STOKES MATRICES OF A REDUCIBLE EQUATION WITH TWO IRREGULAR SINGULARITIES OF POINCARÉ RANK 1 VIA MONODROMY MATRICES OF A REDUCIBLE HUEN TYPE EQUATION

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Abstract. We consider a second order reducible equation having non-resonant irregular singularities at \( x = 0 \) and \( x = \infty \). Both of them are of Poincaré rank 1. We introduce a small complex parameter \( \varepsilon \) that splits together \( x = 0 \) and \( x = \infty \) into four different Fuchsian singularities \( x_L = -\sqrt{\varepsilon}, x_R = \sqrt{\varepsilon} \), and \( x_{LL} = -\frac{1}{\sqrt{\varepsilon}}, x_{RR} = \frac{1}{\sqrt{\varepsilon}} \), respectively. The perturbed equation is a second order reducible Fuchsian equation with 4 different singularities, i.e. a Heun type equation. Then we prove that when the perturbed equation has exactly two resonant singularities of different type, all the Stokes matrices of the initial equation are realized as a limit of the nilpotent parts of the monodromy matrices of the perturbed equation when \( \varepsilon \to 0 \) in the real positive direction. To establish this result we combine a direct computation with a theoretical approach.

Key words: Reducible second order equation, Stokes phenomenon, Irregular singularity, Heun type equations, Monodromy matrices, Regular singularity, Limit

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1. Introduction

With this paper we continue our research on the Stokes phenomenon for reducible linear scalar equations using perturbative approach. In this paper we consider the second order reducible equation

\[
L(y) = L_2(L_1(y)) = 0
\]

where

\[
L_j = \partial - \left( \frac{\alpha_j}{x} + \frac{\beta_j}{x^2} + \gamma_j \right), \quad j = 1, 2, \quad \partial = \frac{d}{dx}
\]

and \( \alpha_j, \beta_j, \gamma_j \in \mathbb{C} \) are arbitrary bounded parameters. In general the equation \((1.1) - (1.2)\) has two irregular singular points over \( \mathbb{C}P^1 \) of Poincaré rank 1: \( x = 0 \) and \( x = \infty \).

Introducing a small complex parameter \( \varepsilon \) we perturb equation \((1.1) - (1.2)\) to the equation

\[
L(y, \varepsilon) = L_2,\varepsilon(L_1,\varepsilon(y)) = 0
\]

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where
\[ L_{j,\varepsilon} = \partial - \frac{\alpha_j}{2} \left( \frac{1}{x - \sqrt{\varepsilon}} + \frac{1}{x + \sqrt{\varepsilon}} \right) - \frac{\beta_j}{2\sqrt{\varepsilon}} \left( \frac{1}{x - \sqrt{\varepsilon}} - \frac{1}{x + \sqrt{\varepsilon}} \right) - \frac{\gamma_j}{2\sqrt{\varepsilon}} \left( -\frac{1}{x - \frac{1}{\sqrt{\varepsilon}}} + \frac{1}{x + \frac{1}{\sqrt{\varepsilon}}} \right) \]
such that
\[ L_{j,\varepsilon} \to L_j \quad \text{as} \quad \varepsilon \to 0. \]

In general the equation \( (1.3) - (1.4) \) is a second order Fuchsian equation having five regular singularities over \( \mathbb{C}P^1 \) – four of them are finite and the fifth is at \( x = \infty \). In this article we restrict our attention to the equation \( (1.3) - (1.4) \) which has exactly four singular points. In particular we study such an equation \( (1.4) \) for which the coefficients of the equation \( (1.1) - (1.2) \) do not depend on the parameter of perturbation \( \varepsilon \). It turns out that there is an unique family of parameters, which satisfies this condition – the family with \( \alpha_1 = 0, \alpha_2 = -2 \) and arbitrary \( \beta_j, \gamma_j \) (see Theorem 2.1). The first order differential operators \( L_j \) and \( L_{j,\varepsilon} \) which determine this particular family become
\[ L_1 = \partial - \left( \frac{\beta_1}{x^2} + \gamma_1 \right), \quad L_2 = \partial - \left( -\frac{2}{x} + \frac{\beta_2}{x^2} + \gamma_2 \right) \]
and
\[ L_{1,\varepsilon} = \partial - \frac{\beta_1}{2\sqrt{\varepsilon}} \left( \frac{1}{x - \sqrt{\varepsilon}} - \frac{1}{x + \sqrt{\varepsilon}} \right) - \frac{\gamma_1}{2\sqrt{\varepsilon}} \left( -\frac{1}{x - \frac{1}{\sqrt{\varepsilon}}} + \frac{1}{x + \frac{1}{\sqrt{\varepsilon}}} \right), \]
\[ L_{2,\varepsilon} = \partial - \left( \frac{\beta_2}{2\sqrt{\varepsilon}} - 1 \right) \frac{1}{x - \sqrt{\varepsilon}} + \left( \frac{\beta_2}{2\sqrt{\varepsilon}} + 1 \right) \frac{1}{x + \sqrt{\varepsilon}} - \frac{\gamma_2}{2\sqrt{\varepsilon}} \left( -\frac{1}{x - \frac{1}{\sqrt{\varepsilon}}} + \frac{1}{x + \frac{1}{\sqrt{\varepsilon}}} \right). \]

Through this paper we call the equation \( (1.1) \) with \( (1.5) \) the initial equation. Since every four distinct points over the Riemann sphere can be fixed by a M"obius transformation at \( 0, 1, \infty \) and \( a \neq 0, 1, \infty \) through this paper we call the equation \( (1.3) \) with \( (1.6) \) the perturbed equation or the Heun type equation. In this paper we assume that the point \( x = 0 \) is a non-resonant irregular point for the initial equation. This assumption is equivalent to the condition
\[ \beta_1 \neq \beta_2. \]

To each singular point of the initial equation, with respect to a given fundamental matrix, we associate the so called Stokes matrices \( St_j^\theta, j = 0, \infty \) corresponding to the singular direction \( \theta \). Due to the reducibility the initial equation admits actual fundamental matrices \( \Phi_0(x, 0) \) at \( x = 0 \) and \( \Phi_\infty(x, 0) \) at \( x = \infty \) with respect to which the Stokes matrices \( St_j^\theta \) have the upper-triangular form
\[ St_j^\theta = \begin{pmatrix} 1 & \mu_j^\theta \\ 0 & 1 \end{pmatrix}. \]
Moreover the reducibility implies that the initial equation can have only one singular direction \( \theta = \arg(\beta_1 - \beta_2) \) at the origin (see Theorem 3.11) and only one singular direction \( \theta = \arg(\gamma_2 - \gamma_1) \) at \( x = \infty \) (see Theorem 3.13). Denote by \( x_R = \sqrt{\varepsilon}, x_L = -\sqrt{\varepsilon}, x_{RR} = \sqrt{\varepsilon}, \)
1/√ε, x_{LL} = −1/√ε the singular points of the perturbed equation. To each of them we associate the so called monodromy matrices M_j(ε) and M_{jj}(ε), j = L, R. Through this paper we call a double resonance these values of the parameters β_j, γ_j, ε for which the solution of the perturbed equation can contain logarithmic terms near exactly two singular points of different type. Under this we mean that one of the point is of the type x_j, j = R, L and the other is of the type x_{jj}, j = R, L. It turns out that the perturbed equation has a double resonance if and only if ε ∈ R^+ (see Proposition 4.2). Due to the reducibility during a double resonance the Heun type equation admits two different fundamental matrices Φ_0(x, ε) and Φ_∞(x, ε) which respect to which the monodromy matrices M_j(ε) and M_{jj}(ε) have the simple form (see Theorem 4.6)

\begin{align}
M_j(ε) &= e^{\frac{πi(λ+T_j B)}{2}} e^{2πi T_j}, \quad M_{jj}(ε) = e^{-πi x_j G} e^{2πi T_j}, \quad j = R, L.
\end{align}

The constant diagonal matrices Λ, B and G come from the initial equation (see Proposition 3.2). The matrices Λ/2 + B/2x_j and −x_{jj}G/2 are their perturbed analog. The matrices T_j and T_{jj} are nilponent

\begin{align}
T_j &= \begin{pmatrix} 0 & d_j \\ 0 & 0 \end{pmatrix}, \quad T_{jj} = \begin{pmatrix} 0 & d_{jj} \\ 0 & 0 \end{pmatrix}, \quad j = L, R
\end{align}

and they measure the existence of logarithmic terms around the singular points. Then the main result of this paper demonstrates in an explicit way that during a double resonance both Stokes matrices S^0_t and S^θ_t of the initial equation are realized as a limit of the matrices e^{2πi T_j} and e^{2πi T_{jj}}, i.e.

\begin{align}
e^{2πi T_j} \longrightarrow S^θ_0, \quad e^{2πi T_{jj}} \longrightarrow S^θ_∞, \quad j = R, L
\end{align}

when √ε → 0.

Glutsyuk make the start for studying the Stokes phenomenon of linear systems with an irregular singularity at the origin by a perturbative approach with the papers [8, 9, 10]. More precisely, he considers a generic perturbation depending on a parameter ε that splits the irregular singularity at the origin of the initial system into Fuchsian singularities. Then Glutsyuk prove that the Stokes operators of the initial system with an irregular singularity at the origin of Poincaré rank k ≥ 1 are limits of the transition operators of the perturbed system. Recently, Lambert, Rousseau, Hurtubise and Klimeš [16, 17, 12, 15] using a different approach extend the set of the values of the parameter of perturbation ε over a whole neighborhood of ε = 0. They study only non-resonant initial system with an irregular singularity at the origin of Poincaré rank k ∈ N. In the case when k = 1 the perturbation splits the origin into two Fuchsina singularity x_L = √ε and x_R = −√ε [16]. In [16] Lambert and Rousseau prove that the monodromy operator acting on the fundamental matrix solutions of the perturbed systems decomposes into the Stokes operator multiplied by the classical monodromy operator acting on the branch of (x − x_L)^{λ/2}B/2x_L (x − x_R)^{λ/2}B/2x_R. In addition, in [17] they prove that the so called unfolded Stokes matrices S^θ_t j, j = L, R of the perturbed system depend analytically on the parameter of perturbation ε and converge when ε → 0 to the Stokes matrices S_t j, j = L, R of the initial system. Later in [15] Klimeš specifies this result expressing the acting of the monodromy operators on the solutions of the perturbed systems by the monodromy matrices M_j(ε), the unfolded Stokes matrices S^θ_t j and the matrices e^{πi(λ+B/x_j)}, j = L, R. In [27] Remy also is interested in Stokes phenomenon for a linear differential system with an irregular singularity at the origin x^{r+1}Y′(x) = A(x)Y(x) and an arbitrary single level r ≥ 1 from a perturbative point of view. But he does not split
the origin into Fuchsian singularities. He considers a regular holomorphic perturbation of the coefficients of the initial system \( x^{r+1}Y'(x) = A^ε(x)Y(x), A^1(x) = A(x) \) which preserves the single level \( r \) of the initial system. He proves that the Stokes-Ramis matrices of the initial system are limits of the convergence of the Stokes-Ramis matrices of the perturbed system.

Parallel to the study of linear systems Ramis [24], Zhang [35], Duval [7] and latter Lambert and Rousseau [18] dealt with the Stokes phenomenon in the confluence of the classical hypergeometric equation and generalized hypergeometric family. In [7] Duval study the family \( D_{p+1,p}(\alpha; \beta) \) of generalized hypergeometric equations for \( p \geq 2 \). This equation has two Stokes matrices \( St_0 \) and \( St_\infty \) and can be obtained from the Fuchsian equation \( D_{p+1,p+1}(\alpha; \beta) \) by a confluence procedure. More precisely, Duval apply such a confluence procedure by the change \( z = t/b \) and then making \( b \to \infty \). The change \( z = t/b \) takes the singular points \( z = 0, 1, \infty \) of the equation \( D_{p+1,p+1}(\alpha; \beta) \) into the points \( 0, b, \infty \). When \( b \to \infty \) in a non-real direction Duval prove, by a direct calculation, that the Stokes matrices can be obtained as limit of the connection matrices linking well chosen fundamental set of solutions around \( b \) and \( \infty \) of the Fuchsian equation. When \( b \to \infty \) in a real direction she express, by a direct calculation, the Stokes matrices as limits of the monodromy matrices around \( b \) and \( \infty \) with respect to a “mixed” basis of solutions. In [32] we consider a particular family of third order linear reducible scalar equation having a non-resonant irregular singularity at the origin of Poincaré rank 1. We prove that the Stokes matrices of the initial equation are limits of the nilpotent parts of the monodromy matrices of a Fuchsian equation with respect to a “mixed” basis of solutions that contains logarithmic terms. This Fuchsian equation was obtained from the initial equation by introducing a small real parameter \( \varepsilon \) that splits the origin into two regular singularities \( x_L = -\sqrt{\varepsilon} \) and \( x_R = \sqrt{\varepsilon} \).

Very recently in [20] Malek is interested in the effect of the unfolding in families of singular PDEs from an asymptotic point of view. He unfolds singularly perturbed differential operator of irregular type \( \varepsilon t^2 \partial_t \) into a family of singular operators \( D_{\varepsilon \alpha}(\partial_t) = (\varepsilon t^2 - \varepsilon^\alpha) \partial_t \) of Fuchsian type. Then he studies how this unfolding changes the asymptotic properties of holomorphic solutions of the unfolded equation in comparison to the ones of the initial equations.

In this paper, as in our previous two papers [32, 33] we deal with reducible scalar equations. This time the initial equation has two non-resonant irregular singular points of Poincaré rank 1, taken at \( x = 0 \) and \( x = \infty \). Introducing a small complex parameter of perturbation \( \varepsilon \) we split the non-resonant singularity at the origin into two finite Fuchsian singularities \( x_R = \sqrt{\varepsilon} \) and \( x_L = -\sqrt{\varepsilon} \). At the same time we also split \( x = \infty \) into two finite Fuchsian singularities \( x_{RR} = 1/\sqrt{\varepsilon} \) and \( x_{LL} = -1/\sqrt{\varepsilon} \). Since this situation is more complicated we restrict our attention to second order equations. Due to the reducibility both initial and perturbed equations admit upper-triangular fundamental matrices, whose off-diagonal element can be expressed as an integral. As in [32] we fully exploit this phenomenon in purpose of explicitly computing the Stokes matrices of the initial equation and the monodromy matrices of the perturbed equation. In fact, in this paper we use two different fundamental matrices \( \Phi_0(x, 0) \) and \( \Phi_\infty(x, 0) \) of the initial equation depending of the path of integration. These solutions correspond to the point \( x = 0 \) and \( x = \infty \). Here we are not interested in the connection problem. The off-diagonal element of the actual \( \Phi_0(x, 0) \) and \( \Phi_\infty(x, 0) \) is obtained from the integral representation by two steps. In the first step we expres the integral as formal power series in \( x \) (resp. \( x^{-1} \)). It turns out that these series are in general divergent. In the second step, utilizing
the summability theory, we lift these formal series to actual solutions. With respect to the actual \( \Phi_0(x,0) \) and \( \Phi_\infty(x,0) \) we explicitly compute the corresponding Stokes matrices \( S_\theta^0 \) and \( S_\theta^\infty \), where \( \theta \) is a singular direction. To each fundamental matrix of the initial equation corresponds a fundamental matrix \( \Phi_j(x,\varepsilon), j = 0, \infty \) of the perturbed equation, such that \( \lim_{\varepsilon \to 0} \Phi_j(x,\varepsilon) = \Phi_j(x,0), j = 0, \infty \). Moreover, this choice of fundamental matrices leads to the so called ”mixed” basis of solutions for which one of the solutions is always an eigenvector of the local monodromy around the singular points (see Remark 3.7).

With respect to exactly these fundamental matrices we explicitly compute the monodromy matrices \( M_j(\varepsilon) \) and \( M_{jj}(\varepsilon), j = R, L \) of the perturbed equation only during a double resonance. Then we explicitly show that during a double resonance the both Stokes matrices of the initial equation are realized as a limit of the nilpotent parts of the suitable monodromy matrices of the perturbed equation when \( \sqrt{\varepsilon} \to 0 \) (see Theorem 5.6). Since we compute the Stokes and monodromy matrices explicitly by hand one can say that our approach is closer to the works of Duval \[7\], Ramis \[24\] and Zhang \[35\]. Moreover, our result almost repeats the result of Duval when \( \beta \) to the works of Duval \[7\], Ramis \[24\] and Zhang \[35\]. Moreover our approach allows us to split more that one irregular singularity into several Fuchsian singularities.

In addition, we find two particular families of parameters \( \beta_j, \gamma_j \) for which the initial equation has trivial Stokes matrices at both singular points. The first family is determined by the condition

\[
\gamma_1 = \gamma_2
\]

and \( \beta_j \)'s are arbitrary such that satisfy the non-resonant condition (1.7). This family is expected. For these values of the parameters the elements of the matrices \( \Phi_0(x,0) \) and \( \Phi_\infty(x,0) \) are defined by well known functions (see Proposition 3.2(3) and Proposition 3.5(1)). It turns out that the matrices \( e^{2\pi i T_j} \) and \( e^{2\pi i T_{jj}}, j = R, L \) are also trivial when \( \gamma_1 = \gamma_2 \) (see from Theorem 1.8(1) to Theorem 1.16(1)). The second family of parameters for which the both Stokes matrices are trivial is a very interesting. We are surprised to find that for parameters \( \beta_j, \gamma_j \) such that satisfy the condition (1.7) and

\[
(1.9) \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (\gamma_2 - \gamma_1)^n (\beta_2 - \beta_1)^n}{n! (n+1)!} = 0
\]

the initial equation also has trivial Stokes matrices at both singular points (see Theorem 3.11 and Theorem 3.13). The cause for this phenomenon is the convergence of the series

\[
\hat{\psi}(x) = \sum_{k=1}^{\infty} \frac{(-1)^k k! S_{k-1}}{((\beta_2 - \beta_1)^k)} x^k \quad \text{and} \quad \hat{\varphi}(x) = \sum_{k=1}^{\infty} \frac{k! S_{k-1}}{(\gamma_2 - \gamma_1)^k} x^{-k}
\]

under condition (1.9). Here \( S_{k-1}, k \in \mathbb{N} \) are the partial sums of the number series (1.9). The series \( \hat{\psi}(x) \) is an element of the formal matrix solution at \( x = 0 \), and the series \( \hat{\varphi}(x) \) includes in the formal matrix solution at \( x = \infty \) (see Proposition 3.2(2) and Proposition...
family of the non-linear equation \( b \) in the mathematical physics: optics, hydrodynamics, gravity. Comparing the coefficients where

\[
(1.13)
\]

In particular, following \([5]\) the initial equation appears as a linearization of the equation

\[
(1.12)
\]

On of the motivation for studying this pair of equations arises from its relation with Heun class of equations. We will only comment the application of the initial equation regarding it

\[
(1.10)
\]

In the same paper \([28]\) Salatich and Slavyanov relate by an antiquantization procedure the DCHE \((1.10)\) with the third Painlevé equation

\[
P_{III}
\]

In section 2.2 repeating their approach we obtain two particular families of the initial equation that satisfy the conditions \((1.11)\) provided that the non-resonant condition \((1.7)\) remains valid.

Recently in several works \([3, 4, 5]\) Buchstaber, Glutsyuk and Tertychnyi study the following family of DCHE

\[
(1.12)
\]

When the parameters are real such that \( \lambda + \mu^2 > 0 \) the equation \((1.12)\) appears as a linearization of the family of nonlinear equations on two-torus that model the Josephson effect in superconductivity. The initial equation coincides with the equation \((1.12)\) if and only if

\[
n = 2, \quad \beta_1 + \beta_2 = -\mu, \quad \gamma_1 + \gamma_2 = \mu, \quad \beta_1 \beta_2 = \gamma_1 \gamma_2 = 0, \quad \gamma_1 = \mu, \quad \beta_1 \gamma_2 + \beta_2 \gamma_1 = \lambda.
\]

In particular, following \([5]\) the initial equation

\[
x^2 y''(x) + (-2x + \mu - \mu x^2) y'(x) - 2\mu x y(x) = 0, \quad \mu \neq 0
\]

appears as a linearization of the equation

\[
(1.13)
\]

where \( \omega > 0 \) is a real parameter, related to \( \mu \) by \( \mu = 1/2\omega \). The equation Jo is a particular family of the non-linear equation

\[
\dot{\varphi}(t) + \sin \varphi(t) = B + A \cos \omega t, \quad A, \omega > 0, \quad B \geq 0.
\]
Here \( \omega > 0 \) is a fixed constant, \( A, B \) are parameters. The last equation arises in several models in physics (Josephson junction in superconductively), mechanics, geometry. Among many results, in the pointed papers the authors study the problems of the existence of a holomorphic solution on \( \mathbb{C} \) of (1.12) and the existence of the eigenfunctions of the monodromy operators with a given eigenvalue.

This article is organized as follows. In the next section we recall the required facts and definitions from the theory of differential equations with irregular singularities, as well as from the theory of Fuchsian equations. We also introduce the perturbed equation as a Heun type equation and define global fundamental matrices of both equations, which we use to build local fundamental matrix solutions. In section 3 we compute the Stokes monodromy matrices of the perturbed equation under the restriction (1.7). In section 4 we compute the monodromy matrices of the perturbed equation provided that condition (1.7) is valid. In the last section 5 we establish the main results of this paper.

2. Preliminaries

2.1. The equation (1.3) - (1.4) as a Heun type equation. In this section we introduce the perturbed equation as a Heun type equation.

In general the equation (1.3) - (1.4) is a second order Fuchsian equation with 5 regular singularities over \( \mathbb{C} \mathbb{P}^1 \) taken at \( x_1 = \infty, x_2 = \sqrt{\varepsilon}, x_3 = -\sqrt{\varepsilon}, x_4 = 1/\sqrt{\varepsilon}, x_5 = -1/\sqrt{\varepsilon} \). The next theorem describes all the cases when the this equation has only four regular singular points.

Theorem 2.1. The equation (1.3) - (1.4) has only four regular singular points over \( \mathbb{C} \mathbb{P}^1 \) if and only if the parameters \( \alpha_j, \beta_j, \gamma_j \) and \( \varepsilon \) satisfy one of the following conditions:

(I.) \( \alpha_1 = 0, \alpha_2 = -2, \) and \( \beta_j \) and \( \gamma_j \) are arbitrary.

(II.) \( \alpha_1 = \alpha_2 = -1, \beta_1 - \beta_2 + \frac{\alpha_1 - \alpha_2}{\varepsilon} \) is arbitrary.

(III.) \( \gamma_1 = -\gamma_2 = -2\sqrt{\varepsilon}, \frac{\varepsilon(\beta_1 - \beta_2) + \sqrt{\varepsilon}(\alpha_1 - \alpha_2)}{\varepsilon(1 - \varepsilon^2)} = 1. \)

(IV.) \( \gamma_2 = -\gamma_1 = -2\sqrt{\varepsilon}, \frac{\varepsilon(\beta_1 - \beta_2) - \sqrt{\varepsilon}(\alpha_1 - \alpha_2)}{\varepsilon(1 - \varepsilon^2)} = -1. \)

(V.) \( \beta_1 = -\alpha_1 \sqrt{\varepsilon}, \beta_2 = -\alpha_2 \sqrt{\varepsilon}, \) and \( \gamma_j \) are arbitrary.

(VI.) \( \beta_1 = -(\alpha_1 - 2)\sqrt{\varepsilon}, \beta_2 = -(\alpha_2 + 2)\sqrt{\varepsilon}, \) and \( \frac{\alpha_1 - \alpha_2}{2} + \frac{\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}(\gamma_2 - \gamma_1) = -1. \)

(VII.) \( \beta_1 = \alpha_1 \sqrt{\varepsilon}, \beta_2 = \alpha_2 \sqrt{\varepsilon}, \) and \( \gamma_j \) are arbitrary.

(VIII.) \( \beta_1 = (\alpha_1 - 2)\sqrt{\varepsilon}, \beta_2 = (\alpha_2 + 2)\sqrt{\varepsilon}, \) and \( \frac{\alpha_1 - \alpha_2}{2} - \frac{\sqrt{\varepsilon}(\gamma_2 - \gamma_1)}{1 - \sqrt{\varepsilon}} = -1. \)

Proof. The transformation \( x = 1/t \) changes the equation (1.3) - (1.4) into the equation

\[
\ddot{y}(t) + \left[ \frac{p_1}{t} + \frac{p_2}{t - \sqrt{\varepsilon}} + \frac{p_3}{t + \sqrt{\varepsilon}} + \frac{p_4}{t - \sqrt{\varepsilon}} + \frac{p_5}{t + \sqrt{\varepsilon}} \right] \dot{y}(t) + \\
\left[ \frac{q_{10}}{t^2} + \frac{q_{11}}{t} + \frac{q_{20}}{(t - \sqrt{\varepsilon})^2} + \frac{q_{21}}{t - \sqrt{\varepsilon}} + \frac{q_{30}}{(t + \sqrt{\varepsilon})^2} + \frac{q_{31}}{t + \sqrt{\varepsilon}} \right] y(t) = 0, \tag{2.14}
\]

where the coefficients \( p_j, q_{j0}, q_{j1} \) depend on the parameters \( \alpha_j, \beta_j, \gamma_j \) and \( \varepsilon \). The equation (2.14) has 5 regular singularities over \( \mathbb{C} \mathbb{P}^1 \) taken at \( t_1 = 0, t_2 = \sqrt{\varepsilon}, t_3 = -\sqrt{\varepsilon}, t_4 = 1/\sqrt{\varepsilon}, t_5 = -1/\sqrt{\varepsilon} \). Each point \( t = t_j, 1 \leq j \leq 5 \) is an ordinary point for the equation
If and only if the coefficients $p_j$, $q_{j0}$ and $q_{j1}$ satisfy the simultaneous conditions
\[
\begin{align*}
p_j &= 0, \\
q_{j0} &= 0, \\
q_{j1} &= 0.
\end{align*}
\]

In particular, the point $t_1 = 0$ is an ordinary point for the equation (2.14) if and only if
\[
\begin{align*}
p_1 &= \alpha_1 + \alpha_2 + 2 = 0, \\
q_{10} &= \alpha_1 (\alpha_2 + 1) = 0, \\
q_{11} &= 2\beta_1 + \alpha_1 \beta_2 + \alpha_2 \beta_1 - \frac{2\gamma_1}{\varepsilon} - \frac{\alpha_1 \gamma_2}{\varepsilon} - \frac{\alpha_2 \gamma_1}{\varepsilon} = 0.
\end{align*}
\]

This system of conditions has two families of solutions. The first one is given by $\alpha_1 = 0$, $\alpha_2 = -2$ and the parameters $\beta_j$, $\gamma_j$ and $\varepsilon$ are arbitrary. This family defines the case (I.). The second family of solutions is defined by $\alpha_1 = \alpha_2 = -1$ and
\[
\beta_1 - \beta_2 + \frac{\gamma_2 - \gamma_1}{\varepsilon} = 0,
\]
which we call the case (II.). In the cases (I.) and (II.) the equation (I.3) - (I.4) admits only finite regular singular points taken at $x_2$, $x_3$, $x_4$ and $x_5$.

Next, the point $t_2 = \sqrt{\varepsilon}$ is an ordinary point for the equation (2.14) if and only if
\[
\begin{align*}
p_2 &= \frac{\gamma_1 + \gamma_2}{2\sqrt{\varepsilon}} = 0, \\
q_{20} &= -\frac{1}{4} (2\sqrt{\varepsilon} - \gamma_2) = 0, \\
q_{21} &= \frac{\gamma_1}{2\sqrt{\varepsilon}} (2\sqrt{\varepsilon} - \gamma_2) + \frac{\gamma_1 \gamma_2}{4\varepsilon} + \frac{\beta_1 \gamma_2 \sqrt{\varepsilon} + \beta_2 \gamma_1 \sqrt{\varepsilon} + \alpha_1 \gamma_2 + \alpha_2 \gamma_1}{2\varepsilon (1 - \varepsilon^2)} = 0.
\end{align*}
\]

This conditions imply that either
\[
\gamma_1 = \gamma_2 = 0, \quad \text{and} \quad \alpha_j, \beta_j, \varepsilon \quad \text{are arbitrary},
\]
or
\[
\gamma_1 = -\gamma_2 = -2\sqrt{\varepsilon}, \quad \frac{\varepsilon (\beta_1 - \beta_2) + \sqrt{\varepsilon} (\alpha_1 - \alpha_2)}{\varepsilon (1 - \varepsilon^2)} = 1.
\]

In the first case when $\gamma_1 = \gamma_2 = 0$ the equation (I.3) - (I.4) admits 3 regular singularities taken at $x_1$, $x_2$ and $x_3$. Because of this reason we do not include it in the list of the statement of the theorem. Note also that in this case the point $x = \infty$ is a regular point for the equation (I.1) - (I.2). We call the second case (III). In this case the equation (I.3) - (I.4) admits regular singular points taken at $x_1$, $x_2$, $x_3$ and $x_5$.

Similarly, the point $t_3 = -\sqrt{\varepsilon}$ is an ordinary point for the equation (2.14) if and only if
\[
\begin{align*}
p_3 &= -\frac{\gamma_1 + \gamma_2}{2\sqrt{\varepsilon}} = 0, \\
q_{30} &= \frac{1}{4} (2\sqrt{\varepsilon} + \gamma_2) = 0, \\
q_{31} &= \frac{\gamma_1}{2\sqrt{\varepsilon}} (2\sqrt{\varepsilon} + \gamma_2) - \frac{\gamma_1 \gamma_2}{4\varepsilon} + \frac{\beta_1 \gamma_2 \sqrt{\varepsilon} + \beta_2 \gamma_1 \sqrt{\varepsilon} - \alpha_1 \gamma_2 - \alpha_2 \gamma_1}{2\varepsilon (1 - \varepsilon^2)} = 0.
\end{align*}
\]

This system of conditions implies either
\[
\gamma_1 = \gamma_2 = 0, \quad \text{and} \quad \alpha_j, \beta_j, \varepsilon \quad \text{are arbitrary}
\]
or
\[
\gamma_2 = -\gamma_1 = -2\sqrt{\varepsilon}, \quad \frac{\varepsilon (\beta_1 - \beta_2) - \sqrt{\varepsilon} (\alpha_1 - \alpha_2)}{\varepsilon (1 - \varepsilon^2)} = -1.
\]

Just before the first case when $\gamma_1 = \gamma_2 = 0$ and $\alpha_j, \beta_j$ are arbitrary non-zero parameters is out of the statement of the theorem. We call the second case the case (IV.).
The second one is defined by
\[ V. \]
We call these two cases the case (1.3) - (1.4) admits regular singular points at \( \varepsilon \), and the case (1.5) - (1.6) do not depend on the parameters of perturbation.

This system admits two families of solutions. The first one is given by
\[ \beta_1 = -\alpha_1 \sqrt{\varepsilon}, \quad \beta_2 = -\alpha_2 \sqrt{\varepsilon}, \quad \text{and} \quad \gamma_j, \varepsilon \quad \text{are arbitrary}. \]

The second one is defined by
\[ \beta_1 = -(\alpha_1 - 2)\sqrt{\varepsilon}, \quad \beta_2 = -(\alpha_2 + 2)\sqrt{\varepsilon}, \quad \frac{\alpha_2 - \alpha_1}{2} + \frac{\sqrt{\varepsilon}}{1 - \varepsilon^2}(\gamma_2 - \gamma_1) = -1. \]

Again we find two families of solutions of this system. The first one is given by
\[ \beta_1 = \alpha_1 \sqrt{\varepsilon}, \quad \beta_2 = \alpha_2 \sqrt{\varepsilon}, \]
and \( \gamma_j, \varepsilon \) are arbitrary. The second family is defined by
\[ \beta_1 = (\alpha_1 - 2)\sqrt{\varepsilon}, \quad \beta_2 = (\alpha_2 + 2)\sqrt{\varepsilon}, \quad \frac{\alpha_2 - \alpha_1}{2} - \frac{\sqrt{\varepsilon}(\gamma_2 - \gamma_1)}{1 - \varepsilon^2} = -1. \]

These are the last two cases – the case (VII.) and the case (VIII.). In both cases the equation (1.3) - (1.4) admits 4 regular singularities taken at \( x_1, x_2, x_4 \) and \( x_5 \).

This ends the proof. \( \square \)

In this paper we study the equation (1.3) - (1.4) under the condition (I.) when the coefficients of the equation (1.1) - (1.2) do not depend on the parameter of perturbation \( \varepsilon \). It is well known that every 4 distinct points over \( \mathbb{CP}^1 \) can be fixed by a Möbius transformation at 0, 1, \( \infty \) and \( a \neq 0, 1, \infty \). Recall that the Heun equation is a second order Fuchsian equation with four singular points over \( \mathbb{CP}^1 \) taken at 0, 1, \( \infty \) and \( a \neq 0, 1, \infty \). Since a Möbius transformation can take the perturbed equation into the Heun equation through this paper we call the perturbed equation Heun type equation. We prefer to work with the perturbed equation rather than work with Heun equation because of the symmetries. The next proposition gives all the more reason for working with the perturbed equation instead of the Heun equation.

**Proposition 2.2.** The Möbius transformation
\[ x = \sqrt{\varepsilon} \frac{t + \frac{1}{1 + \varepsilon}}{t - \frac{1}{1 + \varepsilon}} \]
takes the perturbed equation into the Heun equation

\[
\dot{y} + \left[\left(\frac{\beta_1 + \beta_2}{2\sqrt{\varepsilon}} + 1\right) \frac{1}{t} + \frac{\gamma_1 + \gamma_2}{2\sqrt{\varepsilon}} \frac{1}{t-1} - \frac{\gamma_1 + \gamma_2}{2\sqrt{\varepsilon}} \frac{1}{t - \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2}\right] y + \\
\frac{\beta_1 \beta_2}{4\varepsilon} \frac{1}{t^2} + \frac{\gamma_1}{2\sqrt{\varepsilon}} \left(\frac{\gamma_2}{2\sqrt{\varepsilon}} - 1\right) \frac{1}{(t-1)^2} + \frac{\gamma_1}{2\sqrt{\varepsilon}} \left(\frac{\gamma_2}{2\sqrt{\varepsilon}} + 1\right) \frac{1}{(t - \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2)^2} + \\
\left(\frac{\beta_1 \gamma_1 + \beta_2 \gamma_1}{4\varepsilon} + \frac{\gamma_1}{2\sqrt{\varepsilon}}\right) \frac{4\varepsilon}{(1-\varepsilon)^2} \frac{1}{t} + \left(\frac{\beta_1 \gamma_2 + \beta_2 \gamma_1}{4\varepsilon} + \frac{\gamma_1}{2\sqrt{\varepsilon}}\right) \frac{1}{t - \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2}
\] 

+ \left(\frac{\beta_1 \gamma_2 + \beta_2 \gamma_2}{4\varepsilon} \frac{1}{(1-\varepsilon)^2} - \frac{\gamma_1}{2\sqrt{\varepsilon}} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 + \frac{1+\varepsilon}{2\varepsilon} \gamma_1 \gamma_2\right) \frac{1}{t - \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2} y = 0,
\]

which does not have limit when \(\varepsilon \to 0\).

\[\text{Proof.}\] The proof is straightforward. \(\Box\)

We leave to the reader as an easy exercise to check that there is no Heun equation, obtained from the perturbed equation by a Möbius transformation, that has a limit when \(\varepsilon \to 0\).

2.2. The initial equation and the third Painlevé equation. In \([28]\) Salatich and Slavyanov applying an antiquantization procedure to the DCHE (1.10) obtain a particular case of the third Painlevé equation \(P_{III}\)

\[(2.15)\]

\[\ddot{q} - \frac{\dot{q}^2}{q} + \frac{\dot{q}}{t} - \gamma q^3 - \frac{\alpha q^2 + \beta}{t} - \frac{\delta}{q} = 0,\]

where \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\) are parameters and \(\cdot = \frac{d}{dt}\). In this section following their approach we connect by an antiquantization particular families of the initial equation with particular families of the \(P_{III}\) equation (2.15).

Under the non-resonant condition (1.7) there are four particular families of the initial equation that satisfy the conditions (1.11). They are defined as follows:

1. \(\gamma_2 = \beta_1 = 0, \gamma_1 = 1, c = 2, a = 2, h = -1\) and \(\beta_2 = t \neq 0\) is an arbitrary. Then the corresponding DCHE is

\[(2.16)\]

\[x^2 y''(x) + (-x^2 + 2x - t) y'(x) + (-2x + t) y(x) = 0.\]

2. \(\gamma_2 = \beta_2 = 0, \gamma_1 = 1, c = 2, a = 2, h = 0\) and \(\beta_1 = t \neq 0\) is an arbitrary. Then the corresponding DCHE is

\[(2.17)\]

\[x^2 y''(x) + (-x^2 + 2x - t) y'(x) - 2x y(x) = 0.\]

3. \(\gamma_1 = \beta_1 = 0, \gamma_2 = 1, c = 2, a = h = 0\) and \(\beta_2 = t \neq 0\) is an arbitrary. Then the corresponding DCHE is

\[(2.18)\]

\[x^2 y''(x) + (-x^2 + 2x - t) y'(x) = 0.\]

4. \(\gamma_1 = \beta_2 = 0, \gamma_2 = 1, c = 2, a = 0, h = -1\) and \(\beta_1 = t \neq 0\) is an arbitrary. Then the corresponding DCHE is

\[(2.19)\]

\[x^2 y''(x) + (-x^2 + 2x - t) y'(x) + t y(x) = 0.\]
Note that for the initial equation $c, a, h$ from (1.10) are fixed and $t$ is a varying parameter. Denote
\[ \sigma(x) = x^2, \quad \tau(x) = -x^2 + 2x - t, \quad \omega(x) = -ax. \]

Following [28] we associate with the DCHE the Hamiltonian
\[ H(q, p) = \frac{1}{t} \left[ \sigma(q) p^2 + \tau(q) p + \omega(q) \right]. \]

The Hamiltonian system is written as
\[
\begin{align*}
\dot{q} & = \frac{\partial H}{\partial p} = \frac{2\sigma(q) p + \tau(q)}{t} = \frac{2q^2 - q^2 + 2q - t}{t}, \\
\dot{p} & = -\frac{\partial H}{\partial q} = -\frac{\sigma'(q) p^2 + \tau'(q) p + \omega'(q)}{t} = \frac{2qp^2 - 2q p + 2p - a}{t}.
\end{align*}
\]

Eliminating from this system $p$ we obtain the following non-linear second order equation for $q$
\[ \dot{q} - \frac{q^2}{q} + \frac{\dot{q}}{t} - \frac{q^3}{t^2} - \frac{2(a - 1)q^2}{t^2} - \frac{1}{t} + \frac{1}{q} = 0, \]
which is called the third Painlevé $P_{III'}$ equation [23]. The transformations
\[ t = z^2, \quad q = z^u \]
take the last equation into the particular $P_{III}$
\[ u'' - \frac{u'^2}{u} + \frac{u'}{z} - 4u^3 - \frac{4(2(a - 1)u^2 + 1)}{z} + \frac{4}{u} = 0 \]
with $\gamma = 4, \alpha = 8(a - 1), \beta = 4$ and $\delta = -4$. As a result, the first two families of the initial equation are related to $P_{III}$ equation with
\[ \gamma = -4, \quad \alpha = 8, \quad \beta = 4, \quad \delta = -4. \]
The last two families are related to the $P_{III}$ equation with
\[ \gamma = -4, \quad \alpha = -8, \quad \beta = 4, \quad \delta = -4. \]

These two families of the $P_{III}$ equation fall into the generic case of the third Painlevé equation. It is defined by the condition $\gamma \delta \neq 0$ and it is well studied in [22, 23]. Note that when $a = 0$ the $P_{III'}$ equation has a Riccaty type of solution. Indeed, when $a = 0$ the function $p \equiv 0$ satisfies the second equation of the above Hamiltonian system. Then the function $q$ satisfies the Riccati equation
\[ t \dot{q} = -q^2 + 2q - t. \]

Similarly, when $a = 2$ the $P_{III'}$ equation again has a Riccaty type of solution. This time the function $p = 1$ is a particular solution of the Hamiltonian system. Then the function $q$ satisfies the Riccati equation
\[ t \dot{q} = q^2 + 2q - t. \]
2.3. Global solutions. In [32] we have proved that reducible differential equations by means of the first-order differential equations admit a global fundamental matrix, whose off-diagonal elements are represented by iterated integrals. More precisely,

**Theorem 2.3.** Both initial equation and perturbed equations admit a global fundamental matrix \( \Phi(x, \cdot) \) of the form

\[
\Phi(x, \cdot) = \begin{pmatrix}
\Phi_1(x, \cdot) & \Phi_{12}(x, \cdot) \\
0 & \Phi_2(x, \cdot)
\end{pmatrix}.
\]

The diagonal elements \( \Phi_j(x, \cdot), j = 1, 2 \) are the solutions of the equations \( L_j \cdot u = 0 \) with \( L_{j, \epsilon} \) and \( L_{j, 0} = L_j \) given by. The off-diagonal element is defined as

\[
\Phi_{12}(x, \cdot) = \Phi_1(x, \cdot) \int_{\Gamma(x, \cdot)} \frac{\Phi_2(z, \cdot)}{\Phi_1(z, \cdot)} \, dz.
\]

The paths of integration \( \Gamma(x, \epsilon) \) and \( \Gamma(x, 0) \) are taken from the same base point \( x \) in such a way that \( \Gamma(x, \epsilon) \to \Gamma(x, 0) \) as \( \epsilon \to 0 \), and the matrices \( \Phi(x, \cdot) \) are fundamental matrix solution of the corresponding equations.

As an immediate corollary we define a fundamental set of solutions of both equations.

**Proposition 2.4.** Both initial and perturbed equations possess a fundamental set of solutions of the form

\[
\Phi_1(x, \cdot), \quad \Phi_1(x, \cdot) \int_{\Gamma(x, \cdot)} \frac{\Phi_2(z, \cdot)}{\Phi_1(z, \cdot)} \, dz.
\]

2.4. Irregular singularities. The initial equation is a second order linear differential equation having two irregular singularities. They are taken at the origin and at the infinity point and both singular points are of Poincaré rank 1. In this paper, as in our previous works [32], we utilize summability theory to build actual fundamental matrices at the origin and at the infinity point. Then with respect to these fundamental matrices we compute the corresponding Stokes matrices. In this paragraph we review some definitions, facts and notation from the applications of summability theory to the ordinary differential equations which is required to compute the Stokes matrices of the initial equation. We consider in parallel the summation at the origin and at the infinity point following works of Balser [1] and Ramis [26] for the summation at the origin, and Sauzin [29] for the summation at \( x = \infty \).

We denote by \( \mathbb{C}[[x]] \) the field of formal power series at the origin

\[
\mathbb{C}[[x]] = \left\{ \sum_{n=0}^{\infty} f_n x^n \mid f_n \in \mathbb{C}, \ n \in \mathbb{N} \right\},
\]

and by \( \mathbb{C}[[x^{-1}]] \) the field of formal power series at \( \infty \)

\[
\mathbb{C}[[x^{-1}]] = \left\{ \sum_{n \geq 0} \varphi_n x^{-n} \mid \varphi_n \in \mathbb{C}, \ n \in \mathbb{N} \right\}.
\]

Equipping \( \mathbb{C}[[x]] \) (resp. \( \mathbb{C}[[x^{-1}]] \)) with the natural derivation \( \partial = \frac{d}{dx} \) such that

\[
\partial(\psi \varphi) = \partial(\psi) \varphi + \psi \partial(\varphi), \quad \psi, \varphi \in \mathbb{C}[[x]] \quad \text{(resp. } \psi, \varphi \in \mathbb{C}[[x^{-1}]]\text{)}
\]

we make \( \mathbb{C}[[x]] \) (resp. \( \mathbb{C}[[x^{-1}]] \)) a differential algebra. All singular directions and sectors are defined on the Riemann surface of the natural logarithms.
Definition 2.5. 1. An open sector $S$ with vertex 0 is a set of the form

$$S = S(\theta, \alpha, \rho) = \left\{ x = re^{i\delta} \mid 0 < r < \rho, \theta - \alpha/2 < \delta < \theta + \alpha/2 \right\}.$$ 

An open sector $S_1$ with vertex $\infty$ is a set of the form

$$S_1 = S_1(\theta, \alpha, R) = \left\{ x = re^{i\delta} \mid r > R, \theta - \alpha/2 < \delta < \theta + \alpha/2 \right\}.$$ 

Here $\theta$ is an arbitrary real number (the bisector of the sector), $\alpha$ is a positive real (the opening of the sector) and $\rho$ (resp. $R$) is either a positive real number or $+\infty$ (the radius of the sector).

2. A closed sector $\bar{S}$ with vertex 0 is a set of the form

$$\bar{S} = \bar{S}(\theta, \alpha, \rho) = \left\{ x = re^{i\delta} \mid 0 < r \leq \rho, \theta - \alpha/2 \leq \delta \leq \theta + \alpha/2 \right\}.$$ 

A closed sector $\bar{S}_1$ with vertex $\infty$ is a set of the form

$$\bar{S}_1 = \bar{S}_1(\theta, \alpha, R) = \left\{ x = re^{i\delta} \mid r \geq R, \theta - \alpha/2 \leq \delta \leq \theta + \alpha/2 \right\}.$$ 

Here $\theta$ and $\alpha$ are as before, but $\rho$ (resp. $R$) is a positive real number (never equal to $+\infty$).

Definition 2.6. 1. Let the function $f(x)$ be holomorphic on a sector $S(\theta, \alpha, \rho)$. The (formal) power series $\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n$ is said to represent $f(x)$ asymptotically, as $x \to 0$ in $S$, if for every closed sector $W \subset S$ and all $N \geq 0$ there exist a positive constant $C(N,W)$ such that

$$\left| f(x) - \sum_{n=0}^{N-1} f_n x^n \right| \leq C(N,W) |x|^N, \quad x \in W.$$ 

2. Let the function $\varphi(x)$ be holomorphic on a sector $S_1(\theta, \alpha, R)$. The (formal) power series $\hat{\varphi}(x) = \sum_{n=0}^{\infty} \varphi_n x^{-n}$ is said to represent $\varphi(x)$ asymptotically, as $x \to \infty$ in $S_1$, if for every closed sector $W \subset S_1$ and all $N \geq 0$ there exist a positive constant $K(N,W)$ such that

$$\left| \varphi(x) - \sum_{n=0}^{N-1} \varphi_n x^{-n} \right| \leq K(N,W) |x|^{-N}, \quad x \in W.$$ 

In this case one usually write

$$f(x) \sim \hat{f}(x), \quad x \in S, \quad x \to 0,$$

$$\varphi(x) \sim \hat{\varphi}(x), \quad x \in S_1, \quad x \to \infty.$$ 

A function $f(x)$ (resp. $\varphi(x)$) can have at most one asymptotic series representation as $x \to 0$ (resp. $x \to \infty$) in a given sector $S$ (resp. $S_1$). Moreover, the set $\mathcal{A}(S)$ (resp. $\mathcal{A}(S_1)$) of functions, which are holomorphic on the sector $S$ (resp. $S_1$) and admit asymptotic representation on this sector forms a differential algebra. The maps

$$\mathcal{A}(S) \to \mathbb{C}[[x]] \quad \text{and} \quad \mathcal{A}(S_1) \to \mathbb{C}[[x^{-1}]]$$

$$f(x) \mapsto \hat{f}(x) \quad \text{and} \quad \varphi(x) \mapsto \hat{\varphi}(x)$$

are homomorphisms of differential algebras. The Borel - Ritt Theorem implies that these maps are surjective maps [11, 26]. Unfortunately they are not injective maps.

Among formal power series $\mathbb{C}[[x]]$ (resp. $\mathbb{C}[[x^{-1}]]$) we distinguish the formal power series of Gevrey order 1.
Definition 2.7. A formal power series \( \hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n \) (resp. \( \hat{\varphi}(x) = \sum_{n=0}^{\infty} \varphi_n x^{-1} \)) is said to be of Gevrey order 1 if there exist two positive constants \( C, A > 0 \) such that

\[
|f_n| < C A^n n! \quad \text{for every } n \in \mathbb{N}
\]

(resp. \( |\varphi_n| < C A^n n! \) for every \( n \in \mathbb{N} \)).

We denote by \( \mathbb{C}[[x]]_1 \) and \( \mathbb{C}[[x^{-1}]]_1 \) the sets of all power series at \( x = 0 \) and at \( x = \infty \) of Gevrey order 1. These sets are sub-algebras of \( \mathbb{C}[[x]] \) and \( \mathbb{C}[[x^{-1}]] \), respectively, as commutative differential algebras over \( \mathbb{C} \). [19, 30].

Definition 2.8. The formal Borel transform of a formal power series \( \hat{x} \) is called the formal series

\[
\hat{\mathcal{B}}_1 \hat{x}(\xi) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \xi^n.
\]

Likewise, we introduce the formal Borel transform of a formal power series near \( x = \infty \).

Definition 2.9. The formal Borel transform \( \hat{\mathcal{B}}_1 \) of order 1 of a formal power series \( \hat{x} \) is called the formal series

\[
\hat{\mathcal{B}}_1 \hat{x}(p) = \sum_{n=0}^{\infty} \frac{\varphi_n}{n!} p^n.
\]

If \( \hat{f}(x) \in \mathbb{C}[[x]]_1 \) (resp. \( \hat{\varphi}(x) \in \mathbb{C}[[x^{-1}]]_1 \)) then its formal Borel transform \( \hat{\mathcal{B}}_1 \) converges in a neighborhood of the origin \( \xi = 0 \) (resp. \( p = 0 \)) with a sum \( f(\xi) \) (resp. \( \varphi(p) \)).

Definition 2.10. Let \( f(\xi) \) be analytic and of exponential size at most 1 at \( \infty \), i.e. \( |f(\xi)| \leq A \exp(B|\xi|) \), \( \xi \in \theta \) along a direction \( \theta \) from 0 to \( +\infty e^{i\theta} \). Then, the integral

\[
(L_\theta f)(x) = \int_{0}^{+\infty e^{i\theta}} f(\xi) \exp\left(-\xi x\right) d\left(\frac{\xi}{x}\right)
\]

is said to be Laplace complex transform \( L_\theta \) of order 1 in the direction \( \theta \) of \( f \).

Definition 2.11. Let \( \varphi(p) \) be analytic and of exponential size at most 1 at \( \infty \), i.e. \( |\varphi(p)| \leq A \exp(B|p|) \), \( p \in \theta \) along a direction \( \theta \) from 0 to \( +\infty e^{i\theta} \). Then, the integral

\[
(L_\theta \varphi)(x) = \int_{0}^{+\infty e^{i\theta}} \varphi(p) \exp(p x) dp
\]

is said to be Laplace complex transform \( L_\theta \) of order 1 in the direction \( \theta \) of \( \varphi \).

Let the function \( \varphi(p) \) be analytic and satisfies the estimate

\[
|\varphi(p)| \leq A_0 e^{c_0 |p|}, \quad c_0 \in \mathbb{R}
\]

along a direction \( \theta \) from 0 to \( +\infty e^{i\theta} \). Then the Laplace complex transform \( L_\theta \) in the direction \( \theta \) satisfies some useful properties.

Lemma 2.12. [29] Let \( \varphi \) as above, \( \hat{\varphi} := (L_\theta \varphi) \) and \( c \in \mathbb{C} \). Then each of the functions \( -p \varphi(p), e^{-c p} \varphi(p) \) or \( 1 \ast \varphi(p) \) satisfies estimates of the form (2.22) and

- \((L_\theta(-p \varphi)) = \frac{d\hat{\varphi}}{dx}\),
- \((L_\theta(e^{-c p} \varphi)) = \hat{\varphi}(x + c),\)
- \((L_\theta(1 \ast \varphi)) = x^{-1} \hat{\varphi}(x),\)
Lemma 2.16. Assume that \( x \) is summable of a class of formal power series near respective domain of definition, \([26, 29]\).

Remark 2.15. If \( \hat{\beta} \) is summable in the direction \( \theta \) then

Definition 2.13. The formal power series \( \hat{f} = \sum_{n=0}^{\infty} f_n x^n \) is 1-summable (or Borel summable) in the direction \( \theta \) if there exist an open sector \( V \) bisected by \( \theta \) whose opening is \( > \pi \) and a holomorphic function \( f(x) \) in \( V \) such that for every non-negative integer \( N \)

\[
\left| f(x) - \sum_{n=0}^{N-1} f_n x^n \right| \leq C_{V_1} A_{V_1}^N N! |x|^N
\]
on every closed subsector \( \overline{V}_1 \) of \( V \) with constants \( C_{V_1}, A_{V_1} > 0 \) depending only on \( V_1 \). The function \( f(x) \) is called the 1-sum (or Borel sum) of \( \hat{f}(x) \) in the direction \( \theta \).

If a series \( \hat{f}(x) \) is 1-summable in all but a finite number of directions, we will say that it is 1-summable.

One useful criterion for a Gevrey series of order 1 to be 1-summable is given in terms of Borel and Laplace transforms:

Proposition 2.14. Let \( \hat{f} \in \mathbb{C}[[x]]_1 \) (resp. \( \hat{\varphi} \in \mathbb{C}[[x^{-1}]]_1 \)) and let \( \theta \) be a direction. The following are equivalent:

1. \( \hat{f} \) (resp. \( \hat{\varphi} \)) is 1-summable in the direction \( \theta \).
2. The convergent power series \( \hat{B}_1 \hat{f}(\xi) \) (resp. \( \hat{B}_1 \hat{\varphi}(p) \)) has an analytic continuation \( h \) (resp. \( g \)) in a full sector \( \{ j \in \mathbb{C} | 0 < |j| < \infty, |\arg(j) - \theta| < \epsilon \} \) for \( j = \xi \) (resp. \( j = p \)). In addition, this analytic continuation has exponential growth of order 1 at \( \infty \) on this sector, i.e. \( |h(\xi)| \leq A \exp(B |\xi|) \) (resp. \( |g(p)| \leq A_0 \exp(c_0 |p|) \)). In this case \( f = (\mathcal{L}_\theta h)(x) \) (resp. \( \varphi = (\mathcal{L}_\theta g)(x) \)) is its 1-sum in the direction \( \theta \).

Remark 2.15. If \( \hat{f}(x) \) (resp. \( \hat{\varphi}(x) \)) is convergent, then \( \hat{f}(x) \) (resp. \( \hat{\varphi}(x) \)) is \( k \)-summable in the direction \( \theta \) (any \( k > 0 \) and any \( \theta \)) and the classical sum and \( k \)-sum coincide in their respective domain of definition, \([26, 29]\).

The next result, which can be found in \([33]\) (see Lemma 4.1(I) in \([33]\)) provides 1-summability of a class of formal power series near \( x = 0 \).

Lemma 2.16. Assume that \( \beta_j \neq \beta_i \) and that \( \alpha_j - \alpha_i \notin \mathbb{Z}_{\leq -2} \). The formal power series

\[
\hat{f}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2 + \alpha_j - \alpha_i)^{(n)}}{(\beta_j - \beta_i)^n} x^n
\]
is 1-summable in any direction \( \theta \neq \arg(\beta_i - \beta_j) \) from 0 to \( +\infty e^{i\theta} \). The function

\[
f_\theta(x) = \int_0^{+\infty e^{i\theta}} \left( 1 + \frac{\xi}{\beta_j - \beta_i} \right)^{\alpha_j - \alpha_i - 2} e^{-\frac{\xi}{x}} d \left( \frac{\xi}{x} \right)
\]
defines its 1-sum in such a direction.

Here by the symbol \( (a)^{(n)} \) we denote the rising factorial

\[
(a)^{(n)} = a(a + 1)(a + 2) \ldots (a + n - 1), \quad (a)^{(0)} = 1.
\]
As a differential equation with an irregular point at the origin and an irregular point at 
\( x = \infty \) the initial equation admits unique formal fundamental matrices \( \Phi_0(x) \) and \( \Phi_\infty(x) \) in the form of the theorem of Hukuhata-Turrittin [34, 30]

\[
\Phi_0(x, 0) = \hat{L}(x) x^\Lambda \exp \left( -\frac{B}{x} \right), \quad \Phi_\infty(x, 0) = \hat{Q}(x) \left( \frac{1}{x} \right)^{-\Lambda} \exp(G x) .
\]

Here the matrices \( \Lambda, B \) and \( G \) are diagonal and

\[
\Lambda = \text{diag}(0, -2), \quad B = \text{diag}(\beta_1, \beta_2), \quad G = \text{diag}(\gamma_1, \gamma_2).
\]

Once having formal fundamental matrices we can introduce the so called formal monodromy matrices \( \hat{M}_0 \) and \( \hat{M}_\infty \) at the origin and at the infinity point.

**Definition 2.17.** The formal monodromy matrix \( \hat{M}_0 \) related to the formal fundamental matrix \( \Phi_0(x, 0) \) is defined as

\[
\Phi_0(x, 0) e^{2\pi i}, 0) = \Phi_0(x, 0) \hat{M}_0 .
\]

In the same manner, the formal monodromy matrix \( \hat{M}_\infty \) related to the formal fundamental matrix \( \Phi_\infty(x, 0) \) is defined as

\[
\Phi_\infty(x, 0) e^{-1}, e^{2\pi i}, 0) = \Phi_\infty(x, 0) \hat{M}_\infty .
\]

In particular,

\[
\hat{M}_0 = e^{2\pi i \Lambda} = I_2 = e^{-2\pi i \Lambda} = \hat{M}_\infty ,
\]

where \( I_2 \) is the identity matrix of order 2.

Since the formal monodromy matrices are equal to the identity matrix, in this paper we will present the formal fundamental matrices \( \Phi_0(x, 0) \) and \( \Phi_\infty(x, 0) \) in a slight different form

\[
\Phi_0(x, 0) = \exp(G x) \hat{H}(x) x^\Lambda \exp \left( -\frac{B}{x} \right),
\]

\[
\Phi_\infty(x, 0) = \exp \left( -\frac{B}{x} \right) \left( \frac{1}{x} \right)^{-\Lambda} \hat{P}(x) \exp(G x) .
\]

These special forms of the formal fundamental solutions, as well as, of the actual fundamental solutions allow us to show in an explicit way how these solutions are changed under the perturbation. Here the matrix-function \( x^\Lambda, \exp(G x) \) and \( \exp(-B/x) \) must be regarded as formal function. The entries of the matrices \( \hat{H}(x) \) and \( \hat{P}(x) \) are formal power series in \( x \) and \( x^{-1} \), respectively.

Utilizing the summability theory we relate to the formal fundamental matrices \( \hat{M}_0(x, 0) \) and \( \hat{M}_\infty(x, 0) \) actual fundamental matrices. More precisely,

**Theorem 2.18.** (Hukuhara-Turrittin-Martinet-Ramis) The entries of the matrix \( \hat{H}(x) \) (resp. \( \hat{P}(x) \)) in (2.23) are 1-summable in every non-singular direction \( \theta \). If we denote by \( H_\theta(x) \) (resp. \( P_\theta(x) \)) the 1-sum of \( \hat{H}(x) \) (resp. \( \hat{P}(x) \)) along \( \theta \) obtained from \( \hat{H}(x) \) (resp. \( \hat{P}(x) \)) by a Borel-Laplace transform, then \( \Phi_0^\theta(x, 0) = e^{G x} H_\theta(x) x^\Lambda e^{-B/x} \) (resp. \( \Phi_\infty^\theta(x, 0) = e^{-B/x} x^\Lambda P_\theta(x) e^{G x} \)) is an actual fundamental matrix at the origin (resp. at \( x = \infty \)) of the initial equation.
Let \( \theta \) be a singular direction of the initial equation at the origin. Let \( \theta^+ = \theta + \epsilon \) and \( \theta^- = \theta - \epsilon \), where \( \epsilon > 0 \) is a small number, be two non-singular neighboring directions of the singular direction \( \theta \). Denote by \( \Phi_0^{\theta^+}(x) \) and \( \Phi_0^{\theta^-}(x) \) the actual fundamental matrices at the origin of the initial equation corresponding to the direction \( \theta^+ \) and \( \theta^- \) in the sense of Theorem 2.18. Then

**Definition 2.19.** With respect to the given formal fundamental matrix \( \hat{\Phi}_0(x, 0) \) at the origin the Stokes matrix \( St_0^\theta \in GL_2(\mathbb{C}) \) corresponding to the singular direction \( \theta \) is defined as

\[
St_0^\theta = (\Phi_0^{\theta^+}(x, 0))^{-1} \Phi_0^{\theta^-}(x, 0).
\]

Similarly, let \( \theta \) be a singular direction of the initial equation at the infinity point. Let \( \theta^+ = \theta + \epsilon \) and \( \theta^- = \theta - \epsilon \), where \( \epsilon > 0 \) is a small number, be two non-singular neighboring directions of the singular direction \( \theta \). Denote by \( \Phi_\infty^{\theta^+}(x) \) and \( \Phi_\infty^{\theta^-}(x) \) the actual fundamental matrices at the origin of the initial equation corresponding to the direction \( \theta^+ \) and \( \theta^- \) in the sense of Theorem 2.18. Then

**Definition 2.20.** With respect to the given formal fundamental matrix \( \Phi_\infty(x, 0) \) at the infinity point the Stokes matrix \( St_\infty^\theta \in GL_2(\mathbb{C}) \) corresponding to the singular direction \( \theta \) is defined as

\[
St_\infty^\theta = (\Phi_\infty^{\theta^+}(x, 0))^{-1} \Phi_\infty^{\theta^-}(x, 0).
\]

2.5. **Fuchsian singularities.** We write \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

The considered in this paper Heun type equation is a Fuchsian equation of order 2 with 4 finite singular points taken at \( x_R = \sqrt{\varepsilon}, x_L = -\sqrt{\varepsilon}, x_{RR} = 1/\sqrt{\varepsilon} \) and \( x_{LL} = -1/\sqrt{\varepsilon} \). In this paragraph we briefly introduce the needed facts and definitions from the theory of the Fuchsian equations.

We firstly recall the necessary local theory of Fuchsian singularities following the book of Golubev [11]. Recall that in the case of the scalar differential equations the regular singularity and the Fuchsian singularity are the same notion. With every Fuchsian (regular) singularity of a given \( n \)-order scalar linear differential equation we associate a \( n \)-order algebraic equation, the so called characteristic (or inditial) equation. More precisely, consider an \( n \)-order linear differential equation

\[
y^{(n)}(x) + b_{n-1}(x) y^{(n-1)}(x) + \cdots + b_0(x) y(x) = 0, \quad b_j(x) \in \mathbb{C}(x).
\]

Let \( x = x_0 \in \mathbb{C} \) be a regular singularity for this equation. Recall that due to the theorem of Fuchs this means that all the functions

\[
b_{n-k}(x)(x - x_0)^k
\]

are holomorphic functions at \( x = x_0 \). Then

**Definition 2.21.** 1. The \( n \)-order algebraic equation

\[
\rho(n-1)(n-2) \cdots (n-(n-1)) + c_{n-1} \rho(n-2) \cdots (n-(n-2)) + \cdots + c_1 \rho + c_0 = 0,
\]

where

\[
c_k = \lim_{x \to x_0} b_{n-k}(x)(x - x_0)^k, \quad 0 \leq k \leq n - 1
\]

is called the characteristic (or the inditial) equation at the regular singularity \( x_0 \in \mathbb{C} \). Its roots \( \rho_k, 1 \leq k \leq n \) are called the characteristic exponents at the singularity \( x_0 \).

2. The characteristic equation at the point \( t = 0 \) of the equation obtained after the transformation \( x = 1/t \), is called the characteristic equation at \( x = \infty \). Its roots are called the characteristic exponents at the regular point \( x = \infty \).
Denote by $\rho_i^j$ and $\rho_{i}^{jj}$, $i = 1, 2$, $j = R, L$ the characteristic exponents at the singular points $x_j$ and $x_{jj}$, respectively. In [32], Proposition 4.6 we have proved that if we know the coefficients of the differential operators $L_{j,\varepsilon}$ then we can directly determine the characteristic coefficients at every singular points. The restriction of Proposition 4.6 to the Heun type reducible equation leads to

**Proposition 2.22.** The coefficients of the operators $L_{j,\varepsilon}$ in (1.4) are unique determined only by the characteristic exponents

\[
L_{1,\varepsilon} = \partial - \left( \frac{\rho_1^R}{x - x_R} + \frac{\rho_1^L}{x - x_L} + \frac{\rho_1^{RR}}{x - x_{RR}} + \frac{\rho_1^{LL}}{x - x_{LL}} \right),
\]

\[
L_{2,\varepsilon} = \partial - \left( \frac{\rho_2^R - 1}{x - x_R} + \frac{\rho_2^L - 1}{x - x_L} + \frac{\rho_2^{RR} - 1}{x - x_{RR}} + \frac{\rho_2^{LL} - 1}{x - x_{LL}} \right).
\]

Thanks to Proposition 2.22 the characteristic exponents $\rho_i^j$ and $\rho_{i}^{jj}$, $i = 1, 2$, $j = R, L$ are

\[
\rho_1^R = \frac{\beta_1}{2\sqrt{\varepsilon}}, \quad \rho_2^R = \frac{\beta_2}{2\sqrt{\varepsilon}}, \quad \rho_1^L = -\frac{\beta_1}{2\sqrt{\varepsilon}}, \quad \rho_2^L = -\frac{\beta_2}{2\sqrt{\varepsilon}},
\]

\[
\rho_1^{RR} = -\frac{\gamma_1}{2\sqrt{\varepsilon}}, \quad \rho_2^{RR} = 1 - \frac{\gamma_2}{2\sqrt{\varepsilon}}, \quad \rho_1^{LL} = \frac{\gamma_1}{2\sqrt{\varepsilon}}, \quad \rho_2^{LL} = 1 + \frac{\gamma_2}{2\sqrt{\varepsilon}}.
\]

The exponents differences $\Delta_{ij}^1 = \rho_1^j - \rho_2^j$ and $\Delta_{ij}^{jj} = \rho_1^{jj} - \rho_2^{jj}$ corresponding to the above characteristic exponents are defined as follows,

\[
\Delta_{ij}^R = \frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}}, \quad \Delta_{ij}^L = -\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}},
\]

\[
\Delta_{ij}^{RR} = -1 - \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}}, \quad \Delta_{ij}^{LL} = -1 + \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}}.
\]

The local theory of the Fuchsian singularities ensures a necessary condition for the existence of the logarithmic term near the singular point $x_j$ or $x_{jj}$ in the solution of the Heun type equation. If the exponent difference $\Delta_{ij}^1 \in \mathbb{N}_0$ or $\Delta_{ij}^{jj} \in \mathbb{N}_0$ then the Heun type equation always admits a particular solution $\Phi_1(x, \varepsilon)$ in the form

\[
\Phi_1(x, \varepsilon) = (x - x_j)^{\rho_i^j} h_{1,j}(x), \quad \Phi_1(x, \varepsilon) = (x - x_{jj})^{\rho_{i}^{jj}} h_{1,jj}(x),
\]

where $h_{1,j}(x)$ and $h_{1,jj}(x)$ are holomorphic functions in a neighborhood of the point $x_j$ and $x_{jj}$, respectively. But the second element $\Phi_{12}(x, \varepsilon)$ of the fundamental set of solutions near the point $x_j$ or $x_{jj}$ can contain logarithmic term. Classically, such a Fuchsian singularity $x_j$ (resp. $x_{jj}$) for which $\Delta_{ij}^1 \in \mathbb{N}_0$ (resp. $\Delta_{ij}^{jj} \in \mathbb{N}_0$) is called a resonant Fuchsian singularity [13]. In this paper we restrict our attention to these families of the perturbed equation for which there are exactly two different types of resonant Fuchsian singularities – one of the type $x_j$, $j = R, L$ and one of the type $x_{jj}, j = R, L$. Through this paper we call these values of the parameters $\beta_j, \gamma_j, \varepsilon$ for which the perturbed equation has two different types of resonant Fuchsian singularities just a double resonance. The motivation of studying the Heun type equation with double resonances naturally arises from our previous work [32] where we have shown that during a resonance the Stokes matrices of the initial equation are connected by a limit with the parts of the monodromy matrices of the perturbed equation when $\sqrt{\varepsilon} \to 0$. In the present paper we study the double resonances with the purpose to connect again by a limit both Stokes matrices of the initial equation with suitable monodromy matrices of the perturbed equation.
We finish this paragraph by introducing the notion of the monodromy matrices following the paper of Dubrovin and Mazzocco [6]. Consider the Heun type equation over \( X = \mathbb{C}P^1 - \{x_R, x_L, x_{RR}, x_{LL}\} \). Let \( \Phi(x, \varepsilon) \) be a fundamental matrix of the Heun type equation. It is a multivalued analytic function on the punctured Riemann sphere \( X \). This multivaluedness is described by the monodromy matrices. Let \( x_0 \in X \) be a point that does not lie on the same line with three of the singular points. Note that three of the singular points (therefore all the points) lie on the same line if and only if either \( \sqrt{\varepsilon} \in \mathbb{R} \) or \( \sqrt{\varepsilon} \in i\mathbb{R} \). In fact the so called from us double resonance holds when \( \sqrt{\varepsilon} \in \mathbb{R} \). Let \( \gamma_R, \gamma_L, \gamma_{RR}, \gamma_{LL} \) be simple closed loops starting and ending at the point \( x_0 \), going around the singular points \( x_R, x_L, x_{RR}, x_{LL} \), respectively, in positive direction and not crossing each other. The so chosen loops fix a basis in the fundamental group \( \pi_1(X, x_0) \) with base point at \( x_0 \) of the punctured Riemann sphere \( X \). Let \( \Phi_{\gamma}(x, \varepsilon) \) be the result of the analytic continuation of the fundamental matrix \( \Phi(x, \varepsilon) \) along the loop \( \gamma \in \pi_1(x, x_0) \). Since the Heun type equation is a linear equation, the matrix \( \Phi_{\gamma}(x, \varepsilon) \) is also a fundamental matrix of the same equation. Therefore there is a unique invertable constant matrix \( M_{\gamma}(\varepsilon) \in GL_2(\mathbb{C}) \) such that
\[
\Phi_{\gamma}(x, \varepsilon) = \Phi(x, \varepsilon) M_{\gamma}.
\]
The matrix \( M_{\gamma} \) depends only on the homotopy class \( [\gamma] \) of the loop \( \gamma \).

**Definition 2.23.** The antihomomorphism mapping
\[
\pi_1(X, x_0) \rightarrow GL_2(\mathbb{C})
\]
\( [\gamma] \rightarrow M_{\gamma}(\varepsilon) \)
\( M_{\gamma_1 \gamma_2}(\varepsilon) = M_{\gamma_2}(\varepsilon) M_{\gamma_1}(\varepsilon), \quad M_{\gamma^{-1}}(\varepsilon) = M_{\gamma}^{-1}(\varepsilon) \)
determines monodromy representation of the Heun type equation with respect to the given fundamental matrix.

**Definition 2.24.** The images \( M_j(\varepsilon) = M_{\gamma_j}(\varepsilon) \) of the generators \( \gamma_R, \gamma_L, \gamma_{RR}, \gamma_{LL} \) of \( \pi_1(X, x_0) \) under the monodromy representation are called the monodromy matrices of the Heun type equation.

In Section 4, Theorem 4.1 we will present explicitly fundamental matrices with respect to which we will compute the corresponding monodromy matrices.

2.6. **Symmetries.** The considered in this paper equations can be rewritten as second-order homogeneous equation
\[
y'' + b_1(x, \varepsilon) y' + b_0(x, \varepsilon) y = 0.
\]
(2.24)
The coefficients \( b_j(x, 0) \) of the equation (1.11) - (1.12) are
\[
b_1(x, 0) = -\frac{\alpha_1 + \alpha_2}{x} - \frac{\beta_1 + \beta_2}{x^2} - (\gamma_1 + \gamma_2),
b_0(x, 0) = \frac{\alpha_1 \gamma_2 + \alpha_2 \gamma_1}{x} + \frac{\alpha_1 \alpha_2 + \beta_1 \gamma_2 + \beta_2 \gamma_1}{x^2} + \frac{2 \beta_1 + \alpha_1 \beta_2 + \alpha_2 \beta_1}{x^3} + \frac{\beta_1 \beta_2}{x^4} + \gamma_1 \gamma_2.
\]
The equation (1.1) - (1.2) with coefficients \( b_j(x,0) \) is invariant under the following transformations:

\[
\begin{align*}
\alpha_1 &\rightarrow -\alpha_1, \quad \alpha_2 \rightarrow -\alpha_2 - 2, \quad \beta_j \rightarrow -\gamma_j, \quad \gamma_j \rightarrow -\beta_j, \quad x \rightarrow \frac{1}{x}, \quad y \rightarrow y, \\
\alpha_1 &\rightarrow -\alpha_1, \quad \alpha_2 \rightarrow -\alpha_2 - 2, \quad \beta_j \rightarrow \gamma_j, \quad \gamma_j \rightarrow \beta_j, \quad x \rightarrow -\frac{1}{x}, \quad y \rightarrow y, \\
\alpha_1 &\rightarrow \alpha_1, \quad \alpha_2 \rightarrow \alpha_2, \quad \beta_j \rightarrow -\beta_j, \quad \gamma_j \rightarrow -\gamma_j, \quad x \rightarrow -x, \quad y \rightarrow y.
\end{align*}
\]

Then we have the following obvious result.

**Proposition 2.25.** Let \( \phi(x) \) be a particular solution near \( x = 0 \) of the equation (1.1) - (1.2) with parameters \( \alpha_1 = 0, \alpha_2 = -2 \) and \( \beta_j, \gamma_j, j = 1, 2 \) arbitrary. Then

1. \( \phi(x^{-1}) \) is a particular solution near \( x = \infty \) of the equation (1.1) - (1.2) with parameters \( \alpha_1 = \alpha_2 = 0 \) and \( \beta_j = -\gamma_j, j = 1, 2 \).
2. \( \phi(-x^{-1}) \) is a particular solution near \( x = \infty \) of the equation (1.1) - (1.2) with parameters \( \alpha_1 = \alpha_2 = 0 \) and \( \beta_j = \gamma_j \).
3. \( \phi(-x) \) is a particular solution near the origin of the equation (1.1) - (1.2) with parameters \( \alpha_1 = 0, \alpha_2 = -2 \) and \( -\beta_j, -\gamma_j \).

When \( \alpha_1 = 0, \alpha_2 = -2 \) the transformation

\[ x \rightarrow \frac{1}{x} \]

takes the operators \( L_{J,\varepsilon} \) from (1.6) of the Heun type equation into operators of a Heun type equation

\[
\begin{align*}
L_{1,\varepsilon} &= \partial - \frac{\gamma_1}{2\sqrt{\varepsilon}} \left( \frac{1}{x - \sqrt{\varepsilon}} + \frac{1}{x + \sqrt{\varepsilon}} \right) - \frac{\beta_1}{2\sqrt{\varepsilon}} \left( \frac{1}{x - \frac{1}{\sqrt{\varepsilon}}} - \frac{1}{x + \frac{1}{\sqrt{\varepsilon}}} \right), \\
L_{2,\varepsilon} &= \partial + \frac{\gamma_2}{2\sqrt{\varepsilon}} \left( \frac{1}{x - \sqrt{\varepsilon}} - \frac{1}{x + \sqrt{\varepsilon}} \right) + \left( 1 - \frac{\beta_2}{2\sqrt{\varepsilon}} \right) \frac{1}{x - \frac{1}{\sqrt{\varepsilon}}} + \left( 1 + \frac{\beta_2}{2\sqrt{\varepsilon}} \right) \frac{1}{x + \frac{1}{\sqrt{\varepsilon}}}.
\end{align*}
\]

Thus the Heun type equation is invariant under the following transformation

\[
\begin{align*}
\beta_1 &\rightarrow -\gamma_1, \quad \gamma_1 \rightarrow -\beta_1, \quad 1 - \frac{\beta_2}{2\sqrt{\varepsilon}} \rightarrow \frac{\gamma_2}{2\sqrt{\varepsilon}}, \quad 1 + \frac{\beta_2}{2\sqrt{\varepsilon}} \rightarrow -\frac{\gamma_2}{2\sqrt{\varepsilon}}, \quad x \rightarrow \frac{1}{x}, \quad y \rightarrow y.
\end{align*}
\]

3. **The initial equation**

In this section we will compute the Stokes matrices of the initial equation at \( x = 0 \) and \( x = \infty \). In paragraph 2.2 we introduced a global fundamental matrix of the initial equation. Recall that its element \( \Phi_{12}(x,0) \) is represented as an integral

\[
\Phi_{12}(x,0) = \Phi_1(x,0) \int_{\Gamma(x,0)} \frac{\Phi_2(z,0)}{\Phi_1(z,0)} \, dz,
\]

where \( \Phi_j(x,0), j = 1,2 \) are the solutions of the equations \( L_{j,0}(u) = 0 \). In this paper we use two different fundamental matrices with respect to which we will compute the Stokes
matrices: the matrix $\Phi_0(x,0)$ at $x = 0$ and the matrix $\Phi_\infty(x,0)$ at $x = \infty$. The difference between these matrix solutions is the path of integration $\Gamma(x,0)$. Choosing

$$\Phi_1(x,0) = e^{\gamma_1 x} e^{-\frac{\beta_1}{x}}, \quad \Phi_2(x,0) = \frac{e^{\gamma_2 x} e^{-\frac{\beta_2}{x}}}{x^2},$$

the element $\Phi_{12}(x,0)$ becomes

$$\Phi_{12}(x,0) = \Phi_1(x,0) \int_{\Gamma(x,0)} \frac{e^{(\gamma_2-\gamma_1)z} e^{-\frac{\beta_2-\beta_1}{z}}}{z^2} \, dz.$$

When $\Phi_{12}(x,0)$ is an element of the matrix $\Phi_0(x,0)$ the path $\Gamma(x,0)$ is a path from 0 to $x$, approaching 0 in the direction $\theta = \arg(\beta_2 - \beta_1)$. When $\Phi_{12}(x,0)$ is an element of the matrix $\Phi_\infty(x,0)$ the path $\Gamma(x,0)$ is a path from $+\infty e^{i\theta}$ to $x$, approaching $+\infty e^{i\theta}$ in the direction $\theta = \arg(\gamma_1 - \gamma_2)$.

**Remark 3.1.** When $\beta_1 = \beta_2$ the integral

$$\int_0^x \frac{e^{(\gamma_2-\gamma_1)z}}{z^2} \, dz$$

does not exists. For this reason in this paper we study the initial and perturbed equations under assumption that

$$\beta_1 \neq \beta_2 \quad \text{and} \quad \gamma_i, i = 1, 2 \quad \text{are arbitrary;}$$

In fact, when $\beta_1 = \beta_2$ the point $x = 0$ is a resonant irregular singularity for the initial equation. To study such an irregular singularity we have to use some other representation at the origin of the solution $\Phi_{12}(x,0)$. It seems that an integral representation but with a base point different from 0 to be useful. This problem, as well as its perturbed analog are postponed for further researches.

The integral representation of the element $\Phi_{12}(x,0)$ is unsuitable for the computation of the Stokes matrices. On the other hand, as we show below, this integral is easily represented as formal series at the origin, as well as the infinity point. Then applying summability theory we lift these series to the corresponding actual solutions. With respect to these actual solutions we can easily compute the Stokes multipliers $\mu_{j}^{\theta}, j = 0, \infty$.

We begin by deducing formal fundamental matrices at the origin and at the infinity point from the integrals.

**Proposition 3.2.** Assume that $\beta_1 \neq \beta_2$. Then the initial equation possesses an unique formal fundamental matrix $\hat{\Phi}_0(x,0)$ at the origin in the form

$$\hat{\Phi}_0(x,0) = \exp(Gx) \hat{H}(x) x^{\Lambda} \exp\left(-\frac{B}{x}\right),$$

where

$$G = \text{diag}(\gamma_1, \gamma_2), \quad \Lambda = \text{diag}(0,-2), \quad B = \text{diag}(\beta_1, \beta_2).$$

The matrix $\hat{H}(x)$ is given by

$$\hat{H}(x) = \begin{pmatrix} 1 & \frac{x^2 e^{(\gamma_2-\gamma_1)x}}{\beta_2-\beta_1} - \frac{(\gamma_2-\gamma_1)x^2 \psi(x)}{\beta_2-\beta_1} \\ 0 & \frac{1}{\beta_2-\beta_1} \end{pmatrix}.$$
The element \( \hat{\psi}(x) \) is represented by a formal power series

\[
(3.27) \quad \hat{\psi}(x) = \sum_{k=1}^{\infty} \frac{(-1)^k k! S_{k-1}}{(\beta_2 - \beta_1)^k} x^k
\]

where \( S_{k-1}, k \in \mathbb{N} \) are the partial sums of the following infinite absolutely convergent series of numbers

\[
(3.28) \quad S = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (\gamma_2 - \gamma_1)^n (\beta_2 - \beta_1)^n}{n!(n+1)!}.
\]

Moreover,

1. If \( S \neq 0 \) but \( \gamma_1 \neq \gamma_2 \) then the formal power series \( \hat{\psi}(x) \) is divergent.
2. If \( S = 0 \) but \( \gamma_1 \neq \gamma_2 \) then the power series \( \hat{\psi}(x) \) represents an analytic in \( \mathbb{C} \) function.
3. If \( \gamma_1 = \gamma_2 \) then \( \hat{\psi}(x) = \frac{x}{\beta_2 - \beta_1} \) but the matrix \( \hat{H}(x) \) has the form

\[
(3.29) \quad \hat{H}(x) = \begin{pmatrix} 1 & \frac{x^2}{ \beta_2 - \beta_1} \\ 0 & 1 \end{pmatrix}.
\]

Proof. We have already chosen \( \hat{\Phi}_1(x) \) and \( \hat{\Phi}_2(x) \). Then if \( \gamma_1 \neq \gamma_2 \) for \( \hat{\Phi}_{12}(x) \) we have

\[
\hat{\Phi}_{12}(x) = \hat{\Phi}_1(x) \int_0^x \frac{\hat{\Phi}_2(z)}{\hat{\Phi}_1(z)} \, dz = e^{\gamma_1 x} e^{-\frac{\gamma_1}{x}} \int_0^x \frac{e^{(\gamma_2 - \gamma_1)z} e^{-\frac{\beta_2 - \beta_1}{z}}}{z^2} \, dz
\]

where the integral is taken in the direction \( \arg(\beta_2 - \beta_1) \). In particular case when \( \gamma_1 = \gamma_2 \) the element \( \hat{\Phi}_{12}(x, 0) \) becomes

\[
\hat{\Phi}_{12}(x, 0) = e^{\gamma_1 x} e^{-\frac{\gamma_1}{x}} \int_0^x \frac{e^{-\frac{\beta_2 - \beta_1}{z}}}{z^2} \, dz = \frac{e^{\gamma_1 x} e^{-\frac{\beta_2}{\beta_1}}}{\beta_2 - \beta_1},
\]

where the integral is again taken in the direction \( \arg(\beta_2 - \beta_1) \).

Let \( \gamma_1 \neq \gamma_2 \). The Taylor series at \( z = 0 \) of the function \( e^{(\gamma_2 - \gamma_1)z} \) is

\[
e^{(\gamma_2 - \gamma_1)z} = \sum_{k=0}^{\infty} \frac{(\gamma_2 - \gamma_1)^k}{k!} z^k.
\]

Then it is not so difficult to show that

\[
\int_0^x e^{(\gamma_2 - \gamma_1)z} e^{-\frac{\beta_2 - \beta_1}{z}} \, dz = \sum_{k=0}^{\infty} \frac{(\gamma_2 - \gamma_1)^k}{k!} \int_0^x z^k e^{-\frac{\beta_2 - \beta_1}{z}} \, dz =
\]

\[
= (\beta_2 - \beta_1) S \int_0^x \frac{e^{-\frac{\beta_2 - \beta_1}{z}}}{z} \, dz - x e^{-\frac{\beta_2 - \beta_1}{x}} S \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(\beta_2 - \beta_1)^k} x^k + x e^{-\frac{\beta_2 - \beta_1}{x}} \sum_{k=1}^{\infty} \frac{(-1)^k k! S_{k-1}}{(\beta_2 - \beta_1)^k} x^k,
\]
where $S$ is defined by the series
\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\gamma_2 - \gamma_1)^n(\beta_2 - \beta_1)^n}{n!(n+1)!},
\]
and $S_{k-1}, k \in \mathbb{N}$ are its partial sums. This series of numbers is absolutely convergent and we denote by $S$ its sum.

It turns out that the sum of the first two addends in the above representation of the integral $\int_0^\infty e^{(\gamma_2 - \gamma_1)z} e^{-(\beta_2 - \beta_1)/z} dz$ is equal to zero. Indeed, consider the integral
\[
h(x) = e^{-\beta_1} \int_0^x \frac{e^{-\beta_2 - \beta_1}}{z} dz.
\]
Let us present it in the following form
\[
h(x) = e^{-\beta_2} \int_0^x \frac{e^{(\beta_2 - \beta_1)(\frac{1}{x} - \frac{1}{z})}}{z} dz.
\]
Introducing a new variable $\zeta$ via
\[
(\beta_2 - \beta_1) \left( \frac{1}{x} - \frac{1}{z} \right) = -\zeta
\]
the integral $h(x)$ gets into
\[
h(x) = e^{-\beta_2} \int_0^{\infty(\tau)} \frac{e^{-\zeta}}{\zeta + \beta_2 - \beta_1} d\zeta,
\]
where $\tau = \arg(\beta_2 - \beta_1)$. The analytic continuation of $h(x)$ on the $x$-plane yields an analytic in $x$ function
\[
h_\theta(x) = e^{-\beta_2} \int_0^{+\infty e^{i\theta}} \frac{e^{-\zeta}}{\zeta + \beta_2 - \beta_1} d\zeta
\]
in all directions $\theta$ except along the direction $\arg(\beta_1 - \beta_2)$.

Next, consider the formal power series
\[
\hat{f}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(\beta_2 - \beta_1)^k} x^k,
\]
This series belongs to the class of 1-summable series from Lemma 2.16 for $2 + \alpha_2 - \alpha_1 = 1$ and $\beta_2 - \beta_i = \beta_2 - \beta_1$. Thus for any direction $\theta \neq \arg(\beta_2 - \beta_1)$ the function
\[
f_\theta(x) = \int_0^{+\infty e^{i\theta}} \left( 1 + \frac{\zeta}{\beta_2 - \beta_1} \right)^{-1} e^{-\zeta} d \left( \frac{\zeta}{x} \right) = \frac{\beta_2 - \beta_1}{x} \int_0^{+\infty e^{i\theta}} \frac{e^{-\frac{\zeta}{x}}}{\beta_2 - \beta_1 + \zeta} d\zeta
\]
defines its 1-sum in such a direction. Then, the explicit form of the element $\Phi_{12}(x)$ follows from a combination of the above observations on the integral $h(x)$ and the formal series $\hat{f}(x)$.

Finally, the formal power series
\[
\hat{\psi}(x) = \sum_{k=1}^{\infty} \frac{(-1)^k k! S_{k-1}}{(\beta_2 - \beta_1)^k} x^k = \sum_{k=1}^{\infty} b_k x^k
\]
is convergent only at \( x = 0 \) when \( S \neq 0 \). Indeed, since the series (3.28) is convergent (and \( S \) is its sum), its sequence of partial sums \( \{ S_{k-1} \} \) satisfy the following limits

\[
\lim_{k \to \infty} S_{k-1} = S \neq 0.
\]

Then, for the radius of convergence \( R \) for the power series \( \hat{\psi}(x) \), we have

\[
R = \lim_{k \to \infty} \left| \frac{b_k}{b_{k+1}} \right| = \lim_{k \to \infty} \frac{1}{k+1} \left| \frac{(\beta_2 - \beta_1) S_{k-1}}{S_k} \right| = \lim_{k \to \infty} \frac{|\beta_2 - \beta_1|}{k+1} = 0.
\]

Let now \( S = 0 \). Then \( S_{k-1} \) and \( S_k \) can be rewritten as

\[
S_{k-1} = -\sum_{n=k}^{\infty} \frac{(-1)^{n+1}(\gamma_2 - \gamma_1)^n(\beta_2 - \beta_1)^n}{n!(n+1)!} = -\frac{(-1)^k(\gamma_2 - \gamma_1)^k(\beta_2 - \beta_1)^k}{k!(k+1)!} \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma_2 - \gamma_1)^n(\beta_2 - \beta_1)^n}{(k+1)^{(n)}(k+2)^{(n)}}.
\]

\[
S_k = -\sum_{n=k+1}^{\infty} \frac{(-1)^{n+1}(\gamma_2 - \gamma_1)^n(\beta_2 - \beta_1)^n}{n!(n+1)!} = -\frac{(-1)^{k+1}(\gamma_2 - \gamma_1)^{k+1}(\beta_2 - \beta_1)^{k+1}}{(k+1)!(k+2)!} \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma_2 - \gamma_1)^n(\beta_2 - \beta_1)^n}{(k+2)^{(n)}(k+3)^{(n)}}.
\]

Then for the the radius of convergence \( R \) we obtain

\[
R = \lim_{k \to \infty} \frac{|\beta_2 - \beta_1|}{k+1} \left| \frac{\gamma_2 - \gamma_1}{k} \right| \frac{\beta_2 - \beta_1}{k+1} \frac{(k+1)!}{(k+2)!} \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma_2 - \gamma_1)^n(\beta_2 - \beta_1)^n}{(k+1)^{(n)}(k+2)^{(n)}}.
\]

Since each of the infinite series has a limit 1 when \( k \to \infty \) for \( R \) we find that

\[
R = \lim_{k \to \infty} \frac{k+2}{|\gamma_2 - \gamma_1|} = \infty.
\]

Thus the series \( \hat{\psi}(x) \) is convergent in \( \mathbb{C} \). Unfortunately till now we can not show explicitly which is the analytic in \( \mathbb{C} \) function, that has \( \hat{\psi}(x) \) as a series expansion around \( x = 0 \).

This completes the proof. \( \square \)

Recall that the Bessel function of the first kind \( J_\nu(z) \) of order \( \nu \) can be defined by its series expansion around \( z = 0 \), [31],

\[
J_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{2k}}{k!(n+k+1)}, \quad \text{if} \quad -\nu \notin \mathbb{N},
\]

\[
J_{-\nu}(z) = (-1)^\nu J_\nu(z), \quad \text{if} \quad -\nu \in \mathbb{N}.
\]

Here \( \Gamma(x) \) is the classical Gamma function (see [2]). It is well known that for \( \nu > -1 \) the Bessel function \( J_\nu(z) \) has an infinite number of zeros at that only real zeros. In particular, if \( \nu = 1 \) the first five zeros of \( J_1(z) \) are \( z = 0, z = 3.8317, z = 7.0156, z = 10, 1735, z = 13, 3237 \).

**Remark 3.3.** The condition \( S \neq 0 \) implies that the number \( z = 2\sqrt{(\gamma_2 - \gamma_1)(\beta_2 - \beta_1)} \) is not a zero of the Bessel function \( J_1(z) \).
Remark 3.4. The formal solution $\hat{\Phi}_{12}(x)$ can be obtained as a formal particular solution near $x = 0$ of the following non-homogenous linear ODE

\begin{equation}
(3.31) \quad y'(x) - \left(\frac{\beta_1}{x} + \gamma_1\right) y(x) = \frac{e^{\gamma_2 x} e^{-\frac{\beta_2}{x}}}{x^2}.
\end{equation}

Looking for $\hat{\Phi}_{12}(x)$ in the form $e^{\gamma_2 x} e^{-\frac{\beta_2}{x}} \hat{y}(x)$, we find that $\hat{y}(x)$ must satisfy the equation

\[ x^2 \hat{y}'(x) + \left[(\beta_2 - \beta_1) + (\gamma_2 - \gamma_1) x^2 \right] \hat{y}(x) = 1. \]

The last equation admits an unique formal solution

\[ \hat{y}(x) = \sum_{k=0}^{\infty} a_k x^k, \]

provided that $\beta_2 \neq \beta_1$. The coefficients $a_k$ are recursively constructed from the equation

\[(k - 1) a_{k-1} + (\beta_2 - \beta_1) a_k + (\gamma_2 - \gamma_1) a_{k-2} = 0, \quad k \geq 2,\]

as $a_0 = \frac{1}{\beta_2 - \beta_1}$, $a_1 = 0$. Note that we are not able to give an explicit formula for the coefficients $a_k$. In particular,

\begin{align*}
    a_2 &= \frac{-\gamma_2 - \gamma_1}{(\beta_2 - \beta_1)^2}, \quad a_3 = \frac{2(\gamma_2 - \gamma_1)}{(\beta_2 - \beta_1)^3}, \quad a_4 = -\frac{3!(\gamma_2 - \gamma_1)}{(\beta_2 - \beta_1)^4} + \frac{(\gamma_2 - \gamma_1)^2}{(\beta_2 - \beta_1)^3}, \\
    a_5 &= 4!(\gamma_2 - \gamma_1) - \frac{6(\gamma_2 - \gamma_1)^2}{(\beta_2 - \beta_1)^4}, \quad a_6 = -\frac{5!(\gamma_2 - \gamma_1)}{(\beta_2 - \beta_1)^5} + \frac{6.3!(\gamma_2 - \gamma_1)^2}{(\beta_2 - \beta_1)^4} - \frac{(\gamma_2 - \gamma_1)^3}{(\beta_2 - \beta_1)^4}, \ldots
\end{align*}

As a result $\hat{\Phi}_{12}(x)$ becomes

\[ \hat{\Phi}_{12}(x) = \frac{e^{\gamma_2 x} e^{-\frac{\beta_2}{x}}}{\beta_2 - \beta_1} + x e^{\gamma_1 x} e^{-\frac{\beta_2}{x}} \sum_{k=1}^{\infty} a_{k+1} x^k. \]

Now we represent $e^{\gamma_2 x}$ as

\[ e^{\gamma_2 x} = e^{\gamma_1 x} e^{(\gamma_2 - \gamma_1)x} = e^{\gamma_1 x} \sum_{k=0}^{\infty} \frac{(\gamma_2 - \gamma_1)^k}{k!} x^k. \]

Then we can rewrite $\hat{\Phi}_{12}(x)$ as

\[ \hat{\Phi}_{12}(x) = \frac{e^{\gamma_2 x} e^{-\frac{\beta_2}{x}}}{\beta_2 - \beta_1} + x e^{\gamma_1 x} e^{-\frac{\beta_2}{x}} \sum_{k=1}^{\infty} c_k x^k, \]

where the formal series $\sum c_k x^k$ is the product of the series $\sum \frac{(\gamma_2 - \gamma_1)^k}{k!} x^k$ and $\sum a_{k+1} x^k$. In particular,

\[ c_k = \sum_{s=0}^{k-1} b_s a_{k+1-s}, \quad b_s = \frac{(\gamma_2 - \gamma_1)^s}{s!}. \]

So, the series $\sum_{k=1}^{\infty} c_k x^k$ is nothing but the series $\frac{x^{k-1}}{\beta_2 - \beta_1} \hat{\psi}(x)$ from Proposition 3.2.

We give here this method to illustrate the difficulty of the construction of a suitable particular formal solution even in the case of second order reducible equation. Fortunately, in our case we have an integral expression of $\Phi_{12}(x)$ whose formal representation near the origin gives us a formal solution $\hat{\Phi}_{12}(x)$.

Thanks to Proposition 2.25 we construct a formal fundamental matrix of the initial equation at the point $x = \infty$. 
Proposition 3.5. Let $\beta_1 \neq \beta_2$. Then the initial equation possesses an unique formal fundamental matrix $\hat{\Phi}_\infty(x,0)$ at $x = \infty$ in the form

$$\hat{\Phi}_\infty(x,0) = \exp\left(-\frac{B}{x}\right) \left(\begin{array}{cc} 1 & -\Lambda \hat{P}(x) \exp(Gx) \\
0 & 1 \end{array}\right),$$

where the matrices $B, \Lambda$ and $G$ are introduced by Proposition 3.2. The matrix $\hat{P}(x)$ is defined as follows:

1. If $\gamma_1 = \gamma_2$ then

$$\hat{P}(x) = \left(\begin{array}{cc} 1 & e^{-\beta_2 - \beta_1 x} \\
0 & 1 \end{array}\right).$$

2. If $\gamma_1 \neq \gamma_2$ then

$$\hat{P}(x) = \left(\begin{array}{cc} 1 & -\hat{\phi}(x) x \\
0 & 1 \end{array}\right).$$

The element $\hat{\phi}(x)$ is defined by the formal series

$$\hat{\phi}(x) = \sum_{k=1}^{\infty} \frac{k!}{(\gamma_2 - \gamma_1)^k} x^{-k}.$$

The symbol $S_{k-1}$ is defined by Proposition 3.2.

Moreover,

(a) If $\lim_{k \to \infty} S_{k-1} \neq 0$ then the formal series $\hat{\phi}(x)$ is divergent.

(b) If $\lim_{k \to \infty} S_{k-1} = 0$ then the series $\hat{\phi}(x)$ defines an analytic in $\mathbb{C}P^1 - \{0\}$ function.

Proof. The elements $\hat{\Phi}_j(x)$, $j = 1, 2$ as the solutions of the equations $L_j y = 0$ remain the same as in Proposition 3.2. From Proposition 2.25 it follows that if $\hat{\Phi}_{12}(t)$ is a particular solution at $t = 0$ of the initial equation with parameters $\alpha_1 = \alpha_2 = 0, \beta_j = -\gamma_j$ then $\hat{\Phi}_{12}(x^{-1})$ is a particular solution near $x = \infty$ of our initial equation. For the initial equation with $\alpha_1 = \alpha_2 = 0, \beta_j = -\gamma_j$ the operators $L_j$ become

$$L_j = \partial + \left(\frac{\gamma_j}{t^2} + \beta_j\right), \quad \partial = \frac{d}{dt}.$$

Then the corresponding entries $\hat{\Phi}_j(t)$ are given by

$$\hat{\Phi}_j(t) = e^{-\beta_j t} e^{\frac{\gamma_j}{t}}.$$

Following the method of construction of $\hat{\Phi}_{12}(x)$ in Proposition 3.2 we find that when $\gamma_1 \neq \gamma_2$

$$\hat{\Phi}_{12}(t) = -t e^{-\beta_1 t} \int_0^t e^{-(\beta_2 - \beta_1)z} e^{\frac{\gamma_2 - \gamma_1}{z}} dz = -te^{\frac{\gamma_2}{t}} e^{-\beta_1 \frac{t}{t}} \sum_{k=1}^{\infty} \frac{k! S_{k-1}}{(\gamma_2 - \gamma_1)^k} \frac{t^k}{t^k}.$$
where the integral is taken in the direction \( \arg(\gamma_1 - \gamma_2) \). The sign minus comes from the observation that the transformation \( x = 1/t \) changes the equation
\[
y'(x) - \left(\frac{\beta_1}{x} + \gamma_1\right) y(x) = \frac{e^{\gamma_2 x} e^{-\frac{\beta_2}{x}}}{x^2},
\]
which has \( \hat{\Phi}_{12} \) as a particular solution, into the equation
\[
\hat{y}(t) + \left(\beta_1 + \frac{\gamma_1}{t}\right) y(t) = -e^{\frac{\gamma_2 t}{t^2}} e^{-\beta_2 t}.
\]
The symbol \( S_{k-1} \) is defined by \( \text{Proposition 3.2} \). Then putting \( t = x^{-1} \) in this formula for \( \hat{\Phi}_{12}(t) \) we obtain a particular solution near \( x = \infty \) of our equation.

In the same way as in the proof of \( \text{Proposition 3.2} \) we can show that the series
\[
\hat{\varphi}(x) = \sum_{k=1}^{\infty} \frac{k! S_{k-1}}{(\gamma_2 - \gamma_1)^k} x^{-k}
\]
is divergent when \( \lim_{k \to \infty} S_{k-1} \neq 0 \), and has a radius of convergence \( R = \infty \) when \( \lim_{k \to \infty} S_{k-1} = 0 \).

When \( \gamma_1 = \gamma_2 = \gamma \) the element \( \hat{\Phi}_{12}(t) \) becomes
\[
\hat{\Phi}_{12}(t) = -e^{-\beta_1 t} e^{\frac{\gamma}{t}} \int_0^t e^{-t(\beta_2 - \beta_1)z} d_z = \frac{e^{-\beta_1 t} e^{\frac{\gamma}{t}}}{\beta_2 - \beta_1} \left[ e^{-t(\beta_2 - \beta_1) t} - 1 \right],
\]
where the integral is taken in the real positive axis. Putting \( t = x^{-1} \) in this formula we obtain \( \hat{\Phi}_{12}(x) \) when \( \gamma_1 = \gamma_2 \).

This ends the proof. \( \square \)

In keeping with the theory the divergent power series \( \hat{\psi}(x) \) defined by \( (3.27) \) is 1-summable. More precisely,

**Lemma 3.6.** Let \( \beta_2 \neq \beta_1, \gamma_2 \neq \gamma_1 \). Assume also that \( S \neq 0 \), where \( S \) is given by \( (3.28) \). Then the formal power series \( \hat{\psi}(x) \) defined by \( (3.27) \) is 1-summable in any direction \( \theta \neq \arg(\beta_1 - \beta_2) \). The function
\[
\psi_\theta(x) = \frac{\beta_2 - \beta_1}{x} \int_0^{+\infty} e^{\theta z} \frac{u(\zeta) e^{-\frac{\zeta}{\zeta + \beta_2 - \beta_1}} d\zeta}{(\gamma_2 - \gamma_1) x - 1},
\]
where \( u(\zeta) \) is the analytic in \( \mathbb{C} \) function
\[
u(\zeta) = \sum_{k=0}^{\infty} \frac{(\gamma_2 - \gamma_1)^k}{k! (k+1)!} \zeta^k,
\]
defines its 1-summ in such a direction.

**Proof.** Because of convergence of the sequence \( \{S_k\} \) of the partial sums of the series \( (3.28) \) there exists a positive constant \( B \) such that
\[
|S_k| \leq B \quad \text{for all} \quad k \geq 0.
\]
Then for the coefficients \( b_k = \frac{(-1)^k k! S_{k-1}}{(\beta_2 - \beta_1)^k} \) of the formal series \( \hat{\psi}(x) \) we find the following estimate
\[
|b_k| \leq B A^k k! \quad \text{for all} \quad k \in \mathbb{N},
\]
where \( A = 1/|\beta_2 - \beta_1| \). Therefore the series \( \hat{\psi}(x) \) is of Gevrey order 1.
As a result the formal Borel transform of the series \( \dot{\psi}(x) \)

\[
\psi(\zeta) = \mathcal{B}_1(\dot{\psi})(\zeta) = \sum_{k=1}^{\infty} \frac{(-1)^k S_{k-1}}{(\beta_2 - \beta_1)^k \zeta^k}
\]

is an analytic function near the origin in the Borel \( \zeta \)-plane. Moreover, it turns out that

\[
\psi(\zeta) = -\left( \frac{1}{1 + \frac{\zeta}{\beta_2 - \beta_1}} - 1 \right) \frac{J_1(2\sqrt{(\gamma_1 - \gamma_2)} \zeta)}{\sqrt{(\gamma_1 - \gamma_2)}},
\]

where

\[
J_1(2\sqrt{(\gamma_1 - \gamma_2)} \zeta) = \sum_{k=0}^{\infty} \frac{(\gamma_2 - \gamma_1)^k}{k! (k+1)!} \zeta^k
\]

is the Bessel function of order 1 (see [34]). In particular, the analytic in \( \mathbb{C} \) function \( J_1(2\sqrt{(\gamma_1 - \gamma_2)} \zeta)/\sqrt{(\gamma_1 - \gamma_2)} \) is the classical Gamma function (see [2]). In particular, the analytic in \( \mathbb{C} \) function \( J_1(2\sqrt{(\gamma_1 - \gamma_2)} \zeta)/\sqrt{(\gamma_1 - \gamma_2)} \) is a particular solution near \( \zeta = 0 \) of the following differential equation

\[
\zeta u''(\zeta) + 2u'(\zeta) - (\gamma_2 - \gamma_1) u(\zeta) = 0.
\]

For simplicity we denote by \( u(\zeta) \) the function \( J_1(2\sqrt{(\gamma_1 - \gamma_2)} \zeta)/\sqrt{(\gamma_1 - \gamma_2)} \). Then \( \psi(\zeta) \) becomes

\[
\psi(\zeta) = \frac{\zeta}{\zeta + \beta_2 - \beta_1} u(\zeta).
\]

The function \( \psi(\zeta) \) has an analytic continuation along any ray \( \theta \neq \arg(\beta_1 - \beta_2) \) and

\[
|\psi(\zeta)| \leq A_0 e^{\gamma_2 - \gamma_1 |\zeta|}
\]

along such a ray for an appropriate constant \( A_0 \). Then the complex Laplace transform of the function \( \psi(\zeta) \) gives

\[
\psi_\theta(x) = \mathcal{L}_\theta \psi(x) = \int_0^{+\infty} e^{\theta x} \psi(\zeta) \exp\left(-\frac{\zeta}{x}\right) d\left(\frac{\zeta}{x}\right) =
\]

\[
= \frac{1}{x} \int_0^{+\infty} e^{\theta x} \frac{\zeta u(\zeta) e^{-\frac{\zeta}{x}}}{\zeta + \beta_2 - \beta_1} d\zeta =
\]

\[
= \frac{1}{x} \int_0^{+\infty} e^{\theta x} \frac{u(\zeta) e^{-\frac{\zeta}{x}} d\zeta}{\zeta + \beta_2 - \beta_1} - \frac{\beta_2 - \beta_1 x}{\gamma_2 - \gamma_1} \int_0^{+\infty} e^{\theta x} \frac{u(\zeta) e^{-\frac{\zeta}{x}}}{\zeta + \beta_2 - \beta_1} d\zeta.
\]

The analytic function \( u(\zeta) \) can be regarded as the formal Borel transform of the convergent series

\[
\sum_{k=0}^{\infty} \frac{(\gamma_2 - \gamma_1)^k}{(k+1)!} x^k = \sum_{s=1}^{\infty} \frac{(\gamma_2 - \gamma_1)^{s-1}}{s!} x^{s-1} = \frac{1}{x(\gamma_2 - \gamma_1)} \left( e^{(\gamma_2 - \gamma_1) x} - 1 \right).
\]

Since the Borel sum of an analytic function in \( \mathbb{C} \) gives the same function, we find that for any ray \( \theta \neq \arg(\beta_1 - \beta_2) \) the function

\[
\psi_\theta(x) = -\frac{\beta_2 - \beta_1}{x} \int_0^{+\infty} e^{\theta x} \frac{u(\zeta) e^{-\frac{\zeta}{x}}}{\zeta + \beta_2 - \beta_1} d\zeta + \frac{1}{x(\gamma_2 - \gamma_1)} \left( e^{(\gamma_2 - \gamma_1) x} - 1 \right).
\]
defines the 1 - sum of \( \hat{\psi}(x) \) in such a direction.

This completes the proof. \( \square \)

**Remark 3.7.** Since

\[
\left| \frac{u(\zeta)}{\zeta + \beta_2 - \beta_1} \right| \leq Ae^{\|\gamma_2 - \gamma_1\| |x|}
\]

for an appropriate constant \( A > 0 \), the Laplace integral

\[
\int_{0}^{\infty} e^{it} \frac{u(\zeta)}{\zeta + \beta_2 - \beta_1} d\zeta
\]

exists and defines a holomorphic function in the open disc [19]

\[
D_\theta(|\gamma_2 - \gamma_1|) = \left\{ x \in \mathbb{C} \middle| \Re \left( \frac{e^{it}}{x} \right) > |\gamma_2 - \gamma_1| \right\}.
\]

When we move the direction \( \theta \in I \) continuously the corresponding 1-sums \( \psi_\theta(x) \) stick each other analytically and define a holomorphic function \( \tilde{\psi}(x) \) on a sector

\[
\bigcup_{\theta \in I} \tilde{D}_\theta(|\gamma_2 - \gamma_1|)
\]

whose opening is \( > 2\pi \). Here \( I \) is an open interval in \( \mathbb{R} \) such that

\[
I = (-2\pi + \arg(\beta_1 - \beta_2), \arg(\beta_1 - \beta_2)), \quad \text{if } \arg(\beta_1 - \beta_2) \in [-\pi, 2\pi),
\]

and

\[
I = (\arg(\beta_1 - \beta_2), \arg(\beta_1 - \beta_2) + 2\pi) \quad \text{if } \arg(\beta_1 - \beta_2) \in [0, \pi).
\]

The set \( \tilde{D}_\theta(|\gamma_2 - \gamma_1|) \) is the lifting of \( D_\theta(|\gamma_2 - \gamma_1|) \) on the Riemann surface of the natural logarithm \( \tilde{\mathbb{C}} \), i.e.

\[
\tilde{D}_\theta(|\gamma_2 - \gamma_1|) = \left\{ x = re^{it} \in \tilde{\mathbb{C}} \middle| r < \frac{1}{|\gamma_2 - \gamma_1|}, t \in (\theta - \arccos r|\gamma_2 - \gamma_1|, \theta + \arccos r|\gamma_2 - \gamma_1|) \right\}.
\]

The notation \( e^{it} \) represents a point on \( \tilde{\mathbb{C}} \) which is ”above” the complex number \( e^{it} \). Then the point \( re^{it} \) is the canonical projection of \( re^{it} \) on \( \mathbb{C} \). In this way the points \( re^{it} \) and \( re^{i(t+2\pi m)}, m \in \mathbb{Z} \) are regarded as different points on \( \tilde{\mathbb{C}} \) but with the same projection on \( \mathbb{C} \) (see [29] for details).

The function \( \tilde{\psi}(x) \) is a multivalued function on this sector. It is asymptotic to the series \( \hat{\psi}(x) \) in Gevrey order 1 sense on this sector and defines the 1-sum of this series. In every non-singular direction \( \theta \) the multivalued function \( \tilde{\psi}(x) \) has one value \( \psi_\theta(x) \). Near the singular direction \( \theta = \arg(\beta_1 - \beta_2) \) the function \( \tilde{\psi}(x) \) has two different values: \( \psi_\theta^+(x) = \psi_{\theta+\epsilon}(x) \) and \( \psi_\theta^-(x) = \psi_{\theta-\epsilon}(x) \) where \( \epsilon > 0 \) is a small number.

We have a similar result for the summation of the divergent series \( \tilde{\varphi}(x) \) from (3.34).

**Lemma 3.8.** Let \( \beta_2 \not= \beta_1, \gamma_2 \not= \gamma_1 \). Assume also that \( S \not= 0 \), where \( S \) is given by (3.28). Then the formal power series \( \tilde{\varphi}(x) \) defined by (3.34) is 1 - summable in any direction \( \theta \not= \arg(\gamma_2 - \gamma_1) \). The function

\[
\varphi_\theta(x) = -x \int_{0}^{+\infty} e^{it} \frac{\nu(p) e^{-xp}}{1 - \frac{p}{\gamma_2 - \gamma_1}} dp = -x \frac{e^{-\frac{\beta_2 - \beta_1}{\gamma_2 - \gamma_1}} - 1}{\beta_2 - \beta_1}.
\]
where \( v(p) \) is the analytic in \( \mathbb{C} \) function

\[
v(p) = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta_2 - \beta_1)^k}{k! (k+1)!} p^k,
\]

defines its \( 1 \)-sum in such a direction.

**Proof.** Similar to the series \( \hat{\psi}(x) \) the coefficients of the series \( \hat{\varphi}(x) \) satisfy the following estimate

\[
\left| \frac{k! S_{k-1}}{(\gamma_2 - \gamma_1)^k} \right| \leq B A^k k! \quad \text{for all} \quad k \in \mathbb{N},
\]

where \( A = 1/|\gamma_2 - \gamma_1| \). The constant \( B \) is the same as in the proof of Lemma 3.6. Therefore the series \( \hat{\varphi}(x) \) is of Gevrey order 1. As a result the formal Borel transform of the series \( \hat{\varphi}(x) \)

\[
\varphi(p) = \hat{B}_1 \hat{\varphi}(p) = \sum_{k=0}^{\infty} \frac{(k+1)! S_k}{(\gamma_2 - \gamma_1)^{k+1} k!} p^k = \sum_{k=0}^{\infty} \frac{(k+1) S_k}{(\gamma_2 - \gamma_1)^k} p^k
\]
is an analytic function near the origin in the Borel \( p \)-plane. Observe that

\[
\varphi(p) = \frac{d}{dp} \left( \sum_{k=0}^{\infty} \frac{S_k p^{k+1}}{(\gamma_2 - \gamma_1)^{k+1}} \right) = \frac{dW(p)}{dp}.
\]

The series \( W(p) \) defines the function

\[
W(p) = \left( \frac{1}{1 - \frac{p}{\gamma_2 - \gamma_1}} - 1 \right) \sum_{k=0}^{\infty} \frac{(-1)^k (\beta_2 - \beta_1)^k}{k! (k+1)!} p^k.
\]

As in the proof of Lemma 3.6 the convergent in \( \mathbb{C} \) series defines the Bessel function of order 1

\[
J_1(2\sqrt{(\beta_2 - \beta_1)p}/(\gamma_2 - \gamma_1)p) = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta_2 - \beta_1)^k}{k! (k+1)!} p^k.
\]

For simplicity we denote by \( v(p) \) the analytic in \( \mathbb{C} \) function \( J_1(2\sqrt{(\beta_2 - \beta_1)p}/(\gamma_2 - \gamma_1)p) \).

In particular, the entire function \( v(p) \) satisfies the differential equation

\[
p v''(p) + 2v'(p) + (\beta_2 - \beta_1) v(p) = 0.
\]

Then \( W(p) \) becomes

\[
W(p) = \left( 1 - \frac{1}{1 - \frac{p}{\gamma_2 - \gamma_1}} \right) v(p).
\]

Now applying the properties of the complex Laplace transform (see Lemma 2.12) we find that

\[
(L_\theta \varphi)(x) = \left( L_\theta \frac{dW}{dp} \right)(x) = x (L_\theta W)(x) - W(0) =
\]

\[
x \int_{0}^{+\infty} e^{i\theta} \left( 1 - \frac{1}{1 - \frac{p}{\gamma_2 - \gamma_1}} \right) v(p) e^{-xp} \, dp =
\]

\[
x \int_{0}^{+\infty} v(p) e^{-xp} \, dp - x \int_{0}^{+\infty} e^{i\theta} \frac{v(p) e^{-xp}}{1 - \frac{p}{\gamma_2 - \gamma_1}} \, dp.
\]
Since $v(p)/(1 - \frac{p}{\gamma_2 - \gamma_1})$ has exponential growth $\leq 1$ at $\infty$, i.e.
\[
\left| \frac{v(p)}{1 - \frac{p}{\gamma_2 - \gamma_1}} \right| \leq A_0 e^{\beta_2 - \beta_1 |p|},
\]
along any direction $\theta \neq \arg(\gamma_2 - \gamma_1)$ form 0 to $+\infty e^{i\theta}$ the integral
\[
\int_0^{+\infty e^{i\theta}} \frac{v(p) e^{-xp}}{1 - \frac{p}{\gamma_2 - \gamma_1}} dp
\]
defines a holomorphic function on (see [29])
\[
\pi_\theta = \left\{ x \in \mathbb{C} \mid Re(x e^{i\theta}) > |\beta_2 - \beta_1| \right\}.
\]

The analytic in $\mathbb{C}$ function
\[
v(p) = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta_2 - \beta_1)^k}{k! (k + 1)!} p^k
\]
can be regarded as the formal Borel transform of the convergent series
\[
\sum_{k=0}^{\infty} \frac{(-1)^k (\beta_2 - \beta_1)^k}{(k + 1)!} x^{-k-1} = -\frac{1}{\beta_2 - \beta_1} \sum_{s=1}^{\infty} \frac{(-1)^s (\beta_2 - \beta_1)^s}{s!} x^{-s} = -\frac{1}{\beta_2 - \beta_1} \left[ e^{-\beta_2 - \beta_1} - 1 \right].
\]
Since the Borel sum of an analytic function in $\mathbb{C} \mathbb{P}^1 - \{0\}$ gives the same function, for any ray $\theta \neq \arg(\gamma_2 - \gamma_1)$ the function
\[
\varphi_\theta(x) = -x \int_0^{+\infty e^{i\theta}} \frac{v(p) e^{-xp}}{1 - \frac{p}{\gamma_2 - \gamma_1}} dp - x \frac{e^{\beta_2 - \beta_1} - 1}{\beta_2 - \beta_1}
\]
defines the 1-sum of the formal series $\hat{\varphi}(x)$.

This ends the proof. \hfill \square

**Remark 3.9.** Following [29] we denote by $\tilde{\pi}_\theta$ the lifting of $\pi_\theta$ of the Riemann surface of the natural logarithm $\hat{\mathbb{C}}$, i.e.
\[
\tilde{\pi}_\theta := \left\{ x = re^{it} \in \hat{\mathbb{C}} \mid r > |\beta_2 - \beta_1|, t \in \left( -\theta - \arccos \frac{|\beta_2 - \beta_1|}{r}, -\theta + \arccos \frac{|\beta_2 - \beta_1|}{r} \right) \right\},
\]
Denote also by $\tilde{D}(I, |\beta_2 - \beta_1|)$ the sector on $\hat{\mathbb{C}}$
\[
\tilde{D}(I, |\beta_2 - \beta_1|) := \bigcup_{\theta \in I} \tilde{\pi}_\theta.
\]
Here $I$ is an open interval in $\mathbb{R}$ such that
\[
I = (-2\pi + \arg(\gamma_2 - \gamma_1), \arg(\gamma_2 - \gamma_1)),
\]
if $\arg(\gamma_2 - \gamma_1) \in [-\pi, 2\pi)$, and
\[
I = (\arg(\gamma_2 - \gamma_1), \arg(\gamma_2 - \gamma_1) + 2\pi)
\]
if $\arg(\gamma_2 - \gamma_1) \in [0, \pi)$. When we move the direction $\theta \in I$ continuously the corresponding 1-sums $\varphi_\theta(x)$ stick each other analytically and define a holomorphic function $\hat{\varphi}(x)$ on the sector $\tilde{D}(I, |\beta_2 - \beta_1|)$ of opening $> 2\pi$. On this sector the multivalued function $\hat{\varphi}(x)$ defines the Borel sum of the series $\varphi(x)$ and it is asymptotic to this series in Gevrey order.
1. In every non-singular direction $\theta$ it has one value $\varphi_\theta(x)$. Near the singular direction $\theta = \arg(\gamma_2 - \gamma_1)$ the multivalued function $\tilde{\varphi}(x)$ has two different values: $\varphi^+_\theta = \varphi_{\theta+\epsilon}(x)$ and $\varphi^-_\theta = \varphi_{\theta-\epsilon}(x)$ for a small number $\epsilon > 0$.

Denote by $F(x)$ the actual matrix-function $x^\Lambda \exp(-B/x)$, where the matrices $\Lambda$ and $B$ are defined by Proposition 3.2. The first existence result says

**Proposition 3.10.** Let $\beta_1 \neq \beta_2$.

1. Assume that $S = 0$ where $S$ is defined by (3.28) but $\gamma_1 \neq \gamma_2$. Then the initial equation possesses an unique actual fundamental matrix $\Phi(x,0)$ at the origin in the form

$$\Phi(x,0) = \exp(Gx) H(x) F(x),$$

where $H(x) = \hat{H}(x)$, defined by (3.29), is an analytic in $\mathbb{C}$ matrix-function. The matrix $F(x)$ is the branch of $x^\Lambda \exp(-B/x)$ for $\arg(x)$ and the matrix $G$ is introduced by Proposition 3.2.

2. Assume that $\gamma_1 = \gamma_2$. Then the initial equation possesses an unique actual fundamental matrix at the origin in the form (3.38), where the matrix $H(x) = \hat{H}(x)$ is defined by (3.29). The matrix $F(x)$ is the branch of $x^\Lambda \exp(-B/x)$ for $\arg(x)$ and the matrix $G$ is introduced by Proposition 3.2.

3. Assume that $S \neq 0$ where $S$ is defined by (3.28) but $\gamma_1 \neq \gamma_2$. Then for every non-singular direction $\theta$ the initial equation possesses an unique actual fundamental matrix $\Phi_0^\theta(x,0)$ at the origin of the form

$$\Phi_0^\theta(x,0) = \exp(Gx) H_\theta(x) F_\theta(x), \quad \Phi_0^{\theta+2\pi}(x,0) = \Phi_0^\theta(x,0) \hat{M} = \Phi_0^\theta(x,0).$$

The matrix $F_\theta(x)$ is the branch of $x^\Lambda \exp(-B/x)$ for $\arg(x) = \theta$ and the matrix $G$ is introduced by Proposition 3.2. The matrix $H_\theta(x)$ is defined by

$$H_\theta(x) = \begin{pmatrix} 1 & -\frac{x^2}{\beta_2 - \beta_1} + x^2 \phi_\theta(x) \\ 0 & 1 \end{pmatrix},$$

where

$$\phi_\theta(x) = \int_0^{+\infty} e^{i\theta} \frac{u(\zeta) e^{-\zeta}}{\zeta + \beta_2 - \beta_1} d\zeta$$

with

$$u(\zeta) = \sum_{k=0}^{\infty} \left( \frac{\gamma_2 - \gamma_1}{k!(k+1)!} \right)^k \zeta^k.$$

For the singular direction $\theta = \arg(\beta_1 - \beta_2)$ the initial equation admits two actual fundamental matrices at the origin

$$(\Phi_0^\theta(x,0))^\pm = \Phi_0^{\theta\pm\epsilon}(x,0),$$

where the matrices $\Phi_0^{\theta\pm\epsilon}(x,0)$ are given by (3.39) and $\epsilon > 0$ is a small number.

**Theorem 3.11.** With respect to the actual fundamental matrix at the origin given by Proposition 3.10 and extended by Remark 3.7, the initial equation has one singular direction $\theta = \arg(\beta_1 - \beta_2)$. The corresponding Stokes matrix $St_0^\theta$ is

$$St_0^\theta = \begin{pmatrix} 1 & -2\pi i(\gamma_2 - \gamma_1) S \\ 0 & 1 \end{pmatrix},$$

where $S$ is introduced by (3.28).
Proof. Let $\theta^+ = \theta + \epsilon$ and $\theta^- = \theta - \epsilon$ for a small positive $\epsilon$ be two directions which are slightly to the left and to the right of the singular direction $\theta = \arg(\beta_1 - \beta_2)$. Let $\Phi_{0+}^\theta(x,0)$ and $\Phi_{0-}^\theta(x,0)$ be the associated actual fundamental matrices at $x = 0$ on these directions (in sense of Proposition 3.10).

Then comparing $\Phi_{0+}^\theta(x,0)$ and $\Phi_{0-}^\theta(x,0)$ we find

$$\Phi_{0-}^\theta(x,0) - \Phi_{0+}^\theta(x,0) = \begin{pmatrix} 1 & \mu_0^\theta \\ 0 & 1 \end{pmatrix},$$

where

$$\mu_0^\theta = (\gamma_2 - \gamma_1) e^{\gamma_1 x} e^{-\frac{\beta_2}{x}} [\phi_g^-(x) - \phi_g^+(x)] = 2\pi i (\gamma_2 - \gamma_1) e^{\gamma_1 x} e^{-\frac{\beta_2}{x}} \text{Res} \left( \frac{u(\zeta) e^{-\frac{\zeta}{\zeta + \beta_2 - \beta_1}}}{\zeta + \beta_2 - \beta_1}; \zeta = \beta_1 - \beta_2 \right) = 2\pi i (\gamma_2 - \gamma_1) u(\beta_1 - \beta_2) \Phi_1(x,0).$$

Observe that $u(\beta_1 - \beta_2) = -S$ where $S$ is the sum of the absolutely convergent number series

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1} (\gamma_2 - \gamma_1)^k (\beta_2 - \beta_1)^k}{k! (k+1)!},$$

introduced by (3.28). Now, a direct application of the definition of the Stokes matrix gives us the result.

Note that when either $S = 0$ or $\gamma_1 = \gamma_2$ the Stokes matrix $S_{0}^\theta = I_2$. This phenomenon witness the fact that the series $\hat{\nu}(x)$ from (3.27) is convergent when $S = 0$ or $\gamma_1 = \gamma_2$. Moreover it is in concordance with a theorem on the analytic dependence of the Stokes matrix $S_{0}^\theta$ on the parameter (see [12]).

This ends the proof.

The second existence result states

**Proposition 3.12.** Let that $\beta_1 \neq \beta_2$.

1. Assume that $\gamma_1 = \gamma_2$. Then the initial equation possesses an unique actual fundamental matrix $\Phi_\infty(x,0)$ at $x = \infty$ in the form

$$\Phi_\infty(x,0) = \exp \left( -\frac{B}{x} \right) \left( \frac{1}{x} \right)^{-\Lambda} P(x) \exp(Gx),$$

where $P(x) = \tilde{P}(x)$ is defined by (3.32). The matrices $\Lambda, B$ and $G$ are defined by Proposition 3.2 and $\exp(Gx)$ is the branch of $\exp(Gx)$ for $\arg(x)$.

2. Assume that $\gamma_1 \neq \gamma_2$ but $S = 0$ where $S$ is introduced by (3.28). Then the initial equation possesses an unique actual fundamental matrix $\Phi_\infty(x,0)$ at $x = \infty$ in the form (3.40), where $P(x) = \tilde{P}(x)$, defined by (3.33), is an analytic in $\mathbb{CP}^1 - \{0\}$ matrix-function. The matrices $\Lambda, B$ and $G$ are defined by Proposition 3.2 and $\exp(Gx)$ is the branch of $\exp(Gx)$ for $\arg(x)$.

3. Assume that $\gamma_1 \neq \gamma_2$ but $S \neq 0$ where $S$ is introduced by (3.28). Then for every non-singular direction $\theta$ the initial equation possesses an unique actual fundamental matrix $\Phi_{\infty}^\theta(x,0)$ near $x = \infty$ of the form

$$\Phi_{\infty}^\theta(x,0) = \exp \left( -\frac{B}{x} \right) \left( \frac{1}{x} \right)^{-\Lambda} P_\theta(x) \exp(Gx)_\theta.$$
The matrices $B$, $\Lambda$ and $G$ are given in Proposition 3.2 and $(\exp(Gx))_\theta$ is the branch of $\exp(Gx)$ for $\arg(x) = \theta$. The matrix $P_\theta(x)$ is defined by

$$P_\theta(x) = \begin{pmatrix} 1 & \frac{e^{\beta_2-\beta_1}-1}{\beta_2-\beta_1} + \omega_\theta(x) \\ 0 & 1 \end{pmatrix},$$

where

$$\omega_\theta(x) = \int_0^{+\infty} e^{i\theta} \frac{v(p) e^{-xp}}{1 - \frac{p}{\gamma_2 - \gamma_1}} dp$$

with

$$v(p) = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta_2 - \beta_1)^k}{k!(k+1)!} p^k.$$

For the singular direction $\theta = \arg(\gamma_2 - \gamma_1)$ the initial equation admits two actual fundamental matrices at $x = \infty$

$$(\Phi_{\infty}^\theta(x,0))^\pm = \Phi_{\infty}^{\theta \pm \epsilon}(x,0),$$

where the matrices $\Phi_{\infty}^{\theta \pm \epsilon}(x,0)$ are given by (3.41) and $\epsilon > 0$ is a small number.

**Theorem 3.13.** With respect to the actual fundamental matrix near $x = \infty$ given by Proposition 3.12 and extended by Remark 3.9 the initial equation has one singular direction $\theta = \arg(\gamma_2 - \gamma_1)$. The corresponding Stokes matrix $St_{\infty}^\theta$ is

$$St_{\infty}^\theta = \begin{pmatrix} 1 & 2\pi i(\gamma_2 - \gamma_1)S \\ 0 & 1 \end{pmatrix},$$

where $S$ is introduced by (3.28).

**Proof.** The proof is similar to the proof of Theorem 3.11.

We again observe that, according to expectation, $St_{\infty}^\theta = I_2$ when either $S = 0$ or $\gamma_1 = \gamma_2$. This fact one more time confirms the convergence of the series $\hat{\phi}(x)$ from (3.34) and is in keeping with a theorem on analytic dependence of the Stokes matrices on the parameter (see [12]).

4. **Heun type equation**

In this section we will compute the monodromy matrices of the Heun type equation. The Heun type equation is invariant under the transformation

$$\sqrt{\epsilon} \rightarrow -\sqrt{\epsilon}.$$

So through this section we assume that

(4.42) \[ \arg(\sqrt{\epsilon}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \]

Observe that in this case

$$\arg\left(\frac{1}{\sqrt{\epsilon}}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Observe also that $\arg(\sqrt{\epsilon}) = \arg\left(\frac{1}{\sqrt{\epsilon}}\right)$ if and only if $\sqrt{\epsilon} \in \mathbb{R}^+$, and $\arg(\sqrt{\epsilon}) = \arg\left(-\frac{1}{\sqrt{\epsilon}}\right)$ if and only if $\sqrt{\epsilon} \in i\mathbb{R}^+$.

In the introduction we have denoted by $x_j, j = R, L$ the singular points $\sqrt{\epsilon}$ and $-\sqrt{\epsilon}$, and by $x_{jj}, j = R, L$ the singular points $1/\sqrt{\epsilon}$ and $-1/\sqrt{\epsilon}$. Recall that when $\beta_1 \neq \beta_2, \gamma_1 \neq \gamma_2$ the initial equation admits two Stokes matrices – one of them correspond to the origin and the other corresponds to the infinity point. In order to realize both Stokes matrices
as a limit of two different matrices $e^{2\pi i T_j}$ we concern with this Heun type equation which has exactly two resonant singular points. One of them will be of the type $x_j$, and the other will be of the type $x_{jj}$. Throughout this paper we call such a resonance for which two singular points of different type are resonant, a double resonance. We distinguish 4 different types of double resonances.

- The double resonance of type (A.1). It is defined by the condition
  \[ \Delta_{12}^L = \frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}}, \quad \Delta_{12}^{LL} + 1 = \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \in \mathbb{N}_0. \]

- The double resonance of type (A.2). It is defined by the condition
  \[ \Delta_{12}^L = \frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}}, \quad \Delta_{12}^{RR} + 1 = \frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \in \mathbb{N}_0. \]

- The double resonance of type (A.3). It is defined by the condition
  \[ \Delta_{12}^R = \frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}}, \quad \Delta_{12}^{LL} + 1 = \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \in \mathbb{N}_0. \]

- The double resonance of type (A.4). It is defined by the condition
  \[ \Delta_{12}^R = \frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}}, \quad \Delta_{12}^{RR} + 1 = \frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \in \mathbb{N}_0. \]

In accordance with the initial equation we consider two different fundamental matrix $\Phi_j(x, \varepsilon), j = 0, \infty$ of the perturbed equation. The fundamental matrix $\Phi_0(x, \varepsilon)$ corresponds to the fundamental matrix $\Phi_0(x, 0)$ of the initial equation at the origin. Therefore the integral that defines the function $\Phi_{12}(x, \varepsilon)$ must be taken in the direction $\text{arg}(\beta_2 - \beta_1)$ with the same base point $x$ as in the definition of the matrix $\Phi_0(x, 0)$. The fundamental matrix $\Phi_\infty(x, \varepsilon)$ corresponds to the fundamental matrix $\Phi_\infty(x, 0)$ of the initial equation at $x = \infty$. Therefore its element $\Phi_{12}(x, \varepsilon)$ contains an integral that must be taken in the direction $\text{arg}(\gamma_1 - \gamma_2)$ with the same base point $x$ as in the definition of the matrix $\Phi_\infty(x, 0)$. The next theorem provides the explicit form of the fundamental matrices $\Phi_0(x, \varepsilon)$ and $\Phi_\infty(x, \varepsilon)$ introduced by Theorem 2.3.

**Theorem 4.1.** Assume that $\beta_1 \neq \beta_2$. Then both fundamental matrices $\Phi_0(x, \varepsilon)$ and $\Phi_\infty(x, \varepsilon)$ have the same elements $\Phi_j(x, \varepsilon), j = 1, 2$

\[
\Phi_1(x, \varepsilon) = \left(\frac{x - \sqrt{\varepsilon}}{x + \sqrt{\varepsilon}}\right)^{\frac{\beta_1}{2\sqrt{\varepsilon}}}, \quad \frac{1}{\sqrt{\varepsilon}, \varepsilon - x}^{\frac{\gamma_1}{2\sqrt{\varepsilon}}}, \quad \Phi_2(x, \varepsilon) = \left(\frac{1}{\sqrt{\varepsilon} - x} \right)^{\frac{\beta_2}{2\sqrt{\varepsilon}}}, \quad \frac{1}{\sqrt{\varepsilon}, \varepsilon + x}^{\frac{\gamma_2}{2\sqrt{\varepsilon}}}. \]

The element $\Phi_{12}(x, \varepsilon)$ of the fundamental matrix $\Phi_0(x, \varepsilon)$ is defined as

\[
\Phi_{12}(x, \varepsilon) = \Phi_1(x, \varepsilon) \int_\varepsilon^x \left(\frac{z - \sqrt{\varepsilon}}{z + \sqrt{\varepsilon}}\right)^{\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} + 1} \left(\frac{1}{\sqrt{\varepsilon} - z} \right)^{\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}}} dz,
\]

where $\varepsilon = \sqrt{\varepsilon}$ during a double resonance of type A.1 and A.2. During a resonance of type A.3 or A.4 $\varepsilon = -\sqrt{\varepsilon}$. The integral is taken in the direction $\text{arg}(\beta_2 - \beta_1)$ as the base point $x$ is a point near the origin.
Remark 4.3. When \( \beta_1 = 2 \), the function \( \Phi_{12}(x, \varepsilon) \) always lies on the real positive axis. The function \( \Phi_{12} \) is not defined in this case. To overcome the above problem we have to use some other kind of solution or some other kind of perturbation. For this reason the so called resonant perturbation is out of scope of this paper.

Proof. The statement follows immediately either from the conditions

\[
\Delta_{21}^{RR} - 1 = \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \in \mathbb{N}, \quad \arg\left(\frac{1}{\sqrt{\varepsilon}}\right) = \arg(\gamma_1 - \gamma_2)
\]

that define the double resonance of type A.1 or A.3, or from the conditions

\[
\Delta_{21}^{LL} - 1 = \frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \in \mathbb{N}, \quad \arg\left(-\frac{1}{\sqrt{\varepsilon}}\right) = \arg(\gamma_1 - \gamma_2)
\]

that define the double resonance of type A.2 or A.4. Indeed these conditions imply that

\[
\arg\left(\frac{1}{\sqrt{\varepsilon}}\right) = \arg(\sqrt{\varepsilon}).
\]

But the latter is true if and only if \( \sqrt{\varepsilon} \in \mathbb{R}^+ \). And in particular, \( \varepsilon \in \mathbb{R}^+ \).

In very particular case when \( \gamma_1 = \gamma_2 \) the element \( \Phi_{12}(x, \varepsilon) \) of both matrices \( \Phi_0(x, \varepsilon) \) and \( \Phi_\infty(x, \varepsilon) \) do not contain logarithmic terms. Moreover, one can assume that the path of integration of the element \( \Phi_{12}(x, \varepsilon) \) of \( \Phi_\infty(x, \varepsilon) \) can be taken in any direction, that does not cross the points \( x_j, j = R, L \). In this paper on purpose completeness of the text we fix always this path in the direction \( \arg(\gamma_1 - \gamma_2) \). So when \( \gamma_1 = \gamma_2 \) the direction will be the real positive axis.

This ends the proof. \( \square \)

Remark 4.3. When \( \beta_1 = \beta_2 \) we meet with the same problem as with the initial equation (see Remark 3.1). The function \( \Phi_{12}(x, \varepsilon) \) as an element of the matrix \( \Phi_0(x, \varepsilon) \) becomes

\[
\Phi_{12}(x, \varepsilon) = \Phi_1(x, \varepsilon) \int_x^x \frac{1}{(z - \sqrt{\varepsilon})(z + \sqrt{\varepsilon})} \left(\frac{1}{\sqrt{\varepsilon}} + z\right)^{2z-1} d\zeta
\]

when \( \beta_1 = \beta_2 \). Here \( \varepsilon \) is either \( \sqrt{\varepsilon} \) or \( -\sqrt{\varepsilon} \). But this integral does not exist. Recall that when \( \beta_1 = \beta_2 \) the point \( x = 0 \) is a resonant irregular singular point for the initial equation (see Remark 3.1). To overcome the above problem we have to use some other kind of solution or some other kind of perturbation. For this reason the so called resonant irregular singularities are out of scope of this paper.

Thanks to Proposition 4.2 we fix the paths of integration including in the solution \( \Phi_{12}(x, \varepsilon) \) as follows:
During a double resonance of type A.1 for both fundamental matrices \( \Phi_0(x, \varepsilon) \) and \( \Phi_\infty(x, \varepsilon) \) the path is taken in the real positive axis. In this case \( d_R = d_{RR} = 0 \). In particular \( \arg(\beta_2 - \beta_1) = \arg(\gamma_1 - \gamma_2) = 0 \).

During a double resonance of type A.2 for the fundamental matrix \( \Phi_0(x, \varepsilon) \) the path of integration is taken in the real positive axis. For the matrix \( \Phi_\infty(x, \varepsilon) \) the path is taken in the real negative axis. Here \( d_R = d_{LL} = 0 \). In particular \( \arg(\beta_2 - \beta_1) = \arg(\gamma_2 - \gamma_1) = 0 \).

During a double resonance of type A.3 for the fundamental matrix \( \Phi_0(x, \varepsilon) \) the path of integration is taken in the real negative axis. For the matrix \( \Phi_\infty(x, \varepsilon) \) the path is taken in the real positive axis. Here \( d_L = d_{RR} = 0 \). In particular \( \arg(\beta_1 - \beta_2) = \arg(\gamma_1 - \gamma_2) = 0 \).

During a double resonance of type A.4 for both fundamental matrices \( \Phi_0(x, \varepsilon) \) and \( \Phi_\infty(x, \varepsilon) \) the path is taken in the real negative axis. Here \( d_L = d_{LL} = 0 \). In particular \( \arg(\beta_1 - \beta_2) = \arg(\gamma_2 - \gamma_1) = 0 \).

The following statement describes the behavior of the fundamental matrices \( \Phi_0(x, \varepsilon) \) and \( \Phi_\infty(x, \varepsilon) \) near the singular points.

**Theorem 4.4.** Assume that \( \beta_1 \neq \beta_2 \). Then during a double resonance the fundamental matrix \( \Phi_0(x, \varepsilon) \) of the Heun type equation is represented near the singular points \( x_j, j = R, L \) as follows,

\[
\Phi_0(x, \varepsilon) = (I_L(\varepsilon) + \mathcal{O}(x - x_L)) (x - x_L)^{\frac{1}{2}A + \frac{1}{2x_L}B} (x - x_L)^{T_R}
\]

in a neighborhood of \( x_L \) which does not contain the other singular points, and

\[
\Phi_0(x, \varepsilon) = (I_R(\varepsilon) + \mathcal{O}(x - x_R)) (x - x_R)^{\frac{1}{2}A + \frac{1}{2x_R}B} (x - x_R)^{T_R}
\]

in a neighborhood of \( x_R \) which does not contain the other singular points.

Similarly, during a double resonance the fundamental matrix \( \Phi_\infty(x, \varepsilon) \) of the Heun type equation is represented near the singular points \( x_{jj}, j = R, L \) as follows,

\[
\Phi_\infty(x, \varepsilon) = (I_{LL}(\varepsilon) + \mathcal{O}(x - x_{LL})) (x - x_{LL})^{-\frac{1}{2}B}G (x - x_{LL})^{T_{LL}}
\]

in a neighborhood of \( x_{LL} \) which does not contain the other singular points, and

\[
\Phi_\infty(x, \varepsilon) = (I_{RR}(\varepsilon) + \mathcal{O}(x_{RR} - x)) (x_{RR} - x)^{-\frac{1}{2}G} (x_{RR} - x)^{T_{RR}}
\]

in a neighborhood of \( x_{RR} \) which does not contain the other singular points. The matrices \( I_j(\varepsilon) + \mathcal{O}(x - x_j) \) are holomorphic matrix-functions near the points \( x_j, j = L, R, LL, RR \), respectively. The matrices \( A, B, G \) are the matrices, associated with the initial equation and defined in the previous section. The matrices \( T_j \) and \( T_{jj} \) are given by

\[
T_j = \begin{pmatrix} 0 & d_j \\ 0 & 0 \end{pmatrix}, \quad T_{jj} = \begin{pmatrix} 0 & d_{jj} \\ 0 & 0 \end{pmatrix}.
\]

The elements \( d_j \) and \( d_{jj} \) are defined as follows

\[
d_j = \text{Res} \left( \frac{\Phi_2(x, \varepsilon)}{\Phi_1(x, \varepsilon)}, x = x_j \right), \quad d_{jj} = \text{Res} \left( \frac{\Phi_2(x, \varepsilon)}{\Phi_1(x, \varepsilon)}, x = x_{jj} \right).
\]

**Proof.** The elements \( \Phi_i(x, \varepsilon), i = 1, 2 \) are the same for both fundamental matrices \( \Phi_0(x, \varepsilon) \) and \( \Phi_\infty(x, \varepsilon) \). In a neighborhood of the singular point \( x_j, j = L, R, LL \), which does not contain the other singular point, they have the form

\[
\Phi_i(x, \varepsilon) = (x - x_j)^{m_{i,j}} h_{i,j}(x),
\]
\[
\Phi_{i}(x, \varepsilon) = (x_{RR} - x)^{m_{i,RR}} h_{i,RR}(x)
\]

in a neighborhood of the point \(x_{RR}\), which does not contain the rest singular points. Here
\[
m_{1,L} = -\beta_{1}/2\sqrt{\varepsilon}, m_{2,L} = -\beta_{2}/2\sqrt{\varepsilon} - 1, m_{1,R} = \beta_{1}/2\sqrt{\varepsilon}, m_{2,R} = \beta_{2}/2\sqrt{\varepsilon} - 1, m_{1,LL} = \gamma_{1}/2\sqrt{\varepsilon}, m_{2,LL} = \gamma_{2}/2\sqrt{\varepsilon}, m_{1,RR} = -\gamma_{1}/2\sqrt{\varepsilon}, m_{2,RR} = -\gamma_{2}/2\sqrt{\varepsilon}.
\]
The functions \(h_{i,j}(x)\) are holomorphic in the same neighborhood of the singular points \(x_{j}\).

Consider the element \(\Phi_{12}(x, \varepsilon)\) of the matrix \(\Phi_{0}(x, \varepsilon)\). During a double resonance of type \textbf{A.1} and \textbf{A.2} it has the simple form
\[
\Phi_{12}(x, \varepsilon) = (x - x_{R})^{m_{2,R}+1} u_{12,R}(x),
\]
in a neighborhood of the point \(x_{R}\), which does not contain the point \(x_{L}\). The function \(u_{12,R}(x)\) is a holomorphic function in the same neighborhood of the point \(x_{R}\). At the same time in a neighborhood of the point \(x_{L}\), which does not contain the point \(x_{R}\), the solution \(\Phi_{12}(x,0)\) has the form
\[
\Phi_{12}(x, \varepsilon) = (x - x_{L})^{m_{1,L}} h_{1,L}(x) \left[ d_{L} \log(x - x_{L}) + (x - x_{L})^{-\frac{\beta_{2} - \beta_{1}}{2\sqrt{\varepsilon}}} g_{1,L}(x) \right]
\]
\[
= d_{L} (x - x_{L})^{m_{1,L}} \log(x - x_{L}) h_{1,L}(x) + (x - x_{L})^{-\frac{\beta_{2}}{2\sqrt{\varepsilon}}} f_{1,L}(x).
\]
Here \(g_{1,L}(x)\) and \(f_{1,L}(x)\) are holomorphic functions in the same neighborhood of the point \(x_{L}\).

In the same manner during a double resonance of type \textbf{A.3} and \textbf{A.4} the element \(\Phi_{12}(x, \varepsilon)\) of \(\Phi_{0}(x, \varepsilon)\) is represented in a neighborhood of the point \(x_{L}\), which does not contain the point \(x_{R}\), as
\[
\Phi_{12}(x, \varepsilon) = (x - x_{L})^{m_{2,L}+1} v_{12,L}(x),
\]
where \(v_{12,L}(x)\) is a holomorphic function in the same neighborhood of the point \(x_{L}\). At the same time in a neighborhood of the point \(x_{R}\), which does not contain the point \(x_{L}\), this element is represented as
\[
\Phi_{12}(x, \varepsilon) = (x - x_{R})^{m_{1,R}} h_{1,R}(x) \left[ d_{R} \log(x - x_{R}) + (x - x_{R})^{-\frac{\beta_{2} - \beta_{1}}{2\sqrt{\varepsilon}}} g_{1,R}(x) \right] =
\]
\[
= d_{R} (x - x_{R})^{m_{1,R}} \log(x - x_{R}) h_{1,R}(x) + (x - x_{R})^{-\frac{\beta_{2}}{2\sqrt{\varepsilon}}} f_{1,R}(x).
\]
Here \(g_{1,R}(x)\) and \(f_{1,R}(x)\) are holomorphic functions in the same neighborhood of \(x_{R}\).

Consider now the element \(\Phi_{12}(x, \varepsilon)\) of the fundamental matrix \(\Phi_{\infty}(x, \varepsilon)\). During a double resonance of type \textbf{A.1} and \textbf{A.3} it is represented simply in a neighborhood of the point \(x_{RR}\), which does not contain the point \(x_{LL}\), as
\[
\Phi_{12}(x, \varepsilon) = (x_{RR} - x)^{m_{2,RR}+1} w_{12,RR}(x).
\]
Here \(w_{12,RR}\) is a holomorphic function in the same neighborhood. At the same time in a neighborhood of the point \(x_{LL}\) which does not contain the point \(x_{RR}\) this element is represented as
\[
\Phi_{12}(x, \varepsilon) = (x - x_{LL})^{m_{1,LL}} h_{1,LL}(x) \left[ d_{LL} \log(x - x_{LL}) + (x - x_{LL})^{-\frac{\beta_{2} - \beta_{1}}{2\sqrt{\varepsilon}}} g_{1,LL}(x) \right] =
\]
\[
= d_{LL} (x - x_{LL})^{m_{1,LL}} \log(x - x_{LL}) h_{1,LL}(x) + (x - x_{LL})^{-\frac{\beta_{2}}{2\sqrt{\varepsilon}}} f_{1,LL}(x).
\]
Here \(g_{1,LL}(x)\) and \(f_{1,LL}(x)\) are holomorphic functions in the neighborhood of the point \(x_{LL}\).
In the same manner during a double resonance of type A.2 and A.4 the element $\Phi_{12}(x, \varepsilon)$ of the matrix $\Phi_{\infty}(x, \varepsilon)$ is represented in a neighborhood of the point $x_{RR}$, which does not contain the point $x_{LL}$, as

$$\Phi_{12}(x, \varepsilon) = (x_{RR} - x)^{m_{1,RR}} h_{1,RR}(x) \left[ d_{RR} \log(x_{RR} - x) + (x_{RR} - x)^{-\frac{2\pi i}{2\sqrt{\varepsilon}} + 1} g_{1,RR}(x) \right]$$

$$= d_{RR} (x_{RR} - x)^{m_{1,RR}} \log(x_{RR} - x) h_{1,RR}(x) + (x_{RR} - x)^{-\frac{2\pi i}{2\sqrt{\varepsilon}} + 1} f_{1,RR}(x).$$

Here $g_{1,RR}(x)$ and $f_{1,RR}(x)$ are holomorphic functions in the same neighborhood of $x_{RR}$. At the same time in a neighborhood of the point $x_{LL}$, which does not contain the point $x_{RR}$ this element has the simple form

$$\Phi_{12}(x, \varepsilon) = (x - x_{LL})^{m_{2,LL} + 1} q_{12,LL}(x),$$

where $q_{12,LL}(x)$ is a holomorphic function in the same neighborhood of the point $x_{LL}$.

Any solution $\Phi_{12}(x, \varepsilon)$ that contains logarithmic terms is obtained from a solution $\Phi_{12}(x, \varepsilon)$ defined by the corresponding integral after its analytic continuation along a curve that does not cross a singular point except the base point $\epsilon$.

This ends the proof.

**Remark 4.5.** The matrices $\Lambda/2 + B/2x_j$ and $-x_{jj} G/2$ are expressed in terms of the characteristic exponents $\rho_{i}^{j}$ and $\rho_{j}^{j}$, $i = 1, 2, j = R, L$ as follows,

$$\frac{1}{2} \Lambda + \frac{1}{2x_j} B = \begin{pmatrix} \rho_{i}^{j} & 0 \\ 0 & \rho_{j}^{j} - 1 \end{pmatrix}, \quad \frac{-x_{jj}}{2} G = \begin{pmatrix} \rho_{i}^{jj} & 0 \\ 0 & \rho_{jj}^{jj} - 1 \end{pmatrix}.$$
We finish this indentation explaining the computation of the number $d_{RR}$. In this paper we will apply the same technique used in [32] to compute the residue of the complex function $f(x)$. Namely, let $x_0$ be a pole for $f(x)$ of order $s$, i.e.

$$f(x) = \frac{\varphi(x)}{(x-x_0)^{s}},$$

where $\varphi(x)$ is a holomorphic function in a neighborhood of $x_0$ and $\varphi(x_0) \neq 0$. Then

$$\text{Res}(f(x), x = x_0) = \frac{\varphi(s-1)(x_0)}{(s-1)!}.$$  

Applying formula (4.47) the number $d_{RR}$ becomes the coefficient before $\frac{1}{x-x_{RR}}$ in $\frac{\Phi_2(x, \varepsilon)}{\Phi_1(x, \varepsilon)}$. 

But after integration we get

$$\int_{-1/\sqrt{\varepsilon}}^{x} \frac{d_{RR}}{t - \frac{1}{\sqrt{\varepsilon}}} dt = -d_{RR} \int_{-1/\sqrt{\varepsilon}}^{x} \frac{dt}{\sqrt{\varepsilon} - t} = d_{RR} \log \left( \frac{1}{\sqrt{\varepsilon}} - x \right) - d_{RR} \log \frac{2}{\sqrt{\varepsilon}}.$$ 

So, the number $d_{RR} = \text{Res} \left( \frac{\Phi_2(x, \varepsilon)}{\Phi_1(x, \varepsilon)}, x = x_{RR} \right)$ is exactly the coefficient before $\log(x_{RR} - x)$.

To the end of this section we will compute the non-zero numbers $d_j$ during every type of double resonance. The numbers $d_j$, $j = R, L$ will be computed with respect to the fundamental matrix $\Phi_0(x, \varepsilon)$. The numbers $d_{jj}$, $j = R, L$ will be computed with respect to the fundamental matrix $\Phi_\infty(x, \varepsilon)$.

We start with the computation of the numbers $d_L$ and $d_{LL}$ during a double resonance of type A.1.

**Theorem 4.8.** Let $\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$. Then with respect to the fundamental matrix $\Phi_0(x, \varepsilon)$ for the number $d_L$ we have

1. If $\gamma_1 = \gamma_2$ then $d_L = 0$.
2. If $\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \in \mathbb{N}$ then

$$d_L = -\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \left( 1 + \varepsilon \right)^{\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}}} \frac{2\sqrt{\varepsilon}}{1 - \varepsilon} \sum_{k=1}^{\gamma_1 - \gamma_2} \frac{(2\sqrt{\varepsilon})^{k-1}}{\Gamma(k)\Gamma(k+1)} \frac{\Gamma\left(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + 1\right)}{\frac{1}{1 + \varepsilon}} A,$$

where

$$A = \sum_{s=0}^{k} \left( \frac{k}{s} \right) \frac{\sqrt{\varepsilon}^{k-s + s}}{\Gamma\left(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} + 1 + k + s\right)} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{s}.$$ 

**Proof.** According to (4.44) the number $d_L$ is defined by

$$d_L = \text{Res} \left( \frac{(x - \sqrt{\varepsilon})^{\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - 1}}{(x + \sqrt{\varepsilon})^{\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} + 1}} \left( \frac{1}{\sqrt{\varepsilon}} - x \right)^{\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}}}; x = x_L \right).$$

Then for the number $d_L$ we find that

$$d_L = \frac{1}{\left(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}}\right)!} D_{\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}}} \left( (x - \sqrt{\varepsilon})^{\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - 1} \left( \frac{1}{\sqrt{\varepsilon}} + x \right)^{\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}}}; x = x_L \right).$$
where we have denoted by $D$ the differential operator $\frac{d}{dx}$. Obviously when $\gamma_1 = \gamma_2$ the number $d_L$ becomes zero. Let $\gamma_1 \neq \gamma_2$. Now applying the Leibnitz’s rule

$$D^n (f(x) g(x)) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(f(x)) D^k(g(x)),$$

we obtain the wanted expression for $d_L$. \hfill \Box

**Theorem 4.9.** Let $\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$. Then with respect to the fundamental matrix $\Phi_\infty(x, \varepsilon)$ the number $d_{LL}$ is given as follows,

1. If $\gamma_1 = \gamma_2$ then $d_{LL} = 0$.
2. If $\frac{\beta_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$ then

$$d_{LL} = \frac{2\sqrt{\varepsilon}}{\beta_2 - \beta_1} \left( 1 + \varepsilon \right) \sum_{k=0}^{\gamma_1} \frac{(2\sqrt{\varepsilon})^{1+k}}{\Gamma(\gamma_1)} \frac{\Gamma(\gamma_1)}{A}\left( 1 - \varepsilon \right),$$

where

$$A = \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma(\gamma_1)}{\Gamma\left(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + s\right)} \frac{1 + \varepsilon}{1 - \varepsilon}.$$

**Proof.** Let $\gamma_1 \neq \gamma_2$. To compute the number $d_{LL}$ we apply the same procedure as in the construction of the formal solution of the initial equation near $x = \infty$. Instead of the solution $\Phi_{12}(x, \varepsilon)$ given by Theorem 4.7 we will use the solution $\Phi_{12}(x, \varepsilon)$ built by the operators $L_{j,\varepsilon}$ from by (2.25). Observe that the limit $\varepsilon \to 0$ takes these operators $L_{j,\varepsilon}$ into the operators $L_j$ defined by (3.35). Observe also that after the transformation $x = 1/t$ the function $\Phi_{12}(x, \varepsilon)$ becomes a particular solution of the following non-homogeneous equation

$$\dot{y}(t) + \left[ \frac{\beta_1}{2\sqrt{\varepsilon}} \left( \frac{1}{t - \frac{\beta_1}{\sqrt{\varepsilon}}} + \frac{1}{t + \frac{\beta_1}{\sqrt{\varepsilon}}} \right) + \frac{\gamma_1}{2\sqrt{\varepsilon}} \left( \frac{1}{t - \frac{\beta_1}{\sqrt{\varepsilon}}} - \frac{1}{t + \frac{\beta_1}{\sqrt{\varepsilon}}} \right) \right] y(t)$$

(4.48)

$$= -\frac{1}{\varepsilon} \left( \frac{1}{\frac{\beta_2}{\sqrt{\varepsilon}} - t} \frac{\beta_2}{2\sqrt{\varepsilon}} \right)^{\frac{\beta_2}{\sqrt{\varepsilon}} - 1} \left( \frac{t + \sqrt{\varepsilon}}{t - \sqrt{\varepsilon}} \right)^{\frac{\beta_2}{2\sqrt{\varepsilon}}}. $$

Then the number $d_{LL}$ of the perturbed equation becomes the number $d_L$ of the transformed equation. As a result, we redefine the number $d_{LL}$ as

$$d_{LL} = -\frac{1}{\varepsilon} \text{Res} \left( \frac{t - \frac{\beta_2}{\sqrt{\varepsilon}}}{t + \frac{\beta_2}{\sqrt{\varepsilon}}} ; t = -\frac{\beta_2}{\sqrt{\varepsilon}} \right).$$

Then

$$d_{LL} = -\frac{1}{\varepsilon} \left( \frac{\beta_2}{2\sqrt{\varepsilon}} - 1 \right)! \times \sum_{k=0}^{\gamma_1} \binom{\gamma_1}{k} \left( \frac{t - \frac{\beta_2}{\sqrt{\varepsilon}}}{t + \frac{\beta_2}{\sqrt{\varepsilon}}} \right)^{\frac{\beta_2}{\sqrt{\varepsilon}} - k} D^k \left( \frac{t - \frac{\beta_2}{\sqrt{\varepsilon}}}{t + \frac{\beta_2}{\sqrt{\varepsilon}} + 1} \right) \left( \frac{t + \sqrt{\varepsilon}}{t - \sqrt{\varepsilon}} \right)^{\frac{\beta_2}{2\sqrt{\varepsilon}} - 1} \left( \frac{t + \sqrt{\varepsilon}}{t - \sqrt{\varepsilon}} \right)^{\frac{\beta_2}{2\sqrt{\varepsilon}} + 1}.$$
where we have denoted by $D$ the differential operator $\frac{d}{df}$. Now it is not difficult to show that the number $d_{LL}$ has the pointed form.

When $\gamma_1 = \gamma_2$ the solution $\Phi_{12}(x, \varepsilon)$ given by Theorem 4.11 does not contain $\log(x-x_{LL})$ since it does not have a singularity at the point $x_{LL}$.

This ends the proof. \hfill $\Box$

**Remark 4.10.** We will use the same equation (1.48) to compute the numbers $d_{jj}$ for the rest double resonances.

The next two theorems give the numbers $d_L$ and $d_{RR}$ during a double resonance of type A.2.

**Theorem 4.11.** Let $\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$. Then with respect to the fundamental matrix $\Phi_0(x, \varepsilon)$ for the number $d_L$ we have

1. If $\gamma_1 = \gamma_2$ then $d_L = 0$.
2. If $\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$ then

$$d_L = \frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \left( 1 - \varepsilon \right) \sum_{k=1}^{\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}}} \frac{(-2\sqrt{\varepsilon})^{k-1}}{(k-1)! k!} \left( \frac{\sqrt{\varepsilon}}{1 - \varepsilon} \right)^k \frac{\Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}})}{\Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + 1)} A,$$

where

$$A = \sum_{s=0}^{k} \binom{s}{k} \frac{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} + s)}{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} - k + s + 1)} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^s.$$

**Theorem 4.12.** Let $\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$. Then with respect to the fundamental matrix $\Phi_\infty(x, \varepsilon)$ the number $d_{RR}$ is defined as follows:

1. If $\gamma_1 = \gamma_2$ then $d_{RR} = 0$.
2. If $\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$ then

$$d_{RR} = -\frac{2\sqrt{\varepsilon}}{\beta_2 - \beta_1} \frac{1}{1 - \varepsilon^2} \left( 1 - \varepsilon \right) \sum_{k=0}^{\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}}} \frac{(-1)^k (2\sqrt{\varepsilon})^{k+1}}{k! (k+1)!} \left( \frac{\sqrt{\varepsilon}}{1 - \varepsilon} \right)^k \frac{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} + 1)}{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} - k)} A,$$

where

$$A = \sum_{s=0}^{k} \binom{s}{k} \frac{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} + s + 1)}{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} - k + s + 1)} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^s.$$

In the next two theorems we present explicitly the numbers $d_R$ and $d_{LL}$ during a double resonance of type A.3.

**Theorem 4.13.** Let $\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} \in \mathbb{N}$. Then with respect to the fundamental matrix $\Phi_0(x, \varepsilon)$ for the number $d_R$ we have

1. If $\gamma_1 = \gamma_2$ then $d_R = 0$.
2. If $\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \in \mathbb{N}$ then

$$d_R = \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \left( 1 - \varepsilon \right) \sum_{k=1}^{\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}}} \frac{(-1)^k (2\sqrt{\varepsilon})^{k-1}}{(k-1)! k!} \left( \frac{\sqrt{\varepsilon}}{1 - \varepsilon} \right)^k \frac{\Gamma(\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}})}{\Gamma(\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} - k + 1)} A,$$
where
\[ A = \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} + s)}{\Gamma(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} - k + s + 1)} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^s. \]

**Theorem 4.14.** Let \( \beta_1 - \beta_2 \in \mathbb{N} \). Then with respect to the fundamental matrix \( \Phi_\infty(x, \varepsilon) \) for the number \( d_{LL} \) we have

1. If \( \gamma_1 = \gamma_2 \) then \( d_{LL} = 0 \).
2. If \( \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \in \mathbb{N} \) then

\[ d_{LL} = -\frac{2\sqrt{\varepsilon}}{\beta_1 - \beta_2} \frac{1}{1 - \varepsilon^2} \left( 1 - \varepsilon \right) \frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} \sum_{k=0}^{\gamma_1 - \gamma_2 - 1} \frac{(-2\sqrt{\varepsilon})^{k+1}}{k!(k+1)!} \left( \frac{\sqrt{\varepsilon}}{1 - \varepsilon} \right)^k \frac{\Gamma(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} + 1)}{\Gamma(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} - k)} A, \]

where
\[ A = \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} + s)}{\Gamma(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} - k + s + 1)} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^s. \]

The last two theorems of this subsection give the numbers \( d_R \) and \( d_{RR} \) during a double resonance of type A.4.

**Theorem 4.15.** Let \( \frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} \in \mathbb{N} \). Then with respect to the fundamental matrix \( \Phi_0(x, \varepsilon) \) for the number \( d_R \) we have

1. If \( \gamma_1 = \gamma_2 \) then \( d_R = 0 \).
2. If \( \frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} \in \mathbb{N} \) then

\[ d_R = \frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \frac{1 + \varepsilon}{1 - \varepsilon} \frac{2\sqrt{\varepsilon}}{\beta_1 - \beta_2} \sum_{k=1}^{\gamma_2 - \gamma_1} \frac{(2\sqrt{\varepsilon})^{k-1}}{(k-1)!k!} \left( \frac{\sqrt{\varepsilon}}{1 + \varepsilon} \right)^k \frac{\Gamma(\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}})}{\Gamma(\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} - k + 1)} A, \]

where
\[ A = \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} + s)}{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} - k + s + 1)} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^s. \]

**Theorem 4.16.** Let \( \frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} \in \mathbb{N} \). Then with respect to the fundamental matrix \( \Phi_\infty(x, \varepsilon) \) for the number \( d_{RR} \) we have

1. If \( \gamma_1 = \gamma_2 \) then \( d_{RR} = 0 \).
2. If \( \frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \in \mathbb{N} \) then

\[ d_{RR} = -\frac{2\sqrt{\varepsilon}}{\beta_1 - \beta_2} \frac{1}{1 - \varepsilon^2} \left( 1 - \varepsilon \right) \frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} \sum_{k=0}^{\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} - 1} \frac{(2\sqrt{\varepsilon})^{k+1}}{k!(k+1)!} \left( \frac{\sqrt{\varepsilon}}{1 + \varepsilon} \right)^k \frac{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} + 1)}{\Gamma(\frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} - k)} A, \]

where
\[ A = \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma(\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} + s)}{\Gamma(\frac{\beta_1 - \beta_2}{2\sqrt{\varepsilon}} - k + s + 1)} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^s. \]
5. MAIN RESULTS

In this section we establish the main result of this paper. Our goal is to connect by a radial limit $\sqrt{r} \to 0$ the monodromy matrices $M_j(x, \varepsilon), j = R, L$ and $M_{jj}(x, \varepsilon), j = R, L$ of the Heun type equation with the Stokes matrices of the initial equation. As we mentioned in the introduction in this paper we consider the initial equation with two different fundamental matrix solutions without studying the connection between them. These are the fundamental solution at the origin and the fundamental solution at the infinity point. In the same manner we consider the Heun type equation with two different fundamental matrix solutions $\Phi_0(x, \varepsilon)$ and $\Phi_\infty(x, \varepsilon)$ again without studying their connection. So we can imagine that we deal with two pairs of initial and perturbed equation. The initial equation of the first pair has an irregular singularity at the origin of Poincaré rank $1$ and we split it into two Fuchsian singularities $x_L$ and $x_R$. The initial equation of the second pair has an irregular singularity at $x = \infty$ again of Poincaré rank $1$ and we split it into two new Fuchsian singular points $x_{LL}$ and $x_{RR}$.

In next two paragraphs we express the so called unfolded Stokes matrices of each imaginary pair as a part of monodromy matrices of the same pair during a double resonance.

5.1. The unfolded Stokes matrices $St_j(x)$ as the matrices $e^{2\pi i T_j}$. As we saw in Section 3 the initial equation has only one singular direction $\theta = \arg(\beta_1 - \beta_2)$ at the origin. Following [16, 17] we consider the initial equation together with its Stokes matrix at the origin in the ramified domain of the origin $\{ x \in \mathbb{C}P^1 : \arg(\beta_1 - \beta_2) - \kappa < \arg(x) < \arg(\beta_1 - \beta_2) + \kappa \}$ where $0 < \kappa < \pi/2$. We cover this domain by two open sectors

$$\Omega_{1,0} = \left\{ x = re^{i\delta} \big| 0 < r < \rho, \arg(\beta_1 - \beta_2) - \kappa < \delta < \arg(\beta_1 - \beta_2) + \pi + \kappa \right\},$$

$$\Omega_{2,0} = \left\{ x = re^{i\delta} \big| 0 < r < \rho, -(\arg(\beta_1 - \beta_2) + \pi + \kappa) < \delta < \arg(\beta_1 - \beta_2) + \kappa \right\}.$$

Denote by $\Omega_R$ and $\Omega_L$ the connection components of the intersection $\Omega_{1,0} \cap \Omega_{2,0}$. The radius $\rho$ of the sectors $\Omega_{j,0}$ is so chosen that the only singular points which belong to $\Omega_R$ and $\Omega_L$ to be the points $x_R$ and $x_L$, respectively. Recall that during a double resonance $x_R, x_L, \beta_2 - \beta_1 \in \mathbb{R}$. So, $\arg(\beta_1 - \beta_2) \in \{0, \pi\}$.

Let us extend the actual fundamental matrix solution $\Phi_0(x, 0)$ introduced by Proposition 3.10 to the whole $\Omega_{1,0} \cup \Omega_{2,0}$. From the sectorial normalization theorem of Sibuya [30] if follows that over the sector $\Omega_{1,0}$ the matrix $H_1(x) = H_{\theta}^{+}(x)$ is an analytic matrix function asymptotic at the origin to the matrix $\hat{H}(x)$. Similarly, over the sector $\Omega_{2,0}$ the matrix $H_2(x) = H_{\theta}^{-}(x)$ is an analytic matrix function asymptotic at the origin to the matrix $\hat{H}(x)$. Then the matrix

$$\Psi_j^0(x) = \exp(Gx) H_j(x) F(x), \quad j = 1, 2$$

with the corresponding branch of $F(x)$ is an actual fundamental matrix at the origin over the sector $\Omega_{j,0}$, respectively. Note that if $\arg(\beta_1 - \beta_2) = 0$ we can observe the Stokes phenomenon on $\Omega_R$. If $\arg(\beta_1 - \beta_2) = \pi$ we can observe the Stokes phenomenon over $\Omega_L$.

Let us turn around the origin in the positive sense, starting from the sector $\Omega_{1,0}$. On the first sector $\Omega_{j,0}, j = R, L$ that we cross we can not observe the Stokes phenomenon. On the next sector $\Omega_{j,0}, j = R, L$ that we cross we define the Stokes matrix $St_R$ (resp. $St_L$) as

$$((\Phi_0^0(x, 0))^{+1} (\Phi_{\theta+2\pi}^0(x, 0))^{+1} = (\Psi_j^0(x))^{-1} \Psi_j^0(x) \hat{M} = St_j \hat{M} = St_0^\theta \hat{M} = St_0^\theta,$$

where $St_0^\theta$ is the Stokes matrix defined by Theorem 3.11 and $\hat{M} = I_2$. 


At the same time we consider the Heun type equation on the sectorial domains $\Omega_{1,0}(\varepsilon)$ and $\Omega_{2,0}(\varepsilon)$. They are obtained from the open sectors $\Omega_{1,0}$ and $\Omega_{2,0}$ by making a cut between the singular points $x_L$ and $x_R$ through the real axis. The point $x_0 = 0$ belongs to this cut. When $\varepsilon \to 0$ the sectorial domains $\Omega_{j,0}(\varepsilon)$ tend to the sectors $\Omega_{j,0}, j = 1, 2$, respectively. The sectorial domains $\Omega_{1,0}(\varepsilon)$ and $\Omega_{2,0}(\varepsilon)$ intersect in the sectors $\Omega_L(\varepsilon)$ and $\Omega_R(\varepsilon)$ and along the cut. The singular points $x_j, j = R, L$ belong to this cut.

Let us represent the fundamental matrix $\Phi_0(x, \varepsilon)$ of the Heun type equation in a slightly different form

$$\Phi_0(x, \varepsilon) = G(x, \varepsilon) H(x, \varepsilon) F(x, \varepsilon),$$

where

$$G(x, \varepsilon) = (x - x_{LL})^{-\frac{\beta_1}{2\sqrt{\varepsilon}}} G(x_{RR} - x)^{-\frac{\beta_2}{2\sqrt{\varepsilon}}} G$$

and

$$F(x, \varepsilon) = (x - x_{LL})^{\frac{1}{2}\Lambda + \frac{1}{2\sqrt{\varepsilon}} B} (x - x_{R})^{\frac{1}{2}\Lambda + \frac{1}{2\sqrt{\varepsilon}} B}.$$

The path of integration $\Gamma_0(x, \varepsilon)$ is a path either from $x_R$ to $x$ or from $x_L$ to $x$, taken in the direction $\arg(\beta_2 - \beta_1)$. Analytic continuation of the path $\Gamma_0(x, \varepsilon)$ around the origin in the positive sense yields two branches $h_{12}(x, \varepsilon)$ and $h_{12}^+(x, \varepsilon)$ of the element $h_{12}(x, \varepsilon)$ near the singular direction $\arg(\beta_1 - \beta_2)$. The branch $h_{12}(x, \varepsilon)$ corresponds to a path taken in the direction $\arg(\beta_1 - \beta_2) - \epsilon$, and the branch $h_{12}^+(x, \varepsilon)$ corresponds to a path taken in the direction $\arg(\beta_1 - \beta_2) + \epsilon$. Here $\epsilon > 0$ is a small number. When $\Gamma_0(x, \varepsilon)$ crosses the singular direction $\arg(\beta_1 - \beta_2)$ we rather observe the Stokes phenomenon than the linear monodromy. This phenomenon is measured geometrically by the so called unfolded Stokes matrix [17]. In concordance with the initial equation we fix on the sector $\Omega_{1,0}(\varepsilon)$ the fundamental matrix of the Heun type equation as

$$\Psi_0^0(x, \varepsilon) = G(x, \varepsilon) H_1(x, \varepsilon) F_1(x, \varepsilon),$$

where $H_1(x, \varepsilon) = \{h_{ij}^+(x, \varepsilon)\}^{2}_{i,j=1}$. The matrix $F_1(x, \varepsilon)$ is the branch on $\Omega_{1,0}(\varepsilon)$ of the matrix $F(x, \varepsilon)$. Similarly, on the sector $\Omega_{2,0}(\varepsilon)$ we fix the fundamental matrix $\Phi_0(x, \varepsilon)$ as

$$\Psi_0^0(x, \varepsilon) = G(x, \varepsilon) H_2(x, \varepsilon) F_2(x, \varepsilon),$$

where $H_2(x, \varepsilon) = \{h_{ij}^-(x, \varepsilon)\}^{2}_{i,j=1}$. The matrix $F_2(x, \varepsilon)$ coincides with $F_1(x, \varepsilon)$ on the sector $\Omega_j(\varepsilon), j = R, L$, which contains the non-singular direction $\arg(\beta_2 - \beta_1)$. On the sector $\Omega_j(\varepsilon), j = R, L$ which contains the singular direction $\arg(\beta_1 - \beta_2)$ these two matrices again coincide since $M = I_2$ and therefore $F_2(x, \varepsilon) = F_1(x, \varepsilon) M = F_1(x, \varepsilon)$. Let is turn around the origin in the positive sense starting from the sector $\Omega_{1,0}(\varepsilon)$ and the solution $\Psi_0^0(x, \varepsilon)$ on it. When $\Gamma_0(x, \varepsilon)$ crosses the direction $\arg(\beta_2 - \beta_1)$ we can not observe the Stokes phenomenon on the corresponding $\Omega_j(\varepsilon), j = R, L$. There the solutions $\Psi_0^0(x, \varepsilon)$ and $\Psi_0^1(x, \varepsilon)$ coincide. Now we continue analytically the solution $\Psi_0^0(x, \varepsilon)$. When $\Gamma_0(x, \varepsilon)$ crosses the singular direction $\arg(\beta_1 - \beta_2)$ we already observe the Stokes phenomenon. The jump of the solution $\Psi_0^1(x, \varepsilon)$ to the solution $\Psi_0^0(x, \varepsilon)$ is measured by the unfolding Stokes matrix $St_j(\varepsilon)$

$$(\Psi_0^0(x, \varepsilon))^{-1} \Psi_0^1(x, \varepsilon) = St_j(\varepsilon) M = St_j(\varepsilon),$$
since $\hat{M} = I_2$. In this formula $St_j(\varepsilon) = St_R(\varepsilon)$ if \( \arg(\beta_1 - \beta_2) = 0 \), and $St_j(\varepsilon) = St_L(\varepsilon)$ if \( \arg(\beta_1 - \beta_2) = \pi \).

In Proposition 4.31 of [16] Lambert and Rousseau represent the monodromy matrices $M_j(\varepsilon)$ of the perturbed equation as a product of the unfolded Stokes matrices $St_j(\varepsilon)$ and monodromy matrices of the branch of $F(x, \varepsilon)$. Recently, in his remarkable paper [15] Klimeš provides an explicit expression of the unfolded Stokes matrices $St_j(\varepsilon)$, $j = L, R$ in terms of the monodromy matrices $M_j(\varepsilon)$, $j = L, R$ and the matrices $e^{\pi i(\Lambda^+/x_L)}$, $j = L, R$. Here we present and use the result of Klimeš in our formulation from [32]. As in [32] we consider both equations provided that the matrices $F(x)$ and $F(x, \varepsilon)$ are not changed when we cross the first intersection staring from the sectors $\Omega_1$ and $\Omega_2(\varepsilon)$. In fact since the formal monodromy matrix $\hat{M}$ is equal to the identity matrix $I_2$, the matrices $F(x)$ and $F(x, \varepsilon)$ are not changed even after one tour around the origin.

**Proposition 5.1.** Let $M_j(\varepsilon)$ and $St_j(\varepsilon)$, $j = L, R$ be the monodromy matrices and the unfolded Stokes matrices of the Heun type equation with respect to the fundamental solution $\Psi^0(x, \varepsilon)$ on the sector $\Omega_{1,0}(\varepsilon)$. Then depending on the arg($\beta_1 - \beta_2$) they satisfy the following relations:

1. If \( \arg(\beta_1 - \beta_2) = 0 \) on the sector $\Omega_{1,0}(\varepsilon)$ (upper sector)
   \[
   M_L(\varepsilon) = e^{\pi i(\Lambda^+ + \frac{1}{x_L})} St_L(\varepsilon), \quad M_R(\varepsilon) = St_R(\varepsilon) e^{\pi i(\Lambda^+ + \frac{1}{x_R})}.
   \]

   On sector $\Omega_{2,0}(\varepsilon)$ (lower sector)
   \[
   M_L(\varepsilon) = St_L(\varepsilon) e^{\pi i(\Lambda^+ + \frac{1}{x_L})}, \quad M_R(\varepsilon) = e^{\pi i(\Lambda^+ + \frac{1}{x_R})} St_R(\varepsilon).
   \]

2. If \( \arg(\beta_1 - \beta_2) = \pi \) on the sector $\Omega_{1,0}(\varepsilon)$ (lower sector)
   \[
   M_L(\varepsilon) = St_L(\varepsilon) e^{\pi i(\Lambda^+ + \frac{1}{x_L})}, \quad M_R(\varepsilon) = e^{\pi i(\Lambda^+ + \frac{1}{x_R})} St_R(\varepsilon).
   \]

   On the sector $\Omega_{2,0}(\varepsilon)$ (upper sector)
   \[
   M_L(\varepsilon) = e^{\pi i(\Lambda^+ + \frac{1}{x_L})} St_L(\varepsilon), \quad M_R(\varepsilon) = St_R(\varepsilon) e^{\pi i(\Lambda^+ + \frac{1}{x_R})}.
   \]

As an immediate consequence we have

**Corollary 5.2.** During a double resonance the unfolded Stokes matrices $St_j(\varepsilon)$ and the matrices $e^{2\pi i T_j}$, $j = L, R$ satisfy the following relation
\[
St_L(\varepsilon) = e^{2\pi i T_L}, \quad St_R(\varepsilon) = e^{2\pi i T_R}.
\]

**Proof.** From Proposition 5.1 we have that
\[
M_L(\varepsilon) = St_L(\varepsilon) e^{\pi i(\Lambda^+ + \frac{1}{x_L})}
\]
on the setor $\Omega_{1,0}(\varepsilon)$ (resp. $\Omega_{2,0}(\varepsilon)$) when \( \arg(\beta_1 - \beta_2) = \pi \) (resp. \( \arg(\beta_1 - \beta_2) = 0 \)). On the other hand during a double resonance for the monodromy matrix $M_L(\varepsilon)$ given by [4.35] we have
\[
M_L(\varepsilon) = e^{2\pi i T_L} e^{\pi i(\Lambda^+ + \frac{1}{x_L})} = e^{\pi i(\Lambda^+ + \frac{1}{x_L})} e^{2\pi i T_L}.
\]
Comparing both expressions for the monodromy matrix $M_L(\varepsilon)$ we find that
\[
St_L(\varepsilon) = e^{2\pi i T_L}.
\]
Moreover, this relation remains valid even if
\[
M_L(\varepsilon) = e^{\pi i(\Lambda^+ + \frac{1}{x_L})} St_L(\varepsilon),
\]
since the matrices $e^{2\pi i T_L}$ and $e^{\pi i (\Lambda + \frac{1}{2} B)}$ commute during a double resonance.

In the same manner one can obtain the relation between $St_{R}(\varepsilon)$ and $e^{\pi i (\Lambda + \frac{1}{2} R) B}$. □

5.2. The unfolded Stokes matrices $St_{\varepsilon j}(\varepsilon)$ as the matrices $e^{2\pi i T_{jj}}$. Recall that during a double resonance $x_{RR}, x_{LL}, \gamma_2 - \gamma_1 \in \mathbb{R}$. So we consider the initial and the perturbed equations in a neighborhood of the real infinity point. Due to the symmetries of the initial and the perturbed equations we apply in a neighborhood of the real infinity point the same construction as in the previous paragraph. Recall that the initial equation admits only one singular direction $\theta = \arg(\gamma_2 - \gamma_1)$ at the infinity point. So, now we consider the initial equation together with its Stokes matrix at the infinity point in the ramified domain of the real infinity point $\{x \in \mathbb{C}P^1 : |x| > \rho, \arg(\gamma_2 - \gamma_1) - \kappa < \arg(x) < \arg(\gamma_2 - \gamma_1) + \kappa\}$ where $0 < \kappa < \pi/2$. Just above we cover this domain by two open sectors

$$\Omega_{1,\infty} = \left\{ x = re^{i\delta} | r > \rho, \arg(\gamma_2 - \gamma_1) - \kappa < \delta < \arg(\gamma_2 - \gamma_1) + \pi + \kappa \right\},$$

$$\Omega_{2,\infty} = \left\{ x = re^{i\delta} | r > \rho, -(\arg(\gamma_2 - \gamma_1) + \pi + \kappa) < \delta < \arg(\gamma_2 - \gamma_1) + \kappa \right\}.$$ 

In this paper we consider this covering as a mirror image of the corresponding covering of the origin. The idea is that we consider both equations on the Riemann sphere. Then if we face this covering its right side will be an continuation of the real negative side. Then under $\arg(\gamma_2 - \gamma_1)$ we mean the angle between the point $\gamma_2 - \gamma_1$ and the left real axis in the positive sense with the base point at $x = \infty$. In particular if $\gamma_2 - \gamma_1 < 0$ then the sector $\Omega_{1,\infty}$ is the upper sector. Conversely, if $\gamma_2 - \gamma_1 > 0$ then $\Omega_{1,\infty}$ is the lower sector. Denote by $\Omega_{RR}$ and $\Omega_{LL}$ the connection components of the intersection $\Omega_{1,\infty} \cap \Omega_{2,\infty}$. The radius $\rho$ of the sectors $\Omega_j$ is so chosen that the only singular points which belong to $\Omega_{RR}$ and $\Omega_{LL}$ to be $x_{RR}$ and $x_{LL}$, respectively. Note that due to the mirror image, this time $\Omega_{RR}$ is the left intersection. We denote it again by $\Omega_{RR}$ to underline that it contains the point $x_{RR}$.

Applying the normalization theorem of Sibuya [30] to the solution $\Phi^\infty_\theta(x, 0)$ from Proposition 5.12 we obtain over the sector $\Omega_{1,\infty}$ the analytic matrix function $P_1(x) = P^\infty_\theta(x)$ which is asymptotic at $x = \infty$ to the formal matrix $\hat{P}(x)$. Similarly, over the sector $\Omega_{2,\infty}$ the matrix $P_2(x) = P^\infty_\theta(x)$ is an analytic matrix function asymptotic at $x = \infty$ to the formal matrix $\hat{P}(x)$. Then the matrix

$$\Psi^\infty_j(x) = \exp \left( -\frac{B}{x} \right) \left( \frac{1}{x} \right)^{-\Lambda} P_j(x) \exp(Gx), \quad j = 1, 2$$

with the corresponding branches of $(1/x)^{-\Lambda}$ and $\exp(Gx)$ is an actual fundamental matrix at $x = \infty$ over the sector $\Omega_{j,\infty}$ respectively. If $\arg(\gamma_2 - \gamma_1) = 0$ we can observe the Stokes phenomenon on $\Omega_{RR}$. If $\arg(\gamma_2 - \gamma_1) = \pi$ we can observe the Stokes phenomenon over $\Omega_{LL}$.

Let us turn around $x = \infty$ in the positive sense by analytic continuation. We start from the sector $\Omega_{1,\infty}$ and the solution $\Psi^\infty_1(x)$. On the first sector $\Omega_{jj}, j = R, L$ that we cross we can not observe the Stokes phenomenon. On the next sector $\Omega_{jj}, j = R, L$ that we cross we define the Stokes matrix $St_{jj}, j = R, L$ as

$$(\Psi^\infty_1(x))^{-1} \Psi^\infty_j(x) = St_{jj} = St^\theta_{\infty},$$

where $St^\theta_{\infty}$ is the Stoke matrix computed by Theorem 5.13.

At the same time we consider the perturbed equation on the whole $\Omega_{1,\infty}(\varepsilon) \cup \Omega_{2,\infty}(\varepsilon)$. The sectorial domains $\Omega_{1,\infty}(\varepsilon)$ and $\Omega_{2,\infty}(\varepsilon)$ are obtained from the open sectors $\Omega_{1,\infty}$ and
When \( \gamma \) \( \infty \) relations:

Let \( \text{Proposition 5.3.} \)

The analog of Proposition 5.1 gives us an explicit connection between the monodromy the perturbed equation we can replace the origin by the infinity point in Proposition 5.1.

Then depending on the \( \arg(\gamma_2 - \gamma_1) \) they satisfy the following relations:

1. If \( \arg(\gamma_2 - \gamma_1) = 0 \) on the sector \( \Omega_{1, \infty}(\varepsilon) \) (lower sector)

\[
M_{LL}(\varepsilon) = e^{-\pi i x_{LL}} G St_{LL}(\varepsilon), \quad M_{RR}(\varepsilon) = St_{RR}(\varepsilon)e^{-\pi i x_{RR}} G.
\]
On sector $\Omega_{2,\infty}(\varepsilon)$ (upper sector)

$$M_{LL}(\varepsilon) = S_{LL}(\varepsilon) e^{-\pi i x_{LL} G}, \quad M_{RR}(\varepsilon) = e^{-\pi i x_{RR} G} S_{RR}(\varepsilon).$$

(2) If $\arg(\gamma_2 - \gamma_1) = \pi$ on the sector $\Omega_{1,\infty}(\varepsilon)$ (upper sector)

$$M_{LL}(\varepsilon) = S_{LL}(\varepsilon) e^{-\pi i x_{LL} G}, \quad M_{RR}(\varepsilon) = e^{-\pi i x_{RR} G} S_{RR}(\varepsilon).$$

On the sector $\Omega_{2,\infty}(\varepsilon)$ (lower sector)

$$M_{LL}(\varepsilon) = e^{-\pi i x_{LL} G} S_{LL}(\varepsilon), \quad M_{RR}(\varepsilon) = S_{RR}(\varepsilon) e^{-\pi i x_{RR} G}.$$

As an immediate consequence we have

**Corollary 5.4.** During a double resonance the unfolded Stokes matrices $S_{jj}(\varepsilon)$ and the matrices $e^{2\pi i T_j}$, $j = L, R$ satisfy the following relation

$$S_{LL}(\varepsilon) = e^{2\pi i T_{LL}}, \quad S_{RR}(\varepsilon) = e^{2\pi i T_{RR}}.$$

**Proof.** The proof is similar to the proof of Corollary 5.2.

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5.3. The main results. It turns out that the matrices $e^{2\pi i T_j}$ and $e^{2\pi i T_{jj}}$ have limits when $\sqrt{\varepsilon} \to 0$ radially. In the following lemma we compute the limits of the numbers $d_j$ and $d_{jj}$ given in the previous section.

**Lemma 5.5.** Assume that $\beta_1 \neq \beta_2$. Then during a double resonance the number $d_j$ either is equal to zero or has the following limit when $\sqrt{\varepsilon} \to 0$

$$\lim_{\sqrt{\varepsilon} \to 0} d_j = (\gamma_2 - \gamma_1) \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma_2 - \gamma_1)^k (\beta_2 - \beta_1)^k}{k! (k + 1)!}.$$

Similarly, during a double resonance the number $d_{jj}$ either is equal to zero or has the following limit when $\sqrt{\varepsilon} \to 0$

$$\lim_{\sqrt{\varepsilon} \to 0} d_{jj} = - (\gamma_2 - \gamma_1) \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma_2 - \gamma_1)^k (\beta_2 - \beta_1)^k}{k! (k + 1)!}.$$

**Proof.** We will study in detail only the limit of the numbers $d_L$ and $d_{LL}$ computed in Theorem 4.8 and Theorem 4.9. The limit of the rest numbers $d_j$ and $d_{jj}$ obtained in the previous section is computed in the same manner.

Let us firstly compute the limit of the finite sum $A$ when $\sqrt{\varepsilon} \to 0$ introduced in Theorem 4.8. We can rewrite $A$ as

$$A = \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma\left(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}} + s\right)}{\Gamma\left(\frac{2\gamma_2}{2\sqrt{\varepsilon}}\right)} \frac{\Gamma\left(\frac{2\gamma_2 - s}{2\sqrt{\varepsilon}}\right)}{\Gamma\left(\frac{2\gamma_2 - s}{2\sqrt{\varepsilon}} - 1 - k + s\right)} \frac{\left(\frac{2\gamma_2 - s}{2\sqrt{\varepsilon}}\right)^{s}}{\left(\frac{2\gamma_2 - s}{2\sqrt{\varepsilon}}\right)^{1-k+s}} \left(1 + \varepsilon\right)^{s}.$$ 

Then using the limit

$$\lim_{|z| \to \infty} \frac{\Gamma(z + \alpha)}{\Gamma(z) z^{\alpha}} = 1,$$

we find that

$$\lim_{\sqrt{\varepsilon} \to 0} A = \left(\frac{2}{1 - \varepsilon}\right)^k \left(\frac{\gamma_1 - \gamma_2}{2\sqrt{\varepsilon}}\right)^{k-1}.$$
As a result the number $d_L$ is reduced to the form

$$d_L = (\gamma_2 - \gamma_1) \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\gamma_2 - \gamma_1} \sum_{k=1}^{k} \frac{(2\sqrt{\varepsilon})^{k-1}}{(k-1)!} \frac{(\varepsilon)^k}{(1 + \varepsilon)^k} \frac{2}{1 - \varepsilon} \frac{(\gamma_2 - \gamma_1)}{2\sqrt{\varepsilon}} \frac{\Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}})}{\Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + 1)}.$$

Now, choosing log 1 = 0, it is easy to see that

$$\lim_{\varepsilon \to 0} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\gamma_2 - \gamma_1} = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} (1 - \varepsilon^2)^{-k} = 1.$$

Dropping out these two expressions, we rewriting $d_L$ as

$$d_L = (\gamma_2 - \gamma_1) \sum_{k=1}^{k} \frac{(2\sqrt{\varepsilon})^{k-1} (2\sqrt{\varepsilon})^k}{(k1)!k!} \frac{(\gamma_1 - \gamma_2)^k}{2\sqrt{\varepsilon}} \frac{(\beta_2 - \beta_1)}{2\sqrt{\varepsilon}} \frac{\Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}})}{\Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + 1)}.$$

Again applying the limit (5.49), we find that

$$\lim_{\varepsilon \to 0} d_L = (\gamma_2 - \gamma_1) \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma_2 - \gamma_1)^k (\beta_2 - \beta_1)^k}{k!(k + 1)!}.$$

In the same manner we can rewrite the finite sum $A$ from Theorem 4.9 as

$$A = \sum_{s=0}^{k} \frac{k! + s}{(\beta_2 - \beta_1)^{1+s} \Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + s)} \frac{(\beta_2 - \beta_1)^{1+s} \Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + s)}{\Gamma(\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} - k + s)} = (\varepsilon)^{s}.$$

Again using the limit (5.49) we find that

$$\lim_{\varepsilon \to 0} A = \left( \frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}} \right)^{k+1} \left( \frac{2}{1 - \varepsilon} \right)^k.$$

As a result the number $d_{LL}$ is reduced to

$$d_{LL} = \frac{1}{1 - \varepsilon^2} \sum_{k=0}^{\infty} \frac{(2\sqrt{\varepsilon})^{1+k} (\sqrt{\varepsilon})^k}{(1 + \varepsilon)^k} \frac{(\beta_2 - \beta_1)}{2\sqrt{\varepsilon}} \frac{\Gamma(\frac{2(\gamma_2 - \gamma_1)}{2\sqrt{\varepsilon}} + 1)}{\Gamma(\frac{2(\gamma_2 - \gamma_1)}{2\sqrt{\varepsilon}} - k)}.$$

Now it is not difficult to show that

$$\lim_{\varepsilon \to 0} d_{LL} = (\gamma_1 - \gamma_2) \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma_2 - \gamma_1)^k (\beta_2 - \beta_1)^k}{k!(k + 1)!}.$$

This completes the proof.

In Theorem 4.25 in [17] Lambert and Rousseau prove that the unfolded Stokes matrices $St_j(\varepsilon), j = R, L$ depend analytically on the parameter of perturbation $\varepsilon$ and converge when $\varepsilon \to 0$ to the Stokes matrices $St_j, j = R, L$ of the initial equation. Due to the symmetries we have the same result for the unfolded Stokes matrices $St_{jj}, j = R, L$ and the Stokes matrices $St_{j}, j = R, L$ of the initial equation. Then thanks to Lemma 5.5 Corollary 5.2 and Theorem 4.25 in [17] we state the main result of this paper

**Theorem 5.6.** Assume that $\varepsilon \in \mathbb{R}^+$. Assume also that $\beta_2 - \beta_1, \gamma_2 - \gamma_1 \in \mathbb{R}$ are fixed such that $\beta_1 \neq \beta_2$ and $\frac{\beta_2 - \beta_1}{2\sqrt{\varepsilon}}, \frac{\gamma_2 - \gamma_1}{2\sqrt{\varepsilon}} \in \mathbb{Z}$. Then depending on the position of $\beta_2 - \beta_1$ and $\gamma_2 - \gamma_1$ toward 0 the matrices $T_j, T_{jj}, j = R, L$ of the Heun type equation and the Stokes matrices $St_k, k = 0, \infty$ of the initial equation are connected as follows,
If $\beta_2 - \beta_1, \gamma_2 - \gamma_1 \in \mathbb{R}^+$ then
\[ e^{2\pi i T_L} \rightarrow S^0_\infty, \quad e^{2\pi i T_{RR}} \rightarrow S^0_\infty, \]
when $\sqrt{\varepsilon} \to 0$.

If $\beta_1 - \beta_2, \gamma_1 - \gamma_2 \in \mathbb{R}^+$ then
\[ e^{2\pi i T_R} \rightarrow S^0_0, \quad e^{2\pi i T_{LL}} \rightarrow S^\pi_\infty, \]
when $\sqrt{\varepsilon} \to 0$.

If $\beta_2 - \beta_1, \gamma_1 - \gamma_2 \in \mathbb{R}^+$ then
\[ e^{2\pi i T_L} \rightarrow S^\pi_0, \quad e^{2\pi i T_{LL}} \rightarrow S^\pi_\infty, \]
when $\sqrt{\varepsilon} \to 0$.

If $\beta_1 - \beta_2, \gamma_2 - \gamma_1 \in \mathbb{R}^+$ then
\[ e^{2\pi i T_R} \rightarrow S^0_0, \quad e^{2\pi i T_{RR}} \rightarrow S^0_\infty, \]
when $\sqrt{\varepsilon} \to 0$.

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