An Exposition of a Result in “Conjugate Codes for Secure and Reliable Information Transmission”

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Abstract
An elementary proof of the attainability of random coding exponent with linear codes for additive channels is presented. The result and proof are from Hamada (Proc. ITW, Chendu, China, 2006), and the present material explains the proof in detail for those unfamiliar with elementary calculations on probabilities related to linear codes.

1 Introduction
In this material, the details of the proof of a result in [1], an article prepared for an invited talk, are presented without assuming any prerequisite knowledge. In fact, when the author prepared the manuscript [2], which includes one illustrative application of the method of concatenating ‘conjugate code pairs’ devised in [1, 3], the author thought some (or most) proofs are elementary and straightforward, so that they are not needed for those working in our society of information theory. However, in this article, still more details will be presented to increase the accessibility.

We remark the result and its detailed proof are written so that they can be read without referring to [1]. Specifically, in this material, an elementary proof of the attainability of random coding exponent with linear codes for additive channels is presented. (Of course, many proofs for the attainability of random coding exponent had existed, but the incentive for developing this approach was to design quantum error-correcting codes and codes that can be used in cryptographic protocols. For these purposes, we needed to design codes and decoders under constraints arising from quantum mechanics.)

Thus, this material is supplementary to [1] for those unfamiliar with the elementary approach adopted in [1], but the result treated in this material is
compact, classical, and comprehensible without understanding the main issues treated in [1]. This approach is nothing special, but it may be said to be that of the method of types [4, 5], which requires no prerequisite knowledge, with the very basics of linear codes incorporated.

The aforementioned illustrative application of the method for concatenation is construction of pairs of linear codes \((L_1, L_2)\) with \(L_2^\perp \subseteq L_1\) (‘conjugate code pairs’) that achieve a high information rate on the Shannon theoretic criterion. Such a code pair can be viewed as a succinct representation of the corresponding quantum error-correcting code (QECC). The code construction is explicit in the standard sense that the codes are constructible with polynomial complexity. Another (cryptographic) application, which reflects the original motivation of [1, 2] has been presented in [6].

\section{Corrections and Remark to [1]}

\subsection{Corrections to [1]; Some Apply Also to [2]}

1. p. 149, right column, line 14, ‘ensemble’ should be followed by ‘(multiset)’
2. p. 150, left column, line \(-1\),
   \[ a_n |\mathcal{P}_n|^2 d^{-nE_r(W,r)} \]
   should read
   \[ a_n |\mathcal{P}_n|^2 q^{-nE_r(W,r)} \]
3. p. 150, right column, line \(-9\), ‘parameter \(k\)’ should read ‘the number \(k/n\)’
4. p. 151, left column, line \(-8\), ‘\((y^{(i)} \cdots y^{(N)}_N)\)’ should read ‘\((y^{(1)} \cdots y^{(N)})\)’
5. p. 152, left column, line 1, ‘\((\bigoplus_{i=1}^t C_1^{(i)} \cdot \bigoplus_{i=1}^t C_2^{(i)})\)’ should read
   \[ ' (\bigoplus_{j=1}^t C_1^{(i)} \cdot \bigoplus_{j=1}^t C_2^{(i)})' \]
6. p. 152, left column, Eq. (6),
   \[ M_Q(C_j^{(i)} \setminus \{0_n\}) \leq (|\mathcal{P}_n(\mathbb{F}_q)| - 1)q^{-n(1-r_j)}A \]
   should read
   \[ M_Q(C_j^{(i)} \setminus \{0_n\}) \leq (|\mathcal{P}_n(\mathbb{F}_q)| - 1)q^{-n(1-r_j)}|\mathcal{T}_nQ|A \]

Essentially the same errors as in 1, 2 and 6 exist in Section 4 of [2] (ver. 2), but the contents of Section 4 of [2] are presented below in the corrected form.

\subsection{Remark to [1, 2]}

Note that, in [1, 2], an ensemble has been represented as a multiset, which is similar to a usual set but permits duplicated entries.

Now the author thinks representing an ensemble as an ordered set is more natural, as will be done in the present article.
3 Preliminaries

In this section, we fix our notation, and recall some notions to be used. As usual, \(|a|\) denotes the largest integer \(a'\) with \(a' \leq a\), and \([-a]\) is defined in a different manner \(\frac{1}{2}(a - a')\). An \([n, k]\) linear (error-correcting) code over a finite field \(\mathbb{F}_q\), the finite field of \(q\) elements, is a \(k\)-dimensional subspace of \(\mathbb{F}_q^n\). The dual of a linear code \(C \subseteq \mathbb{F}_q^n\) is \(\{y \in \mathbb{F}_q^n \mid \forall x \in C, \ x \cdot y = 0\}\) and denoted by \(C^\perp\), where \(x \cdot y = xy^\perp\) with \(y^\perp\) being the transpose of \(y\). The zero vector in \(\mathbb{F}_q^n\) is denoted by \(0_n\). The \(n \times n\) identity (resp. zero) matrix is denoted by \(I_n\) (resp. \(O_n\)). For integers \(i \leq j\), we often use the set \([i, j] \cap \mathbb{Z} = \{i, i + 1, \ldots, j\}\), which consists of integers lying in the interval \([a, b] = \{z \in \mathbb{R} \mid a \leq z \leq b\}\).

We can find good codes in an ensemble if the ensemble is ‘balanced’ in the following sense. Suppose \(S = \{C^{(i)}\}_{i=1}^N\) is an ensemble (ordered set) of subsets of \(\mathbb{F}_q^n\). If there exists a constant \(V\) such that \(|\{i \in [1, N] \cap \mathbb{Z} \mid x \in C^{(i)}\}| = V\) for any word \(x \in \mathbb{F}_q^n \setminus \{0_n\}\), the ensemble \(S\) is said to be balanced. (We remark that the ‘balancedness’ is defined in a different manner in \[2\] for ensembles of encoders, not codes.)

The first result was to construct a relatively small balanced ensemble. This result can be found in \[1\, 2\], but it is included in Appendix \[A\, 2\]. With the method of types, we will show that a large portion of a balanced ensemble consists of good codes. While the goodness of codes should be evaluated by the decoding error probability, it is also desirable to quantify the goodness in such a way that the goodness does not depend on characteristics of channels. In view of this, the following proposition is useful.

The next proposition relates the spectrum of a code with its decoding error probability when it is used on an additive memoryless channel.

**Proposition 1** \[8\, \text{Theorem 4}\]. Suppose we have an \([n, \kappa]\) linear code \(C\) over \(\mathbb{F}_q\) such that

\[
M_Q(C) \leq a_n q^{\kappa - n} |T_Q^n|, \quad Q \in \mathcal{P}_n(\mathbb{F}_q) \setminus \{\mathcal{P}_{0_n}\}.
\]
for some \( a_n \geq 1 \). Then, its decoding error probability with the minimum entropy syndrome decoding is upper-bounded by
\[
a_n |\mathcal{P}_n(\mathbb{F}_q)|^2 q^{-nE_{r}(W, r)}
\]
for any additive channel \( W \) of input-output alphabet \( \mathbb{F}_q \), where \( r = \kappa/n \) and \( E_{r}(W, r) \) is the random coding exponent of \( W \) defined by
\[
E_{r}(W, r) = \min_{Q \in \mathcal{P}(\mathbb{F}_q)} [D(Q||W) + |1 - r - H(Q)|^+].
\]
Here, \( D \) and \( H \) denote the relative entropy and entropy, respectively, and \( |x|^+ = \max\{0, x\} \).

For a proof, see Section 4.3. In the simplest case where \( q = 2 \), the premise of the above proposition reads ‘the spectrum of \( C \) is approximated by the binomial coefficients \(|T_n Q|^+\) up to normalization.’

The following lemma shows a large portion of a balanced ensemble \( \{C^{(i)}\}_{i=1}^{N^*} \) is made of good codes (we have applied this fact to ensembles written as \( \{C^{(j)}\}_{j=1}^{N^*} \) in [1,2]).

**Lemma 1** [1, p. 152, left column]. Assume we have a balanced ensemble \( \{C^{(i)}\}_{i=1}^{N^*} \). Let us say an \([n, \kappa]\) code \( C^{(i)} \) is \( A \)-good if
\[
M_Q(C^{(i)}) \leq A(|\mathcal{P}_n(\mathbb{F}_q)| - 1)q^{-n(1-\rho)}|T_Q^n|
\]
for all \( Q \in \mathcal{P}_n(\mathbb{F}_q) \setminus \{P_0\} \), where \( \rho = \kappa/n \). Then, the number of codes that are not \( q^{\epsilon n} \)-good in \( \{C^{(i)}\}_{i=1}^{N^*} \) is at most
\[
z = [N^* q^{-\epsilon n}].
\]

This lemma will be proved in Section 4.2. Note, owing to Proposition 1, for the \( q^{\epsilon n} \)-good codes \( C^{(i)} \) in the above lemma, the decoding error probability is upper-bounded by
\[
a'_n q^{-nE_{r}(W, \rho) - \epsilon},
\]
where \( a'_n = |\mathcal{P}_n(\mathbb{F}_q)|^3 \) is at most polynomial in \( n \).

**4.2 Proof of Lemma 1**

A proof of Lemma 1 will be given, though it may be a routine in information theory. We have a lemma.

**Lemma 2** Assume \( S \) and \( W \) are finite sets, and non-negative numbers \( f_w(x) \) are associate with each pair \((x, w) \in S \times W\). Denote by \( \overline{f}_w \) the average of \( f_w(x) \) over \( S \):
\[
\overline{f}_w = \frac{1}{|S|} \sum_{x \in S} f_w(x).
\]
Then, for any \( a > 0 \), the number of members in \( S \) that fail to satisfy the condition
\[\forall w \in W, \quad f_w(x) \leq \overline{f}_w |W| a\]
is upper-bounded by \( a^{-1}|S| \).
Proof. Let $X$ be a random variable uniformly distributed over $S$. Then, the probability that $X$ fails to satisfy \( \forall w \in W, f_w(X) \leq |W|a \) is upper-bounded as follows:

\[
\Pr\{ \exists w \in W, f_w(X) > |W|a \} \leq \sum_w \Pr\{ f_w(X) > |W|a \} \leq \sum_{w : f_w > 0} \Pr\{ f_w(X) > |W|a \} \leq (|W|a)^{-1} \leq a^{-1},
\]

where the equality (i) and inequality (ii) follow from the fact that $f_w = 0$ implies $f_w(x) = f_w |W|a = 0$ for all $x \in S$, and Markov’s inequality, respectively. Markov’s inequality is included at the end of this subsection with a proof. The lemma immediately follows from (4). □

Proof of Lemma 1. From the fact that \( \{ C^{(i)} \}_{i=1}^{N^*} \) is balanced, it follows

\[
\frac{1}{N^*} \sum_{i=1}^{N^*} M_Q(C^{(i)}) = \frac{q^k - 1}{q^a - 1} |T_Q^a| \leq \frac{q^k}{q^a} |T_Q^a|
\]

for any $Q \in P_n(F_q), Q \neq P_0$. From these, we readily obtain the equality and hence the inequality in (4). Now Lemma 1 follows upon applying Lemma 2 to $S = \{ (C^{(i)}, i) \mid i \in [1, N^*] \cap \mathbb{Z} \}$, where $f_w((C, i)) = M_Q(C), w = Q$ and $W = P_n(F_q) \setminus \{ P_0 \}$. □

Lemma 3 (Markov’s Inequality) For a positive constant $A$, and a random variable $Y$ that takes non-negative values and has a positive mean $\mu$, we have

\[
\Pr\{ Y \geq A\mu \} \leq 1/A.
\]

Proof. We have $\mu = \sum_y P_Y(y) \geq \sum_{y : y \geq \mu A} P_Y(y) \geq \mu A \Pr\{ Y \geq \mu A \} = \mu A \sum_{y : y \geq \mu A} P_Y(y) = \mu A \Pr\{ Y \geq \mu A \}$, which implies the lemma.

The relation $V(q^n - 1) = N^*(q^k - 1)$ immediately follows by counting the pairs $(x, C)$ such that $x \in C \setminus \{0_n\}$ and $C$ is a component of $\{ C^{(i)} \}_{i=1}^{N^*}$ in two ways, and the other equality follows similarly.
4.3 Proof of Proposition

We use the following basic inequality [4, 5, 9]:

\[ \sum_{y \in F_q^n: P_y = Q} P^n(y) \leq q^{-nD(Q\|P)} \]  

(6)

for any \( P \in \mathcal{P}(F_q) \). (Recall \( P^n \) denotes the product of \( n \) copies of \( P \).) The symmetric group on \( \{1, \ldots, n\} \), which is composed of all permutations on \( \{1, \ldots, n\} \), is denoted by \( S_n \). We define an action of \( S_n \) on \( F_q^n \) by

\[ \pi((x_1, \ldots, x_n)) = (x_{\pi(1)}, \ldots, x_{\pi(n)}) \]

for any \( \pi \in S_n \) and \((x_1, \ldots, x_n) \in F_q^n\), and put

\[ \pi(C) = \{ \pi(x) \mid x \in C \}, \quad \pi \in S_n, \ C \subseteq F_q^n. \]

The expectation operation with respect to a random variable \( X \) taking values in \( \mathcal{X} \) is denoted by \( \mathbb{E}_X \):

\[ \mathbb{E}_X f(X) = \sum_{x \in \mathcal{X}} P_X(x) f(x) \]

where \( f \) is a real-valued function on \( \mathcal{X} \).

**Lemma 4** Assume a linear code \( C \subseteq F_q^n \) satisfies

\[ M_Q(C \setminus \{0_n\})/|T^n_Q| \leq a_n q^{-nT}, \quad Q \in \mathcal{P}_n(F_q) \]

with some real numbers \( a_n \geq 1 \) and \( T \). Let \( J \) be a set of coset representatives for \( F_q^n/C \) such that each coset \( D \in F_q^n/C \) has a representative that belongs to \( J \) and that attains the minimum of \( H(P_x), x \in D \) (the resulting decoding is called minimum entropy decoding). Then, we have for any \( P_n \in \mathcal{P}(F_q^n), \)

\[ \mathbb{E}_\pi P_n(\pi(J^c)) \leq a_n|P_n(F_q^n)| \sum_{Q \in \mathcal{P}_n(F_q^n)} P_n(T^n_Q) q^{-n(T-H(Q))^+} \]

where \( c \) denotes complement, \( |t|^+ = \max\{t, 0\} \), and the random variable \( \pi \) is uniformly distributed over \( S_n \).

**Corollary 1** Assume for a linear code \( C \subseteq F_q^n \), \( M_Q(C \setminus \{0_n\}) \) is bounded as in Lemma 4. Then, with \( J \) as in the lemma, we have for any \( P \in \mathcal{P}(F_q) \),

\[ P^n(J^c) \leq a_n|P_n(F_q^n)|^2 q^{-nE(P,T)} \]

where

\[ E(P,T) = \min_{Q \in \mathcal{P}_n(F_q)} [D(Q\|P) + |T - H(Q)|^+] \].

6
A proof of Lemma 4 is given in the next subsection.

Proof of Corollary 1 Clearly, \(E^\pi P_n(\pi(J)^c) = P_n(J^c)\). Then, inserting the estimate of \(P_n(T^c_n)\) in (6) into the bound on \(E^\pi P_n(\pi(J)^c)\) in the lemma, we have

\[
P_n(J^c) \leq a_n |P_n(F_q)| \sum_{Q \in P_n(F_q)} q^{-n[D(Q)||P^c| + |T - H(Q)|^c]}
\]

and hence, the corollary.

Putting \(T = 1 - \kappa/n\) in this corollary, we readily obtain the proposition.

4.4 Proof of Lemma 4

In the proof, \(P_n(F_q)\) is abbreviated as \(P_n\). We will show that \(G = E^\pi P_n(\pi(J)^c)\) is bounded above by the claimed quantity.

Imagine we list up all words in \(\pi(C \setminus \{0_n\})\) for all \(\pi \in S_n\) permitting duplication. Clearly, the number of appearances of any fixed word \(y \in F_n^q\) in the list only depends on its type \(P_y \in P_n\). Namely, for any \(Q \in P_n\), there exists a constant, say \(L_Q\), such that

\[
|\{
\pi \in S_n \mid y \in \pi(C \setminus \{0_n\})\}| = L_Q
\]  
(7)

for any word \(y\) with \(P_y = Q\). Then, counting the number of words of a fixed type \(Q\) in the list in two ways, we have \(|T^c_n| L_Q = |S_n| M_Q(C \setminus \{0_n\})\). Hence, for any type \(Q \in P_n(F_q)\)

\[
\frac{L_Q}{|S_n|} = \frac{M_Q(C \setminus \{0_n\})}{|T^c_n|} \leq a_n q^{-nT}
\]  
(8)

by assumption. From (7) and (8), we have

\[
\frac{|A_y(C \setminus \{0_n\})|}{|S_n|} \leq a_n q^{-nT}
\]

(9)

for any \(y \in F_n^q\), where

\[
A_y(C \setminus \{0_n\}) = \{ \pi \in S_n \mid y \in \pi(C \setminus \{0_n\})\}.
\]

Then, we have

\[
G = \frac{1}{|S_n|} \sum_{\pi \in S_n} \sum_{x \notin J} P_n(x)
\]

\[
= \sum_{x \notin F_n^q} P_n(x) \frac{|\{\pi \in S_n \mid x \notin J\}|}{|S_n|}
\]  
(10)

Since \(x \notin J\) occurs only if there exists a word \(u \in F_n^q\) such that \(H(P_u) \leq H(P_x)\) and \(u - x \in \pi(C \setminus \{0_n\})\) from the design of \(J\) specified above (minimum entropy
coding), it follows

\[ |\{ \pi \in S_n \mid x \notin J \}|/|S_n| \leq \sum_{u \in F_q^n: H(P_u) \leq H(P_x)} a_n q^{-nT} \]

\[ = \sum_{Q' \in P_n: H(Q') \leq H(P_x)} a_n |T^n_{Q'}| q^{-nT} \]

\[ \leq \sum_{Q' \in P_n: H(Q') \leq H(P_x)} a_n q^{nH(Q')-nT} \]

where we have used \((9)\) for the second inequality, and another well-known inequality \([4, 5, 9]\)

\[ \forall Q \in P_n(\mathbb{F}_q), \quad |T^n_Q| \leq q^{nH(Q)} \]

(12)

for the last inequality. Then, using the inequalities \(\min\{at, 1\} \leq a \min\{t, 1\}\) and \(\min\{s + t, 1\} \leq \min\{s, 1\} + \min\{t, 1\}\) for \(a \geq 1, s, t \geq 0\), we can proceed from \((10)\) as follows, which completes the proof:

\[ G \leq \sum_{x \in F_q^n} P_n(x) \min \left\{ \sum_{Q' \in P_n: H(Q') \leq H(P_x)} a_n q^{nH(Q')-nT}, 1 \right\} \]

\[ \leq a_n \sum_{Q \in P_n} P_n(T^n_Q) \min \left\{ \sum_{Q' \in P_n: H(Q') \leq H(Q)} q^{nH(Q')-nT}, 1 \right\} \]

\[ \leq a_n \sum_{Q \in P_n} P_n(T^n_Q) \sum_{Q' \in P_n: H(Q') \leq H(Q)} \min\{q^{-n|T-H(Q')|}, 1\} \]

\[ \leq a_n |P_n| \sum_{Q \in P_n} P_n(T^n_Q) \max_{Q' \in P_n(\mathbb{F}_q): H(Q') \leq H(Q)} q^{-n|T-H(Q')|} \]

\[ = a_n |P_n| \sum_{Q \in P_n} P_n(T^n_Q) q^{-n|T-H(Q)|}. \]

5 Concluding Remarks

In \([1, 3]\) (or \([2]\)), quantum-mechanically compatible pairs of linear codes that are constructible with polynomial complexity were presented. The Calderbank-Shor-Steane quantum codes corresponding to the constructed pairs achieve the so-called Shannon rate. The most novel result among these would be the method for concatenating compatible (conjugate) code pairs, which have been published in \([1]\).

The present material was prepared for explaining the results not included in \([3]\) for those unfamiliar with the elementary combinatorial approach (the method of types with the very basics of linear codes incorporated).

This material might be included somewhere else (possibly in some other context).
A Some Other Contents of [1]

A.1 Compatible (Conjugate) Code Pairs [1]

Consider a pair of linear codes \((C_1, C_2)\) satisfying

\[ C_2^\perp \subseteq C_1, \quad (13) \]

which condition is equivalent to \(C_1^\perp \subseteq C_2\). The following question arises from an issue on quantum error correction: How good both \(C_1\) and \(C_2\) can be under the constraint \((13)\)? This is the subject treated in [1, 3, 2].

We have named a pair \((C_1, C_2)\) with \((13)\) a conjugate code pair in [1]. In what follows, we will use a ‘compatible code pair’ in place of ‘conjugate code pair.’

A.2 Code Ensemble Based on Extension Field [1]

The companion matrix of a polynomial \(f(x) = x^n - f_{n-1}x^{n-1} - \cdots - f_1x - f_0\), which is monic (i.e., of which the leading term has coefficient 1), over \(\mathbb{F}_q\) is defined to be

\[
T = \begin{bmatrix}
0_{n-1} & f_0 \\
& f_1 \\
& \vdots \\
& I_{n-1} \\
& f_{n-1}
\end{bmatrix}.
\]

Let \(T\) be the companion matrix, or its transpose, of a monic primitive polynomial of degree \(n\) over \(\mathbb{F}_q\). Given an \(n \times n\) matrix \(M\), let \(M|_m\) (resp. \(M|_m\)) denote the \(m \times n\) submatrix of \(M\) that consists of the first (resp. last) \(m\) rows of \(M\). We put \(C_1^{(i)} = \{xT^{k_1} | x \in \mathbb{F}_q^{k_1}\}\) and \(C_2^{(i)} = \{x(T^{-1})^{k_2} | x \in \mathbb{F}_q^{k_2}\}\) for \(i = 1, 2, \ldots\), where \(M^t\) denotes the transpose of \(M\). Then, setting

\[
B = B_T = \{(C_1^{(i)}, C_2^{(i)})|_{i=1}^{q^{n-1}}\}, \quad (14)
\]

we have the next lemma.

Lemma 5 [1, Lemma 1]. Let \(T\) be the companion matrix of a monic primitive polynomial of degree \(n\) over \(\mathbb{F}_q\). For integers \(k_1, k_2\) with \(0 \leq n - k_2 \leq k_1 \leq n\) and \(B_T = \{(C_1^{(i)}, C_2^{(i)})|_{i=1}^{q^{n-1}}\}\) constructed as above, any \((C_1^{(i)}, C_2^{(i)})\) is a compatible code pair, and both \(\{C_1^{(i)}|_{i=1}^{q^{n-1}}\}\) and \(\{C_2^{(i)}|_{i=1}^{q^{n-1}}\}\) are balanced.

Remark. It is known (and proved in a self-contained manner in [3, Sections VII]) that the matrix \(T\) has the following property, which are used in the proof of Lemma 5 below: The set \(\{O_n, I_n, T, \ldots, T^{q^{n-2}}\}\) is isomorphic to \(\mathbb{F}_{q^n}\) as a field.

Proof of Lemma 5. The condition \((13)\) is fulfilled since \(T^iT^{-1} = I_n\) implies that the \(C_2^{(i)}\) is spanned by the first \(n - k_2\) rows of \(T^i\). (This is easily seen if
we divide the two matrices on the left-hand side of $T^i T^{-i} = I_n$ into submatrices as in Figure 1.

We can write $C_{1}^{(i)} = \{ y T^i \mid y \in F_q^n, \text{supp} y \subseteq [1,k_1] \cap \mathbb{Z} \}$, where $\text{supp} (y_1, \ldots, y_n) = \{i \mid y_i \neq 0 \}$. Imagine we list up all codewords in $C_{1}^{(i)}$ permitting duplication. Specifically, we list up all $y T^i$ as $y$ and $i$ vary over the range $\{ y \mid y \in F_q^n, \text{supp} y \subseteq [1,k_1] \cap \mathbb{Z} \}$ and over $[1,q^n-1] \cap \mathbb{Z}$, respectively.

With $y \in F_q^n \backslash \{0\}$ fixed, $y T^i, i \in [1,q^n-1] \cap \mathbb{Z}$, are all distinct since $T^i \neq T^j$ implies $y T^i - y T^j = y T^l$ for some $l$ and $y T^l$ is not zero. Hence, any nonzero fixed word in $F_q^n$ appears exactly $q^{k_1} - 1$ times in listing $y T^i$ as above. Namely, the ensemble $\{ C_{1}^{(i)} \}_{i=1}^{q^n-1}$ is balanced. Using $(T^{-i})^t$ in place of $T^i$, we see the ensemble $\{ C_{2}^{(i)} \}_{i=1}^{q^n-1}$ is also balanced, completing the proof.

Lemmas 1 and 5 show the existence of a compatible code pair having exponentially decreasing decoding error probabilities in $B$.

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