Some Properties of Rough Pythagorean Fuzzy Sets

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ABSTRACT
Pythagorean fuzzy sets are advancements of the intuitionistic fuzzy sets and overcome their limitations. In this paper, we exploit the concept of full congruence relation of Pythagorean fuzzy sets and define the lower and upper approximations of Pythagorean fuzzy set. Using the concept of approximations of Pythagorean fuzzy set we introduce the concept of rough Pythagorean fuzzy set.

1. Introduction

Considering the imprecision in decision-making, Zadeh [1] introduced the idea of fuzzy set which has a membership function, $\mu$ that assigns to each element of the universe of discourse, a number from the unit interval $[0, 1]$ to indicate the degree of belongingness to the set under consideration. The notion of fuzzy sets generalises classical sets theory, by allowing intermediate situations between the whole and nothing. In a fuzzy set, a membership function is defined to describe the degree of membership of an element to a class. The membership value ranges from 0 to 1, where 0 shows that the element does not belong to a class, 1 means the element belongs to a class, and other values indicate the degree of membership to a class. For fuzzy sets, the membership function replaced the characteristic function in crisp sets. Nasseri et al. [2–5] conducted a lot of research work in the field of fuzzy set theory and they also used some interesting methods for ranking triangular fuzzy numbers and trapezoidal fuzzy numbers.

Albeit, the concept of fuzzy set theory seems to be inconclusive because of the exclusion of nonmembership function and the disregard for the possibility of hesitation margin. Atanassov [6] critically studied these shortcomings and proposed a concept called intuitionistic fuzzy sets (IFSs). The construct (that is, IFSs) incorporates both membership function, $\mu$ and nonmembership function, $\nu$ with hesitation margin, $\pi$ (that is, neither membership nor nonmembership functions), such that $\mu + \nu \leq 1$ and $\mu + \nu + \pi = 1$. The notion of IFSs provides a flexible framework to elaborate uncertainty and vagueness. The idea of
IFS seems to be resourceful in modelling many real-life situations such as medical diagnosis, career determination, selection process, and multi-criteria decision-making, among others. There are lot of research work done in the field of IFSs [7–13].

There are situations where $\mu + \nu \geq 1$ unlike the cases captured in IFSs. This limitation in IFS naturally led to a construct, called Pythagorean fuzzy sets (PFSs). Pythagorean fuzzy set (PFS) proposed in [14–16] is a new tool to deal with vagueness, considering the membership grade, $\mu$ and the nonmembership grade, $\nu$ satisfying the conditions $\mu + \nu \leq 1$ or $\mu + \nu \geq 1$, and also, it follows that $\mu^2 + \nu^2 + \pi^2 = 1$, where $\pi$ is the Pythagorean fuzzy set index. The construct of PFSs can be used to characterise uncertain information more sufficiently and accurately than IFS. Garg [17] presented an improved score function for the ranking order of interval-valued Pythagorean fuzzy sets (IVPFSs). Based on it, a Pythagorean fuzzy technique for order of preference by similarity to ideal solution (TOPSIS) method, by taking the preferences of the experts, in the form of interval-valued Pythagorean fuzzy decision matrices, was discussed. Other explorations of the theory of PFSs can be found in [18–23].

Pythagorean fuzzy set has attracted renewed attention of many researchers, and subsequently and is used in many application areas such as decision-making, aggregation operators, and information measures. Garg [14] unveiled some new logarithmic operational laws and their aggregation operators for PFS with some applications, and discussed a decision-making problem under Pythagorean fuzzy environment, by proposing some generalised aggregation operators. Garg [15] proposed an improved score function for solving the MCDM problem with partially known weight information, such that the preferences related to the criteria are taken in the form of interval-valued Pythagorean fuzzy sets. Garg [16] developed a new decision-making model with probabilistic information, using the concept of immediate probabilities to aggregate the information under the Pythagorean fuzzy set environment, and defined two new exponential operational laws about IVPFS and their corresponding aggregation operators to be applied in MCDM.

R. Biswas and S. Nanda [24] introduced the concept of the lower and upper approximations of a subgroup of a group. N. Kuroki [25] gave the notion of rough ideal in a semigroup. B. Davvaz [26,27] introduced roughness in rings and the notion of rough prime ideals and rough primary ideals in rings. V. Selvan and G. Senthil Kumar [28] studied the lower and upper approximations of ideals and fuzzy ideals in a semiring. The notion of rough intuitionistic fuzzy ideal in a semigroup was given by J. Gosh et. al., [29].

R. Biswas [24] used the concept of intuitionistic fuzzy set to the theory of groups and studied the intuitionistic fuzzy subgroups of a group. The concept of intuitionistic fuzzy R-subgroup of a near ring was given by Y. H. Yon, Y. B. Jun and K. H. Kim [30].

The notions of fuzzy subnear-ring and ideal were introduced by Abou-Zaid in [31]. In 2001, Kyung Ho Kim and Young Bae Jun [32] in their paper entitled ‘Normal fuzzy R-subgroups in nearrings’ introduced the concept of a normal fuzzy R-subgroup in near-rings and explored some related properties. In [33], Kuncham et al., introduced the fuzzy prime ideal of near-rings.

In this paper, we introduce the concept of the lower and upper approximations of Pythagorean fuzzy sets. The rest of the paper is organised as follows. In Section 2, the preliminaries and some definitions are given and some algebraic structures of Pythagorean fuzzy sets are presented. In Section 3, we study the definition of rough Pythagorean fuzzy sets.
and discuss some important properties of rough Pythagorean fuzzy ideals of a near-ring. Finally, a conclusion is made in Section 4.

2. Preliminaries and Definitions

In this section, we recall the related concepts to the fuzzy sets, the intuitionistic fuzzy sets and the Pythagorean fuzzy sets as the definition of intuitionistic fuzzy set, Pythagorean fuzzy set. We also give some algebraic operations on Pythagorean fuzzy numbers and Pythagorean fuzzy sets.

Definition 2.1: [34] Let \((S, \rho)\) be an approximation space, where \(S\) is the non-empty universe, \(\rho\) is an equivalence relation and \(X\) is a non-empty subset of \(S\). Then the sets

\[ \rho_\leq (X) = \{ x \in S | [x]_\rho \subseteq X \} \]
\[ \rho_\geq (X) = \{ x \in S | [x]_\rho \cap X \neq \emptyset \subset X \} \]

are respectively called the lower approximation and the upper approximation of the set \(X\) with respect to \(\rho\). \(X\) is called \(\rho\)-definable if \(\rho_\leq (X) = \rho_\geq (X)\). If \(\rho_\leq (X) \neq \rho_\geq (X)\), then \(X\) is called rough set with respect to \(\rho\).

Intuitionistic fuzzy sets.

Intuitionistic fuzzy set brought by Atanassov [6] is a development of the traditional fuzzy set, which is an appropriate way to scope with vagueness. It can be defined as follows:

Definition 2.2: [6] Let \(X\) be a fixed set. An intuitionistic fuzzy set (IFS) \(I\) in \(X\) is an expression having the following form

\[ I = \{ (x, \mu_I(x), \nu_I(x)) : x \in X \} \]

where the functions \(\mu_I(x)\) and \(\nu_I(x)\) are the degree of membership and the degree of non-membership of the element \(x \in X\), respectively. Also \(\mu_I : X \rightarrow [0, 1], \nu_I : X \rightarrow [0, 1]\) and \(0 \leq \mu_I(x) + \nu_I(x) \leq 1\), for all \(x \in X\).

The degree of indeterminacy \(\pi_I(x) = 1 - \mu_I(x) - \nu_I(x)\).

In practice, maybe for some reason, the condition \(0 \leq \mu(x) + \nu(x) \leq 1\) is not true. For instance, \(0.4 + 0.7 = 1.1 > 1\) but \(0.4^2 + 0.7^2 < 1\), or \(0.5 + 0.7 = 1.2 > 1\) but \(0.5^2 + 0.7^2 < 1\). To overcome this situation, in 2013 Yager [22] introduced the concept of the Pythagorean fuzzy set.

Definition 2.3: [22] A PFS \(P\) in a finite universe of discourse \(X\) is given by

\[ P = \{ (x, \mu_P(x), \nu_P(x)) | x \in X \} \]

where \(\mu_P(x) : X \rightarrow [0, 1]\) denotes the degree of membership and \(\nu_P(x) : X \rightarrow [0, 1]\) denotes the degree of non-membership of the element \(x \in X\) to the set \(A\) with the condition that \(0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1\).

The degree of indeterminacy \(\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}\) (Figure 1).
2.1. Some Operations on Pythagorean Fuzzy Numbers

Here, we introduce some operations on Pythagorean fuzzy numbers and pythagorean fuzzy sets used in the rest of the paper.

Given three Pythagorean fuzzy numbers (PFNs) \( \alpha = \langle \mu, \nu \rangle \), \( \alpha_1 = \langle \mu_1, \nu_1 \rangle \) and \( \alpha_2 = \langle \mu_2, \nu_2 \rangle \). The basic operations can be defined as follows:

(i) \( \bar{\alpha} = \langle \nu, \mu \rangle \)

(ii) \( \alpha_1 \lor \alpha_2 = \langle \max\{\mu_1, \mu_2\}, \min\nu_1, \nu_2 \rangle \)

(iii) \( \alpha_1 \land \alpha_2 = \langle \min\{\mu_1, \mu_2\}, \max\nu_1, \nu_2 \rangle \)

(iv) \( \alpha_1 \oplus \alpha_2 = \langle \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2 \rangle \)

(v) \( \alpha_1 \otimes \alpha_2 = \langle \mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2} \rangle \)

(vi) \( \lambda \cdot \alpha = \langle \sqrt{1 - (1 - \mu^2)\lambda}, \nu^\lambda \rangle, \lambda > 0 \)

(vii) \( \alpha^\lambda = \langle \mu^\lambda, \sqrt{1 - (1 - \nu^2)^\lambda} \rangle, \lambda > 0 \)

**Definition 2.4 ([22]):** Let \( \alpha_1 = \langle \mu_1, \nu_1 \rangle \) and \( \alpha_2 = \langle \mu_2, \nu_2 \rangle \) be two PFNs; \( S(\alpha_1) = \mu_1^2 - \nu_1^2 \) and \( S(\alpha_2) = \mu_2^2 - \nu_2^2 \) be their score functions; \( H(\alpha_1) = \mu_1^2 + \nu_1^2 \) and \( H(\alpha_2) = \mu_2^2 + \nu_2^2 \) be the accuracy degrees of \( \alpha_1 \) and \( \alpha_2 \), then Yager and Abbasov [22] defined the following:

(1) If \( S(\alpha_1) < S(\alpha_2) \) then \( \alpha_1 \) is smaller than \( \alpha_2 \), that is \( \alpha_1 < \alpha_2 \);
(2) If \( S(\alpha_1) > S(\alpha_2) \) then \( \alpha_1 > \alpha_2 \);
(3) If \( S(\alpha_1) = S(\alpha_2) \) then
(4) if \( H(\alpha_1) < H(\alpha_2) \) then \( \alpha_1 < \alpha_2 \)
(5) if \( H(\alpha_1) > H(\alpha_2) \) then \( \alpha_1 > \alpha_2 \)

If \( H(\alpha_1) = H(\alpha_2) \) then \( \alpha_1 \) and \( \alpha_2 \) represent the same information, that is \( \alpha_1 = \alpha_2 \).
3. Rough Pythagorean Fuzzy Set in a Near-Ring

In this section, we recall some basic definitions of near-ring which was introduced by Abou-Zaid [31].

A near-ring is a non-empty set \( R \) with two binary operations \( + \) and \( \cdot \), satisfying the following axioms:

(i) \((R, +)\) is a group,
(ii) \((R, \cdot)\) is a semigroup,
(iii) \(x.(y + z) = x.y + x.z \) for all \( x, y, z \in R \).

Precisely speaking, it is a left near-ring because it has satisfied left distributive law. We will use the word ‘near ring’ instead of ‘left near ring’.

Note that \( x.0 = 0 \) and \( x.(-y) = -xy \) for all \( x, y \in R \) but in general \( 0.x \neq 0 \) for some \( x \in R \).

**Definition 3.1:** Let \( P = (\mu_P, v_P) \) and \( Q = (\mu_Q, v_Q) \) be two Pythagorean fuzzy sets in a near ring \( R \), then their product is defined as \( P.Q = (\mu_P \mu_Q, v_P v_Q) \), where \( (\mu_P \mu_Q)(x) = \bigvee_{x=yz} [\mu_P(y) \land \mu_P(z)] \) and \( (v_P v_Q)(x) = \bigwedge_{x=yz} [v_P(y) \lor v_P(z)] \) for all \( x \in R \).

**Definition 3.2:** Let \( P = (\mu_P, v_P) \) and \( Q = (\mu_Q, v_Q) \) be two Pythagorean fuzzy sets in a near-ring \( R \), then their sum \( P + Q \) is defined as \( P + Q = (\mu_P + \mu_Q, v_P + v_Q) \), where \( (\mu_P + \mu_Q)(x) = \bigvee_{x=yz} [\mu_P(y) \land \mu_P(z)] \) and \( (v_P + v_Q)(x) = \bigwedge_{x=yz} [v_P(y) \lor v_P(z)] \) for all \( x \in R \).

**Definition 3.3:** A Pythagorean fuzzy set \( P = (\mu_P, v_P) \) in a near-ring \( R \) is said to be a Pythagorean fuzzy subring of \( R \) if for all \( x, y \in R \) satisfies the following properties

(i) \( \mu_P(x - y) \geq \mu_P(x) \land \mu_P(y) \)
(ii) \( \mu_P(xy) \geq \mu_P(y) \lor \mu_P(y) \)
(iii) \( v_P(x - y) \leq v_P(x) \lor v_P(y) \)
(iv) \( v_P(xy) \leq v_P(x) \land v_P(y) \)

**Example 3.1:** Let \( R = \{a, b, c, d\} \) be a set with two binary operations as follows.

\[
\begin{array}{ccccc}
+ & a & b & c & d \\
\hline
a & a & b & c & d \\
b & b & a & d & c \\
c & c & d & b & a \\
d & d & c & a & b \\
\end{array}
\]

Then \((R, +, \cdot)\) is a near-ring.

Let \( PFS P = (\mu_P, v_P) \) in \( R \) defined by \( \mu_P(a) = 0.9, \mu_P(b) = 0.6, \mu_P(c) = \mu_P(d) = 0.3 \) and \( v_P(a) = 0.2, v_P(b) = 0.3, v_P(c) = v_P(d) = 0.6 \).

It can be shown that a Pythagorean fuzzy set \( P = (\mu_P, v_P) \) is a Pythagorean fuzzy subring of \( R \).

**Definition 3.4:** A map \( f \) from a near-ring \( R \) into a near-ring \( S \) is called homomorphism if \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \) for all \( x, y \in R \).
Let $f : R \rightarrow S$ be a homomorphism of near-rings. For any $PFS = (\mu_P, \nu_P)$ in $S$, we define $PFS' = (\mu_{P'}, \nu_{P'})$ in $R$ by $\mu_{P'}(x) = \mu_P(f(x))$, $\nu_{P'}(x) = \nu_P(f(x))$ for all $x \in R$.

**Definition 3.5:** Let $R$ and $R'$ be two near-rings and $f : R \rightarrow R'$ be a homomorphism and $P = (\mu_P, \nu_P)$ and $Q = (\mu_Q, \nu_Q)$ be Pythagorean fuzzy set of $R$ and $R'$, respectively, then the image $f(P)$ and inverse image $f^{-1}(Q)$ are defined as follows:

$$f(A) = (f(\mu_P), f(\nu_P))$$

where

$$f(\mu_P)(y) = \begin{cases} \sup \{\mu_P(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \varnothing; \\ 0, & \text{if } f^{-1}(y) = \varnothing. \end{cases}$$

$$f(\nu_P)(y) = \begin{cases} \inf \{\nu_P(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \varnothing; \\ 0, & \text{if } f^{-1}(y) = \varnothing. \end{cases}$$

and $f^{-1}(Q) = (f^{-1}(\mu_Q), f^{-1}(\nu_Q))$, where $f^{-1}(\mu_Q) = \mu_Q(f(x))$, $f^{-1}(\nu_Q) = \nu_Q(f(x)), \forall x \in R$.

**Definition 3.6:** Let $\rho$ be an equivalence relation in a near-ring $R$, then $\rho$ is called a full congruence relation if $(p, q) \in \rho$ implies $(p + q, x + q)$, $(px, qx)$ and $(xa, xb) \in \rho$ for all $x \in \rho$.

The $\rho$ congruence class containing the element $x \in R$ is denoted by $[x]_\rho$.

**Theorem 3.1:** Let $\rho$ be a full congruence relation on a relation $R$.

If $a, b \in R$ then

(i) $[a]_\rho + [b]_\rho = [a + b]_\rho$

(ii) $[-a]_\rho = [-a]_\rho$

(iii) $[xy]_\rho \subseteq [a]_\rho \cdot [b]_\rho = [ab]_\rho$.

**Proof:** Straight forward.

A full congruence relation $\rho$ on $R$ is called complete if $[ab]_\rho = [a]_\rho \cdot [b]_\rho$, for all $a, b \in R$.

**Definition 3.7:** Let $\rho$ be a full congruence relation in a near-ring $R$ and $P = (\mu_P, \nu_P)$ be a Pythagorean fuzzy set of $R$. Then the Pythagorean fuzzy set $\rho_* = (\rho_*(\mu_P), \rho_*(\nu_P))$ and $\rho^* = (\rho^*(\mu_P), \rho^*(\nu_P))$ are called $\rho$ the lower and $\rho$ the upper approximation of Pythagorean fuzzy set $P$, respectively where $\rho_*(\mu_P)(x) = \bigwedge_{p \in [x]_\rho} \mu_P(p)$, $\rho_*(\nu_P)(x) = \bigvee_{p \in [x]_\rho} \nu_P(p)$ and $\rho^*(\mu_P)(x) = \bigvee_{p \in [x]_\rho} \mu_P(p)$, $\rho^*(\nu_P)(x) = \bigwedge_{p \in [x]_\rho} \nu_P(p)$.

**Definition 3.8:** Let $\rho$ be a full congruence relation in a near-ring $R$ and $P = (\mu_P, \nu_P)$ be a Pythagorean fuzzy set of $R$.

If $\rho_* \neq \rho^*$ on Pythagorean fuzzy set $P$, then this set is called rough Pythagorean fuzzy set and $\rho_*$ and $\rho^*$ are called lower and upper Pythagorean fuzzy set, respectively.

**Theorem 3.2:** Let $\rho$ be a congruence relation on $R$. If $P, Q$ are any two Pythagorean fuzzy sets of $R$, then $\rho^*(P) + \rho^*(Q) \subseteq \rho^*(P + Q)$. 
Proof: Since $\rho$ is a congruence relation on $R$, $[x]_\rho + [y]_\rho \subseteq [x + y]_\rho, \forall x, y \in R$.

Let $P = (\mu_P, v_P)$ and $Q = (\mu_Q, v_Q)$ be any two Pythagorean fuzzy set of $R$.

Then $\rho^*(P) + \rho^*(Q) = (\rho^*(\mu_P) + \rho^*(\mu_Q), \rho^*(v_P) + \rho^*(v_Q))$ and $\rho^*(A + B) = (\rho^*(\mu_P + \mu_Q), \rho^*(v_P + v_Q))$.

To prove $\rho^*(P) + \rho^*(Q) \subseteq \rho^*(P + Q)$ it is enough to prove that for every $x \in R$, $(\rho^*(\mu_P) + \rho^*(\mu_Q))(x) \leq \rho^*(\mu_P + \mu_Q)(x)$ and $(\rho^*(v_P) + \rho^*(v_Q))(x) \geq \rho^*(v_P + v_Q)(x)$.

$$\rho^*(\mu_P + \mu_Q)(x) = \bigvee_{x = y + z} \rho^*(\mu_P)(y) \land \rho^*(\mu_Q)(z)$$

$$= \bigvee_{x = y + z} \left( \bigvee_{p \in [y]_\rho} (\mu_P(p) \land \mu_Q(q)) \right)$$

$$\leq \bigvee_{x = y + z} \left( \bigvee_{p + q \in [y + z]_\rho} (\mu_P(p) \land \mu_Q(q)) \right)$$

$$= \bigvee_{p + q \in [x]_\rho} [\mu_P(p) \land \mu_Q(q)] \text{ since } y + z = x$$

$$= \bigvee_{a \in [x]_\rho, a = p + q} [\mu_P(p) \land \mu_Q(q)]$$

$$= \bigvee_{a \in [x]_\rho} [\mu_P(p) \land \mu_Q(q)]$$

$$= \rho^*(\mu_P + \mu_Q)(x).$$

$$\rho^*(v_P) + \rho^*(v_Q))(x) = \bigwedge_{x = y + z} \rho^*(v_P)(y) \lor \rho^*(v_Q)(z)$$

$$= \bigwedge_{x = y + z} \left( \bigwedge_{p \in [y]_\rho} (v_P(p) \lor v_Q(q)) \right)$$

$$\leq \bigwedge_{p + q \in [y + z]_\rho} (v_P(p) \lor v_Q(q))$$

$$= \bigwedge_{p + q \in [x]_\rho} [v_P(p) \lor v_Q(q)] \text{ since } y + z = x$$

$$= \bigwedge_{a \in [x]_\rho, a = p + q} [v_P(p) \lor v_Q(q)]$$

$$= \bigwedge_{a \in [x]_\rho} [v_P(p) \lor v_Q(q)]$$

$$= \bigwedge_{a \in [x]_\rho} [v_P + v_Q](a)$$

$$= \rho^*(v_P + v_Q)(x).$$

Thus we have, $\rho^*(P) + \rho^*(Q) \subseteq \rho^*(P + Q)$. 
Equality holds, if $\rho$ is a full congruence relation. ■

**Theorem 3.3:** Let $\rho$ be a congruence relation on $R$. If $P, Q$ are any two Pythagorean fuzzy sets of $R$, then $\rho_*(P) + \rho_*(Q) \subseteq \rho_*(P + Q)$.

**Proof:** Since $\rho$ is a congruence relation on $R, [x]_\rho + [y]_\rho \subseteq [x + y]_\rho, \forall x, y \in R$.

Let $P = (\mu_P, v_P)$ and $Q = (\mu_Q, v_Q)$ be any two Pythagorean fuzzy sets of $R$.

Then $\rho_*(P) + \rho_*(Q) = (\rho_*(\mu_P) + \rho_*(\mu_Q), \rho_*(v_P) + \rho_*(v_Q))$ and $\rho_*(P + Q) = (\rho_*(\mu_P + \mu_Q), \rho_*(v_P + v_Q))$.

To show $\rho_*(P) + \rho_*(Q) \subseteq \rho_*(P + Q)$ it is enough to prove that for every $x \in R$, $(\rho_*(\mu_P) + \rho_*(\mu_Q))(x) \leq \rho_*(\mu_P + \mu_Q)(x)$ and $(\rho_*(v_P) + \rho_*(v_Q))(x) \geq \rho_*(v_P + v_Q)(x)$.

\[
(\rho_*(\mu_P) + \rho_*(\mu_Q))(x) = \bigvee_{x = y + z} [\rho_*(\mu_P)(y) \wedge \rho_*(\mu_Q)(z)] \\
= \bigvee_{x = y + z} [(\wedge_{p \in [y]_\rho} \mu_P(p)) \wedge (\wedge_{q \in [z]_\rho} \mu_Q(q))] \\
= \bigvee_{x = y + z} [\wedge_{p \in [y]_\rho, q \in [z]_\rho} (\mu_P(p) \wedge \mu_Q(q))] \\
\leq \bigvee_{x = y + z} [\bigvee_{p + q \in [y + z]_\rho} \mu_P(p) \wedge \mu_Q(q)] \\
= \bigvee_{x = y + z} [\mu_P + \mu_Q](y + z) \\
= \rho_*(\mu_P + \mu_Q)(x).
\]

\[
(\rho_*(v_P) + \rho_*(v_Q))(x) = \bigwedge_{x = y + z} [\rho_*(v_P)(y) \vee \rho_*(v_Q)(z)] \\
= \bigwedge_{x = y + z} [(\vee_{p \in [y]_\rho} v_P(p)) \vee (\vee_{q \in [z]_\rho} v_Q(q))] \\
= \bigwedge_{x = y + z} [\bigvee_{p \in [y]_\rho, q \in [z]_\rho} (v_P(p) \vee v_Q(q))] \\
\leq \bigwedge_{x = y + z} [\bigvee_{p + q \in [y + z]_\rho} \mu_P(p) \vee \mu_Q(q)] \\
= \bigwedge_{x = y + z} [\bigvee_{p + q \in [y + z]_\rho} (v_P + v_Q)(p + q)] \\
= \bigwedge_{x = y + z} [v_P + v_Q](y + z) \\
= \rho_*(v_P + v_Q)(x).
\]

Thus we have, $\rho_*(P) + \rho_*(Q) \subseteq \rho_*(P + Q)$. ■
Theorem 3.4: Let $\rho$ be a congruence relation on $R$. If $P$, $Q$ are any two Pythagorean fuzzy sets of $R$, then $\rho^*(P).\rho^*(Q) \subseteq \rho^*(P, Q)$.

Proof: Since $\rho$ is a congruence relation on $R$, $xy|x \in [p]_\rho, y \in [q]_\rho \subseteq [pq]_\rho, \forall p, q \in R$.

Let $P = (\mu_P, v_P)$ and $Q = (\mu_Q, v_Q)$ be any two Pythagorean fuzzy sets of $R$.

Then $\rho^*(P), \rho^*(Q) = (\rho^*(\mu_P), \rho^*(v_P), \rho^*(v_Q))$ and $\rho^*(P, Q) = (\rho^*(\mu_P, \mu_Q), \rho^*(v_P, v_Q))$.

To show $\rho^*(P), \rho^*(Q) \subseteq \rho^*(P, Q)$ it is enough to prove that for every $x \in R$, $(\rho^*(\mu_P), \rho^*(\mu_Q))(x) \leq \rho^*(\mu_P, \mu_Q)(x)$ and $(\rho^*(v_P), \rho^*(v_Q))(x) \geq \rho^*(v_P, v_Q)(x)$.

Now, for all $x \in R$,

$$(\rho^*(\mu_P), \rho^*(\mu_Q))(x) = \bigvee_{x = yz} [\rho^*(\mu_P)(y) \land \rho^*(\mu_Q)(z)]$$

$$= \bigvee_{x = yz} \left[ (\bigvee_{p \in [y]_\rho} \mu_P(p)) \land (\bigvee_{q \in [z]_\rho} \mu_Q(q)) \right]$$

$$= \bigvee_{x = yz} \left[ \bigvee_{pq \in [x]_\rho} (\mu_P(p) \land \mu_Q(q)) \right]$$

$$\leq \bigvee_{x = yz} \left[ \bigvee_{pq \in [x]_\rho} (\mu_P(p) \land \mu_Q(q)) \right]$$

$$= \bigvee_{pq \in [x]_\rho} [\mu_P(p) \land \mu_Q(q)]$$

$$= \bigvee_{ab \in [x]_\rho} [\mu_P(p) \land \mu_Q(q)]$$

$$= \bigvee_{\alpha \in [x]_\rho} (\mu_P + \mu_Q)(\alpha)$$

$$= \rho^*(\mu_P, \mu_Q)(x).$$

$$(\rho^*(v_P), \rho^*(v_Q))(x) = \bigwedge_{x = yz} [\rho^*(\mu_P)(y) \lor \rho^*(\mu_Q)(z)]$$

$$= \bigwedge_{x = yz} \left[ (\bigwedge_{p \in [y]_\rho} \mu_P(p)) \lor (\bigwedge_{q \in [z]_\rho} \mu_Q(q)) \right]$$

$$= \bigwedge_{x = yz} \left[ \bigwedge_{pq \in [x]_\rho} (\mu_P(p) \lor \mu_Q(q)) \right]$$

$$\leq \bigwedge_{x = yz} \left[ \bigwedge_{pq \in [x]_\rho} (\mu_P(p) \lor \mu_Q(q)) \right]$$

$$= \bigwedge_{pq \in [x]_\rho} [\mu_P(p) \lor \mu_Q(q)]$$

$$= \bigwedge_{ab \in [x]_\rho} [\mu_P(p) \lor \mu_Q(q)]$$

$$= \bigwedge_{\alpha \in [x]_\rho} (\mu_P(p) \lor \mu_Q(q)].$$
= \bigwedge_{\alpha \in [x]_\rho} (\mu_P + \mu_Q)(\alpha) \\
= \rho^*(\mu_P, \mu_Q)(x).

Thus we have, \(\rho^*(P), \rho^*(Q) \subseteq \rho^*(P, Q)\). ■

**Theorem 3.5:** Let \(\rho\) and \(\varphi\) be two congruence relations on \(R\). If \(P\) is a Pythagorean fuzzy set of \(R\), then \((\rho \cap \varphi)^*(P) \subseteq \rho^*(P) \cap \varphi^*(P)\).

**Proof:** Since \(\rho\) and \(\varphi\) are two congruence relations on \(R\), then \(\rho \cap \varphi\) is also a congruence relation on \(R\) and \(\rho \cap \varphi \subseteq \rho, \rho \cap \varphi \subseteq \varphi\).

Let \(P = (\mu_P, \nu_P)\) be a Pythagorean fuzzy set of \(R\).

Then, \((\rho \cap \varphi)^*(P) \subseteq \rho^*(P)\) and \((\rho \cap \varphi)^* \subseteq \rho^*(P)\).

Therefore, \((\rho \cap \varphi)^*(P) \subseteq \rho^*(P) \cap \varphi^*(P)\). ■

**Theorem 3.6:** Let \(\rho\) and \(\varphi\) be two congruence relations on \(R\). If \(P\) is a Pythagorean fuzzy set of \(R\), then \((\rho \cap \varphi)^*(P) \subseteq \rho^*(P) \cap \varphi^*(P)\).

**Proof:** Since \(\rho\) and \(\varphi\) are two congruence relations on \(R\), then \(\rho \cap \varphi\) is also a congruence relation on \(R\) and \(\rho \cap \varphi \subseteq \rho, \rho \cap \varphi \subseteq \varphi\).

Let \(P = (\mu_P, \nu_P)\) be a Pythagorean fuzzy set of \(R\).

Then, \((\rho \cap \varphi)_*(P) \supseteq \rho_*(P)\) and \((\rho \cap \varphi)_* \supseteq \rho_*(P)\).

Therefore, \((\rho \cap \varphi)_*(P) \subseteq \rho_*(P) \cap \varphi_*(P)\). ■

**Theorem 3.7:** Let \(\rho\) be a full congruence relation on \(R\). Then

(a) If \(P\) is a Pythagorean fuzzy subring of \(R\), then \(P\) is an upper rough Pythagorean fuzzy subring of \(R\).

(b) If \(P\) is a Pythagorean fuzzy ideal of \(R\), then \(P\) is an upper rough Pythagorean fuzzy ideal of \(R\).

**Proof:** (a) Let \(P = (\mu_P, \nu_P)\) be a Pythagorean subring of \(R\). Then, \(\rho^*(P) = (\rho^*(\mu_P), \rho^*(\nu_P))\).

Now, for all \(x, y \in R\),

(i)

\[\rho^*(\mu_P)(x - y) = \bigvee_{z \in [x-y]_\rho} \mu_P(z)\]

= \bigvee_{z \in [x]_\rho - [y]_\rho} \mu_P(z), \text{ since } \rho \text{ is full congruence relation of } \rho.

= \bigvee_{p-q \in \rho \cap \rho} \mu_P(p - q)

\geq \bigvee_{p \in [x]_\rho, q \in [y]_\rho} [\mu_P(p) \wedge \mu_P(q)]

= \left(\bigvee_{p \in [x]_\rho} \mu_P(p)\right) \wedge \left(\bigvee_{q \in [y]_\rho} \mu_P(q)\right)

= \rho^*(\mu_P)(x) \wedge \rho^*(\mu_P)(y).
\[
\rho^*(v_P)(x - y) = \bigwedge_{z \in [x - y]} v_P(z)
\]
\[
= \bigwedge_{z \in [x]_\rho - [y]_\rho} v_P(z)
\]
\[
= \bigwedge_{p - q \in [x]_\rho - [y]_\rho_{ho}} v_P(p - q)
\]
\[
\leq \bigwedge_{p \in [x]_\rho, q \in [y]_\rho_{ho}} \left[ v_P(p) \land v_P(q) \right]
\]
\[
= \left( \bigwedge_{p \in [x]_\rho} v_P(p) \right) \lor \left( \bigwedge_{q \in [y]_\rho} v_P(q) \right)
\]
\[
= \rho^*(v_P)(x) \lor \rho^*(v_P)(y).
\]

(ii)

\[
\rho^*(\mu_P)(xy) = \bigvee_{z \in [xy]} \mu_P(z)
\]
\[
\geq \bigvee_{z \in [x]_\rho, [y]_\rho} \mu_P(z)
\]
\[
= \bigvee_{pq \in [x]_\rho [y]_\rho_{ho}} \mu_P(pq)
\]
\[
\geq \bigvee_{p \in [x]_\rho, q \in [y]_\rho_{ho}} \left[ \mu_P(p) \land \mu_P(q) \right]
\]
\[
= \left( \bigvee_{p \in [x]_\rho} \mu_P(p) \right) \land \left( \bigvee_{q \in [y]_\rho} \mu_P(q) \right)
\]
\[
= \rho^*(\mu_P)(x) \land \rho^*(\mu_P)(y).
\]

\[
\rho^*(v_P)(xy) = \bigwedge_{z \in [xy]} v_P(z)
\]
\[
\leq \bigwedge_{z \in [x]_\rho, [y]_\rho} v_P(z)
\]
\[
= \bigwedge_{pq \in [x]_\rho [y]_\rho_{ho}} v_P(pq)
\]
\[
\leq \bigwedge_{p \in [x]_\rho, q \in [y]_\rho_{ho}} \left[ v_P(p) \land v_P(q) \right]
\]
\[
= \left( \bigwedge_{p \in [x]_\rho} v_P(p) \right) \lor \left( \bigwedge_{q \in [y]_\rho} v_P(q) \right)
\]
\[
= \rho^*(v_P)(x) \lor \rho^*(v_P)(y).
\]

Therefore, \( \rho^*(P) \) is a Pythagorean fuzzy subring of \( R \).

Hence, \( P \) is an upper rough Pythagorean fuzzy subring of \( R \).

(b) Let \( P = (\mu_P, v_P) \) be a Pythagorean subring of \( R \).

We prove that \( \rho^*(\mu_P)(xy) \geq \rho^*(\mu_P)(x) \lor \rho^*(\mu_P)(y) \) and \( \rho^*(v_P)(xy) \leq \rho^*(v_P)(x) \land \rho^*(v_P)(y) \) for all \( x, y \in R \).

\[
\rho^*(\mu_P)(xy) = \bigvee_{z \in [xy]} \mu_P(z)
\]
Therefore, $\rho^*(P)$ is a Pythagorean fuzzy ideal of $R$.
Hence, $P$ is an upper rough Pythagorean fuzzy ideal of $R$. 

\[ \rho^*(\nu P)(xy) = \bigwedge_{z \in [xy]} \nu P(z) \]
\[ \leq \bigwedge_{z \in [x]_\rho [y]_\rho} \nu P(z) \]
\[ = \bigwedge_{pq \in [x]_\rho [y]_\rho} \nu P(pq) \]
\[ \leq \bigwedge_{p \in [x]_\rho, q \in [y]_\rho} [\nu P(p) \wedge \nu P(q)] \]
\[ = \left( \bigwedge_{p \in [x]_\rho} \nu P(p) \right) \wedge \left( \bigwedge_{q \in [y]_\rho} \nu P(q) \right) \]
\[ = \rho^*(\nu P)(x) \wedge \rho^*(\nu P)(y). \]

**Theorem 3.8:** Let $\rho$ be a complete congruence relation on $R$. Then

(i) If $P$ is a Pythagorean fuzzy subring of $R$, then $P$ is a lower rough Pythagorean fuzzy subring of $R$.
(ii) If $P$ is a Pythagorean fuzzy ideal of $R$, then $P$ is a lower rough Pythagorean fuzzy ideal of $R$.

**Proof:** Since $\rho$ is a complete congruence relation on $R$, we have $[x]_\rho - [y]_\rho = [x - y]_\rho$ and $[x]_\rho [y]_\rho = [xy]_\rho$, for all $x, y \in R$.

(a) Let $P = (\mu P, \nu P)$ be a Pythagorean fuzzy subring of $R$ and $\rho_0(P) = (\rho_0(\mu P), \rho_0(\nu P))$.
Now for all $x, y \in R$, we have

(i)
\[ \rho_0(\mu P)(x - y) = \bigwedge_{z \in [x - y]_\rho} \mu P(z) \]
\[ = \bigwedge_{z \in [x]_\rho - [y]_\rho} \mu P(z), \]
\[ = \bigwedge_{p - q \in [x]_\rho - [y]_\rho} \mu P(p - q) \]
\[ \geq \bigwedge_{p \in [x]_\rho, q \in [y]_\rho} [\mu P(p) \wedge \mu P(q)] \]
\[
\rho^* (\mu_P(x - y)) = \bigvee_{z \in [x - y]} \nu_P(z)
\]
\[
= \bigvee_{z \in [x]_p - [y]_q} \mu_P(p - q)
\]
\[
\leq \bigvee_{p \in [x], q \in [y]} [\nu_P(p) \lor \nu_P(q)]
\]
\[
= \left( \bigvee_{p \in [x]} \nu_P(p) \right) \lor \left( \bigvee_{q \in [y]} \nu_P(q) \right)
\]
\[
= \rho^* (\nu_P(x)) \lor \rho^* (\nu_P(y)).
\]

(ii)

\[
\rho^* (\mu_P(xy)) = \bigwedge_{z \in [xy]} \mu_P(z)
\]
\[
\geq \bigwedge_{z \in [x]_p [y]_q} \mu_P(pq)
\]
\[
\geq \bigwedge_{p \in [x], q \in [y]} [\mu_P(p) \land \mu_P(q)]
\]
\[
= \left( \bigwedge_{p \in [x]} \mu_P(p) \right) \land \left( \bigwedge_{q \in [y]} \mu_P(q) \right)
\]
\[
= \rho^* (\mu_P(x)) \land \rho^* (\mu_P(y)).
\]

\[
\rho^* (\nu_P(xy)) = \bigvee_{z \in [xy]} \nu_P(z)
\]
\[
\leq \bigvee_{z \in [x]_p [y]_q} \nu_P(pq)
\]
\[
\leq \bigvee_{p \in [x], q \in [y]} [\nu_P(p) \lor \nu_P(q)]
\]
\[
= \left( \bigvee_{p \in [x]} \nu_P(p) \right) \lor \left( \bigvee_{q \in [y]} \nu_P(q) \right)
\]
\[
= \rho^* (\nu_P(x)) \lor \rho^* (\nu_P(y)).
\]

Therefore, \( \rho^* (P) \) is a Pythagorean fuzzy subring of \( R \).
Hence, \(P\) is an upper rough Pythagorean fuzzy subring of \(R\).

(b) Let \(P = (\mu_P, \nu_P)\) be a Pythagorean subring of \(R\). We prove that 
\[
\rho_\ast(\mu_P)(xy) \geq \rho_\ast(\mu_P)(x) \lor \rho_\ast(\mu_P)(y)
\]
and 
\[
\rho_\ast(\nu_P)(xy) \leq \rho_\ast(\nu_P)(x) \land \rho_\ast(\nu_P)(y)
\]
for all \(x, y \in R\). Now, 
\[
\rho_\ast(\mu_P)(xy) = \bigwedge_{z \in [xy]_P} \mu_P(z)
\]
\[
= \bigwedge_{z \in [x]_P[y]_P} \mu_P(z)
\]
\[
\geq \bigwedge_{p \in [x]_P, q \in [y]_P} [\mu_P(p) \lor \mu_P(q)]
\]
\[
= \left( \bigwedge_{p \in [x]_P} \mu_P(p) \right) \lor \left( \bigwedge_{q \in [y]_P} \mu_P(q) \right)
\]
\[
= \rho_\ast(\mu_P)(x) \lor \rho_\ast(\mu_P)(y).
\]
\[
\rho_\ast(\nu_P)(xy) = \bigvee_{z \in [xy]_P} \nu_P(z)
\]
\[
= \bigvee_{z \in [x]_P[y]_P} \nu_P(z)
\]
\[
\leq \bigvee_{p \in [x]_P, q \in [y]_P} [\nu_P(p) \land \nu_P(q)]
\]
\[
= \left( \bigvee_{p \in [x]_P} \nu_P(p) \right) \land \left( \bigvee_{q \in [y]_P} \nu_P(q) \right)
\]
\[
= \rho_\ast(\nu_P)(x) \land \rho_\ast(\nu_P)(y).
\]
Therefore, \(\rho_\ast(P)\) is a Pythagorean fuzzy ideal of \(R\).

Hence, \(P\) is a lower rough Pythagorean fuzzy ideal of \(R\). \(\blacksquare\)

4. Conclusion

The notion of Pythagorean fuzzy sets is a relatively novel mathematical framework in the fuzzy family with higher ability to cope imprecision imbedded in decision-making. In this paper, we have studied the concept of PFS more expressly with relevant illustrations, where necessary. Some important remarks, which differentiated PFSs from IFSs, were drawn. It was observed that every IFS is PFS, but the converse is not always true. Some theorems on PFSs were deduced and proved, especially on the ideas of congruence relations of Pythagorean fuzzy sets. We also extended the concept of rough Pythagorean fuzzy set and prove some important theorem of rough Pythagorean fuzzy sets in a near-ring.

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