Double commutants of multiplication operators on $C(K)$.

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Abstract. Let $C(K)$ be the space of all real or complex valued continuous functions on a compact Hausdorff space $K$. We are interested in the following property of $K$: for any real valued $f \in C(K)$ the double commutant of the corresponding multiplication operator $F$ coincides with the norm closed algebra generated by $F$ and $I$. In this case we say that $K \in \mathcal{DCP}$. It was proved in [1] that any locally connected metrizable continuum is in $\mathcal{DCP}$. In this paper we indicate a class of arc connected but not locally connected continua that are in $\mathcal{DCP}$. We also construct an example of a continuum that is not arc connected but is in $\mathcal{DCP}$.

1. Introduction

The famous von Neumann’s double commutant theorem [2] can be stated the following way. Let $(X, \Sigma, \mu)$ be a space with measure and $f$ be a real-valued element of $L^\infty(X, \Sigma, \mu)$. Let $F$ be the corresponding multiplication operator in $L^2(X, \Sigma, \mu)$, i.e. $(Fx)(t) = f(t)x(t)$ for $x \in L^2(X, \Sigma, \mu)$ and $t$ from a subset of full measure in $X$. Then

$$\{F\}^{cc} = \mathcal{A}_F$$

where $\{F\}^{cc}$ is the double commutant (or bicommutant) of $F$, i.e. $\{F\}^{cc}$ consists of all bounded linear operators on $L^2(X, \Sigma, \mu)$ that commute with every operator commuting with $F$ and $\mathcal{A}_F$ is the closure in the weak (or strong) operator topology of the algebra generated by $F$ and the identity operator $I$.

The generalization on the case of complex multiplication operators (or normal operators on a Hilbert space) is then immediate. Quite naturally arises the question of obtaining similar results for multiplication operators on other Banach spaces of functions. De Pagter and Ricker proved in [3] that von Neumann’s result remains true for spaces $L^p(0, 1), 1 \leq p < \infty$, and more generally for any Banach ideal $X$ in the space of all measurable functions such that $X$ has order continuous norm and $L^\infty(0, 1) \subset X \subseteq L^1(0, 1)$. But they also proved that the double commutant of the operator $T$, $(Tx)(t) = tx(t), x \in L^\infty, t \in [0, 1]$, is considerably larger than the algebra $\mathcal{A}_T$ and consists of all operators of multiplication by Riemann integrable functions on $[0, 1]$. The
last result gives rise to the following question: let \( C(K) \) be the space of all continuous real-valued functions on a Hausdorff compact space \( K \). When is it true that for every multiplication operator \( F \) on \( C(K) \) its double commutant coincides with the algebra \( A_F \)? This property is obviously a topological invariant of \( K \) and we will denote the class of compact Hausdorff spaces that have it as \( \mathcal{DCP} \) (short for double commutant property).

2. Continuums with \( \mathcal{DCP} \) property

In [I] the author proved that if \( K \) is a compact metrizable space without isolated points then the following implications hold.

(1) If \( K \) is connected and locally connected then \( K \in \mathcal{DCP} \).
(2) If \( K \in \mathcal{DCP} \) then \( K \) is connected.

In the presence of isolated points the analogues of the above statements become more complicated (see [I, Theorem 1.15]). To avoid these minor complications and keep closer to the essence of the problem we will assume that the compact spaces we consider have no isolated points.

A simple example (see [I, Example 1.16]) shows that the condition that \( K \) is connected is not sufficient for \( K \in \mathcal{DCP} \).

Example 1. Let \( K \) be the closure in \( \mathbb{R}^2 \) of the set \( \{(x, \sin 1/x) : x \in (0, 1]\} \). Let \( f(x, y) = x, (x, y) \in K \), and let \( F \) be the corresponding multiplication operator. Then it is easy to see (see details in [I, Example 1.16]) that the double commutant \( \{F\}^{cc} \) consists of all operators of multiplication on functions from \( C(K) \) but \( A_F \) consists of operators of multiplication on functions from \( C(K) \) that are constant on the set \( \{(0, y) : y \in [0, 1]\} \).

Therefore the next question is whether the condition that \( K \) is connected and locally connected is necessary for \( K \in \mathcal{DCP} \)? Below we provide a negative answer to this question. In order to consider the corresponding example let us recall the following two simple facts.

Proposition 2. Let \( K \) be a compact Hausdorff space and \( f \in C(K) \). Let \( F \) be the corresponding multiplication operator. Then

(1) The double commutant \( \{F\}^{cc} \) consists of multiplication operators.
(2) The algebra \( A_F \) coincides with the closure of the algebra generated by \( F \) and \( I \) in the operator norm.
Proof. (1) Let $T \in \{F\}^{cc}$. Let $1$ be the function in $C(K)$ identically equal to 1. Clearly for every $a \in C(K)$ the operator $F$ commutes with the multiplication operator $A$ where $Ax = ax, x \in C(K)$. Therefore for any $a \in C(K)$ $T$ commutes with $A$ and $T(a) = T(a1) = TA1 = AT1 = aT1 = (T1)a$. Hence if we take $g = T1$ then $T$ coincides with the multiplication operator $G$ generated by the function $g$.

(2) If $T \in \{F\}^{cc}$ then by part (1) of the proof $T = G$ where $G$ is a multiplication operator by a function $g \in C(K)$. It remains to notice that $\|G\| = \|G1\|_{C(K)}$ and therefore on $\{F\}^{cc}$ the convergence in strong operator topology implies convergence in the operator norm.  

\[\square\]

Corollary 3. Let $f \in C(K)$ and $F$ be the corresponding multiplication operator. The following two statements are equivalent.

1. $\{F\}^{cc} = A_F$.
2. For any $G \in \{F\}^{cc}$ and for any $s, t \in K$ the implication holds

\[f(s) = f(t) \Rightarrow g(s) = g(t),\]

where $g \in C(K)$ is the function corresponding to the operator $G$.

In what follows our main tool will be the following lemma which was actually proved though not stated explicitly in [1] (see [1, Proof of Theorem 1.14]).

Lemma 4. Let $K$ be a compact metrizable space, $f \in C(K)$, and $F$ be the corresponding multiplication operator. Let $G \in \{F\}^{cc}$ and $g$ be the corresponding function from $C(K)$. Let $u, v \in K$ be such that

- $f(u) = f(v)$.
- The points $u$ and $v$ have open, and locally connected neighborhoods in $K$.
- For any open connected neighborhood $U$ of $u$ there is an open interval $I_U$ in $\mathbb{R}$ such that $f(u) \in I_U \subset f(U)$.

Then $g(u) = g(v)$.

We will also need a simple lemma proved in [1, Lemma 1.13]

Lemma 5. Let $K$ be a compact Hausdorff space, $F,G$ multiplication operators on $C(K)$ by functions $f$ and $g$, respectively and $G \in \{F\}^{cc}$. Let $k \in K$ be such that $\text{Int}f^{-1}(\{f(k)\}) \neq \emptyset$. Then $g$ is constant on $f^{-1}(\{f(k)\})$.

Now we are ready to give an example of a metrizable connected compact space $K$ such that $K$ is not locally connected but $K \in DCP$. 
Let $B$ be the well known “broom”.

$$B = \text{cl}\{(x,y) \in \mathbb{R}^2 : x \geq 0, y = \frac{1}{n} x, n \in \mathbb{N}, x^2 + y^2 \leq 1\}.$$ 

**Proposition 6.** $B \in \mathcal{DCP}$. 

*Proof.* Let $f \in C(B)$ and $G \in \{F\}^{cc}$. By part (1) of Proposition 2 $G$ is a multiplication operator. Let $g$ be the corresponding function from $C(K)$. Let $u, v \in B$ and $f(u) = f(v)$. We can assume without loss of generality that $f \geq 0$ and $\min_{k \in B} f(k) = 0$. Let $D = \{k \in B : k = (x,0), 0 < x \leq 1\}$. We will divide the remaining part of the proof into four steps.

1. Assume first that $u, v \in B \setminus D$ and that $0 < f(u) = f(v) < M = \max_{k \in B} f(k)$. For any $m \in \mathbb{N}$ let $B_m = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y = \frac{1}{n} x, n \geq m, x^2 + y^2 \leq 1\}$. Then for any large enough $m$ we have

$$\min_{k \in B_m} f(k) < f(u) < \max_{k \in B_m} f(k).$$

Notice that for every $m \in \mathbb{N}$ the set $B_m$ is a compact, connected and locally connected subset of $B$. Moreover, every point of $B_m$ is a point of local connectedness in $B$ and the set $B_m \setminus \{0,0\}$ is open in $B$. Let $B^1_m = \text{cl}\{k \in B_m : f(k) < f(u)\}$ and $B^2_m = \text{cl}\{k \in B_m : f(k) > f(u)\}$. There are two possibilities. (a) The set $B^1_m \cap B^2_m$ is empty. In this case, because $B_m$ is connected, $f \equiv f(u)$ on some open subset of $B$ and by Lemma 3 we have $g(u) = g(v)$.

(b) $\exists w \in B^1_m \cap B^2_m$. Because $B$ is locally connected at $w$ the pairs of points $(u,w)$ as well as $(v,w)$ satisfy all the conditions of Lemma 4 whence $g(u) = g(v)$.

2. Let $u, v \in B \setminus D$ and $f(u) = f(v) = 0$. There are two possibilities. First: $f \equiv 0$ on some open neighborhood of either $u$ or $v$. Then $g(u) = g(v)$ by Lemma 5. Second: $f$ is not constant on any open neighborhood of either $u$ or $v$. In this case, because $B \setminus D$ is locally connected, we can find sequences $u_n \to u$ and $v_n \to v$ such that $u_n, v_n \in B \setminus D$ and $0 < f(u_n) = f(v_n) < M$, $n \in \mathbb{N}$. Then by the previous step $g(u_n) = g(v_n)$ whence $g(u) = g(v)$. The case $u, v \in B \setminus D$ and $f(u) = f(v) = M$ can be considered similarly.

3. Now we will assume that $u$ and $v$ are arbitrary distinct points of $B$ and that $0 < f(u) = f(v) < M$. Let again $m \in \mathbb{N}$ be so large that inequalities (1) hold. Like on step (I) we have two alternatives (a) and

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1 We can assume of course that $M > 0$ because otherwise $F = 0$ and the statement $\{F\}^{cc} = \mathcal{A}_F$ becomes trivial.
(b). In case (a) we apply again Lemma 5. In case (b) we cannot apply Lemma 4 directly because $B$ might be not locally connected at $u$ and/or at $v$. Therefore we fix $w \in B_1 \cap B_2$ and consider two subcases. (b1). $f$ is constant on some neighborhood of either $u$ or $v$. Then $f(u) = f(v)$ by Lemma 5. (b2). $f$ is not constant on any open neighborhood of $u$ or $v$. Let $u_n \in B \setminus D$ be such that $u_n \to u$. Because $f(w)$ is an inner in $R$ point of the set $f(W)$ where $W$ is an arbitrary connected open neighborhood of $w$ in $B_m$ we can find a sequence of points $u_n \in B_1 \cap B_2$ and consider two subcases. (b1). $f$ is constant on some neighborhood of either $u$ or $v$. Then $f(u_n) = f(u)$. (b2). $f$ is not constant on any open neighborhood of $u$ or $v$. Let $u_n \in B \setminus D$ be such that $u_n \to u$.

(IV). Finally assume that $u$ and $v$ are arbitrary points in $B$ and $f(u) = f(v) = 0$ (the case $f(u) = f(v) = M$ can be considered in the same way). If there is a point $w \in B \setminus D$ such that $f(w) = 0$ then we can proceed as in step (III). Let us assume therefore that $f > 0$ on $B \setminus D$. Let $a \in (0, 1)$ be the smallest number such that $f(a, 0) = 0$. Then for any $n \in N$ such that $n > 1/a$ the set $f\{\{(x, 0): a-1/n \leq x \leq a\}\}$ is an interval $[0, \delta_n]$ where $\delta_n > 0$. Therefore we can find $a_n \in [a-1/n, a)$ and $u_n \in B \setminus D$ such that $u_n \to u$ and $f(u_n) = f(a_n, 0), n \in N, n > 1/a$, whence by step (III) $g((a_n), 0) = g(u_n)$ and therefore $g((a, 0) = g(u)$. Similarly, $g(v) = g((a, 0) = g(u)$ and we are done.

By analyzing the steps of the proof of Proposition 6 and the properties of the broom $B$ we used, we come to the following more general statement that can be proved in exactly the same way as Proposition 6.

Proposition 7. Let $K$ be a compact connected metrizable space. Assume that there are compact subsets $K_m, m \in N$ of $K$ with the properties.

1. $K_m \subsetneq K_{m+1}$.
2. $K_m$ is connected and locally connected.
3. The interior of $K_m$ in $K$ is dense in $K_m, m \in N$.
4. Every point of $K_m$ is a point of local connectedness in $K$.
5. The set $\bigcup_{m=1}^{\infty} K_m$ is dense in $K$.
6. For every point $k \in K \setminus \bigcup_{m=1}^{\infty} K_m$ there is a path in $K$ from $k$ to a point in $\bigcup_{m=1}^{\infty} K_m$.
Then $K \in \mathcal{DCP}$.

**Example 8.** This example is somewhat similar, though topologically not equivalent to the broom. The corresponding compact subspace of $\mathbb{R}^2$ is traditionally called the “bookcase” and is defined as follows.

$$BC = \overline{\bigcup_{n=1}^{\infty} \{(x, 1/n) : x \in [0, 1]\}} \cup \{(0, y) : y \in [0, 1]\} \cup \{(1, y) : y \in [0, 1]\}.$$ 

We claim that $BC \in \mathcal{DCP}$.

**Proof.** For any $m \in \mathbb{N}$ let $BC_m = BC \cap \{(x, y) \in \mathbb{R}^2 : y \geq 1/m\}$. Then the compacts $BC_m$ have properties (1) – (6) from the statement of Proposition 7. □

The conditions of Proposition 7 and the arc connectedness theorem (see [4, Theorem 5.1, page 36]) guarantee that the compact space $K$ satisfying the conditions of that proposition is arc connected. It is not known to the author if the arc connectedness of a metrizable compact $K$ is sufficient for $K \in \mathcal{DCP}$, but as our next example shows it surely is not necessary.

**Proposition 9.** Let $K$ be the union of the square $[-1, 0] \times [-1, 1]$ and the set $\{(x, \sin(1/x)) : 0 < x \leq 1\}$. Then $K \in \mathcal{DCP}$.

**Proof.** Let $f \in C(K)$, $F$ be the corresponding multiplication operator, $G \in \{F\}^{cc}$, and $g \in C(K)$ the function corresponding to $G$. We can assume without loss of generality that $f(K) = [0, M]$ where $M > 0$. Let $E = \{(0, y) : y \in [-1, 1]\}$. Notice that $K$ is locally connected at any point of $K \setminus E$. The set $K \setminus E$ is the union of two disjoint open connected subsets of $K$: $C_1 = [-1, 0] \times [-1, 1]$ and $C_2 = \{(x, \sin(1/x)) : x \in (0, 1]\}$. Like in the proof of Proposition 6 we have to consider several possibilities.

(1) If $u, v \in C_1$ or $u, v \in C_2$ and $0 < f(u) = f(v) < M$. In this case we can prove that $g(u) = g(v)$ in very much the same way as in step (I) of the proof of Proposition 6 by considering the sets $C_{1,m} = [-1 \times -1/m], m \in \mathbb{N}$ (respectively the sets $C_{2,m} = \{(x, \sin(1/x)) : 1/m < x < 1\}$).

(2) Let now assume that $0 < f(u) = f(v) < M$, $u \in C_1$, $v \in C_2$, and at least one of the inequalities holds $f(u) < \sup_{k \in C_1} f(k)$ or $f(v) < \sup_{k \in C_2} f(k)$. Then like in the proof of Proposition 6, we can either find an open subset of $K$ on which $f$ is identically equal
Problem 10.  

1) Is it possible to characterize the metrizable continua from the class $DCP$ in purely topological terms not involving multiplication operators?
(2) In particular, is it true that any metrizable arc connected continuum belongs to $\mathcal{DCP}$?

(3) This question is a special case of the previous one. Let $C$ be the standard Cantor set and

$$K = \{(x, y) : x \in C, y \in [0, 1]\} \cup \{(x, 0) : x \in [0, 1]\}.$$ 

Is it true that $K \in \mathcal{DCP}$? A positive answer to question (3) would be in the author’s opinion a strong indication that the answer to question (2) should also be positive.

References

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