Relevant multi-setting tight Bell inequalities for qubits and qutrits

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In the celebrated paper [J. Phys. A: Math. Gen. 37, 1775 (2004)], D. Collins and N. Gisin presented for the first time a three setting Bell inequality (here we call it CG inequality for simplicity) which is relevant to the Clauser-Horne-Shimony-Holt (CHSH) inequality. Inspired by their brilliant ideas, we obtained some multi-setting tight Bell inequalities, which are relevant to the CHSH inequality and the CG inequality. Moreover, we generalized the method in the paper [Phys. Rev. A 79, 012115 (2009)] to construct Bell inequality for qubits to higher dimensional system. Based on the generalized method, we present, for the first time, a three setting tight Bell inequality for two qutrits, which is maximally violated by nonmaximally entangled states and relevant to the Collins-Gisin-Linden-Massar-Popescu inequality.

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I. INTRODUCTION

Two Bell inequalities (BIs) are said to be relevant if there exist quantum states that violate one of them but not the other. Analogously, an inequality is said to be relevant to a given set of inequalities if there exist quantum states violating it, but not violating any of the inequalities in the set [1]. A long-surviving open question proposed by N. Gisin is that [1]: for a given Hilbert space whose dimension is limited, is there a finite set of inequalities such that no other inequality is relevant with respect to that set? The significance considering this problems are two-folded: On the hand, it helps to reveal the recondite relationship between various multi-setting BIs. On the other hand, since BIs are at the heart of the study of non-locality and many rather nonintuitive quantum phenomena such as quantum key secret sharing [2], quantum communication complexity problems [3], etc., can be measured with BIs of some form, it may contribute a lot to the field of quantum information and computation.

Historically, Bell inequality, which may rule out all local hidden-variables theories, was first proposed by J. S. Bell in 1964 [4] when studying the Einstein-Podolsky-Rosen (EPR) paradox [5]. After his pioneering work, there arose many generalization of the original Bell inequality. Among all of these generalizations, the Clauser-Horne-Shimony-Holt (CHSH) inequality [6], which is a two-setting inequality for two-qubit system, may be the most famous one. The CHSH inequality has many merits: (i) it is tight, i.e., it defines one of the facets of the convex polytope [7] of local-realistic (LR) models; (ii) it violate all the pure two-qubit entangled states; (iii) it is maximally violated by maximally entangled states. Another significant step to generalize the Bell inequality for two $d$-dimensional (qudit) system is made by D. Collins et al., who constructed a CHSH type inequality for arbitrary $d$-dimensional (qudit) systems in 2002, now known as the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [8]. This inequality was shown to be tight [9] but it doesn’t preserve the merit (iii) for CHSH inequality, i.e., it’s maximal violation occurs at the nonmaximally entangled states [10]. It is worthwhile to note that, more recently, a new tight inequality preserve this merit was proposed by S. W. Lee and D. Jaksch [11].

All the inequalities mentioned above belong to the two-setting Bell inequalities, i.e., they are based on the standard Bell experiment, in which each local observer is given a choice between two dichotomic observables. However, one may extend the number of measurement settings. Actually, multi-setting Bell inequalities may have many advantages in many protocols in quantum information theory. Then, to constructing new multi-setting BIs is an important and pressing work.

Let’s go back to the open question mentioned in the first paragraph. The first step concerning this problem was made by D. Collins and N. Gisin. In 2004, they presented for the first time a three setting Bell inequality (here we call it CG inequality for simplicity) which is relevant to the CHSH inequality [12]. In this paper, we generalized the method introduced in Ref. [13] for constructing Bell inequality to higher dimensional systems. Using this method, we obtain some multi-setting tight Bell inequalities, which are relevant to the CHSH inequality and the CG inequality. Moreover, we present, for the first time, a three setting tight Bell inequality for two qutrits, which is maximally violated by nonmaximally entangled states and relevant to the CGLMP inequality.

The article is organized as follows. In Sec. II, we first review the method for constructing Bell inequality for qubits introduced in Ref. [13] and then generalize this method to higher dimensional system. In Sec. III, we presented various multi-setting tight Bell inequalities for two-qubit systems. In Sec. IV, a three-setting tight Bell inequality, which is maximally violated by nonmaximally
entangled state and relevant to the CGLMP inequality, is introduced. Finally in Sec. V, we concluded the article with some final remarks.

II. AN EFFECTIVE METHOD FOR CONSTRUCTING BELL INEQUALITY

Finding all the Bell inequalities for a given number of measurement settings and outcomes is a very difficult problem [14]. There exist various method to construct Bell inequalities [15]. In this section, we first review the method for constructing Bell inequality for qubits introduced in Ref. [13] and then generalize it to higher dimensional systems case. In Ref. [13], the author develop a systematic approach to establish Bell inequalities for qubits based on the Cauchy-Schwarz inequality. Let’s consider the following Bell-type scenario: N spatially separated observers measures independently among M observables with two possible outcomes −1 and 1, determined by some local parameters denoted by $\lambda$. We denote by $X_j(\mathbf{n}_{k_j}, \lambda)$, or $X_{j,k_j}$ for simplicity, the observables on the $j$-th party. Then the correlation function, in the case of a local realistic theory, is defined as $Q(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}, \cdots, \mathbf{n}_{k_N}) = \int_{\Omega} \prod_{j=1}^{N} X_j(\mathbf{n}_{k_j}, \lambda) | \rho(\lambda) \rangle d\lambda$, where $j = 1, 2, \cdots, N$ and $k_j = 1, 2, \cdots, M$. For convenience, we denote the correlation function $Q(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}, \cdots, \mathbf{n}_{k_N})$ as $Q_{k_1,k_2,\cdots,k_N}$. Define the Bell function as:

$$B(\lambda) = \sum_{\chi} C_{\chi} \prod_{j=1}^{N} X_{j,k_j}.$$  

(1)

Note that the symbol $\chi$ associated with $\prod_{j=1}^{N} X_{j,k_j}$ stands for $N$ pairs of indices (one pair for each observer). Obviously, there are $N_{\chi} = (1 + M)^N$ distinct values of $\chi$.

The constant numbers $C_{\chi}$s are coefficients of $\prod_{j=1}^{N} X_{j,k_j}$. In a local realistic theory, for each set of values of $X_j$s, the Bell function corresponds to a number, which is called a “root” of the Bell function $B(\lambda)$. We now rewrite the theorem 1 and 2 in Ref. [13] as a single theorem without proof. Anyone who want to know the details please see Ref. [13].

**Theorem 1.** Let $S_2 = \{ B(\lambda) \mid B^2(\lambda) = 1 \}$, i.e., $B(\lambda)$ must have two distinct “roots” $\Lambda_1 = -1$ and $\Lambda_2 = 1$, then for $\forall B(\lambda) \in S_2$, one has the Bell inequality: $|\langle B(\lambda) \rangle_{LHV} | \leq 1$. In general, if $S_n = \{ B(\lambda) \mid \prod_{j=0}^{n-1} (B - \Lambda_j) = 0, B(\lambda) \notin \bigcup_{k \in \mathbb{Z}, n \geq 3} S_k, n \in \text{integers}, n \geq 3 \}$, which means that $n$ “roots” of $B(\lambda)$ uniformly distribute between $-1$ and 1 with $\Lambda_j = -1 + 2j/(n - 1)$, for $\forall B(\lambda) \in S_n$, one has the Bell inequality: $|\langle B(\lambda) \rangle_{LHV} | \leq 1$.

Based on this theorem, we only need to determine the coefficients $C_{\chi}$ in Eq. (1) so that the Bell function $B(\lambda)$ has only specified distinct “roots”. This can be done by solve a series of equations of $C_{\chi}$s. To generalize this method to $N$-qudit system, where the observable $X_{j,k_j}$ on the $j$-th party has $d$ possible outcomes 0, 1, $\cdots$, $d - 1$, we should use the joint probability instead of the correlation function for the snake of simplicity and convenience. Denote by $P(X_{1,1}, = x_{1,1}, \cdots, X_{N,M_N} = x_{N,M_N})$ the joint probability that the $j$th party measures $X_{j,k_j}$ and obtains the outcome $x_{j,k_j}$, then the general Bell function for $N$-qudit system can be defined as:

$$B(\lambda) = \sum_{\chi} C_{\chi} P(X_{1,1} = x_{1,1}, \cdots, X_{N,M_N} = x_{N,M_N}).$$  

(2)

After this modification of the Bell function, one can also use the theorem 1 to construct Bell inequality for $N$-qudit systems and the procedures are the same as that of qubits case.

This method is efficient to construct various multi-setting Bell inequality. However, like many other method to construct Bell inequalities, it has its own disadvantages: (i) the equations of $C_{\chi}$s are Pan-set equations, i.e., the number of independent equations is less than the number of variables $C_{\chi}$s. Thus it is difficult to find all the solves of this equations; (ii) Some inequalities obtained by this method are trivial, i.e., they cannot be violated in quantum mechanics; (iii) This method gives no information about the tightness of the obtained Bell inequalities. Plentiful numerical results provide an empirical conclusion: the less the number of distinct “roots” of the Bell function, the more likely that the corresponding Bell inequality is tight. (iv) The number of unknown coefficients $C_{\chi}$ grows exponentially with the dimension of the system. Thus, when the system get large, it is very difficult to solve the equations of $C_{\chi}$s. Actually, in order to simplify the calculation, we always choose only these Bell functions which possess some nature symmetry but not the general one as defined in equations (1) or (2).

III. MULTI-SETTING TIGHT BELL INEQUALITY FOR TWO QUBITS

In this section, we shall focus on Bell inequality for two-particle systems. The Bell-type scenario involves only two observers and each of them measures $M$ different local observables of two outcomes $\pm 1$. For simplicity and convenience, we denote $X_{1,1s}$, $X_{2,2s}$ as $A_k$ and $B_k$ ($k = 1, \cdots, M$) respectively. The correlation function $Q(A_i B_j)$, in the case of a local realistic theory, is then the average values of the products $A_iB_j$ over many runs of the experiment. We also denote $Q(A_i)$, $Q(A_j)$ and $Q(B_j)$ as $Q_{1j}$, $Q_{2j}$ and $Q_{0j}$, respectively. Then the famous CHSH inequality:

$$I_{CHSH} = Q_{11} + Q_{12} + Q_{21} - Q_{22} \leq 2.$$  

(3)

holds in any local realistic theory. The CHSH inequality is almost always the most efficient one to prove a quantum state to be nonlocal. The first Bell inequality relevant to the CHSH inequality was proposed by D. Collins and N. Gisin in 2004 [12]. In the form of joint probability, their inequality reads:
It is easy to check that the CG inequality is tight and its corresponding Bell function has four distinct “roots”. For state:  

$$\rho_{cg} = 0.85P(|\psi\rangle) + 0.15P(|01\rangle).$$

(5)

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, one can check that this state does not violate the CHSH inequality. However it does violate the CG inequality and the violation is 0.0129.

Inspired by their brilliant ideas, we find various Bell inequalities, which are relevant to the CHSH inequality and the CG inequality. Our approach to the new Bell inequalities are based on the method introduced in Sec. II. To illuminate how this method works, here we show the details for the constructing of a three-setting BI. Firstly, we define a three-setting Bell function as:

$$B(\lambda) = c_0 + c_1(A_1 + B_1) + c_2(A_2 + B_2) + c_3(A_3 + B_3) + c_4(A_1B_1) + c_5(A_1B_2 + A_2B_1) + c_6(A_1B_3 + A_3B_1) + c_7(A_2B_2) + c_8(A_2B_3 + A_3B_2) + c_9A_3B_3.$$  

(6)

Here, we assume that the Bell function is symmetric under the permutations of $A_i$ and $B_i$. Then, in order to find a inequality whose corresponding Bell function may have four distinct “roots”, we should calculate $Z(\lambda) = (B^2(\lambda) - 1)(B^2(\lambda) - 1/9)$ and simplify the result using the equations $A_i^2 = 1$ and $B_i^2 = 1$. Let all the coefficients in the simplified expression of $Z(\lambda)$ equal to zero and solve the equations. Finally, one obtain a series of solutions for the coefficients $c_j$ ($j = 0, \cdots, 9$). Each solution corresponds to a Bell inequality. However, some of these inequalities are trivial or equivalent, thus we have to rule out them by calculate the quantum violation and check the tightness of each inequality. After some direct calculation, our new three-setting BI is as follows:

$$-8 \leq I_3^4 = Q_{21} + Q_{12} + Q_{31} + Q_{13} + Q_{32} + Q_{23} - Q_{11} - Q_{22} + Q_{10} + Q_{01} - Q_{20} - Q_{02} \leq 4.$$  

(7)

In quantum mechanics, the observables could be spin projections onto unit vectors $A_i = n_{a_i} \cdot \vec{\sigma}$, $B_j = n_{a_j} \cdot \vec{\sigma}$, and for a two-qubit state $\rho$, we have: $Q(A_iB_j) = \text{Tr}[(n_{a_i} \cdot \vec{\sigma} \otimes n_{a_j} \cdot \vec{\sigma}) \rho]$. $Q(A_iB_j) = \text{Tr}[(\mathbb{I} \otimes n_{a_j} \cdot \vec{\sigma}) \rho]$. Here, I is the $2 \times 2$ identity matrix and $\vec{\sigma}$ is the vector of Pauli matrices. Inequality (7) is tight and also have four distinct “roots”. Its maximal quantum violation occurs at the maximally entangled state: $|\psi\rangle_{max}^2 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and the the violation is 5. This inequality is relevant to the CHSH inequality. One can check that the state $\rho_{cg}$ also violate this inequality and the violation is 4.0516. In fact, this inequality is equivalent to the CG inequality.

We also obtain two four-setting tight Bell inequalities. The first one, which has four distinct “roots”, reads:

$$-6 \leq I_4^4 = Q_{11} + Q_{22} + Q_{12} + Q_{21} + Q_{21} + Q_{14} + Q_{14} - Q_{24} - Q_{42} - 2Q_{33} + Q_{31} + Q_{13} + Q_{32} + Q_{23} \leq 6.$$  

(8)

The inequality (8) is relevant to the CHSH inequality and the CG inequality. In fact, we have find numerically many states that do not violate the CHSH (or CG) inequality but violate the inequality (8) and vice versa. Here, for simplicity, we only present a single one that do not violate the CG inequality but violate the inequality (8):

$$\rho_1 = \begin{pmatrix}
0.046125 & -0.057737 & 0.017786i & -0.000649 & -0.092414i & 0.054845 & + & 0.071287i \\
-0.057737 & -0.017786i & 0.146863 & -0.039254 & -0.242031i & 0.428573 & -0.307414 & + & 0.244118i \\
-0.000649 & -0.092414i & 0.039254 & -0.242031i & 0.428573 & -0.307414 & + & 0.244118i \\
0.054845 & -0.071287i & -0.103099 & + & 0.199746i & -0.307414 & -0.244118i & 0.378439
\end{pmatrix}.$$

One may easily check that $\rho_1$ does not violate the CG inequality but do violate the inequality (8) and the violation is 6.33804. The maximal violation of the inequality (8) also occurs at the maximally entangled state $|\psi\rangle_{max}^2$ and the violation is 8.1655. Its threshold visibility is 0.7348, which is larger than that of the CHSH inequality, indicating that this inequality is not as strong as the CHSH inequality. The other new four-setting tight BI with five distinct “roots” reads:

$$-10 \leq I_5^5 = Q_{10} + Q_{01} - Q_{12} - Q_{21} + Q_{33} - Q_{31} + Q_{13} + Q_{32} + Q_{23} - Q_{04} - Q_{40} + Q_{41} + Q_{14} + Q_{43} + Q_{34} \leq 6.$$  

(9)
It is surprising that the maximal quantum violation of this inequality occurs at the nonmaximally entangled state $|\psi\rangle_{2}^{\text{max}} = 0.718824|00\rangle + 0.695192|11\rangle$ and the violation is 7.74134, which is larger than 7.7395, the violation of the maximally entangled state $|\psi\rangle_{2}^{\text{max}}$. One can also check that this inequality is relevant to the CHSH inequality and the inequality (8).

It will be useful for later on to write inequality (8) in the following way:

$$ I_{[4]}^{s} = \begin{pmatrix}
A_1 & A_2 & A_3 & A_4 \\
B_1 & 1 & 1 & 1 \\
B_2 & 1 & 1 & -1 \\
B_3 & 1 & 1 & -2 \\
B_4 & 1 & -1 & 0
\end{pmatrix}. \quad (10) $$

Here, the coefficient in the matrix indicate the coefficients of the corresponding expectation values.

Using the same method, we also found many six-setting Bell inequality for two qubits. Here we present only one of them whose correlation coefficients are regular with respect to the CHSH inequality and inequality (8). In the matrix form, it reads:

$$ I_{[6]}^{e} = \begin{pmatrix}
A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\
B_1 & 1 & 1 & 1 & 1 & 1 \\
B_2 & 1 & 1 & 1 & 1 & -1 \\
B_3 & 1 & 1 & 1 & -2 & 0 \\
B_4 & 1 & 1 & -3 & 0 & 0 \\
B_5 & 1 & 1 & -2 & 0 & 0 & 0 \\
B_6 & 1 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}. \quad (11) $$

Inequality (11) is also tight and relevant to the previous inequalities. Its corresponding Bell function has seven distinct “roots”. From our numerical results, it seems like that multi-setting tight Bell inequalities with different number of distinct “roots” may be always relevant to each other. Inspired by inequality (3), (10) and (11), it is not difficult to guess the general form of a set of even setting Bell inequalities:

$$ I_{[2n]}^{r} = \begin{pmatrix}
A_1 & A_2 & A_3 & \cdots & A_n & A_{n+1} & A_{n+2} & \cdots & A_{2n-2} & A_{2n-1} & A_{2n} \\
B_1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
B_2 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\
B_3 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & -2 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
B_n & 1 & 1 & 1 & \cdots & 1 & 1 & -(n-1) & \cdots & 0 & 0 & 0 \\
B_{n+1} & 1 & 1 & 1 & \cdots & 1 & -n & 0 & \cdots & 0 & 0 & 0 \\
B_{n+2} & 1 & 1 & 1 & \cdots & -(n-1) & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
B_{2n-2} & 1 & 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
B_{2n-1} & 1 & 1 & -2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
B_{2n} & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} \leq n(n + 1). \quad (12) $$

Numerically, the inequality (12) is tight and its corresponding Bell function has $(n^2 + n + 2)/2$ distinct “roots”. In fact, we have checked it for less than 22 settings and believe this is true for arbitrary even settings. However, to prove it analytically seems difficult and we have to leave it open. It is worthwhile to note that the inequality (12) was also obtained in Ref. [1] by searching numerically all possibilities assuming small integer coefficient of the matrix. Its quantum violation and other properties were also discussed there. We constructed this set of Bell inequalities using a fresh method. One of the interests of the inequality (12) is that some of its reduced inequalities are also tight. For instance, if we set $A_6$ and $B_6$ equal to 1, then we get a five-setting Bell inequality reduced from the inequality (11):

$$ I_{[5]}^{rd} = Q_{11} + Q_{12} + Q_{13} + Q_{14} + Q_{15} + Q_{21} + Q_{22} + Q_{23} + Q_{24} + Q_{25} + Q_{31} + Q_{32} + Q_{33} + Q_{34} - 2Q_{35} + Q_{41} + Q_{42} + Q_{43} - 3Q_{44} + Q_{31} + Q_{32} + Q_{51} + Q_{52} - 2Q_{53} + Q_{10} + Q_{01} - Q_{20} - Q_{02} \leq 12. \quad (13) $$

It is easy to check that inequality (13) is also tight. More generally, if we set $A_{2n}$ and $B_{2n}$ in inequality (12) equal to 1, then most of, not all, the reduced inequality $I_{[2n-1]}^{rd} \leq n(n + 1)$ is tight. In fact, we have check to
15 setting and find that only the inequality $I_{3}^{red}$ is not tight, which is surprising. We guess that all the inequalities $I_{2n-1}^{red} \leq n(n+1)$ are tight for all $n \geq 3$. If this is true, then we can obtain a set of odd setting tight Bell inequalities. Another interesting question related is whether we can get a BI reduced form inequality (12) that may stronger than the CHSH inequality?

IV. MULTI-SETTING TIGHT BELL INEQUALITY FOR TWO QUTRITS

Most of multi-setting Bell inequalities are construct for qubits [16, 17]. Up to now, there is few multi-setting Bell inequalities for higher dimensional systems. More recently, Ji et al. introduced a three setting Bell inequality for two-qutrit systems, which is maximally violated by maximally entangled state if local measurements are configured to be mutually unbiased [18]. Here based on the generalized method in Sec. II, we construct two new three-setting Bell inequalities. One of them is tight and relevant to the CGLMP inequality. The other one is not tight but it is as strong as the CGLMP inequality and also relevant to the CGLMP inequality. For two-qutrit system, the CGLMP inequality reduces to:

$$I_{CGLMP}^{(3)} = \left[ P(A_{1} = B_{1}) + P(B_{1} = A_{2} + 1) + P(A_{2} = B_{2} + P(B_{2} = A_{1}) \right] - \left[ P(A_{1} = B_{1} - 1) + P(B_{1} = A_{2}) + P(A_{2} = B_{2} - 1) + P(B_{2} = A_{1} - 1) \right] \leq 2. \quad (14)$$

Here we denote the joint probability $P(A_{i} = B_{j} + m)$ ($i, j = 1, 2$) that the measurements $A_{i}$ and $B_{j}$ have outcomes that differ, modulo three, by $m$: $P(A_{i} = B_{j} + m) = \sum_{n=0}^{\infty} P(A_{i} = a, B_{j} = a + m)$. The inequality (14) is tight [9] and it quantum violation is investigated in Ref. [10].

To find a new Bell inequality, we can use the method introduced in Sec. II. Let’s denote the outcome of $A_{i}$ and $B_{i}$ by $a_{i}$ and $b_{i}$ respectively. Unlike the qubits case, it is more convenient to express the Bell function in terms of joint probability $P(a_{i} = b_{j} \equiv r)$ that the measurements $A_{i}$ and $B_{j}$ have outcomes that differ, modulo three, by $r$: $P(a_{i} = b_{j} \equiv r) = \sum_{a,b} P(a_{i} = a, b_{j} = r-a)$. Then the Bell function is defined as: $B(\lambda) = \sum_{c} c_{ijr} P(A_{i} = B_{j} \equiv r)$. Here we specify $c_{ijr} = c_{jir}$ so that the Bell function is symmetric under the permutations of $A_{i}$ and $B_{i}$. In fact, this assumption is always valid since the two observers are in the same status in the Bell-type scenario for derivation of Bell inequality. Of course, the Bell inequalities corresponding to this kind of Bell functions are symmetric under the permutations of $A_{i}$ and $B_{i}$, too. After a similar procedure as described in Sec. III, we find a new three setting Bell inequality for two qutrits, which is relevant to the CGLMP inequality for two qutrits (namely, the inequality (14)):

$$I_{3}^{(3)} = -2P(a_{1} + b_{1} = 0) + P(a_{1} + b_{1} = 1) + P(a_{1} + b_{1} = 2) + P(a_{1} + b_{2} = 0) - P(a_{1} + b_{2} = 2) + P(a_{2} + b_{1} = 0) - P(a_{2} + b_{1} = 2) + P(a_{1} + b_{3} = 1) - P(a_{1} + b_{3} = 2) + P(a_{3} + b_{1} = 1) - P(a_{3} + b_{1} = 2) + P(a_{2} + b_{3} = 1) - P(a_{2} + b_{3} = 2) + P(a_{3} + b_{2} = 1) - P(a_{3} + b_{2} = 2) + P(a_{3} + b_{3} = 0) - P(a_{3} + b_{3} = 1) \leq 4 \quad (15)$$

It is easy to check that the inequality (15) is tight and has four distinct “roots”. To calculate the quantum violation, we should note that the quantum prediction of the joint probability $P(A_{i} = k, B_{j} = l)$ when $A_{i}$ and $B_{j}$ are measured in the initial state $|\psi\rangle$ is given by:

$$P(A_{i} = k, B_{j} = l) = |\langle kl | U(A) \otimes U(B) | \psi\rangle |^{2}$$

$$= \text{Tr} \{ [U(A)\dagger \otimes U(B)\dagger] \Pi_{k} \otimes \Pi_{l} [U(A) \otimes U(B)] | \psi\rangle \langle \psi | \} \quad (16)$$

where $U(A), U(B)$ are the unitary transformation matrix and $\Pi_{k} = \ket{k}_{k}, \Pi_{l} = \ket{l}_{l}$ are the projectors for systems A and B, respectively. Then for the maximally entangled
\begin{align}
I_3^{(3)} & = -P(a_1 + b_1 = 0) + P(a_1 + b_1 = 1) + 4P(a_1 + b_2 = 0) - 4P(a_1 + b_2 = 1) + 4P(a_2 + b_1 = 0) \\
& - 4P(a_2 + b_1 = 1) - 3P(a_1 + b_3 = 1) + 3P(a_1 + b_3 = 2) - 3P(a_3 + b_1 = 1) + 3P(a_3 + b_1 = 2) \\
& + 7P(a_2 + b_2 = 0) - 7P(a_2 + b_2 = 2) + 3P(a_2 + b_3 = 1) - 3P(a_2 + b_3 = 2) + 3P(a_3 + b_2 = 1) \\
& - 3P(a_3 + b_2 = 2) \leq 14
\end{align}

For the maximally entangled state $|\psi\rangle_3^{\text{max}}$, the quantum violation is 20.1105 and the corresponding threshold visibility is 0.6962 which is the same as that of the CGLMP inequality. Its maximal violation also occurs at a nonmaximally entangled state: $|\phi\rangle_3^{\text{max}} = 0.60297|00\rangle + 0.5641|11\rangle + 0.5641|22\rangle$ and the maximal violation is 5.1803. However, numerical results show that it is relevant to the CGLMP inequality. This is, to our knowledge, the first tight three-setting Bell inequality for two qutrits equivalent to the CGLMP inequality. Using the same method, we also find a three-setting Bell inequality for two qutrits, which is as strong as the CGLMP inequality:

\begin{center}
\textbf{V. CONCLUSION AND REMARKS}
\end{center}

In summary, we generalized the method in [13] to construct Bell inequality for qubits to higher dimensional systems. Using the method, we have obtained some multi-setting tight Bell inequalities, which are relevant to the CHSH inequality and the CGLMP inequality. Moreover, we present, for the first time, a three setting tight Bell inequality for two qutrits, which is maximally violated by nonmaximally entangled states and relevant to the CGLMP inequality. The concept “root” seems very important for Bell inequalities. It plays a vital role to construct and classify Bell inequalities. Besides, from the numerical results, it seems like that if two Bell functions have different numbers of distinct “roots”, then the corresponding tight Bell inequalities may be relevant to each other. A very important question related to the generalized method is: can we use this method to find a general tight Bell inequality for $N$-qudit system? We will investigate this problem subsequently.

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