ON THE MOD $p$ KERNEL OF THE THETA OPERATOR

SHOYU NAGAOKA

(Communicated by Kathrin Bringmann)

Abstract. The theta operator is a generalization of the classical Ramanujan operator to the case of Siegel modular forms. We construct Siegel modular forms for which the images of the theta operator mod $p$ are vanishing.

1. Introduction

In [5], it was reported that the image by the theta operator $\Theta$ of Igusa cusp form $\chi_{35}$ vanishes mod 23, namely

$$\Theta(\chi_{35}) \equiv 0 \pmod{23}$$

holds. This means that if $\det(T) \not\equiv 0 \pmod{23}$, then the corresponding Fourier coefficient $a_{\chi_{35}}(T)$ is divisible by 23. However, the reason why such a phenomenon exists was not clarified. Therefore, we consider whether it can be shown that there exist Siegel modular forms with a similar property (see also [6]).

In this note, we show that we can construct a family of Siegel modular forms whose image of the theta operator is vanishing mod $p$ in the noncuspidal case. The main tool used here is the Siegel-Eisenstein series.

2. Siegel modular forms

For basic facts about Siegel modular forms, we refer the reader to [4].

Let $\mathbb{H}_n$ denote the Siegel upper-half space of degree $n$. The real symplectic group $Sp_n(\mathbb{R})$ acts on $\mathbb{H}_n$ as

$$g \mapsto M(g) = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in Sp_n(\mathbb{R}).$$

The group $\Gamma_n := Sp_n(\mathbb{Z})$ is called the Siegel modular group of degree $n$.

A holomorphic function $F : \mathbb{H}_n \to \mathbb{C}$ is called a Siegel modular form of weight $k(\in \mathbb{Z})$ if $F$ satisfies

$$F(M(g)) = \det(CZ + D)^k F(g) \quad \text{for} \quad M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \Gamma_n.$$

(When $n = 1$, we need the holomorphy at $\infty$.) We denote the space of all such functions by $M_k(\Gamma_n)$.

Received by the editors April 4, 2014 and, in revised form, June 13, 2014 and June 16, 2014. 2010 Mathematics Subject Classification. Primary 11F46; Secondary 11F33.
Any Siegel modular form $F$ admits a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \Lambda_n} a_F(T)e^{2\pi i \text{tr}(TZ)}, \quad Z \in \mathbb{H}_n,$$

where $\Lambda_n$ is a lattice in $\text{Sym}_n(\mathbb{R})$ defined by

$$\Lambda_n := \{ T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z} \}.$$

The space of Siegel modular forms with all the Fourier coefficients in a ring $R \subset \mathbb{C}$ is denoted by $M_k(\Gamma_n)_R$.

**Siegel-Eisenstein series.** A typical example of a Siegel modular form is the Siegel-Eisenstein series. We consider the subgroup $\Gamma_{n,\infty}$ of $\Gamma_n$:

$$\Gamma_{n,\infty} := \left\{ \left( \begin{array}{cc} AB \\ CD \end{array} \right) \in \Gamma_n \mid C = 0_n \right\}.$$

Let $k$ be an even integer such that $k > n + 1$. The (normalized) Siegel-Eisenstein series of weight $k$ and degree $n$ is defined by

$$E_k^{(n)}(Z) := \sum_{M = (C_D)}^{M = (\infty)} \det(CZ + D)^{-k}, \quad Z \in \mathbb{H}_n.$$

It is known that $a_{E_k^{(n)}}(0_n) = 1$ and $E_k^{(n)} \in M_k(\Gamma_n)_{\mathbb{Q}}$. The condition $k > n + 1$ comes from the convergence condition.

**Theta operator.** For a Siegel modular form

$$F = \sum_{0 \leq T \in \Lambda_n} a_F(T)e^{2\pi i \text{tr}(TZ)},$$

the theta operator $\Theta$ is defined by

$$\Theta(F)(Z) := \sum_{0 \leq T \in \Lambda_n} \det(T) \cdot a_F(T)e^{2\pi i \text{tr}(TZ)}$$

(cf. Böcherer and Nagaoka [2]). The case of degree 1 was studied by Ramanujan, and it was used effectively in the theory of $p$-adic modular forms (e.g., cf. [8]). It should be noted that $\Theta(F)$ is not necessarily a Siegel modular form.

### 3. Main result

The following is the main theorem of this note.

**Theorem 3.1.** Assume that $n$ is an even positive integer. Let $p$ be a prime number with $p > n + 3$ and $p \equiv (-1)^n \pmod{4}$, and let $t \geq 1$ be an odd integer. Then there exists a modular form $F \in M_{n/2 + \frac{t-1}{2}}(\Gamma_n)_{\mathbb{Z}(p)}$ satisfying

$$\Theta(F) \equiv 0 \pmod{p} \quad \text{and} \quad F \not\equiv 0 \pmod{p}.$$

In section [4] we see that such $F$ can be constructed by a constant multiple of the Siegel-Eisenstein series.
4. GENERALIZED BERNOULLI NUMBERS

In this section, we summarize some results concerning the generalized Bernoulli numbers. We follow the notation and terminology of [9].

Let \( \chi \) be a Dirichlet character of conductor \( f = f_\chi \). The generalized Bernoulli numbers \( B_{m,\chi} \) are defined by

\[
\sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft} - 1} = \sum_{m=0}^{\infty} B_{m,\chi} \frac{t^m}{m!}.
\]

(c.f. [9], Chapter 4).

In the following, we shall state Carlitz’s result about generalized Bernoulli numbers in the case that \( \chi \) is quadratic.

**Theorem 4.1** (Carlitz [3]). Suppose that \( \chi \) is a primitive and quadratic Dirichlet character of conductor \( f_\chi \).

1. If \( \chi \neq \chi^0 \), then \( f_\chi B_{m,\chi} \) is a rational integer for every \( m \geq 0 \), and if \( f_\chi \) is not a power of a prime, then even \( \frac{1}{m} B_{m,\chi} \) is a rational integer.

2. Suppose that \( f_\chi = p \) is an odd prime integer. Then we have

\[
B_{m,\chi} = \begin{cases} 
-\frac{1}{p} \text{ mod } \mathbb{Z} & \text{if } m = \frac{p-1}{2} \cdot t \text{ for an odd } t \geq 1, \\
0 \text{ mod } \mathbb{Z} & \text{otherwise.}
\end{cases}
\]

Here the congruence \( a \equiv b \text{ mod } \mathbb{Z} \) means that \( a - b \in \mathbb{Z} \).

**Remark 4.2.** Statement (2) is a translation of the result of Carlitz [3], Theorem 3.

5. FOURIER EXPANSION OF THE SIEGEL-EISENSTEIN SERIES

Let \( E_k^{(n)} \) be the Siegel-Eisenstein series of degree \( n \) and weight \( k \) (\( k \): even, \( k > n + 1 \)). We write the Fourier expansion as

\[
E_k^{(n)}(Z) = \sum_{0 \leq T \in \Lambda_n} a_k^{(n)}(T)e^{2\pi i \tau(T)Z}.
\]

For \( S \in \Lambda_m \), the discriminant \( d \) of \( S \) is given by

\[
d(S) := \begin{cases} 
(-1)^{\frac{m}{2}} \text{det}(2S) & \text{if } m \text{: even}, \\
\frac{1}{2}(-1)^{\frac{m-1}{2}} \text{det}(2S) & \text{if } m \text{: odd}.
\end{cases}
\]

Moreover, for even \( m \), \( \chi_S \) is defined by \( \left( \frac{d(S)}{\tau} \right) \).

The following result is essentially due to Böcherer [1].

**Theorem 5.1** (Böcherer [1]). (1) (i) If \( n \) is even, then for \( T \in \Lambda_n \) with \( T > 0 \),

\[
da_k^{(n)}(T) = (-1)^{\frac{n}{2}} \cdot 2^n \cdot \frac{k}{B_k} \prod_{i=1}^{\frac{n-2}{2}} \frac{k - i}{B_{2k-2i}} \frac{B_{k - \frac{n}{2} + i} \cdot \chi_T}{B_{2k-n}} \cdot b_k^{(n)}(T)
\]

for some \( b_k^{(n)}(T) \in \mathbb{Z} \).
(ii) If \( n \) is odd, then for \( T \in \Lambda_n \) with \( T > 0 \),
\[
a_k^{(n)}(T) = 2^n \cdot \frac{k}{B_k} \cdot \frac{n-1}{2} \prod_{i=1}^{k-2i} B_{2k-2i} \cdot c_k^{(n)}(T)
\]
for some \( c_k^{(n)}(T) \in \mathbb{Z} \).

Here \( B_m \) (resp. \( B_{m,\chi} \)) is the \( m \)-th Bernoulli number (resp. generalized Bernoulli number).

(2) If \( n \) is even and \( d(T) \) is square-free, then
\[
b_k^{(n)}(T) = 1
\]
in the expression of (1)(i).

Remark 5.2. Concerning statement (2), we refer the reader to Böcherer [1], p. 283.

6. Proof of the main theorem

We begin by proving the statement of the main theorem for \( t = 1 \). Namely, we prove the existence of \( F \in M_{\frac{n}{2} + \frac{p-1}{2}}(\Gamma_n)Z(p) \) satisfying
\[
\Theta(F) \equiv 0 \pmod{p} \quad \text{and} \quad F \not\equiv 0 \pmod{p},
\]
because we have a modular form \( F_{p-1} \in M_{p-1}(\Gamma_n)Z(p) \) such that \( F_{p-1} \equiv 1 \pmod{p} \) and \( F \cdot F_{p-1} \in M_{\frac{n}{2} + \frac{p-1}{2} + (1+2j)}(\Gamma_n)Z(p) \). The existence of such an \( F_{p-1} \) is guaranteed by the assumption \( p > n + 3 \) (e.g., cf. [2]). Hence we shall prove the main theorem in the case that \( t = 1 \).

For \( n \) and \( p \) in Theorem 3.1, we set
\[
k = \frac{n}{2} + \frac{p-1}{2}
\]
and consider the Eisenstein series
\[
E_k^{(n)} = E_{\frac{n}{2} + \frac{p-1}{2}}.
\]
The assumption \( p > n + 3 \) comes from the convergence condition \( k > n + 1 \).

For this \( k \) and \( T > 0 \), we recall the expression of the Fourier coefficients of \( E_k^{(n)} \):
\[
a_k^{(n)}(T) = (-1)^{\frac{n}{2}} \cdot 2^n \cdot \frac{k}{B_k} \cdot \prod_{i=1}^{\frac{n-2}{2}} B_{k-2i} \cdot \prod_{i=1}^{\frac{n-2}{2}} B_{2k-2i} \cdot b_k^{(n)}(T)
\]
for some \( b_k^{(n)}(T) \in \mathbb{Z} \).

We consider the \( p \)-order of some factor of the right-hand side:
\[
\alpha_p(n, k) := \nu_p \left( \frac{k}{B_k} \cdot \prod_{i=1}^{\frac{n-2}{2}} \frac{k-i}{B_{2k-2i}} \right),
\]
where \( \nu_p \) is the additive valuation normalized as \( \nu_p(p) = 1 \).
Lemma 6.1. \( \alpha_p(n, k) \leq 0. \)

Proof. Since \( k = \frac{n}{2} + \frac{p-1}{2} < p - 1, \) we have

\[
\nu_p \left( k \cdot \prod_{i=1}^{\frac{n-2}{2}} (k - i) \right) = 0.
\]

By the theorem of von Staudt-Clausen, we have

\[
\nu_p(B_k) \geq 0.
\]

Next we check the factors \( B_{2k-2i}. \) Since \( p - 1 < 2k - 2i < 2(p - 1) \quad (1 \leq i \leq \frac{n-2}{2}), \) again by the theorem of von Staudt-Clausen, we obtain

\[
\nu_p(B_{2k-2i}) \geq 0 \quad (1 \leq i \leq \frac{n-2}{2}).
\]

Therefore the equality \( \alpha_p(n, k) \leq 0 \) holds. \( \square \)

We set

\[
F := p^{-\alpha_p(n, k)} \cdot F^{(n)}
\]

and prove that \( F \) satisfies the required properties:

- \( \Theta(F) \equiv 0 \pmod{p}, \)
- \( F \) has \( p \)-integral Fourier coefficients,
- \( F \not\equiv 0 \pmod{p}. \)

Proof of \( \Theta(F) \equiv 0 \pmod{p}. \) We write the Fourier expansion of \( F \) as

\[
F = \sum_{0 \leq T \in \Lambda_n} a_F(T) e^{2\pi i \text{tr}(TZ)}.
\]

It suffices to show that

\[
a_F(T) \equiv 0 \pmod{p} \text{ if } \det(2T) \neq 0 \pmod{p}.
\]

From the definition of \( F, \) the Fourier coefficients \( a_F(T) \) for \( T > 0 \) can be written as

\[
a_F(T) = p^{-\alpha_p(n, k)} \cdot (-1)^{\frac{n}{2}} \cdot 2^n \cdot \frac{k}{B_k} \cdot \prod_{i=1}^{\frac{n-2}{2}} \frac{k - i}{B_{2k-2i}} \cdot \frac{B_{k-\frac{n}{2}, \chi T}}{B_{2k-n}} \cdot b_k^{(n)}(T)
\]

for some \( b_k^{(n)}(T) \in \mathbb{Z} \) (cf. Theorem 5.1). By the definition of \( \alpha_p(n, k), \)

\[
\nu_p \left( p^{-\alpha_p(n, k)} \cdot (-1)^{\frac{n}{2}} \cdot 2^n \cdot \frac{k}{B_k} \cdot \prod_{i=1}^{\frac{n-2}{2}} \frac{k - i}{B_{2k-2i}} \right) = 0.
\]

This implies that

\[
\nu_p(a_F(T)) \geq \nu_p \left( \frac{B_{k-\frac{n}{2}, \chi T}}{B_{2k-n}} \right).
\]

Lemma 6.2.

\[
\frac{B_{k-\frac{n}{2}, \chi T}}{B_{2k-n}} = \frac{B_{\frac{n}{2}, \chi T}}{B_{p-1}} \equiv \begin{cases} 1 \pmod{p} & \text{if } \chi_T = \chi_p = \left( \frac{-1}{p} \right)^{\frac{n}{2}}, \\ 0 \pmod{p} & \text{otherwise}. \end{cases}
\]
Proof. If $\chi_T = \chi_p$, then by the results of Carlitz (Theorem 4.1(2)) and von Staudt-Clausen, we have

$$\begin{cases}
p \cdot B_{\frac{p-1}{2}} \chi_p \equiv -1 \pmod{p}, \\
p \cdot B_{p-1} \equiv -1 \pmod{p},
\end{cases}$$

This implies

$$\frac{B_{\frac{p-1}{2}} \chi_T}{B_{p-1}} \equiv 1 \pmod{p}.$$ 

If $\chi_T \neq \chi_p$, then by Theorem 4.1(1), $\nu_p(B_{\frac{p-1}{2}} \chi_T) \geq 0$. Hence, again by the theorem of von Staudt-Clausen, we obtain

$$\frac{B_{\frac{p-1}{2}} \chi_T}{B_{p-1}} \equiv 0 \pmod{p}.$$ 

From this lemma, we have

$$\nu_p\left(\frac{B_{\frac{p-1}{2}} \chi_T}{B_{p-1}}\right) = 0 \quad \text{if} \quad \chi_T = \chi_p,$$

$$\nu_p\left(\frac{B_{\frac{p-1}{2}} \chi_T}{B_{p-1}}\right) \geq 1 \quad \text{otherwise}.$$ 

If $\det(2T) \not\equiv 0 \pmod{p}$, then $\chi_T \neq \chi_p$. In this case, we obtain

$$\nu_p(a_F(T)) \geq \nu_p\left(\frac{B_{\frac{p-1}{2}} \chi_T}{B_{p-1}}\right) \geq 1,$$

namely, $a_F(T) \equiv 0 \pmod{p}$. Consequently, we have $\Theta(F) \equiv 0 \pmod{p}$.

Proof of the $p$-integrality of $a_F(T)$. For $T > 0$, the $p$-integrality was already proved above. We must confirm the $p$-integrality of $a_F(T)$ for $T$ with lower rank $r$ ($0 < r < n$). For this $T$, there exists some $U \in GL_n(\mathbb{Z})$ such that

$$T[U] = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad 0 < T_1 \in \Lambda_r.$$ 

First, we assume that $r$ is even. In this case, we can write

$$a_F(T) = p^{-\alpha_p(n,k)} \cdot (-1)^{\frac{r+1}{2}} \cdot 2^r \cdot k \cdot \prod_{i=1}^{\frac{r-1}{2}} \frac{k+i}{B_{2k-2i}} \cdot \frac{B_{k-\frac{r}{2},\chi_T}}{B_{2k-r}} \cdot b_k^{(r)}(T_1)$$

for some $b_k^{(r)}(T_1) \in \mathbb{Z}$. Since $\alpha_p(r,k) \geq \alpha_p(n,k)$, $\nu_p(B_{k-\frac{r}{2},\chi_T}) \geq 0$ (by Theorem 4.1), and $-\alpha_p(r,k) + \nu_p(B_{2k-r}) \leq -\alpha_p(n,k)$, we have

$$\nu_p(a_F(T)) \geq -\alpha_p(n,k) + \alpha_p(r,k) + \nu_p\left(\frac{B_{k-\frac{r}{2},\chi_T}}{B_{2k-r}}\right) \geq 0.$$ 

Next, we assume that $r$ is odd. This case is easier than the above. In this case, we see that

$$\nu_p(a_F(T)) \geq -\alpha_p(n,k) + \alpha_p(r,k) \geq 0.$$ 

For any case (including $r = 0$), we obtain

$$\nu_p(a_F(T)) \geq 0 \quad \text{for all} \quad T.$$
Proof of $F \not\equiv 0 \pmod{p}$. We consider the special case of $T$ such that $\det(2T) = p$ (note that $p \equiv (-1)^{\frac{n}{2}} \pmod{4}$) and investigate the corresponding Fourier coefficient $a_F(T)$:

$$a_F(T) = p^{-\alpha_p(n,k)} \cdot (-1)^{\frac{n}{2}} \cdot 2^n \cdot \frac{k}{B_k} \cdot \prod_{i=1}^{\frac{n}{2}} \frac{k - i}{B_{2k-2i}} \cdot \frac{B_{k-\frac{n}{2}} \chi_p}{B_{2k-n}}$$

(cf. Theorem 5.1(2)). Therefore we have

$$\nu_p(a_F(T)) = -\alpha_p(n,k) + \nu_p \left( \frac{k}{B_k} \cdot \prod_{i=1}^{\frac{n}{2}} \frac{k - i}{B_{2k-2i}} \right) + \nu_p \left( \frac{B_{k-\frac{n}{2}} \chi_p}{B_{2k-n}} \right)$$

$$= \nu_p \left( \frac{B_{k-\frac{n}{2}} \chi_p}{B_{2k-n}} \right)$$

(by the definition of $\alpha_p(n,k)$)

$$= 0 \quad \text{(by Lemma 6.2).}$$

This completes the proof of the main theorem.

Remark 6.3. (1) In the case $n = 2$, $\alpha_p(2,k) = 0$ for prime $p$ with $p \equiv 3 \pmod{4}$ because

$$B_{1+\frac{n-1}{2}} = B_{\frac{p+1}{2}} \not\equiv 0 \pmod{p}$$

(e.g., cf. Washington [9], p. 86, Exercises 5.9). Therefore we can set

$$F = E_{\frac{n+1}{2}}^{(2)} (p > 3).$$

In particular,

$$\Theta \left( E_{\frac{n+1}{2}}^{(2)} \right) \equiv 0 \pmod{p}$$

for any prime $p > 3$ with $p \equiv 3 \pmod{4}$ (see also [4], Theorem 7.2).

(2) If we add a condition on $p$ such that $p$ is regular in our discussion, then we do not need the factor “$p^{-\alpha_p(n,k)}$” in the definition of $F$ because $\alpha_p(n,k) = 0$ in this case.

(3) Under the same assumption of the main theorem, we consider the following modular form of degree $n + 1$:

$$G := p^{-\alpha_p(n,k)} E^{(n+1)}_{\frac{n+1}{2} + \frac{p-1}{2}} \in M_{\frac{n+1}{2} + \frac{p-1}{2}}(\Gamma_{n+1}).$$

This form $G$ gives an example of the so-called “mod $p$ singular modular form” studied by Böcherer and Kikuta.

References

[1] Siegfried Böcherer, Über die Fourierkoeffizienten der Siegelschen Eisensteinreihen (German, with English summary), Manuscripta Math. 45 (1984), no. 3, 273–288, DOI 10.1007/BF01158040. MR734842 (86b:11037)

[2] Siegfried Böcherer and Shoyu Nagaoka, On mod $p$ properties of Siegel modular forms, Math. Ann. 338 (2007), no. 2, 421–433, DOI 10.1007/s00208-006-0081-7. MR2302069 (2008d:11041)

[3] L. Carlitz, Arithmetic properties of generalized Bernoulli numbers, J. Reine Angew. Math. 202 (1959), 174–182. MR0109132 (22 #7920)

[4] Helmut Klingen, Introductory lectures on Siegel modular forms, Cambridge Studies in Advanced Mathematics, vol. 20, Cambridge University Press, Cambridge, 1990. MR1046530 (91a:11021)

[5] H. Kodama, T. Kikuta and S. Nagaoka, Note on Igusa’s cusp form of weight 35, to appear in Rocky Mountain J. Math.
[6] H. Kodama and S. Nagaoka, *A congruence relation satisfied by Siegel cusp form of odd weight (Japanese)*, J. School Sci. Eng. Kinki Univ. 49 (2013), 9–15.

[7] Shoyu Nagaoka, *A remark on Serre’s example of $p$-adic Eisenstein series*, Math. Z. 235 (2000), no. 2, 227–250. DOI 10.1007/s002090000134. MR1795506 (2001m:11068)

[8] Jean-Pierre Serre, *Formes modulaires et fonctions zêta $p$-adiques* (French), Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), Springer, Berlin, 1973, pp. 191–268. Lecture Notes in Math., Vol. 350. MR0404145 (53 #7949a)

[9] Lawrence C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1982. MR718674 (85g:11001)

Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan

E-mail address: nagaoka@math.kindai.ac.jp