Decoherence-free linear quantum subsystems

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Abstract—This paper provides a general theory for characterizing and constructing a decoherence-free (DF) subsystem for an infinite dimensional linear open quantum system. The main idea is that, based on the Heisenberg picture of the dynamics rather than the commonly-taken Schrödinger picture, the notions of controllability and observability in control theory are employed to characterize a DF subsystem. A particularly useful result is a general if and only if condition for a linear system to have a DF component; this condition is used to demonstrate how to actually construct a DF dynamics in some specific examples. It is also shown that, as in the finite dimensional case, we are able to do coherent manipulation and preservation of a state of a DF subsystem.

Index Terms—Quantum linear systems, quantum information, decoherence-free subsystem, controllability and observability.

I. INTRODUCTION

Quantum information processing shows maximum performance when it is run based on an ideal closed system. However, in reality any quantum system is an open system interacting with surrounding environment and has to be affected by decoherence, which can seriously degrade the information. Therefore the theory of decoherence-free subspaces (DF subspaces) or more generally decoherence-free subsystems (DF subsystems) [9], [10], [35], [38], [40], [51], [52] is very promising for realizing various key technologies in quantum information science, such as quantum computation [2], [4], [33], [39], [67], [68], memory [3], [34], [48], communication [57], [62], and metrology [17]. It also should be mentioned that some experimental demonstrations have been reported [19], [29], [55], [70].

Let us now review the basic idea of DF (or noiseless) subsystems. Quantum information processing is founded on the unitarity of the ideal closed dynamics of a system of interest [43]. More precisely, the “state” \( \hat{\rho}_c \) of a closed quantum system is driven by an Hermitian operator (Hamiltonian) \( \hat{H} \) and is subjected to the dynamics \( d\hat{\rho}_c/dt = -i[H, \hat{\rho}_c] = -i[\hat{H}\hat{\rho}_c - \hat{\rho}_c\hat{H}] \), which yields the solution \( \hat{\rho}_c(t) = e^{-iHt}\hat{\rho}_c(0)e^{iHt} \) (a detailed description is given in Appendix); this unitary change of \( \hat{\rho}_c(0) \) represents the quantum information processing under consideration, in which case it is often called the gate operation on \( \hat{\rho}_c(0) \). However, in reality any quantum system interacts with environment and does not obey a unitary dynamics. Especially when the interaction is instantaneous and a Markovian approximation can be taken, the system’s state evolves in time with the following master equation:

\[
\frac{d\hat{\rho}}{dt} = -i[H, \hat{\rho}] + \sum_{i=1}^{m} \left( \hat{L}_i \hat{\rho} \hat{L}_i^* - \frac{1}{2} \hat{L}_i^* \hat{L}_i \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_i^* \hat{L}_i \right). \tag{1}
\]

The operator \( \hat{L}_i \) represents the coupling between the system and the \( i \)-th environment field; this second term represents the decoherence. Due to this, the time-evolution of \( \hat{\rho} \) is not unitary, implying that in general coherent manipulation or preservation of an arbitrary state of an open system is impossible. Nevertheless, it is known in the finite dimensional case that we can have a chance to protect a specific part of the state against the environment. That is, if the system matrices \( \hat{H} \) and \( \hat{L}_i \) satisfy specific algebraic conditions and an appropriate initial state \( \hat{\rho}(0) \) is prepared, then in certain basis vectors the system state has a block diagonal form \( \hat{\rho}(t) = \text{diag}(\hat{\rho}_D(t) \otimes \hat{\rho}_D(t), O) \) for all \( t \geq 0 \), with \( \otimes \) denoting the Kronecker product; \( \hat{\rho}_D \) is the state of the DF subsystem, which obeys a unitary time evolution, while the decohered (D) component \( \hat{\rho}_D \) is still subjected to a non-unitary dynamics. That is, the dynamics of these states are respectively given by

\[
\frac{d\hat{\rho}_D(t)}{dt} = -i[H_D, \hat{\rho}_D], \tag{2}
\]

\[
\frac{d\hat{\rho}_D}{dt} = -i[H', \hat{\rho}_D] + \sum_{i=1}^{m} \left( \hat{L}_i \hat{\rho}_D \hat{L}_i^* - \frac{1}{2} \hat{L}_i^* \hat{L}_i \hat{\rho}_D - \frac{1}{2} \hat{\rho}_D \hat{L}_i^* \hat{L}_i \right). \tag{3}
\]

When \( \dim \hat{\rho}_D = 1 \) (i.e. \( \hat{\rho}_D = 1 \)), the space of \( \hat{\rho}_D \) is called the DF subspace. Note that the complement \( \text{diag}(O, \hat{\rho}_D(t)) \) is also decohered and its dynamics is governed by a master equation. Equation (2) means that, by appropriately devising the reduced Hamiltonian \( \hat{H}_D \), we can carry out a desirable unitary gate operation.

Now we come up with the following question; is it possible to construct a DF subsystem in the infinite dimensional case? This question is equivalent to asking the applicability of the above theory for discrete variable (DV) systems to continuous variable (CV) systems [12], [20]. Indeed, despite the recent rapid progress of quantum information science with infinite-dimensional/CV systems, there has not been developed a general theory of DF subsystems; it was only shown in [3] that the infinite dimensional DF subsystems can be characterized in terms of operator algebras, which is though hard to use for engineering purpose. The problem becomes relatively easy if we focus on the case where there is no decohered subspace component, i.e. the case where the decomposition \( \hat{\rho}(t) = \hat{\rho}_D(t) \otimes \hat{\rho}_D(t) \) is possible. Actually in this case some infinite dimensional systems having a DF subsystem are known, in which case the DF component is often called the dark mode; the DF mode is utilized for state generation.
and preservation \cite{26,41,47}, cooling of a mechanical oscillator \cite{16}, and state transfer \cite{58}. However there is no general theory unifying the methods taken in these examples. Particularly what is lacking is the general procedure for constructing a DF subsystem, which has actually been found in the finite dimensional case \cite{13}.

This paper provides a basic framework dealing with a general linear quantum system containing a DF subsystem, when especially there is no decohered subspace component. A linear system is an infinite dimensional open system that describes the dynamics of, for instance, optical modes of a light field \cite{21,28,59,61,65,66}, position of an opto-mechanical oscillator \cite{39,50}, vibration mode of a trapped particle \cite{37}, collective spin component of a large atomic ensemble \cite{24,31,42}, or more generally canonical conjugate pairs of a harmonic oscillator network \cite{22,23,24,30,44,45,60,63}, which all serve as quantum information devices.

The key ideas are twofold. The first main idea is that we focus on the dynamics of finite number of physical quantities (called “observables” in physics) that completely characterize the system of interest, rather than the state that must be defined on the whole infinite dimensional space. This means that we represent the system in the Heisenberg picture, rather than the Schrödinger picture describing the dynamics of a state. Then, the system dynamics is not described by the master equation \cite{1}, but by a quantum stochastic differential equation (QSDE) \cite{5,11,21,27,61}. The QSDE explicitly represents the system-environment coupling in a form that the equation contains the terms of environmental input and output fields. Together with the fact that we are now dealing with the finite number of observables, this critical feature of the QSDE further leads us to have the second main idea. That is, we are now able to naturally define a DF observable as follows; i.e., it is an observable such that its dynamics is not affected by the environmental input field and the behavior of that observable does not appear in the output field. Actually it will be proven in this paper that this definition is equivalent to the original one, for the finite dimensional case. The significant point of this definition is that a DF subsystem is now clearly characterized in terms of the notions of controllability and observability, which are well developed in control theory (e.g., \cite{1,32}). In particular, in the linear case, such controllability and observability properties can be straightforwardly captured by simple matrix algebras that only require us to compute the rank of certain matrices. As a result, based on the QSDE representation of a given linear open quantum system, we obtain a necessary and sufficient condition for that system to have a DF subsystem, which further leads to an explicit procedure for constructing the DF subsystem. This result will be given in Section III.

Once a DF subsystem is constructed, then we should be interested in the stability of the dynamics. In the current context, this is the property that autonomously erases the correlation between the DF and the D parts, and it has been extensively investigated for finite dimensional systems \cite{49,51,52}. In Section IV, particularly in the case of Gaussian systems \cite{18,59}, a simple if and only if criterion that guarantees the stability of a DF subsystem will be provided.

The remaining part of the paper is devoted to the study of some concrete examples of a DF linear subsystem, which is divided into two topics as follows.

(i) (Section V) The first topic deals with coherent manipulation and preservation of a state of a DF subsystem. More specifically, it is demonstrated that, in a specific two-mode linear system having a one-mode DF component, arbitrary linear unitary gate operation and state preservation are possible under relatively mild assumptions. Also, for the same system, it will be shown that preserving a state of the DF subsystem is equivalent to preserving a certain entangled state among the whole two-mode system. In addition, the stability property is examined, showing its importance in realizing robust state manipulation and preservation with the DF subsystem.

(ii) (Section VI) The second is about how to engineer a DF subsystem for a given open linear system. We actually find in two specific examples that, by appropriately devising an auxiliary system coupled to the original one, it is possible to construct a physically meaningful DF subsystem. Therefore, in addition to the practical merit that a DF subsystem can be used for coherent quantum information processing, a DF subsystem is useful in the sense that it simulates an ideal closed quantum system even in a realistic open environment.

We use the following notations: for a matrix \( A = (a_{ij}) \), the symbols \( A^\dagger \), \( A^\top \), and \( A^\# \) represent its Hermitian conjugate, transpose, and elementwise complex conjugate of \( A \), i.e., \( A^\dagger = (a_{ji}^\ast) \), \( A^\top = (a_{ji}) \), and \( A^\# = (a_{ji}^\ast) = (A^\dagger)^\dagger \), respectively. For a matrix of operators, \( A = (\hat{a}_{ij}) \), we use the same notation, in which case \( \hat{a}_{ij}^\dagger \) denotes the adjoint to \( \hat{a}_{ij} \). \( I_n \) denotes the \( n \times n \) identity matrix. \( \mathcal{R} \) and \( \mathcal{I} \) denote the real and imaginary parts, respectively. \( O \) is a zero matrix with appropriate dimension. \( \otimes \) is the tensor product, which is the Kronecker product in the finite dimensional case. \( \text{Ker}(A) = \{ x | Ax = 0 \} \) and \( \text{Range}(A) = \{ y | y = Ax, \forall x \} \) denote the kernel and the range of a matrix \( A \), respectively. Throughout the paper, we set \( \hbar = 1 \).

Some fundamentals of quantum mechanics, e.g., states and observables, are given in Appendix.

II. Preliminaries

A. The QSDE

Let us consider an open system interacting with some environment fields, which are assumed to be independent vacuum fields for simplicity. The time-evolution of an observable of this system is described by a QSDE, as shown below. For a comprehensive introduction to the theory of quantum stochastic calculus, we refer to \cite{11}.

Let \( \hat{a}_i(t) \) be the annihilation operator of the \( i \)-th vacuum field and assume that \( \hat{a}_i(t) \) instantaneously interacts with the system, which means that it satisfies the canonical commutation relation (CCR) \( [\hat{a}_i(s), \hat{a}_j^\dagger(t)] = \delta_{ij}\delta(t - s) \). This relation reminds us the classical white noise; actually the field annihilation process \( \hat{A}_i(t) = \int_0^t \hat{a}_i(s) ds \), which is the quantum version to the classical Wiener process, satisfies the following quantum Ito rule (in the vacuum field) \cite{5,11,21,27,31,61,63}.
in a system whose Hamiltonian and coupling operator are hence the system is of infinite dimensional. Let us here define Note that both the field operator is also obtained; the output field $X(t)$ evolves in the Heisenberg picture to $\hat{X}(t) = \hat{U}^\dagger(t)\hat{X}(0)\hat{U}(t)$, where $\hat{U}(t)$ is the unitary operator from time 0 to $t$, which is constructed from the relation $\hat{U}(t + dt) = \hat{U}(t + dt, t)\hat{U}(t)$ with $\hat{U}(t + dt, t) = \exp[-i\hat{H}_{\text{int}}(t + dt, t)]$. Then, using the above quantum Itô rule we can derive the QSDE of $\hat{X}(t)$:
\[
d\hat{X}(t) = \left[i\hat{H}(t), \hat{X}(t)\right] + \sum_{i=1}^{m} \left(\hat{L}_i(t)\hat{X}(t)\hat{L}_i(t) - \frac{1}{2}\hat{L}_i(t)\hat{L}_i(t)\hat{X}(t)\right)dt + \sum_{i=1}^{m} \left([\hat{X}(t), \hat{L}_i(t)]d\hat{A}_i^\dagger(t) - [\hat{X}(t), \hat{L}_i(t)]d\hat{A}_i(t)\right), \tag{4}
\]
Here, we have added a system Hamiltonian $\hat{H}$ and defined $\hat{H}(t) = \hat{U}^\dagger(t)\hat{H}\hat{U}(t)$ and $\hat{L}_i(t) = \hat{U}^\dagger(t)\hat{L}_i\hat{U}(t)$. The master equation (1) of the system’s unconditional state, $\rho(t)$, is obtained through the time evolution of the mean value of $\hat{X}(t)$ and the relation $\langle \hat{X}(t) \rangle = \text{Tr} [\hat{X}(0)\rho(t)]$. The change of the field operator is also obtained; the output field $\hat{A}^\text{out}_i(t) = \hat{U}^\dagger(t)\hat{A}_i\hat{U}(t)$ after the interaction satisfies
\[
d\hat{A}^\text{out}_i(t) = \hat{L}_i(t)dt + d\hat{A}_i(t). \tag{5}
\]

**B. Quantum linear systems**

In this paper, we consider a system composed of $n$ subsystems. The variable of each subsystem is called the *mode*; particularly in our case the $i$-th mode is specified by the canonical conjugate pairs $(\hat{q}_i, \hat{p}_i)$ satisfying the CCR $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$. Note that both $\hat{q}_i$ and $\hat{p}_i$ are infinite dimensional operators, hence the system is of infinite dimensional. Let us here define the vector of operators $\hat{x} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T$. Then the CCRs $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$ are summarized as
\[
\hat{x}\hat{x}^\dagger - (\hat{x}\hat{x}^\dagger)^\dagger = i\Sigma_n, \tag{6}
\]
where
\[
\Sigma_n = \text{diag}\{\Sigma_1, \ldots, \Sigma_n\}, \quad \Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
($\Sigma_n$ is a $2n \times 2n$ block diagonal matrix.) We are interested in a system whose Hamiltonian and coupling operator are respectively given by
\[
\hat{H} = \hat{x}\hat{x}^\dagger G/2, \quad \hat{L}_i = c_i^\dagger \hat{x},
\]
where $G = G^\dagger \in \mathbb{R}^{2n \times 2n}$ and $c_i \in \mathbb{C}^{2n}$ ($i = 1, \ldots, m$) \cite{60, 61, 63}. Also let us define the canonical conjugate pairs of the noise process:
\[
\hat{Q}_i = (\hat{A}_i + \hat{A}_i^\dagger)/\sqrt{2}, \quad \hat{P}_i = (\hat{A}_i - \hat{A}_i^\dagger)/\sqrt{2i}, \tag{7}
\]
and collect them into a single vector of operators as $\mathcal{V} = (\hat{Q}_1, \hat{P}_1, \ldots, \hat{Q}_m, \hat{P}_m)^T$. Then, from the QSDE \cite{4}, the vector of system variables $\hat{x}(t) = (\hat{q}_1(t), \hat{p}_1(t), \ldots, \hat{q}_n(t), \hat{p}_n(t))^T$, where $\hat{q}_i(t) = \hat{U}^\dagger(t)\hat{q}_i\hat{U}(t)$ and $\hat{p}_i(t) = \hat{U}^\dagger(t)\hat{p}_i\hat{U}(t)$, satisfies the following linear equation:
\[
d\hat{x}(t) = A\hat{x}(t)dt + \Sigma_n C^\dagger \Sigma_m d\mathcal{W}(t). \tag{8}
\]
The coefficient matrices are given by
\[
A = \Sigma_n (G + C^\dagger \Sigma_m C/2) \in \mathbb{R}^{2n \times 2n},
\]
\[
C = \sqrt{2}(\mathcal{R}(c_1), \mathcal{R}(c_1), \ldots, \mathcal{R}(c_m), \mathcal{R}(c_m))^T \in \mathbb{R}^{2m \times 2n}.
\]
This specific structure of the system matrices is due to the unitary evolution of $\hat{q}_i(t)$ and $\hat{p}_i(t)$, which is indeed necessary to satisfy the CCR $[\hat{q}_i(t), \hat{p}_j(t)] = i\delta_{ij}$ for all $t$. Moreover, corresponding to the canonical conjugate representation of the input field \cite{7}, let us define the output process as
\[
\hat{Q}^\text{out}_i = (\hat{A}^\text{out}_i + \hat{A}^\text{out}_i^\dagger)/\sqrt{2}, \quad \hat{P}^\text{out}_i = (\hat{A}^\text{out}_i - \hat{A}^\text{out}_i^\dagger)/\sqrt{2},
\]
and collect them as $\mathcal{V}^\text{out} = (\hat{Q}^\text{out}_1, \hat{P}^\text{out}_1, \ldots, \hat{Q}^\text{out}_m, \hat{P}^\text{out}_m)^T$. Then, from \cite{5} we have
\[
d\mathcal{V}^\text{out}(t) = C\hat{x}(t)dt + d\mathcal{W}(t). \tag{9}
\]
To denote the system variable, in what follows we often omit the time index $t$ and simply use $\hat{x}$; to avoid confusion, the initial value is explicitly written as $\hat{x}(0)$.

Here we remark that the linearly transformed variable $\hat{x}' = T^\dagger \hat{x}$ must satisfy the CCR \cite{6}, which implies that $x'x'^\dagger - (x'x'^\dagger)^\dagger = T^\dagger (xx\dagger - (xx\dagger)^\dagger)T = iT^\dagger \Sigma_n T = i\Sigma_n$. Consequently, in the quantum case, the similarity transformation $T$ must be *symplectic*, meaning that it satisfies
\[
T^\dagger \Sigma_n T = \Sigma_n. \tag{10}
\]
Note further that, in the linear case, the time evolution is also a symplectic transformation, because we can find a symplectic matrix $S$ satisfying $\hat{x}(t) = ST^\dagger \hat{x}(0)$.

Now let $\langle \hat{x} \rangle$ be the mean vector, where the mean operation $\langle \cdot \rangle$ is taken elementwise. The covariance matrix is defined by
\[
V = \langle \Delta \hat{x}\Delta \hat{x}^\dagger + (\Delta \hat{x}\Delta \hat{x}^\dagger)^\dagger \rangle/2, \quad \Delta \hat{x} = \hat{x} - \langle \hat{x} \rangle.
\]
Note that the uncertainty relation $V + i\Sigma_n/2 \geq 0$ holds. Then for the linear system \cite{8}, the time-evolutions of $\langle \hat{x} \rangle$ and $V$ are respectively given by
\[
d\langle \hat{x} \rangle/dt = A\langle \hat{x} \rangle, \quad dV/dt = AV + VA^\dagger + D, \tag{11}
\]
where $D = \Sigma_n^\dagger C^\dagger C\Sigma_n/2$.

**C. Gaussian systems**

A quantum Gaussian state can be characterized by only the mean vector $\langle \hat{x} \rangle$ and the covariance matrix $V$. More specifically, its Wigner function is identical to the Gaussian probabilistic distribution with mean $\langle \hat{x} \rangle$ and covariance $V$. A notable property of a linear system is that it preserves the Gaussianity of the state; that is, if the initial state of the system \cite{8} is Gaussian, then, for all later time $t$ the state is Gaussian with mean $\langle \hat{x}(t) \rangle$ and covariance $V(t)$, whose time evolutions are given by the linear differential equations (11). A unique
steady state of this Gaussian system exists only when $A$ is a Hurwitz matrix, i.e., all the eigenvalues of $A$ have negative real parts. If it exists, the mean vector is $\langle \hat{x}(\infty) \rangle = 0$, and the covariance matrix $V(\infty)$ is given by the unique solution to the algebraic Lyapunov equation $AV(\infty) + V(\infty)A^\top + D = 0$. Note that several non-Gaussian states can be generated in a linear quantum system [69].

D. Controllability and observability

Here we review the notions of controllability and observability, which play the fundamental roles in characterizing various important properties of a system, such as stabilizability of the system via control input or capability of constructing an estimator that continuously monitors the system variables. See e.g. [1], [32] for full description of those theories.

Let us consider the following general linear system:

$$\dot{x} = Ax + Bu, \quad y = Cx,$$  \hspace{1cm} (12)

where $x$ is the system variable of dimension $n$, and $u$ and $y$ are an input and an output with certain dimensions, respectively. Note that $u$ represents any input such as a disturbing noise and a tunable control signal. Now define the controllability matrix $C = (B, AB, \ldots, A^{n-1}B)$ and assume that $\text{rank}(C) = n'$. (Note $n' \leq n$ in general.) Then, there exists a linear coordinate transformation $x \rightarrow x' = (x'_1, x'_2, \ldots, x'_{n'})$ such that the system (12) is transformed to

$$\frac{d}{dt} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} A'_{11} & O \\ A'_{21} & A'_{22} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} + \begin{pmatrix} O \\ B'_2 \end{pmatrix} u, \quad y = (C'_1, C'_2)^\top \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix},$$

where $x'_1$ is of dimension $n-n'$. This equation shows that $x'_1$ is not affected by $u$, hence it is called uncontrollable with respect to (w.r.t.) $u$. Of course, if $n' = n$, or equivalently if $C$ is of full rank, then all the elements of $x$ are affected by $u$ and in this case the system is called controllable. Next, let us define the observability matrix $O = (C^\top, A^\top C^\top, \ldots, (A^\top)^{n-1}C^\top)^\top$ and assume that $\text{rank}(O) = n''$. Then, there exists a linear coordinate transformation that transforms (12) to the system of the following form:

$$\frac{d}{dt} \begin{pmatrix} x''_1 \\ x''_2 \end{pmatrix} = \begin{pmatrix} A''_{11} & A''_{12} \\ O & A''_{22} \end{pmatrix} \begin{pmatrix} x''_1 \\ x''_2 \end{pmatrix} + \begin{pmatrix} B''_1 \\ B''_2 \end{pmatrix} u, \quad y = (O, C'_2)^\top \begin{pmatrix} x''_1 \\ x''_2 \end{pmatrix},$$

where $x''_1$ is of dimension $n - n''$. Clearly, $x''_1$ cannot be seen from the output $y$, hence $x''_1$ is called unobservable w.r.t. $y$. Note here that the term “unobservable” does not mean that $x''_1$ is not a physical quantity.

III. CHARACTERIZATION OF LINEAR DF SUBSYSTEMS

In this section, based on the QSDE representation of the system, we show the if and only if condition for a general linear quantum system to have a DF subsystem, particularly in the case where there is no decohered subspace component. Moreover, we will see that this condition leads to a concrete procedure for constructing the DF subsystem.

Here we remark that in [6], [7] the Heisenberg picture is taken to investigate the quantum error correction, in the general finite dimensional case. This approach differs from ours in that the authors have examined the mean dynamics of the system observables with its environment fields traced out (averaged); that is, unlike the QSDE, the time-evolution equation does not contain the input and output terms, so the notions of controllability and observability are not applicable.

A. The Heisenberg picture description of a DF subsystem

In this subsection, we describe a finite dimensional open system having a DF subsystem, in the Heisenberg picture. As mentioned in Section I, if the system contains a DF subsystem, then its state, which is subjected to the master equation (1), is represented in a specific basis as $\rho(t) = \text{diag}(\hat{\rho}_{DF}(t) \otimes \hat{\rho}_D(t), O)$ with $\hat{\rho}_{DF}$ governed by the unitary dynamics (2). Ticozzi and Viola [51], [52] gave an explicit iff condition for the system to have a DF subsystem; in particular when there is no decohered subspace, the condition is that the system matrices are of the following form:

$$\hat{H} = \hat{H}_{DF} \otimes I_D + I_{DF} \otimes \hat{H}_D, \quad \hat{L}_i = I_{DF} \otimes \hat{L}_{D,i}. \hspace{1cm} (13)$$

Actually, by substituting these matrices with initial state $\hat{\rho}(0) = \hat{\rho}_{DF}(0) \otimes \hat{\rho}_D(0)$ for the master equation (1), we obtain (2) and (3).

Let us now change the picture to the Heisenberg’s one and look at the dynamics of an open system, the QSDE (4). Now we assume that the system matrices are given by (13); then the system observables $\hat{X}_{DF}(t) = \hat{U}^\ast(t)(\hat{X}_{DF}(0) \otimes I_D)\hat{U}(t)$ and $\hat{X}_D(t) = \hat{U}^\ast(t)(I_{DF} \otimes \hat{X}_D(0))\hat{U}(t)$ satisfy the following dynamical equations:

$$d\hat{X}_{DF} = i[\hat{H}_{DF}, \hat{X}_{DF}] dt,$$

$$d\hat{X}_D = i[\hat{H}_D, \hat{X}_D] + \sum_{i=1}^m \left( \hat{L}_{D,i}^\ast \hat{X}_D \hat{L}_{D,i} - \frac{1}{2} \hat{L}_{D,i}^\ast \hat{L}_{D,i} \hat{X}_D \right) dt,$$

$$+ \sum_{i=1}^m \left( [\hat{X}_D, \hat{L}_{D,i}] d\hat{A}_i^\ast - [\hat{X}_D, \hat{L}_{D,i}] d\hat{A}_i \right), \hspace{1cm} (15)$$

where we have omitted the time index $t$. Also, the output equation (5) becomes

$$d\hat{A}_i(\text{out}) = \hat{L}_{D,i}(t) dt + d\hat{A}_i(t). \hspace{1cm} (16)$$

Note that $\hat{L}_{D,i}(t)$ evolves in time according to (15). Thus, the above equations imply that $\hat{X}_{DF}(t)$ is completely isolated from the environment. In other words, $\hat{X}_{DF}(t)$ is not affected by the input field $\hat{A}_i(t)$, and the output field $\hat{A}_i(\text{out})$ does not contain any information about that observable for all $i$. In terms of control theory, therefore, an observable that is decoherence free is uncontrollable w.r.t. $\hat{A}_i(t)$ and unobservable w.r.t. $\hat{A}_i(\text{out})$ for all $i$. 

B. The DF condition for linear systems

In the above subsection we have seen that a DF observable is uncontrollable and unobservable. In the finite dimensional case, this is indeed true for any DF observable, because if \( A \) is necessary and sufficient. This characterization is reasonable from both physics and engineering viewpoints, and thus, it may be taken as a definition for a system to have a DF subsystem even in the infinite dimensional case. In particular, for the linear quantum system \( O \) and \( \theta \), we pose the following formal statement:

**Definition 3.1:** For the linear quantum system \( O \) and \( \theta \), suppose that (i) there exists a subsystem that is uncontrollable w.r.t. \( \mathcal{W} \) and unobservable w.r.t. \( \mathcal{W}^\text{out} \), and further, (ii) its variable satisfies the CCR \( \{ \} \). Then, that subsystem is called the DF subsystem, and its variable \( \hat{x}_\text{DF} \) is called the DF mode.

The requirement for \( \hat{x}_\text{DF} \) to satisfy the CCR property stems from the fact that, in CV quantum information processing, any gate operation acts on the pairs of canonical conjugate observables, and this discards our problem from a simple one where we are interested in a merely uncontrollable and unobservable subsystem. A different view of this requirement is that it corresponds to the physical realizability condition \( \mathcal{W}^\text{end} \) of the DF subsystem; indeed, if the subsystem is completely isolated, it must be a physical system. See \( \mathcal{W} \) where, in a specific optical system, an uncontrollable and unobservable subsystem is constructed using the physical realizability condition.

The purpose here is to fully characterize the linear system having a DF subsystem. As in the classical case, the following controllability matrix \( C \) and the observability matrix \( O \) will play a key role in deriving such a characterization:

\[
\begin{align*}
C &= (\Sigma_n C^T \Sigma_m, A \Sigma_n C^T \Sigma_m, \ldots, A^{2n-1} \Sigma_n C^T \Sigma_m), \\
O &= (C^T, A^T C^T, \ldots, (A^T)^{2n-1} C^T)^T.
\end{align*}
\]

The following lemma will be useful:

**Lemma 3.1:**

\[
\Sigma_n v \in \ker(O) \iff v \in \ker(C^T)
\]

**Proof:** Suppose \( \Sigma_n v \in \ker(O) \). This means that we have \( CA^k \Sigma_n v = 0 \), \( \forall k \geq 0 \), which is further equivalent to \( C(\Sigma_n G)A^k \Sigma_n v = 0 \), \( \forall k \geq 0 \), due to the specific structure of \( A \), i.e., \( A = \Sigma_n (G + C^T \Sigma_m C/2) \). This condition yields

\[
(A^k \Sigma_n C^T \Sigma_m)^T v = \Sigma_m C \Sigma_n (G \Sigma_n^k + C^T \Sigma_m C \Sigma_n / 2)^k v = \Sigma_m C \Sigma_n (G \Sigma_n^k)^k v = 0,
\]

for all \( k \geq 0 \), hence \( v \in \ker(C^T) \). The reverse direction is readily obtained.

This strong relationship between the controllability and observability properties follows from the specific structure of the system matrices mentioned in Section II-B. Interestingly, this relationship can be connected to the CCR as follows. Let us define two observables \( \tilde{r}_1 = v^\top \Sigma_n \hat{x} \) and \( \tilde{r}_2 = v^\top \hat{x} \) with \( v \) a normalized vector in \( \ker(C^T) \). Then from \( \{ \} \) we find

\[
\begin{align*}
[\tilde{r}_1, \tilde{r}_2] &= \tilde{r}_1 \tilde{r}_2 - \tilde{r}_2 \tilde{r}_1 = \tilde{r}_1 \tilde{r}_2 - (\tilde{r}_2 \tilde{r}_1)^\top \\
&= v^\top \Sigma_n [\hat{x} \hat{x}^\top - (\hat{x} \hat{x}^\top)^\top] v = v^\top \Sigma_n (i \Sigma_n) v = i.
\end{align*}
\]

This means that, if an observable is free from the input field, then its canonical conjugate must not appear in the output field. Thus, in the linear case, an uncontrollable physical quantity and an unobservable physical quantity are in the relationship of canonical conjugation.

The following theorem is our first main result:

**Theorem 3.1:** The linear system \( O \) and \( \theta \) has a DF subsystem if and only if

\[
\ker(O) \cap \ker(O \Sigma_n) \neq \emptyset. \quad (17)
\]

Then there always exists a matrix \( T_1 \) satisfying \( \text{Range}(T_1) = \ker(O) \cap \ker(O \Sigma_n) \) and \( T_1^\top \Sigma_n T_1 = \Sigma_i \), and then the DF mode is given by \( \hat{x}_\text{DF} = T_1^\top \hat{x} \).

**Proof:** First recall that the condition (i) in Def. 3.1 means \( \text{Range}(C)^c \cap \ker(O) \neq \emptyset \), where the subscript \( c \) denotes the complement of the set. This condition can be equivalently represented by \( \ker(C^\top) \cap \ker(O) \neq \emptyset \), because the controllability and observability properties are invariant under the similarity transformation of the basis vectors. Now, Lemma 3.1 states that any vector satisfying \( O \Sigma_n v = 0 \) fulfills \( C^\top v = 0 \), and vice versa. This means \( \ker(O \Sigma_n) = \ker(C^\top) \), hence, the condition (i) in Def. 3.1 is equivalent to (17).

Next we prove, by a constructive manner, that there always exists a matrix \( T_1 \) satisfying \( \text{Range}(T_1) = \ker(O) \cap \ker(O \Sigma_n) \) and \( T_1^\top \Sigma_n T_1 = \Sigma_i \). First, let us take a vector \( v_1 \in \ker(O) \cap \ker(O \Sigma_n) \), which readily implies that \( \Sigma_n v_1 \) is also contained in this space. Note that \( v_1 \) and \( \Sigma_n v_1 \) are orthogonal. Next we take a vector \( v_2 \in \ker(O) \cap \ker(O \Sigma_n) \) that is orthogonal to both \( v_1 \) and \( \Sigma_n v_1 \). Then, as before, \( \Sigma_n v_2 \) is also an element of this space, and further, it is orthogonal to \( v_1 \), \( \Sigma_n v_1 \), and \( v_2 \). Repeating the same procedure, we construct \( T_1 = (v_1, \Sigma_n v_1, \ldots, v_k, \Sigma_n v_k) \); this matrix satisfies \( T_1^\top T_1 = I \) and \( \Sigma_n T_1 = T_1 \Sigma_i \), which thus yield \( T_1^\top \Sigma_n T_1 = \Sigma_i \). Hence \( \hat{x}_\text{DF} = T_1^\top \hat{x} \) satisfies the CCR \( \{ \} \), implying that (17) leads to the fulfillment of the condition (ii) in Def. 3.1.

Consequently, (17) is an iff condition for the system to have a DF subsystem, and \( \hat{x}_\text{DF} = T_1^\top \hat{x} \) is the DF mode.

As proven above, (17) is not merely the equivalent conversion of the condition (i) in Def. 3.1; it clarifies the fact that, if there exists an uncontrollable and unobservable subspace, it must contain the pair of vectors \( v \) and \( \Sigma_n v \), which directly leads to the concrete procedure for constructing the DF subsystem. Now we understand that, in Def. 3.1, it is not necessary to additionally require the CCR condition (ii); that is, if there exists an uncontrollable and unobservable mode, it automatically satisfies the CCR.

Below we have an explicit form of the dynamics of the DF and D modes, by actually constructing the linear coordinate transformation from \( \hat{x} \) to \( \hat{x}' = (\hat{x}_\text{DF}, \hat{x}_D)^\top \), where \( \hat{x}_D \) is the complement mode to \( \hat{x}_\text{DF} \). First, a vector \( u_1 \) is taken in such a way that it is orthogonal to all the column vectors of \( T_1 \); then \( \Sigma_n u_1 \) is also orthogonal to all the column vectors of \( T_1 \) and \( u_1 \). Second, we take a vector \( u_2 \) that is orthogonal to both \( u_1 \) and \( \Sigma_n u_1 \) in addition to the condition \( u_2^\top T_1 = 0 \). Repeating this procedure, we have \( T_2 = (u_1, \Sigma_n u_1, \ldots, u_{n-\ell}, \Sigma_n u_{n-\ell}) \). Then, \( T = (T_1, T_2) \) is orthogonal, and it satisfies the symplectic condition \( \{ \} \); i.e.,

\[
\begin{pmatrix}
T_1^\top \\
T_2^\top
\end{pmatrix}
(T_1, T_2) = I_{2n},
\]
Lemma 3.1 that $O T_1 = O \Sigma_n T_1 = 0$ is satisfied; this means the system has a DF subsystem with DF mode $T_1^T \hat{x}$.

This proposition further leads to a convenient result in a particular case:

Corollary 3.1: In addition to the assumptions made in Proposition 3.1, suppose $\text{Ker}(C) = \text{Ker}(C \Sigma_n)$. Then, the system specified by $G$ and the same $C$ has a DF subsystem with mode $T_1^T \hat{x}$, if and only if $C G T_1 = 0$.

Proof: As proven above, $T_1^T \hat{x}$ is a DF mode if and only if $C (\Sigma_n G) T_1 = 0$ holds for all $k \geq 0$. Hence, we further equivalently have $G T_1 = T_1 \Sigma_n Q$ and thus $C G T_1 = 0$.

IV. STABILITY OF DF SUBSYSTEM

In this section, we assume that the system’s initial state is Gaussian, hence the state is always Gaussian and is characterized by only the mean vector and the covariance matrix, as explained in Section II.C. Within this framework, we hereby study a stability property of the general linear DF subsystem. Note that in the finite dimensional case some useful results guaranteeing similar stability properties of a DF subsystem have been obtained in [49], [51], [52].

In order to coherently manipulate a quantum state of the DF subsystem, i.e., $\rho_{DF}$, it has to be separated from that of the D subsystem with mode $\hat{x}_D$; actually, if $\hat{x}_{DF}$ is quantally correlated (i.e., entangled) with $\hat{x}_D$, this means that $\rho_{DF}$ can be affected from the environment. This requirement for separability of the DF and the D states is, in Gaussian case, implied by the condition that the covariance matrix $V$ of the whole system is expressed as $V = \text{diag}(V_{DF}, V_D)$, where $V_{DF}$ and $V_D$ correspond to the covariance matrices of $\hat{x}_{DF}$ and $\hat{x}_D$, respectively. Now, from (11), the covariance matrix of the system (18) changes in time with the following Lyapunov differential equation:

$$\frac{dV}{dt} = (A_1 O \ A_2) V + V (A_1^T O \ A_2^T) + (O O \ D_2)$$

where $A_1 = c_1 G_{DF}$, $A_2 = T_2^T A T_2$, and $D_2 = T_2^T \Sigma_n C^T \Sigma_n T_2 / 2$. Hence, if the solution of (21) takes a block diagonal form $V(t) = \text{diag}(V_{DF}(t), V_D(t))$ in a long time limit without respect to the initial value $V(0)$, this means that the DF dynamics is robust in the sense that an unwanted correlation between the DF and the D modes autonomously decreases and finally vanishes. The following theorem provides a convenient criterion for the system to have this desirable property (eig($A$) denotes any eigenvalue of a matrix $A$).

Theorem 4.1: If $\text{Re}[\text{eig}(\Sigma_n G_{DF}) + \text{eig}(A_2)] < 0$ then $A_1$ and if in a long time limit the solution of (21) takes a form $V(t) = \text{diag}(V_{DF}(t), V_D(t))$, where $V_{DF}(t)$ and $V_D(t)$ correspond to the covariance matrices of $\hat{x}_{DF}(t)$ and $\hat{x}_D(t)$, respectively. Moreover, if $G_{DF} \geq 0$, then the above iff condition is simplified to $\text{Re}[\text{eig}(A_2)] < 0$.

Proof: When partitioning the matrix variable $V$ as $V = (V_1, V_2; V_2^T, V_3)$, the dynamics of $V_2$ is given by:

$$dV_2 / dt = A_1 V_2 + V_2^T A_2^T.$$
We now take a vector form of this matrix differential equation. For this purpose, let \( v_2 \) be a collection of the row vectors of \( V_2 \), which is a real \(4f(n-f)\)-dimensional row vector. Then \( v_2 \) obeys the dynamics of the form \( dv_2/dt = A_0v_2 \) where \( A = A_1 \otimes I_{2(2n-f)} + I_{2f} \otimes A_2 \). Now note that any eigenvector of \( A \) equals \( \text{eig}(A_1) + \text{eig}(A_2) \). Therefore, we have \( v_2 \to 0 \) as \( t \to \infty \), if and only if the real parts of sum of any eigenvalues of \( A_1 = \Sigma_i G_{DF} \) and \( A_2 \) are strictly negative.

To show the second part of the theorem, let us consider the eigen-equation \( \Sigma_i G_{DF} g = \lambda g \) with \( g \) the eigenvector and \( \lambda \) the corresponding eigenvalue. From this we have \( g^\dagger (G_{DF} \Sigma_i G_{DF}) g = \lambda g^\dagger G_{DF} g \), which immediately leads to \( (\lambda + \lambda^*) g^\dagger G_{DF} g = 0 \). If \( G_{DF} \gtrsim 0 \), this can be further expressed as \( (\lambda + \lambda^*) ||G_{DF}g||^2 = 0 \), implying \( \lambda + \lambda^* = 0 \) or \( G_{DF} g = 0 \); in both cases we have \( \Re(\text{eig}(\Sigma_i G_{DF})) = 0 \).

We remark that the specific assumption \( G_{DF} > 0 \) is significant from the viewpoint of state manipulation; recall that the unitary gate operation with a quadratic Hamiltonian \( \hat{H} = \hat{x}^2 \hat{g}^2/2 \) is given by \( \hat{U}(t) = \exp(-i\hat{x}^2 \hat{g}^2/2t) \). Then, it was shown in \( \text{(22)} \) that, if \( G > 0 \), for any \( \epsilon > 0 \) and \( \tau_2 > 0 \), there always exists \( \tau_2 > 0 \) such that \( ||\hat{U}(\tau_1) - \hat{U}(\tau_2)|| < \epsilon \) with \( \| \cdot \| \) an appropriate operator norm. This result means that a unitary operation requiring propagation with negative time can be always realized by a physically realizable unitary operation with positive time; this condition is indeed necessary to implement a desirable unitary operation via a sequence of unitaries generated from a set of available Hamiltonians.

V. COHERENT STATE MANIPULATION AND PRESERVATION ON A DF SUBSYSTEM

Coherent state manipulation and preservation are certainly primary applications of a DF subsystem, as demonstrated extensively in the finite dimensional case. This section presents two simple examples to show how a DF mode can be used in order to achieve these goals in an infinite dimensional linear quantum system.

A. Particles trapped via dissipative coupling

Let us consider a toy system composed of two “particles” trapped in a two-sided optical cavity depicted in Fig. 1. The coupling operator between the particles and the outer environment field is \( \hat{L} = \sqrt{\kappa}(\hat{b}_1 + \hat{b}_2) \), where \( \hat{b}_i = (\hat{q}_i + i\hat{p}_i)/\sqrt{2} \). \( \hat{q}_i \) and \( \hat{p}_i \) represent the position and momentum operators of the \( i \)-th particle. This dissipation effect described by \( \hat{L} \) stems from the coupling of the particles to the single-mode cavity field that immediately decays and can be adiabatically eliminated \( \text{(37)} \); the parameter \( \kappa \) represents the decay rate of this dissipation. For more realistic setups, see \( \text{(16), (26), (58)} \). The system dynamics \( \text{(8)} \) and the output equation \( \text{(9)} \) are now given by

\[ d\hat{x} = A\hat{x}dt + Bd\hat{W}, \quad d\hat{W}^{\text{out}} = C\hat{x}dt + d\hat{W}, \]

where

\[ A = \Sigma G - \frac{\kappa}{2} \left( \begin{array}{ccc} I_2 & I_2 & I_2 \\ I_2 & I_2 & I_2 \\ I_2 & I_2 & I_2 \end{array} \right), \quad B = -\sqrt{\kappa} \left( \begin{array}{c} I_2 \\ O \\ I_2 \end{array} \right), \]

\[ C = \sqrt{\kappa} \left( \begin{array}{c} I_2 \\ O \\ I_2 \end{array} \right). \]

We begin with the assumption that the system does not have its own Hamiltonian (i.e., \( G = 0 \)) and then try to find a DF subsystem. In this case, we have

\[ O = C = \sqrt{\kappa} \left( \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array} \right), \]

\[ O\Sigma C = \sqrt{\kappa} \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right). \]

Therefore, \( v_1 = (1, 0, -1, 0)^T/\sqrt{2} \) and \( \Sigma v_1 = (0, 1, 0, -1)^T/\sqrt{2} \) span the kernels of both of these matrices; i.e., \( \text{span}\{v_1, \Sigma v_1\} = \text{Ker}(O) = \text{Ker}(O\Sigma C) \). Hence, from Theorem \( \text{(5.1)} \), the system has a DF subsystem. In particular, the procedure shown in the proof of Theorem \( \text{(5.1)} \) yields \( T_1 = (v_1, \Sigma v_1) \), thus the DF mode is given by

\[ \hat{x}^{\text{DF}} = T_1^\dagger \hat{x} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \hat{q}_1 - \hat{q}_2 \\ \hat{p}_1 - \hat{p}_2 \end{array} \right). \]

Here let us discuss adding a nonzero matrix \( G \) that does not change the above-obtained DF subsystem. Now \( \text{Ker}(O) = \text{Ker}(O\Sigma C) \), hence from Corollary \( \text{(5.1)} \), \( G \) is required to satisfy \( CGT_1 = 0 \). Then it is immediate to see that \( G \) has to be of the form \( G = (G_1, G_2; G_2, G_1) \) where \( G_1 \) and \( G_2 \) are both \( 2 \times 2 \) symmetric matrices. Therefore, the DF subsystem is driven by the Hamiltonian \( \text{(20)} \) with \( G_{DF} = T_1^\dagger GT_1 \). This means that, if each particle is governed by the Hamiltonian specified by \( G_1 \) and this Hamiltonian can be arbitrarily chosen, we can perform arbitrary linear gate operation on the state of \( \hat{x}^{\text{DF}} \). Also, \( G_{DF} = G_1 - G_2 = O \), we can preserve any state of the DF subsystem.

Next, to evaluate the stability of the DF subsystem, let us explicitly construct the dynamics of both \( \hat{x}^{\text{DF}} \) and \( \hat{x}^{\text{DF}} \), according to the procedure shown above \( \text{(18)} \). To obtain the matrix \( T_2 \), we chose \( u_1 = (1, 0, 1, 0)^T/\sqrt{2} \) as a basis vector orthogonal to \( v_1 \) and \( \Sigma v_1 \). Then \( \Sigma v_1 = (0, 1, 0, 1)^T/\sqrt{2} \) is orthogonal to \( v_1, \Sigma v_1, \) and \( u_1 \), implying \( T_2 = (u_1, \Sigma u_1) \). Hence, the linear coordinate transformation is obtained as

\[ T = (T_1, T_2), \quad T_1 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} I_2 \\ -I_2 \end{array} \right), \quad T_2 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} I_2 \\ I_2 \end{array} \right). \]

The transformed variable \( \hat{x}' = T^\dagger \hat{x} \) then satisfies

\[ d\hat{x}' = A'\hat{x}'dt + B'd\hat{W}, \quad d\hat{W}^{\text{out}} = C'\hat{x}'dt + d\hat{W}, \]

where

\[ A' = \left( \begin{array}{cc} \Sigma(G_1 - G_2) \\ O \end{array} \right), \quad B' = -\sqrt{\kappa} \left( \begin{array}{c} O \\ I_2 \end{array} \right), \quad C' = \sqrt{\kappa}(O, I_2). \]

As expected, the first CCR pair of \( \hat{x}' \), i.e., \( \hat{x}_{DF} \), is neither affected by the incoming noise field nor observed from the output field. Now assume that the state is Gaussian. Then, by
taking large $\kappa$ so that the condition of Theorem 4.4 is satisfied, we can guarantee the stability of the DF subsystem. That is, the off-diagonal block matrix of the covariance matrix $V' = (\Delta\hat{x}'D + (\Delta\hat{x}'D)^\top)/2$ converges to zero as $t \to \infty$, thus in the long time limit the states of DF and D modes are always separated. Hence, with this Gaussian system, arbitrary linear gate operation and state preservation can be carried out in a robust manner, under relatively mild assumptions.

Lastly, under the assumption that the state is Gaussian and the condition $G_{DF} = G_1 - G_2 = 0$, let us examine the state preserved in the whole system. In fact, in this case $\hat{x}_{DF}$ does not change in time, and hence the whole state of $\hat{x}$ is preserved if $\hat{x}_D$ is in a steady state. This condition is achieved by again taking large $\kappa$ so that the coefficient matrix of the drift term of $\hat{x}_D$ is Hurwitz; in this case, $\hat{x}_D$ has a steady Gaussian state with covariance matrix $I_2/2$. As a result, the covariance matrix of the whole state of $\hat{x}'$ is $V'(0) = \text{diag}\{V_{DF}(0), I_2/2\}$, where here the initial time is reset to $t = 0$. The covariance matrix of $\hat{x}$ is then given by

$$V(0) = TV'(0)T^\top = \frac{1}{2} \begin{pmatrix} I_2/2 + V_{DF}(0) & I_2/2 - V_{DF}(0) \\ I_2/2 - V_{DF}(0) & I_2/2 + V_{DF}(0) \end{pmatrix}.$$ 

If the initial state of $\hat{x}_{DF}$ can be set to a squeezed state [12], [20], [21], [56], [61] with covariance matrix $V_{DF}(0) = \text{diag}\{e^{r/2}, e^{-r/2}\}$, then $V(0)$ takes the form

$$V(0) = \frac{1}{4} \begin{pmatrix} 1 + e^r & 1 - e^r & 1 - e^{-r} & 1 - e^{-r} \\ 1 - e^r & 1 + e^r & 1 + e^{-r} & 1 + e^{-r} \\ 1 - e^{-r} & 1 + e^r & 1 + e^{-r} & 1 + e^{-r} \\ 1 - e^{-r} & 1 + e^{-r} & 1 + e^{-r} & 1 + e^{-r} \end{pmatrix}.$$ (23)

As mentioned above, this state does not change in time. Thus the state preserved is a pure entangled Gaussian state called the two-mode squeezed state [12], [20]; actually, the purity for this state is $P = 1/\sqrt{16\det(V(0))} = 1$, and the logarithmic negativity [54], which is a convenient measure for entanglement, is $E_N = r/2 > 0$. This result was found in a specific setup [47], but here we have the following general statement; in general, preservation of a nontrivial state of a DF subsystem can lead to that of an entangled state of the whole system, if the stability of the DF mode is guaranteed. In other words, a system having a stable DF mode can be utilized for preserving an entangled state among the whole system, as well as a state of the DF subsystem.

**B. Particles trapped via dispersive coupling**

The second example shown here deals with the two particles trapped in an optical cavity as before, but with different coupling $\hat{L} = \sqrt{\kappa}/2(\hat{q}_1 + \hat{q}_2)$, where again $\hat{q}_i$ is the position operator of the $i$-th particle. This coupling realizes a continuous-time measurement of $\hat{q}_1 + \hat{q}_2$; see e.g. [14], [15] for how to actually construct this kind of position monitoring scheme. Now $c = \sqrt{\kappa}/2(1, 0, 1, 0)$, hence, when $G = 0$, we have

$$\mathcal{O} = \sqrt{\kappa} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega \Sigma_2 = \sqrt{\kappa} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and thus

$$\text{Ker}(\mathcal{O}) = \text{span}\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \text{Ker}(\Omega \Sigma_2) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$ 

The intersection of these two spaces is the same as that obtained in the previous example, thus we have the same coordinate transformation matrix [22]. However, the dynamical equation of $\hat{x}' = T^\top \hat{x}$ differs from the previous one:

$$d\hat{x}' = A' \hat{x}' dt + B' d\hat{\mathbf{W}}, \quad d\hat{\mathbf{W}} = C' \hat{x}' dt + d\hat{\mathbf{W}},$$

where

$$A' = \begin{pmatrix} \Sigma(G_1 - G_2) \\ O \\ \Sigma(G_1 + G_2) \end{pmatrix}, \quad B' = -\sqrt{2\kappa}(O, E_{22}), \quad C' = \sqrt{2\kappa}(O, E_{11}).$$

Here $E_{11} = \text{diag}\{1, 0\}$ and $E_{22} = \text{diag}\{0, 1\}$. Note that $q_0' = \sqrt{q_1^2 + q_2^2}/\sqrt{2}$ is not affected by the noisy environment while it appears in the output equation. This means that we can extract information about $q_0'$ without disturbing it; that is, $q_0'$ is a quantum non-demolition (QND) observable (see e.g. [61]).

Let us now discuss the state manipulation and preservation with this system. As in the previous example, if two particles have the same and arbitrary Hamiltonian specified by $G_1$, this means that any linear gate operation on the DF mode is possible. However, this system does not satisfy the condition of Theorem 4.4 implying that the system does not have capability of decoupling the DF and the D modes once they become correlated. In this sense, the state manipulation with this DF mode is not robust. Next, to see if the system can preserve a state of the whole system, let us again set the initial state to the Gaussian state with covariance matrix [23]. A notable difference from the previous case is that now the D mode does not have a steady state, and eventually the whole state is a time-varying one with covariance matrix

$$V(0) \to V(t) = V(0) + \frac{1}{4} \begin{pmatrix} \kappa t E_{22} & \kappa t E_{22} \\ \kappa t E_{22} & \kappa t E_{22} \end{pmatrix}.$$ 

The purity of this Gaussian state is $P = 1/\sqrt{16\det(V(t))} = 1/\sqrt{1 + \kappa t}$ and thus converges to zero as $t \to \infty$. That is, any useful state does not remain alive in the long time limit. As a result, due to the lack of the stability property, this system offers neither desirable state manipulation nor preservation.

**VI. ENGINEERING A DF LINEAR SUBSYSTEM**

This section deals with the topic of how to engineer a DF subsystem, when a certain linear open system is given to us. More precisely, the problem is not how to find a DF component in that system, but rather how to synthesize an auxiliary system so that the extended system has a desirable DF mode.

This engineering problem is important for purely exploring the nature of a system of interest, as well as for the application
to quantum information science such as coherent state manipulation and preservation as discussed in the previous section. In fact, any quantum system is essentially an open system and its properties must be perturbed by surrounding environment. Therefore, constructing a subsystem that is exactly isolated from environment is of vital importance.

A. Opto-mechanical oscillator

The first example is an opto-mechanical oscillator depicted in Fig. 2(a); see e.g. [36] for a more detailed description. The system is composed of a two-sided optical cavity where one of the mirrors is movable and acts as an oscillator. Let \( \hat{q}_1 \) and \( \hat{p}_1 \) be the oscillator’s position and momentum operators, respectively. Also define \( \hat{q}_2 = (\hat{b}_2 + \hat{b}_2^\dagger)^1/2 \) and \( \hat{p}_2 = (\hat{b}_2 - \hat{b}_2^\dagger)/\sqrt{2} \) with \( \hat{b}_2 \) the annihilation operator of the cavity mode. The oscillator is driven by the free Hamiltonian \( \hat{H}_0 = \hbar^2 \hat{q}_1^2 + \hat{p}_1^2 / m \) where \( m \) and \( \omega \) represent the mass and the resonant frequency of the oscillator, and further, it is coupled to the cavity mode through the radiation pressure force. The cavity mode evolves according to the Hamiltonian \( \hat{H}_\text{rp} = \epsilon \hat{q}_2 \hat{b}_2^\dagger \hat{b}_2 \); consequently the whole system Hamiltonian is given by \( \hat{H} = \hat{H}_0 + \hat{H}_\text{rp} \). The cavity mode couples to an outer coherent light field at the other mirror in a standard dissipative manner, corresponding to \( \hat{L} = \sqrt{\kappa} \hat{b}_2 \). As a result, the linearized dynamics of the whole oscillator-cavity system is given by

\[
d\hat{x} = A \hat{x} dt + B d\hat{W}, \quad d\hat{W}^\text{out} = C \hat{x} dt + d\hat{W},
\]

where

\[
A = \begin{pmatrix}
0 & 1/m & 0 & 0 \\
-\hbar^2 \omega^2 & 0 & \gamma & 0 \\
0 & 0 & 0 & -\kappa \\
\gamma & 0 & 0 & -\kappa
\end{pmatrix}, \\
B = -\sqrt{2\kappa} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad C = -B^\dagger.
\]

Here \( \gamma \) denotes the radiation pressure strength proportional to \( \epsilon \) and \( \kappa \) denotes the decay rate of the cavity field.

It is immediate to see that \( \mathcal{O} \) is of full rank, and thus the system does not have a DF subsystem. Hence, as posed in the beginning of this section, we try to engineer an appropriate extended system so that it contains a DF subsystem. In particular, we follow the idea of Tsang [53]; an auxiliary system with single mode \( (\hat{q}_3, \hat{p}_3) \) is prepared and directly coupled to the cavity mode through the interaction Hamiltonian \( \hat{H}_\text{int} = -g\hat{q}_2\hat{q}_3 \) with \( g \) the coupling strength. Then the extended system of variable \( \hat{x}_e = (\hat{x}, \hat{q}_3, \hat{p}_3)^\top \) is given by

\[
d\hat{x}_e = A_e \hat{x}_e dt + B_e d\hat{W}, \quad d\hat{W}^\text{out} = C_e \hat{x}_e dt + d\hat{W},
\]

where

\[
A_e = \begin{pmatrix} A & \begin{pmatrix} 0 & 0 \\
0 & 0 \\
0 & 0 & g & 0 \\
0 & 0 & 0 & \mu & 0
\end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu & 0
\end{pmatrix} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
B_e = \begin{pmatrix} B \\
\begin{pmatrix} 0 \end{pmatrix} \end{pmatrix}, \quad C_e = -B^\dagger.
\]

Here the auxiliary system is assumed to have the Hamiltonian \( -\nu \hat{q}_3^2 + \mu \hat{p}_3^2 / 2 \). Also \( O_2 \) denotes the \( 2 \times 2 \) zero matrix. The problem is to find a set of parameters \( (g, \mu, \nu) \) such that the above extended system has a DF component. First, we need that the observability matrix \( \mathcal{O} \) composed of \( A_e \) and \( C_e \) is not of full rank; this requirement yields \( \mu \nu = -\omega^2 \). Next, the iff condition given in Theorem 3.1 imposes us to have \( \mu = 1 / m \). As a result, the parameters must satisfy \( \mu = 1 / m \) and \( \nu = -\hbar \omega^2 \), implying that the Hamiltonian of the auxiliary system is exactly the same as that of the original oscillator. Although we can fabricate this auxiliary system using only some optical devices, a natural situation is that another opto-mechanical oscillator with the same mass and resonant frequency serves as the auxiliary system. The resulting extended system is shown in Fig. 2(b), which has been investigated recently in [20] in the context of quantum memory. Note that the coupling strength \( g \) can be chosen arbitrarily; this strikingly differs from the case [53] where \( g \) is also required to satisfy a certain condition.

The dynamical equations of the DF and the D modes are extended system of variable

\[
\hat{x}_e = (\hat{x}, \hat{q}_3, \hat{p}_3)^\top \text{ is given by}
\]

\[
d\hat{x}_e = A_e \hat{x}_e dt + B_e d\hat{W}, \quad d\hat{W}^\text{out} = C_e \hat{x}_e dt + d\hat{W},
\]

where

\[
A_e = \begin{pmatrix} A & \begin{pmatrix} 0 & 0 \\
0 & 0 \\
0 & 0 & g & 0 \\
0 & 0 & 0 & \mu & 0
\end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu & 0
\end{pmatrix} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
B_e = \begin{pmatrix} B \\
\begin{pmatrix} 0 \end{pmatrix} \end{pmatrix}, \quad C_e = -B^\dagger.
\]

Here the auxiliary system is assumed to have the Hamiltonian \( -\nu \hat{q}_3^2 + \mu \hat{p}_3^2 / 2 \). Also \( O_2 \) denotes the \( 2 \times 2 \) zero matrix. The problem is to find a set of parameters \( (g, \mu, \nu) \) such that the above extended system has a DF component. First, we need that the observability matrix \( \mathcal{O} \) composed of \( A_e \) and \( C_e \) is not of full rank; this requirement yields \( \mu \nu = -\omega^2 \). Next, the iff condition given in Theorem 3.1 imposes us to have \( \mu = 1 / m \). As a result, the parameters must satisfy \( \mu = 1 / m \) and \( \nu = -\hbar \omega^2 \), implying that the Hamiltonian of the auxiliary system is exactly the same as that of the original oscillator. Although we can fabricate this auxiliary system using only some optical devices, a natural situation is that another opto-mechanical oscillator with the same mass and resonant frequency serves as the auxiliary system. The resulting extended system is shown in Fig. 2(b), which has been investigated recently in [20] in the context of quantum memory. Note that the coupling strength \( g \) can be chosen arbitrarily; this strikingly differs from the case [53] where \( g \) is also required to satisfy a certain condition.

The dynamical equations of the DF and the D modes are explicitly obtained as follows. With the parameters determined above, the procedure shown in Section III yields the linear coordinate transformation matrix \( T = (T_1, T_2) \) as

\[
T_1 = \frac{1}{\gamma'} \begin{pmatrix} gI_2 \\
-\nu I_2 \end{pmatrix}, \quad T_2 = \frac{1}{\gamma'} \begin{pmatrix} \gamma I_2 \\
gI_2 \end{pmatrix},
\]

\[1 \text{ The problem considered in [53] is to find the parameter set } (g, \mu, \nu) \text{ so that the input noise field } \hat{Q}(t) \text{ does not appear in the output field } \hat{P}^\text{out}(t), \text{ for the purpose of enhancing the sensitivity to the incoming unknown force acting on the oscillator.} \]
where $\gamma' = \sqrt{\gamma^2 + g^2}$. Hence, the DF mode is

$$\dot{x}_{DF} = T_1^T \dot{x}_e = \frac{1}{\gamma'} \begin{pmatrix} g \dot{q}_1 - \gamma \dot{q}_3 \\ g \dot{p}_1 - \gamma \dot{p}_3 \end{pmatrix},$$

and the dynamics of $\dot{x}'_e = T^T \dot{x}_e = (\dot{x}_{DF}^T, \dot{x}_B^T)^T$ is given by

$$d\dot{x}'_e = A'_e \dot{x}'_e dt + B'_e d\hat{W}, \quad d\hat{W} = C'_e \dot{x}_e dt + d\hat{W},$$

where

$$A'_e = \begin{pmatrix} 0 & 1/m \\ -m \omega^2 & 0 \end{pmatrix}, \quad B'_e = \begin{pmatrix} \frac{O_2}{B} \\ \frac{1}{B} \end{pmatrix}, \quad C'_e = -B'_e^T.$$

Here now the parameter $\gamma$ in $A$ is replaced by $\gamma' = \sqrt{\gamma^2 + g^2}$. This equation shows that the relative coordinate of the oscillators is decoupled from the cavity field; actually, if the radiation pressure force pushes the oscillators along the same direction, then their relative coordinate is independent of the optical path length of the cavity, so it should be decoherence free. Note that, of course, this can happen only when the two oscillators are identical. A notable point is that the DF subsystem is governed by the same Hamiltonian as that of the original oscillator. This means that an ideal closed optomechanical oscillator can be in fact implemented in a realistic open system.

### B. Trapped particles

Let us again consider the system of two particles discussed in Section 4-A, which is now additionally subjected to the following Hamiltonian:

$$\hat{H} = \frac{1}{2} \sum_{j=1,2} \left( \frac{1}{m_j} \dot{p}_j^2 + m_j \omega_j^2 \dot{q}_j^2 \right) + \frac{1}{2} k (\dot{q}_1 - \dot{q}_2)^2, \tag{24}$$

where $m$ and $\omega$ are the mass and the resonant frequency of both of the particles. The particles interact with each other through a harmonic potential with strength $k$, as indicated in the second term of (24). The corresponding $G$ matrix is of the form $G = (G_1, G_2; G_2, G_1)$, thus from the result obtained in Section 4-A this system has a DF component. However, this single-mode DF subsystem cannot, of course, simulate the behavior of the closed two-mode system driven by the Hamiltonian (24). Therefore, let us try to engineer an extended system having a two-mode DF subsystem in which the closed dynamics driven by the Hamiltonian (24) is realized. That is, we want to simulate the closed system with Hamiltonian (24), in an open quantum system.

Let us consider an extended system composed of three particles trapped in a ring-type cavity depicted in Fig. 3. The Hamiltonian is assumed to be

$$\hat{H}_e = \hat{H} + \frac{1}{2} \left( \frac{1}{m_2} \dot{p}_3^2 + m_2 \omega^2 \dot{q}_3^2 \right) + \frac{1}{2} k_2 (\dot{q}_2 - \dot{q}_3)^2 + \frac{1}{2} k_3 (\dot{q}_3 - \dot{q}_1)^2,$$

where $k_j$ denotes the coupling strength between the particles and $\omega$ the resonant frequency of the auxiliary particle; these parameters will be appropriately chosen later. The particles couple to the common single-mode cavity field that can be adiabatically eliminated, which as a result leads to the system-bath interaction described by $L_e = \sqrt{\kappa} (b_1 + b_2 + b_3)$ with $b_i = (\dot{q}_i + i \dot{p}_i)/\sqrt{2}$. The system matrices in this case are given by

$$G = \begin{pmatrix} G_q & 0 \\ 0 & I_3 \end{pmatrix}, \quad G_q = \begin{pmatrix} \omega^2 + k + k_3 & -k & -k_3 \\ -k & \omega^2 + k + k_2 & -k_2 \\ -k_3 & -k_2 & \omega^2 + k_2 + k_3 \end{pmatrix},$$

$$c = \sqrt{\kappa/2(1,1,1,i,i,i)}.$$ Here, for simple notation, the matrices are represented in the basis of $\hat{x} = (\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{p}_1, \hat{p}_2, \hat{p}_3)^T$; this notation is used throughout this subsection. Also, we have assumed $m = 1$ without loss of generality.

If $G = O$, the linear coordinate transformation matrix $T = (T_1, T_2)$ with

$$T_1 = \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix},$$

$$T_2 = \begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix},$$

yields the DF mode $\hat{x}_{DF} = T_1^T \hat{x}$ and the D mode $\hat{x}_D = T_2^T \hat{x}$. Here the goal is to find a condition of $G$ so that the above $\hat{x}_{DF}$ is still a DF mode even when the Hamiltonian specified by $G$ is added. Now Corollary B.1 is applicable, hence $G$ must satisfy $G T_1 = 0$, and this yields $\omega' = \omega$. In this case, it can be actually verified that $\hat{x}_{DF}$ is the DF mode driven by the Hamiltonian $\hat{H}_{DF} = \hat{x}_{DF}^T G_{DF} \hat{x}_{DF} / 2$ where

$$G_{DF} = T_1 G T_1^T = \begin{pmatrix} \omega^2 + 2k + (k_2 + k_3)/2 & \sqrt{3}(k_2 - k_3)/2 \\ \sqrt{3}(k_2 - k_3)/2 & \omega^2 + 2(k_2 + k_3)/2 \end{pmatrix} \oplus I_2.$$

We are particularly interested in the question whether the coupled particles with Hamiltonian (24) can be simulated with the isolated DF mode $\hat{x}_{DF} = T_1^T \hat{x}$. Then, because the two particles are identical, the condition $2k + (k_2 + k_3)/2 = 3(k_2 + k_3)/2$ is necessary. By choosing the coupling strength as $k_2 = \sqrt{3} k_3$ and $k_3 = (2 - \sqrt{3}) k$, we end up with the expression

$$G_{DF} = \begin{pmatrix} \omega^2 + \sqrt{3} k + (3 - \sqrt{3}) k & \omega^2 - (3 - \sqrt{3}) k \\ -(3 - \sqrt{3}) k & \omega^2 + \sqrt{3} k + (3 - \sqrt{3}) k \end{pmatrix} \oplus I_2.$$

Fig. 3. Three particles trapped in a ring-type cavity.
The corresponding Hamiltonian driving the DF mode is thus given by
\[ \hat{H}_{DF} = \frac{1}{2} \sum_{j=1,2} \left[ \hat{p}_j^2 + (\omega^2 + \sqrt{3}k)\hat{q}_j^2 \right] + \frac{1}{2} (3 - \sqrt{3})k (\hat{q}_1^2 - \hat{q}_2^2), \]
where now \( \hat{x}_{DF} = (\hat{q}_1, \hat{q}_2, \hat{p}_1, \hat{p}_2)^T \). Recall that \( m = 1 \) is assumed. We can further prove that \( G_{DF} > 0 \) and also that \( A_2 = T^*_1 AT_2 \) is Hurwitz. Therefore, from Theorem 4.1, the DF mode with the above Hamiltonian is stable. Summarizing, we have constructed a stable DF subsystem of two-mode coupled particles whose Hamiltonian has the same structure as that of the Hamiltonian of the original open system.

VII. Conclusion

In this paper, we have developed a basic theory of general linear DF subsystems, shown some applications, and demonstrated how to actually construct a DF subsystem. The theory contains a general iff condition for an open linear system to have a DF component, to emphasize the usefulness of the general results obtained in the former part of the paper. Several applications of linear DF subsystems, in this paper we have studied only very basic coherent manipulation and preservation of the state of a single-mode DF subsystem. What was additionally focused on is the idea of synthesis of an open linear quantum system having a desirable DF component, to emphasize the usefulness of the general results obtained in the former part of the paper. Application to quantum memory is one such example [64].

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Appendix

This appendix gives some fundamentals of quantum mechanics. In quantum mechanics, any physical quantity takes a random value when measuring it, hence it is essentially a random variable and is called an observable. A biggest difference between quantum observables and classical (i.e., non-quantum) random variables is that in general the former cannot be measured simultaneously; mathematically, this fact is represented by the non-commutativity of two observables \( \hat{A} \) and \( \hat{B} \), i.e., \( [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0 \). This implies that an observable should be expressed by a matrix (for a finite dimensional system) or an operator (for an infinite dimensional system), which is further required to be Hermitian for the realization value to be real. Accordingly, in quantum mechanics, a probabilistic distribution is also represented by a matrix or an operator, and it is called a state. A state \( \hat{\rho} \) has to satisfy \( \hat{\rho} = \rho^* \geq 0 \) and \( Tr(\hat{\rho}) = 1 \). When a system of interest is in a state \( \hat{\rho} \) and we measure an observable \( \hat{A} \), then the measurement result is randomly distributed with mean \( \langle \hat{A} \rangle = Tr(\hat{A}\hat{\rho}) \).

In an ideal situation, the equation of motion of an observable \( \hat{A}(t) \), which is called the Heisenberg equation, is given by \( d\hat{A}(t)/dt = i[\hat{H}, \hat{A}(t)] \), where \( \hat{H} \) is an observable called Hamiltonian. (For simplicity, \( \hat{H} \) is now assumed to be time-independent.) The solution of this equation is \( \hat{A}(t) = U^\dagger(t)\hat{A}(0)U(t) \) with \( U(t) = e^{-i\hat{H}t} \) unitary, hence it is a unitary evolution of \( \hat{A}(0) \). The mean value of the measurement results of \( \hat{A}(t) \) at time \( t \) is given by \( \langle \hat{A}(t) \rangle = Tr(\hat{A}(t)\hat{\rho}) \). Note that the same statistics is obtained if we take a picture that the observable is fixed to \( \hat{A} = \hat{A}(0) \) and the state evolves in time as \( d\hat{\rho}(t)/dt = -i[\hat{H}, \hat{\rho}(t)] \); actually we then have \( \langle \hat{A}(t) \rangle = Tr(\hat{A}(t)\hat{\rho}) = Tr(\hat{A}\hat{\rho}(t)) \). This is called the Schrödinger picture, while in the Heisenberg picture we fix the state and deal with an observable as a dynamical one.

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