Locally Anisotropic Interactions: 
I. Nonlinear Connections in Higher Order Anisotropic Superspaces

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Abstract

Higher order anisotropic superspaces are constructed as generalized vector superbundles provided with compatible nonlinear connection, distinguished connection and metric structures.

1 Introduction

The differential supergeometry have been formulated with the aim of getting a geometric framework for the supersymmetric field theories (see the theory of graded manifolds [10, 24, 25, 23], the theory of supermanifolds [51, 34, 7, 20] and, for detailed considerations of geometric and topological aspects of supermanifolds and formulation of superanalysis, [15, 12, 27, 17, 47, 49]). In this paper we apply the supergeometric formalism for a study of a new class of (higher order anisotropic) superspaces.

The concept of local anisotropy is largely used in some divisions of theoretical and mathematical physics [18, 19, 19, 32] (see also possible applications in physics and biology in [2, 1]). The first models of locally anisotropic (la) spaces (la–spaces) have been proposed by P.Finsler [16] and E.Cartan [13] (early approaches and modern treatments of Finsler geometry and its extensions can be found, for instance, in [36, 3, 4, 28]). In our works [41, 42, 43, 45, 46] we try to formulate the geometry of la-spaces in a manner as to include both variants
of Finsler and Lagrange, in general supersymmetric, extensions and higher dimensional Kaluza–Klein (super)spaces as well to propose general principles and methods of construction of models of classical and quantum field interactions and stochastic processes on spaces with generic anisotropy.

We cite here the works [8, 9] by A. Bejancu where a new viewpoint on differential geometry of supermanifolds is considered. The author introduced the nonlinear connection (N–connection) structure and developed a corresponding distinguished by N–connection supertensor covariant differential calculus in the frame of De Witt [51] approach to supermanifolds in the framework of the geometry of superbundles with typical fibres parametrized by noncommutative coordinates. This was the first example of superspace with local anisotropy. In our turn we have given a general definition of locally anisotropic superspaces (la–superspaces) [12]. We note that in some of our supersymmetric generalizations we are inspired by the R. Miron, M. Anastasiei and Gh. Atanasiu works on the geometry of nonlinear connections in vector bundles and higher order Lagrange spaces [23].

In this work we shall formulate the theory of higher order vector superbundles provided with nonlinear and distinguished connections and metric structures (a generalized model of la–superspaces). Such superbundles contain as particular cases the supersymmetric extensions and various higher order prolongations of Riemann, Finsler and Lagrange spaces.

The paper is organized as follows: Section 2 is a brief review on supermanifolds and superbundles. An introduction into the geometry of higher order distinguished vector superbundles is presented in Section 3. Section 4 and 5 deals respectively with the geometry of nonlinear and linear distinguished connections in vector superbundles and in distinguished vector superbundles. Concluding remarks and discussion are contained in Section 6.

2 Supermanifolds and Superbundles

In this section we establish the necessary terminology on supermanifolds (s–manifolds) [51]. Here we note that a number of different approaches to supermanifolds are broadly equivalent for local considerations. For simplicity, we shall restrict our study only with geometric constructions on locally trivial superspaces.

To build up s–manifolds (see [34, 20, 47]) one uses as basic structures Grassmann algebra and Banach space. A Grassmann algebra is introduced as a real associative algebra A (with unity) possessing a finite (canonical) set of anticommutative generators \( \beta^A, [\beta^A, \beta^B]_+ = \beta^A \beta^B + \beta^B \beta^A = 0 \), where \( \hat{A}, \hat{B},... = 1, 2, ..., \hat{L} \). In this case it is also defined a Z\(_2\)-graded commutative
algebra $\Lambda_0 + \Lambda_1$, whose even part $\Lambda_0$ (odd part $\Lambda_1$) is a $2^{L-1}$-dimensional real vector space of even (odd) products of generators $\beta_\hat{A}$. After setting $\Lambda_0 = R + \Lambda_0'$, where $R$ is the real number field and $\Lambda_0'$ is the subspace of nilpotent elements, the projections $\sigma : \Lambda \to R$ and $s : \Lambda \to \Lambda_0'$ are called, respectively, the body and soul maps.

A Grassmann algebra can be provided with both structures of a Banach algebra and Euclidean topological space by the norm \cite{34}

$$\|\xi\| = \Sigma_{\hat{A}_i}|a^{\hat{A}_1...\hat{A}_k}|, \xi = \Sigma_{r=0}^L a^{\hat{A}_1...\hat{A}_r}\beta_{\hat{A}_1}...\beta_{\hat{A}_r}.$$ 

A superspace is introduced as a product

$$\Lambda^{n,k} = \Lambda_0 \times ... \Lambda_0 \times \Lambda_1 \times ... \Lambda_1$$

which is the $\Lambda$-envelope of a $Z_2$-graded vector space $V^{n,k} = V_0 \otimes V_1 = R^n \oplus R_k$ is obtained by multiplication of even (odd) vectors of $V$ on even (odd) elements of $\Lambda$. The superspace (as the $\Lambda$-envelope) posses $(n + k)$ basis vectors $\{\hat{\beta}_i, \quad i = 0, 1, ..., n - 1, \text{ and } \hat{\beta}_i, \quad i = 1, 2, ..., k\}$. Coordinates of even (odd) elements of $V^{n,k}$ are even (odd) elements of $\Lambda$. We can consider equivalently a superspace $V^{n,k}$ as a $(2^{L-1})(n + k)$-dimensional real vector spaces with a basis $\{\hat{\beta}_i(\Lambda), \hat{\beta}_i(\Lambda)\}$.

Functions of superspaces, differentiation with respect to Grassmann coordinates, supersmooth (superanalytic) functions and mappings are introduced by analogy with the ordinary case, but with a glance to certain specificity caused by changing of real (or complex) number field into Grassmann algebra $\Lambda$. Here we remark that functions on a superspace $\Lambda^{n,k}$ which takes values in Grassmann algebra can be considered as mappings of the space $R^{(2^{L-1})(n + k)}$ into the space $R^{2L}$. Functions differentiable on Grassmann coordinates can be rewritten via derivatives on real coordinates, which obey a generalized form of Cauchy-Riemann conditions.

A $(n, k)$-dimensional s-manifold $\tilde{M}$ can be defined as a Banach manifold (see, for example, \cite{20}) modeled on $\Lambda^{n,k}$ endowed with an atlas $\psi = \{U_{(i)}, \psi_{(i)} : U_{(i)} \to \Lambda^{n,k}, (i) \in I\}$ whose transition functions $\psi_{(i)}$ are supersmooth \cite{34, 20}. Instead of supersmooth functions we can use $G^\infty$-functions \cite{34} and introduce $G^\infty$-supermanifolds ($G^\infty$ denotes the class of superdifferentiable functions). The local structure of a $G^\infty$-supermanifold is built very much as on a $C^\infty$-manifold. Just as a vector field on a $n$-dimensional $C^\infty$-manifold written locally as

$$\Sigma_{i=0}^{n-1} f_i(x^j) \frac{\partial}{\partial x^i},$$

where $f_i$ are $C^\infty$-functions, a vector field on an $(n, k)$-dimensional $G^\infty$-supermanifold $\tilde{M}$ can be expressed locally on an open region $U \subset \tilde{M}$ as

$$\Sigma_{I=0}^{n-1+k} f_I(x^I) \frac{\partial}{\partial x^I} = \Sigma_{i=0}^{n-1} f_i(x^j, \theta^i) \frac{\partial}{\partial x^i} + \Sigma_{i=1}^{k} f_i(x^j, \theta^i) \frac{\partial}{\partial \theta^i}.$$
where  \( x = (\tilde{x}, \theta) = \{x^I = (\tilde{x}^i, \theta^i)\} \) are local (even, odd) coordinates. We shall use indices  \( I = (i, \hat{i}), J = (j, \hat{j}), K = (k, \hat{k}), \ldots \) for geometric objects on  \( \tilde{M} \). A vector field on  \( U \) is an element  \( X \in \text{End}[G^\infty(U)] \) (we can also consider supersmooth functions instead of  \( G^\infty \)-functions) such that

\[
X(fg) = (Xf)g + (-)^{|f||X|}fXg,
\]

for all  \( f, g \) in  \( G^\infty(U) \), and

\[
X(af) = (-)^{|X||a|}aXf,
\]

where  \(|X|\) and  \(|a|\) denote correspondingly the parity (= 0, 1) of values  \( X \) and  \( a \) and for simplicity in this work we shall write  \((-)^{|f||X|}\) instead of  \((-1)^{|f||X|}\).

A super Lie group (sl-group) \[35\] is both an abstract group and a s-manifold, provided that the group composition law fulfills a suitable smoothness condition (i.e. to be superanalytic, for short, \( sa \)) [20].

In our further considerations we shall use the group of automorphisms of  \( \Lambda^{(n,k)} \), denoted as  \( GL(n,k,\Lambda) \), which can be parametrized as the super Lie group of invertible matrices

\[
Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where  \( A \) and  \( D \) are respectively  \((n\times n)\) and  \((k\times k)\) matrices consisting of even Grassmann elements and  \( B \) and  \( C \) are rectangular matrices consisting of odd Grassmann elements. A matrix  \( Q \) is invertible as soon as maps  \( \sigma A \) and  \( \sigma D \) are invertible matrices. A sl-group represents an ordinary Lie group included in the group of linear transforms  \( GL(2^{L-1}(n+k),\mathbb{R}) \). For matrices of type  \( Q \) one defines \[14, 24, 25\] the superdeterminant,  \( \text{sdet}Q \), supertrace,  \( \text{str}Q \), and superrank,  \( \text{srank}Q \).

A Lie superalgebra (sl-algebra) is a  \( \mathbb{Z}_2 \)-graded algebra  \( A = A_0 \oplus A_1 \) endowed with product  \( [,] \) satisfying the following properties:

\[
[I, I'] = -(-)^{|I||I'|}[I', I],
\]

\[
[I, [I', I'']] = [[I, I'], I''] + (-)^{|I||I'|}[I'[I, I'']],
\]

\( I \in A_{|I|}, \quad I' \in A_{|I'|}, \) where  \(|I|, |I'| = 0, 1\) enumerates, respectively, the possible parity of elements  \( I, I' \). The even part  \( A_0 \) of a sl-algebra is a usual Lie algebra and the odd part  \( A_1 \) is a representation of this Lie algebra. This enables us to classify sl-algebras following the Lie algebra classification [22]. We also point out that irreducible linear representations of Lie superalgebra \( A \) are realized in  \( \mathbb{Z}_2 \)-graded vector spaces by matrices \( \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \) for even elements and
for odd elements and that, roughly speaking, \( A \) is a superalgebra of generators of a sl-group.

A sl-module \( W \) (graded Lie module) \([34]\) is introduced as a \( \Lambda \)-module endowed with a product \( \{,\} \) which satisfies the graded Jacobi identity and makes \( W \) into a graded-anticommutative Banach algebra over \( \Lambda \). One calls the Lie module \( G \) the set of the left-invariant derivatives of a sl-group \( G \).

The tangent superbundle (ts-bundle) \( TM \) over a s-manifold \( \tilde{M} \), \( \pi : \tilde{T} \tilde{M} \to \tilde{M} \) is constructed in a usual manner (see, for instance, \([26]\)) by taking as the typical fibre the superspace \( \Lambda^{n,k} \) and as the structure group the group of automorphisms, i.e. the sl-group \( GL(n,k,\Lambda) \).

Let us denote by \( F \) a vector superspace (vs-space) of dimension \((m,l)\) (with respect to a chosen base we parametrize an element \( y \in E \) as \( y = (\hat{y},\zeta) = \{ y^A = (\hat{y}^a,\zeta^\hat{a}) \} \), where \( a = 1,2,...,m \) and \( \hat{a} = 1,2,...,l \). We shall use indices \( A = (a,\hat{a}), B = (b,\hat{b}) \) ... for objects on vs-spaces. A vector superbundle (vs-bundle) \( \tilde{E} \) over base \( \tilde{M} \) with total superspace \( \tilde{E} \), standard fibre \( \hat{F} \) and surjective projection \( \pi_E : \tilde{E} \to \tilde{M} \) is defined (see details and variants in \([12, 49]\)) as in the case of ordinary manifolds (see, for instance, \([26, 29, 30]\)). A section of \( \tilde{E} \) is a supersmooth map \( s : U \to \tilde{E} \) such that \( \pi_E \circ s = id_U \).

A subbundle of \( \tilde{E} \) is a triple \( (\tilde{B}, f, f') \), where \( \tilde{B} \) is a vs-bundle on \( \tilde{M} \), maps \( f : \tilde{B} \to \tilde{E} \) and \( f' : \tilde{M} \to \tilde{M} \) are supersmooth, and (i) \( \pi_E \circ f = f' \circ \pi_B \); (ii) \( f : \pi_B^{-1}(x) \to \pi_E^{-1} \circ f'(x) \) is a vs-space homomorphism.

We denote by

\[ u = (x,y) = (\hat{x},\theta,\hat{y},\zeta) = \{ u^\alpha = (x^I,y^A) = (\hat{x}^i,\theta^\hat{i},\hat{y}^a,\zeta^\hat{a}) = (\hat{x}^i,x^j,\hat{y}^a,y^\hat{a}) \} \]

the local coordinates in \( \tilde{E} \) and write their transformations as

\[ x'' = x''(x^I), \quad \text{srank} \left( \frac{\partial x''}{\partial x^I} \right) = (n,k), \quad (1) \]

\[ y^{A'} = Y_A^{A'}(x)y^A, \quad \text{where} \quad Y_A^{A'}(x) \in G(m,l,\Lambda). \]

For local coordinates and geometric objects on ts-bundle \( T\tilde{M} \) we shall not distinguish indices of coordinates on the base and in the fibre and write, for instance,

\[ u = (x,y) = (\hat{x},\theta,\hat{y},\zeta) = \{ u^\alpha = (x^I,y^I) = (\hat{x}^i,\theta^i,\hat{y}^i,\zeta^i) = (\hat{x}^i,x^j,\hat{y}^i,y^i) \}. \]

We shall use general Greek indices on both type of vs- and ts-bundles.
3 Distinguished Vector Superbundles

Some recent considerations in mathematical physics are based on the so-called k-jet spaces (see, for instance, [39, 38, 5]). In order to formulate a systematic theory of connections and of geometric structures on k-jet bundles, in a manner following the approaches [50] and [29, 30], R. Miron and Gh. Atanasiu [31] introduced the concept of k–osculator bundle for which a fiber of k-jets is changed into a k–osculator fiber representing an element of k–order curve. Such considerations are connected with geometric constructions on tangent bundles of higher order. On the other hand for developments in modern supersymmetric Kaluza–Klein theories (see, for instance, [37]) a substantial interest would present a variant of "osculator" space for which the higher order tangent s–space distributions are of different dimensions. This section is devoted to the definition of such type distinguished vector superbundle spaces.

A vector superspace $F^{<z>}$ of dimension $(m, l)$ is a distinguished vector superspace (dvs–space) if it is decomposed into an invariant oriented direct sum

$$F^{<z>} = F^{(1)} \oplus F^{(2)} \oplus \ldots \oplus F^{(z)},$$

where\( \sum_{p=1}^{z} m(p) = m, \sum_{p=1}^{z} l(p) = l. \)

Coordinates on $F^{<p>}$ will be parametrized as

$$(y^{<p>} = (y^{(1)}, y^{(2)}, \ldots, y^{(p)}) = (\hat{y}^{(1)}, \zeta^{(1)}, \hat{y}^{(2)}, \zeta^{(2)}, \ldots, \hat{y}^{(p)}, \zeta^{(p)}) = \{y^{<A>} = (\hat{y}^{<a>}, \zeta^{<\hat{a}>}) = (\hat{y}^{<a>}, y^{<\hat{a}>})\},$$

where bracketed indices are correspondingly split on $F^{(p)}$–components:

$$< A > = (A^{(1)}, A^{(2)}, \ldots, A^{(p)}), < a > = (a^{(1)}, a^{(2)}, \ldots, a^{(p)})$$

and $$< \hat{a} > = (\hat{a}^{(1)} \hat{a}^{(2)}, \ldots, \hat{a}^{(p)}).$$

For simplicity, we shall also write (2) as

$$< A > = (A^{(1)}, A^{(2)}, \ldots, A^{(p)}), < a > = (a^{(1)}, a^{(2)}, \ldots, a^{(p)})$$

and $$< \hat{a} > = (\hat{a}^{(1)} \hat{a}^{(2)}, \ldots, \hat{a}^{(p)})$$

if this will give not rise to ambiguities.

A distinguished vector superbundle (dvs–bundle) $\tilde{E}^{<z>} = (E^{<z>}, \pi^{<d>}, F^{<d>}, M)$, with surjective projection $\pi^{<z>} : \tilde{E}^{<z>} \rightarrow \tilde{M}$, where $\tilde{M}$ and $\tilde{E}^{<z>}$ are respectively base and total s–spaces and the dv–space $F^{<z>}$ is the standard fibre, is defined in a usual manner (see correspondingly [12, 15, 6, 49] on vector superbundles and [27, 30, 28] on vector bundles).

A dvs–bundle $\tilde{E}^{<z>}$ is constructed as an oriented set of vs–bundles $\pi^{<p>} : \tilde{E}^{<p>} \rightarrow \tilde{E}^{<p-1>}$ (with typical fibers $F^{<p>}, p = 1, 2, \ldots, z$); $\tilde{E}^{<0>} = \tilde{M}$. We shall use index $z (p)$ as to denote the total (intermediate) numbers of consequent vs–bundle coverings of $\tilde{M}$.

Local coordinates on $\tilde{E}^{<p>}$ are denoted as

$$u_{(p)} = (x, y^{<p>}) = (x, y^{(1)}, y^{(2)}, \ldots, y^{(p)}) = \ldots$$
(\hat{x}, \theta, \hat{y}_{<p>}, \zeta_{<p>}) = (\hat{x}, \theta, \hat{y}_{(1)}, \zeta_{(1)}, \hat{y}_{(2)}, \zeta_{(2)}, \ldots, \hat{y}_{(p)}, \zeta_{(p)}) =
\{u^{<\alpha>} = (x^I, y^{<A>}) = (\hat{x}^i, \theta^i, \hat{y}^{<\alpha>}, \zeta^{<\alpha>}) = (\hat{x}^i, x^I, \hat{y}^{<\alpha>}, y^{<\alpha>})\} = \ldots

(in our further considerations we shall consider different variants of splitting indices of geometric objects).

Instead of (1) the coordinate transforms for dvs–bundles \(\{u^{<\alpha>} = (x^I, y^{<A>})\} \rightarrow \{u^{<\alpha'>} = (x'^I, y'^{<A'>})\}\) are given by recurrent maps:

\[
x'^I = x^I(x^I), \quad \text{srank}\left(\frac{\partial x'^I}{\partial x^I}\right) = (n, k),
\]

\[
y'^A_{(1)} = K'^A_{A_1}(x)y^A_{(1)}, K'^A_{A_1}(x) \in G(m_1, l_1, \Lambda),
\]

\[
y'^A_{(p)} = K'^A_{A_p}(u_{(p-1)})y^A_{(p)}, K'^A_{A_p}(u_{(p-1)}) \in G(m_p, l_p, \Lambda),
\]

\[
y'^A_{(z)} = K'^A_{A_z}(u_{(z-1)})y^A_{(z)}, K'^A_{A_z}(u_{(z-1)}) \in G(m_z, l_z, \Lambda).
\]

In brief we write transforms (3) as

\[
x'^I = x^I(x^I), \quad y'^{<A'>} = K'^{<A'>}_{<A>} y^{<A>}.
\]

More generally, we shall consider matrices \(K'^{<\alpha'>}_{<\alpha>} = (K'^I_{<I'}), K'^{<A'>}_{<A>}\), where \(K'^I_{<I'} = \frac{\partial x'^I}{\partial x^I}\).

In consequence the local coordinate bases of the module of ds-vector fields on \(\mathcal{E}^{<z'}>, \Theta(\mathcal{E}^{<z'>})\),

\[
\partial_{<\alpha>} = (\partial, \partial_{<A>}) = (\partial, \partial_{(A_1)}, \partial_{(A_2)}, \ldots, \partial_{(A_z)}) =
\]

\[
\frac{\partial}{\partial u^{<\alpha>}} = \left(\frac{\partial}{\partial x^I}, \frac{\partial}{\partial y^A_{(1)}}, \frac{\partial}{\partial y^A_{(2)}}, \ldots, \frac{\partial}{\partial y^A_{(z)}}\right)
\]

(4)

(the dual coordinate bases are denoted as

\[
d^{<\alpha>} = (d^I, d^{<A>}) = (d^I, d^{(A_1)}, d^{(A_2)}, \ldots, d^{(A_z)}) =
\]

\[
du^{<\alpha>} = (dx^I, dy^{(A_1)}, dy^{(A_2)}, \ldots, dy^{(A_z)})
\]

(5)

are transformed as

\[
\partial_{<\alpha>} = (\partial, \partial_{<A>}) = (\partial, \partial_{(A_1)}, \partial_{(A_2)}, \ldots, \partial_{(A_z)}) \rightarrow \partial_{<\alpha>} =
\]

\[
(\partial, \partial_{<A>}) = (\partial, \partial_{(A_1)}, \partial_{(A_2)}, \ldots, \partial_{(A_z)})
\]
\[
\frac{\partial}{\partial x^i} = K_i^T \frac{\partial}{\partial x^i} + Y^{A'_i}_{(1,0)} \frac{\partial}{\partial y_{A'_1}} + Y^{A'_i}_{(2,0)} \frac{\partial}{\partial y_{A'_2}} + \ldots + Y^{A'_i}_{(z,0)} \frac{\partial}{\partial y_{A'_z}}, \tag{6}
\]
\[
\frac{\partial}{\partial y_{A'_1}} = K_{A'_1} \frac{\partial}{\partial y_{A'_1}} + Y^{A'_2}_{(2,1)} \frac{\partial}{\partial y_{A'_2}} + \ldots + Y^{A'_z}_{(z,1)} \frac{\partial}{\partial y_{A'_z}},
\]
\[
\frac{\partial}{\partial y_{A'_2}} = K_{A'_2} \frac{\partial}{\partial y_{A'_2}} + Y^{A'_3}_{(3,2)} \frac{\partial}{\partial y_{A'_3}} + \ldots + Y^{A'_z}_{(z,2)} \frac{\partial}{\partial y_{A'_z}},
\]
\[
\frac{\partial}{\partial y_{A'_{z-1}}} = K_{A'_{z-1}} \frac{\partial}{\partial y_{A'_{z-1}}} + Y^{A'_z}_{(z,z-1)} \frac{\partial}{\partial y_{A'_z}}.
\]
\[
\frac{\partial}{\partial y_{A'_z}} = K_{A'_z} \frac{\partial}{\partial y_{A'_z}}.
\]

\(Y\)-matrices from (6) are partial derivations of corresponding combinations of \(K\)-coefficients from coordinate transforms (3),

\[
Y_{A'_f}^{A'_p} = \frac{\partial (K_{A'_p}^A y^{A'_p})}{\partial y^{A'_f}}, \quad f < p.
\]

In brief we denote respectively ds-coordinate transforms of coordinate bases (4) and (5) as

\[
\partial_{<a>} = (K_{<a>}^{<a'>} + Y_{<a>}^{<a'>}) \partial_{<a'>} \quad \text{and} \quad d^{<a>} = (K_{<a>}^{<a>} + Y_{<a>}^{<a>}) d^{<a'}},
\]

where matrix \(K_{<a>}^{<a>'}\), its s-inverse \(K_{<a>}^{<a>}'\), as well \(Y_{<a>}^{<a>}'\) and \(Y_{<a>}^{<a>}\) are parametrized according to (6). In order to illustrate geometric properties of some of our transforms it is useful to introduce matrix operators and to consider in explicit form the parametrizations of matrices under consideration. For instance, in operator form the transforms (6)

\[
\partial = \tilde{Y} \partial',
\]

are characterized by matrices of type

\[
\partial = \partial_{<a>} = \begin{pmatrix}
\frac{\partial}{\partial y_{A'_1}} \\
\frac{\partial}{\partial y_{A'_2}} \\
\vdots \\
\frac{\partial}{\partial y_{A'_z}}
\end{pmatrix}, \quad \partial' = \partial_{<a'>} = \begin{pmatrix}
\frac{\partial}{\partial y_{A'_1}} \\
\frac{\partial}{\partial y_{A'_2}} \\
\vdots \\
\frac{\partial}{\partial y_{A'_z}}
\end{pmatrix}
\]
and

\[
\overline{Y} = \tilde{Y}^{<\alpha'>} = \begin{pmatrix}
K_{1}^{I'} & Y_{(1,0)}^{A_1'} & Y_{(2,0)}^{A_2'} & \ldots & Y_{(z,0)}^{A_z'} \\
0 & K_{A_1}^{A_1'} & Y_{(2,1)}^{A_{A_1}'} & \ldots & Y_{(z,1)}^{A_{A_z}'} \\
0 & 0 & K_{A_2}^{A_2'} & \ldots & Y_{(2,2)}^{A_{A_z}'} \\
\vdots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & K_{A_z}^{A_z'}
\end{pmatrix}.
\]

We note that we obtain a supersymmetric generalization of the Miron–Atanasiusiu [31] osculator bundle \((\text{Osc}^z \tilde{M}, \pi, \tilde{M})\) if the fiber space is taken to be a direct sum of \(z\) vector s-spaces of the same dimension \(\dim \mathcal{F} = \dim \tilde{M}\), i.e. \(\mathcal{F}^{<d>} = \mathcal{F} \oplus \mathcal{F} \oplus \ldots \oplus \mathcal{F}\). In this case the \(K\) and \(Y\) matrices from (3) and (6) satisfy identities:

\[
K_{A_1}^{A_1'} = K_{A_2}^{A_2'} = \ldots = K_{A_z}^{A_z'},
\]

\[
Y_{(1,0),A}^{A'} = Y_{(2,1),A}^{A'} = \ldots = Y_{(z,z-1),A}^{A'}.
\]

\[
Y_{(p,0),A}^{A'} = Y_{(p+1,1),A}^{A'} = \ldots = Y_{(z,z-1),A}^{A'}; \quad (p = 2, \ldots, z - 1).
\]

For \(z = 1\) the \(\text{Osc}^1 \tilde{M}\) is the ts-bundle \(T\tilde{M}\).

Introducing projection \(\pi_0 = \pi^{<z>} : \tilde{\mathcal{E}}^{<z>} \to \tilde{M}\) we can also consider projections \(\pi_{p_2}^{p_1} : \tilde{\mathcal{E}}^{<p_1>} \to \tilde{\mathcal{E}}^{<p_2>}\) \((p_2 < p_1)\) defined as

\[
\pi_{s_2}^{s_1}(x, y^{(1)}, \ldots, y^{(p_1)}) = (x, y^{(1)}, \ldots, y^{(p_2)}).
\]

The s-differentials \(d\pi_{p_2}^{p_1} : T(\tilde{\mathcal{E}}^{<p_1>}) \to T(\tilde{\mathcal{E}}^{<p_2>})\) of maps \(\pi_{p_2}^{p_1}\) in turn define vertical dvs-subbundles \(V_{h+1} = \text{Ker}d\pi_{p_2}^{p_1}\) \((h = 0, 1, \ldots, p_1 - 1)\) of the tangent dvs-bundle \(T(\tilde{\mathcal{E}}^{<z>})\) \(\) (the dvs-space \(V_1 = \tilde{V}\) is the vertical dvs-subbundle on \(\tilde{\mathcal{E}}^{<z>}\)).

The local fibres of dvs-subbundles \(V_h\) determines this regular s-distribution \(V_{h+1} : u \in \tilde{\mathcal{E}}^{<z>} \to V_{h+1}(u) \subset T(\tilde{\mathcal{E}}^{<z>})\) for which one holds inclusions \(V_z \subset V_{z-1} \subset \ldots \subset V_1\). The enumerated properties of vertical dvs–subbundles are explicitly illustrated by transformation laws (6) for distinguished local bases.

## 4 Nonlinear Connections in Vector S–Bundles

The purpose of this section is to present an introduction into geometry of the nonlinear connection structures in vector s-bundles. The concept of nonlinear connection (N-connection) was introduced in the framework of Finsler geometry [14, 13, 21] \(\) (the global definition of N-connection is given in [8]). It should be noted that the N-connection (splitting) field could play an important role in modeling various variants of dynamical reduction from higher dimensional to lower dimensional s–spaces with (or not) different types of local
anisotropy. Monographs \[29, 30\] contain details on geometrical properties of N-connection structures in $v$-bundles and different generalizations of Finsler geometry and some proposals (see Chapter XII in \[29\], written by S. Ikeda) on physical interpretation of N-connection in the framework of “unified” field theory with interactions nonlocalized by $v$-dependencies are discussed. We emphasize that N-connection is a different geometrical object from that introduced by using nonlinear realizations of gauge groups and supergroups (see, for instance, the collection of works on supergravity \[37\] and approaches to gauge gravity \[40, 33\]). To make the presentation to aid rapid assimilation we shall have realized our geometric constructions firstly for vs-bundles then (in the next section) we shall extend them for higher order extensions, i.e. for general dvs-bundles.

Let consider the definitions of N-connection structure \[42\] in a vs-bundle $\tilde{E} = (\tilde{E}, \pi_{\tilde{E}}, \tilde{M})$ whose typical fibre is $\hat{F}$ and $\pi^T : T\tilde{E} \to T\tilde{M}$ is the superdifferential of the map $\pi_E$ ($\pi^T$ is a fibre-preserving morphism of the ts-bundle $(T\tilde{E}, \tau_{\tilde{E}}, \tilde{M})$ to $\tilde{E}$ and of ts-bundle $(TM, \tau, \tilde{M})$ to $\tilde{M}$). The kernel of this vs-bundle morphism being a subbundle of $(T\tilde{E}, \tau_{\tilde{E}}, \tilde{E})$ is called the vertical subbundle over $\tilde{E}$ and denoted by $V\tilde{E} = (V\tilde{E}, \tau_{\tilde{E}})$. Its total space is $V\tilde{E} = \bigcup_{u \in \tilde{E}} V_u$, where $V_u = ker\pi^T, \ u \in \tilde{E}$. A vector

\[ Y = Y^\alpha \frac{\partial}{\partial u^\alpha} = Y^I \frac{\partial}{\partial x^I} + Y^A \frac{\partial}{\partial y^A} = Y^i \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a} + Y^\hat{a} \frac{\partial}{\partial \zeta^\hat{a}} \]

is tangent to $\tilde{E}$ in the point $u \in \tilde{E}$ is locally represented as

\[ (u, Y) = (u^\alpha, Y^\alpha) = (x^I, y^A, Y^I, Y^A) = (\tilde{x}^i, \tilde{\theta}^\hat{a}, \tilde{y}^a, \tilde{\zeta}^\hat{a}, \tilde{Y}^i, \tilde{Y}^a, \tilde{Y}^\hat{a}). \]

A nonlinear connection, N-connection, in vs-bundle $\tilde{E}$ is a splitting on the left of the exact sequence

\[ 0 \to V\tilde{E} \to T\tilde{E} \to \tilde{E} / V\tilde{E} \to 0, \tag{7} \]

i.e. a morphism of vs-bundles $N : T\tilde{E} \to V\tilde{E}$ such that $N \circ i$ is the identity on $V\tilde{E}$.

The kernel of the morphism $N$ is called the horizontal subbundle and denoted by

\[ (H\tilde{E}, \tau_{\tilde{E}}, \tilde{E}). \]

From the exact sequence (7) one follows that N-connection structure can be equivalently defined as a distribution $\{E_u \to H_u\tilde{E}, T_u\tilde{E} = H_u\tilde{E} \oplus V_u\tilde{E}\}$ on $\tilde{E}$ defining a global decomposition, as a Whitney sum,

\[ T\tilde{E} = H\tilde{E} \oplus V\tilde{E}. \tag{8} \]
To a given N-connection we can associate a covariant s-derivation on $\tilde{M}$:

$$\nabla_X Y = X^I \{ \frac{\partial Y^A}{\partial x^I} + N_I^A(x, Y) \} s_A, \quad (9)$$

where $s_A$ are local independent sections of $\tilde{\mathcal{E}}$, $Y = Y^A s_A$ and $X = X^I s_I$.

S-differentiable functions $N_I^A$ from (3) written as functions on $x^I$ and $y^A$, $N_I^A(x, y)$, are called the coefficients of the N-connection and satisfy these transformation laws under coordinate transforms (1) in $\mathcal{E}$:

$$N_I^A \frac{\partial x^I'}{\partial x^I} = M_A^I N_I^A - \frac{\partial M_A^I(x)}{\partial x^I} y^A.$$

If coefficients of a given N-connection are s-differentiable with respect to coordinates $y^A$ we can introduce (additionally to covariant nonlinear s-derivation (9)) a linear covariant s-derivation $\hat{D}$ (which is a generalization for vs-bundles of the Berwald connection [11]) given as follows:

$$\hat{D}(\frac{\partial}{\partial x^I}) = \hat{N}_B^I \frac{\partial}{\partial y^B}, \quad \hat{D}(\frac{\partial}{\partial y^A}) = 0,$$

where

$$\hat{N}^A_{BI}(x, y) = \frac{\partial N^A_{I}(x, y)}{\partial y^B} \quad (10)$$

and

$$\hat{N}^A_{BC}(x, y) = 0.$$

For a vector field on $\tilde{\mathcal{E}}$ $Z = Z^I \frac{\partial}{\partial x^I} + Y^A \frac{\partial}{\partial y^A}$ and $B = B^A(y) \frac{\partial}{\partial y^A}$ being a section in the vertical s-bundle $(\mathcal{V}\tilde{\mathcal{E}}, \tau_V, \tilde{\mathcal{E}})$ the linear connection (10) defines s-derivation (compare with (9)):

$$\hat{D}_Z B = [Z^I \frac{\partial B^A}{\partial x^I} + \hat{N}^A_{BI} B^B] + Y^B \frac{\partial B^A}{\partial y^B} \frac{\partial}{\partial y^A}.$$

Another important characteristic of a N-connection is its curvature:

$$\Omega = \frac{1}{2} \Omega^A_{IJ} dx^I \wedge dx^J \otimes \frac{\partial}{\partial y^A}$$

with local coefficients

$$\Omega^A_{IJ} = \frac{\partial N^A_{I}}{\partial x^J} - (-)^{|I|J} \frac{\partial N^A_{J}}{\partial x^I} + N^B_{IJ} \hat{N}^A_{BI} - (-)^{|I|J} N^B_{JI} \hat{N}^A_{BI}, \quad (11)$$

where for simplicity we have written $(-)^{|K||J|} = (-)^{|KJ|}$.

We note that linear connections are particular cases of N-connections locally parametrized as $N^A_I(x, y) = N^A_{BI}(x) x^I y^B$, where functions $N^A_{BI}(x)$, defined on $\tilde{M}$, are called the Christoffel coefficients.
5 N-Connections in DVS-Bundles

In order to define a N-connection in a dvs-bundle $\bar{\mathcal{E}}^{<z>}$ we consider a s-subbundle $N \left( \bar{\mathcal{E}}^{<z>} \right)$ of the ts-bundle $T \left( \bar{\mathcal{E}}^{<z>} \right)$ for which one holds (see [29, 30, 31] respectively for jet and osculator bundles) the Whitney sum (compare with (8))

$$T \left( \bar{\mathcal{E}}^{<z>} \right) = N \left( \bar{\mathcal{E}}^{<z>} \right) \oplus V \left( \bar{\mathcal{E}}^{<z>} \right).$$

$N \left( \bar{\mathcal{E}}^{<z>} \right)$ can be also interpreted as a regular s-distribution (horizontal distribution being supplementary to the vertical s-distribution $V \left( \bar{\mathcal{E}}^{<z>} \right)$) determined by maps $N : u \in \bar{\mathcal{E}}^{<z>} \rightarrow N(u) \subset T_u \left( \bar{\mathcal{E}}^{<z>} \right)$.

The condition of existence of a N-connection in a dvs-bundle $\bar{\mathcal{E}}^{<z>}$ can be proved as in [29, 30, 31]: It is required that $\bar{\mathcal{E}}^{<z>}$ is a paracompact s-differentiable (in our case) manifold.

Locally a N-connection in $\bar{\mathcal{E}}^{<z>}$ is given by its coefficients

$$N_{(01)}^A (u), \left( N_{(02)}^A (u), N_{(12)}^A (u) \right), \ldots \left( N_{(p)}^A (u), N_{(1p)}^A (u), \ldots, N_{(p-1)p-1}^A (u) \right), \ldots,$$

$$N_{(0z)}^A (u), N_{(1z)}^A (u), \ldots, N_{(pz)}^A (u), \ldots, N_{(z-lz)}^A (u), \ldots,$$

where, for instance, $(N_{(0p)}^A (u), N_{(1p)}^A (u), \ldots, N_{(p-1)p-1}^A (u))$ are components of N-connection in vs-bundle $\pi^{<p>} : \tilde{E}^{<p>} \rightarrow \tilde{E}^{<p-1>}$.

Here we note that if a N-connection structure is defined we must correlate to it the local partial derivatives on $\bar{\mathcal{E}}^{<z>}$ by considering instead of local coordinate bases (4) and (5) the so-called locally bases (la-bases)

$$\delta_{<a>} = (\delta^I, \delta^{<A>} ) = \left( \delta^I, \delta^{(A_1)}, \delta^{(A_2)}, \ldots, \delta^{(A_s)} \right) =$$

$$\frac{\partial}{\partial u^{<a>}} = \left( \frac{\partial}{\partial x^I}, \frac{\partial}{\partial y_{(1)}}, \frac{\partial}{\partial y_{(2)}}, \ldots, \frac{\partial}{\partial y_{(z)}} \right)$$

(12)

(13)

with components parametrized as

$$\delta^I = \partial^I - N_{(0)}^A \partial_A - N_{(1)}^A \partial_A - \ldots - N_{(p-1)}^A \partial_A,$$

$$\delta_{A_1} = \partial_{A_1} - N_{(p)}^A \partial_A - N_{(p+1)}^A \partial_A - \ldots - N_{(p-1)}^A \partial_A,$$

(14)
and

$$\delta_{A_2} = \partial_{A_2} - N_{A_2}^A \partial_{A_3} - N_{A_3}^A \partial_{A_4} - \ldots - N_{A_{z-1}}^A \partial_{A_z} - N_{A_z}^A \partial_{A_z},$$



$$\delta_{t_{z-1}} = \partial_{t_{z-1}} - N_{t_{z-1}}^A \partial_{t_{z}},$$

$$\delta_{t_z} = \partial_{t_z},$$

or, in matrix form, as

$$\delta_\bullet = \tilde{N}(u) \times \delta_\bullet,$$

where

$$\delta_\bullet = \delta_{<\alpha>} = \begin{pmatrix} \delta_I \\ \delta_{A_1} \\ \vdots \\ \delta_{A_z} \end{pmatrix}, \quad \delta_\bullet = \delta_{<\alpha>} = \begin{pmatrix} \partial_I \\ \partial_{A_1} \\ \vdots \\ \partial_{A_z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y_{(1)}} \\ \frac{\partial}{\partial y_{(2)}} \\ \vdots \\ \frac{\partial}{\partial y_{(z)}} \end{pmatrix}$$

and

$$\tilde{N} = \begin{pmatrix} 1 & -N_{I}^{A_1} & -N_{I}^{A_2} & \ldots & -N_{I}^{A_z} \\ 0 & 1 & -N_{A_1}^{A_1} & \ldots & -N_{A_1}^{A_z} \\ 0 & 0 & 1 & \ldots & -N_{A_2}^{A_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

In generalized index form we write the matrix (6) as $\tilde{N}_{<\alpha>}$, where, for instance,

$$\tilde{N}_I^I = \delta_{I_I}, \tilde{N}_{B_1}^{A_1} = \delta_{B_1}^{A_1}, \ldots, \tilde{N}_I^{A_1} = -N_{I}^{A_1}, \ldots, \tilde{N}_{A_1}^{A_1} = -N_{A_1}^{A_1}, \tilde{N}_{A_2}^{A_1} = -N_{A_2}^{A_1}, \ldots.$$

So in every point $u \in \mathcal{E}^{<z>}$ we have this invariant decomposition:

$$T_u \left( \mathcal{E}^{<d>} \right) = N_0(u) \oplus N_1(u) \oplus \ldots \oplus N_{z-1}(u) \oplus V_z(u),$$

where $\delta_I \in N_0, \delta_{A_1} \in N_1, \ldots, \delta_{A_{z-1}} \in N_{z-1}, \partial_{A_z} \in V_z$.

We note that for the osculator s-bundle $\left(Osc^2 \tilde{M}, \pi, \tilde{M} \right)$ there is an additional (we consider the N-adapted variant) s-tangent structure

$$J : \Xi \left( Osc^2 \tilde{M} \right) \rightarrow \Xi \left( Osc^2 \tilde{M} \right)$$

defined as

$$\frac{\delta}{\partial y_{(1)}} = J \left( \frac{\delta}{\partial x^I} \right), \quad \ldots \quad \frac{\delta}{\partial y_{(z-1)}} = J \left( \frac{\delta}{\partial y_{(z-2)}} \right), \quad \frac{\partial}{\partial y_{(z)}} = J \left( \frac{\delta}{\partial y_{(z-1)}} \right)$$

(16)
(in this case $I$- and $A$-indices take the same values and we can not distinguish them), by considering vertical $J$-distributions

\[ N_0 = N, N_1 = J (N_0), ..., N_{z-1} = J (N_{z-2}). \]

In consequence, for the la-adapted bases on \( (Osc^z \tilde{M}, \pi, \tilde{M}) \) there is written this N–connection matrix:

\[
N = N^{<J>}_{<I>} = \begin{pmatrix}
1 & -N^{J}_{(1)I} & -N^{J}_{(2)I} & ... & -N^{J}_{(z)I} \\
0 & 1 & -N^{J}_{(1)I} & ... & -N^{J}_{(z-1)I} \\
0 & 0 & 1 & ... & -N^{J}_{(z-2)I} \\
... & ... & ... & ... & ...
\end{pmatrix}.
\] (17)

There is a unique distinguished local decomposition of every $s$–vector $X \in \chi (\tilde{\varepsilon}^{<z>})$ on la-base (12):

\[
X = X^{(H)} + X^{(V_1)} + ... + X^{(V_z)},
\] (18)

by using the horizontal, $h$, and verticals, $v_1, v_2, ..., v_z$, projections:

\[
X^{(H)} = hX = X^I \delta_I, \quad X^{(V_1)} = v_1 X = X^{(A_1)} \delta_{A_1}, ..., \quad X^{(V_z)} = v_z X = X^{(A_z)} \delta_{A_z}.
\]

With respect to coordinate transforms (3) the la-bases (12) and ds–vector components (18) are correspondingly transformed as

\[
\frac{\delta}{\partial x^I'} = \frac{\partial x^I'}{\partial x^I} \frac{\delta}{\partial x^I}, \quad \frac{\delta}{\partial A_p'} = K_{A_p'} A_p \frac{\delta}{\partial y_{(p)}}.
\] (19)

and

\[
X' = \frac{\partial x'}{\partial x^I} X^I, \quad X^{(A_p')} = K_{A_p'} A_p X^{(A_p')}, \forall p = 1, 2, ..., z.
\]

Under changing of coordinates (3) the local coefficients of a nonlinear connection transform as follows:

\[
Y^{<\alpha'>}_{<\alpha>} \tilde{N}^{<\beta'>}_{<\alpha'>} = \tilde{N}^{<\beta'}_{<\alpha'>} (K^{<\beta'>}_{<\alpha'>} + Y^{<\beta'>}_{<\alpha'>})
\]

(we can obtain these relations by putting (19) and (6) into (14) where \( \tilde{N}^{<\beta'>}_{<\alpha'>} \) satisfy \( \delta_{<\alpha'>} = \tilde{N}^{<\beta'>}_{<\alpha'>} \delta_{<\beta'>} \).

For dual la-bases (13) we have these N–connection "prolongations of differentials":

\[
\delta x^I = dx^I,
\]
\[ \delta y^A_1 = dy^A_1 + M^A_1 dx^I, \]
\[ \delta y^A_2 = dy^A_2 + M^A_2 dy^A_1 + M^A_2 dx^I, \]

\[ \delta y^A_s = dy^A_s + M^A_s dy^A_1 + M^A_s dy^A_2 + \ldots + M^A_s dx^I, \]

where \( M^{(*)\bullet}_s \) are the dual coefficients of the N-connection which can be expressed explicitly by recurrent formulas through the components of N–connection \( N^{<I}_{<A>}. \)

To do this we shall rewrite formulas (20) in matrix form:

\[ \delta^* = d^* \times M(u), \]

where

\[ \delta^* = \left( \begin{array}{cccc} \delta x^I & \delta y^A_1 & \delta y^A_2 & \cdots \delta y^A_s \end{array} \right), \quad d^* = \left( \begin{array}{cccc} dx^I & dy^A_1 & dy^A_2 & \cdots & dy^A_s \end{array} \right) \]

and

\[ M = \begin{pmatrix}
1 & M^A_1 & M^A_2 & \cdots & M^A_s \\
0 & 1 & M^A_2 & \cdots & M^A_s \\
0 & 0 & 1 & \cdots & M^A_s \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}, \]

and then, taking into consideration that bases \( \partial_s(\delta^*) \) and \( d^*(\delta^*) \) are mutually dual, to compute the components of matrix \( M \) being s–inverse to matrix \( N \) (see (15)). We omit these simple but tedious calculus for general dvs-bundles and, for simplicity, we present the basic formulas for osculator s–bundle \( (Osc^s \tilde{\mathcal{M}}, \pi, \tilde{\mathcal{M}}) \) when \( J \)-distribution properties (16) and (17) alleviates the problem. For common type of indices on \( \tilde{\mathcal{M}} \) and higher order extensions on \( Osc^s \tilde{\mathcal{M}} \) the dual la-base is expressed as

\[ \delta x^I = dx^I, \]
\[ \delta y^I_1 = dy^I_1 + M^I_1,J dx^J, \]
\[ \delta y^I_2 = dy^I_2 + M^I_2,J dy^I_1 + M^I_2,J dx^J, \]

\[ \delta y^I_s = dy^I_s + M^I_{(s)},J dy^I_{(s-1)} + M^I_{(s)},J dy^I_{(s-2)} + \ldots + M^I_{(s)},J dx^J, \]

with \( M \)-coefficients computed by recurrent formulas:

\[ M^I_{(1),J} = N^I_{(1),J}, \]
\[ M^I_{(2),J} = N^I_{(2),J} + N^I_{(1),K} M^K_{(1),J}, \]
\[ M_I^J(s) = N_I^J(s) + N_I^{(s-1)K} M_K^J + \ldots + N_I^{(2)K} M_K^{(s-2)J} + N_I^{(1)K} M_K^{(s-1)J}. \]

One holds these transformation law for dual coefficients (21) with respect to coordinate transforms (3):

\[
\begin{align*}
M^{(1)}_{KJ} Y^I_{(0,0)K} &= M^{(1)}_{KJ'} Y^{K'}_{(0,0)J} + Y^I_{(1,0)J}, \\
M^{(2)}_{KJ} Y^I_{(0,0)K} &= M^{(2)}_{KJ'} Y^{K'}_{(0,0)J} + M^{(1)}_{KJ'} Y^{K'}_{(1,0)J} + Y^I_{(2,0)J},
\end{align*}
\]

\[
\begin{align*}
M^{(z)}_{KJ} Y^I_{(0,0)K} &= M^{(z)}_{KJ'} Y^{K'}_{(0,0)J} + M^{(z-1)}_{KJ'} Y^{K'}_{(1,0)J} + \ldots + M^{(1)}_{KJ'} Y^{K'}_{(z-1,0)J} + Y^I_{(z,0)J}.
\end{align*}
\]

(the proof is a straightforward regroupation of terms after we have put (3) into (21)).

Finally, we note that curvatures of a N-connection in a dvs-bundle \( \tilde{E}^{<z>} \) can be introduced in a manner similar to that for usual vs-bundles (see (11) by a consequent step by step inclusion of higher dimension anisotropies:

\[
\Omega_{(p)} = \frac{1}{2} \Omega^{A_p}_{(p)\alpha_{p-1}} \delta u^{\alpha_{p-1}} \wedge \delta u^{\beta_{p-1}} \otimes \delta \frac{\partial}{\partial y_{(p)}^A}, p = 1, 2, \ldots, z,
\]

with local coefficients

\[
\Omega^A_{(p)\beta_{p-1}\gamma_{p-1}} = \frac{\delta N^A_{\beta_{p-1}}}{\delta u_{(p-1)}^{\gamma_{p-1}}} - (-)^{1+\beta_{p-1}\gamma_{p-1}} \frac{\delta N^A_{\gamma_{p-1}}}{\delta u_{(p-1)}^{\beta_{p-1}}} + \frac{\delta N^D_{\beta_{p-1}}}{\delta y_{(p)}^A} \hat{N}^A_{\beta_{p-1} \gamma_{p-1}} + (-)^{1+\beta_{p-1}\gamma_{p-1}} \frac{\delta N^D_{\gamma_{p-1}}}{\delta y_{(p)}^A} \hat{N}^A_{\beta_{p-1} \gamma_{p-1}},
\]

where \( \hat{N}^A_{\beta_{p-1} \gamma_{p-1}} = \frac{\delta N^A_{\beta_{p-1}}}{\delta y_{(p)}^A} \) (we consider \( y^{A_0} \approx x^I \)).

6 Discussion

We have explicitly constructed a new class of superspaces with higher order anisotropy. The status of the results in this work and the relevant open questions are discussed as follows.

From the generally mathematical point of view it is possible a definition of a supersymmetric differential geometric structure imbedding both type of supersymmetric extensions of Finsler and Lagrange geometry as well various Kaluza–Klein superspaces. The first type of superspaces, considered as locally anisotropic, are characterized by nontrivial nonlinear connection structures and corresponding distinguishing of geometric objects and basic structure equations. The second type as a rule is associated to trivial nonlinear
connections and higher order dimensions. A substantial interest for further considerations presents the investigations of physical consequences of models of field interactions on higher and/or lower dimensional superspaces provided with N–connection structure.

It worth noticing that higher order derivative theories are one of currently central division in modern theoretical and mathematical physics. It is necessary a rigorous formulation of the geometric background for developing of higher order analytic mechanics and corresponding extensions to classical quantum field theories. Our results do not only contain a supersymmetric extension of higher order fiber bundle geometry, but also propose a general approach to the ”physics” with locally anisotropic interactions. The elaborated in this paper formalism of distinguished vector superbundles highlights a scheme by which supergravitational and superstring theories with higher order anisotropy can be constructed. This is a matter of our further investigations [42, 44].

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References

[1] Antonelli P L and Miron R (eds) 1996 *Lagrange and Finsler Geometry, Applications to Physics and Biology* (Dordrecht, Boston, London: Kluwer Academic Publishers)

[2] Antonelli P L and Zastavniak T J (guest eds) 1994 *Lagrange Geometry, Finsler Spaces and Noise Applied in Biology and Physics*, in Mathematical and Computer Modelling ed Rolin F Y 20. N415 (Plenum Press)

[3] Asanov G S Finsler 1985 *Geometry, Relativity and Gauge Theories* (Dordrecht, Boston, London: D. Reidel Publishing Company)

[4] Asanov G S and Ponomarenko S F 1988 *Finsler Bundle on Space–Time. Associated Gauge Fields and Connections* (Chişinău, Moldova: Știința) [in Russian]

[5] Asanov G S 1989 *Fibered Generalization of the Gauge Field Theory. Finslerian and Jet Gauge Fields* (Moscow: Moscow University) [in Russian]

[6] Barthel W J 1963 *Reine Angew. Math.* 212 120

[7] Bartocci C, Bruzzo U and Hermández–Ruipérez G 1991 *The Geometry of Supermanifolds* (Dordrecht, Boston, London: Kluwer Academic Publishers)

[8] Bejancu A 1990 *A New Viewpoint on Differential Geometry of Supermanifolds, I* (Timişoara, Romania: Timişoara University Press)

[9] Bejancu A 1991 *A New Viewpoint on Differential Geometry of Supermanifolds, II* (Timişoara, Romania, Timişoara University Press)

[10] Berežnij F A and Leites D A 1975 *Doklady Akademii Nauk SSSR* 224 505 [in Russian]; 1975 *Sov. Math. Dokl.* 16 1218

[11] Berwald L 1926 *Math. Z.* 25 40; 1927 *Correction, Math. Z.* 26 176

[12] Bruzzo U and Cianci R 1984 Class. Quant. Grav. 1 213

[13] Cartan E 1935 *Les Espaces de Finsler* (Paris: Hermann)

[14] Cartan E 1936 *Exposés de Géométrie in Series Actualités Scientifiques et Industrielles* 79; reprinted 1971 (Paris : Hermann)

[15] Cianci R 1990 *Introduction to Supermanifolds* (Napoli: Bibliopolis)

[16] Finsler P 1918 *Über Kurven und Flächen in Allgemeiner Räumen* (Göttingen: Dissertation); reprinted 1951 (Basel: Birkhäuser)

[17] Hoyos J, Quiros M, Ramírez Mittelbrunn J and De Uries F J 1984 *J. Math. Phys.* 25 833; 841; 847

[18] Ingarden R S 1976 *Tensor N.S.* 30 201

[19] Ishikawa H 1981 *J. Math. Phys.* 22 995

[20] Jadczic A and Pilch K 1981 *Commun. Math. Phys.* 78 373

[21] Kawaguchi A 1956 *Tensor N. S.* 6 596

[22] Kac V 1977 *Commun.Math.Phys.* 53 31

[23] Konstant B 1977 in *Differential Geometric Methods in Mathematical Physics*, Lecture Notes in Mathematics 570 177
[24] Leites D A 1980 *Usp. Math. Nauk* **35** 3 [in Russian]

[25] Leites D A 1980 *The Theory of Supermanifolds* (Petrozavodsk: URSS) [in Russian]

[26] Leng S 1972 *Differential Manifolds* (Reading, Mass, Addison–Wesley)

[27] Manin Yu I 1984 *Gauge Fields and Complex Geometry* (Moscow: Nauka) [in Russian]

[28] Matsumoto M 1986 *Foundations of Finsler Geometry and Special Finsler Spaces* (Kaisish: Shigaken)

[29] Miron R and Anastasie M 1987 *Vector Bundles. Lagrange Spaces. Application in Relativity* (Bucharest, Romania: Academiei, Romania) [in Romanian]

[30] Miron R and Anastasie M 1994 *The Geometry of Lagrange Spaces: Theory and Applications* (Dordrecht, Boston, London: Kluwer Academic Publishers)

[31] Miron R and Atanasiu Gh 1994 *Compendium sur les Espaces Lagrange D’ordre Supérieur*, Seminarul de Mecanica. Universitatea din Timișoara. Facultatea de Matematică (Timisoara, Romania)

[32] Miron R and Kavaguchi T 1991 *Int. J. Theor. Phys.* **30** 1521

[33] Ponomarev V N, Barvinsky A O and Obukhov Yu N 1985 *Geometrodynamical Methods and Gauge Approach to Gravity Theory* (Moscow: Energoatomizdat) [in Russian]

[34] Rogers A 1980 *J. Math. Phys.* **21** 1352

[35] Rogers A 1981 *J. Math. Phys.* **22** 939

[36] Rund H 1959 *The Differential Geometry of Finsler Spaces* (Berlin: Springer–Verlag)

[37] Salam A and Sezgin E (eds) 1989 *Supergravities in Diverse Dimensions*, vol. 1 and 2, (Amsterdam, Singapore: World Scientific)

[38] Sardanashvily G A 1993 *Gauge Theory in Jet Manifolds* (Palm Harbor: Hadronic Press); 1994 *Five Lectures on the Jet Manifold. Methods in Field Theory* E-print: hep-th/9411089

[39] Saunders D J 1989 *The Geometry of Jet Bundles* (Cambridge: Cambridge Univ. Press)

[40] Tseytlin A A 1982 *Phys. Rev.* **D26** 3327

[41] Vacaru S 1996 *J. Math. Phys.* **37** 508

[42] Vacaru S 1996 *Nonlinear Connections in Superbundles and Locally Anisotropic Supergravity* E-print: gr-qc/9604016

[43] Vacaru S 1996 *Spinors in Multidimensional and Locally Anisotropic Spaces* E-print: gr-qc/9604015

[44] Vacaru S 1996 *Locally anisotropic gravity and strings*, E-print: gr-qc/9604013

[45] Vacaru S and Goncharenko Yu 1995 *Int. J. Theor. Phys.* **34** 1955

[46] Vacaru S and Ostaf S 1996 in *Lagrange and Finsler Geometry* eds. Antonelli P L and Miron R (Dordrecht, Boston, London: Kluwer Academic Publishers) 241

[47] Vladimirov V S and Volovich I V 1984 *Theor. Math. Phys.* **59** 317 [in Russian]

[48] Vlasov A A 1966 *Statistical Distribution Functions* (Moscow: Nauka) [in Russian]
[49] Volovich I V 1975 Doklady Academii Nauk SSSR 269 524 [in Russian]

[50] Yano Y and Ishihara S I 1973 Tangent and Cotangent Bundles. Differential Geometry (New York: Marcel Dekker)

[51] DeWitt B 1984 Supermanifolds (Cambridge: Cambridge University Press)