On sequences of homomorphisms into measure algebras and the Efimov problem

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Joint work with Piotr Borodulin-Nadzieja.
\[ A, B \] — Boolean algebras

\( \mathcal{H}(A, B) \) — the family of all Boolean homomorphisms \( A \rightarrow B \)

\( \kappa \) — a cardinal (finite or infinite)

\( \text{Bor}(2^\kappa) \) — the Borel \( \sigma \)-field on \( 2^\kappa \)

\( \lambda_\kappa \) — the standard product measure on \( 2^\kappa \)

\( \mathcal{N}(\lambda_\kappa) = \{ A \in \text{Bor}(2^\kappa) : \lambda_\kappa(A) = 0 \} \)
Preliminaries

\( A, B \) — Boolean algebras

\( \mathcal{H}(A, B) \) — the family of all Boolean homomorphisms \( A \rightarrow B \)

\( \kappa \) — a cardinal (finite or infinite)

\( \text{Bor}(2^{\kappa}) \) — the Borel \( \sigma \)-field on \( 2^{\kappa} \)

\( \lambda_\kappa \) — the standard product measure on \( 2^{\kappa} \)

\( \mathcal{N}(\lambda_\kappa) = \{ A \in \text{Bor}(2^{\kappa}) : \lambda_\kappa(A) = 0 \} \)

\( M_\kappa = \text{Bor}(2^{\kappa})/\mathcal{N}(\lambda_\kappa) \)

\( M = M_\omega = \text{Bor}([0, 1])/\mathcal{N} \)

\( M_0 = \{0, 1\} \)
Let $V$ be a ground model and $\mathbb{A} \in V$ a Boolean algebra.

$\mathbb{A}$ is a Boolean algebra in any $\mathbb{M}_\kappa$-generic extension $V[G]$. 
Ultrafilters in $V^{M_\kappa}$ induce homomorphisms in $V$

Let $V$ be a ground model and $A \in V$ a Boolean algebra.

$A$ is a Boolean algebra in any $M_\kappa$-generic extension $V[G]$.

Let $\dot{U}$ be an $M_\kappa$-name for an ultrafilter on $A$, that is, $1 \Vdash_{M_\kappa} \text{"$\dot{U}$ is an ultrafilter on $A$"}$.
Ultrafilters in $V^{\mathbb{M}_\kappa}$ induce homomorphisms in $V$

Let $V$ be a ground model and $\mathbb{A} \in V$ a Boolean algebra. $\mathbb{A}$ is a Boolean algebra in any $\mathbb{M}_\kappa$-generic extension $V[G]$.

Let $\dot{U}$ be an $\mathbb{M}_\kappa$-name for an ultrafilter on $\mathbb{A}$, that is,

$$1 \models_{\mathbb{M}_\kappa} \text{“}\dot{U} \text{ is an ultrafilter on } \mathbb{A}\text{”}.$$

In $V$, we define a Boolean homomorphism $\phi_{\dot{U}} : \mathbb{A} \to \mathbb{M}_\kappa$:

$$\phi_{\dot{U}}(A) = [\{A \in \dot{U}\}]$$

for every $A \in \mathbb{A}$. 
Let $V$ be a ground model and $\mathbb{A} \in V$ a Boolean algebra.

$\mathbb{A}$ is a Boolean algebra in any $\mathbb{M}_\kappa$-generic extension $V[G]$.

Let $\phi: \mathbb{A} \rightarrow \mathbb{M}_\kappa$ be a Boolean homomorphism.
Let $V$ be a ground model and $\mathbb{A} \in V$ a Boolean algebra.

$\mathbb{A}$ is a Boolean algebra in any $\mathbb{M}_\kappa$-generic extension $V[G]$.

Let $\phi: \mathbb{A} \to \mathbb{M}_\kappa$ be a Boolean homomorphism.

Define an $\mathbb{M}_\kappa$-name $\dot{U}_\phi$:

$$\dot{U}_\phi = \{(A, \phi(A)) : A \in \mathbb{A}\}.$$

Then:

$$1 \models_{\mathbb{M}_\kappa} "\dot{U}_\phi \text{ is an ultrafilter on } \mathbb{A}".$$
Duality between homomorphisms and ultrafilters

\[ \text{homomorphisms } \phi : A \rightarrow \mathbb{M}_\kappa \]

\[ \uparrow \]

\[ \mathbb{M}_\kappa \text{-names for ultrafilters on } A \]
Duality between homomorphisms and ultrafilters

\[ \text{homomorphisms } \phi : A \to \mathcal{M}_\kappa \]

| \[ \uparrow \]
| \[ \mathcal{M}_\kappa \text{-names for ultrafilters on } A \]

**Fact 1**

If \( \dot{U} \) is an \( \mathcal{M}_\kappa \)-name for an ultrafilter on \( A \), then:

\[ \models_{\mathcal{M}_\kappa} \dot{U}(\phi_{\dot{U}}) = \dot{U}. \]
Duality between homomorphisms and ultrafilters

homomorphisms $\phi : A \to M_\kappa$

\[ \uparrow \]

$M_\kappa$-names for ultrafilters on $A$

Fact 1
If $\dot{\mathcal{U}}$ is an $M_\kappa$-name for an ultrafilter on $A$, then:

$\models_{M_\kappa} \dot{\mathcal{U}}(\phi_{\dot{\mathcal{U}}}) = \dot{\mathcal{U}}.$

Fact 2
If $\phi : A \to M_\kappa$ is a homomorphism, then:

$\phi(\dot{\mathcal{U}}_\phi) = \phi.$
But what about sequences?

Fix $\phi_n, \phi \in \mathcal{H}(A, M_\kappa)$ and let $\dot{U}_n = \dot{U}_{\phi_n}$ and $\dot{U} = \dot{U}_\phi$. 
Fix $\phi_n, \phi \in \mathcal{H}(A, \mathbb{M}_\kappa)$ and let $\dot{U}_n = \dot{U}_{\phi_n}$ and $\dot{U} = \dot{U}_{\phi}$.

**Question**

What can we say about convergence properties of the sequence $(\dot{U}_n)$ in $St(A) \cap V^{\mathbb{M}_\kappa}$? When is it convergent to $\dot{U}$?
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**Question**

What can we say about convergence properties of the sequence \( (\dot{U}_n) \) in \( St(A) \cap V^{\mathbb{M}_\kappa} \)? When is it convergent to \( \dot{U} \)?

**Proposition**

The following conditions are equivalent:

1. \( \models_{\mathbb{M}_\kappa} \text{“}(\dot{U}_n) \text{ converges to } \dot{U}\text{”} \),
2. for every \( A \in \Delta \) it holds:

\[
\phi(A) = \bigwedge_n \bigvee_{m \geq n} \phi_m(A)
\]
But what about sequences?

Fix $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ and let $\dot{U}_n = \dot{U}_{\phi_n}$ and $\dot{U} = \dot{U}_\phi$.

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What can we say about convergence properties of the sequence $(\dot{U}_n)$ in $St(\mathbb{A}) \cap V^{\mathbb{M}_\kappa}$? When is it convergent to $\dot{U}$?

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The following conditions are equivalent:

1. $\models_{\mathbb{M}_\kappa} "(\dot{U}_n) \text{ converges to } \dot{U}"$,

2. for every $A \in \mathbb{A}$ it holds:

$$\phi(A) = \bigwedge_n \bigvee_{m \geq n} \phi_m(A) \quad (= \bigvee_n \bigwedge_{m \geq n} \phi_m(A))$$

If (2) holds, then we say that $(\phi_n)$ converges algebraically to $\phi$. 
Two topologies on $\mathcal{H}(A, M_\kappa)$

Fréchet–Nikodym metric on $M_\kappa$

For every $A, B \in M_\kappa$ put:

$$d_\kappa(A, B) = \lambda_\kappa(A \triangle B).$$
Two topologies on $\mathcal{H}(A, \mathbb{M}_\kappa)$

**Fréchet–Nikodym metric on $\mathbb{M}_\kappa$**

For every $A, B \in \mathbb{M}_\kappa$ put:

$$d_\kappa(A, B) = \lambda_\kappa(A \triangle B).$$

$(\mathbb{M}_\kappa, d_\kappa)$ is a complete metric space.
Two topologies on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

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Pointwise topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

We may endow $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ with the pointwise topology:

$$V(\phi, A, \varepsilon) = \{\psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa) : d_\kappa(\phi(A), \psi(A)) < \varepsilon\},$$

where $\phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$, $A \in \mathbb{A}$, $\varepsilon > 0$. 
Two topologies on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

**Frechet–Nikodym metric on $\mathbb{M}_\kappa$**

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where $\phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$, $A \in \mathbb{A}$, $\varepsilon > 0$.

$\text{St}(\mathbb{A})$ and $\mathcal{H}(\mathbb{A}, \mathbb{M}_0)$ with the pointwise topology are homeomorphic. In particular, $\text{St}(\mathbb{A})$ always embeds into $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$. 
Two topologies on $\mathcal{H}(\mathbf{A}, \mathbb{M}_\kappa)$

**Uniform metric on $\mathcal{H}(\mathbf{A}, \mathbb{M}_\kappa)$**

For every $\phi, \psi \in \mathcal{H}(\mathbf{A}, \mathbb{M}_\kappa)$ put:

$$d_{hom}(\phi, \psi) = \sup\{d_\kappa(\phi(A), \psi(A)) : A \in \mathbf{A}\}.$$
Two topologies on $\mathcal{H}(A, M_\kappa)$

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For every $\phi, \psi \in \mathcal{H}(A, M_\kappa)$ put:

$$d_{\text{hom}}(\phi, \psi) = \sup\{d_\kappa(\phi(A), \psi(A)) : A \in A\}.$$

$d_{\text{hom}}$ is a metric bounded by 1.
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**Fact**

$\mathcal{H}(\mathbb{A}, \mathcal{M}_0)$ with the uniform topology is a discrete space.
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**Uniform metric on $\mathcal{H}(\mathbb{A}, M_\kappa)$**

For every $\phi, \psi \in \mathcal{H}(\mathbb{A}, M_\kappa)$ put:

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$d_{hom}$ is a metric bounded by 1.

**Fact**

$\mathcal{H}(\mathbb{A}, M_0)$ with the uniform topology is a discrete space.

**Convergence of sequences**

1. $(\phi_n)$ converges pointwise to $\phi$ if it converges to $\phi$ in the pointwise topology on $\mathcal{H}(\mathbb{A}, M_\kappa)$.

2. $(\phi_n)$ converges uniformly to $\phi$ if it converges to $\phi$ in the uniform topology on $\mathcal{H}(\mathbb{A}, M_\kappa)$. 
Let \( \phi_n, \phi \in \mathcal{H}(A, \mathcal{M}_\kappa) \).
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If $\kappa = 0$, then algebraic convergence $\iff$ pointwise convergence.
Let $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, M_\kappa)$.

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If $\kappa = 0$, then algebraic convergence $\iff$ pointwise convergence.
Convergence of ultrafilters vs. convergence of homomorphisms

**Proposition**

The following conditions are equivalent:

1. $\vDash_{M_\kappa} \text{“}(\mathcal{U}_n) \text{ converges to } \mathcal{U}\text{”}$,
2. $(\phi_n)$ converges algebraically to $\phi$.

**Corollary**

If $\vDash_{M_\kappa} \text{“}(\mathcal{U}_n) \text{ converges to } \mathcal{U}\text{”}$, then $(\phi_n)$ converges pointwise to $\phi$.

**Theorem**

1. If $\vDash_{M_\kappa} \forall \omega \ni \mathcal{U}_n = \mathcal{U}$, then $(\phi_n)$ converges uniformly to $\phi$.
2. If $(\phi_n)$ converges uniformly to $\phi$, then for almost all $n \in \omega$ there is $p_n \in M_\kappa$ such that $p_n \vDash \mathcal{U}_n = \mathcal{U}$.
Proposition

The following conditions are equivalent:

1. $\models_{\mathcal{M}\kappa} "(\dot{U}_n)\text{ converges to }\dot{U}"$,  
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2. If $(\phi_n)$ converges uniformly to $\phi$, then for almost all $n \in \omega$ there is $p_n \in \mathcal{M}_\kappa$ such that $p_n \models \mathcal{U}_n = \mathcal{U}$. 

Interlude—distinguishing ultrafilters

Let $\mathcal{U}$ and $\mathcal{V}$ be $\mathbb{M}_\kappa$-names for ultrafilters on $\mathbb{A}$ st. $1 \Vdash \mathcal{U} \neq \mathcal{V}$.
Let $\dot{U}$ and $\dot{V}$ be $\mathbb{M}_\kappa$-names for ultrafilters on $\mathbb{A}$ st. 1 $\models \dot{U} \neq \dot{V}$.

There is a maximal antichain $(p_n)$ in $\mathbb{M}_\kappa$ and a sequence $(A_n)$ in $\mathbb{A}$ such that $p_n \models A_n \in \dot{U} \triangle \dot{V}$.
Let $\dot{U}$ and $\dot{V}$ be $\mathbb{M}_\kappa$-names for ultrafilters on $\mathbb{A}$ st. $1 \models \dot{U} \neq \dot{V}$.

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A priori we have no control over the values $\lambda_\kappa(p_n)$...
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**Theorem**

If $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ are $\mathbb{M}_\kappa$-names for ultrafilters on $\mathbb{A}$ st. $1 \models \dot{\mathcal{U}} \neq \dot{\mathcal{V}}$, then for every $\varepsilon > 0$ there exists $p \in \mathbb{M}_\kappa$ and $C \in \mathbb{A}$ such that

- $\lambda_\kappa(p) > 1/4 - \varepsilon$, and
- $p \models C \in \dot{\mathcal{U}} \triangle \dot{\mathcal{V}}$. 
Interlude—distinguishing ultrafilters

Let $\dot{U}$ and $\dot{V}$ be $\mathbb{M}_κ$-names for ultrafilters on $\mathbb{A}$ st. $1 \not\models \dot{U} \neq \dot{V}$.

There is a maximal antichain $(p_n)$ in $\mathbb{M}_κ$ and a sequence $(A_n)$ in $\mathbb{A}$ such that $p_n \models A_n \in \dot{U} \triangle \dot{V}$.

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**Theorem**

If $\dot{U}$ and $\dot{V}$ are $\mathbb{M}_κ$-names for ultrafilters on $\mathbb{A}$ st. $1 \not\models \dot{U} \neq \dot{V}$, then for every $\varepsilon > 0$ there exists $p \in \mathbb{M}_κ$ and $C \in \mathbb{A}$ such that

- $\lambda_κ(p) > 1/4 - \varepsilon$, and
- $p \models C \in \dot{U} \triangle \dot{V}$.

Remark: $1/4$ is optimal!
The Efimov problem

Problem

Does there exists an Efimov space, i.e. an infinite compact Hausdorff space with no non-trivial convergent sequences nor any copies of $\beta\omega$?

No ZFC example is known!
The Efimov problem

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Consistent examples

1. CH (Fedorchuk, Dow–Pichardo-Mendoza, Talagrand)
2. ♦ (Fedorchuk, Kunen–Džamonja, de la Vega, S.–Zdomskyy)
3. $s = \omega_1$ & $c = 2^{\omega_1}$ (Fedorchuk)
4. $\text{cof}(\mathcal{N}^\omega, \subseteq) = s$ & $2^s < 2^c$ (Dow)
5. $b = c$ (Dow–Shelah)
6. $\text{cof}([\text{cof} \mathcal{N}]^\omega, \subseteq) = \text{cof}(\mathcal{N}) < c$ (S.)
7. and in many models obtained by forcing...
Efimov spaces in the random model

Theorem (Dow–Fremlin)
There is an Efimov space in the random model.
Efimov spaces in the random model

Theorem (Dow–Fremlin)

There is an Efimov space in the random model.

If $A \in V$ is a $\sigma$-complete Boolean algebra, then

$$\models_{M_\kappa} \text{"St}(A) \text{ has no non-trivial convergent sequences."}$$
Efimov spaces in the random model

**Theorem (Dow–Fremlin)**

There is an Efimov space in the random model.

If $A \in V$ is a $\sigma$-complete Boolean algebra, then

$$\vdash_{M_\kappa} \text{"St}(A) \text{ has no non-trivial convergent sequences."}$$

**Theorem**

If $A \in V$, then TFAE:

1. $\vdash_{M_\kappa} \text{"St}(A) \text{ has no non-trivial convergent sequences"};$
2. every algebraically convergent sequence in $\mathcal{H}(A, M_\kappa)$ converges uniformly.
Corollary

If, in $V$, $\mathcal{A}$ is such a Boolean algebra that it has size $< \kappa$ and every algebraically convergent sequence in $\mathcal{H}(\mathcal{A}, M_\kappa)$ converges uniformly, then, in $V^{M_\kappa}$, the Stone space $St(\mathcal{A})$ is an Efimov space.
Corollary
If, in $V$, $\mathcal{A}$ is such a Boolean algebra that it has size $< \kappa$ and every algebraically convergent sequence in $\mathcal{H}(\mathcal{A}, M_\kappa)$ converges uniformly, then, in $V^{M_\kappa}$, the Stone space $St(\mathcal{A})$ is an Efimov space.

Corollary
If $\mathcal{A}$ is such a Boolean algebra that $St(\mathcal{A})$ is an F-space, then every algebraically convergent sequence in $\mathcal{H}(\mathcal{A}, M_\kappa)$ is uniformly convergent.
Corollary

If, in $\mathcal{V}$, $\mathcal{A}$ is such a Boolean algebra that it has size $< \kappa$ and every algebraically convergent sequence in $\mathcal{H}(\mathcal{A}, \mathcal{M}_\kappa)$ converges uniformly, then, in $\mathcal{V}^{\mathcal{M}_\kappa}$, the Stone space $St(\mathcal{A})$ is an Efimov space.

Corollary

If $\mathcal{A}$ is such a Boolean algebra that $St(\mathcal{A})$ is an F-space, then every algebraically convergent sequence in $\mathcal{H}(\mathcal{A}, \mathcal{M}_\kappa)$ is uniformly convergent.

In particular, that holds for $\mathcal{A} = \wp(\omega)$ and $\mathcal{A} = \wp(\omega)/\text{Fin}$. 
Two more topologies on $\mathcal{H}(A, M_\kappa)$

**Stone duality**

Let $A$ and $B$ be Boolean algebras.

There is a one-to-one correspondence between homomorphisms $A \to B$ and continuous functions $St(B) \to St(A)$:

$$\mathcal{H}(A, B) \ni \phi \mapsto f_\phi \in C(St(B), St(A))$$

$$f_\phi^{-1}[A] = \phi(A), \quad \forall A \in A$$
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**Stone duality**

Let $A$ and $B$ be Boolean algebras.

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**Borel Fréchet–Nikodym (pseudo)metric on $M_{\kappa}$**

For every $A, B \in Bor(St(M_{\kappa}))$ put:

$$d_{\kappa}^{Bor}(A, B) = \hat{\lambda}_{\kappa}(A \triangle B).$$
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For every $A, B \in Bor(St(M_\kappa))$ put:

$$d_{\kappa}^{Bor}(A, B) = \hat{\lambda}_\kappa(A \triangle B).$$

$d_{\kappa}^{Bor}$ is a pseudometric.
Two more topologies on $\mathcal{H}(A, M_\kappa)$

**Borel pointwise topology on $\mathcal{H}(A, M_\kappa)$**

Sub-basic open sets:

$$V(\phi, A, \varepsilon) = \{ \psi \in \mathcal{H}(A, M_\kappa) : d_{\kappa}^{Bor}(f_\phi^{-1}[A], f_\psi^{-1}[A]) < \varepsilon \},$$

where $\phi \in \mathcal{H}(A, M_\kappa), A \in Bor(St(A)), \varepsilon > 0$. 
Two more topologies on $H(A, M_\kappa)$

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**Pointwise Borel convergence of sequences**

$(\phi_n)$ converges Borel pointwise to $\phi$ if it converges to $\phi$ in the Borel pointwise topology on $H(A, M_\kappa)$. 
Two more topologies on $\mathcal{H}(A, M_\kappa)$

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$(\phi_n)$ converges Borel pointwise to $\phi$ if it converges to $\phi$ in the Borel pointwise topology on $\mathcal{H}(A, M_\kappa)$.

**Borel uniform metric on $\mathcal{H}(A, M_\kappa)$**

For every $\phi, \psi \in \mathcal{H}(A, M_\kappa)$ put:

$$d_{hom}^{Bor}(\phi, \psi) = \sup\{d_\kappa^{Bor}(f_\phi^{-1}[A], f_\psi^{-1}[A]): A \in Bor(St(A))\}.$$
Two more topologies on $\mathcal{H}(A, M_\kappa)$

**Borel pointwise topology on $\mathcal{H}(A, M_\kappa)$**

Sub-basic open sets:

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**Borel uniform metric on $\mathcal{H}(A, M_\kappa)$**

For every $\phi, \psi \in \mathcal{H}(A, M_\kappa)$ put:

$$d^\text{Bor}_\text{hom}(\phi, \psi) = \sup\{d^\text{Bor}_\kappa(f_\phi^{-1}[A], f_\psi^{-1}[A]) : A \in \text{Bor}(\text{St}(A))\}.$$  

$d^\text{Bor}_\text{hom}$ is a metric bounded by 1.
Two (really?) more topologies on $\mathcal{H}(A, M_{\kappa})$

Fact

For every $A$ and $\kappa$, $d_{\text{hom}} = d_{\text{Bor}}^{\text{Bor}}$. 
Two (really?) more topologies on $\mathcal{H}(A, M_\kappa)$

**Fact**

For every $A$ and $\kappa$, $d_{hom} = d_{hom}^{Bor}$.

Let $\phi_n, \phi \in \mathcal{H}(A, M_\kappa)$.

$(\phi_n)$ converges (Borel) uniformly to $\phi$

$\Downarrow$

$(\phi_n)$ converges Borel pointwise to $\phi$
Two (really?) more topologies on $\mathcal{H}(A, \mathbb{M}_\kappa)$

Fact

For every $A$ and $\kappa$, $d_{\text{hom}} = d_{\text{Bor}}^{\text{hom}}$.

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⇓

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$\Uparrow$

$(\phi_n)$ converges algebraically to $\phi$
Topologies on the space of probability measures

\( P(A) \) — finitely additive probability measures on \( A \)

**Norm topology**

\( \mu, \nu \in P(A) \):

\[
d_{\text{var}}(\mu, \nu) = \sup_{A, B \in A, A \lor B = 0} (|\mu(A) - \nu(A)| + |\mu(B) - \nu(B)|)
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**Weak topology**

\( \mu \in P(\mathbb{A}), \varphi \in C(St(\mathbb{A}))^{**}, \varepsilon > 0: \)

\[ V(\mu; \varphi; \varepsilon) = \{ \nu \in P(\mathbb{A}): |\varphi(\mu) - \varphi(\nu)| < \varepsilon \}. \]
Topologies on the space of probability measures

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**Weak* topology**

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Weak* topology — equivalent definition

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Topologies on the space of probability measures

**Weak* topology — equivalent definition**

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**Fact**

Let \( \mu_n, \mu \in P(\mathbb{A}) \). The sequence \((\mu_n)\) converges to \( \mu \) with respect to the weak topology if and only if \((\hat{\mu}_n(B))\) converges to \( \hat{\mu}(B) \) for every Borel set \( B \subseteq St(\mathbb{A}) \).
Homomorphisms and measures

\[ F : \mathcal{H}(A, M_{\kappa}) \rightarrow P(A) \]

\[ F(\phi) = \lambda_{\kappa} \circ \phi \]

\[ F(\phi)(A) = \lambda_{\kappa}(\phi(A)) \]
Homomorphisms and measures

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**Proposition**

\( F \) is:

1. uniformly-norm continuous;
Homomorphisms and measures

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**Proposition**

\( F \) is:

1. uniformly-norm continuous;
2. pointwise-weak* continuous.
Corollary

\( \phi_n, \phi \in \mathcal{H}(\mathbb{A}, M_\kappa) \). Then:

1. if \((\phi_n)\) converges uniformly to \(\phi\), then \((\lambda_\kappa \circ \phi_n)\) converges to \(\lambda_\kappa \circ \phi\) in norm;
Corollary

\( \phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathcal{M}_\kappa) \). Then:

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Corollary

\( \phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathcal{M}_\kappa) \). Then:

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3. if \((\phi_n)\) converges pointwise to \(\phi\), then \((\lambda_\kappa \circ \phi_n)\) converges to \(\lambda_\kappa \circ \phi\) weakly*. 


Uniform countable additivity

A sequence \((\mu_k)\) of Radon probability measures on a compact space \(K\) is \textit{uniformly countably additive} if for every descending sequence \((E_n)\) of Borel sets such that \(\bigcap E_n = \emptyset\) and every \(\varepsilon > 0\) there is \(N \in \omega\) such that \(\mu_k(E_n) < \varepsilon\) for every \(n \geq N\) and \(k \in \omega\).
Homomorphisms and measures

**Uniform countable additivity**

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**Nikodym Convergence Theorem**

Every weakly convergent sequence of Radon measures on a compact space is uniformly countably additive.
Homomorphisms and measures

Uniform countable additivity
A sequence \((\mu_k)\) of Radon probability measures on a compact space \(K\) is \textit{uniformly countably additive} if for every descending sequence \((E_n)\) of Borel sets such that \(\bigcap E_n = \emptyset\) and every \(\varepsilon > 0\) there is \(N \in \omega\) such that \(\mu_k(E_n) < \varepsilon\) for every \(n \geq N\) and \(k \in \omega\).

Nikodym Convergence Theorem
Every weakly convergent sequence of Radon measures on a compact space is uniformly countably additive.

Theorem
\(\phi_n, \phi \in \mathcal{H}(A, M_\kappa)\). If \((\phi_n)\) converges pointwise to \(\phi\) and \((\hat{\lambda}_\kappa \circ f_{\phi_n}^{-1})\) is uniformly countably additive, then \((\phi_n)\) converges Borel pointwise to \(\phi\).
A Boolean algebra $\mathbb{A}$ has the **Grothendieck property** if every weakly* convergent sequence of measures on $\text{St}(\mathbb{A})$ is weakly convergent.

Examples: $\sigma$-complete Boolean algebras
Non-examples: countable Boolean algebras
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Theorem
If $\mathbb{A}$ has the Grothendieck property, then every pointwise convergent sequence in $\mathcal{H}(\mathbb{A}, \mathcal{M}_\kappa)$ is Borel pointwise convergent.
Thank you for your attention!