OPERATOR-VALUED FRAMES

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Abstract. We develop a natural generalization of vector-valued frame theory, which we term operator-valued frame theory, using operator-algebraic methods. This extends work of the second author and D. Han which can be viewed as the multiplicity one case and extends to higher multiplicity their dilation approach. We prove several results for operator-valued frames concerning duality, disjointedness, complementarity, and composition of operator-valued frames and the relationship between the two types of similarity (left and right) of such frames. A key technical tool is the parametrization of Parseval operator-valued frames in terms of a class of partial isometries in the Hilbert space of the analysis operator. We apply these notions to an analysis of multiframe generators for the action of a discrete group $G$ on a Hilbert space. One of the main results of the Han-Larson work was the parametrization of the Parseval frame generators in terms of the unitary operators in the von Neumann algebra generated by the group representation, and the resulting norm path-connectedness of the set of frame generators due to the connectedness of the group of unitary operators of an arbitrary von Neumann algebra. In this paper we generalize this multiplicity one result to operator-valued frames. However, both the parametrization and the proof of norm path-connectedness turn out to be necessarily more complicated, and this is at least in part the rationale for this paper. Our parametrization involves a class of partial isometries of a different von Neumann algebra. These partial isometries are not path-connected in the norm topology, but only in the strong operator topology. We prove that the set of operator frame generators is norm pathwise-connected precisely when the von Neumann algebra generated by the right representation of the group has no minimal projections. As in the multiplicity one theory there are analogous results for general (non-Parseval) frames.

1. Introduction

The mathematical theory of frame sequences on Hilbert space has developed rather rapidly in the past decade. This is true of both the finite dimensional and infinite dimensional aspect of the theory. The motivation has come from applications to engineering as well as from the pure mathematics of the theory.

The theory of finite frames has developed almost as a separate theory in itself, with applications to industry (cf. the recent work [4] of Balans, Casazza, and Edidin...
on signal reconstruction without noisy phase) as well as recently demonstrated
connections to theoretical problems such as the Kadison-Singer problem \[5\].

Important examples of infinite frames are the Gabor (Weyl-Heisenberg) frames
time-frequency analysis and the wavelet frames \[7\]. Some papers dealing with
infinite frames which relate directly or indirectly to this article are \[15, 13, 9, 1, 10, 20, 19, 21\].

Work on this article began in January 1999, when the first named author visited
the second named author at Texas A&M University following the special session on
“The functional and harmonic analysis of wavelets and frames” that took place at
the annual AMS meeting at San Antonio. Our purpose was to develop operator-
theoretic methods for dealing with multiwavelets and multiframes, thus extending
the approach of the AMS Memoir \[15\]. We developed the theory of operator-valued
frames to provide a framework for such problems and we tested this model by
solving a problem concerning norm path-connectedness. It has been brought to our
attention that a few other recent papers in the literature overlap to some extent with
our approach, notably works of Casazza, Kutyniok and Li \[6\] on “fusion frames”,
and also recent work of Bodmann \[2\] on quantum computing and work of W. Sun
\[25\] on \(G\)-frames. These do not deal however with the path-connectedness that we
address. The papers of Kadison on the Pythagorean theorem \[17, 16\] are examples
of works of pure mathematics that several authors have realized are both directly
and indirectly relevant to frame theory.

Also, several papers in the literature deal with frames in Hilbert \(C^*\)-modules,
including one by the same authors of this paper \[11, 12, 19\]. The problems and
framework considered in this paper are of a significantly different nature and there
is no essential overlap.

We note that the key idea in \[15\] was the observation that frames “dilate” to
Riesz bases. This was proven at the beginning of \[15\] (see also \[22\], p. 145),
and was then used to obtain results on Gabor frames, more generally frames generated
by the action of unitary systems, and certain group representations. The dilation
result for the special case of Parseval frames can be simply deduced from Naimark’s
dilation theorem for positive operator-valued measures, in fact from the special case
of Naimark’s Theorem specific to purely atomic positive operator-valued measures.
W. Czaja gives a nice account of this in \[8\], along with some new dilation results.
V. Paulsen gives a nice proof of Naimark’s theorem in \[24\] using the theory of
completely positive mappings. Similarly, we use dilation theory in the present
paper to work with operator-valued frames.

Consider a multiframe generator \(\{\psi_1, \psi_2\}\) for a unitary system \(\mathcal{U}\), that is, two
vectors in a Hilbert space \(H\) for which the collection \(\{U\psi_m \mid U \in \mathcal{U}, m = 1, 2\}\)
forms a frame:

\[
\sum_{U \in \mathcal{U}} (| (x, U\psi_1) |^2 + | (x, U\psi_2) |^2) \leq b \|x\|^2
\]

for some positive constants \(a\) and \(b\) and all \(x \in H\). Set \(H_o := C^2\), choose \(\{e_1, e_2\}\) to
be an orthonormal basis of \(H_o\), define the rank-two operator \(A\) given for \(z \in H\) by
Az := (z, ψ₁)e₁ + (z, ψ₂)e₂ and then denote \( A_U := AU^* \). Then equation (1) holds precisely when

\[ aI \leq \sum_{U \in \mathcal{U}} A_U^* A_U \leq bI, \]

where the series converges in the strong operator topology. In other words, in lieu of considering the two vectors \( \{\psi_1, \psi_2\} \), we can consider the rank-two operator \( A \).

The above is a simple example of an operator-valued frame generator and leads naturally to the more general Definition 2.1 below of an operator-valued frame consisting of operators with ranges in a given Hilbert space \( H_o \), and the frame condition is expressed in terms of boundedness above and below of a series of positive operators converging in the strong operator topology. So, the usual (vector) frames can be seen as operator-valued frames of “multiplicity one”.

It is easy to recover from the operator \( A \) defined above the vectors \( \psi_1 \) and \( \psi_2 \) that were used to define it, and, in general, to decompose (but not uniquely) an operator-valued frame in a (vector) multiframe (see comments after Remark 4.10). However, we expect that this paper will make clear that “assembling” a multiframe in an operator-valued frame is more than just a space-saving bookkeeping device.

Indeed, operator theory techniques allow us to obtain directly for operator-valued frames (and hence for the related vector multiframe) properties known for (vector) frames.

More importantly, however, treating multiframe as operator-valued frames permits us to parametrize them in an explicit and transparent way and thus handle the sometimes major differences that occur when the multiplicity rises above one, and in particular, when it is infinite.

A case in point, and in a sense our best “test” of the usefulness of the notion of operator-valued frames, is the analysis of frame generators for a discrete group (see [15] and Section 6 for a review of the definitions). Han and the second named author proved in [15, Theorem 6.17] that the collection of all the Parseval frame generators for a given unitary representation \( \{G, \pi, H\} \) of a countable group \( G \) is (uniquely) parametrized by the unitary operators of the von Neumann algebra \( \pi(G)' \) generated by the unitaries \( \pi_g \) of the representation. Since the unitary group of any von Neumann algebra is path-connected in the norm topology, the collection of all the Parseval frame generators is therefore also path-connected; i.e., it has a single homotopy class.

As soon as \( \dim H_o > 1 \), the above parametrization is no longer sufficient (see Remark 7.7), and it must be replaced by a new parametrization involving a class of partial isometries of a different von Neumann algebra (see Theorem 7.1, Proposition 7.3).

Furthermore, when \( \dim H_o = \infty \), it is possible to show that the partial isometries involved in this parametrization are not path-connected in the norm topology (they are path-connected in the strong operator topology, though). Nevertheless, we prove in Theorem 8.1 that the collection of operator frame generators is still norm connected, precisely when the von Neumann algebra generated by the left (or right) regular representation of the group has no minimal projections. The key step is provided by Lemma 8.4, where the strong continuity of a certain path of partial isometries is parlayed into the norm continuity of the corresponding path of operator frame generators.
One of the main themes of this article is the analysis of one-to-one parametrizations of operator-valued frames in general, and of operator frame generators for unitary systems and groups in particular. In the process we extend to operator-valued frames many of the properties of vector frames. In more detail:

In Section 2 we define operator-valued frames, their analysis operators and their frame projections, and then prove that the dilation approach of [15] carries over to the higher multiplicity case, i.e., that Parseval operator-valued frames are the compressions of "orthonormal" operator-valued frames, namely collections of partial isometries, all with the same initial projection and with mutually orthogonal ranges spanning the space.

In Section 3 we obtain a one-to-one parametrization of all the operator-valued frames on a certain Hilbert space $H$, with given multiplicity and index set, in terms of a class of operators in the analysis operator Hilbert space (partial isometries if we consider only Parseval frames).

In Section 4 we study two kinds of similarities of operator-valued frames. The similarity obtained by multiplying an operator-valued frame from the right generalizes the usual one in the vector case and inherits its main properties. For higher multiplicity, however, we also have a similarity from the left which has different properties. We characterize the case when two operator-valued frames are similar both from the right and from the left, in terms of the parametrization mentioned above (Proposition 4.8). We also discuss composition of frames, when the range of the operators in one frame matches the domain of the operators in a second frame and we present this notion as the tool to decompose an operator-valued frame into a (vector) multiframe.

In Section 5 we define and parametrize the dual of operator-valued frames and extend to higher multiplicity also the notions of disjoint, strongly disjoint, and strongly complementary frames that were introduced in [15] for the vector case.

In Section 6 we start the analysis of operator frame generators for unitary systems. The notion of local commutant introduced in [9] has a natural analog in the higher multiplicity case (see Proposition 6.2). Unitary representations of discrete groups have an operator frame generator with values in $H_o$ precisely when they are unitarily equivalent to a subrepresentation of the left regular representation with multiplicity $\dim H_o$, i.e., $\lambda \otimes I_o$ with $I_o$ the identity of $B(H_o)$ (Theorem 6.5). This result was previously formulated in terms of (vector) multiframes in [15, Theorem 3.11].

In Section 7 we present parametrizations of operator frame generators for a discrete group representation (Theorem 7.1). As mentioned above, higher multiplicity brings substantial differences with the vector case, which are illustrated by Proposition 7.6.

As already mentioned, Section 8 studies the path-connectedness of the operator frame generators for a unitary representation of a discrete group using von Neumann algebra techniques.

Finally, let us notice explicitly that although, in the applications, frames are mainly indexed by finite or countable index sets and the vectors in a frame belong to finite or separable Hilbert spaces, and similarly, discrete groups are finite or countable, we found that making these assumptions provides no simplification in our proofs (with one very minor exception). Thus we decided to state and prove our results in the general case. The only thing to keep in mind, when the index set
Let $H$ and $H_o$ be Hilbert spaces. A collection $\{A_j\}_{j \in J}$ of operators $A_j \in B(H,H_o)$ indexed by $J$ is called an operator-valued frame on $H$ with range in $H_o$ if the series

$S_A := \sum_{j \in J} A_j^* A_j$

converges in the strong operator topology to a positive bounded invertible operator $S_A$. The frame bounds $a$ and $b$ are the largest number $a > 0$ and the smallest number $b > 0$ for which $aI \leq S_A \leq bI$; i.e., $b = \|S_A\|$ and $a^{-1} = \|S_A^{-1}\|$. If $a = b$, i.e., $S_A = aI$, then the frame is called tight; if $S_A = I$, the frame is called Parseval. $\{\text{rank}(A_j) \mid j \in J\}$ is called the multiplicity of the operator-valued frame. When the reference to $H$, $H_o$, and $J$ is understood, we denote by $\mathcal{F}$ the set of all the operator-valued frames on $H$, with ranges in $H_o$ and indexed by $J$.

If the operator-valued frame has multiplicity one, the operators $A_j$ can be identified through the Riesz Representation Theorem with Hilbert space vectors, and hence in this case an operator-valued frame is indeed a (vector) frame under the usual definition. Explicitly, if $A_j$ is the rank one operator given by $A_j z = (x,x_j)e_j$ for some unit vectors $e_j \in H_o$, some vectors $x_j \in H$ and all $z \in H$, and then $S_A = \sum_{j \in J} A_j^* A_j$; hence $S_A$ is bounded and invertible if and only if $aI \leq S_A \leq bI$ for some $a, b > 0$; namely,

$a\|x\|^2 \leq (S_A x, x) = \sum_{j \in J} |(x,x_j)|^2 \leq a\|x\|^2$

for all $x \in H$. This is precisely the condition that guarantees that $\{x_j\}_{j \in J}$ is a (vector) frame.

Notice that if $\{A_j\}_{j \in J} \in \mathcal{F}$ and if $\{e_m\}_{m \in M}$ is an orthonormal basis of $H_o$, then it is easy to see that $\{A_j^* e_m\}_{(j,m) \in J \times M}$ is a (vector) frame on $H$; i.e., operator-valued frames can be decomposed into (vector) multiframe. We will revisit this decomposition when discussing frame compositions more generally.

The advantage of treating a collection of vectors forming a multiframe as an operator-valued frame is that we can more easily apply to it the formalism of operator theory. This is already evidenced by the next example.

**Example 2.2.** Let $K$ be an infinite dimensional Hilbert space and let $\{V_n\}_{n \in \mathbb{N}}$ be a collection of partial isometries with mutually orthogonal range projections $V_n V_n^*$ summing to the identity and all with the same initial projection $V_n^* V_n = E_o$. Let $P \in B(K)$ be a nonzero projection and let $A_n := V_n^* P \in B(H,H_o)$, where we set $H := PK$, $H_o := E_o K$. Then

$$\sum_{n=1}^{\infty} A_n^* A_n = P(\sum_{n=1}^{\infty} V_n V_n^*) P|_{PK} = P|_{PK} = I,$$

i.e., the sequence $\{A_n\}$ is a Parseval frame with range in $H_o$.

By introducing the *analysis operator* (also called the frame transform, e.g., [15]), we will see that this example is ‘generic’ (see Proposition 2.4 below).
Analysis operator. Given a Hilbert space $H_o$ and an index set $\mathcal{J}$, define the partial isometries
\begin{equation}
L_j : H_o \ni h \mapsto e_j \otimes h \in \ell^2(\mathcal{J}) \otimes H_o,
\end{equation}
where $\{e_j\}$ is the standard basis of $\ell^2(\mathcal{J})$. Then
\begin{equation}
L_j^*L_i = \begin{cases} 
I_o & \text{if } i = j, \\
0 & \text{if } i \neq j
\end{cases}
\end{equation}
and
\begin{equation}
\sum_{j \in \mathcal{J}} L_jL_j^* = I \otimes I_o,
\end{equation}
where $I$ denotes the identity operator on $\ell^2(\mathcal{J})$ and $I_o$ denotes the identity operator on $H_o$ and the convergence is in the strong topology.

Proposition 2.3. For every $\{A_j\}_{j \in \mathcal{J}} \in \mathcal{F}$,
\begin{enumerate}
\item The series $\sum_{j \in \mathcal{J}} L_jA_j$ converges in the strong operator topology to an operator $\theta_A \in B(H, \ell^2(\mathcal{J}) \otimes H_o)$.
\item $S_A = \theta_A^*\theta_A$.
\item $\{A_j\}_{j \in \mathcal{J}}$ is Parseval if and only if $\theta_A$ is an isometry.
\end{enumerate}

Proof. For every $x \in H$,
\begin{equation}
\|\theta_A x\|^2 = \sum_{j \in \mathcal{J}} \|L_jA_j x\|^2 = \sum_{j \in \mathcal{J}} \|A_j x\|^2 = (S_A x, x),
\end{equation}
where the first identity holds because the operators $L_j$ have mutually orthogonal ranges, the second one holds because they are partial isometries, and the third one holds by the definition (2) of $S_A$. These identities and routine arguments prove (i)-(iii). \hfill \Box

Explicitly,
\begin{equation}
\theta_A = \sum_{j \in \mathcal{J}} L_jA_j
\end{equation}
and for every $x \in H$, $\theta_A(x) = \sum_{j \in \mathcal{J}} (e_j \otimes A_j x)$. As a consequence of Proposition 2.3 (ii),
\begin{equation}
\theta_A S_A^{-1/2}
\end{equation}
is an isometry, and hence
\begin{equation}
P_A := \theta_A S_A^{-1/2}\theta_A^*
\end{equation}
is the range projection of $\theta_A S_A^{-1/2}$ and hence of $\theta_A$. Moreover, $\{A_j\}_{j \in \mathcal{J}}$ is Parseval if and only if $\theta_A$ is a projection.

Given $\{A_j\}_{j \in \mathcal{J}} \in \mathcal{F}$, the operator $\theta_A \in B(H, \ell^2(\mathcal{J}) \otimes H_o)$ is called the analysis operator and the projection $P_A \in B(\ell^2(\mathcal{J}) \otimes H_o)$ is called the frame projection of $\{A_j\}_{j \in \mathcal{J}}$.

The analysis operator fully ‘encodes’ the information carried by the operator-valued frame; namely, the frame can be reconstructed from its analysis operator via the identity
\begin{equation}
A_j = L_j^*\theta_A \quad \text{for all } j \in \mathcal{J}.
\end{equation}
Indeed,
\[ L_j^* \theta_A = L_j^* \sum_{i \in J} L_i A_i = \sum_{i \in J} L_j^* L_i A_i = A_j \]

by (4). In particular, two operator-valued frames \( \{ A_j \}_{j \in J} \) and \( \{ B_j \}_{j \in J} \in \mathcal{F} \) are identical if and only if \( \theta_A = \theta_B \). Also,

\[ \theta_A^* = \sum_{i \in J} A_j L_j^*, \]

where the convergence is in the strong topology, because by (9), \( A_j^* = \theta_A^* L_j \) and \( L_j L_j^* \) are mutually orthogonal projections that sum to the identity \( I \otimes I_o \) of \( \ell^2(J) \otimes H_o \). The same argument shows that for any two operator-valued frames \( \{ A_j \}_{j \in J} \) and \( \{ B_j \}_{j \in J} \),

\[ \theta_B^* \theta_A = \sum_{i \in J} B_j^* A_j \quad \text{in the strong operator topology}. \]

Recall now that Parseval (vector) frames were shown in [15, Proposition 1.1] to be compressions of orthonormal bases. The higher multiplicity analog of that result is given by the following proposition.

**Proposition 2.4.** For every \( \{ A_j \}_{j \in J} \in \mathcal{F} \), there is a Hilbert space \( K \) containing \( H \) and \( H_o \), a collection of partial isometries \( \{ V_j \}_{j \in J} \) all with the same domain \( H_o \) and with mutually orthogonal ranges spanning \( K \), and a positive invertible operator \( T \in B(H) \) such that \( A_j = V_j^* T \) for all \( j \in J \). In particular, if \( \{ A_j \}_{j \in J} \) is a Parseval frame, then \( T \) can be chosen to be the projection on \( H \).

**Proof.** Let \( K = \ell^2(J) \otimes H_o \). Identify \( H_o \) with \( \mathbb{C} \otimes H_o \subset K \) and identify \( H \) with its image \( PAK \) under the isomorphism \( \theta_A S_A^{-1/2} \) (see (7)). Then for all \( j \in J \), we identify \( A_j \) with

\[ A_j(\theta_A S_A^{-1/2})^*|_{PAK} = L_j^*(\theta_A S_A^{-1/2} \theta_A^*)|_{PAK}. \]

Clearly, \( T := \theta_A S_A^{-1/2} \theta_A^*|_{PAK} \) is positive and invertible and it is the identity \( PA|_{PAK} \) on \( H \) if and only if \( \{ A_j \}_{j \in J} \) is a Parseval frame. \( \square \)

In analogy with the vector case (see [15, Chapter 1]), we introduce the following terminology.

**Definition 2.5.** An operator-valued frame \( \{ A_j \}_{j \in J} \) for which \( PA = I \otimes I_o \) is called a Riesz frame, and an operator-valued frame that is both Parseval and Riesz, i.e., such that \( \theta_A^* \theta_A = I \) and \( \theta_A \theta_A^* = I \otimes I_o \), is called an orthonormal frame.

**Remark 2.6.** If we identify operator-valued frames with their images in the analysis space \( \ell^2(J) \otimes H_o \), (i.e., up to right unitary equivalence of the frames, in the notation of Section 4 below), then

(i) General frames are the frames of the form \( \{ L_j^* T \}_{j \in J} \) for some positive operator \( T = PTP \) invertible in \( B(P\ell^2(J) \otimes H_o) \) and some projection \( P \in B(\ell^2(J) \otimes H_o) \). The operator \( T \) and the projection \( P \) are uniquely determined (up to Hilbert space isomorphism, i.e., right unitary equivalence of the operator-valued frames).

(ii) Parseval frames are the frames of the form \( \{ L_j^* P \}_{j \in J} \) for some projection \( P \) in \( B(\ell^2(J) \otimes H_o) \).
can be "obtained" from a single operator-valued frame. Consider first Parseval
multiplicity and same index set, operating on the same Hilbert space (upto isomorphism)
operators will carry the same information about the original frame, which can be
will produce a new analysis operator into a "larger" space; however, both analysis
operators will carry the same information about the original frame, which can be
recovered equally well from either of them.

3. Parametrization of operator-valued frames

In this section we show that all the operator-valued frames with the same multi-
plicity and same index set, operating on the same Hilbert space (up to isomorphism)
can be "obtained" from a single operator-valued frame. Consider first Parseval
frames. Following the dilation viewpoint (cf. Proposition 2.4 above), we can im-
merse all these frames in the analysis space $\ell^2(J) \otimes H_o$ by identifying them with
the compression of $\{L_j^*\}_{j \in J}$ to their frame projection. Since all the frame projections
are equivalent (each range is isomorphic to the original Hilbert space of the frame),
the partial isometries implementing these equivalences will provide the link between
the frames. To make this idea precise, and to handle at the same time frames that
are not Parseval, we introduce the following notation.

Given $\{A_j\}_{j \in J} \in \mathcal{F}$, define

$$\mathcal{M}_A := \{ M \in B(\ell^2(J) \otimes H_o) \mid M = MP_A, M^*M|_{P_M^2(J) \otimes H_o} \text{ is invertible} \}.$$  

Equivalently, $\mathcal{M}_A := \{ M \in B(P_M \ell^2(J) \otimes H_o, \ell^2(J) \otimes H_o) \mid M \text{ is left invertible} \}$. If $M \in \mathcal{M}_A$, denote by $(M^*M)^{-1} \in B(P_M \ell^2(J) \otimes H_o)$ the inverse of $M^*M|_{P_M^2(J) \otimes H_o}$.

**Theorem 3.1.** Let $\{A_j\}_{j \in J} \in \mathcal{F}$. For all $\{B_j\}_{j \in J} \in \mathcal{F}$ define

$$\Phi_A(\{B_j\}_{j \in J}) := \theta_B S_A^{-1} \theta_A^*.$$  

Then $\Phi_A : \mathcal{F} \to \mathcal{M}_A$ is one-to-one and onto and $\Phi_A^{-1}(M) = \{L_j^* M \theta_A^*\}_{j \in J}$ for all $M \in \mathcal{M}_A$. If $\{B_j\}_{j \in J} \in \mathcal{F}$ and $M := \Phi_A(\{B_j\}_{j \in J})$, then $\theta_B = M \theta_A$ and

$$V_M := M(M^*M)^{-1/2}$$

is a partial isometry that implements the equivalence $P_B \sim P_A$, i.e., $P_B = V_M V_A^*$
and $P_A = V_M^* V_M$.

**Proof.** Let $\{B_j\}_{j \in J} \in \mathcal{F}$ and let $M := \Phi_A(\{B_j\}_{j \in J})$. Then $MP_A = M$ because $P_A$
is the range projection of $\theta_A$. Moreover,

$$M^*M = \theta_A S_A^{-1} \theta_B^* \theta_B S_A^{-1} \theta_A^* \geq \theta_A S_A^{-1} a I S_A^{-1} \theta_A^* \geq \frac{a}{b} \theta_A S_A^{-1} \theta_A^* \geq \frac{a}{b} P_A,$$

where $a$ is a lower bound for $S_B$ and $b$ is an upper bound for $S_A$. Thus $M^*M$
is invertible in $B(P_A \ell^2(J) \otimes H_o)$ and hence $M \in \mathcal{M}_A$, i.e., $\Phi_A$ maps into $\mathcal{M}_A$.

Assume now that we have $\Phi_A(\{B_j\}_{j \in J}) = \Phi_A(\{C_j\}_{j \in J})$ for two frames in $\mathcal{F}$. Since
$\theta_B = \theta_B S_A^{-1} \theta_A^* \theta_A = M \theta_A$, it follows that $\theta_B = \theta_C$, and by (10), the two frames
coincide, i.e., \( \Phi_A \) is injective. We prove now that \( \Phi_A \) is onto. For any \( M \in \mathcal{M}_A \), define \( \{ B_j := L_j^*M\theta_A \}_{j \in J} \). Then
\[
\sum_{j \in J} B_j^*B_j = \sum_{j \in J} \theta_A^*M^*L_j^*M\theta_A = \theta_A^*M^*M\theta_A \leq \|M\|^2\theta_A^*\theta_A = \|M\|^2S_A.
\]
Similarly,
\[
\sum_{j \in J} B_j^*B_j \geq \|(M^*M)^{-1}\|^{-1}S_A,
\]
which proves that \( \{ B_j \}_{j \in J} \) is an operator-valued frame. Moreover,
\[
\theta_B = \sum_{j \in J} L_jB_j = \sum_{j \in J} L_jL_j^*M\theta_A = M\theta_A.
\]
We have just seen that \( \theta_B = \Phi_A(\{ B_j \}_{j \in J})\theta_A \). Thus \( M\theta_A = \Phi_A(\{ B_j \}_{j \in J})\theta_A \) and hence
\[
M = MP_A = M\theta_A S_A^{-1}\theta_A^* = \Phi_A(\{ B_j \}_{j \in J})\theta_A S_A^{-1}\theta_A^* = \Phi_A(\{ B_j \}_{j \in J}).
\]
This proves that the map \( \Phi_A \) is onto and that \( \Phi_A^{-1}(M) = \{ L_j^*M\theta_A \}_{j \in J} \) for all \( M \) in \( \mathcal{M}_A \). It remains only to compute \( P_B \), which, by definition, is the range projection of \( \theta_B = M\theta_A \). Since \( M = MP_A \) and \( \theta_A \) is one-to-one (recall that \( \theta_A S_A^{-1/2} \) is an isometry), \( P_B \) is the range projection of \( M \).
\[
V_M^*V_M = (M^*M)^{-1/2}M^*M(M^*M)^{-1/2} = P_A
\]
(recall that \( (M^*M)^{-1} \) denotes the inverse of \( M^*M \) in \( B(\ell^2(J) \otimes H_o) \)); hence \( V_M^* \) is a partial isometry. Since \( (M^*M)^{-1/2} \) is invertible, the range of \( V_M \) coincides with the range of \( M \), and hence \( P_B = V_M^*V_M \).

An easy consequence of Theorem 3.1 and its proof is the following:

**Corollary 3.2.** If \( \{ A_j \}_{j \in J} \), \( \{ B_j \}_{j \in J} \), and \( \{ C_j \}_{j \in J} \) are operator-valued frames, then

\[
\Phi_A(\{ C_j \}_{j \in J}) = \Phi_B(\{ C_j \}_{j \in J}) \Phi_A(\{ B_j \}_{j \in J}).
\]

This corollary shows that \( \Phi_A(\{ B_j \}_{j \in J}) \) behave like a partial isometry with initial projection \( P_A \) and range projection \( P_A \). In fact, if both frames are Parseval, \( \Phi_A(\{ B_j \}_{j \in J}) \) is precisely a partial isometry with these initial and range projections.

Since every operator-valued frame is right-similar to a Parseval frame (see Definition 3.1 below), we can focus on Parseval frames. For ease of reference we present in the following corollary the main result of Theorem 3.1 formulated directly for Parseval frames.

**Corollary 3.3.** Given a Parseval operator-valued frame \( \{ A_j \}_{j \in J} \in \mathcal{F} \), then
\[
\{ \{ L_j^*V\theta_A \}_{j \in J} \mid V \in B(\ell^2(J) \otimes H_o), V^*V = P_A \}
\]
is the collection of all Parseval operator-valued frames in \( \mathcal{F} \). The correspondence is one-to-one: if \( V \in B(\ell^2(J) \otimes H_o), V^*V = P_A \), and \( \{ B_j := L_j^*V\theta_A \}_{j \in J} \in \mathcal{F} \), then \( V = \theta_B\theta_A^* \).

**Proof.** We need only to show that when \( \{ A_j \}_{j \in J} \in \mathcal{F} \) is Parseval and \( M \in \mathcal{M}_A \), then the operator-valued frame \( \Phi_A^{-1}(M) := \{ B_j := L_j^*M\theta_A \}_{j \in J} \) is Parseval if and only if \( M \) is a partial isometry. This is clear since \( \theta_B\theta_B^* = M\theta_A\theta_A^*M^* = MP_A^*M^* = MM^* \) and as remarked after equation (8), \( \{ B_j \}_{j \in J} \) is Parseval if and only if \( \theta_B\theta_B^* \) is a projection. Finally, \( V = M = \Phi_A(\{ B_j \}_{j \in J}) = \theta_B\theta_A^* \) since \( S_A^{-1} = I \).
4. Similarity and composition of frames

For operator-valued frames with ranges in $H_o$ where $\dim H_o > 1$ there are two natural distinct notions of similarity, which are called ‘right’ and ‘left’.

**Definition 4.1.** Let $\{A_j\}_{j \in I}, \{B_j\}_{j \in I} \in \mathcal{F}$. We say that:

(i) $\{B_j\}_{j \in I}$ is right-similar (resp., right unitarily equivalent) to $\{A_j\}_{j \in I}$ if there is an invertible operator (resp., unitary operator) $T \in B(H)$ such that $B_j = A_jT$ for all $j \in I$.

(ii) $\{B_j\}_{j \in I}$ is left-similar (resp., left unitarily equivalent) to $\{A_j\}_{j \in I}$ if there is an invertible operator (resp., unitary operator) $R \in B(H_o)$ such that $B_j = RA_j$ for all $j \in I$.

When $\dim H_o = 1$, the left similarity is trivial (a multiplication by a nonzero scalar) and the right similarity is just the usual similarity of the vector frames (corresponding to the operators, i.e., functionals, by the Riesz Representation Theorem).

We leave to the reader the following simple results about right similarity.

**Lemma 4.2.** Let $\{A_j\}_{j \in I} \in \mathcal{F}$ have frame bounds $a$ and $b$, i.e., $aI \leq S_A \leq bI$, let $T \in B(H)$ be an invertible operator, and let $\{B_j\}_{j \in I} := \{A_jT\}_{j \in I}$. Then

(i) $\{B_j\}_{j \in I} \in \mathcal{F}$ and $\frac{1}{\|T\|^2}I \leq S_B \leq b\|T\|^2I$. In particular, if $T$ is unitary, then $\{B_j\}_{j \in I}$ has the same bounds as $\{A_j\}_{j \in I}$. Assuming that $\{A_j\}_{j \in I}$ is Parseval, then $\{B_j\}_{j \in I}$ is Parseval if and only if $T$ is unitary.

(ii) $\theta_B = \theta_AT$ and $S_B = T^*S_AT$.

(iii) $\Phi_A(\{B_j\}_{j \in I}) = \theta_A T S_A^{-1}\theta_A^*$.

Now we characterize right-similar frames.

**Proposition 4.3.** Let $\{A_j\}_{j \in I}, \{B_j\}_{j \in I} \in \mathcal{F}$ and let $M := \Phi_A(\{B_j\}_{j \in I})$. Then the following conditions are equivalent:

(i) $B_j = A_jT$ for all $j \in I$ for some invertible operator $T \in B(H)$, i.e., $\{A_j\}_{j \in I}$ and $\{B_j\}_{j \in I}$ are right-similar.

(ii) $\theta_B = \theta_AT$ for some invertible $T \in B(H)$.

(iii) $M = P_A M P_A$ is invertible in $B(P_A l^2(I) \otimes H_o)$.

(iv) $P_B = P_A$.

If the above conditions are satisfied, the invertible operator $T$ in (i) and (ii) is uniquely determined and $T = S_A^{-1}\theta_A^*\theta_B$.

In the case that $\{A_j\}_{j \in I}$ is Parseval, then $\{B_j\}_{j \in I}$ is Parseval if and only if the operator $T$ in (i), or equivalently in (ii), is unitary.

**Proof.** (i) $\iff$ (ii) One implication is given by Lemma 4.2 and the other is immediate.

(ii) $\implies$ (iii) We have $\theta_B = \theta_AT = (\theta_A T S_A^{-1}\theta_A^*)\theta_A$. Let $N := \theta_A T S_A^{-1}\theta_A^*$. Then

\[
N^*N = (\theta_A T S_A^{-1}\theta_A^*)(\theta_A T S_A^{-1}\theta_A^*) = \theta_A S_A^{-1}T^*S_AT S_A^{-1}\theta_A^* \\
\geq a\|T\|^2\theta_A S_A^{-2}\theta_A^* \geq \frac{a\|T\|^2}{b}\theta_A S_A^{-1}\theta_A^* = \frac{a\|T\|^2}{b}P_A,
\]

and $NP_A = (\theta_A T S_A^{-1}\theta_A^*)P_A = N$. Thus $N \in M_A$ and by the injectivity of the map $\Phi_A$ in Theorem 3.4, $M = \theta_A T S_A^{-1}\theta_A^*$. We now see that also $P_A M = M$, i.e.,
Let $\Phi$ be an operator-valued frame and $J$ be an operator-valued frame having frame bounds $A$ and $B$. Then $J$ is also Parseval and right-similar to $\theta J$. This can also be verified directly from $J$. This holds if $R$ is an isometry.

(iii) Assume that $\{A_j\}_{j \in J}$ is Parseval. Then $\{B_j\}_{j \in J}$ is Parseval if and only if $P_A(I \otimes R)P_A = P_A$ if and only if $P_B = (I \otimes R)P_B(I \otimes R^*)$. In particular, this holds if $R$ is an isometry.

(iv) $P_B$ is unitarily equivalent to $P_A$.

Proof. (i) Obvious.

(ii) For every $R \in B(H_n)$, $h \in H_n$, and $j \in J$, from the definition of $L_j$ we have that $L_jRh = e_j \otimes Rh = (I \otimes R)L_jh$, i.e., $L_jR = (I \otimes R)L_j$. Then

$$
\theta_B = \sum_{j \in J} L_jRA_j = \sum_{j \in J} (I \otimes R)L_jA_j = (I \otimes R)\theta_A.
$$

Consequently, $S_B = \theta_B^*\theta_B = \theta_A^*(I \otimes R^*)\theta_A$. Clearly, $M := (I \otimes R)P_A \in M_A$ and since $\theta_B = (I \otimes R)\theta_A = M\theta_A$, by the injectivity of $\Phi_A$ in Theorem 3.4 it follows that $\Phi_A(\{B_j\}_{j \in J}) = M$. This can also be verified directly from

$$
\Phi_A(\{B_j\}_{j \in J}) = \theta_BS_A^{-1}\theta_A^* = (I \otimes R)\theta_A^*S_A^{-1}\theta_A^* = (I \otimes R)P_A.
$$

(iii) Immediate from (ii).

(iv) Denote by $N(X)$ the null projection of the operator $X$. Then

$$
P_B^+ = [(I \otimes R)(\theta_A)]^+ = [(I \otimes R)(\theta_A^*)^{1/2}]^+ = N \left( (\theta_A^*)^{1/2}(I \otimes R^*) \right).
$$

**Remark 4.4.** (i) For every operator-valued frame $\{A_j\}_{j \in J}$, $\{B_j := A_jS_A^{-1/2}\}_{j \in J}$ is a Parseval operator-valued frame. Thus every operator-valued frame is right-similar to a Parseval operator-valued frame. Every operator-valued frame right unitarily equivalent to $\{B_j\}_{j \in J}$ is also Parseval and right-similar to $\{A_j\}_{j \in J}$.

(ii) The equivalence of (i) and (iv) is the higher multiplicity analog of [15, Proposition 2.6 and Corollary 2.8].

Now we consider left similarity.

**Lemma 4.5.** Let $\{A_j\}_{j \in J}$ be an operator-valued frame having frame bounds $a$ and $b$ and let $R \in B(H_n)$ be an invertible operator. Then

(i) $\{B_j := RA_j\}_{j \in J}$ is an operator-valued frame and $\frac{a}{\|R\|^2}I \leq S_B \leq b\|R\|^2I$.

In particular, if $R$ is unitary, then $\{B_j\}_{j \in J}$ has the same frame bounds as $\{A_j\}_{j \in J}$.

(ii) $\theta_B = (I \otimes R)\theta_A$, $S_B = \theta_A^*(I \otimes R^*)\theta_A$, $P_B = [(I \otimes R)\theta_A]$ is the range projection of $(I \otimes R)\theta_A$, and $\Phi_A(\{B_j\}_{j \in J}) = (I \otimes R)P_A$.

(iii) Assume that $\{A_j\}_{j \in J}$ is Parseval. Then $\{B_j\}_{j \in J}$ is Parseval if and only if $P_A(I \otimes R^*)P_A = P_A$ if and only if $P_B = (I \otimes R^*)P_A(I \otimes R^*)$. In particular, this holds if $R$ is an isometry.

Proof. (i) Obvious.

(ii) For every $R \in B(H_n)$, $h \in H_n$, and $j \in J$, from the definition of $L_j$ we have that

$$
L_jRh = e_j \otimes Rh = (I \otimes R)L_jh,
$$

i.e., $L_jR = (I \otimes R)L_j$. Then

$$
\theta_B = \sum_{j \in J} L_jRA_j = \sum_{j \in J} (I \otimes R)L_jA_j = (I \otimes R)\theta_A.
$$

Consequently, $S_B = \theta_B^*\theta_B = \theta_A^*(I \otimes R^*)\theta_A$. Clearly, $M := (I \otimes R)P_A \in M_A$ and since $\theta_B = (I \otimes R)\theta_A = M\theta_A$, by the injectivity of $\Phi_A$ in Theorem 3.4 it follows that $\Phi_A(\{B_j\}_{j \in J}) = M$. This can also be verified directly from

$$
\Phi_A(\{B_j\}_{j \in J}) = \theta_BS_A^{-1}\theta_A^* = (I \otimes R)\theta_A^*S_A^{-1}\theta_A^* = (I \otimes R)P_A.
$$

(iii) Immediate from (ii).

(iv) Denote by $N(X)$ the null projection of the operator $X$. Then

$$
P_B^+ = [(I \otimes R)(\theta_A)]^+ = [(I \otimes R)(\theta_A^*)^{1/2}]^+ = N \left( (\theta_A^*)^{1/2}(I \otimes R^*) \right).$$
Now \( x \in N \left( (\theta_A \theta^*_A)^{1/2} (I \otimes R^*) \right) \) if and only if \( x \in (I \otimes R^*)^{-1} N \left( (\theta_A \theta^*_A)^{1/2} \right) \); thus
\[
P^1_B = (I \otimes R^{-1})^* P^1_A = [(I \otimes R^{-1})^* P^1_A] \sim [P^1_A (I \otimes R^{-1})] = P^1_A,
\]
where we use the well-known fact \([X] \sim [X^*]\). Since \( P_B \sim P_A \) (e.g., see Theorem 3.1), it follows that \( P_B \) and \( P_A \) are unitarily equivalent. \( \square \)

In Proposition 4.3 we have seen that the invertible operator implementing the right-similarity of two operator-valued frames is uniquely determined and that it must be a unitary operator when both frames are Parseval. The following example shows that neither of these conclusions holds in the case of left-similarities.

**Example 4.6.** Let \( H := \ell^2(\mathbb{J}) \), \( H_o := \mathbb{C}^2 \), \( Q_1, Q_2 \) be two orthogonal (rank-one) projections in \( B(H_o) \), let \( P := I \otimes Q_1 \), and let \( \{A_j := L^*_j P|_{P\ell^2(\mathbb{J}) \otimes H_o}\}_{j \in J} \). By Example 2.2 \( \{A_j\}_{j \in J} \) is a Parseval operator-valued frame. Moreover, for every \( \lambda \neq 0 \), \( R := Q_1 + \lambda Q_2 \) is an invertible operator and
\[
RA_j = RL^*_j P|_{P\ell^2(\mathbb{J}) \otimes H_o} = L^*_j (I \otimes R) P|_{P\ell^2(\mathbb{J}) \otimes H_o} = L^*_j (I \otimes Q_1 + \lambda Q_2) (I \otimes Q_1)|_{P\ell^2(\mathbb{J}) \otimes H_o} = L^*_j P|_{P\ell^2(\mathbb{J}) \otimes H_o} = A_j.
\]

If \( A_j \in H_o \) and \( \{B_j = RA_j\}_{j \in J} \) are two left-similar Parseval operator-valued frames, then \( P_B = (I \otimes R) P_A (I \otimes R^*) \) by Lemma 4.3 and furthermore there is a unitary operator \( U \) for which \( P_B = UP_A U^* \). The following example shows that it can be impossible to choose \( U \in I \otimes B(H_o) \).

**Example 4.7.** Let \( H := \ell^2(\mathbb{J}) \), \( H_o := \mathbb{C}^2 \), \( P_1, P_2 \) be two orthogonal projections in \( B(H) \), \( Q_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q_3 := \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \) and \( R := \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \).

Define \( P := P_1 \otimes Q_1 + P_2 \otimes Q_2 \), \( \{A_j := L^*_j P|_{P\ell^2(\mathbb{J}) \otimes H_o}\}_{j \in J} \), and \( \{B_j := RA_j\}_{j \in J} \). Then by Example 2.2 \( \{A_j\} \) is a Parseval operator-valued frame and \( P_A = P \). Since
\[
P(I \otimes R^*)P = P_1 \otimes (Q_1 R^* RQ_1) + P_2 \otimes (Q_2 R^* RQ_2) = P_1 \otimes Q_1 + P_2 \otimes Q_2 = P,
\]
by Lemma 4.3 (i) and (iii), \( \{B_j\}_{j \in J} \) is also a Parseval operator-valued frame and
\[
P_B = (I \otimes R) \theta_A \theta^*_A (1 \otimes R^*) = (I \otimes R) P(1 \otimes R^*)
= P_1 \otimes RQ_1 R^* + P_2 \otimes RQ_2 R^* = P_1 \otimes Q_1 + P_2 \otimes Q_3.
\]
This implies that if \( P_B = UP_A U^* \) for some unitary \( U \), then \( U \notin I \otimes B(H_o) \). Indeed if \( U = 1 \otimes W \) for some unitary \( W \in B(H_o) \), then by the above computation,
\[
P_B = P_1 \otimes WQ_1 W^* + P_2 \otimes WQ_2 W^* = P_1 \otimes Q_1 + P_2 \otimes Q_3,
\]
which is impossible since \( WQ_1 W^* WQ_2 W^* = 0 \) while \( Q_1 Q_3 \neq 0 \).

Operator-valued frames can be both right and left-similar. The following proposition determines when this occurs.

**Proposition 4.8.** Let \( \{A_j\}_{j \in J} \in \mathbb{F} \), \( R \in B(H_o) \) be an invertible operator, and let \( \{B_j := RA_j\}_{j \in J} \). Then the following conditions are equivalent.

(i) \( \{B_j\}_{j \in J} \) and \( \{A_j\}_{j \in J} \) are right-similar.

(ii) \( (I \otimes R) P_A = F_A(I \otimes R) P_A \) is invertible in \( B(P_A \ell^2(\mathbb{J}) \otimes H_o) \).

(iii) \( F_A(I \otimes R) P_A = 0 \) and \( F_A(I \otimes R^{-1}) P_A = 0 \).

(iv) \( \sum_{j \in J} A^*_j RA_j \) for every \( i \in \mathbb{J} \)

\[
RA_i = A_i S_A^{-1} \sum_{j \in J} A^*_j RA_j \quad \text{for every } i \in \mathbb{J}
\]
and

\[ R^{-1}A_i = A_i S_A^{-1} \sum_{j \in J} A_j^* R^{-1} A_j \quad \text{for every } i \in J. \quad (15) \]

(v)

\[ RA_i = A_i S_A^{-1} \sum_{j \in J} A_j^* R A_j \quad \text{for every } i \in J \]

and

\[ \left( \sum_{j \in J} A_j^* R A_j \right)^{-1} = S_A^{-1} \left( \sum_{j \in J} A_j^* R^{-1} A_j \right) S_A^{-1}. \quad (16) \]

If \( R \) is unitary, then these conditions are also equivalent to

(vi) \((I \otimes R)P_A = P_A(I \otimes R)\).

(vii) \( R \) commutes with \( A_j S_A^{-1} A_j^* \) for all \( i, j \in J \).

**Proof.**

(i) \( \iff \) (ii) Let \( M := \Phi_A(\{B_j\}_{j \in J}) \). Then by Lemma 4.3, \( M = (I \otimes R)P_A \).

By Proposition 4.3, we see that \( \{B_j\}_{j \in J} \) is right-similar to \( \{A_j\}_{j \in J} \) if and only if \( MP_A = P_A M P_A \) is invertible in \( B(\ell^2(J) \otimes H_o) \).

(ii) \( \iff \) (iii) Set

\[ I \otimes R := \begin{pmatrix} P_A(I \otimes R)P_A & P_A(I \otimes R)P_A^+ \\ 0 & P_A(I \otimes R)P_A^+ \end{pmatrix} \]

and

\[ (I \otimes R)^{-1} := \begin{pmatrix} P_A(I \otimes R)^{-1}P_A & P_A(I \otimes R)^{-1})P_A^+ \\ 0 & P_A(I \otimes R)^{-1}P_A^+ \end{pmatrix}. \]

Then it is immediate to verify that \( P_A(I \otimes R)^{-1}P_A = 0 \) if and only if

\[ (P_A(I \otimes R)P_A)(P_A(I \otimes R)^{-1}P_A) = P_A, \]

and if and only if \( P_A(I \otimes R)P_A \) is invertible.

(iii) \( \iff \) (iv) Recall that \( (I \otimes R)L_j = L_j R \) for all \( j \in J \) and that \( P_A = \theta_A S_A^{-1} \theta_A^* \).

Therefore

\[ (I \otimes R)P_A = (I \otimes R)\theta_A S_A^{-1} \theta_A^* = (I \otimes R) \sum_{j \in J} L_j A_j S_A^{-1} \theta_A^* = \sum_{j \in J} L_j R A_j S_A^{-1} \theta_A^* \]

and

\[ P_A(I \otimes R)P_A = \theta_A S_A^{-1} \theta_A^* (I \otimes R) \theta_A S_A^{-1} \theta_A^* \]

\[ \quad \quad = \left( \sum_{j \in J} L_j A_j \right) S_A^{-1} \left( \sum_{i, j \in J} A_i^* L_i^*(I \otimes R) L_j A_j S_A^{-1} \theta_A^* \right) \]

\[ \quad \quad = \left( \sum_{j \in J} L_j A_j \right) S_A^{-1} \left( \sum_{i, j \in J} A_i^* R L_i^* L_j A_j S_A^{-1} \theta_A^* \right) \]

\[ \quad \quad = \left( \sum_{j \in J} L_j A_j \right) S_A^{-1} \left( \sum_{j \in J} A_j^* R A_j S_A^{-1} \theta_A^* \right) \]

Hence \( P_A(I \otimes R)^{-1}P_A = 0 \) if and only if

\[ \sum_{j \in J} L_j R A_j S_A^{-1} \theta_A^* = \left( \sum_{j \in J} L_j A_j \right) S_A^{-1} \left( \sum_{j \in J} A_j^* R A_j S_A^{-1} \theta_A^* \right). \quad (17) \]

By multiplying \( (17) \) on the left by \( L_j^* \) and on the right by \( \theta_A \), we obtain \( (14) \). Conversely, by multiplying \( (14) \) on the left by \( L_i \) and on the right by \( S_A^{-1} \theta_A^* \) and
summing over \( i \in \mathbb{J} \) we obtain back (17). Thus \( P_A^1(I \otimes R^{-1})P_A = 0 \) is equivalent to (14). By the same argument, \( P_A^1(I \otimes R^{-1})P_A = 0 \) is equivalent to (16).

(iv) \( \iff \) (v) Assume that (14) and (15) hold. Then

\[
A_i^* A_i = A_i^* R^{-1} A_i S_A^{-1} \left( \sum_{j \in \mathbb{J}} A_j^* R A_j \right)
\]

for all \( i \), and hence by summing over \( i \in \mathbb{J} \) and then multiplying on the left and on the right by \( S_A^{-1/2} \) we obtain

\[
S_A^{-1/2} \left( \sum_{i \in \mathbb{J}} A_i^* R^{-1} A_i S_A^{-1} \left( \sum_{j \in \mathbb{J}} A_j^* R A_j \right) \right) S_A^{-1/2} = I.
\]

Similarly,

\[
S_A^{-1/2} \left( \sum_{i \in \mathbb{J}} A_i^* R A_i S_A^{-1} \left( \sum_{j \in \mathbb{J}} A_j^* R^{-1} A_j \right) \right) S_A^{-1/2} = I.
\]

Thus

\[
\left( S_A^{-1/2} \left( \sum_{j \in \mathbb{J}} A_j^* R A_j \right) S_A^{-1/2} \right)^{-1} = S_A^{-1/2} \left( \sum_{j \in \mathbb{J}} A_j^* R^{-1} A_j \right) S_A^{-1/2},
\]

and hence (10) holds. Conversely, if (14) and (10) hold, then by multiplying (14) on the right by the right-hand side of (16) we easily obtain (15).

Assume now that \( R \) is unitary.

(iii) \( \iff \) (vi) Obvious.

(vi) \( \implies \) (vii) As seen above in the course of the proof of (iii) \( \iff \) (iv),

\[
(I \otimes R)P_A = \sum_{i,j \in \mathbb{J}} L_{ij} R A_j S_A^{-1} A_i^* L_i^*
\]

and

\[
P_A(I \otimes R) = \sum_{i,j \in \mathbb{J}} L_{ij} A_j S_A^{-1} A_i^* R L_i^*.
\]

Thus

\[
\sum_{i,j \in \mathbb{J}} L_{ij} R A_j S_A^{-1} A_i^* L_i^* = \sum_{i,j \in \mathbb{J}} L_{ij} A_j S_A^{-1} A_i^* R L_i^*.
\]

By multiplying this identity on the left by \( L_j^* \) and on the right by \( L_i \) and recalling (4) we obtain (vii).

(vii) \( \implies \) (iv) For every \( i \in \mathbb{J} \),

\[
A_i S_A^{-1} \sum_{j \in \mathbb{J}} A_j^* R A_j = R A_i S_A^{-1} \sum_{j \in \mathbb{J}} A_j^* A_j = R A_i,
\]

i.e., (14) holds. Since \( R^{-1} = R^* \) also commutes with all \( A_j S_A^{-1} A_i^* \), the same argument shows that (15) too holds.

**Composition of frames.** Let \( A = \{A_j\}_{j \in \mathbb{J}} \) be an operator-valued frame in \( B(H, H_0) \) and \( B = \{B_m\}_{m \in \mathbb{M}} \) be an operator-valued frame in \( B(H_0, H_1) \). Then it is easy to check that \( \{C_{(m,j)} := B_m A_j\}_{m \in \mathbb{M}, j \in \mathbb{J}} \) is an operator-valued frame in \( B(H, H_1) \). The operator-valued frame \( \{C_{(m,j)}\}_{(m,j) \in \mathbb{M} \times \mathbb{J}} \) is called the composition of the frames \( A = \{A_j\}_{j \in \mathbb{J}} \) and \( B = \{B_m\}_{m \in \mathbb{M}} \).

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Proposition 4.9. Let \( \{ C_{(m,j)} := B_mA_j \}_{(m,j) \in \mathbb{M} \times J} \) be the composition of the operator-valued frames \( \{ A_j \}_{j \in J} \) and \( B = \{ B_m \}_{m \in \mathbb{M}}. \) Then

\( \theta_C = (I_J \otimes \theta_B) \theta_A \) and \( S_C = \theta_A^*(I_J \otimes S_B) \theta_A, \) where \( I_J \) is the identity on \( L^2(J). \) In particular, if \( \{ A_j \}_{j \in J} \) and \( \{ B_m \}_{m \in \mathbb{M}} \) are Parseval, then \( \{ C_{(m,j)} \}_{m \in \mathbb{M}, j \in J} \) is Parseval.

(ii) If \( P_A \) commutes with \( (I_J \otimes S_B) \), then \( P_C = (I_J \otimes \theta_B) P_A (I_J \otimes S_B^{-1}) P_A (I_J \otimes \theta_B^*). \)

Proof. (i) Let \( \{ e_j \}_{j \in J} \) and \( \{ f_m \}_{m \in \mathbb{M}} \) denote the standard orthonormal bases of \( \ell^2(J) \) and \( \ell^2(\mathbb{M}), \) respectively. Then for any \( x \in H, \)

\[
(I_J \otimes \theta_B) \theta_A(x) = (I_J \otimes \theta_B) \sum_{j \in J} (e_j \otimes A_j)x = \sum_{j \in J} (e_j \otimes \theta_B(A_j x)) = \sum_{m \in \mathbb{M}, j \in J} (e_j \otimes f_m \otimes B_m A_j x) = \theta_C(x).
\]

The formula for \( S_C \) now follows directly, as well as the Parseval case.

(ii) If \( P_A \) commutes with \( (I_J \otimes S_B) \), then

\[
P_A (I_J \otimes S_B^{-1}) P_A = (P_A (I_J \otimes S_B) P_A)^{-1};
\]

hence \( S_C^{-1} = S_A^{-1} \theta_A^*(I_J \otimes S_B^{-1}) \theta_A S_A^{-1}, \) and thus

\[
P_C = \theta_C S_C^{-1} \theta_C^* = (I_J \otimes \theta_B) \theta_A S_A^{-1} \theta_A^*(I_J \otimes S_B^{-1}) \theta_A S_A^{-1} \theta_A^*(I_J \otimes \theta_B) = (I_J \otimes \theta_B) P_A (I_J \otimes S_B^{-1}) P_A (I_J \otimes \theta_B).
\]

Remark 4.10. (i) If the composition of an operator-valued frame \( \{ B_m \}_{m \in \mathbb{M}} \) with the frame \( \{ A_j \}_{j \in J} \) is the same as the composition with the frame \( \{ A'_j \}_{j \in J} \), then \( \{ A_j \}_{j \in J} = \{ A'_j \}_{j \in J}. \) Indeed, if \( B_m A_j = B_m A'_j \) for all \( m \in \mathbb{M} \) and \( j \in J, \)

\[
\sum_{m \in \mathbb{M}} B_m^* B_m A_j = S_B A_j = S_B A'_j,
\]

and hence \( A_j = A'_j, \) since \( S_B \) is invertible.

(ii) Assume that \( \bigcup_{j \in J} A_j H_o \) is dense in \( H_o. \) Then the composition of \( \{ A_j \}_{j \in J} \) with \( \{ B_m \}_{m \in \mathbb{M}} \) equals the composition of \( \{ A_j \}_{j \in J} \) with \( \{ B'_m \}_{m \in \mathbb{M}} \) if and only if \( \{ B_m \}_{i \in I} = \{ B'_m \}_{m \in \mathbb{M}}. \)

Given an operator-valued frame \( \{ A_j \}_{j \in J} \) with \( A_j \in B(H, H_o) \) and given an arbitrary vector frame \( \{ B_m \}_{m \in \mathbb{M}} \) with \( B_m \in B(H_o, \mathbb{C}), \) we can view the vector frame \( \{ C_{(m,j)} := B_m A_j \}_{(m,j) \in \mathbb{M} \times J} \) to be a "decomposition" of the original operator-valued frame.

This decomposition is of course not unique. For instance, in the discussion after Definition 2.1 we chose \( \{ B_m \}_{m \in \mathbb{M}} \) to be an orthonormal basis \( \{ e_m \}_{m \in \mathbb{M}} \) of \( H_o \) and then \( C_{(m,j)} := B_m A_j \) corresponds to the vector frame \( \{ A_j^* e_m \}_{(m,j) \in \mathbb{M} \times J}. \) If \( \{ B'_m \}_{m \in \mathbb{M}} \) is (right) similar to \( \{ B_m \}_{m \in \mathbb{M}}, \) i.e., \( B'_m = B_m R \) for some invertible operator \( R \in B(H_o) \) and all \( m \in \mathbb{M} \) (i.e., if \( B'_m \) is a Riesz frame (see Remark 2.6), then \( C_{(m,j)} := B_m A_j \) corresponds to the vector frame \( \{ A_j^* R^* e_m \}_{m \in \mathbb{M}, j \in J} \) and provides a vector frame decomposition both of the operator-valued frame \( \{ A_j \}_{j \in J} \) and also of the left-similar operator-valued frame \( \{ R A_j \}_{j \in J}. \) \( \{ A_j^* R^* e_m \}_{(m,j) \in \mathbb{M} \times J} \) is similar to \( \{ A_j^* e_m \}_{(m,j) \in \mathbb{M} \times J} \) only if \( R \) satisfies the conditions of Proposition P: left right. However, if \( T \in B(H) \) is invertible, \( \{ T A_j^* e_m \}_{(m,j) \in \mathbb{M} \times J} \) is by definition
similar to \( \{ A_j^* e_m \}_{(m,j) \in \mathbb{N} \times J} \) and provides a vector decomposition of the operator-valued frame \( \{ A_j T^* \}_{j \in J} \).

5. DUAL FRAMES, COMPLEMENTARY FRAMES, AND DISJOINTNESS

Dual frames. A vector frame \( \{ b_j \}_{j \in J} \) on a Hilbert space \( H \) is said to be the dual of another vector frame \( \{ a_j \}_{j \in J} \) on \( H \) if

\[
x = \sum_{j \in J} (x, a_j) b_j \quad \text{for all } x \in H.
\]

It is well known and easy to see that this condition can be reformulated in terms of the analysis operators \( \theta_b \) and \( \theta_a \) of the two frames as

\[
\theta_b^* \theta_a = I.
\]

In particular, \( \{ S_a^{-1} a_j \}_{j \in J} \) has analysis operator \( \theta_a S_a^{-1} \) and is called the canonical dual of the frame \( \{ a_j \}_{j \in J} \) (other duals are called alternate duals). When \( \{ a_j \}_{j \in J} \) is Parseval, the identity

\[
x = \theta_b^* \theta_a x = \sum_{j \in J} (x, a_j) a_j \quad \text{for all } x \in H
\]

is called the reconstruction formula (see [15 Sections 1.2, 1.3]).

These notions extend naturally to operator-valued frames:

**Definition 5.1.** Let \( \{ A_j \}_{j \in J}, \{ B_j \}_{j \in J} \in \mathcal{F} \). Then \( \{ B_j \}_{j \in J} \) is called a dual of \( \{ A_j \}_{j \in J} \) if \( \theta_B^* \theta_A = I \). The operator-valued frame \( \{ A_j S_A^{-1} \}_{j \in J} \) is called the canonical dual frame of \( \{ A_j \}_{j \in J} \).

**Remark 5.2.** (i) Notice that \( \theta_B^* \theta_A = I \) if and only if \( \theta_a^* \theta_B = I \); i.e., \( \{ B_j \}_{j \in J} \) is a dual of \( \{ A_j \}_{j \in J} \) if and only if \( \{ A_j \}_{j \in J} \) is a dual of \( \{ B_j \}_{j \in J} \) (cf. [15 Proposition 1.13]).

(ii) If we consider two operator-valued frames \( \{ A_j \}_{j \in J} \) with \( A_j \in B(H_1, H_a) \) and \( \{ B_j \}_{j \in J} \) with \( B_j \in B(H_2, H_a) \) with a given unitary \( U : H_1 \to H_2 \), and if we want to keep track of the different Hilbert spaces, then we would define duality between them (relative to the choice of the unitary \( U \)) by asking that \( \theta_B^* \theta_A = U \), or, equivalently, that \( \theta_A^* \theta_B = U^* \).

In general, duals are far from unique. Theorem 5.1 provides a natural way to parametrize the collection of all the dual frames of a given operator-valued frame. Recall that for \( \{ A_j \}_{j \in J}, \{ B_j \}_{j \in J} \in \mathcal{F} \), there is a unique \( M \in \mathcal{B}(\ell^2(J) \otimes H_a) \) such that \( \theta_B = M \theta_A, M = M P_A \) and \( M^* M \) is invertible in \( \mathcal{B}(P_A \ell^2(J) \otimes H_a) \); namely, \( M := \Phi_A(\{ B_j \}_{j \in J}) \). Then in the two-by-two matrix relative to the decomposition \( P_A + P_A^* = I \otimes I_o, M = \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix} \) and \( X^* X + Y^* Y \) is invertible.

**Proposition 5.3.** Let \( \{ A_j \}_{j \in J}, \{ B_j \}_{j \in J} \in \mathcal{F} \) and let \( M := \Phi_A(\{ B_j \}_{j \in J}) \). Then the following conditions are equivalent.

(i) \( \{ B_j \}_{j \in J} \) is a dual of \( \{ A_j \}_{j \in J} \).

(ii) \( \sum_{j \in J} B_j^* A_j = I \), where the convergence is in the strong operator topology.

(iii) \( P_A M P_A = \theta_A S_A^{-2} \theta_A \).

In particular, \( \{ B_j \}_{j \in J} \) is the canonical dual of \( \{ A_j \}_{j \in J} \) if and only if \( M = P_A M P_A \).
Example 5.5. Let \( H := l^2(\mathbb{J}) \), \( H_\theta := \mathbb{C}^2 \), \( Q_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( Q_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \),
and \( R := \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \). Define \( P := I \otimes Q_1 \) and define the operator-valued frames \( \{A_j := L_j^* P_{|P(l^2(\mathbb{J}) \otimes H_\theta)}\}_j \in J \), and \( \{B_j := RA_j\}_j \in J \). Then by Example 2.2, \( \{A_j\} \) is a Parseval operator-valued frame and \( P_A = P \). Now \( M = (I \otimes R)P \), \( PMP = P \) and hence \( \{B_j := RA_j\}_j \in J \) is a dual of \( \{A_j\}_j \in J \) for every \( \lambda \). \( P^\perp MP = I \otimes \begin{pmatrix} 0 & 0 \\ \lambda & 1 \end{pmatrix} \) however; hence every \( \lambda \) defines a different operator-valued frame \( \{B_j\}_j \in J \).

Disjoint frames and complementary frames. The treatment given in [15] Chapter 2 for the vector case generalizes without difficulties to the higher multiplicity case. For the reader’s convenience we present the definitions and one of the key arguments.
Definition 5.6. Let \( \{A_j\}_{j \in J} \) and \( \{B_j\}_{j \in J} \) be two operator-valued frames with \( A_j \in B(H_A, H_a) \) and \( B_j \in B(H_B, H_a) \) for all \( j \in J \). Then the two frames are called:

(i) disjoint if \( \{A_j \oplus B_j\}_{j \in J} \) is an operator-valued frame;

(ii) strongly disjoint (also called orthogonal) if there are two invertible operators \( T_A \in B(H_A) \) and \( T_B \in B(H_B) \) such that \( \{A_j T_A \oplus B_j T_B\}_{j \in J} \) is an Parseval operator-valued frame on \( H_A \oplus H_B \);

(iii) strongly complementary if there are two invertible operators \( T_A \in B(H_A) \) and \( T_B \in B(H_B) \) such that \( \{A_j T_A \oplus B_j T_B\}_{j \in J} \) is a Parseval operator-valued frame on \( H_A \oplus H_B \).

Proposition 5.7. Let \( \{A_j\}_{j \in J} \) and \( \{B_j\}_{j \in J} \) be two operator-valued frames as in the Definition 5.6. Then the two frames are:

(i) disjoint if and only if \( P_A H_J \cap P_B H_J = \{0\} \) and \( P_A H_J + P_B H_J \) is closed;

(ii) strongly disjoint if and only if \( P_A P_B = 0 \) if and only if \( \theta_A \theta_B = 0 \); and

(iii) strongly complementary if and only if \( P_A + P_B = I \otimes I_o \).

Proof. (i) \( \{A_j \oplus B_j\}_{j \in J} \) is an operator-valued frame if and only if \( \{A_j T_A \oplus B_j T_B\}_{j \in J} \) is also an operator-valued frame for any choice of invertible operators \( T_A \) and \( T_B \).

Since right-similarities do not change the frame projections (see Proposition 4.3), we can assume that both frames are Parseval. Since \( (A_j \oplus B_j)(x \oplus y) = A_j x + B_j y \) for all \( x \in H_A, y \in H_B \), a simple computation shows that \( (A_j \oplus B_j)^* z = A_j^* z \oplus B_j^* z \) for all \( z \in H_a \); hence

\[
(A_j \oplus B_j)^*(A_j \oplus B_j)(x \oplus y), (x \oplus y) = (A_j x + B_j y, A_j x + B_j y) = (A_j x, A_j x) + (B_j y, B_j y) + (A_j x, B_j y) + (B_j y, A_j x) = (A_j^* A_j x, x) + (B_j^* B_j y, y) + (A_j^* A_j x, y) + (B_j^* B_j y, x).
\]

Now sum over \( J \) using the fact (see (11)) that all the series converge in the strong operator topology, e.g., \( \sum_{j \in J} A_j^* B_j = \theta_A^* \theta_B \) and similarly for the other series. Thus

\[
\sum_{j \in J} (A_j \oplus B_j)^*(A_j \oplus B_j)(x \oplus y), (x \oplus y) = (\theta_A^* \theta_A x, x) + (\theta_B^* \theta_B y, y) + (\theta_B^* \theta_A x, y) + (\theta_A^* \theta_B y, x) = (\theta_A x + \theta_B y, \theta_A x + \theta_B y).
\]

As in the proof of [15] Theorem 2.9, the projections \( P_A \) and \( P_B \) satisfy the condition in (i) if and only if

\[
a \|\theta_A x + \theta_B y\| \leq \sqrt{\|\theta_A x\|^2 + \|\theta_B y\|^2} = \sqrt{\|x\|^2 + \|y\|^2} = \|x \oplus y\| \leq b \|\theta_A x + \theta_B y\|
\]

for some \( a, b > 0 \). By definition, this is precisely the condition that guarantees that \( \{A_j \oplus B_j\}_{j \in J} \) is an operator-valued frame.

(ii) It is clear that the three conditions \( P_A P_B = 0 \), \( \theta_A^* \theta_B = 0 \), and \( \theta_B^* \theta_A = 0 \) are all equivalent and that they also imply the condition in (i). If these conditions hold, by the proof of (i), \( (S_{A \oplus B}(x \oplus y), (x \oplus y)) = (S_A x, x) + (S_B y, y) \) for all \( x \in H_A, y \in H_B \). In particular, \( S_{A \oplus B} = I_A \otimes I_B \) if (and only if) \( S_A = I_A \) and \( S_B = I_B \); i.e., it is sufficient to choose right-similarities that make the two operator-valued frames Parseval to obtain that their direct sum is also Parseval. Conversely, assume without loss of generality that the direct sum of the frames is already
Parseval and hence both frames are Parseval. Then, again by the computation in the first part of the proof,
\[ \|\theta_A x\|^2 + \|\theta_A y\|^2 = \|x\|^2 + \|y\|^2 = (S_{A \otimes B} (x \oplus y), (x \oplus y)) = \|\theta_A x + \theta_B y\|^2. \]

This clearly implies that the ranges of \( \theta_A \) and \( \theta_B \) are orthogonal.

(iii) From the proof of (ii), it is easy to see that if \( P_A P_B = 0 \), then \( P_{A+B} = P_A + P_B \). The rest of the proof is then obvious.

□

Corollary 5.8. Let \( \{A_j\}_{j \in J} \) be an operator-valued frame on a Hilbert space \( H \), with range in \( H_o \).

(i) Up to right unitary equivalence, the collection of strong complements of \( \{A_j\}_{j \in J} \) is uniquely parametrized by
\[ \{ \{L_j^* T\}_{j \in J} \mid T = P_A^* TP_A \geq 0, T \text{ invertible in } B(P_A^* \ell^2(J) \otimes H_o) \}. \]

(ii) Up to right unitary equivalence, the collection of operator-valued frames strongly disjoint from \( \{A_j\}_{j \in J} \) is uniquely parametrized by
\[ \{ \{L_j^* T\}_{j \in J} \mid P \leq P_A^*, T = PTP \geq 0, T \text{ invertible in } B(P\ell^2(J) \otimes H_o) \}. \]

6. UNITARY SYSTEMS AND GROUPS

General unitary systems and local commutants. In the terminology of [9] and [15], a unitary system \( \mathcal{U} \) on a Hilbert space \( H \) is simply a collection of unitary operators that includes the identity. Following the customary terminology for (vector) frames, we introduce the analogous notion for operator-valued frames:

Definition 6.1. An operator \( A \in B(H, H_o) \) is called an operator frame generator for a unitary system \( \mathcal{U} \) if \( \{AU^*_U\}_{U \in \mathcal{U}} \) is an operator-valued frame. If \( \{AU^*_U\}_{U \in \mathcal{U}} \) is Parseval, \( A \) is said to be a Parseval operator frame generator.

If \( \dim H_o = 1 \), i.e., \( A \) corresponds to a vector \( \psi \), then \( AU^* \) corresponds to the vector \( U\psi \). Recall from [15] Proposition 3.1 that given a wavelet generator for a unitary system \( \mathcal{U} \), i.e., a vector \( \psi \) such that \( \{U\psi\}_{U \in \mathcal{U}} \) is an orthonormal basis, then a vector \( \phi \) is a Parseval frame generator from \( \mathcal{U} \) if and only if \( \phi = V\psi \) for a co-isometry \( V \) such that \( (VV^* - UV^*) = 0 \) for every \( U \in \mathcal{U} \). The local commutant at \( \psi \) is defined in [9] as the collection
\[ C_{\psi}(\mathcal{U}) := \{ T \in B(H) \mid (TU - UT)\psi = 0 \quad \text{for all} \quad U \in \mathcal{U} \}. \]
The multi-dimensional analog of an orthonormal basis is a collection of partial isometries with mutually orthogonal equivalent domains spanning the Hilbert space, or equivalently, an operator-valued frame with frame projection \( \theta_A \theta_A^* = I \).

Proposition 6.2. Suppose that \( A \in B(H, H_o) \) is a frame generator for a unitary system \( \mathcal{U} \) for which \( \theta_A \theta_A^* = I \). Then an operator \( B \in B(H, H_o) \) is a Parseval frame generator for \( \mathcal{U} \) if and only if \( B = AV^* \) for some co-isometry \( V \) such that \( (VV^* - UV^*)A^* = 0 \) for every \( U \in \mathcal{U} \). If \( B = AV^* \) for such a co-isometry, then \( V := \theta_B \theta_A \).

Proof. Assume \( B \) is a Parseval frame generator for \( \mathcal{U} \) and let \( V := \theta_B \theta_A \). Then by the hypotheses, \( \theta_B \theta_A \theta_A^* \theta_B = I \); i.e., \( V \) is a co-isometry and
\[ B = BI = L_I \theta_B = L_I \theta_A \theta_A^* \theta_B = AV^*. \]
Since $AU^* = L_U^* \theta_A$ and $BU^* = L_U^* \theta_B A$ for all $U \in \mathcal{U}$, we have

$$AU^* V^* = L_U^* \theta_A \theta_B = L_U^* \theta_B = BU^* = AV^* U^*.$$  

Moreover, if $AV^* = AW^*$ and $A(WU - UW)^* = 0$ for every $U \in \mathcal{U}$, then also $AU^* V^* = AV^* W^*$ and hence $\theta_A V^* = \theta_A W^*$, whence $V = W$. Conversely, if $V$ is a co-isometry and $A(VU - UV)^* = 0$ for every $U \in \mathcal{U}$, then

$$\sum_{U \in \mathcal{U}} UV A^* AV^* U^* = V \left( \sum_{U \in \mathcal{U}} U A^* A U^* \right) V^* = VV^* = I,$$

whence $AV^*$ is a Parseval frame generator for $\mathcal{U}$. \hfill \Box

**Discrete group representations.** Let $G$ be a discrete group, not necessarily countable and let $\Lambda$ be the left regular representation of $G$ (resp., let $\rho$ be the right regular representation of $G$). Denote by $\mathcal{L}(G) \subset B(\ell^2(G))$ (resp., $\mathcal{R}(G)$) the von Neumann algebra generated by the unitaries $\{\lambda_g\}_{g \in G}$ (resp., $\{\rho_g\}_{g \in G}$). It is well known that both $\mathcal{L}(G)' = \mathcal{R}(G)$ and $\mathcal{R}(G)' = \mathcal{L}(G)$ are finite von Neumann algebras that share a faithful trace vector $\chi_e$, where $\{\chi_g\}_{g \in G}$ is the standard basis of $\ell^2(G)$.

Let $H_\sigma$ be a Hilbert space and $I_\sigma$ be the identity of $B(H_\sigma)$. Then we call $\lambda \otimes id : G \ni g \longrightarrow \lambda_g \otimes I_\sigma$ the left regular representation of $G$ with multiplicity $H_\sigma$. Set $H_G = \ell^2(G) \otimes H_\sigma$.

Given a unitary representation $(G, \pi, H)$, denote by $\pi(G)''$ the von Neumann algebra generated by $\{\pi_g\}_{g \in G}$. Operator frame generators, if any, for the unitary system $\pi_g$, are called generators for the representation. Explicitly:

**Definition 6.3.** Let $(G, \pi, H)$ be a unitary representation of the discrete group $G$ on the Hilbert space $H$. Then an operator $A \in B(H, H_\sigma)$ is called a frame generator (resp. a Parseval frame generator) with range in $H_\sigma$ for the representation if $\{A_g := A \pi_g^{-1}\}_{g \in G}$ is an operator-valued frame (resp. a Parseval operator-valued frame).

Before characterizing those representations that have an operator frame generator and then parametrizing its generators, we need the following preliminary lemma.

**Lemma 6.4.** Let $A$ and $B$ be two generators with range in $H_\sigma$ for a unitary representation $(G, \pi, H)$. Then

(i) $\theta_A \pi_g = (\lambda_g \otimes I_\sigma) \theta_A$ for all $g \in G$.

(ii) $\theta_A^* \theta_B$ is in the commutant $\pi(G)'$ of $\pi(G)''$. In particular, $S_A \in \pi(G)'$ and $A S_A^{-1/2}$ is a Parseval frame generator.

(iii) $\theta_T \theta_B \in \mathcal{R}(G) \otimes B(H_\sigma)$ for any $T \in \pi(G)'$. In particular, we have that $P_A \in \mathcal{R}(G) \otimes B(H_\sigma)$.

(iv) $P_A \sim P_B$, where the equivalence is in $\mathcal{R}(G) \otimes B(H_\sigma)$, i.e., it is implemented by a partial isometry belonging to $\mathcal{R}(G) \otimes B(H_\sigma)$.

**Proof.** (i) For all $g, q \in G$ and $h \in H_\sigma$, one has

$$L_{gq} h = \chi_{gq} \otimes h = \lambda_g \chi_q \otimes h = (\lambda_g \otimes I_\sigma)(\chi_q \otimes h) = (\lambda_g \otimes I_\sigma)L_q h.$$
Thus
\[ \theta_A \pi_g = \sum_{p \in G} L_p A_{\pi_{p^{-1}}} \pi_g = \sum_{p \in G} L_p A_{\pi_{p^{-1}} g} = \sum_{q \in G} L_{gq} A_{\pi_{q^{-1}}} = \sum_{q \in G} (\lambda_g \otimes I_o) L_q A_{\pi_{q^{-1}}} = (\lambda_g \otimes I_o) \theta_A. \]

(ii) For all \( g \in G \) one can apply (i) twice and obtain
\[ \theta_A^* \theta_B \pi_g = \theta_A^* (\lambda_g \otimes I_o) \theta_B = \pi_g \theta_A^* \theta_B. \]

Thus, \( \theta_A^* \theta_B \in \pi(G)' \). In particular, setting \( B = A \) we have \( S_A = \theta_A^* \theta_A \in \pi(G)' \).

Then \( A S_A^{-1/2} \pi_g^{-1} = A \pi_{g^{-1}} S_A^{-1/2} \) for all \( g \in G \), and hence \( A S_A^{-1/2} \) is a Parseval frame generator.

(iii) For all \( g \in G \) and \( T \in \pi(G)' \), applying (i) twice, one obtains
\[ \theta_A T \theta_B^* (\lambda_g \otimes I_o) = \theta_A T \pi_g \theta_B^* = \theta_A \pi_g \theta_B^* = (\lambda_g \otimes I_o) \theta_A T \theta_B^*. \]

Therefore,
\[ \theta_A T \theta_B^* \in (\mathcal{L}(G) \otimes I_o)' = \mathcal{L}(G)' \otimes (I_o)' = \mathcal{R}(G) \otimes B(H_o). \]

Setting \( A = B \) and \( T = S_A^{-1} \), we see that \( P_A = \theta_A S_A^{-1} \theta_A^* \in \mathcal{R}(G) \otimes B(H_o) \).

(iv) By passing if necessary to \( A S_A^{-1/2} \) (resp., \( B S_B^{-1/2} \)) which by (ii) is a Parseval frame generator and by Proposition 1.3 has frame projection \( P_A \) (resp., \( P_B \)), we can assume, without loss of generality, that both \( \theta_A \) and \( \theta_B \) are isometries. Then the partial isometry \( V = \theta_B \theta_A^* \in \mathcal{R}(G) \otimes B(H_o) \) implements the equivalence, i.e., \( V^* V = P_A \) and \( V V^* = P_B \).

Given a countable group, in [13, Theorems 3.8, 3.11, Proposition 6.2], Han and Larson have identified its representations that have a multi-frame (vector) generator with the subrepresentations of its left regular representation with finite multiplicity. Lemma 6.4 can be used to easily reobtain their result with an increase in generality.

**Theorem 6.5.** A unitary representation \((G, \pi, H)\) of a discrete group is unitarily equivalent to a subrepresentation of \( \lambda \otimes \text{id} \) with multiplicity \( H_o \) if and only if \((G, \pi, H)\) has an operator frame generator with range in \( H_o \).

**Proof.** Assume that \((G, \pi, H)\) is unitarily equivalent to, and hence can be identified with, a subrepresentation of \( \lambda \otimes \text{id} \), the restriction \( \lambda \otimes \text{id} |_{PH_o} \) for some projection \( P \in (\mathcal{L}(G) \otimes I_o)' = \mathcal{R}(G) \otimes B(H_o) \). Let \( H := PH_o, \ A := L_e^* P |_H \), and
\[ A_g = L_e^* P |_H (\lambda_g \otimes I_o) P |_H = L_e^* (\lambda_g \otimes I_o) P |_H \quad \text{for all } g \in G. \]

Then
\[ S_A = \sum_{g \in G} A_g^* A_g = \sum_{g \in G} P(\lambda_g \otimes I_o) L_e^* (\lambda_g^{-1} \otimes I_o) P |_H = P(\sum_{g \in G} L_g L_g^*) P |_H = I |_H. \]

This shows that \( A \) is a (Parseval) frame generator for \((G, \pi, H)\).

Conversely, assume that \( A \in B(H, H_o) \) is a frame generator for \((G, \pi, H)\). Then \( \theta_A S_A^{-1/2} \) is an isometry onto the subspace \( P_A H_G, S_A^{-1/2} \) commutes with \( \pi \), and \( P_A \).
commutes with the left regular representation with multiplicity $H_o$ by Lemma 6.4 (ii) and (iii). Then by Lemma 6.4 (i),
\[
\theta_A S_A^{-1/2} \pi_g = \theta_A \pi_g S_A^{-1/2} = (\lambda_g \otimes I_o) \theta_A S_A^{-1/2} = (\lambda_g \otimes I_o) P_A |_{P_A H_G} \theta_A S_A^{-1/2}
\]
for all $g \in G$; i.e., $\pi$ is unitarily equivalent to $\lambda \otimes id \big|_{P_A H_G}$.

From the above proof it is easy to obtain the following:

**Remark 6.6.** If $A \in B(H, H_o)$ is a frame generator for $(G, \pi, H)$, then the equivalence of $(G, \pi, H)$ and $(G, \lambda \otimes id \big|_{P_A H_G}, P_A H_G)$ is implemented by the isometry $\theta_A S_A^{-1/2}$. An isometry $V$ implements this equivalence if and only if $V = \theta_A S_A^{-1/2} U$ for some unitary operator $U \in \pi(G)'$.

It is well known and easy to see that two subrepresentations of the left regular representation with multiplicity $H_o$, $(\lambda \otimes id) P|_{P H_G}$ and $(\lambda \otimes id) Q|_{Q H_G}$, are equivalent if and only if $P \sim Q$ in $\mathcal{R}(G) \otimes B(H_o)$. In other words, the equivalence classes of subrepresentations of the left regular representation with fixed multiplicity $H_o$ are identified with the collection of equivalence classes of projections of the von Neumann algebra $\mathcal{R}(G) \otimes B(H_o)$.

Theorem 6.5 permits us to characterize those operator-valued frames labeled by a discrete group $G$ that have a frame generator. For simplicity’s sake, because of Remark 4.4 we need to consider only Parseval operator-valued frames.

**Proposition 6.7.** Let $G$ be a discrete group and let $\{A_g\}_{g \in G}$ be a Parseval operator-valued frame in $B(H, H_o)$. Then there is a unitary representation $\pi$ of $G$ on $H$ for which $A_g = A_e \pi_{g^{-1}}$ for all $g \in G$ if and only if $A_{gp} A_{gq}^* = A_p^* A_q$ for all $p, q, g \in G$.

**Proof.** Assume $A_g = A_e \pi_{g^{-1}}$ for some unitary representation $\pi$ and for all $g \in G$. Then
\[
A_{gp} A_{gq}^* = A_e \pi_{(gp)^{-1}}(\pi_{(pq)^{-1}})^* A_e^* = A_e \pi_{p^{-1}} \pi_{g^{-1}} \pi_q (\pi_q^{-1})^* A_e^* = A_p^* A_q
\]
for all $p, q, g \in G$. Assume now that $A_{gp} A_{gq}^* = A_p^* A_q$. Then for every $g \in G$, by the proof of Lemma 6.3 (i),
\[
(\lambda_g \otimes I_o) \theta_A \theta_A^* = \sum_{p, q \in G} (\lambda_g \otimes I_o) L_{g p} A_p A_q^* L_{g q}^* (\lambda_g \otimes I_o)^* = \sum_{p, q \in G} L_{g p} A_p A_q^* L_{g q}^* = \sum_{r, s \in G} L_{r} A_{g^{-1}} A_{g^{-1}}^* L_{s} = \theta_A \theta_A^* .
\]
This proves that the projection $P_A = \theta_A \theta_A^*$ is a Parseval operator-valued frame on $\mathcal{R}(G) \otimes B(H_o)$. But then the operator-valued weight $\{A_g\}_{g \in G}$ can be identified with the compression to $P_A$ of the left regular representation $\lambda_g \otimes I_o$, which has an operator frame generator. Explicitly, again by the proof of Lemma 6.3 (i),
\[
A_g = L_g^* \theta_A = L_e P_A (\lambda_g \otimes I_o) P_A \theta_A = A_e \theta_A^* (\lambda_{g^{-1}} \otimes I_o) P_A \theta_A.
\]
Since $(\lambda_g \otimes I_o) P_A$ is a unitary representation of $G$ on the Hilbert space $P_A$, then $\pi_g := \theta_A^* (\lambda_g \otimes I_o) P_A \theta_A$ is a unitary representation of $G$ on the Hilbert space $H$. Thus $A_g := A_e \pi_{g^{-1}}$; i.e., $A_e$ is a frame generator for $(G, \pi, H)$. 

\[\square\]
Remark 6.8. (i) Proposition [6.7] is a generalization of the following known result for group-indexed frames: When \( \dim H_o = 1 \), i.e., in the case of a Parseval vector frame \( \{ x_g \}_{g \in G} \), the necessary and sufficient condition for that frame to have a generator for some unitary representation of \( G \) (necessarily unitarily equivalent to a subrepresentation of the left regular representation) is that

\[
\langle x_{gp}, x_{gq} \rangle = \langle x_p, x_q \rangle \quad \text{for all} \quad p, q, g \in G.
\]

This result can be deduced from the material in Chapter 3 of [15]; however, it was not stated explicitly in that paper. Condition (21) is clearly equivalent to the condition that the range of the analysis operator is invariant under the left regular representation of the group on the analysis space, and so the frame can be obtained from the standard orthonormal basis for this representation by simply projecting, thereby obtaining the required subrepresentation of \( G \). We note that Nga Nguyen has written an exposition of this in a forthcoming article stemming from her thesis research, along with some extensions to frames satisfying this condition which are not necessarily Parseval, where the situation is more complicated. We also note that some special cases, notably where \( G \) is a cyclic group on a finite dimensional Hilbert space, have been independently proven and used by others.

(ii) More is true in a case where the group is abelian (see also Remark 7.5). If \( G \) is abelian, then Cor. 3.14 or Theorem 6.3 of [15] states that all group frames for a unitary representation of \( G \) on the same Hilbert space are unitarily equivalent. So in the abelian case two frames both satisfying the condition (21) are necessarily unitarily equivalent.

(iii) We do not know if there is a similar necessary and sufficient condition for frames indexed by a unitary system, or at least by some structured unitary system, such as a Gabor system.

7. Parametrization of operator frame generators

Theorem 3.11 shows how to parametrize all operator-valued frames with a given multiplicity in terms of a single operator-valued frame. This general result can be applied to parametrize all operator frame generators for a unitary representation of a discrete group in terms of a single operator frame generator.

**Theorem 7.1.** Let \( A \in B(H, H_o) \) be a frame generator for the unitary representation \((G, \pi, H)\).

(i) If \( B(H_G) \ni M = MP_A \) and \( M^*M|_{P_AH_G} \) is invertible in \( B(P_AH_G) \), then \( L_e^*M\theta_A \) is a frame generator for \((G, \pi, H)\) if and only if \( M \in \mathcal{R}(G) \otimes B(H_o) \).

(ii) The collection \( \mathcal{F}_G \) of all the operator frame generators for \((G, \pi, H)\) with the same multiplicity \( H_o \) is uniquely parametrized as

\[
\mathcal{F}_G = \{ L_e^*M\theta_A \mid M \in \mathcal{R}(G) \otimes B(H_o), M = MP_A, M^*M|_{P_AH_G} \text{ is invertible} \}.
\]

(iii) If \( A \) is a Parseval generator, the collection of all the Parseval operator frame generators for \((G, \pi, H)\) with multiplicity \( H_o \) is uniquely parametrized as

\[
\{ L_e^*V\theta_A \mid V \in \mathcal{R}(G) \otimes B(H_o), V^*V = P_A \}.
\]

If \( B = L_e^*V\theta_A \) with \( V \in \mathcal{R}(G) \otimes B(H_o) \) and \( V^*V = P_A \), then \( V = \theta_B\theta_A \).

(iv) If \( A \) is a Parseval generator and \( P \sim P_A \) in \( \mathcal{R}(G) \otimes B(H_o) \), then \( P = P_B \) for \( B = L_e^*V\theta_A \) for some \( V \in \mathcal{R}(G) \otimes B(H_o) \) with \( V^*V = P_A \) and \( VV^* = P \).
Proof: (i) If \( M = MP_A \) is an operator in \( \mathcal{R}(G) \otimes B(H_o) \) and \( M^*M \big|_{P_A H_o} \) is invertible, by Theorem 3.1(ii), \( \{L_g^*M_A\}_{g \in G} \) is an operator-valued frame. But then for all \( g \in G \),
\[
L_g^*M_A = (L_g^* \lambda_{g^{-1}} \otimes I_o)M_A = L_e^*M \lambda_{g^{-1}} \otimes I_o, \theta_A = L_e^*M \theta_A \pi_{g^{-1}},
\]
by Lemma 6.3(i); i.e., \( L_g^*M_A \) is the generator of \( \{L_g^*M_A\}_{g \in G} \). Conversely, assume that \( L_g^*M_A \) is an operator frame generator for \( (G, \pi, H) \); i.e., assume that \( \{L_e^*M_A \pi_{g^{-1}} \}_{g \in G} \) is a frame. For all \( g \in G \) we have
\[
L_g^*M_A \pi_{g^{-1}} = (L_e^* \lambda_{g^{-1}} \otimes I_o)(\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o) \theta_A = L_g^*(\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o) \theta_A.
\]
Set \( N_g := (\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o)P_A \). Then obviously \( N_g = N_gP_A \) and
\[
N_g^*N_g = P_A(\lambda_g \otimes I_o)M^*(\lambda_{g^{-1}} \otimes I_o)(\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o)P_A
= (\lambda_g \otimes I_o)P_A M^*M P_A (\lambda_{g^{-1}} \otimes I_o).
\]
Since by hypothesis \( P_A M^*M P_A \) is invertible in \( B(P_A H_o) \) and since \( \lambda_g \otimes I_o \) commutes with \( P_A \), \( N_g^*N_g \) is also invertible in \( B(P_A H_o) \). But then, by the uniqueness part of Theorem 3.1 \( N_g = N_e = M \); i.e., \( M \) commutes with \( \lambda_g \otimes I_o \) for all \( g \in G \) and hence \( M \in \mathcal{R}(G) \otimes B(H_o) \).

(ii) If \( B \in B(H, H_o) \) is an operator frame generator for \( (G, \pi, H) \), then by Theorem 3.1 \( B \pi_{g^{-1}} = L_g^*M_A \) for some unique \( M = MP_A \) for which \( M^*M \big|_{P_A H_o} \) is invertible. In particular, \( B = L_e^*M_A \) and hence \( M \in \mathcal{R}(G) \otimes B(H_o) \) by the above proof.

(iii) and (iv) The rest of the proof follows by the same arguments and Corollary 4.3.

Special cases of operator frame generators arise from right or left similarities. First we need the following lemma.

Lemma 7.2. Let \( A \in B(H, H_o) \) be an operator frame generator for \( (G, \pi, H) \).

(i) Let \( R \in B(H_o) \) be invertible. Then \( RA \) is an operator frame generator for \( (G, \pi, H) \).

(ii) Let \( T \in B(H) \) be invertible. Then \( \{A \pi_{g^{-1}}T\}_{g \in G} \) has a generator (necessarily \( AT \)) if and only if \( T \in \pi(G)' \).

(iii) Let \( T \in B(H) \) be invertible. If \( T \in \pi(G)' \), then \( AT \) is an operator frame generator for \( (G, \pi, H) \) and \( AT = L_e^*(Y \otimes I_o) \theta_A \) for some invertible operator \( Y \in \mathcal{R}(G) \). If \( T \) is unitary, then \( Y \) can be chosen to be unitary.

Proof. (i) Obvious.

(ii) The sufficiency of the condition is clear. For the necessity, assume that \( \{A \pi_{g^{-1}}T\}_{g \in G} \) has a generator \( B \), i.e., \( A \pi_{g^{-1}}T = B \pi_{g^{-1}} \), for all \( g \in G \). Then \( \theta_A T = \theta_B \) by Lemma 7.2 and hence \( T = S_A^{-1/2} \theta_A \theta_B \in \pi(G)' \) by Lemma 6.3(ii).

(iii) By Lemma 6.3(ii), \( T \) commutes with \( S_A^{-1/2} \); hence
\[
AT = L_e^* \theta_A T = L_e^* \theta_A T S_A^{-1/2} \theta_A^* \theta_A = L_e^*(\theta_A S_A^{-1/2} T S_A^{-1/2} \theta_A^*) \theta_A.
\]
By Lemma 6.2(i) and (ii),
\[
\theta_A S_A^{-1/2} \pi_g S_A^{-1/2} \theta_A^* = \theta_A \pi_{g} S_A^{-1} \theta_A^* = (\lambda_g \otimes I_o) \theta_A S_A^{-1} \theta_A^* = (\lambda_g \otimes I_o) P_A.
\]
Since $\theta_A S_A^{-1/2}$ is a unitary operator in $B(H, P_A H_G)$ and since the unitary group \{\pi_g \mid g \in G\} (resp., $(\lambda_g \otimes I_o) P_A|_{P_A H_G}$) generates the von Neumann algebra $\pi(G)''$ (resp., $P_A(\mathcal{L}(G) \otimes I_o) P_A|_{P_A H_G}$), the map

$$\pi(G)'' \ni X \mapsto \theta_A S_A^{-1/2} XS_A^{-1/2} \theta_A^* \in P_A(\mathcal{L}(G) \otimes I_o) P_A|_{P_A H_G}$$

is a (spatial) isomorphism of von Neumann algebras. Let $Q \otimes I_o$ be the central support of $P_A$, so $Q \in \mathcal{R}(G) \cap \mathcal{L}(G)$. It is well known [18, Proposition 5.5.5] that the map

$$(\mathcal{L}(G) \otimes I_o)(Q \otimes I_o)|_{(Q \otimes I_o)H_G} \ni X \mapsto X P_A|_{P_A H_G} \in P_A(\mathcal{L}(G) \otimes I_o) P_A|_{P_A H_G}$$

is also an isomorphism. Thus $\theta_A S_A^{-1/2} T S_A^{-1/2} \theta_A^* = (Z \otimes I_o) P_A$ for some operator $Z \in \mathcal{L}(G)$ for which $(Z \otimes I_o)(Q \otimes I_o)|_{(Q \otimes I_o)H_G}$ is invertible and then we have $AT = L_e^*(Z \otimes I_o) P_A \theta_A$. By passing if necessary to $ZQ + Q^\perp \in \mathcal{L}(G)$, we can assume without loss of generality that $Z$ is invertible. If $T$ is unitary, we can similarly assume that $Z$ too is unitary.

Recall that the involution $J$, defined by $J(x \chi_e) := x^* \chi_e$ for all $x \in \mathcal{L}(G)$ and then extended to $\ell^2(\mathcal{F})$, establishes the conjugate linear isomorphism of $\mathcal{L}(G)$ and $\mathcal{R}(G)$, $\mathcal{L}(G) \ni x \mapsto JxJ \in \mathcal{R}(G)$. For all $h \in H_o$,

$$(Z^* \otimes I_o)L_c h = Z^* \chi_e \otimes h = JZJ \chi_e \otimes h = (JZJ \otimes I_o)L_c h;$$

hence

$$AT = L_e^*(Z \otimes I_o) \theta_A = L_e^*(JZJ^* \otimes I_o) \theta_A = L_e^*(JZ^* J \otimes I_o) \theta_A.$$  

Let $Y := JZ^* J$ and $M := (Y \otimes I_o) P_A$. Then $Y \in \mathcal{R}(G)$; hence $M \in \mathcal{R}(G) \otimes B(H_o)$ and $M = MP_A$. Furthermore, $Y$ is invertible (unitary if $T$ and hence $Z$ are unitary); hence $M^* M = P_A(Y^* Y \otimes I_o) P_A$ is invertible in $B(P_A H_G)$. Then by Theorem 7.3, $AT = L_e^*(Y \otimes I_o) \theta_A = L_e^* M \theta_A$ is an operator frame generator for $(G, \pi, H)$. 

A reformulation of statement (ii) is that if two operator-valued frames with generators $A$ and $B$ are right-similar, then the (unique) similarity operator must belong to $\pi(G)'$. Using this fact, the characterization of right-similarity for general operator-valued frames carries through easily for operator-valued frames with a generator as follows.

**Proposition 7.3.** Let $A$ and $B$ be frame generators with the values in the same space $H_o$ for a unitary representation $(G, \pi, H)$. Then the following conditions are equivalent:

(i) $B = AT$ for some invertible operator $T \in \pi(G)'$;
(ii) $B \pi_{g^{-1}} = \pi_g T$ for all $g \in G$ for some invertible operator $T \in B(H_o)$;
(iii) $\theta_B = \theta_A T$ for some invertible $T \in B(H)$;
(iv) $P_A \theta_B S_A^{-1} \theta_A^* \theta_A$ is invertible in $B(P_A H_G)$;

(iv') $B = L_e^* M \theta_A$ for some $M \in \mathcal{R}(G) \otimes B(H_o)$ with $M = MP_A$ and such that $P_A M P_A$ is invertible in $B(P_A H_G)$;

(v) $P_B = P_A$. 

Corollary 7.4. Let $A \in B(H, H_o)$ be an operator frame generator for $(G, \pi, H)$. Then all the operator frame generators for $(G, \pi, H)$ are left-similar to $A$ if and only if $P_A \in (\mathcal{L}(G) \cap \mathcal{R}(G)) \otimes I_o$.

Proof. $P_A$ belongs to the center $(\mathcal{L}(G) \cap \mathcal{R}(G)) \otimes I_o$ of $\mathcal{R}(G) \otimes B(H_o)$ if and only if there are no projections $\mathcal{R}(G) \otimes B(H_o)$ that are different but Murray-von Neumann equivalent to it. By Lemma 6.4 and Proposition 7.3, this is equivalent to the condition that any operator frame generator for $(G, \pi, H)$ is left-similar to $A$. □

Remark 7.5. (i) Corollary 7.4 provides a higher multiplicity analog of Proposition 3.13 in [15].

(ii) If the group $G$ is abelian, then so is $\mathcal{L}(G) = \mathcal{R}(G)$ and hence $\mathcal{R}(G) \otimes I_o$ is the center of $\mathcal{R}(G) \otimes B(H_o)$. Thus in particular if $\dim H_o = 1$, $\mathcal{R}(G) \otimes B(H_o)$ is abelian and all operator frame generators for $(G, \pi, H)$ are left-similar. This generalizes Cor. 3.14 (and Theorem 6.3) of [15], which states that, for vector group frames, if $G$ is abelian, then all group frames for a unitary representation of $G$ on the same Hilbert space are unitarily equivalent. (See also Remark 6.8 (i) in the present article.)

To simplify notation, we formulate the next result directly for Parseval operator frame generators.

Proposition 7.6. Let $A, B \in B(H, H_o)$ be Parseval frame generators for $(G, \pi, H)$.

(i) $B = AU$ for some unitary $U \in \pi(G)'$ if and only if $B = L^*_e W \theta_A$ for some unitary $W \in \mathcal{R}(G) \otimes B(H_o)$, again if and only if $P_B = P_A$.

(ii) Let $U \in B(H_o)$ be unitary. Then $B = UA$ if and only if $B = L^*_e (I \otimes U) \theta_A$. If $B = UA$, then $P_B = (I \otimes U) P_A(I \otimes U^*)$.

(iii) $B = AU$ for some unitary $U \in \pi(G)'$ if and only if $B = L^*_e (W \otimes I_o) \theta_A$ for some unitary $W \in \mathcal{R}(G)$. If $B = L^*_e (W \otimes I_o) \theta_A$, then $P_B = (W \otimes I_o) P_A(W^* \otimes I_o)$.

Proof. (i) By Proposition 4.3, $P_B = P_A$ if and only if the operator-valued frames $\{B \pi_{g^{-1}}\}_{g \in G}$ and $\{A \pi_{g^{-1}}\}_{g \in G}$ are right unitarily equivalent. By Lemma 7.2, this unitary equivalence holds if and only if $B = AU$ for some unitary $U \in \pi(G)'$. Also, by Proposition 7.3, $P_B = P_A$ if and only if $\theta_B \theta_A^*$ is a unitary in $B(P_A H_G)$ and hence it is the compression to $P_A H_G$ of a unitary $W \in \mathcal{R}(G) \otimes B(H_o)$ that commutes with $P_A$.

(ii) This is immediate from Lemma 4.5.

(iii) Assume that $B = AU$ for a unitary $U \in \pi(G)'$. In the proof of Lemma 7.2 (iii) we can choose $Z$ to be unitary and hence $W := JZ^* J \in \mathcal{R}(G)$ is also unitary. Then $\theta_B = (W \otimes I_o) \theta_A$; hence $P_B = (W \otimes I_o) P_A(W^* \otimes I_o)$. On the other hand, if $B = L^*_e (W \otimes I_o) \theta_A$ for some unitary $W \in \mathcal{R}(G)$, then by the same argument as in the proof of Lemma 7.2 (iii), we see that

$$B = L^*_e (JW^* J \otimes I_o) \theta_A = L^*_e \theta_A (\theta_A^* (JW^* J \otimes I_o) \theta_A),$$

where $JW^* J \in \mathcal{L}(G)$ and hence $\theta_A^* (JW^* J \otimes I_o) \theta_A$ is a unitary in $\pi(G)'$. □

Remark 7.7. For vector frames ($\dim H_o = 1$), Han and Larson [15, Theorem 6.17] have shown that given a Parseval frame generator $\eta$ for $(G, \pi, H)$, the collection of all the (vector) Parseval frame generators for $(G, \pi, H)$ is parametrized by the
unitary group of $\pi(G)'$; namely, it coincides with $\{U_\eta \mid U \in \pi(G)'', U \text{ is unitary}\}$. This result is also a consequence of Theorem 7.1 since the partial isometry $V$ intertwining $P_A$ and $P_B$ can be extended to a unitary $W$ because the von Neumann algebra $\mathcal{R}(G)$ is finite. However, Proposition 7.6 shows why this result does not hold when $\dim H_o > 1$.

8. Homotopy of operator frame generators

The objective of this section is to prove the following theorem:

**Theorem 8.1.** Let $(G, \pi, H)$ be a unitary representation of a discrete group $G$ and assume that the collection $\mathcal{F}_G$ of all the operator frame generators with range in $H_o$ for $(G, \pi, H)$ is nonempty.

(i) If $\dim H_o < \infty$, then $\mathcal{F}_G$ is path-connected in the norm topology.

(ii) If $\dim H_o = \infty$, then $\mathcal{F}_G$ is path-connected in the norm topology if and only if the von Neumann algebra $\mathcal{R}(G)$ generated by the right regular representation of $G$ is diffuse (i.e. has no nonzero minimal projections).

As we will point out in the proof of the theorem, it is easy to reduce the problem to showing that any two Parseval operator frame generators are homotopic. The latter property is obviously true in the case when $\dim H_o = 1$, because then (by [15, Theorem 6.17], see also Remark 7.7 and Proposition 7.6), the collection of Parseval (vector) frame generators for $(G, \pi, H)$ is parametrized by the unitary group of the von Neumann algebra $\pi(G)'$, which is well known to be path-connected in the norm topology. In the general case, however, by Theorem 7.1 (iii) all Parseval operator frame generators for $(G, \pi, H)$ are parametrized by the partial isometries of the algebra $\mathcal{R}(G) \otimes B(H_o)$ that have the same initial projection, the frame projection of a fixed Parseval operator frame generator. When $\dim H_o = \infty$, this class of partial isometries is not path-connected in the norm topology. It is, however, path-connected in the strong operator topology when $\mathcal{L}(G)$ has no nonzero minimal projections, and this is sufficient for the path-connectedness in the norm topology of the operator frame generators. In order to do that we need to introduce some notation and preliminary results.

Let $V, W$ be partial isometries in $\mathcal{R}(G) \otimes B(H_o)$ with the same initial projection, i.e., $V^*V = W^*W$, and hence with range projections, $VV^*, WW^*$ Murray-von Neumann equivalent ($VV^* \sim WW^*$). We say that

$$V \approx W$$

if there is a norm continuous path of partial isometries

$$\{V(t) \in \mathcal{R}(G) \otimes B(H_o) \mid t \in [0, 1]\}$$

such that $V^*(t)V(t) = V^*V$ for all $t \in [0, 1]$, $V(0) = V$, and $V(1) = W$. Clearly, $\approx$ is an equivalence relation for the class of partial isometries that have the same initial projection. Adapting the proof of [23, Theorem 3.1] with a slight change yields the following equivalent characterization. For the reader’s convenience we sketch the proof.

**Lemma 8.2.** Let $V, W$ be partial isometries in $\mathcal{R}(G) \otimes B(H_o)$ with the same initial projection. Then $V \approx W$ if and only if $VV^*$ and $WW^*$ are unitarily equivalent in $\mathcal{R}(G) \otimes B(H_o)$. 
Proof. If $VV^*$ and $WW^*$ are unitarily equivalent, then $(WW^*)^\perp \sim (V V^*)^\perp$; i.e., there is a partial isometry $Z \in \mathcal{R}(G) \otimes B(H_0)$ with $Z Z^* = (WW^*)^\perp$ and with $Z^* Z = (VV^*)^\perp$. Then a simple computation shows that the operator

$$U := WV^* + Z \in \mathcal{R}(G) \otimes B(H_0)$$

is unitary and that $W = U V$. Since $U$ is homotopic in the norm topology to the identity, choose a norm continuous path of unitaries $U(t) \in \mathcal{R}(G) \otimes B(H_0)$ with $U(0) = I$ and $U(1) = U$. Then $V(t) := U(t)V$ is the required norm continuous path of partial isometries with the same initial projection that joins $V(0) = V$ and $V(1) = W$. This establishes that $V \approx W$. Conversely, if $V \approx W$ and $V(t)$ is a norm continuous path of partial isometries with the same initial projection that joins $V(0) = V$ with $V(1) = W$, then $P(t) := V(t)V(t)^*$ is a norm continuous path of projections joining $P(0) = VV^*$ with $P(1) = WW^*$. It is well known that homotopy of projections implies unitary equivalence. \hfill $\square$

If $\dim H_0 = \infty$, there are partial isometries $V, W \in \mathcal{R}(G) \otimes B(H_0)$ with the same initial projection but with range projections that are not unitarily equivalent, e.g., $V = I \otimes I_0$ and $W = I \otimes Z$ with $Z \in B(H_0)$ a nonunitary isometry. By Lemma 8.2 $V$ and $W$ cannot be joined by a norm continuous path of partial isometries all with the same initial projection. The norm continuity of such a path of partial isometries, however, is only sufficient but is not always necessary for the existence of a norm continuous path of Parseval frame generators joining $L^*_e V$ with $L^*_e W$. The existence of the latter is, in view of Theorem 7.1 (iii), equivalent to the existence of a path of partial isometries $V(t)$ joining $V$ and $W$ for which $L^*_e V(t)$ is norm continuous. It is convenient to denote the existence of such a path by using the following notation:

Let $V, W$ be partial isometries in $\mathcal{R}(G) \otimes B(H_0)$ with the same initial projection. We say that

$$V \sim_W W$$

if there is a path of partial isometries $\{V(t) \in \mathcal{R}(G) \otimes B(H_0) \mid t \in [0, 1]\}$ such that $L^*_e V(t)$ is norm continuous, $V^*(t) V(t) = V^* V$ for all $t \in [0, 1]$, $V(0) = V$, and $V(1) = W$.

Clearly, $\sim_W$ is also an equivalence relation for partial isometries that have the same initial projection and $V \approx W$ implies $V \sim_W W$.

A key ingredient in the proof of Theorem 8.1 is that a finite trace in a von Neumann algebra is strongly continuous (actually, $\sigma$-weakly, but we do not need this here).

Denote by $\tau(X) = (X \chi_e, \chi_e)$ for $X \in \mathcal{R}(G)$ the normalized trace on $\mathcal{R}(G)$. Denote by $E := \mathcal{R}(G) \otimes B(H_0) \rightarrow B(H_0)$ the corresponding slice map, namely, the bounded linear extension of the map $E(X \otimes Y) = \tau(X) Y$ for all $X \in \mathcal{R}(G)$ and all $Y \in B(H_0)$. It is easy to see that the map $E$ is positive, that is, $E(Z) \geq 0$ when $Z \geq 0$, or, equivalently, $E(Z_1) \leq E(Z_2)$ when $Z_1 \leq Z_2$. Also, the map $E$ is normal, that is, $E(Z \gamma) \uparrow E(Z)$ when $Z, \gamma \uparrow Z$, or equivalently, $E$ is $\sigma$-weakly continuous.

The bridge between trace and norm is given by the following lemma.
Lemma 8.3. \( E(Z) = L_e^* Z L_e \) for all \( Z \in \mathcal{R}(G) \otimes B(H_o) \).

Proof. It is enough to verify that the two maps agree on elementary tensors. Indeed, for all \( h \in H_o \) and all \( X \in \mathcal{R}(G) \), \( Y \in B(H_o) \) we have

\[
L_e^* (X \otimes I_o) L_e h = L_e^* (X \chi_e) \otimes h = L_e^* \left( \sum_{g \in G} (X \chi_e, \chi_g) \chi_g \right) \otimes h = L_e^* \left( \sum_{g \in G} (\chi_g \otimes (X \chi_e, h)) = (X \chi_e, h) = \tau(X) h. \right.

Thus \( L_e^* (X \otimes I_o) L_e = \tau(X) I_o \) and hence

\[
L_e^* (X \otimes Y) L_e = L_e^* (I \otimes Y) (X \otimes I_o) L_e = Y L_e^* (X \otimes I_o Y) L_e = \tau(X) Y = E(X \otimes Y). \]

\[ \square \]

Notice that the trace \( \tau \) on \( \mathcal{R}(G) \) is always finite, while the trace \( \tau \otimes tr \) on \( \mathcal{R}(G) \otimes B(H_o) \) is finite only if \( \dim H_o < \infty \). Thus, given two partial isometries with the same initial projection, we want to construct a strongly continuous path of partial isometries that connects them, where the strong convergence occurs in the first component of the tensor product. This will be achieved via the following key lemma.

Lemma 8.4. Assume that \( \dim H_o = \infty \) and that \( \mathcal{R}(G) \) has no minimal projections. Let \( R \) be a projection in \( \mathcal{R}(G) \) with \( R \sim R^\perp \), let \( \{Q_n\}_{1 \leq n \leq N} \) with \( N \leq \infty \) be a collection of infinite projections in \( B(H_o) \), and let \( I = \sum_{n=1}^N F_n \) be a decomposition of the identity into mutually orthogonal central projections \( F_n \in \mathcal{R}(G) \cap \mathcal{L}(G) \). Then there is a path of partial isometries \( \{W(t) \mid t \in [0, 1]\} \) in \( \mathcal{R}(G) \otimes B(H_o) \) such that

(i) \( L_e^* W(t) \) is norm continuous,

(ii) \( W(t)^* W(t) = \sum_{n=1}^N F_n \otimes Q_n \) for all \( t \in [0, 1] \),

(iii) \( W(t) W(t)^* \leq \sum_{n=1}^N F_n \otimes Q_n \) for all \( t \in [0, 1] \),

(iv) \( W(0) = \sum_{n=1}^N F_n \otimes Q_n \), and

(v) \( W(1) W(1)^* = \sum_{n=1}^N R F_n \otimes Q_n \).

Proof. The reduced von Neumann algebra \( R \mathcal{R}(G) R := \mathcal{R}(G) R |_{\mathcal{R}(G)} \) has no minimal projections; thus it contains a strongly continuous increasing net of projections \( \{R(t) \mid t \in [0, 1]\} \) with \( R(0) = 0, R(1) = R \). For instance, \( R(t) \) can be obtained from the spectral resolution of a positive generator of a maximal abelian subalgebra of \( R \mathcal{R}(G) R \). Since \( R^\perp \sim R \), there is a unitary \( U \in \mathcal{R}(G) \) such that \( R^\perp = U R U^* \). Since \( Q_n \) is an infinite projection in \( B(H_o) \), there exist partial isometries \( S_{1,n}, S_{2,n} \in B(H_o) \) such that

\[
S_{1,n}^* S_{1,n} = S_{2,n}^* S_{2,n} = Q_n \quad \text{and} \quad S_{1,n} S_{1,n}^* + S_{2,n} S_{2,n}^* = Q_n.
\]

Notice that \( S_{1,n}, S_{2,n} \) are the generators of the Cuntz algebra \( \mathcal{O}_2 \) represented on \( Q_n H_o \). Define for \( t \in [0, 1] \),

\[
W(t) := \sum_{n=1}^N \left( (R(t) + U R(t) U^*)^\perp F_n \otimes Q_n + R(t) F_n \otimes S_{1,n}^* + R(t) U^* F_n \otimes S_{2,n} \right).
\]
By definition, \( W(0) = \sum_{n=1}^{N} F_n \otimes Q_n \). Since \( R(t) \perp (R(t) + UR(t)U^*)^\perp \) for all \( t \in [0, 1] \), and \( S_{2,n}^* S_{1,n} = S_{1,n}^* S_{2,n} = 0 \) for all \( n \), it follows that

\[
W(t) \ast W(t) = \sum_{n=1}^{N} \left( (R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + (R(t) + UR(t)U^*)^\perp R(t)F_n \otimes Q_n S_{1,n} \right. \\
\left. + (R(t) + UR(t)U^*)^\perp R(t)U^* F_n \otimes Q_n S_{2,n} + R(t)(R(t) + UR(t)U^*)^\perp F_n \otimes S_{1,n}^* Q_n \right. \\
\left. + R(t)F_n \otimes S_{1,n}^* S_{1,n} + R(t)U^* F_n \otimes S_{1,n}^* S_{2,n} \\
+ UR(t)(R(t) + UR(t)U^*)^\perp F_n \otimes S_{2,n}^* Q_n \\
+ UR(t)F_n \otimes S_{2,n}^* S_{1,n} + UR(t)U^* F_n \otimes S_{2,n}^* S_{2,n} \right \setminus \\
= \sum_{n=1}^{N} \left( (R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + R(t)F_n \otimes S_{1,n}^* S_{1,n} \\
+ UR(t)U^* F_n \otimes S_{2,n}^* S_{2,n} \right \setminus \\
= \sum_{n=1}^{N} \left( (R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + R(t)F_n \otimes Q_n + UR(t)U^* F_n \otimes Q_n \right \setminus \\
= \sum_{n=1}^{N} F_n \otimes Q_n.
\]

Using the fact that \( UR(t)U^* \perp (R(t) + UR(t)U^*)^\perp \) and \( UR(t)U^* \perp R(t) \) for all \( t \), a similar computation yields

\[
W(t) \ast W(t) = \sum_{n=1}^{N} \left( (R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + R(t)F_n \otimes S_{1,n}^* S_{1,n} + R(t)F_n \otimes S_{2,n}^* S_{2,n} \right) \\
= \sum_{n=1}^{N} \left( (R(t) + UR(t)U^*)^\perp + R(t) \right) F_n \otimes Q_n \setminus \\
\leq \sum_{n=1}^{N} F_n \otimes Q_n.
\]

In particular,

\[
W(1)W(1)^\ast = \sum_{n=1}^{N} \left( (R + R^\perp)^\perp + R \right) F_n \otimes Q_n = \sum_{n=1}^{N} RF_n \otimes Q_n.
\]

Thus \( \{W(t) \mid t \in [0, 1]\} \) is a path of partial isometries of \( \mathfrak{R}(G) \otimes B(H_o) \) that satisfies conditions (ii), (iii), (iv), and (v). We now show that the condition (i) is also satisfied. Let \( 0 \leq t < t' \leq 1 \) and \( \Delta R := R(t') - R(t) \). Then

\[
W(t') - W(t) = \sum_{n=1}^{N} \left( -\Delta R + URU^* F_n \otimes Q_n + \Delta RF_n \otimes S_{1,n} + \Delta RU^* F_n \otimes S_{2,n} \right).
\]
By using the facts that $\Delta R \perp U\Delta R U^*$ and $Q_nS_{i,n} = S_{i,n}Q_n = S_{i,n}$ for $i = 1, 2$ and all $n$, we obtain

$$
(W(t') - W(t))(W(t') - W(t))^*
= \sum_{n=1}^{N} \left( (\Delta R + U\Delta R U^*)F_n \otimes Q_n - (\Delta R + U\Delta R U^*)\Delta RF_n \otimes S_{1,n}^* \right)
- \sum_{n=1}^{N} \left( (\Delta R + U\Delta R U^*)U\Delta RF_n \otimes S_{2,n}^* - \Delta R(\Delta R + U\Delta R U^*)F_n \otimes S_{1,n} \right)
+ \sum_{n=1}^{N} \left( (\Delta R)^2 F_n \otimes S_{1,n}S_{1,n}^* + \Delta RU\Delta RF_n \otimes S_{1,n}S_{2,n}^* 
- \Delta RU^*(\Delta R + U\Delta R U^*)F_n \otimes S_{2,n} + \Delta RU^*\Delta RF_n \otimes S_{2,n}S_{1,n}^* \right)
+ \sum_{n=1}^{N} \left( \Delta RF_n \otimes S_{2,n}S_{2,n}^* \right)
$$

$$
= \sum_{n=1}^{N} \left( (\Delta R + U\Delta R U^*)F_n \otimes Q_n - \Delta RF_n \otimes S_{1,n}^* - U\Delta RF_n \otimes S_{2,n}^* 
- \Delta RF_n \otimes S_{1,n} + \Delta RF_n \otimes S_{1,n}S_{1,n}^* - \Delta RU^*F_n \otimes S_{2,n} + \Delta RF_n \otimes S_{2,n}S_{2,n}^* \right)
= \sum_{n=1}^{N} \left( (2\Delta R + U\Delta R U^*)F_n \otimes Q_n - \Delta RF_n \otimes (S_{1,n}^* + S_{1,n}) 
- U\Delta RF_n \otimes S_{2,n} - \Delta RU^*F_n \otimes S_{2,n} \right).
$$

Thus,

$$
E((W(t') - W(t))(W(t') - W(t))^*)
= \sum_{n=1}^{N} \left( \tau(\Delta RF_n)(3Q_n - (S_{1,n} + S_{1,n}^*)) - \tau(\Delta RU^*F_n)S_{2,n} - \tau(U\Delta RF_n)S_{2,n}^* \right)
\leq \sum_{n=1}^{N} 7\tau(\Delta RF_n)I_o = 7\tau(\Delta R)I_o.
$$

Hence

$$
\|L^*_cW(t') - L^*_cW(t)\|^2 = \|E((W(t') - W(t))(W(t') - W(t))^*)\| \leq 7\tau(\Delta R).
$$

Since $\Re(G)$ is finite, $\tau(R(t))$ is continuous and hence $L^*_cW(t)$ is norm continuous, which concludes the proof. □

Now we can proceed to prove Theorem 8.1.

Proof. It is well known that in any von Neumann algebra (or, more in general, unital $C^*$-algebra), positive invertible operators are homotopic to the identity. But then, the operator frame generator $A$ for $(G, \pi, H)$ is homotopic to the Parseval operator frame generator $AS_A^{-1/2}$ by Lemma 6.4 (ii). Thus, to prove the path-connectedness in the norm topology of $F_G$, it is enough to prove that the collection of Parseval operator frame generators for $(G, \pi, H)$ is path-connected in the norm topology. By 6.4, this collection is parametrized by

$$
\{L^*_cV\theta_A \mid V \in \Re(G) \otimes B(H_o), V^*V = P_A \}.
$$


and since $\theta_A$ is an isometry, we only need to prove that $V \sim W$ for any two partial isometries $V$ and $W$ in $\mathcal{R}(G) \otimes B(H_o)$ with the same initial projection $P_A$, i.e., $V^*V = W^*W = P_A$.

(i) The algebra $\mathcal{R}(G) \otimes B(H_o)$ is finite because both $\mathcal{R}(G)$ and $B(H_o)$ are finite; hence the equivalence of the projections $VV^*$ and $WW^*$ implies their unitary equivalence. But then $V \approx W$ by Lemma 8.2, and hence $V \sim W$. This proves that $\mathcal{T}_G$ is norm connected.

(ii) We prove first that the condition is necessary. Assume that $\mathcal{R}(G)$ has a nonzero minimal projection $Q$. Then $Q$ belongs to a finite type I subfactor of $\mathcal{R}(G)$. Indeed if $c(Q)$ is the central cover of $Q$, which is the smallest projection in the center $\mathcal{L}(G) \cap \mathcal{R}(G)$ of $\mathcal{R}(G)$ that majorizes $Q$, then $c(Q)$ is minimal in $\mathcal{L}(G) \cap \mathcal{R}(G)$. But then the reduced von Neumann algebra $\mathcal{R}(G)c(Q) := c(Q)\mathcal{R}(G)\|c(Q)\|$ is a factor, it is finite because so is $\mathcal{R}(G)$, and it is of type I because it contains the minimal projection $Q$. Let $\{E_{i,j}\}_{1 \leq i,j \leq n}$ be a set of matrix units for $\mathcal{R}(G)c(Q)$, i.e., $E_{i,j} = E_{j,i}, E_{i,k}E_{h,j} = \delta_{h,k}E_{i,j}$ for all $1 \leq i,j,h,k \leq n$, $\sum_{i=1}^n E_{i,i} = c(Q)$, and $\mathcal{R}(G)c(Q) = \{\sum_{i,j=1}^n c_{i,j}E_{i,j} \mid c_{i,j} \in \mathbb{C}\}$. Then every element $Z$ in the factor $\mathcal{R}(G)c(Q) \otimes B(H_o)$ has the unique matricial form

$$Z = \sum_{i,j=1}^n E_{i,j} \otimes Z_{i,j} \quad \text{for } Z_{i,j} \in B(H_o).$$

Therefore, it is easy to see from Lemma 8.3 that

$$L^*_eZL_e = \frac{1}{n} \sum_{i=1}^n Z_{i,i}.\quad (25)$$

Since $\mathcal{R}(G)c(Q) \otimes B(H_o)$ is an infinite type I factor, there is a proper isometry $V \in \mathcal{R}(G)c(Q) \otimes B(H_o)$, i.e., $V^*V = c(Q) \otimes I_0$ but $VV^* \neq c(Q) \otimes I_0$. To prove that $\mathcal{T}_G$ is not path-connected in the norm topology, it will be enough to show that the two Parseval operator-valued frame generators $L_e^*c(Q) \otimes I_0$ and $L_t^*V$ cannot be connected by any norm continuous path of arbitrary operator-valued frame generators.

Assume otherwise. Then by Theorem 7.1 (ii), there is a path of operators $M(t) \in \mathcal{R}(G) \otimes B(H_o)$ with $M(t)c(Q) \otimes I_0 = M(t)$, $M(t)^*M(t)c(Q) \otimes I_0$ is invertible in $\mathcal{R}(G)c(Q) \otimes B(H_o)$, $M(0) = c(Q)$, $M(1) = V$, and $L_t^*M(t)$ is norm continuous. By (21), $M(t) = \sum_{i,j=1}^n E_{i,j} \otimes M_{i,j}(t)$ for a (unique) collection $M_{i,j}(t) \in B(H_o)$. Then for all $s,t \in [0,1]$, by (25),

$$(L_e^*M(s) - L_e^*M(t))(L_e^*M(s) - L_e^*M(t))^*$$

$$= L_e^* \left( \sum_{i,j=1}^n E_{i,j} \otimes \sum_{k=1}^n (M(s)_{i,k} - M(t)_{i,k})(M(s)_{k,i} - M(t)_{k,i})^* \right) L_e$$

$$= \frac{1}{n} \sum_{i,k=1}^n (M(s)_{i,k} - M(t)_{i,k})(M(s)_{k,i} - M(t)_{k,i})^*.$$

Thus the norm continuity of $L_t^*M(t)$ is equivalent to the norm continuity of each $M_{i,j}(t) \in B(H_o)$ for $1 \leq i,j \leq n$, and the latter is equivalent to the norm continuity of $M(t)$. But then, $M(t)^*M(t)$ is norm continuous and by the norm continuity of the inverse (e.g., see [13, Problem 100]), $(M(t)^*M(t))^{-1}$ is also norm continuous in $\mathcal{R}(G)c(Q) \otimes B(H_o)$. As a consequence, $P(t) := M(t)(M(t)^*M(t))^{-1}M(t)^*$ is a
norm continuous path, and it is immediate to see (cf. Theorem 3.1) that the $P(t)$ are projections. But this is impossible since $P(0) = c(Q)$ and $P(1) = VV^*$ are not unitarily equivalent and hence not homotopic.

We prove now that if $\mathcal{R}(G)$ has no nonzero minimal projections, then $V \sim W$.

By the standard type decomposition of von Neumann algebras (for these and other von Neumann algebra properties, see [18]), there is a (unique) central projection $F(1) \in \mathcal{L}(G) \cap \mathcal{R}(G)$ for which $(\mathcal{L}(G) \cap \mathcal{R}(G))F(1)$ is diffuse, i.e., has no atoms and hence is isomorphic to $L_\infty(\mathbb{R})$, and $(\mathcal{L}(G) \cap \mathcal{R}(G))(F(1))$ is atomic and hence $\mathcal{R}(G)(F(1))$ is a direct sum of type $II_1$ factors. Since it is immediate to verify that $V \sim W$ if and only if both

$$VF(1) \otimes I_o \sim WF(1) \otimes I_o \quad \text{and} \quad (F(1))^\perp \otimes I_o \sim W(F(1))^\perp \otimes I_o,$$

we can consider separately the cases where the center of $\mathcal{R}(G)$ is diffuse and where it is atomic.

Consider first the case where $\mathcal{R}(G)$ has diffuse center. Then there is a strongly continuous increasing net of central projections $F(t) \in \mathcal{L}(G) \cap \mathcal{R}(G)$ such that $F(0) = 0$ and $F(1) = I$. Let

$$V(t) := V(F(t) \otimes I_o) + W(F(t) \otimes I_o).$$

Since $F(t) \otimes I_o$ is in the center of $\mathcal{R}(G) \otimes B(H_o)$, we see that

$$V(t)^*V(t) = V^*V(F(t)^\perp \otimes I_o) + W^*W(F(t) \otimes I_o) = V^*V = P_A$$

for all $t \in [0, 1]$ and $V(0) = V, V(1) = W$. Moreover, for all $s < t \in [0, 1]$, $V(t) - V(s) = V(F(s) - F(t)) \otimes I_o + W(F(t) - F(s)) \otimes I_o = (W - V)(F(t) - F(s)) \otimes I_o$; hence

$$(V(t) - V(s))(V(t) - V(s))^*$$

$$= (F(t) - F(s)) \otimes I_o((W - V)(W - V)^*)(F(t) - F(s)) \otimes I_o$$

$$\leq ||W - V||^2(F(t) - F(s)) \otimes I_o$$

$$\leq 4||F(t) - F(s)||^2 \otimes I_o.$$

But then,

$$||L_e^*V(t) - L_e^*V(s)||^2$$

$$= ||E((V(t) - V(s))(V(t) - V(s))^*)||$$

$$\leq 4||E((F(t) - F(s)) \otimes I_o)||$$

$$= 4\tau((F(t) - F(s))).$$

By the strong (actually $\sigma$-weak) continuity of $\tau$, $L_e^*V(t)$ is norm continuous, and hence $V \sim W$. Following the terminology introduced in [1] for wavelet generators for the unitary system, we say that the path constructed in the case where $\mathcal{R}(G)$ has diffuse center is a direct path.

Consider now the key case where $\mathcal{R}(G)$ has no nonzero minimal projections and the center of $\mathcal{R}(G)$ is atomic. Then the identity $I \in \mathcal{R}(G)$ can be decomposed (uniquely) into a sum $I = \sum_{n=1}^N F_n$ of $N \leq \infty$ mutually orthogonal projections $F_n$ minimal in the center $\mathcal{R}(G) \cap \mathcal{L}(G)$. Notice that since $\mathcal{R}(G)$ has a finite faithful trace
the cases where \( n \) is a factor, and being finite and with no minimal projections, it is of type \( \tau \).

Thus \( F^{(2)} + F^{(3)} + F^{(4)} = I \). Reasoning as above, we can consider separately the cases where \( F^{(2)} = I \) and \( F^{(2)} = 0 \).

Assume first that \( F^{(2)} = I \), i.e.,

\[
W^*WF_o \otimes I_o = V^*VF_n \otimes I_o \neq F_n \otimes I_o \quad \text{for all } n.
\]

Then

\[
(WW^*)^\perp F_n \otimes I_o \neq F_n \otimes I_o \quad \text{and} \quad (VV^*)^\perp F_n \otimes I_o \neq F_n \otimes I_o \quad \text{for all } n.
\]

Since the factor \( \mathcal{R}(G) \otimes B(H_o) \) is infinite, (26) implies that \((VV^*)^\perp F_n \otimes I_o \sim F_n \otimes I_o \) for every \( n \), and hence \((VV^*)^\perp \sim I \otimes I_o \) and similarly, \((WW^*)^\perp \sim I \otimes I_o \). But then \( VV^* \) and \( WW^* \) are unitarily equivalent, hence \( V \approx W \) by Lemma 8.2, and thus \( V \sim W \).

When \( F^{(2)} = 0 \), i.e., \( F^{(3)} + F^{(4)} = I \), we have \( W^*W = V^*V \sim I \otimes I_o \). Since \( \mathcal{R}(G) \) is a direct sum of type \( II_1 \) factors and in every \( II_1 \) factor there are projections equivalent to their orthogonal complement, we can fix a projection \( R \in \mathcal{R}(G) \) with \( R \sim R^\perp \). As in each of the infinite factors we have \( RF_n \otimes I_o \sim R^\perp F_n \otimes I_o \sim F_n \otimes I_o \), it follows that \( R \otimes I_o \sim R^\perp \otimes I_o \sim I \otimes I_o \). Fix a partial isometry \( V_o \in \mathcal{R}(G) \otimes B(H_o) \) with \( V_o^*V_o = V^*V = W^*W \) and \( V_o^*V_o = R \otimes I_o \). We claim that \( V \sim V_o \). The same argument will prove that \( W \sim V_o \) and hence that \( V \sim W \), which will conclude the proof.

Assume next that \( F^{(3)} = I \). Then

\[
VV^* \sim V^*V \sim I \otimes I_o \sim R \otimes I_o \quad \text{and} \quad (VV^*)^\perp \sim I \otimes I_o \sim R^\perp \otimes I_o = (R \otimes I_o)^\perp.
\]

Thus \( VV^* \) and \( V_o^*V_o \) are unitarily equivalent; hence \( V \approx V_o \), and hence \( V \sim V_o \).

Finally consider the case where \( F^{(4)} = I \), namely where \( V^*V \sim I \otimes I_o \) but \((VV^*)^\perp F_n \otimes I_o \neq F_n \otimes I_o \) for every \( n \), which is the crux of the proof. If for a certain \( n \) the projection \((VV^*)^\perp F_n \otimes I_o \) is finite and hence \((\tau \otimes tr)((VV^*)^\perp F_n \otimes I_o) < \infty \), let \( Q_n^\perp \in B(H_o) \) be a finite projection with

\[
tr(Q_n^\perp) > \frac{(\tau \otimes tr)((VV^*)^\perp F_n \otimes I_o)}{\tau(F_n)}.
\]

Then \( Q_n \sim I_o \). Since \( \mathcal{R}(G)F_n \) is a type \( II_1 \) factor, it contains a projection \( R_n \) with trace

\[
\tau(R_n) = \frac{(\tau \otimes tr)((VV^*)^\perp F_n \otimes I_o)}{\tau(F_n)tr(Q_n^\perp)}.
\]
Equivalently,
\[(\tau \otimes tr)(R_n \otimes Q_n^\perp) = (\tau \otimes tr)((VV^*)^\perp F_n \otimes I_o)\]
and hence, again because \(\Re(G)F_n\) is a factor, \(R_n \otimes Q_n^\perp \sim (VV^*)^\perp F_n \otimes I_o\). If for a
certain \(n\) the projection \((VV^*)^\perp F_n \otimes I_o\) is infinite (but still \((VV^*)^\perp F_n \otimes I_o \neq F_n \otimes I_o\)
by the definition of \(F^{(n)}\), there is a projection \(Q_n^\perp \in B(H_o)\) with \(Q_n \sim I_o\) and for
which \(F_n \otimes Q_n^\perp \sim (VV^*)^\perp F_n \otimes I_o\). In this case, set \(R_n := F_n\), so for all \(n\),
\[(VV^*)^\perp F_n \otimes I_o \sim R_n \otimes Q_n^\perp\]
with \(R_n = R_n F_n\). Moreover,
\[(R_n \otimes Q_n^\perp)^\perp F_n \otimes I_o = F_n \otimes Q_n + R_n^\perp F_n \otimes Q_n^\perp \sim F_n \otimes I_o \sim VV^* F_n\]
because \(Q_n \sim I_o\) and \(Q_n^\perp \not\sim I_o\). Thus \(VV^* F_n \otimes I_o\) and \(F_n \otimes Q_n + R_n^\perp F_n \otimes Q_n^\perp\) are
unitarily equivalent for all \(n\) and hence \(VV^*\) and \(\sum_{n=1}^N F_n \otimes Q_n + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp\) are
unitarily equivalent in \(\Re(G) \otimes B(H_o)\). Choose a unitary \(U \in \Re(G) \otimes B(H_o)\)
such that \(UVV^* U^* = \sum_{n=1}^N F_n \otimes Q_n + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp\). Then \(V \approx UV\).

Now apply Lemma 6.4 to the fixed projection \(R \sim R^\perp\), and the sequences of
central projections \(F_n\) and infinite projection \(Q_n\) that we have constructed for
\(1 \leq n \leq N\). Thus we obtain a path of partial isometries \(\{W(t) \mid t \in [0, 1]\}\) in
\(\Re(G) \otimes B(H_o)\), where \(L_t W(t)\) is norm continuous, \(W(t)^* W(t) = \sum_{n=1}^N F_n \otimes Q_n\) and
\(W(t) W(t)^* \leq \sum_{n=1}^N F_n \otimes Q_n\) for all \(t \in [0, 1]\), \(W(0) = \sum_{n=1}^N F_n \otimes Q_n\), and
\(W(1)^* W(1) = \sum_{n=1}^N R F_n \otimes Q_n\). Then let
\[
V(t) := (W(t) + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp) UV \quad \text{for} \quad t \in [0, 1].
\]
Since the initial projections and the range projections of all the partial isometries
\(W(t)\) are orthogonal to \(\sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp\), \(W(t) + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp\) are partial
isometries for all \(t \in [0, 1]\). Therefore \(V(t)\) are also partial isometries belonging to
\(\Re(G) \otimes B(H_o)\), and all have initial projection \(P_A = (UV)^* UV\). Since \(L_t W(t)\) is
norm continuous, so is \(L_t V(t)\). Thus by definition, \(UV = V(0) \sim V(1)\). Moreover,
the range projection of \(V(1)\) is unitarily equivalent to the range projection \(R \otimes I_o\)
of \(V_o\). Indeed,
\[
V(1) V(1)^* = \left(W(1) + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp\right) UVV^* U^* \left(W(1)^* + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp\right)
\]
\[
= W(1) W(1)^* + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp
\]
\[
= \sum_{n=1}^N R F_n \otimes Q_n + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp
\]
\[
\sim R \otimes I_o
\]
and
\[
(V(1) V(1)^*)^\perp = \sum_{n=1}^N R_n^\perp F_n \otimes Q_n + \sum_{n=1}^N R_n^\perp F_n \otimes Q_n^\perp \sim R^\perp \otimes I_o = (R \otimes I_o)^\perp.
\]
But then, $V(1) \approx V_o$. Since we have already established that $V \approx UV \sim e^{V(1)}$, we conclude in this case too that $V \sim e^{V_o}$, thus completing the proof. □

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