ij} = -\partial_i v^\varepsilon \partial_j v^\varepsilon + \frac{1}{2} \delta_{ij} |\nabla v^\varepsilon|^2. \]

We begin by observing that if \( u^\varepsilon \) solves (71) then it holds by direct computation that

\[ \text{div} S^\varepsilon = 0, \]  

where

\[ S^\varepsilon_{ij} = \partial_i u^\varepsilon \partial_j u^\varepsilon - \varepsilon^2 \frac{\delta_{ij}}{\delta(\varepsilon)^2} \partial_i u^\varepsilon \partial_j u^\varepsilon + \frac{\delta_{ij}}{2} \left( \frac{\varepsilon^2}{\delta(\varepsilon)^2} |\nabla u^\varepsilon|^2 + \frac{1}{4\delta(\varepsilon)^2}(1 - |u^\varepsilon|^2)^2 - |\nabla v^\varepsilon|^2 - \frac{(u^\varepsilon + 1)}{\delta(\varepsilon)} \lambda^\varepsilon \right) + \frac{\delta_{ij}}{\delta(\varepsilon)} u^\varepsilon(u^\varepsilon + 1). \]

This is summarized in the following proposition.

**Proposition 8.** Let \( u^\varepsilon \) be a solution to (71). Then

\[ \text{div} S^\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n), \]

where \( S^\varepsilon \) is given by (75).
Proof. A direct computation using the fact that \( u^\varepsilon \in C^2(K) \) for any \( K \subset \subset \mathbb{T}^n \) yields

\[
\text{div} S^\varepsilon = \nabla u^\varepsilon \left(-\frac{\varepsilon^2}{\delta(\varepsilon)} \Delta u^\varepsilon + \frac{1}{\delta(\varepsilon)} u^\varepsilon (1 - (u^\varepsilon)^2) + v^\varepsilon - \lambda_c \right) = 0.
\]

Proposition 9. Let \( \{u^\varepsilon\}_\varepsilon \) be a sequence of solutions to (71) satisfying (72) and

\[
\limsup_{\varepsilon \to 0} |\lambda_c| < +\infty.
\]

For any \( \varepsilon > 0 \) define the 2-tensors \( S^\varepsilon \) as above and \( T^\varepsilon \) by

\[
T^\varepsilon_{ij} = \partial_i u^\varepsilon \partial_j v^\varepsilon - \frac{\delta_{ij}}{2} |\nabla u^\varepsilon|^2.
\]

Then \( T^\varepsilon - S^\varepsilon \) tends to 0 in \( L^1(\mathbb{T}^n) \).

Proof. We once again argue as in [36] for Ginzburg-Landau. From (72) and Proposition 2 the set of \( x \) in \( \mathbb{T}^n \) such that \( u^\varepsilon(x) \geq -1 + \delta(\varepsilon)^{1+\alpha} \) can be covered by a collection of balls \( B_1, \cdots, B_k \) such that

\[
\sum_{i=1}^k r(B_i)^{n-1} \leq C H^{n-1}(\{u^\varepsilon \geq -1 + \delta(\varepsilon)^{1+\alpha}\}).
\]

We denote \( Z_c \) as the union of these balls and and observe that

\[
\lim_{\varepsilon \to 0} \text{Cap}_1(Z_c) = 0.
\]

This follows from the fact that the 1-capacity of a ball \( B(x, r) \) is \( \alpha_n - 1 r^{n-1} \) and the capacity is subadditive so \( \text{Cap}_1(Z_c) \leq C H^{n-1}(\{u^\varepsilon \geq -1 + \delta(\varepsilon)^{1+\alpha}\}) \), which tends to zero by assumption.

The difference between \( S^\varepsilon \) and \( T^\varepsilon \) is

\[
S^\varepsilon - T^\varepsilon = -\frac{\varepsilon^2}{\delta(\varepsilon)^2} \partial_i u^\varepsilon \partial_j v^\varepsilon + \frac{\delta_{ij}}{2} \left( \frac{\varepsilon^2}{\delta(\varepsilon)^2} |\nabla u^\varepsilon|^2 + \frac{1}{4 \delta(\varepsilon)^2} (1 - |u^\varepsilon|^2)^2 - \frac{1}{\delta(\varepsilon)} (u^\varepsilon + 1)\lambda_c \right) + \delta_{ij} \frac{v^\varepsilon (u^\varepsilon + 1)}{\delta(\varepsilon)}.
\]

Thus it is easily seen that

\[
|S^\varepsilon - T^\varepsilon| \leq C \left( \frac{\varepsilon^2}{\delta(\varepsilon)^2} |\nabla u^\varepsilon|^2 + \frac{1}{2 \delta(\varepsilon)^2} (1 - |u^\varepsilon|^2)^2 + \frac{2}{\delta(\varepsilon)} (|v^\varepsilon| + |\lambda_c|)|u^\varepsilon + 1| \right). \tag{79}
\]

Now define the function \( \chi : [0, 1] \to [0, 1] \) to be the affine interpolation between the values \( \chi(-1) = 1, \chi(-1 + \delta(\varepsilon)^{1+\alpha}) = 1 \) and \( \chi(-1/2) = 1/2 \) and \( \chi(0) = 0 \) and \( \chi(1) = -1 \). Multiply (71) by \( \chi(u^\varepsilon) + u^\varepsilon \) and integrating by parts we have

\[
\frac{1}{\delta(\varepsilon)^2} \int_{\mathbb{T}^n} \varepsilon^2 |\nabla u^\varepsilon|^2 (\chi(u^\varepsilon) + 1) - u^\varepsilon (\chi(u^\varepsilon) + u^\varepsilon) (1 - (u^\varepsilon)^2) = -\frac{1}{\delta(\varepsilon)} \int_{\mathbb{T}^n} (\chi(u^\varepsilon) + u^\varepsilon) (u^\varepsilon - \lambda_c) dx. \tag{80}
\]

The set \( \{\chi(u^\varepsilon) = 1\} \) contains the set \( u^\varepsilon(x) \leq -1 + \delta(\varepsilon)^{1+\alpha} \) and therefore \( Z_c \). When \( |u^\varepsilon| \geq 1/2 \), which is true on \( \{\chi(u^\varepsilon) = 1\} \), the left side of (80) can be bounded from below by

\[
\frac{1}{2 \delta(\varepsilon)^2} \int_{Z_c} \varepsilon^2 |\nabla u^\varepsilon|^2 + \frac{1}{4} (1 - (u^\varepsilon)^2)^2.
\]
Indeed on \( \{ \chi(u^\varepsilon) = 1 \} \) we have
\[
|u^\varepsilon|(1 - |u^\varepsilon|) \geq (1 - |u^\varepsilon|^2),
\]
since \( |u^\varepsilon| \geq (1 + |u^\varepsilon|) \) when \( u^\varepsilon \in (-1, -1/2) \). Since \((u^\varepsilon + 1)/\delta(\varepsilon)\) is bounded in \((C^0(\mathbb{T}^n))^*\) and therefore in \(W^{-1,p}\) for \( p \in (1, n/(n - 1))\) by standard embeddings, we conclude that \( \nu_\varepsilon \) is bounded uniformly in \( L^1 \). Then by the definition of \( \chi \) and the fact that \( u^\varepsilon \in (-1, -1+\delta(\varepsilon)^{1+\alpha}) \) where \( \chi(u^\varepsilon) = 1 \) we have
\[
\frac{1}{\delta(\varepsilon)} \left| \int_{\mathbb{T}^n} (\chi(u^\varepsilon) + u^\varepsilon)v_\varepsilon \right| = \frac{1}{\delta(\varepsilon)} \left| \int_{Z^\varepsilon} (1 + u^\varepsilon)v_\varepsilon \right| \leq C\delta(\varepsilon)\alpha\|v_\varepsilon\|_{L^1}. \tag{81}
\]
Combining the above we conclude
\[
\frac{1}{\delta(\varepsilon)^2} \int_{Z^\varepsilon} \varepsilon^2|\nabla u^\varepsilon|^2 + (1 - (u^\varepsilon)^2)^2 = o_\varepsilon(1) \text{ as } \varepsilon \to 0. \tag{82}
\]
Focusing on the remaining terms in (79), it remains to show that
\[
\frac{1}{\delta(\varepsilon)} \int_{Z^\varepsilon} |1 + u^\varepsilon|(\|v_\varepsilon\| + |\lambda_\varepsilon|) = o_\varepsilon(1) \text{ as } \varepsilon \to 0. \tag{83}
\]
This however follows from the definition of \( Z^\varepsilon \):
\[
\frac{1}{\delta(\varepsilon)} \int_{Z^\varepsilon} |u^\varepsilon + 1|(|v_\varepsilon| + |\lambda_\varepsilon|) \leq C(\|v_\varepsilon\|_{L^1} + |\lambda_\varepsilon|)\delta(\varepsilon) \leq C\delta(\varepsilon)^\alpha, \tag{84}
\]
where we’ve used the fact that \( \limsup_\varepsilon |\lambda_\varepsilon| < +\infty \). Finally combining (81) and (82) and using (79) we conclude that
\[
\int_{\mathbb{T}^n \setminus Z^\varepsilon} |T_\varepsilon - S_\varepsilon| \to 0 \text{ as } \varepsilon \to 0,
\]
the desired result. \(\square\)

We now complete the proof of Theorem 4.

*Proof.* Choose a decreasing subsequence \( \{ \varepsilon_k \} \) tending to zero such that \( \sum_k \text{Cap}_1(Z_{\varepsilon_k}) < +\infty \) and let
\[
E_\delta = \bigcup_{k > \frac{1}{2}} Z_{\varepsilon_k}.
\]
Since \( \omega_\varepsilon := (1 + u^\varepsilon)/\delta(\varepsilon) \) is a family of probability measures on \( \mathbb{T}^n \), we have \( \omega_\varepsilon \rightarrow \omega \) weakly in \((C^0(\mathbb{T}^n))^*\) up to a subsequence, and thus \( \omega_\varepsilon \rightarrow \omega \) strongly in \( W^{-1,p} \) for \( p \in (1, n/(n - 1)) \) via the compact embedding \((C^0(\mathbb{T}^n))^* \subset W^{-1,p} \) which follows from the compact embedding \( W^{1,q}(\mathbb{T}^n) \subset C^0(\mathbb{T}^n) \) for \( q > n/(n - 1) \). Now define
\[
F_\delta := E_\delta \cup \tilde{E}_\delta, \tag{85}
\]
where \( \tilde{E}_\delta \) are the sets given by Proposition 3 with \( \kappa = 0 \). Then by subadditivity of capacity 17 we have
\[
\lim_{\delta \to 0} \text{Cap}_1(F_\delta) = 0.
\]
From Proposition 3 we therefore conclude
\[
\nabla v_\varepsilon \rightarrow \nabla v \text{ in } L^2_\delta(\mathbb{T}^n).
\]

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Thus, combining the above, we have

\[ S_\varepsilon - T_\omega \text{ converges to } 0 \text{ in } L^1_\delta(\mathbb{T}^n), \]

where the sets \( F_\delta \) in Definition 2 are given by (85). Thus \( T_\omega \) is divergence free in finite part from Proposition 7.

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