New concise upper bounds on quantum violation of general multipartite Bell inequalities

Elena R. Loubenets

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Last years, bounds on the maximal quantum violation of general Bell inequalities were intensively discussed in the literature via different mathematical tools. In the present paper, we analyze quantum violation of general Bell inequalities via the LqHV (local quasi hidden variable) modelling framework, correctly reproducing the probabilistic description of every quantum correlation scenario. The LqHV mathematical framework allows us to derive for all $d$ and $N$ a new upper bound $(2d - 1)^{N-1}$ on the maximal violation by an $N$-qudit state of all general Bell inequalities, also, new upper bounds on the maximal violation by an $N$-qudit state of general Bell inequalities for $S$ settings per site. These new upper bounds essentially improve all the known precise upper bounds on quantum violation of general multipartite Bell inequalities. For some $S$, $d$ and $N$, the new upper bounds are attainable.
I. INTRODUCTION

Quantum violation of Bell inequalities is now used in many quantum information tasks and is also important for the analysis of nonlocal games strategies in computer science. The most analytically studied cases of quantum violation of specific Bell inequalities refer to the Clauser-Horne-Shimony-Holt (CHSH) inequality and the Mermin-Klyshko inequality. It is also well known that the maximal quantum violation of correlation bipartite Bell inequalities cannot exceed the real Grothendiek’s constant $K^\mathbb{R}_G \in [1.676, 1.783]$ independently of a dimension of a bipartite quantum state and numbers of settings and outcomes per site. But this is not already the case for quantum violation of bipartite Bell inequalities on joint probabilities and last years bounds on the maximal quantum violation of Bell inequalities were intensively discussed in the literature via different mathematical tools.

To our knowledge, the maximal violation by an $N$-qudit quantum state of general Bell inequalities for arbitrary numbers of measurement settings and outcomes at each site admits the following upper bounds.

- $N = 2$: (a) for an arbitrary two-qudit state – the precise upper bound $(2d - 1)$ in Eq. (64) of Ref. 9 and the precise upper bound $2d$ in Proposition 5.2 of Ref. 14; (b) for the two-qudit Greenberger-Horne-Zeilinger (GHZ) state – the upper bound $Cd/\sqrt{\ln d}$, found up to a universal constant in Theorem 0.3 of Ref. 11.

- $N \geq 3$: (c) for the $N$-qudit GHZ state – the precise upper bound $(2^{N-1}(d - 1) + 1)$ in Eq. (58) of Ref. 9; (d) for an arbitrary $N$-qudit state – the precise upper bound $(2^{N-1}d^{N-1} - 2^{N-1} + 1)$ in Eq. (62) of Ref. 9 and the precise upper bound $(2d)^{N-1}$ in comments after Proposition 5.2 in Ref. 14.

In the present paper, we analyze the maximal quantum violation of general Bell inequalities via the LqHV (local quasi hidden variable) modelling framework, introduced and developed in Refs. 9, 18, 19. A general correlation scenario admits a LqHV model if and only if it is nonsignaling. Therefore, the probabilistic description of each quantum correlation scenario admits the LqHV modelling. Moreover, the probabilistic description of all projective $N$-partite joint quantum measurements on an $N$-qudit state can be reproduced via the single LqHV model specified in Ref. 13.
The LqHV mathematical framework allows us to derive a new precise upper bound

\[(2d - 1)^{N-1}\]  

on the maximal violation by an arbitrary N-qudit state of general Bell inequalities for arbitrary numbers of settings and outcomes per site. For \(N = 2\), this new bound reduces to our upper bound (64) in Ref. 9. For all \(N \geq 3\), the new bound (1) essentially improves all the known precise upper bounds for general multipartite Bell inequalities, see in item (d) above.

For the maximal quantum violation of general Bell inequalities for \(S\) settings per site, the new upper bound (1) allows us also to improve due to

\[(2 \min\{d, S\} - 1)^{N-1}\]  

the precise upper bound (62) in Ref. 9 for generalized \(N\)-partite joint quantum measurements and due to

\[\min\{d^{\frac{N-1}{2}}, 3^{N-1}\}, \quad \text{for } S = 2,\]  
\[\min\{d^{\frac{S(N-1)}{2}}, (2 \min\{d, S\} - 1)^{N-1}\}, \quad \text{for } S \geq 3,\]  

the precise upper bound (19) in Ref. 13 for projective \(N\)-partite joint quantum measurements. For some \(d, S\) and \(N\), the upper bounds (2), (3) are attainable, see in Section VI.

The main results of the present paper are formulated by theorem 1 and corollary 1 in Section V.

II. PRELIMINARIES: GENERAL BELL INEQUALITIES

In this section, we shortly recall the notion of a general Bell inequality. The general framework for multipartite Bell inequalities for an arbitrary number of measurement settings and any spectral type of outcomes at each site was introduced in Ref. 22 where specific examples of Bell inequalities are discussed in section 3.

Consider an \(N\)-partite correlation scenario, where each \(n\)-th of \(N \geq 2\) parties performs \(S_n \geq 1\) measurements with outcomes \(\lambda_n \in \Lambda_n\) of any nature and an arbitrary spectral
type. We label each measurement at $n$-th site by a positive integer $s_n = 1, \ldots, S_n$ and each $N$-partite joint measurement, induced by this correlation scenario and with outcomes

$$(\lambda_1, \ldots, \lambda_N) \in \Lambda = \Lambda_1 \times \cdots \times \Lambda_N$$

(4)

by an $N$-tuple $(s_1, \ldots, s_N)$, where $n$-th component specifies a measurement at $n$-th site. For concreteness, we denote by $E_{S, \Lambda}$, $S = S_1 \times \cdots \times S_N$, an $S_1 \times \cdots \times S_N$-setting correlation scenario with outcomes in $\Lambda$ and by $P^{(E_{S, \Lambda})}_{(s_1, \ldots, s_N)}$ a joint probability distribution of outcomes $(\lambda_1, \ldots, \lambda_N) \in \Lambda$ for an $N$-partite joint measurement $(s_1, \ldots, s_N)$ under a scenario $E_{S, \Lambda}$.

An $N$-partite correlation scenario $E_{S, \Lambda}$ is referred to as nonsignaling if, for any two joint measurements $(s_1, \ldots, s_N)$ and $(s_1', \ldots, s_N')$ with common settings $s_{n_1}, \ldots, s_{n_M}$ at some $1 \leq n_1 < \ldots < n_M \leq N$ sites, the marginal probability distributions of distributions $P^{(E_{S, \Lambda})}_{(s_1, \ldots, s_N)}$ and $P^{(E_{S, \Lambda})}_{(s_1', \ldots, s_N')}$, describing measurements at sites $1 \leq n_1 < \ldots < n_M \leq N$, coincide. For details, see section 3 in Ref. 24.

For a correlation scenario $E_{S, \Lambda}$, consider a linear combination

$$B^{(E_{S, \Lambda})}_{\Phi_{S, \Lambda}} = \sum_{s_1, \ldots, s_N} \langle f_{(s_1, \ldots, s_N)}(\lambda_1, \ldots, \lambda_N) \rangle^{(E_{S, \Lambda})}_{E_{S, \Lambda}},$$

(5)

$$\Phi_{S, \Lambda} = \{ f_{(s_1, \ldots, s_N)} : \Lambda \to \mathbb{R} \mid s_n = 1, \ldots, S_n, \ n = 1, \ldots, N \},$$

of averages (expectations) of the most general form

$$\langle f_{(s_1, \ldots, s_N)}(\lambda_1, \ldots, \lambda_N) \rangle^{(E_{S, \Lambda})}_{E_{S, \Lambda}}$$

(6)

$$= \int_{\Lambda} f_{(s_1, \ldots, s_N)}(\lambda_1, \ldots, \lambda_N) P^{(E_{S, \Lambda})}_{(s_1, \ldots, s_N)}(d\lambda_1 \times \cdots \times d\lambda_N),$$

specified for each joint measurement $(s_1, \ldots, s_N)$ by a bounded real-valued function $f_{(s_1, \ldots, s_N)}(\cdot)$ of outcomes $(\lambda_1, \ldots, \lambda_N) \in \Lambda$ at all $N$ sites.

Depending on a choice of a function $f_{(s_1, \ldots, s_N)}$ for a joint measurement $(s_1, \ldots, s_N)$, an average (6) may refer either to the joint probability of events observed at $M \leq N$ sites or, in case of real-valued outcomes, for example, to the expectation

$$\langle \lambda_1^{(s_1)} \cdots \lambda_{n_M}^{(s_{n_M})} \rangle^{(E_{S, \Lambda})}_{E_{S, \Lambda}} = \int_{\Lambda} \lambda_1 \cdots \lambda_{n_M} P^{(E_{S, \Lambda})}_{(s_1, \ldots, s_N)}(d\lambda_1 \times \cdots \times d\lambda_N)$$

(7)

of the product of outcomes observed at $M \leq N$ sites or may have a more complicated form. In quantum information, the product expectation (7) is referred to as a correlation function. For $M = N$, a correlation function is called full.
The probabilistic description of an arbitrary correlation scenario \( \mathcal{E}_{S,A} \) admits a LHV 
(local hidden variable) model if all its joint probability distributions

\[
\left\{ P_{(s_1, ..., s_N)}^{(\mathcal{E}_{S,A})}, \ s_1 = 1, ..., S_n, ..., s_N = 1, ..., S_N \right\}
\]

admit the representation

\[
P_{(s_1, ..., s_N)}^{(\mathcal{E}_{S,A})} (d\lambda_1 \times \cdots \times d\lambda_N)
= \int_{\Omega} P_{1,s_1} (d\lambda_1 | \omega) \cdot \cdots \cdot P_{N,s_N} (d\lambda_N | \omega) \nu_{\mathcal{E}_{S,A}} (d\omega)
\]

via a single probability distribution \( \nu_{\mathcal{E}_{S,A}} (d\omega) \) of some variables \( \omega \in \Omega \) and conditional probability distributions \( P_{n,s_n} (\cdot | \omega) \), referred to as "local" in the sense that each \( P_{n,s_n} (\cdot | \omega) \) at \( n \)-th site depends only on the corresponding measurement \( s_n = 1, ..., S_n \) at this site.

Let a correlation scenario \( \mathcal{E}_{S,A} \) admit an LHV model. Then a linear combination \((5)\) of its averages \((6)\) satisfies the tight LHV constraint (see Theorem 1 in Ref. 22):

\[
B_{\Phi_{S,A}}^{\inf} \leq B_{\Phi_{S,A}}^{\mathcal{E}_{S,A}}|_{lhv} \leq B_{\Phi_{S,A}}^{\sup}
\]

with the LHV constants

\[
B_{\Phi_{S,A}}^{\sup} = \sup_{\lambda^{(s_n)}_n \in \Lambda_n, \forall s_n, \forall n} \sum_{s_1, ..., s_N} f_{(s_1, ..., s_N)} (\lambda^{(s_1)}_1, \ldots, \lambda^{(s_N)}_N),
\]

\[
B_{\Phi_{S,A}}^{\inf} = \inf_{\lambda^{(s_n)}_n \in \Lambda_n, \forall s_n, \forall n} \sum_{s_1, ..., s_N} f_{(s_1, ..., s_N)} (\lambda^{(s_1)}_1, \ldots, \lambda^{(s_N)}_N).
\]

From \((10)\), it follows that, in the LHV case,

\[
\left| B_{\Phi_{S,A}}^{\mathcal{E}_{S,A}}|_{lhv} \right| \leq B_{\Phi_{S,A}}^{lhv} = \max \left\{ B_{\Phi_{S,A}}^{\sup}, \left| B_{\Phi_{S,A}}^{\inf} \right| \right\}.
\]

Note that some of the LHV inequalities in \((10)\) may be fulfilled for a wider (than LHV) class of correlation scenarios. This is, for example, the case for the LHV constraints on joint probabilities following explicitly from nonsignaling of probability distributions. Moreover, some of the LHV inequalities in \((10)\) may be simply trivial, i.e. fulfilled for all correlation scenarios, not necessarily nonsignaling.

Each of the tight LHV inequalities in \((10)\) that may be violated under a non-LHV scenario is referred to as a Bell (or Bell-type) inequality.
III. QUANTUM VIOLATION

Let, under a correlation scenario with $S_n$ measurement settings and outcomes $\lambda_n \in \Lambda_n$ at each $n$-th site, every $N$-partite joint measurement $(s_1, ..., s_N)$ be performed on a quantum state $\rho$ on a complex Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and be described by the joint probability distribution

$$\text{tr}[\rho \{M_{1,s_1}(d\lambda_1) \otimes \cdots \otimes M_{N,s_n}(d\lambda_N)\}], \quad (13)$$

where each $M_{n,s_n}(d\lambda_n)$ is a normalized positive operator-valued (POV) measure, representing on a complex Hilbert space $\mathcal{H}_n$ a quantum measurement $s_n$ at $n$-th site. For a POV measure $M_{n,s_n}$, all its values $M_{n,s_n}(F_n)$, $F_n \subseteq \Lambda_n$, are positive operators on $\mathcal{H}_n$ and $M_{n,s_n}(\Lambda_n) = \mathbb{1}_{\mathcal{H}_n}$. For concreteness, we specify this $S_1 \times \cdots \times S_N$-setting quantum correlation scenario by symbol $\mathcal{E}_{M_{S,\Lambda}}^{(\rho)}$ where

$$M_{S,\Lambda} : = \{M_{n,s_n}, \ s_n = 1, ..., S_n, \ n = 1, ..., N\}, \quad (14)$$

$$S = S_1 \times \cdots \times S_N, \quad \Lambda = \Lambda_1 \times \cdots \times \Lambda_N,$$

is a collection of POV measures (13) at all $N$-sites.

As it is well known, the probabilistic description of a quantum correlation scenario $\mathcal{E}_{M_{S,\Lambda}}^{(\rho)}$ does not need to admit a LHV model. Therefore, under correlation scenarios on an $N$-partite quantum state $\rho$, Bell inequalities (10) may be violated and the parameter $\Upsilon_{S_1 \times \cdots \times S_N}^{(\rho,\Lambda)}$ gives the maximal violation of general $S_1 \times \cdots \times S_N$-setting Bell inequalities for an arbitrary outcome set $\Lambda_n$ at each $n$-th site while the parameter $\Upsilon_{\rho}^{(S_1 \times \cdots \times S_N)}$ – the maximal violation of all general Bell inequalities.
IV. ANALYTICAL UPPER BOUND

We recall that, for every state \( \rho \) on a complex Hilbert space \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \) and arbitrary positive integers \( S_1, \ldots, S_N \geq 1 \), there exists an \( S_1 \times \cdots \times S_N \)-setting source operator \( T_{S_1 \times \cdots \times S_N}^{(\rho)} \) – a self-adjoint trace class operator on the space

\[
(\mathcal{H}_1)^{\otimes S_1} \otimes \cdots \otimes (\mathcal{H}_N)^{\otimes S_N},
\]

satisfying the relation

\[
\text{tr} \left[ T_{S_1 \times \cdots \times S_N}^{(\rho)} \left( \mathbb{I}_{\mathcal{H}_1^{\otimes k_1}} \otimes X_1 \otimes \mathbb{I}_{\mathcal{H}_2^{(s_1-1-k_1)}} \otimes \cdots \otimes \mathbb{I}_{\mathcal{H}_N^{\otimes k_N}} \otimes X_N \otimes \mathbb{I}_{\mathcal{H}_1^{(s_N-1-k_N)}} \right) \right] = \text{tr} \left[ \rho \left( X_1 \otimes \cdots \otimes X_N \right) \right],
\]

\[ k_1 = 0, \ldots, (S_1 - 1), \ldots, k_N = 0, \ldots, (S_N - 1), \]

for all bounded linear operators \( X_1, \ldots, X_N \) on Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_N \), respectively. Here, we set \( \mathbb{I}_{\mathcal{H}_n^{\otimes k}} \otimes X_n \mid_{k=0} = X_n \otimes \mathbb{I}_{\mathcal{H}_n^{\otimes k}} \mid_{k=0} = X_n \).

Due to its definition, an \( S_1 \times \cdots \times S_N \)-setting source operator \( T_{S_1 \times \cdots \times S_N}^{(\rho)} \) constitutes a self-adjoint trace class dilation of a state \( \rho \) to the complex Hilbert space (18). Clearly, \( T_{1 \times \cdots \times 1}^{(\rho)} \equiv \rho \) and \( \text{tr}[T_{S_1 \times \cdots \times S_N}^{(\rho)}] = 1 \).

The analytical bound (53) in Theorem 3 of Ref. 9, derived via the LqHV modeling framework\(^{9,18} \), implies the following statement.

**Proposition 1** Under all generalized \( N \)-partite joint quantum measurements, the maximal violation (16) by a state \( \rho \) of general \( S_1 \times \cdots \times S_N \)-setting Bell inequalities satisfies the relation

\[
1 \leq \Upsilon_{S_1 \times \cdots \times S_N}^{(\rho)} \leq \inf_{T_{S_1 \times \cdots \times S_N}^{(\rho)}} \left\| T_{S_1 \times \cdots \times S_N}^{(\rho)} \right\|_{\text{cov}}, \forall n \]

where: (i) infimum is taken over all source operators \( T_{S_1 \times \cdots \times S_N}^{(\rho)} \) with only one setting at some \( n \)-th site and over all sites \( n = 1, \ldots, N \); (ii) notation \( \left\| \cdot \right\|_{\text{cov}} \) means the covering norm – a new norm introduced for self-adjoint trace class operators by relation (11) in Ref. 9.

By Lemma 1 in Ref. 9, for every self-adjoint trace class operator \( W \) on a tensor product Hilbert space \( \mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m \), its covering norm \( \left\| W \right\|_{\text{cov}} \) satisfies the relation

\[
\left| \text{tr}[W] \right| \leq \left\| W \right\|_{\text{cov}} \leq \left\| W \right\|_1 ,
\]

7
where $\|\cdot\|_1$ is the trace norm and the equality $\|W\|_{\text{cov}} = |\text{tr}[W]|$ is true if a self-adjoint trace class operator $W$ is tensor positive, that is, satisfies the relation $^{20}$

$$\text{tr}[W\{X_1 \otimes \cdots \otimes X_m\}] \geq 0$$  \hspace{1cm} (22)

for all positive bounded operators $X_j$ on $G_j$, $j = 1, \ldots, m$. Every positive trace class operator is tensor positive but not vice versa. For example, the permutation (flip) operator $V_d(\psi_1 \otimes \psi_2) := \psi_2 \otimes \psi_1, \psi_2 \in \mathbb{C}^d$, on $\mathbb{C}^d \otimes \mathbb{C}^d$ is tensor positive but is not positive. Its trace norm is $\|V_d\|_1 = d^2$ while the covering norm $\|V_d\|_{\text{cov}} = d$.

For every source operator $T^{(p)}_{S_1 \times \cdots \times S_N}$, its trace $\text{tr}[T^{(p)}_{S_1 \times \cdots \times S_N}] = 1$, so that, by (21), $\|T^{(p)}_{S_1 \times \cdots \times S_N}\|_{\text{cov}} \geq 1$ and is equal to one: $\|T^{(p)}_{S_1 \times \cdots \times S_N}\|_{\text{cov}} = 1$ if a source operator $T^{(p)}_{S_1 \times \cdots \times S_N}$ is tensor positive.

This and relation (20) imply that if, for an $N$-partite state $\rho$, tensor positive source operators $T^{(p)}_{S_1 \times \cdots \times 1 \times \cdots \times S_N}, n = 1, \ldots, N$, exist for all integers $S_1, \ldots, S_N \geq 1$, then the maximal violation (17) by an $N$-partite state $\rho$ of general Bell inequalities for arbitrary numbers of settings and any spectral type of outcomes at each site is equal to one: $\Upsilon_{\rho} = 1$, so that this $N$-partite quantum state $\rho$ is local in the sense that it satisfies all general Bell inequalities.

Examples of nonseparable $N$-partite quantum states that have tensor positive source operators $T^{(p)}_{S_1 \times \cdots \times 1 \times \cdots \times S_N}, n = 1, \ldots, N$, for all integers $S_1, \ldots, S_N \geq 1$ (and are, therefore, local) are presented in Ref. 30.

V. NEW NUMERICAL UPPER BOUNDS

Let $\rho_{d,N}$ be an arbitrary $N$-qudit quantum state on $\mathcal{H}^{\otimes N}$, where dim $\mathcal{H} = d < \infty$. In order to evaluate via the analytical bound (21) the maximal violation $\Upsilon^{(p_{d,N})}_{S_1 \times \cdots \times S_N}$ by an $N$-qudit state $\rho_{d,N}$ of all general $S_1 \times \cdots \times S_N$-setting Bell inequalities (10), we need to present at least one source operator $T^{(p_{d,N})}_{S_1 \times \cdots \times S_N}$. We first consider the case of a pure state and then, by convexity, extend our result to an arbitrary $\rho_{d,N}$.

A pure $N$-qudit state $|\psi_{d,N}\rangle\langle\psi_{d,N}|$ admits the decomposition

$$|\psi_{d,N}\rangle\langle\psi_{d,N}| = \sum \varsigma_{m_j \cdots k} s_{m_j m_{j_1} \cdots k_1} |e^{(1)}_m\rangle\langle e^{(1)}_m| \otimes |e^{(2)}_j\rangle\langle e^{(2)}_j| \otimes \cdots \otimes |e^{(N)}_k\rangle\langle e^{(N)}_k|,$$  \hspace{1cm} (23)

where $\sum_{m,j,...,k} |\varsigma_{m_j \cdots k}|^2 = 1$ and $\{e^{(n)}_m \in \mathcal{H}, m = 1, \ldots, d\}, n = 1, \ldots, N$, are orthonormal bases.
in $\mathcal{H}$. Introducing the normalized vectors

$$
\phi_{j...k} = \frac{1}{\beta_{j...k}} \sum_m \zeta_{m...k} e_m^{(1)} \in \mathcal{H}, \quad \|\phi_{j...k}\| = 1,
$$

$$
\beta_{j...k} = \left( \sum_m |\zeta_{m...k}|^2 \right)^{1/2}, \quad \sum_{j,...,k_{N-1}} \beta_{j...k}^2 = 1,
$$

we rewrite (23) in the form

$$
|\psi_{d,N}\rangle \langle \psi_{d,N}| = \sum \beta_{j...k} \beta_{j_1...k_1} |\phi_{j...k}\rangle \langle \phi_{j_1...k_1}| \otimes |e_j^{(2)}\rangle \langle e_{j_1}^{(2)}| \otimes \cdots \otimes |e_k^{(N)}\rangle \langle e_{k_1}^{(N)}|.
$$

In view of decomposition (25), let us introduce on the Hilbert space

$$
\mathcal{H} \otimes \mathcal{H}^{\otimes S_2} \otimes \cdots \otimes \mathcal{H}^{\otimes S_N}
$$

the self-adjoint operator

$$
T_{1\times S_2\times \cdots \times S_N}^{(\psi_{d,N})} = \sum \beta_{j...k} \beta_{j_1...k_1} |\phi_{j...k}\rangle \langle \phi_{j_1...k_1}| \otimes W_{j_1j}^{(2,S_2)} \otimes \cdots \otimes W_{k_1k}^{(N,S_N)},
$$

where

$$
2W_{j_1j}^{(n,S_n)} |j \neq j_1 = \frac{\langle e_j^{(n)}| (e_j^{(n)} + e_{j_1}^{(n)}) \rangle \otimes_{S_n} |e_{j_1}^{(n)}\rangle \langle e_j^{(n)} + e_{j_1}^{(n)}| - \langle e_j^{(n)} - e_{j_1}^{(n)}| (e_j^{(n)} - e_{j_1}^{(n)}) \rangle \otimes_{S_n} |e_{j_1}^{(n)}\rangle \langle e_j^{(n)} - e_{j_1}^{(n)}|}{2^{S_n}}
$$

$$
+ i \frac{\langle e_j^{(n)}| (i e_{j_1}^{(n)} + e_j^{(n)}) \rangle \otimes_{S_n} |e_j^{(n)} + i e_{j_1}^{(n)}| - \langle e_j^{(n)} - i e_{j_1}^{(n)}| (i e_{j_1}^{(n)} + e_j^{(n)}) \rangle \otimes_{S_n} |e_{j_1}^{(n)}\rangle \langle e_j^{(n)} + i e_{j_1}^{(n)}|}{2^{S_n}}
$$

are operators on $\mathcal{H}^{\otimes S_n}$ invariant with respect to permutations of spaces $\mathcal{H}$ in $\mathcal{H}^{\otimes S_n}$ and satisfying the relations

$$
\left( W_{j_1j}^{(n,S_n)} \right)^* = W_{j_1j}^{(n,S_n)}, \quad \text{tr}_{\mathcal{H}^{\otimes (S_n-1)}} \left[ W_{j_1j}^{(n,S_n)} \right] = |e_j^{(n)}\rangle \langle e_{j_1}^{(n)}|.
$$

It is easy to verify that the partial trace

$$
\text{tr}_{\mathcal{H}^{\otimes (S_n-1)}} \left[ T_{1\times S_2\times \cdots \times S_N}^{(\psi_{d,N})} \right] = |\psi_{d,N}\rangle \langle \psi_{d,N}|.
$$

Therefore, the self-adjoint operator $T_{1\times S_2\times \cdots \times S_N}^{(\psi_{d,N})}$ constitutes a $1 \times S_2 \times \cdots \times S_N$-setting source operator for a pure state $|\psi_{d,N}\rangle \langle \psi_{d,N}|$. 

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Evaluating due to relation (21) the covering norm of the source operator (27)

\[
\left\| T_{1 \times S_2 \times \ldots \times S_N}^{(\psi_{d,N})} \right\|_{\text{cov}} \leq 1 + \sum_{\substack{(j,\ldots,k) \\ \neq (j_1,\ldots,k_1)}} \beta_{j\ldots k}\beta_{j_1\ldots k_1} \left\{ \delta_{j_1} + 2(1 - \delta_{j_1}) \right\} \times \cdots \times \left\{ \delta_{k_1} + 2(1 - \delta_{k_1}) \right\}
\]

(31)

and taking into account that \( \sum_{j,\ldots,k} \beta_{j\ldots k}^2 = 1 \) and

\[
\sum_{\substack{j,\ldots,l, r \\ \neq r_1,\ldots,k \neq k_1, m}} \beta_{j\ldots kl}\beta_{j_1\ldots k_1} \leq \frac{1}{2} \left( \beta_{j\ldots kl}^2 + \beta_{j_1\ldots k_1}^2 \right) \leq (d - 1)^m;
\]

(32)

also, similar relations for \( m \) non-equal pairs of indices standing at arbitrary places in the sum in (31), we derive

\[
\left\| T_{1 \times S_2 \times \ldots \times S_N}^{(\psi_{d,N})} \right\|_{\text{cov}} \leq \sum_{m=0}^{N-1} \binom{N-1}{m} 2^m (d - 1)^m = (2d - 1)^{N-1},
\]

(33)

where \( \binom{N-1}{m} \) are the binomial coefficients.

From (20), (33) it follows that

\[
\Upsilon_{S_1 \times \ldots \times S_N}^{(\psi_{d,N})} \leq (2d - 1)^{N-1}
\]

(34)

for arbitrary numbers \( S_1,\ldots,S_N \) of settings at all \( N \) sites. By convexity, this upper bound is extended to an arbitrary state \( \rho_{d,N} \).

Taking into account (17) and incorporating also the upper bound Eq. (58) of Ref. 9 on the maximal violation of general Bell inequalities by the \( N \)-qudit GHZ state

\[
\frac{1}{\sqrt{d}} \sum_{m=1}^{d} |e_m\rangle \otimes N,
\]

(35)

we have.

**Theorem 1** For an arbitrary \( N \)-qudit state \( \rho_{d,N} \), the maximal violation \( \Upsilon_{\rho_{d,N}} \) of general Bell inequalities for arbitrary numbers of settings and outcomes at each site admits the upper bound

\[
\Upsilon_{\rho_{d,N}} \leq (2d - 1)^{N-1}
\]

(36)
under all generalized $N$-partite joint quantum measurements. For the $N$-qudit GHZ state \( |\psi_{GHZ,d,N}\rangle \), the maximal violation of general Bell inequalities is upper bounded by

\[
\Upsilon_{\rho_{\text{ghz},d,N}} \leq 2^{N-1}(d-1) + 1.
\]

(37)

Due to the new upper bound (36), the upper bound (62) in Ref. 9 for generalized quantum measurements and the upper bound (19) in Ref. 13 for projective parties’ measurements, we have the following corollary of Theorem 1.

**Corollary 1** For an arbitrary $N$-qudit state $\rho_{d,N}$, the maximal violation $\Upsilon_{\rho_{d,N}}^{S_1 \times \cdots \times S_N}$ of general Bell inequalities for $S$ settings and an arbitrary number of outcomes at each site satisfies the relation

\[
\Upsilon_{\rho_{d,N}}^{(\rho_{d,N})} \leq (2 \min\{d,S\} - 1)^{N-1}
\]

under all generalized $N$-partite joint quantum measurements and the relation

\[
\Upsilon_{2 \times \cdots \times 2}^{(\rho_{d,N})} \leq \min\{d^{\frac{N-1}{2}}, 3^{N-1}\}, \quad \text{for } S = 2,
\]

\[
\Upsilon_{S \times \cdots \times S}^{(\rho_{d,N})} \leq \min\{d^{\frac{S(N-1)}{2}}, (2 \min\{d,S\} - 1)^{N-1}\}, \quad \text{for } S \geq 3
\]

under projective $N$-partite joint quantum measurements.

**VI. DISCUSSION**

For $N = d = S = 2$, the upper bound in (39) gives $\sqrt{2}$ and, in view of the Cirel’son bound\(^2\), is attained on the CHSH inequality.

For $d = S = 2$, $N \geq 3$, the upper bound in (39) is equal to $2^{\frac{N-1}{2}}$ and, due to the results in Refs. 4, 5, it is attained on the Mermin–Klyshko inequality. The latter implies that, for projective $N$-partite joint quantum measurements, the Mermin-Klyshko inequality gives the maximal quantum violation not only among all Bell inequalities on full correlation functions (as it was proved in Ref. 4) but also among all general $N$-partite Bell inequalities for two settings and two outcomes per site.

**Concerning the attainability of the term $3^{N-1}$ in the upper bound $\min\{d^{\frac{N-1}{2}}, 3^{N-1}\}$ in (39).** From Eq. (48) in Ref. 9 and relation $\Upsilon_{S_1 \times \cdots \times S_N}^{(\rho, A)} \leq \Upsilon_{S_1 \times \cdots \times S_N}^{(\rho)}$ for violation parameters
in (15), (16), it follows that, for quantum correlation scenarios $E_{MSA}^{(\rho)}$ on an $N$-partite state $\rho$, the quantum analogs of $S_1 \times \cdots \times S_N$-setting Bell inequalities (10) admit the bounds:

$$B_{\Phi_{SA}}^{\inf} - \frac{\Upsilon^{(\rho)}_{S_1 \times \cdots \times S_N} - 1}{2} (B_{\Phi_{SA}}^{\sup} - B_{\Phi_{SA}}^{\inf})$$

$$\leq B_{\Phi_{SA}}^{(E_{MSA}^{(\rho)})}$$

$$\leq B_{\Phi_{SA}}^{\sup} + \frac{\Upsilon^{(\rho)}_{S_1 \times \cdots \times S_N} - 1}{2} (B_{\Phi_{SA}}^{\sup} - B_{\Phi_{SA}}^{\inf}),$$

where $\Upsilon^{(\rho)}_{S_1 \times \cdots \times S_N}$ is the maximal violation (16) by an $N$-partite state $\rho$ of general $S_1 \times \cdots \times S_N$-setting Bell inequalities.

Consider the Zohren-Gill (ZG) inequalities (31) on joint probabilities

$$1 \leq B_{zg | lhv} \leq 2,$$

constituting the Bell inequalities for the bipartite case with two settings ($N = S = 2$) and $d$ outcomes at each site.

In view of their numerical results on violation of the ZG inequality (31) by two-qudit states of a dimension $d$ in a range from 2 to $10^6$ (see Fig. 1 in Ref. 31), Zohren and Gill conjectured (31) that, for the infinite dimensional optimal bipartite states $\tau_{d,2}$, $d \to \infty$, specified by Fig 2 in Ref. 31, the tight quantum analog of the ZG inequality $B_{zg | lhv} \geq 1$ under projective bipartite joint quantum measurements has the form (32)

$$B_{zg | \tau_{d,2}, d \to \infty} \geq 0.$$  (42)

On the other hand, from (40), (41) it follows

$$B_{zg | \tau_{d,2}} \geq \frac{3 - \Upsilon^{(\tau_{d,2})}_{2 \times 2}}{2}.$$  (43)

This and the tightness for $d \to \infty$ of the quantum analog (42), proved via the numerical results in Ref. 31, imply that, under projective bipartite joint quantum measurements on an optimal state $\tau_{d,2}, d \to \infty$,

$$0 \geq 3 - \Upsilon^{(\tau_{d,2})}_{2 \times 2} |_{d \to \infty} \Leftrightarrow \Upsilon^{(\tau_{d,2})}_{2 \times 2} |_{d \to \infty} \geq 3.$$  (44)

However, for $N = S = 2$, $d \to \infty$, the upper bound (39) under projective measurements reads $\Upsilon^{(\tau_{d,2})}_{2 \times 2} |_{d \to \infty} \leq 3$. This and relation (44) imply

$$\Upsilon^{(\tau_{d,2})}_{2 \times 2} |_{d \to \infty} = 3.$$  (45)
In view of definition (16) of the maximal violation parameter $\Upsilon_{2 \times 2}^{(\tau_{d,2})}$, this means that there must exist a general $2 \times 2$-setting Bell inequality where, under projective bipartite quantum measurements on an optimal state $\tau_{d,2}, d \to \infty$, the term 3 in the upper bound $\min\{\sqrt{d}, 3\}$ in (39) is attained, exactly or almost. Finding such a general Bell inequality is a problem for a future research.

In conclusion, for the maximal quantum violation of general Bell inequalities, we have derived a new precise upper bound (36), reducing for $N = 2$ to our bipartite bound $(2d - 1)$ in Eq. (64) of Ref. 9 and essentially improving for $N \geq 3$ all the known precise upper bounds for general multipartite Bell inequalities listed in item (d) of the Introduction.

Via the upper bounds (38), (39), the new upper bound (36) also essentially improves the known precise upper bounds on the maximal quantum violation of general multipartite Bell inequalities for $S$ settings per site. For some $S, d$ and $N$ discussed above, the upper bounds (38), (39) are attainable.

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REFERENCES

1. On general Bell inequalities, see Section II.
2. B. S. Cirel’son, Quantum generalizations of Bell’s inequality. Letters in Math. Phys. 4, 93–100 (1980).
3. B. S. Tsirelson, Quantum analogues of the Bell inequalities. The case of two spatially separated domains. J. Soviet Math. 36, 557–570 (1987).
4. R. F. Werner and M. M. Wolf, All multipartite Bell correlation inequalities for two dichotomic observables per site. Phys. Rev. A 64, 032112 (2001).
5. V. Scarani, N. Gisin, Spectral decomposition of Bell’s operators for qubits. J. of Physics A: Math. Gen. 34, 6043–6053 (2001).
This follows from the definition of the Grothendieck’s constant $K_{G}^{(R)}$ and Theorem 2.1 in Ref. 3.

D. Perez-Garcia, M. M. Wolf, C. Palazuelos, I. Villanueva, M. Junge, Unbounded violation of tripartite Bell inequalities. Commun. Math. Phys. 279, 455–486 (2008).

M. Junge and C. Palazuelos, Large violation of Bell inequalities with low entanglement. Commun. Math. Phys. 306, 695–746 (2011).

E. R. Loubenets, Local quasi hidden variable modelling and violations of Bell-type inequalities by a multipartite quantum state. J. Math. Phys. 53, 022201 (2012).

J. Briet, T. Vidick. Explicit Lower and Upper Bounds on the Entangled Value of Multiplayer XOR Games. Commun. Math. Phys. 321, 181–207 (2013).

C. Palazuelos, On the largest Bell violation attainable by a quantum state. J. Funct. Analysis 267, 1959–1985 (2014).

E. R. Loubenets, Context-invariant and Local Quasi Hidden Variable (qHV) Modelling Versus Contextual and Nonlocal HV Modelling. Found. Phys. 45, 840–850 (2015).

E. R. Loubenets, On the existence of a local quasi hidden variable (LqHV) model for each $N$-qudit state and the maximal quantum violation of Bell inequalities. Intern. J. of Quantum Information 14, 1640010 (2016).

C. Palazuelos and T. Vidick, Survey on Nonlocal Games and Operator Space Theory. J. Math. Phys. 57, 015220 (2016).

M. Junge, T. Oikhberg, C. Palazuelos, Reducing the number of questions in nonlocal games. J. Math. Phys. 57 (10), 102203 (2016).

That is, Bell inequalities of an arbitrary type – either for correlation functions or for joint probabilities or of a more complicated form, see in Section II.

Throughout the article, the term "precise bound" means that, in this bound, no any universal constant is involved – in contrast to some bounds in Refs. 8, 10, 11, 14 defined up to universal constants.

E. R. Loubenets, Nonsignaling as the consistency condition for local quasi-classical probability modeling of a general multipartite correlation scenario. J. Phys. A: Math. Theor. 45, 185306 (2012).

E. R. Loubenets, Context-invariant quasi hidden variable (qHV) modelling of all joint von Neumann measurements for an arbitrary Hilbert space. J. Math. Phys. 56, 032201 (2015).

For this notion, see Section II.
See Proposition 3 in Ref. 19.

E. R. Loubenets, Multipartite Bell-type inequalities for arbitrary numbers of settings and outcomes per site. J. Phys. A: Math. Theor. 41, 445304 (2008).

On the general framework for the probabilistic description of multipartite correlation scenarios, see Ref. 24.

E. R. Loubenets, On the probabilistic description of a multipartite correlation scenario with arbitrary numbers of settings and outcomes per site. J. Phys. A: Math. Theor. 41, 445303 (2008).

For the main statements on the LHV modelling of a general multipartite correlation scenario, see section 4 in Ref. 24.

Here, the term a tight LHV constraint means that, in the LHV frame, the bounds established by this constraint cannot be improved. On the difference between the terms a tight linear LHV constraint and an extreme linear LHV constraint in case of, for example, the LHV constraints on a linear combination of correlation functions, see the end of Section 2.1 in Ref. 22.

See Proposition 1 in Ref. 9 for an \( N \)-partite case and Proposition 1 in Ref. 28 for a bipartite case.

E. R. Loubenets, Quantum states satisfying classical probability constraints. Banach Center Publ. 73, 325–337 (2006), e-print arXiv:quant-ph/0406139.

See definition 2 in Section II of Ref. 9.

E. R. Loubenets, Full locality of a noisy state for \( N \geq 3 \) nonlocally entangled qudits. E-print arXiv:1611.06723 (2016).

S. Zohren and R. D. Gill, Maximal Violation of the Collins-Gisin-Linden-Massar-Popescu Inequality for Infinite Dimensional States. Phys. Rev. Lett. 100, 120406 (2008).

See Eq. (8) in Ref. 31.

To our knowledge, for the maximal quantum violation \( \Upsilon^{(\rho_{d,N})}_{S_1 \times \cdots \times S_N} \) by a state \( \rho_{d,N} \) of general \( S_1 \times \cdots \times S_N \)-setting Bell inequalities, the precise upper bounds are presented by Eq. (62) in Ref. 9 and by Eq. (19) of Ref. 13. The bipartite upper bound \( \Upsilon^{(\rho_{d,2})}_{S \times S} \leq C \min\{d,S\} \) presented by Eq.(01) in Ref. 15 (also, in Refs. 8, 14) is defined up to a universal constant.

Note also that the upper bounds in Refs. 10, 14 on the maximal quantum violation of general Bell inequalities used in nonlocal games do not need to hold for all general Bell inequalities.