In this article, we first obtain an embedding result for the Sobolev spaces with variable-order, and then we consider the following Schrödinger–Kirchhoff type equations

\[
\left( a + b \int_{\Omega \times \Omega} \frac{\mid \xi(z) - \xi(y) \mid^p}{|z - y|^{N+ps}} \, dx \, dy \right)^{p-1} ( -\Delta_p^{s(\cdot)} ) \xi(x) + \lambda V(x) |\xi|^{p-2} \xi = f(x, \xi), \quad x \in \Omega, \\
\xi = 0, \quad x \in \partial \Omega,
\]

where \(\Omega\) is a bounded Lipschitz domain in \(\mathbb{R}^N\), \(1 < p < +\infty\), \(a, b > 0\) are constants, \(s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (0, 1)\) is a continuous and symmetric function with \(N > s(x, y) p\) for all \((x, y) \in \Omega \times \Omega\), \(\lambda \geq 0\) is a parameter, \((-\Delta_p^{s(\cdot)})\) is a fractional \(p\)-Laplace operator with variable-order, \(V(x) : \Omega \to \mathbb{R}^+\) is a potential function, and \(f(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}\) is a continuous nonlinearity function. Assuming that \(V\) and \(f\) satisfy some reasonable hypotheses, we obtain the existence of infinitely many solutions for the above problem by using the fountain theorem and symmetric mountain pass theorem without the Ambrosetti–Rabinowitz (AR) condition for short condition.

**Keywords:** Schrödinger–Kirchhoff type equations; variable-order fractional Laplacian; fountain theorem; symmetric mountain pass theorem

**MSC:** 35J91; 35A15; 35R11; 35J60

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**1. Introduction and the Main Results**

In this article, we investigate the following Schrödinger–Kirchhoff type equations

\[
(P_p) : \left\{ \begin{array}{ll}
\left( a + b \int_{\Omega \times \Omega} \frac{\mid \xi(z) - \xi(y) \mid^p}{|z - y|^{N+ps}} \, dx \, dy \right)^{p-1} ( -\Delta_p^{s(\cdot)} ) \xi(x) + \lambda V(x) |\xi|^{p-2} \xi = f(x, \xi), & x \in \Omega, \\
\xi = 0, & x \in \partial \Omega,
\end{array} \right.
\]

where \(\Omega\) is a bounded Lipschitz domain in \(\mathbb{R}^N\) and \(1 < p < +\infty\), \(s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (0, 1)\) is a continuous symmetric function with \(N > s(x, y) p\) for all \((x, y) \in \Omega \times \Omega\), \((-\Delta_p^{s(\cdot)})\) is the fractional \(p\)-Laplace operator with variable-order, defined as

\[
(-\Delta_p^{s(\cdot)} \xi)(x) := P.V. \int_{\Omega} \frac{|\xi(x) - \xi(y)|^{p-2} (\xi(x) - \xi(y))}{|x - y|^{N+ps(x, y)}} \, dy, \quad x \in \Omega,
\]

where \(\xi \in C_0^\infty(\Omega)\), and \(P.V.\) stands for the Cauchy principal value. Since \(s(\cdot)\) is a continuous function, \((-\Delta_p^{s(\cdot)})\) is called fractional Laplace operator with variable-order. The usual fractional \(p\)-Laplace operator has been studied extensively by many scholars, see [1–6].

For the variable-order fractional and some important results, we refer to [7–12]. The fractional derivatives of variable-order were introduced by Lorenzo et al. in [7]. Subsequently, Samko et al. generalized the Riemann–Liouville fractional integration and
differentiation to the fractional operator with variable-order, see [9,10] for more details with respect to this topic.

When $p = 2$, the fractional Laplace operator with variable-order was studied by Xiang et al. in [11], they investigated the following Laplacian equations

$$
\begin{cases}
(-\Delta)^{s(\cdot)}\xi + \lambda V(x)\xi = a|\xi|^{p(x)-2}\xi + \beta|\xi|^{q(x)-2}\xi, & x \in \Omega, \\
\xi = 0, & x \in \mathbb{R}\setminus\Omega,
\end{cases}
$$

where $(-\Delta)^{s(\cdot)}$ is the fractional Laplacian operator. First of all, they proved the embedding theorem of variable-order fractional Sobolev space, and then they obtained a multiplicity result for a Schrödinger equation via variational methods.

Note that we also mention the work by Wang et al. in [12]; they also studied the fractional Laplace operator with variable-order, as follows

$$
\begin{cases}
M(|\xi|^2(\cdot))(-\Delta)^{s(\cdot)}\xi + V(x)\xi = \lambda|\xi|^{p(x)-2}\xi + \mu|\xi|^{q(x)-2}\xi, & x \in \Omega, \\
\xi = 0, & x \in \mathbb{R}\setminus\Omega,
\end{cases}
$$

where $M$ is a model of Kirchhoff coefficient, and infinitely many solutions were obtained by using four different critical point theorems. The main feature for this kind of Kirchhoff-type problem is that $M$ could be zero at zero.

In recent decades, many scholars have extensively studied the existence of results for classical Schrödinger equations and fractional Schrödinger equations under reasonable assumptions of $V$ and $f$. We refer the reader to [13–17]. Nyamoradi et al. in [15] studied the Schrödinger–Kirchhoff type equations by variational methods. Note that Teng in [16] established the existence of high or small energy solutions by applying a variant fountain theorem. Especially, in [17] César E. Torres Ledesma studied the existence of multiple solutions for Schrödinger–Kirchhoff type involving the non-homogeneous fractional $p$-Laplacian, which $V$ and $f$ are under some weaker assumptions.

On the other hand, the Kirchhoff equation was introduced by Kirchhoff in [18]. Kirchhoff proposed the following model

$$
\rho \frac{\partial^2 x}{\partial t^2} - \left( \frac{p_0}{h} + \frac{E}{2L} \int_0^1 \left| \frac{\partial \xi}{\partial t} \right|^2 \, dx \right) \frac{\partial^2 \xi}{\partial x^2} = 0,
$$

where $\rho, p_0, h, E, L$ are constants with respect to some physical meanings, respectively. We call $(P_\rho)$ a problem of Kirchhoff type because there is the Kirchhoff term

$$
a + b \int_{\Omega \times \Omega} \frac{|\xi(x) - \xi(y)|^p}{|x - y|^{N+p(x,y)}} \, dx \, dy,
$$

which not only makes the study of $(P_\rho)$ interesting but also becomes more delicate and causes some mathematical difficulties. The literature on Kirchhoff type problems about the existence and multiplicity of solutions is quite large; here we just list a few, for example, [4,6,17,19–24] for further details. It is worth pointing out that if $a > 0$, $b \geq 0$, $\mathcal{M}(t) = a + bt^m$ is non-degenerate Kirchhoff type equations, for example, see [23,24]; if $a = 0$, $b > 0$, $\mathcal{M}(t) = a + bt^m$ is degenerate Kirchhoff type equations, such as, see [6,20]. The fractional Kirchhoff type equation regarding non-local integro-differential operator was first introduced in [19] by Fiscella et al., and they studied the non-negative solutions for this kind of equation as follows

$$
\begin{cases}
\mathcal{M}(|\xi|^2) = f(x, \xi), & \text{in } \Omega, \\
\xi = 0, & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

where
where $\Omega$ is a bounded Lipschitz domain of $\mathbb{R}^N$. Moreover, in [20], Molica et al. investigated a kind of nonlocal fractional Kirchhoff type equations, and three solutions were obtained by applying the critical points theorem.

Indeed, the works of literature on Kirchhoff equations, Schrödinger equations, and their applications are quite large. Kirchhoff equations model several physical and biological systems, for example, population density, see [25–27] for some related works. On the other hand, many scholars are interested in Schrödinger equations, which describe the dynamic behavior of particles in quantum mechanics, see [28], and the standing wave solutions, see [29]. Moreover, fractional Schrödinger–Kirchhoff type equations involving an external magnetic potential were studied in [30].

As is known, the (AR) condition plays a crucial role to guarantee that the Palais–Smale sequences are bounded. In the famous paper [31], Ambrosetti and Rabinowitz introduced the well-known (AR) condition, that is, there exist constants $p_0 < p$ and $0 < M_0$ such that

$$0 < \mu_0 F(x, \xi) \leq f(x, \xi)\xi, \quad \text{for } x \in \mathbb{R}^N, \ |\xi| \geq M_0,$$

where $F(x, \xi) = \int_0^1 f(x, t)dt$. In [32], Servadei et al. obtained the existence and multiplicity of nontrivial solutions and showed that the verification of the Palais–Smale compactness condition depends on the (AR) condition. However, there are a lot of functions where the (AR) condition is not satisfied, an example of such function is

$$f(x, \xi) = |\xi|^{p-2}\xi \log(1 + |\xi|).$$

For this kind of problem, many people have attached much importance to finding new, reasonable conditions instead of the (AR) condition, see, for instance, [5,6,33,34].

Motivated by the above cited works, we find that there are some papers on Kirchhoff equations or Schrödinger equations involving the fractional $p$-Laplace operator; however, there are no results for Schrödinger–Kirchhoff type equations driven by the fractional $p$-Laplace operator with variable-order. Thus, we are devoted to investigating the existence of infinitely many solutions for Schrödinger–Kirchhoff type equations involving a variable-order fractional $p$-Laplace operator by applying the fountain theorem and symmetric mountain pass theorem, respectively.

Our work is different from the previous articles. To the best of our knowledge, this article is the first to discuss the existence of infinitely many solutions for the fractional $p$-Laplacian Schrödinger–Kirchhoff type equations without the (AR) condition. Under the reasonable hypothesis, we first establish an embedding theorem for the variable-order fractional framework, and our results generalize Theorem 1 and Theorem 2 of [15] in some directions. Finally, relative to paper [11], for the case $p = 2$, we can deal with a general Kirchhoff–Schrödinger type equations and the nonlinearity with variable coefficients.

Throughout this paper, for simplicity, $C_i$ and $K_i, i = 1, 2, \ldots, N$ are used in various places to denote distinct constants, and we will specify them whenever it is necessary. Define the function space $C_+(\overline{\Omega})$

$$C_+(\overline{\Omega}) := \{ H(x) \in C(\overline{\Omega}, \mathbb{R}), \ 1 < H^- \leq H(x) \leq H^+ < +\infty \},$$

where $H^- := \min H(x)$ and $H^+ := \max H(x)$ for all $x \in \overline{\Omega}$.

$s(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to (0, 1)$ is a continuous function, satisfying:

(S1): $s(\cdot)$ is symmetric function, that is, $s(x, y) = s(y, x)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

(S2): $0 < s^- := \min_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} s(x, y) < s(x, y) < s^+ := \max_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} s(x, y) < 1$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Regarding the potential function $V(x) : \Omega \to \mathbb{R}$, we assume the following hypothesis:
(V1): $V(x)$ is the continuous function, satisfying $\inf_{x \in \Omega} V(x) > V_0$, where $V_0$ is a positive constant. Moreover, there is $d > 0$ such that the set $\{x \in \Omega : V(x) \leq d\}$ is nonempty and $\text{meas}(\{x \in \Omega : V(x) \leq d\}) < +\infty$.

(V2): There exists positive constant $h$ such that
\[
\lim_{|y| \to \infty} \text{meas}\left(\{x \in B_{R_0}(y) : V(x) \leq \kappa_0\}\right) = 0 \quad \text{for any } \kappa_0 > 0,
\]
where $B_{R_0}(x)$ denotes the open ball of $\Omega$ centered at $x$ and of radius $R_0 > 0$.

Furthermore, the nonlinearity $f(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous Carathéodory function, satisfying:

(F1): There exist positive constants $C_1$ and $C_2$ such that
\[
|f(x,t)| \leq C_1|t|^{p-1} + C_2|t|^{\theta(x)-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},
\]
where $\theta(x) \in C_+(\Omega)$ and $1 < p < \theta(x) < p^*_s = (pN)/(N - p\beta(x))$.

(F2): $\lim_{|t| \to 0} \frac{f(x,t)}{|t|^p} = 0$ for $x \in \Omega$ uniformly.

(F3): There exist positive constants $p^2 < \mu < p^*_s$ and $r$ such that
\[
F(x,t) \leq \frac{1}{\mu} f(x,t)t + C_3|t|^r \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},
\]
where $0 < C_3 < \frac{\theta(p-r)}{\mu}$ and $\inf_{x \in \Omega, |t|=r} F(x,t) := \beta > 0$.

(F4): $f(x, t) = -f(x, t)$ for all $(x, t) \in \Omega \times \mathbb{R}$.

Remark 1. It is obvious that the condition (F3) is weaker than the well-known (AR) condition.

Before stating our main results, we need to present the corresponding variational framework and definition, which plays an important role to solve problem $(P_0)$.

Definition 1. We say that $\xi \in X_0$ is a (weak) solution of Schrödinger–Kirchhoff type equations $(P_0)$, if
\[
\left( a + b \int_{\Omega \times \Omega} \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps(x,y)}} \, dx \, dy \right)^{p-1}
\times \int_{\Omega \times \Omega} \frac{|\xi(x) - \xi(y)|^{p-2}(\xi(x) - \xi(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps(x,y)}} \, dx \, dy
= - \int_\Omega \lambda V(x)|\xi|^{p-2} \xi \varphi \, dx + \int_\Omega f(x, \xi) \varphi \, dx
\]
for any $\varphi \in X_0$, where $X_0$ will be introduction in Section 2.

The functional $I : X_0 \to \mathbb{R}$, which is defined as
\[
I(\xi) := \frac{1}{bp^2} \left( a + b \int_{\Omega \times \Omega} \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps(x,y)}} \, dx \, dy \right)^{p} + \frac{1}{p} \int_\Omega \lambda V(x)|\xi|^p \, dx - \int_\Omega F(x, \xi) \, dx
\]
for all $\xi \in X_0$, where $F(x, \xi) = \int_0^\xi f(x, s) \, ds$. Moreover, if (F1) and (V1) hold, then $I : X_0 \to \mathbb{R}$ is of class $C^1(X_0, \mathbb{R})$ and
\[ \langle I'(\xi), \varphi \rangle := \left( a + b \int_{\Omega \times \Omega} \left| \frac{\xi(x) - \xi(y)}{|x-y|^{N+ps(x,y)}} \right|^p dx dy \right)^{p-1} \]

\begin{align*}
&\times \int_{\Omega \times \Omega} \left| \frac{\xi(x) - \xi(y)}{|x-y|^{N+ps(x,y)}} \right|^{p-2}(\xi(x) - \xi(y)) (\varphi(x) - \varphi(y)) \, dx dy \\
&+ \int_{\Omega} \lambda V(x)|\xi|^{p-2} \varphi \, dx - \int_{\Omega} f(x, \xi) \varphi \, dx
\end{align*}

for any \( \xi, \varphi \in X_0 \). Under our reasonable assumptions, the functional \( I \) is well defined. Hence, \( \xi \in X_0 \) is a (weak) solution of Schrödinger–Kirchhoff type equations \( (P_v) \) if and only if \( \xi \in X_0 \) is a critical point of the functional \( I \).

**Theorem 1.** Let \((S1)\)–\((S2)\), \((V1)\)–\((V2)\), \((F1)\)–\((F4)\) hold, then the problem \((P_v)\) has infinitely many nontrivial weak solutions in \( X_0 \), whenever \( \lambda > 0 \) is sufficiently large.

The remainder of this paper is organized as follows. Some basic knowledge about the Lebesgue spaces with variable exponent and fractional Sobolev spaces with variable exponents and variable-order are given in Section 2. The functional \( I \) satisfying \((PS)\) condition is proved in Section 3. In Section 4, by using the fountain theorem, we prove Theorem 1. Finally, in Section 5, we prove Theorem 1 by applying the symmetric mountain pass theorem.

**2. Preliminary Results**

**2.1. Variable Exponent Lebesgue Spaces**

In this subsection, we recall some preliminary knowledge of generalized Lebesgue spaces with variable exponent. The readers are invited to consult [35–40] for a detailed description.

Let \( \Omega \) be a nonempty Lipschitz domain in \( \mathbb{R}^N \), a measurable function \( \theta(x) \in C_+ (\overline{\Omega}) \), and \( u \) be a measurable real-valued function. We introduce the Lebesgue spaces with variable exponent

\[ L^{\theta(x)}(\Omega) := \left\{ \xi : \xi \text{ is a measurable and } \int_{\Omega} |\xi|^{\theta(x)} dx < \infty \right\} \]

with the norm

\[ \|\xi\|_{L^{\theta(x)}(\Omega)} := \inf \left\{ \chi > 0 : \int_{\Omega} \left| \frac{\xi}{\chi} \right|^{\theta(x)} dx \leq 1 \right\}, \]

then \((L^{\theta(x)}(\Omega), \| \cdot \|_{L^{\theta(x)}(\Omega)})\) is a Banach space (see [38]), called generalized Lebesgue space.

**Lemma 1** (See [40]). The space \((L^{\theta(x)}(\Omega), \| \cdot \|_{L^{\theta(x)}(\Omega)})\) is separable, uniformly convex, reflexive, and its conjugate space is \((L^{\theta'(x)}(\Omega), \| \cdot \|_{L^{\theta'(x)}(\Omega)})\), where \( \theta'(x) \) is the conjugate function of \( \theta(x) \) i.e.,

\[ \frac{1}{\theta(x)} + \frac{1}{\theta'(x)} = 1, \text{ for all } x \in \Omega. \]

For all \( \xi \in L^{\theta(x)}(\Omega), v \in L^{\theta'(x)}(\Omega) \), the Hölder type inequality

\[ \left| \int_{\Omega} \xi v dx \right| \leq \left( \frac{1}{\theta} + \frac{1}{\theta'} \right)^{\frac{1}{p}} \| \xi \|_{L^{\theta(x)}(\Omega)} \| v \|_{L^{\theta'(x)}(\Omega)} \leq 2 \| \xi \|_{L^{\theta(x)}(\Omega)} \| v \|_{L^{\theta'(x)}(\Omega)} \]

holds.
The mapping \( \rho_{\theta(x)} : L^{\theta(x)}(\Omega) \to \mathbb{R} \) is defined:

\[
\rho_{\theta(x)}(\xi) := \int_{\Omega} |\xi|^{\theta(x)} \, dx.
\]

The relation between modular and Luxemburg norm is clarified by the following properties.

**Lemma 2** (See [36]). Suppose that \( \xi_n, \xi \in L^{\theta(x)}(\Omega) \), then the following properties hold

(i) \( \|\xi\|_{L^{\theta(x)}(\Omega)} > 1 \Rightarrow \|\xi\|_{L^{\theta(x)}(\Omega)}^{\theta(x)} \leq \rho_{\theta(x)}(\xi) \leq \|\xi\|_{L^{\theta(x)}(\Omega)}^{\theta(x)} \); 

(ii) \( \|\xi\|_{L^{\theta(x)}(\Omega)} < 1 \Rightarrow \|\xi\|_{L^{\theta(x)}(\Omega)}^{\theta(x)} \leq \rho_{\theta(x)}(\xi) \leq \|\xi\|_{L^{\theta(x)}(\Omega)}^{\theta(x)} \); 

(iii) \( \|\xi\|_{L^{\theta(x)}(\Omega)} < 1 \) (respectively, \( = 1, > 1 \)) \( \iff \rho_{\theta(x)}(\xi) < 1 \) (respectively, \( = 1, > 1 \)); 

(iv) \( \|\xi\|_{L^{\theta(x)}(\Omega)} \to 0 \) (respectively, \( \to \infty \)) \( \iff \rho_{\theta(x)}(\xi_n) \to 0 \) (respectively, \( \to \infty \)); 

(v) \( \lim_{n \to \infty} \|\xi_n - \xi\|_{\theta(x)} = 0 \iff \lim_{n \to \infty} \rho_{\theta(x)}(\xi_n - \xi) = 0 \).

**Remark 2.** Note that for any function \( \theta_1(x), \theta_2(x) \in C_+((\overline{\Omega}) \) and \( \theta_1(x) < \theta_2(x) \), there exists a continuous embedding \( L^{\theta_2(x)}(\Omega) \hookrightarrow L^{\theta_1(x)}(\Omega) \) for any \( x \in \overline{\Omega} \). Especially, when \( \theta(x) \equiv \text{constant} \), the results of Lemmas 1 and 2 still hold.

2.2. Variable-Order Fractional Sobolev Spaces

Let \( 1 < p < +\infty, s(\cdot) \) is a continuous symmetric function, and let the Gagliardo seminorm with variable-order be denoted as

\[
[\xi]_{s(\cdot),p} := \int_{\Omega \times \Omega} \frac{|\xi(x) - \xi(y)|^p}{|x - y|^{N + ps(x,y)}} \, dxdy,
\]

where \( \xi : \Omega \to \mathbb{R} \) is a measurable function. Now, we define the fractional Sobolev spaces with variable-order by

\[
W = W_{s(\cdot),p}(\Omega) := \left\{ \xi \in L^p(\Omega) : [\xi]_{s(\cdot),p} < \infty \right\},
\]

then \((W, \| \cdot \|_W)\) is a reparable, reflexive Banach space and assume that it is endowed with the norm

\[
\|\xi\|_W := \left( \|\xi\|_{L^p(\Omega)}^p + a^{p-1} [\xi]_{s(\cdot),p}^p \right)^{1/p}.
\]

The variable-order fractional critical exponent is defined by

\[
p_{s(\cdot)}^* = \begin{cases} 
\frac{Np}{N - ps(x,y)} & \text{if } ps(x,y) < N, \\
\infty & \text{if } ps(x,y) \geq N.
\end{cases}
\]

We denote by \( W_0 \) the closure of \( C_0^\infty(\Omega) \) in \( W \) and with the norm

\[
\|\xi\|_{W_0} := [\xi]_{s(\cdot),p}^p,
\]

then, the \((W_0, \| \cdot \|_{W_0})\) is also a reparable, reflexive Banach space. \( W_0^* \) denotes the dual space of \( W_0 \).

Now let us give a very crucial lemma, which the proof process is similar to the one of Lemma 2.1 of [11].

**Lemma 3.** Let \( 0 < s_0 < s(\cdot) < s_1 < 1 < p < +\infty \) and \( W_0^s(\Omega) \) \( (j = 0, 1) \) in \( W_0 \) with \( s(x,y) = s_j \), the embeddings \( W_0^{s_j} \hookrightarrow W_0 \hookrightarrow W_0^{s_0} \) are continuous. Moreover, if \( N > p_{s_0}^* \), for any constant exponent \( p \in [1, \frac{Np}{(N - ps_0)}] \), \( W_0 \) can be continuously embedded into \( L^p(\Omega) \).

**Proof.** First, for any \( \xi \in W_0 \), we obtain
\[
\int_{\Omega} \int_{\Omega \setminus |x-y| \geq 1} \frac{|\xi|^p}{|x-y|^{N+ps(x,y)}} dxdy \leq \int_{\Omega} \left( \int_{|z| \geq 1} \frac{1}{|z|^{N+ps(x,y)}} dz \right) |\xi|^p dx
\]
\[
\leq C(N, s(\cdot), p) \|\xi\|_{L^p(p)}.
\]

where we used the fact that the kernel \(1/|z|^{N+ps(x,y)}\) is integrable since \(N + ps(x, y) > N\).

Taking into account the above estimate, it follows

\[
\int_{\Omega} \int_{\Omega \setminus |x-y| \geq 1} \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps(x,y)}} dxdy \leq 2^{p-1} \int_{\Omega} \int_{\Omega \setminus |x-y| \geq 1} \frac{|\xi(x)|^p + |\xi(y)|^p}{|x-y|^{N+ps(x,y)}} dxdy
\]
\[
\leq 2^{p} C(N, s(\cdot), p) \|\xi\|_{L^p(p)}.
\]

hence, for any \(\xi \in W_0^p\) and \(s_0 < s(\cdot)\), we have

\[
\int_{\Omega} \int_{\Omega \setminus |x-y| \geq 1} \frac{|\xi|^p}{|x-y|^{N+ps(x,y)}} dxdy \leq \int_{\Omega} \int_{\Omega \setminus |x-y| \geq 1} \frac{|\xi|^p}{|x-y|^{N+s_0p}} dxdy
\]
\[
\leq \int_{\Omega} \left( \int_{|z| \geq 1} \frac{1}{|z|^{N+s_0p}} dz \right) |\xi|^p dx
\]
\[
\leq C(N, s_0, p) \|\xi\|_{L^p(p)}.
\]

On the other hand

\[
\int_{\Omega} \int_{\Omega \setminus |x-y| \leq 1} \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps(x,y)}} dxdy \leq \int_{\Omega} \int_{\Omega \setminus |x-y| \leq 1} \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps_1}} dxdy
\]
\[
\leq \int_{\Omega} \int_{\Omega \setminus |x-y| \leq 1} \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps_1}} dxdy.
\]

Thus, combining with (3)–(5), we get

\[
\int_{\Omega} \int_{\Omega} \frac{a^{p-1}|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps(x,y)}} dxdy
\]
\[
\leq \int_{\Omega} \int_{\Omega \setminus |x-y| \geq 1} \frac{a^{p-1}|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps(x,y)}} dxdy + \int_{\Omega} \int_{\Omega \setminus |x-y| \leq 1} \frac{a^{p-1}|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps_1}} dxdy
\]
\[
\leq (2^p C(N, s_0, p) + 1) \|\xi\|_{L^p(p)} + \int_{\Omega} \int_{\Omega \setminus |x-y| \leq 1} \frac{a^{p-1}|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps_1}} dxdy
\]
\[
\leq C(N, s_0, p) \left( \|\xi\|_{L^p(p)} + \int_{\Omega} \int_{\Omega \setminus |x-y| \leq 1} \frac{a^{p-1}|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps_1}} dxdy \right).
\]

which gives the desired estimate, up to relabeling the constant \(C(N, s_0, p)\). From this inequality, we can obtain the following continuous embedding

\[W_0^p \hookrightarrow W_0.\]

Similarly, we can also deduce the following continuous embedding

\[W_0 \hookrightarrow W_0^{s_0}.\]

Finally, by Theorem 6.7 and Theorem 6.9 of [41], we know that for any constant exponent \(p \in (1, Np/(N - ps_0))\), the following continuous embedding holds
Therefore, the embedding $W_0^0 \hookrightarrow L^p(\Omega)$. □

**Theorem 2.** Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $1 < p < +\infty$ and $s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (0, 1)$ is a continuous function satisfying (S1) with $N > ps(x, y)$ for all $(x, y) \in \Omega \times \Omega$. Assume that $\vartheta(x) \in C_+(\overline{\Omega})$ such that $\vartheta(x) < p^*_s(x)$ for all $x \in \overline{\Omega}$. Then, there exists $C_\vartheta = C_\vartheta(N, p, s, \vartheta, \Omega)$ such that for any $\xi \in W$, it holds that

$$\|\xi\|_{L^{p(\cdot)}(\Omega)} \leq C_\vartheta \|\xi\|_{W}.$$ 

That is, the space $W$ is continuously embedded in $L^{p(\cdot)}(\Omega)$. Furthermore, this embedding is compact. If $\xi \in W_0$, then there exists a constant $C_\vartheta' = C_\vartheta'(N, p, s, \vartheta, \Omega)$ such that

$$\|\xi\|_{L^{p(\cdot)}(\Omega)} \leq C_\vartheta'\|\xi\|_{W_0}.$$ 

**Proof.** Here the process of proof is similar to [39]. Since $1 < p < +\infty$, $\vartheta(x) \in C_+(\overline{\Omega})$, $s(x, y)$ are continuous symmetric functions, there is a constant $k_1 > 0$ such that

$$\inf_{x \in \Omega} \left\{ \frac{Np}{N - ps(x, x)} - \vartheta(x) \right\} = k_1 > 0 \quad \text{for all } x \in \overline{\Omega}. $$

Hence, we find a positive constant $\epsilon$ and $K$ numbers of disjoint hypercubes $\Omega_i$ such that $\Omega = \bigcup_{i=1}^K \Omega_i$ and $\text{diam}(\Omega_i) < \epsilon$, that verify that

$$\frac{Np}{N - ps(z, y)} - \vartheta(x) = \frac{k_1}{2} > 0 \quad \text{for all } (z, y) \in \Omega_i \times \Omega_i \text{ and } x \in \Omega_i, \ i = 1, 2, \ldots, K. $$

Let $s_i = \inf_{(z, y) \in \Omega_i \times \Omega_i} s(z, y)$ and the fractional Sobolev critical exponent $p^*_s = \frac{Np}{N - s_i}$. Then

$$p^*_s = \frac{Np}{N - s_i} \geq \frac{k_1}{2} + \vartheta(x) \quad (6)$$

for all $x \in \Omega_i$ and $N > s_i p$. According to the Sobolev embedding theorem (Theorem 6.7 and Theorem 6.9 of [41]), there exists a positive constant $k_2$ such that

$$\|\xi\|_{L^{p(\cdot)}(\Omega_i)} \leq k_2 \left( \|\xi\|_{L^p(\Omega)} + \int_{\Omega_i \times \Omega_i} \frac{a^{p-1} |\xi(x) - \xi(y)|^p}{|x - y|^{n + ps_i}} dxdy \right).$$

In addition, from Lemma 3, we obtain

$$\|\xi\|_{L^{p(\cdot)}(\Omega_i)} \leq k_3 \left( \|\xi\|_{L^p(\Omega)} + \int_{\Omega_i \times \Omega_i} \frac{a^{p-1} |\xi(x) - \xi(y)|^p}{|x - y|^{n + ps(x, y)}} dxdy \right). \quad (7)$$

Since $|\xi| = \sum_{i=1}^K |\xi|_{L^p(\Omega)}$, from (6), we get $\vartheta(x) < p^*_s$. There exist $a_i(x) \in C_+(\Omega)$ such that $1/\vartheta(x) = 1/p^*_s + 1/a_i(x)$. By utilizing the Hölder’s inequality, we have

$$\|\xi\|_{L^{p(\cdot)}(\Omega)} \leq k_4 \|\xi\|_{L^{p(\cdot)}(\Omega_i)} \leq k_5 \|\xi\|_{L^{p(\cdot)}(\Omega_i)} \leq k_6 \|\xi\|_{L^{p(\cdot)}(\Omega_i)}. \quad (8)$$

Note that

$$\|\xi\|_{L^{p(\cdot)}(\Omega)} \leq k_6 \sum_{i=1}^K \|\xi\|_{L^{p(\cdot)}(\Omega_i)}. \quad (9)$$
by (7)–(9), we have
\[ \| \xi \|_{L^{\theta(x)}(\Omega)} \leq k_7 \sum_{i=1}^{K} \left( \| \xi \|_{L^{p_i}(\Omega)} + \int_{\Omega \times \Omega} \frac{a^{p_i-1}|\xi(x) - \xi(y)|^p}{|x - y|^{N+ps(x,y)}} \, dx \, dy \right) \leq C_\theta \| \xi \|_{W}, \]
where the constants $C_\theta = C_\theta(N, p, s, \theta, \Omega) > 0$.

Finally, we recall that the previous embedding is compact since in the constant $s_i$ case we have that for subcritical exponents the embedding is compact. This completes the proof. \qed

Moreover, in order to investigate the Schrödinger–Kirchhoff type equations $(P_\rho)$, we consider the variable-order fractional Sobolev linear subspace $X_0$ with potential function, which is defined as follows
\[ X_0 = \left\{ \xi : \xi \in W_{0, \rho} \ \left| \int_{\Omega} V(x) \| \xi \|^p \, dx < +\infty \right. \right\} \]
with the norm
\[ \| \xi \|_{X_0} := \left( \int_{\Omega} V(x) \| \xi \|^p \, dx + \int_{\Omega \times \Omega} \frac{a^{p_i-1}|\xi(x) - \xi(y)|^p}{|x - y|^{N+ps(x,y)}} \, dx \, dy \right)^{1/p}. \]

The mapping $\rho : X_0 \to \mathbb{R}$ is defined:
\[ \rho_{X_0}(\xi) := \int_{\Omega} V(x) \| \xi \|^p \, dx + \int_{\Omega \times \Omega} \frac{a^{p_i-1}|\xi(x) - \xi(y)|^p}{|x - y|^{N+ps(x,y)}} \, dx \, dy. \]

**Lemma 4.** Let $(V_1)$ holds. If $\theta(x) \in [p, p^*_s(x))$, then the embeddings
\[ X_0 \hookrightarrow W_0 \hookrightarrow L^{\theta(x)}(\Omega) \]
are continuous with $\min\{1, V_0\} \| \xi \|^p_{W_0} \leq \| \xi \|^p_{X_0}$ for all $\xi \in X_0$. In particular, there exists a constant $C_0^{\rho}$ such that
\[ \| \xi \|_{L^{\theta(x)}(\Omega)} \leq C_0^{\rho} \| \xi \|_{X_0} \]
for all $\xi \in X_0$. If $\theta(x) \in [p, p^*_s(x)]$, then the embedding $X_0 \hookrightarrow L^{\theta(x)}(B_{R_1})$ is compact for any $R_1 > 0$.

**Proof.** From the definition of $\| \xi \|_{X_0}$ and $(V_1)$, we can get the following inequality
\[ \min\{1, V_0\} \| \xi \|^p_{W_0} \leq \| \xi \|^p_{X_0} \]
for any $\xi > 0$. Combined with Lemma 3, we have $X_0 \hookrightarrow W_0 \hookrightarrow L^{\theta(x)}(\Omega)$. The embeddings are obviously continuous, so
\[ \| \xi \|_{L^{\theta(x)}(\Omega)} \leq C_0^{\rho} \| \xi \|_{X_0}. \]

Fix $R_1 > 0$ and note that
\[ \left( \| \xi \|^p_{L^p(B_{R_1})} + \int_{B_{R_1} \times B_{R_1}} \frac{a^{p_i-1}|\xi(x) - \xi(y)|^p}{|x - y|^{N+ps(x,y)p}} \, dx \, dy \right)^{1/p} \]
is an equivalent norm on $W_0$ and the embedding $X_0 \hookrightarrow W_0$ is continuous. According to Corollary 7.2 of [41], the embedding
\[ W_0 \hookrightarrow L^{\theta(x)}(\Omega) \]
is compact. Therefore, also the embedding $X_0 \hookrightarrow L^{\theta(x)}(\Omega)$ is compact by the first part of the lemma. □

**Lemma 5.** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded open domain, then $(X_0, \| \cdot \|_{X_0})$ is a separable and reflexive Banach space.

**Proof.** First, we prove that $(X_0, \| \cdot \|_{X_0})$ is a Banach space. Set $(V1)$ holds, $\{\xi_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $X_0$, for all $\xi > 0$, there is a natural number $N$ such that if $n, k \geq N$

$$V_0\|\xi_n - \xi_k\|_{L^p(\Omega)} < \|\xi_n - \xi_k\|_{X_0} < \xi. \quad (10)$$

Since $\|\xi\|_{X_0} \geq \|\xi\|_{W_0}$ and $(W_0, \| \cdot \|_{W_0})$ is a Banach space, there is $\xi \in W_0$ such that $\xi_n \rightarrow \xi$ strongly in $W_0$. Therefore, there is a subsequence $\xi_{n_j}$ such that $\xi_{n_j}(x) \rightarrow \xi(x)$ a.e. $x \in \Omega$. From Fatou’s lemma with $\xi = 1$, we have

$$\int_\Omega V(x)|\xi|^p dx \leq \liminf_{n \rightarrow \infty} \int_\Omega V(x)|\xi_n|^p dx \leq \liminf_{n \rightarrow \infty} \int_\Omega V(x)|\xi_n - \xi_{N_0} + \xi_{N_0}|^p dx \leq 2^p \liminf_{n \rightarrow \infty} \left[\int_\Omega V(x)|\xi_n - \xi_{N_0}|^p dx + \int_\Omega V(x)|\xi_{N_0}|^p dx\right] \leq 2^p \liminf_{n \rightarrow \infty} \left[1 + \int_\Omega V(x)|\xi_{N_0}|^p dx\right] \leq \infty, \quad (11)$$

and then $\xi \in X_0$. Combining Fatou’s lemma and (10), we deduce for all $n, n_j \geq N$

$$\rho_{X_0}(\xi_n - \xi) \leq \liminf_{j \rightarrow \infty} \rho_{X_0}(\xi_n - \xi_{n_j}) < \xi, \quad (12)$$

this implies that $\xi_n \rightarrow \xi$ strongly in $X_0$ as $n \rightarrow \infty$. Hence, $(X_0, \| \cdot \|_{X_0})$ is a Banach space.

Next, we show that $(X_0, \| \cdot \|_{X_0})$ is a separable and reflexive space. We define the operator $G : W^{s(x),p}(\Omega) \rightarrow L^p(\Omega) \times L^p(\Omega \times \Omega)$ by

$$G(\xi) = \left(V^{1/p} \xi(x), \frac{a^{-1}p^{s(x)}(y)}{|x-y|^{N+2s(x,y)}}\right),$$

clearly $G(\xi)$ is an isometry, the rest of the proof is similar to Theorem 8.1 of [42]. Thus, we get $(X_0, \| \cdot \|_{X_0})$ is reflexive space (see [42], Proposition 3.20), and we get $(X_0, \| \cdot \|_{X_0})$ is separable space (see [42], Proposition 3.25). □

**Lemma 6.** Let $(V1)$-(V2) hold. Let $\theta(x) \in [p, p^{\ast}_s(x))$ and $\{\xi_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $X_0$. Then there exists $u \in X_0 \cap L^{\theta(x)}(\Omega)$ such that up to a subsequence

$$\xi_n \rightarrow \xi \text{ strongly in } L^{\theta(x)}(\Omega) \text{ as } n \rightarrow \infty.$$  

**Proof.** The proof of $\xi_n \rightarrow \xi$ strongly in $L^p(\Omega)$ is similar to the process of proof Theorem 2.1 in [4], so it is omitted. For $p < \theta(x) < p^{\ast}_s(x)$ there exists $q \in (0,1)$ such that $\frac{1}{\theta(x)} = \frac{q}{p} + \frac{1-q}{p^{\ast}_s(x)}$ and

$$\|\xi_n - \xi\|_{L^{\theta(x)}(\Omega)} \leq \|\xi_n - \xi\|_{L^p(\Omega)} \|\xi_n - \xi\|_{L^{p^{\ast}_s(x)}(\Omega)}^{1-q} \rightarrow 0, \text{ as } n \rightarrow \infty.$$  

Since $\{\xi_n\}_{n \in \mathbb{N}}$ is bounded sequence in $L^{p^{\ast}_s(x)}(\Omega)$, $\xi_n \rightarrow \xi$ strongly in $L^{\theta(x)}(\Omega)$ as $n \rightarrow \infty$. □
Let \( \{e_j\} \subset X_0^* \) and \( \{e_j\} \subset X_0^* \) such that \( X_0 = \text{span}\{e_j : j = 1, 2, ...\} \) and \( X_0^* = \text{span}\{e_j^* : j = 1, 2, ...\} \) and
\[
(e_j^*, e_i) = \begin{cases} 
1, & i \neq j; \\
0, & i = j.
\end{cases}
\]

Set \( X_i = \text{span}\{e_j : j = 1, 2, ...\} \), and define \( A_k = \bigoplus_{i=1}^k X_i \), \( B_k = \bigoplus_{i=k}^\infty X_i \).

Then I has a sequence of critical points \( \xi_k \) such that \( I(\xi_k) \to +\infty \) as \( k \to \infty \).

**Theorem 3** (Fountain Theorem, see [43]). Let \( X_0 \) be a real Banach space and an even functional \( I \in C^1(X_0, \mathbb{R}) \) satisfies the Palais–Smale ((PS) for short) condition, for every \( c > 0 \), and that there is \( k_0 > 0 \), such that for every \( k \geq k_0 \) there exists \( \rho_k > r_k > 0 \), so that the following properties hold:

(i) \( a_k = \max\{I(\xi) : \xi \in A_k, \|\xi\| = \rho_k\} \leq 0 \);

(ii) \( b_k = \inf\{I(\xi) : \xi \in B_k, \|\xi\| = r_k\} \to +\infty \) as \( k \to \infty \).

**Theorem 4** (Symmetric Mountain Pass Theorem, see [44]). Let \( X_0 \) be an infinite dimensional Banach space and let \( I \in C^1(X_0, \mathbb{R}) \) be even, satisfy (PS) condition, and \( I(0) = 0 \). If \( X_0 = A_k \supset B_k \), where \( A_k \) is finite dimensional and \( I \) satisfies:

(i) there exist constants \( \rho, \alpha > 0 \) such that \( I|_{\partial B_k \cap B_k} \geq \alpha \); 

(ii) for any finite dimensional subspace \( \hat{X} \subset X_0 \), there is \( R = R(\hat{X}) > 0 \) such that \( I(\xi) \leq 0 \) on \( \hat{X} \setminus B_R \), then \( I \) possesses an unbounded sequence of critical values.

### 3. Palais–Smale Condition

In what follows, we need the following definition and prove a lemma which will play a critical role.

**Definition 2.** Let \( X_0 \) be a Banach space, \( I \in C^1(X_0, \mathbb{R}) \). We say that \( I \) satisfies the (PS) condition, if any (PS) sequence \( \{\xi_n\} \subset X_0 \) with
\[
I(\xi_n) \to c, \quad I'(\xi_n) \to 0, \quad \text{as} \quad n \to \infty,
\]
possesses a convergent subsequence in \( X_0 \).

Suppose that \( \{\xi_n\}_{n \in \mathbb{N}} \) is a bounded sequence in \( X_0 \). Combining Theorem 2 with Lemma 6, there exists \( \xi \in X_0 \) such that
\[
\xi_n \rightharpoonup \xi \quad \text{in} \quad X_0, \quad \xi_n \to \xi \quad \text{a.e. in} \quad \Omega, \quad \xi_n \to \xi \quad \text{in} \quad L^0(\mathbb{R})(\Omega). \tag{13}
\]

**Lemma 7.** Let the conditions of Theorem 1 hold, then \( I \) satisfies the (PS) condition for large \( \lambda > 0 \).

**Proof. Step 1.** We show that \( \{\xi_n\}_{n \in \mathbb{N}} \) is bounded in \( X_0 \). Assume that \( \{\xi_n\}_{n \in \mathbb{N}} \subset X_0 \) is a sequence, from Definition 2, there exists a positive constant \( C \) such that
\[
|I(\xi_n)| \leq C \quad \text{and} \quad \|I'(\xi_n)\|_{X_0^*} \leq C \tag{14}
\]
for every \( n \in \mathbb{N} \).

We prove this by contrary arguments. Supposing \( \{\xi_n\}_{n \in \mathbb{N}} \) is unbounded in \( X_0 \), that is
\[
\|\xi_n\|_{X_0} \to \infty, \quad \text{as} \quad n \to \infty. \tag{15}
\]
Let $\omega_n := \frac{\xi_n}{\|\xi_n\|_{X_0}}$, then $\omega_n \in X_0$ with $\|\omega_n\|_{X_0} = 1$, $\|\omega_n\|_{s} \leq C_s \|\omega_n\|_{X_0} = C_s$ for $s \in [p, p^*_\mu(\cdot)]$. Set

$$G(t) = F(x, t^{-1}z)t^{\mu} \text{ for all } (x, z) \in \Omega \times \mathbb{R} \text{ and for all } t > 1.$$ 

For $|z| \geq r$ and $t \in [1, \frac{|z|}{r}]$, by (F3), we obtain

$$G'(t) = -\frac{z}{t^2}f(x, t^{-1}z)t^{\mu} + F(x, t^{-1}z)t^{\mu-1}\mu = t^{\mu-1}[\mu F(x, t^{-1}z) - t^{-1}zf(x, t^{-1}z)] \leq C_3 t^{\mu-p-1}|z|^p.$$ 

Then

$$G\left(\frac{|z|}{r}\right) - G(1) = \int_1^{\frac{|z|}{r}} G'(t)dt \leq \int_1^{\frac{|z|}{r}} C_3 t^{\mu-p-1}|z|^p dt = \frac{C_3 |z|^\mu}{(\mu - p)r^{(\mu - p)}} - \frac{C_3 |z|^\mu}{(\mu - p)}.$$ 

when $t = 1$, we have

$$F(x, z) = G(1) \geq G\left(\frac{|z|}{r}\right) - \frac{C_3 |z|^\mu}{(\mu - p)r^{(\mu - p)}} \geq \left(\frac{\beta}{r^\mu} - \frac{C_3}{(\mu - p)r^{(\mu - p)}}\right)|z|^\mu.$$ 

Noting that $0 < C_3 < \frac{\beta(\mu - p)}{r^\mu}$, we have $\frac{\beta}{r^\mu} - \frac{C_3}{(\mu - p)r^{(\mu - p)}}|z|^\mu > 0$. Since $\mu > p^2$, there is a constant $p^2 < \theta < p^*_\mu(\cdot)$ so that $\theta < \mu$, and

$$\lim_{|z| \to \infty} \frac{F(x, \xi)}{|\xi|^\theta} = +\infty. \quad (16)$$ 

In particular, we have

$$\lim_{|z| \to \infty} \frac{F(x, \xi)}{|\xi|^p} = +\infty. \quad (17)$$ 

By (F1) and (F2), we get

$$|F(x, \xi)| \leq C_4 |\xi|^p + C_5 |\xi|^{p(x)}. \quad (18)$$

Combining (16) and (18), for any $M_1 > 0$, there is a positive constant $C(M_1)$ so that

$$|F(x, \xi)| \geq M_1 |\xi|^\theta - C(M_1) |\xi|^p, \text{ for all } (x, \xi) \in \Omega \times \mathbb{R}. \quad (19)$$

Moreover, we have

$$I\left(\xi_n\right) = \frac{1}{\|\xi_n\|_{X_0}^p} \left( a + b \int_{\Omega} \int_{\Omega} \frac{|\xi_n(x) - \xi_n(y)|^p}{|x - y|^{N + p(\mu(\cdot))}}dxdy \right)^p + \frac{1}{\|\xi_n\|_{X_0}^p} \int_{\Omega} \lambda V(x)|\xi_n|^p dx - \int_{\Omega} F(x, \xi_n) dx.$$ 

Since $p^2 < \theta$, we deduce that

$$\lim_{n \to +\infty} \int_{\Omega} \frac{F(x, \xi_n)}{\|\xi_n\|_{X_0}^\theta} dx = 0. \quad (20)$$
Since \( \|\omega_n\|_{X_0} = 1 \), up to subsequences, there exists \( \omega \in X_0 \) such that \( \omega_n \to \omega \) in \( X_0 \), \( \omega_n \to \omega \) a.e. in \( \Omega \), \( \omega_n \to \omega \) in \( L^{p(x)}(\Omega) \). In the case of \( \omega \neq 0 \), setting \( \Omega_0 := \{ x \in \Omega : \omega \neq 0 \} \).

If \( \text{meas}(\Omega) > 0 \), then \( \int_{\Omega} \omega^pdx > 0 \). By (19), we have

\[
\int_{\Omega} F(x, \xi_n)dx > M_1 \|\omega_n\|_{L^p(\Omega)} = \frac{C(M_1)}{\|\xi_n\|_{X_0}} \|\omega_n\|_{L^p(\Omega)}^{\theta - p}.
\]

Therefore

\[
0 = \liminf_{n \to \infty} \left( \int_{\Omega} F(x, \xi_n)dx + C(M_1) \frac{\|\omega_n\|_{L^p(\Omega)}^\theta}{\|\xi_n\|_{X_0}^\theta} \right) > \liminf_{n \to \infty} M_1 \|\omega_n\|_{L^p(\Omega)} > \int_{\Omega} \omega^pdx > 0.
\]

This is a contradiction. Hence \( \text{meas}(\Omega) = 0 \) and as a result \( \omega = 0 \) a.e. in \( \mathbb{R}^N \). Thus, according to (V1), we obtain

\[
\|\omega_n\|_{L^p(\Omega)}^p = \int_{V(x) \geq d} |\omega_n|^pdx + \int_{V(x) < d} |\omega_n|^pdx \leq \frac{1}{A_0} \|\omega_n\|_{X_0} + o(1) \leq \frac{2}{A_0}
\]

for large \( n \). Combining (F1) and (F3), there exists a positive constant \( C_6 \) such that

\[
\mu F(x, \xi) - f(x, \xi) \xi \leq C_6 |\xi|^p,
\]

for all \( (x, \xi) \in \Omega \times X_0 \).

Consequently, we have

\[
0 < \frac{1}{\|\xi_n\|_{X_0}} \left[ \mu I(\xi_n) - \langle I'(\xi_n), \xi_n \rangle \right] = \frac{1}{\|\xi_n\|_{X_0}} \left[ \frac{\mu}{bp_2} \left( a + b[\xi_n]_{i(\cdot),p}^p \right)^p - \frac{\mu}{p} \int_{\Omega} \lambda V(x) |\xi_n|^pdx - \mu \int_{\Omega} F(x, \xi_n)dx \right.
\]

\[
- \left( a + b[\xi_n]_{i(\cdot),p}^p \right)^{p-1} + \frac{\mu}{p} \left( a + b[\xi_n]_{i(\cdot),p}^p \right)^{p-1} \int_{\Omega} \lambda V(x) |\xi_n|^pdx + \int_{\Omega} f(x, \xi_n)\xi_ndx \right]
\]

\[
= \frac{1}{\|\xi_n\|_{X_0}} \left[ \frac{\mu}{bp_2} \left( a + b[\xi_n]_{i(\cdot),p}^p \right)^p - \frac{\mu}{p} \left( a + b[\xi_n]_{i(\cdot),p}^p \right)^{p-1} \int_{\Omega} \lambda V(x) |\xi_n|^pdx + \int_{\Omega} f(x, \xi_n)\xi_ndx \right]
\]

\[
\geq \frac{1}{\|\xi_n\|_{X_0}} \left[ \frac{\mu - p^2}{p^2} \|\xi_n\|_{X_0}^p - \frac{\mu}{p} \int_{\Omega} \lambda V(x) |\xi_n|^pdx - C_6 \int_{\Omega} |\xi_n|^pdx \right]
\]

\[
\geq \frac{1}{\|\xi_n\|_{X_0}} \left[ \frac{\mu - p^2}{p^2} \|\xi_n\|_{X_0}^p - C_6 \int_{\Omega} |\xi_n|^pdx \right]
\]

\[
\geq \frac{\mu - p^2}{p^2} C_6 \lambda d.
\]

Since \( \mu > p^2 \), letting \( \lambda > 0 \) be so large that the term \( \frac{\mu - p^2}{p^2} - \frac{C_6}{\lambda d} \) is positive, we get a contradiction. Therefore, the sequence \( \{\xi_n\}_{n \in \mathbb{N}} \) is bounded in \( X_0 \).

**Step 2.** To show that \( \{\xi_n\}_{n \in \mathbb{N}} \) has a convergent subsequence in \( X_0 \). Then, the sequence \( \{\xi_n\}_{n \in \mathbb{N}} \) has a subsequence, still denoted by \( \{\xi_n\}_{n \in \mathbb{N}} \), and there exists \( \xi \in X_0 \) such that

\[
\xi_n \to \xi,
\]

weakly in \( X_0 \),

this implies

\[
(I'(\xi_n) - I'(\xi), \xi_n - \xi) \to 0, \text{ as } n \to \infty.
\]

Moreover, combining with (F1) and the Hölder inequality, we obtain
\[
\int_{\Omega} (f(x, \xi_n) - f(x, \xi))(\xi_n - \xi)dx \\
\leq \int_{\Omega} \left[ C_1(|\xi_n|^p + |\xi|^p) + C_2(|\xi_n|^p + |\xi|^p)\right] (\xi_n - \xi)dx \\
\leq C_1 \left( ||\xi_n||_{L^p(\Omega)}^p + ||\xi||_{L^p(\Omega)}^p \right) ||(\xi_n - \xi)||_{L^p(\Omega)} \\
+ C_2 \left( ||\xi_n||_{L^p(\Omega)}^p + ||\xi||_{L^p(\Omega)}^p \right) ||(\xi_n - \xi)||_{L^p(\Omega)}' \\
\]

hence, by (13) implies that

\[
\lim_{n \to \infty} \int_{\Omega} (f(x, \xi_n) - f(x, \xi))(\xi_n - \xi)dx = 0. \tag{22}
\]

For convenience, fix \( \psi \in X_0 \) and the linear functional \( \Gamma_\psi \) is defined as

\[
\Gamma_\psi(\nu) = \int_{\Omega \times \Omega} \frac{|\psi(x) - \psi(y)|^{p-2}(\psi(x) - \psi(y))(\nu(x) - \nu(y))}{|x - y|^{N+s(x,y)p}} dx dy
\]

for all \( \nu \in X_0 \). According to the Hölder inequality, \( \Gamma_\psi(\nu) \) is continuous and we obtain

\[
|\Gamma_\psi(\nu)| \leq ||\psi||_{X_0}||\nu||_{X_0}, \quad \text{for all } \nu \in X_0.
\]

Obviously, \( \Gamma_\psi(\nu) \) is bounded. Therefore, (13) gives that

\[
\lim_{n \to \infty} \Gamma_\psi(\nu_n - \nu) = 0 \quad \text{and} \quad \lim_{n \to \infty} \left( [\xi_n]_{s(x,y),p}^p - [\xi]_{s(x,y),p}^p \right) \Gamma_\psi(\nu_n - \nu) = 0. \tag{23}
\]

Since \( \{[\xi_n]_{s(x,y),p} - [\xi]_{s(x,y),p}\} \) is bounded in \( \mathbb{R} \). Hence, combining (13) and (21)–(23), we obtain

\[
o(1) = (l'(\xi_n) - l'(\xi), \xi_n - \xi) \\
= \left( a + b[\xi_n]_{s(x,y),p}^p \right)^{p-1} \\
\times \int_{\Omega \times \Omega} \frac{|\xi_n(x) - \xi_n(y)|^{p-2}(\xi_n(x) - \xi_n(y))(\xi_n(x) - \xi_n(y))(\xi(x) + \xi(y))}{|x - y|^{N+s(x,y)p}} dx dy \\
- \left( a + b[\xi]_{s(x,y),p}^p \right)^{p-1} \\
\times \int_{\Omega \times \Omega} \frac{|\xi(x) - \xi(y)|^{p-2}(\xi(x) - \xi(y))(\xi(x) - \xi(x))(\xi(x) + \xi(y))}{|x - y|^{N+s(x,y)p}} dx dy \\
- \int_{\Omega} \lambda V(x)(|\xi_n|^p - |\xi|^p)\xi_n - |\xi|^p - \xi)dx \\
- \int_{\Omega} (f(x, \xi_n) - f(x, \xi))(\xi_n - \xi)dx
\]
\[ (a + b[\xi_n]_{s(\cdot),p})^{p-1} \times \int_{\Omega} \frac{|\xi_n(x) - \xi_n(y)|^{p-2}(\xi_n(x) - \xi_n(y)) (\xi_n(x) - \xi_n(y) - \xi + \xi(y))}{|x - y|^{N+2s(x,y)p}} \] 

\[ - \left( a + b[\xi_n]_{s(\cdot),p} \right)^{p-1} \times \int_{\Omega} \frac{|\xi_n(x) - \xi(y)|^{p-2}(\xi(x) - \xi(y)) (\xi_n(x) - \xi_n(y) - \xi(x) + \xi(y))}{|x - y|^{N+2s(x,y)p}} \] 

\[ + \left( a + b[\xi_n]_{s(\cdot),p} \right)^{p-1} \times \int_{\Omega} \frac{|\xi_n(x) - \xi(y)|^{p-2}(\xi(x) - \xi(y)) (\xi_n(x) - \xi_n(y) - \xi(x) + \xi(y))}{|x - y|^{N+2s(x,y)p}} \] 

\[ - \int_{\Omega} \Lambda V(x) (|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi) (\xi_n - \xi) dx \] 

\[ - \int_{\Omega} (f(x, \xi_n) - f(x, \xi)) (\xi_n - \xi) dx \] 

\[ = \left( a + b[\xi_n]_{s(\cdot),p} \right)^{p-1} \times \left( \Gamma_{\xi_n}(\xi_n - \xi) - \Gamma_{\xi}(\xi_n - \xi) \right) \] 

\[ - \int_{\Omega} \Lambda V(x) (|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi) (\xi_n - \xi) dx \] 

\[ = \left( a + b[\xi_n]_{s(\cdot),p} \right)^{p-1} \times \left( \Gamma_{\xi_n}(\xi_n - \xi) - \Gamma_{\xi}(\xi_n - \xi) \right) \] 

\[ - \int_{\Omega} \Lambda V(x) (|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi) (\xi_n - \xi) dx + o(1). \]

That is

\[ \lim_{n \to \infty} \left( a + b[\xi_n]_{s(\cdot),p} \right)^{p-1} \times \left( \Gamma_{\xi_n}(\xi_n - \xi) - \Gamma_{\xi}(\xi_n - \xi) \right) \]

\[ - \int_{\Omega} \Lambda V(x) (|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi) (\xi_n - \xi) dx = 0. \]

Note that

\[ (a + b[\xi_n]_{s(\cdot),p})^{p-1} \times (\Gamma_{\xi_n}(\xi_n - \xi) - \Gamma_{\xi}(\xi_n - \xi)) \geq 0 \text{ and } \]

\[ \int_{\Omega} \Lambda V(x) (|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi) (\xi_n - \xi) dx \geq 0 \]

for any \( n \in \mathbb{N} \). Considering \( a, b > 0 \) and (V1), we have
\[
\lim_{n \to \infty} (a + b[\xi_n] \eta(p))^{p-1} \times (\Gamma_\eta(\xi_n - \xi) - \Gamma_\eta(\xi_n - \xi)) = 0 \quad \text{and} \\
\lim_{n \to \infty} \int_\Omega \lambda V(x)(|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi)(\xi_n - \xi) \, dx = 0.
\]

(24)

(25)

Now, we describe the Simon inequalities, that is
\[
|\eta_1 - \eta_2|^p \leq \begin{cases} 
\epsilon_p [(|\eta_1|^{p-2} \eta_1 - |\eta_2|^{p-2} \eta_2)(|\eta_1| - |\eta_2|)]^\frac{p}{2}\rho, & 1 < p < 2, \\
\tilde{\epsilon}_p (|\eta_1|^{p-2} \eta_1 - |\eta_2|^{p-2} \eta_2)(|\eta_1| - |\eta_2|), & p \geq 2,
\end{cases}
\]

for all \( \eta_1, \eta_2 \in \mathbb{R}^N \), where \( c_p > 0 \) and \( \tilde{\epsilon}_p > 0 \) are constants which depend on \( p \). We also need the following elementary inequalities, that is to say
\[
(\epsilon_1 + \epsilon_2)^{\frac{p(2-p)}{2}} \leq \epsilon_1^{\frac{p(2-p)}{2}} + \epsilon_2^{\frac{p(2-p)}{2}}, \quad \text{for all} \quad \epsilon_1, \epsilon_2 \geq 0 \quad \text{and} \quad 1 < p < 2.
\]

(27)

when \( p \geq 2 \), by the well-know Simon inequalities (24) and (26) as \( n \to \infty \), we have
\[
[\xi_n - \xi]_{\Omega(x_0)}^p \leq \int_{\Omega \times \Omega} \frac{|(\xi_n(x) - \xi_n(y) - \xi(x) + \xi(y))|^p}{|x - y|^{N+\kappa(x,y,p)}} \, dxdy = \epsilon_p (\Gamma_\xi(\xi_n - \xi) - \Gamma_\xi(\xi_n - \xi)) = o(1).
\]

Similarly, combining with (V1) and (24) as \( n \to \infty \), we obtain
\[
\|
\xi_n - \xi
\| \leq \epsilon_p \int_\Omega V(x)(|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi)(\xi_n - \xi) \, dx = o(1).
\]

Hence, \( \|
\xi_n - \xi
\|_{X_0} \to 0 \) as \( n \to \infty \), as required.

When \( 1 < p < 2 \), since \( \xi_n \to \xi \) weakly in \( X_0 \), there exists a positive constant \( \kappa_1 \) such that \( [\xi_n]_{\Omega(x_0)} \leq \kappa_1 \) for all \( n \in \mathbb{N} \). Combining with the Simon inequality (26), Hölder inequality and (24) as \( n \to \infty \), we get
\[
[\xi_n - \xi]_{\Omega(x_0)}^p \leq \epsilon_p [\Gamma_\xi(\xi_n - \xi) - \Gamma_\xi(\xi_n - \xi)]^\frac{p}{2} \left( (a + b[\xi_n])^p + (a + b[\xi_n])^p \right)^{\frac{2-p}{2}} \\
\leq \epsilon_p [\Gamma_\xi(\xi_n - \xi) - \Gamma_\xi(\xi_n - \xi)]^\frac{p}{2} \left( (a + b[\xi_n])^p + (a + b[\xi_n])^p \right)^{\frac{2-p}{2}} \\
\leq 2\epsilon_p (a + b\kappa_1)^{\frac{p(2-p)}{2}} (\Gamma_{u_0}(\xi_n - \xi) - \Gamma_\xi(\xi_n - \xi)) = o(1).
\]

Similarly, since \( \xi_n \to \xi \) weakly in \( X_0 \), there exists a positive constant \( \theta_0 \) such that \( \|
\xi_n\|^p \leq \theta_0 \) for all \( n \in \mathbb{N} \). Moreover, combining with the elementary inequality (27), Hölder inequality and (24) as \( n \to \infty \), we obtain
\[
\|
\xi_n - \xi
\| \leq 2\epsilon_p \theta_0 \left( \int_\Omega V(x)(|\xi_n|^{p-2} \xi_n - |\xi|^{p-2} \xi)(\xi_n - \xi) \, dx \right)^{\frac{p}{2}} = o(1).
\]

Therefore, \( \|
\xi_n - \xi
\|_{X_0} \to 0 \) as \( n \to \infty \). Thus, the functional \( I \) satisfies the \((PS)\) condition. \( \square \)
4. Proof of Theorem 1 by Using the Fountain Theorem

**Lemma 8** (see [21]). Assume that \( \xi(x) \in C_+(\Omega) \), \( \xi(x) < p^*(x) \), for any \( x \in \Omega \) and denote \( \beta_k = \sup_{\xi \in B_k, \|\xi\|_{X_0} = 1} \|\xi\|_{C^{+}(\Omega)}, \)

then \( \lim_{k \to \infty} \beta_k = 0. \)

To prove Theorem 1 by using the fountain theorem, we first prove the following two lemmas.

**Lemma 9.** Let the conditions of Theorem 1 hold, then there exist constants \( \rho_k > 0 \) such that

\[
\max_{\xi \in A_k, \|\xi\| = \rho_k} I(\xi) \leq 0.
\]

**Proof.** By (17), there exist \( C_7 > \frac{b \rho^{p-1}}{C_{A_k} p^2 \mu^p \rho^{p-1}}, C_8 > 0 \) such that

\[
|F(x, \xi)| \geq C_7 |\xi|^p - C_8 \text{ for all } x \in \Omega \text{ and } \xi \in X_0. \tag{28}
\]

Since all norms are equivalent on the finite dimensional Banach space \( A_k \), there exists a positive constant \( C_{A_k} \) such that

\[
\|\xi\|_{L^p(\Omega)} \geq C_{A_k} \|\xi\|_{X_0}. \]

Then for \( \|\xi\|_{X_0} = \rho_k \geq 1 \), from (28), we have

\[
I(\xi) := \frac{1}{b^p} \left( a + b |\xi|_{X_0}^p \right)^p + \frac{1}{p} \int_{\Omega} AV(x)|\xi|^p dx - \int_{\Omega} F(x, \xi) dx
\]

\[
\leq \frac{1}{b^p} \left( a + b |\xi|_{X_0}^p \right)^p + \frac{1}{p} \int_{\Omega} AV(x)|\xi|^p dx + C_8 \int_{\Omega} dx - C_7 \int_{\Omega} |\xi|^p dx
\]

\[
\leq \frac{1}{b^p} \left( a + b |\xi|_{X_0}^p \right)^p + \frac{1}{p} \|\xi\|_{X_0}^p + C_8 |\Omega| - C_7 |\xi|_{L^p(\Omega)}^p
\]

\[
\leq \frac{1}{b^p} \left( a + b |\xi|_{X_0}^p \right)^p + \frac{1}{p} \|\xi\|_{X_0}^p + C_8 |\Omega| - C_7 C_{A_k} |\xi|_{X_0}^p
\]

\[
\leq \frac{1}{b^p} \left( a + b |\xi|_{X_0}^p \right)^p + \frac{1}{p} \rho_k^p + C_8 |\Omega| - C_7 C_{A_k} \rho_k^p.
\]

Therefore, for \( p^2 > p > 1 \), there exists \( \rho_k > 1 \) large enough such that

\[
a_k = \max_{\xi \in A_k, \|\xi\| = \rho_k} I(\xi) \leq 0.
\]

This completes the proof. \( \square \)

**Lemma 10.** Let the conditions of Theorem 1 hold, then there exist constants \( r_k \) such that

\[
\inf_{\xi \in B_k, \|\xi\| = r_k} I(\xi) > +\infty.
\]
**Proof.** Note that by Young’s inequality, we have

\[
  a + b|ξ|^p_{I(ξ)} = \left( a^{\frac{p}{p-1}} \right)^{\frac{p}{p-1}} + \left( \frac{b|ξ|^p_{I(ξ)}}{p} \right)^{\frac{1}{p}} \\
  = \left( \frac{p}{p-1} \right)^{-1} \left( \frac{p}{p-1} \times a^{\frac{p}{p-1}} \right)^{\frac{p}{p-1}} + \frac{1}{p} \left( \frac{b|ξ|^p_{I(ξ)}}{p} \right)^{\frac{1}{p}} \\
  \geq a^{\frac{p}{p-1}} \left( \frac{p}{p-1} \times a^{\frac{p}{p-1}} \right)^{\frac{1}{2}} + \frac{1}{p} \left( \frac{b|ξ|^p_{I(ξ)}}{p} \right)^{\frac{1}{p}} \\
  \geq a^{\frac{p}{p-1}} \left( \frac{p}{p-1} \times a^{\frac{p}{p-1}} \right)^{\frac{1}{2}},
\]

hence, we get the following conclusion from the above derivation

\[
  \left( a + b|ξ|^p_{I(ξ)} \right)^{\frac{p}{p}} \geq pba^{\frac{p}{p-1}} |ξ|^p_{I(ξ)}.
\]

(29)

For \( ξ \in B_k \), according to (F1) and (29), we obtain that

\[
  I(u) := \frac{1}{2p} \left( a + b|ξ|^p_{I(ξ)} \right)^{\frac{p}{p}} + \frac{1}{p} \int_{\Omega} \lambda V(x)|ξ|^{p} dx - \int_{\Omega} F(x, ξ) dx \\
  \geq \frac{a^{p-1}|ξ|^p_{I(ξ)}}{p} + \frac{1}{p} \int_{\Omega} \lambda V(x)|ξ|^{p} dx - \int_{\Omega} F(x, ξ) dx \\
  \geq \frac{a^{p-1}|ξ|^p_{I(ξ)}}{p} + \frac{1}{p} \int_{\Omega} \lambda V(x)|ξ|^{p} dx - C_1 \int_{\Omega} |ξ|^{p} dx - C_2 \int_{\Omega} |ξ|^{2p} dx \\
  \geq \frac{1}{p} \|ξ\|_{X_0}^{\frac{p}{p}} \left( \frac{1}{p} - \frac{C_1 C_p}{p} \right) - \frac{C_2}{\theta^2} C_0^\theta \|ξ\|_{X_0}^{\frac{p}{p}} \\
  = \|ξ\|_{X_0}^{\frac{p}{p}} \left( \frac{1}{p} - \frac{C_1 C_p}{p} \right) - \frac{C_2}{\theta^2} C_0^\theta \|ξ\|_{X_0}^{\frac{p}{p}}.
\]

where \( β_k \) is defined as in Lemma 8 and \( 1 - C_1 C_p > 0 \), choosing

\[
  r_k = \left( \frac{1}{p} - \frac{C_1 C_p}{p} \right) \frac{1}{\theta^2} \left( 2pC_2 C_0^\theta \|ξ\|_{X_0}^{\frac{p}{p}} \right)^{\frac{1}{p}}.
\]

Thus, we obtain \( r_k \to +∞ \) as \( k \to +∞ \), thanks to Lemma 8 and the fact that \( 1 < p < \theta^2 \). By the choice of \( r_k \in B_k \) with \( \|ξ\|_{X_0} = r_k \) such that \( r_k > r_k > 0 \), we get

\[
  b_k = \inf_{ξ \in B_k, \|ξ\| = r_k} I(ξ) \geq \frac{1}{2p} \frac{r_k^{p}}{r_k} \to +∞,
\]

as \( k \to +∞ \). This completes the proof. □

**Proof of Theorem 1.** Let \( X_0 \) be a Banach space and the conditions of Theorem 1 hold. First, from Lemma 7 that \( I \) satisfies the (PS) condition. Moreover, according to the condition (F4) that \( I(0) = 0 \) and \( I \) is an even function. Finally, by Lemmas 9 and 10, we deduce that \( I \) satisfies the conditions (i) and (ii) of Theorem 3. Therefore, \( I \) satisfies all the conditions of Theorem 3, we obtain that problem \( (P_0) \) has a sequence of solutions \( ξ_k \) as \( k \to ∞ \). In conclusion, by Theorem 3, the problem \( (P_0) \) has infinitely many nontrivial weak solutions. This completes the proof. □

**5. Proof of Theorem 1 by Applying the Symmetric Mountain Pass Theorem**

In order to prove Theorem 1 by applying the symmetric mountain pass theorem, we first prove the following results.
**Lemma 11.** Let the conditions of Theorem 1 hold, then there exist constants \( \rho, \alpha > 0 \) such that

\[
I|_{\partial B_R \cap B_k} \geq \alpha
\]

**Proof.** From (F2) that for any \( \varepsilon > 0 \) there exists a positive \( \delta = \delta_\varepsilon \) such that

\[
|f(x, \xi)| \leq \frac{\varepsilon}{p} |\xi|^{p-1}, \text{ for all } x \in \Omega \text{ and } |\xi| \leq \delta,
\]

by (F1), we obtain

\[
|f(x, \xi)| \leq \left( C_2 + C_4 \delta^{p-\theta(x)} \right) |\xi|^{{\theta(x)}-1}, \text{ for all } x \in \Omega \text{ and } |\xi| \geq \delta,
\]

combining with the above two inequalities, we have

\[
|f(x, \xi)| \leq \frac{\varepsilon}{p} |\xi|^{p-1} + \left( C_2 + C_4 \delta^{p-\theta(x)} \right) |\xi|^{{\theta(x)}-1}, \text{ for all } x \in \Omega \text{ and } \xi \in X_0.
\]

Therefore, we get

\[
|F(x, \xi)| \leq \frac{\varepsilon}{p} |\xi|^{p} + \frac{C_\xi}{\delta} |\xi|^{{\theta(x)}}, \text{ for all } x \in \Omega \text{ and } \xi \in X_0,
\]

where \( C_\xi = C_2 + C_4 \delta^{p-\theta(x)} \). Thus, using Lemma 4, (29) and (30), we obtain

\[
I(\xi) := \frac{1}{bp^2} \left( a + b [\xi]_\theta^p \right)^p + \frac{1}{p} \int_{\Omega} \lambda V(x)|\xi|^p \, dx - \int_{\Omega} F(x, \xi) \, dx \\
\geq \frac{a \theta^{p-1}}{p} |\xi|^{p} \left( [\xi]_{\theta}^p \right) + \frac{1}{p} \int_{\Omega} \lambda V(x)|\xi|^p \, dx - \int_{\Omega} F(x, \xi) \, dx \\
\geq \frac{a \theta^{p-1}}{p} |\xi|^{p} \left( [\xi]_{\theta}^p \right) + \frac{1}{p} \int_{\Omega} \lambda V(x)|\xi|^p \, dx - \frac{\varepsilon}{p} \int_{\Omega} |\xi|^p \, dx - \frac{C_\xi}{\delta} \int_{\Omega} |\xi|^\theta \, dx \\
\geq \frac{1}{p} |\xi|^{p} \left( \left( \frac{a}{p} - \frac{\varepsilon}{p} \right) |\xi|^{1-1} - \frac{C_\xi}{\delta} \right) \left| \xi \right|^{\theta} \left| \xi \right|^{\theta} - \frac{C_\xi}{\delta} \right) |\xi|^{\theta}.
\]

Taking \( \varepsilon = \frac{1}{2a_p^1} \) and setting

\[
\eta(t) = \frac{1}{2a_p^p} t^{p-1} - \frac{C_\xi}{\delta} t^{\theta-1} - \frac{C_\xi}{\delta} t^{\theta-1} \quad \text{for all } t \in \mathbb{R}^+_0,
\]

there exists a positive \( \rho \) such that \( \max_{t \in \mathbb{R}^+_0} \eta(t) = \eta(\rho) \), since \( \theta^- > p > 1 \), we deduce that

\[
I(\xi) \geq \alpha = \frac{\eta(\rho)}{2} > 0
\]

for all \( \xi \in X_0 \) with \( \|\xi\|_{X_0} = \rho \). \( \square \)

**Lemma 12.** Let the conditions of Theorem 1 hold, then for any finite dimensional subspace \( \tilde{X} \in X_0 \), there is \( R = R(\tilde{X}) > 0 \) such that

\[
I(\xi) \leq 0
\]

on \( \tilde{X} \setminus B_R \).
Proof. We show that for any finite dimensional subspace $\tilde{X}$ of $X_0$, there exists $R_0 = R_0(\tilde{X})$ such that $I(\xi) < 0$ for all $\xi \in X_0 \setminus B_{R_0}(\tilde{X})$, where $B_{R_0}(\tilde{X}) = \{ \xi \in X_0 : \|\xi\| < R_0 \}$. By (F1) and (17), for any $M > \frac{k_{p-1}}{p^{p-1}I_{C}(p)}$, there exists a positive constant $C(M)$ such that

$$|F(x, \xi)| \geq M|\xi|^p - C(M)|\xi|^p$$

for all $x \in \Omega$ and $\xi \in X_0$.

For any finite dimensional subspace $\tilde{X} \subset X_0$, by the equivalence of norms in the finite dimensional space, there exists a positive constant $C(p^2)$ such that

$$\|\xi\|_{L^2(\Omega)} \geq C(p^2)\|\xi\|_{X_0},$$

for all $\xi \in \tilde{X}$, therefore, we get that for all $t \geq 1$ sufficiently large,

$$I(t\xi) := \frac{1}{bp^2}\left(a + b[\xi]^p_{L^p(X)}\right)^p + \frac{1}{p} \int_\Omega \lambda V(x)|t\xi|^p dx - \int_\Omega F(x, t\xi) dx$$

$$\leq \frac{1}{bp^2}\left(a + b[\xi]^p_{L^p(X)}\right)^p + \frac{1}{p} \int_\Omega \lambda V(x)|\xi|^p dx - \int_\Omega \left(M|\xi|^p - C(M)|\xi|^p\right) dx$$

$$\leq \frac{t^p}{bp^2}\left(a + b[\xi]^p_{L^p(X)}\right)^p + \frac{tp}{p} \int_\Omega \lambda V(x)|\xi|^p dx + C(M)t^p \int_\Omega |\xi|^p dx - Mt^p \int_\Omega |\xi|^p dx$$

$$\leq \frac{t^p}{bp^2}\left(a + \frac{b}{ap-1}t^p \|\xi\|_{X_0}^p\right)^p + \frac{tp}{p} \|\xi\|_{X_0}^p + C(M)C_p t^p \|\xi\|_{X_0}^p - MC(p^2)t^p \|\xi\|_{X_0}^p,$$

when $M$ is sufficiently large, we get

$$I(t\xi) = \frac{t^p}{bp^2}\left(a + \frac{b}{ap-1}t^p \|\xi\|_{X_0}^p\right)^p + \frac{tp}{p} \|\xi\|_{X_0}^p$$

$$+ C(M)C_p t^p \|\xi\|_{X_0}^p - MC(p^2)t^p \|\xi\|_{X_0}^p$$

$$\rightarrow -\infty \text{ as } t \rightarrow \infty$$

for all $\xi \in X_0$. Consequently, there exists a positive constant $R_0$ such that $I(\xi) \leq 0$ with $\|\xi\| = R$ and $R \geq R_0$. 

Proof of Theorem 1. Let $X_0$ be a Banach space and the conditions of Theorem 1 hold. First, from Lemma 7 that $I$ satisfies the (PS) condition. Moreover, according to the condition (F4) that $I(0) = 0$ and $I$ is an even function. Finally, by Lemmas 11 and 12, we deduce that $I$ satisfies the conditions (i) and (ii) of Theorem 4. Therefore, $I$ satisfies all the conditions of Theorem 4, we obtain that problem $(P_0)$ has a sequence of solutions $\tilde{\xi}_k$ as $k \rightarrow \infty$. In conclusion, by Theorem 4, the problem $(P_0)$ has infinitely many nontrivial weak solutions. This completes the proof.

6. Conclusions

In this article, we obtain an embedding result for the variable-order fractional Sobolev spaces, and then we investigate the following Schrödinger–Kirchhoff type equations. The existence of infinitely many solutions is obtained by utilizing the fountain theorem and symmetric mountain pass theorem respectively, and the Ambrosetti–Rabinowitz condition is not required.

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