Regular colorings and factors of regular graphs

Anton Bernshteyn\textsuperscript{a,1}, Omid Khormali\textsuperscript{b}, Ryan R. Martin\textsuperscript{c,2}, Jonathan Rollin\textsuperscript{d}, Danny Rorabaugh\textsuperscript{e}, Songling Shan\textsuperscript{f}, Andrew J. Uzzell\textsuperscript{g}

\textsuperscript{a}Department of Mathematics, University of Illinois at Urbana-Champaign
\textsuperscript{b}Department of Mathematical Sciences, University of Montana
\textsuperscript{c}Department of Mathematics, Iowa State University
\textsuperscript{d}Department of Mathematics, Karlsruhe Institute of Technology
\textsuperscript{e}Department of Mathematics and Statistics, Queen’s University
\textsuperscript{f}Department of Mathematics, Vanderbilt University
\textsuperscript{g}Department of Mathematics, University of Nebraska–Lincoln

Abstract

An \((r-1, 1)\)-coloring of an \(r\)-regular graph \(G\) is an edge coloring such that each vertex is incident to \(r-1\) edges of one color and 1 edge of a different color. In this paper, we completely characterize all 4-regular pseudographs (graphs that may contain parallel edges and loops) which do not have a \((3, 1)\)-coloring. An \(\{r-1, 1\}\)-factor of an \(r\)-regular graph is a spanning subgraph in which each vertex has degree either \(r-1\) or 1. We prove various conditions that must hold for any vertex-minimal 5-regular pseudographs without \((4, 1)\)-colorings or without \(\{4, 1\}\)-factors. Finally, for each \(r \geq 6\) we construct graphs that are not \((r-1, 1)\)-colorable and, more generally, are not \((r-t, t)\)-colorable for small \(t\).

Keywords: \(r\)-regular graph, \(\{r-1, 1\}\)-factor, \((r-1, 1)\)-coloring

1. Introduction

A graph with no loops or multiple edges is called simple; a graph in which both multiple edges and loops are allowed is called a pseudograph. Unless specified otherwise, the word “graph” in this paper is reserved for pseudographs. All (pseudo)graphs considered here are undirected and finite. Note that we count a loop twice in the degree of a vertex.

The famous Berge–Sauer conjecture asserts that every 4-regular simple graph contains a 3-regular subgraph \([6]\). This conjecture was settled by Tashkinov in 1982 \([11]\). In fact, he proved that every connected 4-regular pseudograph with either at most two pairs of multiple edges and no loops or at most one pair of multiple edges and at most one loop contains a 3-regular subgraph. Observe that
this cannot hold for all 4-regular pseudographs, because the graph consisting of a single vertex with two loops contains no 3-regular subgraph. The following question remains open.

**Question 1.1.** Which 4-regular pseudographs contain 3-regular subgraphs?

Note that in 1988, Tashkinov [12] classified the values of $t$ and $r$ for which every $r$-regular pseudograph contains a $t$-regular subgraph. Beyond finding regular subgraphs in regular graphs, finding factors—that is, regular spanning subgraphs—in regular graphs is also of special interest. As early as 1891, Petersen [9] studied the existence of factors in regular graphs. Since then numerous results on factors have appeared—see, for example, [2, 5, 7, 10]. The concept of factors can be generalized as follows: for any set of integers $S$, an $S$-factor of a graph is a spanning subgraph in which the degree of each vertex is in $S$. Several authors [1, 3, 8] have recently studied $\{a, b\}$-factors in $r$-regular graphs with $a + b = r$. In particular, Akbari and Kano [1] made the following conjecture:

**Conjecture 1.2.** If $r$ is odd and $0 \leq t \leq r$, then every $r$-regular graph has an $\{r - t, t\}$-factor.

However, Axenovich and Rollin [3] disproved this conjecture. The following theorem summarizes what is known about $\{r - t, t\}$-factors of $r$-regular graphs. (Note that although intended for simple graphs, the result of Petersen [9] applies to pseudographs as well.)

**Theorem 1.3.** Let $t$ and $r$ be positive integers with $t \leq \frac{r}{2}$.

(a) When $r$ is even:

- If $t$ is even, then every $r$-regular graph has a $t$-factor, and thus has an $\{r - t, t\}$-factor (Petersen [9]).
- Every $r$-regular graph of even order has an $\left\{\frac{r}{2} + 1, \frac{r}{2} - 1\right\}$-factor (Lu, Wang, and Yu [8]).
- If $t$ is odd and $t \leq \frac{r}{2} - 2$, then there exists an $r$-regular graph of even order that has no $\{r - t, t\}$-factor ([8]).
- If $t$ is odd, then trivially, no $r$-regular graph of odd order has an $\{r - t, t\}$-factor.

(b) When $r$ is odd and $r \geq 5$:

- If $t$ is even, then every $r$-regular graph has an $\{r - t, t\}$-factor (Akbari and Kano [1]).
- If $t$ is odd and $\frac{r}{2} \leq t$, then every $r$-regular graph has an $\{r - t, t\}$-factor ([1]).
- If $t$ is odd and $(t + 1)(t + 2) \leq r$, then there exists an $r$-regular graph that has no $\{r - t, t\}$-factor (Axenovich and Rollin [3]).

(c) Every 3-regular graph has a $\{2, 1\}$-factor (Tutte [14]).
The first case of Conjecture 1.2 that Theorem 1.3 does not address is when \( r = 5 \) and \( t = 1 \). As we will give much of our attention to this case, we restate it separately.

**Conjecture 1.4.** Every 5-regular graph has a \( \{4,1\} \)-factor.

An \((r-t,t)\)-coloring of an \( r \)-regular graph \( G \) is an edge-coloring (with at least two colors) such that each vertex is incident to \( r-t \) edges of one color and \( t \) edges of a different color. An ordered \((r-t,t)\)-coloring of \( G \) is an \((r-t,t)\)-coloring using integers as colors such that each vertex is incident to \( r-t \) edges of some color \( i \) and \( t \) edges of some color \( j \) with \( i < j \). Bernshteyn [4] introduced \((3,1)\)-colorings as an approach to answering Question 1.1. The advantage of working with \((3,1)\)-colorings is that this notion is “global” (i.e., there is a condition at each vertex), while the presence of a 3-regular subgraph is a “local” notion (a large 4-regular graph can contain a small 3-regular subgraph). Bernshteyn proved the following.

**Theorem 1.5** (Bernshteyn [4]). A connected 4-regular graph contains a 3-regular subgraph if and only if it admits an ordered \((3,1)\)-coloring.

We observe that the notion of an \((r-t,t)\)-coloring of an \( r \)-regular graph generalizes that of an \( \{r-t,t\} \)-factor, because \( \{r-t,t\} \)-factors correspond to \((r-t,t)\)-colorings that use exactly two colors. (In an \( r \)-regular graph with \( 0 < t < r \), \( t \)-factors correspond to ordered \((r-t,t)\)-colorings that use exactly two colors.) Thus, \((r-t,t)\)-colorings provide a common approach to attacking Question 1.1 and Conjecture 1.4. This leads us to ask whether the following weaker version of Conjecture 1.4 holds.

**Question 1.6.** Does every 5-regular graph have a \((4,1)\)-coloring?

For \( r \geq 6 \), the answer to the analogue of Question 1.6 for \((r-1,1)\)-colorings is negative (see Section 4).

Similarly, Theorem 1.5 motivates the following weaker version of Question 1.1.

**Question 1.7.** Which 4-regular graphs have \((3,1)\)-colorings?

The arrows in Figure 1 indicate the relationships among \( t \)-factors, \( \{r-t,t\} \)-factors, ordered \((r-t,t)\)-colorings, \((r-t,t)\)-colorings, and \( t \)-regular subgraphs of \( r \)-regular graphs.

Now we are ready to describe our main results. First, in Section 2, we characterize all 4-regular graphs which are not \((3,1)\)-colorable, which settles Question 1.7. Because the statement of the result requires additional definitions, we postpone it until then (see Theorem 2.1). Then, in Section 3, we make progress toward settling Conjecture 1.4 and Question 1.6 by proving several conditions on vertex-minimal 5-regular graphs without \((4,1)\)-colorings and \( \{4,1\} \)-factors. Finally, in Section 4, we construct relevant examples of \( r \)-regular graphs for \( r \geq 6 \) and various \( t \): some with no \((r-t,t)\)-coloring, others with an \((r-t,t)\)-coloring but no \( \{r-t,t\} \)-factor.
2. (3, 1)-colorings in 4-regular graphs

In this section, we characterize 4-regular graphs that do not admit (3, 1)-colorings.

Let us first establish some terminology. Let $G_1$ and $G_2$ be vertex-disjoint graphs with edges $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. The edge adhesion of $G_1$ and $G_2$ at $e_1$ and $e_2$ is the graph $G = (G_1, e_1) + (G_2, e_2)$ obtained by subdividing edges $e_1$ and $e_2$ and identifying the two new vertices. (See Figure 2.) That is,

\[
\begin{align*}
V(G) &= V(G_1) \cup V(G_2) \cup \{w\}; \\
E(G) &= (E(G_1) \setminus \{e_1\}) \cup (E(G_2) \setminus \{e_2\}) \cup \{u_1w, v_1w, u_2w, v_2w\}.
\end{align*}
\]

![Figure 2: Edge adhesion of two graphs, $G = (G_1, e_1) + (G_2, e_2)$.](image)

The adhesion of a loop to graph $H$ at edge $e = uv \in E(H)$ is the graph $H' = (H, e) + O$ obtained by subdividing $e$ and adding a loop at the new vertex. (See Figure 3.) That is,

\[
\begin{align*}
V(H') &= V(H) \cup \{x\}; \\
E(H') &= (E(H) \setminus \{e\}) \cup \{ux, vx, xx\}.
\end{align*}
\]

![Figure 3: Adhesion of a loop at an edge, $H' = (H, e) + O$](image)
Let $C$ be a (simple) cycle. A double cycle is obtained from $C$ by doubling each edge. We say a double cycle is even (respectively, odd) if it has an even (respectively, odd) number of vertices. (See Figure 4.)

Figure 4: Double cycles (odd on top, even on bottom).

Clearly, double cycles and graphs resulting from edge adhesion of two 4-regular graphs or from the adhesion of a loop to a 4-regular graph are all 4-regular. We are now ready to give the main result of this section.

Theorem 2.1. A connected 4-regular graph is not $(3,1)$-colorable if and only if it can be constructed from odd double cycles via a sequence of edge adhesions.

Remark 2.2. Theorem 2.1 naturally lends itself to a proof by induction. In particular, an equivalent statement is that a connected 4-regular graph is not $(3,1)$-colorable if and only if it is an odd double cycle or obtained from two 4-regular, non-$(3,1)$-colorable graphs by a sequence of edge adhesions.

Before we prove Theorem 2.1, we need to develop a few lemmas.

Lemma 2.3. A double cycle with $n \geq 1$ vertices is $(3,1)$-colorable if and only if $n$ is even.

Proof. Even double cycles have perfect matchings and are thus $(3,1)$-colorable. Assume that there is a $(3,1)$-coloring $c$ of an odd double cycle $G$. Let $G'$ denote the cycle obtained by removing one of the parallel edges between any two adjacent vertices in $G$. Color an edge in $G'$ red if the corresponding edges in $G$ are of the same color under $c$ and blue otherwise. Observe that the edges incident to any vertex in $G'$ are of different colors, since $c$ is a $(3,1)$-coloring of $G$. This is a contradiction since $G'$ is an odd cycle.

Lemma 2.4 (Bernshteyn [4]). If $G$ is a 4-regular graph and there exists a non-double edge $uv$ in $G$ with $u \neq v$ such that $G - \{u, v\}$ is connected, then $G$ is $(3,1)$-colorable.

Lemma 2.5 (Bernshteyn [4]). If $G$ is a 4-regular graph and $G' = (G,e) + O$ for some edge $e \in E(G)$, then either $G$ or $G'$ has a 3-regular subgraph.

Lemma 2.6. Let $G_1$ and $G_2$ be $(3,1)$-colorable 4-regular graphs and let $G_2$ have a loop $vv$. Construct $G$ by subdividing an edge $uw$ in $G_1$, identifying the new vertex with $v$, and removing the loop $vv$, so

\[
V(G) = V(G_1) \cup V(G_2);
E(G) = (E(G_1) \setminus \{uw\}) \cup (E(G_2) \setminus \{vv\}) \cup \{uv, wv\}.
\]

(See Figure 5.) Then $G$ is $(3,1)$-colorable.
Proof. Fix (3, 1)-colorings $c_i$ of $G_i$ for $i \in \{1, 2\}$. Note that $v$ in $G_2$ is incident to only one loop and that the two non-loop edges incident to $v$ have different colors under $c_2$. Without loss of generality, assume that $c_1(uw)$ is equal to the color of one of the non-loop edges incident to $v$. Therefore the colorings $c_1$ and $c_2$ extend to a (3, 1)-coloring of $G$ by coloring the edges $uv$ and $uw$ with color $c_1(uw)$.

\[\square\]

**Corollary 2.7** (to Lemmas 2.5, 2.6). Suppose exactly one of the connected 4-regular graphs $G_1$ and $G_2$ is (3, 1)-colorable. Then for any $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, $(G_1, e_1) + (G_2, e_2)$ is (3, 1)-colorable.

**Proof.** Without loss of generality, we assume that $G_1$ is (3, 1)-colorable and $G_2$ is not. Let $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$. By Theorem 1.5 and Lemma 2.5, the graph $G'_2 = (G_2, e_2) + O$ is (3, 1)-colorable. Applying Lemma 2.6 to $G_1$ and $G'_2$, we see that $(G_1, e_1) + (G_2, e_2)$ is (3, 1)-colorable.

\[\square\]

**Lemma 2.8.** Let $G$ be a 4-regular graph that is not (3, 1)-colorable. If $G$ has a non-double, non-loop edge, then $G$ is not 2-connected.

**Proof.** Let $uv$ be a non-double, non-loop edge, and suppose for contradiction that $G$ is 2-connected. By Lemma 2.4, since $G$ is not (3, 1)-colorable, $G' = G - \{u, v\}$ is disconnected. Since $G$ is 2-connected, neither $u$ nor $v$ is a cut-vertex. Therefore, every component of $G'$ must contain at least one vertex from $N_G(u)$ and at least one vertex from $N_G(v)$. Since the sum of the degrees of the vertices must be even in each component, the 4-regularity of $G$ implies that each component of $G'$ must have an even number of vertices from $N_G(u) \cup N_G(v)$.

Let $N_G(u) \setminus \{v\} = \{u_1, u_2, u_3\}$ and $N_G(v) \setminus \{u\} = \{v_1, v_2, v_3\}$. Without loss of generality, $G'$ is the disjoint union of a component $G_1$ containing $u_1$ and $v_1$ and a subgraph $G_2$ (of one or two components) containing $u_2, u_3, v_2$, and $v_3$.

Let $G'_1 = (G_1 + u_1v_1, u_1v_1) + O$ and $G'_2 = ((G - G_1) + uv, uv) + O$. (See Figure 6.) That is,

\[
\begin{align*}
V(G'_1) &= V(G_1) \cup \{w_1\}; \\
E(G'_1) &= E(G_1) \cup \{u_1w_1, v_1w_1, w_1w_1\}; \\
V(G'_2) &= V(G_2) \cup \{u, v, w_2\}; \\
E(G'_2) &= E(G_2) \cup \{uw_2, uw_3, uv, vw_2, vw_3, uw_2, vw_2, w_2w_2\}.
\end{align*}
\]

By the assumption of 2-connectedness, the vertex $u_1$ is not a cut-vertex of $G$, so $u_1 \neq v_1$ and $G'_1 - \{u_1, w_1\}$ is connected. Thus by Lemma 2.4, $G'_1$ is
Figure 6: Splitting a 2-connected graph into two (3,1)-colorable graphs, from the proof of Lemma 2.8.

(3,1)-colorable. Likewise, $G_2' - \{u, w_2\}$ is connected, so $G_2'$ is (3,1)-colorable. Select (3,1)-coloring $c_i$ of $G_i'$ for $i \in \{1, 2\}$. Note that because of the loops, $c_1(u_1w_1) \neq c_1(v_1w_1)$ and $c_2(uw_2) \neq c_2(vw_2)$. We can assume that $c_1(u_1w_1) = c_2(uw_2)$ and $c_1(v_1w_1) = c_2(vw_2)$. Therefore, the colorings $c_1$ and $c_2$ easily extend to a (3,1)-coloring $c$ of $G$, which is a contradiction.

Lemma 2.9. Let $G$ be a connected 4-regular graph that is not 2-connected. Then $G = (G_1, e_1) + (G_2, e_2)$ for some 4-regular graphs $G_1, G_2$ and edges $e_1 \in E(G_1), e_2 \in E(G_2)$.

Proof. Indeed, let $w \in V(G)$ be a cut-vertex. Now the lemma is implied by the following observation. Since the number of vertices with odd degrees in a graph is always even, $G - w$ consists of exactly two components and each of these components receives exactly two of the edges incident to $w$.

Proof of Theorem 2.1. Consider 4-regular graphs $G_1$ and $G_2$ and edges $e_1$ in $G_1, e_2$ in $G_2$. Any (3,1)-coloring of $(G_1, e_1) + (G_2, e_2)$ yields a (3,1)-coloring of $G_1$ or $G_2$, since the edges obtained by subdividing $e_1$ or $e_2$ are of the same color. Therefore every graph that is obtained from odd double cycles via edge adhesion is not (3,1)-colorable due to Lemma 2.3.

Now let $G$ be a connected 4-regular graph that is not (3,1)-colorable. We use induction on $|V(G)|$ to prove that $G$ is constructed from odd double cycles via edge adhesion. If $|V(G)| = 1$, then $G$ is a double cycle of one vertex and the theorem trivially holds. Assume that $|V(G)| \geq 2$. We may also assume that $G$ contains a non-double edge. Otherwise, if every edge is double, then $G$ is a double cycle, and by Lemma 2.3, $G$ is an odd double cycle, and thus we are done.

If each non-double edge is a loop, then one can easily check that $G$ is not 2-connected. If $G$ has a non-double non-loop edge, Lemma 2.8 implies that it is not 2-connected. By Lemma 2.9, $G = (G_1, e_1) + (G_2, e_2)$ for some 4-regular
graphs $G_1$, $G_2$ and edges $e_1 \in E(G_1)$, $e_2 \in E(G_2)$. Corollary 2.7 implies that either both $G_1$ and $G_2$ are (3,1)-colorable or neither of them is (3,1)-colorable. In the latter case, by the inductive hypothesis, we are done.

Assume that both $G_1$ and $G_2$ are (3,1)-colorable. Let $G'_1 = (G_1, e_1) + O$ and observe that $G$ is obtained from $G'_1$ and $G_2$ as in the statement of Lemma 2.6. Since $G_2$ is (3,1)-colorable, but $G$ is not, Lemma 2.6 implies that $G'_1$ is not (3,1)-colorable. Therefore, by the inductive hypothesis, $G'_1$ is obtained from odd double cycles via edge adhesion. Since $G'_1$ contains a loop and at least two vertices, it is not a double cycle. Thus, $G'_1 = (G'_{11}, e'_{11}) + (H, f)$, where both $G'_{11}$ and $G'_{12}$ are not (3,1)-colorable. Note that, without loss of generality, $G'_{11}$ does not contain the subdivided edge $e_1$, and so $G = (G'_{11}, e'_{11}) + (H, f)$ for some graph $H$ and edge $f$ in $H$. Since both $G$ and $G'_{11}$ are not (3,1)-colorable, neither is $H$ by Corollary 2.7. We have shown that $G$ is obtained from two graphs that are not (3,1)-colorable via edge adhesion, and so the inductive step is complete.

3. (4,1)-colorings and {4,1}-factors in 5-regular graphs

In this section, we make progress toward settling Conjecture 1.4 and Question 1.6. In particular, we show that if $G$ is a vertex-minimal counterexample to Conjecture 1.4, then $G$ must satisfy a large number of structural conditions. We show that similar conditions must hold for any vertex-minimal graph that gives a negative answer to Question 1.6.

A set $S$ of edges of a connected graph $G$ is called an edge cut if $G - S$ is disconnected. An edge cut $S$ is minimal provided $G - (S \setminus \{e\})$ is connected for each edge $e \in S$. An edge cut of size 1 is called a bridge. Note that a minimal edge cut does not contain loops.

Most of the following results are obtained using reductions to smaller graphs. We also use a corollary of Tutte’s 1-Factor Theorem.

Theorem 3.1 (Tutte [13]). A graph $G$ has a 1-factor if and only if the number of connected components of $G - S$ of odd order is at most $|S|$ for every vertex set $S \subseteq V(G)$.

Corollary 3.2. Every 2k-edge-connected $(2k + 1)$-regular graph has a 1-factor.

In Section 3.1, we prove our results about (4,1)-colorings. In Section 3.2, we prove our results about {4,1}-factors.

3.1. 5-regular graphs without (4,1)-colorings

We begin by showing that a vertex-minimal 5-regular graph with no (4,1)-coloring must satisfy several connectivity conditions. An edge-coloring $c$ of $G$ extends an edge-coloring $c'$ of $G'$ if $c(e) = c'(e)$ for all $e \in E(G) \cap E(G')$.

Theorem 3.3. Let $G$ be a vertex-minimal 5-regular graph without a (4,1)-coloring.

(a) $G$ is connected.
(b) $G$ is not 4-edge-connected, i.e., contains an edge cut on 3 edges.

(c) $G$ has no minimal edge cut of size 2.

(d) $G$ does not have two bridges.

(e) Each bridge in $G$ has (precisely) one endpoint incident to two loops.

(f) The edges of any minimal edge cut of size 3 in $G$ have a vertex in common, and this vertex is incident to a loop.

Proof. (a) This follows from vertex-minimality.

(b) This is a consequence of Corollary 3.2 with $k = 2$.

(c) Assume $\{uv, wx\}$ is a minimal edge cut, so $G - \{uv, wx\}$ is disconnected, but $G - uv$ and $G - wx$ are both connected. Then $G - \{uv, wx\}$ has precisely two components, $G_1$ and $G_2$, and, without loss of generality, $u, w \in V(G_1)$ and $v, x \in V(G_2)$. We obtain 5-regular graphs $G'_1 = G_1 + uw$ and $G'_2 = G_2 + vx$ by adding a new edge (possibly a loop or parallel edge) to each component. By the assumption of vertex-minimality, both graphs $G'_1$ and $G'_2$ have $(4, 1)$-colorings. Consider such colorings $c_i$ of $G'_i$ for $i \in \{1, 2\}$ such that $c_1(uw) = c_2(vx) = 1$. Note that all edges of $E(G) \setminus \{uv, wx\}$ are contained in exactly one of $G'_1$ or $G'_2$. So we obtain a $(4, 1)$-coloring of $G$ by coloring $uv$ and $wx$ with color 1 and all other edges according to $c_1$ and $c_2$, a contradiction.

(d) Assume $uv$ and $wx$ are bridges in $G$. Then $G - \{uv, wx\}$ has three components. Without loss of generality, assume that $u$ and $w$ are contained in the same component. We obtain two 5-regular graphs by adding the edges $uw$ and $wx$ (possibly loops or parallel edges). The proof proceeds exactly as in (c).

(e) If there is a bridge with both endpoints incident to two loops, then there are no other edges and the graph is easily $(4, 1)$-colorable. Assume that there is a bridge $uv$ with each endpoint incident to at most one loop. Then $G - uv$ has two components $G_1$ and $G_2$, each with at least 2 vertices. We obtain a 5-regular graph from $G_1$ (respectively, $G_2$) by adding a new vertex incident to two loops and to $u$ (respectively, to $v$). Both graphs have $(4, 1)$-colorings by assumption of vertex-minimality. Much as before, we obtain a $(4, 1)$-coloring of $G$ by choosing the same color for the new edges incident to $u$ and $v$, a contradiction.

(f) Consider distinct edges $uv$, $wx$ and $yz$ forming a minimal edge cut of size 3. First observe that a vertex which is incident to all three edges is incident to a loop due to statements (c) and (d) of this theorem. Thus assume that there is no such vertex. Removing the three edges from $G$ yields exactly two components $G_1$ and $G_2$, each with at least 2 vertices. Without loss of generality assume $u, w$ and $y$ are in $G_1$ and $v, x$ and $z$ are in $G_2$.

Let $\mathcal{H}_i$ denote the set of all 5-regular graphs that contain $G_i$ as a subgraph and have one more vertex than $G_i$, $i \in \{1, 2\}$. Note that $\mathcal{H}_i \neq \emptyset$. We consider each $H \in \mathcal{H}_i$ with a fixed copy $K = K(H)$ of $G_i$ and call edges in $E(H) \setminus E(K)$ new if they are incident to vertices of $K$. By assumption all graphs in $\mathcal{H}_1 \cup \mathcal{H}_2$
are \((4, 1)\)-colorable. Assume first that, for all \(i \in \{1, 2\}\), there is a graph \(H \in \mathcal{H}_i\) having a \((4, 1)\)-coloring where all \((2 \text{ or } 3)\) new edges are of the same color. Then, much as before, we obtain a \((4, 1)\)-coloring of \(G\), a contradiction.

So, assume that for any graph \(H \in \mathcal{H}_2\) and for any \((4, 1)\)-coloring of \(H\) the new edges in \(H\) are not all of the same color. Consider a \((4, 1)\)-coloring \(c_1\) of the graph in \(\mathcal{H}_1\) obtained from \(G_1\) by adding a new vertex \(p\) incident to one loop and connected to \(u, w\) and \(y\) by new edges. Without loss of generality assume that \(c_1(up) = c_1(wp) \neq c_1(yp)\). Further consider a \((4, 1)\)-coloring \(c_2\) of the graph in \(\mathcal{H}_2\) obtained from \(G_2\) by adding by adding a new vertex \(q\) incident to two loops and edges \(vx\) and \(qz\). Then \(c_2(vx) \neq c_2(qz)\) by assumption. Therefore we obtain a \((4, 1)\)-coloring of \(G\) from \(c_1\) and \(c_2\) as before, a contradiction.

Now we prove a number of conditions involving loops, parallel edges, or forbidden subgraphs (see Figure 7).

**Theorem 3.4.** Let \(G\) be a vertex-minimal 5-regular graph without a \((4, 1)\)-coloring.

(a) \(G\) does not contain a 4-regular subgraph with at least 2 vertices.

(b) \(G\) does not have 3 parallel edges.

(c) \(G\) does not contain a path of length three consisting of double edges.

(d) No vertex of \(G\) that has a loop is incident to a double edge.

(e) No vertices with loops are adjacent.

(f) \(G\) contains at least 5 loops.

(g) There do not exist \(u_1, u_2, u_3, v_1, v_2, v_3 \in V(G)\) such that the \(u_i\) have loops and such that for each \(i\) and \(j\), \(u_i\) is adjacent to \(v_j\) (that is, there is no \(K_{3,3}\) with one loop on each vertex of one side of the vertex partition).

(h) No vertex is adjacent to more than 3 vertices with loops.

(i) No 4-vertex subgraph of \(G\) has 8 or more edges.

---

Figure 7: From Theorem 3.4 (b, c, d, e, g, h), forbidden subgraphs in a vertex-minimal 5-regular graph with no \((4, 1)\)-coloring.
Proof. (a) Suppose for contradiction that $G$ has a 4-regular subgraph $H$ with at least 2 vertices. Let $F$ denote the set of edges $uv$ in $G$ with $u \in V(H)$ and $v \notin V(H)$. We obtain a 5-regular graph $G'$ from $G$ by removing the vertices of $H$ and adding some new edges between the vertices of degree less than 5 and, if there is an odd number of such vertices, one new vertex with two loops. Let $F'$ denote the set of new edges in $G'$, except for the loops incident to the new vertex, if such exists.

By vertex-minimality, $G'$ has a $(4, 1)$-coloring $c$. We extend this to a coloring of $G$ as follows. Assign a color $k$ not used by $c$ to all edges in $H$ and a color different from $k$ to all edges in $E(G) \setminus E(H)$ having both endpoints in $H$. Each edge in $F$ shares a vertex with least one edge in $F'$. Consider an injective map $f : F \to F'$ such that $e$ and $f(e)$ have a common vertex for all $e \in F$. Then color each $e \in F$ with color $c(f(e))$. This coloring is a $(4, 1)$-coloring of $G$, a contradiction.

(b) Assume that there are at least three edges between vertices $u$ and $v$. Let $F$ denote the set of edges incident to $u$ or $v$ but not both. Observe that $G$ has at least 3 vertices, as the 2-vertex 5-regular graphs are easily $(4, 1)$-colorable. Consider the 5-regular graph $G'$ obtained from $G$ by removing $u$ and $v$ and adding a matching between the (remaining) neighborhood of $u$ and the (remaining) neighborhood of $v$, possibly creating parallel edges and loops. By assumption $G'$ has a $(4, 1)$-coloring $c$. We extend this to a coloring of $G$ by coloring the edges in $F$ with the colors of the corresponding new edges under $c$. Then $u$ and $v$ are either both incident to edges of the same color only, or both incident to an edge of one color and an edge of a second color. In either case we can color the parallel edges between $u$ and $v$ such that we obtain a $(4, 1)$-coloring of $G$, a contradiction.

(c) Let $v_1$, $v_2$, $v_3$, and $v_4$ denote the vertices of a double path in $G$. Let $u_2$ be the other neighbor of $v_2$ and $u_3$ the other neighbor of $v_3$. We assume that $u_2 \neq v_3$ and $u_3 \neq v_2$ due to part (b) of this Theorem. Remove $v_2$ and $v_3$ from $G$, add two edges between $v_1$ and $v_4$, and add an edge (possibly a loop or multiple edge) between $u_2$ and $u_3$. Let $G'$ denote the resulting graph, which, by hypothesis, has a $(4, 1)$-coloring. We consider several cases (see Figure 8).

First, suppose both edges between $v_1$ and $v_4$, as well as the edge $u_2u_3$, have color 1. Then in $G$, we give color 1 to all edges incident to $v_2$ or $v_3$ except for one of the edges between $v_2$ and $v_3$, to which we give color 2.

Second, suppose the edges between $v_1$ and $v_4$ have color 1 and the edge $u_2u_3$ has color 2. Then in $G$, we give color 2 to $u_2v_2$ and $u_3v_3$ and color 1 to all other edges incident to $v_2$ or $v_3$.

Third, suppose $u_1u_2$ and one of the edges between $v_1$ and $v_3$ have color 1, while the other has color 2. Then in $G$ we give color 2 to one of the edges between $v_1$ and $v_2$ and to one of the edges between $v_3$ and $v_4$. We give color 1 to all other edges incident to $v_2$ or $v_1$.

Fourth, suppose all three edges have different colors. Then there are two subcases to consider. If one of the edges between $v_1$ and $v_4$ is the only edge of its color that is incident to both $v_1$ and $v_4$, then we may instead give it the same
Figure 8: Extending the coloring of \( G' \) to \( G \) in the proof of Theorem 3.4 (c).

color as \( u_2u_3 \) and so reduce the problem to the previous case. Assume, then, that the edges between \( v_1 \) and \( v_4 \) have colors 1 and 2, that \( v_3 \) is incident to four edges with color 1, that \( v_4 \) is incident to four edges with color 2, and that \( u_2u_3 \) has color 3. When we color \( G \), we give color 1 to one of the edges between \( v_1 \) and \( v_2 \), color 2 to one of the edges between \( v_3 \) and \( v_4 \), and color 3 to all other edges incident to \( v_2 \) or \( v_3 \).

We have shown that in all four cases, we may extend a \((4,1)\)-coloring of \( G' \) to a \((4,1)\)-coloring of \( G \), which is a contradiction.

(d) Suppose to the contrary that \( u \) is a vertex with a loop and that there is a double edge between \( u \) and some other vertex \( v \). Observe that \( v \) cannot have a loop; if it did, then \( u \) and \( v \) would each have exactly one neighbor outside of \( \{u, v\} \). This is a minimal edge cut of size 2, which contradicts Theorem 3.3 (c).

Thus, \( u \) sends one edge to a vertex \( w \) outside of \( \{u, v\} \), while \( v \) sends three, to vertices \( x, y, \) and \( z \). We remove \( u \) and \( v \) from \( G \) and create a 5-regular graph \( G' \) by adding edges \( e = wx \) and \( f = yz \). (As usual, we may create loops or multiple edges.) By hypothesis, \( G' \) has a \((4,1)\)-coloring. If \( e \) and \( f \) both have color 1, then we may extend the coloring to \( G \) by coloring all edges incident to \( u \) or to \( v \) with color 1, except for one edge between \( u \) and \( v \), to which we give color 2. If \( e \) has color 1 and \( f \) has color 2, then we extend the coloring to \( G \) by giving color 1 to both \( uw \) and \( vx \) and color 2 to all other edges incident to \( u \) or \( v \). In either case, we have a contradiction.

(e) Suppose that \( u \) and \( v \) are adjacent vertices with loops. By part (d), there is exactly one edge between \( u \) and \( v \). Observe that neither \( u \) nor \( v \) can have
two loops. Indeed, if both have two loops, then $G'/\{u, v\}$ is a component and obviously has a $(4,1)$-coloring. If, say, $v$ has two loops but $u$ has only one, then $u$ sends two edges to vertices outside of $\{u, v\}$. These edges form a minimal edge cut of size 2, which contradicts Theorem 3.3 (e).

Thus, we may assume that $u$ and $v$ are incident to only one loop each and hence both send two edges to vertices outside of $\{u, v\}$. Delete $u$ and $v$ and form a new 5-regular graph $G'$ by adding a matching between the (remaining) neighborhood of $u$ and the (remaining) neighborhood of $v$. By hypothesis, $G'$ has a $(4,1)$-coloring. Then we obtain a $(4,1)$-coloring of $G$ much as in part (b), a contradiction.

(f) If $X$ and $Y$ are disjoint subsets of $V(G)$, let $e(X, Y)$ denote the number of edges between $X$ and $Y$. Since $G$ does not admit a $(4,1)$-coloring, it does not contain a perfect matching. This means that there is a set $S \subset V(G)$ such that the number of components of $G - S$ of odd order is strictly greater than $|S|$. Let $C_1, \ldots, C_t$ be the components of $G - S$ of odd order. Then $5|V(C_i)| = 2|E(C_i)| + e(S, C_i)$ and hence $e(S, C_i)$ is odd, $1 \leq i \leq t$. Similarly, there is an even number of edges between $S$ and a component of $G - S$ of even order. Therefore $|S| \equiv t \pmod{2}$, and thus $t \geq |S| + 2$.

Recall from Theorem 3.3 (e, f) that if $G$ contains a bridge, then one of the endpoints of the bridge is incident to two loops, and if $G$ contains a minimal edge cut of size 3, then its edges share an endpoint which is incident to a loop. Therefore, either $e(S, C_i) \geq 5$ or there is only one vertex in $C_i$ and this vertex is adjacent to $\ell \geq 1$ loops. In the latter case, $e(S, C_i) + 2\ell = 5$. Let $C = \bigcup_{i=1}^{t} C_i$ and let $k$ be the total number of loops in $G$. Then

$$5t \leq |e(S, C)| + 2k \leq 5|S| + 2k$$

and hence

$$k \geq \frac{5}{2}(t - |S|) \geq \frac{5}{2} \cdot 2 = 5,$$

as desired.

(g) By part (a), we may assume that the $v_i$ form an independent set, because if, say, $v_1 v_2$ were an edge in $G$, then $\{v_1, v_2, u_1, u_2, u_3\}$ would induce a 4-regular subgraph of $G$. Delete all of the $u_i$ and the $v_i$ and form a new 5-regular graph $G'$ by adding a matching $M = \{e_{12}, e_{23}, e_{31}\}$ among the neighborhoods of the $v_i$ such that each edge $e_{ij}$ (which may be a loop) has one endpoint in $N(v_i)$ and the other in $N(v_j)$.

By hypothesis, $G'$ has a $(4,1)$-coloring $c$. When we extend this coloring to $G$, we will give $c(e_{ij})$ to one edge incident to $v_i$ and to one edge incident to $v_j$. Furthermore, we will give to each vertex $v_i$ an ordered triple $(a_1, a_2, a_3) := (c(v_i u_1), c(v_i u_2), c(v_i u_3))$. There are three cases we must consider (see Figure 9).

First, suppose that all of the $e_{ij}$ have color 1. In this case, we give $v_1$ the triple $(2, 1, 1)$, $v_2$ the triple $(1, 2, 1)$, and $v_3$ the triple $(1, 1, 2)$. Additionally, we give color 1 to the loop at each $u_i$.

Second, suppose that the $e_{ij}$ have exactly two colors. Without loss of generality, let $c(e_{12}) = c(e_{31}) = 1$ and $c(e_{23}) = 2$. Observe that in $G$, $v_1$ is incident
to two edges with color 1, while $v_2$ and $v_3$ are each incident to one edge with color 1 and one edge with color 2. We give the triple $(1, 1, 2)$ to $v_1$, $(2, 2, 2)$ to $v_2$, and $(1, 1, 1)$ to $v_3$. Additionally, we give color 1 to the loops at $u_1$ and $u_2$ and color 2 to the loop at $u_3$.

Third, suppose that all of the $e_{ij}$ have different colors. Without loss of generality, let $c(e_{12}) = 1$, $c(e_{23}) = 2$, and $c(e_{31}) = 3$. In this case, we give the triple $(1, 1, 1)$ to $v_1$ and $(2, 2, 2)$ to both $v_2$ and $v_3$. Additionally, we give color 2 to the loop at each $u_i$.

In all three cases, we have produced a $(4, 1)$-coloring of $G$, which is a contradiction.

(h) Suppose that $u \in V(G)$ is adjacent to vertices $v_1$, $v_2$, $v_3$, and $v_4$, all of which have loops. By part (e), the $v_i$ form an independent set. By part (d), there is only one edge between $u$ and each $v_i$. Delete the $v_i$ and form a new 5-regular graph $G'$ by adding two loops at $u$, and, for each $i$ such that $v_i$ has only one loop, adding an edge $e_i$ between the two vertices of $N(v_i) \setminus \{u\}$. By hypothesis, $G'$ has a $(4, 1)$-coloring. We extend this coloring to $G$ as follows: for each $v_i$ with only one loop, we give all edges incident to $v_i$, except for $uv_i$, the same color as $e_i$. We then give each $uv_i$ the color of the loops incident to $u$ in $G'$, which we may assume is a new color. Finally, if any of the $v_i$ have two loops, we give these loops a color different from the color of $uv_i$. Thus, $G$ has a $(4, 1)$-coloring, which is a contradiction.

(i) The proof of this statement is computationally assisted but can be checked by hand with extensive case work. An exhaustive search shows that there exist only seven graphs on 4 vertices with at least 8 edges and with maximum degree 5 that satisfy parts (a–e) of this theorem (see Figure 10). Moreover, all seven graphs have exactly 8 edges.

Let $H$ be a subgraph of $G$ with 4 vertices and 8 edges. Since $G$ is 5-regular, there are 4 edges between $H$ and $G - H$. Let $U = \{u, v, w, x\}$ be the multiset of vertices in $H$, where the multiplicity of a vertex in $U$ equals the number of edges between the vertex and $G - H$. Let $U' = \{u', v', w', x'\}$ be the corresponding neighbors in $G - H$.
In each of the seven present graphs, \( H \) has a 1-factor \( M \) and there are two distinct vertices in \( U \), \( u \) and \( v \) (without loss of generality), so that \( H - \{u, v\} \)
has a non-loop edge \( e \). Define the 5-regular graph \( G' = (G - H) + \{u'v', w'x'\} \),
which has fewer vertices than \( G \). By our assumption of vertex-minimality, \( G' \)
has a \((4, 1)\)-coloring \( c \). We can then define a \((4, 1)\)-coloring of \( G \) as follows.
Let \( c_1 = c(u'v') \) and \( c_2 = c(w'x') \). If \( c_1 = c_2 \), use a new color for the edges in the
one-factor \( M \) and use \( c_1 \) for all other edges incident to a vertex in \( H \). If \( c_1 \neq c_2 \),
use \( c_1 \) for \( uu' \), \( vv' \), and \( e \) and use \( c_2 \) for all other edges incident to a vertex in \( H \).

![Figure 10: 4-vertex, 8-edge graphs with the vertices in \( U \) labeled and an edge in \( H - \{u, v\} \)
dashed, from the proof of Theorem 3.4 (i).](image)

### 3.2. 5-regular graphs without \( \{4, 1\} \)-factors

The results in this subsection are very similar to those in the previous subsection, so we will omit some of the proofs. Notice first that every statement of
Theorem 3.3 also holds for vertex-minimal graphs without \( \{4, 1\} \)-factors because
the proofs do not require the use of more than two colors.

**Theorem 3.5.** Let \( G \) be a vertex-minimal 5-regular graph without a \( \{4, 1\} \)-factor.

(a) \( G \) is connected.

(b) \( G \) is not 4-edge-connected, i.e., contains an edge cut on 3 edges.

(c) \( G \) has no minimal edge cut of size 2.

(d) \( G \) does not have two bridges.

(e) Each bridge in \( G \) has (precisely) one endpoint incident to two loops.

(f) The edges of any minimal edge cut of size 3 in \( G \) have a vertex in common,
and this vertex is incident to a loop.

Most of the statements in Theorem 3.4 also hold for vertex-minimal graphs
without \( \{4, 1\} \)-factors. We discuss the differences between Theorems 3.4 and 3.6
in Remark 3.7 below.
**Theorem 3.6.** Let $G$ be a vertex-minimal 5-regular graph without a $\{4,1\}$-factor.

(a) $G$ does not contain a copy of $K_4$.

(b) $G$ has no parallel non-loop edges.

(c) No vertices with loops are adjacent.

(d) $G$ contains at least 5 loops.

(e) There do not exist $u_1, u_2, u_3, v_1, v_2, v_3 \in V(G)$ such that the $u_i$ have loops and such that for each $i$ and $j$, $u_i$ is adjacent to $v_j$.

**Remark 3.7.** Here, we elaborate on the relationships between the statements in Theorems 3.4 and 3.6. First, the proofs of Theorem 3.4 (a, h) do not work for factors, since in each case, we may need three colors to create the contradictory $(4,1)$-coloring of $G$.

Next, the proof of Theorem 3.6 (a) given below does not work for general $(4,1)$-colorings, because there might be three edges of color 1 and one edge of color 2 incident to one endpoint of the new edge and with four edges of color 3 incident to the other endpoint. (This corresponds to the last configuration in Figure 11, but with a third color assigned to the four lower edges.) It is not hard to show that it is impossible to extend this coloring to a $(4,1)$-coloring of the original graph.

Next, we can improve the condition of Theorem 3.4 (b) to prohibit double edges: see Theorem 3.6 (b). (We cannot improve the statement for $(4,1)$-colorings, because, if we try to follow the proof of Theorem 3.6 (b) given below, we may obtain three different colors from the smaller graph $G'$, making extension to a $(4,1)$-coloring of $G$ impossible.) So, the analogous statements to Theorem 3.4 (c, d) for factors are merely special cases of forbidding parallel non-loop edges. Similarly, with no parallel non-loop edges and, by Theorem 3.5 (d, e), at most one double loop, the analogous statement to Theorem 3.4 (i) is immediate.

Finally, the proofs of Theorem 3.4 (e, f, g), which correspond to Theorem 3.6 (c, d, e), work for $\{4,1\}$-factors in exactly the same way, so we will not give the proofs.

**Proof of Theorem 3.6.** (a) Let $K = \{u_1, u_2, v_1, v_2\}$ denote the vertices of a copy of $K_4$ in $G$. We obtain a graph $G'$ by removing all vertices in $K$ from $G$ and adding two adjacent new vertices $u$ and $v$. Then, for each $x \notin K$, we add an edge $xu$ for each edge $xu_i$ in $G$ and an edge $xv$ for each edge $xv_i$ in $G$, $i \in \{1, 2\}$. (Note that this may create multiple edges.)

The new graph $G'$ is 5-regular and has fewer vertices than $G$. By the assumption of vertex-minimality, it has a $\{4,1\}$-factor. This $\{4,1\}$-factor extends to a $\{4,1\}$-factor of $G$ regardless of the colors of the edges incident to $u$ and $v$ (see Figure 11). This is a contradiction.
(b) Assume that there are at least two edges between $u$ and $v$. Consider the 5-regular graph $G'$ obtained from $G$ by removing $u$ and $v$ and adding a matching between $N(u) \setminus \{v\}$ and $N(v) \setminus \{u\}$ (possibly creating parallel edges and loops). By assumption, $G'$ has a $\{4, 1\}$-factor $F$. We can extend $F$ to a $\{4, 1\}$-factor of $G$ by adding some of the edges between $u$ and $v$ to $F$, which is a contradiction.

4. $r$-Regular Graphs for $r \geq 6$

In this section we give a negative answer to the analogue of Question 1.6 for $r \geq 6$. More generally, for each odd $t$ and each even $r$, as well as for each odd $t$ and each odd $r \geq (t+2)(t+1)$, we construct an $r$-regular graph with no $(r-t, t)$-coloring. Note that for even $t$, every $r$-regular graph has a $(r-t, t)$-coloring and for odd $t \leq \frac{r}{2}$ and even $r$ every $r$-regular graph has a $(r-t, t)$-coloring due to Theorem 1.3.

**Theorem 4.1.** Let $r$ and $t$ be positive integers with $t \leq \frac{r}{2}$ odd. If $r$ is even or $r \geq (t+2)(t+1)$, then there exists an $r$-regular graph that is not $(r-t, t)$-colorable.

Observe that this is the same upper bound on odd $r$ as in Theorem 1.3(b) (due to [3]) for the existence of $r$-regular graphs without $\{r-t, t\}$-factors.

**Proof.** First, if $r$ is even, then the $r$-regular graph with one vertex and $\frac{r}{2}$ loops has no $(r-t, t)$-coloring, since $t$ is odd.

Now suppose that $r \geq (t+2)(t+1) \geq 6$ is odd. Let $G$ be a graph on vertices $v, u, u_1, \ldots, u_{t+1}$ with $t+2$ edges between $v$ and $u_i$ and $\frac{r-t-2}{2}$ loops incident to $u_i$, $1 \leq i \leq t+1$, and $r-(t+2)(t+1) \geq 0$ edges between $v$ and $u$ and $\frac{(t+2)(t+1)}{2}$ loops incident to $u$. Observe that $G$ is $r$-regular. Suppose that $G$ admits an $(r-t, t)$-coloring and observe that in any such coloring, there is an
such that all $t + 2$ edges between $v$ and $u_i$ are of the same color. However, this is a contradiction, because there is no coloring of the loops incident to this $u_i$ such that there are exactly $t$ edges of another color incident to $u_i$, as $t$ is odd.

Now we will exhibit $r$-regular graphs of even order that have $(r - 1, 1)$-colorings but not $(r - 1, 1)$-factors. The constructions are similar to constructions in [8].

**Theorem 4.2.** For every even $r \geq 6$ there exists an $(r - 1, 1)$-colorable $r$-regular graph of even order without an $(r - 1, 1)$-factor.

**Proof.** Note that $K_{r+1}$ has an odd number of vertices and thus does not have an $(r - 1, 1)$-factor, as $r - 1$ is odd. However, there is an $(r - 1, 1)$-coloring with 3 colors. Indeed color a copy of $K_r$ in red, $r - 1$ of the remaining edges blue and the last edge green.

If $\frac{r}{2}$ is odd, then let $G_1, \ldots, G_{\frac{r}{2}}$ be vertex-disjoint copies of $K_{r+1} - e$. Form a graph $G$ from the union of $G_i$ by connecting all vertices of degree $r - 1$ in the $G_i$ to a new vertex $u$. Then $G$ has an even number of vertices and is $r$-regular. Moreover there is an $(r - 1, 1)$-coloring with 3 colors. Indeed start coloring the edges incident to $u$ and extend the coloring to each $G_i$, 1 $\leq$ $i$ $\leq$ $\frac{r}{2}$, using the coloring of $K_{r+1}$ given above. Assume that $G$ has an $(r - 1, 1)$-factor, i.e., an $(r - 1, 1)$-coloring in two colors. Then there is an $i$, 1 $\leq$ $i$ $\leq$ $\frac{r}{2}$, such that both edges between $G_i$ and $u$ are of the same color. This yields an $(r - 1, 1)$-coloring of $K_{r+1}$ in two colors, a contradiction.

If $\frac{r}{2}$ is even, then let $t = 3(\frac{r}{2} - 1)$. Let $G_1, \ldots, G_t$ be vertex-disjoint copies of $K_{r+1} - e$. Form a graph $G$ from the union of the $G_i$ and a disjoint copy of $K_3$ with vertex set $\{u_0, u_1, u_2\}$ by connecting both vertices of degree $r - 1$ in $G_i$ to $u_j$ if $j(\frac{r}{2} - 1) < i \leq (j + 1)(\frac{r}{2} - 1)$. Then $G$ has an even number of vertices and is $r$-regular. One can show that $G$ has an $(r - 1, 1)$-coloring but no $(r - 1, 1)$-factor with arguments similar to those given above.

**5. Concluding Remarks**

Here we state a number of open problems related to our work. Recall from the Introduction that Tashkinov [11] showed that every 4-regular graph with no multiple edges and at most one loop contains a 3-regular subgraph. It is not known whether the restriction on the number of loops is necessary.

**Question 5.1.** Does every 4-regular graph with no multiple edges have a 3-regular subgraph?

Let us note that Question 5.1 is open even for the class of 4-regular graphs with no multiple edges and at most two loops.

Our next question concerns $(r - 1, 1)$-colorings with a bounded number of colors. Bernshteyn [4] showed that if $G$ is a 4-regular graph that has a $(3, 1)$-coloring, then $G$ has a $(3, 1)$-coloring that uses at most three colors.
Question 5.2. Is there a positive integer $K$ such that every 5-regular graph has a $(4,1)$-coloring using at most $K$ colors?

Question 5.2 lies “between” Conjecture 1.4 and Question 1.6 in the following sense. An affirmative answer to Question 5.2 clearly gives an affirmative answer to Question 1.6. On the other hand, as observed in the Introduction, Conjecture 1.4 implies an affirmative answer to Question 5.2 with $K = 2$. Let us also note that none of the proofs of the statements in Theorems 3.3 and 3.4 required more than three colors.

Our final question concerns ordered $(r - 1,1)$-colorings.

Question 5.3. For $r \geq 5$, if $G$ is an $r$-regular graph with an $(r - 1)$-regular subgraph, does $G$ admit an ordered $(r - 1,1)$-coloring?

As observed in the Introduction, the converse to this statement always holds (see Figure 1). Also, Theorem 1.5 implies that the corresponding statement is true for $r = 4$.

Acknowledgments

We are grateful to Maria Axenovich, Sogol Jahanbekam, Yunfang Tang, Claude Tardif, and Torsten Ueckerdt for helpful conversations.

References

[1] S. Akbari and M. Kano. {$k, r-k$}-factors of $r$-regular graphs. Graphs Combin., 30(4):821–826, 2014.

[2] J. Akiyama and M. Kano. Factors and factorizations of graphs: Proof techniques in factor theory, volume 2031 of Lecture Notes in Mathematics. Springer, Heidelberg, 2011.

[3] M. Axenovich and J. Rollin. Brooks type results for conflict-free colorings and $(a,b)$-factors in graphs. Discrete Math., 338(12):2295–2301, 2015.

[4] A. Yu. Bernshteyn. 3-regular subgraphs and (3,1)-colorings of 4-regular pseudographs. J. Appl. Ind. Math., 8(4):458–466, 2014.

[5] B. Bollobás, A. Saito, and N. C. Wormald. Regular factors of regular graphs. J. Graph Theory, 9(1):97–103, 1985.

[6] J. A. Bondy and U. S. R. Murty. Graph theory with applications. American Elsevier Publishing Co., Inc., New York, 1976.

[7] F. R. K. Chung and R. L. Graham. Recent results in graph decompositions. In Combinatorics (Swansea, 1981), volume 52 of London Math. Soc. Lecture Note Ser., pages 103–123. Cambridge Univ. Press, Cambridge-New York, 1981.
[8] H. Lu, D. G. L. Wang, and Q. Yu. On the existence of general factors in regular graphs. *SIAM J. Discrete Math.*, 27(4):1862–1869, 2013.

[9] J. Petersen. Die Theorie der regulären graphs. *Acta Math.*, 15(1):191–220, 1891.

[10] M. D. Plummer. Graph factors and factorization: 1985–2003: a survey. *Discrete Math.*, 307(7–8):791–821, 2007.

[11] V. A. Tashkinov. 3-regular subgraphs of 4-regular graphs. *Mat. Zametki*, 36(2):239–259, 1984.

[12] V. A. Tashkinov. Regular parts of regular pseudographs. *Mat. Zametki*, 43(2):263–275, 1988.

[13] W. T. Tutte. The factorization of linear graphs. *J. London Math. Soc.*, 22:107–111, 1947.

[14] W. T. Tutte. The subgraph problem. *Ann. Discrete Math.*, 3:289–295, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).