Mixing and relaxation dynamics of the Henon map at the edge of chaos

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Abstract

The mixing properties (or sensitivity to initial conditions) and relaxation dynamics of the Henon map, together with the connection between these concepts, have been explored numerically at the edge of chaos. It is found that the results are consistent with those coming from one-dimensional dissipative maps. This constitutes the first verification of the scenario in two-dimensional cases and obviously reinforces the idea of weak mixing and weak chaos.

Key words: Nonextensive thermostatistics, Henon map, dynamical systems

1 Introduction

Nonlinear one-dimensional dissipative systems at their critical points (like chaos threshold, bifurcation points, etc) have become one of the vivid areas of research since 1997, after pioneering work of Tsallis et al. [1]. One of the relevant reason for this, in our opinion, was that they were able to constitute one of the first methods of inferring the entropic index \( q \) of nonextensive entropy [2]

\[
S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}, \tag{1}
\]

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and consequently determining the proper thermostatistics which the system under consideration obeys, from the microscopic dynamics of the system. This method depends on the measure of the divergence of initially nearby orbits. More precisely, the effect of any uncertainty on initial conditions exhibits, for periodic and chaotic orbits, an exponential temporal evolution as 

\[ \xi(t) = \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} \sim \exp(\lambda t) \]

where \( \lambda \) is the standard Lyapunov exponent, \( \Delta x(0) \) and \( \Delta x(t) \) are the uncertainties at times 0 and \( t \) (\( \lambda < 0 \) for periodic cases, and \( \lambda > 0 \) for chaotic ones); whereas at critical points (e.g., chaos threshold) this feature is related to a power-law type such as

\[ \xi(t) = [1 + (1 - q_{mix})\lambda_{mix} t]^{1/(1-q_{mix})} \]  

(2)

where \( \lambda_{mix} \) is a generalized version of the Lyapunov exponent (here, \textit{mix} stands for \textit{mixing} and the aim of this notation will be transparent soon). This equation corresponds to power-law growth of the upper bounds of a complex time dependence of \( \xi(t) \) and these upper bounds allow us to infer the proper value of \( q_{mix} \) for the system under consideration. Let us call this Method I of finding the proper \( q_{mix} \) of a given dynamical system.

The second method (Method II) is based on the geometrical aspects of the critical attractor of the dynamical system at the chaos threshold. Scaling arguments have shown that the appropriate \( q_{mix} \) is related to the multifractal structure of the critical dynamical attractor by [3]

\[ \frac{1}{1-q_{mix}} = \frac{1}{\alpha_{min}} - \frac{1}{\alpha_{max}} \]  

(3)

where \( \alpha_{min} \) and \( \alpha_{max} \) are the end points of the multifractal singularity spectrum of the critical attractor and characterize the most concentrated and most rarefied regions respectively on the attractor. This fascinating relation relates power-law sensitivity or mixing (left-hand side) with purely geometrical quantities (right-hand side) and constitutes another independent method for inferring the proper \( q_{mix} \) of the dynamical system under study.

Finally, a third method (Method III) has been proposed for obtaining \( q_{mix} \), dealing with the entropy increase rates [4,5]. This method is the one where nonextensive entropy \( S_q \) enters directly and consequently it provides also a justification for the use of notation \( q \) within the two previous methods. The procedure for Method III is the following: We first partition the phase space into \( W \) cells, then we choose one of these cells (randomly or not) and locate \( N \) initial conditions all inside this cell. At any time \( t \), we have \( N_i(t) \) points in each cell (naturally \( \sum_{i=1}^{W} N_i(t) = N \)) and thus we can define a set of probabilities through \( p_i(t) = N_i(t)/N \) (\( \forall i \)), which enable us to calculate the entropy \( S_q(t) \). Using this procedure, one can define a generalized version of the Kolmogorov-Sinai entropy as

\[ K_q \equiv \lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} S_q(t)/t \]

and the proper \( q_{mix} \).
value of a given system is the one for which the entropy production $K_q$ is finite. If $q < q_{\text{mix}}$ ($q > q_{\text{mix}}$), $K_q$ diverges (vanishes).

These three independent methods (from now on referred to as mixing-based approach) have already been tested and verified with numerical calculations for a number of one-dimensional dissipative systems such as the logistic map [1], z-logistic map family [6], the circle map [3], $z$-circular map family [7], asymmetric logistic map family [8] and the single site map [9]. Quite remarkably, it is found that all three methods give one and the same value for $q_{\text{mix}}$ of a given system. It is worth mentioning here that all these numerical findings have been supported very recently by an analytical work [10].

We can now turn our attention to the second approach, which will be referred to as relaxation-based approach. This approach has been developed by one of us and his collaborators [11] and it is based on investigating numerically the critical temporal evolution of the volume of the phase space occupied by an ensemble of initial conditions spread over the entire phase space. In that sense, this approach differs from the mixing-based approach, where a set of initial conditions spread in the vicinity of the inflection point is used. In the beginning of time, all cells are occupied and $S_q$ is bounded by the equiprobability as $S_q = (W^{1-q} - 1)/(1 - q)$. As time evolves, the number of occupied cells (say $W_{\text{occ}}$) starts contracting as $[1 + (q_{\text{rel}} - 1)t/\tau_q]^{1/(1-q_{\text{rel}})}$, here $q_{\text{rel}} > 1$, $\text{rel}$ stands for relaxation and $\tau_q > 0$ is a characteristic relaxation time. In [11], $z$-logistic map family has been analyzed and $q_{\text{rel}}(t \to \infty)$ values for $z$ parameter has been obtained.

The existence of two different classes of $q$ values coming from these two different approaches (namely, mixing-based and relaxation-based approaches) is also present in the results recently obtained for fully developed turbulence [12,13]. Therefore, all these findings naturally force us to ask an intriguing question: are these classes connected to each other or not? The affirmative answer to this question came very recently by Borges at al. [14]. Their procedure goes as follows: Beginning with $N$ points located inside a single cell (similar to Method III explained before), the dynamics of the system (for $t \to \infty$) leads it to a state with a stationary value of $S_{q_{\text{mix}}}$ — this is the attractor of the system for the particular conditions of $W$, $N$ and the initial cell chosen. The path that the system evolves towards its attractor changes as the initial cell is changed. Some of the initial cells spread the points very fast, increasing the entropy to values much higher than that of the attractor (there is an overshooting in the time evolution of $S_{q_{\text{mix}}}$). Consequently, for these specific cases, the attractor is achieved from above. The procedure consists in observing the rate in which the system goes to its attractor, for those initial cells which yield large overshoots. Specifically, the time evolution of $\Delta S_{q_{\text{mix}}} \equiv S_{q_{\text{mix}}}(t) - S_{q_{\text{mix}}} (\infty)$ is followed. The procedure is repeated for increasing values of $W$ (and correspondingly increasing values of $N$). A
A power-law regime appears and becomes more pronounced for increasing $W$. The slope (in a log-log plot) of the range of the $\Delta S_{q_{mix}}$ curve that follows a power-law is identified with $1/(q_{rel}(W) - 1)$ (with $q_{rel} > 1$). It is worth to stress that $q_{rel} = q_{rel}(W)$, i.e., the rate in which the system reaches its final state depends on the coarse graining ($W$) adopted. The connection between the mixing-based and the relaxation-based approaches is established by the scaling law

$$q_{rel}(\infty) - q_{rel}(W) \propto W^{-|q_{mix}|}.$$  (4)

In this work, our aim is to provide further support to this connection by analyzing, for the first time, a two-dimensional dissipative system within this context. To accomplish this task, we will focus on the Henon map [15]

$$x_{t+1} = 1 - ax_t^2 + y_t$$
$$y_{t+1} = bx_t$$  (5)

where $a$ and $b$ are map parameters. This map reduces to the standard logistic map when $b = 0$, whereas it becomes conservative when $b = 1$; in between these two cases, it is a two-dimensional dissipative map. Specifically, we focus on small values of the $b$ parameter such as $b = 0.001, 0.01, 0.1, \ldots$. This map has been studied within the mixing-based approach using the three different methods and it is found that the $q_{mix}$ value for all values of $b$ parameter coincides with that of the logistic case, as expected [16]. Here, we first investigate the Henon map within the relaxation-based approach and then establish the connection between these two different approaches following the same lines of [14].

2 Numerical Results

As introduced in the previous Section, for the relaxation approach, we follow the dynamical evolution of an ensemble of initial conditions uniformly distributed over the phase space. In practice, a partition of the phase space on $W$ cells of equal size is performed and a set of $N$ initial copies of system is followed whose initial conditions are uniformly distributed over the entire phase space. Our results for the temporal evolution of the volume occupied by the ensemble are given in Fig. 1 for various $b$ values. As it is evident, after the power-law contraction of the volume occupied by the ensemble sets up, the slope $[\mu = -1/(1 - q_{rel})]$ is independent from the $b$ values and coincides with that of the logistic map (namely, $\mu = 0.71$ hence $q_{rel} = 2.41$).
Now we are at the position to establish the connection between this result and the one obtained in [16]. Fig. 2 shows the temporal evolution of $S_{q_{\text{mix}}}$ ($q_{\text{mix}} = 0.2445$) for different values of $W$, at the edge of chaos for $b = 0.1$. (It would be recommended that $N \gg W$, but we adopted here $N = W$, due to restricted computer capabilities. Of course, we compared this case with $N = 10W$, and we found that the differences are negligible.) It becomes evident that the limits $\lim_{W \to \infty}$ and $\lim_{t \to \infty}$ are non-commutable. If the former is taken first, the system will never relax to a stationary state (equilibrium is but a particular case), which leads to the conclusion that reaching equilibrium depends on the coarse graining adopted, and ultimately, depends on the observer! Equilibrium is only possible with incomplete knowledge ($W < \infty$). To arrive to the scaling law stated by Eq. (4), we must take first the temporal limit, and then $W \to \infty$. When we consider the downhill side of these curves and take the slopes of $\Delta S_q$ versus time (as explained earlier), then we end up with Fig. 3, where the verification of the scaling (4) is evident. On the other hand, the extrapolation value of $q_{\text{rel}}(W \to \infty)$ is computed as $\approx 2$, which is different from the logistic case ($\approx 2.4$). This discrepancy may be originated, most probably, from the fact that $b = 0.1$ value is too large for analyzing, as it is also observed from the deterioration seen in Fig. 1 of [16] (Method I) for increasing values of the $b$ parameter. A careful investigation of the effect of increasing $b$ values to $q_{\text{rel}}(W \to \infty)$ would be a subject of an extensive computational effort.

3 Conclusions

After the analysis of two-dimensional dissipative Henon map at the chaos threshold using three distinct methods (Methods I, II and III) [16], it became evident that the scenario, which we called here mixing-based approach, is valid not only for one-dimensional dissipative map families, but also for two-dimensional cases. At this point, what missing in the literature were (i) to study the relaxation-based approach for a two-dimensional dissipative system making use of the same procedure applied so far to one-dimensional cases [11], and (ii) to verify the connection (given by the scaling relation (4)) between the mixing-based and relaxation-based approaches for a two-dimensional dissipative system. This work constitutes the first step along this line and provides some further elements that reinforce the idea of weak mixing.

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Figure Captions

**Figure 1** - Time evolution of the volume occupied by the ensemble for various values of $b$ parameter. Thick line corresponds to a straight line with $\mu = 0.71$.

**Figure 2** - Time evolution of $S_{q_{mix}}$, with $q_{mix} = 0.2445$ (log-log scale). The Henon map is at its chaos threshold for $b = 0.1$, that is, $a_c = 1.26359565$. Points correspond to $W = 700 \times 700$, $1000 \times 1000$, $2000 \times 2000$, $4000 \times 4000$ and $5000 \times 5000$.

**Figure 3** - Scaling law for the Henon map (Eq. (4)), at the chaos threshold with $b = 0.1$. Correlation coefficient is $\approx 0.99$. To improve the correlation, it becomes necessary to use greater values of $W$ and $N$, that means better computer machinery. Dashed line indicates extrapolated value of $q_{rel}(W \to \infty) \approx 1.987$ (that is different from the value of the logistic ($b = 0$) map: $q_{rel}(W \to \infty) \approx 2.4$).
Fig 1

$W(t)_{\text{occupied}} / W$

$t$

$W = 10000 \times 10000$

$b = 0.05$

$b = 0.01$

$b = 0$

$N = 130000$
