We study constrained Hamiltonian systems by utilizing general forms of time discretization. We show that for explicit discretizations, the requirement of preserving the canonical Poisson bracket under discrete evolution imposes strong conditions on both allowable discretizations and Hamiltonians. These conditions permit time discretizations for a limited class of Hamiltonians, which does not include homogeneous cosmological models. We also present two general classes of implicit discretizations which preserve Poisson brackets for any Hamiltonian. Both types of discretizations generically do not preserve first class constraint algebras. Using this observation, we show that time discretization provides a complicated time gauge fixing for quantum gravity models, which may be compared with the alternative procedure of gauge fixing before discretization.

I. INTRODUCTION

In all approaches to quantum gravity the idea of a discrete structure of spacetime at the Planck scale is fundamental. In lattice approaches, such as Regge calculus, dynamical triangulations, and other path integral methods, discreteness in space and time is built in right from the start. In canonical methods, such as loop quantum gravity, discreteness of space arises as a rather natural consequence of Dirac quantization, and it is possible that a form of discrete time evolution may emerge from a regulated action of the Hamiltonian constraint. This is already manifested to some extent in spin foam models, which may be viewed as "covariantizations" of loop quantum gravity models.

In attempts to construct a path integral for quantum gravity, a standard starting point is a classical non-Hamiltonian discretization using triangulations. It is interesting to ask whether a phase space path integral for quantum gravity can be defined starting from a classical discretization that is Hamiltonian. If possible, this would provide a regularization that is more closely tied to the classical theory than triangulations or spin foam methods. There are two main approaches to the problem: (i) fix all coordinate gauge conditions classically and discretize the reduced Hamiltonian system, or (ii) attempt a discretization of the constrained system and study its consequences for the Hamiltonian structure. The primary requirement for canonical quantization of a discrete system is to ensure that the fundamental Poisson bracket is preserved under discrete classical evolution. In the second approach, the consequences of discretization on the constraint algebra also require scrutiny.

These questions have been studied to some extent in the literature. Discrete evolution schemes for unconstrained Hamiltonian dynamics appear in, with further refinements in, where schemes preserving the Poisson bracket are presented. More recently, the applicability of discrete evolution schemes to constrained Hamiltonian theories have been discussed, with a view to applicability to quantum gravity.

From elementary numerical methods, it is known that there are implicit and explicit approaches to discretization of differential equations. Although either approach may be useful for studying classical evolution, explicit methods are more natural for quantization. This is because discrete Hamiltonians that are functions of canonical variables at more than one time instant are effectively non-local in time, so it is unclear whether canonical quantization makes sense. It is therefore necessary to study classical discretizations for which the discrete Hamiltonian does not depend on more than one time point. This may restrict somewhat the potential applicability of implicit schemes.

For the purpose of developing a fundamental lattice approach to quantum gravity starting from a discretization of Hamilton’s equations, it is necessary to ensure that canonical structures are not affected by the procedures used. This is because classical preservation of Poisson brackets is a requirement for a clean approach to canonical quantization. The main purpose of this paper is to investigate this aspect. We study certain issues associated with discrete time evolution. We do not address space discretization, since our main concern is the extent to which time discretization can maintain Hamiltonian structures for quantization. Specifically we address the following questions: (i) What are the general classes of discretization? (ii) What are the conditions required to maintain the fundamental Poisson brackets under evolution for a given Hamiltonian system? (iii) Are there restrictions on allowed Hamiltonians? (iv) How does the approach work in simple systems, and (v) How viable is the approach for quantum gravity?

Our starting points are general forms of explicit and implicit discretizations containing arbitrary functions of phase
space variables. In Section II we derive the necessary conditions on these functions to preserve the fundamental Poisson bracket for both types of schemes. We also study the consequences of discretization for constraint evolution for the parametrized particle. In Section III we apply the approaches to the parametrized scalar field, FRW cosmological models, and BF theory. In the final section we discuss some consequences of our results for applicability to quantum gravity.

II. DISCRETE HAMILTONIAN EVOLUTION

As for differential equations, there are two general approaches that may be utilised to obtain a discretization of Hamilton's equations that preserve the canonical Poisson bracket. The explicit schemes, as the name suggests, give explicit functions of the form

\[ p_{n+1} = f(q_n, p_n), \quad q_{n+1} = g(q_n, p_n) \]  

for evolution. This is in contrast to implicit schemes, where the general equations are of the form

\[ F(p_{n+1}, q_{n+1}, p_n, q_n) = 0, \quad G(p_{n+1}, q_{n+1}, p_n, q_n) = 0. \]  

We consider both types of discretization, and outline the conditions under which the schemes preserve Poisson brackets, and thus have a utility for quantization at the discrete level.

A. Explicit schemes

Consider a one particle system with canonical phase space variables \((q, p)\), and arbitrary Hamiltonian \(H(q, p)\). A first attempt at time discretization of Hamilton's equations with step \(\epsilon\) might be the simplest finite differencing scheme

\[ q_{n+1} = q_n + \epsilon \frac{\partial H(q_n, p_n)}{\partial p_n}, \quad p_{n+1} = p_n - \epsilon \frac{\partial H(q_n, p_n)}{\partial q_n}. \]  

However there is an immediate problem with this: it does not preserve the Poisson bracket under evolution for even the standard Hamiltonian \(H(q, p) = p^2/2m + V(q)\); the order \(\epsilon^2\) term does not vanish unless \(V(q) = 0\). To rectify this it is necessary to introduce higher order terms in \(\epsilon\) in the defining equations of the discretization. This might be done using explicit or implicit finite differencing schemes. We choose a general explicit approach and study its consequences. The reasons are simplicity and potential utility for canonical quantization.

The defining equations for the discrete evolution we study are the following:

\[ q_{n+1} = q_n + \epsilon \frac{\partial H(q_n, p_n)}{\partial p_n} + \epsilon^2 \alpha(q_n, p_n), \quad p_{n+1} = p_n - \epsilon \frac{\partial H(q_n, p_n)}{\partial q_n} + \epsilon^2 \beta(q_n, p_n). \]  

\(\alpha, \beta\) are at this stage arbitrary functions. The Poisson bracket of the evolved variables is

\[ \{q_{n+1}, p_{n+1}\} = 1 - \epsilon \left[ \{q_n, \frac{\partial H}{\partial q_n}\} + \{p_n, \frac{\partial H}{\partial p_n}\} \right] + \epsilon^2 \left[ \{\alpha, p_n\} + \{q_n, \beta\} - \left\{ \frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial q_n} \right\} \right] + \epsilon^3 \left[ \left\{ \frac{\partial H}{\partial p_n}, \beta \right\} - \{\alpha, \frac{\partial H}{\partial q_n}\} \right] + \epsilon^4 \{\alpha, \beta\}. \]  

This equation displays the conditions required for preserving the canonical Poisson bracket. The first order term in \(\epsilon\) vanishes identically because the Hamiltonian is a function of the discrete phase space variables at a single time point. (See below for discussion of a discretization where this is not the case.) The remaining task is to see what functions
$H$, $\alpha$, and $\beta$ lead to vanishing of the three higher order terms in $\epsilon$. It is obviously better to fix $H$ by looking at systems of interest, and then attempt solutions for $\alpha$ and $\beta$.

In order to obtain a general procedure for solving these conditions, it is useful to consider first the Hamiltonian independent $\epsilon^4$ term. Its solutions can be classified into one of the following four cases: (i) $\alpha = 0$, (ii) $\beta = 0$, (iii) $\alpha(q_n), \beta(q_n)$ or $\alpha(p_n), \beta(p_n)$, and (iv) $\alpha = \beta \neq 0$. For a given Hamiltonian, the other conditions can be written out explicitly.

The restrictions that arise may be seen by considering the standard Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

The conditions have a unique solution: Case (ii) gives

$$\alpha(q) = \frac{1}{m} \frac{\partial V(q)}{\partial q}$$

The remaining cases (i),(iii) and (iv) do not give a solution. This result is surprising given that there are numerous possibilities for discretizing classical differential equations. The lesson is that preserving canonical Poisson brackets imposes rather strong conditions on allowable discretizations.

As a further illustration consider the parametrized particle. This case is also useful for seeing what happens to the constraints under discrete evolution. The canonical variables are $(t, p')$ and $(q, p)$, which depend on a time parameter $\lambda$. The Hamiltonian constraint is

$$H := p' + \frac{p^2}{2m} + V(q) = 0$$

Following (5), discrete evolution with lapse function $N$ is defined by

$$q_{n+1} = q_n + N\epsilon \frac{p_n}{m} + \epsilon^2 \alpha(q_n, p_n)$$
$$p_{n+1} = p_n - N\epsilon \frac{\partial V(q_n)}{\partial q_n} + \epsilon^2 \beta(q_n, p_n).$$

From these equations it is clear that preservation of the Poisson bracket goes through in the same way as for the unparametrized case, with $\beta = 0$ and $\alpha$ given by (5) up to a factor of $N$.

The second check is the evolution of constraints using (10). Is $H_{n+1} = H(t_{n+1}, p'_{n+1}, q_{n+1}, p_{n+1}) = H_n$? A straightforward computation using the evolution equations gives

$$H_{n+1} = p'_n + \frac{1}{2m} \left[ p_n - \epsilon N \partial V + \epsilon^2 \beta \right]^2 + V(q_{n+1})$$
$$= H_n + \frac{\epsilon^2}{2m} \left[ N^2 (\partial V)^2 + \beta^2 + 2p_n \beta + 2m \alpha \right] + O(\epsilon^3),$$

where $\partial V = \partial V(q_n)/\partial q_n$. From this it is clear that $H_{n+1}$ does not equal $H_n$: the Taylor expansion of the potential contains terms that cannot be cancelled by any choice of the functions $\alpha, \beta$ (which are in any case effectively fixed by the requirement of preserving the canonical Poisson bracket). The discretization has made the constraint second class.

At this stage there are two possible ways to proceed: (i) impose the Hamiltonian constraint strongly since it is not preserved in time, or (ii) fix the free function $N$ such that the additional terms on the right hand side of (11) vanish. The latter amounts to $N$ becoming a fixed function of $n$, and hence the time step $\epsilon N$ also becomes a function of $n$. Thus, making the evolution consistent via (ii) requires a variable time step in this sense, which concomitantly means fixing the Lagrange multiplier.

From the theory of constrained Hamiltonian systems it is possible to see that these two choices are in fact equivalent: Gauge fixing leads to first class constraints becoming second class pairs with the gauge conditions. Since we started with a first class system (the parametrized particle), which became second class as a result of time discretization, we can ask what time gauge condition has been implicitly imposed by the discretization that led to the Hamiltonian constraint not being preserved in time.
To answer this question let us recall the procedure for the continuous case in the canonical theory, where time gauge fixing is accomplished by setting a function of the phase space variables to equal the time parameter. Consider a phase space function $f(t,p^i,q,p)$ and impose the gauge condition
\[
\chi := \lambda - f = 0. \tag{12}
\]
This fixes parameter time $\lambda$. Preserving the gauge condition under evolution requires
\[
\frac{d\chi}{d\lambda} = \frac{\partial\chi}{\partial\lambda} + \{\chi,NH\} = 0. \tag{13}
\]
This fixes the lapse as a function of $f$
\[
N = \frac{1}{\{f,H\}} \tag{14}
\]
Conversely, fixing $N$ first leads via the above equations to a connection with the canonical gauge fixing function $f$. Thus a discretization may also be determined by fixing the lapse, which may be viewed as imposing a variable time step. This of course, is what is routinely done in numerical relativity, with the difference that in that case it is not necessary to preserve the canonical Poisson bracket for quantization.

For the discrete evolution we have defined, this procedure may be used to extract the gauge fixing condition. This is done by first imposing $H_{n+1} = 0$ strongly since it is second class. This gives an equation for $N$. The gauge fixing function $f$ is then determined by solving \[\text{(14)},\] which in general is rather messy. For example, for the above discretization of the parametrized particle, it is not possible to solve for $N$ to all orders in $\epsilon$ due to the presence of all orders of $N$ in $H_{n+1}$. (This is because the expansion of $V(q_{n+1})$ in powers of $\Delta q_n = q_{n+1} - q_n$ contains powers of $N$.) Thus only an approximate solution for $N$ is possible, up to a specified order in $\epsilon$.

The lessons from this section are two-fold: (i) The conditions on allowable explicit time discretizations are rather strong if canonical structures are to be preserved. (ii) For constrained theories with time reparametrization invariance, time discretization is equivalent to rather complicated gauge fixing conditions.

### B. Implicit schemes

There are implicit schemes for time discretizations discussed in the literature. We mention two examples below before introducing some generalizations. One of these uses a leap-frog numerical scheme to construct a consistent Hamiltonian evolution \[\text{(9)}.\] The other method is implicit and uses a "Hamiltonian" that depends on phase space before introducing some generalizations. One of these uses a leap-frog numerical scheme to construct a consistent Hamiltonian evolution \[\text{(9)}.\]

The leap-frog discretization scheme for the standard Hamiltonian $p^2/2m + V(q)$ is \[\text{(9)}.\]

\[
q_{n+1} = q_n + \epsilon \frac{p_n}{m} - \frac{\epsilon^2}{2m} \frac{\partial V(q_n)}{\partial q_n}, \quad p_{n+1} = p_n - \frac{\epsilon}{2} \left( \frac{\partial V(q_n)}{\partial q_n} + \frac{\partial V(q_{n+1})}{\partial q_{n+1}} \right). \tag{15}
\]
This is an implicit scheme since the $V(q_{n+1})$ term in the $p_{n+1}$ equation is determined by first computing $q_{n+1}$. Without this term it is, up to factors, the same as the one we derived above from the general form \[\text{(9)}.\] It preserves the canonical Poisson bracket under evolution.

The non-local Hamiltonian scheme \[\text{(9)}.\] is rather unusual. Evolution equations are derived from a discrete action, which may be viewed as a function of coordinates $q_n$ and velocities $v_n$:
\[
S[q_n,v_n] = \sum_n L_n(q_n,v_n) = \sum_n \left[ v_n(q_{n+1} - q_n) - \epsilon H(v_n,q_n) \right]. \tag{16}
\]
This resembles a Hamiltonian form. The conjugate momentum is defined by $p_{n+1} := \partial L_n/\partial q_{n+1} = v_n$. This leads to the following discrete equations for the canonically conjugate pair $(q_n,p_n)$ (Eqsns. (37,43-44) of Ref. \[\text{(9)}\]):
\[
q_{n+1} = q_n + \epsilon \frac{\partial H(q_n,p_{n+1})}{\partial p_{n+1}}, \quad p_{n+1} = p_n - \epsilon \frac{\partial H(q_n,p_{n+1})}{\partial q_n} \tag{17}
\]
It is in the sense manifested by these equations that the discretised phase space Hamiltonian is non-local in time.

The requirement of preserving the Poisson bracket for more general implicit schemes is at first sight more complicated than the procedure we followed for the explicit case. The computation of Poisson brackets requires implicit differentials assuming the validity of the implicit function theorem for the discretization. For a given discretization this leads to restrictions on the form of the Hamiltonian. It therefore appears that a general result may be difficult to obtain, and that one is stuck with a case by case analysis.

There is however a general approach for finding a large class of implicit schemes that are automatically consistent. This involves (i) viewing discrete evolution as a canonical transformation from the variables \((q_n, p_n)\) to the variables \((q_{n+1}, p_{n+1})\), and (ii) requiring that the canonical transformation resemble a discretization of Hamilton’s equations. This approach gives two general classes of implicit discretization starting from a given Hamiltonian \(H(q, p)\).

To see how this approach works it is useful to consider each of the four classes of generating functions for canonical transformations. As a first example consider a generating function of the form

\[
I(q_{n+1}, p_n) = -q_{n+1}p_n + \epsilon \, F(q_{n+1}, p_n) + \epsilon^2 \, G(q_{n+1}, p_n) + \cdots
\]  

The standard canonical transformation rules give the equations

\[
q_{n+1} = q_n + \epsilon \, \frac{\partial F}{\partial p_n} + \epsilon^2 \, \frac{\partial G}{\partial p_n} + \cdots
\]

\[
p_{n+1} = p_n - \epsilon \, \frac{\partial F}{\partial q_{n+1}} - \epsilon^2 \, \frac{\partial G}{\partial q_{n+1}} + \cdots.
\]

The discretization prescription is to fix the functional form of \(F\) to be that of the given Hamiltonian \(H(q_n, p_n)\) with the replacement \(q_n \rightarrow q_{n+1}\), ie set

\[
F = H(q_{n+1}, p_n).
\]

The result is an automatically consistent class of implicit discretizations where the function \(G\) is arbitrary. The limit \(\epsilon \to 0\) gives the continuous time Hamilton equations.

The second example of this type is provided by the generating function

\[
I(q_n, p_{n+1}) = -q_n p_{n+1} + \epsilon \, F(q_n, p_{n+1}) + \epsilon^2 \, G(q_n, p_{n+1}) + \cdots \tag{18}
\]

The transformation equations are now

\[
q_{n+1} = q_n + \epsilon \, \frac{\partial F}{\partial p_{n+1}} + \epsilon^2 \, \frac{\partial G}{\partial p_{n+1}} + \cdots
\]

\[
p_{n+1} = p_n - \epsilon \, \frac{\partial F}{\partial q_n} - \epsilon^2 \, \frac{\partial G}{\partial q_n} + \cdots \tag{19}
\]

These again give the continuum Hamilton equations in the \(\epsilon \to 0\) limit with \(F = H(q_n, p_{n+1})\) for arbitrary \(G\). (A special case of this scheme arises from the action viewpoint discussed in \[\text{[3]}\].) The remaining two types of transformations with generating functions depending on the pairs \((q_n, q_{n+1})\) and \((p_n, p_{n+1})\) do not give a form resembling discretized Hamilton equations, so the two cases discussed above appear to exhaust this approach for generating consistent implicit schemes.

As for the explicit scheme of the last section, it readily verified that constraints are not preserved generically with these two types of implicit discretization. For the parametrized particle \[\text{[1]}\] for example, the evolution equations with the first method \[\text{[1]}\] (without the \(\epsilon^2\) terms, and with lapse \(N\)) are

\[
q_{n+1} = q_n + N \epsilon \, \frac{p_n}{m},
\]

\[
p_{n+1} = p_n - N \epsilon \, \frac{\partial V(q_{n+1})}{\partial q_{n+1}},
\]

\[
p_{n+1} = p_n, \quad t_{n+1} = t_n + N \epsilon.
\]

Substitution of these into \(H_n = p_n^2 + p_{n+1}^2/2m + V(q_n)\) shows that \(H_{n+1} \neq H_n\).

In summary, we have seen that both the explicit and implicit schemes are quite restrictive in their respective settings, although with the latter applicable to any Hamiltonian. In particular, the schemes we have considered are two-step, involving only variables at times \(n\) and \(n+1\). Multi-step general schemes cannot be represented as canonical transformations in any simple way for the implicit case, and checking of Poisson bracket preservation appears to be much more tedious for the explicit case. It is also apparent from the simple example of the parametrized particle that constraints are not preserved under discrete time evolution.
III. APPLICATIONS

It is useful to see whether there are viable implementations of the explicit and implicit discretization discussed in
the last section. In the following we consider a few examples of models with Hamiltonian constraints to see if there are
solutions for the functions \(\alpha, \beta\) in (5) that preserve the canonical Poisson bracket. To do this we follow the procedure
described above. We also examine the time gauge fixing conditions introduced by the implicit discretizations (19-22).

A. The parametrized scalar field

As a first application we look at the parametrized scalar field in a curved spacetime. Its Hamiltonian is

\[
H = \int N \left[ p^t + \frac{1}{2\sqrt{q}} \Pi^2 + \frac{1}{2} \sqrt{q} q^{ab} \partial_a \phi \partial_b \phi + \sqrt{q} V(\phi) \right]
\]

where the canonical coordinates are \((t, p^t)\) and \((\phi, \Pi)\), and \(q_{ab}\) is the spatial background metric. The discrete evolution
equations are

\[
\begin{align*}
\phi_{n+1} &= \phi_n + \epsilon N \frac{\Pi}{\sqrt{q}} + \epsilon^2 \alpha(\phi_n, \Pi_n) \\
\Pi_{n+1} &= \Pi_n + \epsilon \left[ \partial_a \left( \sqrt{q} q^{ab} \partial_b \phi N \right) - V' N \sqrt{q} \right] + \epsilon^2 \beta(\phi_n, \Pi_n) \\
t_{n+1} &= t_n + N \epsilon \\
p_{n+1} &= p_n.
\end{align*}
\]

From this one calculates \(\{\phi_{n+1}, \Pi_{n+1}\}\) to try to find functions \(\alpha\) and \(\beta\) such that \(\{\phi_{n+1}, \Pi_{n+1}\} = 1\). It turns out that
this is not possible except for the special case \(\phi = \phi(t)\), where it reduces to the case of the parametrized particle (9).

B. Cosmological models

We consider two models here, FRW coupled to a scalar field and deSitter space. These serve to illustrate the main
problem with discretization arising from the Hamiltonian constraint of general relativity: the kinetic term contains a
mixture of gravitational configuration and momentum variables, which makes it difficult to find a discretization that
preserves the canonical commutation rules under evolution for the explicit scheme.

For both models we use geometrodynamics phase space variables \((q_{ab}, \tilde{\pi}^{ab})\), and the parametrization

\[
q_{ab} = a(t) \ e_{ab} \quad \tilde{\pi}^{ab} = \frac{1}{3} P(t) \ e^{ab}
\]

where \(e_{ab}\) is the flat Euclidean 3–metric. (The main features of the analysis are unchanged for the triad-connection
phase space variables.)

The Hamiltonian constraint for flat FRW minimally coupled to a scalar field is

\[
H(a, p) = -\frac{1}{6} a^{1/2} p^2 + \frac{1}{2} a^{-3/2} \Pi^2 a^{3/2} V(\phi),
\]

where \(V(\phi)\) is a potential for the scalar field \(\phi\), and \(\Pi\) is its canonical momentum. For De Sitter space, the potential
is replaced by the cosmological constant \(\Lambda\).

The discrete evolution equations obtained using (5) are

\[
a_{n+1} = a_n - \epsilon N \frac{1}{3} a_n^{1/2} p_n + \epsilon^2 \alpha(a_n, p_n)
\]

for both the FRW and deSitter case. The momentum evolution equation is

\[
p_{n+1} = p_n + \epsilon N \left[ \frac{1}{12} a_n^{-1/2} p_n^2 + \frac{3}{2} a_n^{5/2} \Pi^2 - \frac{3}{2} \sqrt{a_n} V(\phi) \right] + \epsilon^2 \beta(a_n, p_n)
\]

for the FRW case, and

\[
p_{n+1} = p_n + \epsilon N \left[ \frac{1}{12} a_n^{-1/2} p_n^2 + \frac{3}{2} a_n^{5/2} \Lambda \right] + \epsilon^2 \beta(a_n, p_n)
\]
for deSitter. (We ignore the evolution equations for the scalar field, since these are not relevant for the point we wish
to make.) The main observation again is that computing \{a_{n+1}, p_{n+1}\} shows that there is no choice for \(\alpha(a_n, p_n)\) and
\(\beta(a_n, p_n)\) which preserves the Poisson bracket to all non-vanishing orders in \(\epsilon\).

Thus we conclude that this general form of discretization is not applicable to models derived from general relativity.

C. BF theory

This example of a fully constrained dynamics is of particular interest in the context of spin foam models. Once
again we focus on time discretization, which is what is relevant for preserving the Poisson bracket under evolution.

**Abelian case:** the continuum Hamiltonian is given by

\[
H = \int \left( \lambda^a F_{ab} + \mu \partial_a E_a \right) d^3x
\]  

(31)

with the two constraints \(F_{ab} = 0\) and \(\partial_a E_a = 0\) and Lagrange multipliers \(\lambda\) and \(\mu\). The discrete evolution equations for the canonical variables \(A_a\) and \(E_a\) are

\[
\begin{align*}
A_{a,n+1} &= A_{a,n} - \epsilon \partial_a \mu_n, \\
E_{a,n+1} &= E_{a,n} + \epsilon 2\epsilon^{ab} \partial_b \lambda_n.
\end{align*}
\]

(32)

(33)

From these it is obvious that the Poisson brackets are trivially preserved, even without considering terms of order \(\epsilon^2\).

**Non-Abelian case:** The only difference is that the canonical variables are now Lie-algebra valued. Thus the Hamiltonian is

\[
H = \int (\lambda^a F_{ab} + \mu^i D_a E_{ai}) d^3x
\]

(34)

where \(D_a = \partial_a + A_a\) is the covariant derivative. As a result, the discrete evolution now becomes non-trivial:

\[
\begin{align*}
A_{a,n+1}^i &= A_{a,n}^i - \epsilon D_a \mu_n^i + \epsilon^2 \alpha \left( A_{a,n}^i, E_{ai} \right) \\
E_{ai,n+1} &= E_{ai,n} + \epsilon \left[ 2\epsilon^{ab} D_b \lambda_n^i + \epsilon^{ijk} E_{aj,n} \epsilon_{ijk} \right] + \epsilon^2 \beta \left( A_{a,n}^i, E_{ai} \right).
\end{align*}
\]

(35)

(36)

As a consequence of the additional terms compared to the Abelian case, it is straightforward to check that there is
no solution for \(\alpha\) and \(\beta\) that leads to a preservation of the Poisson bracket.

For the implicit schemes the Poisson bracket is of course preserved by the general argument given in the last
section. The constraint associated with time evolution in BF theory is a combination of the two "internal symmetry"
constraints, as occurs for example in 2+1 gravity \[10\]. Thus it is expected that there is some partial breaking of these
symmetries due to time discretization. Given this, it is useful to see what are the (partial) gauge fixing conditions
induced by the time discretization.

Denoting the continuum gauge conditions by \(f(A, E) = 0\) and \(g(A, E) = 0\), the relation between these and the
lagrange multipliers \(\lambda^i\) and \(\mu^i\) are given by the requirement that the gauges are preserved in time. Schematically this is

\[
\{ f, F[\lambda] + G[\mu] \} = 0, \quad \{ g, F[\lambda] + G[\mu] \} = 0.
\]

(37)

For the discretization, one can in principle extract the functions \(\lambda^i\) and \(\mu^i\) by setting the time evolved constraints to
zero, i.e. \(G_{n+1}^i = G_i(A_{n+1}, E_{n+1}) = 0\) and \(F_{n+1}^i = F_i(A_{n+1}, E_{n+1}) = 0\). One can then attempt to solve Eqn. \[37\] for
the gauge conditions \(f\) and \(g\). To get an idea of what the resulting equations look like, the first task is to extract \(\lambda^i\) and \(\mu^i\) from the evolved constraints. For the first implicit scheme \[19\] for example, where \(H = H[A_{n+1}, E_n]\), these equations are

\[
\begin{align*}
G_{n+1}^i &= \lambda^i + \epsilon \left[ -\frac{\partial H}{\partial A_{n+1}^i} + \epsilon^{ijk} \left( E_{n}^k \frac{\delta H}{\delta E_n^j} - A_{n+1}^j \frac{\delta H}{\delta A_{n+1}^k} \right) \right] - \epsilon^2 \epsilon^{ijk} \frac{\delta H}{\delta E_n^j} \frac{\delta H}{\delta E_n^k} = 0 \\
F_{n+1}^i &= \lambda^i + \epsilon \left[ \frac{\partial H}{\partial E_n^i} + 2 \epsilon^{ijk} A_{n}^j \frac{\delta H}{\delta E_n^k} \right] + \epsilon^2 \epsilon^{ijk} \frac{\delta H}{\delta E_n^j} \frac{\delta H}{\delta E_n^k} = 0
\end{align*}
\]

(38)

(39)
up to order $\epsilon$, where we have suppressed $\epsilon^{ab}$ and the spatial indices $(a, b \cdots)$. Inserting

$$\frac{\delta H}{\delta E_n^i} = \epsilon^{ijk} A_{n+1}^j A_{n+1}^k$$

(40)

and

$$\frac{\delta H}{\delta A_{n+1}^i} = 2 \left( \partial A_n^i + \epsilon^{ijk} A_{n+1}^A \lambda_n^A \right)$$

(41)

into the equations above, one can in principle solve for $\lambda$ and $\mu$, and substitute in (37) to get the functions $f$ and $g$. These equations give an indication of the rather complex gauge fixing induced by this discretization.

Note that although we are dealing here with time discretization only, this may still affect the first class nature of the purely internal symmetry constraints: the first class property is preserved automatically under evolution if $G_n^i = 0$ follows from $G_n^i = 0$, without fixing any Lagrange multipliers. We have seen above that this is not the case. Therefore, although time discretization does not affect the internal symmetry algebra at fixed time ($n$), it nevertheless makes the constraints second class with respect to evolution in time.

From the above examples, the pattern of calculation is clear for application to quantum gravity. It is unlikely that explicit schemes will work due to non-conservation of the Poisson bracket. The two classes of implicit schemes can be pursued to some extent at least, in the connection-triad canonical variables. A first attempt might be to keep space continuous and discretize time for the general form of the constraints with the Barbero-Immirzi parameter. It is possible there are choices of this parameter that simplify the form of the constraints and gauge fixing conditions.

IV. DISCUSSION

We have studied general classes of explicit and implicit time discretizations of Hamilton’s equations containing arbitrary functions. The general forms were motivated by simplicity, and relative ease in establishing the primary criterion that the Poisson bracket is preserved under discrete evolution.

The explicit class of schemes we studied preserves the Poisson bracket for only a limited class of Hamiltonians. Therefore its use for quantum gravity is rather limited. The two classes of implicit schemes have more potential utility in that these apply to any Hamiltonian. However the resulting gauge conditions are quite complicated.

Our approach appears to exhaust explicit and implicit two-step discretization schemes that preserve the Poisson bracket. In both cases, continuum first class constraints are not preserved in time. Therefore gauge fixing conditions are implicit in the discretization, and can in principle be explicitly recovered. Given this one can ask what is the utility of this lattice approach to gauge fixing, versus the alternative path of gauge fixing in the continuum theory followed by discretization (where preservation of constraints is obviously not an issue). For example, the problem of time is "solved" by gauge fixing in both approaches. Furthermore, at the level of quantization, the old question of whether different gauge fixings lead to the same quantum gravity model would reemerge in a lattice disguise.

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