Anticipated backward stochastic Volterra integral equations *

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Abstract

In this paper, we introduce a new type of integral equations called anticipated backward stochastic Volterra integral equations (anticipated BSVIEs). The generators of these BSVIEs involve the future values of the solution \((Y, Z)\). We obtain the existence and uniqueness theorem and a multi-dimensional comparison theorem for the solutions of these equations. In the comparison theorem, the generator functions are allowed to contain the anticipated term of \(Z\), and do not need any monotonicity assumptions on the anticipated term of \(Y\).

keywords: Anticipated backward stochastic Volterra integral equation, Backward stochastic Volterra integral equation, Comparison theorem

1 Introduction

Stochastic Volterra integral equations (SVIEs, for short) were introduced by Berger and Mizel [1], and developed to the anticipating SVIEs in [2], [3]. Backward stochastic Volterra integral equations (BSVIEs, for short) were introduced in [9], [12], [13]. In more details, let \((\Omega, \mathcal{F}, P, \mathcal{F}_t, t \geq 0)\) be a complete stochastic basis such that \(\mathcal{F}_0\) contains all \(P\)-null elements of \(\mathcal{F}\) and suppose that the filtration is generated by a \(d\)-dimensional standard Brownian motion \(W = W(t)_{t \geq 0}\). Let \((Y(\cdot), Z(\cdot, \cdot))\) be the unique adapted \(M\)-solution to the following backward stochastic integral equation:

\[
Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T],
\]

where \(g : \Omega \times \Delta \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m\) and \(\psi : \Omega \times [0, T] \rightarrow \mathbb{R}^m\) are given maps with \(\Delta = \{(t, s) \in [0, T]^2 \mid t \leq s\}\). Such an equation is of BSVIEs introduced by Yong [12] and [13]. And there are many applications of BSVIEs, such as in risk management [14], capital allocations [15], mathematical finance and stochastic optimal controls [16]. It will become more and more clear for BSVIEs to increasingly bring out incontestable significance in the theory and applications.

On the other hand, an important special case of BSVIE (1.1) is as follows:

\[
Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s), \quad t \in [0, T],
\]

where \(f : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m\) and \(\xi\) is an \(\mathcal{F}_T\)-measurable \(L^p\)-integral random variable \((p > 1)\). This is the integral form of a so-called backward stochastic differential equation (BSDE,

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for short); see [1], [5], [8], etc., for systematic discussions. A simple glance tells us that BSDE (1.1) is a natural generalization of BSDE (1.2).

Recently, Peng and Yang [6] introduced the following type of anticipated BSDE:

\[
\begin{align*}
    -dY_t &= f(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)})dt - Z_t dW_t, \quad 0 \leq t \leq T; \\
    Y_t &= \xi_t, \quad Z_t = \eta_t, \quad T \leq t \leq T + K, \\
\end{align*}
\]

where \( \xi, \eta \) are given adapted stochastic processes, and \( \delta(\cdot), \zeta(\cdot) \) are given nonnegative deterministic functions. There have been a lot of developments and applications of the anticipated BSDEs, see [4], [8], [16], etc. These tempt us to introduce the following new type of BSDEs:

\[
\begin{align*}
    Y(t) &= \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Y(t, s + \delta(s)), Z(s, t + \zeta(s)), Z(s + \zeta(s), t))ds \\
    &\quad - \int_t^T Z(t, s)dW(s), \quad t \in [0, T]; \\
    Y(t) &= \psi(t), \quad t \in [T, T + K]; \\
    Z(t, s) &= \eta(t, s), \quad (t, s) \in [0, T + K]^2 \setminus [0, T]^2. \\
\end{align*}
\]

See Section 3 for detailed discussions. We call the equation (1.4) an anticipated backward stochastic Volterra integral equation (ABSVIE, for short). One can note that, comparing with BSDE (1.1), the distinct development of ABSVIE (1.4) is that the generator of ABSVIE (1.4) involves not only the present values of solutions but also the future ones of solutions.

In this paper, we prove the existence and uniqueness of the solution to ABSVIE (1.4) under proper assumptions. The method used to prove the existence and uniqueness theorem (Theorem 4.1 below) is much more convenient than the four steps method in Yong [13]. Since the comparison theorem is a fundamental tool, which plays an important role in the theory and applications of BSDEs, we also prove a comparison theorem for multi-dimensional ABSVIEs, which covers one of the main results in Wang and Yong [11]. It’s worthy pointing out that, in the comparison theorem (Theorem 4.1 below), the generator functions are allowed to contain the anticipated term of \( Z \), and do not need any monotonicity assumptions on the anticipated term of \( Y \), that generalizes the comparison theorem in Peng and Yang [6]. Similar to BSDE (1.1) and the anticipated BSDE (1.3), ABSVIE (1.4) can also be applied in mathematical finance, risk management, especially in the field of stochastic optimal controls. About this topic, we will give some further studies in the coming future researches.

This paper is organized as follows. In Section 2, we introduce some preliminaries. Section 3 is devoted to the proof of the existence and uniqueness of the solutions to ABSVIEs. In Section 4, a comparison theorem for multi-dimensional ABSVIEs is given.

## 2 Preliminaries

Let \((\Omega, \mathcal{F}, P, \mathcal{F}_t, t \geq 0)\) be a complete stochastic basis such that \( \mathcal{F}_0 \) contains all \( P \)-null elements of \( \mathcal{F} \) and suppose that the filtration is generated by a \( d \)-dimensional standard Brownian motion \( W = W(t)_{t \geq 0} \). Denote the norm in \( \mathbb{R}^m \) by \(| \cdot |\) and \( \mathbb{R}^{m \times m}_d = \{ A = (a_{ij}) \in \mathbb{R}^{m \times m} | a_{ij} = 0, i \neq j \} \). Given \( T > 0 \), and let \( K \geq 0 \) be a constant. We will use the following notations:

- \( \Delta = \{(t, s) \in [0, T]^2 | 0 \leq t < s \leq T \} \), \( \bar{\Delta} = \{(t, s) \in [0, T + K]^2 | 0 \leq t < s \leq T + K \} \);
- \( L^2(\mathcal{F}_t, \mathbb{R}^m) = \{ \mathbb{R}^m \text{-valued } \mathcal{F}_t \text{-measurable random variables such that } E[|\xi|^2] < \infty \} \);
- \( L^2(0, T; \mathbb{R}^m) = \{ \mathbb{R}^m \text{-valued and } \mathcal{F}_t \text{-adapted stochastic processes such that } E\int_0^T |\psi(t)|^2 dt < \infty \} \);
- \( L^2(0, T; \mathbb{R}^{m \times m}_d) = \{ \mathbb{R}^{m \times m}_d \text{-valued and } \mathcal{F}_t \text{-adapted processes such that } E\int_0^T |\psi(t)|^2 dsdt < \infty \} \);
- \( L^2(0, T + K; \mathbb{R}^{m \times m}_d) = \{ \mathbb{R}^{m \times d}_d \text{-valued and } \mathcal{F}_t \text{-adapted processes such that } E\int_0^{T+K} \int_0^{T+K} |\eta(t, s)|^2 dsdt < \infty \} \).
For any $\beta \geq 0$, let $H^{2,\beta}_\Delta[0,T+K]$ be the space of all pairs $(Y(\cdot), Z(\cdot, \cdot))$ such that $Y: [0,T+K] \to \mathbb{R}^m$ is $\mathcal{F}$-adapted, $Z: \Delta \times \Omega \to \mathbb{R}^{m \times d}$, with $s \mapsto Z(t,s)$ being $\mathcal{F}$-adapted on $[t,T+K]$, and

$$
\|(Y(\cdot), Z(\cdot, \cdot))\|_{H^{2,\beta}_\Delta[0,T+K]} \equiv \left[ E \int_0^{T+K} \left( e^{\beta t} |Y(t)|^2 + \int_t^{T+K} e^{\beta s} |Z(t,s)|^2 ds \right) dt \right]^{\frac{1}{2}} < \infty.
$$

Clearly, $H^{2,\beta}_\Delta[0,T+K]$ is a Hilbert space. It is easy to see that for any $\beta \geq 0$, the norm $\| \cdot \|_{H^{2,\beta}_\Delta[0,T+K]}$ is equivalent to $\| \cdot \|_{H^{2,\beta}_\Delta[0,T+K]}$. Thus, actually,

$$
\| \cdot \|_{H^{2,\beta}_\Delta[0,T+K]} \equiv \| \cdot \|_{H^{2,\beta}_\Delta[0,T+K]}, \; \forall \beta > 0.
$$

The following propositions can be found in [10, 11, 13]. Our Proposition 2.1 is a special case of Lemma 3.1 in [10], which provides a helpful estimate in BSVEs. Proposition 2.2 is Theorem 3.7 of Yong [13], a basic result of BSVEs: an existence and uniqueness theorem. Proposition 2.3 is Theorem 3.4 of Wang and Yong [11], which is a comparison theorem of BSVEs.

**Proposition 2.1.** We consider the following simple BSVE

$$
Y(t) = \psi(t) + \int_t^T g(t,s) ds - \int_t^T Z(t,s) dW(s), \; t \in [0,T], \tag{2.1}
$$

where $\psi(\cdot) \in L^2_T(0,T;\mathbb{R}^m)$, $g: \Delta \times \Omega \to \mathbb{R}^m$ be $\mathcal{F}_T$-measurable such that $s \mapsto g(t,s)$ is $\mathcal{F}$-progressively measurable for all $t \in [0,T]$, and $E \int_0^T \int_t^T e^{\beta s} |g(t,s)|^2 ds dt < \infty$. Then (2.1) admits a unique adapted solution $(Y(\cdot), Z(\cdot, \cdot)) \in H^{2,\beta}_\Delta[0,T]$, and we have the following estimate:

$$
E \int_0^T \left( e^{\beta t} |Y(t)|^2 + \int_t^T e^{\beta s} |Z(t,s)|^2 ds \right) dt \leq C e^{\beta T} E \int_0^T |\psi(t)|^2 dt + C \beta E \int_0^T \int_t^T e^{\beta s} |g(t,s)|^2 ds dt. \tag{2.2}
$$

Hereafter $C$ is a positive constant which may be different from line to line.

**(BV1)** Assume $g(\omega, t, s, y, z, \vartheta): \Omega \times \Delta \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$, such that $s \mapsto g(t,s,y,z,\vartheta)$ is $\mathcal{F}$-progressively measurable and $(t,s) \mapsto g(t,s,0,0,0)$ is uniformly bounded. Moreover, it holds

$$
|g(t,s,y,z,\vartheta) - g(t,s,y',z',\vartheta')| \leq L(t,s) (|y-y'| + |z-z'| + |\vartheta-\vartheta'|),
$$

$\forall (t,s) \in \Delta$, $y,y' \in \mathbb{R}^m$, $z,z',\vartheta,\vartheta' \in \mathbb{R}^{m \times d}$, a.s.,

where $L : \Delta \to \mathbb{R}$ is a deterministic function such that the following holds:

$$
\sup_{t \in [0,T]} \int_t^T L(t,s)^{2+\varepsilon} ds < \infty, \; \varepsilon > 0.
$$

**Proposition 2.2.** Let (BV1) holds. Then for any $\psi(\cdot) \in L^2_T(0,T;\mathbb{R}^m)$, the following BSVE admits a unique adapted $M$-solution $(Y(\cdot), Z(\cdot, \cdot)) \in H^2[0,T]$,

$$
Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(t,s),Z(s,t)) ds - \int_t^T Z(t,s) dW(s), \; t \in [0,T].
$$

For $i = 0,1$, we assume that $g^i = g^i(\omega, t, s, y, z) : \Omega \times \Delta \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ satisfy the following conditions:

**(BV2)** Let $s \mapsto g^i(t,s,y,z)$ is $\mathcal{F}$-progressively measurable, $(y,z) \mapsto g^i(t,s,y,z)$ is uniformly Lipschitz, $(t,s) \mapsto g^i(t,s,0,0)$ is uniformly bounded.
Proposition 2.3. Assume \( g^i(\omega, t, s, y, z) : \Omega \times \Delta \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m \) satisfy (BV2), \( i = 0, 1 \). Let \( (Y^i(\cdot), Z^i(\cdot, \cdot)) \in \mathcal{H}^2_\Delta [0, T] \) be respectively the solutions of BSVIEs as follows:

\[
Y^i(t) = \psi^i(t) + \int_t^T g^i(t, s, Y^i(s), Z^i(t, s))ds - \int_t^T Z^i(t, s)dW(s), \quad t \in [0, T].
\]

Suppose \( g(\omega, t, s, y, z) : \Omega \times \Delta \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m \) is measurable, \( s \mapsto g(t, s, y, z) \) is \( \mathcal{F} \)-progressively measurable, \( (y, z) \mapsto g(t, s, y, z) \) is uniformly Lipschitz, \( y \mapsto g(t, s, y, z) \) is nondecreasing, such that

\[
g^0(t, s, y, z) \leq g^1(t, s, y, z) \leq g^1(t, s, y, z), \quad \forall (t, y, z) \in [0, s] \times \mathbb{R}^m \times \mathbb{R}^{m \times d}, \text{ a.s., a.e. } s \in [0, T].
\]

Moreover, \( \mathcal{F}_s(t, s, y, z) \) exists and

\[
\mathcal{F}_{s1}(t, s, y, z), \ldots, \mathcal{F}_{sz}(t, s, y, z) \in \mathbb{R}^{m \times m}, \quad \forall (t, y, z) \in [0, s] \times \mathbb{R}^m \times \mathbb{R}^{m \times d}, \text{ a.s., a.e. } s \in [0, T].
\]

Then for any \( \psi^3(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \) satisfying

\[
\psi^0(t) \leq \psi^1(t), \quad \text{a.s., } t \in [0, T],
\]

the corresponding unique adapted solution \( (Y^i(\cdot), Z^i(\cdot, \cdot)) \in \mathcal{H}^2_\Delta [0, T] \) satisfies

\[
Y^0(t) \leq Y^1(t), \quad \text{a.s., } t \in [0, T].
\]

3 Existence and uniqueness theorem

We now consider a new form of BSVIEs as follows:

\[
\begin{aligned}
Y(t) &= \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t), Y(s + \delta(s)), Z(t, s + \zeta(s)), Z(s + \zeta(s), t))ds \\
&\quad - \int_t^T Z(t, s)dW(s), \quad t \in [0, T]; \\
Y(t) &= \psi(t), \quad t \in [T, T + K]; \\
Z(t, s) &= \eta(t, s), \quad (t, s) \in [0, T + K]^2 \setminus [0, T]^2.
\end{aligned}
\]

(3.1)

where \( \delta(\cdot) \) and \( \zeta(\cdot) \) are two \( \mathbb{R}^+ \)-valued continuous functions defined on \([0, T]\) such that:

(i) There exists a constant \( K \geq 0 \) such that, for all \( s \in [0, T] \),

\[
s + \delta(s) \leq T + K; \quad s + \zeta(s) \leq T + K.
\]

(ii) There exists a constant \( M \geq 0 \) such that, for all non-negative and integrable \( g_1(\cdot), g_2(\cdot, \cdot), \)

\[
\int_t^T g_1(s + \delta(s))ds \leq M \int_t^{T+K} g_1(s)ds, \quad t \in [0, T];
\]

\[
\int_t^T g_2(t, s + \delta(s))ds \leq M \int_t^{T+K} g_2(t, s)ds, \quad (t, s) \in \Delta.
\]

We call equation (3.1) the anticipated BSVIE.

Definition 3.1. A pair of \( (Y(\cdot), Z(\cdot, \cdot)) \) is called an adapted \( M \)-solution of ABSVIE (3.1) if it holds in the usual Itô’s sense for almost all \( t \in [0, T + K] \). And in addition, the following holds:

\[
Y(t) = EY(t) + \int_0^t Z(t, s)dW(s), \quad t \in [0, T].
\]
Definition 3.2. A pair of \((Y(\cdot), Z(\cdot, \cdot))\) is called an adapted solution of the following simple \(ABSVIE\) (3.2), if (3.2) holds in the usual Itô’s sense

\[
\begin{cases}
Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Y(s + \delta(s)), Z(t, s + \zeta(s)))ds \\
- \int_t^T Z(t, s) dW(s), \ t \in [0, T]; \\
Y(t) = \psi(t), \ t \in [T, T + K]; \\
Z(t, s) = \eta(t, s), \ (t, s) \in \Delta \setminus \Delta.
\end{cases}
\] (3.2)

Assume that for all \((t, s) \in \Delta, g(t, s, \omega, y, z, x, \xi, \eta, \zeta) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times d} \times L^2(F_r; \mathbb{R}) \times L^2(F_r; \mathbb{R}^{m \times d}) \rightarrow L^2(F_s; \mathbb{R}^m), \) where \(r, r' \in [s, T + K], \) and \(g\) satisfies the following conditions:

(H1) There exist a constant \(L > 0,\) such that for all \((t, s) \in \Delta, g(t, s, y, z, x, \xi, \eta, \zeta) \in \mathbb{R}^{m \times d},\) \(\xi(\cdot), \xi'(\cdot) \in L^2_E(s, T + K; \mathbb{R}^m),\) \(\eta(t, \cdot), \eta'(t, \cdot), \zeta(t, \cdot), \zeta'(t, \cdot) \in L^2_E(s, T + K; \mathbb{R}^{m \times d}),\) \(r, r' \in [s, T + K],\) we have

\[
|g(t, s, y, z, x, \xi(r), \eta(t, r'), \zeta(r', t)) - g(t, s, y, z, x, \xi'(r), \eta(t, r'), \zeta'(r', t))| \leq L \left(|y - y'| + |z - z'| + |x - x'| + E^{\mathbb{F}}[|\xi(r) - \xi'(r)| + |\eta(t, r') - \eta'(t, r')| + |\zeta(r', t) - \zeta'(r', t)|]\right).
\]

(H2) \(E \int_0^T \int_t^T |g_0(t, s)|^2 ds dt < \infty,\) where \(g_0(t, s) = g(t, s, 0, 0, 0, 0, 0, 0).\)

To establish the well-posedness of the anticipated \(BSVIE\) (3.1), we introduce the following space. For any \(\beta \geq 0,\) we let \(\mathcal{H}^{2, \beta}[0, T + K]\) be the space of all pairs \((Y(\cdot), Z(\cdot, \cdot))\) such that \(Y : [0, T + K] \rightarrow \mathbb{R}^m\) is \(\mathcal{F}\)-adapted, \(Z : [0, T + K] \times [0, T + K] \rightarrow \mathbb{R}^{m \times d},\) with \(s \mapsto Z(t, s)\) being \(\mathcal{F}\)-adapted on \([0, T + K],\) and

\[
\|(Y(\cdot), Z(\cdot, \cdot))\|_{\mathcal{H}^{2, \beta}[0, T + K]} \equiv \left[ E \int_0^{T + K} \left( e^{\beta t} |Y(t)|^2 + \int_t^{T + K} e^{\beta s} |Z(t, s)|^2 ds \right) dt \right]^{\frac{1}{2}} < \infty.
\]

Different from \(\mathcal{H}^{2, \beta}_\Delta[0, T + K]\) defined in the previous section, we see that \(Z(\cdot, \cdot)\) is defined on \([0, T + K] \times [0, T + K].\) Similar to \(\mathcal{H}^{2, \beta}_\Delta[0, T + K],\) we have that \(\mathcal{H}^{2, \beta}[0, T + K]\) is a Hilbert space, and

\[
\| \cdot \|_{\mathcal{H}^{2, \beta}[0, T + K]} = \| \cdot \|_{\mathcal{H}^{2, \beta}_\Delta[0, T + K]} \equiv \| \cdot \|_{\mathcal{H}^{2, \beta}[0, T + K]} , \ \forall \beta > 0.
\]

Next, let \(\mathcal{M}^{2, \beta}[0, T + K]\) be the set of all \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^{2, \beta}[0, T + K]\) such that

\[
Y(t) = EY(t) + \int_0^t Z(t, s) dW(s), \ t \in [0, T].
\] (3.3)

For any \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{M}^{2, \beta}[0, T + K],\) one can show that

\[
E \int_0^T \left( e^{\beta t} |Y(t)|^2 + \int_0^T e^{\beta s} |Z(t, s)|^2 ds \right) dt \leq 2E \int_0^T \left( e^{\beta t} |Y(t)|^2 + \int_t^T e^{\beta s} |Z(t, s)|^2 ds \right) dt,
\] (3.4)

since from (3.3), one has

\[
E \int_0^T \int_0^t e^{\beta s} |Z(t, s)|^2 ds dt \leq E \int_0^T e^{\beta t} |Y(t)|^2 dt.
\]

We now state and prove a well-posedness theorem for \(ABSVIE\) (3.1).
Theorem 3.3. Suppose that \( g \) satisfies (H1) and (H2), and \( \delta, \zeta \) satisfy (i) and (ii). Then for any given terminal conditions \( \psi(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}; \mathbb{R}^m) \) and \( \eta(\cdot, \cdot) \in L^2_{\mathbb{F}}([0, T + K]; \mathbb{R}^{m \times d}) \), the anticipated BSVIE \((3.5)\) admits a unique adapted \( M \)-solution \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[0, T + K] \).

Proof. For any \((y(\cdot), z(\cdot, \cdot)) \in \mathcal{M}^2[0, T + K]\), consider the following BSVIE:

\[
\begin{align*}
Y(t) &= \psi(t) + \int_t^T \bar{g}(t, s) ds - \int_t^T Z(t, s) dW(s), \ t \in [0, T]; \\
Y(t) &= \psi(t), \ t \in [T, T + K]; \\
Z(t, s) &= \eta(t, s), \ (t, s) \in [0, T + K]^2 \setminus [0, T]^2,
\end{align*}
\]

where

\[
\bar{g}(t, s) = g(t, s, y(s), z(s), t, y(s) + \delta(s), z(s + \zeta(s), z(s + \zeta(s), t)), (t, s) \in [0, T]^2.
\]

From Proposition 2.1 we know that \((3.5)\) admits a unique adapted solution \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2_{\Delta}[0, T + K]\). We define \(Z(\cdot, \cdot)\) on \(\Delta^c\) from the following:

\[
Y(t) = EY(t) + \int_0^t Z(t, s) dW(s), \ t \in [0, T].
\]

Then \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{M}^2[0, T + K]\) is an adapted \( M \)-solution to Eq. \((3.5)\). Thus, the map \((y(\cdot), z(\cdot, \cdot)) \mapsto (Y(\cdot), Z(\cdot, \cdot))\) is well-defined. By the estimate \((2.2)\) in Proposition 2.1 one has

\[
E \int_0^T \left( e^{\beta t} |Y(t)|^2 + \int_t^T e^{\beta s} |Z(t, s)|^2 ds \right) dt \leq C e^{\beta T} E \int_0^T |\psi(t)|^2 dt + \frac{C}{\beta} E \int_0^T \int_t^T e^{\beta s} |\bar{g}(t, s)|^2 ds dt.
\]

For the second term, from (H1) we have

\[
E \int_0^T \int_t^T e^{\beta s} |\bar{g}(t, s)|^2 ds dt \leq 5L^2 E \int_0^T \int_t^T e^{\beta s} (|g_0(t, s)|^2 + |y(s)|^2 + |z(t, s)|^2 + |z(s, t)|^2
\]

\[
+ 3|y(s + \delta(s))|^2 + 3|z(t, s + \zeta(s)|^2 + 3|z(s + \zeta(s), t)|^2) ds dt
\]

\[
\leq 5L^2 E \int_0^T \int_t^T e^{\beta s} |g_0(t, s)|^2 ds dt + 10L^2(T + 1) E \int_0^T \left( e^{\beta t} |y(t)|^2 + \int_0^T e^{\beta s} |z(t, s)|^2 ds \right) dt
\]

\[
+ 30L^2 (T + 1) E \int_0^{T + K} \left( e^{\beta t} |y(t)|^2 + \int_t^{T + K} e^{\beta s} |z(t, s)|^2 ds \right) dt
\]

\[
\leq 5L^2 E \int_0^T \int_t^T e^{\beta s} |g_0(t, s)|^2 ds dt
\]

\[
+ 60L^2 (T + 1)(T + K + 1) E \int_0^{T + K} \left( e^{\beta t} |y(t)|^2 + \int_0^{T + K} e^{\beta s} |z(t, s)|^2 ds \right) dt.
\]

Hence

\[
E \int_0^T \left( e^{\beta t} |Y(t)|^2 + \int_t^T e^{\beta s} |Z(t, s)|^2 ds \right) dt \leq C e^{\beta T} E \int_0^T |\psi(t)|^2 dt + \frac{C}{\beta} E \int_0^T \int_t^T e^{\beta s} |g_0(t, s)|^2 ds dt
\]

\[
+ \frac{C}{\beta} E \int_0^{T + K} \left( e^{\beta t} |y(t)|^2 + \int_t^{T + K} e^{\beta s} |z(t, s)|^2 ds \right) dt.
\]
If \((Y_i(\cdot), Z_i(\cdot, \cdot))\) is the corresponding adapted M-solution of \((y_i(\cdot), z_i(\cdot, \cdot))\) to BSVIE (3.3), \(i = 1, 2\), note (3.4), then
\[
E \int_0^{T+K} \left( e^{\beta t}|Y_1(t) - Y_2(t)|^2 + \int_0^{T+K} e^{\beta s}|Z_1(t, s) - Z_2(t, s)|^2 ds \right) dt \leq \frac{C}{\beta} E \int_0^{T+K} \left( e^{\beta t}|y_1(t) - y_2(t)|^2 + \int_0^{T+K} e^{\beta s}|z_1(t, s) - z_2(t, s)|^2 ds \right) dt.
\]
Let \(\beta = 2C + 1\), then \((y(\cdot), z(\cdot, \cdot)) \rightarrow (Y(\cdot), Z(\cdot, \cdot))\) is a contraction on \(M^2[0, T + K]\). This completes the proof.

**Corollary 3.4.** Suppose that \(g = g(t, s, y, z, \xi, \eta)\) satisfies (H1) and (H2), and \(\delta, \zeta\) satisfy (i) and (ii). Then for any given terminal conditions \(\psi(\cdot) \in L^2_{\mathcal{F}_{T+K}}(0, T + K; \mathbb{R}^m)\) and \(\eta(\cdot, \cdot) \in L^2_{\mathcal{F}}(\Delta; \mathbb{R}^{m \times d})\), the anticipated BSVIE (3.2) admits a unique adapted solution \((Y(\cdot), Z(\cdot, \cdot))\) \(\in \mathcal{H}_{\Delta}^2[0, T + K]\).

### 4 A comparison theorem for ABSVIEs

In this section we prove a comparison theorem for ABSVIEs of the following type: For \(i = 0, 1, 2\),
\[
\begin{align*}
Y^i(t) &= \psi^i(t) + \int_t^T g^i(t, s, Y^i(s), Z^i(s, t), Y^i(s + \delta(s)), Z^i(s + \zeta(s))) ds \\
&\quad - \int_t^T Z^i(t, s)dW(s), \quad t \in [0, T]; \\
Y^i(t) &= \psi^i(t), \quad t \in [T, T + K]; \\
Z^i(t, s) &= \eta^i(t, s), \quad (t, s) \in \Delta \setminus \Delta.
\end{align*}
\]
(4.1)

For ABSVIEs of the above form, we need only the values \(Z^i(t, s)\) of \(Z^i(\cdot, \cdot)\) for \((t, s) \in \Delta\) and the notation of M-solution is not necessary. For the generator \(g^i(\cdot)\) of ABSVIE (4.1), we adopt the following assumptions.

Assume that for all \((t, s) \in \Delta, g^i(\cdot) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L^2(\mathcal{F}_r; \mathbb{R}^m) \times L^2(\mathcal{F}_r; \mathbb{R}^{m \times d}) \rightarrow L^2(\mathcal{F}_r; \mathbb{R}^m), \) and \(g^i(\cdot)\) satisfies the following conditions:

**(H3)** There exist a constant \(L > 0\), such that for all \((t, s) \in \Delta, y, y', \xi, \eta, \xi', \eta' \in \mathbb{R}^{m \times d}, \xi(\cdot), \xi'(\cdot) \in L^2_F(s, T + K; \mathbb{R}^m), \eta(\cdot, \cdot), \eta'(\cdot, \cdot) \in L^2_{\mathcal{F}_r}(s, T + K; \mathbb{R}^{m \times d}), \) we have
\[
|g^i(t, s, y, z, \xi(\cdot), \eta(\cdot, \cdot)) - g^i(t, s, y', z', \xi'(\cdot), \eta'\cdot, \cdot))| \leq L \left[ |y - y'| + |z - z'| + E^{\mathcal{F}_r}[|\xi(\cdot) - \xi'(\cdot)| + |\eta(\cdot, \cdot) - \eta'(\cdot, \cdot)|]\right].
\]

**(H4)** \(E \int_0^T \int_t^T ||g^i(t, s, 0, 0, 0, 0)||^2 ds dt < \infty.\)

It is known that under (i), (ii) and (H3), (H4) hold. For any \(\psi^i(\cdot) \in L^2_{\mathcal{F}_T}(0, T + K; \mathbb{R}^m), \eta^i(\cdot, \cdot) \in L^2_{\mathcal{F}_T}(\Delta; \mathbb{R}^{m \times d}), \) ABSVIE (4.1) admits a unique adapted solution \((Y^i(\cdot), Z^i(\cdot, \cdot)) \in \mathcal{H}_{\Delta}^2[0, T + K].\)

**Theorem 4.1.** Let (i), (ii) and (H3), (H4) hold. Suppose for all \((t, s) \in \Delta, \eta : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L^2(\mathcal{F}_r; \mathbb{R}^m) \times L^2(\mathcal{F}_r; \mathbb{R}^{m \times d}) \rightarrow L^2(\mathcal{F}_r; \mathbb{R}^m), \) where \(r, r' \in [s, T + K]\), satisfies (H3), (H4), and \(\forall (t, s, z, \xi) \in \Delta \times \mathbb{R}^{m \times d} \times L^2(\mathcal{F}_r; \mathbb{R}^m) \times L^2(\mathcal{F}_r; \mathbb{R}^{m \times d}), \) \(\eta(t, s, z, \xi, \eta)\) is nondecresing such that
\[
g^i(t, s, y, z, \xi, \eta) \leq \mathcal{G}(t, s, y, z, \xi, \eta) \leq g^i(t, s, y, z, \xi, \eta),
\]
\(\forall (t, y, z, \xi, \eta) \in [0, s] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L^2(\mathcal{F}_r; \mathbb{R}^m) \times L^2(\mathcal{F}_r; \mathbb{R}^{m \times d}), \) a.s., a.e. \(s \in [0, T],\)
Moreover, \( \overline{g}(t, s, y, z, \xi, \eta) \) exists and
\[
\overline{g}_{1i}(t, s, y, z, \xi, \eta), \ldots, \overline{g}_{ij}(t, s, y, z, \xi, \eta) \in \mathbb{R}^{m \times m},
\]
\[\forall (t, y, z, \xi, \eta) \in [0, s] \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d} \times L^{2}(\mathcal{F}_{r}; \mathbb{R}^{m}) \times L^{2}(\mathcal{F}_{r}; \mathbb{R}^{m \times d}), \ a.s., \ a.e. \ s \in [0, T].\]

Then, for any \( \psi^{i}(\cdot) \in L_{\mathcal{F}_{T}}^{2}(0, T + K; \mathbb{R}^{m}) \) satisfying
\[\psi^{0}(t) \leq \psi^{1}(t), \ a.s., \ t \in [0, T + K], \quad (4.2)\]
the corresponding unique adapted solution \((Y^{i}(\cdot), Z^{i}(\cdot, \cdot)) \in \mathcal{H}^{\Delta}_{\overline{\Delta}}[0, T + K] \) of ABSVIE \((4.1)\) satisfy
\[Y^{0}(t) \leq Y^{1}(t), \ a.s., \ t \in [0, T + K].\]

**Proof.** Let \( \overline{\psi}(\cdot) \in L_{\mathcal{F}_{T}}^{2}(0, T + K; \mathbb{R}^{m}), \overline{\pi}(\cdot, \cdot) \in L_{\mathcal{F}_{T}}^{2}(\overline{\Delta}; \mathbb{R}^{m \times d}) \) such that
\[\psi^{0}(t) \leq \overline{\psi}(t) \leq \psi^{1}(t), \ a.s. \ t \in [0, T + K].\]

Let \((\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot)) \in \mathcal{H}^{\Delta}_{\overline{\Delta}}[0, T + K] \) be the unique adapted solution to the following ABSVIE:
\[
\begin{cases}
\overline{Y}(t) = \overline{\psi}(t) + \int_{t}^{T} \overline{g}(t, s, \overline{Y}(s), \overline{Z}(t, s), \overline{Y}(s + \delta(s)), \overline{Z}(t, s + \zeta(s)))ds \\
- \int_{t}^{T} \overline{Z}(t, s)dW(s), \ t \in [0, T]; \\
\overline{Y}(t) = \overline{\psi}(t), \ t \in [T, T + K]; \\
\overline{Z}(t, s) = \overline{\pi}(t, s), \ (t, s) \in \overline{\Delta} \setminus \Delta.
\end{cases}
(4.3)
\]
Set \( \overline{Y}_{0}(\cdot) = Y^{1}(\cdot) \) and consider the following ABSVIE:
\[
\begin{cases}
\overline{Y}_{1}(t) = \overline{\psi}(t) + \int_{t}^{T} \overline{g}(t, s, \overline{Y}_{1}(s), \overline{Y}_{0}(s + \delta(s)), \overline{Z}(t, s + \zeta(s)))ds \\
- \int_{t}^{T} \overline{Z}_{1}(t, s)dW(s), \ t \in [0, T]; \\
\overline{Y}_{1}(t) = \psi(t), \ t \in [T, T + K]; \\
\overline{Z}_{1}(t, s) = \pi(t, s), \ (t, s) \in \overline{\Delta} \setminus \Delta.
\end{cases}
\]
Let \((\overline{Y}_{1}(\cdot), \overline{Z}_{1}(\cdot, \cdot)) \in \mathcal{H}^{\Delta}_{\overline{\Delta}}[0, T + K] \) be the unique adapted solution to the above equation. Since
\[
\begin{cases}
\overline{g}(t, s, y, z, \overline{Y}_{0}(s + \delta(s)), \overline{Z}(t, s + \zeta(s))) \leq g^{1}(t, s, y, z, \overline{Y}_{0}(s + \delta(s)), \overline{Z}(t, s + \zeta(s))), \\
(t, y, z) \in [0, s] \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d}, \ a.s., \ a.e. \ s \in [0, T]; \\
\overline{g}_{1i}(t, s, y, z, \overline{Y}_{0}(s + \delta(s)), \overline{Z}(t, s + \zeta(s))) \in \mathbb{R}^{m \times m}, \\
(t, y, z) \in [0, s] \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d}, \ a.s., \ a.e. \ s \in [0, T], \ 1 \leq j \leq d; \\
\overline{\psi}(t) \leq \psi^{1}(t), \ a.s. \ t \in [0, T + K].
\end{cases}
\]
By Proposition 2.3, we obtain that
\[\overline{Y}_{1}(t) \leq \overline{Y}_{0}(t), \ a.s. \ t \in [0, T + K].\]

Next, we consider the following BSVIE:
\[
\begin{cases}
\overline{Y}_{2}(t) = \overline{\psi}(t) + \int_{t}^{T} \overline{g}(t, s, \overline{Y}_{2}(s), \overline{Z}_{2}(t, s), \overline{Y}_{1}(s + \delta(s)), \overline{Z}(t, s + \zeta(s)))ds \\
- \int_{t}^{T} \overline{Z}_{2}(t, s)dW(s), \ t \in [0, T]; \\
\overline{Y}_{2}(t) = \psi(t), \ t \in [T, T + K]; \\
\overline{Z}_{2}(t, s) = \pi(t, s), \ (t, s) \in \overline{\Delta} \setminus \Delta.
\end{cases}
\]
Let \((\tilde{Y}_2(\cdot), \tilde{Z}_2(\cdot, \cdot)) \in H^2_\Delta [0, T + K]\) be the adapted solution to the above equation. Now, since \(\xi \mapsto \mathcal{g}(t, s, y, z, \xi, \eta)\) is nondecreasing, we have
\[
\mathcal{g}(t, s, y, z, \tilde{Y}_1(s + \delta(s)), \tilde{Z}(t, s + \zeta(s))) \leq \mathcal{g}(t, s, y, z, \tilde{Y}_0(s + \delta(s)), \tilde{Z}(t, s + \zeta(s))),
\]
\[(t, y, z) \in [0, s] \times \mathbb{R}^m \times \mathbb{R}^{m \times d}, \ a.s., \ a.e. \ s \in [0, T].\]

Hence, similar to the above, we obtain
\[
\tilde{Y}_2(t) \leq \tilde{Y}_1(t), \quad a.s. \ t \in [0, T + K].
\]

By induction, we can construct a sequence \(\{\tilde{Y}_k(\cdot), \tilde{Z}_k(\cdot, \cdot)\}_{k \geq 1} \in H^2_\Delta [0, T + K]\) such that
\[
\begin{cases}
\tilde{Y}_k(t) = \tilde{\mathcal{g}}(t) + \int_t^T \tilde{\mathcal{g}}(t, s, \tilde{Y}_k(s), \tilde{Z}_k(t, s), \tilde{Y}_{k-1}(s + \delta(s)), \tilde{Z}(t, s + \zeta(s)))ds \\
- \int_t^T \tilde{Z}_k(t, s)dW(s), \quad t \in [0, T]; \\
\tilde{Y}_k(t) = \tilde{\mathcal{g}}(t), \quad t \in [T, T + K]; \\
\tilde{Z}_k(t, s) = \mathcal{g}(t, s), \quad (t, s) \in \Delta \setminus \Delta.
\end{cases}
\]

Similarly, we deduce
\[
Y^1(t) = \tilde{Y}_0(t) \geq \tilde{Y}_1(t) \geq \tilde{Y}_2(t) \cdots, \quad a.s. \ t \in [0, T + K].
\]

Next we will show that the sequence \(\{\tilde{Y}_k(\cdot), \tilde{Z}_k(\cdot, \cdot)\}_{k \geq 1}\) is Cauchy in the Banach space \(H^2_\Delta [0, T + K]\). To show this, we introduce an equivalent norm of the space \(H^2_\Delta [0, T + K]\) as
\[
\| (Y(\cdot), Z(\cdot, \cdot)) \|_\beta := E \int_0^{T+K} \left( e^{\beta t} |Y(t)|^2 + \int_t^{T+K} e^{\beta s} |Z(t, s)|^2 ds \right) dt.
\]

By utilizing an estimate in Proposition 2.1, we have
\[
E \int_0^T \left( e^{\beta t} |\tilde{Y}_k(t) - \tilde{Y}_{k-1}(t)|^2 + \int_t^T e^{\beta s} |\tilde{Z}_k(t, s) - \tilde{Z}_{k-1}(t, s)|^2 ds \right) dt \\
\leq \frac{C}{\beta} E \int_0^T \int_t^T e^{\beta s} \mathcal{g}(t, s, \tilde{Y}_k(s), \tilde{Z}_k(t, s), \tilde{Y}_{k-1}(s + \delta(s)), \tilde{Z}(t, s + \zeta(s))) \\
- \mathcal{g}(t, s, \tilde{Y}_{k-1}(s), \tilde{Z}_{k-1}(t, s), \tilde{Y}_{k-2}(s + \delta(s)), \tilde{Z}(t, s + \zeta(s))) |^2 ds dt \\
\leq \frac{C}{\beta} E \int_0^T \int_t^T e^{\beta s} (|\tilde{Y}_k(s) - \tilde{Y}_{k-1}(s)|^2 \\
+ |\tilde{Z}_k(t, s) - \tilde{Z}_{k-1}(t, s)|^2 + |\tilde{Y}_{k-1}(s + \delta(s)) - \tilde{Y}_{k-2}(s + \delta(s))|^2) ds dt \\
\leq \frac{C}{\beta} E \int_0^T \left( e^{\beta t} |\tilde{Y}_k(t) - \tilde{Y}_{k-1}(t)|^2 + \int_t^T e^{\beta s} |\tilde{Z}_k(t, s) - \tilde{Z}_{k-1}(t, s)|^2 ds \right) dt \\
+ \frac{C}{\beta} E \int_0^{T+K} e^{\beta t} |\tilde{Y}_{k-1}(t) - \tilde{Y}_{k-2}(t)|^2 dt.
\]

Hence
\[
(1 - \frac{C}{\beta}) E \int_0^T \left( e^{\beta t} |\tilde{Y}_k(t) - \tilde{Y}_{k-1}(t)|^2 + \int_t^T e^{\beta s} |\tilde{Z}_k(t, s) - \tilde{Z}_{k-1}(t, s)|^2 ds \right) dt \\
\leq \frac{C}{\beta} E \int_0^{T+K} e^{\beta t} |\tilde{Y}_{k-1}(t) - \tilde{Y}_{k-2}(t)|^2 dt.
\]
Note that the constant $C > 0$ in the above can be chosen independent of $\beta \geq 0$. Thus by choosing $\beta = 3C$, we obtain

$$E \int_0^{T+K} \left( e^{\beta t} |\tilde{Y}_k(t) - \tilde{Y}_{k-1}(t)|^2 + \int_t^{T+K} e^{\beta s} |\tilde{Z}_k(t,s) - \tilde{Z}_{k-1}(t,s)|^2 ds \right) dt$$

$$\leq \frac{1}{2} E \int_0^{T+K} e^{\beta t} |\tilde{Y}_{k-1}(t) - \tilde{Y}_{k-2}(t)|^2 dt$$

$$\leq \frac{1}{2} E \int_0^{T+K} \left( e^{\beta t} |\tilde{Y}_{k-1}(t) - \tilde{Y}_{k-2}(t)|^2 + \int_t^{T+K} e^{\beta s} |\tilde{Z}_{k-1}(t,s) - \tilde{Z}_{k-2}(t,s)|^2 ds \right) dt$$

$$\leq \left( \frac{1}{2} \right)^{k-2} E \int_0^{T+K} \left( e^{\beta t} |\tilde{Y}_2(t) - \tilde{Y}_1(t)|^2 + \int_t^{T+K} e^{\beta s} |\tilde{Z}_2(t,s) - \tilde{Z}_1(t,s)|^2 ds \right) dt.$$

It follows that $\{(\tilde{Y}_k(\cdot), \tilde{Z}_k(\cdot),\cdot)\}_{k \geq 1}$ is Cauchy in the Banach space $H^2_0[0,T+K]$. Denote their limits by $\tilde{Y}(\cdot)$ and $\tilde{Z}(\cdot)$, respectively. Then $(\tilde{Y}(\cdot), \tilde{Z}(\cdot),\cdot) \in H^2_0[0,T+K]$ and

$$\lim_{k \to \infty} \left( E \int_0^{T+K} e^{\beta t} |\tilde{Y}_k(t) - \tilde{Y}(t)|^2 dt + E \int_0^{T+K} \int_t^{T+K} e^{\beta s} |\tilde{Z}_k(t,s) - \tilde{Z}(t,s)|^2 ds dt \right) = 0,$$

also we have

$$\begin{cases} \tilde{Y}(t) = \tilde{\omega}(t) + \int_t^T \tilde{\eta}(t,s,\tilde{Y}(s),\tilde{Z}(s)) ds + \int_t^T \tilde{Z}(s) d\tilde{W}(s), & t \in [0,T]; \\
\tilde{Y}(t) = \tilde{\omega}(t), & t \in [T,T+K]; \\
\tilde{Z}(t,s) = \tilde{\eta}(t,s), & (t,s) \in \tilde{\Delta} \setminus \tilde{\Delta}. \end{cases}$$

Connecting the above equation with the equation (4.3), let $\beta = 3C + 2$ and by using the estimate (2.2) again, similar to the equation (4.4) and (4.1) we obtain

$$E \int_0^{T+K} \left( e^{\beta t} |\tilde{Y}(t) - Y(t)|^2 + \int_t^{T+K} e^{\beta s} |\tilde{Z}(t,s) - Z(t,s)|^2 ds \right) dt \leq \frac{1}{2} E \int_0^{T+K} e^{\beta t} |\tilde{Y}(t) - Y(t)|^2 dt.$$

Hence we have

$$\tilde{\eta}(t) = \tilde{Y}(t) \leq \tilde{Y}_0(t) = Y^0(t), \quad \text{a.s. } t \in [0,T+K].$$

Similarly, we can prove that

$$Y^0(t) \leq \tilde{\eta}(t), \quad \text{a.s. } t \in [0,T+K].$$

Therefore, our conclusion follows.

**Remark 4.2.** one may note that, Theorem 4.11 covers one of the main results (Theorem 3.4) in Wang and Yong [11].

**Remark 4.3.** In Theorem 4.11 the generator functions $g^0$ and $g^1$ are allowed to contain the anticipated term of $Z$, and do not need any monotonicity assumptions on the anticipated term of $Y$, that generalizes the comparison theorem (Theorem 5.1) in Peng and Yang [10].

For a special case when $g^0$ and $g^1$ are independent of the anticipated term of $Z$, we easily get the following comparison result.
Theorem 4.4. Suppose that \( g^i = g^i(t, s, y, z, \xi), \ i = 0, 1, \) satisfy (H3) and (H4), and \( \delta, \zeta \) satisfy (i) and (ii). Let \((Y^i(\cdot), Z^i(\cdot, \cdot)) \in L^2_\mathcal{F}(0, T + K; \mathbb{R}^m) \times L^2_\mathcal{F}(\Delta; \mathbb{R}^{m \times d})\) be the unique adapted solution to the following ABSVIEs respectively:

\[
\begin{align*}
Y^i(t) &= \psi^i(t) + \int_t^T g^i(t, s, Y^i(s), Z^i(t, s), Y^i(s + \delta(s))) ds - \int_t^T Z^i(t, s) dW(s), \ t \in [0, T]; \\
Y^i(t) &= \psi^i(t), \ t \in [T, T + K].
\end{align*}
\] (4.5)

Let \( \gamma = \gamma(t, s, y, z, \xi) \) satisfies the same conditions as in Theorem 4.1. Then for any \( \psi^i(\cdot) \in L^2_\mathcal{F}(0, T + K; \mathbb{R}^m) \) satisfying

\[\psi^0(t) \leq \psi^1(t), \ \text{a.s., } t \in [0, T + K],\]

the corresponding unique adapted solution \((Y^i(\cdot), Z^i(\cdot, \cdot)) \in L^2_\mathcal{F}(0, T + K; \mathbb{R}^m) \times L^2_\mathcal{F}(\Delta; \mathbb{R}^{m \times d})\) of ABSVIEs (4.3) satisfy

\[Y^0(t) \leq Y^1(t), \ \text{a.s., } t \in [0, T + K].\]

Example 1. Let \( g^0(t, s, \xi(r), \eta(t, t')) = -E^\mathcal{F}_s[|\xi(r)| + |\eta(t, t')|] - \ln 2, g^1(t, s, \xi(r), \eta(t, t')) = E^\mathcal{F}_s[|\xi(r)| + |\eta(t, t')| + \pi]. \) If we choose \( \gamma(t, s, \xi(r), \eta(t, t')) = E^\mathcal{F}_s[|\xi(r)| + \eta(t, t') + 1, \) then it’s easy to check that \( g^0, g^1 \) and \( \gamma \) satisfy the assumptions of Theorem 4.1. So if the terminal condition (4.2) holds, we can derive \( Y^0(t) \leq Y^1(t), \ \text{a.s. } t \in [0, T + K].\)

Acknowledgements

References

[1] M. Berger, V. Mizel, Volterra equation with Itô integrals, I, II, J. Integral Equations 2 (1980) 187-245, 319-337.

[2] E. Pardoux, P. Protter, Stochastic Volterra equations with anticipating coefficients, Ann. Probab. 18 (1990) 1635-1655.

[3] E. Alòs, D. Nualart, Anticipating stochastic Volterra equations, Stochastic Processes. Appl. 72 (1997) 73-95.

[4] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 4 (1990) 55-61.

[5] N. El Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance, Math. Finance 7 (1997) 1-71.

[6] S. Peng, Z. Yang, Anticipated backward stochastic differential equations, Ann. Probab. 37 (2009) 877-902.

[7] Z. Yang, R.J. Elliott, A converse comparison theorem for anticipated BSDEs and related non-linear expectations, Stochastic Process. Appl. 123 (2013) 275-299.

[8] F. Zhang, Comparison theorems for anticipated BSDEs with non-Lipschitz coefficients, J. Math. Anal. Appl. 416 (2014) 768-782.

[9] J. Lin, Adapted solution of a backward stochastic nonlinear Volterra integral equations, Stoch. Anal. Appl. 20 (2002) 165-183.

[10] Y. Shi, T. Wang, Solvability of general backward stochastic Volterra integral equations, J. Korean Math. Soc. 49 (2012) 1301-1321.
[11] T. Wang, J. Yong, Comparison theorems for some backward stochastic Volterra integral equations, Stochastic Processes. Appl. 125 (2015) 1756-1798.

[12] J. Yong, Backward stochastic Volterra integral equations and some related problems, Stochastic Process. Appl. 116 (2006) 779-795.

[13] J. Yong, Well-posedness and regularity of backward stochastic Volterra integral equations, Probab. Theory Related Fields 142 (2008) 21-77.

[14] J. Yong, Continuous-time dynamic risk measures by backward stochastic Volterra integral equations, Appl. Anal. 86 (2007) 1429-1442.

[15] K. Eduard, O. Ludger, Classical differentiability of BSVIEs and dynamic capital allocations, (2014) Available at SSRN: [http://ssrn.com/abstract=2379500](http://ssrn.com/abstract=2379500)

[16] L. Chen, Z. Wu, Maximum principle for the stochastic optimal control problem with delay and application, Automatica 46 (2010) 1074-1080.