Linear Diophantine equations in Piatetski-Shapiro sequences

by

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1. Introduction. Let \( \lfloor x \rfloor \) denote the integer part of \( x \in \mathbb{R} \). For a non-integral \( \alpha > 0 \), the sequence \( (\lfloor n^\alpha \rfloor)_{n=1}^\infty \) is called the Piatetski-Shapiro sequence with exponent \( \alpha \). Let \( \text{PS}(\alpha) = \{ \lfloor n^\alpha \rfloor : n \in \mathbb{N} \} \). We say that an equation \( f(x_1, \ldots, x_n) = 0 \) is solvable in \( \text{PS}(\alpha) \) if there are infinitely many pairwise distinct tuples \( (x_1, \ldots, x_n) \in \text{PS}(\alpha)^n \) satisfying this equation. In this article, we investigate the solvability in \( \text{PS}(\alpha) \) of linear Diophantine equations

\[
ax + by = cz
\]

for all fixed \( a, b, c \in \mathbb{N} \). For example, the solvability of the equation \( y = \theta x + \eta \) for \( \theta, \eta \in \mathbb{R} \) with \( \theta \notin \{0, 1\} \) has been studied by Glasscock [Gla17, Gla20]. He asserts that if the equation \( y = \theta x + \eta \) has infinitely many solutions \( (x, y) \in \mathbb{N}^2 \), then for Lebesgue-a.e. \( \alpha > 1 \) it is solvable or not in \( \text{PS}(\alpha) \) according as \( \alpha < 2 \) or \( \alpha > 2 \). As a direct consequence, for Lebesgue-a.e. \( 1 < \alpha < 2 \), the equation \( z = (a/c)x + (b/c) \) is solvable in \( \text{PS}(\alpha) \) for all \( a, b, c \in \mathbb{N} \) with \( \gcd(a, c) \mid b \). In other words, the equation (1.1) with \( \gcd(a, c) \mid b \) is solvable in \( \text{PS}(\alpha) \). On the other hand, for \( \alpha > 2 \), we did not know at all whether the equation (1.1) is solvable in \( \text{PS}(\alpha) \) or not.

Our main result provides an answer to this question. We consider the set of \( \alpha \) in a short interval \( [s, t] \subset (2, \infty) \) such that (1.1) is solvable. The following theorem asserts that the Hausdorff dimension of this set is positive. To state the theorem, let \( \{x\} \) be the fractional part of \( x \in \mathbb{R} \), and \( \dim_H(X) \) the Hausdorff dimension of \( X \subseteq \mathbb{R} \) (the definition will be recalled in Section 2).

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Theorem 1.1. Let \( a, b, c \in \mathbb{N} \). For all positive real numbers \( 2 < s < t \),

\[
\dim_{H}(\{ \alpha \in [s,t] : ax + by = cz \text{ is solvable in } \text{PS}(\alpha) \}) \geq \begin{cases} 
\left( s + \frac{s^3}{(2 + \{ s \} - 2^{-1}[s])(2 - \{ s \})} \right)^{-1} & \text{if } a = b = c, \\
2 \left( s + \frac{s^3}{(2 + \{ s \} - 2^{-1}[s])(2 - \{ s \})} \right)^{-1} & \text{otherwise.}
\end{cases}
\]

Note that the lower bound in either case is greater than \( 1/s^3 \) for all \( 2 < s < t \). The positivity of the Hausdorff dimension implies that this set is uncountable for any closed interval \([s,t] \subset (2, \infty)\). Moreover, we can easily prove the following:

Corollary 1.2. For any closed interval \( I \subset (2, \infty) \), the set of \( \alpha \in I \) such that \( ax + by = cz \) is solvable in \( \text{PS}(\alpha) \) is uncountable and dense in \( I \).

In particular, for \( a = b = 1, c = 2 \), a pairwise distinct tuple \((x,z,y)\) satisfying (1.1) forms an arithmetic progression of length 3. Therefore Corollary 1.2 implies

Corollary 1.3. For any closed interval \( I \subset (2, \infty) \), the set of \( \alpha \in I \) such that \( \text{PS}(\alpha) \) contains infinitely many arithmetic progressions of length 3 is uncountable and dense in \( I \).

There are some related works on arithmetic progressions and Piatetski-Shapiro sequences. It is an exercise to show that for all \( 1 < \alpha < 2 \), the set \( \text{PS}(\alpha) \) contains arbitrarily long arithmetic progressions (consisting of consecutive elements). Frantzikinakis and Wierdl [FW09] proved that any set of positive integers with positive upper density contains arbitrarily long arithmetic progressions whose common difference belongs to \( \text{PS}(\alpha) \) for all non-integral \( \alpha > 1 \) (here we say that \( A \subseteq \mathbb{N} \) has positive upper density if \( \lim_{N \to \infty} |A \cap \{1, \ldots, N\}|/N > 0 \)). This result is an extension of Szemerédi’s theorem [Sze75]. Furthermore, the second author and Yoshida [SY19] gave another extension of Szemerédi’s theorem to Piatetski-Shapiro sequences by showing that for any \( A \subseteq \mathbb{N} \) with positive upper density, the set \( \{[n^\alpha] : n \in A\} \) with \( 1 < \alpha < 2 \) contains arbitrarily long arithmetic progressions. They also posed a question:

Question 1.4 ([SY19, Question 13]). Is it true that

\[
\sup \{ \alpha \geq 1 : \text{PS}(\alpha) \text{ contains arbitrarily long arithmetic progressions} \} = 2?
\]

We do not get any answer to this question here, but surprisingly, by Corollary 1.3 the supremum of \( \alpha \) such that \( \text{PS}(\alpha) \) contains infinitely many arithmetic progressions of length 3 is positive infinity. Glasscock also posed a related question for the equation (1.1) with \( a = b = c = 1 \).
Question 1.5 ([Gla17 Question 6]). Does there exist an $\alpha_S > 1$ with the property that for Lebesgue-a.e. or all $\alpha > 1$, the equation $x + y = z$ is solvable or not in $\text{PS}(\alpha)$ according as $\alpha < \alpha_S$ or $\alpha > \alpha_S$?

By Corollary 1.2, the case with “all $\alpha > 1$” in Question 1.5 is false since the supremum of $\alpha > 0$ such that (1.1) is solvable in $\text{PS}(\alpha)$ is positive infinity. However, the case with “Lebesgue-a.e.” in Question 1.5 is still open.

The rest of the article is organized as follows. First in Section 2 we define the discrepancy of the sequences and the Hausdorff dimension, and describe some known useful results. In Sections 3 and 4, we prove a series of lemmas. Finally we provide a proof of Theorem 1.1.

Notation. Let $\mathbb{N} = \{1, 2, \ldots\}$. For $x \in \mathbb{R}$, let $[x]$ denote the integer part of $x$, $\{x\}$ denote the fractional part of $x$, and $\lceil x \rceil$ denote the minimum integer $n$ such that $x \leq n$. A tuple $(x_1, \ldots, x_k) \in \mathbb{R}^k$ is called pairwise distinct if $\#\{x_1, \ldots, x_k\} = k$. Let $\sqrt{-1}$ denote the imaginary unit, and define $e(x)$ by $e^{2\pi \sqrt{-1} x}$ for all $x \in \mathbb{R}$.

2. Preparations. For all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, define

$$\{x\} = (\{x_1\}, \ldots, \{x_d\}).$$

Let $(x_n)_{1 \leq n \leq N}$ be a sequence composed of $x_n \in \mathbb{R}^d$ for all $1 \leq n \leq N$. We define the discrepancy $D(x_1, \ldots, x_N)$ of $(x_n)_{n=1}^N$ by

$$\sup_{0 \leq a_i < b_i \leq 1} \left| \#\{n \in \mathbb{N} \cap [1, N] : \{x_n\} \in \prod_{i=1}^d [a_i, b_i]\} - \prod_{i=1}^d (b_i - a_i) \right|.$$

In order to evaluate an upper bound for the discrepancy, we use the following inequality which was shown by Koksma [Kok50] and Szüsz [Szu52] independently: there exists $C_d > 0$ which depends only on $d$ such that for all $K \in \mathbb{N}$, we have

$$D(x_1, \ldots, x_N) \leq C_d \left( \frac{1}{K} + \sum_{0 < \|k\|_\infty \leq K} \frac{1}{\nu(k)} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi \sqrt{-1} \langle k, x_n \rangle} \right| \right),$$

where we let $\langle \cdot, \cdot \rangle$ denote the standard inner product and define

$$\|k\|_\infty = \max(|k_1|, \ldots, |k_d|), \quad \nu(k) = \prod_{i=1}^d \max(1, |k_i|).$$

This inequality is sometimes referred as the Erdős–Turán–Koksma inequality. We refer the readers to [DT97, Theorem 1.21] for more details on discrepancies and a proof of (2.1). This inequality reduces the estimate of the
discrepancy to that of exponential sums. Furthermore, the exponential sum is evaluated by the following lemma.

**Lemma 2.1 (van der Corput's kth derivative test).** Let $f(x)$ be real and have continuous derivatives up to $k$th order, where $k \geq 4$. Let $\lambda_k \leq f^{(k)}(x) \leq h\lambda_k$ (or the same for $-f^{(k)}(x)$). Let $b - a \geq 1$. Then there exists $C(h, k) > 0$ such that

$$\left| \sum_{a < n \leq b} e^{2\pi\sqrt{-1}f(n)} \right| \leq C(h, k)((b - a)\lambda_k^{1/(2^k - 2)} + (b - a)^{1 - 2^{-k}}\lambda_k^{-1/(2^k - 2)}).$$

**Proof.** See Titchmarsh’s book [Tit86, Theorem 5.13].

We next introduce the Hausdorff dimension. For every $U \subseteq \mathbb{R}$, write the diameter of $U$ as $\text{diam}(U) = \sup_{x, y \in U} |x - y|$. Fix $\delta > 0$. For all $F \subseteq \mathbb{R}$ and $s \in [0, 1]$, we define

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{j=1}^{\infty} \text{diam}(U_j)^s : F \subseteq \bigcup_{j=1}^{\infty} U_j, (\forall j \in \mathbb{N}) \text{diam}(U_j) \leq \delta \right\},$$

and $\mathcal{H}^s(F) = \lim_{\delta \to 0^+} \mathcal{H}_\delta^s(F)$ is called the $s$-dimensional Hausdorff measure of $F$. Further,

$$\dim_H(F) = \inf \{ s \in [0, 1] : H^s(F) = 0 \}$$

is called the *Hausdorff dimension* of $F$. Note that the Hausdorff dimension can be defined on all metric spaces, but we use only $\mathbb{R}$ in this article. By the definition, the following basic properties hold:

- (Monotonicity) for all $F \subseteq E \subseteq \mathbb{R}$, we have $\dim_H(F) \leq \dim_H(E)$;
- (Countable stability) if $F_1, F_2, \ldots \subseteq \mathbb{R}$ is a countable sequence of sets, then $\dim_H(\bigcup_{n=1}^{\infty} F_n) = \sup_{n \in \mathbb{N}} \dim_H(F_n)$.

We refer the readers to Falconer’s book [Fal14] for more details on fractal dimensions. In order to prove Theorem 1.1 we construct a general Cantor set which is a subset of the set of all $\alpha$ such that (1.1) is solvable in $\text{PS}(\alpha)$. In [Fal14, (4.3)], we can see a general construction of Cantor sets and a technique to evaluate their Hausdorff dimension as follows: Let $[0, 1] = E_0 \supseteq E_1 \supseteq \cdots$ be a decreasing sequence of sets, with each $E_k$ a union of a finite number of disjoint closed intervals called $k$th level basic intervals, with each interval of $E_k$ containing at least two intervals of $E_{k+1}$, and with the maximum length of $k$th level intervals tending to 0 as $k \to \infty$. Then let

$$F = \bigcap_{k=0}^{\infty} E_k.$$

**Lemma 2.2 ([Fal14, Example 4.6(a)])**. Suppose in the general construction (2.2) each $(k - 1)$st level interval contains at least $m_k \geq 2$ kth level
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intervals \((k = 1, 2, \ldots)\) which are separated by gaps of at least \(\delta_k\), where \(0 < \delta_{k+1} < \delta_k\) for each \(k\). Then

\[
\dim_H(F) \geq \lim_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \delta_k)}.
\]

Since the Hausdorff dimension is stable under similarity transformations, the initial interval \(E_0\) may be taken to be an arbitrary closed interval. Moreover, let \(E^0_k\) be the set of interior points of \(E_k\) for all \(k \in \mathbb{N}\). Then the Hausdorff dimension of \(\bigcap_{k=0}^{\infty} E^0_k\) is equal to that of \(\bigcap_{k=0}^{\infty} E_k\). To see why, let \(N_k\) be the boundary of \(E_k\), that is, the set of all end points of \(k\)th level intervals. We easily see that

\[
N := F \setminus \bigcap_{k=0}^{\infty} E^0_k \subset \bigcup_{k=0}^{\infty} N_k =: N_{\infty}.
\]

Since each \(N_k\) is a finite set, \(N_{\infty}\) is countable. By monotonicity, and the fact that the Hausdorff dimension of a countable set is 0, we get

\[
0 \leq \dim_H(N) \leq \dim_H(N_{\infty}) = 0,
\]

that is, \(\dim_H(N) = 0\). Therefore by countable stability,

\[
\dim_H(F) = \max \left\{ \dim_H\left(\bigcap_{k=0}^{\infty} E^0_k\right), \dim_H(N) \right\} = \dim_H\left(\bigcap_{k=0}^{\infty} E^0_k\right).
\]

To summarize this discussion, we have the following:

**Lemma 2.3.** Let \(E_0\) be any open interval, and let \(E_0 \supseteq E_1 \supseteq \cdots\) be a decreasing sequence of sets, with each \(E_k\) a union of a finite number of disjoint open intervals, and with the maximum length of \(k\)th level intervals tending to 0 as \(k \to \infty\). Suppose each \((k-1)\)st level interval contains at least \(m_k \geq 2\) \(k\)th level intervals \((k = 1, 2, \ldots)\) which are separated by gaps of at least \(\delta_k\), where \(0 < \delta_{k+1} < \delta_k\) for each \(k\). Then

\[
\dim_H\left(\bigcap_{k=0}^{\infty} E_k\right) \geq \lim_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \delta_k)}.
\]

**3. Lemmas I.** We write \(O(1)\) for a bounded quantity. If this bound depends only on some parameters \(a_1, \ldots, a_n\), then for instance we write \(O_{a_1, \ldots, a_n}(1)\). As is customary, we often abbreviate \(O(1)X\) and \(O_{a_1, \ldots, a_n}(1)X\) to \(O(X)\) and \(O_{a_1, \ldots, a_n}(X)\) respectively for a non-negative quantity \(X\). We also write \(f(X) \ll g(X)\) and \(f(X) \ll_{a_1, \ldots, a_n} g(X)\) if \(f(X) = O(g(X))\) and \(f(X) = O_{a_1, \ldots, a_n}(g(X))\) respectively, where \(g(X)\) is non-negative.

Let us consider the solvability of the equation (1.1). In this and subsequent sections, we fix \(a, b, c, d \in \mathbb{N}\) with \(d \geq 2\) and \(\beta, \gamma \in \mathbb{R}\) with \(d < \beta < \gamma < d + 1\). Unless it causes confusion, we do not indicate their dependence
hereafter. Take a large parameter \( x_0 = x_0(a, b, c, d, \beta, \gamma) > 0 \). For all integers \( x \geq x_0 \), we define

\[
J_{a,b,c}(x) = \begin{cases}
\left( \left( \frac{b}{cx^2 \log x} + \frac{a}{c} \right)^{1/\gamma} x, \left( \frac{a}{c} \right)^{1/\gamma} x \right)_N & \text{if } c < a, \\
\left( \left( \frac{a}{c - b(x^2 \log x)^{-1}} \right)^{1/\beta} x, \left( \frac{a}{c} \right)^{1/\gamma} x \right)_N & \text{if } a < c, \\
\left( 2^{1/\gamma} \left( x + \frac{1}{x |\log x|} \right), 2^{1/\beta} x \right)_N & \text{if } a = b = c,
\end{cases}
\]

where we let \((s, t)_N\) denote \((s, t) \cap \mathbb{N}\), and set \( x \mathbb{N} = \{xn : n \in \mathbb{N}\} \). Note that \( J_{a,b,c}(x) \) is non-empty if \( x_0 \) is sufficiently large. When \( a = c \) and \( b \neq c \), \( J_{a,b,c}(x) \) is not defined above, but this case comes down to the case when \( a \neq c \) by switching the roles of \((a, x)\) and \((b, y)\). Thus the three cases in the definition of \( J_{a,b,c}(x) \) are sufficient.

**Lemma 3.1.** Assume that \( a \neq c \). Then there exists \( C > 0 \) such that for all integers \( x \geq x_0 \) and for all \( z \in J_{a,b,c}(x) \), we can find \( \alpha = \alpha(x, z) \in (\beta, \gamma) \) such that \( ax^\alpha + b = cz^\alpha \), and

\[
(3.1) \quad \left| \alpha - \frac{\log(a/c)}{\log(z/x)} \right| \leq \frac{C}{x^2 \log x}.
\]

**Proof.** Fix any \( x \geq x_0 \) and \( z \in J_{a,b,c}(x) \). For all \( u \in \mathbb{R} \), consider the continuous function \( f(u) = ax^u + b - cz^u \). We consider two cases.

**Case a > c.** Let

\[
\alpha_0 = \frac{\log(a/c)}{\log(z/x)}, \quad \alpha_1 = \frac{\log(a/c + b/(cx^2 \log x))}{\log(z/x)}.
\]

Then \( z \in J_{a,b,c}(x) \) implies \( \beta < \alpha_0 < \alpha_1 < \gamma \). It follows that \( f(\alpha_0) = b > 0 \). By taking a larger \( x_0 \) if necessary, we have

\[
f(\alpha_1) = x^{\alpha_1}(a + bx^{-\alpha_1} - c(z/x)^{\alpha_1}) \leq x^{\alpha_1}(a + b/(x^2 \log x)) - c(z/x)^{\alpha_1} = 0.
\]

Therefore, by the intermediate value theorem, there exists a zero \( \alpha = \alpha(x, z) \) of \( f \) such that \( \beta < \alpha_0 \leq \alpha \leq \alpha_1 < \gamma \). Since \( \log(1+u) \leq u \) for all \( u \in (-1, \infty) \), we have

\[
|\alpha_1 - \alpha_0| = \frac{\log(1 + b/(ax^2 \log x))}{\log(z/x)} \leq \frac{b}{ax^2 \log x} \cdot \frac{1}{\log(z/x)}.
\]

From this inequality and \( 1/\log(z/x) \ll a,c,\gamma 1 \), we obtain (3.1).

**Case c > a.** Let

\[
\alpha_0 = \frac{\log(c/a)}{\log(x/z)}, \quad \alpha_1' = \frac{\log(c/a - b/(ax^2 \log x))}{\log(x/z)}.
\]
Since $z \in J_{a,b,c}(x)$, we have $\beta < \alpha_1 < \alpha_0 < \gamma$ and $x \ll_{a,b,c,\beta,\gamma} z$. Then by the calculation in Case $a > c$, $f(x_0) = b > 0$. Further, $x \ll z$ implies $z^{-\alpha_1} \leq z^{-\beta} \ll x^{-\beta}$. Thus if $x_0$ is sufficiently large, we have $z^{-\alpha_1} \leq 1/(x^2 \log x)$, which yields

$$f(\alpha_1) = z^{\alpha_1}(a(x/z)^{\alpha_1} + bz^{-\alpha_1} - c) \leq z^{\alpha_1}(a(x/z)^{\alpha_1} + b/(x^2 \log x) - c) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x, z)$ of $f$ such that $\beta < \alpha_1 \leq \alpha \leq \alpha_0 < \gamma$. Since $|\log(1 - u)| \leq 2u$ for all $u \in (0, 1/2)$, we have

$$|\alpha_0 - \alpha_1| = \left| \frac{\log(1 - b/(cx^2 \log x))}{\log(x/z)} \right| \leq \frac{2b}{cx^2 \log x} \cdot \frac{1}{\log(x/z)}$$

provided $x_0$ is sufficiently large. From this inequality and $1/\log(x/z) \ll_{a,c,\gamma} 1$, we obtain (3.1). \quad \Box

**Lemma 3.2.** There exists $C > 0$ such that for all integers $x \geq x_0$ and $z \in J_{1,1,1}(x)$, we can find $\alpha = \alpha(x, z) \in (\beta, \gamma)$ such that $x^\alpha + (x + (x \lfloor \log x \rfloor)^{-1})^\alpha = z^\alpha$, and

$$|\alpha - \frac{\log 2}{\log(z/x)}| \leq \frac{C}{x^2 \log x}.$$ 

**Proof.** Take any $x \geq x_0$ and $z \in J_{1,1,1}(x)$. For all $u \in \mathbb{R}$, consider the continuous function $f(u) = x^u + (x + (x \lfloor \log x \rfloor)^{-1})^u - z^u$, and set

$$\alpha_0 = \frac{\log 2}{\log(z/x)}, \quad \alpha_1 = \frac{\log 2}{\log(\frac{z}{x + (x \lfloor \log x \rfloor)^{-1}})}.$$

By $z \in J_{1,1,1}(x)$, we get $\beta < \alpha_0 < \alpha_1 < \gamma$. By the definitions of $\alpha_0$ and $\alpha_1$, we have

$$f(\alpha_0) > z^{\alpha_0}\left(\frac{1}{2} + \frac{1}{2} - 1\right) = 0, \quad f(\alpha_1) < z^{\alpha_1}\left(\frac{1}{2} + \frac{1}{2} - 1\right) = 0.$$

Therefore, by the intermediate value theorem, there exists a zero $\alpha = \alpha(x, z)$ of $f$ such that $\alpha_0 \leq \alpha \leq \alpha_1$. Further, we deduce (3.2) since

$$|\alpha_1 - \alpha_0| \leq \frac{\gamma^2}{\log 2} \log\left(1 + \frac{1}{x^2 \log x}\right) \leq \frac{\gamma^2}{\log 2} \cdot \frac{1}{x^2 \log x}. \quad \Box$$

**Lemma 3.3.** Let $\varepsilon > 0$ be an arbitrarily small real number. For all $X, Y, Z \in \mathbb{N}$, and $\alpha \in \mathbb{R}$ with $\beta < \alpha < \gamma$, if

$$aX^\alpha + bY^\alpha = cZ^\alpha,$$

then there exists $n_0 \in \mathbb{N}$ such that

$$a\lfloor (n_0 X)^\alpha \rfloor + b\lfloor (n_0 Y)^\alpha \rfloor = c\lfloor (n_0 Z)^\alpha \rfloor,$$

$$\max\{(n_0 X)^\alpha, (n_0 Y)^\alpha, (n_0 Z)^\alpha\} < 1/2,$$
Therefore

Hence

This yields

\[ n_0 \ll \varepsilon (X + Y)^{\gamma^2/((2+\beta) - 2^{1-\beta})(2-\gamma)) + \varepsilon. \]

**Proof.** Choose \( X, Y, Z \in \mathbb{N} \) and \( \alpha \) with \( \beta < \alpha < \gamma \) satisfying (3.3). For all \( n \in \mathbb{N} \),

\[ c[(nZ)^\alpha] = c(nZ)^\alpha - c\{(nZ)^\alpha\} = a\{(nX)^\alpha\} + b\{(nY)^\alpha\} + \delta(n), \]

where we define \( \delta(n) = a\{(nX)^\alpha\} + b\{(nY)^\alpha\} - c\{(nZ)^\alpha\} \). Let

\[ A = \{ n \in \mathbb{N} : |\delta(n)| < 1, \max\{(nX)^\alpha\}, \{(nY)^\alpha\}, \{(nZ)^\alpha\} < 1/2 \}, \]

and note that any \( n \in A \) satisfies (3.4) and (3.5). Let us show the existence of \( n \in A \) satisfying (3.6). Take a small \( \xi = \xi(d, \beta, \gamma, \varepsilon) > 0 \) and take a sufficiently large parameter \( R = R(a, b, c, d, \beta, \gamma, \varepsilon) \). Set

\[ N = \lceil R(X + Y)^{\gamma^2/((2+\beta) - 2^{1-\beta})(2-\gamma)) + \varepsilon \rceil, \]

and set \( \psi = \{ \beta \} - 2 + (2^{d+2} - 2)(1/2^d - 2\xi) \). Since

\[ \psi = 2 + \{ \beta \} - 2^{1-\beta} + O(\xi), \]

we have \( 0 < \psi < \beta < \alpha \) for \( \xi \) small enough. Moreover, we let \( L(h_1, h_2) = (h_1X^\alpha + h_2Y^\alpha)/c \).

**Case 1.** We first discuss the case when

\[ \lvert L(h_1, h_2) \rvert \geq N^{-\psi} \]

for all \( h_1, h_2 \in \mathbb{Z} \) with \( 0 < \max(|h_1|, |h_2|) \leq N^\xi \). In this case, define

\[ A_1 = \left\{ n \in \mathbb{N} : 0 \leq \{(nX)^\alpha/c\} < \frac{1}{4ac}, 0 \leq \{(nY)^\alpha/c\} < \frac{1}{4bc} \right\}. \]

Then we have \( A_1 \subseteq A \). Indeed, take any \( n \in A_1 \). We see that

\[ (nX)^\alpha = c\{(nX)^\alpha/c\} + c\{(nX)^\alpha/c\}. \]

Since the first term on the right-hand side of (3.11) is an integer and the second term belongs to \( [0, 1] \) by \( n \in A_1 \), we get \( \{(nX)^\alpha\} = c\{(nX)^\alpha/c\} \). This yields \( \{(nX)^\alpha\} < 1/(4a) \). Similarly, \( \{(nY)^\alpha\} < 1/(4b) \). Further,

\[ \{(nZ)^\alpha\} \leq a\{(nX)^\alpha/c\} + b\{(nY)^\alpha/c\} < a\{(nX)^\alpha/c\} + b\{(nY)^\alpha/c\} < \frac{1}{2c}. \]

Hence

\[ |\delta(n)| \leq a\{(nX)^\alpha\} + b\{(nY)^\alpha\} + c\{(nZ)^\alpha\} < \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1. \]

Therefore \( A_1 \subseteq A \).

We now evaluate the distribution of \( A_1 \). Let \( D_1(N) \) be the discrepancy of the sequence \( (\{(nX)^\alpha/c\}, \{(nY)^\alpha/c\})_{N<n\leq2N} \). Then (2.1) with \( K = \lfloor N^\xi \rfloor \) implies that

\[ D_1(N) \ll N^{-\xi} + \sum_{0 < \|\{(h_1, h_2)\|_\infty \leq N^\xi} \frac{1}{\nu(h_1, h_2)} \left| \frac{1}{N} \sum_{N<n\leq2N} e(L(h_1, h_2)n^\alpha) \right|. \]
For all \( u \in \mathbb{R} \), define \( f(u) = L(h_1, h_2)u^\alpha \). For each \( N < u \leq 2N \),
\[
|L(h_1, h_2)|N^{\alpha-(d+2)} \ll |f^{(d+2)}(u)| \ll |L(h_1, h_2)|N^{\alpha-(d+2)}.
\]
Therefore Lemma 2.1 with \( k = d + 2 \) yields
\[
\frac{1}{N} \sum_{N < n \leq 2N} e(L(h_1, h_2)n^\alpha)
\ll \left( |L(h_1, h_2)|N^{\alpha-(d+2)} \right)^{1/(2d+2-2)} + \frac{|L(h_1, h_2)|N^{\alpha-(d+2)} - 1/(2d+2-2)}{N^{1/2d}} \ll \left( L(N^\xi, N^\xi)N^{\gamma-2} \right)^{1/(2d+2-2)} + \frac{N^{(2-\{\beta\}+\psi)/(2d+2-2)}}{N^{1/2d}},
\]
where in the last inequality we used \( \alpha - d < \{\gamma\} \) and \( d + 2 - \alpha < 2 - \{\beta\} \).
By the definition of discrepancy, it follows that \( (2-\{\beta\}+\psi)/(2d+2-2) - 1/2d = -2\xi \).
Then
\[
\frac{1}{N} \sum_{N < n \leq 2N} e(L(h_1, h_2)n^\alpha) \ll ((X + Y)^\gamma N^{\gamma-2+\xi})^{1/(2d+2-2)} + N^{-2\xi}.
\]
Therefore, since
\[
\sum_{0 < \| (h_1, h_2) \|_\infty \leq N^\xi} \frac{1}{\nu(h_1, h_2)} \ll (\log N^\xi)^2 \ll \xi N^\xi/(2d+2-2),
\]
we have
\[
(3.12) \quad D_1(N) \ll \xi N^{-\xi} + ((X + Y)^\gamma N^{\gamma-2+2\xi})^{1/(2d+2-2)}.
\]
Let \( E_1(N) \) be the right-hand side of (3.12). By the definition of discrepancy,
\[
\frac{\#(A_1 \cap (N, 2N])}{N} = \frac{1}{16abc^2} + O_\xi(E_1(N)).
\]
By (3.7), we have
\[
(3.13) \quad (X + Y)^\gamma N^{\gamma-2+2\xi} \ll R^{\gamma-2+2\xi}(X + Y)^\epsilon.
\]
Here
\[
\epsilon = \gamma + \{\gamma\} - 2 + 2\xi \left( \frac{\gamma^2}{(2 + \{\beta\} - 2^{1-\lceil\beta\rceil})(2 - \{\gamma\})} + \varepsilon \right)
= \gamma \left( 1 - \frac{\gamma}{2 + \{\beta\} - 2^{1-\lceil\beta\rceil}} \right) - \varepsilon(2 - \{\gamma\}) + O(\xi)
\leq \gamma \cdot \frac{2 + \{\beta\} - \gamma}{2 + \{\beta\} - 2^{1-\lceil\beta\rceil}} - \varepsilon(2 - \{\gamma\}) + O(\xi) < 0
\]
for \( \xi \) small enough. This yields
\[
E_1(N) \ll \xi R^{-\xi} + R^{\{\gamma\}-2+2\xi)/(2d+2-2)}.
\]
Therefore if $\xi$ is sufficiently small and $R$ is sufficiently large, then
\[
\frac{1}{16abc^2} + O_\xi(E_1(N)) \geq \frac{1}{32abc^2}.
\]
Hence, in this case, \(#(A \cap (N, 2N]) \geq #(A_1 \cap (N, 2N]) \geq N/(32abc^2) > 0\),
which implies that there exists $n_0 \in A$ satisfying (3.6).

**Case 2.** We next discuss the case when (3.9) is false, that is, there exist $h_1, h_2 \in \mathbb{Z}$ with $0 < \max(|h_1|, |h_2|) \leq N^\xi$ such that
\[
|L(h_1, h_2)| < N^{-\psi}.
\]
We observe that $h_1$ and $h_2$ are non-zero and have opposite signs, since if not, then $1/c \leq |L(h_1, h_2)| < N^{-\psi}$, which causes a contradiction when $R$ is sufficiently large. Thus we may assume that $h_1 < 0 < h_2$ by multiplying both sides of (3.14) by $|(−1)|$ if necessary. Let $h'_1 = −h_1$ and $\theta = L(h_1, h_2)/h_2$.

In the case $\theta \geq 0$, let
\[
A_2 = \left\{ n \in [1, N^{\psi/\alpha}/(8bc)^{1/\alpha}] \cap \mathbb{N}: 0 \leq ((nX)^{\alpha}/(ch_2)) < \frac{1}{8abcN^\xi} \right\};
\]
then $A_2 \subseteq A$. To see why, suppose $n \in A_2$. Then
\[
(nX)^\alpha/c = h_2\{((nX)^\alpha/(ch_2))\} + h_2\{(nX)^\alpha/(ch_2)\},
\]
where the first term is an integer and the second term belongs to $[0, 1)$. This yields $\{(nX)^\alpha/c\} = h_2\{(nX)^\alpha/(ch_2)\}$. Thus we obtain $0 \leq \{(nX)^\alpha/c\} < 1/(4ac)$. Further, since
\[
(nY)^\alpha/c = \frac{h'_1}{ch_2}(nX)^\alpha + n^\alpha\theta = h'_1\{((nX)^\alpha/(ch_2))\} + h'_1\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta,
\]
$h'_1\{((nX)^\alpha/(ch_2))\} \in \mathbb{Z}$, 
\[
0 \leq h'_1\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta < \frac{1}{8bc} + \frac{1}{8bc} = \frac{1}{4bc},
\]
we have $\{(nY)^\alpha/c\} = h'_1\{(nX)^\alpha/(ch_2)\} + n^\alpha\theta$ and $0 \leq \{(nY)^\alpha/c\} < 1/(4bc)$. Hence, $A_2 \subseteq A_1 \subseteq A$.

We next evaluate the distribution of $A_2$. Let $V = N^{\psi/\alpha}/(2(8bc)^{1/\alpha})$, and $D_2(N)$ be the discrepancy of the sequence $((nX)^\alpha/(ch_2))_{V < n \leq 2V}$. Then by (2.1) with $K = [N^{2\xi}]$
\[
D_2(N) \ll \frac{1}{N^{2\xi}} + \sum_{0 < |h| \leq N^{2\xi}} \frac{1}{|h|} \left| \sum_{V < n \leq 2V} e((h/(ch_2))X^\alpha n^\alpha) \right|.
\]
From Lemma 2.1 with $k = d + 2$ we deduce
\[
D_2(N) \ll \frac{1}{N^{2\xi}}
\]
\[
+ \sum_{0 < |h| \leq N^{2\xi}} \frac{1}{|h|} \left( \left( \frac{|h|X^\alpha}{ch_2} \right)^{1/(2d+2)-2} \right) + \left( \frac{|h|X^\alpha}{ch_2} \right)^{1/(2d+2)-2} \right) \right).
\]

We see that

\[
\sum_{0 < |h| \leq N^{2\xi}} \frac{1}{|h|} \left( \frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2} \right)^{1/(2d+2-2)} \\
\leq (X^\gamma V^{\{\gamma\}-2})^{1/(2d+2-2)} \cdot 2 \sum_{1 \leq h \leq N^{2\xi}} h^{-1+1/(2d+2-2)} \\
\ll (X^\gamma V^{\{\gamma\}-2})^{1/(2d+2-2)} \cdot N^{2\xi/(2d+2-2)}.
\]

In addition, since \(d - \alpha < 0\) and \(h_2 \leq N^\xi\), we see that

\[
\sum_{0 < |h| \leq N^{2\xi}} \frac{1}{|h|} \left( \frac{|h|X^\alpha}{ch_2} V^{\alpha-d-2} \right)^{-1/(2d+2-2)} V^{1/2d} \\
\leq \left( \frac{ch_2}{X^\alpha} \right)^{1/(2d+2-2)} V^{(2+d-\alpha)/(2d+2-2)-1/2d} \cdot 2 \sum_{h=1}^\infty h^{-1-1/(2d+2-2)} \\
\ll N^{\xi} \cdot V^{1/(2d+1)-1/2d} = N^{\xi} V^{(-1+2^{-d})/(2d+1-1)}.
\]

Hence

\[
D_2(N) \ll \frac{1}{N^{2\xi}} + (X^\gamma N^{2\xi} V^{\{\gamma\}-2})^{1/(2d+2-2)} + N^{\xi} V^{(-1+2^{-d})/(2d+1-1)} \\
\ll \frac{1}{N^{2\xi}} + (X^\gamma N^{2\xi + \psi(\{\gamma\})/(\gamma(2d+1-1))} \\
\ll N^{\xi} V^{\{\gamma\}-2}/\gamma \ll R^{2\xi + \psi(\{\gamma\})/(\gamma(2d+1-1))}.
\]

Let \(E_2(N)\) be the right-hand side. Now by (3.7), we have

\[
(3.16) \quad X^\gamma N^{2\xi + \psi(\{\gamma\})} \gamma \ll R^{2\xi + \psi(\{\gamma\})/(\gamma(2d+1-1))}.
\]

Here

\[
e' = \gamma + \left( 2\xi + \frac{\psi}{\gamma} (\{\gamma\} - 2) \right) \left( \frac{\gamma^2}{(2 + \{\beta\} - 2^1-[\beta])} (2 - \{\gamma\}) + \varepsilon \right) \\
= \gamma - \gamma \cdot \frac{2 + \{\beta\} - 2^1-[\beta]}{2 + \{\beta\} - 2^1-[\beta]} + O(\xi) - \varepsilon \cdot \frac{\psi}{\gamma} (2 - \{\gamma\}) + O(\xi) \\
= -\varepsilon \cdot \frac{\psi}{\gamma} (2 - \{\gamma\}) + O(\xi),
\]

where we have used (3.8). This implies that for \(\xi\) small enough,

\[
E_2(N) \ll N^{-2\xi} + (R^{2\xi + \psi(\{\gamma\})} \gamma (X + Y)^e')^{1/(2d+2-2)} \\
+ N^{\xi + \psi(-1+2^{-d})/(\gamma(2d+1-1))} \\
\ll N^{-2\xi}.
\]

Therefore, by making \(\xi\) smaller and \(R\) larger if necessary, we get

\[
\frac{\#(A_2 \cap (V, 2V))}{V} = \frac{1}{8abcN^\xi} + O(E_2(N)) \geq \frac{1}{16abcN^\xi} > 0.
\]
Hence, there exists \( n_0 \in A \) such that
\[
n_0 \ll_{\varepsilon} \frac{(X + Y)^{\psi/\alpha} \gamma^2}{((2+\{\beta\} - 2^{-\lfloor \beta \rfloor})(2-\{\gamma\}) + \varepsilon},
\]
which implies the inequality (3.6) since \( \psi < \alpha \). In the case \( \theta < 0 \), let \( \theta' = L(h_1, h_2)/h_1 > 0 \). By switching the roles of \((\theta, X^{\alpha})\) and \((\theta', Y^{\alpha})\), and by a similar argument to the case \( \theta \geq 0 \), we also find \( n_0 \in A \) satisfying (3.6). \( \blacksquare \)

**Lemma 3.4.** For all \( \alpha > 0 \) and \( X, Y, Z \in \mathbb{N} \), define
\[
\eta(\alpha, X, Y, Z) = \min \left\{ \frac{\log([W^{\alpha}] + 1)W^{-\alpha}}{\log W} : W = X, Y, Z \right\}.
\]
For all \( \alpha > 0 \) and \( X, Y, Z \in \mathbb{N} \), if \( a[X^{\alpha}] + b[Y^{\alpha}] = c[Z^{\alpha}] \), then for all \( \tau \in (\alpha, \alpha + \eta(\alpha, X, Y, Z)) \), we have
\[
a[X^{\tau}] + b[Y^{\tau}] = c[Z^{\tau}].
\]

**Proof.** The claim is clear since we observe that
\[
[X^{\alpha}] = [X^{\tau}], \quad [Y^{\alpha}] = [Y^{\tau}], \quad [Z^{\alpha}] = [Z^{\tau}]
\]
for all \( \tau \in (\alpha, \alpha + \eta(\alpha, X, Y, Z)) \). \( \blacksquare \)

**4. Lemmas II.** Let \( 2 \leq \beta < \gamma \), and let \( a, b, c \in \mathbb{N} \) as in the previous section. Let \( x_0 > 0 \) be a large parameter. For each \( x \geq x_0 \), let \( K(x) \subseteq \mathbb{N} \) be a non-empty finite set. For each \( x \geq x_0 \) and \( z \in K(x) \), let \( \theta(x, z) \) and \( \ell(x, z) \) be positive real numbers, and define an interval \( I(x, z) = (\theta(x, z), \theta(x, z) + \ell(x, z)) \). For each \( x \geq x_0 \), define
\[
G_x = \bigcup_{z \in K(x)} I(x, z).
\]

Let us consider the following conditions:

(C1) for all integers \( x \geq x_0 \), \( K(x) \) does not contain any multiples of \( x \);
(C2) for all integers \( x \geq x_0 \) and \( z \in K(x) \), if \( z \neq \max K(x) \), then \( z + 1 \in K(x) \) or \( z + 2 \in K(x) \);
(C3) there exists \( Q_1 > 0 \) such that for all \( x \geq x_0 \),
\[
\max(\inf \{|\beta - \alpha| : \alpha \in G_x\}, \inf \{|\gamma + x^{-2} - \alpha| : \alpha \in G_x\}) \leq Q_1 x^{-1};
\]
(C4) there exists a real number \( \kappa \in (0, \infty) \setminus \{1\} \) such that for all \( x \geq x_0 \) and \( z \in K(x) \),
\[
\theta(x, z) = \frac{\log \kappa}{\log(z/x)} + O\left(\frac{1}{x^2 \log x}\right);
\]
(C5) there exist \( Q_2, Q_3 > 0 \) and \( q > 2 \) such that for all \( x \geq x_0 \) and \( z \in K(x) \),
\[
Q_2 x^{-q} \leq \ell(x, z) \leq Q_3 x^{-\beta};
\]
(C6) for all integers \( x \geq x_0 \), \( G_x \subseteq (\beta, \gamma + x^{-2}) \);
(C7) for all integers $x \geq x_0$ and $z \in K(x)$, there exists a pairwise distinct tuple $(X(x, z), Y(x, z), Z(x, z)) \in \mathbb{N}^3$ such that for all $\tau \in I(x, z)$,

$$a[X(x, z)^\tau] + b[Y(x, z)^\tau] = c[Z(x, z)^\tau], \quad X(x, z) \geq x.$$ 

**Proposition 4.1.** Suppose that there exist $x_0, K(x), \theta(x, z)$, and $\ell(x, z)$ satisfying (C1) to (C7). Let $q$ be as in (C5). Then

$$\dim_H(\{\alpha \in [\beta, \gamma]: ax + by = cz \text{ is solvable in PS(\alpha)}\}) \geq 2/q.$$

**Remark 4.2.** The idea of the proof of Proposition 4.1 comes from the proof of Jarník's theorem in Falconer's book [Fal14, Theorem 10.3]. Jarník's theorem states that for every $q > 2$ the set of $\alpha \in [0, 1]$ such that there exist infinitely many $x, z \in \mathbb{N}$ with $|\alpha - z/x| \leq x^{-q}$ has Hausdorff dimension $2/q$.

The goal of this section is to prove Proposition 4.1. Suppose that there exist $x_0, K(x), \theta(x, z)$, and $\ell(x, z)$ satisfying (C1) to (C7), and choose such $x_0, K(x), \theta(x, z)$, and $\ell(x, z)$. Let $Q_1, Q_2, Q_3, \kappa, q$ be as in (C3) to (C5). Let $x_1 > 0$ and $U_1 > 0$ be large parameters depending on $a, b, c, d, \beta, \gamma, Q_1, Q_2, Q_3, \kappa, x_0, q$. Below we do not indicate the dependence of those parameters. Let $p$ denote a variable running over prime numbers.

**Lemma 4.3.** There exists $B_1 > 0$ such that for all $p \geq x_1$ and distinct $z, z' \in K(p)$, the intervals $I(p, z)$ and $I(p, z')$ are separated by a gap of at least $B_1 p^{-1}$ if $x_1$ is sufficiently large.

**Proof.** By (C4) and (C6), for all $p \geq x_1$ and $z \in K(p)$, we have

$$(4.1) \quad \frac{\beta}{2} \leq \frac{\log \kappa}{\log(z/p)} \leq 2\gamma$$

if $x_1$ is sufficiently large. This implies that

$$(4.2) \quad p \ll z \ll p.$$ 

By (C4) and the inequalities (4.1) and (4.2), there exists $B_0 > 0$ such that

$$|\theta(p, z) - \theta(p, z')| = \left|\frac{\log \kappa}{\log z} - \frac{\log \kappa}{\log z'} + O\left(\frac{1}{p^2 \log p}\right)\right| \geq \frac{\log \kappa}{\log z} \left|\log z - \log z'\right| + O\left(\frac{1}{p^2 \log p}\right) \geq \frac{\beta^2}{4|\log \kappa|} \log\left(\frac{z + 1}{z}\right) + O\left(\frac{1}{p^2 \log p}\right) \geq B_0 p^{-1}$$

for all $p \geq x_1$ and all $z, z' \in K(p)$ with $z < z'$. Further, since $\ell(p, z) \leq Q_3 p^{-2}$ by (C5), there exists $B_1 > 0$ such that for all $p \geq x_1$ and distinct $z, z' \in K(p)$, the intervals $I(p, z)$ and $I(p, z')$ are separated by a gap of at least

$$(4.3) \quad B_0 p^{-1} - Q_3 p^{-2} \geq B_1 p^{-1}$$

if $x_1$ is sufficiently large. \[\square\]
Now we call the open interval $I(p, z)$ ($z \in K(p)$) a basic interval of $G_p$ for all $p \geq x_1$. For each $U \geq U_1$, define
\[ H_U = \bigcup_{U < p \leq 2U} G_p. \]
For all $U < p \leq 2U$, we also call a basic interval of $G_p$ a basic interval of $H_U$.

**Lemma 4.4.** There exist $B_2, B_3 > 0$ such that for any $U \geq U_1$, all distinct basic intervals of $H_U$ are separated by gaps of at least $B_2 U^{-2}$, and the length of each basic interval of $H_U$ is at least $B_3 U^{-q}$ if $U_1$ is sufficiently large.

**Proof.** We take distinct prime numbers $p$ and $p'$ with $U < p, p' \leq 2U$. Then, for all $z \in K(p)$ and $z' \in K(p')$, the condition (C4), the inequality (4.1), and the mean value theorem imply that
\[ |\theta(p, z) - \theta(p', z')| \geq \left| \frac{\log \kappa}{\log(z/p)} - \frac{\log \kappa}{\log(z'/p')} \right| + O\left( \frac{1}{U^2 \log U} \right) \]
\[ \geq \frac{\beta^2}{4|\log \kappa|} \left| \frac{z}{p} - \frac{z'}{p'} \right| \min\left( \frac{p}{z}, \frac{p'}{z'} \right) + O\left( \frac{1}{U^2 \log U} \right). \]
We may assume that $p'/z' > p/z$. By (C1), $z$ and $p$ are coprime, which yields
\[ |zp' - z'p| \geq 1. \]
Therefore
\[ \left| \frac{z}{p} - \frac{z'}{p'} \right| \min\left( \frac{p}{z}, \frac{p'}{z'} \right) = \left| \frac{z}{p} - \frac{z'}{p'} \right| \frac{p}{z} \geq \frac{1}{p'z} \gg U^{-2} \]
by the inequalities (4.2) and $U < p, p' \leq 2U$. Therefore for all $U \geq U_1$,
\[ |\theta(p, z) - \theta(p', z')| \gg U^{-2} \]
if $U_1$ is sufficiently large. Further, for all $U < p \leq 2U$ and $z \in K(p)$, we deduce by (C5) that $\ell(p, z) \ll U^{-\beta}$, where $\beta \geq 2$. Hence there exists $D_1 > 0$ such that for all distinct prime numbers $U < p, p' \leq 2U$, and all $z \in K(p)$ and $z' \in K(p')$, the intervals $I(p, z)$ and $I(p', z')$ are separated by gaps of at least $D_1 U^{-2}$. By combining this with Lemma 4.3, there exists $D_2 > 0$ such that distinct basic intervals of $H_U$ are separated by gaps of at least $D_2 U^{-2}$. Furthermore by (C5), for all $U < p \leq 2U$ and $z \in K(p)$, we have $Q_2 \cdot 2^{-q} U^{-q} \leq \ell(p, z)$. In conclusion, we find that all distinct basic intervals of $H_U$ are separated by gaps of at least $B_2 U^{-2}$, and have length of at least $B_3 U^{-q}$, where we let $B_2 = D_2$ and $B_3 = Q_2 \cdot 2^{-q}$. ■

**Lemma 4.5.** There exists $B_4 > 0$ such that the following statement holds: for every $U \geq U_1$, if an open interval $I \subset (\beta, \gamma + p^{-2})$ satisfies
\[ 3B_4 / \text{diam}(I) < U < p \leq 2U, \]
then the open interval $I$ completely includes at least
\[ \frac{U^2}{6B_4 \log U} \cdot \text{diam}(I) \] basic intervals of $H_U$. 
Proof. By (C4), (4.1), and (4.2), there exists $D_3 > 0$ such that for every $z \in K(p)$ and the minimum $z' \in K(p)$ with $z' > z$,

$$|\theta(p, z) - \theta(p, z')| = \left|\frac{\log \kappa}{\log(z/p)} - \frac{\log \kappa}{\log(z'/p)} + O\left(\frac{1}{p^2 \log p}\right)\right| \leq \frac{4\gamma^2}{|\log \kappa|} \cdot \frac{1}{z} \cdot |z - z'| + O\left(\frac{1}{p^2 \log p}\right) \leq D_3 p^{-1}. \tag{4.7}$$

Here we apply (C2) when we deduce the last inequality. From (C3), (C6) and (4.7), there exists $B_4 > 0$ such that

$$(\beta, \gamma + p^{-2}) \subseteq (\beta, \beta + B_4 p^{-1}) \cup \bigcup_{z \in K(p)} (\theta(p, z), \theta(p, z) + B_4 p^{-1}) \cup (\gamma + p^{-2} - B_4 p^{-1}, \gamma + p^{-2}).$$

Therefore for all $U \geq U_1$ and $U < p \leq 2U$, any open interval $I \subset (\beta, \gamma + p^{-2})$ satisfying (4.5) completely includes at least $B_4^{-1} p \cdot \text{diam}(I) - 2 \geq (3B_4)^{-1} U \cdot \text{diam}(I)$ basic intervals of $G_p$. Hence, by the prime number theorem, the open interval $I$ completely includes at least (4.6) basic intervals of $H_U$ for a large enough $U_1$.

Proof of Proposition 4.1. Let $B_3$ and $B_4$ be constants as in Lemma 4.4 and Lemma 4.5 respectively. Let $u_1 = \max(U_1, 2)$. For every $k = 2, 3, \ldots$, we put

$$u_k = \max(u_{k-1}^k, \lfloor 3(B_4/B_3) u_{k-1}^q \rfloor),$$

and $B_5 = B_3/(6B_4)$. Let $E_1$ be the open interval $(\beta, 2\gamma)$. For every $k = 2, 3, \ldots$, let $E_k$ be the union of basic intervals of $H_{u_k}$ which are completely included by $E_{k-1}$. Let $F$ be the intersection of all $E_k$'s. Define $m_1 = 1$, and for $k \geq 2$, define

$$m_k = \frac{u_k^2}{6B_4 \log u_k} B_3 u_{k-1}^{-q} = B_5 \frac{u_k^2 u_{k-1}^{-q}}{\log u_k}.$$

Lemma 4.4 implies that each $(k - 1)$st level interval of $F$ has length at least $B_3 u_{k-1}^{-q}$. Then, by Lemma 4.5 each $(k - 1)$st level interval of $F$ contains at least $m_k$ $k$th level intervals. In addition, by Lemma 4.4, disjoint $k$th level intervals of $F$ are separated by gaps of at least $\delta_k = B_2 u_k^{-2}$. Therefore, Lemma 2.3 implies that

$$\dim_H(F) \geq \lim_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(\delta_k m_k)} = \lim_{k \to \infty} \frac{2 \log u_{k-1} + \log(B_5^{k-2} u_1^{-q} (u_2 \cdots u_{k-2})^{2-q} (\log u_2)^{-1} \cdots (\log u_{k-1})^{-1})}{q \log u_{k-1} + \log \log u_k - \log(B_2 B_5)}.$$

Since $u_k \geq u_{k-1}$ for all $k \geq 2$, we have $\log u_k \geq k! \log u_1$ and $u_k \geq u_{k-1}$. Further, for $k \geq 1$ large enough, we have $u_k = u_{k-1}^k$. Thus for $k \geq 1$ large
enough, we see that
\[ 2 \log u_{k-1} = 2k^{-1} \log u_k, \quad q \log u_{k-1} = qk^{-1} \log u_k, \]
\[ | \log(B_{k-2}^k u_{k-1}^{-q} (u_2 \cdots u_{k-2})^{2-q} \log u_2)^{-1} \cdots (\log u_{k-1})^{-1}) | \ll \log u_{k-2}. \]
Therefore, since \( \log u_{k-2}/\log u_k = 1/(k(k-1)) \to 0 \) as \( k \to \infty \), we get
\[ \dim_H(\bigcap_{k=1}^\infty E_k) \geq \frac{2}{q}. \]

We finally show that for any \( \tau \in F \), the equation \( ax + by = cz \) is solvable in \( \text{PS}(\tau) \) and \( \tau \in [\beta, \gamma] \). If this claim is true, we get the conclusion of Proposition 4.1 by the monotonicity of the Hausdorff dimension.

Take any \( \tau \in F \). It is clear that \( \tau \in [\beta, \gamma] \) since the condition (C6) yields \( H_{u_k} \subseteq (\beta, \gamma + u_k^{-2}) \), which implies \( F \subseteq [\beta, \gamma] \). Further, by (C7), for all \( k > 1 \), there exist a prime number \( u_k < p_k \leq 2u_k \) and \( z_k \in K(p_k) \) such that we find a pairwise distinct tuple \( (X(p_k, z_k), Y(p_k, z_k), Z(p_k, z_k)) \in \mathbb{N}^3 \) such that
\[ a[X(p_k, z_k)^\tau] + b[Y(p_k, z_k)^\tau] = c[Z(p_k, z_k)^\tau], \quad X(p_k, z_k) \geq p_k. \]
Since \( X(p_k, z_k) \geq p_k \geq u_k \to \infty \) as \( k \to \infty \), the equation \( ax + by = cz \) is solvable in \( \text{PS}(\tau) \). ■

5. Proof of Theorem 1.1 Fix any \( a, b, c \in \mathbb{N} \). Without loss of generality, we may assume that either \( a \neq c \) or \( a = b = c = 1 \). Let \( \varepsilon > 0 \) be arbitrarily small. Let \( d = [s] \) and choose real numbers \( \beta, \gamma \) with \( d \leq s \leq \beta < \gamma < \min(t, d+1) \). Let \( x_0 = x_0(a, b, c, d, \beta, \gamma) \) be as in Section 3. By the monotonicity of the Hausdorff dimension, we have
\[ \dim_H(\{ \alpha \in [s, t] : ax + by = cz \text{ is solvable in } \text{PS}(\alpha) \}) \]
\[ \geq \dim_H(\{ \alpha \in [\beta, \gamma] : ax + by = cz \text{ is solvable in } \text{PS}(\alpha) \}). \]
Take \( \alpha(x, z) \) as in Lemmas 3.1 and 3.2. Let \( K(x) = J_{a,b,c}(x), \theta(x, z) = \alpha(x, z) \). We give \( \ell(x, z) \) later. Let us check the conditions (C1) to (C7), and apply Proposition 4.1.

Case \( a > c \). By Lemma 3.1, for all \( x \geq x_0 \) and \( z \in J_{a,b,c}(x) \) we have
\[ ax^{\alpha(x,z)} + b = cz^{\alpha(x,z)}. \]
Thus by Lemma 3.3, there exists \( n_0 = n_0(x, z) \in \mathbb{N} \) such that
\[ a[(n_0x)^\alpha] + b[n_0^\alpha] = c[(n_0z)^\alpha], \]
\[ \max\{\{n_0x^\alpha\}, \{n_0^\alpha\}, \{n_0z^\alpha\}\} < 1/2, \]
\[ n_0 \ll \varepsilon x^{\gamma^2/(2+\{\beta\}-2^{-1-\{\beta\}}\{2-\{\gamma\}\})+\varepsilon}. \]
Define \( \eta \) as in Lemma 3.4. Let \( \ell(x, z) = \eta(\alpha(x, z), n_0x, n_0, n_0z) \). The condition (C1) is clear from the definition of \( J_{a,b,c}(x) \). The condition (C2) is also clear since we find at most one multiple of \( x \) among any three consecutive
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Let \( x_0 \geq 3 \). Lemma 3.1 implies (C4). By Lemma 3.4 for each \( x \geq x_0 \) and \( z \in J_{a,b,c}(x) \), each \( \tau \in (\alpha(x,z), \alpha(x,z) + \ell(x,z)) \) satisfies
\[
a\lfloor (n_0 x)^\tau \rfloor + b\lfloor n_0^\tau \rfloor = c\lfloor (n_0 z)^\tau \rfloor, \quad n_0 x \geq x.
\]
Therefore we have (C7). Let us prove (C3), (C5), (C6).

We show (C3). Let \( x \) be an integer with \( x \geq x_0 \). For each \( i \in \{1, 2\} \), let
\[
z_{1,i} = \left\lfloor \left( \frac{b}{cx^2 \log x} + \frac{a}{c} \right)^{1/\gamma} x \right\rfloor + i, \quad z_{2,i} = \lfloor (a/c)^{1/\beta} x \rfloor - i.
\]
Note that \( J_{a,b,c}(x) \) does not contain multiples of \( x \). Thus we do not know whether \( z_{1,i}, z_{2,i} \in J_{a,b,c}(x) \) for each \( i \in \{1, 2\} \). However, by (C2), there exist \( i_1, i_2 \in \{1, 2\} \) such that \( z_{1,i_1}, z_{2,i_2} \in J_{a,b,c}(x) \). Lemma 3.1 implies that
\[
\alpha(x, z_{1,i_1}) = \frac{\log(a/c)}{\log(z_{1,i_1}/x)} + O\left( \frac{1}{x^2 \log x} \right).
\]
Here we have
\[
\log(z_{1,i_1}/x) = \log\left( \left( \frac{b}{cx^2 \log x} + \frac{a}{c} \right)^{1/\gamma} + O(x^{-1}) \right)
= \frac{1}{\gamma} \log(a/c) + \log \left( 1 + O\left( \frac{b}{a \gamma x^2 \log x} \right) + O(x^{-1}) \right)
= \frac{1}{\gamma} \log(a/c) + O(x^{-1}).
\]
Therefore
\[
\alpha(x, z_{1,i_1}) = \frac{\log(a/c)}{\gamma \log(a/c) + O(x^{-1})} + O\left( \frac{1}{x^2 \log x} \right) = \gamma + O(x^{-1}).
\]
Similarly, we have \( \alpha(x, z_{2,i_2}) = \beta + O(x^{-1}) \). Hence we obtain (C3).

We next show (C5). For all \( x \geq x_0 \) and \( z \in J_{a,b,c}(x) \), we have \( x < z \) by the definition of \( J_{a,b,c}(x) \). Recall that
\[
\ell(x, z) = \eta(\alpha(x, z), n_0 x, n_0, n_0 z) = \frac{\log((\lfloor W^\alpha \rfloor + 1)W^{-\alpha})}{\log W},
\]
where \( W \) is one of \( n_0 x, n_0, \) or \( n_0 z \). From \( \beta < \alpha(x, z) \), we have \( \ell(x, z) \leq \log(1 + (n_0 x)^{-\beta}) \leq x^{-\beta} \). Further, by the facts \( (5.3), (5.4) \), \( 1 < x < z \ll x \), and \( \alpha < \gamma \), we have
\[
\ell(x, z) \geq \frac{\log(1 + 2^{-1}W^{-\alpha})}{\log W} \gg \frac{1}{(n_0 z)\gamma \log(n_0 z)} \gg \epsilon \cdot x^{-q},
\]
where
\[
q = q(\beta, \gamma, \epsilon) = (\gamma + \epsilon) \left( \frac{\gamma^2}{(2 + \{\beta\} - 2^{1-\lfloor\beta\rfloor})(2 - \{\gamma\})} + 1 + \epsilon \right).
\]
Therefore (C5) holds (with $Q_3 = 1$). The remaining condition (C6) is clear since $\beta < \alpha(x, z) < \gamma$ and $\alpha(x, z) + \ell(x, z) < \gamma + x^{-2}$ by (C5) (with $Q_3 = 1$).

Case $c > a$. Define $n_0 = n_0(x, z)$ and $\ell(x, z)$, $q(\beta, \gamma, \varepsilon)$ the same way as in Case $a > c$. The condition (C1) is clear since $z < x$ by the definition of $J_{a,b,c}(x)$. The condition (C2) is also clear since $J_{a,b,c}(x)$ forms a set of consecutive integers. Lemma 3.1 implies (C4). Similarly to the discussion in Case $a > c$, we have (C5)–(C7). To show (C3), let $x$ be an integer with $x \geq x_0$. Let

$$z_1 = \left(\frac{a}{c - b(x^2 \log x)^{-1}}\right)^{1/\beta} + 1, \quad z_2 = [(a/c)^{1/\gamma}x] - 1.$$ 

We observe that $z_1, z_2 \in J_{a,b,c}(x)$ if $x_0$ is sufficiently large. Lemma 3.1 implies that $\alpha(x, z_1) = \beta + O(x^{-1})$ and $\alpha(x, z_2) = \gamma + O(x^{-1})$. This gives (C3).

Case $a = b = c = 1$. By Lemma 3.2 for all $x \geq x_0$ and $z \in J_{1,1,1}(x)$, by letting $X = X(x, z) = x^2\log x$, $Y = Y(x, z) = x^2\log x + 1$, $Z = Z(x, z) = zx\log x$, we have

$$X^{\alpha(x,z)} + Y^{\alpha(x,z)} = Z^{\alpha(x,z)}.$$ 

Therefore, from Lemma 3.3 there exists $n_0 = n_0(x, z) \in \mathbb{N}$ such that

$$[(n_0X)^{\alpha}] + [(n_0Y)^{\alpha}] = [(n_0Z)^{\alpha}],$$

$$\max\{(n_0X)^{\alpha}, (n_0Y)^{\alpha}, (n_0Z)^{\alpha}\} < 1/2,$$

$$n_0 \ll_{\varepsilon} (X + Y)^{\gamma^2/((2 + \{\beta\} - 2^{1 - \{\beta\}})(2 - \{\gamma\})) + \varepsilon}.$$

Defining $r = r(\gamma, \beta, \varepsilon) = \gamma^2/((2 + \{\beta\} - 2^{1 - \{\beta\}})(2 - \{\gamma\})) + \varepsilon$, we obtain

$$n_0 \ll_{\varepsilon} x^{(2+\varepsilon)r}.$$ 

Let $\ell(x, z) = \eta(\alpha(x, z), n_0X, n_0Y, n_0Z)$ be as in Lemma 3.4.

The condition (C1) is clear since $x < z < 2x$ by the definition of $J_{1,1,1}(x)$. The condition (C2) is also clear since $J_{1,1,1}(x)$ forms a set of consecutive integers. Lemma 3.2 implies (C4). By Lemma 3.4 for all $x \geq x_0$ and $z \in J_{1,1,1}(x)$, each $\tau \in (\alpha(x, z), \alpha(x, z) + \ell(x, z))$ satisfies

$$[(n_0X)^{\tau}] + [(n_0Y)^{\tau}] = [(n_0Z)^{\tau}], \quad n_0X \geq x.$$ 

Therefore (C7) holds. It remains to prove (C3), (C5), and (C6).

Let us show (C3). Take any integer $x \geq x_0$. Let

$$z_1 = [2^{1/\gamma}(x + (x[\log x])^{-1})] + 1, \quad z_2 = [2^{1/\beta}x] - 1.$$ 

It follows that $z_1, z_2 \in J_{1,1,1}(x)$ if $x_0$ is sufficiently large. Then Lemma 3.2 implies that $\alpha(x, z_1) = \gamma + O(x^{-1})$ and $\alpha(x, z_2) = \beta + O(x^{-1})$. Therefore we have (C3).
We next show (C5). Let \( x \) be an integer with \( x \geq x_0 \) and \( z \in J_{1,1,1}(x) \). It is clear that \( x < z \) and \( X(x) < Y(x) < Z(x, z) \). Recall that

\[
\ell(x, z) = \eta(\alpha(x, z), n_0X, n_0Y, n_0Z) = \frac{\log \left(\left\lfloor W^\alpha \right\rfloor + 1\right) W^{-\alpha}}{\log W},
\]

where \( W \) is one of \( n_0X, n_0Y, \) or \( n_0Z \). Therefore, as \( \beta < \alpha \), we have \( \ell(x, z) \leq \log(1 + (n_0Z)^{-\beta}) \leq Z^{-\beta} \leq x^{-\beta} \). Further, from (5.5), (5.6) and \( \alpha < \gamma \), we obtain

\[
\ell(x, z) \geq \frac{\log(1 + 2^{-1}W^{-\alpha})}{\log W} \gg \frac{1}{(n_0Z)^\gamma \log(n_0Z)} \gg \epsilon x^{-2(\epsilon)(\gamma + \epsilon)(r+1)}.
\]

Hence, (C5) holds. The condition (C6) is clear since \( \beta < \alpha(x, z) < \gamma \) and \( \alpha(x, z) + \ell(x, z) < \gamma + x^{-2} \) by (C5).

To summarize the above discussion, define

\[
D_{a,b,c}(\beta, \gamma, \epsilon) = \begin{cases} 
2 & \text{if } a = b = c, \\
\frac{2}{(2 + \epsilon)(\gamma + \epsilon)(r(\beta, \gamma, \epsilon) + 1)} & \text{otherwise}.
\end{cases}
\]

Cases \( a > c, c > a, a = b = c = 1 \) and Proposition 4.1 imply that

\[
\dim_H(\{\alpha \in [\beta, \gamma] : ax + by = cz \text{ is solvable in } \text{PS}(\alpha)\}) \geq D_{a,b,c}(\beta, \gamma, \epsilon).
\]

Therefore, by (5.1) and by letting \( \epsilon \to +0, \gamma \to \beta, \beta \to s \), we have

\[
\dim_H(\{\alpha \in [s, t] : ax + by = cz \text{ is solvable in } \text{PS}(\alpha)\}) \geq D_{a,b,c}(s, s, 0).
\]

By the definitions of \( q \) and \( r \), we get the conclusion of Theorem 1.1.

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