Frequency-weighted $H_2$-optimal Model Order Reduction via Projection

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Summary
In projection-based model order reduction, a reduced-order approximation of the original full-order system is obtained by projecting it onto a reduced subspace that contains its dominant characteristics. The problem of frequency-weighted $H_2$-optimal model order reduction is to construct a local optimum of the squared $H_2$-norm of the weighted error transfer function. In this paper, a projection-based model order reduction algorithm is proposed that constructs reduced-order models that nearly satisfy the first-order optimality conditions for the frequency-weighted $H_2$-optimal model order reduction problem. It is shown that as the order of the reduced model is increased, the deviation in the satisfaction of the optimality conditions decays. Numerical methods to improve the computational efficiency of the proposed algorithm are also discussed. Three numerical examples are presented to demonstrate the efficacy of the proposed algorithm.

KEYWORDS:
$H_2$-optimal, frequency-weighted, model order reduction, nearly optimal, projection, suboptimal

1 | INTRODUCTION
The complexity of the modern-day dynamic systems has been growing rapidly with each passing day. The direct simulation of the high-order mathematical models that describe large-scale dynamic systems requires a huge amount of computational resources, which are limited due to the high economic cost of the memory resources. To address this issue, model order reduction (MOR) algorithms are used to obtain reduced-order models (ROM) that are computationally cheaper to simulate and analyze, but they closely mimic the original high-order models. The ROM can then be used as a surrogate in the design and analysis with tolerable approximation error [3][8][9].

Projection-based MOR is a large family of algorithms wherein the original high-order model is projected onto a reduced subspace that contains its dominant characteristics. The sense of dominance determines the specific type of MOR procedure. Most of the projection-based MOR methods require solutions of some Lyapunov or Sylvester equations to construct the ROM [3]. During the last two decades, several computationally efficient low-rank methods for the solution of Lyapunov and Sylvester equations have been proposed like [1][21][26][31]. By using these methods for solving large-scale linear matrix equations, a ROM of the original large-scale model can be constructed within the admissible time for most of the projection-based MOR algorithm.

Balanced truncation (BT) [28] is among the most important projection-based MOR methods, which is known for its stability preservation, high fidelity, and apriori error bound expression [17]. In BT, the original model is projected onto the dominant eigenspace of the product of controllability and observability gramians. The computation of controllability and observability gramians requires the solution of two high-order Lyapunov equations, which is a computationally expensive task in large-scale settings. However, their low-rank solutions can be obtained cheaply, which extends the applicability of BT
to large-scale systems [43]. In some situations like reduced-order controller design, it is required that the MOR procedure ensures less weighted approximation error. This necessitates the inclusion of frequency-weights in the MOR algorithm. In [17], BT is generalized to incorporate frequency weights in the approximation criterion, which leads to frequency-weighted BT (FWBT). The original model in FWBT is projected onto the dominant eigenspace of the product of frequency-weighed controllability and frequency-weighed observability gramians. Again, the applicability of FWBT can be extended to large-scale systems by using low-rank solutions of linear matrix equations [1]. Several modifications and extensions to FWBT are reported in the literature to ensure additional properties like stability [41] and passivity [46]; see [19, 29] for a detailed survey.

The Krylov subspace-based methods are another important class of projection-based MOR methods. The full-order system in these methods is projected onto the subspace of generalized controllability and observability matrices, which leads to moment matching, i.e., the ROM matches some moments of the original transfer function at some selected frequency points [4]. Among the gold standards of these methods is the iterative rational Krylov algorithm (IRKA) [20, 40], which constructs a local optimum for the $H_2$-optimal MOR problem, i.e., the ROM is a local optimum of the squared $H_2$-norm of the error transfer function. Unlike BT, IRKA [20, 40] does not require the solutions of large-scale Lyapunov equations. Thus it is computationally efficient and can handle large-scale systems. IRKA is heuristically generalized to the frequency-weighted scenario in [21] and [45]. The algorithms proposed in [2] and [45] ensure less $H_2$-norm of the weighted error transfer function; however, they do not seek to construct local optimum for the frequency-weighted $H_2$-optimal MOR problem.

In [22], the first-order optimality conditions for the single-sided case of frequency-weighted $H_2$-optimal MOR are derived, and an algorithm based on Lyapunov and Riccati equations is proposed to satisfy these conditions. This algorithm is numerically trackable only for small-scale systems. The optimality conditions derived in [22] are shown equivalent to the tangential interpolation conditions in [10], and a Krylov subspace-based iterative algorithm is proposed, which nearly satisfies these conditions. In [47], an iteration-free Krylov subspace-based algorithm is presented, which exactly satisfies a subset of the optimality conditions while guaranteeing the stability of the ROM at the same time.

The first-order optimality conditions for the double-sided case of the frequency-weighted $H_2$-optimal MOR are derived in [16, 33, 44]. The algorithms presented to generate the local optimum in [16, 23, 27, 33, 37, 44] are not feasible for large-scale systems due to high computational cost associated with nonlinear optimization. In [47], a Krylov subspace-based algorithm, i.e., frequency-weighted iterative tangential interpolation algorithm (FWITIA) [47], is proposed that ensures less frequency-weighted $H_2$-norm of the error transfer function in the double-side case. However, the connection with the optimality conditions derived in [33] is not investigated, which has motivated the work in this paper.

In this paper, we consider the double-sided case of the frequency-weighted $H_2$-optimal MOR problem within the projection framework. The main motivation for seeking a projection-based solution is to avoid nonlinear optimization and benefit from the efficient sparse-dense Sylvester equation solver, particularly formulated for the projection-based $H_2$-MOR algorithms [5]. We show that the exact satisfaction of the optimality conditions is inherently not possible within the projection framework. However, the optimality conditions can be nearly satisfied, and the deviation in the satisfaction of the optimality conditions are also discussed. Then, a projection-based iterative algorithm is proposed that solves sparse-dense Sylvester equations in each iteration to construct the ROM. Upon convergence, the ROM nearly satisfies first-order optimality conditions for the double-sided case of the frequency-weighted $H_2$-optimal MOR problem. Moreover, it is shown that FWITIA also seeks to satisfy the optimality conditions derived in [33]. The efficacy of the proposed algorithm is highlighted by considering one illustrative and two benchmark numerical examples.

2 | PRELIMINARIES

Let us denote the $n^{\text{th}}$-order stable linear time-invariant system with $m$ inputs and $p$ outputs as $H(s)$, which has the following equivalence with its state-space realization

$$
H(s) = C(sI - A)^{-1} B + D
$$

(1)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$. The important mathematical notations used throughout the text are given in Table 1.

2.1 | Problem Setting

In a large-scale setting, the order $n$ of (1) is high, the matrices $(A, B, C)$ are sparse, $m \ll n$, and $p \ll n$. Let us denote the $r^{\text{th}}$-order approximation of $H(s)$ as $\tilde{H}(s)$, which has the following equivalence with its state-space realization

$$
\tilde{H}(s) = \tilde{C}(sI - \tilde{A})^{-1} \tilde{B} + D
$$

where $\tilde{A} \in \mathbb{R}^{r \times r}$, $\tilde{B} \in \mathbb{R}^{r \times m}$, $\tilde{C} \in \mathbb{R}^{p \times r}$, and $r \ll n$. In projection-based MOR, the state-space matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$
are obtained as the following
\[ \begin{align*}
\hat{A} &= \hat{W}^T A \hat{V}, \\
\hat{B} &= \hat{W}^T B, \\
\hat{C} &= C \hat{V}
\end{align*} \] (2)
where \( \hat{V} \in \mathbb{R}^{nxr}, \hat{W} \in \mathbb{R}^{nxr} \) and \( \hat{W}^T \hat{V} = I \) wherein the columns of \( \hat{V} \) span \( r \)-dimensional subspace along the kernel of \( \hat{W}^T \), and \( \Pi = \hat{V} \hat{W}^T \in \mathbb{R}^{nxn} \) is an oblique projection onto that subspace.

Let us denote the error transfer function as \( E(s) \), which has the following equivalence with its state-space realization
\[ E(s) = H(s) - \hat{H}(s) = C(sI - A_r)^{-1} B_e \]
where
\[ A_e = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C_e = \begin{bmatrix} C & -\hat{C} \end{bmatrix}. \]

Let us denote the input and output weights as \( W_i(s) \) and \( W_o(s) \), respectively, which have the following equivalence with their state-space realizations
\[ W_i(s) = C(sI - A_i)^{-1} B_i + D_i, \]
\[ W_o(s) = C(sI - A_o)^{-1} B_o + D_o \]
where \( A_i \in \mathbb{R}^{nxn}, B_i \in \mathbb{R}^{nxr}, C_i \in \mathbb{R}^{mxn}, D_i \in \mathbb{R}^{nxp}, A_o \in \mathbb{R}^{nxn}, B_o \in \mathbb{R}^{nxr}, C_o \in \mathbb{R}^{mxn}, \) and \( D_o \in \mathbb{R}^{nxp} \). Also, assume that \( W_i(s) \) and \( W_o(s) \) are stable.

Further, let us denote the weighted error transfer function as \( E_w(s) \), which has the following equivalence with its state-space realization
\[ E_w(s) = W_o(s) E(s) W_i(s) = C_o(sI - A_w)^{-1} B_w, \]
where
\[ A_w = \begin{bmatrix} A & 0 & BC_i & 0 \\ 0 & \hat{A} & \hat{B}C_i & 0 \\ 0 & 0 & A_j & 0 \\ B_j C - B_o \hat{C} & 0 & A_o \end{bmatrix}, \quad B_w = \begin{bmatrix} B \hat{D}_i \\ \hat{B}_{D_i} \\ B_i \\ B_o \hat{C} \\ 0 \end{bmatrix}, \quad C_w = \begin{bmatrix} D_j C - D_o \hat{C} & 0 & C_o \end{bmatrix}. \]

Let us denote the controllability and observability gramians of the realization \( (A_w, B_w, C_w) \) as \( P_w \) and \( Q_w \), respectively, which solve the following Lyapunov equations
\[ \begin{align*}
A_w P_w + P_w A_w^T + B_w B_w^T &= 0, \\
A_w^T Q_w + Q_w A_w + C_w^T C_w &= 0.
\end{align*} \] (4, 5)

\( P_w \) and \( Q_w \) can be partitioned according to the structure of the realization in (2) as the following
\[ P_w = \begin{bmatrix} P & P_{12} & P_{13} & P_{14} \\ P_{12}^T & \hat{P} & \hat{P}_{23} & \hat{P}_{24} \\ P_{13}^T & \hat{P}_{23} & \hat{P} & \hat{P}_{34} \\ P_{14}^T & \hat{P}_{24} & \hat{P}_{34} & P_o \end{bmatrix}, \quad Q_w = \begin{bmatrix} Q & Q_{12} & Q_{13} & Q_{14} \\ Q_{12}^T & \hat{Q} & \hat{Q}_{23} & \hat{Q}_{24} \\ Q_{13}^T & \hat{Q}_{23} & \hat{Q} & \hat{Q}_{34} \\ Q_{14}^T & \hat{Q}_{24} & \hat{Q}_{34} & \hat{Q}_{44} \end{bmatrix}. \]

In the frequency-weighted \( H_2 \)-MOR, a ROM \( \hat{H}(s) \) is sought, which ensures that the squared \( H_2 \)-norm of \( E_w(s) \) is small, i.e.,
\[ \min_{H(s)} \| E_w(s) \|^2_{H_2}. \]

The \( H_2 \)-norm of \( E_w(s) \) is the energy of the its impulse response and is related to its controllability and observability gramians as the following
\[ \| E_w(s) \|^2_{H_2} = \text{tr}(C_w P_w C_w^T ) = \text{tr}(D_o C P C^T D_o^T ) + \text{tr}(D_o C \hat{P} \hat{C}^T D_o^T - 2 D_o C P_{12} \hat{C}^T D_o^T + C_o P_o C_o^T ) + 2 D_o C P_{23} \hat{C}^T D_o^T - 2 D_o C \hat{P} \hat{C}^T D_o^T ) \]
\[ = \text{tr}(\hat{B}_w^T Q_w B_w) + \text{tr}(D_o^T B_d^T Q B D_i + D_i^T B_d^T \hat{Q} B D_i + D_i^T B_w^T Q_w B_w + D_i^T B_w^T Q O_1 B D_i + D_i^T B_o^T Q O_2 B D_i + 2 D_i^T B_w^T Q_1 B_i + 2 D_i^T B_d^T Q_2 B_i + D_i^T B_o^T Q O_3 B D_i + D_i^T B_o^T Q O_4 B D_i). \]

\( \hat{H}(s) \) is a local optimum of \( \| E_w(s) \|^2_{H_2} \) if it satisfies the following first-order optimality conditions
\[ \frac{\partial}{\partial \hat{A}} \| E_w(s) \|^2_{H_2} = 0 \Rightarrow \hat{X} + X = 0, \] (6)
\[ \frac{\partial}{\partial \hat{B}_d} \| E_w(s) \|^2_{H_2} = 0 \Rightarrow \hat{Y} D_d D_d^T + Y = 0, \] (7)
\[ \frac{\partial}{\partial C_o} \| E_w(s) \|^2_{H_2} = 0 \Rightarrow D_o^T D_o Z + Z = 0 \] (8)

where
\[ \hat{X} = Q_{12}^T P_{12} + \hat{Q} \hat{P}, \quad \hat{Y} = Q_{12}^T B + \hat{Q} \hat{B}, \]
\[ Z = C P_{12} - C P, \quad X = Q_{23}^T D_{23} + Q_{24}^T P_{24} \]
\[ Y = (Q_{12}^T P_{13} + \hat{Q} P_{23} + Q_{23}^T P_{13} + Q_{24}^T P_{23}^T) C_i^T + Q_{23} B_d D_i^T, \]
\[ Z = -B_o^T (Q_{14}^T P_{12} + \hat{Q} \hat{P} + Q_{14}^T P_{24}^T + Q_{24}^T P_{24}^T) D_o^T C_a^T P_{24} \]

2.2 || FWITIA

FWITIA [47] constructs a ROM that tends to satisfy the following conditions
\[ \frac{\partial}{\partial \hat{B}} \| W_o(s) E(s) \|^2_{H_2} = 2 (Q_{12}^T B + \hat{Q} \hat{B}) = 0, \] (9)
\[ \frac{\partial}{\partial C_o} \| E(s) W_i(s) \|^2_{H_2} = 2 (C P_{12} - \hat{C} \hat{P}) = 0. \] (10)
Let $\bar{H}(s)$ has simple poles and the following pole-residue form

$$\bar{H}(s) = \sum_{i=1}^{m} \frac{\tilde{l}_i e^{\tilde{\rho}_i s}}{s - \tilde{\lambda}_i} + D.$$  

Now define $F[H(s)] = C_f(s I - A_f)^{-1} B_f$ and $G[H(s)] = C_s(s I - A_s)^{-1} B_s$ wherein

$$A_f = \begin{bmatrix} A & BC \tilde{r} \\ 0 & A \end{bmatrix}, \quad A_s = \begin{bmatrix} A & 0 \\ B_0 C & A_s \end{bmatrix},$$

$$B_f = \begin{bmatrix} P_{13} C^T + B D_i D^T \tilde{l}_i \\ P_{12} C_i + B_i D_i^T \end{bmatrix}, \quad B_s = \begin{bmatrix} B \\ B_0 D \end{bmatrix},$$

$$C_f = \begin{bmatrix} C^T_i D_i^T \\ C^T \end{bmatrix}^T, \quad C_s = \begin{bmatrix} \bar{Q}_{13} B_0 + C^T D_o^T D_o^T \tilde{l}_i \\ \bar{Q}_{12} B_0 + C^T D_0^T D_0 \end{bmatrix}^T.$$  

(11)

(12)

(13)

The conditions (9) and (10) can be satisfied if the following tangential interpolation conditions are satisfied

$$F[H(-\tilde{\lambda}_i)] \tilde{r}_i = \tilde{F} [\bar{H}(-\tilde{\lambda}_i)] \tilde{r}_i,$$

$$\tilde{l}_i^T G[H(-\tilde{\lambda}_i)] = \tilde{l}_i^T G[\bar{H}(-\tilde{\lambda}_i)].$$

(14)

(15)

The poles $\tilde{\lambda}_i$ and residues $\tilde{r}_i$ of $\bar{H}(s)$ are not known a priori. Thus the interpolation points and tangential directions are initialized arbitrarily, and after every iteration, the interpolation points $\lambda_i$ are updated as $-\tilde{\lambda}_i$, and the tangential directions $(c_i, b_i)$ are updated as the residues $(\tilde{l}_i, \tilde{r}_i)$. The rational Krylov subspaces to satisfy the tangential interpolation conditions (14) and (15) are computed as

$$\text{Ran} \begin{bmatrix} V_a \\ W_b \end{bmatrix} = \text{span} \{ (\sigma_i I - A_f)^{-1} B_f b_i \},$$

$$\text{Ran} \begin{bmatrix} W_a \\ W_b \end{bmatrix} = \text{span} \{ (\sigma_i I - A_s^T)^{-1} C_s^T c_i \}.$$  

(16)

(17)

$\bar{V}$ and $\bar{W}$ are set as $\text{Ran}(\bar{V}) \supset \text{Ran}(V_a)$ and $\text{Ran}(\bar{W}) \supset \text{Ran}(W_b)$. $\bar{V} = \text{orth}(\bar{V})$, $\bar{W} = \text{orth}(\bar{W})$, $\bar{W} = \bar{W}(\bar{V}^T \bar{W})^{-1}$. The algorithm is stopped when the relative change in $\lambda$ stagnates, and a ROM $\hat{H}(s)$ that nearly satisfies the interpolation conditions (14) and (15) is achieved.

### 3.1 Limitation of Projection Framework

Let us assume at the moment for simplicity that $D_i D_i^T, D_0 D_o^T, \tilde{P}$, and $\bar{Q}$ are invertible. Then it can be noted from the optimality conditions (7) and (8) that the optimal choices of $\tilde{B}$ and $\bar{C}$ should satisfy the following

$$\tilde{B} = -\bar{Q}^{-1} Q_{12} B - \bar{Q}^{-1} Y (D_i D_i^T)^{-1},$$

$$\bar{C} = C P_{14} \tilde{P}^{-1} + (D_o^T D_o)^{-1} Z \tilde{P}^{-1}.$$  

$\tilde{B}$ and $\bar{C}$ are computed as $\bar{V}^T B$ and $C \bar{V}$, respectively, in the projection framework. This suggests that $\bar{V}$ and $\bar{W}$ should be selected as $\bar{V} = P_{12} \tilde{P}^{-1}$ and $\bar{W} = -Q_{12} \bar{Q}^{-1} (\text{note that } \tilde{P}$ and $\bar{Q}$ are symmetric). Moreover, this selection ensures that $\bar{Y} = 0$ and $\bar{Z} = 0$ without assuming that $D_i D_i^T$ and $D_0 D_o^T$ are invertible. Since $P_{12}, Q_{12}, \tilde{P},$ and $\bar{Q}$ depend on the unknown $(\tilde{A}, \tilde{B}, \tilde{C})$, the problem is nonconvex. Nevertheless, if such a solution is found within the projection framework, $\bar{X} = 0$ due to the oblique projection condition $\bar{W}^T \bar{V} = I$. The deviations in the satisfaction of optimality conditions (6) and (8) are then quantified by $X, Y$, and $Z$. If $X = 0, Y = 0,$ and $Z = 0$, the problem can be solved within the projection framework by finding the reduction subspaces that ensure $V = P_{12} \tilde{P}^{-1}$, $W = -Q_{12} \bar{Q}^{-1}$, and $W^T V = I$. The reduction subspaces $\bar{V}$ and $\bar{W}$ have no influence on $P_{13}, P_i, Q_{14}$, and $Q_o$. There seems no straightforward way to influence $P_{23}, P_{24}, P_{34}, Q_{23}, Q_{24}$, and $Q_{34}$ using $\bar{V}$ and $\bar{W}$ so that $X, Y,$ and $Z$ become zeros. Further, it is shown in [24][38][39] that the nonzero cross-terms $P_{13}, P_{23}, Q_{14},$ and $Q_{24}$ are inherent to the frequency-weighted MOR problem, and the effect of frequency-weights vanishes when these are zeros. Therefore, at best, we can ensure $X = 0, Y = 0,$ and $Z = 0$ within the projection framework.

FRTTIA is not motivated by the optimality conditions (6)-(8). Instead, it follows the system theory perspective that ensuring small $\|E(s)W(s)\|_{H_2}$ and $\|W_o(s)E(s)\|_{H_2}$ generally ensures that $\|E_w(s)\|_{H_2}$ is also small. Note that $H_2$-norm, unlike $H_{\infty}$-norm, does not enjoy submultiplicative property, and hence,

$$\|E_w(s)\|_{H_2} \leq \|E(s)W(s)\|_{H_2} + \|W_o(s)E(s)\|_{H_2}$$

does not hold in general. However, it can be shown that $\bar{Y} = 0$ and $\bar{Z} = 0$ is equivalent to ensuring that the gradients of the additive components of $\|E_w(s)\|_{H_2}$ with respect to $\bar{B}$ and $\bar{C}$ become zero. This is established in Lemma 1.

**Lemma 1.** Let us split $\|E_w(s)\|_{H_2}^2$ into its additive components as $\|E_w(s)\|_{H_2}^2 = J_1 + J_2 = J_3 + J_4$ where

$$J_1 = \text{tr}(D_o^T C P C^T D_i^T + D_i C \tilde{C} P C^T D_i^T - 2D_o C P_{12} C^T D_i^T),$$

$$J_2 = \text{tr}(C_o P^T C + 2D_o C P_{14} C^T - 2D_o \tilde{C} P_{24} C^T),$$

$$J_3 = \text{tr}(D_i^T B_i^T Q B D_i + D_i^T B_i^T \tilde{Q} B D_i + 2D_i^T B_i^T Q_{12} \tilde{B} D_i),$$

$$J_4 = \text{tr}(B_i^T Q B_i + 2D_i^T B_i^T Q_{13} B_i + D_i^T B_i^T \tilde{Q} B Q_{23} B_i).$$
Then \( \frac{\partial}{\partial \Delta_1} J_1 = 0 \) and \( \frac{\partial}{\partial \Delta_2} J_3 = 0 \) when \( \bar{Z} = 0 \) and \( \bar{Y} = 0 \), respectively.

**Proof.** Let us denote the first-order derivative of \( J_1 \) with respect to \( \hat{B} \) as \( \Delta \hat{B}_{J_1} \) and the differential of \( \hat{B} \) as \( \Delta \hat{B} \). Then

\[
\Delta \hat{B}_{J_1} = tr(2D_I^T \Delta_{\hat{B}} \hat{B} D_I + 2D_I^T B^T Q_{12} \Delta \hat{B} D_i)
\]

\[
= tr\left( \left( 2D_I^T D_I^T \hat{B} \hat{Q} + 2D_I^T B^T Q_{12} \right) \Delta \hat{B} \right).
\]

Since \( \Delta \hat{B}_{J_1} = tr\left( \left( \frac{\partial}{\partial \Delta_1} J_1 \right)^T \Delta \hat{B} \right) \), \( \frac{\partial}{\partial \Delta_1} J_3 = 2 \bar{Y} D_I D_I^T \). Thus when \( \bar{Y} = 0, \frac{\partial}{\partial \Delta_1} J_3 = 0 \).

Now, let us denote the first-order derivative of \( J_1 \) with respect to \( \hat{C} \) as \( \Delta \hat{C}_{J_1} \) and the differential of \( \hat{C} \) as \( \Delta \hat{C} \). Then

\[
\Delta \hat{C}_{J_1} = tr(2D_o \Delta \hat{C} \hat{P} \hat{C}^T D_o^T - 2D_o \sum P_{12} \Delta \hat{C} D_o^T)
\]

\[
= tr\left( \left( 2 \hat{P} \hat{C}^T D_o^T - 2 \sum P_{12} \hat{C} D_o^T \right) \Delta \hat{C} \right).
\]

Since \( \Delta \hat{C}_{J_1} = tr\left( \left( \frac{\partial}{\partial \Delta_1} J_1 \right)^T \Delta \hat{C} \right) \), \( \frac{\partial}{\partial \Delta_1} J_1 = 2 D_I^T D_o \bar{Z} \). Thus when \( \bar{Z} = 0, \frac{\partial}{\partial \Delta_1} J_1 = 0 \). This completes the proof.

Although this was not recognized in [47], it is evident now that FWITIA is not completely heuristic in terms of seeking that \( ||E_n(s)||_2^2 \) is small. Note that FWITIA seeks to ensure that \( \bar{Y} = 0 \) and \( \bar{Z} = 0 \) by satisfying the tangential interpolation conditions (14) and (15). However, the interpolation conditions (14) and (15) require \( F[H(s)] \) and \( G[H(s)] \) to maintain the structure of \( F[H(s)] \) and \( G[H(s)] \) given in (11)-(13), which is not possible in general. Therefore, FWITIA may not satisfy the interpolation conditions (14) and (15) exactly, and thus \( \bar{Y} \approx 0 \) and \( \bar{Z} \approx 0 \) upon convergence.

An interesting parallel should be noted that the nonzero cross-terms \( P_{13}, P_{23}, Q_{14}, \) and \( Q_{24} \) inhibit FWBT to inherit the stability preservation and Hankel singular values retention properties of BT [18, 39]. Here also, the nonzero cross-terms inhibit the projection framework to satisfy the first-order optimality conditions exactly, which is possible in the standard \( H_2 \)-optimal MOR problem case.

### 3.2 Frequency-weighted \( H_2 \)-optimal MOR

As seen already that to ensure \( \bar{X} = 0, \bar{Y} = 0, \) and \( \bar{Z} = 0 \), we need to find the reduction subspaces \( \bar{V} = P_{12} \bar{P}^{-1} \) and \( \bar{W} = -Q_{12} \bar{Q}^{-1} \) that ensure the oblique projection condition \( \bar{W}^T \bar{V} = I \). By expanding the Lyapunov equations (4) and (5) according to the structure of \( (A_{\mu}, B_{\mu}, C_{\mu}) \) in (5), it can be noted that \( P_{23}, \bar{P}, P_{12}, \bar{Q}, Q_{12} \) and \( Q_{12} \) solve the following Sylvester equations

\[
\bar{A}P_{23} + P_{23}A^T + \bar{B}_C P_{12} + D_I B_I^T = 0,
\]

\[
\bar{A} \bar{P} + P_{12} A^T + \bar{B}_C P_{23}^T + P_{23} C_I B_I^T + \bar{B}_D D_I^T \bar{B} = 0.
\]

If \( \bar{V} \) is obtained using the oblique projection \( \bar{V} = P_{12} \bar{P}^{-1} \bar{Q}^{-1} \) with \( \bar{P} = P_{12} \bar{P}^{-1} \) and \( \bar{W} = -Q_{12} \bar{Q}^{-1} \), the equation (2) and the equations (10)-(21) can be viewed as two coupled system of equations, i.e.,

\[
(\bar{A}, \bar{B}, \bar{C}) = \bar{f}(P_{12}, Q_{12}, \bar{P}, \bar{Q}),
\]

\[
(P_{12}, Q_{12}, \bar{P}, \bar{Q}) = g(\bar{A}, \bar{B}, \bar{C}).
\]

Clearly, the fixed points of \( (\bar{A}, \bar{B}, \bar{C}) = \bar{f}(g(\bar{A}, \bar{B}, \bar{C})) \) ensure that \( \bar{X} = 0, \bar{Y} = 0, \) and \( Z = 0 \) if the condition for oblique projection \( \bar{W}^T \bar{V} = I \) is also satisfied. We now show in the next theorem that these fixed points satisfy the optimality conditions (6)-(8) if \( CP_{13} - \bar{C} P_{23} = 0 \) and \( B^T Q_{14} + B^T Q_{24} = 0 \).

**Theorem 1.** Let \( \bar{A} \) be Hurwitz and \( (\bar{A}, \bar{B}, \bar{C}) \) be a fixed point of \( (\bar{A}, \bar{B}, \bar{C}) = \bar{f}(g(\bar{A}, \bar{B}, \bar{C})) \). Also, let that \( \bar{P} \) and \( \bar{Q} \) are invertible at the fixed point, and the fixed point is obtained by using oblique projection \( \bar{P}_{12} \bar{P}^{-1} \bar{Q}^{-1} \) with \( \bar{P} = P_{12} \bar{P}^{-1} \) and \( \bar{W} = -Q_{12} \bar{Q}^{-1} \). Then \( (\bar{A}, \bar{B}, \bar{C}) \) satisfies the first-order optimality conditions (6)-(8) provided \( CP_{13} - \bar{C} P_{23} = 0 \) and \( B^T Q_{14} + B^T Q_{24} = 0 \).

**Proof.** We need to show that when \( CP_{13} - \bar{C} P_{23} = 0 \) and \( B^T Q_{14} + B^T Q_{24} = 0 \), \( X = 0, Y = 0, \) and \( Z = 0 \) at the fixed points of \( (\bar{A}, \bar{B}, \bar{C}) = f(g(\bar{A}, \bar{B}, \bar{C})) \). By expanding the Lyapunov equation (5), one can note that \( P_{34} \) and \( P_{24} \) satisfy the following Sylvester equations

\[
A_P P_{34} + P_{34} A_P^T + (P_{13}^T C_T^{-1} - P_{23}^T C_T^T) B_o^T = 0,
\]

\[
A_P P_{24} + P_{24} A_P^T + \bar{B}_C P_{34} + (P_{12}^T C_T - \bar{P} C_T) B_o^T = 0.
\]

Since \( \bar{Z} = 0 \) and \( CP_{13} - \bar{C} P_{23} = 0 \), we get

\[
P_{34} A_P + P_{34} A_P^T = 0 \quad \text{and} \quad A_P P_{24} + P_{24} A_P^T + \bar{B}_C P_{34} = 0.
\]

Thus \( P_{34} = 0 \) and \( P_{24} = 0 \).

It can be noted by expanding the Lyapunov equation (7) that \( Q_{24} P_{23} \) solve the following Sylvester equations

\[
A_T Q_{24} + Q_{24} A_T = C_T (B_o^T Q_{14} + B^T Q_{24}) = 0,
\]

\[
A_T Q_{23} + Q_{23} A_T = -C_T (B_o^T Q_{14} + (\bar{Q} B + Q_{12} B) C_I) = 0.
\]

Now, since \( \bar{Y} = 0 \) and \( B^T Q_{14} + B^T Q_{24} = 0 \), we get

\[
A_T P_{24} + Q_{24} A_T = 0 \quad \text{and} \quad A_T Q_{23} + Q_{23} A_T = -C_T (B_o^T Q_{14} + B^T Q_{24}) = 0.
\]

Thus \( Q_{24} = 0, Q_{23} = 0, \) and resultantly \( X = 0 \).
Further, since $\hat{W}^T \hat{V} = I$, $\hat{Q} = -Q_{12}^T \hat{V}$ and $\hat{P} = \hat{W}^T P_{12}$. Then $Y$ and $Z$ become
\[ Y = Q_{12}^T (P_{13} - \hat{V} P_{23}) C_1^T = 0 \]
\[ Z = -B_{12}^T (Q_{14}^T + \hat{Q}_{24}^T \hat{W}^T) P_{12} = 0. \]
Since $P_{13} - \hat{C} P_{23}$ is $B^T Q_{14} + \hat{B}^T Q_{24}$ is $0$, $P_{13} - \hat{V} P_{23} = 0$ and $Q_{14}^T + Q_{24}^T \hat{W}^T = 0$. Thus $Y = 0$ and $Z = 0$. This completes the proof.

Remark 1. When $W_1(s) = I$, $Y = 0$, and the fixed point of $(\hat{A}, \hat{B}, \hat{C}) = f(g(\hat{A}, \hat{B}, \hat{C}))$ satisfies the optimality condition exactly if $\hat{Q}$ is invertible at the fixed point. Similarly, when $W_2(s) = I$, $Z = 0$, and the fixed point of $(\hat{A}, \hat{B}, \hat{C}) = f(g(\hat{A}, \hat{B}, \hat{C}))$ satisfies the optimality condition exactly if $\hat{P}$ is invertible at the fixed point.

Note that $\hat{P}$ and $\hat{Q}$ do not change the subspaces $\hat{V} = P_{12} \hat{P}^{-1}$ and $\hat{W} = -Q_{12} \hat{Q}^{-1}$ but only transform the basis of the subspaces $P_{12}$ and $-Q_{12}$. Thus we can construct $\hat{V}$ and $\hat{W}$ as $\hat{V} = P_{12}$ and $\hat{W} = -Q_{12}$. By doing so, the invertibility of $\hat{P}$ and $\hat{Q}$ is no more required, which otherwise makes the problem quite restrictive. If the ROM is obtained using the oblique projection $\Pi = -P_{12} Q_{12}^T$, with $\hat{V} = P_{12}$ and $\hat{W} = -Q_{12}$, the equation 2 and the equations 16-21 can be viewed as two coupled system of equations, i.e.,
\[ (\hat{A}, \hat{B}, \hat{C}) = f_1(P_{12}, Q_{12}) \text{ and } (P_{12}, Q_{12}) = g_1(\hat{A}, \hat{B}, \hat{C}). \]

In the next theorem, we show that the fixed points of $(\hat{A}, \hat{B}, \hat{C}) = f_1(g_1(\hat{A}, \hat{B}, \hat{C}))$ satisfy the optimality conditions 6-8 if $CP_{13} - \hat{C} P_{23} = 0$ and $B^T Q_{14} + \hat{B}^T Q_{24} = 0$.

Theorem 2. Let $\hat{A}$ be Hurwitz and $(\hat{A}, \hat{B}, \hat{C})$ be a fixed point of $(\hat{A}, \hat{B}, \hat{C}) = f_1(g_1(\hat{A}, \hat{B}, \hat{C}))$ obtained by using oblique projection $\Pi = -P_{12} Q_{12}^T$ with $\hat{V} = P_{12}$ and $\hat{W} = -Q_{12}$. Then $(\hat{A}, \hat{B}, \hat{C})$ satisfies the first-order optimality conditions 6-8 provided $CP_{13} - \hat{C} P_{23} = 0$ and $B^T Q_{14} + \hat{B}^T Q_{24} = 0$.

Proof. Since the fixed points of $(\hat{A}, \hat{B}, \hat{C}) = f_1(g_1(\hat{A}, \hat{B}, \hat{C}))$ are obtained by using oblique projection $\Pi = -P_{12} Q_{12}^T$, the following holds $CP_{13} - \hat{C} P_{23} = 0$, $Q_{14}^T B + \hat{B} = 0$, and $Q_{14}^T P_{12} + I = 0$ at the fixed points. We first show that $\hat{X} = 0$, $\hat{Y} = 0$, and $\hat{Z} = 0$ if the fixed points of $(\hat{A}, \hat{B}, \hat{C}) = f_1(g_1(\hat{A}, \hat{B}, \hat{C}))$ by using the projection $\Pi = \hat{W}^T \hat{V}$, which simultaneously also provides good Petrov-Galerkin approximations of $P_{13}$ and $Q_{14}$ as $\hat{W} P_{13} \approx P_{13}$ and $\hat{W} Q_{14} \approx Q_{14}$.

Remark 2. We can now state that $X = 0$, $Y = 0$, and $Z = 0$. From theorem 1, we know that when $\hat{Y} = 0$, $\hat{Z} = 0$, $P_{13} = \hat{V} P_{23}$, and $Q_{14} = -\hat{W} Q_{24}$, $P_{34} = 0$, $Q_{23} = 0$ and $X = 0$. Further, since $P = I$, and $\hat{Q} = I$, $Y$ and $Z$ become
\[ Y = (Q_{12}^T P_{13} + P_{23}) C_1^T \text{ and } Z = B_{12}^T (Q_{14}^T P_{12} + P_{24}) \]
Since $P_{13} - \hat{V} P_{23} = 0$ and $Q_{14} + \hat{W} Q_{24} = 0$, $Y = 0$ and $Z = 0$. This completes the proof.

By expanding the Lyapunov equations (4) and (5), one can note that $P_{13}$ and $Q_{14}$ solve the following Sylvester equations
\begin{align*}
AP_{13} + P_{13} A^T + B(C P_{12} + D B) &= 0, \quad (22) \\
A^T Q_{14} + Q_{14} A + C^T (B^T Q_{24} + D^T C) &= 0. \quad (23)
\end{align*}
Thus $\hat{V} P_{13}$ and $-\hat{W} Q_{14}$, as given in the following sense
\[ \text{ran}(\hat{A} \hat{V} P_{23} + \hat{V} P_{23} A^T + B(C P_{12} + D B) \hat{V}) \subseteq \text{ran}(\hat{W}) \]
\[ \text{ran}(\hat{A}^T \hat{W} Q_{24} + \hat{W} Q_{24} A + C^T (B^T Q_{24} + D^T C) \hat{W}) \subseteq \text{ran}(\hat{V}) \]
In general, $P_{13} \neq \hat{V} P_{23}$ and $Q_{14} \neq -\hat{W} Q_{24}$, and therefore, $P_{13} - \hat{C} P_{23} \neq 0$ and $B^T Q_{14} + \hat{B}^T Q_{24} \neq 0$. To achieve a nearly optimum ROM, we need to find fixed points of $(\hat{A}, \hat{B}, \hat{C}) = f_1(g_1(\hat{A}, \hat{B}, \hat{C}))$ by using the projection $\Pi = \hat{W}^T \hat{V}$, which simultaneously also provides good Petrov-Galerkin approximations of $P_{13}$ and $Q_{14}$ as $\hat{V} P_{23} \approx P_{23}$ and $\hat{W} Q_{24} \approx Q_{24}$.

Remark 3. When $W_2(s)$ and $W_1(s)$ are co-inner and inner functions, respectively, $P_{13} = 0$, $P_{23} = 0$, $Q_{14} = 0$, and $Q_{24} = 0$. Thus $CP_{13} - \hat{C} P_{23} = 0$ and $B^T Q_{14} + \hat{B}^T Q_{24} = 0$, and the fixed points of $(\hat{A}, \hat{B}, \hat{C}) = f_1(g_1(\hat{A}, \hat{B}, \hat{C}))$ satisfy the optimality conditions 6-8 exactly.
3.2.1 Deviation in the Optimality Conditions

We now show that as the order of the ROM increases, the fixed points of \( (\tilde{A}, \tilde{B}, \tilde{C}) = f_1(g_1(\tilde{A}, \tilde{B}, \tilde{C})) \) implicitly ensures that \( P_{13} \approx \tilde{V} P_{23} \) and \( Q_{14} \approx -\tilde{W} Q_{24} \). Thus the deviation in the satisfaction of the optimality conditions (6)-(8) decays as the order of ROM increases.

Note,

\[
||E(s)W_i(s)||_{H_2}^2 = \text{trace}(CP_CT - 2CP_{12}C^T + \tilde{C}\tilde{P}\tilde{C}^T).
\]

When \( \tilde{Z} = 0 \), \( ||E(s)W_i(s)||_{H_2}^2 \) becomes

\[
||E(s)W_i(s)||_{H_2}^2 = \text{trace}(CP_CT - \tilde{C}\tilde{P}\tilde{C}^T) = \text{trace}(C(P - V\tilde{P}\tilde{V})C^T).
\]

Thus as the order of ROM increases and \( ||E(s)W_i(s)||_{H_2}^2 \) decreases, \( \tilde{P} = \tilde{V}\tilde{P}\tilde{V} \) approaches \( P \). Also,

\[
||W_i(s)E(s)||_{H_2}^2 = \text{trace}(B^TQB + 2B^TQ_{12}\tilde{B} + \tilde{B}^T\tilde{Q}\tilde{B}).
\]

When \( \tilde{Y} = 0 \), \( ||W_i(s)E(s)||_{H_2}^2 \) becomes

\[
||W_i(s)E(s)||_{H_2}^2 = \text{trace}(B^TQB - \tilde{B}^T\tilde{Q}\tilde{B}) = \text{trace}(B^T(Q - \tilde{W}\tilde{Q}\tilde{W})B).
\]

As the order of ROM increases and \( ||W_i(s)E(s)||_{H_2}^2 \) decreases, \( \tilde{Q} = \tilde{W}\tilde{Q}\tilde{W} \) approaches \( Q \).

3.3 Frequency-weighted \( H_2 \)-suboptimal MOR

Motivated by the results of the last subsection, we propose an algorithm, which computes the fixed points of \( (\tilde{A}, \tilde{B}, \tilde{C}) = f_1(g_1(\tilde{A}, \tilde{B}, \tilde{C})) \) to nearly satisfy the optimality conditions (6)-(8). The fixed points of \( (\tilde{A}, \tilde{B}, \tilde{C}) = f_1(g_1(\tilde{A}, \tilde{B}, \tilde{C})) \) can be found by using the fixed point iteration algorithm with an additional constraint that \( P_{13} \) and \( Q_{14} \) satisfy the oblique projection condition \( \tilde{W}^T\tilde{V} = I \). To ensure that \( \tilde{W}^T\tilde{V} = I \), most of the \( H_2 \)-optimal MOR algorithms use the correction equation \( \tilde{W} = \tilde{W}(\tilde{V}^T\tilde{W})^{-1} \). Theoretically, it does ensure that \( \tilde{W}^T\tilde{V} = I \), however, it becomes numerically unstable even for small systems. A more robust approach is to take the geometric interpretation of \( \tilde{W}^T\tilde{V} = I \), i.e., the columns of \( \tilde{W} \) and \( \tilde{V} \) form biorthogonal basis of a subspace in \( \mathbb{R}^n \). Therefore, we use biorthogonal Gram-Schmidt method (steps 6-11 of Algorithm 1) to ensure that \( \tilde{W}^T\tilde{V} = I \) for better numerical properties. The pseudo code of our approach is given in Algorithm [11] which is referred to as the frequency-weighted \( H_2 \)-suboptimal MOR algorithm (FWHMOR)

Remark 4. FWHMOR provides approximations of \( P \) and \( Q \) as \( \tilde{P} \) and \( \tilde{Q} \), respectively, which can be used in FWBT to save some computational cost by avoiding the computation of large-scale Lyapunov equations (24) and (25).
Algorithm 1 FWHMOR

Input: Original system: \((A, B, C)\); Input weight: \((A_i, B_i, C_i)\); Output weight: \((A_o, B_o, C_o)\), Initial guess: \(\hat{A}, \hat{B}, \hat{C}\).
Output: ROM \((\tilde{A}, \tilde{B}, \tilde{C})\).

1. Compute \(P_i\) and \(Q_o\) by solving
   \[
   A_iP_i + P_iA_i^T + B_iB_i^T = 0, \\
   A_o^TQ_o + Q_oA_o + C_o^TC_o = 0.
   \]

2. Compute \(P_{13}\) and \(Q_{14}\) from the equations (22) and (23), respectively.

3. while (not converged) do

4. Compute \(P_{23}\) and \(Q_{24}\) from the equations (16) and (19), respectively.

5. Compute \(P_{12}\) and \(Q_{12}\) from the equations (18) and (21), respectively.

6. for \(i = 1, \ldots, r\) do

7. \(v = P_{12}(\ldots, i), v = \prod_{k=1}^i (I + P_{12}(\ldots, k)Q_{12}(\ldots, k)^T)v\).

8. \(w = -Q_{12}(\ldots, i), w = \prod_{k=1}^i (I + Q_{12}(\ldots, k)P_{12}(\ldots, k)^T)w\).

9. \(v = \frac{v}{\|v\|}, u = \frac{w}{\|w\|}, v = \frac{v}{w} v\).

10. \(\tilde{V}(\ldots, i) = v, \tilde{W}(\ldots, i) = w\).

11. end for

12. \(\hat{A} = W^TAV, \hat{B} = W^TB, \hat{C} = C\hat{\bar{V}}\).

13. end while

\(R^*\) from the right, one can note that the following Sylvester equations hold

\[
S^T R^* Q_{24} + R^* Q_{24} A_o - L_o^T (B_o^T Q_o + D_o^T C_o) = 0, \\
A_o^TQ_{12}R + Q_{12}RS - C_o^T B_o^T Q_{24}R - Q_{14}B_oL_o \\
- C_o^T D_o^T D_o L_o = 0.
\]

Due to uniqueness, \(W_o = -Q_{12}R\) and \(W_b = -R^* Q_{24}\). Since \(R\) only changes the basis of \(V_o\) and \(W_o\), the columns of \(\tilde{V}\) and \(\tilde{W}\) in FWHMOR span the same subspaces as spanned by \(V_o\) and \(W_o\), respectively, in FWITIA. Therefore, the ROM constructed by FWHMOR satisfies the tangential interpolation conditions [14] and [15] upon convergence like FWITIA, provided \(P_{13} = \tilde{V}P_{23}\) and \(Q_{14} = -\tilde{W}Q_{24}\). However, there are some notable numerical differences between FWHMOR and FWITIA. FWHMOR does not require \(H(s)\) and \(\hat{H}(s)\) to have simple poles, unlike FWITIA. Also, FWHMOR does not require \(\hat{A}\) to be diagonalizable, unlike FWITIA. Thus FWITIA can be considered equivalent to FWHMOR if \(H(s)\) and \(\hat{H}(s)\) have simple poles, and \(\hat{A}\) is diagonalizable in every iteration. The spectral factorization of \(\hat{A}\) in every iteration of FWITIA may cause numerical ill-conditioning [5]. Moreover, FWITIA uses the correction equation \(\tilde{W} = \tilde{W}(\tilde{V}^T \tilde{W})^{-1}\) to ensure the oblique projection condition \(\tilde{W}^T \tilde{V} = I\), whereas FWHMOR uses numerically more stable biorthogonal Gram-Schmidt [5] to achieve that. In short, FWHMOR is numerically more general and stable algorithm than FWITIA, though both span the same subspaces. Moreover, the results of subsection 3.2 provide the theoretical foundation for FWITIA as well in terms of seeking to satisfy the optimality conditions [6]-[8]. Therefore, FWITIA is no more a heuristic generalization of [40] but an interpolation framework for the frequency-weighted \(H_s\)-optimal MOR problem.

3.3.2 Computational Aspects

We now discuss some computational aspects of FWHMOR to be considered for its efficient numerical implementation.

Initial Guess: The initial guess of the ROM can be made arbitrarily, for instance, by direct truncation of the original state-space realization. However, a good choice of the initial ROM generally has a positive impact on the performance of the fixed point iteration methods. Therefore, it is recommended to generate the initial guess by using the eigensolver in [34]. Since the mirror images of the poles with large residues have a big contribution to the \(H_s\)-norm, the initial guess can be generated with the eigensolver in [34] by projecting \(H(s)\) onto the dominant eigenspace of \(A\). Another option is to compute the initial ROM by using the low-rank approximation methods in [11] such that it provides good Petrov-Galerkin approximations of \(P_{13}\) and \(Q_{14}\) as \(\tilde{V}P_{23}\) and \(-\tilde{W}Q_{24}\). This ensures that \(||P_{13} - \tilde{V}P_{23}||\) and ||\(Q_{14} + \tilde{W}Q_{24}\)|| are small to begin with.

Convergence and Stopping Criteria: Like in most of the \(H_s\)-optimal MOR algorithms, the convergence is not guaranteed in FWHMOR. Therefore, a good stopping criterion is required to stop the algorithm in case it does not converge within admissible time. The stopping criterion should have two main properties: (i) It should be easily computable (ii) It should quickly indicate that the error has dropped appreciably. These two properties make sure that the computation of stopping criteria is not a computational burden in itself, and it can save computational effort by indicating that the algorithm is not improving the accuracy of ROM any further. The connection between FWITIA and FWHMOR has been established in this subsection. Thus the relative change in eigenvalues of \(\hat{A}\) can be used as the stopping criterion. Due to the small size of \(\hat{A}\), this can be achieved accurately and cheaply using \(QZ\)-method. \(||\hat{\bar{X}}||_2\) can also be used as a stopping criterion. The computation of \(\hat{P}\) and \(\hat{Q}\) in \(||\hat{\bar{X}}||_2\) requires solutions of two small-scale Lyapunov equations, i.e., (17) and (20), which can be done cheaply. Also, note that from a pragmatic perspective, achieving less \(||E_o(s)||^2_{H_s}\) is the main objective and not the local optimum in itself. Thus the stopping criterion can be based directly on the error itself. However, the computation of \(||E_o(s)||^2_{H_s}\) in each iteration is an expensive operation in a large-scale setting. We have discussed in subsection 3.2 that \(\bar{Y} = 0\) and \(\bar{Z} = 0\) essentially minimize...
Our algorithm with FWBT. Further, we replace MOR problem [19]. Therefore, we compare the performance and is considered a gold standard for the frequency-weighted selection for MOR in [12]. Although the ROM constructed by frequency-weighted MOR problem, and the third example is a theoretical results of the paper. The second example is a illustrative one, which is presented to aid convenient repeatability and validation of all examples. The first example is an illustrative one, which is

In this section, FWHMOR is tested on three numerical examples. The first example is an illustrative one, which is presented to aid convenient repeatability and validation of all the theoretical results of the paper. The second example is a frequency-weighted MOR problem, and the third example is a controller reduction problem. The original high-order models in the last two examples are taken from the benchmark collection for MOR in [12]. Although the ROM constructed by FWBT is not optimal in any norm, it offers supreme accuracy and is considered a gold standard for the frequency-weighted MOR problem [19]. Therefore, we compare the performance of our algorithm with FWBT. Further, we replace \( P \) and \( Q \) in FWBT with \( \tilde{P} \) and \( \tilde{Q} \), respectively, to perform approximate FWBT, which we refer to as Approximate-FWBT (A-FWBT) wherein \( \tilde{P} \) and \( \tilde{Q} \) are generated by FWHMOR.

Experimental Setup and Hardware: In all examples, FWHMOR is initialized arbitrarily. The mirror images of the poles of the initial guess used in FWHMOR are selected as interpolation points, and its residues are selected as tangential directions in FWITIA. This ensures a fair comparison between FWITIA and FWHMOR, as both algorithms are expected to behave similarly with this selection. The relative change in the poles of the ROM is used as the stopping criterion with a tolerance of \( 1 \times 10^{-2} \). The Lyapunov and Sylvester equations are solved using MATLAB’s ‘lyap’ command. The experiments are performed using MATLAB 2016 on a computer with a 2GHz i7 processor, 16GB random access memory, and Windows 10 operating system.

### Illustrative Example:

Consider a 6th order system with the following state-space realization

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-5.4545 & 4.5455 & 0 & -0.0545 & 0.0455 & 0 \\
10 & -21 & 11 & 0.1 & -0.21 & 0.11 \\
0 & 5.5 & -6.5 & 0 & 0.055 & -0.065 \\
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0.0909 & 0.4 & -0.5 \\
\end{bmatrix}^T,
\]

\[
C = \begin{bmatrix}
2 & -2 & 3 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

Let the input and output frequency weights be the following

\[
A_i = \begin{bmatrix}
-2 & -4.375 \\
8 & 0 \\
\end{bmatrix},
B_i = \begin{bmatrix}
2 \\
0 \end{bmatrix}^T,
C_i = \begin{bmatrix}
1 \\
0 \end{bmatrix}.
\]

\[
A_o = \begin{bmatrix}
-5 & -9.375 \\
16 & 0 \\
\end{bmatrix},
B_o = \begin{bmatrix}
2 \\
0 \end{bmatrix}^T,
C_o = \begin{bmatrix}
2.5 & 0 \\
\end{bmatrix}.
\]

The initial guess used in FWHMOR is the following

\[
\tilde{A}^{(0)} = \begin{bmatrix}
0.0332 & 5.4109 \\
-4.8283 & -0.2998 \\
\end{bmatrix},
\]

\[
\tilde{B}^{(0)} = \begin{bmatrix}
-0.0747 & -0.2958 \\
\end{bmatrix}^T,
\]

\[
\tilde{C}^{(0)} = \begin{bmatrix}
1.0117 & -0.2599 \\
\end{bmatrix}.
\]

Both FWHMOR and FWITIA converge in 4 iterations. The reduction subspaces in FWHMOR are the following

\[
\tilde{V} = \begin{bmatrix}
0.2132 & -0.0046 \\
-0.9666 & 0.0623 \\
0.2671 & -0.0341 \\
0.14 & 0.348 \\
-1.3601 & -1.6698 \\
0.6732 & 0.5017 \\
\end{bmatrix}
\quad \text{and} \quad
\tilde{W} = \begin{bmatrix}
0.4167 & -0.337 \\
-0.7398 & 0.6556 \\
0.5269 & -0.3808 \\
0.0139 & 0.2199 \\
-0.0325 & -0.4411 \\
0.0137 & 0.2622 \\
\end{bmatrix},
\]

which construct the following ROM

\[
\tilde{A} = \begin{bmatrix}
0.4059 & 1.6956 \\
-15.6668 & -0.6719 \\
\end{bmatrix},
\tilde{B} = \begin{bmatrix}
-0.0186 & -0.2875 \\
\end{bmatrix}^T,
\]

\[
\tilde{C} = \begin{bmatrix}
3.1608 & -0.2362 \\
\end{bmatrix}.
\]
The reduction subspaces in FWITIA are the following

\[
\begin{bmatrix}
0.0086 & 0.1932 \\
-0.063 & -0.9053 \\
0.0272 & 0.2621 \\
-0.1994 & -0.1223 \\
0.9381 & -0.0251 \\
-0.2746 & 0.2427
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0.0084 & 0.4674 \\
-0.1161 & -0.827 \\
-0.0698 & 0.5935 \\
-0.4109 & 0.0283 \\
0.8306 & -0.0621 \\
-0.4857 & 0.0304
\end{bmatrix}
\]

which construct the following ROM

\[
\hat{A} = \begin{bmatrix}
1.4570 & 25.1669 \\
-1.1444 & -1.7230
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
0.5377 & -0.0374
\end{bmatrix}^T,
\]

\[
\hat{C} = [0.2248 \ 2.9833].
\]

One can verify by using MATLAB’s command \(T = mldivide(V, V')\) that the reduction subspaces and the ROMs generated by FWITIA and FWHMOR are related to each other with the similarity transformation \(T = \begin{bmatrix}
0.0275 & 0.8906 \\
-0.5842 & -0.7104
\end{bmatrix}\).

This numerically confirms the results of [3, 1]. The deviations in the optimality conditions [2, 3, 4] and the interpolation conditions [14] and [15] (which are denoted by \(\mathcal{F}\) and \(\mathcal{G}\), respectively) for both ROMs are tabulated in Table 2. It can be noted that deviations are so small that these ROMs can be considered as local optima for all practical purposes. Moreover, FWITIA and FWHMOR also provide good approximations of \(P\) and \(Q\). The \(H_2\) and \(H_\infty\)-norms of the weighted error transfer function \(E_w(s)\) are compared with FWBT in Table 5.

| Deviation         | FWITIA     | FWHMOR    |
|-------------------|------------|-----------|
| \(||X + X||_2\)   | 2.90 \times 10^{-4} | 1.88 \times 10^{-4} |
| \(||Y D_t D_t^\top + Y||_2\) | 1.19 \times 10^{-4} | 1.06 \times 10^{-4} |
| \(||D_t^\top D_o Z + Z||_2\) | 2.26 \times 10^{-4} | 1.46 \times 10^{-5} |
| \(||\mathcal{F}||_2\)   | 6.96 \times 10^{-4} | 6.96 \times 10^{-4} |
| \(||\mathcal{G}||_2\)   | 2.13 \times 10^{-5} | 2.13 \times 10^{-5} |
| \(||P_{13} - \hat{V} P_{23}||_2\) | 0.0946 | 0.0946 |
| \(||Q_{14} + WQ_{24}||_2\) | 0.1097 | 0.1096 |
| \(||P - V PV^T||_2\)     | 0.0419 | 0.0419 |
| \(||Q - WQW^T||_2\)     | 0.2247 | 0.2247 |

| Technique | \(||E_w(s)||_{H_2}\) | \(||E_w(s)||_{H_\infty}\) |
|-----------|---------------------|---------------------|
| FWBT      | 0.0080              | 0.0471              |
| FWITIA    | 0.0061              | 0.0471              |
| FWHMOR    | 0.0061              | 0.0471              |
| A-FWBT    | 0.0061              | 0.0471              |

**Clamped Beam:** Consider the 348\(^{th}\) clamped beam model from the benchmark collection of [12]. The input weight is a 4\(^{th}\) order band-pass filter with the passband [5, 10] rad/sec, which is designed using MATLAB’s command \(butter(2,[5,10],’s’).\) The output weight is also a 4\(^{th}\) order band-pass filter with the passband [10, 25] rad/sec, which is designed using MATLAB’s command \(butter(2,[10,25],’s’).\) A 5\(^{th}\) order ROM is obtained by using FWBT, FWITIA, FWHMOR, and A-FWBT. The \(H_2\) and \(H_\infty\)-norms of the weighted error transfer function \(E_w(s)\) are tabulated in Table 4. It can be seen that FWHMOR and A-FWBT construct accurate ROMs. The singular values of \(E(s)\) within [5, 25] rad/sec are plotted in Figure 1. It can be seen that FWHMOR and A-FWBT ensure good accuracy within the desired frequency region.

| Technique | \(||E_w(s)||_{H_2}\) | \(||E_w(s)||_{H_\infty}\) |
|-----------|---------------------|---------------------|
| FWBT      | 0.3399              | 0.4418              |
| FWITIA    | 0.2479              | 0.2417              |
| FWHMOR    | 0.2478              | 0.2408              |
| A-FWBT    | 0.2478              | 0.2408              |

**International Space Station:** Consider the 270\(^{th}\) order international space station model from the benchmark collection of [12] as the plant \(P(s).\) An \(H_\infty\)-controller \(K(s)\) is designed using MATLAB’s \(ncfyn\) command wherein the loop shaping filter is specified as \(\frac{20}{s+1.5} I_{3\times3}.\) The resulting controller is a 260\(^{th}\) order controller, which is reduced to 2\(^{nd}\) order controller \(\hat{K}(s)\) based on the closeness of the closed-loop transfer function criterion [29], wherein the input and output weights...
are specified as
\[ W(s) = (I + P(s)K(s))^{-1} \text{ and} \]
\[ W_0(s) = (I + P(s)K(s))^{-1}P(s). \]
The \( H_2 \)- and \( H_{\infty} \)-norms of \( E_w(s) = W(s)(K(s) - \hat{K}(s))W_0(s) \) are tabulated in Table 5. It can be noted that FWHMOR and A-FWBT ensure good accuracy.

### TABLE 5 Weighted Error

| Technique | \( ||E_w(s)||_{H_2} \) | \( ||E_w(s)||_{H_{\infty}} \) |
|-----------|-----------------|-----------------|
| FWBT      | 0.1361          | 4.7385          |
| FWITIA    | 0.0066          | 0.0951          |
| FWHMOR    | 0.0066          | 0.0950          |
| A-FWBT    | 0.0066          | 0.0950          |

5 | CONCLUSION

We addressed the problem of frequency-weighted \( H_2 \)-MOR within the projection framework. It is shown that although the first-order optimality conditions for the problem can not be inherently met within the projection framework, the deviation in the optimality conditions decays as the order of the ROM increases. A fixed point iteration algorithm is proposed, which generates a nearly (local) optimal ROM. The oblique projection in the proposed algorithm is computed by solving sparse-dense Sylvester equations, which can be solved efficiently. The proposed algorithm is tested on three numerical examples to highlight its effectiveness. The numerical results validate the theory developed in the paper.

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