Schwinger, Pegg and Barnett and a relationship between angular and Cartesian quantum descriptions.

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Abstract

From a development of an original idea due to Schwinger, it is shown that it is possible to recover, from the quantum description of a degree of freedom characterized by a finite number of states (i.e., without classical counterpart) the usual canonical variables of position/momentum and angle/angular momentum, relating, maybe surprisingly, the first as a limit of the later.

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I. INTRODUCTION

Quantum mechanics has a lot of intriguing aspects. Definitely, one of those aspects is the fact that there’s still a handful of fundamental questions about it, over which debate have not ceased after so many years. Among these questions is the problem of the quantum phase, which a few years ago had an important chapter (but not the final, it seems) in its history with the approach due to Pegg and Barnett [1].

Within the broad grasp of the Pegg and Barnett formalism (PB), there is the particular and important problem of one dimensional angular coordinates in quantum mechanics. This specific problem is less problematic than the question of the phase as a whole, but nevertheless it is also ‘solved’ (or re-solved) within the procedure of PB. In this article, I shall relate the PB approach, in this particular context, to an idea presented by Schwinger, and from this relation, although relatively simple, seems to emerge quite interesting results.

Schwinger’s original idea was to recover a usual Cartesian degree of freedom (e.g., a degree of freedom endowed with a canonically related pair of observables of position and linear momentum) from a degree of freedom described by a finite set of states (that is,
without classical counterpart) through a limiting process. Here, I extend his discussion, showing that the Cartesian degree of freedom can in fact be recovered by an infinite number of limiting processes. The referred relationship comes from noting that a limiting element of those infinitely many processes which work for the Cartesian case reproduces exactly the Pegg-Barnett approach for the angle/angular momentum case. So, in this sense, a circle would be the limit of a line and not the opposite.

There’s a conceptual bonus in the Schwinger procedure to obtain the quantum description of a Cartesian degree of freedom. Schwinger’s approach to finite and discrete’s degrees of freedom is, in its roots [2], by nature laid over quantum mechanical concepts: quantum state, incompatible observables and unitary transformations. Once that, starting from this, one obtains the quantum description of degrees of freedom with classical counterpart, it is then as explicit as it is possible that no quantization of classical quantities must be necessarily involved in such descriptions. The PB approach to angular coordinates can also be seen by the same perspective and therefore shares this virtue. If one can see both descriptions (Cartesian and angular) as different manifestations of a same situation, then there might be room for new interpretations of the ultimate physical meaning of such mathematical structures.

II. THE SCHWINGER UNITARY OPERATORS BASES AND THE DISCRETE GENESIS OF THE CANONICAL VARIABLES

Long time ago Schwinger has noticed that one can obtain a complete basis in operator space out of a pair of unitary operators $U$ and $V$, which act on each other sets of $N$ eigenvectors as

$$V^s|u_n⟩ = |u_{n-s}⟩, \quad U^s|v_n⟩ = |v_{n+s}⟩, \quad n = 0, 1, ...N - 1$$

where a cyclic notation is understood,

$$|u_k⟩ \equiv |u_{k(\text{mod}N)}⟩ \quad |v_m⟩ \equiv |v_{m(\text{mod}N)}⟩.$$

(2)

The operators have the roots of unity as eigenvalues

$$U|u_k⟩ = \exp\left[\frac{2\pi i}{N}k\right]|u_k⟩, \quad V|v_k⟩ = \exp\left[\frac{2\pi i}{N}k\right]|v_k⟩,$$

and therefore

$$U^N = V^N = 1.$$  

(4)

The pair also obeys Weyl algebra

$$U^jV^l = \exp\left[\frac{2\pi i}{N}jl\right]V^lU^j,$$

and its eigenvectors are connected by a discrete Fourier transform

$$⟨v_k|u_n⟩ = \frac{1}{\sqrt{N}}\exp\left[-\frac{2\pi i}{N}kn\right].$$

(6)
which means that the two sets of states carry a maximum degree of incompatibility. It must be kept clear that this construction is absolutely general, as Schwinger obtains all results above from the mere existence of a complete family (with a finite number) of eigenstates of a given abstract operator.

Schwinger has realized that the pair of operators \( \{U, V\} \) could be used to define a basis in operator space (as will be discussed in more detail in a following work) and has also noticed that, if one goes from this discrete finite dimensional case to a usual continuous degree of freedom, the ordinary position-momentum description is recovered.

To further extend Schwinger’s original idea (which he concisely explored in just a few lines), first it must be introduced a scaling factor

\[
\epsilon = \sqrt{\frac{2\pi}{N}},
\]

which will become infinitesimal as \( N \to \infty \). Then, two Hermitian operators \( \{P, Q\} \), (for simplicity, odd \( N \)'s will be considered, as even values only require only a little more care and a heavier notation),

\[
P = \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} j \epsilon^\delta p_0 |v_j\rangle \langle v_j|,
\]

\[
Q = \sum_{j'=-\frac{N-1}{2}}^{\frac{N-1}{2}} j' \epsilon^{2-\delta} q_0 |u_{j'}\rangle \langle u_{j'}|,
\]

constructed out of the projectors of the eigenstates of \( U \) and \( V \). \( \delta \) is a free parameter which might assume any value in the open interval \((0, 2)\) (the original Schwinger discussion is equivalent to setting \( \delta = 1 \)). \( \{p_0, q_0\} \) are real parameters that might carry units of momentum and position, respectively, and \( \epsilon^\delta p_0 \) and \( \epsilon^{2-\delta} q_0 \) are the distance between successive eigenvalues of the \( P \) and \( Q \) operators. With the help of these, we can rewrite the Schwinger operators as

\[
V = \exp \left[ \frac{i\epsilon^{2-\delta} P}{p_0} \right] \quad U = \exp \left[ \frac{i\epsilon^\delta Q}{q_0} \right].
\]

Let also both eigenstate sets be relabeled as

\[
|v_j\rangle \equiv |p\rangle \quad |u_{j'}\rangle = |q\rangle, \quad \text{with } q = q_0 \epsilon^{2-\delta} j' \text{ and } p = p_0 \epsilon^\delta j.
\]

With that,

\[
P = \sum_{p=-\frac{N-1}{2} \epsilon^\delta p_0}^{\frac{N-1}{2} \epsilon^\delta p_0} p |p\rangle \langle p| \quad Q = \sum_{q=-\frac{N-1}{2} \epsilon^{2-\delta} q_0}^{\frac{N-1}{2} \epsilon^{2-\delta} q_0} q |q\rangle \langle q|,
\]

and Eqs. (3) now will read

\[
\exp \left[ \frac{i p' Q}{p_0 q_0} \right] |p\rangle = |p + p'|\]

and

\[\text{(12)}\]
if \( \{p', q'\} \) are defined following the recipe of (11).

The equations above have a clear analogy with the usual relations between position and momentum, apart from the fact that only discrete values of the parameters are allowed and that the cyclic conditions (Eq.(2)) are still holding.

The \( N \to \infty \) limit can now be easily performed. For \( \delta \) assuming any value in the open interval \((0, 2)\), both Hermitian operators defined on Eqs.(11) will feature an unbounded and continuous spectrum, as the limit leads them to

\[
P = \int_{-\infty}^{\infty} p |p\rangle \langle p| dp \quad Q = \int_{-\infty}^{\infty} q |q\rangle \langle q| dq
\]

and Eqs. (12,13) now will be valid for any real numbers \( \{p, q, p', q'\} \). It must be observed that, in the way they are obtained, the labels \( \{p, q\} \) span the set of all rational numbers, which is a proper subset of the set of real numbers. On the other hand, every real number can be written as the limit of an infinite sequence of rational numbers. Then the expression

\[
\exp \left[ \frac{iq' P}{p_0 q_0} \right] | q\rangle = | q - q'\rangle.
\]

might converges to any real eigenvalue and its associated eigenvector. This is enough to ensure that the hole usual Hilbert space of usual canonical variables is recovered. Also, after the limit is performed the cyclic condition becomes irrelevant, and the familiar relations are easily recovered from their discrete counterparts

\[
Q | q\rangle = q | q\rangle, \quad \langle q' | q\rangle = \delta (q' - q), \quad -\infty \leq q', q \leq \infty
\]

\[
P | p\rangle = p | p\rangle, \quad \langle p' | p\rangle = \delta (p' - p), \quad \langle p | q\rangle = \frac{1}{\sqrt{2\pi p_0 q_0}} \exp \left( \frac{ipq}{p_0 q_0} \right).
\]

Therefore the results for a degree of freedom endowed with a usual position-momentum canonical pair of variables are completely reproduced, provided that the product of the parameters \( p_0 q_0 \) is set to \( \hbar \).

The \( \epsilon^{2-\delta} \) and \( \epsilon^\delta \) factors, roughly speaking, control how ‘fast’ (as \( N \) increases) one will not be able to identify the distance between labels of consecutive eigenvalues. The result above is then rather peculiar, as it states that how you perform this limit doesn’t affect the final result. The usual canonical variables would be recovered anyway.

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1 This limit has to be taken carefully, but it works as if the limiting value of the discrete projector \( |p\rangle \langle p| \) is, after the limit, \( |p\rangle \langle p| dp \). Just think on what happens to the resolution of unity to be sure of that.

2 The author would like to thank one of the anonymous referees for drawing attention to this point.
But things can get different if you consider the extreme situation $\delta = 0$ (or $\delta = 2$, which is equivalent). In this case one of the variables is not scaled at all and what follows is almost identical to the Pegg-Barnett scheme (for simplicity, the reference angle is set to zero). One would have

$$V = \exp \left[ i\frac{2M}{m_0} \right] \quad U = \exp \left[ i\frac{\Theta}{\theta_0} \right]$$

(18)

where

$$M = \sum_{j=-\frac{N}{2}}^{\frac{N}{2}} jm_0|v_j\rangle\langle v_j| \quad \Theta = \sum_{j'=-\frac{N}{2}}^{\frac{N}{2}} \epsilon^2 j'\theta_0|u_{j'}\rangle\langle u_{j'}|.$$  

(19)

If desired, the exponential of the angle operator might be used instead of the operator itself, for the well known reasons given in [4]. The pair $\{m_0, \theta_0\}$ may carry different dimensional units. Let (again) both eigenstates sets be relabeled as

$$|v_j\rangle \equiv |m\rangle \quad |u_{j'}\rangle = |\theta\rangle,$$

with $\theta = \theta_0\epsilon^2 j'$ and $m = m_0j$.  

(20)

In the $N \to \infty$ limit one would have

$$M = \sum_{m=-\infty}^{\infty} m|m\rangle\langle m| \quad \Theta = \int_{-\pi}^{\pi} \theta|\theta\rangle\langle \theta|d\theta.$$  

(21)

$$\Theta \mid \theta \rangle = \theta \mid \theta \rangle, \quad \langle \theta' \mid \theta \rangle = \delta \left( \theta' - \theta \right), \quad -\pi \leq \theta', \theta \leq \pi$$

(22)

$$M \mid m \rangle = m \mid m \rangle, \quad \langle m' \mid m \rangle = \delta_{m',m}, \quad -\infty \leq m' , m \leq \infty$$

(23)

$$\langle \theta \mid m \rangle = \frac{1}{\sqrt{2\pi m_0\theta_0}} \exp \left( \frac{i\theta m}{m_0\theta_0} \right).$$

(24)

The cyclic notation becomes meaningless to the $| m \rangle$ states in the $N \to \infty$ limit, as this label gets unbounded. In the $| \theta \rangle$ states, however, it takes naturally into account the boundary conditions one good set of angle states must have, i.e.,

$$| \theta \rangle \equiv | \theta \left( \text{mod} \ 2\pi \right) \rangle,$$

(25)

and the action of the angle shift operator naturally obeys the boundary condition. But it has to be stressed that (as in the Pegg-Barnett scheme), the range of the variable $\theta$ is confined to $[0, 2\pi)$ by definition, and cyclicity modulo $2\pi$ is only matter of notation. Therefore, and maybe surprisingly, the usual results for angle-angular momentum variables are recovered from the same discrete root from which the position-momentum results also emerged. Again, the product $m_0\theta_0$ must be set to $\hbar$. $\theta_0$ is not expected to be a dimensional unit but must be related to how one is measuring the angle.
III. CONCLUSIONS

The basic result here was to show that the two kinds of canonical variables defined on degrees of freedom with classical counterpart can be obtained from a description of a degree of freedom without classical counterpart. In a pragmatic sense, one could say that the Pegg-Barnett formalism for the angle/angular momentum case was seen as an extension of the Schwinger approach to quantum Cartesian variables. In addition, the discussion which led to those results have interesting aspects of its own.

One of those aspects is the role of the scaling factors in the limiting process. In the first part of the discussion, where the parameter \( \delta \) is free to vary in the open interval \((0, 2)\), the initial discrete variables are changed to a position/momentum like description, still discrete and with contour conditions holding prior to effectively considering the limit. The parameter \( \delta \) controls the distance between successive eigenvalues of the Hermitian operators \( P \) and \( Q \), and the greater the one, the smaller the other, in such a way that their product is fixed. The infinite and continuum limit of these variables is the position-linear momentum pair. Schwinger had already stated that this would happen for \( \delta = 1 \), and what is surprising is that it happens to any value of \( \delta \) in the open interval \((0, 2)\).

In the second part, we consider \( \delta \) in one extreme of the interval previously considered, \((\delta = 0)\). Variables are now changed to an angle/angular momentum like description. The limit to continuum in this case only affects one of the variables (in the discrete/continuous sense) and the angle/angular momentum operators and eigenstates are promptly recovered, basically reproducing the PB scheme. The first interesting thing is that, in this sense, angle/angular momentum variables are a limiting case of Cartesian variables and not the opposite. One also sees that, for a finite number of states, there is no fundamental distinction between angular or Cartesian coordinates, or better, between the variables that will be identified with angular or Cartesian coordinates after the limit is taken, as representations (19) and (11) (prior to the \( N \to \infty \) limit) can always be connected by a simple transformation. The possibility of this transformation is only lost after the limiting process.

As a parallel remark, there is nothing on the simple steps that led from discrete to continuous variables that constrains the product of \( p_0q_0 \) to \( \hbar \). In fact, there’s no (technical) reason for this product to have the same value in both situations. We know from experience that this happens, but it could be the case that \( \hbar \) had a dependence on the number of states allowed to the system (but fortunately it seems that is not).

In the sense above, one could say that it is not the geometry of a given system that impose different quantum variables (in a quantization procedure over a infinite line or over a ring), but, rather, that are the different limiting cases of genuine discrete quantum descriptions that suit different geometries. The author cannot refrain himself from remarking that even the physical validity of this limit might be still put into discussion [3].

After this work was finished the author became aware of reference [6], which discusses in great detail a similar limit to continuum, from a mathematical point of view.

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REFERENCES

[1] D.T. Pegg and S. M. Barnett, Europhys. Lett. 6, 483 (1988).
[2] J. Schwinger, Quantum Kinematics and Dynamics, (Benjamim, New York, 1970), Chaps 1,2,3.
[3] A. Bohm, The Rigged Hilbert Space and Quantum Mechanics, (Springer Lecture notes in Physics, Vol. 78, 1978).
[4] P. Carruthers and M. M. Nieto, Rev. of Mod. Phys. 40, 411 (1968).
[5] B. Leaf, Found. of Physics 12, 583 (1982).
[6] L. Barker, Journal of Functional Analysis 186, 153 (2001).