Towards a semiclassical justification of the ‘effective random matrix theory’ for transport through ballistic chaotic quantum dots

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The scattering matrix $S$ of a ballistic chaotic cavity is the direct sum of a ‘classical’ and a ‘quantum’ part, which describe the scattering of channels with typical dwell time smaller and larger than the Ehrenfest time, respectively. According to the ‘effective random matrix theory’ of Silvestrov, Goorden, and Beenakker [Phys. Rev. Lett. 90, 116801 (2003)], statistical averages involving the quantum-mechanical scattering matrix are given by random matrix theory. While this effective random matrix theory is known not to be applicable for quantum interference corrections to transport, which appear to subleading order in the number of scattering channels $N$, it is believed to correctly describe quantum transport to leading order in $N$. We here partially verify this belief, by comparing the predictions of the effective random matrix theory for the ensemble averages of polynomial functions of $S$ and $S^\dagger$ of degree 2, 4, and 6 to a semiclassical calculation.

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I. INTRODUCTION

For electrons moving in a ballistic conductor with chaotic classical dynamics there is a minimal time after which the wave nature of electrons becomes apparent: the Ehrenfest time $\tau_E$. The Ehrenfest time $\tau_E$ is the time it takes for classical trajectories initially a ‘quantum distance’ apart to separate and reach a ‘classical distance’. Here, the quantum distance is the Fermi wavelength $\hbar/p_F$, where $p_F$ is the Fermi momentum, whereas the classical distance can be taken to be the system size $L$ or the width $W$ of contacts connecting the sample to source and drain reservoirs. For chaotic classical dynamics with Lyapunov exponent $\lambda$, $\tau_E$ is then determined by the condition $W = (\hbar/p_F) \exp(\lambda \tau_E)$, so that

$$\tau_E = \frac{1}{\lambda} \ln \frac{p_F W}{\hbar}.$$  \hspace{1cm} (1)

If the Ehrenfest time $\tau_E$ is small in comparison to the mean dwell time $\tau_D$ in the conductor and other time scales relevant for quantum transport (such as the dephasing time or the period of an applied AC bias) — which is true for most experimentally realized ballistic quantum dots —, the time threshold it poses for quantum processes is irrelevant, which explains why quantum signatures do not distinguish ballistic conductors from their disordered counterparts. Indeed, quantum transport in ballistic quantum dots with $\tau_E \ll \tau_D$ and quantum transport in disordered quantum dots are both described by random matrix theory.

Recently, there has been considerable interest in the theoretical question what happens if the Ehrenfest time exceeds the mean dwell time $\tau_D$. In this regime significant differences between the manifestations of quantum mechanics in ballistic chaotic and disordered conductors can occur. It is important to distinguish the effect of the Ehrenfest time on quantum phenomena that involve the splitting of trajectories only and quantum interference phenomena, which involve the divergence of classical trajectories initially a quantum distance apart and their joining again. Examples of the former are shot noise\textsuperscript{5,6,7,8,9} and the excitation gap induced by the proximity to a superconductor\textsuperscript{10,11,12,13}. Examples of the latter are weak localization\textsuperscript{14,15,16} and the excitation gap induced by the proximity to a superconductor\textsuperscript{10,11,12,13}. The dot's classical phase space can be divided into a part containing classical trajectories with dwell time shorter than $\tau_E$ and a part with classical trajectories with dwell time longer than $\tau_E$. Correspondingly, the dot's $N \times N$ scattering matrix $S(\varepsilon)$ is written as the direct sum of a scattering matrix $S_{cl}(\varepsilon)$ for $N_c$ ‘classical channels’ and a scattering matrix $S_q(\varepsilon)$ for $N_q$ ‘quantum channels’, with $N = N_c + N_q$.

$$S(\varepsilon) = \begin{pmatrix} S_{cl}(\varepsilon) & 0 \\ 0 & S_q(\varepsilon) \end{pmatrix}.$$  \hspace{1cm} (2)

The $N_c$-dimensional ‘classical scattering matrix’ $S_{cl}(\varepsilon)$ represents fully deterministic scattering from the quantum dot, where all probability intensity is concentrated around one classical trajectory. The scattering phase shifts in $S_{cl}(\varepsilon)$ follow from the dwell times of the corresponding classical trajectories. On the other hand, $S_q(\varepsilon)$ represents quantum scattering, where the probability intensity is divided over a large number of scattering modes in all contacts. Writing

$$S_q(\varepsilon) = e^{i\tau_E \tilde{S}_q(\varepsilon)},$$  \hspace{1cm} (3)

in order to factor out a trivial energy dependence arising from the fact that all trajectories contribution to $S_q$ have...
a minimal dwell time $\tau_{E}^{[14,15]}$. Silvestrov, Goorden, and 
Beenakker proposed that the statistical distribution of 
$S_{q}(\epsilon)$ is that of random matrix theory \cite{12} provided $N_{q}$ is 
large.

Since the original proposal of Ref. \cite{12} it has been un-
derstood that this ‘effective random matrix theory’ does 
not provide a faithful description of all signatures of 
quadratic transport. For example, the effective random 
matrix theory predicts that the weak localization cor-
rection to the conductance of a ballistic quantum dot is 
Ehrenfest-time independent, whereas both microscopic 
semiclassical theory and numerical simulations find an 
exponential dependence on $\tau_{E}/\tau_{D}^{[14,15,16,19]}$. Weak local-
ization is a quantum interference effect, which arises as a 
correction to subleading order in the total channel num-
ber $N$. The effective random matrix theory has been 
successful in predicting and explaining Ehrenfest-time de-
pendences that appear to leading order in $N$, such as shot 
noise \cite{8,23,24}, the proximity-induced gap in a quantum dot 
coupled to a superconductor \cite{10,11,12,13,23}, the density of 
transmission eigenvalues \cite{24} and the probability distribution 
of lifetimes of quasibound states in chaotic quantum dots \cite{25}.

Whereas the effective random matrix theory has been 
compared to the results of accurate numerical simu-
lations in all four examples mentioned above, the micro-
scopic verifications of the effective random matrix theory 
are limited to a theoretical construction of the decomposi-
tion \cite{24} and to the comparison of the effective random 
matrix theory and microscopic calculations of the $\tau_{E}$ de-
pendence of the shot noise power of a chaotic quantum 
dot \cite{6,7,8,9} and the density of states of a quantum dot 
coupled to a superconductor (an ‘Andreev quantum dot’) for 
energies much larger than the proximity-induced energy gap \cite{24}. (For energies comparable to the energy gap, the 
theory of Ref. \cite{11} makes use of an ansatz that is similar 
in spirit to the ansatz of effective random matrix theory.) 
The comparison to a microscopic calculation of shot noise 
amounts to a test of the effective random matrix theory 
for traces of polynomial functions of the scattering matrix 
$S$ and its hermitian conjugate $S^{\dagger}$ of degree 4. The 
comparison to the density of states of an Andreev quantum 
dot at high energies addresses the statistics of polynomials of 
$S_{q}(\epsilon)S_{q}^{\dagger}(-\epsilon)$ of arbitrary degree, but not 
of $S_{q}$. The aim of this article is to provide the next step 
towards a microscopic verification of the effective rank 
matrix theory by comparing a semiclassical theory of the 
ensemble average of a trace of a degree-six polynomial 
function of $S$ and $S^{\dagger}$ with the predictions of the 
effective random matrix theory. Since the hypothesis of 
the effective random matrix was formulated after the mi-
croscopic calculations of the Ehrenfest-time dependence 
of shot noise and the density of states in an Andreev quantum dot, we believe that this calculation is the first nontrivial 
test of the effective random matrix theory.

In Sec. III below we review the predictions of the ef-
fective random matrix theory for the averages of traces of 
polynomials of $S$ and $S^{\dagger}$ of degree two, four, and six. 
We follow with a semiclassical calculation of these aver-
gages in Sec. III and conclude in Sec. IV. Details of the 
calculations can be found in the two appendices.

II. PREDICTIONS OF THE EFFECTIVE 
RANDOM MATRIX THEORY

In our calculation, we consider a ballistic quantum dot 
coupled to electron reservoirs through ballistic point con-
tacts. Hence, the scattering matrix $S$ acquires a block 
structure $S = S_{ij}$, where the indices $i$ and $j$ label the 
point contacts. The dimension of the block $S_{ij}$ is $N_{i} \times N_{j}$, 
where $N_{i}$ is the number of channels in the $i$th point con-
tact. Each block $S_{ij}$ can be decomposed into a ‘classical’ 
and a ‘quantum’ scattering matrix as in Eq. (2).

We are interested in averages of the form

\begin{equation}
Q_{2} = \frac{1}{N} \langle \text{tr} S_{ij}(\epsilon_{1})S_{ij}^{\dagger}(\epsilon_{2}) \rangle,
\end{equation}

\begin{equation}
Q_{4} = \frac{1}{N} \langle \text{tr} S_{ij}(\epsilon_{1})S_{kj}(\epsilon_{2})S_{kl}(\epsilon_{3})S_{lj}^{\dagger}(\epsilon_{4}) \rangle,
\end{equation}

\begin{equation}
Q_{6} = \frac{1}{N} \langle \text{tr} S_{ij}(\epsilon_{1})S_{kj}(\epsilon_{2})S_{kl}(\epsilon_{3})S_{lm}(\epsilon_{4})S_{mn}(\epsilon_{5})S_{np}(\epsilon_{6}) \rangle.
\end{equation}

The polynomials $Q_{2}$ and $Q_{4}$ describe, e.g., the 
conductance and shot noise power of a quantum dot with 
metal contacts \cite{26} whereas polynomials of higher degree 
are necessary for a theory of transport and equilibrium 
properties of quantum dots with superconducting 
contacts \cite{13}. According to the effective random matrix 
theory, these averages have the structure

\begin{equation}
Q_{n} = \left[ 1 - e^{-F_{n}(1,...,n)\tau_{E}/\tau_{D}} \right] Q_{n}^{cl} + e^{-F_{n}(1,...,n)\tau_{E}/\tau_{D}} Q_{n}^{RMT},
\end{equation}

where the function $F_{n}$ is defined as

\begin{equation}
F_{n}(1,...,n) = 1 + \frac{i\tau_{E}}{\hbar} \sum_{j=1}^{n} (-1)^{j} \epsilon_{j},
\end{equation}

and $Q_{n}^{cl}$ and $Q_{n}^{RMT}$ are obtained from Eq. (4) by replacing 
$S$ by $S_{q}$ and $\tilde{S}_{q}$ and $N$ by $N_{q}$ and $N_{q}$, respectively.
The classical averages $Q_{n}^{cl}$ read \cite{12}

\begin{equation}
Q_{n}^{cl} = \frac{N_{i}N_{j}}{N^{2}F_{2}(1,2)},
\end{equation}

\begin{equation}
Q_{4}^{cl} = \frac{N_{i}N_{j}\delta_{ik}\delta_{jl}}{N^{2}F_{4}(1,2,3,4)},
\end{equation}

\begin{equation}
Q_{6}^{cl} = \frac{N_{i}N_{j}\delta_{ik}\delta_{jl}\delta_{lm}\delta_{mn}}{N^{2}F_{6}(1,2,3,4,5,6)}.
\end{equation}

The random matrix average of the trace of a degree two 
polynomial of $S$ and $S^{\dagger}$ is equal to the classical average \cite{27}

\begin{equation}
Q_{2}^{RMT} = Q_{2}^{cl}.
\end{equation}

However, the higher-order random matrix averages are 
different. For $Q_{4}$ one has \cite{28}.
where the random matrix average \( Q_6 \) reads

\[
Q_{6}^{\text{RMT}} = \frac{N_i N_j N_k \delta_{i k}}{N^4 F_2(1, 2) F_2(3, 4)} + \frac{N_i N_j N_k \delta_{i j}}{N^4 F_2(3, 2) F_2(1, 4)} - \frac{N_i N_j N_k N_l F_2(1, 2, 3, 4)}{N^4 F_2(1, 2) F_2(3, 2) F_2(3, 4) F_2(1, 4)}
\]

where the dots \( \ldots \) refer to the simultaneous cyclic permutations \((1, 2, i, j) \rightarrow (3, 4, k, l) \rightarrow (5, 6, m, n)\).

Together, Eqs. (3), (4), (5), (6), and (10) specify the prediction of the effective random matrix theory for the averages of the traces of polynomials of the scattering matrix and its hermitian conjugate for polynomials up to degree 6.

III. SEMICLASSICAL CALCULATION

We now perform a semiclassical calculation of the averages \( Q_2, Q_4, \) and \( Q_6 \) defined in Eq. (1) above and show that they agree with the predictions of the effective random matrix theory as described in the previous section. Although semiclassical calculations of \( Q_2 \) and \( Q_4 \) exist, see Refs. 27, 30, 31, 32 and 44, respectively, we briefly review their derivation in order to establish the context for the calculation of \( Q_6 \). (We are not aware of a semiclassical calculation of the energy dependence of \( Q_4 \), however.)

In our calculation, we take the limit \( \hbar \to 0 \) while keeping the ratios \( \tau_E/\tau_D \) and \( N_i/N \), and the products \( \tau_D \delta_{i j}/\hbar \) fixed. The latter condition implies that the functions \( F_n \) defined in Eq. (3) remain constant in the limiting procedure. Note that the channel numbers \( N_i \) diverge in this limit. The divergence of the channel numbers does not affect \( Q_2, Q_4, \) or \( Q_6 \), however, because these depend on the ratios \( N_i/N \) only. Since \( \tau_E \propto \ln(1/\hbar) \), see Eq. (1), the condition that the ratio \( \tau_E/\tau_D \) is kept constant in the limiting procedure implies that the dwell time \( \tau_D \) diverges as well. The divergence of \( \tau_D \) removes any dependence on the non-universal short-time dynamics in the quantum dot.

Starting point of our calculation is an expression of the dot’s scattering matrix \( S \) as a sum over classical trajectories \( \alpha^{30,31,32} \)

\[
(S_{ij})_{mn} = \left( \frac{\pi \hbar}{2 W_i W_j} \right)^{1/2} \sum_{\alpha} A_{\alpha} e^{i S_{\alpha}/\hbar},
\]

where \( i \) and \( j \) label the exit and entrance leads, respectively, and \( m \) and \( n \) label the propagating modes in these leads. The widths of the entrance and exit contacts are \( W_i \) and \( W_j \), respectively. The trajectory \( \alpha \) connects the entrance contact \( j \) to the exit contact \( i \). The components \( p_{\perp} \) and \( p'_{\perp} \) of its momentum perpendicular to the lead axis upon entrance and exit, respectively, are compatible with that of the modes \( n \) and \( m \) in the corresponding leads,

\[
p_{\perp} = \pm \pi \hbar n/W_j, \quad n = 1, \ldots, N_j,
p'_{\perp} = \pm \pi \hbar m/W_i, \quad m = 1, \ldots, N_i.
\]

(Here and in the remainder of this article, primed variables refer to the exit contact.) Further, \( S_{\alpha} \) is the classical action of trajectory \( \alpha \) and \( A_{\alpha} \) is its stability amplitude. The latter is defined as

\[
A_{\alpha} = \left| \frac{\partial p'_{\perp}}{\partial y} \right|^{-1/2},
\]

where \( y \) is the coordinate perpendicular to the axis of the entrance contact, see Fig. 1 and the partial derivative is taken at constant \( p_{\perp} \). The classical action \( S_{\alpha} \) is a function of \( p_{\perp} \) and \( p'_{\perp} \). The spatial coordinates \( y \) and \( y' \) of the trajectory \( \alpha \) upon entrance and exit can be expressed as derivatives of \( S_{\alpha} \),

\[
y = \frac{\partial S_{\alpha}}{\partial p_{\perp}}, \quad y' = -\frac{\partial S_{\alpha}}{\partial p'_{\perp}}.
\]

For simplicity of notation, the Maslov index and other phase shifts are included in \( S_{\alpha} \).

We are interested in the scattering matrix \( S_{ij} \) at different values of the energy \( \varepsilon \). The energy \( \varepsilon \) enters in Eq. (1) through the energy dependence of the action \( S_{\alpha} \),

\[
S_{\alpha}(\varepsilon) = S_{\alpha}(\varepsilon_0) + (\varepsilon - \varepsilon_0) t_{\alpha},
\]

where \( \varepsilon_0 \) is a reference energy and \( t_{\alpha} \) is the duration of the trajectory \( \alpha \). We neglect the energy dependence of the stability amplitudes \( A_{\alpha} \).

Using Eq. (1), the traces \( Q_{\alpha}, n = 2, 4, 6, \) are expressed as double, quadruple, and sixfold summations over clas-
coordinates for a Poincaré surface of section taken in the interior of the quantum dot are coordinates \( s \) and \( u \) taken along the stable and unstable directions in phase space. In view of this, we replace the trajectory sum \( \sum' \), which is taken over trajectories with specified transverse momentum components \( p_{\perp} \) and \( p'_\perp \) at entrance and exit contacts, by a trajectory sum over trajectories with specified stable and unstable phase space coordinates \( s \) and \( u' \) at entrance and exit contacts, respectively. Referring to App.\( \text{A} \) for details, we find that the traces \( Q_2, Q_4, \) and \( Q_6 \) can also be expressed as

\[
Q_2 = \frac{1}{2\pi\hbar} \int ds_1 du'_1 \sum_{\alpha_1,\alpha_2} A_1 A_2 e^{i(S_1-S_2)/\hbar},
\]

\[
Q_4 = \frac{1}{(2\pi\hbar)^2} \int ds_1 du'_1 ds_3 du'_3 \times \sum_{\alpha_1,\ldots,\alpha_4} A_1 \ldots A_4 e^{i(S_1-\ldots-S_4)/\hbar},
\]

\[
Q_6 = \frac{1}{(2\pi\hbar)^3} \int ds_1 du'_1 ds_3 du'_3 ds_5 du'_5 \times \sum_{\alpha_1,\ldots,\alpha_6} A_1 \ldots A_6 e^{i(S_1-\ldots-S_6)/\hbar},
\]

where the classical trajectories \( \alpha_\mu, \mu = 1, \ldots, n \) satisfy the conditions

\[
s_1 = s_2, \ldots, s_{n-1} = s_n,
\]

\[
u'_2 = u'_3, \ldots, \nu'_n = u'_1,
\]

with \( n = 2, 4, 6 \). Further, in Eqs. \( \text{(19)} - \text{(21)} \), the stability amplitudes are defined as

\[
A_\alpha = \left| \frac{\partial u'}{\partial u} \right|^{-1/2},
\]

and the classical actions \( S_\alpha(s,u') \) are Legendre transforms of the original classical actions \( S_\alpha(p_{\perp}, p'_\perp) \), so that

\[
\frac{\partial S_\alpha}{\partial s} = u_\alpha, \quad \frac{\partial S_\alpha}{\partial u'} = -s'_\alpha.
\]

For the interpretation of Eqs. \( \text{(19)} - \text{(21)} \), one should keep in mind that the phase space coordinates \( s \) and \( u \) are defined only locally. That means that the coordinate transformation \( (s,u) \rightarrow (s,u') \) can only be made for pairs of trajectories that enter or exit the quantum dot at nearby phase space points. This poses no problems for our calculation, because only classical trajectories that exit the quantum dot at nearby positions and with close momenta have an action difference \( \Delta S \) that varies sufficiently slowly as a function of the phase space coordinates to give a finite contribution to \( Q_n \), \( n = 2, 4, 6 \). Indeed, the explicit calculations of \( Q_2, Q_4, \) and \( Q_6 \) in the following subsections show that the entire contribution to \( Q_n \) comes from trajectories for which the differences of the stable or unstable phase space coordinates are small.
A. Calculation of $Q_2$

The leading contribution to $Q_2$ arises from equal trajectories $\alpha_1 = \alpha_2$. In that case, the action difference

$$\Delta S = S_1 - S_2 = (\epsilon_1 - \epsilon_2)t,$$

where $t$ is the common duration of the trajectories $\alpha_1$ and $\alpha_2$. The stability amplitudes are equal, $A_1 = A_2 = A$. The factor $A_1^2 A_2^{-1} = A^2$ in Eq. (11) provides the Jacobian necessary to replace the integration over the unstable phase space coordinate $u'$ at the exit contact by an integration over the unstable phase space coordinate $u$ at the entrance contact. This way, $Q_2$ is expressed in terms of an integration over the phase space coordinates $s$ and $u$ at a Poincaré surface of section taken anywhere along the classical trajectory $\alpha_1 = \alpha_2$. Following the latter strategy, we write $Q_2$ as

$$Q_2 = \int dq \int_0^\infty dt_1 dt_2 P_i(t_1) P_j(t_2) e^{i(t_1 - \epsilon_2)(t_1 + t_2)/\hbar}/2\pi\hbar(t_1 + t_2)N,$$

where $q$ refers to the phase space coordinate at which the reference surface of section is taken, $P_i(t_1)$ selects only those $q$ for which the classical propagation of $q$ ends up in contact $i$ after time $t_1$, and $P_j(t_2)$ selects $q$ for which the classical propagation of the time-reversed of $q$ ends at contact $j$ after a time $t_2$. We divided by $t = t_1 + t_2$ to cancel a spurious contribution from the freedom to choose the reference surface of section at an arbitrary point along the trajectory $\alpha_1 = \alpha_2$. Replacing $P_i$ and $P_j$ by classical probabilities to reach the contacts $i$ and $j$ for an arbitrary phase space point $q$,

$$P_i(t) = \frac{N_i}{N^2_{TD}} e^{-t/\tau_D}, \quad P_j(t) = \frac{N_j}{N^2_{TD}} e^{-t/\tau_D},$$

the integration over $q$ contributes the total phase space volume $2\pi\hbar N\tau_D$ of the quantum dot and we find the well-known result

$$Q_2 = \frac{N_i N_j}{N^2 F_2(1, 2)}$$

upon integration over $t_1$ and $t_2$.

B. Calculation of $Q_4$

The trace $Q_4$ is expressed as a quadruple sum over classical trajectories, see Eq. (20). The typical configuration of four trajectories that contributes to $Q_4$ was pointed out in Refs. 30, 31, 32. The trajectories $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$, which are paired close to entrance and exit, have a small-angle encounter at which the pairing of the trajectories is interchanged, see Fig. 2. Before arriving at the encounter, $\alpha_1$ and $\alpha_2$, and $\alpha_3$ and $\alpha_4$ are paired. After the encounter, $\alpha_2$ and $\alpha_3$, and $\alpha_4$ and $\alpha_1$ are paired. The encounter may reside fully inside the quantum dot, as in Fig. 2, so that the pairs of trajectories are uncorrelated when they exit and enter the quantum dot, or the small-angle encounter may touch one or two of the lead openings, as in Figs. 3 and 4 respectively, so that exit or entrance of the four trajectories is correlated. While the importance of such small-angle encounters of classical trajectories was first realized for weak localization, they play a crucial role in all semiclassical theories of quantum transport. Following the same reasoning as in Fig. 2, we need to sum over $\alpha_1$ and $\alpha_3$ only. In order to parameterize $\alpha_1$ and $\alpha_3$ we take a Poincaré surface of section chosen at an arbitrary point during the encounter. We then use the coordinate $q$ of a reference point at the Poincaré surface of section, as well as the differences $s_3 - s_1$ and $u_3 - u_1$ of the stable and unstable phase space coordinates to parameterize $\alpha_1$ and $\alpha_3$. The trajectories $\alpha_1$ and $\alpha_2$, and $\alpha_3$ and $\alpha_4$ originate from the same contact and with the same stable phase stable phase space coordinate, hence

$$s_1 = s_2, \quad s_3 = s_4.$$  \hspace{1cm} (29)

Similarly,

$$u_4 = u_1, \quad u_2 = u_3.$$  \hspace{1cm} (30)

Since the trajectory pairs have the same stable and unstable phase space coordinates at the entrance and exit contacts, respectively, the conditions (29) and (30) hold for encounters that do not touch the contacts as well as for encounters that touch the contacts.

We first discuss the case of Fig. 2 in which the encounter resides fully inside the quantum dot. We closely follow Ref. 33 in which the same configuration of trajectories was considered in the limit $\tau_D/\tau_E \to 0$. The encounter region is defined as the segment of the trajectories for which $|s_1 - s_3| < c$ and $|u_1 - u_3| < c$, where $c \sim (p_F W)^{1/2}$ is a classical cut-off below which the classical dynamics can be linearized. The precise choice of

![FIG. 2: Left: Four classical trajectories that give the random matrix contribution to the average $Q_4$. These trajectories have a small-angle encounter which fully resides inside the quantum dot. Right: Schematic drawing of the encounter together with the definitions of the various times used in the text. The encounter region is shown thick.](image-url)
In this equation, the factor \( \tau_D \) is given by Eq. (33) with \( \lambda = \frac{1}{\lambda} \ln \frac{c^2}{(u_3 - u_1)(s_3 - s_1)} \). Hence, the duration of the encounter is
\[
\tau_{\text{enc}} = \frac{1}{\lambda} \ln \frac{c^2}{(u_3 - u_1)(s_3 - s_1)},
\]
where \( \lambda \) is the Lyapunov exponent for the classical dynamics in the quantum dot. We parameterize the durations \( t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_3}, \) and \( t_{\alpha_4} \) of the four stretches connecting the encounter region with the lead openings,
\[
\begin{align*}
t_{\alpha_1} &= t_{\text{enc}} + t_{12} + t_{41}, \\
t_{\alpha_2} &= t_{\text{enc}} + t_{12} + t_{23}, \\
t_{\alpha_3} &= t_{\text{enc}} + t_{34} + t_{23}, \\
t_{\alpha_4} &= t_{\text{enc}} + t_{34} + t_{41}.
\end{align*}
\]
Schematically, the definitions of \( t_{\text{enc}} \) and of the times \( t_{12}, t_{23}, t_{34}, \) and \( t_{41} \) are shown in the right panel of Fig. 2.
The action difference \( \Delta S \) is the symplectic area enclosed by the four trajectories, plus a contribution from the energy differences,
\[
\Delta S = (s_3 - s_1)(u_3 - u_1) + t_{\text{enc}}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4) + t_{12}(\varepsilon_1 - \varepsilon_2) + t_{23}(\varepsilon_3 - \varepsilon_2) + t_{34}(\varepsilon_3 - \varepsilon_4) + t_{41}(\varepsilon_1 - \varepsilon_4).
\]
Note that the enclosed phase space areas are conserved along the motion of the trajectories. In particular, this means that the action difference \( \Delta S \) is independent of where the Poincaré surface of section is chosen.

Integrating over the position of the Poincaré surface of section, we then find
\[
Q_4 = \int dt_{12} dt_{23} dt_{34} t_{41} P_j(t_{12}) P_k(t_{23}) P_l(t_{34}) P_i(t_{41}) \times \int d(u_3 - u_1) d(s_3 - s_1) \frac{\tau_D e^{\Delta S/k - \tau_{\text{enc}}/\tau_D}}{2\pi \hbar \tau_{\text{enc}}}.
\]
In this equation, the factor \( t_{\text{enc}} \) in the denominator cancels a spurious contribution to the integral from the freedom to choose the Poincaré surface of section anywhere along the encounter region. The classical probabilities \( P_i(t_{41}), P_j(t_{12}), P_k(t_{23}), \) and \( P_l(t_{34}) \) are defined as in Eq. (27). The factor \( e^{\Delta S/k - \tau_{\text{enc}}/\tau_D} \) is the probability that the trajectories do not exit the quantum dot during the encounter stretch. (If they do, the encounter touches the lead opening. This case is treated separately below.)

Taking \( u_3 - u_1 \) to be positive (while adding a factor 2 in Eq. (32)), we perform the variable change
\[
u_3 - u_1 = c/\sigma, \quad s_3 - s_1 = c x_0.
\]
With the new integration variables, the integration domain is \(-1 < x < 1\) and \(1 < \sigma < 1/|x|\). Further, \( t_{12} = t_{34} = 0 \).

\[
t_{\text{enc}} = \lambda^{-1} \ln(1/|x|).
\]
Integrating over \( t_{12}, t_{23}, t_{34}, \) and \( t_{41} \), and \( \sigma \), and performing a partial integration to \( x \), we then find
\[
Q_4 = -\frac{N_n N_j N_k N_l F_4(1, 2, 3, 4)}{N_4 F_2(1, 2) F_2(3, 2) F_2(3, 4) F_2(1, 4)} \times \int_0^1 dx x F_4(1, 2, 3, 4)/\lambda \tau_0 \frac{2 \sin(x r)}{\pi x},
\]
where we omitted a term that is an oscillating function of \( r = c^2/\hbar \) in the limit \( \hbar \to 0 \). The remaining integral over \( x \) can be evaluated in the limit \( \hbar \to 0 \) at fixed \( \tau_0/\tau_D \), and we arrive at the final result
\[
Q_4 = -\frac{N_n N_j N_k N_l F_4(1, 2, 3, 4)}{N_4 F_2(1, 2) F_2(3, 2) F_2(3, 4) F_2(1, 4)} \times e^{-\tau_0 F_4(1, 2, 3, 4)/\tau_D},
\]
where the Ehrenfest time \( \tau_E \) is defined as, cf. Eq. (1),
\[
\tau_E = \frac{1}{\lambda} \ln r = \frac{1}{\lambda} \ln \frac{c^2}{\hbar}.
\]
It is important to notice that the precise value of the phase-space cut-off \( c \) enters only through the Ehrenfest time \( \tau_E \). However, since the Ehrenfest time appears in the combination \( \tau_E/\tau_0 \) only and since \( \tau_0 \to \infty \) in the classical limit taken here, any change of \( c \) by a factor of order unity will not affect the final result. It is because of this that we did not need to carefully specify the classical length scale \( W \) entering into the cut-off \( c \).

If the encounter region touches one of the lead openings, as in Figs. 3a and b, a slight modification of the above calculation is called for. For definiteness, we consider an encounter that touches the entrance lead opening, as shown in Fig. 3a. This implies \( j = l \), so that the corresponding contribution to \( Q_4 \) is proportional to \( \delta_{jl} \). The action difference \( \Delta S \) is given by Eq. (33) with \( t_{12} = t_{34} = 0 \). Since the trajectories \( \alpha_1 \) and \( \alpha_3 \) must be

\[
\begin{align*}
t_{\alpha_1} &= t_{\text{enc}} + t_{12} + t_{41}, \\
t_{\alpha_2} &= t_{\text{enc}} + t_{12} + t_{23}, \\
t_{\alpha_3} &= t_{\text{enc}} + t_{34} + t_{23}, \\
t_{\alpha_4} &= t_{\text{enc}} + t_{34} + t_{41}.
\end{align*}
\]
correlated upon entry for the configuration shown in Fig. 3b, the encounter time \( t_{\text{enc}} \) is bounded by

\[
0 < t_{\text{enc}} = \frac{1}{\lambda} \ln \frac{c}{\left| u_3 - u_1 \right|} < \frac{1}{\lambda} \ln \frac{c}{s_3 - s_1}.
\]  

(39)

Then, proceeding as in the calculation of \( Q_4 \) and integrating over \( t_{23} \) and \( t_{41} \), we find

\[
Q_4^{(3)} = \frac{N_i N_j N_k \delta_{jl}}{N^3 F_2(3, 2, 2)} \int dt_{\text{enc}} P_j(t_{\text{enc}})
\times \int_{c}^{e} \frac{d(s_1 - s_3) d(u_1 - u_3)}{2 \pi \hbar t_{\text{enc}}}
\times e^{i(s_3 - s_1)(u_3 - u_1)/\hbar + it_{\text{enc}}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)/\hbar}.
\]  

(40)

In order to perform the integrations in Eq. (40), we take

\[
u_3 - u_1 = \frac{c}{\sigma}, \quad s_3 - s_1 = c \sigma.
\]  

(41)

With the new integration variables, the integration domain is \(-1 < x < 1, 1 < \sigma < e^{\lambda t_{\text{enc}}}, \) and \( 0 < t_{\text{enc}} < \lambda^{-1} \ln(1/|x|) \). Hence, with \( r = c^2/\hbar \) and integrating over \( t_{\text{enc}} \) and \( \sigma \), the integral (41) becomes

\[
N_i N_j N_k \delta_{jl} \int dt_{\text{enc}} P_j(t_{\text{enc}})
\times \int_{0}^{1} dx 2 \sin(xr) e^{iF_1(1, 2, 3, 4)/\lambda \tau_D}
\times N^3 F_2(3, 2, 2) F_2(1, 4)
\times e^{-\tau_D F_1(1, 2, 3, 4)/\tau_D},
\]  

(42)

where, again, we omitted terms proportional to \( \sin r \).

Similarly, for \( Q_4^{(3)} \) one finds

\[
N_i N_j N_k \delta_{ik} \int dt_{\text{enc}} P_i(t_{11}) P_j(t_{22})
\times \int_{0}^{1} dx 2 \sin(xr) e^{iF_1(1, 2, 3, 4)/\lambda \tau_D}
\times N^3 F_2(3, 2, 2) F_2(1, 2)
\times e^{-\tau_D F_1(1, 2, 3, 4)/\tau_D}.
\]  

(43)

Finally, if the encounter touches both lead openings, one has

\[
Q_4^{(4)} = \tau_D \int dt_{1} dt_{2} P_i(t_{11}) P_j(t_{22})
\times \int_{0}^{1} dx 2 \sin(xr) e^{i\Delta S/\hbar}
\times N^3 F_2(3, 2, 2) F_2(1, 2)
\times e^{-\tau_D F_1(1, 2, 3, 4)/\tau_D},
\]  

(44)

where \( P_i \) and \( P_j \) are classical probabilities given in Eq. 27. \( t_{1} \) and \( t_{2} \) are the propagation times between the surface of section and the exit contact and the entrance contact and the surface of section, respectively, and \( t_{\text{enc}} = t_{1} + t_{2} \). The action difference \( \Delta S \) is given by Eq. 33 with \( t_{12} = t_{23} = t_{34} + t_{41} = 0 \). In order to ensure that \( \alpha_1 \) and \( \alpha_3 \) are correlated upon entrance as well as exit, we require

\[
t_1 < \frac{1}{\lambda} \ln \frac{c}{s_3 - s_1}, \quad t_2 < \frac{1}{\lambda} \ln \frac{c}{u_3 - u_1}.
\]  

(45)

Performing integrations to \( t_1 - t_2 \) and \( u_3 - u_1 \) and abbreviating \( x = (s_3 - s_1) \), one finds

\[
Q_4^{(4)} = \delta_{ik} \delta_{jl} \int dt_{\tau_D} e^{-t/\tau_D + i\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4}
\times \int_{0}^{1} dx 2 \sin(xr) e^{i\Delta S/\hbar}.
\]  

(46)

The integral over \( x \) has to be performed in the limit \( r \rightarrow \infty \) at fixed ratio \( \tau_D/t \). Since

\[
\int_{0}^{1} dx 2 \sin(xr) e^{i\Delta S/\hbar} = \theta(\tau_D - t),
\]  

(47)

if \( r \rightarrow \infty \), where \( \theta(x) = 1 \) if \( x > 0 \) and \( 0 \) otherwise, we find

\[
Q_4^{(4)} = \frac{N_i N_j N_k \delta_{ik} \delta_{jl}}{N^3 F_2(3, 2, 2) F_2(1, 2)} e^{-\tau_D F_1(1, 2, 3, 4)/\tau_D}.
\]  

(48)

Together, Eqs. 37, 42, 43, and 48 reproduce the prediction of the effective random matrix theory.

We note that there is a close connection between the appearance of Kronecker deltas involving the contact indices \( i, j, k, \) and \( l \), and the configurations of classical trajectories that contribute to \( Q_4 \): each encounter region that touches a lead opening gives rise to a Kronecker delta and, conversely, each Kronecker delta derived from an encounter that touched a lead opening. Hence, the four contributions to \( Q_4 \) — the third line in Eq. 17 and the three terms in Eq. 10 —, each of which have a different product of Kronecker deltas, are in a one-to-one correspondence with the four configurations of classical trajectories shown in Figs. 2b, 3b, 3d, and 4b.

We should point out that, without dependence on the energy arguments, \( i.e. \), after setting \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 \), the semiclassical calculation of \( Q_4 \) was first performed by Whitney and Jacquod.\textsuperscript{23} The same results can also be inferred from an earlier calculation by Agam, Aleiner, and Larkin, who used a different formalism.\textsuperscript{24} The energy dependence of the four terms we calculate here is non-trivial, however, and clearly reveals the structure of the classical trajectories underlying the four different terms in the average.
While our calculation of $Q_6^{(2)}$ closely followed that of Braun et al., our calculation of the remaining three contributions to $Q_6$ differs considerably from that of Ref. 33. Braun et al. do not consider encounters that touch the lead openings. Instead, they calculate the remaining contributions to $Q_4$ using the ‘diagonal approximation’ for the trajectory sums. Although this approximation gives the correct result if $\tau_E \ll \tau_D$, it is based on a different class of trajectories than the ones we considered (see also Ref. 35). In the diagonal approximation, one considers the case that the four trajectories $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are pairwise equal (rather than pairwise close), but without imposing any relation between the two pairs. For example, one admits trajectories $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$ where $\alpha_2$ and $\alpha_3$ exit with the same transverse momentum $p_\perp$ but at a classically different spatial coordinate $y$. When summed over the full family of trajectories, such configurations appear with rapidly oscillating phases, and their net contribution vanishes.\(^{9,16,41}\) These rapidly oscillating phases disappear only if all four trajectories pass through the contact at equal positions and angles (up to quantum uncertainties), as is the case for the configurations shown in Figs. 5a and b.

**C. Calculation of $Q_6$**

The generic configuration of classical trajectories that contributes to $Q_6$ is shown in Fig. 5. Instead of dealing with trajectories with one small-angle encounter, one now has to consider configurations of trajectories with two small-angle encounters or of trajectories with a ‘three-encounter’\(^{34,35,36}\) A three-encounter arises when two encounters overlap, so that all six trajectories are within a phase space $\Delta$ simultaneously. In order to avoid confusion, we refer to non-overlapping encounters of two pairs of trajectories as ‘two-encounters’ for the remainder of this section. We first consider the case that all encounters fully reside inside the quantum dot. The case that encounters touch the lead openings will be discussed afterwards.

Without encounters that touch the lead openings, one distinguishes two types of contributions to $Q_6$: the contribution from the case in which the trajectories undergo two separate two-encounters, and the contribution from the case in which the six trajectories involved undergo a three-encounter. These are shown in Fig. 5a and 5b, respectively. The configuration shown in Fig. 5a shows only one configuration with two two-encounters. There are two more configurations, which are obtained by the cyclic permutations $(1,2,i,j) \rightarrow (3,4,k,l) \rightarrow (5,6,m,n)$.

Because the two two-encounters in Fig. 5a do not overlap, their contribution $Q_6^{(4)}$ to $Q_6$ factorizes. The stretch connecting the trajectories $\alpha_1$ and $\alpha_2$ to their joint entrance contact $j$ contributes a factor $N_2/N_F(1,2)$; the other stretches connecting the encounters to the contacts contribute similar factors to $Q_6^{(4)}$. Further, using the results of the previous subsection, we find that the two-encounter of the trajectories $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in Fig. 5a contributes a factor $-F_2(1,2,3,4) \exp(-\tau_E F_4(1,2,3,4)/\tau_D)$. Similarly, the two-encounter involving the trajectories $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6$ contributes a factor $-F_2(1,4,5,6) \exp(-\tau_E F_4(1,4,5,6)/\tau_D)$. Finally, the stretch of the trajectories $\alpha_1$ and $\alpha_4$ that connects the two two-encounters in Fig. 5a contributes a factor $1/F_2(1,4)$. Using that $F_2(1,2,3,4) + F_2(1,4,5,6) = F_2(1,2,3,4,5,6) + F_2(1,4)$, we find that $Q_6^{(4)}$ is given by

\[
Q_6^{(4)} = \frac{N_1N_2N_3N_4N_5N_6 e^{-\tau_E F_6(1,2,3,4,5,6)/\tau_D}}{N_0 F_2(1,2) F_2(3,2) F_2(3,4) F_2(5,4) F_2(5,6) F_2(1,6)} \left[ \frac{F_4(1,2,3,4) F_4(1,4,5,6)}{F_2(1,4)} e^{-\tau_E F_4(1,4)/\tau_D} + \ldots \right],
\]

where the dots . . . represent two more terms obtained by the cyclic permutations $(1,2,i,j) \rightarrow (3,4,k,l) \rightarrow (5,6,m,n)$. As before, the superscript $(4)$ refers to the figure that shows the corresponding configuration of classical trajectories.

Although the calculation of $Q_6^{(4)}$ is similar in spirit, it proves rather cumbersome. Referring to App. B for details,
we find

\[
Q_6^{(5)} = - \frac{N_i N_j N_k N_l N_m N_n e^{-\tau_E F_6(1, 2, 3, 4, 5, 6)/\tau_D}}{N^6 F_2(1, 2) F_2(3, 2) F_2(3, 4) F_2(5, 4) F_2(5, 6) F_2(1, 6)} \left\{ \frac{F_4(1, 2, 3, 4) F_4(1, 4, 5, 6)}{F_2(1, 4)} \left[ 1 - e^{-\tau_E F_2(1, 4)/\tau_D} \right] + \ldots \right\}
\]

Combining these two contributions, we find that the terms proportional to \(\exp(-2\tau_E/\tau_D)\) cancel, so that the final result is uniformly proportional to \(\exp(-\tau_E/\tau_D)\).

\[
Q_6^{(6)} = \frac{N_i N_j N_k N_l N_m N_n e^{-\tau_E F_6(1, 2, 3, 4, 5, 6)/\tau_D}}{N^6 F_2(1, 2) F_2(3, 4) F_2(5, 6) F_2(1, 6) F_2(3, 2) F_2(5, 4)} \times \left[ \frac{F_4(1, 2, 3, 4) F_4(1, 4, 5, 6)}{F_2(1, 4)} + \frac{F_4(3, 4, 5, 6) F_4(3, 2, 1, 6)}{F_2(3, 6)} + \frac{F_4(5, 6, 1, 2) F_4(5, 4, 3, 2)}{F_2(5, 2)} + F_6(1, 2, 3, 4, 5, 6) \right].
\]

In order to compute the contribution to \(Q_6\) of encounters that touch the lead openings, we have to consider the configurations of classical trajectories shown in Fig. 6. Figure 6a shows classical trajectories with one three-encounter where one encounter touches the lead opening. Figure 6d shows a configuration of classical trajectories with a three-encounter that partially touches both lead openings. The contributions from (a) and (b) have one Kronecker delta involving the lead indices, the contributions of (c), (d), and (e) have two Kronecker deltas, the contribution of (f) is zero, and the contribution of (g) has four Kronecker deltas.
configuration with two encounters,

\[
Q_6^{(b)} = -\frac{N_j N_i e^{-\tau_E F_0(1,2,3,4,5,6)/\tau_D}}{N^3 F_2(1,2) F_2(3,4) F_2(5,6)} \left[ \frac{N_i N_k F_4(1,2,3,4) \delta_{im} e^{-\tau_E F_2(1,4)/\tau_D}}{N^2 F_2(2,3) F_2(1,4)} + \ldots \right] + \left[ \frac{N_i N_k F_4(3,4,5,2) \delta_{jn} e^{-\tau_E F_2(5,2)/\tau_D}}{N^2 F_2(3,4) F_2(5,2)} + \ldots \right],
\]

Details of the calculation of the three-encounter are again left to the appendix. The result is

\[
Q_6^{(c)} = -\frac{N_j N_i N_\delta e^{-\tau_E F_0(1,2,3,4,5,6)/\tau_D}}{N^3 F_2(1,2) F_2(3,4) F_2(5,6)} \left[ \frac{N_i N_k F_4(1,2,3,4) \delta_{ijm} (1 - e^{-\tau_E F_2(1,4)/\tau_D})}{N^2 F_2(2,3) F_2(1,4)} + \ldots \right] + \left[ \frac{N_i N_k F_4(3,4,5,2) \delta_{ijn} (1 - e^{-\tau_E F_2(5,2)/\tau_D})}{N^2 F_2(3,4) F_2(5,2)} + \ldots \right].
\]

Combining both contributions, we again find that the terms proportional to \(e^{-2\tau_E/\tau_D}\) cancel, so that

\[
Q_6^{(b)} = \frac{N_i N_j N_k N_i N_\delta_{ijm} F_4(1,2,3,4)}{N^4 F_2(1,2) F_2(3,4) F_2(5,6) F_2(1,4)} + \frac{N_i N_j N_k N_\delta_{ijn} F_4(3,4,5,2)}{N^4 F_2(3,2) F_2(3,4) F_2(5,4) F_2(1,6) F_2(5,2)} + \ldots
\]

There are two different types of contributions to \(Q_6\) with two Kronecker deltas: Two Kronecker deltas that both involve lead indices of entrance (or exit) leads, or a product of two Kronecker deltas, where one involves lead indices of the entrance lead and one involves lead indices of the exit lead. Only a configuration of trajectories with a three-encounter contribute to the former, see Fig. 6b, whereas configurations of trajectories with a three-encounter as well as configurations of trajectories with two two-encounters contribute to the latter, see Fig. 6c and e. Referring to the appendix for calculational details regarding the configurations of trajectories with a three-encounter, here we simply quote the results,

\[
Q_6^{(b)} = \left[ \frac{N_i N_j N_k N_\delta_{ijm} F_4(1,2,3,4)}{N^4 F_2(1,2) F_2(3,4) F_2(5,6) F_2(1,4)} + \frac{N_i N_j N_k N_\delta_{ijn} F_4(3,4,5,2)}{N^4 F_2(3,2) F_2(3,4) F_2(5,4) F_2(1,6) F_2(5,2)} + \ldots \right] e^{-\tau_E F_0(1,2,3,4,5,6)/\tau_D},
\]

Finally, we have to consider the case that there is correlated escape of one pair and one triplet of trajectories and correlated escape of two triplets of trajectories, see Figs. 6f and g, respectively. One verifies that there is no contribution to \(Q_6\) for the configuration shown in Fig. 6f. For the configuration shown in Fig. 6g one finds

\[
Q_6^{(c)} = \left[ \frac{N_i N_j N_k N_\delta_{ijm} F_4(1,2,3,4)}{N^4 F_2(1,2) F_2(3,4) F_2(5,6) F_2(1,4)} + \frac{N_i N_j N_k N_\delta_{ijn} F_4(3,4,5,2)}{N^4 F_2(3,2) F_2(3,4) F_2(5,4) F_2(1,6) F_2(5,2)} + \ldots \right] e^{-\tau_E F_0(1,2,3,4,5,6)/\tau_D},
\]

Together, Eqs. (51), (54), (55), (56), and (57) combine to the effective random matrix theory ansatz for \(Q_6\).

**IV. CONCLUSION**

In this article we calculated the Ehrenfest-time dependence of the ensemble averages of polynomial functions \(Q_n\) of the \(N \times N\) scattering matrix \(S\) of a ballistic chaotic quantum dot and its hermitian conjugate \(S^\dagger\), for polynomials of degree \(n = 2, 4,\) and 6. For all cases our calculations of the leading-in-\(N\) averages agreed with the effective random matrix theory of Silvestrov, Goorden, and Beenakker.

The detailed semiclassical calculations of \(Q_2\) and \(Q_4\) were included in this article because they are a prerequisite for the semiclassical calculations of \(Q_6\). For the calculation of \(Q_4\) a few guiding principles would have been sufficient, however. These are: (i) the classical trajectories contributing to \(Q_4\) have at most one small-angle encounter, (ii) each encounter lasts one Ehrenfest time \(\tau_E\), and (iii) \(Q_4\) becomes equal to the classical limit \(Q_4^{(cl)}\) if \(\tau_E\) is much larger than the mean dwell time \(\tau_D\), whereas \(Q_4\) equals the random matrix limit \(Q_4^{\text{RMT}}\) if \(\tau_E \ll \tau_D\). From the first two guiding principles, both of which are well established in the literature and the exponential distribution of classical dwell times, one concludes that \(Q_4\) is of the form

\[
Q_4 = A + B \exp(-\tau_E/\tau_D).
\]
The third guiding principle then fully determines the coefficients $A$ and $B$ and, hence, $Q_6$.

No such shortcut exists for the calculation of $Q_6$. The relevant configurations of semiclassical trajectories have up to two encounters, hence we expect a general form

$$Q_6 = A + B \exp(-\tau_E/\tau_D) + C \exp(-2\tau_E/\tau_D). \quad (59)$$

The detailed calculation of this article was needed to show that the coefficient $C = 0$. In this sense, the semiclassical calculation of $Q_6$ provides the first nontrivial test of the effective random matrix theory.

The fact that the coefficient $C$ in Eq. (59) is zero is at the heart of the effective random matrix theory ansatz: It implies that (i) all trajectories with dwell times longer than $\tau_E$ are treated equally and (ii) that $\tau_E$ is the only time marking the boundary between classical and quantum effects. Whereas our calculation indirectly verifies these claims for leading-in-$N$ averages of low-degree polynomials of $S$ and $S^\dagger$, they are not true for subleading-in-$N$ averages. For subleading-in-$N$ averages, more than one integer multiple of $\tau_E$ may determine quantum effects. Known examples are weak localization, to which the predictions of effective random matrix theory was obtained only after a tedious calculation showing that the coefficient $C$ in Eq. (59) above is zero. The possibility of the appearance of a term $\propto \exp(-2\tau_E/\tau_D)$ caused a large increase in complexity upon going from polynomials of degree four to polynomials of degree six. We have searched for a simplification in our calculation that would allow us to cancel all contributions $\propto \exp(-2\tau_E/\tau_D)$ at an earlier point in the calculation. However, since the terms proportional to $\exp(-2\tau_E/\tau_D)$ were obtained from qualitatively different configurations of classical trajectories, we have not been able to find such a simplification. Without such a simplification, the task of extending the present calculation to polynomials of arbitrary degree seems too daunting to accomplish.

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FIG. 7: Poincaré surface of section of lead opening, together with the standard coordinates $y$, $p_\perp$, as well as the phase space coordinates $s$, $u$, taken along the stable and unstable directions in phase space. The trajectory pair $\alpha_1$, $\alpha_2$, which have equal perpendicular momentum components $p_{\perp,1} = p_{\perp,2}$ are mapped to a trajectory pair $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ with equal stable phase space coordinates, $s_1 = s_2$. The unstable phase space coordinates do not change in the mapping.

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**APPENDIX A: TRAJECTORY SUM WITH FIXED COORDINATES $s$ AND $u'$**

In this appendix we show that one can replace the standard formulation in which the traces $Q_n$ are expressed in terms of integrals over classical trajectories with specified transverse momentum components at entrance and exit contacts by an integral over classical trajectories with specified stable and unstable phase space coordinates at entrance and exit contacts, respectively.

We first consider a single integral over the transverse momentum $p_\perp$ in the entrance contact,

$$K(p_{\perp,1}, p_{\perp,2}) = \frac{1}{2\pi\hbar} \int dp_\perp \sum_{\alpha_1, \alpha_2} A_1 A_2 e^{i(S_1 - S_2)/\hbar}, \quad (A1)$$

where the trajectory $\alpha_n$ has transverse momentum $p_\perp$ at its entrance contact,

$$p_{\perp,1} = p_{\perp,2} = p_\perp, \quad (A2)$$

and transverse momentum $p_{\perp,\mu}$ at its exit contact, $\mu = 1, 2$. The stability amplitude is given by Eq. (13). The classical action $S_\perp$ is a function of $p_\perp$ and $p_{\perp,\mu}, \mu = 1, 2$.

In order to rewrite $K$ in terms of an integral over classical trajectories with a fixed stable phase space coordinate $s$ upon entrance into the quantum dot, we consider a Poincaré surface of section taken at the entrance contact, see Fig. 7. There are two sets of coordinates to label...
trajectories piercing this Poincaré surface of section: the spatial and momentum coordinates $y$ and $p_\perp$, and the stable and unstable phase space coordinates $s$ and $u$, see Fig. 4 The coordinate axes for $s$ and $u$ do not need to be perpendicular, but the coordinates are normalized such that $dsdu = dydp_\perp$. The phase space points where $\alpha_1$ and $\alpha_2$ pierce the Poincaré surface of section are indicated in the figure.

To each pair of two sufficiently close classical trajectories $\alpha_1$, $\alpha_2$ we now assign another pair $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, where the trajectory $\tilde{\alpha}_\mu$ is obtained from $\alpha_\mu$ by slightly changing the boundary condition in the entrance contact while keeping the boundary condition upon exit the same. This implies that the unstable phase space coordinates of $\alpha_\mu$ and $\tilde{\alpha}_\mu$ are equal, $\mu = 1, 2$. In order to uniquely define the pair $\tilde{\alpha}_1, \tilde{\alpha}_2$, we require that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ have the same stable phase space coordinate $s$ upon entry, and that $p_\perp, \tilde{1} = p_\perp, 2 - p_\perp, 3$. The construction of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ is shown in Fig. 7.

For this construction, one verifies that the action difference of the original and primed trajectory pair is not changed,

$$S_1 - S_2 = \tilde{S}_1 - \tilde{S}_2,$$

(A3)

provided we replace the action $S$, which is a function of $p_\perp$, by the action $\tilde{S}$, which is a function of the stable phase space coordinate $s$, obtained by Legendre transform of the original action $S$. Also, using the normalization $dsdu = dp_\perp dy$ and Eq. (A2), one finds

$$\int dp_\perp \left| \frac{\partial p'_\perp 1}{\partial y_1} \right|^{-1/2} \left| \frac{\partial p'_\perp 2}{\partial y_2} \right|^{-1/2} \ldots =$$

$$\int ds \left| \frac{\partial p'_\perp 1}{\partial u_1} \right|^{-1/2} \left| \frac{\partial p'_\perp 2}{\partial u_2} \right|^{-1/2} \ldots, \quad (A4)$$

where

$$s = s_1 = s_2. \quad (A5)$$

The same procedure can be applied to integrations over the transverse momentum $p'_\perp$ in the exit contacts. In this case, one replaces the integral over $p'_\perp$ by an integral over the unstable phase space coordinate $u'$, and replaces pairs of trajectories with equal $p'_\perp$ by pairs of trajectories with equal $u'$. The result of this program is precisely Eqs. (10)-(21) of the main text.

**APPENDIX B: THREE-ENCOUNTER OF CLASSICAL TRAJECTORIES**

1. Three-encounter fully inside the quantum dot

In this appendix we describe the details of the calculation of $Q_6$. This is the contribution to $Q_6$ from a three-encounter that fully resides inside the quantum dot.

![FIG. 8: Definition of the four measures of the encounter duration used in the text: $t_{enc}$ is the total duration of the three-encounter, $t'_{enc}$ is the duration of the part of the encounter that all trajectories involved in the three-encounter are within a phase-space distance $c$, and $t_5$ and $t_6$ are the durations of the encounter stretches where one pair of trajectories has already diverged from the remaining two pairs.](image)

In order to calculate $Q_6$, we take a Poincaré surface of section at a point that all six trajectories are within a phase space distance $c$. We parameterize the trajectories using phase space coordinates $s$ and $u$ along the stable and unstable directions in phase space. At the Poincaré surface of section, one has

$$s_1 = s_2, \quad s_3 = s_4, \quad s_5 = s_6$$

$$u_2 = u_3, \quad u_4 = u_5, \quad u_6 = u_1. \quad (B1)$$

(These equalities continue to hold if the three-encounter touches the lead opening.) The action difference $\Delta S$ reads

$$\Delta S = (u_3 - u_1)(s_3 - s_1) + (u_5 - u_1)(s_5 - s_3) + \varepsilon_1 t_1 - \varepsilon_2 t_2 + \ldots - \varepsilon_6 t_6. \quad (B2)$$

In order to parameterize the durations of the six trajectories involved, we separate each duration $t_\mu$, $\mu = 1, 2, \ldots, 6$, into segments before and after the encounter and one segment of duration $t_{\mu, enc}$ inside the encounter region. Similarly, we define

$$\Delta S_{enc} = (u_3 - u_1)(s_3 - s_1) + (u_5 - u_1)(s_5 - s_3) + \varepsilon_1 t_{1, enc} - \varepsilon_2 t_{2, enc} + \ldots - \varepsilon_6 t_{6, enc}. \quad (B3)$$

Integrating over the time segments outside the encounter, we then arrive at
The factor $t_{\text{enc}}$ is the total duration of the encounter (the time that at least two trajectories are close together),

$$t_{\text{enc}} = \frac{1}{\lambda} \ln \frac{c}{\min(|s_3 - s_1|, |s_5 - s_1|, |s_3 - s_5|)} + \frac{1}{\lambda} \ln \frac{c}{\min(|u_3 - u_1|, |u_5 - u_1|, |u_3 - u_5|)},$$

and $t'_{\text{enc}}$ is the time that all three trajectories are close together, see Fig. 8.

$\text{The factor } t'_{\text{enc}} \text{ in the denominator cancels a spurious contribution to the integral arising from the freedom to choose the Poincaré surface of section at any point during three-encounter. The integration domain in Eq. (B5) is given by the conditions max}(|u_3 - u_1|, |u_5 - u_1|, |u_3 - u_5|) < c \text{ and max}(|s_3 - s_1|, |s_5 - s_1|, |s_3 - s_5|) < c.$

We rewrite the integral (B5) using $s = \max(|s_3 - s_1|, |s_5 - s_1|, |s_3 - s_5|)$, $s' = \min(|s_3 - s_1|, |s_5 - s_1|, |s_3 - s_5|)$, $u = \max(|u_3 - u_1|, |u_5 - u_1|, |u_3 - u_5|)$ and $u' = \min(|u_3 - u_1|, |u_5 - u_1|, |u_3 - u_5|)$ as our integration variables. After some tedious algebra, one arrives at

$$I = \tau_D \int_0^{c} ds du \int_0^{s/2} ds' \int_0^{u/2} du' \frac{e^{-t'_{\text{enc}} F_4(1,2,3,4,5,6)/\tau_D}}{(\pi h)^3}$$

$$\times \left\{ e^{-F_4(1,2,3,4) t_s/\tau_D} - F_4(1,2,3,4) t_s/\tau_D + e^{-F_4(1,2,3,4) t_u/\tau_D} - F_4(1,2,3,4) t_u/\tau_D + \ldots \right\}$$

$$\times \left( \cos \frac{us' + u's - us}{h} + \cos \frac{us + u's - u's'}{h} \right)$$

$$+ 2 \left[ e^{-F_4(1,2,3,4) t_u/\tau_D} - F_4(5,4,1,6) t_u/\tau_D + \ldots \right] \left( \cos \frac{us' - u's + u' + su'}{h} \right),$$

where the dots indicate cyclic permutations $(1, 2) \rightarrow (3, 4) \rightarrow (5, 6)$, and with

$$t'_{\text{enc}} = \frac{1}{\lambda} \ln \frac{c}{u} + \frac{1}{\lambda} \ln \frac{c}{s'},$$

$$t_s = \frac{1}{\lambda} \ln \frac{s}{s'},$$

$$t_u = \frac{1}{\lambda} \ln \frac{u}{u'},$$

The total duration of the encounter is $t_{\text{enc}} = t'_{\text{enc}} + t_s + t_u$.

We then perform the variable change

$$s = c/\sigma, \quad s' = c z'/\sigma,$$

$$u = c y/\sigma, \quad u' = c y y'/\sigma.$$  \hspace{1cm} (B10)

With these new variables, the integration over $\sigma$ can be done and cancels the factor $t'_{\text{enc}}$ in the denominator. Writing $r = c^2/h$, the remaining integral then reads

$$I = \frac{\lambda \tau_D r^2}{2} \int_0^{1/2} dx' dy' \int_0^{1} y dy y F_4(1,2,3,4,5,6)/\lambda \tau_D$$

$$\times \left\{ [(x')^2 F_4(1,2,5,6)/\lambda \tau_D (y')^2 F_4(1,6,3,2)/\lambda \tau_D + (x')^2 F_4(1,2,3,4)/\lambda \tau_D (y')^2 F_4(1,6,3,2)/\lambda \tau_D + \ldots]$$

$$\times \cos (yr(x' + y' - 1)) + \cos (yr(1 + x'y' - y')) + \cos (yr(1 + x'y' - x' y')) + \cos (yr(1 - x' y')) + 2 [(x')^2 F_4(1,2,3,4)/\lambda \tau_D (y')^2 F_4(5,4,1,6)/\lambda \tau_D + \ldots] \cos (yr(x' + y' - x' y') + \cos (yr(x' - y'))) \right\}.$$  \hspace{1cm} (B11)
In order to evaluate Eq. (B11), we write each product \((x')^i(y')^j\) as

\[(x')^i(y')^j = (1/2)^{i+j} + [(x')^i - (1/2)^i](1/2)^j + [(y')^j - (1/2)^j](1/2)^i + [(x')^i - (1/2)^i][y')^j - (1/2)^j]\]

(B12)

and evaluate each of the four terms in Eq. (B12) separately. In Eq. (B12), the exponents \(\epsilon_1\) and \(\epsilon_2\) represent the exponents \(F_4(\epsilon_1, \epsilon_2) = \lambda\) in Eq. (B11). The separation in Eq. (B12) makes sense, because both \(\epsilon_1\) and \(\epsilon_2\) are sent to zero if the classical limit \(h \to 0\) at fixed ratio \(\tau_E/\tau_D\) is taken. Hence, the first term in Eq. (B12) approaches unity, whereas the other terms are nonzero only if \(x'\) or \(y'\) are small, or both.

The four terms in Eq. (B12) are evaluated separately. For the first term in Eq. (B12), the integrals to be done exactly. Omitting a prefactor \((1/2)^i\epsilon+j\), which is sent to 1 in the classical limit taken here, and abbreviating \(\eta = F_6(1, 2, 3, 4, 5, 6)/\lambda\tau_D\), we find that the integrals from the first term give

\[I_1 = \frac{6\lambda\tau_D r^2}{\pi^2} \int_0^{1/2} dx'dy' \int_0^1 dy'y' \left[ \cos(\eta yr(x' + y' - 1)) + \cos(\eta(1 + x'y' - y')) \right. \]
\[\left. + \cos(\eta(1 + x'y' - x')) + \cos(\eta(1 - x'y')) + \cos(\eta(x' + y' - x'y')) + \cos(\eta(x' - y')) \right] \]
\[= \frac{6\lambda\tau_D}{\pi^2} \int_0^1 dy'y' \left[ -g_c(ry) \cos(ry) + g_s(ry) \sin(ry) \right], \]

(B13)

where

\[g_c(v) = \int_0^v dw \frac{1 - \cos w}{w}, \quad g_s(v) = \int_0^v dw \frac{\sin w}{w}. \]

(B14)

Then, performing a partial integration to \(y\) and omitting terms that are oscillating with \(r\), we find

\[I_1 = \frac{6\lambda\tau_D \eta}{\pi^2} \int_0^1 dy'y' \left[ g_c(ry) \cos(ry) - g_s(ry) \sin(ry) \right], \]

(B15)

In this expression we can take the limits \(r \to \infty\) and \(\eta \to 0\), keeping \(r^{-\eta} = e^{-\tau_E F_6(1, 2, 3, 4, 5, 6)/\tau_D}\) fixed. We then find

\[I_1 = -F_6(1, 2, 3, 4, 5, 6) e^{-\tau_E F_6(1, 2, 3, 4, 5, 6)/\tau_D}. \]

(B16)

For the second term in Eq. (B12), we first perform the integration to \(y'\), with the result

\[I_2 = \frac{\lambda\tau_D r^2}{\pi^2} \int_0^{1/2} dx'dy' \int_0^1 dy'y' \left[ (x')^i - (1/2)^i \right] \left[ \cos(\eta yr(x' + y' - 1)) + \cos(\eta(1 + x'y' - y')) \right. \]
\[\left. + \cos(\eta(1 + x'y' - x')) + \cos(\eta(1 - x'y')) + \cos(\eta(x' + y' - x'y')) + \cos(\eta(x' - y')) \right] + \ldots \]
\[= \frac{\lambda\tau_D}{\pi^2} \int_0^{1/2} dx' \int_0^1 dy'y' \left[ \frac{\partial}{\partial y} \right] \left[ \cos(\eta yrx') - \cos(ry) + (1 - x') \cos(\eta(1 - x')) \right] \]
\[+ \ldots, \]

(B17)

where we abbreviated \(\epsilon = F_4(1, 2, 3, 4)/\lambda\tau_D\). Performing a partial integration to \(y\), omitting oscillating terms, we then find

\[I_2 = \frac{\lambda\tau_D}{\pi^2} \int_0^{1/2} \frac{dy'}{y'} \cos(r y') \frac{(x')^i - (1/2)^i}{1 - x'} r(x') \]
\[- F_6(1, 2, 3, 4, 5, 6) \frac{1}{\pi^2} \int_0^{1/2} dx' \frac{dy'}{y'} \left[ \frac{\partial}{\partial y} \right] \left[ \cos(\eta yrx') - \cos(ry) + (1 - x') \cos(\eta(1 - x')) \right] \]
\[+ \ldots. \]

(B18)

The integral in the first line is of order \(1/r\) and can be neglected in the limit \(r \to \infty\). The integral in the second line is of order \(\epsilon \propto 1/\lambda\tau_D\), so that it can also be neglected. This is best seen by evaluating the integral after expanding \((x')^i - (1/2)^i \approx \epsilon \ln(2x')\) and replacing \(y\) by 1, which gives a finite value of the integral. Hence, we conclude that, in the classical limit \(r \to \infty\) at fixed \(r^{-1/\lambda\tau_D}\), we have

\[I_2 = 0. \]

(B19)
Similarly, one finds that the third term in Eq. (B11) does not contribute to \( I \).

For the remaining contribution \( I_4 \), we note that the only nonzero contribution can come from small \( x' \) and small \( y' \). Neglecting \( x' \) and \( y' \) with respect to unity, one finds that the contribution from the first term between brackets \( \{ \ldots \} \) in Eq. (B11) gives zero. In the same approximation, after two partial integrations the second term gives

\[
I_4 = \frac{4\lambda \tau_D \epsilon_1 \epsilon_2}{\pi^2} \int_0^{1/2} \frac{dx' dx'}{x' y'} \int_0^{1/2} dy y^{-1} (x')^{\epsilon_1} (y')^{\epsilon_2} \sin(y x') \sin(y y') + \ldots,
\]

where \( \epsilon_1 = F_4(1, 2, 3, 4)/\lambda \tau_D \) and \( \epsilon_2 = F_4(5, 4, 1, 6) \) (or cyclic permutations), and \( \eta = F_6(1, 2, 3, 4, 5) / \lambda \tau_D \). We then shift variables \( x' = x''/y \) and \( y' = y''/y \) and integrate over \( y \),

\[
I_4 = \frac{4\lambda \tau_D \epsilon_1 \epsilon_2}{\pi^2 (\epsilon_1 + \epsilon_2 - \eta)} \int_0^{1/2} \frac{dx'' dy''}{x'' y''} (x'')^{\epsilon_1} (y'')^{\epsilon_2} \left[ (\max(x'', y''))^{\eta - \epsilon_1 - \epsilon_2} - (1/2)^{\eta - \epsilon_1 - \epsilon_2} \right] \sin(x x'') \sin(y y'') + \ldots
\]

where we used that \( F_4(1, 2, 3, 4) + F_4(5, 4, 1, 6) - F_6(1, 2, 3, 4, 5, 6) = F_2(1, 4) \). As before, the dots \( \ldots \) refer to the two additional terms obtained by cyclic permutation \((1, 2) \rightarrow (3, 4) \rightarrow (5, 6) \) of the expression written above.

The final result for \( Q_6^{(6)} \) is found by substituting \( I = I_1 + I_4 \) into Eq. (B11).

### 2. Three-encounter that touches the lead openings

If the three-encounter touches the lead openings, escape of the trajectories involved in the encounter is no longer uncorrelated. In order to deal with three-encounters that touch the lead openings, the initial part of the calculation of the previous subsection has to be modified. However, the final integrals will all be of the form of the integrals \( I_1, I_2, \) and \( I_4 \) calculated above. Here we discuss the configurations of Figs. 6b–g proceeds in a similar manner. In the discussion here we further limit ourselves to the labeling of trajectories as shown in the figure. For Fig. 6b, this means that we consider the exit of trajectories \( \alpha_3 \) and \( \alpha_5 \) to be correlated, whereas the exit of trajectory \( \alpha_3 \), as well as all entrances are uncorrelated. There are five more contributions to \( Q_6 \): two of these arise from configurations where the exit of \( \alpha_1 \) and \( \alpha_3 \) or the exit of \( \alpha_3 \) and \( \alpha_5 \) is correlated; three more arise from configurations with correlated entry of two trajectories, and uncorrelated exits. For Fig. 6c, we take the exits of all trajectories to be correlated, whereas the entrances are not. There is another contribution obtained by reversing the roles of entrance and exit.

In order to describe a three-encounter that touches the lead opening as shown in Fig. 6b, we again take a Poincaré surface of section at a point that all six trajectories are within a phase space distance \( c \). In the configuration of Fig. 6b, this ‘central’ part of the three-encounter does not touch a lead opening. We ensure that \( \alpha_3 \) has moved away from the other trajectories before arriving at the lead opening by requiring

\[
\min(|u_1 - u_3|, |u_5 - u_3|) > |u_1 - u_5|.
\]

The escape of \( \alpha_1 \) and \( \alpha_5 \) is correlated if

\[
0 < t' < t_u,
\]

where

\[
t_u = \frac{1}{\lambda} \ln \left( \frac{\max(|u_3 - u_1|, |u_4 - u_5|)}{|u_5 - u_1|} \right),
\]

and \( t' \) is the duration of the stretch of the encounter immediately adjacent to the lead opening, during which \( \alpha_1 \) and \( \alpha_5 \) are correlated with each other, but not with \( \alpha_3 \), see Fig. 6b. We then find

\[
Q_6^{(6)} = \frac{N_1 N_2 N_3 N_4 N_5 N_6 \delta_{\text{enc}}}{N^5 F_2(1, 2) F_2(3, 4) F_2(5, 6) F_2(3, 2)} I,
\]

where

\[
I = \int d(s_3 - s_1) d(s_5 - s_1) d(u_3 - u_1) d(u_5 - u_1) \times \frac{dt' e^{3A S_{\text{enc}} - t_{\text{enc}}/\tau_D}}{(2\pi \hbar)^2 t_{\text{enc}}^2},
\]
with
\[ t_{\text{enc}}' = t' + t_{\text{enc}}' + t_s, \]
\[ t_{\text{enc}}' = \frac{1}{\lambda} \ln \frac{c}{\max(|u_3 - u_1|, |u_3 - u_5|)} \]
\[ + \frac{1}{\lambda} \ln \frac{c}{\max(|s_1 - s_3|, |s_1 - s_5|, |s_3 - s_5|)} \]
\[ t_s = \frac{1}{\lambda} \ln \frac{c}{\min(|s_1 - s_3|, |s_1 - s_5|, |s_3 - s_5|)}. \]  

The action difference $\Delta S_{\text{enc}}$ is given by Eq. (B3) above.

Rewriting the integral in terms of the variables \( s = \max(|s_1 - s_3|, |s_1 - s_5|, |s_3 - s_5|), s' = \min(|s_1 - s_3|, |s_1 - s_5|, |s_3 - s_5|), u = \max(|u_3 - u_1|, |u_3 - u_5|), \) and \( u' = |u_5 - u_1|, \) and integrating over \( t', \) we arrive at

\[ I = \tau_D \int_0^c ds \int_0^{s/2} ds' \int_0^u du \int_0^{u/2} du' \frac{e^{-t_{\text{enc}}' F_4(1,2,3,4,5,6)/\tau_D}}{(\pi \hbar)^2 t_{\text{enc}}' F_4(5, 4, 1, 6)} \left[ 1 - e^{-t_s F_4(1,4,5,6)/\tau_D} \right] \]
\[ \times \left\{ e^{-t_s F_4(1,2,3,4,5,6)/\tau_D} + e^{-t_s F_4(3,4,5,6)/\tau_D} \right\} \]
\[ \times \left( \cos \frac{us + u's' - u's}{\hbar} + \cos \frac{us - u's' - u's}{\hbar} + \cos \frac{us - u's + u's'}{\hbar} \right) \]
\[ + 2 e^{-t_s F_4(1,2,3,4)/\tau_D} \left( \cos \frac{us' - u's}{\hbar} + \cos \frac{us' + u's - u's'}{\hbar} \right) \].

The remainder of the calculation is identical to that of the previous subsection: We write
\[ (x')^\epsilon = 1 + ((x')^\epsilon - (1/2)^\epsilon), \]
where \( \epsilon \) represents $F_4(1,2,5,6)/\lambda \tau_D$, $F_4(3,4,5,6)/\lambda \tau_D$, or $F_4(1,2,3,4)/\lambda \tau_D$, and evaluate the two terms in Eq. (B32) separately. The calculation of the first term is identical to that of $I_3$, see Eqs. [B17]–[B19], and vanishes. The calculation of the second term is identical to that of $I_4$, see Eq. (B31) above. We thus find
\[ I = -\frac{F_4(1,2,3,4)}{F_2(1,4)} e^{-\tau_D F_4(1,2,3,4,5,6)/\tau_D} \left[ 1 - e^{-\tau_D F_4(1,4)/\tau_D} \right]. \]

Substitution of Eq. (B33) into Eq. (B32) and addition of the remaining five configurations of trajectories obtained by relabeling or by interchanging entrance and exit gives Eq. (B34).

Proceeding in a similar manner for the case shown in Fig. 6c, we find
\[ G_v^6 = \frac{N_i N_j N_k \delta_{ik} \delta_{im}}{N^4 F_2(1, 2) F_2(3, 4) F_2(5, 6)} I, \]  
where
\[ I = \tau_D \int d(s_3 - s_1) d(s_5 - s_1) d(u_3 - u_1) d(u_5 - u_1) \]
\[ \times \frac{d t_{\text{enc}}' e^{i \Delta S_{\text{enc}}/\hbar t_{\text{enc}}' \tau_D}}{(2 \pi \hbar)^2 t_{\text{enc}}' \tau_D}. \]
In this integral, the contribution from the term 1 in Eq. (B34) in the previous subsection. We thus find integrals identical to the integrals $I_{1}$ and $I_{2}$ considered in the previous subsection. We thus find

$$t_{\text{enc}} = t_{\text{enc}}' + \frac{1}{\lambda} \ln \frac{\max(|s_{1} - s_{3}|, |s_{1} - s_{5}|, |s_{3} - s_{5}|)}{\min(|s_{1} - s_{3}|, |s_{1} - s_{5}|, |s_{3} - s_{5}|)}\tag{B36}$$

We require

$$0 < t_{\text{enc}}' < \frac{1}{\lambda} \ln \frac{c}{\max(|s_{1} - s_{3}|, |s_{1} - s_{5}|, |s_{3} - s_{5}|)} + \frac{1}{\lambda} \ln \frac{c}{\max(|u_{1} - u_{3}|, |u_{1} - u_{5}|, |u_{3} - u_{5}|)}\tag{B37}$$

In order to ensure that the three-encounter indeed touches the lead opening at a point that all trajectories involved in the encounter are separated by phase space distances less than the cut-off $c$. The integration over $t_{\text{enc}}'$ can be done immediately. Rewriting the integral in terms of the variables $s = \max(|s_{1} - s_{3}|, |s_{1} - s_{5}|, |s_{3} - s_{5}|)$, $s' = \min(|s_{1} - s_{3}|, |s_{1} - s_{5}|, |s_{3} - s_{5}|)$, $u = \max(|u_{1} - u_{3}|, |u_{1} - u_{5}|, |u_{3} - u_{5}|)$, and $u' = \min(|u_{1} - u_{3}|, |u_{1} - u_{5}|, |u_{3} - u_{5}|)$, and performing the variable change

$$s = c/\sigma, \quad s' = c x'/\sigma, \quad u = c y/\sigma, \quad u' = c y'/\sigma, \tag{B38}$$

we arrive at the integral

$$I = e^{-\tau_{D}F_{6}(1,2,3,4,5,6)/\tau_{D}}\tag{B40}$$

Substituting this result into Eq. (134) and adding what one obtains after interchanging entrance and exit leads, one arrives at Eq. (135).

The remaining configurations of trajectories with a three-encounter (Figs. 6 and 7) are treated in the same manner.

References

1. I. L. Aleiner and A. I. Larkin, Phys. Rev. B 54, 14423 (1996).
2. A. I. Larkin and Y. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 55, 2262 (1968) [Sov. Phys. JETP 28, 1200 (1969)].
3. G. M. Zaslavsky, Phys. Rep. 80, 157 (1981).
4. L. P. Kouwenhoven, C. M. Marcus, P. L. McEuen, S. Tarucha, R. M. Westervelt, and N. S. Wingreen, in Mesoscopic Electron Transport, edited by L. L. Sohn, L. P. Kouwenhoven, and G. Schön (Kluwer, Dordrecht, 1997), vol. 345 of NATO ASI Series E.
5. C. W. J. Beenakker, Rev. Mod. Phys. 69, 731 (1997).
6. O. Agam, I. Aleiner, and A. Larkin, Phys. Rev. Lett. 85, 3153 (2000).
7. P. G. Silvestrov, M. C. Goorden, and C. W. J. Beenakker, Phys. Rev. B 67, 241301 (2003).
8. J. Tworzydło, A. Tajic, H. Schomerus, and C. W. J. Beenakker, Phys. Rev. B 68, 115313 (2003).
9. R. S. Whitney and P. Jacquod, Phys. Rev. Lett. 94, 116801 (2005).
10. I. Adagideli and C. W. J. Beenakker, Phys. Rev. Lett. 89, 237002 (2002).
11. M. G. Vavilov and A. I. Larkin, Phys. Rev. B 67, 115335 (2003).
12. P. G. Silvestrov, M. C. Goorden, and C. W. J. Beenakker, Phys. Rev. Lett. 90, 116801 (2003).
13. C. W. J. Beenakker, Lect. Notes Phys. 667, 131 (2005), cond-mat/0406018.
14. I. Adagideli, Phys. Rev. B 68, 233308 (2003).
15. S. Rahav and P. W. Brouwer, Phys. Rev. Lett. 95, 056806 (2005).
16. P. Jacquod and R. S. Whitney, cond-mat/0512662 (2005).
17. J. Tworzydło, A. Tajic, and C. W. J. Beenakker, Phys. Rev. B 69, 165318 (2004).
18. P. Jacquod and E. V. Sukhorukov, Phys. Rev. Lett. 92, 116801 (2004).
19. P. W. Brouwer and S. Rahav, cond-mat/0512095 (2005).
20. I. L. Aleiner and A. I. Larkin, Phys. Rev. E 55, 1243(R) (1997).
21. C. Tian and A. I. Larkin, Phys. Rev. B 70, 035305 (2004).
22. J. Tworzydło, A. Tajic, H. Schomerus, P. W. Brouwer, and C. W. J. Beenakker, Phys. Rev. Lett. 93, 186806 (2004).
23. M. C. Goorden, P. Jacquod, and C. W. J. Beenakker, Phys. Rev. B 72, 064526 (2005).
24. H. Schomerus and P. Jacquod, J. Phys. A 38, 10663 (2005).
25. H. Schomerus and J. Tworzydło, Phys. Rev. Lett. 93, 154102 (2004).
26. Y. M. Blanter and M. Büttiker, Phys. Rep. 336, 1 (2000).
27. R. Blümel and U. Smilansky, Phys. Rev. Lett. 60, 477 (1988).
28. M. L. Polianski and P. W. Brouwer, J. Phys. A 36, 3215.
(2003).

29 P. W. Brouwer, A. Lamacraft, and K. Flensberg, Phys. Rev. B 72, 075316 (2005).
30 R. A. Jalabert, H. U. Baranger, and A. D. Stone, Phys. Rev. Lett. 65, 2442 (1990).
31 H. U. Baranger, R. A. Jalabert, and A. D. Stone, Phys. Rev. Lett. 70, 3876 (1993).
32 H. U. Baranger, R. A. Jalabert, and A. D. Stone, Chaos 3, 665 (1993).
33 P. Braun, S. Heusler, S. Müller, and F. Haake, cond-mat/051192 (2005).
34 S. Heusler, S. Müller, P. Braun, and F. Haake, Phys. Rev. Lett. 96, 066804 (2006).
35 S. Müller, S. Heusler, P. Braun, F. Haake, and A. Altland, Phys. Rev. Lett. 93, 014103 (2004).
36 S. Müller, S. Heusler, P. Braun, F. Haake, and A. Altland, Phys. Rev. E 72, 046207 (2005).

37 K. Richter and M. Sieber, Phys. Rev. Lett. 89, 206801 (2002).
38 D. Spehner, J. Phys. A 36, 7269 (2003).
39 M. Turek and K. Richter, J. Phys. A 36, L455 (2003).
40 Strictly speaking, one should also include trajectories for which transverse momenta components are opposite, cf. Eq 12. However in that case the summation over trajectories contains a rapidly oscillating phase, so that their net contribution to the sum vanishes in the classical limit, see, e.g., Refs. 37, 41.
41 S. Rahav and P. W. Brouwer, Phys. Rev. Lett. 96, 196804 (2006).
42 C. Tian, A. Altland, and P. W. Brouwer, cond-mat/0605051 (2006).
43 S. Rahav and P. W. Brouwer, unpublished (2006).