Multiplicity of bounded solutions to the \( k \)-Hessian equation with a Matukuma-type source

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Abstract

The aim of this paper is to deal with the \( k \)-Hessian counterpart of the Laplace equation involving a nonlinearity studied by Matukuma. Namely, our model is the problem

\[
\begin{aligned}
S_k(D^2 u) &= \lambda \frac{|x|^{\mu - 2}}{(1 + |x|^2)^{\frac{\mu}{2}}} (1 - u)^q \quad \text{in} \quad B,

u &< 0 \quad \text{in} \quad B,

u &= 0 \quad \text{on} \quad \partial B,
\end{aligned}
\]

where \( B \) denotes the unit ball in \( \mathbb{R}^n \), \( n > 2k \) \( (k \in \mathbb{N}) \), \( \lambda > 0 \) is an additional parameter, \( q > k \) and \( \mu \geq 2 \). In this setting, through a transformation recently introduced by two of the authors that reduces problem (1) to a non-autonomous two-dimensional generalized Lotka-Volterra system, we prove the existence and multiplicity of solutions for the above problem combining dynamical-systems tools, the intersection number between a regular and a singular solution and the super and subsolution method.

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1 Introduction

The classical Matukuma equation

\[\Delta u + \frac{u^q}{1 + |x|^2} = 0 \quad \text{in} \quad \mathbb{R}^3,\]

was proposed by T. Matukuma \[18\] as a mathematical model for a globular cluster of stars, where \( q > 1 \) is a parameter and \( u > 0 \) stands for the gravitational potential. This equation has been extensively studied in the literature, see e.g. \[1, 3, 4, 13, 17, 21\].

A more general model was proposed by J. Batt, W. Faltenbacher and E. Horst \[4\] which contains as a particular case the equation

\[\Delta u + \frac{|x|^\mu - 2}{(1 + |x|^2)^{\frac{\mu}{2}}} u^q = 0 \quad \text{in} \quad \mathbb{R}^3,\]

where \( \mu > 0 \) is an additional parameter, see e.g. \[4, 17\] and the references therein. Recently, an extensive study of solutions to the preceding equation has been extended to higher \((n > 3)\) dimensions, see \[30\]. It is well-known that this kind of equation admits three different types of positive radial solutions depending on the parameters \( \mu \), \( q \) and \( n \) (in case \( n > 3 \)). In particular, they admit the so-called \( E \)-solutions which are characterized by \( \lim_{r \to 0} u(r) < \infty \), see \[5, 30\].

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The aim of this paper is to study radially symmetric bounded solutions to the Matukuma equation in the framework of the \(k\)-Hessian operator. More precisely, we consider the question of the existence and multiplicity of radially symmetric bounded solutions of the problem

\[
\begin{cases}
S_k(D^2u) = \lambda \frac{|x|^{n-2}}{(1+|x|^2)^\frac{n}{2}} (1-u)^q & \text{in } \Omega, \\
u < 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1}
\]

where \(\lambda > 0\) is an additional parameter, \(q > k, \mu \geq 2\), and \(\Omega\) is a suitable bounded domain in \(\mathbb{R}^n\). We point out that answers to the above questions raised for (1) are, to our knowledge, unknown in the literature. We handle the existence and multiplicity of radially symmetric bounded solutions combining dynamical-systems tools with the approach of the intersection number between a regular solution and a singular solution. To this end, we first use a new transformation recently introduced in [24], which reduces the radial version of (1) (denoted \((P_\lambda)\)) to a two-dimensional non-autonomous Lotka-Volterra system (denoted \((MS_{q,\mu})\)). This non-autonomous system can be considered as an asymptotically autonomous system in the sense of Thieme [25]. Thus we focus on the corresponding limiting systems, particularly in case \(t \to -\infty\) (denoted \((LV S_{q,\mu})\)), which allows us to obtain two relevant exponents for system \((MS_{q,\mu})\): They are the Tso and Joseph-Lundgren type exponents. It is worth to mention that system \((LV S_{q,\mu})\) matches up with an autonomous Lotka-Volterra system obtained for studying problem (1) with a power weight on the right hand side equal to \(|x|^{n-2}\). See [24] for more details.

The flow of system \((MS_{q,\mu})\) is analyzed from the corresponding limiting systems; two critical points (denoted \((\hat{x}, \hat{y})\) and \(P_\lambda(n - 2 + \mu, 0)\)) of \((MS_{q,\mu})\), which also are critical points of \((LV S_{q,\mu})\), are the key to obtaining a singular solution and a bounded solution for problem \((P_\lambda)\). More precisely, we show that the orbits of \((MS_{q,\mu})\) starting at the critical point \(P_\lambda(n - 2 + \mu, 0)\) are characterized by the existence of bounded solutions to problem \((P_\lambda)\) (see Proposition 4.2 below). On the other hand, the orbits of \((MS_{q,\mu})\) starting at the critical point \((\hat{x}, \hat{y})\) yield with existence of a singular solution to \((P_\lambda)\) for some \(\lambda > 0\) (denoted \(\hat{\lambda}\)). This new parameter \(\lambda\) is essential to obtaining the multiplicity of radially symmetric bounded solutions to (1).

A general existence result of solutions to (1) is obtained basically by the super and subsolutions method. See Section 3.

The paper is organized as follows. In Section 2 we briefly describe the \(k\)-Hessian operator and introduce some basic definitions. Theorems 2.1 and 2.2 which are our main results, are established in this section. In Section 3 we prove a general existence result of classical solutions of (1) (see Lemma 3.1) and we use this result to prove Theorem 2.1. In Section 4 we obtain a proper non-autonomous Lotka-Volterra System from which we construct a singular solution, using the contraction mapping theorem. In Section 5 we study the intersection number between a regular solution and a singular solution of suitable equations. Finally, in Section 6 we prove Theorem 2.2.

2 Preliminaries and main results

The \(k\)-Hessian operator \(S_k\) is defined as follows. Let \(k \in \mathbb{N}\) and let \(\Omega\) be a suitable bounded domain in \(\mathbb{R}^N\). Let \(u \in C^2(\Omega), 1 \leq k \leq n\), and let \(\Lambda = (\lambda_1, \lambda_2, ..., \lambda_n)\) be the eigenvalues of the Hessian matrix \((D^2u)\). Then the \(k\)-Hessian operator is given by the formula

\[S_k(D^2u) = P_k(\Lambda) = \sum_{1 \leq i_1 < ... < i_k \leq n} \lambda_{i_1} ... \lambda_{i_k},\]

where \(P_k(\Lambda)\) is the \(k\)-th elementary symmetric polynomial in the eigenvalues \(\Lambda\). This operator has a long history, see e.g. [3, 9, 20, 28, 29, 33, 34] and the references therein. Note that they include
the usual Laplace operator ($k = 1$). Recently, this class of operators has attracted renewed interest, see e.g. [7, 12, 14, 15, 22, 23, 31, 32, 35, 36].

Let $\Omega = B$ be the unit ball in $\mathbb{R}^n$, which is an admissible domain for $S_k$. Then the $k$-Hessian operator when acting on radially symmetric functions can be written as $S_k(D^2 u) = c_{n,k} r^{1-n} (r^{n-k}(u')^k)'$, where $r = |x|$, $x \in \mathbb{R}^n$ and $c_{n,k}$ is defined by $c_{n,k} = \binom{n}{k}/n$.

Thus we can write (1) in radial coordinates, i.e.,

\[
\begin{cases}
    c_{n,k} r^{1-n} (r^{n-k}(u')^k)' = \lambda \frac{r^{n-2}}{(1+r^2)^2} (1-u)^q, & 0 < r < 1, \\
    u(r) < 0, & 0 \leq r < 1, \\
    u'(0) = 0, & u(1) = 0.
\end{cases}
\]

We introduce the space of functions $\Phi_k^0$ defined on $(0,1)$ for problem $(P_\lambda)$:

\[
\Phi_k^0 = \{ u \in C^2((0,1)) \cap C^1([0,1]) : (r^{n-i}(u')^i)' \geq 0 \text{ in } (0,1), i = 1, ..., k, u'(0) = u(1) = 0 \}.
\]

Note that the functions in $\Phi_k^0$ are non-positive on $[0,1]$. However, if $(r^{n-i}(u')^i)' > 0$ for every $i = 1, \ldots, k$, then every function in $\Phi_k^0$ is negative and strictly increasing on $(0,1)$. This in turn implies, as we are looking for solutions of $(P_\lambda)$ in $\Phi_k^0$, that the parameter $\lambda$ must be positive.

**Definition 2.1.** Let $\lambda > 0$. We say that a function $u \in C([0,1])$ is:

(i) a classical solution of $(P_\lambda)$ if $u \in \Phi_k^0$ and the first equality in $(P_\lambda)$ holds;

(ii) an integral solution of $(P_\lambda)$ if $u$ is absolutely continuous on $(0,1)$, $u(1) = 0$, $\int_0^1 r^{n-k}(u'(r))^{k+1} dr < \infty$ and the equality

\[
c_{n,k} r^{n-k}(u'(r))^k = \lambda \int_0^r s^{n-1} \frac{s^{\mu-2}}{(1+s^2)^2} (1-u(s))^q ds, \text{ a.e. } r \in (0,1),
\]

holds whenever the integral exists.

The concept of integral solution was introduced in [10] for a more general class of radial operators, see e.g. [10] and the references therein. The standard concept of weak solution is equivalent in this case to the notion of integral solution, see [10, Proposition 2.1].

We recall the version of the method of super and subsolutions for (1), see [33, Theorem 3.3] for more details.

**Definition 2.2.** A function $u \in \Phi^k(B) := \{ u \in C^2(B) \cap C(\overline{B}) : S_i(D^2 u) \geq 0 \text{ in } B, i = 1, ..., k \}$ is called a subsolution (resp. supersolution) of (1) if

\[
\begin{cases}
    S_k(D^2 u) \geq 0 \text{ (resp. $\leq$)} & \frac{|x|^{n-2}}{(1+|x|^2)^2} (1-u)^q \text{ in } B, \\
    u \leq 0 \text{ (resp. $\geq$)} & \text{ on } \partial B.
\end{cases}
\]

Note that the trivial function $u \equiv 0$ is always a supersolution.

The following concept is needed to establish a general result on the existence of solutions to problem (1).

**Definition 2.3.** We say that a function $v$ is a maximal solution of (1) if $v$ is a solution of (1) and, for each subsolution $u$ of (1), we have $u \leq v$.

This notion of maximal solution was recently introduced in [24] to prove existence results, see also [24].

3
Now we state our first main result concerning the existence and non-existence of solutions to problem \((P_\lambda)\).

**Theorem 2.1.** Let \(n > 2k\), \(q > k\) and \(\mu \geq 2\). There exists \(\lambda^* > 0\) such that problem \((P_\lambda)\) admits a maximal bounded solution for \(\lambda \in (0, \lambda^*)\), at least one possibly unbounded integral solution for \(\lambda = \lambda^*\) and no classical solutions for all \(\lambda > \lambda^*\). Additionally,

\[
\lambda^* \geq d(\mu) \left( \frac{n}{k} \right) \left( \frac{2k}{q-k} \right)^k \left( \frac{q-k}{q} \right)^q,
\]

where the positive constant \(d(\mu)\) is given by

\[
d(\mu) = \begin{cases} 
\frac{1}{(\frac{q}{2})^\frac{n}{2}} & \text{if } \mu = 2, \\
\frac{1}{(\frac{q}{2})^\frac{n}{2}} \frac{1}{(\frac{q}{2})^\frac{\mu-2}{2}} & \text{if } 2 < \mu \leq 4, \\
\frac{1}{(\frac{q}{2})^\frac{n}{2}} \frac{1}{(\frac{q}{2})^\frac{\mu-2}{2}} & \text{if } \mu > 4.
\end{cases}
\]

Next, in order to state our second main result we introduce two relevant exponents. Let \(\sigma \geq 0\). From now on we shall denote by

\[
q^*(k, \sigma) = \frac{(n+2)k + \sigma(k+1)}{n-2k}
\]

and

\[
q_{\text{JL}}(k, \sigma) := \begin{cases} 
\frac{k(k+1)n-k^2(2-\sigma)+2k+\sigma-2\sqrt{k(2k+\sigma)((k+1)n-k(2-\sigma))}}{k(k+1)n-2k^2(k+3)-2k\sigma-2\sqrt{k(2k+\sigma)(k+1)n-k(2-\sigma))}}, & n > 2k + 8 + \frac{4\sigma}{k}, \\
\infty, & 2k < n \leq 2k + 8 + \frac{4\sigma}{k},
\end{cases}
\]

the Tso and Joseph-Lundgren type exponents, respectively.

The generalized Joseph-Lundgren exponent, \(q_{\text{JL}}(k, \sigma)\), was recently obtained in [24] in connection with the multiplicity of radial bounded solutions of a \(k\)-Hessian equation involving a weight of the form \(|x|^{\sigma}\). We point out that, for \(k = 1\) and \(\sigma = 0\), \(q_{\text{JL}}(1,0)\) coincides with the classical Joseph-Lundgren exponent \(10\). When \(k > 1\) and \(\sigma = 0\), this exponent also appears in a large class of problems with nonlinear radial operators including the usual Laplace, \(p\)-Laplace and \(k\)-Hessian operators [19, 20]. See also [23] for the case of the \(k\)-Hessian operator.

Now we state our second main result.

**Theorem 2.2.** Let \(n > 2k\), \(q > k\) and \(\mu \geq 2\). Assume that \(q^*(k, \mu - 2) < q < q_{\text{JL}}(k, \mu - 2)\). Then there exists a positive constant \(\bar{\lambda} < \lambda^*\) such that for each \(N \geq 1\), there is an \(\varepsilon > 0\) such that if \(|\lambda - \bar{\lambda}| < \varepsilon\), then \((P_\lambda)\) has at least \(N\) solutions. In particular, if \(\lambda = \bar{\lambda}\), then \((P_\lambda)\) has infinitely many solutions.

It is remarkable that the exponents \(q^*(k, \mu - 2)\) and \(q_{\text{JL}}(k, \mu - 2)\) have the same role for problem \((P_\lambda)\) with different types of weights on the right hand side, either \(\frac{r^{\mu-2}}{(1+r^2)^\frac{\mu}{2}}\) or \(r^{\mu-2}\), see [24] Theorem 3.1 (I) for the last case. On the other hand, even though the corresponding parameter \(\bar{\lambda}\) also plays the same role in both cases, this value of the parameter \(\lambda\) is different in each case. See Lemma 4.4 below for the definition of \(\bar{\lambda}\), and see [24] Theorem 3.1 (I) for its definition in case of \(r^{\mu-2}\). In Section 4 we discuss further these relationships.

3 Existence and non-existence of solutions of problem \((P_\lambda)\)

In this section we prove a general existence result of classical solutions of problem \((P_\lambda)\). We begin with
Lemma 3.1. Let \( n > 2k, q > k, \mu \geq 2 \) and \( \lambda_0 > 0 \). Assume that there exists a classical solution of

\[
\begin{cases}
c_nkr^{1-n} (r^{n-k}(w')^k)' = \lambda_0 \frac{x^{\mu-2}}{(1+|x|^2)^{\frac{q}{2}}} (1-w)^q, & 0 < r < 1, \\
w < 0, & 0 \leq r < 1, \\
w'(0) = 0, w(1) = 0.
\end{cases}
\]

Then, for every \( \lambda \in (0, \lambda_0) \), problem \( (P_\lambda) \) has a classical maximal bounded solution. Moreover, the classical maximal bounded solutions form a decreasing sequence as \( \lambda \) increases.

Proof. Fix \( \lambda \in (0, \lambda_0) \) and define the functions

\[
g(t) = [\lambda_0(1+t)^q]^{1/k} \quad \text{and} \quad \tilde{g}(t) = [\lambda(1+t)^q]^{1/k}, \quad \text{for all} \quad t \geq 0.
\]

Set \( \Phi(s) = \tilde{h}^{-1}(h(s)) \) (\( s \leq 0 \)) with \( h \) and \( \tilde{h} \) given by

\[
h(s) = \int_s^0 \frac{1}{g(-t)} \, dt \quad \text{and} \quad \tilde{h}(s) = \int_s^0 \frac{1}{\tilde{g}(-t)} \, dt, \quad s \leq 0.
\]

Since \( q > k \), \( \lim_{s \to -\infty} h(s) \) exists and hence \( \Phi \) is bounded by \([23, \text{Lemma 2.1 (i)-(ii)}]\). Next, by \([2]\) and the convexity of \( \Phi \) \([23, \text{Lemma 2.1 (iii)}]\), we have

\[
S_k(D^2\Phi(w)) = c_nkr^{1-k}(\Phi'(w)w')^{k-1} \left( \Phi''(w)(w')^2 + \Phi'(w)w'' + \frac{n-k}{k} \Phi'(w)w' \right)
\geq c_nkr^{1-k}(\Phi'(w)w')^{k-1} \left( w'' + \frac{n-k}{k} w' \right)
= (\Phi'(w))^k S_k(D^2w) = \frac{(\tilde{g}(\Phi(w)))^k}{(\tilde{g}(-w))^k} S_k(D^2w) = \lambda \frac{x^{\mu-2}}{(1+|x|^2)^{\frac{q}{2}}} (1-\Phi(w))^q.
\]

Therefore \( \Phi(w) \) is a bounded subsolution of \( (P_\lambda) \) and thus, by the method of super and subsolutions, we have, by \([33, \text{Theorem 3.3}]\), a solution \( u \in L^\infty((0,1)) \) of \( (P_\lambda) \) with \( \Phi(w) \leq u \leq 0 \). Now, to prove that \( (P_\lambda) \) admits a maximal solution, we consider \( u_1 \) as the solution of

\[
(Q) \quad \begin{cases}
S_k(D^2u_1) = \lambda \frac{|x|^{\mu-2}}{(1+|x|^2)^{\frac{q}{2}}} \quad \text{in} \quad B, \\
u_1 = 0 \quad \text{on} \quad \partial B.
\end{cases}
\]

Note that \( u_1 \in \Phi^k_0(B) \) since \( \mu \geq 2 \). As \( u \) is in particular a subsolution of \( (Q) \), we have \( u \leq u_1 \) on \( B \) by the comparison principle \([27]\). Next, we define \( u_i \) (\( i = 2, 3, \ldots \)) as the solution of

\[
S_k(D^2u_i) = \lambda \frac{|x|^{\mu-2}}{(1+|x|^2)^{\frac{q}{2}}} (1-u_{i-1})^q \quad \text{in} \quad B, \\
u_i = 0 \quad \text{on} \quad \partial B.
\]

Note that \( u_i \in \Phi^k_0(B) \) (\( i = 2, 3, \ldots \)) since \( \mu \geq 2 \). Using again the comparison principle we obtain an increasing sequence \( \{u_i\} \), which is bounded from below by \( u \) and by 0 from above. Hence, we can pass to the limit to obtain a classical solution \( u_{\max} \) of \( (P_\lambda) \), which is maximal since the recursive sequence \( \{u_i\} \) does not depend on the subsolution \( u \). Now let \( \lambda_1 < \lambda_2 \) and \( u_{\lambda_1}, u_{\lambda_2} \) be maximal solutions of \( (P_\lambda) \) (\( i = 1, 2 \)), respectively. Since \( u_{\lambda_2} \) is a subsolution of \( (P_\lambda_1) \), we have \( u_{\lambda_2} \leq u_{\lambda_1} \) by the maximality of \( u_{\lambda_1} \).
### 3.1 Proof of Theorem 2.1

Fix $\mu \geq 2$. For $R > 1$, let $B_R$ be a ball centered at zero with radius $R$ such that $\overline{B} \subset B_R$, and let $\eta$ be the solution of

$$
\begin{cases}
S_k(D^2 \eta) = 1 & \text{in } B_R, \\
\eta = 0 & \text{on } \partial B_R.
\end{cases}
$$

Then there exists a constant $\beta$ such that $\eta < \beta < 0$ on $\partial B$. Set $M = \max_{x \in \overline{B}} |\eta(x)|$, $C = C(\mu) = \max_{r \in [0,1]} \frac{x^{\mu-2}}{(1+r^2)^{\frac{\mu}{2}}(1+|x|^2)^{\frac{1}{2}}} > 0$ and take $\lambda < C^{-1}(1 + M)^{-\frac{\mu}{2}}$. Then

$$
S_k(D^2 \eta) = 1 > \lambda C (1 + M)^{\frac{\mu}{2}} \geq \lambda C (1 - \eta)\frac{|x|^{\mu-2}}{(1 + |x|^2)^{\frac{1}{2}}} (1 - \eta)^{\frac{\mu}{2}} \text{ in } B.
$$

By [33, Theorem 3.3], for every $\lambda \in (0, C^{-1}(1 + M)^{-\frac{\mu}{2}})$ there exists a solution $u_\lambda$ of $(P_\lambda)$. Thus we may define $\lambda^* = \sup\{\lambda > 0 : \text{there is a solution } u_\lambda \in C^2(B) \text{ of (1)}\}$. (3)

Then $\lambda^* > 0$.

To see that $\lambda^*$ is finite, we consider the inequality

$$
\Delta u \geq C(n, k)[S_k(D^2 u)]^{\frac{1}{n}},
$$

which holds for every $u \in \Phi^k(B)$, see e.g. [33, Proposition 2.2, part (4)] and the comments therein. Consider now the eigenvalue problem

$$
(E_m) \begin{cases}
-\Delta u = \lambda m(x)u & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
$$

where $m(x) := h(|x|)^{\frac{1}{k}}$. It is known that problem $(E_m)$ has a first eigenvalue, $\lambda_{1,m} > 0$, associated with an eigenfunction $\phi_{1,m} > 0$, see e.g. [2, Theorem 0.6]. It is not difficult to see that there exists a constant $M > 0$ such that for every $u < 0$, we have $(1 - u)^{\frac{1}{k}} \geq M |u|^{\frac{1}{C(n,k)}}$. Let $\lambda \in (0, \lambda^*)$ and let $u$ be a solution of problem $(P_\lambda)$. Then, using (4), we obtain

$$
\lambda_{1,m} \int_B |u|m(x)\phi_{1,m} = - \int_B \Delta \phi_{1,m}|u| = - \int_B Du D\phi_{1,m} \geq M \lambda^{\frac{1}{k}} \int_B |u|m(x)\phi_{1,m},
$$

which in turn implies that $\lambda \leq \left(\frac{\lambda_{1,m}}{M}\right)^{k}$. Thus $\lambda^*$ is finite.

Now, let $\lambda \in (0, \lambda^*)$. Then $u_\lambda$ is a maximal bounded solution of $(P_\lambda)$ by Lemma 3.1 applied to $\lambda_0 \in (\lambda, \lambda^*)$.

Next let $\lambda_i$ be an increasing sequence such that $\lambda_i \to \lambda^*$ as $i \to +\infty$ and let $u_{\lambda_i}$ be a maximal solution of $(P_{\lambda_i})$. By Lemma 3.1, for all $r \in [0,1]$ we have $u_{\lambda_{i+1}}(r) \leq u_{\lambda_i}(r) \leq 0$. On the other hand, integrating the first equation in $(P_{\lambda_i})$, we obtain

$$
u_{\lambda_i}(r) = - \int_1^r \left[ c_{n,k}^{-1} r^{-k-n} \int_0^r s^{n-1} \frac{s^{\mu-2}}{(1+s^2)^{\frac{\mu}{2}}} \lambda_i (1-u_{\lambda_i}(s))^{\frac{\mu}{2}} ds \right]^{\frac{1}{k}} dr.
$$

Applying the monotone convergence theorem twice, we conclude that

$$
u^*(r) := \lim_{i \to +\infty} u_{\lambda_i}(r), \text{ exists a.e. } r \in (0,1)
$$
To obtain a Lotka-Volterra system, we set

\[ u = \frac{1}{C} \int_0^r \left[ \int_0^s \frac{\lambda^* (1 - u^*(s))^q}{1 + s^2} ds \right]^{\frac{1}{q}} dr, \quad a.e. \ r \in (0, 1). \]

The assertion concerning the non-existence of solutions follows directly from the definition of \( \lambda^* \).

Next we obtain a lower bound for \( \lambda^* \) in Theorem 2.1. The function \( v(x) = \frac{1}{q-\mu} \left( |x|^2 - 1 \right) \) satisfies

\[ S_k(D^2 v) = n c_{n,k} (2k)^k (q-k)^{-k} \geq \binom{n}{k} (2k)^k \left( \frac{q-k}{q} \right)^{-q} (1-v)^q \geq C^{-1} \binom{n}{k} (2k)^k \left( \frac{q-k}{q} \right)^{-q} h(r)(1-v)^q, \]

where the constant \( C = C(\mu) \) is given by

\[ C = \max_{r \in [0,1]} h(r) = \max_{r \in [0,1]} \frac{r^{\mu-2}}{(1 + r^2)^{\frac{\mu}{2}}} = \begin{cases} 1 & \text{if } \mu = 2, \\ \frac{1}{2} & \text{if } 2 < \mu \leq 4, \\ 0 & \text{if } \mu > 4. \end{cases} \]

Hence \( v \) is a subsolution of \( (P_{\lambda}) \) for all \( \lambda \leq C^{-1} \binom{n}{k} (2k)^k \left( \frac{q-k}{q} \right)^{-q} \). Since \( v_0 \equiv 0 \) is a supersolution and \( v \leq v_0 \), for any such \( \lambda \) there exists a solution of \( (P_{\lambda}) \). By the first statement of Theorem 2.1, this shows that

\[ \lambda^* \geq C^{-1} \binom{n}{k} \left( \frac{2k}{q-k} \right)^k \left( \frac{q-k}{q} \right)^q. \]

Setting \( d(\mu) = C^{-1} \) we conclude the proof of Theorem 2.1. \( \square \)

4 Existence of a singular solution of \( (P_{\lambda}) \)

In this section we obtain a singular solution of \( (P_{\lambda}) \) which is derived from a proper non-autonomous Lotka-Volterra system.

4.1 A non-autonomous Lotka-Volterra system

We start considering the radial problem

\[
\begin{align*}
(r^{n-k}(u')^k)' &= c_{n,k}^{-1} r^{n-1} f(r, u), \quad 0 < r < 1, \\
u(r) &< 0, \quad 0 \leq r < 1, \\
u'(0) &= 0, \quad u(1) = 0.
\end{align*}
\]

Let \( u \) be a solution of (5) and set \( w = u - 1 \). Then \( w \) is a solution of

\[
\begin{align*}
(r^{n-k}(u')^k)' &= r^{n-1} c_{n,k}^{-1} f(r, w + 1), \quad (r > 0).
\end{align*}
\]

To obtain a Lotka-Volterra system, we set

\[
\begin{align*}
x(t) &= r^k c_{n,k}^{-1} f(r, w + 1), \quad y(t) = r w - w', \quad r = e^t, \\
f(r, w + 1) &= \lambda h(r)(-w)^q.
\end{align*}
\]

where \( w' = dw/dr \). We point out that this change of variable is well-known in the case \( k = 1 \), see e.g. [5] [6] [30] [37]. In the framework of the \( k \)-Hessian operator, this transformation has been recently introduced in [24]. Further, for

\[ f(r, w + 1) = \lambda h(r)(-w)^q \]
we see that such a change of variables becomes optimal for equation (6) since, depending on the weight $h$, we obtain either an autonomous or a non-autonomous Lotka-Volterra system. More precisely, after some calculation one can see that the pair of functions $(x(t), y(t))$ solves the following non-autonomous Lotka-Volterra system:

$$(LV S_{q,\rho}) \quad \begin{cases}
\frac{dx}{dt} = x [\rho(t) - x - qy], \\
\frac{dy}{dt} = y \left[ -\frac{n-2k}{k} + \frac{1}{k} + y \right],
\end{cases}$$

where $\rho(t) = n + r \frac{h'(r)}{h(r)}$ and $r = e^t$.

Note that we can recover the function $w$ by the formula

$$w = - \left[ e_{n,k}^{-1} \lambda r^2 h(r) \right]^{-\frac{1}{\gamma}} (xy^k)^{\frac{1}{\gamma}}. \quad (9)$$

Now, in order to transform the problem $(P\lambda)$ into the system $(LV S_{q,\rho})$, we set

$$h(r) = \frac{r^{n-2}}{(1 + r^2)^{\frac{n}{2}}}$$

thus obtaining the non-autonomous dynamical system:

$$(MS_{q,\rho}) \quad \begin{cases}
\frac{dx}{dt} = x \left[ n - 2 + \frac{\mu}{1 + e^t} \right] - x - qy, \\
\frac{dy}{dt} = y \left[ -\frac{n-2k}{k} + \frac{1}{k} + y \right].
\end{cases}$$

Since the limits $\lim_{t \to \pm \infty} \rho(t) = \lim_{t \to \pm \infty} (n - 2 + \frac{\mu}{1 + e^t}) =: \rho_{\pm}$ exist, we may consider the system $(MS_{q,\rho})$ as an asymptotically autonomous system in the sense of Thieme (see [25]). Thus we can describe the flow of $(MS_{q,\rho})$ from the autonomous systems $(LV S_{q,\rho_{\pm}})$.

### 4.2 The linearization of $(LV S_{q,\rho_{\pm}})$ at the stationary points

The following decompositions:

$$\rho(t) = n - 2 + \mu - \frac{\mu e^t}{1 + e^t}$$

$$(10)$$

and

$$\rho(t) = n - 2 + \frac{\mu e^t}{1 + e^t}.$$  

$$(11)$$

are useful when $t \to -\infty$ and $t \to +\infty$, respectively.

Now let $P = (a, b)$ be a stationary point of $(LV S_{q,\rho_{\pm}})$, if we introduce the coordinates $\tilde{x} := x - a$ and $\tilde{y} := y - b$, then using e.g. (10) we can write $(MS_{q,\rho})$ as a time-dependent perturbation of $(LV S_{q,\rho_{\pm}})$:

$$(12)$$

$$\begin{pmatrix}
\frac{d\tilde{x}}{dt} \\
\frac{d\tilde{y}}{dt}
\end{pmatrix} = A \begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix} + \begin{pmatrix}
-x^2 - qxy \\
\frac{\bar{q} \tilde{x}}{\bar{k}} + \tilde{y}^2
\end{pmatrix} + \begin{pmatrix}
-(\tilde{x} + a) \left( \frac{\mu e^t}{1 + e^t} \right) \\
0
\end{pmatrix}$$

with

$$A := \begin{pmatrix}
-n + 2 + \mu - 2a - qb & -qa \\
\frac{k}{k} & \frac{\bar{q} + 2b - \frac{n-2k}{k}}{\bar{k}}
\end{pmatrix},$$

Case $(LV S_{q,\rho_{\pm}})$: The critical points of $(LV S_{q,\rho_{\pm}})$ are $P_1(0,0)$, $P_2(0,\frac{n-2k}{k})$, $P_3(n - 2 + \mu, 0)$, and

$$(13)$$

$$\begin{pmatrix}
q(n - 2k) - k(n - 2 + \mu), \\
\frac{2k - 2 + \mu}{q - k}
\end{pmatrix} := (\tilde{x}, \tilde{y}).$$
Note that, under the assumptions $2k < n$, $k < q$ and $2 \leq \mu$, the first three critical points belong to $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ and they are saddle points. The fourth critical point $(\hat{x}, \hat{y})$ belongs to the interior of $\mathbb{R}_+^2$ if, and only if, $q > k(n - 2 + \mu)/(n - 2k)$. Further, $(\hat{x}, \hat{y})$ is a stable node for $q > q^*(k, \mu - 2) = (n+2k+\mu-2)(k+1)$. It is not difficult to see that the (bounded) orbit $(x(t), y(t))$ of $(LV S_{q,\rho})$ starts at $P_3(n - 2 + \mu, 0)$. See [24].

Case $(LV S_{q,\rho})$: Considering the decomposition of $\rho(t)$ as in (14), we obtain
\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{pmatrix} = A \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
-x^2 - qxy \\
x^2 + y^2
\end{pmatrix}
\]
with
\[
A := \begin{pmatrix}
1 - 2a - q^2 & 2a - q^2 \\
\frac{b}{k} & \frac{2b - n - 2k}{k}
\end{pmatrix}
\]

The critical points of $(LV S_{q,\rho})$ are $P_1(0, 0), P_2(0, \frac{n - 2k}{k}), P_3(n - 2, 0)$, and
\[
\begin{pmatrix}
\frac{q(n - 2k) - k(n - 2)}{q - k} & \frac{2k - 2}{q - k}
\end{pmatrix} := (\hat{x}, \hat{y}).
\]

We point out that $(LV S_{q,\rho})$ has four critical points whenever that $k > 1$. In the case $k = 1$ the critical point $(\hat{x}, \hat{y})$ coincides with $P_3(n - 2, 0)$. Further, the first three critical points are saddle points and $(\hat{x}, \hat{y})$ is a stable focus provided that $q > q^*(k, -2)$ where
\[
q^*(k, -2) := \frac{nk - 2}{n - 2k}.
\]

Note that $q^*(k, -2) > k(n - 2)/(n - 2k)$. Further, for all $q \geq k(n - 2)/(n - 2k)$, $(\hat{x}, \hat{y})$ belongs to $\mathbb{R}_+^2$ and for $q \geq q^*(k, \mu - 2)$ both critical points $(\hat{x}, \hat{y})$ and $(\hat{x}, \hat{y})$ belong to the interior of $\mathbb{R}_+^2$.

### 4.3 The flow of $(MS_{q,\mu})$

Note that $\rho(t) = n - 2 + \mu - \frac{\mu q^2}{\mu + q}$ is strictly decreasing from $\rho(-\infty) = n - 2 + \mu$ to $\rho(+\infty) = n - 2$ on $\mathbb{R} := [-\infty, +\infty]$. We define
\[
S(t, x, y) := \rho(t) - x - qy, \quad t \in \mathbb{R},
\]
\[
W(x, y) := \frac{n - 2k}{k} + \frac{x}{k} + y
\]
and we write $(MS_{q,\mu})$ in the form
\[
x' = xS(t, x, y) \\
y' = yW(x, y).
\]

For functions $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, we let
\[
F_0 := F^{-1}\{0\}, \quad F_+ := F^{-1}\{\mathbb{R}^+\}, \quad F_- := F^{-1}\{\mathbb{R}^-\}.
\]

Now $S_0(t), \quad t \in \mathbb{R}$, is the straight line $y = -\frac{1}{q}x + \frac{\rho(t)}{q}$ with intercepts $(\rho(t), 0)$ and $(0, \frac{\rho(t)}{q})$. It moves downward in the closed strip $Z$ between $S_0(-\infty)$ and $S_0(+\infty)$ when $t$ runs from $-\infty$ to $+\infty$. On the other hand, $W_0$ is the fixed straight line $y = \frac{n - 2k}{k} - \frac{x}{k}$. Furthermore, we define
\[
G(x, y) := x + \frac{(n - 2k)(q + 1)}{k - 1} \left( \frac{k}{n - 2k}y - 1 \right).
\]

Note that $G_0$ is the straight line $y = -\frac{k + 1}{k(q + 1)}x + \frac{n - 2k}{k}$ with intercepts $\left( \frac{(n - 2k)(q + 1)}{k + 1}, 0 \right)$ and $(0, \frac{n - 2k}{k})$. 

\[9\]
Lemma 4.1. Let \( \varphi \) be a solution of \( (MS_{q,\mu}) \) on \( (T_0, T) \). Let \( q \geq q^*(k, \mu - 2) \).

i) If \( \varphi(t_0) \in G_- \cup G_0 \) for \( t_0 \geq -\infty \), then \( \varphi(t) \in G_- \) for \( t > t_0 \).

ii) If \( \varphi(t_0) \in G_+ \cup G_0 \) for \( t_0 > -\infty \), then \( \varphi(t) \in G_+ \) for \( -\infty \leq t < t_0 \).

Proof. We have

\[
\frac{d}{dt} G(\varphi(t)) = x' + \frac{k(q+1)}{k+1} y'
\]

\[
= x(\rho(t) - x - qy) + \frac{k(q+1)}{k+1} y \left( -\frac{n-2k}{k} + \frac{x+y}{k} \right).
\]

If \( \varphi(t) \in G_0 \), then \( -x = \left( \frac{(n-2k)(q+1)}{k+1} \right) \left( \frac{k}{n-2\pi} y - 1 \right) \), whence

\[
\frac{d}{dt} G(\varphi(t))|_{\varphi(t)\in G_0} = x \left[ \rho(t) + \frac{k(q+1)}{k+1} y - \frac{(n-2k)(q+1)}{k+1} - (q+1) y + \frac{q+1}{k+1} y \right]
\]

\[
= x \left[ \rho(t) - \frac{(n-2k)(q+1)}{k+1} \right] < 0 \text{ for all } t.
\]

Next we link the solution of the initial value problem

\[
\begin{aligned}
(x^n - k(w')^k)' &= r^{n-k} \frac{1}{c_{n,k} \lambda} \frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}} (-w')^q, \quad r > 0, \\
w(0) &= w_0 \in (-\infty, 0), \\
w'(0) &= 0
\end{aligned}
\]

with a property of the orbits of system \( (MS_{q,\mu}) \).

Proposition 4.2. Suppose that \( q \geq q^*(k, \mu - 2) \). Then there exists a unique global solution \( w \) of \( (10) \) in the regularity class \( C^2(0, \infty) \cap C^1[0, \infty) \). Furthermore, the function \( w \) defined by \( (9) \) is the unique solution of \( (10) \) if, and only if, the orbit \( (x(t), y(t)) \) of system \( (MS_{q,\mu}) \) given by \( (7) \) starts at the point \( P_3(n-2+\mu, 0) \).

Proof. Let \( w_0 \) be an arbitrary negative number. Defining \( B(r) = \int_0^r s^{\frac{\alpha-\beta(s+1)}{\beta + 1}} (a(s)s^\gamma)ds, r > 0 \), with \( \alpha = n-k, \beta = k, \gamma = n-1, a(s) = c_{n,k}^\frac{1}{\lambda} \frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}} \) and \( \theta = [(\gamma+1)(\beta+1) - (\alpha-\beta)(q+1)]/(\beta+1) \), we obtain \( B(r) \leq 0 \) for \( r > 0 \) if, and only if, \( q \geq q^*(k, \mu - 2) \). Then the global existence of \( (16) \) follows from [11] Theorem 4.1. The uniqueness follows by a simple application of the contraction mapping principle, as in [23].

Next, let \( w(r) \) be the solution of \( (10) \). By \( (7) \) and \( (11) \), the function \( y = y(t) \) satisfies

\[
\lim_{t \to -\infty} y(t) = \lim_{r \to 0^+} r \frac{w'(r)}{-w(r)} = 0
\]

and for \( x = x(t) \), we have \( \lim_{t \to -\infty} x(t) = \lim_{r \to 0} \lambda \frac{1}{c_{n,k} \lambda} \frac{r^{n-k}}{(1+r^2)^{\frac{n}{2}}} w(r), \) with \( h(r) = \frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}} \). Now, by the equation in \( (16) \) and L'Hôpital's rule, we have

\[
\lim_{r \to 0} \frac{r^k h'(r)}{w'(r)^k} = \lim_{r \to 0} \frac{r^k h'(r) + nr^{n-1} h(r)}{r^k w'(r)^k}, \quad \text{lim}_{r \to 0} \frac{r^k h(r)}{w'(r)^k} = \frac{\rho(0)}{c_{n,k} \lambda} = \frac{n-2+\mu}{c_{n,k} \lambda}.
\]
Thus
\[ \lim_{t \to -\infty} x(t) = n - 2 + \mu. \] (18)

From (17) and (18) we conclude that
\[ \lim_{t \to -\infty} (x(t), y(t)) = (n - 2 + \mu, 0) = P_3(n - 2 + \mu, 0). \] (19)

Conversely, suppose that \( (x(t), y(t)) \to P_3(n - 2 + \mu, 0) \) as \( t \to -\infty \). Restricting the general linearization (12) to \( \tilde{x} = x - \rho_- \) with \( \rho_- = n - 2 + \mu \) and \( \tilde{y} = y \), we obtain

\[
\begin{cases}
\tilde{x}' = -\rho_- \tilde{x} - q \rho_- y - \tilde{x}^2 - q \tilde{x} y - \mu(\tilde{x} + \rho_-) g(t), \\
\tilde{y}' = y \left( \tilde{\mu} + \frac{\tilde{x}}{k} + y \right),
\end{cases}
\] (20)

where \( g(t) := \frac{e^{2t}}{1 + e^t} \) and \( \tilde{\mu} := \frac{2k + \mu - 2}{k} \). Define \( \varepsilon(t) := \frac{\tilde{x}(t)}{k} + y(t) \). Then, for some \( t_0 \)

\[ y(t) = y(t_0) e^{-\tilde{\mu} t} e^{\tilde{\mu} t + f_t \varepsilon(s) ds}. \]

Since \( \varepsilon(t) \to 0 \) as \( t \to -\infty \), there exists \( \delta > 0 \) small enough such that \( |\varepsilon(t)| \leq \delta < \tilde{\mu} \) for \( t \leq t_0 \). Thus

\[ u(t) \leq C_0 e^{(\tilde{\mu} - \delta) t}, \quad C_0 := y(t_0) e^{-(\tilde{\mu} - \delta) t_0}. \]

If \( \varepsilon \) were integrable at \(-\infty\), then for some \( C > 0 \),

\[ y(t) = C e^{\tilde{\mu} t} e^{\int_{-\infty}^t \varepsilon(s) ds}, \quad t \leq t_0. \] (21)

Since \( \bar{x}(t) \to 0 \), we may assume \( \frac{\tilde{\mu} k}{2} + \bar{x} \geq 0 \). By (20), we have

\[
\frac{1}{2} (\bar{x}^2)' = - (\rho_- + \bar{x}) \bar{x}^2 - q \bar{x}^2 y - \mu \bar{x}^2 g(t) - \rho_- (qy + \mu g(t)) \bar{x}
\leq - (\rho_- + \bar{x}) \bar{x}^2 + \frac{\rho_-}{2} [(qy + \mu g(t))^2 + \tilde{x}^2]
\leq \rho_- (q^2 y^2 + \mu^2 g^2(t)).
\]

Then
\[
\bar{x}^2(t) \leq 2 \rho_- \left( q^2 \int_{-\infty}^t y^2(s) ds + \mu^2 \int_{-\infty}^t g^2(s) ds \right)
\leq 2 \rho_- \left( \frac{q^2 C_0^2}{2(\tilde{\mu} - \delta)} e^{2(\tilde{\mu} - \delta) t} + \mu^2 e^{4t} \right).
\] (22)

Thus \( \bar{x}(t) = O(e^{\min(2, \tilde{\mu} - \delta) t}) \). We have shown that \( \varepsilon \) is integrable at \(-\infty\). Hence using (21)

\[
y(t) = C e^{\tilde{\mu} t} \left[ 1 + O \left( \int_{-\infty}^t \left( \frac{\bar{x}(s)}{k} + y(s) \right) ds \right) \right]
= C e^{\tilde{\mu} t} \left[ 1 + O(e^{2t}) \right].
\]

Together with (22) we get \( \bar{x}(t) = O(e^{2t}) \), and using (20)

\[
\begin{align*}
\bar{x}' &= - (n - 2 + \mu) \bar{x} - \mu (n - 2 + \mu) e^{2t} - q(n - 2 + \mu) C e^{\tilde{\mu} t} + O(e^{4t}) \\
&= - \mu (n - 2 + \mu) e^{(\mu + 1) t} - q(n - 2 + \mu) C e^{(k + 1) \mu + nk - 2} t + O(e^{(n + 2 + \mu) t})
\end{align*}
\]

\[
\begin{align*}
\bar{x}(t) &= - \frac{\mu(n - 2 + \mu)}{\mu + n} e^{2t} - \frac{q(n - 2 + \mu)}{(k + 1) \mu + nk - 2} C e^{\tilde{\mu} t} + O(e^{4t}) \\
x(t) &= n - 2 + \mu - \frac{\mu(n - 2 + \mu)}{\mu + n} e^{2t} - \frac{q(n - 2 + \mu)k}{(k + 1) \mu + nk - 2} C e^{\tilde{\mu} t} + O(e^{4t}),
\end{align*}
\]
and using (21),

\[ y(t) = Ce^{\mu t} \left[ 1 - \frac{\mu(n - 2 + \mu)}{2k(\mu + n)} e^{2t} - \frac{q(n - 2 + \mu) - [(k + 1)\mu + nk - 2]}{\mu[(k + 1)\mu + nk - 2]} Ce^{\mu t} + O(e^{4t}) \right]. \]

The previous expressions for \( x, y \) together with (11) imply that

\[ \left[-w(r)\right]^{q-k} = \frac{C_{n,k}}{\lambda}(1 + r^2) \frac{r}{2} e^{-2(2k+\mu-2)x} \ln r \left[y(\ln r)\right]^k \]

\[ \rightarrow \frac{C_{n,k}}{\lambda}(n - 2 + \mu) C^k := \left[-w_0\right]^{q-k} \ (r \to 0). \]

On the other hand, differentiating the function in (9) with respect to \( t \), we have

\[ w'(r) = \frac{1}{q-k} \frac{w(r)}{r} \left[-2 + \frac{\mu}{1+e^{2t}} - (q-k)y(t) - r \frac{h'(r)}{h(r)} \right] \]

\[ = -w(r) \frac{y(\ln r)}{r} \rightarrow 0 \ (r \to 0). \]

Hence the function \( w \) defined by (9) is the unique solution of problem (10) by the first statement of this proposition. This concludes the proof. \( \blacksquare \)

Now we are in position to construct a singular solution of \((P_\lambda)\). We begin with the following technical lemma.

**Lemma 4.3.** Suppose that \( q > q^*(k, \mu - 2) \). Then there exists a \( t_0 \in \mathbb{R} \) such that \((MS_{q,\mu})\) admits a solution \((x(t), y(t)) \in C^1(-\infty, t_0)\) satisfying

\[ (x(t), y(t)) \to (\hat{x}, \hat{y}) \text{ as } t \to -\infty. \]

**Proof.** Let \( \bar{x}(t) := x(t) - \hat{x}, \bar{y}(t) := y(t) - \hat{y}, \) and \( \bar{X}(t) := (\bar{x}(t), \bar{y}(t))^T \). Then, by (12), \( \bar{X} \) satisfies

\[ \frac{d}{dt} \bar{X} = A_0 \bar{X} + S(t, \bar{X}), \]

where

\[ A_0 := \left( \begin{array}{cc} n - 2 + \mu - 2\hat{x} & -q\hat{x} \\ \hat{x} & \hat{y} + 2\hat{y} - \frac{n - 2k}{k} \end{array} \right) = \left( \begin{array}{cc} -\hat{x} & -q\hat{x} \\ \hat{x} & \hat{y} \end{array} \right), \]

\[ S(t, \bar{X}) := \left( \begin{array}{c} \bar{S}(t, \bar{X}) \\ \bar{W}(\bar{X}) \end{array} \right), \]

\[ \bar{S}(t, \bar{X}) := \frac{\bar{x}^2}{k} - q\bar{x}\bar{y} + (\bar{x} + \hat{x}) \frac{\mu e^{2t}}{1 + e^{2t}}, \]

\[ \bar{W}(\bar{X}) := \frac{\bar{x}\bar{y}}{k} + \bar{y}^2. \]

Let \( X := C((-\infty, t_0), \mathbb{R}^2) \) and let \( \varepsilon > 0 \) be small. Here \( t_0 \) and \( \varepsilon \) will be chosen later. We define the ball \( B_\varepsilon := \{ X \in X : \|X\|_X := \sup\{\|X(t)\|_{\mathbb{R}^2} : t \in (-\infty, t_0)\} < \varepsilon\} \). We show that the Lipschitz constants of \( \bar{S} \) and \( \bar{W} \) are small. Let \( X_1 := (x_1, y_1)^T, X_2 := (x_2, y_2)^T \in B_\varepsilon, \) and \( O := (0, 0)^T. \)
When $t$ is large and negative, we have $\mu e^{2t}/(1 + e^{2t}) < \varepsilon$, and whence

\[
|\bar{S}(t, X) - \bar{S}(t, X)| \leq |x_1 - x_2||x_1 + x_2| + q(|x_1 - x_2||y_1| + |y_1 - y_2||x_2|) + |x_1 - x_2|\frac{\mu e^{2t}}{1 + e^{2t}} \\
\leq C\varepsilon(|x_1 - x_2| + |y_1 - y_2|),
\]

(23)

\[
|\bar{W}(t) - \bar{W}(t)| \leq \frac{1}{k}|x_1 - x_2||y_1| + |y_1 - y_2||x_2| + |y_1 - y_2|y_1 + y_2|
\leq C\varepsilon(|x_1 - x_2| + |y_1 - y_2|),
\]

(24)

\[
|\bar{S}(t, O)| = |\bar{x}|\frac{\mu e^{2t}}{1 + e^{2t}},
\]

(25)

\[
|\bar{W}(O)| = 0.
\]

(26)

By $\mathcal{F}(\bar{X}(t))$ we define

\[
\mathcal{F}(\bar{X}(t)) := \int_{-\infty}^{t} e^{(t-\tau)A_0}S(\tau, \bar{X}(\tau))d\tau.
\]

We find a solution of the equation $\bar{X}(t) = \mathcal{F}(\bar{X}(t))$ in $B_{\varepsilon}$ if $\varepsilon > 0$ is small and $t_0$ is large and negative. As seen in Section 1.2, $A_0$ has two eigenvalues with negative real parts when $q > q^*(k, \mu - 2)$. Therefore, there exists an $\alpha > 0$ such that $\|e^{tA_0}\|_{\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)} \leq Ce^{-\alpha t}$. If $t < t_0$, then by (23) and (24), we have

\[
\|\mathcal{F}(X_1(t)) - \mathcal{F}(X_2(t))\|_{\mathbb{R}^2} \leq \int_{-\infty}^{t} \left\|e^{(t-\tau)A_0}(S(\tau, X_1(\tau)) - S(\tau, X_2(\tau)))\right\|_{\mathbb{R}^2} d\tau \\
\leq \int_{-\infty}^{t} Ce^{-\alpha(t-\tau)}d\tau C\varepsilon \|X_1 - X_2\|_{\mathbb{R}^2} \\
\leq \frac{C\varepsilon}{\alpha} \|X_1 - X_2\|_{\mathbb{R}^2}.
\]

We can now choose $\varepsilon > 0$ sufficiently small if $t_0$ is large and negative. Therefore, there exists a $t_0 \in \mathbb{R}$ such that

\[
\|\mathcal{F}(X_1) - \mathcal{F}(X_2)\|_{\mathbb{R}^2} \leq \frac{1}{2} \|X_1 - X_2\|_{\mathbb{R}^2} \text{ for } X_1, X_2 \in B_{\varepsilon}.
\]

Now from (25) and (26) we see that $\|S(t, O)\|_{\mathbb{R}^2} = o(1)$ as $t \to -\infty$. Thus, $\|\mathcal{F}(O)\|_{\mathbb{R}^2} = o(1)$ as $t \to -\infty$. We have

\[
\|\mathcal{F}(\bar{X})\|_{\mathbb{R}^2} \leq \|\mathcal{F}(\bar{X}) - \mathcal{F}(O)\|_{\mathbb{R}^2} + \|\mathcal{F}(O)\|_{\mathbb{R}^2} \leq \frac{1}{2} \varepsilon + o(1) < \varepsilon
\]

(27)

provided that $t_0$ is large and negative. Hence $\mathcal{F}$ is a contraction mapping on $B_{\varepsilon}$. It follows from the contraction mapping theorem that $\mathcal{F}$ has a unique fixed point in $B_{\varepsilon}$ which is a solution of $\bar{X}(t) = \mathcal{F}(\bar{X}(t))$. When $t_0$ is large and negative, $\varepsilon > 0$ can be chosen arbitrarily small. By (27) and the uniqueness of the fixed point in $B_{\varepsilon}$, we conclude that $\|\bar{X}(t)\|_{\mathbb{R}^2} \to 0$ as $t \to -\infty$. Thus, $\bar{X}(t)$ is the desired solution.

In the following lemma we define in terms of the orbit $(x(t), y(t))$ obtained in Lemma 4.3 the value $\tilde{\lambda}$ associated with a singular solution $\bar{u}(r)$.

**Lemma 4.4.** Suppose that $q > q^*(k, \mu - 2)$. There exists $\lambda = \tilde{\lambda} > 0$ such that the problem

\[
\begin{cases}
(r^{n-k}(u'(r))^k)' = r^{n-1}c_{n,k}^{-1}(1 + u(r))^{\frac{n}{2}}, & 0 < r < 1, \\
u(r) < 0, & 0 < r < 1, \\
u(0) = 0,
\end{cases}
\]

(28)
has a singular solution \( \hat{u}(r) \) that satisfies

\[
\hat{u}(r) = - \left( \frac{c_{n,k} \lambda r^k}{r} \right) \frac{1}{r} - \frac{2k-2}{q-k} \left( 1 + o(1) \right) \text{ as } r \to 0.
\]

**Proof.** Let \((x(t), y(t))\) be given by Lemma 4.3 and set \(h(r) = \frac{r^{\mu-2}}{(1+r^2)^{\frac{q}{2}}}\). Since \((x(t), y(t)) \to (\hat{x}, \hat{y})\) as \(t \to -\infty\), \( \Xi \) yields

\[
u(r) := 1 + w(r) \\
= 1 \left[ (n-1) \lambda r^k h(r) \right]^{-\frac{1}{r}} \left( x(t) y(t)^k \right)^{\frac{1}{r}} \rightarrow -\infty \text{ as } r \to 0.
\]

Then \(u(r)\) is a singular solution of (28). Differentiating \(w(r)\) with respect to \(r\), we have

\[
\left( c_{n,k}^{-1} \lambda \right)^{-\frac{1}{r}} w'(r) = \left[ r^{2k} h(r) \right]^{-\frac{1}{r}} r^{-1} \frac{1}{q-k} \left( x(t) y(t)^k \right)^{-\frac{1}{r}} \left\{ 2k + \frac{r^{\mu-2} h'(r)}{h(r)} - \left( \frac{x(t)}{x(t)} + k \frac{y'(t)}{y(t)} \right) \right\}
\]

\[
= r^{-\frac{2k+1}{q-k}} h(r)^{-\frac{1}{r}} \frac{1}{q-k} \left( x(t) y(t)^k \right)^{-\frac{1}{r}} \left\{ r^{\mu-2} h'(r) \frac{1}{h(r)} + 2 - \frac{\mu}{1 + e^{2t}} + (q-k) y(t) \right\}.
\]

On the other hand, \(|r^{\mu-2} h'(r)/h(r)| \leq C\) for small \(r > 0\), we have

\[
|r^{n-k}(w'(r))^k| \leq r^{n-k} r^{\mu-2 - \frac{k+1}{q-k}} Cr^{-\frac{1(n-k)}{q-k}} \rightarrow 0 \text{ as } r \to 0.
\]

Now, \((MS_{q,\mu})\) has orbits on the \(x\)-axis and \(y\)-axis, the uniqueness of a solution of \((MS_{q,\mu})\) shows that \((x(t), y(t))\) is not on the \(x\)-axis nor on the \(y\)-axis. Hence \(x(t) > 0\) and \(y(t) > 0\) as long as the solution \((x(t), y(t))\) exists. By (31) we see that

\[
w(r) < 0.
\]

Integrating the equation in (28) over \([s, r]\), we have

\[
r^{n-k}(w'(r))^k - s^{n-k}(w'(s))^k = \int_s^r \tau^{n-1} c_{n,k} \lambda h(\tau)(\tau-w(\tau))^d \tau.
\]

Letting \(s \to 0\), (31) yields

\[
r^{n-k}(w'(r))^k = \int_0^r \tau^{n-1} c_{n,k} \lambda h(\tau)(\tau-w(\tau))^d \tau.
\]

By simple calculation, we see that the integrand is integrable near 0.

Next we define

\[
\bar{r} := \sup \{ \delta > 0 : \text{ The solution } w(r) \text{ of (30) exists for } 0 < r < \delta \}.
\]

We show by contradiction that \(\bar{r} = \infty\).

Suppose to the contrary, i.e., \(\bar{r} < \infty\). Then, by (32) and (33), we see that \((w'(r))^k \geq 0\). Since \(w(r) \to -\infty\) as \(r \to 0\), there exists an \(r_0 > 0\) such that \(w'(r_0) > 0\). Since \(w'\) is continuous and \(w\) cannot be 0, we see that

\[
w'(r) > 0 \text{ for } 0 < r < \bar{r}.
\]
Integrating again the equation in (28), but now over \([r_0, \bar{r}]\), we have
\[
r^{n-k}(w'(r))^k = r_0^{n-k}(w'(r_0))^k + \int_{r_0}^{\bar{r}} r^{n-1} e^{-1} c_{n,k} \lambda h(\tau)(-w(\tau))^q d\tau
\]
\[
> r_0^{n-k}(w'(r_0))^k.
\]
Thus, for all \(r_0 < r < \bar{r}\), we have
\[
w'(r) > \left(\frac{r_0}{r}\right)^{\frac{n-k}{k}} w'(r_0) > 0.
\]
A last integration gives
\[
0 > w(r) > w(r_0) + \int_{r_0}^{\bar{r}} \left(\frac{r_0}{r}\right)^{\frac{n-k}{k}} w'(r_0) d\tau > -\infty.
\]
Hence \(\lim_{r\to\bar{r}} w(r)\) exists. On the other hand, differentiating the function in (9) with respect to \(r\), we have
\[
w'(r) = \frac{1}{q-k} \frac{w(r)}{r} \left[-2 + \frac{\mu}{1+e^t} - (q-k) y(t) - \frac{h'(r)}{h(r)} \right]
\]
\[
= -w(r) \frac{y(\ln r)}{r}.
\]
Since \((\hat{x}, \hat{y}) \in G_-\) and all solutions starting in \(G_+\) remain in \(G_-\) by Lemma 4.1, we conclude that \(\lim_{r\to\bar{r}} w'(r)\) exists and \(w'(\bar{r}) > 0\). Since \(w(\bar{r}) < 0\), \(w'(\bar{r}) \neq 0\), and \(w\) satisfies (5), \(w(r)\) can be locally defined as the solution of (6) in a right neighborhood of \(r = \bar{r}\). This contradicts the definition of \(\bar{r}\), and therefore \(\bar{r} = \infty\).

Since \(w(r) < 0\) and \(w'(r) > 0\) for all \(r > 0\), (7) yields that \((x(t), y(t))\) can be defined for all \(t \in \mathbb{R}\). We now define \(\hat{\lambda} := 2^{\mu/2} c_{n,k} x(0) y(0)^{k}\). Then \(w(1) = -1\) and \(u(1) = 0\). This solution \(u(r)\) of (28) with \(\lambda = \hat{\lambda}\) is denoted by \(\hat{u}(r)\). Then (29) follows from (50) and \((\hat{x}, \hat{y}(\hat{u}))\) is a desired singular solution. \(\square\)

## 5 Intersection number

In this section we study the intersection number between a regular and a singular solution of suitable equations. These results will be used in the next section to prove Theorem 2.2 on the multiplicity of solutions of problem \((P_\lambda)\).

Let \(\hat{\lambda}\) be as in Lemma 4.4 and consider the problem
\[
\left\{
\begin{aligned}
(r^{n-k}(U')^{k})' &= c_{n,k} \hat{\lambda} r^{n+\mu-3} (-U)^q, \quad r > 0, \\
U(0) &= -1, \\
U'(0) &= 0.
\end{aligned}
\right.
\]
(35)

Let
\[
\hat{U}(r) := -\left[ c_{n,k} \hat{\lambda} y^{k} \right]^{\frac{1}{1-\gamma}} r^{-\frac{1}{\gamma}},
\]
where \(\gamma := \hat{\gamma}^{-1} = (q-k)/(2k+\mu-2)\). It is easy to see that \(\hat{U}(r)\) is a singular solution of the first equation in (35). Moreover, this equation is of the Emden-Fowler type, which corresponds to the system \((LV_{S_{\rho, \rho,-}})\) with \(h(r) = r^{\mu-2}\). In this case, replacing the stationary point \((x, y) = (\hat{x}, \hat{y})\) and \(\lambda = \hat{\lambda}\) in (9), we obtain the function \(\hat{U}(r)\). Compare with the value \(\hat{\lambda}\) defined in [24] Theorem 3.1 and its corresponding singular solution.

The following lemma shows that the singular solution \(\hat{U}(r)\) crosses infinitely many times the regular solution of (35).

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Proposition 5.1. Suppose that \( q^*(k, \mu - 2) < q < q_{JL}(k, \mu - 2) \). Let \( U(r) \) be the unique solution of (35). Then
\[
Z_{(0, \infty)}[\tilde{U}(\cdot) - U(\cdot)] = \infty,
\]
where \( Z_I[\varphi(\cdot)] \) denotes the number of the zeros of the function \( \varphi(\cdot) \) in the interval \( I \subset \mathbb{R} \), i.e.,
\[
Z_I[\varphi(\cdot)] := \sharp \{ r \in I : \varphi(r) = 0 \}.
\]

Proof. By the local analysis at the point \((\hat{x}, \hat{y})\) (see [24, Section 6]), we see that this point is a stable spiral for \( q^*(k, \mu - 2) < q < q_{JL}(k, \mu - 2) \). The orbit \((x(t), y(t))\) of \((LV_{S_{q,n+\mu-2}})\) starts from the point \((n + \mu - 2, 0)\) at \( t = -\infty \) and rotates around the point \((\hat{x}, \hat{y})\) counterclockwise. Therefore there exists a sequence \( \{t_n\}_{n=1}^{\infty} \) such that \( t_1 < t_2 < \cdots, y(t_n) = \hat{y} \) for all \( n \) and \( x(t_2) < x(t_4) < \cdots < \hat{x} < \cdots < x(t_3) < x(t_1) \). Let \( r_n := e^{\alpha n} \). By (9) with \( h(r) = r^{\mu - 2} \), we have
\[
\frac{\tilde{U}(r_n)}{U(r_n)} = \left[ \frac{\hat{x} \hat{y}^k}{x(t_n)(y(t_n))^k} \right]^{\frac{1}{\mu - 2}} < 1, \quad \text{if } n \in \{1, 3, \ldots\},
\]
\[
> 1, \quad \text{if } n \in \{2, 4, \ldots\},
\]
and therefore \( Z_{(0, \infty)}[\tilde{U}(\cdot) - U(\cdot)] = \infty. \)

Lemma 5.2. Let \( \tilde{u}(r) \) be the singular solution obtained in Lemma 4.2. Define \( \tilde{w}(r) = \tilde{u}(r) - 1 \) and \( (F_\alpha \tilde{w})(r) = \frac{1}{\alpha} \tilde{w}(\frac{r}{\alpha}) \) for \( r > 0 \) and \( \alpha > 0 \). Then
\[
(F_\alpha \tilde{w})(r) \to \tilde{U}(r) \text{ in } C_{loc}(0, \infty) \text{ as } \alpha \to \infty.
\]

Proof. Let \( I \subset (0, \infty) \) be an arbitrary compact interval. From Lemma 4.4 we see that
\[
\tilde{w}(r) = -\left[ c_{n,k} \hat{x} \hat{y}^k \right]^{\frac{1}{\lambda}} r^{\alpha} \left( 1 + \theta \left( \frac{r}{\alpha^\gamma} \right) \right),
\]
where \( \theta(r) \) satisfies \( \limsup_{r \to 0} \theta(r) = 0 \). Therefore
\[
\theta \left( \frac{r}{\alpha^\gamma} \right) \to 0 \text{ in } C(I) \text{ as } \alpha \to \infty.
\]

Using this convergence, we have
\[
\frac{1}{\alpha} \tilde{w} \left( \frac{r}{\alpha^\gamma} \right) = -\frac{1}{\alpha} \left[ c_{n,k} \hat{x} \hat{y}^k \right]^{\frac{1}{\lambda}} \left( \frac{r}{\alpha^\gamma} \right)^{-\frac{1}{\gamma}} \left( 1 + \theta \left( \frac{r}{\alpha^\gamma} \right) \right)
\]
\[
= -\left[ c_{n,k} \hat{x} \hat{y}^k \right]^{\frac{1}{\lambda}} r^{\frac{1}{\alpha^\gamma}} \left( 1 + \theta \left( \frac{r}{\alpha^\gamma} \right) \right)
\]
\[
\to \tilde{U}(r) \text{ in } C(I) \text{ as } \alpha \to \infty.
\]
Since \( I \) was chosen arbitrarily, \( 36 \) holds.

Lemma 5.3. Let \( w(r, \alpha) \) be the solution of the problem
\[
\begin{cases}
(r^{\alpha - k}(w')^k)' = r^{\alpha - 1} c_{n,k} \frac{\hat{x} \hat{y}^k}{(1 + r^2)^{\alpha/2}} (-w)^n, & r > 0, \\
w(0, \alpha) = -\alpha, \\
w_r(0, \alpha) = 0. 
\end{cases}
\]
Let \( (F_\alpha w)(r, \alpha) := \frac{1}{\alpha} w \left( \frac{r}{\alpha^\gamma}, \alpha \right) \). Then
\[
(F_\alpha w)(r, \alpha) \to U(r) \text{ in } C_{loc}[0, \infty) \text{ as } \alpha \to \infty.
\]
Proof. Let \( I \subset [0, \infty) \) be an arbitrary compact interval including 0. Let \( \bar{w}(r, \alpha) := (F_\alpha w)(r, \alpha) \). Then \( \bar{w}(r, \alpha) \) satisfies
\[
\begin{aligned}
(r^{n-k}(w')^k)' &= r^{n-1}c_{n,k}^{-1} \hat{\lambda} \frac{s^{n+\mu-3}}{(1 + \alpha - 2\gamma s^2)\mu/2} (-\bar{w})^q, \quad r > 0, \\
\bar{w}(0, \alpha) &= -1, \\
\bar{w}_r(0, \alpha) &= 0.
\end{aligned}
\]

Since \(-\alpha \leq \bar{w}(r, \alpha) \leq 0\) for \( r \geq 0 \), we see that \(-1 \leq \bar{w}(r, \alpha) \leq 0\) for \( r \geq 0 \). In particular, \( \bar{w} \) is uniformly bounded in \( I \). Integrating the first equation above over \([0, r]\), we have
\[
r^{n-k}(\bar{w}')^k = \int_0^r c_{n,k}^{-1} \hat{\lambda} \frac{s^{n+\mu-3}}{(1 + \alpha - 2\gamma s^2)\mu/2} (-\bar{w})^q ds. \tag{39}
\]

Then
\[
|\bar{w}_r(r, \alpha)| \leq \left( r^{-n+k} \int_0^r c_{n,k}^{-1} \hat{\lambda} s^{n+\mu-3} ds \right)^{1/k} \leq \left( \frac{c_{n,k}^{-1} \hat{\lambda}}{n + \mu - 2} \right)^{1/k} r^{k\mu - 2}.
\]

Here \( \bar{w}(r, \alpha) \) is equicontinuous in \( I \). By the Ascoli-Arzelà theorem, there exist a sequence \( \{\alpha_j\} \) diverging to \(+\infty\) and \( \bar{w}^*(r) \in C(I) \) such that
\[
\bar{w}(r, \alpha_j) \to \bar{w}^*(r) \text{ in } C(I) \text{ as } j \to \infty. \tag{40}
\]

By (39),
\[
\bar{w}(r, \alpha_j) = -1 + \int_0^r \left( t^{-n+k} \int_0^t c_{n,k}^{-1} \hat{\lambda} s^{n+\mu-3} \frac{(-\bar{w}(s, \alpha_j))^q ds}{(1 + \alpha_j - 2\gamma s^2)\mu/2} \right)^{1/k} dt.
\]

Letting \( j \to \infty \), we have
\[
\bar{w}^*(r) = -1 + \int_0^r \left( t^{-n+k} \int_0^t c_{n,k}^{-1} \hat{\lambda} s^{n+\mu-3} (-\bar{w}^*(s))^q ds \right)^{1/k} dt \text{ for } r \in I, \tag{41}
\]

since the following two convergences are uniform on \( I \):
\[
\left( t^{-n+k} \int_0^t c_{n,k}^{-1} \hat{\lambda} s^{n+\mu-3} \frac{(-\bar{w}(s, \alpha_j))^q ds}{(1 + \alpha_j - 2\gamma s^2)\mu/2} \right)^{1/k} \to \left( t^{-n+k} \int_0^t c_{n,k}^{-1} \hat{\lambda} s^{n+\mu-3} (-\bar{w}^*(s))^q ds \right)^{1/k}.
\]

The equality (41) indicates both that \( \bar{w}^*(r) \in C^2(I^i) \cap C^1(I) \) and \( \bar{w}^*(r) \) is the solution of the problem
\[
\begin{cases}
(r^{n-k}((\bar{w}^*)')^k)' = c_{n,k}^{-1} \hat{\lambda} r^{n+\mu-3} (-\bar{w}^*)^q, & r \in I, \\
\bar{w}^*(0, \alpha) = -1, \\
\bar{w}^*_r(0, \alpha) = 0,
\end{cases}
\]

where \( I^i \) denotes the set of the interior points of \( I \). Therefore \( \bar{w}^*(r) = U(r) \) for \( r \in I \). Since \( I \) can be chosen arbitrarily, (39) implies (38). \( \square \)
Lemma 5.4. Suppose that $q^*(k, \mu - 2) < q < q_JL(k, \mu - 2)$. Then
\[
Z_{[0, 1]}[\hat{w}(\cdot) - w(\cdot, \alpha)] \to \infty \text{ as } \alpha \to \infty.
\] (42)

Proof. Since $q^*(k, \mu - 2) < q < q_JL(k, \mu - 2)$, Proposition 5.1 states that
\[
Z_{[0, \infty)}[\tilde{U}(\cdot) - U(\cdot)] = \infty.
\] (43)

Let $U_1, U_2$ be solutions of the equation in (39). We have
\[
k^{r-n-k}U_2^{k-1}(U_2 - U_1)'' + \{kr^{n-k}U_1'V_1 + (n-k)r^{n-k-1}V_2\}(U_2 - U_1)' = r^{n+\mu-3}c_{n,k}^3\lambda V_3(U_2 - U_1),
\] (44)
where $V_1, V_2, V_3$ are continuous function of $r$. We set $U_1 := U$ and $U_2 := \tilde{U}$. Since $U_2' \neq 0$ for $r > 0$, the ODE (44) is of second order. By the uniqueness of the solution of ODEs, each zero of $\tilde{U}(\cdot) - U(\cdot)$ is simple. The zero set of $\tilde{U}(\cdot) - U(\cdot)$ does not have an accumulation point, and hence each zero is isolated. Because of this fact and (43), for every large neighborhood of the zero of $\tilde{V}$ where $R > 0$ such that $1 + \bar{w}(\cdot) - w(\cdot, \alpha) \geq N$, we have $\{0, R\} \supset [0, 1]$, whence
\[
Z_{[0, 1]}[\hat{w}(\cdot) - w(\cdot, \alpha)] \geq Z_{[0, 1]}[\hat{w}(\cdot) - w(\cdot, \alpha)] \geq N.
\]
for large $\alpha > 0$. The number $N$ can be chosen arbitrarily large, whence (42) holds.

6 Proof of Theorem 2.2

Proof. Let $w(r, \alpha)$ be the solution of (37). Then $\hat{w}(r, \alpha) := (\lambda/\alpha)^{1/(q-k)}w(r, \alpha)$ satisfies
\[
\begin{cases}
(r^{n-k}(\hat{w}')')' = r^{n-1}c_{n,k}^1\lambda^{r^n-2}(1+r^2)^{\gamma+2}(-\hat{w})^q, & r > 0, \\
\hat{w}(0, \alpha) = - (\lambda^{1/(q-k)})^\alpha, \\
\hat{w}_r(0, \alpha) = 0.
\end{cases}
\]

Further, $-(\lambda/\alpha)^{1/(q-k)} \leq \hat{w}(r, \alpha) < 0$ for $r \geq 0$, and $\hat{w}_r(0, \alpha) > 0$ for $r > 0$. Let $u(r, \alpha) = 1 + \hat{w}(r, \alpha)$. Then $u$ satisfies the equation in (P$_\lambda$). If $\alpha > (\lambda/\bar{\lambda})^{1/(q-k)}$, then $u(0, \alpha) < 0$. Since $u$ is increasing, $u$ is a solution of (P$_\lambda$) if and only if $u(1, \alpha) = 0$. This equation is equivalent to
\[
\lambda = \lambda(-w(1, \alpha))^{q-k}.
\] (45)
Since \(0 = \tilde{u}(1) = 1 + \tilde{w}(1)\), we have \(\tilde{w}(1) = -1\). We now study \(Z_{[0,1]}[\tilde{w}(\cdot) - w(\cdot, \alpha)]\), which we call the intersection number. By Lemma 5.4

\[
Z_{[0,1]}[\tilde{w}(\cdot) - w(\cdot, \alpha)] \to \infty \text{ as } \alpha \to \infty.
\] (46)

Using the same argument used in the proof of Lemma 5.4 we can easily show that each zero of \(\tilde{w}(\cdot) - w(\cdot, \alpha)\) in \((0, \infty)\) is simple. Let \(\alpha > 0\) be fixed. Since \(\tilde{w}(0) - w(0, \alpha) = \infty\), the zero set is uniformly away from the origin. The coefficient of the second derivative of the ODE which \(\tilde{w} - w\) satisfies is uniformly away from zero on a compact interval in \((0, \infty)\). Since the zero set does not have an accumulation point, we have \(Z_{[0,1]}[\tilde{w}(\cdot) - w(\cdot, \alpha)] < \infty\). Now each zero depends continuously on \(\alpha\). Therefore the intersection number on \([0,1]\) is preserved if a zero does not go out from the boundary of \([0,1]\) and if another zero does not come from the boundary. Since \(\tilde{w}(0) - w(0, \alpha) = \infty\), a zero cannot go out or come from \(r = 0\). By 45, we see that a zero comes from \(r = 1\) infinitely many times. Therefore, there exists a sequence \(\{\alpha_n\}_{n=1}^{\infty}\) such that \(\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots \to \infty\), \(\tilde{w}(1) - w(1, \alpha_n) = 0\), and \(Z_{[0,1]}[\tilde{w}(\cdot) - w(\cdot, \alpha)] = n\). Since \(\tilde{w}(1) = -1\), we have \(w(1, \alpha_n) = -1\) for every \(n \geq 1\). Now, if \(w(1, \alpha) \neq -1\) and \(Z_{[0,1]}[\tilde{w}(\cdot) - w(\cdot, \alpha)]\) is odd, then \(w(1, \alpha) < -1\). On the other hand, if \(w(1, \alpha) \neq -1\) and \(Z_{[0,1]}[\tilde{w}(\cdot) - w(\cdot, \alpha)]\) is even, then \(w(1, \alpha) > -1\). Since \(w(1, \alpha)\) is continuous in \(\alpha\), \(w(1, \alpha)\) oscillates around \(-1\) infinitely many times as \(\alpha \to \infty\). If \(\lambda = \bar{\lambda}\), then 45 is indefinitely many solutions. For each \(N \geq 1\), there exists an \(\varepsilon > 0\) such that if \(|\lambda - \bar{\lambda}| < \varepsilon\), whence 45 has at least \(N\) solutions. Thus, the conclusion holds. \(\square\)

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