Hamiltonian systems related to Invariant Metrics

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Abstract. Via a non degenerate symmetric bilinear form we identify the coadjoint representation with a new representation which acts on the Lie algebra. By inducing the canonical symplectic structure of the coadjoint orbits to the orbits of the new representation, we study Hamiltonians on the orbits by determining the Hamiltonian vector fields for invariant functions and we find commuting conditions under the corresponding Lie-Poisson bracket.

1. Introduction

One of the more succesfully algebraic methods used in analytical Mechanics to investigate Hamiltonian systems is related to the well known Theorem of Adler-Kostant-Symes (AKS-Theorem) [1, 8, 18] (see for example [7]). By using an ad-invariant non-degenerate symmetric bilinear form it is possible to identify the coadjoint representation from the Lie group $G$ into $\text{Aut}(g^*)$ with the adjoint representation from $G$ into $\text{Aut}(g)$ and so to induce a symplectic form on the orbits of the adjoint representation. The AKS-Theorem describes the solutions of the Hamiltonian systems corresponding to ad-invariant functions. Moreover it asserts that the ad-invariant functions are in involution with respect to the corresponding Lie-Poisson bracket. From this started point this scheme were extended by other authors to several situations and applications (see for instance [5, 12, 13, 16, 17]).

In this work we consider a representation which is equivalent to the coadjoint one, by choosing a left invariant pseudo Riemannian metric on the Lie group, that is, a non-degenerate symmetric bilinear form on the corresponding Lie algebra (not necessarily ad-invariant as in the original version of the AKS-scheme). This allows to induce a symplectic structure on the orbits and after that it is natural the study of the corresponding Hamiltonian systems. As in the original case one can derive the respective formulas for the Hamiltonian vector fields, Hamiltonian systems and also prove involution conditions for the invariant functions.

The advantage of working on the Lie algebra lies on the possibility of using the algebra deriving from the Lie bracket. A essential ingredient for the application of the original version of the AKS scheme is the existence of an ad-invariant metric. This is always possible for semisimple Lie algebras, since the Killing form is the natural candidate to be used but it becomes an obstacle for non semisimple Lie groups. In fact it is not longer true that any non semisimple Lie algebra could be equipped with such metric and the algebraic structure of Lie algebras admitting such a metric is rather strong (see for instance [4, 10, 11, 14]).

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In the final section we show some examples. We consider basically nilpotent Lie algebras in low dimensions and by fixing a certain decomposition we concentrate in finding invariant functions. For nilpotent Lie algebras Corwin and Greenleaf [3] exposed a method to find generators of the ring of rational invariant functions. Notice that while the invariant functions are invariant by equivalent representations this could not be the case for the rational invariant ones.

We hope that the present approach helps to bring another insight to different geometrical problems by the use of this tool.

We will assume through this work that Lie groups and their corresponding Lie algebras are real and finite dimensional. The Lie algebra of a Lie group $G$ will be identified with the left invariant vector fields of $G$ and denoted with greek letters as usual.

## 2. Preliminaries

A Lie group $G$ acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation $Ad : G \to Aut(\mathfrak{g})$ given by $Ad(g)X = dI(g)_eX$ for all $g \in G$, $X \in \mathfrak{g}$ where $I(g) : G \to G$ is defined by $I(g)h = ghg^{-1}$ for all $g, h \in G$.

We use de notation $ad_XY$ for $[X,Y]$, where $X,Y$ are elements of the Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$, i. e. the space of linear functionals on $\mathfrak{g}$. The coadjoint representation from $G$ into $Aut(\mathfrak{g}^*)$ is defined by:

$$ (Ad^\ast(g)\phi)(Y) = \phi(Ad(g^{-1})Y) \quad \forall g \in G, \ Y \in \mathfrak{g}, \ \phi \in \mathfrak{g}^*. $$

Under this action of $G$, any point $\phi \in \mathfrak{g}^*$ sweeps out an orbit denoted by $O_\phi = \{Ad^\ast(g^{-1})\phi, \ \text{for all } g \in G\}$. The tangent vectors to the orbit at a point $\phi \in \mathfrak{g}^*$ are the induced vector fields defined as usual by

$$ X^*_\phi = \frac{d}{ds}|_{s=0} Ad^\ast(sX)\phi = -\phi \circ ad_X, \quad X \in \mathfrak{g}, \ \phi \in \mathfrak{g}^*,. $$

Each orbit can be equipped with a symplectic form $\omega$ as follows: let $X, Y$ be vectors in $\mathfrak{g}$ and let $X^*, Y^*$ be the induced vectors on $\mathfrak{g}^*$, then the 2-form $\omega$ given by

$$ \omega_\phi(X^*, Y^*) = \phi([X,Y]) \quad X,Y \in \mathfrak{g}, \ \phi \in \mathfrak{g}^* \quad (1) $$

makes every orbit a symplectic manifold, in fact $\omega$ is closed and non-degenerate (see for example [15]).

Whenever $(M, \omega)$ denotes a symplectic manifold and $f : M \to \mathbb{R}$ is a smooth function on $M$, then there is exactly one vector field $X_f$ on $M$ with the property:

$$ v_p(f) = df_p(v) = \omega_p(v, X_f) \quad \forall p \in M, \ \forall v \in T_pM. $$

The vector field $X_f$ is called the Hamiltonian vector field associated to the function $f$, and $f$ is said to be the Hamiltonian (or the Hamiltonian function). Moreover, the Hamiltonian system for the function $f$ is

$$ \frac{dx}{dt} = X_f(x(t)) $$

for $x : \mathbb{R} \to M$ a curve on $M$.

Recall that a metric on a Lie group $G$ is a map $\langle \cdot, \cdot \rangle$, which for every $g \in G$ $\langle \cdot, \cdot \rangle_g$ is a (not necessarily definite) metric on $T_gG$ and such it is smooth in the usual sense, that is, if $X, Y$ denote vector fields on $G$, the function $g \to \langle X, Y \rangle_g$ is smooth.
We are interested in left invariant metrics, that is, those such that left multiplication by elements of the group are isometries. In this situation the metric is determined by its value at the tangent space of the identity element $T_eG$. Therefore one identifies a left invariant metric on $G$ with a non-degenerate symmetric bilinear form on the Lie algebra $\mathfrak{g}$.

If $\langle , \rangle$ denotes such a bilinear form, to each $\phi \in \mathfrak{g}^*$ it corresponds exactly one $Z \in \mathfrak{g}$ such that $\phi = \ell_Z$, where $\ell_Z(Y) = \langle Z, Y \rangle$. The map $Z \mapsto \ell_Z$ is a linear isomorphism which allows to transport the coadjoint representation to a representation $\tau$ on the Lie algebra, which is given by

$$\langle \tau(g)X, Y \rangle = \langle X, Ad(g^{-1})Y \rangle.$$ 

In particular, making use of the exponential map one has

$$\langle \tau(\exp X)Y, Z \rangle = \langle Y, Ad(\exp -X)Z \rangle = \langle Y, \sum_{j=0}^{\infty} (-ad_X)^j Z \rangle,$$

and so

$$\tau(\exp X)Y = \sum_{j=0}^{\infty} (-ad_X)^j Y$$

(2)

where $ad_X^t : \mathfrak{g} \to \mathfrak{g}$ is the the transpose of the adjoint map, thus it is the endomorphism defined by $\langle ad_X^t Y, Z \rangle = \langle Y, ad_X Z \rangle$. Notice that (2) holds in a neighborhood of $0 \in \mathfrak{g}$.

**Remark 1** In the special case when $G$ is an $n$-step nilpotent simply connected Lie group then the exponential map $\exp : \mathfrak{g} \to G$, is a diffeomorphism from the Lie algebra $\mathfrak{g}$ onto $G$ and so the representation $\tau$ is given by:

$$\tau(\exp X)Y = \sum_{j=1}^{n} (-ad_X^j)Y.$$

If $U \in \mathfrak{g}$, the orbit of the action through this point is denoted by $G \cdot U$ and the isotropy by $G_U$:

$$G \cdot U = \{ \tau(g)U, \forall g \in G \} \quad G_U = \{ g \in G / \tau(g)U = U \}.$$

A vector field at the orbit is induced by a vector field $Y \in \mathfrak{g}$ and at $U \in \mathfrak{g}$ it has the form

$$\tilde{Y}_U = \frac{d}{ds}_{|s=0} \tau(\exp sY)U = \frac{d}{ds}_{|s=0} \sum_{j=1}^{n} (-ad_{sY}^j)U = -ad_Y^1 U.$$

The Lie algebra of the isotropy subgroup $G_U$ is the set of elements

$$L(G_U) = \{ Y \in \mathfrak{g} \text{ such that } 0 = \frac{d}{ds}_{|s=0} \tau(\exp sY)U \} = \{ Y \in \mathfrak{g} / ad_Y^1 U = 0 \}.$$

Every orbit of the representation $\tau$ has a natural symplectic structure which is induced by the canonical symplectic structure of the orbits of the coadjoint representation (see 1). Explicitely for $U \in \mathfrak{g}$ and $Y, Z \in \mathfrak{g}$ the symplectic form in $M$ at the point $U$ is given by

$$\omega_U(\tilde{Y}, \tilde{Z}) = \langle U, [Y, Z] \rangle.$$

Our aim is to find the Hamiltonian vector field $X_H$ for $H$ the restriction to a orbit of a function $f : \mathfrak{g} \to \mathbb{R}$. 


Definition 1 If \( \langle \cdot, \cdot \rangle \) denotes a non-degenerate bilinear form on \( \mathfrak{g} \) and \( f : \mathfrak{g} \to \mathbb{R} \) is a function on \( \mathfrak{g} \), the gradient of \( f \) at the point \( U, \nabla f(U) \), is defined by
\[
\langle \nabla f(U), Y \rangle = df(Y) \quad \text{for all } U, Y \in \mathfrak{g}.
\]

Let \( H = f|_M \) denote the restriction of a function \( f : \mathfrak{g} \to \mathbb{R} \) to an orbit \( M = G \cdot U \) of the representation \( \tau \). The question is to determine the Hamiltonian vector field associated to \( H \).

Let \( U \in \mathfrak{g} \) and let \( Y \in \mathfrak{g} \) which induces a vector field at the orbit \( M = G \cdot U \), then
\[
dH_U(\tilde{Y}_U) = \langle \nabla f(U), \tilde{Y}_U \rangle = \langle \nabla f(U), -ad^*_Y U \rangle = -\omega_U(Y, \nabla f).
\]
On the other hand
\[
dH_U(\tilde{Y}_U) = \langle U, [Y, X_H] \rangle,
\]
comparing this with the previous relation one gets that
\[
X_H(U) = -\tilde{\nabla} f(U)(U) = ad^{\dagger}_{\nabla f(U)}(U).
\]

Example. Let \( \mathfrak{h} \) be a Lie algebra with a metric \( \langle \cdot, \cdot \rangle \) and let \( f : \mathfrak{h} \to \mathbb{R} \) be the function given by \( f(V) = \langle Q, V \rangle \) for a fixed point \( Q \in \mathfrak{h} \). Then the gradient of \( f \) is \( \nabla f(V) = Q \). In fact,
\[
\langle \nabla f(V), Y \rangle = df(Y) = \frac{d}{dt}_{|t=0} f(tQ + sY) = \frac{d}{dt}_{|t=0} \langle Q, V + sY \rangle = \langle Q, Y \rangle
\]
Consider now the orbits by the representation \( \tau \) with the symplectic form \( \omega \) above defined. The Hamiltonian vector field associated to \( H = f|_M \) results
\[
X_H(U) = ad^\dagger_Q(U)
\]
and the Hamiltonian system for \( H \)
\[
\begin{cases}
x' = ad_Q(x) \\
x(0) = P
\end{cases}
\]
for any fixed element \( P \in \mathfrak{h} \)
has the solution
\[
x(t) = -\tau(\exp tQ)P.
\]
In particular if the metric \( \langle \cdot, \cdot \rangle \) is ad-invariant, then the solution \( x \) coincides with \( x(t) = Ad(\exp tQ)P \) (compare with [7] chapter 2).

Definition 2 A smooth function \( f : \mathfrak{g} \to \mathbb{R} \) is said to be \( \tau \)-invariant if
\[
f(\tau(g)U) = f(U) \quad \text{for every } g \in G, \text{ and } U \in \mathfrak{g}.
\]

Remark 2 A \( \tau \)-invariant function is trivial on the orbits \( G \cdot U \) for all \( U \in \mathfrak{g} \). However we shall make use of them later, in a context where they are not trivial.

Remark 3 A \( \tau \)-invariant function is actually a coadjoint invariant function. In fact, by construction the representation \( \tau : G \to \text{End}(\mathfrak{g})^* \) is equivalent to the coadjoint representation \( Ad^* : G \to \text{End}(\mathfrak{g}^*) \), since there is a linear isomorphism \( T : \mathfrak{g} \to \mathfrak{g}^* \) such that \( \tau(g) = T^{-1} \circ Ad(g)^* \circ T \) (explicitly \( T = \ell \)). Now if \( f : \mathfrak{g} \to \mathbb{R} \) is a \( \tau \)-invariant function, then
\[
f(\tau(g)x) = f(T^{-1} \circ Ad(g)^* \circ Tx) = f(x) = f(T^{-1} \circ Tx),
\]
now \( \tilde{f} := f \circ T^{-1} : \mathfrak{g}^* \to \mathbb{R} \) is by the relation above a \( Ad^* \)-invariant function. Moreover the map \( f \to \tilde{f} \) is a bijection.
Proposition 1 Let $f$ be a $\tau$-invariant function, then the following relations hold:

(i) $\nabla f(\tau(g)U) = Ad(g)\nabla f(U)$ for every $g \in G$ and $U \in \mathfrak{g}$;

(ii) $ad_{\nabla f(U)}^g U = 0$ for all $U \in \mathfrak{g}$.

Proof. (i) For the first assertion, differentiate $f(\tau(g)U) = f(U)$ with respect to $U$:

$$\frac{d}{ds}|_{s=0} f(\tau(g)(U + sV)) = \frac{d}{dt}|_{t=0} f(U + sV)$$

which gives

$$df_{\tau(g)U}(\tau(g)V) = \langle \nabla f(\tau(g)U), \tau(g)V \rangle = \langle Ad(g^{-1})\nabla f(\tau(g)U), V \rangle$$

On the other hand,

$$df_U(V) = \langle \nabla f(U), V \rangle \quad \text{for all } V$$

and so we obtain,

$$Ad(g^{-1})\nabla f(\tau(g)U) = \nabla f(U).$$

(ii) Let $U \in \mathfrak{g}$, since $f$ is $\tau$ invariant $f(\tau(\exp sY)U) = f(U)$ for all $Y \in \mathfrak{g}$ and for $s$ near 0. If we differentiate at $s = 0$ we obtain,

$$0 = \frac{d}{ds}|_{s=0} f(\tau(\exp sY)U) = df_U(\tilde{Y}_U)$$

$$= \langle \nabla f(U), -ad_Y U \rangle$$

$$= \langle -ad_Y \nabla f(U), U \rangle$$

$$= \langle ad(\nabla f(U))Y, U \rangle$$

$$= \langle Y, ad(\nabla f(U)) \rangle U \quad \forall Y \in \mathfrak{g}$$

and that means

$$ad(\nabla f(U))(U) = 0 \quad \text{for all } U \in \mathfrak{g}.$$

\[\blacksquare\]

3. A involution condition

A symplectic structure on a differentiable manifold $(M, \omega)$ induces a Poisson bracket on $C^\infty M$. In fact, for smooth functions $H_1, H_2 : M \to \mathbb{R}$ the Poisson bracket $\{H_1, H_2\}$ of them is a new function on $M$ defined as

$$\{H_1, H_2\}(m) = \omega_m(X_{H_1}, X_{H_2}).$$

Notice that $C^\infty M$ is an associative algebra and $\{ , \}$ is a Lie bracket on $C^\infty M$ satisfying the Leibniz rule.

Two functions $f, g$ are said to be in involution if $\{f, g\} = 0$.

Let $(M, \{ , \})$ denote a differentiable manifold equipped with a Poisson bracket and let $f : M \to \mathbb{R}$ be a fixed function on $M$, $f$ is called completely integrable if there exist $n$ functions in $C^\infty M$ such that

- $\{f, fi\} = 0 = \{fi, fj\}$ for all $i, j = 1, \ldots, n$;
- The differentials $df_1, \ldots, df_n$ are linearly independent on every point on a open set invariant under the flow of $X_f$.

Let $\langle , \rangle$ denote a metric on the Lie algebra $\mathfrak{g}$, that is, $\langle , \rangle$ is a non degenerate symmetric bilinear form on $\mathfrak{g}$. For a subset $v \subseteq \mathfrak{g}$ one defines the orthogonal subspace $v^\perp$ as

$$v^\perp = \{x \in \mathfrak{g} : \langle x, v \rangle = 0 \text{ for all } v \in v\}.$$
Note that $v^\perp$ is not necessarily complementary to $v$ unless the metric is definite, however it always holds $\dim g = \dim v + \dim v^\perp$.

Assume there are Lie subalgebras $g_+$ and $g_-$ in $g$ such that
\[ g = g_+ \oplus g_- \]
as direct sum of vector subspaces, then $g$ can be decomposed as a direct sum of subspaces $g = g^\perp_+ \oplus g^\perp_-$.

**Proposition 2** Suppose $g$ has a non degenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle$ and that $g_+, g_-$ are Lie subalgebras of $g$ such that $g$ decomposes into
\[ g = g_+ \oplus g_- \]
as direct sum of linear subspaces.

i) The maps $g^\perp_+ \to g_-^\perp$ and $g^\perp_- \to g_+^\perp$ defined by $X \to \ell_{X|g_-}$ and $X \to \ell_{X|g_+}$ respectively are linear isomorphisms. So the coadjoint action of $G_-$ on $g^\perp_-$ (respectively from $G_+$ on $g^\perp_+$) induces an action of $G_-$ onto $g^\perp_+$ (and another one of $G_+$ onto $g^\perp_-$), denoted by $\bar{\tau}$.

ii) Let $\pi_{g^\perp_\pm}$ be the projection of $g$ onto $g^\perp_\pm$ with respect to the splitting $g = g^\perp_+ \oplus g^\perp_-$. If $X \in g^\perp_+$, $g_- \in G_-$ then
\[ \bar{\tau}(g_-)X = \pi_{g^\perp_+(\bar{\tau}(g_-)X)} \]

iii) The infinitesimal vector field induced by $Y \in g_-$ at the point $X \in g^\perp_+$ is
\[ \bar{Y}(X) = \pi_{g^\perp_-(\bar{Y}(X))}(-ad^Y Y) \]

**Proof.** i) is a consequence from linear algebra. Let $X \in g^\perp_+$, $V \in g_-$, $g_- \in G_-$, a direct computation gives \[ Ad^*(g_-)\ell_X(V) = \langle X, Ad(g_-)^{-1}V \rangle \]
which proves ii) and iii) follows from ii) by taking $g_- = \exp(tY)$ and differentiate at $t = 0$. 

An orbit $G_- \cdot X \subseteq g^\perp_+$ of the $\bar{\tau}$-action is the set
\[ G_- \cdot X = \{ \bar{\tau}(g_-)X, g_- \in G_- \} \]
and the isotropy subgroup at $X$ is the set
\[ G_X = \{ g_- \in G_- : \bar{\tau}(g_-)X = X \} \]
whose Lie algebra is
\[ L(G_X) = \{ Y_- \in g_- : \langle ad^Y X, Z \rangle = 0 \text{ for all } Z \in g_- \}. \]

The orbit $G_- \cdot X \subseteq g^\perp_+$ can be equipped with a symplectic structure which is induced by the symplectic structure on the coadjoint orbits, namely
\[ \omega_X(\bar{Y}, \bar{Z}) = \langle X, [Y, Z] \rangle \]
for $X \in g^\perp_+$, and $Y, Z \in g_-$. 

Notation: Given a vector $Y \in g$ we denote by $Y_-$ (resp. $Y_+$) the projection of $Y$ onto $g_-$ with respect to the decomposition $g = g_- \oplus g_+$. 

Theorem 1 Let \( g, g_+, g_- \) and \( \langle , \rangle \) be as in Proposition 2. Let \( M := G_- \times X \subset g_+^\perp \) be a \( G_- \)-orbit, \( \omega \) the orbit symplectic structure and \( f : g \rightarrow \mathbb{R} \) a smooth function. Then:

(i) The Hamiltonian vector field of \( H = f|_M \) at \( U \in M \) is

\[
X_H(U) = -\nabla f(U)(U) = \pi_{g_+}^{-1}(ad_{\nabla f(U)}^U(U))
\]

(ii) If \( f : g \rightarrow \mathbb{R} \) is \( \tau \)-invariant, then the Hamiltonian equation for \( H = f|_M \) is

\[
\frac{du}{ds} = -ad_{\nabla f_-}(u)u = ad_{\nabla f_+}(u)u
\]

(iii) Let \( f_1, f_2 : g \rightarrow \mathbb{R} \) be \( \tau \)-invariant functions and let \( H_i = f_i|_M, i=1,2 \), the restrictions respectively. Then

\[
\{H_1, H_2\} = 0
\]

(iv) Assume the multiplication map from \( G_+ \times G_- \rightarrow G \) defined by \( (g_+, g_-) \rightarrow g_+g_- \) is a diffeomorphism, and that \( f : g \rightarrow \mathbb{R} \) is \( \tau \)-invariant. Then the initial value problem

\[
\begin{cases}
\frac{du}{ds} = -ad_{\nabla f_+}(u)u \\
u(0) = U_0
\end{cases}
\]  

(4)

can be solved by factorization. In fact, if we write

\[
\exp(t\nabla f(U_0)) = g_+(t)g_-(t) \in G_+ \times G_-
\]

then

\[
u(t) = \tau(\exp -t\nabla f_+(U_0))U_0
\]

is the solution of (4).

Proof.

Let \( U \in g_+^\perp, Y \in g_-, \) then

\[
dH_U(\nabla f(U)) = \langle \nabla f(U), \nabla f(U) \rangle = \langle \nabla f(U), \pi_{g_+}^{-1}(ad_{\nabla f(U)}^U) \rangle
\]

\[
= \langle \nabla f_-(U), -ad_{\nabla f_+}(U) \rangle
\]

\[
= \langle \pi(Y)\nabla f_-(U), U \rangle
\]

But on the other hand

\[
dH_U(\nabla f(U)) = \langle U, [Y, X_H] \rangle
\]

which implies

\[
X_H(U) = -\nabla f_-(U)(U) = \pi_{g_+}^{-1}(ad_{\nabla f_-}(U))(U).
\]

If \( f \) is \( \tau \)-invariant then by Proposition 1 we have

\[
0 = ad_{\nabla f(U)}^U = ad_{\nabla f_+(U)}^U + ad_{\nabla f_-}(U)
\]

Now

\[
X_H(U) = \pi_{g_+}^{-1}(ad_{\nabla f_+}(U)) = -\pi_{g_+}^{-1}(ad_{\nabla f_+}(U))
\]

and we compute for \( Y \in \mathfrak{g}_+ \)

\[
\langle ad_{\nabla f_+}(U)Y, Y \rangle = \langle U, ad_{\nabla f_+(U)}Y \rangle = 0
\]
that means that $\text{ad}_{\nabla H}^t U \in g_+^1 \forall U$, and so

$$X_H(U) = -\pi_{g_+^1} (\text{ad}_{\nabla f_+^t}^t U) = -\text{ad}_{\nabla f_+^t}^t U = \text{ad}_{\nabla f_-^t}^t U$$

which proves (2).

To prove (3), let $U \in M \subset g_+^1$. Then we have

$$\{H_1, H_2\} = \langle U, [\nabla f_1^-(U), \nabla f_2^-(U)] \rangle = \langle \text{ad}_{\nabla f_1^-}^t U, \nabla f_2^-(U) \rangle$$
$$= \langle -\text{ad}_{\nabla f_1^+}^t U, \nabla f_2^-(U) \rangle = \langle U, -[\nabla f_1^+(U), \nabla f_2^-(U)] \rangle$$
$$= \langle \text{ad}_{\nabla f_2^-}^t U, \nabla f_1^+(U) \rangle$$
$$= -\langle U, [\nabla f_2^+(U), \nabla f_1^+(U)] \rangle = 0$$

It remains to prove (4). First note that $\text{ad}_{\nabla f(U_0)}^t U_0 = 0$ implies

$$\tau(\exp t \nabla f(U_0)) U_0 \equiv U_0$$

Write $g(t) = \exp t \nabla f(U_0) = g_+(t) g_-(t)$. Since $\tau(g_+ g_-) U_0 \equiv U_0$, then $\tau(g_+^{-1}) U_0 = \tau(g_-) U_0$. But since

$$g'(t) = \frac{d}{ds} \bigg|_{s=0} \exp s \nabla f(U_0) \exp t \nabla f(U_0) = dR_g \nabla f(U_0)$$

one immediately obtains

$$dR_{g^{-1}} g' = \nabla f(U_0).$$

We compute directly to get

$$\nabla f(\tau(g_+^{-1}) U_0) = \text{Ad}(g_+^{-1}) \nabla f(U_0) = dL_{g_+^{-1}} dR_{g_+} dR_{g_-}^{-1} g'$$
$$= dL_{g_+^{-1}} dR_{g_+} dR_{g_-}^{-1} (dR_{g_+} g'_+ + dL_{g_+})$$
$$= dL_{g_+^{-1}} (g'_+) + dR_{g_-}^{-1} (g'_-) \in g_+ \oplus g_-$$

So $\nabla f_+ (\tau(g_+^{-1}) U_0) = dL_{g_+^{-1}} (g'_+)$. But

$$u' = -\text{ad}_{dL_{g_-}^{-1}}(g'_+) u = -\text{ad}_{\nabla f_+}^t (u)$$

which says that $u$ is a solution of (4).

4. Examples

The topic of having enough functions in involution is a question to solve when we try to apply the last theorem in the previous section. In the case of nilpotent Lie groups, Corwin and Greenleaf [3] developed a method to construct the generators of the ring of rational $\tau$-invariant functions. Below we show some examples in low dimensions, essentially in nilpotent cases, where the theory of the previous section was applied.

In dimension four there are solvable Lie algebras admitting an ad-invariant metric, however no nilpotent Lie algebra can be equipped with an ad-invariant metric unless it is abelian (see for instance [11]). Solvable Lie algebras with ad-invariant metrics were applied in the study of quadratic Hamiltonian in $\mathbb{R}^n$ ([12], [13]).

We know that the cardinal of the set of invariant functions is invariant by equivalent representations. However the rational one is not invariant. In fact as relation (3) shows by moving from one representation to a equivalent one, one could lose the rationality of the function.
Example i) Consider the three step nilpotent Lie algebra $\mathfrak{g}$ generated by the elements $e_1, e_2, e_3, e_4$, the Lie bracket relations given by:

$$[e_4, e_1] = e_2 \quad [e_4, e_2] = e_3$$

and the metric defined by

$$\langle e_2, e_2 \rangle = 1 = \langle e_4, e_4 \rangle = -\langle e_1, e_3 \rangle.$$

Then for $x = \sum x_ie_i$, $y = \sum y_ie_i$, the $\tau$-action is the following one

$$ad^\tau_x y = (y_1x_2 - x_1y_2)e_1 + (x_4y_1 - x_1y_4)e_2 + (x_4y_2 - y_4x_1)e_3.$$

Take the following decomposition of $\mathfrak{g}$ into a direct sum of subalgebras, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where

$$\mathfrak{g}_1 = \text{span}\{e_4, e_2, e_3\} \quad \mathfrak{g}_2 = \text{span}\{e_1\}.$$

It is not difficult to see that

$$\mathfrak{g}_1^\perp = \text{span}\{e_3\} \quad \mathfrak{g}_2^\perp = \text{span}\{e_1, e_2, e_4\}.$$

As we said $G_1$ acts on $\mathfrak{g}_2^\perp$. The orbits at a point $y = (y_1, y_2, y_3, y_4)$ have dimension zero if $y_1 = 0$ and have dimension two if $y_1 \neq 0$. So for example for the element $p = (1, 0, 0, 0)$ the orbit at $p$ is the set

$$M = \{(1, u, 0, v) / u, v \in \mathbb{R}\}.$$

By following the method developed by Corwin and Greenleaf we obtain two functions which generate the ring of rational $\tau$-invariant functions. In fact, for a point $x = \sum x_ie_i$ they are

$$f_1(x) = x_1 \quad f_2(x) = x_1x_3 - \frac{1}{2}x_2^2$$

The gradient of the function $f_2$ is $\nabla f_2(U) = -u_1e_1 - u_2e_2 - u_3e_3$ and so the Hamiltonian system for $f_2$ with initial value $u(0) = (u_1^0, u_2^0, u_3^0, u_4^0)$ is

$$\begin{cases} \frac{du}{dt} = -u_1u_2e_4 \\ u(0) = (u_1^0, u_2^0, u_3^0, u_4^0) \end{cases}$$

This system has the solution $u(t) = (u_1^t, u_2^t, u_3^t, u_4^t + u_3^0u_2^0t)$.

For $x \in M$ the function $f_2$ take the value $f_2(x) = -2u^2$, which is not constant at the orbit. Moreover the set $N = \{y \in M : f_2(y) = c\}$ for a constant $c$, is empty if $c > 0$, is the set $\{(1, 0, 0, v)\}$ if $c = 0$ and is the set $\{(1, \pm\sqrt{-\frac{c}{2}}, 0, v)\}$ for $c < 0$ and that means $N$ is never compact.

Example ii) In this example we work on a fixed Lie algebra. We show that it is possible to obtain a different number of generators of the ring of rational invariant functions by changing the metric.

Let $\mathfrak{g}$ be the two-step nilpotent Lie algebra generated by $Z, X, Y, X^*, Y^*, Z^*$ with the relations

$$[X, Y] = Z \quad [Z^*, X] = Y^* \quad [Z^*, Y] = -X^*$$

and the metric given by

$$\langle X^*, X \rangle = 1 = \langle Y^*, Y \rangle = \langle Z, Z \rangle = \langle Z^*, Z^* \rangle.$$


In this situation the ring of rational $\tau$-invariant functions is generated by

$$ p_1(u) = y, \quad p_2(u) = x, \quad p_3(u) = z^*, \quad p_4(u) = xx^* + zz^* + yy^* $$

where $u = zZ + xX + yY + z^*Z^* + x^*X^* + y^*Y^*$. We observe that this Lie algebra admits an $\text{ad}$-invariant metric defined by

$$ \langle X^*, X \rangle = 1 = \langle Y^*, Y \rangle = \langle Z^*, Z \rangle. $$

and in this case the ring of rational $\text{ad}$-invariant functions is generated by $p_1, p_2, p_3, p_4$ as above. Furthermore, if we take the metric which makes the basis $Z, X, Y, X^*, Y^*, Z^*$ an orthonormal basis, then the ring of rational $\tau$-invariant functions is generated by $p_1, p_2, p_3$ as above.

**Acknowledgments**

The author is grateful to V. Bangert for his support during the stay at the Universität zu Freiburg, where part of this work was written.

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