On best approximations in Banach spaces from the perspective of orthogonality

Debmalya Sain1 · Saikat Roy2

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Abstract
We study best approximations in Banach spaces via Birkhoff–James orthogonality of functionals. To exhibit the usefulness of Birkhoff–James orthogonality techniques in the study of best approximation problems, some algorithms and distance formulae are presented. As an application of our study, we obtain some crucial inequalities, which also strengthen the classical Hölder’s inequality. The relevance of the algorithms and the inequalities are discussed through concrete examples.

Keywords Birkhoff–James orthogonality · Weak∗ continuous functionals · Best approximations · Inequalities

Mathematics Subject Classification 46B28 · 46B20

1 Introduction

The purpose of the present article is to study best approximations in Banach spaces, from the perspective of Birkhoff–James orthogonality of linear functionals. Recently, such a study has been carried out in the context of smooth, strictly convex, reflexive Banach spaces in [15]. The current work strengthens those ideas in further detail and in a more general setting. To demonstrate the applicability of the results developed in this article, we obtain some distance formulae in certain special cases, which also give rise to some important classes of inequalities.

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Saikat Roy
saikatroy.cu@gmail.com

Debmalya Sain
saindebmalya@gmail.com

1 Department of Mathematics, Indian Institute of Science, Bangalore, Karnataka 560012, India
2 Department of Mathematics, National Institute of Technology Durgapur, Durgapur, West Bengal, India
The symbol $\mathbb{X}$ denotes a Banach space. Unless otherwise specified, we work only with real Banach spaces. Let $\theta$ denote the zero vector of any vector space, other than the scalar field $\mathbb{R}$. Let $B_\mathbb{X}$ and $S_\mathbb{X}$ denote the closed unit ball and the unit sphere of $\mathbb{X}$, respectively. We denote the collection of all extreme points of $B_\mathbb{X}$ by $\text{Ext}(B_\mathbb{X})$. Recall that $\mathbb{X}$ is said to be strictly convex if $\text{Ext}(B_\mathbb{X}) = S_\mathbb{X}$. The topological dual of $\mathbb{X}$ is denoted by $\mathbb{X}^*$. Note that the Banach space $\mathbb{X}$ can always be embedded into $\mathbb{X}^{**}$ via the canonical isometric isomorphism $\psi$. Given any $f \in \mathbb{X}^*$, the norm attainment set of $f$, denoted by $M_f$, is defined by

$$M_f := \{x \in S_\mathbb{X} : |f(x)| = \|f\|\}.$$ 

Let $\mathbb{L}(\mathbb{X})$ denote the collection of all bounded linear operators on $\mathbb{X}$, endowed with the usual operator norm. For any linear operator $T \in \mathbb{L}(\mathbb{X})$, the range of $T$ and the kernel of $T$ are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. In similar spirit, we denote the kernel of any $f \in \mathbb{X}^*$ by $\mathcal{N}(f)$. Given any natural number $m$, let $\mathbb{H}$ denote the Hilbert space $\mathbb{R}^m$, equipped with the usual dot product $\langle \cdot, \cdot \rangle$. Members of $\mathbb{L}(\mathbb{H})$ are identified as matrices in the usual way. Given any $T \in \mathbb{L}(\mathbb{H})$, let $[T]$ denote the matrix representation of $T$ with respect to the standard ordered basis of $\mathbb{H}$. Let $T^*$ denote the Hilbert adjoint of $T$. Evidently, $[T^*] = [T]'$, where $[T]'$ denotes the transpose of the matrix $[T]$.

Approximation theory is an extensive field of research due to its diversified applications in many branches of Science. Given any element $x \in \mathbb{X}$ and a subspace $\mathbb{Y}$ of $\mathbb{X}$, distance between $x$ and $\mathbb{Y}$, denoted by $\text{dist}(x, \mathbb{Y})$, is defined by $\text{dist}(x, \mathbb{Y}) := \inf\{\|x - y\| : y \in \mathbb{Y}\}$. An element $y_0 \in \mathbb{Y}$ is said to be a best approximation to $x$ out of $\mathbb{Y}$ if $\text{dist}(x, \mathbb{Y}) = \|x - y_0\|$. The existence and the uniqueness of the best approximation cannot be guaranteed, in general. However, the existence of best approximation(s) is evidently assured for finite-dimensional subspaces. Moreover, in case of a strictly convex Banach space, the best approximation is unique, whenever it exists.

Birkhoff–James orthogonality is of essential importance in understanding the geometry of a Banach space [1, 2, 5–7]. Given any two elements $x, y \in \mathbb{X}$, $x$ is said to be Birkhoff–James orthogonal to $y$, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\| \forall \lambda \in \mathbb{R}$. It is not difficult to see that $y_0 \in \mathbb{Y}$ is a best approximation to $x$ out of $\mathbb{Y}$ if and only if $(x - y_0) \perp_B \mathbb{Y}$. Given any non-zero element $x \in \mathbb{X}$, a member $f \in \mathbb{X}^*$ is called a support functional of $B_\mathbb{X}(\theta, \|x\|)$ at $x$, if $\|f\| = 1$ and $f(x) = \|x\|$. The point $x$ is called smooth if the support functional of $B_\mathbb{X}(\theta, \|x\|)$ at $x$ is unique. Bhatia and Šemrl completely characterized Birkhoff–James orthogonality of matrices in [1]. Based on this rudimentary result, the authors provided some distance formulae in the same article. One may consult [4, 9] for a study of best approximations and orthogonality of matrices. We refer the readers to [10, 13, 16, 18] for some current works involving the geometry of operator spaces and orthogonality of operators in Banach space setting. Some recent developments on best approximations to compact operators can be found in [15], where the central themes are semi-inner-products [3, 8, 14] and operator orthogonality.

The current article presents a comprehensive approach to address the problem of finding best approximation(s) to a given point $x$ out of a subspace $\mathbb{Y}$, in its
full generality. After recalling some basic facts in Sect. 2, we build up the theoretical background of our work in Sects. 3 and 4. The results presented in Sect. 4 should be viewed as generalizations of the results obtained in [15] and will be mentioned accordingly. The integral theme of our development is Birkhoff–James orthogonality of functionals. Application of Birkhoff–James orthogonality not only reduces the computational difficulties to resolve the above mentioned problem but also strengthens the classical duality principle [15, Section 4]. An extra advantage of employing the concept of Birkhoff–James orthogonality (over that of the classical duality principle) is that it provides an easy way out to compute the all possible best approximation(s) to \( x \) out of \( \mathcal{V} \). Indeed, we devote Sect. 5 to show the applicability of the results, developed in the preceding sections, in context of the said problem. We obtain concrete solutions to some problems regarding best approximations and provide certain distance formulae under specific assumptions, which also produce some interesting inequalities, including a finite-dimensional strengthening of the classical Hölder’s inequality.

2 Preliminaries

In this section, we mention some known facts that will be used extensively in the next two sections. We begin with a simple proposition which has important applications in the study of topological vector spaces.

**Proposition 2.1** [12, Lemma 3.9] Suppose that \( g_1, g_2, \ldots, g_m \) and \( f \) are linear functionals on a vector space \( \mathcal{X} \). Let

\[
\mathcal{W} = \bigcap_{i=1}^{m} \mathcal{N}(g_i) = \{ x \in \mathcal{X} : g_1(x) = g_2(x) = \cdots = g_m(x) = 0 \}.
\]

Then the following three conditions are equivalent:

(i) There exist scalars \( \lambda_1, \lambda_2, \ldots, \lambda_m \), such that

\[
f = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_m g_m.
\]

(ii) There exists \( \gamma < \infty \), such that

\[
|f(x)| \leq \gamma \max_{1 \leq i \leq m} |g_i(x)| \quad (x \in \mathcal{X}).
\]

(iii) \( f(x) = 0 \) for every \( x \in \mathcal{W} \).
Suppose that \( \tau \) is a topology on a vector space \( \mathbb{X} \), such that every one point set in \( \mathbb{X} \) is closed and the vector space operations on \( \mathbb{X} \) are continuous with respect to the topology \( \tau \). Then, the vector space \( \mathbb{X} \) equipped with the topology \( \tau \) is called a topological vector space. The topological vector space \( \mathbb{X} \) is called locally convex if there exists a local base at \( \theta \), whose members are convex. Every topological vector space enjoys an important separation property:

**Proposition 2.2** [12, Theorem 1.10] Let \( \mathbb{X} \) be a topological vector space. Let \( K \) and \( C \) be subsets of \( \mathbb{X} \), such that \( K \) is compact and \( C \) is closed with \( K \cap C = \emptyset \). Then, there exists a neighborhood \( V \) of \( \theta \), such that

\[
(K + V) \cap (C + V) = \emptyset.
\]

Given a Banach space \( \mathbb{X} \), \( \mathbb{X}^* \) equipped with the weak* topology is a locally convex topological vector space. Moreover, every linear functional on \( \mathbb{X}^* \) that is weak* continuous is of the form \( \psi(x) = \langle x, y \rangle \) for some \( y \in \mathbb{X} \), where \( \psi \) denotes the canonical embedding of \( \mathbb{X} \) into \( \mathbb{X}^{**} \). We refer the readers to the standard text [12] for more information in this regard. Weak* topology on \( \mathbb{X}^* \) has a crucial compactness property known as the Banach–Alaoglu Theorem:

**Theorem 2.3** (Banach–Alaoglu) Let \( \mathbb{X} \) be a normed linear space. Then, the closed unit ball \( B_{\mathbb{X}^*} \) of \( \mathbb{X}^* \) is compact with respect to the weak* topology on \( \mathbb{X}^* \).

We next present a classical result which is a variant of the geometric Hahn–Banach Theorem and is popularly known as the Mazur Theorem. In the following theorem, we do not require the topological vector space to be locally convex.

**Theorem 2.4** [19, Theorem 18.2] Let \( \mathbb{X} \) be a topological vector space and let \( E \) be a linear subspace of \( \mathbb{X} \). Let \( V \) be a convex open subset of \( \mathbb{X} \), such that

\[
E \cap V = \emptyset.
\]

Then, there exists a closed hyperplane \( H \) of \( \mathbb{X} \), such that

\[
E \subseteq H, \quad H \cap V = \emptyset.
\]

Let \( \mathbb{X} \) be a Banach space and let \( \mathcal{W} \) be any non-trivial subspace of \( \mathbb{X} \). Let \( f \) be any member of \( \mathbb{X}^* \). A member \( f_0 \) of \( \mathbb{X}^* \) is said to be a Hahn–Banach extension of \( f \big|_{\mathcal{W}} \) if

\[
f \big|_{\mathcal{W}} = f_0 \big|_{\mathcal{W}} \text{ and } \|f \big|_{\mathcal{W}}\| = \|f_0\|.
\]

The next two propositions are about some basic facts regarding real \( \ell_p \) spaces, where \( p \in (1, \infty) \). Given any \( 1 \leq p \leq \infty \), \( q \) is said to be the conjugate to \( p \) if \( q = 1 \) (\( q = \infty \)), whenever \( p = \infty \) (\( p = 1 \)), and, \( q = \frac{p}{p-1} \), whenever \( 1 < p < \infty \).

**Proposition 2.5** Let \( p \in (1, \infty) \). Then, the dual of \( \ell_p \) is isometrically isomorphic to \( \ell_q \), and for any \( c = (c_1, c_2, \ldots) \in \ell_q \), the corresponding member \( f_c \in \ell_p^* \) is given by

\[
f_c(x) = \sum_{n=1}^{\infty} c_n x_n.
\]
Proposition 2.6 Let $p \in (1, \infty)$ and let $\mathbf{a} = (a_1, a_2, \ldots) \in \ell_p^*$ be non-zero. Let $\mathbf{c} = (c_1, c_2, \ldots) \in \ell_q$ be such that $\mathbf{c}$ corresponds to the support functional of $\mathbf{a}$ in $\ell_p^*$. Then, for each $k \in \mathbb{N}$, $c_k$ is given by

$$c_k = \frac{\text{sgn}(a_k)|a_k|^{p-1}}{\|\mathbf{a}\|_p^{p-1}}.$$ 

3 Orthogonality of functionals

Our aim in this section is to obtain a characterization of Birkhoff–James orthogonality of weak* continuous functionals, which is of paramount importance in the context of our current work on best approximations. We would like to mention that the said characterization can also be obtained by modifying Theorem 2.1 of [11]. However, we present a complete proof of the same, for the convenience of the readers.

We need the following proposition, which completely describes the norm attainment sets of weak* continuous functionals, to serve our purpose.

Proposition 3.1 Let $\mathcal{X}$ be a Banach space and let $f \in \mathcal{X}^{**}$ be weak* continuous. Then, $M_f = \pm D$, where $D$ is a non-empty compact path connected subset of $B_{\mathcal{X}}^{**}$ with respect to the weak* topology on $\mathcal{X}^{*}$.

Proof Denote the canonical embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$ by $\psi$. Since $f$ is weak* continuous, there exists $x_0 \in \mathcal{X}$, such that $\psi(x_0) = f$. By the Hahn–Banach Theorem, there exists a support functional of $B_{\mathcal{X}}(\theta, \|x_0\|)$, say $x_0^*$, at $x_0$. Observe that

$$\psi(x_0)(x_0^*) = f(x_0^*) = x_0^*(x_0) = \|x_0\| = \|\psi(x_0)\| = \|f\|.$$ 

Therefore, $M_f \neq \emptyset$. Let $D$ be a subset of $B_{\mathcal{X}}^{**}$, defined by

$$D = \{x^* \in B_{\mathcal{X}}^{**} : f(x^*) = \|f\| \}.$$ 

Evidently, $M_f = \pm D$ and $D$ is non-empty, since $M_f$ is non-empty. In addition, it is easy to see that $D$ is a closed subset of $B_{\mathcal{X}}^{**}$. It now follows from the Banach–Alaoglu Theorem that $D$ is a compact subset of $B_{\mathcal{X}}^{**}$, with respect to the weak* topology on $\mathcal{X}^{*}$. Note that for any $x_1^*, x_2^* \in D$, $tx_1^* + (1-t)x_2^* \in D$ for all $t \in [0, 1]$. Since $\mathcal{X}^{*}$ equipped with the weak* topology is a topological vector space; therefore, the mapping $t \mapsto tx_1^* + (1-t)x_2^*$ is continuous. Consequently, $D$ is a path connected subset of $B_{\mathcal{X}}^{**}$. This completes the proof. 

Based on Proposition 3.1, we now obtain a characterization of orthogonality of weak* continuous functionals. Note that the following characterization also includes [15, Theorem 3.2].
Theorem 3.2. Let $\mathbb{X}$ be a Banach space and let $f, g \in \mathbb{X}^{**}$ be weak* continuous. Then, $f \perp_B g$ if and only if $M_f \cap \mathcal{N}(g) \neq \emptyset$.

Proof. We only prove the necessary part as the proof of the sufficient part is trivial. Suppose on the contrary that $M_f \cap \mathcal{N}(g) = \emptyset$. By Proposition 3.1, $M_f = \pm D$, where $D$ is a non-empty compact connected subset of $\mathbb{B}_\mathbb{X}$, with respect to the weak* topology on $\mathbb{X}^*$. It now follows from the connectedness of $D$ that either $f(x) \cdot g(x) > 0$ for all $x \in D$, or $f(x) \cdot g(x) < 0$ for all $x \in D$. Without loss of generality, let $f(x) \cdot g(x) > 0$ for all $x \in D$. Let $p : \mathbb{B}_\mathbb{X} \times [-1, 1] \to \mathbb{R}$ be defined by

$$p(x, \lambda) = |f(x) + \lambda g(x)| \quad \forall (x, \lambda) \in \mathbb{B}_\mathbb{X} \times [-1, 1].$$

Obviously, $p$ is continuous. In addition, for each $y \in M_f$ there exist a weak* open set $U_y$ containing $y$ and $\delta_y \in (0, 1)$, such that

$$p(\bar{y}, \lambda) < ||f|| \quad \forall (\bar{y}, \lambda) \in U_y \times (-\delta_y, 0).$$

On the other hand, for each $z \in \mathbb{B}_\mathbb{X} \setminus M_f$ there exist a weak* open set $V_z$ containing $z$ and $\delta_z \in (0, 1)$, such that

$$p(\bar{z}, \lambda) < ||f|| \quad \forall (\bar{z}, \lambda) \in V_z \times (-\delta_z, \delta_z).$$

Therefore, the collection $\{U_y : y \in M_f\} \cup \{V_z : z \in \mathbb{B}_\mathbb{X} \setminus M_f\}$ forms a weak* open cover for $\mathbb{B}_\mathbb{X}$. Due to the compactness of $\mathbb{B}_\mathbb{X}$ (with respect to the weak* topology on $\mathbb{X}^*$), there exist natural numbers $k_1, k_2$, such that

$$\mathbb{B}_\mathbb{X}^* \subseteq \left( \bigcup_{i=1}^{k_1} U_{y_i} \right) \cup \left( \bigcup_{j=1}^{k_2} V_{z_j} \right).$$

Let $0 < \mu_0 < \min \left\{ \min \left\{ \{ \delta_{y_i} \}_{j=1}^{k_1} \} \right\}, \min \left\{ \{ \delta_{z_j} \}_{j=1}^{k_2} \right\} \right\}$. Since $f - \mu_0 g$ is weak* continuous, by Proposition 3.1, $M_{f - \mu_0 g} \neq \emptyset$. Let $x_0 \in M_{f - \mu_0 g}$. Then it follows from the choice of $\mu_0$ that

$$||f - \mu_0 g|| = |(f - \mu_0 g)(x_0)| < ||f||.$$

This is a contradiction to the fact that $f \perp_B g$. This completes the proof. \hfill \Box

As an application of Theorem 3.2 we have the following corollary:

Corollary 3.2.1 (James characterization of Birkhoff–James orthogonality) Let $\mathbb{X}$ be a Banach space and let $x, y \in \mathbb{X}$. Then $x \perp_B y$ if and only if there exists $x^*_0 \in S_{\mathbb{X}^{**}}$, such that $x^*_0(x) = ||x||$ and $x^*_0(y) = 0$.

Proof. Denote the canonical embedding of $\mathbb{X}$ into $\mathbb{X}^{**}$ by $\psi$. Let $\psi(x) = f$ and $\psi(y) = g$. Then, $x \perp_B y$ if and only if $f \perp_B g$. Since $f, g \in \mathbb{X}^{**}$ are weak* continuous, it follows from Theorem 3.2 that $x \perp_B y$ if and only if $M_f \cap \mathcal{N}(g) \neq \emptyset$. Let $x^*_0 \in M_f \cap \mathcal{N}(g)$. Then
$x_0^*(x) = f(x_0^*) = \|f\| = \|x\|$ and $x_0^*(y) = g(x_0^*) = 0$.

This completes the proof. □

4 Birkhoff–James orthogonality and best approximations

Let $\mathbb{X}$ be a reflexive, strictly convex Banach space. One of the fundamental ideas in [15] was to identify the Banach space $\mathbb{X}$ with its double dual $\mathbb{X}^{**}$ and then treat the best approximation problem in $\mathbb{X}^{**}$ by employing Birkhoff–James orthogonality techniques. Unfortunately, the above idea does not work if $\mathbb{X}$ is not reflexive. This lacuna can be overcome by identifying $\mathbb{X}$ to the space of all weak$^*$ continuous functionals on $\mathbb{X}^*$. The following theorem provides a necessary and sufficient condition regarding the best approximation problem in the space of all weak$^*$ continuous functionals. In that sense, the result is a generalization of Theorem 3.4 and Theorem 3.5 of [15]. In addition, note that we do not require the strict convexity of $\mathbb{X}$.

Theorem 4.1 Let $\mathbb{X}$ be a Banach space and let $f \in \mathbb{X}^{**}$ be weak$^*$ continuous. Let $\mathbb{Y}$ be a subspace of $\mathbb{X}^{**}$, such that each member of $\mathbb{Y}$ is weak$^*$ continuous and $f \notin \mathbb{Y}$. Let $g_0 \in \mathbb{Y}$. Then, $g_0$ is a best approximation to $f$ out of $\mathbb{Y}$ if and only if for every finite-dimensional subspace $Z$ of $\mathbb{Y}$ containing $g_0$, $\bigcap_{g \in Z} \mathcal{N}(g) \cap M_{f-g_0} \neq \emptyset$.

**Proof** We first prove the necessary part. Suppose on the contrary that there exists a finite-dimensional subspace $Z$ of $\mathbb{Y}$ containing $g_0$, such that

$$\bigcap_{g \in Z} \mathcal{N}(g) \cap M_{f-g_0} = \emptyset.$$ 

Let $\{h_j : 1 \leq j \leq k\}$ be a basis of $Z$. Consequently, $\bigcap_{g \in Z} \mathcal{N}(g) = \bigcap_{j=1}^k \mathcal{N}(h_j)$. Also, it follows from Proposition 3.1 that $M_{f-g_0} = \pm D$, where $D$ is a compact convex subset of $B_{\mathbb{X}}$, with respect to the weak$^*$ topology on $\mathbb{X}^*$. Since $\mathbb{X}^*$ equipped with the weak$^*$ topology is a locally convex topological vector space and $\bigcap_{g \in Z} \mathcal{N}(g)$ is a closed subset of $\mathbb{X}^*$ disjoint from $D$, there exists a convex neighbourhood $V$ of $\theta$, such that

$$(D + V) \cap \left(\bigcap_{g \in Z} \mathcal{N}(g) + V\right) = \emptyset.$$ 

Since $D$ is convex, so is $D + V$. In particular, $D + V$ is an open convex subset of $\mathbb{X}^*$, disjoint from $\bigcap_{g \in Z} \mathcal{N}(g)$. Therefore, it follows from Theorem 2.4 that there exists a closed hyperplane $H$ of $\mathbb{X}^*$, such that

$$\bigcap_{g \in Z} \mathcal{N}(g) \subseteq H, \quad H \cap (D + V) = \emptyset.$$ 

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Let $h : \mathbb{X}^* \to \mathbb{R}$ be a linear functional, such that $N(h) = H$. It now follows from Proposition 2.1 that $h \in \text{span}\{h_j : 1 \leq j \leq k\} \subseteq \mathbb{X}$. In particular, $h$ is weak* continuous. In addition, note that

\[ M_{f-g_0} \cap N(h) = \emptyset. \]

Theorem 3.2 ensures that $f - g_0 \perp_B h$. However, this is a contradiction to the fact that $g_0$ is a best approximation to $f$ out of $\mathbb{Y}$.

To prove the sufficient part of the theorem, let $g_1 \in \mathbb{Y}$ be arbitrary and let $\mathbb{Z} = \text{span}\{g_0, g_1\}$. Then it follows from the hypothesis of the theorem that

\[ \bigcap_{g \in \mathbb{Z}} N(g) \cap M_{f-g_0} \neq \emptyset. \]

In particular, $M_{f-g_0} \cap N(g_1) \neq \emptyset$. It now follows from Theorem 3.2 that $f - g_0 \perp_B g_1$. Since $g_1 \in \mathbb{Y}$ was chosen arbitrarily, we have that $(f - g_0) \perp_B \mathbb{Y}$. Consequently, $g_0 \in \mathbb{Y}$ is a best approximation to $f$ out of $\mathbb{Y}$ and the proof follows. $\Box$

Whenever $\mathbb{Y}$ is finite-dimensional, the above theorem takes a simpler form. We record this as a corollary. The proof of the corollary follows directly from Theorem 4.1, and therefore, it is omitted.

**Corollary 4.1.1** Let $\mathbb{X}$ be a Banach space and let $f, g_1, g_2, \ldots, g_m \in \mathbb{X}^{**}$ be weak* continuous. Let $g_1, g_2, \ldots, g_m$ be linearly independent and $f \notin \mathbb{Y}$, where $\mathbb{Y} = \text{span}\{g_1, g_2, \ldots, g_m\}$. Then, $g_0 \in \mathbb{Y}$ is a best approximation to $f$ out of $\mathbb{Y}$ if and only if $\bigcap_{i=1}^m N(g_i) \cap M_{f-g_0} \neq \emptyset$.

As an immediate application of Theorem 4.1, we now obtain a distance formula in the space of all weak* continuous functionals on $\mathbb{X}^*$. The following distance formula can be regarded as a strengthened version of Theorem 3.6 of [15].

**Theorem 4.2** Let $\mathbb{X}$ be a Banach space and let $f \in \mathbb{X}^{**}$ be weak* continuous. Let $\mathbb{Y}$ be a subspace of $\mathbb{X}^{**}$, such that each member of $\mathbb{Y}$ is weak* continuous and $f \notin \mathbb{Y}$. Suppose that $g_0 \in \mathbb{Y}$ is a best approximation to $f$ out of $\mathbb{Y}$. Then, for any finite-dimensional subspace $\mathbb{Z}$ of $\mathbb{Y}$ containing $g_0$,

\[ \|f - g_0\| = \text{dist}(f, \mathbb{Y}) = \max \left\{ |f(x)| : x \in \bigcap_{g \in \mathbb{Z}} N(g) \cap S_{\mathbb{X}^*} \right\}. \]

**Proof** It follows from Theorem 4.1 that $\bigcap_{g \in \mathbb{Z}} N(g) \cap M_{f-g_0} \neq \emptyset$. In addition, note that

\[ (f - g_0)|_{\bigcap_{g \in \mathbb{Z}} N(g)} = f. \quad (1) \]

Fix some $x_0 \in \bigcap_{g \in \mathbb{Z}} N(g) \cap M_{f-g_0}$. Then, it is easy to see that
\[ \|f - g_0\| = |(f - g_0)(x_0)| = |f(x_0)| \quad \text{ (using (1)).} \]

We now claim that

\[ |f(x_0)| = \max \left\{ |f(x)| : x \in \bigcap_{g \in \mathbb{Z}} \mathcal{N}(g) \cap S_{X^*}. \right\}. \]

Indeed, if there exists \( y_0 \in \bigcap_{g \in \mathbb{Z}} \mathcal{N}(g) \cap S_{X^*} \), such that \( |f(y_0)| > |f(x_0)| \), then we obtain that

\[ \|f - g_0\| \geq |f(y_0)| > |f(x_0)| = |(f - g_0)(x_0)| = \|f - g_0\|, \]

which is a contradiction. This completes the proof. \( \square \)

Assuming \( \mathcal{Y} \) to be finite-dimensional in the above theorem, we have the following distance formula:

**Corollary 4.2.1** Let \( X \) be a Banach space and let \( f, g_1, g_2, \ldots, g_m \in X^{**} \) be weak\(^*\) continuous for some \( m \in \mathbb{N} \). Let \( g_1, g_2, \ldots, g_m \) be linearly independent and let \( f \notin \mathcal{Y} \), where \( \mathcal{Y} = \text{span}\{g_1, g_2, \ldots, g_m\} \). Then,

\[ \text{dist}(f, \mathcal{Y}) = \max \left\{ |f(x)| : x \in \bigcap_{i=1}^{m} \mathcal{N}(g_i) \cap S_{X^*}. \right\}. \]

**Proof** A standard compactness argument ensures that there exist \( \alpha_i \in \mathbb{R} \), where \( 1 \leq i \leq m \), such that \( \sum_{i=1}^{m} \alpha_i g_i \) is a best approximation to \( f \) out of \( \mathcal{Y} \). Now, arguing as in Theorem 4.2, we get the desired formula. \( \square \)

As mentioned in the introduction, Birkhoff–James orthogonality techniques provide some genuine insights in determining best approximation(s) to a given point out of a subspace. Indeed, the following result completely characterizes best approximation(s) to a given point out of a finite-dimensional subspace in the setting of weak\(^*\) continuous functionals.

**Theorem 4.3** Let \( X \) be a Banach space and let \( f, f_0, g_1, g_2, \ldots, g_m \in X^{**} \) be weak\(^*\) continuous for some \( m \in \mathbb{N} \). Let \( g_1, g_2, \ldots, g_m \) be linearly independent and let \( f \notin \mathcal{Y} \), where \( \mathcal{Y} = \text{span}\{g_1, g_2, \ldots, g_m\} \). Then, \( f - f_0 \) is a best approximation to \( f \) out of \( \mathcal{Y} \) if and only if \( f_0 \) is a Hahn–Banach extension of \( f|_{\mathcal{W}} \), where \( \mathcal{W} = \bigcap_{i=1}^{m} \mathcal{N}(g_i) \).

**Proof** We first prove the necessary part. Suppose on the contrary that \( f|_{\mathcal{W}} \neq f_0|_{\mathcal{W}} \). Then there exists \( x_0 \in \mathcal{W} \), such that \( f(x_0) \neq f_0(x_0) \). However, it then follows from Proposition 2.1 that \( f - f_0 \notin \mathcal{Y} \), which is a contradiction. Next, suppose that \( \|f|_{\mathcal{W}}\| \neq \|f_0\| \). Then \( \|f\|_{\mathcal{W}} \neq \|f_0\| \). We now claim that \( M_{f_0} \cap \mathcal{W} = \emptyset \). Indeed, if \( y_0 \in M_{f_0} \cap \mathcal{W} \), then it follows that
\|f_0\| = |f_0(y_0)| = |f(y_0)| \leq \|f\|_W < \|f_0\|.

which is absurd. Therefore, \(M_{f_0} \cap \mathcal{W} = \emptyset\), as expected. Then, Corollary 4.1.1 ensures that \(f - f_0\) is not a best approximation to \(f\) out of \(\mathcal{Y}\), which is a contradiction.

We now prove the sufficient part. Since \(\|f\|_W = f_0\|_W\), \(N(f - f_0)\) contains \(\mathcal{W}\). It now follows from Proposition 2.1 that \(f - f_0 \in \mathcal{Y}\). Next, consider any \(y_1 \in M_{f|_W}\). Then

\[\|f\|_W = |f(y_1)| = |f_0(y_1)| = \|f_0\|.

Thus, \(y_1 \in M_{f_0}\) and \(M_{f|_W} \subseteq M_{f_0}\). Consequently, \(M_{f_0} \cap \mathcal{W} \neq \emptyset\). Therefore, applying Corollary 4.1.1, we obtain that \(f - f_0\) is a best approximation to \(f\) out of \(\mathcal{Y}\). This completes the proof. \(\square\)

We would like to remark here that the above theorem can also be stated in terms of norm attainment sets of weak* continuous functionals.

**Remark 4.1** Let \(\mathbb{X}\) be a Banach space and let \(f, g_1, g_2, \ldots, g_m \in \mathbb{X}^{**}\) be weak* continuous for some \(m \in \mathbb{N}\). Let \(g_1, g_2, \ldots, g_m\) be linearly independent and let \(f \not\in \mathcal{Y}\), where \(\mathcal{Y} = \text{span}\{g_1, g_2, \ldots, g_m\}\). Let us consider the set

\[\mathcal{B} := \{h \in \mathbb{X}^{**} : f - h\text{ is a best approximation to } f\text{ out of } \mathcal{Y}\}.

Then \(\mathcal{B}\) is precisely the collection of those weak* continuous functionals which are the extensions of \(f|_{\bigcap_{i=1}^m N(g_i)}\) and whose norm attainment sets contain the norm attainment set of the restriction of \(f\) to \(\bigcap_{i=1}^m N(g_i)\). In other words, for any \(h \in \mathcal{B}\)

\[h|_W = f|_W\text{ and } M_{f|_W} \subseteq M_h, \text{ where } \mathcal{W} = \bigcap_{i=1}^m N(g_i).

It is obvious that given any Banach space \(\mathbb{X}\), the best approximation problems in \(\mathbb{X}\) can be treated as the best approximation problems in the space of all weak* continuous functionals on \(\mathbb{X}^*\). Therefore, the foregoing results can also be rephrased in terms of best approximation problems in \(\mathbb{X}\). As an evidence of this, we present the following theorem which is essentially a variant of Theorem 4.1. However, we refrain ourselves from doing analogous treatment to the remaining results, to avoid monotony.

**Theorem 4.4** Let \(\mathbb{X}\) be a Banach space and let \(x_0 \in \mathbb{X}\). Let \(\mathcal{Y}\) be a subspace of \(\mathbb{X}\), such that \(x_0 \not\in \mathcal{Y}\) and \(y_0 \in \mathcal{Y}\). Then, \(y_0\) is a best approximation to \(x_0\) out of \(\mathcal{Y}\) if and only if for every finite-dimensional subspace \(\mathbb{Z}\) of \(\mathcal{Y}\) containing \(y_0\), \(\bigcap_{z \in \mathbb{Z}} N(\psi(z)) \cap M_{\psi(x_0 - y_0)} \neq \emptyset\), where \(\psi : \mathbb{X} \to \mathbb{X}^{**}\) denotes the canonical embedding.

We have seen in Theorem 3.2 that there is a deep connection between Birkhoff–James orthogonality and the norm attainment set of a given functional. Since every weak* continuous functional is norm attaining, there is a scope to employ
Birkhoff–James orthogonality techniques in all of the preceding results. However, Birkhoff–James orthogonality is not so straightforward for the functionals that do not attain their norms. Consequently, the same techniques cannot be applied for functionals not attaining their respective norms. We end this section with a result which addresses this issue. In addition, note that the result is valid in any Banach space, real or complex.

**Theorem 4.5** Let $\mathbb{X}$ be a Banach space and let $f \in \mathbb{X}^*$. Let $\mathbb{Y}$ be a subspace of $\mathbb{X}^*$, such that $f \notin \mathbb{Y}$ and $g_0 \in \mathbb{Y}$. Then $g_0$ is a best approximation to $f$ out of $\mathbb{Y}$ if and only if for every finite-dimensional subspace $\mathbb{Z}$ of $\mathbb{Y}$ containing $g_0$, the following holds true:

$$
\left\| (f - g_0) \big|_{\mathbb{Z}} \right\| = \| f - g_0 \|.
$$

**Proof** We first prove the necessary part. Suppose on the contrary that there exists a finite-dimensional subspace $\mathbb{Z}$ of $\mathbb{Y}$ containing $g_0$, such that

$$
\left\| (f - g_0) \big|_{\mathbb{Z}} \right\| < \| f - g_0 \|.
$$

Let $\{h_j : 1 \leq j \leq k\}$ be a basis of $\mathbb{Z}$. Then it is straightforward to check that

$$
\bigcap_{g \in \mathbb{Z}} \mathcal{N}(g) = \bigcap_{j=1}^{k} \mathcal{N}(h_j).
$$

Let $f_0 : \bigcap_{j=1}^{k} \mathcal{N}(h_j) \to \mathbb{R}$ be defined by

$$
f_0(x) = f(x) \quad \forall x \in \bigcap_{j=1}^{k} \mathcal{N}(h_j).
$$

By the Hahn–Banach Theorem, $f_0$ possesses a linear extension $\tilde{f}_0 : \mathbb{X} \to \mathbb{R}$, such that

$$
\left\| \tilde{f}_0 \right\| = \| f_0 \| = \left\| f \big|_{\bigcap_{j=1}^{k} \mathcal{N}(h_j)} \right\|.
$$

Now, consider the linear functional $(f - \tilde{f}_0) : \mathbb{X} \to \mathbb{R}$. Since $(f - \tilde{f}_0)$ vanishes identically on $\bigcap_{j=1}^{k} \mathcal{N}(h_j)$, it follows from Proposition 2.1 that $(\tilde{f}_0 - f) \in \text{span}\{h_j : 1 \leq j \leq k\} \subseteq \mathbb{Y}$. On the other hand,
\[ \|f - (f - \tilde{f}_0)\| = \|\tilde{f}_0\| = \|f\| \cap \bigcap_{j=1}^{N(h_j)} \mathcal{N}(h_j) = \|f\| \cap \bigcap_{g \in \mathbb{Z}^N} \mathcal{N}(g) = \|(f - g_0)\| \cap \bigcap_{g \in \mathbb{Z}^N} \mathcal{N}(g) < ||f - g_0||, \]

where the second last equality follows from the fact that \( f(x) = (f - g_0)(x) \) for all \( x \in \bigcap_{g \in \mathbb{Z}^N} \mathcal{N}(g) \). However, this is a contradiction to the fact that \( g_0 \) is a best approximation to \( f \) out of \( \mathbb{Y} \).

We now prove the sufficient part. Let \( g_1 \in \mathbb{Y} \) be arbitrary and let \( \mathbb{Z} = \text{span}\{g_0, g_1\} \). Then, it follows from the hypothesis of the theorem that

\[ \|f - g_1\| \geq \|(f - g_1)\| \cap \bigcap_{g \in \mathbb{Z}^N} \mathcal{N}(g) = \|(f - g_0)\| \cap \bigcap_{g \in \mathbb{Z}^N} \mathcal{N}(g) = \|f - g_0\|. \]

Thus, \( g_0 \) is a best approximation to \( f \) out of \( \mathbb{Y} \) and this completes the proof. \( \square \)

5 Some applications and distance formulae

In this section, we exhibit some interesting applications and examples to the theories developed in the preceding sections. Let us begin with an algorithm that generalizes the Problem given in [15].

**Problem 5.1** Let \( \mathbb{X} \) be a Banach space and let \( m \in \mathbb{N} \). Let \( x_0, y_1, y_2, \ldots, y_m \in \mathbb{X} \) be such that \( y_1, y_2, \ldots, y_m \) are linearly independent and \( x_0 \notin \mathbb{Y} \), where \( \mathbb{Y} = \text{span}\{y_1, y_2, \ldots, y_m\} \). Then, find best approximation(s) to \( x_0 \) out of \( \mathbb{Y} \) and compute \( \text{dist}(x_0, \mathbb{Y}) \).

Corollary 4.1.1, Corollary 4.2.1 and Theorem 4.3 allow us to approach the problem in the following three steps:

Step 1: We embed \( \mathbb{X} \) into its double dual \( \mathbb{X}^{**} \) via the canonical isometric isomorphism \( \psi \). Let \( \psi(x_0) = f \) and \( \psi(y_i) = g_i \) for \( 1 \leq i \leq m \). Let \( \mathbb{Z} = \text{span}\{g_1, g_2, \ldots, g_m\} \). Evidently, the above problem is equivalent to finding the best approximation(s) to \( f \) out of \( \mathbb{Z} \) and computing \( \text{dist}(f, \mathbb{Z}) \). Since \( f, g_1, g_2, \ldots, g_m \) are weak\(^*\) continuous, the criteria of Corollary 4.1.1, Corollary 4.2.1 and Theorem 4.3 are satisfied.

Step 2: Let \( \mathbb{W} = \bigcap_{i=1}^{m} \mathcal{N}(g_i) \). We now consider the following two cases:
Case I: $\mathcal{W}$ is one-dimensional. Consider any non-zero $z \in \mathcal{W}$. Then, it follows from Corollary 4.2.1 that

$$\text{dist}(x_0, \mathcal{W}) = \text{dist}(f, \mathcal{W}) = \max \{|f(x)| : x \in \mathcal{W} \cap S_{X^*}\} = \frac{1}{\|z\|} |f(z)|.$$ 

Case II: $\mathcal{W}$ is not one-dimensional. Consider $\mathcal{W} \cap \ker f$ and find some non-zero $u \in \mathcal{W}$, such that $u \perp_B (\mathcal{W} \cap \ker f)$. Note that the existence of such an $u$ is always guaranteed, since $M_{ij} \neq \emptyset$. Then, we have that

$$\max \{|f(x)| : x \in \mathcal{W} \cap S_{X^*}\} = \frac{1}{\|u\|} |f(u)|.$$ 

In other words,

$$\text{dist}(x_0, \mathcal{W}) = \text{dist}(f, \mathcal{W}) = \max \{|f(x)| : x \in \mathcal{W} \cap S_{X^*}\} = \frac{1}{\|u\|} |f(u)|.$$ 

Step 3: Let us consider the following collection:

$$\Lambda := \{f - f_0 \in X^{**} : f_0 \text{ is a Hahn–Banach extension of } f|_{\mathcal{W}}\}.$$ 

It is not difficult to see that $\Lambda \neq \emptyset$ and $\Lambda \subseteq \mathcal{Z}$. It follows from Theorem 4.3 that $\Lambda$ is precisely the collection of best approximation(s) to $f$ out of $\mathcal{Z}$.

Thus, we completely obtain the solution of the above problem, as $\psi^{-1}(\Lambda)$ is precisely the collection of best approximation(s) to $x_0$ out of $\mathcal{Y}$.

The algorithm presented in Problem 5.1 is particularly advantageous for $\ell^n_1$ and $\ell^n_\infty$ spaces, for $n \in \mathbb{N}$. This is because the dual of $\ell^n_1$ ($\ell^n_\infty$) is $\ell^n_1$ ($\ell^n_\infty$) and if any member $x^*$ of $\ell^n_1$ ($\ell^n_\infty$) corresponds to a member $(a_1, a_2, \ldots, a_n)$ of $\ell^n_1$ ($\ell^n_\infty$), then the action of $x^*$ on any member $(b_1, b_2, \ldots, b_n)$ of $\ell^n_1$ ($\ell^n_\infty$) is given by the formula:

$$x^*(b_1, b_2, \ldots, b_n) = \sum_{i=1}^{n} a_i b_i.$$ 

We elaborate this in more detail in the following problem:

**Problem 5.2** Let $X = \ell^n_1$ or $\ell^n_\infty$ for some $n \in \mathbb{N}$. Let $x_0, y_1, y_2, \ldots, y_m \in X$, where $1 \leq m < n$ be, such that $y_1, y_2, \ldots, y_m$ are linearly independent and $x_0 \not\in \mathcal{Y}$, where $\mathcal{Y} = \text{span}\{y_1, y_2, \ldots, y_m\}$. Compute $\text{dist}(x_0, \mathcal{Y})$.

Step 1: Let $x_0 = (x_1, x_2, \ldots, x_n)$ and let $y_i = (y_{i1}, y_{i2}, \ldots, y_{im})$, where $1 \leq i \leq m$. Let $\psi(x_0) = f$ and $\psi(y_i) = g_i$, where $1 \leq i \leq m$. Let $\mathcal{Z} = \text{span}\{g_1, g_2, \ldots, g_m\}$. Evidently, the above problem is equivalent to computing $\text{dist}(f, \mathcal{Z})$.

Step 2: Note that

$$\bigcap_{i=1}^{m} \mathcal{N}(g_i) = \mathcal{W} = \left\{(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n : \sum_{j=1}^{n} y_{ij} u_j = 0; 1 \leq i \leq m\right\}.$$
Therefore, applying Corollary 4.2.1, we obtain
\[
\text{dist}(x_0, \mathcal{Y}) = \text{dist}(f, \mathcal{Z}) = \max \left\{ |f(x)| : x \in \mathcal{W} \cap S_{\mathcal{X}^*} \right\}.
\]

In other words, the problem of computing \(\text{dist}(x_0, \mathcal{Y})\) reduces to the problem of finding the absolute maximum of \(f\) on the unit sphere of the solution space of the system of linear equations:
\[
\sum_{j=1}^{n} y_j u_j = 0; \quad 1 \leq i \leq m.
\]

Note that the non-triviality of the solution space is guaranteed by the existence of best approximation(s).

We now present an example to illustrate the utility of the above problem:

**Example 5.2.1** Let \(\mathcal{X} = \ell_1^4\) and let \(\mathcal{Y} = \text{span}\{y_1, y_2\} \subseteq \mathcal{X}\), where \(y_1 = (1, 2, 0, 0)\), \(y_2 = (-1, 0, 2, 0)\). Let \(x_0 = (1, 1, 1, 1)\). Then it is trivial to see that \(x_0 \notin \mathcal{Y}\). Our aim is to calculate \(\text{dist}(x_0, \mathcal{Y})\) and to find a best approximation to \(x_0\) out of \(\mathcal{Y}\).

Let \(\psi : \ell_1^4 \to \ell_\infty^4\) denote the canonical isometric isomorphism. Let \(\psi(x_0) = f\) and \(\psi(y_1) = g_1, \psi(y_2) = g_2\).

Let \(\mathcal{Z} = \text{span}\{g_1, g_2\}\). It is easy to see that
\[
\mathcal{N}(g_1) \cap \mathcal{N}(g_2) = \text{span}\left\{(1, -\frac{1}{2}, \frac{1}{2}, 0), (0, 0, 0, 1)\right\}.
\]

We now follow the procedure, as described in Case II of Problem 5.1. Therefore, we find
\[
\bigcap_{i=1}^{2} \mathcal{N}(g_i) \cap \mathcal{N}(f),
\]
which is given by
\[
\text{span}\left\{(1, -\frac{1}{2}, \frac{1}{2}, 1), (1, -\frac{1}{2}, \frac{1}{2}, -1)\right\}.
\]

Note that \(\bigcap_{i=1}^{2} \mathcal{N}(g_i) \cap \mathcal{N}(f) \cap S_{\mathcal{X}^*} = \{0\}\). Thus,
\[
\text{dist}(x_0, \mathcal{Y}) = \text{dist}(f, \mathcal{Z}) = \max \left\{ |f(x)| : x \in \bigcap_{i=1}^{2} \mathcal{N}(g_i) \cap S_{\mathcal{X}^*} \right\} = 2.
\]

Next, we find a best approximation to \(f\) out of \(\mathcal{Z}\). Define \(f_0 : \bigcap_{i=1}^{2} \mathcal{N}(g_i) \to \mathbb{R}\) by
\[
f_0(x_1, x_2, x_3, x_4) = f\bigg|_{\bigcap_{i=1}^{2} \mathcal{N}(g_i)} (x_1, x_2, x_3, x_4)
\]
\[
= x_1 + x_4 \quad \forall (x_1, x_2, x_3, x_4) \in \bigcap_{i=1}^{2} \mathcal{N}(g_i).
\]

Let \(\tilde{f}_0 : \ell_\infty^4 \to \mathbb{R}\) be defined by
$$\tilde{f}_0(x_1, x_2, x_3, x_4) = x_1 + x_4 \quad \forall (x_1, x_2, x_3, x_4) \in \ell^4_\infty.$$  

Then we have that

$$\|\tilde{f}_0\| = \|f_0\| = \left\| f \right\|_{\bigcap_{i=1}^{\infty} \mathcal{N}(g)} = 2.$$  

Consequently, $\tilde{f}_0$ is a Hahn–Banach extension of $f|_{\bigcap_{i=1}^{\infty} \mathcal{N}(g)}$. Therefore, it follows from Theorem 4.3 that $f - \tilde{f}_0$ is a best approximation to $f$ out of $\mathbb{Z}$. Thus, $\psi^{-1}(f - \tilde{f}_0) = (0, 1, 1, 0)$ is a best approximation to $x_0$ out of $\mathcal{Y}$.

In light of Problem 5.2, we can say that best approximation problems in context of the $\ell^n_\infty$ and $\ell^1_n$ spaces reduce to the problem of maximizing a functional to the unit sphere of the solution space of a system of homogeneous linear equations, and the problem becomes trivial if $m = n - 1$. Indeed, in that case all we need to do is to solve a system of homogeneous linear equations. On the other hand, we have the following explicit distance formulae whenever $n = 2$:

**Proposition 5.3** Let $\mathbb{X} = \ell^2_1$, and let $x = (a, b) \in \mathbb{X}$. Let $y = (c, d) \neq (0, 0)$ be, such that $x \notin \mathcal{Y}$, where $\mathcal{Y} = \text{span}\{y\}$. Then

$$\text{dist}(x, \mathcal{Y}) = \frac{2|ad - bc|}{|d| + |c| + ||d| - |c||}.$$  

**Proof** Let $\psi : \ell^2_1 \longrightarrow \ell^2_\infty$ be the canonical isometric isomorphism. Let $\psi(x) = f$ and let $\psi(y) = g$. Clearly

$$\mathcal{N}(g) = \{(u_1, u_2) \in \mathbb{R}^2 : cu_1 + du_2 = 0\}.$$  

Assume that $d \neq 0$. Then it follows that $(1, -\frac{c}{d}) \in \mathcal{N}(g)$. Let $\alpha \in \mathbb{R}$ be, such that

$$\left\| \alpha \left(1, -\frac{c}{d}\right) \right\|_\infty = 1.$$  

It can be shown without any difficulty that

$$|\alpha| = \frac{2|d|}{|d| + |c| + ||d| - |c||}.$$  

Applying Corollary 4.2.1, we obtain that

$$\text{dist}(x, \mathcal{Y}) = \left| f \left(\alpha \left(1, -\frac{c}{d}\right)\right) \right| = \frac{2|ad - bc|}{|d| + |c| + ||d| - |c||}.$$  

Similarly, if $c \neq 0$, one can show that

$$\text{dist}(x, \mathcal{Y}) = \frac{2|ad - bc|}{|d| + |c| + ||d| - |c||}.$$  

This completes the proof.  

$\square$
Proposition 5.4  Let $\mathbb{X} = \ell_2^\infty$, and let $\mathbf{x} = (a, b) \in \mathbb{X}$. Let $\mathbf{y} = (c, d) \neq (0, 0)$ be, such that $\mathbf{x} \not\in \mathbb{Y}$, where $\mathbb{Y} = \text{span}\{\mathbf{y}\}$. Then
\[
\text{dist}(\mathbf{x}, \mathbb{Y}) = \frac{|ad - bc|}{|c| + |d|}.
\]

Proof  Let $\psi : \ell_2^\infty \to \ell_1^\infty$ be the canonical isometric isomorphism. Let $\psi(\mathbf{x}) = f$ and let $\psi(\mathbf{y}) = g$. Clearly
\[
\mathcal{N}(g) = \{(u_1, u_2) \in \mathbb{R}^2 : cu_1 + du_2 = 0\}.
\]
Assume that $d \neq 0$. Then it follows that $(1, -\frac{c}{d}) \in \mathcal{N}(g)$. Let $\alpha \in \mathbb{R}$ be, such that $\left\|\alpha \left(1, -\frac{c}{d}\right)\right\|_1 = 1$. It can be shown without any difficulty that $|\alpha| = \frac{|d|}{|c| + |d|}$. Applying Corollary 4.2.1, we obtain that
\[
\text{dist}(\mathbf{x}, \mathbb{Y}) = \left|f \left(\alpha \left(1, -\frac{c}{d}\right)\right)\right| = \frac{|ad - bc|}{|c| + |d|}.
\]

Similarly, if $c \neq 0$, one can show that
\[
\text{dist}(\mathbf{x}, \mathbb{Y}) = \frac{|ad - bc|}{|c| + |d|}.
\]

This completes the proof. $\square$

In view of Propositions 5.3, 5.4 and Theorem 3.8 of [15], the proof of the following result is obvious.

Proposition 5.5  Let $\mathbb{X} = \ell_p^2$, $1 \leq p \leq \infty$; and let $\mathbf{x} = (a, b) \in \mathbb{X}$. Let $\mathbf{y} = (c, d) \neq (0, 0)$ be, such that $\mathbf{x} \not\in \mathbb{Y}$, where $\mathbb{Y} = \text{span}\{\mathbf{y}\}$. Then
\[
\text{dist}(\mathbf{x}, \mathbb{Y}) = \frac{|ad - bc|}{\|\langle c, d \rangle\|_q},
\]
where $q$ is conjugate to $p$.

The next theorem presents a sufficient condition for the uniqueness of the best approximation in finite-dimensional real polyhedral Banach space. Recall that a finite-dimensional real Banach space is called polyhedral if $\text{Ext}(B_{\mathbb{X}})$ is finite. For more information on the local structure of finite-dimensional real polyhedral Banach spaces, we refer the readers to [17].

Theorem 5.6  Let $\mathbb{X}$ be an $n$-dimensional real polyhedral Banach space and let $x, y_1, y_2, \ldots, y_m \in \mathbb{X}$; $1 \leq m < n$. Let $y_1, y_2, \ldots, y_m$ be linearly independent and $x \not\in \mathbb{Y}$, where $\mathbb{Y} = \text{span}\{y_1, y_2, \ldots, y_m\}$. Let $\psi : \mathbb{X} \to \mathbb{X}^{**}$ denote the canonical isometric isomorphism. Suppose that $\bigcap_{i=1}^m \mathcal{N}(\psi(y_i)) \cap S_{\mathbb{X}^*}$ contains only smooth point(s) of $B_{\mathbb{X}^*}$. Then, the best approximation to $x$ out of $\mathbb{Y}$ is unique.
Proof Let $\psi(x) = f$ and let $\psi(y_i) = g_i$ for each $1 \leq i \leq m$. Clearly, $f \notin \mathbb{Z}$, where $
abla = \text{span}\{g_1, g_2, \ldots, g_m\}$. It is enough to show that the best approximation to $f$ out of $\nabla$ is unique. Suppose that $\sum_{i=1}^{m} \alpha_i g_i$ and $\sum_{i=1}^{m} \beta_i g_i$ are best approximations to $f$ out of $\nabla$ for real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m$. Note that $\bigcap_{i=1}^{m} \mathcal{N}(g_i) \cap S_{X^*} \subseteq F \cup (-F)$ for some facet $F$ of $B_{X^*}$. Indeed, if $x_1 \in \left( \bigcap_{i=1}^{m} \mathcal{N}(g_i) \cap S_{X^*} \right) \cap F_1$ and $x_2 \in \left( \bigcap_{i=1}^{m} \mathcal{N}(g_i) \cap S_{X^*} \right) \cap F_2$ for some distinct facets $F_1$ and $F_2$ of $B_{X^*}$, with $F_1 \neq -F_2$, then there exists $t \in (0, 1)$, such that $\frac{tx_1 + (1-t)x_2}{\|tx_1 + (1-t)x_2\|} \in \bigcap_{i=1}^{m} \mathcal{N}(g_i) \cap S_{X^*}$ is a non-smooth point of $B_{X^*}$. Let $g_0$ be a best approximation to $f$ out of $\nabla$. Then, it follows from Corollary 4.1.1 that

$$\bigcap_{i=1}^{m} \mathcal{N}(g_i) \cap M_{f-g_0} \neq \emptyset.$$  

Since $\bigcap_{i=1}^{m} \mathcal{N}(g_i) \cap S_{X^*}$ contains only smooth points of the facets $F$ and $-F$, it follows that $f - g_0 = \lambda h$, where $h \in S_{X^*}$ is the unique support functional corresponding to the facet $F$ [17] and $\lambda \in \mathbb{R}$ is non-zero. However, this proves that

$$f - \sum_{i=1}^{m} \alpha_i g_i = \lambda_1 h$$ and $$f - \sum_{i=1}^{m} \beta_i g_i = \lambda_2 h,$$

where $\lambda_1, \lambda_2$ are non-zero real numbers with $|\lambda_1| = |\lambda_2|$. Suppose that $\lambda_1 = -\lambda_2 = \lambda_0$. Then, it follows that

$$f = \frac{1}{2} \sum_{i=1}^{m} (\alpha_i + \beta_i) g_i,$$

which is a contradiction, since $f \notin \mathbb{Z}$. Therefore, we must have $\lambda_1 = \lambda_2$. Consequently, $\sum_{i=1}^{m} \beta_i g_i = \sum_{i=1}^{m} \alpha_i g_i$ and the best approximation is unique. This completes the proof of the theorem. \hfill $\square$

The converse of the above theorem need not be true. The following example illustrates such a situation:

Example 5.6.1 Let $X = \ell^3_1$ and let $x = (0, 0, 1)$. We identify $X^{**}$ to the dual of $\ell^3_\infty$ and let $\psi : X \rightarrow X^{**}$ denote the canonical isometric isomorphism. Let $\psi(x) = f$ and $\psi(y) = g$. Then, it is not difficult to see that

$$f(x_1, x_2, x_3) = \frac{1}{2} (x_2 + x_3) \text{ and } g(x_1, x_2, x_3) = x_3 \quad \forall (x_1, x_2, x_3) \in \ell^3_\infty.$$  

Clearly

$$\mathcal{N}(g) \cap S_{X^*} = \left\{ (x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}, \max\{|x_1|, |x_2|\} = 1 \right\},$$

which also contains non-smooth points of $B_{X^*}$. Our aim is to show that the best approximation to $x$ out of $\text{span}\{y\}$ is unique.
Let $\lambda_0 \varrho$ be the best approximation to $f$ out of $\text{span}\{g\}$. Then, it follows from Corollary 4.2.1 that

$$\text{dist}(f, \text{span}\{g\}) = \max \left\{ \frac{1}{2} |(x_2 + x_3)| : x_3 = 0, \max\{|x_1|, |x_2|\} = 1 \right\} = \frac{1}{2}.$$ 

Define $f_0 : \mathcal{N}(g) \to \mathbb{R}$ by

$$f_0(x_1, x_2, x_3) = f|_{\mathcal{N}(g)}(x_1, x_2, x_3) = \frac{1}{2} x_2 \quad \forall (x_1, x_2, x_3) \in \mathcal{N}(g).$$

Next, we define $\tilde{f}_0 : \ell_\infty^3 \to \mathbb{R}$ by

$$\tilde{f}_0(x_1, x_2, x_3) = \frac{1}{2} x_2 \quad \forall (x_1, x_2, x_3) \in \ell_\infty^3.$$ 

Now, it is not difficult to see that

$$\left\| \tilde{f}_0 \right\| = \|f_0\| = \left\| f|_{\mathcal{N}(g)} \right\| = \frac{1}{2}.$$ 

Consequently, $\tilde{f}_0$ is a Hahn–Banach extension of $f|_{\mathcal{N}(g)}$. Therefore, it follows from Theorem 4.3 that $f - \tilde{f}_0 = 1/2 \varrho$ is a best approximation to $f$ out of $\text{span}\{g\}$.

For any $\lambda \in \mathbb{R}$, consider the linear functional $(f - \lambda \varrho) : \ell_\infty^3 \to \mathbb{R}$, given by

$$(f - \lambda \varrho)(x_1, x_2, x_3) = \frac{1}{2} x_2 + (\frac{1}{2} - \lambda)x_3 \quad \forall (x_1, x_2, x_3) \in \ell_\infty^3.$$ 

Observe that if $(\frac{1}{2} - \lambda) > 0$, choosing $(x_0, y_0, z_0) \in \text{Ext}(B_{\ell_\infty^3})$, such that $y_0z_0 = 1$, we get $|(f - \lambda \varrho)(x_0, y_0, z_0)| > \frac{1}{2}$. Again, if $(\frac{1}{2} - \lambda) < 0$, choosing $(x_0, y_0, z_0) \in \text{Ext}(B_{\ell_\infty^3})$, such that $y_0z_0 = -1$, we get $|(f - \lambda \varrho)(x_0, y_0, z_0)| > \frac{1}{2}$. Therefore, the only solution of $\lambda_0$ for which $\|f - \lambda_0 \varrho\| = \frac{1}{2}$ is $\lambda_0 = \frac{1}{2}$ and the best approximation to $f$ out of $\text{span}\{g\}$ is unique. Consequently, the best approximation to $x$ out of $\text{span}\{y\}$ is unique.

As an application of the ideas developed in Theorems 4.1 and 4.2.1, it is also possible to explore the following invariant distance problem:

**Problem 5.7** Let $(\mathcal{X}_1, \| \cdot \|_1)$ and $(\mathcal{X}_2, \| \cdot \|_2)$ be two Banach spaces with $\mathcal{X}_1 \subset \subset \mathcal{X}_2$. Let $\mathcal{Y}$ be a finite-dimensional vector subspace of $\mathcal{X}_1 \cap \mathcal{X}_2$ and let $x_0 \in (\mathcal{X}_1 \cap \mathcal{X}_2) \setminus \mathcal{Y}$. Then, find a necessary and sufficient condition on $x_0$ and $\mathcal{Y}$, such that

$$\text{dist}_1(x_0, \mathcal{Y}) = \text{dist}_2(x_0, \mathcal{Y}).$$

In the following theorem, we provide a complete solution to the above problem when $\mathcal{X}_1 = \ell_{p_1}$ and $\mathcal{X}_2 = \ell_{p_2}$, where $1 < p_1, p_2 < \infty$ and $p_1 \neq p_2$.

**Theorem 5.8** Let $1 < p_1 < p_2 < \infty$. Let $x, y_1, \ldots, y_m \in \ell_{p_1} \cap \ell_{p_2}$ be, such that $y_1, y_2, \ldots, y_m$ are linearly independent with $x \notin \mathcal{Y}$, where $\mathcal{Y} = \text{span}\{y_1, y_2, \ldots, y_m\}$. Let $x_0$ and $y_0$ be the best approximations to $x$ out of $\mathcal{Y}$ in $\ell_{p_1}$ and $\ell_{p_2}$, respectively.
Then \( \text{dist}_{p_1}(x, \mathbb{V}) = \text{dist}_{p_2}(x, \mathbb{V}) \) if and only if \( x - x_0 = x - y_0 = \lambda e_j \), for some non-zero \( \lambda \in \mathbb{R} \) and \( j \in \mathbb{N} \), and \( \mathbb{V} \subseteq \{(\lambda_1, \lambda_2, \ldots) \in \ell_{p_1} \cap \ell_{p_2} : \lambda_j = 0\} \).

**Proof** We only prove the necessary part as the sufficient part of the theorem is trivial. Let \( \psi_1 : \ell_{p_1} \rightarrow \ell_{q_1}^* \) and \( \psi_2 : \ell_{p_2} \rightarrow \ell_{q_2}^* \) denote the canonical isometric isomorphisms, where \( q_1 \) and \( q_2 \) are conjugates to \( p_1 \) and \( p_2 \), respectively. Given any \( \eta \in \ell_{p_1} \), \( \eta \) is also a member of \( \ell_{p_2} \) and by Proposition 2.5,

\[
\psi_1(\eta)|_{\ell_{q_2}} = \psi_2(\eta).
\]

Let \( Z_1 = \text{span}\{\psi_1(y_i) : 1 \leq i \leq m\} \) and \( Z_2 = \text{span}\{\psi_2(y_i) : 1 \leq i \leq m\} \). It follows from the hypothesis of the theorem that \( \text{dist}_{q_1}(\psi_1(x), Z_1) = \text{dist}_{q_2}(\psi_2(x), Z_2) \). Let

\[
\bigcap_{i=1}^m N(\psi_1(y_i)) = \mathcal{W}_1, \quad \bigcap_{i=1}^m N(\psi_2(y_i)) = \mathcal{W}_2.
\]

Applying Corollary 4.2.1, we have that

\[
\text{dist}_{q_1}(\psi_1(x), Z_1) = \max \left\{ |\psi_1(x)(z)| : z \in \mathcal{W}_1 \cap S_{q_1} \right\} = \text{dist}_{q_2}(\psi_2(x), Z_2)
\]

\[
= \max \left\{ |\psi_2(x)(z)| : z \in \mathcal{W}_2 \cap S_{q_2} \right\} = \lambda \text{ (say)}.
\]

Since \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are strictly convex, there exist \( u_0 \in S_{q_1} \) and \( v_0 \in S_{q_2} \), such that

\[
M_{\psi_1(x)}|_{\mathcal{W}_1} = \{ \pm u_0 \}, \quad M_{\psi_2(x)}|_{\mathcal{W}_2} = \{ \pm v_0 \}.
\]

Obviously, \( |\psi_1(x)(u_0)| = |\psi_2(x)(v_0)| \). Also, it is not difficult to see that \( v_0 \in \ell_{q_1} \) and \( \|v_0\|_{q_1} \leq 1 \). Now, it follows from (2) that \( \psi_1(y_i)(v_0) = \psi_1(y_i)|_{\ell_{q_2}}(v_0) = \psi_2(y_i)(v_0) = 0 \) for each \( 1 \leq i \leq m \). Therefore, \( v_0 \in \mathcal{W}_1 \). If \( \|v_0\|_{q_1} < 1 \), consider some \( \mu_0 > 1 \), such that \( \mu_0 v_0 \in S_{q_1} \). Then

\[
|\psi_1(x)(\mu_0 v_0)| = \mu_0 |\psi_1(x)(v_0)| = \mu_0 |\psi_2(x)(v_0)| = \mu_0 |\psi_1(x)(u_0)| > |\psi_1(x)(u_0)|,
\]

which is a contradiction. Therefore, \( \|v_0\|_{q_1} = 1 \). Also, \( u_0 = \pm v_0 \), as otherwise \( \psi_1(x)|_{\mathcal{W}_1} \) would attain norm at two pair of points. Without loss of generality, let \( u_0 = v_0 \) and \( \psi_1(x)(u_0) = \psi_2(x)(v_0) = \lambda \). Since \( q_1 \neq q_2 \), we must have \( u_0 = v_0 = \pm e_j \) for some \( j \in \mathbb{N} \). Let \( u_0 = v_0 = e_j \) and let \( h_1, h_2 \) be the (unique) best approximations to \( \psi_1(x) \), \( \psi_2(x) \) out of \( Z_1 \) and \( Z_2 \), respectively. Obviously, \( \psi_1(x_0) = h_1 \) and \( \psi_2(y_0) = h_2 \).

By Theorem 4.3, \( \psi_1(x - x_0) \) and \( \psi_2(x - y_0) \) are the Hahn–Banach extensions of \( \psi_1(x)|_{\mathcal{W}_1} \) and \( \psi_2(x)|_{\mathcal{W}_2} \), respectively. Since \( \ell_{q_1} \) and \( \ell_{q_2} \) are strictly convex, we have

\[
M_{\psi_1(x-x_0)} = M_{\psi_2(x-y_0)} = \{ \pm e_j \}.
\]

Therefore, \( \frac{1}{r} \psi_1(x - x_0) \) and \( \frac{1}{s} \psi_2(x - y_0) \) are the (unique) support functionals at \( e_j \). Again it follows from Proposition 2.6 that
Now, applying Proposition 2.5, we have that
\[ \frac{1}{\lambda} \psi_1(x - x_0)(z) = z_j \quad \forall \ z = (z_1, z_2, \ldots) \in \ell_{q_1}, \]
\[ \frac{1}{\lambda} \psi_2(x - y_0)(z) = z_j \quad \forall \ z = (z_1, z_2, \ldots) \in \ell_{q_2}. \]

Let \( \{ \rho \} = J(e_j) \). Since \( e_j \perp_B V \), we have
\[ \rho(y) = y_j = 0 \quad \forall \ y = (y_1, y_2, \ldots) \in V. \]

Consequently, \( V \subseteq \{ (\lambda_1, \lambda_2, \ldots) \in \ell_{p_1} \cap \ell_{p_2} : \lambda_j = 0 \} \) and the proof follows.

It can be seen from [15, Remark 3.10] that best approximation problems give rise to a family of inequalities in context of \( \ell_p \) spaces \((1 < p < \infty)\). Our next goal is to find the said family of inequalities in a more general setting. The following result is the first step towards achieving the said goal.

**Theorem 5.9** Let \( n \in \mathbb{N} \) and let \( X_i = \ell_{p_i}^{m_i} \), where \( m_i \) are natural numbers and \( 1 \leq p_i \leq \infty \) for each \( 1 \leq i \leq n \). Let
\[ X = \bigoplus_{i=1}^n p X_i \quad \text{for some} \ 1 \leq p \leq \infty. \]

Let \( \sum_{i=1}^n m_i = m \) and let \( T \) be an \( m \) by \( m \) non-zero matrix. Then, for any \( x_0 \in X \) the following holds true:
\[
\min \{ ||x_0 - y||_X : y \in \mathcal{R}(T) \} = \max \left\{ \frac{1}{||z||_X} |\langle x_0, z \rangle| : z \in N(T^*) \setminus \{ \theta \} \right\}. \quad (3)
\]

**Proof** We begin the proof with the observation that the existence of the minimum of the set \( \{ ||x_0 - y||_X : y \in \mathcal{R}(T) \} \) is guaranteed by the existence of best approximation(s) to \( x_0 \) out of \( \mathcal{R}(T) \).

Let \( q, q_1, q_2, \ldots, q_n \) be the conjugates to \( p, p_1, p_2, \ldots, p_n \), respectively. Clearly, for each \( 1 \leq i \leq n \)
\[ X_i^* = \ell_{q_i}^{m_i}, \ \text{and} \ X^* = \bigoplus_{i=1}^n q X_i^*. \]

Also, for any member \( x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in X^* \)
\[ x^*(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i^*(x_i) \quad \forall \ (x_1, x_2, \ldots, x_n) \in \bigoplus_{i=1}^n p X_i. \]
Clearly, $\mathbb{X}$ and $\mathbb{R}^m$ are isomorphic as vector spaces. Let $\{e_j\}_{j=1}^m$ denote the standard ordered basis of $\mathbb{X}$. Observe that $\mathcal{R}(T) = \text{span}\{y_1, y_2, \ldots, y_m\}$, where $y_j = T(e_j)$ for all $1 \leq j \leq m$. Therefore

$$[T] = [y_1' \ y_2' \cdots \ y_m'].$$

Let $x_0 = (u_1, u_2, \ldots, u_n)$, where $u_i \in \mathbb{X}_i$ for all $1 \leq i \leq n$. In addition, let

$$y_j = (w_{j1}, w_{j2}, \ldots, w_{jn}), \quad \text{where} \quad w_{ji} \in \mathbb{X}_i \quad \forall \ 1 \leq i \leq n.$$

Let $\psi : \bigoplus_{i=1}^n \mathbb{X}_i \rightarrow \left( \bigoplus_{i=1}^n \mathbb{X}_i^* \right)^*$ denote the canonical isometric isomorphism. Let

$$\psi(x_0) = f_0 \quad \text{and} \quad \psi(y_j) = g_j \quad \forall \ 1 \leq j \leq m.$$

Then it is not difficult to see that for each $1 \leq j \leq m$,

$$g_j(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n \langle w_{ji}, x_i \rangle \quad \forall \ (x_1, x_2, \ldots, x_n) \in \bigoplus_{i=1}^n \mathbb{X}_i^*,$n

$$f_0(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n \langle u_i, x_i \rangle \quad \forall \ (x_1, x_2, \ldots, x_n) \in \bigoplus_{i=1}^n \mathbb{X}_i^*.$$n

Let $\mathbb{Z} = \text{span}\{g_1, g_2, \ldots, g_m\}$. Note that $\bigcap_{j=1}^m \mathcal{N}(g_j)$ is a subspace of $\bigoplus_{i=1}^n \mathbb{R}^m_i$. Identifying each member $z = (z_1, z_2, \ldots, z_n)$ of $\bigoplus_{i=1}^n \mathbb{X}_i^*$ as a member of $\bigoplus_{i=1}^n \mathbb{R}^m_i \cong \mathbb{R}^m$, we then have

$$\bigcap_{j=1}^m \mathcal{N}(g_j) = \left\{ (z_1, \ldots, z_n) \in \bigoplus_{i=1}^n \mathbb{R}^m_i : \sum_{i=1}^n \langle w_{1i}, z_i \rangle = \cdots = \sum_{i=1}^n \langle w_{mi}, z_i \rangle = 0 \right\}.$$n

In other words

$$\bigcap_{j=1}^m \mathcal{N}(g_j) = \left\{ z \in \mathbb{R}^m : [y_1' \ y_2' \cdots \ y_m']z = \theta \right\} = \{ z \in \mathbb{R}^m : T^*z = \theta \} = \mathcal{N}(T^*).$$n

We now consider the following two cases:

Case I: Let $x_0 \in \mathcal{R}(T)$. Then $\min\{\|x_0 - y\|_\infty : y \in \mathcal{R}(T)\} = 0$. Since $x_0 \in \text{span}\{y_1, y_2, \ldots, y_m\}$, we have that $f_0 \in \mathbb{Z}$. It now follows from Proposition 2.1 that

$$\mathcal{N}(T^*) = \bigcap_{j=1}^m \mathcal{N}(g_j) \subseteq \ker f_0.$$n

Therefore, $\max \left\{ \frac{1}{\|x_0\|_\infty} \|f_0(z)\| : z \in \bigcap_{j=1}^m \mathcal{N}(g_j) \cap \text{span} \mathbb{X}_i^* \right\} = 0$. Now, applying (4), we have that $\max \left\{ \frac{1}{\|x_0\|_\infty} |\langle x_0, z \rangle| : z \in \mathcal{N}(T^*) \setminus \{\theta\} \right\} = 0$ and the equality (3) follows.

Case II: Let $x_0 \not\in \mathcal{R}(T)$ and let $y \in \mathcal{R}(T)$. Then, applying Corollary 4.2.1 we have

$\square$ Birkhäuser
Now, applying (4), we have that
\[ \|x_0 - y\|_X = \|\psi(x_0 - y)\|_{X^*} = \|f_0 - \psi(y)\|_{X^*} \geq \text{dist}(f_0, Z) \]
\[ = \max \left\{ \frac{1}{\|z\|_{X^*}}|\langle x_0, z \rangle| : z \in \mathcal{N}(T^*) \setminus \{\theta\} \right\} \]
\[ = \max \left\{ \frac{1}{\|z\|_{X^*}}|\langle f_0(z), z \rangle| : z \in \mathcal{N}(T^*) \setminus \{\theta\} \right\} \].

Since \( y \in \mathcal{R}(T) \) was chosen arbitrarily, we obtain
\[ \min \{ \|x_0 - y\|_X : y \in \mathcal{R}(T) \} \geq \max \left\{ \frac{1}{\|z\|_{X^*}}|\langle x_0, z \rangle| : z \in \mathcal{N}(T^*) \setminus \{\theta\} \right\} . \]

However, the above inequality is necessarily an equality, since
\[ \text{dist}(x_0, \mathcal{R}(T)) = \text{dist}(f_0, Z) = \min \{ \|x_0 - y\|_X : y \in \mathcal{R}(T) \} . \]

This completes the proof. \( \square \)

The above result can also be stated in the form of the following inequality:

**Corollary 5.9.1** Let \( n \in \mathbb{N} \) and let \( X_i = \ell^{m_i}_{\ell^{p_i}} \), where \( m_i \) are natural numbers and \( 1 \leq p_i \leq \infty \) for each \( 1 \leq i \leq n \). Let
\[ X = \bigoplus_{i=1}^{n} X_i \quad \text{for some} \ 1 \leq p \leq \infty . \]
Let \( \sum_{i=1}^{n} m_i = m \) and let \( T \) be an \( m \) by \( m \) non-zero matrix. Let \( x_0 \in X \). Then, for any \( y \in \mathcal{R}(T) \)
\[ \|x_0 - y\|_X \geq \frac{1}{\|z\|_{X^*}}|\langle x_0, z \rangle| , \]
for all \( z \in \mathcal{N}(T^*) \setminus \{\theta\} \).

As an application of Theorem 5.9, we have the following:

**Theorem 5.10** Let \( x_0 = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \) and let \( \{m_1, m_2, \ldots, m_n\} \subseteq \mathbb{N} \) be, such that \( m = \sum_{i=1}^{n} m_i \). Set \( s_0 = 0 \) and \( s_k = \sum_{i=1}^{k} m_i \) for each \( 1 \leq k \leq n \). Let \( \{p, p_1, \ldots, p_n\} \subseteq (1, \infty) \) and let \( q, q_1, q_2, \ldots, q_n \) be conjugates to \( p, p_1, p_2, \ldots, p_n \), respectively. Then, for any \( a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m \setminus \{\theta\} \),
\[
\left(\sum_{k=1}^{n} \left( \sum_{j=s_{k-1}+1}^{s_k} |x_j - \lambda a_j|^p k \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \leq \left( \sum_{k=1}^{n} \left( \sum_{j=s_{k-1}+1}^{s_k} |b_j|^q k \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
\]

(5)

for all \((b_1, b_2, \ldots, b_m) \in W \setminus \{\theta\}\) and \(\lambda \in \mathbb{R}\), where

\[W = \left\{(z_1, z_2, \ldots, z_m) \in \mathbb{R}^m : \sum_{i=1}^{m} a_i z_i = 0\right\}.

Moreover, the above inequality is optimal.

**Proof** Let \(\mathbb{H}\) denote the Hilbert space \(\mathbb{R}^m\) equipped with the usual dot product. Let \(\mathcal{X} = \text{span}\{(a_1, a_2, \ldots, a_m)\}\). Therefore, we have that \(W = \mathcal{X}^\perp\). Let for each \(1 \leq i \leq n:\)

\[\mathcal{X}_i = e^m_{p_i} \quad \text{and} \quad \mathcal{X} = \bigoplus_{i=1}^{n} \mathcal{X}_i.

Then \(\mathcal{X}\) and \(\mathbb{R}^m\) are isomorphic as vector spaces. Let \(\{e_j\}_{j=1}^{m}\) denote the standard ordered basis of \(\mathcal{X}\). Define \(T : \mathbb{H} \to \mathbb{H}\) in such a way that

\[\mathcal{R}(T) = \text{span}\{T(e_j) : 1 \leq j \leq m\} = \mathcal{X}.

Then it is easy to see that \(\mathcal{N}(T^*) = \mathcal{X}^\perp\). It now follows from Theorem 5.9 that

\[
\min\{\|x_0 - y\|_{\mathcal{X}} : y \in \mathcal{R}(T)\} = \max\left\{\frac{1}{\|z\|_{\mathcal{X}^*}} |\langle x_0, z \rangle| : z \in \mathcal{N}(T^*) \setminus \{\theta\}\right\},
\]

Now, inequality (5) is obtained by removing minimum and maximum from both sides, and expressing \(\|x_0 - y\|_{\mathcal{X}}\) and \(\|z\|_{\mathcal{X}^*}\), in their explicit forms. Moreover, the inequality is optimal. This completes the proof. \(\square\)

The inequality (5) obtained in Theorem 5.10 exhibits a stronger version of Hölder’s inequality in finite-dimensional case. We explain this in more detail in the following remark:

**Remark 5.1** We consider two non-zero elements \(u = (u_1, u_2, \ldots, u_m)\) and \(v = (v_1, v_2, \ldots, v_m)\) in \(\mathbb{R}^m\), with \(m \geq 2\). Let \(x_0 = (u_1, u_2, \ldots, u_m)\) and \(\overline{v} = (b_1, b_2, \ldots, b_m)\), where

\[b_j = \begin{cases} 
\text{sgn}(u_j v_j)v_j & \text{if } u_j \neq 0, \ 1 \leq j \leq m, \\
v_j & \text{if } u_j = 0, \ 1 \leq j \leq m.
\end{cases}
\]
Let \( \mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m \setminus \{ \theta \} \) be, such that \( \sum_{j=1}^{m} a_j b_j = 0 \). Consider a subset \( \{m_1, m_2, \ldots, m_n\} \) of natural numbers, such that \( m = \sum_{i=1}^{m} m_i \). Now, considering \( p = p_1 = p_2 = \cdots = p_n \) and \( \lambda = 0 \) in Theorem 5.10, we obtain that

\[
\left( \sum_{j=1}^{m} |u_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{m} |b_j|^q \right)^{\frac{1}{q}} \geq \left| \sum_{j=1}^{m} u_j b_j \right|.
\]

Since \( \sum_{j=1}^{m} |b_j|^q = \sum_{j=1}^{m} |v_j|^q \) and \( \sum_{j=1}^{m} u_j b_j = \sum_{j=1}^{m} |u_j v_j| \), on simplification, we get

\[
\sum_{j=1}^{m} |u_j v_j| \leq \left( \sum_{j=1}^{m} |u_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{m} |v_j|^q \right)^{\frac{1}{q}}.
\]

We end this section with an example involving a particular type of minimization problem. It is worth mentioning that such kind of problems are difficult to handle from the computational point of view. The reader is invited to verify this claim, using any method that seems fit, including the use of computer programming. However, employing the duality techniques developed in this work from the perspective of orthogonality, we can immediately solve these problems via trivial computations.

**Example 5.10.1** Let \( \mathbf{x}_0 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}) \in \mathbb{R}^{10} \) and let \( \{a_i\}_{i=1}^{10} \) be real variables. Let us consider the following minimization problem:

\[
\min_{A_i} \left\{ \begin{array}{l}
|\alpha_1 - A_1|^5 + (|\alpha_2 - A_2|^7 + |\alpha_3 - A_3|^7)^{\frac{5}{7}} \\
+ (|\alpha_4 - A_4|^3 + |\alpha_5 - A_5|^3 + |\alpha_6 - A_6|^3)^{\frac{5}{3}} \\
+ (|\alpha_7 - A_7|^{11} + |\alpha_8 - A_8|^{11})^{\frac{5}{11}} + (|\alpha_9 - A_9|^9 + |\alpha_{10} - A_{10}|^9)^{\frac{5}{9}}
\end{array} \right\}^{\frac{1}{5}},
\]

where \( A_i (1 \leq i \leq 10) \) are given by

\[
A_1 = -9a_1 + 7a_2 + 9a_3 + 9a_4 + 8a_5 + 9a_6 + 6a_7 + 7a_8 + 8a_9 + 9a_{10},
\]

\[
A_2 = a_1 - 4a_3 + a_4 + 3a_5 + 3a_6 - 2a_7 + 6a_8 + 9a_9 + 9a_{10},
\]

\[
A_3 = a_1 + 5a_2 + 3a_3 + 6a_4 + 4a_5 - 8a_6 + 9a_7 + 7a_8 + 2a_9 + 2a_{10},
\]

\[
A_4 = 3a_1 + 9a_2 - 4a_3 + 3a_4 + 8a_5 + 2a_6 + 5a_7 + 9a_8 + a_9 + 7a_{10},
\]

\[
A_5 = 6a_1 + 5a_2 + 7a_3 - a_4 + 6a_5 + a_6 + 8a_7 + 8a_8 - 6a_9 + 5a_{10},
\]

\[
A_6 = 8a_1 + 3a_2 + 8a_3 - a_4 + 2a_5 + a_7 + 3a_8 + 9a_9 + 5a_{10},
\]

\[
A_7 = 2a_2 + 8a_3 - 7a_4 + 3a_5 + 4a_7 + 2a_8 + 6a_{10},
\]

\[
A_8 = 7a_1 + 7a_2 + a_3 + 5a_4 + 2a_6 + a_7 + a_8 + 4a_9 + 9a_{10},
\]

\[
A_9 = 9a_1 + a_2 + 9a_3 + 5a_4 - 3a_5 + 7a_6 + 5a_7 + 3a_8 + 9a_9 + 8a_{10},
\]

\[
A_{10} = 12a_1 + \frac{13}{2}a_2 + 13a_3 - \frac{3}{2}a_4 + \frac{5}{2}a_5 + \frac{11}{2}a_6 + \frac{11}{2}a_7 + \frac{9}{2}a_8 + 11a_9 + 14a_{10}.
\]
In view of Theorem 5.9, we obtain the solution of the above problem in the following three steps:

Step I: The above minimization problem has five summands:

- first summand: $|\alpha_1 - A_1|^5$
- second summand: $(|\alpha_2 - A_2|^7 + |\alpha_3 - A_3|^7)^{\frac{5}{7}}$
- third summand: $(|\alpha_4 - A_4|^3 + |\alpha_5 - A_5|^3 + |\alpha_6 - A_6|^3)^{\frac{5}{3}}$
- fourth summand: $(|\alpha_7 - A_7|^{11} + |\alpha_8 - A_8|^{11})^{\frac{5}{11}}$
- fifth summand: $(|\alpha_9 - A_9|^9 + |\alpha_{10} - A_{10}|^9)^{\frac{5}{9}}$

We assume $\mathcal{X}_1 = \ell_1^p$ for any $p_1 \in (1, \infty)$, $\mathcal{X}_2 = \ell_2^7$, $\mathcal{X}_3 = \ell_3^2$, $\mathcal{X}_4 = \ell_{11}^2$, $\mathcal{X}_5 = \ell_9^2$.

Finally, let $\mathcal{X} = \bigoplus_{i=1}^{10} \mathcal{X}_i$.

Step II: Let

$$
\begin{align*}
\mathbf{y}_1 &= (-9, 1, 1, 3, 6, 8, 0, 7, 9, 12), & \mathbf{y}_2 &= \left(7, 0, 5, 9, 5, 3, 2, 7, 1, \frac{13}{2}\right), \\
\mathbf{y}_3 &= (9, -4, 3, -4, 7, 8, 1, 9, 13), & \mathbf{y}_4 &= \left(9, 1, 6, 3, -1, -1, -7, 0, 5, -\frac{3}{2}\right), \\
\mathbf{y}_5 &= \left(8, 3, 4, 8, 6, 0, 3, 5, -3, \frac{5}{2}\right), & \mathbf{y}_6 &= \left(9, 3, -8, 2, 1, 2, 0, 2, 7, \frac{11}{2}\right), \\
\mathbf{y}_7 &= \left(6, -2, 9, 5, 8, 1, 4, 1, 5, \frac{11}{2}\right), & \mathbf{y}_8 &= \left(7, 6, 5, 7, 8, 3, 2, 1, 3, \frac{9}{2}\right), \\
\mathbf{y}_9 &= (-8, 0, 0, 1, -6, 9, 0, 4, 9, 11), & \mathbf{y}_{10} &= (8, 9, 2, 7, 5, 5, 6, 9, 8, 14).
\end{align*}
$$

Define $T: \mathbb{R}^{10} \to \mathbb{R}^{10}$ in such a way that

$$
[T] = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5, \mathbf{y}_6, \mathbf{y}_7, \mathbf{y}_8, \mathbf{y}_9, \mathbf{y}_{10}].
$$

It is not difficult to see that $\mathcal{N}(T^*) = \text{span}\{(0, 0, 0, 0, 0, 1, 1, 1, 1, -2)\}$.

Step III: Clearly, the given problem is equivalent to finding the minimum of the collection:

$$
\{\|\mathbf{x}_0 - y\|_\mathcal{X} : y \in \mathcal{R}(T)\}.
$$

Now, using Theorem 5.9, it is easy to see that

$$
\begin{align*}
\min\{\|\mathbf{x}_0 - y\|_\mathcal{X} : y \in \mathcal{R}(T)\} &= \left\{1 + 2^{\frac{25}{32}} + \left(1 + 2^8\right)^{\frac{10}{9}}\right\}^{-\frac{4}{3}} |\alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 - 2\alpha_{10}|.
\end{align*}
$$

Therefore, we have obtained the complete solution to the minimization problem (6).

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Declarations

Conflict of interest The authors declare none.

References

1. Bhatia, R., Šemrl, P.: Orthogonality of matrices and some distance problems. Linear Algebra Appl. 287, 77–85 (1999)
2. Birkhoff, G.: Orthogonality in linear metric spaces. Duke Math. J. 1, 169–172 (1935)
3. Giles, J.R.: Classes of semi-inner-product spaces. Trans. Am. Math. Soc. 129, 436–446 (1967)
4. Grover, P.: Orthogonality to matrix subspaces, and a distance formula. Linear Algebra Appl. 445, 280–288 (2014)
5. James, R.C.: Orthogonality and linear functionals in normed linear spaces. Trans. Am. Math. Soc. 6(1), 265–292 (1947)
6. James, R.C.: Inner product in normed linear spaces. Bull. Am. Math. Soc. 53, 559–566 (1947)
7. James, R.C.: Reflexivity and the sup of linear functionals. Israel J. Math. 13, 289–300 (1972)
8. Lumer, G.: Semi-inner-product spaces. Trans. Am. Math. Soc. 100, 29–43 (1961)
9. Lau, K.K., Riha, W.O.J.: Characterization of best approximations in normed linear spaces of matrices by elements of finite-dimensional linear subspaces. Linear Algebra Appl. 35, 109–120 (1981)
10. Paul, K., Sain, D., Jha, K.: On strong orthogonality and strictly convex normed linear spaces. J. Inequal. Appl. 2013, 242 (2013)
11. Roy, S., Senapati, T., Sain, D.: Orthogonality of bilinear forms and application to matrices. Linear Algebra Appl. 615, 104–111 (2021)
12. Rudin, W.: Functional Analysis, 2nd edn. McGraw-Hill, Inc., New York (1991)
13. Sain, D.: Birkhoff–James orthogonality of linear operators on finite dimensional Banach spaces. J. Math. Anal. Appl. 447, 860–866 (2017)
14. Sain, D.: On the norm attainment set of a bounded linear operator and semi-inner-products in normed spaces. Indian J. Pure Appl. Math. 51, 179–186 (2020)
15. Sain, D.: On best approximations to compact operators. Proc. Am. Math. Soc. 149, 4273–4286 (2021)
16. Sain, D., Paul, K.: Operator norm attainment and inner product spaces. Linear Algebra Appl. 439, 2448–2452 (2013)
17. Sain, D., Paul, K., Bhunia, P., Bag, S.: On the numerical index of polyhedral Banach spaces. Linear Algebra Appl. 577, 121–133 (2019)
18. Sain, D., Paul, K., Mal, A.: A complete characterization of Birkhoff–James orthogonality in infinite dimensional normed space. J. Oper. Theory 80(2), 399–413 (2018)
19. Treves, F.: Topological Vector Spaces, Distributions and Kernels. Academic Press, New York (1967)