Stability and instability of standing-wave solutions to one-dimensional quadratic-cubic Klein–Gordon equations

Daniele Garrisi

Abstract. We study the stability of standing-waves solutions to a scalar non-linear Klein–Gordon equation in dimension one with a quadratic-cubic non-linearity. Orbits are obtained by applying the semigroup generated by the negative complex unit multiplication on a critical point of the energy constrained to the charge.

Mathematics Subject Classification. Primary 35Q55, 47J35.

Keywords. Stability, Sturm-Liouville, Klein–Gordon equation.

Introduction

This work aims to classify the stability and instability of standing-wave solutions to the non-linear Klein–Gordon equation which can be written in the form

\[
\frac{du}{dt} = JE'(u(t)),
\]

where \( E \) is a functional defined on \( X_r := H^1_1(\mathbb{R}; \mathbb{C}) \times L^2(\mathbb{R}; \mathbb{C}) \), the space of complex-valued functions which are radially symmetric with respect to the origin, and \( J \) is an unbounded linear transformation \( J : X^*_r \to X_r \). We will apply the abstract method devised in \([14,15]\), and in \([28,29]\). We define the set

\[
\text{Orb}(u_\omega) := \{ T(s\omega)u_\omega \mid s \in \mathbb{R} \}
\]

where \( T(s) \) is a continuous semi-group of operators acting on \( X_r \). The equation to which we would like to apply this method is the quadratic-cubic one-dimensional non-linear Klein–Gordon equation

\[
(\partial^2_{tt} - \partial^2_{xx} + m^2)\phi + G'(|\phi|)\frac{\phi}{|\phi|} = 0 \quad \text{(NLKG)}
\]

This work has been supported by the PDE Research Group of the Department of Mathematical Sciences of the University of Nottingham Ningbo China and funded by the FoSE New Researchers Grant.
where
\[ G(s) = -as^3 + bs^4, \quad s \in \mathbb{R} \]  
and \( m, a \) and \( b \) are positive real numbers. A standing-wave is a solution to (NLKG) which can be written as
\[ \phi(t, x) := e^{-i\omega t} R(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R} \]
where \( \omega \) is a real number and \( R \) is a real-valued function of class \( H^1_r(\mathbb{R}; \mathbb{R}) \).

Dispersive equations with competing powers non-linearities have been proposed for several applications. In [4], a one-dimensional cubic-quintic non-linear Schrödinger equation arises from boson gas interaction. The non-linear Klein-Gordon equation models the field equation for spin-0 particles, [19, §2]; for a cubic-quintic non-linear Klein–Gordon equation, [28] proved that there are stable and unstable standing-waves when \( \omega \) gets close to \( 13/16 \) and 1, respectively, a result anticipated by the numerical inspection in [3]. We also quote the work of G. Rosen, [27], for a quintic non-linearity in dimension three.

The functional \( E \) of (1) is defined as
\[
E(\phi, \psi) := \frac{1}{2} \int_{-\infty}^{+\infty} |\psi(x)|^2 dx + \frac{1}{2} \int_{-\infty}^{+\infty} |\phi'(x)|^2 dx + \frac{m^2}{2} \int_{-\infty}^{+\infty} |\phi(x)|^2 dx + \int_{-\infty}^{+\infty} G(|\phi(x)|) dx.
\]
We also define
\[
Q(\phi, \psi) := -\text{Im} \int_{-\infty}^{+\infty} \psi(x)\overline{\phi(x)} dx.
\]

In this work, we are going to verify the assumptions stated in [14], which are necessary to the application of their abstract framework. It is known that one of the key ingredients is the construction of a one-parameter family \( u: (\omega_1, \omega_2) \rightarrow X_r \) such that
\[
E'(u_\omega) = \omega Q'(u_\omega).
\]

Such construction is done in Proposition 6. We use the approach devised in [10, 16, 24] and properties of one-dimensional elliptic equations proved in [9]. In Theorem 2 of Chapter 3, we prove [14, Assumption 3]. Therefore, according to [14, Theorem 2], to determine the stability of the set (2) it is sufficient to study the convexity of the function
\[
d(\omega) := E(u_\omega) - \omega Q(u_\omega).
\]
We introduce the following real-valued function defined on \((0, +\infty)^3\)
\[
\tau(a, b, m) := \frac{2m^2b}{a^2}.
\]
In the main Theorem, we prove that \( \tau \) alone is sufficient to classify the stability of all the standing-waves for every \( m > 0 \) and non-linearity \(-3as^2 + 4bs^3\) with \( a, b > 0 \).

**Theorem 1.** (Stability and instability) There exist \( \tau_* > 1 \) such that

(i) if \( \tau(a, b, m) \geq \tau_* \), then \( \text{Orb}(u_\omega) \) is stable for every \( \omega \in (\omega^*, m) \).
(ii) if $1 < \tau(a, b, m) < \tau_*$, there are $\omega_1(\tau)$ and $\omega_2(\tau)$ such that $\text{Orb}(u_\lambda)$ is stable for
\[
\omega \in (\omega_*, \omega_1(\tau)) \cup (\omega_2(\tau), m)
\]
and unstable for
\[
\omega_1(\tau) \leq \omega \leq \omega_2(\tau).
\]
(iii) if $0 < \tau(a, b, m) \leq 1$, there exists $\omega_3(\tau)$ such that $\text{Orb}(u_\lambda)$ is stable for
\[
\omega \in (\omega_*, \omega_3(\tau))
\]
and unstable for
\[
\omega \in [\omega_3(\tau), m).
\]

The constant $\tau_*$ is an irrational number as it will be clear from the construction. Numerical approximations show that $\tau_* \in (1.13, 1.14)$, while the proof that $\tau_* > 1$ follows from simple properties of an auxiliary function (labelled $k_2$ in the proof of Theorem 1). Though different differential equations and systems are studied, for the technique deployed, other references, as [14, Examples A,C,D,E], [24,25] and [1], are worth to be mentioned. In particular, [25] addresses the quadratic-cubic non-linear Schrödinger equation, in the case where $a < 0$ and $b > 0$. Calculations done in Theorem 1 take advantage on the assumption that $G$ is a cubic-quartic non-linearity. We can recall a few more cases, where the convexity of $d$ has been studied in its entire domain, as the non-linear Klein–Gordon equation with a pure-power non-linearity, in [28], and the one-dimensional non-linear Schrödinger equation in [1,24,25].

For the sake of clarity, we remark that the set in (2) is different from the ground-state, defined as
\[
\Gamma := \{e^{-is\omega}u_\omega(\cdot - y) \mid (s, y) \in \mathbb{R}^2\}
\]
which includes argument translation together with a phase change. The stability of the ground-state for the non-linear Klein-Gordon equation has been studied in [7], while we refer to [5,8,13,23] and [20] for the non-linear Schrödinger equation. In these works, the Concentration-Compactness Lemma of [21,22] is an essential ingredient.

In Chapter 1, we are going to introduce further operators necessary to state the assumptions of [14], and provide definitions of quantities already mentioned in (1) and (2), namely $J$ and $T$. In Chapter 2 we construct the one-parameter family $u_\omega$, while in Chapter 3 we address the spectral properties the Hessian of $E - \omega Q$. The conclusions of the first three chapters hold as long as $G$ is $C^3$, while in Chapter 4, we take full advantage of the quadratic-cubic structure of the non-linearity, which allows us to find an exact form for $d(\omega)$.

1. Preliminary notations

We write the non-linear Klein–Gordon equation in the Hamiltonian form (1). We also prove that the assumptions made in the introduction of [14] on $J,
For every $z \in \mathbb{C}$, as and $\lim_{T \to 5^1}$ where the authors proved that traveling waves are not stable. We set $JI$ on $(\xi, \eta)$ for every $\alpha, \beta$. There exists a sequence $(\psi_n, \phi_n) \in L^2_r \times H^1_r$ such that $I_0(\psi_n, \phi_n) = \xi_n$ and $\lim_{n \to \infty} (\phi_n, -\psi_n) = (\alpha, \beta)$. For every $(h, k) \in X_r$ we have $(\psi_n, h_2) + (\phi_n, k_2) = (\xi_n, (h, k))$. Taking the limit, we obtain $\xi = I_0(-\beta, \alpha)$, proving that $\xi \in D(J)$. Also, $J\xi = u$. Therefore, $J$ is closed. Given $\xi = I_0(\psi_1, \phi_1)$ and $\eta = I_0(\psi_2, \phi_2)$, there holds

\[
\langle \xi, J\eta \rangle = \langle \xi, JI_0(\psi_1, \phi_1) \rangle = \langle \xi, (\phi_2, -\psi_2) \rangle
\]

\[
= \langle I_0(\psi_1, \phi_1), (\phi_2, -\psi_2) \rangle = (\psi_1, (\phi_2, -\psi_2)_2 - (\phi_1, \psi_2)_2
\]

\[
= -[(\psi_2, \phi_1)_2 - (\phi_2, \psi_1)_2] = -\langle I_0(\psi_2, \phi_2), (\phi_1, -\psi_1) \rangle
\]

\[
= -\langle I_0(\psi_2, \phi_2), JI_0(\psi_1, \phi_1) \rangle = -\langle \eta, J\xi \rangle.
\]

Finally, $J$ is onto. Given $u = (\phi, \psi) \in H^1_r \times L^2_r$, $I(-\psi, \phi) \in D(J)$ and $JI(-\psi, \phi) = u$. \hfill $\square$
The semigroup $T$ We define $T(s): X_r \rightarrow X_r$ as $T(s)(\phi, \psi) = e^{-is}(\phi, \psi)$ for every $s \in \mathbb{R}$.

**Proposition 2.** $T(s)J = JT^*(-s)$ for every $s \in \mathbb{R}$.

**Proof.** The two unbounded operators have the same domain $D(J)$. Given $\xi \in D(J)$, there exists $(\psi, \phi) \in L^2_r \times H^1_r$ such that $\xi = I_0(\psi, \phi)$. Then

$$\langle T^*(-s)\xi, (h, k) \rangle = \langle \xi, (e^{is}h, e^{is}k) \rangle = (\psi, e^{is}h)_2 + (\phi, e^{is}k)_2 = (e^{-is}\psi, h)_2 + (e^{-is}\phi, k)_2.$$

Therefore,

$$JT^*(-s)I_0(\psi, \phi) = JI_0(e^{-is}\psi, e^{-is}\phi) = (e^{-is}\phi, -e^{is}\psi) = e^{-is}(\phi, -\psi) = T(s)JI_0(\psi, \phi).$$

$\square$

The operator $B$ We define the bounded operator $B: X_r \rightarrow X_r^*$ as $B(u) := I_0(i\psi, -i\phi)$.

**Proposition 3.** For every $u, v \in X_r$, there holds

(i) $Q(u) = \frac{1}{2}\langle Bu, u \rangle$

(ii) $JB(u) = T'(0)u$

(iii) $\langle Bu, v \rangle = \langle Bv, u \rangle$.

**Proof.** (i)

$$\frac{1}{2}\langle Bu, u \rangle = \frac{1}{2}\langle B(\phi, \psi), (\phi, \psi) \rangle = \frac{1}{2}(i\psi, \phi)_2^2 - \frac{1}{2}(i\phi, \psi)_2^2$$

$$= \frac{1}{2}\text{Re} \int_{-\infty}^{\infty} i\psi \bar{\phi} dx + \frac{1}{2}\text{Re} \int_{-\infty}^{\infty} -i\bar{\phi} \psi dx$$

$$= -\frac{1}{2}\text{Im} \int_{-\infty}^{\infty} \psi \bar{\phi} dx + \frac{1}{2}\text{Re} \int_{-\infty}^{\infty} -i\bar{\phi} \psi dx$$

$$= -\frac{1}{2}\text{Im} \int_{-\infty}^{\infty} \psi \bar{\phi} dx + \frac{1}{2}\text{Re} \int_{-\infty}^{\infty} i\psi \bar{\phi} dx$$

$$= -\frac{1}{2}\text{Im} \int_{-\infty}^{\infty} \psi \bar{\phi} dx - \frac{1}{2}\text{Im} \int_{-\infty}^{\infty} \bar{\phi} \psi dx = Q(u).$$

(ii) $T'(0) \in \mathcal{L}(X_r, X_r)$ is the infinitesimal generator of the semigroup $T$ and $T'(0)u = -iu$. We obtain the equality between the two operators from

$$JB(u) = JI_0(i\psi, -i\phi) = (-i\phi, -i\psi) = T'(0)u.$$

(iii) We set $u := (\phi_1, \psi_1)$ and $v := (\phi_2, \psi_2)$. Then,

$$\langle Bu, v \rangle = (i\psi_1, \psi_2)_2 - (i\phi_1, \psi_2)_2 = -(\psi_1, i\psi_2)_2 + (\phi_1, i\psi_2)_2$$

$$= (i\psi_2, \phi_1)_2 - (i\phi_2, \psi_1)_2 = \langle Bv, u \rangle.$$

$\square$
2. Regularity assumptions

In this section, we check the regularity assumptions listed in the introduction of [14], and construct a one-parameter family \( u_\omega \) of solutions to \( E'(u_\omega) = \omega Q'(u_\omega) \). The whole construction depends on results of differentiability of Nemitski operators, and compactness of certain linear operators. We summarize these results in the next proposition.

**Proposition 4.** (i) Given \( g \) in \( C^1(\mathbb{R}) \) such that \( g(0) = 0 \), the map \( \phi \mapsto g \circ \phi \) is \( C(H^1, H^1) \) to bounded domains of \( \mathbb{R} \).

(ii) if \( g \) is \( C^2(\mathbb{R}) \) and \( g(0) = g'(0) = 0 \), then \( \phi \mapsto g \circ \phi \) is \( C^1(H^1, H^1) \) and

\[
\langle \mathcal{F}'(\phi_0), \phi \rangle = g'(\phi_0) \phi
\]

where \( \mathcal{F}(\phi) := g \circ \phi \).

(iii) Given \( a \in C^1_0(\mathbb{R}, \mathbb{R}) \), the linear operator mapping \( \phi \) to \( \mathcal{F}(\lambda)(a(x)\phi(x)) \) is \( \mathcal{L}_c(H^1, H^1) \) for every \( \lambda < 0 \).

(iv) \( E \) and \( Q \) are \( C^2(H^1, \mathbb{C}) \). Moreover, given \( u_0 := (\phi_0, \psi_0) \)

\[
\langle E'(u_0), (\phi, \psi) \rangle = (\phi'_0, \phi'_2) + (\psi_0, \psi_2) + \left( G'(|\phi_0|) \frac{\phi_0(x)}{|\phi_0(x)|}, \phi \right)_2
\]

\[
\langle Q'(u_0), (\phi, \psi) \rangle = (\phi_0, i\psi_2) + (\phi, i\psi_2)_2
\]

for every \( (\phi, \psi) \in X_r \) and

\[
\langle E''(u_0)(\phi, \psi), (h, k) \rangle = (\phi', h'_2) + (\psi, k') + m^2(\phi, h)_2
\]

\[
+ \left( G''(|\phi_0|) \text{Re}(\phi), h \right)_2 + (|\phi_0|^{-1}G'(|\phi_0|) i\text{Im}(\phi), h')_2
\]

and

\[
\langle Q''(u_0)(\phi, \psi), (h, k) \rangle = (\phi, ik)_2 + (h, i\psi)_2.
\]

In (i) and (ii) and (iv), the proof follows from the application of ideas illustrated in [2, Theorem 2.6]. The quoted theorem proves \( C(L^p, L^{op}) \) regularity, with \( \Omega \) bounded domain of \( \mathbb{R}^n \). However, their technique can be adapted to our setting, by taking advantage of the bounded inclusion \( H^1_x \subseteq L^\infty \). In (iii), roughly speaking, the multiplication by \( a \) allows to reduce to bounded domains of \( \mathbb{R} \), where it is known that the resolvent of the Laplacian is compact. In the remainder of this section, we check that Assumptions 1, 2 of [14]. For the convenience of the reader, we are also going to repeat here these assumptions.

**Assumption 1.** For every \( u_0 \in X_r \), there exists \( t_0 > 0 \) and a solution \( u \) to (1) on the interval \([0, t_0)\) such that

(a) \( u(0) = u_0 \)

(b) \( E(u(t)) = E(u_0) \) and \( Q(u(t)) = Q(u_0) \) for every \( 0 \leq t < t_0 \).

According to [12, §3], (NLKG) is locally well-posed in \( X_r \), meaning that for every initial datum \((\phi_0, \psi_0)\), there exists a unique solution \( \phi: [0, T) \times \mathbb{R} \to \mathbb{C} \) to (NLKG) such that

\[
\phi(t, \cdot) \in C_t H^1_x([0, T) \times \mathbb{R}) \cap C_t^1 L^2_x([0, T) \times \mathbb{R}).
\]

Moreover, \( E \) and \( Q \) are constant on the trajectory \((\phi(t, \cdot), \partial_t \phi(t, \cdot))\).
Assumption 2. There exist real $\omega_1 < \omega_2$ and a mapping $\omega \mapsto u_\omega$ from $(\omega_1, \omega_2)$ into $X_r$ such that for every $\omega \in (\omega_1, \omega_2)$,

(a) $u_\omega$ is $C^1$ at $\omega$
(b) $E'(u_\omega) = \omega Q'(u_\omega)$
(c) $u_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2)$
(d) $T'(0)u_\omega \neq 0$.

We cover the proof of the existence of this mapping and the four statements (a-d) into the next four propositions. In Proposition 5, we are also going to include properties of $u_\omega$ which are going to turn out to be useful in the next chapters.

Proposition 5. Given $a, b$ and $m$ positive real numbers, there exists $\omega_* > 0$ and one-parameter family $u_\omega := (R_\omega, -i\omega R_\omega)$ such that

(i) $(\omega_*, m) \ni \omega \mapsto u_\omega \in X_r$
(ii) $R_\omega$ is positive, symmetric-decreasing and $xR'_\omega(x) < 0$ unless $x = 0$.

Proof. We define a function $R_\omega$ on an open interval of $\mathbb{R}$ containing the origin as the solution to the initial value problem

\begin{align*}
R''_\omega(x) - G'(R_\omega(x)) - (m^2 - \omega^2)R_\omega(x) &= 0, \\
R_\omega(0) &= R_\omega(\omega), \\
R'_\omega(0) &= 0
\end{align*}

(9)

where $R_\omega(\omega)$ is the first positive solution to the equation

\begin{equation}
V(s) := -\frac{2G(s)}{s^2} = m^2 - \omega^2.
\end{equation}

(10)

From [9, Theorem 5], such solution extends to a positive strictly symmetric-decreasing function on $\mathbb{R}$. Moreover, from [9, Remark 6.3] the functions $R_\omega, R'_\omega$ and $R''_\omega$ have exponential decay, implying that $R_\omega$ is in $H^2_r(\mathbb{R}, \mathbb{R})$. In order to ensure that $R_\omega(\omega)$ exists, we restrict to $\omega > \omega_*$,

\begin{equation}
\omega_* := (m^2 - \sup(V))^{\frac{1}{2}} = \sqrt{m^2 - \frac{a^2}{2b}}
\end{equation}

(11)

which is the smallest $\omega$ such that (10) has at least one solution when $\tau \geq 1$. When $\tau \leq 1$, $\omega_* = 0$. We define $u_\omega := (R_\omega, -i\omega R_\omega)$.

Proposition 6. The function $u: (\omega_*, m) \to X_r$ is $C^1((\omega_*, m), X_r)$.

Proof. We can rely on the argument of [29, Lemma 20], provided adaptation to the spatial dimension $N = 1$ is done. We fix $\omega_0 \in (\omega_*, m)$, $R_0 := R_{\omega_0}$. The equation (9) can be rewritten as

\begin{equation}
R_0 = -\mathcal{A}(\lambda_0)G'(R_0)
\end{equation}

where $\lambda_0 := \omega_0^2 - m^2$. Therefore, it is convenient to define the function

\begin{equation}
\mathcal{A}: (\omega_0^2 - m^2, 0) \times H^1_r(\mathbb{R}, \mathbb{R}) \to H^1_r(\mathbb{R}, \mathbb{R})
\end{equation}

\begin{equation}
\mathcal{A}(\lambda, \phi) := \phi + \mathcal{A}(\lambda)G'(\phi).
\end{equation}

The proof of the regularity of $R_\omega$ takes several steps.

(i) $\mathcal{A}$ is well defined. Since $G'$ is $C^1$ and $G'(0) = 0$, the function $G'(\phi)$ is in $H^1_r$ from (i) of Proposition 4. Since $\mathcal{A}(\lambda)H^1_r \subseteq H^1_r$, we have $\phi + \mathcal{A}(\lambda)G'(\phi) \in H^1_r$.

(ii) It is differentiable at every point $(\lambda_0, \phi_0)$. More precisely,
\begin{equation}
\langle G'(\lambda, \phi_0), (\lambda, \phi) \rangle = \phi + \lambda \mathcal{R}(\lambda_0)^2 G'(\phi_0) + \mathcal{R}(\lambda_0) G''(\phi_0) \phi.
\end{equation}

In fact,
\begin{align*}
\mathcal{G}(\lambda_0 + \lambda, \phi_0 + \phi) &= \phi_0 + \phi + (\mathcal{R}(\lambda_0) + \lambda \mathcal{R}(\lambda_0)^2 + o(\lambda))(G'(\phi_0 + \phi)) \\
&= \phi_0 + \phi + (\mathcal{R}(\lambda_0) + \lambda \mathcal{R}(\lambda_0)^2 + o(\lambda))(G'(\phi_0) + G''(\phi_0) \phi + o(\phi)) \\
&= \mathcal{G}(\lambda_0, \phi_0) + \phi + \lambda \mathcal{R}(\lambda_0)^2 G'(\phi_0) + \mathcal{R}(\lambda_0) G''(\phi_0) \phi + f
\end{align*}

where
\begin{equation}
f = o(\lambda)G'(\phi_0 + \phi) + \lambda \mathcal{R}(\lambda_0)^2 (G''(\phi_0) \phi + o(\phi)) + \mathcal{R}(\lambda_0) o(\phi).
\end{equation}

The first equality follows from [18, p. 174, Theorem 6.7 of §III] and the resolvent equation, [18, p. 36, Eq. (5.5) of §I]. Since $G'$ is $C^2$, and $G'(0) = G''(0) = 0$, the second equality follows from (ii) of Proposition 4, that is the notation $o(\phi)$ applies in the sense of the $H^1$ norm. From $\mathcal{R}(\lambda_0) \in \mathcal{L}(H^1, H^1)$ and (iii) of Proposition 4, $f$ is $o(\lambda, \phi)$. (iii). $\mathcal{G}$ is in $C^1((\omega^2 - m^2, 0) \times H^1, H^1)$. This follows from the continuity of the resolvent, [18, Theorem 6.7 of §III], and (i) of Proposition 4.

(iv). $\partial_\phi \mathcal{G}(\omega^2_0 - m^2, R_0) \in GL(H^1)$. Here, the conclusions apply to a specific point in $H^1$, namely $R_0$, and the compactness result of (iii) of Proposition 4 is used for the first time. Since $G'(0) = G''(0) = 0$ and $R_0$ decays to zero exponentially the operator
\begin{equation}
\partial_\phi \mathcal{G}(\omega^2_0 - m^2, R_0) = \phi + \mathcal{R}(\lambda_0) G''(R_0) \phi
\end{equation}
is a compact perturbation of $I_{H^1}$, by (iii) of Proposition 4. Therefore, it is a Fredholm operator of index zero, by [18, p. 238, Theorem 5.26 of §IV]. We can show that $\partial_\phi \mathcal{G}(\omega^2_0 - m^2, R_0)$ is injective. In fact, given $\phi \in H^1_r$ such that
\begin{equation}
\phi + \mathcal{R}(\omega^2_0 - m^2) G''(R_0) \phi = 0,
\end{equation}
there holds
\begin{equation}
-\phi'' + (m^2 - \omega^2_0) \phi + G''(R_0) \phi = 0. \tag{12}
\end{equation}

In $H^1(\mathbb{R}; \mathbb{C})$, the solutions to (12) are multiples of $R'_0$. However, $R'_0$ is an odd function. Therefore, $\phi \equiv 0$. Since the Fredholm index is zero, the operator is also surjective. Therefore, it is invertible.

Conclusions (i-iv) allow us to apply the Implicit Function Theorem. There exists an open interval $\omega^2_0 - m^2 \in J_0 \subseteq (\omega^2_* - m^2, 0)$ and $\phi: J_0 \to H^1_r$ such that $\mathcal{G}(\lambda, \phi_\lambda) = 0$ for every $\lambda \in J_0$, and $\phi_{\omega^2_0 - m^2} = R_0$. Therefore,
\begin{equation}
-\phi''_\lambda - \lambda \phi_\lambda - G'(\phi_\lambda) = 0. \tag{13}
\end{equation}

To conclude, we prove that $\phi_{\omega^2_0 - m^2}$ coincides with $R_\omega$: the former is regular, $R_\omega$ will as also be regular. There exists $\delta > 0$ such that for $\lambda \in (\omega^2_0 - m^2 - \delta, \omega^2_0 - m^2 + \delta) \cap J_0$, there holds $\phi_\lambda(0) > 0$, because $\phi_{\omega^2_0 - m^2}(0) = R_0(0) > 0$. From [9, Theorem 5], there exists only one even, positive solution, vanishing at infinity to (13). Therefore, $\phi_{\omega^2 - m^2} = R_\omega$. Thus, $R_\omega$ is $C^1((\omega_*, m), H^1_r)$ and $u_\omega \in C^1((\omega_*, m), X_r)$ as claimed. In fact, since the second component is a multiple of the first, the regularity is $C^1((\omega_*, m), H^1_r \times H^1_r)$. \hfill \Box

From Proposition 6, $u \in C^1((\omega_*, m), X_r)$, giving (a) of Assumption 2.
Proposition 7. For every \( \omega \in (\omega_*, m) \), there holds \( (E - \omega Q)'(u_\omega) = 0 \).

Proof. The statement follows by substituting \((\phi_0, \psi_0)\) with \((R_\omega, -i\omega R_\omega)\) in the first-order derivatives of \(E\) and \(Q\) in (iv) of Proposition 4 for every \( \omega \in (\omega_*, m) \). Notice that from (i) of Proposition 4, \( R_\omega \) does not have zeroes. Therefore, \( \phi_0/|\phi_0| \equiv 1 \) is always defined on \((-\infty, +\infty)\). \( \square \)

We proved (b) of Assumption 2. We are going to address (c) and (d) in the next proposition.

Proposition 8. \( u_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2) \) and \( T'(0)u_\omega \neq 0 \).

Proof. Since \( T'(0) \) is bounded, \( D(T'(0)^3) = X_r \). Also, for every \( \omega \in (\omega_*, m) \), there holds \( T'(0)^2u_\omega = (-R_\omega, i\omega R_\omega) \). Since \( R_\omega \) is \( H^2_\omega \), we have
\[
(T'(0)^2u_\omega, (h, k))_{L^2} = (R_\omega' - R_\omega, h_2) + (\omega R_\omega, k_2)
\]
\[
= (R_\omega'' - R_\omega, h_2) + (\omega R_\omega, k_2)
\]

Since \( R_\omega \in H^2_\omega \), \( I(T'(0)^2u_\omega) \in I_0(L^2_r \times H^1_r) \). Therefore, \( I(T'(0)^2u_\omega) \in D(J) \), proving the first of the two statements. The second one follows from
\[
T'(0)u_\omega = (-iR_\omega, -\omega R_\omega) \neq 0 \quad \square
\]

Remark 1. Although it is clear from its definition, we wish to stress that the operator \( I \) appearing in [14, Assumption 2] and in Proposition 7 is the Riesz representation of linear functionals defined in (7) and not \( I_0 \) defined in (8).

Remark 2. When \( \omega = \omega_* \), the constant function \( R_{\omega_*} \equiv R_*(\omega_*) = (2b)^{-1}a \) solves the initial value problem (9), but it is not a square integrable function. When \( \omega = m \), \( R_*(m) = b^{-1}a \). This the “zero-mass” of the problem (9), whose existence in \( H^1_r \cap C^2 \) is guaranteed by [9, §5]. However, since \( R_*(\omega) = R_\omega(0) \to 0 \) as \( \omega \to m \), there is no convergence \( R_\omega \to R_m \) in \( H^1_r \). In conclusion, \((\omega_*, m)\) is a maximal interval of existence of a regular one-parameter family.

3. The spectrum of the Hessian

In this section, we prove [14, Assumption 3] on the spectrum of the Hessian of \( E - \omega_0 Q \). Given \( \omega_0 \in (\omega_*, m) \), the Hessian of \( E - \omega_0 Q \) is defined as the bounded operator in \( \mathcal{L}(X_r, X_r) \) such that
\[
H_{\omega_0}(\phi, \psi) := I^{-1}((E'' - \omega_0 Q'')(\phi, \psi)),
\]
where \( I \) is the Riesz isomorphism from \( X_r \) to \( X'_r \).

Assumption 3. For each \( \omega_0 \in (\omega_*, m) \), \( H_{\omega_0} \) has exactly one negative simple eigenvalue and has its kernel spanned by \( T'(0)u_{\omega_0} \), and the rest of its spectrum is positive and bounded away from zero.

To prove the last assumption, we exploit the result of Sturm-Liouville Theory of [17, p. 228, §10.4] with the following extension: if the assumption that \( u \) has two zeroes is replaced by the assumption that \( u, v \in C^2_0(\mathbb{R}) \), then one can still conclude that \( v \) has at least one zero in \((-\infty, +\infty)\).
Lemma 1. $H_{\omega_0}$ is a self-adjoint, bounded and Fredholm operator of index zero on $X_r$.

Proof. To simplify the notation, we set $H := H_{\omega_0}$. It is convenient to have an explicit expression of the Hessian. Given $(\phi, \psi) \in X_r$, we set $H(\phi, \psi) =: (H_1(\phi, \psi), H_2(\phi, \psi))$. Using $(0, k) \in X_r$ as test vector in the formula for the second derivative in Proposition 4, we obtain

$$H_2(\phi, \psi) = \psi + \omega_0 \phi.$$ 

Using $(h, 0)$ with $h \in H^2_r(\mathbb{R}, \mathbb{C})$, the equality

$$-H''_1 + m^2 H_1 = -\phi'' + m^2 \phi + G''(R_0)Re(\phi) + iR_0^{-1}G'(R_0)Im(\phi) - \omega_0 \psi$$

follows. Applying $\mathcal{R}(-m^2)$ to both sides of the equality, one obtains

$$H_1(\phi, \psi) = \phi + \mathcal{R}(-m^2)(G''(R_0)Re(\phi) + iR_0^{-1}G'(R_0)Im(\phi) - i\omega_0 \psi).$$

Since $H^2_r \times L^2_r$ is a dense subset of $X_r$, the two equalities for $H_1$ and $H_2$ hold on $X_r$. Since $R_0$ vanishes at infinity, both $G''(R_0)$ and $iR_0^{-1}G'(R_0)$ are $C_0^1(\mathbb{R}, \mathbb{C})$. Therefore, the Hessian is a compact perturbation of the operator $A(\phi, \psi) := (\phi - \mathcal{R}(-m^2)(i\omega_0 \psi), i\omega_0 \phi + \psi)$

by (iii) of Proposition 4. $A$ is in $GL(X_r)$. Given $(\alpha, \beta) \in X_r$ the equation $A(\phi, \psi) = (\alpha, \beta)$ can be solved as follows: the second component reads $\psi = \beta - i\omega_0 \phi$. A substitution in the first component gives

$$(I_{H_1^+} - \omega_0^2 \mathcal{R}(-m^2))\phi = \alpha + \omega_0 \mathcal{R}(-m^2)i\beta.$$ 

The operator on the lefthand-side is bounded and invertible in $H^1_r$. In fact, by merely checking operators composition, through the resolvent equation [18, p. 36] one can deduce that the inverse is $I_{H_1^+} + \omega_0^2 \mathcal{R}(\omega_0^2 - m^2)$. Therefore,

$$\phi = (I_{H_1^+} + \omega_0^2 \mathcal{R}(\omega_0^2 - m^2))(\alpha + \omega_0 \mathcal{R}(-m^2)i\beta)$$

and

$$\psi = \beta - i\omega_0(I_{H_1^+} + \omega_0^2 \mathcal{R}(\omega_0^2 - m^2))(\alpha + \omega_0 \mathcal{R}(-m^2)i\beta).$$

Since $A \in GL(X_r)$, it is a Fredholm operator of index zero. Since $H$ is a compact perturbation of a Fredholm operator of index zero, by [18, Theorem 5.26 of §IV], it is Fredholm operator with index zero. 

In the next theorem, we prove Assumption 3.

Theorem 2. For every $\omega_0 \in (\omega_*, m)$,

(a) the kernel of $H$ is spanned by $T'(0)u_{\omega_0}$

(b) the operator $H$ has exactly one negative simple eigenvalue

(c) the rest of its spectrum is positive and bounded away from zero.

Proof. $T'(0)u_{\omega_0} \in \text{ker}(H)$ follows from the remarks preceding [14, Eq. (2.18)].

Now, given an element $(\phi, \psi)$ in the kernel of the Hessian, from (iv) of Proposition 4, we have

$$0 = \langle (E - \omega_0 Q)''(\phi, \psi), (0, k) \rangle = \langle \psi, k \rangle_2 - \omega_0 \langle \phi, ik \rangle_2$$

for every $k \in L^2_r$. Therefore,

$$\psi = -i\omega_0 \phi.$$
Now, we apply the second derivative to \((ih, 0) \in X_r\) for every real-valued function \(h \in H^1_r\). From
\[
\text{Re}(\phi ih) = \text{Im}(\phi)h, \quad \text{Re}(\phi' ih') = \text{Im}(\phi)'h',
\]
it follows
\[
0 = \langle (E - \omega_0 Q)'(\phi, \psi), (ih, 0) \rangle
\]
\[
= (\text{Im}(\phi)', h')_2 + m^2(\text{Im}(\phi), h)_2 + (R_0^{-1} G'(R_0)\text{Im}(\phi), h)_2 - \omega_0(ih, i\psi)_2
\]
\[
= (\text{Im}(\phi)', h')_2 + m^2(\text{Im}(\phi), h)_2 + (R_0^{-1} G'(R_0)\text{Im}(\phi), h)_2 - \omega_0(ih, i\omega_0\phi)_2
\]
\[
= (\text{Im}(\phi)', h')_2 + m^2(\text{Im}(\phi), h)_2 + (R_0^{-1} G'(R_0)\text{Im}(\phi), h)_2 - \omega_0^2(\text{Im}(\phi), h)_2
\]
\[
= (L_-(\text{Im}(\phi)), h)_2.
\]
\[\text{(14)}\]

\(L_-\) is the unbounded operator with domain \(H^2_r \subseteq L^2_r\) defined as
\[
L_-(f) = -f'' + R_0^{-1} G'(R_0)f + (m^2 - \omega_0^2)f.
\]

Therefore, \(L_-(\text{Im}(\phi)) = 0\). The kernel of \(L_-\) has dimension one. Since \(R_0\) is in \(\ker(L_-)\), there exists \(\mu \in \mathbb{R}\) such that \(\text{Im}(\phi) = \mu R_0\). Now, we apply the second derivative to \((h, 0) \in X_r\) for every function \(h \in H^1(\mathbb{R}, \mathbb{R})\). Therefore,
\[
0 = \langle (E - \omega_0 Q)'(\phi, \psi), (h, 0) \rangle
\]
\[
= (\text{Re}(\phi)', h')_2 + m^2(\text{Re}(\phi), h)_2 + (G''(R_0)\text{Re}(\phi), h)_2 - \omega_0(h, i\psi)_2
\]
\[
= (\text{Re}(\phi)', h')_2 + m^2(\text{Re}(\phi), h)_2 + (G''(R_0)\text{Re}(\phi), h)_2 - \omega_0(h, i \cdot (-i\omega_0\phi))_2
\]
\[
= (\text{Re}(\phi)', h')_2 + m^2(\text{Re}(\phi), h)_2 + (G''(R_0)\text{Re}(\phi), h)_2 - \omega_0^2(\text{Re}(\phi), h)_2
\]
\[
= (L_+(\text{Re}(\phi)), h)_2,
\]
where \(L_+\) is the unbounded operator with domain \(H^2 \subseteq L^2\) defined as
\[
L_+(f) = -f'' + G''(R_0)f + (m^2 - \omega_0^2)f.
\]

Therefore, \(L_+(\text{Re}(\phi)) = 0\). From the remarks right after (12) it follows \(\text{Re}(\phi) = 0\). In conclusion,
\[
(\phi, \psi) = \mu (iR_0, \omega_0R_0) = -\mu(-iR_0, -\omega_0R_0) = -\mu T'(0)u_\omega.
\]

(b). From the [30, Proposition 4.2] and [14, p. 187], the operator \(L_+\) has exactly one negative, simple eigenvalue. We use the notation \(-\alpha_{\omega_0}^2\) and \(\chi_{\omega_0} \in H^2_r(\mathbb{R}, \mathbb{R})\) for this eigenvalue and the corresponding eigenvector. There holds
\[
(H(\chi_{\omega_0}, -i\omega_0\chi_{\omega_0}),(\chi_{\omega_0}, -i\omega_0\chi_{\omega_0}))_{X_r} = \alpha_{\omega_0}^2\|\chi_{\omega_0}\|_2^2 < 0.
\]

Therefore, there exists at least one negative eigenvalue. We prove that this eigenvalue is unique. Let \((\phi, \psi)\) be an eigenvector with eigenvalue \(\lambda < 0\), that is
\[
H(\phi, \psi) = \lambda(\phi, \psi).
\]

Taking the \(X_r\) inner product with \((0, k)\), we obtain
\[
(\psi, k)_2 - \omega_0(\phi, ik)_2 = \lambda(\psi, k)_2
\]
for every \(k \in L^2_r\). Then
\[
\psi = \frac{i\omega_0}{\lambda - 1} \phi. \tag{15}
\]
Taking the $X_r$ inner product with $(i \text{Im}(\phi), 0)$, from (14), we obtain
\[
\lambda \|\text{Im}(\phi')\|_2^2 + m^2 \lambda \|\text{Im}(\phi)\|_2^2 = (L_-(\text{Im}(\phi)), \text{Im}(\phi))_2
\]
implies $(L_-(\text{Im}(\phi), \text{Im}(\phi)))_2 \leq 0$. From [30, §3] $L_- \geq 0$. Therefore, $\text{Im}(\phi) = 0$, because $\lambda < 0$. Therefore, the imaginary part of the first component of every eigenvector is zero. We take the inner product with $(h, 0)$, where $h \in H^1_r(\mathbb{R}, \mathbb{R})$. Then
\[
\lambda(\phi', h')_2 + m^2 \lambda(\phi, h)_2 = (H(\phi, \psi), (h, 0))_{X_r}
\]
\[
= (\phi', h') + m^2(\phi, h)_2 + (G''(R_0)\phi, h)_2 - \omega_0(h, i\psi)_2
\]
\[
= (\phi', h') + m^2(\phi, h)_2 + (G''(R_0)\phi, h)_2
\]
\[
- \omega_0(\lambda - 1)^{-1}(h, i \cdot i \omega_0 \phi)_2
\]
\[
= (\phi', h') + m^2(\phi, h)_2 + (G''(R_0)\phi, h)_2
\]
\[
+ \omega_0^2(\lambda - 1)^{-1}(\phi, h)_2.
\]
Therefore, $\phi$ satisfies the second-order differential equation
\[
K_\lambda \phi'' - F''(R_0)\phi - G_\lambda \phi = 0
\]
where
\[
K_\lambda := 1 - \lambda, \quad G_\lambda := -\left(\frac{\omega_0^2}{1 - \lambda} + m^2 \lambda\right).
\]
Suppose that there are two eigenvectors $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ corresponding to negative eigenvalues $\lambda_1 \leq \lambda_2 < 0$. Clearly, $1 - \lambda_1 \geq 1 - \lambda_2 > 1$. That is, $K_{\lambda_1} \geq K_{\lambda_2} > K_0$. Since
\[
\frac{dG_\lambda}{d\lambda} = -\frac{\omega_0^2}{(1 - \lambda)^2} - m^2 < 0
\]
we also have $G_{\lambda_1} \geq G_{\lambda_2} > G_0 := -\omega_0^2$. Now, suppose that $\lambda_1 < \lambda_2$, that is $K_{\lambda_2} < K_{\lambda_1}$. Then from [17, p. 228, §10.4], $\phi_2$ has a zero in $x_* \in (-\infty, +\infty)$. Since $\phi_2$ is even, $|x_*|$ is also a zero of $\phi_2$. Taking the derivative in (9), we obtain
\[
(R_0'')'' - F''(R_0)R_0' + \omega_0^2 R_0' = 0.
\]
Since $\lambda_2 < 0$, we have $K_{\lambda_2} > K_0$ and $G_{\lambda_2} > G_0$. Again, from [17, p. 228, §10.4], $R_0'$ has a zero $x_{**} \in ([x_*], +\infty)$. This contradicts (b) of Proposition 5, according to which $R_0'(x) < 0$ for every $x > 0$. Therefore, $\lambda_1 = \lambda_2 =: \lambda$ and $\phi_1 = k\phi_2$ for some $k \in \mathbb{R}$. From (15),
\[
(\phi_1, \psi_1) = \left(\phi_1, \frac{i \omega_0}{\lambda - 1} \phi_1\right) = \left(k\phi_2, k \frac{i \omega_0}{\lambda - 1} \phi_2\right)
\]
\[
= k \left(\phi_2, \frac{i \omega_0}{\lambda - 1} \phi_2\right) = k(\phi_2, \psi_2)
\]
showing that the unique negative eigenvalue is also simple. We denote it by $\lambda_{\omega_0}$.

(c). Since $\lambda_{\omega_0}$ is the unique negative eigenvalue, the complement of $\{0, \lambda_{\omega_0}\}$ in $\sigma(H)$ is positive. From Lemma 1, $H$ is a self-adjoint Fredholm operator. From [26, Lemma], there exists $a > 0$ such that $\sigma(H) \cap (-a, a)$ is a finite
set of eigenvalues. Then, the essential spectrum is bounded away from the origin. □

4. Stability and instability

Since [14, Assumption 3] is satisfied, by [14, Theorem 2], stability and instability of \((R_0, -i\omega_0 R_0)\) depends on the study of the convexity of the function \(d\), defined in (5), in a neighbourhood of \(\omega_0\). We have

\[
d'(\omega) = (E - \omega Q)'(u_\omega) - Q(u_\omega) = -Q(u_\omega) \\
= -(i \cdot -i\omega R_\omega, R_\omega)_2 = -\omega \|R_\omega\|_2^2 =: -\sigma(\omega).
\]

The third equality follows from (4). Then, we will inspect the sign of the derivative \(\sigma\). Calculations of the next proof take advantage on the assumption that \(G\) is a cubic-quartic non-linearity, as defined in (3).

Proof of Theorem 1. We divide the proof in two steps. In the first one, we evaluate \(d'\). In the second, we study the sign of \(d''\).

First step. We multiply (9) by \(2R'_\omega\) and integrate. Since \(R_\omega\) vanishes at infinity, we have

\[
R'_\omega(x)^2 = (m^2 - \omega^2)R_\omega(x)^2 + 2G(R_\omega(x)).
\]

From (ii) of Proposition 5, \(R_\omega\) is symmetric. Therefore, we restrict it to the interval \((0, +\infty)\). Still, from (ii) of Proposition 5, \(R'_\omega < 0\) on \((0, +\infty)\). Therefore, we can write

\[
R'_\omega(x) = -\sqrt{(m^2 - \omega^2)R_\omega(x)^2 + 2G(R_\omega(x))}.
\]

From the remarks preceding this proof, we have

\[
-d'(\omega) = \sigma(\omega) = 2\omega \int_0^\infty R_\omega(x)^2 \, dx \\
= 2\omega \int_0^\infty \frac{R_\omega(x)^2 R'_\omega(x) \, dx}{\sqrt{(m^2 - \omega^2)R_\omega(x)^2 + 2G(R_\omega(x))}} \\
= 2\omega \int_0^\infty \frac{R_\omega(x) R'_\omega(x) \, dx}{\sqrt{m^2 - \omega^2 - 2aR_\omega(x) + 2bR_\omega(x)^2}} \\
= \omega \int_0^R s ds \frac{1}{\sqrt{m^2 - \omega^2 - 2a \frac{s}{\sqrt{m^2 - \omega^2}} + 2b s^2}} \\
= \omega \int_0^R s ds \frac{1}{\sqrt{m^2 - \omega^2 - 2a \frac{s}{\sqrt{m^2 - \omega^2}} + 2b s^2}} \\
+ \frac{\omega a}{2b} \int_0^R \frac{1}{\sqrt{m^2 - \omega^2 - 2a \frac{s}{\sqrt{m^2 - \omega^2}} + 2b s^2}} =: A + B. \tag{16}
\]

The fifth equality follows from the substitution \(R_\omega(x) = s\) on the interval \((0, +\infty)\). Since \(R_+ (\omega)\) is the first positive zero to \(m^2 - \omega^2 = 2aR_+ (\omega) - 2bR_+ (\omega)^2\), there holds

\[
\left| R_+ (\omega) - \frac{a}{2b} \right| = \frac{a^2}{4b^2} - m^2 - \omega^2 = \frac{a^2}{4b^2} (1 - \alpha^2 (\omega)) \tag{17}
\]

where

\[
\alpha (\omega) = \left( \frac{2b(m^2 - \omega^2)}{a^2} \right)^{\frac{1}{2}}. \tag{18}
\]
In order to find a suitable integration by substitution, we rearrange the argument of the square root
\[ m^2 - \omega^2 - 2as + 2bs^2 \]
\[ = 2aR_*(\omega) - 2bR_*(\omega)^2 - 2as + 2bs^2 \]
\[ = 2b \left( \left( s - \frac{a}{2b} \right)^2 - \left( R_*(\omega) - \frac{a}{2b} \right)^2 \right). \tag{19} \]

From (19), we obtain
\[ A = \frac{\omega}{\sqrt{2b}} \int_0^{R_*(\omega)} \frac{(s - \frac{a}{2b}) \, ds}{\sqrt{(s - \frac{a}{2b})^2 - (R_*(\omega) - \frac{a}{2b})^2}} \]
\[ = \frac{\omega}{\sqrt{2b}} \left[ \sqrt{(s - \frac{a}{2b})^2 - (R_*(\omega) - \frac{a}{2b})^2} \right]_0^{R_*(\omega)} = -\frac{\omega a}{(2b)^{\frac{3}{2}}} \alpha(\omega) \]

and
\[ B = \frac{\omega a}{(2b)^{\frac{3}{2}}} \int_0^{R_*(\omega)} \frac{ds}{\sqrt{(s - \frac{a}{2b})^2 - (R_*(\omega) - \frac{a}{2b})^2}} \]
\[ = \frac{\omega a}{(2b)^{\frac{3}{2}}} \left[ \ln \left| s - \frac{a}{2b} + \sqrt{(s - \frac{a}{2b})^2 - (R_*(\omega) - \frac{a}{2b})^2} \right| \right]_0^{R_*(\omega)} \]
\[ = \frac{\omega a}{(2b)^{\frac{3}{2}}} \ln \left| R_*(\omega) - \frac{a}{2b} \right| - \frac{\omega a}{(2b)^{\frac{3}{2}}} \ln \left| \frac{a}{2b} + \sqrt{\frac{a^2}{4b^2} - (R_*(\omega) - \frac{a}{2b})^2} \right| \]
\[ = \frac{\omega a}{(2b)^{\frac{3}{2}}} \ln \left| \frac{a}{2b} \sqrt{1 - \alpha^2(\omega)} \right| - \frac{\omega a}{(2b)^{\frac{3}{2}}} \ln \left| \frac{a}{2b} + \frac{a}{2b} \alpha(\omega) \right| \]
\[ = \frac{\omega a}{2(2b)^{\frac{3}{2}}} \ln \left( 1 + \alpha(\omega) \right) - \frac{\omega a}{2(2b)^{\frac{3}{2}}} \ln \left( 1 - \alpha(\omega) \right)^2 \]
\[ = \frac{\omega a}{2(2b)^{\frac{3}{2}}} \ln \left( \frac{1 + \alpha(\omega)}{1 - \alpha(\omega)} \right). \]

From (16),
\[ \sigma(\omega) = A + B = -\frac{\omega a}{(2b)^{\frac{3}{2}}} \alpha(\omega) + \frac{\omega a}{2(2b)^{\frac{3}{2}}} \ln \left( \frac{1 + \alpha(\omega)}{1 - \alpha(\omega)} \right). \]

Second step. In order to study the sign of the derivative of $\sigma$, we represent it as the composite function of $\alpha$, which is a strictly decreasing and surjective function from the interval $(\omega_*, m)$ to $(0, 1)$. One can check that $\alpha(\omega_*) = 1$ and $\alpha(m) = 0$ directly from (17) and the definition of $\omega_*$ in (11). From (18) we have
\[ \omega = \frac{a}{\sqrt{2b}} \sqrt{\tau - \alpha(\omega)^2}. \]  

(20)

Therefore,

\[ \sigma(\omega) = \frac{a^2}{8b^2} k_1(\alpha(\omega)), \quad \sigma'(\omega) = \frac{a^2}{8b^2} k_1'(\alpha(\omega)) \alpha'(\omega) \]  

(21)

where

\[ k_1(\alpha) = \sqrt{\tau - \alpha^2} \left( \ln \left( \frac{1 + \alpha}{1 - \alpha} \right) - 2\alpha \right), \quad \alpha \in (0, 1). \]

We have

\[ k_1'(\alpha) = -\frac{\alpha}{\sqrt{\tau - \alpha^2}} \ln \left( \frac{1 + \alpha}{1 - \alpha} \right) + \frac{2\alpha^2 \sqrt{\tau - \alpha^2}}{1 - \alpha^2} + \frac{2\alpha^2}{\sqrt{\tau - \alpha^2}}. \]

Then,

\[ k_1'(\alpha) \sqrt{\tau - \alpha^2} \left( \frac{1 - \alpha^2}{2\alpha^2} \right) = \tau - \frac{1 - \alpha^2}{2\alpha} \ln \left( \frac{1 + \alpha}{1 - \alpha} \right) + 1 - 2\alpha \]

\[ = \tau - \left[ \frac{1 - \alpha^2}{2\alpha} \ln \left( \frac{1 + \alpha}{1 - \alpha} \right) - 1 + 2\alpha \right] \]

\[ =: \tau - k_2(\alpha). \]

From (21), it follows

\[ \sigma'(\omega) = \frac{a^2}{4b^2} \cdot \frac{\alpha^2 \alpha'(\omega)}{(1 - \alpha^2) \sqrt{\tau - \alpha(\omega)^2}} (\tau - k_2(\alpha(\omega))). \]  

(22)

We define

\[ \tau_* := \sup_{\alpha \in (0, 1)} k_2(\alpha). \]

The behaviour of \( k_2 \) at the endpoints is \( \lim_{\alpha \to 0} k_2 = 0 \) and \( \lim_{\alpha \to 1} k_2 = 1 \) proving that \( \tau_* \geq 1 \). Also,

\[ k_2'(\alpha) = -\frac{1}{2} \left( 1 + \frac{1}{\alpha^2} \right) \ln \left( \frac{1 + \alpha}{1 - \alpha} \right) + \frac{1}{\alpha} + 4\alpha \]

Since \( \lim_{\alpha \to 1} k_2'(\alpha) = -\infty \), we can infer that \( \tau_* > 1 \).

**Conclusions**

The case \( \tau \geq \tau_* \). Suppose that the constants \( a, b, m \) in (3) and (6) are such that \( \tau(a,b,m) > \tau_* = \sup(k_2) \). Since \( \tau > k_2(\alpha(\omega)) \) on \( (\omega_*, m) \), from (22) \( \sigma' \) is negative on \( (\omega_*, m) \). Therefore \( d''(\omega) > 0 \). From [14, Theorem 2], the orbit (2) is stable for every \( \omega \in (\omega_*, m) \). When \( \tau(a,b,m) = \tau_* = \sup(k_2) \), \( d'' \) has at least one zero, as \( \tau_* > \max\{k_2(0), k_2(1)\} \), implying that \( \sup(k_2) \) is achieved in the interior of \( 0 \leq \alpha \leq 1 \). We can show that \( d'' \) has exactly one zero. In fact,

\[ \alpha^2 k_2' = f_1 + f_2, \quad f_1 := -\frac{1}{2} (1 + \alpha^2) \ln \left( \frac{1 + \alpha}{1 - \alpha} \right), \quad f_2 := \alpha + 4\alpha^3. \]

Since \( f_1 \) is a strictly monotonically decreasing function and \( f_2 \) is a strictly monotonically increasing function, \( \alpha^2 k_2' \) has at most one zero on \( (0, 1) \). Then \( k_2' \) has at most one zero on \( (0, 1) \), which is the maximum point of \( k_2 \). Set

\[ \alpha_d := \arg\max(k_2). \]  

(23)
Since \( k_2(\alpha) < \tau_* \) for \( \alpha \neq \alpha_d \), there holds \( d''(\omega) > 0 \) unless \( \omega = \omega_d := \alpha^{-1}(\alpha_d) \). From [14, Theorem 2] the orbit (2) is stable for every \( \omega \), as \( d \) is strictly convex in a neighbourhood of \( \omega_d \).

The case \( 1 < \tau < \tau_* \). If the non-linearity (3) satisfies \( \tau(a, b, m) < \tau_* \), cases of instability may occur. Since \( 1 < \tau \), there are two values \( \gamma_1(\tau) > \alpha_d > \gamma_2(\tau) \) such that

\[
k_2(\gamma_1) = k_2(\gamma_2) = \tau,
\]
and \( k_2 < \tau \) on \( (0, \gamma_2) \cup (\gamma_1, 1) \) and \( k_2 > \tau \) on \( (\gamma_2, \gamma_1) \). Therefore, (2) is stable on \( (\omega_*, \omega(\tau)) \cup (\omega(\tau), m) \) and unstable on \( \omega(\tau) \leq \omega \leq \omega_2(\tau) \), where

\[
\omega_i(\tau) = \alpha^{-1}(\gamma_i(\tau))
\]
for \( i = 1, 2 \).

The case \( 0 < \tau \leq 1 \). This is the case where stability and instability subsets of \( (\omega_*, m) \) are both connected. If \( \tau > 0 \), there is a unique \( \gamma_3 = \gamma_3(\tau) \) such that

\[
k_2(\gamma_3) = \tau
\]
and \( k_2(\gamma_3) < \tau \) if \( \alpha < \gamma_3 \) and \( k_2(\gamma_3) > \tau \) if \( \alpha > \gamma_3 \). We set

\[
\omega_3 := \alpha^{-1}(\gamma_3(\tau)).
\]

Therefore, from [14, Theorem 2] (2) is stable on \( (\omega_*, \omega_3(\tau)) \) and unstable on the interval \( [\omega_3(\tau), m) \).

\[\square\]

**Proposition 9.** The frequencies \( \omega_1, \omega_2 : (1, \tau_*) \to (0, +\infty) \) are smooth, monotonic strictly decreasing and increasing respectively; \( \omega_3 : (0, 1] \to (0, +\infty) \) is strictly decreasing. Moreover,

\[
\lim_{\tau \to 1^+} \omega_1(\tau) = 0.
\]

and

\[
\lim_{\tau \to 1^-} \omega_3(\tau) = \lim_{\tau \to 1^+} \omega_2(\tau) = \alpha_m,
\]

where \( \alpha_m \) is the unique point in \( (0, 1) \) such that \( k_2(\alpha_m) = 1 \).

**Proof.** We restrict \( k_2 \) to open intervals where it is invertible. Since the function \( k_2 : (\alpha_m, \alpha_d) \to (1, \tau_*) \) is invertible and \( k'_2 > 0 \), from (24) \( \gamma'_2 > 0 \). From (25) and (18), \( \omega'_2 < 0 \). Since the function \( k_2 : (0, \alpha_m] \to (0, 1] \) is invertible and strictly increasing, from (24) \( \gamma'_3 > 0 \). From (27) and (18), \( \omega'_3 < 0 \). Since \( k_2 \) is invertible on the interval \( (0, \alpha_d) \), we also have \( \lim_{\tau \to 1^+} \gamma_2(\tau) = \gamma_3(1) = \alpha_m \). Therefore, from (20)

\[
\omega_3(1) = \lim_{\tau \to 1^+} \omega_2(\tau) = \frac{a}{\sqrt{2b}} \sqrt{1 - \alpha_m^2}.
\]

Finally, since \( k_2 : (\alpha_d, 1) \to (1, \tau_*) \) is invertible and \( k'_2 < 0 \), from (26), \( \gamma'_1 < 0 \). From (25) and (18), \( \omega'_1 > 0 \). From (24), \( \lim_{\tau \to 1^+} \gamma_1(\tau) = 1 \). From (18), \( \omega_1(\tau) \to 0 \). \[\square\]
Remark 3. The case $\tau = 0$ does not occur due to the restrictions set on $a, b$ and $m$. If we allowed $\tau = 0$, we would obtain a quadratic Klein-Gordon equation, as the non-linearity is the pure power $-3a\omega^2$. This case has been covered in the calculations of [28, p. 325] which apply to the one-dimensional cases as well, even if the author set the restriction $n \geq 3$ in the introduction of the paper. We highlight the sharp change in the stability scenario, as in the cases (ii) and (iii) of Theorem 1, orbits become unstable as $\omega$ increases, while in the pure-power case, orbits become stable as $\omega$ increases.

Remark 4. In general, when one of the two parameters ($a, b$) is fixed and the other one converges to zero, the stability of the limiting non-linearity changes completely. The case $a > 0$ and $b \to 0$ has been described already in Remark 3. When $b > 0$ is fixed and $a \to 0$, $\tau \to +\infty$, thus exceeds $\tau^*$ and all the waves are stable, according to (i) of Theorem 1. However, when $a = 0$, the elliptic equation in (9) becomes

$$R''_\omega - 4bR^3_\omega - (m^2 - \omega^2)R_\omega = 0$$

for which non-trivial solutions in $H^1_r$ do not exist. On the contrary, one can multiply the equation by $2R'_\omega$, integrate and evaluate at $x = 0$, and obtain

$$R^2_\omega(x) - 2bR^4_\omega(x) - (m^2 - \omega^2)R^2_\omega(x) = 0.$$  

As $R_\omega$ is symmetric, $R'_\omega(0) = 0$. Therefore, $-2bR^4_\omega(0) - (m^2 - \omega^2)R^2_\omega(0) = 0$. Thus, $R_\omega \equiv 0$. Therefore, there are not standing-waves the stability one can study.

Data availability statement This manuscript does not come with extra data.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] Adami, Riccardo, Cacciapuoti, Claudio, Finco, Domenico, Noja, Diego: Variational properties and orbital stability of standing waves for NLS equation on a star graph. J. Differ. Equ. 257(10), 3738–3777 (2014)
[2] Ambrosetti, Antonio, Prodi, Giovanni: A primer of nonlinear analysis, Cambridge Studies in Advanced Mathematics, vol. 34. Cambridge University Press, Cambridge (1993)
[3] Anderson, David LT.: Stability of time-dependent particlelike solutions in nonlinear field theories. ii. J. Math. Phys. 12(6), 945–952 (1971)
[4] Barashenkov, I.V., Gocheva, A.D., Makhan’kov, V.G., Puzynin, I.V.: Stability of the soliton-like “bubbles”, Phys. D 34(1-2), 240–254 (1989)
[5] Bartsch, Thomas, Jeanjean, Louis, Soave, Nicola: Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$. J. Math. Pures Appl. (9) **106**(4), 583–614 (2016)

[6] Bartsch, Thomas, Wang, Zhi-Qiang., Wei, Juncheng: Bound states for a coupled Schrödinger system. J. Fixed Point Theory Appl. **2**(2), 353–367 (2007)

[7] Bellazzini, J., Benci, V., Bonanno, C., Micheletti, A.M.: Solitons for the nonlinear Klein-Gordon equation. Adv. Nonlinear Stud. **10**(2), 481–499 (2010)

[8] Bellazzini, J., Benci, V., Ghimenti, M., Micheletti, A.M.: On the existence of the fundamental eigenvalue of an elliptic problem in $\mathbb{R}^N$. Adv. Nonlinear Stud. **7**(3), 439–458 (2007)

[9] Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Rational Mech. Anal. **82**(4), 313–345 (1983)

[10] Garrisi, D., Georgiev, V.: Orbital stability and uniqueness of the ground state for the non-linear Schrödinger equation in dimension one. Discrete Contin. Dyn. Syst. **37**(8), 4309–4328 (2017)

[11] Garrisi, Daniele: On the orbital stability of standing-wave solutions to a coupled non-linear Klein-Gordon equation. Adv. Nonlinear Stud. **12**(3), 639–658 (2012)

[12] Ginibre, J., Velo, G.: The global Cauchy problem for the nonlinear Klein-Gordon equation. II. Ann. Inst. H. Poincaré Anal. Non Linéaire **6**(1), 15–35 (1989)

[13] Gou, Tianxiang, Jeanjean, Louis: Existence and orbital stability of standing waves for nonlinear Schrödinger systems, Nonlinear. Analysis **144**, 10–22 (2016)

[14] Grillakis, Manoussos, Shatah, Jalal, Strauss, Walter: Stability theory of solitary waves in the presence of symmetry. I. J. Funct. Anal. **74**(1), 160–197 (1987)

[15] Grillakis, Manoussos, Shatah, Jalal, Strauss, Walter: Stability theory of solitary waves in the presence of symmetry. II. J. Funct. Anal. **94**(2), 308–348 (1990)

[16] Iliev, Iliya D., Kirchev, Kiril P.: Stability and instability of solitary waves for one-dimensional singular Schrödinger equations. Differ. Integral Equ. **6**(3), 685–703 (1993)

[17] Ince, E.L.: Ordinary Differential Equations. Dover Publications, New York (1944)

[18] Kato, Tosio: Perturbation Theory for Linear Operators, Classics in Mathematics. Springer-Verlag, Berlin (1995)

[19] Lee, T.D.: Particle physics and introduction to field theory, Contemporary Concepts in Physics, vol. 1, Harwood Academic Publishers, Chur, (1981), Translated from the Chinese

[20] Li Houwang, Zou Wenming: Normalized ground states for semilinear elliptic systems with critical and subcritical nonlinearities, J. Fixed Point Theory Appl. **23**(3), Paper No. 43, 30 (2021)

[21] Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(2), 109–145 (1984)

[22] Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. II. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(4), 223–283 (1984)

[23] Liu, Chuangye, Nguyen, Nghiem V., Wang, Zhi-Qiang.: Existence and stability of solitary waves of an $m$-coupled nonlinear Schrödinger system. J. Math. Study **49**(2), 132–148 (2016)
[24] Maeda, Masaya: Stability and instability of standing waves for 1-dimensional nonlinear Schrödinger equation with multiple-power nonlinearity. Kodai Math. J. 31(2), 263–271 (2008)

[25] Ohta, Masahito: Stability and instability of standing waves for one-dimensional nonlinear Schrödinger equations with double power nonlinearity. Kodai Math. J. 18(1), 68–74 (1995)

[26] Phillips, John: Self-adjoint Fredholm operators and spectral flow. Canad. Math. Bull. 39(4), 460–467 (1996)

[27] Rosen, Gerald: Particlelike solutions to nonlinear scalar wave theories. J. Math. Phys. 6, 1269–1272 (1965)

[28] Shatah, Jalal: Stable standing waves of nonlinear Klein–Gordon equations. Comm. Math. Phys. 91(3), 313–327 (1983)

[29] Shatah, Jalal, Strauss, Walter: Instability of nonlinear bound states. Comm. Math. Phys. 100(2), 173–190 (1985)

[30] Weinstein, Michael I.: Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm. Pure Appl. Math. 39(1), 51–67 (1986)

Daniele Garrisi
Room 324, Sir Peter Mansfield Building Department of Mathematical Sciences
University of Nottingham Ningbo China
199 Taikang East Road
Ningbo 315100
China
e-mail: daniele.garris@nottingham.edu.cn

Accepted: March 17, 2023.