Double grazing bifurcations of the non-smooth railway wheelset systems

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Abstract There are numerous non-smooth factors in railway vehicle systems, such as flange impact, dry friction, creep force, and so on. Such non-smooth factors heavily affect the dynamical behavior of the railway systems. In this paper, we investigate and mathematically analyze the double grazing bifurcations of the railway wheelset systems with flange contact. Two types of models of flange impact are considered, one is a rigid impact model and the other is a soft impact model. First, we derive Poincaré maps near the grazing trajectory by the Poincaré-section discontinuity mapping (PDM) approach for the two impact models. Then, we analyze and compare the near grazing dynamics of the two models. It is shown that in the rigid impact model the stable periodic motion of the railway wheelset system translates into a chaotic motion after the grazing bifurcation, while in the soft impact model a pitchfork bifurcation occurs and the system tends to the chaotic state through a period doubling bifurcation. Our results also extend the applicability of the PDM of one constraint surface to that of two constraint surfaces for autonomous systems.

Keywords Railway wheelset · Non-smooth · PDM · Double grazing bifurcations

1 Introduction

As an important means of transportation, railway vehicles play an important role in a country’s economy and in our way of life. For improvement in the speed of railway vehicles, it is necessary to consider many factors in the research of vehicle dynamics. The non-smooth dynamics, being at the core in railway wheelset systems, is a very important topic, which has been paid considerable attention by researchers [1].

In railway vehicle system, the non-smooth factors mainly come from suspension system and wheel-rail contact system, such as dry friction, flange impact, non-smooth creeps between rail and wheel, and so on [1]. The non-smooth factors induced abundant dynamic phenomena of the railway vehicle system. Huilgol [2] derived the lateral and yaw motion equations of a railway axle and investigated the dynamics of the system.
Kaas-Petersen [3] investigated the dynamics of a railway bogie with flange contact, being the first to identify chaos in railway vehicle systems. Thereby, Meijaard et al. [4] did, in fact, detect chaos in the railway wheelset system. Later, Knudsen et al. [5] considered flange contact to analyze the dynamics of a railway wheelset system by path following routines. The results show that the system has both symmetric and asymmetric oscillations and chaotic motion. Then, True and his coworkers [6,7] continue a numerical investigation of the dynamical behavior and the effect of parameters on the dynamics of the railway wheelset system that was started by Knudsen et al. [5]. Ahmadian and Yang [8,9] performed a bifurcation analysis of railway wheelset and bogie systems with flange contact and non-smooth yaw dampers. It is shown that the yaw damping have a mixed effect on the critical speed and flange contact contribute significantly to the hunting behavior. True and Asmund [10] investigated lateral dynamics of a railway freight wagon wheelset with dry damping. They found that the action of dry friction completely changes the bifurcation diagram, and that the longitudinal component of the dry friction damping force destabilizes the wagon. Hoffmann [11,12] considered the dry friction and leaf spring non-smoothness to investigate the dynamics of the European two-axle railway freight wagons with UIC standard suspension. Gao et al. [13], based on the bifurcation and stability theory, investigated symmetric and asymmetric bifurcation behaviours of a bogie system with non-smooth wheel-rail contact relation. True et al. [14] compared the resulting solutions that are found with three different strategies of handling the non-smoothness that occur in railway vehicle dynamics. As a side work, they tested several integrators, both explicit and implicit ones, and evaluated their performances and compared with respect to accuracy, and computation time.

Grazing impact arises in non-smooth systems when the flow is tangent to the boundary at zero velocity in phase space [15,16]. Nordmark [17] investigated the motion near the grazing impact of a single degree of freedom impact oscillators by periodically forced, introduced the concept of discontinuity mapping and constructed the normal form map for grazing bifurcation of the system for the first. Chin et al. [18,19] studied the dynamics of the Nordmark map after one parameter unfolding in detail. They found that there exist complex phenomena of the mapping such as period adding cascade. The topological and statistical properties of chaotic attractors of the Nordmark map are studied in [20,21]. Fredriksson and Nordmark [22] extended the previous investigations [17] to many degrees of freedom systems and gave a stability criterion for a local attractor near the grazing point. di Bernardo et al. [23] extended the results presented in [17] to n-dimensional piecewise-smooth systems and derived normal form maps for grazing bifurcation. For additional literature about grazing bifurcation, see, e.g., [24,25].

In spite of those research works, detailed investigation of grazing bifurcation in the railway vehicle systems is still missing. In particular, in most papers about grazing bifurcation in vibro-impact systems [26–28], only one constraint and a drive by harmonic excitation were considered. For such systems, it is convenient to use a local zero-time discontinuous mapping (ZDM) approach to study the near grazing dynamics. However, the differential equations of lateral motion of vehicle systems are usually autonomous, and with bilateral symmetry constraints. Therefore, the existing literature results cannot be directly applied. In this work, we consider two-degree-of-freedom autonomous railway wheelset system with bilateral constraints. A local Poincaré-section discontinuity mapping (PDM) approach technique is used to investigate the grazing bifurcation of the system.

The remaining of this paper is organized in the following way. In Sect. 2, the dynamical model of the railway wheelset is described. In Sect. 3, we construct the Poincaré mapping of the dynamics near the grazing for two impact models of railway wheelset. In Sect. 4, numerical simulations are carried out to validate the theoretical analysis. In Sect. 5, we present the conclusions. In “Appendix”, we describe the PDM calculation of the soft impact model.

## 2 Dynamical model of the railway wheelset

We consider the nonlinear dynamics of a single railway wheelset at a constant running speed on a straight track with flange. The wheel profiles are conical and the conicity is \( \lambda \). The railway wheelset is connected to the truck frame by a set of longitudinal and lateral springs, with no damping, as shown in Fig. 1. The railway wheelset has two degrees of freedom. Using the Newton–Euler method, the motion equations of the railway wheelset without flange impact are as follows.
The flange contact can be considered as a rigid impact model. The equations governing the non-smooth railway wheelset systems are as follows [5,6]:

\[
\begin{align*}
    m\dddot{y} + 2k_1y + 2F_y &= 0, \\
    I\ddot{\varphi} + 2k_2d_1^2\varphi + 2aF_x &= 0,
\end{align*}
\]

where \( m \) is the mass of the wheelset, \( I \) is the roll moment of inertia of the wheelset, \( d_1 \) is half of the primary yaw spring arm, \( a \) is half of the track gauge, \( r_0 \) is the wheel radius, \( k_1 \) and \( k_2 \) are lateral stiffness and yaw stiffness of primary suspension, respectively, \( F_x \) and \( F_y \) are the longitudinal and lateral creep forces, respectively.

The flange contact force \( F_T(y) \) is approximated by a linear spring with deadband [5,6], as shown in Fig. 2, which is described by

\[
F_T(y) = \begin{cases} 
    k_0(y - \delta), & y > \delta, \\
    0, & -\delta \leq y \leq \delta, \\
    k_0(y + \delta), & y < -\delta.
\end{cases}
\]

Then, the motion equations of the railway wheelset with flange impact are as follows [5,6]:

\[
\begin{align*}
    m\dddot{y} + 2k_1y + 2F_y + F_T(y) &= 0, \\
    I\ddot{\varphi} + 2k_2d_1^2\varphi + 2aF_x &= 0.
\end{align*}
\]

The dynamics of the system (2.6) has been investigated in detailed by Knudsen [5,6] based on numerical method, it is shown that there are abundant dynamical phenomena in the system (2.6), such as symmetry and asymmetry oscillations and chaos motion. However, the detailed investigation of grazing bifurcation for the systems (2.6) is missing. In particular, the dynamics of rigid flange impact model are not considered in Ref. [5,6].

Let \( y = x_1, \dot{y} = x_2, \varphi = x_3, \dot{\varphi} = x_4 \). Then Eq. (2.6) can be rewritten as the equivalent four-dimensional autonomous dynamical system as follows:

\[
\begin{align*}
    x_1 &= x_2, \\
    x_2 &= -\frac{2k_1}{m}x_1 - \frac{2}{m}F_y - \frac{1}{m}F_T(x_1), \\
    x_3 &= x_4, \\
    x_4 &= -\frac{2k_2d_1^2}{I}x_3 - \frac{2a}{I}F_x,
\end{align*}
\]

where \( V \) is usually taken as a control parameter.

The systems (2.7) can be written in the general form:

\[
\dot{X} = \begin{cases} 
    F_2(X, V), & H_1(X, V) > 0, \\
    F_1(X, V), & 0 \leq H_2(X, V) \leq 0, \\
    F_3(X, V), & H_2(X, V) < 0,
\end{cases}
\]

where \( X = (x_1, x_2, x_3, x_4)^T \) are the state variables of the system, \( H_1(X, V) = x_1 - \delta, H_2(X, V) = x_1 + \delta \) are the impact surface functions and

\[
\begin{align*}
    F_2(X, V) &= \begin{pmatrix} 
        -\frac{2k_1}{m}x_1 - \frac{2}{m}F_y - \frac{k_0}{m}(x_1 - \delta) \\
        x_2 \\
        x_4 \\
        -\frac{2k_2d_1^2}{I}x_3 - \frac{2a}{I}F_x
    \end{pmatrix}, \\
    F_1(X, V) &= \begin{pmatrix} 
        -\frac{2k_1}{m}x_1 - \frac{2}{m}F_y \\
        x_2 \\
        x_4 \\
        -\frac{2k_2d_1^2}{I}x_3 - \frac{2a}{I}F_x
    \end{pmatrix}, \\
    F_3(X, V) &= \begin{pmatrix} 
        -\frac{2k_1}{m}x_1 - \frac{2}{m}F_y - \frac{k_0}{m}(x_1 + \delta) \\
        x_2 \\
        x_4 \\
        -\frac{2k_2d_1^2}{I}x_3 - \frac{2a}{I}F_x
    \end{pmatrix}.
\end{align*}
\]

2.2 Rigid impact model

The flange contact can be considered as a rigid impact system. The equations governing the non-impact motions for \( -\delta < y < \delta \) are given by Eq.
Fig. 1 Model of railway wheelset and the coordinate system

Fig. 2 Flange contact force versus lateral displacement

\[(2.1)\]

\[m \ddot{y} + 2k_1 y + 2F_y = 0,\]

\[I \ddot{\varphi} + 2k_2 d_1^2 \varphi + 2a F_x = 0.\]

When \(y = \pm \delta\), the system satisfies the Newton’s law of collisions, that is

\[\dot{y}_+ = -e \dot{y}_-,\]

where \(0 < e < 1\) is the coefficient of restitution, and ‘+’ represents the instantaneous moment after impact and ‘−’ represents the instantaneous moment before impact.

Let \(y = x_1, \dot{y} = x_2, \varphi = x_3, \dot{\varphi} = x_4\). Then Eq. (2.1) can be rewritten as the equivalent four-dimensional autonomous dynamical system as follows:

\[x_1 = x_2,\]
\[\dot{x}_2 = -\frac{2k_1}{m} x_1 - \frac{2}{m} F_y,\]
\[\dot{x}_3 = x_4,\]
\[\dot{x}_4 = -\frac{2k_2 d_1^2}{I} x_3 - \frac{2a}{I} F_x.\]

The systems (2.10) can be written in the general form:

\[\dot{X} = F_1(X, V),\]

where \(X = (x_1, x_2, x_3, x_4)^T\) are the state variables of the system, and

\[F_1(X, V) = \begin{pmatrix} x_2 \\ -\frac{2k_1}{m} x_1 - \frac{2}{m} F_y \\ x_4 \\ -\frac{2k_2 d_1^2}{I} x_3 - \frac{2a}{I} F_x \end{pmatrix}.\]

3 Grazing bifurcation of the railway wheelset

Assume that there is a double grazing orbit, denoted by \(\Gamma\), in the system (2.1), and that the grazing point of right constraint A is marked as \(X^*_1\) with \(X^*_1 = (x_{1A}^*, x_{2A}^*, x_{3A}^*, x_{4A}^*)\). The impact surface \(\Sigma_1\) of constraint A is defined as

\[\Sigma_1 = \{ X \in D(X^*_1) | H_1(X, V) = x_1 - \delta \}. \quad (3.1)\]

Due to the existence of the symmetry, the grazing point of the left constraint B is marked as \(X^*_2\) with \(X^*_2 = (x_{1B}^*, x_{2B}^*, x_{3B}^*, x_{4B}^*)\). The impact surface \(\Sigma_2\) of constraint B is defined as

\[\Sigma_2 = \{ X \in D(X^*_2) | H_2(X, V) = x_1 + \delta \}. \quad (3.2)\]

According to [16], the double grazing periodic motion satisfies the conditions

\[H_1(X_1^*, V^*) = 0,\]
\[v_1(X_1^*, V^*) = \frac{\partial H_1}{\partial t}(\phi_1(X_1^*, V^*, 0), V^*) = H_{1,X} F_1(X_1^*, V^*) = 0, \quad (3.3)\]
\[a_1(X_1^*, V^*) = \frac{\partial^2 H_1}{\partial t^2}(\phi_1(X_1^*, V^*, 0), V^*) = H_{1,X} F_{1,X} F_1(X_1^*, V^*) < 0,\]

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respectively. Where $M$ and $g$ govern by the vector field $\phi(X^*_2, V^*, 0, V^*)$

$$H_2(X^*_2, V^*) = 0, \quad v_2(X^*_2, V^*) = \frac{\partial H_2}{\partial t}(\phi(X^*_2, V^*, 0, V^*)) = H_{2, x} F_1(X^*_2, V^*) = 0, \quad (3.4)$$

$$a_2(X^*_2, V^*) = \frac{\partial^2 H_2}{\partial t^2}(\phi(X^*_2, V^*, 0, V^*)) = H_{2, x} F_1(X^*_2, V^*) > 0. \quad (3.5)$$

We define two Poincaré surfaces

$$\Pi_1 = \{X_1 \in D_1(X^*_1)|v(X_1, V) = H_{1, x} F_1(X_1, V) = 0\}, \quad (3.6)$$

$$\Pi_2 = \{X_2 \in D_2(X^*_2)|v(X_2, V) = H_{2, x} F_1(X_2, V) = 0\}, \quad (3.7)$$

and two maps

$$P_1 : \Pi_1 \mapsto \Pi_2, \quad P_2 : \Pi_2 \mapsto \Pi_1, \quad (3.8)$$

where the flow maps are used to construct $P_1$ and $P_2$ governed by the vector field $F_1$ alone. That is, near the grazing points $X^*_1$ and $X^*_2$, we map points using only trajectories of the flow $\phi_1$ which generated by the vector field $F_1$, where $F_1$ represent the vector field without impact with the constraint. So the Poincaré maps $P_1$ and $P_2$ are defined by linearization

$$P_1(X, V) = X^*_2 + N_1(X - X^*_1) + M_1(V - V^*) + h.o.t., \quad (3.9)$$

$$P_2(X, V) = X^*_1 + N_2(X - X^*_2) + M_1(V - V^*) + h.o.t., \quad (3.10)$$

respectively, where $N_1 = \frac{\partial P_1}{\partial X}|_{X = X^*_1, V = V^*}, M_1 = \frac{\partial P_1}{\partial V}|_{X = X^*_1, V = V^*}, N_2 = \frac{\partial P_2}{\partial X}|_{X = X^*_2, V = V^*}, M_2 = \frac{\partial P_2}{\partial V}|_{X = X^*_2, V = V^*}. N_1$ and $N_2$ are $3 \times 3$ matrices, $M_1$ and $M_2$ are $3 \times 1$ matrices.

We next construct Poincaré maps in small neighborhoods of grazing points for the soft and the rigid railway wheelset impact models.

3.1 The normal form map for soft impact model

By Eq. (5.16) of the Appendix A, the discontinuity mapping near grazing of the soft impact railway wheelset system can be defined as follows,

$$PDM_1(X_1) = \begin{cases} \begin{pmatrix} X_1 \\ X_1 + \beta_1(X_1, V) \sqrt{\frac{-2H_1(X_1, V)}{a_1^2}} + \frac{2F_1^*({a_1^*}_1 - {a_2^*}_1)}{a_1^2} \end{pmatrix} \; \text{if} \; H_1(X_1, V) < 0, \\ \beta_2(V - V^*) \sqrt{\frac{-2H_1(X_1, V)}{a_1^2}} + h.o.t. \; \text{if} \; H_1(X_1, V) \geq 0, \end{cases} \quad (3.11)$$

where $\beta_1 = 2(F_2^* - F_1^*) + \frac{2F_1^*({a_1^*}_1 - {a_2^*}_1)}{a_1^2}, \beta_2 = 2(F_2^* - F_1^*) + \frac{2F_1^*({a_1^*}_1 - {a_2^*}_1)}{a_1^2}$. The superscript ‘∗’ denotes quantities evaluated at the grazing point ($X^*_1, V^*$), and ‘h.o.t’ represents the high-order terms about ($X, V, t$). We also define,

$$PDM_2(X_2) = \begin{cases} \begin{pmatrix} X_2 \\ X_2 + \beta_3(X_2, V) \sqrt{\frac{-2H_2(X_2, V)}{a_2^2}} + \frac{2F_2^*({a_1^*}_2 - {a_2^*}_2)}{a_2^2} \end{pmatrix} \; \text{if} \; H_2(X_2, V) > 0, \\ \beta_4(V - V^*) \sqrt{\frac{-2H_2(X_2, V)}{a_2^2}} + h.o.t. \; \text{if} \; H_2(X_2, V) \leq 0, \end{cases} \quad (3.12)$$

where $\beta_3 = 2(F_3^* - F_1^*) + \frac{2F_1^*({a_1^*}_1 - {a_2^*}_1)}{a_2^2}, \beta_4 = 2(F_3^* - F_1^*) + \frac{2F_1^*({a_1^*}_1 - {a_2^*}_1)}{a_2^2}$. Then the Poincaré map $P$ from $\Pi_1$ to itself can be written as

$$P = P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X). \quad (3.13)$$

For $X_1 \in \Pi_1$, we analyze the Poincaré map in the following way.

(1) For the case $H_1(X_1, V) < 0$,

$$X_2 = P_1 \circ PDM_1(X_1) \quad = X_2^* + N_1(PDM_1(X_1) - X_1^*) + M_1(V - V^*) + h.o.t. \quad = X_2^* + N_1(X_1 - X_1^*) + M_1(V - V^*) + h.o.t. \quad (3.14)$$

(i) If $H_2(X_2, V) > 0$, then

$$P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1) \quad = P_2 \circ PDM_2(X_2) \quad = X_1^* + N_2(X_2) \quad - X_2^* + M_2(V - V^*) + h.o.t. \quad (3.15)$$
(ii) If $H_2(X_2, V) \leq 0$, then

$$P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$

$$= P_2 \circ PDM_2(X_2)$$

$$= X_1^* + N_2(X_2 + \beta_3(X_2 - X_2^*))$$

$$+ \beta_4(V - V^*)$$

$$\sqrt{-2H_2(X_2, V)}$$

$$\frac{1}{a_3^*} - X_2^*$$

$$+ M_2(V - V^*) + h.o.t.$$ \hspace{1cm} (3.16)

(2) For the case $H_1(X_1, V) \geq 0$,

$$P(X_1) = P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$

$$= \begin{cases} 
X_1^* + N_2(X_2 - X_2^*) + M_2(V - V^*) + h.o.t., & H_1(X_1, V) < 0, H_2(X_2, V) > 0,
X_1^* + N_2(X_2 + \beta_3(X_2 - X_2^*)) + \beta_4(V - V^*) \sqrt{-2H_2(X_2, V)} a_3^* - X_2^*, & H_1(X_1, V) < 0, H_2(X_2, V) \leq 0,
X_1^* + N_2(X_2 + \beta_3(X_2 - X_2^*)) + \beta_4(V - V^*) \sqrt{-2H_2(X_2, V)} a_3^* - X_2^*, & H_1(X_1, V) \geq 0, H_2(X_2, V) > 0,
X_1^* + N_2(X_2 - X_2^*) + M_2(V - V^*) + h.o.t., & H_1(X_1, V) \geq 0, H_2(X_2, V) \leq 0.
\end{cases}$$

$$= \begin{cases} 
X_1^* + N_1(X_1 + \beta_1(X_1 - X_1^*)) + \beta_2(V - V^*) \sqrt{-2H_1(X_1, V)} a_1^* - X_1^*, & H_1(X_1, V) < 0, H_2(X_2, V) > 0,
X_1^* + N_1(X_1 + \beta_1(X_1 - X_1^*)) + \beta_2(V - V^*) \sqrt{-2H_1(X_1, V)} a_1^* - X_1^*, & H_1(X_1, V) < 0, H_2(X_2, V) \leq 0,
X_1^* + N_1(X_1 + \beta_1(X_1 - X_1^*)) + \beta_2(V - V^*) \sqrt{-2H_1(X_1, V)} a_1^* - X_1^*, & H_1(X_1, V) \geq 0, H_2(X_2, V) > 0,
X_1^* + N_1(X_1 - X_1^*) + M_1(V - V^*) + h.o.t., & H_1(X_1, V) \geq 0, H_2(X_2, V) \leq 0.
\end{cases}$$

$$X_2 = P_1 \circ PDM_1(X_1)$$

$$= X_2^* + N_1(PDM_1(X_1) - X_1^*)$$

$$+ M_1(V - V^*) + h.o.t.$$

$$= X_2^* + N_1(X_1 + \beta_1(X_1 - X_1^*))$$

$$\sqrt{-2H_1(X_1, V)}$$

$$\frac{1}{a_1^*} - X_1^*$$

$$+ \beta_2(V - V^*) \sqrt{-2H_1(X_1, V)} a_1^* - X_1^*$$

$$+ M_1(V - V^*) + h.o.t.$$ \hspace{1cm} (3.17)

(i) If $H_2(X_2, V) > 0$, then

$$P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$

$$= P_2 \circ PDM_2(X_2)$$

$$= X_1^* + N_2(X_2 - X_2^*)$$

$$+ M_2(V - V^*) + h.o.t.$$ \hspace{1cm} (3.18)

(ii) If $H_2(X_2, V) \leq 0$, then

3.2 The normal form map for rigid impact model

According to Ref. [16], the discontinuity mapping near grazing of the rigid impact railway wheelset system can be defined as follows:

$$PDM_1(X_1)$$

$$= \begin{cases} 
X_1, & H_1(X_1, V) < 0, H_2(X_2, V) > 0,
X_1 + \beta_5 \sqrt{-2H_1(X_1, V)} a_1^* + h.o.t., & H_1(X_1, V) \geq 0,
\end{cases}$$

$$PDM_2(X_2)$$

$$= \begin{cases} 
X_2, & H_2(X_2, V) > 0,
X_2 + \beta_6 \sqrt{-2H_2(X_2, V)} a_2^* + h.o.t., & H_2(X_2, V) \leq 0,
\end{cases}$$

where $\beta_5 = -(Wa_1^* - (v_{1X}^* W) F_1(X_1^*, V^*))$, $v_{1X}^* = (H_2 F_1) x|_{X=X_1^*, V=V^*}$, $v_{2X}^* W = -(1 + e)$, $W = (0, 0, 0)^T$. Furthermore,

$$PDM_2(X_2)$$

$$= \begin{cases} 
X_2, & H_2(X_2, V) > 0,
X_2 + \beta_6 \sqrt{-2H_2(X_2, V)} a_2^* + h.o.t., & H_2(X_2, V) \leq 0,
\end{cases}$$

where $\beta_6 = -(Wa_2^* - v_{2X}^* W F_1(X_2^*, V^*))$ and $v_{2X}^* = (H_2 F_1) x|_{X=X_2^*, V=V^*}$.
Then the Poincaré map $P$ from $\Pi_1$ to itself can be written as

$$P(X) = P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X).$$  \hspace{1cm} (3.22)$$

For $X_1 \in \Pi_1$, we analyze the Poincaré map in the following way.

(1) For the case $H_1(X_1, V) < 0$,

$$X_2 = P_1 \circ PDM_1(X_1)$$
$$= X_2^* + N_1(PDM_1(X_1) - X_1^*)$$
$$+ M_1(V - V^*) + h.o.t$$
$$= X_2^* + N_1(X_1 - X_1^*)$$
$$+ M_1(V - V^*) + h.o.t.$$ \hspace{1cm} (3.23)

(i) If $H_2(X_2, V) > 0$, then

$$P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$
$$= P_2 \circ PDM_2(X_2)$$
$$= X_1^* + N_2(X_2 - X_2^*)$$
$$+ M_2(V - V^*) + h.o.t.$$  \hspace{1cm} (3.24)

(ii) If $H_2(X_2, V) \leq 0$, then

$$P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$
$$= P_2 \circ PDM_2(X_2)$$
$$= X_1^* + N_2(X_2 + \beta_6$$
$$\sqrt{\frac{-2H_2(X_2, V)}{a_2^*} - X_2^*})$$
$$+ M_2(V - V^*) + h.o.t.$$ \hspace{1cm} (3.25)

(2) For the case $H_1(X_1, V) \geq 0$,

$$X_2 = P_1 \circ PDM_1(X_1)$$
$$= X_2^* + N_1(PDM_1(X_1) - X_1^*)$$
$$+ M_1(V - V^*) + h.o.t$$
$$= X_2^* + N_1(X_1 + \beta_5 \sqrt{\frac{-2H_1(X_1, V)}{a_1^*} - X_1^*})$$
$$+ M_1(V - V^*) + h.o.t.$$ \hspace{1cm} (3.26)

(i) If $H_2(X_2, V) > 0$, then

$$P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$
$$= P_2 \circ PDM_2(X_2)$$
$$= X_1^* + N_2(X_2 - X_2^*)$$
$$+ M_2(V - V^*) + h.o.t.$$ \hspace{1cm} (3.27)

(ii) If $H_2(X_2, V) \leq 0$, then

$$P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$
$$= P_2 \circ PDM_2(X_2)$$
$$= X_1^* + N_2(X_2 + \beta_6$$
$$\sqrt{\frac{-2H_2(X_2, V)}{a_2^*} - X_2^*})$$
$$+ M_2(V - V^*) + h.o.t.$$  \hspace{1cm} (3.28)

According to (3.23–3.28), for every point $X_1 \in D_1(X_1^*)$ on the Poincaré surface $\Pi_1$, we have

$$P(X_1) = P_2 \circ PDM_2 \circ P_1 \circ PDM_1(X_1)$$
$$\begin{cases} 
X_1^* + N_2(X_2 - X_2^*) + M_2(V - V^*) + h.o.t, & H_1(X_1, V) < 0, H_2(X_2, V) > 0, \\
X_1^* + N_2(X_2 + \beta_6 \sqrt{\frac{-2H_2(X_2, V)}{a_2^*} - X_2^*}) + M_2(V - V^*) + h.o.t, & H_1(X_1, V) < 0, H_2(X_2, V) \leq 0, \\
X_1^* + N_2(X_2 - X_2^*) + M_2(V - V^*) + h.o.t, & H_1(X_1, V) \geq 0, H_2(X_2, V) > 0, \\
X_1^* + N_2(X_2 + \beta_6 \sqrt{\frac{-2H_2(X_2, V)}{a_2^*} - X_2^*}) + M_2(V - V^*) + h.o.t, & H_1(X_1, V) \geq 0, H_2(X_2, V) \leq 0. 
\end{cases}$$
In this Section, we apply the discontinuity mapping approach to derive explicit expressions for the composite Poincaré map $P(X_1, V)$ in the neighborhood of selected grazing periodic orbit. Here the shooting method is used to obtain grazing periodic orbit. The corresponding parameters values are taken from Table 1. The grazing points of periodic trajectory $\Gamma$ are

$X_1^* \approx (0.009100, 0, 0.000811, -0.013250)^T$,  
$X_2^* \approx (-0.009100, 0, -0.000811, 0.013250)^T$,  
$V^* \approx 10.056027 \text{ m/s}$.

Figure 3 is the phase portrait of the double grazing periodic orbit $\Gamma$. The blow up in the neighborhood of grazing points of Fig. 3 are shown in Fig. 4. Figure 5 is the time history of the displacement for the grazing periodic orbit $\Gamma$.

4.1 Numerical simulations for soft impact model

For soft impact, the composite Poincaré map is

\[
\begin{align*}
X_1^{**} &= \gamma_1 \Delta X + \gamma_2 \Delta V + h.o.t, \\
X_2^{**} &= \gamma_1 \Delta X + \gamma_2 \Delta V + \\
\left( \gamma_3 \Delta X + \gamma_4 \Delta V \right) \sqrt{\gamma_5 \Delta X + 0.000856 \Delta V^T} + h.o.t, \\
X_1^{**} &= \gamma_1 \Delta X + \gamma_2 \Delta V + \gamma_6 \Delta X \Delta X^T + h.o.t, \\
X_2^{**} &= \gamma_1 \Delta X + \gamma_2 \Delta V + \gamma_6 \Delta X \Delta X^T + h.o.t,
\end{align*}
\]

where $\Delta X = X_1 - X_1^{**}$, $\Delta V = V - V^*$, $\Delta X^T$ is the first component of $\Delta X$, $\Delta V^T$ is the first component of $\Delta V$, and

\[
\begin{align*}
\gamma_1 &= \begin{pmatrix} 0.009100 \\ 0.000810 \\ -0.013249 \end{pmatrix}, \\
\gamma_2 &= \begin{pmatrix} 0.952259 & -0.351723 & -0.053662 \\ 0.085117 & -0.031439 & -0.004797 \\ -1.38605 & 0.511948 & 0.078108 \end{pmatrix}, \\
\gamma_3 &= \begin{pmatrix} 0.001804 \\ 0.000065 \\ -0.003700 \end{pmatrix}, \\
\gamma_4 &= \begin{pmatrix} -3.77022 \\ -0.336997 \end{pmatrix}, \\
\gamma_5 &= \begin{pmatrix} 0.952770 \\ -0.351912 \end{pmatrix}, \\
\gamma_6 &= \begin{pmatrix} -4.400.76 \end{pmatrix}, \\
\gamma_7 &= \begin{pmatrix} 1.93875e7 \end{pmatrix}, \\
\gamma_8 &= \begin{pmatrix} 1.73293e6 \end{pmatrix}.
\end{align*}
\]

Figure 6 shows the bifurcation diagrams of the system: (a) is obtained by direct numerical simulation and (b) is obtained by the PDM method for the soft impact model. It is shown that the qualitative properties of the two cases are almost the same, i.e., after the grazing bifurcation it has a pitchfork bifurcation and translates to chaos through period doubling. Figure 7 shows the motion at $V = 10.0560 \text{ m/s}$ before the grazing bifurcation, from the picture we can see that the system is in periodic motion which confirmed by the Poincaré maps and the largest Lyapunov exponent. Figures 8 and 9 show the motion of the system at $V = 10.0561 \text{ m/s}$ and $V = 10.0565 \text{ m/s}$ after grazing bifurcation, respectively. Figure 8 shows that the system is in period-1 motion at $V = 10.0561 \text{ m/s}$ and Fig. 9 shows that the system is in chaotic motion at $V = 10.0565 \text{ m/s}$.
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4.2 Numerical simulations for rigid impact model

For rigid impact, the composite Poincaré map is

\[
P = \begin{cases}
X_{i}^{**} + \gamma_1' \Delta X + \gamma_2' \Delta V + h.o.t, & \text{if } H_1(X_1, V) < 0, H_2(X_2, V) > 0,

X_{i}^{**} + \gamma_1' \Delta X + \gamma_2' \Delta V + \gamma_3' \sqrt{\gamma_4' \Delta X + 0.000856 \Delta V^1 + h.o.t}, & \text{if } H_1(X_1, V) < 0, H_2(X_2, V) \leq 0,

X_{i}^{**} + \gamma_1' \Delta X + \gamma_2' \Delta V + \gamma_5' \sqrt{\Delta X^1} + h.o.t, & \text{if } H_1(X_1, V) \geq 0, H_2(X_2, V) > 0,

\gamma_3' \sqrt{\gamma_4' \Delta X + 0.000856 \Delta V^1 + 0.036963 \sqrt{\Delta X^1} + h.o.t}, & \text{if } H_1(X_1, V) \geq 0, H_2(X_2, V) \leq 0,
\end{cases}
\]

where

\[
X_{i}^{**} = \begin{pmatrix}
0.009100 \\
0.000810 \\
-0.013249
\end{pmatrix}, \\
\gamma_1' = \begin{pmatrix}
0.952259 & -0.351723 & -0.053662 \\
0.085117 & -0.031439 & -0.004797 \\
-1.38605 & 0.511948 & 0.078108
\end{pmatrix}, \\
\gamma_2' = \begin{pmatrix}
0.001804 \\
0.000065 \\
-0.003700
\end{pmatrix}, \\
\gamma_3' = \begin{pmatrix}
-0.036963 \\
-0.003304 \\
0.053801
\end{pmatrix}, \\
\gamma_4' = \begin{pmatrix}
0.952770 \\
-0.351912 \\
-0.053691
\end{pmatrix}^T, \\
\gamma_5' = \begin{pmatrix}
-0.036943 \\
-0.003302 \\
0.053772
\end{pmatrix}.
\]
Fig. 6 Bifurcation diagrams for the soft impact model

Fig. 7 a Phase portrait; b Poincaré maps; c the largest Lyapunov exponent at $V = 10.0560\ m/s$
through grazing bifurcation, and the bifurcation diagram given by the PDM method compares well with that given by direct numerical simulation method. The motion of the rigid and soft impact model are the same before grazing bifurcation because the system has the same governing equation (2.1). So, from Fig. 7 we can see that the system is in periodic motion. Figure 11a, c, d gives the phase portrait, Poincaré maps and the largest Lyapunov exponent at $V = 10.0563 \, m/s$, respectively. Figure 11b is a local blow up of Fig. 11a. It shows that the system is in chaotic oscillations.

4.3 Effect of parameters

In this section, we investigate the effect of the wheelset system parameters on the grazing bifurcation. Suspension parameters and wheel-rail contact geometric parameters have important effects on the grazing critical speed of railway wheelset system. The conicity of wheels is an important geometric parameter of wheel-rail contact which has a significant influence of dynamics for the wheelset systems. From Fig. 12 we can see that the grazing critical speed $V^*$ decreases as the
Figure 9  a Phase portrait; b blow up of phase portrait (a); c Poincaré maps; d the largest Lyapunov exponent at \( V = 10.0565 \text{m/s} \)

value of wheel conicity \( \lambda \) increases. Figure 13 shows the influence of lateral stiffness on the grazing critical velocity. From Fig 13 we can see that grazing critical speed \( V^* \) increases as the value of lateral stiffness \( k_1 \) increases. Next, we take the wheel conicity \( \lambda = 0.15 \) and the other parameters are fixed at their standard values as defined in Table 1. From Fig. 12 we can obtain the corresponding grazing critical velocity value \( V^* \approx 5.702148 \text{m/s} \). The grazing points of periodic trajectory are

\[
X_1^* \approx (0.009100, 0, 0.000811, -0.023364)^T, \\
X_2^* \approx (-0.009100, 0, -0.000811, 0.023364)^T,
\]

Figure 14 show the bifurcation diagrams by using direct numerical simulation (Fig. 14a) and the PDM method (Fig. 14b) for the soft impact model as wheel conicity \( \lambda = 0.15 \). And Fig. 15 show the bifurcation diagrams by using direct numerical simulation (Fig. 15a) and the PDM method (Fig. 15b) for the rigid impact model as wheel conicity \( \lambda = 0.15 \). From Figs. 14 and 15 we can see that the bifurcation diagram given by the PDM method compares well with that given by direct numerical simulation method for the two impact models and that the near grazing dynamics are similar to that of the wheel conicity \( \lambda = 0.05 \).
Fig. 10  Bifurcation diagrams for the rigid impact model

Fig. 11  a Phase portrait; b blow up of phase portrait (a); c Poincaré maps; d the largest Lyapunov exponent at $V = 10.0563 \text{ m/s}$
Fig. 12 The grazing curve on the parameters plane ($\lambda$, $V^*$)

Fig. 13 The grazing curve on the parameters plane ($k_1$, $V^*$)

5 Conclusions

In this paper, the double grazing bifurcations of two classes of non-smooth railway wheelset models are investigated. The non-smoothness at the railway wheelset systems mainly come from flange impact. We consider two types of flange impact models. To understand and explain the dynamical behavior of the railway wheelset systems caused by the grazing bifurcations, the PDM approach is used to construct the Poincaré maps of the systems near grazing orbit. Numerical simulation results indicate that the Poincaré-section discontinuity mapping approach is valid for the railway wheelset system with both rigid and soft impact.

Table 1 Values of parameters [5]

| Parameter                                           | Value                  |
|-----------------------------------------------------|------------------------|
| Mass of the wheelaxle                               | $m = 1022 \text{ kg}$  |
| Roll moment of inertia of wheelset                  | $I = 678 \text{ kg m}^2$ |
| Half of the track gauge                             | $a = 0.716 \text{ m}$  |
| Distance from centre of gravity to $k_2$            | $d_1 = 0.620 \text{ m}$ |
| Shear modulus                                       | $G = 808 \text{ MN m}^{-2}$ |
| Major semiaxis of contact ellipse                   | $a_c = 6.578 \text{ mm}$ |
| Minor semiaxis of contact ellipse                   | $b_c = 3.937 \text{ mm}$ |
| Dead band                                           | $\delta = 9.1 \text{ mm}$ |
| Spring constant                                     | $k_0 = 14.6 \text{ MN m}^{-1}$ |
| Lateral stiffness of primary suspension             | $k_1 = 1.823 \times 10^4 \text{ Nm}^{-1}$ |
| Yaw spring stiffness of primary suspension          | $k_2 = 0 \text{ Nm}^{-1}$ |
| Lateral wheel-rail contact parameter                | $\Psi = 0.54219$       |
| Longitudinal wheel-rail contact parameter           | $\Phi = 0.60252$       |
| Centred wheel rolling radius                        | $r_0 = 0.4572 \text{ m}$ |
| Wheel conicity                                      | $\lambda = 0.05$       |
| Coefficient of friction                             | $\mu = 0.15$           |
| $N$ is the vertical force between wheel and rail    | $\mu N = 10 \text{ KN}$ |

It is shown that the chaotic motion appears after grazing bifurcation in the rigid impact model of the railway wheelset system, which corresponds to one of the scenarios described in Nordmark [17]. In the soft impact model with symmetric bilateral constraints, the system has a pitchfork bifurcation and tends to chaos through period doubling after grazing bifurcation. Such phenomenon is different from that of unilateral constraint system. In unilateral constraint system, it translates directly to chaos through period doubling [30]. The results of this paper could provide potential reference for dynamical optimal design of railway vehicle systems. We consider one parameter grazing bifurcation of the railway wheelset in this work. The degenerate grazing bifurcation of the system will be considered in our future work.
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Fig. 14  Bifurcation diagrams for the soft impact model as $\lambda = 0.15$

Fig. 15  Bifurcation diagrams for the rigid impact model as $\lambda = 0.15$

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Appendix: Calculating the PDM of the soft impact model

Suppose that a dynamical system discontinuity can be represented by state space discontinuity surface $D$, and that the system has a single discontinuity boundary $\Sigma = \{ X \in D | H(X, V) = 0 \}$. Consider the $n$ degree of freedom soft impact autonomous system

$$
\dot{X} = \begin{cases} 
F_1(X, V), & H(X, V) < 0, \\
F_2(X, V), & H(X, V) \geq 0.
\end{cases}
$$

(5.1)
where $X \in \mathbb{R}^{2n}$, $V \in \mathbb{R}$, $F_1, F_2 : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ are supposed to be sufficiently smooth. The flows $\phi_i(X, V, t)$, ($i = 1, 2$) generated by each of the vector fields can be defined as the quantities that satisfy
\[
\frac{\partial \phi_i(X, V, t)}{\partial t} = F_i(\phi_i(X, V, t)), \phi_i(X, V, 0) = X.
\]

Expanding each of the system flows $\phi_i(X, V, t)$ in a Taylor series about the grazing point $(X^*, V^*, 0)$, we obtain
\[
\phi_i(X, V, t) = X^* + [(X - X^*) + F_i^*(t)] t
\]
\[+ \left[ \frac{1}{2} F_{i,X}^* F_i^* t^2 + F_{i,V}^* (V - V^*) t \right] + h.o.t
\]
\[= X + F_i^* t + \left[ \frac{1}{2} F_{i,X}^* F_i^* t^2 + F_{i,V}^* (V - V^*) t \right] + h.o.t.
\]

where $F_i^* = F_i(X^*, V^*)$, $F_{i,X}^* = \frac{\partial F_i(X, V)}{\partial X} |_{X=X^*,V=V^*}$, $F_{i,V}^* = \frac{\partial F_i(X, V)}{\partial V} |_{X=X^*,V=V^*}$ ($i = 1, 2$).

We define the Poincaré section as
\[
\Pi_N = \{ X \in D | V = \frac{\partial H(X, V)}{\partial t} = 0 \}.
\]

Suppose that the state point $X_0 \in \Pi_N$ represents the initial point in the neighborhood of grazing point $X^*$. In order to construct the Poincaré-section discontinuity mapping (PDM), we evolve the flow $\phi_1$ backward starting from point $X_0$ until it intersects the discontinuity boundary $\Sigma$ at point $X_1$. The time it takes from $X_0$ to $X_1$ is denoted by $-\delta_1$. The forward evolution using $\phi_2$ by a time $\delta_2$ until it intersects $\Sigma$ again at point $X_2$. Finally, using $\phi_1$ backwards until it reaches the Poincaré section $\Pi_N$ again at point $X_3$ after a time $\delta_3$.

The PDM is the map $X_0 \mapsto X_3$, and it has the form:
\[
\text{PDM}(X_0, V) = \begin{cases} 
X_0, & H(X_0, V) < 0, \\
\phi_1(\phi_2(\phi_1(X_0, V, -\delta_1), V, -\delta_2), V, -\delta_3), & H(X_0, V) \geq 0.
\end{cases}
\]

Since the point $X_1$ is on $\Sigma$, we have $H(X_1, V) = 0$. Expanding $H(X_1, V)$ with respect to $\delta_1$, we have
\[
H(X_1, V) = H(\phi_1(X_0, V, -\delta_1), V)
\]
\[= H(X_0, V) - v_1(X_0, V)\delta_1
\]
\[+ \frac{1}{2} a_1(X_0, V)\delta_1^2 + h.o.t
\]
\[= 0,
\]

where $v_1(X, V) = \frac{\partial H(X, V)}{\partial t} = H_X F_1$ is the generalized velocity and $a_i(X, V) = \frac{\partial^2 H(X, V)}{\partial \delta_i^2} = H_X F_{i,X} F_1$ is generalized acceleration ($i = 1, 2$).

By Eq. (5.7) and that $v_1(X_0, V) = 0$, we have
\[
\delta_1 = \sqrt{\frac{-2H(X_0, V)}{a_1}} + h.o.t.
\]

**Step 2.** From $X_1$ to $X_2$.

According to Eq. (5.3), we have
\[
X_2 = \phi_2(X_1, V, \delta_2)
\]
\[= X_1 + F_2^* \delta_2 + \frac{1}{2} F_{2,X}^* F_{2,V}^* \delta_2
\]
\[+ F_{2,X}^* (X_1 - X^*) \delta_2 + F_{2,V}^* (V - V^*) \delta_2 + h.o.t.
\]

Since the point $X_2$ is in the surface $\Sigma$, we have $H(X_2, V) = 0$. Expanding $H(X_2, V)$ with respect to $\delta_2$, we have
\[
H(X_2, V) = H(\phi_2(X_1, V, \delta_2), V)
\]
\[= H(X_1, V) + v_2(X_1, V)\delta_2
\]
\[+ \frac{1}{2} a_2(X_1, V)\delta_2^2 + h.o.t
\]
\[= 0.
\]

According to Eq. (5.10) and $H(X_1, V) = 0$, we have
\[
\delta_2 = \frac{-2 v(X_1, V)}{a_2^*} + h.o.t
\]
\[= \frac{2}{a_2} [a_1^* + a_{1,X}^* (X_0 - X^*) + a_{1,V}^* (V - V^*)] \delta_1
\]
\[+ h.o.t.
\]

**Step 3.** From $X_2$ to $X_3$.

According to Eq. (5.3), we have
\[
X_3 = \phi_1(X_2, V, -\delta_3)
\]
\[= X_2 - F_1^* \delta_3
\]
\[+ \frac{1}{2} F_{1,X}^* F_{1,V}^* \delta_3^2 - F_{1,X}^* (X_2 - X^*) \delta_3
\]
\[+ F_{1,V}^* (V - V^*) \delta_3 + h.o.t.
\]
Since the point $X_3$ is on the Poincaré section $\Pi_N$, it means $v_1(X_3, V) = 0$. Expanding $v_1(X_3, V)$ with respect to the grazing point $(X^*, V^*, 0)$, and according to Eqs. (5.6) and (5.9) we obtain

$$v_1(X_3, V) = v_1(\phi(X_2, V, -\delta_3), V)$$
$$= \frac{1}{2}a_1^* F_1^* \delta_3^2 + \frac{1}{2}a_2^* F_2^* \delta_3^2$$
$$- [a_1^* + a_1^* (X_0 - X^*) + a_1^* V (V - V^*)] \delta_1$$
$$+ [a_2^* + a_2^* (X_0 - X^*) + a_2^* V (V - V^*)] \delta_1$$
$$- a_1^* F_1^* \delta_2 [a_1^* + a_1^* (X_0 - X^*)] \delta_1$$
$$+ a_1^* (V - V^*) + a_2^* F_2^* \delta_1$$
$$= 0.$$ (5.13)

Suppose $\delta_3$ has the form:

$$\delta_3 = \alpha_1 \delta_1 + \alpha_2 \delta_2 + \alpha_3 (X_0 - X^*) \delta_1 + \alpha_4 (V - V^*) \delta_1$$
$$+ \alpha_5 (X_0 - X^*) \delta_2 + \alpha_6 (V - V^*) \delta_2 + \alpha_7 \delta_1 \delta_2$$ (5.14)
$$+ \alpha_8 \delta_1^2 + \alpha_9 \delta_2^2 + h.o.t.$$

Substituting Eq. (5.14) into Eq. (5.13) and comparing the coefficients of the same terms, yields

$$\alpha_1 = -1, \alpha_2 = 1, \alpha_3 = \alpha_4 = \alpha_8 = 0, \alpha_5 = \frac{a_2^* X_0 - a_1^* X_1}{a_1^*}, \alpha_6 = \frac{a_2^* V - a_1^* V_1}{a_1^*}, \alpha_7 = \frac{1}{a_1^*} (a_2^* F_2^* - a_1^* F_1^*),$$
$$\alpha_9 = \frac{1}{a_1^*} (a_2^* F_1^* - a_2^* F_2^*).$$

Substituting Eqs. (5.8), (5.11), (5.14), (5.6) and (5.9) into (5.12), we obtain

$$X_3 = X_0 + 2 \left[ (F_{2,1}^* - F_{1,1}^*) + \frac{F_1^* (a_1^* X_1 - a_2^* X_1)}{a_1^*} \right] (X_0 - X^*)$$
$$\sqrt{-2H(X_0, V) \frac{a_1^*}{a_1^*}}$$
$$+ 2 \left[ (F_{2,2}^* - F_{1,1}^*) + \frac{F_1^* (a_1^* V_1 - a_2^* V_1)}{a_1^*} \right] (V - V^*)$$
$$\sqrt{-2H(X_0, V) \frac{a_1^*}{a_1^*}} + h.o.t.$$ (5.15)

So, the expression of $PDM(X_0, V)$ is

$$X_0 + 2 \left[ (F_{2,2}^* - F_{1,1}^*) + \frac{F_1^* (a_1^* X_1 - a_2^* X_1)}{a_1^*} \right] (X_0 - X^*) \sqrt{-2H(X_0, V) \frac{a_1^*}{a_1^*}}$$
$$+ 2 \left[ (F_{2,2}^* - F_{1,1}^*) + \frac{F_1^* (a_1^* V_1 - a_2^* V_1)}{a_1^*} \right] (V - V^*) \sqrt{-2H(X_0, V) \frac{a_1^*}{a_1^*}} + h.o.t.$$ (5.16)

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