A method for constructing Weierstrass elliptic function solutions and their degenerated solutions of the mKdV equation

Sirendaoreji*
Mathematical Science College, Inner Mongolia Normal University, Huhhot 010022, Inner Mongolia, P. R. China

Abstract
A Weierstrass type projective Riccati equation expansion method is proposed by using the Weierstrass elliptic function solutions of the projective Riccati equations and the conversion formulas which transform the Weierstrass elliptic functions into the hyperbolic and the trigonometric functions. The Weierstrass elliptic function solutions and their degenerated solutions including the solitary wave and the periodic wave solutions of the mKdV equation are obtained by using the proposed method.

Key words: 35C07; 35Q53; 35A23; 74J35.

1 Introduction
Traveling wave solutions of nonlinear evolution equations (NLEEs) can help people to understand and explain nonlinear phenomena in many areas of applied science, such as plasma physics, optical fiber, fluid mechanics, biology and chemistry, etc. Therefore, the problem of how to find new methods for seeking traveling wave solutions of NLEEs has become a hot topic in soliton theory. In the past few decades, various direct methods, such as the tanh–function method[1], the auxiliary equation method[2], the Riccati equation expansion method[3], the unified Riccati equation expansion method[4], the Jacobi elliptic function expansion method[5], the Weierstrass elliptic function method[6, 7, 8, 9, 10], the projective Riccati equation expansion method[11, 12, 13, 14, 15], and others have been proposed to find exact traveling wave solutions of NLEEs.

Among them, the projective Riccati equation expansion method usually can give some new traveling wave solutions of NLEEs which are different from those solutions obtained by other direct methods. In Ref.[16], two groups of Weierstrass elliptic function solutions of the projective Riccati equations were used to construct the Weierstrass elliptic function solutions of the Burgers equation and a nonlinear dispersive–dissipative equation. Also in Refs.[?], the Weierstrass elliptic function solutions of the modified Korteweg–de Vries (mKdV) equation were obtained by using the projective Riccati equation expansion method. Especially, in Ref.[17] the author attempt to transform the Weierstrass elliptic function solutions of the mKdV equation into the hyperbolic and trigonometric function

*E-mail: siren@imnu.edu.cn
solutions using the following conversion formula

\[ \wp(z, g_2, g_3) = e_2 - (e_2 - e_3) \text{cn}^2 \left( \sqrt{e_1 - e_3} z, m \right), m = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}} \]  

which depends on the roots \( e_1, e_2, e_3 \) \( (e_1 \geq e_2 \geq e_3) \) of the third order polynomial equation \( P(w) \equiv w^3 - g_2 w - g_3 = 0 \). It is no doubt that the formula (1) can be used to convert the Weierstrass elliptic function solutions of NLEEs into the Jacob elliptic function solutions, and then the hyperbolic and trigonometric function solutions of NLEEs can be obtained by taking the limit of modulus \( m \to 1 \) and \( m \to 0 \). But it is already pointed out in [19, 20, 21] that the solutions obtained by this process may be incorrect.

In order to overcome this difficulty, in Ref. [21] we have constructed four new conversion formulas which can directly convert the Weierstrass elliptic function solutions of NLEEs into the hyperbolic and trigonometric function solutions. More importantly, these formulas do not depend on the roots of the above third order polynomial equation, not need to use the Jacobi elliptic functions in the converting process, and can ensure that the converted solutions of NLEEs are correct.

This indicates that if we can find some new Weierstrass elliptic function solutions of the projected Riccati equations, then we can use these Weierstrass elliptic function solutions and our conversion formulas to give some new solitary wave and trigonometric periodic wave solutions of NLEEs. Therefore, this paper aims to give more Weierstrass elliptic function solutions of the projective Riccati equations and use our conversion formulas to propose a direct method which we called the Weierstrass type projective Riccati equation expansion method. Finally, we shall take the mKdV equation as an illustrative example to show the effectiveness of our method.

This paper is organized as follows. In the next section, four groups of Weierstrass elliptic function solutions of the projective Riccati equations are constructed. And by using these Weierstrass elliptic function solutions and our conversion formulas, the called Weierstrass type projective Riccati equation expansion method is proposed to find exact traveling wave solutions to NLEEs. In Sec. 3 the proposed method is applied to construct the Weierstrass elliptic function solutions, the solitary and periodic wave solutions of the mKdV equation. The conclusions are given in Sec. 4.

2 Weierstrass type projective Riccati equation expansion method

The Weierstrass elliptic function \( w = \wp(\xi, g_2, g_3) \) is defined as the inverse function of the Weierstrass elliptic integral [22, 23]

\[ \xi = \int_{\infty}^{w} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \]  

or the solution of the following nonlinear ordinary differential equation (ODE)

\[ \left( \frac{dw}{d\xi} \right)^2 = P(w) = 4w^3 - g_2 w - g_3, \]  

where the real parameters \( g_2, g_3 \) are called invariants.
In the following we shall consider the projective Riccati equations of the form

\[
\begin{align*}
F'(\xi) &= pF(\xi)G(\xi), \\
G'(\xi) &= q + pG^2(\xi) - rF(\xi),
\end{align*}
\]  

(4)

where \( F, G \) are unknown functions of the variable \( \xi \) and \( p, q, r \) are constants.

By using the direct assumption approach we can construct the Weierstrass elliptic function solutions of Eqs. (4) as following

\[
\begin{align*}
F(\xi) &= \frac{q}{6r} + \frac{2}{pr} \wp(\xi, g_2, g_3), \\
G(\xi) &= \frac{12\wp'(\xi, g_2, g_3)}{p[pq + 12\wp(\xi, g_2, g_3)]}, \\
G^2(\xi) &= -\frac{q}{p} + \frac{2r}{p} F(\xi),
\end{align*}
\]  

(5)

\[
\begin{align*}
F(\xi) &= \frac{5q}{6r} + \frac{5q^2p}{72r\wp(\xi, g_2, g_3)}, \\
G(\xi) &= \frac{-q\wp'(\xi, g_2, g_3)}{[pq + 12\wp(\xi, g_2, g_3)] \wp(\xi, g_2, g_3)}, \\
G^2(\xi) &= -\frac{q}{p} + \frac{2r}{p} F(\xi) - \frac{24r^2}{20pq} F^2(\xi),
\end{align*}
\]  

(6)

\[
\begin{align*}
F(\xi) &= \frac{(2 + q)[pq + 12\wp(\xi, g_2, g_3)]}{r[12p + pq + 12\wp(\xi, g_2, g_3)]}, \\
G(\xi) &= \frac{\wp'(\xi, g_2, g_3)}{p(\wp(\xi, g_2, g_3) + \frac{p}{2} + \frac{pq}{12})^2 - \frac{p^2}{4}}, \\
G^2(\xi) &= -\frac{q}{p} + \frac{2r}{p} F(\xi) - \frac{r^2(q + 4)}{p(q + 2)^2} F^2(\xi),
\end{align*}
\]  

(7)

\[
\begin{align*}
F(\xi) &= \frac{q(p^2 + 2)[pq + 12\wp(\xi, g_2, g_3)]}{pr[12q + p^2q + 12\wp(\xi, g_2, g_3)]}, \\
G(\xi) &= \frac{q\wp'(\xi, g_2, g_3)}{p(\wp(\xi, g_2, g_3) + \frac{q}{2} + \frac{pq^2}{12})^2 - \frac{q^2}{4}}, \\
G^2(\xi) &= -\frac{q}{p} + \frac{2r}{p} F(\xi) - \frac{pr^2(p^2 + 4)}{q(p + 2)^2} F^2(\xi),
\end{align*}
\]  

(8)

where the invariants \( g_2 \) and \( g_3 \) are given by

\[
g_2 = \frac{p^2q^2}{12}, \quad g_3 = \frac{p^3q^3}{216}.
\]  

(13)

Here the solutions (5) and (7) are previously known\[16, 17, 18\], but the solutions (9) and (11) are new.

We find that the Weierstrass elliptic function can degenerate to the hyperbolic and
trigonometric function by using the following conversion formulas \[21\]

\[\wp(\xi, \theta) = \frac{\theta}{12} - \frac{\theta^3}{4 \text{sech}^2\left(\frac{\sqrt{\theta}}{2} \xi\right)}, \theta > 0, \quad (14)\]

\[\wp(\xi, \theta) = \frac{\theta}{12} + \frac{\theta^3}{4 \text{csch}^2\left(\frac{\sqrt{\theta}}{2} \xi\right)}, \theta > 0, \quad (15)\]

\[\wp(\xi, \theta) = \frac{\theta}{12} - \frac{\theta^3}{4 \text{sec}^2\left(\frac{\sqrt{-\theta}}{2} \xi\right)}, \theta < 0, \quad (16)\]

\[\wp(\xi, \theta) = \frac{\theta}{12} - \frac{\theta^3}{4 \text{csc}^2\left(\frac{\sqrt{-\theta}}{2} \xi\right)}, \theta < 0, \quad (17)\]

where \(\theta\) is a real number.

A given NLEE with respect to two variables \(x\) and \(t\) is of the form

\[P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, \cdot \cdot \cdot) = 0, \quad (18)\]

where the subscripts denote the partial derivatives, \(P\) is a polynomial in unknown function \(u(x, t)\) and its derivatives.

The Weierstrass type projective Riccati equation expansion method proposed here follows the following five steps.

\textbf{Step 1:} Making the wave transformation

\[u(x, t) = u(\xi), \xi = x - \omega t, \quad (19)\]

we may exchange the Eq. (18) into the following ODE

\[H(u, u', u'', \cdot \cdot \cdot) = 0, \quad (20)\]

where primes denote the derivatives with respect to \(\xi\) and the wave speed \(\omega\) is a constant to be determined later.

\textbf{Step 2:} Assume that the Eq. (20) has the truncated formal series solution

\[u(\xi) = a_0 + \sum_{i=1}^{n} F^{i-1}(\xi) (a_i F(\xi) + b_i G(\xi)), \quad (21)\]

where \(F(\xi)\) and \(G(\xi)\) are the Weierstrass elliptic function solution of the projective Riccati Eqs. (4), \(n\) is an integer number which can be determined by balancing the highest order derivative terms with the highest power nonlinear terms in (20), \(a_0, a_i, b_i (i = 1, 2, \cdot \cdot \cdot , n)\) are undetermined constants and \(a_0^2 + b_0^2 \neq 0\).

\textbf{Step 3:} Substituting (21), (4) together with one of the relations (6), (8), (10), (12) into (20) and equating the coefficients of like powers of \(F^i(\xi)G^j(\xi)\) to zero yields a set of algebraic equations. Solving this set of algebraic equations with the aid of Maple or Mathematica we can determine the values of \(\omega, a_0\) and \(a_i, b_i (i = 1, 2, \cdot \cdot \cdot , n)\).

\textbf{Step 4:} Putting each solutions of the algebraic equations obtained in \textbf{Step 3} together with the Weierstrass elliptic function solutions \(F(\xi), G(\xi)\) of the projective Riccati equations into (21) and using (19), we obtain the Weierstrass type traveling wave solutions of the Eq. (18).

\textbf{Step 5:} Taking the conversion formulas (14)–(17) into the Weierstrass elliptic function type traveling wave solutions obtained in \textbf{Step 4}, we get the solitary wave and the periodic wave solutions of the Eq. (18).
3 Solutions of the mKdV equation

Now let us consider the mKdV equation

\[ u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \quad (22) \]

where \( \alpha, \beta \) are constants.

Taking the wave transformation (19) into (22) we get the following ODE

\[ -\omega u'(\xi) + \alpha u^2(\xi)u'(\xi) + \beta u'''(\xi) = 0. \quad (23) \]

By using the homogenous balance method we can determine that \( n = 1 \). Thus the solution of Eq. (23) can be chosen as

\[ u(\xi) = a_0 + a_1 F(\xi) + b_1 G(\xi), \quad (24) \]

where \( F(\xi), G(\xi) \) are the solution of the projective Riccati equations (4), \( a_0, a_1, b_1 \) are undermined constants.

(1) Taking (24) with (4), (6) into (23) and setting the coefficients of \( F_i G_j (i = 1, 2, 3; j = 0, 1) \) to zero we obtain a set of algebraic equations

\[
\begin{align*}
\alpha a_1^2 p &= 0, \\
6\alpha a_1 a_0 b_1 r - 2\alpha a_1^2 b_1 q + \frac{2\alpha b_1^3 r^2}{p} + 3\beta b_1 r^2 p &= 0, \\
-2\alpha a_1 a_0 b_1 q - \frac{\alpha b_1^2 q r}{p} - \beta b_1 r p q + \alpha a_1^2 b_1 r - \omega b_1 r &= 0, \\
2\alpha a_0 a_1 b_1 r + 4\alpha a_1 b_1^2 r + 6\alpha a_1^2 q - \omega a_1 p &= 0.
\end{align*}
\]

Solving this set of algebraic equations with use of Maple we get

\[ a_0 = 0, a_1 = 0, b_1 = \varepsilon p \sqrt{-\frac{3\beta}{2\alpha}}, \omega = \frac{\beta p q}{2}, \varepsilon = \pm 1. \quad (25) \]

Substituting (25), (4), (19) into (24) we obtain the Weierstrass type traveling wave solution of the mKdV equation

\[ u(x, t) = \frac{12\varepsilon \sqrt{3\beta}}{pq + 12\varepsilon (x - \frac{\beta p q}{2} t, g_2, g_3)}, \quad (26) \]

where the invariants \( g_2, g_3 \) are determined by (13).

On substituting the conversion formulas (11)–(17) with \( \theta = -p q \) into Eq. (26), we find that the Weierstrass elliptic function solution (26) degenerate to the following solitary wave and periodic wave solutions of the mKdV equation

\[
\begin{align*}
u_1^{(1)}(x, t) &= -\varepsilon \sqrt{-\frac{6\beta p q}{\alpha}} \tanh \frac{-p q}{2}(x - \frac{\beta p q}{2} t), \quad pq < 0, \alpha \beta < 0, \\
u_2^{(1)}(x, t) &= -\varepsilon \sqrt{-\frac{6\beta p q}{\alpha}} \coth \frac{-p q}{2}(x - \frac{\beta p q}{2} t), \quad pq < 0, \alpha \beta < 0,
\end{align*}
\]

5
Using Maple to solve the above algebraic equations we obtain

\[ u_3^{(1)}(x,t) = \frac{\varepsilon}{2} \sqrt{-\frac{6\beta pq}{\alpha}} \tan \frac{\sqrt{pq}}{2} (x - \frac{\beta pq}{2} t), pq > 0, \alpha \beta < 0, \]

\[ u_4^{(1)}(x,t) = -\frac{\varepsilon}{2} \sqrt{-\frac{6\beta pq}{\alpha}} \cot \frac{\sqrt{pq}}{2} (x - \frac{\beta pq}{2} t), pq > 0, \alpha \beta < 0, \]

where \( p, q \) are free parameters.

(2) Taking (24) with (4), (8) into (23) and setting the coefficients of \( F^i G^j \) \((i = 1, 2, 3, 4, 5; j = 0, 1)\) to zero leads the following set of algebraic equations

\[
\begin{align*}
\alpha a_1^3 p - & \frac{144\beta a_1 p^2 r^2 + 72\alpha b_1^2 a_1 r^2}{25q} = 0, \\
\frac{3456\beta b_1 p r^4}{625q^2} - & \frac{72\alpha a_1^2 b_1 r^2}{25q} + \frac{576\alpha b_1^3 r^4}{625q^2 p} = 0, \\
4\alpha a_1 b_1^2 r + & 6\beta a_1 p^2 r + 2\alpha a_1^2 a_0 p - \frac{48\alpha b_1^2 a_0 r^2}{25q} = 0, \\
5\alpha a_1^2 b_1 r - & \frac{72\alpha b_1^3 r^3}{25q p} - \frac{288\beta b_1 p r^3 + 96\alpha b_1 a_0 a_1 r^2}{25q} = 0, \\
- 2\alpha a_1 a_0 b_1 q - & \frac{\alpha b_1^3 r q}{p} - \beta b_1 r p q + \alpha a_1^2 b_1 r - \omega b_1 r = 0, \\
6\alpha a_1 a_0 b_1 r + & \frac{24\omega b_1 r^2}{25q} - \frac{24\alpha a_1^2 b_1 r^2}{25q} - 2\alpha a_1^2 b_1 q + \frac{74\alpha b_1^3 r^2}{25p} + \frac{171\beta b_1 p r^2}{25} = 0, \\
\alpha a_1^2 a_1 p + 2\alpha a_0 b_1^2 r - & \alpha a_1^2 b_1 q - \beta a_1 p^2 q - \omega a_1 p = 0.
\end{align*}
\]

Using Maple to solve the above algebraic equations we obtain

\[ a_0 = \frac{5\varepsilon \sqrt{\alpha \omega}}{3\alpha}, b_1 = 0, q = \frac{16\omega}{9p\beta}, r = -\frac{5\varepsilon \alpha \sqrt{\alpha \omega}}{9p\beta}, \varepsilon = \pm 1, \] \hspace{1cm} (27)

where \( \alpha \omega > 0 \) and \( p \) is a free parameter.

Inserting (27), (7) and (19) into (24) we obtain the following Weierstrass type traveling wave solution of the mKdV equation

\[ u(x,t) = -\frac{\varepsilon \omega (32\omega + 81\beta \varphi(x - \omega t, g_2, g_3))}{81\beta \sqrt{\alpha \omega} \varphi(x - \omega t, g_2, g_3)}, \]

\hspace{1cm} (28)

where \( \alpha \omega > 0, \) the invariants \( g_2, g_3 \) are given by

\[ g_2 = \frac{64\omega^2}{243\beta^2}, g_3 = \frac{512\omega^3}{19683\beta^3}. \]

(29)

Substituting the conversion formulas (14)–(17) with \( \theta = -qp = -\frac{16\omega}{9p \beta} \) into (28) and using (29) we obtain the solitary wave like solutions and the periodic wave solutions of the mKdV equation as following

\[ u_1^{(2)}(x,t) = \frac{\varepsilon \omega \left( 5 \cosh^2 \frac{2}{3} \sqrt{-\frac{\alpha \omega}{\beta}} (x - \omega t) + 9 \right)}{3\sqrt{\alpha \omega} \left( \cosh^2 \frac{2}{3} \sqrt{-\frac{\alpha \omega}{\beta}} (x - \omega t) - 3 \right)}, \alpha \omega > 0, \beta \omega < 0, \]
where $\omega$ is a free parameter.

(3) Inserting (24) with (4), (10) into (23) and setting the coefficients of $F^i G^j$ ($i = 1, 2, 3, 4; j = 0, 1$) to zero we obtain the following set of algebraic equations

\[
\begin{cases}
\alpha a_1^3 p - \frac{24 \beta a_1 p^2 r^2 + 3 \alpha b_1^2 a_1 r^2 q + 6 \beta a_1 p^2 r^2 q + 12 \alpha b_1^2 a_1 r^2}{(q + 2)^2} = 0, \\
\alpha a_0^2 b_1 r - \omega b_1 r - 2 \alpha a_1 a_0 b_1 q - \frac{\alpha b_1^2 r q}{p} - 2 \beta b_1 r q = 0, \\
4 \alpha a_1 b_1^2 r + 2 \alpha a_0^2 a_0 p + 6 \beta a_1 p^2 r - \frac{2 \alpha b_1^2 a_0 r^2 q + 8 \alpha b_1^2 a_0 r^2}{(q + 2)^2} = 0, \\
5 \alpha a_1^2 b_1 r - \frac{14 \beta b_1 p r^3 q + 4 \alpha b_1 a_0 a_1 r^2 q + 16 \alpha b_1 a_0 a_1 r^2 + 56 \beta b_1 p r^3}{(q + 2)^2} - 3 \alpha b_1^3 r^3 q + 12 \beta b_1^3 r^3 \frac{p(q + 2)^2}{p(q + 2)^2} = 0, \\
6 \beta b_1 p r^4 q^2 + 48 \beta b_1 p r^4 q + 96 \beta b_1 p r^4 + \frac{\alpha b_1^3 r^4 q^2 + 8 \alpha b_1^3 r^4 q + 16 \alpha b_1^3 r^4}{(q + 2)^4} = 0, \\
-12 \alpha a_2^2 b_1 r^2 + 3 \alpha a_1^2 b_1 r^2 q \frac{p(q + 2)^2}{p(q + 2)^2} = 0, \\
\frac{\alpha b_1^2 r^2 q^2 + 4 \alpha b_1^2 r^2 q + 2 \alpha b_1^2 r^2}{p(q + 2)^2} - 2 \alpha a_2^2 b_1 q + 6 \beta b_1 p r^2 + 6 \alpha a_1 a_0 b_1 r \frac{p(q + 2)^2}{p(q + 2)^2} + 4 \beta b_1 p r^2 q^2 + 16 \beta b_1 p r^2 q - 4 \alpha a_1 b_1 r^2 q - \alpha a_1 b_1 r^2 q + 4 \omega b_1 r^2 \frac{(q + 2)^2}{(q + 2)^2} = 0, \\
\alpha a_0^2 a_1 p + 2 \alpha a_0 b_1^2 r - \alpha a_1 b_1^2 q - \beta a_1 p^2 q - \omega a_1 p = 0.
\end{cases}
\]

Solving this set of algebraic equations with aid of Maple we obtain

\[
a_0 = \varepsilon (q + 2) \sqrt{\frac{3 \omega}{\alpha (q^2 + 4q + 12)}}, \quad a_1 = -\frac{2 \sqrt{3} \varepsilon (q + 4) r}{(q + 2) \alpha \sqrt{\frac{\omega (q^2 + 4q + 12)}{\alpha}}}.
\]

\[
b_1 = 0, \quad p = \frac{2 \omega (q + 4)}{(q^2 + 4q + 12) \beta}.
\]

From which we calculate that

\[
g_2 = \frac{q^2 \omega^2 (q + 4)^2}{3(q^2 + 4q + 12)^2 \beta^2}, \quad g_3 = \frac{q^3 \omega^3 (q + 4)^3}{27(q^2 + 4q + 12)^3 \beta^3}.
\]

(31)
Substituting (30) with (31), (9) into (24) we get the Weierstrass type traveling wave solution of the mKdV equation

\[ u(x, t) = -\frac{\sqrt{3}\omega}{\sqrt{\alpha\omega(q^2 + 4q + 12)}} \left[ \frac{6\beta(q + 6)(q^2 + 4q + 12)\varphi(\xi, g_2, g_3) + \omega(q + 4)(q^2 - 6q - 24)}{6(q^2 + 4q + 12)\beta\varphi(\xi, g_2, g_3) + \omega(q + 12)(q + 4)} \right], \tag{32} \]

where \( \xi = x - \omega t \) and the invariants \( g_2, g_3 \) are given by (31).

Taking the conversion formulas (14)–(17) with \( \theta = -q\beta = -\frac{2\omega(q + 4)q}{(q^2 + 4q + 12)\beta} \) into (32) leads the solitary and periodic wave solutions of the mKdV equation

\[ u_1^{(3)}(x, t) = \varepsilon \sqrt{\frac{3\omega}{\alpha(q^2 + 4q + 12)}} \left( \frac{4(q + 2)\cosh^2 \eta - q(q + 6)}{4\cosh^2 \eta + q} \right), \]
\[ u_2^{(3)}(x, t) = -\varepsilon \sqrt{\frac{3\omega}{\alpha(q^2 + 4q + 12)}} \left( \frac{4(q + 2)\sinh^2 \eta + q(q + 6)}{4\sinh^2 \eta - q} \right), \]
\[ \eta = \frac{1}{2} \sqrt{\frac{2\omega(q + 4)q}{(q^2 + 4q + 12)\beta}} (x - \omega t), \alpha \omega > 0, \beta \omega(q + 4)q < 0, \]
\[ u_3^{(3)}(x, t) = \varepsilon \sqrt{\frac{3\omega}{\alpha(q^2 + 4q + 12)}} \left( \frac{4(q + 2)\cos^2 \zeta - q(q + 6)}{4\cos^2 \zeta + q} \right), \]
\[ u_4^{(3)}(x, t) = \varepsilon \sqrt{\frac{3\omega}{\alpha(q^2 + 4q + 12)}} \left( \frac{4(q + 2)\sin^2 \zeta - q(q + 6)}{4\sin^2 \zeta + q} \right), \]
\[ \zeta = \frac{1}{2} \sqrt{\frac{2\omega(q + 4)q}{(q^2 + 4q + 12)\beta}} (x - \omega t), \alpha \omega > 0, \beta \omega(q + 4)q > 0, \]

where \( q \) and \( \omega \) are free parameters.

(4) Taking (24) with (4), (12) into (23) and setting the coefficients of \( F^iG^j \) (i =
1, 2, 3, 4; \ j = 0, 1) to zero we obtain the following set of algebraic equations

\[
\begin{align*}
\alpha a^2 b_1 r - \omega b_1 r - 2\alpha a_1 a_0 b_1 q - \frac{\alpha b^2 r q}{p} - \beta b_1 r p q &= 0, \\
2\alpha a^2 b_1 p - \frac{2\alpha b^2 a_0 p^4 r^2 + 8\alpha b^2 a_0 p^2 r^2}{q(p^2 + 2)^2} + 4\alpha a_1 b_1^2 r + 6\beta a_1 p^2 r &= 0, \\
\alpha a_1^2 p - \frac{3\alpha b^2 a_1 p^4 r^2 + 12\alpha b^2 a_1 p^2 r^2 + 6\beta a_1 p^6 r^2 + 24\beta a_1 p^4 r^2}{q(p^2 + 2)^2} &= 0, \\
\frac{3\alpha b^3 p^3 r^3 + 12\alpha b^3 p r^3 + 12\beta b_1 p^5 r^3 + 48\beta b_1 p^3 r^3 + 4\alpha b_1 a_0 a_1 p^4 r^2 + 16\alpha b_1 a_0 a_1 p^2 r^2}{q(p^2 + 2)^2} &= 0, \\
- 5\alpha a^2 b_1 r &= 0, \\
6\beta b_1 p^9 r^4 + 48\beta b_1 p^7 r^4 + 96\beta b_1 p^5 r^4 + 8\alpha b^3 p^5 r^4 + 16\alpha b^3 p^3 r^4 + \alpha b^3 p^7 r^4 &= 0, \\
- 3\alpha a^2 b_1^2 p^4 r^2 + 12\alpha a^2 b_1 p^2 r^2 &= 0, \\
\frac{\omega b_1 p^4 r^2 + 4\omega b_1 p^2 r^2 - 4\alpha a_1 b_1^2 p^2 r^2 - \alpha a_1 b_1^2 p^4 r^2}{q(p^2 + 2)^2} - 2\alpha a_1^2 b_1 q + \frac{2\alpha b^3 r^2}{p} + 3\beta b_1 r^2 p &= 0, \\
6\alpha a_1 a_0 b_1 r + \alpha a_3 b_1^3 r^2 + 4\beta a_1 b_1^3 r^2 + 2\beta a_1 b_1^5 r^2 + 16\beta b_1 r^3 p &= 0, \\
\alpha a^2 a_1 p + 2\alpha a_0 a_1^2 r - \alpha a_1 b_1^2 q - \beta a_1 p^2 q - \omega a_1 p &= 0.
\end{align*}
\]

Solving this set of algebraic equations with use of Maple we find that

\[
\begin{align*}
\alpha_0 &= 0, \ b_1 = p \sqrt{-\frac{3\beta}{2\alpha}}, \ q = \frac{2\omega}{p\beta}, \ r = \frac{2\varepsilon(p^2 + 2)a_1}{p^2\beta} \sqrt{\frac{\alpha\omega}{3p^2 + 12}}, \\
\alpha_0 &= 0, \ b_1 = -p \sqrt{-\frac{3\beta}{2\alpha}}, \ q = \frac{2\omega}{p\beta}, \ r = \frac{2\varepsilon(p^2 + 2)a_1}{p^2\beta} \sqrt{\frac{\alpha\omega}{3p^2 + 12}}.
\end{align*}
\]

(33) and (34)

where \(\varepsilon = \pm 1\).

From (33) and (34), the invariants \(g_2\) and \(g_3\) are calculated to be

\[
g_2 = \frac{\omega^2}{3\beta^2}, \ g_3 = \frac{\omega^3}{27\beta^3}.
\]

(35)

Taking (33) and (34) with (35), (19) into (24), respectively, we obtain the Weierstrass type traveling wave solution of the mKdV equation

\[
u(x, t) = \varepsilon \omega \sqrt{\frac{3(p^2 + 4)}{\alpha \omega}} \left( \frac{36\beta \varepsilon \sqrt{\frac{2\beta}{p^2 + 4}} \psi' (\xi) + 12\beta p \left(3\beta \phi (\xi) + \omega \phi (\xi) + \omega^2 p\right)}{36\beta^2 p^2 \psi^2 (\xi) + 12\beta \omega (p^2 + 6) \phi (\xi) + \omega^2 (p^2 + 12)} \right),
\]

(36)

\[
u(x, t) = \varepsilon \omega \sqrt{\frac{3(p^2 + 4)}{\alpha \omega}} \left( \frac{-36\beta \varepsilon \sqrt{\frac{2\beta}{p^2 + 4}} \psi' (\xi) + 12\beta p \left(3\beta \phi (\xi) + \omega \phi (\xi) + \omega^2 p\right)}{36\beta^2 p^2 \psi^2 (\xi) + 12\beta \omega (p^2 + 6) \phi (\xi) + \omega^2 (p^2 + 12)} \right),
\]

(37)

where \(\phi (\xi) = \phi (x - \omega t, g_2, g_3), \ \alpha \omega > 0, \ \beta \omega < 0\) and the invariants \(g_2\) and \(g_3\) are given by (35).
Because $\beta \omega < 0$ in ($36$) and ($37$), so $\theta = -qp = -\frac{2\omega}{\beta} > 0$. This indicates that the Weierstrass elliptic function solutions ($36$) and ($37$) can only degenerate to the solitary wave solutions of the mKdV equation. Therefore, by substituting the conversion formulas ($14$) and ($15$) with $\theta = -qp = -\frac{2\omega}{\beta}$ into ($36$) and ($37$) we obtain the solitary wave like solutions of the mKdV equation as follows

$$u_1^{(4)}(x, t) = -\sqrt{\frac{3(p^2 + 4)}{\alpha \omega}} \left( \frac{4\sqrt{\frac{\omega^2}{p^2+4}} \sinh \xi \cosh \xi - \epsilon \omega p}{4 \cosh^2 \xi + p^2} \right),$$

$$u_2^{(4)}(x, t) = -\sqrt{\frac{3(p^2 + 4)}{\alpha \omega}} \left( \frac{4\sqrt{\frac{\omega^2}{p^2+4}} \sinh \xi \cosh \xi - \epsilon \omega p}{4 \sinh^2 \xi - p^2} \right),$$

$$u_3^{(4)}(x, t) = \sqrt{\frac{3(p^2 + 4)}{\alpha \omega}} \left( \frac{4\sqrt{\frac{\omega^2}{p^2+4}} \sinh \xi \cosh \xi + \epsilon \omega p}{4 \cosh^2 \xi + p^2} \right),$$

$$u_4^{(4)}(x, t) = \sqrt{\frac{3(p^2 + 4)}{\alpha \omega}} \left( \frac{4\sqrt{\frac{\omega^2}{p^2+4}} \sinh \xi \cosh \xi + \epsilon \omega p}{4 \sinh^2 \xi - p^2} \right),$$

$$\xi = \frac{1}{2} \sqrt{-\frac{2\omega}{\beta}} (x - \omega t), \alpha \omega > 0, \beta \omega < 0,$$

where $\omega$ is a free parameter.

The above traveling wave solutions of the mKdV equation have not been obtained in Refs. $[17, 18]$, and the solutions $u_i^{(j)}$ ($i = 1, 2, 3, 4; j = 2, 3, 4$) cannot be obtained by using other direct methods. More importantly, these solutions are very considerable in physics. For example, the solutions $u_1^{(1)}$ and $u_2^{(2)}$ are the kink type and bell type solitary waves for $\varepsilon = -1$, and for $\varepsilon = 1$ they express the anti–kink type and anti–bell type solitary waves, respectively. The solutions $u_3^{(3)}$ and $u_2^{(3)}$ are the bell type and anti–bell type solitary waves for $\varepsilon = 1$ and $\varepsilon = -1$, respectively. The solutions $u_1^{(4)}$ and $u_3^{(4)}$ are the anti–kink type and kink type solitary waves. $u_3^{(i)}, u_4^{(i)}$ ($i = 2, 3$) are the trigonometric periodic solutions and other solutions are singular solutions. In order to observe the profiles of these solutions, the kink and anti–kink type solutions, the bell and anti–bell type solutions and the periodic solutions are shown in Fig.1, Fig.2 and Fig.3. But the plots of those singular solutions are omitted.

4 Conclusions

In the present paper, the Weierstrass type projective Riccati equation expansion method is proposed to construct Weierstrass elliptic function solutions of NLEEs. At the same time, the conversion formulas are also used to transform these Weierstrass elliptic function solutions into the hyperbolic and trigonometric function solutions of NLEEs. Our method is more powerful than the other direct algebraic methods, and it can be regarded as an extension of the projective Riccati equation expansion method. In order to explain our method more clear we need to point out the following four points.

(1) The Weierstrass type projective Riccati equation expansion method can give more types of new traveling wave solutions of NLEEs that cannot be obtained by other direct
Figure 1: The plots of the kink type and anti–kink type solitary wave solutions. (a) Kink type solution $u_1^{(1)}$ with $p = -1.25, q = 1.25, \alpha = -1, \beta = 1.35, \varepsilon = -1$, (b) Anti–kink type solution $u_1^{(1)}$ with $p = -1.25, q = 1.25, \alpha = -1, \beta = 1.35, \varepsilon = 1$, (c) Anti–kink type solution $u_1^{(4)}$ with $p = 2, \alpha = 1, \beta = -1.25, \omega = 1.25, \varepsilon = -1$, (d) Kink type solution $u_3^{(4)}$ with $p = 2, \alpha = 1, \beta = -1.25, \omega = 1.25, \varepsilon = -1$. 
Figure 2: The plots of the bell type and anti–bell type solitary wave like solutions. (a) Bell type solution $u_2^{(2)}$ with $\alpha = 1, \beta = -1.25, \omega = 1.025, \varepsilon = -1$, (b) Anti–bell type solution $u_2^{(2)}$ with $\alpha = 1, \beta = -1.25, \omega = 1.025, \varepsilon = 1$. (c) Ant–bell type solution $u_1^{(3)}$ with $q = -2, \alpha = 1, \beta = 1.25, \omega = 1.025, \varepsilon = -1$, (d) Bell type solution $u_1^{(3)}$ with $q = -2, \alpha = 1, \beta = 1.25, \omega = 1.025, \varepsilon = 1$. (e) Anti–bell type solution $u_2^{(3)}$ with $q = -2, \alpha = 1, \beta = -1.25, \omega = 1.025, \varepsilon = -1$, (f) Bell type solution $u_2^{(3)}$ with $q = -2, \alpha = 1, \beta = 1.25, \omega = 1.025, \varepsilon = -1$. 12
Figure 3: The plots of the periodic solutions with $\alpha = 0.25, \beta = 0.5, \omega = 0.995$. (a) Solution $u_3^{(2)}$ with $\varepsilon = 1$, (b) Solution $u_4^{(2)}$ with $\varepsilon = 1$, (c) Solution $u_3^{(3)}$ with $q = 2, \varepsilon = -1$, (d) Solution $u_4^{(3)}$ with $q = 2, \varepsilon = -1$. 
methods. As shown above, in the case of the mKdV equation, by using the Weierstrass type projective Riccati equation expansion method we have obtained the kink type, the bell type, the anti–kink type, the anti–bell type, the periodic and the singular solitary wave and periodic solutions, etc. However, other direct methods cannot give these types of solutions at the same time.

(2) Compared with the previously known conversion formulas, our conversion formulas (14)–(17) don’t require the roots of \( p(w) = 0 \), can convert the Weierstrass elliptic function solutions of NLEEs into the hyperbolic and trigonometric function solutions in a straightforward way, and can ensure that the obtained solutions are correct.

In addition, the conversion formulas (14)–(17) also can be used in other Weierstrass elliptic function methods to transform the Weierstrass elliptic function solutions into the hyperbolic and trigonometric function solutions.

(3) Although we have concerned with the mKdV equation, our method can be applied to construct the exact solitary wave and periodic wave solutions of a wide class of NLEEs.

(4) In addition to the projective Riccati equations, we find that other auxiliary equations also possess the Weierstrass elliptic function solutions. Therefore, the Weierstrass type expansion method can also be proposed for these auxiliary equations which can lead us to establish a systematic Weierstrass elliptic function method for solving NLEEs. So our idea is significance to extend the application area of the Weierstrass elliptic functions.

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