Q-SYSTEMS AS CLUSTER ALGEBRAS II: CARTAN MATRIX OF FINITE TYPE AND THE POLYNOMIAL PROPERTY

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ABSTRACT. We define the cluster algebra associated with the Q-system for the Kirillov-Reshetikhin characters of the quantum affine algebra $U_q(\hat{g})$ for any simple Lie algebra $g$, generalizing the simply-laced case treated in [Kedem 2007]. We describe some special properties of this cluster algebra, and explain its relation to the deformed Q-systems which appeared on our proof of the combinatorial-KR conjecture. We prove that the polynomiality of the cluster variables in terms of the “initial cluster seeds”, including solutions of the Q-system, is a consequence of the Laurent phenomenon and the boundary conditions. We also give a formulation of both Q-systems and generalized T-systems as cluster algebras with coefficients. This provides a proof of the polynomiality of solutions of generalized T-systems with appropriate boundary conditions.

1. Introduction

The Q-system is a recursion relation for the characters of certain finite-dimensional representations of the quantum affine algebra $U_q(\hat{g})$ or the Yangian $Y(g)$, where $g$ is a simple Lie algebra. Q-systems were introduced in [15] for the classical algebras. They were later generalized by [10] for the exceptional algebras and later to more complicated situations, such as twisted quantum affine algebras [11] and double affine algebras [12].

The special modules related to Q-systems are called [17, 16] Kirillov-Reshetikhin modules. The fact that their characters satisfy the Q-system was proved by Kirillov and Reshetikhin [15] for $g = A_r$, by Nakajima [18] for the simply-laced algebras and by Hernandez [13, 12] in further generality.

Cluster algebras were introduced by Fomin and Zelevinsky [4] in 2000, and are a very general algebraic tool which has since been applied in various algebraic, combinatorial and geometric contexts. In particular, they have been used to study Y-systems [6], which are related to Q-systems in the sense that both can be derived from T-systems [17]. The T-systems are a consequence of the fusion relations for Yangian or quantum algebra modules.

The form of the Q-system suggests that it should be possible to recast it as part of a cluster algebra. The first step in this reformulation, for the case where $g$ is simply-laced, was derived in [14]. In the current article, we give a generalization of this case to non-simply laced $g$. In this formulation, the Q-system appears in a very simple and easily generalizable form. We note that it does not appear to be directly related to the cluster algebra coming from the Y-system studied by Fomin and Zelevinsky [6].

In [3], we proved a combinatorial identity (“the $M = N$” conjecture of [10]) which implies the proof of the combinatorial Kirillov-Reshetikhin conjecture. In our proof, we introduced what we called the deformed Q-system, depending on an increasing number of formal variables. A specialization of this system can be expressed as the Q-system with general boundary conditions.

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Our proof of the $M = N$ identity depends crucially on the fact that the KR-characters are polynomials in the fundamental KR-characters. This can be rephrased in terms of the cluster algebra, by considering a specialization of the initial cluster variables to the special point which gives the KR-characters at the other nodes of the cluster graph. We call this specialization of the initial parameters the KR point.

On the other hand, it is known that cluster variables obey the Laurent phenomenon [5]. We show that, under the specialization of the cluster variables to the KR point, this becomes what we called in [14] the strong Laurent phenomenon. That is, the cluster variables on the entire cluster graph, not just the subgraph corresponding to the $Q$-system, are polynomials in the initial cluster variables.

The paper is organized as follows. We recall the definition of $Q$-systems for any simple Lie algebra in Section 2, as well as the definition of normalized cluster algebras without coefficients. Section 3 deals with the formulation of each $Q$-system as a subgraph in a cluster algebra. In Section 3.1, we review the results of [14] about the formulation of $Q$-systems as cluster algebras for simply-laced Lie algebras. In Sections 3.2 and 3.3, we formulate the cluster algebras corresponding to the non-simply laced simple Lie algebras. In Section 4, we prove that the special boundary conditions which give solutions of the $Q$-system as characters of Kirillov-Reshetikhin modules imply the polynomiality of the cluster variables as functions of the seed variables at the boundary node. Section 5 is a discussion of the results. Appendix A is a reformulation of the $Q$-system as a cluster algebra with coefficients, addressing the technical point of subtraction-free expressions. In the body of the paper, this is done through a renormalization of the cluster variables, which is not always generalizable. Appendix B is a discussion of the formulation of generalized $T$-systems as cluster algebras, with two main examples, both of which have the polynomiality property.

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2. Definitions

2.1. The $Q$-system.

2.1.1. KR-modules. Let $g$ be a simple Lie algebra of rank $r$ and let $I_r = \{1, ..., r\}$ be the parameterizing set for the simple roots of $g$. Let $C$ be the Cartan matrix of $g$.

For any such algebra, there is a corresponding quantum affine algebra $U_q(\hat{g})$ and a Yangian algebra $Y(g)$. In the study of the finite-dimensional modules of these algebras, there are certain special modules, called Kirillov-Reshetikhin (KR) modules [15]. These can be defined in terms of their Drinfeld polynomials [2], for example, which have a particularly simple form for these modules.

The modules are parameterized by $g$-highest weights of the form $k\omega_\alpha$, that is, a multiple of one of the fundamental weights, as well as a complex parameter $\zeta$. We refer to such a module by $W_{\alpha,k}(\zeta)$ where $\alpha \in I_r$, $k \in \mathbb{Z}_+$, and $\zeta \in \mathbb{C}^*$. These are finite-dimensional modules, which...
are analogs of evaluation modules of affine algebras, which are obtained from inducing from the irreducible representation of the finite-dimensional subalgebra $\mathfrak{g}$, which does not depend on $\zeta$. Unlike such evaluation modules, however, they are not necessarily irreducible when restricted to the subalgebra $U_q(\mathfrak{g}) \subset U_q(\hat{\mathfrak{g}})$ or $\mathfrak{g} \in \hat{Y}(\mathfrak{g})$, except in special cases, such as when $\mathfrak{g} \neq A_r$. The idea is that $W_{a,k}(\zeta)$ is the “smallest” finite-dimensional $\mathfrak{g}[t]$-module, with largest component $V(k\omega_{\alpha})$ (the irreducible $\mathfrak{g}$-module), which deforms to a Yangian module.

KR-modules have been extensively studied in recent years, and much is known about their properties. In particular, their decomposition, and the decomposition of their tensor products, into irreducible $U_q(\mathfrak{g})$-modules is known in terms of explicit multiplicity formulas. These formulas can be derived [15, 10, 13, 3] from the $Q$-system, a recursion relation for the characters $Q_{\alpha,k} = \text{char} \left( \text{Res}_{U_q(\hat{\mathfrak{g}})}^{U_q(\mathfrak{g})} W_{a,k}(\zeta) \right)$, in addition to a certain asymptotic property (see (C) of Theorem 8.1 of [10]). We leave out the parameter $\zeta$ in what follows, because it only affects the action of the affine part of the algebra and not $U_q(\mathfrak{g})$ or the characters $Q_{\alpha,k}$.

The formulas for the decomposition coefficients and the statement that the characters of KR-modules satisfy the $Q$-system were both known (until they were proven) as the Kirillov-Reshetikhin conjecture.

2.1.2. $Q$-systems. We introduce the $Q$-system in two steps. First, let us write down a completely general relation for a family of variables

$$\{Q_{\alpha,k} \mid \alpha \in I_r, k \in \mathbb{Z}\}.$$ 

Let $\mathfrak{g}$ be a simple Lie algebra with Cartan matrix $C$. We denote the simple roots $\alpha$ by the corresponding integers in $I_r = \{1, \ldots, r\}$. The $Q$-system associated with $\mathfrak{g}$ is a recursion relation of the form

$$Q_{\alpha,k+1} Q_{\alpha,k-1} = Q_{\alpha,k}^2 \prod_{\beta \sim \alpha} T^{(\alpha,\beta)}_k, \quad \alpha \in I_r, \quad k \in \mathbb{Z},$$

where $\alpha \sim \beta$ means that $\alpha$ is connected to $\beta$ in the Dynkin diagram, and

$$T^{(\alpha,\beta)}_k = \prod_{i=0}^{[C_{\alpha,\beta}]-1} Q_{\beta,\lfloor t_{k+i} \alpha \rfloor},$$

where $[a]$ is the integer part of $a$. Here, $t_\alpha$ are the integers which symmetrize the Cartan matrix. Namely, $t_r = 2$ for $B_r$, $t_\alpha = 2$ ($\alpha < r$) for $C_r$, $t_3 = t_4 = 2$ for $F_4$ and $t_2 = 3$ for $G_2$, and $t_\alpha = 1$ otherwise.

This is a recursion relation which can be used to express any $Q_{\alpha,k}$ in terms of $2r$ initial data points. Usually, one chooses the initial conditions $\{Q_{\alpha,0}, Q_{\alpha,1} \mid \alpha \in I_r\}$. Then

$$Q_{\alpha,k+1} = \frac{Q_{\alpha,k}^2 \prod_{\beta \sim \alpha} T^{(\alpha,\beta)}_k}{Q_{\alpha,k-1}}, \quad k \geq 1,$$

and

$$Q_{\alpha,k-1} = \frac{Q_{\alpha,k}^2 \prod_{\beta \sim \alpha} T^{(\alpha,\beta)}_k}{Q_{\alpha,k+1}}, \quad k \leq 0.$$
Consider Equation \((2.3)\) in the particular case of the initial conditions that for all \(\alpha\), \(Q_{\alpha,0} = 1\) and \(Q_{\alpha,1} = \text{Char} W_{\alpha,1}\), the character of the fundamental Kirillov-Reshetikhin module, with highest \(\mathfrak{g}\)-weight \(\omega_\alpha\). In this case, the solutions \(Q_{\alpha,k}\) of \((2.3)\) are known to be the characters of KR-modules \(W_{\alpha,k}\) with highest \(\mathfrak{g}\)-weight \(k\omega_\alpha\) \([18, 13]\). The recursion relation \((2.3)\) with these initial conditions is what is usually called the \(Q\)-system in the literature.

We do not impose these initial conditions in the definition of the cluster algebra. As we will see, this is a special (singular) point in the parameter space for this algebra. One way to see that it is singular is that equation \((2.4)\) cannot be used, in this case, to define all the variables \(Q_{\alpha,k}\) with \(k < 0\) because, for example, \(Q_{\alpha,-1} = 0\) in this case.

More general initial conditions, of the form
\[(2.5)\]

\[Q_{\alpha,0} = b_\alpha, \quad Q_{\alpha,1} = a_\alpha\]

for some formal variables \(a_\alpha, b_\alpha\) give a solution of (a special case of) the “deformed \(Q\)-system” introduced in \([3]\). We will explain this point in Section 5.

2.1.3. Normalized \(Q\)-systems. In this paper we deal with cluster algebras without coefficients. To see that it is possible to reformulate the \(Q\)-system in this way, a version of the following Lemma was given in \([14]\):

**Lemma 2.1.** There exist complex numbers \(\epsilon_1, ..., \epsilon_r\) (all of them roots of 1) such that the variables \(R_{\alpha,i} = \epsilon_\alpha Q_{\alpha,i}\) satisfy the normalized \(Q\)-system
\[(2.6)\]

\[R_{\alpha,k+1}R_{\alpha,k-1} = R_{\alpha,k}^2 + \prod_{\beta \sim \alpha} \tilde{T}^{(\alpha,\beta)}_{k}\]

where
\[\tilde{T}^{(\alpha,\beta)}_k = \prod_{i=0}^{\lfloor C_{\alpha,\beta} \rfloor} R_{\beta,1}^{t_{\alpha}(1+i) - t_{\alpha}(k+i)}\]

**Proof.** From Equation \((2.2)\) we note that the total degree of \(Q_{\beta,j'}\) (for various \(j'\)) in \(\mathcal{T}^{(\alpha,\beta)}_k\) is always \(\lfloor C_{\alpha,\beta} \rfloor\), where \(C\) is the Cartan matrix of \(\mathfrak{g}\). Therefore \(R_{\alpha,k} = \epsilon_\alpha Q_{\alpha,k}\) satisfy the following system:

\[R_{\alpha,k+1}R_{\alpha,k-1} = R_{\alpha,k}^2 - \left( \prod_{\beta} \epsilon_{\beta}^{C_{\alpha,\beta}} \right) \prod_{\beta \sim \alpha} \tilde{T}^{(\alpha,\beta)}_k\]

where \(\tilde{T}\) is just \(T\) with the variables \(Q\) replaced by \(R\).

Thus, if the variables \(\epsilon_\alpha\) satisfy

\[\prod_{\beta=1}^{r} \epsilon_{\beta}^{C_{\alpha,\beta}} = -1\]

then \((2.6)\) is satisfied. But since \(C\) is a Cartan matrix of finite type, it is invertible, and this system has a solution. Namely if we set \(\epsilon_{\beta} = e^{i\pi \mu_\beta}\), then \(\mu_\alpha = \sum_\beta C^{-1}_{\beta,\alpha}\). \(\square\)

In fact, we find \(\epsilon_4^4 = 1\) in general. We note that this choice of normalization is done in order to make contact with cluster algebras without coefficients below. We do not believe that it is essential for the final result.

We will refer to \((2.6)\) as the normalized \(Q\)-system or simply the \(Q\)-system when it is clear that we are considering the evolution of the variables \(R\).
2.2. Cluster algebras. We use here the definition of [7] to define a cluster algebra without coefficients.

**Definition 2.2.** A cluster algebra. Fix $n$ and define a labeled $n$-ary tree $T_n$, with nodes labeled by some parameter $t$ and $n$ edges labeled $1,\ldots,n$ emanating from each node.

To each node $t$ we associate a seed $(x,B)_t$, where $x = (x_1,\ldots,x_n)$ (the cluster variables) and $B$ (the mutation matrix) is a skew-symmetric $n \times n$ matrix with integer coefficients.

If the node $t$ is connected to the node $t'$ by an edge labeled by $k$, then

$$
\mu_k(x_{t'}) = \mu_k(x_t, B_t) = \begin{cases} 
  x_i, & i \neq k; \\
  x_i^{-1}(\prod_j x_j^{[B_{jk}]_+} + \prod_j x_j^{-[B_{jk}]_+}), & i = k.
\end{cases}
$$

(2.7)

$$
\mu_k(B_{ij}) = \begin{cases} 
  -B_{ij}, & i = k \text{ or } j = k; \\
  B_{ij} + \text{sgn}(B_{ik})[B_{ik}B_{kj}]_+, & \text{otherwise}.
\end{cases}
$$

(2.8)

where $[n]_+$ denotes the positive part of $n$ and

$$\text{sgn}(n) = \begin{cases} 
  0, & n = 0; \\
  1, & n > 0; \\
  -1, & n < 0.
\end{cases}$$

One can check the following properties of cluster mutations:

- The mutations (2.8) preserve the skew-symmetry of $B$.
- If $B_{ij} = 0$ then the mutations $\mu_i$ and $\mu_j$ commute.
- If $B_{ij} = 0$ and $k,l$ are two integers distinct from both $i$ and $j$, then

$$
\mu_i \circ \mu_j(B_{kl}) = B_{kl} + \text{sgn}(B_{ki})[B_{ki}B_{kj}]_+ + \text{sgn}(B_{kj})[B_{kj}B_{jl}]_+.
$$

(2.9)

This statement can be extended to any finite sequence of commuting mutations.

- Mutations are defined in such a way that $\mu_2^2 = 1$. Thus, cluster graphs are not oriented.

**Definition 2.3.** We define $A$ to be the family of variables $\{R_{\alpha,i}\}_{i \in \mathbb{Z}, \alpha \in I}$ which satisfy the normalized $Q$-system (2.6).

In what follows we are interested in subgraphs of $T_n$ which encode the $Q$-system, that is, the cluster variables at the nodes are elements of $A$ and the mutations between nodes are $Q$-system evolutions.

More precisely, we will define the quotient graphs $G_\theta$, by identifying nodes corresponding to the same cluster variables and mutation matrices. These are infinite graphs, but they have certain periodicity properties and are therefore easy to describe explicitly.

Our starting goal is to find a graph $G_\theta$ for each algebra, and the corresponding cluster variables, which encode the full $Q$-system, in the following sense: The union over all nodes of $G_\theta$ of the cluster variables is the full set $A$.

Of course once the seed $(x,B)$ is given at any node $t$, then the cluster algebra is defined, that is, the cluster at any other node is defined via the evolution equations. Therefore, our definition gives a cluster algebra on the full tree $T_n$. We will have something to say about special properties of the cluster variables on $T_n$ in Section 4 but thus far they have no representation-theoretical interpretation to our knowledge.
This definition of $\mathcal{G}_g$ allows us to encode the $Q$-system inside a cluster algebra. The graphs $\mathcal{G}_g$ which we describe are usually subgraphs of a larger component of $\mathbb{T}_n$ with the same property, and therefore our definition is not unique. There may be other nodes of $\mathbb{T}_n$ with cluster variables in $\mathcal{A}$ which are not in $\mathcal{G}_g$. This happens if the rank of $g$ is greater than two. We will give the example of $A_3$ to make this point clearer in the discussion.

3. The $Q$-systems as cluster algebras for simple $g$

Let $C$ be a Cartan matrix of a simple Lie algebra $g$ of rank $r$, and let $B$ be the $2r \times 2r$ matrix defined in the block form:

$$B = \begin{pmatrix} A & -C^t \\ C & 0 \end{pmatrix},$$

where $A = C^t - C$. This is a skew-symmetric matrix of integers.

For each $k \in \mathbb{Z}$, define $x[k] = (x_1, \ldots, x_{2r})$, where

$$x_\alpha = \begin{cases} R_{\alpha,2t_\alpha k}, & \alpha \leq r \\ R_{\alpha-r,2t_\alpha k+1}, & r < \alpha \leq 2r \end{cases}$$

Here, $t_\alpha = 2$ for the short roots of $B_r$, $C_r$ and $F_4$, $t_\alpha = 3$ for the short root of $G_2$, and $t_\alpha = 1$ otherwise.

In our cluster algebra graph, nodes labeled by $k \in \mathbb{Z}$ have associated with them cluster $(x[k], B)$. We will prove the following theorem:

**Theorem 3.1.** There exists a cluster graph $\mathcal{G}_g$, which includes all nodes labeled by $k \in \mathbb{Z}_+$, with corresponding cluster variables $x[k]$ as in (3.2) and $B$ as in (3.1), such that all mutations in the graph are $Q$-system evolutions. Moreover, the set $\mathcal{A}$ is equal to the union of the cluster variables over all nodes in $\mathcal{G}_g$.

We will describe such a graph for each algebra $g$ below.

In all cases considered below, if the mutations $\{\mu_{i_1}, \ldots, \mu_{i_n}\}$ commute starting from node $t$ (that is, if $B_{t_{i_k},t_{j_l}} = 0$ for all $k, l$ at the node $t$), then we may act with these mutations in any order, as long as they are distinct, and nodes corresponding to acting with a certain set of mutations, regardless of order, can be identified with each other in the quotient graph $\mathcal{G}_g$ (this is similar in spirit to the bipartite graph of [7]). These mutations give rise to an n-simplex (e.g. a cube in the case $n = 3$) in $\mathcal{G}_g$.

We will in general describe only the cluster variables reached via certain compound mutations $\mu^{(i)} = \mu_{i_1} \circ \cdots \circ \mu_{i_n}$, where the corresponding mutations commute. The cluster graph $\mathcal{G}_g$ is recovered by replacing a compound mutation with the corresponding n-simplex.

3.1. The simply laced case. This case was considered in [14]. We review the results here in order to introduce the notations.

If $g$ is simply-laced, the Cartan matrix is symmetric, $C^t = C$, and therefore the matrix $A$ vanishes and $B$ is block-diagonal:

$$B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}.$$

The graph $\mathcal{G}_g$ has a particularly simple form in this case. It includes the special nodes $k$ ($k \in \mathbb{Z}_+$) with cluster variables

$$x[k] = (R_{1,2k}, \ldots, R_{r,2k}, R_{1,2k+1}, \ldots, R_{r,2k+1}).$$
In addition, it includes the special nodes \( k' \), which are reached from \( k \) by the compound mutation 
\[
\mu^{(1)} = \mu_1 \circ \cdots \circ \mu_r.
\]
These commute due to the block-diagonal structure of \( B \). The following was proved in [14]:

**Theorem 3.2.** For \( k \in \mathbb{Z}_+ \), Let \( x[k] \) be as in (3.4) and \( B \) be the matrix in (3.3). Then

1. At the node \( k \), the mutations \( \mu_1, \ldots, \mu_r \) commute among themselves, and evolution according to these mutations corresponds to the \( Q \)-system evolutions \( R_{\alpha, 2k} \mapsto R_{\alpha, 2k+2}, \alpha \in I_r \).
2. At the same node, the mutations \( \mu_{r+1}, \ldots, \mu_{2r} \) also commute among themselves, and applying them to the variables \( x[k] \) corresponds to the \( Q \)-system evolution \( R_{\alpha, 2k+1} \mapsto R_{\alpha, 2k-1}, \alpha \in I_r \).
3. Define the compound mutation
\[
\mu^{(I_r)} = \mu_1 \circ \cdots \circ \mu_r
\]
and denote the node reached by this compound mutation acting on the cluster variables at \( k \) by \( k' \). Then the mutation matrix \( B' = \mu^{(I_r)}(B) \) is
\[
B' = -B = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}.
\]
and
\[
\mu^{(I_r)}(x[k]) = (R_{1, 2k+2}, \ldots, R_{r, 2k+2}, R_{1, 2k+1}, \ldots, R_{r, 2k+1}) = x[k'].
\]
4. At the point \( k' \), the mutations \( \mu_{r+1}, \ldots, \mu_{2r} \) commute among themselves, the evolution according to these mutations corresponds to the \( Q \)-system evolution \( R_{\alpha, 2k+1} \mapsto R_{\alpha, 2k+3}, \alpha \in I_r \).
5. The compound mutation
\[
\mu^{(I_r')} = \mu_{r+1} \circ \cdots \circ \mu_{2r}
\]
maps the seed at \( k' \) to the seed at \( k+1 \) with cluster variables \( x[k+1] \) and mutation matrix \( B \).

This theorem is proved by direct calculation.

Let us illustrate some examples of graphs which correspond to \( G_0 \).

**Example 3.3.** The graph \( G_{sl_2} \) is linear:

\[
\begin{array}{ccccccc}
\cdots & 2 & 1 & 2 & 1 & \cdots \\
(k-1)' & k & k' & k+1 & (k+1)' \\
\end{array}
\]

The cluster variables at the primed and unprimed nodes are described by \( x[k] = (R_{1,2k}, R_{1,2k+1}) \) and \( x[k'] = (R_{1,2k+2}, R_{1,2k+1}) \) of Theorem 3.2. The numbers above each edge correspond to the mutations \( \mu_1 \) and \( \mu_2 \). This is the full tree \( T_2 \).

**Example 3.4.** The graph \( G_{sl_3} \) is slightly more interesting:
The cluster algebra has four variables, and four possible mutations, which are the labels along the edges. The special nodes $k$ and their primed versions appear on the center line of the graph. The other nodes have intermediate cluster variables.

The full cluster algebra graph of $\mathfrak{sl}_3$ would include, for example, an edge labeled 4 emanating from the node $\mu_1(k)$. However the cluster mutation in this direction is not a $Q$-system evolution.

**Example 3.5.** The graph $G_{\mathfrak{sl}_4}$ described in Theorem 3.2 is as follows:

The graphs $G_{\mathfrak{g}}$ which we describe are not, in general, the complete subgraph of $T_n$ corresponding to $Q$-system evolutions, except when $\text{rank}(\mathfrak{g}) \leq 2$. We chose the graphs $G_{\mathfrak{g}}$ so that the union of all cluster variables at their nodes corresponds to the complete set of normalized characters of the $KR$-modules. These graphs are not unique, and their extensions are of interest also (see section 5.2).

Clearly, the cluster graphs $G_{\mathfrak{g}}$ are difficult to draw as the rank becomes high. Therefore, in treating the higher-rank algebras, it is useful to adopt a pictorial notation which is similar in spirit to the bipartite graph notation of [7].

First, we define the notion of a compound mutation to be a product of $n$ distinct commuting mutations. Let $I = \{i_1, \ldots, i_n\}$ be the index set of these mutations, and define $\mu(I) = \mu_{i_1} \circ \cdots \circ \mu_{i_n}$. The order of mutations is irrelevant by definition. We may also use the notation $\mu(I) = \prod_{i \in I} \mu_i$. Obviously, since $\mu_i^2 = 1$, we preserve the property that $(\mu(I))^2 = 1$ for the product of commuting notations.

The edge:
corresponds to the compound mutation \( t' = \mu^{(1)}(t) \). In the graph \( G_{\mathfrak{g}} \), this corresponds to an \( n \)-simplex with node \( t \) at one end and node \( t' \) at the other. It is a diamond in the case for \( G_{\mathfrak{sl}_3} \) or a cube in the case of \( G_{\mathfrak{sl}_4} \).

Thus the graph \( G_{\mathfrak{g}} \) for \( \mathfrak{g} \) simply-laced with the sequence of nodes and compound mutations as follows:

\[
\begin{array}{ccccccc}
\cdots & \Pi' & \Pi & \Pi' & \Pi & \cdots \\
(k-1)' & k & k' & k+1 & (k+1)'
\end{array}
\]

where \( \Pi = I_r \) and \( \Pi' = \{r+1, \ldots, 2r\} \). The cluster variables and mutation matrices at the nodes \( k, k' \) are given by Theorem 3.2.

3.2. The \( Q \)-system for the algebras \( B_r, C_r, F_4 \). In the case of non-simply laced algebras, we describe the graph \( G_{\mathfrak{g}} \) for algebras of types \( B_r, C_r, F_4 \) separately from \( G_2 \).

Let \( \mathfrak{g} \) be a simple algebra of type \( B_r, C_r, F_4 \). In the Dynkin diagram of \( \mathfrak{g} \), let \( \ell \) be the long root connected to the short root \( s \):

\[
\begin{align*}
\text{\( B_r \):} & \quad \begin{array}{c}
1 & 2 & \cdots & r-2 & r-1 & r
\end{array} \\
\text{\( C_r \):} & \quad \begin{array}{c}
1 & 2 & \cdots & r-2 & r-1 & r
\end{array} \\
\text{\( F_4 \):} & \quad \begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{align*}
\]

In each case, in the normalized \( Q \)-system (2.6) we have

\[
\begin{align}
\bar{\gamma}_{2k+1}^{(s,\ell)} &= R_{\ell,k}R_{\ell,k+1} = R_{\ell,k}^{[\alpha / 2]}R_{\ell,k+1}^{[\alpha / 2]}; \quad (3.5) \\
\bar{\gamma}_{2k}^{(s,\ell)} &= R_{\ell,k}^2 = R_{\ell,k}^{[\alpha / 2]}; \quad (3.6) \\
\bar{\gamma}_{2k}^{(\ell,s)} &= R_{s,2k} = R_{s,2k}^{[\alpha / 2]}. \quad (3.7)
\end{align}
\]

In case \((\alpha, \beta)\) are not the pair \((\ell, s)\) or \((s, \ell)\) then

\[
\bar{\gamma}_k^{(\alpha, \beta)} = R_{\beta,k}^{[\alpha / 2]}. \quad (3.8)
\]

Define the sets

- \( \Pi' = I'_r = \{r+1, \ldots, 2r\} \);
- \( \Pi_< \): The subset in \( I_r \) corresponding to the short roots of \( \mathfrak{g} \);
- \( \Pi_\geq \): The subset in \( I_r \) corresponding to the long roots of \( \mathfrak{g} \);
- \( \Pi'_< = \{\alpha + r \mid \alpha \in \Pi_<\} \);
- \( \Pi'_\geq = \{\alpha + r \mid \alpha \in \Pi_\geq\} \);

**Theorem 3.6.** The graph \( G_{\mathfrak{g}} \) is obtained from the sequence
Here, the cluster at the node labeled $k$ is $x[k]$ as in (3.2) with mutation matrix $B$ as in (3.1). All the compound mutations are products of mutually commuting mutations, and the cluster variables at each node are in $A$.

To prove this Theorem, we need only list the cluster variables $(x[k^{(i)}], B^{(i)})$ at $k^{(i)}$—at intermediate nodes, the cluster variables are a mixture of cluster variables from $k^{(i-1)}$ and $k^{(i)}$. Note that in this subsection, we read the superscript of $k$ modulo 5, so $k^{(0)} := k$ etc.

We then observe that the relevant columns of $B^{(i)}$ describe $Q$-system evolutions when applied to the relevant entries of $x[k^{(i)}]$. We use the fact that when $\mu_i$ and $\mu_j$ commute, the $j$th column of $\mu_i(B)$ is equal to the $j$th column of $B$.

We recall that at the node $k$, we have $x[k] = (x_1, \ldots, x_{2r})$, where
\[
x_\alpha = \begin{cases} R_{\alpha,2t_k}, & \alpha \in I_r; \\ R_{\alpha,2t_k+1}, & \text{otherwise.} \end{cases}
\]
Moreover, according to (3.1), at the node $k$ we have
\[
B = \begin{pmatrix} A & -C^t \\ C & 0 \end{pmatrix}.
\]
Here, the submatrix $A$ is no longer the zero matrix. Instead, it has zeros everywhere except for two entries, $A_{\ell,s} = -1$ and $A_{s,\ell} = 1$. Thus, the mutations $\{\mu_\alpha : \alpha \in \Pi_< \}$ commute among themselves, as do the all mutations $\{\mu_i : i \in \Pi'\}$.

It is clear from the form of the $Q$-system (see Equations (3.5) and (3.8)) that the mutations $\{\mu_\alpha : \alpha \in \Pi_< \}$ describe the evolutions $\mu_\alpha : R_{\alpha,2k} \mapsto R_{\alpha,2k+2}$ ($\alpha \in \Pi_<)$, This is the same evolution as in the simply-laced case if $\alpha \neq s$, and for $\alpha = s$, it follows from (3.5).

Thus, edges labeled by $\alpha \in \Pi_$ connected to node $k$ lead to nodes in which all the clusters are in $A$. The compound mutation $\mu_i(B)$ leads to the node $k^{(1)}$.

(We also note that the mutations columns $r+1, \ldots, 2r$ describe the $Q$-system evolutions $R_{\alpha,2k+1} \mapsto R_{\alpha,2k-1}$, from Equations (3.6), (3.7) and (3.8); However due to the periodic structure of the graph, we do not need to consider this evolution separately).

### Lemma 3.7

1. The cluster variables at the node $k^{(1)}$ are $x[k^{(1)}] = (x_1, \ldots, x_{2r}) = \mu_{\Pi_<}(x[k])$ with
\[
x_\alpha = \begin{cases} R_{\alpha,2k}, & \text{if } \alpha \text{ is a short root;} \\ R_{\alpha,2k+2}, & \text{if } \alpha \text{ is a long root;} \\ R_{\alpha,2t_k+1}, & \text{if } \alpha > r; \end{cases}
\]
2. The mutation matrix $B^{(1)}$ at $k^{(1)}$ has the following form:
\[
B^{(1)} = \begin{pmatrix} \alpha & -C^t \\ D & 2A \end{pmatrix},
\]
where
\[
D_{ij} = \begin{cases} C_{ij}, & \text{if } i \text{ and } j \text{ are both long roots;} \\ -C_{ij}, & \text{if } j \text{ is short;} \\ 0, & \text{otherwise.} \end{cases}
\]
Proof. The first statement follows from the discussion above. The second statement involves several cases.

- Because the mutations $\mu_\alpha$ with $\alpha < r$ and $\alpha \in \Pi_<$ mutually commute, their effect on the columns and rows of $B$ indexed by $\{\alpha : \alpha \in \Pi_<\}$ is only to change their sign. Thus,

$$B^{(1)}_{i,\alpha} = -B^{(1)}_{\alpha,i} = -B_{i,\alpha}, \quad \alpha \in \Pi_<.$$  

This accounts for the appearance of $-A$ in the upper left hand corner of $B^{(1)}$, as well as the columns of the matrix $D$ which correspond to short roots.

- Let $\beta \in \Pi_>$ be a long root and consider the change column $\beta$ of $B$. If $i \notin \Pi_<$, then

$$B^{(1)}_{i,\beta} = B_{i,\beta} + \sum_{\alpha \in \Pi_<} \text{sgn}(B_{i,\alpha})[B_{i,\alpha}B_{\alpha,\beta}]_+.$$  

But $B_{\alpha,\beta} = 0$ unless $(\alpha, \beta) = (s, \ell)$. In that case, $B_{s,\ell} = 1$. We have that $B_{i,s} > 0$ if and only if $i = s + r$, in which case it is equal to 2. Thus, we have

$$B^{(1)}_{s+r,\ell} = 0,$$

and all other elements in the long-root columns of the lower left hand corner of $B^{(1)}$ are unchanged. This gives the matrix $D$ in this quadrant, and $-D^t$ in the upper right block of $B^{(1)}$ since mutations preserve skew-symmetry.

- Finally, elements in the lower right quadrant mutate as follows:

$$B^{(1)}_{i+r,j+r} = \sum_{\alpha \in \Pi_<} \text{sgn}(B_{i+r,\alpha})[B_{i+r,\alpha}B_{\alpha,j+r}]_+ = 2A.$$  

By examination of the matrix $B^{(1)}$ we conclude that at the node $k^{(1)}$, the mutations $\{\mu_\alpha : \alpha \in \Pi_>\}$ all commute among themselves.

Lemma 3.8. (1) The cluster variables $x[k^{(2)}] = (x_1, \ldots, x_{2r}) = \mu^{(\Pi_>)}(x[k^{(1)}])$ are

$$x_\alpha = \begin{cases} R_{2t,\alpha-k+2} & \alpha \leq r; \\ R_{2t,\alpha-k+1} & \alpha > r. \end{cases}$$

(2) The mutation matrix $B^{(2)}$ has the form

$$\begin{pmatrix} A & C - A \\ -C^t & -A \end{pmatrix}.$$  

Proof. Again, by inspection, the columns corresponding to long roots in the matrix $B^{(1)}$ correspond to the evolution of the $Q$-system according to Equations (3.7) and (3.8). This proves the first statement in the Lemma.

We look at the evolution of the matrix $B^{(1)}$ under $\mu^{(\Pi_>}$. 

- If $\alpha \in \Pi_>$ then $B^{(2)}_{i,\alpha} = -B^{(1)}_{i,\alpha}$. This accounts for the appearance of $A$ in the upper left-hand block of $B^{(2)}$ and for the columns corresponding to the long roots in the lower left-hand quadrant of $B^{(2)}$. 

• If $\alpha \in \Pi_<$ and $i > r$ then
\[
B^{(2)}_{i,\alpha} = B^{(1)}_{i,\alpha} + \sum_{\beta \in \Pi_>^+} \text{sgn}(B^{(1)}_{i,\beta})[B^{(1)}_{i,\beta}B^{(1)}_{\beta,\alpha}]_+
\]

The summation has a contribution only when $(\beta, \alpha) = (\ell, s)$ in which case $B^{(1)}_{\ell,s} = 1$. Then the only non-vanishing contribution occurs if $i = \beta + r$, in which case $B^{(1)}_{\beta+r,\beta} = 2$ and $B_{\ell+r,s} = 3$. This gives the matrix in the lower left quadrant of $B^{(2)}$.

• Finally we have
\[
B^{(2)}_{i+r,j+r} = 2A_{i,j} + \sum_{\alpha \in \Pi_>} \text{sgn}(B^{(1)}_{i+r,\alpha})[B^{(1)}_{i+r,\alpha}B^{(1)}_{\alpha,j+r}]_+ = 2A_{ij}.
\]

\[\square\]

At this point, it is evident that the mutations $\{\mu_{\alpha+r} : \alpha \in \Pi_<\}$ commute among themselves, due to the form of the matrix $2A$ appearing in the lower right quadrant of $B^{(2)}$. Moreover, the corresponding columns of $B^{(2)}$ give the $Q$-system evolution for $R_{\alpha,2k+1} \mapsto R_{\alpha,2k+3}$ if $\alpha$ is short. As a result we have

**Lemma 3.9.**  
1. The cluster variables at $k^{(3)}$ are $\mu^{(\Pi_<)}(x[k^{(2)}]) = x[k^{(3)}] = (x_1, ..., x_i)$ where
\[
x_{\alpha+r} = R_{\alpha,4k+3}, \quad \alpha \in \Pi_<,
\]
and the other cluster variables are unchanged from $x[k^{(2)}]$.

2. The mutation matrix $B^{(3)} = -B^{(1)}$

The proof is by a calculation analogous to the previous two lemmas.

The columns $\{\alpha : \alpha \in \Pi_<\}$ of $-B^{(1)}$ describe the mutations corresponding to the evolutions $R_{\alpha,4k+2} \mapsto R_{\alpha,4k+4}$ as they are identical to the corresponding columns in the matrix $B$. The mutations corresponding to the short roots again commute. We have the compound evolution $\mu^{(\Pi_<)}$ which brings us to the node $k^{(4)}$, with

**Lemma 3.10.**  
1. The cluster variables at the point $k^{(4)}$ are $\mu^{(\Pi_<)}(x[k^{(3)}]) = x[k^{(4)}] = (x_1, ..., x_{2r})$, with
\[
x_\alpha = \begin{cases} 
R_{\alpha,2\alpha(k+2)}, & \alpha \leq r \\
R_{\alpha,2\alpha k+1}, & \alpha - r \in \Pi_> \\
R_{\alpha,2\alpha k+3}, & \alpha - r \in \Pi_<.
\end{cases}
\]

2. The mutation matrix at the node $k^{(4)}$ is $B^{(4)} = -B$.

Again, the proof is analogous to that of the previous lemmas.

Finally we note that at the node $k^{(4)}$, not only do the mutations $\{\mu_{\alpha+r} : \alpha \in \Pi\}$ all commute among themselves, the corresponding columns of $B^{(4)}$ describe the evolution of the $Q$-system for any $R_{\alpha,i}$ with $i$ odd. Therefore the compound evolution $\mu^{(\Pi)}$ makes sense at this node. The node $k^{(5)}$ is reached from node $k^{(4)}$ by acting with $\mu^{(\Pi)}$. In fact, we find that the node $k^{(5)}$ is the node $k+1$ in $S_6$.

**Lemma 3.11.**  
1. The cluster variables $x[k^{(5)}] = \mu^{(\Pi)}(x[k^{(4)}]) = x[k+1]$, where $x[k]$ for any $k \in \mathbb{Z}$ was defined in equation \((3.2)\).

2. The mutation matrix at the node $k^{(5)}$ is $B$. 


The proof is by direct calculation. Thus, there is a periodicity to the graph $G$, where the matrix $B^{(i)}$ is repeated along the “band” corresponding to the subgraph every fifth step.

3.3. The $Q$-system for $G_2$ as a cluster algebra. In this case the normalized $Q$-system (2.6) has terms $\mathcal{T}_{\alpha,k}$ as follows:

$$
\tilde{T}_{1,2}^{(1,2)} = R_{2,3k}, \\
\tilde{T}_{3k}^{(2,1)} = R_{1,k}^3, \\
\tilde{T}_{3k+1}^{(2,1)} = R_{1,k}^2 R_{1,k+1}, \\
\tilde{T}_{3k+2}^{(2,1)} = R_{1,k} R_{1,k+1}^2.
$$

Again we will describe a graph $G_{G_2}$, a quotient graph of a subgraph of $T_4$, which contains at its nodes cluster variables with all the normalized characters $\{R_{\alpha,k} : \alpha \in I_2, k \in \mathbb{Z}_+\}$.

The graph $G_{G_2}$ contains all nodes labeled by $k \in \mathbb{Z}_+$, and also the nodes $k^{(i)}$ with $i = 1, \ldots, 6$. At these nodes we have cluster variables $x[k^{(i)}]$ and mutation matrices $B^{(i)}$. We will describe the clusters at those nodes explicitly. As in the other cases, we choose to evolve along nodes of the graph when the corresponding mutations describe one of the $Q$-system equations. The graph $G_{G_2}$ is the following graph:

With the notation of the previous subsection, we can also depict this graph as follows:

The following can be easily verified by explicit calculation:
Theorem 3.12. The mutations along the graph $\mathcal{G}_{G_2}$ describe the evolution of the $Q$-system of $G_2$, the cluster variables at each point $k^{(i)}$ are defined as follows:

| $i$ | $x[k^{(i)}]$ | $B^{(i)}$ |
|-----|--------------|-----------|
| 0   | $(R_{1,2k}, R_{2,6k}, R_{1,2k+1}, R_{2,6k+1})$ | $\begin{pmatrix} 0 & -2 & -2 & 3 \\ 2 & 0 & 1 & -2 \\ 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \end{pmatrix}$ |
| 1   | $(R_{1,2k}, R_{2,6k+2}, R_{1,2k+1}, R_{2,6k+1})$ | $\begin{pmatrix} 0 & 2 & -2 & -1 \\ -2 & 0 & -1 & 2 \\ 2 & 1 & 0 & -2 \\ 1 & -2 & 2 & 0 \end{pmatrix}$ |
| 2   | $(R_{1,2k}, R_{2,6k+2}, R_{1,2k+1}, R_{2,6k+3})$ | $\begin{pmatrix} 0 & -C \\ C' & -A \end{pmatrix}$ |
| 3   | $(R_{1,2k+2}, R_{2,6k+4}, R_{1,2k+1}, R_{2,6k+3})$ | $-B^{(2)}$ |
| 4   | $(R_{1,2k+2}, R_{2,6k+4}, R_{1,2k+1}, R_{2,6k+5})$ | $-B^{(1)}$ |
| 5   | $(R_{1,2k+2}, R_{2,6k+6}, R_{1,2k+1}, R_{2,6k+5})$ | $-B$ |
| 6   | $x[k+1] = (R_{1,2k+2}, R_{2,6k+6}, R_{1,2k+3}, R_{2,6k+7})$ | $B$ |

Note that $k^{(0)} := k$ and $k^{(6)} = k + 1$ as the cluster variables at that node correspond to the point $k + 1$.

To prove this theorem, one simply compares the $Q$-system evolution along in the direction of the label of the edge emanating from the node $k^{(i)}$, and the corresponding columns of the mutation matrices at the points $k^{(i)}$. That is, if an edge labeled $n$ emanates from $k^{(i)}$ in the graph $\mathcal{G}_{G_2}$, then the $n$th variable of $x[k^{(i)}]$ satisfies the $Q$-system described by column $n$ of $B^{(i)}$. The “diamonds” in $\mathcal{G}_{G_2}$ occur when two mutations commute. Since this is a finite system it can be done by a direct calculation.

4. Polynomiality phenomenon

We now consider the specialization of the cluster algebra to the point $\{Q_{\alpha,0} = 1\}_\alpha$ (so that $R_{\alpha,0} = c_{\alpha}$), which we call the Kirillov-Reshetikhin point, because solutions to the $Q$-system with this boundary condition, for $m > 0$, are characters of Kirillov-Reshetikhin modules.

Note that at the KR point, $Q_{\alpha,-1} = R_{\alpha,-1} = 0$ for all $\alpha$. However due to the Laurent phenomenon, which states, among other things, that all cluster variables are Laurent polynomials in $R_{\alpha,0}$ and $R_{\alpha,1}$, all cluster variables are well-defined at this point. They have no singularity since at the KR point $R_{\alpha,0} \neq 0$.

In our proof of the combinatorial Kirillov-Reshetikhin conjecture [3], a crucial ingredient was the following Lemma:

**Lemma 4.1.** For any algebra $g$, any solution to the KR $Q$-system $Q_{\alpha,m}$ is a polynomial in the $r$ variables $\{Q_{\alpha,1}\}$.

This follows from the fact that KR-modules are all in the Grothendieck group of the trivial and fundamental KR-modules.

One of the reasons for recasting the $Q$-system as a cluster algebra is the availability of the Laurent phenomenon theorem. It implies a purely algebraic proof for the polynomiality lemma.
(The idea for this proof is due to Sergei Fomin.) We will explain the general situation which gives rise to this property in the $Q$-system.

Consider a cluster algebra with a special node, which we call 0 or the origin, with cluster variables 
\[ x = (a_1, \ldots, a_n; b_1, \ldots, b_m) = (a, b) \]
and mutation matrix $B$ such that 
\[ B_{ij} = 0 \quad \text{if } 1 \leq i, j \leq n. \]

We note that, as a result of this assumption about the form of the $B$-matrix, we have 
\[ \mu_i(a_i) = \frac{N_i(b)}{a_i}, \quad 1 \leq i \leq n, \]
where 
\[ N_i(b) = \prod_j b_j^{[B_{j+n,i}]} + \prod_j b_j^{-[B_{j+n,i}]} \]
is a function of $\{b_1, \ldots, b_m\}$ only, because $B_{j,i} = 0$ if $i, j \leq n$.

We assume that there is a special point in the parameter space $b$, which we call the KR point in analogy with the $Q$-system situation, such that for all $i \leq n$, $\mu_i(a_i) = 0$. Evaluation of $b$ at the KR point means the evaluation at the point $N_i(b) = 0 \ (i \leq n)$. This point depends only on $b$, not on $a$, which are left as formal variables.

We will show that for a cluster variable $y(a, b)$ at any node of the cluster tree $T_{m+n}$, considered as a function of $(a, b)$ (a Laurent polynomial, due to the Laurent property), terms with singularities at any $a_i = 0$ vanish at the KR point.

Lemma 4.2. Let 
\[ y(a, b) = b^{-m}P(a, b) \]
where $P(a, b)$ is a polynomial in $b$ but may have some singularities in $\{a_i\}$. That is, 
\[ P(a, b) = \sum_{n \in \mathbb{Z}^r} C_n(b)a^n, \]
with only finitely many non-zero terms, and where $C_n(b)$ are polynomials in the variables $b$. Then $C_n(b)$ is divisible by 
\[ \prod_{j: n_j < 0} N_j(b)^{-n_j}. \]

Proof. By change of variables to $\mu_i(a_i) = a'_i$ for all $i \leq n$, we have that 
\[ P(a, b) = \sum_{n \in \mathbb{Z}^r} C_n(b) \prod_{j=1}^n (a'_j)^{-n_j}N_j(b)^{n_j}. \]
Any terms with $n_j < 0$ have a positive power of $N_j(b)$ in the denominator. These must cancel with factors in $C_n(b)$, for each $n$ which has negative components, because the Laurent property states that a cluster variable cannot be a rational function in the variables $b$. Furthermore, there can be no cancellations between terms with different $n$, since the $a'_j$ are independent variables. Thus, $C_n(b)$ is divisible by $N_j(b)^{-n_j}$ for each negative $n_j$. \qed
Remark 4.3. It was pointed out to us after this paper was completed that this Lemma follows from Lemma 4.2 of [1].

Thus we have what we called “strong Laurent phenomenon” in [14] or the polynomiality property:

Corollary 4.4. With the cluster algebra as defined above, at the evaluation of \( b \) at a point such that \( \mu_i(a_i) = N_i(b) = 0 \), all cluster variables are polynomials in the variables \( \{a_i\} \).

Proof. \( C_n(b) \) is divisible by \( N_j(b)^{-n_j} \) for each negative \( n_j \). Therefore, it vanishes at the KR point, where \( N_j(b) = 0 \) for each \( j \). These are all the terms which have poles at \( a_j = 0 \). \( \square \)

Lemma 4.1 is a special case of this statement (with a rearrangement of the indices). We have \( m = n = r \), \( a_i = Q_{1,i} \) and \( b_i = Q_{1,0} \). The indices of the matrix \( B \) are accordingly rearranged.

Note that, in all the algebras introduced so far, we had the property that \( B_{ij} = 0 \) if \( r < i, j \leq 2r \), which is the analog of the condition on \( B \) above.

We conclude that the Laurent phenomenon is responsible for the polynomiality property of the characters \( Q_{\alpha,i} \) in terms of \( Q_{\alpha,1} \) for the KR \( Q \)-system. This property is quite general and is not related to the appearance of a Cartan matrix in the mutation matrix, only on the block structure of the mutation matrix, and the specific nature of the KR point in the parameter space.

Moreover, we can conclude much more.

Corollary 4.5. Any cluster variable at any node of the full cluster graph \( T_{2r} \) extending \( \mathcal{G}_0 \) as defined in Section 3 is a polynomial in \( R_{\alpha,1} \) after specialization of the parameters to the KR point, \( R_{\alpha,0} = \epsilon_\alpha \).

We do not, at this point, have a representation-theoretical interpretation for this phenomenon.

5. Discussion

5.1. Deformed Q-systems and the cluster algebra. In [3], we have defined deformed Q-systems depending on infinite families of parameters \( u = \{u_\alpha, u_{\alpha,1}, u_{\alpha,2}, \ldots\}_{\alpha \in I_r} \), as well as the extra parameters \( a = \{a_1, a_2, \ldots\} \) in the non-simply-laced case. The deformed Q-system is defined as follows:

\[
(5.1) \quad u_{\alpha,n} Q_{\alpha,n+1} Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta \sim \alpha} U_n^{(\alpha,\beta)}, \quad \alpha \in I_r, \; n \geq 1,
\]

where

\[
U_n^{(\alpha,\beta)} = \prod_{i=0}^{[C_{\alpha,\beta}]^1-1} Q_{\beta,\left\lfloor \frac{n+i}{\alpha} \right\rfloor}, \quad (\alpha, \beta) \neq (s, \ell)
\]

and

\[
U_n^{(\alpha,\beta)} = a_\alpha (1+\lfloor \frac{n}{\alpha} \rfloor)^{-n} \prod_{i=0}^{[C_{\alpha,\beta}]^1-1} Q_{\beta,\left\lfloor \frac{n+i}{\alpha} \right\rfloor}, \quad (\alpha, \beta) = (s, \ell).
\]

The deformed Q-system (5.1) is subject to the initial conditions that \( Q_{\alpha,0} = 1 \) and \( Q_{\alpha,1} = u_\alpha^{-1} \), for all \( \alpha \in I_r \).

The deformed variables \( Q(u,a) \) were introduced as a tool for proving the \( M = N \) identity of [10], yielding fermionic expressions for the tensor product multiplicities of KR-modules.
We may view the cluster variables $R_{\alpha,k}$ of the cluster graph $\mathcal{G}_g$ of Section 3 as specializations of the deformed characters $Q(u,a)$ (after normalization). This amounts to relaxing the initial condition on the $Q$-system, which is the specialization to the KR point.

Let $\varphi$ denote the evaluation
$$
\varphi(a_i) = 1, \quad \varphi(u_{\alpha,i}) = 1, \quad i \geq 2,
$$
and
$$
\varphi(a_1) = u_{s,1}
$$
where $s$ the unique short root connected to a long root.

The evaluated solutions of the deformed $Q$-system
$$
\overline{Q}(\{u_{\alpha}, u_{\alpha,1}\}_{\alpha \in I_r}) := \varphi(Q(u,a))
$$
obey a system of equations identical to the $Q$-system (2.1) for all $k \geq 1$, except for the difference that $Q_{\alpha,0}$ always comes with a prefactor $u_{\alpha,1}$.

Therefore, we may absorb the parameter $u_{\alpha,1}$ into the definition of $Q_{\alpha,0}$, namely by introducing
$$
Q_{\alpha,k}(\{u_{\beta,1}, u_{\beta,-1}\}_{\beta \in I_r}) := u_{\alpha,1}^0 \overline{Q}(\{u_{\beta}, u_{\beta,1}\}_{\beta \in I_r}).
$$
Then $Q_{\alpha,k}$ obey the $Q$-system (2.1) for $k \geq 1$, with the initial conditions replaced by $Q_{\alpha,0} = u_{\alpha,1}$ and $Q_{\alpha,1} = u_{\alpha,-1}$. This is identical to (2.5).

Upon the renormalization of the variables as in Lemma 2.1, we deduce that the specialized deformed KR characters are related to the variables $R_{\alpha,k}$ via:

$$
R_{\alpha,k} = \epsilon_{\alpha}^{\delta_{k,0}} \varphi\left(Q_{\alpha,k}(u,a)\right).
$$

We note that without the evaluation $\varphi$, the variables $Q_{\alpha,m}(u,a)$ are not obviously part of a cluster algebra. For example, they do not have the Laurent property.

In [3], we derived and used a crucial “substitution invariance” property of the deformed KR characters $Q$ which, after the specialization $\varphi$, reads as follows for the $Q$’s:

$$
Q_{\alpha,k+t_{\beta,j}}(\{Q_{\beta,0}, Q_{\beta,1}\}) = Q_{\alpha,k}(\{Q_{\beta,t_{\beta,j}}, Q_{\beta,t_{\beta,j}+1}\}).
$$

We showed that this condition is equivalent to the $Q$-system, with appropriate initial conditions.

When rephrased in terms of $R$’s, with $j = 2m$ even, this is nothing but the property of translational invariance of the cluster graph $\mathcal{G}_g$. Namely, the expression of cluster variables at any node $n$ of $\mathcal{G}_g$ in terms of that at cluster variables at the node 0 is the same as that of the cluster variables at the node $n + m$ in terms of the cluster variables at the node $m$. This is obvious from the structure of the graph.

Equation (5.3) is actually slightly more general, as it also includes translations by an odd integer $j$. For instance, the case $j = 1$ corresponds in the cluster formulation to the $k \to k'$ (“half-”) translation in the simply-laced case, and to $k \to k'(3)$ in the $G_2$ case. At the node $k'$ (or $k(3)$ for $G_2$), the mutation matrix is indeed $JBJ$, where $J = \begin{pmatrix} 0 & I_{r \times r} \\ I_{r \times r} & 0 \end{pmatrix}$, hence the nodes $k$ and $k'$ (or $k$ and $k'(3)$ for $G_2$) are equivalent upon interchanging the even and odd cluster variables $x \to Jx$, and the half-translation invariance property (5.3) for odd $j$ follows.

5.2. Extended graphs. As remarked above, generally there are larger subgraphs of $\mathbb{T}_{2r}$ which correspond to $Q$-system evolutions.
5.2.1. **Graphs corresponding to Q-system evolutions.** In Example 3.5 for $A_3$, we did not describe edges which do not correspond to $Q$-system evolutions. But in addition this graph does not describe all edges which are $Q$-system evolutions. For example, $\mu_6$ acting on $\mu_2 \circ \mu_3(x[k], B)$ is a $Q$-system evolution, which results in the cluster variable

$$x = (R_{1,2k}, R_{2,2k+2}, R_{3,2k+2}, R_{1,2k+1}, R_{2,2k+1}, R_{3,2k+3}).$$

Similarly, $\mu_4$ acting on $\mu_1 \circ \mu_2(x[k], B)$ is a $Q$-system evolution.

We can describe the complete graph corresponding to $Q$-system evolutions in the case of $A_3$:

![Graph of Q-system evolutions](image)

5.3. **Generalizations and open questions.** As mentioned in the introduction, there are generalizations of $Q$-systems to Cartan matrices corresponding to affine algebras [12]. Moreover there are generalizations of $Q$-systems to twisted quantum affine algebras. It is reasonable to expect that these can be recast in the cluster algebra language, although Lemma 2.1 does not, in general, work for affine Cartan matrices.

$Q$-systems are obtained from $T$-systems by taking the combinatorial limit, which amounts to ignoring the dependence on the spectral parameter [15, 17]. The $T$-system is a discrete Hirota equation in the case of $A_n$, and in general has an integrability property. As a result, $Q$-systems inherit some remarkable properties, such as the existence of conserved quantities and associated linear systems. We will address this property in a future publication.

An interesting question arises from the $M = N$ identity which was proved in [3]. This is a combinatorial identity which relates a restricted sum over products of binomial coefficients to an unrestricted one (see [10] for the original conjecture).

For example, the $M = N$ identity has the following form for $A_1$:

$$\sum_{m \in \mathbb{Z}_+^k : p_i \geq 0} \prod_i \binom{m_i + p_i}{m_i} = \sum_{m \in \mathbb{Z}_+^k} \prod_i \binom{m_i + p_i}{m_i}$$
where for a choice of non-negative integers \( n \in \mathbb{Z}^k_+ \), \( p_i = \sum_j \min(i, j)(n_j - 2m_j) \) and the sum is taken over \( m \in \mathbb{Z}^k_+ \) such that \( \sum_j j(n_j - 2m_j) = l \in \mathbb{Z}_+ \).

The restricted sum on the left hand side is manifestly positive, whereas the unrestricted sum on the right is an alternating sum. In [3], we have recast this identity in terms of generating functions and their power series expansions in order to prove the original conjecture of [10]. The \( M = N \) identity is a property of the following generating function:

\[
Z_{l;n}^{(k)}(u) = \frac{Q_1(u)Q_k(u)^{l+1}}{Q_{k+1}(u)^{l+1}} \prod_{i=1}^k \frac{Q_i(u)^{n_i}}{u_i},
\]

where \( Q_m(u) \) are solutions of the deformed \( Q \)-system. The \( M = N \) identity is the following statement: The constant term of \( Z \) in \( u \), after evaluation at \( u_1 = \cdots = u_k = 1 \), is equal to the constant term in \( u \) of the power series part of \( Z \) in each of the \( u_i \), after evaluation at the same point.

Our proof relies only on the polynomiality property of the solutions to the KR \( Q \)-system. It would be interesting to understand this identity more fully in the context of the properties of the associated cluster algebras.

**Appendix A. \( Q \)-systems with coefficients**

In this paper, we considered only \( Q \)-systems which correspond to some finite-type Cartan matrix. For this reason, it is always possible to renormalize the variables \( Q_{\alpha,k} \) which appear in the \( Q \)-system (2.1) so as to eliminate the minus sign appearing on the right hand side, due to Lemma 2.1. This is necessary in order to conform with the usual definition of a cluster mutation, which is a subtraction-free expression.

Such renormalization is not always possible, for example for certain affine-type Cartan matrices, and therefore in general, it is preferable to reformulate the system in terms of a cluster algebra with coefficients [7], in such a way that a specialization of the values of the coefficients reproduces the original \( Q \)-system (2.1).

It is possible to formulate this by introducing the coefficients \( \{q_1, \ldots, q_r\} \) as extra cluster variables which do not mutate, and then writing a “\( Q \)-system with coefficients” by replacing the minus sign on the right hand side of Equation (2.1) with these coefficients.

The resulting \( Q \)-systems with coefficients are

\[
(A.1) \quad Q_{\alpha,k+1}Q_{\alpha,k-1} = Q_{\alpha,k}^2 + q_{\alpha} \prod_{\beta \sim \alpha} \gamma_{k}^{(\alpha,\beta)},
\]

with \( \gamma_{k}^{(\alpha,\beta)} \) as in equation (2.2). When \( q_1 = \cdots = q_r = -1 \), this reproduces the original \( Q \)-system (2.1).

For each simple Lie algebra, the systems (A.1) correspond again to a cluster algebra on the subgraphs \( \mathcal{G}_r \) introduced in Section 3. Each mutation along edges of \( \mathcal{G}_r \) is one of the equations in the system of equations (A.1).

At the node \( k \) (in the notation of Section 3) of the graph \( \mathcal{G}_r \), we have the augmented cluster variables

\[
x = (Q_{\text{even}}, Q_{\text{odd}}; q_1, \ldots, q_r),
\]
where $Q_{\text{even}}$ is the collection of elements $(Q_{\alpha,2\iota,k})_{\alpha \in I}$, and $Q_{\text{odd}} = (Q_{\alpha,2\iota,k+1})_{\alpha \in I_r}$. Thus, $x$ is a collection of $3r$ elements, the last $r$ of which are the coefficients $q_1, \ldots, q_r$, which do not mutate.

The exchange matrix is, in principle, of size $3r \times 3r$ but we need only specify the first $2r$ columns, because the coefficients $\{q_\alpha\}_{\alpha \in I_r}$, which are the last $r$ variables of $x$, do not mutate.

The relevant $3r \times 2r$ exchange matrix, consisting of the first $2r$ columns, is denoted by $\tilde{B}$ and we refer to it as the augmented exchange matrix.

**Definition A.1.** The augmented exchange matrix $\tilde{B}$ at node $k$ is a $3r \times 2r$-matrix with entries as follows: The first $2r$ rows of $\tilde{B}$ coincide with those of $B$, while the last $r$ rows have entries

$$\tilde{B}_{2r+i,\alpha,\beta} = -\delta_{\alpha,\beta} = -\tilde{B}_{2r+i,\alpha,\beta}, \quad \alpha, \beta \in I_r.$$

We claim that the cluster mutations

$$(A.2) \quad \mu_i : x_i \mapsto x_i^{-1} \left( \prod_{1 \leq j \leq 3r} x_{ji}^{[\tilde{B}_{ji}]} + \prod_{1 \leq j \leq 3r} x_{ji}^{-[\tilde{B}_{ji}]} \right), \quad 1 \leq i \leq 2r,$$

along the edges of the cluster subgraph $\mathcal{G}_r$ coincide with the relations (A.1). In fact we have already shown this for the submatrix $B$ in Section 3. We need only show that the augmentation of $\tilde{B}$ has the correct evolution. This can be seen either algebraically or graphically.

**Example A.2.** Let us illustrate this structure in the simply-laced case. In this case, at node $k$, the augmented exchange matrix has the block form

$$\tilde{B} = \begin{bmatrix} 0 & -C \\ C & 0 \\ -I & I \end{bmatrix},$$

where $I$ is the $r \times r$ identity matrix and $C$ is the Cartan matrix.

In the subgraph $\mathcal{G}_r$, mutations along the edges labeled $1, \ldots, r$ are $Q$-system evolutions corresponding to the evolution of variables with even indices. These commute, and since $\tilde{B}_{2r+i,\alpha,\beta} = -\delta_{\alpha,\beta}$, the coefficient $q_\alpha$ is introduced in the second term of the right hand side of (A.2).

Thus, the mutation $\mu_\alpha$ with $1 \leq \alpha \leq r$ has the effect of changing the sign of $\tilde{B}_{2r+i,\alpha,\alpha}$ and leaving the rest of the block $(\tilde{B}_{ij})_{2r+1 \leq i \leq 3r}$ unchanged.

It was shown in Section 3 that $\mu_1 \circ \cdots \circ \mu_r(B) = -B$.

Moreover, the block $(\tilde{B}_{ij})_{2r+1 \leq i \leq 3r}$ also changes sign, because

$$\mu_1 \circ \cdots \circ \mu_r(B_{2r+1,\alpha,\beta}) = B_{2r+1,\alpha,\beta} + \sum_{\gamma=1}^{r} B_{2r+1,\alpha,\gamma} [B_{2r+1,\gamma,\beta} + B_{2r+1,\gamma,\beta}]$$

$$= B_{2r+1,\alpha,\beta} + \sum_{\gamma=1}^{r} (-\delta_{\alpha,\gamma}) \delta_{\gamma,\beta}$$

$$= -\delta_{\alpha,\beta} = -B_{2r+1,\alpha,\beta}.$$

Thus, we have that $\mu_1 \circ \cdots \circ \mu_r(\tilde{B}) = -\tilde{B}$ at the node $k'$.

Similarly, the effect of acting with the mutations $\mu_{r+1}, \ldots, \mu_{2r}$ on $\tilde{B}'$ again changes its overall sign.
In general, it is easiest to illustrate this part of the evolution of \( \tilde{B} \) graphically. We represent a skew-symmetric integer matrix as a quiver, where the nodes are enumerated by the rows of the matrix, and if \( A_{ij} = m > 0 \) then the node \( i \) is connected to the node \( j \) by an arrow from \( i \) to \( j \) labeled by \( m \).

In the case of the cluster algebra without coefficients, the quivers have nodes \( \alpha \in \{1, \ldots, r\} \), corresponding to variables \( Q_{\text{even}} \) with even indices, and \( \{\emptyset = r + \alpha, \alpha \in I_r\} \) corresponding to the variables with odd indices. Their connectivity is determined by the exchange matrix \( B \), which depends on the details of the Cartan matrix.

To add the coefficients \( q_\alpha \), we consider an extended quiver corresponding to the matrix \( \tilde{B} \), extended by skew-symmetry and with a vanishing diagonal block connecting the coefficient nodes. The extended quiver thus has \( r \) extra nodes, corresponding to the coefficients \( q_1, \ldots, q_r \). Their connectivity to the other nodes does not depend on the details of the Cartan matrix or \( B \).

For each \( \alpha \), node \( q_\alpha \) is connected only to nodes \( \alpha \) and \( \emptyset \). At the node labeled by \( k \) in \( G_r \), the connectivity of these three nodes is illustrated by the subquiver in the left hand side of the figure below. The mutations \( \mu_\alpha \) and \( \mu_\emptyset \) are the only ones which act on it nontrivially, as follows:

After acting with \( \mu_\alpha \) or \( \mu_\emptyset \) on the quiver on the left, all arrows are reversed, and vice versa.

Now we note that for both cases of simply-laced Lie algebras and the non simply-laced ones, the evolution from the node \( k \) to the node \( k + 1 \) always involves a sequence where \( \mu_\alpha \) acts first, then \( \mu_\emptyset \), in this order, once (in the case of ADE), twice (in the case of the short roots of B,C,F) or three times (in the case of the short root of \( G_2 \)). Thus, we return to the original configuration on the left hand side of the picture each time, for each of the triples \( \alpha, \emptyset, q_\alpha \).

This proves that the rows corresponding to \( q_\alpha \) in the mutation matrix \( \tilde{B} \) are identical at the nodes \( k \) and \( k + 1 \), and hence so is the full exchange matrix.

It is left only to check that, when progressing from smaller to larger \( k \) (i.e. to the “right” in the graph \( G_r \)), the coefficient \( q_\alpha \) always appears only in the second term. This is again due to the order of acting with the even and odd mutations in the graph.

We conclude that the exchange relation with coefficients (A.1) is preserved under evolutions in the same subgraph \( G_r \) which describes \( Q \)-system evolutions, for any of the Cartan matrices corresponding to simple Lie algebras.

**Appendix B. Generalized T-systems, bipartite cluster algebras and the polynomial property**

There is a larger class of recursion relations which satisfy the same polynomiality property, again due to the same argument as in Section 4. In this section, we formulate these relations, which generalize the \( T \)-systems of quantum spin chains [15, 17], as cluster algebras.
For clarity of presentation, we only consider systems corresponding to simply-laced Lie algebras \( \mathfrak{g} \) here. These are ones which have the simple bipartite graph similar to \( \mathcal{G}_r \).

The \( Q \)-systems considered in this paper are the “combinatorial limit” of such \( T \)-systems, which are the fusion relations satisfied by the transfer matrices of generalized Heisenberg spin chains. The combinatorial limit is obtained by dropping one of the parameters, corresponding in the original system to the spectral parameter.

It is known \([13]\) that the solutions to the \( T \)-systems, given appropriate boundary conditions, are the \( q \)-characters of quantum affine algebras \([8]\). These boundary conditions are precisely the ones which ensure that the cluster variables have the polynomiality property.

There is also a similar class of such systems which appears in representation theory, as shown in \([9]\) (section 18.2). As for the \( T \)-systems of simply-laced Lie algebras, this example also has a “bipartite property”. These are mentioned in the second example below. Interestingly, these systems also are presented in \([9]\) with the precise boundary conditions which ensure polynomiality.

In general, \( T \)-systems for non-simply laced Lie algebras \([17]\) are not bipartite, in the same way that \( Q \)-systems for non-simply-laced algebras are not, and have a more complicated structure for the graph \( \mathcal{G}_r \). In order to keep the discussion clear, we limit ourselves to bipartite systems in this Appendix.

**Definition B.1. Generalized \( T \)-systems.** We consider the index set \( I_r \times \mathbb{Z} \) and a matrix \( A \) (the “incidence matrix”) with rows and columns parametrized by this index set, \( A = (A_{\alpha,\beta}^{ij})_{i,j \in \mathbb{Z}} \), with entries in \( \mathbb{Z}_+ \). We define a generalized \( T \)-system with coefficients to be the recursion relation of the form

\[
T_{\alpha,j;k+1}T_{\alpha,j;k-1} = T_{\alpha,j+1;k}T_{\alpha,j-1;k} + q_\alpha \prod_{j' \neq \alpha} (T_{\beta,j';k})^A_{\beta,\alpha}^{j,j'}, \quad \alpha \in I_r, \quad j, k \in \mathbb{Z}.
\]

This is a bipartite system, in the following sense. We view Equation (B.1) as an evolution in the direction of \( k \). This means that (1) the evolution replaces \( T_{\alpha,j;k-1} \) by \( T_{\alpha,j;k+1} \), preserving the parity of \( k \) and (2) the right hand side of (B.1) depends only on variables of the opposite parity in \( k \). The second property is the characteristic of bipartite systems.

Such a property holds for \( Q \)-systems with a symmetric Cartan matrix. We give two examples of \( T \)-systems with this property. The first example appeared in \([15]\) and was generalized in \([17]\). It has generalizations to other Cartan matrices, which we do not consider.

**Example B.2.** The \( T \)-systems satisfied by the \( q \)-characters of Kirillov-Reshetikhin modules have the following form, in the simply-laced case:

\[
T_{\alpha,j;k+1}T_{\alpha,j;k-1} = T_{\alpha,j+1;k}T_{\alpha,j-1;k} - \prod_{\beta \sim \alpha} T_{\beta,j;k}^{[-C_{\beta,\alpha}]}, \quad \alpha \in I_r, j, k \in \mathbb{Z},
\]

where \( C \) is the (symmetric) Cartan matrix corresponding to a simple, simply-laced Lie algebra \( \mathfrak{g} \).

**Remark B.3.** We have made the identification \( T_{\alpha,j;k} = T_{\alpha,k}(u) \) (for rational \( R \)-matrices) where \( j = 2u + c \) for some complex number \( C \), where \( T_{\alpha,k}(u) \) is the transfer matrix corresponding to the Kirillov Reshetikhin module with highest weight \( k\omega_\alpha \) and spectral parameter \( u \). Similarly, for trigonometric \( R \)-matrices, \( j \) is related to \( \log \) of the spectral parameter with a complex shift.

The \( q \)-characters of the KR-modules of the quantum affine algebra of \( \mathfrak{g} \), with highest \( \mathfrak{g} \)-weight \( k\omega_\alpha \), satisfy this recursion relation subject to the boundary conditions \( T_{\alpha,j;0} = 1 \) and \( T_{0,j;k} = T_{r+1,j;k} = 1 \) \([13]\).
Example B.4. Consider some arbitrary acyclic quiver with nodes labeled by $I_r$ and the corresponding quiver matrix $\Gamma = (\Gamma_{ij})_{i,j \in I_r}$, such that $\Gamma_{i,j} = k > 0$ if there are $k$ arrows pointing from node $i$ to node $j$, and $\Gamma$ is a skew-symmetric matrix.

To such a quiver there corresponds a recursion relation of the form (B.1)

\[ f_{i,a,b} f_{i,a,b-1} = f_{i,a,b} f_{i,a-1,b-1} - \prod_{j:i \to j} f_{j,a,b} \prod_{j:j \to i} f_{j,a-1,b-1}. \]

Here, the product over $i : i \to j$ has $k$ factors if $\Gamma_{ij} = k > 0$, and so forth.

Defining $f_{i,a,b} = : T_{i,a+b+1,b-a}$, this recursion relation can be written in the form of a $T$-system

\[ T_{\alpha,j,k+1} T_{\alpha,j,k-1} = T_{\alpha,j+1,k} T_{\alpha,j-1,k} - \prod_{\beta}(T_{\beta,j+1,k} T_{\beta,j-1,k}). \]

Define the matrix $A$ as follows: $A_{\beta,\alpha}^{j+1} = [\Gamma_{\alpha,\beta}]_+, A_{\beta,\alpha}^{j-1} = [\Gamma_{\beta,\alpha}]_+, and all other entries of $A$ vanishing. Then the last equation takes the form (B.1)

\[ T_{\alpha,j,k+1} T_{\alpha,j,k-1} = T_{\alpha,j+1,k} T_{\alpha,j-1,k} - \prod_{\beta}(T_{\beta,j',k})^{A_{j,j'}^{\beta,\alpha}}. \]

This last relation is obviously equation (B.1) specialized so that all the coefficients $q_\alpha = -1$. We remark that the way in which the identification of the indices was made above, there is a parity restriction on the sum $j + k$. This is not essential to the discussion below, as the two parities of $j + k$ actually decouple in the associated cluster algebra.

B.1. Formulation of generalized $T$-systems as cluster algebras. The bipartite $T$-systems of Equation (B.1) can be formulated as cluster algebras of infinite rank, if the matrix $A$ satisfies some mild conditions (see the Lemma below). The model for this description is the $Q$-system in the previous section for the simply-laced case.

First, we specify a cluster variable (including coefficients) at some node in the cluster graph, which we label $k$, and an augmented exchange matrix $B$, which includes the submatrix $A$. Equation (B.1) is an exchange relation $\mu_{\alpha,j} : T_{\alpha,j,k-1} \mapsto T_{\alpha,j,k+1}$, and the parity of $k$ is preserved under any mutation. Thus, the exchange matrix $B$ consists of two column-sets, each labeled by $I_r \times \mathbb{Z}$, corresponding to even and odd variables ($k$ is fixed):

\[ T_{\text{even}} = \{T_{\alpha,j,2k} : \alpha \in I_r, j \in \mathbb{Z}\}, \]
\[ T_{\text{odd}} = \{T_{\alpha,j,2k+1} : \alpha \in I_r, j \in \mathbb{Z}\}. \]

We also have $r$ coefficients $\{q_\alpha : \alpha \in I_r\}$ which do not mutate. At the node labeled $k$ in the cluster tree, the cluster variable is

\[ x = (T_{\text{even}}, T_{\text{odd}}; q_1, \ldots, q_r). \]

Let $A$ be a matrix as in Definition (B.1) and let $P$ be the matrix on the same index set $I_r \times \mathbb{Z}$ defined as

\[ P_{\alpha,\beta}^{i,j} = \delta_{\alpha,\beta}(\delta_{i,j+1} + \delta_{i,j-1}). \]
Define $C = P - A$, then the exchange matrix $B$ on the “doubled” index set is

$$B = \begin{pmatrix} 0 & -C' \\ C & 0 \end{pmatrix}$$

at the node $k$.

In order to distinguish between the index sets for the “even” and “odd” variables, we will refer to the first as $(\alpha, j)$ and the latter as $(\overline{\alpha}, j)$ in analogy with the formulation of the $Q$-system. Even mutations act according to the first set of columns, and odd mutations act via the second half.

The augmented matrix $\tilde{B}$ is defined by adding $r$ rows to $B$, which we label simply by the indices $q_1, \ldots, q_r$. For example, the entry $B_{q_\alpha, (\overline{\beta}, j)}$ corresponds to the row labeled by $q_\alpha$ and column $(\overline{\beta}, j)$ of an even variable, whereas $B_{q_\alpha, (\overline{\beta}, j)}$ is in the column corresponding to the odd variable $(\overline{\beta}, j)$. Thus, at the node $k$,

$$\tilde{B}_{q_\alpha, (\overline{\beta}, j)} = -\delta_{\alpha, \overline{\beta}} = -\tilde{B}_{q_\alpha, (\overline{\beta}, j)}.$$  

We can write the $T$-system (B.1) as

$$T_{\alpha; k+1} T_{\alpha; k-1} = \prod_{j'} T_{\alpha, j'; k}^{[C_{j', j}^+]_\alpha} + q_\alpha \prod_{\alpha, j'} T_{\beta, j'; k}^{[-C_{j', j}^+]_{\alpha}}.$$  

This is identical to the exchange relation $\mu_{\alpha,j} : T_{\alpha; j; k-1} \mapsto T_{\alpha; j; k+1}$ given by the matrix $\tilde{B}$ above when $k$ is odd, and $\mu_{\alpha,j} : T_{\alpha; j; k+1} \mapsto T_{\alpha; j; k-1}$ if $k$ is even.

The mutations $\mu_{\alpha,j}$ commute with each other for different $(\alpha, j)$. Define the node $k'$ to be the node reached from $k$ via the compound mutation

$$\mu_{\text{even}} = \prod_{\alpha \in I_r, j \in \mathbb{Z}} \mu_{\alpha,j}.$$  

The cluster variable at this node is $x' = (T'_{\text{even}}, T'_{\text{odd}}, q_1, \ldots, q_r)$, where $T'_{\text{even}} = \{T_{\alpha, 2k+2} : \alpha \in I_r, j \in \mathbb{Z}\}$, and the rest of the entries are as in $x$ at node $k$.

**Definition B.5.** The “bipartite” subgraph $\mathcal{G}_r$ of the full tree associated with the cluster algebra is the following: It is the subgraph containing the node labeled $k \in \mathbb{Z}$, as well as all nodes obtained from it by mutations along all distinct even edges, or alternatively all distinct odd edges.

The node $k'$ in $\mathcal{G}_r$ (respectively $(k-1)'$) is the node reached from node $k$ by the sequence of all even (respectively odd) mutations. The node $k+1$ in $\mathcal{G}_r$ is the node reached from $k'$ by the sequence of all odd mutations. The graph is thus extended for all $k \in \mathbb{Z}$.

The cluster algebra structure holds if we require that the exchange relation given by the matrix $\tilde{B}'$ at the node $k'$ be consistent with the evolution $\mu_{\alpha,j} : T_{\alpha; j; 2k+1} \mapsto T_{\alpha; j; 2k+3}$ given by equation (B.1). The mutations are consistent with the recursion (B.1) if and only if $\tilde{B}' = -B$. This condition is satisfied if the three conditions of the following lemma hold.

**Lemma B.6.** The mutations along $\mathcal{G}_r$ of the cluster algebra defined by the seed $(x, \tilde{B})$ at node $k$, with $x$ as in Equation (B.3) and $\tilde{B}$ is as above, restricted to the edges of the subgraph $\mathcal{G}_r$, are each described by one of the recursion relations (B.1), with the matrix $C = P - A$, where $P$ is given by Equation (B.4) and $A$ is a matrix with non-negative entries, provided that:

1. The matrices $A$ and $P$ satisfy the condition $P A^T - AP = 0$. (Given condition 3, this implies that $A$ and $P$ commute.)
(2) The matrix $P$ is such that $\sum_k P_{\alpha \beta}^{kj} = 2 \delta_{\alpha, \beta}$. 
(3) The matrix $A$ is symmetric.

Proof. These three conditions are necessary and sufficient to ensure that $\tilde{B}' = -\tilde{B}$.

Under the compound mutation $\mu_{\text{even}}$, the even rows and columns change sign. We are left with two other blocks of $\tilde{B}'$ whose mutation needs to be checked: The odd columns-odd rows, and the odd columns and the coefficient rows.

Condition (1) comes from requiring that $(\tilde{B}')_{\alpha, \beta}^{jl} = 0$. To see this, consider

$$(\tilde{B}')_{\alpha, \beta}^{jl} = 0 + \sum_{\gamma, k} \text{sign}(B_{\alpha, \gamma}^{jk}) [B_{\gamma, \gamma}^{jk} B_{\gamma, \beta}^{kl}] +$$

$$= \sum_{\gamma, k} \text{sign}(C_{\alpha, \gamma}) [-C_{\alpha, \gamma}^{jk} C_{\beta, \gamma}^{kl}] +$$

$$= (PA^t - AP)_{\alpha \beta}^{jl}$$

Condition (2) is the result of requiring that $(\tilde{B}')_{q_\alpha, (\beta, \iota)}^{jl} = -\tilde{B}_{q_\alpha, (\beta, \iota)}$. It is easiest to illustrate this graphically (see below), but also,

$$(\tilde{B}')_{q_\alpha, (\beta, \iota)}^{jl} = \delta_{\alpha, \beta} + \sum_{\gamma, k} \text{sign}(B_{q_\alpha, (\gamma, \iota)}^{jk}) [B_{q_\alpha, (\gamma, \iota)}^{jk} B_{\gamma, \beta}^{kl}] +$$

$$= \delta_{\alpha, \beta} - \sum_{\gamma, k} [\delta_{\alpha, \gamma} C_{\beta, \gamma}^{ik}] +$$

$$= \delta_{\alpha, \beta} - \sum_k P_{\beta, \alpha}^{ik}.$$  

Requiring that $(\tilde{B}')_{q_\alpha, (\beta, \iota)}^{jl} = -\delta_{\alpha, \beta}$ gives condition (2) of the Lemma.

Condition (3) appears because we require the evolution $\mu_{\pi,j}$ to have the same form at $k'$ as the evolution $\mu_{\alpha,j}$ at the node $k$. That is, the evolution of even and odd variables is the same, as the matrix $A$ in equation (B.1) does not depend on $k$. Therefore, since the upper right block of $\tilde{B}'$ is equal to $C^t = P - A^t$, we find that $A = A^t$. \hfill $\Box$

We illustrate the last statement of the proof graphically in terms of the quiver graph corresponding to $\tilde{B}$. In fact, we only need to consider the cluster variables in $x$ which are connected to $q_\alpha$. These are $\{T_{\alpha,j;2k} : j \in \mathbb{Z}\}$, for some fixed $k$ and $\alpha$. (In the system of [9] there is a restriction on the parity of $j + k$ but this is not a necessary assumption for our discussion here, as the two parities decouple).

Thus, we need only consider a “slice” of the quiver graph with constant $\alpha$. The quiver graph of this slice is an infinite double strip and one extra node, has the following arrows at the node $k$ (to keep the picture readable, we omit the arrows between the empty circles and $q_\alpha$ as they are clearly decoupled from what happens to the quiver with solid circles. They are connected in exactly the same way as the solid circles):
Note that the sum of the vertical and horizontal coordinates has a different parity for solid and empty circles. In the slice of constant $\alpha$ they are not connected by any arrows.

Now suppose we act with $\prod_j \mu_{\alpha,j}$ on this quiver. Then all the arrows reverse. This is seen from the following sequence of mutations on a piece of the quiver:

All the arrows are reversed in the last picture. This shows that these entries of the matrix $\tilde{B}'$ indeed have the opposite sign.
We conclude that generalized $T$-systems of the form (B.1) are subsets of cluster algebras, so we can apply the Lemmas of Section 4:

**Lemma B.7.** In the case where there is a boundary condition on the recursion relation (B.1) such that $T_{\alpha,j;k_0-1} = 0$ for all $j$ and $\alpha$, with some fixed $k_0$, then all cluster variables are polynomials in the variables $\{T_{\alpha,j;k_0} : \alpha \in I_r, j \in \mathbb{Z}\}$.

It turns out that in both the examples given in this appendix, such a boundary condition holds with $k_0 = 0$. This gives a simple explanation of the phenomenon of polynomiality of the solutions of the recursion found in [9].

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