CHARACTERIZATIONS OF STABILIZABLE SETS
FOR SOME PARABOLIC EQUATIONS IN $\mathbb{R}^n$

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Abstract. We consider the parabolic type equation in $\mathbb{R}^n$:
\[
(\partial_t + H)y(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n; \quad y(0, x) \in L^2(\mathbb{R}^n),
\]
where $H$ can be one of the following operators: (i) a shifted fractional Laplacian; (ii) a shifted Hermite operator; (iii) the Schrödinger operator with some general potentials. We call a subset $E \subset \mathbb{R}^n$ as a stabilizable set for (0.1), if there is a linear bounded operator $K$ on $L^2(\mathbb{R}^n)$ so that the semigroup $\{e^{-t(H-\chi_E K)}\}_{t \geq 0}$ is exponentially stable. (Here, $\chi_E$ denotes the characteristic function of $E$, which is treated as a linear operator on $L^2(\mathbb{R}^n)$.)

This paper presents different geometric characterizations of the stabilizable sets for (0.1) with different $H$. In particular, when $H$ is a shifted fractional Laplacian, $E \subset \mathbb{R}^n$ is a stabilizable set for (0.1) if and only if $E \subset \mathbb{R}^n$ is a thick set, while when $H$ is a shifted Hermite operator, $E \subset \mathbb{R}^n$ is a stabilizable set for (0.1) if and only if $E \subset \mathbb{R}^n$ is a set of positive measure. Our results, together with the results on the observable sets for (0.1) obtained in [1, 18, 24, 32], reveal such phenomena: for some $H$, the class of stabilizable sets contains strictly the class of observable sets, while for some other $H$, the classes of stabilizable sets and observable sets coincide. Besides, this paper gives some sufficient conditions on the stabilizable sets for (0.1) where $H$ is the Schrödinger operator with some general potentials.

1. Introduction

1.1. Notation. Let $\mathbb{N} := \{0, 1, 2, \ldots \}$ and let $\mathbb{N}^+ := \{1, 2, \ldots \}$. Write $C(\cdots)$ for a positive constant that depends on what are enclosed in the brackets. Use $\| \cdot \|_{L^2(\mathbb{R}^n)}$ to denote the operator norm on $L^2(\mathbb{R}^n)$. Use respectively $\| \cdot \|_{L^2(\mathbb{R}^n)}$ and $(\cdot, \cdot)$ to denote the usual norm and the usual inner product in $L^2(\mathbb{R}^n)$. Write respectively $|\cdot|$ and $(\cdot, \cdot)_{\mathbb{R}^n}$ for the usual norm and the usual inner product in $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, write $[x]$ for the integer part of $x$. Given a subset $E \subset \mathbb{R}^n$, write $|E|$ for its Lebesgue measure in $\mathbb{R}^n$ (if it is measurable); write $\chi_E$ for its characteristic function. Given $L > 0$ and $x \in \mathbb{R}^n$, write $Q_L(x)$ for the closed cube (in $\mathbb{R}^n$) centered at $x$ and of side-length $L$; write $B(x, L)$ for the closed ball (in $\mathbb{R}^n$) centered at $x$ and of radius $L$, while use $B^c(x, L)$ to denote the complement of $B(x, L)$ in $\mathbb{R}^n$. Given a function $V$ over $\mathbb{R}^n$, write $V_-(x) := \max\{-V(x), 0\}$, $x \in \mathbb{R}^n$. Given a polynomial $P$, write $\deg P$ for its degree. Given a linear operator $H$ on a Hilbert space, we write $\sigma(H)$ for the spectrum of $H$. Use $\widehat{\cdot}$ and $\mathcal{F}^{-1}$ to denote the Fourier transform and its inverse, respectively. Write $(-\Delta)^{\frac{s}{2}}$ (with $s > 0$) for the fractional Laplacian defined by

\[
(-\Delta)^{\frac{s}{2}} \varphi := \mathcal{F}^{-1}(|\xi|^s \hat{\varphi} (\xi)), \quad \varphi \in C_0^\infty (\mathbb{R}^n).
\]

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1.2. Equation. The subject of this paper is related to the stabilizability for the parabolic type equation in $\mathbb{R}^n$:

$$(\partial_t + H)y(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad y(0, x) \in L^2(\mathbb{R}^n),$$

(1.1)

where the operator $H$ has one of the following forms:

(i) The first form is as:

$$H = (-\Delta)^{\frac{s}{2}} - c,$$

(1.2)

where $s > 0$ and $c \in \mathbb{R}$.

(ii) The second form is as:

$$H = -\Delta + |x|^2 - c,$$

(1.3)

where $c \in \mathbb{R}$.

(iii) The third form is as:

$$H = -\Delta + V(x),$$

(1.4)

where the real-valued potential $V$ satisfies one of the following two conditions:

*Condition I* The function $V$ is locally integrable so that for some $\delta \in (0, 1),

$$\int_{\mathbb{R}^n} V_-(x)|\varphi|^2 dx \leq \delta \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx, \text{ when } \varphi \in C_0^\infty(\mathbb{R}^n).$$

*Condition II* The function $V$ is locally bounded and measurable so that

$$\lim_{|x| \to \infty} V(x) = \infty.$$

Several notes on the above equation are given in order.

(a) The form $H$ given by (1.4) is a generalization of that given by (1.3). The reasons that we consider them as two different cases are as follows: First, our results for (1.1) with (1.3) are much more delicate than those for (1.1) with (1.4); Second, our methods to study them are totally different.

(a) We call (1.1) with (1.2) a shifted fractional heat equation (a fractional heat equation, for short). It is well known that in the case (1.2), $(-H)$ is self-adjoint and generates an analytic semigroup $\{e^{-tH}\}_{t \geq 0}$ satisfying

$$\|e^{-tH}\|_{L(L^2(\mathbb{R}^n))} = e^{ct}, \text{ when } t \geq 0.$$  

(1.5)

(a) We call (1.1) with (1.3) a heat equation associated with a shifted Hermite operator. It follows by [40] that in the case (1.3), $(-H)$ is self-adjoint and generates an analytic semigroup $\{e^{-tH}\}_{t \geq 0}$ satisfying

$$\|e^{-tH}\|_{L(L^2(\mathbb{R}^n))} = e^{(c-n)t}, \text{ when } t \geq 0.$$  

(1.6)

(a) We call (1.1) with (1.4) a heat equation with a potential. The Schrödinger operator with potentials satisfying either *Condition I* or *Condition II* is important and has been widely studied. When $V$ satisfies *Condition I*, $(-H)$ is self-adjoint and generates an analytic semigroup (see, e.g., [36, Theorem X.17]). When $V$ satisfies *Condition II*, $(-H)$ is self-adjoint, generates an analytic semigroup and has a discrete spectrum (see [7, Theorem 3.1]).
Let $n \geq 3$ and let

$$V(x) := -\left(\frac{n-2}{2}\right)^2 \frac{\delta}{|x|^2}, \quad x \in \mathbb{R}^n,$$

where $\delta \in (0, 1)$. Then $V$ satisfies Condition I. This can be derived from the classical Hardy inequality:

$$\int_{\mathbb{R}^n} |\nabla \varphi(x)|^2 \, dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|\varphi(x)|^2}{|x|^2} \, dx, \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

For the above $V$, the corresponding equation (1.1) with (1.4) is not exponentially stable. Indeed, by [7, Theorem 4.1], we have

$$\|e^{-tH}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = 1 \text{ for each } t \geq 0.$$

Let $a > 0$ and $c \in \mathbb{R}$, and let

$$V(x) := |x|^a - c, \quad x \in \mathbb{R}^n. \quad (1.7)$$

Clearly, the above $V$ satisfies Condition II. For this $V$, when $c \geq \lambda_1$ (the first eigenvalue of the operator $-\Delta + |x|^a$), the corresponding equation (1.1) with (1.4) is not exponentially stable.

### 1.3. Concepts

To give our main results, we need the following concepts about the equation (1.1):

- **(b1)** A measurable set $E \subset \mathbb{R}^n$ is called a *stabilizable set* for (1.1), if there is a linear bounded operator $K$ on $L^2(\mathbb{R}^n)$ so that for some constants $M > 0$ and $\omega > 0$,

$$\|e^{-t(H-xEK)}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq M e^{-\omega t} \text{ for all } t \geq 0. \quad (1.8)$$

When $E \subset \mathbb{R}^n$ is a stabilizable set for (1.1), we say that the equation (1.1) is stabilizable over $E \subset \mathbb{R}^n$.

- **(b2)** A measurable set $E \subset \mathbb{R}^n$ is called a thick set, if there is $\gamma > 0$ and $L > 0$ so that

$$|E \cap Q_L(x)| \geq \gamma L^n \text{ for each } x \in \mathbb{R}^n. \quad (1.9)$$

Some notes on the above concepts are given in order.

- **(c1)** The concept of stabilizable sets seems to be new for us. It links with the stabilizability and is comparable to the concept of observable sets given in [43], i.e., a measurable set $E \subset \mathbb{R}^n$ is called an observable set for (1.1), if for every $T > 0$, there is $C = C(E,T) > 0$ so that when $y$ solves (1.1),

$$\|y(T, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C \left(\int_0^T \int_E |y(t,x)|^2 \, dx \, dt\right)^{1/2}. \quad (1.10)$$

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The equation (1.1) is said to be exponentially stable, if there is $M > 0$ and $\omega > 0$ so that

$$\|e^{-tH}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq M e^{-\omega t} \text{ for all } t \geq 0.$$
There have been several common methods to study the stabilizability for control systems, such as LQ theory and Lyapunov functions (see, for instance, \cite{10, 20, 44, 45}). We would like to mention the recent work \cite{41} which gives a new characterization of the stabilizability in terms of a weak observability inequality (see Theorem 2.1 in the current paper). It plays an important role in our studies.

To our best knowledge, the concept of thick sets arose from studies of the uncertainty principle (see, for instance, \cite{9}, p. 5, or \cite{17}, p. 113)

### 1.4. Aim and motivation

First, the concept of stabilizable sets connects with the stabilizability, while the concept of observable sets links with the null controllability. Since the null controllability implies the stabilizability (see, e.g., \cite{45}, p. 227), we see that if $E$ is an observable set for (1.1), then it is a stabilizable set for (1.1). Second, the observable sets for heat equations have been studied in, for instance, \cite{1, 2, 5, 11–13, 18, 21, 24, 25, 27–32, 34, 35, 43}. Especially, according to \cite{43, Theorem 1.1} (see also \cite{13}), the observable sets for (1.1), where $H = -\Delta$, are characterized by thick sets, while according to \cite{1, Remark 1.13}, the observable sets for (1.1), with (1.2) where $s > 1$ and $c \in \mathbb{R}$, are also characterized by thick sets. Thus, we naturally ask the following question:

**How to characterize stabilizable sets for (1.1)?**

The answer for the above question may give geometric characterizations of stabilizable sets for (1.1). Such characterizations, along with the characterizations on observable sets obtained in \cite{1, 18, 24, 32}, may give an explicit gap between the stabilizability and the null controllability from the perspective of control regions. The studies on the above problem may lead us to a new subject about the stabilizability of PDEs. To our best knowledge, the above problem has not been touched upon.

The aim of this paper is to answer the above question.

### 1.5. Main results

The first main result concerns (1.1) with (1.2).

**Theorem 1.1.** Let $E \subset \mathbb{R}^n$. Then the following statements are equivalent:

(i) The set $E$ is a thick set.

(ii) The set $E$ is a stabilizable set for (1.1) with (1.2) where $s > 0$ and $c \geq 0$.

Several notes on Theorem 1.1 are given in order.

(d1) We explain why there is the restriction $c \geq 0$ in (ii) of Theorem 1.1: From (1.5), we see that when $c < 0$, the equation (1.1), with (1.2) where $s > 0$, is exponentially stable, while when $c \geq 0$, it is not. Thus, when studying the stabilization for (1.1) with (1.2), we only need to focus on the case that $c \geq 0$ and $s > 0$.

(d2) Since each observable set is a stabilizable set, the class of stabilizable sets contains the class of observable sets. On the other hand, for (1.1) with (1.2) where $s > 1$ and $c \in \mathbb{R}$, it was proved in \cite{1, Remark 1.13} that the observable sets are characterized by thick sets. Then by Theorem 1.1, we see that the classes of stabilizable sets and observable sets coincide for this case.

(d3) In the case that $s \in (0, 1)$ and $c \geq 0$, it follows from Theorem 1.1 that the stabilizable sets for (1.1) with (1.2) are characterized by thick sets, while in the case that either $s \in (0, 1)$, $c \in \mathbb{R}$ and $n \geq 1$ or $s = 1$, $c \in \mathbb{R}$ and $n = 1$, it follows from \cite[Theorem 3 & Remark 7]{18} and \cite[Theorem 1.1]{24} that some thick sets (for
instance, $B^c(x, R)$ with $x \in \mathbb{R}^n$ and $R > 0$ are not observable sets for the equation (1.1) with (1.2). Hence, for the equation (1.1) with (1.2) where either $s \in (0, 1)$, $c \geq 0$ and $n \geq 1$ or $s = 1$, $c \geq 0$ and $n = 1$, the class of stabilizable sets contains strictly the class of observable sets. We would like to mention what follows: For the case where $s = 1$, $c \in \mathbb{R}$ and $n \geq 2$, it is still open whether there is a thick set which is not an observable set for (1.1) with (1.2), to our best knowledge (see [18, 24]).

(d) To prove $(i) \Rightarrow (ii)$, we built up an abstract criteria for the stabilizability, i.e., Lemma 2.2. It can be viewed as a variant of Lebeau-Robbiano strategy and has independent significance.

The second main result concerns with the equation (1.1) with (1.3).

**Theorem 1.2.** Let $E \subset \mathbb{R}^n$. Then the following statements are equivalent:

(i) The set $E$ has a positive measure.

(ii) The set $E$ is a stabilizable set for (1.1) with (1.3) where $c \geq n$.

Some remarks about Theorem 1.2 are given as follows:

(e) We explain why there is the restriction $c \geq n$ in (ii) of Theorem 1.2: From (1.6), we see that when $c < n$, the equation (1.1) with (1.3) is exponentially stable, while when $c \geq n$, it is not. Thus, when studying the stabilization for (1.1) with (1.3), we only need to focus on the case that $c \geq n$.

(e) For the equation (1.1) with (1.3) where $c \geq n$, the class of observable sets is strictly contained in the class of stabilizable sets. Indeed, given a set of positive measure in a half space of $\mathbb{R}^n$, we see from [32, Theorem 1.10] that it is not an observable set, while we find by Theorem 1.2 that it is a stabilizable set.

(e) From Theorem 1.1 and Theorem 1.2, we see that the geometric characterizations for (1.1), with (1.2) and (1.3) respectively, are different. The reason is that the spectral inequalities for $H$, given by (1.2) and (1.3) respectively, are different (see Lemma 3.1 and Lemma 3.2).

1.6. Some sufficient conditions on stabilizable sets. The following two results concern sufficient conditions on stabilizable sets for the equation (1.1) with (1.4).

**Theorem 1.3.** Suppose that $V$ satisfies Condition I. Then every thick set is a stabilizable set for (1.1) with (1.4).

**Theorem 1.4.** Suppose that $V$ satisfies Condition II. Then each nonempty open set is a stabilizable set for (1.1) with (1.4).

Some notes on Theorem 1.3 and Theorem 1.4 are given in order.

(f) From (a5) and (a6) in Subsection 1.2, we see that when $V$ verifies either Condition I or Condition II, the equation (1.1) with (1.4) may not be exponentially stable. Thus, it makes sense to study the stabilization for (1.1) with (1.4).

(f) We cannot get a sufficient and necessary condition on the stabilizable sets for (1.1) with (1.4), since we are not able to get the desired spectral inequality for the operator $H$ given by (1.4). Thus we cannot use the above-mentioned Lemma 2.2 to prove either Theorem 1.3 or Theorem 1.4.
We use different approaches to show Theorem 1.3 and Theorem 1.4: When $V$ satisfies Condition I, we can treat $V$ as a small perturbation of $-\Delta$ in the sense of the quadratic form. Thus, we only need the spectral inequality for the Laplacian in the proof of Theorem 1.3; When $V$ satisfies Condition II, we can use a unique continuation property of the elliptic equation $Hu = 0$, as well as the techniques used in [3] (see also in [4]), to prove Theorem 1.4.

As a comparison, we mention some sufficient conditions on the observable sets for (1.1) with (1.4). First, for the case that $V(x) = |x|^{2k}$, $x \in \mathbb{R}^n$, with $k \in \mathbb{N}^+$, it was obtained in [12, 32] that when $k \geq 2$, the cone $E := \{x \in \mathbb{R}^n : |x| \geq r_0, x/|x| \in \Theta_0\}$ (Here, $r_0 > 0$ and $\Theta_0$ is a nonempty open subset of $S^{n-1}$.) is an observable set, while when $k = 1$, the above cone is no longer an observable set; it was proved in [5] that when $k = 1$, every thick set is an observable set. Second, for the case that $V$ is a real valued analytic potential vanishing at infinity, it was proved in [21] that each thick set is an observable set.

Besides, we would like to mention the recent work [11] where observable sets were studied for (1.1) with (1.4) where potentials are bounded and time-dependent.

1.7. Plan of the paper. The rest of the paper is organized as follows: Section 2 presents an abstract criteria for the stabilization. Section 3 gives two spectral inequalities. Section 4 proves Theorem 1.1 and Theorem 1.2 with the aid of lemmas built up in Section 2 and Section 3. Section 5 shows Theorem 1.3 and Theorem 1.4 via different approaches.

2. An abstract criteria for stabilization

This section presents a criteria for the stabilizability from the perspective of spectral inequalities. First of all, we introduce the next Theorem 2.1 which is a direct consequence of [41, Theorem 1] and plays an important role in our studies.

**Theorem 2.1 ([41]).** Let $E$ be a measurable subset of $\mathbb{R}^n$. Then $E$ is a stabilizable set for (1.1) if and only if there is $T > 0$, $\alpha \in (0, 1)$ and $C = C(E, T) > 0$ so that when $y$ solves (1.1),

$$\|y(T, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C \left( \int_0^T \int_E |y(t, x)|^2 dx dt \right)^{1/2} + \alpha \|y(0, \cdot)\|_{L^2(\mathbb{R}^n)}. \quad (2.1)$$

We call (2.1) a weak observability inequality. Before stating the above-mentioned criteria, we explain its main connotations: The Lebeau-Robbiano strategy (see [22]) leads to the observability inequality, equivalently the null controllability, for the heat equation. This strategy has been generalized to abstract settings in some Hilbert spaces (see [6, 32, 33, 39]) and in some Banach spaces (see [16]). In essence, it is a combination of the Lebeau-Robbiano spectral inequality (see [22]) and a dissipative inequality. A key condition in this combination is as: the decay rate in the dissipative inequality is strictly larger than the growth rate in the spectral inequality. Since the gap between the null controllability and the stabilizability is explicitly given by (1.10) and (2.1), we find that
the aforementioned key condition can be relaxed in the studies of the stabilizability. This relaxed condition leads to an abstract criteria for the stabilizability in the next Lemma 2.2. It plays an important role in the proofs of both Theorem 1.1 and Theorem 1.2.

Lemma 2.2. Let $H$ be a self-adjoint operator so that $(-H)$ generates a $C_0$ semigroup $\{e^{-tH}\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$. Let $E$ be a measurable set in $\mathbb{R}^n$ and let $\{\pi_k\}_{k \in \mathbb{N}^+}$ be a family of orthogonal projections on $L^2(\mathbb{R}^n)$. If there are positive constants $c_1, c_2, a, b, M$ so that for each $k \in \mathbb{N}^+$, the spectral inequality

$$ \|\pi_k \varphi\|_{L^2(\mathbb{R}^n)} \leq e^{c_1 k^a} \|\varphi\|_{L^2(E)}, \text{ when } \varphi \in L^2(\mathbb{R}^n) $$  \hspace{1cm} (2.2)

and the dissipative inequality

$$ \|(1 - \pi_k)(e^{-tH} \varphi)\|_{L^2(\mathbb{R}^n)} \leq M e^{-c_2 k^b} \|\varphi\|_{L^2(\mathbb{R}^n)}, \text{ when } \varphi \in L^2(\mathbb{R}^n), \ t > 0 $$  \hspace{1cm} (2.3)

hold, then $E$ is a stabilizable set for (1.1).

Remark 2.3. In Lemma 2.2, the family $\{\pi_k\}_{k \in \mathbb{N}^+}$ can be replaced by a family $\{\pi_k\}_{1 \leq k \in \mathbb{R}}$ of orthogonal projections on $L^2(\mathbb{R}^n)$.

Remark 2.4. We compare Lemma 2.2 with the abstract Lebeau-Robbiano strategy in the following manner: First, the abstract Lebeau-Robbiano strategy says that (see e.g. in [33, Theorem 2.2], as well as [6, Theorem 2.1]) the observability inequality (1.10) holds, if

$$ (2.2) \text{ and } (2.3) \text{ are true, and } a < b. $$  \hspace{1cm} (2.4)

In (2.4), the condition $a < b$ is necessary. Indeed, there are examples showing that if (2.2) and (2.3) hold for some $E \subset \mathbb{R}^n$, but $a \geq b$, then the observability inequality (1.10) is not true for any $T > 0$ (see Remark 4.1 for details). Second, Lemma 2.2 says: to get (2.1), we only need (2.2) and (2.3), but not the condition $a < b$. The advantage we can take from this is as follows: We are allowed to afford more cost in the spectral inequality when considering smaller $E$. Thus, the class of stabilizable sets might contain strictly the class of observable sets. These will be discussed in detail in Subsection 3.3.

Proof of Lemma 2.2. According to Theorem 2.1, to prove that $E$ is a stabilizable set for (1.1), it suffices to show

$$ \|e^{-tH} \varphi\|_{L^2(\mathbb{R}^n)} \leq C \left( \int_0^T \|e^{-tH} \varphi\|_{L^2(E)}^2 \, dt \right)^{1/2} + \alpha \|\varphi\|_{L^2(\mathbb{R}^n)}, \text{ when } \varphi \in L^2(\mathbb{R}^n) $$  \hspace{1cm} (2.5)

for some $C > 0$, $T > 0$ and $\alpha \in (0, 1)$. Before proving (2.5), we give some preliminaries. First, since $(-H)$ generates a $C_0$ semigroup in $L^2(\mathbb{R}^n)$, there is $\delta_0 \geq 0$ and $M' > 0$ so that

$$ \|e^{-tH}\|_{\text{L}(L^2(\mathbb{R}^n))} \leq M' e^{\delta_0 t}, \ t \geq 0. $$

Without loss of generality, we can assume that $M' \leq M$ where $M$ is given by (2.3). Then the shifted operator$^2$

$$ \tilde{H} = H + \delta_0 I $$  \hspace{1cm} (2.6)

satisfies

$$ \|e^{-t\tilde{H}}\|_{\text{L}(L^2(\mathbb{R}^n))} \leq M, \ t \geq 0. $$  \hspace{1cm} (2.7)

$^2$Here $I$ is the identity operator on $L^2(\mathbb{R}^n)$. 
Second, with $a, b, c_1, c_2, M$ given in (2.2)-(2.3), we set
\[
\gamma := 2^{b+a} > 1, \quad N := \max \left\{ 2, \frac{2^{b+2\delta}d_0}{c_2} \right\}, \quad (2.8)
\]
\[
C(M, \gamma) := M^2 + \frac{\gamma - 1}{8M^2} \left( \frac{4M^4}{\gamma} \right)^{\gamma/(\gamma-1)}, \quad D(M, N) := \frac{e^{-2c_1(2N)^{a/b}}}{8M^2N}, \quad (2.9)
\]
\[
A := \frac{2^{b+1}}{c_2} \ln \left( 1 + \frac{25C(M, \gamma)}{D(M, N)} \right), \quad \tau_0 := \frac{3A}{2N}. \quad (2.10)
\]

Third, we define a function:
\[
g(\tau) := \frac{\tau}{4M^2} \exp \left\{ -2c_1 \left[ \left( A/\tau \right)^{\frac{a}{b}} \right] \right\}, \quad \tau \in (0, \tau_0). \quad (2.11)
\]

Recall that for each $x \in \mathbb{R}$, $[x]$ denotes the integer part of $x$.

We now prove (2.5) by two steps$^3$:

**Step 1.** We prove the following recurrence inequality: when $\tau \in (0, \tau_0)$ and $\varphi \in L^2(\mathbb{R}^n)$,
\[
g(\tau)\|e^{-\tau \hat{H}} \varphi\|_{L^2(\mathbb{R}^n)}^2 - g(\tau/2)\|\varphi\|_{L^2(\mathbb{R}^n)}^2 \leq \int_{\tau/2}^\tau \|e^{-t \hat{H}} \varphi\|_{L^2(E)}^2 dt + \alpha_0 \tau \|\varphi\|_{L^2(\mathbb{R}^n)}^2, \quad (2.12)
\]
where
\[
\alpha_0 := D(M, N) \exp \{-c_2 2^{-(b+1)}A\}/50. \quad (2.13)
\]

First, we observe that when $k \in \mathbb{N}^+$, $\varphi \in L^2(\mathbb{R}^n)$ and $t \in (0, \tau_0)$,
\[
\frac{1}{2}e^{-2c_1k^a}\|e^{-t \hat{H}} \varphi\|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{2}e^{-2c_1k^a} \left( \|\pi_k e^{-t \hat{H}} \varphi\|_{L^2(\mathbb{R}^n)}^2 + \| (1 - \pi_k) e^{-t \hat{H}} \varphi\|_{L^2(\mathbb{R}^n)}^2 \right)
\leq \frac{1}{2} \|\pi_k e^{-t \hat{H}} \varphi\|_{L^2(E)}^2 + \frac{1}{2}e^{-2c_1k^a} \| (1 - \pi_k) e^{-t \hat{H}} \varphi\|_{L^2(\mathbb{R}^n)}^2
\leq \|e^{-t \hat{H}} \varphi\|_{L^2(E)}^2 + (1 + \frac{1}{2}e^{-2c_1k^a}) \| (1 - \pi_k) e^{-t \hat{H}} \varphi\|_{L^2(\mathbb{R}^n)}^2
\leq \|e^{-t \hat{H}} \varphi\|_{L^2(E)}^2 + M^2 (1 + \frac{1}{2}e^{-2c_1k^a}) e^{-2c_2t \delta} \|\varphi\|_{L^2(\mathbb{R}^n)}^2, \quad (2.14)
\]

In (2.14), we used the fact that each $\pi_k$ is an orthogonal projection on $L^2(\mathbb{R}^n)$ on Line 1; we used the spectral inequality (2.2) on Line 2; we used the Cauchy Schwartz inequality and the fact that $\|\varphi\|_{L^2(E)} \leq \|\varphi\|_{L^2(\mathbb{R}^n)}$ on Line 3; we used the dissipative inequality (2.3) and the fact that $e^{-\delta t} \leq 1$ when $0 < t < \tau_0$ on Line 4.

$^3$Some ideas are borrowed from [6, Theorem 2.1], see also [33, Theorem 2.2].
Next, we arbitrarily fix $\tau \in (0, \tau_0)$ and $\varphi \in L^2(\mathbb{R}^n)$. Integrating both sides of (2.14) with respect to $t$ on $(\frac{\tau}{2}, \tau)$, using (2.7), we find that for all $k \in \mathbb{N}^+$,

$$
\frac{\tau}{4M^2} e^{-2c_1k^a} \| e^{-\tau \tilde{H}} \varphi \|_{L^2(\mathbb{R}^n)}^2 \leq \frac{1}{2} e^{-2c_1k^a} \int_{\frac{\tau}{2}}^{\tau} \| e^{-t \tilde{H}} \varphi \|_{L^2(\mathbb{R}^n)}^2 dt
$$

$$
\leq \int_{\frac{\tau}{2}}^{\tau} \| e^{-t \tilde{H}} \varphi \|_{L^2(\mathbb{R}^n)}^2 dt + M^2 \left(1 + \frac{1}{2} e^{-2c_1k^a}\right) \int_{\frac{\tau}{2}}^{\tau} e^{-2c_2tk^b} \| \varphi \|_{L^2(\mathbb{R}^n)}^2 dt
$$

$$
\leq \int_{\frac{\tau}{2}}^{\tau} \| e^{-t \tilde{H}} \varphi \|_{L^2(\mathbb{R}^n)}^2 dt + M^2 \left(1 + \frac{1}{2} e^{-2c_1k^a}\right) \frac{\tau e^{-c_2\tau k^b}}{2} \| \varphi \|_{L^2(\mathbb{R}^n)}^2.
$$

(2.15)

In the last inequality of (2.15), we used the estimate:

$$
\int_{\frac{\tau}{2}}^{\tau} e^{-2c_2tk^b} dt < \frac{\tau}{2} e^{-c_2\tau k^b}.
$$

Set

$$
k(\tau) := \left[ \left( \frac{A}{\tau} \right)^{\frac{1}{b}} \right].
$$

(2.16)

By (2.11) and (2.16), it follows that

$$
\frac{\tau}{4M^2} e^{-2c_1k(\tau)^a} = g(\tau).
$$

(2.17)

Meanwhile, by (2.8), (2.10) and (2.16), it follows that $k(\tau) \in \mathbb{N}^+$. Thus, we have (2.15) where $k = k(\tau)$.

We now claim

$$
M^2 \left(1 + \frac{1}{2} e^{-2c_1k(\tau)^a}\right) \frac{\tau e^{-c_2\tau k(\tau)^b}}{2} \leq g(\tau/2) + \alpha_0 \tau.
$$

(2.18)

When (2.18) is proved, the desired (2.12) follows from (2.15) (with $k = k(\tau)$), (2.17) and (2.18) at once.

The rest of this step is to show (2.18). For this purpose, we first claim

$$
M^2 B \left(1 + \frac{x}{2}\right) \leq \frac{1}{8M^2} x^\gamma + \alpha_0, \quad \text{when} \ x \in (0, 1),
$$

(2.19)

where

$$
B := \frac{e^{-c_22^{-\gamma}A}}{2} \in (0, 1).
$$

(2.20)

To this end, we define a function:

$$
F(x) := \frac{1}{8M^2} x^\gamma - \frac{M^2 B}{2} x + \alpha_0 - M^2 B, \quad x > 0.
$$

A direct computation shows

$$
F'(x) < 0, \quad \text{when} \ x \in (0, x_0); \quad F'(x) > 0, \quad \text{when} \ x \in (x_0, \infty),
$$

where

$$
x_0 := \left( \frac{4M^4 B}{\gamma} \right)^{\frac{1}{\gamma}}.
$$
Then we have
\[
\min_{x > 0} F(x) = F(x_0) \\
= \alpha_0 - M^2 B - \frac{(\gamma - 1) \left(\frac{4M^4 B}{\gamma}\right)^{\frac{1}{\gamma - 1}}}{8M^2} \\
\geq \alpha_0 - \left(M^2 + \frac{\gamma - 1}{8M^2} \left(\frac{4M^4}{\gamma}\right)^{\gamma/(\gamma - 1)}\right) B \\
> 0.
\]  
(2.21)

In (2.21), for the first inequality, we used the fact that \(B^{\frac{1}{\gamma - 1}} < B\) (which follows from facts that \(B \in (0, 1)\) and \(\gamma/(\gamma - 1) > 1\)); for the last inequality, we first see from (2.13) and (2.20) that it is equivalent to
\[
A > \frac{2^{b+1}}{c_2} \ln \frac{25(M^2 + \frac{\gamma - 1}{8M^2}(\frac{4M^4}{\gamma})^{\gamma/(\gamma - 1)})}{D(M, N)},
\]  
(2.22)

then we obtained (2.22) from (2.9) and (2.10). Now (2.19) follows from (2.21) at once.

We next use (2.19) to show (2.18). Indeed, since
\[
\left(A/\tau\right)^{\frac{1}{b}} < 2\left[(A/\tau)^{\frac{1}{b}}\right] = 2k(\tau),
\]
we find that
\[
k(\tau)^{b} > \frac{A}{2^{b+1}} \text{ and } 2^{\frac{1}{b}+1}k(\tau) > \left(\frac{2A}{\tau}\right)^{\frac{1}{b}} > k(\tau/2),
\]
which yields
\[
e^{-c_2\tau k(\tau)^{b}} < e^{-c_22^{-b-a}} \text{ and } e^{-2c_1k(\tau)^a} < e^{-2c_1k(\tau/2)^a}.
\]

These, along with (2.19) (where \(x := e^{-2c_1k(\tau)^a} \in (0, 1)\)) and (2.17), lead to (2.18).

Hence the proof of (2.12) is completed.

**Step 2. We prove (2.5) with the aid of (2.12) and a telescopic series method.**

It deserves mentioning that the similar method has already been successfully applied to obtain observability inequalities in \([2, 6, 14, 15, 33, 34, 43]\).

Arbitrarily fix \(t \in (0, \tau_0)\) and \(\varphi \in L^2(\mathbb{R}^n)\). We set
\[
t_j = 2^{-j}t, \quad j \in \mathbb{N}.
\]  
(2.23)

Applying (2.12), where \(\tau = 2^{-1}t_j\) and \(\varphi\) is replaced by \(e^{-t_j+1}\hat{H}\varphi\) we get
\[
g(2^{-1}t_j)\|e^{-t_j}\hat{H}\varphi\|_{L^2(\mathbb{R}^n)}^2 - g(2^{-1}t_{j+1})\|e^{-t_j+1}\hat{H}\varphi\|_{L^2(\mathbb{R}^n)}^2 \\
\leq \int_{2^{-1}t_j}^{2^{-1}t_{j+1}} \|e^{-s}\hat{H}\varphi\|_{L^2(V)}^2 ds + \alpha_0 2^{-1}t_j \|\varphi\|_{L^2(\mathbb{R}^n)}^2 \\
\leq \int_{t_{j+1}}^{t_j} \|e^{-s}\hat{H}\varphi\|_{L^2(V)}^2 ds + \alpha_0 \tau_0 2^{-(j+1)} \|\varphi\|_{L^2(\mathbb{R}^n)}^2.
\]  
(2.24)
Summing up (2.24) from \( j = 0 \) to \( j = m \in \mathbb{N}^+ \) leads to
\[
g(t/2)\|e^{-t\tilde{H}} \varphi\|^2_{L^2(\mathbb{R}^n)} - g(2^{-1}t_{m+1})\|e^{-t_{m+1}\tilde{H}} \varphi\|^2_{L^2(\mathbb{R}^n)} 
\leq \int_{t_{m+1}}^t \|e^{-s\tilde{H}} f\|^2_{L^2(E)} ds + \alpha_0 \tau_0 (1 - 2^{-m+1}) \|\varphi\|^2_{L^2(\mathbb{R}^n)}. \tag{2.25}
\]

Meanwhile, three facts are given in order. First, it follows by (2.11) that
\[ g(t) \to 0, \text{ as } t \to 0; \]

Second,
\[ t_{m+1} = \frac{t}{2^{m+1}} \to 0, \text{ as } m \to \infty; \]

Third,
\[ \|e^{-t_{m+1}\tilde{H}} \varphi\|_{L^2(\mathbb{R}^n)} \leq M \|\varphi\|^2_{L^2(\mathbb{R}^n)}, \text{ when } m \in \mathbb{N}. \]

Combining the above facts together, and sending \( m \to \infty \) in (2.25), we find that when \( t \in (0, \tau_0) \) and \( \varphi \in L^2(\mathbb{R}^n) \),
\[
g(t/2)\|e^{-t\tilde{H}} \varphi\|^2_{L^2(\mathbb{R}^n)} \leq \int_0^t \|e^{-s\tilde{H}} \varphi\|^2_{L^2(E)} ds + \alpha_0 \tau_0 \|\varphi\|^2_{L^2(\mathbb{R}^n)}. \tag{2.26}
\]

Finally, we will get (2.5) from (2.26). Indeed, by (2.8) and (2.10), we have \( \frac{A}{N} < \tau_0 \). Thus by (2.26) (with \( t = \frac{A}{N} \)) and (2.6), we see that when \( \varphi \in L^2(\mathbb{R}^n) \),
\[
\|e^{-\frac{A}{N} H} \varphi\|^2_{L^2(\mathbb{R}^n)} = \frac{e^{2A\delta_0}}{N} \|e^{-\frac{A}{N} H} \varphi\|^2_{L^2(\mathbb{R}^n)} 
\leq \frac{e^{2A\delta_0}}{g(\frac{A}{2N})} \int_0^{\frac{A}{2N}} \|e^{-sH} \varphi\|^2_{L^2(E)} ds + \beta \|\varphi\|^2_{L^2(\mathbb{R}^n)}, \tag{2.27}
\]

where
\[ \beta := \frac{8NM^2\alpha_0 \tau_0}{A} \exp \left\{ 2c_1 (2N)^\frac{N}{2} + \frac{2A\delta_0}{N} \right\}. \]

Meanwhile, by (2.9), (2.10), (2.13) and the fact:
\[ e^{\frac{2A\delta_0}{N}} \leq e^{c_2 2^{-(b+1)}A}, \]
we deduce that
\[ 0 < \beta < 1. \]

This, along with (2.27), leads to (2.5) where
\[
\alpha = \sqrt{\beta} \in (0, 1), \ T = \frac{A}{N}, \ C = \sqrt{\frac{e^{2A\delta_0}}{g(\frac{A}{2N})}}.
\]

Hence, we finish the proof of Lemma 2.2. \( \square \)
Remark 2.5. Lemma 2.2 can be extended into what follows: (The proof is very similar
to that of Lemma 2.2, we omit the details.)

Let $X$ and $U$ be two real Hilbert spaces. Let $H$ be a self-adjoint operator on
$X$ so that $(-H)$ generates a $C_0$ semigroup $\{e^{-tH}\}_{t \geq 0}$ on $X$. Let $B$ be a linear bounded operator from $U$ to $X$. Let $\{\pi_k\}_{k \in \mathbb{N}^+}$ be a family of orthogonal projections on $X$. If there are positive
constants $c_1, c_2, a, b$ and $M$ so that for each $k \in \mathbb{N}^+$,
\[
\|\pi_k \varphi\|_X \leq e^{c_1 k^2} \|B^* \pi_k \phi\|_U, \text{ when } \varphi \in X,
\]
and
\[
\|(1 - \pi_k)(e^{-tH} \varphi)\|_X \leq Me^{-c_2 t k^2} \|\varphi\|_X, \text{ when } \varphi \in X, \ t > 0,
\]
then there is $\alpha \in (0,1)$, $T > 0$ and $C > 0$ so that
\[
\|e^{-TH} \varphi\|_X \leq C \left( \int_0^T \|B^* e^{-tH} \varphi\|_U^2 \|\varphi\|_X^2 \mathrm{d}t \right)^{1/2} + \alpha \|\varphi\|_X \text{ for all } \varphi \in X.
\]

According to ([41, Theorem 1]), the inequality (2.30) is equivalent to the stabilization
of the control system:
\[
(\partial_t + H)y(t) = Bu(t), \ t \geq 0, \text{ with } u \in L^2(0,\infty; U).
\]

3. Two spectral inequalities

Generally, in applications of Lemma 2.2, it is easier to verify the dissipative inequality
(2.3) than the spectral inequality (2.2). In this section, we present spectral inequalities
for the shifted fractional Laplacian and the shifted Hermite operator respectively. They
not only play important roles in the proofs of Theorem 1.1 and Theorem 1.2, but also
may have independent significance. We start with introducing some spectral projections.

3.1. Spectral projections. First, we consider the operator $H$ given by (1.2). Define a
family of orthogonal projections $\{\pi_k\}_{1 \leq k \in \mathbb{R}}$ in the following manner: For each $k \in \mathbb{R}$ with
$k \geq 1$, let $\pi_k : \varphi (\in L^2(\mathbb{R}^n)) \rightarrow \pi_k \varphi (\in L^2(\mathbb{R}^n))$ be given by
\[
\pi_k \varphi = \begin{cases} 
\mathcal{F}^{-1} \left( \chi(|\xi|^s - c \leq k) \hat{\varphi}(\xi) \right), & \text{if } k + c > 0, \\
0, & \text{if } k + c \leq 0,
\end{cases}
\]
where $\chi(|\xi|^s - c \leq k)$ denotes the characteristic function of the set $\{\xi \in \mathbb{R}^n : |\xi|^s - c \leq k\}$. It deserves mentioning that the above $\pi_k$ is exactly the usual spectral projection $P_{(-\infty,k]}(H)$, associated with $H$ given by (1.2).

We next consider the operator $H$ given by (1.3). Recall several known facts on the
Hermite operator $H_0 := -\Delta + |x|^2$:

Fact one (see [40]) We have $\sigma(H_0) = \{2k + n, \ k \in \mathbb{N}\}$. Thus by the spectral theorem
(see, e.g., [7, p. 412]), we see
\[
H_0 = -\Delta + |x|^2 = \sum_{k=0}^{\infty} (2k + n) P_k,
\]
where $P_k$ denotes the orthogonal projection onto the linear space spanned by the eigenfunctions of $H_0$ associated with the eigenvalue $2k + n$.

**Fact two (see [40])** For each $k \in \mathbb{N}$, let

$$
\varphi_k(x) = \left(2^k k! \sqrt{\pi}\right)^{-\frac{1}{2}} H_k(x) e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R},
$$

where $H_k$ is the Hermite polynomial given by

$$
H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}), \quad x \in \mathbb{R}.
$$

For each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ ($\alpha_i \in \mathbb{N}$), we define the following Hermite function by the tensor product:

$$
\Phi_\alpha(x) = \prod_{i=1}^n \varphi_{\alpha_i}(x_i), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
$$

Then for each $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$, $\Phi_\alpha$ is an eigenfunction of $H_0$ corresponding to the eigenvalue $2k + n$, and $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ forms a complete orthonormal basis in $L^2(\mathbb{R}^n)$.

**Fact three** The authors in [5] built up spectral inequalities for finite combinations of Hermite functions when $E$ is (i) an open subset; (ii) a weakly thick set (see (3.24) for the definition); (iii) a thick set.

We now define a family of orthogonal projections $\{\pi_k\}_{k \in \mathbb{N}^+}$ (associated with $H$ given by (1.3)) in the following manner: For each $k \in \mathbb{N}^+$, let $\pi_k : \varphi (\in L^2(\mathbb{R}^n)) \to \pi_k \varphi (\in L^2(\mathbb{R}^n))$ be given by

$$
\pi_k \varphi = \begin{cases} 
\sum_{0 \leq j \leq (k+c-n)/2} P_j \varphi, & \text{if } k + c \geq n, \\
0, & \text{if } k + c < n,
\end{cases}
$$

where $P_j$ is given by (3.2). The above $\pi_k$ is exactly the usual spectral projection $P_{(-\infty, k]}(H)$, associated with $H$ given by (1.3).

### 3.2. Spectral inequalities.

**Lemma 3.1.** Let $H$ be given by (1.2) and let $\{\pi_k\}_{1 \leq k \in \mathbb{R}}$ be defined by (3.1). If $E$ is a thick set in $\mathbb{R}^n$, then there is $C > 0$ so that

$$
\| \pi_k \varphi \|_{L^2(\mathbb{R}^n)} \leq e^{Ck^\frac{1}{2}} \| \pi_k \varphi \|_{L^2(E)}, \quad \text{when } \varphi \in L^2(\mathbb{R}^n) \text{ and } k \geq 1.
$$

**Proof.** Let $s > 0$ and $c \in \mathbb{R}$. Arbitrarily fix $k \in \mathbb{R}$ so that $k \geq 1$. In the case that $k + c \leq 0$, (3.7) is clearly true. We now consider the case that $k + c > 0$. By (3.1), we see that the support of $\pi_k \varphi$ is contained in the ball $B(0, (k+c)^{1/s})$. Then by the Logvinenko-Sereda theorem (see [17], [19, Theorem 1] or [43, Lemma 2.1]), we can find $C_1 > 0$ so that

$$
\| \pi_k \varphi \|_{L^2(\mathbb{R}^n)} \leq e^{C_1(1+(k+c)^{\frac{1}{s}})} \| \pi_k \varphi \|_{L^2(E)}, \quad \text{when } \varphi \in L^2(\mathbb{R}^n).
$$

Since $k \geq 1$, (3.7) follows from (3.8) at once. This completes the proof of Lemma 3.1. \[ \square \]

**Lemma 3.2.** Let $H$ be given by (1.3) and let $\{\pi_k\}_{k \in \mathbb{N}^+}$ be given by (3.6). If $E$ is a subset of positive measure in $\mathbb{R}^n$, then there is $C > 0$ so that

$$
\| \pi_k \varphi \|_{L^2(\mathbb{R}^n)} \leq e^{\frac{C}{2} k \ln k^{+} + Ck} \| \pi_k \varphi \|_{L^2(E)}, \quad \text{when } \varphi \in L^2(\mathbb{R}^n) \text{ and } k \in \mathbb{N}^+.
$$
Proof. We borrowed some idea from [5]. Arbitrarily fix \( c \in \mathbb{R} \) and \( k \in \mathbb{N}^+ \). From (3.6), we see that it suffices to consider the non-trivial case
\[ k \geq k_0 := \max\{1, n - c\}. \]
According to (3.6), (3.2), (3.3) and (3.5), the range of \( \pi_k \) is as
\[ \mathcal{E}_k = \text{span}\left\{ \Phi_\alpha : |\alpha| \leq \frac{k + c - n}{2}, \alpha \in \mathbb{N}^n \right\}. \]
(3.10)
To proceed, we recall the following two results from [5]: First, it follows by [5, Lemma 4.2] that there is \( c_n > 0 \) independent of \( k \) so that
\[ \| \varphi \|_{L^2(\mathbb{R}^n)} \leq \frac{2}{\sqrt{3}} \| \varphi \|_{L^2(B(0,c_n\sqrt{k+c+1}))} \], \quad \text{when} \ \varphi \in \mathcal{E}_k. \]
(3.11)
(It deserves mentioning that though the above \( \mathcal{E}_k \) is slightly different from that in [5, Lemma 4.2], the conclusion is still true. This can be verified easily.) Second, it follows from [5, Lemma 4.4] that if \( \omega \subset \mathbb{R}^n \) satisfies \(|\omega \cap B(0,R)| > 0\), then each polynomial \( P = P(x_1, \ldots, x_n) \), with \( \deg P = d \), satisfies
\[ \| P \|_{L^2(B(0,R))} \leq \frac{2^{d+1}}{\sqrt{3}} \sqrt{\frac{4|B(0,R)|}{|\omega \cap B(0,R)|}} F\left( \frac{|\omega \cap B(0,R)|}{|B(0,R)|} \right) \| P \|_{L^2(\omega \cap B(0,R))}, \]
(3.12)
where
\[ F(t) := \left( \frac{1 + (1 - \frac{1}{4})^{1/n}}{1 - (1 - \frac{1}{4})^{1/n}} \right)^d, \quad t \in (0, 1]. \]
(3.13)
We now prove (3.9). Arbitrarily fix a measurable subset \( E \subset \mathbb{R}^n \), with \( |E| > 0 \). Then there is \( \varepsilon_0 > 0 \) and \( R_0 > 0 \) so that
\[ |E \cap B(0,R)| \geq \varepsilon_0, \quad \text{when} \quad R \geq R_0. \]
(3.14)
Meanwhile, it follows by (3.3) that for each \( \varphi \in \mathcal{E}_k \), there is a polynomial \( P_k \), with \( \deg P_k \leq k \), so that
\[ \varphi(x) = P_k(x)e^{-\frac{|x|^2}{2}}, \quad x \in \mathbb{R}^n. \]
(3.15)
Set
\[ k_1 := \max\{k_0, (R_0/c_n)^2 - c\}. \]
(3.16)
There are only two possibilities for the above fixed \( k \): either \( k > k_1 \) or \( k \leq k_1 \). We organize the rest of the proof by two steps.

Step 1. We consider the case that \( k > k_1 \).

Given \( \varphi \in \mathcal{E}_k \), we have
\[ \| \varphi(x) \|_{L^2(\mathbb{R}^n)} \leq \frac{2}{\sqrt{3}} \| P_k(x) \|_{L^2(B(0,c_n\sqrt{k+c+1}))} \]
\[ \leq C \varepsilon_0^{-\frac{1}{2}} (k + c + 1)^{\frac{d}{2}} F(t_k) \| P_k \|_{L^2(E \cap B(0,c_n\sqrt{k+c+1}))} \]
\[ \leq C \varepsilon_0^{1/2} (k + c + 1)^{\frac{d}{2}} e^{\frac{c^2(k+c+1)}{2}} F(t_k) \| \varphi \|_{L^2(E)}, \]
(3.17)
where
\[ t_k := \frac{|E \cap B(0, c_n \sqrt{k + c + 1})|}{|B(0, c_n \sqrt{k + c + 1})|} \in (0, 1]. \tag{3.18} \]

In (3.17), for the first inequality on Line 1, we used (3.11), (3.15) and the inequality: \( \|e^{-|\cdot|^2}\|_{L^\infty} \leq 1 \) (which is trivial); for the second inequality on Line 2, we used (3.12) and (3.14) (note that by (3.16), one has \( c_n \sqrt{k + c + 1} > R_0 \) when \( k > k_1 \)); for the third inequality on Line 3, we used (3.15) and the inequality:
\[ e^{\frac{|x|^2}{2}} \leq e^{\frac{i}{2} (k+1)} , \text{ when } x \in B(0, c_n \sqrt{k + c + 1}). \]

We now estimate the upper bound of \( F(t_k) \). By (3.13) where \( d = k \), we have
\[ F(t_k) \leq 2^k \left( 1 - \left( 1 - \frac{t_k}{4} \right)^{1/n} \right)^{-k}. \tag{3.19} \]

To proceed, two facts are given in order. First, the following function is decreasing:
\[ t \mapsto \left( 1 - \left( 1 - \frac{t}{4} \right)^{1/n} \right)^{-k}, \text{ when } t \in (0, 1]; \]

Second, it follows by (3.14) and (3.18) that
\[ c_0 k^{-n} \leq t_k \leq 1, \]
where \( c = c(n) > 0 \) depends only on the dimension \( n \). Combining these facts together, we obtain from (3.19) that
\[ F(t_k) \leq 2^k \left( 1 - \left( 1 - \frac{c_0}{4} k^{-n} \right)^{1/n} \right)^{-k} \leq C k^{n/2}, \tag{3.20} \]
where \( C = C(\varepsilon_0, n) > 0 \) is independent of \( k \). Since \( \pi_k \varphi \in \mathcal{E}_k \) for each \( \varphi \in L^2(\mathbb{R}^n) \), it follows by (3.17) and (3.20) that
\[ \|\pi_k \varphi\|_{L^2(\mathbb{R}^n)} \leq e^{\frac{k}{2} \ln k + C k} \|\pi_k \varphi\|_{L^2(\mathcal{E})}, \text{ when } \varphi \in L^2(\mathbb{R}^n), \ k > k_1, \tag{3.21} \]
where \( C > 0 \) is independent of \( k \).

**Step 2. We consider the case that \( k \leq k_1 \).**

Since \( |E| > 0 \), it follows by (3.12) that if \( \varphi \in \mathcal{E}_k \) satisfies \( \|\varphi\|_{L^2(\mathcal{E})} = 0 \), then \( \varphi = 0 \) over \( \mathbb{R}^n \). This shows that \( \|\cdot\|_{L^2(\mathcal{E})} \) is a norm on \( \mathcal{E}_k \). On the other hand, we see from (3.10) that the subspace \( \mathcal{E}_k \) is of finite dimension. Hence, the norm \( \|\cdot\|_{L^2(\mathcal{E})} \) is equivalent to the norm \( \|\cdot\|_{L^2(\mathbb{R}^n)} \). In particular, there is \( C = C(k_1, E) > 0 \), independent of \( k \), so that
\[ \|\pi_k \varphi\|_{L^2(\mathbb{R}^n)} \leq C \|\pi_k \varphi\|_{L^2(\mathcal{E})}, \text{ when } \varphi \in L^2(\mathbb{R}^n), \ k \leq k_1. \tag{3.22} \]

Finally, the spectral inequality (3.9) follows from (3.21) and (3.22). Thus, we finish the proof of Lemma 3.2. \( \square \)
3.3. **Comparison of spectral inequalities.** This subsection concerns the difference between two spectral inequalities given by Lemma 3.1 and Lemma 3.2 respectively.

We first consider the case that $H = H_0 = -\Delta + |x|^2$, which is (1.3) with $c = 0$. Let $\pi_k$, with $k \in \mathbb{N}^+$, be the projection (associated with the above $H$) given by (3.6) where $c = 0$. Let $E \subset \mathbb{R}^n$. With regard to the spectral inequality:

$$\|\pi_k \varphi\|_{L^2(\mathbb{R}^n)} \leq C(k, E)\|\pi_k \varphi\|_{L^2(E)}, \text{ when } \varphi \in L^2(\mathbb{R}^n), \ k \geq 1, \quad (3.23)$$

the following interesting phenomena were revealed in the recent work [5, Theorem 2.1]:

- If $E$ is a non-empty open set, then one has (3.23) with $C(k, E) = e^{ck \ln k}$ for some $C = C(E, n) > 0$; (It deserves mentioning that according to our Lemma 3.2, (3.23) with $C(k, E) = e^{ck \ln k}$ is also true when $E$ is any set of positive measure.)
- If $E$ is a weakly thick set in the following sense:

$$\liminf_{R \to \infty} \frac{|E \cap B(0, R)|}{|B(0, R)|} > 0, \quad (3.24)$$

then one has (3.23) with $C(k, E) = e^{ck}$ for some constant $C = C(E, n) > 0$;
- If $E$ is a thick set, then one has (3.23) with $C(k, E) = e^{c \sqrt{k}}$ for some $C = C(E, n) > 0$.

**From these, we see that different geometry of $E$ may lead to different growth order of $C(k, E)$ in terms of $k$.**

We next consider the case that $H = (-\Delta)^{\frac{s}{2}}$ with $s > 0$, which is (1.2) with $c = 0$. Let $\pi_k$, with $k \in \mathbb{N}^+$, be the projection (associated with $H$) defined by (3.1) where $c = 0$. Let $E$ be a subset of $\mathbb{R}^n$. With regard to the spectral inequality (3.23) in this case, we have what follows:

- According to the Logvinenko-Sereda theorem (see [17, p.113]), the spectral inequality (3.23) holds for some $C(k, E)$ if and only if $E$ is a thick set;
- The above conclusion and Lemma 3.1 show further that the spectral inequality (3.23) holds for $C(k, E) = e^{C(1+ k^{\frac{1}{2}})}$ if and only if $E$ is a thick set.

**From these, we see that the geometry of $E$ can influence the spectral inequality (3.23) only in the manner: when $E$ is a thick set, (3.23) holds, while when $E$ is not a thick set, (3.23) is not true.**

We now explain what causes the above-mentioned difference: In the case when $H$ is given by (1.3) (with $c = 0$), $\mathcal{E}_k$ (the range of $\pi_k$ associated with $H$) is of finite dimension, and is spanned by finite many Hermite functions. Thus, for any subset $E$ of positive measure, the norms $\| \cdot \|_{L^2(\mathbb{R}^n)}$ and $\| \cdot \|_{L^2(E)}$ are equivalent. This leads to the spectral inequality for any subset of positive measure. On the other hand, in the case that $H = (-\Delta)^{\frac{s}{2}}$ with $s > 0$, which is (1.2) with $c = 0$, the corresponding subspace $\mathcal{E}_k$ is clearly not of finite dimension. Moreover, it is translation invariant in the sense that if $g(x) \in \mathcal{E}_k$, then $g(x - x_0) \in \mathcal{E}_k$ for all $x_0 \in \mathbb{R}^n$ (since the support of $\widehat{g}(\cdot - x_0)$ is the same as that of $\widehat{g}(\cdot)$). Then by testing the spectral inequality to the function $g(\cdot - x_0)$, where $x_0$ is arbitrarily taken from $\mathbb{R}^n$, one can prove that $E$ must satisfy the condition (1.9). These will be exploited in detail later. (See Step 3 in the proof of Theorem 1.1.)
4. Proof of main results

In this section, we will prove Theorem 1.1 and Theorem 1.2 with the help of Lemma 2.2, Lemma 3.1 and Lemma 3.2.

4.1. Proofs of Theorem 1.1. First of all, we recall that the operator $H$ given by (1.2) is self-adjoint and $(-H)$ generates an analytic semigroup. Now we divide the proof into two steps.

Step 1. We show that (i) $\Rightarrow$ (ii).

Suppose that $E \subset \mathbb{R}^n$ is a thick set. To show (ii), we let $H$ be given by (1.2), with arbitrarily fixed $c \geq 0$ and $s > 0$. Let $\pi_k$, with $k \in \mathbb{N}^+$, be the projection (associated with the aforementioned $H$) given by (3.1). According to Lemma 3.1, the above $\{\pi_k\}_{k \in \mathbb{N}^+}$ and $E$ satisfy the spectral inequality (2.2), with $a = 1/s$ and $c_1 = C$ (where $C$ is given in (3.7)). Meanwhile, by the Plancherel theorem, we see that for each $k \in \mathbb{N}^+$ and each $\varphi \in L^2(\mathbb{R}^n)$,

$$\left\| (1 - \pi_k)(e^{-tH}\varphi) \right\|_{L^2(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}\left( \chi(|\xi|^s - c > k) e^{-t(|\xi|^{2s}-c)}\hat{\varphi}(\xi) \right) \right\|_{L^2(\mathbb{R}^n)} \leq e^{-tk}\left\| \varphi \right\|_{L^2(\mathbb{R}^n)}, \quad t \geq 0,$$

which leads to the dissipative inequality (2.3) with $M = b = c_2 = 1$.

Finally, applying Lemma 2.2 to the above $H$, $\{\pi_k\}_{k \in \mathbb{N}^+}$ and $E$ leads to (ii).

Step 2. We show that (ii) $\Rightarrow$ (i).

Let $E \subset \mathbb{R}^n$ be measurable. According to Theorem 2.1 in Section 2, it suffices to show what follows:

Statement A. If there is $C > 0$, $T > 0$ and $\alpha \in (0, 1)$ so that for each $\varphi \in L^2(\mathbb{R}^n)$,

$$\left\| e^{-TH}\varphi \right\|_{L^2(\mathbb{R}^n)} \leq C \left( \int_0^T \int_E \left| (e^{-sH}\varphi)(x) \right|^2 \, dx \, dt \right)^{1/2} + \alpha \left\| \varphi \right\|_{L^2(\mathbb{R}^n)},$$

where $H = (-\Delta)^{\frac{s}{2}} - c$ with some $s > 0$ and $c \geq 0$, then $E$ is a thick set.

Before proving Statement A, we mention that a similar statement (i.e., (4.2), with $\alpha = c = 0, s = 2$) is proved in [43, Theorem 1.1]. The new ingredient here is that we shall add a parameter to eliminate the impact of the term $\alpha \left\| \varphi \right\|_{L^2(\mathbb{R}^n)}$ in (4.2).

We now show Statement A. Suppose that there is $s > 0$ and $c \geq 0$ so that (4.2) holds for $H = (-\Delta)^{\frac{s}{2}} - c$. Given $x_0 \in \mathbb{R}^n$, we define a function

$$u(t, x; l) := e^{ct}(t + l)^{-\frac{a}{2}} g\left( \frac{x - x_0}{(t + l)^{\frac{a}{2}}} \right), \quad t \geq 0, x \in \mathbb{R}^n,$$

where $l > 0$ is a parameter (which will be determined later) and $g$ is the inverse Fourier transform of $e^{-|\cdot|^s}$, i.e.,

$$g(x) = (\mathcal{F}^{-1}e^{-|\cdot|^s})(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-|\xi|^s} \, d\xi, \quad x \in \mathbb{R}^n.$$
Three facts on the above $g$ are given in order. First, it is clear that $g$ is a smooth function; Second, [8, Theorem 2.1] gives the following point-wise estimate: for some $C_1 > 0$,
\[ |g(x)| \leq \frac{C_1}{(1 + |x|^2)^{\frac{n+2}{4}}}, \quad x \in \mathbb{R}^n; \] (4.5)
Third, it follows from (4.4) that each $u(t, x; l)$ (with $l > 0$) satisfies the following fractional heat equation:
\[ (\partial_t + (-\Delta)^{\frac{s}{2}})u(t, x; l) = cu(t, x; l), \quad t > 0, x \in \mathbb{R}^n; \quad u(0, x; l) = \varphi(x; l), \quad x \in \mathbb{R}^n, \]
with
\[ \varphi(x; l) := l^{-\frac{s}{2}}g \left( \frac{x - x_0}{l^2} \right), \quad x \in \mathbb{R}^n. \]
(It is clear that $\varphi(\cdot; l) \in L^2(\mathbb{R}^n).$)

By a direct computation, we have that for some absolute constant $C_2 > 0$,
\[ \|u(t, \cdot; l)\|_{L^2(\mathbb{R}^n)} = C_2(t + l)^{-\frac{n}{2}}e^{ct}, \quad \text{when} \quad t \geq 0, \quad l > 0. \] (4.6)
Using (4.2) (where $\varphi(\cdot) = \varphi(\cdot; l)$), (4.6) and the identity:
\[ (e^{-tH}) \varphi(x; l) = e^{ct}e^{-t(-\Delta)^{\frac{s}{2}}} \varphi(x; l), \quad l > 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \] (4.7)
we deduce that when $l > 0$
\[ C_2(T + l)^{-\frac{n}{2}} \leq C e^{CT} \left( \int_0^T \int_E \left| e^{-t(-\Delta)^{\frac{s}{2}}} \varphi(x; l) \right|^2 \mathrm{d}x \mathrm{d}t \right)^{1/2} + C_2\alpha l^{-\frac{n}{2}}. \] (4.8)

Let
\[ l_0 := \frac{T}{\left( \frac{2}{1+\alpha} \right)^{\frac{2n}{2n} - 1}}. \] (4.9)
Then a direct calculation leads to
\[ (T + l_0)^{-\frac{n}{2}} - \alpha l_0^{-\frac{2n}{2n}} - \frac{1 - \alpha}{2} l_0^{-\frac{n}{2}} > 0. \]
This, along with (4.8) (where $l = l_0$), yields
\[ C_3^2 \leq C_3^2 e^{2cT} \int_0^T \int_E \left| e^{-t(-\Delta)^{\frac{s}{2}}} \varphi(x; l_0) \right|^2 \mathrm{d}x \mathrm{d}t, \] (4.10)
where
\[ C_3 := \frac{C_2(1 - \alpha)l_0^{-\frac{2n}{2n}}}{2} > 0. \]
Related to (4.10), we have the following observations: First, for each $L > 0,$
\[
\int_0^T \int_E \left| e^{-t(-\Delta)^{\frac{s}{2}}} \varphi(x; l_0) \right|^2 \mathrm{d}x \mathrm{d}t \nonumber
\]
\[
= \int_0^T \int_{E \cap B^c(x_0, L)} \left| e^{-t(-\Delta)^{\frac{s}{2}}} \varphi(x; l_0) \right|^2 \mathrm{d}x \mathrm{d}t + \int_0^T \int_{E \cap B(x_0, L)} \left| e^{-t(-\Delta)^{\frac{s}{2}}} \varphi(x; l_0) \right|^2 \mathrm{d}x \mathrm{d}t. \] (4.11)
Second, when \( t \in [0, T] \) and \( L > 0 \),
\[
\int_{B^c(x_0, L)} \left| e^{-t(-\Delta)^{\frac{3}{4}}} \varphi(x; l_0) \right|^2 \, dx = \int_{B^c(x_0, L)} |e^{-ct} u(t, x; l_0)|^2 \, dx \\
\leq C_1^2 (t + l_0)^{-\frac{2n}{s}} \int_{B^c(x_0, L)} \left( 1 + \frac{|x - x_0|^2}{(t + l_0)^{2/s}} \right)^{-n-s} \, dx \\
\leq C_1^2 (t + l_0)^{-\frac{2n}{s}} \int_{|y| > L/(t + l_0)^{1/s}} (1 + |y|^2)^{-n-s} \, dy \\
\leq C_1^2 l_0^{-\frac{2n}{s}} \int_{|y| > L/(T + l_0)^{1/s}} (1 + |y|^2)^{-n-s} \, dy. \quad (4.12)
\]

(In (4.12), for the equality on Line 1, we used (4.7); for the inequality on Line 2, we used (4.3) and (4.5); for the inequality on Line 3, we used the change of variable \( x - x_0 = (t + l_0)^{1/s} y \); for the inequality on Line 4, we used the fact that \( 0 \leq t \leq T \).) Third, by (4.12) and the fact that \( (1 + |\cdot|^2)^{-n-s} \in L^1(\mathbb{R}^n) \), we can choose \( L_0 = L_0(s, T, n, \alpha) > 0 \) so that
\[
C_2 e^{2cT} \int_0^T \int_{E \cap B^c(x_0, L_0)} \left| e^{-t(-\Delta)^{\frac{3}{4}}} \varphi(x; l_0) \right|^2 \, dx \, dt \leq \frac{C_3^2}{2}. \quad (4.13)
\]

Now, combining (4.10), (4.11) and (4.13) together, we see
\[
\frac{C_2^2}{2} \leq C_2 e^{2cT} \int_0^T \int_{E \cap B(x_0, L_0)} \left| e^{-t(-\Delta)^{\frac{3}{4}}} \varphi(x; l_0) \right|^2 \, dx \, dt. \quad (4.14)
\]

Meanwhile, it follows from (4.3), (4.5) and (4.7) that when \( x \in B(x_0, L_0) \) and \( t \in (0, T) \),
\[
\left| e^{-t(-\Delta)^{\frac{3}{4}}} \varphi(x; l_0) \right| = e^{-ct} |u(t, x; l_0)| \leq C_1 l_0^{-\frac{n}{4}}. \quad (4.15)
\]

From (4.14) and (4.15), we find
\[
\frac{C_2^2}{2} \leq |E \cap B(x_0, L_0)| TC_2 C_4^2 e^{2cT} l_0^{-\frac{2n}{s}}.
\]

This shows that for some \( C_4 > 0 \) independent of \( x_0 \in \mathbb{R}^n \),
\[
|E \cap B(x_0, L_0)| \geq C_4. \quad (4.16)
\]

Because
\[
B(x_0, L_0) \subset Q_{L_1}(x_0) \quad \text{with} \quad L_1 := 2L_0,
\]
we see from (4.16) that for some \( \gamma > 0 \),
\[
|E \cap Q_{L_1}(x_0)| \geq \gamma L_1^n.
\]

Since \( x_0 \) can be arbitrarily taken from \( \mathbb{R}^n \), the above shows that \( E \) is a thick set in \( \mathbb{R}^n \).

Hence, we finish the proof of Theorem 1.1. \( \square \)
4.2. Proof of Theorem 1.2. First of all, we recall that the operator $H$ given by (1.3) is self-adjoint and $(-H)$ generates an analytic semigroup. We organize the proof by the following two steps:

Step 1. We show that $(i) \Rightarrow (ii)$.

Suppose that $E \subset \mathbb{R}^n$ is a subset of positive measure. Let $H$ be given by (1.3), with an arbitrarily fixed $c \geq n$. Let $\pi_k$, with $k \in \mathbb{N}^+$, be the projection (associated with the aforementioned $H$) given by (3.6). According to Lemma 3.2, $\{\pi_k\}_{k \in \mathbb{N}^+}$ and $E$ satisfy the spectral inequality (2.2) for some $a > 1$ and $c_1 > 0$. Meanwhile, by the spectral theorem (see, e.g., [7, p. 412]) and (3.6), it follows that when $\varphi \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}^+$,

$$
\|(1 - \pi_k)(e^{-tH}\varphi)\|_{L^2(\mathbb{R}^n)} = \left\| \sum_{j > (k+c-n)/2} e^{-(2j+n-c)t} P_j \varphi \right\|_{L^2(\mathbb{R}^n)} 
\leq e^{-kt}\|\varphi\|_{L^2(\mathbb{R}^n)}, \quad t > 0.
$$

From (4.17), we see that $H$ and $\{\pi_k\}_{k \in \mathbb{N}^+}$ satisfy the dissipative inequality (2.3) with $M = b = c_2 = 1$.

Now, by applying Lemma 2.2 to the above $H$, $\{\pi_k\}_{k \in \mathbb{N}^+}$ and $E$, we get $(ii)$.

Step 2. We show that $(ii) \Rightarrow (i)$.

Suppose that $E$ is a stabilizable set for (1.1) with (1.3) where $c \geq n$ is arbitrarily fixed. Then by Theorem 2.1 in Section 2, there is $\alpha \in (0, 1)$, $T > 0$ and $C > 0$ so that

$$
\|e^{-TH}\varphi\|_{L^2(\mathbb{R}^n)} \leq C\left( \int_0^T \int_E |e^{-tH}\varphi|^2 \, dx \, dt \right)^{1/2} + \alpha\|\varphi\|_{L^2(\mathbb{R}^n)} \quad \text{for all } \varphi \in L^2(\mathbb{R}^n),
$$

where $H$ is given by (1.3). Meanwhile, it follows from (3.5) that $\Phi_0$, defined by

$$
\Phi_0(x) = \pi^n/4 e^{-|x|^2/2}, \quad x \in \mathbb{R}^n,
$$

is the $L^2$-normalized eigenfunction of the Hermite operator $H_0 = -\Delta + |x|^2$ corresponding to the eigenvalue $\lambda = n$. Thus, we have

$$
(e^{-tH}\Phi_0)(x) = e^{-(n-c)t}\Phi_0(x), \quad \text{when } t \geq 0, \ x \in \mathbb{R}^n.
$$

This, along with (4.18) where $\varphi = \Phi_0$, yields

$$
0 < 1 - \alpha \leq e^{(c-n)T} - \alpha \leq C\pi^n/4T^{1/2}e^{(c-n)T}\left( \int_E e^{-|x|^2} \, dx \right)^{1/2}, \quad \text{for all } \varphi \in L^2(\mathbb{R}^n),
$$

which leads to $|E| > 0$.

Hence, we complete the proof of Theorem 1.2. \hfill \square

Remark 4.1. With the aid of the proof of Theorem 1.1, we will see that the condition $0 < a < b$ in (2.4) cannot be dropped in general. Here are two counterexamples:

- Let $H$ be given by (1.2) where $c = 0$ and $0 < s < 1$. Let $\pi_k$, with $k \in \mathbb{N}^+$, be the projection (associated with $H$) given by (3.1) (where $c = 0$). Let

$$
E := \{ x \in \mathbb{R}^n : |x| \geq 1 \}.
$$

Clearly, it is a thick set in $\mathbb{R}^n$. Moreover, from Step 1 in the proof of Theorem 1.1, we see that in this case, (2.2) (with the above $E$ and $a = 1/2$) and (2.3) (where
Let $H$ be given by $(1.3)$ where $c = 0$. Let $\pi_k$, with $k \in \mathbb{N}^+$, be the projection (associated with $H$) given by $(3.6)$ (where $c = 0$). Let

$$E := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0 \}.$$

Then it follows from [5, Theorem 2.1] that $(2.2)$ (with the above $E$ and $a = 1$) and $(2.3)$ (with $b = 1$) are true. Thus we have $a = b = 1$ in this case. However it was proved in [32, Theorem 1.10] that any half space is not an observable set for $(1.1)$ with $(1.3)$.

Besides, we would like to mention what follows: In the critical case $a = b = 1$, the observability inequality can still hold if one has a further logarithmic assumption in the spectral inequality $(2.2)$. We refer to [12, Theorem 5] for this refined version of Lebeau-Robbiano strategy.

5. Proofs of Theorem 1.3 and Theorem 1.4

In this section, we will use different approaches to prove Theorem 1.3 and Theorem 1.4. These approaches provide explicit expressions for the feedback operator $K$ in $(1.8)$.

5.1. Proof of Theorem 1.3. By $(b_1)$ in Subsection 1.3, Theorem 1.3 is an immediate consequence of the next Lemma 5.1.

**Lemma 5.1.** Assume that $E$ is a thick set in $\mathbb{R}^n$. Suppose that $H = -\Delta + V(x)$ where $V$ satisfies Condition I. Then there is $\omega > 0$ so that

$$\| e^{-t(H+\chi_E)} \|_{L(L^2(\mathbb{R}^n))} \leq e^{-\omega t} \text{ for each } t \geq 0.$$  

**Proof.** Notice that $H + \chi_E$ is closed and its domain is dense in $L^2(\mathbb{R}^n)$. Thus, according to the Hille-Yosida theorem, it suffices to show what follows: There exists $\omega > 0$ so that

$$\| (H + \chi_E + \lambda)^{-1} \|_{L(L^2(\mathbb{R}^n))} \leq \frac{1}{\lambda + \omega} \text{ for each } \lambda \in (-\omega, \infty). \quad (5.1)$$

To this end, we first claim that there exists $\omega > 0$ so that

$$((H + \chi_E)\varphi, \varphi) \geq \omega \int_{\mathbb{R}^n} |\varphi|^2 \text{ for each } \varphi \in C_0^\infty(\mathbb{R}^n). \quad (5.2)$$

For this purpose, three observations are given in order. First, since $V$ holds Condition I, there is $\delta \in (0, 1)$ so that

$$((H + \chi_E)\varphi, \varphi) \geq (1 - \delta) \int_{\mathbb{R}^n} |
abla \varphi|^2 dx + \int_E |\varphi|^2 dx \text{ for each } \varphi \in C_0^\infty(\mathbb{R}^n). \quad (5.3)$$

Second, given $N \geq 1$, let $\pi_N$ be the projection given by $(3.1)$ where $s = 1, c = 0, k = N$. Then we have

$$\widehat{\pi_N \varphi}(\xi) = \chi_{\{ |\xi| \leq N \}} \widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n, \varphi \in L^2(\mathbb{R}^n). \quad (5.4)$$
Since $E$ is a thick set, by Lemma 3.1 (where $H = (-\Delta)^{1/2}$ and $\pi_k$ is given by (3.1) with $s = 1, c = 0, k = N$), we can find some $C = C(E, n) > 0$ so that for each $N \geq 1$,

$$\|\pi_N \varphi\|_{L^2(\mathbb{R}^n)} \leq e^{CN} \|\pi_N \varphi\|_{L^2(E)} \quad \text{for each } \varphi \in L^2(\mathbb{R}^n).$$

(5.5)

Third, let $N \geq 1$ satisfy

$$\omega := \min \left\{ (1 - \delta)N^2 - 2, \frac{1}{2}e^{2CN} \right\} > 0.$$

Then we have that when $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\text{RHS (5.3)} \geq (1 - \delta) \int_{\mathbb{R}^n} |(1 - \pi_N)\nabla \varphi|^2 dx + \frac{1}{2} \int_{E} |\pi_N \varphi|^2 dx - 2 \int_{E} |(1 - \pi_N)\varphi|^2 dx$$

$$\geq \left( (1 - \delta)N^2 - 2 \right) \int_{\mathbb{R}^n} |(1 - \pi_N)\varphi|^2 dx + \frac{1}{2}e^{-2CN} \int_{\mathbb{R}^n} |\pi_N \varphi|^2 dx$$

$$\geq \omega \int_{\mathbb{R}^n} |\varphi|^2 dx.$$  \hfill (5.6)

(Here, RHS (5.3) denotes the right hand side of (5.3).) In (5.6), for the inequality on Line 1, we used the inequality:

$$\int_{\mathbb{R}^n} |\nabla \varphi|^2 dx = \int_{\mathbb{R}^n} |(1 - \pi_N)\nabla \varphi|^2 dx + \int_{\mathbb{R}^n} |\pi_N \nabla \varphi|^2 dx \geq \int_{\mathbb{R}^n} |(1 - \pi_N)\nabla \varphi|^2 dx$$

and the inequality$^4$:

$$\int_{E} |\varphi|^2 dx \geq \frac{1}{2} \int_{E} |\pi_N \varphi|^2 dx - 2 \int_{E} |(1 - \pi_N)\varphi|^2 dx;$$

for the inequality on Line 2, we used the fact

$$\int_{\mathbb{R}^n} |(1 - \pi_N)\nabla \varphi|^2 dx = \int_{\mathbb{R}^n} \chi_{(\|\xi\| \geq N)} |\xi|^2 |\hat{\varphi}|^2 d\xi \geq N^2 \int_{\mathbb{R}^n} |(1 - \pi_N)\varphi|^2 dx$$

and the spectral inequality (5.5). Now by (5.3) and (5.6), we get (5.2).

Finally, by (5.2), we have that when $\lambda > -\omega$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$0 \leq (\lambda + \omega) \int_{\mathbb{R}^n} |\varphi|^2 \leq ((H + \chi_E + \lambda)\varphi, \varphi) \leq \|(H + \chi_E + \lambda)\varphi\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^2(\mathbb{R}^n)}.$$ 

This shows that when $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$(\lambda + \omega)\|\varphi\|_{L^2(\mathbb{R}^n)} \leq \|(H + \chi_E + \lambda)\varphi\|_{L^2(\mathbb{R}^n)}.$$

Using a density argument to the above leads to (5.1). Thus we finish the proof of Lemma 5.1. \hfill $\square$

$^4$It follows from the fact that $\varphi = \pi_N \varphi + (1 - \pi_N)\varphi$ and the elementary inequality: $|a+b|^2 \geq \frac{1}{2}|a|^2 - 2|b|^2$, with $a = \pi_N \varphi$ and $b = (1 - \pi_N)\varphi$. 
5.2. **Proof of Theorem 1.4.** Suppose that $V$ satisfies Condition II. Then the operator $H = -\Delta + V$ is self-adjoint and has compact resolvent in $L^2(\mathbb{R}^n)$ (see [7]). Thus, 

$$\sigma(H) = \{\lambda_j\}_{j \in \mathbb{N}^+} \text{ with } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$ 

Let $\varphi_j$, with $j \in \mathbb{N}^+$, be the $L^2$-normalized eigenfunction of $H$ with respect to $\lambda_j$, i.e.,

$$H\varphi_j = (-\Delta + V)\varphi_j = \lambda_j \varphi_j \text{ and } \|\varphi_j\|_{L^2(\mathbb{R}^n)} = 1, \quad j = 1, 2, \cdots. \quad (5.7)$$

Then $\{\varphi_j\}_{j \in \mathbb{N}^+}$ forms a complete orthonormal basis in $L^2(\mathbb{R}^n)$.

If $\lambda_1 > 0$, then the conclusion in Theorem 1.4 is clearly true. Thus, without loss of generality, we can assume $\lambda_1 \leq 0$ from now on. Hence, there is $N \in \mathbb{N}^+$ so that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq 0 < \lambda_{N+1} \leq \lambda_{N+2} \leq \cdots. \quad (5.8)$$

Before continuing with the proof of Theorem 1.4, we give several lemmas concerning with the eigenfunctions of $H$.

**Lemma 5.2.** Assume that $V$ satisfies Condition II. Then each eigenfunction of $H = -\Delta + V$ is continuous on $\mathbb{R}^n$.

**Proof.** This is a direct consequence of [38, Theorem C.2.4, p.497].

**Lemma 5.3.** Suppose that $V \in L^\infty_{loc}(\mathbb{R}^n)$. Let $u \in H^2_{loc}(\mathbb{R}^n)$ be a solution to the equation:

$$\Delta u = V(x)u \quad \text{in} \quad \mathbb{R}^n.$$ 

If $u$ vanishes on a nonempty open set, then $u$ is identically zero.

**Proof.** See, e.g., [37].

To state the next lemma, we introduce the following notation: Given a nonempty open set $E \subset \mathbb{R}^n$, we define the matrix

$$A := A_E = \begin{pmatrix}
(\varphi_1, \varphi_1)_E & (\varphi_1, \varphi_2)_E & \cdots & (\varphi_1, \varphi_N)_E \\
(\varphi_2, \varphi_1)_E & (\varphi_2, \varphi_2)_E & \cdots & (\varphi_2, \varphi_N)_E \\
\vdots & \vdots & \ddots & \vdots \\
(\varphi_N, \varphi_1)_E & (\varphi_N, \varphi_2)_E & \cdots & (\varphi_N, \varphi_N)_E
\end{pmatrix}, \quad (5.9)$$

where

$$(\varphi_i, \varphi_j)_E := \int_E \varphi_i(x)\varphi_j(x)dx, \quad 1 \leq i, j \leq N. \quad (5.10)$$

(Recall $\varphi_j$, with $j \in \mathbb{N}^+$, is given by (5.7).)

**Lemma 5.4.** Suppose that $V$ satisfies Condition II. Let $A$ be given by (5.9), associated with a nonempty open subset $E \subset \mathbb{R}^n$. Then

$$\det A \neq 0. \quad (5.11)$$

**Remark 5.5.** (i) In a slightly different setting, Lemma 5.4 was given in [4, p.42], without proof. For the reader’s convenience, we provide a detailed proof here. (ii) We mention that a quantitative lower bound for the norm of the matrix $A$ (where $V = 0$) is obtained in [26, Lemma 3.1] with the aid of the Lebeau-Robbiano spectral inequality. However, for the current case, we don’t have the corresponding spectral inequality. (At least, we do not find it in any published paper.)
Proof of Lemma 5.4. By contradiction, we suppose that (5.11) is not true. Then there exists \( 0 \neq z = (z_1, z_2, \cdots, z_N)^T \in \mathbb{R}^N \) so that \( Az = 0 \), which gives

\[
z^T A z = 0. \tag{5.12}
\]

By (5.9) and (5.10), we can rewrite (5.12) as

\[
\int_E \left| \sum_{1 \leq j \leq N} z_j \varphi_j(x) \right|^2 \, dx = 0. \tag{5.13}
\]

Meanwhile, according to Lemma 5.2, the function \( \sum_{1 \leq j \leq N} z_j \varphi_j(x) \) is continuous on \( \mathbb{R}^n \). This, together with (5.13), leads to

\[
\sum_{1 \leq j \leq N} z_j \varphi_j(x) = 0, \quad \text{when } x \in E. \tag{5.14}
\]

With respect to \( \lambda_1, \ldots, \lambda_N \), there are only two possibilities: either they are the same or at least two of them are different.

In the first case that they are the same, we have \( \lambda_1 = \lambda_2 = \cdots = \lambda_N = \lambda \) for some \( \lambda \in \mathbb{R} \). Then \( (-\Delta + V) \varphi_j = \lambda \varphi_j \) over \( \mathbb{R}^n \) for all \( j = 1, \ldots, N \), consequently,

\[
(-\Delta + V) \left( \sum_{1 \leq j \leq N} z_j \varphi_j \right) = \lambda \left( \sum_{1 \leq j \leq N} z_j \varphi_j \right) \text{ over } \mathbb{R}^n. \tag{5.15}
\]

Since \( \sum_{1 \leq j \leq N} z_j \varphi_j \in H^2_{\text{loc}}(\mathbb{R}^n) \) and because \( \sum_{1 \leq j \leq N} z_j \varphi_j = 0 \) over \( E \) (see (5.14)), by (5.15), we can apply Lemma 5.3 to conclude that

\[
\sum_{1 \leq j \leq N} z_j \varphi_j = 0 \text{ over } \mathbb{R}^n. \tag{5.16}
\]

Meanwhile, since \( \varphi_1, \ldots, \varphi_N \) are linearly independent, it follows from (5.16) that \( z_1 = z_2 = \cdots = z_N = 0 \), which contradicts \( z \neq 0 \).

In the second case that at least two among \( \lambda_1, \ldots, \lambda_N \) are different, we can let \( \mu_1, \ldots, \mu_\ell \) (with \( \ell \in [2, N] \) and \( \mu_1 < \cdots < \mu_\ell \)) be all distinct values among \( \{\lambda_1, \ldots, \lambda_N\} \). Then by (5.8), we can decompose \( \{\lambda_1, \ldots, \lambda_N\} \) as \( \ell \) groups:

\[
\underbrace{\lambda_{j_0 + 1} = \cdots = \lambda_{j_1}}_{= \mu_1} < \underbrace{\lambda_{j_1 + 1} = \cdots = \lambda_{j_2}}_{= \mu_2} < \cdots < \underbrace{\lambda_{j_{\ell - 1} + 1} = \cdots = \lambda_{j_\ell}}_{= \mu_\ell}
\]

where \( j_0 = 0 \) and \( j_\ell = N \). Thus, we can rewrite (5.14) as:

\[
\sum_{k=1}^\ell \phi_k = 0 \text{ over } E, \tag{5.17}
\]

with

\[
\phi_k = \sum_{j=j_{k-1}+1}^{j_k} z_j \varphi_j, \quad k = 1, 2, \ldots, \ell. \tag{5.18}
\]
Since \( \varphi_j, j = j_{k-1} + 1, \ldots, j_k \), are the eigenfunctions corresponding to the same eigenvalue \( \mu_k \), we deduce from (5.18) that for \( k = 1, 2, \ldots, \ell \),

\[
(-\Delta + V)\phi_k = \mu_k \phi_k \quad \text{over } E.
\] (5.19)

Since \( E \) is open, acting \((-\Delta + V)\) on both sides of (5.17), using (5.19) and the continuity of \( \phi_k \), we obtain that

\[
\mu_1 \phi_1 + \cdots + \mu_\ell \phi_\ell = 0 \quad \text{over } E.
\] (5.20)

Similarly, we can also obtain that when \( m = 0, 1, \ldots, \ell - 1 \),

\[
\mu_1^m \phi_1 + \cdots + \mu_\ell^m \phi_\ell = 0 \quad \text{over } E.
\] (5.21)

Since \( \mu_1, \ldots, \mu_\ell \) are distinct, we have

\[
\det \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\mu_1 & \mu_2 & \cdots & \mu_\ell \\
\vdots & \vdots & \ddots & \vdots \\
\mu_1^{\ell-1} & \mu_2^{\ell-1} & \cdots & \mu_\ell^{\ell-1}
\end{pmatrix} \neq 0.
\]

(The left hand side is the Vandermonde determinant.) This, together with (5.21), gives

\[
\phi_1 = \cdots = \phi_\ell = 0 \quad \text{over } E.
\] (5.22)

Now, by (5.18) and (5.22), we can use the similar method as that used in the first case to see that \( z_1 = z_2 = \cdots = z_N = 0 \), which contradicts \( z \neq 0 \).

Hence, we end the proof of Lemma 5.4. \( \square \)

We now return to the proof of Theorem 1.4. Let \( E \subset \mathbb{R}^n \) be a non-empty open subset. According to Lemma 5.4, the matrix \( \mathbb{A} \) defined by (5.9) is invertible. Thus, we can define an operator \( K : \psi \in L^2(\mathbb{R}^n) \to K\psi \in L^2(\mathbb{R}^n) \) by

\[
(K\psi)(x) := \rho \begin{pmatrix}
\left(\psi, \varphi_1\right) \\
\left(\psi, \varphi_2\right) \\
\vdots \\
\left(\psi, \varphi_N\right)
\end{pmatrix}^{-1} \begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x) \\
\vdots \\
\varphi_N(x)
\end{pmatrix} \quad \text{for each } x \in \mathbb{R}^n,
\] (5.23)

where \( \rho = \lambda_1 - 1 \). Consider the closed-loop system:

\[
y_t - \Delta y + Vy = \chi_E Ky, \quad t > 0; \quad y(0, \cdot) \in L^2(\mathbb{R}^n).
\] (5.24)

We will treat any solution to (5.24) as a function from \([0, \infty)\) to \(L^2(\mathbb{R}^n)\), denoted by \( y(t), t \geq 0 \).

According to (b) in Subsection 1.3, to get Theorem 1.4, it suffices to show the following two assertions:

Assertion 1 \( K \) is a linear bounded operator on \( L^2(\mathbb{R}^n) \).

Assertion 2 There is \( C > 0 \) and \( \omega > 0 \) so that each solution \( y \) to (5.24) satisfies

\[
\|y(t)\|_{L^2(\mathbb{R}^n)} \leq Ce^{-\omega t}\|y(0)\|_{L^2(\mathbb{R}^n)} \quad \text{for each } t \geq 0.
\]
We first show Assertion 1. By (5.23), it is clear that $K$ is linear. Since $\varphi_j$, $j = 1, \ldots, N$, are orthonormal, it follows from (5.23) that

$$
\|K\psi\|_{L^2(\mathbb{R}^n)} = |\rho| \left\| \mathbb{A}^{-1} \begin{pmatrix} (\psi, \varphi_1) \\ (\psi, \varphi_2) \\ \vdots \\ (\psi, \varphi_N) \end{pmatrix} \right\|_{L^2(\mathbb{R}^n)}
$$

$$
\leq |\rho| \|\mathbb{A}^{-1}\| \left( \sum_{j=1}^{N} |(\psi, \varphi_j)|^2 \right)^{1/2}
$$

(5.25)

which shows that $K$ is bounded. This leads to Assertion 1.

We next show Assertion 2. Taking the inner product (in $L^2(\mathbb{R}^n)$) in (5.24) with $\varphi_j$, $1 \leq j \leq N$, we find

$$
\frac{d}{dt} \begin{pmatrix} (y(t), \varphi_1) \\ (y(t), \varphi_2) \\ \vdots \\ (y(t), \varphi_N) \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix} \begin{pmatrix} (y(t), \varphi_1) \\ (y(t), \varphi_2) \\ \vdots \\ (y(t), \varphi_N) \end{pmatrix} = \mathbb{B} \begin{pmatrix} (y(t), \varphi_1) \\ (y(t), \varphi_2) \\ \vdots \\ (y(t), \varphi_N) \end{pmatrix}, \quad t \geq 0,
$$

(5.26)

where

$$
\mathbb{B} = \rho \begin{pmatrix} (\varphi_1, \varphi_1)_E & (\varphi_1, \varphi_2)_E & \cdots & (\varphi_1, \varphi_N)_E \\ (\varphi_2, \varphi_1)_E & (\varphi_2, \varphi_2)_E & \cdots & (\varphi_2, \varphi_N)_E \\ \vdots & \vdots & \ddots & \vdots \\ (\varphi_N, \varphi_1)_E & (\varphi_N, \varphi_2)_E & \cdots & (\varphi_N, \varphi_N)_E \end{pmatrix} \mathbb{A}^{-1} = \begin{pmatrix} \rho & 0 & \cdots & 0 \\ 0 & \rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho \end{pmatrix}.
$$

Since $\rho = \lambda_1 - 1$, it follows from (5.26) that

$$
\frac{d}{dt} \begin{pmatrix} (y(t), \varphi_1) \\ (y(t), \varphi_2) \\ \vdots \\ (y(t), \varphi_N) \end{pmatrix} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 + \lambda_1 - \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 + \lambda_1 - \lambda_N \end{pmatrix} \begin{pmatrix} (y(t), \varphi_1) \\ (y(t), \varphi_2) \\ \vdots \\ (y(t), \varphi_N) \end{pmatrix}, \quad t \geq 0.
$$

(5.29)

Let $P_{\leq N}$ be the projection operator defined by

$$
P_{\leq N}\varphi := \sum_{1 \leq j \leq N} (\varphi, \varphi_j) \varphi_j, \quad \varphi \in L^2(\mathbb{R}^n).$$
Write

\[ P_{>N} := I - P_{\leq N}, \quad \text{with } I \text{ the identity operator on } L^2(\mathbb{R}^n). \]

Since \( \lambda_1 \leq \lambda_j \), when \( 2 \leq j \leq N \), we get from (5.28) that

\[ \| P_{\leq Ny(t)} \|_{L^2(\mathbb{R}^n)} \leq e^{-t} \| P_{\leq Ny(0)} \|_{L^2(\mathbb{R}^n)}, \quad \text{when } t \geq 0. \]

(5.30)

It remains to estimate \( \| P_{>Ny(t, \cdot)} \|_{L^2(\mathbb{R}^n)} \). To this end, we take the inner product (in \( L^2(\mathbb{R}^n) \)) in (5.24) with \( (y(t), \varphi_j) \varphi_j \) to see that when \( j \geq N + 1 \),

\[ \frac{1}{2} \frac{d}{dt} |(y(t), \varphi_j)|^2 + \lambda_j |(y(t), \varphi_j)|^2 = (y(t), \varphi_j)(Ky(t), \varphi_j), \quad t \geq 0. \]

(5.31)

Meanwhile, by the Cauchy-Schwartz inequality, we get that when \( j \geq N + 1 \),

\[ |(y(t), \varphi_j)(Ky(t), \varphi_j)| \leq \frac{\lambda_j}{2} |(y(t), \varphi_j)|^2 + \frac{1}{2\lambda_j} |(Ky(t), \varphi_j)|^2, \quad t \geq 0. \]

(5.32)

Taking the sum in (5.31) over \( j \geq N + 1 \), using (5.32) and the fact \( \lambda_j \geq \lambda_{N+1} \), we infer

\[
\begin{align*}
\frac{d}{dt} \| P_{>Ny(t)} \|_{L^2(\mathbb{R}^n)}^2 & \leq \frac{1}{\lambda_{N+1}} \| P_{>Ny(t)} \|_{L^2(\mathbb{R}^n)}^2 \\
& \leq \frac{1}{\lambda_{N+1}} \|Ky(t)\|_{L^2(\mathbb{R}^n)}^2 \\
& \leq \frac{1}{\lambda_{N+1}} \|P_{\leq Ny(t)}\|_{L^2(\mathbb{R}^n)}^2, \quad t \geq 0.
\end{align*}
\]

(5.33)

(In the last step of (5.33), we used (5.25).) Applying the Gronwall inequality to (5.33) and using (5.30), we deduce that when \( t \geq 0 \),

\[
\begin{align*}
\| P_{>Ny(t)} \|_{L^2(\mathbb{R}^n)}^2 & \leq e^{-\lambda_{N+1}t} \| P_{>Ny(0)} \|_{L^2(\mathbb{R}^n)}^2 \\
& + \frac{1}{\lambda_{N+1}} \|P_{\leq Ny(0)}\|_{L^2(\mathbb{R}^n)}^2 \int_0^t e^{-\lambda_{N+1}(t-s)} e^{-2s} ds \\
& \leq e^{-\lambda_{N+1}t} \| P_{>Ny(0)} \|_{L^2(\mathbb{R}^n)}^2 + C_1 e^{-C_2t} \| P_{\leq Ny(0)} \|_{L^2(\mathbb{R}^n)}^2
\end{align*}
\]

(5.34)

for some \( C_1, C_2 > 0 \). By (5.30) and (5.34), we obtain

\[ \|y(t)\|_{L^2(\mathbb{R}^n)}^2 \leq C_3 e^{-C_4t} \|y(0)\|_{L^2(\mathbb{R}^n)}^2, \quad t \geq 0 \]

for some \( C_3, C_4 > 0 \). This leads to Assertion 2.

Thus, we complete the proof of Theorem 1.4.

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References

[1] P. Alphonse, J. Bernier, Smoothing properties of fractional Ornstein-Uhlenbeck semigroups and null-controllability. arXiv preprint arXiv:1810.02629, (2018).

[2] J. Apraiz, L Escauriaza, G. Wang, C. Zhang, Observability inequalities and measurable sets. J. Eur. Math. Soc., 16 (2014) 2433–2475.

[3] V. Barbu, R. Triggiani, Internal stabilization of Navier-Stokes equations with finite dimensional controllers. Indiana Univ. Math. J., 53 (2004), 1443–1494.

[4] V. Barbu, Stabilization of Navier-Stokes Flows, Springer, New York, 2010.

[5] K. Beauchard, P. Jaming, K. Pravda-Starov, Spectral inequality for finite combinations of Hermite functions and null-controllability of hypoelliptic quadratic equations. arXiv preprint arXiv:1804.04895, (2018).

[6] K. Beauchard, K. Pravda-Starov, Null-controllability of hypoelliptic quadratic differential equations. J. Éc. polytech. Math., 5 (2018) 1–43.

[7] F. A. Berezin, M. A. Shubin, The Schrödinger equation, Mathematics and its Applications, Kluwer Academic Publishers 66 (1991).

[8] R. M. Blumenthal, R. K. Getoor, Some theorems on stable processes. Trans. Amer. Math. Soc., 95 (1960) 263–273.

[9] A. Bonami, B. Demange, A survey on uncertainty principles related to quadratic forms. Collect. Math., (2006) 1–36.

[10] J. M. Coron, Control and Nonlinearity. American Mathematical Society, Vol. 136. Mathematical Surveys and Monographs, Providence (2007).

[11] Y. Duan, L. Wang, C. Zhang, Observability inequalities for the heat equation with bounded potentials on the whole space. arXiv preprint arXiv:1910.04340 (2019)

[12] T. Duyckaerts, L. Miller, Resolvent conditions for the control of parabolic equations. J. Funct. Anal., 263 (2012) 3641–3673

[13] M. Egidi, I. Veselić, Sharp geometric condition for null-controllability of the heat equation on $\mathbb{R}^d$ and consistent estimates on the control cost. Arch. Math., 111 (2018) 85–99.

[14] L. Escauriaza, S. Montaner, C. Zhang, Observation from measurable sets for parabolic analytic evolutions and applications. J. Math. Pure et Appl. 104 (2015) 837–867.

[15] L. Escauriaza, S. Montaner, C. Zhang, Analyticity of solutions to parabolic evolutions and applications. SIAM J. Math. Anal. 49 (2017) 4064–4092.

[16] D. Gallaun, C. Seifert, M. Tautenhahn, Sufficient criteria and sharp geometric conditions for observability in Banach spaces. arXiv preprint arXiv:1905.10285v1, (2019).

[17] V. Havin, B. Jöricke, The Uncertainty Principle in Harmonic Analysis. Springer Science and Business Media, 2012.

[18] A. Koenig, Non-null-controllability of the fractional heat equation and of the Kolmogorov equation. arXiv preprint arXiv:1804.10581, (2018).

[19] O. Kovrijkine, Some results related to the Logvinenko-Sereda Theorem. Proc. Amer. Math. Soc., 129 (2001) no. 10, 3037–3047.

[20] I. Lasiecka, R. Triggiani, Control theory for partial differential equations: continuous and approximation theories, I: Abstract parabolic systems, Encyclopedia of Mathematics and its Applications 74, Cambridge University Press, 2000.
[21] G. Lebeau, I. Moyano, Spectral Inequalities for the Schrödinger operator. arXiv preprint arXiv:1901.03513v1, (2019).
[22] G. Lebeau, L. Robbiano, Contrôle exact de l’équation de la chaleur. Commun. Partial Diff. Eq., 20 (1995) 335–356.
[23] G. Lebeau, E., Zuazua, Null-controllability of a system of linear thermoelasticity, Arch. Rational Mech. Anal., 141 (1998) 297-329.
[24] P. Lissy, A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation. 2020. hal-02420212v2.
[25] Q. Lü, Bang-bang principle of time optimal controls and null controllability of fractional order parabolic equations. Acta Math. Sin. (Engl. Ser.) 26 (2010) 2377–2386.
[26] Q. Lü, G. Wang, On the existence of time optimal controls with constraints of the rectangular type for heat equations. SIAM J. Control Optim. 49 (2011) 1124–1149.
[27] S. Micu, E. Zuazua, On the lack of null-controllability of the heat equation on the half-line. Trans. Amer. Math. Soc., 353 (2001) 1635–1659.
[28] S. Micu, E. Zuazua, On the lack of null-controllability of the heat equation on the half space. Port. Math., 58 (2001) 1–24.
[29] S. Micu, E. Zuazua, On the controllability of a fractional order parabolic equation. SIAM J. Control Optim., 44 (2006) no. 6, 1950–1972.
[30] L. Miller, On the null-controllability of the heat equation in unbounded domains. B. Sci. math., 129 (2005) 175–185.
[31] L. Miller, On the controllability of anomalous diffusions generated by the fractional Laplacian. Math. Control Signals Syst., 18 (2006) no. 3, 260–271.
[32] L. Miller, Unique continuation estimates for sums of semiclassical eigenfunctions and null-controllability from cones. Preprint, (2009)
[33] L. Miller, A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. Discrete Contin. Dynam. Systems 14 (2010) no. 4, 1465–1485.
[34] K. D. Phung, G. Wang, An observability estimate for parabolic equations from a measurable set in time and its applications. J. Eur. Math. Soc., 15 (2013) 681–703.
[35] Y. Privat, E. Trélat, E. Zuazua, Optimal shape and location of sensors for parabolic equations with random initial data. Arch. Rational Mech. Anal. 216 (2015) 921-981.
[36] M. Reed, B. Simon, Methods of modern mathematical physics, vol. II: Fourier analysis and self-adjointness. Academic Press, New York, 1975.
[37] M. Schechter, B. Simon, Unique continuation for Schrödinger operators with unbounded potential. J. Math. Anal. Appl., 77 (1980) 482-492.
[38] B. Simon, Schrödinger semigroups. Bull. Amer. Math. Soc., 7 (1982) 447–526.
[39] G. Tenenbaum, M. Tucsnak, On the null-controllability of diffusion equations. ESAIM Control Optim. Calc. Var., 17 (2011) no. 4, 1088–1100.
[40] B. C. Titchmarsh, Eigenfunction expansions associated with second order differential operators, Part I. Second edition. Oxford: Oxford Univ. Press (1969).
[41] E. Trélat, G. Wang, Y. Xu, Characterization by observability inequalities of controllability and stabilization properties. Pure and Applied Analysis 2 (2020) 93–122.
[42] G. Wang, $L^\infty$-null controllability of the heat equation and its consequence for the time optimal control problem. SIAM J. Control Optim. 47 (2008) 1701–1720.
[43] G. Wang, M. Wang, C. Zhang, Y. Zhang, Observable set, observability, interpolation inequality and spectral inequality for the heat equation in $\mathbb{R}^n$. J. Math. Pures Appl., 126 (2019) 144–194.

[44] G. Wang, Y. Xu, Periodic Feedback Stabilization for Linear Periodic Evolution Equations. Springer Briefs in Mathematics, Springer, (2016).

[45] J. Zabczyk, Mathematical control theory: an introduction, Birkhäuser, Boston, MA, 1995.

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