We study the spectrum of low-lying eigenmodes of the kinetic operator for scalar particles, in the color adjoint representation of Yang-Mills theory. The kinetic operator is the covariant Laplacian, plus a constant which serves to renormalize mass. In the pure gauge theory, our data indicates that the interval between the lowest eigenvalue and the mobility edge tends to infinity in the continuum limit. On these grounds, it is suggested that the perturbative expression for the scalar propagator may be misleading even at distance scales that are small compared to the confinement scale. We also measure the density of low-lying eigenmodes, and find a possible connection to multi-critical matrix models of order \( m = 1 \).

I. INTRODUCTION

In recent years it has been recognized that kinetic operators in confining gauge theories may have a low-lying spectrum of localized eigenstates \([1–3]\). By “kinetic operator” we mean, e.g., a Euclidean Dirac operator or covariant Laplacian operator in a background gauge field, with the possible addition of a constant representing a mass term. The “mobility edge” is a point in the spectrum between an interval of spatially localized eigenmodes, and the bulk of eigenmodes which are extended over the full volume.\(^{1}\)

In condensed matter physics, it has long been known that the Hamiltonian of an electron moving in a stochastic potential has a low-lying spectrum of localized eigenstates \([4]\). If the energy at the mobility edge is higher than that of the Fermi surface, then the material is an insulator \([5]\). This is because the propagation of a wavepacket is essentially an interference effect among energy eigenstates which are spatially extended. If there are no available extended modes, then there is no particle propagation through the medium.

This condensed matter example motivates us to ask the following question: In a confining lattice gauge theory, what is the interval in GeV (Dirac operator) or \((\text{GeV})^2\) (covariant Laplacian) between the lowest eigenvalue of kinetic operator, and the mobility edge, as we take the continuum limit? If this interval shrinks to zero in physical units (but infinity in lattice units) in the continuum limit. The consequences of this fact for the propagation of scalar particles was not so clear. In our present paper we extend our study to the full spectrum of localized states. Our objective is to (i) find how the average interval \(\Delta \lambda_{\text{mob}}\) between the lowest eigenvalue and the mobility edge varies with coupling; and (ii) compute the density of eigenmodes in this interval. With those results in hand, we discuss possible consequences for the two-point function of adjoint scalar fields in confining gauge theories.

II. SCALING OF THE LOCALIZATION INTERVAL

The kinetic operator for scalar fields, in the adjoint representation of the SU(2) gauge group, is given by

\[
K_{xy}^{ab} = -D_{xy}^{ab} + m_0^2 \delta^{ab} \delta_{xy} \tag{2.1}
\]

where

\[
D_{xy}^{ab} = \sum_{\mu} \left[ U_{\mu}^{ab}(x) \delta_{x,x+\hat{\mu}} + U_{\mu}^\dagger{}^{ab}(x-\hat{\mu}) \delta_{x,x-\hat{\mu}} - 2 \delta^{ab} \delta_{xy} \right] \tag{2.2}
\]

is the covariant lattice Laplacian, with link variables

\[
U_{\mu}^{ab} = \frac{1}{2} \text{Tr}[\sigma^a U_{\mu}(x) \sigma^b U_{\mu}^\dagger(x)] \tag{2.3}
\]

in the adjoint representation, and \(m_0^2\) may be taken negative. We compute numerically the low-lying eigenvalues and eigenmodes of the covariant Laplacian operator

\[
-\Delta_{xy}^{ab} \phi_y^{ab}(x) = \lambda_y \phi_y^{ab}(x) \tag{2.4}
\]

in thermalized lattice configurations, and from these we compute the inverse participation ratio (IPR) of each eigenmode

\[
\text{IPR}_n = V \sum_x |\phi_n^a(x) \phi_n^b(x)|^2 \tag{2.5}
\]

\(^{1}\) In fact there is usually an interval of localized states at both the lower and the upper ends of the spectrum, and therefore two mobility edges.
From our previous work [2] we know that there is some range of eigenvalues of the covariant Laplacian, lying between the lowest eigenvalue $\lambda_0$ and the mobility edge $\lambda_{mob}$, whose corresponding eigenmodes are localized, and whose IPR’s grow linearly with lattice volume $V$. For $\lambda_n > \lambda_{mob}$ the eigenmodes are extended, and the IPR’s are all of $O(1)$ at large volumes.

At a given coupling $\beta$, then the data points at different volumes tend to cluster around the same straight line at small $\Delta \lambda$, as seen in Figs. 1-2, at $\beta = 2.35$, and Figs. 3-4 at $\beta = 2.5$. There does not appear to be a strong volume dependence in the data at $\Delta \lambda < \Delta \lambda_{mob}$. Above the mobility edge we would expect $R_n \to 0$ in the infinite volume limit, since in that limit the IPR’s of localized and extended states tend to infinity and to finite values, respectively. Fitting a straight line

\begin{equation}
    f(\Delta \lambda) = 1 - \frac{\Delta \lambda}{\Delta \lambda_{int}}
\end{equation}

to the combined volume data at small $\Delta \lambda$, the intercept $\Delta \lambda_{int}$ with the x-axis is a reasonable measure of the width of the peak in the $R$ vs. $\Delta \lambda$ data. Even if $\Delta \lambda_{int}$ is not precisely the same as $\Delta \lambda_{mob}$, we expect these quantities to scale with $\beta$ in the same way. The result, for $\beta \in [2.2, 2.5]$ is shown in Table I, and we see that the data is consistent with

\begin{equation}
    \Delta \lambda_{int} \approx \frac{0.045(3) \text{GeV}}{a}
\end{equation}
TABLE I: The intercept $\Delta \lambda_{int}$ vs. $\beta$, as determined from the best linear fit (2.9) to data in the intervals and volumes shown. $\Delta \lambda_{int}$ and $\Delta \lambda_{int}^\alpha$ are in units of GeV$^2$ and GeV, respectively.

| $\beta$ | volumes  | fitting interval | $\Delta \lambda_{int}$ | $\Delta \lambda_{int}^\alpha$ |
|--------|----------|----------------|------------------------|------------------------|
| 2.20   | $12^4, 14^4, 16^4$ | [0.0, 0.030] | 0.038(1) | 0.042 |
| 2.30   | $12^4, 14^4, 16^4, 18^4$ | [0.0, 0.040] | 0.051(1) | 0.044 |
| 2.35   | $14^4, 16^4, 18^4, 20^4, 24^4$ | [0.0, 0.045] | 0.065(1) | 0.047 |
| 2.40   | $14^4, 16^4, 18^4, 20^4, 24^4$ | [0.0, 0.050] | 0.076(2) | 0.047 |
| 2.50   | $16^4, 18^4, 20^4, 24^4$ | [0.0, 0.060] | 0.099(4) | 0.043 |

Since $\Delta \lambda_{int} \sim \Delta \lambda_{mob}$, the implication is that

$$\lim_{a \to 0} \Delta \lambda_{mob} = \infty \quad (2.11)$$

We reserve discussion of this result to section IV, below.

### III. DENSITY OF LOCALIZED STATES

The covariant Laplacian operator in adjoint representation can be thought of as a $3V \times 3V$ random matrix, where the factor of 3 comes from the color index. We are interested in computing the density of localized eigenstates at the low end of the spectrum. Suppose this has the form

$$\rho(\lambda) = \kappa(\lambda - \lambda_0)^\alpha \quad (3.1)$$

where we normalize the density of states such that

$$\int_{\lambda_0}^{\lambda_{max}} d\lambda \rho(\lambda) = 3 \quad (3.2)$$

Then the number of eigenmodes with eigenvalues less than $\lambda$ is

$$n(\lambda) = V \int_{\lambda_0}^{\lambda} d\lambda' \rho(\lambda') \quad (3.3)$$

A standard method for determining $\alpha$ is based on the observation that under the eigenvalue rescaling

$$z = V^{1/(1+\alpha)}(\lambda - \lambda_0) \quad (3.4)$$

the corresponding eigenvalue number $n(z)$ becomes independent of $V$. This is readily verified: the number of eigenvalues $\delta n(\lambda)$ in an interval $\delta \lambda$ around $\lambda$ is

$$\delta n(\lambda) = V \rho(\lambda) \delta \lambda \quad (3.5)$$

Rescale $z = V^\alpha(\lambda - \lambda_0)$. Then, using the assumed form (3.1) for $\rho(\lambda)$, we have

$$\delta n(z) = \kappa V^{1-\alpha} z^\alpha \delta z \quad (3.6)$$

Choosing $p = 1/(1+\alpha)$, $\delta n(z)$ is volume independent. So in order to compute the density of states, we compute $n(\lambda)$ at a variety of lattice volumes, and look for a constant $\alpha$ such that, under the rescaling (3.4), the curves for $n(z)$ computed at each lattice volume coincide.

In numerical simulations, we evaluate the first $N_{ev}$ eigenvalues of the covariant Laplacian in a set of $N_{conf}$ independent thermalized configurations. The resulting $N_{conf} \times N_{ev}$ eigenvalues of all $N_{conf}$ configurations are then sorted from lowest to highest regardless of configuration or eigenmode number. Then, if $\lambda_m$ is the $m$-th eigenvalue in the ordered list of eigenvalues, we identify

$$n(\lambda_m) = \frac{m}{N_{conf}} \quad (m = 1, 2, \ldots, N_{conf} \times N_{ev}) \quad (3.7)$$

The maximum value is $n(\lambda_{max}) = N_{ev}$, where $\lambda_{max}$ is the largest eigenvalue in the list. Other values of $n(\lambda)$ for $\lambda < \lambda_{max}$ and $\lambda \neq \lambda_m$ are obtained by interpolation (data points are connected by lines in the figures shown). The numerical results for $\beta = 2.5$ on $L^4$ lattices at $L = 14, 16, 20, 24$ are shown in Fig. 5.

![Figure 5](image)

FIG. 5: Cumulative number of eigenmodes $n(\lambda)$ with eigenvalue less than $\lambda$ at $\beta = 2.5$. Data shown is for lattice volumes from $14^4$ to $24^4$.

In Fig. 5 the $n(\lambda)$ curve is different for each lattice volume. This is, of course, not surprising, since as volume increases, so does the number of eigenvalues in any given interval $\Delta \lambda$. Since we are interested in checking universality under rescaling, right at the very end of the spectrum, we concentrate on the region where $n(\lambda) < 10$. In Figs. 6-8 we show $n(z)$, under the rescaling (3.4), for $\alpha = 1, 2, 4$ respectively. The data seems compatible with universality at $\alpha = 2$, in which case we would have

$$n(z) = \kappa \frac{z^\alpha}{3} \quad (3.8)$$

Fig. 9 shows a best cubic fit (solid line) to the $\alpha = 2$ scaled data at all volumes, in the interval $z \in [0, 2]$; $\kappa$ is determined to be 3.8. The cubic curve appears to fit the data at $\beta = 2.5$ quite well, strengthening the case that $\alpha = 2$ is the correct exponent.

The exponent $\alpha = 2$ may be of some significance. According to ref. [6], even-integer exponents $\alpha = 2m$ are obtained in simple large-N hermitian matrix models with polynomial potentials, when the coupling constants in the potential are...
Let \( \lambda_i \) be the \( i \)-th eigenvalue of the covariant Laplacian in the \( n \)-th lattice configuration, with \( \lambda_i > \lambda_{i-1} \). The set of all \( \{ \lambda_i^n \} \) of all configurations is then sorted in ascending order, and we denote by \( N_i^n \) the location (from 1 to \( N_{\text{conf}} \times N_{\text{ev}} \)) of \( \lambda_i^n \) in the sorted list. For the \( n \)-th configuration, the eigenvalue spacing \( s_i^n \) between neighboring eigenvalues in the unfolded spectrum is defined as

\[
s_i^n = \frac{N_{i+1}^n - N_i^n}{N_{\text{conf}}}
\]

(i.e. regardless of \( i \), \( n \)). In the case of the Dirac operator [8], the distribution \( P(s) \) is well described by a certain Wigner distribution. For the spectrum of the covariant Laplacian, our numerical results show that eigenvalue spacing distribution of the unfolded spectrum is also well described by one of the Wigner distributions, namely, the orthogonal distribution

\[
P(s) = \frac{\pi}{2} s^2 e^{-\frac{\pi}{2} s^2},
\]

as seen in Fig. 10 for \( \beta = 2.35 \) on an 18\(^4\) lattice volume. We have also checked that eq. \( (3.10) \) gives a good fit to the eigenvalue spacing distribution at \( \beta = 2.4 \) and \( \beta = 2.3 \).

Returning to the density of states \( \rho(\lambda) \) of the original spectrum, we find that the quadratic (\( \alpha = 2 \)) power behavior continues well past \( \lambda_{\text{mq}} \), but does not persist throughout the spectrum. Fig. 9 showed an excellent cubic fit to the cumulative \( n(z) \) data, but this was for low-lying eigenmodes with \( n(z) < 12 \). In Figs. 11 and 12 we show all of the data for \( n(z) \), obtained at \( \beta = 2.5 \) on 16\(^4\) and 20\(^4\) lattices, together with the cubic fit to the data at all volumes. The vertical line in each graph indicates the value of \( z \) which corresponds, at each volume, to \( \lambda - \lambda_0 = 0.05 \). In both cases the data for \( n(z) \) begins to diverge away from the cubic fit at about this value of...
\( \lambda - \lambda_0 \). In physical units, \( \Delta \lambda = (\lambda - \lambda_0)/a^2 \) at \( \lambda - \lambda_0 = 0.05 \) is roughly two and a half times our estimate for \( \Delta \lambda_{\text{int}} \) at this coupling.

Given the density of states we can compute the phase-space volume, in physical units, taken up by states below the mobility edge. For purposes of comparison we begin with the free case, where it is easy to show that the density of states in \( D = 4 \) dimensions is \( \rho(\lambda) = \lambda/\pi^2 \). The total number of states in the eigenvalue interval from \( \lambda = 0 \) to \( \lambda_{\text{stop}} = \Lambda a^2 \) is

\[
N_\Lambda = \sum_{\lambda \leq \Lambda a^2} 1 = V \int_0^{\Lambda a^2} d\lambda \frac{\lambda}{\pi^2}
\]

(3.11)

where \( a \) is the lattice spacing, and \( \Lambda \) is some fixed number in units of \( (\text{GeV})^2 \). Now divide by the physical volume, to get the number of states per unit volume, \( \eta \), which lie in the eigenvalue interval \( \lambda/a^2 < \Lambda \):

\[
\eta \equiv \frac{N_\Lambda}{V_{\text{phys}}} = \frac{1}{a^4} \int_0^{\Lambda a^2} d\lambda \frac{\lambda}{\pi^2} = \frac{1}{\pi^2} \int_0^\Lambda d\lambda_{\text{phys}} \lambda_{\text{phys}} = \frac{1}{\pi^2} \Lambda^2 \]

(3.12)

Then if \( \Lambda \) is a finite cutoff in physical units, \( \eta \) is also finite, even as \( a \to 0 \).

We make the same computation for localized states of the covariant Laplacian, which lie in the eigenvalue interval \( \lambda \in [\lambda_0, \lambda_{\text{mob}}] \)

\[
\eta \equiv \frac{N_{\lambda_{\text{mob}}}}{V_{\text{phys}}} = \frac{1}{a^4} \int_{\lambda_0}^{\lambda_{\text{mob}}} d\lambda \kappa(a)(\lambda - \lambda_0)^2 = \frac{\kappa(a)(\lambda_{\text{mob}} - \lambda_0)^3}{3 a^4}
\]

(3.13)

Denoting eigenvalues in physical units as \( \lambda_{\text{phys}} \)

\[
\lambda = \lambda_{\text{phys}} a^2
\]

(3.14)

we have

\[
\eta = \frac{\kappa(a)(\lambda_{\text{mob}} - \lambda_0)^3}{3 a^4} \]

(3.15)

According to the data in the previous section,

\[
\lambda_{\text{mob}}^{\text{phys}} - \lambda_0^{\text{phys}} = M/a
\]

(3.16)

where \( M \approx 0.045 \text{ GeV} \). This means that

\[
\eta = \frac{1}{a} \frac{\kappa(a)}{3} M^3
\]

(3.17)
The question is then how the factor $\kappa$ in the density of states (3.1) varies as a function of $a$. In the fit to the $n(z)$ data at $\beta = 2.5$, we found $\kappa \approx 3.8$. Repeating the same analysis at $\beta = 2.35$, we find $\kappa \approx 2.8$. Thus $\kappa$ does not seem to decrease with smaller lattice spacing; if anything, it increases.

From this data, we draw a remarkable conclusion: The number of states below the mobility edge, per unit physical volume, diverges in the continuum limit. This means that if we compute the contribution to any observable, keeping only a finite number of the lowest scalar field eigenmodes per unit volume, then all of the contributing states would be localized states in the continuum limit. This is an unexpected feature of adjoint scalar fields in confining gauge theories, and suggests that the particle propagator is dominated, at scales which are large compared to the lattice spacing, by the localized eigenstates. In the next section we present a second argument leading to the same conclusion.

IV. LOCALIZATION AND THE SCALAR PROPAGATOR

In a free theory, the eigenmodes of scalar and fermionic kinetic operators are simply plane wave states, and this is also the starting point of the weak-coupling perturbative approach to gauge theories. In perturbation theory, it is assumed that the eigenmodes of kinetic operators such as the covariant Laplacian, in a suitable (e.g. Landau) gauge, are approximately plane wave states, at least for Euclidean momenta which are large compared to $A_{QCD}$. On the other hand, we have seen in section II that for the adjoint covariant Laplacian, the interval between the lowest eigenvalue $\lambda_0$ and the mobility edge $\lambda_{mob}$ tends to infinity, in physical units, in the continuum limit. What this implies is that the contribution of finite momenta extended modes to the scalar particle propagator is negligible, unless we are willing to tolerate a mass subtraction which would introduce tachyonic modes into the theory.

On the lattice, the scalar particle propagator in the quenched (no scalar loop) approximation has the form

$$G^{ab}(x-y) = \sum_n \frac{\phi_n^a(x)\phi_n^b(y)}{\lambda_n + m_0^2}$$

where the VEV is evaluated in the pure gauge theory with some appropriate gauge choice. By “finite momentum extended modes”, we mean extended eigenmodes ($\lambda > \lambda_{mob}$) whose Fourier components are negligible for momenta $|\vec{p}|/a < P$, where $P$ is a momentum cutoff which is large in physical units, but small compared to $1/a$. Consider the contribution to the scalar propagator, in the continuum limit, due to these extended, finite momenta eigenmodes. Denoting this contribution by $G'(x-y)$, we have in physical units

$$G_{phys}^{ab}(x-y) = \frac{1}{a^2} \sum'_n \frac{\phi_n^a(x)\phi_n^b(y)}{\lambda_n + m_0^2}$$

where $\sum'$ denotes the sum over the finite momenta, extended eigenmodes. The number of such eigenmodes cannot exceed the number of lattice momenta $\vec{p}$ satisfying the restriction $|\vec{p}|/a < P$. This number is of order $a^4V\Delta V_p^{phys}$, where $\Delta V_p^{phys} = \frac{1}{2}P^4$ is the momentum space volume in physical units, and $V$ is the lattice volume in lattice units. We can easily estimate the magnitude of $G_{phys}^{ab}$ from

$$\sum'_n \sim a^4V\Delta V_p^{phys}$$

$$\phi_n^a(x)\phi_n^b(y) \sim \frac{1}{V}$$

so that

$$G^{ab}(x-y) \sim \frac{\Delta V_p^{phys}}{(\lambda' + m_0^2)/a^2}$$

where $\lambda'$ is the magnitude of a typical eigenmode in the range of eigenmodes summed by $\sum'$. On the other hand, in the continuum limit, $\lambda'/a^2 \rightarrow \infty$, and therefore the contribution to the scalar propagator from finite momentum extended modes is negligible, for any $m_0^2 > 0$. This is by no means special to confining gauge theories; it is also true in QED. The resolution in QED is to allow for a negative bare mass term $m_0^2 < 0$, adjusted so that $\lambda_0 + m_0^2 = O(a^2)$. But for the adjoint scalar in Yang-Mills theory, this choice of counterterm is inadequate. The problem is that for eigenvalues $\lambda_n > \lambda_{mob}$ contributing to $G'(x-y)$

$$\frac{\lambda_n + m_0^2}{a^2} > \frac{\lambda_0 + m_0^2}{a^2} + \Delta\lambda_{mob}$$

Therefore, even if we chose a counterterm such that $\lambda_0^2 + m_0^2 = 0$, the denominator in eq. (4.4) would still be of order $\Delta\lambda_{mob} \sim 1/a$, which means that $G' \rightarrow 0$ in the continuum limit. The only way to avoid this is to choose a counterterm such that $\lambda_{mob} + m_0^2 = O(a^2)$. But in that case, the eigenvalues $\lambda_n + m_0^2$ of the kinetic operator for the localized eigenmodes $\lambda_n < \lambda_{mob}$ are negative, i.e. tachyonic.

In a quenched theory the bare mass constant can be chosen at will, and tachyon modes in the scalar kinetic term are not excluded. However, in a well-defined field theory with a dynamical scalar field there can be no true tachyon modes; these only appear in perturbative calculations around a false vacuum state. Thus the quenched scalar propagator can only approximate the scalar propagator in the unquenched theory when the lowest eigenmode $\lambda_0 + m_0^2$ of the kinetic operator is, on average, greater than zero.

In D=4 dimensions, SU(2) gauge-Higgs theory with the scalar field in the adjoint representation (aka the Georgi-Glashow model) is known to have two distinct phases: a confinement phase and a Higgs phase [9]. Our quenched calculation would be relevant as an approximation to the scalar propagator of SU(2) gauge-Higgs theory in the confined phase. In this theory

$$Z = \int DU \int D\phi e^{-(S_{YM}+S_0)}$$

2 In contrast, a gauge-Higgs theory with the scalar in the fundamental representation has only one phase, and the asymptotic string tension is vanishing for all finite gauge and Higgs couplings (cf. ref. [10]).
we have assuming that it is possible to choose some values for the system remains in the confined phase, the effect of matter loops on the vacuum state is not very drastic; and (ii) the main effect of the \( \gamma \phi^4 \) term on the scalar particle propagator is to renormalize the mass term.

In gauge-Higgs theory, the \( \Lambda_n \) must, from their definition, be positive semi-definite; the quenched approximation (essentially \( \langle \Lambda_n \rangle \approx \langle \Lambda_n + m_0^2 \rangle \)) can only be relevant to the unquenched theory for \( \langle \Lambda_n + m_0^2 \rangle \geq 0 \). In that case, as we have seen, the contribution of finite momentum extended modes to the scalar propagator is negligible, and it is the localized modes which dominate scalar particle propagation at distances which are large compared to the lattice scale. This would mean that ordinary weak-coupling perturbation theory goes very wrong for adjoint scalar particles, even at distance scales which are quite small compared to the confinement scale.\(^3\)

V. CONCLUSIONS

In a previous article [2] we had found something odd about the spectrum of the covariant Laplacian in the adjoint representation: the lowest eigenmodes appear to be localized in a volume which shrinks to zero, in physical units, in the continuum limit. In the present work we have extended our study to the full interval of localized states, and have found other surprising features: First, the range of eigenvalues \( \Delta \lambda_{n,m} \) of the localized eigenmodes tends to infinity, in physical units, in the continuum limit. Secondly, the density of eigenmodes rises from zero quadratically, up to the mobility edge (suggestive of a connection to multi-critical matrix models of degree \( m = 1 \)), and the number of localized eigenmodes per unit physical volume is infinite in the continuum limit. We must add a caveat that these conclusions rest on the results of numerical simulations of SU(2) lattice gauge theory, carried out at couplings between \( \beta = 2.2 \) and \( \bar{\beta} = 2.5 \), and lattice volumes up to \( 24^4 \). It is, of course, not excluded that the trend in the data could change at higher couplings and/or larger volumes.

An infinite range of eigenvalues between the lowest eigenvalue and the mobility edge has an interesting consequence. If mass counterterms leading to tachyonic modes are excluded, then the quenched scalar particle propagator is dominated by localized modes; there is a negligible contribution from extended eigenmodes corresponding to finite physical momenta. Exclusion of tachyonic modes is necessary if the quenched propagator is to be a reasonable approximation to the scalar propagator in a gauge theory with dynamical scalar fields. Again there is a caveat: We do not really know if an infinite range of localized eigenmodes persists in the unquenched theory. This will require further, computationally more intensive, investigations.

If the propagator for adjoint scalar fields is completely dominated by localized states, when evaluated in the confined phase of gauge-Higgs theory in some appropriate gauge, then this would raise some doubts about the validity of perturbation theory, at least as applied to adjoint scalar fields. Although one naturally expects weak-coupling perturbation theory to

\(^3\) Another short-distance phenomenon which is missed by perturbation theory is the excess lattice action found on thin P-vortex sheets and monopole lines; c.f. ref. [11].
break down at a distance scale comparable to the confinement scale, it is generally believed that this procedure should provide correct answers for short-distance quantities. If, however, the scalar propagator is dominated, even at short distances, by localized eigenmodes, then weak-coupling perturbation theory may be misleading. This is an interesting (and obviously radical) possibility, which calls for further investigation.

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APPENDIX: ALTERNATIVE FITS

In addition to the linear fit of IPR to \( \lambda \), we have explored some other possible fits and fitting procedures for estimating \( \lambda_{mob} \). In particular we have tried to fit various fractional powers of the IPR to \( \lambda \), and to bin the data in different ways. As an example, we have subdivided the interval between minimal and maximal value of \( \lambda \) into 10 subintervals and calculated the average IPR\(^{1/p} \) of the eigenvalues in each interval. We associate, with each average IPR\(^{1/p} \), a value of \( \lambda \) in the center of the corresponding \( \lambda \) interval, and then fit these data points to

\[
\left( \frac{\text{IPR}}{L^4} \right)^{1/p} = A \lambda + B \quad (A.1)
\]

The x-axis intercept \( \lambda_{int} \)

\[
\lambda_{int} = -\frac{B}{A} \quad (A.2)
\]

is taken to be an estimate of the mobility edge \( \lambda_{mob} \approx \lambda_{int} \).

A plot of \( (\text{IPR}/L^4)^{1/4} \) vs. \( \lambda \) at \( \beta = 2.4 \), on a 20\(^4 \) lattice is shown in Fig. 13; the data obtained at \( \beta = 2.4 \) on \( 16^4, 20^4, 24^4 \) lattice volumes are displayed together in Fig. 14. We should note that these plots tend to underestimate \( \lambda_0 \), in that the range of \( \lambda \) (which is divided into ten intervals) depends on the highest and lowest values of \( \lambda_n \) found in all lattice configurations. This means that lattice configurations in which the lowest eigenvalues are well below the average may introduce data points in the graph whose lowest \( \lambda \) value is also well below the average \( \lambda_0 \). Nevertheless, a linear fit of IPR\(^{1/p} \) vs. \( \lambda \), with \( p \neq 1 \), might conceivably be superior to a linear fit of IPR vs. \( \lambda \) in the neighborhood of the mobility edge, and this in turn would give a better estimate for \( \lambda_{mob} \). We have concluded, however, after some experimentation with different fits, that the data is not really adequate to convincingly determine the optimal value of \( p \) near the mobility edge.

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