KUREPA’S CONJECTURE ON THE LEFT FACTORIAL FUNCTION IS TRUE

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Abstract. We answer Kurepa’s conjecture on the left factorials in affirmative.

1. Introduction

In 1971, Kurepa [9] introduced the left factorial function denoted as \(!n\) for \(n \in \mathbb{N}\) known as Kurepa’s function:

\[ !n = \sum_{i=0}^{n-1} i! \]

where he conjectured that \(\gcd(n!, !n) = 2\) for all \(n \geq 2\). As well, in his original paper [9], Kurepa showed that there is an infinite number of values \(n\) for which the conjecture is true. Since the appearance of [9] it has been a large number of publications related to Kurepa’s function, where the authors mostly tried to find equivalent reformulation of the conjecture in order to approach its solution closer (e.g. [4, 8, 10, 11]). The review of the problem with relevant references to the subject can be found in [5, 8].

Despite the numerous attempts (e.g. [3] for one of them), this conjecture hitherto has not been answered. Its formulation is in the list of unsolved problems in [7] mentioned under V44. The numerical confirmation of this conjecture was provided by many authors for very large series of \(n\). For instance, in the attempt to disprove the conjecture, [1] provided extensive numerous calculations up to \(n \approx 2^{34}\) with further additional analysis in [2]. The recent paper [5] provided a massive numerical study of Kurepa’s conjecture for the set of \(!n\) up to \(!116447 \approx 116446! \approx 1.045 \times 10^{539361}\). The history of the earlier numerical analysis can be found in [1, 5, 8] as well.

In the present note, we affirm the truth of the conjecture, and our proof is very elementary.
2. Proof of the conjecture

Recall some basic and elementary properties of \( n! \). We have \( 1! = 1 \), \( 2! = 2 \), and \( n! \) is even for all \( n \geq 2 \). For any \( n \geq 3 \) we have \((n - 1)! < n! < n!\). The last is true since \( 0! + \sum_{k=1}^{n-1} kk! = n! \) for \( n \geq 1 \) (e.g. [6, page 2] or [4]).

Using this fact, we have

\[
\frac{n!}{3!} = \frac{1}{3!} \sum_{k=0}^{2} k! + \frac{1}{3!} \sum_{k=3}^{n-1} k! = I_1 + I_2.
\]

The term \( I_1 \) is less than 1, while the second term, \( I_2 \), is integer. Hence, the \( 3 \) is not a divisor of \( n! \). For \( n \geq 4 \), the \( 4 \) is not a divisor either, since \( 4! = 10 \), and \((n - 1)! \) (mod 4) \( \equiv 0 \). It is also easily seen that for \( n \geq 5 \), the \( 5 \) is not a divisor, since \( 5! = 120 \), and \((n - 1)! \) (mod 5) \( \equiv 0 \). So, for the numbers \( 3, 4, \) and \( 5 \) the answer is elementary.

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**Theorem 2.1.** For \( n \geq 3 \)

\[
\frac{n!}{n} \notin \mathbb{N}.
\]

**Proof.** Write

\[
\frac{n!}{n} = (n - 2) + (n - 2)! + (n - 1)!
\]

assuming that \( n \geq 3 \). For the last two terms we have

\[
(n - 2)! + (n - 1)! = [(n - 2)!][(n - 1) + 1] = [(n - 2)!]n,
\]

which means that \( (n - 1)(n - 2)/n \in \mathbb{N} \), and the original problem reduces to the following: for \( n \geq 3 \) prove

\[
\frac{!(n - 2)}{n} \notin \mathbb{N}.
\]

The following solution to the problem is mostly algebraic. We introduce a polynomial function

\[
F_{i,n-1}(z) = \prod_{k=1}^{n-1}(z - k), \quad i \leq n,
\]

where \( F_{n,n-1}(z) \) is set to 1. The polynomial function \( F_{i,n-1}(z) \) is an extension of the factorial function, since \( F_{0,n-1}(n) = n! \), and for any \( i \leq n \) we have \( F_{i,n-1}(n) = (n - i)! \). The other polynomial function

\[
L_{i,n-1}(z) = \sum_{k=i}^{n} F_{k,n-1}(z), \quad i \leq n,
\]

is an extension of the left factorial function, since \( L_{i,n-1}(n) = !(n - i + 1) \).

We prove the following statement. The polynomial equation \( L_{3,n-1}(z) = 0 \) has no zero roots. Recall that a polynomial equation

\[
z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0
\]
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has a zero root if and only if \( a_0 = 0 \). In our case we have the equation

\[
L_{3,n-1}(z) = \prod_{k=3}^{n-1} (z-k) + \prod_{k=4}^{n-1} (z-k) + \ldots + \prod_{k=n-2}^{n-1} (z-k) + (z-n+1) + 1 = 0,
\]

and we are to prove that this equation has no zero roots. Setting \( z = 0 \) yields

\[
L_{3,n-1}(0) = (-1)^{n-3} \prod_{k=3}^{n-1} k + (-1)^{n-4} \prod_{k=4}^{n-1} k + \ldots + (-1)^{n-1} \prod_{k=n-3}^{n-1} k + (n-2)(n-1) + 1.
\]

It is readily seen from this expression that \( L_{3,n-1}(0) \) is positive, if \( n \) is odd, and it is negative, otherwise. Hence \( L_{3,n-1}(0) \neq 0 \). The last means that \( !(n-2) = L_{3,n-1}(n), n \geq 3, \) cannot be presented as \( nP(n) \), where \( P(z) \) is some polynomial function with integer coefficients, and hence the required statement follows. With this statement \( L_{3,n-1}(n)/n = !(n-2)/n \notin \mathbb{N}, n \geq 3, \) and the theorem is proved. \( \square \)

Statements and declarations

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