Scenes from a Monopoly: Quickest Detection of Ecological Regimes

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Abstract

We study the stochastic dynamics of renewable resources under the threat of ecological regime shifts. We establish a Pareto optimal framework of regime shift detection under uncertainty that minimizes the delay with which economic agents become aware of the shift. We integrate ecosystem surveillance in the formation of optimal resource extraction policies. We fully solve the case of a profit-maximizing monopolist and provide the conditions that determine whether an adverse regime shift can lead to an aggressive or a precautionary extraction policy, depending on the interaction between market demand, resource scarcity and detection time. We apply our framework to the case of the Cantareira water reservoir in São Paulo, Brazil, and study the events that led to its depletion and the consequent water supply crisis.

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1 Introduction

The dynamic management of renewable resources requires making decisions under ecological uncertainty, defined by Pindyck (2002) as uncertainty over the evolution of the relevant ecosystem. Stochastic bio-economic models traditionally capture this uncertainty by describing environmental fluctuations as idiosyncratic shocks affecting the stock of natural resources. However, another way ecological uncertainty can manifest itself is by means of regime shifts, broadly identifiable as abrupt changes in the structure of the resource ecosystem such as the underlying population dynamics or the resource’s ability to regenerate (Biggs et al. (2009)). Regime shifts can cause substantial changes in the provision of ecosystem services, and can have significant impacts on both economic systems and the well-being of populations (Stern (2006)). Their occurrence has been extensively documented as a consequence of both natural and anthropogenic factors such as climate change and environmental overexploitation (Lindenmayer et al. (2011), Österblom et al. (2007)). Furthermore, the relevance of regime shifts is reflected in the contemporary resource market, where firms are investing substantially in monitoring resource stocks as ecological extreme events are on the rise.

There is an extensive literature studying the impact of stochasticity on firm extraction and harvesting activities, dating from Pindyck (1984) and Reed (1988) up to Saphores (2003) and Alvarez and Koskela (2007). An emerging literature focuses on resource management under potential regime shifts, intended as structural changes in ecosystem dynamics: Polasky et al. (2011) shows how the threat of a regime shift can yield a precautionary effect, while Ren and Polasky (2014) show that either precautionary or aggressive policies can occur depending on the parametric structure of the problem. Baggio and Fackler (2016), de Zeeuw and He (2017), Costello et al. (2019), Arvaniti et al. (2019), Crépin and Nævdal (2020) and Kvamsdal (2022) extend the analysis to endogeneity, reversibility and observability of the regime shift. Sakamoto (2014) and Diekert (2017) explicitly consider a strategic environment and show how the potential occurrence of a catastrophic regime shift can facilitate cooperation between competing economic agents.

This paper aims to reconcile the existing literature on regime shifts with the role played by market structure first pioneered by Pindyck (1987). The main challenge in responding optimally to a regime shift is precisely the dual nature of the ecological uncertainty that economic agents face: the exact moment at which the shift occurs is unknown ex ante, and it is often not an easy feat to immediately disentangle structural changes in the ecosystem from idiosyncratic environmental fluctuations. This problem is further amplified by the fact that ecosystems are in constant evolution, and multiple large-scale changes to their structure are often caused or accelerated by economic activity. Crépin et al. (2012) highlight the importance of adaptive resource management in understanding the likelihood of regime shifts and their consequent impacts on human well-being, and Barrett (2013) shows how uncertainty over tipping points thresholds, rather than over their consequences, can cause coordination between economic agents to collapse. The main contribution of our framework is the integration of environmental surveillance and regime shift detection in a model of natural resource extraction. Environmental monitoring is a common practice in real-world resource management: for example, the Norwegian company Aker BioMarine, a global krill monopoly, uses drones to collect, process and transmit density and distribution on the krill biomass.1 These techniques, combined with in situ

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1https://www.akerbiomarine.com/news/aker-biomarine-pioneering-machine-learning-for-operational-decision-making
measurements, constitute the most effective ways for efficient management and controlled exploitation of natural resources. In ecology, using real-time remote sensing data is increasingly common, especially with indicators of approaching thresholds or impending collapse in ecosystems ([Batt et al. (2013), Carpenter et al. (2014)]). Our framework is therefore particularly relevant as it sheds light on how firms operate within modern-day resource markets, in which monitoring resource stocks takes an increasingly central role as drastic ecosystem changes become more frequent.

The potential occurrence of regime shifts can substantially alter constraints and incentives faced by economic agents who extract natural resources. In this paper we first characterize the losses stemming from regime shifts under ecological uncertainty, which manifest in the delay with which the agents become aware of their occurrence. Minimizing this delay requires the agents to be able to detect the presence of a regime shift in the quickest time possible, and therefore we establish a novel framework that minimizes the efficiency loss caused by incomplete observability of the environmental conditions in which agents operate. The problem involves the search for a way to deduce the occurrence of a general change in the drift of the controlled stochastic process that drives the natural resource evolution, and is formulated as an optimal stopping problem. This class of problems are known as quickest detection problems. Originated in the Brownian disorder literature pioneered by Shiryaev (1963, 1996), detection methods have found multiple applications throughout the statistical and econometric literature, from Krämer et al. (1988) and Ploberger and Krämer (1992) to Horváth and Trapani (2022). We extend the work by Moustakides (2004) on drift changes in martingales to general Itô diffusions: to our knowledge, ours is the first paper to integrate this framework in a continuous-time optimization problem in economics, and particularly in the regime shifts and renewable resources literature. More importantly, we show Pareto optimality of our framework for any resource-extracting economic agent, regardless of the criterion used to obtain extraction policies.

In order to understand the impact of regime shifts on agents’ incentives, and especially within our framework of quickest detection, it is of substantial importance to include in the analysis the dynamics of the market in which extractive firms operate. We therefore integrate the surveillance procedure in the maximization problem of a resource-extracting monopolist, such that the firm maximizes its profits with respect to the resource dynamics over a time horizon given by the detection time. We find that upon detecting an adverse regime shift and at high stock levels, the monopolist can potentially adopt an aggressive extraction policy due to a combination of a growth effect, driven by the declining market value of a marginal unit of in situ stock, and a horizon effect, driven by the quicker detection of a regime shift of larger magnitude. At low resource stock levels, on the other hand, the firm’s policy always follows the sign of the change in resource growth. A negative regime shift, therefore, implies a precautionary extraction policy due to the emergence of resource scarcity. Furthermore, we show how the magnitude of regime shifts plays a significant role in how firms adapt their extraction policies to adverse events. This role stems from the fact that in our framework structural changes affect both the resource dynamics and the expected detection time, and thus the time horizon over which agents optimize. This can lead to the emergence of a scenario where a substantial adverse regime shift may still lead to an aggressive extraction strategy despite declining demand for the resource.

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2 For further details we refer to Poor and Hadjiliadis (2008) and Tartakovsky et al. (2014)
3 As an example of the crucial role of demand, Myanmar Timber Enterprise (MTE), a state-run company holding monopoly over harvesting and sale of timber, has directly been responsible for the loss of more than 13,000 square miles of tree cover between 2001 and 2018 due to the lucrative and increasing demand for teak. https://www.nationalgeographic.com/history/article/after-century-logging-myanmar-struggles-preserve-teak-groves
We then apply our framework to the case of the Cantareira water reservoir, a large-scale system of interconnected reservoirs which serves the Metropolitan Area of São Paulo in Brazil. The reservoir is managed by Companhia de Saneamento Básico do Estado de São Paulo (SABESP), a water and waste management company acting as a semi-public natural monopoly. In early 2013, the reservoir’s stored water volume began decreasing sharply to the point that by 2014 it barely covered a month’s supply of the population requirement. SABESP only realized the critical state of the reservoir in January 2014 and began to reduce withdrawals, but by July 2014 its operational capacity was depleted, leaving a densely populated area inhabited by more than 25 million people in a devastating water crisis. Using daily data on reservoir volume, water pumping, rainfall and river inflows, we show how the depletion was caused by a catastrophic regime shift in the reservoir dynamics, and estimate the structural parameters pre- and post-shift via particle filtering. We find that the implementation of our detection procedure could have allowed the water monopolist to detect its occurrence more than six months ahead of the delayed time at which it changed its pumping policy. We further show counterfactual evidence of how adjusting the policy at the detection time could have substantially delayed depletion, if not avoided it altogether, and therefore could have drastically dampened the severe impact of the water supply crisis on the population.

The remainder of the paper is structured as follows. Section 2.1 formalizes the resource dynamics, sets up the detection procedure and proves its Pareto optimality. Section 2.2 solves the monopolist’s maximization problem within a sequential framework of regime shift detection. Section 3 explores the characteristics of the solution to the firm’s problem, and shows the different policy responses to a regime shift. Section 4 presents the application of our framework to the case of the Cantareira water reservoir, and section 5 concludes.

2 The Model

2.1 Resource dynamics, regime shifts and quickest detection

We start by modeling the stochastic dynamics of a renewable resource extracted by an economic agent. Let $X_t$ be the resource stock available at time $t$, which behaves according to the stochastic differential equation (SDE)

$$dX_t = (\mu_t - q_t)dt + \sigma_tdW_t, \quad X_0 = x_0$$

(1)

where $q_t \in \mathbb{R}^+$ is the extraction policy, $\sigma_t = \sigma(X_t, t)$ is the intensity of noise in the evolution of the resource stock, $\mu_t = \mu(X_t, t)$ is the process that drives the resource growth and $X_t \geq 0$. Finally, $W_t$ is the standard Brownian motion in the filtered probability space $(\mathbb{R}, \mathcal{F}_t, P)$. The processes $\mu_t, q_t, \sigma_t$ are adapted to the same filtration $\mathcal{F}_t$, and satisfy the standard requirements for existence and uniqueness of a weak solution for (1).4

In order to capture the regime shift that the resource dynamics can undergo, we describe two alternative scenarios faced by the agent: one in which the resource evolves according to equation (1),

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4See Oksendal (2013) for all further details
and an alternate in which the stock’s ability to regenerate (the drift) changes. This is consistent with Polasky et al. (2011), who define regime shift as a change in the system dynamics such as the intrinsic growth rate or the carrying capacity of the resource. The evolution for the resource stock then becomes

\[ dX_t = (\mu_t + \lambda_t - q_t) dt + \sigma_t dW_t, \]  

(2)

where \( \lambda_t = \lambda(X_t, t) \in \mathbb{R} \) is the change in resource growth, also adapted to \( \mathcal{F}_t \). If \( \lambda_t < 0 \ \forall t \), the growth rate of the resource is reduced and it undergoes a negative (adverse) regime shift, and vice versa. The regime shift can be made dependent on antecedent factors and we can write \( \lambda_t := \lambda(X_t, t, \Theta) \), where \( \Theta \) is the information set the agent has when it starts monitoring the resource stock. By antecedent factors we imply any process adapted to the filtration \( \mathcal{F}_0 \), which does not vary during the surveillance period and contributes to the knowledge the agent has at \( t = 0 \) on the magnitude of the regime shift \( \lambda_t \). The set \( \Theta \) can be constructed to include the agent’s extraction policy adopted before the initial observation time, allowing us to study a framework in which past extractive activity can determine future changes in resource growth.

We therefore want to study the scenario in which at a given change point in time \( \theta \), which is happening with certainty but at time unknown, the SDE driving the resource stock will switch between drifts:

\[
\begin{align*}
  dX_t = \begin{cases} 
    (\mu_t - q_t) dt + \sigma_t dW_t & t < \theta \\
    (\mu_t + \lambda_t - q_t) dt + \sigma_t dW_t & t \geq \theta.
  \end{cases}
\end{align*}
\]  

(3)

Note that since the occurrence of \( \theta \) is certain, the question faced by the agent is not if a regime shift will occur but rather when. The agent now faces two sources of uncertainty when choosing the extraction policy that maximizes its profits. The first source is given by the Brownian motion \( W_t \) calibrated by the diffusion coefficient \( \sigma_t \), which represents the fluctuations inherent to the natural randomness of environmental conditions. The second source is the uncertainty over the timing \( \theta \) of the shift, at which the resource’s drift changes from \( \mu_t \) to \( \mu_t + \lambda_t \). Whilst being unknown to the agent ex ante, this change point would be immediately inferable in absence of fluctuations. In presence of fluctuations, however, the agent needs to be able to distinguish the structural change in the drift from idiosyncratic noise.

We now need to establish from a decision maker’s perspective the importance of adjusting to a regime shift in the ecosystem as quickly as possible. Why should any economic agent undertake any supplementary analysis in order to infer whether the regime shift has actually occurred? The key element behind the optimality of implementing a detection framework, thus for economic agents to better understand the state of the ecosystem they operate in, lies in the losses caused by delays in detecting the regime shift. Let us formalize this point. The extraction policy \( q \) is chosen by the agent according to a specific criterion \( J(q) := J(q, x) \). From Section 2.2 onwards, we will assume the agent to be a monopolist firm that follows a profit-maximizing criterion, but this framework applies to any optimizing economic decision maker, regardless of the nature of the criterion. For a social planner, this criterion would be expressed in terms of welfare, for a risk-averse individual it would be in terms of utility drawn from resource use. This criterion is only assumed to be bounded, continuous and differentiable at least once with respect to every argument.\(^\text{5}\) The extraction policy \( q^\mu := q^\mu(x, t) \) is

\(^{5}\text{More precisely, we require } J \text{ to be Lipschitz continuous.} \)
chosen by the agent such that it maximizes the expected discounted criterion \( J(q) \) within a given time horizon \( \tau \), i.e.

\[
q^\mu \to \sup_{q \in \mathcal{Q}} \mathbb{E} \int_0^\tau e^{-\rho t} J(q) \, dt \quad \text{s.t. (1),}
\]

where \( \mathcal{Q} := Q(x,t) \) is the non-empty set of Markovian admissible controls in feedback form such that \( \mathbb{E} \int_0^\tau |e^{-\rho t} J(q)| \, ds < \infty \) for all \( t < \tau \), \( q \in \mathcal{Q} \) and \( \rho > 0 \) is the discount rate. In the next section we shall discuss in detail the form of this optimal policy, but for the moment we stop at positing the policy exists and is progressively measurable with respect to \( \mathcal{F}_t \). Let us now assume that the regime shift occurs in the dynamics of \( X_t \) as shown in (3), but the agent only realizes the occurrence of the shift at \( \tau > \theta \), thus with a delay \( \tau - \theta \). This implies that there exists an extraction policy \( q^\lambda := q^\lambda(x,t) \) in the time interval \( [\theta, \tau] \) that achieves the supremum of the discounted criterion function, i.e.

\[
q^\lambda \to \sup_{q} \mathbb{E} \int_\theta^\tau e^{-\rho(t-\theta)} J(q) \, dt \quad \text{s.t.} \quad dX_t = (\mu_t + \lambda_t - q) dt + \sigma_t dW_t.
\]

Because of the delay, however, in this time interval the agent will continue to extract according to the policy \( q^\mu \) that achieves the supremum of the optimization problem constrained by the pre-regime shift resource dynamics \( dX_t = (\mu_t - q^\mu) dt + \sigma_t dW_t \). We can then establish the following result.

**Proposition 1.** If \( \theta \) is unobservable and the agent adjusts extraction to the post-regime shift resource dynamics with a delay \( \tau > \theta \), she incurs a positive expected loss:

\[
\mathbb{E} \int_\theta^\tau e^{-\rho(t-\theta)} L(X_t, t) dt := \mathbb{E} \int_\theta^\tau e^{-\rho(t-\theta)} \left( J(q^\lambda) - J(q^\mu) \right) dt > 0,
\]

where \( J(q^\lambda) \) represents the criterion function for the period \( [\theta, \tau] \) evaluated at the extraction adjusted exactly at the regime shift occurrence \( \theta \), and \( J(q^\mu) \) the one obtained by adjusting at \( \tau \). The loss is increasing in the length of the delay. The same result applies for false alarms \( (\tau < \theta) \).

**Proof:** See Appendix A.1.

The rationale behind this result is intuitive: the losses, whether in terms of profits, utility or welfare, are generated by the fact that the agent chooses its extraction policy by maximizing a criterion which hinges upon the continuous observation of the evolution of the resource \( X_t \) as given by (3). If the changepoint \( \theta \) was observable, the agent would immediately adjust extraction in order to adapt to the post-shift resource growth \( \mu_t + \lambda_t \). On the other hand, if the agent realizes the occurrence of the regime shift at a time \( \tau > \theta \) and only then adjusts extraction, thus with a delay, within the time interval \( [\theta, \tau] \) between the actual change point \( \theta \) and the adjustment time \( \tau \), the agent is de facto extracting a “wrong” quantity, optimal for the pre-regime shift dynamics of the resource but suboptimal for the post-shift ones. Figure 1 presents a schematic representation of this phenomenon. In Appendix A.1 we characterize explicitly the stochastic dynamics of the loss function \( L(x,t) \) in terms of the gradients of the Hamilton-Jacobi-Bellman equations associated to the respective optimization problems.

The problem now involves the minimization of the delay \( \tau - \theta \), which implies finding a strategy to detect the change in drift of \( X_t \) in the quickest time possible via sequential observations, as seen in (3). In order to solve this problem the agent searches for a “rule” (an optimal stopping time) \( \tau \) adapted to the filtration \( \mathcal{F}_t \), at which one can conclude the change point \( \theta \) has been reached and the
regime shift has happened, so she may reassess its extraction decisions according to the new regime in which it operates. As delays are costly, this search requires the optimization of the tradeoff between two measures, one being the delay between the time a change occurs and it is detected i.e. \((\tau - \theta)^+\), and the other being a measure of the frequency of false alarms for events of the type \((\tau < \theta)\). The agent minimizes the worst possible detection delay over all possible realizations of paths of \(X_t\) before the change and over all possible change points \(\theta\). This problem is formalized as

\[
\inf_{\tau} J(\tau) := \sup_{\theta} \text{ess sup} \mathbb{E}_\theta[|(\tau - \theta)^+| |\mathcal{F}_\theta] 
\]

This class of problems is usually comprised of three elements: a controlled stochastic process under observation (the evolution of the renewable resource), an unknown change point at which the properties of the process change (a regime shift), and a decision maker observing the process. In this search process there has to be an expected “time to first false alarm”, which represents the minimum time that the agent is willing to wait before reassessing its decisions when no shift is yet detected. This is to include in the process the fact that the signal (the change in the drift due to a regime shift) can be drowned in a noisy environment that does not allow for its detection within a “reasonable” time frame. This constraint is formalized in the following way:

\[
\mathbb{E}_\theta=\infty[\tau] \geq T, 
\]

for when the regime shift is a constant \(\lambda \in \mathbb{R}\), or equivalently via a divergence-type criterion

\[
\mathbb{E}_\theta=\infty\left[\int_0^\tau \frac{\lambda^2}{2} dt\right] \geq T
\]

for when the regime shift is time-varying (note how (6) essentially reduces to (5) for a constant \(\lambda\)). Constraints (5) and (6) are to be interpreted in the sense that since false alarms are costly as well, as shown in Proposition 1, if a negligible regime shift is expected the agent would be willing to tolerate uncertainty longer. This is because for a “small” \(\lambda\), the difference between the pre- and post-shift problems is also negligible and therefore so is the loss incurred within the delay. In either case, \(T := T(\Theta)\) is decided \textit{ex ante} by the agent and it can be interpreted both as a measure of tolerance to ecological uncertainty as well as a measure of the “quality” of the detection system, since it bounds the expected delay in the detection under a false alarm, i.e. the minimum waiting time the agent faces when \(\theta = \infty\) (the change point never occurs) before reassessing extraction decisions. This quantity depends on the \textit{ex ante} information the agent has on the magnitude of the regime shift. The optimal procedure to determine \(\tau\) is given by the following Proposition.
Proposition 2: The stopping time that solves (4) at which it is optimal for the agent to reassess its extraction decisions due to a regime shift \( \lambda_t \) under the constraint (6) is given by

\[
\tau(\lambda_t, \nu) = \inf_t \left\{ t \geq 0; u_t - \inf_{0 \leq s \leq t} \left\{ u_s \right\} \geq \nu \right\},
\]

where \( u_t \) is given by

\[
u_t = \log dQ_{\theta=0} \frac{dQ_{\theta=\infty}}{\theta} = \int_0^t \lambda(s, \hat{X}_s, \Theta) \sigma(s, \hat{X}_s) d\hat{X}_s + \frac{1}{2} \int_0^t \frac{\lambda(s, \hat{X}_s, \Theta)^2}{\sigma(s, \hat{X}_s)^2} ds.
\]

The process \( u_t \) is the logarithm of the Radon-Nikodym derivative between the probability measure post-regime shift and the measure pre-regime shift of the process \( \hat{X}_t \), which is given by

\[
\hat{X}_t = \int \sigma(t, x)^{-1} dx \bigg|_{x=X_t},
\]

and \( Q \) is the probability measure under which \( \hat{X}_t<\theta \) is a martingale. The threshold \( \nu \in \mathbb{R}^+ \) is set such that it solves \( e^{\nu} - \nu - 1 = T \), is unique for each choice of \( T \), and the constraint (6) is binding with equality.

Propositions 1 and 2 are very general. A specific application that has direct implications for our following framework yields the following Lemma:

Lemma 1: For the scenario in which the uncontrolled resource stock evolves as a drifted Brownian motion with constant diffusion coefficient \( \sigma \in \mathbb{R} \) and at \( t = \theta \) the drift changes to \( \lambda := \lambda(\Theta) \in \mathbb{R} \), the optimal stopping time under the constraint (5) is given by (7), where \( u_t = \frac{\lambda}{\sigma^2} X_t - \frac{\lambda^2}{2\sigma^2} t \) under the martingale measure \( Q \) for the pre-regime resource stock \( X_t \), and the threshold is the solution of the equation \( \frac{2\sigma^2}{\lambda^2} (e^{\nu} - \nu - 1) = T \). Furthermore, the expected delay of detection is given by

\[
\mathbb{E}_Q[\tau(\lambda, \nu)] = \frac{2\sigma^2}{\lambda^2} (e^{-\nu} + \nu - 1).
\]

Proofs: See Appendix A.2.

Monitoring continuously the stock of the resource \( X_t \) can be a costly procedure for any agent. However, obtaining a constant stream of data on the resource stock level \( X_t \) is required in order for the agent to constantly update and obtain its optimal extraction policy in (1). This cost can be included straightforwardly in the criterion \( J \) by adding fixed costs. Once this is included, the added costs of undertaking the detection procedure are negligible. On the other hand, the loss the agent incurs in not implementing the procedure is nonzero and increasing in the delay, as shown by (18). Implementing a detection procedure that minimizes this delay, and therefore our framework, is Pareto optimal.

The procedure essentially involves first observing the process given by the log-likelihood ratio (the
Radon-Nikodym derivative) of the stock process or equivalently its unit-diffusive transformation (note that we are under the measure $Q$) under the two regimes and comparing it with its minimum observed value. This can be interpreted simply by noticing that if the two regimes are very similar (for example, if $|\lambda|$ is very small in the constant diffusion coefficient case), then the Radon-Nikodym derivative between the two measures will be often close to unity. In this case the process $u_t - \inf_{0 \leq s \leq t} u_s$, which is known as a cumulative sum (CUSUM) process, is always greater or equal to zero and represents the “strength” of the regime shift, will be most of the time close to zero. This implies that unless the diffusion coefficient is very small it will be difficult to detect the presence of such a small drift change. If on the other hand the two regimes are rather different, then one should be able to detect more easily when the regime changes, and the observation process should reflect this change as it increases.

At the stopping time $\tau$, therefore, the agent will detect the change in drift, which is a change from a $Q$-martingale to a $Q$-sub/supermartingale. Note that in the drifted Brownian motion case, the larger the change in drift $\lambda$, the smaller the threshold $\nu$ and the “earlier” one expects the CUSUM process to hit the threshold after the change occurred. If $\lambda$ is very small, then $\nu$ will be very large and the agent may wait for much longer before detecting a change of regime. Since the constraint is binding, the effective time period in which the agent optimizes is therefore between $t = 0$ and the final time given by a combination of $T$ and $\tau(\lambda_t, \nu)$, the expected time to first reassessment plus the delay of detection. The “tolerance” $T$ is chosen by the agent; however, $\tau(\lambda_t, \nu)$ is a random variable. Since the agent knows the average delay time of detection, as given by (8), it can assume as time horizon the sum of the expectations of both change-point and delay, which is equivalent to taking an *ex ante* time interval $E_{\theta=\infty}[\tau] + E[\tau(\lambda_t, \nu)]$. In the baseline detection case the agent has an uniform/uninformative prior on the time of the regime shift $\theta$. Bayesian extensions of quickest detection problems that include prior beliefs on the change point time have been studied, among others, by Gapeev and Shiryaev (2013): we leave the complex yet important application of these methods to our framework for future research.

### 2.2 The resource-extracting monopolist’s problem

We will now focus on the relevant scenario for our application, which is a monopolist firm that chooses resource extraction according to a profit-maximizing criterion. We consider a risk-neutral monopolist facing a linear inverse demand function of the form $p(q) = a - bq$, with a cost function defined as $cq + F$, where $a > c \geq 0$, $F \geq 0$ and $b > 0$. The extraction rate is chosen by the firm in order to maximize the expected value of the sum of discounted profits under the resource dynamics (3), and the profit function takes the form

$$\Pi(q) = [(a - bq)q - cq - F]$$

(9)

We assume a profit function not directly depending on the stock level $X$ but only on the extracted quantity: this implies a marginal cost function linear in extraction, rather than the stock level, and fixed operating costs. This assumption can be relaxed, at the expense of an optimal extraction function only available in numerical form. Furthermore, linear demand is commonly chosen in the context of

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6Cost functions of this form also allow us to flexibly model a natural monopoly since the average cost $AC = c + \frac{F}{q}$ is decreasing in output. It is easy to show that the choice of a quadratic cost function of the form $cq^2$ yields essentially equivalent results.
water pricing, as done by Tsur (2020) and Chakravorty et al. (2022). A natural monopoly such as a water utility company is a relevant illustration of our model as there are high fixed costs and relatively small variable costs.

The firm’s detection strategy presented in Proposition 1 applies to a very general framework: however, it assumes that the optimal extraction policy $q$ is known to the monopolist (i.e. is $\mathcal{F}_t$-adapted). In order to obtain a tractable form for the optimal $q$, we restrict our attention to the scenario described by Lemma 1, where the regime shift $\lambda$ and diffusion coefficient $\sigma$ are constants. We choose the diffusion coefficient $\sigma$ to be independent of the state $X_t$ (i.e. a drifted Brownian motion) in order to include the possibility that the exogenous environmental shocks may drive the resource to extinction, something that log-normal fluctuations in a geometric Brownian motion by construction cannot represent. The drift, furthermore, can equivalently be extended to include seasonality effects, as long as known to the firm, exogenous and periodic (i.e. independent from the initial observation time). This choice allows us to obtain a fully explicit solution, which is useful in order to analyze the firm’s response to the regime shift by means of comparative statics. Once again our results remain robust to this modeling choice. In Section OA.1 of the Online Appendix we examine the model for a stock process driven by a geometric Brownian motion, an isoelastic demand function and a stock-dependent marginal cost function, solve it explicitly up to two equations (an algebraic equation for the parameters and an ordinary differential equation for the time-dependent component) and show how the salient characteristics of our framework remain unaffected.

The simplest way of modeling a regime shift is to assume that the shift occurs only once, however, as pointed out by Sakamoto (2014), such ecological shifts are better modeled as open-ended processes. Within a multi-regime setting, in which the firm detects multiple regime changes throughout subsequent periods, the stochastic control problem of the firm will read:

$$
\sup_{q \in Q} \sum_{i=0}^{\infty} \mathbb{E}_{\tau_i} \int_{\tau_i}^{\tau_{i+1}} \Pi(q) e^{-\rho t} dt
$$

$$
dX_t = \begin{cases} 
(\mu + \sum_{j=0}^{i} \lambda_j - q) dt + \sigma dW_t, & t \in [\tau_i, \tau_{i+1}) \\
(\mu + \sum_{j=0}^{i} \lambda_j + \lambda_{i+1} - q) dt + \sigma dW_t, & t \geq \tau_{i+1}, \ i \in \mathbb{N}.
\end{cases}
$$

$$
X_t \geq 0 \ \forall t, \ X_0 = x_0 \in \mathbb{R}^+, \tau_0 = 0
$$

$$
\tau_{i+1} = \mathbb{E}_{\theta=\infty}[\tau] + \mathbb{E}[\tau(\lambda_{i+1}, \nu)]
$$

where $i \in \mathbb{N}$ are the different periods, and the extraction policy exists among the class of admissible Markov controls $Q$. Here $\lambda_0$ is exogenous and $\tau_i, \lambda_i$ are the subsequent periods and relative changes in resource growth, and $\lambda_{i+1} : \mathbb{R}^+ \rightarrow \mathbb{R}$. We assume $\tau_0 = 0$ for simplicity. Here we formalize the structure of the firm’s extraction decisions in a sequential manner, where the firm assumes a constant $\lambda$ for each period.\footnote{An explicit dependence of the stopping time $\tau$ on $\lambda$ would make the control variable $q$ and the detection threshold simultaneous, and the problem intractable. In order to circumvent this issue whilst maintaining the endogeneity of the regime shifts, we model the firm to detect a change in drift $\lambda$ which is determined by extraction in the previous period} Once solved, this problem will yield a piecewise continuous control in the feedback form $q(X_t, t)$. Even though the structure of the underlying resource dynamics is simple, the presence of endogenous prices due to the presence of a monopoly makes the problem faced by the firm a more
involved one, particularly due to the presence of the various constraints. Because of the positivity constraint on $X_t$, the value function $V(t, x)$ is not necessarily always differentiable, especially because of the necessary boundary conditions $V(t, 0) = 0, q^*(t, 0) = 0$. In Appendix A.3 we show how the monopolist’s value function $V$, defined as the sequence of overall future expected discounted profits up to each detection point, is a viscosity solution of the Hamilton-Jacobi-Bellman equation associated with the optimization problem (10). The optimal extraction policy is therefore a weak solution of (10), and is presented in the following Proposition.

**Proposition 3.** The extraction policy that solves the monopolist’s problem (10) is the sequence $\{q^*_i(t, x)\}, i \in \mathbb{N}^+$ of the optimal policies for each time period $i : t \in [\tau_i, \tau_{i+1}]$. The optimal policy for each period $i$ is the sum of two components: the first is a constant part equal to the extraction policy of a static monopolist, and the second is a variable part that depends on both time and resource stock. The form of the policy is the following:

$$q^*_i(t, x) = q^m - q''_i(t, x), \quad x > 0, \quad q^*_i(t, 0) = 0$$

(11)

$$q^m = \frac{a - c}{2b},$$

$$q''_i(t, x) = \frac{\sigma^2 \psi'(x)}{\psi(x)} e^{-\rho(\tau_{i+1} - t)} \quad \forall t \in [\tau_i, \tau_{i+1}], i \in \mathbb{N}^+$$

where

$$\psi(x) = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}, \quad \alpha_{1,2} = \frac{-A^i \pm \sqrt{(A^i)^2 - 4BC}}{\sigma^2},$$

(12)

and the coefficients $c_1, c_2$ are set by the boundary conditions. The constants $A^i, B$ and $C$ are given by

$$A^i = \mu + \sum_{j=0}^{i} \lambda_j - q_m, \quad B = \frac{1}{4b}, \quad C = b q_m^2 - F.$$  

(13)

The resource rent for the monopolist is given by the derivative of the firm’s value function with respect to the resource stock, which is given by

$$V_x(t, x) = \frac{\sigma^2 \psi'(x)}{2B \psi(x)} e^{-\rho(\tau_{i+1} - t)},$$

(14)

which is the value of a marginal unit of in situ stock.

**Proof:** See Appendix A.3.

The optimal extraction (11) consists of two parts. The first one, $q^m$, is the quantity at which the monopolist’s marginal revenue equals marginal cost, and is the profit maximizing extraction policy of a static monopolist. The second part not only consists of demand but is variable and explicitly dependent on state $X_t$ and modulated by the distance between present and the detection time, representing the time horizon of the firm. Observe that $V_x$ here is the rent which is the scarcity value (or the market value) of the marginal unit of in situ stock. Coherently with the literature, $q^*_i$ decreases when the resource rent rises.

Figure 2 shows a simulation of the detection procedure where the resource stock is hit by an adverse regime shift $\lambda \in \mathbb{R}^-$ at $\theta = 26.6$, which is subsequently detected successfully by the firm at
Figure 2: Negative regime shift detection and subsequent extraction adjustment at $\tau(\lambda)$, with over-extraction due to detection delay. Numerical simulation done with a Shoji-Ozaki discretization for the time-dependent drift. The regime shift happens at $\theta = 26.6$. Parameter values: $\mu = 2$, $\lambda = -1$, $\sigma = 1$, $a = 4$, $b = 0.2$, $c = 0.5$, $F = 1$.

time $\tau(\lambda)$, which occurs when the Q-CUSUM process hits the threshold $\nu$. From the figure one can see how the delay in detection $\tau(\lambda) - \theta$ can yield resource over-extraction, since the “real” growth rate of the resource stock shifts to $\mu - \lambda$ at $\theta$, but the monopolist continues to apply an extraction policy which does not account for this change. Once the regime shift is detected at $\tau(\lambda)$, the extraction policy is updated and the resource stock is extracted optimally once again. Lastly, Figure 2 shows the counterfactual dynamics of the resource stock in absence of the detection procedure, where the firm continues to apply the “wrong” extraction policy, $v$ yielding a substantially lower stock level.

Figure 3 shows the potential variability in detection times by allowing the shift to happen at a given $\theta$ as in Figure 2, simulating 200 independent trajectories with identical structural parameters and then for each applying the detection procedure. For explanatory purposes we increase the variance of the idiosyncratic fluctuations $\sigma$ to 1.5, which shows how in comparison to Figure 2 the average detection time is higher: this is because the same signal (the shift $\lambda$) becomes increasingly masked by noise, thus making the detection process more delayed. Figure 3 shows how this simulation yields an average detection time $\tau_{ave} = 58.08$, which implies an average delay of 31.4, and the shaded area reports its 95% confidence interval.

The choice of $T$, mean time to the first false alarm, is left for the firm to choose and is therefore arbitrary. It is however a quantity that plays a key role in our framework, as it regulates the threshold at which the CUSUM process triggers the detection and consequently the time horizon in which the firm operates. It is therefore important to understand the effect of varying $T$ on both resource and extraction dynamics, and establish a set of criteria for its choice. Figure 4 shows how the choice of $T$ has a concave effect on the detection time. The red line in Figure 4 presents Monte Carlo estimates of the average detection time, as well as its 95% confidence band, using the same parameters as Figure 2 except for a lower variance of fluctuations. The line is obtained via varying $T$ on a grid between 20 and
Figure 3: Monte Carlo simulation of 200 regime shift detections and extraction adjustments for the same parameters as in Figure 2 except for an increased $\sigma = 1.5$, with a common change point at $\theta = 26.6$. The shaded area represents the 95% CI for the average detection time $\tau_{\text{ave}}$. Only 20 paths are plotted for visual clarity.

100, and for each point running 300 simulations of the optimally controlled $X$ with relative extraction $q^*$ evaluated at its respective $T$-dependent time horizon. The regime shift occurs at $\theta = T/2$ for each value of $T$ on the grid, which is equivalent to assuming uniform priors on $\theta$. One can see that whilst $\tau_{\text{ave}}$ is increasing in $T$, its sensitivity is limited and drops drastically for large $T$. This result is robust to varying parametric choices.

We now illustrate the emergence of extinction risk for a general time interval $[\tau_i, \tau_{i+1}]$ as a function of past extraction decisions. After $\tau_i$, the firm reassesses its optimal policy to $q^*_i(t, x)$ as the dynamics of the resource stock are now given by

$$dX_t = (\mu + \sum_{j=0}^i \lambda_j - q^*_i(t, X_t))dt + \sigma dW_t, \quad t \in [\tau_i, \tau_{i+1}]$$

Let us suppose that $\mu - \sum_{j=0}^i \lambda_j > 0$. This means that the resource stock, in absence of firm extraction, has a net tendency to regenerate at least until the upcoming change point. The firm knows the magnitude of the upcoming regime shift, as it depends on the past extraction decisions. The firm at this point begins the detection process for the next change of regime. If $\lambda_{i+1} < 0$, the firm will realize the future emergence of depletion (extinction) risk if

$$\mu + \sum_{j=0}^{i+1} \lambda_j < 0,$$

realizing that at the next detection time $\tau_{i+1}$ the new regime will be one in which the drift of the resource stock process will be negative, meaning that the resource will have a net tendency to be
Figure 4: Sensitivity of the detection process to changing the mean false alarm constraint $T$ against the firm’s time horizon. The red curve shows the Monte Carlo mean detection time obtained by letting $T$ vary on a grid between 20 and 100, for each simulating 300 optimally controlled trajectories for the resource stock and undertaking the detection procedure. The curve for the average stopping time estimates and their 95% confidence bands are interpolated. The black dashed line represents the time horizon $E_Q[\tau]$ used for the firm optimization. The regime shift occurs at $\theta = T/2$ for each value of $T$. Parameter values as in Figure 2 except for $\sigma = 0.5$.

Due to the sequential nature of the detection process and the stochastic dynamics of the resource, there is no steady state in our model. The system as well as its probability density are both non stationary and switching at random times, and hence optimal extraction must be specified for every state that can possibly occur. The monopolist’s extraction policy is therefore an ex ante policy. The time $\theta$ at which the regime changes, however, is a random variable: the firm therefore will use the expected detection time (8) to evaluate the boundary conditions and simultaneously undertake the detection procedure. If the threshold $\nu$ is reached before the expected detection time $\tau_{i+1}$, then the firm switches to the subsequent period with the modified drift, since the regime shift has been detected. If the threshold has not yet been reached, the firm continues the optimal extraction assuming the same underlying resource dynamics with an infinitesimal time interval as horizon.

It is also possible to introduce the stopping time itself as the final time, $\tau_{i+1} = \tau(\lambda(q_i), \nu)$, thus directly joining real-time detection and firm optimization. This choice would leave the properties of the model unchanged, as it can be shown with standard arguments that the value function $W$ of this problem can be rewritten as an infinite-time version of the value function $V$ of the problem (10) with a stochastic discount factor $v$:

$$W(t, x, v) = e^{-v}V(t, x),$$
in the augmented state space \((x, v) \in \mathbb{R}^2\), where \(v\) is the solution of
\[
   dv = \Pr[\tau(\lambda_i, \nu) \in (t, t + dt)|\tau(\lambda_i, \nu) \geq t]dt.
\]
The problem is more involved but the form of the extraction policy remains unchanged, and at each detection time \(t = \tau(\lambda_i, \nu)\) the firm will switch to the next period. As our specification is equivalent to uniform priors on the unobservable change point \(\theta\) and this distribution is unaffected by within-period firm choices, the problem only involves the well-known distributional properties of running minima. We leave a more comprehensive exploration of this aspect for future research, potentially joined by the dependence of \(v\) on extraction and the extension of our framework to the Bayesian formulation mentioned in Section 2.1.

3 Response to adverse regime shifts

Having established the optimal extraction policy in Proposition 2, we can now study the effects of an adverse regime shift on the firm’s extraction decisions. We choose to focus on \(\lambda \in \mathbb{R}^-\) because we deem this to be of greater relevance. For a general time period \(i = [\tau_i, \tau_{i+1}]\), we define an aggressive extraction strategy as one where for all levels of stock there exists a time in the new regime where \(q_{i+1}^*(t, x) > q_i^*(\tau_{i+1}, x)\) where \(q_i^*\) is the extraction policy for the the pre-shift period \([\tau_i, \tau_{i+1}]\) and \(q_{i+1}^*\) is the policy for the post-regime shift period. On the other hand, a precautionary strategy is one where \(q_{i+1}^*(t, x) < q_i^*(\tau_{i+1}, x)\) for all \(t \in [\tau_{i+1}, \tau_{i+2}]\). In intuitive terms, an aggressive strategy occurs when the firm increases its extraction policy as a response to a structural change, and vice versa for a precautionary one. Applying this definition to our framework, we want to study how negative changes in drift (the regime shift) affect the extraction policy:

\[
   \frac{\partial q^*(t, x)}{\partial \mu} < 0 \quad \text{or} \quad \frac{\partial q^*(t, x)}{\partial \mu} > 0, \quad \forall \ t, x \in \mathbb{R}^+ \tag{15}
\]

This question does not have a univocal answer. A novel result generated by our model lies in the fact that a regime shift can ultimately have both aggressive and precautionary effects on firm extraction: this is because a regime shift affects simultaneously the resource drift, the extraction policy and the expected detection time. We first examine the firm’s “instantaneous” response to a regime shift, i.e. the change in extraction policy between periods at exactly the detection time. We will approach the analysis by studying the changes in firm policy for both high \((x \to \infty)\) and low \((x \to 0)\) resource stocks, and infer the nature of the general response from these boundary points.

**High resource stock levels:** We begin by observing that both the extraction policy and resource rent have the following limiting behavior:

\[
   \lim_{x \to \infty} q_i^*(t, x) = q^m - \alpha_1 e^{-\rho(\tau_{i+1} - t)}, \quad \lim_{x \to \infty} V_x(t, x) = \alpha_1 e^{-\rho(\tau_{i+1} - t)}. \tag{16}
\]
where $\alpha_1$ is given in (12) and is a combination of the model parameters. This result shows that for any time $t \in [\tau_i, \tau_{i+1}]$, there is a maximum extraction level given by a fixed amount generated entirely by demand, minus a term which incorporates the dynamics of the resource stock and the time horizon of the firm. Furthermore, for large enough resource stocks, the resource rent only varies with the expected detection time. Using (16) we can show that for high stock levels the nature of the change in extraction policy at detection time is determined as follows:

$$q^*_i(\tau_{i+1}, X^*_i) < q^*_i(\tau_{i+1}, X^*_i) \iff \tilde{\alpha}_1 e^{-\rho \tau_{i+1}} > \alpha_1,$$

where $X^*_i$ is the level of optimally controlled stock observed at the detection of the regime shift and $\tilde{\alpha}_1$ is evaluated at the “new” post-shift drift. If this inequality holds, the immediate effect of the regime shift is precautionary extraction, and aggressive otherwise. Therefore, the key determinant of the inequality is the total change in $\alpha_1$ due to a change in $\mu$: using (16) and the envelope theorem, this is determined by

$$\frac{\partial V_x(t, x)}{\partial \mu} > 0.$$  

The derivative is positive if $q^* > \sqrt{F/b}$, and goes to zero as $\mu$ increases. This is an intuitive result: in a situation where there is “enough” resource stock in the ecosystem not to introduce any scarcity concerns in the firm’s extraction policy, then the effect of a regime shift on the resource rent simply mirrors the change in resource growth. If the resource growth is large enough, then the resource rent is unaffected by the regime shift.

We highlight two contrasting effects, the combination of which ultimately determines the instantaneous firm response to the regime shift: a \textit{growth} effect, which depends on the magnitude and sign of $\partial V_x/\partial \mu$, and a \textit{horizon} effect which is due to $e^{-\rho \tau_{i+1}} < 1$ and depends on how long ahead the firm is planning and thus on the magnitude of the regime shift. The first case is given by $\partial V_x/\partial \mu > 0$. In this case, if the regime shift is an adverse one and $\partial \mu < 0$, we observe an aggressive growth effect due to declining resource rent, leading to increased extraction. Note that this scenario occurs with combinations of high demand and low marginal costs, and is further amplified by the horizon effect resulting in an overall aggressive extraction strategy. The second case occurs when $\partial V_x/\partial \mu = 0$. Here even though the growth effect is absent, the horizon effect still applies, resulting in declining rent and consequently an aggressive extraction strategy. Note that for high $\mu$ (i.e. for rapidly growing resource stock), the effect will be purely driven by the horizon effect since the resource rent is unaffected by the shift. Lastly, for $\partial V_x/\partial \mu < 0$, in presence of an adverse regime shift there is a precautionary growth effect contrasted by the horizon effect. The overall policy of the firm will depend on the model parameters. Note that a large negative regime shift will result in quicker detection and thus a strengthened horizon effect.

\textbf{Low resource stock levels:} we now examine the case of low stocks. We know due to the boundary condition (and the viscosity argument) that both policies are tied at 0 for $x = 0$. In general, as stocks decrease, resource rent increases and extraction decreases due to the scarcity effect imposed by the concavity of the value function. For $dx$ in proximity of $x = 0$ one can show straightforwardly that the variable part of the extraction policy is very close to $\frac{c_1 \alpha_1 + c_2 \alpha_2}{c_1 + c_2}$. The derivative with respect to $\mu$ of this term is now a complicated expression of all parameters (including both $\mu$ and $\tau_{i+1}$) but a numerical analysis is a simple exercise, and shows how the overall change in policy due to a regime shift for low
levels of stock always follows the sign of the change in drift. A negative regime shift, therefore, implies a precautionary extraction policy. When stock levels are low and growth rates are adversely affected, the scarcity effect dominates and adverse regime shifts always imply a precautionary approach.

We can now characterize the overall effect of the monopolist’s response to adverse regime shifts. If at high stocks the response is aggressive, then since \( q^* \) is monotonically increasing in \( x \) by standard arguments the function \( q^*_i(\tau_{i+1}, x) - q^*_{i+1}(\tau_{i+1}, x) \) has a root somewhere in \( \mathbb{R}^+ \): this point \( X_{th}^* \) is the threshold stock level observed at the detection time where the firm decision is unaffected by the shift. Stock levels on either side of it will define whether the instantaneous change in policy by the firm is aggressive or precautionary. Since \( q^* \) is monotonically increasing in time, it is important to study whether the policy becomes aggressive in the course of the following period and, if so, when. If there exists an optimally controlled stock level \( X^*_f \) at which \( q^*_i(\tau_{i+1}, X^*_f) = q^*_i(\tau_{i+1}, X^*_{i+1}) \), then there will be an indifference curve on the \((x \times t) \in \mathbb{R}^+ \times \mathbb{R}^+ \) space so that the firm does not change its policy, and any marginal increase in stock level for a given time on the curve involves a switch to an aggressive policy. This curve is identified by the solution of the equation that imposes the pre-shift and post-shift resource rents to be the same:

\[
V^i_x(x, \tau_{i+1}) - V^{i+1}_x(x, t) = 0.
\]

Let us illustrate the emergence of this phenomenon. We choose a range of values for the model parameters which are meant to be largely illustrative. Suppose the ecosystem undergoes a regime shift of magnitude \( \lambda = -10 \) resulting in a modified drift \( \tilde{\mu} = \mu + \lambda = 20 \). In Figure 5, the green shaded regions depict the evolution of the firm’s optimal extraction policies in the pre-shift period \([0, \tau_1]\). Once the regime shift regime is detected, the firm updates its assessment as depicted by the orange
shaded region. For clarity of exposition, we calibrate $T$ as to obtain the same expected detection times $E[\tau_1] = E[\tau_2] = 40$. Figure 5 shows the emergence of both precautionary and aggressive behaviors, depending on both stock level $X$ and time $t$. For the parameters given in Figure 5 we find $X_{th}^* = 19.82$ and for stock levels at $\tau_1$ below this threshold an instantaneous precautionary behaviour is observed. One can observe that at stock levels in the vicinity of $X_{th}^*$ for which the firm adopts a precautionary strategy at detection, over the course of the new regime extraction eventually outpaces the previous levels yielding an aggressive approach. In Section OA.3 of the Online Appendix we present another illustration where a large adverse regime shift may still lead to an aggressive extraction strategy despite declining demand (the case of “green” consumers). In other words, not all regime shifts are equal.

4 Empirical application: catastrophic regime shift in the Cantareira water reservoir

The main piece of evidence motivating our paper is that ecological regime shifts are indeed often observed in the dynamics of renewable resources. A scenario in particular captures the essence of our framework well: the case of one of the world’s largest water reservoirs, the Cantareira system. The Cantareira reservoir is an ensemble of six reservoirs connected by channels and pipelines serving the Metropolitan Area of São Paulo (MASP) in Brazil, which is one of the largest metropolitan areas in the world. The Cantareira system is managed by Companhia de Saneamento Básico do Estado de São Paulo (SABESP), a water and waste management company acting as a semi-public natural monopoly.

In early 2013, the volume experienced a sharp decrease and the operational capacity of the reservoir was subsequently depleted. This depletion occurred despite the preceding rainy season being one of the heaviest recorded in recent times. At one point, the city’s main Cantareira reservoir was down to 5%, which barely covered a month’s supply of the population’s requirements. SABESP realized the critical state of the reservoir only in January 2014 and began to reduce withdrawals, but by July 2014 the operational capacity of the reservoir was depleted. Since then, water withdrawal has been done by pumping of the so-called “strategic reserve” or “dead volume”, as well as starting to drill underground to extract groundwater. Dead volume pumping involves extracting the water that remains at the bottom of the reservoir, an often-criticized practice as it is considered dangerous due to the increasingly stagnant nature of the water as well as the presence of harmful elements.\(^8\)

This shortage led to an unprecedented crisis of water supply faced by MASP in 2014, which left the 20 million inhabitants of the area at risk of catastrophic drought whose long-terms impact are still felt to this day.\(^9\) Coutinho et al. (2015) study the close-to-depletion reservoir dynamics via a tipping-point transition approach. The reasons behind this catastrophic outcome have clear roots in environmental changes. The expansion of deforestation activities into the Amazon basin has increased pollution, severely reduced the upstream water sources and reduced rainfall.\(^10\) Some of the causes, however, can also be found in the economic decisions behind the crisis such as SABESP’s poor water management with fragile pipes and ageing infrastructures.\(^11\)

\(^8\)http://www.brasilwire.com/cantareira/
\(^9\)http://www.riotimesonline.com/brazil-news/rio-politics/will-brazils-sao-paulo-run-out-of-water-cantareira-system-reaches-the-worst-level-in-the-last-5-years/
\(^10\)http://www.theguardian.com/cities/2017/nov/28/sao-paulo-water-amazon-deforestation
\(^11\)https://theconversation.com/sao-paulo-water-crisis-shows-the-failure-of-public-private-partnerships-39483
This scenario is therefore an ideal test for our framework. The first question is whether a regime shift consistent with our model (10) actually happened. The second question is whether SABESP could have detected the regime shift by applying our framework as in Proposition/Lemma 1 before early 2014, which is when the firm started reacting to the rapidly depleting reservoir. If this is the case, the last question we want to ask ourselves is whether the reservoir depletion could have been avoided, or at least delayed, if SABESP would have reacted to the regime shift by adjusting its outflow policy at the detection time.

Figure 6 illustrates the reservoir dynamics. Observe that despite the inter-annual trend, a clear seasonal fluctuation is present in the rainfall (darker shaded region) which is reflected in the volume of stored water or the percentage of operational volume, as shown by the blue line. In early 2010 and 2011 outflow (the blue line, equivalent to $q_t$ in our framework) had to be suddenly increased as a consequence of high river inflow, in order to allow the reservoir volume to stay within its maximum capacity. Around early 2013 the reservoir volume suddenly began a sharp decline, initiated by a reduction in inflow and rainfall. In 2015 inflow and rainfall increased, but despite a small increase in the reservoir volume, it stabilized at a level well below the necessary operational capacity which remained persistent, thus generating enough scarcity to plunge the region in the aforementioned water crisis. As inflow we use daily data from the rivers Jaguari, Cachoeira, Atibainha and Piva, as well as the upstream Águas Claras reservoir, obtained from the public bulletins on SABESP’s website. The firm “extraction” $q^*_t$ is the outflow from the reservoir allowed daily by SABESP, which equals to river outflow plus the amount of water pumped to be sold for regional consumption. Rainfall is obtained as the daily precipitation levels (mm). We input inflow, rainfall and extraction and estimate the following model:
Table 1: Parameter estimates and their standard errors for the pre- and post-shift models for the reservoir dynamics (17) using the observed river inflow, rainfall and outflow (chosen by the firm) as given inputs. There is no evidence of $\beta$ and $\gamma$ changing before and after the considered date for the regime shift $\theta = \text{January 1st, 2013}$, whilst there is a strong evidence of a subsequent negative regime shift $\lambda$ that dominates the pre-existing drift.

|                | $\beta$     | $\gamma$   | $\lambda$ | $\sigma$   |
|----------------|-------------|------------|------------|------------|
| (pre-shift)    | 42781.41    | 0.658      | -          | 202094.3   |
|                | (4214.25)   | (0.009)    | -          | (38110.37) |
| (post-shift)   | 42668.86    | 0.618      | -66885.78  | 176511.4   |
|                | (1715.6)    | (0.026)    | (8552.4)   | (25324.43) |

where $dt$ is assumed as one day to match the daily data and $\theta$ is assumed to happen at the beginning of March 2013, when the reservoir volume suddenly began a sharp decline, whilst water inflow and outflow kept at its previous levels. Varying this change point in an interval of $\pm$ 2 months yields equivalent estimates. There is no evidence of the variance of the fluctuations being dependent on $X$, nor of $q^*$ varying linearly with reservoir volume, and therefore we estimate the model (17) which is consistent with our theoretical framework (10). We then estimate $\beta, \gamma, \lambda$ and $\sigma$ by means of joint particle filtering for both state and parameters. All details on the simulation, filtering procedure and parameter estimation is reported in Section OA.4 of the Online Appendix. Table 2 reports the coefficient estimates and their standard errors of each filtered parameter distribution.

Estimations of both models show that the rainfall coefficients as well as the variance of fluctuations do not change significantly between the pre- and post-shift periods: rainfall parameters are essentially unaffected, and $\sigma$ is the parameter that varies the most (from ca. $2 \times 10^6$ cubic meters to $1.7 \times 10^6$), and even though the difference is not statistically significant this change shows how the dynamics of the reservoir after the regime shift become increasingly driven by the deterministic part. This is shown in Figure 7, which plots the fitted pre- and post-shift models against the observed reservoir volume. For plotting convenience, both model fit and volume are expressed as percentages with respect to the reservoir’s capacity ranging from a maximum of $1.2695 \times 10^9 m^3$ to a minimum volume of $9.8155 \times 10^6 m^3$ below which the pumping of the dead volume is activated. The figure shows how the model estimated in (17), shown by the dotted red line, tracks well the data (black line) whilst remaining firmly within a 95% confidence band obtained by sampling 1000 coefficient sets from their reconstructed distributions, and running 1000 simulations for each set. The slight decay in fit in early 2011 for the pre-shift model, seen in the upper panel of Figure 7, leaves open the possibility of the presence of multiple regime shifts, the other happening around end 2009-early 2010. The bottom panel of Figure 7 shows how the post-regime model in (17) fits very well the data, as well as how the regime shift manifests by the emergence of a dominant deterministic force that drives the reservoir volume to the point where the operational capacity is exhausted. Furthermore, estimating the pre-shift model using the post-shift
Figure 7: Fit of the model pre-shift (above panel) and post-shift (below). The shaded areas represent rainfall (darker) and river inflows (lighter). The black line is the observed reservoir volume, the dashed red line is the reservoir volume obtained tracking (17) with the particle filter and the estimated coefficients. Monte Carlo 95% confidence/plausibility bands obtained by sampling 1000 coefficient sets from their filtered distributions and running 1000 simulations for each sample. The dashed blue line in the upper panel represents the normalized maximum reservoir capacity (1.269 km$^3$).
Table 2: Regime shift detection times with increasing firm tolerances $T$, if the detection process had been activated on $t_0 = \text{January 1st, 2013}$.

| $T$  | $\nu$  | $\tau$ (detection date) | $\tau - t_0$ (days) |
|------|--------|--------------------------|----------------------|
| 30   | 24.67  | 16-06-2013               | 166                  |
| 100  | 25.874 | 16-06-2013               | 166                  |
| 500  | 27.484 | 23-06-2013               | 173                  |
| 1000 | 28.177 | 23-06-2013               | 173                  |
| 5000 | 29.786 | 29-06-2013               | 179                  |
| 10000| 30.479 | 29-06-2013               | 179                  |

data results in a substantially worse fit: a likelihood ratio test between the model estimated in Table 2 and the same model without $\lambda$ yields a p-value of $1.5 \times 10^{-3}$, which provides further evidence of the regime shift. We remain open-minded on the interpretation of what caused the emergence of $\lambda$: all evidence points towards a story of climate change exacerbated by deforestation around the upstream Amazon basins, leading to droughts and rising temperatures. This is captured in our framework by a deterministic force towards extinction, broadly conceivable as a reduced environmental suitability for the pre-existing volume levels of the reservoir.

Upon estimation of $\mu$, $\lambda$ and $\sigma$, we now turn to the remaining two questions of interest with respect to our framework. The first question is whether SABESP could have detected the regime shift. We therefore implement the detection procedure as done in Proposition 1 and in particular Lemma 1, as our model shows evidence of a constant diffusion coefficient. Assuming SABESP started the detection process in real time on January 1st, 2013 (the choice of initial date is entirely irrelevant), the detection problem essentially involves observing the running minimum/cumulative sum process over the reservoir volume $X_t$ under the appropriate change of measure, and detecting the presence of a regime shift once this process hits the threshold $\nu$. Note that the change of measure is effectively the way to account for all observable effects of inflow and rainfall on reservoir volume, and estimate whether a new force ($\lambda$) has emerged that transformed the “residual” fluctuations in a supermartingale.

This threshold, however, depends on the firm’s tolerance/distance to the first false alarm $T$. In order to account for this, we undertake the detection process for the substantially different values of $T = [30, 100, 500, 5000, 10000]$, which imply “impatiences” (first times to false alarm) ranging between 30 days and 27 years, representing most values a firm could reasonably assume. Given our Monte Carlo simulations shown in Figure 4, we expect a concave effect of $T$ on the detection time $\tau$: this is indeed the case. Table 2 presents the different thresholds for the detection process with varying $T$, and the corresponding detection dates and times it would have taken for SABESP to detect the shift. The detection dates range between June 6th and 29th of 2013, more than six months before the point at which SABESP started adjusting its outflow policy.

We can now address the question of whether the reservoir depletion could have been avoided, or at least delayed, if SABESP would have reacted to the regime shift by adjusting its outflow policy at the detection time. For the Cantareira reservoir, SABESP started decreasing outflow in mid-January 2014, then decreased the usable capacity limit by 18.5% on the 15th of June and by a 10.7% on October 23rd. We therefore apply the equivalent outflow strategy at the average detection time (June 22nd, 2013), and obtain Monte Carlo trajectories of the model simulating 5000 trajectories keeping every other input and parameter unchanged, with confidence bands obtained as before. Figure 8 shows how
the simulated volume (dashed red line) remains above 25% of the maximum reservoir capacity even at the times when SABESP started pumping from its strategic reserves. It is certainly too ambitious a claim to say that the adoption of our detection framework could have avoided the water crisis. There is, however, clear evidence of how the reservoir depletion should have at least been delayed, as the regime shift that happened in 2013 could have been detected by SABESP and its outflow strategy could have been adjusted earlier. Furthermore, pumping from the dead volume could have been avoided, improving the quality of the water supplied to the population as well as saving substantial amounts of public funding poured in the company in order to tap the deepest levels of the reservoir. In a situation where every day of shortage involved rationing water for millions of people, to the point that some citizens took to drilling through their basements to reach groundwater, the adoption of our framework can help natural monopolies to better understand and manage the ecosystem in which they operate.

5 Concluding Remarks

In this paper we study the stochastic dynamics of a renewable resource subject to ecological regime shifts. The occurrence of such shifts can substantially alter the constraints faced by economic agents who extract natural resources. We first characterize the losses stemming from the delay with which agents become aware of a regime shift and subsequently establish a novel framework of ecosystem surveillance that minimizes the efficiency loss caused by incomplete observability of the environmental conditions in which agents operate. We show Pareto optimality of our framework for any resource-extracting economic agent, regardless of the criterion used to obtain extraction policies. We integrate the detection procedure in the maximization problem of a resource-extracting monopolist, such that the firm optimizes with respect to the resource dynamics over a time horizon given by the detection
time. Finally, we apply our framework to the case of the Cantareira water reservoir, a large-scale system of interconnected reservoirs which serves the Metropolitan Area of São Paulo in Brazil.

To conclude, some caveats are in order. Whilst the framework we propose for the quickest detection of ecological regimes is general and optimal for any economic agent, the monopolist’s problem is intentionally simplified. This choice enables us to study in depth the solution and to fully characterize the monopolist’s response to regime shifts. Lastly, although a pure monopoly is relatively rare within resource markets, our results can be used as a first step towards richer competition structures.

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### A Appendix

#### A.1 Proof of Proposition 1

Because of the delay $|\tau - \theta|$, the firm chooses an extraction
\[ q^\mu \rightarrow \sup_{q \in Q} \mathbb{E} \int_0^\tau e^{-\rho t} J(q) dt \]
\[ \text{s.t. } dX_t = (\mu_t - q)dt + \sigma_t dW_t. \]

where \( J(q) := J(q, x), Q := Q(x, t) \) is the non-empty set of Markovian admissible controls in feedback form such that \( \mathbb{E} \int_t^\tau |e^{-\rho s} J(q)| ds < \infty \) for all \( t < \tau \) and \( q \in Q \). We call this extraction policy \( q^\mu \) as it assumes there has not been yet the regime shift and the optimization is undertaken under the pre-shift dynamic constraint. Define the bounded set \( J^*(q, \tau) \) as the supremum of the maximization problem (i.e. the total volume of maximized criterion units: profits, welfare or utilis) achieved with policy \( q^\mu \) over a finite period \([0, \tau]\), which is a nonempty set of real numbers bounded above and below. However, the overall “real” supremum of the maximization problem, which corresponds to an observable \( \theta \) at which the agent switches policy, is achieved by the following:

\[
\left( \sup_q \mathbb{E} \int_0^\theta e^{-\rho t} J(q) dt \right) + \left( \sup_q \mathbb{E} \int_\theta^\tau e^{-\rho t} J(q^\lambda) dt \right) := J^*(q^\mu, \theta) + J^*(q^\lambda, \tau - \theta)
\]

where \( J^*(q^\mu, \theta) \) is the supremum set of the problem (i.e. the maximized profits) up to \( \theta \) under the constraint with drift \( \mu - q \) generated by the optimal policy \( q^\mu \), and \( J^*(q^\lambda, \tau - \theta) \) is the supremum set of the problem between \( \theta \) and \( \tau \) generated by the policy \( q^\lambda \) under the constraint with drift \( \mu_\tau + \lambda_\tau - q^\lambda \). This is shown using the additive property of the supremum over bounded nonempty sets:

\[
J^*(q^*, \tau) = \sup_q \mathbb{E} \int_0^\tau e^{-\rho t} \Pi(q) dt
\]
\[ \text{s.t. } dX_t = \begin{cases} 
(\mu_t - q)dt + \sigma_t dW_t & t < \theta \ (dX_t) \\
(\mu_t + \lambda_t - q)dt + \sigma_t dW_t & t \geq \theta \ (dX^\lambda_t) 
\end{cases} \]

\[
= \sup_q \mathbb{E} \left( \int_0^\theta e^{-\rho t} J(q) dt + \int_\theta^\tau e^{-\rho t} J(q) dt \right)
\]
\[ \text{s.t. } dX_t \]

\[
= \sup_q \mathbb{E} \left( \int_0^\theta e^{-\rho t} J(q) dt \right) + \sup_q \mathbb{E} \left( \int_\theta^\tau e^{-\rho t} J(q) dt \right)
\]
\[ \text{s.t. } dX_t \]

\[
= \sup_q \left( \int_0^\theta \int_0^\theta e^{-\rho t} J(q) P(dX) dt \right) + \sup_q \left( \int_\theta^\tau e^{-\rho t} J(q) P(dX^\lambda) dt \right)
\]

where \( q^* \) is the optimal policy that switches from \( q \) to \( q^\lambda \) exactly at \( \theta \), and \( P(x) \) is the probability measure induced by the diffusion \( x \). Using the dominated convergence and Fubini theorems, both suprema terms are bounded and positive for all \( t \in [0, \tau] \). However, between \( \theta \) and \( \tau \) any admissible policy \( \tilde{q} \in Q, q \neq q^\lambda \) will not achieve the supremum, and \( \Pi(q, \tau - \theta) < \Pi^*(q^\lambda, \tau - \theta) \). Since this is valid for any \( \tilde{q} \), it follows that \( \Pi(q^\mu, \tau - \theta) - \Pi^*(q^\lambda, \tau - \theta) < 0 \) for any \( \tau \geq \theta \). We can then write


\[ E \int_0^T e^{-\rho t} \left( J(q^\lambda) - J(q) \right) dt = \int_0^T (J^*(q^\lambda, t) - J(q, t)) dt > 0. \] (18)

Since the quantity \( J^*(q^\lambda, t) - J(q^\mu, t) \) is positive for any \( t \in [\theta, \tau] \) and the admissible control set \( Q \) includes all \( \mathcal{F}_t \)-progressively measurable functions, the detection delay \( (\tau - \theta) \) induces a loss for the agent, expressed in the same units as \( J \), which is increasing in the length of the delay itself. Due to the feedback form of the Markovian policies \( q^\lambda := q^\lambda(X_t, t) \) and \( q := q(X_t, t) \) we define \( J(q^\lambda) - J(q) \) as a continuous function \( L(X_t, t) \). One can repeat the same proof as before for \( \tau < \theta \) using \( q^\mu \) as optimal and the proof is complete. \( \square \)

We further note that the instantaneous loss function \( L_t \) is a function of both stock \( X_t \) and time \( t \). Omitting arguments for clarity, its behavior in the interval \( t \in [0, \tau] \) is defined by the stochastic differential equation

\[ \mathbb{E} dL(X_t, t) = \left[ V^\lambda_x \mathcal{A}^\lambda[q^\lambda] - V_x \mathcal{A}^\lambda[q] + \frac{\sigma^2}{2} \left( J^\lambda_{qq} q^{\lambda,2}_{xx} - J_{qq} q^2_{xx} \right) \right] dt \] (19)

under the filtration \( \mathcal{F}_t \), obtained using standard Itô calculus and the optimality condition in the respective Hamilton-Jacobi-Bellman equations associated to the value function \( V \) of the respective optimization problems, which read \( q^i = J^{-1}_q(V^i_x) \), where \( i = \{0, \lambda\} \), the subscripts in the drift and diffusion coefficients indicate partial derivatives, and the operator \( \mathcal{A}^\lambda \) is the infinitesimal generator of the controlled resource stock given by

\[ \mathcal{A}^\lambda[q] := (\mu_t + \lambda_t - q^\lambda)\phi_x + \frac{\sigma^2}{2} \phi_{xx} + \phi_t. \]

All the extraction policy terms \( q \) have the \( \lambda \) exponent in order to represent whether the policy is evaluated at the post-shift drift \( \mu_t + \lambda_t \) or not. Lastly, the terms \( V^\lambda, V \) are the solutions of the Hamilton-Jacobi-Bellman partial differential equation

\[ 0 = -\rho V^i + \sup_q \{J(q) + \mathcal{A}^i[V]\} \]

for the post-shift and pre-shift problems, respectively. The terms \( V^\lambda_x, V_x \), therefore, indicate the resource rents evaluated at the respective optimal extraction policies \( q^\lambda, q \). The SDE starts at \( L(X_\theta, \theta) > 0 \) since (18) applies for all times. By optimality of \( q^\lambda, V^\lambda \) is a martingale while \( V \) is a supermartingale. It then follows that \( L(X_t, t) > 0 \) in \( t \in [\theta, \tau] \) almost surely.

Equation (19) has an intuitive interpretation: the deterministic part of the instantaneous loss evolves according to two differential terms. The first is the difference between the instantaneous expected change in extraction, \( \mathcal{A}^\lambda q^\lambda = \mathbb{E}[dq^\lambda] \) of the “theoretical” extraction policy \( q^\lambda \) with the suboptimal policy \( q \) generated by the detection delay \( \tau \), expressed in units of resource rent \( V_x \). The inefficiency is generated by the fact that the policy \( q \) is applied to a resource stock that evolves according to a process that is not what the agent considers in its optimization, i.e. \( \mathcal{A}^\lambda q \). Intuitively, this represents why the extraction policy the agent applies in the interval \( [\theta, \tau] \) is wrong: the chosen policy \( q \) is optimal for a resource stock that grows deterministically as \( \mu_t \), and is applied to a resource stock that however
grows at the post-shift rate $\mu_t + \lambda_t$.

### A.2 Proof of Proposition 2

In the period before $\theta$, the dynamics of the resource $X_t$ are determined by the SDE

$$dX_t = (\mu_t - q_t)dt + \sigma_t dW_t.$$  \hspace{1cm} (20)

under the triple $(\mathbb{R}^+, \mathcal{F}, P)$. Define now the transformation, sometimes called the Lamperti transform, given by

$$\tilde{X}_t := g(t, X_t) = \int \sigma(t, x)^{-1} dx \bigg|_{x=X_t}.$$  

Under the standard conditions of existence of a solution for (20), $g(.)$ maps one-to-one with the state space of $X$ for all $t$ and thus this integral exists. One can then transform the original stock process in one with an unit diffusion. First, a straightforward application of Itô’s lemma to $g_x(t, x) = 1/\sigma(t, x)$ yields

$$d\tilde{X}_t = \left( g_t(t, x) + (\mu(t, x) + q(t, x))g_x(t, x) + \frac{1}{2} \sigma^2(t, x)g_{xx}(t, x) \right) dt + \sigma(t, x)g_x(t, x) dW_t$$

$$= \left( g_t(t, x) + (\mu(t, x) + q(t, x))\sigma(t, x)^{-1} + \frac{1}{2} \sigma^2(t, x)g_{xx}(t, x) \right) dt + dW_t.$$

Now notice that $g_{xx}(t, x) = -\sigma_{xx}(t, x)/\sigma(t, x)^2$ and that $X_t = g^{-1}(t, \tilde{X}_t)$. We can then rewrite the previous expression as

$$d\tilde{X}_t = \left( g_t(t, g^{-1}(t, \tilde{X}_t)) + q(t, g^{-1}(t, \tilde{X}_t)) + \frac{\mu(t, g^{-1}(t, \tilde{X}_t))}{\sigma(t, g^{-1}(t, \tilde{X}_t))} \right) dt + dW_t$$

$$= \tilde{\mu}(t, g^{-1}(t, \tilde{X}_t)) dt + dW_t,$$

Now, Girsanov theory tells us that the process

$$M_t = \exp \left( - \int_0^t \tilde{\mu}(s, g^{-1}(s, \tilde{X}_s)) dW_s - \frac{1}{2} \int_0^t \tilde{\mu}(s, g^{-1}(s, \tilde{X}_s)) ds \right)$$

is a $P$-martingale. Therefore, the process

$$\tilde{W}_t = W_t + \int_0^t \tilde{\mu}(t, g^{-1}(t, \tilde{X}_t)) ds$$

is a $Q$-Brownian motion, where one obtains the new probability measure by $Q = \mathbb{E}_P(M_t)$. The process $\tilde{X}_t$ therefore admits the representation

$$\tilde{X}_t = x_0 + \int_0^t d\tilde{W}_s$$
and is therefore a Brownian motion under the measure $Q$. Using the same procedure as before to the post-shift resource process, the firm’s detection problem now becomes

$$d\tilde{X}_t = \begin{cases} d\tilde{W}_t & t < \theta \\ \tilde{X}_t + d\tilde{W}_t & t \geq \theta, \end{cases}$$

where

$$\tilde{X}_t = \frac{\lambda(t, g^{-1}(t, \tilde{X}_t), \Theta)}{\sigma(t, g^{-1}(t, \tilde{X}_t))}.$$

We first notice that one can only focus on the constraints that bind with equality: if $E_{\infty}[\tau] > T$ and equivalently for (6), one could set a stopping time that achieves the constraint with equality, without increasing the detection delay by randomizing between $\tau$ and 0. We can therefore restrict our attention to the $F_t$-adapted stopping rules which satisfy (5) and (6) with equality. This problem was first studied by Shiryaev (1963), and Moustakides (2004) shows that the stopping rule that solves the optimization problem (4) for a general $F_t$-adapted process as regime shift, with a modified divergence-type/entropic criterion to account for a general process $\lambda_t$ given by

$$J(\tau) := \sup_{\theta} \text{ess sup} \mathbb{E}_\theta \left[ 1_{\tau > \theta} \int_\theta^\tau \frac{1}{2} \lambda_t^2 dt | F_\theta \right]$$

is given by the stopping time

$$\tau(\lambda_t, \nu) = \inf \{ t \geq 0; CS_t \geq \nu \}.$$

The process $CS_t$ is the cumulative sum process defined as the running minimum of the difference at any instant $s \leq t$ between the Radon-Nikodym derivative of the post- and pre-regime shift martingale measures for $\tilde{X}$, $u_t$, and its minimum obtained value up to that instant, namely

$$CS_t(\lambda_t) = u_t(\lambda_t) - \inf_{0 \leq s \leq t} u_t(\lambda_t) \geq 0.$$

where

$$u_t = \log \frac{dQ_{\theta=0}}{dQ_{\theta=\infty}} = \int_0^t \lambda(s, \tilde{X}_s, \Theta) d\tilde{X}_s + \frac{1}{2} \int_0^t \lambda(s, \tilde{X}_s, \Theta)^2 ds.$$ 

(21)

and the threshold $\nu$ is chosen to satisfy the constraint (6). It can be shown that

$$E_{\infty} \left[ 1_{\tau(\nu) > \tau} \int_\theta^{\tau(\nu)} \frac{1}{2} \lambda_t^2 dt | F_\tau \right] = e^{\nu} - \nu - e^{CS_\tau} + CS_\tau$$

for all $\tau \in \mathbb{R}^+$, and thus from the definition of (6) we have that at $\tau = 0$ the optimal threshold is given by the value of $\nu$ that solves

$$e^{\nu} - \nu - 1 = T.$$

This concludes the proof of Proposition 1.

For Lemma 1, the case of the drifted Brownian motion, the proof is more straightforward. First, notice that the Lamperti transform is simply given by $1/\sigma$ and the regime shift is given by $\lambda/\sigma$. Due to
the prior information $\Theta$, the agent knows the magnitude of $\lambda$ and the detection problem reverts exactly to the Brownian disorder problem studied by Shiryaev (1996) and in the case of multiple drifts by Hadjiliadis and Moustakides (2006). The Brownian disorder is essentially the detection of the change between a martingale and a sub/supermartingale, depending on the sign of $\lambda$. This requires that the extraction decisions, that define both sign and magnitude of the change in resource growth, be set strictly before the time of the initial condition on $X$ (here normalized to 0, i.e. $X_0$). One can easily see that for a constant $\lambda$ (21) becomes

$$u_t(\lambda(q_{ex})) = \log \frac{dQ_{\theta=0}}{dQ_{\theta=\infty}} = \frac{\lambda^2}{2\sigma^2} t + \frac{\lambda}{\sigma^2} \tilde{W}_t = \frac{\lambda}{\sigma^2} X_t - \frac{\lambda^2}{2\sigma^2} t.$$

The CUSUM statistic process is then given by the difference at any instant $s \leq t$ between $u_t$ and its minimum obtained value up to that instant, namely

$$CS_t(\lambda) = u_t(\lambda) - \inf_{0 \leq s \leq t} u_t(\lambda) \geq 0,$$

and the threshold equation is immediately obtained by noticing that for constant $\lambda$ in order to revert to the constraint (6) one needs to multiply (5) by $2\sigma^2/\lambda^2$. The delay function can be obtained as follows, which also helps to shed further light on the equivalence between (5) and (6): define

$$d(\nu) = e^{-\nu} + \nu - 1$$

and then the function $h(z) = (2\sigma^2/\lambda^2)(d(\nu) - d(z))$. Similarly as before one can write the expected delay (i.e. the expectation over every $\tau$) for constant $\lambda$ as

$$E_\tau \left[ \mathbb{1}_{\tau(\nu) > \theta} \int_0^{\tau(\nu)} \frac{1}{2} \lambda^2 t d\theta | \mathcal{F}_\theta \right] = 2\sigma^2 \frac{\lambda^2}{\lambda^2} (d(\nu) - d(\tau)) = h(\tau).$$

Then $h$ satisfies the following:

$$h'(z) + h''(z) = -1, \quad h'(0) = h(\nu) = 0.$$

With this, expanding $h(CS_t)$ under the measure Q with Itô’s lemma, for any stopping time finite in expectation we have that

$$E[h(CS_\tau)] = h(CS_0) - E[\tau].$$

Since for the stopping time $\tau = \tau(\lambda, \nu)$ (i.e. the one that solves our problem) we have that $CS_{\tau(\lambda, \nu)} = \nu$ and since the CUSUM process starts at 0, the formula for the delay immediately follows and the proof is complete.

### A.3 Proof of Proposition 3

Let us begin with the first time period, with the instantaneous drift given by $\mu$ and the firm optimizes in $t \in [0, \tau_1]$ where $\tau_1 = E_{\theta=\infty}[\tau] + E[\tau(\lambda_0, \nu)]$. At time $t = 0$ the firm believes that the resource
is driven by a diffusion process with the natural growth rate $\mu$. At a random time $\theta$, there is an initial exogenous change, $\lambda_0$, in the resource dynamics. The first change is exogenous so as to start the process of subsequent adjustment. Until the detection of this change, the firm operates in an environment where the resource evolves according to the process

$$dX_t = (\mu - q_0(t, X_t))dt + \sigma dW_t, \quad t \in [0, \tau_1].$$

The final time which the firm uses as a reference for its decisions is given by

$$\tau_1 = E_{\theta=\infty}[\tau] + E[\tau(\lambda_0, \nu)] = T + \frac{2\sigma^2}{\lambda_0^2}(e^{-\nu} + \nu - 1)$$

where the threshold $\nu$ solves $\frac{2\sigma^2}{\lambda_0^2}(e^{\nu} - \nu - 1) = T$. Within this time interval $[0, \tau_1]$, the value of the firm is given by

$$V(t, X_t) = \sup_{q \in Q} E_t \int_t^{\tau_1} \Pi(q) e^{-\rho s} ds \quad \text{s.t.} \quad dX_t = (\mu - q)dt + \sigma dW_t, \quad X_t \geq 0, t \in [0, \tau_1].$$  \hspace{1cm} (22)

where the control set is given by $Q = \{q : \mathbb{R}^+ \to \mathbb{R}^+, q \geq 0, \text{bounded and } F_t\text{-adapted}\}$. What we first want to achieve is to show that the value function $V$ is a weak solution of the optimization problem (22) to deal with the constraint $X_t \geq 0$, which does not allow the value function to be always continuous or differentiable. Furthermore, between periods the drift of the constraint changes so the value function changes as well. In all that follows we will use as a reference Fleming and Soner (2006). We first write the Hamilton-Jacobi-Bellman equation for this problem in terms of its infinitesimal generator. Define the set $\mathcal{D} \in C([0, \tau_1] \times \mathbb{R})$. Then $V(t, x) \in \mathcal{D}$ is a classical solution of the optimization problem (22) if it satisfies the equation

$$- \frac{\partial}{\partial t} V + \mathcal{A}[V(t, .)](x) = 0,$$  \hspace{1cm} (23)

where $\mathcal{A}$ is the generator of the HJB equation. Now, define a continuous function $\mathcal{H}$ (the Hamiltonian) such that

$$\mathcal{A}[\phi](x) = \mathcal{H}(t, x, D\phi(x), D^2\phi(x))$$

and consider the equation

$$- \frac{\partial}{\partial t} W(t, x) + \mathcal{H}(t, x, DW(t, x), D^2W(t, x)) = 0.$$  \hspace{1cm} (24)

Following the fundamental work by Crandall and Lions (1981), a function $V(t, x) \in C([0, \tau_1] \times \mathbb{R})$
is a viscosity subsolution of (24) if for all \( v \in C^\infty(D) \)

\[
- \frac{\partial}{\partial t} v(\bar{t}, \bar{x}) + \mathcal{H}(\bar{t}, \bar{x}, Dv(\bar{t}, \bar{x}), D^2v(\bar{t}, \bar{x})) \leq 0
\]

for every point \((\bar{t}, \bar{x})\) which is a local maximum of \(V - v\). Similarly, \(V(t, x)\) is a viscosity supersolution of (24) if for all \( v \in C^\infty(D) \)

\[
- \frac{\partial}{\partial t} v(\bar{t}, \bar{x}) + \mathcal{H}(\bar{t}, \bar{x}, Dv(\bar{t}, \bar{x}), D^2v(\bar{t}, \bar{x})) \geq 0.
\]

for every point \((\bar{t}, \bar{x}) \in D\) which is a local minimum of \(V - v\). The function \(V(t, x)\) is a viscosity solution of the equation (24) if it is both a viscosity subsolution and a viscosity supersolution. This implies that the function \(V(t, x)\) is a weak solution of the optimization problem (22). Let us now show that \(V\) is a viscosity solution of our problem (22).

Let \(v \in C^2([0, \tau_1] \times \mathbb{R})\), let \(V - v\) be maximized at the point \((\bar{t}, \bar{x}) \in ([0, \tau_1] \times \mathbb{R})\) and let us fix an optimal control (extraction rate) \(q \in Q\). Let \(X(\cdot) = X(\cdot; t, q)\) be the controlled stochastic process that drives the resource stock. For every time \(\tau > \bar{t}\) for which \(X_\tau > 0\), we have, using Itô’s lemma and Bellman’s principle of optimality,

\[
0 \leq \frac{\mathbb{E}_{\bar{t}}[V(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x}) - V(\tau, x(\tau)) + v(\tau, x(\tau))]}{\tau - \bar{t}}
\]

\[
0 \leq \frac{1}{\tau - \bar{t}} \mathbb{E}_{\bar{t}} \left[ \int_{\bar{t}}^{\tau} \Pi(t, x, q) dt - v(\bar{t}, \bar{x}) + v(\tau, x(\tau)) \right].
\]

This implies

\[
0 \leq v_t(\bar{t}, \bar{x}) + \Pi(\bar{t}, x, q) + v_x(\mu + q) + \frac{\sigma^2}{2} v_{xx}
\]

for all \(q \in Q\): we can then write

\[
0 \leq v_t(\bar{t}, \bar{x}) + \sup_{q \in Q} \left[ \Pi(\bar{t}, x, q) + v_x(\mu - q) + \frac{\sigma^2}{2} v_{xx} \right]
\]

\[
0 \leq v_t - \mathcal{H}(\bar{t}, \bar{x}, Dv(\bar{t}, \bar{x}), D^2v(\bar{t}, \bar{x})).
\]

This proves that \(V\) is a viscosity subsolution of the problem (22). Proceeding similarly proves that \(V\) is a viscosity supersolution of the problem: if \(V - v\) attains a minimum at \((\bar{t}, \bar{x})\) then for any \(\epsilon > 0\) and \(\tau > \bar{t}\) we can find a control \(q \in Q\) such that

\[
0 \geq -\epsilon(\tau - \bar{t}) + \mathbb{E} \left[ \int_{\bar{t}}^{\tau} \Pi(t, x, q) dt - v(\bar{t}, \bar{x}) + v(\tau, x(\tau)) \right]
\]

which implies
$$\epsilon \geq \frac{1}{\tau - \bar{t}} \mathbb{E}_t \left[ \int_{\bar{t}}^\tau \Pi(t,x,q)dt - v(\bar{t},\bar{x}) + v(\tau,x(\tau)) \right].$$

Proceeding equivalently as before, one shows that $V$ is a viscosity supersolution of (22). We can conclude that $V$ is a viscosity solution of (22). We can conclude that $V$ is a viscosity solution of (22). Note that for every time $\tau_e \in [0, \tau_1]$ for which $X_\tau > 0$, since for optimality we have $\Pi_q(.,q^*) - V_x = 0$ and $\Pi$ is continuous and twice differentiable in $q$, it can be easily shown that the inequalities of the definition of sub- and supersolution are satisfied with equality, which means that $V(t,x)$ is also a classical solution of (23) for each $t = \tau_e$. We now need to deal with the positivity constraint. Given the “feasible” set $D' = ([0, \tau_1] \times O \subset \mathbb{R}^+), we cannot impose that the value function $V(t,x)$ nor its gradient $\partial_x V(t,x)$ are differentiable (or continuous, for that matter) at 0 at the boundary of $\partial D'$. Following Fleming and Soner (2006), we need to impose a boundary inequality, which does not require $V$ nor the boundary $\partial D'$ to be differentiable at 0. This implies that the value function $V(t,0)$ must be a viscosity subsolution of (22). Following the previous definitions, we must have

$$v_t(t,0) \leq \mathcal{H}(t,0,Dv,D^2v)$$

$$\leq \sup_{q \in Q} \left\{ \Pi(t,x,q) + v_x(0)(\mu - q) + v_{xx}(0)\frac{\sigma^2}{2} \right\}$$

(25)

for all continuous functions for which $V - v$ is locally maximized around $x = 0$. Given a natural boundary condition given by the fact that when the resource is zero, the extraction must be zero and consequently the objective $\Pi$ must be zero. Since $V - v$ has to be maximized around 0, we have

$$\mathcal{H}(t,0,a_a) \geq \mathcal{H}(t,0,v_x(t,0),v_{xx}(t,0)) \quad \forall a \geq v_x(t,0).$$

The proof is simple, one just needs to write $\mathcal{H}(t,0,a,a_x) = \sup_{q \in Q} \Pi(t,x,q) + \alpha(\mu + q) + a_x\frac{\sigma^2}{2}$ and use $\alpha \geq v_x(t,0)$ to show the inequality holds. Given this result, condition (25) is easily seen to be satisfied by $V(t,0) = 0$, which we choose because of its immediate intuitive economic interpretation. Note that the other natural boundary condition imposes $q(t,0) = 0$, so this restriction needs to be imposed on the value function gradient at 0, and unconstrained otherwise. Note that this is simply the static optimality condition that imposes marginal revenue equalling marginal cost. We therefore can say that the constrained viscosity solution given by

$$V(t,0) = 0$$

$$V_x(t,0) = p'(0) - c'(0)$$

$$V(t,x) \text{ solves } V_t - \mathcal{H}(t,x,DV(t,x),D^2V(t,x)) = 0 \quad (t,x) \in [0, \tau_1] \times \mathbb{R}^+]$$

is a solution to the problem (22). The last part we need to show is uniqueness, and thus need the examine the characteristics of the objective function and the stock dynamics. The constant drift and diffusion coefficients are trivially continuous and bounded. The set $Q$ is bounded below by 0 since $q(t,0) = 0$ and above by the fact that $q$ is decreasing in $V_x$ and $q \geq 0$ and is thus a compact set.
Regarding the cost function, the technical definition of a natural monopoly is that the cost function is subadditive, implying \( c(\sum_i q_i) \leq \sum_i c(q_i) \). Hence it is always cheaper to produce \( \sum_i q_i \) units of output using a single firm than using two or more firms (see for example Baumol et al. (1982)). This implies that \( \Pi(t, x, q) \) is continuous and bounded on \( \mathbb{R}^+ \times \mathbb{R}^+ \times Q \), as well as its partial derivatives \( \Pi_t, \Pi_x, \Pi_{xx} \). The conditions of Theorem 4.4 of Fleming and Soner (2006) are then satisfied and thus \( V(t, x) \) is unique. Since this viscosity result applies for any time period \([\tau_i, \tau_{i+1}]\), the value function across all periods is equal to the value function for each period, and at each changepoint \( \tau_i \) the solution is obtained as a so-called _envelope solution_ of a super- and a subsolution of (26), using Theorem 2.14 in Bardi et al. (1997).

Let us now then search for a classical solution within the set \((t, x) \in [0, \tau_1] \times \mathbb{R}^+\). The HJB equation for the firm’s optimization problem reads

\[
0 = V_t - \rho V + \max_{q \in Q} \left\{ (a - bq)q - cq - F - qV_x \right\} + \mu V_x + \frac{\sigma^2}{2} V_{xx}.
\]

Equation (26) implies an optimal extraction policy given by

\[
q^*(t, X_t) = \left[ \frac{a - c - V_x}{2b} \right]_+.
\]

Note that this implies that in order for extraction to stay positive, \( V_x \leq a - c \), meaning the resource rent cannot exceed the demand intercept parameter minus the marginal cost. This is clearly a consequence of the assumption of linear demand, which results in a quadratic criterion. It will be clear in what follows that the solution will be naturally constrained by the boundary conditions to satisfy this requirement. Substituting in (26) and grouping terms, we obtain the following nonlinear partial differential equation:

\[
0 = V_t - \rho V + AV_x + BV_x^2 + \frac{\sigma^2}{2} V_{xx} + C
\]

where the constants \( A, B \) and \( C \) are given by

\[
A = \mu - \frac{a - c}{2b}, \quad B = \frac{1}{4b}, \quad C = \frac{(a - c)^2}{4b} - F.
\]

The natural boundary conditions of this problem are given by the viscosity solution formulation (26) as well as economic intuition,

\[
V(t, x) = 0 \text{ for } x < 0, \quad V(t, 0) = 0, \quad q(t, 0) = 0.
\]

Because of the homogeneous nature of the profit function, we guess a general solution of the HJB equation of the form \( e^{-\rho(t-\tau)}V(x) = V(t, x) \) and we linearize it with the nonlinear change of variable

\[
V'(x) = \frac{\sigma^2 \psi'(x)}{2B \psi(x)}
\]

...
where \( \psi(.) \) is a general twice differentiable function on \( \mathbb{R} \). By this linearization, one can easily obtain the general solution

\[
\psi_g(x) = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}.
\]

where \( \alpha_{1,2} = -A \pm \sqrt{A^2 - 4BC} \) and \( \alpha_2 < \alpha_1 \). The coefficients \( \alpha_{1,2} \) are given by the boundary conditions (28), after noticing that \( V(t, 0) = 0 \) implies \( \psi(0) = 1 \). Note that care must be taken to include the "guess" constraint (29) in the coefficient formulation to keep the particular solution separable in \( t \) and \( x \): this is achieved by noticing that naturally \( q(\tau, 0) = q^m - \sigma^2 \psi'(0) \) where \( q^m = \frac{a - c}{2b} \). The requirement that \( q \) is zero at zero stock is further enforced by the viscosity argument at the right limit to the boundary. Note that \( q^m \) is the solution of the static first-order condition for the monopolist that maximizes pointwise in time, given by \( \partial_q \Pi(q) = 0 \). Furthermore, the particular solution can be computed in closed form, but its expression is rather lengthy and therefore omitted. Obtaining these terms is a very simple matter from a numerical perspective so we refer to the particular solution as \( \psi(x) \). It is then immediate to show that the optimal extraction policy in feedback form is therefore

\[
q_0^*(t, x) = q^m - \sigma^2 \psi'(x) \frac{e^{-\rho(\tau_1 - t)}}{\psi(x)}.
\]

From this expression we also obtain the resource rent for the monopolist:

\[
V_x = 2b\sigma^2 \psi'(x) \frac{e^{-\rho(\tau_1 - t)}}{\psi(x)}
\]

This solves the firm’s optimization problem for \( t \in [0, \tau_1] \). Once the firm detects the change then it will reassess decisions with the new drift \( \mu + \lambda_0 \). The optimization problem will therefore be equivalent to (22), with \( \mu = \mu + \lambda_0 \) and now between \( \tau_1 \) and \( \tau_2 \), where \( \tau_2 \) is the “new” expected time for the subsequent regime shift. Rescaling the new time index to the interval 0 and \( \tau_2 = \tau_2 - \tau_1 \), by the same arguments one obtains the same form of the solution as the previous time interval as

\[
q_1^*(t, x) = q^m - \sigma^2 \psi'(x) \frac{e^{-\rho(\tau_2 - t)}}{\psi(x)}
\]

where \( \psi \) is evaluated at \( \tilde{\theta} = \mu + \lambda_0 + q^m \). Similarly one can show this process is equivalent for a general time interval \( i = [\tau_i, \tau_{i+1}] \), and we have a solution of the HJB equation for each period.

The last step we need is a verification theorem. This is obtained by “stopping” the HJB at an arbitrary time \( \tau \). Define \( v(t, x) \) our candidate function that solves the HJB equation. Use Itô’s lemma on the discounted function we obtain

\[
v(\tau, X_\tau) = e^{-\rho \tau} v(X_\tau) = v(x) + \int_0^\tau e^{-\rho s} \mathcal{A}[v](X_s) ds + \int_0^\tau v_{xx}(X_s) dW_s
\]

where \( \mathcal{A} \) is the HJB generator. Taking expectations and adding the objective \( e^{-\rho t} \Pi(q_t) \):

\[
v(x) = \mathbb{E} \int_0^\tau e^{-\rho s} \Pi(q_s) ds + e^{-\rho \tau} \mathbb{E} v(X_\tau) - \mathbb{E} \int_0^\tau e^{-\rho s} [\mathcal{A}[v](X_s) + \Pi(q_s)] ds.
\]
Since by assumption $v$ is a solution of the HJB equation, we have

$$A[v](X_s) + \Pi(q_s) \geq 0,$$

for all times $s \in [0, \tau]$. If we choose $\tau = t$, we have the inequality

$$v(x) \leq \mathbb{E} \int_0^\tau e^{-\rho t} \Pi(q_s) ds + e^{-\rho \tau} v(X_t)$$

for all choices of control $q$. Since we established $q^*_t \in Q$ as the controls for which $v$ the HJB is solved, then

$$v(x) = \mathbb{E} \int_0^\tau e^{-\rho t} \Pi(q^*_s) ds + e^{-\rho \tau} v(X_t).$$

Then $v(x) = V(x)$ for all $t \in [\tau_i, \tau_{i+1}]$, $v(t, x) = V(t, x)$ and $q^*$ solves (10), and the proof is complete.

## B Resource stock as a geometric process

Let us now show that the characteristics of our model remain unadulterated if one chooses a geometric process for the uncontrolled resource stock. We therefore have

$$dX_t = (\mu X_t - q_t) dt + \sigma X_t dW_t,$$

where $q_t = q(X_t, t)$. The firm needs to monitor the controlled stock for a change in $\mu$, which now translates in the following problem, shown in period 1 for simplicity:

$$dX_t = \begin{cases} (\mu X_t - q_t) dt + \sigma X_t dW_t & t < \theta \\ ([\mu + \lambda] X_t - q_t) dt + \sigma X_t dW_t & t \geq \theta. \end{cases}$$

(32)

The detection problem is now a detection of a regime shift yielding a change in the growth rate of the (uncontrolled) resource stock. The first question that we need to address is how does the detection procedure apply in this framework, and whether it can be optimally applied here as well. The firm now thus faces the following sequential optimization problem:

$$\sup_{q \in Q} \sum_{i=0}^{\infty} \mathbb{E}_{\tau_i} \int_{\tau_i}^{\tau_{i+1}} \Pi(q_t, X_t)e^{-\rho t} dt$$

$$dX_t = \begin{cases} \left[ \left(\mu + \sum_{j=0}^{i} \lambda_j \right) X_t - q_t \right] dt + \sigma X_t dW_t, & t \in [\tau_i, \tau_{i+1}) \\ \left[ \left(\mu + \sum_{j=0}^{i} \lambda_j + \lambda_{i+1} \right) X_t - q_t \right] dt + \sigma X_t dW_t, & t \geq \tau_{i+1}, \ i \in \mathbb{N}. \end{cases}$$

$$X_t \geq 0 \ \forall t, \tau_0 = 0$$

$$\tau_i = \mathbb{E}_{\infty}[\tau] + \mathbb{E}[\tau(\lambda_i, \nu)]$$

(33)

Due to the geometric nature of the dynamics, the positivity constraint on $X$ is naturally satisfied, the value function of the optimization problem is smooth and does not require the use of the viscosity
machinery within each period \([\tau_i, \tau_{i+1}]\) (the requirement of the viscosity envelope at each changepoint, however, still applies). However, it imposes the variance of the fluctuations to be dependent on resource stock, meaning that extinction cannot be driven by environmental stochasticity alone. Let us start with the optimization process and let each \(\tau_i\) as given. The HJB equation for this problem whose solution yields the firm’s value function \(V = V(x, t)\) of problem (34) for each period \(i\) is

\[-V_t + \rho V = \sup_q [p(q)q - c(q, x) + -qV_x] + (\mu + \sum_{j=0}^{i} \lambda_j) x V_x + \frac{\sigma^2}{2} x^2 V_{xx}\]

which implies the optimal extraction policy is given by

\[q^* \text{s.t. } p(q) + p'(q)q = V_x + c(q, x).\]

In order to obtain an explicit solution for this nonlinear problem, as well as to show how our detection framework integrates well with other functional specifications, we choose an isoelastic demand function \(q(p) = b p^{-\eta}\) and marginal extraction \(c(X_t) = c X^{-1/\eta}\), where \(b, c \in \mathbb{R}^+\) and \(\eta > 1\), as done in Pindyck (1987) in a stationary setting. It can be shown after lengthy but straightforward computations that the solution of the HJB reads

\[V(x, t) = A_g \phi(t)x^{(\eta-1)/\eta}\]

where \(A_g\) is a parameter that solves

\[A_g^{1/\eta} \left( A_g \frac{\eta - 1}{\eta} + c \right) = \left[ \left( \rho - \mu - \sum_{j=1}^{i-1} \lambda_j \right) b \frac{\eta - 1}{\eta} + \frac{\sigma^2}{2b} \left( \frac{\eta - 1}{\eta} \right)^2 \right]^{1/\eta}\]

and \(\phi\) is the solution to the nonlinear ordinary differential equation

\[\phi_t' = -\phi_t \left[ \frac{\eta - 1}{\eta} \left( \mu + \sum_{j=1}^{i-1} \lambda_j \right) - \frac{\eta - 1}{\eta^2} \frac{\sigma^2}{2} - \rho \right] - \frac{1}{A_g} \left( \frac{b}{\eta - 1} \right) \left( \frac{\eta - 1}{\eta} A_g \phi_t - c \right)^{1-\eta}\]

equipped with the boundary condition \(\phi(\tau_{i+1}) = 1\). The natural boundary condition \(q(0, t) = 0\) is immediately seen to be trivially satisfied. Note that for parameter sets such that \(A_g\) is large (or for large \(\eta\)), the time component in the value function is the simple exponential discounting factor \(\phi = c_1 e^{-\hat{\rho}(\tau_{i+1} - t)}\), where the discount factor \(\hat{\rho}\) now includes all structural parameters except \(b\) and \(c\). Showing how at the detection time that the solution is simply the stationary solution (i.e. the solution of the HJB where \(V_t = 0\)). Given that \(X\) is now a geometric process, \(\mu\) and \(\sigma\) are rates and reasonably smaller than unity, likely to be of a magnitude around \(10^{-1}/10^{-2}\). Note also that in the the degenerate case \(b = \eta - 1\) and then letting \(\eta \to 1\) we have \(\phi = c_1 e^{\hat{\rho}(\tau_{i+1} - t)} + \frac{1}{A_g \rho}\). The optimal extraction policy in feedback form is then given by
\[ q^*(X_t, t) = b \left( \frac{\eta - 1}{\eta} A_g \phi(t) + c \right)^{-\eta} X_t = B_t X_t, \]

which is linear in \( X \). Whilst there no is fully explicit solution for \( A_g \) and \( \phi \), obtaining it numerically is a simple matter and it can be shown straightforwardly (albeit not analytically) that the characteristics of this solution studied in the main paper still apply, and especially both the tradeoff between change in resource growth rate and firm time horizon due to the detection delay. This is intuitively seen by observing that the extraction policy is linear in \( X \), therefore changes in drift affect the coefficient \( A_g \) in the same direction. The time-dependent component, however, can move in the opposite direction since the magnitude of the shift affects \( \tau_{i+1} \) as well as the parameters. The firm’s detection problem is therefore applied to the following resource dynamics:

\[
\frac{dX_t}{X_t} = \left\{ \begin{array}{ll}
(\mu + \sum_{j=0}^{i} \lambda_j - B_t) dt + \sigma dW_t & t < \theta \\
(\mu + \sum_{j=0}^{i} \lambda_j + \lambda - B_t) dt + \sigma dW_t & t \geq \theta.
\end{array} \right.
\]

Since the optimal extraction policy is linear in \( X \), the regime shift is equivalent to a Brownian disorder problem when the resource stock is expressed in growth rates, since the shift \( \lambda(q_i) \) enters linearly at \( \theta \). It’s therefore obvious that Proposition 1 holds equivalently in this framework by applying the detection procedure with constant parameters shown in Lemma 1 to \( \log(X_t) \), albeit with a slightly different change of measure (we have \( B_t \) instead of the \( q_t \) in (32)) chosen such that the pre-shift process \( \log(X_t) \) is a martingale.

C Extinction/depletion risk

We define the emergence of catastrophe risk as the situation in which the instantaneous drift of the resource stock \( X_t \) is negative in period \( i \) at any time \( t \in [\tau_i, \tau_{i+1}] \):

\[
\mu + \sum_{j=0}^{i-1} \lambda_j - q^m + \sigma^2 \int_{\tau_i}^{t} \frac{\psi(X_s)}{\psi(X_t)} e^{-\rho(\tau_{i+1}-s)} ds < 0,
\]

which implies that \( P(\lim_{t \to \infty} X_t = 0) = 1 \). At this moment the firm may have to reassess its extraction policies, due to the fact that the resource growth rate has been affected by its past extraction decisions to a point where extinction is likely. In fact, the probability of the resource being zero in infinite time is unity, which means that the resource eventually will be depleted. We can then study the first passage time to depletion. The firm can now exploit the non-stationary nature of the time intervals in which it operates: a first immediate analysis should be what happens if it stops extracting. Normalizing time to \( \tau_i = 0 \), we define the probability of extinction as

\[
\phi(x) = \Pr\left[ \inf_{t \in \mathbb{R}^+} X_t \leq 0 \bigg| X_0 = X^*_0, q^*(t; X_t) = 0 \right]
\]

and the first time to catastrophe as
\( \tau_c = \inf \{ t \mid X_t \leq 0, X_0 = X^*_\tau, q^*(t, X_t) = 0 \}. \)

Then \( X_t \) follows simply a drifted Brownian motion and the problem is equivalent of finding where a standard Brownian motion crosses the line \( x - \mu - \sum_{j=0}^i \lambda_j \) (remember that \( \mu + \sum_{j=0}^i \lambda_j \) is negative). It’s a well-known stochastic analysis problem, and the firm stops extracting the expected time to catastrophe is

\[
E\tau_c = \frac{X^*_\tau}{\mu + \sum_{j=0}^i \lambda_j}.
\]

(36)

and the probability of extinction is \( \phi(x) = \exp \left( -\frac{2(\mu + \sum_{j=0}^i \lambda_j)}{\sigma^2} x \right) \). With no extraction, the first passage time to catastrophe \( \tau_{cat} \), for a resource stock starting at \( X^*_\tau \), will be distributed according to the following density:

\[
P\{ \tau_{cat} \in dt \} = \frac{X^*_\tau}{\sqrt{2\pi\sigma^2t^3}} \exp \left( -\frac{(X^*_\tau - (\mu + \sum_{j=0}^i \lambda_j)t)^2}{2\sigma^2t} \right) dt,
\]

\[
= IG \left( \frac{X^*_\tau}{\mu + \sum_{j=0}^i \lambda_j}, \left( \frac{X^*_\tau}{\sigma} \right)^2 \right),
\]

(37)

which follows an inverse Gaussian distribution, as seen in Figure 9. If \( E\tau_c \leq \tau_i + 1 \), on average the resource will be depleted within the detection period even if the firm stops extracting altogether: we are therefore in a situation of potential irreversible catastrophe, where even the most precautionary of extraction behavior cannot avoid the resource from being expected to deplete. In other words, since extraction always reduces the drift, (36) and the density (37) give the upper bound on all first times to catastrophe. Since as we have shown before deviation from the optimal policy is costly,
the firm will then continue its extraction policy until extinction. If the Because of the stochastic
fluctuations, the firm cannot know with certainty whether the first passage time will happen before
the next regime change, but it can have an average measurement of its probability. If \( \mathbb{E}\tau_c \geq \tau_{i+1} \), so
if \( X_{\tau_i} > \mu \tau_{i+1} \), catastrophe is on average avoidable within the first detection period if the firm stops
extraction, therefore the firm can study whether its optimal extraction policy allows to avoid it as
well. In other words, the firm wants to check whether

\[
\mathbb{E}\tau_c \leq \tau_{i+1},
\]

\[
\tau_c = \inf \left\{ t | X_t \leq 0, t \in [0, \tau_{i+1}], X_0 = X^*_\tau_i \right\}.
\]

Define \( \psi(t) = \psi(t; X_{\tau_i}, 0) \) the density function of the first time to catastrophe: then we have that

\[
1 - \psi(t) = 1 - \phi(0, t),
\]

where \( \phi(x, t) \) is the probability that the optimally controlled resource stock \( X^*_t \) hits the absorbing
barrier at 0, and can be written as

\[
\phi(x, t) = \Pr \left\{ \inf_{s \in [t, \tau_{i+1}]} X^*_s \leq 0 \mid X_t = x \right\},
\]

for \( 0 \leq t \leq \tau_{i+1} \). The firm therefore has to solve the Kolmogorov forward equation given by

\[
\frac{\partial}{\partial t} \phi(x, t) + \frac{\partial}{\partial x} \phi(x, t) \left( \mu + \sum_{j=1}^{i-1} \lambda_j - q^*(t, x) \right) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \phi(x, t) = 0
\]

with absorbing boundary conditions given by

\[
\begin{cases}
\phi(x, \tau_{i+1}) = 1 & x \leq 0 \\
\phi(x, \tau_{i+1}) = 0 & x > 0, \\
\phi(0, t) = 1, \\
\phi(t, \infty) = 0.
\end{cases}
\]

The KFE for this problem has no closed form solution, given the dependence of the extraction policy
on both \( x \) and \( t \), and needs to be solved numerically with standard methods. Once the solution is
obtained, the firm can recover the density of the first time to catastrophe \( \tau_c \) from (38) and compute
its numerical first moment: if \( \mathbb{E}\tau_c \geq \tau_{i+1} \) the firm continues its optimal extraction policy.

### D Reaction to adverse regime shifts: the case of green consumers

An illustration allows us to shed further light on the role played by the regime shift magnitude together
with market demand: the case of “green” consumers. The resource has an intrinsic growth of \( \mu = 35 \) and
Figure 10: Reaction to an adverse regime shift can lead to both precautionary and aggressive extraction policies, even if consumer demand declines (“green” consumers). Orange surface: extraction policy for a moderate decline in resource growth $\lambda = -5$, precautionary policy. Green surface: extraction policy for a substantial adverse regime shift $\lambda = -17.5$, both precautionary and aggressive policy. Parameters: $c = 3$, demand parameters of $a = 15, b = 0.5$, $F = 30$ and $\rho = 0.02$. Suppose the ecological system undergoes a regime shift of magnitude $\lambda = -5$. Additionally, consumers preoccupied about the environment reduce their demand by changing $\tilde{a} = 14$. In Figure 10, the green shaded region depicts the evolution of the firm’s optimal extraction policies up until the adjustments at the calibrated $\tau_1 = 40, E[\tau_2] = 50$. Therefore, within the first interval $[0, \tau_1]$, the extraction levels reflect the firm’s assumption that the resource is growing at its natural rate of growth as shown in Proposition 2. Once this regime is detected at $\tau_1$ and along with the reduced demand, the firm updates its assessment and the precautionary growth prevails amplified by the demand effect, resulting in a precautionary extraction strategy. Suppose now the same ecological system undergoes a regime shift of a much larger magnitude $\lambda = -17.5$, halving the original resource growth and yielding a quicker expected detection $E[\tau_2] \sim 40$, *ceteris paribus*. In this case the precautionary growth effect, driven by the change in drift and lower demand, is dominated by the horizon effect strengthened by the larger magnitude of the regime shift, resulting in aggressive extraction strategy. This mechanism is illustrated in Figure 10, and represents the urgency that a firm faces in dealing with upcoming regime shifts, together with the fact that *not all regime shifts are equal*. In other words, a perilous adverse regime shift may still lead to an aggressive extraction strategy even when demand decreases precautionarily.

E Particle filtering and parameter estimation

When analyzing data that results from a dynamic system, and particularly when dealing with sequential observations of a process driven by a stochastic differential equation, both problems of state filtering and parameter estimation become relevant. A way to solve this problem is to treat parameters
as hidden states of the system, as in the method first developed by Liu and West (2001). There have been multiple improvements since their important contribution, but we find that the implementation of the basic setup is sufficient for what is required by our framework. The setup for the joint state and parameter estimation can be formulated in the following way, where $\theta$ is our vector of parameters of interest:

$$
X_{t+1}|X_t \sim p(X_{t+1}|X_t, \theta)
$$

$$
Y_t|X_t \sim p(Y_t|X_t, \theta)
$$

$$
x_0 \sim p(x_0|\theta)
$$

$$
\theta \sim p(\theta), \quad t \in [0, \ldots, T].
$$

In this Bayesian setup the unknown parameters $\theta$ are treated as random quantities, and therefore we have deal with the conditional densities $p(\cdot|\cdot, \theta)$ jointly with assuming a prior distribution $p(\theta)$. The joint posterior for both state and parameters is given by the smoothing problem

$$
p(X_{0:t+1}|Y_{0:t+1}) = \frac{p(Y_{t+1}|X_{0:t+1}, Y_{1:t}, \theta) p(X_{t+1}|X_{0:t}, Y_{1:t}, \theta)}{p(Y_{t+1}|Y_{1:t})} p(X_{0:t}, \theta|Y_{1:t}),
$$

where $Y$ starts at 1 because $Y_0 = X_0$. The posterior distribution of parameters, which is what we are ultimately interested in, can be written using the Chapman-Kolmogorov equation for a Markovian process as

$$
p(\theta|X_{0:t}, Y_{1:t}) \propto p(\theta) p(X_0|\theta) \prod_{k=1}^{t} p(X_k|X_{k-1}, \theta) p(Y_k|X_k, \theta)
$$

which is evaluated via the filtering procedure. The filtering density of the “current” state $X_t$ and the parameter vector $\theta$ is given by

$$
p(X_{t+1}, \theta|Y_{1:t+1}) = \frac{p(Y_{t+1}|X_{t+1}, \theta) p(X_{t+1}|Y_{1:t}, \theta)}{p(Y_{t+1}|Y_{1:t})} p(\theta|Y_{1:t}),
$$

and one can thus approach filtering conditional on parameters, which is a well-known fact. Filtering is thus the task of estimating recursively in time the sequence of marginal posteriors, and needs to be done for both states and parameters in order to estimate parameters. We assume a Gaussian measurement density, i.e. $p(Y_t|X_t, \theta) = N(X_t, \sigma^2_y, \theta)$, where $\sigma^2_y = 3.5 \times 10^5$ is calibrated ex ante via trajectory matching. This has the intuitive interpretation of $Y$ being the real reservoir volume with an additional classical measurement error centered on the observation of reservoir volume $X$, which helps in giving some leeway in fitting the model to the data. However, even in a linear Gaussian measurement system and in the presence of Gaussian fluctuations, the nonlinear and non-Gaussian nature of $X_t$ due to inflow, rainfall and outflow ($q^*$), choices such as the Kalman filter are likely to yield imperfect approximations of the true dynamics and thus we choose a particle approach in order to be completely agnostic on the distributional nature of $p(X_{t+1}|X_t, \theta)$. The approach by Liu and West fixes the common issue of particle decay and filter degeneracy due to fixed parameters by means of approximating the posterior $p(\theta|Y_{1:t})$ with a particle set $(X^*_t, \theta^*_t, w_t^*)$. They propose to estimate the posterior distribution for $\theta$ via a Gaussian kernel density estimation. This implies approximating the
parameters’ transition density with a Gaussian density:

\[ p(X_{t+1}, \theta|Y_{1:t+1}) \propto \sum_{i}^{N} p(Y_{t+1}|X_{t+1}, \theta_{t+1})p(X_{t+1}|X_{t}^{i}, \theta_{t+1})w_{t}^{i}N(\theta_{t+1}|m_{t}^{i}, V_{t}) \]

The advantage introduced is that the conditional variance is the Monte Carlo posterior variance \( V_{t} \) (i.e. independent of \( \theta_{t} \)), and the Gaussian kernel depends on a linear combination of particles and empirical mean of past particles \( m_{t}^{i} = a\theta_{t}^{i} + (1-a)\hat{\theta}_{t} \). The original Liu and West approach has the advantages of being relatively simple whilst avoiding particle decay and overcoming the issue of degeneracy due to fixed parameters in the simulation. In all what follows we use the smoothing \( a = 0.1 \). The algorithm we employ is the following:

1) Obtain an initial set of 1000 particles \( (X_{t}^{i}, \theta_{t}^{i}, w_{t}^{i}) \), \( i = 1, \ldots, 1000 \). Calculate the conditional mean \( \mu_{t+1} = \mathbb{E}[X_{t+1}|X_{t}^{i}, \theta_{t}^{i}] \) and \( m_{t}^{i} = a\theta_{t}^{i} + (1-a)\hat{\theta}_{t} \). We start with the Dirac mass \( p(x_{0}|\theta) = \delta(x_{0}) \).

2) Simulate an “index” variable through importance sampling \( j \propto w_{t}^{j}p(Y_{t+1}|\mu_{t+1}, m_{t}^{j}) \) with \( j = 1 : 1000 \). Sample a new parameter vector \( \theta_{t+1}^{j} \) from the k-th normal component of the kernel density \( \theta_{t+1}^{j} \sim N(\theta_{t+1}|m_{t}^{j}, (1-a)\hat{\theta}_{t}V_{t}) \).

3) Simulate the new states: \( X_{t}^{i+1} = p(X_{t+1}|X_{t}^{i}, \theta_{t+1}^{j}) \) via standard Euler-Maruyama methods for the SDE, using a daily interval as \( \Delta t (1/365) \).

4) Update particle weights: \( w_{t+1}^{j} \propto \frac{p(Y_{t+1}|X_{t+1}^{j}, \theta_{t+1}^{j})}{p(Y_{t+1}|X_{t+1}^{i}, \theta_{t+1}^{i})} \).

5) Repeat steps a large enough number of times to produce a final posterior reconstruction \( X_{t+1}^{j}, \theta_{t+1}^{k} \).

In procedures such as this one, which can be expensive computationally, it is common knowledge that the quality of the starting guess can sometimes make a substantial difference. We thus start by calibrating our model with simulations of the SDE (17) via trajectory matching until a reasonable result (i.e. non-explosive simulated behavior, relative gradient and flex points matching with the data) is achieved. We then run a first pass of the estimation algorithm using flat (uniform) priors centered on this guess in order to reduce the dimensions of the parameter space, with a wide enough interval (between \( 10^{4} \) and \( 10^{6} \) for \( \beta \), between 0.5 and 1 for \( \gamma \) since simulations show that \( \gamma > 1 \) always generates an explosive behavior, between \( 10^{6} \) and \( -10^{6} \) for \( \lambda \) and between \( 10^{4} \) and \( 10^{7} \) for \( \sigma \)). The main results are then obtained by using log-normal priors for \( \beta, \gamma, \sigma \) each with mean and variance the (log) mean and variance of the previous run’s posteriors, and equivalently Gaussian priors for \( \lambda \) as we do not want to restrict it to be either positive or negative. Reconstructed posterior densities are shown in Figure 12. The results of pre- and post-regime shift estimations are reported in the main body of the paper. An immediate set of checks for quality of the estimation is examining effective number of particles used and conditional log-likelihood at every step: Figure 11 shows how particle decay and likelihood only drop where the filter cannot track the data as well as in the other times (around late 2010 - early 2011), whilst still remaining solidly within an acceptable confidence zone. Lastly, we note that other techniques could equivalently be used, in particular approximate likelihood methods such as simulated maximum likelihood, but in particular for the pre-shift model given the long time series for the Cantareira reservoir that we use for our analysis (daily observations, 2007 to 2013) we prefer the particle filter approach as it’s much lighter computationally.
Figure 11: Effective number of particles and conditional log-likelihood at every step from the estimation of pre-shift (above) and post-shift models (below).
Figure 12: Filtered posterior densities for the model parameters.