Phantom-like effects in an asymmetric brane embedding with induced gravity and the Gauss–Bonnet term in the bulk

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Abstract

We construct an asymmetric braneworld embedding with induced gravity on the brane, where stringy effects are taken into account by the incorporation of the Gauss–Bonnet (GB) term into the bulk action. We derive the effective Friedmann equation of the brane and then investigate possible realization of the phantom-like behavior in this setup. We show that in the absence of the GB term in the bulk action (a pure induced gravity scenario), the phantom-like behavior in the asymmetric case can be realized in smaller redshifts than the corresponding symmetric case. We also show that in the general case with the curvature effect, the phantom-like behavior can be realized in two subcases: in the symmetric subcase and in an asymmetric branch of the solutions. In either case, this phantom-like behavior happens without introducing any phantom fields either on the brane or in the bulk.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

An ever increasing number of observational data indicate that our universe is currently in a phase of accelerating expansion [1]. A simple way to explain the cosmic acceleration is by introducing a cosmological constant, with an equation of state parameter \( \omega = -1 \), and then investigating the so-called \( \Lambda \)CDM model. This model is favored by recent observations, but has some important difficulties such as the fine-tuning and coincidence problems and the unknown origin of the cosmological constant [2]. To accommodate the cosmic speedup, some other types of unknown energy components (the so-called dark energy) with negative pressure have been proposed [3]. However, understanding the nature of the dark energy is one of the fundamental problems of modern theoretical cosmology [4]. There is another approach for realizing the late-time acceleration: modifying the geometric part of the gravitational theory. This proposal can be realized in braneworld scenarios [5], string inspired scenarios [6], \( f(R) \) gravity [7] and so on.

Over the last few decades, there has been much interest in the extra dimensional theories after the modification of the old Kaluza–Klein picture, where the extra dimensions must be sufficiently compact. These recent developments are based on the idea that ordinary matter and gauge fields could be confined to a three-dimensional (3D) world (3-brane), while gravity and possibly non-standard matter are free to propagate in the entire extra dimensional spacetime (the bulk). The cosmological braneworld solutions exhibit many interesting and unusual properties such as modifying the gravity at early or late times. The former issue is presented in the RSII model [8], where a positive tension brane is embedded in an anti-de Sitter (AdS5) bulk, and the latter case is associated with the braneworld models with an induced gravity term in the brane action. The Dvali–Gabadadze–Porrati (DGP) scenario [5] is a subclass of these models where a tensionless brane is embedded in an infinite Minkowski bulk. In this
model, gravity leaks off the brane into extra dimension at large scales. Gravity leakage at late times causes the cosmic speedup due to the weakening of gravity on the brane, without the need for introducing a dark energy component [9].

Most braneworld scenarios assume a $Z_2$-symmetric brane which is motivated by a model of M-theory proposed by Horava and Witten [10]. However, some recent papers examine the more general models that are not directly derived from M-theory, relaxing the mirror symmetry across the brane (see for instance [11–16]). In the asymmetric case (without $Z_2$-symmetry), the parameters in the bulk, such as the gravitational and cosmological constants, can differ on either side of the brane. A distinct property of the asymmetric brane model is that they can admit self-accelerating solutions even without the need to introduce an induced gravity term in the brane action [15, 16].

Since the braneworld scenarios are motivated by string theories, it is natural to include some extra terms such as the Gauss–Bonnet (GB) term in the 5D field equations. The GB extension of general relativity (GR) has been motivated from a string theoretical point of view as a version of higher-dimensional gravity, since this sort of modification also appears in low-energy effective actions in this context [17]. The GB term leads to second-order gravitational field equations linear in the second derivatives in the bulk metric which is ghost free, the property of curvature invariant of the GB term [18]. The inclusion of the GB term in the bulk action results in a variety of novel phenomena that certainly affect the cosmological dynamics of this generalized braneworld setup [19].

Recently, well-known astronomical observations with WMAP7 have indicated that the equation of state parameter of dark energy can be less than $-1$ and can even have a transient behavior [20]. A simple way of explaining this phenomenon is to consider a non-canonical phantom dark energy [21] that introduces new theoretical facilities and challenges in this field. Phantom fields are a type of scalar fields with negative sign for the kinetic energy term. Indeed, phantom fields suffer from instabilities due to violation of the null energy condition, and a phantom universe eventually ends up with a Big Rip singularity [22]. Thus, it follows immediately that there must be some alternative approaches for realizing a phantom-like behavior without introducing any phantom field in the model. By phantom-like behavior, we mean the growth of the effective dark energy density with cosmic time and, at the same time, the effective equation of state parameter should always stay less than $-1$. In other words, the normal branch of the DGP model that cannot explain the self-acceleration has the key property that the brane is extrinsically curved so that shortcuts through the bulk allow gravity to screen the effects of the brane energy momentum contents at the Hubble parameters $H \sim r_c^{-1}$, where $r_c$ is the crossover distance. Since in this case $H(t)$ is a decreasing function of the cosmic time, the effective dark energy component is increasing with time and therefore we observe a phantom-like behavior without introducing any phantom matter that violates null energy condition and suffers from several theoretical problems. In this regard, it has been shown that the normal, non-self-accelerating branch of the DGP scenario has the potential to explain the phantom-like behavior without introducing any phantom fields in the brane [23]. This type of analysis has then been extended by several authors [24]. The phantom-like behavior of 4D $f(R)$ gravity was studied in [25].

With these preliminaries, in this paper we assume a braneworld model with induced gravity whose bulk action includes, in addition to the familiar Einstein term, a GB contribution. We relax the mirror symmetry of the embedding and derive the field equations on the brane. In section 2, we derive the bulk solution, which in general has the Schwarzschild–anti-de Sitter form. Using the generalized junction conditions, we derive the effective Einstein equations in the bulk and brane. The absence of mirror symmetry and the presence of the GB term lead to a novel and very complicated Friedmann equation in the brane that will be interpreted in several interesting subcases. Section 3 deals with the cosmological dynamics of this asymmetric braneworld setup in the absence of the GB curvature effect. We will show that a late-time accelerated expansion can be deduced in all branches of the scenario. Also, an effective phantom-like behavior can be obtained in some branches of this model. On the other hand, we will show that in the asymmetric case of this pure induced gravity scenario, the phantom-like behavior can be realized in smaller redshifts than the symmetric case. In section 4, we consider the general case with the GB term and show that it is possible to realize the phantom-like behavior by justifying some conditions governing the field equations. Finally, the summary and conclusions are presented in section 5.

2. Equations of motion

In this section we consider an asymmetric braneworld model with the GB contribution in the bulk and the induced gravity term on the brane. The action of this model is given as follows:

$$S = S_{\text{bulk}} + S_{\text{brane}},$$

where

$$S_{\text{bulk}} = \sum_{i=1,2} M_i^3 \int_{B_i} d^5x \sqrt{-g} \left( R_i - 2\Lambda_i + \alpha_i L_{\text{GB}}^i \right)$$

$$- 2M_1^3 \int_{ab} d^4x \sqrt{-h} \left[ K_{ij} + 2\alpha_i \left( J - 2G^{ab}K_{ab} \right) \right]$$

and

$$S_{\text{brane}} = \int_{\text{brane}} d^4x \sqrt{-h} \left( m^2R - 2\sigma + \mathcal{L}_M \right).$$

Here, $S_{\text{bulk}}$ is the action of the bulk, $S_{\text{brane}}$ is the brane action and $S$ is the total action. $B_i$ ($i = 1, 2$) is corresponding to two bulk spaces on either side of the brane with 5D gravitational and cosmological constants $M_i^3$ and $\Lambda_i$, respectively. $R_i$ is the scalar curvature of the bulk metric $g_{ij}^{(i)}$, $\alpha_i > 0$ is the GB coupling and $L_{\text{GB}}^i$ is the GB term in the bulk defined by

$$L_{\text{GB}}^i = R^2 - 4R_{ab}R^{ab} + \mathcal{R}_{abcd}\mathcal{R}^{abcd}.$$

The variable $h_{ab}$ is the induced metric on the brane and is given by $h_{ab} = g_{ab} - n_a n_b$, where $n^a$ is the outward pointing unit normal to $\partial B_i$. $K_i = K_{ab}^i h^{ab}$ is the trace of the extrinsic
The curvature of the brane in the bulk, $G_{ab}^{\text{ind}}$ is the 4D Einstein tensor on the brane and $J$ is the trace of

$$J_{ab} = \frac{1}{3} \left( 2K K_{ac} K^c_b + K_{ac} K^c_{db} - 2K_{ac} K_d K_{ab} - 2K_{ac} K^c_{db} - K^2 K_{ab} \right).$$

(4)

The variable $R$ is the induced scalar curvature on the brane and $\sigma$ and $L_M$ are the tension and matter Lagrangians of the brane, respectively.

Variation of the action gives the following field equations,

$$\nabla^i G^j_{iab} + \alpha H_{ab} + \Lambda \delta_{ab} = \frac{1}{2M^4} S_{ab} \delta(y),$$

(5)

where $H_{ab}$ is the Lovelock tensor defined by

$$H_{ab} = R R_{ab} - R_{ac} R^c_b - 2 R_{cd} R_{abed} + R_a c d e R_{bcde} - \frac{1}{2} g_{ab} L_{GB}$$

(6)

and $S_{ab}$ is the contribution from the brane located at $y = 0$ and has the following form [26]:

$$S_{ab} = T_{ab}^{\text{brane}} - \sigma h_{ab} - m^2 G_{ab}.$$  

(7)

The stress energy in the brane is $T_{ab}^{\text{brane}} = \frac{3L_M}{3M} - h_{ab} L_M$. Using the Gauss–Codazzi equation, the energy momentum of the matter on the brane is conserved [27] and therefore $\nabla^a T_{ab} = 0$, where $\nabla^a$ is the covariant derivative on the brane associated with the induced metric $h_{ab}$. Note that this relation is a consequence of the absence of matter in the bulk.

### 2.1. Bulk solution

At this stage, let us relax the index $i$ since the following analysis will apply to both subspaces of the bulk manifold. We will save this index for when it is necessary. For a homogeneous and isotropic brane, the bulk metric can be written, using the generalized Birkhoff's theorem, in the following form [28]:

$$ds^2 = -f(\alpha) \, dt^2 + \frac{da^2}{f(\alpha)} + a^2 \gamma_{ij} \, dx^i \, dx^j.$$  

(8)

Here, $\gamma_{ij}$ is a 3D metric of a space with constant curvature $k = -1, 0, 1$. With this metric, the $(i, i)$ and $(r, r)$ or $(a, a)$ components of the field equations (5) are given as [29, 30]

$$(a^2 + 4\alpha k - 4\alpha) \frac{d^2 f}{da^2} - 4\alpha \left( \frac{df}{da} \right)^2 + 2(f + \Lambda a^2 - k) = 0$$  

(9)

and

$$\alpha(3af - 2k) \frac{df}{da} - 3fa^2 - \Lambda a^4 + 3ka^2 = 0,$$  

(10)

respectively. Note that the latter equation acts as a constraint equation. The solution of these equations is given as [29]

$$f(\alpha) = k + \frac{a^2}{4\alpha} \left( \frac{1}{3} \sqrt{1 + \frac{4}{3} \alpha \Lambda + 8\alpha \mu \frac{\Lambda}{a^2}} \right),$$  

(11)

where $\mu \geq 0$ is an arbitrary constant related to the black hole mass by the relation $M_{BH} = 3M^3 V \mu$, where $V$ is the volume of the 3D space [28]. The solution for $f(a)$ has two branches due to two signs in front of the square root in equation (11). For the negative branch, the solution has the general relativistic limit as $\alpha \rightarrow 0$, but for the positive branch, there is no general relativistic limit (see [29] and references therein). For $\mu \neq 0$, the metric (8) faces two classes of singularities: The first is an essential singularity at $a = 0$. For the negative branch, this singularity is covered by an event horizon if $k < 0$ or $k = 1$ and $\mu \geq 2\alpha$. For such cases, the event horizon ($a = a_0$) is

$$a_0^2 = \frac{3\alpha L}{\Lambda} + \sqrt{9k^2 + 12k^2 \alpha \Lambda - 6\mu \Lambda \frac{\Lambda}{a^2}}.$$  

However, this is not the case for the positive branch. So, to discard this naked singularity, we must cut off the spacetime at some small values of $a$. This can be done by introducing a second brane at $a \sim M^{-1}c_1$ [29]. The second class of singularities, the so-called branch singularities, occurs when $a = a_0 := [-\alpha \mu/(1 + 4\alpha \Lambda)]^{1/3} > 0$, in which the term inside the square root in equation (11) vanishes. In order to avoid this singularity, one requires that $1 + 4\alpha \Lambda \geq 0$ and $\alpha \mu \geq 0$. From this relation, since $\mu \geq 0$, we must have $\alpha \geq 0$, which is consistent with string theory considerations.

### 2.2. The Friedmann equation on the brane

We consider the location of the brane as $t = \tau(\tau)$ and $a = a(\tau)$, which is parameterized by the proper time $\tau$ of the brane. Then the induced metric on the brane is given by

$$ds^2 = -d\tau^2 + a(\tau)^2 \gamma_{ij} \, dx^i \, dx^j.$$  

(12)

In this equation, $\tau$ and $a(\tau)$ correspond to the cosmic time and the scale factor of the Friedmann–Robertson–Walker (FRW) universe, respectively. Note that since the brane coincides with both boundaries, the metric on the brane is well defined only when $a(\tau) = a_1(\tau) = a(\tau)$. The Hubble parameter on the brane is defined by $H = \frac{\dot{a}}{a}$. The tangent vector at a point on the brane can be written as follows,

$$u = \dot{\tau} \frac{\partial}{\partial \tau} + \dot{a} \frac{\partial}{\partial a},$$  

(13)

where $i = 1, 2$ corresponds to two sides of the brane and the dot marks a derivative with respect to the proper time $\tau$. The normal vector to the brane is

$$n^{(i)} = \theta_i (-\dot{a}(\tau), \dot{\tau}(\tau), 0),$$

where $\theta_i = \pm 1$. For $\theta_i = 1$, $B_i$ corresponds to $0 < a < a_i(\tau)$, whereas for $\theta_i = -1$, $B_i$ corresponds to $a_i(\tau) < a < \infty$. With this definition, the conditions $n^{(i)} u^{(i)} = 0$ and $n^{(i)} n^{(i)} = 1$ are satisfied. Normalization of $n^{(i)}$ imposes a constraint equation so that

$$-f(\alpha) \dot{a}^2 + \frac{a^2}{f(\alpha)} = -1.$$  

(14)

The dynamics of the brane are given by the generalized junction conditions for a GB braneworld gravity [31]

$$[K_{ab}] - h_{ab} \left[ K \right] + 2(3[\alpha J_{ab}] - h_{ab} [\alpha J])$$  

$$- 2P_{abcd} \left[ \alpha K^{ab} \right] = -S_{ab},$$  

(15)
where

\[ P_{abcd} = R_{abcd} + 2h_{a(d}R_{c)b} + 2h_{b[c}R_{d]a} + R h_{a[c}h_{d)b} \]  

(16)
is the divergence-free part of the Riemann tensor of the brane which can be constructed from the induced metric on the brane, \( h_{ab} \). Note that by definition, \( [X] \equiv X_2 R_2 - X_1 R_1 \), and \( X_{ab} \) is defined by equation (7). We take the brane matter to be a perfect fluid, so \( T_{ab} = (\rho + p)h_{a}u_{b} + \rho h_{ab} \), where \( \rho \) and \( p \) are the energy density and pressure of the perfect fluid, respectively. The extrinsic curvature has the following non-vanishing components:

\[(K_i)_{ab}u^a u^b = \theta_1(h_1i_1)^{-1} \left( \dot{a} + \frac{h_1}{2} \right) \]  

(17)

and

\[(K_i)_{ij}' = (K_i)_{ij}' = -\theta_1 h_1i_1. \]  

(18)

Now, the \((\tau, \tau)\) component of equation (15) can be recast as follows:

\[ \sum \theta_i M_i^3 f_i \frac{i_1}{a} \left[ 1 - \frac{4}{3} \left( \frac{f_i}{a} \right)^2 + 4a \left( H^2 + \frac{k}{a^2} \right) \right] = \frac{\rho + \sigma}{3} - m^2 \left( H^2 + \frac{k}{a^2} \right). \]  

(19)

Using the constraint equation (14), this equation can be rewritten in the following form:

\[ H^2 + \frac{k}{a^2} = \frac{\rho + \sigma}{3m^2} + \frac{1}{m^2} \sum_{i=1,2} \theta_i M_i^3 \left[ f_i(a) + \frac{i_1}{3} \right] \left( 3k - f_i(a) \right) \]  

\[ \times \left[ 1 + \frac{8a}{3} H^2 + 4a \frac{i_1}{3} \left( \frac{3k - f_i(a)}{a^2} \right) \right]. \]  

(20)

This is a very complicated equation for cosmological dynamics on the brane in which the \( Z_2 \)-symmetry is relaxed. The case where the bulk is symmetric is equivalent to the geometrical setting in which the brane is just a boundary of the bulk space; this is one way of justifying this mirror symmetry [13]. In the case of \( Z_2 \)-symmetry, there are only two ways of bounding the bulk by the brane resulting in the usual DGP-like branches of the model. We note that if one adopts a \( Z_2 \)-symmetry across the brane, equation (20) reduces to

\[ H^2 + \frac{k}{a^2} = \frac{\rho + \sigma}{3m^2} + \frac{1}{r_c} \left( H^2 + \frac{k}{a^2} + \frac{\chi}{4a} \right) \]  

\[ \times \left[ 1 + \frac{8a}{3} H^2 + 4a \frac{i_1}{3} \left( \frac{3k - f_i(a)}{a^2} \right) \right], \]  

(21)

where \( r_c \) is the DGP crossover scale and

\[ \chi \equiv 1 \mp \left[ 1 + \frac{4}{3} \alpha \Lambda + 8a \frac{\mu}{a^2} \right]. \]

Equation (21) gives the well-known DGP or DGP–GB Friedmann equations in appropriate limits (see for instance [19]). In what follows, we consider some special and simpler cases of this equation to study possible realization of the phantom-like behavior on the brane.

2.3. A pure induced gravity brane (\( \alpha = 0 \)) without \( Z_2 \)-symmetry

In this section, we consider a pure induced gravity scenario in the absence of mirror symmetry and the GB contribution (\( \alpha = 0 \)). Then we investigate the phantom-like behavior of this model. In this case \( f(a) \) has the following form:

\[ f(a) = k \pm \left( \frac{\Lambda}{6} \frac{a^2 + \mu}{a^2} \right). \]  

(22)

Only the negative sign of this relation is well defined, so we take this branch. Using equations (14) and (22), the field equation (20) can be recast in the following form:

\[ H^2 + \frac{k}{a^2} = \frac{\rho + \sigma}{3m^2} + \frac{1}{m^2} \sum_{i=1,2} \theta_i M_i^3 \left[ H^2 + \frac{k}{a^2} - \frac{2A_1}{6} \right] \]  

(23)

In what follows, we ignore the dark radiation term \((\mu/a^4)\), because we investigate the cosmological implications of the model at the late epochs and this term decays at late times very fast. This term is important when one treats perturbations on the brane. To investigate the cosmological implications, we consider the Friedmann equation (23) in some special cases. Note that the general case has been studied in [13].

2.3.1. The case with \( \Lambda_1 = \Lambda_2 = \Lambda_0, M_1 \neq M_2 \). In this case, the Friedmann equation (23) has a general solution as follows,

\[ H^2 + \frac{k}{a^2} = \frac{\rho + \sigma}{3m^2} + 2 \left( \frac{1}{l_1} + \frac{1}{\epsilon l_2} \right)^2 \]  

\[ \times \left[ 1 + \left( \frac{l_1l_2}{l_1 + \epsilon l_2} \right)^2 \left( \frac{\rho + \sigma}{3m^2} + \frac{\Lambda_0}{6} \right) \right], \]  

(24)

where we have defined the length scale \( l_1 = 2m^2/M_1^2 \) and \( \epsilon = \theta_1/\theta_2 \). Also we have assumed that \( M_1 > M_2 > 0 \). We note that for the \( Z_2 \)-symmetric case, these solutions are a generalization of the brane 1 and brane 2 branches studied in [32]. However, in our scenario we have two extra branches due to asymmetric embedding of the brane. In this regard we can define an effective cosmological constant on the brane as follows,

\[ H^2 + \frac{k}{a^2} = \frac{\rho}{3} \cdot \frac{\Lambda_{eff}}{3}, \]  

(25)

where we have assumed a flat FRW brane and

\[ \frac{\Lambda_{eff}}{3} = \frac{\sigma}{3m^2} + 2 \left( \frac{1}{l_1} + \frac{1}{\epsilon l_2} \right)^2 \]  

\[ \times \left[ 1 + \left( \frac{l_1l_2}{l_1 + \epsilon l_2} \right)^2 \left( \frac{\rho + \sigma}{3m^2} + \frac{\Lambda_0}{6} \right) \right]. \]  

(26)

This equation indicates that a late-time behavior can be deduced in all branches of the scenario. Choosing the lower sign in the right-hand side of equation (26) leads to a very interesting result. Indeed, in this case the effective cosmological constant can be decomposed into two distinct
parts: the first term in the rhs of (26) can be considered as a cosmological constant on the brane and the second term has a screening effect. The screening effect is due to the induced gravity term on the brane that leads to an increase of the effective cosmological constant with cosmic time. So, there are two branches of the scenario that give us an opportunity to investigate a phantom-like behavior on the brane. By phantom-like behavior, we mean the growth of the effective dark energy density with cosmic time and the effective equation of state parameter that must be less than $-1$ and, at the same time, the Hubble rate $H$ must be negative to avoid a Big Rip-type singularity in the future. To realize a phantom-like behavior, we rewrite the Friedmann equation (24) for the lower sign in a dimensionless form

$$E^2(z) = \frac{\dot{H}^2}{H_0^2} = \Omega_m (1 + z)^3 + \Omega_\sigma + 2A$$

$$-2\sqrt{A} \sqrt{\Omega_m (1 + z)^3 + \Omega_\sigma + \Omega_{\Lambda_b} + A.} \quad (27)$$

where we have defined $A = \Omega_i + \Omega_t + 2\epsilon \sqrt{\Omega_i \Omega_t}$ and the cosmological parameters are defined as $\Omega_m = \frac{\rho_m}{3m H_0^2}$, $\Omega_\sigma = \sqrt{\frac{\sigma}{3m H_0^2}}$, $\Omega_i = \frac{1}{\tau H_0^2}$, $\Omega_{\Lambda_b} = -\frac{\lambda_b}{\Omega_{\Lambda_b} H_0^2}$, and $\rho_0$ and $H_0$ are the present values of the matter energy density and Hubble parameter, respectively. Note that a constraint equation is imposed on the dimensionless equation (27) at redshift $z = 0$, so that

$$1 = \Omega_m + \Omega_\sigma + 2A - 2\sqrt{A} \sqrt{\Omega_m + \Omega_\sigma + \Omega_{\Lambda_b} + A.} \quad (28)$$

To investigate the phantom-like behavior, we rewrite the standard Friedman equation as follows:

$$H^2 = \frac{1}{3m^2} \left( \rho + \rho_{\text{eff}}^{(\text{DE})} \right). \quad (29)$$

where $\rho$ is the energy density of the standard matter and $\rho_{\text{eff}}^{(\text{DE})}$ is energy density corresponding to dark energy. Taking the negative sign of equation (24) and comparing it with equation (29) leads to the following relation for $\rho_{\text{eff}}^{(\text{DE})}$:

$$\rho_{\text{eff}}^{(\text{DE})} = \sigma + 3m^2 \left( \frac{1}{l_1} + \frac{1}{l_2} \right)^2 \left[ 1 - \frac{2 + \left( \frac{l_1 l_2}{l_1 + l_2} \right)^2 \left( \rho + \sigma + \frac{\Lambda_b}{3m^2} - \frac{\lambda_b}{\sigma} \right)}{1 + \frac{l_1 l_2}{l_1 + l_2} \left( \frac{\sigma}{3m^2} - \frac{\lambda_b}{\sigma} \right)} \right]. \quad (30)$$

Taking the time derivative of equation (24) and using the continuity equation for standard matter on the brane, that is, $\dot{\rho} + 3H (\rho + p) = 0$ (with $p = 0$), the Hubble rate can be deduced as follows:

$$\dot{H} = -\frac{\rho}{2m^2} \left( 1 - \frac{1}{\sqrt{1 + \left( \frac{l_1 l_2}{l_1 + l_2} \right)^2 \left( \rho + \sigma + \frac{\Lambda_b}{3m^2} - \frac{\lambda_b}{\sigma} \right)}} \right). \quad (31)$$

This relation shows that $\dot{H} < 0$. By computing the time derivative of equation (29) and using the fact that $\dot{H} < 0$, we find that $\rho_{\text{eff}} > 0$. Note that $\rho_{\text{eff}}^{(\text{DE})}$ satisfies the energy conservation equation

$$\rho_{\text{eff}} + 3H (1 + \omega_{\text{eff}}) \rho_{\text{eff}} = 0, \quad (32)$$

where the effective equation of state parameter of dark energy can be expressed as follows:

$$\omega_{\text{eff}} = -\frac{\Omega_m (1 + z)^3}{(E^2 - \Omega_m (1 + z)^3) \sqrt{1 + \frac{1}{A} \left( \rho_{\text{eff}} + \Omega_\sigma + \Omega_{\Lambda_b} \right)}}. \quad (33)$$

Figure 1 shows that, for both the symmetric ($\theta_1 = \theta_2 = -1$) and asymmetric ($\theta_1 = -\theta_2 = 1$) cases, the effective equation of state parameter stays in the phantom region. However, there is no smooth crossing of the phantom divide line, $\omega_{\text{eff}} = -1$, in this setup. Indeed, to have a smooth crossing of the phantom divide line, a canonical scalar field should be added to the brane action (see the first reference of [23]). It is important to note that the asymmetric effect leads to breakdown of the effective phantom-like picture in smaller redshifts relative to the phantom-like behavior on the symmetric brane. The deceleration parameter is given by $q = -(1 + \frac{\dot{E}}{H_0 E})$ where

$$\dot{E} = -\frac{\rho}{2m^2} \left( 1 - \frac{1}{\sqrt{1 + \left( \frac{l_1 l_2}{l_1 + l_2} \right)^2 \left( \rho + \sigma + \frac{\Lambda_b}{3m^2} - \frac{\lambda_b}{\sigma} \right)}} \right) \times \left( 1 - \frac{A}{A + \Omega_m (1 + z)^3 + \Omega_\sigma + \Omega_{\Lambda_b}} \right) < 0. \quad (34)$$

This relation implies that the deceleration parameter can never be less than $-1$. Consequently, there is no super-acceleration and big rip singularity in this model. Figure 2 shows the
2.3.2. The case with $\alpha = 0$ and $m = 0$. The case without induced gravity ($m = 0$) has been studied thoroughly in the literature (see for instance [15] and also [12]). Here we give a brief review of this case for completeness and, in addition, comment on the possible variation of the fundamental scales. It has been shown that for $\theta_1 \theta_2 = 1$ this model corresponds to the generalized Randall–Sundrum (RS) model (with $\theta_1 = \theta_2 = 1$) and the inverse RS model ($\theta_1 = \theta_2 = -1$). However, for $\theta_1 \theta_2 = 1$ this case cannot lead to a de Sitter phase and therefore it cannot be accounted for the late-time acceleration of the universe. For $\theta_1 \theta_2 = -1$, the Friedmann equation (23) with $m = 0$ has the following form [12]:

$$H^2 + \frac{k}{a^2} = \frac{M_1^6 \Lambda_1 - M_2^6 \Lambda_2}{6(M_1^6 - M_2^6)} + \frac{M_1^6 + M_2^6}{9(M_1^6 - M_2^6)} (\rho + \sigma)^2$$

$$\pm \frac{2M_1^6 M_2^6}{9(M_1^6 - M_2^6)} (\rho + \sigma)\frac{3}{2} (\Lambda_1 - \Lambda_2)(M_1^6 - M_2^6) \right)^{1/2}\] (35)

In a certain range of parameters, this model is cosmologically equivalent to an induced gravity scenario in the presence of mirror symmetry. When

$$\rho + \sigma \ll (\rho + \sigma)_{max} = \frac{2M_1^6 M_2^6}{(M_1^6 + M_2^6)} \sqrt{(\Lambda_1 - \Lambda_2)}$$

the Friedmann equation approximates to [12, 15]

$$H^2 + \frac{k}{a^2} \approx \frac{\rho + \sigma}{3m_b^2} + \Lambda_{eff} \frac{6}{1 + 8\alpha},$$

where

$$m_b^2 = \frac{(M_1^6 - M_2^6)^2}{M_1^6 M_2^6} \sqrt{\frac{3}{2M_1^6 - M_2^6}}(\Lambda_1 - \Lambda_2)$$

and

$$\Lambda_{eff} = \frac{M_1^6 \Lambda_1 - M_2^6 \Lambda_2}{(M_1^6 - M_2^6)}.$$

Since the effective cosmological constant in this case cannot evolve with time, it is impossible to realize a phantom-like behavior in this situation. However, if we consider some sort of evolving matter fields in the bulk (such as canonical scalar fields), which can be different on either side of the brane, it is possible essentially to realize a phantom-like behavior on the brane in this case. Also one can realize the phantom-like behavior in this case even by incorporation of the possibility to have varying fundamental constants on either sides of the brane. The importance of the possible realization of the phantom-like effect in this manner lies in the fact that this phantom-like behavior will occur in the absence of the induced gravity. If this is actually the case, it will be significant progress in this field. These conjectures are under investigation and will be reported in our future work.

2.4. The case with $\alpha \neq 0$.

2.4.1. A stringy model with induced gravity. In a stringy model with induced gravity, there is no bare cosmological constant in the bulk action ($\Lambda_i = 0$), there is no tension in the brane action ($\sigma = 0$) and the GB parameter is always positive ($\alpha_i > 0$). It is important to note that in the presence of the mirror ($Z_2$) symmetry, this case has been dubbed as the GBIG scenario and its cosmological dynamics and effective phantom-like behavior have been studied extensively in the literature (see [19] and [33]).

For the positive sign in equation (11), the bulk metric contains

$$f(a) = k + \frac{a^2}{2\alpha_i}$$

Note that this branch is not well defined at $\alpha = 0$. For the negative sign of equation (11), the only allowed value of $k$ is $k = 1$; therefore $f(a) = 1$. Since we are interested in a flat FRW universe, we take only the positive branch in our analysis. In this case, the Friedmann equation is given in the following form,

$$H^2 = \frac{\rho + \sigma}{3m_b^2} + \frac{2}{3} \sum_{i=1,2} \theta_i \left( \frac{1}{2\alpha_i} + H^2[1 + 8\alpha_i H^2] \right).$$
where \( \ell_i \) has been introduced in the previous section. One of the simplest cosmological models that can exhibit the late-time acceleration of the universe is a de Sitter spacetime. So, it is worth seeking such a solution in our model. To do this, we set \( \rho = 0 \), which is an appropriate choice for late times.

Then the Friedmann equation (38) is simplified to

\[
\lim_{z \to -1} H^2(z) = H_0^2 = \frac{2}{3} \sum_{i=1,2} \frac{\theta_i}{\ell_i} \sqrt{\frac{1}{2a_i} + H^2(1 + 8a_i H^2)}. \tag{39}
\]

This equation implies that this model can evolve to a de Sitter phase at late times for two branches with \( \theta_1 = \theta_2 = 1 \) and \( \theta_1 \theta_2 = -1 \) with \( M_1 > M_2 \) for the latter case. Note that for \( m = 0 \), as has been shown by Padilla [15], only the asymmetric branch can exhibit cosmic acceleration. Here, the effect of the induced gravity term leads to a de Sitter phase for the symmetric branch with \( \theta_1 = \theta_2 = 1 \). To investigate the phantom mimicry of this model, similarly to the previous section we define an effective cosmological constant on the brane due to extra dimensional effects as follows:

\[
\frac{\Lambda_{\text{eff}}}{3} = \frac{\sigma}{3m^2} + \frac{2}{3} \sum_{i=1,2} \frac{\theta_i}{\ell_i} \sqrt{\frac{1}{2a_i} + H^2(1 + 8a_i H^2)}. \tag{40}
\]

For a phantom accelerating phase, \( \omega(z) < -1 \), that by definition [32]

\[
\omega(z) = \frac{2q(z) - 1}{3[1 - \Omega_m(z)]}, \quad q(z) = \frac{d \log H(z)}{d \log(1 + z)} - 1, \tag{41}
\]

\[
\Omega_m(z) = \frac{\Omega_m(1 + z)^3}{E^2(z)}.
\]

This condition is equivalent to \( \Omega_m(z) > \frac{2}{3} \frac{d \log H(z)}{d \log(1 + z)} \) and \( \Lambda_{\text{eff}} > 0 \), where the dot marks differentiation with respect to the cosmic time. The expansion rate \( H \) is given by

\[
H = \frac{\rho}{2m^2} \left[ 1 - \frac{1}{3} \sum_{i=1,2} \frac{\theta_i}{\ell_i} \left( \sqrt{\frac{1}{2a_i} + H^2} \right)^{-1} \right]. \tag{42}
\]

For

\[
\sum_{i=1,2} \frac{\theta_i}{\ell_i} \left( \sqrt{\frac{1}{2a_i} + H^2} \right) < 0,
\]

we find that \( H < 0 \). Since the deceleration parameter is related to \( H \) via the relation \( q = (-1 + \frac{H}{H'}) \), with \( H < 0 \) we find that \( q > -1 \). As a result, there is no superacceleration in this case. Since \( H < 0 \), the effective cosmological constant increases with time:

\[
\dot{\Lambda}_{\text{eff}} = \frac{2}{3} H \dot{H} \sum_{i=1,2} \frac{\theta_i}{\ell_i} \left( \sqrt{\frac{1}{2a_i} + H^2} \right) > 0. \tag{44}
\]

On the other hand, \( \omega_{\text{eff}} \) can be obtained by virtue of the continuity equation

\[
\omega_{\text{eff}} = - \left( 1 + \frac{\dot{\Lambda}_{\text{eff}}}{\Lambda_{\text{eff}}} \right) < -1. \tag{45}
\]

Note that the phantom-like behavior can be realized in this case for \( \theta_1 = \theta_2 = -1 \) and also for one of the mixed branches (i.e. \( \theta_1 \theta_2 = -1 \)).

2.4.2. The general case with \( \Lambda_i \neq 0 \) and \( \sigma \neq 0 \). Now, we consider a more general case. For a spatially flat universe without the dark radiation term, the solutions for the bulk metric are given by

\[
f_i(a) = \frac{a^2}{4a_i} \left( 1 \mp \sqrt{1 + \frac{4}{3} a_i \Lambda_i} \right). \tag{46}
\]

Then, the Friedmann equation (20) can be rewritten in the following form:

\[
H^2 = \frac{\rho + \sigma}{3m^2} + \frac{2}{3} \sum_{i=1,2} \frac{\theta_i}{\ell_i} \sqrt{1 + 4a_i H^2} \mp \sqrt{1 + \frac{4}{3} a_i \Lambda_i} \times \left( 1 + 4a_i H^2 \mp \frac{1}{2} \sqrt{1 + \frac{4}{3} a_i \Lambda_i} \right). \tag{47}
\]

There are some constraints imposed on this equation. One of these constraints is \( \Lambda_i < -\frac{1}{3} a_i \). Note that since the cosmological constant in the bulk is negative, this constraint is satisfied naturally for positive \( a_i \)’s. On the other hand, by choosing the negative sign in relation (46), the square root of equation (47) should be positive. Therefore, we find that

\[
1 + 4a_i H^2 - \sqrt{1 + \frac{4}{3} a_i \Lambda_i} > 0. \tag{48}
\]

It can be checked easily that this condition is satisfied too. Now, by adopting a strategy much similar to the procedure that has led us to equation (26), the effective cosmological constant in this setup can be defined as follows,

\[
\frac{\Lambda_{\text{eff}}}{3} = \frac{\sigma}{3m^2} + \frac{2}{3} \sum_{i=1,2} \frac{\theta_i}{\ell_i} \sqrt{1 + 4a_i H^2} \mp \sqrt{1 + \frac{4}{3} a_i \Lambda_i} \times \left( 1 + 4a_i H^2 \mp \frac{1}{2} \sqrt{1 + \frac{4}{3} a_i \Lambda_i} \right). \tag{49}
\]

In an analogous manner to the previous subsections, the conditions for realization of the phantom-like behavior are given as follows:

\[
H = -\frac{\rho}{2m^2} \left[ 1 - \frac{8}{3} \sum_{i=1,2} \frac{\theta_i \sqrt{a_i}}{l_i} \sqrt{\frac{1 + 4a_i H^2 + \frac{2}{3} \sqrt{1 + \frac{4}{3} a_i \Lambda_i}}{1 + 4a_i H^2 + \sqrt{1 + \frac{4}{3} a_i \Lambda_i}}} \right]^{-1} < 0 \tag{50}
\]

and

\[
\dot{\Lambda}_{\text{eff}} = \frac{16}{3} H \dot{H} \sum_{i=1,2} \frac{\theta_i \sqrt{a_i}}{l_i} \frac{1 + 4a_i H^2 + \frac{2}{3} \sqrt{1 + \frac{4}{3} a_i \Lambda_i}}{1 + 4a_i H^2 + \sqrt{1 + \frac{4}{3} a_i \Lambda_i}} > 0 \tag{51}
\]
if the condition
\[
\sum_{i=1,2} \theta_i \sqrt{\alpha_i} \left( \frac{1 + 4\alpha_i H^2}{1 + 4\alpha_i H^2 \mp \sqrt{1 + \frac{4}{3} \alpha_i \Lambda_i}} \right) < 0 \quad (52)
\]
is satisfied. As an important result, in this general case the phantom-like behavior can be realized with both signs of equation (47) in the symmetric case (with \(\theta_1 = \theta_2 = -1\)). This can happen also in one of the asymmetric branches (with \(\theta_1 \theta_2 = -1\)) without introducing a phantom field in the brane or bulk action.

3. Summary and conclusion

In this paper, we have assumed a braneworld model with induced gravity whose bulk action includes, in addition to the familiar Einstein term, a GB contribution. We have relaxed the mirror symmetry of the embedding, so the gravitational and cosmological constants and even the GB parameter can be different on either side of the brane. We have derived the bulk solutions, which in general have the Schwarzschild–anti–de Sitter form. Using the generalized junction condition, we have derived the effective Einstein equation in the bulk and the brane. The absence of mirror symmetry and the presence of the curvature effects due to the GB correction term lead to a complicated Friedmann equation for the cosmological dynamics on the brane. To gain some intuition on the cosmological dynamics, we have considered the scenario in some special cases. Firstly, we considered the case of a pure induced gravity scenario (in the absence of the GB contribution in the bulk). We have shown that in this case, an effective cosmological constant can be defined on the brane that is screened, but the screening term reduces in time for two branches of the scenario so that the value of the effective cosmological constant increases with cosmic time. Thus the phantom-like behavior can be realized in smaller redshifts than the symmetric case.

This pure induced gravity scenario, the phantom-like behavior can be realized in two branches of the scenario so that the value of the effective cosmological constant can be defined on the brane (this branch corresponds to the Brane\(^1\) solution of [32]) and the other originates from pure asymmetric effects (with \(\theta_1 \theta_2 = -1\)). We have shown that, in the asymmetric case of this pure induced gravity scenario, the phantom-like behavior can be realized in smaller redshifts than the symmetric case. Secondly, we have also considered the general case with GB curvature correction and shown that it is possible to realize the phantom-like behavior in this case by justifying some conditions on the field equations of the scenario.

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