The minimal and the new minimal supersymmetric Grand Unified Theories on noncommutative space-time

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Abstract
We construct noncommutative versions of both the minimal and the new minimal supersymmetric Grand Unified Theories (GUTs). The enveloping-algebra formalism is used to carry out such constructions. The beautiful formulation of the Higgs sector of these noncommutative theories is a consequence of the fact that, in the GUTs at hand, the ordinary Higgs fields can be realized as elements of the Clifford algebra \( \mathbb{C}l_{10}(\mathbb{C}) \). In the noncommutative supersymmetric GUTs we formulate, supersymmetry is linearly realized by the noncommutative fields; but it is not realized by the ordinary fields that define those noncommutative fields via the Seiberg–Witten map.

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1. Introduction

Let us begin by saying that by canonical noncommutative space-time—or simply noncommutative space-time—we mean the noncommutative space defined by \( [X^\mu, X^\nu] = i\omega^{\mu\nu} \), where \( \omega^{\mu\nu} \) is a c-number. We shall assume that Lorentz indices are raised and lowered with the Minkowski metric \((-++,+++,+++)\).

The formulation of gauge theories on canonical noncommutative space-time that are deformations of ordinary gauge theories for arbitrary gauge groups in arbitrary unitary representations demands, as yet, using the enveloping-algebra formalism. This formalism was set up in [1–3] and put to use in the construction of the noncommutative Standard Model [4], a noncommutative deformation of the ordinary Standard Model with no new degrees of freedom—see [5–7] for other noncommutative extensions of the ordinary Standard Model. The enveloping-algebra formalism was also employed [8] to formulate GUTs (Grand Unified Theories) in the SU(5) and SO(10) gauge group cases. The nontrivial issue of constructing
Yukawa terms—for SO(10) and E6—within the enveloping-algebra framework was tackled in [9]. Outside the enveloping-algebra formalism, the formulation of noncommutative gauge theories for SO(N) groups was also discussed in [10].

In the enveloping-algebra formalism, the noncommutative gauge fields belong to the universal enveloping algebra of the Lie algebra of the ordinary gauge group and they are defined in terms of the ordinary fields by means of the Seiberg–Witten map. Let us recall that the Seiberg–Witten map maps ordinary gauge orbits into noncommutative gauge orbits. When the Seiberg–Witten map is computed as a formal power series in the noncommutativity matrix parameter \( \omega^{\mu\nu} \), the action of the noncommutative theory is expressed as a formal power series in \( \omega^{\mu\nu} \) with coefficients that are integrated polynomials in the ordinary fields and their derivatives. Many theoretical properties—e.g., renormalizability [11–16], gauge anomalies [17, 18], existence of noncommutative deformations of ordinary instantons and monopoles [19–21]—of the noncommutative gauge theories so defined have been studied by taking the first few terms of the appropriate \( \omega^{\mu\nu} \)-expanded actions. Some phenomenological properties of the noncommutative gauge theories at hand have been analyzed in [22–27].

The UV/IR mixing effects [28] that occur in the \( \omega^{\mu\nu} \)-unexpanded noncommutative field theories cannot be exhibited in the noncommutative gauge theory constructed by defining the Seiberg–Witten map as a series expansion in \( \omega^{\mu\nu} \), unless some re-summation of an infinite number of terms in powers of \( \omega^{\mu\nu} \) is carried out: a daunting task. Fortunately, for the enveloping-algebra formalism to work [3] it is not necessary that the Seiberg–Witten be given by a formal series expansion in the noncommutativity matrix \( \omega^{\mu\nu} \). Indeed, the enveloping-algebra formalism works equally well if one considers the Seiberg–Witten map as being given by an expansion in the number of ordinary fields, thus leaving its dependence on \( \omega^{\mu\nu} \) exact. Hence, if one wants to study noncommutative UV/IR effects in theories defined within the enveloping-algebra formalism one should use this \( \omega^{\mu\nu} \)-exact Seiberg–Witten map. This was done for the first time in [29] were it was shown, in the U(1) case with fermions in the adjoint, that if the \( \omega^{\mu\nu} \) dependence of the Seiberg–Witten is handled exactly, then, there is a UV/IR mixing phenomenon in the noncommutative theory defined within the enveloping-algebra formalism. The analysis of the UV/IR mixing effects was later extended [30] to fermions in the fundamental representation coupled to U(1) gauge fields. The UV/IR mixing that occurs in the one-loop propagator of adjoint fermions coupled to U(1) fields and its very interesting implications on neutrino physics has been deeply analyzed in [31–34]—see [35] for a recent short review. It is worth mentioning that the cohomological technics developed in [36, 37]—see also [38]—are extremely helpful [39] in the computation of the \( \omega^{\mu\nu} \)-exact expansion of the Seiberg–Witten map in the number of fields.

Ordinary (i.e., on Minkowski space-time) SO(10) GUTs—see [40] for a status review—provide appealing extensions of the Standard Model, for the 16 spinor representation of SO(10) unifies—with each family—the fermionic matter of the Standard Model plus a right-handed neutrino. This is in addition to the unification of the interactions. They also yield tiny neutrino masses through the see-saw mechanism. The minimal supersymmetric GUT [41, 42] has also other nice features such as \( b-\tau \) unification and leads to realistic phenomenology if split supersymmetry is at work [43]. Another way to iron out the problems that the original minimal supersymmetric SO(10) GUT gave rise to is to include in it a Higgs in the 120 irrep of SO(10). This proposal was put forward in [44], were the theory was named the new minimal supersymmetric GUT. An extensive analysis of the new minimal supersymmetric GUT has been presented in [45].

The purpose of this paper is to formulate the corresponding counterparts of the minimal supersymmetric and the new minimal supersymmetric GUTs, which we have just mentioned,
on canonical noncommutative space-time. Two preliminary comments are in order. First, these GUTs are particularly adequate for their generalization to noncommutative space-time, for all the Higgs fields in them have—a beautiful interpretation as appropriate elements of the Clifford algebra $\mathbb{C}l_{10}(\mathbb{C})$; and recall that associative unital algebras are key mathematical objects in the noncommutative geometry [46]. This is a feature not shared with SO(10) GUTs carrying Higgs fields in the 16, 54, etc… irreps of SO(10) [47, 48], which nonetheless should admit noncommutative versions within the enveloping-algebra formalism. Second, it is known [49, 50] that the supersymmetry of the effective U(1) supersymmetric DBI action for open strings ending on D-branes in the presence of a constant Neveu–Schwarz $B_{\mu\nu}$ field is a nonlinearly realized supersymmetry when the DBI action is written in terms of the ordinary gauge field and its superpartners, whereas it is a linearly realized supersymmetry when that action is expressed, upon using the Seiberg–Witten map, in terms of corresponding noncommutative fields. Hence, when formulated in terms of ordinary fields, the supersymmetry of the noncommutative U(1) theory is not the supersymmetry of the corresponding ordinary theory, which is obtained by setting the noncommutativity parameter to zero. It also happens [50] that noncommutative U(N) super-Yang–Mills has a linearly realized supersymmetry if the theory is expressed in terms of noncommutative fields, and yet that supersymmetry has a nonlinear realization when, upon using the Seiberg–Witten map, ordinary fields are chosen to formulate the theory. If we have SU(N), the supersymmetric invariance of the noncommutative supersymmetric theory is linearly realized in terms of the noncommutative fields, but cannot be realized by using the ordinary fields that define the former noncommutative fields via the Seiberg–Witten map—we shall see that this very situation occurs for the noncommutative GUTs that we shall construct. Let us also mention that in the SU(N) case the one-loop UV divergent radiative corrections preserve, up to first order in $\omega^{\mu\nu}$, the structure of classical action that is consistent with having linearly realized supersymmetry when the action is expressed in terms of the noncommutative fields—see [51] for details. It would thus appear that some nice properties of ordinary supersymmetric theories are still maintained through its—although hidden—noncommutative linear realization.

The layout of this paper is as follows. In section 2, we discuss how to obtain the field content and action of the noncommutative minimal supersymmetric GUT from the noncommutative new minimal supersymmetric GUT. Section 3 is a summary of the field content and action of the new minimal supersymmetric GUT on ordinary Minkowski space-time. We formulate the theory in terms of ordinary superfields in the Wess–Zumino gauge and interpret its Higgs superfields as elements of $\mathbb{C}l_{10}(\mathbb{C})$, for this is most suitable for its noncommutative generalization. Section 4 is devoted to the construction of our noncommutative counterpart of the new minimal supersymmetric GUT by using the enveloping-algebra formalism. In section 5 of the paper, we make some comments on the fact that in the noncommutative theory formulated in the previous section supersymmetry, which is linearly realized by the noncommutative fields, is not realized by the corresponding ordinary fields.

2. The noncommutative minimal supersymmetric Grand Unified Theory

The action of the noncommutative minimal supersymmetric GUT is obtained from the action of the noncommutative new minimal supersymmetric GUT by removing from the latter the Higgs superfield that is constructed from the ordinary Higgs field transforming under the 120 irrep of SO(10). Hence, we shall move on directly to the construction of the noncommutative new minimal supersymmetric GUT.
3. The new minimal supersymmetric Grand Unified Theory on Minkowski space-time

The new minimal supersymmetric GUT was introduced in [44]—see also [45]. Let us spell out its superfield content: first, three—one for each family in the Standard Model—chiral scalar superfields, \( \Phi_f^{(10)}, f = 1, 2, 3 \), transforming under the 16 irrep of SO(10), where \( \Phi_f^{(10)} \)'s contain the fermion fields of the Standard Model plus a right-handed neutrino; second, five Higgs chiral scalar superfields, \( \Phi_{1,210}^{(210)}, \Phi_{1,210}^{(120)}, \Phi_{1,126}^{(120)}, \Phi_{1,126}^{(120)} \) and \( \Phi_{1,126}^{(120)} \), transforming, respectively, under the 210, the 10, the 126, the 126 and the 120 irreps of SO(10). The indices \( i_1, i_2, \ldots \) run from 1 to 10, and \( \Phi_{i_1i_2i_3i_4}^{(210)}, \Phi_{i_1i_2i_3i_4}^{(120)}, \Phi_{i_1i_2i_3i_4}^{(120)} \) and \( \Phi_{i_1i_2i_3i_4}^{(120)} \) are totally antisymmetric SO(10) tensors with regard to its \( i_1, i_2, \ldots \) indices. Further, \( \Phi_{(126)}^{(126)} \) and \( \Phi_{(126)}^{(126)} \) satisfy the following duality equations:

\[
\Phi_{k_{i_1i_2i_3i_4}}^{(126)} = -\frac{i}{4!} \varepsilon_{ik_1k_2k_3k_4} \Phi_{i_1i_2i_3i_4}^{(126)},
\]

\[
\Phi_{k_{i_1i_2i_3i_4}}^{(126)} = \frac{i}{4!} \varepsilon_{ik_1k_2k_3k_4} \Phi_{i_1i_2i_3i_4}^{(126)}.
\]

Finally, there is the vector superfield, \( V \), taking values in the appropriate—see below—representation of SO(10). In the Wess–Zumino gauge, \( V \) reads

\[
V = -\theta \sigma^\mu \bar{\theta} a_\mu + i \bar{\theta} \bar{\sigma}^\lambda \theta \lambda + \frac{1}{8} \bar{\theta}^2 \bar{\sigma}^2 D.
\]

Here we shall adopt the supersymmetry conventions of [52].

Let \( \Gamma^i \) denote the Dirac matrices in ten Euclidean dimensions. These matrices generate the Clifford algebra \( \mathbb{C}_{10}(\mathbb{C}) \). We shall see later that a noncommutative version of the new minimal supersymmetric GUT can be constructed in a very smart way by using the \( \mathbb{C}_{10}(\mathbb{C}) \) Clifford-algebra-valued Higgs superfields

\[
\Phi_{(210)}^{(210)} = \Gamma^i \Gamma^j \Gamma^k \Phi_{(210)}^{(210)}, \quad \Phi_{(10)}^{(10)} = \Gamma^i \Phi_{(10)}^{(10)}, \quad \Phi_{(126)}^{(126)} = \Gamma^i \Gamma^j \Gamma^k \Phi_{(126)}^{(126)},
\]

\[
\Phi_{(126)}^{(126)} = \Gamma^i \Gamma^j \Gamma^k \Phi_{(126)}^{(126)}, \quad \Phi_{(120)}^{(120)} = \Gamma^i \Gamma^j \Gamma^k \Phi_{(120)}^{(120)},
\]

rather than the SO(10) tensor superfields \( \Phi_{(210)}^{(210)}, \Phi_{(10)}^{(10)}, \Phi_{(126)}^{(126)}, \Phi_{(126)}^{(126)}, \Phi_{(120)}^{(120)} \) and \( \Phi_{(120)}^{(120)} \), which give rise to the former.

From now on, the symbol \( V \) will stand for the vector superfield in the Wess–Zumino gauge whose supersymmetric components take values in the 16 \( \mathbb{T} \) representation of SO(10):

\[
V = \frac{1}{2} \Sigma^i V^{ij}, \quad \Sigma^i = \frac{1}{4!} \left[ \Gamma^i, \Gamma^j \right], \quad i, j = 1 \ldots 10,
\]

\[
V^{ij} = -\theta \sigma^\mu \bar{\theta} a_\mu^i + i \bar{\theta} \bar{\sigma}^\lambda \theta \lambda^{ij} + \frac{1}{8} \bar{\theta}^2 \bar{\sigma}^2 D^{ij}.
\]

\( V^{ij} \) carry the 45 irrep of SO(10). Below, we shall use the notations

\[
a_\mu = \frac{1}{2} \Sigma^i a_\mu^i, \quad \lambda = \frac{1}{2} \Sigma^i \lambda^{ij}, \quad \bar{\lambda} = \frac{1}{2} \Sigma^i \bar{\lambda}^{ij}, \quad D = \frac{1}{2} \Sigma^i D^{ij}.
\]

Let us introduce now the chiral coordinate \( y = x + i \theta \sigma^\mu \bar{\theta} a_\mu \). Let \( \Lambda \) be the chiral superfield defined as follows:

\[
\Lambda = \frac{1}{2} \Lambda^i \Sigma^i,
\]

\[
\Lambda^{(y)} = -2 i \theta \sigma^\mu \bar{\theta} a_\mu(y) - 2 \theta \bar{\sigma}^\lambda \theta \lambda(y),
\]

where \( \bar{\xi} \) is an infinitesimal spinor. Then, the supersymmetry transformation of the vector superfield we have introduced—recall that we have chosen the Wess–Zumino gauge—reads

\[
\delta_{\xi} V = (\xi \bar{Q} + \bar{\xi} \bar{Q}) V + \delta_{\Lambda} V,
\]

where

\[
Q_\alpha = \frac{\partial}{\partial a_\mu^\alpha} - i \sigma^\mu_{\alpha \beta} \bar{\theta} a_\mu \partial_\beta, \quad \bar{Q}_\alpha = -\frac{\partial}{\partial a_\mu^\alpha} + i \theta \sigma^\mu_{\alpha \beta} \partial_\beta
\]
and $\delta_\lambda V$ is given by the following compensating gauge transformation:

$$
\delta_\lambda V = \frac{i}{2} \mathcal{L}_V (\Lambda + \tilde{A}) + \frac{i}{2} \mathcal{L}_V \coth \mathcal{L}_V (\Lambda - \tilde{A}), \quad \mathcal{L}_V F = [V, F].
$$

(3.6)

In the Wess–Zumino gauge, the supersymmetry transformation of the scalar superfield $\Phi_f^{(16)}$ reads

$$
\delta_W^W \Phi_f^{(16)} = (\xi Q + \tilde{\xi} \bar{Q}) \Phi_f^{(16)} + \delta_\lambda \Phi_f^{(16)}, \quad \delta_\lambda \Phi_f^{(16)} = -i\lambda \Phi_f^{(16)}.
$$

And last, but not least, the CI$_{10}(\mathbb{C})$ Clifford-algebra-valued Higgs superfields in (3.1) transform under supersymmetry in the Wess–Zumino gauge as follows:

$$
\delta_W^W \Phi^{(H)} = (\xi Q + \tilde{\xi} \bar{Q}) \Phi^{(H)} + \delta_\lambda \Phi^{(H)}, \quad \delta_\lambda \Phi^{(H)} = -i[\Lambda, \Phi^{(H)}],
$$

where $\Phi^{(H)}$ stands for any of the scalar superfields defined in (3.1).

The superfields $V, \Phi_f^{(16)}, \Phi^{(10)}, \Phi^{(120)}, \Phi^{(126)}$ and $\Phi^{(120)}$ give a redundant characterization of the physical system, for there is still the invariance under the following gauge transformation:

$$
\delta_\Omega V = \frac{i}{2} \mathcal{L}_V (\Omega + \tilde{\Omega}) + \frac{i}{2} \mathcal{L}_V \coth \mathcal{L}_V (\Omega - \tilde{\Omega}),
$$

$$
\delta_\Omega \Phi_f^{(16)} = -i\Omega \Phi_f^{(16)}, \quad \delta_\Omega \Phi^{(H)} = -i[\Omega, \Phi^{(H)}], \quad H = 210, 10, 126, \overline{126}, 120.
$$

Note that $\Omega = \frac{1}{2} \Omega^{ij}(\gamma) \Sigma^i \Sigma^j, \Omega^{ij}(x)$ being infinitesimal real functions.

Let us define the action, $S$, of the new minimal GUT in terms of the superfields introduced above:

$$
S = S_{\text{SYM}} + S_{V\Phi} + S_{\text{spot}},
$$

where

$$
S_{\text{SYM}} = \frac{1}{64\pi} \text{Im} \left\{ \tau \text{ Tr} \int d^4x \, d^2\theta \, W^{a} W_a \right\}, \quad \tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2},
$$

$$
S_{V\Phi} = \int d^4x \, d^2\theta \, d^2\bar{\theta} \sum_f (\Phi^{(16)})_f \bar{\Phi}^{(16)} + \sum_f \frac{1}{s(H)} \text{Tr}((\Phi^{(H)})^\dagger e^{2V} \Phi^{(H)} e^{-2V}),
$$

$$
S_{\text{spot}} = \int d^4x \, d^2\theta \, [W_{\text{matter}} + W_{\text{Higgs}}] + \text{h.c.},
$$

(3.7)

with

$$
W_a = -\frac{i}{2} \tilde{D}^2 (e^{-2V} D_a e^{2V})
$$

and with $H$ running over the 210, the 10, the 126, the $\overline{126}$ and the 120 irreps of SO(10). In (3.7), the coefficients $s(H)$ are symmetry factors with values $s(210) = 1/32(1/4!)^2, s(10) = 1/32, s(126) = -1/64(1/51)^2 , s(\overline{126}) = -1/64(1/51)^2$ and $s(120) = -(1/31)^2$. $W_{\text{matter}}$ and $W_{\text{Higgs}}$ in (3.7) denote the superpotentials, which read

$$
W_{\text{matter}} = \sum_{f,f'} \left\{ \mathcal{Y}^{(10)}_f \Phi^{(10)}_f \Phi^{(16)}_f + \mathcal{Y}^{(120)}_f \Phi^{(120)}_f \Phi^{(16)}_f + \mathcal{Y}^{(126)}_f \Phi^{(126)}_f \Phi^{(16)}_f \right\}
$$

and

$$
W_{\text{Higgs}} = \frac{M^{(210)}}{64(4!)} \text{Tr} \Phi^{(210)} \Phi^{(210)} - \frac{M^{(126)}}{32(5!)} \text{Tr} \Phi^{(126)} \Phi^{(126)} + \frac{M^{(10)}}{64} \text{Tr} \Phi^{(10)} \Phi^{(10)}
$$

$$
- \frac{M^{(120)}}{64(3!)} \text{Tr} \Phi^{(120)} \Phi^{(120)} + \lambda_1 \text{ Tr} \Phi^{(210)} \Phi^{(210)} + \lambda_2 \text{ Tr} \Phi^{(120)} \Phi^{(120)} + \lambda_3 \text{ Tr} \Phi^{(120)} \Phi^{(120)} + \lambda_5 \text{ Tr} \Phi^{(120)} \Phi^{(120)}
$$

$$
+ \lambda_4 \text{ Tr} \Phi^{(120)} \Phi^{(120)} + \lambda_7 \text{ Tr} \Phi^{(120)} \Phi^{(120)} + \lambda_8 \text{ Tr} \Phi^{(120)} \Phi^{(120)}.
$$

(3.8)
In the superpotential $W_{\text{matter}}$, the chiral superfield $\Phi^{(16)}_f$, $f = 1, 2, 3$, is defined as follows:

$$\Phi^{(16)}_f = \Phi^{(16)}_f \cdot B, \quad B = \prod_{j=\text{odd}} \Gamma^j.$$ 

The action of $\delta^{WZ}_\xi$ and of $\delta_\Omega$ on $\Phi^{(16)}_f$ read

$$\delta^{WZ}_\xi \Phi^{(16)}_f = (\xi \bar{Q} + \bar{\xi} Q) \Phi^{(16)}_f + \delta_\lambda \Phi^{(16)}_f, \quad \delta_\lambda \Phi^{(16)}_f = i \Phi^{(16)}_f \lambda,$$

respectively.

For further reference, we shall close this section with the expansion in supersymmetric components of the chiral scalar superfields $\Phi^{(16)}_f$, $f = 1, 2, 3$ and $\Phi^{(H)}$, $H = 210, 10, 126, \overline{126}, 120$, defined above:

$$\Phi^{(16)}_f = A^{(16)}_f(y) + \sqrt{2} \theta \psi^{(16)}_f(y) + \theta^2 F^{(16)}_f(y), \quad f = 1, 2, 3,$$

$$\Phi^{(16)}_t = \tilde{A}^{(16)}_t(y) + \sqrt{2} \tilde{\theta} \tilde{\psi}^{(16)}_t(y) + \tilde{\theta}^2 \tilde{F}^{(16)}_t(y), \quad f = 1, 2, 3,$$

$$\Phi^{(H)} = A^{(H)}(y) + \sqrt{2} \theta \psi^{(H)}(y) + \theta^2 F^{(H)}(y), \quad H = 210, 10, 126, \overline{126}, 120.$$ 

(3.9)

$A^{(16)}_f, \tilde{A}^{(16)}_t$ and $F^{(16)}_f$ transform under the 16 irrep of SO(10). $A^{(16)}_f = A^{(16)}_f B$, $\tilde{A}^{(16)}_t = \tilde{A}^{(16)}_t B$ and $F^{(16)}_f = F^{(16)}_f B$. By construction, each $A^{(16)}_f$, $\tilde{A}^{(16)}_t$ and $F^{(16)}_f$ transforms like the conjugate spinor representation of SO(10). $A^{(H)}$, $\psi^{(H)}$ and $F^{(H)}$ take values in the Clifford algebra $\mathbb{C}_{10}(\mathbb{C})$ and are constructed from the appropriate components of the corresponding SO(10) antisymmetric tensor superfields.

### 4. The noncommutative new minimal supersymmetric Grand Unified Theory

Here we shall put forward a supersymmetric noncommutative deformation of the new minimal supersymmetric GUT. This will be a noncommutative field theory on the noncommutative superspace defined by the triplet $(X^\mu, \theta^a, \bar{\theta}^\dot{a})$ satisfying the following equations:

$$[X^\mu, X^\nu] = i \omega^{\mu\nu}, \quad \{\theta^a, \theta^\beta\} = 0, \quad \{\theta^a, \bar{\theta}^\dot{b}\} = 0,$$

$$\{\bar{\theta}^\dot{a}, \bar{\theta}^\dot{b}\} = 0, \quad [X^\mu, \theta^a] = 0, \quad [X^\mu, \bar{\theta}^\dot{a}] = 0.$$ 

(4.1)

Let $\xi^a$ and $\bar{\xi}^{\dot{a}}$ be infinitesimal Grassmann numbers. Then, the previous set of equations is invariant under supertranslations, and thus defined as

$$X^\mu = X^\mu + i \theta \sigma^a \xi^a - i \bar{\xi} \sigma^{\dot{a}} \bar{\theta}^{\dot{a}} - \theta^a \sigma^a \xi^a, \quad \theta^a = \theta^a + \xi^a, \quad \bar{\theta}^\dot{a} = \bar{\theta}^\dot{a} + \bar{\xi}^{\dot{a}}.$$ 

(4.2)

Hence, one is naturally led to understand supersymmetry as realized by supertranslations—modulo gauge transformations, if the Wess–Zumino gauge is chosen—of suitable fields defined on the noncommutative superspace introduced above. These suitable fields on our noncommutative superspace—which we shall call noncommutative superfields—will be obtained by taking any ordinary superfield and promoting its components to the category of noncommutative fields. Thus, we shall leave unchanged the Grassmann structure of the superfields. This is in harmony with the fact that there is no deformation of the Grassmann algebra introduced in (4.1).

### 4.1. The noncommutative vector superfield and the super-Yang–Mills action

Taking (3.2) as the starting point, we introduce first the noncommutative vector superfield in the Wess–Zumino gauge, $\tilde{V}$, of our theory:

$$\tilde{V} = -\theta \sigma^a \bar{\theta}^\dot{a} \tilde{A}_a + i \theta \tilde{\sigma}^a \tilde{\theta}^\dot{a} - i \tilde{\theta}^2 \bar{\theta} \lambda + \frac{1}{2} \theta \tilde{\theta}^2 \tilde{D}. $$

(4.3)
The components \( \hat{a}_\mu, \hat{\lambda}, \tilde{\lambda} = \hat{\lambda}^1 \) and \( \hat{D} \) are noncommutative fields which—recall that we are dealing with a simple gauge group: SO(10)—are to be constructed from their ordinary counterparts by using the formalism put forward in [1, 3, 8]. That is, \( \hat{a}_\mu, \hat{\lambda}, \tilde{\lambda} \) and \( \hat{D} \) are functions of \( a_\mu, \lambda, \tilde{\lambda}, D \)—in (3.2) and (3.3)—and \( \omega^{\mu\nu} \) that solve the following Seiberg–Witten map equations:

\[
\begin{align*}
\hat{s}\hat{\Omega} &= s\hat{\Omega}, \\
\hat{s}\hat{a}_\mu &= s\hat{a}_\mu, \\
\hat{s}\hat{\lambda}_\mu &= s\hat{\lambda}_\mu, \\
\hat{s}\hat{\lambda}_\mu &= s\hat{\lambda}_\mu, \\
\hat{s}\hat{\hat{D}} &= s\hat{\hat{D}}.
\end{align*}
\tag{4.4}
\]

The symbol \( s \) denotes the ordinary BRS operator, which is defined as follows:

\[
s\Omega = -i\Omega\Omega,
\]

\[
sa_\mu = \partial_\mu\Omega + [a_\mu, \Omega] = D_\mu\Omega,
\]

\[
s\lambda_\mu = i[\lambda_\mu, \Omega],
\]

\[
s\tilde{\lambda}_\mu = i[\tilde{\lambda}_\mu, \Omega].
\]

\( s\hat{D} = i[D, \Omega] \).

\( \hat{s} \) denotes the noncommutative BRS operator, which acts on the noncommutative fields; thus,

\[
\begin{align*}
\hat{s}\hat{a}_\mu &= \partial_\mu\hat{\Omega} + [\hat{a}_\mu, \hat{\Omega}], \\
\hat{s}\hat{\lambda}_\mu &= i[\hat{\lambda}_\mu, \hat{\Omega}], \\
\hat{s}\hat{\tilde{\lambda}}_\mu &= i[\hat{\tilde{\lambda}}_\mu, \hat{\Omega}],
\end{align*}
\]

\( s\hat{\hat{D}} = i[\hat{\hat{D}}, \hat{\Omega}] \).

\( \hat{s}\hat{\Omega} = s\hat{\Omega} \) in (4.4). One obtains a solution to (4.4) by particularizing the general formulae in [39] to the case at hand.

Let us stress that our definition of noncommutative vector superfield as a function of the ordinary fields in the gauge supermultiplet, \( (a_\mu, \lambda, D) \), is quite in keeping with the fact that \( (a_\mu, \lambda_\mu, D) \) and \( (a_\mu + \delta_\Omega a_\mu, \lambda_\mu + \delta_\Omega \lambda_\mu, D + \delta_\Omega D) \) characterize the same field configuration, when \( \delta_\Omega a_\mu, \delta_\Omega \lambda_\mu \) and \( \delta_\Omega D \) denote infinitesimal gauge transformations. Indeed, one can show that

\[
\hat{V}[a_\mu + \delta_\Omega a_\mu, \lambda_\mu + \delta_\Omega \lambda_\mu, D + \delta_\Omega D] = \hat{V}[a_\mu, \lambda_\mu, D] + \hat{\delta}_\Omega \hat{V}[a_\mu, \lambda_\mu, D],
\]

where

\[
\hat{\delta}_\Omega \hat{V} = \frac{i}{2} \hat{\mathcal{L}}_\psi (\hat{\Omega} + \hat{\tilde{\Omega}}) + \frac{i}{2} \hat{\mathcal{L}}_\psi \coth \hat{\mathcal{L}}_\psi (\hat{\Omega} - \hat{\tilde{\Omega}}),
\]

\( \hat{\mathcal{L}}_\psi F = [\hat{V}, F] \).

In (4.6) and (4.7), \( \hat{\Omega} \) denotes the chiral superfield which is obtained from \( \hat{\Omega}(x) \) in (4.5) by replacing \( y^\mu \) with the chiral coordinate \( y^\mu = y^\mu + \theta^a \sigma^a \tilde{\theta} \), \( \hat{\Omega}(x) \), which is the image under the Seiberg–Witten map of \( \Omega \), defines the noncommutative gauge transformations of \( \hat{a}_\mu, \hat{\lambda}_\mu \) and \( \hat{D} \):

\[
\hat{\delta}_\Omega \hat{a}_\mu = \partial_\mu \hat{\Omega} + [\hat{a}_\mu, \hat{\Omega}],
\]

\[
\hat{\delta}_\Omega \hat{\lambda}_\mu = [\hat{\lambda}_\mu, \hat{\Omega}],
\]

\[
\hat{\delta}_\Omega \hat{\hat{D}} = [\hat{\hat{D}}, \hat{\Omega}].
\]

A final comment regarding the superfield gauge transformation in (4.7). Let \( \hat{\Omega}(y) \) be such that \( \hat{\Omega}(x) = \hat{\Omega}(x) \) which does not depend on \( \theta \) or on \( \tilde{\theta} \). Then, for such an \( \hat{\Omega}(y) \), the transformation in (4.7) is the most general gauge transformation of the superfield in the Wess–Zumino gauge which gives a vector superfield in the Wess–Zumino gauge.

Let us now define the supersymmetry transformations of \( \hat{V} \) introduced above. It is plain that a supertranslation—see (4.2)—acting on \( \hat{V} \) is generated by \( \xi Q + \xi \hat{Q} \), with \( Q_\mu \) and \( \hat{Q}_\mu \) as given in (3.5). As in the ordinary case, \( (\xi Q + \xi \hat{Q}) \hat{V} \) contains more components than a vector superfield in the Wess–Zumino gauge does, but, analogously to the ordinary case, these extra components are not physical since they can be set to zero by an appropriate (field-dependent) noncommutative superfield gauge transformation. Hence, we define the
infinitesimal supersymmetry transformation of the noncommutative vector superfield as follows:

$$\delta^\text{WZ}_\xi \hat{\nabla} = (\xi Q + \xi \hat{Q}) \hat{\nabla} + \delta_\lambda \hat{\nabla},$$  

(4.8)

where $\lambda(y)$ is the chiral superfield

$$\lambda(y) = -2i \theta \sigma^\nu \xi \hat{a}_\mu(y) - 2 \theta^2 \xi \lambda(y)$$  

(4.9)

and where the noncommutative superfield gauge transformation $\delta^\text{WZ}_\lambda \hat{\nabla}$ is obtained by replacing $\hat{Q}$ with $\lambda$ in (4.7). Of course, $\delta^\text{WZ}_\lambda \hat{\nabla}$ as defined in the previous equations looks like the ordinary $\delta^\text{WZ}_\lambda \hat{\nabla}$ in (3.4) and (3.6) comes from the fact that we are not deforming the Grassmann part of the superspace.

From (4.8) one readily deduces the action of $\delta^\text{WZ}_\xi$ on the components, $(\hat{a}_\mu, \hat{\lambda}_\alpha, \hat{\theta})$, of $\hat{\nabla}$:

$$\delta^\text{WZ}_\xi \hat{a}_\mu = -i \bar{\xi} \hat{a}_\mu + i \xi \hat{a}_\mu \lambda,  
\delta^\text{WZ}_\xi \hat{\lambda}_\alpha = (\sigma^\mu \xi)_a \hat{f}_{\mu \nu} + i \bar{\xi} \hat{\theta} D,  
\delta^\text{WZ}_\xi \hat{\theta} = -i \theta \sigma^\nu \xi \hat{\theta} \sigma^\mu \lambda + D \hat{\theta},$$  

(4.10)

where $\hat{f}_{\mu \nu} = \partial_\mu \hat{a}_\nu - \partial_\nu \hat{a}_\mu + i[\hat{a}_\mu, \hat{a}_\nu]$, and $D = \partial_\mu \hat{\lambda}_\alpha + i[\hat{a}_\mu, \hat{\lambda}_\alpha]$. It is worth mentioning that $\delta^\text{WZ}_\xi \hat{a}_\mu$, $\delta^\text{WZ}_\xi \hat{\lambda}_\alpha$ and $\delta^\text{WZ}_\xi \hat{\theta}$ are well-defined functions of the infinitesimal gauge orbit of $(\hat{a}_\mu, \hat{\lambda}_\alpha, D)$, for

$$\delta^\text{WZ}_\xi \hat{X}_{\mu} = \delta^\text{WZ}_\xi \hat{a}_\mu + i \theta \sigma^\nu \delta^\text{WZ}_\xi \hat{\lambda}_\alpha + D + \delta^\text{WZ}_\xi \hat{\theta} = \delta^\text{WZ}_\xi \hat{D},$$  

(4.11)

where $\hat{X}_{\mu} = \hat{a}_\mu$, $\hat{\lambda}_\alpha$, and $\delta^\text{WZ} \hat{\theta}$ generates an infinitesimal ordinary gauge transformation.

It can be seen that if $\hat{\theta}$, $\hat{\lambda}_\alpha$, and $\hat{D}$ are solutions to the equations (4.4), then

$$\delta^\text{WZ}_\xi \hat{a}_\mu = \delta^\text{WZ}_\xi \hat{a}_\mu + \delta^\text{WZ}_\xi \hat{\lambda}_\alpha,  
\delta^\text{WZ}_\xi \hat{\theta} = \hat{\theta} + \delta^\text{WZ}_\xi \hat{\theta},  
\delta^\text{WZ}_\xi \hat{D} = \delta^\text{WZ}_\xi \hat{D} + \delta^\text{WZ}_\xi \hat{\theta} D$$  

(4.12)

are also solutions to the equations (4.4), satisfying the conditions

$$\delta^\text{WZ}_\xi \hat{a}_\mu[\omega = 0] = \delta^\text{WZ}_\xi \hat{a}_\mu[\omega = 0],  
\delta^\text{WZ}_\xi \hat{\lambda}_\alpha[\omega = 0] = \delta^\text{WZ}_\xi \hat{\lambda}_\alpha[\omega = 0],  
\delta^\text{WZ}_\xi \hat{\theta} D[\omega = 0] = \delta^\text{WZ}_\xi \hat{\theta} D[\omega = 0].$$

Note that $\delta^\text{WZ}_\xi \hat{a}_\mu = \delta^\text{WZ}_\xi \hat{a}_\mu[\omega = 0]$, $\delta^\text{WZ}_\xi \hat{\lambda}_\alpha = \delta^\text{WZ}_\xi \hat{\lambda}_\alpha[\omega = 0]$ and $\delta^\text{WZ}_\xi \hat{\theta} D[\omega = 0] = \delta^\text{WZ}_\xi \hat{\theta} D[\omega = 0]$, and also note that $\delta^\text{WZ}_\xi$ gives—just set $\omega^{\mu \nu} = 0$—the ordinary supersymmetry transformations in the Wess–Zumino gauge in (4.10). The reader should bear in mind that $\delta^\text{WZ}_\xi \hat{a}_\mu$, $\hat{\lambda}_\alpha$, $\hat{D}$ is the same for the fields in $(\hat{a}_\mu, \hat{\lambda}_\alpha, \hat{D})$ as for their transformed fields $\hat{a}_\mu', \hat{\lambda}_\alpha'$ and $\hat{D}'$ in (4.12). It is thus clear that imposing invariance under the noncommutative supersymmetry transformations in (4.10) will be compatible with demanding ordinary gauge invariance, and, hence, with asking for noncommutative gauge invariance for SO(10).

Now, using the definitions in (4.10), it is not difficult to show that

$$[\delta^\text{WZ}_\xi, \delta^\text{WZ}_\xi] \hat{X} = -2i (\xi_1 \sigma^\mu \xi_2 - \xi_2 \sigma^\mu \xi_1) \partial_\mu \hat{X} + \delta_\lambda \hat{X},$$  

(4.13)

where $\hat{X}$ stands for any of the fields in $(\hat{a}_\mu, \hat{\lambda}_\alpha, \hat{D})$, $\hat{\lambda}$ is given by

$$\hat{\lambda} = 2i (\xi_1 \sigma^\mu \xi_2 - \xi_2 \sigma^\mu \xi_1) \hat{a}_\mu$$  

(4.14)

and

$$\delta^\text{WZ}_\lambda \hat{a}_\mu = \partial_\mu \hat{\lambda} + i[\hat{a}_\mu, \hat{\lambda}],  
\delta^\text{WZ}_\lambda \hat{\lambda}_\alpha = i[\hat{\lambda}_\alpha, \hat{\lambda}],  
\delta^\text{WZ}_\lambda \hat{\theta} D = [\hat{D}, \hat{\lambda}],$$

are noncommutative gauge transformations. From equation (4.13) one draws the conclusion that the space of solutions, $(\hat{a}_\mu, \hat{\lambda}_\alpha, \hat{D})$, of the Seiberg–Witten map equations in (4.4) carries a representation of the $\mathcal{N} = 1$ supersymmetry algebra; a representation which is linear modulo noncommutative gauge transformations.
We are now ready to introduce the noncommutative super-Yang–Mills action, $S_{\text{NCSYM}}$, of our noncommutative new minimal and minimal supersymmetric GUTs. Firstly, we restrict ourselves to solutions $\hat{\Omega}$, $\hat{a}_\mu \hat{\lambda}_a$ and $\hat{D}$ to (4.4) which satisfy
\[
\hat{\Omega}[\omega=0] = \Omega, \quad \hat{a}_\mu[\omega=0] = a_\mu, \quad \hat{\lambda}_a[\omega=0] = \lambda_a, \quad \hat{\bar{\lambda}}_a[\omega=0] = \bar{\lambda}_a, \quad \hat{D}[\omega=0] = D.
\]
Secondly, we use this triplet $(\hat{a}_\mu, \hat{\lambda}_a, \hat{D})$ and equation (4.3) to construct the corresponding noncommutative $\hat{V}$, with noncommutative field strength given by
\[
\hat{W}_a = -\frac{1}{4} \hat{D}^2 (e^{-2\hat{V}} \star D_a e^{2\hat{V}}),
\]
which transforms under the noncommutative gauge transformations in (4.7) as follows:
\[
\delta_\Omega \hat{W}_a = -i[\hat{\Omega}, \hat{W}_a].
\]
Finally, $S_{\text{NCSYM}}$ is defined as follows:
\[
S_{\text{NCSYM}} = \frac{1}{64\pi} \text{Im} \left\{ \tau \text{ Tr} \int d^4x d^2\theta \, \hat{W}^a \star \hat{\bar{W}}_a \right\},
\]
where $\tau = \frac{\alpha_{\text{EM}}}{2\pi} + \frac{4\pi i}{g^2}$. $S_{\text{NCSYM}}$ is manifestly invariant under the noncommutative supersymmetry transformation in (4.8) and the noncommutative gauge transformation in (4.7). Obviously, one reaches the same conclusion if one expresses first $S_{\text{NCSYM}}$ in terms of the fields in the noncommutative supermultiplet $(\hat{a}_\mu, \hat{\lambda}_a, \hat{D})$ and then one uses (4.10) and (4.5).

Let us close this subsection by pointing out that the action $S_{\text{NCSYM}}$ in (4.15) does not suffer from the ambiguity related with the choice of representations of the gauge fields which occurs in the noncommutative Standard Model defined via the Seiberg–Witten map, if, as we do, one assumes that the gauge fields (and its supersymmetric partners) are elements of the Clifford algebra $\mathbb{C}l_{10}(\mathbb{C})$. Of course, the fact that only this representation of the gauge fields is needed to formulate the covariant derivative of the matter fields is a consequence of the fact that in our models only the 16, the 10, the 126, the $\overline{126}$, the 210 and the 120 representations of $\text{SO}(10)$ occur, and that these representations naturally arise from the representation theory of $\mathbb{C}l_{10}(\mathbb{C})$. The fact that GUTs help solving the ambiguity problem related with the choice of representations of gauge fields was first pointed out in [8].

4.2. The noncommutative matter and noncommutative Higgs superfields and their interactions

In this subsection we shall apply the ideas put forward in the previous section to the construction of the noncommutative superfields that we shall take as the noncommutative counterparts of the ordinary matter superfields $\Phi_f^{(1)}, \ f = 1, 2, 3$, and the ordinary Higgs superfields $\Phi^{(H)}, \ H = 210, 10, 126, \overline{126}, 210$, in (3.9). Then, we shall easily build their noncommutative interactions with the vector superfield of the previous subsection and also construct the noncommutative superpotential. Thus we shall generalize $S_{\nu \Phi}$ and $S_{\text{spot}}$ in (3.7) to the noncommutative case.

Let us introduce the following chiral superfields:
\[
\hat{\Phi}_f^{(16)} = \hat{A}_f^{(16)}(y) + \sqrt{2} \partial \hat{\psi}_f^{(16)}(y) + \theta^2 \hat{F}_f^{(16)}(y), \ f = 1, 2, 3,
\]
\[
\hat{\Phi}_f^{(16)} = \hat{\zeta}_f^{(16)}(y) + \sqrt{2} \partial \hat{\psi}^{(16)}(y) + \theta^2 \hat{\hat{F}}_f^{(16)}(y), \ f = 1, 2, 3,
\]
\[
\hat{\Phi}^{(H)} = \hat{\tilde{A}}^{(H)}(y) + \sqrt{2} \partial \hat{\tilde{\psi}}^{(H)}(y) + \theta^2 \hat{\tilde{F}}^{(H)}(y), \quad H = 210, 10, 126, \overline{126}, 210,
\]
where $\hat{A}_f^{(16)}$, $\hat{\psi}_f^{(16)}$, $\hat{F}_f^{(16)}$, $\hat{\zeta}_f^{(16)}$, $\hat{\psi}^{(16)}$, $\hat{\hat{F}}_f^{(16)}$, $\hat{\tilde{A}}^{(H)}$, $\hat{\tilde{\psi}}^{(H)}$ and $\hat{\tilde{F}}^{(H)}$ are noncommutative fields, which we shall define below by using the enveloping-algebra formalism of [1, 3, 8].
Firstly, $\tilde{A}_f^{(16)}$, $\tilde{\psi}_f^{(16)}$ and $\tilde{F}_f^{(16)}$ are functions of the corresponding ordinary fields, $A_f^{(16)}$, $\psi_f^{(16)}$ and $F_f^{(16)}(x)$—see (3.9), the ordinary gauge field $a_\mu$—see (3.3)—and $\omega^\mu_\nu$ that solve the following Seiberg–Witten equations in the BRS form:

$$\hat{s}\tilde{A}_f^{(16)} = s\tilde{A}_f^{(16)}, \quad \hat{s}\tilde{\psi}_f^{(16)} = s\tilde{\psi}_f^{(16)}, \quad \hat{s}\tilde{F}_f^{(16)} = s\tilde{F}_f^{(16)}. \quad (4.17)$$

The action of the BRS operators $\hat{s}$—noncommutative—and $s$—ordinary—on the corresponding fields is defined as follows:

$$\hat{s}\tilde{A}_f^{(16)} = -i\hat{\Omega} \star \tilde{A}_f^{(16)}, \quad \hat{s}\tilde{\psi}_f^{(16)} = -i\hat{\Omega} \star \tilde{\psi}_f^{(16)}, \quad \hat{s}\tilde{F}_f^{(16)} = -i\hat{\Omega} \star \tilde{F}_f^{(16)},$$

$$s\tilde{A}_f^{(16)} = -i\Omega A_f^{(16)}, \quad s\tilde{\psi}_f^{(16)} = -i\Omega \tilde{\psi}_f^{(16)}, \quad s\tilde{F}_f^{(16)} = -i\Omega \tilde{F}_f^{(16)} \quad (4.18)$$

where $\hat{\Omega}$ is the very same noncommutative object which occurs in (4.5).

Secondly, $\tilde{A}_f^{(16)}$, $\tilde{\psi}_f^{(16)}$ and $\tilde{F}_f^{(16)}$ are also functions of the corresponding ordinary fields, $A_f^{(16)}$, $\psi_f^{(16)}$ and $F_f^{(16)}(x)$—see (3.9), the ordinary gauge field $a_\mu$—see (3.3)—and $\omega^\mu_\nu$ which satisfy

$$\hat{s}\tilde{z}_f^{(16)} = s\tilde{z}_f^{(16)}, \quad \hat{s}\tilde{\psi}_f^{(16)} = s\tilde{\psi}_f^{(16)}, \quad \hat{s}\tilde{F}_f^{(16)} = s\tilde{F}_f^{(16)}. \quad (4.19)$$

The BRS operators $\hat{s}$ and $s$ act thus on the corresponding fields in the previous set of equation:

$$\hat{s}\tilde{A}_f^{(16)} = i\tilde{A}_f^{(16)} \star \tilde{\Omega}, \quad \hat{s}\tilde{\psi}_f^{(16)} = i\tilde{\psi}_f^{(16)} \star \tilde{\Omega}, \quad \hat{s}\tilde{F}_f^{(16)} = i\tilde{F}_f^{(16)} \star \tilde{\Omega},$$

$$s\tilde{A}_f^{(16)} = i\tilde{A}_f^{(16)} \star \tilde{\Omega}, \quad s\tilde{\psi}_f^{(16)} = i\tilde{\psi}_f^{(16)} \star \tilde{\Omega}, \quad s\tilde{F}_f^{(16)} = i\tilde{F}_f^{(16)} \star \tilde{\Omega}, \quad (4.20)$$

where, again, $\tilde{\Omega}$ is the very same noncommutative object which enters (4.5).

Finally, $\tilde{A}^{(H)}$, $\tilde{\psi}^{(H)}$ and $\tilde{F}^{(H)}$, in (3.9), the ordinary gauge field $a_\mu$—see (3.3)—and $\omega^\mu_\nu$ that solve the following Seiberg–Witten map equations in the BRS form:

$$\hat{s}\tilde{A}^{(H)} = s\tilde{A}^{(H)}, \quad \hat{s}\tilde{\psi}^{(H)} = s\tilde{\psi}^{(H)}, \quad \hat{s}\tilde{F}^{(H)} = s\tilde{F}^{(H)}. \quad (4.21)$$

where now

$$\hat{s}\tilde{A}^{(H)} = -i[\hat{\Omega}, \tilde{A}^{(H)}], \quad \hat{s}\tilde{\psi}^{(H)} = -i[\hat{\Omega}, \tilde{\psi}^{(H)}], \quad \hat{s}\tilde{F}^{(H)} = -i[\hat{\Omega}, \tilde{F}^{(H)}] \quad (4.22)$$

It is plain that the construction of $\tilde{\Phi}_f^{(16)}$, $\tilde{\Phi}_f^{(H)}$ and $\tilde{\Phi}^{(H)}$ yields noncommutative superfields that are well defined on the infinitesimal gauge orbit of the ordinary fields they are functions of. Indeed, one readily sees that

$$\tilde{\Phi}_f^{(16)}[a_\mu + \delta_\Omega a_\mu; A_f^{(16)}, \psi_f^{(16)} + \delta_\Omega \psi_f^{(16)}, F_f^{(16)} + \delta_\Omega F_f^{(16)}] = \tilde{\Phi}^{(16)} + \delta_\Omega \tilde{\Phi}^{(16)},$$

$$\tilde{\Phi}_f^{(H)}[a_\mu + \delta_\Omega a_\mu; A^{(H)}, \psi^{(H)} + \delta_\Omega \psi^{(H)}, F^{(H)} + \delta_\Omega F^{(H)}] = \tilde{\Phi}^{(H)} + \delta_\Omega \tilde{\Phi}^{(H)}, \quad (4.23)$$

where

$$\delta_\Omega \tilde{\Phi}^{(16)} = -i\hat{\Omega} \star \tilde{\Phi}^{(16)}, \quad \delta_\Omega \tilde{\Phi}^{(H)} = i\tilde{\Phi}^{(H)} \star \hat{\Omega}, \quad \delta_\Omega \tilde{\Phi}^{(H)} = -i[\hat{\Omega}, \tilde{\Phi}^{(H)}]_1.$$

In (4.8), we have defined the action of the noncommutative supersymmetry operator in the Wess–Zumino gauge, $\tilde{\delta}_k^{WZ}$, on the vector superfield $\tilde{V}$. This definition leads to the following
where $\hat{A}$ is the chiral noncommutative superfield given in (4.9).

The action of $\delta_{WZ}^\xi$ on the components of the noncommutative matter and Higgs superfields can be worked out from (4.24). One obtains, thus,

\begin{equation}
\begin{aligned}
\delta_{WZ}^\xi A_f^{(16)} &= \sqrt{2} \xi \, \psi_f^{(16)}, \\
\delta_{WZ}^\xi \bar{\psi}_f^{(16)} &= \sqrt{2} i (\sigma^\mu \bar{\xi})_a D_\mu A_f^{(16)} + \xi_\alpha \bar{F}_f^{(16)}, \\
\delta_{WZ}^\xi F_f^{(16)} &= i \sqrt{2} \bar{\xi} \sigma^\mu D_\mu \psi_f^{(16)} + 2 i \bar{\xi} \bar{\lambda} \sigma^\mu A_f^{(16)}, \\
\delta_{WZ}^\xi \hat{A}_f^{(16)} &= \sqrt{2} \xi \bar{\psi}_f^{(16)}, \\
\delta_{WZ}^\xi \hat{\bar{\psi}}_f^{(16)} &= \sqrt{2} i (\sigma^\mu \bar{\xi})_a D_\mu \hat{A}_f^{(16)} + \xi_\alpha \hat{F}_f^{(16)}, \\
\delta_{WZ}^\xi \hat{F}_f^{(16)} &= i \sqrt{2} \bar{\xi} \sigma^\mu D_\mu \hat{\bar{\psi}}_f^{(16)} + 2 i \bar{\xi} \bar{\lambda} \sigma^\mu \hat{A}_f^{(16)}, \\
\delta_{WZ}^\xi \hat{\bar{A}}_f^{(16)} &= \sqrt{2} \xi \bar{\psi}_f^{(16)}, \\
\delta_{WZ}^\xi \hat{\bar{\psi}}_f^{(16)} &= \sqrt{2} i (\sigma^\mu \bar{\xi})_a D_\mu \hat{\bar{A}}_f^{(16)} + \xi_\alpha \hat{\bar{F}}_f^{(16)}, \\
\delta_{WZ}^\xi \hat{\bar{F}}_f^{(16)} &= i \sqrt{2} \bar{\xi} \sigma^\mu D_\mu \hat{\bar{\psi}}_f^{(16)} + 2 i \bar{\xi} \bar{\lambda} \sigma^\mu \hat{\bar{A}}_f^{(16)}),
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
D_\mu A_f^{(16)} &= \partial_\mu A_f^{(16)} + i \hat{a}_\mu \psi_f^{(16)}, \\
D_\mu \psi_f^{(16)} &= \partial_\mu \psi_f^{(16)} + i \hat{a}_\mu \bar{\psi}_f^{(16)}, \\
D_\mu \hat{A}_f^{(16)} &= \partial_\mu \hat{A}_f^{(16)} - i \hat{a}_\mu \bar{\psi}_f^{(16)} - i \psi_f^{(16)} \bar{a}_\mu, \\
D_\mu \hat{\bar{A}}_f^{(16)} &= \partial_\mu \hat{\bar{A}}_f^{(16)} + i [\bar{a}_\mu, \hat{\bar{A}}_f^{(16)}], \\
D_\mu \hat{\bar{\psi}}_f^{(16)} &= \partial_\mu \hat{\bar{\psi}}_f^{(16)} + i [\bar{a}_\mu, \hat{\bar{\psi}}_f^{(16)}].
\end{aligned}
\end{equation}

Let us recall—see (4.10) and (4.11)—that the action of $\delta_{WZ}^\xi$ on the components of $\hat{V}$ is well defined on the infinitesimal gauge orbit of the ordinary fields these components depend upon. This state of affairs also occurs for the components of the noncommutative matter, $\hat{F}_f^{(16)}$, and $\hat{\Phi}_f^{(16)}$, and Higgs superfields, $\hat{\Phi}_f^{(H)}, H = 210, 10, 126, \overline{126}, 120$, constructed above. Indeed, if $\hat{\psi}$ stands for any of those noncommutative components, then

\begin{equation}
\delta_{WZ}^\xi \hat{\psi}[a_\mu, \bar{a}_\mu; \psi + \delta_{\Omega} \psi] = \hat{\delta}_{\Omega} (\delta_{WZ}^\xi \hat{\psi}),
\end{equation}

where $\hat{\delta}_{\Omega} (\delta_{WZ}^\xi \hat{\psi}) = -i \hat{\Omega} \sigma^\mu \hat{\psi}$, for the components of $\hat{F}_f^{(16)}$; $\hat{\delta}_{\Omega} (\delta_{WZ}^\xi \hat{\psi}) = -i \delta_{WZ}^\xi \hat{\psi} \bar{\sigma}_{\Omega}$, if $\hat{\psi}$ denotes any component of $\hat{F}_f^{(16)}$, and $\hat{\delta}_{\Omega} (\delta_{WZ}^\xi \hat{\psi}) = -i [\bar{\Omega}, \delta_{WZ}^\xi \hat{\psi}]$, when they are the components of $\hat{\Phi}_f^{(H)}$, the ones we are dealing with. $\delta_{\Omega}$ generates the ordinary infinitesimal gauge transformations. We then conclude that there is no obstruction to demand gauge invariance and invariance under the supersymmetry transformations in (4.25) at the same time.

Let $\hat{\psi}$ denote any of the noncommutative component fields in $\hat{\psi}_f^{(16)}, \hat{\bar{\psi}}_f^{(16)}, \hat{F}_f^{(16)}, \hat{\bar{A}}_f^{(16)}, \hat{\bar{\psi}}_f^{(16)}, \hat{\bar{F}}_f^{(16)}$, $(\hat{A}_f^{(16)}, \psi_f^{(16)}, \hat{F}_f^{(16)})$ or $(\hat{A}_f^{(H)}, \hat{\psi}_f^{(H)}, \hat{F}_f^{(H)})$. Then, using the fact that $\hat{\psi}$ solves the appropriate Seiberg–Witten map equations in (4.17), (4.19) or (4.21), it is not difficult to show that

\begin{equation}
\hat{\psi} = \hat{\psi} + \delta_{WZ}^\xi \hat{\psi}
\end{equation}

solves the same Seiberg–Witten map equation as $\hat{\psi}$. Of course, at $\omega^{\mu\nu} = 0$, $\hat{\psi}$ and $\hat{\bar{\psi}}$ differ by an ordinary supersymmetry transformation in the Wess–Zumino gauge of $\hat{\psi}[\omega = 0]$.

We have seen that the spaces of solutions of the Seiberg–Witten map equations in (4.17), (4.19) and (4.21) are constituted, respectively, by the noncommutative matter, $(\hat{A}_f^{(16)}, \hat{\psi}_f^{(16)}, \hat{F}_f^{(16)}), (\hat{A}_f^{(H)}, \hat{\psi}_f^{(H)}, \hat{F}_f^{(H)})$, and Higgs, $(\hat{A}_f^{(H)}, \hat{\psi}_f^{(H)}, \hat{F}_f^{(H)})$, triplets. On these
spaces of solutions $\tilde{\delta}_N^{WZ}$ acts according to the formulae in (4.25). Let us show now that each of these spaces of solutions carries a representation of the $N = 1$ supersymmetry algebra. Taking into account the definitions in (4.25), one shows that

$$\left[\delta_N^{WZ}, \tilde{\delta}_N^{WZ}\right] \hat{\phi} = -2i(\xi_1 \sigma^\mu \xi_2 - \xi_2 \sigma^\mu \xi_1) \hat{\phi}_\mu + \delta_{\Lambda} \hat{\phi},$$

where $\hat{\phi}$ stands for any of the fields in the noncommutative matter and Higgs triplets we are dealing with and

$$\hat{\Lambda} = 2i(\xi_1 \sigma^\mu \xi_2 - \xi_2 \sigma^\mu \xi_1) \hat{\phi}_\mu.$$

Of course, this is the same $\hat{\Lambda}$ as for the noncommutative gauge supermultiplet $(\hat{a}_\mu, \hat{\lambda}_\sigma, \hat{D})$; see equations (4.13) and (4.14). The noncommutative gauge transformation $\delta_{\Lambda} \hat{\phi}$ is given by

$$\begin{align*}
\delta_{\Lambda} \hat{\phi} &= -i \Lambda \star \hat{\phi}, & \text{if} & \hat{\phi} \in (\hat{A}_f^{(16)}, \hat{\psi}_f^{(16)}, \hat{F}_f^{(16)}), \\
\delta_{\Lambda} \hat{\phi} &= i \Lambda \star \hat{\phi}, & \text{if} & \hat{\phi} \in (\hat{\tilde{A}}_f^{(16)}, \hat{\tilde{\psi}}_f^{(16)}, \hat{\tilde{F}}_f^{(16)}), \\
\delta_{\Lambda} \hat{\phi} &= -i [\hat{\Lambda}, \hat{\phi}]. & \text{if} & \hat{\phi} \in (\hat{A}^{(H)}, \hat{\psi}^{(H)}, \hat{F}^{(H)}).
\end{align*}$$

We are now ready to introduce the noncommutative deformations, say $S_{\tilde{V}_\Phi}$ and $S_{\text{spot}}$, of $S_{V\Phi}$ and $S_{\text{spot}}$ in (3.7). But first, we impose the following conditions on the components of the noncommutative matter superfields $\hat{\Phi}_f^{(16)}$, $\tilde{\Phi}_f^{(16)}$ and $\tilde{\Phi}_f^{(H)}$:

$$\begin{align*}
\hat{A}_f^{(16)}[\omega = 0] &= \hat{A}_f^{(16)}, & \hat{\psi}_f^{(16)}[\omega = 0] &= \hat{\psi}_f^{(16)}, & \hat{F}_f^{(16)}[\omega = 0] &= \hat{F}_f^{(16)}, \\
\tilde{A}_f^{(16)}[\omega = 0] &= \tilde{A}_f^{(16)}, & \tilde{\psi}_f^{(16)}[\omega = 0] &= \tilde{\psi}_f^{(16)}, & \tilde{F}_f^{(16)}[\omega = 0] &= \tilde{F}_f^{(16)}, \\
\hat{A}_f^{(H)}[\omega = 0] &= \hat{A}_f^{(H)}, & \hat{\psi}_f^{(H)}[\omega = 0] &= \hat{\psi}_f^{(H)}, & \hat{F}_f^{(H)}[\omega = 0] &= \hat{F}_f^{(H)}.
\end{align*}$$

(4.26)

Furnished with the noncommutative superfields $\hat{\Phi}_f^{(16)}$, $\tilde{\Phi}_f^{(16)}$ and $\tilde{\Phi}_f^{(H)}$ whose components satisfy the conditions in (4.26) and the noncommutative vector superfield $\tilde{V}$ employed to define $S_{\text{NC}SYM}$ in (4.15), we define $S_{\tilde{V}_\Phi}$ and $S_{\text{spot}}$ in terms of them as follows:

$$\begin{align*}
S_{\tilde{V}_\Phi} &= \int d^4x \; d^2\theta \; d^2\bar{\theta} \sum_f \left( \gamma^{(10)}_{ff'} \hat{\Phi}_f^{(16)} \star \hat{\Phi}_{f'}^{(16)} + \sum_{H} \frac{1}{s(H)} \text{Tr}((\hat{\Phi}_f^{(H)})^* \star \hat{\Phi}_f^{(H)} \star \hat{\Phi}_f^{(H)} \star e^{-2\tilde{V}}) \right), \\
S_{\text{spot}} &= \int d^4x \; d^2\theta \; [W_{\text{matter}} + W_{\text{Higgs}}] + \text{h.c.},
\end{align*}$$

(4.27)

where

$$W_{\text{matter}} = \sum_{f,f'} \left\{ \gamma^{(10)}_{ff'} \hat{\Phi}_f^{(16)} \star \hat{\Phi}_{f'}^{(16)} + \gamma^{(120)}_{ff'} \hat{\Phi}_f^{(16)} \star \hat{\Phi}_{f'}^{(16)} \right\},$$

and

$$W_{\text{Higgs}} = \frac{M_2^{(120)}}{64(4!)} \text{Tr} \hat{\Phi}_f^{(210)} \star \hat{\Phi}_{f'}^{(210)} - \frac{M_2^{(126)}}{32(5!)} \text{Tr} \hat{\Phi}_f^{(126)} \star \hat{\Phi}_{f'}^{(126)} + \frac{M_2^{(10)}}{64} \text{Tr} \hat{\Phi}_f^{(10)} \star \hat{\Phi}_{f'}^{(10)} + \sum_{H} \left[ \lambda_1 \text{Tr} \hat{\Phi}_f^{(120)} \star \hat{\Phi}_{f'}^{(120)} + \lambda_2 \text{Tr} \hat{\Phi}_f^{(120)} \star \hat{\Phi}_{f'}^{(120)} + \lambda_3 \text{Tr} \hat{\Phi}_f^{(120)} \star \hat{\Phi}_{f'}^{(120)} + \lambda_4 \text{Tr} \hat{\Phi}_f^{(120)} \star \hat{\Phi}_{f'}^{(120)} + \lambda_5 \text{Tr} \hat{\Phi}_f^{(120)} \star \hat{\Phi}_{f'}^{(120)} \right],$$

(4.28)
It is apparent that $S_{\hat{Y}}$ and $S_{\hat{\alpha}}$ are invariant under the noncommutative supersymmetry transformations in (4.24) and the gauge transformations in (4.23). When $S_{\hat{Y}}$ and $S_{\hat{\alpha}}$ are expressed in terms of the components of the noncommutative superfields, the corresponding invariance is given by the transformations in (4.10) and (4.25), on the one hand, and (4.5), (4.18), (4.20) and (4.22), on the other hand.

We would like to stress that $S_{\hat{Y}}$ and $S_{\hat{\alpha}}$ in (4.27) and (4.28) almost look like the naive deformations of their corresponding ordinary counterparts $S_{Y}$ and $S_{\alpha}$, which are displayed in (3.7) and (3.8). This likeness we have pointed out partially stems from the fact that the components of the Higgs superfields, $\Phi^{(H)}$, in (4.16) take values in the Clifford algebra $\mathbb{C}l_{10}(\mathbb{C})$. Note that the doubling that occurs in some of the terms in $W_{Higgs}$ is due to the fact that given three functions $f_1$, $f_2$ and $f_3$,

$$\int d^3 x \, f_1 \ast f_2 \ast f_3 \neq \int d^3 x \, f_1 \ast f_3 \ast f_2,$$

unless two of them are equal.

Finally, it is easy—although lengthy—to express $S_{\hat{Y}}$ and $S_{\hat{\alpha}}$ in terms of the ordinary fields. To do so, one first obtains explicit expressions for $A^{(16)}_{\hat{Y}}, \psi^{(16)}_{\hat{Y}}, F^{(16)}_{\hat{Y}}, A^{(16)}_{\hat{\alpha}}, \psi^{(16)}_{\hat{\alpha}}, F^{(16)}_{\hat{\alpha}}, \tilde{A}^{(16)}_{\hat{Y}}, \tilde{\psi}^{(16)}_{\hat{Y}}, \tilde{F}^{(16)}_{\hat{Y}}$ and $\tilde{A}^{(16)}_{\hat{\alpha}}, \tilde{\psi}^{(16)}_{\hat{\alpha}}, \tilde{F}^{(16)}_{\hat{\alpha}}$ in terms of the corresponding ordinary fields: the reader has only to particularize the general expressions in [39] to the case at hand. Then, one substitutes those expressions in (4.27) and (4.28) and does the lengthy arithmetic.

5. Final comments

In this paper, we have formulated the minimal and new minimal supersymmetric GUTs on canonical (i.e., $[X^\mu, X'^\nu] = i\omega^{\mu\nu}$) noncommutative space-time by using the enveloping-algebra formalism. Taking advantage of the Seiberg–Witten map, we have constructed noncommutative superfields in the Wess–Zumino gauge out of the ordinary components of the corresponding ordinary superfields. Thus, supersymmetry is linearly realized explicitly in terms of the noncommutative fields. However, unlike in the $U(n)$ case in the fundamental representation, the noncommutative supersymmetry transformations in (4.10) cannot be generated by applying the Seiberg–Witten map to an $\omega$-deformed transformation of the ordinary fields. Indeed, it can be shown—as in [50]—that the equations

$$\hat{a}_\mu + \hat{\delta}_\xi a_\mu = \hat{a}_\mu[a_\mu + \hat{\delta}_\xi a_\mu, \lambda_\alpha + \hat{\delta}_\xi \lambda_\alpha, D + \hat{\delta}_\xi D],$$

$$\hat{\lambda}_\alpha + \hat{\delta}_\xi \lambda_\alpha = \hat{\lambda}_\alpha[a_\mu + \hat{\delta}_\xi a_\mu, \lambda_\alpha + \hat{\delta}_\xi \lambda_\alpha, D + \hat{\delta}_\xi D],$$

$$\hat{D} + \hat{\delta}_\xi \hat{D} = \hat{D}[a_\mu + \hat{\delta}_\xi a_\mu, \lambda_\alpha + \hat{\delta}_\xi \lambda_\alpha, D + \hat{\delta}_\xi D]$$

are not satisfied by any $\hat{\delta}_\xi a_\mu$, $\hat{\delta}_\xi \lambda_\alpha$ and $\hat{\delta}_\xi D$ in the Lie algebra of $SO(10)$, if $\hat{a}_\mu[\cdots, \lambda_\alpha, \cdots]$ and $\hat{D}[\cdots, \cdots]$ define Seiberg–Witten maps. Analogously, it is not difficult to see that the noncommutative supersymmetry transformations of the noncommutative Higgses and their superpartners in (4.25) cannot be generated from variations of the corresponding ordinary fields as follows:

$$\hat{A}^{(H)} + \hat{\delta}_\xi \hat{A}^{(H)} = \hat{A}^{(H)}[a_\mu + \hat{\delta}_\xi a_\mu, A^{(H)} + \hat{\delta}_\xi A^{(H)}, \psi^{(H)} + \hat{\delta}_\xi \psi^{(H)}, F^{(H)} + \hat{\delta}_\xi F^{(H)}],$$

$$\hat{\psi}_\alpha^{(H)} + \hat{\delta}_\xi \hat{\psi}_\alpha^{(H)} = \hat{\psi}_\alpha^{(H)}[a_\mu + \hat{\delta}_\xi a_\mu, A^{(H)} + \hat{\delta}_\xi A^{(H)}, \psi^{(H)} + \hat{\delta}_\xi \psi^{(H)}, F^{(H)} + \hat{\delta}_\xi F^{(H)}],$$

$$\hat{F}^{(H)} + \hat{\delta}_\xi \hat{F}^{(H)} = \hat{F}^{(H)}[a_\mu + \hat{\delta}_\xi a_\mu, A^{(H)} + \hat{\delta}_\xi A^{(H)}, \psi^{(H)} + \hat{\delta}_\xi \psi^{(H)}, F^{(H)} + \hat{\delta}_\xi F^{(H)}],$$

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where $\hat{A}^{(H)}[\cdot, \cdot, \cdot, \cdot]$, $\hat{\psi}^{(H)}[\cdot, \cdot, \cdot, \cdot]$ and $\hat{F}^{(H)}[\cdot, \cdot, \cdot, \cdot]$ give Seiberg–Witten maps. In summary, the supersymmetry of our noncommutative SO(10) theories is not realized by the ordinary fields, but, recall that it is linearly realized by the noncommutative fields. Let us point out that in the U(n) case—in the fundamental representation or its siblings—such realization of the supersymmetry transformations in terms of ordinary fields exists, but it is at the cost of being a nonlinear $\omega$-dependent transformation—see [50].

It is thus clear that if one uses ordinary fields—the fields that create and destroy leptons, quarks, photons, gluons, etc—to formulate, via the Seiberg–Witten map, our SO(10) supersymmetric theories on noncommutative space-time, the picture that emerges as regards to the supersymmetry properties of those ordinary fields differs radically from the picture that materializes when those very fields are used to formulate the corresponding supersymmetric theories on ordinary Minkowski space-time. Indeed, when space-time is noncommutative there is no supersymmetry in terms of the ordinary fields, although there is a hidden supersymmetry that reveals itself when the noncommutative fields are used. It is too early to say whether this absence of supersymmetry for the ordinary fields in the noncommutative theory can be accepted¹ as a supersymmetry breaking mechanism relevant for the description of Nature: if so, it would be the noncommutative character of space-time that breaks through interactions the supersymmetry carried by ordinary fields when $\omega^{\mu\nu} = 0$. It is clear that more understanding of the properties of the theories at hand is needed before a verdict is issued. It should be noticed that the logarithmic UV/IR mixing phenomena of noncommutative supersymmetric theories [53] may be a key to interpreting as a phenomenologically relevant supersymmetry breaking mechanism the fact that supersymmetry is not realized by the ordinary fields in the noncommutative theory; otherwise, the lower the energy, the closer we would be to $\omega^{\mu\nu} = 0$, where supersymmetry is realized (linearly) by the ordinary fields. Hence, it would seem right to think that in defining the GUTs introduced above the Seiberg–Witten map should not be understood as a formal power series expansion in $\omega^{\mu\nu}$, but in an $\omega^{\mu\nu}$-exact form way—see [39] for the appropriate formulae. Let us point out that the $\omega$-exact Seiberg–Witten map is not a polynomial in the $\ast$-product, so there may be UV/IR mixing even though the gauge group is simple.

It is plain that there are many issues—UV/IR mixing, renormalizability, vacua, . . . —regarding the noncommutative GUTs we have introduced above that should be studied to gain more understanding of the properties of these theories. In particular, it is an open problem to see whether our noncommutative GUTs fit in F-theory—or more generally in the String Theory framework—where the SO(10) group occurs naturally and where noncommutativity effects have been unveiled [54].

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References

[1] Madore J, Schraml S, Schupp P and Wess J 2000 Eur. Phys. J. C 16 161 (arXiv:hep-th/0001203)
[2] Jurco B, Schraml S, Schupp P and Wess J 2000 Eur. Phys. J. C 17 521 (arXiv:hep-th/0006246)
[3] Jurco B, Moller L, Schraml S, Schupp P and Wess J 2001 Eur. Phys. J. C 21 383 (arXiv:hep-th/0104153)
[4] Calmet X, Jurco B, Schupp P, Wess J and Wohlgenannt M 2002 Eur. Phys. J. C 23 363 (arXiv:hep-ph/0111115)

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