Statistical Inference for Large-dimensional Tensor Factor Model by
Weighted/Unweighted Projection

Yong He *, Lingxiao Li *, Lorenzo Trapani †

June 22, 2022

Tensor Factor Models (TFM) are appealing dimension reduction tools for high-order large-dimensional time series, and have wide applications in economics, finance and medical imaging. Two types of TFM have been proposed in the literature, essentially based on the Tucker or CP decomposition of tensors. In this paper, we propose a projection estimator for the Tucker-decomposition based TFM, and provide its least-square interpretation which parallels to the least-square interpretation of the Principal Component Analysis (PCA) for the vector factor model. The projection technique simultaneously reduce the dimensionality and the magnitudes of the idiosyncratic error matrix, thus leading to an increase of signal-to-noise ratio. We derive a faster convergence rate of the projection estimator than that of the naive PCA-based estimator, under mild conditions which allow the idiosyncratic noise to have weak cross-correlations and weak autocorrelations. Further motivated by the least-squares interpretation, we propose a robust version by utilizing a Huber-loss function, which leads to an iterative weighted projection technique. Extensive numerical studies are conducted to investigate the empirical performance of the proposed (weighted) projection estimator relative to the state-of-the-art ones. The simulation results shows that the projection estimator performs better than the non-projection estimators, and the weighted projection estimator performs much better than the existing ones in the heavy-tailed case.

* Institute of Financial Studies, Shandong University, China; e-mail: heyong@sdu.edu.cn, lilingxiao@mail.sdu.edu.cn
† School of Economics, University of Nottingham, UK; e-mail: Lorenzo.Trapani@nottingham.ac.uk
1 Introduction

Tensors, a.k.a high-order arrays, has emerged as one of the most active areas in both statistics and machine learning. Tensors arise in numerous scientific research areas involving neuroimaging analysis (Zhou et al., 2013; Ji et al., 2021; Chen et al., 2021a), macroeconomic indicators (Chen et al., 2020a, 2021c), financial data (Han et al., 2020b; He et al., 2021b) and more. High-order tensors often impose significant computational challenges due to the consequent high-dimensionality. A typical example would be functional MRI data, typically consisting of hundreds of thousands of voxels observed over time. Most existing literature on tensor focus on independent and identically distributed \((i.i.d.)\) tensor data or tensor data with \(i.i.d.\) noise. while tensor time series data are rarely studied in the literature. In the current paper, we model the tensor time series by factor model and thus achieve great dimension reduction for analyzing tensors.

1.1 Closely Related Literature Review

Factor model has been one of the most powerful tools for extracting common dependence among multi-dimensional features thereby reducing dimensions greatly. In the past decades, classical vector factor models have been extensively studied in the communities of statistics and economics, see for example the seminal work by Bai and Ng (2002) and Stock and Watson (2002) and some subsequent representative work by Bai (2003), Onatski (2009), Ahn and Horenstein (2013), Fan et al. (2013), and Trapani (2018), Aït-Sahalia and Xiu (2017), Aït-Sahalia et al. (2020). Noticeably, there exists some work on relaxing the restrictive moment conditions on idiosyncratic errors, see the endeavors by Yu et al. (2019), Chen et al. (2021b) and He et al. (2022).

Significant efforts have been paid in developing methodologies and establishing theories for vector factor models, while in contrast, there exists limited literature on factor models for matrix/tensor-valued time series. Wang et al. (2019) for the first time proposed Matrix Factor Model (MFM) exploiting the double
low-rank structure of matrix-valued observations. Then Chen and Fan (2021) proposed an $\alpha$-PCA method and Yu et al. (2022) proposed a projected estimation method for MFM. He et al. (2021a) proposed a strong rule to determine whether there is a factor structure of matrix time series. See also some extensions of MFM such as the constrained matrix factor model by Chen et al. (2020b), the threshold matrix factor model in Liu and Chen (2019), the online change point model by He et al. (2021c) and applications of MFM by Chen and Chen (2020); Gao et al. (2021). As for Tensor Factor Model (TFM), it’s still in its infancy and there exist only a handful of work. The seminal work is by Chen et al. (2021c), which proposed to analyze high dimensional dynamic tensor time series in the form:

$$X_t = M_t + E_t, \; t = 1, \cdots, T,$$  

(1.1)

where $X_1, \ldots, X_T \in \mathbb{R}^{p_1 \times p_2 \times \cdots \times p_K}$ are the observed tensor time series, $M_t$ and $E_t$ are the signal and noise components of $X_t$, respectively. We are interested in recovering the unknown signal tensor $M_t$ from the tensor time series data. The handful of work on tensor factor model divides into two types: TFM based on Tucker decomposition (Chen et al., 2021c; Han et al., 2020a,b) or CP decomposition (Han et al., 2021). In this article, we will focus on the Tucker-decomposition type, that’s

$$X_t = F_t \times_1 A_1 \times_2 \cdots \times_K A_K + E_t = F_t \times_{k=1}^K A_k + E_t, \; t = 1, \cdots, T,$$

where $A_k \in \mathbb{R}^{p_k \times r_k}$ is the $k$-th mode-wise loading matrix, $F_t \in \mathbb{R}^{r_1 \times \cdots \times r_K}$ is the latent comment factor tensor with $r_d \ll p_d$, $E_t$ is the $p_1 \times p_2 \times \cdots \times p_K$ idiosyncratic error tensor. All the existing Tucker-decomposition based TFM assume that the factors accommodate all dynamics, making the idiosyncratic noise “white” with no autocorrelation but allowing substantial contemporary cross-correlation among the error process, and the estimation of the loading space is done by an eigen-analysis of the nonzero auto-covariance matrices (Chen et al., 2021c). In this article, for theoretical analysis and algorithms, we consider the other type of models, which assumes that a common factor must have impact on almost all series, but allows the idiosyncratic noise to have weak cross-correlations and weak autocorrelations, and principle
component analysis (PCA) of the sample covariance matrix is typically used to estimate the spaces spanned by the row/column loading matrices.

1.2 Contributions and Structure of the Paper

In this work, we first prove the equivalence between minimizing the least squares loss under identifiability condition and a similar iterative projection method by Han et al. (2020a). In other word, we provide the least squares interpretation of the iterative projection for tensor factor model, which parallels to the least squares interpretation of traditional PCA for vector factor models (Fan et al., 2013) and of Projection Estimation for matrix factor models (He et al., 2021b). We first propose a naive estimator based on the eigen-analysis of the mode-wise sample covariance matrix, referred to as initial estimator in the following. We then provide an iterative algorithm for estimating the factor loading matrices \( \{A_k, k = 1, \ldots, K\} \) in TFM. We provide the theoretical convergence rates for the initial estimator and the one-step iteration projection estimators, which indicates that the projection technique improves the estimation accuracy attributed to the increased signal-to-noise ratio. Further motivated by the least squares formulation, we further propose a robust estimation method for TFM by replacing the least squares loss with the Huber loss. We also propose an efficient iterative “weighted” projection algorithm to solve the corresponding optimization problem. As far as we know, this is the first work on robust analysis of TFM.

The rest of the article is organized as follows. In Section 2, we first formulate the estimation of factor loading matrices and the factor score tensor by minimizing the least squares loss under the identifiability condition and give the KKT condition to the optimization problem, from which it naturally leads to a projection estimation algorithm. In Section 3, we investigate the theoretical properties of the initial estimator and the one-step iteration estimator under mild conditions. In Section 4, we provide robust estimators by considering the Huber loss function and present detailed algorithm to obtain the minimizers. We also propose robust estimators for the pair of factor numbers. In Section 5, we conduct thorough numerical studies to illustrate the advantages of the RTFA method and the robust iterative eigenvalue-ratio method over the state-of-the-art methods. We discuss possible future research directions and conclude the article in Section
The proofs of the main theorems and additional details are collected in the supplementary materials.

2 Methodology

First of all, we introduce the notations used throughout the study. For a tensor $\mathcal{X} \in \mathbb{R}^{p_1 \times p_2 \times \cdots \times p_K}$, the mode-$k$ product with a matrix $\mathbf{A} \in \mathbb{R}^{d \times p_k}$, denoting as $\mathcal{X} \times_k \mathbf{A}$, is a tensor of size $p_1 \times \cdots \times p_k \times d \times p_{k+1} \times \cdots \times p_K$. Elementwise, $(\mathcal{X} \times_k \mathbf{A})_{i_1, \cdots, i_k-1, j, i_k+1, \cdots, i_K} = \sum_{i_{k} = 1}^{p_k} x_{i_1, \cdots, i_{k-1}, j, i_{k+1}, \cdots, i_K} a_{j, i_k}$. The mode-$k$ unfolding matrix of $\mathcal{X}$ is denoted by $\text{mat}_k(\mathcal{X})$ and arranges all $p_k$ mode-$k$ fibers of $\mathcal{X}$ to be the columns to get a $p_k \times (p_1 \cdots p_k - 1) \times p_k + 1 \times \cdots \times p_K$ matrix. For a matrix $\mathbf{A}$, $\mathbf{A}^\top$ is the transpose of $\mathbf{A}$, $\text{Tr}(\mathbf{A})$ is the trace of $\mathbf{A}$, and $\|\mathbf{A}\|_F$ is the Frobenious norm of $\mathbf{A}$. $\mathbf{A} \otimes \mathbf{B}$ denotes the kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$. $\mathbf{I}_k$ represents a $k$-order identity matrix.

2.1 Least Squares and Projection Estimation

Let $\mathcal{X}_t$ be a $p_1 \times p_2 \times \cdots \times p_K$ tensor observed at time point $t$. The tensor factor model is

$$
\mathcal{X}_t = \mathcal{F}_t \times_1 \mathbf{A}_1 \times_2 \cdots \times_K \mathbf{A}_K + \mathcal{E}_t = \mathcal{F}_t \times_{k=1}^{K} \mathbf{A}_k + \mathcal{E}_t, \ t = 1, \cdots, T, \quad (2.1)
$$

where $\mathbf{A}_k$ is the $p_k \times r_k$ loading matrix, $\mathcal{F}_t$ is the $r_1 \times \cdots \times r_K$ comment factor tensor, $\mathcal{E}_t$ is the $p_1 \times p_2 \times \cdots \times p_K$ idiosyncratic component, and $\mathcal{M}_t = \mathcal{F}_t \times_{k=1}^{K} \mathbf{A}_k$ is the signal component. In order to ensure the identifiability of the model, without loss of generality, it is assumed that $\mathbf{A}_k^\top \mathbf{A}_k / p_k = \mathbf{I}_k$ for $k = 1, 2, \cdots, K$.

Under the identifiability condition, $\{\mathbf{A}_k, k = 1, \cdots, K\}$ is estimated by minizing the following least squares loss:

$$
L_1(\mathbf{A}_1, \cdots, \mathbf{A}_K, \mathcal{F}_t) = \frac{1}{T} \sum_{t=1}^{T} \|\mathcal{X}_t - \mathcal{F}_t \times_{k=1}^{K} \mathbf{A}_k\|_F^2 \quad \text{s.t.} \quad \frac{1}{p_k} \mathbf{A}_k^\top \mathbf{A}_k = \mathbf{I}_{r_k}, k = 1, \cdots, K. \quad (2.2)
$$
In the right-hand form of (2.2),

\[
\|X_t - F_t \times \bigotimes_{k=1}^{K} A_k\|_F^2 = \|\text{mat}_k(X_t) - \text{mat}_k(F_t) \times \bigotimes_{j \in [K] \setminus \{k\}} A_j\|_F^2 = \|\text{mat}_k(X_t) - A_k \cdot \text{mat}_k(F_t)(\bigotimes_{j \in [K] \setminus \{k\}} A_j)\|_F^2, 
\]

where \( \bigotimes_{j \in [K] \setminus \{k\}} A_j = A_K \otimes A_{K-1} \otimes \cdots \otimes A_{k+1} \otimes A_{k-1} \otimes \cdots \otimes A_1 \). Let \( B_k = \bigotimes_{j \in [K] \setminus \{k\}} A_j \), then \( B_k \) satisfies that

\[
B_k^T B_k/p_k = I_{r_k} \quad \text{for all} \quad k \in \{1, \ldots, K\}, \quad \text{where} \quad p_k = p_1 p_2 \cdots p_K, \quad r_k = r/r_k, \quad r = r_1 r_2 \cdots r_K. 
\]

For ease of notation, abbreviates \( \text{mat}_k(X_t) \) as \( X_{k,t} \), and \( \text{mat}_k(F_t) \) as \( F_{k,t} \), the right-hand side of the above equation (2.2) can be simplified to

\[
\frac{1}{T} \sum_{t=1}^{T} \|X_t - F_t \times \bigotimes_{k=1}^{K} A_k\|_F^2 = \frac{1}{T} \sum_{t=1}^{T} \|X_{k,t} - A_k F_{k,t} B_k^T\|_F^2 := L_1(A_k, B_k, F_{k,t}) 
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left[ \text{Tr}(X_{k,t}^T X_{k,t}) - \frac{1}{p_k} \text{Tr}(X_{k,t}^T A_k A_k^T X_{k,t} B_k B_k^T) \right]. \tag{2.3}
\]

First, we assume that \( \{A_k, B_k\} \) is given and solve the optimization problem on \( F_{k,t} \). Let \( \partial L_1(A_k, B_k, F_{k,t})/\partial F_{k,t} = 0 \) for each time point \( t \), it can be obtained that

\[
F_{k,t} = \frac{1}{p_k} A_k^T X_{k,t} B_k. 
\]

Exchanging \( F_{k,t} = A_k^T X_{k,t} B_k/p_k \) in loss function \( L_1(A_k, B_k, F_{k,t}) \), we have

\[
L_1(A_k, B_k) = \frac{1}{T} \sum_{t=1}^{T} \left[ \text{Tr}(X_{k,t}^T X_{k,t}) - \frac{1}{p_k} \text{Tr}(X_{k,t}^T A_k A_k^T X_{k,t} B_k B_k^T) \right] 
\]

\[
\text{s.t.} \quad \frac{1}{p_k} A_k^T A_k = I_{r_k}, \quad \frac{1}{p_k} B_k^T B_k = I_{r_k}. \tag{2.4}
\]

Finally, the following Lagrangian function can be obtained:

\[
\min_{\{A_k, B_k\}} L_3 = L_1(A_k, B_k) + \text{Tr} \left[ \Theta \left( \frac{1}{p_k} A_k^T A_k - I_{r_k} \right) \right] + \text{Tr} \left[ \Lambda \left( \frac{1}{p_k} B_k^T B_k - I_{r_k} \right) \right], \tag{2.5}
\]
where the Lagrange multipliers $\Theta$ and $\Lambda$ are Symmetric matrices.

According to the KKT condition, let

$$\frac{\partial L_1}{\partial A_k} = -\frac{1}{T} \sum_{t=1}^{T} \frac{2}{p} X_{k,t} B_k B_k^T X_{k,t} A_k + \frac{2}{p_k} A_k \Theta = 0,$$

$$\frac{\partial L_1}{\partial B_k} = -\frac{1}{T} \sum_{t=1}^{T} \frac{2}{p} X_{k,t}^T A_k A_k^T X_{k,t} B_k + \frac{2}{p_k} B_k \Lambda = 0.$$  

Then the following equations hold:

$$\begin{cases}
(M_k)^T p_k \sum_{t=1}^{T} X_{k,t} B_k B_k^T X_{k,t}) A_k = A_k \Theta, & \text{or} & M_k A_k = A_k \Theta, \\
(M_{-k})^T p_k \sum_{t=1}^{T} X_{k,t} A_k A_k^T X_{k,t}) B_k = B_k \Lambda, & M_{-k} B_k = B_k \Lambda,
\end{cases} \tag{2.6}$$

where

$$M_k = \frac{1}{T p_k} \sum_{t=1}^{T} X_{k,t} B_k B_k^T X_{k,t}, \quad M_{-k} = \frac{1}{T p_k} \sum_{t=1}^{T} X_{k,t} A_k A_k^T X_{k,t}.$$  

Assuming that the first $r_k$ eigenvalues of $M_k$ are $\{\lambda_{k,1}, \lambda_{k,2}, \cdots, \lambda_{k,r_k}\}$ and the corresponding eigenvectors are $\{u_{k,1}, \cdots, u_{k,r_k}\}$, we have $\Theta = \text{diag}(\lambda_{k,1}, \lambda_{k,2}, \cdots, \lambda_{k,r_k})$, $A_k = \sqrt{p_k}(u_{k,1}, \cdots, u_{k,r_k})$ satisfies the KKT condition. As the estimation of $A_k$ relies on the unknown projection matrix $B_k$, or equivalently $\{A_j, j \neq k\}$, a natural solution is to replace it with a consistent initial estimator $\hat{B}_k$. The choices of $\hat{B}_k$ will be discussed later. Denote $\tilde{Y}_{k,t} = \frac{1}{p_k} X_{k,t} \hat{B}_k$, and let

$$\tilde{M}_k = \frac{1}{T p_k} \sum_{t=1}^{T} \tilde{Y}_{k,t} \tilde{Y}_{k,t}^T,$$

and the eigenvectors of $\tilde{M}_k$ are $\{\tilde{u}_{k,1}, \cdots, \tilde{u}_{k,r_k}\}$, the projected estimator would be $\tilde{A}_k = \sqrt{p_k}(\tilde{u}_{k,1}, \cdots, \tilde{u}_{k,r_k})$. Then we alternate the procedure for $k = 1, \ldots, K$, and obtain the projected estimators $\{\tilde{A}_k, k = 1, 2, \cdots, K\}$.

From the above analysis, we can see that minimizing the least squares naturally leads to a projection algorithm, which achieves simultaneous reduction of the dimensionality and the magnitudes of the idiosyncratic error matrix. The projection algorithm for each fixed $k$ can viewed from the perspective of a matrix factor model and is exactly the same as in Yu et al. (2022). We summarize the projection procedure in
Algorithm 1. The projection method can be implemented recursively by plugging in the newly estimated \( \{\tilde{A}_k, k = 1, \ldots, K\} \) to replace \( \{\hat{A}, k = 1, \ldots, K\} \) in Step 2 and iterating Steps 2-4. Theoretical analysis of the recursive solution is challenging. The simulation results show that the projection estimators with a single iteration perform sufficiently well compared with the recursive method. Actually, with \( T \asymp p_1 \asymp p_2 \cdots \asymp p_K \) and \( \{\hat{A}, k = 1, \ldots, K\} \) chosen suitably, it can be proved that the projected estimator \( \tilde{A}_k \) converges to \( A_k \) after rotation at rate \( O_p(1/\sqrt{Tp_k}) \) in terms of the averaged squared errors, which is the optimal rate even when all the other loading matrices are known in advance.

2.1.1 Initial projection matrices

Abbreviates mat \(_k(E_t)\) as \( E_{k,t} \), then model (2.1) turns into

\[
X_{k,t} = A_k F_{k,t} B_k^T + E_{k,t}. \tag{2.7}
\]

The columns of \( X_{k,t} \) can be written in the form of a vector factor model as

\[
x_{k,t,j} = A_k f_{k,t,j} + e_{k,t,j} := A_k \tilde{f}_{k,t,j} + e_{k,t,j}, \quad t = 1, \ldots, T, \quad j = 1, \ldots, p_k - k, \tag{2.8}
\]

where \( \tilde{f}_{k,t,j} = F_{k,t} B_{k,j}^T \). Therefore, to estimate \( A_k \), we view each column as an individual vector observation and use the conventional PCA method for vector time series. In detail, define the scaled mode-wise sample covariance matrix as

\[
\tilde{M}_k = \frac{1}{Tp} \sum_{t=1}^{T} \sum_{j=1}^{p_k - k} x_{k,t,j} x_{k,t,j}^T = \frac{1}{Tp} \sum_{t=1}^{T} X_{k,t} X_{k,t}^T.
\]
When the columns of $A_k$ are orthogonal and under other mild conditions, approximately

$$
\widehat{M}_k \approx \frac{1}{p_k} A_k \left( \frac{1}{Tp_k} \sum_{t=1}^{T} \sum_{j=1}^{p-k} f_{k,t,j} f_{k,t,j}^\top \right) A_k^\top + \frac{1}{Tp_k} \sum_{t=1}^{T} \sum_{j=1}^{p-k} e_{k,t,j} e_{k,t,j}^\top.
$$

The term $T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{r_k} \hat{f}_{k,t,j} \hat{f}_{k,t,j}^\top$ converges to a positive definite matrix while the error terms are asymptotically negligible under certain conditions. As a result, the leading $r_k$ eigenvalues of $\widehat{M}_k$ outstands.

By Davis-Kahan’s $\sin(\Theta)$ theorem (e.g. Davis and Kahan (1970)), the $r_k$ leading eigenvectors of $\widehat{M}_k$ share the same column space as that of $A_k$ asymptotically. Therefore, we estimate the loading matrix $A_k$ by $\widehat{A}_k = \sqrt{p_k} \hat{Q}_k$, where $\hat{Q}_k$ is a matrix with columns being the $r_k$ leading eigenvectors of $\widehat{M}_k$. Thus the estimator for $B_k$ can be naturally chosen as $\widehat{B}_k = \otimes_{j=k+1}^{K} \widehat{A}_j \otimes_{j=1}^{k-1} \widehat{A}_j, k = 1, \cdots, K$. Some other choices of initial estimates of $A_k$ are admissible as long as two sufficient conditions in (3.1) in the following section are fulfilled. For simplicity, we only demonstrate theoretically that the above initial estimators work.

**Algorithm 1** Least squares method for estimating loading spaces

**Input:** tensor datas \( \{X_t, t = 1, \cdots , T\} \), factor numbers \( r_1, \cdots , r_K \)

**Output:** factor loading matrices \( \{A_k, k = 1, \cdots , K\} \)

1. obtain the initial estimators $\widehat{A}_k, k = 1, \cdots , K$;
2. project to reduce dimensions by defining $\hat{Y}_{k,t} = \frac{1}{p_k} X_{k,t} \hat{B}_k$, where $\hat{B}_k = \otimes_{j=k+1}^{K} \widehat{A}_j \otimes_{j=1}^{k-1} \widehat{A}_j, k = 1, \cdots , K$;
3. given \( \{\hat{Y}_{k,t}, k = 1, \cdots , K\} \), define $\widehat{M}_k = (Tp_k)^{-1} \sum_{t=1}^{T} (\hat{Y}_{k,t}) \hat{Y}_{k,t}^\top$, set $\widehat{A}_k$ as $\sqrt{p_k}$ times the matrix with columns being the first $r_k$ eigenvectors of $\widehat{M}_k$;
4. Output the projection estimators as $\{A_k, k = 1, \cdots , K\}$.

### 2.2 Estimation of Factor Numbers

We have two ways to estimate the factor numbers, the first one is based on the simple mode-wise sample covariance matrix $\widehat{M}_k$, the second one based on the mode-wise sample covariance matrix of the projected data, i.e., $\widehat{M}_k$. In detail, with the idea of eigenvalue-ratio, estimators of the factor numbers \( \{r_k, k = 1, \cdots , K\} \) can be defined as:
\[
\hat{r}_k^{\text{PE-ER}} = \arg\max_{j \leq r_{\text{max}}} \frac{\lambda_j(\widetilde{M}_k)}{\lambda_{j+1}(\widetilde{M}_k)}, \quad \hat{r}_k^{\text{PE-ER}} = \arg\max_{j \leq r_{\text{max}}} \frac{\lambda_j(\widetilde{M}_k)}{\lambda_{j+1}(\widetilde{M}_k)}, \quad k = 1, \ldots, K, \tag{2.10}
\]

where \(r_{\text{max}}\) is a predetermined positive constant greater than \(\{r_k, k = 1, \ldots, K\}\).

If the common factors are strong enough, the first \(r_k\) eigenvalues of \(\widetilde{M}_k (\widetilde{M}_k)\) will be well separated from the others, and (2.10) will take the maximum value at \(j = r_k\). One problem to calculate \(\widetilde{M}_k\) is that \(\hat{B}_k\) must be predetermined, which means \(r_k\) (or equivalently \(r_j, j \neq k\)) must be given first. Empirically, both \(r_k\) and \(r_{-k}\) (or equivalently \(r_j, j \neq k\)) are unknown. To address the problem, we propose to determine the numbers of factors by the following iterative Algorithm 2.

**Algorithm 2** Projected estimation of the numbers of factor

**Input:** tensor data \(\{X_t, t = 1, \ldots, T\}\), maximum number \(r_{\text{max}}\), maximum iterative step \(m\)

**Output:** factor numbers \(\{\hat{r}_k^{\text{PE-ER}}, k = 1, \ldots, K\}\)

1. initialize: \(r_k^{(0)} = r_{\text{max}}, k = 1, \ldots, K\);
2. given \(r_k^{(0)}\), obtain the initial estimators \(\hat{A}_k, k = 1, \ldots, K\) and set \(\hat{A}_k^{(0)} = \hat{A}_k\);
3. for \(s = 1, \ldots, m\), compute \(\hat{B}_k^{(s)} = \otimes_{j=k+1}^{K} \hat{A}_j^{(s-1)} \otimes_{j=1}^{k} \hat{A}_j^{(s-1)}\);
4. compute \(\widetilde{M}_k^{(s)}\), obtain \(\hat{r}_k^{(s)}\) for \(k = 1, \ldots, K\);
5. renew \(\hat{A}_k^{(s)}\) as \(\sqrt{p_k}\) times the matrix with columns being the first \(\hat{r}_k^{(s)}\) eigenvectors of \(\widetilde{M}_k^{(s)}\);
6. Repeat steps 3 to 5 until \(\hat{r}_k^{(s)} = \hat{r}_k^{(s-1)}, \forall k \in \{1, \ldots, K\}\), or reach the maximum number of iterations;
7. Output the last step estimator as \(\{\hat{r}_k^{\text{PE-ER}}, k = 1, \ldots, K\}\).

3 Theoretical results

3.1 Technical Assumptions

Prior to presenting the theoretical properties of the initial estimators and the projected estimators, we first give some assumptions, which can be viewed as higher-order extensions of those adopted for large-dimensional matrix factor model by Chen et al. (2020a) and Yu et al. (2022).

**Assumption 1.** (Alpha Mixing) The vectorized factor series \(\{\text{Vec}(F_t)\}\) and noise series \(\{\text{Vec}(E_t)\}\) are \(\alpha\)-mixing. A vector process \(\{z_t, t = 0, \pm 1, \pm 2, \ldots\}\) is \(\alpha\)-mixing if, for some \(\gamma > 2\), the mixing coefficients satisfy
the condition that
\[ \sum_{h=1}^{\infty} \alpha(h)^{1-2/\gamma} < \infty \]
where \( \alpha(h) = \sup_{t, sup_{A \in F_{t-\infty}} A \in F_{t+h}} |P(A \cap B) - P(A)P(B)| \) and \( F_{t}^{\text{e}} \) is the \( \sigma \)-field generated by \( \{z_{t} : \tau \leq t \leq s\} \).

**Assumption 2.** (Factor Matrix) The factor matrix satisfies \( \mathbb{E}(\text{Vec}(F_{t})) = 0, \mathbb{E}||\text{Vec}(F_{t})||^{4} \leq c < \infty \) for some constant \( c > 0 \), and
\[ \frac{1}{T} \sum_{t=1}^{T} F_{k,t}F_{k,t}^{\top} \xrightarrow{p} \Sigma_{k}, \]
where \( \Sigma_{k} \) is a \( r_{k} \times r_{k} \) positive definite matrix with distinct eigenvalues and spectral decomposition \( \Sigma_{k} = \Gamma_{k}A_{k}\Gamma_{k}^{\top}, k = 1, 2, \cdots, K \). The factor numbers \( \{r_{k}, k = 1, 2, \cdots, K\} \) are fixed as \( \min\{p_1, p_2, \cdots, p_K\} \rightarrow \infty \).

**Assumption 3.** (Loading Matrix) Positive constants \( \{\bar{a}_{k}, k = 1, 2, \cdots, K\} \) exist such that \( \|A_{k}\|_{\text{max}} \leq \bar{a}_{k} \). As \( \min\{p_1, p_2, \cdots, p_K\} \rightarrow \infty \), \( \|p_{k}^{-1}A_{k}^{\top}A_{k} - I_{r_{k}}\| \rightarrow 0 \).

The \( \alpha \)-mixing condition in Assumption 1 allows weak temporal correlations for both the vectorized factor and noise series. Assumption 2 shows that the mode-\( k \) unfolding factor matrices have finite fourth moments for all \( k = 1, \ldots, K \), and their second-order sample moments converge to positive definite matrices \( \Sigma_{k} \). \( \Sigma_{k} \) are assumed to have distinct eigenvalues to ensure the identifiability of eigenvectors. We assume strong factor conditions and model identifiability condition in Assumption 3. Assumptions 1-3 are standard and common in the literature and these assumptions can be viewed as direct extensions of the those by Chen et al. (2020a) and Yu et al. (2022) from matrix (order 2 tensor) factor model to higher-order tensor factor model.

**Assumption 4.** (Weak Correlation of Noise \( \mathcal{E}_{t} \) Across Column, Row, and Time) A positive constant \( c < \infty \)
exists such that

1. \( \mathbb{E} e_{t,i_1i_2...i_K} = 0, \mathbb{E} \left( e_{t,i_1i_2...i_K}^8 \right) \leq c. \)

2. for any \( t \in [T], i \in [p_k], j \in [p_{-k}], \)

   (1) \( \sum_{s=1}^{T} \sum_{l=1}^{p_k} \sum_{h=1}^{p_{-k}} \left| \mathbb{E} \left( e_{t,ij}^k e_{s,ih} \right) \right| \leq c, \)

   (2) \( \sum_{l=1}^{p_{-k}} \left| \mathbb{E} \left( e_{t,ij}^k e_{t,ih} \right) \right| \leq c. \)

3. for any \( t \in [T], i, l, j, h_1 \in [p_k], j, h_1 \in [p_{-k}], \)

   (1) \( \sum_{s=1}^{T} \sum_{l_2=1}^{p_k} \sum_{h_2=1}^{p_{-k}} \left| \text{Cov} \left( e_{t,ij}^k e_{t,ij}^k, e_{s,ih_2} e_{s,ih_2}^k \right) \right| \leq c, \quad \sum_{s=1}^{T} \sum_{l_2=1}^{p_k} \sum_{h_2=1}^{p_{-k}} \left| \text{Cov} \left( e_{t,ij}^k e_{t,ij}^k, e_{s,ih_1} e_{s,ih_1}^k \right) \right| \leq c, \quad \sum_{s=1}^{T} \sum_{l_2=1}^{p_k} \sum_{h_2=1}^{p_{-k}} \left| \text{Cov} \left( e_{t,ij}^k e_{t,ij}^k, e_{s,ih_2} e_{s,ih_2}^k \right) \right| \leq c. \)

   (2) \( \sum_{s=1}^{T} \sum_{l_2=1}^{p_k} \sum_{h_2=1}^{p_{-k}} \left( \left| \text{Cov} \left( e_{t,ij}^k e_{t,ih_1} e_{s,ij}^k e_{s,ih_2} \right) \right| + \left| \text{Cov} \left( e_{t,ij}^k e_{t,ih_1} e_{s,ij}^k e_{s,ih_2} \right) \right| \right) \leq c. \)

Assumption 4.1 assumes that the noises have finite eighth moment. Assumption 4.2.(1) allows weak dependence of noise in each mode and time dimension, and Assumption 4.2.(2) further controls the weak dependence of noises in each mode. Assumption 4.3 controls the second-order correlation among the elements of mode-\( k \) unfolding matrices of the noise tensors.

**Assumption 5.** (Weak Dependence Between Factor \( F_i \) and Noise \( \mathcal{E}_t \)) There exists a constant \( c > 0 \), such that,

1. for any deterministic vectors \( \mathbf{v} \) and \( \mathbf{w} \), satisfying \( \| \mathbf{v} \| = 1 \) and \( \| \mathbf{w} \| = 1 \), \( \mathbb{E} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left( \mathbf{F}_{k,t}^\top \mathbf{v} \mathbf{E}_{k,t} \mathbf{w} \right) \right)^2 \leq c \)

2. for any \( i, l_1 \in [p_k] \) and \( j, h_1 \in [p_{-k}], \)

   (1) \( \left\| \sum_{h_2=1}^{p_{-k}} \mathbb{E} \left( \tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{ih_2} \right) \right\|_{\text{max}} \leq c, \quad \left\| \sum_{l_2=1}^{p_k} \mathbb{E} \left( \tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{l_2j} \right) \right\|_{\text{max}} \leq c, \)

   (2) \( \left\| \sum_{l_2=1}^{p_k} \sum_{h_2=1}^{p_{-k}} \text{Cov} \left( \tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{ih_2}, \tilde{\zeta}_{l_2j} \otimes \tilde{\zeta}_{l_2h_2} \right) \right\|_{\text{max}} \leq c, \quad \left\| \sum_{l_2=1}^{p_k} \sum_{h_2=1}^{p_{-k}} \text{Cov} \left( \tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{l_2j}, \tilde{\zeta}_{ih_2} \otimes \tilde{\zeta}_{l_2h_2} \right) \right\|_{\text{max}} \leq c, \)

where \( \tilde{\zeta}_{ij} = \text{Vec} \left( \sum_{t=1}^{T} \mathbf{F}_{k,t} e_{t,ij}^{(k)} / \sqrt{T} \right) \).
Taking \( \mathbf{v}^\top \mathbf{E}_{k,t} \mathbf{w} \) as a random variable with zero mean and bounded variance, Assumption 5.1 shows that the temporal correlation of \( \mathbf{F}_{k,t} \mathbf{v}^\top \mathbf{E}_{k,t} \mathbf{w} \) is also weak. Assumption 5.2 controls the high-order correlation between the vectorized factor and noise series. Assumption 5 automatically satisfies provided that the noises are independent across time and independent of the factor series, given Assumptions 1 to 4.

### 3.2 Theorems on initial projection matrices

The following theorem establishes the convergence rate of the initial projection estimators discussed in Section 2.1.1.

**Theorem 3.1.** Under Assumptions 1 to 5, for any \( k \in [K] \), there exist \( r_k \times r_k \) matrices \( \hat{\mathbf{H}}_k \) satisfying \( \hat{\mathbf{H}}_k \hat{\mathbf{H}}_k^\top \overset{p}{\to} \mathbf{I}_{r_k} \), such that

\[
\frac{1}{p_k} \left\| \mathbf{A}_k - \mathbf{A}_k \hat{\mathbf{H}}_k \right\|_F^2 = O_p \left( w_k \right),
\]

where \( w_k = \frac{1}{p_k^2} + \frac{1}{T p_{-k}} \).

The theoretical convergence rates for the initial estimators incorporates the results for matrix factor model developed in Yu et al. (2022) as special cases. The estimation error bounds for projected estimators clearly depend on the accuracy of the initial projection matrices. In the following theoretical analysis of the projected estimators, we first give sufficient conditions for general initial projection matrices such that the projection procedure is guaranteed to work.

### 3.3 Theorems on projection estimators

We first give the following conditions for the convergence rate of the initial estimators to ensure that the projection estimation procedure works.

**(Sufficient Condition)** For \( k = 1, 2, \cdots, K \), there exist \( r_{-k} \times r_{-k} \) matrices \( \hat{\mathbf{H}}_{-k} \) satisfying \( \hat{\mathbf{H}}_{-k} \hat{\mathbf{H}}_{-k}^\top \overset{p}{\to} \mathbf{I}_{r_{-k}} \) and

\[
\begin{align*}
\text{(a).} \quad & \frac{1}{p_{-k}} \left\| \hat{\mathbf{B}}_k - \mathbf{B}_k \hat{\mathbf{H}}_k \right\|_F^2 = O_p \left( w_{-k} \right) \\
\text{(b).} \quad & \frac{1}{p_k} \left\| \frac{1}{T p_{-k}} \sum_{s=1}^{T} \mathbf{E}_{k,s} \left( \hat{\mathbf{B}}_k - \mathbf{B}_k \hat{\mathbf{H}}_{-k} \right) \mathbf{F}_{k,s}^\top \right\|_F^2 = O_p \left( m_{-k} \right),
\end{align*}
\]

(3.1)
where $\hat{B}_k = \otimes_{j \neq k} \hat{A}_j$, and $w_{-k}, m_{-k} \to 0$ as $T, p_1, p_2, \ldots, p_K$ go to infinity simultaneously.

The following theorem presents the rate of convergence of the projection estimators under the sufficient condition in (3.1).

**Theorem 3.2.** Under Assumptions 1 to 5 and Sufficient Condition 3.1, for any $k \in [K]$, there exist matrices $\tilde{H}_k$, satisfying $\tilde{H}_k^\top \tilde{H}_k / p_1 \rightarrow I_{r_k}$, such that

$$\frac{1}{p_k} \left\| \hat{A}_k - A_k \tilde{H}_k \right\|_F^2 = O_p (\bar{w}_k),$$

as $T, p_1, \ldots, p_K$ go to infinity simultaneously, and

$$\bar{w}_k = \frac{1}{T p_{-k}} + \frac{1}{p^2} + w_{-k} \left( \frac{1}{p_k} + \frac{1}{T p_k} \right) + m_{-k}.$$

The theorem shows that the convergence rates of the projection estimators are dependent on the convergence rates of the initial estimators. In fact, the initial estimators presented in Section 2.1.1 satisfies the sufficient condition with $w_{-k}$ and $m_{-k}$ given in the lemma below.

**Lemma 3.3.** For the initial estimators $\{\hat{A}_k\}, (k = 1, 2, \ldots, K)$ presented in 2.1.1, under Assumptions 1 to 5, there exist $r_{-k} \times r_{-k}$ matrices $\hat{H}_{-k}$ satisfying $\hat{H}_{-k}^\top \hat{H}_{-k} / p_{-k} \rightarrow I_{r_{-k}}$, such that

(a). $\frac{1}{p_{-k}} \left\| \hat{B}_k - B_k \hat{H}_k \right\|_F^2 = O_p (w_{-k}),$

(b). $\frac{1}{p_k} \left\| \frac{1}{T p_{-k}} \sum_{s=1}^T E_{k,s} \left( \hat{B}_k - B_k \hat{H}_{-k} \right) F_{k,s}^\top \right\|_F^2 = O_p (m_{-k}),$

where $w_{-k} = \sum_{j \neq k} \left( \frac{1}{p_j} + \frac{1}{T p_{-j}} \right)$, $m_{-k} = \sum_{j \neq k} \left( \frac{1}{T p_{-j}} + \frac{1}{T p_2} \right)$.

Provided that the initial estimators in 2.1.1 are adopted as the initial projection matrices, the one-step further projection technique is guaranteed to derive refined estimators with faster convergence rates, which is given in the following corollary.

**Corollary 3.1.** For the projected estimator $\tilde{A}_k$ in section 2.1, under Assumptions 1 to 5, there exist matrices
\( \tilde{\mathbf{H}}_k \), satisfying \( \tilde{\mathbf{H}}_k^T \tilde{\mathbf{H}}_k/p_1 \xrightarrow{p} \mathbf{I}_{r_k} \), such that

\[
\frac{1}{p_k} \left\| \tilde{\mathbf{A}}_k - \mathbf{A}_k \tilde{\mathbf{H}}_k \right\|_F^2 = O_p(\tilde{w}_k),
\]

as \( T, p_1, \cdots, p_K \) go to infinity simultaneously, and

\[
\tilde{w}_k = \frac{1}{T p_{-k}} + \frac{1}{p^2} + \sum_{j \neq k} \left( \frac{1}{T p_j} + \frac{1}{T^2 p_{-j}} + \frac{1}{p_k p_j} \right).
\]

By comparing the convergence rates of the initial estimators in Theorem 3.1, the corollary shows that the projected estimators achieve faster convergence rates than the initial estimators.

In fact, instead of setting the estimator of \( \mathbf{B}_k \) as \( \hat{\mathbf{B}}_k = \otimes_{j \neq k} \hat{\mathbf{A}}_j \), we can choose another estimator \( \hat{\mathbf{B}}_k^* \) as \( \sqrt{p_{-k}} \) times the matrix with columns being the first \( r_{-k} \) eigenvectors of \( \hat{\mathbf{M}}_{-k} = (T p)^{-1} \mathbf{X}_{k,t}^T \mathbf{X}_{k,t} \), which is motivated by the matrix factor model form in (2.7), and similar algorithm as Algorithm 2 can be proposed to obtain the projected estimators by replacing \( \hat{\mathbf{B}}_k \) with \( \hat{\mathbf{B}}_k^* \) and we denote the resulted projection estimators as \( \tilde{\mathbf{A}}_k^* \), see detailed procedure in Algorithm 3. By the results in Yu et al. (2022), the projection estimator \( \tilde{\mathbf{A}}_k^* \) satisfies that

\[
\frac{1}{p_k} \left\| \tilde{\mathbf{A}}_k^* - \mathbf{A}_k \tilde{\mathbf{H}}_k \right\|_F^2 = O_p(\tilde{w}_k^*),
\]

as \( T, p_1, \cdots, p_K \) go to infinity simultaneously, and

\[
\tilde{w}_k^* = \frac{1}{T p_{-k}} + \frac{1}{p^2} + \frac{1}{T^2 p_k^2}.
\]

If we assume that \( p_1 \asymp p_2 \asymp \cdots \asymp p_K \asymp p' \), when \( K = 2 \), \( \tilde{w}_k = O_p((p')^{-4} + (T p')^{-1}) \) and \( \tilde{w}_k^* = \tilde{w}_k \), due to \( \hat{\mathbf{B}}_k^* = \hat{\mathbf{B}}_k \) in this case. When \( K = 3 \), \( \tilde{w}_k = O_p((p')^{-6} + (T p'^2)^{-1}) \) and \( \tilde{w}_k^* = \tilde{w}_k \) still holds. When \( K > 3 \), \( \tilde{w}_k = O_p((p')^{-6} + (T p'^2)^{-1}) \), and \( \tilde{w}_k^* = O_p((p')^{-2K} + (T p')^{-2} + (T p'^{(K-1)})^{-1}) \). It shows that when \( K \) and \( T \) are large enough, the estimator \( \tilde{\mathbf{A}}_k^* \) will have faster converge rate than \( \tilde{\mathbf{A}}_k \). But to obtain \( \hat{\mathbf{B}}_k^* \), one needs to
Algorithm 3 Projected method for estimating loading spaces

**Input:** tensor data \( \{X_t, t = 1, \cdots, T\} \), factor numbers \( r_1, \cdots, r_K \)

**Output:** factor loading matrices \( \{\hat{A}_k, k = 1, \cdots, K\} \)

1: obtain the initial estimators \( \{\hat{A}_k, k = 1, \cdots, K\} \) and \( \{\hat{B}_k, k = 1, \cdots, K\} \) by \( \sqrt{p_k} \) times the matrix with columns being the first \( r_k \) eigenvectors of \( \hat{M}_{k} = (T\hat{p})^{-1}X_{k,t}^{\top}X_{k,t} \);

2: project to reduce dimensions by defining \( \hat{Y}_{k,t} = \hat{p}_{k}^{-1}X_{k,t}\hat{B}_k, k = 1, \cdots, K \);

3: given \( \{\hat{Y}_{k,t}, k = 1, \cdots, K\} \), define \( \hat{M}_k^* = (T\hat{p}_k)^{-1}\sum_{t=1}^{T} \hat{Y}_{k,t}^{\top}\hat{Y}_{k,t}^* \), obtain \( \tilde{A}_k^* \) as \( \sqrt{p_k} \) times the matrix with columns being the first \( r_k \) eigenvectors of \( \hat{M}_k^* \);

perform eigen-decomposition of a very large scale \( p_k \times p_k \) matrix, and its computational burden will be much heavier than calculating the Kronecker product of \((K - 1)\) small scale matrices of dimension \( p_k \times r_k \).

### 3.4 Consistency of the estimators for factor numbers

In this section, we establish the consistency of the propose estimators for the pair of factor numbers. We first focus on the estimators based on the simple mode-wise sample covariance matrix \( \hat{M}_k \). The following theorem shows that the \( \hat{r}_k^{\text{IE-ER}} \) defined in (2.10) are consistent.

**Theorem 3.4.** Under Assumptions 1 to 5, when \( \min \{r_1, r_2, \cdots, r_K\} > 0, \min \{T, p_1, p_2, \cdots, p_K\} \rightarrow \infty \) and \( r_{\text{max}} \) is a predetermined constant no smaller than \( \max \{r_1, r_2, \cdots, r_K\} \), then for any \( k \in [K] \)

\[
\Pr(\hat{r}_k^{\text{IE-ER}} = r_k) \rightarrow 1.
\]

In Algorithm 2, we choose to calculate eigenvalue-ratio of the projection version \( \hat{M}_k \) rather than the initial version \( \hat{M}_k \), as \( \hat{M}_k \) is more accurate. The consistency of the estimator outputted by the iterative algorithm is guaranteed by the following theorem.

**Theorem 3.5.** Under Assumptions 1 to 5, when \( \min \{r_1, r_2, \cdots, r_K\} > 0, \min \{T, p_1, p_2, \cdots, p_K\} \rightarrow \infty \) and \( r_{\text{max}} \) is a predetermined constant no smaller than \( \max \{r_1, r_2, \cdots, r_K\} \), if for all \( j \in [K]\{k\}, \hat{r}_j^{(s-1)} \in [r_j, r_{\text{max}}] \) for some \( s \) in the iterative Algorithm 2, then

\[
\Pr(\hat{r}_k^{(s)} = r_k) \rightarrow 1.
\]
The theorem states that as long as we choose $r_k^{(0)}$ larger than real $r_k$ at the beginning, the iterative method will yield consistent estimators of the number of factors. The algorithm is computationally very fast because it has a large probability to stop within finite steps.

## 4 Extension: Robust Tensor Factor Analysis

The statistical procedures based on least squares method usually perform poorly for heavy-tailed data. To address the problem, it is natural to consider replacing the least square loss function with the Huber loss function, that is, consider the following optimization problem:

$$
\min_{\{A_1, \ldots, A_K, F_t\}} L_2(A_1, \ldots, A_K, F_t) = \frac{1}{T} \sum_{t=1}^T H_\tau(\|X_t - F_t \times_{k=1}^K A_k\|_F) = \frac{1}{T} \sum_{t=1}^T H_\tau(\|X_{k,t} - A_k F_{k,t} B_k^\top\|_F),
$$

s.t. $\frac{1}{p_k} A_k A_k^\top = I_{r_k}$, $\frac{1}{p_{-k}} B_k B_k^\top = I_{r_{-k}}$, $k = 1, 2, \ldots, K$

(4.1)

where the Huber loss function is defined as

$$
H_\tau(x) = \begin{cases} 
\frac{1}{2} x^2, & |x| \leq \tau, \\
\tau |x| - \frac{1}{2} \tau^2, & |x| > \tau.
\end{cases}
$$

Then the Huber loss function $L_H = H_\tau(\|X_{k,t} - A_k F_{k,t} B_k^\top\|_F)$ can be expressed as

$$
\begin{cases} 
\frac{1}{2} \text{Tr}(X_{k,t}^\top X_{k,t}) - 2 \text{Tr}(X_{k,t}^\top A_k F_{k,t} B_k^\top) + p \text{Tr}(F_{k,t}^\top F_{k,t}), & \|X_{k,t} - A_k F_{k,t} B_k^\top\|_F^2 \leq \tau^2, \\
\tau \sqrt{\text{Tr}(X_{k,t}^\top X_{k,t}) - 2 \text{Tr}(X_{k,t}^\top A_k F_{k,t} B_k^\top) + p \text{Tr}(F_{k,t}^\top F_{k,t})} - \frac{1}{2} \tau^2, & \|X_{k,t} - A_k F_{k,t} B_k^\top\|_F^2 > \tau^2.
\end{cases}
$$
As same as the least square case, for each time point \( t \), let \( \partial H_\tau / \partial F_{k,t} = 0 \), then it can be obtained that 
\[
F_{k,t} = A_k^T X_{k,t} B_k / p.
\]
Substituting \( F_{k,t} = A_k^T X_{k,t} B_k / p \) in \( \mathcal{L}_H \), we can also further obtain that

\[
\mathcal{L}_H = \left\{ \begin{array}{ll}
\frac{1}{2} [\text{Tr}(X_{k,t}^T X_{k,t}) - \frac{1}{p} \text{Tr}(X_{k,t}^T A_k A_k^T X_{k,t} B_k B_k^T)] + \| X_{k,t} - A_k F_{k,t} B_k^T \|_F^2 \leq \tau^2, \\
\tau \sqrt{\text{Tr}(X_{k,t}^T X_{k,t}) - \frac{1}{p} \text{Tr}(X_{k,t}^T A_k A_k^T X_{k,t} B_k B_k^T)} - \frac{1}{2} \tau^2, & \| X_{k,t} - A_k F_{k,t} B_k^T \|_F^2 > \tau^2.
\end{array} \right.
\] (4.2)

When \( \| X_{k,t} - A_k F_{k,t} B_k^T \|_F^2 \leq \tau^2 \), the following equalities hold:

\[
\frac{\partial H_\tau}{\partial A_k} = -\frac{1}{p} X_{k,t} B_k B_k^T X_{k,t}^T A_k, \quad \frac{\partial H_\tau}{\partial B_k} = -\frac{1}{p} X_{k,t}^T A_k A_k^T X_{k,t} B_k.
\]

When \( \| X_{k,t} - A_k F_{k,t} B_k^T \|_F^2 > \tau^2 \), the following equalities hold:

\[
\frac{\partial H_\tau}{\partial A_k} = -\frac{\tau}{p} \frac{X_{k,t} B_k B_k^T X_{k,t}^T A_k}{\sqrt{\text{Tr}(X_{k,t}^T X_{k,t}) - \frac{1}{p} \text{Tr}(X_{k,t}^T A_k A_k^T X_{k,t} B_k B_k^T)}}
\]
\[
\frac{\partial H_\tau}{\partial B_k} = -\frac{\tau}{p} \frac{X_{k,t}^T A_k A_k^T X_{k,t} B_k}{\sqrt{\text{Tr}(X_{k,t}^T X_{k,t}) - \frac{1}{p} \text{Tr}(X_{k,t}^T A_k A_k^T X_{k,t} B_k B_k^T)}}
\]

The Lagrangian function is as follows:

\[
\min_{(A_k, B_k)} \mathcal{L}_2 = L_2(A_k, B_k) + \text{Tr} \left[ \Theta (\frac{1}{p} A_k A_k - I_{r_k}) \right] + \text{Tr} \left[ A (\frac{1}{p - k} B_k B_k - I_{r_k}) \right],
\] (4.3)

where the Lagrange Multipliers \( \Theta \) and \( A \) are symmetric matrices. According to KKT condition, we have

\[
\frac{\partial \mathcal{L}_2}{\partial A_k} = -C_k A_k + \frac{2}{p_k} A_k \Theta = 0, \quad \frac{\partial \mathcal{L}_2}{\partial B_k} = -D_k B_k + \frac{2}{p - k} B_k A = 0,
\] (4.4)

where

\[
C_k = \frac{1}{T} \sum_{t=1}^{T} w'_{k,t} X_{k,t} B_k B_k^T X_{k,t}^T, \quad D_k = \frac{1}{T} \sum_{t=1}^{T} w'_{k,t} X_{k,t}^T A_k A_k^T X_{k,t}.
\]
and the weights $w'_{k,t}$ are

$$w'_{k,t} = \begin{cases} \frac{1}{p}, & \|X_{k,t} - A_k F_{k,t} B_k^\top\|_F^2 \leq \tau^2, \\ \frac{1}{\tau} \frac{1}{p} \sqrt{\text{Tr}(X_{k,t}^\top X_{k,t}) - \frac{1}{p} \text{Tr}(X_{k,t}^\top A_k A_k^\top X_{k,t} B_k B_k^\top)}, & \|X_{k,t} - A_k F_{k,t} B_k^\top\|_F^2 > \tau^2, \end{cases}$$

By redefining symbols, we obtain that

$$M^w_k A_k = A_k \Theta, \quad M^w_k B_k = B_k \Lambda,$$

where

$$M^w_k = \frac{1}{T p - k} \sum_{t=1}^T w_{k,t} X_{k,t} B_k B_k^\top X_{k,t}^\top, \quad M^{-w}_k = \frac{1}{T p} \sum_{t=1}^T w_{k,t} X_{k,t}^\top A_k A_k^\top X_{k,t},$$

and the weights $w_{k,t}$ are

$$w_{k,t} = \begin{cases} \frac{1}{2}, & \|X_{k,t} - A_k F_{k,t} B_k^\top\|_F^2 \leq \tau^2, \\ \frac{1}{\tau} \frac{1}{2} \sqrt{\text{Tr}(X_{k,t}^\top X_{k,t}) - \frac{1}{p} \text{Tr}(X_{k,t}^\top A_k A_k^\top X_{k,t} B_k B_k^\top)}, & \|X_{k,t} - A_k F_{k,t} B_k^\top\|_F^2 > \tau^2. \end{cases}$$

Assuming that the first $r_k$ eigenvalues of $M^w_k$ are \{\lambda^w_{k,1}, \lambda^w_{k,2}, \ldots, \lambda^w_{k,r_k}\} and the corresponding eigenvectors are \{u^w_{k,1}, \ldots, u^w_{k,r_k}\}, we have $\Theta = \text{diag}(\lambda^w_{k,1}, \lambda^w_{k,2}, \ldots, \lambda^w_{k,r_k})$, $A_k = \sqrt{p_k}(u^w_{k,1}, \ldots, u^w_{k,r_k})$ satisfies the KKT condition. As the estimation of $A_k$ relies on the unknown $\{A_j, j = 1, \ldots, K\}$, the following algorithm 4 gives an iterative method to obtain the robust estimators of $\{A_k, k = 1, 2, \ldots, K\}$. The initial estimators $\hat{A}_k$ of $A_k$ is still selected as $\sqrt{p_k}$ times the matrix with columns being the first $r_k$ eigenvectors of $\hat{M}_k = \frac{1}{T p} \sum_{t=1}^T X_{k,t} X_{k,t}^\top$.

In Algorithm 4, the number of factors are given in advance. Empirically, the factor numbers are unknown and need to be estimated. We also propose robust estimators of the factor numbers which works well in the presence of heavy-tailed data. The proposed method is based on eigenvalue-ratios of the weighted mode-wise sample covariance matrix of projected data, i.e. $M^w_k$.
empirically. Therefore, we propose to determine the numbers of factors by the following iterative Algorithm

\begin{algorithm}
\caption{Robust Tensor Factor Analysis}
\textbf{Input:} tensor data \(\{\mathcal{X}_t, t = 1, \cdots, T\}\), factor numbers \(r_1, \cdots, r_K\)
\textbf{Output:} factor loading matrix \(\{\hat{A}_k^w, k = 1, \cdots, K\}\)
\begin{enumerate}
\item obtain the initial estimators \(\{\hat{A}_k^{(0)} = \hat{A}_k, k = 1, \cdots, K\}\);
\item compute \(\hat{B}_k^{(s)} = \otimes_{j=k+1}^{K} \hat{A}_j^{(s-1)} \otimes_{j=1}^{k-1} \hat{A}_j^{(s-1)}\);
\item calculate weights \(w_{k,t}^{(s)}\) with \(\hat{A}_k^{(s-1)}, \hat{B}_k^{(s-1)}\);
\item compute \(\hat{M}_k^{w(s)}\), renew \(\hat{A}_k^{(s)}\) as \(\sqrt{p_k}\) times the matrix with columns being the first \(r_k\) eigenvectors of \(\hat{M}_k^{w(s)}\);
\item Repeat steps 2 to 4 until convergence, output the last step estimators as \(\{\hat{A}_k^w, k = 1, \cdots, K\}\).
\end{enumerate}
\end{algorithm}

\begin{algorithm}
\caption{Robust estimation of the numbers of factor}
\textbf{Input:} tensor data \(\{\mathcal{X}_t, t = 1, \cdots, T\}\), maximum number \(r_{\text{max}}\), maximum iterative step \(m\)
\textbf{Output:} factor numbers \(\{\hat{r}_k^{\text{RTFA-ER}}, k = 1, \cdots, K\}\)
\begin{enumerate}
\item initialize: \(\hat{r}_k^{(0)} = r_{\text{max}}, k = 1, \cdots, K\);
\item obtain the initial estimators \(\{\hat{A}_k^{(0)} = \hat{A}_k, k = 1, \cdots, K\}\);
\item for \(s = 1, \cdots, m\), compute \(\hat{B}_k^{(s)} = \otimes_{j=k+1}^{K} \hat{A}_j^{(s-1)} \otimes_{j=1}^{k-1} \hat{A}_j^{(s-1)}\);
\item use \(\hat{A}_k^{(s-1)}\) and \(\hat{B}_k^{(s)}\) to calculate weights \(w_{k,t}^{(s)}\);
\item compute \(\hat{M}_k^{w(s)}\), obtain \(\hat{r}_k^{(s)}\) by 4.7;
\item renew \(\hat{A}_k^{(s)}\) as \(\sqrt{p_k}\) times the matrix with columns being the first \(\hat{r}_k^{(s)}\) eigenvectors of \(\hat{M}_k^{w(s)}\);
\item Repeat steps 3 to 6 until \(\hat{r}_k^{(s)} = \hat{r}_k^{(s-1)}, \forall k \in \{1, \cdots, K\}\), or up to the maximum number of iterations, output the last step estimator as \(\{\hat{r}_k^{\text{RTFA-ER}}, k = 1, \cdots, K\}\).
\end{enumerate}
\end{algorithm}

One problem to calculate \(M_k^w\) is that \(B_k\) must be predetermined, which means \(r_{-k}\) (or equivalently \(r_j, j \neq k\)) must be given first. However, both \(r_k\) and \(r_{-k}\) (or equivalently \(r_j, j \neq k\)) are unavailable empirically. Therefore, we propose to determine the numbers of factors by the following iterative Algorithm 5.

\section{Simulation Study}

\subsection{Data generation}

In this section, we investigate the finite sample performance of the proposed weighted/unweighted iterative projection methods. We compare the performances of initial estimator (IE), the projected estimator (PE),
and the robust tensor factor analysis (RTFA) in terms of estimating the loading matrices and the number of factors. The tensor observations are generated following the order 3 tensor factor model:

\[ X_t = F_t \times_1 A_1 \times_2 A_2 \times_3 A_3 + \mathcal{E}_t. \]

We set \( r_1 = r_2 = r_3 = 3 \), draw the entries of \( A_1 \), \( A_2 \) and \( A_3 \) independently from uniform distribution \( U(-1, 1) \), and let

\[
\begin{align*}
\text{Vec}(F_t) &= \phi \times \text{Vec}(F_{t-1}) + \sqrt{1 - \phi^2} \times \epsilon_t, \epsilon_t i.i.d \sim \mathcal{N}(0, I_{r_1 r_2 r_3}) \\
\text{Vec}(E_t) &= \psi \times \text{Vec}(E_{t-1}) + \sqrt{1 - \psi^2} \times \text{Vec}(U_t)
\end{align*}
\]

where \( U_t \) is from a tensor normal distribution or a tensor-t distribution. If \( U_t \) is from a tensor normal distribution \( \mathcal{T} \mathcal{N}(\mathcal{M}, \Sigma_1, \Sigma_2, \Sigma_3) \), then \( \text{Vec}(U_t) \sim \mathcal{N}(\text{Vec}(\mathcal{M}), \Sigma_3 \otimes \Sigma_2 \otimes \Sigma_1) \); If \( U_t \) is from a tensor-t distribution \( t_{\nu}(\mathcal{M}, \Sigma_1, \Sigma_2, \Sigma_3) \), then \( \text{Vec}(U_t) \) will be from a multivariate t distribution \( t_{\nu}(\mathcal{M}, \Sigma_3 \otimes \Sigma_2 \otimes \Sigma_1) \).

In our study, we set \( \mathcal{M} = 0 \), \( \Sigma_k \) to be the matrix with 1 on the diagonal, and \( 1/p_k \) on the off-diagonal, for \( k = 1, 2, 3 \). The parameters \( \phi \) and \( \psi \) control for temporal correlations of \( F_t \) and \( E_t \). By setting \( \phi \) and \( \psi \) unequal to zero, common factors have cross-correlations, and idiosyncratic noises have both cross-correlations and weak autocorrelations. For RTFA, the Huber loss threshold parameter \( \tau \) is set as the median of \( ||X_t - F_t \times_1 \hat{A}_1 \times_2 \hat{A}_2 \times_3 \hat{A}_3||_F \), where \( \hat{A}_1 \), \( \hat{A}_2 \) and \( \hat{A}_3 \) are the initial estimators of \( A_1 \), \( A_2 \) and \( A_3 \).

In section 5.2, it is assumed that factor numbers are known, and the performance of estimating the number of factors is investigated in section 5.3. All the following simulation results are based on 1000 repetitions.

### 5.2 Verifying the convergence rates for loading spaces

In this section, we compare the performance of RTFA and IE, PE in estimating loading spaces. We consider the following three settings:

**Setting A:** \( p_1 = p_2 = p_3 = 10, \phi = 0.1, \psi = 0.1 \)

**Setting B:** \( p_1 = 100, p_2 = p_3 = 10, \phi = 0.1, \psi = 0.1 \)
Table 1: Averaged estimation errors and standard errors of loading spaces for Settings A, B and C under Tensor Normal distribution over 1000 replications. “RTFA”: robust tensor factor analysis method. “IE”: initial estimation method. “PE”: projection estimation method.

| p_1 | p_2 | p_3 | Evaluation | T     | RTFA            | IE            | PE            |
|-----|-----|-----|------------|-------|-----------------|----------------|----------------|
| 10  | 10  | 10  |  \mathcal{D}(\hat{A}_1, A_1) | 20    | 0.04445(0.01725) | 0.19696(0.13760) | 0.04438(0.01716) |
|     |     |     |            | 50    | 0.02864(0.01116) | 0.17826(0.12431) | 0.02856(0.01112) |
|     |     |     |            | 100   | 0.02197(0.00902) | 0.17284(0.12419) | 0.02192(0.00899) |
|     |     |     |            | 200   | 0.01756(0.00840) | 0.17134(0.12410) | 0.01751(0.00838) |
| 100 | 10  | 10  |  \mathcal{D}(\hat{A}_2, A_2) | 20    | 0.04311(0.01496) | 0.18730(0.12483) | 0.04301(0.01498) |
|     |     |     |            | 50    | 0.02906(0.01290) | 0.18430(0.13372) | 0.02903(0.01297) |
|     |     |     |            | 100   | 0.02211(0.01050) | 0.17359(0.12322) | 0.02206(0.01049) |
|     |     |     |            | 200   | 0.01770(0.00959) | 0.17733(0.13090) | 0.01766(0.00958) |
| 20  | 20  | 20  |  \mathcal{D}(\hat{A}_1, A_1) | 20    | 0.04011(0.00586) | 0.04101(0.00618) | 0.04005(0.00587) |
|     |     |     |            | 50    | 0.02528(0.00355) | 0.02592(0.00379) | 0.02524(0.00355) |
|     |     |     |            | 100   | 0.01794(0.00249) | 0.01858(0.00272) | 0.01791(0.00249) |
|     |     |     |            | 200   | 0.01261(0.00179) | 0.01328(0.00205) | 0.01259(0.00179) |
| 20  | 20  | 20  |  \mathcal{D}(\hat{A}_2, A_2) | 20    | 0.01255(0.00356) | 0.17784(0.12591) | 0.01253(0.00355) |
|     |     |     |            | 50    | 0.00795(0.00218) | 0.17458(0.12275) | 0.00794(0.00218) |
|     |     |     |            | 100   | 0.00575(0.00166) | 0.17192(0.12221) | 0.00569(0.00164) |
|     |     |     |            | 200   | 0.00404(0.00110) | 0.16810(0.12153) | 0.00410(0.00119) |

Setting C: \( p_1 = p_2 = p_3 = 20, \ \phi = 0.1, \ \psi = 0.1 \)

Due to the identifiability problem of factor model, the performance of the above three methods is evaluated by comparing the distance between the estimated loading space and the true loading space, which is

\[ \mathcal{D}(\hat{A}_k, A_k) = \left(1 - \frac{1}{r_k} \text{Tr} \left( \hat{Q}_k \hat{Q}_k^\top Q_k Q_k^\top \right) \right)^{1/2}, \ k = 1, 2, 3 \]
Table 2: Averaged estimation errors and standard errors of loading spaces for Settings A, B and C under Tensor $t_3$ distribution over 1000 replications. “RTFA”: robust tensor factor analysis method. “IE”: initial estimation method. “PE”: projection estimation method.

| $p_1$ | $p_2$ | $p_3$ | $T$ | Evaluation | RTFA | IE | PE |
|-------|-------|-------|-----|------------|------|----|----|
| 10    | 10    | 10    | 20  | $\mathcal{D}(\hat{A}_1, A_1)$ | 0.08911(0.10895) | 0.44766(0.14407) | 0.17002(0.19217) |
|       |       |       | 50  |             | 0.05549(0.07106) | 0.44145(0.13638) | 0.12357(0.16391) |
|       |       |       | 100 |             | 0.04140(0.03959) | 0.43303(0.13715) | 0.10423(0.14996) |
|       |       |       | 200 |             | 0.03753(0.03426) | 0.43529(0.13104) | 0.09306(0.13894) |
|       |       |       | 20  | $\mathcal{D}(\hat{A}_2, A_2)$ | 0.08677(0.11153) | 0.41913(0.15010) | 0.17309(0.20789) |
|       |       |       | 50  |             | 0.05451(0.06867) | 0.41607(0.14370) | 0.12462(0.17448) |
|       |       |       | 100 |             | 0.04234(0.04514) | 0.41588(0.13714) | 0.10152(0.15377) |
|       |       |       | 200 |             | 0.03891(0.03961) | 0.41551(0.13440) | 0.09310(0.13894) |
| 10    | 10    | 10    | 20  | $\mathcal{D}(\hat{A}_3, A_3)$ | 0.08633(0.11077) | 0.41161(0.14998) | 0.16928(0.20479) |
|       |       |       | 50  |             | 0.05198(0.06565) | 0.40574(0.14673) | 0.12322(0.17727) |
|       |       |       | 100 |             | 0.04311(0.04849) | 0.41484(0.13887) | 0.10252(0.15360) |
|       |       |       | 200 |             | 0.03627(0.03221) | 0.42030(0.13658) | 0.08937(0.13966) |

where $Q_k$ and $\hat{Q}_k$ are the left singular-vector matrices of the true loading matrix $A_k$ and its estimator $\hat{A}_k$.

The distance is always in the interval $[0, 1]$. When $A_k$ and $\hat{A}_k$ span the same space, the distance $\mathcal{D}(\hat{A}_k, A_k)$ is equal to 0, while is equal to 1 when the two spaces are orthogonal.

Table 1 and table 2 shows the averaged estimation errors with standard errors in parentheses under Settings A, B and C for tensor normal distribution and tensor $t_3$ distribution respectively. First, we focus on the case where the idiosyncratic noises follow tensor normal distribution. Table 1 shows that all three
methods benefit from the increase in dimensions and sample size $T$. For fixed sample size $T$, when $p_k$ is small, RTFA and PE perform better than IE in estimating $A_k$. It is worth noting that RTFA and PE behave similarly in the case of tensor normal idiosyncratic noises. This is because that RMFA is based on eigen-analysis of the weighted mode-wise sample covariance matrices of projected data, while PE is based on unweighted ones, and they both share the benefits brought by projection. Under the tensor normal idiosyncratic noises case, the weights in RMFA tends to be equal and thus RMFA and PE perform almost the same.

Then consider the case where the idiosyncratic noises follow the tensor $t_3$ distribution. Table 2 shows that, although all methods benefit from increased dimensions and sample size $T$, RTFA outperforms IE and PE in all combinations of dimensions and sample sizes. This shows that for heavy-tailed data, the weights of mode-wise sample covariance matrices of the projected data in RTFA play important role. In summary, RTFA is a robust method for estimating loading spaces. When the tensor data is heavy-tailed, RTFA performs significantly better than IE and PE. When the data is light-tailed, RTFA has similar performance to PE and can be used as a safe replacement of the existing methods.

5.3 Estimating the numbers of factors

In this section, we investigate the empirical performances of the proposed Robust iterative Eigenvalue-Ration based procedure (RTFA-ER), along with the unweighted version (PE-ER) and the initial version (IE-ER). We consider the following setting here:

**Setting D:** $p_1 = p_2 = p_3 = 30$, $\phi = 0.1$, $\psi = 0.1$

Table 3 presents the frequencies of exact estimation over 1000 replications under Settings A, C and D by different methods. We set $r_{max} = 8$ for IE-ER and PE-ER and RTFA-ER. Under the tensor normal case, the performance of all three methods improve as the dimensions increase. The estimation accuracy of RTFA-ER and PE-ER is comparable, and both are higher than IE-ER. When the data tails become heavier, the performance of all methods deteriorates with larger extent of heaviness of tails, especially IE-ER. The performance of RTFA-ER and PE-ER improves with dimension $\{p_1, p_2, p_3\}$ and sample size $T$ when the data
Table 3: The frequencies of exact estimation of the numbers of factors under Settings A, C and D over 1000 replications. “RTFA-ER”: the proposed robust iterative eigenvalue-ratio based method. “IE-ER”: initial eigenvalue-ratio based method. “PE-ER”: projected iterative eigenvalue-ratio based method.

| $p_1 = p_2 = p_3$ | Distribution | $T$ | RTFA-ER | IE-ER | PE-ER |
|-------------------|--------------|-----|---------|-------|-------|
| 10 normal         | 20           | 0.410 | 0.049  | 0.403 |
|                   | 50           | 0.437 | 0.071  | 0.430 |
|                   | 100          | 0.465 | 0.069  | 0.453 |
|                   | 200          | 0.472 | 0.077  | 0.460 |
| $t_5$             | 20           | 0.533 | 0.059  | 0.501 |
|                   | 50           | 0.641 | 0.035  | 0.590 |
|                   | 100          | 0.656 | 0.037  | 0.622 |
|                   | 200          | 0.688 | 0.019  | 0.646 |
| $t_3$             | 20           | 0.194 | 0.023  | 0.135 |
|                   | 50           | 0.245 | 0.023  | 0.146 |
|                   | 100          | 0.304 | 0.023  | 0.180 |
|                   | 200          | 0.338 | 0.016  | 0.195 |
| 20 normal         | 20           | 0.997 | 0.675  | 0.997 |
|                   | 50           | 1.000 | 0.731  | 0.999 |
|                   | 100          | 1.000 | 0.728  | 1.000 |
|                   | 200          | 1.000 | 0.724  | 1.000 |
| $t_5$             | 20           | 0.989 | 0.496  | 0.987 |
|                   | 50           | 0.996 | 0.463  | 0.993 |
|                   | 100          | 0.998 | 0.437  | 0.997 |
|                   | 200          | 1.000 | 0.377  | 1.000 |
| $t_3$             | 20           | 0.875 | 0.031  | 0.713 |
|                   | 50           | 0.949 | 0.008  | 0.770 |
|                   | 100          | 0.972 | 0.004  | 0.849 |
|                   | 200          | 0.979 | 0.005  | 0.883 |
| 30 normal         | 20           | 1.000 | 0.970  | 1.000 |
|                   | 50           | 1.000 | 0.982  | 1.000 |
|                   | 100          | 1.000 | 0.981  | 1.000 |
|                   | 200          | 1.000 | 0.981  | 1.000 |
| $t_5$             | 20           | 1.000 | 0.900  | 0.999 |
|                   | 50           | 1.000 | 0.922  | 0.997 |
|                   | 100          | 1.000 | 0.934  | 0.998 |
|                   | 200          | 1.000 | 0.946  | 0.999 |
| $t_3$             | 20           | 0.970 | 0.149  | 0.863 |
|                   | 50           | 0.995 | 0.099  | 0.921 |
|                   | 100          | 0.997 | 0.078  | 0.939 |
|                   | 200          | 1.000 | 0.042  | 0.966 |

is heavy-tailed, and RTFA-ER consistently outperforms PE-ER and tends to converge to 1 fast, which shows that RTFA-ER is a more robust estimation method.

6 Discussion

Tensor factor model is a powerful tool for dimension reduction of high-order tensors and is drawing growing attention in the last few years. In this paper, we propose a projection estimation method for Tucker-decomposition based Tensor Factor Model (TFM). We also provide the least squares interpretation of the iterative projection for TFM, which parallels to the least squares interpretation of traditional PCA for vector
factor models (Fan et al., 2013) and of Projection Estimation for matrix factor models (He et al., 2021b). We establish the theoretical properties for the one-step iteration projection estimators, and faster convergence rates are achieved by the projection technique compared with the naive Tensor PCA method due to the elevated signal-to-noise ratio. We also propose a robust estimation method for TFM by utilizing the Huber loss. An efficient iterative “weighted” projection algorithm is proposed to derive the robust estimators. The theoretical analysis of the robust estimators are challenging and we leave it as our future work.

Acknowledgements

He’s work is supported by NSF China (12171282,11801316), National Statistical Scientific Research Key Project (2021LZ09), Young Scholars Program of Shandong University, Project funded by China Postdoctoral Science Foundation (2021M701997) and the Fundamental Research Funds of Shandong University.

References

Ahn, S.C., Horenstein, A.R., 2013. Eigenvalue ratio test for the number of factors. Econometrica 81, 1203–1227.

Aït-Sahalia, Y., Kalnina, I., Xiu, D., 2020. High-frequency factor models and regressions. Journal of Econometrics 216, 86–105.

Aït-Sahalia, Y., Xiu, D., 2017. Using principal component analysis to estimate a high dimensional factor model with high frequency data. Journal of Econometrics 201, 388–399.

Bai, J., 2003. Inferential theory for factor models of large dimensions. Econometrica 71, 135–171.

Bai, J., Ng, S., 2002. Determining the number of factors in approximate factor models. Econometrica 70, 191–221.

Chen, E.Y., Chen, R., 2020. Modeling dynamic transport network with matrix factor models: with an application to international trade flow. arXiv:1901.00769.
Chen, E.Y., Fan, J., 2021. Statistical inference for high-dimensional matrix-variate factor models. Journal of the American Statistical Association (just-accepted), 1–44.

Chen, E.Y., Fan, J., Li, E., 2020a. Statistical inference for high-dimensional matrix-variate factor model. arXiv:2001.01890.

Chen, E.Y., Tsay, R.S., Chen, R., 2020b. Constrained factor models for high-dimensional matrix-variate time series. Journal of the American Statistical Association 115, 775–793.

Chen, H., Guo, Y., He, Y., Ji, J., Liu, L., Shi, Y., Wang, Y., Yu, L., Zhang, X., 2021a. Simultaneous differential network analysis and classification for matrix-variate data with application to brain connectivity. Biostatistics, in press.

Chen, L., Dolado, J.J., Gonzalo, J., 2021b. Quantile factor models. Econometrica 89, 875–910.

Chen, R., Yang, D., Zhang, C.H., 2021c. Factor models for high-dimensional tensor time series. Journal of the American Statistical Association, in press.

Davis, C., Kahan, W.M., 1970. The rotation of eigenvectors by a perturbation. III. SIAM Journal on Numerical Analysis 7, 1–46.

Fan, J., Liao, Y., Mincheva, M., 2013. Large covariance estimation by thresholding principal orthogonal complements. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 75, 603–680.

Gao, Z., Yuan, C., Jing, B.Y., Wei, H., Guo, J., 2021. A two-way factor model for high-dimensional matrix data. arXiv:2103.07920.

Han, Y., Chen, R., Yang, D., Zhang, C., 2020a. Tensor factor model estimation by iterative projection. arXiv:2006.02611.

Han, Y., Zhang, C.H., Chen, R., 2020b. Rank determination in tensor factor model. 2011.07131.

Han, Y., Zhang, C.H., Chen, R., 2021. Cp factor model for dynamic tensors. arXiv:2110.15517.
He, Y., Kong, X., Trapani, L., Yu, L., 2021a. One-way or two-way factor model for matrix sequences? arXiv:2110.01008.

He, Y., Kong, X., Yu, L., Zhang, X., 2022. Large-dimensional factor analysis without moment constraints. Journal of Business & Economic Statistics 40, 302–312.

He, Y., Kong, X., Yu, L., Zhang, X., Zhao, C., 2021b. Matrix factor analysis: From least squares to iterative projection. arXiv:2112.04186.

He, Y., Kong, X.B., Trapani, L., Yu, L., 2021c. Online change-point detection for matrix-valued time series with latent two-way factor structure. arXiv:2112.13479.

Ji, J., He, Y., Liu, L., Xie, L., 2021. Brain connectivity alteration detection via matrix-variate differential network model. Biometrics 77, 1409–1421.

Liu, X., Chen, E., 2019. Helping effects against curse of dimensionality in threshold factor models for matrix time series. arXiv:1904.07383.

Onatski, A., 2009. Testing hypotheses about the number of factors in large factor models. Econometrica 77, 1447–1479.

Stock, J.H., Watson, M.W., 2002. Forecasting using principal components from a large number of predictors. Journal of the American statistical association 97, 1167–1179.

Trapani, L., 2018. A randomised sequential procedure to determine the number of factors. Journal of the American Statistical Association 113, 1341–1349.

Wang, D., Liu, X., Chen, R., 2019. Factor models for matrix-valued high-dimensional time series. Journal of Econometrics 208, 231–248.

Yu, L., He, Y., Kong, X., Zhang, X., 2022. Projected estimation for large-dimensional matrix factor models. Journal of Econometrics 229, 201–217.
Yu, L., He, Y., Zhang, X., 2019. Robust factor number specification for large-dimensional elliptical factor model. Journal of Multivariate analysis 174, 104543.

Zhou, H., Li, L., Zhu, H., 2013. Tensor regression with applications in neuroimaging data analysis. Journal of the American Statistical Association 108, 540–552.
A Proof of Theorem 3.1

A.1 Proof of Theorem 3.1

Proof. According to the definition,

\[
\hat{M}_k = \frac{1}{Tp} \sum_{t=1}^{T} X_{k,t} X_{k,t}^T
\]

\[
= \frac{1}{Tp} \sum_{t=1}^{T} (A_k F_{k,t} B_k^T + E_{k,t}) (A_k F_{k,t} B_k^T + E_{k,t})^T
\]

\[
= \frac{1}{Tp} \left( \sum_{t=1}^{T} A_k F_{k,t} B_k^T B_k F_{k,t}^T A_k + \sum_{t=1}^{T} A_k F_{k,t} E_{k,t}^T + \sum_{t=1}^{T} E_{k,t} B_k F_{k,t}^T A_k + \sum_{t=1}^{T} E_{k,t} E_{k,t}^T \right)
\]

\[
=: I + II + III + IV.
\]

By the fact that \(T^{-1} \sum_t F_{k,t} F_{k,t}^T \to \Sigma_k, p^{-1} B_k B_k \to I_{r_k}, \|A_k\|_F^2 \asymp \|\hat{A}_k\|_F^2 \asymp p_k\) and the diagonal entries of \(\hat{A}_k\) converge to some positive constants (in Lemma D.1), we have

\[
\|\hat{H}_k\| = O_p(1), \|\hat{H}_k\|_F = O_p(1)
\]
Next, in Lemma A.3, we have

\[ \frac{1}{p_k} \| II \hat{A}_k \|_F^2 = O_p \left( \frac{1}{T_{p-k}} \right) \]

\[ \frac{1}{p_k} \| III \hat{A}_k \|_F^2 = O_p \left( \frac{1}{T_{p-k}} \right) \]

\[ \frac{1}{p_k} \| IV \hat{A}_k \|_F^2 = O_p \left( \frac{1}{T_{p-k}} \right) + o_p(1) + \frac{1}{p_k} \| \hat{A}_k - A_k \hat{H}_k \|_F^2. \]

Hence,

\[ \frac{1}{p_k} \| \hat{A}_k - A_k \hat{H}_k \|_F^2 = \frac{1}{p_k} \| (II + III + IV) \hat{A}_k \hat{A}_k^{-1} \|_F^2 \]

\[ \lesssim \frac{1}{p_k} \left( \| II \hat{A}_k \|_F^2 + \| III \hat{A}_k \|_F^2 + \| IV \hat{A}_k \|_F^2 \right) \]

\[ = O_p \left( \frac{1}{T_{p-k}} + \frac{1}{p_k^2} \right). \]

To complete the proof, it remains to show that \( \hat{H}_k \hat{H}_k \rightarrow I_{r_k} \). By Cauchy-Schwarz inequality,

\[ \left\| \frac{1}{p_k} A_k^T (\hat{A}_k - A_k \hat{H}_k) \right\|_F^2 \leq \left\| \frac{A_k}{p_k} \right\|_F^2 \left\| \frac{\hat{A}_k - A_k \hat{H}_k}{p_k} \right\|_F^2 = o_p(1) \]

\[ \left\| \frac{1}{p_k} A_k^T (\hat{A}_k - A_k \hat{H}_k) \right\|_F^2 \leq \left\| \frac{\hat{A}_k}{p_k} \right\|_F^2 \left\| \frac{\hat{A}_k - A_k \hat{H}_k}{p_k} \right\|_F^2 = o_p(1). \]

Note that \( p_k^{-1} \hat{A}_k^T \hat{A}_k = I_{r_k} \) and \( p_k^{-1} \hat{A}_k^T \hat{A}_k \rightarrow I_{r_k} \), then

\[ I_{r_k} = \frac{1}{p_k} \hat{A}_k^T A_k \hat{H}_k + o_p(1) = \hat{H}_k^T \hat{H}_k + o_p(1) \]

which concludes Theorem 3.1.

### A.2 Theoretical Lemmas

**Lemma A.1.** Under Assumptions 1 to 5, for all \( k \in [K] \), as \( \min \{ T, p_1, p_2, \cdots, p_K \} \rightarrow \infty \), we have

1. \( \sum_{t=1}^T \mathbb{E} \left\| E_{k,t}^T A_k \right\|_F^2 = O(Tp), \sum_{t=1}^T \mathbb{E} \left\| E_{k,t} B_k \right\|_F^2 = O(Tp) \)

2. \( \mathbb{E} \left\| \sum_{t=1}^T F_{k,t} B_k^T E_{k,t} \right\|_F^2 \leq O(Tp), \mathbb{E} \left\| \sum_{t=1}^T F_{k,t}^T A_k E_{k,t} \right\|_F^2 \leq O(Tp) \)

3. \( \mathbb{E} \left\| \sum_{t=1}^T F_{k,t} B_k^T E_{k,t} \right\|_F^2 \leq O(Tp), \mathbb{E} \left\| \sum_{t=1}^T F_{k,t}^T A_k E_{k,t} B_k \right\|_F^2 \leq O(Tp) \)

(3) it holds that for any \( i \leq p_1 \),

\[ \mathbb{E} \left\| \sum_{t=1}^T E_{k,t} e_{i,t}^{(k)} \right\|_F^2 = O(Tp + T^2 p_{-k}^2) \]

\[ \mathbb{E} \left\| \sum_{t=1}^T A_k^T E_{k,t} e_{i,t}^{(k)} \right\|_F^2 = O(Tp + T^2 p_{-k}^2) \]

**Proof.** As \( \{ r_k, k = 1, \cdots, K \} \) are fixed constants, without loss of generality, we assume \( r_1 = r_2 = \cdots = r_K = \)
1 in this proof.

(1). By Assumption 4.2

$$\sum_{t=1}^{T} \mathbb{E} \left[ \left\| \mathbf{f}_{t}^{T} \mathbf{A}_{k} \right\|_{F}^{2} \right] = \sum_{t=1}^{T} \mathbb{E} \left( \sum_{j} \left( e_{t,j}^{(k)} \mathbf{A}_{k} \right)^{2} \right) = \sum_{t=1}^{T} \sum_{j} \mathbb{E} \left( \sum_{i} e_{t,j}^{(k)} A_{k,i} \right)^{2} \leq a_{k}^{2} \sum_{t=1}^{T} \sum_{j} \sum_{i} \sum_{i_1} \mathbb{E} \left| e_{t,j}^{(k)} e_{t,i_1}^{(k)} \right| \leq (Tp).$$

$$\sum_{t=1}^{T} \mathbb{E} \left[ \left\| \mathbf{e}_{t} \right\|_{F}^{2} \right] = O(Tp) \text{ holds similarly.}$$

(2). The results hold directly by Assumption 5.1.

(3). On one hand, use Assumptions 4 then we have

$$\mathbb{E} \left[ \left\| \sum_{t=1}^{T} \mathbf{e}_{t,i} \right\|^{2} \right] = \mathbb{E} \left( \sum_{i_1} \left( \sum_{t} \sum_{j} e_{t,i_1}^{(k)} e_{t,i_1}^{(k)} \right)^{2} \right)$$

$$= \sum_{i_1} \mathbb{E} \left( \sum_{t} \sum_{j} e_{t,i_1}^{(k)} e_{t,i_1}^{(k)} e_{t,i_1}^{(k)} e_{t,i_1}^{(k)} \right)^{2} + \sum_{i_1} \mathbb{E} \left( \sum_{t} \sum_{j} e_{t,i_1}^{(k)} e_{t,i_1}^{(k)} \right)^{2} \leq c_{1} Tp + c_{2} \sum_{t,s,j_1,j_2} \left( \sum_{i_1} \mathbb{E} e_{t,i_1}^{(k)} e_{t,i_1}^{(k)} \right) \leq c_{1} Tp + c_{2} c_{3} T^{2} p_{-k}^{2} = O(Tp + T^{2} p_{-k}^{2}).$$

On the other hand, assume \( r_{k} = 1 \) so that

$$\mathbb{E} \left[ \left\| \sum_{t=1}^{T} \mathbf{e}_{t}^{(k)} \right\|^{2} \right]$$

$$= \mathbb{E} \left( \sum_{t} \sum_{i_1} \sum_{j} \left( A_{k,i_1} e_{t,j}^{(k)} e_{t,j}^{(k)} - A_{k,i_1} e_{t,j}^{(k)} e_{t,j}^{(k)} \right) \right)^{2} + \left( \sum_{t} \sum_{i_1} \sum_{j} \mathbb{E} A_{k,i_1} e_{t,j}^{(k)} e_{t,j}^{(k)} \right)^{2} \leq \sum_{t,s,j_1,j_2} \left| A_{k,i_1} A_{k,i_2} \right| \left| \mathbb{E} e_{t,j_1}^{(k)} e_{t,j_1}^{(k)} e_{i_1}^{(k)} e_{i_1}^{(k)} \right| \left( \sum_{t} \sum_{i_1} \sum_{j} \left| A_{k,i_1} \right| \left| \mathbb{E} e_{t,j_1}^{(k)} e_{t,j_1}^{(k)} \right| \right)^{2} \leq O(Tp + T^{2} p_{-k}^{2})$$

which concludes the lemma.
Lemma A.2. Under Assumptions \(1\) to \(5\), as \(\min \{T, p_1, p_2, \ldots, p_K\} \to \infty\),

\[
\lambda_j \left( \hat{\mathbf{M}}_k \right) = \begin{cases} 
\lambda_j (\Sigma_k) + o_p(1), & j \leq r_k, \\
O_p \left( \frac{1}{\sqrt{T p_{-k}}} + \frac{1}{p_k} \right), & j > r_k 
\end{cases}
\]

where \(\hat{\mathbf{M}}_k = (1/Tp)\mathbf{X}_{k,t}\mathbf{X}_{k,t}^T\).

**Proof.** Recall that by equation (A.1), \(\hat{\mathbf{M}}_k = \mathcal{I} + \mathcal{I}\mathcal{I} + \mathcal{I}\mathcal{I}\mathcal{I} + \mathcal{I}\mathcal{I}\mathcal{I}\). We will study the spectral norms of these four terms and show that \(\mathcal{I}\) is the main term.

Firstly, by Assumptions \(2\) and \(3\), we have

\[
\frac{1}{Tp_{-k}} \sum_{t} \mathbf{F}_{k,t} \mathbf{B}_k^\top \mathbf{B}_k \mathbf{F}_{k,t}^\top \overset{p}{\to} \Sigma_k, \\
\mathcal{I} \overset{p}{\to} p_k^{-1} \mathbf{A}_k \Sigma_k \mathbf{A}_k^\top
\]

while the leading \(r_k\) eigenvalues of \(p_k^{-1} \mathbf{A}_k \Sigma_k \mathbf{A}_k^\top\) are asymptotically equal to those of \(\Sigma_k\). Hence, \(\lambda_j (\mathcal{I}) = \lambda_j (\Sigma_k) + o_p(1)\) for \(j \leq r_k\) while \(\lambda_j (\mathcal{I}) = 0\) for \(j > r_k\) because \(\text{rank}(\mathcal{I}) \leq r_k\).

Secondly by Cauchy-Schwartz inequality and Lemma A.1.(2),

\[
\|\mathcal{I}\mathcal{I}\| \leq \frac{1}{Tp} \|\mathbf{A}_k\|_F \left\| \sum_{t=1}^T \mathbf{F}_{k,t} \mathbf{B}_k^\top \mathbf{E}_{k,t}^\top \right\| \leq \frac{1}{\sqrt{Tp_k}} \left\| \frac{1}{\sqrt{Tp}} \sum_{t=1}^T \mathbf{F}_{k,t} \mathbf{B}_k^\top \mathbf{E}_{k,t}^\top \right\| \leq O \left( \frac{1}{\sqrt{Tp_{-k}}} \right)
\]

Similarly, \(\|\mathcal{I}\mathcal{I}\mathcal{I}\| \leq O_p \left( \frac{1}{\sqrt{Tp_{-k}}} \right)\).

Lastly for \(\mathcal{I}\mathcal{I}\mathcal{I}\mathcal{I}\), denote \(\mathbf{U}_\mathbf{E} = (Tp)^{-1} \sum_t \mathbf{E} \left( \mathbf{E}_{k,t} \mathbf{E}_{k,t}^\top \right)\), then by Assumption 4.3(1), we have

\[
\mathbb{E} \|\mathcal{I}\mathcal{I}\mathcal{I}\mathcal{I} - \mathbf{U}_\mathbf{E}\|_F^2 = \frac{1}{T^2 p^2} \sum_{i_1, i_2} \mathbb{E} \left( \sum_{t, i} (e_{t,i_1} e_{t,i_2} - \mathbb{E} e_{t,i_1} e_{t,i_2}) \right)^2 \leq \frac{1}{T^2 p^2} \sum_{i_1, i_2} \sum_{t, j_1, j_2} |\text{Cov} (e_{t,i_1, j_1}, e_{t,i_2, j_2}, e_{t,i_1, j_2}, e_{t,i_2, j_1})| \leq \frac{c}{Tp_{-k}}.
\]

Meanwhile, by Assumption 4.2(1), for any \(i \leq p_k\),

\[
\sum_{i_1} \left| \sum_{t} \sum_{j} \mathbb{E} (e_{t,i_1, j} e_{t,i_2, j}) \right| \leq \sum_{t, j} \sum_{i_1} |\mathbb{E} (e_{t,i_1, j})| \leq c T p_{-k}
\]

4
Hence, $\|U_E\|_1 = \|U_E\|_\infty \leq O(p_1^{-1})$, which further implies $\|U_E\| \leq O(p_1^{-1})$. Therefore,

$$\|IV\| = O_p \left( \frac{1}{\sqrt{T}} + \frac{1}{p_k} \right).$$

The lemma holds with Weyl's inequality.

**Lemma A.3.** Recall that by equation (A.1), $\hat{\mathbf{M}}_k = \mathbf{I} + \mathbf{II} + \mathbf{III} + IV$. Under Assumptions 1 to 5, as $\min \{T, p_1, p_2, \ldots, p_K\} \to \infty$, it holds that

$$\frac{1}{p_k} \|IV\|_F^2 = O_p \left( \frac{1}{T^{p_k}} \right) \leq O_p \left( \frac{1}{T^{p_k}} \right).$$

Proof. Firstly, by equation (A.1),

$$\frac{1}{p_k} \|IV\|_F^2 = \frac{1}{p_k} \left\| \frac{1}{T} \sum_{t=1}^{T} \mathbf{A}_k \mathbf{F}_{k,t} \mathbf{B}_k^\top \mathbf{E}_{k,t}^\top \right\|_F^2 \\
\leq \frac{\|\mathbf{A}_k\|_F^2}{p_k} \left\| \frac{1}{T} \sum_{t=1}^{T} \mathbf{F}_{k,t} \mathbf{B}_k^\top \mathbf{E}_{k,t}^\top \right\|_F^2 \|\hat{\mathbf{A}}_k\|_F^2 \\
\leq O_p \left( \frac{1}{T^{p_k}} \right).$$

Similarly,

$$\frac{1}{p_k} \|IV\|_F^2 \leq \frac{1}{p_k} \left\| \frac{1}{T} \sum_{t=1}^{T} \mathbf{E}_{k,t} \mathbf{B}_k \mathbf{F}_{k,t}^\top \right\|_F^2 \|\hat{\mathbf{A}}_k\|_F^2 \|\mathbf{A}_k\|_F^2 = O_p \left( \frac{1}{T^{p_k}} \right).$$

Lastly, use Lemma A.1.1 (3) and $\|\hat{\mathbf{H}}_k\|_F^2 = O_p(1)$,

$$\frac{1}{p_k} \|IV\|_F^2 = \frac{1}{p_k} \left\| \sum_{t=1}^{T} \left( \hat{\mathbf{A}}_k - \mathbf{A}_k \hat{\mathbf{H}}_k + \mathbf{A}_k \hat{\mathbf{H}}_k \right)^\top \mathbf{E}_{k,t} \mathbf{E}_{k,t}^\top \right\|_F^2 \\
= \frac{1}{p_k} \left\| \sum_{t=1}^{T} \frac{1}{T} \mathbf{A}_k^\top \mathbf{E}_{k,t} \mathbf{e}_{t,i} + \frac{1}{T} \sum_{t=1}^{T} \left( \hat{\mathbf{A}}_k - \mathbf{A}_k \hat{\mathbf{H}}_k \right)^\top \mathbf{E}_{k,t} \mathbf{e}_{t,i} \right\|_F^2 \\
= O_p \left( \frac{1}{p_k} + \frac{1}{T} \right) + O_p \left( \frac{1}{p_k} + \frac{1}{T^{p_k}} \right) \times \frac{1}{p_k} \|\hat{\mathbf{A}}_k - \mathbf{A}_k \hat{\mathbf{H}}_k\|_F^2.$$

The lemma follows.
B  Proof of Theorem 3.2

B.1  Proof of Theorem 3.2

Proof. Note that by definition, we have

\[ \tilde{M}_k = \frac{1}{T p_k p_{-k}} \sum_{t=1}^{T} X_{k,t} \tilde{B}_k \tilde{B}_k^\top X_{k,t} \]

\[ = \frac{1}{T p_k p_{-k}} \sum_{t=1}^{T} (A_k F_{k,t}^\top B_k + E_{k,t}) \tilde{B}_k \tilde{B}_k^\top (B_k F_{k,t}^\top A_k^\top + E_{k,t}) \]

\[ = \frac{1}{T p_k p_{-k}} \sum_{t=1}^{T} A_k F_{k,t}^\top B_k B_k^\top B_k^\top F_{k,t}^\top A_k^\top + \frac{1}{T p_k p_{-k}} \sum_{t=1}^{T} E_{k,t} \tilde{B}_k \tilde{B}_k^\top E_{k,t}^\top \]

\[ + \frac{1}{T p_k p_{-k}} \sum_{t=1}^{T} A_k F_{k,t}^\top B_k B_k^\top E_{k,t}^\top + \frac{1}{T p_k p_{-k}} \sum_{t=1}^{T} E_{k,t}^\top \tilde{B}_k \tilde{B}_k^\top E_{k,t} \]

\[ := \mathcal{V} + \mathcal{VI} + \mathcal{VII} + \mathcal{VIII}. \]

Define \( \tilde{A}_k = \text{diag}(\lambda_1(\tilde{M}_k), \lambda_2(\tilde{M}_k), \ldots, \lambda_r(\tilde{M}_k)) \), then

\[ \tilde{A}_k A_k = \tilde{M}_k \tilde{A}_k. \]

Define \( \tilde{H}_k = (T p_k p_{-k})^{-1} \sum_{t=1}^{T} F_{k,t}^\top B_k \tilde{B}_k B_k \tilde{B}_k^\top A_k^\top A_k^{-1} \), then

\[ \tilde{A}_k - A_k \tilde{H}_k = (\mathcal{VI} + \mathcal{VII} + \mathcal{VIII}) \tilde{A}_k A_k^{-1}. \]

We will show that the diagonal entries of \( \tilde{A}_k \) converge to some distinct positive constants in Lemma D.2, then \( \| \tilde{H}_k \| = O_p(1) \). Further in Lemma B.2 we will have

\[ \frac{1}{p_k} \| \mathcal{VI} \tilde{A}_k \|_F^2 = O_p \left( m - k + \frac{1}{T p_{-k}} \right), \]

\[ \frac{1}{p_k} \| \mathcal{VII} \tilde{A}_k \|_F^2 = O_p \left( m - k + \frac{1}{T p_{-k}} \right), \]

\[ \frac{1}{p_k} \| \mathcal{VIII} \tilde{A}_k \|_F^2 \lesssim \frac{1}{T p} + \frac{1}{p^2} + \left( \frac{1}{p_k} + \frac{1}{T p_{-k}} \right) w_{-k}^2 + o_p(1) \times \frac{1}{p_k} \| \tilde{A}_k - A_k \tilde{H}_k \|_F^2. \]
Hence,
\[
\frac{1}{p_k} \| \tilde{A}_k - A_k \tilde{H}_k \|_F^2 = \frac{1}{p_k} \| (\mathcal{VI} + \mathcal{VII} + \mathcal{VIII}) \tilde{A}_k \tilde{A}_k^{-1} \|_F^2 \\
\leq \frac{1}{p_k} \left( \| \mathcal{VI} \tilde{A}_k \|_F^2 + \| \mathcal{VII} \tilde{A}_k \|_F^2 + \| \mathcal{VIII} \tilde{A}_k \|_F^2 \right) \\
\leq \frac{1}{Tp_{-k}} + \frac{p}{p^2} + w_{-k}^{-1} \left( \frac{1}{p_k^2} + \frac{1}{Tp_k} \right) + m_{-k}.
\]

To complete the proof, it remains to show that \( \tilde{H}_k^\top \tilde{H}_k \xrightarrow{p} I_k \). By Cauchy-Schwartz inequality,
\[
\left\| \frac{1}{p_k} A_k^\top \left( \tilde{A}_k - A_k \tilde{H}_k \right) \right\|_F^2 \leq \frac{\| A_k \|_F^2}{p_k} \frac{\| \tilde{A}_k - A_k \tilde{H}_k \|_F^2}{p_k} = o_p(1)
\]
\[
\left\| \frac{1}{p_k} \tilde{A}_k^\top \left( \tilde{A}_k - A_k \tilde{H}_k \right) \right\|_F^2 \leq \frac{\| \tilde{A}_k \|_F^2}{p_k} \frac{\| \tilde{A}_k - A_k \tilde{H}_k \|_F^2}{p_k} = o_p(1).
\]

Note that \( p_k^{-1} \tilde{A}_k^\top \tilde{A}_k = I_r \) and \( p_k^{-1} A_k^\top A_k \to I_r \), then
\[
I_r = \frac{1}{p_k} \tilde{A}_k^\top A_k \tilde{H}_k + o_p(1) = \tilde{H}_k^\top \tilde{H}_k + o_p(1)
\]
which concludes Theorem 3.2.

B.2 Theoretical Lemmas

**Lemma B.1.** Under Assumptions 1 to 5 and the sufficient condition, as \( \min \{ T, p_1, p_2, \cdots, p_K \} \to \infty \), for \( j \leq r_k \) we have
\[
\lambda_j \left( \tilde{M}_k \right) = \lambda_j \left( \Sigma_k \right) + o_p(1),
\]
where \( \tilde{M}_k = (Tp_k)^{-1} \sum_{t=1}^T \tilde{Y}_{k,t} \tilde{Y}_{k,t}^\top \) with \( \tilde{Y}_{t} = p_{-k}^{-1} X_{k,t} \tilde{B}_k \).

**Proof.** Recall that by equation (B.1), \( \tilde{M}_k = \mathcal{V} + \mathcal{VI} + \mathcal{VII} + \mathcal{VIII} \). Firstly, without loss of generality we
assume \( r_1 = r_2 = \cdots = r_K = 1 \), then

\[
\mathcal{V} = \frac{1}{Tp_k p_{-k}^2} A_k \left( \sum_{t=1}^{T} F_{k,t} B_k^T \hat{B}_k \hat{B}_k^T B_k F_{k,t} \right) A_k^T
\]

\[
= \frac{1}{Tp_k p_{-k}^2} A_k \left( \sum_{t=1}^{T} F_{k,t} B_k^T (B_k \hat{H}_{-k} + \hat{B}_k - B_k \hat{H}_{-k}) (B_k \hat{H}_{-k} + \hat{B}_k - B_k \hat{H}_{-k})^T B_k F_{k,t} \right) A_k^T
\]

\[
= \frac{1}{Tp_k p_{-k}^2} A_k \left( \sum_{t=1}^{T} F_{k,t} B_k^T \hat{B}_k \hat{H}_{-k} \hat{B}_k^T B_k F_{k,t} \right) A_k^T
\]

\[
+ \frac{1}{Tp_k p_{-k}^2} A_k \left( \sum_{t=1}^{T} F_{k,t} B_k^T (\hat{B}_k - B_k \hat{H}_{-k}) (\hat{B}_k - B_k \hat{H}_{-k})^T B_k F_{k,t} \right) A_k^T
\]

\[
+ \frac{1}{Tp_k p_{-k}^2} A_k \left( \sum_{t=1}^{T} F_{k,t} B_k^T (\hat{B}_k - B_k \hat{H}_{-k}) (\hat{B}_k - B_k \hat{H}_{-k})^T B_k F_{k,t} \right) A_k^T
\]

Note that \( \hat{H}_{-k} \hat{H}_{-k}^T \overset{p}{\sim} I_{r_{-k}} \), then by sufficient condition and Weyl's inequality, \( \lambda_j(I) = \lambda_j(p_k^{-1} A_k \Sigma_k A_k^T) + o_p(1) \) for \( j \leq k_2 \). The leading \( r_k \) eigenvalues of \( p_k^{-1} A_k \Sigma_2 A_k^T \) asymptotically converge to \( \Sigma_k \) as \( p_k \to \infty \).

Hence, \( \lambda_j(\mathcal{V}) = \lambda_j(\Sigma_k) + o_p(1) \) for \( j \leq k_2 \). Since \( \operatorname{rank}(\mathcal{V}) \leq r_k \), \( \lambda_j(\mathcal{V}) = 0 \) for \( j > k_2 \).

Secondly, let \( r_1 = r_2 = \cdots = r_K = 1 \), then

\[
\| \mathcal{V} \mathcal{I} \| = \frac{1}{Tp_k p_{-k}^2} \left\| \sum_t \mathcal{E}_{k,t} \hat{B}_k \hat{B}_k^T B_k F_{k,t} A_k^T \right\|_F
\]

\[
\leq \frac{1}{Tp_k p_{-k}^2} \left\| \sum_t \mathcal{E}_{k,t} \hat{B}_k F_{k,t} A_k^T \right\|_F
\]

\[
\leq \frac{1}{Tp_k p_{-k}^2} \left\| \sum_t \mathcal{E}_{k,t} (\hat{B}_k - B_k \hat{H}_{-k}) F_{k,t} \right\|_F \| A_k \|_F + \frac{1}{Tp_k p_{-k}} \left\| \sum_t \mathcal{E}_{k,t} B_k \hat{H}_{-k} F_{k,t} A_k^T \right\|_F
\]

\[
= O_p \left( \sqrt{m_{-k}} + \frac{1}{\sqrt{T p_{-k}}} \right)
\]

Similarly,

\[
\| \mathcal{V} \mathcal{I} \mathcal{I} \| = O_p \left( \sqrt{w_{-k}} + \frac{1}{\sqrt{T p_{-k}}} \right)
\]
Lastly,
\[ \|VIII\| = \frac{1}{T_p p_{-k}^2} \left\| \sum_t E_{k,t} \hat{B}_k \hat{B}_k^\top E_{k,t}^\top \right\|_F \]
\[ \leq \frac{1}{T_p p_{-k}^2} \sum_t \left\| E_{k,t} \hat{B}_k \right\|_F^2 \]
\[ \lesssim \frac{1}{T_p p_{-k}^2} \sum_t \left\| E_{k,t} \right\|_F^2 \left\| \hat{B}_k - \hat{B}_k \hat{H}_{-k} \right\|_F^2 + \frac{1}{T_p p_{-k}^2} \sum_t \left\| E_{k,t} \hat{B}_k \hat{H}_{-k} \right\|_F^2 \]
\[ = O_p \left( w_{-k} + p_{-k}^2 \right) = o_p(1) \]

The lemma then holds by Weyl's theorem.

**Lemma B.2.** Under Assumptions 1 to 5 and the sufficient condition, as \( \min \{ T, p_1, p_2, \ldots, p_K \} \to \infty \), it holds that
\[ \frac{1}{p_k} \|VII \tilde{A}_k\|_F^2 = O_p \left( m_{-k} + \frac{1}{T_{p-k}} \right), \]
\[ \frac{1}{p_k} \|VII \tilde{A}_k\|_F^2 = O_p \left( m_{-k} + \frac{1}{T_{p-k}} \right), \]
\[ \frac{1}{p_k} \|VIII \tilde{A}_k\|_F^2 \lesssim \frac{1}{T_p} + \frac{1}{p^2} + \left( \frac{1}{p_k} + \frac{1}{T_p} \right) w_{-k}^2 + o_p(1) \times \frac{1}{p^2} \left\| \tilde{A}_k - A_k \tilde{H}_k \right\|_F^2. \]

**Proof.**

Firstly, with \( r_1 = r_2 = \cdots = r_K = 1 \), by Lemma A.1(2) and sufficient condition (b), we have
\[ \frac{1}{p_k} \|VII \tilde{A}_k\|_F^2 = \frac{1}{p_k} \left\| \sum_{t=1}^T E_{k,t} \hat{B}_k \hat{B}_k^\top E_{k,t}^\top \right\|_F^2 \]
\[ \lesssim \frac{1}{p_k} \left\| \sum_{t=1}^T E_{k,t} \hat{B}_k \hat{B}_k^\top \right\|_F^2 \]
\[ \lesssim \frac{1}{p_k} \left\| \sum_{t=1}^T E_{k,t} \left( \hat{B}_k - \hat{B}_k \hat{H}_k \right) \hat{B}_k^\top \right\|_F^2 + \frac{1}{p_k} \left\| \sum_{t=1}^T E_{k,t} \hat{B}_k \hat{H}_k \hat{B}_k^\top \right\|_F^2 \]
\[ = O_p \left( m_{-k} + \frac{1}{T_{p-k}} \right) . \]

Similarly,
\[ \frac{1}{p_k} \|VII \tilde{A}_k\|_F^2 = O_p \left( m_{-k} + \frac{1}{T_{p-k}} \right) . \]
Lastly, some simple calculations lead to

\[
\mathcal{VIII} \tilde{A}_k = \frac{1}{T_p k p_{-k}} \sum_{t=1}^{T} \mathbb{E}_{k,t} \hat{B}_k \hat{B}_k^\top \mathbb{E}_{k,t}^\top \tilde{A}_k
\]

\[
= \frac{1}{T_p k p_{-k}} \sum_{t=1}^{T} \mathbb{E}_{k,t} \left( \hat{B}_k - B_k \tilde{H}_{-k} + B_k \hat{H}_{-k} \right) \left( \hat{B}_k - B_k \tilde{H}_{-k} + B_k \hat{H}_{-k} \right)^\top \mathbb{E}_{k,t}^\top \tilde{A}_k
\]

\[
= \frac{1}{T_p k p_{-k}} \left( \sum_{t=1}^{T} \mathbb{E}_{k,t} \hat{B}_k \hat{H}_{-k} \hat{H}_{-k} \hat{B}_k^\top \mathbb{E}_{k,t}^\top \tilde{A}_k + \sum_{t=1}^{T} \mathbb{E}_{k,t} \left( \hat{B}_k - B_k \tilde{H}_{-k} \right) \hat{H}_{-k}^\top B_k^\top \mathbb{E}_{k,t}^\top \tilde{A}_k \right)
\]

\[
+ \sum_{t=1}^{T} \mathbb{E}_{k,t} \left( \hat{B}_k - B_k \tilde{H}_{-k} \right)^\top \mathbb{E}_{k,t}^\top \tilde{A}_k
\]

\[
:= \mathcal{L} \mathcal{X} + \mathcal{X} + \mathcal{X} \mathcal{L}.
\]

For \( \mathcal{L} \mathcal{X} \), assume that \( r_1 = r_2 = \cdots = r_K = 1 \),

\[
\mathbb{E} \left\| \sum_{t} \mathbb{E}_{k,t} B_k B_k^\top \mathbb{E}_{k,t}^\top \tilde{A}_k \right\|_F^2 \leq \| B_k \|_F^2 \mathbb{E} \left\| \mathbb{E}_{k,t} B_k A_k^\top \mathbb{E}_{k,t} \right\|_F^2 = \| B_k \|_F^2 \mathbb{E} \left\| \sum_{t} B_k^\top \epsilon_{t,i}^{(k)} \epsilon_{t,i}^{(k)} A_k \right\|_F^2
\]

while for any \( i, j \),

\[
\mathbb{E} \left\| \sum_{t} B_k^\top \epsilon_{t,i}^{(k)} \epsilon_{t,j}^{(k)} A_k \right\|_F^2 \leq \mathbb{E} \left\| \sum_{t} \left( B_k^\top \epsilon_{t,i}^{(k)} \epsilon_{t,j}^{(k)} A_k - B_k^\top \epsilon_{t,i}^{(k)} A_k \right) \right\|_F^2 + \mathbb{E} \left\| \sum_{t} B_k^\top \epsilon_{t,i}^{(k)} \epsilon_{t,j}^{(k)} A_k \right\|_F^2
\]

\[
\leq \sum_{t,s,i_1,j_1,i_2,j_2} \text{Cov} ( \epsilon_{t,i_1,j_1}^{(k)}, \epsilon_{t,i_2,j_2}^{(k)}) + \left( \sum_{t,s,i_1,j_1} \mathbb{E} \epsilon_{t,i_1,j_1}^{(k)} \right)^2 \leq O (T_p + T^2).
\]

( by Assumptions 4.3(2) and 4.2(2) )

(B.2)

Hence,

\[
\frac{1}{p_k} \| \mathcal{L} \mathcal{X} \|_F^2 = \frac{1}{p_k} \left\| \sum_{t=1}^{T} \mathbb{E}_{k,t} B_k \tilde{H}_{k} \mathbb{E}_{k,t}^\top \tilde{A}_k \right\|_F^2
\]

\[
\lesssim \frac{1}{T^2 p_k p_{-k}} \left( \left\| \sum_{t=1}^{T} \mathbb{E}_{k,t} B_k^\top \mathbb{E}_{k,t}^\top \tilde{A}_k \right\|_F^2 + \left( \sum_{t} \| \mathbb{E}_{k,t} B_k \|_F^2 \right)^2 \| \tilde{A}_k - A_k \|_F^2 \right)
\]

\[
\lesssim \frac{1}{T p} + \frac{1}{p_k p_{-k}} \| \tilde{A}_k - A_k \|_F^2.
\]

Next, for \( \mathcal{X} \), assume that \( r_1 = r_2 = \cdots = r_K = 1 \),

\[
\left\| \sum_{t} \mathbb{E}_{k,t} \left( \hat{B}_k - B_k \tilde{H}_{-k} \right) B_k^\top \mathbb{E}_{k,t} A_k \right\|_F^2 \leq \sum_{t,i,j} \left\| \sum_{t} \epsilon_{t,i}^{(k)} B_k^\top \mathbb{E}_{k,t} A_k \right\|_F^2 \| \hat{B}_k - B_k \tilde{H}_{-k} \|_F^2.
\]
while for any $i, j$,

$$
\mathbb{E} \left\| \sum_t e_{t,ij}^{(k)} B_k^T E_{k,t}^T \right\|_F^2 \\
\lesssim \sum_{i_{1},i_{2},j_{1}} \sum_{s_{1},s_{2}} \left| \text{Cov} \left( e_{t,i_{1}}^{(k)} e_{t,i_{2}}^{(k)} e_{t,s_{1}}^{(k)} e_{t,s_{2}}^{(k)} \right) \right| + \left( \sum_t \sum_{i_{1},j_{1}} \left| e_{t,i_{1}}^{(k)} e_{t,i_{1}}^{(k)} \right| \right)^2 \\
= O \left( Tp + T^2 \right). \quad ( \text{by Assumption 4.2(1) and 4.3(2)} )
$$

Hence,

$$
\frac{1}{p_k} \| \mathcal{X} \|_F^2 \lesssim \frac{1}{Tp_k p_k^2} \sum_t \mathbf{E}_{k,t} \left( \hat{B}_k - B_k \tilde{\mathbf{H}}_{-k} \right) \tilde{\mathbf{H}}_{-k}^T B_k^T E_{k,t}^T \left( \tilde{A}_k - A_k \tilde{\mathbf{H}}_k \right) \left\|_F^2 \\
+ \frac{1}{p_k} \| B_k^T \|_F^2 \sum_t \left\| \mathbf{E}_{k,t} \left( \hat{B}_k - B_k \tilde{\mathbf{H}}_{-k} \right) \tilde{\mathbf{H}}_{-k}^T B_k^T E_{k,t}^T \tilde{A}_k \tilde{\mathbf{H}}_k \right\|_F^2 \\
\lesssim \frac{1}{T^2 p_k^4 p_k^2} \left( \sum_t \| \mathbf{E}_{k,t} \|_F^2 \sum_t \left\| \mathbf{E}_{k,t} \left( \hat{B}_k - B_k \tilde{\mathbf{H}}_{-k} \right) \|_F \right| \tilde{A}_k - A_k \tilde{\mathbf{H}}_k \|_F \right)^2 \\
+ \sum_t \| \mathbf{E}_{k,t} \tilde{B}_k \|_F^2 \sum_t \mathbb{E}_{k,t} \| \hat{B}_k - B_k \tilde{\mathbf{H}}_{-k} \|_F^2 \| \tilde{A}_k - A_k \tilde{\mathbf{H}}_k \|_F^2 \\
\lesssim \frac{1}{T^2 p_k^4 p_k^2} \left( \sum_t \| \mathbf{E}_{k,t} \tilde{B}_k \|_F^2 + \sum_t \left\| \mathbf{E}_{k,t} \left( \hat{B}_k - B_k \tilde{\mathbf{H}}_{-k} \right) \right\|_F^2 \sum_t \| \mathbf{E}_{k,t} \left( \hat{B}_k - B_k \tilde{\mathbf{H}}_{-k} \right) \|_F \| \tilde{A}_k - A_k \tilde{\mathbf{H}}_k \|_F^2 \\
+ \left( \sum_t \| \mathbf{E}_{k,t} \tilde{B}_k A_k^T \mathbf{E}_{k,t} \|_F^2 \right) + \sum_t \| \mathbf{A}_k^T \mathbf{E}_{k,t} \|_F^2 \left\| \hat{B}_k - B_k \tilde{\mathbf{H}}_{-k} \right\|_F^2 \| \tilde{A}_k - A_k \tilde{\mathbf{H}}_k \|_F^2 \\
\lesssim \left( \frac{1}{Tp} + \frac{1}{p^2} \right) w_{-k} + \left( \frac{1}{p_k^2} + \frac{1}{Tp_k} \right) w_{-k}^2 + \left( \frac{1}{p_k} + w_{-k} \right) w_{-k} \frac{1}{p_k} \| \tilde{A}_k - A_k \tilde{\mathbf{H}}_k \|_F^2.
$$

( by equation B.2 and Lemma A.1(3) )
Combine the above results so that

\[
\frac{1}{p_k} \left\| \mathcal{V} \mathcal{T} \mathcal{I} \mathcal{I} \tilde{A}_k \right\|_F^2 \lesssim \frac{1}{T^p} + \frac{1}{p^2} + \left( \frac{1}{p_k^2} + \frac{1}{T^p} \right) w_{2,k}^2 + o_p(1) \times \frac{1}{p_k} \left\| \tilde{A}_k - A_k \tilde{H}_k \right\|_F^2
\]

and the lemma holds.

C Proof of Lemma 3.3

Proof. (a.) For

\[
\tilde{B}_k - B_k \tilde{H}_{-k} = \tilde{A}_k \otimes \cdots \otimes \tilde{A}_{k+1} \otimes \tilde{A}_{k-1} \otimes \cdots \otimes \tilde{A}_1 - (\otimes_{j \neq k} A_j) \left( \otimes_{j \neq k} \tilde{H}_j \right)
\]

\[
= \left( \tilde{A}_K - A_K \tilde{H}_K + A_K \tilde{H}_K \right) \otimes \cdots \otimes \left( \tilde{A}_{k+1} - A_{k+1} \tilde{H}_{k+1} + A_{k+1} \tilde{H}_{k+1} \right)
\]

\[
\otimes \left( \tilde{A}_{k-1} - A_{k-1} \tilde{H}_{k-1} + A_{k-1} \tilde{H}_{k-1} \right) \otimes \cdots \otimes \left( \tilde{A}_1 - A_1 \tilde{H}_1 + A_1 \tilde{H}_1 \right)
\]

\[- \otimes_{j \neq k} (A_j \tilde{H}_j)
\]

\[
= \left( \tilde{A}_K - A_K \tilde{H}_K \right) \otimes \left( \otimes_{j \neq k} (A_j \tilde{H}_j) \right) + A_K \tilde{H}_K \otimes \left( \tilde{A}_{K-1} - A_{K-1} \tilde{H}_{K-1} \right) \otimes \left( \otimes_{j \neq k, K-1, K} \tilde{A}_j \right)
\]

\[+ \cdots + \otimes_{j \neq 1,k} (A_j \tilde{H}_j) \otimes \left( \tilde{A}_1 - A_1 \tilde{H}_1 \right)
\]

Hence,

\[
\frac{1}{p_k} \left\| \tilde{B}_k - B_k \tilde{H}_{-k} \right\|_F^2 \lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| (\otimes_{i \in (j+1,K), i \neq k} A_i \tilde{H}_i) \otimes (\tilde{A}_j - A_j \tilde{H}_j) \otimes (\otimes_{i \in (1,j-1), i \neq k} A_i) \right\|_F^2
\]

\[
\lesssim \frac{1}{p_k} \sum_{j \neq k} \left( \prod_{i \neq j,k} \| A_i \|_F^2 \right) \left\| \tilde{A}_j - A_j \tilde{H}_j \right\|_F^2
\]

\[
\lesssim \sum_{j \neq k} w_j
\]

(b.) Recall that by equation (A.1), \( \tilde{M}_k = I + \mathcal{I} \mathcal{T} + \mathcal{I} \mathcal{I} \mathcal{T} + \mathcal{I} \mathcal{V} \) and \( \tilde{A}_k - A_k \tilde{H}_k = (\mathcal{I} \mathcal{T} + \mathcal{I} \mathcal{I} \mathcal{T} + \mathcal{I} \mathcal{V}) \tilde{A}_k \tilde{A}_k^{-1} \),
then
\[
\frac{1}{p_k} \left\| \frac{1}{T_{p-k}} \sum_s E_{k,s} \left( \hat{B}_k - B_k \hat{H}_{-k} \right) F_{k,s}^\top \right\|_F^2
\]
\[
\lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T} \sum_s E_{k,s} \left( \left( \otimes i \in (j+1, K), i \neq k \frac{1}{p_i} A_i \hat{H}_i \right) \otimes \frac{1}{p_j} \left( \hat{A}_j - A_j \hat{H}_j \right) \otimes \left( \otimes i \in (1, j-1), i \neq k \frac{1}{p_i} A_i \right) \right) F_{k,s}^\top \right\|_F^2
\]
\[
\lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T} \sum_s E_{k,s} \left( \left( \otimes i \in (j+1, K), i \neq k \frac{1}{p_i} A_i \right) \otimes \frac{1}{p_j} \left( \hat{A}_j - A_j \hat{H}_j \right) \otimes \left( \otimes i \in (1, j-1), i \neq k \frac{1}{p_i} A_i \right) \right) F_{k,s}^\top \right\|_F^2
\]
\[
\lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T_{p-k}} \sum_s E_{k,s} \left( \left( \otimes i \in (j+1, K), i \neq k \frac{1}{p_i} A_i \right) \otimes \left( III + IIII + IV \right) \hat{A}_j \otimes \left( \otimes i \in (1, j-1), i \neq k \frac{1}{p_i} A_i \right) \right) F_{k,s}^\top \right\|_F^2
\]
\[
\lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T_{p-k}} \sum_s E_{k,s} \left( \left( \otimes i \in (j+1, K), i \neq k \frac{1}{p_i} A_i \right) \otimes \left( III + IIII + IV \right) \hat{A}_j \otimes \left( \otimes i \in (1, j-1), i \neq k \frac{1}{p_i} A_i \right) \right) F_{k,s}^\top \right\|_F^2
\]
\[
+ \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T_{p-k}} \sum_s E_{k,s} \left( \left( \otimes i \in (j+1, K), i \neq k \frac{1}{p_i} A_i \right) \otimes \left( III + IIII + IV \right) \hat{A}_j \otimes \left( \otimes i \in (1, j-1), i \neq k \frac{1}{p_i} A_i \right) \right) F_{k,s}^\top \right\|_F^2
\]
\[
+ \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T_{p-k}} \sum_s E_{k,s} \left( \left( \otimes i \in (j+1, K), i \neq k \frac{1}{p_i} A_i \right) \otimes \left( III + IIII + IV \right) \hat{A}_j \otimes \left( \otimes i \in (1, j-1), i \neq k \frac{1}{p_i} A_i \right) \right) F_{k,s}^\top \right\|_F^2
\]
\[
:= \mathcal{XIII} + \mathcal{XIII} + \mathcal{XIV}
\]

For \( \mathcal{XIII} \),
\[
\mathcal{XIII} \lesssim \frac{1}{T^2 p_k p_{-k}} \sum_{j \neq k} \left\| \frac{1}{T} \sum_t F_{j,t} B_j^\top E_{j,t} \hat{A}_j \right\|_F^2 \left\| \sum_s E_{k,s} B_k F_{k,s}^\top \right\|_F^2
\]
\[
\lesssim \frac{1}{T^2 p_k p_{-k}} \sum_{j \neq k} \left\| \sum_t F_{j,t} B_j^\top E_{j,t} A \right\|_F^2 + \left\| \sum_t F_{j,t} B_j^\top E_{j,t} \right\|_F \left\| \hat{A}_j - A_j \hat{H}_j \right\|_F^2 \left\| \sum_s E_{k,s} B_k F_{k,s}^\top \right\|_F^2
\]
\[
\lesssim \frac{1}{T^2 p_k p_{-k}} \sum_{j \neq k} \left( 1 + \frac{p_j}{T_{p-j}} \right).
\]
For $X_{III}$, assume that $r_1 = r_2 = \cdots = r_K = 1$,

$$
\mathbb{E} \left\| s \mathbf{E}_{k,s} \left( \otimes_{i \in (j+1,K), i \neq k} A_i \right) \otimes \left( \frac{1}{T p} \sum_t \mathbf{E}_{j,t} \mathbf{B}_{j,t}^\top \right) \otimes \left( \otimes_{i \in (1,j-1), i \neq k} A_i \right) \right\|_F^2
\lesssim \frac{1}{T^2 p^2} \sum_{i=1}^{p_k} \mathbb{E} \left( \sum_{t,s} \mathbf{E}_{s,t} \sum_{n} e_n^{(k)} a_{1,i_1} a_{2,i_2} \cdots e_n^{(j)} b_{j,m} a_{K,i_K} F_{j,t} F_{k,s}^\top \right)^2
\lesssim \frac{1}{T^2 p^2} \sum_{i=1}^{p_k} \mathbb{E} \left( \sum_{t,s} \sum_{n} e_n^{(k)} a_{1,i_1} a_{2,i_2} \cdots e_n^{(j)} b_{j,m} a_{K,i_K} F_{j,t} F_{k,s}^\top \right)^2
\lesssim \frac{1}{p_j} \sum_{i=1}^{p_k} \left( \sum_{i_1} \sum_{i_2} \cdots \sum_{i_j} \sum_{i_{K-1}} \sum_{m=1}^{p_j} \sum_{i_K} \sum_{m=1}^{p_j} |E_{i_1,i_2,\cdots,i_j,i_{K-1}}| \right)^2 + \frac{1}{p^2} \sum_{i=1}^{p_k} \left( \sum_{i_1} \sum_{i_2} \cdots \sum_{i_j} \sum_{i_{K-1}} \sum_{m=1}^{p_j} \sum_{i_K} \sum_{m=1}^{p_j} |E_{i_1,i_2,\cdots,i_j,i_{K-1}}| \right)^2
= O \left( \frac{1}{p_j} + \frac{1}{p_k} \right).
$$

where $n = i_1 + (i_2 - 1)p_1 + \cdots + (i_{K-1} - 1)p_1 \cdots p_{K-1}$. Hence,

$$
X_{III} \lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T p-k} \sum_s \mathbf{E}_{k,s} \left( \otimes_{i \in (j+1,K), i \neq k} A_i \right) \otimes \left( \frac{1}{T p} \sum_t \mathbf{E}_{j,t} \mathbf{B}_{j,t}^\top \right) \otimes \left( \otimes_{i \in (1,j-1), i \neq k} A_i \right) \right\|_F^2 \left\| A_j \tilde{A}_j \right\|_F^2
\lesssim \frac{1}{T^2 p_k p^2 - k} \sum_{j \neq k} \left( p_j + \frac{p_j^2}{p_k} \right).
$$

For $X_{IV}$, assume that $r_1 = r_2 = \cdots = r_K = 1$,

$$
X_{IV} \lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T p-k} \sum_s \mathbf{E}_{k,s} \left( \otimes_{i \in (j+1,K), i \neq k} A_i \right) \otimes \left( \frac{1}{T p} \sum_t \mathbf{E}_{j,t} \mathbf{B}_{j,t}^\top \right) \otimes \left( \otimes_{i \in (1,j-1), i \neq k} A_i \right) \right\|_F^2
\lesssim \frac{1}{p_k} \sum_{j \neq k} \left\| \frac{1}{T p-k} \sum_s \mathbf{E}_{k,s} \mathbf{F}_{k,s}^\top \right\|_F^2 ||A_K||_F^2 \cdot ||A_1||_F^2 || \frac{1}{T p} \sum_t \mathbf{E}_{j,t} \mathbf{B}_{j,t}^\top \right\|_F^2 + \left| \frac{1}{T p} \mathbf{E}_{j,t} \mathbf{B}_{j,t}^\top \right|_F^2 \left| \tilde{A}_j - A_j \tilde{A}_j \right|_F^2
\lesssim \frac{1}{T^2 p_k p^2 - k} \sum_{j \neq k} \left( \frac{1}{T p^2} + \frac{p_j^2}{T} \right).
$$

Hence,

$$
\frac{1}{p_k} \left\| \frac{1}{T p-k} \sum_s \mathbf{E}_{k,s} \left( \tilde{B}_k - \hat{B}_k \hat{H}_k \right) \right\|_F^2 = O_p \left( \frac{1}{T^2 p_k} \sum_{j \neq k} (p_j^2 + T p^2 - j) \right).
$$
D Proof of Theorem 3.4

Proof. By Lemma D.1, we actually have

\[ \lambda_j \left( \hat{M}_k \right) = \begin{cases} 
\lambda_j \left( \Sigma_k \right) + o_p(1), & j \leq r_k, \\
O_p \left( \frac{1}{\sqrt{T p_{-k}}} + \frac{1}{p_k} \right), & j > r_k.
\end{cases} \]

Let \( \delta = \max \left\{ (T p_{-k})^{-1/2}, (p_k)^{-1} \right\} \), then

\[
\max_{j \leq r_{k-1}} \frac{\lambda_j \left( \hat{M}_k \right)}{\lambda_{j+1} \left( \hat{M}_k \right) + c\delta} = O_p(1), \\
\max_{j \geq r_{k+1}} \frac{\lambda_j \left( \hat{M}_k \right)}{\lambda_{j+1} \left( \hat{M}_k \right) + c\delta} \leq O_p(1), \\
\frac{\lambda_j \left( \hat{M}_k \right)}{\lambda_{j+1} \left( \hat{M}_k \right) + c\delta} \bigg|_{j=r_k} \geq c\delta^{-1} \rightarrow \infty,
\]

which concludes the consistency.

**Lemma D.1.** Under Assumptions 1 to 5, as \( \min \{ T, p_1, p_2, \cdots, p_K \} \rightarrow \infty \),

\[ \lambda_j \left( \hat{M}_k \right) = \begin{cases} 
\lambda_j \left( \Sigma_k \right) + o_p(1), & j \leq r_k, \\
O_p \left( \frac{1}{\sqrt{T p_{-k}}} + \frac{1}{p_k} \right), & j > r_k.
\end{cases} \]

where \( \hat{M}_k = (1/T) X_{k,t} X_{k,t}^\top \).

**Lemma D.2.** Under Assumptions 1 to 5 and the sufficient condition, as \( \min \{ T, p_1, p_2, \cdots, p_K \} \rightarrow \infty \), for \( j \leq r_k \) we have

\[ \lambda_j \left( \hat{M}_k \right) = \lambda_j \left( \Sigma_k \right) + o_p(1), \]

where \( \hat{M}_k = (T p_{-k})^{-1} \sum_{t=1}^{T} \hat{Y}_{k,t} \hat{Y}_{k,t}^\top \) with \( \hat{Y}_t = p_{-k}^{-1} X_{k,t} \hat{B}_k \).