Oscillations near separatrix for perturbed Duffing equation

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Abstract

A periodic perturbation generates a complicated dynamics close to separatrices and saddle points. We construct an asymptotic solution which is close to the separatrix for the unperturbed Duffing’s oscillator over a long time. This solution is defined by a separatrix map. This map is obtained for any order of the perturbation parameter. Properties of this map show an instability of a motion for the perturbed system.

Introduction

We consider an equation for the perturbed Duffing’s oscillator:

\[ u'' + 2u - 2u^3 = \varepsilon \cos(\omega t + \Phi_0). \]  

(1)

Here \( \varepsilon \) is a small positive parameter, \( \omega \) and \( \Phi_0 \) are constants.

The goal of the paper is to construct an asymptotic solution for (1) which is close to the separatrix for the unperturbed Duffing’s oscillator over a long time. Formally the words \textit{asymptotic solution} mean that there exists \( \varepsilon_0 = \text{const} > 0 \), such that a following asymptotic series

\[ U(t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U_n(t), \]  

(2)

gives a residual as \( o(\varepsilon^n) \), \( \forall n \in \mathbb{N}, \) as \( \varepsilon \in (0, \varepsilon_0) \) when \( U(t, \varepsilon) \) is being substituted into (1).

Our plane is to concentrate on asymptotic solution with a separatrix as leading term:

\[ U_0(t) = \tanh(t + \tilde{t}). \]  

(3)

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A similar problem was studied by A. Poincare [1]. He considered a special case of the three body problem. Later V.K. Mel’nikov established that the separatrices split under a perturbation and he calculated a gap between the separatrices of perturbed equation [2]. N.N.Filonenko, R.Z.Sagdeev and G.M. Zaslavskii [3] obtained a separatrix map for canonical variables in Hamiltonian systems.

The problem for the perturbation of a separatrix is very important for a capture into a resonance. Therefore the problem of dynamics near the separatrix was studied in the point of view of the capture. A.I. Neishtadt [4] calculated a measure of trajectories which cross the separatrix and are captured into a resonance. A.V.Timofeev studied the dynamics near the separatrix close to the saddle [5].

A change of the angle variable when the solution crosses the separatrix was obtained by J.R.Cary and R.T.Scodjie [6]. D.C.Diminie and R.Haberman studied a separatrix crossing near a saddle-center bifurcation and a pitchfork bifurcation [7]. Full asymptotic expansions for the problem of the separatrix crossing near the saddle-center bifurcation was obtained in [8, 9].

In this work following new results are obtained. It is constructed the map for the dynamics near the separatrix for any power of the perturbation parameter. It is shown that the phase shift over a circle near the separatrix circle is defined by a term of the order \(\varepsilon^2\) on the previous circle. The constructed map shows that there exists a manifold of \(\varepsilon \in (0, \varepsilon_0)\) for \(\forall \varepsilon_0 > 0\), such that the asymptotic solution has more than \(N\) circles near the separatrix for \(\forall N \in \mathbb{N}\). A Cantor set gives an example of such manifolds.

This work has following structure. In section 1 the main problem and results are presented in formal form. Section 2 contains the derivation of the map for the dynamics near the separatrix. In section 3 the consequences of the separatrix dynamics are presented.

1 Main problem and results

1.1 A trial solution

Let us consider an asymptotic solution in the form (2). Main problem is to construct a bounded asymptotic solution of such type as \(t \in (0, -N \ln(\varepsilon))\) for any \(N \in \mathbb{N}\), and \(\varepsilon \in (0, \varepsilon_0)\).

A simplest result is following.

**Theorem 1** There exists two parametric asymptotic solution of (1) in form
\((2)\) and \((3)\) when
\[
\frac{1}{2} \ln(\varepsilon) \ll t \ll -\frac{1}{2} \ln(\varepsilon).
\]

Higher-order terms of \((2)\) are
\[
U_n(t) = A_n e^{2t} + B_n e^{-2t} + W_n^-(t),
\]
as \(t \to -\infty\), where
\[
W_n^-(t) = O(e^{-2nt}), \quad t \to -\infty,
\]
and \(W_n^-(t)\) has an asymptotic expansion into series of powers of \(e^t\) as \(t \to \infty\), such that it does not contains the terms \(C_1 e^{2t}\) and \(C_2 e^{-2t}\) where \(C_1\) and \(C_2\) are some constants. This means that all parameters for \(U_n(t)\) are \(A_n^-\) and \(B_n^-\). Therefore the parameters of the asymptotic solution are:
\[
A^- = \sum_{n=1}^{\infty} \varepsilon^n A_n^-,
\]
\[
B^- = \sum_{n=1}^{\infty} \varepsilon^n B_n^-.
\]

Parameter \(A^-\) can be excluded from the solution by time shift in main term of the asymptotic solution \((3)\) and \(\Phi_0\) in the perturbation term of \((1)\). Therefore we take a case \(A_n^- = 0\). Parameter \(B^-\) is a distance between the asymptotic solution and the separatrix of unperturbed equation \((1)\).

1.2 Numeric simulations and challenge for the analytic studies

There are two scenarios for a prolongation of the trial solution. When \(t = O(-\ln(\varepsilon)/2)\) the trajectory of the trial solution closes to the saddle point \((1,0)\). This trajectory is able to turn to the lower separatrix, which goes to another saddle point \((-1,0)\). The different way for the trajectory lies near the separatrix which goes from \((1,0)\) to \((+\infty, +\infty)\). The similar changes are possible near the saddle \((-1,0)\).
Figure 2: The left figure shows the dependence of the oscillating time on $\varepsilon$ for the solution of the Cauchy problem $u(0) = 0, u'(0) = 1$ for (1) where $\omega = 1, \Phi_0 = \pi$. The solutions was studied on the interval $t \in (0, 100)$ for the values of the perturbation parameter $\varepsilon \in (0.001, 0.1)$. The right figure shows the scaled structure of the peak near $\varepsilon = 0.08$.

These cases are shown on the figure

Let us concentrate to the trajectories which oscillates between the saddles $(-1, 0)$ and $(1, 0)$. For this we should to find manifolds for the parameters of a solution and the perturbation $\varepsilon$. These manifolds have a complicated structure. For example one can see the dependency of the life-time for oscillating asymptotic solution on the perturbation parameter $\varepsilon$. This shown on the right figure 2. On left picture 2 one can see the thin structure of the peak near $\varepsilon = 0.08$.

This numeric simulations take a challenge for analytic studies. Our goals are to calculate the asymptotic solutions, to find a dependency for the trajectory on the parameters and to give the formulas for the manifolds of the parameters for the solutions with an oscillating behaviour.

1.3 Results

The asymptotic solution which is defined in Theorem 1 will be used as a trial solution and we will prolong it to the large time. The asymptotic behaviour of this solution has the same form:

$$U_n(t) = A_n^+ e^{-2t} + B_n^+ e^{2t} + W_n^+(t),$$

as $t \to \infty$, where

$$W_n^+(t) = O(e^{-2nt}), \quad t \to +\infty,$$
We will show if $B_1^+ < 0$, then the solution (2), (3) is bounded when
\[ \frac{1}{2} \ln(\varepsilon) \ll t \ll -\ln(\varepsilon) \]
and when $1 \ll t \ll -\ln(\varepsilon)$ the solution has a following asymptotic expansion:
\[ u(t, \varepsilon) = -\tanh(\theta) + \sum_{n=1}^{\infty} \varepsilon^n u_n(\theta), \]
\[ \theta = t + \frac{1}{2} \ln(\varepsilon) + \frac{1}{2} \ln \left(-\frac{1}{16} B_1^+\right), \]
as $\theta \to \mp \infty$, where
\[ u_n(\theta) = a_n^\pm e^{\mp 2\theta} + b_n^\pm e^{\mp 2\theta} + w_n^\pm(\theta), \]
where $w_n^\pm(\theta)$ has the same properties as $W_n^\pm(t)$ in Theorem 1.

The following theorem gives conditions for the prolongation of previous theorem on $N$-circles near the separatrix of the unperturbed equation.

**Theorem 2** The asymptotic solution of (1) from Theorem 1 may be extended on the interval $-k \ln(\varepsilon) \ll t \ll -(k + \frac{1}{2}) \ln(\varepsilon)$, $k = 0, 1, \ldots, [\frac{N}{2}]$ and has a form
\[ u(t, \varepsilon) = (-1)^{k-1} \tanh(t_k) + \sum_{n=0}^{\infty} \varepsilon^n u_n^k(t_k), \]
where $u_n^k(t_k)$ has the following asymptotic behaviour
\[ u_n^k(t_k) \sim \sum_{n=-\infty}^{n} e^{2\omega t_k} \left( \sum_{l=0}^{2n} \sum_{m=0}^{n+1} \left[ \hat{u}_n^{k,\pm}(\omega t_k) \cos(m\omega t_k) + \hat{v}_n^{k,\pm}(\omega t_k) \sin(m\omega t_k) \right] \right), \]
as $t_k \to \pm \infty$. The condition of extendability is $(-1)^m [\sigma_1(m) + \Delta \sigma_1(m)] < 0$ for $m = 1, \ldots, N, N \in \mathbb{N}$. Here the parameter $\Delta \sigma_1(k)$ is following
\[ \Delta \sigma_1(k) = \frac{\pi}{16 \cosh(\pi \omega/2)} \cos(\psi_k), \]
and $t_k, \psi(k)$ and $\sigma_1(k)$ are defined by the recurrent sequence:
\[ t_1 = t, \quad t_{k+1} = t_k + \frac{\omega}{2} \ln(\varepsilon) + \psi(k), \]
\[ \sigma_1(1) = B_k^- = 0, \quad \psi(1) = \phi, \quad \chi_k(1) = A_k^- = 0, \]
\[ \sigma_1(n+1) = -32 \frac{\sigma_{k+1}(n) + \Delta \sigma_{k+1}(n)}{\sigma_1(n)}, \]
\[ \psi(n+1) = -\frac{\omega}{2} \left( \ln \left( \frac{1}{16} (\sigma_1(n) + \Delta \sigma_1(n)) \right) - \ln(2) \right) + \psi(n), \]
\[ \chi_1(n) = -2, \quad \chi_{k+1}(n) = -\frac{1}{32} (\sigma_1(n) + \Delta \sigma_1(n)) (\chi_k(n) + \Delta \chi_{k-1}(n)). \]
Here

\[ \Delta \sigma_k(n) = (u_n^k)_{1,0,0}^+ - (u_n^k)_{1,0,0}^-; \]
\[ \Delta \chi_k(n) = (u_n^k)_{-1,0,0}^+ - (u_n^k)_{-1,0,0}^-; \]

**Corollary 1** The main term of the asymptotic expansion for \( u(t, \varepsilon) \) depends on \( B_j^-, k = 1, \ldots, j \) as \( t_k = O(1) \). This means that the asymptotic expansion is unstable with respect to a small correction of the parameters.

**Corollary 2** There exist a set of the parameters \( \{A_k^-\}_{k=1}^\infty, \{B_k^-\}_{k=1}^\infty \) such that the solution has an oscillating behaviour near the separatrices as \( \delta = \omega \ln(\varepsilon) \) belongs by a Cantor set.

## 2 Separatrix dynamics

In this section theorem 1 is proved in three steps. First step is a construction of the asymptotic expansion which is valid near the saddle points \( u = \pm 1 \). On the second step we obtain an asymptotic expansion which is valid close to the separatrices \( u = \pm \tanh(t) \). On last step we match these asymptotic expansions and construct the uniform asymptotic solution, which is valid over all domains close to the separatrices and the saddles.

### 2.1 Separatrix asymptotic expansion

Let us to construct an asymptotic solution in the form:

\[ u(t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U_n(t). \tag{4} \]

The leading term of the expansion is a separatrix solution

\[ U_0 = \tanh(t) \]

of the unperturbed equation:

\[ U_0'' + 2U_0 - 2U_0^3 = 0. \]

We will study the asymptotic expansion where the main term is the separatrix solution of

\[ U_n'' + 2U_n - 6U_0^2U_n = f_n, \tag{5} \]

where

\[ f_1 = \cos(\omega t + \phi_0), \]
and when \( n > 1 \) \( f_n \) is a polynomial of third order with respect to \( u_j, u_k \) and \( u_l \) for \( j + k + l = n \).

There are two linear independent solutions of the linearized equation:

\[ v'' + 2v - 6U_0^2v = 0. \]

There solutions are:

\[ v_1 = \frac{1}{\cosh^2(t)}, \quad v_2 = \frac{\sinh(4t)}{32\cosh^2(t)} + \frac{\sinh(2t)}{4\cosh^2(t)} + \frac{3t}{8\cosh^2(t)}. \]

Wronskian of the functions \( v_1 \) and \( v_2 \) is equal to unit.

A general solution of (5) has a following form:

\[ U_n(t) = v_1(t) \int_{t_0}^t dt f_n(t)v_2(\tilde{t}) - v_2(t) \int_{t_0}^\tilde{t} df_n(t)v_1(\tilde{t}) + A_1 v_1 + B_1 v_2. \tag{6} \]

Here \( t_0 \) is a constant and \( A_1, B_1 \) are parameters of the solution.

**Lemma 1** Let

\[ f_n \sim \sum_{k=-\infty}^{n-1} e^{2kt} \left( \sum_{l=0}^{2(n-1)} t^l \left[ \sum_{m=0}^{n} \left( F_{k,l,m}^\pm \cos(m\omega t) + H_{k,l,m}^\pm \sin(m\omega t) \right) \right] \right), \quad t \to \pm \infty, \]

then

\[ U_n(t) \sim \sum_{k=-\infty}^{n} e^{2kt} \left( \sum_{l=0}^{2n} t^l \left[ \sum_{m=0}^{n+1} \left( \tilde{U}_{k,l,m}^\pm \cos(m\omega t) + \tilde{V}_{k,l,m}^\pm \sin(m\omega t) \right) \right] \right), \quad t \to \pm \infty. \]

To prove this lemma one should use asymptotic behaviours for \( v_{1,2} \), substitute the formula for the \( f_n \) into (6) and integrate.

Denote \( A_n^\pm = \tilde{U}_{1,0,0}^\pm / 4 \) and \( B_n^\pm = 16\tilde{U}_{1,0,0}^\pm \). These parameters define the solution for a large time. The changes of these parameters are:

\[ \Delta A_n = A_n^+ - A_n^-, \quad \Delta B_n = B_n^+ - B_n^- . \]

The value of \( \Delta B_1 \) is defined by Melnikov’s integral:

\[ \Delta B_1 = B_1^+ - B_1^- = \int_{-\infty}^{\infty} \frac{\cos(\omega t + \Phi_0)}{\cosh^2(t)} \, dt - \int_{-\infty}^{\infty} \frac{\sin(\omega t)\sin(\Phi_0)}{\cosh(\pi\omega / 2)} \cdot \cos(\Phi_0) \cdot \frac{\pi}{\cosh(\pi\omega / 2)} . \]

The following formula gives \( \Delta A_1 \):

\[ \Delta A_1 = \lim_{s \to +\infty} \left[ -\sin(\Phi_0) \left( \int_{-s}^{s} v_2(t) \sin(\omega t) \, dt - \frac{2\sin(\omega s) - \omega \cos(\omega s) e^{2s}}{8(\omega^2 + 4)} \right) \right] . \]
2.1.1 Validity of the separatrix asymptotic expansion

The separatrix asymptotic expansion (2) is valid until:

$$\frac{\varepsilon U_{n+1}}{U_n} \ll 1.$$  

It yields the bounds for the independent variable $t$:

$$\varepsilon \exp(2t) \ll 1, \quad |t| \ll -\frac{1}{2} \ln(\varepsilon)$$  

and the parameters of the solution $A_n^+$ and $B_n^+$:

$$A_n^+ \ll \varepsilon, \quad B_n^+ \ll \varepsilon \quad \forall n \in \mathbb{N}.$$  

**Lemma 2** There exists an asymptotic solution for equation (1) in the form (2), when $|t| \ll -\frac{1}{2} \ln(\varepsilon)$.

2.2 Saddle asymptotic expansion

The asymptotic expansion has a form:

$$u(t, \varepsilon) = \pm 1 + \sum_{n=1}^{\infty} \varepsilon^{n/2} u_n^+(\tau) \quad \tau = t + \tau_0 \quad (7)$$

near the saddle points $u = \pm 1$.

We use the sigh ‘+’ for the expansion near $u = 1$ and the sign ‘-’ for the expansion near $u = -1$.

The correction terms are defined by the following equation:

$$u_n^+'' - 4u_n^+ = f_n^\pm,$$

where $f_1^\pm \equiv 0$, $f_2^\pm = \cos(\omega t - \omega \tau_0 + \phi_0) \pm 6(u_1^\pm)^2$ and $f_n^\pm$ is a polynomial of order 3, which is defined by the correction terms with the indexes $j, k, l$ such that $j + k + l = n$ when $n \geq 3$. In a general case $f_n^\pm$ is a finite sum of powers of $e^\tau$, sine, cosine and independent variable $\tau$. A general formula for $n$-th correction term has a form:

$$u_n^\pm = \alpha_n^\pm e^{-2\tau} + \beta_n^\pm e^{2\tau} + w_n^\pm(\tau). \quad (8)$$

Here $w_n^\pm(\tau)$ has not the terms $C_1 e^{2\tau}$ and $C_2 e^{-2\tau}$ for $\forall C_1, C_2 = const$ as $\tau \to \pm \infty$.  

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First and second corrections are:

\[ u_1^\pm = \alpha_1^\pm \exp(-2\tau) + \beta_1^\pm \exp(2\tau); \]

\[ u_2^\pm = \alpha_2^\pm \exp(-2\tau) + \beta_2^\pm \exp(2\tau) + \frac{(\alpha_1^\pm)^2}{12} \exp(-4\tau) + \frac{(\beta_1^\pm)^2}{12} \exp(4\tau) - \frac{1}{2} \alpha_1^\pm \beta_1^\pm - \frac{1}{4 + \omega^2} \cos(\omega \tau + \phi_0 - \omega \tau_0). \]

The \( n \)-th correction term is estimated for large \( \tau \):

\[ u_n^\pm = O(\exp(\pm 2n|\tau|)), \quad n > 1, \quad \tau \to \pm \infty. \]

The domain of validity for the saddle asymptotic expansions is defined by the inequality:

\[ \varepsilon^{1/2} u_{n+1}^\pm \ll 1, \quad \tau \to \pm \infty. \]

It yields:

\[ |\tau| \ll -\frac{1}{4} \ln(\varepsilon). \]

**Lemma 3** There exists an asymptotic solution in form (7) as \( \varepsilon \to 0 \), where \( u_n(\tau) \) has form (8) and \( |\tau| \ll -\frac{1}{4} \ln(\varepsilon) \).

**2.2.1 Matching of the asymptotic expansions**

The matching of asymptotic expansions \( U(t, \varepsilon) \) and \( u(\tau, \varepsilon) \) yields:

\[ \alpha_{2n} = 0, \quad \beta_{2n} = 0, \]

One obtains recurrent formulas for the correction terms with odd indexes:

\[ -2 \exp(-2t) = \varepsilon^{1/2} \alpha_1^+ \exp(-2\tau), \quad \alpha_1^+ = -2, \]

\[ \tau = t + \tau_0, \quad \tau_0 = \frac{1}{4} \ln(\varepsilon). \]

Then:

\[ \frac{1}{16} B_1^+ \exp(2t) = \beta_1^+ \varepsilon^{-1/2} \exp(2\tau), \quad \beta_1^+ = \frac{1}{16} B_1^+. \]

For higher-order terms as \( t \to \infty \) one obtains:

\[ 4A_n^+ \exp(-2t) = \varepsilon^{1/2} \alpha_{2n+1}^+ \exp(-2\tau), \]

\[ \alpha_{2n+1}^+ = 4A_n^+; \]

\[ \frac{1}{16} B_n^+ \exp(2t) = \beta_{2n-1}^+ \varepsilon^{-1/2} \exp(2\tau), \]

\[ \beta_{2n-1}^+ = \frac{1}{16} B_n^+, \quad n \in \mathbb{N}. \]
The sign of $\beta^+_1$ depends on the parameter $\phi_0$:

$$
\beta^+_1 = \frac{1}{16} B_1^+ + \frac{1}{16} \cos(\phi_0) \frac{1}{\cosh(\pi \omega/2)}
$$

If $\beta^+_1 > 0$ or the same $\cos(\phi_0) > -16 B_1^- \cosh(\pi \omega/2)$ then the asymptotic solution of (11) goes to infinity, otherwise when $\cos(\phi_0) < -16 B_1^- \cosh(\pi \omega/2)$ the solution of (11) is close to the separatrix which goes to the left saddle.

**Theorem 3** There exists an asymptotic solution of (11), such that this solution has a form (2) when $\frac{1}{2} \ln(\varepsilon) \ll t \ll -\frac{1}{2} \ln(\varepsilon)$ and has form (7) when $-\frac{1}{4} \ln(\varepsilon) \ll t \ll -\frac{1}{2} \ln(\varepsilon)$.

### 2.3 Lower separatrix branch

Let us consider the case when $\beta^+_1 < 0$, then the main term of an asymptotic solution as $\tau \to \infty$ is the separatrix which goes from one saddle $(1, 0)$ on the plane $(u, u')$ to another saddle $(-1, 0)$. This separatrix is:

$$
u_0 = -\tanh(t).
$$

The asymptotic expansion, which is close to the lower separatrix has a similar form as the asymptotic expansion close to the upper separatrix:

$$u(\theta, \varepsilon) = u_0(\theta) + \sum_{n=0}^{\infty} \varepsilon^n u_n(\theta). \quad (9)
$$

It is easy to see that the asymptotic solutions (9) has the same form as (4), where one should use the variable $\theta$ instead of $t$.

Equation for $n$-th correction term has a following form:

$$u''_n + 3u_n - 6u_0^2 u_n = f_n. \quad (10)
$$

Here $f_n$ is a polynomial of 3-d order with respect to terms $u_j, u_k, u_l$, such that $j + k + l = n$, and

$$f_n \sim \sum_{k=-\infty}^{n-1} e^{2k\theta} \left( \sum_{l=0}^{2(n-1)} \theta^l \left[ \sum_{m=0}^{n} \left( F_{k,l,m}^\pm \cos(m\omega \theta) + H_{k,l,m}^\pm \sin(m\omega \theta) \right) \right] \right), \quad \theta \to \pm \infty.
$$

The general formula for the solution of (10) yields:

$$u_n(\theta) \sim \sum_{k=-\infty}^{n} e^{2k\theta} \left( \sum_{l=0}^{2n} \theta^l \left[ \sum_{m=0}^{n+1} \left( \tilde{u}_{k,l,m}^\pm \cos(m\omega \theta) + \tilde{v}_{k,l,m}^\pm \sin(m\omega \theta) \right) \right] \right), \quad \theta \to \pm \infty.
$$

Denote $a_n^\pm = \tilde{u}_{-1,0,0}^\pm / 4$ and $b_n^\pm = 16 \tilde{u}_{1,0,0}^\pm$. These parameters define the parameters of the solution for the large time. The changes of these parameters are:

$$\Delta a_n = a^-_n - a^+_n, \quad \Delta b_n = b^-_n - b^+_n.$$
2.3.1 Validity of separatrix expansion

The lower separatrix expansion is valid until:

\[ \frac{\varepsilon u_{n+1}}{u_n} \ll 1. \]

It yields:

\[ \varepsilon \exp(2\theta) \ll 1, \quad |\theta| \ll -\frac{1}{2} \ln(\varepsilon). \]

Lemma 4 There exists an asymptotic solution of (1) in form (9) as

|\theta| \ll -\frac{1}{2} \ln(\varepsilon).

2.3.2 Matching of lower separatrix asymptotic expansion and asymptotic expansion close to right saddle

Let \( \beta_1^+ < 0 \). In this case one can match the asymptotic expansion in the right saddle point and the lower separatrix asymptotic expansion (9).

The matching yields:

\[ \varepsilon^{1/2} \beta_1^+ \exp(2\tau) = -2 \exp(2\theta) \quad \text{and} \quad \varepsilon^{1/2} \alpha_1^+ \exp(-2\tau) = \varepsilon 4a_1^+ \exp(-2\theta), \]

as \( \tau \to \infty \) and \( \theta \to -\infty \). Here \( \beta_1^+ < 0 \), as a result we obtain:

\[ 2\tau + \frac{1}{2} \ln(\varepsilon) + \ln(-\beta_1^+) = 2\theta + \ln 2. \]

Or the same

\[ \theta = \tau + \frac{1}{4} \ln(\varepsilon) + \frac{1}{2} \ln(-\beta_1^+) - \frac{1}{2} \ln 2. \]

A substitution of \( \theta \) and reductions give a following formula:

\[ \frac{1}{2} \ln(\varepsilon) + \ln(\alpha_1^+) - 2\tau = \ln(\varepsilon) + 2 \ln(2) + \ln(a_1^+) - 2\tau - \frac{1}{2} \ln(\varepsilon) - \ln(-\beta_1^+) - \ln(2), \]

or

\[ \alpha_1^+(-\beta_1^+) = 8a_1^+, \quad a_1^+ = \frac{1}{8} \alpha_1^+(-\beta_1^+). \]

It is easy to see that \( \alpha_1^+ = -2 \). A following computation gives formulas for \( \beta_3^+ \) and \( b_1^+ \):

\[ \varepsilon^{3/2} \beta_3^+ \exp(2\tau) = \frac{1}{16} \varepsilon b_1^+ \exp(2\theta). \]

As a result we obtain:

\[ \ln(\beta_3^+) + 2\tau = -4 \ln(2) + \ln(b_1^+) + 2\tau + \ln(-\beta_1^+) - \ln(2), \]
or

\[ \beta_3^+ = \frac{1}{32} b_1^+ (-\beta_1^+), \quad b_1^+ = 32 \beta_3^+ / \beta_1^+. \]

The same way leads us to formulas for the higher order terms:

\[ \varepsilon^{(2n+1)/2} a_{2n-1}^+ \exp(-2\tau) = 4\varepsilon^n a_n^+ \exp(-2\theta), \]

\[ \alpha_{2n-1}^+ (-\beta_1^+) = 8a_n^+, \quad a_n^+ = \frac{1}{8} \alpha_{2n-1}^+ (-\beta_1^+). \]

\[ \varepsilon^{(2n+1)/2} \beta_{2n+1}^+ \exp(2\tau) = \frac{1}{16} \varepsilon b_n^+ \exp(2\theta), \]

\[ \frac{\beta_{2n+1}^+}{(-\beta_1^+)} = \frac{1}{32} b_n^+, \quad b_n^+ = -32 \frac{\beta_{2n+1}^+}{\beta_1^+}. \]

It is convenient to write out the formulas which connect the parameters of the asymptotic expansions near the upper and the lower separatrices:

\[ a_1^+ = \frac{1}{16} (B_1^- + \Delta B_1), \]

\[ a_{n+1}^+ = -\frac{1}{32} (A_n^+ + \Delta A_n) (B_1^- + \Delta B_1), \]

\[ b_n^+ = -32 \frac{B_{n+1}^- + \Delta B_{n+1}}{B_1^- + \Delta B_1}, \quad n \in \mathbb{N}; \]

\[ \theta = t + \frac{1}{2} \ln(\varepsilon) + \frac{1}{2} \ln \left( -\frac{1}{16} (B_1^- + \Delta B_1) \right) - \frac{1}{2} \ln(2), \]

\[ \phi = -\frac{\omega}{2} \left[ \ln(\varepsilon) + \ln \left( -\frac{1}{16} (B_1^- + \Delta B_1) \right) - \ln(2) \right] + \Phi. \]

### 2.4 Neighborhood of left saddle point

#### 2.4.1 Matching of saddle asymptotic expansion and lower separatrix asymptotic expansion

The matching of the expansions near left saddle point yields:

\[ 2 \exp(-2\theta) = \varepsilon^{1/2} a_1^- \exp(-2\sigma), \]

\[ \frac{\varepsilon}{16} b_1^- \exp(2\theta) = \varepsilon^{1/2} \beta_1^- \exp(2\sigma), \]

\[ \varepsilon a_1^- \exp(-2\theta) = \varepsilon^{3/2} a_3^- \exp(-2\sigma). \]

As a result one obtain formulas for the parameters of the saddle asymptotic expansion:

\[ -2\theta = \frac{1}{2} \ln(\varepsilon) - 2\sigma, \]
\[
\begin{align*}
\alpha_1^- &= 2, \\
4a_1^- &= \alpha_3^-, \\
\beta_1^- &= \frac{1}{16}b_1^-.
\end{align*}
\]

A matching yields formulas for higher-order terms:

\[
\varepsilon^{(2n+1)/2}(\alpha_{2n+1}^- \exp(-2\sigma)+\beta_{2n+1}^- \exp(2\sigma)) = \varepsilon^n(4a_n^- \exp(-2\theta)+\frac{1}{16}b_n^- \exp(2\theta)).
\]

Therefore one obtains formulas for the coefficients of the asymptotic expansion:

\[
\alpha_{2n-1}^- = 4a_n^-, \quad 16\beta_{2n+1}^- = b_n^-.
\]

Here one can obtain an explicit formula for the changing of coefficient \(b_n\). For example,

\[
\Delta b_1 = b_1^- - b_1^+ = \frac{\pi}{\cosh(\pi \omega/2)} \cos \left( -\frac{\omega}{2}(\ln(\varepsilon) + \ln(-\beta_1^+) - \ln(2)) + \Phi \right).
\]

As a result we obtain the following theorem.

**Theorem 4** If \(\beta_1^+ < 0\) then there exists an asymptotic solution of \((1)\) when \(\frac{1}{2} \ln(\varepsilon) \ll t \ll 1\) with sub sign "$+$" when \(-\frac{1}{2} \ln(\varepsilon) \ll t \ll -\frac{1}{2} \ln(\varepsilon)\) and (7) with sign "$-$" when \(-\frac{3}{4} \ln(\varepsilon) \ll t \ll -\ln(\varepsilon)\).

**2.5 Next circle near separatrix**

Let us consider an oscillation near the separatrix. Denote parameters of asymptotic solution by sequence numbers:

\[
\begin{align*}
t_0 &= t, \quad \theta_0 = \theta, \quad \tau_0 = \tau, \\
\sigma_0 &= \sigma, \quad B_n^+(0) = B_n^+, \quad b_n^+(0) = b_n^+, \\
\Phi(0) &= \Phi, \quad \phi(0) = \phi.
\end{align*}
\]

The parameters of the asymptotic expansion for following circles may being calculated by formulas:

if \(B_1^-(m) + \Delta B_1(m) < 0\) and \(n \geq 2\):

\[
\begin{align*}
\theta(m) &= t(m) + \frac{1}{2} \ln(\varepsilon) + \frac{1}{2} \ln \left( \frac{1}{16}(B_1^-(m) + \Delta B_1(m)) \right) - \frac{1}{2} \ln(2), \\
b_n^+(m) &= -32 \frac{B_{n+1}^-(m) + \Delta B_{n+1}(m)}{B_1^-(m) + \Delta B_1(m)}; \quad (12)
\end{align*}
\]

\[
\begin{align*}
a_1^+ &= \frac{1}{32} \left( B_1^-(m) + \Delta B_1(m) \right), \\
a_n^+(m) &= -\frac{1}{32} \left[ A_{n-1}(m) + \Delta A_{n-1}(m) \right] \left( B_1^-(m) + \Delta B_1(m) \right);
\end{align*}
\]
if \( b_{1}^{+}(m) + \Delta b_{1}(m) > 0: \)

\[
t(m + 1) = \theta(m) + \frac{1}{2} \ln(\varepsilon) + \ln \left( \frac{1}{16} (b_{1}^{+}(m) + \Delta b_{1}(m)) \right) - \frac{1}{2} \ln(2),
\]

\[
B_{n}^{-}(m + 1) = -32 \frac{b_{n+1}^{+}(m) + \Delta b_{n+1}(m)}{b_{1}^{+}(m) + \Delta b_{1}(m)}, \tag{13}
\]

\[
A_{n}^{-}(m) = -\frac{1}{32} \left[a_{n-1}^{+}(m) + \Delta a_{n-1}(m)\right] \left(b_{1}^{+}(m) + \Delta b_{1}(m)\right);
\]

In particular,

\[
B_{1}^{+}(m) = B_{1}^{-}(m) + \frac{\pi}{16} \frac{\cos(\Phi(m))}{\cosh(\frac{\pi \omega}{2})},
\]

\[
b_{1}^{-}(m) = b_{1}^{+}(m) + \frac{\pi}{16} \frac{\cos(\phi(m))}{\cosh(\frac{\pi \omega}{2})}. \tag{14}
\]

For given parameter \( \Phi(0) \) we obtain following discreet dynamics over a near-separatrix circle:

\[
\Phi(m + 1) = \omega(t(m + 1) - \theta(m + 1)) + \phi(m)
\]

\[
= -\frac{\omega}{2} \left( \ln(\varepsilon) + \ln \left( \frac{1}{16} (b_{1}^{+}(m) + \Delta b_{1}(m)) - \ln(2) \right) \right) + \phi(m), \tag{15}
\]

\[
\phi(m) = \omega(\theta(m) - t(m)) + \Phi(m) = -\omega(\ln(\varepsilon) + \frac{1}{2} \ln(\frac{1}{16} (B_{1}^{-}(m) + \Delta B_{1}(m)) - \frac{1}{2} \ln(2)) + \Phi(m).
\]

It is easy to see that \( \Phi(m) \) and \( \phi(m) \) have the same order:

\[
\phi(m) = -\omega \left( \frac{1}{2} + m \right) \ln(\varepsilon) + O(1),
\]

\[
\Phi(m) = -m \omega \ln(\varepsilon) + O(1).
\]

The shown formulas give a dependency on the sequence \( \{B_{n}^{-}\}_{n=0}^{\infty} \) for dynamics of the solution for times \( t = O(N \ln(\varepsilon)) \) for \( \forall N \in \mathbb{N} \).

**Lemma 5** For \( \forall \varepsilon \in (0, \varepsilon_{0}) \), where \( \varepsilon = \text{const} > 0 \) there exists a sequence \( \{B_{n}^{-}\}_{n=0}^{\infty} \) such that \( B_{1}^{-}(k) < 0 \) and \( b_{1}^{+} > 0 \) \( \forall k \).

The sequence of the lemma and inequalities (12) and (13) give

**Theorem 5** For \( \forall \varepsilon \in (0, \varepsilon_{0}) \), where \( \varepsilon = \text{const} > 0 \) there exists the sequence \( \{B_{n}^{-}\}_{n=0}^{\infty} \) such that the asympotic solution oscillates near the separatrices as \( t = O(N \ln(\varepsilon)) \) for all \( \forall N \in \mathbb{N} \).
3 Discreet dynamical system

Let us consider the discreet dynamical systems which defines by separatrix
map (12), (13). It is convenient to change this system by the following way.
Let us define

\[ \sigma_k(2n-1) \equiv B_k^-(n-1), \quad \sigma_k(2n) \equiv b_k^+(n-1), \quad n \in \mathbb{N}; \]
\[ \Delta \sigma_k(2n-1) \equiv \Delta B_k(n), \quad \Delta \sigma_k(2n) \equiv \Delta b_k(n-1); \]
\[ \chi_k(2n-1) \equiv A_k^-(n-1), \quad \chi_k(2n) \equiv a_k^+(n-1), \quad k \geq 2, \quad n \in \mathbb{N}; \]
\[ \Delta \chi_k(2n-1) \equiv \Delta A_k(n-1), \quad \Delta \chi_k(2n) \equiv \Delta a_k(n-1); \]
\[ \psi(2n-1) \equiv \Phi(n) + n\omega \ln(\varepsilon), \quad \psi(2n) \equiv \phi(n) + \left( n + \frac{1}{2} \right) \omega \ln(\varepsilon). \]

This formulas is prolonged on the next step if \((\sigma_1(n) + \Delta \sigma_1(m))(−1)^n > 0). The formulas for a discreet dynamical system are:

\[ \sigma_k(1) = B_k^-, \quad \chi_k(1) = A_k^-, \quad \psi(1) = \Phi, \]
if \((\sigma_1(m) + \Delta \sigma_1(m))(−1)^m > 0), \quad \forall m \leq n, \text{ then} \]
\[ \sigma_k(n+1) = -32 \frac{\sigma_{k+1}(n) + \Delta \sigma_{k+1}(n)}{\sigma_1(n) + \Delta \sigma_1(n)}, \]
\[ \psi(n+1) = -\frac{\omega}{2} \left( \ln \left( \frac{1}{16}(\sigma_1(n) + \Delta \sigma_1(n)) \right) - \ln(2) \right) + \psi(n), \]
\[ \chi_1(2n) = 2, \quad \chi_1(2n-1) = -2, \]
\[ \chi_k(n+1) = -\frac{1}{32} \left( \chi_{k-1}(n) + \Delta \chi_{k-1}(n) \right) \left( \sigma_1(n) + \Delta \sigma_1(n) \right). \]

Those formulas give a nonlinear discreet dynamical system. The non-
linearity defines by the terms \(\Delta \sigma_k(n), \Delta \chi_k(n)\) or the same \(\Delta B_k(n), \Delta b_k(n), \Delta A_k(n), \Delta a_k(n)\).

3.1 Generalized Bernoulli shift

The separatrix map defines a Bernoulli shift for parameters of this map. Formulas (12) and (13) show that the \(\sigma_k(n+1)\) depends on \(\sigma_{k+2}(n)\). It means that \((n)\)-th correction on the \((m+1)\)-th oscillation depends on \((n+2)\)-th correction on the \(m\)-th oscillation near the separatrix. This is typical property of the Bernoulli shift. This property shows the loss of the accuracy for the approximation of the motion and the instability the solution with respect to initial data.
3.2 Cantor manifold

When we study the asymptotic solution we say that there exists \( \varepsilon_0 > 0 \) such that for some \( \varepsilon \in (0, \varepsilon_0) \) there exists studying asymptotic solution. In this subsection we turn the study by the other side. We concentrate on the problem for the structure of the manifold \( \varepsilon \in (0, \varepsilon_0) \) which gives the oscillations near the separatrices.

The following condition

\[
\cos(\psi(n) - \omega \frac{n - 1}{2} \ln(\varepsilon)) < -16\sigma_1(n) \cosh \left( \frac{\pi \omega}{2} \right)
\]  

(16)

defines the possibility to prolong of the discreet dynamical system on \( (n+1) \)-th step. Define \( \delta = -\omega \ln(\varepsilon) \), then \( \delta \in (\delta_0, \infty) \) where \( \delta_0 = -\omega \ln(\varepsilon_0) \) and \( 0 < \varepsilon_0 < 1 \). Let \( \sigma_1(n) \) is such that

\[
\arccos(-16\sigma_1(n) \cosh(\frac{\pi \omega}{2})) - \psi(n) = \pm \frac{\pi}{3} + 2\pi k.
\]

Then (16) defines the parameter \( \delta \)

\[
\forall k > \left[ \frac{\omega(n - 2)}{4\pi} \ln(\varepsilon_0) \right] + 1.
\]

Let

\[
\frac{n - 1}{2} \delta \in (-\pi + 2\pi k, \pi + 2\pi k) \in (\delta_0, \infty),
\]

then when

\[
\frac{n - 1}{2} \delta < -\frac{\pi}{3} + 2\pi k, \text{ and } \frac{n - 1}{2} \delta > \frac{\pi}{3} + 2\pi k
\]

the discrete dynamical system is prolonged by the next step.

Let us consider sequences \( \alpha \in \mathbb{N} \) such that

\[
-16\sigma(l) \cosh(\frac{\pi \omega}{2}) > 1, \text{ as } l > n_\alpha, \text{ } l < 3n_\alpha - 2
\]

and \( n_{\alpha+1} = 3n_\alpha - 2 \),

\[
\arccos(-16\sigma_1(n_{\alpha+1}) \cosh(\frac{\pi \omega}{2})) - \psi(n_{\alpha+1}) = \pm \frac{\pi}{3} + 2\pi k,
\]

\[
\forall k > \left[ \frac{\omega(n_{\alpha+1} - 1)}{4\pi} \ln(\varepsilon_0) \right] + 1.
\]

Let \( \varepsilon \in (0, \varepsilon_0) \), then the set of \( \delta \), for which the discrete dynamical system is prolonged, is the Cantor set on any \( (-\pi + 2\pi, k\pi + 2\pi k) \).

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