Error Bounds for Repeat-Accumulate Codes Decoded via Linear Programming

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Abstract

We examine regular and irregular repeat-accumulate (RA) codes with repetition degrees which are all even. For these codes and with a particular choice of an interleaver, we give an upper bound on the decoding error probability of a linear-programming based decoder which is an inverse polynomial in the block length. Our bound is valid for any memoryless, binary-input, output-symmetric (MBIOS) channel. This result generalizes the bound derived by Feldman et al., which was for regular RA(2) codes.

Keywords: Coding theory, repeat-accumulate codes, linear-programming (LP) decoding, upper bound, error performance.

I. INTRODUCTION

Since the discovery [10] of Turbo codes in 1993, there has been much focus on understanding why they perform superbly as they do. The discovery of Turbo codes also sparked an abundance of research into LDPC codes which were originally discovered by Gallager [11]. This vast study of Turbo and LDPC codes, as well as their many variations, has mainly been with respect to two types of decoders: Optimal maximum-likelihood (ML) and sub-optimal iterative message-passing algorithms. The latter have been extensively researched with several variations of the decoding algorithm, producing in some cases an accurate understanding of the decoder performance.

Recently, a novel decoding scheme based on linear programming (LP) was proposed. Initially, an LP-based decoder was proposed for Turbo codes by Feldman et al. [13] with an explicit performance bound given for repeat-accumulate (RA) codes, a variant of Turbo codes. Later, another LP-based decoder was proposed for LDPC codes by the same authors [12]. These results, among others, have been well-summarized in [1]. Further results for the LP decoder of LDPC codes include the characterization of pseudocodewords, and in particular, minimum-weight pseudocodewords (e.g., [8], [9]); results for the binary symmetric channel (BSC) on the error-correction capability (e.g., [5], [16]), and others.

One interesting property of the LP decoder is the ML certificate property. That is, that whenever the LP decoder outputs a codeword, it is guaranteed to be the ML codeword. Iterative message-passing algorithms do not share this property. On the other hand, iterative algorithms have in some cases the
advantage of lower decoding complexity, as compared to LP decoding. However, for LDPC codes this advantage is all but eliminated (see [14], [15]).

Compared to iterative decoding, there has so far been less research on LP-based decoding. While the first analytic result for LP decoding [13] has been for the case of RA codes, most of the results for LP decoding thereafter refer to LDPC codes. In [13], regular RA(2) codes were examined, based on flow theory and graph-theoretical arguments. Halabi and Even [6] have proposed a better bound on RA(2) codes which is based on a more careful examination of the underlying graph-theoretical nature of the problem. Irregular RA code ensembles have been shown to achieve excellent performance under iterative message-passing decoding. For example, in the BEC there are known capacity-achieving sequences of codes (see e.g., [7]). Motivated by these results for iterative decoding, we examine regular and irregular RA codes under LP decoding. We show how to extend the results of [13] to regular RA(q) codes for even q and to irregular RA ensembles where all repetition degrees are even. The essential novelty in this work is the application of Euler’s (graph-theoretic) theorem to an appropriately defined (hyperpromenade) graph.

The remainder of the paper is organized as follows. Preliminary material is given in Section II. Section III contains the derivation of our error bound for regular RA codes. A discussion of these results as well as their extension to irregular RA codes appears in Section IV. Section V concludes the paper.

II. Preliminaries

In this section, we give our nomenclature and some necessary preliminary material. Our notations largely follow those of Feldman [1]. In the rest of the paper, we will deal exclusively with repeat-accumulate (RA) codes. These codes were proposed by Divsalar et al. in [3], in which regular code ensembles were defined, and later generalized to irregular ensembles in [4]. Repeat-accumulate codes feature a simple encoder structure and are known to have good decoding performance under iterative message-passing decoding. The encoder of a regular RA(q) code, shown in Fig. 1, takes an input block of k bits, applies a q-fold repetition code to obtain a block of n = qk bits, interleaves the block and finally feeds it into a rate-1 accumulator. The accumulator is a recursive convolutional encoder with one memory element which outputs at time index t simply the mod-2 sum of the inputs up to time t. The code rate in this case is $R = \frac{1}{q}$. In an irregular code, the number of times a bit is repeated (or its repetition degree) is not constant. The fraction of bits which are repeated a certain number of times by the encoder is known as the degree distribution, and it is usually expressed either in vector form or in polynomial form.

Our analysis will focus on transmission of regular or irregular RA codes over memoryless, binary-input, output-symmetric (MBIOS) channels. We denote by $x_i \in \{0, 1\}, i = 1, \ldots, k$ the i’th bit of the codeword to be transmitted and by $y_i = y(x_i)$ the channel modulation of the i’th bit. Since the channel is
memoryless, the \( i \)’th received symbol \( \tilde{y}_i \) depends only on \( y_i \) by the conditional probability law \( P(\tilde{y}_i|y_i) \) imposed by the channel.

The log-likelihood ratio (LLR) \( \gamma_i \) is defined as

\[
\gamma_i = \ln \left( \frac{P(\tilde{y}_i|y_i = y(0))}{P(\tilde{y}_i|y_i = y(1))} \right)
\]

(1)

Example 1: In the binary symmetric channel (BSC) the channel input alphabet is binary, and so we have \( y_i = x_i \). The log-likelihood ratio is \( \gamma_i = \ln \left( \frac{1-p}{p} \right) \) if \( \tilde{y}_i = 0 \) and \( \gamma_i = \ln \left( \frac{p}{1-p} \right) \) if \( \tilde{y}_i = 1 \).

Example 2: Consider the binary-input additive white gaussian noise (AWGN) channel. Following conventional notation, we map bit 0 to +1 and bit 1 to −1, i.e., we have \( y_i = 1 - 2x_i \). In the AWGN channel we have

\[
\tilde{y}_i = y_i + z_i
\]

where \( z_i \) is a normally-distributed random variable, \( z_i \sim \mathcal{N}(0, \sigma^2) \). The LLR in the AWGN channel is easily shown to be \( \gamma_i = \frac{2\tilde{y}_i}{\sigma^2} \).

It is convenient for purposes of analysis to rescale the LLR. In the BSC, the rescaling enables to have \( \gamma_i = 1 \) if \( \tilde{y}_i = 0 \) and \( \gamma_i = -1 \) if \( \tilde{y}_i = 1 \). In the AWGN channel, rescaling allows us to express \( \gamma_i = \tilde{y}_i \).

A. A Linear Program to Decode Repeat-Accumulate Codes

We are interested in the performance of a linear-programming (LP) decoder for RA codes. To make our presentation self-contained, we briefly present the linear program proposed by Feldman [1].

First, we look at the accumulator section of the encoder, assuming at this stage that it were the entire encoder. The accumulator is a rate-1 convolutional encoder, and has a state diagram and trellis as shown in Fig. 2. The trellis \( T \) features connections or edges describing transitions between states from successive time intervals, which are labeled according to the output of the accumulator. Each edge also has a ’type’ which depends upon the input bit triggering the transition. Note that the trellis contains an extra layer used to terminate the code. Adding the extra bit to force the encoder back to the zero-state incurs a small loss in the code rate, but makes analysis more convenient, since each codeword corresponds to a ”cycle” rather than an arbitrary path in \( T \). All edges have a direction (i.e., forward in time). Also, every edge \( e \in T \) is assigned a cost \( \gamma_e = \gamma_i \) where the index \( i \) is selected according to the trellis segment containing the edge. Consequently, it can be shown that finding the ML codeword is equivalent...
to finding the minimum-cost path traversing across the trellis. If indeed the accumulator were the entire code, there would be no additional constraints on the path, and finding the one with minimal cost could be accomplished, for example, by using the Viterbi algorithm.

We now take the effect of the (possibly irregular) repetition code and interleaver into account. Assume that input bit $x_t$ has repetition degree $q_t$. If so, we would expect the inputs into the accumulator at some set of indices $X_t \triangleq \{i^1, i^2, \ldots, i^{q_t}\}$ to be identical (obviously, this set depends on the interleaver). Translating this into trellis terms, we require that for all $t = 1, \ldots, k$ we have the same type of edge at all layers $i \in X_t$. Any path satisfying this requirement is called an agreeable path.

A linear program to decode RA codes (RALP) was defined by Feldman [1] as follows:

RALP: minimize $\sum_{e \in T} \gamma_e f_e$ s.t.

\[ \sum_{e \in \text{out}(s^0)} f_e = 1 \quad (2) \]

\[ \sum_{e \in \text{out}(s)} f_e = \sum_{e \in \text{in}(s)} f_e \quad \forall s \in T \setminus \{s^0, s^n\} \quad (3) \]

\[ x_t = \sum_{e \in I_t} f_e \quad \forall i \in X_t, \ t = 1, \ldots, k \quad (4) \]

\[ 0 \leq f_e \leq 1 \quad \forall e \in T \]

where $\text{in}(s)$ is the set of edges entering node $s$, $\text{out}(s)$ is the set of edges exiting node $s$, and $I_t = \{(s^0_{i-1}, s^1_i), (s^1_{i-1}, s^0_i)\}$ is the pair of "input-1" edges entering layer $i$. Equation (2) ensures that one unit of flow is sent across the trellis. Equation (3) enforces flow conservation at each node, i.e., that whatever flow enters must also exit. The agreeability constraints are imposed by equation (4). These constraints
say that a feasible flow must have, for all $X_t$, $t = 1, \ldots, k$, the same amount $x_t$ of total flow on input-1 edges at every segment $i \in X_t$.

In order to use RALP as a decoder, one should solve the LP problem above on the trellis with edge costs $\gamma_e$ defined by the received vector $\tilde{y}$, thus obtaining an optimum point $(f^*, x^*)$. If $f^*$ is integral (i.e., all values are 0 or 1), $x^*$ is output as the decoded information word. If not, the output is "error". We refer to this algorithm as the RALP decoder. It can be shown that this decoder has the ML certificate property: whenever it finds a codeword, it is guaranteed to be the ML codeword.

III. AN ERROR BOUND FOR REGULAR RA(q) CODES WITH EVEN q

In this section, we derive an upper bound on the decoding error probability of the RALP decoder. For simplicity, we deal in this section exclusively with regular codes. This is an extension of the results of [1], which applied to RA(2) codes, to the case of RA(q) codes for even $q$. For the purpose of analysis, we define an auxiliary graph which contains subgraphs called hyperpromenades which carry a meaning similar to error events in convolutional codes. The structure of these hyperpromenades suggests a design of an interleaver. We show how to design a suitable interleaver, and show that the RALP decoder has an inverse-polynomial error rate (in the blocklength $n$) when this interleaver is used. Our discussion will not depend initially on the repetition degrees being even; we will only require this assumption later on.

Let $\Theta$ be a weighted undirected graph with $n$ vertices $(g_1, \ldots, g_n)$ connected in a line. We call these edges Hamiltonian, as they form a Hamiltonian path along the graph. We associate a cost (weight) $c[g_i, g_{i+1}]$ with each Hamiltonian edge $(g_i, g_{i+1})$, equal to the cost added by decoding code bit $i$ to the opposite value of the transmitted codeword. Formally, we have

$$c[g_i, g_{i+1}] = \gamma_i (1 - 2x_i)$$

where $x_i$ is the $i$'th codeword bit, and $\gamma_i$ is the log-likelihood ratio of code bit $i$, as defined in [1]. In the BSC, we have $c[g_i, g_{i+1}] = +1$ if $x_i = y_i = \tilde{y}_i$ and $c[g_i, g_{i+1}] = -1$ if $y_i \neq \tilde{y}_i$. Naturally, the decoder does not know the costs $c[g_i, g_{i+1}]$; they are used solely as a means for analysis. In addition to the Hamiltonian edges described above, $\Theta$ contains also hyperedges connected between the vertices. A $q$-hyperedge is an edge connecting $q$ vertices, and is formally defined as an unordered $q$-tuple of vertices from the graph. We connect a total of $k$ hyperedges, where hyperedge $t$ contains the vertices within the index set $X_t$ ($t = 1, \ldots, k$). Note that according to this setting, exactly one hyperedge is connected to every vertex. In [1], where the authors consider the case $q = 2$, these extra edges form a matching on the vertices of the graph. Extending this nomenclature to any $q$, we will call them matching hyperedges. These edges are defined to have zero cost in the auxiliary graph.

An atom path $\mu(\sigma, \tau)$ is a walk which begins at vertex $g_\sigma$ and finishes at vertex $g_\tau$, using Hamiltonian edges only. Therefore, if $\sigma < \tau$, we have $\mu(\sigma, \tau) = (g_\sigma, g_{\sigma+1}, \ldots, g_{\tau-1}, g_\tau)$. A hyperpromenade $\Psi$ is a set of atom paths, possibly with multiple copies of the same atom path in the set. The set $\Psi$ is also
required to satisfy a certain "agreeability" constraint. Formally, define, for each segment $i$ in the trellis where $1 \leq i \leq n$, the following multiset $B_i$:

$$B_i = \{ \mu \in \Psi : \mu = \mu(\sigma, \tau), \text{ where } i = \sigma \text{ or } i = \tau \}$$

Note that if multiple copies of some $\mu(\sigma, \tau)$ exist in $\Psi$, then $B_i$ contains multiple copies as well. We say that $\Psi$ is a hyperpromenade if, for all $t = 1 \ldots, k$, where $X_t = \{t^1, t^2, \ldots, t^q\}$, we have

$$|B_{t^1}| = |B_{t^2}| = \cdots = |B_{t^q}|$$

**Example 3:** As an example, consider the auxiliary graph illustrated in Figure 3. In this auxiliary graph of an RA(4) code, the multiset

$$\Psi = \{ \mu(1, 2), \mu(1, 2), \mu(3, 10), \mu(4, 5), \mu(4, 12), \mu(5, 7), \mu(6, 11), \mu(7, 12), \mu(8, 9), \mu(8, 9) \}$$

is a hyperpromenade.

The cost of every atom path $\mu(\sigma, \tau)$ is equal to the sum of the costs of its edges. The cost of a hyperpromenade is equal to the sum of the costs of the atom paths it contains, including repeated ones.

We have the following theorem ([1, Theorem 6.13]).

**Theorem 1:** For any regular RA(q) code, the RALP decoder succeeds if all hyperpromenades have positive cost. The RALP decoder fails if there is a hyperpromenade with negative cost.

This theorem was stated in [1], and was therein proved for the case of $q = 2$. The same proof applies also for $q > 2$. However, in order to use this result we need to show how to construct graphs $\Theta$ which yield good interleavers for RA(q) codes; we will show that these graphs have a small probability of having a negative-cost hyperpromenade. A key metric in our analysis will be the girth of the auxiliary graph. As the auxiliary graph contains hyperedges as well as regular edges (thus it is a hypergraph), the notion of girth needs to be extended. Define a path $p = (p_0, \ldots, p_k)$ in the auxiliary graph to be a series of vertices where every two consecutive vertices are connected by an edge or a hyperedge. The
only exception is that the same (hyper)edge may not be traveled two times in a row; this means that U-turns are not allowed, and also roundabouts within a hyperedge (e.g., if \(i_1, i_2, i_3 \in X_t\) for some \(t\), then \((g_{i_1} \rightarrow g_{i_2} \rightarrow g_{i_3})\) is not a valid path). Aside from this restriction, a path may repeat vertices, edges and hyperedges. Path length is measured in edges, so the path \(p = (p_0, \ldots, p_k)\) has length \(k\). A cycle is a path that begins and ends in the same vertex. The girth of a hypergraph is thus the length of its shortest cycle. We further define a simple path (resp. simple cycle) to be a path (cycle) which does not repeat Hamiltonian edges but may repeat hyperedges.

Our first step is to show that an auxiliary graph \(\Theta\) with high girth can be constructed, thus implying the existence of appropriate interleavers. For the case of \(q = 2\), Erdős and Sachs [17] (see also [2]) have shown a construction for such an interleaver. The following result is an extension to \(q \geq 3\) using a similar technique. While our subsequent error bound is valid only for even \(q\), this restriction need not be imposed yet.

**Theorem 2:** Let \(n = qk\) be the block length of a regular RA\((q)\) code with \(q \geq 3\) and \(n \geq q^4\). Then one may construct for this code an auxiliary graph which is a Hamiltonian line plus \(kq\)-hyperedges which form a matching, so that the auxiliary graph has girth no less than \(g = \lfloor \log_q n \rfloor - 1\).

Proof: See appendix A.

Denote the interleaver produced by this approach by \(\pi_E\). The next step is to study the auxiliary graph of an RA code which uses \(\pi_E\) as an interleaver. We focus on the underlying nature of hyperpromenades in this graph.

A study of the structure of hyperpromenades. First, we point out that in the case of \(q = 2\), it was shown by Feldman [1] that every hyperpromenade is equivalent to a cycle in \(\Theta_1\). This observation simplifies the analysis, as one must deal solely with simple cycles in the auxiliary graph. In our case where \(q > 2\), this is not necessarily true. Therefore, our conclusions must be based only on the definition of a hyperpromenade.

Our goal will be to provide an upper bound on the probability that the auxiliary graph contains a negative-cost hyperpromenade. Let \(\Psi\) be any hyperpromenade in \(\Theta\). We construct a graph called the hyperpromenade graph \(\Theta_\Psi\) as follows:

1) For every atom path \(\mu(\sigma, \tau) \in \Psi\), draw in \(\Theta_\Psi\) two vertices, labeled \(\sigma\) and \(\tau\), according to the endpoints of the atom path. Connect the two vertices by an edge. If \(\mu(\sigma, \tau)\) appears more than once in \(\Psi\), we will have multiple replicas of this structure, accordingly.
2) Merge all vertices with the same label into one vertex. At this stage, the graph may no longer be simple, i.e., there may be vertex pairs connected by more than one edge.
3) Add the matching hyperedges to the graph.

\(^1\)in fact, this was the original definition of a promenade, to which the hyperpromenade reduces for the case \(q = 2\).
By this construction, it is obvious that one can reconstruct any hyperpromenade given its hyperpromenade graph, i.e., there is a $1 - 1$ relation between hyperpromenades and their graphs. We further assign a cost to every edge $(\sigma, \tau)$ in $\Theta_\Psi$ as follows: $c[(\sigma, \tau)] = c[\mu(\sigma, \tau)]$ where $c[\mu(\sigma, \tau)]$ is the cost of the atom path in $\Theta$. Hyperedges in $\Theta_\Psi$ are assigned zero cost. With this definition, the total cost of the edges in $\Theta_\Psi$ is the same as the cost of the hyperpromenade. We note that the hyperpromenade graph may or may not be connected (in the sense that there is a path between any two of its vertices). If it is, we call the hyperpromenade connected. This property is different from the connectedness of the auxiliary graph, since now vertices common to different atom paths are ignored unless they are at the endpoints. As an example, we draw the hyperpromenade graph of $\Psi$ from Example 3 in Figure 4. In this example, the hyperpromenade is not connected and has two connected components.

Let $\Psi$ be a hyperpromenade which is not connected, and consider the corresponding graph, $\Theta_\Psi$. It is easy to verify that each of its connected components is the hyperpromenade graph of a valid hyperpromenade. Therefore, $\Psi$ can be partitioned into disjoint connected hyperpromenades $\Psi_1, \Psi_2, \ldots, \Psi_M$ satisfying $c[\Psi] = c[\Psi_1] + \cdots + c[\Psi_M]$. If $\Psi$ is a negative-cost hyperpromenade, it thus must have a component with negative cost. Therefore, by Theorem 1, the probability that the RALP decoder fails is the same as the probability of having a connected hyperpromenade with negative cost.

The next step is to establish that it is enough to look at simple paths and cycles which are contained in a connected negative-cost hyperpromenade. In the following, we assume $n = q^{4l+1}$ for some integer $l$ to avoid floors and ceilings, although our arguments do not rely on this.

**Theorem 3:** Let $\Theta$ be the auxiliary graph of a regular RA$\langle q \rangle$ code with $q$ even. Assume $\Theta$ has girth at least $g$, where $g \triangleq \log_q n - 1$ (by Theorem 2, this girth is attainable). If there exists a hyperpromenade $\Psi$ in $\Theta$ with $c[\Psi] \leq 0$, then there exists a simple path or cycle $Y$ in $\Theta$ that contains $\frac{g}{2}$ Hamiltonian edges, and has cost $c[Y] \leq 0$.

![Fig. 4. The hyperpromenade graph $\Theta_\Psi$ of the hyperpromenade from Example 3.](image)
Proof: Let $\Psi$ be a hyperpromenade with $c[\Psi] \leq 0$. By the discussion above, we may assume w.l.o.g. that $\Psi$ is connected. First, we will show that there is a cycle $\tilde{H} = (\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{|\tilde{H}|} = \tilde{h}_0)$ in $\Theta$ which has $c[\tilde{H}] = c[\Psi]$, where $c[\tilde{H}]$ is measured along the edges of $\tilde{H}$. Draw the hyperpromenade graph $\Theta_{\Psi}$, and contract the matching hyperedges. The result is a graph, with no hyperedges, where vertex $\sigma$ has degree $q|B_{\sigma}|$. Since $q$ is even, all degrees are even and we can find an Eulerian tour $C$ in $\Theta_{\Psi}$, i.e., a simple cycle which passes through all the edges. Since every edge in $\Theta_{\Psi}$ has the same cost as its corresponding atom path in $\Psi$, we have $c[C] = c[\Psi]$. By adding back the matching hyperedges and tracing along the atom paths making up $\Psi$, we get from $C$ the desired cycle $\tilde{H}$ in $\Theta$ with $c[\tilde{H}] = c[\Psi]$. Now, contract the matching hyperedges in $\tilde{H}$. Denote by $H = (h_0, h_1, \ldots, h_{|H|} = h_0)$ the contracted version of $\tilde{H}$.

No two matching hyperedges share an endpoint, and by definition the same hyperedge cannot be used twice in a row. Therefore, at most every other edge of $H$ is a matching hyperedge. Thus, and since $\tilde{H}$ is a cycle,

$$|H| \geq \frac{1}{2}|\tilde{H}| \geq \frac{1}{2}g$$

Write out the cost of $H$ explicitly as

$$c[H] = \sum_{i=0}^{|H|-1} c[h_i, h_{i+1}]$$

Let $H_i = (h_{i}, \ldots, h_{i+\frac{g}{2}})$ be a subsequence of $H$ containing $g/2$ edges, and let

$$c[H_i] = \sum_{j=i}^{i+\frac{g}{2}-1} c[h_j, h_{j+1}]$$

$H_i$ must be a simple path (or a simple cycle), i.e., it can have no repeated Hamiltonian edges; otherwise, by adding the matching hyperedges back into $H$, this would imply the existence of a cycle in $\Theta$ of length less than $g$. Note that

$$c[\tilde{H}] = c[H] = \left(\frac{1}{2g}\right)^{|H|-1} \sum_{i=0}^{|H|-1} c[H_i]$$

since every edge is counted exactly $\frac{1}{2g}$ times. Now, if $c[H] = c[\Psi] \leq 0$, then there must be a simple path or cycle $H_i$ such that $c[H_i] \leq 0$. Adding back the matching hyperedges, we get the desired simple path or cycle $Y$.

Theorem 3 asserts that if there are no negative-cost simple paths or cycles, then there is no corresponding negative-cost hyperpromenade. It is also the first point in our derivation which requires to use the assumption that $q$ is even. We will now use this to get an error bound for LP decoding under the BSC. This bound extends [1, Theorem 6.5].

Theorem 4: Consider a regular RA($q$) code ($q$ even) with block length $n$, and $\pi_E$ constructed in the proof of Theorem 2 as an interleaver. Assume that the code is transmitted over the BSC. Let $\epsilon > 0$ and contracting the hyperedge unites all vertices it connects into one vertex, retaining any other edges connected to the original vertices.
be some positive number. If the transition probability $p$ satisfies $p < q^{-4\left(\epsilon + 1 + \frac{1}{2} \log_q (4q-2)\right)}$, then when decoded using the RALP decoder, the code has word error probability

$$\text{WEP} < K (\log_q n) \cdot n^{-\epsilon} \quad (7)$$

where $K$ is a positive constant.

**Proof:** By theorems [1] and [3] the decoder will succeed if all simple paths or cycles in $\Theta$ with $\frac{1}{2} g = \frac{1}{2} (\log_q n - 1)$ (this equality is attained by definition of $\pi_E$) Hamiltonian edges have positive cost. We claim there are at most $n (2q - 1)^{\frac{1}{2} g}$ simple paths and cycles with $\frac{1}{2} g$ Hamiltonian edges. To see this, build a simple path or cycle by choosing any vertex $g_i$ and traversing a simple path beginning with a Hamiltonian edge. There are at most two choices for the first edge. If, after traversing the Hamiltonian edge, we arrive at a vertex $g_i$, then from $g_i$ we can choose to proceed along the second Hamiltonian edge connected to it, or traverse a hyperedge. If a hyperedge is traversed, there are at most $2 (q - 1)$ possible choices for the next Hamiltonian edge. This gives a total of $2q - 1$ choices for the second Hamiltonian edge. Proceeding in this manner, we see that there are no more than $(2q - 1)^{\frac{1}{2} g}$ simple paths or cycles beginning from the vertex $g_i$. Choosing an arbitrary starting vertex gives a total of no more than $n (2q - 1)^{\frac{1}{2} g}$ possible simple paths or cycles.

In the BSC, each Hamiltonian edge has cost $-1$ or $1$. Therefore, in any simple path or cycle $Y$, at least half of the edges must have cost $-1$ in order to have $c[Y] \leq 0$. Consequently, we have

$$\Pr \left( c[Y] \leq 0 \right) = \sum_{k = \frac{1}{2} g}^{\frac{1}{2} g} \left( \frac{\frac{1}{2} g}{k} \right) p^{k} (1 - p)^{\frac{1}{2} g - k} \leq \frac{1}{4} \left( \frac{\frac{1}{2} g}{\frac{1}{2} g} \right) p^{\frac{1}{2} g} \quad (8)$$

Applying the union bound over all possible choices of $Y$ (i.e., paths with $\frac{1}{2} g = \frac{1}{2} (\log_q n - 1)$ Hamiltonian edges), we have

$$\text{WEP} \leq n (2q - 1)^{\frac{1}{2} g} \frac{1}{4} g \left( \frac{\frac{1}{2} g}{\frac{1}{2} g} \right) p^{\frac{1}{2} g}$$

$$\leq \frac{1}{4} (\log_q n) \cdot n^{1 + \frac{1}{2} \log_q (2q-1)} n^{\frac{1}{2} \log_q 2} n^{\frac{1}{2} \log_q p} \cdot \frac{1}{4}$$

$$= K (\log_q n) \cdot n^{-\epsilon}$$

where $K = \frac{1}{4} p^{-\frac{1}{4}}$.

A similar result for the AWGN channel, which extends [1] Theorem 6.6] follows.

**Theorem 5:** Consider a regular RA$(q)$ code ($q$ even) with block length $n$, and $\pi_E$ constructed in the proof of Theorem [2] as an interleaver. Assume that the code is transmitted over the binary-input AWGN. Let $\epsilon > 0$ be some positive number. If the noise variance satisfies $\frac{1}{\sigma^2} > 4 \ln q \left( 1 + \epsilon + \frac{1}{2} \log_q (2q - 1) \right)$, then the code has, when decoded using the RALP decoder, word error probability

$$\text{WEP} < K \sqrt{\frac{1}{\log_q n - 1}} n^{-\epsilon} \quad (9)$$
where $\tilde{K}$ is a positive constant.

**Proof:** In the AWGN channel, each Hamiltonian edge in the auxiliary graph has cost

$$c[g_i, g_{i+1}] = \gamma_i (1 - 2x_i) = \gamma_i y_i = (y_i + z_i) y_i = 1 + \xi_i$$

where $\xi_i \sim \mathcal{N}(0, \sigma^2)$. Therefore, if $Y$ is any simple path or cycle with $\frac{1}{2} g = \frac{1}{2} (\log_q n - 1)$ Hamiltonian edges, we have $c[Y] = \frac{g}{2} + Z$, where

$$Z \sim \mathcal{N} \left(0, \frac{g}{2} \sigma^2\right)$$

For a random variable $X \sim \mathcal{N}(0, s^2)$, we have the following inequality: for all $x > 0$,

$$\Pr(X \geq x) \leq \frac{s}{x\sqrt{2\pi}} e^{-\frac{x^2}{2s^2}}$$

Using (10) we get

$$\Pr(c[Y] \leq 0) = \Pr(Z \geq \frac{g}{2}) \leq \sqrt{\frac{\sigma^2}{\pi g}} e^{-\frac{g^2}{4\sigma^2}}$$

As in Theorem 4, using the union bound over all possible choices of $Y$ gives

$$\text{WEP} \leq n(2q - 1)^{\frac{1}{2}g} \cdot \Pr(c[Y] \leq 0) \leq n(2q - 1)^{\frac{1}{2}g} \cdot \sqrt{\frac{\sigma^2}{\pi g}} e^{-\frac{g^2}{4\sigma^2}} = \sqrt{\frac{\sigma^2}{\pi (\log_q n - 1)}} \cdot n^{1 + \frac{1}{2} \log_q(2q - 1) - \frac{\log_q e}{4\sigma^2}} \leq \tilde{K} \sqrt{\frac{1}{\log_q n - 1}} n^{-\epsilon}$$

where $\tilde{K} \triangleq \sqrt{\frac{\sigma^2}{\pi}}$.

A similar result for general MBIOS channels is given here in Theorem 6. The proof follows the lines of the proofs of theorems 4 and 5 and is omitted.

**Theorem 6:** Consider a regular RA$(q)$ code ($q$ even) with block length $n$, and $\pi_E$ constructed in the proof of Theorem 2 as an interleaver. When transmitted over an MBIOS channel and decoded using the RALP decoder, the code has word error probability WEP satisfying

$$\text{WEP} < n(2q - 1)^{\frac{1}{2}g} \cdot \Pr \left(\sum_{i=1}^{\frac{1}{2}(\log_q n - 1)} \tilde{z}_i \leq 0\right)$$

where $\tilde{z}_i$, $i = 1, \ldots, \frac{1}{2}(\log_q n - 1)$ are i.i.d. random variables with cumulative distribution function

$$\Pr(\tilde{z}_i \leq z) = \Pr(\gamma_i \leq z|x_i = 0)$$

We note that the BSC and AWGN error bounds in (7) and (9) can be derived as special cases of (12).
TABLE I
BSC transition probability thresholds ensuring vanishing error probability, as derived from Theorem 4.

| $q$ | Threshold |
|-----|-----------|
| 4   | $2 \cdot 10^{-5}$ |
| 6   | $1.6 \cdot 10^{-6}$ |
| 8   | $2.7 \cdot 10^{-7}$ |

IV. DISCUSSION AND NUMERICAL RESULTS

In the last section, we have given explicit bounds on the decoding error probability for the RALP decoder. While these bounds apply to regular RA($q$) codes with even $q$, it is possible to extend the error bounds to irregular RA codes, where all repetition degrees are even. This is apparent if we note that the proofs of Theorems 1 and 3–6 do not make any assumption on the regularity of the code. The fact that all repetition degrees must be even is required in the proof of Theorem 3 (this, both for the regular and irregular cases). However, we are unable to provide an extension of Theorem 2 to irregular codes. In other words, the construction of an interleaver which yields an auxiliary graph with girth $g = \log_q n - 1$ does not easily extend to the irregular case. Still, whatever girth may be achieved for a specific auxiliary graph of an irregular RA code can be used to apply Theorem 6 to any MBIOS channel. That is, if a girth $g'$ can be achieved for an irregular graph (rather than $g = \log_q n - 1$ when $\pi_E$ is used as an interleaver), we would have that

$$\text{WEP} < n(2q_{\text{max}} - 1)^{\frac{1}{2}g'} \cdot \Pr \left( \sum_{i=1}^{\frac{1}{2}g'} \tilde{z}_i \leq 0 \right)$$

(13)

instead of (12), where $q_{\text{max}}$ now denotes the maximum repetition degree; the quantity $n(2q_{\text{max}} - 1)^{\frac{1}{2}g'}$ is a revised bound on the number of possible simple paths and cycles with $\frac{1}{2}g'$ Hamiltonian edges in the irregular graph. It particularizes to the expression in the proof of Theorem 4 in the special case of a regular code.

In Table I we give the thresholds for the BSC transition probability for which the error bound (7) decays to zero, for some choices of $q$. It is apparent that the threshold worsens as $q$ increases. This is in contrast to our expectation that coding performance should improve with the reduction of coding rate. Obviously, one cause for this is that having a negative cost path or cycle with $\frac{1}{2}g'$ Hamiltonian edges is only a necessary condition for decoding failure. It would be reasonable to conjecture that, as $q$ increases, many structural restrictions other than this condition must exist in order to have a negative-cost hyperpromenade. Also, the reliance on a union bound over all possible simple paths and cycles undermines the tightness of the bound. Furthermore, if we were to examine an irregular RA code, it can be seen that the bound in (13) would yield the same result for the irregular code as well as for a regular code with repetition degree $q_{\text{max}}$. Thus the possible improvement (increase) in the coding rate obtained by reducing the repetition degrees of some information symbols is not reflected in our bound.

We further note that the improvement over Feldman’s work presented in [6] for $q = 2$ does not seem
to extend to \( q > 2 \). This is due to the following. In [6], an improvement for \( q = 2 \) is obtained by a careful characterization of cycles in the auxiliary graph; since for \( q = 2 \) every cycle is a promenade, (thus finding a cycle is a sufficient condition for identifying a promenade) the study in [6] successfully captures all error events necessary for an upper bound. The distinction between the \( q = 2 \) case and \( q > 2 \) is that in the latter case, not every cycle is a valid hyperpromenade. Consequently, analysis of cycles alone is insufficient in order to obtain an upper bound on the decoding error probability for \( q > 2 \).

V. Summary

We have presented an upper bound on the word error probability of regular and irregular RA codes transmitted over MBIOS channels and decoded by the RALP decoder. This bounding technique extends the one presented by Feldman [1] for regular RA(2) codes to the case of regular RA(q) codes and to irregular RA codes with even repetition degrees. Our technique essentially relies on applying Euler’s graph-theoretic theorem to an appropriately-defined graph (i.e., the hyperpromenade graph).

APPENDIX A

PROOF OF THEOREM 2

Theorem 2 Let \( n = qk \) be the block length of a regular RA(q) code, \( q \geq 3 \) and \( n \geq q^4 \). Then one may construct for this code an auxiliary graph which is a Hamiltonian line plus \( kq \)-hyperedges which form a matching, so that the auxiliary graph has girth no less than \( g = \lfloor \log_q n \rfloor - 1 \).

In the proof we will assume \( n \) is a power of \( q \) to avoid using floor and ceiling notations. The proof easily extends to the general case.

Proof: Let \( H \) be a Hamiltonian cycle with \( n = q^{g+1} \) vertices. Let \( E_0 \) be the set of edges in \( H \), and \( V \) the set of vertices (this is denoted by \( H = (V, E_0) \)). Let \( D \) denote the set of all possible \( q \)-hyperedges, and let \( A \subseteq D \) satisfy the following conditions

1) No vertex is incident with more than one \( q \)-hyperedge in \( A \).
2) The girth of \( H_A = (V, E_0 \cup A) \) is not less than \( g \).

Then we shall show that if \( |A| < q^g \), there exists \( A^+ \subseteq D \) such that \( |A^+| = |A| + 1 \) and \( A^+ \) satisfies [1] and [2]. By repeatedly applying this result we obtain that there exists some set \( A \) with \( |A| = q^g \) satisfying the above two conditions.

Let \( d_A \) be the distance function in \( H_A \). Let \( V_2(A) \subseteq V \) denote the set of vertices with degree 2 in \( H_A \), i.e., those which are not incident with any \( q \)-hyperedge in \( A \). Given that \( |A| < q^g \), it follows that \( V_2(A) \) has at least \( q \) members. If some set of \( q \) vertices \( p_1, p_2, \ldots, p_q \in V_2(A) \) is such that \( d_A(p_i, p_j) \geq g - 1 \) for all \( i, j \in \{1, \ldots, q\}, i \neq j \), then the set \( A^+ = A \cup \{p_1p_2\ldots p_q\} \) satisfies the required conditions.

Suppose there is no such set of \( q \) vertices. Define \( t \) to be the maximum number of vertices \( p_1, \ldots, p_t \in V_2(A) \) such that \( d_A(p_i, p_j) \geq g - 1 \) for all \( i, j \in \{1, \ldots, t\}, i \neq j \). By our assumption we have that
1 \leq t \leq q - 1. Select vertices \( p_1, \ldots, p_t \) which achieve this maximum. Let

\[
D_r(z) = \{ v \in V | d_A(z, v) \leq r \}
\]

(14)

We claim that

\[
V_2(A) \subseteq U' \triangleq D_{g-1}(p_1) \cup D_{g-1}(p_2) \cup \ldots \cup D_{g-1}(p_t)
\]

(15)

This is easily seen, as follows. Suppose there is a vertex \( p_{t+1} \in V_2(A) \setminus U' \). Then the set \( p_1, \ldots, p_t, p_{t+1} \) is such that \( d_A(p_i, p_j) \geq g - 1 \) for all \( i, j \in \{1, \ldots, t+1\} \); this is in contradiction with the definition of \( t \).

Set \( p_1, \ldots, p_t \) according to the definition above, and choose \( p_{t+1}, \ldots, p_q \in V_2(A) \) arbitrarily. For any \( x \in V_2(A) \) the set \( D_{g-1}(x) \) has size at most

\[
1 + 2 + 2q + 2q^2 + \cdots + 2q^{q-2} = 1 + 2q^{q-1} - 1
\]

(16)

Consequently, if \( U \triangleq D_{g-1}(p_1) \cup D_{g-1}(p_2) \cup \ldots \cup D_{g-1}(p_q) \), then

\[
|U| \leq |D_{g-1}(p_1)| + |D_{g-1}(p_2)| + \ldots + |D_{g-1}(p_q)| \leq q + 2q^{q-1} - 1
\]

(17)

Let \( W = V \setminus U \). Since \( |V| = q^{q+1} \), it follows from the preceding inequality that

\[
|W| \geq q^{q+1} - q - 2q^{q-1} - 1
\]

(18)

Let \( \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_q \in W \) be arbitrary vertices, and let \( \tilde{U} \triangleq D_{g-1}(\tilde{p}_1) \cup D_{g-1}(\tilde{p}_2) \cup \ldots \cup D_{g-1}(\tilde{p}_q) \). This situation is depicted in Figure 5.

We have that \( D_{g-1}(\tilde{p}_1) \) has size at most

\[
1 + (1 + q) + (1 + q)q + (1 + q)q^2 + \cdots + (1 + q)q^{q-2} = 1 + (1 + q)q^{q-1} - 1
\]

(19)

and therefore \( |\tilde{U}| \) satisfies

\[
|\tilde{U}| \leq q + q(1 + q)q^{q-1} - 1 (a) \leq |W|
\]

(20)

where (a) stems from plugging in the expression from (18) and applying some algebra \(^3\). Now, (20) implies that there exist vertices \( s_1, s_2, \ldots, s_q \in W \) such that any pair \( i \neq j \) has \( d_A(s_i, s_j) \geq g - 1 \).

To see this, note that one may select \( s_1, s_2, \ldots, s_q \in W \) sequentially, as follows: first select \( s_1 \in W \) arbitrarily, then select for \( i = 2, \ldots, q \)

\[
s_i \in W \setminus \bigcup_{j=1}^{i-1} D_{g-1}(s_j)
\]

(21)

arbitrarily; (20) ensures that the set in the RHS of (21) is nonempty. We further have by definition of \( W \) that none of the vertices \( s_1, s_2, \ldots, s_q \) are in \( V_2(A) \). Therefore, for every \( s_i, i = 1, \ldots, q \) there is a distinct hyperedge \( s_is_1^{(2)} \ldots s_i^{(q)} \). Consider these \( q \)-hyperedges, \( s_1s_1^{(2)} \ldots s_1^{(q)}, s_2s_2^{(2)} \ldots s_2^{(q)}, \ldots, s_qs_q^{(2)} \ldots s_q^{(q)} \).

\(^3\)one needs to assume here that \( q \geq 3 \) and \( g \geq 3 \); \( g \geq 3 \) follows from the assumption that \( n \geq q^4 \).
Since all vertices in $W$ have distance at least $g$ from $p_1, p_2, \ldots, p_q$, it follows that $s_j^{(i)}$, $2 \leq i \leq q$, $1 \leq j \leq q$ all have distance at least $g-1$ from $p_1, p_2, \ldots, p_q$. Therefore, the set

$$A^+ = A \cup \left\{ p_1 s_1^{(2)} \ldots s_1^{(q)}, p_2 s_2^{(2)} \ldots s_2^{(q)}, \ldots, p_q s_q^{(2)} \ldots s_q^{(q)} \right\}$$

$$\setminus \left\{ s_1 s_1^{(2)} \ldots s_1^{(q)}, s_2 s_2^{(2)} \ldots s_2^{(q)}, \ldots, s_q s_q^{(2)} \ldots s_q^{(q)} \right\}$$

satisfies the required conditions.

Once we have built a circle of vertices and a $q$-fold matching between them, the theorem follows by removing one of the edges along the circle; this is the nonexistent edge in the graph $\Theta$ between the first and last nodes. Removing this edge does not reduce the girth of the auxiliary graph, and completes the desired construction.

**Discussion.** The proof of the theorem incurs bounding the size of the neighborhood sets $D_{g-1}()$. We could have tightened the bounds we used, e.g. in (16), by noting that if a matching edge is traversed, the next level neighbor can be only one of two choices (a Hamiltonian edge must be used next). This would have replaced equations (16) and (19) with more elaborate, albeit precise expressions. Consequently, the girth bound would have improved by a constant factor at most. This is of lesser importance than the ultimate behavior of the girth of the graph which is logarithmic in the block length. We thus omit this refinement.

Our proof uses a construction to show it is possible to build the desired graph. We note that this
construction contains degrees of freedom which can lead to different results.

The complexity of the proposed construction. We will show that the complexity of constructing the matching above is polynomial in \( n \), and in particular that it is no more than \( O(n^{q+2}) \) in time and space. To show this, we go over the stages of the construction and bound their complexity (some technical details are omitted).

1) The basic iteration step in the construction involves adding a hyperedge to the graph. This step is performed \( k = n/q \) times. We therefore examine the worst-case complexity of this basic step and multiply the result by \( k \).

2) In each iteration, construct for every vertex \( x \in V \) the set \( D_{g-1}(x) \), in the form of a list. This entails a complexity of no more than \( n \cdot q^g = O(n^2) \), since \( q^g \) is an upper bound on the size of \( D_{g-1}(x) \) (see Eq. (19)).

3) Using the lists constructed in step (2), we need to determine if there exists a set of vertices \( p_1, p_2, \ldots, p_q \in V_2(A) \) such that \( d_A(p_i, p_j) \geq g - 1, i \neq j \). It can be seen that the complexity of this step is no more than \( O(nq^{q+1}) \). This bound includes the complexity associated with finding the vertices \( p_1, \ldots, p_t \) defined in the construction. If \( t = q \), the iteration step ends. If not, we need to construct the set \( W \) and find the vertices \( s_1, s_2, \ldots, s_q \in W \) such that any pair \( i \neq j \) has \( d_A(s_i, s_j) \geq g - 1 \).

4) The construction of the set \( W \) makes use of the precalculated lists of neighbors, and can be seen to have complexity no more than \( O(n) \).

5) The final step requires finding vertices \( s_1, s_2, \ldots, s_q \in W \) such that any pair \( i \neq j \) has \( d_A(s_i, s_j) \geq g - 1 \). Using the sequential construction described above, the worst-case complexity can be seen to be \( O(n^2) \). Once the vertices are found, the matching hyperedges are added and the iteration step ends.

The total complexity of constructing a high-girth interleaver is thus no more than

\[
\frac{n}{q} \left( O(n^2) + O(n^{q+1}) + O(n^q) + O(n^2) \right) = O(n^{q+2})
\]

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