I. INTRODUCTION

Black holes (BHs) are the most important astrophysical objects predicted by general relativity (GR) and modified gravity theories. In Einstein-Maxwell theory, due to the no-hair theorem\(^1\)\(^2\), BHs have only three parameters, which are its mass, angular momentum, and charge. It means that a BH in GR is the simplest compact object. On the other hand, to solve the dark energy or the matter problem, several gravitational theories beyond GR have been advocated. They introduce additional degrees of freedom as the dark side of our Universe.

In some modified gravity theories, BHs can have additional parameters corresponding to the additional fields. Such BHs are called hairy BHs. Since several properties of hairy BHs are different from those of BHs in GR, observations using gravitational waves\(^5\), BH shadows\(^6\), and motion of galaxies\(^7\) can give us constraints on the scalar field to the electromagnetic field strength and the double-dual Riemann tensor \(F_{\mu\nu}, F_{\mu\nu}F_{\alpha\beta}\) in a scalar-vector-tensor theory. In our model, the scalarization can be realized under the curved background with a non-trivial electromagnetic field, such as Reissner-Nordström Black Holes (RN BHs). Firstly, we investigate the stability of the constant scalar field around RN BHs in the model, and show that the scalar field can suffer a tachyonic instability. Secondly, the bound state solution of the test scalar field around a RN BH and its stability are discussed. Finally, we construct scalarized BH solutions, and investigate their stability.

We present spontaneous scalarization of charged black holes (BHs) which is induced by the coupling of the scalar field to the electromagnetic field strength and the double-dual Riemann tensor \(F_{\mu\nu}, F_{\mu\nu}F_{\alpha\beta}\) in a scalar-vector-tensor theory. In our model, the scalarization can be realized under the curved background with a non-trivial electromagnetic field, such as Reissner-Nordström Black Holes (RN BHs). Firstly, we investigate the stability of the constant scalar field around RN BHs in the model, and show that the scalar field can suffer a tachyonic instability. Secondly, the bound state solution of the test scalar field around a RN BH and its stability are discussed. Finally, we construct scalarized BH solutions, and investigate their stability.

1. The spacetime is spherically symmetric and static spacetime with the asymptotic flatness.
2. \(F\), \(\tilde{F}\) and the gradient of the scalar field vanish in the asymptotic region.
3. The scalar and vector fields have the same symmetries with the metric.
4. The norm of the Noether current \(J_{\phi\mu}J_{\phi}^{\mu}\) is finite on and outside the BH horizon.
5. The theory has the canonical kinetic term of the scalar field \(X \subset f_2(X, F, \tilde{F}, Y)\).
6. The functions \(G_{i=3,4,5}(X)\), \(f_{i=3,4}(X)\) and \(\tilde{f}_3(X)\) are analytic at \(X = 0\).
7. \(f_2(X, F, \tilde{F}, Y)\) is analytic at \(X = 0, Y = 0, F = 0, \) and \(\tilde{F} = 0\).
8. \(f_3(X)\) vanishes at \(X = 0\).

Here, \(G_i\) and \(f_i\) are functions of the SVT theory, and \(F = F_{\mu\nu}F_{\mu\nu}\), \(\tilde{F} = F_{\mu\nu}F_{\mu\nu}\) (see Appendix A). \(F_{\mu\nu}\) and \(\tilde{F}_{\mu\nu}\) are the field strength of the vector field, and its dual field (i.e. \(F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\), \(\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\rho\nu\sigma} F_{\rho\sigma}\)). \(X\) is a kinetic...
term of the scalar field $\Phi$ (i.e. $X = -\frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi$). Under these assumptions, the scalar field around BHs must be constant (see Appendix\textsuperscript{[3]} for proof). The class of the SVT theory which satisfies the above conditions does not possess BH solutions with a scalar hair. On the other hand, BH solutions in the class which does not satisfy the conditions may have a scalar hair. As an example, hairy BH solutions were constructed in the theory in which $f_3(0)$ does not vanish.\textsuperscript{[16]} The scalar field in the hairy BHs decays fast in the far region. Thus, it may be difficult to distinguish these hairy BHs from BHs in GR at the asymptotic region. The other possibility in which hairy BHs can be realized is violation of the shift symmetry. Although it is expected that there are many types of hairy BH solutions in the theory, we focus on the hairy BH solutions associated with spontaneous scalarization.

Spontaneous scalarization is the mechanism through which a relativistic compact object (either a BH or a star) in GR becomes unstable in the strong gravity regime, and spontaneously obtains a nonzero scalar hair with a nontrivial profile of the scalar field. Following the pioneering work by Damour and Esposito-Farese \textsuperscript{[17, 18]} spontaneous scalarization has been studied mainly in the context of a relativistic star, where an instability of a relativistic star in GR is caused by a tachyonic conformal coupling to matter. Because of the existence of the threshold value for the tachyonic coupling \textsuperscript{[19]}, which is marginally consistent with constraints from binary-pulsar observations \textsuperscript{[20]} (see \textsuperscript{[21]} for a review), spontaneous scalarization occurs only in a highly compact star, and hence the difference from GR appears only in the strong gravity regimes.

On the other hand, models for spontaneous scalarization of BHs have been proposed and studied very recently. The first models of BH scalarization were based on the Einstein-scalar-Gauss-Bonnet theory\textsuperscript{[22–27]}. In these models, Schwarzschild solutions with the vanishing scalar field can be solutions if the coupling between the Gauss-Bonnet term and the scalar field satisfies the certain condition, and the scalar field suffers from the tachyonic instability in the vicinity of the BH horizon if the coupling constant is larger than the threshold value, and it is then expected that scalarized BH solutions are realized as the final state of the instability. The scalarized rotating BH solutions were also constructed in the same models\textsuperscript{[28]}. It was also discussed which class of ST theories can realize the spontaneous scalarization in the context of the Horndeski theory\textsuperscript{[29]}.

It was also discussed whether models of spontaneous scalarization of BHs can be successfully embedded into realistic cosmology, in the case that the scalar field has a cosmological origin. The authors of \textsuperscript{[30]} applied the scalar-Gauss-Bonnet model of scalarization to inflation, and mentioned that a catastrophic instability develops and inflation is spoiled. The authors of \textsuperscript{[31]} discussed the possibility that the scalar energy is sub-dominated in Universe, and concluded that tuning of cosmological initial data is needed in order to keep the scalar field subdominant during cosmic evolution.

Spontaneous scalarization can be realized in other models. The simple example is SVT theories with coupling $f(\Phi) F_{\mu\nu} F_{\mu\nu}$ or $f(\Phi) F_{\mu\nu}^2 F_{\mu\nu}$\textsuperscript{[32–34]}. In this model, scalarization can occur for the Reissner-Nordström (RN) BH which satisfies a certain condition. Scalarization of the ST theory coupled to Born-Infeld non-linear electromagnetic field was also discussed\textsuperscript{[35]}

In this paper, we discuss the possibility of BH scalarization induced by the other type of a coupling in the SVT theory. From the SVT theory without an Ostrogradsky ghost, the invariant quantities of the vector field which can induce the tachyonic instability of the scalar field are $F_{\mu\nu} F_{\mu\nu}$, $F_{\mu\nu}^2 F_{\mu\nu}$, and $L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$ where $L^{\mu\nu\alpha\beta} = \frac{1}{4} \varepsilon^\gamma^\delta \varepsilon^\alpha^\beta^\gamma^\delta \overline{R}_{\rho\sigma\gamma\delta}$ is double-dual Reimann tensor.\textsuperscript{[2]} It was shown that the first two invariants can induce scalarization\textsuperscript{[32–34]}. In this paper, we discuss the possibility of scalarization which is induced by $L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$. As a simple toy model, we consider the following action:

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} R - \frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi - \frac{1}{4} F_{\mu\nu}^\alpha F_{\mu\nu}^\alpha \right. $$

$$+ H(\Phi) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right),$$

where $g_{\mu\nu}$ is the spacetime metric, $R$ is the Ricci scalar associated with $g_{\mu\nu}$, $g$ is the determinant of the metric, and $\nabla_\mu$ is the covariant derivative with respect to $g_{\mu\nu}$. $\Phi$ is the scalar field. $H(\Phi)$ is a function of $\Phi$ which satisfies

$$H(0) = 0,$$

$$\frac{H'(0) = 0}{H'(0) = 0},$$

where $H'(\Phi) = \partial_\Phi H(\Phi)$. It characterizes the non-linearity of the scalar field. The condition for $H(\Phi)$ ensures that the RN BH with vanishing scalar field is a solution, and scalarization of a RN BH is realized. Since the model is subclass of the SVT theory, an Ostrogradsky ghost does not appear. The coupling in the action Eq.(1) originates from the function $f_3(\Phi, X)$ in Eq.(A1), and we call the model the $f_3$ model. The SVT theory for spontaneous scalarization of charged BHs with analytic model functions has to be the model which has the coupling $H(\Phi) h(F_{\mu\nu} F_{\mu\nu} F_{\mu\nu}^\alpha)$, where $h$ is function of $F_{\mu\nu} F_{\mu\nu}$ and $F_{\mu\nu}^\alpha F_{\mu\nu}^\alpha$\textsuperscript{[32–34, 65]}, or the model which has the coupling $H(\Phi) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$ in the $f_4$ model.

This paper is organized as follows: In section II, we investigate the condition in which the tachyonic instability of the scalar field around RN BH is realized. In section III, using the test field analysis, bound states around RN BHs are constructed, and the stability is investigated. In section IV, we construct the scalarized BH solutions in the $f_4$ model.

In this paper, we use units $(16\pi G = c = 1)$ and mostly plus signature of the metric.

\textsuperscript{2} $\varepsilon^\alpha^\beta^\gamma^\delta = \frac{1}{\sqrt{-g}} E^\alpha^\beta^\gamma^\delta$ is covariant anti-symmetric tensor where $E^\alpha^\beta^\gamma^\delta$ is the Levi-Civita tensor with $E^{0123} = 1$. 

It's important to note the use of various mathematical symbols and notations used throughout the text, as they are integral to understanding the physics being discussed. These symbols represent quantities and relationships within the context of the scalar-field theory and the scalarization of black holes.
Let us consider the following integration:

\[ \int_V \nabla^\mu \nabla_\mu \Phi + H'(\Phi)L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 0. \]  

(4)

Let us consider the following integration:

\[ \int_V d^4x \sqrt{-g} \left( H'(\Phi)\nabla^2 \Phi + H'(\Phi)^2 L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \right) = 0. \]  

(5)

Using the divergence theorem, we have

\[ \int_V d^4x \sqrt{-g} \left( - H''(\Phi)\nabla^\mu \Phi \nabla_\mu \Phi + H'(\Phi)^2 L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \right) \]

\[ = - \int_{\partial V} d^3x \sqrt{\tilde{g}} H'(\Phi)\nabla_\mu \Phi; \]  

(6)

where \( H''(\Phi) = \partial_\Phi^2 H(\Phi) \). From the spacetime symmetry and the asymptotic flatness, the right hand side must vanish. Outside the horizon, \( \nabla^\mu \Phi \nabla_\mu \Phi \) is positive. Therefore, if \( H''(\Phi)L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} < 0 \) inside \( V \), each term in the integrand of the left hand side in Eq. (5) must vanish, and the scalar field has to be constant.

B. Tachyonic instability of scalar field around RN BHs in the \( f_4 \) model

As is discussed in the previous subsection, the sign of \( L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \) is important for a non-trivial scalar field profile. In this subsection, we treat the scalar field as a test field, and consider the stability of the scalar field around the RN BH solution with the vanishing scalar field.

The RN BH solution is a spherically symmetric charged BH, whose metric and vector field are given as

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(7)

where

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]  

(8)

Here, we assume \( Q < M \). In this spacetime, we obtain the expression

\[ L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \frac{16Q^2(1 - f(r))}{r^6}. \]  

(9)

Since the BH event horizon is \( r_+ = M + \sqrt{M^2 - Q^2} \), \( L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \) is positive outside the horizon. Therefore, if \( H''(0) < 0 \), the scalar field must be constant under the RH BH.

Let us see this statement from another point of view. We consider a perturbation \( \delta \Phi \) of the scalar field under the vanishing scalar profile. For simplicity, we neglect the metric and vector perturbations. From the Eq. (4), the equation of motion for the perturbation is

\[ \nabla_{(RN)}^\mu \nabla_\mu \delta \Phi - m_{eff}^2 \delta \Phi = 0, \]  

(10)

where the effective mass \( m_{eff} \) is defined as

\[ m_{eff}^2 := -H''(0)L_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \]  

(11)

If \( m_{eff}^2 \) is positive, the scalar field is linearly stable, and it corresponds to the no-hair condition which is derived in the previous subsection. On the other hand, if \( m_{eff}^2 \) is negative, the scalar field suffers from a tachyonic instability, and it is expected that the solution with the non-trivial scalar field exists as the end point of the instability.

C. Stability analysis around the RN BHs

We have investigated the test scalar field around the RN BH, and shown that if \( H''(0) > 0 \), the scalar field may become unstable. In this subsection, we discuss the
linear stability of the scalar field around a RN BH by checking the behavior of the effective potential. Following the standard procedure, we separate the variables as follows:

\[
\Phi(t, r, \theta, \phi) = \sum_{l,m} \sigma_{lm}(t, r) Y_{lm}(\theta, \phi),
\]

(13)

where \( Y_{lm}(\theta, \phi) \) is the spherical harmonics. From the Eq.(11), we obtain the equation for \( \sigma_{lm} \) as

\[
-\frac{\partial^2 \sigma_{lm}}{\partial t^2} + \frac{\partial^2 \sigma_{lm}}{\partial r^2} = U_{\text{eff}} \sigma_{lm},
\]

(14)

where \( r^* \) is the tortoise coordinate of a RN BH:

\[
dr^* = \frac{dr}{f_{(\text{RN})}(r)},
\]

(15)

and the effective potential \( U_{\text{eff}} \) is given as

\[
U_{\text{eff}} = f_{(\text{RN})}(r) \left( -\frac{16H''(0)Q^2(1 - f_{(\text{RN})}(r))}{r^6} + \frac{l(l+1)}{r^2} + \frac{f'_{(\text{RN})}(r)}{r} \right),
\]

(16)

where \( f'_{(\text{RN})}(r) = \frac{df_{(\text{RN})}(r)}{dr} \). The typical profile of the effective potential is depicted in Fig.2. As expected, there is the region in which the effective potential is negative around the event horizon, and it implies that there may be an unstable scalar mode.

To verify the existence of RN BH solutions for which the stability is not ensured, we use the method which is discussed in [36]. In the frequency domain, the equation of motion Eq.(15) becomes

\[
-\frac{d^2 \sigma_{lm}}{dr^*_2} + U_{\text{eff}}(r) \sigma_{lm} = \omega^2 \sigma_{lm},
\]

(17)

where \( \sigma_{lm}(t, r) = e^{i\omega t} \tilde{\sigma}_{lm}(r) \). To investigate the linear stability which means that \( \omega^2 \) is nonnegative for all modes, we focus on the following equation for \( \tilde{\varphi}(r^*) \):

\[
\left( -\frac{d^2}{dr^*_2} + U_{\text{eff}} \right) \tilde{\varphi} = 0.
\]

(18)

As discussed in [36], if the solution \( \tilde{\varphi}(r^*) \) of Eq.(18) does not cross zero, the scalar field is linearly stable. Using the method, we found that there is a parameter region in which the constant scalar field solution on a RN BH can be unstable (see Fig.3).

III. TEST FIELD ANALYSIS

So far, we have investigated the stability of the constant scalar profile around the RN BH, and found that there are parameter regions in which the \( l = 0 \) mode of the scalar field perturbation is unstable. This result implies that there are bound states for a negative effective potential in the unstable parameter region. We expect that the state is the final state of the instability. In this section, we construct the profile of the bound state and investigate its parameter dependence.

A. Bound states of the test scalar field around the RN BHs.

The static and spherically symmetric profiles of the scalar field around a RN BH are solutions of the following

\[
M^2 \left( \frac{f_{(\text{RN})}}{M^2} \right) \tilde{\varphi} = 0.
\]

FIG. 2. The effective potential for \( H''(0)/M^2 = 2 \) and \( l = 0 \).
\[
\partial^2 \Phi + \left( \frac{2}{r} + \frac{f'_\text{(RN)}(r)}{f\text{(RN)}(r)} \right) \partial_r \Phi + 16Q^2 \frac{H'(\Phi(r))}{r^6} \left( -1 + \frac{1}{f\text{(RN)}(r)} \right) = 0.
\]

(19)

| \eta/M^2 | Q_s/(M\Phi_H) |
|----------|----------------|
| 0 nodes  | 0.128 | 0.193 |
| 1 node   | 0.661 | -0.243 |
| 2 nodes  | 1.669 | 0.271 |

TABLE I. The typical parameter set of the bound states in the quadratic model with \( Q/M = 0.99 \).

We can solve this equation with the regularity boundary condition on the event horizon. Assuming that the bound state is the final state of the instability around the vanishing profile \( \Phi = 0 \), we impose \( \Phi(r \to \infty) = 0 \) as the boundary condition. The scalar charge \( Q_s \) is defined as

\[
Q_s = -r^2 \Phi'(r)|_{r \to \infty}.
\]

(20)

From here, we specify the coupling function \( H(\Phi) \). As a first step, we consider the simplest coupling:

\[
H(\Phi) = \frac{\eta}{2} \Phi^2,
\]

(21)

where \( \eta \) is a constant. This model is named the quadratic model. As we will show in the next subsection, the bound state of this coupling is unstable. In order to stabilize the bound state, the nonlinear effect of \( H(\Phi) \) plays an important role. In particular, the existence of the region in which \( H''(\Phi) \) is negative is crucial. Then, we also consider the following coupling function as an example model:

\[
H(\Phi) = \frac{\eta}{2} \left( 1 - e^{-\Phi^2} \right).
\]

(22)

This model is named the exponential model.

1. The quadratic model

In the quadratic model, the equation of motion for the scalar field becomes linear, and the normalization factor of the field is irrelevant as long as the backreaction can be ignored. We found that for each electric charge \( Q \), there is a solution which is characterized by the number of nodes. The typical profiles of the scalar field are depicted in Fig.4, and the corresponding parameter sets are summarized in Table I. In the figure, \( \Phi \) is normalized such that the scalar field on the horizon is unity. Fig.5 shows the relation between the electric charge \( Q/M \) and the coupling \( \eta/M^2 \) for each number of nodes. From the plot, we find that there are solutions with 0, 1, and 2 nodes for any \( Q/M \), and there may be solutions with any nodes for any \( Q/M \). We found that the line of the solution with 0 nodes in the figure is almost the same as the critical line of Fig.3.

2. The exponential model

In the exponential model, the equation of motion for the scalar field is nonlinear, and the relevant free parameters are \( \eta/M^2, Q/M, \Phi_H \), where \( \Phi_H \) is the value of the scalar field on the event horizon. The typical profile of the scalar field is depicted in Fig.6, and the corresponding parameter sets are summarized in Table II. For fixed coupling \( \eta/M^2 \) and electric charge, \( \Phi_H \) becomes large as the number of node decreases. As \( \eta/M^2 \) or \( Q/M \) is increased, the solution with higher nodes can be constructed. Figure 7 shows the relation between \( \Phi_H \) and the charge-to-mass ratio \( Q/M \) for different number of nodes and coupling \( \eta/M^2 \). From the plot, for fixed coupling \( \eta/M^2 \) and nodes, \( \Phi_H \) is an increasing function for the charge-to-
FIG. 5. The relation between the coupling $\eta/M^2$ and the electric charge $Q/M$ for different number of nodes in the quadratic model.

FIG. 6. The typical behavior of the non-trivial profile of the scalar field around a RN BH for $Q/M = 0.99$ in the exponential model. The corresponding parameter sets are summarized in Table II.

|                                | $\Phi_H$ | $Q_\ast/M$ |
|--------------------------------|----------|------------|
| 0 nodes, $\eta/M^2 = 1$       | 1.80     | 0.734      |
| 0 nodes, $\eta/M^2 = 10$      | 2.49     | 1.93       |
| 1 node, $\eta/M^2 = 10$      | 2.34     | -1.64      |
| 2 nodes, $\eta/M^2 = 10$     | 2.19     | 1.38       |

TABLE II. The typical parameter set of the bound states in the exponential model with $Q/M = 0.99$.

mass ratio. There is the critical value $Q_c(\eta/M^2, n)/M$ for a solution with $n$ nodes in which $\Phi_H$ vanishes. It means that if $Q/M < Q_c(\eta/M^2, n)/M$, the solution with $n$ nodes does not exist. We have scanned the parameter region of $(Q/M, \eta/M^2)$ in which the bound state exists, and found that the critical line above which there is the bound state almost coincides with the line of Fig. 3. In this model, the scalar charge can be of order one, which means that the large difference from GR appears in the asymptotic region.

B. Stability analysis of the bound state

We discuss the stability of the bound state, which is constructed in the previous subsection, around a RN BH. Let us denote the bound state as $\Phi_0(r)$, and consider the perturbation around the state:

$$\Phi(t, r, \theta, \phi) = \Phi_0(r) + \sum_{l,m} \sigma_{lm}(t, r) Y_{lm}(\theta, \phi),$$

(23)

where $Y_{lm}(\theta, \phi)$ is the spherical harmonics. From the equation of motion for the scalar field, we obtain the equation for $\sigma_{lm}$ as

$$-\frac{\partial^2 \sigma_{lm}}{\partial t^2} + \frac{\partial^2 \sigma_{lm}}{\partial r^2} = U_{\text{eff}} \sigma_{lm},$$

(24)

where the effective potential $U_{\text{eff}}$ is given as

$$U_{\text{eff}} = f_{(\text{RN})}(r) \left( \frac{16H''(\Phi_0(r))Q^2(1 - f_{(\text{RN})}(r))}{r^6} + \frac{l(l + 1)}{r^2} + \frac{f'_{(\text{RN})}(r)}{r} \right).$$

(25)
1. The quadratic model

In the quadratic model, the effective potential is the same as one of the perturbations around the constant scalar field, and hence the stability condition remains the same. Therefore, the bound state with \( H(\Phi) = \frac{\eta^2}{2} \Phi^2 \) is unstable against the radial perturbations.

2. The exponential model

In the exponential model, the effective potential of the radial perturbation around the bound state on the RN BH is shown in Fig. 8. The positive effective potential ensures the stability of the bound state. From Eq. (25), in order that the effective potential becomes positive everywhere, \( H''(\Phi_0) \) must be negative around the horizon. The condition in which \( H''(\Phi_0) \) is negative around the horizon is \( \sqrt{2} < |\Phi_0| \). Figure 8 implies that there is parameter sets in which the effective potential becomes positive everywhere. Therefore, the nonlinearity of \( H(\Phi) \) can stabilize the bound state.

IV. SCALARIZED SOLUTION

In this section, we construct the scalarized BH solutions as the final state of spontaneous scalarization and discuss the stability.

A. Construction

To construct the solutions, let us focus on the equations of motion. Due to the \( U(1) \) symmetry, the equation of motion for the vector field can be expressed in the divergence form:

\[
\nabla \mu (F_{\mu\nu} - 4H(\Phi)L^{\alpha\beta\mu\nu}F_{\alpha\beta}) = 0.
\]

We assume that the spacetime is spherically symmetric, whose metric is given as

\[
ds^2 = -a(r)dt^2 + \frac{dr^2}{b(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

and \( \Phi = \Phi(r) \). Varying the action with respect to the metric and the scalar field, we have the set of the ordinary differential equations:

\[
\begin{align*}
b' - \frac{1 - b}{r} + \frac{rb\Phi'^2}{4} + \frac{rbA_t'^2}{4a} + \frac{2bH(\Phi)A_t'^2}{ra} (1 - b) &= 0, \\
a' - \frac{1 - b}{r} - \frac{r\Phi'^2}{4} + \frac{rA_t'^2}{4a} - \frac{2H(\Phi)A_t'^2}{ra} (1 - 3b) &= 0, \\
\Phi'' + \left( \frac{2}{r} + \frac{a'}{2a} + \frac{b'}{2b} \right) \Phi' + \frac{4A_t'^2H'(\Phi)}{r^2a} (1 - b) &= 0.
\end{align*}
\]

Since the equation for the vector field is written as the divergence form as Eq. (26), we can integrate it and obtain

\[
A_t' = -\sqrt{\frac{a}{b}} \sqrt{\frac{2Q}{r^2 - 8(b - 1)\Phi}},
\]

where \( Q \) is an integration constant.

We substitute Eq. (31) for Eqs. (28)-(30). Then, we have three equations for \( a, b, \) and \( \Phi \), and can solve them numerically. In the vicinity of the horizon, the metric functions behave as \( a(r), b(r) \propto (r - r_H) \). From the regularity condition, the behavior of the fields around the horizon is given by:
\[ \Phi(r) = \Phi_H - \frac{16Q^2H'(\Phi_H)}{r_H(r_H^2 + 8H(\Phi_H))(r^2 + r_H^2 + 8H(\Phi_H))}(r - r_H) + O((r - r_H)^2), \]  
\[ a(r) = a_1(r - r_H) + O((r - r_H)^2), \]  
\[ b(r) = \frac{-Q^2 + r_H^2 + 8H(\Phi_H)}{r_H(r_H^2 + 8H(\Phi_H))}(r - r_H) + O((r - r_H)^2), \]  

where \( \Phi_H \) is the value of the scalar field on the event horizon, and \( a_1 \) is a free parameter.

We impose the asymptotically flat boundary condition for the metric, and the vanishing vector and scalar fields at the infinity. From the equations, the asymptotic behaviors of the fields are given as

\[ \Phi(r) = \frac{Q_s}{r} + \frac{MQ_s}{r^2} + O\left(\frac{1}{r^3}\right), \]  
\[ a(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + O\left(\frac{1}{r^3}\right), \]  
\[ b(r) = 1 - \frac{2M}{r} + \frac{4Q^2 + Q_s^2}{4r^2} + O\left(\frac{1}{r^3}\right), \]  
\[ A_t(r) = \frac{2Q}{r} + O\left(\frac{1}{r^3}\right), \]

where \( M \) and \( Q_s \) are integration constants, which correspond to mass and scalar charge. From Eq. (38), \( Q \) corresponds to the electric charge.

We numerically integrate the equations of motion under the boundary conditions around the horizon and at the infinity, and construct the scalarized BH solutions in the quadratic and the exponential models.

### 1. The quadratic model

The typical profile of the fields is depicted in Fig. 9 and the parameters of the solutions are summarized in Table III. These solutions can be overcharged. We found that the 0 node solution has the biggest \( Q/M \) for fixed \( \eta/M^2 \) and \( Q/\sqrt{\eta} \). Since nodeless solutions are more likely to be stable, we focus on nodeless solutions. The relation between \( Q/M \) and \( Q/\sqrt{\eta} \) and the relation between \( \Phi_H \) and \( Q/\sqrt{\eta} \) in nodeless solution are plotted in Fig 10. The figure shows that \( Q/M \) as the function of \( Q/\sqrt{\eta} \) has a maximum value, and \( Q/M \) approaches 1 for larger \( Q/r_H \). Furthermore, for the large coupling \( \eta/r_H^2 \), the maximum value of the charge-to-mass ratio becomes large. There is a critical parameter \( Q_c/\sqrt{\eta} \) for \( Q/\sqrt{\eta} \), above which we can not construct the solution, numerically. We could not determine whether there are solutions above the critical parameter, or not. On the other hand, there is also the critical value below which the scalarized BH solution does not exist, and on which the scalar field on the event horizon becomes zero.

In the case of the scalarization induced by the term \( F_{\mu \nu} F^{\mu \nu} \), there is the critical line in the parameter region of the coupling and \( Q/M \) [32, 33]. For a fixed value of coupling, there is no scalarized BH solution whose charge-to-mass ratio is larger than the critical line, on which the solution is extremal. On the other hand, in our model, the BH solution which has the maximum \( Q/M \) is not the extremal BH for a fixed value of \( \eta/r_H^2 \). It is expected

| \( \eta/r_H \) | \( Q/M \) | \( Q_s/M \) |
|----------------|--------|--------|
| 0 nodes, \( \eta/r_H = 1.0 \) | 1.09 | 0.6522 |
| 1 node, \( \eta/r_H = 1.0 \) | 1.01 | -0.269 |
| 2 nodes, \( \eta/r_H = 1.0 \) | 1.00 | 0.9029 |
| 0 nodes, \( \eta/r_H = 0.1 \) | 1.01 | 0.164 |

FIG. 9. The typical profile of the scalarized BH solutions for \( Q/r_H = 1.0 \) for quadratic model. The corresponding parameter sets of each solutions are summarized in Table III.
that there is the critical line on the parameter plane of $(\eta/r_H^2, Q/M)$, which corresponds to the maximum value of $Q/M$ for a fixed value of coupling. We leave it as a future work.

Finally, we shortly mention the expected properties of the extremal BH in the $f_4$ model. From the Eqs. (32)-(34), the extremal BH, in which the coefficient of $r - r_H$ vanishes, must satisfy the following condition:

$$-Q^2 + r_H^2 + 8H(\Phi_H) = 0.$$  \hspace{1cm} (39)

When this condition is satisfied, the denominator of the coefficient of $r - r_H$ in $\Phi$ vanishes. To construct the extremal BH solution with finite scalar field, there are two possibilities. One possibility is that $H'(\Phi_H)$ is zero, and one particular solution is $\Phi_H = 0$. Then, the solution does not have a nontrivial scalar field. The other possibility is that our ansatz of expansion for the field (i.e. Eqs. (32)-(34)) is not valid. It would be possible to construct the extremal BH by generalization of the ansatz. We also leave the construction of the extremal BH in the $f_4$ model as a future work.

|    | $Q/M$ | $Q_s/M$ |
|----|-------|---------|
| 0 nodes, $\eta/r_H^2 = 1.0$ | 1.07 | 0.523 |
| 1 node, $\eta/r_H^2 = 1.0$ | 1.01 | -0.197 |
| 2 nodes, $\eta/r_H^2 = 1.0$ | 1.00 | 0.0902 |
| 0 nodes, $\eta/r_H^2 = 0.1$ | 1.00 | 0.113 |

TABLE IV. The typical parameter sets for the scalarized BH solutions for the exponential model.

2. The exponential model

The typical behavior of the fields is depicted in Fig. 11. These behaviors are almost the same as the quadratic model except around the horizon. The relation between $Q/M$ and $Q/\sqrt{\eta}$ is depicted in Fig. 12. The critical parameter above which we could not construct the solutions numerically is smaller than the quadratic case, and we could not determine whether solutions exit beyond the critical value, again. Furthermore, there is a solution whose $\Phi_H$ becomes zero for each value of $\eta$. If the $Q/\sqrt{\eta}$ is smaller than the value of the solution, there is no scalarized BH solution.

FIG. 10. The sequence of the scalarized BH solutions for fixed value $\eta/r_H^2$ in the quadratic model. Top : The relation between $Q/\sqrt{\eta}$ and $Q/M$ for nodeless solution. Dotted line is $Q/M = 1$. Bottom : The relation between $Q/\sqrt{\eta}$ and the scalar field on the horizon.

FIG. 11. The typical behavior of the scalarized BH solutions with $Q/r_H = 1$ in the exponential model. The corresponding parameter sets are summarized in Table IV.
The sequence of the scalarized BH solutions for the fixed value $\eta/r_H^2$ in the exponential model. Top: The relation between $Q/\sqrt{\eta}$ and $Q/M$ for nodeless solution. Dotted line is $Q/M = 1$. Bottom: The relation between $Q/\sqrt{\eta}$ and the scalar field on the horizon.

**B. Stability**

We can check the stability of the scalarized BH solutions by calculating the effective potential. Let us consider the perturbation around the scalarized BH solutions $\Phi_s$, which is constructed by solving Eqs. (28)-(30). Here, we only consider the stability against the perturbation of the scalar field:

$$\Phi(t, r, \theta, \phi) = \Phi_s(r) + \sum_{l,m} \frac{\sigma_{lm}(t, r)}{r} Y_{lm}(\theta, \phi).$$

(40)

From the equation of motion for the scalar field, we have the equation for $\sigma_{lm}$:

$$- \frac{\partial^2 \sigma_{lm}}{\partial t^2} + \frac{\partial^2 \sigma_{lm}}{\partial r_*^2} = U_{\text{eff}} \sigma_{lm},$$

(41)

where $dr_* = dr/\sqrt{ab}$. The effective potential $U_{\text{eff}}(r)$ is given as

$$U_{\text{eff}} = \frac{l(l+1)}{r^2} a + \frac{1}{2r}(a'b + ab') - H''(\Phi_s) \frac{16Q^2}{r^2} (1 - b).$$

(42)

The typical form of the effective potential for each model is plotted in Fig. 13 and Fig. 14. In the exponential model, the effective potential for the nodeless solution is positive everywhere outside the horizon as shown in Fig. 14. As is expected from the bound state analysis in Sec. III B, the solution is stable for the $l = 0$ mode scalar perturbation. Therefore, we conclude the scalarized BH solution in the exponential model can be the end state of scalarization.

**V. SUMMARY AND DISCUSSION**

In this paper, we have investigated spontaneous scalarization of charged BHs in the scalar-vector-tensor(SVT) theory which contains the coupling of the scalar field to the double dual Riemann tensor and the field strength (i.e. $L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$). We described this model as the $f_4$ model. First, we investigated the stability of the test scalar field around a RN BH with the trivial scalar profile,
and showed that the coupling induces the tachyonic instability in the vicinity of the BH event horizon if the BH charge or the coupling is sufficiently large and $H''(0) > 0$. The parameter region in which the scalarization is induced is depicted in Fig.3

Secondly, using the test field approximation, the bound state of the scalar field around a RN BH was constructed. To construct the profile, we consider the quadratic and the exponential models. The quadratic model is the simplest model in which the scalarization can be realized. The exponential model is a non-trivial example model. Performing the stability analysis, we showed that the bound state of the quadratic model is unstable. On the other hand, we also showed that the nonlinear effect of the scalar field in the exponential model can stabilize the bound state.

Finally we constructed the scalarized BH solutions. The solutions are characterized by the BH mass, electric and/or magnetic fields [32]. In our model, scalarization can be induced in Minkowski spacetime [37]. The parameter region in which the scalarization is induced in the vicinity of the BH event horizon if the BH charge or the coupling is sufficiently large and $H''(0) > 0$. Thus, scalarized BHs can be overcharged.

The $f_4$ model has different properties from the previous models of scalarization [32]. In the previous model, the origin of scalarization is the nontrivial profile of $F_{\mu\nu} F_{\mu\nu}$. So, scalarization can be induced in Minkowski spacetime with electric and/or magnetic fields [32]. In our model, the trigger of scalarization is the non-trivial profile of $L_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$. The term vanishes in Minkowski spacetime. For scalarization of the $f_4$ model, the curved spacetime is crucial.

In this paper, we consider scalarization of the $f_4$ model around RN BHs, and showed that scalarization is induced. In an astrophysical setup, scalarization may realize under the Kerr BH with the external magnetic field, which may be approximated by Wald’s solution [37]. Therefore, it is expected that the difference between the $f_4$ model and Einstein-Maxwell theory with minimal coupling may appear in such a setup. This task is beyond the scope of this paper.

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**Appendix A: Scalar-vector-tensor theories with $U(1)$ symmetry**

In this appendix, we summarize the most general SVT theory with $U(1)$ symmetry whose equations of motion are of the second order [8]. The scalar, vector and tensor fields are denoted by $\Phi$, $A_\mu$, and $g_{\mu\nu}$, respectively.

The action of the theory can be expressed in the following form

$$S[g, \Phi, A] = \int d^4 x \sqrt{-g} \left( \sum_{i=3}^{5} \mathcal{L}_{\text{ST}}^i + \sum_{i=2}^{4} \mathcal{L}_{\text{SVT}}^i \right)$$

where $\mathcal{L}_{\text{ST}}^i (i = 3, 4, 5)$ are the Lagrangians of the ST sector, and $\mathcal{L}_{\text{SVT}}^i (i = 2, 3, 4)$ are the Lagrangians of the SVT sector. The action of the ST part corresponds to the Horndeski theory [10–12], which is given as

\begin{align}
\mathcal{L}_3^\text{ST} &= G_3(\Phi, X) \Phi^\mu, \\
\mathcal{L}_4^\text{ST} &= G_4(\Phi, X) R + G_{4,X}(\Phi, X) (\Phi^\mu)^2 - \Phi^{\mu\nu} \Phi^\nu, \\
\mathcal{L}_5^\text{ST} &= G_5(\Phi, X) G_{\mu\nu} \Phi^\mu \Phi^\nu - \frac{G_{5,X}(\Phi, X)}{6} \left( (\Phi^{\mu\nu})^3 - 3(\Phi^{\mu\nu})(\Phi^{\rho\nu}\Phi^\rho) + 2\Phi^{\mu\nu} \Phi^{\rho\nu} \Phi^\rho \right),
\end{align}

where $G_{i=3,4,5}(\Phi, X)$ are arbitrary functions of $\Phi$ and $X$, and $G_{i,X}(\Phi, X) = \partial_X G_i(\Phi, X)$. Here, $\Phi_{\alpha\beta} = \nabla_\alpha \nabla_\beta \Phi$ is the 2nd covariant derivative of $\Phi$, and $X = \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi$ is the canonical kinetic term of $\Phi$, $R$, and $G_{\mu\nu}$ are the Ricci scalar and the Einstein tensor.

The SVT parts of the theory are given as

\begin{align}
\mathcal{L}_2^\text{SVT} &= f_2(\Phi, X, F, \tilde{F}, Y), \\
\mathcal{L}_3^\text{SVT} &= M_3^{\mu\nu} \Phi_{\mu\nu}, \\
\mathcal{L}_4^\text{SVT} &= M_4^{\mu\nu\alpha\beta} \Phi_{\mu\alpha} \Phi_{\nu\beta} + f_4(\Phi, X) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta},
\end{align}

where $F$, $\tilde{F}$, and $Y$ are defined as

\begin{align}
F &= F^{\mu\nu} F_{\mu\nu}, \\
\tilde{F} &= \tilde{F}^{\mu\nu} F_{\mu\nu}, \\
Y &= \nabla_\mu \tilde{F} \nabla_\nu \Phi \Phi^{\alpha\nu} F^{\mu\alpha},
\end{align}

4 The $G_2$ term of Horndeski theory is included in the $f_2$ term of scalar-vector-tensor sector.
and $M_3^{\mu\nu}$, $M_4^{\mu\nu\alpha\beta}$, and $L^{\mu\nu\alpha\beta}$ are given as

$$M_3^{\mu\nu} = \left( f_3(\Phi, X) g_{\mu\nu} + \tilde{f}_3(\Phi, X) \nabla_\mu \Phi \nabla_\nu \Phi \right) \tilde{F}^\mu \tilde{F}^\nu,$$

(A11)

$$M_4^{\mu\nu\alpha\beta} = \left( \frac{1}{2} f_4(\Phi, X) + \tilde{f}_4(\Phi) \right) \tilde{F}^{\mu\nu} F^{\alpha\beta},$$

(A12)

$$L^{\mu\nu\alpha\beta} = \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \epsilon^{\rho\gamma\delta} R_{\rho\gamma\delta}.$$

(A13)

Here, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor, $\epsilon^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{g}} E^{\alpha\beta\gamma\delta}$ is the covariant anti-symmetric tensor, and $E^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor with $E^{0123} = 1$. $F_{\mu\nu}$ is the field strength (i.e. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$), and $F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$ is the dual of the field strength. $f_2(\Phi, X, \tilde{F}, \tilde{F}, Y)$ is an arbitrary function of $\Phi$, $X$, $\tilde{F}$, and $Y$. $f_3(\Phi, X)$, $\tilde{f}_3(\Phi, X)$, and $f_4(\Phi, X)$ are arbitrary functions of $\Phi$ and $X$, respectively. $\tilde{f}_4(\Phi)$ is an arbitrary function of $\Phi$. $f_4(X, \Phi, X)$ is the derivative of $f_4(\Phi, X)$ with respect to $X$.

The action Eq.(A1) gives a framework of the most general SVT theory with 2nd order equations of motion which has $U(1)$ symmetry. This theory includes several subclasses of theory which has the scalar, vector, and tensor fields without an Ostrogradsky ghost. One of the important subclasses is the theory which has the shift symmetry:

$$\Phi \rightarrow \Phi + c.$$  

(A14)

The theory with the shift symmetry is given by eliminating the $\Phi$ dependence as follows:

$$G_{i=3,4,5}(\Phi, X) \rightarrow G_{i=3,4,5}(X),$$

(A15)

$$f_2(\Phi, X, \tilde{F}, \tilde{F}, Y) \rightarrow f_2(X, \tilde{F}, \tilde{F}, Y),$$

(A16)

$$f_{i=3,4}(\Phi, X) \rightarrow f_{i=3,4}(X),$$

(A17)

$$\tilde{f}_3(\Phi) \rightarrow \tilde{f}_3(X),$$

(A18)

$$\tilde{f}_4(\Phi) \rightarrow \text{constant.}$$

(A19)

By virtue of the shift symmetry, the equation of motion of the scalar field can be written as divergence form:

$$\nabla_\mu J_{\Phi}^\mu = 0,$$

(A20)

where $J_{\Phi}^\mu$ is the Noether current associated with the shift symmetry.

Appendix B: No hair theorem for shift symmetric SVT theory with $U(1)$ symmetry

In this section, we only consider the theory which possesses the shift symmetry Eq.(A14), and discuss an extension of the BH no-hair theorem of the shift symmetric Horndeski theory[14] to the subclass of the SVT theory with the shift symmetry.

We prove the no-hair theorem of the shift symmetric SVT theory without a cosmological constant under the following assumptions:

1. The spacetime is spherically symmetric and static spacetime with the asymptotic flatness.
2. $F$, $\tilde{F}$, and the gradient of the scalar field vanish in the asymptotic region.
3. The scalar and vector fields have the same symmetries with the metric.
4. The norm of the Noether current $J_{\Phi\mu} J_{\Phi}^\mu$ is finite on and outside the BH horizon.
5. The theory has the canonical kinetic term of the scalar field $X \subset f_2(X, F, \tilde{F}, Y)$.
6. The functions $G_{i=3,4,5}(X)$, $f_{i=3,4}(X)$ and $\tilde{f}_3(X)$ are analytic at $X = 0$.
7. $f_2(X, F, \tilde{F}, Y)$ is analytic at $X = 0$, $Y = 0$, $F = 0$, and $\tilde{F} = 0$.
8. $f_3(X)$ vanishes at $X = 0$.

Since we assume that the action does not have a cosmological constant, $f_2(0, 0, 0, 0, 0) = 0$, and $f_2$ vanishes at the infinity. The no-hair theorem states that under the above conditions, BHs can not have a nontrivial profile of the scalar field and are solutions of the following vector-tensor theory:

$$S_{\text{SVT}} = \int d^4 x \sqrt{-g} \left( G_4(0, 0) R + f_2(0, 0, F, \tilde{F}, 0) + f_4(0, 0) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right).$$

(B1)

This theory contains RN BHs, Born-Infeld BHs[38], and BH solutions of the generalized Einstein-Maxwell theory derived by Horndeski[39]. The assumptions 1-7 are simple extensions from the assumptions of the no hair theorem of the shift symmetric Horndeski theory, and the condition 8 is an additional one.

To prove the no hair theorem, we focus on the general property of the current $J_{\Phi}^\mu$. In this section, we use the following metric ansatz:

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + g^2(r) r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(B2)

where $f(r)$ and $g(r)$ are functions of $r$. Without loss of generality, we assume $g(r) > 0$. The location $r_H$ of the BH horizon is dictated by $f(r_H) = 0$.

Following the same discussion as the shift symmetric Horndeski theory[14], the assumptions 1 and 3 imply that the nontrivial component of the current $J_{\Phi}^\mu$ is radial direction $J_{\Phi}^r$, and, the assumption 4 implies that

$$J_{\Phi}^r = 0$$

(B3)

on and outside of the BH horizon. For further discussion, we focus on the expression of the current $J_{\Phi}^\phi$. The
expression of $J_{s}^{r}$ is expressed as

$$J_{s}^{r} = C_{f2Y} f_{2Y}(X, F, \tilde{F}, Y) + C_{f2X} f_{2X}(X, F, \tilde{F}, Y) + C_{f3}(X) + C_{f3X} f_{3X}(X) + C_{f4X} f_{4X}(X) + C_{f4XX} f_{4XX}(X) + C_{G3X} G_{3X}(X) + C_{G4X} G_{4X}(X) + C_{G4XX} G_{4XX}(X) + C_{G5X} G_{5X}(X) + C_{G5XX} G_{5XX}(X),$$

(B4)

where $X = -\frac{1}{2} f(r) \Phi'^{2}$, $\tilde{F} = 0$, $F = -2 A_{t}^{2}$, and $Y = -f(r) A_{t}^{2} \Phi'^{2}$. The coefficients $C_{f2Y}$, $C_{f2X}$, $C_{f3}$, $C_{f3X}$, $C_{f4X}$, $C_{f4XX}$, $C_{G3X}$, $C_{G4X}$, $G_{4XX}$, $G_{5X}$, and $C_{G5XX}$ are expressed as

\[
\begin{align*}
C_{f2Y} &= -F(r) f(r) \Phi'(r), \\
C_{f2X} &= f(r) \Phi'(r), \\
C_{f3} &= \frac{F(r) f(r) (rg'(r) + g(r))}{rg(r)}, \\
C_{f3X} &= \frac{2F(r) X f(r) (rg'(r) + g(r))}{rg(r)}, \\
C_{f4X} &= \frac{F(r) f(r) \Phi'(r) \left(3f(r) (rg'(r) + g(r))^2 - 2\right)}{r^2 g(r)^2}, \\
C_{f4XX} &= \frac{F(r) X f(r) (rg'(r) + g(r))^2 \Phi'(r)}{r^2 g(r)^2}, \\
C_{G3X} &= X(r) \left(-f'(r) - \frac{4f'(r) (rg'(r) + g(r))}{rg(r)}\right), \\
C_{G4X} &= -\frac{2f(r) \Phi'(r) \left(r^2 g(r)^2 f'(r) (rg'(r) + g(r))^2 + f(r) (rg'(r) + g(r))^2 - 1\right)}{r^2 g(r)^2}, \\
C_{G4XX} &= -\frac{4X(r) f(r) (rg'(r) + g(r)) \Phi'(r) (rg'(r) f'(r) + f(r) (rg'(r) + g(r)))}{r^2 g(r)^2}, \\
C_{G5X} &= X(r) f'(r) \left(3f(r) (rg'(r) + g(r))^2 - 1\right), \\
C_{G5XX} &= -\frac{2X(r)^2 f(r) f'(r) (rg'(r) + g(r))^2}{r^2 g(r)^2}.
\end{align*}
\]

(B5)\(\text{to}\) (B15)

Since the coefficient $C_{f3}$ does not vanish for $\Phi' = 0$, the condition in which the solution with a trivial scalar field exists is $f_{3}(0) = 0$, which is the assumption 8. The expression of $X$, $F$, and $\tilde{F}$ ensures that these scalar functions are finite on and outside of BH horizon.

We focus on the assumptions 6 and 7. The assumptions imply that the functions $G_{i=3,4,5}(X)$, $f_{2}(X, F, \tilde{F}, Y)$, $f_{i=3,4}(X)$, and $\tilde{f}_{3}(X)$ can be expressed in terms of Taylor series:

\[
\begin{align*}
G_{i=3,4,5}(X) &= \sum_{N \geq 0} \frac{G_{i=3,4,5}^{(N)}}{N!} X^{N}, \\
f_{2}(X, F, \tilde{F}, Y) &= \sum_{N, M \geq 0} \frac{f_{2}^{(N,M)}}{N! M!} X^{N} Y^{M}, \\
f_{i=3,4}(X) &= \sum_{N \geq 0} \frac{f_{i=3,4}^{(N)}}{N!} X^{N}, \\
\tilde{f}_{3}(X) &= \sum_{N \geq 0} \frac{\tilde{f}_{3}^{(N)}}{N!} X^{N},
\end{align*}
\]

(B16)\(\text{to}\) (B19)
where \( f_3^{(0)} = 0 \). Let us focus on the asymptotic region. In the asymptotic region, the metric functions behave as \( f(r), g(r) = 1 + \mathcal{O}(\frac{1}{r}) \), and \( F, \tilde{F} \) decay as some power of \( \frac{1}{r} \). Therefore, the form of the current in the asymptotic region is given as

\[
J^r \simeq \alpha \Phi',
\]
where

\[
\alpha = f_2^{(1,0)}(0, 0).
\]

The assumption 5 implies that \( \alpha \) does not vanish, and \( \Phi' \) must be zero in the asymptotic region. From Eq. (B14), the current behaves as

\[
J^r = S(\Phi'(r), F(r), f(r), g(r)) \Phi'(r),
\]
where \( S \) is a non-linear function of \( \Phi'(r), F(r), f(r), g(r) \), and approaches to \( \alpha \) at infinity. Moving inward from the infinity, \( S \) varies continuously, and it implies that the solution of \( J^r = 0 \) is \( \Phi' = 0 \) everywhere outside the horizon. This is the no hair theorem of the shift symmetric SVT theory with \( U(1) \) symmetry.

### Appendix C: Validity of the test field analysis.

In this appendix, the validity of the test field analysis used in Sec. III is discussed. The analysis can be valid under the limit of small \( G \) in which the Einstein-Maxwell theory is the leading order and the effect of the scalar field is the next order. To discuss the limit explicitly, let us introduce \( \bar{A}_\mu \) and \( \bar{H}(\Phi) \) which are defined as

\[
\bar{A}_\mu \equiv \frac{1}{\sqrt{G}} \bar{A}_\mu, \quad \bar{H}(\Phi) \equiv G H(\Phi).
\]

Using these variables, the action is expressed as

\[
S = \int d^4x \sqrt{-\bar{g}} \left[ \frac{1}{16\pi} R - \frac{1}{4} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} - \frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi + \bar{H}(\Phi) \bar{I}^{\mu\alpha\beta} \bar{F}_{\mu\nu} \bar{F}_{\alpha\beta} \right],
\]
where \( \bar{F}_{\mu\nu} = \nabla_\mu \bar{A}_\nu - \nabla_\nu \bar{A}_\mu \). Taking the limit of small \( G \) with fixing the metric and \( \bar{A}_\mu \), the leading order action becomes the Einstein-Maxwell theory, and the 1st order equation is the Klein-Gordon equation with the coupling. This limit corresponds to the test field analysis in Sec. III.
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