Exact Convergence Rate Analysis of the Independent Metropolis-Hastings Algorithms

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Abstract

A well-known difficult problem regarding Metropolis-Hastings algorithms is to get sharp bounds on their convergence rates. Moreover, different initializations may have different convergence rates so a uniform upper bound may be too conservative to be used in practice. In this paper, we study the convergence properties of the Independent Metropolis-Hastings (IMH) algorithms on both general and discrete state spaces. Under mild conditions, we derive the exact convergence rate and prove that different initializations of the IMH algorithm have the same convergence rate. In particular, we get the exact convergence speed for IMH algorithms on general state spaces under certain conditions. Connections with the Random Walk Metropolis-Hastings (RWMH) algorithm are also discussed, which solve the conjecture proposed by Atchadé and Perron [AP07] using a counterexample.

1 Introduction

Markov chain Monte Carlo methods, such as the Metropolis-Hastings algorithms [MRR+53] [Has70], the Gibbs sampler [GG84] [GS90], have revolutionized statistics, especially statistical computing and Bayesian methodology. When facing a complicated probability distribution, no matter discrete or continuous, low or high dimensional, MCMC methods provide powerful simulation tools to draw samples from the target distribution. The idea of learning a system by simulating it with random sampling has been proven to be useful in almost every quantitative subject of study. This makes MCMC methods ubiquitous in many areas including physics, biology, chemistry, and computer science. In particular, the Metropolis algorithm is recognized as the ‘top 10 algorithms’ in the 20th century [Cip00].

Mathematically speaking, given a target distribution \( \pi(x) \), the MCMC algorithm generates a Markov chain \( \Phi \equiv \{ \Phi_1, \Phi_2, \cdots \} \) whose stationary distribution is designed to be \( \pi \). Under mild conditions, classical Markov chain theory guarantees that the Markov chain \( \Phi \) will converge to its stationary distribution when the chain is run long enough.

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However, the convergence result mentioned above does not give any quantitative convergence rate analysis. Therefore, a central part of the MCMC theory is to establish convergence rate analysis for MCMC algorithms. Unfortunately, getting sharp bounds for the convergence rates is known to be very difficult in Markov chain theory, especially for general state Markov chains. Consider a Markov chain \( \pi(x) \) with stationary distribution \( \pi(x) \) in \( \mathbb{R}^d \), most of the existing theoretical works are established under the notion of geometric ergodicity. Roughly speaking, a Markov chain with transition probability \( P \) is said to be geometrically ergodic if the following inequality holds for \( \pi \)-almost everywhere \( x \):

\[
\| P^n(x, \cdot) - \pi(\cdot) \| \leq C(x)r^n,
\]

where \( r \) is a constant between 0 and 1, \( \| \cdot \| \) is some distance metric between two probability measures, usually taken to be the total variation (TV) distance.

Formula 1.1 guarantees that, for almost all initializations, the convergence rate of \( \Phi \) can be uniformly upper bounded by a constant \( r \). However, one can ask several natural questions regarding 1.1, for example:

- (Q1) Is the convergence rate \( r \) given by 1.1 sharp? Formula 1.1 gives an upper bound for the convergence rate, but this upper bound may be far away from a sharp speed.

- (Q2) Does every point \( x \) have the same convergence rate? It is tempting to believe that all the points have the same convergence rate, however, as we will see shortly, this is not true in general. Therefore, even if one could get a ‘sharp rate’ for 1.1, this still only tells the ‘slowest’ convergence rate among all the initial states. Suppose there exists a Markov chain, such that the convergence rate at one point (say \( x_1 \)) equals 0.001, while the convergence rate at another point (say \( x_2 \)) equals 0.999. Then the bound \( r \) given by 1.1 would be practically useless when one starts the chain at \( x_1 \). Therefore, a more precise but perfectly natural question would be inquiring the convergence rates \( r(x) \) for different initializations.

We believe both Q1 and Q2 would be of both theorist and practitioner’s interest. First of all, most of the present MCMC convergence theories focus on the upper bound of the convergence speed. Thus it is natural to ask for a sharp upper bound, or a useful lower bound. Secondly, an a priori bound for the convergence rate would be very helpful for users to evaluate the computation cost, or choose a fast MCMC algorithm among many candidates before implementing them all. Lastly, it seems Q2 has not been seriously discussed and studied before. At first glance, 1.1 seems to suggest that every point has the same convergence rate \( r \). However, it only shows every point has convergence rate upper-bounded by \( r \). The following example shows different initializations may have significant differences in their convergence speeds.

Example 1 (Markov Chain on Three Points). Let \( \Phi \) be a Markov chain on state space \( X = \{x_1, x_2, x_3\} \) with transition matrix:

\[
P = \begin{pmatrix}
1/3 & 1/3 & 1/3 \\
2/3 & 0 & 1/3 \\
0 & 2/3 & 1/3
\end{pmatrix}
\]
Then it is clear that the stationary distribution \( \pi \) of \( \Phi \) is the uniform distribution on \( \mathcal{X} \), i.e., \( \pi(x_i) = \frac{1}{3} \) for every \( i \). Meanwhile, since \( P(x_1, \cdot) = \pi(\cdot) \), the chain will converge after one step when starting at \( x_1 \), i.e., \( \|P^n(x_1, \cdot) - \pi(\cdot)\| = 0 \) for any \( n \geq 1 \). However, it is clear that the convergence rates at \( x_2 \) and \( x_3 \) are both strictly greater than 0, therefore different initializations may have different convergence rates.

Admittedly, Example 1 is an artificial example just for illustration. But as we can see in Section 5, even for Random-walk Metropolis-Hastings algorithms (RWMH), the same phenomenon may occur as well. We do not know if the fact ‘different initializations have different convergence rates’ is true in general. If so, then such insight may shed light on practical applications as a careful selection of initialization would greatly increase the algorithm’s efficiency. Therefore, we believe Q2 is an interesting and meaningful question to investigate.

However, getting precise answers for Q1 and Q2 are known to be very difficult due to the complicated nature of Markov chains. Sharp convergence rate analysis is very rare for MCMC algorithms on general state spaces, even for toy models. In this paper, we focus on studying the convergence of the Independent Metropolis-Hastings (IMH) algorithm which is a widely-used special class of MCMC methods. We show that both Q1 and Q2 can be answered precisely for IMH chains on both discrete and general state spaces. For example, under mild conditions, we could calculate the non-asymptotic convergence speed for IMH algorithms on general state spaces exactly, with exact convergence rate and exact constant. We could also show that, though in general different initializations may have different convergence rates, all the initializations converge to stationary at the same rate for IMH algorithms. Our work extends the previous results of Smith and Tierney [ST96] on general state spaces and Liu [Liu96] on discrete state spaces.

### 1.1 Our contributions

In this paper, we consider the IMH chain on general and discrete state space separately. Our target is to answer the questions Q1 and Q2 proposed in Section 1. Our results are summarized in Table 1.

| State Space \( \mathcal{X} \) | (Exact) Convergence Rate | (Exact) Convergence Speed | Convergence Rate for Different Initializations |
|-------------------------------|--------------------------|--------------------------|---------------------------------------------|
| Continuous                    | \( 1 - \frac{1}{w^*} \) (Thm 3, 4) | \( (1 - \frac{1}{w^*})^n \) (Thm 3) | All equals \( 1 - \frac{1}{w^*} \) (Thm 5) |
| Discrete                      | \( 1 - \frac{1}{w^*} \) (Thm 7) | \( (1 - \pi_1)(1 - \frac{1}{w^*})^n, (1 - \frac{1}{w^*})^n \) (Thm 7) | All equals \( 1 - \frac{1}{w^*} \) (Thm 8) |

In short, our results include:

1. **(Q1) Exact convergence rate analysis for the IMH algorithm:**
   - For IMH on general state spaces, we prove that, under certain conditions, we have the following ‘exact convergence’ result:
     \[
     \sup_x \|P^n(x, \cdot) - \pi\|_{TV} = (r^*)^n
     \] (1.2)
where \( r^* = 1 - \frac{1}{w^*} \) is a computable constant which will soon be clear (defined in Section 3.1). It is worth mentioning that formula 1.2 gives an equality instead of an upper or lower bound, so it completely characterizes the convergence speed for the IMH chain.

- For IMH on discrete state space, we prove that:
  \[
  c(r^*)^n \leq \sup_x \| P^n(x, \cdot) - \pi \|_{TV} \leq (r^*)^n,
  \]
  where \( r^* \) is the same as above, \( c = \pi_1 \) is a computable constant which will be defined in Section 4.

2. (Q2) Convergence rate analysis with different initializations:

- For IMH on general state spaces, we prove that, under certain conditions, for every \( x \in X \), the induced probability measure \( P^n(x, \cdot) \) converges to \( \pi \) at the same rate.
- For IMH on discrete state space, we prove that \( P^n(x, \cdot) \) converges to \( \pi \) at the same rate for all \( x \in \mathcal{X} \) without any further assumptions.

3. We discuss the similarity and difference between IMH algorithms and RWMH algorithms. We provide examples to show that RWMH do not enjoy some of the nice properties as IMH. In particular, we solve the conjecture proposed by Atchadé and Perron [AP07] by means of a counterexample.

Theorem 3 is somewhat surprising as it gives a simple, clean formula for the exact speed of convergence, with lower bound matches perfectly with the upper bound under certain conditions. To our best knowledge, this is the first time we could get the exact speed for a general state space Markov chain. Theorem 7 is the discrete variant of Theorem 3. It is interesting to find out that the exact result in Theorem 3 for general state space does not hold when the state space is discrete, though the convergence rate is still the same. Theorem 5 and Theorem 8 are also interesting as they show the IMH chain converges at the same rate for an arbitrary initialization, which gives pretty precise quantitative results on their convergence rates.

1.2 Organization

The rest of this paper is organized as follows. In Section 2, we introduce our notations and review previous works. Section 3 analyzes the IMH chain on general state spaces. Section 4 analyzes the IMH chain on discrete state space. Connections between IMH and RWMH algorithms as well as some future directions are discussed in Section 5.

2 Preliminaries

2.1 The Metropolis-Hastings algorithm

Let \((\mathcal{X}, \mathcal{F}, \pi)\) be a probability space, equipped with \(\sigma\)-algebra \(\mathcal{F}\) and a probability measure \(\pi\). In practice, \(\mathcal{X}\) usually stands for one of the following four possibilities:
1. $\mathcal{X}$ is a finite set. For example, $\mathcal{X} = \{1, 2, \cdots, n\}$.

2. $\mathcal{X}$ is a countably infinite set. For example, $\mathcal{X} = \mathbb{N}$.

3. $\mathcal{X}$ is the Euclidean space, i.e., $\mathcal{X} = \mathbb{R}$ or $\mathbb{R}^p$.

4. $\mathcal{X}$ is a subset of the Euclidean space. For example, $\mathcal{X} = \mathbb{R}^+$ or $\mathcal{X} = \{x \in \mathbb{R}^p : \|x\| \leq c\}$ for some constant $c$.

When $\mathcal{X}$ is a finite or countably infinite set, $\mathcal{F}$ is usually taken as $2^\mathcal{X}$ and thus all the subsets of $\mathcal{X}$ are measurable. When $\mathcal{X}$ is the Euclidean space or a subset of the Euclidean space, $\mathcal{F}$ is usually taken as the Borel $\sigma$-algebra. The four possibilities of $\mathcal{X}$ cover almost all the practical applications of our interest. Therefore, without further specification, we shall assume the measurable space $\mathcal{X}$ falls into one of the four categories mentioned above henceforth. Throughout this paper, $\mathcal{X}$ is called ‘state space’ as it also serves as the state space for the Markov chains. Naturally, we say ‘discrete state space’ when $\mathcal{X}$ finite or countably infinite, and ‘continuous state space’ or ‘general state space’ when $\mathcal{X}$ is the Euclidean space or a subset of the Euclidean space. In particular, when $\mathcal{X}$ is a general state space, we further assume the probability measure $\pi$ has a density on $\mathcal{X}$. We slightly abuse the notation and use $\pi$ to denote both the probability mass function (pmf) on discrete state spaces and probability density function (pdf) on general state spaces.

Given a target probability distribution $\pi$, the Metropolis-Hastings (MH) algorithm is designed to draw samples from $\pi$ approximately. The details of the Metropolis-Hastings algorithm are described in Algorithm 1.

**Algorithm 1 Metropolis-Hastings Algorithm (MH)**

**Input:** initial setting $x$, number of iterations $T$, Markov transition kernel $q$

1. for $t = 1, \cdots T$ do
2. Propose $x' \sim q(x, x')$
3. Compute
   $$a = \frac{q(x', x)\pi(x')}{q(x, x')\pi(x)}$$
4. Draw $r \sim \text{Uniform}[0, 1]$
5. If $(r < a)$ then set $x = x'$.
   Otherwise, leave $x$ unchanged.
6. end for

Algorithm 1 is arguably the most popular class of Markov chain Monte Carlo (MCMC) methods. It is first proposed by Metropolis et al. [MRR+53] in 1953 to simulate a liquid in equilibrium with its gas phase. Hastings generalized the algorithm in 1970 [Has70] and thus simulations following this scheme are called the Metropolis-Hastings algorithm. The idea of MH algorithm is to construct a Markov chain $\Phi$ with stationary distribution $\pi$. If we run $\Phi$ for a long time and collect the samples $\Phi_1, \Phi_2, \cdots$, then the distribution of $\Phi_n$ will be very close to our target $\pi$ for large enough $n$ under mild conditions.

It is worth mentioning that one of the most appealing features of the MH (and other MCMC) algorithm is that, one can know $\pi$ up to a normalizing constant. In many practical
applications such as Bayesian computation, the target distribution $\pi$ is often of the following form:

$$\pi(x) = \frac{\pi_u(x)}{c},$$

where $c = \int_{\mathcal{X}} \pi_u$ is referred to as the ‘normalizing constant’. In practice, often one can easily evaluate $\pi_u$ but evaluating the integral $\int_{\mathcal{X}} \pi_u$ is difficult. The MH algorithm is still feasible under this setting as the following simple identity holds for Step 3 of Algorithm 1:

$$\frac{q(x', x)\pi(x')}{q(x, x')\pi(x)} = \frac{q(x', x)\pi_u(x')}{q(x, x')\pi_u(x)}.$$

Therefore the ‘acceptance ratio’ is irrelevant to the normalizing constant $c$. This is one reason that MCMC methods turn out to be extremely helpful in sampling complicated distributions and become very popular in the Bayesian statistical community.

As mentioned in Algorithm 1, one needs to specify a Markov transition kernel $q$ before running the MH algorithm. Some popular choices of the proposals include:

1. Independent proposal: Here $q(x, y) = q(y)$ which does not depend on the current state of the Markov chain. For example, $q = \mathcal{N}(0, \sigma^2)$ for $\sigma > 0$.

2. Random walk proposal: Here $q(x, y) = q(|x - y|)$. For example, at each state $x$, the proposal may be chosen as $\mathcal{N}(x, \sigma^2)$ for $\sigma > 0$.

3. Metropolis-adjusted Langevin algorithm (MALA): For example, at each state $x$, the proposal may be chosen as $\mathcal{N}(x + \frac{\delta}{2} \nabla \log \pi(x), \delta)$ for $\delta > 0$.

An MH algorithm using an independent proposal is called the Independent Metropolis-Hastings (IMH) algorithm, which is the main focus of our paper.

### 2.2 Markov chain convergence

Let $\Phi = \{\Phi_1, \Phi_2, \ldots\}$ be a Markov chain on state space $\mathcal{X}$ with transition kernel $P$. In the general state space setting, it is further assumed that $P(x, \cdot)$ is absolute continuous with density $p(x, y)$ with respect to the Lebesgue measure $\lambda$, except perhaps for an atom $P(x, \{x\}) > 0$.

A probability measure $\pi$ is called the **stationary distribution** of $\Phi$ if

$$\pi P = \pi,$$

where $\pi P$ is the induced probability measure defined by

$$(\pi P)(A) = \int_{x \in A} \pi(dx) P(x, A) \quad (2.1)$$

Furthermore, the Markov chain $\Phi$ is called **reversible** if it satisfies a detailed-balance equation, that is:

$$\pi(x) p(x, y) = \pi(y) p(y, x),$$
when $\mathcal{X}$ is continuous, or
\[ \pi(x)P(x,y) = \pi(y)P(y,x), \]
when $\mathcal{X}$ is discrete.

By design, Metropolis-Hastings algorithms are reversible Markov chains with target distribution $\pi$ as stationary distribution.

Now we are ready to discuss the convergence properties of a Markov chain. Let $P^n(x,\cdot)$ be the probability measure after running the Markov chain $\Phi$ for $n$ steps with starting point $x$. The distance between $P^n(x,\cdot)$ with $\pi$ is measured by total variation distance, which is defined as follows:

**Definition 1.** Let $\mu, \nu$ be two probability measures on the same sigma-algebra $\mathcal{F}$ of subsets of a sample space $\Omega$, the total variation distance between $\mu$ and $\nu$ is defined as:
\[
\|\mu - \nu\|_{TV} = \max_{A \in \mathcal{F}} |\mu(A) - \nu(A)|
\]

One might guess $\Phi$ will eventually converge to its stationary distribution $\pi$. In other words, $\|P^n(x,\cdot) - \pi\|_{TV} \to 0$. However, this is not true without any further assumption. In fact, the stationary distribution of a Markov chain may not even be non-unique. For example, consider the naive Markov chain which never moves, i.e., $P(x,\{x\}) = 1$, then any probability distribution on $\mathcal{X}$ satisfies equation 2.1 trivially and is thus a stationary distribution. Therefore, we need to following two technical assumptions, which are generally satisfied in practice.

**Definition 2 (\(\phi\)-irreducible).** A Markov Chain $\Phi$ is $\phi$-irreducible if there exists a non-zero $\sigma$-finite measure $\phi$ on $\mathcal{X}$ such that for all measurable $A \subset \mathcal{X}$ with $\phi(A) > 0$, for all $x \in \mathcal{X}$, there exists positive integer $n = n(x,A)$ such that
\[ P^n(x,A) > 0. \]

Roughly speaking, $\phi$-irreducible ensures every subset with positive measure will be reached from anywhere in the state space.

**Definition 3 (Aperiodic).** A Markov Chain $\Phi$ is aperiodic if there does not exist $d \geq 2$ and disjoint subsets $\mathcal{X}_1, \cdots, \mathcal{X}_d \subset \mathcal{X}$ such that $P(x,\mathcal{X}_{(i+1) \mod d}) = 1$ for all $x \in \mathcal{X}_i$ and $1 \leq i \leq d$.

Aperiodicity ensures the Markov chain will not periodically explore the state space.

Assuming a Markov chain is irreducible and aperiodic, the following well-known theorem guarantees the Markov chain will converge to its unique stationary distribution.

**Theorem 1** (Chapter 13 in [MT12]). If a Markov chain $\Phi$ on a state space with countably generated $\sigma$-algebra is $\phi$-irreducible and aperiodic, and has a stationary distribution $\pi$, then
\[ \|P^n(x,\cdot) - \pi\|_{TV} \to 0 \]
as $n \to \infty$ for $\pi$-a.e. $x$.  

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Aperiodicity and irreducible are generally satisfied in practical applications of MH algorithms, at least after some simple modifications. Therefore we will assume all the Markov chains mentioned in this paper are irreducible, aperiodic, and reversible.

Theorem 1 provides guarantees the convergence of an irreducible, aperiodic Markov chain, but does not tell us anything about the convergence speed.

A Markov chain $\Phi$ with transition kernel $P$ is said to be uniformly ergodic if

**Definition 4 (Uniformly Ergodic).**

$$\sup_x \|P^n(x, \cdot) - \pi\|_{TV} \leq Cr^n$$ (2.2)

for $C > 0$ and $0 < r < 1$.

When $X$ is a continuous, unbounded space, then a Markov chain may often fail to be uniformly ergodic. Therefore, the following concept ‘geometrically ergodic’ is needed. $\Phi$ is said to be geometrically ergodic if

**Definition 5 (Geometrically Ergodic).**

$$\|P^n(x, \cdot) - \pi\|_{TV} \leq C(x)r^n,$$ (2.3)

for some $0 < r < 1$.

Roughly speaking, both uniform ergodicity and geometric ergodicity guarantee the Markov chain converges to its stationary distribution at a geometric rate for $\pi$-a.e. initialization position $x$. However, it is a well-known difficult problem to give sharp bounds on the right-hand side of 2.2 and 2.3. In the next section, we will review some existing works on bounding the convergence rates of MH algorithms.

### 2.3 Related works

There are numerous studies concerning the convergence rate of Markov chains. The techniques for bounding convergence rates are significantly different between Markov chains on general and discrete state spaces.

When $X$ is finite (but possibly very large), it is clear from Definition 4 and Definition 5 that every geometrically ergodic chain must be uniformly ergodic, as we could take the supremum of $C(x)$ in formula 2.3 over the state space $X$. There are many techniques including eigenanalysis, group theory, geometric inequalities, Fourier analysis, multicommodity flows for getting upper and lower bounds of the convergence rates. These techniques can sometimes turn into very sharp bounds for the right-hand side of 2.2, see Bayer and Diaconis [BD+92], Diaconis and Shahshahani [DS81], Liu [Liu96] for inspiring examples. We also refer the readers to Diaconis [Dia09], which gives an extensive review of the existing tools and results.

For general state Markov chains, uniform ergodicity is often ‘too good to be true’ when the state space is continuous and unbounded (see, for example, Theorem 3.1 of Mengerson and Tweedie [MT+96]). Even geometrically ergodic is not at all trivially satisfied by usual Markov chains. General conditions for geometric ergodicity are established in the seminal works of
Rosenthal [Ros95], Mengerson and Tweedie [MT+96], and Roberts and Tweedie [RT96]. More refined estimates for explicit $C(x)$ and $r$ are often referred to as ‘honest bounds’ and are mostly developed under the ‘drift-and-minorization’ framework by Rosenthal [Ros95]. However, choosing a suitable ‘drift function’ is often difficult in practical applications, and the estimates given by the ‘drift-and-minorization’ framework are usually very conservative. There are very few examples of sharp rates of convergence for the Metropolis algorithm, even for toy models. Therefore, natural questions like Q1 and Q2 mentioned in the introduction usually can not be answered. For more about the convergence of Markov chains on general state spaces, we refer the readers to an excellent survey by Roberts and Rosenthal [RR04].

For IMH algorithms, Liu [Liu96] has computed explicitly the eigenvalues and eigenvectors of the Markov transition matrix on a discrete state space. For IMH algorithms on general state spaces, Mengerson and Tweedie have shown that uniform ergodicity and geometric ergodicity are equivalent. They have also provided the necessary and sufficient condition for the IMH algorithm being uniformly ergodic in Theorem 2.1 of [MT+96]. The exact transition probabilities for the IMH algorithm are derived by Smith and Tierney [ST96]. The whole spectrum of the IMH algorithm is derived by Atchadé and Perron [AP07]. Quantitative convergence bounds for non-geometric IMH algorithms are studied by Roberts and Rosenthal [RR11].

3 IMH on general state spaces

Now we are ready to state and prove our main results. Suppose one wish to sample from a probability distribution with density $\pi$ on the general state space $\mathcal{X}$. Let $p$ be a probability density function that we are able to sample from. Throughout this section, we assume both $p$ and $\pi$ are continuous therefore it makes sense to discuss the behavior of our Markov chain at every point $x$ and avoids the measurability issues. We further assume that the support of $\pi$ is contained in that of $p$, i.e., $\text{Supp}(\pi) \subset \text{Supp}(p)$. The IMH algorithm is defined as follows:

\begin{algorithm}[H]
\caption{Independent Metropolis-Hastings Algorithm (IMH)}
\textbf{Input:} initial setting $x$, number of iterations $T$, proposal distribution $p$
\begin{algorithmic}[1]
\State \textbf{for} $t = 1, \cdots T$ \textbf{do}
\State Propose $x' \sim p(x')$
\State Compute \hspace{1cm} $a(x, x') = \frac{p(x)\pi(x')}{p(x')\pi(x)}$
\State Draw $r \sim \text{Uniform}[0, 1]$
\State \textbf{If} ($r < a$) \textbf{then} set $x = x'$. \textbf{Otherwise}, leave $x$ unchanged.
\State \textbf{end for}
\end{algorithmic}
\end{algorithm}

This section is divided into two parts. In Section 3.1, we give the exact convergence speed of the IMH algorithm, which answers Q1 proposed in Section 1. In Section 3.2, we provide the answer of Q2 for the IMH algorithm.
3.1 Exact convergence speed

Let \( w \) be the ratio between densities \( \pi \) and \( p \), i.e.,

\[
   w(x) = \frac{\pi(x)}{p(x)}.
\]

Then it’s clear that the acceptance ratio \( a \) can be written as

\[
   a(x, x') = \frac{w(x')}{w(x)}.
\]

The following theorem is proved by Mengerson and Tweedie, which gives the necessary and sufficient conditions for IMH being geometrically ergodic.

**Theorem 2** (Mengerson and Tweedie [MT+96], Theorem 2.1). With all the notations as above, we have

1. Let \( \Phi = \{\Phi_1, \Phi_2, \ldots\} \) be the Markov chain associated with the IMH algorithms, with target density \( \pi \) and proposal density \( p \). Then the followings are equivalent:
   - \( \Phi \) is uniformly ergodic
   - \( \Phi \) is geometrically ergodic
   - Function \( w(x) \) is uniformly bounded, i.e., there exists \( M > 0 \) such that \( w(x) \leq M \) for all \( x \in \mathcal{X} \)

2. Suppose \( w(x) \leq M \) for all \( x \in \mathcal{X} \), then for any \( n \geq 1 \)

\[
   \|P^n(x, \cdot) - \pi\|_{TV} \leq (1 - \frac{1}{M})^n \tag{3.1}
\]

Theorem 2 shows \( w \) uniformly upper bounded is the necessary and sufficient condition for the IMH algorithm being geometrically ergodic. In particular, uniformly ergodic and geometrically ergodic are equivalent for the IMH chain. It is therefore convenient to define:

**Definition 6.**

\[
   d_{IMH}(t) = \max_{x \in \mathcal{X}} \|P^t(x, \cdot) - \pi\|_{TV}.
\]

\( d_{IMH}(t) \) describes the distance to the equilibrium distribution in the worst-case scenario. It is worth mentioning that \( d(t) \) is also called the ‘maximal variation distance’ and is widely used in the Markov chain literature. For example, a primary objective of analyzing Markov chains on discrete state spaces is to bound \( d(t) \), see Levin et al. [LP17], Aldous and Diaconis [AD87] for discussions.

For the IMH chain, an immediate corollary for Theorem 2 is:

**Corollary 1.** Let \( w^* = \sup_x w(x) \), then for any \( n \geq 1 \)

\[
   d_{IMH}(n) \leq (1 - \frac{1}{w^*})^n \tag{3.2}
\]
Corollary 1 gives the best possible upper bound of Theorem 2. Our next theorem shows, suppose $w^*$ can be attained at some $x^* \in \mathcal{X}$, then $(1 - \frac{1}{w^*})^n$ is also a lower bound and is thus the exact convergence speed of the IMH chain.

**Theorem 3.** With all the notations as above, assume that there exists $x \in \mathcal{X}$ such that $w(x^*) = w^*$, then we have:

$$d_{\text{IMH}}(n) = (1 - \frac{1}{w^*})^n \quad (3.3)$$

We will show both $d_{\text{IMH}}(n) \leq (1 - \frac{1}{w^*})^n$ and $d_{\text{IMH}}(n) \geq (1 - \frac{1}{w^*})^n$ holds. Though the first part is already shown in Mengerson and Tweedie [MT+96], we provide give a coupling proof here the sake of completeness.

**Proof of the upper bound in Theorem 3.** We prove the upper bound by a coupling argument. We define a coupling of the IMH algorithm to be a pair of two stochastic processes $(\Phi, \tilde{\Phi}) = (\Phi_i, \tilde{\Phi}_i)_{i=0}^{\infty}$ such that both $\Phi$ and $\tilde{\Phi}$ are IMH chains with proposal density $p$, though they may have different initial distributions. We further require the two chains stay together after they visit the same state, i.e.:

$$\text{If } \Phi_i = \tilde{\Phi}_i \text{ for some } i, \text{ then for any } j \geq i: \Phi_j = \tilde{\Phi}_j. \quad (3.4)$$

Moreover, suppose $\tilde{\Phi}$ starts at stationary, in other words,

$$\tilde{\Phi}_0 \sim \pi, \quad (3.5)$$

then it is clear that $\tilde{\Phi}_i$ is also distributed according to $\pi$. Then the following inequality is standard in Markov chain theory, see Levin et al. [LP17] Theorem 5.2, or Rosenthal [Ros95] page 16 for a proof.

**Fact.** Let $(\Phi, \tilde{\Phi})$ be an arbitrary coupling satisfying 3.4 and 3.5. Let $T$ be the first time the two chains meet:

$$T = \min\{n, \Phi_n = \tilde{\Phi}_n\}. \quad (3.6)$$

Then we have:

$$\|P^n(x, \cdot) - \pi\|_{TV} \leq \mathbb{P}(T \geq n) \quad (3.7)$$

To use inequality 3.7, it suffices to design a coupling between $\Phi$ and $\tilde{\Phi}$ and bound the tail probability of meeting time $T$.

In our IMH algorithm example, given any current state $x$, the actual transition density from $x$ to $y$ is given by:

$$P(x, y) = p(y) \min\{\frac{w(y)}{w(x)}, 1\} \geq \frac{p(y)w(y)}{w^*} = \frac{\pi(y)}{w^*}. \quad (3.8)$$

Therefore, the transition kernel $P$ satisfies the following minorization condition:

$$P(x, \cdot) \geq \frac{1}{w^*} \pi(\cdot) \quad (3.8)$$

for any $x \in \mathcal{X}$. 

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Inequality 3.8 allows us to split \( P(x, \cdot) \) into a mixture of two probability distributions:

\[
P(x, \cdot) = \frac{1}{w^*} \pi(\cdot) + (1 - \frac{1}{w^*}) q_{\text{res}}(x, \cdot) \tag{3.9}
\]

where \( q_{\text{res}}(x, \cdot) \) is the residual measure defined by

\[
q_{\text{res}}(x, A) = \pi(A) - \frac{1}{w^*} P(x, A) \tag{3.10}
\]

for any measurable set \( A \).

Now we are ready to define our coupling. At time \( k \), let \( x, x' \) be two arbitrary current states of \( \Phi \) and \( \tilde{\Phi} \), respectively. Assume \( x' = x \), then sample \( \Phi_{k+1} \) from \( P \) and set \( \Phi_{k+1} = \tilde{\Phi}_{k+1} \). Otherwise, flip a coin with head probability \( \frac{1}{w^*} \), if it comes up heads, sample \( \Phi_{k+1} \) according to \( \pi \) and set \( \Phi_{k+1} = \tilde{\Phi}_{k+1} \). Otherwise, sample \( \Phi_{k+1} \) and \( \tilde{\Phi}_{k+1} \) from \( q_{\text{res}}(x, \cdot) \) and \( q_{\text{res}}(x', \cdot) \) independently. It is not hard to verify the pair \((\Phi, \tilde{\Phi})\) defined above is a coupling satisfying 3.4 and 3.5. Meanwhile, at each step before \( \Phi \) and \( \tilde{\Phi} \) meets, the meeting probability equals \( \frac{1}{w^*} \). Therefore the following inequality holds:

\[
P(T \geq n) \leq (1 - \frac{1}{w^*})^n. \tag{3.11}
\]

Combining 3.7 with 3.11, we have

\[
d_{\text{IMH}}(n) \leq (1 - \frac{1}{w^*})^n, \tag{3.12}
\]

as desired. \( \square \)

The lower bound proof requires the following two lemmas. The first lemma shows the total variation distance is lower bounded by a geometric series with the ‘rejection probability’ as the common ratio.

**Lemma 1.** Let \( R(x) \) be the ‘rejected probability’ given current state \( x \), more precisely,

\[
R(x) = \int_X \left( 1 - \min\left\{ \frac{w(y)}{w(x)}, 1 \right\} \right) p(y) dy \tag{3.13}
\]

Then

\[
\| P^n(x, \cdot) - \pi \|_{TV} \geq R^n(x) \tag{3.14}
\]

**Proof of Lemma 1.** Given a Markov chain starting at \( x \), then it is clear that the probability of staying at \( x \) after \( n \) steps is no less than \( R^n(x) \). Therefore, we have

\[
P^n(x, \{x\}) \geq R^n(x).
\]

Meanwhile, it is clear that \( \pi(\{x\}) = 0 \) as \( \pi \) is a continuous distribution. We have:

\[
\| P^n(x, \cdot) - \pi \|_{TV} = \max_{A \subset X} \| P^n(x, A) - \pi(A) \| \geq R^n(x) - 0 = R^n(x),
\]

as desired. \( \square \)
The next lemma gives the formula for computing $R(x)$

**Lemma 2.** With all the notations as above, we have

$$
R(x) = 1 - \frac{\int_{y \in C(w(x))} \pi(y) dy}{w(x)} - \int_{y \in C^c(w(x))} p(y) dy
$$

(3.15)

where $C(w) \doteq \{ z : w(z) \leq w \}$ is a subset of $X$ with ratio no larger than $w$.

In particular

$$
R(x^*) = 1 - \frac{1}{w^*}
$$

(3.16)

**Proof of Lemma 2.** For fixed $x$, we could split $X$ into $C(w(x))$ and $C^c(w(x))$, straightforward calculation gives:

$$
R(x) = \int_X (1 - \min \{ \frac{w(y)}{w(x)}, 1 \}) p(y) dy
$$

$$
= 1 - \int_{y \in C(w(x))} \frac{p(y) w(y) dy}{w(x)} - \int_{y \in C^c(w(x))} p(y) dy
$$

$$
= 1 - \int_{y \in C(w(x))} \frac{\pi(y) dy}{w(x)} - \int_{y \in C^c(w(x))} \pi(y) dy.
$$

In particular, $C(w(x^*)) = C(w^*) = X$ by definition, therefore

$$
R(x^*) = 1 - \frac{\int_{y \in X} \pi(y) dy}{w^*} = 1 - \frac{1}{w^*},
$$

which completes our proof.

The lower bound of Theorem 3 is immediate after combining Lemma 1 and Lemma 2.

**Proof of the lower bound in Theorem 3.** Combining Lemma 1 and Lemma 2, we have:

$$
d_{\text{IMH}}(t) = \max_{x \in X} \| P^t(x, \cdot) - \pi \|_{TV}
$$

$$
\geq \| P^t(x^*, \cdot) - \pi \|_{TV}
$$

$$
\geq R^n(x^*)
$$

$$
= (1 - \frac{1}{w^*})^n,
$$

which completes the proof.

**Remark 1.** Another way of showing the lower bound is as follows. It is not hard to verify that $P(x^*, \cdot) = R(x^*) \delta_{x^*} + \frac{1}{w^*} \pi(\cdot)$, where $\delta_{x^*}$ is the point mass at $x^*$. As $\pi$ is invariant under the action of $P$, i.e., $\pi P = \pi$, we have:

$$
P^2(x^*, \cdot) = R(x^*) P(x^*, \cdot) + \frac{1}{w^*} \pi(\cdot)
$$

$$
= R^2(x^*) \delta_{x^*} + (1 - R^2(x^*)) \pi(\cdot)
$$
The last step uses the equality $R(x^*) = 1 - \frac{1}{w^*}$. By induction, we have

$$P^n(x^*, \cdot) = R^n(x^*)\delta_{x^*} + (1 - R^n(x^*))\pi(\cdot),$$

and therefore $\|P^n(x^*, \cdot) - \pi\|_{TV} = R^n(x^*)\|\delta_{x^*} - \pi\|_{TV} = (1 - \frac{1}{w^*})^n$.

Theorem 3 gives the exact converge speed as the upper bound matches with the lower bound exactly. This result closes the gap in the literature regarding the convergence speed of IMH algorithm. For example, Smith and Tierney [ST96] proved that $d_{IMH}(t)$ is upper bounded by a term of order at most $(1 - \frac{1}{w^*})^n$ (see page 5 of [ST96]). Similar result is also proved by Tierney ([Tie94], Corollary 4). Mengerson and Tweedie ([MT+96], Theorem 2.1) proves $d_{IMH}(t) \leq (1 - \frac{1}{w^*})^n$. But it seems none of the previous results contain the lower bound explicitly.

The main assumption in Theorem 3 is that $w^*$ is attained at some $x^* \in X$. When the supremum cannot be actually attained, the exact formula 3.3 does not hold. But our next result shows $d_{IMH}(n)$ still decays at the rate of $1 - \frac{1}{w^*}$.

**Theorem 4.** Assume $w^* = \sup_x w(x) < \infty$ and $w(x) < w^*$ for all $x$. Then

1. For any $\epsilon > 0$, we have

$$d_{IMH}(n) = (1 - \frac{1}{w^*} - \epsilon)^n \leq d_{IMH}(n) \leq (1 - \frac{1}{w^*})^n$$

2.

$$\lim_{n \to \infty} \frac{\log d_{IMH}(n)}{n} = 1 - \frac{1}{w^*}$$

**Remark 2.** Formula 3.18 shows the decay rate of $d_{IMH}(n)$ is $1 - \frac{1}{w^*}$. It is worth mentioning that 3.18 is slightly weaker than $d_{IMH}(n) = \Theta((1 - \frac{1}{w^*})^n)$. For example, sequences such as $(1 - \frac{1}{w^*})^n/n$ satisfies 3.18 but is not $\Theta((1 - \frac{1}{w^*})^n)$.

**Proof.** We only prove the first part of Theorem 4 as the second part is then immediate. Moreover, it suffices to prove the left part of inequality 3.17, as the right part is proved in Theorem 3.

For any $\epsilon > 0$, choose $\delta > 0$ such that $\delta < \frac{\epsilon w^*}{1+\epsilon w^*}$. By definition, there exists $x_\delta$ such that $w(x_\delta) > w^* - \delta$. Then $R(x_\delta)$ can be bounded by 3.13

$$R(x_\delta) \geq \int_X (1 - \frac{w(y)}{w(x_\delta)})p(y)dy = 1 - \frac{1}{w(x_\delta)} > 1 - \frac{1}{w^* - \delta} \geq (1 - \frac{1}{w^*} - \epsilon)$$

By Lemma 1, we have:

$$d_{IMH}(t) = \max_{x \in X} \|P^t(x, \cdot) - \pi\|_{TV} \geq \|P^t(x_\delta, \cdot) - \pi\|_{TV} \geq R^n(x_\delta) = (1 - \frac{1}{w^*} - \epsilon)^n,$$

which finishes our proof.
Now we provide some concrete examples to help illustrate the use of Theorem 3 and
Theorem 4. The next example is discussed in Smith and Tierney [ST96], and Jones and
Hobert [JH01].

**Example 2 (Exponential target with exponential proposal).** Suppose the target distribution
is \( \text{Exp}(1) \), that is, \( \pi(x) = e^{-x} \) and \( X = [0, \infty) \). Consider the IMH chain \( \Phi \) with an \( \text{Exp}(\theta) \) proposal, that is, \( p(x) = \theta e^{-\theta x} \). Therefore the weight function has the following form

\[
 w(x) = \frac{\pi(x)}{p(x)} = \frac{1}{\theta} e^{-(1-\theta)x}.
\]

It is not hard to verify that \( \sup w(x) < \infty \) is equivalent to \( 0 < \theta \leq 1 \). Therefore, when
\( \theta \in (0, 1] \), the algorithm is uniformly ergodic; otherwise, it is not geometrically ergodic.

For any fixed \( \theta \in (0, 1] \), we have \( w^*(\theta) = \frac{1}{\theta} \) which is attained at \( x = 0 \). Therefore we
could apply Theorem 3 and conclude that:

\[
 d_{\text{IMH}}(\theta, n) = (1 - \theta)^n.
\]

This also shows the ‘optimal choice’ of \( \theta \) is \( \theta = 1 \), which is obviously true as \( \theta = 1 \) corresponds
to sampling from the stationary distribution \( \text{Exp}(1) \) directly.

For any arbitrary fixed \( \theta \) and accuracy \( \epsilon \), we could also solve the equation

\[
 \epsilon = (1 - \theta)^n.
\]

for \( n \) to derive the number of steps that is necessary and sufficient for the IMH chain to
converge within \( \epsilon \)-accuracy. In this case we have

\[
 n = \frac{\log \epsilon}{\log(1 - \theta)} \quad (3.20)
\]

steps are both necessary and sufficient. Let the accuracy \( \epsilon \) be fixed at 0.01, then 3.20 gives
us \( n = 6.64 \) when \( \theta = 0.5 \); \( n = 43.71 \) when \( \theta = 0.1 \); and \( n = 458.21 \) when \( \theta = 0.01 \). The
relationship between \( n \) and \( \theta \) is also shown in Figure 1. As expected, when \( \theta \) is getting closer
to 1, the IMH algorithm is converging faster.
The next example is a higher-dimensional example. It is related to Bayesian statistics, where the target distribution is the posterior distribution given prior and data.

**Example 3** (Bayesian inference with Multinomial Likelihood). Let \( \vec{x} = (x_1, \cdots, x_K) \) be a sample of size \( N = \sum_{i=1}^{K} x_i \) from a \( K \)-category multinomial distribution with parameter \( \vec{\theta} = (\theta_1, \cdots, \theta_K) \). In other words, the likelihood of is of the following form

\[
p_{\vec{\theta}}(\vec{x}) = \frac{N!}{\prod_{i=1}^{K} x_i!} \theta_i^{x_i}.
\]  

(3.21)

Let \( \pi(\vec{\theta}) \) be a fixed prior on the \( K \)-simplex \( \Theta \), i.e.,

\[
\Theta = \{ \theta \in \mathbb{R}^K : \sum_{i=1}^{K} \theta_i = 1 \text{ and } \theta_i \geq 0 \text{ for all } i \}.
\]

Given data \( x \), it is of our interest to sample from the posterior distribution \( \pi(\vec{\theta}|\vec{x}) \) which has density

\[
\pi(\vec{\theta}|\vec{x}) \propto \pi(\vec{\theta}) p_{\vec{\theta}}(\vec{x}).
\]

In this example, we assume the prior is chosen as a Dirichlet distribution, with parameter \( \vec{\alpha} \), i.e.,

\[
\pi(\vec{\theta}) \propto \theta_1^{\alpha_1 - 1} \cdots \theta_K^{\alpha_K - 1}.
\]  

(3.22)
Since Dirichlet distribution is the conjugate prior of the multinomial likelihood, standard calculation shows the posterior is still a Dirichlet distribution, with parameter \( \alpha + \bar{x} = (\alpha_1 + x_1, \cdots, \alpha_K + x_K) \).

Consider the IMH chain \( \Phi \) with a Uniform(\( \Theta \)) proposal. Therefore the weight function can be calculated as:

\[
w(\bar{\theta} | \bar{x}) = \frac{\pi(\bar{\theta} | \bar{x})}{p(\bar{\theta})} = \frac{(N + \alpha - 1)!}{(K - 1)! \prod_{i=1}^{K} (x_i + \alpha_i - 1)!} \prod_{i=1}^{K} \theta_{i}^{x_i + \alpha_i - 1}, \tag{3.23}
\]

where \( \alpha \doteq \sum_i \alpha_i \).

It is clear that the maximum of \( w(\bar{\theta} | \bar{x}) \) is attained at

\[
\theta^* = \left( \frac{x_1 + \alpha_1 - 1}{N + \alpha - K}, \cdots, \frac{x_K + \alpha_K - 1}{N + \alpha - K} \right),
\]

therefore we have:

\[
w^* = \frac{\pi(\bar{\theta} | \bar{x})}{p(\bar{\theta})} = \frac{(\alpha + N - 1)!}{(K - 1)! \prod_{i=1}^{K} (x_i + \alpha_i - 1)!} \prod_{i=1}^{K} (x_i + \alpha_i - 1)^{x_i + \alpha_i - 1} \frac{(x + \alpha - K)^{x + \alpha - K}}. \tag{3.24}
\]

Therefore, we can apply Theorem 3 and conclude that:

\[
d_{IMH}(n | \bar{x}) = (1 - \frac{1}{w^*})^n,
\]

where \( w^* \) is defined in 3.24.

We can also investigate the asymptotic behavior of 3.24. Assume \( K \) is fixed, and

\[
\bar{x}/N \to \bar{p} = (p_1, \cdots, p_K)
\]

when \( N \to \infty \). Here \( \bar{p} \) can be understand as the true (but unknown) parameter for the multinomial distribution. When the number of trials \( N \) is large enough, the law of large numbers guarantees the proportion of each category converges to \( \bar{p} \). Sterling’s formula gives the following approximation of \( w^* \):

\[
w^* \sim \frac{1}{(K - 1)!} \sqrt{\frac{N^{K-1}}{(2\pi)^{K-1} \prod_{i=1}^{K} P_i}}. \tag{3.25}
\]

Therefore, the number of steps for convergence with \( \epsilon \)-accuracy is approximately:

\[
n(\epsilon) = \frac{\log \epsilon}{\log (1 - \frac{1}{w^*})} \sim (\log \epsilon)w^* \sim \frac{\log \epsilon}{\sqrt{(2\pi)^{K-1} \prod_{i=1}^{K} P_i}} N^{\frac{K-1}{2}},
\]

so \( \Theta(N^{\frac{K-1}{2}}) \) steps are necessary and sufficient for convergence.

When \( K = 2 \), the model is known as Beta-Binomial example. Suppose we choose prior \( \pi = \text{Uniform}(\Theta) \), where \( \Theta = \{(p, q) | 0 \leq p \leq 1, q = 1 - p \} \). The relation between \( n(0.01) \) and \( N \) is shown in Figure 2. As shown in Figure 2, the algorithm converges faster when \( p \) is closer to 0.5, and \( n(\epsilon) \) grows sublinearly with respect to \( N \).
Remark 3. If one uses some non-conjugate priors such as truncated Gaussian priors, then the calculation for \( w^* \) is much more complicated. The asymptotics can still go through using a comparison argument, so \( \Theta(N^{k-1/2}) \) steps are still necessary and sufficient for convergence. Detailed calculations of the Beta likelihood with non-conjugate priors can be found in Section 4.2.1 of Wang [Wan20].

![Figure 2: Number of steps for the IMH algorithm to converge, \( \epsilon = 0.01 \)](image)

Now we briefly conclude this part, in this part we provide answer to our first question (Q1) proposed in Section 1 for the IMH algorithm on general state spaces. Our results can be briefly summarized below:

1. If \( w^* = \infty \), then the IMH chain is not geometrically ergodic, thus the chain does not converge at a geometric rate.

2. If \( w^* < \infty \) and can be attained at some \( x^* \in \mathcal{X} \), then we can derive the following exact converge speed:

\[
d_{\text{IMH}}(n) = (1 - \frac{1}{w^*})^n
\]

3. If \( w^* < \infty \) but cannot be attained at any \( x \in \mathcal{X} \), then the exact convergence rate is still \((1 - \frac{1}{w^*})\), though Formula 3.3 may not hold.
3.2 Convergence rate at every point

In the Section 3.1, we have shown the maximum distance between the IMH chain and its stationary distribution decays at the rate of \((1 - \frac{1}{w^*})\). However, as we mentioned in Section 1, this only gives an upper bound for the convergence rate of among all possible initializations. To be more precise, we define the following functions to describe the convergence rate at a single point \(x\).

**Definition 7.** Let

\[
 r_-(x) = \liminf_{n \to \infty} \frac{\log \|P^n(x, \cdot) - \pi\|_{TV}}{n}
\]

and

\[
 r_+(x) = \limsup_{n \to \infty} \frac{\log \|P^n(x, \cdot) - \pi\|_{TV}}{n}.
\]

In particular, if \(r_-(x) = r_+(x)\) for some \(x \in X\), we define the convergence rate at \(x\) to be

\[
 r(x) = \lim_{n \to \infty} \frac{\log \|P^n(x, \cdot) - \pi\|_{TV}}{n}.
\]

The following corollary is immediate using Lemma 1, Theorem 3, and Theorem 4

**Corollary 2.** With all the notations as above, we have

- For every \(x \in X\),
  \[ R(x) \leq r_-(x) \leq r_+(x) \leq 1 - \frac{1}{w^*}. \]

- \(\sup_x r_+(x) = \sup_x r_-(x) = 1 - \frac{1}{w^*}\)

- If there is \(x^* \in X\) such that \(w(x^*) = w^*\), then \(r(x^*) = 1 - \frac{1}{w^*}\).

Corollary 2 gives a lower bound on \(r_-(x)\) as well as an upper bound on \(r_+(x)\). However, the lower bound does not match the upper lower unless \(x = x^*\). For example, let \(x_n\) be a sequence of points such that \(w(x_n) \to \inf_x w(x)\), then it is clear from 3.13 that \(R(x_n) \to 0\). However, the upper bound is a constant \(1 - \frac{1}{w^*}\) so the two bounds does not match with each other.

It is therefore natural to ask whether \(r(x)\) always exists, if so, what is the value of \(r(x)\)? Our next result shows, for the IMH algorithm, every initialization converge at the same rate, which is \(1 - \frac{1}{w^*}\).

**Theorem 5.** Suppose \(w^* < \infty\). If one of the following two assumptions holds:

- \(w^* = w(x^*)\) for some \(x^* \in X\), and both \(p\) and \(\pi\) are locally Lipschitz.

- \(w^* > w(x)\) for any \(x \in X\), and \(w\) is Lipschitz continuous.

Then for every \(x \in X\):

\[
 r(x) = r_-(x) = r_+(x) = 1 - \frac{1}{w^*}. \tag{3.26}
\]

In other words, the convergence rate for every \(x\) equals \(1 - \frac{1}{w^*}\).
To prove Theorem 5, we need to first introduce the previous result by Smith and Tierney [ST96] on the exact transition probabilities for the IMH chain. Let $\tilde{\Pi}$ and $\tilde{P}$ be two probability distribution functions (PDF) on $\mathbb{R}^+$ defined by:

$$\tilde{\Pi}(w) = \pi(C(w)) = \int_{y \in C(w)} \pi(y) dy$$
$$\tilde{P}(w) = \pi(C(w)) = \int_{y \in C(w)} p(y) dy,$$

and let $\tilde{\Pi}(dw)$ and $\tilde{P}(dw)$ denote the corresponding probability measure on Borel sets of $\mathbb{R}^+$. Furthermore, let function $\lambda : \mathbb{R}^+ \to \mathbb{R}$ be

$$\lambda(w) = \int_{v \leq w} (1 - \frac{v}{w}) \tilde{P}(dv) = \tilde{P}(w) - \frac{\tilde{\Pi}(w)}{w}. \quad (3.27)$$

Comparing 3.27 with 3.13, it is not hard to verify that $\lambda(w) = 1 - \frac{1}{w}$ for $w \geq w^*$, and $\lambda(w(x))$ equals the rejection probability $R(x)$ for any $x \in \mathcal{X}$.

The $n$-step transition probability of the IMH algorithm is given by Theorem 1 of Smith and Tierney [ST96]:

**Theorem 6** (Theorem 1 of Smith and Tierney [ST96]). The $n$-step transition kernel for the IMH chain is given by:

$$P^n(x, dy) = T_n(\max\{w(x), w(y)\})\pi(y)dy + R^n(w(x))\delta_x(dy), \quad (3.28)$$

where $T_n : \mathbb{R}^+ \to \mathbb{R}$ is defined by:

$$T_n(w) = \int_w^\infty \frac{n\lambda^{n-1}(v)}{v^2} dv. \quad (3.29)$$

Using Theorem 6, the proof of Theorem 5 is given as follows:

**Proof of Theorem 5.** We prove Theorem 5 under the first assumption. That is, both $p$ and $\pi$ are locally Lipschitz continuous, and $w^*(x) = w(x^*)$ for some $x^* \in \mathcal{X}$. By Corollary 2, it suffices to prove $r_-(x) \geq (1 - \frac{1}{w^*})$. We prove this by contradiction. Suppose there exist an $\epsilon > 0$ and some $x_0 \in \mathcal{X}$ such that

$$r_-(x_0) \leq 1 - \frac{1}{w^* + \epsilon},$$

then we claim $w(x_0) \leq w^* - \epsilon$. Otherwise we have

$$R(x_0) \geq \int_{\mathcal{X}} (1 - \frac{w(y)}{w(x_0)})p(y)dy = 1 - \frac{1}{w(x_0)} > 1 - \frac{1}{w^* - \epsilon},$$

which contradicts with $R(x_0) \leq r_-(x_0)$

Given $w(x_0) \leq w^* - \epsilon$, for any $y$ with $w(y) \geq w^* - \epsilon$, Theorem 6 gives us:

$$P^n(x_0, dy) = T_n(w(y))\pi(y)dy. \quad (3.30)$$
Notice that, we have the following estimate for $T_n(w)$:

$$T_n(w) = \int w^\star \frac{n\lambda^{n-1}(v)}{v^2} dv + \int_{w^\star}^{\infty} \frac{n(1 - \frac{1}{v})^{n-1}}{v^2} dv$$

$$= \int w^\star \frac{n\lambda^{n-1}(v)}{v^2} dv + 1 - \left(1 - \frac{1}{w^\star}\right)^n$$

$$\leq n\lambda^{n-1}(w^\star) \int_{w^\star}^{w^\star} \frac{1}{v^2} dv + 1 - \left(1 - \frac{1}{w^\star}\right)^n$$

$$= n(1 - \frac{1}{w^\star})^{n-1} \frac{w^\star - w}{ww^\star} + 1 - \left(1 - \frac{1}{w^\star}\right)^n$$

Let $D_n = \{y \in X : w(y) > \max\{1, w^\star - \frac{w^\star - 1}{2n}\}\}$ be a subset of $X$. Then for any $y \in D_n$,

$$T_n(w(y)) \leq 1 - \frac{1}{2}(1 - \frac{1}{w^\star})^n$$

Therefore, when $n$ is large enough such that $w^\star - \frac{w^\star - 1}{2n} > w^\star - \epsilon$, by 3.30 we have

$$\|P^n(x_0, \cdot) - \pi\|_{TV} \geq |P^n(x_0, D_n) - \pi(D_n)| \geq \frac{1}{2} \pi(D_n)(1 - \frac{1}{w^\star})^n$$

Now we claim there exists a universal constant $c > 0$ such that

$$\pi(D_n) \geq \frac{c}{n}$$

If the claim is true, then $r_-(x_0) \geq 1 - \frac{1}{w^\star} - \lim_n \log n \log \frac{c}{2n} = 1 - \frac{1}{w^\star}$, as desired.

To prove the claim, notice that the function $w$ is also locally Lipschitz at $x^\star$ as both $\pi$ and $p$ are locally Lipschitz. Therefore, there exists some $c' > 0$ such that when $y \in [x^\star - \frac{c'}{n}, x^\star + \frac{c'}{n}]$, $w(y) \geq w^\star - \frac{w^\star - 1}{2n}$. For $n$ large enough such that $w^\star - \frac{w^\star - 1}{2n} > 1$, we have

$$\pi(D_n) \geq \int_{y \in [x^\star - \frac{c'}{n}, x^\star + \frac{c'}{n}]} \pi(y) dy \geq \frac{c}{n},$$

where the last inequality uses the continuity of $\pi$.

Proof of Theorem 5 under the second assumption is almost the same as above. The only difference is that we need to find a sequence $x_n$ such that $w(x_n) \to w^\star$, and thus we need to further assume both $p$ and $\pi$ are Lipschitz continuous.

We believe the extra assumption requiring $\pi$ and $p$ locally Lipschitz continuous (or $w$ Lipschitz continuous) is reasonable and is satisfied in almost all the practical assumptions. On the one hand, in most of the practical situations, both $\pi$ and $p$ are continuously differentiable, which implies locally Lipschitz continuous. On the other hand, we do not know if Theorem 5 is true in full generality. It is an outstanding open problem to prove (or disprove) $r(x) = 1 - \frac{1}{w^\star}$ for all $x$ without the ‘locally Lipschitz’ assumption.

We can apply Theorem 5 to Example 2 and Example 3 to obtain the convergence rate at every point.
Example 4 (Exponential target with exponential proposal, continued). With all the settings same as Example 2, since both \( p \) and \( \pi \) are continuously differentiable and are thus locally Lipschitz, we apply Theorem 5 to conclude the convergence rate for every \( x \in [0, \infty) \) equals \( 1-\theta \). In Section 5 of Smith and Tierney [ST96], they have calculated the transition probability via formula 3.28 explicitly, and have:

\[
\Pr(\Phi_n > y | \Phi_0 = x) = \left(1 + \frac{(1-\theta)^n}{n\theta}\right) e^{-y} + o\left(\frac{(1-\theta)^n}{n}\right).
\]

The dominant term of total variation distance is also \((1-\theta)^n\), which agrees with our result.

Example 5 (Bayesian inference with Multinomial likelihood, continued). With all the settings same as Example 2, since both \( p \) and \( \pi \) are continuously differentiable and are thus locally Lipschitz, we apply Theorem 5 to conclude the convergence rate for every \( \theta \in \Theta \) equals

\[
1 - \frac{1}{w^*},
\]

where

\[
w^* = \frac{(\alpha + N - 1)!}{(K-1)! \prod_{i=1}^{K} (x_i + \alpha_i - 1)!} \frac{\prod_{i=1}^{K} (x_i + \alpha_i + 1)}{(x + \alpha - K)^{x + \alpha - K}}
\]
as we have calculated in 3.24.

Now we conclude this section. In Section 3.2 we answer Q2 for the IMH algorithms on general state spaces. Our results suggest, under mild conditions, the IMH algorithm converges at the same rate (which is \( 1 - \frac{1}{w^*} \) as shown in 3.1) for every \( x \in \mathcal{X} \). Combining Section 3.1 with Section 3.2, we can answer Q1 and Q2 together in the following way: For an IMH chain on a general state space, the exact converge rate equals \( 1 - \frac{1}{w^*} \), and every point has the same convergence rate. In particular, if \( w^* \) can be attained at some \( x^* \in \mathcal{X} \), then the exact convergence speed for \( x^* \) equals \((1 - \frac{1}{w^*})^n\).

4 IMH on discrete state spaces

Let \( \mathcal{X} = \{1, 2, \ldots, n\} \) be a finite discrete sample space. Let \( \pi \) and \( p \) be the target and proposal probability mass function (PMF), respectively. Again, we define the ratio function \( w = \frac{\pi}{p} \) the same as Section 3. Without loss of generality, we assume \( w \) is non-increasing, i.e.,

\[
w_1 \geq w_2 \cdots \geq w_n.
\]

In other words, \( w^* = w_1 \), the supremum of function \( w \) is attained at the first state.

The IMH algorithm 2 has the following transition probability kernel:

\[
P(i, j) = \begin{cases} 
p_j \min\left\{1, \frac{w_j}{w_i}\right\} & \text{if } j \neq i \\
p_i + \sum_k p_k \max\left\{0, 1 - \frac{w_k}{w_i}\right\} & \text{if } j = i
\end{cases}
\]  

(4.1)

It is tempting to believe that all the results in Section 3 still hold for discrete state space. However, our next result shows, though the exact convergence rate is still \( 1 - \frac{1}{w^*} \), Theorem 3 does not hold.
4.1 Exact convergence rate

**Theorem 7.** With all the notations as above, we have:

\[(1 - \pi_1)(1 - \frac{1}{w^*})^t \leq d_{IMH}(t) \leq (1 - \frac{1}{w^*})^t \quad (4.2)\]

**Remark 4.** It is worth mentioning that the upper bound is shown by Liu [Liu96], page 117. But the lower bound is not contained in [Liu96].

**Proof.** The right part of inequality 4.2 is already proved in Theorem 3. For the left part, by 4.1, we immediately have:

\[P(1, \cdot) = (1 - \frac{1}{w_1})\delta_1 + \frac{1}{w_1}\pi.\]

Therefore, using the fact \(\pi P = \pi\), we have:

\[P^t(1, \cdot) = (1 - \frac{1}{w_1})^t\delta_1 + (1 - (1 - \frac{1}{w_1})^t)\pi.\]

Then we have

\[d_{IMH}(t) = \max_{x \in X} \|P^t(x, \cdot) - \pi\|_{TV} \geq \|P^t(1, \cdot) - \pi\|_{TV} \geq (1 - \frac{1}{w_1})^t \|\delta_1 - \pi\|_{TV} = (1 - \pi_1)(1 - \frac{1}{w_1})^t,\]

which finishes the proof.

The following corollary is immediate:

**Corollary 3.** The convergence rate \(r\) (defined in Definition 7) at \(x = 1\) equals \(1 - \frac{1}{w_1}\).

**Proof.** Taking the logarithm on both sides of inequality 4.2 and the result follows. \qed

Our next result shows inequality 4.2 is sharp in the sense that it cannot be improved without further assumptions.

**Proposition 1.** There exist two Markov chains \(\Phi_1\) and \(\Phi_2\) such that \(d_{IMH}(\Phi_1, t) = (1 - \frac{1}{w_1})^t\) and \(d_{IMH}(\Phi_2, t) = (1 - \pi_1)(1 - \frac{1}{w_1})^t\).

**Proof.** Let \(\Phi_1\) be the Markov chain with target distribution uniform over \(\{1, 2, \cdots, K\}\) and proposal distribution uniform over \(\{1, 2, \cdots, 2K\}\). In this case \(w_1 = w^* = 2\). Straightforward calculation gives us:

\[\|P^t(i, \cdot) - \pi\|_{TV} = (1 - \frac{1}{K})0.5^t \quad \text{for } i \in \{1, \cdots, K\}\]

\[\|P^t(j, \cdot) - \pi\|_{TV} = 0.5^t \quad \text{for } j \in \{K + 1, \cdots, 2K\}\]

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Therefore, $d_{\text{IMH}}(\Phi_1, t) = (1 - \frac{1}{w_1})^t$.

Let $\Phi_2$ be a Markov chain on \{1, 2, \ldots, K + 1\} with target distribution:

$$\pi(i) = \begin{cases} 
\frac{1}{2} & \text{if } i = 1 \\
\frac{1}{2K} & \text{if } i \neq 1,
\end{cases}$$

and proposal distribution

$$q(i) = \begin{cases} 
\frac{1}{4} & \text{if } i = 1 \\
\frac{3}{4K} & \text{if } i \neq 1,
\end{cases}$$

Then it is clear that $w^* = 2$. Meanwhile,

$$\|P^t(1, \cdot) - \pi\|_{TV} = (1 - \pi_1)0.5^t = 0.5^{t+1}. \quad (4.3)$$

$$\|P^t(j, \cdot) - \pi\|_{TV} = 0.5 - P^t(j, 1) = 0.5^{t+1}. \quad (4.4)$$

$$\|P^t(j, \cdot) - \pi\|_{TV} = 0.5 - P^t(j, 1) = 0.5^{t+1}. \quad (4.5)$$

Therefore, $d_{\text{IMH}}(\Phi_2, t) = (1 - \pi_1)(1 - \frac{1}{w_1})^t$. \hfill \Box

As we can see in Theorem 3, Theorem 7, and Proposition 1, the ‘maximal total variation distance’ $d_{\text{IMH}}$ has different behavior between general and discrete state spaces. For chains on general state spaces, $d_{\text{IMH}}(t) = (1 - \frac{1}{w_1})^t$ can be calculated out explicitly. For chains on discrete state spaces, however, we can only bound $d_{\text{IMH}}$ between $(1 - \pi_1)(1 - \frac{1}{w_1})^t$ and $(1 - \frac{1}{w_1})^t$. An intuitive way of understanding this discrepancy is: As we mentioned in Remark 1 and Theorem 7, for both discrete and continuous state space, a common lower bound for $d_{\text{IMH}}(t)$ is

$$(1 - \pi(\{x^\star\}))(1 - \frac{1}{w_1})^t,$$

and a common upper bound is

$$(1 - \frac{1}{w_1})^t.$$ If $\mathcal{X}$ is a general state space, then any probability density on $\mathcal{X}$ is atomless. Therefore we have $\pi(\{x^\star\}) = 0$ and the lower bound matches the upper bound. However, for discrete state $\mathcal{X}$, $\pi(\{x^\star\}) = \pi_1$ not necessarily equals 0, therefore the lower bound does not match the upper bound.

In conclusion, in this part, we provide answer to our first question (Q1) proposed in Section 1 for the IMH algorithm on general state spaces. We show the exact convergence rate is still $1 - \frac{1}{w_1}$. However, unlike the general state space case, the ‘exact convergence speed’ for the IMH algorithm is not always $(1 - \frac{1}{w^*})^t$. We have $(1 - \pi_1)(1 - \frac{1}{w^*})^t \leq d_{\text{IMH}}(n) \leq (1 - \frac{1}{w^*})^t$ instead.

### 4.2 Convergence rate at every point

In this section, we focus on proving the following theorem, which shows the convergence rate $r(x)$ equals $1 - \frac{1}{w^*}$ for every $x \in \mathcal{X}$.
Theorem 8. Let $\Phi$ be an IMH chain on discrete finite state space $X = \{1, \cdots, n\}$. Then for every $x \in X$:

$$r(x) = r_-(x) = r_+(x) = 1 - \frac{1}{w^*}.$$  \hspace{1cm} (4.6)

In other words, the convergence rate for every $x$ equals $1 - \frac{1}{w^*}$.

To prove Theorem 8, we first review the following result from Liu [Liu96], which gives all the eigenvalues and the corresponding eigenvectors of the Markov transition kernel $P$ (defined in 4.1).

Theorem 9 (Theorem 2.1 of Liu [Liu96]). Let $\Phi$ be the IMH chain with transition kernel $P$ given by 4.1. Then all the eigenvalues of the transition matrix are:

$$\lambda_0 = 1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0,$$  \hspace{1cm} (4.7)

where

$$\lambda_k = \sum_{i=k}^{n} (p_i - \frac{\pi_i}{w_k}).$$

The right eigenvector corresponding to $\lambda_k$ ($k > 0$) is

$$v_k = (0, 0, \cdots, 0, S_{\pi}(k+1), -\pi_k, \cdots, -\pi_k)^T,$$  \hspace{1cm} (4.8)

where there are $k-1$ zero entries and $S_{\pi}(j) = \pi_j + \pi_{j+1} + \cdots + \pi_n$ for every $j$. Moreover, $\langle v_i, v_j \rangle_{\pi} = 0$ for $i \neq j$.

For self-containedness, we provide a short proof of Theorem 9, which is slightly different from Liu’s original proof. We start with the following lemma and then show that Theorem 9 can be viewed as a special case of this lemma.

Lemma 3 (Rank-one perturbation). Let $A$ be a $n \times n$ matrix with (not necessarily distinct) eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $u_1, \cdots, u_n$ be the corresponding (right) eigenvectors. Then for any $u \in \mathbb{R}^n$, all the eigenvalues of the following matrix

$$B = A + uu^T$$

are

$$\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n + uu^T.$$  \hspace{1cm} (4.9)

If $uu^T \neq -\lambda_n$, then the corresponding eigenvectors of $B$ can be written as:

$$\tilde{u}_j = \begin{cases} u_n & \text{if } j = n \\ u_j - \frac{\lambda_j - uu^T u_n}{\lambda_n + uu^T u_n} u_n & \text{if } j \neq n \end{cases}$$

Proof. The proof is straightforward calculation so we omit it here. \hfill $\square$

Remark 5. When $uu^T u_n = -\lambda_n$, the eigenvectors of $B$ can still be derived without too much effort. Since Lemma 3 is enough to include Theorem 9 as a special case, we do not cover any further extension here.
\textbf{Proof of Theorem 9.} Notice that the transition kernel $P$ can be decomposed as the summation of an upper-diagonal matrix and a rank-one matrix

$$P = D + ep^\top$$  \hspace{1cm} (4.9)

where

$$D = \begin{pmatrix}
\lambda_1 & \frac{\pi_2}{w_1} - p_2 & \cdots & \frac{\pi_n}{w_1} - p_{n-1} & \frac{\pi_n}{w_1} - p_n \\
0 & \lambda_2 & \cdots & \frac{\pi_n}{w_2} - p_{n-1} & \frac{\pi_n}{w_2} - p_n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{\pi_n}{w_{n-1}} - p_{n-1} & \frac{\pi_n}{w_{n-1}} - p_{n-1} \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix},$$

$$e = (1, 1, \ldots, 1)^\top,$$

$$p = (p_1, p_2, \ldots, p_n)^\top.$$  

Since $e$ is a common eigenvector of both $P$ and $ep^\top$ with respect to eigenvalue 1, it is an eigenvector of $D$ with respect to eigenvalue 0. We can apply Lemma 3 and the result follows immediately. \hfill \Box

Now we are ready to prove Theorem 8.

\textbf{Proof of Theorem 8.} Let $f_k$ be the normalized eigenfunction of $v_k$, that is:

$$f_k = v_k / \langle v_k, v_k \rangle_\pi.$$  

Then the spectral theorem of reversition transition matrix gives us (see Lemma 12.16 of Levin et al. [LP17] for a proof):

$$\left\| \frac{P^t(x, \cdot)}{\pi(\cdot)} - 1 \right\|_{2,\pi}^2 = \sum_{k=1}^{m-1} f_k(x)^2 \lambda_k^{2t} \geq f_1(x)^2 \lambda_1^{2t}$$  \hspace{1cm} (4.10)

Notice that $\lambda_1 = 1 - \frac{1}{w_1} = 1 - \frac{1}{w^*}$ and $v_1 = (1 - \pi_1, -\pi_1, \ldots, -\pi_1)$ does not have any zero entry. We conclude

$$\left\| \frac{P^t(x, \cdot)}{\pi(\cdot)} - 1 \right\|_{2,\pi} \geq c(\pi)(1 - \frac{1}{w^*})^t$$  \hspace{1cm} (4.11)

for every $x$, where $c(\pi)$ is a positive constant depending only on $\pi$. Therefore, for every $x \in \mathcal{X}$, the $L_2$ convergence between $P^t(x, \cdot)$ to $\pi$ equals $1 - \frac{1}{w^*}$ for every $x$.

Now it remains to prove the $L_1$ convergence rate also equals $1 - \frac{1}{w^*}$ for every $x$. Notice that:

$$\|P^t(x, \cdot) - \pi\|_{\text{TV}} = \frac{1}{2} \left\| \frac{P^t(x, \cdot)}{\pi(\cdot)} - 1 \right\|_{1,\pi} = \frac{1}{2} \sum_{y \in \mathcal{X}} \left| \frac{P^t(x, y)}{\pi(y)} - 1 \right| \pi(y) \geq \frac{\pi_*}{2} \max_y \left| \frac{P^t(x, y)}{\pi(y)} - 1 \right|,$$  \hspace{1cm} (4.12)
where \( \pi_* = \min_y \pi(y) \), and
\[
\left\| \frac{P^t(x, \cdot)}{\pi(\cdot)} - 1 \right\|_{2, \pi} = \sqrt{\sum_{y \in \mathcal{X}} \left( \frac{P^t(x, y)}{\pi(y)} - 1 \right)^2 \pi(y)} \leq \max_y \left| \frac{P^t(x, y)}{\pi(y)} - 1 \right| \tag{4.13}
\]

Combining 4.11, 4.12, and 4.13, we have:
\[
\| P^t(x, \cdot) - \pi \|_{TV} \geq \frac{\pi_*}{2} \left\| \frac{P^t(x, \cdot)}{\pi(\cdot)} - 1 \right\|_{2, \pi} \geq \frac{\pi_* c(\pi)}{2} (1 - \frac{1}{w^*})^t \tag{4.14}
\]
for any \( x \in \mathcal{X} \).

Taking logarithm on both sides of 4.14 and we have
\[
r^{-}(x) \geq 1 - \frac{1}{w^*}.
\]

Since Theorem 7 gives us \( r^+(x) \leq 1 - \frac{1}{w^*} \), we have \( r(x) = 1 - \frac{1}{w^*} \), as desired.

Theorem 5 and Theorem 7 shows that, the IMH chain has the same convergence rate for every initialization \( x \in \mathcal{X} \), regardless of the sample space is continuous or discrete. This answers Q2 in Section 1.

5 Connections with Random-walk Metropolis-Hastings algorithms

In this section, we discuss the connections between the convergence of IMH and RWMH algorithms as well as some open problems. This section mainly focuses on algorithms on general state spaces, though many results are still true for discrete state space.

5.1 Geometric ergodicity of RWMH algorithms

In Section 3, Lemma 1, we have shown
\[
\| P^n(x, \cdot) - \pi \|_{TV} \geq R^n(x)
\]
for the IMH algorithm. It is clear that the same result is true for the RWMN algorithm. Therefore, we have the next corollary:

**Corollary 4** (Also proved in Roberts and Tweedie [RT96], Theorem 5.1 and Proposition 5.1). For any Metropolis-Hastings algorithm, if the supremum of rejection probability is 1. In other words,
\[
\text{esssup}_{x \in \mathcal{X}} R(x) = 1. \tag{5.1}
\]

Then the algorithm is not geometrically ergodic.
Corollary 4 shows that the rejection probability is bounded away from 1 is a necessary condition for the geometric ergodicity. For the IMH algorithm, however, this is not only the necessary but also the sufficient condition for the geometric ergodicity. In Atchadé and Perron [AP07], it is shown that Condition 5.1 is the necessary and sufficient condition for the geometric ergodicity of the MH algorithm if the absolutely continuous part of the Markov chain’s transition kernel is compact. They have also conjectured (see page 78 of [AP07]) that even without the compactness assumption, Condition 5.1 is still the necessary and sufficient condition for the geometric ergodicity of the MH algorithm.

Our next result disproves the conjecture of Atchadé and Perron by a counterexample.

**Theorem 10.** There exists a RWMH chain $\Phi$ such that

$$\text{esssup}_{x \in \mathcal{X}} R(x) < 1$$

but $\Phi$ is not geometrically ergodic.

**Proof.** Consider the RWMH algorithm with Cauchy distribution as target, i.e.,

$$\pi(x) = \frac{1}{\pi(1 + x^2)},$$

and proposal distribution $p(x, \cdot) \sim U[x - 1, x + 1]$. It is not hard to check that

$$\text{esssup}_{x \in \mathcal{X}} R(x) = R(0) = 1 - \frac{1}{2} \int_{-1}^{1} \frac{1}{1 + s^2} ds = 1 - \frac{\pi}{2} < 1.$$

However, for any $x_0 \in \mathbb{R}$, the support of $P^n(x_0, \cdot)$ is contained in $[x_0 - n, x_0 + n]$. Therefore,

$$\|P^n(x_0, \cdot) - \pi\|_{TV} \geq \pi((-\infty, x_0 - n]) + \pi([x_0 + n, \infty))$$

$$\geq \int_{|x_0|+n}^{\infty} \frac{1}{\pi(1 + x^2)} dx$$

$$\geq \int_{|x_0|+n}^{\infty} \frac{1}{2\pi x^2} dx$$

$$= \frac{1}{2\pi(|x_0| + n)}$$

Therefore $\|P^n(x_0, \cdot) - \pi\|_{TV}$ has order at least $\frac{1}{n}$ for every $x_0 \in \mathbb{R}$, which is of polynomial decay. Thus $\Phi$ does not satisfy 1.1 and is therefore not geometrically ergodic.

Therefore, Condition 5.1 is necessary but not sufficient for the geometric ergodicity of RWMH algorithms. In contrast, the sufficient conditions for the geometric ergodicity of the RWMH algorithm usually require the target distribution has a sufficiently light tail. We refer the readers to Mengerson and Tweedie [MT+96], and Roberts and Tweedie [RT96] for more related results.
5.2 Convergence rate at every point of RWMH algorithms

It is shown by Theorem 5 and Theorem 7 that the convergence rate at every point of the IMH algorithms is the same. However, this is not true for RWMH algorithms. Our next example shows the RWMH algorithm may have different convergence rates for different initializations.

Example 6. Consider a RWMH chain $\Phi$ with uniform target distribution $\pi = U[-1, 1]$ and proposal distribution $U[x - \delta, x + \delta]$ for some constant $1 \leq \delta < 2$. Let $T$ be the smallest integer such that $\Phi_T \neq \Phi_0$. Suppose $\Phi_0 \in [1 - \delta, \delta - 1]$, then it is clear that $\Phi_t$ is distributed according to stationary distribution for all $t \geq T$. Therefore, for any $x_0 \in [1 - \delta, \delta - 1]$,

$$\|P^n(x_0, \cdot) - \pi\|_{TV} \leq \mathbb{P}(T > n) = (1 - \delta^{-1})^n$$

However, for any $y_0 > \delta - 1$ or $y < 1 - \delta$, we have

$$\|P^n(y_0, \cdot) - \pi\|_{TV} \geq R^n(y_0) = \left(\frac{\delta - 1 + |y|}{2\delta}\right)^n > (1 - \delta^{-1})^n$$

Therefore, different initializations have different convergence rates. In particular, if one starts the chain at 0, then the convergence rate is $(1 - \delta^{-1})$. However, if one starts the chain at 1 or $-1$, then the convergence rate is no less than 0.5. Therefore, if users choose to start in the range $[1 - \delta, \delta - 1]$, the algorithm will converge much faster than starting at $[-1, 1 - \delta]$ or $[\delta - 1, 1]$.

Example 6 gives a special RWMH algorithm which has different convergence rates on different initializations. Alas, little is known about the convergence rate with different initializations, therefore we do not know if this phenomenon exists for general RWMH algorithms. As we have explained in Section 1, we believe this is worth studying and should be of both theorist and practitioner’s interest.

In conclusion, almost all the previous results on MCMC convergence focus on establishing the uniform upper bound on the convergence rate. Though helpful for providing insights, these bounds are usually conservative and hard to use in practice. Moreover, these results neglect the fact that different initializations may have different convergence rates. In this paper, we give the exact convergence rate for the convergence rate of the IMH algorithm, and show that all the initializations have the same convergence rate on both general and discrete state space. However, examples indicate that our results can not be directly generalized to other Metropolis-Hastings algorithms such as RWMH. This opens up a wide area of research and we hope the reader could help take it further.

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