TORIC REGULATORS
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INTRODUCTION

In the mid-19th century, Dirichlet (for quadratic fields) and then Dedekind defined a regulator map relating the units in the ring of integers of an algebraic number field of finite degree over $\mathbb{Q}$ with $r_1$ real embeddings and $2r_2$ complex embeddings to a lattice of codimension one in a Euclidean space of dimension $r_1 + r_2$. They then showed how a determinant formed from this map and other invariants of the field are related to values of zeta and $L$-functions, known as Dirichlet’s class number formula ([LD68, Supplemente, V, §§ 183 and 184]). Since then, the term “regulator” has been applied to many such maps in number theory and algebraic geometry such as higher algebraic $K$-theory of number fields, Abel-Jacobi maps for algebraic cycles, and more generally, motivic cohomology. In most cases, the source of the regulator map is a group of interest that is deemed to be difficult to compute, and the target somewhat easier to compute. A very general form of this circle of ideas can be found in Beilinson’s conjectures relating motivic cohomology of a smooth projective variety over a number field to real Deligne cohomology [Bei85] and values of $L$-functions, and their refinement by Bloch-Kato [BK90]. There are $p$-adic analogues of these conjectures, where real Deligne cohomology is replaced by a suitable $p$-adic cohomology theory such as syntomic or log-syntomic cohomology, and a conjectural relationship with values of $p$-adic $L$-functions. In the special but important case of a variety with totally degenerate reduction over a $p$-adic field $K$ (please see below for definitions), this paper seeks to tie many of the $p$-adic conjectures and some of the known results together under the guise of what we call toric regulators, which relate motivic cohomology with $p$-adic tori (quotient of a multiplicative torus by a finitely generated free abelian group). These tori may be compact or not.

In [RX07b, RX07a], the second named author and Xarles studied a class of varieties $X$ over a local field $K$ with what they termed totally degenerate reduction. In [RX07b] they studied the étale cohomology of $X$ with $\mathbb{Z}_l$-coefficients and showed that for all $l$ these are, up to finite torsion and cotorsion, extensions of direct sums of Tate twists. In [RX07a] they used this result to define $p$-adic intermediate Jacobians, which are $p$-adically uniformized tori, together with Abel-Jacobi maps from the Chow group of algebraic cycles that are homologically equivalent to zero. The first example of these is the Tate elliptic curve $E_q$, which is given rigid analytically by $\mathbb{G}_m/q^{\mathbb{Z}}$ with $q$ of absolute value less than 1 in $K$. In this case, the intermediate Jacobian is just $K^\times/q^{\mathbb{Z}}$, and the Abel-Jacobi map is essentially the identity. More generally, for a $p$-adically uniformized curve $X$, their work recovers the $p$-adic uniformization of the Jacobian in a purely algebro-geometric way. The Abel-Jacobi map defined in loc. cit. should agree with that provided by Manin-Drinfeld [MD73], although they do not prove that in their paper.
The work [RX07b, RX07a] raises the very natural question of defining toric regulators in higher motivic cohomology, and that is the main purpose of this paper. We will define higher intermediate Jacobians $H^{k+1}_T(X, Z(r))$, given by the quotient of an algebraic torus by periods, and construct, assuming a certain natural conjecture, regulator maps into them, toric regulators.

The toric regulator is a refinement of the regulator of Sreekantan [Sre10b], which is a map

$$r_D : H^{k+1}_{\mathcal{M}}(X, Z(r)) \to H^{k+1}_D(X, \mathbb{Q}(r))$$

where the group on the right is the cohomology of a certain cone defined by Consani [Con98]. It is a finite dimensional $\mathbb{Q}$-vector space. We construct a valuation map (see (2.3))

$$H^{k+1}_T(X, Z(r)) \to H^{k+1}_D(X, Z(r))$$

and conjecture that after tensoring with $\mathbb{Q}$, the Sreekantan regulator is the valuation of the toric regulator.

Another interesting feature of the theory of the toric regulator is the relation with the syntomic regulator. Following the case of $CH^1$ of curves, where the toric regulator is the Abel-Jacobi map and the syntomic regulator is its logarithm, one expects that “the syntomic regulator is the logarithm of the toric regulator.” We formulate this assertion precisely and prove it. From the point of view of the syntomic theory, this adds the interesting assertion that the syntomic regulator can be “exponentiated.” This sometimes allows us to guess formulas for the toric regulator, and we will describe one such guess, but without presenting the syntomic motivation, for brevity.

The Tate elliptic curve is in some sense the original toric regulator. The paper [RX07b, RX07a] may be viewed as a purely algebro-geometric way using $p$-adic Hodge theory to interpret and generalize Tate’s analytic theory and its generalization to curves of higher genus and abelian varieties by Mumford [Mum72b, Mum72a]. Another example is provided by $K_2$ of a curve $X$ with totally degenerate reduction, i.e., a Mumford curve. In this case, it turns out a regulator into an algebraic torus has already been developed, and termed the rigid analytic regulator by Pál [Pál10b] (we prove this except at the prime $p$). We also compare the log of the rigid analytic regulator with the syntomic regulator, as computed in [Bes18].

For the product of two Mumford curves, we explain a conjectural formula for the toric regulator, whose motivation is syntomic.

A question left for future work is the relation between the toric regulator and $L$-functions. Because the syntomic regulator is the logarithm of the toric regulator and is related to special values of $p$-adic $L$-functions, we are looking for such special values that may be “exponentiated.” There have been several instances of such a phenomenon, starting with the refined Birch and Swinnerton-Dyer conjecture of Mazur and Tate [MT87] and its descendents, especially in Darmon’s work on Stark-Heegner points [BD94, BD96, BD98, Dar98, Dar01]. These conjectures concern rational points on elliptic curves in terms of the Tate parameterization at a prime of split multiplicative reduction, and so fit perfectly with the toric regulator in this case. These conjectures inspired in turn the refined $p$-adic Stark conjecture of Gross [Gro88]. There is one example in higher K-theory, which is due to Pál [Pál10a] and Kondo and Yasuda [KY12], providing an $L$-function and a regulator formula in the case of $K_2$ of the the Drinfeld modular curve, in analogy with Beilinson’s work in the classical case [Bei85] and [BD14, Bru10, Nik10] in the $p$-adic case on...
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$K_2$ of a modular curve. Note that to relate this with the toric regulator, one has to either import the result to the number field case or extend the toric regulator to the function field case (the theory of [RX07b] at the prime $p$ currently relies on $p$-adic Hodge theory).

Another possible source of examples is the Sreekantan regulator. This is expected to be connected with $L$-values [Sre10b], and the example of $K_1$ of the product of two Drinfeld modular curves has been worked out by Sreekantan [Sre10a]. Because of the relation between the toric and Sreekantan regulators mentioned above, the sought after $L$-functions should be such that their valuation is the corresponding $L$-function of Sreekantan.

This work began while the first author was on sabbatical at Arizona State University, continued while the second author visited the first at Oxford University, and then during two visits of the first author to Wayne State University. It was completed while the first author was on sabbatical at the Georgia Institute of Technology and then a member of IHÉS. We thank all of these institutions and the Raymond and Beverly Sackler Foundation, whose fellowship supported the stay at IHÉS. The first author is currently supported by a grant from the Israel Science Foundation number 912/18.

1. The toric regulator

Let $K$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $R$, uniformizer $\pi$ and residue field $F$. Let $G$ be the absolute Galois group $G = \text{Gal}(\bar{K}/K)$. Let $X$ be a smooth, projective, geometrically connected variety over $K$ which has totally degenerate reduction in the sense of [RX07b]. Let $k$ and $r$ be non-negative integers such that

\begin{equation}
    k - 2r \leq -1
\end{equation}

In this section we define the toric regulator

$$H_{\text{M}}^{k+1}(X, Z(r))_0 \xrightarrow{\text{reg}_i} H_T^{k+1}(X, Z(r))$$

from the motivic cohomology of $X$ to the toric higher intermediate Jacobian of $X$. The subindex 0 on motivic cohomology refers to homologically trivial classes. This is only relevant when $k + 1 = 2r$, in which case $H_{\text{M}}^{k+1}(X, Z(r)) = CH^r(X)$ are Chow groups. In this particular case the toric regulator is constructed in detail in [RX07a]. We will therefore mostly concentrate on the case of strong inequality in (1.1).

Let $l$ be a prime number. As a consequence of the construction of étale realization functors [Ivo07], one gets an étale regulator map (a rational coefficients version was known for a long time but we will need integral coefficients),

\begin{equation}
    \text{reg}_i : H_{\text{M}}^{k+1}(X, Z(r))_0 \to H^1(K, M_l(r)) \text{ with } M_l = H^k_{\text{et}}(X \otimes_K \bar{K}, Z_l)
\end{equation}

(see for example [Rio06]). Note that we have the restriction (1.1) as otherwise the motivic cohomology groups vanish.

As $X$ has totally degenerate reduction we have the following result of Raskind and Xarles (see [RX07b, Cor. 1 and Theorem 3] as well as [RX07a, Theorem 3] summarizing both the $l \neq p$ and $l = p$ cases).
Theorem 1.1. There exist finitely generated abelian groups $T^i_j$ and, for each $l$, a filtration $W_\bullet$ on the $\mathbb{Z}_l$-module $M_l$ and isogenies

$$\text{Gr}_i^W M_l/\text{tor} \rightarrow \begin{cases} T^{-i}_l/\text{tor} \otimes \mathbb{Z}_l(\frac{k-i}{2}) & k + i \text{ even} \\ 0 & \text{otherwise,} \end{cases}$$

which are isomorphisms for almost all $l$.

Let us recall the construction of the groups $T^i_j$ [RX07b, Section 3]. By assumption, $X$ is the generic fiber of a proper $\mathcal{O}_K$-scheme $\mathcal{X}$ with strictly semi-stable reduction and special fiber $Y$, which decomposes as a union $Y = \bigcup_{i=1}^n Y_i$. Set, for each subset of indices $I \subset \{1, \ldots, n\}$,

$$Y_I = \bigcap_{i \in I} Y_i .$$

Let $\bar{Y}_I := Y_I \otimes \bar{F}$ and let $\bar{Y}^{(m)}$ be the disjoint union of $\bar{Y}_I$ over all subset $I$ of size $m$. We first define groups $C^{i,k}_j = CH^{i+j-k}(\bar{Y}^{(2k-i+1)})$ for each triple $(i, j, k)$ such that $k \geq \max(0, i)$ and 0 otherwise. Then we set

$$C^i_j = \bigoplus_k C^{i,k}_j .$$

To make a complex out of these groups, we define the following maps: For a subset of indices $I$ of size $m+1$ and an integer $0 < r \leq m + 1$, we define $I_r$ to be the subset obtained from $I$ by deleting the $r$’th index. There is an obvious inclusion map $\rho_r : Y_I \rightarrow Y_{I_r}$ and we define maps,

$$\theta_{i,m} = \sum_{r=1}^{m+1} (-1)^{r-1} \rho_r^\ast : CH^i(\bar{Y}^{(m)}) \rightarrow CH^i(\bar{Y}^{(m+1)}) ,$$

$$\delta_{i,m} = \sum_{r=1}^{m+1} (-1)^r \rho_{r+} : CH^i(\bar{Y}^{(m+1)}) \rightarrow CH^{i+1}(\bar{Y}^{(m)}) ,$$

$$d' = \bigoplus_{k \geq \max(0,i)} \theta_{i+j-k,2k-i+1} ,$$

$$d'' = \bigoplus_{k \geq \max(0,i)} \theta_{i+j-k,2k-i} ,$$

and finally

$$d^i_j = d' + d'' : C^i_j \rightarrow C^{i+1}_j .$$

Then we define

$$T^i_j := H^i(C^i_j) .$$

The monodromy map

$$N : T^i_j \rightarrow T^{i+2}_{j-1}$$

is induced by the map $C^i_j \rightarrow C^{i+2}_{j-1}$ which is the identity on the common factors. The composed map

$$N^i : T^{-i}_{j+i} \rightarrow T^i_j$$

is an isogeny for $i \geq 0$ [RX07b, Proposition 1], implying that $N$ is injective for negative $i$ and surjective for positive $i$ after tensoring with $\mathbb{Q}$.
Remark 1.2. The numerical condition (1.1) on $k$ and $r$ imply that $T_r^{k - 2r} \to T_r^{k + 2 - 2r}$ is injective (after tensoring with $\mathbb{Q}$) while $T_r^{k + 1 - 2r} \to T_r^{k + 3 - 2r}$ is injective except when $k - 2r = -1$ in (1.1), i.e., the case of cycles.

There exists a pairing [RX07b, p. 274]

$$(1.5) \quad (\cdot, \cdot) : T_j^d \times T_{d - j}^l \to \mathbb{Z}$$

coming from the intersection pairing and inducing a duality on the torsion free quotients [RX07b, Proposition 1]. We further have the relation

$$(1.6) \quad (Nx, y) = -(x, Ny).$$

Let us now make the following simplifying assumption.

Assumption 1.3. The Galois group of the residue field $F$ acts trivially on all the groups $C_j^{i,k}$.

This can always be achieved after a finite field extension. It is tedious but possible to keep track of the Galois action if this were not the case, and would be important if we considered e.g. towers of field extensions of $K$. Let us further use the following terminology.

Definition 1.4. We say that a map defined for each prime $l$ is an almost injection (respectively, almost surjection) if its kernel (respectively, cokernel) is finite for all $l$ and 0 for almost all $l$. We say it is an almost isomorphism if it is both an almost injection and an almost surjection.

To proceed with the construction of the toric regulator, we will want, as in [RX07a], to isolate from $M_l(r)$ the subquotient which is an extension of $T_r^* \otimes \mathbb{Z}_l$ by $T_r^* \otimes \mathbb{Z}_l(1)$, which, with our indexing, is the the subquotient $W_{2r - k} M_l(r)/W_{2r - k - 4} M_l(r)$. In order to do this, we will use the groups $H_{l}^3$ of Bloch-Kato ([BK90], §3). Recall that when $l \neq p$ $H_{l}^3 = H^3$, while when $l = p$ we have for a $\mathbb{Q}_p$-representation $V$ that $H_{l}^3(K, V) = \ker : H^1(K, V) \to H^1(K, V \otimes B_{dR})$ and for a $\mathbb{Z}_p$-module $g$ cohomology classes are the ones that become $g$ after tensoring with $\mathbb{Q}$. The regulator $\text{reg}_{l}$ from (1.2) takes values in $H_{l}^3(K, M_l(r))$. This is tautological for $l \neq p$ and follows from the work of Nekovář and Nizioł [NN16] when $l = p$ (see Section 4 (4.1) and the following statement and (4.3)).

Proposition 1.5. There exists an integer $n_0$ and well defined maps

$$\text{reg}_l : H_{M}^{k+1}(X, \mathbb{Z}(r))_0 \to H_{l}^3(K, W_{2r - k} M_l(r))$$

such that for any prime $l \neq p$ and any $\alpha \in H_{M}^{k+1}(X, \mathbb{Z}(r))$ we have

$$\text{reg}_l(\alpha) = n_0 x, \quad \text{with} \quad x \in H_{l}^3(K, W_{2r - k} M_l(r)),$$

$$\iota_{2r - k}(x) = n_0 \text{reg}_l(\alpha),$$

with

$$\iota_r : W_r M_l \to M_l$$

the obvious injection.

Proof. The quotient $M_l(r)/W_{2r - k} M_l(r)$ is, up to torsion, an iterated extension of copies of $\mathbb{Z}_l(j)$ for $j < 0$. The $H^0$ of these groups is trivial, and by ([BK90] Example 3.9), we have $H_{l}^3(\mathbb{Z}_l(j)) = H^0(\mathbb{Q}_l/\mathbb{Z}_l(j))$, which are finite and killed by a fixed integer that is independent of $l$. \qed
Projecting on the quotient by $W_{2r-k-4}M_l(r)$ we obtain a map

\[(1.7) \quad H^{k+1}_X(X, Z(r))_0 \to H^1_g(K, W_{2r-k}M_l(r)/W_{2r-k-4}M_l(r)).\]

Now we can proceed in a similar manner to the proof of Proposition 1.5. The Galois module $W_{2r-k}M_l(r)/W_{2r-k-4}M_l(r)$ gives us an extension class in

\[\text{Ext}^1(W_{2r-k}M_l(r)/W_{2r-k-2}M_l(r), W_{2r-k-2}M_l(r)/W_{2r-k-4}M_l(r)).\]

This is almost isomorphic to $\text{Ext}^1(W_{k+2-2r} \otimes Z_l(1), W_{k+2-2r} \otimes Z_l(1))$ and so after multiplying by an integer $n_1$ we get Galois modules $M_l'$ with a short exact sequence

\[(1.8) \quad 0 \to T^k_{r-1} \otimes Z_l(1) \to M_l' \to T^k_{r-1} \otimes Z_l \to 0,
\]

and after multiplying again by an integer $n_2$ we get a regulator map

\[(1.9) \quad \text{reg}^{''}_l : H^{k+1}_X(X, Z(r))_0 \to H^1_g(K, M_l').\]

We now consider boundary maps in the long cohomology sequence coming from (1.8). In degree 0 we use Kummer theory to get the map

\[(1.10) \quad \tilde{N}_l : T^k_{r-2r} \otimes Z_l \to H^1_g(K, T^k_{r-1} \otimes Z_l(1)) \cong T^k_{r-1} \otimes K^{\times(1)},\]

where $K^{\times(1)}$ is the $l$-completion of $K^{\times}$. This is essentially the monodromy pairing considered by Raskind and Xarles [RX07a, p 6064] (although they only define it in some cases). Let’s call this the augmented monodromy (at $l$).

Suppose now that $l \neq p$. The $l$-part of the tame inertia group is isomorphic to $Z_l(1)$ as a Frobenius module and we identify the two for convenience. The following is well known.

**Lemma 1.6.** The following diagram commutes

\[\begin{array}{ccc}
K^{\times} & \xrightarrow{\text{Kummer}} & H^1(K, Z_l(1)) \\
\downarrow \text{val} & & \downarrow \text{val} \\
\mathbb{Z} & = & \mathbb{Z}_l
\end{array}\]

where the vertical map on the right is obtained by restriction to $Z_l(1)$.

From this Lemma and the relation between the monodromy on étale cohomology and on the $T$’s the following is easy for $l \neq p$. For $l = p$ it will be proved in Proposition 4.5.

**Corollary 1.7.** The map

\[T^k_{r-2r} \otimes Z_l \xrightarrow{\tilde{N}_l} T^k_{r-1} \otimes K^{\times(1)} \xrightarrow{\text{val}} T^k_{r-1} \otimes Z_l \]

is just the monodromy map (1.3) with the appropriate indexing tensored with $Z_l$.

By Remark 1.2 the composed map is almost injective.

**Lemma 1.8.** The obvious map from $K^{\times}$ to the pushout of

\[\prod_l K^{\times(l)} \xrightarrow{\text{val}} \prod_l Z_l \]

is just the monodromy map (1.3) with the appropriate indexing tensored with $Z_l$. 
is an isomorphism. For \( l \neq p \) we have the short exact sequence

\[
0 \to (F_\times)_{l\text{-torsion}} \to K^{\times(l)} \overset{\text{val}}{\to} \mathbb{Z}_l \to 0.
\]

Proof. This is well-known and follows from the fact that the group of units in the ring of integers in \( K \) is compact and complete with respect to its subgroups of finite index. This is not the case for a “larger” nonarchimedean valued field such as \( \mathbb{C}_p \). \( \square \)

Corollary 1.9. There is an augmented monodromy map

\[
T_{r-2}^k \overset{\text{aug}}{\to} T_{r-1}^{k+2-2r} \otimes K^{\times}
\]

which gives the augmented monodromy at \( l \) (1.10) after \( l \)-completion for each \( l \).

We can finally define one of the main objects of this paper.

Definition 1.10. The higher toric intermediate Jacobian of \( X \) in degree \( k+1 \) and twist \( r \) is defined by

\[
H_{T}^{k+1}(X, \mathbb{Z}(r)) := \text{coker} \left( T_{r}^{k-2r} \overset{\text{aug}}{\to} T_{r-1}^{k+2-2r} \otimes K^{\times} \right).
\]

Proposition 1.11. The boundary map in the long exact cohomology sequence of (1.8),

\[
H_{g}^{1}(K, T_{r}^{k-2r} \otimes \mathbb{Z}_l) \to H_{g}^{2}(K, T_{r-1}^{k+2-2r} \otimes \mathbb{Z}_l(1)),
\]

is almost injective.

Proof. Suppose first that \( l \neq p \). By local Tate duality and by the duality induced by the pairing 1.5 and the relation with the monodromy operator given in (1.6) we see that this map is almost dual to the map

\[
H^{0}(K, T_{d-r+1}^{2r-k-2} \otimes \mathbb{Z}_l) \to H^{1}(K, T_{d-r}^{2r-k} \otimes \mathbb{Z}_l(1))
\]

obtained from the dual of (1.6), but this is again the augmented monodromy map, this time in the range where after applying the valuation it is almost surjective, hence is almost surjective by Lemma 1.8. For the case \( l = p \) see the syntomic theory of Section 4, in particular, Theorem 4.6. \( \square \)

Corollary 1.12. The group \( H_{g}^{1}(K, M_{l}') \) is almost isomorphic to

\[
\text{coker} \left( T_{r}^{k-2r} \otimes \mathbb{Z}_l \overset{\text{aug}}{\to} T_{r-1}^{k+2-2r} \otimes K^{\times(l)} \right).
\]

Definition 1.13. The toric regulator completed at \( l \) is the map \( \text{reg}_{l}^{d} \) defined as the composition

\[
H_{M}^{k+1}(X, \mathbb{Z}(r)) \overset{\text{reg}''}{\to} H_{g}^{1}(K, M_{l}') \overset{n_{3}}{\to} \text{coker} \left( T_{r}^{k-2r} \otimes \mathbb{Z}_l \overset{\text{aug}}{\to} T_{r-1}^{k+2-2r} \otimes K^{\times(l)} \right)
\]

of the regulator \( \text{reg}'' \) from (1.9) and multiplication by an integer \( n_{3} \) which is done to eliminate the difference between the two groups in Corollary 1.12.

The main object of this work is now given by the following result.
Theorem 1.14. Suppose conjecture 1 is true. Then there exists a unique map, called the toric regulator,

\[ H_{M}^{k+1}(X, \mathbb{Z}(r)) \xrightarrow{\text{reg}} H_{T}^{k+1}(X, \mathbb{Z}(r)), \]

such that for each prime \( l \) the composed map

\[ H_{M}^{k+1}(X, \mathbb{Z}(r)) \xrightarrow{\text{reg}} H_{T}^{k+1}(X, \mathbb{Z}(r)) = \coker \left( T_{r}^{k-2r} \xrightarrow{N} T_{r-1}^{k+2-2r} \otimes K^{\times} \right) \]

\[ \otimes \mathbb{Z}_{l} \xrightarrow{\text{val}} \coker \left( T_{r}^{k-2r} \otimes \mathbb{Z}_{l} \xrightarrow{N_{l}} T_{r-1}^{k+2-2r} \otimes K^{\times(l)} \right) \]

is the toric regulator completed at \( l \) of Definition 1.13.

Proof. By Lemma 1.8 it suffices to show the existence of a map

\[ r_{D} : H_{M}^{k+1}(X, \mathbb{Z}(r)) \to \coker \left( T_{r}^{k-2r} \xrightarrow{N} T_{r-1}^{k+2-2r} \right) \otimes \mathbb{Q} \]

such that for any prime \( l \) the diagram

(1.12)

\[ \begin{array}{ccc}
H_{M}^{k+1}(X, \mathbb{Z}(r)) & \xrightarrow{\text{reg}_{l}} & \coker \left( T_{r}^{k-2r} \otimes \mathbb{Z}_{l} \xrightarrow{N_{l}} T_{r-1}^{k+2-2r} \otimes K^{\times(l)} \right) \\
\downarrow \text{r}_{D} & & \downarrow \text{val} \\
\coker \left( T_{r}^{k-2r} \xrightarrow{N} T_{r-1}^{k+2-2r} \right) \otimes \mathbb{Q} & \xrightarrow{\text{val}} & \coker \left( T_{r}^{k-2r} \otimes \mathbb{Q}_{l} \xrightarrow{N_{l} \otimes \mathbb{Q}_{l}} T_{r-1}^{k+2-2r} \otimes \mathbb{Q}_{l} \right) 
\end{array} \]

commutes. In Section 2 we will show how the work of Sreekantan [Sre10b] gives precisely such a map and Conjecture 1 will say that the diagram (1.12) commutes. This proved the theorem. \( \square \)

Remark 1.15. We hope to prove Conjecture 1 in future work. It is somewhat problematic that the toric regulator is only defined after multiplication by an integer, thereby erasing finer roots of unity information. Fortunately, in various situations no such multiplication is required. We could conjecture that in fact the resulting regulator has a canonical root, but we have no evidence to support this.

2. The relation with the regulator of Sreekantan

In [Sre10b] Sreekantan constructs a new type of regulator and conjectures relations with special values of \( L \)-functions in the function field case. We will conjecture that, in a very precise sense, the Sreekantan regulator is exactly the toric regulator followed by the valuation map. As explained in Section 1, this is an important step in actually showing the existence of the toric regulator.

Let us recall the setup for Sreekantan’s work. Let \( X \) be smooth and proper over \( K \) with semi-stable reduction. In his setup we are not assuming that the reduction is completely degenerate. Sreekantan starts with a variety over a global field but this is for the sake of getting results about \( L \)-functions and he is really only interested in the completion at a finite prime for computing the regulator.

Sreekantan defines certain cohomology groups, for which he first recalls work of Consani. Consani defines [Con98, (3.13)] groups \( K^{i,j,k} \). In fact, it is easy to check...
(see also Observation 1 on p. 273 of [RX07b], noting that Consani’s convention is the same as that of [BGS97] and [GNA90]) that we have

\[ K_{i,j,k} = C_{i,k}^{j,k} \otimes \mathbb{Q}. \]

There are differentials and a monodromy operator which are the same as [RX07b] and Consani defines (immediately following (3.1))

\[ K_{i,j} = \bigoplus_k K_{i,j,k} = C_{i,j}^{i} \otimes \mathbb{Q}. \]

We make the following definition after Consani.

**Definition 2.1** (Consani [Con98]). The Consani complex with twist \( r \) is the complex

\[ C(r) = \text{Cone}(N : K^{s-2r,s-d} \to K^{s-2r+2,s-d}) = \text{Cone}(N : C_r^{s-2r} \to C_r^{s-2r+2}) \otimes \mathbb{Q}. \]

For the normalization of the cone here see [Con98, p. 331].

**Definition 2.2.** The Deligne cohomology group \( H^{k+1}_D(X, \mathbb{Q}(r)) \) is the \( k+1-2r \) cohomology of the Consani complex \( C(r) \).

Comparing with Sreekantan the reader will observe that we have removed the \( v \)-notation, which was to indicate working with the completion of the global \( X \) at the finite prime \( v \).

**Proposition 2.3.** Suppose now that \( X \) has totally degenerate reduction. Then there is a long exact sequence

\[ \cdots \to T_{r-1}^{k-2} \otimes \mathbb{Q} \to T_r^{k+2-2} \otimes \mathbb{Q} \to H^{k+1}_D(X, \mathbb{Q}(r)) \to T_r^{k+1-2} \otimes \mathbb{Q} \to T_r^{k+3-2} \otimes \mathbb{Q} \to \cdots . \]

**Proof.** This follows immediately from the definition of the Consani complex as a cone. \( \square \)

**Proposition 2.4.** Suppose that the \( \bar{Y}_i \) are toric or cellular varieties (which is the case for all known examples of varieties with totally degenerate reduction). Then we have an isomorphism

\[ H^{k+1}_D(X, \mathbb{Q}(r)) \cong CH^{r-1}(Y, 2r - k - 2) \otimes \mathbb{Q} \]

with Bloch’s higher Chow groups.

**Proof.** This is a combination of results in [Con98]: Lemma 3.1 (see also p. 341 where the lemma is used but without precise reference in our context) and equation (2.3), which relies in turn on Conjecture 2.1, which is known for toric or cellular varieties. Unfortunately, it is not known at present that this follows directly from the definition of totally degenerate, although it is expected to be true for any smooth projective variety over a finite field. \( \square \)

**Definition 2.5.** The Sreekantan regulator is a map

\[ r_D : H^{k+1}_M(X, \mathbb{Z}(r)) \to H^{k+1}_D(X, \mathbb{Q}(r)) \]

which is the composition of the boundary map (effectively in motivic homology)

\[ H^{k+1}_M(X, \mathbb{Z}(r)) \cong CH^r(X, 2r - k - 1) \xrightarrow{\partial} CH^{r-1}(Y, 2r - k - 2) \otimes \mathbb{Q}, \]

with the isomorphism (2.1).
By Remark 1.2, unless \( k + 1 = 2r \), the map \( T^{k+1-2r}_r \to T^{k+3-2r}_{r-1} \) is injective after tensoring with \( \mathbb{Q} \), giving an isomorphism

\[
H^{k+1}_D(X, \mathbb{Q}(r)) \cong \text{coker} (T^{k-2r}_r \to T^{k+2-2r}_{r-1}) \otimes \mathbb{Q}.
\]

Thus, the following result is quite natural.

**Conjecture 1.** Assume \( k + 1 < 2r \). Then, with \( r_D \) as in Definition 2.5, diagram (1.12) commutes.

As noted in Section 1, the existence of the toric regulator follows from this conjecture. In addition, we get the following obvious relation between the toric regulator and the Sreekantan regulator: Define the valuation map

\[
\text{val} : H^{k+1}_T(X, \mathbb{Z}(r)) \to H^{k+1}_D(X, \mathbb{Q}(r))
\]

as the composition of the valuation map and the isomorphism (2.2).

**Corollary 2.6.** Assuming Conjecture 1 the following diagram commutes.

\[
\begin{array}{ccc}
H^{k+1}_M(X, \mathbb{Z}(r)) & \xrightarrow{\text{reg}} & H^{k+1}_T(X, \mathbb{Z}(r)) \\
\downarrow & & \downarrow \text{val} \\
H^{k+1}_D(X, \mathbb{Q}(r)) & &
\end{array}
\]

**Remark 2.7.** Note that Sreekantan does not deal with the case of cycles at all. If we try to argue by analogy, we should expect that in the case \( k + 1 = 2r \) the composed map \( CH^r(X) \to H^2_D(X, \mathbb{Q}(r)) \to T^0_r \) factors via the cycle class map and so the analogous Sreekantan regulator maps on homologically trivial cycles \( CH^r(X)_0 \) again to \( \text{coker} (T^{-1}_{r-1} \to T^1_{r-1}) \), but as this last map is (essentially) bijective, the corresponding regulator is trivial (unlike the toric regulator).

3. \( K_2 \) of curves and the rigid analytic regulator of Pál

In this section we assume that \( X \) is of dimension 1. We recall from [Bes17, Section 4] the setup that will be used here and also in some parts of Section 4. By assumption \( X \) is the generic fiber of a proper \( \mathcal{O}_X \) scheme \( X' \) with (Zariski) semi-stable reduction

\[
Y = \bigcup_i Y_i.
\]

In particular, locally near an intersection point \( Y_i \cap Y_j \) there are coordinates \( x, y \) satisfying

\[
xy = \pi, \ Y_i = (x), \ Y_j = (y)
\]

(here, \((f)\) denotes the divisor of the rational function \( f \)).

For simplicity we will assume that components \( Y_i \) and \( Y_j \) intersect at at most one point.

Let \( \Gamma(X) \) be the dual graph of \( Y \) with vertices \( V \) and edges \( E \) (this is of course an abuse of notation as it really depends on the particular model). The vertices correspond to the components \( Y_v \) while the edges are ordered pairs of intersecting components \( (Y_v, Y_w) \) oriented from \( v \) to \( w \), so that an edge \( e \) has tail \( e^+ = v \) and head \( e^- = w \). For such an edge we denote by \( -e \) the same edge with reverse orientation.
The reduction map $X \to Y$ allows us to split $X$ into rigid analytic domains $U_v = \text{red}^{-1} Y_v$ which are wide open spaces in the sense of Coleman. These then intersect along annuli corresponding bijectively to the unoriented edges of $\Gamma(X)$. Indeed, in terms of the coordinates $x, y$ appearing in (3.2) the annulus corresponding to the edge $(Y_i, Y_j)$ gets mapped via $x$ (or $y$) to the rigid analytic space $A(|\pi|, 1)$ with
\[(3.3) \qquad A(r, s) := \{ z \in \bar{K}, \; r < |z| < s \} .\]
An orientation of an annulus fixes a sign for the residue along this annulus and we match oriented edges with oriented annuli as in [Bes17, Definition 4.6]. We use the same notation for the edge and for the associated oriented annulus.

At this point we recall some facts about graph cohomology and harmonic cochains on graphs. For a slightly expanded version the reader may consult [Bes17, Section 4]. For a graph $\Gamma = (V, E)$ and an abelian group $A$ we define 0 and 1 cochains on $\Gamma$ with values in $A$ by

\[C^0(\Gamma, A) = \{ f : V \to A \} , \quad C^1(\Gamma, A) = \{ f : E \to A , \; f(-e) = -f(e) \} .\]

We have a differential
\[d : C^0(\Gamma, A) \to C^1(\Gamma, A) , \quad df(e) = f(e^+) - f(e^-)\]
and the graph cohomology $H^1(\Gamma, A)$ is the cokernel of $d$. We have a pointwise product of $c, d \in C^1(\Gamma, A)$, assuming that $A$ is a ring, defined by
\[c \cdot d = \sum_{e \in E(G)/\pm} c(e) \cdot d(e) ,\]
where the sum is over unoriented edges. The kernel of the dual differential
\[d^* : C^1(G, A) \to C^0(G, A) , \quad d^* f(v) = \sum_{e^+ = v} f(e) ,\]
is the space $\mathcal{H}(\Gamma, A)$ of harmonic cochains with values in $A$. The injection $\mathcal{H}(\Gamma, A) \hookrightarrow C^1(\Gamma, A)$ induces a map
\[(3.4) \quad \mathcal{H}(\Gamma, A) \to H^1(\Gamma, A) .\]
which is an isomorphism if $A$ is a $\mathbb{Q}$-vector space. With $A$ a ring again, the space $\mathcal{H}(\Gamma, A)$ is orthogonal to $\text{Im} d$, hence induces a pairing
\[(3.5) \quad \mathcal{H}(\Gamma, A) \times H^1(\Gamma, A) \to A .\]
There is an unoriented version of the above, which will be needed for what follows. This requires fixing an orientation of each edge. Then we can redefine 1-cochains with values in $A$ as functions from unoriented edges to $A$. The differential and the dual differential are defined as above but using the fixed orientation on each edge and the pointwise product is unchanged. It is trivial that this construction produced isomorphic graph cohomologies.

**Proposition 3.1.** Let $X$ be as above, with dual graph $\Gamma$. We have isomorphisms
\[T^{-1}_1 \cong \mathcal{H}(\Gamma, \mathbb{Z}) ,\]
\[T^1_0 \cong H^1(\Gamma, \mathbb{Z}) .\]
With respect to these isomorphisms the monodromy map $N : T_{-1} \to T_0$ corresponds to the map (3.4) and the product $T_{-1} \times T_0 \to \mathbb{Z}$ to the pairing (3.5) induced by the pointwise product.

**Proof.** We have

$$C_1^{-1} = \bigoplus_{k \geq 0} C_{1,k}^{-1} = \bigoplus_{k \geq 0} \mathcal{C}H^{-k}(\bar{Y}^{(2k+2)}) = \mathcal{C}H^0(\bar{Y}^{(2)})$$

$$C_1^0 = \bigoplus_{k \geq 0} C_{1,k}^0 = \bigoplus_{k \geq 0} \mathcal{C}H^{1-k}(\bar{Y}^{(2k+1)}) = \mathcal{C}H^1(\bar{Y}^{(1)})$$

$$C_1^{-2} = \bigoplus_{k \geq 0} C_{1,k}^{-2} = \bigoplus_{k \geq 0} \mathcal{C}H^{1-k}(\bar{Y}^{(2k+3)}) = 0$$

$$C_0^0 = \bigoplus_{k \geq 1} C_{0,k}^0 = \bigoplus_{k \geq 1} \mathcal{C}H^{1-k}(\bar{Y}^{(2k)}) = \mathcal{C}H^0(\bar{Y}^{(2)})$$

$$C_0^2 = \bigoplus_{k \geq 2} C_{0,k}^2 = \bigoplus_{k \geq 2} \mathcal{C}H^{2-k}(\bar{Y}^{(2k-1)}) = 0$$

$$C_0^0 = \bigoplus_{k \geq 0} C_{0,k}^0 = \bigoplus_{k \geq 0} \mathcal{C}H^{-k}(\bar{Y}^{(2k+1)}) = \mathcal{C}H^0(\bar{Y}^{(1)})$$

(3.6)

As all the components of $\bar{Y}^{(1)}$ are projective lines, their $\mathcal{C}H^1$'s, as well as the $\mathcal{C}H^0$ of the components of $\bar{Y}^{(2)}$ are isomorphic to $\mathbb{Z}$ via the degree map, and the differential $C_1^{-1} \to C_0^0$ is the alternating sum of pushforward maps, which are clearly just the identity map on the appropriate $\mathbb{Z}$ summands. The chosen numbering of the components gives an orientation on all the edges. The unoriented edge $(i, j)$ gets the orientation $(i, j)$ with $i < j$. With this the isomorphisms (3.6) are clear in the unoriented version. Similarly, The $\mathcal{C}H^0$ of each component of $\bar{Y}^{(1)}$ is isomorphic to $\mathbb{Z}$ and the map $C_0^0 \to C_0^0$ is an alternating sum of pullbacks, which are again the identities on the corresponding $\mathbb{Z}$ components, giving the identification of the monodromy operator, again in the unoriented version. The identification of the pairing is clear.

**Corollary 3.2.** We have $H^2_\mathcal{T}(X, \mathbb{Z}(2)) \cong \mathcal{H}(\Gamma, \mathbb{K}^\times)$ and the toric regulator in this case is therefore a map

$$\text{reg}_t : H^2_\mathcal{M}(X, \mathbb{Z}(2)) \to \mathcal{H}(\Gamma, \mathbb{K}^\times).$$

We now recall the (somewhat reformulated) definition of the Pál rigid analytic regulator [Pál10b]. We start by working over $\mathbb{C}_p$ and with closed annuli $A[r, s]$ instead of open ones (3.3).

**Definition 3.3.** View the annulus $e = A[r, s]$ as embedded in $\mathbb{P}^1$ in the obvious way, and let $D = D(r)$ be the disc $\{|z| < r\}$ inside $\mathbb{P}^1$. For rational functions $f, g$ on $\mathbb{P}^1$ which have no poles or zeros on $e$, set

$$t_e(f, g) = \prod_{x \in D} t_x(f, g) \in \mathbb{C}_p^\times,$$

where $t_x$ is the tame symbol at the point $x$. Let $f, g$ be invertible rigid analytic functions on $A[r, s]$. Let $f_n$ and $g_n$ be sequences of rational functions on $\mathbb{P}^1$ that converge to $f$ and $g$ respectively on $A[r, s]$. Then set

$$t_e(f, g) = \lim_{n \to \infty} t_e(f_n, g_n) \in \mathbb{C}_p.$$
For an open annulus $e = A(r, s)$ define $t_e$ to be $t_e^r$ for any smaller closed annulus. Finally, for an oriented (open) annulus $e$ and rigid analytic functions $f, g$ on $e$, define $t_e(f, g)$ by choosing an orientation preserving identification of $e$ with $A(r, s)$.

**Theorem 3.4.** The quantity $t_e(f, g)$ is well defined and in $\mathbb{C}^\times$. Furthermore we have the following:

1. $t_e(f, g) = t_e(g, f)^{-1}$.
2. $t_e(f, 1 - f) = 1$.
3. Let $U$ be the complement in $\mathbb{P}^1(\mathbb{C})$ of the union of a finite number of disjoint closed balls $D[r_i]$ and let $e_i$ be annuli $A(r_i, r_i + \varepsilon)$ where $\varepsilon$ is chosen sufficiently small so that the $e_i$’s are disjoint. Let $f, g$ be invertible meromorphic functions on $U$ which are invertible on the $e_i$. Then the following residue theorem holds:

$$\prod_{x \in U} t_x(f, g) \cdot \prod_i t_{e_i}(f, g) = 1.$$ 

**Proof.** For the closed disc $A[r, s]$ what we defined here is what Pál defines, in the course of proving [Pál10b, Theorem 2.2], as $t_D$ for the boundary component $D(r)$ of the connected rational subdomain $A[r, s]$. In particular, it is well defined. Lemma 3.4 in [Pál10b] shows that the choice of a closed annulus inside an open annulus does not matter and Pál also shows [Pál10b, Theorem 3.11] that the construction commutes with morphisms of domains, which shows that it is independent of the choice of parameterization. Thus, the index is well defined. Switching the orientation corresponds to using $D = \{ s < |z| \}$ and [Pál10b, Theorem 3.2 (iii)] immediately gives (1) (Pál works here already with elements of $K_2$ but the result certainly specializes to what we have here). The formula in Theorem 2.2 (iv) there immediately implies (2) and the residue Theorem is easily deduced from (iii) of Theorem 3.2. \qed

**Lemma 3.5.** Let $\deg_e : \mathcal{O}(e)^\times \to \mathbb{Z}$ be defined by $\deg_e f = \text{res}_e d \log f$. Then, for a constant $c \in \mathbb{C}^\times$, $t_e(c, f) = e^{\deg_e(f)}$.

**Proof.** This holds for rational functions $f$ with a degree function equaling the number of zeros in the disc $D(r)$ minus the number of poles, which is clearly the same as our degree, by [Pál10b]. It then follows in general by continuity. \qed

To define the Pál regulator we first consider functions $f, g \in \mathcal{C}(X)^\times$ having no poles or zeros on any $e \in E$. We can then define $\text{reg}_p(\{f, g\}) \in \mathcal{C}(\Gamma, \mathbb{C}^\times)$ by

$$\text{reg}_p(\{f, g\})(e) = t_e(f, g).$$

Next, if $\alpha = \sum \{ f_i, g_i \}$ is a formal combination of symbols and all the functions $f_i, g_i$ are invertible on all $e \in E$, and all its tame symbols are 1, then

$$\text{reg}_p(\alpha) = \prod \text{reg}_p(\{f_i, g_i\})$$

is defined and the residue theorem implies that it is in $\mathcal{H}(\Gamma, \mathbb{C}^\times)$. Suppose now that all the $f_i, g_i$ are in $K(X)^\times$ and all tame symbols are 1, but without assuming they are invertible on the $e$’s. By making a finite field extension we can make sure that all points where any of these functions have values in $\{0, 1, \infty\}$ are defined over $K$ and, maintaining a semi-stable model by possibly blowing up, we get for the new graph $\Gamma'$ that none of these points is in any of $e \in E'$ and we obtain $\text{reg}_p(\alpha) \in \mathcal{H}(\Gamma', \mathbb{C}^\times)$. Now, (2) of Theorem 3.4 shows that $\text{reg}_p$ factors via $K_2(K(X))$. Note that $\Gamma'$
is obtained from $\Gamma$ by subdividing edges and it is easily seen that $\mathcal{H}(\Gamma, \mathbb{C}^\times_p) \cong \mathcal{H}(\Gamma, \mathbb{C}^\times_p)$. Finally, [Pa10b, Proposition 4.2] shows that, for functions in $K(X)$, $\text{reg}_p$ in fact takes values in some finite extension and compatibility with Galois actions shows that in fact takes values in $K^\times$. To summarize, we get a map

$$\text{reg}_p : K_2(X) \to \mathcal{H}(\Gamma, K^\times).$$

**Theorem 3.6.** The map $\text{reg}_p$ equals the map $\text{reg}_l$ from (3.7) except possibly at the prime $p$. In other words, $\text{reg}_p$ gives $\text{reg}_l$ upon completion at $l$ for all primes $l \neq p$ (conjecturally also for $l = p$).

**Proof.** The proof is inspired by the work of Asakura [Asa06, page 279]. The toric regulator completed at a prime $l$ is the map

$$H^2_{\text{ét}}(X, \mathbb{Z}(2)) \to H^1(K, H^1_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Z}(2))) \to H^1(K, T_{1,-1} \otimes \mathbb{Z}_l(1)) \to T_{1,-1} \otimes K^\times(l)$$

and so we first need to understand the map $H^1_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Z}_l(1)) \to T_{1,-1} \otimes \mathbb{Z}_l$ from which the second left most map above is derived by twisting once and taking Galois cohomology.

**Proposition 3.7.** Suppose $l \neq p$. Let $J$ be the Jacobian of $X$. The map

$$H^1_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Z}_l(1)) \cong T_l(J) \to T_{1,-1} \otimes \mathbb{Z}_l$$

is the limit of the maps

$$J[l^n] \to C^1(\Gamma, \mathbb{Z}/l^n)$$

defined as follows: let $[D] \in J[l^n]$ by the class of a divisor $D$. Then $l^nD$ is the divisor of a function $f$ and the map sends $[D]$ to $(e \mapsto \text{res}_e(d \log(f)) \pmod{l^n})$.

**Proof.** We begin by observing that in the map described here, we could just as well replace $[D]$ by a torsion class in $\text{Pic}(X/\bar{K})$, represented by some Čech cocycle, which is then the boundary of $(g_K)$ and map to the residue of $d \log g_k$ on an annulus, as this is independent of $k$ modulo $n$.

The map we need is

$$H^1_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Z}/l^n(1)) = H^1_{\text{ét}}(Y, \mathbb{R}^1\Psi J/l^n(1)) \to H^1_{\text{ét}}(Y, \mathbb{R}^1\Psi Z/l^n(1)[1]),$$

with $\mathbb{R}^1\Psi$ the functor of “nearby cycles”, followed by the identification

$$\mathbb{R}^1\Psi Z/l^n(1) \cong \mathbb{A}_2^\times Z/l^n,$$

where, for any $k$, $a_k : \bar{Y}^{(k)} \to Y$ is the obvious injection, and so we attempt to understand this last identification following [Sai03, Proposition 1.2]. We have morphisms $X \xrightarrow{1} X \xleftarrow{\bar{Y}^{(k)}} Y$. Then, according to Saito, we have isomorphisms,

$$\theta' : a_k^*Z/l^n \xrightarrow{\sim} \mathbb{R}^k j_! Z/l^n (k)$$

(3.8)

(3.9)

$$0 \to \mathbb{R}^k \Psi Z/l^n \xrightarrow{\bar{\theta}} \mathbb{R}^{k+1} j_! Z/l^n(1) \to \mathbb{R}^{k+1} \Psi Z/l^n(1) \to 0,$$

whose definition we will recall in a moment. To make sense out of these formulas, note that we identify sheaves on $Y$ with sheaves on $\bar{Y}$ with an action of the Galois group. Since there is no $\bar{Y}^{(3)}$, the isomorphism above gives $\mathbb{R}^2 j_! Z/l^n = 0$ and the short exact sequence for $k = 2$ gives $\mathbb{R}^2 \Psi Z/l^n = 0$. Then for $k = 1$ the short exact sequence yields an isomorphism

$$\mathbb{R}^1 \Psi Z/l^n(1) \xrightarrow{\theta} \mathbb{R}^2 j_! Z/l^n(2).$$
from which we get the required identification by composing with the inverse of \( \theta' \). Next we recall how the thetas are defined, focusing on a neighborhood of a point in \( Y^{(2)} \) locally given by an equation \( xy = \pi \). The special fiber \( Y \) is defined there by \( \pi \) and the two components of \( Y \) passing through the point are given by the additional equation \( x = 0 \) and \( y = 0 \). Let \( i_1 \) be the embedding of \( Y_1 \) into \( X \), \( j_1 : X - Y_1 \rightarrow X \) the obvious open immersion, and \( a' : Y_1 \rightarrow Y \). Saito defines a map

\[
\theta_i : a'_i \mathbb{Z}/l^n \rightarrow i'_i \mathbb{R}^1 j_* \mathbb{Z}/l^n(1)
\]

to be the map sending 1 to the Kummer image of the local generator of \( \mathcal{O}(Y_1) \). On our local neighborhood, these are clearly the Kummer images of \( x \) and \( y \) respectively, which we will denote by \((x)\) and \((y)\). Then, following Saito, the map

\[
\theta' = \sum \theta_i : a'_i \mathbb{Z}/l^n \rightarrow i'_i \mathbb{R}^1 j_* \mathbb{Z}/l^n(1)
\]
is an isomorphism and the isomorphisms (3.8) are deduced from it by taking cup products. Finally, the map

\[
i^* \mathbb{R}^k j_* \mathbb{Z}/l^n \rightarrow \mathbb{R}^k \Psi Z/l^n
\]
is surjective and the map \( \bar{\theta} \) is induced by the map \( i^* \mathbb{R}^k j_* \mathbb{Z}/l^n \rightarrow i^* \mathbb{R}^{k+1} j_* \mathbb{Z}/l^n(1) \) obtained by cup product with the Kummer class of \( \pi \), which, on our neighborhood, is \((x)+(y)\). We finally obtain the map we want, locally on our chosen neighborhood as the composition

\[
(3.10)
\]

\[
\mathbb{R}^1 \Psi Z/l^n(1) \xrightarrow{\text{HIT}} i^* \mathbb{R}^1 j_* \mathbb{Z}/l^n(1) \xrightarrow{((x)+(y))_1} i^* \mathbb{R}^2 j_* \mathbb{Z}/l^n(2) \xrightarrow{(x)_1(y)_1} a_2 Z/l^n
\]

where the last map is an isomorphism that needs to be inverted. From this it is clear that the Kummer images of \( x \) and \( y \) go to \( \pm 1 \) respectively. Rigidifying and taking the residue of the corresponding \( d \log \)'s on the resulting annulus would obviously give the same. So, finally, our required map would map a torsion class in the picard group to the cocycle \((g_k)\) and then apply (3.10) to the Kummer image of \( g_k \). This is now clearly the same as the map claimed in the Proposition. □

Switching to \( K_2 \) for convenience, we can now concentrate on a single annulus \( e \) and compute the map

\[
(3.11)
K_2(X)/l^n \rightarrow H^1(K, \text{Pic}(X)[l^n] \otimes \mu_{l^n}) \xrightarrow{\text{res}_e} H^1(K, \mu_{l^n}) \rightarrow K^\times/(K^\times)^l^n
\]

where \( \mu \) denotes roots of unity and \( \text{res}_e \) is the “e component” of the map in Proposition 3.7. Let \( A = \mathcal{O}(e) \). Furthermore, identify \( e \) with some \( A(r, s) \subset \mathbb{P}^1 \). Let \( B \) be the ring of rational functions on \( \mathbb{P}^1 \) regular on \( e \) and let \( D = D(e) \). We have a map \( \text{Spec}(A) \rightarrow X \) and using the pullback via this map we can write the map (3.11) as the composition of \( K_2(X)/l^n \rightarrow K_2(A)/l^n \) with

\[
(3.12)
K_2(A)/l^n \rightarrow H^1(K, \text{Pic}(A)[l^n] \otimes \mu_{l^n}) \xrightarrow{\text{res}_e} H^1(K, \mu_{l^n}) \rightarrow K^\times/(K^\times)^l^n.
\]

For \( f \in B \) we have

\[
\text{res}_e d\log(f) = \sum_{x \in D} \text{res}_x d\log(f)
\]

and therefore the restriction of (3.12) to \( K_2(B)/l^n \) is the sum over \( x \in D \) of the maps

\[
K_2(B)/l^n \rightarrow H^1(K, \text{Pic}(B)[l^n] \otimes \mu_{l^n}) \xrightarrow{\text{res}_x} H^1(K, \mu_{l^n}) \rightarrow K^\times/(K^\times)^l^n.
\]
According to [Asa06, Lemma 4.3 (2)] for each \( x \in D \) this last map is just the tame symbol at \( x \) modulo \((K^\times)^n\). It is easy to see that if \( f, g \in A \) are sufficiently well approximated by \( f_i, g_i \in B \), then their images in \( K_2(A)/l^n \) become identical. The theorem follows easily. \( \square \)

4. The relation with the syntomic regulator

In this section we will analyze the construction of the toric regulator at the prime \( l = p \), after tensoring with \( \mathbb{Q} \). We will see that the logarithm of the toric regulator may be computed using the syntomic regulator of Nekovář and Nizioł. We will then check our description of the toric regulator for \( K_2 \) of curves via the Pál regulator using the computation of the syntomic regulator in this case by the first named author [Bes18].

Let \( X \) be a smooth variety over a \( p \)-adic field \( K \). Then, Nekovář and Nizioł [NN16, Theorem 5.9] show that the regulator map,

\[
\text{reg}_p : H^{k+1}_\mathcal{M}(X, \mathbb{Z}(r))_0 \to H^1(K, H^k_\mathcal{et}(X \otimes_K \bar{K}, \mathbb{Q}_p(r))) ,
\]

factors via the subgroup \( H^1_{\text{st}}(K, H^k_\mathcal{et}(X \otimes_K \bar{K}, \mathbb{Q}_p(r))) \), where, for a de Rham representation \( V \) of \( G \), \( H^1_{\text{st}}(V) \) is semi-stable cohomology, which may be interpreted as the group of Yoneda extensions in the category of potentially semi-stable representation and may be computed in terms of the complex \( C^\mathbf{st}_n(V) \) of [Nek93, 1.19],

\[
C^\mathbf{st}_n(V) : D_{\text{st}}(V) \xrightarrow{(\varphi - 1, N, -i)} D_{\text{st}}(V) \oplus D_{\text{et}}(V) \oplus \text{DR}(V)/F^{N+1-p\varphi+0} D_{\text{et}}(V) .
\]

Here, \( D_{\text{st}} \) and \( \text{DR} \) are the functors defined by Fontaine: \( D_{\text{st}}(V) \) is a \( K_\varnothing \)-vector space, where \( K_\varnothing \) is the maximal unramified extension of \( \mathbb{Q}_p \) inside \( K \), equipped with a linear nilpotent operator \( N \) (called monodromy) and a semi-linear (with respect to the unique lift of Frobenius on \( K_\varnothing \) operator \( \varphi \) (Frobenius), satisfying the relation

\[
N\varphi = p\varphi N ,
\]

and \( \text{DR}(V) \) is a filtered \( K \)-vector space, and there is an isomorphism \( D_{\text{st}}(V) \otimes K \to \text{DR}(V) \). We remark that the map between \( H_{\text{st}} \), computed in terms of the complex \( C^\mathbf{st}_n \), and Galois cohomology, is a manifestation of the Bloch-Kato exponential map [BK90, Definition 3.10].

Assume now that \( X \) has totally degenerate reduction. Let \( V = H^k_{\mathcal{et}}(X \otimes_K \bar{K}, \mathbb{Q}_p(r)) \). We first note that as the reduction of \( X \) is semi-stable, the representation \( V \) is semi-stable. It follows [Nek93, 1.24 (3)] that

\[
H^1_{\text{st}}(K, V) = H^1_{\mathcal{et}}(V) .
\]

Let us compute \( H^1_{\mathcal{et}}(K, V) \). Let \( D = D_{\text{et}}(V) \). Then, as a \( K_\varnothing \)-vector space with a Frobenius and a monodromy operator it equals, by the semi-stable conjecture of Fontaine, proved by Tsuji [Tsu99] to Hyodo-Kato cohomology \( H^k(Y^\times/W^\times) \). It follows from [RX07b] that we have a slope decomposition

\[
D = \bigoplus_{i+j=k} T_{j}^{i-j} \otimes K_\varnothing(r - j)
\]

and the monodromy operator is compatible with the one defined on the \( T \)'s. For simplicity let us renumber this as follows: We have a vector space decomposition

\[
D = \bigoplus_{i} D^i , \ D^i \cong T^i \otimes K_\varnothing ,
\]

where
where, with respect to the rational structure provided by $T^i$, the Frobenius $\phi$ acts by $p'\sigma$, so that $(D^i)^{\phi=p'} = T^i \otimes \mathbb{Q}_p$. The monodromy maps are induced by the ones defined before $N : T^i \to T^{i-1}$. Note that by remark 1.2 the map $N : T^0 \to T^{-1}$ is injective after tensoring with $\mathbb{Q}$. The filtration is compatible with the filtration on the individual terms, where $\text{DR}(\mathbb{Q}_p(i)) = F^{-i} \supset F^{-i-1} = 0$. We can now compute the semi-stable cohomology of $V$.

**Proposition 4.1.** We have a short exact sequence

\[
0 \to \text{DR}(V)/F^0 \to H^1_{\text{st}}(K, V) \to (T^{-1}/NT^0) \otimes \mathbb{Q}_p \to 0
\]

and a map $H^1_{\text{st}}(V) \to \text{DR}(V)/(F^0 + T^0 \otimes \mathbb{Q}_p)$ such that the composition

\[
\text{DR}(V)/F^0 \to H^1_{\text{st}}(K, V) \to \text{DR}(V)/(F^0 + T^0 \otimes \mathbb{Q}_p)
\]

is the projection. Both of these maps are functorial.

**Proof.** Let us begin by computing the cohomology of the complex $C^*_{\text{st}}(V)$, which is obtained from $C_{\text{st}}(V)$ by dropping the de Rham component,

\[
D_{\text{st}}(V) \xrightarrow{(p-1,N)} D_{\text{st}}(V) \oplus D_{\text{st}}(V) \xrightarrow{N+1-p'\sigma} D_{\text{st}}(V).
\]

We can decompose this last complex according to slopes

\[
C^*_{\text{st}}(V) = \bigoplus_i C^*_{\text{st}}(V)
\]

with

\[
C^*_{\text{st}}(V) : D^i \xrightarrow{(p'\sigma-1,N)} D^i \oplus D^{i-1} \xrightarrow{N+1-p'\sigma} D^{i-1}.
\]

Since $p'\sigma - 1$ is bijective unless $i = 0$, we immediately see that $H^0(C^*_{\text{st}}) = 0$ unless $i = 0$, and also for $i = 0$ since $N : T^0 \to T^{-1}$ is injective after tensoring with $\mathbb{Q}$. Consider next $H^1$. Suppose $(x, y) \in D^i \oplus D^{i-1}$ represents an element in $H^1(C^*_{\text{st}})$. If $i \neq 0$ we may use the bijectivity of $p'\sigma - 1$ to assume $x = 0$ and the equation on $y$ becomes $(p'\sigma - 1)y = 0$ so $y = 0$ as well. Thus, $H^1(C^*_{\text{st}}) = 0$ unless $i = 0$. In this last case we can write explicitly

\[
H^1(C^*_{\text{st}}) = \frac{\{(x, y), x \in D^0, y \in D^{-1}, Nx = (\sigma - 1)y\}}{\{(\sigma - 1)z, Nz, z \in D^0\}}
\]

and we have the following.

**Lemma 4.2.** We have an isomorphism $(T^{-1}/NT^0) \otimes \mathbb{Q}_p \cong H^1(C^*_{\text{st}})$ given by $u \mapsto (0, u)$.

**Proof.** The map is clearly well defined and injective. Surjectivity amounts to the statement that any element in $H^1(C^*_{\text{st}})$ has a representative $(0, y)$, i.e., that any representative $(x, y)$ has $x \in \text{Im} \sigma - 1$. This is true because $N$ is defined over $\mathbb{Q}$ and is injective.

**Remark 4.3.** The above Lemma may be interpreted for an extension

\[
0 \to T^{-1} \otimes \mathbb{Q}_p(1) \to V \to T^0 \otimes \mathbb{Q}_p \to 0
\]

as saying that the map $H^1_{\text{st}}(K, T^{-1} \otimes \mathbb{Q}_p(1)) \to H^1_{\text{st}}(K, V)$ is surjective.
We have an obvious short exact sequence of complexes
\[ 0 \to \text{DR}(V)/F^0[1] \to C^\bullet_{\text{st}}(V) \to C^\bullet_{\text{st}}(V) \to 0 \]
and the associated long exact sequence together with the computation of the cohomology of \( C^\bullet_{\text{st}}(V) \) immediately gives the short exact sequence (4.4). To define the map \( H^1_{\text{st}}(V) \to \text{DR}(V)/(F^0 + T^0 \otimes \mathbb{Q}_p) \) start with a representative \((x, y, d) \in D_{\text{st}}(V) \oplus D_{\text{st}}(V) \oplus \text{DR}(V)/F^0 \) and use the computation of the cohomology of \( C^\bullet_{\text{st}}(V) \) to see that it is equivalent to a representative with \( x = 0 \). The \( d \) component of this representative is now unique up to an element of \( T^0 \otimes \mathbb{Q}_p \) and this gives the map. The composed map (4.5) is clearly the projection. \( \square \)

Consider now the case \( V = \mathbb{Q}_p(1) \), so that \( T^{-1} = \mathbb{Z} \) while \( T^0 = 0 \). Clearly, in this case the map \( H^1_{\text{st}}(K, \mathbb{Q}_p(1)) \to \text{DR}(V)/F^0 = K \) splits the short exact sequence (4.4) and we have an isomorphism \( H^1_{\text{st}}(K, \mathbb{Q}_p(1)) \cong K \otimes \mathbb{Q}_p \). Nekovář proves the following result.

**Proposition 4.4** ([Nek93, 1.35]). The composed map \( K^\times \xrightarrow{\text{Kummer}} H^1_{\text{st}}(K, \mathbb{Q}_p(1)) \cong K \otimes \mathbb{Q}_p \) is given by \( x \to (\log(x), v(x)) \).

**Proposition 4.5.** For the short exact sequence (4.4) the composed map
\[ T^0 \otimes \mathbb{Q}_p \cong H^0_{\text{st}}(K, T^0 \otimes \mathbb{Q}_p) \to H^1_{\text{st}}(K, T^{-1} \otimes \mathbb{Q}_p(1)) \to T^{-1} \otimes K^\times(1) \xrightarrow{\text{reg}} T^{-1} \otimes \mathbb{Q}_p \]
is just the monodromy map.

**Proof.** By Proposition 4.4 we can compute the map by projecting on the cohomology of the complexes \( C^\bullet_{\text{st}} \), where the result is easy. \( \square \)

**Theorem 4.6.** The toric regulator at \( p \) exists. Furthermore we have the following commutative diagram, where \( V = H^k_{\text{et}}(X \otimes K, \mathbb{Q}_p(r)) \),
\[
\begin{array}{ccc}
H^{k+1}_{\text{M}}(X, \mathbb{Z}(r))_0 & \xrightarrow{\text{log}} & H^1_{\text{st}}(K, V) \\
\downarrow & & \downarrow \\
H^{k+1}_{\text{T}}(X, \mathbb{Z}(r)) & \xrightarrow{\text{log}} & T^{-1} \otimes K/T^0 \otimes \mathbb{Q}_p.
\end{array}
\]
Here, the vertical map on the right is a projection relative to the subspace \( \oplus_{i \leq -2} T^i \otimes K \).

**Proof.** Consider the quotient \( V' = V/W_{2r-k}V \). As this does not have non-positive \( D^i \)'s, and as \( \text{DR}(V') = F^0 \) (since this is true for all the Tate subquotients), we easily see that \( H^1_{\text{st}}(K, V') = 0 \). Thus, we may again do the factoring, as in Section 1, of the regulator into \( H^1_{\text{et}}(K, W_{2r-k}V) \), and then project to \( H^1_{\text{et}}(K, V'') \) with \( V'' = W_{2r-k}V/W_{2r-k-4}V \). Furthermore, the map \( H^1_{\text{et}}(K, V'') \to H^1_{\text{et}}(K, T^0 \otimes \mathbb{Q}_p) \) is 0 because \( \text{DR}(\mathbb{Q}_p) = F^0 \) and by Remark 4.3. Thus, the toric regulator exists at \( p \) as in Section 1. The commutativity of the diagram in the theorem is now straightforward from the following commutative diagram
\[
\begin{array}{ccc}
H^1_{\text{et}}(K, T^{-1} \otimes \mathbb{Q}_p(1)) & \xrightarrow{\text{log}} & H^1_{\text{et}}(K, V'') \\
\downarrow & & \downarrow \\
T^{-1} \otimes K & \to & \text{DR}(V'')/(F^0 + T^0 \otimes \mathbb{Q}_p)
\end{array}
\]
and the fact that the composition of the Kummer map with the vertical map on
the left is just the log map by Nekovář’s result 4.4.

Nekovář and Nizioł define the syntomic regulator,
\begin{equation}
\text{reg} : H^{k+1}_M(X, \mathbb{Z}(r)) \to H^{k+1}_\text{syn}(X, r),
\end{equation}
into syntomic cohomology groups. These groups are constructed in such a way that
there is a spectral sequence
\begin{equation}
E_2^{p,q} = H^p_\text{et}(K, H^q_\text{ét}(X \otimes K, \mathbb{Q}_p(r))) \Rightarrow H^{p+q}_\text{syn}(X, r).
\end{equation}
One easily deduces from this spectral sequence and the syntomic regulator a map
\begin{equation}
\text{reg} : H^{k+1}_M(X, \mathbb{Z}(r))_0 \to H^1_\text{ét}(K, H^k_\text{ét}(X \otimes K \bar{K}, \mathbb{Q}_p(r))),
\end{equation}
which is the same as the map (4.1). The syntomic regulator is computed without
étale cohomology, using a mixture of de Rham and (log) crystalline cohomology
constructions, and so is more computable, at least in principle, using a kind of
“p-adic differential geometry” approach. Indeed, in the good reduction case the
syntomic regulator has been defined for a long time and has been computed in
several cases, primarily by the first named author [Bes00, BdJ03, BdJ12, Bes12].
Recently, some of these results have been extended to the semi-stable reduction
case [Bes18].
We can summarize the results and comments of this section to this point by the
motto “The log of the toric regulator is computed from the syntomic regulator”. To
end this section we illustrate this by revisiting the case of $K_2$ of a totally degenerate
curve and showing how the syntomic regulator is indeed the logarithm of the Pál
rigid analytic regulator.
Let $X$ be as in Section 3 and consider $k = 1$, $r = 2$ again. We have
\begin{equation}
V = H^1_\text{ét}(X \otimes K \bar{K}, \mathbb{Q}_p(2))
\end{equation}
. Recall that in this case $T^0 = 0$. We have $\text{DR}(V) \cong H^1_\text{DR}(X/K)$ and $F^0 \text{DR}(V) = F^2 \text{DR}(X/K) = 0$. According to Theorem 4.6 the logarithm of the toric regulator
map agrees with the composed map
\begin{equation}
\text{reg}_p : H^2_M(X, \mathbb{Z}(2)) \xrightarrow{\text{res}_{\text{syn}}} H^1_M(K, V) \to H^1_\text{DR}(X/K) \to T^{-1} \otimes K
\end{equation}
We identify the vector space on the right hand side with the space $\mathcal{H}(\Gamma, K)$ of $K$-
valued harmonic cochains on the dual graph. As expected from our motto, we get the
following result
\begin{theorem}
Let $X$ be as above. Then the diagram
\begin{equation}
\begin{array}{ccc}
H^2_M(X, \mathbb{Z}(2)) & \xrightarrow{\text{reg}_p} & T^{-1} \otimes K^x \\
\downarrow{\text{reg}_{\text{res}_{\text{syn}}}} & & \downarrow{\log} \\
T^{-1} \otimes K
\end{array}
\end{equation}
commutes, with $\text{reg}_p$ the Pál regulator and $\text{reg}_p$ the p-adic regulator from (4.9).
\end{theorem}
\begin{proof}
Suppose that an element $\alpha$ of $K_2(X)$ restricts to an element $\sum \{f_i, g_i\}$ in
$K_2$ of the function field of $X$. In [Bes18] the first named author proved a formula
for the cup product with a cohomology class $[\omega]$ of the image of $\alpha$ under
\begin{equation}
H^2_M(X, \mathbb{Z}(2)) \xrightarrow{\text{res}_{\text{syn}}} H^1_M(K, V) \to H^1_\text{DR}(X/K).
\end{equation}
One checks easily that the projection $H^1_{st}(K, V) \to H^1_{\text{dR}}(X/K)$ defined there coincides with the one we have been using. This formula was valid when $[\omega]$ is in the kernel of the monodromy operator $N$ ($X$ can be any curve with semi-stable reduction).

To explain the formula and to complete the proof we first analyze $H^1_{\text{dR}}(X/K)$ in a bit more detail. We have the weight decomposition

$$H^1_{\text{dR}}(X/K) = T^{-2} \otimes K \oplus T^{-1} \otimes K = H^1(\Gamma, K) \oplus \mathcal{H}(\Gamma, K).$$

With respect to this decomposition the monodromy operator vanishes on $H^1(\Gamma, K)$ and maps $\mathcal{H}(\Gamma, K)$ on $H^1(\Gamma, K)$ via (3.4). The cup product makes both summands isotropic and gives the pointwise product (3.5) otherwise. Consequently, if $\chi$ and $\text{pr}$ denote the projections on $H^1(\Gamma, K)$ and $\mathcal{H}(\Gamma, K)$ respectively, and we have $[\omega] \in \text{Ker } N = H^1(\Gamma, K)$, $\beta \in H^1_{\text{dR}}(X/K)$, then

$$[\omega] \cup \text{pr } \beta = \chi([\omega]) \cdot \text{pr} (\beta).$$

We can finally introduce the formula of [Bes18]. The formula expresses the cup product in term of expressions assigned to the individual symbols $\{f, g\}$. The expression that we need is the one in Proposition 3.7, which is the same as the expression for the regulator by the main theorem [Bes18, Theorem 1.2]. This expression is

$$\sum_v \langle \log(f), F_v \rangle \cdot \langle \log(g), (v - z)_1 \rangle - \sum_e \chi(e) \cdot \langle \log(f), \log(g) \rangle_e.$$

Here, $Z$ is a subset containing all the singularities of $f$ and $g$, but in any case, when all the components of the reduction are projective lines, the first term vanishes because all the “triple indices” appearing in the sum vanish by [BdJ12, Proposition 8.4]. Thus, the projection of the regulator on $\mathcal{H}(\Gamma, K)$ is the map obtained by sending $\{f, g\}$ to $e \mapsto \langle \log(f), \log(g) \rangle_e$.

Therefore, the following Lemma completes the proof.

**Lemma 4.8.** For rigid analytic functions on the annulus $e$ we have $\langle \log(f), \log(g) \rangle_e = \log(t_e(f, g))$.

**Proof.** Suppose, after identifying $e$ with $A(r, s)$, that $f$ and $g$ are rational functions on $\mathbb{P}^1$. By [Bes00, Proposition 4.10] we have

$$\langle \log(f), \log(g) \rangle_e = \sum_{x \in D(r)} \langle \log(f), \log(g) \rangle_x.$$

At each point in $D = D(r)$ we have, essentially by definition,

$$\langle \log(f), \log(g) \rangle_x = \log t_x(f, g).$$

Thus, the result is true for rational $f$ and $g$ and then is true in general by the definition of the Pál regulator and by continuity.

We close this section with a conjecture, which is suggested by the relation between the toric regulator with both the syntomic and the Sreekantan regulator (but note that in this conjecture we do not need to make any assumptions about the reduction, other than being semi-stable).

**Conjecture 2.** The composition

$$H^{k + 1}_M(X, \mathbb{Z}(r)) \rightarrow H^{k + 1}_M(K, H^1_{\text{dR}}(X \otimes K \bar{K}, \mathbb{Q}_p(r))) \to H^1(C_{\text{ns}}^*(H^1_{\text{dR}}(X \otimes K \bar{K}, \mathbb{Q}_p(r))))$$

factors via the Sreekantan regulator.
5. A conjectural formula for $K_1$ of surfaces

One nice feature of the relation with the syntomic regulator developed in the preceding section is that just as we are able to test formulas for the toric regulator by taking their logarithms and comparing with the syntomic regulator, we can look at formulas for the syntomic regulator and attempt to exponentiate them to get conjectural formulas for the toric regulator. In this section we present a conjecture for the toric regulator for $K_1$ of surfaces, which is suggested by the corresponding result of the first named author in the syntomic case [Bes12]. Unfortunately, the formula we need is the analogue of the one in [Bes12] for the semi-stable reduction case, and it is still conjectural. As it will further take some work to introduce the results we will not present the motivation here and describe the conjecture without it.

We first recall some facts about Mumford curves. Let $X$ be such a curve, given as $G \backslash \mathbb{H}$ where $\mathbb{H}$ is the Drinfeld upper half plane and $G$ is some Schottky group. Let $\Gamma = (V, E)$ be the corresponding dual graph, which is the quotient $G \backslash \mathbb{T}$, with $\mathbb{T}$ the tree of $H$. Let $l$ be a prime. The filtration on $M = H_1^{et}(X \otimes \overline{K}, \mathbb{Z}(1))$ takes the form of a short exact sequence

\[ 0 \to H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{Z}_l(1) \to M \to H^1(\Gamma, \mathbb{Z}_l(1)) \to 0. \]

The augmented monodromy map $\tilde{N} : \mathcal{H}(\Gamma, \mathbb{Z}) \to H^1(\Gamma, K^\times)$ has the following description.

**Proposition 5.1.** Let $\alpha \in \mathcal{H}(\Gamma, \mathbb{Z})$. Then there exist a unique $\omega_\alpha \in \Omega^1(X/K)$ such that for each $e \in E$ we have $\text{res}_e(\omega_\alpha) = \alpha(e)$. Furthermore, on each $U_v$ there exists $f^*_v \in \mathcal{O}(U_v)^\times$ such that $d \log(f^*_v) = \omega_\alpha|_{U_v}$.

This appears in [Col00] without a clear indication of the proof (Coleman shows the existence of $\omega_\alpha$ and mentions it is locally a $d \log$, presumably relying on the theory of theta functions as we will do). We show how it follows from [GvdP80] (or [MD73]). We start by defining Theta functions.

**Definition 5.2 ([GvdP80, II, (2.3) and (2.3.5)])**. Let $\gamma \in G$. The theta function $u_\gamma$ is defined, up to a multiplicative constant, as follows: Pick any $x \in \mathbb{H}$ and let $w_{\gamma,x}$ be any function on $\mathbb{P}^1$ whose divisor is $\gamma(x) - x$. Then

\[ u_\gamma := \prod_{\delta \in \mathcal{G}} \delta^* w_{\gamma,x}. \]

This function is independent of the choice of $x$ as shown in [GvdP80, II, (2.3.4)]

One can normalize the function by normalizing $w$ in an obvious way, but we will not do this. The function $u_\gamma$ has no zeros or poles. It has a constant factor of automorphy, meaning that $\mu(\delta, \gamma) := \delta^* u_\gamma / u_\gamma$ is constant. As a consequence the one form $\omega_\gamma := d \log(u_\gamma)$ is $G$ invariant and holomorphic, descending to a holomorphic one form on $X$.

We compute the residues of $\omega_\gamma$. On $\mathbb{H}$ it is clear that for an edge $\tilde{e}$ of $\mathbb{T}$ the residue of $d \log(u_{\gamma,x})$ on $\tilde{e}$ is non-zero if and only if $\tilde{e}$ sits on the path between the vertex $v_x$, corresponding to the domain where $x$ resides, and $\gamma(v_x)$, and it is $\pm 1$ depending
on its orientation compared with that of the path. Averaging on \( G \), we immediately get that the residue of \( \omega_\gamma \) on an edge \( e \in E(\Gamma) \) is
\[
\sum_{\delta \in G} \text{res}_e \delta \log(w_{\gamma,x}) , \quad \text{\( \tilde{e} \) any lift of \( e \).}
\]
Now we recall that as the graph \( \Gamma \) is finite, the space \( H_1(\Gamma, \mathbb{Z}) \) of harmonic forms on \( \Gamma \) can be identified with the first homology of \( \Gamma \) with coefficients in \( \mathbb{Z} \), which is, by definition,
\[
H_1(\Gamma, \mathbb{Z}) := \ker(\mathbb{Z}[E] \xrightarrow{d} \mathbb{Z}[V]) , \quad d(e) = (e^+) - (e^-).
\]
Furthermore, the Hurewicz isomorphism
\[
(5.2) \quad \text{Hur : } \mathcal{G}^{ab} = H_1(\mathcal{G}, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})
\]
can be evaluated on \( \gamma \in \mathcal{G} \) by choosing a vertex \( v \) and pushing down the path from \( v \) to \( \gamma(v) \) from \( T \) to \( \Gamma \). This immediately gives

**Proposition 5.3.** The harmonic cocycle \( e \mapsto \text{res}_e u_\gamma \) is Hur(\( \gamma \)).

**Proof of Proposition 5.1.** By [GvdP80, Chapter VI (4.2) Proposition] the functions \( \omega_\gamma \) span \( \Omega^1(X) \). The result follows easily. \( \square \)

Because for each \( e \in E \) the function \( f_{\alpha}^+ \) and \( f_{\alpha}^- \) \( \text{d} \log \) to the same form on \( e \), their quotient is a constant \( c_\alpha(e) \in K^\times \). This is a cocycle on \( \Gamma \) with values in \( K^\times \) and its image in \( H^1(\Gamma, K^\times) \) is uniquely determined by \( \alpha \). By pairing with harmonic cocycles we get a bilinear form
\[
\mu' : H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \to K^\times
\]

**Proposition 5.4.** With the Hurewicz isomorphism (5.2) we have \( \mu' = \mu \).

**Proof.** Using Proposition 5.3 what we need to prove is that for the form \( \omega = \omega_\gamma = \text{d} \log(u_\lambda) = \text{d} \log(u) \), the homomorphism \( \delta \mapsto \delta^* u/u \) in \( H^1(\mathcal{G}, K^\times) \) represents the same cohomology class as the map \( e \mapsto \text{d} \log \omega|_{U^+} - \text{d} \log \omega|_{U^-} \) in \( H^1(\Gamma, K^\times) \). But both are clearly the image of \( \omega \) under the boundary map in the short exact sequence
\[
1 \to K^\times \to \mathcal{O} \xrightarrow{\text{dlog}} \Omega \to 0
\]
either as sheaves on \( X \) or, taking global sections on \( \mathcal{H} \), as \( \mathcal{G} \)-modules. \( \square \)

Because \( \mu \) is symmetric by [MD73, Theorem 1] we get.

**Corollary 5.5.** the form \( \mu' \) is symmetric.

**Corollary 5.6.** We have \( \check{N}(\alpha) = c_\alpha \).

**Proof.** This is because \( \mu \) gives the periods for the \( p \)-adic uniformization of the Jacobian of \( X \) by [MD73]. \( \square \)

We will from now onward normalize the choice of \( f_{\alpha}^+ \) in such a way that
\[
(5.3) \quad c_\alpha \in \mathcal{H}(\Gamma, K^\times).
\]
This may require multiplying by some integer.

Suppose now we have two Mumford curves \( X_i, \ i = 1, 2 \), with corresponding graphs \( \Gamma_i \), and we consider the surface \( X = X_1 \times X_2 \). We want a formula for the toric regulator
\[
\text{reg}_\mathcal{G} : H_M^3(X, \mathbb{Z}(2)) \to H_T^3(X, \mathbb{Z}(2))
\]
given on elements of the form
\[ \Theta = \sum \langle C_j, g_j \rangle , \]
where \( C_j \subseteq X \) are curves and \( g_j \) is a rational function on \( C_j \) such that the divisors of \( g_j \) cancel on \( X \).

Let us first compute \( H^3_T(X, \mathbb{Z}(2)) \). Putting aside the uninteresting terms corresponding to \( H^0 \otimes H^2 \), the main contribution to \( H^3_T(X, \mathbb{Z}(2)) \) is
\[ M = M_1 \otimes M_2 , \quad M_i = H^1_{\text{et}}(X_i \otimes \mathbb{K}, \mathbb{Z}_l(1)) . \]

Taking the tensor product of the short exact sequences of the form (5.1) corresponding to \( M_i \), we get on \( M \) a 3-step filtration and we may consider the interesting quotient \( M' \) having the following short exact sequence,
\[ 0 \to (\mathcal{H}(\Gamma_1, \mathbb{Z}) \otimes H^1(\Gamma_2, \mathbb{Z}) \otimes \mathcal{H}(\Gamma_2, \mathbb{Z}) \otimes H^1(\Gamma_1, \mathbb{Z})) \otimes \mathbb{Z}_l(1) \to M' \to \mathcal{H}(\Gamma_1, \mathbb{Z}) \otimes \mathcal{H}(\Gamma_2, \mathbb{Z}) \to 0 , \]
and we may apply a projection \( P \) on one of the summands on the left to get an extension
\[ 0 \to \left( \mathcal{H}(\Gamma_1, \mathbb{Z}) \otimes H^1(\Gamma_2, \mathbb{Z}) \right) \otimes \mathbb{Z}_l(1) \to M'' \to \mathcal{H}(\Gamma_1, \mathbb{Z}) \otimes H^1(\Gamma_2, \mathbb{Z}) \to 0 \]

The associated augmented monodromy is
\[ (5.5) \quad \text{id}_{\mathcal{H}(\Gamma_1, \mathbb{Z})} \otimes \tilde{\mathcal{N}}_2 : \mathcal{H}(\Gamma_1, \mathbb{Z}) \otimes \mathcal{H}(\Gamma_2, \mathbb{Z}) \to \mathcal{H}(\Gamma_1, \mathbb{Z}) \otimes H^1(\Gamma_2, \mathbb{Z}) \otimes K^\times \]
and we get a regulator
\[ P \text{reg}_1 : H^3_{\mathcal{M}}(X, \mathbb{Z}(2)) \to PH^3_T(X, \mathbb{Z}(2)) . \]

We use the duality between graph cohomology and harmonic cocycles to view the resulting intermediate Jacobian \( PH^3_T(X, \mathbb{Z}(2)) \), which is the cokernel of (5.5), as bilinear forms \( H^1(\Gamma_1, \mathbb{Z}) \times \mathcal{H}(\Gamma_2, \mathbb{Z}) \to K^\times \) modulo those forms which are obtained from bilinear forms \( H^1(\Gamma_1, \mathbb{Z}) \times H^1(\Gamma_2, \mathbb{Z}) \to \mathbb{Z} \) by composing in the second coordinate with \( \tilde{N}_2 : \mathcal{H}(\Gamma_2, \mathbb{Z}) \to H^1(\Gamma_2, \mathbb{Z}) \otimes K^\times \) (to be precise, we need to compose with the dual of \( \tilde{N}_2 \) but this is the same by Corollary 5.5).

To construct the required form out of the element (5.4) we are going to further assume that for each index \( j \) the curve \( C_j \) has semi-stable reduction, the projections \( \pi_i : C_j \to X_i \) are finite for \( i = 1, 2 \) and that they give maps of graphs between the corresponding dual graphs. Let \( \alpha \in \mathcal{H}(\Gamma_2, \mathbb{Z}) \), \( \beta \in H^1(\Gamma_1, \mathbb{Z}) \). Identify \( \beta \) with a harmonic representative (which may require again multiplying by a fixed integer, and let \( (f_{\alpha, \varepsilon})_{\varepsilon \in V_2} \) be the corresponding functions as in Proposition 5.1, normalized as in (5.3). We pick an orientation for the edges of \( \Gamma_2 \). Consider a curve \( C_j \) with a rational function \( g_j \) on it. The map induced by \( \pi_2 \) on graphs determines an orientation on the edges of \( \Gamma_{C_j} \). Furthermore, we get pulled-back function \( h^w = \pi_2^* f_{\alpha}(w) \) for each \( w \in V_{C_j} \). Define
\[
\text{reg}_2(\Theta)(\beta, \alpha)_j = \sum_{e \in E_{C_j}} t_e(g_j, h^{\varepsilon_+})^{\beta(\pi_1(e))} ,
\]
\[
\text{reg}_2(\Theta)(\beta, \alpha) = \sum_j \text{reg}_2(\Theta)(\beta, \alpha)_j .
\]

Note that the only place where the orientation of the graph enters is when deciding on the function \( h^{\varepsilon_+} \). Let us check that this gives a well defined element of \( PH^3_T(X, \mathbb{Z}(2)) \), fixing a single \( C = C_j \) and \( g = g_j \). Because of the harmonicity condition there is no ambiguity in \( \beta \). Since we also imposed harmonicity in the
construction of $f^\alpha_\alpha$ these function, and consequently the functions $h^w$, are defined up to a single multiplicative factor $c$. Correspondingly, the terms $t_c(g, h^e)$ will change by $t_c(g, c) = e^{-\deg_e g}$ by Lemma 3.5, with $e \mapsto \deg_e(g) = \text{res}_e d \log g$. This last quantity belongs to $dC_0^0(\Gamma, \mathbb{Z})$ by [BZ17, Lemma 2.1] and thus the regulator does not change.

Now we check what happens if we change the orientation. Suppose we change the orientation for one $e \in E_2$. That changes the orientation for all edges in $e' \in \pi^{-1}_2 e$ and for each of these $h^e$ is replaced by $h^{e'}/\tilde{N}_{2}(\alpha)(e)$. Thus, the change is given by some quadratic form evaluated on $\beta$ and $\tilde{N}_{2}(\alpha)$ as required.

**Conjecture 3.** We have $P_{\text{reg}} = \text{reg}_\alpha$.

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