ON RAMIFIED COVERS OF THE PROJECTIVE PLANE I:
SEGRE’S THEORY AND CLASSIFICATION IN SMALL DEGREES
(WITH AN APPENDIX BY EUGENII SHUSTIN)

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ABSTRACT. We study ramified covers of the projective plane \( \mathbb{P}^2 \). Given a smooth surface \( S \) in \( \mathbb{P}^n \) and a generic enough projection \( \mathbb{P}^n \to \mathbb{P}^2 \), we get a cover \( \pi : S \to \mathbb{P}^2 \), which is ramified over a plane curve \( B \). The curve \( B \) is usually singular, but is classically known to have only cusps and nodes as singularities for a generic projection.

Several questions arise: First, What is the geography of branch curves among all cuspidal-nodal curves? And second, what is the geometry of branch curves; i.e., how can one distinguish a branch curve from a non-branch curve with the same numerical invariants? For example, a plane sextic with six cusps is known to be a branch curve of a generic projection iff its six cusps lie on a conic curve, i.e., form a special 0-cycle on the plane.

We start with reviewing what is known about the answers to these questions, both simple and some non-trivial results. Secondly, the classical work of Beniamino Segre gives a complete answer to the second question in the case when \( S \) is a smooth surface in \( \mathbb{P}^3 \). We give an interpretation of the work of Segre in terms of relation between Picard and Chow groups of 0-cycles on a singular plane curve \( B \). We also review examples of small degree.

In addition, the Appendix written by E. Shustin shows the existence of new Zariski pairs.

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1. Introduction

Let $S$ be a non-singular algebraic surface of degree $\nu$ in the complex projective space $\mathbb{P}^r$. One can obtain information on $S$ by projecting it from a generically chosen linear subspace of codimension 3 to the projective plane $\mathbb{P}^2$. The ramified covers of the projective plane one gets in this way were studied extensively by the Italian school (in particular, by Enriques, who called a surface with a given morphism to $\mathbb{P}^2$ a “multiple plane”, and later by Segre, Zariski and others.) The main questions of their study were which curves can be obtained as branch curves of the projection, and to which extent the branch curve determines the pair $(S, \pi : S \to \mathbb{P}^2)$. In the course of this study, Zariski studied the fundamental groups of complements of plane curves and in particular introduced what later became known as Enriques-Zariski-Van Kampen theorem (see Zariski’s foundational paper [6]). It was also discovered by the Italian school that a branch curve of generic projections of surfaces in characteristics 0 has only nodes and cusps as singularities, though we were not able to trace a reference to a proof from that era (but see [63] for a modern proof). Segre-Zariski criterion (see [6] and also Zariski’s book [22]) for a degree 6 plane curve with 6 cusps to be a branch curve is well known and largely used, but Segre’s generalization ([8]), where he gave a necessary and sufficient condition for a plane curve to be a branch curve of a ramified cover of a smooth surface in $\mathbb{P}^3$ in terms of adjoint linear systems to the branch curve, was largely forgotten (see Theorem 4.32).

Two recent surveys, by D’Almeida ([41]) and Val. S. Kulikov ([60]), were written on Segre’s generalization. However, our approach is different, for we give an interpretation of Segre’s work in terms of studying various equivalence relations of 0-cycles on nodal-cuspidal curves. We also emphasize the logic of passing from adjoint curves for a plane singular curve to rational objects which become regular on the normalization of this curve, (what would be called ”weakly holomorphic functions” in analytic geometry [18, Chapter VI], and control geometry of its space models.

The geometry of ramified covers in dimension two is very different from the geometry in dimension one. In dimension one, for any set of points $B$ in the projective line $\mathbb{P}^1$ we always have many possible (non isomorphic) ramified covers of $\mathbb{P}^1$ for which $B$ is the branch locus. In terms of monodromy data, the fundamental group $G = \pi_1(\mathbb{P}^1 - B)$ is free, and thus admits many epimorphic representations $G \to \text{Sym}_\nu$ for multiple values of $\nu$; and, moreover, every such a representation is actually coming from a ramified cover due to the Riemann-Grauert-Remmert theorem (see Theorem 2.2). However, in dimension two Chisini made a surprising conjecture (circa 1944, cf. [14]) that the pair $(S, \pi : S \to \mathbb{P}^2)$, where $\pi$ is generic, can be uniquely determined by the branch curve $B$, if $\deg \pi \geq 5$ (and in the case of generic linear projections, if this curve is of sufficiently high degree). This conjecture was proved only recently by Kulikov ([52], [65]). In terms of monodromy data, by Grauert-Remmert theorem, even though it is true that every representation $\rho : \pi_1(\mathbb{P}^2 - B) \to \text{Sym}_\nu$ comes from a ramified cover $S \to \mathbb{P}^2$ of degree $\nu$ with a normal surface $S$, one has to ensure certain “local” conditions on the representation $\rho$ which ensure that $S$ is non-singular, which sharply reduce the number of admissible representations into the symmetric group. In fact, the Chisini’s conjecture implies that once the degree of the ramified cover sufficiently high, there is only one such representation.

The structure of the paper is as follows. Sections 2 and 3 contain preliminary material. In section 2 we recall some facts about ramified coverings and Grauert-Remmert theorems, and in the following section 3 we look at $V(d, c, n)$ (resp. $B(d, c, n)$) the variety of degree $d$ plane curves (resp.
branch curves) with $c$ cusps and $n$ nodes (in addition, we prove the following interesting fact: in coordinates $(d, c, \chi)$, with geometric Euler characteristic $\chi$, the duality map $(d, c, \chi) \rightarrow (d^*, c^*, \chi^*)$ becomes a linear reflection). We also recall a number of necessary numerical conditions for a curve to be a branch curve. In the main section, Section 4 we re-establish the results of Segre for smooth surfaces in $\mathbb{P}^3$ and discuss the geography of surfaces with ordinary singularities and their branch curves. Looking at the variety of nodal cuspidal plane curves, we also compute the dimension of the component which consists of branch curves of smooth surfaces in $\mathbb{P}^3$ (see Subsection 4.4). In Section 5 we classify admissible (i.e. nodal-cuspidal irreducible) branch curves of small degree. Appendix A is written by Eugenii Shustin, where new Zariski pairs are found. Each Zariski pair consists of a branch curve of a smooth surface in $\mathbb{P}^3$ and a nodal–cuspidal curve which is not a branch curve. In the other Appendices we recall some facts on the Picard and Chow groups of nodal cuspidal curves we use, and on the bisecants to complete intersection curves in $\mathbb{P}^3$.

In the subsequent papers (see [67]) we will deal with an analogue of the Segre theory for surfaces with ordinary singularities in $\mathbb{P}^3$, and also give a combinatorial reformulation of the Chisini conjecture.

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2. Ramified covers

In this section we start with a general discussion on branched coverings, continuing afterwards to investigation of surfaces and generic projections.

2.1. Étale covers. Let $S$ be a scheme (of finite type) over $\mathbb{C}$, and $S_{an}$ be the corresponding analytic space. Let $Et^f_S$ be the category of finite étale schemes over $S$, and $Et^f_{S_{an}}$ be the category of finite étale complex-anaytic spaces over $S_{an}$. One can verify (cf. [16] and [17, XI.4.3]) that the “analytization” functor

$$a_S : Et^f_S \rightarrow Et^f_{S_{an}}$$

is faithfully flat. The following Grauert-Remmert theorem generalizes the so-called Riemann existence theorem in case $\dim S = 1$:

**Theorem 2.1** (Grauert-Remmert). If $S$ is normal, then $a_S$ is equivalence of categories.

2.2. Ramified covers of complex analytic spaces. Let $X$ be a complex analytic space, $Y \subset X$ be a closed analytic subspace in $X$, and $U = X - Y$ be the complement. Assume that $U$ is dense in $X$.

**Theorem 2.2** (Grauert-Remmert). If $X$ is normal, then the restriction functor

$$\text{res}_U : \text{(normal analytic covers of } X \text{ étale over } U) \longrightarrow \text{(étale analytic covers of } U)$$

is an equivalence of categories.
For other formulations of Theorem 2.2 and the proof, see [21, Proposition 12.5.3, Theorem 12.5.4].

We say that \( f : X' \to X \) is a ramified cover branched over \( Y \) if \( f|_U \) is étale and the ramification locus of \( f \) (i.e. \( \text{supp}(\Omega^1_{X'/X}) \)) is contained in \( Y \). Note that even if \( X \) is smooth, we still have to allow ramified covers \( X' \to X \) with normal singularities in order to get an essentially surjective restriction functor, as seen in the following example. Let \( X = \mathbb{P}^2, Y = (xy = 0), U = X - Y, \) and \( f : U' \to U \) be a degree 2 unramified cover given by the monodromy representation \( \pi^a_1(U) \simeq \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/2 \) which sends both generators to the generator of \( \mathbb{Z}/2 \). It is easily seen in this example that \( f \) cannot be extended to a ramified cover \( X' \to X \) with smooth \( X' \), but if we allow normal singularities one gets canonical extension given in coordinated by \( z^2 = xy \), a cone with \( A_1 \) singularity.

2.3. Ramified covers of \( \mathbb{P}^2 \). From now on, we restrict ourselves to \( \text{char} = 0 \). Let \( S \) be a smooth surface in \( \mathbb{P}^r = \mathbb{P}(V) \). Let \( W \subset V \) be a codimension 3 linear subspace such that \( \mathbb{P}(W) \cap S = \emptyset \) and let \( p \) the resulting projection map \( p : \mathbb{P}(V) \to \mathbb{P}(V/W) \) and \( \pi : S \to \mathbb{P}^2 \) its restriction to \( S \). It is clear that \( \pi \) is a finite morphism of degree equals to \( \text{deg} S \).

for a generic choice of \( W \), \( \pi \) is called a generic projection map and the following is classical (see, for example, [15], [26] and [63]):

(i) \( \pi \) is ramified along an irreducible curve \( B \subset \mathbb{P}^2 \) which has only nodes and cusps as singularities;
(ii) The ramification divisor \( B^* \subset S \) is irreducible and smooth, and the restriction \( \pi : B^* \to B \) is a resolution of singularities;
(iii) \( \pi^{-1}(B) = 2B^* + \text{Res} \) for some residual curve \( \text{Res} \) which is reduced.

Remark 2.3. Note that not every ramified cover \( S \) of \( \mathbb{P}^2 \) with a branch curve \( B \subset \mathbb{P}^2 \) can be given as a restriction of generic linear projection \( \mathbb{P}^r \to \mathbb{P}^2 \) (to a smooth surface \( S \)). See, for example, Remark 5.7.

Remark 2.4. Note that generically cusps to not occur in a generic projection of a smooth space curve, but do occur for the projection of a ramification curve of surfaces, already in the basic example of smooth surfaces in \( \mathbb{P}^3 \) and its projection to \( \mathbb{P}^2 \). Consider, for example, the case of a smooth surface \( S \) in \( \mathbb{P}^3 \) and its generic projection to \( \mathbb{P}^2 \). Since the branch curve \( B \) is the projection of the ramification curve \( B^* \) which is a space curve, it generically has double points corresponding to bisecants of \( B^* \) containing the projection center \( O \). The cuspidal points are somewhat more unusual for projections of smooth space curves, since they do not occur in the projections of generic smooth space curves. However, the projections of generic ramification curves have cusps. To give a typical example, consider a family of plane (affine) cubic curves \( z^2 - 3az + x = 0 \) in the \((x,z)\)-plane, where \( a \) is a parameter. The real picture is the following: for \( a > 0 \) the corresponding cubic parabola has 2 extremum point, for \( a = 0 \) one inflection point and for \( a < 0 \) no real extremums; the universal family in the \((x,z,a)\) space is the so-called real Whitney singularity, and projection to the “horizontal” \((x,a)\) plane gives a semi-cubic parabola \( a^2 - x^3 \) with a cusp. In other words, substituting \( y = -3a \), we see that the affine cubic surface \( S \) can be considered as the “universal cubic polynomial” in \( z, p(z) = z^3 + y \cdot z + x = 0 \), and its discriminant \( \Delta = 27y^2 + 4x^3 \) has an \( A_2 \) singularity, which is a cusp. (Recall that in general a discriminant of a polynomial of degree \( n \) with \( a_{n-1} = 0 \) has singularity of type \( A_{n-1} \).)

3. Moduli of branch curves and their geography

The geography of surfaces was introduced and studied by Bogomolov-Miyaoka-Yau, Persson, Bombieri, Catanese and more. Parallel to the terminology of geography of surfaces, we will use the term geography of branch curves for the distribution of branch curves in the variety of nodal-cuspidal curves. Subsection 3.1 recalls few facts on nodal-cuspidal degree \( d \) curves with \( c \) cusps.
and \( n \) nodes and introduces a more natural coordinate to work with: \( \chi \) – the Euler characteristic. The main subsection is Subsection 3.2, which compares the geography of branch curves in the \((d, c, \chi)\) coordinates to the geography of surfaces in \((c_1^2, c_2)\) coordinates. Subsection 3.3 constructs the variety of branch curves.

### 3.1. Severi-Enriques varieties of nodal-cuspidal curves.

**Notation 3.1.** For a triple \((d, c, n) \in \mathbb{N}^3\) let \(V(d, c, n)\) be the variety of plane curves of degree \(d\) with \(c\) cusps and \(n\) nodes as their only singularities.

It is easy to prove that \(V(d, c, n)\) is a disjoint union of locally closed subschemes of \(\mathbb{P}^N\), where \(N = \frac{1}{2}d(d + 3)\).

A curve \(C \in V(d, c, n)\), has arithmetic genus \(p_a\) and geometric genus \(g = p_g\) when

\[(1)\quad p_a = \frac{1}{2}(d - 1)(d - 2),\]

\[(2)\quad g = p_a - c - n = \frac{1}{2}(d - 1)(d - 2) - c - n,\]

and we let \(\chi\) to be the topological Euler characteristics of the normalization of \(C\)

\[(3)\quad \chi = 2 - 2g.\]

We shall use the coordinates \((d, c, \chi)\) instead of \((d, c, n)\) since many formulas, such as Plücker formulas, become linear in these coordinates. Note that one can present \(n\) in terms of \((d, c, \chi)\) as follows:

\[
n = \frac{1}{2}(d - 1)(d - 2) - c + \frac{1}{2}\chi - 1 = \frac{1}{2}d(d - 3) - c + \frac{1}{2}\chi.
\]

Let \(C \in V(d, c, n)\) be a Plücker curve, i.e., a curve that its dual \(C^\vee\) is also a curve in some \(V(d^*, c^*, n^*)\) (Note that this is an open condition in \(V(d, c, n)\) and that \((C^\vee)^\vee = C\).) Then the following Plücker formulas hold:

\[(4)\quad d^* = d(d - 1) - 3c - 2n,\]

\[(5)\quad g^* = g^\ast\]

where \(g^*\) is the geometric genus of \(C^\vee\).

The formula from \(c^*\) can be induced from Equations \(2\), \(4\) for \(C^\vee\), i.e.

\[c^* = 3d^2 - 6d - 8c - 6n.\]

#### 3.1.1. Linearity of the Plücker formulas

The Plücker formulas become linear in the \((d, c, \chi)\) coordinates (and also the formulas for the Chern classes of a surface whose branch curve \(B \in V(d, c, \chi)\). See Lemma 3.9 and 3.10), which is the primarily reason we want to consider them. Namely,

\[(6)\quad d^* = 2d - c - \chi,\]

\[(7)\quad c^* = 3d - 2c - 3\chi,\]

\[(8)\quad \chi^* = \chi.\]

in other words, in these coordinates projective duality is given by a linear transformation

\[D = \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ 0 & 0 & 1 \end{pmatrix} \]
which is diagonalizable with eigenvalues \((-1, 1, 1)\) where the eigenvector \(d - c - \chi = d^* - d\) corresponds to the eigenvalue \((-1)\), i.e., gives a reflection in the lattice \(\mathbb{Z} \oplus \mathbb{Z} \oplus 2\mathbb{Z}\). We hope to explain this phenomenon elsewhere.

The fact that the invariants \(d, c, n\) and \(g\) of the curve are not negative implies, in the \((d, c, \chi)\) coordinates, the following inequalities:

\[
\begin{align*}
(n \geq 0) & \Rightarrow 2c - \chi \leq d(d - 3), \\
(g \geq 0) & \Rightarrow \chi \leq 2, \\
(d^* \geq 0) & \Rightarrow c + \chi \leq 2d, \\
(c^* \geq 0) & \Rightarrow 2c + 3\chi \leq 3d.
\end{align*}
\]

Zariski also proved ([9, Section 3]) the following inequality

\[
c < \frac{1}{2}(d - \beta)(d - \beta - 3) + 2,
\]

where \(\beta = [(d - 1)/6]\). His proof uses the computation of the virtual dimension of complete linear system of curves of order \(d - \beta - 3\) passing through the \(c\) cusps of \(C\). (see also [22, Chapter VIII]). But his inequality is stronger than the ones given by Plucker formulas only for small \(d\)'s; we use it once for \(d = 8\) when classifying branch curves of small degree (see Section 5).

**Remark 3.2.** For a nodal–cuspial curve \(C \in V(d, c, n)\) we have the following inequality

\[
2c + n \leq (d - 1)^2
\]

or, in \((d, c, \chi)\) coordinates:

\[
2c + \chi \leq d^2 - d + 2,
\]

which is induced from intersecting two generic polars of \(C\) and Bézout theorem.

![Figure 1: Geography of admissible plane curves in the \((c, n)\)-plane for large \(d\).](image)

The dashed line is where the expected dimension of \(\{\text{family of degree } d \text{ curves with } n \text{ nodes and } c \text{ cusps}\} = \frac{1}{2}d(d + 3) - n - 2c = 0\).
Figure 2: Geography of admissible plane curves in the \((c, \chi)\)-plane for large \(d\).

The dashed line is where the expected dimension of \(\{\text{family of degree } d \text{ curves with } n \text{ nodes and } c \text{ cusps}\} = 3d - \frac{1}{2} \chi - c = 0\).

For more obstructions on the existence of singular plane curves and a recent survey on equisingular families and, in particular, nodal-cuspidal curves see [59].

3.2. Geography of branch curves.

**Notation.** Let \(B(d,c,n)\) be the subvariety in \(V(d,c,n)\) consisting of branch curves of generic linear projections to \(\mathbb{P}^2\). We discuss it in Subsection 3.3.

Let \(B \in B(d,c,n)\) be the branch curve of a generic linear projection \(\pi: S \to \mathbb{P}^2\) for a smooth irreducible projective surface \(S \subset \mathbb{P}^r\). Let \(\nu = \deg \pi\) and \(g = p_g(B)\) be the geometric genus of \(B\). An important invariant of \(B\) is the fundamental group of its complement \(\pi_1(\mathbb{P}^2 - B)\).

**Remark 3.3.** Let \(C \in V(d,c,n)\). If \(c = 0\), i.e., \(C\) is a nodal curve, then, by Zariski-Deligne-Fulton’s theorem, the fundamental group \(\pi_1(\mathbb{P}^2 - C)\) of the complement of \(C\) is abelian ([22], [64], [31]). This theorem was proved by Zariski under the assumption that the Severi variety of nodal curves is irreducible (this was assumed to be established by Severi, but later was found to be mistaken). The correct proof of the irreducibility of the Severi variety \(V(d,0,n)\) was given by Harris [37], which completed Zariski’s proof. Independent proofs were given later by Deligne and Fulton ([64], [31]) and others.

We begin with a consequence from Nori’s result on fundamental groups of complements plane curves. Though the proof is known, we bring it as it is enlightening and brings together various aspects of the subject.

**Lemma 3.4** (Nori [31]). Let \(B \in B(d,c,n)\). Then \(6c + 2n \geq d^2\).

**Proof.** Let \(\psi: \pi_1(\mathbb{P}^2 - B) \to Sym_{\nu}\) be the monodromy representation, sending each generator to a permutation, which describes the exchange of the sheets. Since \(\pi: S \to \mathbb{P}^2\) is a generic projection, the image \(H = \text{Im}(\psi)\) is generated by transpositions. As \(S\) is irreducible, \(H\) is acts transitively on a set of \(n\) points. This implies that \(H = Sym_{\nu}\) and thus \(\psi\) is an epimorphism. Thus for \(\nu > 2\), the fundamental group \(\pi_1(\mathbb{P}^2 - B)\) is not abelian and therefore \(c > 0\) (by Remark 3.3).
Nori proved \([34]\) that for a cuspidal plane curve \(C\) with \(d^2 > 6c + 2n\) and \(c > 0\) the fundamental group of the complement \(\pi_1(\mathbb{P}^2 - C)\) is abelian. Thus, by the above discussion, \(\nu = 2\). However, there is no smooth double cover of \(\mathbb{P}^2\) ramified over a singular \(C\) (indeed, locally \(S\) would be isomorphic to the singular cone \(z^2 = xy\) in a formal neighborhood of a node of \(C\) and to the singular surface \(z^2 = x^2 - y^3\) in a formal neighborhood of a cusp). This implies that Nori’s condition cannot hold for a branch curve. Therefore

\[(14)\quad 6c + 2n - d^2 \geq 0,\]
or in \((d, c, \chi)\) coordinates:
\[4c + \chi - 3d \geq 0.\]

In the spirit of the above Lemma, we have the following result of Shimada:

**Lemma 3.5** (Shimada \([51]\)). Let \(B \in B(d, c, n)\). Then \(2n < d^2 - 5d + 8\).

**Proof.** Let \(C \in V(d, c, n)\). By \([51]\), if \(2n \geq d^2 - 5d + 8\) then \(\pi_1(\mathbb{P}^2 - C)\) is abelian. However, for a branch curve \(B \in B(d, c, n)\), the corresponding fundamental group is not abelian, and we have

\[(15)\quad 0 < \frac{1}{2}(d^2 - 5d + 8) - n,\]
or, in \((d, c, \chi)\) coordinates:
\[2c - \chi - 2d + 8 > 0.\]

The following conditions on \(c\) and \(n\) are less obvious than the previous Lemmas:

**Lemma 3.6.**

\[(16)\quad c = 0 \mod 3, \quad n = 0 \mod 4\]
or in \((c, \chi)\) coordinates:
\[c = 0 \mod 3, \quad \chi = 2c - d(d - 3) \mod 8\]

**Proof.** see \([40]\). □

### 3.2.1. Geography of branch curves in \((d, c, \chi)\) versus geography of surfaces in \((c_1^2, c_2)\).

Let \(S\) be a smooth algebraic surface and \(\pi : S \to \mathbb{P}^2\) be a generic ramified cover. Let \(B\) be the branch curve of \(\pi\), \(B \in B(v)\) for some vector \(v \in L\), and let \(\nu = \deg \pi\). It is well known that \(d \geq 2\nu - 2\) but for the convenience of the reader we bring the proof of this fact.

**Lemma 3.7.**

\[(17)\quad d \geq 2\nu - 2.\]

**Proof.** Denote by \(\pi : S \to \mathbb{P}^2\) the projection map and let \(C = f^{-1}(l)\) for a generic line \(l\). The curve \(C\) is irreducible and smooth. Applying the Riemann-Hurwitz’s formula to the map \(\pi|_C : C \to l\), we get \(2g(C) - 2 = -2\nu + d\), since all the ramification points of the map \(\pi|_C\) are of ramification index 2, and \((C^2)_S = \nu\), which implies
\[d = 2\nu - 2 + 2g(C) \geq 2\nu - 2\]

A different proof will be given in Subsection \([17]\) when we discuss the geometry of surfaces with ordinary singularities in \(\mathbb{P}^3\). □

**Remark 3.8.** Note that the proof of the above Lemma implies that the degree of a branch curve is even.
We want to express the Chern invariants $c_2^2(S)$ and $c_2(S)$ in terms of $(d, c, \chi)$ and, equivalently, in terms of $(d, c, n)$, so we give 2 formulas for each invariant.

**Lemma 3.9.** (see [52])

\[
(18) \quad c_2^2(S) = 9\nu - \frac{9}{2}d - \frac{1}{2}\chi
\]

\[
(19) \quad c_2^2(S) = 9\nu - \frac{9}{2}d + \left(\frac{(d-1)(d-2)}{2} - n - c\right) - 1
\]

**Proof.** Let $\pi : S \to \mathbb{P}^2$ be the ramified cover, $R = B^*$, the ramification curve and $C = f^{-1}(l)$ for $l$ a generic line.

First, we want to compute $[R]^2$ and $[C]^2$.

By Riemann-Hurwitz, $K_S = -3f^*([l]) + [R] = -3[C] + [R]$. As $\pi : R \to B$ is a normalization of the branch curve $B$, we apply adjunction formula to $R$ we get

\[
2g - 2 = (K_S + [R]) \cdot R = (\cdot 3[C] + 2[R]) \cdot R = -3[C] \cdot R + 2[R] \cdot R = -3f^*[l] \cdot R + 2[R]^2 =
\]

\[
= -3[l] \cdot f_*[R] + 2[R]^2 = -3\deg B + 2[R]^2 = -3d + 2[R]^2
\]

and thus

\[
[R]^2 = \frac{3}{2}d + g - 1.
\]

We also have

\[
(20) \quad [C]^2 = f^*[l] \cdot [C] = [l] \cdot f_*[C] = [l] \cdot (\deg f*[l]) = \nu[l]^2 = \nu.
\]

We can now compute $c_2^2(S)$:

\[
c_2^2(S) = K_S^2 = (3[C] + [R])^2 = 9[C]^2 - 6[C] \cdot [R] + [R]^2 = 9\nu - 6d + \frac{3}{2}d + g - 1 =
\]

\[
= 9\nu - \frac{9}{2}d + g - 1 = 9\nu - \frac{9}{2}d - \frac{1}{2}\chi
\]

The expression in $(d, c, n)$-coordinates follows easily. \hfill \Box

**Lemma 3.10.** (see [52])

\[
(21) \quad c_2(S) = 3\nu - \chi - c, \text{ or}
\]

\[
(22) \quad c_2(S) = 3\nu + d^2 - 3d - 3c - 2n.
\]

**Proof.** To compute $c_2(S)$ we use the usual trick of considering a pencil of lines in $\mathbb{P}^2$ passing through a generic point $p \in \mathbb{P}^2$ and its corresponding preimage with respect to $\pi : S \to \mathbb{P}^2$ – the Lefshetz pencil $C_t$ of curves on $S$.

We then apply the following formula on $C_t$

\[
c_2(S) = \chi(S) = 2\chi(\text{generic fiber}) + \#(\text{singular fibers}) - (\text{self-intersection of } C_t)
\]

(see, for example, [28], section 4.2).\hfill

The generic fiber of $C_t$ is a ramified cover of a line $l$ with $d$ simple ramification points (i.e. ramification index 2 at every point), and thus $\chi(\text{generic fiber}) = 2\nu - d$ by the Riemann–Hurwitz formula.

The number of singular fibers in the pencil $C_t$ is clearly equal to the degree $d^*$ of the curve $B^\vee$ (the dual to the branch curve $B$), which by the Plücker formulas for $B$ satisfies $d^* = d(d-1) - 3c - 2n$. The self-intersection $[C_t]^2$ of the fiber equals to $\nu$ (by (20)). Thus
\[
c_2(S) = \chi(S) = 2(2\nu - d) + d^r - \nu = 2(2\nu - d) + (d(d - 1) - 3c - 2n) - \nu = \\
3\nu + d^2 - 3d - 3c - 2n = 3\nu - \chi - c.
\]

\[\square\]

**Remark 3.11.** Equation (21) can be written as an analog to Riemann-Hurwitz formula for the map \(S \to \mathbb{P}^2\)

\[
c_2(S) - \nu c_2(\mathbb{P}^2) = -\chi - c,
\]
as Iversen described in [20].

**Remark 3.12.** Inverting the formulas above, we get \(n\) and \(c\) in terms of \(c_1^2, c_2, \nu\) and \(d\):

\[
n = -3c_1^2(S) + c_2(S) + 24\nu + \frac{d^2}{2} - 15d,
\]
\[
c = 2c_1^2(S) - c_2(S) - 15\nu + 9d
\]
We use these formulas below in Subsection 4.7.1.

The next two results are rather surprising, as one gets an inequality for the branch curve which is independent of the degree of the projection:

**Lemma 3.13.** (see, e.g., the introduction of [35]) Let \(B \in B(d, c, n)\) a branch curve of a linear projection to \(\mathbb{P}^2\) of a surface of general type, where \(d, c, n, \chi\) and \(\nu\) as above. Then

\[
5\chi + 6c - 9d \leq 0
\]
or, equivalently, in \((d, c, n)\) coordinates:

\[
10n + 16c - 5d^2 + 6d \leq 0
\]

*Proof.* Substituting the expressions for \(c_1^2(S)\) and \(c_2(S)\) (from Lemmas 3.9, 3.10) in terms of \(\nu\) and \((d, c, \chi)\) and \((d, c, n)\) into the Bogomolov inequality \(c_1^2(S) \leq 3c_2(S)\), we get the desired inequality. \[\square\]

There is, however, an inequality which is true for every branch curve, restricting the sum of the nodes and the cusps (though it is weaker than inequality (23)).

**Lemma 3.14.** Let \(B \in B(d, c, n)\). Then

\[
15d - 5\chi - 6c > 0
\]
or

\[
10n + 16c < 5d^2.
\]

*Proof.* Note that Nemirovski’s inequality (see [54]) for branch curves is

\[
\frac{3d - \chi}{3d - \chi - c} < 6
\]
or, equivalently

\[
15d - 5\chi - 6c > 0.
\]

\[\square\]

**Remark 3.15.** The variety \(B(d, c, n)\) is not necessarily connected. See, for example, [66], where it is proven that \(B(48, 168, 840)\) has at least two disjoint irreducible components.
3.2.2. **Chisini’s conjecture.** The following theorem was known as Chisini’s Conjecture, by now proved by Victor Kulikov (see [52], [65]):

**Theorem 3.16.** Let $B$ be the branch curve of generic projection $\pi : S \to \mathbb{P}^2$ of degree at least 5. Then $(S,f)$ is uniquely determined by the pair $(\mathbb{P}^2,B)$.

Kulikov proved this conjecture for generic covers of degree greater than 11 and for generic linear projections of degree greater than 4. Kulikov considered two surfaces $S_1, S_2$ ramified over the same branch curve, and studied the fibre product $S_1 \times_{\mathbb{P}^2} S_2$, proving that the normalization of this fibre product contradicts Hodge’s Index Theorem if $(S_1,f_1)$ is not isomorphic to $(S_2,f_2)$.

**Remark 3.17.** We want to mention that a version of a Generalized Chisini’s conjecture also exists, for surfaces with normal isolated singular points:

**Conjecture 3.18.** Let $f_i : S_i \to \mathbb{P}^2$, $i = 1, 2$ be two generic coverings with the same branch curve $B$ where $S_i$ can have singular points, denoted as Sing $S_i$. Assume $f_1(Sing S_1) = f_2(Sing S_2)$. Then either there exists a morphism $\phi : S_1 \to S_2$ s.t. $f_1 = f_2 \circ \phi$ or $(f_1,f_2)$ is an exceptional pair.

See [57] for the definition of an exceptional pair. This theorem was partially proven by V. S. Kulikov and Vik. S. Kulikov for $f_1, f_2$ generic $m$-canonical coverings, for $m \geq 5$ (see [55]) or when $\text{max}(\text{deg } f_1, \text{deg } f_2) \geq 12$ or $\text{max}(\text{deg } f_1, \text{deg } f_2) \leq 4$ (see [57]).

**Remark 3.19.** One of the theorems induced from the proof of the Chisini’s conjecture was the fact that a class of certain factorization associated to the branch curve $B$ (i.e. the Braid Monodromy Factorization) determines the diffeomorphism type of $S$ as a smooth 4-manifold. We refer the reader to [33], [39] for an introduction of this factorization, and to Kulikov and Teicher’s proof [34] of the above theorem.

3.2.3. **Representation-theoretic reformulation.** Let $G_i$ (resp. $\Gamma_i$) be the local fundamental group of $\mathbb{P}^2 - B$ at the neighborhood of a cusp (resp. a node) of $B$. Note that each $G_i$ is isomorphic to the group with presentation $\{a, b : aba = bab\}$ and every $\Gamma_i$ is isomorphic to the group with presentation $\{a, b : ab = ba\} = \mathbb{Z}^2$.

Let $l$ be a line in $\mathbb{P}^2$ in generic position with $B$, $p_i$ ($i = 1, \ldots , d$) be the intersection points of $B$ and $l$, $p_r$ be a generically chosen point in $l$ and $\gamma_i$ be a small loop around $p_i$ starting and ending at $p_r$. The map $Free_d \to \pi_1(\mathbb{P}^2 - B)$ sending generators of $Free_d$ to $[\gamma_i]$ is epimorphic by Zariski–Van Kampen theorem, and the classes $[\gamma_i]$ are called geometric generators of $\pi_1(\mathbb{P}^2 - B)$.

It is well known (see [44] or [52] Proposition 1) that given a ramified cover $S \to \mathbb{P}^2$, the monodromy map $\varphi : \pi_1(S) \to \text{Sym}_\nu$ satisfies the following three conditions:

(i) for each geometric generator $\gamma$, the image $\varphi(\gamma)$ is a transposition in $\text{Sym}_\nu$;
(ii) for each cusp $q_i$, the image of the two geometric generators of $G_i$ is two non-commuting transpositions in $\text{Sym}_\nu$;
(iii) for each node $p_i$, the images of two geometric generators of $\Gamma_i$ are two different commuting transpositions in $\text{Sym}_\nu$.

The inverse assertion is a group theoretic reformulation on the Chisini’s theorem ([44]):

**Proposition 3.20.** The map associating the monodromy representation with each ramified cover $S \to \mathbb{P}^2$ gives an isomorphism of the set of the isomorphism classes of generic ramified covers of $\mathbb{P}^2$ of degree $\nu$ with the branch curve $B$ and the set of isomorphism classes of epimorphisms $\varphi : \pi_1(\mathbb{P}^2 - B) \to \text{Sym}_\nu$ satisfying the conditions (i), (ii) and (iii) above, with respect to the action of $\text{Sym}_\nu$ on the set of such representations by inner automorphisms.

3.3. **Construction of the variety of branch curves $B(d,c,n)$.** Let $V = V(d,c,n)$ be the Severi-Enriques subvariety in $|dh|$ of degree $d$ plane curves with $n$ nodes and $c$ cusps. Let $B = B(d,c,n) \subseteq V$ the subset consists of branch curves. In this subsection we show that $B(d,c,n)$ is a subvariety
of $V(d,c,n)$. Although it is standard, we have not found it in the literature, though references to its existence can be found in [24] or in [61]. The following lemma proves that the variety of branch curves of ramified covers is a union of connected components of $V$. Using the same techniques in the following proof, and the fact that the Chisini’s conjecture is proven (for generic linear projections), one can prove that also $B$ is a union of connected components of $V$.

**Lemma 3.21.** Over the field $k = \mathbb{C}$, every connected component $V_i$ of $V$ either does not contain branch curves of generic covers at all, or every curve $C \in V_i$ is a branch curve of a generic cover. Explicitly, every component of $B$ is a connected component of $V$.

**Proof.** Let us fix a connected component $V_1$ of $V = V(d,c,n)$, let $p \in V_1$, and let $C$ be the corresponding plane curve. Take $q \in V_1$, $q \neq p$ and choose a path $I = [0,1] \to V_1$ connecting $p$ and $q$. Let us denote $G_C = \pi_1(\mathbb{P}^2 - C)$, $G_{C_t} = \pi_1(\mathbb{P}^2 - C_t)$ with $C_t \in V_1, t \in I$ where $C_1$ corresponds to $q$. As these curves are equisingular, we get an identification of fundamental groups

$$G_{C_t} \cong G_C.$$ 

For every $t \in I$. Consider the group $\text{Hom}(G_C, \text{Sym}_N)$ and its subgroup $\text{Hom}_{\text{geom}}(G_C, \text{Sym}_N)$ of geometric homomorphisms – i.e., homomorphisms which satisfy the conditions (i),(ii),(iii) above – which can be empty. From the above identification, we get a canonical set bijection from $\text{Hom}(G_{C_t}, \text{Sym}_N) \to \text{Hom}(G_C, \text{Sym}_N)$ preserving the set of geometric homomorphisms. In particular, $\text{Hom}_{\text{geom}}(G_C, \text{Sym}_N)$ is empty if and only if $\text{Hom}_{\text{geom}}(G_{C_1}, \text{Sym}_N)$ is empty, and thus $C$ is a branch curve if and only if $C_1$ is. Therefore $B(d,c,n)$ is a union of connected components of $V$ and thus it is a subvariety. 

**Remark 3.22.** We want to describe here on the action of the fundamental group $\pi_1(V)$ on $G = \pi_1(\mathbb{P}^2 - C)$. Let $p \in V$, $C$ be the corresponding degree $d$ plane curve and $U = \mathbb{P}^2 - C$. A loop $\gamma : I \to V$ (starting and ending at $p$), induces an automorphism of the group $G = \pi_1(U)$, and thus an automorphism of the set of representations $\text{Hom}(\pi_1(U), \text{Sym}_N)$ which preserves the set of geometric representations $\text{Hom}_{\text{geom}}(\pi_1(U), \text{Sym}_N)$. To describe it more explicitly, note that we can choose a line $l \subset \mathbb{P}^2$ in generic position to every $C_t$, $t \in I$ (since the set of lines in special position to a fixed curve in $\mathbb{P}^2$ forms a dual curve in the dual plane, and thus the space of lines which are special to some $C_t$ is of real codimension 1 in the dual plane). Note that $l \cap U \cong \mathbb{P}^1 - \{d \text{ points}\}$. Let us now choose a base point $a_+$ on $l$ not belonging to any of the curves $C_t$, and a “geometric basis” $\Gamma$ of $\pi_1(U, a_+) = \pi_1(\mathbb{P}^2 - C, a_+)$, which gives an epimorphism

$$e(\Gamma) : \pi_1(l \cap U, a_+) \to \pi_1(U, a_+).$$

Recall that the group of classes of diffeomorphism of $(\mathbb{P}^1 - d \text{ points})$ modulo diffeomorphisms homotopic to identity can be identified with the commutator of the braid group $B_d' = \text{Braid}_d/\text{Center}(\text{Braid}_d)$ (see e.g. [33]). A loop $\gamma \in \pi_1(V)$ gives a diffeomorphism of $l \cap U$, which in turn induces an automorphism of $\pi_1(l \cap U)$, i.e. an element in $\text{Aut}(\pi_1(l \cap U))$ or equivalently, an element $b_+ \in B_d'$. It follows that there is a natural diagram

$$\begin{array}{ccc}
\pi_1(V) & \xrightarrow{\alpha} & \text{Aut } \pi_1(U) \\
\downarrow{\beta} & & \downarrow{\text{Aut}(\text{Hom}_{\text{geom}}(\pi_1(U), \text{Sym}_N))} \\
\text{Aut } \pi_1(l \cap U) & \cong & B_d'
\end{array}$$

and a commutative triangle:

$$\begin{array}{ccc}
\pi_1(V) & \xrightarrow{\alpha} & \text{Im}(\alpha) \subseteq \text{Aut } \pi_1(U) \\
\downarrow{\beta} & & \downarrow{\text{Im}(\beta) \subseteq B_d'} \\
\text{Im}(\beta) & \subseteq & B_d'
\end{array}$$
An element $b_\gamma \in B'_d$ which is the image of $\gamma$ admits a decomposition of $b_\gamma$ into a product of canonical generators of $B'_d$; i.e. $b_\gamma = x_1^{\pm 1} \cdot \ldots \cdot x_k^{\pm 1}$. Since $\pi_1(l \cap U) = \text{Free}_d = \langle y_1, \ldots, y_d \rangle$, we can describe explicitly the action of each $x_i$ on $\text{Aut}(\text{Free}_d)$:

$$x_i(y_j) = y_j \text{ if } j \neq i, i + 1$$

$$x_i(y_i) = y_{i+1}$$

$$x_i(y_{i+1}) = y_{i+1}^{-1} \cdot y_i \cdot y_{i+1}.$$  

Thus, the action of an element $\gamma \in \pi_1(V)$ on the group $G = \pi_1(U, a_s)$ can be expressed as a map on the generators $\{y_i\}$ of $G$: $(y_i \mapsto b_\gamma(y_i) = (x_1^{\pm 1} \cdot \ldots \cdot x_k^{\pm 1}) \cdot y_i)$ where $x_j \cdot y_i$ is given by the above action. Note that this action is non-trivial in general, and thus $\pi_1(V, p)$ acts generically non-trivially on the set of good covers $S \to \mathbb{P}^2$ ramified over a given curve $C$. However, in a situation when such a cover is unique up to a deck transformation, like in the case of a high degree ramified cover, (due to Chisini’s conjecture), this action reduces to the action of the deck transformation group $\text{Aut}(S/\mathbb{P}^2)$ which is the trivial group, for geometric reasoning.

4. Surfaces in $\mathbb{P}^3$

Let $X$ be a smooth surface in $\mathbb{P}^3$ and $p : \mathbb{P}^r \to \mathbb{P}^2$ be generic projection; we decompose $p$ as a composition of projections $\mathbb{P}^r \to \mathbb{P}^3 \to \mathbb{P}^2$ such that $S = p_1(X)$ is smooth or has ordinary singularities in $\mathbb{P}^3$. We begin in section 4.1 with the examination of branch curves of smooth surfaces in $\mathbb{P}^3$ and proceed to singular surface in section 4.7.

4.1. Smooth surfaces in $\mathbb{P}^3$. Our goal here is to reformulate and give a more modern proof to a result of Segre [3] published in 1930. Segre proved that the set of singular points of the branch curve of a smooth surface in $\mathbb{P}^3$ is a special 0-cycle with respect to some linear systems on $\mathbb{P}^2$, i.e., it lies on some curves of unexpectedly low degree. (We remind that a curve passing through the singularities of a given one is called adjoint curve. See Definition 4.7.) For example, if deg $S = 3$, we get the following result Zariski published in 1929 (cf. [3]): the variety of plane 6-cuspidal sextics has two disjoint irreducible components. Every curve in the first component is a branch curve of a smooth cubic surface and all its six cusps are lying on a conic, while the second component does not contain any branch curves. (Miraculously, this condition does not define a subvariety of positive codimension in the variety of all plane curves of degree 6 with 6 cusps, but rather selects one of its two irreducible components, which was probably the most surprising discovery of Zariski concerning this variety.)

In the following paragraphs we recall Segre’s method for constructing some adjoint curves to branch curves of ramified covers. The main result is the following: a nodal–cuspidal curve $B$ is a branch curve iff there are two adjoint curves of (some particular) low degree passing through all the singularities of $B$ (see Theorem 4.32). Though this result was presented in [60] (by Val. S. Kulikov) and in [11] (by J. D’Almeida), our point of view is different, as we emphasize the relations between the Picard and Chow groups of 0–cycles of the singularities of the branch curve. We also investigate the connections between adjoint curves and the sheaf of weakly holomorphic rational functions on a nodal–cuspidal curve $C$. We hope that the study of the Picard group of branch curves and the study of adjunction with values in sheaf of weakly holomorphic rational functions (see [13]) gives a new understanding of the work of Segre.

Let $S$ be a smooth surface of degree $\nu$ in $\mathbb{P}^3$, and let $\pi : \mathbb{P}^3 \to \mathbb{P}^2$ be a projection from a point $O$ which is not on $S$. Let $B \subset \mathbb{P}^2$ be a branch curve of $\pi$. It is easy to see that the degree of $B$ is $d = \nu(\nu - 1)$: indeed, $B$ is naturally a discriminant of a homogeneous polynomial of degree $\nu$ in one variable. The curve $B$ is in general singular, however, for a generic projection it has only nodes and cusps as singularities (see e.g. [53]).
Assume now that $S$ is given by a homogeneous form $f(x_0,\ldots,x_3)$ of degree $\nu$, and $O = (O_0,\ldots,O_3)$ is a point in $\mathbb{P}^3$ which is not on $S$. The polar surface $Pol_O(S)$ is given by the degree $\nu-1$ form $\sum O_if_i$, where $f_i = \frac{\partial f}{\partial x_i}$. The following lemma is well known:

**Lemma 4.1.** Let $\pi : S \to \mathbb{P}^2$ be the projection with center $O$. The ramification curve $B^*$ of $\pi$ is the intersection of $S$ and the first polar surface $Pol_O(S)$.

Indeed, the intersection of $S$ and $Pol_O(S)$ consists of such points $p$ on $S$ that the tangent plane to $S$ at $p$, $T_pS$, contains the point $O$. This implies that the line joining $O$ and $p$ intersects $S$ with multiplicity at least 2 at $p$.

Note that this gives yet another proof that $\deg B^* = \deg S \cdot \deg(Pol_O(S)) = \nu(\nu - 1)$.

**Notation:**

1. $H \in A_2\mathbb{P}^3$ is a class of a hyperplane in $\mathbb{P}^3$;
2. $h \in A_1\mathbb{P}^2$ is a class of a line in $\mathbb{P}^2$;
3. $\ell^* = H|_{B^*}$, $\ell^* \in A_0B^*$;
4. $\ell = h|_B$, $\ell \in A_0B$;

We also denote

5. $S'_O = Pol_O(S) \subset \mathbb{P}^3$, and
6. $S''_O = Pol''_O(S)$ is the second polar surface to $S$ w.r.t. the point $O$; it is given by a homogeneous form $f'' = (\sum O_i \frac{\partial f}{\partial x_i})^2f = \sum O_iO_jf_{ij}$ of degree $\nu - 2$.

7. We call a 0-subscheme with length 1 at every point a 0-cycle.
8. Let $P \subset B$ be the 0-cycle of nodes on $B$, and $P^*$ be its preimage on $B^*$. Note that $\deg P^* = 2\deg P$, as can be seen from Lemma 4.2.
9. Let $Q \subset B$ be the 0-cycle of cusps on $B$, and $Q^*$ be its preimage on $B^*$. Note that $\deg Q^* = \deg Q$ (see Lemma 4.2).
10. $\xi$ be the 0-cycle of singularities of $B$.

From now on we assume that $O$ is chosen generically for a given surface $S$. It follows that $B^*$ is smooth, and $B$ has only nodes and cusps as singularities. Already in the 19th century the number of nodes and cusps of a branch curve was computed for a smooth surface of a given degree.

**Lemma 4.2** (Salmon [1]). (a) There is one-to-one correspondence between bisecant lines for $B^*$ passing through $O$ and nodes of $B$. Moreover, the number of bisecant lines through $O$ does not depend on $S$, and is equal to

$$n = n(\nu) = \frac{1}{2}\nu(\nu - 1)(\nu - 2)(\nu - 3)$$

(b) If $Q^*$ is the set of points $q$ on $B^*$ such that the tangent line $T_qB^*$ contains the point $O$, then the set $Q = \pi(Q^*)$ is the set of cusps of $B$.

(c) Moreover, $Q^*$ is the scheme-theoretic intersection of $B^*$ and the second polar surface $S''_O$. In other words, they intersect transversally at each point of $Q^*$, and $B^* \cap S'' = Q^*$. In particular, the class $[Q^*]$ in $A_0B^*$ is equal to $(\nu - 2)l^*$.

(d) It follows that $\deg Q$ does not depend on a choice of the surface $S$, and is equal to

$$c = c(\nu) = \nu(\nu - 1)(\nu - 2)$$

**Proof.** (a) The first statement is geometrically clear; for the second see [1] art. 275, 279. Yet another proof is given below, in Proposition 4.3. See also [15], Chapter IX, sections 1.1,1.2 for a way to induce the formula for the number of bisecant of a complete intersection curve in $\mathbb{P}^3$ (i.e. the number $n + c$). For (b), see [1] art. 276. (c) is a straightforward computation, and (d) follows from (c).
Lemma 4.3. Let $\ell \in A_0(B)$ be the class of a plane section on $B$. Then

$$[Q] = (\nu - 2)\ell \quad \text{in} \quad A_0(B),$$

(2) The equality above can be lifted to $\text{Pic} B$: there is a Cartier divisor $Q_0$ such that $\text{can}(Q_0) = Q$ with respect to the canonical map

$$\text{can} : \text{Cartier}(B) \to \text{Weil}(B)$$

associating Weil divisor with a Cartier divisor, and

$$[Q_0] = (\nu - 2)\ell$$

in $\text{Pic}(B)$.

Proof. We have $Q = \pi_*(Q^*)$, and $Q^* = B^* \cap S''_O$. Since $[S''_O] = (\nu - 2)H$ in $A_2\mathbb{P}^3$, we have $[Q^*] = (\nu - 2)\ell^*$ in $A_0B^*$, and thus

$$[Q] = [\pi_*(Q^*)] = \pi_*([Q^*]) = (\nu - 2)\pi_*\ell^* = (\nu - 2)\ell$$

in $A_0B$.

To see that $\pi_*\ell^* = \ell$ it is enough to consider a hyperplane in $\mathbb{P}^3$ containing the point $O$.

(2) Consider the rational function $r = f''_O/H^{(\nu - 2)}$, where $f''_O$ is by definition the equation of the second polar $\text{Pol}^2(O, S)$, and $H$ is an equation of a generic hyperplane containing the projection center $O$. Since the curves $B^*$ and $B$ are birational, $r$ can be considered as a rational function on $B$, where it gives the desired linear equivalence.

Remark 4.4. Note that both cusps and nodes on a curve are associated with Cartier divisors on the curve, even though these Cartier divisors are not positive. For example, on the affine cuspidal curve $C$ given by the equation $y^2 - x^3 = 0$ the divisor $(y/x) = 3[0] - 2[0] = [0]$ is a principle Cartier, but since $y/x$ is not in the local ring of the point $[0]$, it is not locally given a section of the sheaf $\mathcal{O}_C$. For the nodal curve $C$ given by the equation $xy = 0$, the divisor $\left(\frac{y - x^2}{y - 2x}\right) = 3[0] - 2[0] = [0]$ is also a principle Cartier, though not positive.

4.1.1. Example: smooth cubic surface in $\mathbb{P}^3$. Let $S$ be a smooth cubic surface in $\mathbb{P}^3$. Then Lemma 4.2 imply that $B$ is a plane curve with 6 cusps and no other singularities, and Lemma 4.3 implies that

$$[Q] = \ell$$

in $A_0B$. $Q$ is, of course, not a line section of the curve $B$; the linear equivalence above implies that the map

$$\mathbb{P} H^0(\mathbb{P}^2, \mathcal{O}(1)) \simeq \mathbb{P} H^0(B, \mathcal{O}(1)) \to |\ell|$$

is not epimorphic, where $|\ell|$ is the set of all Weil divisors linearly equivalent to a generic line section of $B$. Even though $Q$ is associated with a Cartier divisor $b/a$, this Cartier divisor is not positive.

It is well known that 6 points in general position on $\mathbb{P}^2$ do not lie on a conic. As for the 6 cusps $Q$ on the branch curve we have the following result of Zariski and Segre (see [6], [8]).

Corollary 4.5. All 6 cusps of a degree 6 plane curve $B$ which is a branch curve of a smooth cubic surface lie on a conic.

Remark 4.6. Explicit construction of a branch curve of a cubic. By change of coordinates a cubic surface $S$ is given by the equation

$$f(z) = z^3 - 3az + b,$$
where \(a\) and \(b\) are homogeneous forms in \((x, y, w)\) of degrees 2 and 3, and the projection \(\pi\) is given by \((x, y, w, z) \mapsto (x, y, w)\). In these coordinates the ramification curve is given by the ideal \((f, f') = (f, z^2 - a) = (z^3 - \frac{1}{2}b, z^2 - a)\) and the branch curve \(B\) is given by the discriminant
\[
\Delta(f) = b^2 - 4a^3
\]
In particular, one can easily see that it has 6 cusps at the intersection of the plane conic defined by \(a\) and the plane cubic defined by \(b\), as illustrated on the Figure 3. It is also clear that the conic defined by \(a\) coincides with one constructed in Corollary 4.5.

![Figure 3: The branch curve of a smooth cubic surface](image)

The ideal of \(Q^*\) is equal to \((f, f', f'') = (f, f', z) = (a, b, z)\). Note that \(z\) equal to \(\frac{b}{2a}\) as a rational section of \(O_B^*(1)\). We want to explicate the linear equivalence of \(Q^*\) and the intersection of \(B^*\) with the “vertical” plane (one containing the point \(O\)). For this, let \(l(x, y, w)\) be a linear form in \(x, y, w,\) and consider the rational function on \(B^*\)
\[
\phi = z \frac{b}{2al},
\]
Then \(\phi\) gives the linear equivalence
\[
0 = (\phi) = (b) - (a) - (l) = 3Q - 2Q - (l) = Q - (l),
\]
which gives an explicit proof that \([Q] = \ell\) in \(A_0B\). (We used the fact that cubic \(b\) is tangent to \(B\) at the cusps, while conic \(a\) is not.) This example has a “natural” continuation in example 4.30.

### 4.2. Adjoint curves to the branch curve.

We begin with the definition of an *adjoint* curve. This type of curves will play an essential role when studying branch curve.

**Definition 4.7.** Given a plane curve \(C\), a second curve \(A\) is said to be adjoint to \(C\) if it contains each singular point of \(C\) of multiplicity \(r\) with multiplicity at least \(r - 1\). In particular, \(A\) is adjoint to a nodal-cuspidal curve \(C\) if it contains all nodes and all cusps of \(C\).

For more on adjoint curves see [2, § 7], [15, Chapter II, § 2], or [29] for a more recent survey.

Below, following Segre, we construct more adjoint curves to \(B\) (i.e. \(W, L, L_1\)) and relate them to the geometry of \(B^*\) in \(\mathbb{P}^3\).

We continue this subsection with Proposition 4.8 from [8] and we bring its proof for the convenience of the reader.

**Proposition 4.8.** (a) One has in \(A_0(B)\)
\[
2[P] + 3[Q] = \nu(\nu - 2)\ell.
\]

(b) The equality above can be lifted to \(\text{Pic} B\): there are Cartier divisors \(P_1\) and \(Q_1\) such that in \(\text{Pic} B\):
\[
\text{can}(P_1) = 2P, \quad \text{can}(Q_1) = 3Q, \quad \text{and}

\quad [P_1] + [Q_1] = \nu(\nu - 2)\ell.\]
In fact, $Q_1$ is the canonically defined “tangent” Cartier class $Q_\tau$.

**Proof.** (following Segre [8]).

Let us choose a plane $\Pi$ in $\mathbb{P}^3$ not containing the point $O$, and consider the projection with center $O$ as a map to $\Pi$. Let us also choose a generic point $O' = (O'_0, O'_1, O'_2) \in \Pi$, and let $B' = \text{Pol}_O(B)$ be the polar curve of $B$, defined as follows: if $B$ is given by the homogeneous form $g(x_0, x_1, x_2)$ of degree $d = \nu(\nu - 1)$, then $\text{Pol}_O(B) = \{\sum_{i=0}^{2} O'_i \frac{\partial g}{\partial x_i} = 0\}$. Note that the first polar $B' = \text{Pol}_O B$ is adjoint to $B$.

It is clear that

\[(B \cap B') = 2P + 3Q + R,\]

where $R$ (for “residual”) is the set of non-singular points $p$ on $B$ such that the tangent line to $B$ at $p$ contains $O'$, and thus

\[2P + 3Q + R]' = (d - 1)\ell\]

in $A_0B$ (Here we used the fact that $O'$ is generic, in particular, it does not belong to $B$ and to the union of tangent cones to $B$ at nodes and cusps.)

Let $R'$ be the preimage of $R$ on $B'$. We claim that $R' = B' \cap S'_{O'}$, where $S'_{O'} = \text{Pol}_{O'}(S)$. Indeed, if $p \in R$, then $T_pB$ contains the point $O'$, and if $p'$ is the preimage of $p$ on $B'$, then the tangent space to $S$ at $p'$ can be decomposed into a direct sum of the line $l$ joining $p$ and $p'$ (and containing $O$) and the tangent line $T_{p'}B'$ which projects to the tangent line $T_pB$, (as illustrated on Figure 4 below).

![Figure 4: $R' = B' \cap S'_{O'}$](image)

It follows that

\[[R] = \pi_*([R']) = \pi_*([B' \cap S'_{O'}]) = \pi_*((\nu - 1)\ell') = (\nu - 1)\ell\]

in $A_0B$, and thus

\[2P + 3Q = (2P + 3Q + R) - [R] = (d - 1)\ell - (\nu - 1)\ell = (d - \nu)\ell = \nu(\nu - 2)\ell\]

The proof of the second part is parallel, as the Weil divisors $2P$ and $3Q$ can be lifted to $\text{Pic} B$.

Note that this gives yet another proof for the formula for the number of nodes $n = n(\nu)$.

**Proposition 4.9.** There exist a (unique) curve $W$ in the plane $\Pi$ of degree $\nu(\nu - 2)$ such that in $A_0(B)$

\[[W \cap B] = 2[P] + 3[Q].\]
Proof. By the previous proposition, the cycle $2P + 3Q$ is in the linear system $|\nu(\nu - 2)\ell|$ on $B$. Note that $2P + 3Q$ is actually a Cartier divisor (see Remark 4.4). Now, since $\deg B = \nu(\nu - 1)$ is greater than $\nu(\nu - 2)$, there is a restriction isomorphism

$$0 \to H^0(\Pi, \mathcal{O}(\nu(\nu - 2))) \to H^0(B, \mathcal{O}(\nu(\nu - 2))) \to 0$$

which completes the proof. \hfill \Box

Note that $W$ is an adjoint curve to $B$ which is tangent to $B$ at each cusp of $B$.

Proposition 4.10. Let $a = (\nu - 1)(\nu - 2)$.

(1) We have

$$[2P + 2Q] = a\ell$$

In $A_0(B)$.

(b) The equality above can be lifted to $\text{Pic}(B)$: there are Cartier divisors $P_2$ and $Q_2$ such that in $\text{Pic}(B)$:

$$\text{can}(P_2) = 2P,$$
$$\text{can}(Q_2) = 2Q,$$
and

$$[P_2] + [Q_2] = a\ell$$

(2) There is a (unique) curve $L$ of degree $a$ such that

$$[L \cap B] = 2P + 2Q.$$  

Proof. (1) We have

$$[2P + 2Q] = [2P + 3Q] - [Q] = \nu(\nu - 2)\ell - (\nu - 2)\ell = (\nu - 1)(\nu - 2)\ell = a\ell.$$ 

The computation in $\text{Pic}(B)$ is parallel: we let $P_2 = P_1$ and $Q_2 = Q_1 - Q_0$.

(2) Note that $(\nu - 1)(\nu - 2) < \deg B$, which completes the proof. \hfill \Box

Note that $L$ is an adjoint curve to $B$ which is not tangent to $B$ at the cusps of $B$.

Notation 4.11. Let $\zeta_L$ be the Cartier divisor on $B$ given by restricting the equation of $L$ to $B$. Recall that $\zeta_L$ is supported on the 0–cycle of singularities $\xi$.

Definition 4.12. Let $V(d, c, n)$ be the variety of plane curves of degree $d$ with $c$ cusps and $n$ nodes, abd let $B(d, c, n)$ be the subvariety in $V(d, c, n)$ consisting of branch curves of all ramified covers of $\mathbb{P}^2$.

Example 4.13. By substituting $\nu = 3$ and $\nu = 4$ we get the classical example of a sextic with six cusps on a conic we discussed above, and the example of a degree 12 curve with 24 cusps and 12 nodes, all of them are on a sextic:
(1) The branch curve $C$ of a smooth cubic surface in $\mathbb{P}^3$ is a sextic with six cusps, $C \in B(6, 6, 0)$. We have $\deg L = 2$; two different constructions of this conic was given above in Corollary 4.5 and Remark 4.6. See also Figure 3 above.
(2) The branch curve $C$ of a smooth quartic surface in $\mathbb{P}^3$ is of degree 12, and has 24 cusps and 12 nodes, i.e., $C \in B(12, 24, 12)$. We have $\deg L = 6$. Moreover, the 24 cusps lie on the intersection of a quartic and a sextic curves (see e.g. [8]).
4.3. Adjoint curves and Linear systems. We start with the following easy Lemma:

**Lemma 4.14.** (Adjunction for a flag \((\xi, \zeta, K, P)\)) Assume we are given a flag of \(4\) (arbitrary) schemes \((\xi, \zeta, K, P)\). Then there is a diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
J_{\xi,\zeta} & \rightarrow & J_{\xi,\zeta} \\
\downarrow & & \downarrow \\
0 & \rightarrow & J_K \\
\downarrow & & \downarrow \\
J_\xi & \rightarrow & J_{\xi,K} \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

where \(J_X = J_{X,P}\), and \(X\) is either \(\xi, \zeta,\) or \(K\).

**Corollary 4.15.** Coming back to our standard notations, let \(K\) be a plane curve, \(\xi \subset \zeta \subset K\) be a flag of 0-subschemes on \(K\) such that \(\zeta\) is given by a positive Cartier divisor, and \(\xi = \text{supp} \zeta\). In this case \(J_{\zeta,K} = \mathcal{O}_K(-\zeta)\). Then, given an integer \(n < \deg K\), the diagram above gives isomorphisms

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(\mathbb{P}^2, J_{\xi,\zeta}(n)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(K, J_{\xi,\zeta}(n)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(\mathbb{P}^2, J_{\xi}(n)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(K, J_{\xi,K}(n)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(\mathbb{P}^2, J_{\zeta}(n)) \\
\downarrow & & \downarrow \\
\end{array}
\]

**Corollary 4.16.** Assume that there is an integer \(a\) and a positive Cartier divisor \(\zeta = \zeta_0\) on \(K\) such that there is a linear equivalence \(\zeta_0 \sim al\), where \(l\) is the class of a line section on \(K\). (Such is the case of a branch curve and the class \(\zeta_L\) constructed above.)

Then, setting \(n = a + i\), we get isomorphisms

\[
j_{K,\zeta_0}(i): H^0(\mathbb{P}^2, J_{\zeta_0}(a + i)) \rightarrow H^0(K, \mathcal{O}_K(i))
\]

for every \(i \geq 0\).

In other words, adjoint curves on \(\mathbb{P}^2\) with given tangent conditions at the singularities of the curve \(K\) correspond to homogeneous forms on \(K\) with given tangent conditions at the singularities of the curve \(K\) correspond to homogeneous forms on \(K\).

We only need this isomorphism for \(i = 0\); it implies that there is a curve \(L_0 \in H^0(\mathbb{P}^2, J_{\zeta_0}(a))\) of degree \(a\) corresponding to the element \(1 \in H^0(K, \mathcal{O}_K)\), and \(\zeta_0\) is locally given by the equation of \(L_0\). (This is exactly the case of a branch curve \(K = B\), where \(\zeta_0 = \zeta_L\) and \(L_0 = L\).)

The isomorphism \(j_{K,\zeta_0}(i)\) is given by \(h \mapsto \frac{h}{f_{L_0}}\), where \(f_{L_0}\) is an equation of the curve \(L_0\).
Our next goal is to study curves of various degrees $n > a$ containing the $0$-cycle $\xi$ but restricting to different Cartier divisors with support on $\xi$, not necessarily coinciding with $\zeta_0$. Assume that we are given a positive Cartier divisor $\zeta_1$ on $K$; we will study adjoint curves restricting to $K$ as $\zeta_1$.

Note that $J_{\zeta_1, K} = \mathcal{O}_K(-\zeta_1)$, and consider the restriction map

$$\text{res}_K : H^0(\mathbb{P}^2, J_{\zeta_1}(a + i)) \to H^0(K, \mathcal{O}_K(-\zeta_1)(a + i))$$

To introduce notations we need to recall some basic facts about lineal equivalence of Cartier divisors. Assume that we are given two positive Cartier divisors $D_1$ and $D_2$ on a scheme $X$ and a linear equivalence $D_1 - D_2 = (r)$ for a meromorphic function $r$. We realize both $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$ as subsheaves of the sheaf $\mathcal{M}_X$ of meromorphic functions on $X$, and describe the isomorphism $\mathcal{O}_X(D_1) \to \mathcal{O}_X(D_2)$ given by the function $r$ explicitly. Locally, on a small enough affine open set $U \subset X$, $U \simeq \text{Spec } A$, $D_1$ and $D_2$ are given by equations $f_1$ and $f_2$, $f_i \in A$, $f_1/f_2 = r$ in the full ring of fractions $\mathcal{M}_A$ of $A$, and $\mathcal{O}(D_i)$ is given by the $A$-submodule $\frac{a}{f_i}A$ in $\mathcal{M}_A$, $i = 1, 2$.

The isomorphism $j_r : \frac{1}{f_1}A \to \frac{1}{f_2}A$, $a/f_1 \mapsto r \cdot (a/f_1) = a/f_2$ gives rise to an automorphism of the sheaf $\mathcal{M}_X$ given by the multiplication by $r$. Thus, globally, the sheaf automorphism $j_r : \mathcal{M}_X \to \mathcal{M}_X$ given by the multiplication by $r$ takes $\mathcal{O}(D_1)$ to $\mathcal{O}(D_2)$.

Now, using the linear equivalence $\zeta_0 \sim al$ on $K$, we get an isomorphism

$$j_r : \mathcal{O}_K(-\zeta_1)(a + i) \to \mathcal{O}_K(-\zeta_1)(\zeta_0)(i) \simeq \mathcal{O}_K(\zeta_0 - \zeta_1)(i)$$

given by multiplication with the rational function

$$r = \frac{f_1}{f_{L_0}}$$

where $f_1$ is an equation of a line $l$, and $f_{L_0}$ is the equation of $L_0 \in H^0(\mathbb{P}^2, J_{\zeta_1}(a))$. Thus the image for an adjunction belongs to the sheaf $\mathcal{O}_K(\zeta_0 - \zeta_1) \otimes \mathcal{O}_K(i)$, which is the sheaf of meromorphic functions on $K$ with zeroes at $\zeta_1$ and poles at $\zeta_0$, shifted by $i$.

Since we want to study adjoint curves to $K$, we are interested in positive Cartier divisors of the form $\zeta_1 = \zeta_1^\xi + \zeta_1^{\text{res}}$, where $\zeta_1^\xi$ is supported on $\xi$, i.e., $\text{supp}(\zeta_1^\xi) = \text{supp}(\zeta_0) = \xi$, and $\zeta_1^{\text{res}}$ (res for "residual") is supported on the set of smooth points of $K$. Note that the sections of the sheaf $\mathcal{O}_K(\zeta_0 - \zeta_1)$ can locally be given by $r = h_1/h_0 \cdot g$, where $h_i$ is the local equation for the Cartier divisor $\zeta_i$, and $g$ is regular, i.e., $g \in \mathcal{O}_{K, p}$.

Thus we introduce the following module and sheaf:

**Definition 4.17.** For a commutative ring $A$, we define an $A$-submodule $R_A$ in the full ring of fractions $\mathcal{M}_A$,

$$A \subset R_A \subset \mathcal{M}_A,$$

as the set of all fractions $r = g_1/g_0$ such that $\text{ord}_p(g_1) \geq \text{ord}_p(g_0)$ for each height one ideal $p$ of $A$.

Given a scheme $X$, one can define the sheaf $\mathcal{R}_X$; this sheaf is the subsheaf of the sheaf of meromorphic functions $\mathcal{M}_X$ given locally by fractions $r = g_1/g_0$ such that $\text{ord}_Z(r) = \text{ord}_Z(g_1) - \text{ord}_Z(g_0) \geq 0$ for each codimension one subvariety $Z$ of $X$.

The sheaf $\mathcal{R}_X$ coincides with the structure sheaf $\mathcal{O}_X$ at the set of smooth points of $X$, and there is a filtration

$$\mathcal{O}_X \subset \mathcal{R}_X \subset \mathcal{M}_X.$$ 

Moreover, we have the following easy Lemma:

**Lemma 4.18.** The normalization $\mathcal{N}_A$ of $A$ in the full ring of fractions $\mathcal{M}_A$ is a submodule of $R_A$. I.e., there is a filtration

$$A \subset \mathcal{N}_A \subset R_A \subset \mathcal{M}_A$$

Note that sheaf $\mathcal{N}_X$ coincides with the sheaf $\pi_*(\mathcal{O}_{X^*})$, the pushforward of the structure sheaf along the normalization $X^* \to X$. 
Combining this all together, we get an adjunction sequence

$$a_{K,i,\xi} : J_{\xi,P}(a+i) \xrightarrow{\text{res}_K} \mathcal{O}_K(-\xi)(a+i) \rightarrow \mathcal{O}_K(\xi_0 - \xi_1)(i) =$$

$$= \mathcal{O}_K(\xi_0 - \xi_1)(i)(-\xi_1^{\text{res}}) \subset \mathcal{O}_K(\xi_0 - \xi_1)(i) \subset R_K(i)$$

and, taking union over all positive Cartier divisors $\xi_1$, we finally get our main adjunction

$$(28) a_{K,i} : J_{\xi,P}(a+i) \xrightarrow{\epsilon} R_K(i),$$

where

$$r = f_1^i / f_L.$$

Now we study the image of the map $a_{K,i}$.

**Definition 4.19.** Let $C$ be a plane curve. We say that a line $l$ containing a cuspidal or nodal point $p$ of $C$ is strictly tangent to $C$ at $p$ if $l$ intersects $C$ with multiplicity 3 at $p$.

We also say that a curve $C_1$ containing $p$ is strictly tangent to $C$ at the nodal or cuspidal point $p$ of $C$ if $C_1$ intersects $C$ with multiplicity at least 3 at $p$.

Assume from now on that the adjoint curve $L_0$ to $K$ is not (strictly) tangent to $K$ at its singular points, and does not intersect $K$ elsewhere.

We want to introduce a sheaf of rational functions with denominator vanishing exactly along $L_0$. This sheaf is clearly the image of the adjunction map $a_K$ defined above.

**Definition 4.20.** Let $R_{L_0}^K$ be a subsheaf of $R_K$ consisting of sections $r$ which can be given by $r = f / f_{L_0}$, where $f$ is a homogeneous polynomial on $\mathbb{P}^2$, and $f_{L_0}$ is an equation of the curve $L_0$.

**Proposition 4.21.** If $K$ is a nodal-cuspidal curve and $L_0$ is an adjoint curve not tangent to $K$ at the singularities of $K$ and not intersecting it elsewhere, then the natural inclusion

$$R_{L_0}^K \subset R_K$$

is an equality. Moreover, they both coincide with the sheaf $\pi_* \mathcal{O}_K^*$. 

**Proof.** The proof follows easily from the fact that nodal and cuspidal singularities of curves are resolved by a single blow-up, and, moreover, we can take $t = f_1 / f_0$ or $t = f_1 / f_{L_0}$ as a local coordinate on the resolution, where $f_1$ and $f_0$ vanish at the singular points of $K$ and have separated tangents to $K$ at the singularities of $K$. In this way, both of the sheaves are equal to $\pi_* \mathcal{O}_K^*$, and thus they coincide. \qed

**Remark 4.22.** This proposition is an example for the analytic theory of weakly holomorphic functions and universal denominator theorem (see, for example, [15]) in case our base field is the field of complex numbers. In this case the equation of the adjoint curve $L_0$ works as the universal denominator for the sheaf of weakly holomorphic functions at each point of $K$.

Combining the proposition above and the construction of the adjunction map $a_{K,i}$ (which is essentially a division by the equation of $L_0$), we get the following theorem:

**Theorem 4.23.** For a nodal-cuspidal curve $K$ and an adjoint curve $L_0$ as above, the map $a_{K,i}$ is epimorphic onto $R_K(i)$, and there is an exact sequence

$$0 \rightarrow J_{K,P}(a+i) \xrightarrow{a_{K,i}} J_{\xi,P}(a+i) \xrightarrow{\epsilon} R_{L_0}^K(i) \rightarrow 0$$

$$\downarrow$$

$$R_K(i)$$
In other words, adjoint curves of degree $a + i$ to the curve $K$ on the plane induce rational functions on the curve $K$ for which $\text{ord}_p(r) \geq 0$ for each point $p \in K$.

The map $a_{K,i}$ is an isomorphism modulo ideal spanned by the equation of $K$.

Passing to the global sections for $a + i < \deg K$, we get the following theorem:

**Theorem 4.24.** For $a + i < \deg K$, there are isomorphisms

$$\bigoplus H^0(\mathbb{P}^2, J_\xi(a + i)) \cong \bigoplus H^0(K, R_K(i)) \cong \bigoplus H^0(K^*, \mathcal{O}_{K^*}(i))$$

For higher degrees $i \geq \deg K - a$ one can modify these isomorphisms readily to get a correct version including adjoint curves containing $K$ as a component.

**Proof.** This theorem follows immediately from Theorem 4.23 and Proposition 4.21 if we take into account the projection formula for $\pi : K^* \to K$,

$$\pi_*(\mathcal{O}_{K^*}(i)) \cong \pi_*(\mathcal{O}_{K^*} \otimes \pi^* \mathcal{O}_K(i)) \cong \pi_*(\mathcal{O}_{K^*}) \otimes \mathcal{O}_K(i) \cong R_K \otimes \mathcal{O}_K(i).$$

The meaning of the theorem is that plane curves through $\xi$ exactly correspond to homogeneous functions on $K^*$.

**Remark 4.25. (Graded algebras interpretation)** Assume we are given a smooth space curve $K^*$ not contained in a plane in $\mathbb{P}^3$ and a projection $p : K^* \to K$ to a plane curve $K$. Since $K^*$ is birational to $K$, in order to reconstruct $K^*$ from $K$, we have to say what is the "vertical coordinate $z$" on $K^*$ in terms of $K$. Since $K^*$ and $K$ are birational, the regular (holomorphic) objects on $K^*$ are rational (meromorphic) objects on $K$, and thus we should have an equality of the form $z = f_{n+1}/f_n$ for some integer $n$ and plane curves $f_n$ and $f_{n+1}$ of degrees $n$ and $n + 1$.

More precisely, let $S = \bigoplus S_i$, $S_i = H^0(K, \mathcal{O}(i))$ be the graded algebra of homogeneous functions on $K$, and $T$ be the graded algebra of homogeneous functions on $K^*$. The inclusion $S \to T$ gives an isomorphism of fraction fields $\mathbb{Q}(S) \cong \mathbb{Q}(T)$, since $K$ and $K^*$ are birational. Now $T_1 = S_1 \oplus kz$ for some element ("vertical coordinate") $z \in T_1$; since $T_1 \subset \mathbb{Q}(T) \cong \mathbb{Q}(S)$, we would have

$$z = \frac{f_{n+1}}{f_n}$$

for some integer $n$ and plane curves $f_n$ and $f_{n+1}$ of degrees $n$ and $n + 1$, both passing through the singularities of $K$.

**Corollary 4.26.** As in the previous remark, assume that we are given a smooth space curve $K^*$, a projection $p : \mathbb{P}^3 \to \mathbb{P}^2$ with center $O$ not on $K^*$ such that $K = p(K^*)$ is a nodal-cuspidal curve, and an adjoint curve $L_0$ of degree $a$ to $K$ which is smooth at the singularities of $K$ and is not (strictly) tangent to $K$ there.

Then the "vertical coordinate" $z$ on $K^*$, $z \in H^0(K^*, \mathcal{O}_{K^*}(1))$, is the image of a uniquely defined plane curve $L_1$ of degree $a + 1$ under the adjunction map $a_{K,1}$ defined by the formula (28).

In other words, we can choose $n = a$ in the remark above, and

$$z = \frac{f_{L_1}}{f_{L_0}},$$

where $f_C$ is an equation of a plane curve $C$, $C = L_0$ or $L_1$, $\deg L_1 = a + 1$, and the curve $L_1$ is not a union of $L_0$ and a line, i.e., is a "new" adjoint curve.

The curves $L_0$ and $L_1$ are smooth at the points of $\xi$ and have different tangents at every point $p \in \xi$.

**Proof.** There are two ways to prove it. First, this statement is a corollary of the theorem 4.24. The fact $L_1$ is "new", i.e., not a union of $L_0$ and a line, follows from the fact that $z$ is "new", i.e., does not come from a linear form on $\mathbb{P}^2$ (explicitly, $z \in H^0(\mathbb{P}^3, \mathcal{O}(1)) \cong H^0(K^*, \mathcal{O}_{K^*}(1))$). The fact that
L_1 is smooth at the singularities of K follows from the fact that the fraction \( z = f_{L_1}/f_L \) resolves the singularities of K.

A more straightforward proof is the following: let S be the graded homogeneous algebra of K and T be the graded homogeneous algebra of K^*; and consider the element \( t = z \cdot f_{L_0} \) of T_{a+1}. It is enough to prove that t actually belongs to S_{a+1}, since then we can let \( f_{a+1} = t \) and \( z = f_{a+1}/f_{L_0} \). Now this is an easy local computation for each singular point of K, since the exact sequence

\[
0 \to S_{a+1} \to T_{a+1} \to T_{a+1}/S_{a+1} \to 0
\]

is obtained from the sheaf exact sequence

\[
0 \to \mathcal{O}_K(a+1) \to p_* \mathcal{O}_K^*(a+1) \to F(a+1) \to 0,
\]

where F is by definition the factor sheaf \( p_* \mathcal{O}_K^*/\mathcal{O}_K \), by passing to global sections:

\[
0 \to H^0(K, \mathcal{O}_K(a+1)) \xrightarrow{p^*} H^0(K^*, \mathcal{O}_K^*(a+1)) \to \text{coker } p^* \to 0
\]

Since the factorsheaf F is a product of sheaves supported at singular points of K, this makes computing the image of t in \( H^0(K, F(a+1)) \) an easy local computation at nodes and cusps.

The intuitive meaning of this computation is that \( f_{L_0} \) vanishes at the singularities of K, which implies that \( t = z f_{L_0} \) is a regular (holomorphic) object on K, and thus belongs to S_{a+1}. \( \square \)

In particular, this is the case when \( K = B \) is a branch curve of a smooth surface S in \( \mathbb{P}^3 \), where \( \xi \) is the 0–cycle of singularities of K. In this case we can take \( L = L_0, a = (\nu - 1)(\nu - 2) \), where \( \nu = \text{deg } S \). Segre refers to the existence of the second adjoint curve L_1 as something known from the Cayley’s ”monoïde construction” (see [3, pg. 278]).

Remark 4.27. Summarizing what is written above, the branch curve B has an adjoint curve L of degree equal to a. In this case, we have

\[ z = \frac{f_{L_1}}{f_L} \]

The curves L and L_1 are smooth at the points of \( \xi = P + Q \) and have separated tangents at every point \( p \in \xi \).

Remark 4.28. Note that if the plane nodal-cuspidal curve K has two adjoint curves of degrees n and n + 1 with separated tangents at Sing K for any integer n, then K is the image of a smooth space curve K^* under the projection from \( \mathbb{P}^3 \), but it is only n = a = (\nu - 1)(\nu - 2) that K may actually be a branch curve of a surface projection.

Remark 4.29. We the following isomorphisms:

\[
H^0(\mathbb{P}^2, J_\xi(a+1)) \simeq H^0(K^*, \mathcal{O}_K^*(1)) \simeq H^0(\mathbb{P}^3, \mathcal{O}(1)).
\]

I.e., linear forms on K^* correspond to adjoint curves of degree equal to a + 1 on K.

Example 4.30. For a cubic surface \( f = z^3 - 3az + b \) the branch curve \( B = b^2 - 4a^3 \). The six cusps of B are given by the intersection of a conic and a cubic \( (a = b = 0) \), and in this case \( L = a \) is a conic in general position to B at the cusps, the cubic \( W = b \) is strictly tangent to B at the cusps (see definition 4.19), and both of them do not intersect B elsewhere. We claim that \( L_1 = W \) in this case. Indeed, we have on B^*

\[ f = z^3 - 3az + b = 0, \]

\[ f' = 3(z^2 - a) = 0 \]

and thus

\[ z = \frac{1}{2} \frac{b}{a} \]

on B^*. It follows that L_1 is given by b.
Remark 4.31. In the previous example we can choose the curve \( L_1 \) as any of the curves \( W + l_0L \), where \( l_0 \) is a linear form on \( \mathbb{P} \) (perhaps 0). An easy computation shows that \( L_1 \) is strictly tangent to \( K \) at \( q \in Q \) iff \( l_0 \) contains the point \( q \) (or if \( l_0 = 0 \), but even in this case \( l_0 \) the curves \( L \) and \( L_1 \) have different tangents at \( q \).

4.4. Segre’s theorem. Consider again a smooth surface \( S \) in \( \mathbb{P}^3 \) and a projection \( \pi : S \to \mathbb{P}^2 \) with a center \( O \in \mathbb{P}^3 - S \). Let \( B \) be the branch curve of \( p \), and \( \xi \) be the 0-cycle of singularities of \( B \).

Consider now the graded vector space \( \oplus \mathcal{H}^0(\mathbb{P}^2, J_\xi(n)) \). It follows from the Segre’s computation that \( a = (\nu - 1)(\nu - 2) \) is the smallest integer such that there are adjoint curves of degree \( a \) to \( B \). The vector space \( \mathcal{H}^0(\mathbb{P}^2, J_\xi(a)) \) is one-dimensional and generated by the the curve \( L \). Let \( \zeta_L = L|_B \) be the corresponding divisor class in \( \text{Pic}(B) \). Note that for \( n = a \) the class \( \zeta_L \) gives a canonical lifting of \( 2\xi = 2P + 2Q \) to \( \text{Pic} B \), and thus \( \mathcal{H}^0(\mathbb{P}^2, J_\xi(a)) \cong \mathcal{H}^0(\mathbb{P}^2, J_\xi(a)) \). We have

\[
\zeta_L \in |al|,
\]
\[
[\zeta_L] = 2\xi \quad \text{in} \quad A_0(B),
\]
\[
k = kL \sim H^0(\mathbb{P}^2, J_{\zeta_L}(a)) \sim H^0(\mathbb{P}^2, J_\xi(a)),
\]
\[
H^0(\mathbb{P}^2, J_{\zeta_L}(a)) \cong H^0(B, \mathcal{O}_B(-\zeta_L)(a)) \cong H^0(B, \mathcal{O}_B)
\]

Now \( L \) is smooth at the points of \( \xi \) and is not strictly tangent to \( B \) at these points by Remark 4.27 and thus \( \zeta_L \) is given by a tangent vector to \( p \) at each point \( p \in \xi \), which follows from the description of Cartier divisors supported at nodes and cusps. The picture for the branch curve of a smooth cubic surface is drawn below.

Figure 5: Cartier divisor \( \zeta_L \)

Segre proves that this data is sufficient to reconstruct the surface \( S \):

Theorem 4.32 (Segre). A plane curve \( B \) of degree \( d = \nu(\nu - 1) \) is a branch curve of a smooth surface of degree \( \nu \) in \( \mathbb{P}^3 \) if and only if

1. \( B \) has \( n = \frac{1}{2} \nu(\nu - 1)(\nu - 2)(\nu - 3) \) nodes;
2. \( B \) has \( c = \nu(\nu - 1)(\nu - 2) \) cusps;
3. There are two curves, \( L \) of degree \( a = (\nu - 1)(\nu - 2) \) and \( L_1 \) of degree \( a + 1 \), which both contain the 0-cycle \( \xi \) of singularities of \( B \) and have separated tangents at the points of \( \xi \).

Proof. The necessity of these conditions was proved in the preceding sections. We now prove that they are sufficient.

Let \( B \) be such a curve in the plane \( \mathbb{P}^2 \). First, since \( L \) is adjoint to \( B \), the 0-cycle associated with the scheme-theoretic intersection \( L \cap B \) contains \( 2\xi = 2P + 2Q \), but by conditions of the theorem

\[
\deg B \cdot \deg L = 2 \deg \xi = \nu(\nu - 1)^2(\nu - 2)
\]
It follows that the 0-cycle associated with $L \cap B$ is 
\[ |L \cap B| = 2P + 2Q. \]

Let us denote $\xi = P + Q$. It follows immediately that $2\xi$ is in the linear system $|a\ell|$ on $B$, where $|\ell|$ is the linear system associated with the given plane embedding of $B$. In particular, we conclude that 
\[ \xi \in \frac{1}{2}a \cdot \ell \]

Note also that $|L_1 \cap B| = 2P + 2Q + R$, where $\deg R = d = \nu(\nu - 1)$.

Now the space $H^0(\mathbb{P}^2, J_\xi(a + 1))$ contains a 4-dimensional subspace of the form $kf_1 + kxf + kxf + kxf$, where $f_1$ is the equation of $L_1$ and $f$ is the equation of $L$. (Recall that $k$ is our base field.)

Now consider the linear system on $B$ given by restriction of $(f_1, x, y, w) = kL_1 \oplus H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes kL$. It has $\xi$ as a set of base points. It follows that it defines a rational map 
\[ \phi : B - \xi \to \mathbb{P}^3. \]

Let $\pi : B^* \to B$ be the normalization of $B$. We claim that the rational map $\phi$ can be lifted to give a regular map $\phi^* : B^* \to \mathbb{P}^3$. Indeed, we have the following lemma:

**Lemma (A).** Let $B$ be a plane nodal-cuspidal curve with the set of singularities $\xi$, and let $f \in H^0(B, J_\xi(j))$ and $f_1 \in H^0(B, J_\xi(j + 1))$ be non-zero elements determining adjoint curves $C = Z(f)$ and $C_1 = Z(f_1)$ on the plane, such that $T_pC \neq T_pC_1$ at any point $p \in \xi$.

Let 
\[ \Omega = kf_1 \oplus H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes kf = (f_1, x, y, w). \]

Then the rational map $\phi_\Omega : B \to \mathbb{P}^3$ can be resolved as 
\[
\begin{array}{ccc}
B^* & \longrightarrow & \mathbb{P}^3 \\
\downarrow & & \downarrow \text{pr} \\
B & \longrightarrow & \mathbb{P}^2
\end{array}
\]

where $\pi : B^* \to B$ is the normalization of $B$.

Note that $T_pC \neq T_pC_1$ implies that $f_1 \notin H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes kf$, and also that $\Omega \to T_pC$ is epimorphic at every point $p \in \xi$.

**Proof.** It is clear that we only have to verify the statement at nodes and cusps of $B$ as well as smooth points $p$ on $B$ such that $f_1(p) = f(p) = 0$.

For a node $p$ we can choose coordinates in the local ring of $\mathbb{P}^2$ at $p$ such that $B$ is given by the equation $xy = 0$.

Assume that $f_1$ is given by the equation $a_{1,0}x + a_{0,1}y + (\text{order 2 terms})$, and $f$ is given by the equation $b_{1,0}x + b_{0,1}y + (\text{order 2 terms})$. Note that $\phi_\Omega = (f_1, f, y, w) = (f_1/f, x, y, w)$. One can easily see that $\phi_\Omega$ maps the point $p$ on the branch $(y = 0)$ of $B$ to $a_{1,0}/b_{1,0}$, and the same point on the branch $(x = 0)$ to $a_{0,1}/b_{0,1}$. Thus, if $a_{1,0}b_{0,1} - a_{0,1}b_{1,0} \neq 0$, then $\phi_\Omega$ can be lifted to a regular map $B^* \to \mathbb{P}^3$ with a smooth image in the neighborhood of $p$.

In the same way, in a neighborhood of a cups $B$ can be given by the local equation $y^2 - x^3 = 0$, and thus 
\[ f_1/f = \frac{a_{1,0}x + a_{0,1}y + (\text{order 2})}{b_{1,0}x + b_{0,1}y + (\text{order 2})} = \frac{a_{1,0} + a_{0,1}t + (\text{order 2})}{b_{1,0} + b_{0,1}t + (\text{order 2})}, \]

where $t = y/x$ is the coordinate on the exceptional divisor in the resolution of the cusp. Now it is clear that if $a_{1,0}/b_{1,0} \neq a_{0,1}/b_{0,1}$, then $\phi_\Omega$ lifts to an embedding of the exceptional divisor and thus the normalization of the curve as well.
If now \( p \) is a smooth point of \( B \) such that \( f_1(p) = f(p) = 0 \), then it is a standard fact that the map \((B - p) \to \mathbb{P}^3\) can be uniquely extended to the map \( B \to \mathbb{P}^3 \) in a neighborhood of the point \( p \), since \( \mathbb{P}^3 \) is proper. (Note also that we do not have any such points in the application of this Lemma below, due to the intersection multiplicity computation for \( C_1 \) and \( C \).) \( \square \)

This gives a non-singular model \( C \subset \mathbb{P}^3 \), and a projection \( \pi : C \to B \) with some center \( O \). Note that if we start from a given ramification curve \( B^* \), the curve we reconstruct from \( B \) coincides with \( B^* \).

**Lemma (B).** If \( B \) is a branch curve of the generic projection \( \pi : S \to \mathbb{P}^2 \), where \( S \) is a smooth surface in \( \mathbb{P}(V) \cong \mathbb{P}^3 \), and \( B^* \) is the ramification curve of \( \pi \), then there is an isomorphism \( \mathbb{P}(V) \to \mathbb{P}(kL_1 \oplus H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes kL) \) which takes \( B^* \subset \mathbb{P}(V) \to C \). In other words, the linear system \((f_1, xf, yf, wf)\) reconstructs the curve \( B^* \).

The idea of the proof is, as in the previous lemma, to set \( z = f_1/f \) on \( B^* \).

Recall that preimages of the nodes of \( B \) belong to the bisecant lines to \( B^* \) containing the point \( O \), and preimages of cusps belong to the tangent lines to \( B^* \) containing the point \( O \). Considering tangent lines to \( B^* \) as a limiting case of bisecants to \( B^* \), we see that \( B^* \) has

\[
n + c = \frac{1}{2}(\nu - 1)^2(\nu - 2)
\]

of bisecants (and tangents) containing the point \( O \), which belong to a cone of order \((\nu - 1)(\nu - 2)\) above \( L \) with vertex \( O \).

**Lemma (C).** \( B^* \) does not belong to a surface of degree \( m < \nu - 1 \).

**Proof.** Assume that \( S_1 \) is such a surface of degree \( m \); we can assume that it is irreducible. Consider \( S_1' = \text{Pol}_O(S_1) \). First, if \( S_1 \) is smooth, note that \( S_1' \) contains the preimage of the 0-cycle of cusps \( Q^* \), since at each point \( q \in Q^* \), the tangent line \( l \) to \( B^* \) is contained in \( T_qS_1 \), and also \( l \) contains \( O \), since \( q \) projects to a cusp of \( B \). It follows that \( q \in S_1 \cap S_1' \). Secondly, if \( S_1 \) is not smooth, then \( S_1' \) still contains \( q \).

However, then it follows that the number of cusps \( c \leq \nu(\nu - 1) \cdot (m - 1) \), which contradicts to assumption that \( c = \nu(\nu - 1)(\nu - 2) \). \( \square \)

We now have to prove that the model \( B^* \) we constructed is a complete intersection of a surface \( S \) of degree \( \nu \) and its polar \( \text{Pol}_O(S) \) of degree \( \nu - 1 \) with respect to the (fixed) point \( O \) which is the center of the projection \( \pi : B^* \to B \). For these, following Segre, we apply the following theorem belonging to Halphen (See [3, pg. 359]):

**Theorem (Halphen).** Let \( C \) be a space curve of order \( a \cdot b \) in \( \mathbb{P}^3 \) s.t. \( a < b \) which has \( \frac{1}{2}a(a-1)b(b-1) \) bisecants all lying on a cone of degree \((a - 1) \cdot (b - 1) \). Assume also that \( C \) is not on a surface of degree smaller than \( a \). Then \( C \) is a complete intersection of two surfaces of degree \( a \) and \( b \).

The inverse statement to the Halphen’s theorem is easy; see [1] art. 343 or [15, Chapter IX, sections 1.1, 1.2].

Alternatively, instead of invoking Halphen’s theorem, one can invoke a theory of Gruson and Peskine, as it is done by D’Almeida in [11]; we cite his reasoning for the convenience of the reader:

**Lemma (D).** [11, pg. 231] The curve \( B^* \) constructed above is a complete intersection of two surfaces of degrees \( \nu \) and \( \nu - 1 \).

**Proof.** To prove the lemma, we introduce first the following definition:

**Definition 4.33.** Given a space curve \( C \), we define its index of speciality as

\[
s(C) = \max\{n : h^1(C, \mathcal{O}_C(n)) \neq 0\}.
\]
Now we state the following Speciality Theorem of Gruson and Peskine \cite{27}:

Let $C$ be an integral curve in $\mathbb{P}^3$ of degree $d$, not contained in a surface of degree less than $t$. Let $s = s(C)$. Then $s \leq t + \frac{4}{t} - 4$, with equality holding if and only if $C$ is a complete intersection of type $(t, \frac{4}{t})$ (and thus $\mathcal{O}_C(s)$ is special, i.e., $h^1(\mathcal{O}_C(s)) \neq 0$).

Let now $p : B^* \rightarrow B$ be the projection from the point $O$. The conductor of the structure sheaf $\mathcal{O}_{B^*}$ in $\mathcal{O}_B$ is $Ann(p_*\mathcal{O}_{B^*}/\mathcal{O}_B)$, which by duality is isomorphic to $Ann(\omega_B/p_*\omega_{B^*})$ (see e.g. \cite{19} Chapter 8]). By the definition of the conductor, we get that $Ann(\omega_B/p_*\omega_{B^*}) = \text{Hom}(\omega_B, p_*(\omega_{B^*})) = p_*(\omega_{B^*}) \otimes \omega_B$. It is well known that for a nodal-cuspidal curve, $H$ is a global section of the conductor sheaf iff $H$ passes through the nodes and the cusps of the curve (see e.g. \cite{32} Proposition 3.1).

By Serre duality, for all $i$, $H^1(\mathcal{O}_{B^*}(i)) = H^0(\omega_{B^*}(-i))$. Thus, the minimal degree of the curve containing the singular points of $B$ is

$$\nu(\nu - 1) - 3 - s(B^*).$$

Indeed, for a curve to pass through the singular points of $B$, the conductor has to have sections, i.e. $p_*(\omega_{B^*}) \otimes (\omega_B)^{-1}$ has sections. Since we know that the minimal degree of the curve containing the singular points of $B$ is $(\nu - 1)(\nu - 2)$, we get $s(B^*) = 2\nu - 5$.

As $B^*$ does not lie on any surface of degree $\nu - 2$ (by Lemma (C)), then the Speciality Theorem shows that $B^*$ is a complete intersection of two surfaces of degrees $\nu$ and $\nu - 1$ (taking $t = \nu - 2, d = \nu(\nu - 1)$).

Either way, by results of Halphen or Gruson-Peskine, the curve $B^*$ is a complete intersection of two surfaces, say, $S^\nu$ and $F^{\nu-1}$ of degrees $\nu$ and $\nu - 1$.

We still have to prove that $B^*$ can be written as an intersection of a surface of degree $\nu$ and its polar with respect to the given point $O$.

Let $W = H^0(\mathbb{P}^3, J_{B^*}(\nu))$ be the linear system of surfaces of degree $\nu$ containing $B^*$,

$$W = kS \oplus (H^0(\mathbb{P}^3, \mathcal{O}(1)) \otimes kF),$$

as for any complete intersection of type $(\nu, \nu - 1)$. For a point $t \in \mathbb{P}W$, let $S_t$ be the corresponding surface of degree $\nu$ containing $B^*$. (here we also denoted by $S$ and $F$ some particular equations for the surfaces $S$ and $F$, even though they are defined only up to $G_m$ action).

Consider now the linear map

$$\partial_O : W = H^0(\mathbb{P}^3, J_{B^*}(\nu)) \rightarrow H^0(\mathbb{P}^3, \mathcal{O}(\nu - 1)),$$

which maps $f$ to $\text{Pol}_O f = \sum O_i \partial_i f$, its polar with respect to the fixed point $O$. We claim that $\partial_0$ is injective. Indeed, if $\partial_0(f) = 0$, then $f$ vanishes on a cone of degree $\nu$, containing the curve $B^*$. Note that $F^{\nu-1}$ vanishes on $B^*$ but also gives a degree $\nu - 1$ form on every line generator of the cone ($f = 0$), which implies that the projection map $B^* \rightarrow B$ has degree $\nu - 1$, which is not the case.

Now, for every $t \in \mathbb{P}(W)$ and the corresponding surface $S_t$ of degree $\nu$, consider the triple intersection

$$\eta_t = S_t \cap F^{\nu-1} \cap \text{Pol}_O S_t$$

First, we have $S_t \cap F = B^*$. Let $R_t = S_t \cap \text{Pol}_O S_t$. $R_t$ is a ramification curve for the surface $S_t$ with respect to the projection with the given center $O$. We have $\eta_t = B^* \cap R_t$.

Note that $Q^* \subset R_t$ for every $t$, since $B^*$ belongs to $S_t$ and has all tangent lines at the points of $Q^*$ contain the projection center $O$. 

Thus for every $t$ either the polar surface $\text{Pol}_O S_t$ contains the curve $B^*$, or we have a decomposition of 0-cycles on $B^*$ of the form

$$\eta_t = Q^* + r_t$$

Also note that $Q^* \subseteq B^* \cap \text{Pol}_O F$ by the same geometric argument, i.e., since at the points of $Q^*$ the tangent lines to $B^*$ contain the projection center $O$, these points are on the intersection of $B^*$ with $O$-polar of every surface containing $B^*$. But since these two 0-cycles have the same degree, they coincide. It follows that $Q^* \in |(\nu - 2)h|$ on the curve $B^*$, where $h$ is a class of hyperplane section.

Now, since $\eta_t \in |\partial_0 S_t| = |(\nu - 1)h|$ on $B^*$, we have $r_t \in |h|$ on $B^*$ whenever $\text{Pol}_O S_t$ intersects non-trivially with the curve $B^*$, i.e., does not contain it.

Since $B^*$ is complete intersection, it is linearly normal (which follows easily from the cosideration of Koszul complex). It follows that $r_t$ gives a map

$$W \to H^0(B^*, O(1))$$

from the 5-dimensional space $W$ to the 4-dimensional vector space $H^0(B^*, O(1)) \simeq H^0(\mathbb{P}^3, O(1))$.

Such a map must have a kernel, and let $S_0$ be the corresponding surface in the linear system $|W|$. It follows that $\text{Pol}(O, S_0)$ contains the curve $B^*$, and thus $B^* = S_0 \cap \text{Pol}(O, S_0)$, i.e., $B^*$ is a ramification curve for the projection of the surface $S_0$ to $\mathbb{P}^2$ with the given center $O$. This finishes the proof.

\[\square\]

Remark 4.34. We generalize Segre’s theory for smooth surfaces in $\mathbb{P}^N$, $N > 3$, in the subsequent paper [67].

Let us notice that the 0-cycle of singularities of the branch curve $B$ is special. We would like to emphasize this in the next subsection.

4.5. Special 0-cycles. Let $\xi$ be a 0-cycle in $\mathbb{P}^2$. Define the superabundance of $\xi$ (relative to degree $n$ curves) as:

$$\delta(\xi, n) = h^1 J_\xi(n)$$

We have the following

Lemma 4.35. If $\deg \xi \leq \dim |nh|$, then

$$\dim |nh - \xi| = (\dim |nh| - \deg \xi) + \delta(\xi, n),$$

in other words, $\delta(\xi, n)$ is the speciality index of the 0-cycle $\xi$ with respect to the linear system $|nh|$.

Also note that

$$\delta(\xi, n + 1) \leq \delta(\xi, n).$$

Let now $\xi = P + Q$ - the zero cycle of singularities of $B$, and, as before, $a = (\nu - 1)(\nu - 2)$.

Proposition 4.36. (Speciality index of $\xi$) There are following identities for the speciality index of $\xi$:

$$\delta(\xi, a) = \frac{1}{2}(\nu - 1)(\nu - 2)(2\nu - 5)$$

$$\delta(\xi, a + 1) = \frac{1}{2}(\nu - 3)(2\nu^2 - 7\nu + 4)$$

In particular, the 0-cycle $\xi$ is special with respect to $|ah|$ for all surfaces of degree at least 3, and special with respect to $|(a + 1)h|$ for all surfaces of degree at least 4.
Proof. For the expected dimension \( \text{vdim} |J_\xi(a)| \) we have

\[
\text{vdim} |J_\xi(a)| = \dim |ah| - \deg \xi = \frac{1}{2}a(a + 3) - \frac{1}{2}a(\nu - 1)^2(\nu - 2)
\]

Since \( a = (\nu - 1)(\nu - 2) \), we get

\[
\text{vdim} |J_\xi(a)| = \frac{1}{2}(\nu - 1)(\nu - 2)(5 - 2\nu)
\]

Since, by definition of speciality index,

\[
\dim |J_\xi(d)| = \text{vdim} |J_\xi(d)| + \delta(\xi, d)
\]

and since \( |J_\xi(a)| = \{L\} \), we get the first equality.

The proof of the second formula is parallel; we use isomorphism \( |J_\xi(a + 1)| \simeq \mathbb{P}^3(\mathbb{P}^3, \mathcal{O}(1)) \) (see Remark [123]).

Example 4.37 (6-cuspidal sextic). Let \( \xi_6 \) be a 0-cycle of degree 6 on a plane which is an intersection of conic and cubic curves in \( \mathbb{P}^2 \), given by a degree 2 (resp. 3) polynomial \( f_2 \) (resp. \( f_3 \)). Note that generic 0-cycle of degree 6 is not like this, because generic 6 points do not belong to a conic. Note that for \( \xi_6 \) given by \((f_2, f_3)\) there is a Koszul resolution

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(n - 5) \xrightarrow{f_3} \mathcal{O}_{\mathbb{P}^2}(n - 2) \oplus \mathcal{O}_{\mathbb{P}^2}(n - 3) \xrightarrow{[-f_2, f_3]} \mathcal{O}_{\mathbb{P}^2}(n) \to 0
\]

An easy computation shows that \( \delta(\xi_6, 2) = 1, \delta(\xi_6, 3) = 0, \delta(\xi_6, 4) = 0 \) (see Subsection [116] for the definition of \( \delta(\cdot, \cdot) \)), and that \( H^0J_{\xi_6}(2) = kf_2, \ H^0J_{\xi_6}(3) = kf_3 + kxf_2 + kxf_2 + kxf_2 \). Note that we start the computation from \( n = 2 \), since \( \delta(\xi, 1) = 3 \) is not a defect w.r.t. the linear system. Also note that For a generic 0-cycle \( \xi \) of degree 6, \( \delta(\xi, 2) = 0 \), otherwise it would lie on conic.

4.6. Dimension of \( B(d, c, n) \). In this subsection, let \( d(\nu) = \nu(\nu - 1), c(\nu) = \nu(\nu - 1)(\nu - 2), n(\nu) = \frac{1}{4}\nu(\nu - 1)(\nu - 2)(\nu - 3) \). Motivated by Segre’s theory and the Chisini conjecture, We want to compute the dimension of the component \( B_3(\nu) \) of \( B(d(\nu), c(\nu), n(\nu)) \) which consists of branch curves of smooth surfaces in \( \mathbb{P}^3 \) of degree \( \nu \) with respect to generic projection.

Let \( S(\nu) \) be the variety parameterizes smooth surfaces in \( \mathbb{P}^3 \) of degree \( \nu \). It is well known that \( \dim S(\nu) = \frac{1}{4}(\nu + 1)(\nu + 2)(\nu + 3) - 1 \). Let \( B \in B_3(\nu) \), a branch curve in the plane \( \Pi \) of a smooth surface \( S \) in \( \mathbb{P}^3 \) of degree \( \nu \), when projected from the point \( O = (0 : 0 : 0 : 1) \) (we work with the coordinates \( x : y : w : z \)). Now, \( B \) is also the branch curve of a smooth surface \( S' \) iff there is a linear transformation in \( PGL_4(\mathbb{C}) \) that fixes that point \( O \), fixes the plane \( \Pi \) (with coordinates \( x : y : w \)) and takes \( S \) to \( S' \). It is easy to see that the dimension of this subgroup of transformations \( G \) is 5 (in \( GL_4(\mathbb{C}) \)), but as we are in a projective space, \( \dim P(G) = 4 \).

By the Chisini’s conjecture (proven completely for a generic projection, see [63]), the branch curve \( B \) determines the surface uniquely up to an action of \( P(G) \). Thus,

\[
\text{dim } S(\nu) - 4 = \dim B_3(\nu).
\]

Denote by \( V(\nu) = \frac{1}{4}d(\nu)(d(\nu) + 3) - n(\nu) - 2c(\nu) \) the virtual dimension of a family of degree \( d(\nu) \) curves with \( n(\nu) \) nodes and \( c(\nu) \) cusps.

Example 4.38. (1) For \( \nu = 3, 4 \), \( \dim B_3(\nu) = S(\nu) - 4 = V(\nu) \), as expected (as for these branch curves, \( c(\nu) < 3d(\nu) \). See [22, p. 219]).

(2) For \( \nu \geq 5, \dim B_3(\nu) = S(\nu) - 4 > V(\nu) \). This gives examples of nodal cuspidal curves, whose characteristic linear series is incomplete (for other examples see e.g. Wahl [24]).
4.7. Projecting surfaces with ordinary singularities. We bring here a short subsection on surfaces in \( \mathbb{P}^3 \) with ordinary singularities, as we use it in the next section, where we classify branch curves of small degree. The generalization of Segre’s theory for these surfaces will be presented in \[67\].

It is classical that (see e.g. \[28\]) any projective surface in characteristics 0 can be embedded in \( \mathbb{P}^3 \) in such a way that its image has at most so-called ordinary singularities, i.e., a double curve with some triple and pinch points on it. Any projection \( S \subset \mathbb{P}^n \to \mathbb{P}^2 \) can be factorized then as a composition of projections \( S \subset \mathbb{P}^n \to \mathbb{P}^3 \to \mathbb{P}^2 \) such that the image \( S_1 \) of \( S \) in \( \mathbb{P}^3 \) has ordinary singularities. However, if we project \( S \) to \( \mathbb{P}^3 \) first, and then from \( \mathbb{P}^3 \) to \( \mathbb{P}^2 \), we get an extra component of the branch curve, which would be the image of the double curve.

Assume now that we are given a degree \( \nu \) surface \( S \subset \mathbb{P}^3 = \mathbb{P}(V) \) with ordinary singularities and a point \( O \) not on \( S \). Let \( E^* \) be the double curve of \( S \). Consider the projection map \( \pi : S \to \mathbb{P}(V/l_O) \cong \mathbb{P}^2 \). We define the ramification curve \( B^* \) of the projection as an intersection of \( S \) and the polar surface \( S'_O \). (To justify this definition, one can check that \( S \cap S'_O \) is the support of the sheaf \( \Omega^1_{S/\mathbb{P}^2} \).)

One can now see that \( B^* \) can be decomposed as

\[
B^* = B^*_{\text{res}} + F^*,
\]

where \([E^*] = 2[E^*] \) when \([F^*] \) is the Weil divisor associated with the 1-dimensional Cartier divisor \( 2[E^*] \). Note that \( B^*_{\text{res}} \) in its intersection with the smooth locus of \( S \) is set-theoretically the set of smooth points \( p \) on \( S \) such that the tangent plane \( T_p(S) \) contains \( O \). (To be more careful, \( B^*_{\text{res}} \) is the scheme-theoretical support of the kernel sheaf of the canonical map \( \Omega^1_{S/\mathbb{P}^2} \to i^*i^*\Omega^1_{S/\mathbb{P}^2} \to 0 \), where \( i \) is the embedding of \( F^* \) to \( S \). For a different scheme-theoretic description of \( E^* \) and \( B^*_{\text{res}} \), see \[26\] Section 2).

It follows that the branch curve \( B \) can also be decomposed as

\[
B = B_{\text{res}} + 2E,
\]

where \( E \) is the image of \( E^* \).

Let \( e = \deg E^* \) and \( d = \deg B_{\text{res}} = \nu(\nu - 1) - 2e \). Now a generic hyperplane section of \( S, S \cap H \), is a plane curve of degree \( \nu \) with nodes at the finite set \( E^* \cap H \), and thus there is a restriction

\[
0 \leq e \leq \frac{(\nu - 1)(\nu - 2)}{2},
\]

since the number of nodes of a plane curve can not exceed its arithmetic genus.

It follows that the pair \((\nu, d)\) satisfies

\[
2(\nu - 1) \leq d \leq \nu(\nu - 1),
\]

as illustrated on Figure 6 below.
What is important here is that for a given \( d \) there is only a finite number of possible \( \nu \)'s such that a plane curve \( C \) of degree \( d \) can be a pure branch curve of degree \( \nu \) surface in \( \mathbb{P}^3 \) with ordinary singularities.

As before, we define \( Q^* \) to be an intersection of \( B^* \) and the second polar surface \( S''_O \), i.e., as an intersection of \( S, S'_O \) and \( S''_O \). However, for a singular surface \( S \) not all points of \( Q^* \) form cusps on the branch curve. This is shown, for example, at [15, Chapter IX, section 3.1].

**Notation 4.39.** Denote by \( v^* \in E^* \) a point, such that the tangent plane to \( S \) at \( v^* \) contains the center of projection \( O \). These points are called *vertical points* (or points of immersion) and we denote the set of such points as \( V^* \).

Denote by \( T^* \) the set of triple points of \( E^* \), and by \( t \) the number of these points. Let also \( P^* \) be the set of pinch points of \( E^* \) and let \( p \) be the number of these points.

**Remark 4.40.** Note that the number of pinch points \( p \) is always positive (see [30]). We will use this fact to prove the inexistence of branch curves in \( V(8, 12, 0) \) in Section 5.

The following Lemma is proved at [15] Chapter IX, sections 3.1, 3.2]. This Lemma is the base for generalizing Segre’s theory for singular surfaces, a generalization which will be presented in [67].

**Lemma 4.41.** (1) \( Q^* = S'^O \cap B^* \) can be decomposed as

\[
Q^* = (S'^O \cap 2E^*) + Q^*_{res}
\]

Note that the images of \( (S'^O \cap E^*) \) under the projection are smooth points on \( B_{res} \).

(2) points in \( B^*_{res} \cap E^* \) do not form cusps of the branch curve, i.e., their images are smooth points on \( B_{res} \). Explicitly,

\[
B^*_{res} \cap E^* = P^* + V^*. 
\]

(3) \( S''_O \cap E^* \) can be decomposed as

\[
S''_O \cap E^* = V^* + 3T^* 
\]

and \( S''_O \cap B^*_res \) can be decomposed as

\[
S''_O \cap B^*_res = V^* + Q^*_{res}
\]
Remark 4.42. Denote by \( e^* \) the degree of \( E^\nu \) the dual curve of \( E \) in \( \mathbb{P}^2 \). Given a surface \( S \) in \( \mathbb{P}^3 \), we can express the number of nodes and cusps of its branch curve \( B_{\text{res}} \) by terms of \( \nu, e, e^* \) and \( t \). The following result is proved at [15] Chapter IX, section 3:
\[
\begin{align*}
  c &= \nu(\nu - 1)(\nu - 2) - 3e(\nu - 2) + 3t, \\
  n &= \frac{1}{2} \nu(\nu - 1)(\nu - 2)(\nu - 3) - 2e(\nu - 2)(\nu - 3) - 2e^* - 12t + 2e(e - 1).
\end{align*}
\]

Remark 4.43. Let \( u \) be the number of components of \( E^* \), and \( g = \sum_{i=1}^{u} g_i \) the geometric genus of \( E^* \). By [28], pp. 624, 628 we can express \( c_1^2, c_2 \) and the number of pinch points \( p \) by terms of \( \nu, e, t \) and \( (g - u) \):
\[
\begin{align*}
  c_1^2 &= \nu(\nu - 4)^2 - 5\nu e + 24e + 4(g - u) + 9t, \\
  c_2 &= \nu^2(\nu - 4) + 6\nu + 24e - 7\nu e + 8(g - u) + 15t, \\
  p &= 2e(\nu - 4) - 4(g - u) - 6t.
\end{align*}
\]

4.7.1. Examples. We survey the well known examples of surfaces of degree 3 and 4 in \( \mathbb{P}^3 \) with ordinary singularities and use the results from Remarks 3.12 and 4.42 in order to calculate the number of nodes and cusps of the branch curve \( B_{\text{res}} \) of the surface \( S \). These numbers can be expressed in terms of \( c_1^2(S), c_2(S), \deg(S) \) and \( \deg(B_{\text{res}}) \) or in terms of \( \nu, e, e^* \) and \( t \).

Degree 3 surfaces

We know from the inequality above that \( 0 \leq \deg E^* = e \leq 1 \), in other words, the only cubic surfaces with ordinary singularities are those with double line.

(1) \( e = 0 \). This is a smooth cubic surface, with the branch curve \( B \) being a 6-cuspidal sextic.

(2) \( e = 1 \).

Such a surface has a double line, and thus \( d = \deg B_{\text{res}} = 4 \). Since we consider only generic projections, we can choose coordinates \((x, y, w, z)\) in \( \mathbb{P}^3 \) in such a way that the projection center \( O = (0, 0, 0, 1) \) and the double line \( E^* = l^* \) is given by equations \((z = w = 0)\). In these coordinates the projection is given by the (rational) map \((x, y, w, z) \mapsto (x, y, w)\), and \( E = l \) is the “line at infinity” \((w = 0)\) in the “horizontal” plane \((z = 0)\).

It is easy to see that such a cubic surface can be given by a degree 3 form
\[
f = z^3 + a_1 z^2 + b_1 wz + c_1 w^2,
\]
where \((a_1, b_1, c_1)\) are homogeneous forms in \((x, y)\) of degree 1.

One can see from the definition of the normal cone ([28]) that the normal cone to \( l^* \) in \( S \) is given by the degree 2 part of \( f \) in \((z, w)\), i.e., by the form
\[
[f]_2 = a_1 z^2 + b_1 wz + c_1 w^2
\]

We can consider \([f]_2\) as a section of \( \mathcal{O}(1, 2) \) on the ruled surface
\[
\mathbb{P} N_{l^*/\mathbb{P}3} \simeq l^* \times \mathbb{P}^1.
\]

Note that \([f]_2\), being a quadratic form of the variables \((z, w)\) with coefficients in \( k[x, y] \), degenerates in the zeroes of its discriminant \( \Delta([f]_2) = b_1^2 - 4a_1 c_1 \). It follows that there are 2 points \( p_1 \) and \( p_2 \) on \( l^* \) where this quadratic form degenerates into a double line, which proves that a cubic surface with a double line has 2 pinch points.

Note also that \( B_{\text{res}} \cap l^* \) consists of such points \( p \) on \( l^* \) such that one of the normal lines to \( l^* \) in \( S \) at \( p \) is the “vertical” line (one that contains the point \( O \)): it is the only point of immersion. In the normal plane to the line \( l^* \) with coordinates \((z, w)\) this vertical line is given by the equation \((w = 0)\). It follows that such points \( p \) are exactly those where \( a_1 \)}
vanishes. This gives just one point \( p_0 \), different from the two pinch points \( p_1 \) and \( p_2 \) defined above, and a decomposition

\[
S''_O \cap B^*_\text{res} = p_0 + Q^*_\text{res}
\]

We have \( \deg(S''_O \cap B^*_\text{res}) = \deg B^*_\text{res} = 6 - 2 = 4 \), and thus \( \deg Q^*_\text{res} = 3 \). It follows that the pure branch curve \( B^*_\text{res} \) has 3 cusps. Note also that \( B^*_\text{res} \) has no nodes, since a plane quartic with 3 cusps is rational and can not have any other singularities; i.e., we obtain a point \([B^*_\text{res}]\) in \( B(4,3,0) \).

**Degree 4 surfaces**

We should have \( 0 \leq e = \deg E^* \leq 3 \).

1. \( e = 0 \). This is the case of a smooth quartic surface with degree 12 branch curve, which belongs to \( B(12,24,12) \).

2. \( e = 1 \). Let \( S \) be a quartic surface with a double line \( l^* \). We have

\[
B^* = 2l^* + B^*_\text{res},
\]

where \( d = \deg B^*_\text{res} = 4 \cdot 3 - 2 = 10 \).

Arguing as above we can see that the normal cone to \( l^* \) in \( S \) can be given by the equation

\[
a_2z^2 + b_2wz + c_2w^2 = 0
\]

for some homogeneous forms \((a_2, b_2, c_2)\) of degree 2 of variables \((x, y)\). It follows that \( S \) has 4 pinch points on the line \( l^* \), and the intersection of \( l^* \), \( B^*_\text{res} \) and \( S''_O \) consists of two (different) points \( p_1, p_2 \) which are the points of immersion. It follows that

\[
S''_O \cap B^*_\text{res} = p_1 + p_2 + Q^*_\text{res},
\]

where \( \deg Q^*_\text{res} = \deg(S''_O \cap B^*_\text{res}) - 2 = 18 \).

It is known that a quartic surface with a double line is the image of \( \mathbb{P}^2 \) blown up at 9 points (see [23, pg. 632]). Computing its Chern invariants \( c_2^2 \) and \( c_2 \) and using the formulas from Remark 3.12 one can check that the number of cusps is indeed 18, and the number of nodes is 8. Thus \([B^*_\text{res}] \in B(10,18,8)\). Alternatively, since \( e^* = t = 0 \), by remark 4.42 we find out that indeed \([B^*_\text{res}] \in B(10,18,8)\).

**Remark 4.44.** It is easy to see from the above the classical fact that for a singular surface of degree \( \nu \) in \( \mathbb{P}^3 \) with a double line, the number of pinch points is \( p = 2(\nu - 2) \) and the number of the vertical points is \( \nu - 2 \).

3. \( e = 2 \). In this case we have \( \deg E^* = 2 \). A curve of degree 2 in \( \mathbb{P}^3 \) is either a smooth conic contained in a plane, or a union of two skew lines, or a union of two intersecting lines, or a double line. By definition, the last two curves can not be double curves of a surface with ordinary singularities. Both of the two remaining cases are actually realized, as explained, for example, in [23].

If the double curve is a smooth conic, then it is classical that the surface \( S \) is a projection to \( \mathbb{P}^3 \) of the intersection of two quadrics in \( \mathbb{P}^4 \), (cf. [23]), and one can check (using remark 3.12 or 4.42) that the branch curve is in \( B(8,12,4) \).

If the double curve is a union of 2 skew lines, then it is known that \( S \) is a ruled surface over elliptic curve (cf, say, [23], who deduces it from the classification of surfaces with \( q = 1 \)).

From this classification one can conclude now that \( c_2(S) = c_1(S) = 0 \), and, using the same formulas as before, that the branch curve \( B^*_\text{res} \) gives a point in \( B(8,12,8) \).

4. \( e = 3 \). A double curve \( E^* \) of a surface \( S \) with ordinary singularities is either smooth, or has some triple points. Thus \( E^* \) can be either (a) a rational space cubic, or (b) a non-singular plane cubic, or (c) a union of a conic and a non-intersecting line, or (d) a union of 3 skew
lines, or (e) union of 3 lines intersecting in a point. It is explained, for example, in [28], that only cases (a) and (e) are realized.

In the case (a) the surface \( S \) is the projection of \( \mathbb{P}^1 \times \mathbb{P}^1 \) embedded with the linear system \( |\ell_1 + 2\ell_2| \) to \( \mathbb{P}^5 \), and the branch curve is in \( B(6,6,4) \). We discuss this case in details in Subsection 5.3.

In the case (e) the surface \( S \) is the projection of the 2-Veronese-embedded \( \mathbb{P}^2 \) in \( \mathbb{P}^5 \), and the branch curve is in \( B(6,9,0) \); see the discussion in Subsection 5.

5. Classification of singular branch curves in small degrees

For a smooth curve of even degree \( d \) defined by the equation \( \{ f_B = 0 \} \), let \( \pi : S \to \mathbb{P}^2 \) a degree \( \nu \) cover ramified over \( B \) which is generic in sense of Subsection 2.3. We have that \( d \) is even (see Remark 3.8). By Zariski-Van Kampen theorem \([11]\), \( \pi_1(\mathbb{P}^2 - B) \simeq \mathbb{Z}/d\mathbb{Z} \) which is abelian. Since the monodromy representation of a generic cover into the symmetric group should be epimorphic (see the proof of Lemma 3.4), we conclude that \( \pi \) is of degree 2, i.e., isomorphic to the double cover given by \( z^2 = f_B \) in the total space of the line bundle \( O_{\mathbb{P}^2}(d/2) \).

**Remark 5.1.** As was stated in subsection 2.3 we study generic linear projections \( p : \mathbb{P}^N = \mathbb{P}(V) \to \mathbb{P}(W) \) (where \( W \subset V \) be a codimension 3 linear subspace) where \( \mathbb{P}(W) \cap S = \emptyset \). Explicitly, if \( S \) is a surface in \( \mathbb{P}^3 \), then the projection is from a point \( O \notin S \). However, see remark 5.7.

All possible non-smooth branch curves of degrees 4 and 6 are known and we list them in the next paragraphs. For each case we give examples (and sometimes complete classification) of coverings with a given branch curve. We then give all the numerical possible singular degree 8 branch curves (see Theorem 5.5).

We denote \( \langle a, b \rangle = (aba)(bab)^{-1}, \ a^b = bab^{-1} \), and for the rest of this section we will use the coordinates \((d, c, n)\) in the variety of the nodal–cuspidal curves.

Degree 4 singular branch curves

There is only one branch curve of degree 4, as the following has to be satisfied: \( 4|n, 3|c, \) and the geometric genus \( g(B) = (d - 1)(d - 2)/2 - n - c \geq 0 \). It follows that the only possibility is \( (c = 3, n = 0) \). This unique curve is the famous complexification of the classical deltoid curve, which is a cycloid with 3 cusps, i.e., the trace of a point on a circle of radius \( 1/3 \) rotating within a circle of radius 1. It is not hard to show that all other curves in \( V(4,3,0) \) are obtained from the deltoid by linear transformation, since the dual curve belongs to \( V(3,0,1) \), which is an irreducible space.

Zariski computed the braid monodromy for a deltoid using elliptic curves \([6] \) and proved that \( \pi_1(\mathbb{P}^2 - B) \) is isomorphic to the group with presentation

\[
\left\{ a, b : \langle a, b \rangle = 1, a^2 b^2 = 1 \right\},
\]

where the notation \( \langle a, b \rangle \) was introduced above. This is the dicyclic group of order 12. The monodromy representation is the obvious one: \( a \mapsto (1,2), b \mapsto (2,3) \).

Zariski \([6] \) noted that the discriminant of a cubic surface \( S \) in \( \mathbb{P}^3 \) with a double line is a plane curve of degree 6 which is a union of double line (the image of the double line of \( S \)) and a quartic curve (which is straightforward), and moreover proved that the residual quartic has 3 cusps. Thus the variety \( B(4,3,0) \) is not empty and thus \( B(4,3,0) = V(4,3,0) \).
Degree 6 singular branch curves

(i) The cases \((c = 0, n > 0)\) and \(c = 3\) are not realized.

For \(c = 0\), \(\pi_1(\mathbb{P}^2 - B)\) is abelian (by Remark 3.3), and there are no generic covers ramified over \(C\), as we argued in Lemma 3.4. In the second case, \(c = 3\), Nori’s result we cited (see Equation (14)) implies that the group \(\pi_1(\mathbb{P}^2 - B)\) is also abelian.

It follows that there are no branch curves with these \((d, c, n)\) triples, even though the corresponding varieties \(V(6, 0, n)\) and \(V(6, 3, n)\) are not empty.

(ii) \((c = 6, n = 0)\): This case was studied by Zariski, as we discussed in the introduction to Section 4.1. If \(S\) is a smooth cubic surface in \(\mathbb{P}^3\), then the branch curve \(B\) of a generic projection of \(S\) to \(\mathbb{P}^2\) is in \(V(6, 6, 0)\), and Segre’s result we discussed (or a direct computation) shows that these 6 cusps lie on a conic. See subsection 4.1.1 and Corollary 4.5.

Zariski proved the inverse statement: \(C \in V(6, 6, 0)\) is a branch curve if and only if 6 cusps of \(C\) lie on a conic. It follows from Zariski’s work that branch curves form one of the connected components of \(V(6, 6, 0)\); and, moreover, Zariski proved the existence of other connected components. Degtyarev proved in [61] that \(V(6, 6, 0)\) has exactly two irreducible components.

Zariski also proved [6] that for \(B \in B(6, 6, 0)\) the group \(\pi_1(\mathbb{P}^2 - B)\) is isomorphic to \(\mathbb{Z}/2 \times \mathbb{Z}/3\), whereas for \(C \in V(6, 6, 0) \setminus B(6, 6, 0)\) the group \(\pi_1(\mathbb{P}^2 - C)\) is isomorphic to \(\mathbb{Z}/2 \oplus \mathbb{Z}/3\).

Remark 5.2. As a generalization of the above result, Moishezon [33] proved that the fundamental group of the complement of \(B\) in \(\mathbb{P}^2\) is isomorphic to the quotient \(\text{Braid}_n/\text{Center}(\text{Braid}_n)\) of the braid group \(\text{Braid}_n\) by its center.

(iii) \((c = 6, n = 4)\). Consider the surface \(S = \mathbb{P}^1 \times \mathbb{P}^1\) embedded to \(\mathbb{P}^5\) by linear system \(|\ell_1 + 2\ell_2|\). Then \(S\) is of degree 4 in \(\mathbb{P}^5\), and the image of its generic projection to \(\mathbb{P}^3\) is a quartic with a rational normal curve (the twisted cubic) as its double curve (see [28], pg. 631). The branch curve \(B\) of \(S\) is in \(B(6, 6, 4)\) as can be seen from Remark 3.12 or from Remark 4.42 and it is known [14] that the fundamental group \(\pi_1(\mathbb{P}^2 - B)\) is braid group of the sphere with 3 generators (see [58] for an explicit calculation). Note that \(V(6, 6, 4)\) is irreducible since it is dual to \(V(4, 0, 3)\).

(iv) \((c = 9, n = 0)\). First, we describe the variety \(V = V(6, 9, 0)\). For a curve \(B \in V\), its dual is a smooth plane cubic; this gives an isomorphism of \(V\) and an open subset in the linear system of plane cubics \([3h]\) consisting of smooth curves. It follows immediately that \(V\) is irreducible.

In this case \(B(6, 9, 0) = V(6, 9, 0)\): there is a direct classical construction of a cover with a given branch curve \(C \in V(6, 9, 0)\) from the dual smooth cubic, discussed in Remark 5.4. Moreover, every curve in \(V(6, 9, 0)\) is a branch curve of exactly four different ramified coverings, the construction of which was given by Chisini ([14]). This is the only counterexample to the Chisini’s conjecture (see subsection 3.2.2).

More precisely, we have the following proposition:

Proposition 5.3. (a) Given a sextic \(B\) with 9 cusps and no nodes, there are four covers having \(B\) as a branch curve. Three of them are degree 4 maps \(\mathbb{P}^2 \to \mathbb{P}^2\), obtained as three various projections of Veronese-embedded \(\mathbb{P}^2\) in \(\mathbb{P}^5\), and the fourth one is of degree 3. The construction of the fourth is given in Remark 5.4.

(b) The fundamental group \(\pi(\mathbb{P}^2 - B)\) has exactly 4 non-equivalent representations into symmetric groups \(\text{Sym}_n\) for all \(n\) which rise to smooth generic covers ramified over \(B\).

Proof. (a) See [36].

(b): Note that \(G = \pi_1(\mathbb{P}^2 - B)\) was already calculated by Zariski in [12], showing that:

\[ G \cong \ker(B_3(T) \to H_1(T)), \]
where $B_3(T)$ is the braid group of the torus. We compute it here in a different method, using the degeneration techniques explained in [42]. Let $S$ be the image of the Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$; the branch curve $B$ of a generic projection $S \to \mathbb{P}^2$ belongs to $\mathcal{V}(6,9,0)$ (see e.g. [38]). Since $\mathcal{V}(6,9,0)$ is irreducible, it is enough to look at $B$. Now $S$ can be degenerated into a union of four planes is $\mathbb{P}^5$, with combinatorics shown on the Figure 7 below, as explained in [42].

![Figure 7: degeneration of $V_2$](image)

Using the techniques of [43], [45], one can prove that $G$ has a presentation with generators $\{\gamma_1, \gamma_1', \gamma_2, \gamma_2', \gamma_3, \gamma_3'\}$ and relations

$$\{\langle \gamma_2, \gamma_1 \rangle, \langle \gamma_2, \gamma_1' \rangle, \langle \gamma_1', \gamma_2' \rangle, \langle \gamma_3, \gamma_1 \rangle, \langle \gamma_3, \gamma_1' \rangle, \langle \gamma_3, \gamma_1'' \rangle, \langle \gamma_2', \gamma_3 \rangle, \langle \gamma_2', \gamma_3' \rangle, \langle \gamma_2', \gamma_3'' \rangle, \gamma_2^{-1} \cdot \gamma_2'' \gamma_1'' \gamma_3, \gamma_3^{-1} \cdot \gamma_3'' \gamma_1'' \gamma_3, \gamma_2^{-1} \cdot \gamma_2'' \gamma_1'' \gamma_3, \gamma_3^{-1} \cdot \gamma_3'' \gamma_1'' \gamma_3, \gamma_2^{-1} \cdot \gamma_2'' \gamma_1'' \gamma_3, \gamma_3^{-1} \cdot \gamma_3'' \gamma_1'' \gamma_3, \gamma_1'' \gamma_2'' \gamma_3' \gamma_3' \gamma_2\}$$

Using GAP [62] one can prove that $G$ is actually generated by the set $\{\gamma_1', \gamma_2, \gamma_3, \gamma_3'\}$ and having the following relations

$$\begin{align*}
\gamma_3^{-1} & \cdot \gamma_2 \gamma_3 \gamma_2^{-1} \cdot \gamma_2' \\
\gamma_1^{-1} & \cdot \gamma_3' \gamma_1^{-1} \cdot \gamma_3' \gamma_1' \\
\gamma_3 & \gamma_3' \gamma_1^{-1} \cdot \gamma_3' \gamma_1' \\
\gamma_2 & \gamma_3' \gamma_1^{-1} \cdot \gamma_3' \gamma_1' \\
\gamma_3' & \gamma_1^{-1} \cdot \gamma_3' \gamma_1' \\
\gamma_1 & \gamma_3' \gamma_1^{-1} \cdot \gamma_3' \gamma_1'
\end{align*}$$

It follows that it suffices to look for the homomorphisms $G \to \text{Sym}_n$, when $n = 3, 4, 5$, since 4 transpositions can generate at most symmetric group on 5 letters. Note also that the homomorphisms, in order to correspond to generic covers, have to satisfy Proposition [32, 20].

Using GAP again, one shows that the only epimorphisms are the following:

$$\pi_1(\mathbb{P}^2 - B) \to \text{Sym}_3 :$$

1. $\{\gamma_1', \gamma_2, \gamma_3, \gamma_3'\} \to \{(1,3), (2,3), (1,2), (1,2)\}$
2. $\{\gamma_1', \gamma_2, \gamma_3, \gamma_3'\} \to \{(2,4), (2,3), (3,4), (1,2)\}$
3. $\{\gamma_1', \gamma_2, \gamma_3, \gamma_3'\} \to \{(1,3), (2,3), (3,4), (1,2)\}$
4. $\{\gamma_1', \gamma_2, \gamma_3, \gamma_3'\} \to \{(2,4), (2,3), (1,2), (1,2)\}$

and there are no epimorphisms to $\text{Sym}_5$. □

**Remark 5.4.** We recall a construction of Chisini (see [14] or [35] Section 3]). Let $B \subset \mathbb{P}^2$ a curve with nodes and cusps only, such that its dual $A = B^\vee \subset (\mathbb{P}^2)^\vee$ is a smooth curve of degree $d$. Let

$$\Sigma = \{(\lambda, y) \in (\mathbb{P}^2)^\vee \times \mathbb{P}^2 : \lambda(y) = 0, \lambda \in A\}$$

and $\psi : \Sigma \to \mathbb{P}^2$ be the projection to the second factor. For a given point $y \in \mathbb{P}^2 - B$, the line $l_\lambda$ in $\mathbb{P}^2$ is not tangent to $A$ (i.e. intersects $A$ in $d$ distinct points) iff $\psi^{-1}(y)$ has $d$ points. Hence $\Sigma$ is a degree $d$ covering of $\mathbb{P}^2$ with $B$ as the branch curve. For $d = 3$ we get the fourth example in Proposition [5,3], a degree 3 ramified cover of $\mathbb{P}^2$ branched along a 9-cuspidal sextic. Note that this does not contradict to the fact that the only plane curves which are branch curves of projections of smooth cubic surfaces in $\mathbb{P}^3$ are sextics with six
Degree 8 singular branch curves

**Theorem 5.5.** The only degree 8 singular branch curves of generic linear projections have either 9 cusps and 12 nodes or 12 cusps and 4 nodes.

**Proof.** By simple calculations (see subsection 5.13.2 for the obstructions), one can conclude that there are only finite number of possibilities for $c$ and $n$ for a degree 8 singular branch curve.

It was proven by Zariski-Deligne-Fulton theorem 3.3 on nodal curves that the case $(c = 0, n > 0)$ cannot be realized as a branch curve. The cases $(c = 3, n > 0), (c = 6, n > 0), (c = 9, n = 0)$ and $(c = 9, n = 4)$ are ruled out as branch curves by Nori’s theorem 3.1 (though the corresponding nodal–cuspidal varieties are not empty). Moreover, the case $(c = 18, n = 0)$ cannot be realized even as nodal–cuspidal curve: By the Zariski’s inequality (13), the number of cusps of a degree 8 curve should be less than 16, and thus $V(8, 18, 0)$ is empty. By considering the dual curve, it’s easy to see that also $V(8, 15, 4)$ is empty.

We are left to show that there are no degree 8 branch curves with $(c, n) = (9, 8), (12, 0), (15, 0), (12, 8)$ (although the corresponding nodal–cuspidal varieties are not empty since the genus of these curves is less than 5).

(i) $(c = 9, n = 8)$

Assume that there exists a surface $S \subset \mathbb{P}^3$ such that its branch curve is $B \in B(8, 9, 8)$, such that $\tilde{S}$ is its smooth model in $\mathbb{P}^5$ (i.e. $\tilde{S} \to S$ by generic projection). Since $d \geq 2\nu - 2$, $\nu = 3, 4$ or 5. By the examples in subsection 4.17.1 we see that $\nu = 5$ and thus $e = 6$. In this case, by Lemma 3.9 and 3.10, we can see that $c_1^2(\tilde{S}) = c_2(\tilde{S}) = 12$. The degree 6 double curve (of the quintic) cannot lie on a hyperplane for degree reasons. Therefore the canonical system $|K_{\tilde{S}}|$ is empty (since it is the pull-back of the linear system cut out by hyperplanes passing through the double curve. See [28] pp. 627). Therefore $p_g(\tilde{S}) = 0$. But since $c_1^2 = c_2 = 12$, we get that $\chi(\mathcal{O}_{\tilde{S}}) = 2$. Thus $2 = 1 - q + p_g$ or $q = -1$ – contradiction. Thus $B(8, 9, 8)$ is empty.

(ii) $(c = 12, n = 0)$ Assume that there exist $B \in B(8, 12, 0)$. s.t. it is the branch curve of a surface $S$ in $\mathbb{P}^3$. By the same argument as in case (i) we see that $\nu = 5$ and thus $e = 6$. In this case, by Lemma 3.9 and 3.10 we can see that $c_1^2(S) = 17, c_2(S) = 19$. So by Remark 4.43 we can find that the number of pinch points $p = 0$ – but this cannot happen, by Remark 4.40. Therefore $B(8, 12, 0)$ is empty.

**Remark 5.6.** Note that Zariski proved ([5]) that the twelve cusps of $C \subset V(8, 12, 0)$ cannot be the intersection of a cubic and a quartic curves. We conjecture that this restriction is directly linked to the fact that $B(8, 12, 0)$ is empty.

**Remark 5.7.** Considering a quartic surface $S$ with a double line in $\mathbb{P}^3$, we can project it from a generic smooth point $O \in S$. The resulting branch curve will be a curve in $V(8, 12, 0)$ (see [10]). However, we do not consider this projection as generic. Note that this phenomena happens also in other cases. For example, a branch curve in $V(10, 18, 0)$ of a generic projection does not exist, but if we project a smooth quartic surface in $\mathbb{P}^3$ from a point on the quartic, the branch curve of this projection would be in $V(10, 18, 0)$.

(iii) $(c = 15, n = 0)$ As in cases (i) and (ii), we can see that a surface $S$ with such a branch curve could only be a quintic in $\mathbb{P}^3$ with a degree 6 double curve $E^*$ with 3 triple points (by Remark 4.42). Considering $\Pi$ – the plane passing through these three points – and looking at $E^* \cdot \Pi$, we see that $E^* \subset \Pi$. However, $\deg \Pi \cap S = 5$, so such a surface does not exist.
Note that in this case Zariski demonstrated in (5) that the 15 cusps cannot lie on a quartic curve.

Remark 5.8. The nonexistence in cases (ii) and (iii) can also be proven by the method indicated in (i).

(iv) \((c = 12, n = 8)\). The variety \(V(8,12,8)\) is irreducible, since it is dual to the \(V(4,0,2)\). If \(S\) is a quartic surface in \(\mathbb{P}^3\) which double curve is a union of two skew lines, then we prove in Subsection 4.7 that a branch curve of \(S\) is in \(B(8,12,8)\). By [12], \(\pi_1(\mathbb{P}^2 - B) \simeq \ker(B_1(T) \to H_1(T))\). However the double curve of an image a smooth surface in \(\mathbb{P}^N\) in \(\mathbb{P}^3\) is an irreducible curve (unless the surface is the Veronese surface, where in this case the double curve is a union of three lines. See e.g. [25, Theorem 3]). Thus \(B\) is not a branch curve of generic linear projection.

We now shall construct degree 8 branch curves with \((c,n) = (9,12), (12,4)\). With this we covered all the possible numerics for the possible number of nodes and cusps of a degree 8 branch curve.

(I) \((c = 9, n = 12)\). First, note that \(V(8,9,12)\) is irreducible, since it is dual to the variety \(V(5,0,6)\).

Now note that if we consider the Hirzebruch surface \(F_1\) embedded into \(\mathbb{P}^6\) by \(|2f + s|\) (where \(f\) is the class of a fiber and \(s\) is the class of a movable section, so that \(f^2 = 1, f \cdot s = 1, s^2 = 1\).) A projection of this model of \(F_1\) to \(\mathbb{P}^2\) factorizes as a composition of a projection to \(\mathbb{P}^3\), where the image of \(F_1\) is a quintic surface with a double curve of degree 6, and a projection from \(\mathbb{P}^3 \to \mathbb{P}^2\). One can check that the branch curve \(B\) of the resulting map has 9 cusps and 12 nodes (see [16] or Remark 3.12) and that \(\pi_1(\mathbb{P}^2 - B)\) is isomorphic to the braid group of the sphere with 4 generators (see [7]).

(II) \((c = 12, n = 4)\). We do not know whether \(V(8,12,4)\) is irreducible. If \(S\) is a smooth intersection of two quadrics in \(\mathbb{P}^4\), then the branch curve of a projection of \(S\) to \(\mathbb{P}^2\) is in \(B(8,12,4)\), see Subsection 4.7 for the details. By [49], \(\pi_1(\mathbb{C}^2 - B) \simeq \text{Braid}_4/\langle [x_2, x_2^x x_3] \rangle\).

6. Appendix A : New Zariski pairs (By Eugenii Shustin)

6.1. Introduction. Along Lemma 3.21. The family \(B(d,c,n)\) of the plane branch curves of degree \(d\) with \(c\) cusps and \(n\) nodes as their only singularities consists of entire components of \(V(d,c,n)\), the space parameterizing all irreducible plane curves of degree \(d\) with \(c\) cusps and \(n\) nodes as their only singularities. In the particular case of the branch curves of generic projections of smooth surfaces of degree \(\nu \geq 3\) in \(\mathbb{P}^3\) onto the plane, one has (cf. 1 and Lemma 4.2)

\[
\begin{align*}
d(\nu) &= \nu(\nu - 1), \quad c(\nu) = \nu(\nu - 1)(\nu - 2), \quad n(\nu) = \frac{1}{2}\nu(\nu - 1)(\nu - 2)(\nu - 3) .
\end{align*}
\]

The celebrated Zariski result [6] says that, in the case \(\nu = 3, d = 6, c = 6, n = 0\), the variety \(V(6,6,0)\) contains a component which is disjoint with \(B(6,6,0)\). This suggests

Conjecture 6.1. For each \(\nu \geq 3\), the variety \(V(d(\nu),c(\nu),n(\nu))\) contains a component disjoint with \(B(d(\nu),c(\nu),n(\nu))\).

Here we confirm this conjecture for few small values of \(\nu\).

Theorem 6.2. Conjecture 6.1 holds true for \(3 \leq \nu \leq 10\).

We prove Theorem 6.2 explicitly constructing curves \(C \subset V(d(\nu),c(\nu),n(\nu))\setminus B(d(\nu),c(\nu),n(\nu))\). Our construction is based on the patchworking method as developed in [50]. It seems that this method does not allow one to cover sufficiently large values of \(\nu\).

6.2. Construction.
6.2.1. The main idea. The variety $V(d, c, n)$ is said to be $T$-smooth at $C \in V(d, c, n)$ if the germ of $V(d, c, n)$ at $C$ is the transverse intersection in $|O_{\mathbb{P}^2}(d)|$ of the germs at $C$ of the smooth equisingular strata corresponding to individual singular points of $C$ (cf. [17]). In particular, this implies that, for any subset $S \subset \text{Sing}(C)$, there exists a deformation $C_t^S$, $t \in (\mathbb{C}, 0)$, such that $C_0^S = C$, $C_t^S \in V(d, c', n')$, where $c'$, $n'$ are the numbers of cusps and nodes in $\text{Sing}(C) \setminus S$, respectively, and, furthermore, the deformation $C_t^S$ smooths out all the singular points of $C$ in $S$. Clearly, the $T$-smooth part $V^T(d, c, n)$ of $V(d, c, n)$ is an open subvariety (if not empty) of $V(d, c, n)$.

We derive Theorem 6.2 from

**Proposition 6.3.** (1) Let $\nu \geq 3$. If $V^T(d(\nu), c(\nu), n(\nu) + 1) \neq \emptyset$, then

$$V(d(\nu), c(\nu), n(\nu)) \setminus B(d(\nu), c(\nu), n(\nu)) \neq \emptyset.$$  

(2) If $3 \leq \nu \leq 10$, then $V^T(d(\nu), c(\nu), n(\nu) + 1) \neq \emptyset$.

6.2.2. Proof of Proposition 6.3. If $\nu \geq 5$ then as noticed in Subsection 4.6, $B_3(d(\nu), c(\nu), n(\nu))$ is not $T$-smooth, since a $T$-smooth family must have the expected dimension. On the other hand, smoothing out one node of a curve $C \in V^T(d(\nu), c(\nu), n(\nu) + 1)$, we obtain an element of $V^T(d(\nu), c(\nu), n(\nu))$, a component whose dimension differs from that of $B_3(d(\nu), c(\nu), n(\nu))$.

This reasoning does not cover the case of $\nu = 3$ and 4. We then provide another argument which, in fact, works for all $\nu \geq 3$.

Let $\nu \geq 3$, $C \in V^T(d(\nu), c(\nu), n(\nu) + 1)$. We intend to show that there is a nodal point $p \in C$ such that $\text{Sing}(C) \setminus \{p\}$ is not contained in a plane curve of degree $d''(\nu) = (\nu - 1)(\nu - 2)$. This is enough, since by [8] (see also Proposition 4.2), all the singular points of a branch curve $D$ of $V(d(\nu), c(\nu), n(\nu))$, a component whose dimension differs from that of $B_3(d(\nu), c(\nu), n(\nu))$.

We prove the existence of the required node $p \in C$ arguing for contradiction. Let $p_1, p_2$ be some distinct nodes of $C$ and let $C_1 \supset \text{Sing}(C) \setminus \{p_1\}$, $C_2 \supset \text{Sing}(C) \setminus \{p_2\}$ be some curve of degree $d''(\nu)$. Since $d(\nu) \cdot d''(\nu) = 2(c(\nu) + n(\nu))$, the curves $C_1$ and $C_2$ are non-singular along their intersection with $C$, in particular, they are reduced. Furthermore, $p_1 \not\in C_1$ and $p_2 \not\in C_2$. Let $D$ be the (possibly empty) union of the common components of $C_1$ and $C_2$ with deg $D = k$, $0 \leq k < d(\nu)$. So, $C_1 = D D_1$, $C_2 = D D_2$, where the curves $D_1, D_2$ of degree $d(\nu) - k$ have no component in common. By Bezout’s theorem $D \cap C$ consists of $kd(\nu)/2$ points of $\text{Sing}(C)$, and

$$D_i \cap C = \text{Sing}(C) \setminus \{D \cap C \cup \{p_i\}\}, i = 1, 2.$$ 

Take two distinct generic straight lines $L_1$, $L_2$ through $p_1$. By Noether’s $AF + BG$ theorem (see, for instance, [23]),

- if $k \leq d''(\nu) + 1 - d(\nu)/2$, then there are polynomials $A_1, A_2$ of degree $d''(\nu) + 2 - k$ and polynomials $B_1, B_2$ of degree $2d''(\nu) + 2 - d(\nu) - 2k$ such that $D_2 L_i^2 = A_i D_1 + B_i C$, $i = 1, 2$,
- if $k \geq d''(\nu) + 2 - d(\nu)/2$, then there are polynomials $A_1, A_2$ of degree $d''(\nu) + 2 - k$ such that $D_2 L_i^2 = A_i D_1$, $i = 1, 2$.

The latter case is impossible since $D_2 L_i^2$ and $D_1$ have no component in common. In the former case, we obtain that $D_1$ divides $D_2 L_i^2 (L_2 B_2 - L_1 B_1)$, and hence divides $L_2^2 B_2 - L_1^2 B_1$. In view of

$$\deg D_1 = (\nu - 1)(\nu - 2) - k > 2 + (\nu - 1)(\nu - 4) - 2k = \deg (L_1^2 B_2 - L_2^2 B_1),$$ 

we conclude that $L_2^2 B_2 = L_1^2 B_1$, in particular, $L_1^2$ divides $B_1$, but then $L_2^2$ divides $A_1$ too, and hence $D_2^2 = A_1 D_1 + B_1 C$ contrary to the fact that $p_2 \not\in D_2$ and $p_2 \in D_1 \cap C$.

---

1In the case of nodes and cusps, the smoothness of these equisingular strata always holds.

2For the sake of notation we denote a plane curve and its defining homogeneous polynomial (given up to a constant factor) by the same symbol, no confusion will arise.
6.2.3. Proof of Proposition 6.3(2). We suppose that $\nu \geq 4$. Using the patchworking construction of [50, Theorem 3.1], we obtain curves in $V^T(12, 24, 16)$, $V^T(20, 63, 67)$, $V^T(30, 126, 191)$, $V^T(42, 216, 435)$, $V^T(56, 336, 902)$, $V^T(72, 504, 1550)$, and $V^T(90, 720, 2526)$, what suffices for our purposes in view of Proposition 6.3(1), since
\[
d(4) = 12, \quad c(4) = 24, \quad n(4) = 12 < 16,
\]
\[
d(5) = 20, \quad c(5) = 60 < 63, \quad n(5) = 60 < 67,
\]
\[
d(6) = 30, \quad c(6) = 120 < 126, \quad n(6) = 180 < 191,
\]
\[
d(7) = 42, \quad c(7) = 210 < 216, \quad n(7) = 420 < 435,
\]
\[
d(8) = 56, \quad c(8) = 336, \quad n(8) = 840 < 902,
\]
\[
d(9) = 72, \quad c(9) = 504, \quad d(9) = 1512 < 1550,
\]
\[
d(10) = 90, \quad c(10) = 720, \quad n(10) = 2520 < 2526.
\]

Referring to [50] for details, we only recall that the patchworking construction uses a convex lattice subdivision of the triangle $T_0 = \text{conv}\{(0,0),(0,d),(d,0)\}$. Pieces $\Delta_1, \ldots, \Delta_N$ of the subdivision will serve as Newton polygons of polynomials in two variables (called *block polynomials*) which define curves with nodes and cusps leaving in the respective toric surfaces: $C_k \subset \text{Tor}(\Delta_k)$, $k = 1, \ldots, N$. Along [50, Theorem 4.1], the patchworking construction can be performed under the following sufficient conditions:

(C1) Any two block polynomials have the same coefficients along the common part of their Newton polygons, and the truncations of block polynomials to any edge is a nondegenerate (quasihomogeneous) polynomial.

(C2) The adjacency graph of the pieces of the subdivision can be oriented without oriented cycles so that if, for each polygon $\Delta_k$, $1 \leq k \leq N$, we mark its sides which correspond to the arcs of the adjacency graph coming inside the polygon and denote by $D_k \subset \text{Tor}(\Delta_k)$ the union of the unmarked toric divisors, then the number of cusps of any component $C$ of $C_k$ is less than $CD_k$.

By [50, Theorems 3.1 and 4.1], the resulting curve with the Newton triangle $T_0$ belongs to $V^T(d, c, n)$, and the numbers $c$ and $n$ are obtained by summing up the numbers of cusps and nodes over all the block curves.

Condition (C2) formulated above is, in fact, sufficient for the following transversality property defined in [50, Definition 2.2] and used in the patchworking construction of [50, Theorem 3.1]: the variety $V_C$ consisting of the curves in the linear system $|C|$ on $\text{Tor}(\Delta_k)$ which are equisingular to $C$ and intersect $D'_k$ at the same points as $C$, where $D'_k$ is the union of the marked toric divisors of $\text{Tor}(\Delta_k)$, is smooth at $C$ of codimension $2c(C) + n(C) + CD'_k$, where $c(C), n(C)$ are the numbers of cusps and nodes of $C$. We shall call this property the *$T$-smoothness relative to $D'_k$*. Let $L$ be a coordinate line in $\mathbb{P}^2$ corresponding to a side $\sigma$ of $T_0$. Then the patchworking construction produces a curve, where the variety $V(d, c, n)$ is $T$-smooth relative to $L$, if one replaces condition (C2) by the following one:

(C2') The adjacency graph of the pieces of the subdivision can be oriented without oriented cycles so that if, for each polygon $\Delta_k$, $1 \leq k \leq N$, we mark its sides which correspond to the arcs of the adjacency graph coming inside the polygon or are contained in $\sigma$, and denote by $D_k \subset \text{Tor}(\Delta_k)$ the union of the unmarked toric divisors, then the number of cusps of any component $C$ of $C_k$ is less than $CD_k$.

\(^3\text{Convexity means that the subdivision lifts up to a graph of a convex function linear on each subdivision polygon and having a break along each common edge.}\)
We consider the subdivisions of $T_{12}$, $T_{20}$, $T_{30}$, $T_{42}$, $T_{56}$, $T_{72}$, and $T_{90}$ shown in Figures 1(b), 2(b), 3, where all the slopes are 0, $-1$, or $\infty$. We leave to the reader an easy exercise to check that these subdivisions are convex. Next we describe the block polynomials.

(1) If $d = 12$, we take the block polynomial $F_1(x, y)$ with Newton triangle $T_0$ defining a curve $C_1 \in V(6, 6, 4)$ (such a curve is dual to an irreducible quartic with three nodes). The other block polynomials are obtained via affine automorphisms of $\mathbb{Z}^2$ which interchange two adjacent triangles of the subdivision keeping their common side fixed. Thus, the patchworking construction gives a curve $C_{12} \in V^T(12, 24, 16)$.

(2) If $d = 20$, we take the block polynomial $F_2(x, y)$ with Newton triangle $T_0$ defining a curve $C_2 \in V(6, 9, 0)$ (such a curve is dual to a non-singular cubic). The other block polynomials with the Newton triangles with side length 6 are obtained from $F_2$ by suitable reflections. For any other polygon in the given subdivision, we take a block polynomial splitting into the product of linear polynomials, defining (reducible) nodal curves, and satisfying (C1). It is easy to check that the condition (C2) holds, and thus, one obtains a curve $C_{20} \in V^T(20, 63, 67)$.

(3) If $d = 30$ or 42, for each triangle in the subdivision having side length 6 and intersecting with a coordinate axis, we take the block polynomial obtained from $F_2$ as described above, and, for any other polygon of the subdivision, we take a suitable polynomial splitting into the product of linear polynomials. This gives us the curves $C_{30} \in V^T(30, 126, 191)$ and $C_{42} \in V^T(42, 216, 435)$.

(4) If $d = 56$ we use the block polynomial $F_3$ defining a curve $C_0 \in V(9, 16, 10)$ which is obtained via a slight modification of the construction of a curve $C'_0 \in V(9, 20, 0)$ in [50, Section 4.3]. We consider the subdivision of $T_0$ shown in Figure 2(a) (cf. [50, Figure 2]) and take the following block curves: those with two symmetric Newton quadrangles have 8 cusps each (as in [50, Proof of Theorem 4.3]), the block curve with Newton triangle has one node, and the block curve with Newton square splitting onto 6 lines, has 9 nodes.

Now we subdivide the triangle $T_{56}$ as shown in Figure 2(b) and take the following block curves:

- The block curves with the triangles intersecting with the coordinate axes and the triangle marked with asterisk are defined by the polynomial $F_3$ and its appropriate transforms,
- each other block curve is defined by a polynomial splitting into linear factors.

Observe that the conditions (C1) and (C2) can be satisfied in this situation, which, finally, gives us a curve in $V^T(56, 336, 902)$.

(5) Observe that the construction of step (3) gives a curve $C_{30}$ at which the variety $V(30, 126, 191)$ is $T$-smooth relative to the $y$-axis. Clearly, this $T$-smoothness property at $C_{30}$ hold relatively to almost all lines in $\mathbb{P}^2$, and hence by an appropriate coordinate change we can make $V(30, 126, 191)$ to be $T$-smooth at $C_{30}$ relative to each of the coordinate lines. Let $F_4(x, y)$ be a defining polynomial of $C_{30}$.

Consider the subdivision of $T_{72}$ presented in Figure 3(a). For the triangle incident to the origin, we take the above polynomial $F_4(x, y)$, and, for the other triangles with side length 30, we take the transforms of $F_4$ as described in step (1). For the other polygons of the subdivision we take appropriate polynomials splitting into linear factors. The relative $T$-smoothness of $V(30, 126, 191)$ at $C_{30}$ ensures condition (C2); hence the patchworking procedure is performable, and it gives a curve $C_{72} \in V^T(72, 504, 902)$.

(6) In the case $d = 90$ we need a curve $C'_{30} \in V(30, 120, 199)$ at which the variety $V(30, 120, 199)$ is $T$-smooth relatively to each of the coordinate lines. Assuming that such a curve exists, we take its defining polynomial $F_5(x, y)$ and spread it through all the triangles in the subdivision shown in Figure 3(b) except for the right-most one (marked by asterisk). For the latter triangle and for the parallelogram we take suitable polynomials splitting into linear factors. Again, due to the relative
$T$-smoothness of $V(30, 120, 199)$ at $C'_30$, the condition (C2) holds true, and hence the patchworking procedure gives a curve in $V(90, 720, 2526)$.

The required curve $C'_30$ can be constructed using the modified construction of step (3) and a generic coordinate change afterwards. The modification is as follows: for the upper and the rightmost triangles with side length 6 in the subdivision shown in Figure (c), we take polynomials defining curves in $V(6, 6, 4)$ (instead of $V(6, 9, 0)$ as in the original construction of step (3)). Curves of degree 6 with 6 cusps and 4 nodes do exist: they are dual to rational quartics with 3 nodes. To ensure condition (C1), we need the sextics as above which cross one of the coordinate lines along a prescribed configuration of 6 points. Notice that, given a straight line $L \subset \mathbb{P}^2$, the varieties $V(6, 6, 4)$ and $V(6, 9, 0)$ are $T$-smooth relative to $L$ at each curve crossing $L$ transversally (it immediately follows from condition (C2')). In particular, this yields that the rational maps $V(6, 6, 4) \to \text{Sym}^6(L)$ and

Figure 1. Patchworking construction: The case $\nu = 4, 5, 6, 7$
Figure 2. Patchworking construction: The case $\nu = 8$

$V(6, 9, 0) \to \text{Sym}^6(L)$ defined by $C \mapsto C \cap L$ are dominant. Hence we can choose polynomials defining curves a curve in $V(6, 6, 4)$ and a curve in $V(6, 9, 0)$ so that the considered patchworking data will meet condition (C1).

7. Appendix B: Picard and Chow groups for nodal-cuspidal curves

In this Appendix we remind the reader the connections between Cartier and Weil divisors and the connection of the Picard and Chow groups on a nodal–cuspidal curve with $c$ cusps and $n$ nodes. This connection is implicit in Segre [5], and here we recall the explicit formulation.
Let $B$ a nodal-cuspidal plane curve, with $B^*$ its normalization in $\mathbb{P}^3$. First, by definitions of Pic and $A_0$, we have for $B$

$$0 \to Cart.\ Princ.\ B \to Weil.\ Princ.\ B \to 0$$

$$0 \to G_S \to Cartier\ B \to Weil\ B \to 0$$

$$0 \to G_S \to Pic\ B \to A_0B \to 0$$

where the canonical map $Pic\ B \to A_0B$ is the map induced by associating the class of a Weil divisor with each Cartier divisor on $B$, $S$ is the set of singular points of $B$, and $G_S$ is the subgroup of the group of Cartier divisors on $B$ such that their associated Weil divisors are trivial. Note that the map $Cartier\ B \to Weil\ B$ is surjective since $B$ is a nodal-cuspidal curve.

Secondly, there is an exact sequence

$$0 \to H^0Q_S \to Pic\ B \xrightarrow{\pi_*} Pic\ B^* \to 0$$

where $Q_S = \pi_* (\mathcal{O}_{B^*})/\mathcal{O}_B^* = \prod_{p \in Sing\ B} \left( \prod_{p^* \sim p} \mathcal{O}_{p^*}^* / \mathcal{O}_p^* \right)$.

We also have the following excision diagram:
where \( \xi = P + Q, \xi^* = P^* + Q^* \), \( U = B - \xi \), \( U^* = B^* - \xi^* \), and the map \( A_0\xi^* \to A_0\xi \) can be described as \( \mathbb{Z}^{P^*} \oplus \mathbb{Z}^{Q^*} \to \mathbb{Z}^P \oplus \mathbb{Z}^Q \) which is the factorization of \( \mathbb{Z}^{P^*} \) by a subgroup generated by \( (p_1^* - p_2^*) \) for a preimage of each node \( p \) of \( B \). (For a different proof that the map \( A_0B^* \to A_0B \) is epimorphic, see [48, Example 1.9.5]). Denote \( T = \mathbb{Z}^P / \sim \).

Combining the two diagrams together, we get

\[
\begin{array}{c}
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\downarrow \\
0 \\
\downarrow \\
\downarrow \\
A_0\xi^* \\
\downarrow \\
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\downarrow \\
A_0\xi \\
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A_0\xi^* \to A_0B^* \to A_0U^* \to 0 \\
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A_0\xi \to A_0B \to A_0U \to 0 \\
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A_0\xi^* \to A_0B^* \to A_0U^* \to 0 \\
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A_0\xi \to A_0B \to A_0U \to 0 \\
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0
\end{array}
\]

Where the last column is induced from the fact that \( \text{Pic } B^* \simeq A_0B^* \) and from the exact sequence (35)

\[
0 \to T \to A_0B^* \to A_0B \to 0.
\]

Note that the map \( H^0Q_S \to G_S \) is injective, and cannot be surjective, otherwise the map \( \text{Pic } B^* \to A_0B \) would be an isomorphism. Thus \( T' \cong T \).

8. Appendix C : Bisecants to a complete intersection curve in \( \mathbb{P}^3 \)

We note here that the inverse statement to Theorem 4.4 is easy. Explicitly, we have the following Theorem:

Let \( C \) be a curve in \( \mathbb{P} = \mathbb{P}^3 \) which is a complete intersection of type \((\mu, \nu)\), and let \( a \) be a point in \( \mathbb{P} \) which is not on \( C \) such that \( C \) does not admit any 3-secants through \( a \). Then \( C \) has \( \frac{1}{2}\mu\nu(\mu-1)(\nu-1) \) bisecants passing through \( a \).

Remark 8.1. The above theorem gives a direct proof that the number of nodes of the branch curve \( B \) is indeed \( n = \frac{1}{2}\nu(\nu-1)(\nu-2)(\nu-3) \) (recall that \( B^* \) is a complete intersection of \( S \) and \( \text{Pol}_O S \), i.e., of type \((\nu, \nu-1)\) and that the line \( Oq^* \) for each \( q^* \in Q^* \) is also considered as a bisecant of \( B^* \).
Proof. Our proof is essentially a reformulation of a proof by Salmon, [11 art. 343]. See also [15, Chapter IX, sections 1.1.1.2] for another way to induce this formula. Consider the moduli space $M$ of data $\{l \in \mathbb{P} \text{ which is bisecant for } C, \text{ a point } p' \in l \cap C, \text{ a point } p \in l, p \notin C\}$. (see the following Figure)

It is clear that the line $l$ can be reconstructed uniquely from $p'$ and $p$ as $l_{p,p'}$, and thus $M$ can be embedded into $\mathbb{P} \times \mathbb{P}$, $((l,p',p) \mapsto (p',p))$. For a point $(l,p',p)$ in $M$, let $q$ be a point in $l \cap C$ different from $p'$. Then there is a number $t \in k$ such that $q = p' + tp$. If $C$ is given by 2 equations $u, v$, then we have $u(q) = 0, v(q) = 0$. Let us write $u(q) = u(p' + tp) = u_0(p') + tu_1(p',p) + \cdots + t^\mu u_\mu(p',p)$, where $u_i$ is of degree $\mu - i$ in $p'$ and $i$ in $p$. In the same way we can write $v(q) = v(p' + tp) = v(p') + tv_1(p',p) + \cdots + t^\nu v_\nu(p',p)$.

Let $R(p',p)$ be the resultant of $a(p',p,t)$ and $b(p',p,t)$ in $t$. It has (see the the Sylvester definition of resultant) bidegree $((\mu - 1)(\nu - 1), \mu \nu - 1)$ in $(p',p)$. Lemma: Let $U \subset \mathbb{P} \times \mathbb{P} = \{(p',p) : p' \neq p\}$. Then $U \cap (R = 0) \cap (C \times \mathbb{P}) = M$

Indeed, let $(p',p)$ be such that $R(p',p) = 0, p' \neq p$. Then there is a number $t \in k$ such that $a(p',p,t) = 0, b(p',p,t) = 0$. Let $q = p' + tp$. We have $u(q) = u(p') + ta(p',p) = u(p')$ $v(q) = v(p') + tb(p',p) = v(p')$.

Thus $q \in C$ iff $p' \in C$. It follows that $(p',p)$ is in $M$ iff $p' \in C$. This proves the lemma. It follows now that

$$R \cap (C \times a) = M \cap (\mathbb{P} \times a) = \{\text{bisecants through } a \text{ to } C \text{ with a marked point } p' \text{ in } l \cap C\}$$

The order of this set is equal to $\deg p C \cdot \deg C = (\mu - 1)(\nu - 1)\mu \nu$. Since $C$ does not have any 3-secants through $a$, it follows that the number of bisecants through $a$ is one half of the number above. \hfill \Box

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