Truncated Radial Oscillators with a Bound State in the Continuum via Darboux Transformations

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Abstract. The radial oscillator with zero angular momentum is used to construct a short-range model by cutting-off the potential at a given radius \( r = b \), and by substituting it with a constant potential for \( r > b \). The new potential, called truncated radial oscillator, admits both bound and scattering states. It is shown that the appropriate Darboux transformation leads to new exactly solvable models that have the entire energy spectrum of the truncated radial oscillator plus a new discrete energy eigenvalue. The latter defines a square-integrable wave function for the new system although it is embedded in the scattering regime of the energy spectrum. The new potentials are radial and such that their asymptotic behavior coincides with the profile predicted by von Neumann and Wigner for a potential to admit an eigenvalue in the continuum.

1. Introduction

In 1929 von Neumann and Wigner proposed a method to construct potentials admitting a square integrable eigenfunction (which is given as an ansatz) in the continuum regime of eigenvalues [1]. As the resulting potentials are oscillatory and go to zero as \( 1/r \), it was not clear their possible correspondence with the systems studied in those times. In addition, the method has the limitation that only one eigenfunction can be analytically known (the one which is proposed as ansatz), the other solutions –if any– would be obtained by directly solving the Schrödinger equation. As it would be expected, the potentials derived by the von Neumann-Wigner method were considered as simple toy models that could not occur in nature. However, such systems has been explored with new techniques in connection with wave propagation and semiconductor heterostructures [2–10]. Indeed, the possibility of finding bound states in the continuum in diverse systems that can be studied in the laboratory has been recently communicated [5]. Yet, the possibilities of getting the entire set of eigenfunctions and eigenvalues for the von Neumann-Wigner potentials are wider if the Darboux transformation is taken into account. Such transformation permits to pair the energy spectrum of two different potentials, one of them already provided and assumed as exactly solvable, and is in the mathematical foundations of the factorization method in quantum mechanics [11]. Applied to the harmonic oscillator, the Darboux transformation produces a wide family of one-dimension potentials with the same equidistant energy spectrum as the former [12], and shows the applicability of the supersymmetric formulation of quantum mechanics [13]. Over the years, the method opened the searching of new exactly solvable models in soliton theory [14] and quantum mechanics [11,15],
including non-Hermitian systems with all-real spectra [16–19].

In this work we show that a class of von Neumann-Wigner potentials can be obtained as a Darboux deformation of the truncated radial oscillator, the latter is a short-range potential defined by the radial oscillator in \( r \in [0, b] \), with \( b > 0 \), and a constant potential in \( r \in (b, \infty) \). Our approach is based on a degenerated version of the second-order Darboux transformation introduced in [8,9], and uses a scattering state as the transformation function.

In Section 2 we present a short description of the von Neumann-Wigner method to construct bound states in the continuum. Section 3 revisits the basics of the Darboux transformation we are dealing with. In Section 4 the general solution for the truncated radial oscillator is presented. In sections 5 and 6 the degenerated version of the Darboux transformation is used to deform the truncated radial harmonic oscillator. It is also determined the condition for the existence of the bound state in the continuum. Finally, some concluding remarks are given in section 6.

2. The von Neumann-Wigner proposal

In [1], von Neumann and Wigner investigated an alternative mechanism to obtain information from the stationary Schrödinger equation. Instead providing a potential and looking for the corresponding eigenvalues and eigenfunctions, they assumed the existence of a square-integrable eigenfunction—belonging to a given eigenvalue in the continuum—and pondered about the profile of the potential(s) that may admit such a solution. Below, for the sake of completeness, we provide a brief description of the von Neumann-Wigner method.

Consider the stationary radial Schrödinger equation with zero angular momentum in dimensionless form,

\[
V(r) = E + \frac{\nabla^2 \psi(r)}{\psi(r)}. \tag{1}
\]

The main idea of the method is to assume the existence of a solution which can be written as the product of the free particle solution and a modulation function \( f(r) \). That is,

\[
\psi(r) = f(r) \frac{\sin kr}{kr}, \tag{2}
\]

with \( E = k^2 \). The substitution of (2) into (1) leads to the next expression for the potential:

\[
V(r) = 2k \cot kr \frac{f'(r)}{f(r)} + \frac{f''(r)}{f(r)}, \quad \text{where} \quad \frac{d}{dr} = \frac{d}{dr}. \tag{3}
\]

Then, we look for a function \( f(r) \) such that (i) guarantees a potential \( V(r) \) free of singularities and (ii) the wave function \( \psi(r) \) associated to the eigenvalue \( E = k^2 \) is square integrable.

To satisfy condition (i) the zeros of \( f(r) \) and \( f(r) \sin kr \) must be compensated with zeros (at least of the same order) of \( f'(r) \) and \( f'(r) \), respectively. With this in mind, it may be proven that the appropriate modulation function acquires the form [1]:

\[
f(r) = \frac{1}{A^2 + [2kr - \sin 2kr]^2}, \tag{4}
\]

where \( A \) is a nonzero real constant.

The von Neumann-Wigner potential \( V(r) \) and its corresponding bound state in the continuum \( \psi(r) \) are obtained after substituting (4) into (2) and (3), respectively. They are shown in Figure 1 for \( E = k^2 = 1 \) and \( A = 1 \).

The possibilities for the function \( f(r) \) are not limited to (4), see for example [2]. However, it is remarkable that any \( f(r) \) fulfilling (i)–(ii) gives rise to potentials (3) with the same asymptotic oscillatory behavior [4]:

\[
\lim_{r \to \infty} V(r) \to a \frac{\sin br}{r} + O\left(\frac{1}{r^2}\right). \tag{5}
\]
If $|b| < |a|$, a bound state in the continuum (BIC) appears at a very specific value of the energy, in the present case for $E = b^2/4$. The latter is determined by the oscillatory frequency of the potential. The physical interpretation of this phenomenon is that it appears as a result of the destructive interference caused by the multiple maxima and minima of the potential.

3. Degenerated Darboux transformation

The Darboux transformation is a mathematical tool widely used to find analytical solutions of differential equations that are paired with an already known exactly solvable equation, see e.g. [14, 16, 20]. To summarize the method to be applied in our case, let us consider the radial part of the stationary Schrödinger equation with angular momentum $l = 0$,

$$\hat{H}\psi_E(r) = E\psi_E(r), \quad \text{where} \quad \hat{H} = -\frac{d^2}{dr^2} + V_0(r). \quad (6)$$

With the purpose of constructing the transformation, we define the first order differential operator

$$\hat{A} = \frac{d}{dr} - \beta(r), \quad (7)$$

where $\beta$ is a real valued function to be determined. Applying (7) to (6) we get

$$\left[\hat{H} - 2\beta'(r)\right] \hat{A}\psi_E(r) + \psi_E(r) \frac{d}{dr} \left(V_0 - \beta'(r) - \beta^2(r)\right) = E\hat{A}\psi_E(r). \quad (8)$$

If we choose the $\beta$-function in such a way that it satisfies the next Riccati equation

$$-\beta'(r) + V_0(r) - \beta^2(r) = \epsilon, \quad (9)$$

with $\epsilon$ a real constant. This assumption for $\beta(r)$ guarantees the covariance of the Schrödinger equation under the action of operator $\hat{A}$. To linearize the above nonlinear differential equation we use

$$\beta(r) = \frac{d}{dr} \ln \varphi(r), \quad (10)$$

which maps the Riccati equation into a Schrödinger equation type for the initial potential $V_0(r)$ and the function $\varphi(r)$:

$$-\frac{d^2}{dr^2} \varphi(r) + V_0(r) \varphi(r) = \epsilon \varphi(r). \quad (11)$$
Therefore, the function $\varphi(r)$ is nothing more than a solution, not necessarily physical, of the Schrödinger equation with eigenvalue $\epsilon$, so we take the next notation

$$\varphi(r) \equiv \psi_\epsilon(r).$$  \hfill (12)

In the context of the Darboux transformation the latter function is called “transformation function”, and its corresponding eigenvalue is known as “factorization constant”.

Therefore,

$$\psi_E(r) \rightarrow \psi_E^{(1)}(r) = \frac{W(\psi_\epsilon, \psi_E)}{\psi_\epsilon},$$

$$V_0(r) \rightarrow V_1(r) = V_0(r) - 2 \frac{d}{dr} \left( \frac{\psi'_\epsilon}{\psi_\epsilon} \right).$$

Here $W(\psi_\epsilon, \psi_E)$ stands for the Wronskian of $\psi_\epsilon$ and $\psi_E$. The strength of the Darboux transformation lies in the fact that the solutions of the Schrödinger equation for the potential $V_0$ permit to construct the solutions of a new system characterized by the potential $V_1(r)$. However, as the transformation function appears in the denominator of the new potential (and in the corresponding solutions), it has to be free of nodes. To avoid such restriction is usual to consider higher order transformations, which are equivalent to iterate the Darboux transformation at will. In the second order case, with $\psi_E^{(2)}(r)$ the eigenfunction of the Schrödinger equation for the potential $V^{(2)}(r)$ and $E$ the corresponding eigenvalue, one has the transformation rule:

$$\psi_E(r) \rightarrow \psi_E^{(2)}(r) = \frac{W(\psi_{\epsilon_1}, \psi_{\epsilon_2}, \psi_E)}{W(\psi_{\epsilon_1}, \psi_{\epsilon_2})},$$  \hfill (13)

$$V_0(r) \rightarrow V^{(2)}(r) = V_0(r) - 2 \frac{d^2}{dr^2} \ln W(\psi_{\epsilon_1}, \psi_{\epsilon_2}).$$  \hfill (14)

where $\psi_{\epsilon_1}$ and $\psi_{\epsilon_2}$ are solutions of the initial Schrödinger equation corresponding to the eigenvalues $\epsilon_1$ and $\epsilon_2$, respectively. Thus, we only have to guarantee that the Wronskian of these solutions does not have zeros.

We are interested in a special case of the second order Darboux transformation, which is obtained when the transformation functions are degenerated. With this aim we first consider that the eigenvalue $\epsilon_2$ is very close to $\epsilon_1$. That is, $\epsilon_2 = \epsilon_1 + \eta$, with $|\eta| \ll 1$. Therefore we can write function $\psi_{\epsilon_2}$ in a Taylor series expansion around $\epsilon_1$,

$$\psi_{\epsilon_1+\eta}(r) = \psi_{\epsilon_1}(r) + \frac{1}{1!}(\partial_{\epsilon_1}\psi_{\epsilon_1}) \eta + \frac{1}{2!}(\partial_{\epsilon_1}^2 \psi_{\epsilon_1}) \eta^2 + \ldots,$$  \hfill (15)

where $\partial_{\epsilon_1} \equiv \frac{\partial}{\partial \epsilon_1}$. Substituting (15) into (13) and (14), and considering that the determinant of a matrix with equal columns is zero, after calculating the limit $\eta \rightarrow 0$, we obtain a degenerate version of the second-order Darboux transformation:

$$\psi_E^{(2)}(r) = \frac{W(\psi_{\epsilon_1}, \partial_{\epsilon_1} \psi_{\epsilon_1}, \psi_E)}{W(\psi_{\epsilon_1}, \partial_{\epsilon_1} \psi_{\epsilon_1})},$$  \hfill (16)

$$V^{(2)}(r) = V_0(r) - 2 \frac{d^2}{dr^2} \ln W(\psi_{\epsilon_1}, \partial_{\epsilon_1} \psi_{\epsilon_1}).$$  \hfill (17)

The above transformation can be implemented by using a scattering state as transformation function, even if it has a large number of nodes \cite{7–9,21}.

The approach presented in this section will be used to construct a von Neumann-Wigner potential as the Darboux transformation of a truncated radial oscillator.
4. Truncated radial harmonic oscillator

Let us consider the radial stationary Schrödinger equation, with zero angular momentum,

$$- \frac{d^2}{dr^2} \psi_E(r) + V_0(r) \psi_E = E \psi_E,$$

for a truncated harmonic oscillator (see Figure 2a), defined as

$$V_0(r) = \begin{cases} 
  r^2 - b^2, & r \leq b \\
  0, & b < r 
\end{cases}$$

(19)

The general solution can be written as follows

$$\psi_E(r) = \begin{cases} 
  A \, _1F_1(a, 3/2, r^2) r e^{-r^2/2}, & r \leq b \\
  B e^{ikr} + C e^{-ikr}, & b < r 
\end{cases}$$

(20)

where $F_1$ is the confluent hypergeometric function, $a$ is a parameter that depends on both the energy $E = k^2$ and the cutoff $a = (1/4)(3 - k^2 - b^2)$, and $A$, $B$, $C$, are constants to be determined. Next we will find the discrete spectrum of the system and its corresponding square integrable wave functions.

![Figure 2: (a) The truncated radial harmonic oscillator for $b = 4$ and their corresponding bound states. (b) The corresponding scattering state for $k = 1$.](image)

4.1. Bound states

We restrict the analysis to the case $E = k^2 < 0$, for which $k \equiv i\kappa = i\sqrt{|E|}$. Demanding the continuity of the solution at the cut-off $b$, together with the boundary conditions, we get

$$\psi_E(r) = A \begin{cases} 
  _1F_1(a, 3/2, r^2) r e^{-r^2/2}, & r \leq b \\
  _1F_1(a, 3/2, b^2) b e^{(b^2/2 - \kappa b)c^{k^{-2}r}}, & b < r 
\end{cases}$$

(21)

In addition, demanding the continuity of the first derivative of the solution, we get the transcendental algebraic equation

$$\frac{(3 + \kappa^2 - b^2)b^2}{3} \, _1F_1 \left( a + 1, \frac{5}{2}; b^2 \right) - (b^2 - b\kappa - 1) \, _1F_1 \left( a, \frac{3}{2}; b^2 \right) = 0,$$

(22)

which can be solved after providing the parameters of the system. Equation (22) establishes the quantization rule for the bound states of the system. We solved it numerically, the discrete spectrum so obtained is reported in Table 1 for a specific value of the cut-off.
Table 1: Discrete spectrum of the truncated harmonic oscillator for $b = 4$.

| Bound states        | Energy |
|---------------------|--------|
| Ground state        | -12.999 |
| First excited state | -9.000 |
| Second excited state| -5.005 |

4.2. Scattering states

In this case, let us restrict the energy values to the interval $E > 0$ in (20) and, after taking the continuity conditions at the cut off, we obtain

$$
\psi_E(r) = A \begin{cases} 
1F_1(a, 3/2; r^2) e^{-r^2/2}, & r \leq b \\
D(k, b) e^{ikr} + D^*(k, b) e^{-ikr}, & b < r
\end{cases},
$$

where $k^2 = E$, and

$$
D(k, b) = \frac{A}{6k} e^{-\frac{1}{2}b(b+2ik)} \left\{ 3[i(b^2 - 1) + bk^2] \, _1F_1 \left( a, \frac{3}{2}; b^2 \right) + ib^2(k^2 + b^2 - 3) \, _1F_1 \left( a + 1, \frac{5}{2}; b^2 \right) \right\}
$$

Figure 2b shows a conventional scattering state for a specific value of the energy. Notice that this solution has an infinity number of zeros. Nevertheless, it is useful in the degenerated second order Darboux transformation we are going to develop.

5. Darboux deformed truncated harmonic oscillator

As transformation function we shall use the scattering state

$$
\psi_q(r) = \begin{cases} 
A_i \, _1F_1 \left( a_q, \frac{3}{2}; r^2 \right) e^{-\frac{r^2}{2}}, & r \leq b \\
A_e \sin[qr + \delta], & b < r
\end{cases},
$$

which is a solution belonging to the eigenvalue $\epsilon = q^2$. In the above expression $A_i$ and $A_e$ are arbitrary constants and $a_q = \frac{3-q^2-b^2}{4}$. The continuity and differentiability of (24) are fulfilled for concrete values of $q$, $b$, and $\delta$ (see Figure 3). Such values are determined from the transcendental algebraic expression

$$
b^2 \left( 3 + q^2 - b^2 \right) 3 \, _1F_1 \left( a_q + 1, \frac{5}{2}; b^2 \right) - [b^2 + qb \cot(qb + \delta) - 1] \, _1F_1 \left( a, \frac{3}{2}; b^2 \right) = 0.
$$

However, it is important to mention that the wave functions of the new potential are not necessarily continuous at the cut-off. The latter because equation (16) implies to evaluate second order derivatives of the transformation function.

5.1. Darboux deformed potential

Since the solution of the initial system is used just as an input for the degenerated Darboux transformation, at the present stage we do not worry about the continuity of the transformation.
Differentiable function

Non-differentiable function

Figure 3: (a) Continuous and differentiable scattering state for the values obtained by solving Eq. (25) with $b = 4$, $\delta = 0$ and $q = 0.880871$ (b) Non-differentiable scattering state at the cut-off for $b = 4$, $\delta = 0$ and $q = 4$.

We look for the conditions such that the deformed functions are continuous and differentiable. In such a case, instead of using the transformation function (24) we will use

$$\psi_q(r) = \begin{cases} 1 _{1} F_{1} \left( a_{q} \frac{3}{2}; r^{2} \right) r e^{-\frac{r^{2}}{2}}, & r \leq b \\ \sin(qr + \delta), & b < r \end{cases}.$$  \hspace{1cm} (26)

Here we have omitted the coefficients $A_i$ and $A_e$. Notice that we can substitute the partial derivatives with respect to $\epsilon = q^{2}$ by partial derivatives with respect to $q$ in (16) and (17). The latter because the related expressions involve quotients of wronskians and, therefore, the proportional term that results from applying the chain rule does not appear in the final result. With this in mind we can introduce (26) into (17) to obtain the Darboux deformed potential

$$V^{(2)}(r) = \begin{cases} V_{<}(r), & r \leq b \\ 32q^{2} \frac{\sin(qr + \delta) - q(r + \gamma_{0}) \cos(qr + \delta)}{(\sin(2(qr + \delta) - 2q(r + \gamma_{0}))^{2})} \sin(qr + \delta), & b < r \end{cases},$$  \hspace{1cm} (27)

where $\gamma_0 = \partial_q \delta$, and

$$V_{<} = r^{2} - b^{2} + 4 + 4 \sum_{n=0}^{\infty} \frac{(a_{q})_{n} \Gamma'(a_{q} + n)}{(3/2)_{n} \Gamma(a_{q} + n)n!} d^{2} \ln \left\{ W_{1} F_{1} \left( a_{q} \frac{3}{2}, r^{2} \right), r^{2n} \right\}.$$  \hspace{1cm} (28)

Here, $(a_{q})_{n}$ and $\Gamma$ are the Pochhammer symbol and the Gamma function, respectively. The symbol $'$ denotes partial derivative with respect to $q$. An in-depth review of the calculation of partial derivatives of hypergeometric functions with respect to their parameters can be found in [22].

Figure (4) shows the potential (27) for some specific parameter values. As we can see, it has a divergence at $r = 0$, a discontinuity at the cut-off, and goes to zero as $r \to \infty$. To be concrete, for $\delta = 0$ and $\gamma_{0} \neq 0$ one has

$$V^{(2)}(r) = -4q \frac{\sin(2qr)}{r} + O \left( \frac{1}{r^{2}} \right),$$

which is precisely the asymptotic behavior of the von Neumann-Wigner potentials, see Eq. (5). The latter means that the potential we have constructed should admit a bound state in the continuum corresponding to the eigenvalue of the transformation function.
6. Darboux deformed wave functions

The general solutions of the new system are given by

\[ \psi^{(2)}_k(r) = \begin{cases} A\psi_<(k, r), & r \leq b \\ BF^+(k, r) + CF^-(k, r), & b < r \end{cases} \tag{29} \]

where,

\[ \psi_<(k, r) = r e^{-\frac{2}{3} \sum_{m=0}^{\infty} \frac{(a)_m \Gamma'(a_q+m)}{(3/2)_m \Gamma(a_q+m)m!} W \left[ 1F_1 \left( a_q, \frac{3}{2}, r^2 \right), r^{2m}, 1F_1 \left( a, \frac{3}{2}, r^2 \right) \right]} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma'(a_q+n)}{(3/2)_n \Gamma(a_q+n)n!} W \left[ 1F_1 \left( a_q, \frac{3}{2}, r^2 \right), r^{2n} \right] \tag{30} \]

and

\[ F^\pm(k, r) = \frac{2\gamma q(k^2 - q^2) - (k^2 + q^2)\sin 2(qr + \delta) \pm 4ikq\sin^2(qr + \delta) e^{\pm ikr}}{\sin 2(qr + \delta) - 2q(r + \gamma_0)} e^{\mp ikr}. \tag{31} \]

6.1. Deformed bound states

Now we look for the conventional bound states of the deformed system \( E = k^2 < 0 \), then we take \( k = i\kappa = i\sqrt{|E|} \). First of all we have to notice that the solution remains regular at the origin of coordinates because \( \psi_\kappa(r = 0) = 0 \), while its behavior at \( r \to \infty \) is determined by the functions \( F^\pm \),

\[ \lim_{r \to \infty} F^\pm(\kappa, r) \to \left[ -(\kappa^2 + q^2) + O \left( \frac{1}{r} \right) \right] e^{\mp i\kappa r}. \]

Therefore, to satisfy the square integrable condition we must take \( C = 0 \) in (29). With this in mind and by imposing the continuity of the solution at the cut-off point, we get the bound states of the Darboux deformed system

\[ \psi^{(2)}_{\text{bound}}(r) = A \begin{cases} \psi^{(2)}_<(\kappa, r), & r \leq b \\ \psi^{(2)}(\kappa, b) F^+(\kappa, r), & b < r \end{cases} \tag{32} \]
As we can see in Figure 5, the bound states of the Darboux deformed potential (Figure 4) corresponding to the discrete spectrum showed in Table 1, fulfill the Sturm theorem, that is, the ground state is node-less while the first and second excited states have one and two zeros, respectively. For the first two states, the maximum amplitude of probability is in the region of greatest depth of the potential, while the maximum amplitude of probability for the most excited state is in the region near the cut-off point.

Figure 5: The three bound states corresponding to the discrete spectrum showed in Table 1 and the potential showed in Figure 4. The graphs are shown as if the zero of the wave function were in its corresponding eigenvalue.

### 6.2. Deformed scattering states

Now we explore the continuous regime, that is, the case $E = k^2 > 0$. Starting from equation (29) and imposing the corresponding continuity conditions at the cut-off point

$$
[BF^+(k, r) + CF^-(k, r)]_{r=b} = [A\psi_<(k, r)]_{r=b},
$$

$$
\left[ B\frac{d}{dr}F^+(k, r) + C\frac{d}{dr}F^-(k, r) \right]_{r=b} = [A\frac{d}{dr}\psi_<(k, r)]_{r=b},
$$

we get the next relationships between the coefficients

$$
B = -A \frac{z_1^*}{z_2 - z_2^*}, \quad C = A \frac{z_1}{z_2 - z_2^*},
$$

where

$$
z_1 = \left[ \psi_<(k, r) \frac{d}{dr}F^+(k, r) - F^+(k, r) \frac{d}{dr}\psi_<(k, r) \right]_{r=b},
$$

and

$$
z_2 = \left[ F^-(k, r) \frac{d}{dr}F^+(k, r) \right]_{r=b}.
$$

Here we have used that $\psi_<(k, r)$ is a real valued function and $[F^+(k, r)]^* = F^-(k, r)$. Therefore, the scattering solution of the deformed system, for $k \neq q$, can be written as follows

$$
\psi_s^{(2)}(r) = A \begin{cases} 
\psi_<(k, r), & r \leq b \\
\frac{\text{Im} \left\{ \left[ \psi_<(k, r) \frac{d}{dr}F^+(k, r) - F^+(k, r) \frac{d}{dr}\psi_<(k, r) \right]_{r=b} F^-(k, r) \right\}}{\text{Im} \left\{ F^-(k, r) \frac{d}{dr}F^+(k, r) \right\}_{r=b}}, & b < r
\end{cases},
$$

\[34\]
where \( \text{Im}(z) \) denotes the imaginary part of the complex number \( z \). Figure 6(a) shows a scattering state of the deformed system for some values of the parameters. Up to this point we have used the degenerated Darboux transformation for constructing the bound and scattering states of the deformed potential showed in Figure 4, which has a von Neumann-Wigner behavior at \( r \to \infty \). Therefore, it is time to examine what happens with the eigenvalue \( k = q \) which, according to the asymptotic behavior of the potential, could correspond to a bound state in the continuum.

7. Existence of a bound state in the continuum

The Darboux deformed solutions of the arrival systems are written in terms of the Wronskian scheme, the scattering solution corresponding to \( k = q \) is mapped in principle to the trivial solution. However, since we have the liberty to chose the arbitrary constant \( A \) in (34), we can to apply again the L’Hopital rule (in this case the rule was applied four times) to obtain the limit of the solution

\[
\lim_{k \to q} A\psi_{<}(k,r) = A' \lim_{k \to q} \frac{\partial^2 \psi_{<}(k,r)}{\partial r^2} = f(r).
\]  

The explicit form of \( f(r) \) is very long and not important at the moment, the only important thing to keep in mind is that \( f(r = 0) = 0 \), which guarantees the regularity of the solution at the origin of coordinates (see Figure 6). On the other hand, for \( r \geq b \), we can to apply again the L’Hopital rule (in this case the rule was applied four times) to obtain the limit of the solution at \( k \to q \),

\[
\lim_{k \to q} \frac{\text{Im} \left\{ \left[ \psi_{<}(k,r) \partial_r F^+(k,r) - F^+(k,r) \partial_r \psi_{<}(k,r) \right]_{r=b} \right\}}{\psi_{<}(k,b) [\text{Im} \left[ F^-(k,r) \partial_r F^+(k,r) \right] ]_{r=b}} = \frac{f_1(q,b,r)}{f_2(q,b)}.
\]

The explicit form of the term \( f_2(q,b) \) again is not important, all we have to keep in mind is that it is different from zero when \( k = q \). On the other hand, the term \( f_1(q,b,r) \) is given by

\[
f_1(q,b,r) = \left[ \partial^2_{k} \psi_{<}(k,r) \partial_r \text{Re} [F^+(k,r)] - \partial^3_{k,k,r} \psi_{<}(k,r) \text{Re} [F^+(k,r)] \right]_{k=q,r=b} g_1(q,r)
+ 2 \left[ \partial^2_{k} \psi_{<}(k,r) \partial^2_{k,r} \text{Im} [F^+(k,r)] - \partial^2_{k,r} \psi_{<}(k,r) \partial_k \text{Im} [F^+(k,r)] \right]_{k=q,r=b} g_2(q,r)
\]
Here we have used the notation $\partial_{a_1,...,a_n} = \frac{\partial^n}{\partial a_1...\partial a_n}$, while the functions $g_i(q, r)$, are given by

\begin{align*}
g_1(q, r) &= \frac{(2 \sin qr - 2r) \sin [2(qr + \delta)] - 2q(r + 4\gamma_0) \cos qr + qr^2 \cos [2(qr + \delta)] + 2\gamma_0 \sin qr}{\sin 2qr - 2q(r + \gamma_0)} \\
g_2(q, r) &= \frac{2q(r + 2\gamma_0) \cos qr + qr \cos [2(qr + \delta)] - 2\gamma_0(qr + r) + \sin [2qr + \gamma_0]}{-2q(r + \gamma_0)}, \\
g_3(q, r) &= \frac{2q \sin (qr + \delta)}{\sin [2(qr + \delta)] - 2q(r + \gamma_0)}.
\end{align*}

Therefore,

\begin{align*}
\lim_{r \to \infty} g_1(q, r) &\to qr[\cos qr + 3 \cos (qr + \delta)], \\
\lim_{r \to \infty} g_2(q, r) &\to -q[\cos qr + \cos (qr + \delta)], \\
\lim_{r \to \infty} g_3(q, r) &\to 0.
\end{align*}

From the last expressions we can notice that in order to get a square integrable wave function for $k = q$, the coefficients for $g_1(q, r)$ and $g_2(q, r)$ must be simultaneously zero. Taking into account that

$$[\text{Re}[F^+(k, r)]]_{k=q} = \gamma_0 [\partial_k \text{Im}[F^+(k, r)]]_{k=q} = g_3(q, r),$$

this assumption is true for the values of $q$ that fulfills the next algebraic equation

$$[-\partial^2_k \psi_<(k, r) \partial_k g_3(q, r) + \partial^2_k, r \psi_<(k, r) g_3(q, r)]_{k=q, r=b} = 0. \quad (36)$$

This above equation is analogous to the quantization rule of conventional bound states. In the sense that the nature of the deformed interaction region, define the energy for which the bound state in the continuum appears. In this case for $b = 4$, $\delta = 0$ and $\gamma_0 = 1$, the bound state in the continuum appears for a value of $q = 1.65996$. Figure 6 shows the graph for this peculiar state. As we can see it goes to zero as $r \to \infty$. In this work we do not implement an analytical demonstration of the convergence for the integral of the square modulus of the bound state in the continuum in the interval $[0, \infty)$, but we calculate numerically the corresponding normalization constant, which is 0.216948 for the used parameter values. The physical interpretation of these results underlies that a phenomenon of destructive interference is created between the different parts of the dispersed wave and the multiple crests of the potential, with the net effect that the wave amplitude is cancelled for very large values of $r$ and the wave function becomes normalizable.

8. Concluding remarks

We have shown that the Darboux transformation is useful to solve the problem of finding potentials with a bound state in the continuum (BIC) of energies, the latter originally proposed by von Neumann and Wigner [1]. In contrast with the von Neumann-Wigner method, where
the wave-function of the BIC is the only one that can be analytically determined, our approach permits the determination of all the energy eigenvalues and the corresponding eigenfunctions. The eigenfunctions representing conventional bound states are square-integrable and satisfy the Sturm oscillation theorem. In turn, the BIC deserves special attention since it belongs to the energy eigenvalue of the transformation function, which has been selected among the scattering states of the initial potential, so that the scattering spectrum of the Darboux-deformed potentials reported here lack such energy. The model described above was applied to the truncated radial oscillator, composited by a radial oscillator potential with zero angular momentum in $r \in [0, b]$ and a constant potential in $r \in (b, \infty)$. The energy of the BIC has been obtained from an algebraic equation that depends on the parameters of the initial potential. Its presence may be interpreted as the net effect of the wave function interaction with the multiple maxima and minima of the von Neumann-Wigner potentials, which makes the probability density to be concentrated in a finite region of space, even though they are in the scattering regime of energies. The approach presented in this work is not restricted to radial-like potentials but it can be applied to any exactly solvable truncated potential.

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