Dichotomy and bounded solutions of dynamical systems in the Hilbert space.

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Abstract. For a general discrete dynamics on a Banach and Hilbert spaces we give a necessary and sufficient conditions of the existence of bounded solutions under assumption that the homogeneous difference equation admits an exponential dichotomy on the semi-axes. We consider the so called resonance (critical) case when the uniqueness of solution is disturbed. We show that admissibility can be reformulated in the terms of generalised or pseudoinvertibility. As an application we consider the case when the corresponding dynamical system is e-trichotomy.

Introduction.

Statement of the problem.

Consider the following equation

$$x_{n+1} = A_n x_n + h_n, n \in \mathbb{Z},$$  \hspace{1cm} (1)

where $A_n : \mathcal{H} \to \mathcal{H}$ is a set of bounded operators which has a bounded inverse and acts from the Hilbert space $\mathcal{H}$ onto itself. Suppose that

$$A = (A_n)_{n \in \mathbb{Z}} \in l_{\infty}(\mathbb{Z}, \mathcal{L}(\mathcal{H})), h = (h_n)_{n \in \mathbb{Z}} \in l_{\infty}(\mathbb{Z}, \mathcal{H}).$$

It means that

$$|||A||| = \sup_{n \in \mathbb{Z}} ||A_n|| < +\infty, |||h||| = \sup_{n \in \mathbb{Z}} ||h_n|| < +\infty.$$  

We need to establish conditions of the existence of bounded solutions of the equation (1).

Corresponding homogeneous equation has the following form:

$$x_{n+1} = A_n x_n.$$  \hspace{1cm} (2)

It should be noted that any solution of the homogeneous equation can be represented as: $x_m = \Phi(m, n)x_n, m \geq n,$ where $\Phi(m, n) = A_{m-1}A_{m-2}...A_{n+1}$ if $m > n,$ and $\Phi(m, m) = I.$

Clearly that $\Phi(m, 0) = A_{m-1}A_{m-2}...A_0, U(m) := \Phi(m, 0), U(0) = I.$

Traditionally, the mapping $\Phi(m, n)$ is called evolution operator of the problem (2). Suppose that the equation (2) is e-dichotomous on the semi axes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projectors $P$ and $Q$ in the space $\mathcal{H}$ respectively, i.e.

$$\exists k_1 \geq 1; 0 < \lambda_1 < 1; \exists P(P^2 = P) :$$

$$||U(n)PU^{-1}(m)|| \leq k_1 \lambda_1^{n-m}, n \geq m$$

$$||U(n)(E - P)U^{-1}(m)|| \leq k_1 \lambda_1^{m-n}, m \geq n$$
for any \(m, n \in \mathbb{Z}_+\) (dichotomy on \(\mathbb{Z}_+\));

\[
\exists k_2 \geq 1; 0 < \lambda_2 < 1; \exists Q(Q^2 = Q): \\
||U(n)QU^{-1}(m)|| \leq k_2 \lambda_2^{n-m}, n \geq m \\
||U(n)(E - Q)U^{-1}(m)|| \leq k_2 \lambda_2^{m-n}, m \geq n
\]

for any \(m, n \in \mathbb{Z}_-\) (dichotomy on \(\mathbb{Z}_-\)).

**Theorem 1.** Suppose that the homogeneous equation is e-dichotomous on the semi-axes \(\mathbb{Z}_+\) and \(\mathbb{Z}_-\) with projectors \(P\) and \(Q\) respectively and the operator \(D = P - (E - Q)\) is generalised invertible. Solutions of the equation (1) bounded on the whole axis exist if and only if the following condition is true

\[
\sum_{k=-\infty}^{+\infty} H(k+1)h_k = 0. \quad (3)
\]

Under condition (3) the set of bounded solutions has the following form:

\[
x_n(c) = U(n)PP_{N(D)}c + (G[h])(n), \quad (4)
\]

where

\[
G[h](n) = U(n)Z(n),
\]

\[
Z(n) = \begin{cases} 
\sum_{k=0}^{n-1} PU^{-1}(k+1)h_k - \sum_{k=n}^{+\infty} (E - P)U^{-1}(k+1)h_k + \\
+PD^{-1}[\sum_{k=0}^{+\infty} (E - P)U^{-1}(k+1)h_k + \\
+ \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ n \geq 0 \\
\sum_{k=-\infty}^{n-1} QU^{-1}(k+1)h_k - \sum_{k=n}^{+\infty} (E - Q)U^{-1}(k+1)h_k + \\
+(E - Q)D^{-1}[\sum_{k=0}^{+\infty} (E - P)U^{-1}(k+1)h_k + \\
+ \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ n \leq 0,
\end{cases}
\]

is generalised Green’s operator on \(\mathbb{Z}\) with following properties:

\[
(G[h])(0+0) - (G[h])(0-0) = \sum_{k=-\infty}^{+\infty} H(k+1)h_k = 0,
\]

\[
(LG[h])(n) = h_n, \ n \in \mathbb{Z},
\]

where

\[(Lx)(n) := x_{n+1} - A_n x_n : l_\infty(\mathbb{Z}, B) \to l_\infty(\mathbb{Z}, B), \]

\[
H(n+1) = P_{N(D^*)}QU^{-1}(n+1) = P_{N(D^*)}(E - P)U^{-1}(n+1), D_- \text{ is generalised invertible to operator } D, P_{N(D)} \text{ and } P_{N(D^*)} \text{ are projectors which project } B \text{ onto kernel } N(D) \text{ and cokernel } N(D^*) \text{ of operators } D \text{ and } D^* \text{ respectively.} \]
Proof. A general solution of the problem (1), bounded on the semi axes has the form:

\[
x_n(\xi) = \begin{cases}
  U(n)P\xi + \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \sum_{k=n}^{\infty} U(n)(E - P)U^{-1}(k+1)h_k, & n \geq 0 \\
  U(n)(E - Q)\xi + \sum_{k=-\infty}^{-1} U(n)QU^{-1}(k+1)h_k - \sum_{k=n}^{\infty} (E - Q)U^{-1}(k+1)h_k, & n \leq 0.
\end{cases}
\]

(5)

Prove that solution (5) is bounded on the semi axes \((\mathbb{Z}_+, \mathbb{Z}_-)\). Really, for any \(n \geq 0\) we have \(A_n U(n)P\xi = A_{n-1}...A_0 P\xi = U(n+1)P\xi\). Then \(x_{n+1} = A_n x_n\) and \(x_n\) is solution of the homogeneous equation (2) on \(\mathbb{Z}_+\). Further

\[
A_n \left( \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \sum_{k=n}^{\infty} U(n)(E - P)U^{-1}(k+1)h_k \right) + h_n = \\
= \sum_{k=0}^{n-1} U(n+1)PU^{-1}(k+1)h_k - \sum_{k=n}^{\infty} U(n+1)(E - P)U^{-1}(k+1)h_k + h_n = \\
= \sum_{k=0}^{n} U(n+1)PU^{-1}(k+1)h_k - \sum_{k=n+1}^{\infty} U(n+1)(E - P)U^{-1}(k+1)h_k + h_n - \\
- U(n+1)PU^{-1}(n+1)h_n - U(n+1)(E - P)U^{-1}(n+1)h_n = \\
= x_{n+1}(\xi).
\]

Prove that the represented solution is bounded on the semi axes. Estimate series:

\[
|| \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k || \leq ||h|| \sum_{k=0}^{n-1} ||U(n)PU^{-1}(k+1)|| \leq \\
\leq ||h|| \sum_{k=0}^{n-1} k_1 \lambda_1^n \lambda_{k+1} = ||h|| \lambda_1 \sum_{k=0}^{n-1} \lambda_1^{-(k+1)} = \\
= ||h|| \lambda_1 \frac{\frac{1}{\lambda_1}((\frac{1}{\lambda_1})^{n-1} - 1)}{\frac{1}{\lambda_1} - 1} < \infty,
\]

and

\[
|| \sum_{k=n}^{\infty} U(n)(E - P)U^{-1}(k+1)h_k || \leq ||h|| \sum_{k=n}^{\infty} k_1 \lambda_1^{k+1-n} = \\
= k_1 \lambda_1^{-n}(||h|| \sum_{k=n}^{\infty} \lambda_1^k = k_1 \lambda_1^{-n+1} \cdot \frac{\lambda_1^n}{1 - \lambda_1} < \infty.
\]

Boundedness of solution on \(\mathbb{Z}_-\) can be proved in such a way.
Find condition which guarantees that the solution (5) will be bounded on the whole integer axis. It will be if and only if when
\[ x_{0+}(\xi) = x_{0-}(\xi). \]

Substitute corresponding expressions we obtain
\[
P\xi - \sum_{k=0}^{\pm\infty} (E - P)U^{-1}(k+1)h_k = (E - Q)\xi + \sum_{k=\infty}^{-1} QU^{-1}(k+1)h_k.
\]

Consider the following element
\[
g = \sum_{k=0}^{\pm\infty} (E - P)U^{-1}(k+1)h_k + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k.
\]

Obtain the following operator equation
\[
D\xi = g. \tag{6}
\]

Since \( D \) is normally resolvable then as it is known \(^3\) a necessary and sufficient condition of solvability of the equation (6) is following:
\[
P_{N(D^*)}g = 0. \tag{7}
\]

Since \( DP_{N(D)} = 0 \), then we have that \( PP_{N(D)} = (E - Q)P_{N(D)}. \) Since \( P_{N(D^*)}D = 0 \) we have \( P_{N(D^*)}Q = P_{N(D^*)}(E - P). \) Due to the equality condition (7) can be rewritten as
\[
\sum_{k=\infty}^{+\infty} P_{N(D^*)}QU^{-1}(k+1)h_k = 0,
\]
or
\[
\sum_{k=-\infty}^{+\infty} P_{N(D^*)}(E - P)U^{-1}(k+1)h_k = 0.
\]

In such a way we prove the condition (3). If the condition (3) is true then \( \xi = D^*g + PP_{N(D)}c \), for any \( c \in B. \)

Direct substitution in the representation (5) gives us that the set of solutions bounded on the whole axis \( Z \) has the form (4). \( d \)-normal and \( n \)-normal operators play important role in the theory of boundary value problems.

For such class of operators we can obtain theorems with some refinements.

**Theorem 2.** Suppose that conditions of the theorem 1 are true and bounded operator \( D = P - (E - Q) \) is \( d \) - normal. Bounded on the whole axis solutions of the equation (7) exist if and only if \( d \) linearly independent conditions are true
\[
\sum_{k=-\infty}^{+\infty} H_d(k+1)h_k = 0. \tag{8}
\]
If the condition (8) is true then solutions bounded on the whole axis have the form (4), where
\[
H_d(n) = [P_{N(D^*)}Q]_{U^{-1}}(n) = [P_{N(D^*)}(E - P)]_{U^{-1}}(n),
\]
\[d \leq m \ (m = \dim coker(D) < \infty), \ d = \dim(P_{N(D^*)}Q).
\]

**Proof.** It should be noted that the operator \(P_{N(D^*)}\) is finite dimensional (since it is \(d\)-normal), and then the operator \(P_{N(D^*)}Q\) is finite dimensional (\((R(P_{N(D^*)}Q) \subset R(P_{N(D^*)}))\)).

**Theorem 3.** Suppose that conditions of the theorem 1 are true and the operator \(D = P - (E - Q)\) is \(n\)-normal. Then bounded on the whole axis solutions of the equation (1) exist if and only if the condition (3) is true. Under condition (3) the equation (1) has \(r\)-parametric set of bounded solutions
\[
x_n(c_r) = U(n)[PP_{N(D)}]_{U^{-1}}c_r + (G[h])(n),
\]
where \(r \leq n \ (n = \dim ker(D))\).

**Proof.** Since \(D\) is \(n\)-normal operator, then its kernel has finite dimension. It follows that the operator \(P_{N(D)}\) is finite dimensional and then the operator \(PP_{N(D)}\) is finite dimensional (\((R(PP_{N(D)}) \subset R(P_{N(D)}))\)). If \(\dim N(D) = n\), then
\[
\dim(PP_{N(D)}B) = \dim((E - Q)PP_{N(D)}) = r \leq n.
\]

**Theorem 4.** Suppose that condition of the theorem 1 is hold and the operator \(D = P - (E - Q)\) is Fredholm. Then bounded solutions of the equation (1) exist if and only if \(d\)-conditions (8) are true. Under condition (8) the equation (1) has \(r\)-parametric set of bounded solutions
\[
x_n(c_r) = U(n)[PP_{N(D)}]_{U^{-1}}c_r + (G[h])(n),
\]
where \(r \leq n \ (n = \dim ker(D)), \ d \leq m \ (m = \dim coker(D))\).

**Corollary.** Suppose that under condition of the theorem 1 \([P, Q] = PQ - QP = 0\) and \(PQ = Q\). It is so called e-trichotomy case of the equation (2) on \(\mathbb{Z}\). In this case nonhomogeneous equation (1) has at least one solution on \(\mathbb{Z}\) for any \(h \in l_{\infty}(\mathbb{Z}, B)\).

**Proof.** From the equality \(P_{N(D^*)}D = 0\) and \(DP = (P - (E - Q))P = QP = Q\) follow that \(P_{N(D^*)}Q = P_{N(D^*)}DP = 0\). From here we have solvability \(\forall h \in l_{\infty}(\mathbb{Z}, B)\).

**Corollary 1.** Under conditions of theorem 1 and additive condition \([P, Q] = PQ - QP = 0\) and \(PQ = Q = P\) nonhomogeneous equation (1) has unique bounded on \(\mathbb{Z}\) solution for any \(h \in l_{\infty}(\mathbb{Z}, B)\).
Remark. In this case considering system is e-dichotomous on the whole axis $\mathbb{Z}$. In the finite dimensional case the same result is well known [5]. The theorem 1 under less restrictive assumptions gives possibility to find the set of solutions.

Corollary 2. Suppose that $[P, Q] = 0$. Then operator $D$ has Moore-Penrose pseudo-invertible $D^+$ which is equal to $D$. In this case we have the following variants: 1 a.) equation (1) has solutions if and only if the following condition is true:

$$1 \text{ b.) under condition the set of bounded solutions has the form}$$

2 a.) equation (1) has bounded quasisolutions if and only if the following condition is true:

$$2 \text{ b.) under condition the set of bounded solutions has the form}$$

Proof. If $[P, Q] = 0$ then

$$DDD = (P - (I - Q))^3 = P^3 - 3P^2(I - Q) + 3P(I - Q)^2 - (I - Q)^3 =$$

$$P - 3(P - PQ) + 3P(I - Q) - (I - Q) = P - 3P + 3PQ + 3P - 3PQ - I + Q = P - I + Q = D.$$ 

From the equality we obtain that $D = D^+$.

In the Hilbert space we have more variants for the solutions.

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