FRACTIONAL CALCULUS, COMPLETELY MONOTONIC FUNCTIONS, A GENERALIZED MITTAG-LEFFLER FUNCTION AND PHASE-SPACE CONSISTENCY OF SEPARABLE AUGMENTED DENSITIES

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ABSTRACT

Under the separability assumption on the augmented density, a distribution function can be always constructed for a spherical population with the specified density and anisotropy profile. Then, a question arises, under what conditions the distribution constructed as such is non-negative everywhere in the entire accessible subvolume of the phase-space. We rediscover necessary conditions on the augmented density expressed with fractional calculus. The condition on the radius part \( R(r^2) \) – whose logarithmic derivative is the anisotropy parameter – is equivalent to \( w^{-1}R(w^{-1}) \) being a completely monotonic function whereas the condition on the potential part is stated as its derivative up to the order not greater than \( \frac{1}{2} - \beta_0 \) being non-negative (where \( \beta_0 \) is the central limiting value for the anisotropy parameter). We also derive the set of sufficient conditions on the separable augmented density for the non-negativity of the distribution, which generalizes the condition derived for the generalized Cuddeford system by Ciotti & Morganti (2010) to arbitrary separable systems. This is applied for the case when the anisotropy is parameterized by a monotonous function of the radius of [Baes & Van Hese (2007)]. The resulting criteria are found based on the complete monotonicity of generalized Mittag-Leffler functions.

1. MODELS FOR SPHERICAL DYNAMICAL SYSTEM

1.1. Distribution function

Suppose that \( \mathcal{F}(r, \nu | t) \) is a phase-space distribution so that

\[
\int_S \mathcal{F}(r, \nu | t) \, dr \, d\nu = \text{number of tracers at phase} - \text{space volume} \text{ at time } t.
\]

Here \( r \) is the position vector in the configuration space and \( \nu = \hat{r} \) is the velocity. We only consider the system in equilibrium and thus the distribution must be time-independent. The distribution of a spherically symmetric population in a steady state is also invariant under transforms in \( SO(3) \) so that \( \mathcal{F}(r, \nu | t) = \mathcal{F}(r, \nu | t, ||\nu||) \) where \( r = ||r|| \) and \( \hat{r} = r/r \) are the radial distance and unit vector while \( \nu = v \cdot \hat{r} \) and \( \nu_\theta = v \cdot \nu \sin \theta \) are the radial and tangential velocities. If we adopt the canonical spherical polar coordinate \((r, \theta, \phi)\), they are given by

\[
v = ||\nu|| = (v_r^2 + v_\theta^2)^{1/2}; \quad \nu_\theta = ||\nu_\theta|| = (v_\theta^2 + v_\phi^2)^{1/2},
\]

where \((v_r, v_\theta, v_\phi)\) are the velocity components projected onto the associated orthonormal basis.

In order for the distribution to be indeed time-independent, it must be invariant under dynamic evolutions of tracers, that is, the distribution is a time-independent solution to the Boltzmann transport equation. For typical stellar dynamical applications, the trajectory of each tracer is its orbit under the external potential, which may or may not be self-consistently generated by the tracer population. The transport equation for this case results in the collisionless Boltzmann equation (CBE), whose solution is completely characterized by the theorem due to J. Jeans [1]. The Jeans theorem indicates that if the given time-independent spherically-symmetric distribution function (df) is a solution to CBE with a generic static spherical potential \( \Phi(r) \), it must be in the form of

\[
\mathcal{F} = \mathcal{F}(E, L^2)
\]

where

\[
E = \Phi(r) - \frac{1}{2}v^2; \quad L = ||L|| = rv_{\phi},
\]

are the two isotropic isolating integrals admitted by all such potentials, namely, the specific binding energy and the magnitude of the specific angular momentum, respectively. Here,

\[
\Psi(r) \equiv \left\{ \begin{array}{ll}
\Phi(r_{\text{out}}) - \Phi(r) & \text{if } r_{\text{out}} \text{ is finite} \\
\Phi(\infty) - \Phi(r) & \text{if } r_{\text{out}} = \infty \text{ and } |\Phi(\infty)| < \infty \\
-\Phi(r) & \text{if } r_{\text{out}} = \infty \text{ and } |\Phi(\infty)| \rightarrow \infty
\end{array} \right.
\]

is the relative potential with respect to the boundary \( r_{\text{out}} \). The system not bounded by a finite boundary radius is represented by \( r_{\text{out}} = \infty \) with \( \Phi(\infty) = \lim_{r \rightarrow \infty} \Phi(r) \). If \( r_{\text{out}} \) or \( \Phi(\infty) \) is finite, then \( \mathcal{F}(E < 0, L^2 = 0) = 0 \) because by definition \( E \geq 0 \) for all tracers bound to the system (and bounded by \( r \leq r_{\text{out}} \)).

1.2. Augmented densities of a spherical system

Integrating the spherical two-integral \( df(E, L^2) \) over the velocity space results in a bivariate function of \( \Psi \) and \( r^2 \),

\[
N(\Psi, r^2) \equiv \iiint d\nu \mathcal{F}(E = \Psi - \frac{1}{2}v^2, L^2 = r^2 v_{\phi}^2), \quad (1)
\]

which is referred to as the augmented density (AD). The integral here is formally over the whole velocity subspace, but if \( r_{\text{out}} \) or \( \Phi(\infty) \) is finite, it is essentially within the sphere \( v^2 \leq 2\Psi \) since \( \mathcal{F}(E < 0, L^2 = 0) = 0 \) for these cases. With \( \Psi(r) \) specified, the AD yields the local density \( \nu(\tau) \) via

\[
\nu(\tau) = N(\Psi(r), r^2).
\]

Similarly, the augmented moment functions are given by

\[
m_{k,a}(\Psi, r^2) \equiv \iint d\nu \nu^k \mathcal{F}(E = \Psi - \frac{1}{2}v^2, L^2 = r^2 v_{\phi}^2)
\]

\[
= 4\pi \int dv_\theta dv_\phi v_\phi^{2n+1} \mathcal{F}(E - \nu^2 + v_\theta^2 + \frac{v_\phi^2}{2}, r^2 v_{\phi}^2) \quad (2a)
\]

[1] Ludwig Eduard Boltzmann (1844-1906)
[2] Sir James Hopwood Jeans (1877-1946)
Changing the integration variables to $(\mathcal{E}, L^2)$, these are represented to be a set of integral transformations of the df,

$$m_{k,n} = \frac{2\pi}{r^{2n+2}} \int \mathcal{E} dL^2 K^{k,n} L^{2n} \mathcal{F}(\mathcal{E}, L^2)$$

$$= \frac{2\pi}{r^{2n+2}} \int_{E \geq E_0, L^2 \geq 0} dE dL^2 \Theta(K) K^{k,n} L^{2n} \mathcal{F}(\mathcal{E}, L^2).$$

Here $\Theta(x)$ is the Heaviside unit-step function and

$$E_0 = \begin{cases} 0 & \text{if } r_{\text{out}} \text{ or } \Phi(\infty) \text{ is finite} \\ \infty & \text{if } \lim_{r \to \infty} \Psi(r) = -\Phi(\infty) \to -\infty \end{cases}$$

is the lower bound of the binding energy. The transform kernel and the domain in $(\mathcal{E}, L^2)$ space over which the integral is performed are given by

$$\mathcal{K}(\mathcal{E}, L^2; \Psi, r^2) \equiv 2(\Psi - \mathcal{E}) - r^{-2} L^2,$$

$$\mathcal{T} \equiv \{(\mathcal{E}, L^2) \mid E \geq E_0, L^2 \geq 0, \mathcal{K} \geq 0 \}.$$

Note $\mathcal{K}$ is $r^2$ expressed as a function of 4-tuple $(\mathcal{E}, L^2, \Psi, r^2)$.

2. MATHEMATICAL PRELIMINARY

2.1. Fractional calculus

Definition 2.1 The Riemann-Liouville integral operator of arbitrary non-negative real order $\lambda \geq 0$ is given by

$$\mathcal{I}_x^\lambda f \equiv \begin{cases} \frac{1}{\Gamma(\lambda)} \int_a^x (x-y)^{\lambda-1} f(y) dy & (\lambda > 0) \\ f(x) & (\lambda = 0) \end{cases},$$

where $\Gamma(\lambda)$ is the gamma function.

This is a trivial generalization of the Cauchy formula for repeated integrations. For $0 < \lambda < 1$, this is also recognized as the generalized Abel transform with the classical case corresponding to the $\lambda = \frac{1}{2}$ case. We also define

Definition 2.2 the fractional derivative for $\lambda \geq 0$ such that

$$\mathcal{D}_x^\lambda f \equiv \begin{cases} \frac{d^{|\lambda|}}{dx^{|\lambda|}} \mathcal{I}_x^{|\lambda|} f & (0 < |\lambda| < 1) \\ \frac{d^{|\lambda|} f}{dx^{|\lambda|}} |_{y=x} = f^{(|\lambda|)}(x) & (|\lambda| = 0) \end{cases}$$

where $[\lambda], |\lambda|$, and $|\lambda| = \lambda - [\lambda]$ are the integer ceiling, the integer floor and the fractional part of $\lambda$, respectively.

Note equation (3) is a generalization of the differentiation for positive real order as equation (3) of the integration. These definitions extend to include a negative index using

Definition 2.3 for arbitrary real $\lambda$

$$\mathcal{I}_x^{-\lambda} f = \mathcal{I}_x^{-|\lambda|} f \quad \text{and vice versa.}$$

The basic composite rule for the Riemann-Liouville operators is that, for any pair of non-negative reals $\lambda$ and $\xi$,

$$\mathcal{I}_x^\xi \mathcal{I}_x^{-\lambda} f = \mathcal{I}_x^{\xi - \lambda} f,$$

which may be shown by direct calculations using the Fubini theorem and the Euler integral of the first kind for the beta function, that is,

$$\int_a^x dy (x-y)^{\xi-1} \int_a^y dw (w-y)^{-\lambda-1} f(w)$$

$$= \int_a^x dw f(w) \int_a^x dy (x-y)^{\xi-1}(y-w)^{-\lambda-1}$$

$$= \int_a^x dw f(w) \int_0^1 dt (1-t)^{\xi-1}t^{\lambda-1}.$$

Next for any real $\lambda$ and a non-negative integer $n$

$$\frac{d}{dx} \mathcal{I}_x^\lambda f = \mathcal{I}_x^{\lambda-1} f; \quad \frac{d^n}{dx^n} \mathcal{I}_x^\lambda f = \mathcal{I}_x^{\lambda-n} f.$$

Here the latter follows the former ($n = 1$) by means of induction. The $n = 1$ case is proven by direct differentiation of equation (3) for $\lambda > 1$ and the fundamental theorem of calculus for $\lambda = 1$ while the same case with $\lambda < 1$ is essentially trivial from the definitions of fractional derivatives in equations (3) and (5). Together they also indicate that

$$\mathcal{D}_x^\xi \mathcal{I}_x^\lambda f = \mathcal{I}_x^{\lambda-\xi} f \quad (\xi \leq \lambda),$$

$$\mathcal{D}_x^\xi \mathcal{I}_x^\lambda f = \mathcal{I}_x^{\lambda-\xi} f \quad (\xi \geq \lambda),$$

for non-reals $\lambda, \xi \geq 0$ and arbitrary function $f(x)$, provided that all the integrals in their respective definitions absolutely converge. Next we observe for $\lambda \geq 0$

$$\mathcal{I}_x^{\lambda+1} f' = \mathcal{I}_x^{\lambda} f - \frac{(x-a)^\lambda f(a)}{\Gamma(\lambda + 1)},$$

thanks to the fundamental theorem of calculus ($\lambda = 0$) and integration by part. By means of induction, this generalizes to

$$\mathcal{I}_x^{\lambda+n} f(a) = \mathcal{I}_x^{\lambda} f - \sum_{k=0}^{n-1} \frac{(x-a)^{\lambda+k} f^{(k)}(a)}{\Gamma(\lambda + k + 1)},$$

where $n$ is any non-negative integer, and we also find that

$$\frac{d^n}{dx^n} \mathcal{I}_x^\lambda f = \mathcal{I}_x^{\lambda-n} f + \sum_{k=0}^{n} \frac{(x-a)^{\xi-k} f^{(n-k)}(a)}{\Gamma(1 + \lambda - k)},$$

for $\lambda \geq 0$ and any non-negative integer $n$. The last implies that fractional derivatives in equation (4) are alternatively given by

$$\mathcal{D}_x^\lambda f = \frac{d^{|\lambda|}}{dx^{|\lambda|}} \mathcal{I}_x^{|\lambda|} f + \sum_{k=0}^{n-1} \frac{(x-a)^{\xi-k} f^{(n-k)}(a)}{\Gamma(1 + \lambda - k)},$$

where $\lambda > 0$ and $n = 0, 1, \ldots, [\lambda]$.

Using these and equation (10), we can also derive that

$$\mathcal{I}_x^\xi \mathcal{I}_x^{-\lambda} f = \mathcal{I}_x^{\xi - \lambda} f - \sum_{k=1}^{[\lambda]} \frac{C_{\xi,\lambda}^{[\lambda]} f^{(k)}(a)}{(x-a)^{\xi-k}},$$

$$\mathcal{D}_x^\xi \mathcal{I}_x^{-\lambda} f = \mathcal{I}_x^{\xi - \lambda} f - \sum_{k=1}^{[\lambda]} \frac{C_{\xi,\lambda}^{[\lambda]} f^{(k)}(a)}{(x-a)^{\xi-k}}.$$
for non-negative reals \( \lambda, \xi \geq 0 \) and arbitrary function \( f(x) \), provided again that all the integrals in their respective definitions absolutely converge. Here \( C_{\xi,k}^{\pm} \) are given by

\[
C_{\xi,k}^{\pm} = \frac{1}{\Gamma(1+\xi-k)} \left\{ \begin{array}{l}
\frac{(\xi)_k}{(1+\xi)_k} \quad (+\text{case}) \\
\frac{(-1)^{\xi+k}(\delta)_{\xi+k}}{(1-\delta)_{\xi+k}} \quad (-\text{case})
\end{array} \right.
\]

where \( 0 \leq \delta = \xi - [\xi] < 1 \) is the fractional part of \( \xi \), and

\[
(a)_n = \prod_{j=1}^n (a+j) ; \quad (a)_n^\pm = \prod_{j=1}^n (a-1+j)
\]

are the Pochhammer symbol. The falling product \( (a)_n^- \) follows the combinatorist’s convention whereas the rising one \( (a)_n^+ \) does the analyst’s. Note these are related to each other,

\[
(a)_n^- = (-1)^n (a)_n^+ ; \quad (a)_n^+ = (a-n+1)_n^+.
\]

and also to the gamma functions,

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} ; \quad (a)_n^- = \frac{\Gamma(1+a)}{\Gamma(1+a-n)}.
\]

The last may be used to generalize the Pochhammer symbol for non-integer \( n \). Together equations \( 8 \) and \( 12 \) provide the generalization of equation \( 6 \) for any pair of reals \( \xi \) and \( \lambda \).

**Lemma 2.4** for real \( \lambda \) and \( \alpha > 0 \),

\[
\mathcal{L}^{\alpha}_x f(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} f(x) = \mathcal{L}^{\alpha}_x f(x).
\]

This is formally a generalization of the result, namely

\[
\frac{d^\alpha}{dx^\alpha} = (a)_n^{-\alpha-n} (n = 0, 1, \ldots)
\]

although the last is in fact valid for any \( \alpha \).

We formalize an obvious but important fact, namely

**Lemma 2.5** for \( \lambda > 0 \) and \( x > a \), if \( f \geq 0 \) in \( [a, x] \), then \( \mathcal{L}^{\alpha}_x f(x) \geq 0 \). Moreover \( \mathcal{L}^{\alpha}_x f(x) \) is finite provided that the support of \( f \) in \( (a, x) \) has non-zero measure.

Next, if \( a \) is finite, then for \( \xi > 0 \)

\[
\mathcal{L}^{\xi}_x f(x) = \frac{(x-a)^{\xi}}{\Gamma(\xi)} \int_0^x f(t) (x-t)^{\xi-1} dt.
\]

while for \( 0 < \lambda = -\xi < 1 \) and \( n = 1 \), equation \( 11 \) results in

\[
\mathcal{L}^{\xi}_x f = \mathcal{L}^{\xi}_x f = \frac{f(a)}{\Gamma(1-\lambda)} (x-a)^{-\lambda} + \mathcal{L}^{\xi}_{x-\lambda} f.
\]

It then follows that

**Lemma 2.6** for \( a \neq \pm \infty \),

\[
\lim_{x \to a} \mathcal{L}^{\xi}_x f(x) = \frac{f(a)}{\Gamma(\xi + 1)}.
\]

which is valid for \( \xi \geq 0 \) if \( f(x) \) is right-continuous at \( x = a \) or for \( \xi \geq -1 \) if \( f(x) \) is right-differentiable at \( x = a \). Equation \( 15 \) for \( \xi = 0 \) is equivalent to the definition of the right-continuity while for \( \xi = -1 \), it becomes \( \lim_{x \to a^+} f(x) = 0 \) which holds if \( f(x) \) is finite. Equation \( 15 \) implies that

\[
f^{(n)}(x) = (-1)^n \int_0^x f^{(n+1)}(t) dt = \frac{(-1)^n}{x^{n+1}} \int_0^x ds s^n e^{-\gamma s} \phi(\frac{s}{x}).
\]

10 Leo August Pochhammer (1841-1920)

**Corollary 2.7** if \( f(x) \) is right-continuous at \( x = a \) and \( f(a) \) is finite, then \( \mathcal{L}^{\lambda}_x f(x) = 0 \) for \( \lambda > 0 \).

Next we examine the behaviors of fractional calculus operators under the Laplace transform. For this, we first note a general property of the Laplace transform of the derivative,

\[
s^{\alpha+1} \mathcal{L} \left[ f(x) \right] = \mathcal{L} \left[ f^{(\alpha+1)}(x) \right] + \sum_{j=0}^n s^j f^{(\alpha-j)}(0),
\]

which is valid given that \( \lim_{s \to \infty} s^{\alpha} f^{(0)}(x) = 0 \) for sufficiently large \( s \) (which is required for the Laplace transform to converge). Equation \( 16 \) is proven via integration by part,

\[
\int_0^\infty e^{-sx} \frac{df(x)}{dx} = -f(0) + s \int_0^\infty dx \frac{e^{-sx} f(x)}{d^2 \mathcal{L}^{-\lambda}_x f(x)}
\]

for \( n = 0 \) and the induction completes its proof for any non-negative integer. In order to generalize equation \( 16 \) to include the fractional derivative, we next consider for \( \lambda \geq 0 \)

\[
\int_0^\infty dx e^{-sx} \int_0^\infty dy (x-y)^{\lambda-1} f(y)
\]

with the Euler integral of the second kind for the gamma function, we find that

\[
s^{\lambda} \mathcal{L} \left[ \mathcal{L}^{\alpha}_x f(x) \right] = \mathcal{L}^{\alpha+1}_x [f(x)].
\]

The Laplace transform of an arbitrary real-order derivative is then found by combining equations \( 16 \) and \( 17 \).

2.2. Post–Widder formula

**Theorem 2.8 (Post–Widder)** If \( \phi(t) \) is continuous for \( t > 0 \) and there exist real \( \lambda A > 0 \) and \( \lambda b > 0 \) such that

\[
e^{-b t} |\phi(t)| \leq A \quad \text{for } t > 0,
\]

then the Laplace transform

\[
f(x) = \mathcal{L} \left[ \phi(t) \right] \equiv \int_0^\infty dt e^{-sx} \phi(t).
\]

converges and is infinitely differentiable for \( x > b \). Moreover, \( \phi(t) > 0 \) may be inverted from \( f(x) \) using the differential inversion formula (Post [1930], Widder [1941]),

\[
\phi(t) = \mathcal{L}^{-1} \left[ f(x) \right] = \lim_{n \to \infty} \left( \frac{-1}{n!} \int_0^\infty \frac{f^{(n+1)}(t)}{t} dt \right).
\]

In literature, the last formula is typically named after E. Post or together with D. Widder. A rigorous proof, which is beyond the scope of this paper, may be found in a standard text on the Laplace transform. However, its heuristic justifications abound and are easy to observe. For instance, direct calculations using equation \( 13 \) indicate that

\[
f^{(n)}(x) = (-1)^n \int_0^\infty dt t^n e^{-\gamma t} \phi\left( \frac{t}{x} \right).
\]

11 Pierre-Simon Laplace (1749-1827)
12 Emil Leon Post (1897-1954)
13 David Vernon Widder (1898-1990)
and thus we find that
\[
\frac{(-1)^n}{n!} \binom{n}{r} f^{(n)}(rx) = \int_0^\infty ds P(s; n) \phi'(\frac{r}{n}s),
\]
where
\[P(s; n) = \frac{s^n}{n!} e^{-s}\]
is the probability density of the Poisson distribution with a mean of \(s = n\). It follows that as \(n \to \infty\), the relative dispersion decreases and so \(\phi(st/n) \to \phi(st/n) = \phi(t)\), which results in the Post–Widder formula. Note however that the convergence of equation (15) by itself does not necessarily imply that \(f(x)\) is the Laplace transformation of \(\phi(t)\), which is rather a partial condition for the formula to be valid.

2.3. Completely monotonic functions

**Definition 2.29** A smooth function \(f(x)\) of \(x > 0\) is said to be completely monotonic (cm henceforth) if and only if
\[
(-1)^n f^{(n)}(x) \geq 0 \quad (x > 0, \; n = 0, 1, 2, \ldots).
\]
The definition extends to \(x \geq 0\) if \(f(x)\) is right-continuous at \(x = 0\). Some basic properties of cm functions are:

**Lemma 2.10** Let \(f\) and \(g\) be cm. Then,

1. \((-1)^n f^{(n)}(x) \geq 0\) for any non-negative integer \(n\) is cm.
2. If \(F \geq 0\) in \((0, \infty)\) and \(f = -F'\), then \(F\) is cm.
3. \(\int_0^\infty f(y) dy\) is a cm function of \(y\) if it converges.
4. \(af + bg\) where \(a\) and \(b\) are non-negative constants is cm.
5. \(f \cdot g\) is cm.
6. If \(F > 0\) in \((0, \infty)\) and \(f = F'\), then \(g \circ F\) is cm.
7. \(\exp(f)\) is cm.

Items 1 and 2 are essentially trivial from Definition 2.29 and item 3 is simply a particular case of item 2. Item 4 follows the linearity of differentiations while item 5 is shown using the Leibniz rule, that is, (here \(\binom{n}{k}\) is the binomial coefficient)
\[
(-1)^n \frac{d^n}{dx^n} (f \cdot g) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^k f}{dx^k} (-1)^{n-k} \frac{d^{n-k} g}{dx^{n-k}}.
\]
The last two may be shown using the Faà di Bruno formula (i.e., the general chain rule),
\[
(g \circ F)^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} F^{(k)}(t) \cdot B_{n,k}[f(t), f'(t), \ldots, f^{(n-k)}(t)]
\]
where \(F'(t) = f(t)\) and \(B_{n,k}\) is the Bell polynomial, that is,
\[
B_{n,k}(x_0, \ldots, x_{n-k}) = \sum_{(j_0, j_1, \ldots)} \frac{n!}{j_0! j_1! \cdots} \left(\frac{x_0}{1!}\right)^{j_0} \left(\frac{x_1}{2!}\right)^{j_1} \cdots.
\]
Here the summation is over all sequences \((j_0, j_1, \ldots)\) of non-negative integers constrained such that
\[
\sum_{m=0}^{n} j_m = k; \quad \sum_{m=0}^{(m+1)} j_m = n.
\]
Note then
\[
\sum_{m=0}^{n} m j_m = n - k
\]
and thus \(j_m \geq 0\) indicates that \(j_m = 0\) for \(m > n - k\) (n.b., if otherwise, \(\sum_{m=0}^{n} m j_m > n - k\), which is contradictory). The property 6 follows this because
\[
n - k - \sum_{m=0}^{n} j_{2m+1} = 2 \sum_{m=0}^{n} m (j_{2m} + j_{2m+1})
\]
is even. That is to say, if \(f\) is cm, the parity of the Bell polynomial in equation (22) is \((-1)^{n-k}\), and thus, given that \(g\) is also cm, the parity of every term in the sum on the right-hand side of equation (22) is \((-1)^n\). Equation (22) also indicates that
\[
\frac{d^n \exp(f(t))}{dt^n} = \exp(f(t)) \cdot B_n[f'(t), f''(t), \ldots, f^{(n-k)}(t)]
\]
where \(B_n\) is the \(n\)-th complete Bell polynomial, that is,
\[
B_n(x_1, \ldots, x_n) = \sum_{n=1}^{n} B_n(x_0, \ldots, x_{n-k}).
\]
Note
\[
n - \sum_{m=0}^{n} j_{2m} = 2 \sum_{m=0}^{n} m (j_{2m-1} + j_{2m}).
\]
is even. Hence if \(f\) is cm, the parity of the complete Bell polynomial in equation (23) is \((-1)^n\) and so \(\exp(f)\) is cm.

The archetypal example of a cm function is \(f(x) = e^{-x}\). Other elementary examples of cm functions include:

1. \(f(t) = t^{-\delta} (t > 0)\) is cm if and only if \(\delta \geq 0\).
2. \(f(t) = \ln(1 + t^{-\delta})\) is cm.

These are proven through
\[
\frac{d^n \ln(1 + x)}{dx^n} = (-1)^n x^{-n} \prod_{k=1}^{n} \frac{1}{x_k} - \frac{1}{(1 + x)^{n+1}},
\]
and
\[
\frac{d^{n+1} \ln(1 + x^{-1})}{dx^{n+1}} = (-1)^{n+1} x^{-n+1} \prod_{k=1}^{n+1} \frac{1}{x_k} - \frac{1}{(1 + x)^{n+1}}.
\]
Following this and Lemma 2.10 are

**Corollary 2.11** Let \(g(t)\) be cm, then both \(t^{-\delta} g(t)\) with \(\delta \geq 0\) and \(g(t^p)\) with \(0 < p \leq 1\) are cm.

**Proof.** The first is obvious thanks to Lemma 2.10. The last follows Lemma 2.10 with \(F(t) = t^p\) since \(F' = pt^{p-1}\) for \(0 < p \leq 1\) is cm. Q.E.D.

**Corollary 2.12** For \(0 < p \leq 1\) and \(a, b \geq 0\), these are cm:
\[
f(t) = t^{-a}(1 + t^p)^{-b}; \quad f(t) = t^{-a}(1 + t^p)^b.
\]

**Proof.** Let \(F(t) = c + t^p\). Then \(F' = pt^{p-1}\) is cm for \(0 < p \leq 1\). Hence first \(f \circ F(t) = (1 + t^p)^{-b}\) with \(c = 1\) and \(g(w) = w^{-b}\) for \(0 < p \leq 1\) and \(b \geq 0\) is cm. Next, with \(c = 0\) and \(g(w) = b \ln(1 + w^{-1})\), we find that \(g \circ F(t) = b \ln(1 + t^{-p})\) is cm for \(0 < p \leq 1\) and \(b \geq 0\), and so is \((1 + t^{-p})^b = \exp[b \ln(1 + t^{-p})].\)

The fundamental result characterizing cm functions (Bernstein 1928, Widder 1941) is due to S. Bernstein 18.

**Theorem 2.13** (Hausdorff–Bernstein–Widder) A smooth function \(f(x) > 0\) is completely monotonic if and only if \(f(x) = \int_0^\infty e^{-xt} d\mu(t)\) where \(\mu(t)\) is the Borel measure on \([0, \infty)\), that is, there exists a non-negative distribution \(\phi(t) \geq 0\) of \(t > 0\) such that equation (17) holds.

\[1^8\text{Сергей Натанович Бернштейн (1880-1968)}\]
The ‘if’-part is elementary since
\[ f^{(n)}(x) = (-1)^n \int_0^\infty dt t^n e^{-tx} \phi(t) = (-1)^n \mathcal{L}^{-1} \{ t^n \phi(t) \}. \]

Although the complete proof of the ‘only if’-part is beyond our scope, the partial proof follows the Post–Widder formula. That is, if the inverse Laplace transform \( \phi(t) = \mathcal{L}^{-1}_{>0} \{ f(x) \} \) of a cm function \( f(x) \) is well-defined, then equation (19), provided that it converges, indicates that \( \phi(t) \) must be non-negative in the positive real domain.

### 2.4. Miscellaneous

We note an additional auxiliary relation, which will be used throughout this paper: that is, for any non-negative integer \( n \) and arbitrary differentiable function \( f(x) \),
\[ \left( x^2 \frac{d}{dx} \right)^n (xf) = x^{n+1} \frac{d^n}{dx^n} (xf), \tag{25} \]
which may be proven via the induction on \( n \) (see [An2011b] theorem A3). In fact this is also equivalent to a lemma
\[ x^n f_{(n+1)}(x) = \frac{d}{dx} \left[ x^n f_{(n)}(x) \right] \tag{26} \]
where
\[ f_{(n)}(x) = \frac{d^n[x^n f(x)]}{dx^n}. \]

This lemma may be proven directly via
\[ f_{(n+1)}(x) = \frac{d}{dx^k} \left[ x^k \frac{d}{dx} (x^n f(x)) \right] = \frac{d}{dx^k} \left[ x^k f + x \left( \frac{d}{dx} (x^n f(x)) \right) \right] = \frac{d}{dx^k} (x^n f(x)) \]
\[ = \left( x^n f(x) \right) dx + \sum_{k=0}^n \binom{n}{k} x^k \frac{d}{dx} \left[ x^{n-k} f(x) \right] \]
\[ = \left( 1 + n + x \right) \frac{d}{dx} \left[ x^n f(x) \right] = \frac{d}{dx} \left[ x^n f(x) \right] \]
where we also used that \( d^k x/dx^k = 0 \) if \( k \geq 2 \) and the Leibniz rule (eq. 21). The theorem in equation (25) implying the lemma in equation (26) has been shown in [An2011b corollary A4] whereas the opposite implication may be deduced because the induction step for the proof of equation (25) follows equation (26) as
\[ \frac{d^{n+1}}{dx^{n+1}} (x^{n+1} f(x)) = \frac{d}{dx} \left[ (x^{n+1}) \frac{d}{dx} (xf) \right] = \frac{d}{dx} \left[ x^n \frac{d}{dx} (xf) \right]. \]

Fractional calculus also generalizes the lemma in equation (26) generalizes. In particular, for a non-negative integer \( n \) and \( 0 \leq \delta < 1 \),
\[ \frac{d^n}{dx^n} \left( x^\delta f(x) \right) = \frac{1}{x^n} \frac{d}{dx} \left[ x^n \frac{d}{dx} (x^{\delta} f(x)) \right] = \frac{1}{x^n} \left( x^\delta \frac{d}{dx} (x^n f(x)) \right). \]

Note that for the \( \delta = 0 \), the last results in equation (20). The middle for the same case is consistent with the fundamental theorem of calculus given equation (26) indicating
\[ x^{n+1} f_{(n)}(x) = \int_0^x y^n f_{(n+1)}(y) \, dy + \int_0^x y^n f_{(n+1)}(y) \, dy \tag{28} \]
provided that \( f_{(n)}(0) \) is finite. Equations (26) and (27) imply

**Corollary 2.14** for a non-negative integer \( n \), if \( f_{(n+1)}(x) \geq 0 \) for \( x > 0 \), then \( g(x) = \int_0^x f_{(n+1)}(y) \, dy \geq 0 \) for \( x > 0 \) and \( n \mu \leq n + 1 \).

In fact, the successive applications of this with a descending subscript furthermore suggest that, if \( f_{(n)}(x) \geq 0 \) for \( x > 0 \) and a non-negative integer \( n \), it follows that \( g_n(x) = \int_0^x f_{(n+1)}(y) \, dy \geq 0 \) for \( x > 0 \) and any \( \mu \mu \leq n \).

**Corollary 2.14** with an integer \( \mu \) may be generalized alternatively, namely,

**Theorem 2.15** for a non-negative integer \( n \), if \( x^n f_{(n+1)}(x) \) is cm, then \( x^n f_{(n+1)}(x) \) is also cm.

**Proof.** If \( x^n f_{(n+1)}(x) \) is cm, then by the Bernstein theorem, there exists a non-negative function \( h(u) \geq 0 \) for \( u > 0 \) such that
\[ x^n f_{(n+1)}(x) = \int_0^\infty du e^{-xu} h(u). \]

The complete monotonicity of \( x^n f_{(n+1)} \) can then be shown directly using equation (25), which indicates that
\[ x^n f_{(n+1)} = x^{n-1} \int_0^x dy y^n f_{(n+1)}(y) = \int_0^x dy y^{n-1} \int_0^\infty du e^{-xu} h(u), \]
\[ \frac{d^n}{dx^n} (x^n f_{(n+1)}(x)) = (-1)^k \int_0^1 dt t^{n+k-1} \int_0^\infty du e^{-xu} h(u). \]

Finally, we also note

**Lemma 2.16** for a non-negative integer \( n \), if \( f^{(n+1)}(a) \) is finite and \( f^{(0)}(a) = \cdots = f^{(n)}(a) = 0 \), then \( \delta x^{n+\delta} f(a) = 0 \) for \( 0 \leq \delta < 1 \).

**Proof.** Here we assume \( a = 0 \), but the similar argument holds for any finite “a” accompanied by a simple translation. First,
\[ \int_0^x f(x)^{\delta} \, dx = \frac{x^{\delta}}{(1 - \delta)} \int_0^1 \frac{f(x)^{\delta}}{(1 - t^\delta)} \, dt; \tag{29a} \]
\[ \delta x^{n+\delta} f = \int_0^1 \frac{1}{(1 - \delta)} \int_0^1 \frac{f(y)^n}{(1 - t^\delta)} \, dy \, dt. \tag{29b} \]

Here the latter follows the former because
\[ \frac{d^n}{dx^n} [x^{\delta} f(x)] = x^{\delta} f^{(n+1)}(x) \]
\[ \frac{dy}{dx} = f^{(n+1)}(x) \]
\[ \delta x^{n+\delta} f = \int_0^1 \frac{1}{(1 - \delta)} \int_0^1 \frac{f(y)^n}{(1 - t^\delta)} \, dy \, dt. \]

Finally, given the Leibniz rule,
\[ \frac{d^n}{dy^n} [y^{\delta} f(y)] = y^{\delta} f^{(n+1)}(y) \]
\[ + (1 - \delta) \sum_{k=0}^n (-1)^n \binom{n+1}{k} (\delta)_{n-k} y^{n+k} f^{(k)}(y), \]
which identically vanishes for \( y = 0 \) if the condition part of Lemma 2.16 with \( a = 0 \) holds. Here the conclusion follows as the integrand of equation (29b) with \( x = 0 \) is also zero. Q.E.D.
3. FRACTIONAL CALCULUS ON THE AUGMENTED DENSITY

Al 2014 has shown that the Abel transformation of the augmented moment function of an anisotropic spherical system results in a similar integral transformation of the df as equation (29) but with different powers on \( K \) and \( L^2 \). This result generalizes by means of the fractional calculus. The goal of this section is to establish them (see eqs. 36 and 39) for any pair of non-negative reals \( 0 \leq \mu \leq \xi \).

We start by considering to apply the integral operator of equation (3) to equation (25) on \( \Psi \) or \( r^2 \). In fact, we can establish more general results. With

\[
\mathcal{J}_r(\Psi, r^2) = \int_{r} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2)
\]

where the \( \Psi \) and \( r^2 \) dependencies of the integrable function \( G = G(E, L^2) \) are only through the two integrals of motion \( E \) and \( L^2 \) (henceforth these trivial arguments of \( G \) will be suppressed for the sake of brevity), the Fubini theorem implies

\[
\begin{align*}
& \mathcal{J}_r(\Psi, r^2) = \int_{E,E_s,L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2) \mathcal{F}^{\lambda} \Theta(\mathcal{K}), \\
& \mathcal{J}_r(\Psi, r^2) = \int_{E,E_s,L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2) \mathcal{F}^{\lambda} \Theta(\mathcal{K}), \quad r \geq 2, \quad \lambda > 0,
\end{align*}
\]

Through direct calculations that are basically identical to that of Al 2014 appendix A except for different arguments of the Euler integral for the beta function, we find that

\[
\begin{align*}
& r \lambda \mathcal{J}_r(\Psi, r^2) = \frac{\lambda}{\Gamma(\lambda)} \int_{E, E_s, L^2 \geq 0} \mathcal{F}(E - E_s)^{\lambda - 1} \left[ \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2) \right]^\lambda, \\
& r \lambda \mathcal{J}_r(\Psi, r^2) = \frac{\lambda}{\Gamma(\lambda)} \int_{E, E_s, L^2 \geq 0} \mathcal{F}(E - E_s)^{\lambda - 1} \left[ \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2) \right]^\lambda, \quad r \geq 2, \quad \lambda > 0.
\end{align*}
\]

Hence, we have established that

\[
\begin{align*}
& \mathcal{J}_r(\Psi, r^2) = \frac{\Gamma(s + 1)}{2\Gamma(s + A + 1)} \int_{E, E_s, L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2), \\
& \mathcal{J}_r(\Psi, r^2) = \frac{\Gamma(s + 1)}{2\Gamma(s + A + 1)} \int_{E, E_s, L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2), \quad s > 0,
\end{align*}
\]

which are valid for any \( s > -1 \) and \( \lambda > 0 \), provided that all integrals on the right-hand sides converge.

We next find differentiations of the integral transform \( \mathcal{J}_r \), namely (here \( X = \Psi r^2 \))

\[
\frac{\partial \mathcal{J}_r}{\partial X} = \left\{ \begin{array}{ll}
\frac{1}{\xi} \int_{0}^{\xi} \mathcal{L} \mathcal{K} \mathcal{G}(\Psi - L^2/r^2, L^2) & (s > 0) \\
\frac{1}{\xi} \int_{0}^{\xi} \mathcal{L} \mathcal{K} \mathcal{G}(\Psi - L^2/r^2, L^2) & (s = 0)
\end{array} \right. .
\]

The \( \lambda \)-factor for the \( s = 0 \) case is due to

\[
\delta(\mathcal{K}) = \frac{1}{\lambda} \left[ \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(\Psi - L^2/r^2, E) \right]
\]

where \( \delta(x) = \Theta(x) \) is the Dirac delta. In addition,

\[
L^2 = \left\{ \begin{array}{ll}
2r^2 \Psi & \text{if } E_0 = 0 \\
2r^2 \Psi & \text{if } E_0 = -\infty
\end{array} \right.
\]

Given that

\[
\frac{\partial \mathcal{K}}{\partial \Psi^2} = 0 \quad \frac{\partial \mathcal{K}}{\partial r^2} = \frac{L^2}{r^4},
\]

equation (31) suggests that for an integer \( n \geq 0 \) and \( s > -1 \),

\[
\begin{align*}
& \frac{\partial^n \mathcal{J}_r}{\partial \Psi^n} = \left\{ \begin{array}{ll}
\frac{2^n(s)^n}{\Gamma(1 + s)} \int_{E, E_s, L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(\Psi - L^2/r^2, L^2) & (n > s + 1) \\
\frac{1}{\Gamma(1 + s)} \int_{E, E_s, L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(\Psi - L^2/r^2, L^2) & (n = s + 1)
\end{array} \right.,
\end{align*}
\]

Equations (30a, 32b) and \( N = n_0, n_0 \) expressed as an integral transformation of the df as in equation (31) result in

\[
\begin{align*}
& \frac{\partial^n \mathcal{J}_r}{\partial \Psi^n} = \left\{ \begin{array}{ll}
\frac{\Gamma(s + 1)}{\Gamma(s + A + 1)} \int_{E, E_s, L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2) & (n > s + 1) \\
\frac{\Gamma(s + 1)}{\Gamma(s + A + 1)} \int_{E, E_s, L^2 \geq 0} \mathcal{E} \mathcal{L} \mathcal{K} \mathcal{G}(E, L^2) & (n = s + 1)
\end{array} \right.,
\end{align*}
\]

where \( n \) is again a non-negative integer and \( \xi \geq \frac{1}{r^2} \). Both equations further generalize from an integer \( n \) to a real \( \mu \leq \xi \) using fractional order derivatives, and it can also be shown that they are in fact valid for \( \xi \geq 0 \) if the extended definition in equation (5) is adopted.
In particular, to generalize equation (33), we first find that

$$
+ \frac{1}{\varepsilon_0} \partial_{r^2} \left[ \int_0^r \int r \, d\sigma \, d\tau \, \frac{N}{r^{2\gamma - 1}} \right] = \frac{2\pi^2}{2} \frac{\mathcal{K}^{\xi - 1}}{r^{2\gamma - 1}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

(35)

for $\xi \geq \frac{1}{2}$ and $\lambda \geq 0$, which follows equation (30). The generalization of equation (33) is arrived by applying equation (32), that is, for any reals $0 \leq \mu \leq \xi$ and $\xi \geq \frac{1}{2}$ (the latter restriction that $\xi \geq \frac{1}{2}$ will be dropped later in this section),

$$
+ \frac{1}{\varepsilon_0} \partial_{r^2} \left[ \int_0^r \int r \, d\sigma \, d\tau \, \frac{N}{r^{2\gamma - 1}} \right] = \frac{2^{1+\gamma/2} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{K}^{\xi - 1}}{r^{2\gamma - 1}} \mathcal{F}(E, L^2) \quad (\mu < \xi)
$$

$$
+ \frac{2^{1+\gamma/2} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)}{r^{2\gamma - 1}} \quad (\mu = \xi)
$$

(36)

provided that the integrals converge. Equation (30) for $\xi = \frac{1}{2}$ now reduces to

$$
\rho^2 \frac{\partial}{\partial \rho} \mathcal{N} = \frac{2^{1+\gamma/2} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{K}^{\xi - 1}}{r^{2\gamma - 1}} \mathcal{F}(E, L^2) \quad (\mu < \xi)
$$

$$
\frac{\sqrt{2\pi} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)}{r^{2\gamma - 1}} \quad (\mu = \xi)
$$

(37)

Here setting $\mu = \frac{1}{2} - \xi$ results in equation (34) with $n = 0$ given that $+ \frac{1}{\varepsilon_0} \partial_{\rho} \mathcal{N} = \frac{\partial}{\partial \rho} \mathcal{N}$ is valid for any real $\xi \geq 0$ (n.b., $0 \leq n \leq \xi$ and so if $0 \leq \xi \leq \frac{1}{2}$, then $n = 0$).

A similar generalization of equation (34) from an integer $n$ to a real $\mu$ (cf., eq. (25) and the extension of equation (36) to $\xi \geq 0$ are possible although demonstrating them through direct calculations is comparatively nontrivial. Instead, we derive the generalization of equation (34) following an indirect route. Let us first consider combining equation (30) with $G = \mathcal{F}, \mu = s + 1 > 0$ and $\lambda = 1 - \delta$ where $\delta = \mu - \mu$, and equation (33) with $n = 0$ and $\xi = \mu > 0$ results in

$$
+ \frac{1}{\varepsilon_0} \partial_{r^2} \int_0^r \int r \, d\sigma \, d\tau \, \frac{N}{r^{2\gamma - 1}} = \frac{2^{1+\gamma/2} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{K}^{\xi - 1}}{r^{2\gamma - 1}} \mathcal{F}(E, L^2) \quad (\mu < \xi)
$$

(38)

for $\mu > 0$ and $0 < \delta < 1$. Next equation (32b) indicates that

$$
\left( \frac{r^2}{\partial \rho^2} \right)^{\mu+1} \frac{1}{2} \int_{r^2}^\infty d\sigma \, d\tau \, \left( \frac{r^2 \partial_\rho \mathcal{F}_{\xi \mu}}{\varepsilon_0} \right) = \frac{\pi^2}{2} \frac{1}{r^{2\gamma - 1}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

for a non-negative integer $n = \lfloor \mu \rfloor$. However,

$$
\left( \frac{r^2}{\partial \rho^2} \right)^{\mu+1} \frac{1}{2} \int_{r^2}^\infty d\sigma \, d\tau \, \left( \frac{r^2 \partial_\rho \mathcal{F}_{\xi \mu}}{\varepsilon_0} \right) = 2^{\lfloor \mu \rfloor + 1} \left( \frac{r^2 \partial_\rho \mathcal{F}_{\xi \mu}}{\varepsilon_0} \right)
$$

thanks to equation (28), and consequently, we find that

$$
+ \frac{1}{\varepsilon_0} \partial_{r^2} \left( \frac{r^2 \partial_\rho \mathcal{F}_{\xi \mu}}{\varepsilon_0} \right) = \frac{\pi^2}{2} \frac{1}{r^{2\gamma - 1}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

(38a)

This is also consistent with the case $n = \xi$ of equation (34), again thanks to equation (25). That is to say, equation (38a) is actually valid for any $\mu \geq 0$ including integer values.

Finally, consider applying the integral operator in equation (3) on $\Psi$ to equation (38a), as in

$$
+ \frac{1}{\varepsilon_0} \partial_{r^2} \left( \frac{r^2 \partial_\rho \mathcal{F}_{\xi \mu}}{\varepsilon_0} \right) = \frac{\pi^2}{2} \frac{1}{r^{2\gamma - 1}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

(38b)

It then follows that for $0 \leq \mu < \xi$

$$
\frac{\Gamma(\xi - \mu)}{(2\pi)^2} \frac{1}{\varepsilon_0} \partial_{r^2} \left( \frac{r^2 \partial_\rho \mathcal{F}_{\xi \mu}}{\varepsilon_0} \right) = \frac{1}{(2\pi)^2} \frac{1}{r^{2\gamma + 2}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

(38c)

Equations (38a) and (38c) together, that is,

$$
\frac{1}{\varepsilon_0} \partial_{r^2} \int_0^r \int r \, d\sigma \, d\tau \, \frac{N}{r^{2\gamma - 1}} = \frac{1}{(2\pi)^2} \frac{1}{r^{2\gamma + 2}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

(39)

constitute the generalization of equation (34) from an integer $n$ to a real $\mu$, which is valid for any pair of $\mu$ and $\xi$ with $0 \leq \mu \leq \xi$. For $0 \leq \mu \leq \xi \leq \frac{1}{2}$, the indices transform $(\mu, \xi) \rightarrow (\frac{1}{2} - \xi, 1 - \mu)$ sends equation (39) to (38c) given equation (5). Equations (36) and (39) thus are both valid for any real pair $\mu$ and $\xi$ with $0 \leq \mu \leq \xi$.

In fact, both results and also equation (35) are different manifestations of the same result, that is to say,

$$
\frac{1}{\varepsilon_0} \partial_{r^2} \int_0^r \int r \, d\sigma \, d\tau \, \frac{N}{r^{2\gamma - 1}}
$$

(40)

$$
\frac{1}{(2\pi)^2} \frac{1}{r^{2\gamma + 2}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

(41)

$$
= \frac{1}{(2\pi)^2} \frac{1}{r^{2\gamma + 2}} \int_{r^2}^\infty d\sigma \, d\tau \, \mathcal{F}(E, L^2)
$$

(42)

which are valid for any real pair $(\lambda, \xi)$ such that $\lambda + \xi + \frac{1}{2} \geq 0$. 
4. MOMENT SEQUENCES AND AUGMENTED DENSITIES

Consider the moment sequence of the df in \((E, L^2)\) space restricted along \(K = 0\), given as in

\[ F_\mu(\Psi, r^2) = \frac{(2\pi)^2}{(2\pi)^\nu} \int_0^L dL \int_0^L \int_0^L \mathcal{F}(\Psi - \frac{L^2}{2\pi^r}, L^2) \]  

(41a)

\[ = \left\{ \begin{array}{ll}
\frac{\mu^{\nu+1}}{2\pi^2} \nu_0 \eta_0 \mathcal{F}(\eta_0, \eta_0) & (E_0 = 0, L^2 = 2\pi^r) \\
0 & (E_0 = -\infty, L^2 = \infty)
\end{array} \right.
\]

where

\[ \mathcal{F}(Y; \Psi, r^2) = (2\pi)^2 \mathcal{F}(\Psi - Y, 2\pi^r)Y. \]  

(41b)

Then equations (36) and (37) indicate that

\[ F_\mu = \frac{1}{\xi_0} \int_0^ inf \int_0^{N(\Psi, r^2)} \mathcal{F}(\Psi, r^2) \frac{\partial N}{\partial \Psi} d\Psi \]  

(42)

In particular, if \(\mu\) is a positive integer, this results in

\[ F_0 = \frac{1}{\sqrt{\pi}} \int_0^ inf \int_0^{N(\Psi, r^2)} \frac{\partial \mathcal{F}}{\partial \Psi} \frac{d\Psi}{\sqrt{\pi - Q}} \]  

where \(n = 1, 2, \ldots\). That is to say, a set of fractional calculus chains the AD directly determine the entire moment sequences along a fixed sectional line in \((E, L^2)\) space. In other words, the AD is similar to the moment generating function (or the characteristic function) for the df as a probability density. With varying \((\Psi, r^2)\), the \(K = 0\) lines eventually sweep the whole accessible \((E, L^2)\) space, and thus \(N(\Psi, r^2)\) in principle uniquely determine the two-integral df, \(f(E, L^2)\). A few explicit inversion algorithms from \(N(\Psi, r^2)\) to \(f(E, L^2)\) are already available in the literature utilizing either the known inverse of named integral transforms (see e.g., Lynden-Bell 1962; Dejonghe 1986; Baes & Van Hese 2007) or complex contour integrals (see e.g., Hunter & Qian 1993; An 2011a).

Since the definition of the AD in equation (11) provides the explicit formula from \(f(E, L^2)\) to \(N(\Psi, r^2)\), the knowledge of \(N(\Psi, r^2)\) is therefore mathematically equivalent to knowing \(f(E, L^2)\). Once the potential \(\Psi = \Psi(r)\) is specified, the specification of the AD thus completely determine a unique spherical dynamic system in equilibrium. Although this approach to the df \(f(E, L^2)\) through the AD \(N(\Psi, r^2)\) is advantageous as the observables constrain the AD more directly than the df, this procedure suffers a significant drawback in that the df recovered as such is indeed, that is, non-negative everywhere in all accessible subvolume of the phase-space – the “phase-space consistency”, which is the subject of the reminder of this paper following the current chapter.

Next, we consider what information on the physical properties of the system is sufficient to specify a unique AD. First, we find from equation (39) that the (augmented) velocity moments of the even orders are related to the AD as in

\[ m_{k,n}(\Psi, r^2) = \frac{2^{k+n}G(k + 1 + \nu)}{\sqrt{\pi}^{2^{k+n}+2}} \left( \frac{\partial}{\partial r^2} \right)^n \left( \frac{\partial^2}{\partial \Psi^2} \right)^{k+n} \mathcal{F}(\Psi, r^2) \]

\[ = 2^{k+n} \left( \frac{1}{\nu} \right)^n \int_0^ inf \int_0^{N(\Psi, r^2)} \frac{d\Psi}{\sqrt{\pi - Q}} \left( \frac{d}{dr} \right)^n \left( \frac{d}{dr} \right)^n \mathcal{F}(\Psi, r^2) \]  

(43)

Here note that \(\left( \frac{d}{dr} \right)^n \mathcal{F}(\Psi, r^2) = \Gamma(k + 1 + \nu) / \sqrt{\pi}\). This is basically equation (13) of Dejonghe & Merritt (1992) – see also equation (8) of Baes & Van Hese (2007), equation (A2) of Van Hese et al. (2009), equation (5c) of An (2011b) and so on. Equation (43) indicates that, given potential \(\Psi(r)\), specifying the AD completely fixes every (in principle observable) non-vanishing velocity moment such that

\[ m_{k,n}(\Psi, r^2) = \frac{m_{k,n}(\Psi, r^2)}{N(\Psi, r^2)} \]  

Conversely, equation (45) for \((k, n) = (\mu + 1, 0)\), that is, \(m_{\mu+1,0} = 2^{\mu+1} \int_0^{N(\Psi, r^2)} \mathcal{F}(\Psi, r^2)\), at a fixed \(r\) reduces to

\[ M_{\mu}(r) = \frac{\mu^{\nu+1}}{2\pi^{2\mu+1}} \int_0^{N(\Psi, r^2)} \mathcal{F}(\Psi, r^2) \]  

(44a)

In other words, given the knowns of the local density \(\nu(r)\) and the potential \(\Psi(r)\), the infinite set of the radial velocity moments in every order consists in the moment sequence of the AD considered as a distribution of \(\nu - \) over the compact support if \(\xi_0 = 0\) or the half-open interval \([0, \infty)\) if \(\xi_0 = -\infty\) at fixed \(r\). The problem is closely related to the Haudsolf (1962) for \(\xi_0 = 0\) or the Stieltjes (1912) for \(\xi_0 = -\infty\) moment problem. With the infinite sequence of the radial velocity moments as functions of \(r\), the AD can then be uniquely determined at least formally by such means as e.g., the Hilbert basis or the Laplace and/or Fourier transform (cf., the moment generating function and the characteristic function) etc.

The final information required for the full specification of the system is then the determination of the potential. Clearly the potential may be determined through the Poisson equation \(\nabla^2 \Phi = 4\pi G \rho\), which under the spherical symmetry reduces to

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = -4\pi G \rho (r) \]  

(45)

Hence if \(\Gamma \equiv \rho (r) / \nu (r)\) is assumed to be constant, \(\Psi(r)\) can be fixed by solving the ordinary differential equation on \(\Psi(r)\) that results from setting \(\nu = N(\Psi, r^2)\) in equation (45). Alternatively, from equation (43), we deduce for \(k \geq 1\) that

\[ \frac{\partial m_{k,n}(\Psi, r^2)}{\partial \Psi} = (2k - 1) m_{k-1,n}; \]

\[ \frac{\partial (\nu^{2n+1} m_{k,n})}{\partial r^2} = (k - \frac{1}{2}) \nu \int_0^{N(\Psi, r^2)} \frac{d\Psi}{\sqrt{\pi - Q}} \]  

(46a)

19 Felix Hausdorff (1868-1942)
20 Thomas Joannes Stieltjes (1856-1894)
21 David Hilbert (1862-1943)
22 Jean Baptiste Joseph Fourier (1768-1830)
The total radial derivative of $m_{k,n}$ for $k \geq 1$ then results in
\[
\frac{d m_{k,n}}{dr} = \frac{2 m_{k,n}}{r} \left( \frac{\partial \log(r^{2n+2} m_{k,n})}{\partial \log r^2} - (n + 1) \right) + \frac{d \Psi}{dr} \frac{d m_{k,n}}{d \Psi}.
\]

With $\Psi = \Psi(r)$ and $m_{k,n}[\Psi(r), r^2] = \frac{\partial r^2}{\partial r_0}$, this may be solved for $\frac{d \Psi}{dr}$ if the required velocity moments as a function of $r$ are known. For the simplest case $(k, n) = (1, 0)$, this reduces to the spherical (second-order steady-state) Jeans equation,
\[
1 \frac{d}{dr} \left( \frac{\nu v_r^2}{v_r^2} + \frac{2 v_r^2 - v_r^2}{r} \right) = \frac{d \Psi}{dr},
\]
that is, the spherically-symmetric hydrostatic equilibrium equation with an anisotropic velocity dispersion tensor.

5. NECESSARY CONDITION FOR SEPARABLE AUGMENTED DENSITIES

In the following, we limit our concern to the cases for which the potential and the radius dependencies of the AD are multiplicatively separable such that
\[
N(\Psi, r^2) = P(\Psi)R(r^2).
\]

In addition to mathematical expediency, this assumption is also notable because under the separability assumption in equation (48), the radius part $R(r)$ of the AD alone can uniquely specify the so-called Binney anisotropy parameter,
\[
\beta(r) = 1 - \frac{\nu v_r^2}{2 v_r^2} = 1 - \frac{m_{0,1}[\Psi(r), r^2]}{2 m_{1,0}[\Psi(r), r^2]},
\]

such that Dejonghe [1986], Ojan & Hunter [1995], Baes & Van Hese [2007], An [2011b], see also van der Marel [1994] as $R^{-1}$ being the integrating factor of the Jeans equation, i.e., eq. (47)
\[
\beta(r) = \frac{-d \log R(r^2)}{d \log r^2}, 
\]

Some applications are found e.g., in Baes & Van Hese [2007] while An [2011b] discusses further implications of the separability assumption.

5.1. The radius part

With a separable AD given by equation (48), equation (59) indicates that (hereafter $x \equiv r^2$),
\[
\frac{\partial}{\partial x} [x^2 \partial^2 \frac{\xi}{\partial x^2} N] = \frac{\partial}{\partial x} \frac{\xi}{\partial x} P(\Psi) \frac{\partial^2}{\partial x^2} [x^2 R(x)] \geq 0
\]
for $\mu \leq \xi$ whereas $\frac{\partial}{\partial x} \frac{\xi}{\partial x} P > 0$ for $\xi \geq \frac{1}{2}$. Therefore,
\[
\frac{\partial}{\partial x} \frac{\xi}{\partial x} (x^2 R) \geq 0 \quad (x > 0, \mu \geq 0).
\]

This is actually equivalent to the condition
\[
R_{(n)}(x) \equiv \frac{d^n [x^2 R(x)]}{dx^n} \geq 0 \quad (x > 0, n = 0, 1, 2, \ldots),
\]
which is necessary for the corresponding df to be non-negative as noted by An [2011b]. It is clear that equation (52) implies equation (53) as the latter is a restriction of the former for an integer $\mu = n$. The opposite implication follows Corollary [2.14]; equation (53) for a positive integer $n$ implies equation (52) for $\mu \in \{n - 1, n\}$ and thus equation (52) for $\mu \geq 0$ follows equation (53) for all positive integers $n$.

We find another equivalent necessary condition, such that (Dejonghe 1986; Qian & Hunter 1995; An 2011b), see also van der Marel [1994] as $R^{-1}$ being the integrating factor of the Jeans equation, i.e., eq. (47)
\[
\beta(r) = \frac{-d \log R(r^2)}{d \log r^2},
\]

Some applications are found e.g., in Baes & Van Hese [2007] while An [2011b] discusses further implications of the separability assumption.

5.2. The potential part

Van Hese et al. (2011a) proved that, given equation (48),
\[
P^{(k)}(\Psi) \geq 0 \quad (k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1)
\]
where $\beta_0$ is the limit of the anisotropy parameter at the center, is necessary for the df to be non-negative. We shall show that this generalizes incorporating fractional derivatives.

First, we generalize the result of An [2011b] to include arbitrary real order derivatives. This is trivial since the inverse Abel transform is just a particular fractional derivative as defined in equation (2). If the AD is given as equation (48), equation (56) reduces to
\[
\frac{\partial}{\partial x} \frac{\xi}{\partial x} \left( \frac{R(x)}{x^{\xi - 1/2}} \right) \geq 0 \quad (x > 0)
\]
for $\mu \leq \xi$. Since $R(x) \geq 0$ is trivially necessary, $I_1^{j\beta}(x^{-1}R) > 0$ for $x > 0$ and any $\lambda \geq 0$ unless $R(x) = 0$ almost everywhere in $x \equiv x^2 \in [0, \infty)$ (Lemma 2.5), which will not be considered here. Consequently, equation (61) implies that

$$0 < \frac{\int_0^1 \frac{R}{x}}{x} < \infty \quad \Rightarrow \quad \frac{\partial \partial^n}{\partial P} \geq 0 \quad (\mu \leq \lambda + \frac{1}{2}). \quad (62)$$

With $A = 0$, this indicates that $\frac{\partial \partial^n}{\partial P} \geq 0$ for any $\mu \leq \frac{1}{2}$. The condition for $\mu \leq 0$ is trivial because $\frac{\partial \partial^n}{\partial P} = + \frac{1}{\partial a\partial} P$ while $P(\Psi) \geq 0$. For $\lambda < 0$ on the other hand, equation (62) implies that, if $x^{-\beta}R(x)$ dx is integrable over $x = \infty$, then $\frac{\partial \partial^n}{\partial P} \geq 0$ for any $\mu \leq \lambda + \frac{1}{2}$ and all accessible $\Psi$ is necessary for the existence of a non-negative df. Alternatively, for a fixed $\mu \geq \beta$, equation (62) suggests that $\frac{\partial \partial^n}{\partial P} \geq 0$ is necessary for the df to be non-negative if there exists $\lambda \geq \mu - \frac{1}{2}$ such that $\frac{\partial \partial^n}{\partial P}(x^{-\beta}R)$ is well-defined.

Equation (62) however is inconclusive whether $\frac{\partial \partial^n}{\partial P} \geq 0$ is necessary for a non-negative df given $R(x) \sim x^{-\beta}$ with $\beta < 1$ as $x \to 0$ while this is necessary if we were to extend the result of Van Hese et al. (2011). For this, we first note that if $f(t)$ is right-continuous at $t = a$,

$$\lim_{\varepsilon \to 0^+} \int_a^b h'(t) (t-a)^{-\varepsilon} dt = \lim_{\varepsilon \to 0^+} h(t) = h(a) \quad (a < b). \quad (63)$$

This applied to the left-hand side of equation (36) reduces to

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon - (\xi - \eta - \varepsilon)} \frac{1}{\partial \partial n}(N x^{\beta-1/2}) = \frac{\partial \partial^n}{\partial P}(1 - \eta) \quad (64a)$$

where $\eta < 1$ and $\partial \partial^n = \lim_{x \to 0} x^\beta P(\Psi, x)$. (64b)

Equation (36) overall then results in

$$\frac{\partial \partial^n}{\partial P} \partial \partial^n = 2 \frac{\beta - \eta}{\partial a\partial} \Gamma(1 - \eta) \frac{\partial \partial^n}{\partial P} \partial \partial^n \geq 0, \quad (64c)$$

where

$$\partial \partial^n = \lim_{x \to 0} x^\beta P(\Psi, x). \quad (64d)$$

For $\mu < \frac{3}{2} - \eta$, this is derived with the limit $\varepsilon \to (\xi - \eta)^-$ while maintaining $\mu < \xi < \frac{3}{2} - \eta$. For $\mu = \frac{1}{2} - \eta$ on the other hand, the same limit is taken with $\varepsilon = \xi$. Therefore, this is valid for $\mu \leq \frac{1}{2} - \eta$ and $\eta < 1$, provided that $\frac{\partial \partial^n}{\partial P} \partial \partial^n (x^{\beta-1/2})$ is well-defined for $\xi < \frac{3}{2} - \eta$ (n.b., the integrability of the same for $\varepsilon = \frac{1}{2} - \eta$ is actually required for its validity). The non-negativity of equation (64c) follows the non-negativity of $\partial \partial^n (L^2)$. Of particular interest are equation (64c) for $\mu = 0$ and $\frac{1}{2} - \eta$,

$$\partial \partial^n = 2 \frac{\beta - \eta}{\partial a\partial} \Gamma(1 - \eta) \frac{\partial \partial^n}{\partial P} \partial \partial^n \geq 0, \quad (65a)$$

$$\partial \partial^n = \frac{\partial \partial^n}{\partial P} \partial \partial^n \Gamma(1 - \eta), \quad (65b)$$

that is, explicit formulae for $\partial \partial^n(\Psi)$ and $\partial \partial^n(\Psi)$ from each other.

For a separable AD given as in equation (48), we have,

$$\partial \partial^n(\Psi) = R_\Psi(\Psi) \quad \Rightarrow \quad \partial \partial^n = \lim_{x \to 0^+} x^\beta R(x), \quad (66)$$

Therefore, equation (64c) indicates that

$$0 < \partial \partial^n < \infty \quad \Rightarrow \quad \frac{\partial \partial^n}{\partial P} \partial \partial^n \geq 0 \quad (\mu \leq \frac{1}{2} - \eta). \quad (67)$$

That is to say, if there exists $\eta < 1$ such that $\partial \partial^n(\Psi)$ is a (non-zero) positive finite constant, then $\frac{\partial \partial^n}{\partial P} \partial \partial^n \geq 0$ for any $\mu \leq \frac{1}{2} - \eta$. This actually encompasses equation (62), which is seen as follows: If $\partial \partial^n(\Psi)$ is non-zero finite for $\eta < 1$, then we basically find that $R \sim x^{-\eta}$ as $x \to 0$. Hence $\int_0^1 \frac{R}{x}$ converges for $\lambda < 1 - \eta$, and so if $\mu \leq \lambda + \frac{1}{2}$, then $\mu < \frac{3}{2} - \eta$.

For example, with a constant anisotropy system given by

$$R(x) = x^{-\beta}, \quad \partial \partial^n = 1 \quad (68)$$

the convergence condition reduces to

$$\int_0^1 \frac{R}{x} = \frac{1}{\Gamma(1 - \lambda - \eta)} \quad (69)$$

which converges if $0 < \lambda < 1 - \beta$. It follows that equation (62) indicates that $\frac{\partial \partial^n}{\partial P}(\Psi) \geq 0$ for $\mu \leq \lambda + \frac{1}{2} < \frac{3}{2} - \beta$ is necessary for the df to be non-negative whereas equation (64c) suggests the same for $\mu \leq \frac{3}{2} - \beta$.

6. SUFFICIENT CONDITIONS FOR PHASE-SPACE CONSISTENCY IN TERMS OF SEPARABLE AUGMENTED DENSITIES

Recently, Van Hese et al. (2012) derived the necessary and sufficient condition for the df with $\xi_0 = 0$ to be non-negative, expressed in terms of the integro-differential constraints on the AD. They achieved this by reducing the problem to the Hausdorff moment problem, according to which the df is non-negative if and only if the moment sequence in equation (41a) is a completely monotone sequence $\mathcal{C}$. Since the moment sequence are generated by the AD using equation (42), the monotone sequence condition is expressible in terms of finite differences of integro-differential operations on the AD.

With a separable AD, they have derived a simpler sufficient (but not necessary) condition given as a union of conditions, each of which only involves the potential or the radius part separately but not together. Here we derive an alternative sufficient condition for a separable AD to be resulted from a non-negative df, which turns out to be equivalent to that of Van Hese et al. (2012). The derivation here is based on the properties of cm functions and also uses the Laplace transform extensively. In this section, we only consider the case that $\xi_0 = 0$ and $L^2 = 2 \partial^2 \Psi$, that is, the df has a compact support and $\partial^2(\mathcal{E} < 0, L^2) = 0$.

6.1. Inversion of a separable augmented density for the distribution function

As it has been shown by Hunter & Oiar (1993), see also An (2011a), inverting equation (23) for $\partial^2(\mathcal{E}, L^2)$ is formally equivalent to recovering the two-integral even df, $\partial^2(\mathcal{E}, J^2)$ from the axisymmetric density $\partial^2(\mathcal{E}, L^2)$. The findings of the preceding section together with the inversion of Lyden-Bell (1962) who utilized the Laplace transform for the latter problem suggest that the function $\partial(t)$ defined by equation (58) must be directly related to the underlying df, $\partial^2(\mathcal{E}, L^2)$. We investigate this connection in the following.
Following Lynden-Bell (1962), we apply the Laplace transform on \( \Psi \) to equation (25):

\[
\mathcal{L}\left[ \Psi(L, r^2) \right] = \int_0^\infty d\Psi e^{-x\Psi} N(\Psi, r^2) = \frac{2\pi}{r^2} \int_{E < 0, L^2 \geq 0} dE dL^2 \mathcal{F}(E, L^2) \int_0^\infty d\Psi e^{-x\Psi} \frac{\Theta(\mathcal{K})}{\sqrt{\mathcal{K}}}.
\]

(69a)

The inner integral in the right-hand side reduces to

\[
\int_0^\infty d\Psi e^{-x\Psi} \frac{\Theta(\mathcal{K})}{\sqrt{\mathcal{K}}} = \exp\left(-8 \mathcal{E} - \frac{s L^2}{2 \mathcal{E}} \right) \int_0^\infty d\mathcal{K} \frac{2}{\sqrt{\mathcal{E}}} e^{-4\sqrt{\mathcal{K}}/\mathcal{E}} = \sqrt{\frac{r}{2\mathcal{E}}} e^{-4\mathcal{E}/\mathcal{E}} e^{-s^2/4}.
\]

(69b)

and consequently we find that

\[
\mathcal{L}\left[ \Psi \right] = \sqrt{\frac{2\pi}{s^2 r^2}} \int_0^\infty dL^2 e^{-4\sqrt{L^2}/s} \int_0^\infty d\mathcal{E} e^{-s\mathcal{E}} \mathcal{F}(E, L^2).
\]

(69c)

Substituting variables, \( t = \frac{1}{2}sL^2 \) and \( w = r^2 \), this reduces to

\[
\left( \frac{s}{2\pi} \right)^{\frac{1}{2}} \mathcal{L}\left[ \frac{\Psi(w^{-1})}{w} \right] = \mathcal{L}\left[ \int_0^\infty d\Psi e^{-s\mathcal{E}} \mathcal{F}(E, \frac{2t}{s}). \right]
\]

(70)

If the AD is separable (eq. 48), then \( w^{-1}\Psi(w^{-1}) = P(\Psi) \mathcal{R}(w) \) where \( \mathcal{R}(w) \) is as defined in equation (55) and so the left-hand side becomes

\[
\frac{s^2 P(s)}{(2\pi)^2} \mathcal{R}(w) = \mathcal{L}\left[ \frac{s^2 P(s)}{(2\pi)^2} \phi(t) \right].
\]

(71)

Here \( P(s) \) is the Laplace transformation of \( P(\Psi) \),

\[
P(s) \equiv \int_0^\infty d\Psi e^{-s \Psi} P(\Psi).
\]

(72)

We have also used \( \mathcal{R}(w) = \mathcal{L}\left[ \phi(t) \right] \). Given that the inverse Laplace transformation is unique, equating the right-hand sides of equations (70) and (71) results in

\[
\frac{s^2 P(s)}{(2\pi)^2} \phi(t) = \mathcal{L}\left[ \frac{s^2 P(s)}{(2\pi)^2} \mathcal{F}(E, L^2) \right].
\]

(73a)

Finally reinventing \( t = \frac{1}{2}sL^2 \) leads to

\[
\frac{s^2 P(s)}{(2\pi)^2} \phi\left( \frac{s L^2}{2} \right) = \mathcal{L}\left[ \frac{s^2 P(s)}{(2\pi)^2} \mathcal{F}(E, L^2) \right].
\]

(73b)

The df is then recovered via the inverse Laplace transform,

\[
\mathcal{F}(E, L^2) = \mathcal{L}^{-1}\left[ \frac{s^2 P(s)}{(2\pi)^2} \phi\left( \frac{s L^2}{2} \right) \right].
\]

(74)

6.2. Sufficient condition on a separable augmented density

According to the Bernstein theorem, the df in equation (74) is non-negative if and only if the left-hand side of equation (73b) is a cm function of \( s > 0 \) for all accessible values of \( L^2 \). However \( P(s) \) defined in equation (72) is already cm since \( P(\Psi) \geq 0 \). Hence Lemma 2.10 suggests that \( s^2 \phi(sL^2/2) \) is a cm function of \( s > 0 \) for any \( L^2 \geq 0 \) in fact a sufficient condition for the non-negativity of the df. Equivalently, since

\[
\lim_{t \to \infty} \left[t^{\gamma-1} \phi(t)\right] = \left[\frac{L^2}{2}\right]^{\gamma-1} \left[\frac{d^m}{dE^m}\right] \left[ s^2 \phi\left( \frac{s L^2}{2} \right) \right].
\]

(75)

the condition is also equivalent to that \( t^{\gamma} \phi(t) \) is cm. Unfortunately, this is too severe to be physically relevant, which may be inferred from the constant anisotropy model given by equation (68). With this model, we find for \( \beta < 1 \)

\[
R(\omega)(x) = \left(1 - \beta\right)^{\frac{x}{\lambda}}; \quad \phi(t) = \frac{1}{\rho^p(1 - \beta)}
\]

(76)

where we have used

\[
\lim_{n \to \infty} \left[\frac{(1 + z)^n}{n!} \right] = \frac{1}{\Gamma(1 + z)}
\]

(77)

to find \( \phi(t) \) using equation (58). The condition thus reduces to

\[
(-1)^n \frac{d^n}{dx^n}[x^{\beta-1}P(x)] \geq 0 \quad (t > 0, n = 0, 1, 2, \ldots),
\]

(79)

for \( t > 0 \) and all non-negative integers \( n \), which cannot be satisfied for any constant \( \beta < 1 \).

Nevertheless, the preceding discussion extends to yield useful sufficient conditions. That is, for any fixed \( \lambda \), the conditions that

\[
(-1)^n \frac{d^n}{dx^n}[x^{\beta-1}P(x)] \geq 0 \quad (s > 0, n = 0, 1, 2, \ldots),
\]

(79)

are jointly sufficient to imply equation (73b) being cm and consequently the non-negativity of the df. With increasing \( \lambda \), the constraint in equation (79) tightens whereas the condition in equation (80) becomes strictly weaker. In other words, with a larger \( \lambda \), the smaller subset of functions \( P(\Psi) \) will lead to \( s^2 \mathcal{P}(s) \) being cm. At the same time if \( \phi(t) \) satisfies equation (80) for a fixed \( \lambda = \lambda_0 \), the same condition for any larger \( \lambda \geq \lambda_0 \) automatically holds. Both of these are easily inferred from Corollary 2.11.

6.2.1. The condition on \( R(x) \) equivalent to eq. (80)

Both conditions can also be translated into the direct constraints on the behaviors of \( P(\Psi) \) and \( R(r^2) \). For the radius part, we use that \( \phi(t) \) may be given by equation (58). Note that the existence of \( \phi(t) \) and the validity of equation (58) as well as its non-negativity, that is, \( \phi(t) \geq 0 \) for \( t > 0 \) are all necessary. Substituting equation (58) into the left-hand side of equation (80) results in

\[
(-1)^n \frac{d^n}{dx^n}[x^{\beta-1}R(x)] \geq 0, \quad \lambda = \lambda_0.
\]

(81)

Consequently, provided that this limit converges, equation (80) is equivalent to insisting that there exists a sufficiently large integer \( m \) such that, for all integers \( k \geq 3m \),

\[
(-1)^n \frac{d^n}{dx^n}[x^{\beta-1}R(x)] \geq 0, \quad \lambda = \lambda_0.
\]

(82)

24 If the Laplace transform of \( \phi(t) \) exists, then \( \phi(t) \) cannot diverges faster than \( t^{-1} \) as \( t \to 0 \). Consequently, \( \lim_{t \to 0} t^{\gamma} \phi(t) \to 0 \) and thus \( t^{\gamma} \phi(t) \) cannot be cm because the limit suggests that \( t^{\gamma} \phi(t) \) should be negative or increasing in some interval \( t \in (0, \epsilon_0) \) where \( \epsilon_0 > 0 \).
That is to say, $x^{1/3 - j}R_{R_0}(x)$ being cm for all sufficiently large integers $k$ is necessary and sufficient for $\phi(t)$ derived from the same $R(x)$ to satisfy equation (80), provided that the limit converges. In fact, equation (80) is equivalent to equation (82) for not only all sufficiently large integers but also all non-negative integers $k$, thanks to Theorem 2.15 which indicates that $x^{1/3 - j}R_{m+1}(x)$ being cm implies $x^{1/3 - j}R_{m}(x)$ being also cm. Successive arguments with descending $k$ then establish that $x^{1/3 - j}R_{R_0}(x)$ being cm for all sufficiently large integers $k$ implies that $x^{1/3 - j}R_{R_0}(x)$ is a cm function for all non-negative integers $k$ (the opposite implication is trivial). Note that the condition as stated in equation (85) for all non-negative integers $k$ has already been noted by Van Hese et al. (2012).

6.2.2. The condition on $P(\Psi)$ equivalent to eq. (79)

The explicit constraints on $P(\Psi)$ resulted from equation (79) are derived by means of fractional calculus. We first find, from equations (16) and (17), that (n.b., $\frac{d^j}{df} \Psi(0) = 0$ for $\xi > 0$ from Corollary 2.)

$$s^j P(s) = \sum_{j=0}^{\infty} \lambda^j P(\lambda)$$

$$= \frac{1}{s^{j+1}} P(\Psi) = \int_0^1 \frac{e^{-s P(\Psi)}}{s^{j+1}} d\Psi$$

where $\mu = [\lambda]$ and $0 \leq \lambda = 1 - \mu < 1$. This indicates that

$$\frac{d^j}{df} \Psi(0) = 0 \quad \Psi > 0$$

$$\frac{d^{j-1}}{df} \Psi(0) = 0 \quad \Psi > 0$$

$$\frac{d^j}{df} \Psi(0) = 0$$

for $\lambda \geq 0$ is a sufficient condition for $s^j P(s)$ to be cm. Note, provided that $P(\Psi)$ is right-continuous at $\Psi = 0$, that $\frac{d^j}{df} P(0) = 0$ (cf., eq. (27)), which will thus be taken as granted. Consequently, equation (85) for $0 \leq \lambda < 1$ is essentially an empty condition. For $\lambda = 1$, equation (85) reduces to $P(\Psi) > 0$. For a positive integer $\lambda = m + 1$ on the other hand, the condition is equivalent to

$$P(m+1) = 0 \quad \text{and} \quad P(0) = \cdots = P(0) = 0.$$  

For $0 < \lambda < 1$, the condition in equation (85) may also be replaced with the same boundary condition as equation (86).

In particular, thanks to Lemma 2.16, $P(0) = \cdots = P(0) = 0$ implies $\frac{d^j}{df} P(0) = 0$ for $0 < \lambda < 1$. Consequently, it follows that for $\lambda \geq 1$,

$$P(0) = \cdots = P(0) = 0$$

actually implies equations (85) if $\delta = 0$, they are identical. Therefore, equations (84) and (85) together also exist in a sufficient condition for $s^j P(s)$ to be cm at a fixed $\lambda$. The condition expressed with equation (85) is useful because equation (11) then indicates that equation (84) is equivalent to

$$\frac{d^j}{df} \Psi(0) = \frac{1}{(1-\delta)} \frac{1}{(1-\delta)^{1/3-\lambda}} \Psi(0) \geq 0$$

where $n$ is any non-negative integer greater than $\lambda$.

Again, the joint condition of equations (84) and (85) becomes strictly stronger as $\lambda$ increases in accordance with the rejection on the complete monotonicity of $s^j P(s)$. This is seen using equation (12) for $0 \leq \lambda \leq \lambda$ under the condition of equation (85) or (85a),

$$\frac{d^j}{df} \Psi(0) = \frac{1}{(1-\delta)} \frac{1}{(1-\delta)^{1/3-\lambda}} \Psi(0) \geq 0$$

That is, $\frac{d^j}{df} P(\Psi) > 0$ implies $\frac{d^j}{df} P(\Psi) > 0$ for $0 \leq \xi \leq 1$. The similar implications of equation (85) with descending $\lambda$ are trivial.

6.3. The constant anisotropy model

As an illustrative example, let us consider the constant anisotropy model with $\beta < 1$ (see Appendix B for the $\beta = 1$ case) given by in equation (65). Given equation (76), equations (80) and (82) now reduce to

$$(-1)^n \frac{d^n \Psi(\beta)}{d\beta^n} = \frac{1}{(1-\beta)^{1/3-n}} \geq 0;$$

$$(-1)^n \frac{d^n \Psi(\beta)}{d\beta^n} = \frac{1}{(1-\beta)^{1/3-n}} \geq 0.$$

Thus $\beta + \lambda \geq \lambda$ and $\beta < 1$ is sufficient for these to be satisfied. If $\lambda = m + 1$ is a positive integer and $\beta \geq \lambda - m$, then equation (86) is sufficient for the existence of a non-negative df (cf., Cotti & Morganti 2011). Our result furthermore implies for any real $\beta > 1$ that if $\beta - \lambda < 1$, then equations (84) and (85a) constitutes a sufficient condition.

With a fixed $\beta < 1$, this indicates that, if there exists $\lambda \geq 1 - \beta$ such that equations (84) and (85a) hold, then the corresponding $P(\Psi)$ guarantees the phase-space consistency. This also implies $\frac{d^j}{df} P(0) = 0$ for $0 \leq \xi \leq 3$ while Sect. 5.2 indicates that, for the same system, $\frac{d^j}{df} P(0) = 0$ for $0 < \mu < 1$. Thus $\frac{d^j}{df} P(0) = 0$ is the necessary and sufficient condition for a non-negative df. In fact, here $P(\Psi) = P_0(\Psi)$ and $F(E,E^2) = \frac{\alpha(E)}{E^2}$ where $P_0(\Psi)$ and $\alpha(E)$ are as defined in equations (64) and (65c) with $\eta = \beta$, and so equation (65) results in the inversion formula,

$$F(E,E^2) = \frac{\alpha(E)}{E^2} \geq 0.$$  

This is simply the generalized Eddington inversion formula (see e.g., Evans & An 2006) for constant anisotropy systems. That $\frac{d^j}{df} P(\Psi) > 0$ is necessary and sufficient for the existence of a non-negative df is a trivial consequence of the inversion formula.

7. FAMILY OF MONOTONIC ANISOTROPY PARAMETERS

Consider the anisotropy parameterized to be (c.f., Baes & Van Hese 2007, An 2011b),

$$\beta(r) = \frac{\beta_1 r^{2z} + \beta_2 r^{2z}}{r^{2z} + r^{2z}} \geq 0.$$  

If the spherical system is characterized by a separable AD as in equation (43), this follows the radial function (see eq. 50).

$$R(x) = x^{\beta_1} (1 + x^{2z})^{-\xi} \quad \text{where} \quad s \xi = \beta_2 - \beta_1;$$

$$R(u) = u^{\beta_1} (1 + u^{2z})^{-\xi} \quad \text{where} \quad s \xi = \beta_2 - \beta_1.$$  

Sir Arthur Stanley Eddington (1882-1944)
where \( x = r^2/R^2 \) (i.e., \( r_a = 1 \)), which does not affect the following discussion. Note \( R_{11}(x) \geq 0 \) for \( x > 0 \) restricts \( \beta_1, \beta_2 \leq 1 \). [2011b] also provides an elementary proof that if \( 0 < s \leq 1 \) and \( \beta_1, \beta_2 \leq 1 \), equation (90b) satisfies equation (53). The same is deduced from the complete monotonicity of \( R(w) \) for \( 0 < s \leq 1 \) and \( \beta_1, \beta_2 \leq 1 \) (Corollary 2.12), too.

The situation for \( s > 1 \) however is inconclusive: on one hand, if \( \beta_2 = 1 > \beta_1 \), it is easy to show that \( R''(w) < 0 \) for \( 0 < w < (s-1)/2(1-\beta_1) \) and so the condition fails for \( s > 1 \) whereas [2011b] on the other hand has found that the condition is met for all \( s > 0 \) if \( \zeta = (\beta_1 - \beta_2)/s \) is non-negative integer. It appears that for a fixed \( s > 1 \), there exists a proper subset of parameter combinations \( \beta_1, \beta_2 \leq 1 \) that satisfies the necessary condition of equation (53), but we have not been able to establish the concrete criteria. The necessary condition on the potential part discussed in Sect. 5.2 on the other hand is straightforward. That is, given \( R(x) \) of equation (90b), the potential part \( P(\Psi) \) must satisfy \( \frac{\partial E}{\partial x} \Psi = 0 \) for any \( \lambda \leq \frac{1}{2} \) in order for the df to be non-negative. Here also note \( \beta_1 \leq 1 \) and thus \( \frac{\partial E}{\partial x} \Psi > 0 \) for any \( \lambda \leq \frac{1}{2} \).

For \( 0 \leq x < 1 \), the binomial expansion of equation (90b) and the subsequent term-by-term differentiation indicate that

\[
R(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\zeta)^k}{k!} x^{k-\beta_1};
\]

\[
R(0)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\zeta)^k}{k!} (1 - \beta_1 + sk)^+ x^{k-\beta_1}.
\]

It follows equations (53) and (77) that

\[
\phi(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (\zeta)^k}{k!} \lim_{n \to \infty} \frac{(1 - \beta_1 + sk)^+}{n! n^{\beta_1 - 1}} = \Gamma(1 - \beta_1 + sk) \Gamma(1 - \beta_1),
\]

where \( E_{p,b}(z) \) is the extended generalization of the Mittag-Leffler function \( E_p(z) \) introduced by [1977], see also [2011]. Although the derivation here is essentially formal (see Appendix A for proper treatments) as it that we have not properly considered the issue of the convergence, the result is in fact valid given that \( \beta_1 < 1 \) (for \( \beta_1 = 1 \), see Appendix B) as is found in equation (100). Next we briefly detour to examine properties of generalized Mittag-Leffler functions necessary to derive sufficient conditions in Sect. 6 for the phase-part given that the radial part is given by equation (90b).

### 7.1. Generalized Mittag-Leffler function

**Definition 7.1** Let us consider a particular generalized hypergeometric function defined to be

\[
E_{p,b}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)^k}{\Gamma(pk + b + k)} \frac{z^k}{k!} (p > 0).
\]

Note that the Stirling[26] approximation suggests

\[
\lim_{n \to \infty} \frac{\Gamma(n)}{\Gamma(n + x)} = \begin{cases} 1 & (x > 0) \\ 0 & (x = 0) \\ \infty & (x < 0), \end{cases}
\]

and so the ratio test for equation (93) with \( p > 0 \)

\[
\lim_{k \to \infty} \left| \frac{1 + k - \zeta \Gamma(pk + b) + \Gamma(pk + b + p)}{k + 1} \right| = 0
\]

indicates that the infinite series for \( p > 0 \) absolutely converges for all \( z \). It follows that \( E_{p,b}(z) \) with \( p > 0 \) is an entire function of \( z \). This is indeed a generalization of the Mittag-Leffler function since

\[
E_{p,b}(z) = E_{p,b}(z); \quad E_{p,1}(z) = E_{p,1}(z) = E_{p}(z)
\]

where \( E_{p}(z) \) and \( E_{p,b}(z) \) are the classical Mittag-Leffler function and its generalization by [1950]. If \( p = 1 \) on the other hand, this reduces to the Kummer\[20\] confluent hypergeometric function of the first kind, that is,

\[
E_{1,b}(z) = \frac{1}{\Gamma(1)} \Gamma(\lambda; 1); \quad E_{1,1}(z) = \frac{1}{\Gamma(1)} \Gamma(\lambda; 1)\Gamma(\lambda; 1),
\]

and also the Fox\[20\] H-function,

\[
E_{1,b}(z) = \frac{1}{\Gamma(1)} \Gamma(\lambda; 1) \Gamma(\lambda; 1)\Gamma(\lambda; 1).
\]

Next, the term-by-term integration indicates that for \( \lambda > 0 \)

\[
E_{p,b}(z) = \frac{1}{\Gamma(1)} \Gamma(\lambda; 1) \Gamma(\lambda; 1)\Gamma(\lambda; 1).
\]

Finally \( E_{p,b}(z) \) with \( \lambda \neq 0 \) is also a particular case of the Wright\[30\] generalized hypergeometric function \( \Psi_1 \),

\[
E_{1,b}(z) = \frac{1}{\Gamma(1)} \Gamma(\lambda; 1)\Gamma(\lambda; 1)\Gamma(\lambda; 1).
\]

and using (c.f., Lemma 2.4 and the binomial expansion)

\[
E_{p,b}(z) = \frac{1}{\Gamma(1)} \Gamma(\lambda; 1)\Gamma(\lambda; 1)\Gamma(\lambda; 1).
\]

this then leads to the integral representation

\[
E_{p,b}(z) = \frac{1}{\Gamma(1)} \Gamma(\lambda; 1)\Gamma(\lambda; 1)\Gamma(\lambda; 1).
\]

Here the integral loop \( \Omega \) is the same as usual for the Mittag-Leffler function, that is, it starts and ends at \( -\infty \), and loops around the circle \( \{ |z| = \zeta \} \) in positive sense. This may also be independently proven using the Hankel\[31\] loop integral

\[
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{0}^{(0^+)} \frac{e^{\xi}d\xi}{\xi^{z}},
\]

for the reciprocal gamma function (and also using the binomial expansion), similarly to the classical case. Equation (96)
implies the asymptotic expansion as \( z \to +\infty, \)

\[
E_{p,b}^d(z^n) \sim \frac{\lambda^n z^n}{p^\Delta(\lambda)}
\]

\[
E_{p,b}^d(-z) = \sum_{k=0}^{\infty} \frac{(-1)^k\lambda^k}{k!\Gamma(b-p\lambda-pk)z^{b+k}} \sim \frac{1}{\Gamma(b-p\lambda)z^{b}}
\]

(97)

If \( \xi = -1 \) is a non-negative integer, the series in equation (93) terminates after the finite number of terms and thus reduces to a polynomial on \( z \) in particular, \( E_{p,b}^0(z) = 1/\Gamma(b) \) is constant. In general, if \( \xi = -\lambda \geq 0 \), an alternative expression with the Fox H-function is also derived by separating the sum up to \( k = [\xi] \). That is, equation (93) is alternatively given by

\[
E_{p,b}^\xi(z) = \frac{\xi!}{\Gamma(kp+b)} \sum_{k=0}^{\lfloor \xi \rfloor} \frac{(1-\delta)^{\xi}}{k!} \sum_{k=0}^{\infty} \frac{(1-\delta)^k}{\Gamma(pk+k[\xi]+1)!} \xi^k
\]

where \( \delta = \xi - [\xi] \). For \( 0 < \delta < 1 \), the last infinite sum here results in

\[
\Gamma(1-\delta) \sum_{k=0}^{\infty} \frac{(1-\delta)^{\xi}}{\Gamma(pk+\delta)}(k+\mu+1)! = \left\{ \begin{array}{ll} \frac{(1-\delta)^{\xi}}{(\delta+1)\Gamma(\mu+1)!} & \text{for } \delta > 0 \\
0 & \text{for } \delta = 0 \end{array} \right.
\]

\[
= \frac{1}{\Gamma(1-\delta)} \sum_{k=0}^{\infty} \frac{(1-\delta)^{\xi}}{(\delta+1)\Gamma(\mu+1)!} z^k
\]

(98)

The convergent integration path for the last Fox H-function with \( 0 < \delta < 1 \) is always chosen such that it runs from \( c-i\omega \) to \( c+i\omega \) with \( 0 < c < 1 - \delta \) whereas such straight paths do not exist for equation (93) with \( \lambda < 0 \). Next, we find an extension of equation (93) for a negative \( \lambda = -\xi < 0 \),

\[
E_{p,b}^\xi(z) = \sum_{k=0}^{\mu} \frac{\xi!}{\Gamma(kp+b)} \sum_{k=0}^{\infty} \frac{(1-\delta)^{\xi}}{k!} \sum_{k=0}^{\infty} \frac{(1-\delta)^k}{\Gamma(pk+k[\xi]+1)!} \xi^k
\]

where \( \mu = [\xi] \) and \( \delta = \xi - \mu \).

Finally we observe additional operational properties that

\[
\frac{d^n E_{p,b}^d(z^n)}{dz^n} = (-1)^n(\lambda)^n E_{p,b}^{d+n}(z^n)(-z)^n
\]

(99a)

\[
\frac{d^n E_{p,b}^d(-z)}{dz^n} = \frac{1}{(1-\lambda)^n} \left[ E_{p,b}^{d+n}(-z) - \sum_{k=0}^{n-1} \frac{(n-\lambda)^k z^k}{k!\Gamma(b-pn+p+k)} \right].
\]

(99b)

for a non-negative integer \( n \). The last holds given that \( 1-\lambda)^n \neq 0 \). In addition, using \( (\lambda)^n(\lambda+k) = (\lambda)^{k+1} = \lambda(\lambda+1)^n \), we also find that

\[
\frac{d^2 E_{p,b}^d(z^n)}{dz^2} = \frac{1}{(1-\lambda)^n} \left[ E_{p,b}^{d+n}(-z) - \sum_{k=0}^{n-1} \frac{(n-\lambda)^k z^k}{k!\Gamma(b-pn+p+k)} \right].
\]

(99c)

7.2. Sufficient conditions for the phase-space consistency of eq. (90b) with \( 0 < s \leq 1 \)

Now we consider sufficient conditions on the AD to guarantee the phase-space consistency (Sect. 6) with the radial function given by equation (92) with \( 0 < s \leq 1 \) and \( \pi_0 = 0 \). In Sect. 6.3, we have argued that for \( \beta_1 = \beta_2 < 1 \), if there exists \( 0 < \beta_1 = \beta_2 < 1 \), such that \( 0 < \beta_1 < P(0) \geq 0 \) and \( P(0) = \cdots = P(\beta_1-1) = 0 \), then the df with \( \pi_0 = 0 \) recovered from the particular \( P(\Psi) \) and \( R(\xi) \) is non-negative everywhere. This follows from the fact that \( \pi^\xi(\phi(t)) = \pi^\xi(\delta(t) \Gamma(1-\beta_1) \text{ is cm if and only if } \delta \geq 1/2 \). As with \( \pi^\xi(\phi(t)) = \pi^\xi(1-\beta_1 \Gamma(1-\beta_1) \text{ cm for } \beta_1 < 1 \), the discussion in Sect. 6.3 on sufficient conditions for constant \( \beta_1 \) separable AD can carry over here essentially verbatim if we can establish the set of results regarding the complete monotonicity of the generalized Mittag-Leffler functions of the form of \( \pi^\xi \).

We first note that the leading term of \( E_{p,b}^\xi(\beta_1-\beta_2) \) for \( \tau \to 0 \) is given by the positive constant

\[
E_{p,b}^\xi(\beta_1-\beta_2)(0) = \frac{1}{\Gamma(1-\beta_2)} > 0
\]

for \( \beta_1 < 1 \), which indicates that \( \pi^\xi \) for \( \beta_1 < 1 \), \( s > 0 \) and \( a > 0 \) must be increasing in some interval \((0, c)\) where \( c > 0 \). On the other hand, equation (97) suggests that

\[
\lim_{\tau \to 0} \pi^\xi(\beta_1-\beta_2)(\tau^p) = \frac{1}{\Gamma(1-\beta_2)} > 0
\]

are positive finite for \( \beta_1 < 1 \). It follows that, as \( \tau \to +\infty \), we have \( \pi^\xi \) for \( \beta_1, \beta_2 < 1 \), \( s > 0 \) and \( a > 0 \) are \( \beta_2 - \beta_1 \). That is to say, if \( a > \beta_2 - \beta_1 \), then \( \pi^\xi \) for \( \beta_1, \beta_2 < 1 \) and \( s > 0 \) must be increasing in some non-empty subintervals of \((0, c)\) where \( c > 0 \). Together, these observations imply that \( \pi^\xi \) for \( \beta_1, \beta_2 < 1 \), \( s > 0 \) cannot be cm if \( a < \min(0, \beta_2 - \beta_1) \). Although it is tempting to hypothesis by analogy to the constant \( \beta_1 \) case such that \( \pi^\xi \) for \( \beta_1, \beta_2 < 1 \), \( s > 0 \) and \( a > 0 \) are cm for \( a < \min(0, \beta_2 - \beta_1) \), we have only been able to prove this under a restriction that \( \beta_1 > \beta_2 \) or \( \beta_2 < 1 \) while for \( 1 < s < \beta_2 < 1 \) we only manage to find a more restrictive condition \( a \leq s \) (n.b., \( a < s + 1 - \beta_1 < \beta_2 - \beta_1 < 0 \)) for the complete monotonicity of \( \pi^\xi \).

We next apply the similar discussion in Sect. 6.3 and find that for \( R(\xi) \) given by equation (92) with \( \beta_1 < 1, \beta_2 \leq 1 \), and \( 0 < s < 1 \), if there exists \( 1 < s < \beta_1 < \beta_2 < 1 \) such that equations (82) and (85a) hold for \( \rho(\Psi) \), then the resulting AD, \( P(\Psi)R(\tau^p) \) guarantees the existence of a non-negative df, unless \( 1 < s < \beta_2 < 1 \) or \( s < 1 \). For \((s, \beta_2) = (1,1)\), which results in \( E_{p,b}^{1-\beta_1}(t) = \frac{1}{\Gamma(1-\beta_1)} + \frac{1}{\Gamma(1-\beta_1)} = \frac{1}{\Gamma(1-\beta_1)} \), the condition for a positive integer \( \lambda = m + 1 \geq 1 \) follows from \( E_{p,b}^{1-\beta_1}(t) \) reproduces that of (Costt & Morgant, 2010), for the generalized Cattell system to result from a non-negative df. If \( 1 < s < \beta_2 < 1 \) on the other hand, we at this point only find a slightly restrictive sufficient condition with \( 1 \geq \frac{1}{\Gamma(1-\beta_1)} > \frac{1}{2} - \beta_2 > \frac{1}{\Gamma(1-\beta_1)} \geq \frac{1}{2} - \beta_2 > \frac{1}{2} - \beta_1 > \frac{1}{2} \) (n.b., \( \beta_1 - s < 1 - s < \beta_2 < 1 \)).

In the rest of this section, we proceed to prove that \( \pi^\xi \) for \( \beta \) for
Theorem 7.3
If \( \beta_1 < 1, \beta_2 \leq 1 \), and \( 0 < s \leq 1 \) (but not \( 1 - s < \beta_2 < \beta_1 < 1 \)) is cm. First, if \( \beta_1 = \beta_2 < 1 \), then \( E_{\beta_1-\beta_2}(t^r) = 1/\Gamma(1-\beta_1) \) and so is trivial (see Sect. 6.3). Next note, if we can prove that \( z^{\min(0,1)}E_{\beta}(\cdot) \) is cm for \( b > 0, b \geq p+1, \) and \( 0 < p \leq 1 \), then the desired result follows Corollary 7.2 in the following, we prove the complete monotonicity of \( E_{\beta}(\cdot) \) for \( 0 < p \leq b \) and \( z^{k}E_{\beta}(\cdot) \) for \( b > 0 \) and \( \lambda < 0 \). The further restriction, \( \beta_2 \leq 1 - s \) (i.e., \( p+1 \leq b \)) on the latter case meanwhile occurs naturally. We first introduce a lemma.

**Lemma 7.2** If \( 0 < p \leq 1, b > 0, \) and \( b \geq p+1, \) then \( E_{\beta}(\cdot) \) is cm function of \( w > 0 \). In general, for \( b > 0, \)
\[
\int_0^{\infty} dt e^{-wt} t^{p-1} E_{\beta}(t^r) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda)^{t_k}}{\Gamma(pk + b)} \int_0^{\infty} dt e^{-wt} t^{pk + b - 1} = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda)^{t_k}}{k!w^{pk + b}} = \frac{1}{w^p} \left( 1 + \frac{1}{w^p} \right)^{-1} \tag{100} \]
By Corollary 2.12 this is a cm function of \( w > 0 \) for \( 0 < p \leq 1 \) either if \( b \geq 0 \) and \( \lambda \leq 0 \) or if \( b - p \lambda \geq 0 \) and \( \lambda \geq 0 \). Then from the Bernstein theorem, if \( 0 < p < 1, b > 0, \) and \( b \geq p+1, \) then \( t^{p-1} E_{\beta}(t^r) \geq 0 \) for \( t > 0 \) and so \( E_{\beta}(\cdot) \) is cm function of \( w > 0 \).

The first half of the desired result is now trivial, that is,

**Theorem 7.3** If \( 0 < p \leq 1 \) and \( 0 < p \lambda \leq b, \) then \( E_{\beta}(\cdot) \) is a cm function of \( z > 0 \).

This follows equation (99a). Note that if \( b \geq p+1 \), then \( b - p \lambda \geq 0 \) and \( \lambda > 0 \) and thus Lemma 2.12 together with (\( \lambda)^{t_k} \) \( > 0 \) for \( \lambda > 0 \) completes the proof. As noted, Theorem 7.3 implies

**Corollary 7.4** For \( E_0 = 0 \) and \( R(x) \) given by equation (90b) with \( 0 < s \leq 1 \) and \( \beta_1 < \beta_2 \leq 1, \) if there exists \( 3 \lambda \geq 1/2 - \beta_1, \) such that \( \gamma \partial^{\beta_1} P \geq 0 \) and \( P(0) = \cdots = P(\lambda^{1-1}(0)) = 0, \) then the df inverted from \( P(\Psi)(r^2) \) is non-negative.

This actually extends to \( \beta_1 \leq \beta_2 \leq 1 \) (Sect. 6.3). Also note that if \( P(0) = \cdots = P(\lambda^{1-1}(0)) = 0, \) the \( \gamma \partial^{\beta_1} P \geq 0 \) is the necessary and sufficient condition for the phase-space consistency given \( \Theta_0 = 0 \) and \( R(x) \) with \( 0 < s \leq 1 \) and \( \beta_1 \beta_2 \leq 1. \)

For the second half, we first find

**Theorem 7.5** If \( 0 < p \leq 1, b > 0, \) and \( \xi > 0, \) then \( s^{-\xi}E_{\beta}(\cdot) \) is a cm function of \( z > 0. \)

**Corollary 7.6** For \( E_0 = 0 \) and \( R(x) \) in equation (90b) with \( 0 < s \leq 1 \) and \( b \geq \beta_1 \leq 1, \) if there exists \( 3 \lambda \geq 1/2 - \beta_1 + sn \) where \( n = \lceil (\beta_1 - \beta_2)/s \rceil \) such that \( \gamma \partial^{\beta_1} P \geq 0 \) and \( P(0) = \cdots = P(\lambda^{1-1}(0)) = 0, \) then the df inverted from \( P(\Psi)(r^2) \) is non-negative.

If \( \xi = \mu \) is a non-negative integer, this is trivial since
\[
z^{\mu}E_{\beta}(\cdot) = \sum_{k=0}^{\mu} \binom{\mu}{k} z^{-(\mu-k)} \Gamma(b + pk), \tag{101} \]
with every coefficient being positive. Next, equation (99a) for \( \xi = -\xi \leq 0 \) and \( n = \lceil \xi \rceil \) results in
\[
\frac{d^\xi E_{\beta}(\cdot)}{d\xi} = (1 - e_\xi^+ \partial^{\xi} E_{\beta}(\cdot)), \tag{102a} \]
where \( 0 \leq \epsilon = [\xi] - \xi < 1. \) Now equation (99b) indicates that
\[
(1 - e_\xi^+ \partial^{\xi} E_{\beta}(\cdot)) E_{\beta}(\cdot) \equiv E_{\beta}(\cdot) - \sum_{k=0}^{\xi-1} \left( \frac{\xi}{k} \right) \frac{z^k}{\Gamma(b + pk)} \tag{102b} \]
which is consistent with equation (12). If \( \xi > 0, \) this reduces to (note then that \( \xi \geq 1 \))
\[
z^{-\xi}E_{\beta}(\cdot) = \sum_{k=0}^{\xi-1} \frac{\xi}{k} s^{-(\xi-k)} \Gamma(b + pk) \int_0^1 du \frac{1}{1 - w^{(\xi-k)}E_{\beta}(\cdot) (\xi_0)} \tag{102c} \]
while Theorem 7.3 indicates that \( f(z) = E_{\beta}(\cdot) \) is cm function of \( b + p\xi \) \( - p \epsilon > 0 \) is finally, we are able to prove

**Theorem 7.7** If \( 0 < p \leq 1, \xi > 0, b > 0, \) and \( b \geq p(1 - \xi), \) then \( z^{-\xi}E_{\beta}(\cdot) \) is a cm function of \( s > 0. \)

**Corollary 7.8** For \( E_0 = 0 \) and \( R(x) \) given by equation (90b) with \( 0 < s \leq 1, b \leq \beta_1 < 1, \) and \( b \geq 1 - s, \) if there exists \( 3 \lambda \geq 1/2 - \beta_2 \) such that \( \gamma \partial^{\beta_1} P \geq 0 \) and \( P(0) = \cdots = P(\lambda^{1-1}(0)) = 0, \) then the df inverted from \( P(\Psi)(r^2) \) is non-negative.

If \( \xi \) is a positive integer, this is the same as Theorem 7.5.

For general cases, we note equation (99c) results in
\[
\frac{d\xi E_{\beta}(\cdot)}{d\xi} = -\xi E_{\beta}(\cdot) = -\xi \frac{\xi E_{\beta}(\cdot)}{\xi+1}, \tag{103} \]
where \( \xi \geq 1 \) and \( 0 \leq \epsilon = [\xi] - \xi < 1. \) Then theorem 7.7 indicates that if \( 0 < p \leq 1, b > 0, \) and \( \xi > 1, \) then \( z^{-\xi}E_{\beta}(\cdot) \) and subsequently \( z^{-\xi}E_{\beta}(\cdot) \) are cm. Theorem 7.8 on the other hand suggests that if \( 0 < p \leq 1, \xi < 1, \) and \( b \geq p(1 - \xi), \) then \( E_{\beta}(\cdot) \) is cm. Hence, if \( 0 < p \leq 1, \xi > 0, b > 0, \) and \( b \geq p(1 - \xi), \) the derivative of \( z^{-\xi}E_{\beta}(\cdot) \) is given by a cm function multiplied by a negative constant. It follows Lemma 2.10 that \( z^{-\xi}E_{\beta}(\cdot) \) is a cm function of \( z > 0. \)

8. SUMMARY

We have shown that the fractional calculus operations (eqs. 3[4] and 5) applied to the bivariate augmented density (eq. 11) result in a set of the integral transformations of the two-integral distribution function (eqs. 36[39] and 41). Equation 40 with \( \lambda + \xi + \beta + \frac{1}{2} = 0 \) indicates that the set of fractional calculus operations on the augmented density \( \Psi, r^2 \) listed in equation (42) provides with the complete moment sequence of the distribution function along \( K(E, L^2; \Psi, r^2) = 0 \)
as shown in equation (41a). We infer from this that the augmented density that ensures the non-negativity of the distribution function may be deduced by analogy to the classical moment problem in probability theory (Van Hese et al. 2012). We have also found that equation (40) for a non-negative integer \( \lambda > 0 \) and \( \xi = 0 \) consists in the complete moment sequence of the augmented density at a fixed \( r \) considered as a probability density on \( \Psi \) – which is possible because the augmented density is also non-negative in all accessible \( r \) and \( \Psi \). Comparing this sequence to the velocity moments resulting from the given distribution function (eq. [26]), we deduce that the augmented density (and subsequently the distribution function) is uniquely specified given the potential \( \Psi(r) \) and the density profile \( \nu(r) \) once the infinite set of the radial velocity moments in every order (equivalently the complete radial velocity distribution) as a function of the radius are available (cf., Dejonghe & Merritt 1992).

Given \( \lambda + \xi + \frac{1}{2} \geq 0 \), all the integrands in the right-hand sides of equation (40) are non-negative because the distribution function \( F(\xi, L^2) \) must be non-negative in the whole accessible subspace volume \( T \). This non-negativity implies that it is necessary for the integrally-differential operations on the augmented density \( N(\Psi, r^2) \) given in the left-hand side of equation (40) to be also non-negative, provided that the integrals involved in their definitions are all convergent. This introduces the set of necessary conditions on the augmented density for the non-negativity of the distribution function. If the augmented density is multiplicatively separable into functions of the potential and the radius dependencies like equation (40), this results in the condition stated by An (2011b), that is, equation (53) for the radius part of the augmented density. We have also discovered a few equivalent statements of this condition, notably equation (59) and the function \( R(u) \) defined in equation (55) being completely monotonic and so on. The same argument for the potential part of a separable augmented density on the other hand recovers the conditions derived by Van Hese et al. (2011b) and An (2011a). They were further generalized with fractional calculus to indicate that:

\[
\frac{\partial}{\partial \mu} \Psi \leq 0 \quad \text{for all accessible } \Psi \text{ if } \mu \leq 1, \text{ where there exists } 3 \beta \geq 2 \mu \text{ such that } \lim_{r \to 0} r^{-2 \beta} R(r^2) \text{ is well-defined or } 3 \beta \leq 2 - \mu \text{ such that } \lim_{r \to 0} r^{2 \beta} R(r^2) \text{ is non-zero and finite.}
\]

With separable augmented densities, the distribution function may be inverted from the augmented density by means of the inverse Laplace transform as in equation (74). The non-negativity of the distribution function corresponding to a separable augmented density is guaranteed if the Laplace transformation of the distribution function given in equation (73b) is a complete monotonic function of \( s > 0 \) for any \( L^2 \geq 0 \.

We have shown from this that the set of joint conditions composed of equation (52) with all non-negative integer pairs \( n \) and \( k \) for the radius part \( R(r^2) \) of the augmented density and equations (54) and (55a) for the potential part \( \Psi(r) \) of the same is sufficient to imply the non-negativity of the corresponding distribution function. This last set of sufficient conditions is equivalent to that of Van Hese et al. (2012), which was derived from the argument following the application of the Hausdorff moment problem.

This manuscript is basically an extended version of An et al. (2012).
The expression for the coefficients is found using the binomial series expansion of equation (A3b) for $0 \leq x < 1$,

$$R_n(x) = \sum_{k=0}^{n} \tilde{a}_{n,k} \sum_{j=0}^{\infty} \frac{(-1)^j (\zeta + k)^j}{j!} x^{(k+j)-\beta_1},$$

where we have used $\tilde{a}_{n,k} = 0$ if $k > n$. All the Pochhammer symbols without any directional specification hereafter are interpreted to represent the rising product, i.e., $(a)_n = (a)_n^+$. Matching the coefficients for the same power of $x$ in equations (A5a) and (A7) leads to $[n.b., (\zeta)_n(\zeta + m)_n = (\zeta)_n]$

$$(1 - \beta_1 + sk)n = \sum_{m=0}^{k} \frac{(-1)^m k! \tilde{a}_{n,m}}{(m - k)!} (\zeta)_m (1 - \beta_1 + sx)_n|_{x=0}.$$  

(5b) Although its derivation assumed $0 \leq x < 1$, this is valid regardless. The right-hand side is in the form of the binomial transformation and thus by its involutory inversion

$$\tilde{a}_{n,m} = (\zeta)_m \frac{\sum_{k=0}^{m} \frac{(-1)^k}{k! (m - k)!} (1 - \beta_1 + sk)_n}{\Delta^m_1 - (1 - \beta_1 + sx)_n}|_{x=0},$$

(5c) That is, $\tilde{a}_{n,k}$ is the $k$-th order forward finite difference of $(1 - \beta_1 + sx)_n = \prod_{i=1}^{n} (1 - \beta_1 + sx)_n$ at $x = 0$. Since $(1 - \beta_1 + sx)_n$ is an $n$-th order polynomial of $x$, we have $\Delta^k_1(1 - \beta_1 + sx)_n|_{x=0} = 0$ if $k > n$. The formula for $\tilde{a}_{n,k}$ is found from equation (A4b), and the Chu-Vandermonde identity

$$\sum_{k=0}^{n} \binom{n}{k} (s)_k (t)_n = (s + t)_n,$$

(A6a) or equivalently the Gauss hypergeometric identity

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} \frac{(b)_k}{(c)_k} = \sum_{k=0}^{n} \frac{(-n)(b)_k}{k!(c)_k} = \frac{(c - b)_n}{(c)_n}.$$  

(A6c) For the $s = 1$ case, from equations (A5c), (A6b), and

$$(1 - \beta_1)_n(1 - \beta_1 + k)_n = (1 - \beta_1)_n(1 - \beta_1 + n)_k$$

(A7a) we find that $(\beta_1 < 1)$

$$\frac{\tilde{a}_{n,m}}{(1 - \beta_1)_n} = \frac{(\beta_2 - \beta_1)_m}{m!(1 - \beta_1)_m},
\frac{\tilde{a}_{n,m}}{(1 - \beta_1)_n} = \frac{(\beta_2 - \beta_1)_m}{m!(1 - \beta_1)_m},
\frac{(1 - \beta_2)_k}{(1 - \beta_1)_k}.$$  

(A7b) (A7c)

Here $\zeta = \beta_2 - \beta_1$ since $s = 1$. It is notable as it indicates that $\tilde{a}_{n,k} \geq 0$, and with $(1 - \beta_1)_k(n - k) = (1 - \beta_1)_n$ that

$$\tau_n(y) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} (1 - \beta_1 + k)_n(1 - \beta_2)_k y^k \geq 0$$  

(A7d) for any non-negative integer $n$ and all $x = y > 0$.

Next, we consider the Mellin transform for $0 < z < \lambda$

$$\varphi(z) = \int_{0}^{\infty} dy y^{-z} \frac{\alpha_n}{(1 + y)^z} = \int_{0}^{\infty} dy y^{-z} \frac{\alpha_n}{(1 + y)^z}$$

(A8a) $= \int_{0}^{\infty} du (1 - u)^{-z-1} u^{-z-1} \alpha_n = \sum_{k=0}^{n} \frac{\Gamma(z + k)\Gamma(\lambda - z)}{\Gamma(\lambda)} \tilde{a}_{n,k}$

$$= \frac{\Gamma(z)\Gamma(\lambda - z)}{\Gamma(\lambda)} \sum_{k=0}^{n} \frac{(1 - \beta_1 + sx)^{k}}{k!} \Delta^k_1(1 - \beta_1 + sx)_n|_{x=0}.$$  

This simplifies for $\lambda = \zeta$ utilizing the Newton series

$$f(z) = \int_{0}^{\infty} dy y^{-z} \frac{\alpha_n}{(1 + y)^z} = \int_{0}^{\infty} dy y^{-z} \frac{\alpha_n}{(1 + y)^z}$$

(A8b) if $f(x)$ is an $n$-th order polynomial, the formula is exact after the summation up to $k = n$. Since $\Delta^k_1(1 - \beta_1 + sx)_n|_{x=0} = 0$ for $k > n$ with the $n$-th order polynomial $(1 - \beta_1 + sx)_n$,

$$(1 - \beta_1 - sz)_n = \sum_{k=0}^{n} \frac{(1 - \beta_2)_k}{k!} \Delta^k_1(1 - \beta_1 + sx)_n|_{x=0}.$$  

(A8c) and therefore with $\lambda = \zeta > z > 0$,

$$\Gamma(z)\varphi(z) = \Gamma(z)\Gamma(\zeta - z)(1 - \beta_1 - sz)_n.$$  

(A8d) By means of the inverse Mellin transformation, $R_n = \Re_n$ is then expressible to be a Mellin–Barne\textsuperscript{3} type integral

$$R_n(x) = \frac{1}{2\pi i x^{\beta_1}} \int_{C} \frac{dz}{z^{\beta_1}} \Gamma(z)\Gamma(\zeta - z)(1 - \beta_1 - sz)_n.$$  

(A8e) Although this is actually reducible to algebraic functions on $x$ as in equation (A3b), it is also the $\text{box}$ H-function ($H^{1,2}_{2,2}$ in particular) and further reduces to the Meijer G-function ($G^{1,1}_{1,1,n+1}$) and the hypergeometric function ($F_{n+1,1}^{}$), the last of which would be formally equivalent to equation (A9).

The function $\phi(t)$ is found from equation (A8b)

$$\phi(t) = \frac{1}{2\pi i} \int_{C} \frac{dz}{z^{\beta_1}} \Gamma(z)\Gamma(\zeta - z)(1 - \beta_1 - sz)_n$$

(A9a) where we have used equation (77). If $0 < \zeta \leq (1 - \beta_1)/s$ (n.b., $\zeta = (\beta_2 - \beta_1)/s$ and $\beta_2 \leq 1$), the convergent integration

32 Zhi Shiji (1270-1330)  
33 Alexandre-Théophile Vandermonde (1735-1796)  
34 Johann Carl Friedrich Gauss (1777-1855)  
35 Robert Hjalmar Mellin (1854-1933)  
36 Sir Isaac Newton (1642-1727)  
37 Ernest William Barnes (1874-1953)  
38 Cornelis Simon Meijer (1904-1974)
Lemma B.1
For $c, \lambda > 0$,\[ \int_0^\infty x^{\lambda-1} \, dx = e^c ; \quad \lim_{\lambda \to 0^+} e^c = 1 .\]

Next, it follows that

$$\lim_{\lambda \to 0^+} \int_0^\infty x^{\lambda-1} \, dx = e^c .$$

Theorem B.2
For $F(x) = f(x) - \ell$ where $\ell = \lim_{x \to 0^+} f(x)$,

$$\lim_{\lambda \to 0^+} \lambda \int_0^{\infty} x^{\lambda-1} |F(x)| \, dx = 0 \quad (c > 0)$$

proof. First by the definition of $\lim_{x \to 0^+} f(x)$, we find that for any $\epsilon > 0$, there exists $\delta > 0$ such that, if $0 < x < \delta$, then $|f(x) - \ell| = |F(x)| < \epsilon$. Now if $0 < c \leq \delta$, then for any $\lambda > 0$

$$0 \leq \int_0^\infty x^{\lambda-1} |F(x)| \, dx < e \int_0^\infty x^{\lambda-1} \, dx .$$

If $c > \delta > 0$ on the other hand,

$$0 \leq \int_0^\infty x^{\lambda-1} |F(x)| \, dx = \int_0^\delta x^{\lambda-1} |F(x)| \, dx + \int_\delta^\infty x^{\lambda-1} |F(x)| \, dx < e \int_0^\delta x^{\lambda-1} \, dx + \sup_{0 < x < \delta} |F(x)| \int_\delta^\infty x^{\lambda-1} \, dx .$$

Note here that $\int_\delta^\infty x^{\lambda-1} \, dx$ is finite. Consequently, provided that $f(x)$ is bounded in $(0, c)$, we find from both cases that

$$0 \leq \lim_{\lambda \to 0^+} \lambda \int_0^\infty x^{\lambda-1} |F(x)| \, dx < e$$

where $c > 0$ and we have used Lemma B.1 Q.E.D.

It immediately follows that

**Corollary B.3**
For $F(x) = f(x) - \ell$ where $\ell = \lim_{x \to 0^+} f(x)$,

$$\lim_{\lambda \to 0^+} \lambda \int_0^{\infty} x^{\lambda-1} F(x) \, dx = 0 \quad (c > 0) ,$$

$$\lim_{\lambda \to 0^+} \lambda \int_0^{\infty} x^{\lambda-1} f(x) \, dx = \ell \quad (c > 0) .$$

Equation (6.63) trivially follows this with the change of integration variable $x = t - a$. Formally this is interpreted to be

$$\lim_{\lambda \to 0^+} \lambda x^{\lambda-1} = \delta(x) ; \quad \lim_{a \to 0} \frac{1}{x^\lambda \Gamma(1-a)} = \delta(x)$$

where $\delta(x)$ is the Dirac delta, provided that $f(x)$ is right-continuous.

B.2. The $\beta = 1$ constant anisotropy model

Let us consider the df given by

$$F(E, L^2) = \frac{f(E) \delta(L^2)}{\sqrt{2\pi}^2}$$

where $f(E)$ is an arbitrary function of $E$. This df corresponds to the spherical system entirely built by radial orbits, that is, the $\beta = 1$ constant anisotropy model. Given that $C(L^2) = 2 (\Psi - E)$, the corresponding AD is found to be

$$N(\Psi, r^2) = \frac{1}{\sqrt{2\pi}} \int \frac{r^2}{\sqrt{2(\Psi - E)}} f(E) \, dE = r^{2-\gamma} e^{f(\Psi)}$$

which is separable as in equation (4.8) with

$$P(\Psi) = e^{f(\Psi)} \quad R(x) = x^{-1} .$$

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39 Paul Adrien Maurice Dirac (1902-1984)
The AD is inverted to the df using the fractional derivative,
\[ f(\xi) = e_{\xi}^t \Delta_{\xi} P(\xi) \geq 0, \]  
(B2d)
whose non-negativity is also the necessary and sufficient condition for the phase-space consistency. Note that this is consistent with the results of Sect. 6.3 applicable for \( \beta \leq 1 \) as is \( R(x) \) here the natural limit of the constant anisotropy model in equation (63) to \( \beta = 1 \). Furthermore, we find for \( \lambda = n + \delta > 0 \) and \( n = [\lambda] \) that
\[ x_n^{1-\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty \left( \frac{x}{y} \right)^{\lambda-1} e^{-x^n} \, dx, \]
(B3a)
while \( x_n^{1-\lambda} = \frac{1}{\Gamma(\delta)} \int_0^\infty \left( \frac{x}{y} \right)^{\delta-1} e^{-x^n} \, dx \). Hence, \( R(x) = x^{-1} \) satisfies the necessary condition in equation (53). Moreover, equations (56), (39), and (40) still hold with non-trivial conditions indicating
\[ e_{\xi}^t \Delta_{\xi} e_{\xi}^t P = e_{\xi}^t P(\xi), \]
(B4)
whose non-negativity for \( \mu \leq \frac{\beta}{2} \) is the same necessary condition for \( P(\xi) \) discussed in Sect. 5.2.

From \( R(x) = x^{-1} \), we find that \( R(w) = 1 \) and its inverse Laplace transformation at least formally is given by \( \phi(t) = \delta(t) \). Although equation (50) is strictly then trivial as \( \delta(t) = 0 \) for \( t > 0 \), this interpretation of equation (50) seems improper considering that the Dirac delta is not differentiable at \( t = 0 \). Equation (52) on the other hand reduces to \( x_n^{1-\lambda} \) being cm since \( R(0)(x) = R(x) = x^{-1} \) and \( R(0)(y) = 0 \) for any positive integer \( n \). The sufficient condition following this, that is, equations (53) and (55b) for \( \frac{\lambda}{\beta} \leq \frac{1}{2} \) is in fact a proper one, as is the natural case limiting the constant anisotropy model for \( \beta = 1 \). It appears that for \( R = x^{-1} \) as \( x \to 0 \) and \( \lim_{x \to 0} R \) being nonzero finite), we may consider \( \phi(t) \sim t^{-1} \) for \( t \to 0 \) for the purpose of applying equation (50).

B.3. Equation (50b) with \( \beta_1 = 1 \)

The discussion in Sect. 5 on necessary conditions is valid inclusively for \( \beta_1 \leq 1 \). That is, equation (53) with \( \beta_1 = 1 \) still requires to satisfy equation (53) — if \( 0 < p \leq 1 \), this is automatically met — in order for the df to be non-negative whereas the potential dependent part is restricted to be \( e_{\xi}^t \Delta_{\xi} e_{\xi}^t P \geq 0 \) for the phase-space consistency.

The complication arises however for \( \beta_1 = 1 \) in regards to sufficient conditions discussed in the preceding section. The main difficulty is due to the fact that \( \lim_{\xi \to 0} \int_{-\infty}^{\infty} R(x) \, dx = \) non-zero. While this indicates \( \phi \sim t^{-1} \) for \( t \to 0 \), the particular behavior is incompatible with the convergence of the integral. The formal solution follows adopting equation (B1). In addition, the limit of equation (59) with \( R = x^{-1} \) is identically zero for any \( x > 0 \) and so the function \( \phi(t) \) defined via the formal limit of equation (58) with \( R(x) \) in equation (50b) takes the same value as that with \( \phi(t) + \delta(t) \) for all \( t > 0 \) (that is to say, the Post–Widder formula is technically valid). In other words, the function \( \phi(t) \) derived in equation (52) with \( \beta_1 = 1 \) is in fact the inverse Laplace transform of \( R(w) - 1 \) and the ‘true’ inverse transformation of \( R(w) \) with \( \beta_1 = 1 \) is given by \( \phi(t) + \delta(t) \). For example, since \( 1/\Gamma(0) = 0 \), the \( k = 0 \) term in the power series defining the generalized Mittag-Leffler function \( E_{\alpha,\beta}(z) \) does not contribute. Hence, equation (50b) can in fact be well-defined for \( b = 0 \) case too. In particular,
\[ \int_0^{\infty} e^{-x^n} x^{-\lambda} \, dx = \sum_{k=1}^{\infty} (-1)^k \left( \frac{z}{k!} \right)^{\lambda-1} = \left( 1 + \frac{1}{w^n} \right)^{-\lambda} - 1. \]

Since \( (1 + w^{-n})^{-\lambda} \geq 1 \) for \( w > 0 \) and \( \zeta \leq 0 \), it follows that, if \( 0 < p \leq 1 \) and \( \zeta \leq 0 \), this is also cm and \( E_{\alpha,\beta}(z) \) is in fact a proper one,
\[ \zeta < 0 \quad \text{and so} \quad (1 + w^{-n})^{-\lambda} \geq 1 \]
if \( p > 0 \). Given that \( L_{w^{-n}}[\delta(t)] = 0 \), we also find from this that
\[ L_{w^{-n}}[\delta(t) + t^{-1} E_{\alpha,\beta}(z)] = \int_{w^{-n}}^{\infty} e^{-x^n} \, dx = \sum_{k=1}^{\infty} (-1)^k \left( \frac{z}{k!} \right)^{\lambda-1} = (1 + w^{-n})^{-\lambda} - 1. \]

For the specific discussion concerning sufficient conditions for the non-negativity of the df, we basically consider
\[ P(\Psi)R(r^2) = P(\Psi)R(0) + r^{-2} P(\Psi) \]
where \( R(0) = R(x) - x^{-1} \). The corresponding df (\( E(0) = 0 \)) would be
\[ F(\xi, L^2) = \int_{\xi}^{\infty} \frac{e^{-\beta t} P(t)}{(2\pi)^{3/2}} \delta(t) \, dt \]
and it is obvious that corresponding sufficient condition is to \( \beta_1 = 1 \). Note the condition \( \beta_1 = 0 \) and \( P(0) = \cdots = \beta_1 = 1 \) implies \( \beta_1 = 1 \) for the purpose of applying equation (50).

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