Structure change of Cooper pairs in color superconductivity *
— Crossover from BCS to BEC ? —

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We discuss a possibility of transition from color superconductivity of the standard BCS type at high density, to Bose-Einstein Condensation (BEC) of Cooper pairs at lower density. Examining two-flavor QCD over a wide range of baryon density, we found the size of a Cooper pair becomes small enough to be comparable to the averaged quark-quark distance at lower density. We also consider the same problem in two-color QCD.

1. Introduction

There are two reasons for expecting that color superconductivity at moderate density could be qualitatively different from the usual weak-coupling superconductivity in metal. They are both related to distinct properties of QCD. As usual, Cooper instability is induced by quark-quark attractive interaction, but what is different in quark matter is that it is effective in principle for all of the quarks inside the Fermi sea. This is because the interaction itself has an attractive channel due to color factor. This property is in clear contrast to the electron superconductivity where the Coulomb interaction is repulsive, and the attractive force by the phonon exchange exists only in a small region $|\epsilon_k - \epsilon_F| < \omega_D$ restricted by the Debye cutoff $\omega_D$. At high baryon density, the attractive force is given by one gluon exchange and becomes weak because the coupling constant at typical momentum scale (the chemical potential $\mu$) becomes small due to the asymptotic freedom. This ensures weak-coupling treatment of the color superconductivity, and much of efforts has been done in this direction [1]. However, this implies, at lower densities, the effective attractive interaction becomes large, which will invalidate the naive weak-coupling treatment. These two properties (absence of analogue of $\omega_D$ and the infrared enhancement of the QCD coupling) suggest that as the density becomes low, Cooper pairing will not be restricted only to a vicinity of the Fermi surface and become much more drastic phenomena. Besides, the Cooper pair itself will change into a tightly bound state with small size. Therefore, it is quite natural to expect that at low density (but still above the critical density of the deconfinement transition) the color superconductivity will turn into “strong-coupling superconductivity” or even “Bose-Einstein Condensation” of tightly bound Cooper pairs. To investigate this possibility is our main purpose of this talk. In the following, we will first discuss two-flavor case over a wide range of baryon density with a single model [2]. Structural change of a Cooper pair is best studied by

*Based on the work with H. Abuki and T. Hatsuda [1] and the on-going collaboration with G. Baym and T. Hatsuda [2].
computing its wavefunction (quark correlation in the color superconductor) which is easily obtained once we know the momentum dependence of a superconducting gap. We will also consider the same problem in the two-color case [2].

2. Gap equation

A common field-theoretic strategy of superconductivity is the Nambu-Gor’kov formalism which uses a two component Dirac spinor \( \Psi = \left( \begin{array}{c} \psi \\ \psi^c \end{array} \right) \), \( \psi^c = C\psi^T \). The extended Fermion propagator \( iS(x - y) \equiv \langle T\Psi(x)\bar{\Psi}(y) \rangle \) is now a \( 2 \times 2 \) matrix. The Schwinger-Dyson equation for self energy \( \Sigma = S_0^{-1} - S^{-1} \) is written as

\[
\Sigma(k) = \left( \begin{array}{cc} M(k) & \Delta(k) \\ \gamma^0 \Delta(k)^\dagger \gamma^0 & M(-k) \end{array} \right) = \int \frac{d^4q}{(2\pi)^4} \frac{g^2}{\Lambda^2} \Gamma^a S(q) \Gamma^a D^{\mu\nu}(k - q),
\]

where we ignore quark mass, \( D^{\mu\nu} = \delta^{ab}D_{\mu\nu} \) is the gluon propagator in medium (which includes Debye screening for electric gluons and Landau damping for magnetic), \( S(q) \) is the full quark propagator, and \( \Gamma^a \) is the quark-gluon vertex, which is taken to be a bare one \( \Gamma^a = \text{diag}(\gamma^a T^a, -\gamma^a (T^a)^T) \). For \( g^2 \) in eq. (1), we use a momentum dependent coupling \( g^2(q, k) \) in the “improved ladder approximation” [3] \( (\beta_0 = (11N_c - 2N_f)/3) \):

\[
g^2(q, k) = \frac{16\pi^2}{\beta_0} \frac{1}{\ln((p^2_{\text{max}} + p^2)/\Lambda^2)}, \quad p_{\text{max}} = \text{max}(q, k),
\]

where \( p^2 \) plays a role of a phenomenological infrared regulator. At high momentum, \( g^2 \) shows the same logarithmic behavior as the usual running coupling with \( \Lambda \) identified with \( \Lambda_{\text{QCD}} \). We adopt \( \Lambda = 400 \text{ MeV} \) and \( p^2 = 1.5 \text{ } \Lambda^2 \) which are determined to reproduce the low energy meson properties for \( N_f = 2, N_c = 3 \) vacuum.

Performing the angular and frequency integrals leads to a gap equation with momentum dependence only. Once we obtain the momentum dependent gap \( \Delta(q) \), we can compute the wavefunction of a Cooper pair (or, \( q-q \) correlation function) in momentum space:

\[
\varphi(q) = \frac{\Delta(q)}{2\sqrt{(q - \mu)^2 + |\Delta(q)|^2}}.
\]

The size of Cooper pairs (the coherence length) is defined as the root mean square radius of coordinate space wavefunction \( \varphi(r) \). Recall that in a typical type-I superconductor in metals, the size of a Cooper pair \( \sim \Delta^{-1} \) is much larger than the typical scale \( \sim k_F^{-1} \) (the ratio is \( k_F/\Delta \sim 10^4 \)), because there is a clear scale hierarchy, \( \Delta \ll \omega_D \ll k_F \). On the other hand, since there is no intrinsic scale \( \omega_D \) in QCD, scale hierarchy at extremely high density simply reads \( \Delta \sim \mu e^{-c/g} \ll k_F \sim \mu \). At lower densities, however, such scale separation becomes questionable for \( g \) is not small.

3. Momentum dependent gap and size of a Cooper pair in \( N_f = 2, N_c = 3 \) [1]

When \( N_f = 2 \) and \( N_c = 3 \), the most attractive channel is the flavor anti-symmetric, color anti-symmetric and \( J = 0^+ \) channel \( \Delta(k) = (\tau_2^{\text{flavor}}) \lambda_2^{\text{(color)}} C\gamma_5 \{\Delta_+(k)\Lambda_+(\hat{k}) + \Delta_-(k)\Lambda_-(\hat{k})\} \), where \( \tau_2 \) is the Pauli matrix acting on the flavor space, \( \lambda_2 \) is a Gell-Mann
matrix, and $C$ is the charge conjugation. $\Lambda_\pm(\mathbf{k}) \equiv (1 \pm \mathbf{k} \cdot \mathbf{\alpha})/2$ is the projector on positive (+) and negative (−) energy quarks. We ignore the effects of chiral condensate.

In Fig. 1(a), we show the gap $\Delta_+(k = |\mathbf{k}|)$ as a solution of the gap equation for a wide range of densities. This result tells us the following things. At high density, (i) there is a sharp peak at the Fermi surface, and (ii) the gap decays rapidly but is nonzero for momentum far away from the Fermi surface. The property (i) is similar to the standard BCS but (ii) is not, which is due to the absence of the Debye cutoff in the gluonic interaction. On the other hand, at low density, (iii) the sharp peak at the Fermi surface disappears, and (iv) color superconductivity at low density is not a phenomenon just around the Fermi surface. This last point can be said after a close look at the effects of each contribution to the gap and computing the occupation number. The result shows that the Fermi surface is diffuse substantially at low density [1].

Computing the size of a Cooper pair, one finds that it is less than 4 fm at the lowest density shown in Fig. 1 ($\mu = 800$MeV) [1]. This is small enough from the usual sense, because the ratio $\xi_c/d_q$ of the Cooper pair size $\xi_c$ to typical length of the system $d_q$, i.e., averaged inter-quark distance for free quarks $d_q = (\pi^2/2)^{1/3}/\mu$, is less than 10, in contrast to $10^5$ at the highest density (Fig. 1(b)). If we further extrapolate the curve in Fig. 1(b) to lower chemical potential and are still in the deconfined phase, tightly bound Cooper pairs ($\xi_c/d_q \sim 1$) may seem to appear. Therefore, loosely bound large Cooper pairs similar to the BCS superconductivity in metals are formed at extremely high density, while at lower density, the size of a Cooper pair is small enough. This smooth transition from $\xi_c/d_q \gg 1$ to $\xi_c/d_q \sim 1$ as $\mu$ decreases is analogous to the crossover from the BCS-type superconductor to the BEC of tightly bound Cooper pairs [5].

**Figure 1.** (a): $\Delta_+(k)$ as a function of $k/\mu$ for various densities $\mu = 2^n\Lambda$ with $n = 1, 2, 3, 12$. (b): Ratio of the coherence length to the average inter-quark distance as a function of $\mu$.

### 4. Two color QCD with two flavors [2]

It is interesting to study the same problem the same way in 2-color QCD [2], where the diquark $\Delta(p) = (\tau_2^{(\text{color})} \tau_2^{(\text{flavor})} \gamma_5)\Delta_\Lambda(p)$ is a gauge invariant “baryon” and at low density, it corresponds to the Nambu-Goldstone (NG) boson of a spontaneous broken enlarged chiral symmetry group [3]. Therefore, bosonic-like description of diquark is expected to
be appropriate at least at low density. The formalism developed in Sect. 2 is directly applicable to this problem, but we here retain the chiral condensate $M \neq 0$. Then the Schwinger-Dyson equation (1) forms coupled equations for $M(p)$ and $\Delta(p)$:

$$M(p) = \frac{3}{8} g^2 \int \frac{d^3 q}{(2\pi)^3} D_{\nu\nu}(p-q) \frac{M(q)}{\sqrt{q^2 + M(q)^2}} \left(1 - n_-(q) - n_+(q)\right),$$

$$\Delta(p) = \frac{3}{8} g^2 \int \frac{d^3 q}{(2\pi)^3} D_{\nu\nu}(p-q) \frac{1}{2} \left[\frac{1}{\epsilon_-(q)} + \frac{1}{\epsilon_+(q)}\right] \Delta(q),$$

where $\epsilon_{\pm}(q) = \sqrt{(E \mp \mu)^2 + \Delta^2}$ is the quasi-particle energy (with $E(q) = \sqrt{q^2 + M^2(q)}$) and $n_{\pm} = \{1 - (E \pm \mu)/\epsilon_{\pm}\}/2$ is the occupation number. It is important to notice that, in the limit $\mu \to 0$, these gap equations have the same form which is invariant under the mixture of vector $(M(p), \Delta(p))$:

$$\left(\begin{array}{c} M(p) \\ \Delta(p) \end{array}\right) \approx \frac{3}{8} g^2 \int \frac{d^3 q}{(2\pi)^3} D_{\nu\nu}(p-q) \frac{1}{\sqrt{q^2 + M^2 + \Delta^2}} \left(\begin{array}{c} M(q) \\ \Delta(q) \end{array}\right) \quad (\mu \to 0).$$

This observation is consistent with the Pauli-Gürsey symmetry which is present at $\mu = 0$ and leads to a strong consequence. For simplicity, we consider the chiral limit. We know that, at $\mu = 0$, chiral symmetry is broken $M \equiv M_0 \neq 0$ and the diquark condensate is zero $\Delta = 0$ (there is no Fermi sphere). Consequently, a tightly $q\bar{q}$ bound state “pion” is generated as a NG boson, i.e., “the NG pion”. On the other hand, if the chemical potential is slightly nonzero $\mu \neq 0$, the symmetry in eq. (6) is weakly broken so that the nonzero diquark condensate $\Delta \simeq M_0 \neq 0$ is favored and the chiral condensate is zero $M = 0$. The associated NG boson corresponds to a tightly-bound diquark state “the NG baryon”. Therefore, low density corresponds to the BEC-like region. If the quark mass is nonzero, the exchange between $(M \neq 0, \Delta = 0, \text{NG pion})$ at $\mu = 0$ and $(M = 0, \Delta \neq 0, \text{NG baryon})$ at $\mu \neq 0$ occurs smoothly. On the other hand, at very high density, we can ignore the chiral condensate, and the gap equation for $\Delta(p)$ becomes the same as the previous 3 color case up to the prefactor. Thus, as the density is increased, the bosonic-like description should be replaced by the usual BCS-type description. There must be a transition from BEC to BCS as the density is increased. More evidences of this transition are now under investigation [2].

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