Existence of Schrödinger Evolution with Absorbing Boundary Condition

Lawrence Frolov∗†, Stefan Teufel‡§, and Roderich Tumulka‡¶

February 19, 2025

Abstract

Consider a non-relativistic quantum particle with wave function inside a region Ω ⊂ R³, and suppose that detectors are placed along the boundary ∂Ω. The question how to compute the probability distribution of the time at which the detector surface registers the particle boils down to finding a reasonable mathematical definition of an ideal detecting surface; a particularly convincing definition, called the absorbing boundary rule, involves a time evolution for the particle’s wave function ψ expressed by a Schrödinger equation in Ω together with an “absorbing” boundary condition on ∂Ω first considered by Werner in 1987, viz., ∂ψ/∂n = iκψ with κ > 0 and ∂/∂n the normal derivative. We provide here a discussion of the rigorous mathematical foundation of this rule. First, for the viability of the rule it plays a crucial role that these two equations together uniquely define the time evolution of ψ; we point out here how, under some technical assumptions on the regularity (i.e., smoothness) of the detecting surface, the Lumer-Phillips theorem implies that the time evolution is well defined and given by a contraction semigroup. Second, we show that the collapse required for the N-particle version of the problem is well defined. We also prove that the joint distribution of the detection times and places, according to the absorbing boundary rule, is governed by a POVM.

Key words: detection time in quantum mechanics, Lumer-Phillips theorem, time observable, arrival time in quantum mechanics, contraction semigroup, Schrödinger equation.

∗Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA
†E-mail: laf230@math.rutgers.edu
‡Mathematisches Institut, Eberhard-Karls-Universität, Auf der Morgenstelle 10, 72076 Tübingen, Germany.
§E-mail: stefan.teufel@uni-tuebingen.de
¶E-mail: roderich.tumulka@uni-tuebingen.de

1
1 Introduction

Suppose an ideal detecting surface is placed along the boundary $\partial \Omega$ of an open region $\Omega \subset \mathbb{R}^3$ in physical space, and a non-relativistic quantum particle is prepared at time 0 with wave function $\psi_0$ with support in $\Omega$. Let $Z = (T, X) \in [0, \infty) \times \partial \Omega$ be the random time and location of the detection event; we write $Z = \infty$ if no detection event ever occurs. What is the probability distribution of $Z$? As we have argued elsewhere \cite{17}, there is a simple rule for computing this distribution that is particularly convincing, called the absorbing boundary rule; its equations were first considered by Werner \cite{22}. According to this rule, $\psi$ evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi$$

in $\Omega$ with potential $V : \Omega \to \mathbb{R}$ and boundary condition

$$\frac{\partial \psi}{\partial n}(x) = i \kappa(x) \psi(x)$$

at every $x \in \partial \Omega$, where $\partial/\partial n$ is the outward normal derivative on the surface, i.e.,

$$\frac{\partial \psi}{\partial n}(x) := n(x) \cdot \nabla \psi(x)$$

with $n(x)$ the unit vector perpendicular to $\partial \Omega$ at $x \in \partial \Omega$ pointing outside $\Omega$, and $\kappa(x) \geq 0$ are given values of dimension 1/length that characterize the type of ideal detector (wave number of sensitivity).

Then, the absorbing boundary rule asserts,

$$\text{Prob}_{\psi_0}(t_1 \leq T < t_2, X \in B) = \int_{t_1}^{t_2} dt \int_B d^2x \ n(x) \cdot j^\psi(t)(x)$$

for any $0 \leq t_1 < t_2$ and any measurable set $B \subseteq \partial \Omega$, with $d^2x$ the surface area element and $j^\psi$ the probability current vector field defined by $\psi$, which is

$$j^\psi = \frac{\hbar}{m} \text{Im} \psi^* \nabla \psi.$$  

Note that the boundary condition \cite{22} implies that the current $j^\psi$ is always outward-pointing on $\partial \Omega$, i.e., $n(x) \cdot j^\psi(x) \geq 0$, so \cite{22} is an “absorbing” boundary condition, and one should expect $\|\psi_t\|$ not to be constant but to be a decreasing function of $t$. It is taken for granted in \cite{4} that $\|\psi_0\| = 1$. Finally, to complete the statement of the absorbing boundary rule, the probability that no detection ever occurs is

$$\text{Prob}_{\psi_0}(Z = \infty) = 1 - \int_0^\infty \int_{\partial \Omega} d^2x \ n(x) \cdot j^\psi_t(x) = \lim_{t \to \infty} \|\psi_t\|^2.$$  


Among other things, in this paper we deduce from the Lumer-Phillips theorem \[\text{[14, 12, 7]}\] that \((1)\) and \((2)\) define a unique, autonomous time evolution for \(\psi\), provided \(\kappa(x) \geq 0\), see Theorem 1 below. (If \(\kappa(x) < 0\) then the boundary condition \((2)\) is not absorbing but emitting, that is, there is a current coming out of the boundary; in this case, we would not expect a unique autonomous time evolution of \(\psi\) to exist. For boundary points \(x\) with \(\kappa(x) = 0\) the boundary condition is a Neumann boundary condition and thus reflecting.)

As we will explain, it follows further that if \(\kappa(x) \geq 0\) everywhere, then the probability distribution given by \((4)\) and \((6)\) can be defined for every \(\psi_0 \in L^2(\Omega, \mathbb{C})\), and can be expressed in terms of a POVM (positive-operator-valued measure). Also, we treat not only dimension 3, but directly the obvious generalization to any dimension \(d \in \mathbb{N}\).

In the presence of more than one particle in \(\Omega\), the wave function must be collapsed appropriately when the first particle reaches \(\partial \Omega\) and triggers a detector, and we have developed and discussed the appropriate equations in \[18\]. The \(N\)-particle Schrödinger equation in \(\Omega^N\) gets supplemented by the appropriate boundary condition on \(\partial(\Omega^N)\), which is

\[
 n_i(x_i) \cdot \nabla_i \psi(x_1, \ldots, x_N) = i \kappa(x_i) \psi(x_1, \ldots, x_N) \quad \text{when } x_i \in \partial \Omega.
\]  

Suppose that at time \(T_1\), the first detector gets triggered, in fact at location \(X^1\) by particle number \(I^1\). Now particle number \(I^1\) gets absorbed and removed from consideration, and the wave function replaced by the conditional wave function

\[
\psi'(x') = N \psi_{T_1}(x', x_{I^1} = X^1)
\]  

with \(x' \in \Omega^{N-1}\) any configuration of the remaining \(N-1\) particles and \(N\) the appropriate normalizing factor. If \(\psi\) is symmetric or anti-symmetric under permutations (as it would have to be for identical particles) then so will be \(\psi'\). The process now repeats according to the corresponding equations for \(N-1\) particles.

For this process to be well-defined, we need to explain what exactly \((8)\) means and why \(\psi'\) is a well-defined vector in \(L^2(\Omega^{N-1})\); the difficulty comes from the fact that a general element of \(L^2(\Omega^N)\), such as \(\psi_{T_1}\), does not have well-defined values on a set of measure 0, such as the set where \(x_{I^1} = X^1\). This point will be addressed by Theorem 2 and its proof.

As steps towards Theorem 2, we prove in Theorem 5 in Section 6.1 that conditional wave functions in general have a well-defined distribution, and provide in Theorem 6 the POVM for an experiment done on a conditional wave function. These theorems can be applied also in other contexts independently of absorbing boundary conditions.

The analogous question of existence of solutions arises for the Dirac equation instead of the Laplacian, together with a suitable absorbing boundary condition \[19\]; some results on this question are given in \[16\], and we plan to investigate it further in a future work. We also leave open here, for the Laplacian, the case of unbounded regions \(\Omega\), and limit our theorems to bounded ones. For further discussion of the absorbing boundary rule, see \[17, 18, 5, 21, 10\]. For an overview of other proposals for the detection time
distribution in quantum mechanics, see [13]. Boundary conditions are also used for
defining zero-range interactions; concerning the existence of solutions, see, e.g., [1, 8].

The remainder of this paper is structured as follows: In Section 2, we describe our
theorems about absorbing boundaries. In Sections 3–6, we give the proofs. Our theorems
about conditional wave functions are stated and proven in Section 6.1.

2 Results

2.1 Single Particle

We simply write \( L^2(\Omega) \) for \( L^2(\Omega, \mathbb{C}) \). Our first theorem expresses that the Schrödinger
equation with boundary condition (2) actually defines a unique time evolution; the
theorem can be formulated as follows.

**Theorem 1.** Suppose that \( d \in \mathbb{N} \), that \( \Omega \subset \mathbb{R}^d \) is an open bounded region with Lipschitz
boundary \( \partial \Omega \), that \( \kappa : \partial \Omega \to [0, \infty) \) is in \( L^\infty(\partial \Omega) \), and that \( V : \Omega \to \mathbb{R} \) is in \( L^\infty(\Omega) \).
Then there exists a dense subspace \( D(H) \subset L^2(\Omega) \) such that

1. for \( \psi \in D(H) \), \( \nabla^2 \psi \) is defined as an element of \( L^2(\Omega) \), and \( \psi \bigg|_{\partial \Omega} \) and \( \partial_n \psi \bigg|_{\partial \Omega} \) are
defined as elements of \( L^2(\partial \Omega) \);
2. for \( \psi \in D(H) \), \( \partial_n \psi = i\kappa \psi \) on \( \partial \Omega \);
3. for every \( \psi_0 \in D(H) \), the initial-boundary value problem

\[
\begin{align*}
    i\hbar \partial_t \psi &= \left(-\frac{\hbar^2}{2m} \nabla^2 + V\right) \psi & \text{in } \Omega \\
    \psi &= \psi_0 & \text{at } t = 0 \\
    \partial_n \psi &= i\kappa \psi & \text{on } \partial \Omega 
\end{align*}
\]  

admits a unique global-in-time solution

\[
\psi \in C([0, \infty), D(H)) \cap C^1([0, \infty), L^2(\Omega)) . \tag{10}
\]

Moreover, the operator \( H = -\frac{\hbar^2}{2m} \nabla^2 + V : D(H) \to L^2(\Omega) \) generates a strongly con-
tinuous contraction semigroup \( \bar{W}_t = \exp(-iHt/\hbar) : L^2(\Omega) \to L^2(\Omega) \). Thus, for every
\( \psi_0 \in L^2(\Omega) \), there is a unique global-in-time solution \( \psi \in C([0, \infty), L^2(\Omega)) \) in the sense
of contraction semigroups, given by \( \psi_t = W_t \psi_0 \) (and the two senses of solution agree for
\( \psi_0 \in D(H) \)). If, moreover, \( \Omega \) has a \( C^2 \) boundary and \( \kappa \in C^1(\partial \Omega, [0, \infty)) \), then we can
canonically characterize

\[
D(H) = \{ \psi \in H^2(\Omega) : \partial_n \psi = i\kappa \psi \text{ on } \partial \Omega \} . \tag{11}
\]

We give the proof in Section 3. Here, “Lipschitz boundary” means that each point
in \( \partial \Omega \) has a neighborhood \( U \) such that \( U \cap \partial \Omega \) is, in some Cartesian coordinate system,
the graph of a Lipschitz function \( f \), i.e., \( U \cap \partial \Omega = \{ x : x_d = f(x_1, \ldots, x_{d-1}) \} \); a “\( C^2 \) boundary” is one for which \( f \) is twice continuously differentiable; \( H^2(\Omega) \) denotes the second Sobolev space of \( \Omega \), i.e., the space of \( \psi_0 \in L^2(\Omega) \) whose second distributional derivatives lie in \( L^2(\Omega) \). The terminology “contraction” means that \( \|W_t\psi\| \leq \|\psi\| \); “semigroup” means that \( W_tW_s = W_{t+s} \) for \( t, s \geq 0 \) and \( W_0 = I \) (the identity operator); “strongly continuous” means that \( \lim_{t \to 0} \|W_t\psi_0 - \psi_0\| = 0 \) for every \( \psi_0 \in L^2(\Omega) \). Since \( W_t \) is in general not unitary, \( \|\psi_t\| \) is in general smaller than \( \|\psi_0\| \) for \( t > 0 \) and for \( \|\psi_0\| = 1 \) has the physical meaning of

\[
\|\psi_t\|^2 = \text{Prob}_{\psi_0}(T > t).
\]

The spectrum of a contraction \( W \) lies in the closed unit disk \( \{ z \in \mathbb{C} : |z| \leq 1 \} \) in the complex plane; however, \( W \) is not necessarily diagonalizable. The generator \( H \) of a contraction semigroup has spectrum in the lower half plane \( \{ z \in \mathbb{C} : \text{Im} z \leq 0 \} \); again, \( H \) need not be diagonalizable. In the present case neither \( W_t \) nor \( H \) are unitarily diagonalizable (they are not normal operators, i.e., do not commute with their adjoints), as we show in Remark 5 in Section 4. At least in some cases, \( H \) can be diagonalized, but the eigenfunctions are not mutually orthogonal [20].

Intuitively speaking, the key difference between the two regularity conditions on \( \partial \Omega \), Lipschitz and \( C^2 \), is that Lipschitz allows for edges and corners (as long as they have positive opening angles, such as for a cube) while \( C^2 \) does not. A need for the weaker Lipschitz regularity assumption arises from the \( N \)-particle case, in which the domain in configuration space is \( \Omega^N \), which will have edges and corners even if \( \Omega \) has none (like a sphere); that is, \( \Omega^N \) will not have a \( C^2 \) boundary even if \( \Omega \) does, but still a Lipschitz boundary.

The next question that arises is whether the probability distribution (4) is well defined for a general \( \psi \). The difficulty comes from the fact that (4) involves evaluating \( \psi_t \) on the boundary \( \partial \Omega \), and \( \psi_t \) may fail to be continuous; since a general element \( \psi_t \) in \( L^2(\Omega) \) is an equivalence class of functions modulo arbitrary changes on a set of volume 0, and since \( \partial \Omega \) has volume 0, it is not well defined what \( \psi_t \) is on \( \partial \Omega \). A solution to this problem can be summarized as follows.

**Corollary 1.** Under the assumptions of Theorem 4 there is a POVM \( E(\cdot) \) on \([0, \infty) \times \partial \Omega \cup \{ \infty \} \) acting on \( L^2(\Omega) \) such that the probability distribution

\[
\text{Prob}_{\psi_0}(Z \in \cdot) = \langle \psi_0 | E(\cdot) | \psi_0 \rangle
\]

(defined for every \( \psi_0 \in L^2(\Omega) \) with \( \|\psi_0\| = 1 \)) agrees with (4) and (6) with \( d - 1 \) dimensional surface integrals for \( \psi_0 \in D(H) \) (for which (4) and (6) are well defined). \( E(\cdot) \) has the property that for every \( \psi_0 \in L^2(\Omega) \), the restriction of the measure \( \langle \psi_0 | E(\cdot) | \psi_0 \rangle \) to \([0, \infty) \times \partial \Omega \) is absolutely continuous (i.e., possesses a density) relative to the measure \( dt \cdot d^{d-1}x \).

We have included a proof in Section 5 making use of a strategy of Werner [22].
Remark 1. The proof of Theorem 1 also shows that, in fact, $D(H) \subset H^{3/2}(\Omega)$ for Lipschitz boundary and $\kappa \in L^\infty$ while, as mentioned in the theorem, $D(H) \subset H^2(\Omega)$ for $C^2$ boundary and $C^1 \kappa$; thus, the regularity of $\psi_t$ is lower in general for Lipschitz boundary and bounded $\kappa$ than for $C^2$ boundary and $C^1 \kappa$; even for initial data $\psi_0$ from the space $C_c^\infty(\Omega)$ of smooth functions with compact support (which vanish on $\partial \Omega$ because $\Omega$ is open and so the support cannot touch $\partial \Omega$), $\psi_t$ will generically reside only in $H^{3/2}(\Omega)$ for $t > 0$. Of course, the $C^2$ case is also of interest because it allows for a more explicit characterization of $D(H)$.

2.2 Many Particles

We now turn to the case of $N$ particles. Theorem 1 yields, after we replace $\Omega \to \Omega^N$ and $d \to Nd$, a well-defined time evolution up to the first detection event for any $\psi_0 \in L^2(\Omega^N)$ and a POVM on $[0, \infty) \times \partial(\Omega^N) \cup \{\infty\}$ that we will denote by $\tilde{E}$. However, it is not $\tilde{E}$ that gets measured by the detectors but really only a certain marginal of it that we will denote by $E'$: that is because the point of arrival on $\partial(\Omega^N)$ is an $N$-particle configuration that includes not just the position $x_i \in \partial \Omega$ of the particle, say $i$, that first hits $\partial \Omega$, but also the positions $x_j \in \Omega$ of all other particles at that time, while only $x_i$ gets measured but not $x_j$.

In more detail, the boundary $\partial(\Omega^N)$ consists of $N$ faces $F_i$ corresponding to $x_i$ lying on the boundary $\partial \Omega$ while the other $x_j$'s may remain in the interior of $\Omega$ (which is $\Omega$, as we take $\Omega$ to be an open set); more precisely,

$$\partial(\Omega^N) = \bigcup_{i=1}^N F_i \cup \bigcup_{i,j=1}^N F_{ij}$$

(14)

with

$$F_i := \{(x_1, \ldots, x_N) \in \Omega^N : x_i \in \partial \Omega, x_j \in \Omega \forall j \neq i\}$$

(15)

$$F_{ij} := \{(x_1, \ldots, x_N) \in \Omega^N : x_i \in \partial \Omega, x_j \in \partial \Omega\}.$$  

(16)

Note that $F_{ij}$ (an edge between two faces) has measure 0 in $\partial(\Omega^N)$, that the $F_i$ are mutually disjoint, and each $F_i$ is disjoint from each $F_{jk}$. Given that $Z \neq \infty$, the outcome $(T, X)$ is by Corollary 1 continuously distributed in $[0, \infty) \times \partial(\Omega^N)$, so $X$ lies with probability 1 in one of the $F_i$, say $F_i$. We write $(T^1, X^1, I^1)$ for $(T, X, I)$ and introduce, for easier notation, the permutation function $p : \cup_i F_i \to \partial \Omega \times \{1, \ldots, N\} \times \Omega^{N-1}$ given by

$$p(x) = (x_i, i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \text{ for } x = (x_1, \ldots, x_N) \in F_i.$$  

(17)

\footnote{We can also allow the particles to have different masses $m_i$, as Theorem 1 remains true when different masses are introduced for different components of the vector $x$. But for simplicity, we take the masses to be equal.}
Note that \( p \) is bijective and measure-preserving. We define the POVM \( E' \) on \([0, \infty) \times \partial \Omega \times \{1, \ldots, N\} \cup \{\infty\}\) by

\[
E'(\{\infty\}) = \tilde{E}(\{\infty\})
\]

\[
E'(dt \times B) = \tilde{E}(dt \times p^{-1}(B \times \Omega^{N-1}))
\]

for any measurable \( B \subseteq \partial \Omega \times \{1, \ldots, N\} \). That is, away from \( \infty \), \( E' \) is the marginal of \( \tilde{E} \) that corresponds to ignoring the \( x_j \) other than \( x_i \in \partial \Omega \).

For example, for \( N = 2 \) particles in \( d = 1 \) dimension with \( \Omega = (-1, 1) \), e.g., \( p(-1, x) = (-1, 1, x) \) and \( p(x, -1) = (-1, 2, x) \) for \( -1 < x < 1 \), and \( E'(dt \times \{1\} \times \{2\}) = \tilde{E}(dt \times \Omega \times \{1\}) \) is the operator for computing the probability that particle 2 gets detected first, in fact at time \( T^1 \in dt \) in the location \( X^1 = -1 \). Notice that the permutation \( (I^1, \ldots, I^N) \) of the particles according to the order of detection is random, but the mapping \( p \) is not.

**Corollary 2.** Let \( N \in \mathbb{N} \), and let \( \Omega \subseteq \mathbb{R}^d \), \( \kappa \), and \( V : \Omega^N \rightarrow \mathbb{R} \) be as in Theorem 1. Then the \( N \)-particle Schrödinger equation \((\ref{schroedinger})\) in \( \Omega^N \) and boundary condition \((\ref{boundary})\) define a contraction semigroup \((W_t)_{t \geq 0}\) on \( L^2(\Omega^N) \) and a POVM \( \tilde{E} \) on \([0, \infty) \times \partial(\Omega^N) \cup \{\infty\}\).

Furthermore, for any \( \psi_0 \in L^2(\Omega^N) \) with \( \|\psi_0\| = 1 \) and conditionally on \( Z \neq \infty \), the joint distribution of \( T^1, X^1, I^1 \) exists, is absolutely continuous, and is given by \( \langle \psi_0|E'(-\cdot)|\psi_0\rangle/\langle \psi_0|I - E'(-\{\infty\})|\psi_0\rangle \) with \( E' \) defined in \((18), (19)\).

Next, we construct the entire process of \( N \) detections; the crucial step is to guarantee the existence of the collapsed wave function. We also introduce another piece of notation: when we remove a particle, it is relevant to keep track of which particles remain, and to this end we introduce an index set \( \mathcal{I} \) with \( N \) elements for labeling the particles. The notation \( \Omega^{\mathcal{I}} \) means the set of all mappings from \( \mathcal{I} \) to \( \Omega \); i.e., all configurations in \( \Omega \) with labels from \( \mathcal{I} \); put differently, \( \Omega^{\mathcal{I}} \) is the same as \( \Omega^N \) but with the components labeled by elements of \( \mathcal{I} \) instead of \( 1, \ldots, N \). The initial wave function will be a function on \( \Omega^{\mathcal{I}} \), and it is clear what is meant by the Laplacian on \( \Omega^{\mathcal{I}} \).

**Theorem 2.** Let \( \psi_i \in L^2(\Omega^\mathcal{I}) \) follow the \( N \)-particle evolution with boundary condition \((\ref{boundary})\) under the assumptions of Theorem 1, and \( \|\psi_0\| = 1 \). Given that \( T^1 < \infty \) and \( I^1 = i \), it has probability 1 that \( \psi' \) as in \((\ref{wavefunction})\) is a well defined element of \( L^2(\Omega^{\mathcal{I}'}) \) with \( \mathcal{I}' = \mathcal{I} \setminus \{i\} \). Concerning the iteration of the procedure for \( \psi' \), suppose that for every index set \( \mathcal{I}' \subseteq \mathcal{I} \) we are given a bounded potential \( V_{\mathcal{I}'} : \Omega^{\mathcal{I}'} \rightarrow \mathbb{R} \) that applies whenever only the particles with labels in \( \mathcal{I}' \) are still around. Then the joint distribution of all detections \( Z^k = (\Delta T^k, X^k, I^k) \) with waiting times \( \Delta T^k = T^k - T^{k-1} \) (where \( T^0 = 0 \)), detected places \( X^k \), and labels \( I^k \) (or \( Z^k = \infty \)) exists as a measure on \(([0, \infty) \times \partial \Omega \times \mathcal{I} \cup \{\infty\})^N \), is locally absolutely continuous, and is defined by a POVM \( E_{\mathcal{I}'} \) acting on \( L^2(\Omega^{\mathcal{I}'}) \).

It is understood that if \( Z^k = \infty \) then also \( Z^{k+1} = \infty \); in particular, not every element of \(([0, \infty) \times \partial \Omega \times \mathcal{I} \cup \{\infty\})^N \) can actually occur (also, every \( i \in \mathcal{I} \) can come up at
most once in the sequence). A natural choice of \( V_J \) for \( x \in \Omega \) would be, for example,

\[
V_J(x) = \sum_{i \in J} e_i V_1(x_i) + \sum_{i,j \in J, i \neq j} e_i e_j V_2(x_i - x_j)
\]  

(20)

with arbitrary constants \( e_i \) such as charges.

### 3 Proof of Theorem 1

Theorem 1 is an application of the Lumer-Phillips theorem \cite{14, 12, 7} for contraction semigroups, which is a variant of the Hille-Yosida theorem \cite{7}. For our purpose the following version of this theorem due to Phillips \cite{14} is most convenient.

**Theorem 3** (Lumer-Phillips Theorem for Contraction Semigroups). Let \( H \) be a closed linear operator defined on a dense linear subspace \( D(H) \) of a Hilbert space \( \mathcal{H} \). Moreover, assume that \(-iH\) is dissipative, i.e., that for all \( \psi \in D(H) \),

\[
\text{Re}(\langle \psi, -iH\psi \rangle_{\mathcal{H}}) \leq 0,
\]

and that \(-iH\) admits no dissipative extensions. Then \(-iH/\hbar\) generates a strongly continuous semigroup of contractions \( W_t = \exp(-iHt/\hbar) \) that preserves the domain of \( H \), i.e., \( W_t : D(H) \to D(H) \) for all \( t \geq 0 \). A sufficient condition for a closed dissipative operator \(-iH\) to admit no dissipative extensions is that \( iH^* \) is dissipative as well.

Our goal is to prove that the linear operator defined by equation (11) satisfies the assumptions of the Lumer-Phillips Theorem on the Hilbert space \( L^2(\Omega) \). We first treat \( C^2 \) boundaries and \( C^1 \kappa \), and afterward turn to the case of weaker assumptions on \( \partial \Omega \) and \( \kappa \).

#### 3.1 \( C^2 \) Boundaries

To begin with, we need to explain what it means to set \( \partial_n \psi = i\kappa \psi \) on \( \partial \Omega \). For generic \( \psi \in L^2(\Omega) \), there is no unambiguous way to define the restriction of the wave function or its derivative to a set of measure zero. Luckily, there is a classical result that provides a unique restriction map for wave functions residing in \( H^2(\Omega) \).

**Lemma 1.** \cite[Theorem 8.3]{7} For \( \Omega \subset \mathbb{R}^d \) a bounded \( C^2 \) domain, the restriction map \( \psi \mapsto (\psi|_{\partial \Omega}, \partial_n \psi|_{\partial \Omega}) \) defined for \( \psi \in C^\infty(\Omega) \to H^{3/2}(\partial \Omega) \times H^{1/2}(\partial \Omega) \) admits a continuous extension, also denoted \( \psi \mapsto (\psi|_{\partial \Omega}, \partial_n \psi|_{\partial \Omega}) \), that is surjective for \( H^2(\Omega) \to H^{3/2}(\partial \Omega) \times H^{1/2}(\partial \Omega) \) and admits a continuous right inverse.

This lemma gives rigorous meaning to the domain \( D(H) \) as in (11) of the operator \( H \). To prove \(-iH\) is maximally dissipative, we first note that \( H \) is a particular extension of a minimal operator \( H_0 := H|_{C^\infty(\Omega)} \). The operator \(-iH_0\) is skew-symmetric on its domain,
and it is well known that any dissipative extension of a skew-symmetric operator must be some restriction of the “maximal” operator \(-iH^*_0\), where \(H^*_0\) is the operator adjoint of \(H_0\). We are thus motivated to define the operator \(H_\kappa\)

\[
D(H_\kappa) := \{ \psi \in D(H^*_0) : \partial_n \psi = i\kappa \psi \text{ on } \partial \Omega \}, \quad H_\kappa \psi := H^*_0 \psi
\]

with the intention of later proving that \(H_\kappa = H\) and that this extension of \(H_0\) generates a \(C_0\) contraction semigroup. However, this definition for \(H_\kappa\) does not make sense unless we can define unambiguous restriction maps for wave functions residing in \(D(H^*_0)\).

**Lemma 2.** [2 Theorem 8.3.9 and Corollary 8.3.11] For \(\Omega \subset \mathbb{R}^d\) a bounded \(C^2\) domain, the restriction map \(\psi \mapsto (\psi|_{\partial \Omega} : \partial_n \psi|_{\partial \Omega})\) defined for \(\psi \in H^2(\Omega) \to H^{3/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)\) admits a continuous extension to \(D(H^*_0) \to H^{-1/2}(\partial \Omega) \times H^{-3/2}(\partial \Omega)\). In addition, for \(\psi \in D(H^*_0)\) and \(v \in H^2(\Omega)\), we have the integration by parts formula

\[
\langle v, -iH^*_0 \psi \rangle_{L^2(\Omega)} + \langle -iH^*_0 \psi, v \rangle_{L^2(\Omega)} = \frac{i\hbar}{2m} \int_{\partial \Omega} d^{d-1}x \left( v^* \partial_n \psi - (\partial_n v^*) \psi \right).
\]

Lastly, the kernels of these restriction operators are subsets of \(H^2(\Omega)\), i.e., if \(\psi \in D(H^*_0)\) satisfies the Dirichlet boundary condition \(\psi|_{\partial \Omega} = 0\) or the Neumann boundary condition \(\partial_n \psi|_{\partial \Omega} = 0\) then \(\psi \in H^2(\Omega)\).

For \(D(H_\kappa)\) to be well-defined, we now only need that multiplication by \(\kappa\) is well-defined from \(H^{-1/2}(\partial \Omega) \to H^{-3/2}(\partial \Omega)\). From the product rule we know that multiplication by a \(C^1\) function \(\kappa\) on a compact set such as \(\partial \Omega\) defines a bounded linear map from \(H^1(\partial \Omega) \to H^1(\partial \Omega)\). Let \(\kappa' : H^{-1}(\partial \Omega) \to H^{-1}(\partial \Omega)\) denote the Banach space adjoint of multiplication by \(\kappa\). We show that \(\kappa' \xi\) is simply \(\kappa \xi\) for \(\xi \in L^2(\partial \Omega)\), since \(\kappa' \xi\) acts on elements \(\chi \in H^1(\partial \Omega)\) according to

\[
\langle \chi, \kappa' \xi \rangle_{H^1(\partial \Omega) \times H^{-1}(\partial \Omega)} = \langle \kappa \chi, \xi \rangle_{H^1(\partial \Omega) \times H^{-1}(\partial \Omega)} = \int_{\partial \Omega} d^{d-1}x (\kappa \chi)^* \xi = \int_{\partial \Omega} d^{d-1}x \chi^*(\kappa \xi).
\]

Since \(L^2(\partial \Omega)\) is dense in \(H^{-1}(\partial \Omega)\), multiplication by \(\kappa\) extends uniquely to the bounded operator \(\kappa'\) on \(H^{-1}(\partial \Omega)\). Interpolation (see, e.g., [11 Theorems 5.1 and 7.7]) then implies that multiplication by \(\kappa\) defines bounded linear operators from \(H^s(\partial \Omega) \to H^s(\partial \Omega)\) for \(s \in [-1, 1]\), in particular \(\kappa : H^{-1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \subseteq H^{-3/2}(\partial \Omega)\) is bounded.

Now that we have a rigorous meaning for \(H_\kappa\), we can verify that it satisfies the conditions of the Lumer-Phillips theorem. To show \(H_\kappa\) is closed we suffice to show \(D(H_\kappa)\) is closed in the graph norm. Since \(D(H_\kappa)\) is the kernel of a bounded linear operator \(\psi \mapsto (\partial_n \psi|_{\partial \Omega} - i\kappa \psi|_{\partial \Omega})\) which maps \(D(H^*_0) \to H^{-3/2}(\partial \Omega)\), \(D(H_\kappa)\) is closed with respect to the graph norm of \(H^*_0\). But \(H_\kappa\) is a restriction of \(H^*_0\) so their graph norms are the same on this domain and it follows \(H_\kappa\) is closed.

To show that \(H_\kappa = H\), let \(\psi \in D(H_\kappa)\). Then \(\partial_n \psi|_{\partial \Omega} = i\kappa \psi|_{\partial \Omega} \in H^{-1/2}(\partial \Omega)\). It is known [9 Lemma 3.2] that \(\psi \mapsto \partial_n \psi|_{\partial \Omega}\) is a surjective operator from \(H^{1/2}(\Omega) \cap \)
\[ D(H^*_\kappa) \to H^{-1}(\partial\Omega) \] (and also from \( H^{3/2}(\Omega) \cap D(H^*_\kappa) \to L^2(\partial\Omega) \)), so there exists some \( \phi \in H^{1/2}(\Omega) \cap D(H^*_\kappa) \) such that \( \partial_n(\psi - \phi) \big|_{\partial\Omega} = 0 \). So \( \psi - \phi \in H^2(\Omega) \) which implies \( \psi \in H^{1/2}(\Omega) \). We may now repeat the same argument: \( \partial_n\psi \big|_{\partial\Omega} = i\kappa\psi \big|_{\partial\Omega} \in L^2(\Omega) \), so there exists some \( \phi \in H^{3/2}(\Omega) \cap D(H^*_\kappa) \) such that \( \partial_n(\psi - \phi) \big|_{\partial\Omega} = 0 \), which returns \( \psi \in H^{3/2}(\Omega) \). Repeating the argument once more but now applying Lemma [1] returns \( \psi \in H^2(\Omega) \) as desired. We henceforth refer to \( H_\kappa \) as \( H \).

We may now show that \( H \) is dissipative by applying the integration by parts formula \([23]\). For \( \psi \in D(H) \), we have that

\[
2 \operatorname{Re} \langle \psi, -iH\psi \rangle_{L^2(\Omega)} = 2 \operatorname{Re} \langle \psi, -iH^*_0\psi \rangle_{L^2(\Omega)} = \frac{i\hbar^2}{2m} \int_{\partial\Omega} d^{d-1}x \left( \psi^* \partial_n\psi - (\partial_n\psi^*) \psi \right) = -\frac{\hbar^2}{m} \int_{\partial\Omega} d^{d-1}x \kappa(x) |\psi|^2(x) \leq 0
\]

since \( \kappa(x) \geq 0 \) on \( \partial\Omega \). To prove \( -iH \) admits no dissipative extensions, it is sufficient to show that its adjoint \( iH^* \) is also dissipative. We prove this by deriving an explicit expression for \( D(H^*) \). We first note that since \( H \) extends the minimal operator \( H_0 \), its adjoint \( H^* \) is extended by the maximal operator \( H^*_\kappa \). By definition of the operator adjoint, \( \phi \in D(H^*_0) \) is an element of \( D(H^*) \) if and only if

\[
\langle \psi, -iH^*_0\phi \rangle_{L^2(\Omega)} + \langle -iH\psi, \phi \rangle_{L^2(\Omega)} = 0
\]

for all \( \psi \in D(H) \). Applying the integration by parts formula, we see that this holds if and only if

\[
\int_{\partial\Omega} d^{d-1}x \left( \psi^* \partial_n\phi - (\partial_n\psi^*) \phi \right) = \int_{\partial\Omega} d^{d-1}x \psi^* (\partial_n\phi + i\kappa\phi) = 0
\]

for all \( \psi \in D(H) \). Split surjectivity of the restriction operators implies that for every \( \xi \in H^{3/2}(\partial\Omega) \) there exists a wave function \( \psi \in H^2(\Omega) \) with \( \langle \psi \big|_{\partial\Omega}, \partial_n\psi \big|_{\partial\Omega} \rangle = (\xi, i\kappa\xi) \). In other words, for every \( \xi \in H^{3/2}(\partial\Omega) \) there exists a \( \psi \in D(H) \) with \( \psi \big|_{\partial\Omega} = \xi \). It follows that \( \phi \in D(H^*) \) if and only if

\[
\int_{\partial\Omega} d^{d-1}x \xi^* (\partial_n\phi + i\kappa\phi) = 0
\]

for all \( \xi \in H^{3/2}(\partial\Omega) \), which is a dense subspace, so the domain of the adjoint must be given by

\[
D(H^*) = \{ \phi \in D(H^*_\kappa) : \partial_n\phi = -i\kappa\phi \text{ on } \partial\Omega \}.
\]

One can now repeat the same steps to prove \( D(H^*) \subset H^2(\Omega) \) and apply the integration by parts formula to show dissipativity of \( iH^* \). This completes the proof of Theorem \([1]\) for \( C^2 \) boundaries and \( C^1 \kappa \). We now turn to the case of weaker assumptions.
3.2 Lipschitz Boundaries

The extension to the Lipschitz boundary case requires a significant amount of technical machinery. Fortunately, the theory of closed extensions of \((-\nabla^2 + V)|_{C^\infty_0(\Omega)}\) on Lipschitz domains has received a lot of attention in recent years, and the result that we seek is already known to the literature. We begin by again defining the \textit{minimal operator} as \(H_0 := \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)|_{C^\infty_0(\Omega)}\) and stating the relevant notion of restriction to Lipschitz boundaries \(\partial\Omega\).

\textbf{Lemma 3.} \[3\] \textit{Lemma 3.1 and Lemma 3.2} For \(\Omega \subset \mathbb{R}^d\) a bounded Lipschitz domain, the restriction map \(\psi \mapsto (\psi|_{\partial\Omega}, \partial_n \psi|_{\partial\Omega})\) defined for \(\psi \in C^\infty(\Omega) \to H^1(\partial\Omega) \times L^2(\partial\Omega)\) admits a continuous extension \(\psi \mapsto (\psi|_{\partial\Omega}, \partial_n \psi|_{\partial\Omega})\) for \(D(H_0^*) \cap H^{3/2}(\Omega) \to H^1(\partial\Omega) \times L^2(\partial\Omega)\).

We introduce \(H_\kappa\) as the extension of \(H_0\) consisting of wave functions satisfying the absorbing boundary condition

\[ D(H_\kappa) := \{ \psi \in D(H_0^*) \cap H^{3/2}(\Omega) : \partial_n \psi = i\kappa \psi \text{ on } \partial\Omega \}, \quad H_\kappa \psi := H_0^* \psi. \tag{32} \]

One also has an integration by parts formula for elements of \(H_\kappa\).

\textbf{Lemma 4.} \[4, \text{Theorem 4.1}\] \textit{For } \(\Omega \subset \mathbb{R}^d\) \textit{a bounded Lipschitz domain and } \(\psi, v \in D(H_0^*) \cap H^{3/2}(\Omega)\), \textit{we have the integration by parts formula}

\[ \langle v, -iH_0^* \psi \rangle_{L^2(\Omega)} + \langle -iH_0^* v, \psi \rangle_{L^2(\Omega)} = \frac{i\hbar^2}{2m} \int_{\partial\Omega} d^{d-1}x \left( v^* \partial_n \psi - (\partial_n v^*) \psi \right). \tag{33} \]

\textbf{Theorem 4.} \[4, \text{Theorem 4.7 and Corollary 4.8} \[3, \text{Theorem 6.21}\] \textit{Let } \(\Omega \subset \mathbb{R}^d\) \textit{be a bounded Lipschitz domain and } \(\kappa : \partial\Omega \to [0, \infty)\) \textit{in } \(L^\infty(\partial\Omega)\). \textit{Then the operator } -iH_\kappa \textit{defined by (32) is maximally dissipative. Hence, } -iH_\kappa / \hbar \textit{generates a strongly continuous semigroup of contractions } W_t = \exp(-iH_\kappa t / \hbar) \textit{that preserves the domain of } H_\kappa, \textit{i.e.,}

\( W_t : D(H_\kappa) \to D(H_\kappa) \) \textit{for all } \(t \geq 0\).

\textbf{Remark 2.} \textit{Theorem 4.7 and Corollary 4.8 of } \[4\] \textit{apply to a different set of extensions of } \(H_0\), \textit{those corresponding to the boundary condition } \(\partial_n \psi|_{\partial\Omega} = \nu \psi|_{\partial\Omega}\) \textit{for } \(\nu \in L^\infty(\partial\Omega, \mathbb{R})\). \textit{However, Theorem 6.21 of } \[3\] \textit{and the remark following it imply a generalization of the work in } \[4\] \textit{to boundary conditions of the form } \(\partial_n \psi|_{\partial\Omega} = (i\kappa + \nu) \psi|_{\partial\Omega}\) \textit{with } \(\nu \in L^\infty(\partial\Omega, \mathbb{R}), \kappa \in L^\infty(\partial\Omega, [0, \infty))\).

\textit{We should also note that the terminology used by the authors of } \[4, 3\] \textit{is slightly different than ours. They say an operator } \(H\) \textit{is} accumulative \textit{if } \text{Im}(H \psi, \psi)_{H} \leq 0 \textit{while taking the first slot in the inner product linear and the second slot anti-linear. This is equivalent to saying } -iH \textit{is dissipative in our notation where the first slot of the inner product is anti-linear and the second slot is linear.} \quad \diamond \]

\textit{Now all statements of Theorem } \[4\text{ follow.}
4 Remarks

Remark 3. For $\psi_0 \in D(H)$, the Lumer-Phillips theorem provides the existence of a solution $\psi_t \in H^2(\Omega)$ for $t \geq 0$ to the initial boundary value problem (9). The uniqueness of this solution is straightforward. Let $\psi_0 \in H^2(\Omega), \psi'_t \in H^2(\Omega)$ be a solution to (9) for $t \geq 0$. By linearity $\phi_t := W_t \psi_0 - \psi'_t$ solves (9) with initial condition $\phi_0 = 0$. But the boundary condition implies $\frac{d}{dt} ||\phi_t||_{L^2(\Omega)} \leq 0$ for all $t \geq 0$, hence $\phi_t = 0$ for all time and the two solutions are equal.

Remark 4. Suppose we replace $i\kappa$ in the boundary condition (2) by $\nu + i\kappa$ with $\nu \in C^1(\partial \Omega, \mathbb{R})$, so that (2) becomes

$$\frac{\partial \psi}{\partial n}(x) = (\nu(x) + i\kappa(x))\psi(x).$$

Then this boundary condition is still absorbing, i.e., one that forces the current to point outward. One can repeat the proof of Theorem 1 with this boundary condition to show this also generates a contraction semigroup.

Remark 5. Unlike self-adjoint Hamiltonians, $H$ is not unitarily diagonalizable when $\kappa(x) > 0$ on a set of $\mathbb{R}$s of positive measure in $\partial \Omega$, as we prove below. (We note also that the Hamiltonian of the discrete version of the absorbing boundary rule for a quantum particle on a lattice is easily checked to be non-normal ($HH^* \neq H^*H$), and thus not unitarily diagonalizable.) It seems that, at least in many cases, a complete set of (generalized, non-normalizable) eigenfunctions exists, but they are not mutually orthogonal.

Recall that an operator $A$ in $\mathcal{H}$ is unitarily diagonalizable if and only if there is a generalized orthonormal basis, i.e., a unitary isomorphism $U : \mathcal{H} \to L^2(S, \mu)$ for some measure space $(S, \mu)$, such that $M = UAU^{-1}$ is the multiplication operator by some function $f : S \to \mathbb{C}$. The domain $D(M)$ on which the graph of $M$ is closed is given by

$$D(M) = \left\{ \psi \in L^2(S, \mu) : \int_S |f(s)\psi(s)|^2 \mu(ds) < \infty \right\}.$$

Since the adjoint $T^*$ of any operator $T$ with domain $D(T)$ is defined on the domain

$$D(T^*) = \left\{ \psi \in \mathcal{H} : \exists \phi \in \mathcal{H} : \forall \chi \in D(T) : \langle \psi | T \chi \rangle = \langle \phi | \chi \rangle \right\}$$

(and given there by $T^*\psi = \phi$), the adjoint $M^*$ of a multiplication operator $M$ has domain $D(M^*) = D(M)$ and is given there by multiplication by $f^*$. When $\kappa(x) > 0$ on a set of positive measure, then, as the proof of Theorem 1 has shown, $H$ has domain $D(H)$ different from $D(H^*)$, see (31), while the graph of $H$ is closed, so it follows that $H$ cannot be unitarily diagonalizable.
5 Proof of Corollary 1

For any \( \psi_0 \in D(H) \), also \( \psi_t = \exp(-iHt/\hbar)\psi_0 \) lies in \( D(H) \). Moreover, for any \( \psi \in D(H) \), \( n(x) \cdot j_\psi(x) = (\hbar \kappa(x)/m)|\psi(x)|^2 \) on \( \partial \Omega \), and the restriction of \( \psi \) to \( \partial \Omega \) is well defined as an element of \( L^2(\partial \Omega, d^{d-1}x) \). It follows that for \( \psi_0 \in D(H) \) with \( \|\psi_0\| = 1 \), (4) and (6) together define a probability distribution on \( [0, \infty) \times \partial \Omega \cup \{\infty\} \).

Now define, for \( \psi_0 \in D(H) \), \( J\psi_0 \) to be the function on \( [0, \infty) \times \partial \Omega \) such that \( J\psi_0(t, \cdot) \) is \( \sqrt{\hbar \kappa(x)/m} \) times the restriction of \( \psi_t \) to \( \partial \Omega \). Then \( J\psi_0 \in L^2([0, \infty) \times \partial \Omega, dt \, d^{d-1}x) \), and

\[
\|J\psi_0\|^2 = \frac{\hbar}{m} \int_0^\infty dt \int_{\partial \Omega} d^{d-1}x \kappa(x) |\psi_t(x)|^2 = \|\psi_0\|^2 - \lim_{t \to \infty} \|\psi_t\|^2 \leq \|\psi_0\|^2. \tag{37}
\]

(Note that \( \lim_{t \to \infty} \|\psi_t\|^2 \) exists because \( t \mapsto \|\psi_t\|^2 \) is a non-negative, decreasing function.) The fact \( \|J\psi_0\| \leq \|\psi_0\| \) means that \( J : D(H) \to L^2([0, \infty) \times \partial \Omega, dt \, d^{d-1}x) \) is a bounded operator with operator norm no greater than 1 (i.e., a contraction). Thus, \( J \) possesses a unique bounded extension to \( L^2(\Omega) \), which we will also denote by \( J \).

For arbitrary \( \psi_0 \in L^2(\Omega) \) (outside \( D(H) \)) with \( \|\psi_0\| = 1 \), \( |J\psi_0(t, x)|^2 \) is the joint probability density of \( T \) and \( X \), and \( 1 - \|J\psi_0\|^2 = \text{Prob}_{\psi_0}(Z = \infty) \); that is, the distribution of \( Z \) is well defined. The POVM \( E \) is given on \( [0, \infty) \times \partial \Omega \) by

\[
E(\cdot) = J^* P(\cdot) J, \tag{38}
\]

where \( P \) is the natural PVM (projection-valued measure) on \( L^2([0, \infty) \times \partial \Omega, dt \, d^{d-1}x) \). (The natural PVM on \( L^2(S, \mu) \) associates with every measurable subset \( B \) of a measure space \( (S, \mu) \) the projection to the subspace consisting of the functions vanishing outside \( B \).) The definition of \( E \) is completed by setting \( E(\{\infty\}) = I - J^* J \), which is a positive operator by (57). It follows that \( E([0, \infty) \times \partial \Omega \cup \{\infty\}) = I \), so \( E \) is a POVM, and that (13) agrees with (4) and (6) for \( \psi_0 \in D(H) \). It also follows that \( E(\{\infty\}) = \lim_{t \to \infty} W_t^* W_t \) because \( W_t^* W_t = E([t, \infty) \times \partial \Omega \cup \{\infty\}) \).

Concerning the last sentence of Corollary 1 the absolute continuity of the measure \( \langle \psi_0 | E(\cdot) | \psi_0 \rangle \) on \( [0, \infty) \times \partial \Omega \) is visible from the fact that \( |J\psi_0(t, x)|^2 \) is its density.

6 Several Particles

The main new issue about the case of several particles is whether the collapsed wave function \( \psi' \) in (8) is well defined. To this end, we begin with some general considerations about conditional wave functions.

6.1 Conditional Wave Functions

Let \( S(H) \) denote the unit sphere of the Hilbert space \( H \), which we consider with its Borel \( \sigma \)-algebra.
In general, for a wave function $\psi(a,b)$ of two variables $a, b$, the conditional wave function $\psi'$ is defined as follows: insert for $a$ a random value $A$ whose distribution is the appropriate marginal of $|\psi|^2$, and then normalize. Thus, $\psi'$ is a random function of the single variable $b$. More generally, we can consider a function $\psi(a)$ with values in some Hilbert space $\mathcal{H}_b$ (including the special case that $\mathcal{H}_b$ are the $L^2$ functions of the variable $b$), and then we want that $\psi'$ is a random variable with values in $S(\mathcal{H}_b)$. One question that arises is whether $\psi(A)$ might be 0; we show that this happens with probability 0. Another question arises from the fact that an element of an $L^2$ space is strictly speaking not a function but an equivalence class of functions that can differ on a set of measure 0; we show that $\psi'$ is “almost uniquely” defined. This is done in the next theorem.

**Theorem 5.** Let $\mathcal{A}$ be a measure space such that $L^2(\mathcal{A})$ is separable, $\mathcal{H}_b$ another separable Hilbert space, $\psi \in L^2(\mathcal{A}, \mathcal{H}_b) = L^2(\mathcal{A}) \otimes \mathcal{H}_b$ with $\|\psi\| = 1$, $\tilde{\psi} : \mathcal{A} \rightarrow \mathcal{H}_b$ a representative of $\psi$, and $A$ an $\mathcal{A}$-valued random variable with distribution density $\|\tilde{\psi}(a)\|_b^2$. Then $\psi' := \tilde{\psi}(A)/\|\tilde{\psi}(A)\|_b$ is almost surely well defined as an element of $S(\mathcal{H}_b)$, and the probability distribution of the pair $(A, \psi')$ in $\mathcal{A} \times S(\mathcal{H}_b)$ does not depend on the choice of $\tilde{\psi}$.

**Proof.** We first verify that $L^2(\mathcal{A}, \mathcal{H}_b) = L^2(\mathcal{A}) \otimes \mathcal{H}_b$ in the sense that they are canonically isomorphic; the argument is a variant of one in [13, p. 51]. Given ONBs $\{\alpha_i\}$ of $L^2(\mathcal{A})$ and $\{\beta_j\}$ of $\mathcal{H}_b$, the functions $\gamma_{ij}(a) = \alpha_i(a) \beta_j$ lie in $L^2(\mathcal{A}, \mathcal{H}_b)$ and are orthonormal. To see that they form an ONB, suppose that $f \in L^2(\mathcal{A}, \mathcal{H}_b)$ and

$$0 = \langle f | \gamma_{ij} \rangle_{L^2(\mathcal{A}, \mathcal{H}_b)}$$

$$= \int_{\mathcal{A}} da \langle f(a) | \gamma_{ij}(a) \rangle_b$$

$$= \int_{\mathcal{A}} da \alpha_i(a) \langle f(a) | \beta_j \rangle_b.$$  

Since $\{\alpha_i\}$ is an ONB, it follows that $0 = \langle f(a) | \beta_j \rangle_b$ for almost every $a$, and since $\{\beta_j\}$ is an ONB and countable that $f(a) = 0$ for almost every $a$.

Therefore, $\gamma_{ij} \mapsto \alpha_i \otimes \beta_j$ maps an ONB to an ONB, so its unique continuous linear extension is a unitary isomorphism from $L^2(\mathcal{A}, \mathcal{H}_b)$ to $L^2(\mathcal{A}) \otimes \mathcal{H}_b$.

Now pick any function $\tilde{\psi}$ belonging to the equivalence class of functions that $\psi$ is and let $A$ be a random variable taking values in $\mathcal{A}$ with $|\tilde{\psi}|^2$ distribution, i.e.,

$$\text{Prob}(A \in S) = \int_S da \|\tilde{\psi}(a)\|_b^2.$$ (42)

for all measurable subsets $S$ of $\mathcal{A}$. For the purpose of inserting $A$, we first leave normalization aside and set $\psi_* := \tilde{\psi}(A)$. If we had picked another function $\tilde{\psi}$ instead of $\tilde{\psi}$, then $\hat{\psi}$ would differ from $\tilde{\psi}$ on a set of measure 0 in $\mathcal{A}$. Thus, the distribution $\hat{\psi}(A) = \psi(A)$ with probability 1. Thus, the distribution of $(A, \psi_*)$ is a well-defined probability measure in $\mathcal{A} \times \mathcal{H}_b$. 

14
Next we focus on normalization: $A$ has probability 1 to be such that the norm in $\mathcal{H}_b$ of $\psi_*$ is non-zero. After all, $A$ has probability 0 by (12) to lie in the set of $a$ values with $\|\psi(a)\|^2_b = 0$. Thus, $\psi_*$ can be normalized, i.e., $\mathcal{N} := 1/\|\psi_*\|$ and $\psi' = \mathcal{N}\psi_*$ exists. Since in any Hilbert space $\mathcal{H}$ the normalization mapping $\mathcal{H} \setminus \{0\} \to \mathcal{S}(\mathcal{H})$, $\phi \mapsto \phi/\|\phi\|$ is continuous, it is Borel-measurable. Thus, the distribution of $(A,\psi')$ is defined on $A \times \mathcal{S}(\mathcal{H}_b)$ and is independent of the choice of $\psi$ within the equivalence class that is $\psi$.

\begin{remark}
For $\mathcal{H}_b = L^2(\mathcal{B})$ for some measure space $\mathcal{B}$, we obtain, since $L^2(A) \otimes L^2(\mathcal{B}) = L^2(A \times \mathcal{B})$ with the product measure, that from an $L^2$ function $\psi(a,b)$ we can form the conditional wave function $\psi'(b) = \mathcal{N}\psi(A,b)$, and the pair $(A,\psi')$ has a well-defined probability distribution in $A \times \mathcal{S}(L^2(\mathcal{B}))$.
\end{remark}

\begin{remark}
For a function $\psi(a,b,c)$ of three variables, first conditioning on the $a$ variable (i.e., inserting $A$ and normalizing) to obtain $\psi'(b,c)$ and then conditioning on the $b$ variable to obtain $\psi''(c)$ yields the same distribution for $(A,B,\psi'')$ as first conditioning on the $b$ variable and then on $a$, and the same distribution as conditioning on the pair $(a,b)$.
\end{remark}

\begin{remark}
Theorem 5 remains true when we replace the fixed Hilbert space $\mathcal{H}_b$ by a measurable bundle $\mathcal{H}_b(a)$ of Hilbert spaces, that is, if $\mathcal{H}_b$ depends on $a \in A$; then (a representative of) $\psi$ is a measurable cross-section of this bundle.
\end{remark}

The next theorem is concerned with an experiment with POVM $E$ done on a conditional wave function and asserts that the joint distribution of the outcome $B$ of the experiment and the $A$ on which we conditioned is defined by a product POVM. We begin with the definition of the latter: For any two POVMs $E, F$ on measurable spaces $A, B$ acting on Hilbert spaces $\mathcal{H}_a, \mathcal{H}_b$, a product POVM is a POVM $G$ on $A \times B$ acting on $\mathcal{H}_a \otimes \mathcal{H}_b$ such that for any measurable subsets $A, B$,

$$G(A \times B) = E(A) \otimes F(B).$$

(43)

Note that on subsets $C$ of $A \times B$, $G(C)$ does not have to be a tensor product. If $G$ exists and is unique, we write $E \otimes F$ for $G$.

\begin{proposition}
The product POVM is unique if $\mathcal{H}_a$ and $\mathcal{H}_b$ are separable.
\end{proposition}

\begin{proof}
For $\alpha \in \mathcal{H}_a$ and $\beta \in \mathcal{H}_b$,

$$\langle \alpha \otimes \beta | E(A) \otimes F(B) | \alpha \otimes \beta \rangle = \langle \alpha | E(A) | \alpha \rangle_a \langle \beta | F(B) | \beta \rangle_b$$

(44)

is a product of finite measures and thus always extends uniquely to a measure on $A \times B$. Likewise for $\alpha' \neq \alpha$ and $\beta' \neq \beta$, $\langle \alpha' \otimes \beta' | E(A) \otimes F(B) | \alpha \otimes \beta \rangle$ extends uniquely to a complex measure on $A \times B$. By separability, $\psi \in \mathcal{H}_a \otimes \mathcal{H}_b$ can be written as a countable series $\psi = \sum_{ij} c_{ij} \alpha_i \otimes \beta_j$ using ONBs, so the complex measure $\langle \psi' | G(C) | \psi \rangle$ can be expanded into a convergent series of complex measures and therefore is determined uniquely (while the convergence of the series would not be obvious if nothing is known about the existence of $G$).
\end{proof}
We conjecture that $G$ exists for any two POVMs, but as far as we know this has been proved only under the additional assumption that $\mathcal{A}$ and $\mathcal{B}$ are standard Borel spaces; see Corollary 7 in [6]. However, we do not need this assumption for our result:

**Theorem 6.** Let $\mathcal{A}$ be a measure space such that $L^2(\mathcal{A})$ is separable, and let $P$ be the natural PVM on $\mathcal{A}$ acting on $L^2(\mathcal{A})$. Let $\mathcal{H}_b$ be another separable Hilbert space and $E$ a POVM on the measurable space $\mathcal{Z}$ acting on $\mathcal{H}_b$. Then the product POVM $P \otimes E$ exists, and if a quantum experiment with POVM $E$ and random outcome $Z$ is done on the conditional wave function $\psi'$, the joint distribution of $(A, Z)$ is $\langle \psi | P \otimes E | \psi \rangle$.

**Proof.** The distribution of $A$ is $\langle \psi | P \otimes I | \psi \rangle$. Given $A$ and thus also $\psi'$, the distribution of $Z$ is $\langle \psi' | E | \psi' \rangle$. Thus, a joint distribution exists and is given by, for any measurable $S \subseteq \mathcal{A} \times \mathcal{Z}$ and using the notation $S_a = \{ z \in \mathcal{Z} : (a, z) \in S \}$,

$$\text{Prob}_{\psi}( (A, Z) \in S ) = \int_{\mathcal{A}} da \| \psi(a) \|^2 \left( \frac{\psi(a)}{\| \psi(a) \|} \right) E(S_a) \frac{\psi(a)}{\| \psi(a) \|} \right)_{b}$$

(45)

$$= \int_{\mathcal{A}} da \langle \psi(a) | E(S_a) | \psi(a) \rangle_{b}$$

(46)

(independently of the choice of representative of $\psi$). Since for fixed $S$, this is a bounded quadratic form in $\psi$, it is $\langle \psi | G(S) | \psi \rangle$ for some operator $G(S)$, and $\sigma$-additivity of probabilities implies weak $\sigma$-additivity of $G(S)$, which is sufficient for a POVM. For $S = S' \times S''$, this reduces to $\langle \psi | P(S') \otimes E(S'') | \psi \rangle$, so $G$ is the product POVM $P \otimes E$. □

### 6.2 Proof of Corollary 2

We apply Theorem 1 replacing $d \to Nd$ and $\Omega \to \Omega^N$ and obtain that the time evolution of $\psi$ in $\Omega^N$ exists for all $t > 0$.

Let $J_N$ denote the $J$ operator of Section 5, defined now for the $Nd$-dimensional case; $J_N \psi_0$ is for arbitrary $\psi_0 \in L^2(\Omega^N)$ a well-defined element of

$$L^2([0, \infty) \times \partial(\Omega^N)) \cong \oplus_i L^2([0, \infty) \times F_i)$$

$$\cong L^2([0, \infty) \times \partial \Omega \times \{ 1, \ldots, N \} \times \Omega^{N-1}) ,$$

(47)

(48)

where $\cong$ means isometrically isomorphic, and the last isomorphism is defined by the permutation $p$ of the variables as in (17). A point in the “configuration space” is now $(T^1, X^1, I^1, x')$, and the joint density of $T^1, X^1, I^1$ is the appropriate marginal of $|J_N \psi_0|^2$ (ignoring the other $x_j$’s). Corollary 2 follows by repeating the same steps as in the proof of Corollary 1.

---

2A measurable space is called a standard Borel space if and only if it is isomorphic as a measurable space to a complete separable metric space with its Borel $\sigma$-algebra. This is not a strong restriction as it includes most spaces that one considers in practice, such as countable unions of closed subsets of manifolds.
6.3 Proof of Theorem 2

Replacing \( \{1, \ldots, N\} \) by \( \mathcal{F} \) with \( N \) elements, we obtain operators \( J_{\mathcal{F}} \) and the permutation function \( p : \cup_{i \in \mathcal{F}} F_i \to \cup_{i \in \mathcal{F}} \partial \Omega \wedge \Omega_{\mathcal{F}\setminus\{i\}} \) now given by \( p(x) = (x_i, (x_j : j \in \mathcal{F} \setminus \{i\})) \) whenever \( x \in F_i \). Further, we define the unitary isomorphism

\[
U_p : L^2 \left( [0, \infty) \times \partial (\Omega_{\mathcal{F}}) \right) \to \bigoplus_{i \in \mathcal{F}} L^2 \left( [0, \infty) \times \partial \Omega \times \Omega_{\mathcal{F}\setminus\{i\}} \right)
\]

(49)

\[
U_p(t, x) = (t, p(x))
\]

(50)
in analogy to (48). The desired \( \psi' \) of (3) then is the conditional wave function of \( U_p J_{\mathcal{F}} \psi_0 \) as in Theorem 6 with Remark 8 with \( \mathcal{F} = [0, \infty) \times \partial \Omega \wedge \mathcal{F}, A = (T^1, X^1, I^1), \mathcal{H}_b(i) = L^2(\Omega_{\mathcal{F}\setminus\{i\}}) \), and the \( b \) variable corresponds to \( x' = (x_j : j \in \mathcal{F} \setminus \{i\}) \). Given that \( Z \neq \infty, \psi' \) is with probability 1 a well-defined element of \( \mathcal{H}_b(i) \) with \( i = I^1 \).

Now we iterate the procedure, apply \( \psi' \) the Schrödinger equation for the \( N - 1 \) particles of \( \mathcal{F}' = \mathcal{F} \setminus \{i\} \) with potential \( V_{\mathcal{F}'} \), and the absorbing boundary condition (7); if no detection occurs, we write \( Z^2 = \infty = Z^3 = \ldots = Z^N \), otherwise \( Z^2 = (\Delta T^2, X^2, I^2) \) with \( \Delta T^2 \) the waiting time until detection; then repeat.

It remains to show that the distribution of \( Z = (Z^1, \ldots, Z^N) \) comes from a POVM \( E_{\mathcal{F}} \). This follows from Theorem 6 and \( E_{\mathcal{F}} \) can be specified recursively by setting, for \( \mathcal{F} \subseteq \mathcal{F} \),

\[
E_{\mathcal{F}}(\{(\infty \ldots \infty)\}) = I - J_{\mathcal{F}}^* J_{\mathcal{F}}
\]

(51)

\[
E_{\mathcal{F}}(B) = J_{\mathcal{F}}^* U_p^* \left( \bigoplus_{i \in \mathcal{F}} \left( P_{[0, \infty) \times \partial \Omega \setminus \{i\}} \otimes E_{\mathcal{F}\setminus\{i\}} \right) \right)(B) U_p J_{\mathcal{F}}
\]

(52)

for any measurable \( B \) outside the sequences starting with \( \infty \), that is, for \( B \subseteq ([0, \infty) \times \partial \Omega \wedge \mathcal{F}) \times ([0, \infty) \times \partial \Omega \wedge \mathcal{F} \cup \{\infty\})^{\#\mathcal{F}-1} \). Here, \( P_{\mathcal{F}} \) is the natural PVM on \( \mathcal{F} \). The end of the recursion is that, for every 1-element set \( \mathcal{F} = \{j\} \),

\[
E_{\{j\}}(\{(\infty)\}) = I - J_{\{j\}}^* J_{\{j\}}
\]

(53)

\[
E_{\{j\}}(B) = J_{\{j\}}^* \left( P_{[0, \infty) \times \partial \Omega}(B) \right) J_{\{j\}}
\]

(54)

for \( B \subseteq [0, \infty) \times \partial \Omega \).

Acknowledgments. We thank Sascha Eichmann and Julian Schmidt for helpful discussions and two anonymous referees for pointing out errors in an earlier version of this article.

Declarations

Funding. This work received no funding.
Conflict of interests. The authors declare no conflict of interest.
Availability of data and material. Not applicable.
Code availability. Not applicable.
References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden: *Solvable models in quantum mechanics*. Berlin: Springer (1988)

[2] J. Behrndt, S. Hassi, and H. de Snoo: *Boundary Value Problems, Weyl Functions, and Differential Operators*. Cham: Birkhäuser (2020)

[3] J. Behrndt and M. Langer: Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples. Pages 121–160 in S. Hassi, H. de Snoo, and F. Szafraniec (editors): *Operator Methods for Boundary Value Problems*, Cambridge University Press (2012)

[4] J. Behrndt and T. Micheler: Elliptic differential operators on Lipschitz domains and abstract boundary value problems. *Journal of Functional Analysis* **267**(10): 3657–3709 (2014)

[5] V. Dubey, C. Bernardin, and A. Dhar: Quantum dynamics under continuous projective measurements: non-Hermitian description and the continuum space limit. *Physical Review A* **103**: 032221 (2021) [http://arxiv.org/abs/2012.01196](http://arxiv.org/abs/2012.01196)

[6] D. Dürr, S. Goldstein, R. Tumulka, and N. Zanghì: Quantum Hamiltonians and Stochastic Jumps. *Communications in Mathematical Physics* **254**: 129–166 (2005) [http://arxiv.org/abs/quant-ph/0303056](http://arxiv.org/abs/quant-ph/0303056)

[7] K.-J. Engel and R. Nagel: *One-parameter semigroups for linear evolution equations*. Berlin: Springer (2000)

[8] L. Frolov, S. Leigh, and S. Tahvildar-Zadeh: On the relativistic quantum mechanics of a photon between two electrons in 1+1 dimensions. *Letters in Mathematical Physics* **115**: 9 (2025) [http://arxiv.org/abs/2312.06019](http://arxiv.org/abs/2312.06019)

[9] F. Gesztesy and M. Mitrea: A description of all self-adjoint extensions of the Laplacian and Krein-type resolvent formulas on non-smooth domains. *Journal d’Analyse Mathématique* **113**: 53–172 (2011)

[10] S. Goldstein, R. Tumulka, and N. Zanghì: Arrival Times Versus Detection Times. *Foundations of Physics* **54**: 63 (2024) [http://arxiv.org/abs/2405.04607](http://arxiv.org/abs/2405.04607)

[11] J.L. Lions and E. Magenes: *Non-Homogeneous Boundary Value Problems and Applications*. Berlin, Heidelberg: Springer (1972)

[12] G. Lumer and R.S. Phillips: Dissipative operators in a Banach space. *Pacific Journal of Mathematics* **11**(2): 679–698 (1961)

[13] J.G. Muga and R. Leavens: Arrival Time in Quantum Mechanics. *Physics Reports* **338**: 353 (2000)
[14] R.S. Phillips: Dissipative operators and hyperbolic systems of partial differential equations. *Transactions of the American Mathematical Society* **90**(2): 193–254 (1959)

[15] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*, revised and enlarged edition. San Diego: Academic Press (1980)

[16] S. Tahvildar-Zadeh and S. Zhou: Detection Time of Dirac Particles in One Space Dimension. Pages 187–201 in A. Bassi et al. (editors), *Physics and the Nature of Reality*, Cham: Springer (2024) [http://arxiv.org/abs/2112.07366](http://arxiv.org/abs/2112.07366)

[17] R. Tumulka: Distribution of the Time at Which an Ideal Detector Clicks. *Annals of Physics* **442**: 168910 (2022) [http://arxiv.org/abs/1601.03715](http://arxiv.org/abs/1601.03715)

[18] R. Tumulka: Detection Time Distribution for Several Quantum Particles. *Physical Review A* **106**: 042220 (2022) [http://arxiv.org/abs/1601.03871](http://arxiv.org/abs/1601.03871)

[19] R. Tumulka: Detection Time Distribution for the Dirac Equation. Preprint (2016) [http://arxiv.org/abs/1601.04571](http://arxiv.org/abs/1601.04571)

[20] R. Tumulka: Absorbing Boundary Condition as Limiting Case of Imaginary Potentials. *Communications in Theoretical Physics* **75**: 015103 (2023) [http://arxiv.org/abs/1911.12730](http://arxiv.org/abs/1911.12730)

[21] R. Tumulka: On a Derivation of the Absorbing Boundary Rule. *Physics Letters A* **494**: 129286 (2024) [http://arxiv.org/abs/2310.01343](http://arxiv.org/abs/2310.01343)

[22] R. Werner: Arrival time observables in quantum mechanics. *Annales de l’Institute Henri Poincaré, section A* **47**: 429–449 (1987)