AVERAGE OF GEOMETRIC STRUCTURES IN FINSLER SPACES WITH LORENTZIAN SIGNATURE

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Abstract. Given the class of Finsler spaces with Lorentzian signature \((M, L)\) on a manifold \(M\) endowed with a timelike vector field \(X\) satisfying \(g_{(x,y)}(X,X) < 0\) at any point \((x,y)\) of the slit tangent bundle, a metric defined on \(M\) of signature \(n-1\) is associated to the fundamental tensor \(g\).

Furthermore, an affine, torsion free connection is associated to the Chern connection determined by \(L\). The definition of the average connection does not make use of \(X\). Therefore, there is no direct relation between these two averaged objects.

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1. Introduction

Positive definite Finsler spaces are natural generalizations of Riemannian spaces such that the associated norm function the typical Riemannian assumption of quadratic dependence on the coordinate velocities is dropped [6]. The theory of positive definite Finsler spaces has been developed in the form of sound geometric theories that reduce to the Riemannian theory if the above quadratic restriction in the norm is imposed [4, 3, 5] and where fundamental results from Riemannian geometry are conveniently generalized to the Finslerian case [1].

For positive definite signatures, the theory of average Finsler structures provides a map from the Finsler category to the Riemannian or affine categories, depending on the object considered, by conveniently tracing out the \(y\)-dependence of relevant Finslerian geometric object [10]. Other theories of averaging Finsler metrics have appeared in the literature. In the particular context of Szavo’s metrizability theorem [15], a theory of averaging the fundamental tensor of a Finsler structure was discussed in the work of Vincze in [16]. In a different formulation, another theory of averaging the Finsler metric is fundamental for the methods applied by Matveev and Troyanov in [12].

The sister theory of Finsler spaces with Lorentzian structures appears naturally in the form of relativistic Finslerian generalizations of Lorentzian spacetimes [9]. In contrast with the positive case, for Lorentzian signature

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there is no currently an unique theoretical paradigm were all the questions and specific models found in the literature can be formulated. Indeed, several theories of Finsler structures with Lorentzian or indefinite metrics have been developed in the literature (see [14] for a review of different recent approaches), the main problems related with the appearance of singularities for the metrics and the impossibility to consider within the same formalism relevant particular cases.

In this context, one can consider how to generalize the averaging operation to Finsler spaces with Lorentzian signature. Due to the naturally of the methods in the positive definite case and the interpretation of coarse grained operation in the Lorentzian signature case, this problem is of interest from the point of view of differential geometry and also for its applications in extensions of general relativity. However, the direct adaptation of the procedure as in positive Finslerian geometry leads to singularities of the averaged metric where in principle, the original Finsler metric was regular, a problem that spoils the construction of significant connections from the average metric.

In this work we consider how to define a meaningful averaging operation for the fundamental tensor and for the Chern connection determined by a Finsler structure with Lorentzian signature. This question is investigated in the framework of J. Beem theory of Finsler spaces with indefinite [2] of signature $n - 1$. For this purpose, we have restricted the attention to metric structures allowing the existence of a vector field $X$ timelike respect to any direction of the tangent space (see the relation (3.1)). Such condition is stronger than the condition of time orientation in the Finslerian sense [8], but can be motivated physically as we will discuss below. To define the average of the Chern connection the method is more direct, but it faces the problem of the non-compactness of the unit hyperboloid. Besides this complication, the formulation obtained in the Finslerian case is a direct generalization of the positive definite case.

The structure of this work is the following. In section 2 we introduce the necessary notions of Finsler geometry, including the averaging method as discussed in reference [10]. The fundamental concepts of Finsler spaces with Lorentzian signature that we will need are also introduced, according to [2] [8]. In section 3, we introduce our definition of average Finsler metric for structures of Lorentzian signature. Basically, the method consists to associate to the initial Finsler structure with Lorentzian signature a positive definite Lagrange space structure [13], perform the averaging operation for such positive structure and then obtain a metric of signature $n - 1$ defined over $M$ by the method discussed in conventional Lorentzian geometry (see for instance [11], section 2.6). It is only in the first step of this process that the condition (3.1) on $X$ is necessary. Section 4 deals with the average of the Chern connection. Differently than in the positive case, in Lorentzian
signature the non-compactness of the unit hyperboloid introduces an extra-difficulty. Two ways to overcome this difficulty are briefly discussed.

2. Geometric framework

2.1. Geometric framework for average of Finsler metrics and connections. We introduce the average operation of Finsler metrics and connections following [10]. Let \( M \) be a differentiable manifold of dimension \( n = \text{dim} M \) and \( TM \) the tangent bundle of \( M \). Local coordinates \((U, x)\) on \( M \) induce local natural coordinates \((TU, x', y')\) on \( TM \). The slit tangent bundle is \( N = TM \setminus \{0\} \), where 0 is the zero section of \( TM \); \( N_x \subset N \) is the fiber over \( x \in M \). \( \pi^*TM \to N \) will denote the pull-back bundle.

**Definition 2.1.** A Finsler space on the manifold \( M \) is a pair \((M, F)\) where \( F \) is a non-negative, real function \( F : M \to [0, \infty[ \) such that

- It is smooth in the slit tangent bundle \( N \),
- Positive homogeneity holds: \( F(x, \lambda y) = \lambda F(x, y) \) for every \( \lambda \in [0, +\infty[ \),
- Strong convexity holds: the Hessian matrix

\[
 g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad i, j = 1, \ldots, n
\]

is positive definite on \( N \).

The \( n \)-form \( d^n y \) is defined in local coordinates by the expression

\[
d^n y = \sqrt{|\det g(x, y)|} \delta y^1 \wedge \cdots \wedge \delta y^n,
\]

where \( \det g(x, y) \) is the determinant of the fundamental tensor

\[
 (g)_{ij} = g_{ij}(x, y), \quad i, j = 1, \ldots, n.
\]

and \( \{\delta y^1, \ldots, \delta y^n\} \) is a local dual frame to the holonomic frame \( \{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\} \). \( \{\delta y^1, \ldots, \delta y^n\} \) is constructed using the Finsler norm \( F \). The details can be found in [1] for the positive case and in [8] for the case of signature \( n - 1 \). Note that in [1], there are certain factors powers on \( F \) up or down in the definition of the above quantities respect to our definitions. Such factors are put on hand to meet certain homogeneous properties for the local basis and geometric quantizes. Such factors do not appear in the Lorentzian case, since in this case the condition \( F = 0 \) can lead to un-necessary apparent singularities in the local expression of geometric objects [8]. We stick to this last formulation. For a positive definite Finsler space, the indicatrix over \( x \) is the compact submanifold \( I_x := \{y \in T_x M \text{ s.t. } F(x, y) = 1\} \hookrightarrow T_x M \).

Since \( d^n y \) is invariant by local diffeomorphisms on \( TM \), it defines a section of \( \Lambda^n N \). For each embedding \( i_x : I_x \hookrightarrow N_x \) one consider the volume form on \( I_x \) given by

\[
d\text{vol}_x := i_x^*(d^n y \cdot l),
\]
where \( l = y^i \frac{\partial}{\partial y^i} \), and \( d^n y \cdot l \) is the corresponding contraction; \( i_x^* \) is the pull-back on \( T^* N_x \). Then the volume function \( \text{vol}(I_x) \) is defined by the expression

\[
\text{vol} : M \to \mathbb{R}^+, \quad x \mapsto \text{vol}(I_x) = \int_{I_x} |\psi|^2(x, y) \, dvol_x,
\]

where the weight factor \( |\psi|^2 : TM \setminus \{0\} \to \mathbb{R}^+ \) is a homogenous of degree zero in \( y \), positive, smooth function on the tangent bundle. The average of a function \( f \in \mathcal{F}(I) \) is defined by the expression

\[
\langle f \rangle_\psi(x) := \frac{1}{\text{vol}(I_x)} \int_{I_x} |\psi|^2(x, y) \, f(x, y) \, dvol_x.
\]

Given the Finsler space \( (M, F) \) let us consider the matrix with components

\[
h_{ij}(\psi, x) := \langle g_{ij}(x, y) \rangle_\psi, \quad i, j = 1, \ldots, n,
\]

for each point \( x \in M \).

**Proposition 2.2.** Let \( (M, F) \) be a Finsler space. Then \( \{h_{ij}(\psi, x)\}_{i,j=1}^n \) are the components of a Riemannian metric

\[
h_\psi(x) = h_{ij}(\psi, x) \, dx^i \otimes dx^j, \quad i, j = 1, \ldots, n.
\]

The proof of this result can be found in [10]. This is the fundamental result on which rest the notion of average Finsler metric that we will generalize to certain metric structures with signature \( n-1 \), although the methodology cannot be transferred directly to the Lorentzian signature case.

An analogous construction can be applied to the connections determined by a Finsler structure. For instance to the Chern connection, the Cartan’s connection and other geometric objects. The details of these constructions can be found in [10]. We collect here the result that will be considered for generalization to the Lorentzian case,

**Theorem 2.3.** Let \( \nabla \) be a linear connection of the vector bundle \( \pi^* TM \to N \). Then there is defined an affine connection on \( M \) determined by the covariant derivative of each section \( Y \in \Gamma TM \) along each direction \( X \in T_x M \),

\[
(\nabla)_X Y := \langle \nabla_{i_0} \nabla_{i_v}(X) \pi^* Y \rangle, \quad v \in TU_x \setminus \{0\},
\]

for each \( X \in T_x M \) and \( Y \in \Gamma TM \), where \( U_x \) is an open neighborhood of \( x \in M \).

This theorem defines the averaged Chern connection. In adequate local frames, the connection coefficients of the averaged connection are the average of the connection coefficients of the Chern connection.
2.2. Geometric framework for Finsler spaces with Lorentzian signature. Following J. Beem \[2\] but using the notation from \[8\], we introduce the basic notation and fundamental notions of Finsler spaces with Lorentzian signature theory. As we mentioned before, we formulate the theory for \(n\)-dimensional manifolds. Thus we start with the following

**Definition 2.4.** A Finsler spaces with Lorentzian signature is a pair \((M, L)\) where

1. \(M\) is an \(n\)-dimensional real, second countable, Hausdorff \(C^\infty\)-manifold.
2. \(L : N \rightarrow \mathbb{R}\) is a real smooth function such that
   a. \(L(x, \cdot)\) is positive homogeneous of degree two in the variable \(y\),
   \[ L(x, ky) = k^2 L(x, y), \quad \forall k \in ]0, \infty[ , \]
   b. The vertical Hessian
   \[ g_{ij}(x, y) = \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \]
   is non-degenerate and with signature \(n - 1\) for all \((x, y) \in N\).

Direct consequences of this definition and Euler’s theorem for positive homogeneous functions are the following relations,

\[ \frac{\partial L(x, y)}{\partial y^k} y^k = 2 L(x, y), \quad \frac{\partial L(x, y)}{\partial y^i} = g_{ij}(x, y)y^j, \quad L(x, y) = \frac{1}{2} g_{ij}(x, y)y^i y^j. \]

Furthermore, the function \(L\) determines an homogeneous of degree one function \(F\) by the relation \(F := L^{1/2}\). One main difference with the positive definite Finsler case is that the function \(F\) is not real valued in the whole \(TM\).

Given a Finsler spaces with Lorentzian signature \((M, L)\), a vector field \(X \in \Gamma TM\) is timelike if \(L(x, X(x)) < 0\) at all point \(x \in M\) and a curve \(\lambda : I \rightarrow M\) is timelike if the tangent vector field is timelike in the sense that \(L(\lambda(s), \dot{\lambda}(s)) < 0\). A vector field \(X \in \Gamma TN\) is lightlike if \(L(x, X(x)) = 0\), \(\forall x \in M\) and a curve is lightlike if its tangent vector field is lightlike. Similar notions are for spacelike. A curve is causal if either is timelike and has constant speed \(g_{\lambda}(\dot{\lambda}, \dot{\lambda}) := L(\lambda, \dot{\lambda}) = g_{ij}(\dot{\lambda}, T)\dot{\lambda}^i \dot{\lambda}^j\) or if it is lightlike. There are two sub-manifolds of the tangent space that play a significant role in the theory. The first is the unit hyperboloid bundle,

\[ \Sigma_x := \{ y \in T_x M \text{ st. } L(x, y) = -1 \}. \]

The second is the tangent light cone at \(x\),

\[ L_x := \{ y \in T_x M \text{ st. } L(x, y) = 0 \}. \]

Both manifolds, \(\Sigma_x\) and \(L_x\) are non-compact. Also, they are sub-manifolds of co-dimension 1 of \(N_x\).
The Cartan tensor is defined here as a tensor field of components
\[
C_{ijk} := \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad i, j, k = 1, ..., n,
\]
slightly different than in the way it is formulated in positive definite Finsler geometry \[4\]. As mentioned before, we do not use factors of \( F \) to make the Cartan tensor homogeneous of degree zero. However, the fundamental tensor \( g \) is still of 0-homogeneity in \( y \)-variables. Therefore, by Euler’s theorem
\[
C_{(x,y)}(y,\cdot,\cdot) = \frac{1}{2} g^k \frac{\partial g_{ij}}{\partial y^k} = 0.
\]

Chern connection for Finsler spaces with Lorentzian signature is defined in similar terms as in the case of positive definite Finsler spaces. We introduce here the connection following the index-free formulation that can be found in \[8\].

**Proposition 2.5.** Let \( h(X) \) and \( v(X) \) be the horizontal and vertical lifts of \( X \in \Gamma TM \) to \( TN \), and \( \pi^*g \) the pull back-metric. The Chern connection is characterized by

1. The almost \( g \)-compatibility metric condition is equivalent to
\[
\nabla_{v(X)} \pi^*g = 2C(\hat{X},\cdot,\cdot), \quad \nabla_{h(X)} \pi^*g = 0, \quad \hat{X} \in \Gamma TN.
\]

2. The torsion-free condition:
   
   (a) Null vertical covariant derivative of sections of \( \pi^*TM \):
\[
\nabla_{V(X)} \pi^*Y = 0,
\]
   for any vertical component \( V(X) \) of \( X \).

   (b) Let us consider \( X, Y \in TM \) and their horizontal lifts \( h(X) \) and \( h(Y) \). Then
\[
\nabla_{h(X)} \pi^*Y - \nabla_{h(Y)} \pi^*X - \pi^*([X,Y]) = 0.
\]

The connection coefficients \( \Gamma^i_{jk}(x,y) \) of the Chern connection are constructed in terms of the fundamental tensor components and the Cartan tensor components. The detail of how they are constructed are not necessary for the results described in this work.

3. **Definition of the average of the metric tensor for Finsler spaces with Lorentzian signature**

The direct extension to Lorentzian signature of the average metric tensor for positive definite Finsler metrics by means of (2.7) is not a good strategy, because the average of the components \( \langle g_{ij}(x,y) \rangle_\psi \) does not provide a well-defined, non-singular metric tensor. Instead, the procedure that we follow for the generalization requires of three steps and is the following.

1. Let us consider that \( M \) is endowed with a vector field \( \mathcal{X} \in \Gamma TM \) such that
\[
g_{(x,y)}(\mathcal{X},\mathcal{X}) < 0, \quad \forall (x,y) \in TM.
\]
This is a stronger condition than the requirement of existence of a timelike vector field in the Finslerian sense, namely,

\[(3.2) \quad L(x, T(x)) = g(x, T(x))(T(x), T(x)) < 0.\]

Given the vector field \(\mathcal{X}\) such that (3.1) holds good, for the fundamental tensor \(g_{ij}(x, y)\) we define

\[(3.3) \quad \tilde{g}_{(x, y)}(Y, Z) := g(x, y)(Y, Z) - 2g(x, y)(\mathcal{X}, Y)g(x, y)(\mathcal{X}, Z).\]

This is a generalization of the inverse transformation which associates to a given Riemannian structure, which always can be found for paracompact manifolds, a metric with Lorentzian signature \([11]\). Elementary algebraic manipulations leads to the expression

\[g(x, y)(\mathcal{X}, \mathcal{X}) \tilde{g}_{(x, y)}(Y, Z) = g(x, y)(\mathcal{X}, Y)g(x, y)(Y, Z) - 2g(x, y)(\mathcal{X}, Y)g(x, y)(\mathcal{X}, Z).\]

In the case \(\mathcal{X} = Y = Z\), it leads to the relation

\[(3.4) \quad \tilde{g}_{(x, y)}(Y, Z) := g(x, y)(Y, Z).\]

Hence if \(Y, Z\) take values in an orthogonal basis for the bilinear form \(g(x, y)\), the same will be true for \(\tilde{g}_{(x, y)}\).

The relations (3.4) and (3.5) imply that \(\tilde{g}_{(x, y)}\) has positive signature, for each \(u = (x, y)\). Note that, although \(\tilde{g}\) is not the fundamental tensor of a positive definite Finsler space, but it is the fundamental tensor determining a Lagrange space \([13]\).

2. It is remarkable that the averaging method can also be applied to positive definite Lagrange spaces, since the proposition 2.2 and the formula (2.6) applies directly to the fundamental tensor \(g_{ij}(x, y)\). We apply the averaging operation with the weight function \(\psi = 1\). Then we have the following

**Proposition 3.1.** Let \((M, \tilde{g})\) be the Lagrange space given by the expression (3.3) and a vector field \(\mathcal{X}\) such that the condition (3.1) holds good. Then the components \((\tilde{g}_{ij})\) determine a Riemannian metric \((\tilde{g})\) on \(M\).

3. We can define now a metric with signature \(n - 1\) defined over \(M\) from the Riemannian metric \((\tilde{g})\). We need to assume that there is a non-zero everywhere vector field \(V \in \Gamma TM\), but we can assume that such a \(V\) is indeed the vector field \(\mathcal{X}\) introduced in adopting the condition (3.1). Therefore, we have the following
Proposition 3.2. Let \((M, L)\) be a Finsler spaces with Lorentzian signature, \(X\) a vector field such that condition (3.1) holds good. Then the metric \(\langle \tilde{g} \rangle\) determined by the components whose expression is
\[
\langle \tilde{g} \rangle(x,y) = \langle \tilde{g} \rangle(x,y)(X,Y) - 2 \langle \tilde{g} \rangle_x(X,Y) \langle \tilde{g} \rangle_x(X,Z)
\]
is a Lorentzian metric on \(M\).

This result suggests our definition of average Finsler spaces with Lorentzian signature,

Definition 3.3. Given a Finsler spaces with Lorentzian signature \((M, L)\) and a vector field \(X\) as in (3.1). The average Finsler spaces with Lorentzian signature metric \(\langle \tilde{g} \rangle\) is given by the expression (3.6).

It is direct that when the initial spacetime is Lorentzian (\(g\) is defined over \(M\)), then \(X\) as in (3.1) exists when the spacetime is time ordered [8]. Indeed, one has in the Lorentzian case that \(\langle \tilde{g} \rangle = \langle \tilde{g} \rangle = g\).

3.1. Interpretation of the average Finsler spaces with Lorentzian signature structure. We would like to remark that the condition (3.1) is much stronger than the usual time orientation condition (3.1). One can heuristically justify the condition (3.1) if we think the Finsler spaces with Lorentzian signature structure \((M, L)\) as a small deformation from a time orientable Lorentzian structure. If \(X\) is the time orientation of the structure, then \(L(x, X(x)) = g(x, X(x)) \langle X(x), X(x) \rangle < 0\). Since \(g(x,y)\) is homogeneous of degree zero in \(y\)-variable, it lives in sphere bundle \(SM\), where each projective sphere is defined as the aggregate of equivalence classes
\[
S_xM = \{y = \lambda y_0, \lambda \in \mathbb{R}^+, y_0 \in N_x\}.
\]
Each sphere \(S_xM\) is compact. Hence, if for each \(x \in M\) the condition \(L(x, X(x)) < 0\) holds, then for small Finslerian perturbations of Lorentzian metrics the condition \(g(x,y)(X(x), X(x)) < 0\) with \(y \in N\) is also to be expected. By accepting the condition (3.1) we are accepting that \(g(x,y)\) and the Lorentzian metric \(g(x, X(x))\) do not differ too much.

The fundamental characteristic of the average Finsler structures with Lorentzian signature structure is that, as the Riemannian metric \(\tilde{g}\), it depends on the timelike vector field \(X\) in the Finslerian sense \(X \in \Gamma TM\) and on the timelike vector field respect to the metric \(\langle \tilde{g} \rangle\) \(V = X \in \Gamma TM\). This makes the metric \(\langle \tilde{g} \rangle\) to depend on the vector field \(X\).

4. On the average connection

Let \((M, F)\) be a positive definite Finsler space. As it has been discussed in [10], there is defined the Chern connection \(\nabla\) and its average connection \(\langle \nabla \rangle\). It can be shown that the average connection \(\langle \nabla \rangle\) is an affine connection on \(M\) [10].
The construction of the averaged Chern connection can be extended to the case of signature $n - 1$. If one applies the same procedure as in the case of positive definite Finsler space, after implementing the adequate changes in the measure used in the averaging method when passing from positive definite to signature $n - 1$, then the connection coefficients in a particular holonomic basis for $T_xM$ are of the form

$$\langle \Gamma_{jk}^i \rangle \psi(x) := \frac{1}{\text{vol}(I_x)} \int_{\Sigma_x} |\psi|^2(x, y) \Gamma_{jk}^i(x, y) \text{dvol}_x,$$

where the connection coefficients $\Gamma_{jk}^i(x, y)$ are the connection coefficients of the Chern connection. The submanifold $\Sigma_x \subset N_x$ is non-compact. Therefore, in order to ensure the convergence of the integral, the measure $|\psi|^2(x, y)$ must converge in the limit $y$-coordinates going to infinity conveniently. In general, it is a difficult task to find an universal measure $|\psi|^2$ with such characteristics. Hence we need to assume that $\psi$ depends on the particular details of the given function $L$.

**Example 4.1.** Of particular interest are Finsler spaces with Lorentzian signature where the connection coefficients of the Chern connection do not depend upon the $y$-variable. Examples of such spacetimes are Berwald type spacetimes [9]. Fixed a normal coordinate system on $TM$, let us consider the following sub-manifold of $T_xM$,

$$P_x(r) = \{ y \in T_xM \text{ s.t. } |y| \leq r \}.$$ 

The integral operation is the regularized integral (4.1),

$$\langle \Gamma_{jk}^i \rangle \psi_r(x) := \frac{1}{\text{vol}(P_x(r))} \int_{P_x(r)} |\psi|^2(x, y) \Gamma_{jk}^i(x, y) \text{dvol}_x = \Gamma_{jk}^i(x),$$

where

$$\text{vol}(P_x(r)) = \int_{P_x(r)} |\psi|^2(x, y) \text{dvol}_x.$$ 

In the limit $r \to \infty$ the relation

$$\langle \Gamma_{jk}^i \rangle \psi_r(x) \to \langle \Gamma_{jk}^i \rangle \psi(x), \quad i, j, k = 1, ..., n$$

holds good, showing that the limit $r \to +\infty$ in the integral operation is well defined and equal to $\Gamma_{jk}^i(x)$.

One way to avoid this difficulty is to define the averaging operation of metrics as an integral operation on a subset of the projective sphere $S_xM$. Since $S_xM$ is compact, there is no need of a special convergence property for the measure. Therefore, we can take $|\psi(x, y)| = 1$. On each projective sphere bundle $S_xM$ there is defined an induced volume form $\text{dvol}_x'$ determined by the volume form $\text{dvol}_x$ defined on $N_x$. Let us consider the canonical projection $p : N_x \to S_xM$ and let us note the relation $\dim(N_x) = \dim(S_xM) - 1$. Then $\text{dvol}_x'$ is defined as the unique volume form on $S_xM$ such that the
pull-back $p^*d\text{vol}'_x$ is equal to the contraction $l_x \cdot d\text{vol}_x$, where $l_x = y^k \frac{\partial}{\partial y^k}|_x$.

Then one can define the following average

\begin{equation}
\langle \Gamma^i_{jk}(x) \rangle := \frac{1}{\text{vol}(I_x)} \int_{S_x M} \Gamma^i_{jk}(x, y) \text{vol}'_x.
\end{equation}

(4.2)

It is easy to see that, as in the positive definite case, the torsion of the averaged Chern connection given by either the connection coefficients (4.1) or by (4.2), is zero. However, in general the averaged Chern connection is not the Levi-Civita connection of the average metric $\langle \tilde{g} \rangle$, even when the Finsler spaces with Lorentzian signature is of Berwald type.

5. Conclusion and outlook

We have shown how to extend the averaging operations of metrics and connections from positive definite Finsler structures to certain Finsler structures with signature $n - 1$. In the case of the metric structure, the main requirement added for the construction to work is the existence of a vector field $\mathcal{X}$ such that the condition (3.1) is fulfilled. In the general case, however, the problem of defining a natural average operation remains still open.

We have also discussed the problem of averaging the Chern connection in the setting of metrics with signature $n - 1$. In this case, the procedure does not need the existence or not of such vector field $\mathcal{X}$ constrained by (3.1).

Since the procedures for averaging the metric and the connection are rather different, one should not expect a direct translation of the fundamental characterization of Berwald spaces as given in [7], namely, that a space is of Berwald type iff the averaged Chern connection is the Levi-Civita connection of the average metric. It is an interesting problem, due to the relevance of Berwald spacetimes for physics, to know if there is a definition of the average connection in for signature $n - 1$ such that the characterization given in [7] still holds.

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