Black hole uniqueness theorems and new thermodynamic identities in eleven dimensional supergravity

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Received 23 April 2012, in final form 6 August 2012
Published 28 August 2012
Online at stacks.iop.org/CQG/29/205009

Abstract
We consider stationary, non-extremal black holes in 11-dimensional supergravity having isometry group \( \mathbb{R} \times U(1)^8 \). We prove that such a black hole is uniquely specified by its angular momenta, its electric charges associated with the 7-cycles in the manifold, together with certain moduli and vector valued winding numbers characterizing the topological nature of the spacetime and group action. We furthermore establish interesting, non-trivial, relations between the thermodynamic quantities associated with the black hole. These relations are shown to be a consequence of the hidden \( E_8(8) \) symmetry in this sector of the solution space, and are distinct from the usual ‘Smarr-type’ formulas that can be derived from the first law of black hole mechanics. We also derive the ‘physical process’ version of this first law applicable to a general stationary black hole spacetime without any symmetry assumptions other than stationarity, allowing in particular arbitrary horizon topologies. The work terms in the first law exhibit the topology of the horizon via the intersection numbers between cycles of various dimension.

PACS numbers: 04.65.e-, 11.30.Pb, 12.60.Jv
(Some figures may appear in colour only in the online journal)

1. Introduction

Among the various supergravity theories, 11-dimensional supergravity [12] plays a special role. It lives in the highest possible spacetime dimension (in signature \((-,-,+,\ldots,+)\)), is related to most maximally supersymmetric lower dimensional supergravity theories via compactification and truncation, and has many intriguing connections to the ten-dimensional superstring theories. It is therefore, obviously, of considerable interest to map out the space of stationary black hole solutions in 11-dimensional supergravity, subject to various interesting
asymptotic ‘boundary’ conditions, such as asymptotically flat, asymptotically Kaluza–Klein, asymptotically anti-deSitter, etc. Unfortunately, stated in this generality, this seems an almost intractable problem, because it includes by definition all such solutions in any compactification of the theory. Even worse, it would also include solutions with very low amount of symmetry. There is evidence e.g. from the ‘blackfold approach’ [17] that such solutions exist in higher dimensional gravity theories, but it seems unlikely that one will be able to write down some analytic expression for them.

For this reason, it seems reasonable to restrict oneself from the outset to more special stationary black hole solutions which are either (a) static, or (b) are ‘algebraically special’ in a suitable 11-dimensional sense, or (c) have fermionic symmetries, i.e. solutions to the appropriate ‘Killing spinor’ equation in 11-dimensional supergravity, or (d) have a considerable amount of bosonic symmetries, i.e. vector fields Lie-deriving the solution. Concerning (a), it seems plausible that one can classify all such solutions e.g. via the methods of [27–29, 46], at least in the case of asymptotically flat boundary conditions (in the 11-dimensional sense). It is also conceivable that a modification/generalization of this method could be applied asymptotically Kaluza–Klein boundary conditions, but this remains to be seen. (b) A general notion of algebraically special solutions in higher dimensions, based on the Weyl-tensor, has been proposed by [11, 56], and it has been demonstrated that this notion is useful in principle to find/classify various special solutions e.g. in vacuum Einstein gravity, see e.g. [30, 54]. They include the near horizon limits [45] of higher dimensional extremal black holes, although not black holes themselves. It seems likely that this strategy, complemented by a suitable condition onto the 4-form field strength, could be applied also to 11-dimensional supergravity, but presumably the same restrictions would apply. (c) This program is pursued e.g. in papers [31, 26, 25]. A complete classification was achieved in five-dimensional minimal supergravity [22, 57] (see also [60] for four-dimensions), which is, in some ways [49], a simpler cousin of 11-dimensional supergravity. However, in the latter case, the classification programme has yet to be completed.

In this paper, we will consider (d). In some sense the most stringent, and symmetric, assumption is that the solution be invariant under the Abelian group \( \mathbb{R} \times U(1)^8 \), where the factor \( \mathbb{R} \) corresponds to the asymptotically timelike symmetry (stationarity). It is elementary to see that these assumptions restrict the asymptotic region of the spacetime \( M \) to be of the form \( \sim \mathbb{R}^s \times T^{10-s} \), where the number of asymptotically large spatial dimensions \( s \) is either \( s = 1, 2, 3, 4 \). What makes this symmetry assumption special is that, as has been known for a long time [43, 50], the field equations for the bosonic fields then possess a large number of ‘hidden’ symmetries, parameterized by the exceptional real Lie-group \( E_{8(+8)} \). These hidden symmetries are useful in several ways.

(i) They make it possible, in principle, to generate new solutions from old ones, e.g. via the powerful variant of the ‘inverse scattering method’ [2], suitably generalized to the present situation.

(ii) As we will demonstrate in section 3, the hidden symmetries, together with other ideas, make it possible to derive a uniqueness theorem for black holes along similar lines as the classical results [6, 7, 33, 47, 58, 4]. A naive expectation would be that these are uniquely characterized by their asymptotic quantities, i.e. mass \( m \), angular momenta \( J_i \), and the various charges \( Q[C_7] \) associated with different 7-cycles \( C_7 \) in the asymptotic region. However, since this is already false in five-dimensional pure gravity [14–16], it must necessarily be false also in 11-dimensional supergravity, since the respective solutions can be trivially lifted to ones in this theory. Nevertheless, generalizing a result of
we will show that one can define a collection of vector valued ‘winding numbers’ \( \{v_J \in \mathbb{Z}^8 \} \) associated with the action of \( U(1)^8 \), which encode the topology of \( \mathcal{M} \), together with a collection of ‘moduli’ \( \{l_I \in \mathbb{R}_+ \} \). Furthermore, we prove that each connected component of the solution space, characterized by these data and the asymptotic quantities \( m, J, Q[C_r] \), consists of at most one solution.

(iii) The hidden symmetries also make it possible to derive certain general, non-trivial relations between the thermodynamic quantities in the class of solutions under consideration. One example of such a formula is the generalization of the well-known ‘Smarr relation’. In four-dimensional Einstein–Maxwell theory, this relation is \( \frac{1}{2\pi} k \omega_h = m - 2\Omega J - \Phi Q - \Psi P \), where \( k, \omega_h, \Omega, \Phi, \Psi, Q, P \) are respectively, the surface gravity, horizon area, horizon angular velocity, horizon electrostatic/magnetostatic potential and electric/magnetic charge. We will give an appropriate version of this relation for the black holes in 11-dimensional supergravity under consideration. More interestingly, there exist further non-trivial relations of this sort. In section 4, we will derive, using the \( E_{(k+l)h} \) hidden symmetry, e.g. the formula

\[
0 = -\frac{1}{4} \epsilon^{ijklmnpq} \Phi_{lq} \Phi_{mnp} P_{pj} + 9 \delta^{ij} J_k - 8 m \Omega^j, \tag{1}
\]

or

\[
- (\delta^{jl} - \Omega^j \Omega^l) \left( \delta^{km} - \Omega^k \Omega^m \right) P_{lm} = -\frac{1}{4\pi} \Psi^{k} \kappa \omega_h + 2 \Psi^{km} \Phi_{mn} Q^{jk} + 4 \Psi^{lk} \Phi_{lm} Q^{jk} - 8 \Psi^{[lj]} \Phi_{[kl]} Q^{jk} + 4 \Psi^{jk} \Phi_{lm} Q^{lm} \tag{2}
\]

where \( \Omega^j, \Phi_{ij}, \Psi^{ij} \) and \( Q^{ij}, P_{ij} \) are appropriate generalizations of the angular velocities, electric/magnetic potentials and electric/magnetic charges associated with the various cycles in the horizon manifold\(^2\). We also derive similar other relations of this nature, the more detailed analysis of which we leave to another paper. We emphasize that all these relations are found using only the consequences of the hidden symmetry, and are not obtained from particular, explicit, solutions of the kind that we consider.

Since we are dealing with thermodynamic relations in (iii), it is reasonable to also give the appropriate version of the most basic one, namely the first law of black hole mechanics. We will do this in section 2. Unlike in the rest of this paper, we are assuming here only that the black hole under consideration is stationary, but not that it is also invariant under \( U(1)^8 \). In particular, one has, in principle, the possibility that the horizon manifold might be of a rather general type\(^3\). The version of the first law for 11-dimensional supergravity—indeed, independent of any by-hand symmetry assumptions—is\(^4\)

\[
\frac{1}{8\pi} \kappa \delta \omega_h = \delta m - \sum_{i=1}^{8} \Omega^j \delta J_i - \sum_{r,s} (I^{-1})^{rs} \Phi[C_r] \delta Q[C_s] \tag{3}
\]

where \( C_r \) resp. \( C_s \subset \mathcal{M} \) now run over the various 7-cycles resp. 2-cycles in the horizon submanifold, and where \( I_{rs} \in \mathbb{Z} \) is the matrix of their intersection numbers.

Actually, as we recall, there are strictly speaking two interpretations of the first law, which are in effect different mathematical theorems. The difference between these interpretations concerns the nature of the variations for which the first law holds. In the more restricted,

\(^2\) We will restrict ourselves in section 4 to a horizon of topology \( S^2 \times T^7 \), compare footnote 3. The indices are related to the cycles in this manifold and run from 1, \ldots , 7.

\(^3\) When the isometry group contains \( U(1)^8 \), the horizon topology is restricted to the entries in table 1. In the general case one only knows that the horizon manifold is of ‘positive Yamabe type’ [21].

\(^4\) Throughout the rest of the paper, we set the coefficient in front of the action to be 1 rather than 1/16\( \pi r \). This will result in trivial changes in the prefactors in the thermodynamic relations.
original, version [1] one is considering only variations within the space of stationary solutions. In the more general physical process version [24, 64], one considers also non stationary variations which satisfy the linearized equations of motion on the given black hole background, and which settle down, at late times, to a perturbation towards another stationary black hole. In this paper, we will demonstrate that the first law holds in this physical process sense.

Notations and conventions. Our conventions for the metric and Riemann tensor follow those of [63]. We also use standard notations for differential forms; our conventions are recalled in appendix. a, b, . . . , are 11-dimensional spacetime indices, while i, j, 1, J are indices labeling the various Killing fields that we assume. Throughout, we set the conventional prefactor of \(1/16\pi\) in front of the action equal to 1 for simplicity.

2. First law of black hole mechanics

2.1. Covariant phase space method

The bosonic fields in 11 dimensional supergravity are a Lorentizan metric \(g\) and a 3-form field \(A\) on an oriented 11-dimensional spacetime manifold \(M\). All fermionic superpartners are set zero throughout the paper. The field equations for the bosonic fields follow from the Lagrange 11-form \(L\), given by

\[
L = R \ast 1 - 2 F \wedge \ast F - \frac{4}{3} A \wedge F \wedge F, \tag{4}
\]

where \(F = \text{d}A\) is the associated field strength 4-form. In this paper, we are interested in stationary black hole solutions in this theory, and in the present section we would like to derive the physical process version of the first law of black hole mechanics. A convenient formalism to derive such relations is the covariant phase space method of [65]. This formalism applies to any Lagrangian \(L\) which is constructed locally and in a diffeomorphism covariant way out of a metric and tensor fields on an \(n\)-dimensional manifold \(M\), and their derivatives. To save writing, we denote these collectively by \(\psi\); in the above Lagrangian, \(\psi \equiv (g, A)\). The basic relations in the covariant phase space formalism are readily derived as follows. One considers one-parameter families of field configurations \(\psi_\lambda\), and writes \(\delta\psi = \frac{d}{d\lambda}\psi_\lambda\) for the tangent (‘variation’) at a given \(\lambda\), e.g. \(\lambda = 0\). For the above Lagrangian, \(\delta\psi = (\delta g, \delta A)\). The variation of the Lagrange \(n\)-form may always be written as

\[
\delta L(\psi) = E(\psi) \cdot \delta\psi + d\theta(\psi, \delta\psi), \tag{5}
\]

where \(E\) are the Euler–Lagrange equations, and where \(d\theta\) corresponds to the ‘partial integrations’ that one would carry out if the variation was performed under an integral sign. Let \(X\) be any vector field on \(M\). Then the ‘Noether current’ is the \((n-1)\)-form defined by

\[
J_X(\psi) = \theta(\psi, L_X(\psi)) - i_X L(\psi), \tag{6}
\]

where \(L_X\) is the Lie-derivative, and where \(i_X\) is the operator that contracts the vector field into the first index of the differential form. When the Euler–Lagrange equations \(E = 0\) hold, we have

\[
dJ_X(\psi) = 0, \tag{7}
\]

by a one line calculation using the formula \(L_X = i_X \text{d} + \text{d}i_X\) for the action of the Lie-derivative on a differential form. Since this is an identity that holds for any \(X\), one can prove [41] that

We remark that [59] has also derived a physical process version in Einstein-p-form theory, but he gives the work terms involving the various electric charges only implicitly, and not in the form above.

In particular, we are assuming in this section that \(A\) is globally defined on \(M\). Therefore, there are no magnetic charges as automatically \(\int_C F = \int_C \text{d}A = 0\) for any closed 4-cycle \(C\).
there must always exist a \((n - 2)\)-form \(Q_x\), called ‘Noether charge’, locally constructed from the fields and their derivatives, such that \(dQ_x = \mathcal{J}_x\). When the Euler–Lagrange equation do not hold, one can prove [41] that there is an \((n - 1)\) form \(C_x\), locally constructed from the fields \(\psi\) and their derivatives, and from \(X\) but not its derivatives, such that
\[
\mathcal{J}_x = C_x + dQ_x.
\]

Clearly, \(C_x = 0\) when the Euler–Lagrange equations hold, so \(C_x\) corresponds to the constraints of the theory.

One normally focuses on solutions and manifolds \(\mathcal{M}\) obeying certain asymptotic conditions. A typical condition is that \(\mathcal{M}\) contains an ‘asymptotic region’ \(\mathcal{M}_{\text{asymptotic}}\) diffeomorphic to \(\mathbb{R}^{n-1,1}\) minus some ‘interior’, and that the metric approaches the standard flat Minkowski metric \(g_0\) at a suitable rate in this asymptotic region, whereas the other fields also obey corresponding suitable fall-off conditions. In the case of 11-dimensional supergravity, \(\psi \to \psi_0\), where \(\psi_0 = (g_0, 0)\) consists of the Minkowski metric and the trivial 3-form field. This background configuration obviously has symmetries, \(\mathcal{L}_X \psi_0 = 0\), consisting of the Killing vector fields of Minkowski space. These generate the asymptotic symmetry group \(SO(n - 1, 1) \times \mathbb{R}^p\). Other asymptotic conditions may also be considered. For example, asymptotic Kaluza–Klein boundary conditions state that there is an asymptotic region of \(\mathcal{M}\) modeled on \(\mathbb{R}^{s-1} \times T^{n-r-1}\). This background carries the natural flat product metric \(g_0\), which is the direct product of an \(s + 1\)-dimensional flat Minkowski metric and a flat metric on the torus \(T^{n-r-1}\), together with a suitable background 3-form field \(A_0\), which is Lie-derived by the Killing fields of \(g_0\). In that case, the asymptotic symmetry group is \(U(1)^{s-r-1} \times SO(s, 1) \times \mathbb{R}^{r+1}\). The precise definitions and asymptotic conditions in these cases are given in appendix C.

With each asymptotic symmetry, one can associate in a natural way a corresponding conserved quantity \(H_X(\psi)\) [65] in the following way. Let \(\psi\) be solution to the Euler–Lagrange equations satisfying the asymptotic conditions, and let \(\delta \psi\) be a variation satisfying the linearized Euler–Lagrange equations (around \(\psi\)) in the asymptotic region of \(\mathcal{M}\), but not necessarily in the interior. The conserved quantity associated with \(X\) is defined by its variation through the formula
\[
\delta H_X(\psi) = \int_{\Sigma} \delta Q_X(\psi) - \int_{\Sigma} \theta(\psi, \delta \psi),
\]

\(^7\) Here we have a choice which of the non-diffeomorphic flat metrics on the torus we would like to choose. The ‘moduli space’ of such metrics is \(SL(n - s - 1, \mathbb{R})/SL(n - s - 1, \mathbb{Z})\), the local coordinates of which include the sizes of the torus in the various diameters. In this paper, we will choose a fixed flat metric, but more generally, one could leave the particular choice unspecified. This would result in additional ‘tension-type’ terms in the first law, as discussed e.g. in [44].
be $C_{\Sigma} \cong S^{n-1} \times T^{n-r-1}$. We then also have the vector fields $X = \partial/\partial \phi_i$ along the generators of the torus $T^{n-r-1}$ and corresponding conserved charges.

We now apply Stokes’ theorem to equation (9), to obtain\(^8\)

$$\delta H_{\Sigma} = \int_{\Sigma} d(\delta Q_X - i_X \theta) + \int_{\Sigma} \delta Q_X - i_X \theta$$

(10)

where $\partial \Sigma$ denotes any interior boundary or other asymptotic ends of $\Sigma$—if none of those are present, that term is simply set equal to zero. In this paper, we will have in mind the situation where $\Sigma$ is a surface stretching between a cross section $\mathcal{B} = \partial \Sigma$ of the event horizon of a black hole and infinity. We next use equation (8) to replace

$$d\delta Q_X(\psi) = d\delta J_X(\psi) = \delta C_X(\psi),$$

(11)

and we also use the formula [41]

$$\delta J_X(\psi) = \omega(\psi, \delta \psi, \mathcal{L}_X \psi) + d i_X \theta(\psi, \delta \psi),$$

(12)

where $\omega$ is the symplectic current $(n-1)$ form which depends on a pair of variations via

$$\omega(\psi; \delta_1 \psi, \delta_2 \psi) = \delta_1 \theta(\psi, \delta_2 \psi) - \delta_2 \theta(\psi, \delta_1 \psi) - \theta(\psi, (\delta_1 \delta_2 - \delta_2 \delta_1) \psi).$$

(13)

Now suppose that $X$ Lie-derives the solution $\psi$, i.e. is in particular a Killing field of the metric. Then $\omega(\psi, \delta \psi, \mathcal{L}_X \psi) = 0$, and using equations (12), (11) in equation (10) gives

$$\delta H_{\Sigma} = -\int_{\Sigma} \delta C_X + \int_{\Sigma} \delta Q_X - i_X \theta.$$  

(14)

This equation holds whenever $\psi$ is a solution to the Euler–Lagrange equations satisfying the asymptotic conditions, which is Lie-derived by $X$, and for any variation $\delta \psi$ satisfying the asymptotic conditions, and satisfying the linearized Euler–Lagrange equations near infinity (not necessarily the interior). The relation (14) will be the basis for the derivation of the first law of black hole mechanics in the next subsection. There, we will also use that, under the same conditions,

$$d\delta C_X(\psi) = d\delta J_X(\psi) - d^2 Q_X(\psi) = d\omega(\psi; \delta \psi, \mathcal{L}_X \psi) - d^2 i_X \theta(\psi, \delta \psi) = 0,$$

(15)

i.e. $\delta C_X$ is a conserved current.

In the case of 11-dimensional supergravity, we find in appendix A that the Noether charge respectively constraints are concretely given by

$$Q_X = -\bullet dX - 4 i_X A \wedge q + \frac{1}{4} i_X A \wedge A \wedge F,$$

$$C_X = 2 \bullet f_X + 4 i_X A \wedge \bullet j.$$  

(16)

Here, we have identified $X$ (not necessarily a Killing field) in the first term on the right side of $Q_X$ with a 1-form, and we have introduced the ‘electric’ charge density 7-form $q$ by

$$q = \bullet F + F \wedge A,$$

(17)

Furthermore, $f_X$, a 1-form, is obtained by contracting the Euler–Lagrange equation for the metric $g$ into $X$; concretely $f_X = (G_{ab} - T_{ab}) X^a \, dx^b$, where $T_{ab}$ is the stress tensor, see (A.8). Also, $j$, an 8-form, is the Euler–Lagrange equation for $A$. The explicit form of $j$ is given in equation (A.7); in fact, $j$ may also be written as

$$\bullet j = dq.$$  

(18)

\(^8\) Here and throughout the rest of the paper, the orientation of $\Sigma$ is fixed by the $n-1$ form defined as $\epsilon_{a_1, \ldots, a_n} = -n \, l_{a_0} \epsilon_{a_1, \ldots, a_n}$, where $r^a$ is the future directed timelike normal to $\Sigma$. An orientation on $\partial \Sigma$ is fixed by $\epsilon_{a_1, \ldots, a_{n-1}} = +(n-1) \, r^a \epsilon_{a_1, \ldots, a_{n-1}}$, where $r^a$ is the spacelike normal to $\partial \Sigma$ pointing toward the interior of $\Sigma$.\(\)
$j$ is interpreted as the ‘electric’ current density and $q$ as the charge density. $f_X$ is physically interpreted as minus the flux vector of non-gravitational energy across the horizon as seen by an observer following the flow lines of $X$. Of course, when the Euler–Lagrange equations hold, $j = 0 = f_X$.

### 2.2. Derivation of first law

After these preliminaries, we now derive the ‘physical process version’ of the first law of black hole mechanics for 11-dimensional supergravity. We consider solutions $(\mathcal{M}, g, A)$ representing a stationary black hole, satisfying either asymptotically flat or Kaluza–Klein boundary conditions. The asymptotically timelike Killing field is denoted by $\partial/\partial t$, so $\mathcal{L}_{\partial/\partial t} g = 0 = \mathcal{L}_{\partial/\partial t} A$. In the asymptotic region, $t$ is equal to the time-coordinate in an asymptotically Cartesian coordinate system. We will only be concerned with the exterior of the black hole, also called the ‘domain of outer communication’, and defined more precisely by

$$\mathcal{M}_{\text{exterior}} = \Gamma(\mathcal{M}_{\text{asymptotic}}) \cap I^+(\mathcal{M}_{\text{asymptotic}})$$

where we mean the causal past/future of the asymptotic region. In the following we will usually write simply $\mathcal{M}$ again for the exterior. The future and past event horizon are then the boundary components $\partial \mathcal{M} = \mathcal{H}^+ \cup \mathcal{H}^-$ lying respectively to the future/past of the asymptotic region. By construction, they are smooth null surfaces, which may be connected (single black hole), or disconnected (multiple black holes). For definiteness, we will restrict ourselves to single black hole spacetimes, although all of our arguments will equally apply to multiple black holes as well with trivial modifications. The situation is illustrated by the following Penrose diagram of the higher dimensional Schwarzschild/black string spacetime.

In the figure, the exterior region $\mathcal{M}_{\text{exterior}}$ is shaded.

As is common, we restrict ourselves to the consideration of metrics $g$ which are smooth everywhere, including an open neighborhood of the horizon $\mathcal{H}$. The same is also required for the field strength $F$. However the potential, $A$, while required to be smooth away from $\mathcal{H}$, is allowed to be singular on $\mathcal{H}$; we only demand that the pull-back of $A$ to $\mathcal{H}$ be smooth away from the bifurcation surface $\mathcal{B}_0$, and that the pull-back to $\mathcal{B}_0$ be smooth.\(^9\)

The restriction of $\partial/\partial t$ to the event horizon $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ may either point along the null generators, or not. (Note that this notion of `(non-)rotating` is logically distinct from

\(^{9}\) The nature of this requirement can be illustrated in Einstein–Maxwell theory [23]. The 1-form field $A$ in the Reissner–Nordström solution is given by $A = -(Q/r) \, dt$, which in Kruskal-type coordinates is $A = -(Q/2\pi r)(U^{-1} \, dU - V^{-1} \, dV)$, where $\mathcal{H}^+ = \{V = 0\}$, $\mathcal{H}^- = \{U = 0\}$. Clearly, the restriction to either $\mathcal{H}^+$ is singular, but the pull-back is smooth away from the bifurcations surface $U = 0 = V$. A gauge can be adapted so that $A$ becomes smooth near $\mathcal{H}$, but then it is either no longer Lie-derived by $\partial/\partial t$, or it does not drop to zero near infinity, as required by our asymptotic conditions.
whether the angular momenta vanish or not.) In the first case, the black hole is said to be non-rotating; otherwise it is said to be rotating. If the black hole spacetime is rotating, asymptotically flat, non-extremal, globally hyperbolic, and analytic, then the ‘rigidity theorem’ [36, 48, 37] states that\(^{10}\) there exist \(\Omega \geq 1\) further vector fields \(\xi_i\) which commute, 

\[
[\xi_i, \xi_j] = 0 = [\xi_i, \partial/\partial t],
\]

which Lie-derive the fields, 

\[
L_{\xi_i} g = 0 = L_{\xi_i} A,
\]

which have \(2\pi\) periodic flows, and such that 

\[
K = \frac{\partial}{\partial t} + \Omega^1 \xi_1 + \cdots + \Omega^N \xi_N
\]  

(20)

is a Killing field which is tangent to the generators of the horizon \(\mathcal{H}\). The constants \(\Omega^i\) are referred to as the ‘angular velocities’ of the horizon. For a non-rotating black hole, \(\Omega^i = 0\), and \(K = \partial/\partial t\). One can show [63] that the surface gravity, \(\kappa\), defined by 

\[
\nabla_K K = \kappa K
\]  

(21)

is constant (and positive) over \(\mathcal{H}\). Since the Killing fields \(\xi_i\) must belong to the five-dimensional Cartan subalgebra of the Lie algebra of the asymptotic symmetry group \(SO(10,1) \times \mathbb{R}^{11}\), it is clear that \(N \leq 5\) in the asymptotically flat case. In the asymptotically Kaluza–Klein case, or for non-analytic stationary solutions—if these should exist—we do not have as yet an analogue of the rigidity theorem, so we simply assume the existence of the additional Killing fields \(\xi_i\). Note that in this case, the asymptotic symmetry group is \(U(1)^{10-s} \times SO(s, 1) \times \mathbb{R}^{s+1}\), whose Cartan subalgebra has dimension \(10 - \lfloor \frac{s^2}{2} \rfloor\), i.e. it is larger.

We now come to the derivation of the ‘physical process’ version of the first law. We take a ten-dimensional surface \(\Sigma_0\) in (14) going between a cross section \(\mathcal{B}_0\) of the horizon to infinity as indicated in the above figure. We also take \(X = K\) in (14), and use that \(H_{\partial/\partial t} = m, H_{\xi_i} = -J_i\) are the mass resp. angular momenta. This gives us 

\[
\delta m - \Omega^i \delta J_i = -\int_{\Sigma_0} \delta C_K + \int_{\partial B} (\delta Q_k - i_k \theta).
\]  

(22)

As equation (14), this equation will hold if the variation \((\delta g, \delta A)\) satisfies the linearized Euler–Lagrange equations near infinity. We assume this for the rest of the section. Furthermore, we assume: (i) the variation \((\delta g, \delta A)\) vanishes in an open neighborhood of \(\mathcal{B}_0\), (ii) The non-gravitational part of the stress-energy, \(t_{ab}\), and the non-‘electromagnetic’ part of the current, \(j^{abc}\) (cf equation (A.7)), have compact support on a later surface \(\Sigma_1\) as shown in the next figure, and \((\delta g, \delta A)\) approach a perturbation to another stationary black hole at a sufficiently fast rate. The physical meaning of these requirements is that (i) the black hole is initially unperturbed near the horizon, (ii) all matter and charge eventually fall into the black hole, and the perturbed black hole settles down to another stationary black hole.

Using (i), we immediately see that the second term on the right side of (2.2) is zero. Using (ii) and the fact that \(\delta \delta C_K = 0\) for a Killing field \(K\), we can write the first term on the right \(^{10}\) See especially [37] for the treatment of actions with Chern–Simons type terms.
side as an integral over $\mathcal{H}^+$. Thus,

$$\delta m = \sum_{i=1}^{N} \Omega_i \delta J_i = -2 \int_{\mathcal{H}^+} \delta C_k = -2 \int_{\mathcal{H}^+} \star \delta f_k - 4 \int_{\mathcal{H}^+} i_k A \land \delta (\star j) + \delta (i_k A) \land \star j$$

$$= -2 \int_{\mathcal{H}^+} \star \delta f_k - 4 \int_{\mathcal{H}^+} i_k A \land d\delta q. \quad (23)$$

In the second step we have used the concrete expression for the constraints in 11-dimensional supergravity, see equations (16), whereas in the third step we used that $\star j = dq = 0$ for the electromagnetic current (17) of the background solution. We next evaluate the terms on the right side. First, as we will argue momentarily, the pull back of the 2-form $i_k A$ to $\mathcal{H}^+$ is closed, $d i_k A = 0$. Therefore, the second term on the right side of (23) can be written as

$$\int_{\mathcal{H}^+} i_k A \land d\delta q = \int_{\mathcal{H}^+} d(i_k A \land \delta q)$$

$$= -\int_{\mathcal{H}^+} i_k A \land \delta q + \int_{\mathcal{H}^+} i_k A \land \delta q \quad (24)$$

via Stokes’ theorem, where $\mathcal{B}_1$ is a cross section of the horizon as indicated in the figure, which is ‘later’ than the support of $\delta j$. Since the variation vanishes by assumption (i) near $\mathcal{B}_0$, the first integral on the right side is zero. To write the second integral in more recognizable form, we choose basis of cycles

$$C_r \in H_1(\mathcal{B}_1, \mathbb{Z}), \quad C_s \in H_2(\mathcal{B}_1, \mathbb{Z}), \quad \text{I}_{rs} := \#(C_r \cap C_s) \in \mathbb{Z}, \quad (25)$$

in the nine-dimensional horizon cross section $\mathcal{B}_1$. $I_{rs}$ denotes the matrix of intersection numbers, i.e. the number of intersection points counted with ± signs determined by the relative orientations. Then setting

$$\Phi[C_r] = -\int_{C_r} i_k A, \quad r = 1, \ldots, \dim H_2(\mathcal{B}_1, \mathbb{Z})$$

$$Q[C_s] = 4 \int_{C_s} q, \quad s = 1, \ldots, \dim H_7(\mathcal{B}_1, \mathbb{Z}) \quad (26)$$

and using de-Rahm’s theorem, the integral (24) becomes

$$-4 \int_{\mathcal{H}^+} i_k A \land \delta j = \sum_{r,s} (I^{-1})^{rs} \Phi[C_r] \delta Q[C_s]. \quad (27)$$

The number $\Phi[C_r]$ is interpreted as the ‘electrostatic potential’ of the horizon associated with the 2-cycle $C_r$ of the horizon cross section, whereas $Q[C_s]$ is interpreted as the ‘electric’ charge associated with the 7-cycle $C_s$. It remains to be shown that the pull back of $d i_k A$ to $\mathcal{H}^+$ vanishes. We have

$$di_k A = -i_k dA + \mathcal{L}_k A = -i_k F, \quad (28)$$

so we need to show that $i_k F = 0$ when pulled back to $\mathcal{H}^+$. Let $\kappa$ be a vector field tangent to affinely parameterized null geodesic generators of $\mathcal{H}^+$. It is evidently proportional to $K$ at every point of $\mathcal{H}^+$. By equation (21), if $U$ is the affine parameter, so that $\kappa = \partial / \partial U$ in suitable coordinates, the relation is in fact $K = \kappa U \partial / \partial U$. The Raychaudhuri equation states that

$$\frac{d}{dU} \theta = -\frac{1}{9} \theta^2 - \sigma_{ab} \sigma^{ab} - R_{ab} k^a k^b$$

$$= -\frac{1}{9} \theta^2 - \sigma_{ab} \sigma^{ab} - T_{ab} k^a k^b \quad (29)$$

11 The choice of orientations was specified in footnote 8.
where $T_{ab}$ is the stress tensor of the 3-form field, see equation (A.8), and where in the second line we used Einstein’s equation. Now, for a stationary black hole one finds by the same argument as used in the area theorem that $\vartheta$, the expansion of the geodesic congruence generated by $k$, vanishes on $\mathcal{H}^+$. Consequently, since all three terms on the right side are non-positive, we must have $\vartheta = T_{ab} k^a k^b$ on $\mathcal{H}^+$. By equation (A.8), this gives $F_{abc} F^c_{\alpha \beta} k^\alpha k^\beta = 0$ on $\mathcal{H}^+$, which in turn implies that $i_k F = k \wedge \alpha$ for some 2-form $\alpha$, where $k$ has been identified with a 1-form via $g$. Viewed as a 1-form, $k$ has vanishing pull-back to $\mathcal{H}^+$, therefore so has $i_k F$, which completes the argument.

We now evaluate the first integral on the right side in our balance equation (23). Using the definition $f_K = (G_{ab} - T_{ab}) X^a \, dx^b$, his may be written alternatively as

$$-2 \int_{\mathcal{H}^+} \text{d} f_k = 2 \kappa \int_{\mathcal{H}^+} U \, \vartheta (R_{ab} k^a k^b - T_{ab} k^a k^b) \epsilon \tag{30}$$

where $\epsilon$ is the positively oriented ten-dimensional volume element on $\mathcal{H}^+$ defined by $k \wedge \epsilon = - \star 1$. For the variation of the Ricci tensor component $R_{ab} k^a k^b$, we obtain from the variation of the Raychaudhuri equation

$$\delta (R_{ab} k^a k^b) = \delta R_{ab} k^a k^b + 2R_{ab} k^a \delta k^b = - \frac{d}{dU} \delta \vartheta + 2T_{ab} k^a \delta k^b. \tag{31}$$

We may assume that we are in a gauge such that $\delta k^a$ is proportional to $k^a$. Then from $T_{ab} k^a k^b = 0$ on the horizon, it follows that $T_{ab} k^a \delta k^b = 0$ on the horizon. Using this, and the formula (A.8) for $T_{ab}$, we similarly find

$$\delta (T_{ab} k^a k^b) = \frac{1}{2} k^a \delta F_{abcd} k^c F^d_{\kappa \lambda} - \delta k^a F_{abcd} k^b F_{\kappa \lambda} = \frac{1}{2} k^a \delta F_{abcd} k^b F_{\kappa \lambda} - \delta k^a F_{abcd} k^b F_{\kappa \lambda} = 0, \tag{32}$$

using $\delta g_{ab} k^a \propto k_b$. Thus, we have shown that the first integral in the balance equation (23) is

$$-2 \int_{\mathcal{H}^+} \text{d} f_k = - 2 \kappa \int_{\mathcal{H}^+} U \frac{d}{dU} \delta \vartheta \epsilon = 2 \kappa \delta \omega_h, \tag{33}$$

where the second equality follows from the calculation in section 6.2 of [66], and where $\delta \omega_h$ is the variation of the area of a horizon cross section at asymptotically late times$^{12}$. Combining equations (23), (27), (33), we get

$$2 \kappa \delta \omega_h = \delta m - \Omega^2 \delta J_s - \sum_{i,s} (J^{-1})^a \Phi [C_i] \delta Q [C_i]. \tag{34}$$

This is the desired first law of black hole mechanics in 11-dimensional supergravity. Evidently, it has the same general form as the first law in Einstein–Maxwell theory, but the detailed expression of the term involving ‘electric’ potentials and charges depends on the topology of the nine-dimensional compact horizon cross section in question. It enters via the matrix of intersection numbers $I_{rs}$ between the 2-cycles and 7-cycles inside the horizon cross section.

3. Black hole uniqueness theorem

We now prove a uniqueness theorem for stationary asymptotically Kaluza–Klein black holes having eight commuting rotational vector fields $\xi_1, \ldots, \xi_8$ which Lie-derive $g$ and $A$, and which also commute with the action of the asymptotically time-like Killing field $\vartheta/\partial t$. Hence, from now we assume that the isometry group is $\mathbb{R} \times U(1)^8$. This is consistent with an asymptotic region of the form $\mathbb{R}^{1,1} \times T^{10-s}$ for $s = 1, 2, 3, 4$ large spatial dimensions. For definiteness, we will stick to the case $s = 4$.

$^{12}$ To make the integral over $\mathcal{H}^+$ converge, we are assuming at this stage our assumption (ii) that the perturbation settles down to a perturbation to another stationary black holes at a sufficiently fast rate.
3.1. Structure of orbit space, Weyl–Papapetrou form

As a first step towards a uniqueness theorem, we have to better understand the global nature of the spacetime $M$, its topology, and the global nature of the action of $U(1)^8$. Furthermore, as in the vacuum theory, it is essential to use the symmetries and field equations in order to construct particularly useful coordinates. First, to set up some notation, we introduce the eight-dimensional Gram matrix of the rotational Killing fields, denoted by

$$f_{ij} = g(\xi_i, \xi_j).$$

We say that a point $P \in \mathcal{M}$ is on an ‘axis’ if there is a linear combination

$$\sum_{i=1}^{8} v^i \xi_i \bigg|_P = 0.$$  (35)

It is not difficult to see [39] that $v = (v^1, \ldots, v^8)$ must be in $\mathbb{Z}^8$ up to some overall rescaling, and we fix that rescaling by the requirement $\text{g.c.d.}(v^1, \ldots, v^8) = 1$, where g.c.d. is the greatest common divisor. The particular linear combination will in general depend on the point $P$. The first major step is the following theorem.

**Theorem 1** (Weyl–Papapetrou-form). The metric can be brought into Weyl–Papapetrou form (37) away from the horizon $\mathcal{H}$ and any axis of rotation.

$$g = -\frac{r^2}{\det f} \frac{dr^2}{r^2} + e^{-\nu}(dr^2 + dz^2) + f_{ij}(d\phi^i + w^i \, dt)(d\phi^j + w^j \, dt),$$

where $\phi^i$ are $2\pi$-periodic coordinates such that the rotational Killing fields $\xi_1, \ldots, \xi_8$ take the form $\xi_i = \partial / \partial \phi^i$, and where $t$ is a coordinate such that the timelike Killing field takes the form $\partial / \partial t$. In other words, the metric functions $f_{ij}, w^i, \nu$ are independent of $t, \phi^1, \ldots, \phi^8$, and only depend on $z \in \mathbb{R}, r > 0$.

**Proof.** The proof of this statement is given for the vacuum theory in $n$ dimensions in [39] relying e.g. on global results such as topological censorship [20, 10, 19] and results on spaces with torus actions generalizing those of [52, 53]; here we will only outline the (minor) modifications that have to be made in the case of 11-dimensional supergravity. The proof consists of essentially four ingredients. (a) The distribution of subspaces $\text{span}(\partial / \partial t, \xi_1, \ldots, \xi_8) \perp \subset T \mathcal{M}$ is locally integrable for a solution to the equations of motion in 11-dimensional supergravity. This expresses that the metric (37) has no ‘cross terms’ between $\partial / \partial t$ and the $\nu$-function. The axis of rotation $t$. In other words, the metric functions $f_{ij}$, $w^i$, $\nu$ are independent of $t, \phi^1, \ldots, \phi^8$, and only depend on $z \in \mathbb{R}, r > 0$.

(b) The function $r$ on $\mathcal{M}$ defined by

$$r^2 = -\det \begin{pmatrix} g(\partial_t, \partial_t) & g(\partial_t, \xi_i) \\ g(\xi_i, \partial_t) & g(\xi_i, \xi_j) \end{pmatrix}$$

is globally defined on $\mathcal{M}$, except at the axis and the horizon. In particular, the right side is positive everywhere on $\mathcal{M}$, except at these places, where it is zero. This expresses that the span of $\partial / \partial t, \xi_1, \ldots, \xi_8$ is everywhere a timelike distribution of nine-dimensional subspaces of $T \mathcal{M}$, except at the horizon or the axis. (d) The function $r$ is a harmonic function on the orbit space $\mathcal{M} \equiv \mathcal{M} / G$. This enables one to define $z$ as the conjugate harmonic function on $\mathcal{M}$, so $\mathcal{M}$ is parameterized by $\mathcal{M} = \{(r, z) \mid r > 0\}$ away from the axis and the horizon. Since $(r, z)$ is a harmonically conjugate pair, the induced metric on the orbit space is $e^\nu (dr^2 + dz^2)$ for
some conformal factor $e^\nu$. Statements (a)–(d) are equivalent to the statement that the metric takes the form (37).

The claims (a),(d) are local in nature and follow from the equations of motion. By contrast, the claims (b),(c) are global in nature. Their proof involves the equations of motion, but also global techniques from topology. The global properties (b),(c) in particular imply that the coordinate system can be constructed globally, away from the horizon and the axis.

The proof of (a) is standard for $n$-dimensional vacuum general relativity and four-dimensional Einstein–Maxwell theory. Here we give the proof in the case of 11-dimensional supergravity. For definiteness, we repeat the statement:

**Lemma 1.** The distribution of subspaces $(\text{span}(\partial/\partial t, \xi_1, \ldots, \xi_8))^\perp \subset T\mathcal{M}$ is locally integrable for a solution to the equations of motion in 11-dimensional supergravity.

**Proof.** Let us denote the Killing fields collectively as $\xi_I$, $I = 0, \ldots, 8$, with $\xi_0 = \partial/\partial t$, and $\xi_I, I = 1, \ldots, 8$ denoting the remaining rotational Killing fields. In view of the ‘differential forms’ of Frobenius’ theorem, we must prove that $d\xi_I = \alpha^j I \wedge \xi_I$ for some 1-forms $\alpha^j I$, or what is the same

$$0 = \xi_0 \wedge \cdots \wedge \xi_8 \wedge d\xi_I$$

for any $I$. Here, we have identified the Killing vectors with 1-forms via the metric. Consider the Noether charge 9-form $Q_{\xi_I}$. Since $\xi_I$ Lie-derives the fields $g, A$, it follows from equation (6) and $dQ_{\xi_I} = {\mathcal{J}}_{\xi_I}$ that

$$dQ_{\xi_I} = -i_{\xi_I} L.$$  

(40)

We now contract all Killing fields $\xi_0, \ldots, \xi_8$ into both sides of this equation. The Killing field $\xi_I$ gets contracted twice into the form $L$ on the right side, so we get zero:

$$0 = i_{\xi_0} \cdots i_{\xi_8} dQ_{\xi_I}$$

$$= i_{\xi_0} \cdots i_{\xi_8} d(-\bullet d\xi_I - 4 i_{\xi_I} A \wedge \eta_q + \frac{1}{2} i_{\xi_I} A \wedge A \wedge F)$$

$$= d i_{\xi_0} \cdots i_{\xi_8} (\bullet d\xi_I) + i_{\xi_0} \cdots i_{\xi_8} (-4 d(i_{\xi_I} A) \wedge q)$$

$$- 4 i_{\xi_I} A \wedge dq + \frac{1}{2} d(i_{\xi_I} A) \wedge A \wedge F + \frac{1}{2} i_{\xi_I} A \wedge F \wedge F + 4 i_{\xi_I} F \wedge F \wedge A - \frac{3}{4} A \wedge i_{\xi_I} F \wedge F + \frac{3}{4} i_{\xi_I} A \wedge F \wedge F).$$

(41)

Here, we have used the concrete expression for the Noether charge (16), and we have used repeatedly the fact that $i_{\xi_I} d + di_{\xi_I} = \mathcal{L}_{\xi_I}$, together with the fact that $\mathcal{L}_{\xi_I}$ annihilates any expression formed out of $g, A$, and the $\xi_K$, by $\mathcal{L}_{\xi_I} \xi_K = [\xi_I, \xi_K] = 0$. We have also used $dA = F$ and $dq = 0$, by the equations of motion, see equation (17). We now carry out the contractions $i_{\xi_I}$ into the expression in parenthesis on the right side. When four $i_{\xi_I}$ hit the 4-form $F$, we get zero, again by

$$i_{\xi_0} i_{\xi_2} i_{\xi_3} i_{\xi_4} F = -i_{\xi_1} d(i_{\xi_2} i_{\xi_3} i_{\xi_4} A) = -\mathcal{L}_{\xi_1} (i_{\xi_2} i_{\xi_3} i_{\xi_4} A) = -i_{\xi_2} i_{\xi_3} i_{\xi_4} \mathcal{L}_{\xi_1} A = 0.$$  

(42)

Hence, the only term on the right side of (41) which potentially is not zero is the one where precisely seven insertion operators hit the charge 7-form $q$. To see that such a term vanishes as well, we note, that by $dq = 0$, we have

$$d (i_{\xi_1} \cdots i_{\xi_7} q) = i_{\xi_1} \cdots i_{\xi_7} dq = 0,$$

(43)

so the scalar function $i_{\xi_1} \cdots i_{\xi_7} q$ is constant on $\mathcal{M}$. However, by the fall-off conditions imposed on the field $A$ (see appendix C), it vanishes at infinity, so it must be equal to zero. Hence, we conclude from equation (41) that the scalar function $\bullet (d\xi_I \wedge \xi_0 \wedge \cdots \wedge \xi_8)$ must be
constant on \( \mathcal{M} \). Since there must be at least one point \( P \in \mathcal{M} \) where one linear combination of the \( \xi_j \)'s vanishes by the orbit space theorem, see below, it follows that this quantity must in fact be zero, hence (39) follows.

A more detailed version of statement (b), which elucidates also the nature of the action of the rotational isometry group \( U(1)^8 \), is given in the following theorem, which is proved in the same way as that in [39] for \( n \)-dimensional vacuum general relativity.

**Theorem 2** (Orbit space theorem). If one assumes the isometry group \( G = \mathbb{R} \times U(1)^8 \), then the orbit space \( \hat{\mathcal{M}} = \mathcal{M} / G \) is homeomorphic to an upper half plane \( \{(r, z) \mid r > 0\} \). Furthermore, the boundary \( r = 0 \) can be thought of as divided up into a collection of intervals \( (\infty, z_1) \), \((z_1, z_2)\), \ldots, \((z_{n}, +\infty)\), each of which either represents the orbit space \( \hat{\mathcal{M}} = \mathcal{M} / G \) of the horizon (one interval per horizon component, if multiple horizons are present), or an axis in the spacetime where a linear combination \( \sum v_j^i \psi_i \) of the rotational Killing fields vanishes. The quantity \( v_j \in \mathbb{Z}^8 \) is a vector associated with the \( J \)th interval which necessarily has integer entries. For adjacent intervals \( J \) and \( J+1 \) (not including the horizon), there is a compatibility condition stating that the collection of minors \( \mu_{kl} \in \mathbb{Z}, 1 \leq k < l \leq 8 \) given by

\[
\mu_{kl} = \left| \det \begin{pmatrix} v^i_{J+1} & v^i_J \\ v^i_{J+1} & v^i_J \end{pmatrix} \right| \tag{44}
\]

have greatest common divisor g.c.d.\( \{\mu_{kl}\} = 1 \).

**Remark.** As in the vacuum theory, the relation between \( r, z \), and the asymptotically Cartesian (spatial) coordinates \( x_1, \ldots, x_r \) in the asymptotically KK-region (see section 2.2) is

\[
(r, z) \sim \begin{cases} \left( \sqrt{x_1^2 + x_2^2}, x_3 \right), & \text{if } s = 3 \\
\left( \frac{1}{\sqrt{(x_1^2 + x_2^2)(x_3^2 + x_4^2)}}, \frac{1}{2} (x_1^2 + x_2^2 - x_3^2 - x_4^2) \right), & \text{if } s = 4. \end{cases} \tag{45}
\]

Statement (c) can be proved in the same way as in vacuum general relativity, for a proof see [8]. (d) holds whenever the theory can be locally dimensionally reduced to a certain kind of sigma model on a symmetric space, as proved in [5]. This type of sigma model reduction is recalled for 11-dimensional supergravity in section 3.2.

Some examples of interval structures in five-dimensional vacuum general relativity are summarized in the following table.

| Interval lengths | Vectors (labels) | Horizon |
|------------------|------------------|---------|
| Myers–Perry      | \( \infty, l_1, \infty \) | (1, 0), (0, 0), (0, 1) | \( S^1 \) |
| Black ring       | \( \infty, l_1, l_2, \infty \) | (1, 0), (0, 0), (1, 0), (0, 1) | \( S^2 \times S^1 \) |
| Black Saturn     | \( \infty, l_1, l_2, l_3, \infty \) | (1, 0), (0, 0), (0, 1), (0, 0), (0, 1) | \( S^2 \) and \( S^3 \times S^1 \) |
| Black string     | \( \infty, l_1, l_2, \infty \) | (1, 0), (0, 0), (1, 0) | \( S^3 \times S^2 \) |
| Black di-ring    | \( \infty, l_1, l_2, l_3, l_4, \infty \) | (1, 0), (0, 0), (1, 0), (0, 0), (1, 0), (0, 1) | \( 2 \cdot (S^3 \times S^2) \) |
| Orthogonal di-ring | \( \infty, l_1, l_2, l_3, l_4, \infty \) | (1, 0), (0, 0), (0, 1), (0, 0), (0, 1) | \( 2 \cdot (S^3 \times S^2) \) |
| Minkowski        | \( \infty, \infty \) | (1, 0), (0, 1) | — |

In this table, the interval \( (0, 0) \) corresponds to a horizon. The explicit form of the metric may be found in [51] (Myers–Perry), [55, 15] (black ring), [13] (Saturn), [40] (di–ring), and [42] (orthogonal di–ring). Of course, all these solutions can be lifted trivially to solutions in 11-dimensional supergravity; the vectors \( v_j \) would be turned into eight-dimensional vectors by filling the remaining six components with 0's.
The sequence of vectors $v_j$ encodes the entire information about the topology of $\mathcal{M}$, and the nature of the group action. In particular, the horizon topology is specified. More precisely if $v_{h+1}$ resp. $v_{h+1}$ are associated with the intervals adjacent to a horizon interval \((z_h, z_{h+1})\) we can say for example the following. Let $\mu_{kl} \in \mathbb{Z}$, $1 \leq k < l \leq 8$ be the integers defined from these two vectors as in equation (44), and set $p = \text{g.c.d.}(\mu_{kl})$. This parameter is related to the different horizon topologies by table 1. Note also that the first and last vector $v_0, v_N$ in the above solutions is always $(1, 0)$ resp. $(0, 1)$. This corresponds to the fact that these five-dimensional solutions are asymptotically flat in all five directions. In the case of 11-dimensional supergravity with five asymptotically Minkowskian dimensions, we would instead have $(1, 0, 0, 0, 0, 0, 0, 0, 0)$ and $(0, 1, 0, 0, 0, 0, 0, 0, 0)$. For a more detailed discussion of the interval structure see \cite{39, 32}. This finishes our review of the orbit space theorem, and we now explain how to actually construct the Weyl–Papapetrou coordinates (37).

### 3.2. Sigma model reduction and divergence identities

It is well-known that the field content of 11-dimensional supergravity can be reorganized into that of a gravitating sigma model into a certain coset when the theory is dimensionally reduced from 11 dimensions to three dimensions \cite{43}. This formulation can, and will, be used in the proof of our black hole uniqueness theorem below, and also in section 4. Therefore, we briefly review the construction, following the treatment given in \cite{50}. The formulation relies on the introduction of certain scalar potentials, and we now describe a conceptually simple way of defining these which also makes manifest their relation to the global conserved quantities of the theory, used later.

First, consider the closed ‘electric’ charge 7-form $q$, see equation (17). Contracting this into six out of the eight rotational Killing fields generating the action of $U(1)^8$, we get a 1-form, $\xi_{i_1} \cdots \xi_{i_6} q$. This 1-form is immediately seen to be closed using the identity $\xi_i d + d\xi_i = L_{\xi_i}$ on forms together with the fact that $L_{\xi_i}$ annihilates any tensor field that is constructed from $g, A, \xi_j$ and their covariant derivatives. Hence, at least locally, there exists a scalar function $\chi_{i_1, \ldots, i_6}$ such that

\[
d\chi_{i_1, \ldots, i_6} = i_{\xi_{i_1}} \cdots i_{\xi_{i_6}} q.
\]

Next, consider the Noether charge 9-form $Q_{\xi_i}$, see equation (16). By exactly the same argument as in the proof of lemma 1, we see that the 1-form $i_{\xi_{i_1}} \cdots i_{\xi_{i_6}} Q_{\xi_i}$ is closed. Hence, at least locally, there exists a scalar function $\chi_i$ such that

\[
d\chi_i = i_{\xi_1} \cdots i_{\xi_N} Q_{\xi_i}.
\]

Even though $\mathcal{M}$ is not simply connected, the ‘twist potentials’ $\chi_i, \chi_{i_1, \ldots, i_6}$ are in fact defined globally. This follows from the fact that, since they are invariant under the action of the isometry group $\mathbb{R} \times U(1)^8$, the defining equations can be viewed as equations for closed 1-forms on the simply connected orbit space $\mathcal{M} = \{(r, z) \mid r > 0\}$. A third set of scalars is defined by contracting three rotational Killing fields into the three form $A$,

\[
\chi_{i_1, i_2, i_3} = i_{\xi_{i_1}} i_{\xi_{i_2}} i_{\xi_{i_3}} A.
\]
It turns out that the field equations for the solutions \((g, A)\) invariant under \(\mathbb{R} \times U(1)^3\) can be written entirely in terms of the 128 scalars \((f_{ij}, x_i, x_{i1}, \ldots, x_{ik}; A_{ik}; k)\); in fact, they can be thought of as parameterizing a single field that is valued in the coset space \(E_{8(8)}/SO(16)\) which has precisely this dimension. This construction is also useful for us, so we review it following [50]. Recall that the real\(^{13}\) Lie-algebra \(\mathfrak{e}_{8(8)}\) contains \(\mathfrak{s}(9)\) as a subalgebra. As a vector space, it is given by

\[
\mathfrak{e}_{8(8)} = \mathfrak{s}(9) \oplus (\mathbb{R}^9)^{\mathfrak{c}^3} \oplus (\mathbb{R}^{9\mathfrak{c}})^{\mathfrak{c}^3}.
\]

(49)

This is also how the adjoint representation of \(\mathfrak{e}_{8(8)}\) decomposes when restricted to \(\mathfrak{s}(9)\). The corresponding \(80+84+84=248\) generators are \((e^J, e^{IJK}, e_{IJK})\) where the \(e^J\) generate \(\mathfrak{s}(9)\), and where capital Roman letters \(I, J, \ldots, \) run from \(1, \ldots, 9\). The star symbol on \(e^{IJK}\) is part of the name of the generator, and does not mean any kind of conjugation or dual. Relations and other relevant basic facts about this Lie-algebra are recalled in appendix B. Let us define the 8-bein \(e_i^\alpha\) by \(f_{ij} = \delta_{ij} e_i^\alpha e_j^\beta\), and form the following \(SL(9)\) matrix:

\[
V = \begin{pmatrix} e_i^\alpha & \text{det} e^{-1}(x_i + \frac{1}{720} \epsilon^{jkl} A_{ijjk} x_j x_{jkl}) \\
0 & \text{det} e^{-1} \end{pmatrix}.
\]

(50)

We also define the \(\mathfrak{e}_{8(8)}\) valued function \(v\) by

\[
v = e^{IJK} A_{IJK} + \frac{1}{360} e^{IJK} e^{IJKLMNPQR} X^{LMNPQR} \epsilon^{IJKLMNPQR}
\]

(51)

where \(A_{IJK} = 2\sqrt{3} A_{ijk}\) when \(I = i, J = j, K = k\) are between \(1, \ldots, 8\), and zero otherwise, as well as similarly \(X^{LMNPQR} = 2\sqrt{3} X_{LMNPQR}\) if all indices are between \(1, \ldots, 8\) and zero otherwise. We can now form the 248-dimensional matrix \(V\) valued in the adjoint representation of the group \(E_{8(8)}\) by

\[
V = \exp(\text{ad}(v)) \text{Ad}(V).
\]

(52)

Here, in the last expression, \(V \in SL(9)\) has been viewed as an element of \(E_{8(8)}\) in accordance with the decomposition (49), and where, as is common, ‘Ad’ refers to the adjoint representation of the group on its Lie-algebra, whereas ‘ad’ to the adjoint representation of the Lie-algebra on itself. Let \(\tau\) be the involution on \(E_{8(8)}\) given in appendix B, and define the matrix

\[
M = V \tau (V)^{-1}.
\]

(53)

Then it is shown in [50] that the equations of motion for \(M\) on the orbit space \(\hat{M} = \{(r, \varphi) | r > 0\}\) derived from the action

\[
I = \int_{\hat{M}} r |M^{-1} dM|_\hat{g}^\alpha \hat{d} \nu
\]

(54)

are exactly equivalent to those that can be derived for the 128 scalars \((f_{ij}, x_i, x_{i1}, \ldots, x_{ik}; A_{ik}; k)\) from the 11-dimensional supergravity Lagrangian\(^{15}\). In the above formula, \(k(X, Y) = -\text{Tr}(\text{ad}(X)\text{ad}(Y))\) indicates the Cartan–Killing form and \(d \nu\) is the integration element of the orbit space metric \(\hat{g}\).

The action \(I\) can be viewed as that of nonlinear sigma model in the symmetric space \(E_{8(8)}/SO(16)\), as follows. First, recall that a symmetric space is defined generally as a triple \((G, H, \tau)\), where \(G\) is a Lie-group with involution\(^{16}\) \(\tau\), and \(H\) is a Lie-subgroup of \(G\) satisfying \(G^\tau \subset H \subset G^\tau\), with a superscript \(\tau\) denoting the elements invariant under \(\tau\), and with

\(^{13}\)Unless stated otherwise, all Lie-groups and Lie-algebras in this paper are real, e.g. \(SL(9)\) means \(SL(9, \mathbb{R})\), etc.

\(^{14}\)It can be shown that \(v\) is nilpotent, \(\exp(v)^2 = 0\), so the exponential is in fact a polynomial in the components of \(v\) of degree 4.

\(^{15}\)To make the identification with the quantities used in [50], we should identify their potentials \(\psi_i\) resp. \(\psi'^i\) with \(x_i + \frac{1}{720} \epsilon^{ijkl} A_{ijjk} x_j x_{jkl}\) resp. \(\frac{1}{720} \epsilon^{ijkl} A_{ijjk} x_j x_{jkl}\).

\(^{16}\)An involution is a homorphism \(g \mapsto \tau(g)\) of \(G\) such that \(\tau^2 = \text{id}\) for all \(g \in G\).
the subscript 0 denoting the connected component of the identity. Given a symmetric space, one can define the principal H-bundle \( G \to X = G/H \) (right cosets). This principal fiber bundle has a global section defined by \( X \ni gH \to g \tau (g)^{-1} \in G \). If \( G \) is semi-simple (as in our example \( G = E_{8(+8)} \)) then it carries a natural metric from the Cartan–Killing form on \( g \), and the pull-back of this metric via the above global section then gives a metric \( \hat{g} \) on \( X = G/H \). It is a general theorem about symmetric spaces that if \( G \) is simply connected and non-compact, and \( H \) maximally compact, then (a) that metric \( \hat{g} \) is Riemannian, and (b) it has negative sectional curvature, see theorem 3.1 of [34]. By negative curvature, one means more precisely the following. Let \( R_{ABCD} \) be the Riemann tensor of \( \hat{g}_{AB} \). The Riemann tensor is always anti-symmetric in \( AB \) and \( CD \), and symmetric under the exchange of \( AB \) with \( CD \). Thus, it can be viewed as a symmetric, bilinear form \( \text{Riem}: \wedge^{2}T_{o}X \times \wedge^{2}T_{o}X \to \mathbb{R} \) in each tangent space of \( X \), where \( \sigma = gH \) denotes an element of \( X \). We say that \((X, \hat{g})\) has negative sectional curvature (or simply, is ‘negatively curved’) if this bilinear form only has negative eigenvalues. In other words, there is a \( c > 0 \) such that, for any anti-symmetric 2-tensor \( \omega \) we have \( \text{Riem}(\omega, \omega) \leq -c \| \omega \|^2 \), or in components,

\[
R_{ABCD}^{\omega} \omega^{CD} \leq -c \hat{g}_{AB} \hat{g}_{CD} \omega^{AB} \omega^{CD}.
\]  

We are precisely in this case, if \( G = E_{8(+8)} \), if \( \tau \) is defined as in appendix B, and if \( H = \text{SO}(16) = G_0 \). In fact, writing as above \( \hat{V} \in E_{8(+8)} \) and \( M = \tau \hat{V}^{-1} \), the metric on \( X \) is by construction equal to \( \hat{g} = -\text{Tr}(M^{-1}dM \otimes M^{-1}dM) \), and \( I \) is consequently equal to the action of a nonlinear sigma model from the upper half plane \( \hat{\mathcal{M}} \), taking values in \( X \).

Suppose now that \( \sigma_\lambda : \hat{\mathcal{M}} \to X \) is a family of solutions to the sigma model equations of motion, where \( \sigma_\lambda = \hat{V}_\lambda \cdot \text{SO}(16) \) is the right coset of the matrix \( \hat{V}_\lambda \in E_{8(+8)} \) of the solution given above in equation (52). Let \( \delta \sigma_\lambda = \frac{d}{d\lambda} \sigma_\lambda : \hat{\mathcal{M}} \to \sigma_\lambda^*TX \) be the linearization at any fixed \( \lambda \), interpreted as the infinitesimal displacement of the two-dimensional ‘worldsheet’ in \( X \) swept out by \( \sigma_\lambda \). Then, one can derive from the sigma-model field equation the following equation for \( \delta \sigma_\lambda \equiv \delta \sigma :

\[
\frac{1}{r} \hat{\nabla} (r \hat{\nabla} \delta \sigma^A) = R_{ABCD}^\lambda (\delta \sigma^B) \cdot (\delta \sigma^D) \delta \sigma^C.
\]  

Here \( \hat{\nabla} \) is the natural derivative operator in the bundle \( \sigma^*TX \to \hat{\mathcal{M}} \) that is inherited from the derivative operator of the metric \( \hat{g} \) on \( X \). This equation may be viewed as the generalization of the ‘geodesic deviation equation’ in \( X \) from curves to surfaces.

Now assume that \([0, 1] \ni \lambda \mapsto \sigma_\lambda \) connects two given solutions \( \sigma_0, \sigma_1 \) of the sigma model equations coming from two corresponding matrix functions \( \hat{V}_0, \hat{V}_1 \). Define the scalar valued function \( S \) on \( \hat{\mathcal{M}} \) by

\[
S(\lambda) := \int_0^1 \left( G_{AB}(\sigma_\lambda) \delta \sigma_\lambda^A \delta \sigma_\lambda^B \right) (\lambda) \, d\lambda \geq 0.
\]  

Then it is straightforward to derive from equation (56) the following formula

\[
\frac{1}{r} \hat{\nabla} (r \hat{\nabla} S) = \int_0^1 d\lambda \left( G_{AB}(\sigma_\lambda) \hat{\nabla} \delta \sigma_\lambda^A \cdot \hat{\nabla} \delta \sigma_\lambda^B - R_{ABCD}(\sigma_\lambda) (\delta \sigma_\lambda^A \cdot \delta \sigma_\lambda^B) \delta \sigma_\lambda^C \right)
\geq \int_0^1 d\lambda \left( G_{AB} \hat{\nabla} \delta \sigma_\lambda^A \cdot \hat{\nabla} \delta \sigma_\lambda^B + c \ G_{AB} \delta \sigma_\lambda^A \delta \sigma_\lambda^B \delta \sigma_\lambda^C \right) \geq 0,
\]  

where to get the key \( \geq 0 \) relations we have used (a) that the target space is Riemannian, and (b) that it is negatively curved. A slightly different way of writing this differential inequality, which is useful in the next section, is to define a fictitious \( \mathbb{R}^3 \) parameterized by
Proof. It is clear that the spacetime manifolds \( \mathcal{M} \) must be diffeomorphic and that the action of the group \( \mathbb{R} \times U(1)^8 \) must be equivalent for all \( \lambda \). The orbit space theorem characterizes the action of \( U(1)^8 \). Therefore, the solutions must have the same winding numbers \( \{ v_\lambda \} \). Of course, the orbit space theorem makes no statement about the dynamical fields, so we do not know what is the relation between \( (g_\lambda, A_\lambda) \) for different \( \lambda \). For this one-parameter family, consider the function \( S : \mathbb{R}^3 \setminus [z\text{-axis}] \to \mathbb{R} \) defined above in equation (57). (Recall that the coordinates \( (x, y, z) \) of this \( \mathbb{R}^3 \) are related to the coordinates of the orbit space \( \mathcal{M} = \{(r, z) \mid r > 0\} \) by \( x = r \cos \varphi, y = r \sin \varphi \), where \( \varphi \) is an angle that does
not have any straightforward relation to the coordinates on the spacetime $\mathcal{M}$. The function $S$ is sub-harmonic, equation (59), and non-negative. We wish to apply ‘Weinstein’s lemma’ [67] to $S$.

**Lemma 2 (Weinstein’s lemma).** Let $S(x, y, z) \geq 0$ be a bounded function on $\mathbb{R}^3$ which is continuous on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ and which is a solution to $(\partial^2 + \partial^2 + \partial^2)S \geq 0$, in the distributional sense. Then $S = \text{const.}$

To apply this lemma, we need to verify that our $S$ is uniformly bounded, including near infinity and the $z$-axis. At this stage we need the assumptions about the mass, interval structure, angular momenta, and charges. First, let us write down explicitly the function $S = \int_0^1 d\lambda s_\lambda$. We put $\frac{d}{d\lambda}(g_\lambda, A_\lambda) \equiv (\delta A, \delta g)$, $(g_\lambda, A_\lambda) \equiv (g, A)$. Then relying on calculations in [49], we find, with $\sigma = V \cdot SO(16)$ as above:

$$s_\lambda = G_{AB}(\sigma, g_\lambda) A^{A_\lambda}_{\mu} A^{B}_{\nu}$$

$$= 240 \left( \frac{1}{4} f^{ik} f^{jl} \delta f_{ik} \delta f_{jl} + (\delta \log \det f)^2 + \frac{1}{3} f^{il} f^{jm} f^{kn} \delta A_{ijk} \delta A_{ilm} 
+ 10 f^{mn} \delta \chi_{m} \cdots n \right) \frac{e f_{ijkl} \delta A_{ijkl}}{10} 
\times \left( \frac{1}{60} \delta \chi_{m1} \cdots m_6 = \frac{1}{3} \delta A_{[m_1 m_2 m_3]} A_{m_4 m_5 m_6} \right) 
+ \frac{1}{2} \delta \chi_{J_1 + 1} \left( \delta \chi_{J_1 + 2} \right) \frac{1}{54} e^{k_1 \cdots k_8 A_{jk_1 k_2} A_{k_3 k_4} \delta A_{k_5 k_6 0} + \frac{1}{360} e^{k_1 \cdots 8 A_{jk_1 k_2} \delta \chi_{k_3 \cdots k_8}} \times \left( \delta \chi_{J_1 + 1} \frac{1}{54} e^{k_1 \cdots k_8 A_{ij_1 j_2} A_{j_3 j_4} \delta A_{j_5 j_6 0} + \frac{1}{360} e^{k_1 \cdots 8 A_{ij_1 j_2} \delta \chi_{j_3 \cdots j_8}} \right) \right).$$

(60)

Clearly, $S$ is a sum of squares, as it has to be. To show that it is uniformly bounded, we thus need to show that each of the terms is uniformly bounded individually on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$. By construction, $S$ is smooth, so we need to make sure that it does not blow up anywhere near the $z$-axis, nor at infinity. The behavior near infinity can be controlled in a straightforward manner because the asymptotic behavior of the fields $(g, A)$, hence $(\delta g, \delta A)$ is known. Maybe the only point to note here is that we have to use the precise relation between the coordinates $r, z$ and the asymptotically Cartesian coordinates in the spacetime $\mathcal{M}$, which is given above in equation (45). The asymptotic behaviors of $A_{jk}, f_{ij}$ viewed as functions on $\mathbb{R}^3$, then follow immediately from the behavior of the fields in the original asymptotically Cartesian coordinates. Likewise, it is straightforward to determine the behaviors of the potentials $\chi, \chi_{i \cdots k}$ by integrating up the defining relations (47) resp. (46). If this is done, then it is found that $\chi_j$, hence $S$, is uniformly bounded for large $r^2 + z^2$. The details of this arguments are very similar to those given in [39] in the vacuum theory, so we do not give them here.

The more tricky part is to control the behavior of $\chi_j$, hence $S$, near the $z$-axis in $\mathbb{R}^3$. Recall that the $z$-axis is the union of intervals $[z_j, z_{j+1}]$, where each interval represents points in the original spacetime $\mathcal{M}$ that are either on a horizon, or which are on an axis of rotation.

**Axis of rotation.** On a point $P \in \mathcal{M}$ on an axis of rotation, an integer linear combination $v_j \xi_p = 0$, or equivalently $f_{ij} v_j \xi_p = 0$, where $r, z$ are the Weyl–Papapetrou coordinates of $P$. Furthermore, unless $P$ is a ‘turning point’, $v_j$ spans the null space of $f_{ij}$. On the other hand, if $P$ is a turning point, then it lies on the boundary, say $z = z_j$, of an interval, and the null space of $f_{ij}$ is two-dimensional and spanned by $v_j, v_{j-1}$. Owing to the condition on subsequent vectors $v_j, v_{j-1}$ stated in the orbit space theorem, one can see [39] that there exists a $SL(8, \mathbb{Z})$ matrix $B^{ij}$ such that

$$B^{ij} v_j = (1, 0, 0, 0, 0, 0, 0, 0), \quad B^{ij} v_{j-1} = (0, 1, 0, 0, 0, 0, 0, 0).$$

(61)
By redefining the rotational Killing fields of the spacetime if necessary by this matrix as \( \xi_i \to B^j_i \xi_j \) globally (corresponding to the conjugation of the action of \( U(1)^8 \) on \( \mathcal{M} \) by an inner automorphism), we may assume that the vectors \( v_i^J, v_{i-1}^J \) take the above simple form.

Then it follows that the components of \( f_{ij}, f^{ij} \) have the following behavior near a point \((r, z)\) where \( z \) is in an interval \((z_J, z_{J+1})\) representing an axis of rotation:

\[
\begin{align*}
 f_{ij} &= \begin{cases} 
 O(r^2) & \text{if } i = 1 \text{ or } j = 1, \\
 O(1) & \text{otherwise,}
\end{cases} \\
 f^{ij} &= \begin{cases} 
 O(r^{-2}) & \text{if } i = j = 1, \\
 O(1) & \text{otherwise.}
\end{cases}
\end{align*}
\] (62)

Likewise, since \( \xi_1 \) vanishes on our axis of rotation, we have

\[
A_{ijk} = \begin{cases} 
 O(r^2) & \text{if } i = 1, \text{ or } j = 1, \text{ or } k = 1 \\
 O(1) & \text{otherwise.}
\end{cases}
\] (63)

We also need the behavior of the other potentials \( \chi_i, \chi_{ij}, \ldots, \chi_k \) for \((r, z)\) approaching the interval \((z_J, z_{J+1})\). This is slightly more tricky and requires using the information about the asymptotic charges. Consider first \( \chi_i \), with defining relation (47). It follows from the fact that some linear combination of the \( \xi_i \) vanishes on each axis of rotation, that the left side of this relation is \( = 0 \), hence \( d\chi_i = 0 \) on any axis of rotation, i.e. on any interval \((z_J, z_{J+1})\) marked in red in the following figure. Hence, \( \chi_i = \text{const.} \) on the red line, but not the horizon, marked in blue. Since we are free to add constants to \( \chi_i \), we may e.g. assume that \( \chi_i = \pm \text{const} \) to the left/right of the horizon. This constant may be computed as follows. Consider a curve \( \hat{\gamma} \) as in the figure, going from a point \( z' \) the right of the horizon to another point \( z \) to the left of the horizon.

It is the equivalence class of a nine-dimensional cycle \( C_9 \) in \( \mathcal{M} \) under the quotient by \( U(1)^8 \), i.e. \( \hat{\gamma} = C_9/U(1)^8 \). (The topology of this \( C_9 \) depends on the integer vectors associated with the intervals that \( z, z' \) are in, respectively, see table 1.) Now we compute

\[
(2\pi)^8 X_i \bigg|_{(z', r = 0)} = (2\pi)^8 \int_{\hat{\gamma} = C_9/U(1)^8} d\chi_i = \int_{C_9} d\chi_i \wedge d\phi^1 \wedge \cdots \wedge d\phi^8 = \int_{C_9} Q_{\xi_i} = \int_{C_9} Q_{\xi_i} - i\xi_i \theta = H_{\xi_i} = -J_i,
\] (64)

where \( Q_{\xi_i} \) is the Noether charge, and where we used (47), together with the definition of the ADM-conserved quantity \( J_i \) associated with the Killing fields \( \xi_i = \partial/\partial \phi^i \). Hence, we conclude that

\[
\chi_i(z, r) = \pm \frac{1}{2} (2\pi)^8 J_i + O(r^2).
\] (65)

Here \( \pm \) is chosen depending on whether \( z \) is to the left/right of the horizon. Consequently, because we are assuming that \( J_i \) are the same for our solutions, we have

\[
\delta \chi_i = O(r^2),
\] (66)
near any interval representing an axis, i.e. the red lines. Consider next the potentials \( \chi_{l_1, \ldots, l_s} \) with defining relation (46). It follows from the fact that \( \xi_1 \) vanishes on \((z_J, z_{J+1})\) that the left side of this relation is \( = 0 \) when one of the \( l_k \) is equal to 1. Therefore \( d\chi_{l_1, \ldots, l_t} = 0 \) on the interval \((z_J, z_{J+1})\), i.e. \( \chi_{l_1, \ldots, l_t} = 0 \) must be constant there. We now set this constant in relation to the electric charge of an appropriate 7-cycle, just as we did with the angular momenta before. Let us add constants to \( \chi_{l_1, \ldots, l_t} \) in such a way that these potentials tend to 0 as \( r = 0, z \to +\infty \). Then, consider a curve \( \hat{\gamma} \) connecting \( z' \) at infinity and \( z \in (z_J, z_{J+1}) \) as in the previous figure. This curve corresponds to the image of a 7-cycle under the quotient \( \hat{\gamma} = C_7/U(1)^6 \), where the subgroup \( U(1)^6 \) is generated by \( \xi_1, \ldots, \xi_6 \). We now get

\[
\left(2\pi\right)^6 \chi_{12, \ldots, 6} \bigg|_{(z', r=0)} = \int_{(z'=C_7/U(1)^6)} d\chi_{12, \ldots, 6}
\]

\[
= \int_{C_7} d\chi_{12, \ldots, 6} \land d\phi^1 \land \cdots \land d\phi^6
\]

\[
= \int_{C_7} q = Q[C_7].
\]

(67)

Since the electric charges associated with any cycle are assumed to be the same for the solutions, we find for all \( i, j, \ldots, k = 1, \ldots, 8 \):

\[
\delta \chi_{i j \ldots k} = \begin{cases} O(r^2) & \text{if } i = 1, \text{ or } j = 1, \text{ or } \ldots, \text{ or } k = 1 \\ O(1) & \text{otherwise.} \end{cases}
\]

(68)

This concludes our analysis of the potentials near an axis of rotation. Combining equations (62), (63), (66), (68) with (60), one immediately sees that \( s_j \), hence \( S \), is bounded near the interval \((z_J, z_{J+1})\). Of course, this analysis applies to the interior of any boundary interval representing an axis. It may also be shown by the same type of analysis as in [39] that the same holds true at the turning points, i.e. where two intervals representing an axis intersect.

**Horizon.** The analysis of the behavior of \( s_j \), hence \( S \), on the horizon interval \((z_h, z_{h+1})\) is not problematical, since there is no linear combination of the \( \xi_i \)'s which vanishes there. Hence, the Gram matrix of these Killing fields, \( f_{ij} \) must be non-singular on the horizon interval. Hence, by contrast to the intervals associated with an axis of rotation, all fields appearing in \( s_j \) have a continuous limit as the interval is approached, and \( s_h \) hence remains bounded. Some care is however required at the endpoints \( z_h, z_{h+1} \). Here one has to be alert that near these points, the Weyl–Papapetrou coordinates give a rather distorted picture of the spacetime geometry, as they are not smooth there. Consequently, one has to rule out the possibility that \( s_j \), hence \( S \), might have very direction dependent—and possibly singular—limits as \( z_h, z_{h+1} \) is approached. The comprehensive, rather tedious, analysis of this point was given in [39] in the vacuum case, where it was shown that \( s_j \) remains bounded no matter from what direction these points are approached. That analysis also carries over, with very few modifications, to the present case, so we omit it here.

In summary, we have shown that \( S \) is uniformly bounded on \( \mathbb{R}^3 \), including crucially the \( z \)-axis and infinity. Hence, by Weinstein’s lemma, \( S = \text{const.} \), and since \( S \) decays in some directions, \( S = 0 \), and hence \( s_j = 0 \). Hence, \( \delta \chi_i, \delta \chi_{i j \ldots k}, \delta A_{j k_l}, \delta f_{i j} \) all vanish (for any \( \lambda \)), and hence the fields \( \chi_i, \chi_{i j \ldots k}, A_{j k_l}, f_{i j} \) associated with the two solutions \((g_0, A_0)\) and \((g_1, A_1)\) agree. One must still show that the solutions agree themselves. This is seen as follows. First, it follows from the duality relation (46) that the functions \( w_j = f_{j i} w_i \) in the Weyl–Papapetrou form (37) can be obtained in terms of the scalar potentials parameterizing the matrix \( M \) as

\[
d w_j = -r \frac{1}{\det f} \left( d\chi_j + \frac{1}{54} e^{k_1 \cdots k_s} A_{j k_1 k_2} A_{k_3 k_4 k_5} dA_{k_6 k_7 k_8} + \frac{1}{360} e^{k_1 \cdots k_s} A_{j k_1 k_2} d\chi_{k_3 \cdots k_8} \right).
\]

(69)
Then it follows that \( w \) must agree for the two solutions. The function \( \nu \) in (37) can be recovered from the matrix \( M \) (see (53)) using equation (3.32) in [5], so \( g_0 = g_1 \). It also follows from the duality relation (47) that the components \( A_{ij} \) can be obtained then by integrating

\[
\begin{align*}
\mathrm{d}A_{ij} &= -\frac{1}{720} r \frac{\det f}{\det f_{ij}} \epsilon_{l_1l_2l_3l_4l_5l_6} \epsilon_{l_1l_2l_3l_4l_5l_6} \hat{A}(\mathrm{d}f_{i+l,j+l}) A_{ij}, \\
&\quad + \frac{1}{36} r \frac{\det f_{ij}}{\det f} \epsilon_{m_1m_2m_3m_4m_5m_6} \epsilon_{m_1m_2m_3m_4m_5m_6} \hat{A}(\mathrm{d}f_{i+m,j+m}) A_{ij},
\end{align*}
\]

where the subscript ‘0’ here refers to the contraction with the Killing field \( \xi_0 = \partial/\partial t \). Hence, these components of the \( A \)-field agree for the two solutions. The 1-form components \( A_{ij} \), \( A_0 i \) and 2-form components \( A_0, A_i \) are likewise seen to agree using the field equations for the \( A \)-field. This completes the proof. \( \Box \)

4. Mass formulas

4.1. Komar-type expressions for \( m, J_i \), and Smarr formula

Above, we have given a general formula for the ADM-type conserved quantity \( H_X \) associated with any asymptotic symmetry \( X \), see equation (9). Although that expression can be integrated to give a completely explicit formula very similar to the standard expressions in the case of four-dimensional vacuum general relativity [63], we will use here another, ‘Komar-type’, formula. This formula is less general because it holds only if the asymptotic symmetry \( X \) in question is an actual symmetry of the solution under consideration, i.e. Lie-derives \( (g, A) \).

As in the previous section, we assume that the spacetime is stationary, with asymptotically time-like Killing field \( \xi_0 = \partial/\partial t \), together with \( N \) commuting rotational Killing fields \( \xi_i \) as in equation (20).

**Lemma 3.** Let \( X \) be any vector field which Lie-derives the solution \( (g, A) \). Then the following 9-form \( \alpha_X \) is closed:

\[
\alpha_X = 0,
\]

\[
\alpha_X = -\star \mathrm{d}X - \frac{3}{4} i_X A \wedge q - \frac{1}{4} A \wedge \star (F \wedge X).
\]

Furthermore, the mass \( m = H_{\xi_0/\partial t} \) and angular momenta \( J_i = -H_{\xi_i} \) can be expressed as

\[
m = \frac{9}{8} \int_C \alpha_{\partial t},
\]

\[
J_i = -\int_C \alpha_{\xi_i},
\]

where, \( C \) is any nine-dimensional cycle\(^{17}\) cobordant to spatial infinity.

**Remark.** The cycle \( C \) may be chosen so that \( \xi_i \) is tangent. Then the formula for the angular momentum reduces to \( J_i = \int_C \star \mathrm{d}\xi_i \), which is the standard ‘Komar-type’ expression in the case of vacuum general relativity. Similarly, if \( A = 0 \), then the formula for the mass becomes \( m = -\frac{3}{8} \int_C \star \mathrm{d}\xi_0 \), which is the Komar mass formula in 11-dimensional vacuum general relativity. The factor of \(-9/8\) is the analogue in 11-dimensions of the factor \(-2\) discrepancy between the Komar mass and angular momentum expression in four dimensions [63].

Before we give a proof of this lemma, let us apply it to get a ‘Smarr-type’ formula in 11-dimensional supergravity. Consider the Killing vector field \( K \) as in equation (20), which is tangent to the null generators of the horizon \( \mathcal{H} \). By Stokes theorem, since \( \alpha_K \) is a closed 9-form, we get \( \int_{\mathcal{H}} \alpha_K = \int_{\infty} \alpha_K \), where \( \mathcal{H} \) is the bifurcation surface of the horizon, and where \( \infty \) is a cross section at infinity. Using the lemma, this gives

\[
\int_{\mathcal{H}} \alpha_K = \frac{8}{9} m - \sum_i \Omega_i J_i.
\]

\(^{17}\) The orientations are chosen as in footnote 8.
We now evaluate the integral on the left side. A standard calculation \cite{63} using equation (21) gives \( \int_{\mathcal{H}} *dK = -2\kappa \omega_h \), where \( \omega_h \) is the horizon area. Also, note that \( i_k A = -\Phi \) is the electrostatic ‘potential’ on the horizon, which we have already shown in section 2 to be a closed 2-form on \( \mathcal{H} \), hence \( \mathcal{B} \). Finally, note that since \( K \) itself vanishes on \( \mathcal{B} \), since \( A \) has vanishing pull-back\(^{18}\) to \( \mathcal{B} \), since \( K \) vanishes on \( \mathcal{B} \), and since \( *F \) is smooth, we have \( A \land i_k *F = 0 \) when pulled back to \( \mathcal{B} \). Consequently, defining the charge for a 7-cycle \( C_7 \subset \mathcal{B} \) as above in (26) by \( \Phi[C_7] \), and the electrostatic potential associated with a 2-cycle \( C_2 \subset \mathcal{B} \) by \( \Phi[C_2] \), the left side of (73) evaluates to

\[
2\kappa \omega_h + \frac{2}{3} \sum I_{iJ} (U^{-1})^i J_i = \int_{\mathcal{H}} \omega_K,
\]

where \( I_{iJ} \) is the intersection matrix. Hence, the Smarr formula is

\[
9 \cdot 2 \kappa \omega_h = 8 m - 9 \Omega \mathcal{J} - 6 (U^{-1})^i J_i \Phi[C_7].
\]

It can also be obtained directly from the first law by considering a variation as in the proof of the lemma.

**Proof of lemma 3.** We first note that if we scale the solution \((g, A)\) as \( \phi \equiv (\lambda^2 g, \lambda^3 A) \) for a constant \( \lambda \), then the Lagrangian of 11-dimensional supergravity changes as \( L(\phi) \equiv \lambda^9 L(\phi) \).

Variation along this one-parameter family of rescaled field configurations hence gives, using equation (5), \( \mathcal{L} = d\theta \), where \( \theta \) is evaluated on the variation \( \delta \phi = (2 g, 3 A) \). Using the explicit form of \( \theta \) given in appendix A gives that \( \theta = -\frac{1}{4} d(\lambda \land *F) \), whenever \( (g, A) \) satisfies the equations of motion. Now suppose additionally that a vector field \( \lambda \) Lie-derives this solution. Then from equation (6), together with \( i_\lambda d + d\lambda = \mathcal{L} \lambda \) and the fact that \( \mathcal{L} \lambda \) annihilates any tensor fields built from \((g, A)\):

\[
d\Phi_q = -i_\lambda \mathcal{L} q = \frac{\lambda}{2} i_\lambda (d(A \land *F) - A \land *F - A \land *F) \equiv -\frac{1}{4} d(\lambda A \land *F - A \land *F \land X).
\]

Write the difference between the two sides as \( d\alpha_X = 0 \). Using the explicit form \( \Phi_q \), see equation (9), we find that \( \alpha_X \) is given by the formula claimed in the lemma.

We now show the formulas for \( m, J_i \). First consider the angular momenta. Let us choose a cross section at infinity such that \( \xi_i \) is tangent. Then the term involving \( i_\xi \) in the formula (9) vanishes, and we get \( \delta H_\xi = \delta \int_\infty \Phi_q \), hence \( -J_i = H_{\xi_i} = \int_\infty \Phi_q \). But if \( \xi_i \) is tangent to the cross section at infinity, the last expression is also equal to \( \int_\infty \Phi_q = \int_\infty \alpha_q \), and because \( \alpha_q \) is closed, we may deform the cross section at infinity in the last expression to any other 9-cycle \( C \). This proves the formula for \( J_i \) in the lemma.

To prove the formula for \( m \), consider a one-parameter family of diffeomorphisms which acts in the asymptotic region as a dilatation \( f_\lambda : x^\mu \mapsto \lambda^{-1} x^\mu \), where \( x^\mu \) are the asymptotically Cartesian coordinates at infinity—of course this is not an isometry in non-trivial cases. Let \( \psi = f_\lambda \phi \), where \( \phi \) is the rescaled solution defined above. Then this solution is again asymptotically flat in the sense described in appendix C; the dilatation ensures that the metric asymptotes to the Minkowski metric in the asymptotic region, with conformal factor equal to \( \lambda^{-1} \). Note that \( (f_\lambda^{-1})_* \frac{\partial}{\partial \lambda} \mapsto \frac{\partial}{\partial \lambda} \) in the asymptotic region, hence the vector field \( Y \) generating \( f_\lambda \) has commutator \( [Y, \frac{\partial}{\partial \lambda}] \mapsto \frac{\partial}{\partial \lambda} \). We now consider the defining relation (9) for \( \delta H_{\partial/\partial \lambda} \) under the variation along the family \( \psi_\lambda \). On the one hand, we have \( (\psi_\lambda \equiv \psi) \)

\[
\frac{d}{d\lambda} H_{\partial/\partial \lambda}(\psi_\lambda)|_{\lambda=1} = \frac{d}{d\lambda} [H_{\partial/\partial \lambda}(f_\lambda^* \phi) + H_{\partial/\partial \lambda}(\phi_\lambda)|_{\lambda=1} = \frac{d}{d\lambda} [H_{\partial/\partial \lambda}(f_\lambda^* \phi) + \lambda^2 H_{\partial/\partial \lambda}(\phi)|_{\lambda=1} = \frac{d}{d\lambda} [\lambda^{-1} H_{\partial/\partial \lambda}(\phi) + \lambda^2 H_{\partial/\partial \lambda}(\phi)|_{\lambda=1} = (-1 + 9) H_{\partial/\partial \lambda}(\psi) = 8 m.
\]

\(^{18}\) See footnote 9.
On the other hand, noting that \( \delta \psi = \delta \phi + \mathcal{L}_Y \psi \) with \( \delta \phi = (2g, 3A) \), and using the scaling behavior of \( Q \partial/\partial t \), we have

\[
\frac{d}{d\lambda} H_{\partial/\partial t}(\psi_\lambda) \bigg|_{\lambda=1} = \int_\infty [9Q_{\partial/\partial t}(\psi) - i\partial \theta(\psi; \delta \phi)]
\]

\[
+ \int_\infty [\mathcal{L}_Y Q_{\partial/\partial t}(\psi) - i\partial \theta(\psi; \mathcal{L}_Y \psi)]
\]

\[
= 9 \int_\infty \alpha_{\partial/\partial t} + \int_\Sigma \alpha_{\partial/\partial t} \mathcal{L}_Y \mathcal{J}_{\partial/\partial t} \psi - i\partial \theta(\psi; \mathcal{L}_Y \psi)
\]

\[
= 9 \int_\infty \alpha_{\partial/\partial t} + \int_\Sigma \alpha(\psi; \mathcal{L}_Y \psi, \mathcal{L}_Y \mathcal{J}_{\partial/\partial t} \psi)
\]

\[
= 9 \int_\infty \alpha_{\partial/\partial t}.
\]

(78)

To go to the second equality, we used the explicit expression for \( \alpha_{\partial/\partial t} \) coming from equation (76), and we used Stokes theorem to convert the second integral to that over a slice \( \Sigma \). We may assume that \( Y \) vanishes on the inner boundary, so there is no contribution from there. We have also used \( dQ_{\partial/\partial t} = \mathcal{J}_{\partial/\partial t} \) for the Noether current. In the third line, we have used identity (12), and in the last line we used that \( \partial \) Lie-derives \( \psi \) by assumption. Combining formulas (77), (78), we get the formula for \( m \) in the lemma. \( \square \)

4.2. Conservation laws and mass formulas

The Smarr relation (75) derived in the previous subsection is universal in that it holds for any stationary black hole solution in the theory, with no extra symmetry assumptions. In this subsection we will show that if one makes further by-hand symmetry assumptions of the nature made in the uniqueness theorem, then one can derive further non-trivial relations between the horizon area, mass, angular momenta, electric charge etc. and the corresponding potentials such as angular velocities of the horizon, electric potentials of the horizon, etc. Unlike the Smarr relation, they are not quadratic in the thermodynamic quantities.

As a difference to previous sections, we will allow in this section \( F \)'s which satisfy the Bianchi identity \( dF = 0 \) and field equations\(^\text{19}\), but which cannot be written globally as \( F = dA \). This means that there can be non-zero magnetic charges, which will now also enter the thermodynamic formulas. As in section 3, we are assuming that the black hole spacetime is stationary with 8 additional mutually commuting rotational Killing fields, which we call \( \xi_1, \ldots, \xi_8 \). We assume, for definiteness that the spacetime is asymptotically Kaluza–Klein, with 4 asymptotically large dimensions, i.e. \( \mathcal{M} \cong \mathbb{R}^{3,1} \times T^7 \) in the asymptotic region. Without loss of generality we assume a labeling of the rotational Killing fields such that \( \xi_8 \) is a rotation in the asymptotically large dimensions, i.e. a rotation in the \( \mathbb{R}^{3,1} \) near infinity, while \( \xi_1, \ldots, \xi_7 \) are rotations in the \( T^7 \) near infinity. As a restriction on the class of solutions that we consider, we further assume that the black hole horizon is not rotating in the \( 8 \)-direction. In other words, if we let \( K \) be the linear combination of the Killing fields pointing along the null-generators of \( \mathcal{M} \), then we assume

\[
\xi_0 := K = \frac{\partial}{\partial t} + \Omega^1_\chi_1 + \ldots + \Omega^7_\chi_7
\]

(79)

so that \( \Omega^8 = 0 \). Thus, the horizon is non-rotating in the asymptotically large dimensions, although it can rotate in the extra-dimensions.

\(^{19}\) Note that the field equations (although not the action) only refer to the gauge invariant field strength \( F \).
To make the analysis below simpler, we also impose yet further, by-hand conditions onto the relationship between the fields \((g, F)\) and the symmetries. Our first condition is that \(\xi_8\) is hypersurface-orthogonal, or in other words

\[
\xi_8 \wedge d\xi_8 = 0,
\]

(80)

where as usual, we identify vector fields with 1-forms using the metric. We can see from the Komar expression for the angular momentum (72) that this implies \(J_k = 0\). The second condition is

\[
i_{\xi_j} i_{\xi_k} i_{\xi_l} F = 0, \quad \text{for } i, j, k = 1, \ldots, 7.
\]

(81)

Of course, the fields \((g, F)\) are, by assumption, also Lie-derived by any of the vector fields \(\xi_0, \xi_1, \ldots, \xi_8\). The seven distinguished rotational Killing fields tangent to the extra dimensions \(T^7\) in the asymptotic region are denoted by \(\xi_i\), where unprimed lower case indices run from \(i = 1, \ldots, 7\). We also adopt the convention that lower case indices run between \(i' = 0, \ldots, 7\), and primed upper case indices from \(I' = 0, \ldots, 7\). As we have already described in section 3.1, the horizon topology may, in general, be either one of the possibilities given in table 1. But in this section, we will assume that

\[
\mathcal{R} \cong S^2 \times T^7,
\]

(82)

and we also assume that \(\xi_8\) is tangent to the \(S^2\)-factor on the horizon\(^{20}\).

As in the previous section, we get from the symmetries \(X\) closed 9-forms \(\alpha_X\), \(X = \xi_0, \ldots, \xi_8\). To derive these closed forms, it turns out that there are many more closed forms, whose existence essentially follows from the sigma model formulation of this theory. We will employ these closed forms to derive our thermodynamic relations\(^{21}\). It turns out that it is convenient to use a ‘non-compact’ modification of the sigma model formulation described in section 3.2. The difference is that we now use the eight Killing fields \(\xi_0, \ldots, \xi_7\) (spanning a timelike(!) subspace\(^{22}\) in each tangent space), rather than previously \(\xi_1, \ldots, \xi_8\) (spanning a spacelike(!) subspace). To define the modified sigma model, we consider the eight-dimensional Gram matrix

\[
f_{ij} = g(\xi_i, \xi_j),
\]

(83)

which, unlike (35), is not positive definite, but has Lorentzian signature \((-1, +10)\). Similarly, we introduce the scalar potentials \(\chi_{ij}'\), \(\chi_{ijkl}''\) by formulas completely analogous to (46) and (47). We introduce an 8-bein by \(f_{ij}' = g_{ij} e'_{ij} e'_{jk} e'_{kl}\), and write

\[
V' = \begin{pmatrix}
eu^{ij}_{\alpha\beta} \det e'_{ij} e'_{ij} e'_{ij} e'_{ij} e'_{ij} e'_{ij} e'_{ij} e'_{ij} e'_{ij} e'_{ij} \end{pmatrix}.
\]

(84)

And we define the \(\xi_{8(1+8)}\)-valued function \(v'\) by\(^{23}\)

\[
v' = e^{ijk} A_{ijk}' - \frac{1}{360} \epsilon_{ijk} \epsilon_{ijmnpqrs} X_{lmnpqrs},
\]

(85)

where \(A_{ijk}' = -2\sqrt{3} A_{ij} e^{k} \) when \(i' = i, j' = j, k' = k\) are between \(0, \ldots, 7\), and zero if \(i' = 9\) or \(j' = 9\) or \(k' = 9\), as well as similarly \(X_{lmnpqrs} = 2\sqrt{3} X_{lmnpqrs} = 2\sqrt{3} X_{lmnpqrs}\) if all subscripts

\(^{20}\)In the language of the interval structure introduced in section 3.1, this amounts to saying that the vectors \(v'_{i+1}\), \(v'_{i+1}\) associated with the intervals adjacent to the horizon interval are both equal to \((0, 0, 0, 0, 0, 0, 0)\).

\(^{21}\)Relations of similar nature were previously derived in [35] for non-rotating black holes in Einstein–Maxwell–dilaton theory in four dimensions.

\(^{22}\)Note that \(\partial \alpha_i\) itself may be spacelike (ergoregion), timelike (asymptotic region) or null (horizon, ergosurface).

\(^{23}\)Note that, although the potential \(A\) may not be globally defined, the components \(A_{ijk}\) may be defined globally by the identity \(dA_{ijk} = 2\sqrt{3} \xi_j i_k + i_j k - i_k j\), because that identity may be considered on the simply connected orbit manifold \(\hat{M}\).
are between 0, ..., 7, and zero otherwise. We can now form the 248-dimensional matrix $V'$ valued in the adjoint representation of the group $E_{8(8)}$ by

$$V' = \exp(\text{ad}(V')) \text{Ad}(V'),$$

(compare equation (52)), and we also define $N$ by

$$N = V' \tau'(V')^{-1},$$

(compare equation (53)). In this equation, $\tau'$ is now the involution of $E_{8(8)}$ defined by equation (B.5) in appendix B. The subgroup fixed by $\tau'$ is the non-compact version $\text{Spin}^\ast(16)$ of $SO(16)$, so $V'$ may now be thought of as parameterizing the symmetric space $E_{8(8)}/\text{Spin}^\ast(16)$. The non-compact character of the subgroup $\text{Spin}^\ast(16)$ fixed under the involution $\tau'$ may be traced back to the timelike character of the subspace spanned by $K = \xi_0, \xi_1, \ldots, \xi_7$.

The matrix function $N$ satisfies the equations of motion of the corresponding sigma model, by complete analogy to the matrix $M$ defined before. It can easily be deduced from the equations of motion for $N$, and standard relations for Killing fields, that if we set

$$\omega = \ast(K \wedge N^{-1} \text{d}N),$$

then $\omega$ is a closed 9-form, valued in the adjoint representation of $\varepsilon_{8(8)}$,

$$d\omega = 0.$$  \hfill (88)

Hence, each of the 248 components $(\omega_{IF'}, \omega_{*IF'K'}, \omega_{IJK'})$ in

$$\omega = \text{ad}(\omega_{IF'} e^J_{~I} + \omega_{IF'K'} e^{J'K'L} + \omega_{*IF'K'} e^{I*_{J'K'L}})$$

is a scalar valued, closed 9-form on $\mathcal{M}$. (Here, the star $\ast$ on the symbols is part of the name and does not mean any kind of conjugation or dual.) Their concrete expression is very lengthy and is given in (D.5), (D.6), (D.7) of appendix D. The ‘conservation laws’ $d\omega = 0$ are, in essence, the Noether currents of the hidden symmetry $E_{8(8)}$ of the field equations, which is made manifest in the sigma model formulation.

We now integrate $0 = d\omega$ over a ten-dimensional surface $\Sigma$ going from the bifurcation surface $\mathcal{B}$ of the horizon to spatial infinity. Using Stokes theorem, we then get the 80 relations

$$\int_{\mathcal{B}} \omega_{IF'} = \int_{\infty} \omega_{IF'},$$

the 84 relations

$$\int_{\mathcal{B}} \omega_{IF'K'} = \int_{\infty} \omega_{IF'K'},$$

and the 84 relations

$$\int_{\mathcal{B}} \omega_{*IF'K'} = \int_{\infty} \omega_{*IF'K'}.$$  \hfill (93)

The key step is now to relate the quantities on both sides with the thermodynamic quantities characterizing the black hole. These are as follows. For the mass $m$ and angular momenta $J_i$ we may use the Komar expressions (72). The electric charge $Q[C_7]$ associated with a 7-cycle was defined above in equation (26). The 7-cycles lying in $\mathcal{H} \equiv \mathbb{R} \times (S^2 \times T^7)$ carrying a non-zero electric charge are $C_7 \equiv S^2 \times C_5$, where $C_5 \subseteq T^7$ is an embedded 5-torus. The position of $C_5$ is characterized by five generators $\xi_i, \ldots, \xi_5$, so the 7-cycle $C_7$ is characterized by $[i_1, \ldots, i_5]$. We use the shorthand

$$Q_{klmnr} = (2\pi)^2 \int_{[klmnr]} 4 q.$$  \hfill (94)
We also define \( Q^i j \) = \( \frac{1}{5!} \epsilon^{ijklmn} Q_{ijklmn} \), which obviously contains the same information. Using the relation \( \mathcal{L}_\xi = i_\xi d + d i_\xi \), the Bianchi identity \( d F = 0 \), and the fact that \( \xi_i, \xi_j, K \) Lie-derive \( F \), it follows that \( d(i_k i_j i_k F) = 0 \) on \( \mathcal{H} \), hence there is a scalar function \( \Phi_{ij} \) such that

\[
\Phi_{ij} = -i_k i_j i_k F.
\]

Furthermore, we already showed in section 2 that \( i_k F = 0 \) on \( \mathcal{H} \), so \( \Phi_{ij} \) is constant on \( \mathcal{H} \). The constant is fixed by demanding that \( \Phi_{ij} \) vanishes at infinity. Finally, we consider the non-trivial 4-cycles in \( \mathcal{H} \) of the form \( C^4 \equiv SL \times C_2 \), where \( C_2 \subset T^3 \) is an embedded 2-torus. Its position is characterized by two generators \( \xi_i, \xi_j \), so the corresponding 4-cycle is characterized by \( [i_1 i_2] \). The associated magnetic charges are then defined as

\[
P_{ij} = (2\pi)^5 \int_{[i_1 i_2]} F.
\]

If \( F \) happens to be equal \( dA \) for a globally defined 3-form \( A \), then of course the magnetic charges are zero. Now take five Killing fields \( \xi_i, \ldots, \xi_5 \), and form the 1-form \( i_k i_j i_k i_l i_l \). Because \( q \) is closed, so is this 1-form, and we get, by the same argument as above, a scalar function which we may call

\[
d\Psi^{ij} = \frac{1}{5!} \epsilon^{ijklmn} i_k i_j i_l i_m q,
\]

where we mean the seven dimensional totally antisymmetric tensor (recall that lower case Roman indices \( i = 1, \ldots, 7 \) in this section). Actually, \( \Psi^{ij} \), the magnetic potentials, are again constant on \( \mathcal{H} \); one proves this by the exactly the same argument as just given for the electric potentials. Thus, in summary, our thermodynamical quantities are the mass \( M \), \( \Omega \), the angular velocities of the horizon, \( \Omega \), the electric/magnetic potentials \( \Phi_{ij} \), \( \Psi^{ij} \), the electric/magnetic charges \( Q^{ij}, P_{ij} \), the horizon area \( \omega_h \), and the surface gravity \( \kappa \). We now give relations between them following from equations (91), (92), (93); the lengthy calculations are outlined in appendix D.

Evaluating equation (92) with the choice \( I^j = 0, J^j = j, K = k \) gives

\[
- d_{ij} d_{km} Q^{lm} = -2 \Phi_{jk \kappa} \omega_h^{\kappa} + 2 \Psi^{m n} \Phi_{mn} P_{jk} + 4 \Phi_{[jk} \Psi_{lm]} P_{jm} + 4 \Phi_{jk} \Psi^{lm} P_{lm}.
\]

Here, we remind the reader of our convention that lower case Roman indices go from \( i = 1, \ldots, 7 \), and we have also introduced the 7 \( \times \) 7 matrix \( d^{ij} \) by

\[
d^{ij} = \delta^{ij} - \Omega^i \Omega^j.
\]

with \( d_{ij} \) denoting its inverse. Next, evaluating equation (93) with \( I^j = 9, J^j = j, K = k \) gives

\[
- d^{ij} d_{km} P_{lm} = -2 \Psi^{jk \kappa} \omega_h^{\kappa} + 2 \Psi^{mn} \Phi_{mn} Q^{jk} + 4 \Psi^{ij[k} \Psi^{\ell m]} P_{lm} - 8 \Psi^{ij[k} \Psi_{lm]} P_{jm} + 4 \Psi^{jk} \Phi_{lm} Q^{lm}.
\]

Taking \( I^j = 0 = J^j \) in equation (91) gives

\[
9 \cdot 2 \lambda \omega_h = 8 m - 9 \Omega J^j + 12 \Phi_{ij} Q^{ij} + 6 \Psi^{ij} P_{ij},
\]

which is the Smarr relation (75), already derived earlier in the absence of magnetic charges. Taking \( I^j = 0, J^j = j \) in equation (91) gives

\[
- \frac{1}{4} \epsilon^{ijklmn} \Phi_{jk} \Phi_{mn} P_{pq} = -9 \delta^{ij} J_k + 8 m \Omega^j.
\]

Taking \( I^j = i \) and \( J^j = 9 \) in equation (91) gives

\[
0 = \frac{1}{6} \epsilon^{ijklmn} \Psi^{jk} Q^{mn} + 4 \Phi_{ij} \Psi^{jk} J_k.
\]

\[24\] Even though \( \mathcal{H} \) is not simply connected, the forms under consideration may be viewed as forms on the simply connected orbit space \( M \), so there is no difference between closed and exact invariant 1-forms.
Taking $I' = 0, J' = 9$ in equation (91) gives
\[
\Phi_{mn} \Psi^{mn} \kappa A h = 4 \Phi_{pq} \Phi_{pl} Q^{lm} + 4 \Phi_{pq} \Psi^{pl} \Psi^{mn} P_{lm}.
\] (104)

Finally, taking $I' = i, J' = j$ in equation (91) gives
\[
4 \pi d^{lm} \tau_{lm} + \Omega^i J_i = -4 \Psi^{im} P_{im} + \frac{3}{2} \delta^i_j \Psi^{mn} P_{mn} - 4 \Phi_{im} Q^{im} + \frac{3}{2} \delta^i_j \Phi_{mn} Q^{mn}.
\] (105)

Here, the $7 \times 7$ constant matrix $\tau_{mn}$ is defined by the relation
\[
f_{ij} = \delta_{ij} - \frac{1}{(2\pi)^{7/2}} \frac{\tau_{ij}}{R} + O(R^{-2}),
\] (106)

where $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the standard radial coordinate in the large dimensions ($\mathbb{R}^{3,1}$) relative to an asymptotically Cartesian coordinate system. It is physically interpreted as the ‘tension tensor’ of the seven asymptotically small dimensions$^{25}$ ($T^7$).

We emphasize that all these thermodynamic formulas have not been obtained from a particular explicit solution, but from the general structure of the equations of motion (hidden symmetries) for the class of solutions which have the indicated symmetries. We leave for the future a more detailed analysis of these relations.

5. Conclusions

In this paper, we have considered stationary solutions in 11-dimensional supergravity theory. We first derived the first law of black hole mechanics, valid for arbitrary horizon topologies, and arbitrary stationary black holes, without additional symmetry assumptions. We then specialized to stationary (asymptotically Kaluza–Klein) solutions whose isometry group is (or contains) $\mathbb{R} \times U(1)^8$. In this case, we were able to associate with each such solution a collection of moduli and generalized winding numbers which encode the topology of the solution and the action of the isometries. Furthermore, for each given set of moduli, generalized winding numbers, angular momenta, and electric type charges, we proved a black hole uniqueness theorem. 

The proof of this theorem makes use of the known sigma-model formulation of the field equations of 11-dimensional supergravity when it is ‘dimensionally reduced’. The sigma model formulation also has another application explored in this paper, namely it gives an interesting set of relations between the electric/magnetic charges and potentials, angular momenta and velocities, mass, area and surface gravity. These, rather non-trivial, relations generalize the well-known Smarr-type formulas. We believe that there is a relation between these formulas and the formulas given in [3] for the solutions corresponding to ‘nilpotent orbits’. However, this remains to be worked out.

Acknowledgments

I would like to thank RM Wald for discussions about the first law in the presence of magnetic charges, and H Nicolai for discussions about coset formulations of supergravity and nilpotent orbits. The author is supported by ERC starting grant no QC & C 259562.

$^{25}$ These are the ‘thermodynamic potentials’ which would be conjugate to the deformations of the asymptotic metric on $T^7$ if we would loosen our boundary conditions to allow such. They would give rise to a corresponding term in the first law, see e.g. [44] for an example.
Appendix A. Calculation of the Noether charge and constraints

In this appendix, we derive the expressions for the Noether charge and constraints quoted in section 2.1. The Lagrange 11-form is given in components by

\[ L_{a_1 \ldots a_{11}} = \left( R - \frac{1}{12} F_{bcede} F^{bcede} + \frac{1}{2 \cdot 5!} \epsilon^{b_1 \ldots b_{11}} \varepsilon_{a_1 \ldots a_{11}} F_{b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10}} A_{b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10}} \right) \]

where the epsilon tensor is the natural volume element defined from the metric relative to a given orientation of \( \mathcal{M} \). The 10-form \( \theta \) is given by

\[ \theta_{a_1 \ldots a_{10}} = \epsilon_{a_1 \ldots a_{10}} v^a, \]

and

\[ v_d = g^{bc} (\nabla_d g_{ba} - \nabla_a g_{bc}) - \frac{1}{4} F_{d b c e} \delta A_{b c e} + \frac{1}{5!} \epsilon_{d b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10}} \delta A_{b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10}} \]

From (6), one obtains the following expression for the components of the Noether current

\[ J_{a_1 \ldots a_{10}} = -2 \epsilon_{a_1 \ldots a_{10}} \nabla_a (\nabla_i^d X^c) \]

where we have defined

\[ t_{ab} = G_{ab} - T_{ab} \]

and

\[ j^{bcd} = \nabla_a F^{abcd} - \frac{1}{5!} \epsilon^{bcd f_1 \ldots f_8} F_{f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8} \]

and where

\[ T_{ab} = \frac{1}{4} F_{acde} F^{cde}_{b} - \frac{1}{12} g_{ab} F_{def} F^{def} + \frac{1}{5!} \epsilon^{cdefgh} (\star F)_{acdef} (\star F)_{bgh} \]

is the ‘electromagnetic’ stress tensor. \( t_{ab} \) is interpreted as the non-gravitational stress energy tensor, because it is just the difference between the Einstein tensor and the electromagnetic stress tensor. \( j^{abc} \) is interpreted as the non-electromagnetic current. Of course, \( t_{ab} = 0 \Rightarrow j^{abc} \)

When the equations of motion hold, therefore, we can read off the constraints and Noether charge from the expression for the Noether current as:

\[ C_{a_1 \ldots a_{10}} = \epsilon_{a_1 \ldots a_{10}} (2 t_{a b} X^b - 2 j^{abc} A_{b c} X^c) \]

\[ Q_{a_2 \ldots a_{10}} = \epsilon_{a_2 \ldots a_{10}} \nabla_a (\nabla_i^d X^c) - F^{a_2 \ldots a_{10}} + \frac{1}{5!} \epsilon^{a_2 \ldots a_{10} f_1 \ldots f_8} A_{f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8} X^c \]

These expressions can be conveniently rewritten in differential forms notation if we define the components of \( f_x \) by

\[ f_a = t_{ab} X^b, \]

and

\[ \epsilon^{a_2 \ldots a_{10} f_1 \ldots f_8} = \frac{1}{5!} \epsilon_{a_2 \ldots a_{10}} j^{a_2 \ldots a_{10}} \]

as usual.
Then we get
\[ Q_X = - \star dX - 4 i_X A \wedge q + \frac{4}{3} i_X A \wedge A \wedge F, \]
\[ C_X = 2 \star f_X + 4 i_X A \wedge \star f. \quad (A.12) \]
where \( \star = dq \). These are the formulas claimed in the main text. As in the main text, we use the standard operators \((d\alpha)_{a_1 \ldots a_p} = pV_{[a_1} a_2 \ldots a_p] \) (the exterior derivative), \( (\alpha \wedge \beta)_{a_1 \ldots a_p b_1 \ldots b_q} = \frac{[\alpha \beta]}{pq!}\delta_{a_1 \ldots a_p b_1 \ldots b_q} \) (the wedge product), \((\star \alpha)_{a_1 \ldots a_p} = \frac{1}{(m-p)!}\epsilon_{a_1 \ldots a_p b_1 \ldots a_m} \alpha^{b_1 \ldots b_q} \) (the Hodge dual), and \((i_X \alpha)_{a_1 \ldots a_p} = X^a a_{a_1 \ldots a_p} \) (the interior derivative).

Appendix B. Basic facts and definition of \( \mathfrak{E}_{8(8)} \)

The exceptional, real, Lie-algebra \( \mathfrak{e}_{8(8)} \), is a particular real form of the complex exceptional Lie-algebra \( \mathfrak{e}_8 \). This semi-simple Lie-algebra can be characterized by a Cartan-matrix with corresponding generators and relations, but for our purposes, another set of generators is more suitable. These are often referred to as ‘Freudenthal’s realization’ [18], and are denoted by \( \mathfrak{e}'_I, \mathfrak{e}_J, \mathfrak{e}^{ILJK}, I, J, K = 1, \ldots, 9 \). They are subject to the following relations. Both \( \mathfrak{e}_{ILJK}, \mathfrak{e}^{ILJK} \) are totally antisymmetric in the indices, and \( \mathfrak{e}'_I = 0 \). The 80 basis elements \( \mathfrak{e}'_I \) generate the Lie-algebra \( \mathfrak{sl}(9) \),
\[ [\mathfrak{e}'_I, \mathfrak{e}'_J] = \delta^J_L \mathfrak{e}'_L - \delta^K_J \mathfrak{e}'_L. \quad (B.1) \]
The following relations manifest how the adjoint representation \( \mathfrak{sl}(8,\mathbb{R}) \) splits (49) under the restriction to \( \mathfrak{sl}(9) \),
\[ [\mathfrak{e}'_{ILJK}, \mathfrak{e}^L_M] = \delta^L_I \mathfrak{e}^*_{MKJ} + \delta^L_J \mathfrak{e}^*_{IMK} + \delta^L_K \mathfrak{e}^*_{IMJ}. \quad (B.2) \]

The remaining brackets are
\[ [\mathfrak{e}^*_{ILJK}, \mathfrak{e}^M_{LMN}] = \frac{1}{360\sqrt{3}} \mathfrak{e}^*_{ILJKLMNPQR} \mathfrak{e}^*_{PQR}, \]
\[ [\mathfrak{e}^*_{ILJK}, \mathfrak{e}^*_{LMN}] = \frac{1}{360\sqrt{3}} \epsilon^*_{ILJKLMNPQR} \mathfrak{e}^*_{PQR}, \]
\[ [\mathfrak{e}^*_{ILJK}, \mathfrak{e}^*_{MN}] = \frac{1}{6} \delta^L_I \delta^M_J \mathfrak{e}^*_{NK}. \quad (B.3) \]

The real span of these generators is by definition the Lie algebra \( \mathfrak{e}_{8(8)} \), whereas the complex span is \( \mathfrak{e}_8 \). Its dimension is 80+8+8+4=248.

The Lie-algebra \( \mathfrak{e}_{8(8)} \) has several involutions, i.e. Lie-algebra automorphisms \( \tau \) (meaning \( \tau(\mathfrak{X}, Y) = [\tau X, (\tau Y)] \)) such that \( \tau^2 = \text{id} \). In this paper, we consider two of them. The first one is defined by
\[ \tau(\mathfrak{e}'_I) = -\delta^{IL} \mathfrak{e}'_L, \quad \tau(\mathfrak{e}^{ILJK}) = \delta^{IL} \delta^J L \delta^K e_{LMN}, \quad \tau(\mathfrak{e}_{ILJK}) = \delta_{IL} \delta_{LJ} \delta_{K} e_{MN}, \quad (B.4) \]
where \( \delta_{IJ} = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1) \) is the 9-dimensional Euclidean metric. The second one is
\[ \tau'(\mathfrak{e}'_I) = -\eta_{IL} \mathfrak{e}'_L, \quad \tau'(\mathfrak{e}^{ILJK}) = \eta^{IL} \eta^{JM} \eta^{KN} e_{MN}, \quad \tau'(\mathfrak{e}_{ILJK}) = \eta_{IL} \eta_{LJ} \eta_{K} e_{MN}, \quad (B.5) \]
where \( \eta_{IJ} = \text{diag}(1, 1, 1, 1, 1, 1, 1, -1) \) is a nine-dimensional flat pseudo–Riemannian metric. The elements left invariant by an automorphism automatically form a subalgebra. In the case of \( \tau \), this can be seen to be \( \mathfrak{so}(16) \), whereas in the case of \( \tau' \), this can be seen to be26 \( \mathfrak{so}^*'(16) \). The connected Lie-group corresponding to \( \mathfrak{e}_{8(8)} \) is denoted by \( \mathbb{E}_{8(8)} \). The

26 It is fairly obvious that this Lie-algebra, \( \mathfrak{g}' \), must be a real form of \( \mathfrak{C} \otimes \mathfrak{so}(16) \). To see that it must in fact be \( \mathfrak{so}^*'(16) \), one can verify that the restriction of the Cartan–Killing form of \( \mathfrak{e}_{8(8)} \) to \( \mathfrak{g}' \) has signature \((-64, +56)\). By identifying the generators corresponding to the 64 negative signs, one sees that these correspond to the Lie-algebra \( \mathfrak{u}(8) \), which is a maximal compact sub-algebra of \( \mathfrak{so}^*'(16) \).
corresponding group automorphisms are denoted, by abuse of notation, by the same symbols \( \tau, \tau' \). The triples \((E_8(+8), SO(16), \tau)\) resp. \((E_8(+8), \text{Spin}'(16), \tau')\) form symmetric spaces. The subgroups clearly have dimension 120, so the dimension of the coset spaces \(E_8(+8)/SO(16)\) and \(E_8(+8)/\text{Spin}'(16)\) is hence 128.

### Appendix C. Asymptotic conditions

Asymptotically KK-boundary conditions are in more detail as follows: We assume that a subset \( M \) of \( \mathbb{R}^8 \) is diffeomorphic to the Cartesian product of \( \mathbb{R}^8 \) with a ball removed—corresponding to the asymptotic region of the large spatial dimensions—and \( \mathbb{R} \times T^{D-r-1} \)—corresponding to the time-direction and small dimensions. We will refer to this region as the asymptotic region and call it \( \mathcal{M}_{\text{asymptotic}} \). The metric is required to behave in this region like

\[
g = -dt^2 + \sum_{i=1}^{s} dx_i^2 + \sum_{i=1}^{10-s} d\phi_i^2 + O(R^{-r+2}),
\]

where \( O(R^{-\alpha}) \) stands for metric components that drop off faster than \( R^{-\alpha} \) in the radial coordinate \( R = \sqrt{x_1^2 + \cdots + x_s^2} \), with \( \alpha \)th derivatives in the coordinates \( x_1, \ldots, x_s \), dropping off at least as fast as \( R^{-\alpha-k} \). These terms are also required to be independent of the coordinate \( t \), which together with \( x_i \) forms the standard Cartesian coordinates on \( \mathbb{R}^{8+s} \). The remaining coordinates \( \phi_i \) are \( 2\pi \)-periodic and parameterize the torus \( T^{D-r-1} \). The timelike Killing field is assumed to be equal to \( \partial/\partial t \) in \( \mathcal{M}_{\text{asymptotic}} \). We also require that the 3-form field has asymptotic behavior

\[
A = \sum_{i,j,k=1}^{10-s} O(1) \, d\phi^i \wedge d\phi^j \wedge d\phi^k + \sum_{i,j=1}^{10-s} \sum_{\mu=0}^{s} O(R^{-r+2}) \, dx^\mu \wedge d\phi^i \wedge d\phi^j
\]

\[
+ \sum_{i=1}^{10-s} \sum_{\mu=0}^{s} O(R^{-r+2}) \, dx^\mu \wedge dx^i \wedge d\phi^j + \sum_{\mu,\nu,\sigma=0}^{s} O(R^{-r+2}) \, dx^\mu \wedge dx^\nu \wedge dx^\sigma,
\]

with all components independent of \( t \). In sections 4, 3 we make the more restrictive assumption that the first term on the right side is \( O(R^{-r+1}) \). This is done mainly for simplicity. Otherwise, the asymptotic values for \( A_{ijk} \) at infinity appear as additional parameters in the thermodynamic relations. We call spacetimes satisfying these properties ‘asymptotically Kaluza–Klein spacetimes’\(^{27}\).

Unfortunately, in order to make many of the arguments in the body of the paper in a consistent way, one has to make certain further technical assumptions about the global nature of \((\mathcal{M}, g)\) and the action of the symmetries. Our assumptions are in parallel to those made by Chrusciel and Costa in their study \([9]\) of four-dimensional stationary black holes. The requirements are (a) that \( \mathcal{M} \) contains an acausal, spacelike, connected hypersurface \( \Sigma \) asymptotic to the \( t = 0 \) surface in the asymptotic region, whose closure has as its boundary \( \partial \Sigma = \mathcal{B} \) a cross section of the horizon. We always assume \( \mathcal{B} \) to be compact and connected. (b) We assume that the orbits of \( \partial/\partial t \) are complete. (c) We assume that the horizon is non-degenerate. (d) We assume that \( \mathcal{M} \) is globally hyperbolic. In order to use the rigidity theorem in section 2, and to prove the orbit space theorem in section 3.1, it is necessary to assume (e) that the spacetime, the metric, and the group action are analytic, rather than only smooth.

\(^{27}\) For the axisymmetric spacetimes considered in this paper, one can derive more precise asymptotic expansions, as explained in \([39]\) for the example of the vacuum field equations.
Appendix D. Formulas for $\omega$

In this section, we give the concrete expression for the closed 9-forms $\omega_{\ell'\ell}^{f^I,\ell'}$, $\omega_{\ell'\ell'}^{f\ell'K}$, $\omega_{\ell'\ell''}^{f\ell''K}$ defined by equation (88). As in section 4, we use the index conventions that lower case primed indices run between $\ell' = 0, 1, \ldots, 7$, lower case unprimed indices run between $i = 1, \ldots, 7$, and upper case primed indices run between $I' = 0, 1, \ldots, 7, 9$. And again, we assume a labeling of the Killing fields such that $\xi_0 = K$ is the Killing field tangent to the null generators $\mathcal{H}$, such that $\xi$ is tangent to the $S^2$ factor of $\mathcal{H} \cong \mathbb{R} \times S^3 \times T^7$, and such that $\xi_1, \ldots, \xi_7$ are tangent to the extra dimensions $\cong T^7$ in the asymptotic region. We define

$$\psi^{f\ell'\ell''} = -\frac{\sqrt{3}}{360} \epsilon^{f\ell'\ell''}\delta^i_{i'} \delta^j_j \delta^k_k \psi^{ij'} \psi^{jk'} \psi^{ki'},$$

and as before $A_{f\ell'\ell''} = -2\sqrt{3}A_{f\ell'\ell''}$ when all indices are between 0, $\ldots$, 7 and 0 if one index is $= 9$. Let us then define the following 9-forms $k_{\ell'\ell''}$,

\[
k_{\ell'\ell''} = \begin{cases} 
    \frac{1}{\text{det} f'} f_{f'f''} f_{f''f} e^{m'n'p'} & \text{if } \xi_0 = K \\
    \frac{1}{180} f_{f'f''} f_{f''f} e^{m'n'p'} & \text{if } \xi_0 = \sqrt{3}K \\
    0 & \text{if } \xi_0 = \sqrt{7}K
\end{cases}
\]

where $U_f = \chi_f + \frac{1}{\sqrt{70}} e^{f_1 \ldots f_7} A_{f_1 f_2} A_{f_3 f_4} A_{f_5 f_6} f_{f'f''} f_{f''f} e^{m'n'p'}$. Furthermore, the 9-forms $k_{f\ell'\ell''}$ are defined by

\[
k_{f\ell'\ell''} = -2\sqrt{3} i \ast (F \wedge \xi_0 \wedge \xi_{\ell'} \wedge \xi_{\ell''})
\]

and the 9-forms $k_{f'f''}$ are defined by

\[
k_{f'f''} = -2\sqrt{3} i \ast (F \wedge \xi_0 \wedge \xi_{\ell'} \wedge \xi_{\ell''})
\]

In these formulas, we have, as usual, identified the vector fields $\xi_\ell$ with 1-forms using the metric. The formulas for the 9-forms $\omega_{\ell'\ell'}^{f\ell'}, \omega_{\ell'\ell''}^{f\ell'K}, \omega_{\ell'\ell''}^{f\ell''K}$ are then:

\[
\omega_{\ell'\ell'}^{f\ell'} = \frac{1}{12} \left( A_{fPfQ} \psi^{LQQ} \delta^M_{\ell'} + A_{MPQ} \psi^{PQQ} \delta^L_{\ell'} \right) k_{L'M'} + \frac{1}{432\sqrt{3}} \epsilon^{fPQQR}\delta^L_{\ell'} (A_{fPQ}\delta^M_{\ell'} - A_{MPQ} \psi^{PQQ} \delta^M_{\ell'}) k_{L'M'}
\]

\[
\omega_{\ell'\ell''}^{f\ell'K} = \frac{1}{12} \left( A_{fPfQ} \psi^{LQQ} \delta^M_{\ell'} + A_{MPQ} \psi^{PQQ} \delta^L_{\ell'} \right) k_{L'M'} + \frac{1}{432\sqrt{3}} \epsilon^{fPQQR}\delta^L_{\ell'} (A_{fPQ}\delta^M_{\ell'} - A_{MPQ} \psi^{PQQ} \delta^M_{\ell'}) k_{L'M'}
\]

\[
\omega_{\ell'\ell''}^{f\ell''K} = \frac{1}{12} \left( A_{fPfQ} \psi^{LQQ} \delta^M_{\ell'} + A_{MPQ} \psi^{PQQ} \delta^L_{\ell'} \right) k_{L'M'} + \frac{1}{432\sqrt{3}} \epsilon^{fPQQR}\delta^L_{\ell'} (A_{fPQ}\delta^M_{\ell'} - A_{MPQ} \psi^{PQQ} \delta^M_{\ell'}) k_{L'M'}
\]
In order to obtain these expressions, we had to use the definitions of then by definition obtained as condition implies

Furthermore,

and finally,

In order to obtain these expressions, we had to use the definitions of \( V' \) and of \( \nu' \), perform the Lie-algebra exponential to get \( V' = e^{\ad(\nu')} \Ad(V') \), then get \( N = V' r(t) - 1 \), from which \( \omega = \star (K \wedge N^{-1} \wedge \mathcal{N}) \). The Lie-algebra exponential, defined by its infinite power series, truncates at polynomial order 4 because \( \ad(\nu') \) is nilpotent of order 5. (In this part of the calculation, we are relying on formulas given in [49].) We have also used the geometric condition (80) in the following ways: If we let \( g(\xi_8, \tilde{\xi}_i) = w'_i \), then that condition implies \( w'_i = 0 \). In combination with the definitions of the potentials \( \chi'_i, \chi'_{ij}, \ldots, \chi'_{ij\ldots k} \), there follow the relations (viewed as relations on \( \hat{\mathcal{M}} = \mathcal{M}/(\mathbb{R} \times U(1)^3) \)):

\[ 0 = d\chi'_j + 2A'_{ijk} d\nu'_{jk} - \frac{1}{54} \epsilon^{ijk\nu\mu\alpha'} \epsilon_{\mu'\nu'\alpha'} A_{ij\nu'k} A_{\nu'\alpha'} d\nu_{\alpha'} \]

\[ 0 = dA'_{ij} \]

\[ d\tau_{ijk} = - \frac{r}{\det f'} \int_{f'} \int_{\tilde{\mathcal{M}}} \Phi \Phi^{\nu'} A_{ij} d\nu_{\alpha'} \]

\[ + \frac{1}{36} \epsilon^{ijk\nu\mu\alpha'} A_{ij\nu'k} A_{\nu'\alpha'} d\nu_{\alpha'} \]  

(D.8)
where \( \varphi^{ij} = \frac{1}{720} \epsilon^{jklmnpqr} k_{jl}^{\prime} \). These formulas were used to simplify the expressions for \( k_{i}^{\prime} r, k_{i}^{\prime} J, k_{i}^{\prime} K \). We may also use (81) to set to zero many terms in the expressions for \( \omega \). In particular, the first relation implies, together with the constancy of \( \chi_{i}^{\prime} \) on \( \mathcal{B} \) (cf (47)) and of \( A_{0i}^{\prime} \), that \( U_{i} = \chi_{i}^{\prime} + A_{i}^{\prime} \varphi^{ij} \) is constant over \( \mathcal{B} \).

Next, we integrate the currents \( \omega \) over the nine-dimensional horizon cross section \( \mathcal{B} \), or over a nine-dimensional cross section at infinity. One obtains the following expressions.

\[
\begin{align*}
\int_{\mathcal{B}} k_{0}^{\prime} & = 2 \kappa A_{h}^{2}, \\
\int_{\mathcal{B}} k_{i}^{\prime} & = -J_{i}, \\
\int_{\mathcal{B}} k_{9}^{\prime} & = -J_{i} \Psi^{jk} \Phi_{jk}, \\
\int_{\mathcal{B}} k_{0}^{\prime} & = 4 \Psi^{ij} \Phi_{jk} A_{h}, \\
\int_{\mathcal{B}} k_{i}^{\prime} & = 0, \\
\int_{\infty} k_{0}^{\prime} & = \frac{8}{9} m - \Omega \Psi, \\
\int_{\infty} k_{i}^{\prime} & = -J_{i}, \\
\int_{\infty} k_{9}^{\prime} & = \delta_{i}^{j} J_{i} - \frac{8}{9} m \Omega^{i}, \\
\int_{\infty} k_{i}^{\prime} & = 4 \pi d^{mn} \tau_{mn} + \Omega^{i} J_{i}, \\
\int_{\infty} k_{9}^{\prime} & = 0,
\end{align*}
\]

where we have not displayed several components that are not needed in the text. It has been used that the constant value of \( U_{0} \) on \( \mathcal{B} \) is \( U_{0} = -\Psi^{ij} \Phi_{ij} \), by showing that \( \chi_{i}^{\prime} = O(r^{2}) \) near \( \mathcal{B} \). This can be seen by integrating the defining relation (compare (47)) for \( d \chi_{i}^{\prime} = i_{0i}, \ldots, i_{07}, Q_{i}, \) over a suitable curve \( \hat{\gamma} \) in \( \mathcal{M} \) from the horizon to infinity, and by applying the same kind of argument as in the proof of the uniqueness theorem in section 3. A similar argument, using the first equation in (D.8) then also shows that \( U_{i} = O(r^{2}) \) near \( \mathcal{B} \) for \( i = 1, \ldots, 7 \). We also have

\[
\begin{align*}
\int_{\mathcal{B}} k_{0j} & = 2 \sqrt{3} P_{jk}, \\
\int_{\mathcal{B}} k_{09k} & = 0, \\
\int_{\mathcal{B}} k_{0j} & = 2 \sqrt{3} \Phi_{mn} \varphi_{mn} P_{jk}, \\
\int_{\mathcal{B}} k_{i}^{\prime} & = 0, \\
\int_{\infty} k_{j} & = 2 \sqrt{3} P_{jk}, \\
\int_{\infty} k_{9k} & = 0,
\end{align*}
\]

\( \text{(D.9)} \)

where we have not displayed several components that are not needed in the text. It has been used that the constant value of \( U_{0} \) on \( \mathcal{B} \) is \( U_{0} = -\Psi^{ij} \Phi_{ij} \), by showing that \( \chi_{i}^{\prime} = O(r^{2}) \) near \( \mathcal{B} \). This can be seen by integrating the defining relation (compare (47)) for \( d \chi_{i}^{\prime} = i_{0i}, \ldots, i_{07}, Q_{i}, \) over a suitable curve \( \hat{\gamma} \) in \( \mathcal{M} \) from the horizon to infinity, and by applying the same kind of argument as in the proof of the uniqueness theorem in section 3. A similar argument, using the first equation in (D.8) then also shows that \( U_{i} = O(r^{2}) \) near \( \mathcal{B} \) for \( i = 1, \ldots, 7 \). We also have

\[
\begin{align*}
\int_{\mathcal{B}} k_{0j} & = 2 \sqrt{3} P_{jk}, \\
\int_{\mathcal{B}} k_{09k} & = 0, \\
\int_{\mathcal{B}} k_{0j} & = 2 \sqrt{3} \Phi_{mn} \varphi_{mn} P_{jk}, \\
\int_{\mathcal{B}} k_{i}^{\prime} & = 0, \\
\int_{\infty} k_{j} & = 2 \sqrt{3} P_{jk}, \\
\int_{\infty} k_{9k} & = 0,
\end{align*}
\]

\( \text{(D.9)} \)
\[ \int_{\infty} k_{0jk} = -2\sqrt{3} d_{jm} d_{kn} Q_{mn}, \]
\[ \int_{\infty} k_{ijk} = 0, \tag{D.10} \]
as well as
\[ \int_{B} k_{j} = 2\sqrt{3} \Phi_{mn} \Psi_{mn} Q_{jk}, \]
\[ \int_{B} k_{0} = 0, \]
\[ \int_{B} k_{0j} = 2\sqrt{3} Q_{jk}, \]
\[ \int_{B} k_{j} = 0, \]
\[ \int_{\infty} k_{jk} = -2\sqrt{3} d_{jm} d_{kn} P_{mn}, \]
\[ \int_{\infty} k_{0j} = 0, \]
\[ \int_{\infty} k_{0} = 2\sqrt{3} Q_{jk}, \]
\[ \int_{\infty} k_{ij} = 0. \tag{D.11} \]

To obtain these expressions, we have used the Komar expressions (72) for m, J, which in several cases helps one to read off the interpretation of the surface integrals at infinity. In several of these expressions, have also used the fact that, at B, we have
\[ f_{i'}^{j'} = \begin{cases} O(r^{-2}) & \text{if } i' = 0 \text{ or } j' = 0, \\ O(1) & \text{otherwise,} \end{cases} \]
\[ f_{i}^{j} = \begin{cases} O(r^{-2}) & \text{if } i = 0 = j, \\ O(1) & \text{otherwise.} \end{cases} \tag{D.12} \]

These relations follow from the definition of r (cf (38)) combined with the fact that K becomes null on B, and combined with \( g(\xi, K) = 0 \) from equation (80). We have also used that \( \chi'_{i} = O(r^2) \) near B. We have furthermore used from the definitions of the electric and magnetic potentials at the horizon (95), and (97), that
\[ \Phi_{jk} = -A_{0jk}, \quad \Psi_{jk} = -\psi_{jk}. \tag{D.13} \]

At infinity, we have also used relations like
\[ f_{i}^{j'} = \begin{cases} 1 + O(R^{-1}) & \text{if } i' = 0 = j, \\ \Omega^j + O(R^{-1}) & \text{if } i = i' = j', \\ \delta^{ij} - \Omega^i \Omega^j + O(R^{-1}) & \text{if } i = i', j' = j, \end{cases} \tag{D.14} \]

which follow from the asymptotically Kaluza–Klein boundary conditions together with (20). We have also used that the electric/magnetic potentials are of order \( O(R^{-1}) \) near infinity.

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