FELDER’S ELLIPTIC QUANTUM GROUP AND ELLIPTIC HYPERGEOMETRIC SERIES ON THE ROOT SYSTEM $A_n$

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ABSTRACT. We introduce a generalization of elliptic 6j-symbols, which can be interpreted as matrix elements for intertwiners between corepresentations of Felder’s elliptic quantum group. For special parameter values, they can be expressed in terms of multivariable elliptic hypergeometric series related to the root system $A_n$. As a consequence, we obtain new biorthogonality relations for such series.

1. Introduction

In Baxter’s solution of the eight-vertex model, a decisive step is the introduction of the eight-vertex-solid-on-solid (8VSOS) model, the two models being related by a vertex-face transformation [B]. The $R$-matrix of the 8VSOS model satisfies a modified version of the quantum Yang–Baxter equation known as the quantum dynamical Yang–Baxter (QDYB) equation. The QDYB equation includes as a special case the hexagon relation for 6j-symbols, which was first found by Wigner around 1940 [Wi].

Two currently active research areas arising from the 8VSOS model are dynamical quantum groups and elliptic hypergeometric functions. For surveys, see [ES] and [GR, Chapter 11] or [S4], respectively. The aim of the present work is to find a new and unexpected connection between these two areas.

To explain the notion of a dynamical quantum group, we recall the FRST (Faddeev–Reshetikhin–Sklyanin–Takhtajan) construction [RTF], which associates a bialgebra (in many cases a Hopf algebra) to any quantum $R$-matrix. Physically interesting quantities can then be studied by algebraic methods. It is not obvious how to generalize the FRST construction to dynamical $R$-matrices. A first step was taken by Felder [F], who, rather than defining a dynamical quantum group as an algebra, defined a category of its representations. In the case of Baxter’s dynamical $R$-matrix, pertaining to the 8VSOS model, this category was studied in detail by Felder and Varchenko [FV1]. Later, Etingof and Varchenko [EV1] gave a solid algebraic foundation for dynamical quantum groups as $\h$-Hopf algebroids. By Felder’s quantum group, we mean the $\h$-Hopf algebroid obtained by applying the generalized FRST construction of Etingof and Varchenko to Baxter’s dynamical $R$-matrix.

In [D], Date et al. applied a process of fusion to the 8VSOS model, thereby obtaining more general dynamical $R$-matrices known as elliptic 6j-symbols. The
classical $6j$-symbols of quantum mechanics can be expressed as hypergeometric sums, that is, as $\sum_k a_k a_{k+1}/a_k$ a rational function of $k$. Elliptic $6j$-symbols are given by more general sums, with $a_{k+1}/a_k$ an elliptic function of $k$. Frenkel and Turaev [FT] realized that this was the first example of a completely new class of special functions, elliptic hypergeometric functions. Another relatively recent class of special functions are hypergeometric functions on root systems, which first appeared in the context of $6j$-symbols of unitary groups [AJJ, CCB]. Similarly as for elliptic hypergeometric series, examples appeared in the physics literature long before their nature as generalized hypergeometric series was emphasized, in this case by Holman, Biedenharn and Louck [HBL]. Recently, many authors have considered elliptic hypergeometric functions on root systems, see e.g. [DS1, DS2, KN, Ra, RS, R2, R3, RoS, S1, S2, S3, SW]. (To be precise, we think here mainly of integrals and sums of type I or Dixon-type, as opposed to type II or Selberg-type.) A major recent development is the appearance of elliptic hypergeometric integrals on root systems in the context of supersymmetric quantum field theories [DO, SV], which has led to many new conjectures. In spite of the origin of hypergeometric functions on root systems in the representation theory of unitary groups, little has been written about connections with Lie or quantum groups, and in the elliptic case nothing seems to be known before the present work.

In view of their common origin in Baxter’s 8VSOS model, one would expect direct relations between Felder’s quantum group and elliptic hypergeometric functions. Such relations were obtained in [KNR], and in a somewhat different way in [KnN]. Felder’s quantum group has a one-parameter family of $(N+1)$-dimensional irreducible corepresentations $V_N(z)$. In [KnN], Koelink and van Norden considered pairings of the form $\langle M^M_{st}(w), M^N_{uw}(z) \rangle$, where $M^M_{st}(w)$ denotes a matrix element of $V_M(w)$, and $\langle \cdot, \cdot \rangle$ is the cobraiding on the quantum group. These pairings are matrix elements of the natural intertwiner

$$V_N(z) \hat{\otimes} V_M(w) \rightarrow V_M(w) \hat{\otimes} V_N(z)$$

($\hat{\otimes}$ is a modified tensor product appropriate for dealing with Hopf algebroids). It was shown that such pairings can be identified with the elliptic $6j$-symbols of [D].

In the present paper, we consider rather than (1.1) the intertwiner

$$(V_1(z_1) \hat{\otimes} \cdots \hat{\otimes} V_1(z_N)) \hat{\otimes} (V_1(w_1) \hat{\otimes} \cdots \hat{\otimes} V_1(w_M))$$

$$\rightarrow (V_1(w_1) \hat{\otimes} \cdots \hat{\otimes} V_1(w_M)) \hat{\otimes} (V_1(z_1) \hat{\otimes} \cdots \hat{\otimes} V_1(z_N)).$$

(1.2)

In a natural basis of pure tensors, the matrix elements of this intertwiner are partition functions of the 8VSOS model with fixed boundary conditions. However, we will consider the same intertwiner in a different basis, when the matrix elements can be considered as generalized $6j$-symbols.

The comodule $V_N(z)$ is a quotient of

$$V_1(z) \otimes V_1(qz) \otimes \cdots \otimes V_1(q^{N-1}z)$$
(\(q\) is a parameter of the \(R\)-matrix). Accordingly, when \(w_j = q^{j-1}\omega\) and \(z_j = q^{j-1}\zeta\), our generalized \(6j\)-symbols reduce to the elliptic \(6j\)-symbols of \([D]\). One of our main results is that, if only \(w\) is specialized to a geometric progression, then the generalized \(6j\)-symbols can be expressed in terms of elliptic hypergeometric series related to the root system \(A_n\), of the type studied in \([KN, R2, R3, S1, S2]\). This connection allows us to obtain new biorthogonality relations for such series. Surprisingly, although degenerate cases first appeared in the study of \(SU(n)\), we obtain more general functions using an elliptic deformation of mere \(SU(2)\).

The original motivation for the present study was not the link to hypergeometric series on root systems, which in fact came as a surprise. Rather, it is part of an ongoing project to develop harmonic analysis on dynamical quantum groups, with Felder’s quantum group as the main example. In particular, we believe that some of our findings will be useful for constructing a Haar functional on Felder’s quantum group, and for obtaining a more concrete version of the construction of solutions to the \(q\)-Knizhnik–Zamolodchikov–Bernard equation due to Varchenko and co-workers \([FTV1, FTV2, FV2, MV]\). Finally, in view of the recent appearance of elliptic hypergeometric integrals in quantum field theory mentioned above, one may speculate that the biorthogonal system of Theorem \(7.2\) or related systems with continuous biorthogonality measures, has a role to play in that context.

The plan of the paper is as follows. In \(\S2\) we give preliminaries on dynamical quantum groups. This includes some new definitions and results, in particular on unitary cobraidings on \(h\)-Hopf algebroids. In \(\S3\) we recall the definition of Felder’s quantum group, and collect some elementary though useful results on its cobraiding. In particular, Corollary \(3.7\) is a key result that should have some independent interest. In \(\S4\) we introduce the corepresentations and bases that we will use. In \(\S5\) we discuss a function that appears as a building block of our generalized \(6j\)-symbols. In terms of the 8VSOS model, it is the domain wall partition function; it can also be identified with elliptic weight functions of Tarasov and Varchenko \([TV]\). In \(\S6\) we introduce generalized \(6j\)-symbols and study their main properties. In particular, in Theorems \(6.2\) and \(6.13\) we give explicit expressions for these symbols. Although these formulas may seem complicated, they are natural extensions of Racah’s expression for the classical \(6j\)-symbol as a \(4F_3\) hypergeometric sum. In \(\S7\) we consider the specialization of generalized \(6j\)-symbols that leads to elliptic hypergeometric series on the root system \(A_n\). As an application, in Theorem \(7.2\) we obtain an explicit biorthogonality relation for such series. Finally, an Appendix contains generalities on unitary symmetries of cobraided \(h\)-Hopf algebroids.

**Acknowledgements:** I thank Jonas Hartwig for many discussions, and Vitaly Tarasov for illuminating correspondence.

## 2. Dynamical quantum groups

This Section contains preliminaries on \(h\)-Hopf algebroids, most of which can be found in \([EV1, KoN, KR, R1]\).
2.1. \(\mathfrak{h}\)-Hopf algebroids. Let \(\mathfrak{h}^*\) be a finite-dimensional complex vector space. The notation is motivated by examples where \(\mathfrak{h}^*\) is the dual of the Cartan subalgebra of a semisimple Lie algebra. In the case of interest to us, \(\mathfrak{h}^* = \mathbb{C}\).

We denote by \(M_{\mathfrak{h}^*}\) the field of meromorphic functions on \(\mathfrak{h}^*\), and by \(T_\alpha, \alpha \in \mathfrak{h}^*\), the shift operators \((T_\alpha f)(\lambda) = f(\lambda + \alpha)\) acting on \(M_{\mathfrak{h}^*}\). Moreover, \(D_{\mathfrak{h}}\) denotes the algebra of difference operators \(\sum_i f_i T_{\beta_i}, f_i \in M_{\mathfrak{h}^*}, \beta_i \in \mathfrak{h}^*\), acting on \(M_{\mathfrak{h}^*}\).

An \(\mathfrak{h}\)-algebra is a complex associative algebra with 1, which is bigraded over \(\mathfrak{h}^*\), that is, \(A = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} A_{\alpha\beta}\), with \(A_{\alpha\beta} A_{\gamma\delta} \subseteq A_{\alpha + \gamma, \beta + \delta}\). Moreover, there are two algebra embeddings \(\mu_l, \mu_r : M_{\mathfrak{h}^*} \to A_{00}\) (the left and right moment maps), such that

\[
\mu_l(f)a = a\mu_l(T_\alpha f), \quad \mu_r(f)a = a\mu_r(T_\beta f), \quad a \in A_{\alpha\beta}, \quad f \in M_{\mathfrak{h}^*}.
\]

A morphism of \(\mathfrak{h}\)-algebras is an algebra homomorphism which preserves the bigrading and moment maps.

When \(A\) and \(B\) are \(\mathfrak{h}\)-algebras, \(\widetilde{A \otimes B}\) denotes the quotient of \(\bigoplus_{\alpha,\beta,\gamma \in \mathfrak{h}^*} A_{\alpha\beta} \otimes B_{\gamma\delta}\) by the relations \(\mu_r(f)a \otimes b = a \otimes \mu_l(f)b\). The multiplication \((a \otimes b)(c \otimes d) = ac \otimes bd\), the bigrading \(A_{\alpha\beta} \otimes B_{\gamma\delta} \subseteq (\widetilde{A \otimes B})_{\alpha\beta}\) and the moment maps

\[
\mu_l(f)(a \otimes b) = \mu_l(f)a \otimes b, \quad \mu_r(f)(a \otimes b) = a \otimes \mu_r(f)b
\]

make \(\widetilde{A \otimes B}\) an \(\mathfrak{h}\)-algebra.

The bigrading \(fT_{\beta} \in (D_{\mathfrak{h}})_{\beta\beta}\) and the moment maps \(\mu_l(f) = \mu_r(f) = fT_0\) equip \(D_{\mathfrak{h}}\) with the structure of an \(\mathfrak{h}\)-algebra. It provides a unit object for the tensor product \(\widetilde{\otimes}\), namely,

\[
x \simeq x \otimes T_{-\beta} \simeq T_{-\alpha} \otimes x, \quad x \in A_{\alpha\beta},
\]

and, under the identifications \((\ref{eq:1})\),

\[
(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}.
\]

An \(\mathfrak{h}\)-bialgebroid is an \(\mathfrak{h}\)-algebra \(\Delta : A \to A \otimes A\) (the coproduct) and \(\varepsilon : A \to D_\mathfrak{h}\) (the counit), such that

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta
\]

and, under the identifications \((\ref{eq:1})\),

\[
(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}.
\]

An \(\mathfrak{h}\)-Hopf algebroid is an \(\mathfrak{h}\)-bialgebroid equipped with a \(\mathbb{C}\)-linear map \(S : A \to A\) (the antipode), such that \(S(A_{\alpha\beta}) \subseteq A_{-\beta,-\alpha}, S(\mu_r(f)) = \mu_l(f), S(\mu_l(f)) = \mu_r(f), S(ab) = S(b)S(a), S(1) = 1\),

\[
\Delta \circ S = \sigma \circ (S \otimes S) \circ \Delta, \quad \varepsilon \circ S = S^{D_\mathfrak{h}} \circ \varepsilon,
\]

\[
m \circ (\text{id} \otimes S) \circ \Delta(a) = \mu_l(\varepsilon(a)1),
\]

\[
m \circ (S \otimes \text{id}) \circ \Delta(a) = \mu_r(T_\alpha(\varepsilon(a)1)), \quad a \in A_{\alpha\beta},
\]

where \(m\) denotes multiplication, \(\sigma(a \otimes b) = b \otimes a\), and where \(S^{D_{\mathfrak{h}}}\) is the antiautomorphism of \(D_{\mathfrak{h}}\) defined by

\[
S^{D_{\mathfrak{h}}}(f) = f, \quad S^{D_{\mathfrak{h}}}(T_\alpha) = T_{-\alpha}.
\]
These axioms are far from independent, see [KR].

We will use Sweedler’s notation

$$\sum_{(a)} a' \otimes \cdots \otimes a^{(n)}$$

for the iterated coproduct of $a$. Then, (2.2) may be written

$$x = \sum_{(x)} \mu_l(\varepsilon(x')1)x'' = \sum_{(x)} \mu_r(\varepsilon(x'')1)x'.$$

(2.6)

Moreover, assuming that each $a^{(i)}$ belongs to some bigraded component $A_{\alpha_i\beta_i}$, we write

$$\omega_{i,i+1}(a) = \beta_i = \alpha_{i+1}.$$ For instance, if $a \in A_{\alpha\beta}$, then

$$(\text{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \text{id})\Delta(a) = \sum_{(a)} a' \otimes a'' \otimes a^{(3)},$$

where $a' \in A_{\alpha,\omega_{12}(a)}$, $a'' \in A_{\omega_{12}(a),\omega_{23}(a)}$, $a^{(3)} \in A_{\omega_{23}(a),\beta}$.

2.2. Corepresentations. An $h$-space is an $h^*$-graded vector space over $M_{h^*}$, $V = \bigoplus_{\alpha \in h^*} V_{\alpha}$. A morphism of $h$-spaces is an $M_{h^*}$-linear and grade-preserving map.

If $A$ is an $h$-bialgebroid and $V$ an $h$-space, $A \hat{\otimes} V$ denotes the quotient of

$$\bigoplus_{\alpha,\beta \in h^*} A_{\alpha\beta} \otimes V_{\beta}$$

by the relations $\mu_r(f)a \otimes v = a \otimes f v$. The grading $A_{\alpha\beta} \hat{\otimes} V_{\beta} \subseteq (A \hat{\otimes} V)_\alpha$ and the $M_{h^*}$-linear structure $f(a \otimes v) = \mu_l(f)a \otimes v$ make $A \hat{\otimes} V$ an $h$-space. The identification

$$f T_{-\alpha} \otimes v \simeq f v, \quad v \in V_{\alpha},$$

(2.7)

gives an $h$-space isomorphism $D_h \hat{\otimes} V \simeq V$.

A corepresentation of $A$ on $V$ is an $h$-space morphism $\pi : V \rightarrow A \hat{\otimes} V$ such that

$$(\Delta \otimes \text{id}) \circ \pi = (\text{id} \otimes \pi) \circ \pi, \quad (\varepsilon \otimes \text{id}) \circ \pi = \text{id},$$

(2.8)

using (2.7) in the second equality. If $(e_x)_{x \in X}$ is a homogeneous basis for $V$ over $M_{h^*}$, $e_x \in V_{\omega(x)}$, then one can introduce matrix elements $t_{xy} \in A_{\omega(x)\omega(y)}$ by

$$\pi(e_x) = \sum_{y \in X} t_{xy} \otimes e_y.$$ (2.9)

In terms of matrix elements, (2.8) takes the form

$$\Delta(t_{xy}) = \sum_{z \in X} t_{xz} \otimes t_{zy}, \quad \varepsilon(t_{xy}) = \delta_{xy} T_{-\omega(x)}.$$ (2.10)

Given two $h$-spaces $V$ and $W$, their tensor product $V \hat{\otimes} W$ is defined as $V \otimes W$ modulo the relations

$$T_{-\alpha}(f)v \otimes w = v \otimes f w, \quad v \in V_{\alpha}, \quad f \in M_{h^*}.$$
It is an h-space with $V_\alpha \otimes W_\beta \subseteq (V \otimes W)_{\alpha+\beta}$ and $f(v \otimes w) = fv \otimes w$. If $V$ and $W$ are corepresentation spaces, then so is $V \otimes W$ under

$$\pi_{V \otimes W} = (m \otimes \text{id}) \circ \sigma_{23} \circ (\pi_V \otimes \pi_W),$$

where $\sigma_{23}(a \otimes v \otimes b \otimes w) = a \otimes b \otimes v \otimes w$. Equivalently, in terms of matrix elements,

$$\pi_{V \otimes W}(e_x^V \otimes e_y^W) = \sum_{ab} t_{xa}^V t_{yb}^W \otimes e_a^V \otimes e_b^W.$$  \hspace{1cm} (2.11)

2.3. **FRST construction.** Let $X$ be a finite index set and $\omega : X \to h^*$ an arbitrary function. Let $R = (R_{ac}^{bd})_{a,b,c,d \in X}$ be a matrix, whose elements $R_{ac}^{bd}(\lambda, z)$ are meromorphic functions of $(\lambda, z) \in h^* \times \mathbb{C}^\times$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We refer to $\lambda$ as the dynamical parameter and $z$ as the spectral parameter. Moreover, we require that

$$R_{ac}^{bd} \neq 0 \implies \omega(a) + \omega(c) = \omega(b) + \omega(d).$$  \hspace{1cm} (2.12)

To any such $R$ one may associate an $h$-bialgebroid $A$. As a complex algebra, it is generated by two copies of $M_{h^*}$, whose elements we write as $f(\lambda)$ and $f(\mu)$, respectively, together with generators $(L_{xy}(z))_{x,y \in X, z \in \mathbb{C}^\times}$. The defining relations are

$$f(\lambda)L_{xy} = L_{xy}f(\lambda + \omega(x)), \quad f(\mu)L_{xy} = L_{xy}f(\mu + \omega(y)), \quad f(\lambda)g(\mu) = g(\mu)f(\lambda);$$

$$\sum_{xy} \lim_{t \to z/w} (t - z/w)^N R_{ac}^{xy}(\lambda, t)L_{xb}(z)L_{yd}(w)$$

$$= \sum_{xy} \lim_{t \to z/w} (t - z/w)^N R_{ac}^{xy}(\mu, t)L_{cy}(w)L_{ax}(z),$$  \hspace{1cm} (2.13)

for $N \in \mathbb{Z}$ such that all the limits exist. If each $R_{ac}^{bd}(\lambda, z)$ is holomorphic in $z$, then (2.13) reduces to

$$\sum_{xy} R_{ac}^{xy}(\lambda, z/w)L_{xb}(z)L_{yd}(w) = \sum_{xy} R_{ac}^{xy}(\mu, z/w)L_{cy}(w)L_{ax}(z).$$

The bigrading $f(\lambda), f(\mu) \in A_{00}, L_{xy} \in A_{\omega(x)\omega(y)}$, the moment maps $\mu_{\lambda}(f) = f(\lambda), \mu_{\mu}(f) = f(\mu)$, the coproduct

$$\Delta(L_{ab}(z)) = \sum_{x \in X} L_{ax}(z) \otimes L_{xb}(z), \quad \Delta(f(\lambda)) = f(\lambda) \otimes 1, \quad \Delta(f(\mu)) = 1 \otimes f(\mu)$$

and the counit

$$\varepsilon(L_{ab}(z)) = \delta_{ab} T_{-\omega(a)}, \quad \varepsilon(f(\lambda)) = \varepsilon(f(\mu)) = f$$

make $A$ an $h$-bialgebroid.
2.4. QDYB equation. The FRST construction is of particular interest when \( R \) is a dynamical \( R \)-matrix, meaning that

\[
\sum_{xyz} R^{xy}_{de}(\lambda - \omega(f), z_1/z_2) R^{az}_{xf}(\lambda, z_1/z_3) R^{bc}_{yz}(\lambda - \omega(a), z_2/z_3)
\]

\[
= \sum_{xyz} R^{yz}_{ef}(\lambda, z_2/z_3) R^{xc}_{dz}(\lambda - \omega(y), z_1/z_3) R^{ab}_{xy}(\lambda, z_1/z_2). \tag{2.14}
\]

A dynamical \( R \)-matrix satisfying

\[
\sum_{xy} R^{xy}_{ab}(\lambda, z_1/z_2) R^{de}_{yx}(\lambda, z_2/z_1) = \delta_{ac} \delta_{bd} \tag{2.15}
\]

is called \textit{unitary}.

Let \( V \) be the complex vector space with basis \((v_x)_{x \in X}\). We identify \( R \) with the map \((\mathfrak{h}^* \times \mathbb{C}^\times) \to \text{End}_\mathfrak{h}(V \otimes V)\) given by

\[
R(\lambda, z)(v_x \otimes v_y) = \sum_{ab} R^{xy}_{ab}(\lambda, z)v_a \otimes v_b
\]

(the subscript in \( \text{End}_\mathfrak{h} \) refers to the condition \((2.12)\)). Then, \((2.14)\) and \((2.15)\) can be written in coordinate-free form as

\[
R^{12}(\lambda - h^{(3)}, z_1/z_2) R^{13}_{UW}(\lambda, z_1/z_3) R^{23}(\lambda - h^{(1)}, z_2/z_3)
\]

\[
= R^{23}(\lambda, z_2/z_3) R^{13}_{UV}(\lambda - h^{(2)}, z_1/z_3) R^{12}(\lambda, z_1/z_2),
\]

\[
R^{12}(\lambda, z_1/z_2) R^{21}_{UV}(\lambda, z_2/z_1) = \text{id},
\]

where the notation is explained by the example

\[
R^{12}(\lambda - h^{(3)}, z_1/z_2)(v_a \otimes v_b \otimes v_c) = \sum_{xy} R^{xy}_{ab}(\lambda - \omega(c), z_1/z_2)(v_x \otimes v_y \otimes v_c).
\]

More generally, one may consider the equations

\[
\mathcal{R}^{12}_{UW}(\lambda - h^{(3)}) \mathcal{R}^{13}_{UW}(\lambda) \mathcal{R}^{23}_{UW}(\lambda - h^{(1)}) = \mathcal{R}^{23}_{UV}(\lambda) \mathcal{R}^{13}_{UV}(\lambda - h^{(2)}) \mathcal{R}^{12}_{UV}(\lambda),
\]

\[
\mathcal{R}^{12}_{UV}(\lambda) \mathcal{R}^{21}_{UV}(\lambda) = \text{id},
\]

for operators \( \mathcal{R}_{UV} : \mathfrak{h}^* \to \text{End}_\mathfrak{h}(U \otimes V) \), where \( U, V \) and \( W \) are \( \mathfrak{h}^* \)-graded complex vector spaces. Equivalently, if

\[
\mathcal{R}_{UV}(\lambda)(u_x \otimes v_y) = \sum_{ab} \mathcal{R}^{xy}_{ab}(\lambda; U, V)(u_a \otimes v_b),
\]

one has

\[
\sum_{xyz} \mathcal{R}^{xy}_{de}(\lambda - \omega(f); U, V) \mathcal{R}^{az}_{xf}(\lambda; U, W) \mathcal{R}^{bc}_{yz}(\lambda - \omega(a); V, W)
\]

\[
= \sum_{xyz} \mathcal{R}^{yz}_{ef}(\lambda; V, W) \mathcal{R}^{xc}_{dz}(\lambda - \omega(y); U, W) \mathcal{R}^{ab}_{xy}(\lambda; U, V), \tag{2.16}
\]
\[ \sum_{xy} R_{ab}^{xy}(\lambda; U, V) R_{cd}^{dz}(\lambda; V, U) = \delta_{ac} \delta_{bd}. \quad (2.17) \]

The QDYB equation is equivalent to the star-triangle relation for certain generalized ice models, see [B] for the case \( \mathfrak{h}^* = \mathbb{C} \) and [JM, JMO] for examples involving higher rank Lie algebras. In the case when the spectral parameter is absent, it is the hexagon relation for 6j-symbols, in the case \( \mathfrak{h}^* = \mathbb{C} \) going back to Wigner [W].

2.5. Cobraidings. For the following definition, see [R1, Def. 3.16].

**Definition 2.1.** A **cobraiding** on an \( \mathfrak{h} \)-bialgebroid \( A \) is a \( \mathbb{C} \)-bilinear map \( \langle \cdot, \cdot \rangle : A \times A \to D_\mathfrak{h} \) such that, for any \( a, b, c \in A \) and \( f \in M_{\mathfrak{h}^*}, \)

\[ \langle A_{\alpha\beta}, A_{\gamma\delta} \rangle \subseteq (D_\mathfrak{h})_{\alpha + \gamma, \beta + \delta}, \]

\[ \langle \mu_r(f)a, b \rangle = \langle a, \mu_l(f)b \rangle = f \circ \langle a, b \rangle, \quad (2.18a) \]
\[ \langle a\mu_l(f), b \rangle = \langle a, b\mu_r(f) \rangle = \langle a, b \rangle \circ f, \quad (2.18b) \]

\[ \langle ab, c \rangle = \sum_{(c)} \langle a, c' \rangle T_{\omega_{12}(c)} \langle b, c'' \rangle, \quad (2.19a) \]
\[ \langle a, bc \rangle = \sum_{(a)} \langle a'', b \rangle T_{\omega_{12}(a)} \langle a', c \rangle, \quad (2.19b) \]

\[ \langle a, 1 \rangle = \langle 1, a \rangle = \varepsilon(a), \quad (2.20) \]

\[ \sum_{(a)(b)} \mu_l(\langle a', b' \rangle 1) a''b'' = \sum_{(a)(b)} \mu_r(\langle a'', b'' \rangle 1) b'a'. \quad (2.21) \]

In particular,

\[ \langle A_{\alpha\beta}, A_{\gamma\delta} \rangle \neq 0 \implies \alpha + \gamma = \beta + \delta. \quad (2.22) \]

The following definition is motivated by Proposition 2.5 below.

**Definition 2.2.** A cobraiding is called **unitary** if

\[ \varepsilon(ab) = \sum_{(a)(b)} \langle b', a' \rangle T_{\omega_{12}(a) + \omega_{12}(b)} \langle a'', b'' \rangle. \quad (2.23) \]

To verify that a cobraiding is unitary, the following facts are useful.

**Lemma 2.3.** The equation (2.23) always holds if \( a \) or \( b \) is in \( \mu_l(M_{\mathfrak{h}^*})\mu_r(M_{\mathfrak{h}^*}) \). Moreover, if (2.23) holds for the pairs \( (a_1, b) \) and \( (a_2, b) \), then it holds for \( (a_1a_2, b) \), and if (2.23) holds for the pairs \( (a, b_1) \) and \( (a, b_2) \), then it holds for \( (a, b_1b_2) \).
Proof. We will only prove the last assertion, the second one being similar and the first one straight-forward. Thus, we must prove that

$$\varepsilon(abc) = \sum_{(a)(b)(c)} \langle b', a' \rangle T_{\omega_{12}(a)+\omega_{12}(b)+\omega_{12}(c)} \langle a'', b'' \rangle.$$

By (2.19), the right-hand side equals

$$\sum_{(a)(b)(c)} \langle b', a' \rangle T_{\omega_{12}(a)} \langle c', a'' \rangle T_{\omega_{23}(a)+\omega_{12}(b)+\omega_{12}(c)} \langle a''(4), b'' \rangle T_{\omega_{34}(a)} \langle a'(3), c'' \rangle.$$

Since $\langle c', a'' \rangle \in M_3 T_{\omega_{12}(c)+\omega_{23}(a)}$ and $\langle a''(4), b'' \rangle \in M_3 T_{\omega_{34}(a)-\omega_{12}(b)}$, this may be written

$$= \sum_{(a)(b)} \langle b', a' \rangle T_{\omega_{12}(a)+\omega_{12}(b)} \langle a''(3), b'' \rangle T_{\omega_{23}(a)+\omega_{12}(c)} \varepsilon(c) \varepsilon(a') \varepsilon(c)$$

$$= \sum_{(a)(b)} \langle b', a' \rangle T_{\omega_{12}(a)+\omega_{12}(b)} \langle \mu_1(\varepsilon(a'')1)a''(3), b'' \rangle \varepsilon(c)$$

$$= \sum_{(a)(b)} \langle b', a' \rangle T_{\omega_{12}(a)+\omega_{12}(b)} \langle a''(1), b'' \rangle \varepsilon(c) = \varepsilon(abc),$$

where we used first (2.23) for the pair $(a, c)$, then (2.18a), then (2.6), and finally (2.23) for the pair $(a, b)$. □

Cobraided $\mathfrak{h}$-bialgebroids arise naturally from dynamical $R$-matrices, see [KoN, Remark 3.4] and, for the case without spectral parameter, [R1, Cor. 3.20].

**Proposition 2.4.** Let the dynamical $R$-matrix $R$ and the $\mathfrak{h}$-bialgebroid $A$ be related as in (2.3). Assume that the matrix elements $R_{ac}(\lambda, z)$ are meromorphic in $\lambda$ for each $z \in \mathbb{C}^\times$. Then, there exists a cobraiding on $A$ defined by

$$\langle L_{ab}(w), L_{cd}(z) \rangle = R_{ac}(\lambda, w/z) T_{c(a)-w(c)}.$$  (2.24)

**Proposition 2.5.** If $R$ is unitary in the sense of (2.15), then the cobraiding described in Proposition 2.4 is unitary.

**Proof.** By Lemma 2.3, it is enough to verify that (2.23) holds for $a \mapsto L_{ab}(z)$ and $b \mapsto L_{cd}(w)$. This amounts to the identity

$$\delta_{ab} \delta_{cd} T_{-\omega(a)-\omega(c)} = \sum_{x,y} \langle L_{cx}(w), L_{ay}(z) \rangle T_{\omega(a)+\omega(y)} \langle L_{yb}(z), L_{xd}(w) \rangle$$

$$= \sum_{x,y} R_{cx}^{xy}(\lambda, z/w) R_{by}^{yx}(\lambda, w/z) T_{c(a)-w(c)},$$

which is indeed equivalent to (2.15). □
Proposition 2.6. If $A$ is an $\mathfrak{h}$-Hopf algebroid with a unitary cobrading, then
\[
\langle S(x), y \rangle = T_{\alpha}(y, x)T_{\beta}, \quad x \in A_{\alpha\beta}.
\]

Proof. Expanding $x$ and $y$ using the first expression in (2.6) gives
\[
\langle S(x), y \rangle = \sum (x)(y) \langle S(x'')\mu_r(\varepsilon(x')1), \mu_l(\varepsilon(y')1)y'' \rangle.
\]
Using, respectively, (2.18), (2.23), (2.19a), (2.4) and (2.20), this may be written
\[
\sum (x)(y) \langle S(x''), \mu_r(\varepsilon(x'y)1) \rangle T_{\alpha} \sum (x)(y) \langle S(x''), y'' \rangle T_{\omega_{12}}(y) \langle S(x''), y'' \rangle,
\]
which is obtained by applying $\langle \cdot, y' \rangle$ to both sides of (2.21). □

The following consequence was taken as an axiom in [R1]. In the unitary case, we find that it holds automatically.

Corollary 2.7. In an $\mathfrak{h}$-Hopf algebroid equipped with a unitary cobrading,
\[
\langle a, b \rangle = S^{D_{\mathfrak{h}}} \left( \langle S(a), S(b) \rangle \right).
\]

2.6. Algebraic construction of $R$-matrices. By Proposition 2.4, a dynamical $R$-matrix gives a cobraided $\mathfrak{h}$-bialgebroid. Conversely, cobraided $\mathfrak{h}$-bialgebroids can be used to obtain solutions to the general QDYB equation (2.16). We will consider two closely related examples of such constructions. In both cases, our starting point is the identity
\[
\sum (a)(b)(c) \langle a'', c' \rangle T_{\omega_{12}(c)} \langle b'', d'' \rangle T_{\omega_{12}(a) + \omega_{12}(b)} \langle a', b' \rangle
\]
which is obtained by applying $\langle \cdot, c \rangle$ to both sides of (2.21).
For the first example, consider a cobrained \( \mathfrak{h} \)-bialgebroid obtained from a dynamical \( R \)-matrix as in Proposition 2.3. Introducing the notation

\[
\bar{L}_{ab}(z) = L_{a_1b_1}(z_1) \cdots L_{a_nb_n}(z_n), \quad a, b \in X^n, \quad z \in (\mathbb{C}^\times)^n,
\]

we write

\[
\langle \bar{L}_{ab}(w), \bar{L}_{cd}(z) \rangle = \delta_{\omega(a)+\omega(c),\omega(b)+\omega(d)} \mathcal{Z}^{bd}_{ac}(\lambda; w, z) T_{-\omega(a)-\omega(c)}, \tag{2.28}
\]

where \( \omega(a) = \sum_i \omega(a_i) \). Substituting \( a \mapsto \bar{L}_{da}(u) \), \( b \mapsto \bar{L}_{eb}(w) \) and \( c \mapsto \bar{L}_{fc}(z) \) in (2.26) yields (2.16) in the form

\[
\sum_{xyz} \mathcal{Z}^{xy}_{de}(\lambda - \omega(f); u, w) \mathcal{Z}^{az}_{xf}(\lambda; u, z) \mathcal{Z}^{bc}_{yz}(\lambda - \omega(a); w, z) = \sum_{xyz} \mathcal{Z}^{yz}_{xf}(\lambda; w, z) \mathcal{Z}^{zc}_{dz}(\lambda - \omega(y); u, z) \mathcal{Z}^{ab}_{xy}(\lambda; u, w).
\]

Here, \( u, w \) and \( z \) may be vectors of different dimension. Moreover, in the unitary case we may let \( a \mapsto \bar{L}_{bd}(z) \) and \( b \mapsto \bar{L}_{ac}(w) \) in (2.23), which gives (2.17) in the form

\[
\sum_{xy} \mathcal{Z}^{xy}_{ab}(\lambda; w, z) \mathcal{Z}^{dc}_{xy}(\lambda; z, w) = \delta_{ac} \delta_{bd}.
\]

For the second example, consider a general cobrained \( \mathfrak{h} \)-bialgebroid \( A \), not necessarily obtained via the FRST construction. For each corepresentation \( U \) of \( A \), fix homogeneous basis elements \( e^U_x \) and matrix elements \( t^U_{xy} \) as in (2.9). We may then write

\[
\langle t^U_{ab}, t^V_{cd} \rangle = \delta_{\omega(a)+\omega(c),\omega(b)+\omega(d)} \mathcal{R}^{bd}_{ac}(\lambda; U, V) T_{-\omega(a)-\omega(c)}.
\]

Replacing \( a \mapsto t^U_{da}, b \mapsto t^V_{eb}, c \mapsto t^V_{fc} \) in (2.26) gives the QDYB equation (2.16). Moreover, assuming that the cobrading is unitary, substituting \( a \mapsto t^U_{bd}, b \mapsto t^V_{ac} \) in (2.23) gives (2.17).

The quantity \( \mathcal{R}^{bd}_{ac}(\lambda; U, V) \) is a matrix element for the action of \( t^U_{cd} \) in the representation dual to \( U \), see [R1] §3.2]. It is also a matrix element for the natural intertwiner \( \Phi : V \hat{\otimes} U \to U \hat{\otimes} V \). To be precise [R1 Prop. 3.18], \( \Phi \) is given by

\[
\Phi(e^V_c \otimes e^U_a) = \sum_{bd} \mathcal{R}^{bd}_{ac}(\lambda; U, V) e^V_b \otimes e^V_d.
\]

2.7. **Lattice models.** The quantity \( \mathcal{Z}^{bd}_{ac} \) introduced in (2.28) is the partition function for the lattice model with Boltzmann weights \( R \) and fixed boundary conditions. Though this fact should be expected, it seems not to have been discussed in the literature in the present setting, cobraidings on \( \mathfrak{h} \)-bialgebroids having been defined only recently [R1]. Since we will occasionally refer to the lattice model interpretation, we provide a brief explanation here.

Consider the finite lattice obtained by intersecting \( m \) vertical and \( n \) horizontal lines. The intersection points are called vertices. Including edges and faces at the boundary, each vertex is surrounded by four edges and four faces.
As in \[2.3\] let \(X\) be a finite set, \(\mathfrak{h}^*\) a complex vector space and \(\omega : X \to \mathfrak{h}^*\) an arbitrary function. We label all edges with elements of \(X\) and all faces with elements of \(\mathfrak{h}^*\), so that around each vertex there is a label configuration as in the left part of Figure 1.

We require that the face labelling is obtained from the edge labelling as follows. The top left face has label 0, and as we move east or south, crossing an edge labelled \(x\), the face label increases by \(\omega(x)\). This is possible if and only if the edge labels around each vertex satisfy

\[
\omega(a) + \omega(c) = \omega(b) + \omega(d). \tag{2.30}
\]

If \(\omega\) is injective, the edge labels are conversely determined by the face labels. A labelling satisfying these rules will be called a state.

In our main case of interest, \(\mathfrak{h}^* = \mathbb{C}\), \(X = \{\pm\}\), \(\omega(\pm) = \pm 1\). One may then picture edges labelled + as arrows going up or right and edges labelled − as arrows going down or left. The condition (2.30) is the ice rule, saying that each vertex has two incoming and two outgoing edges. In the left part of Figure 2, we give an example with \(m = 3\) and \(n = 2\).

To obtain a statistical model we must specify a weight function on the states. To this end, we fix parameters \(w_1, \ldots, w_m, z_1, \ldots, z_n \in \mathbb{C}^\times\) and \(\lambda \in \mathfrak{h}^*\). Consider the vertex at the intersection of the \(i\):th horizontal line from the top with the \(j\):th vertical line from the left. If the ambient labelling is as in Figure 1 then this vertex is assigned weight \(R_{ae}^{bd}(\lambda - \alpha, w_j/z_i)\). The weight of a state is then the product of the weights of all vertices. As an example, the state in Figure 1 has
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weight

\[ R_{+}^{\pm}(\lambda, w_{1}/z_{1})R_{+}^{\pm}(\lambda - 1, w_{2}/z_{1})R_{+}^{\pm}(\lambda - 2, w_{3}/z_{1}) \]

\[ \times R_{+}^{\pm}(\lambda + 1, w_{1}/z_{2})R_{+}^{\pm}(\lambda, w_{2}/z_{2})R_{+}^{\pm}(\lambda - 1, w_{3}/z_{2}). \]

We have described the model as a vertex model. In the physics literature, an alternative description as a face model is more common. This corresponds to passing to the dual lattice, interchanging the roles of vertices and faces. In this process, horizontal edges become vertical and vice versa. The face labels end up at vertices, but we move each one to its neighbouring face in the south-east direction. Labels along the east and south boundary are lost, but these do not enter in the partition functions, and can in any case be recovered from the edge labelling. In this dual picture, the Boltzmann weights correspond to interaction round a face rather than a vertex, see the right part of Figures 1 and 2 (in Figure 2, arrows have been rotated 90° clockwise).

We will now consider the model with arbitrary fixed boundary conditions. Given \( a, b \in X^{m}, c, d \in X^{n} \), suppose the boundary edges are labelled as in Figure 3. We may then introduce the partition function

\[ Z_{bd}^{ac}(\lambda; w; z) = \sum_{\text{states with fixed boundary}} \text{weight(state)}. \tag{2.31} \]

We claim that this definition agrees with (2.28). To see why, we use (2.19a) on a decomposition

\[ \langle \vec{L}_{ab}(w), \vec{L}_{cd}(z) \rangle = \langle \vec{L}_{a'b'}^{w'}(w'), \vec{L}_{a''b''}(w''), \vec{L}_{cd}(z) \rangle \]

(where \( w = (w', w'') \) and so on), obtaining the relation

\[ Z_{ac}^{bd}(\lambda; w; z) = \sum_{x \in X^{m}} Z_{a'c}^{bx}(\lambda; w'; z) Z_{a''d}^{b''}(\lambda - \omega(b'); w''; z) \]

where \( Z \) is as in (2.28). Similarly, (2.19b) gives

\[ Z_{ac}^{bd}(\lambda; w; z) = \sum_{x \in X^{m}} Z_{x'c}^{bd}(\lambda; w; z') Z_{ac'}^{x'd'}(\lambda - \omega(c'); w; z''). \]

It is clear that the partition function defined in (2.31) satisfies the same relations, corresponding to a vertical or horizontal splitting of the lattice. Thus, the equivalence of the two definitions follows by induction on \( m \) and \( n \).

3. Felder's elliptic quantum group

3.1. Felder's quantum group as an \( \hbar \)-Hopf algebroid. We will study a particular \( \hbar \)-Hopf algebroid, with \( \hbar^{*} = \mathbb{C} \), constructed from the \( R \)-matrix of the 8VSOS model [B]. The corresponding quantum group was introduced by Felder [F] on the level of its representations, and further studied by Felder and Varchenko [FV1], before it was included in the \( \hbar \)-Hopf algebroid framework of Etingof and Varchenko [EV1].
Let $p$ and $q$ be fixed parameters, with $|p| < 1$ and $q \neq 0$. We fix a choice of $\log q$ and write $q^\lambda = e^{\lambda \log q}$ for $\lambda \in \mathbb{C}$. Compared to the conventions of many authors, we replace $q^2$ by $q$.

We will use the notation

$$\theta(x) = \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x),$$

$$\theta(x_1, \ldots, x_n) = \theta(x_1) \cdots \theta(x_n),$$

$$(x)_k = \theta(x)\theta(qx) \cdots \theta(q^{k-1}x),$$

$$(x_1, \ldots, x_n)_k = (x_1)_k \cdots (x_n)_k,$$

for theta functions and elliptic Pochhammer symbols. We will freely use elementary identities such as

$$(x)_k = (-1)^k q^{(k)} x^k (q^{1-k}/x)_k,$$

see [GR, §11.2].

Let $\mathfrak{h} = \mathbb{C}$, $X = \{\pm\}$, and $\omega(\pm) = \pm 1$. We will often identify $\pm 1 = \pm$. Then,

$$R(\lambda, z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$a(\lambda, z) = \frac{\theta(z, q^{\lambda+2})}{\theta(qz, q^{\lambda+1})}, \quad b(\lambda, z) = \frac{\theta(q, q^{-\lambda-1}z)}{\theta(qz, q^{-\lambda-1})},$$

$$c(\lambda, z) = \frac{\theta(q, q^{\lambda+1}z)}{\theta(qz, q^{\lambda+1})}, \quad d(\lambda, z) = \frac{\theta(z, q^{-\lambda})}{\theta(qz, q^{-\lambda})},$$

satisfies the QDYB equation (2.14) and the unitarity relation (2.15).

Applying the FRST construction of [2.3] one obtains an $\mathfrak{h}$-bialgebroid $A$. The generators will be denoted

$$\alpha(z) = L_{++}(z), \quad \beta(z) = L_{+-}(z), \quad \gamma(z) = L_{-+}(z), \quad \delta(z) = L_{--}(z).$$
In analogy with (2.27), we use notation such as
\[
\vec{\alpha}(z) = \alpha(z_1) \cdots \alpha(z_n), \quad z \in (\mathbb{C}^\times)^n.
\]
We refer to [KNR] for an explicit list of relations, noting only that
\[
L_{ab}(z)L_{ab}(w) = L_{ab}(w)L_{ab}(z), \quad a, b \in \{\pm\},
\]
and that
\[
\begin{align}
\gamma(w)\alpha(z) &= a(\lambda, z/w)\alpha(z)\gamma(w) + b(\lambda, z/w)\gamma(z)\alpha(w), \\
\alpha(w)\gamma(z) &= c(\lambda, z/w)\alpha(z)\gamma(w) + d(\lambda, z/w)\gamma(z)\alpha(w).
\end{align}
\]

3.2. Singular cobraiding. Since \(R(\lambda, z)\) is singular at \(z = q^{-1}\), Proposition 2.5 cannot be applied, so \(A\) does not strictly speaking have a unitary cobraiding. An apparent solution to this problem is to use instead of (2.24) the definition
\[
\langle L_{ab}(w), L_{cd}(z) \rangle_{\text{reg}} = \theta(qw/z)P_{ac}(\lambda, w/z)T_{-\omega(a)-\omega(c)}.
\]
This gives a \textit{bona fide} cobraiding \(\langle \cdot, \cdot \rangle_{\text{reg}}\) on \(A\), which satisfies a modified unitarity axiom, see [H]. However, trying to extend this cobraiding to the \(\mathfrak{h}\)-Hopf algebroid \(\mathcal{E}\), singularities reappear. We prefer to stick to the definition (2.24), which leads to a \textit{singular} cobraiding, defined on a subspace of \(A \times A\). Explicitly, the singular
The cobrading axioms then give a meaning to expressions of the form valid. For generic spectral parameters, the cobrading and unitarity axioms remain valid. The following result is then easily proved by induction.

\[
\begin{pmatrix}
\langle \alpha(w), \alpha(z) \rangle & \langle \alpha(w), \beta(z) \rangle & \langle \alpha(w), \gamma(z) \rangle & \langle \alpha(w), \delta(z) \rangle \\
\langle \beta(w), \alpha(z) \rangle & \langle \beta(w), \beta(z) \rangle & \langle \beta(w), \gamma(z) \rangle & \langle \beta(w), \delta(z) \rangle \\
\langle \gamma(w), \alpha(z) \rangle & \langle \gamma(w), \beta(z) \rangle & \langle \gamma(w), \gamma(z) \rangle & \langle \gamma(w), \delta(z) \rangle \\
\langle \delta(w), \alpha(z) \rangle & \langle \delta(w), \beta(z) \rangle & \langle \delta(w), \gamma(z) \rangle & \langle \delta(w), \delta(z) \rangle \\
\end{pmatrix}
= 
\begin{pmatrix}
T_{-2} & 0 & 0 & a(\lambda, w/z)T_0 \\
0 & b(\lambda, w/z)T_0 & 0 & 0 \\
0 & c(\lambda, w/z)T_0 & 0 & 0 \\
d(\lambda, w/z)T_0 & 0 & 0 & T_2 \\
\end{pmatrix}.
\]

The cobrading axioms then give a meaning to expressions of the form

\[
\langle f(\lambda, \mu)L_{ab1}(w_1) \cdots L_{ab_{bn}}(w_m), g(\lambda, \mu)L_{c1d1}(z_1) \cdots L_{cndn}(z_n) \rangle,
\]

where \( w_i/z_j \notin p^2q^{-1} \) for all \( i, j \). As long as we restrict to the subspace spanned by such expressions, the properties of unitary cobraidings discussed in [25] remain valid.

In [KnN], the singular cobrading was extended to \( \mathcal{E} \), by defining

\[
\langle L_{ab}(w), \det^{-1}(z) \rangle = \delta_{ab} q^{-\frac{1}{2}} \frac{\theta(w/z)}{\theta(w/qz)} T_{-a},
\]

\[
\langle \det^{-1}(w), L_{ab}(z) \rangle = \delta_{ab} q^{-\frac{1}{2}} \frac{\theta(w/z)}{\theta(w/qz)} T_{-a},
\]

\[
\langle \det^{-1}(w), \det^{-1}(z) \rangle = q \frac{\theta(w/qz)}{\theta(w/qz)}.
\]

It is easy to see that, for generic spectral parameters, the cobrading and unitarity axioms remain valid. The following result is then easily proved by induction.

**Lemma 3.1.** For generic \((x, w, y, z) \in (\mathbb{C}^\times)^k \times (\mathbb{C}^\times)^l \times (\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n\),

\[
\langle \det^{-1}(x) \overrightarrow{L_{ab}(w)}, \det^{-1}(y) \overrightarrow{L_{cd}(z)} \rangle = q^{km-\frac{1}{2}(kn+lm)} \prod_{1 \leq i \leq k, 1 \leq j \leq m} \frac{\theta(x_i/y_j)}{\theta(qx_i/z_j)} \prod_{1 \leq i \leq l, 1 \leq j \leq n} \frac{\theta(w_i/y_j)}{\theta(w_i/qy_j)} \langle \overrightarrow{L_{ab}(x)}, \overrightarrow{L_{cd}(z)} \rangle.
\]

The existence of the antipode is related to the following symmetry, which in physical terms is the crossing symmetry for the partition function. Here and below, we write

\[
|x| = x_1 + \cdots + x_n, \quad x \in \{\pm\}^n = \{\pm1\}^n.
\]
Lemma 3.2. For \((w, z) \in (\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n\) generic, \(a, b \in \{\pm\}^m\) and \(c, d \in \{\pm\}^n\),

\[
\langle \tilde{L}_{ab}(w), \tilde{L}_{cd}(z) \rangle = (-1)^{\frac{1}{2}|a|} q^{-\frac{1}{2} \text{min}} \prod_{1 \leq i \leq n, 1 \leq j \leq m} \frac{\theta(z_i/w_j)}{\theta(z_i/q w_j)} 
\times \prod_{j=1}^{n} \frac{F(\lambda - c_1 - \cdots - c_j)}{F(\lambda - b_1 - \cdots - b_j - |b|)} T_{-|c|}(\tilde{L}_{-a_{\text{op}}, -c_{\text{op}}}(q^{-1}z), \tilde{L}_{ab}(w))T_{-|d|},
\]

with \(F\) as in (3.3) and where we write \((x_1, \ldots, x_n)^{\text{op}} = (x_n, \ldots, x_1)\).

Proof. Let \(x = \tilde{L}_{cd}(z)\) and \(y = \tilde{L}_{ab}(w)\) in Proposition 2.6. It is straightforward to check that

\[
S(\tilde{L}_{ab}(z)) = (-1)^{\frac{1}{2}|a|} q^{-\frac{1}{2} \text{min}} \prod_{j=1}^{n} \frac{F(\mu + a_{j+1} + \cdots + a_n)}{F(\lambda + b_{j+1} + \cdots + b_n)} 
\times \tilde{L}_{ab}(z^{\text{op}}),
\]

Using also Lemma 3.1 one arrives at the desired result. \(\square\)

Finally, we mention the following useful algebra symmetries.

**Proposition 3.3.** There exists an algebra antiautomorphism \(\phi\) of \(E\) defined on the generators by \(\phi(f(\lambda)) = f(-\lambda - 2)\), \(\phi(f(\mu)) = f(-\mu - 2)\), \(\phi(\tilde{L}_{ab}(z)) = \tilde{L}_{ab}(z^{-1})\), \(\phi(\det^{-1}(z)) = \det^{-1}(q^{-1}z^{-1})\). It satisfies \(\phi \circ S = S^{-1} \circ \phi\),

\[
(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi, \quad \phi^{D_h} \circ \varepsilon \circ \phi = \varepsilon,
\]

where \(\phi^{D_h}\) is the algebra antiautomorphism of \(D_h\) defined by \(\phi^{D_h}(f(\lambda)) = f(-\lambda - 2)\), \(\phi^{D_h}(T_n) = T_{-n}\). Moreover, there exists an algebra automorphism \(\psi\) of \(E\) defined by \(\psi(f(\lambda)) = f(-\lambda - 2)\), \(\psi(f(\mu)) = f(-\mu - 2)\), \(\psi(\tilde{L}_{ab}(z)) = \tilde{L}_{-a,-b}(z)\), \(\psi(\det^{-1}(z)) = \det^{-1}(z)\). It satisfies \(S \circ \psi = \psi \circ S\),

\[
(\psi \otimes \psi) \circ \Delta = \Delta \circ \psi, \quad \psi^{D_h} \circ \varepsilon \circ \psi = \varepsilon,
\]

where \(\psi^{D_h}\) is the algebra automorphism of \(D_h\) defined by \(\psi^{D_h}(f(\lambda)) = f(-\lambda - 2)\), \(\psi^{D_h}(T_n) = T_{-n}\).

Proposition 3.3 can be proved in a straightforward manner. The most tedious part is to verify that, in order to prove (3.7) and (3.8), it is enough to check them for a set of generators. In the Appendix, this is done in a systematic way.

Note that \(\phi \circ \phi = \psi \circ \psi = \text{id}\) and \(\psi \circ \phi = \phi \circ \psi\). If we write \(x^* = \psi(\phi(x)) = \phi(\psi(x))\), then

\[
\langle x, y \rangle = S^{D_h}(\langle y^*, x^* \rangle),
\]

where \(S^{D_h} = \psi^{D_h} \circ \phi^{D_h}\) is as in (2.5). The \(\mathbb{C}\)-linear involution \(*\) is a slight modification of the \(\mathbb{C}\)-antilinear involution used in [KNR].
3.3. Elementary properties of the cobraiding. We conclude §3.1 with some useful identities involving the singular cobraiding on Felder’s quantum group \( \mathcal{E} \).

**Lemma 3.4.** For generic \((w, z) \in (\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n\),

\[
\langle \tilde{\alpha}(w), \tilde{\alpha}(z) \rangle = T_{-n-m}, \tag{3.10a}
\]

\[
\langle \tilde{\alpha}(w), \tilde{\delta}(z) \rangle = \frac{\left( q^{\lambda+2+n-m} \right)_m}{\left( q^{\lambda+2-m} \right)_m} \prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} T_{-m}, \tag{3.10b}
\]

**Proof.** To prove (3.10a), we use (2.19a) to write

\[
\langle \tilde{\alpha}(w), \tilde{\alpha}(z) \rangle = \sum_{b \in \{\pm\}^n} \langle \alpha(w_1), \tilde{L}_{+b}(z) \rangle T_{|b|} \langle \tilde{\alpha}(w_2, \ldots, w_m), \tilde{L}_{b^+}(z) \rangle.
\]

By (2.22), only the term with \( b_1 = \cdots = b_n = + \) is non-zero, so

\[
\langle \tilde{\alpha}(w), \tilde{\alpha}(z) \rangle = \langle \alpha(w_1), \tilde{\alpha}(z) \rangle T_n \langle \tilde{\alpha}(w_2, \ldots, w_m), \tilde{\alpha}(z) \rangle.
\]

By induction on \( m \), this reduces the proof of (3.10a) to the case \( m = 1 \). In that special case, we similarly use (2.19b) to write

\[
\langle \alpha(w), \alpha(z) \rangle = \langle \alpha(w), \alpha(z_1) \rangle T_1 \cdots T_1 \langle \alpha(w), \alpha(z_n) \rangle = T_{n-1}.
\]

The identity (3.10b) now follows using Lemma 3.2. \( \square \)

**Lemma 3.5.** For \( a \in \{\pm\}^n, x \in \mathcal{E} \) and generic \((w, z) \in (\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n\),

\[
\langle \tilde{\alpha}(w)x, \tilde{L}_{-a}(z) \rangle = \prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \left( q^{\lambda+2+n-m} \right)_m T_{-m} \langle x, \tilde{L}_{-a}(z) \rangle, \tag{3.11a}
\]

\[
\langle \tilde{L}_{+a}(z), x\tilde{\alpha}(w) \rangle = \langle \tilde{L}_{+a}(z), x \rangle T_{-m}, \tag{3.11b}
\]

\[
\langle \tilde{L}_{+a}(z), x\tilde{\delta}(w) \rangle = \prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{\theta(z_j/w_i)}{\theta(qz_j/w_i)} \langle \tilde{L}_{+a}(z), x \rangle \left( q^{\lambda+2+m} \right)_n T_m. \tag{3.11c}
\]

**Proof.** Similarly as in the proof of (3.10a),

\[
\langle \tilde{\alpha}(w)x, \tilde{L}_{-a}(z) \rangle = \sum_{b \in \{\pm\}^n} \langle \tilde{\alpha}(w), \tilde{L}_{-b}(z) \rangle T_{|b|} \langle x, \tilde{L}_{ba}(z) \rangle
\]

\[
= \langle \tilde{\alpha}(w), \tilde{\delta}(z) \rangle T_{-n} \langle x, \tilde{L}_{-a}(z) \rangle.
\]

Thus, (3.11a) follows from (3.10b). The other statements are proved similarly. \( \square \)

**Proposition 3.6.** For \( a, b, c, d \in \{\pm\}^n \) and generic \( z \in (\mathbb{C}^\times)^n \),

\[
\langle \tilde{L}_{ab}(z), \tilde{L}_{cd}(z) \rangle = \delta_{ad} \delta_{bc} T_{-|a|-|c|}.
\]
Proof. We proceed by induction over $n$. The case $n = 1$ follows from (3.4). Writing $\tilde{a} = (a_1, \ldots, a_{n-1})$ for $a \in \mathbb{C}^n$, (2.19) gives

$$\left\langle \tilde{L}_{ab}(z), \tilde{L}_{cd}(z) \right\rangle = \sum_{x \in \{\pm\}^n} \left\langle \tilde{L}_{\tilde{a}b}(\tilde{z}), \tilde{L}_{cx}(z) \right\rangle T_{|x|} \left\langle L_{a_n b_n}(z_n), \tilde{L}_{x d}(z) \right\rangle$$

$$= \sum_{x, y \in \{\pm\}^n} \left\langle \tilde{L}_{\tilde{g}b}(\tilde{z}), \tilde{L}_{\tilde{c}x}(\tilde{z}) \right\rangle T_{|y|} \left\langle \tilde{L}_{\tilde{a}g}(\tilde{z}), L_{c_n x_n}(z_n) \right\rangle$$

$$\times T_{|x|} \left\langle L_{y_n b_n}(z_n), \tilde{L}_{x d}(\tilde{z}) \right\rangle T_{|y|} \left\langle L_{a_n y_n}(z_n), L_{x_n d_n}(z_n) \right\rangle.$$

By the induction hypothesis, all terms vanish except those with $y = x$, and the expression simplifies to

$$\delta_{\tilde{b}c} \delta_{a_n d_n} T_{-|\tilde{c}|} \sum_{x \in \{\pm\}^n} \left\langle \tilde{L}_{\tilde{a}b}(\tilde{z}), L_{c_n x_n}(z_n) \right\rangle T_{|x|} \left\langle L_{x_n b_n}(z_n), \tilde{L}_{x d}(\tilde{z}) \right\rangle T_{-a_n}$$

$$= \delta_{\tilde{b}c} \delta_{a_n d_n} T_{-|\tilde{c}|} \sum_{(u', v') \in \{\pm\}^2} \langle T_{\omega_{12}(u) + \omega_{12}(v)} \rangle u', v'' T_{-a_n},$$

where $u = L_{c_n b_n}(z_n), v = \tilde{L}_{\tilde{a}d}(\tilde{z})$. Applying the unitarity axiom (2.23) completes the proof.

The following corollary will be extremely useful.

Corollary 3.7. For $a, c \in \{\pm\}^m$, $b, d \in \{\pm\}^n$, and $(w, z) \in (\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n$ generic,

$$\left\langle \tilde{L}_{+a}(w) \tilde{L}_{+b}(z), \tilde{L}_{+d}(z) \tilde{L}_{+c}(w) \right\rangle = \left\langle \tilde{L}_{ca}(w), \tilde{L}_{db}(z) \right\rangle T_{-m-n}. \quad (3.12)$$

Proof. Straight-forward expansion using (2.19) gives

$$\left\langle \tilde{L}_{+a}(w) \tilde{L}_{+b}(z), \tilde{L}_{+d}(z) \tilde{L}_{+c}(w) \right\rangle$$

$$= \sum_{e, f \in \{\pm\}^m, g, h \in \{\pm\}^n} \left\langle \tilde{L}_{fa}(w), \tilde{L}_{dg}(z) \right\rangle T_{|f|} \left\langle \tilde{L}_{+f}(w), \tilde{L}_{ce}(w) \right\rangle$$

$$\times T_{|e| + |g|} \left\langle \tilde{L}_{hb}(z), \tilde{L}_{g+}(z) \right\rangle T_{|h|} \left\langle \tilde{L}_{+h}(z), \tilde{L}_{e+}(w) \right\rangle.$$

By Proposition 3.6, the only non-vanishing term in the sum is

$$\left\langle \tilde{L}_{ca}(w), \tilde{L}_{db}(z) \right\rangle T_{|c|} \left\langle \tilde{L}_{+c}(w), \tilde{L}_{+c}(w) \right\rangle T_{m+n} \left\langle \tilde{L}_{+b}(z), \tilde{L}_{b+}(z) \right\rangle T_{n} \left\langle \tilde{L}_{++}(z), \tilde{L}_{++}(w) \right\rangle$$

$$= \left\langle \tilde{L}_{ca}(w), \tilde{L}_{db}(z) \right\rangle T_{-m-n},$$

where we also used (3.10a).

In terms of the 8VSOS model, the right-hand side of (3.12) gives the partition function on a rectangular lattice, with arbitrary fixed boundary conditions, while the left-hand side gives the partition function on a square lattice, with the east and south boundary fixed as domain walls (see §5).
4. Embedded corepresentations

Fixing a non-negative integer \( N \), we identify subsets \( S \subseteq \{1, \ldots, N\} \) with elements of \( \{\pm\}^N \) through

\[
S_i = \begin{cases} 
+ & i \in S, \\
- & i \notin S.
\end{cases}
\]

For \( S \subseteq \{1, \ldots, N\} \) and \( z \in (\mathbb{C}^\times)^N \), we introduce the elements

\[
e_S(z) = \tilde{\gamma}(z_S)\tilde{\alpha}(z_S), \quad E_S(z) = \tilde{L}_S(z).
\]

Here, we use notation such as \( \tilde{\alpha}(z_S) = \prod_{i \in S} \alpha(z_i) \), which is well-defined in view of (3.1).

**Lemma 4.1.** For generic \( z \in (\mathbb{C}^\times)^N \),

\[
\text{span}_{f \in M_{h^*}, S \subseteq [N]} \{ \mu_l(f)e_S(z) \} = \text{span}_{f \in M_{h^*}, S \subseteq [N]} \{ \mu_l(f)E_S(z) \}.
\]

**Proof.** Iterating the commutation relations (3.2) will expand \( e_S(z) \) as a sum of the \( E_T \), with coefficients in \( \mu_l(M_{h^*}) \). Though (3.2) is not applicable when \( qz/w \in p_{\tilde{z}}^\times \), that obstruction does not arise for generic \( z \). Conversely, iterating (3.2) in the form

\[
\alpha(z)\gamma(w) = \frac{1}{a(\lambda, z/w)} \gamma(w)\alpha(z) - \frac{b(\lambda, z/w)}{a(\lambda, z/w)} \gamma(z)\alpha(w)
\]

will expand \( E_S(z) \) as a sum of the \( e_T \), as long as \( z \) is generic.

\[\square\]

**Remark 4.2.** If \( S = \{s_1 < \cdots < s_m\} \) and \( T = \{t_1 < \cdots < t_m\} \), write \( S \leq T \) if \( s_i \leq t_i \) for all \( i \). It is then easy to check that, for generic \( z \), \( E_S \in \text{span}_{f \in M_{h^*}, T \supseteq S} \{ \mu_l(f)e_T(z) \} \), \( e_S \in \text{span}_{f \in M_{h^*}, T \supseteq S} \{ \mu_l(f)E_T(z) \} \).

We denote by \( V(z) \) the space (4.1), viewed as an \( h \)-space with scalar multiplication \( f v = \mu_l(f)v \) and grading corresponding to the left grading in \( E \), that is, \( e_S(z) \), \( E_S(z) \in V_{2|S|-N}(z) \). Since

\[
\Delta(E_S(z)) = \sum_{T \subseteq [N]} \tilde{L}_{ST}(z) \otimes E_T(z),
\]

\( \Delta|_{V(z)} \) is a corepresentation of \( E \).

Next, we introduce the dual elements

\[
f_S(z) = \tilde{\alpha}(z_S)\tilde{\beta}(z_S^c), \quad F_S(z) = \tilde{L}_{+S}(z).
\]

Similarly as in Lemma 4.1, one can show that, for generic \( z \),

\[
\text{span}_{f \in M_{h^*}, S \subseteq [N]} \{ \mu_r(f)f_S(z) \} = \text{span}_{f \in M_{h^*}, S \subseteq [N]} \{ \mu_r(f)F_S(z) \}.
\]

This space will be denoted \( W(z) \).
Proposition 4.3. For generic \( z \in (\mathbb{C}^*)^N \),
\[
\langle F_T(z), E_S(z) \rangle = \delta_{ST} T_{-m},
\]
\[
\langle f_T(z), e_S(z) \rangle = \delta_{ST} A_{S,z} T_{-m},
\]
where \( m = |S| \) and where
\[
A_{S,z}(\lambda) = \frac{(q^{\lambda+2N-2m})_m}{(q^{\lambda+2m})_m} \prod_{i \in S,j \in S^c} \frac{\theta(z_i/z_j)}{\theta(q(z_i/z_j))}.
\]

Proof. The first identity is a special case of Proposition 3.6.

As for (4.3), we first note that (2.22) implies that \( \langle f_T(z), e_S(z) \rangle \) vanishes unless \( |S| = |T| \). Using first (3.11b) and then (3.11a), we may pull out all factors involving \( \alpha \). This leads to an expression containing the factor \( \prod_{i \in T,j \in S^c} \theta(z_i/z_j) \), which vanishes unless \( T \subseteq S \). Thus, we may assume \( T = S \), in which case we obtain
\[
\langle f_S(z), e_S(z) \rangle = A_{S,z} T_{-m} \langle \tilde{\beta}(z_S), \tilde{\gamma}(z_S) \rangle T_{-m},
\]
where, by Proposition 3.6 \( \langle \tilde{\beta}(z_S), \tilde{\gamma}(z_S) \rangle = 1. \) \( \square \)

Corollary 4.4. For generic \( z \in (\mathbb{C}^*)^N \), \( (e_S(z))_{S \subseteq [N]} \) and \( (E_S(z))_{S \subseteq [N]} \) are bases for the space \( V(z) \) over \( M_h^* \). In particular, \( \dim_{M_h^*} V(z) = 2^N \). Moreover, any \( x \in V(z) \) can be written
\[
x = \sum_{S \subseteq [N]} \mu_i(\langle F_S, x \rangle 1) E_S(z) = \sum_{S \subseteq [N]} \mu_i(\tilde{A}_{S,z}^{-1} \langle f_S, x \rangle 1) e_S(z).
\]
Similarly, any \( x \in W(z) \) can be written
\[
x = \sum_{S \subseteq [N]} \mu_r(\langle x, E_S \rangle 1) F_S(z) = \sum_{S \subseteq [N]} \mu_r(\tilde{A}_{S,z}^{-1} \langle x, e_S \rangle 1) f_S(z).
\]

Proof. Any \( x \in V(z) \) can be written \( x = \sum_S C_S(\lambda) E_S(z) \), for some \( C_S \in M_h^* \). Proposition 4.3 then gives \( C_S = \langle F_S, x \rangle 1 \). In particular, the expansion is unique, so \( (E_S(z))_{S \subseteq [N]} \) form a basis. Similar arguments apply for the other cases. \( \square \)

Since \( E_S(z) \) form a basis of \( V(z) \), (2.11) exhibits \( \tilde{L}_{ST}(z) \) as a matrix element of that corepresentation. By (2.11), the fact that the matrix elements factor means that
\[
V(z) \simeq V(z_1) \otimes \cdots \otimes V(z_N)
\]
as corepresentations.

We conclude with two results that will be needed later.

Lemma 4.5. For \( (w,z) \in (\mathbb{C}^*)^n \times (\mathbb{C}^*)^N \) generic, \( u \in W(z) \), \( v \in V(z) \) and \( a, b, c, d \in \{\pm\}^n \),
\[
\langle \tilde{L}_{ab}(w) u, \tilde{L}_{cd}(w) v \rangle = \delta_{ad} \delta_{bc} T_{-|b|} \langle u, v \rangle T_{-|a|},
\]
\[
\langle u \tilde{L}_{ab}(w), v \tilde{L}_{cd}(w) \rangle = \delta_{ad} \delta_{bc} \langle u, v \rangle T_{-|a|-|b|}.
\]
Proof. We may choose \( u = f(\mu)\vec{L}_{+x}(z) \) and \( v = g(\lambda)\vec{L}_{y+}(z) \). The result then follows from Proposition 3.6.

Lemma 4.6. For generic \( z \in (\mathbb{C}^\times)^N \),

\[
\Delta(\vec{\alpha}(z)) = \sum_{S \subseteq [N]} \frac{1}{A_{S,z}(\rho)} \vec{\alpha}(z_S)\vec{\beta}(z_{S^c}) \otimes \vec{\gamma}(z_{S^c})\vec{\alpha}(z_S) \tag{4.6}
\]

\[
\Delta(\vec{\gamma}(z)) = \sum_{S \subseteq [N]} \frac{1}{A_{S,z}(\rho)} \vec{\gamma}(z_S)\vec{\delta}(z_{S^c}) \otimes \vec{\gamma}(z_{S^c})\vec{\alpha}(z_S), \tag{4.7}
\]

where \( A_{S,z} \) is as in (4.4) and we write \( f(\rho) = f(\mu) \otimes 1 = 1 \otimes f(\lambda) \).

Proof. By (4.2) and Corollary 4.4.

\[
\Delta(\vec{\alpha}(z)) = \sum_{S \subseteq [N]} F_S(z) \otimes E_S(z) = \sum_{S,T \subseteq [N]} F_S(z) \otimes \mu_l(A_{T,z}^{-1}(f_T, E_S)1)e_T(z)
= \sum_{S,T \subseteq [N]} \mu_r(A_{T,z}^{-1}(f_T, E_S)1)F_S(z) \otimes e_T(z) = \sum_{T \subseteq [N]} \mu_r(A_{T,z}^{-1})f_T(z) \otimes e_T(z),
\]

which is (4.6).

It is clear from the defining relations that \( \eta(f(\mu)) = f(\mu), \eta(\alpha(z)) = \gamma(z), \eta(\beta(z)) = \delta(z) \) extends to an algebra isomorphism (though not an \( h \)-algebra isomorphism) between subalgebras of \( E \). It is easy to check that \( (\eta \otimes \text{id}) \circ \Delta = \Delta \circ \eta \). Applying \( \eta \otimes \text{id} \) to (4.6) then gives (4.7). \( \square \)

5. Elliptic weight functions

In contrast to the pairings between \( \vec{\alpha} \) and \( \vec{\delta} \) considered in Lemma 3.4, the pairing \( \langle \vec{\beta}(w), \vec{\gamma}(z) \rangle \) is not given by an elementary product. By the discussion in §2.7, it can be identified with the partition function of the 8VSOS model with domain wall boundary conditions, see Figure 4. It is also a special case of the elliptic weight functions introduced in [TV], see further [FTV1, FTV2, FV2, MV].

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{domain_wall.png}
\caption{Domain wall boundary conditions.}
\end{figure}

The following identity is essentially obtained in [TV], although neither the co-braiding nor the relation to the domain wall partition function are discussed there. Below, we give a simple proof using properties of the co-braiding. For the same
identity in the context of the 8VSOS model, see [PRS] [R4]. In [R4], we also obtained an alternative expression, analogous to the Izergin–Korepin determinant for the six-vertex model.

**Proposition 5.1.** For generic \((w, z) \in (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\),

\[
\langle \tilde{\beta}(w), \tilde{\gamma}(z) \rangle = \frac{\theta(q)^n}{(q^{-\lambda}-n)_n} \Phi(w; z; q^{-\lambda}),
\]

where

\[
\Phi(w; z; a) = \sum_{\sigma \in S_n} \prod_{1 \leq i < j \leq n} \frac{\theta(q z_{\sigma(i)}/z_{\sigma(j)}) \theta(w_i/z_{\sigma(j)})}{\theta(z_{\sigma(i)}/z_{\sigma(j)}) \theta(q w_i/z_{\sigma(j)})} \prod_{j=1}^{n} \frac{\theta(aq^{-j}w_j/z_{\sigma(j)})}{\theta(q w_j/z_{\sigma(j)})}.
\]  

(5.1)

Note that \(\Phi(w; z; a)\) has poles only at \(qw_i/z_j \in q^Z\); the singularities at \(z_i/z_j \in q^Z\) cancel in the symmetrization.

**Proof.** We write \(\langle \tilde{\beta}(w), \tilde{\gamma}(z) \rangle = \langle \tilde{\beta}(w_S), \tilde{\gamma}(z_S) \rangle\) for \(S \subseteq [n]\). Applying first (2.19) and (4.7), then (3.11b) and (3.11c), we obtain

\[
\langle \tilde{\beta}(w), \tilde{\gamma}(z) \rangle = \sum_{T \subseteq [n], |T| = m} \langle \tilde{\beta}(w_S), \tilde{\gamma}(z_T) \rangle T_{2m-n} A_{T, z}^{-1} \langle \tilde{\beta}(w_{S^c}), \tilde{\gamma}(z_{T^c}) \rangle T_{S, \gamma(z_{T^c})} T_{z, \gamma(z_T)}
\]

where \(m = |S|\).

This is amenable to iteration. For \([n] = S_1 \sqcup \cdots \sqcup S_N\) (disjoint union),

\[
\langle \tilde{\beta}(w), \tilde{\gamma}(z) \rangle = \sum_{|n| = T_1 \sqcup \cdots \sqcup T_N, |T_i| = |S_i|, 1 \leq i \leq N} \prod_{1 \leq k < l \leq N} \left( \prod_{i \in S_k, j \in T_l} \frac{\theta(w_i/z_j)}{\theta(q w_i/z_j)} \prod_{i \in T_k, j \in T_l} \frac{\theta(q z_i/z_j)}{\theta(z_i/z_j)} \right) \times \prod_{j=1}^{n} \langle \beta(w_{S_j}), \gamma(z_{T_j}) \rangle (\lambda + \sum_{k=1}^{j-1} |S_k|).
\]  

(5.2)

Consider the case \(N = n, S_j = \{j\}\). Then, \(T_j = \{\sigma(j)\}\) for some \(\sigma \in S_n\), so that \(\langle \tilde{\beta}(w_{S_j}), \tilde{\gamma}(z_{T_j}) \rangle = b(\lambda, w_j/z_{\sigma(j)})\). This yields the desired identity. \(\square\)

Since it may have some independent interest, we rewrite (5.2) in terms of \(\Phi\).
Corollary 5.2. For any decomposition \([n] = S_1 \sqcup \cdots \sqcup S_N\),

\[
\Phi(w; z; a) = \sum_{[n]=T_1 \sqcup \cdots \sqcup T_N} \prod_{1 \leq k < \ell \leq N} \left( \prod_{i \in S_k, j \in T_\ell} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \prod_{i \in T_k, j \in T_\ell} \frac{\theta(qz_i/z_j)}{\theta(z_i/z_j)} \right) \\
\times \prod_{j=1}^n \Phi(w_{S_j}; z_{T_j}; q^{-\sum_{i=1}^{j-1} |S_i|} a).
\]

Choosing \(x = \vec{\beta}(w)\) and \(y = \vec{\gamma}(z)\) in (3.8) gives

\[
\langle \vec{\beta}(w), \vec{\gamma}(z) \rangle(\lambda) = \langle \vec{\gamma}(w), \vec{\beta}(z) \rangle(-\lambda - 2).
\]

This proves the following fact.

Corollary 5.3. For generic \(w, z \in (\mathbb{C}^\times)^n\),

\[
\langle \vec{\gamma}(w), \vec{\beta}(z) \rangle = \frac{\theta(q)^n}{(q^{\lambda+2-n})^n} \Phi(w; z; q^{\lambda+2}).
\]

The function \(\Phi\) has some symmetries, which can be explained in terms of symmetries of the algebra \(\mathcal{E}\).

Corollary 5.4. The function \(\Phi\) satisfies

\[
\Phi(w; z; a) = \Phi(z^{-1}; w^{-1}; a)
\]

\[
= q^{-n} a^n \prod_{i,j=1}^n \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \Phi(w^{-1}; qz^{-1}; q^{n+2} a^{-1})
\]

\[
= q^{-n} a^n \prod_{i,j=1}^n \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \Phi(z; qw; q^{n+2} a^{-1}),
\]

where we use the notation \(z^{-1} = (z_1^{-1}, \ldots, z_n^{-1})\).

Proof. Choosing \(x = \vec{\beta}(w)\) and \(y = \vec{\gamma}(z)\) in (3.9) gives

\[
\langle \vec{\beta}(w), \vec{\gamma}(z) \rangle(\lambda) = \langle \vec{\beta}(z^{-1}), \vec{\gamma}(w^{-1}) \rangle(\lambda).
\]

This shows the equality between the first and second member. The equality of the first and third member is a special case of Lemma 3.2. Alternatively, it can be obtained from (5.1), replacing \(\sigma(i)\) by \(\sigma(n+1-i)\) and \(w_i\) by \(w_{n+1-i}\). The last equality follows by combining the other two.

If we specialize \(w\) or \(z\) to a geometric progression, \(\Phi\) factors.
Lemma 5.5. One has
\[ \Phi(w; z; a) \bigg|_{z_j = q^{j-1} \zeta} = (q)_n \frac{\theta(q)^n \prod_{j=1}^n \theta(q^{-n} aw_j / \zeta)}{\theta(qw_j / \zeta)}, \]
\[ \Phi(w; z; a) \bigg|_{w_j = q^{j-1} \omega} = (q)_n \frac{\theta(q)^n \prod_{j=1}^n \theta(q^{-1} aw / z_j)}{\theta(q^{n} \omega / z_j)}. \]

Proof. By symmetry, to prove the first identity we may put \( z_j = q^{n-j} \zeta \) in (5.1). Then, the sum reduces to the single term with \( \sigma = \text{id} \). The second identity follows using Corollary 5.4.

The function \( \Phi \) appears in the following commutation relations.

Lemma 5.6. Let \( S \subseteq [N] \) with \(|S| = m\). Then, for generic \( z \in (\mathbb{C}^\times)^N\),
\[ \bar{a}(z_S) \bar{\gamma}(z_{S^c}) = \sum_{T \subseteq [N], |T| = m} C_{S,T,z}(\lambda) \bar{\gamma}(z_{T^c}) \bar{a}(z_T), \tag{5.3} \]
where
\[ C_{S,T,z}(\lambda) = \frac{\theta(q)^n (q^{\lambda+2+n-m})_m}{(q^{\lambda+2+N-2m})_m} \prod_{i \in T, j \in T^c} \frac{\theta(qz_i / z_j)}{\theta(z_i / z_j)} \Phi(z_{T \setminus S}; z_{S^c} T; q^{\lambda+2+n-m}), \]
with \( n = |S \setminus T| = |T \setminus S| \). In the same notation,
\[ \bar{\beta}(z_{S^c}) \bar{a}(z_S) = \sum_{T \subseteq [N], |T| = m} D_{S,T,z}(\mu) \bar{a}(z_T) \bar{\beta}(z_{T^c}), \tag{5.4} \]
where
\[ D_{S,T,z}(\mu) = \frac{\theta(q)^n (q^{\mu+2-m})_m}{(q^{-\mu+m-N})_n (q^{\mu+2+N-2m})_m} \prod_{i \in T, j \in T^c} \frac{\theta(qz_i / z_j)}{\theta(z_i / z_j)} \times \Phi(z_{T \setminus S}; z_{S^c} T; q^{-\mu+m+n-N}). \]

Proof. Choosing \( x = \bar{a}(z_S) \bar{\gamma}(z_{S^c}) \) in Corollary 4.4 gives
\[ \bar{a}(z_S) \bar{\gamma}(z_{S^c}) = \sum_{T \subseteq [N]} \mu_T \left( \left( A_{T,z}^{-1}(\bar{a}(z_T) \bar{\beta}(z_{T^c}), \bar{a}(z_S) \bar{\gamma}(z_{S^c})) \right) \bar{\gamma}(z_{T^c}) \bar{a}(z_T) \right) \tag{5.5} \]
By Lemma 4.5, Corollary 3.7 and Corollary 5.3
\[ \langle \bar{a}(z_T) \bar{\beta}(z_{T^c}), \bar{a}(z_S) \bar{\gamma}(z_{S^c}) \rangle \]
\[ = \langle \bar{a}(z_{S \cap T}) \bar{a}(z_{T \setminus S}) \bar{\beta}(z_{S \cap T^c}), \bar{a}(z_{S \cap T}) \bar{\gamma}(z_{T \setminus S}) \bar{\gamma}(z_{S \cap T^c}) \rangle \]
\[ = T_{-[S \cap T]} \langle \bar{a}(z_{T \setminus S}), \bar{\beta}(z_{S \cap T}), \bar{a}(z_{S \cap T}) \bar{\gamma}(z_{T \setminus S}) \bar{\gamma}(z_{S \cap T^c}) \rangle T_{-[S \cap T]} \]
\[ = T_{-[S \cap T]} \langle \bar{\gamma}(z_{T \setminus S}), \bar{\beta}(z_{S \cap T}), T_{-[S \cap T]} \rangle \]
\[ = \delta_{|S|, |T|} (q^{\mu+2-m})_n \Phi(z_{T \setminus S}; z_{S^c} T; q^{\lambda+2+n-m}) T_{-2m}. \]
Plugging this into (5.5) yields (5.3). The identity (5.4) can be proved similarly, or be derived from (5.3) by applying $S \circ \psi$, with $\psi$ as in Proposition 3.3. □

6. Generalized elliptic 6j-symbols

6.1. Definition and main properties. In §4 we observed that $E_S(z)$ form a basis for the corepresentation $V(z)$, and that $\tilde{L}_{ST}(z)$ are the corresponding matrix elements. As we have seen in §2.6–2.7, the cobraidings $\langle \tilde{L}_{ST}(w), \tilde{L}_{UV}(z) \rangle$ give a dynamical $R$-matrix, which can be identified with the partition function for the 8VSOS model with fixed boundary conditions.

We are interested in the dynamical $R$-matrix corresponding to the alternative basis $e_S(z)$. For generic $z \in (\mathbb{C}^\times)^N$, define matrix elements $M_{ST}(z)$ by

$$\Delta(e_S(z)) = \sum_{T \subseteq [N]} M_{ST}(z) \otimes e_T(z). \quad (6.1)$$

For generic $(w, z) \in (\mathbb{C}^\times)^M \times (\mathbb{C}^\times)^N$, we then write

$$\langle M_{ST}(w), M_{UV}(z) \rangle = R_{ST}^{TV}(\lambda; w; z) T_{M+N-2|S|-2|U|}. \quad (6.2)$$

We will refer to $R_{ST}^{TV}$ as a generalized 6j-symbol. Note that it vanishes unless $|S| + |U| = |T| + |V|$. By (2.29) and (4.5), $R_{ST}^{TV}$ is a matrix element of the natural intertwiner between the corepresentations (1.2).

Although it is initially defined for generic values of $(w, z)$, $R_{ST}^{TV}$ extends to non-generic values by analytic continuation, and we are particularly interested in such degenerations. For instance, when $w_j = q^{j-1}\omega$ and $z_j = q^{j-1}\zeta$, it reduces to the elliptic 6j-symbols of Date et al. [D]. To understand this, let $V_N(\zeta)$ denote the right-hand side of (4.1) for $z_i = q^{i-1}\zeta$. Using that $\gamma(z)\alpha(qz) = \alpha(z)\gamma(qz)$, one finds that $\dim V_N(\zeta) = N + 1$. A basis for $V_N(\zeta)$ is $(v^N_N(\zeta))_{s=0}^N$, where $v^N_N(\zeta) = E_{|N-s+1,N|}(z) = e_{|N-s+1,N|}(z)$. One can then introduce matrix elements $M^N_{ST}(\zeta)$ by

$$\Delta(v^N_N(\zeta)) = \sum_{t=0}^N M^N_{ST}(\zeta) \otimes v^N_t(\zeta).$$

In [KoN], the pairing $\langle M^M_{ST}(\omega), M^N_{UV}(\zeta) \rangle$ was expressed as an elliptic hypergeometric function, which can be identified with an elliptic 6j-symbol. We will find analogous formulas for the more general pairing $\langle M_{ST}(w), M_{UV}(z) \rangle$. Our approach is different from that of [KoN]: in particular, we do not need any explicit expression for the matrix elements. Instead, we make a more efficient use of formal properties of the cobrading.

As was explained in §2.6, the symbol $R_{ST}^{TV}$ satisfies versions of the QDYB equation and the unitarity relation. It seems worth stating these fundamental properties explicitly.
Proposition 6.1. For $u \in (\mathbb{C}^\times)^L$, $w \in (\mathbb{C}^\times)^M$, $z \in (\mathbb{C}^\times)^N$, $Q, R \subseteq [L]$, $S, T \subseteq [M]$ and $U, V \subseteq [N]$ with $|Q| + |S| + |U| = |R| + |T| + |V|$, 

$$
\sum_{X \subseteq [L], Y \subseteq [M], Z \subseteq [N] \atop |X|+|Y|=|R|+|T| \atop |Y|+|Z|=|S|+|U|} \mathcal{R}^{XY}_{RT} (\lambda + N - 2|V|; u, w) \mathcal{R}^{QZ}_{XY} (\lambda; u, z) \mathcal{R}^{SU}_{YZ} (\lambda + L - 2|Q|; w, z)
$$

Moreover, for $w \in (\mathbb{C}^\times)^M$, $z \in (\mathbb{C}^\times)^N$, $S, T \subseteq [M]$ and $U, V \subseteq [N]$ with $|S|+|U| = |T|+|V|$, 

$$
\sum_{X \subseteq [L], Y \subseteq [M] \atop |X|+|Y|=|S|+|U|} \mathcal{R}^{XY}_{SU} (\lambda; w, z) \mathcal{R}^{VT}_{YX} (\lambda; z, w) = \delta_{ST} \delta_{UV}.
\tag{6.3}
$$

6.2. An explicit formula and further properties. Our first main result is the following expression for generalized $6j$-symbols.

Theorem 6.2. One has

$$
\mathcal{R}^{TV}_{SU} (\lambda; w; z) = 
\frac{(q^{\lambda+2+M+N-2L})_{|S|}}{(q^{\lambda+2+M-2|T|})_{|T|}} \frac{(q^{\lambda+2+M+N-2L})_{|V|}}{(q^{\lambda+2+M-2|T|-|V|})_{|V|}} \prod_{i \in T, j \in T^c \setminus X} \frac{\theta(q w_i / w_j)}{\theta(w_i / w_j)} 
$$

\begin{align*}
&\times \sum_{X \subseteq [L], Y \subseteq [M], Z \subseteq [N] \atop |X|+|Y|=|S|+|U| \atop |Y|-|X|=L-M} \theta(q)^{|U|-|V|-2|Y|} \frac{(q^{\lambda+2+M-N-|U|-|Y|})_{|Y|}}{(q^{\lambda+2+M-2|T|-|Y|})_{|Y|}} \prod_{i \in T, j \in T^c \setminus X} \frac{\theta(q w_i / w_j)}{\theta(w_i / w_j)} \\
&\times \prod_{i \in T \cap X} \frac{\theta(q z_i / w_j)}{\theta(w_i / w_j)} \prod_{i \in T \cap Y, j \in T^c \setminus Y} \frac{\theta(q z_i / z_j)}{\theta(z_i / z_j)} \prod_{i \in T \cap Y, j \in T^c \setminus Y} \frac{\theta(q w_i / z_j)}{\theta(w_i / z_j)} \\
&\times \prod_{i \in S \cap X} \frac{\theta(q w_i / z_j)}{\theta(z_i / z_j)} \prod_{i \in S \cap Y} \frac{\theta(q w_i / z_j)}{\theta(w_i / z_j)} \\
&\times \sum_{X \subseteq [L], Y \subseteq [M], Z \subseteq [N] \atop |X|+|Y|=|S|+|U| \atop |Y|-|X|=L-M} \theta(q)^{|U|-|V|-2|Y|} \frac{(q^{\lambda+2+M-N-|U|-|Y|})_{|Y|}}{(q^{\lambda+2+M-2|T|-|Y|})_{|Y|}} \prod_{i \in T, j \in T^c \setminus X} \frac{\theta(q w_i / w_j)}{\theta(w_i / w_j)} \\
&\times \prod_{i \in T \cap X} \frac{\theta(q z_i / w_j)}{\theta(w_i / w_j)} \prod_{i \in T \cap Y, j \in T^c \setminus Y} \frac{\theta(q z_i / z_j)}{\theta(z_i / z_j)} \prod_{i \in T \cap Y, j \in T^c \setminus Y} \frac{\theta(q w_i / z_j)}{\theta(w_i / z_j)} \\
&\times \Phi(w_{S \cap X}; z_{U \cap Y}; q^{\lambda+2+M-N-|U|-|Y|}) \Phi(w_{T \cap X}; z_{V \cap Y}; q^{-\lambda+|T|-|X|}),
\end{align*}

where $w \in (\mathbb{C}^\times)^M$, $z \in (\mathbb{C}^\times)^N$ and 

$$
L = |S| + |U| = |T| + |V|.
\tag{6.5}
$$

Before proving Theorem 6.2, we discuss some interesting consequences. First of all, using also Corollary 5.4, it is straight-forward to deduce the following symmetries.
Corollary 6.3. In the notation above,
\[ R_{SU}^{TV}(\lambda; w; z) = R_{V; T}^{SU}(\lambda + M + N - 2L; z^{-1}; w^{-1}) \]
\[ = \frac{G_{U,z}(\lambda + N - 2|U|)G_{S,w}(\lambda + M + N - 2L)}{G_{T,w}(\lambda + M - 2|T|)G_{V,z}(\lambda + M + N - 2L)} R_{V; T}^{SU}(\lambda + M + N - 2L; w; z), \]
where
\[ G_{S,z}(\lambda) = (-1)^{|S|} q^{-\binom{|S|}{2}} q^{-\frac{1}{2}N\lambda(q^{\lambda - 1} - q^{-1})} \prod_{i \in S, j \in S^c} \frac{\theta(z_i / z_j)}{\theta(qz_i / z_j)}. \] 

One can give a more instructive proof of Corollary 6.3 using the algebra symmetries of Proposition 3.3. For instance, one has
\[ e_S(z)^* = \delta(z^{-1}) \bar{\beta}(z^{-1}) = M_{S^c, \emptyset}(z^{-1}), \]
where the second equality follows from Lemma 4.6. Applying \( \Delta \) to this equality, using (2.10) and \( (\ast \otimes \ast) \circ \Delta = \Delta \circ \ast \), gives \( M_{ST}(z)^* = M_{S^c, \emptyset}(z^{-1}) \). Choosing \( x = M_{ST}(w) \) and \( y = M_{UV}(z) \) in (3.9) then yields the first equality in Proposition 6.3. Similarly, it follows from (6.10) below that
\[ (\phi \circ S)(M_{ST}(z)) = \frac{G_{S,z}(-\mu - 2 + N - 2|S|)}{G_{T,z}(-\lambda - 2 + N - 2|T|)} \det^{-1}(z^{-1}) M_{T, S^c}(qz^{-1}). \]

(Here and in several places below, the letter \( S \) is used both for the antipode and for a set; we hope this will not confuse the reader.) Then, the identity
\[ \langle M_{ST}(w), M_{UV}(z) \rangle = \psi^{D_h}\left( ((\phi \circ S)(M_{UV}(z)), (\phi \circ S)(M_{ST}(w))) \right) \]
leads to the equality of the first and third member (for the computation, one needs Lemma 3.1). The remaining symmetry follows by combining the other two.

In special situations, the expression (6.4) simplifies.

Corollary 6.4. If any one of the four conditions \(|V| < |S \setminus T|, |U| < |T \setminus S|, |S^c| < |U \setminus V| \) or \(|T^c| < |V \setminus U| \) holds, then \( R_{SU}^{TV} \) vanishes identically. If either \(|V| = |S \setminus T|, |U| = |T \setminus S|, |S^c| = |U \setminus V| \) or \(|T^c| = |V \setminus U| \), then \( R_{SU}^{TV} \) is given by an elementary factor times a product of two elliptic weight functions. Finally, if either \( S^c = \emptyset, T^c = \emptyset, U = \emptyset \) or \( V = \emptyset \), then \( R_{SU}^{TV} \) is given by an elementary factor times a single elliptic weight function.

Proof. With \( X \) and \( Y \) as in (6.4),
\[ |S^c \cap T^c| \geq |X| = |Y| + |S^c| - |U| \geq |S^c| - |U|, \]
\[ |U \cap V| \geq |Y| = |U| - |S^c| + |X| \geq |U| - |S^c|. \]
Thus, if \(|U| < |T \setminus S|\) or \(|S^c| < |U \setminus V|\), the sum is empty. Since \(|S| + |U| = |T| + |V|\), the remaining part of the first statement follows. By the same argument, if any one of the eight equalities stated hold, the sum has only one term.

As an example, to be used later,

\[
\mathcal{R}^T_{SU}(\lambda; w; z) = \theta(q)^{|U|} \prod_{i \in T, j \in U} \frac{\theta(qw_i/z_j)}{\theta(w_i/z_j)} \prod_{i \in T \setminus S, j \in T^c} \frac{\theta(qw_i/w_j)}{\theta(w_i/w_j)} \prod_{i \in T, j \in [N]} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \times \frac{(q^{\lambda+2+M-N-2|T|})^{|S|}}{(q^{\lambda+2+M-2|T|})^{|T|}} \Phi(w_T; z_U; q^{\lambda+2+M-N-|U|})
\]

(6.7)

if \(S \subseteq T\) with \(|T \setminus S| = |U|\), and vanishes else.

Finally, the following fact is needed in \(\S 7.3\).

**Corollary 6.5.** Suppose there exists \((i, j) \in T \times T^c\) with \(w_j = qw_i\). Then, either \((i, j) \in S \times S^c\) or \(\mathcal{R}^T_{SU}(\lambda; w; z)\) vanishes identically. Similarly, if \((i, j) \in V \times V^c\) and \(z_j = qz_i\), then either \((i, j) \in U \times U^c\) or \(\mathcal{R}^T_{SU}(\lambda; w; z)\) vanishes.

**Proof.** Consider the expression (6.4). In the first situation, if \(j \in T^c \setminus X\) then the factor \(\prod_{i \in T, j \in T^c \setminus X} \theta(qw_i/w_j)\) vanishes. Thus, we may assume \(j \in X\); in particular, \(j \in S^c\). Then, if \(i \in S^c \setminus X\) the factor \(\prod_{i \in S^c \setminus X, j \in \bar{X}} \theta(qw_i/w_j)\) vanishes. Thus, non-vanishing terms exist only when \(i \in S\) \((i \in X\) would contradict \(i \in T\) and \(j \in S^c\). The proof of the second statement is similar. \(\Box\)

### 6.3. **Proof of Theorem 6.2.**

The key to the proof of Theorem 6.2 is the following result.

**Proposition 6.6.** Let \((w, z) \in (\mathbb{C}^\times)^M \times (\mathbb{C}^\times)^N\) be generic, and let \(a \in W(w), b \in W(z), c \in V(z)\) and \(d \in V(w)\). Then,

\[
\langle ab, cd \rangle = \sum_{(d', c')T_{\omega_{12}(c)+\omega_{12}(d)}} \langle a, d' \rangle T_M(b, c') T_{-M}.
\]

(6.8)

**Proof.** We may choose \(a = f(\mu)F_T(w), b = g(\mu)F_V(z), c = h(\lambda)E_U(z), d = k(\lambda)E_S(w)\), where \(f, g, h, k \in M_{h^*}\). It is easy to check that the functions \(f, g, h, k\) cancel from both sides of (6.8), and we are reduced to proving

\[
\langle \tilde{L}_T(w) \tilde{L}_V(z), \tilde{L}_U(z) \tilde{L}_S(w) \rangle = \sum_{X \subseteq [M], Y \subseteq [N]} \langle \tilde{L}_{UX}(w), \tilde{L}_{UY}(z) \rangle T_{[X]+[Y]} \times \langle \tilde{L}_T(w), \tilde{L}_U(z) \rangle T_M \langle \tilde{L}_V(z), \tilde{L}_S(w) \rangle T_{-M}.
\]

By Proposition 3.6, the right-hand side equals \(\langle \tilde{L}_T(w), \tilde{L}_U(z) \rangle T_{-M-N}\). Thus, the result follows from Corollary 3.7. \(\Box\)

The following result is a transformed version of Corollary 3.7, where basis vectors of the form \(E_S\) and \(F_S\) have been replaced by \(e_S\) and \(f_S\). Just as for Corollary 3.7, the right-hand side is a partition function for a square with two domain walls.
Corollary 6.7. In the notation above,
\[
\mathcal{R}^{TV}_{SU}(\lambda; w; z) = \frac{1}{A_{T,w}(\lambda)A_{V,z}(\lambda + M - 2|T|)} \times \langle \tilde{\alpha}(w_T)\tilde{\beta}(w_T)\tilde{\alpha}(z_V)\tilde{\beta}(z_V)\tilde{\gamma}(z_{V^c})\tilde{\gamma}(w_{S^c})\tilde{\alpha}(w_S)\tilde{\gamma}(w_S)\rangle 1.
\]

Proof. Choose \(a = f_T(w), b = f_V(z), c = e_U(z)\) and \(d = e_S(w)\) in Lemma 6.6. Using (4.3), we obtain
\[
\langle f_T(w)f_V(z), e_U(z)e_S(w) \rangle
\]
\[
= \sum_{X \subseteq [M], Y \subseteq [N]} \langle M_{SX}(w), M_{UY}(z) \rangle T_{2|X|+|Y|} \times (f_T(w), e_X(w)) T_M \langle f_V(z), e_Y(z) \rangle T_{-M}
\]
\[
= \langle M_{ST}(w), M_{UV}(z) \rangle T_{2|T|+|V|} A_{T,w} T_{M-2|T|} A_{V,z} T_{-M-2|V|},
\]
which gives the desired result after simplification. \(\square\)

Proof of Theorem 6.2. In Corollary 6.7, apply (5.4) to the factor \(\tilde{\beta}(w_{T^c})\tilde{\alpha}(z_V)\) and (5.3) to the factor \(\tilde{\alpha}(z_U)\tilde{\gamma}(w_{S^c})\). Each resulting term is of the form (4.3), with \(z\) replaced by \((w, z)\). The only non-vanishing terms are those where \(\tilde{\beta}(w_{T^c})\tilde{\alpha}(z_V)\) is replaced (up to a multiplier) by \(\tilde{\alpha}(w_{T^c \setminus X}, z_Y)\tilde{\beta}(w_X, z_{V \setminus Y})\) and simultaneously \(\tilde{\alpha}(z_U)\tilde{\gamma}(w_{S^c})\) by \(\tilde{\gamma}(w_X, z_{U \setminus Y})\tilde{\alpha}(w_{S^c \setminus X}, z_Y)\), for some \(X \subseteq S^c \cap T^c, Y \subseteq U \cap V\). All in all, this gives
\[
\mathcal{R}^{TV}_{SU}(\lambda; w; z) = \sum_{X \subseteq S^c \cap T^c, Y \subseteq U \cap V \atop |Y| - |X| = L - M} \frac{A_{(X^c,Y^c),w}(\lambda)}{A_{T,w}(\lambda)A_{V,z}(\lambda + M - 2|T|)} \times C_{(0,U), (S^c \setminus X, Y), (w_{S^c}, z_Y)}(\lambda + N - |U|) D_{(0,V), (T^c \setminus X, Y), (w_{T^c}, z_Y)}(\lambda - |T|),
\]
which yields (6.4) after simplification. \(\square\)

6.4. Asymmetric identities. We have seen that (6.4) displays all symmetries of generalized 6j-symbols given in Corollary 6.3. From the viewpoint of special functions, it is interesting to obtain less symmetric expressions, since symmetries then correspond to non-trivial transformation formulas.

Recall the symmetry \(\phi\) defined in Proposition 3.3. Clearly, \(\phi\) restricts to an \(h\)-space isomorphism \(V(z^{-1}) \rightarrow V(z)\). In particular,
\[
\phi(e_S(z^{-1})) = \tilde{\alpha}(z_S)\tilde{\gamma}(z_{S^c}), \quad S \subseteq [N],
\]
form a basis for \(V(z)\). Similarly,
\[
\phi(f_S(z^{-1})) = \tilde{\beta}(z_{S^c})\tilde{\alpha}(z_S), \quad S \subseteq [N],
\]
form a basis of \(W(z)\).

The following lemma can be proved similarly as (4.3).
Lemma 6.8. For generic $z \in (\mathbb{C}^\times)^N$,

$$\langle \phi(f_T(z^{-1})), \phi(e_S(z^{-1})) \rangle = \delta_{ST} B_{S,z} T_{-2m},$$

where $m = |S|$ and where

$$B_{S,z}(\lambda) = q^{m(N-m)} \frac{(q^{\lambda+1}-m)_{N-m}}{(q^{\lambda+1})_{N-m}} \prod_{i \in S, j \in S} \frac{\theta(z_i/z_j)}{\theta(qz_i/z_j)}. \quad (6.9)$$

The following result is then obtained similarly as Corollary 6.7.

Lemma 6.9. Let $(w, z) \in (\mathbb{C}^\times)^M \times (\mathbb{C}^\times)^N$ be generic. Then, for $S, T \subseteq [M]$ and $U, V \subseteq [N]$,

$$\langle \phi(M_{ST}(w^{-1})), M_{UV}(z) \rangle = \frac{1}{B_{T,w}(\lambda) A_{V,z}(\lambda + M - 2|T|)} \langle \phi(f_T(w^{-1})) f_V(z), e_U(z) \phi(e_S(w^{-1})) \rangle T_{M+N}.$$ 

We need to relate the action of $\phi$ and $S$, first on basis vectors and then on matrix elements.

Lemma 6.10. One has

$$S(e_S(z)) = (-1)^N q^{\frac{1}{2}N\lambda + \binom{N}{2}} \frac{G_{S,z}(\mu) \det(q^{-1}z)}{(q^{\lambda+1})_N} \phi(M_{\emptyset S^c}(qz^{-1})),

where $G$ is as in (6.6).

Proof. Using (3.5), we can write

$$S(e_S(z)) = (-1)^N q^{\frac{1}{2}N\lambda + \binom{N}{2}} \frac{G_{S,z}(\mu) \det(q^{-1}z)}{(q^{\lambda+1})_N} \phi(M_{\emptyset S^c}(qz^{-1})),

which leads to

$$\phi(M_{\emptyset S^c}(qz^{-1})) = \frac{1}{A_{S,z}(\mu) A_{S^c q^{-1}}(-\mu - 2 + N - 2m)} \delta(q^{-1}z S) \delta(q^{-1}z S^c)$$

$$= q^{m(N-m)} \frac{(q^{\mu+1})_m}{(q^{\mu+1+m-N})_m} \prod_{i \in S, j \in S^c} \frac{\theta(qz_i/z_j)}{\theta(q^{-1}z S) \delta(q^{-1}z S).$$

Combining these facts yields the desired result. □
Corollary 6.11. For \( z \in (\mathbb{C}^\times)^N \),

\[
S(M_{ST}(z)) = \frac{G_{S,z}(\mu)}{G_{T,z}(\lambda)} \det^{-1}(q^{-1}z)\phi(M_{T^{-c}S^{-c}}(qz^{-1})). \tag{6.10}
\]

**Proof.** By (2.3), Lemma 6.10 and (3.6),

\[
\sum_{T \subseteq [N]} S(e_T(z)) \otimes S(M_{ST}(z)) = \Delta(S(e_S(z))) = \sum_{T \subseteq [N]} S(e_T(z)) \otimes \frac{G_{S,z}(\mu)}{G_{T,z}(\lambda)} \det^{-1}(q^{-1}z)\phi(M_{T^{-c}S^{-c}}(qz^{-1})).
\]

Since, by Corollary 4.4, the elements \( e_T(z) \) are linearly independent over \( \mu_l(M_{h^*}) \) and \( S \) is invertible, \( S(e_T(z)) \) are linearly independent over \( \mu_r(M_{h^*}) \). It follows that the identity above holds termwise. \( \square \)

We can now obtain the following variation of Corollary 6.7.

Corollary 6.12. Let \( (w, z) \in (\mathbb{C}^\times)^M \times (\mathbb{C}^\times)^N \) be generic. Then,

\[
\mathcal{R}_{SU}^{TV}(\lambda; w; z) = (-1)^{|U|+|V|} q^\binom{|V|}{2} N(M-L)+\frac{1}{2}|U||(U|+1) 
\times \prod_{i \in T, j \in T^c} \frac{\theta(qw_i/w_j)}{\theta(w_i/w_j)} \prod_{i \in V, j \in V^c} \frac{\theta(qz_i/z_j)}{\theta(z_i/z_j)} \prod_{i \in [M], j \in [N]} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} 
\times \frac{(q^{\lambda+2-|T|})_{N+|T|-|U|}(q^{\lambda+1+N-2|U|})_{|U|}}{(q^{\lambda+1+M-2|T|-|V|})_{|V|}(q^{\lambda+2+M-2|T|})_{|T|}(q^{\lambda+2+M-2|T|})_{N-|V|}} T_{N-2|U|} 
\times \langle \bar{\beta}(q^{-1}z_U)\bar{\alpha}(q^{-1}z_{U^c}, w_T)\bar{\beta}(w_{T^c}), \bar{\gamma}(w_{S^{-c}})\bar{\alpha}(w_S, q^{-1}z_{V^c})\bar{\gamma}(q^{-1}z_V) \rangle 1,
\]

with \( L \) as in (6.5).
Proof. Using, respectively, Proposition \[\text{2.6}\] Corollary \[\text{6.11}\] Lemma \[\text{3.1}\] and Lemma \[\text{6.9}\] gives

\[
\langle M_{ST}(w), M_{UV}(z) \rangle = T_{N-2|U} \langle S(M_{UV}(z)), M_{ST}(w) \rangle T_{N-2|V} \\
= T_{N-2|U} \left( \frac{G_{U,z}(\mu)}{G_{V,z}(\lambda)} \sqrt{q^{-1}} \phi(M_{V \cap U^c}(qz^{-1})), M_{ST}(w) \right) T_{N-2|V} \\
= \frac{G_{U,z}(\lambda + N - 2|U|)}{G_{V,z}(\lambda + M + N - 2|T| - 2|V|)} q^{\frac{1}{2}M_N} \prod_{i \in [M], j \in [N]} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \\
\times T_{N-2|U} \langle \phi(f_{U^c}(qz^{-1})) f_T(w), e_S(w) \phi(e_{V^c}(qz^{-1})) \rangle T_{M+2N-2|V},
\]

which simplifies to the given expression. \[\square\]

Theorem 6.13. In the notation above, \( R_{ST}^{TV}(\lambda; w; z) \) can be expressed as

\[
(-1)^{|U|+|V|} q^{(|V|/2) + N(M-L) + \frac{1}{2} |U|(|U|+1)} \prod_{i \in V, j \in V^c} \frac{\theta(qz_i/z_j)}{\theta(z_i/z_j)} \prod_{i \in [M], j \in [N]} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \\
\times \frac{(q^{\lambda+1+N-2|U|})_{|U|}}{(q^{\lambda+N-2|U|})_{|V|} (q^{\lambda+2+M-2|T|})_{|T|} (q^{\lambda+2+M-2|V|})_{N-|V|}} \\
\times \sum_{X \subseteq S \cap T, Y \subseteq [N] \atop |X| + |Y| = |S| + N - |V|} \theta(q)^{|U \cap Y|+|V \cap Y|} \prod_{i \in X, j \in S \setminus X} \frac{\theta(qw_i/w_j)}{\theta(w_i/w_j)} \prod_{i \in T \setminus X, j \in T^c} \frac{\theta(z_j/w_i)}{\theta(z_j/w_i)} \\
\times \prod_{i \in X, j \in Y^c} \frac{\theta(q^2w_i/z_j)}{\theta(qw_i/z_j)} \prod_{i \in S \cap T^c, j \in Y} \frac{\theta(qz_j/w_i)}{\theta(z_j/w_i)} \prod_{i \in S \cap T^c \cap Y^c, j \in Y} \frac{\theta(qz_j/w_i)}{\theta(qz_j/w_i)} \\
\times \phi(q^{-1}z_{U \cap Y}; (w_T \setminus X, q^{-1}z_{U \cap Y^c}); q^{-\lambda-N+|U|+|U \cap Y|}) \\
\times \phi(q^{-1}z_{V \cap Y}; (w_S \setminus X, q^{-1}z_{V \cap Y^c}); q^{\lambda+2+M-2|T|-|V|+|V \cap Y|})
\]
or alternatively as

\[
(-1)^{|S|+|T|} q^{\binom{|S|}{2}} \frac{(q^2)^{|S|-2|T|}}{\theta(q^2/w_j)} \prod_{i \in T, j \in T^e} \frac{\theta(qw_i/w_j)}{\theta(w_i/w_j)} \prod_{i \in [M], j \in [N]} \frac{\theta(w_i/z_j)}{\theta(qw_i/z_j)} \\
\times \frac{(q^{\lambda+1+N-2|U|})_{M-|S|}}{(q^{\lambda+1-|T|})_{M-|T|}(q^{\lambda+2+M-2|T|})_{|T|}(q^{\lambda+2+M-2|T|})_{N-|V|}} \\
\times \sum_{X \subseteq U^c \cap V^c, Y \subseteq [M]} \frac{\theta(q)^{|S \cap Y|+|T \cap Y|}}{\theta(z_j/z_j)} \prod_{i \in U^c \setminus X, j \in X} \frac{\theta(z_j/qw_i)}{\theta(z_j/w_i)} \prod_{i \in V^c \setminus X} \frac{\theta(z_j/w_i)}{\theta(qz_j/z_j)} \\
\times \prod_{i \in Y^c, j \in X} \frac{\theta(q^2w_i/z_j)}{\theta(qw_i/z_j)} \prod_{i \in Y^c \setminus V^c, j \in X} \frac{\theta(z_j/qw_i)}{\theta(z_j/w_i)} \prod_{i \in U^c \setminus V^c, j \in X} \frac{\theta(z_j/w_i)}{\theta(qz_j/z_j)} \\
\times \Phi((z_{V^c \setminus X}, qw_{S \cap Y^c}); qw_{S \cap Y^c}; q^{-\lambda-M-N+|S|+2|U|+|S \cap Y^c|}) \\
\times \Phi((z_{U^c \setminus X}, qw_{U^c \setminus Y^c}); qw_{U^c \setminus Y^c}; q^{\lambda+2-|T|+|T \cap Y^c|}).
\]

**Proof.** Consider the factor

\[
\langle \bar{\beta}(q^{-1}z_U)\bar{\alpha}(q^{-1}z_{U^c}, w_T), \bar{\gamma}(w_{S^c})\bar{\alpha}(w_S, q^{-1}z_{V^c})\gamma(q^{-1}z_V) \rangle
\]

from Corollary 6.12. Similarly as in the proof of Theorem 6.2, commuting \(\beta\) to the right and \(\gamma\) to the left, it can be written

\[
\sum_{X \subseteq S \cap T, Y \subseteq [N]} \frac{A_{(X,Y),(w,q^{-1})}(\lambda)}{|X|+|Y|=|S|+|N|-|V|} \\
\times \frac{A_{(S,V^c),(X,Y),(w,q^{-1})}(\lambda+M-|S|)D_{(T,U^c),(X,Y),(w_T,q^{-1})}(\lambda)T_2|V|-2|S|-2N.}
\]

Inserting this into Lemma 6.12 and simplifying, we obtain the first expression. The second expression then follows using the first symmetry of Proposition 6.3.

An interesting summation formula follows by choosing \(V = \emptyset\) in the second expression of Theorem 6.13, the value of the sum being known from (6.7). From the results of (7.7) it will be clear that this identity generalizes the elliptic \(A_n\) Jackson summation of [R2], see (7.6) below.
Corollary 6.14. Suppose that \( S, T \subseteq [M] \) and \( U \subseteq [N] \) with \( |S| + |U| = |T| \). Then, if \( S \subseteq T \),

\[
\sum_{X \subseteq U^c, Y \subseteq [M]} \frac{\theta(q^{S \cap Y} + T \cap Y)}{\prod_{i \in X, j \in X} \theta(qz_i/z_j)} \frac{\theta(qz_i/z_j)}{\theta(z_i/z_j)} \frac{\theta(qw_i/w_j)}{\theta(w_i/w_j)} \\
\times \prod_{i \in Y^c, j \in X} \theta(q^2 w_i/z_j) \prod_{i \in Y \cap U^c} \theta(qz_j/w_i) (q^{\lambda + 2 - |T| + |T \cap Y|})_{N+|S|-|T \cap Y|} \\
\times \Phi((z_{X^c}, qw_S \cap Y); qw_{S \cap Y}; q^{-\lambda - M - N + |T| + |U| + |S \cap Y|}) \\
\times \Phi((z_{U \setminus X}, qw_{T \cap Y}); qw_{T \cap Y}; q^{\lambda + 2 - |T| + |T \cap Y|}) \\
= (-1)^{|U|} q^{(M - |T|)|S| - \frac{1}{2}|U|(|U| + 1)} \theta(q)^{|U|} \frac{(q^{\lambda + 1 - |T|})_{M-|T|} (q^{\lambda + 2 - M - 2|T|})_{N+|S|}}{(q^{\lambda + 1 - N - 2|U|})_{M-|S|}} \\
\times \prod_{i \in S \setminus T} \theta(w_i/w_j) \prod_{i \in T \setminus [N]} \theta(qw_i/z_j) \prod_{i \in T \setminus [S \setminus U]} \theta(qw_i/w_j) \times \Phi(w_{T \setminus S}; z_U; q^{\lambda + 2 - N - |U|});
\]

otherwise the left-hand side vanishes.

7. Hypergeometric series

7.1. Preliminaries on elliptic hypergeometric series. Elliptic 6j-symbols can be expressed in terms of the elliptic hypergeometric series \(_{12}V_{11}\), where, in general,

\[
m+5V_{m+4}(a; b_1, \ldots, b_m) = \sum_{y=0}^{\infty} \frac{\theta(aq^{2y})}{\theta(a)} (a, b_1, \ldots, b_m)_y q^y.
\]

For an introduction to elliptic hypergeometric functions, the reader is referred to [GR Chapter 11] or [S4]. The series arising from elliptic 6j-symbols are terminating and balanced. Terminating means that \( b_1 = q^{-N} \), with \( N \) a non-negative integer, so that the summation is restricted to \( 0 \leq y \leq N \). When \( m = 2k + 1 \), which is the only case of interest to us, balanced means that

\[
b_1 \cdots b_{2k+1} = a^k q^{k-1}.
\]

Two important results for such series are the elliptic Bailey transformation

\[
_{12}V_{11}(a; q^{-N}, b, c, d, e, f, g) = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_N}{(\lambda q, \lambda q/ef, aq/e, aq/f)_N} \\
\times \frac{(aq, aq/bd, aq/ cd)_N}{(aq/b, aq/c, aq/d, aq/bcd)_N} (7.1)
\]

where \( \lambda = qa^2/bcd \) and \( a^3 q^{N+2} = bcd ef g \), and the elliptic Jackson summation

\[
_{10}V_9(a; q^{-N}, b, c, d, e) = \frac{(aq, aq/bc, aq/bd, aq/ cd)_N}{(aq/b, aq/c, aq/d, aq/bcd)_N},
\]
where $a^2q^{N+1} = bcde$. These identities were first obtained by Frenkel and Turaev [FT], though with some restriction on the parameters they are implicit in [D].

We now turn to the multiple series defined by

$$V^m_n(a; b_1, \ldots, b_{m+2}; c_1, \ldots, c_{m+n+2}; z_1, \ldots, z_n)$$

where

$$
\begin{align*}
V^m_n(a; b_1, \ldots, b_{m+2}; c_1, \ldots, c_{m+n+2}; z_1, \ldots, z_n) &= \sum_{y_1, \ldots, y_n \geq 0} \frac{\Delta(zq^y)}{\Delta(z)} q^{y|y|} \prod_{i=1}^n \frac{\theta(a z_i q^{y+|y|})}{\theta(a z_i)} \prod_{i=1}^n \frac{\theta(b_i)}{\theta(a q/c_i)_{|y|}} \\
&\times \prod_{i=1}^n \prod_{j=1}^n \frac{\prod_{j=1}^n (q z_i / z_j)_{y_i} \prod_{j=1}^{m+n+2}(aq z_i / b_j)_{y_i}}{\prod_{j=1}^n (c_j z_i)_{y_i}}.
\end{align*}
$$

(7.2)

These identities were first obtained by Frenkel and Turaev [FT], though with some restriction on the parameters they are implicit in [D].

$V^m_n$ does not change under the scaling $a \mapsto ta$, $c_j \mapsto tc_j$, $z_j \mapsto z_j/t$. This redundancy of notation is convenient but must be kept in mind. Note also that

$$V^m_n(a; b_1, \ldots, b_{m+2}; c_1, \ldots, c_{m+3}; z) = 2m+10 V^m_n(az; b_1, \ldots, b_{m+2}, c_1 z, \ldots, c_{m+3} z).$$

In particular, $V^1_n = 12 V^1_1$ is the series related to elliptic 6j-symbols.

We call the series $V^m_n$ balanced when the parameters satisfy

$$b_1 \cdots b_{m+2} c_1 \cdots c_{m+n+2} z_1 \cdots z_n = q^{m+1} a^{m+2}.$$
Kajihara and Noumi [KN] and the present author [R3] independently proved the transformation formula

$$
\sum_{y_1,\ldots,y_n \geq 0 \atop y_1+\cdots+y_n = N} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^{n} \frac{\prod_{j=1}^{m} (a_j z_k)^{y_k}}{\prod_{k=1}^{m} (w_j z_k)^{y_k}} \prod_{j=1}^{m+n} (a_j z_k)^{y_k}
$$

$$
\Delta(w) \prod_{k=1}^{m} \frac{\prod_{j=1}^{m} (w_j z_k)^{y_k}}{\prod_{j=1}^{m} (w_k z_j)^{y_k}} \prod_{j=1}^{m+n} (w_k a_j)^{y_k},
$$

(7.4)

where $w_1 \cdots w_m = z_1 \cdots z_n a_1 \cdots a_{m+n}$. This is a discrete analogue of an integral transformation of Rains [Ra]; the latter was recently given a quantum field theory interpretation by Dolan and Osborn [DO].

Eliminating one of the summation variables on each side of (7.4) yields a transformation between series of type $V_{n-1}^{m-1}$ and $V_{m-1}^{n-1}$. Replacing $m$, $n$ by $m+1$ and $n+1$, and applying a standard argument of analytic continuation, the resulting identity takes the form

$$
V_n^m \left( a; b, c, \frac{aq}{w_1}, \ldots, \frac{aq}{w_m}, \frac{q^{-N_1}}{z_1}, \ldots, \frac{q^{-N_n}}{z_n}, q^{M_1} w_1, \ldots, q^{M_m} w_m, d, e; z_1, \ldots, z_n \right)
$$

$$
= c^{|N|-|M|} \frac{\lambda q d, \lambda q e |_{M}(aq/cd, aq/ce)|_N}{(aq/cd, aq/ce)|_M(aq/d, aq/e)|_N}
$$

$$
\times \prod_{j=1}^{m} \frac{(\lambda q w_j/b, \lambda q w_j/c)_M}{(\lambda q w_j/c, \lambda q w_j)_M} \prod_{j=1}^{n} \frac{(aq z_j/b, aq z_j/c)_N}{(aq z_j/b, aq z_j/c)_N} V_n^m \left( \lambda; b, c, \frac{\lambda q}{z_1}, \ldots, \frac{\lambda q}{z_n}; q^{-M_1} w_1, \ldots, q^{-M_m} w_m, q^{N_1} z_1, \ldots, q^{N_n} z_n, \frac{1}{d}, \frac{1}{e}; w_1, \ldots, w_m \right),
$$

(7.5)

where $\lambda = bc/aq = q^{1/2} |N|^{-1/2} |M|^{-1/2} |d/e|.$

The case $m = n = 1$ of (7.5) is a $1_2 V_{11}$ transformation that is different from (7.1), but can be obtained as a consequence of that result. When $m = 0$, the function $V^n_0$ on the right-hand side of (7.5) should be interpreted as 1, and we obtain the multivariable elliptic Jackson summation [R2 Cor. 5.3]

$$
V_n^0 \left( a; b, c; q^{-N_1}/z_1, \ldots, q^{-N_n}/z_n, d, e; z_1, \ldots, z_n \right)
$$

$$
= c^{|N|} \frac{(aq/cd, aq/ce)_N}{(aq/d, aq/e)_N} \prod_{j=1}^{n} \frac{(aq z_j/b, aq z_j/c)_N}{(aq z_j/b, aq z_j/c)_N},
$$

(7.6)

where $a^2 q^{1/2} = bcde.$
Another multivariable elliptic Jackson summation is obtained in [RoS]; see [Sc, Thm. 4.1] for the case $p = 0$:

$$
\sum_{y_1,\ldots,y_n=0}^{N_1,\ldots,N_n} \frac{\Delta(xq^y)}{\Delta(x)} \frac{q^{\frac{a}{2}|y|}}{\theta(a)} \prod_{i=1}^{n} \frac{(aq^{1+|N|}/e x_i)_{|y|-y_i}(d/x_i)_{|y|}(e x_i)_{y_i}}{(d/x_i)_{|y|-y_i}(aq^{1+|N|-N_i}/e x_i)_{|y|}(aq x_i/d)_{y_i}} \\
\times \frac{(a, b, c)_{|y|}}{(aq^{1+|N|}, aq/b, aq/c)_{|y|}} \prod_{i,j=1}^{n} \frac{q^{-N_i x_i/x_j}y_i}{(q x_i/x_j)_{y_i}} = \left(\frac{aq, aq/bc, aq/c|N|}{aq/b, aq/c, aq/c|N|}\right) \prod_{i=1}^{n} \left(\frac{aq x_i/bd, aq x_i/cd|N_i}{aq x_i/d, aq x_i/bcd|N_i}\right),
$$

(7.7)

where $a^2 q^{|N|+1} = bcde$.

7.2. **Hypergeometric series from generalized $6j$-symbols.** The expressions in Theorems 6.2 and 6.13 are generalizations of elliptic hypergeometric representations for elliptic $6j$-symbols. In view of the discussion in §6.1, to recover the latter one should choose $S = [M - s + 1, M], T = [M - t + 1, M], U = [N - u + 1, N], V = [N - v + 1, N]$ and specialize $z_j = q^{j-1} \zeta, w_j = q^{j-1} \omega$. In Theorem 6.2 the factor $\prod_{i \in S \setminus X, j \in Y} \theta(qw_i/w_j)$ then vanishes unless $X = [1, x]$, while the factor $\prod_{i \in Y, j \in U \setminus Y} \theta(qz_i/z_j)$ vanishes unless $Y = [N - y + 1, N]$. Since $x$ and $y$ are related by $y - x = s + u - M$, the expression reduces to a single sum. Similar reductions occur for the two expressions of Theorem 6.13. One can check that all three sums are of type $V_1^1 = 12V_{11}$, and that the equality of the three expressions follows from known transformation formulas for such series. This is the case considered in [KoN]. We will now explain how to generalize these results to include series of type $V_m^n$.

First, we let

$$w_j = q^{j-1} \omega, \quad S = [M - s + 1, M], \quad T = [M - t + 1, M] \quad (7.8)$$
in Theorem 6.2. As above, we may write $X = [1, x]$. By Lemma 5.3, the elliptic weight functions factor, and we find that $R_{ST}^T(\lambda; w; z)$ equals

$$
\frac{(q)_{M-s}(q^{\lambda+2+M+N-2L})_s}{(q)_L(q^{\lambda+2+M-2K})_L(q^{\lambda+2+M+N-2L})_{|V|}} \prod_{(i,j) \in (V \times V^c) \setminus (U \times U^c)} \frac{\theta(qz_i/z_j)}{\theta(qz_i/z_j)} \prod_{i \in U \cap V^c} \frac{\theta(q^{\lambda+1+M+N-L-|U|}|\omega/z_i)}{\theta(q^{\lambda+1+M+N-L-|U|}|z_i)} \prod_{i \in U^c \cap V} \frac{\theta(q^{\lambda+1+M-N-L-|U|}|\omega/z_i)}{\theta(q^{\lambda+1+M-N-L-|U|}|z_i)} 
$$

\times \prod_{i \in U \cap V^c} \frac{\theta(q^{\lambda+1+M+N-L-|U|}|\omega/z_i)}{\theta(q^{\lambda+1+M+N-L-|U|}|z_i)} \prod_{i \in U^c \cap V} \frac{\theta(q^{\lambda+1+M+N-L-|U|}|\omega/z_i)}{\theta(q^{\lambda+1+M+N-L-|U|}|z_i)} \prod_{i \in Y \cap V^c} \frac{\theta(q^{1-L-M-|U|}|\omega/z_i)}{\theta(q^{1-L-M-|U|}|z_i)} \prod_{i \in U \cap V^c} \frac{\theta(q^{1-L-M-|U|}|\omega/z_i)}{\theta(q^{1-L-M-|U|}|z_i)} \prod_{i \in U^c \cap V} \frac{\theta(q^{1-L-M-|U|}|\omega/z_i)}{\theta(q^{1-L-M-|U|}|z_i)}.
$$

Next, we specialize

$$
z_{U \cap V} = (\eta_1, \ldots, \eta_1 q^{k_i-1}, \ldots, \eta_m, \ldots, \eta_m q^{k_m-1}), \quad (7.9a)
$$

$$
z_{U \cap V^c} = (q^{-1-h_i} \xi_1^{-1}, \ldots, \xi_1^{-1}, \ldots, q^{-1-l_n} \xi_n^{-1}, \ldots, \xi_n^{-1}), \quad (7.9b)
$$

so that

$$
N + |k| = |U| + |V| + |l|. \quad (7.10)
$$

We stress that this is not a restriction on the variables $z_i$, since the general case is included as $k_i \equiv l_i \equiv 1$. Then, the only non-vanishing terms are those where

$$
Y = [y_1 + 1, k_1] \cup \cdots \cup [y_m + 1, k_m],
$$

with $0 \leq y_i \leq k_i$, so that $|Y| = |k| - |y|$. One may check that

$$
\prod_{i \in Y \cap V^c} \frac{\theta(qz_i/z_j)}{\theta(z_i/z_j)} = (-1)^{|y||q^{|y||k|-|y|}} \Delta(q^{|y|}) \Delta(\eta^{|y|}) \prod_{i,j=1}^m \frac{(q^{-k_i} \eta_i / \eta_j)_{y_i}}{(q^{k_i} \eta_i / \eta_j)_{y_i}},
$$

cf. [KN] or [R2] §7. After simplification, the result takes the following form.
Corollary 7.1. Assuming (7.8) and (7.9), $R_{SU}^{TY}(\lambda; w; z)$ equals

\[
\frac{(q)_{M-S}(q)_{L-|k|}}{(q)_{L}(q)_{M+|k|-L}} \frac{(q^{\lambda+2M+N-2L})_{S}(q^{\lambda+2N-|V|-|k|})}{(q^{\lambda+2M-N-2t})_{S}(q^{\lambda+2M+N-2L})_{V}(q^{\lambda+2M-N-2L})_{V}|V|-|k|} \\
\times \prod_{i \in U \cap \cap U \cap V \cap V} \frac{\theta(qz_i/z_j)}{\theta(z_i/z_j)} \prod_{i \in U \cap \cap V \cap V} \left( \frac{\theta(q^{\lambda+S+N-L-|U|}|z_i/\omega)}{\theta(q^{M}\omega/z_i)} \prod_{j=1}^{m} \frac{\theta(\eta_j q^{k_j}/z_i)}{\theta(\eta_j/z_i)} \right) \\
\times \prod_{i \in U \cap \cap V \cap V} \left( \frac{\theta(q^{-\lambda+2M-N-2t})_{S}(q^{\lambda+2M-N-2t})_{V}}{\theta(q^{M}\omega/z_i)} \prod_{i=1}^{n} \theta(q^{k_j}z_i) \prod_{j=1}^{m} (q^{k_j}/\omega)_{k_i} \right) \\
\times \prod_{i=1}^{n} \left( \frac{(q^{M+|k|-L}\omega z_i)_{L}}{(q^{M}\omega z_i)_{L}} \sum_{0 \leq y_1, \ldots, y_n \leq |k|} \Delta(q^{y}) \frac{\Delta(q^{y})}{\Delta(\eta)} \right) \\
\times \prod_{i=1}^{m} \left( \frac{\theta(q^{L-M-|k|+|y|+y,\eta_i}/\omega)}{\theta(q^{L-M-|k|})_{L}} \prod_{j=1}^{n} \frac{(q^{L-M-|k|+|y|+y,\eta_i}/\omega)_{y_i}}{(q^{L-M-|k|+|k|})_{L}} \right) \\
\times \prod_{i=1}^{n} \left( \frac{(q^{L-M-|k|+|y|+y,\eta_i}/\omega)_{y_i}}{(q^{L-M-|k|+|y|+y,\eta_i}/\omega)_{y_i}} \prod_{j=1}^{m} \frac{(q^{k_j}z_i)_{y_i}}{(q^{k_j}/\omega)_{y_i}} \prod_{j=1}^{m} \frac{(q^{k_j}z_i)_{y_i}}{(q^{k_j}/\omega)_{y_i}} \right).
\]

In the notation (7.2), the sum in Corollary (7.1) can be written

\[
V_m \left( \frac{q^{L-M-|k|}}{\omega}; q^{L-M-|k|}, q^{1+L-|k|}, \ldots, q^{L-M-|k|}, q^{1+L-M-|k|}, q^{1+L-M-|k|}, \ldots, q^{1+L-M-|k|}; q^{k_1}, q^{k_1}, \ldots, q^{k_1}, q^{k_2}, q^{k_2}, \ldots, q^{k_2}, \ldots, q^{k_m}, q^{k_m}, \ldots, q^{k_m}; q^{\xi_1}, q^{\xi_1}, \ldots, q^{\xi_1}, q^{\xi_2}, q^{\xi_2}, \ldots, q^{\xi_2}, \ldots, q^{\xi_n}, q^{\xi_n}, \ldots, q^{\xi_n} \right).
\]

Note that, since $L = s + |U| = t + |V|$ and (7.10) holds, the series is balanced. Making the same specialization in the second expression of Theorem 6.13 one finds that $R_{SU}^{TY}(\lambda; w; z)$ is an elementary factor times

\[
V_n \left( q^{L-|k|}; q^{L-M-|k|}, q^{1+L-|k|}, q^{1+L-M-|k|}, q^{1+L-M-|k|}, q^{1+L-M-|k|}, \ldots, q^{1+L-M-|k|}; q^{k_1}, q^{k_1}, \ldots, q^{k_1}, q^{k_2}, q^{k_2}, \ldots, q^{k_2}, \ldots, q^{k_m}, q^{k_m}, \ldots, q^{k_m}; q^{\xi_1}, q^{\xi_1}, \ldots, q^{\xi_1}, q^{\xi_2}, q^{\xi_2}, \ldots, q^{\xi_2}, \ldots, q^{\xi_n}, q^{\xi_n}, \ldots, q^{\xi_n} \right).
\]

The fact that these two expressions agree is an instance of (7.5). Thus, we have obtained an algebraic proof of this transformation. (Although we only obtain (7.5) under an additional discreteness condition on the parameters, that condition can be removed by analytic continuation similarly as in the proof of [R2, Cor. 5.3].)
In [7.3] it will be convenient to use the expression obtained by replacing \(y_i\) by \(k_i - y_i\) in Corollary [7.1]. When \(L \leq M\), the condition \(|y| \leq |k| + M - L\) is trivially satisfied, and we find that \(\mathcal{R}_{SU}^{TV}(\lambda; w; z)\) equals

\[
q^{s+N-M-|V|}|k| (q)_M (q)_{-s} (q)_{L} (q^\lambda+2+M+N-2L) (q^\lambda+2+M-2t)t (q^\lambda+2+M+N-2L)|V| (q^\lambda-2t-M)|V| \times \prod_{i \in U \cap N, j \in U \cap N^c} \frac{\theta(q z_i / z_j)}{\theta(z_i / z_j)} \prod_{i \in U \cap N^c} \left( \frac{\theta(q^\lambda+1+M+N-L-|U| \omega / z_i)}{\theta(q^\lambda+1+M-N-L-|U| \omega / z_i)} \prod_{j=1}^{m} \frac{\theta(\eta_j q^{k_j} / z_i)}{\theta(\eta_j / z_i)} \right) \\
\times \prod_{i \in U \cap N^c} \left( \frac{\theta(q^{-\lambda-1+L} \omega / z_i)}{\theta(q^\lambda \omega / z_i)} \right)^n \prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{(\eta_i \xi_j)_{k_i + t_i}}{(\eta_i \xi_j)_{k_i}} \times V_n (q^{M-L} \omega, q^{M-L+1} \xi_1 \omega, \ldots, q^{M-L+n} \xi_n \omega; \omega, q^{M+1} \omega, q / \xi_1, \ldots, q / \xi_n, \eta_1, \ldots, \eta_m; q^{-k_1} / \eta_1, \ldots, q^{-k_m} / \eta_m). (7.11)
\]

This expression remains valid for \(L > M\), if interpreted as

\[
\frac{1}{(q)_{M-L}} \sum_{0 \leq |y| \leq k_i} \frac{\left(\cdots\right)}{(q^{1+M-L})_{|y|}} = \sum_{0 \leq |y| \leq k_i, |y| \geq L-M} \frac{\left(\cdots\right)}{(q)_{M-L+|y|}}.
\]

### 7.3. Biorthogonal functions.

We have seen that, under appropriate specialization of the parameters, \(\mathcal{R}_{SU}^{TV}\) can be written in terms of the multiple elliptic hypergeometric series \(V_n\). Making similar specializations in Proposition [6.1], one obtains new results for such series. We will only consider the unitarity relation [6.3], and show that it leads to a system of biorthogonal functions of type \(V_n\), generalizing the functions of type \(V_1 = 12V_{11}\) studied by Spiridonov and Zhedanov [SZ].

The functions that we will describe depend, apart from \(p\) and \(q\), on \(2n + 3\) parameters \(a, b, c, x_1, \ldots, x_n, N_1, \ldots, N_n\), with \(N_i\) non-negative integers. Fixing all these parameters, let

\[
f_{u_1, \ldots, u_n}(y_1, \ldots, y_n) = V_n \left( \frac{a}{b}; aq^{y_1}, c q^{u_1}, \frac{aq^{1+|N_1|}x_1}{b}, \ldots, \frac{aq^{1+N_n}x_n}{b}, \frac{1}{b^2 c}, \frac{q^{-u_1}}{x_1}, \ldots, \frac{q^{-y_n}}{x_n}, \frac{q^{-u_1}}{x_1}, \ldots, \frac{q^{-u_n}}{x_n}; x_1, \ldots, x_n \right),
\]

for \(0 \leq |u_i| \leq k_i\).
where it is assumed that $u_i$ are integers with $0 \leq u_i \leq N_i$.

Note that, by (7.5), $g_{u_1,\ldots,u_n}(y_1,\ldots,y_n)$ can alternatively be expressed as an elementary factor times

$$
V_n^m \left( \frac{q^{-[N]}b}{a}; \frac{q^{-[y]-[N]}}{a}, \frac{q^{-[u]-[N]}}{c}, \frac{q^{1-[N]}b}{ax_1}, \ldots, \frac{q^{1-[N]}b}{ax_n}; \frac{q^{[N]}b^2c}{a}, q^{y_n}x_1, \ldots, q^{y_n}x_n, \right).
$$

Although it may seem more natural to normalize $g$ to be the latter $V_n^m$-series (without any prefactor), we prefer the definition (7.12) since it exhibits a simpler dependence on the variables $y_i$.

**Theorem 7.2.** The functions defined above satisfy the biorthogonality relations

$$
\sum_{y_1,\ldots,y_n=0}^{N_1,\ldots,N_n} w(y)f_u(y)g_v(y) = \delta_{u,v}\Gamma_u,
$$

where

$$
w(y) = \frac{\Delta(xqy)}{\Delta(x)} q^{[y]} \frac{\theta(aq^2|y|)}{\theta(a) q^{1-[N]}|y|} \prod_{i,j=1}^{n} \frac{q^{-N_i}x_i/x_j}{(qx_i/x_j)^{y_i}} \times \prod_{i=1}^{n} \frac{\theta(bq|y|-y_i/x_i)(b/x_i)|y|q^{[N]}ax_i/b}{\theta(b/x_i)(bq^{1-[N]}x_i)/y_i(aq^2x_i/b)^{y_i}},
$$

$$
\Gamma_u = c^{[N]} q^{[N]^2-[u]} \frac{\Delta(x)}{\Delta(xq^u)} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)^{u_i}}{(q^{-N_i}x_i/x_j)^{u_i}} \frac{(aq)^N(q^{-u}-[N]/c)(q^{-u}-|u|)(q^{1-2[u]}/c)|u|}{(aq/b, bcq^{[N]})_{[N]}},
$$

$$
\times \prod_{i=1}^{n} \left( \frac{\theta(q^{-u}ax_i/bc)}{\theta(q^{-u}ax_i/bc)} \frac{(x_i/aq^{1-[N]}x_i/b^2c)_{N_i}(q^{u_i}+q^{[N]}ax_i/b)_{N_i-u_i}}{(x_i/b, q^{-u}ax_i/bc)_{N_i} (q^{1+u}ax_i/b)_{N_i-u_i}} \right).
$$

When $n = 1$, the biorthogonal functions in Theorem 7.2 reduce to the one-variable functions of Spiridonov and Zhedanov [SZ]. We will briefly discuss two features that distinguish the one- and multivariable case. First of all, in self-explaining notation, repeated use of (7.11) yields that $g_{u}(y; a, b, c, x, N)$ equals an elementary prefactor times $f_u(y; a, b, c, x/q, N)$. Thus, the system is “almost” orthogonal in the sense that $f_u$ and $g_u$ are related by a parameter shift. In the multivariable case, no analogous relation seems to exist.
Another peculiar property of the one-variable case is that $f_u$ and $g_u$ can be viewed as rational functions. To see this, note that

$$f_u(y) = \sum_{k=0}^{u} C_k \frac{(aq^y, q^{-y})_k}{(q^{1-y}/b, aq^{-y+1}/b)_k},$$

with $C_k$ independent of $y$. It follows from classical facts on elliptic functions that $f_u$ is rational in the variable

$$\frac{\theta(sq^y, sq^{-y}/a)}{\theta(tq^y, tq^{-y}/a)},$$

with $s$ and $t$ arbitrary generic parameters. (Geometrically, identifying antipodal points on a complex torus gives the Riemann sphere.) In the multivariable case, there seems to be no analogue of this rational parametrization.

Before explaining how Theorem 7.2 can be obtained from our findings above, we indicate a direct proof. We will use an explicit matrix inversion found in [RoS]; see [Sc] for the case $p = 0$. Namely, for $k, l, m$ multi-indices with $l_i \leq k_i \leq m_i$, $i = 1, \ldots, n$, let

$$A_{mk}(a, b) = \frac{(abq^{2|k|})_m - |k| \prod_{i=1}^{n} (aq^{k_i - |k|}/x_i)_m - |k| \prod_{i=1}^{n} (q^{1+k_i - k_j x_i/x_j})_{m - k_i}}{\prod_{i=1}^{n} (bx_i q^{1+k_i + |k|})_{m - k_i} \prod_{i,j=1}^{n} (q^{1+k_i - k_j x_i/x_j})_{m - k_i}}.$$

$$B_{kl}(a, b) = (-1)^{|k| - |l|} q^{(|k| - |l|)/2} \frac{\theta(abq^{2|k|}) \prod_{i=1}^{n} \theta(aq^{l_i - |k_i|}/x_i)}{\theta(abq^{2|k|}) \prod_{i=1}^{n} \theta(aq^{k_i - |k_i|}/x_i)} \times \frac{(abq^{1+|l| + |k|})_m - |l| \prod_{i=1}^{n} (aq^{1+|l| - k_i}/x_i)_m - |l| \prod_{i,j=1}^{n} (q^{1+l_i - l_j x_i/x_j})_{k_i - l_i}}{\prod_{i=1}^{n} (bx_i q^{1+k_i + |k|})_{k_i - l_i} \prod_{i,j=1}^{n} (q^{1+l_i - l_j x_i/x_j})_{k_i - l_i}}.$$

Then, $B = A^{-1}$, that is, the equivalent identities

$$\sum_{k} A_{mk}(a, b) B_{kl}(a, b) = \delta_{lm} = \sum_{k} B_{mk}(a, b) A_{kl}(a, b)$$

hold. In fact, the first relation is equivalent to the case $aq = ce$ of (7.6), while the second one is the case $aq = bc$ of (7.7).

Let $C_s$ be an arbitrary sequence, labelled by multi-indices $s$ such that $0 \leq s_i \leq N_i$, $i = 1, \ldots, n$. Then,

$$\sum_{y} \sum_{s} C_s A_{us}(a, b) A_{ys}(c, d) \sum_{t} \frac{1}{C_{N-t}} B_{N-t,u}(a, b) B_{N-t,y}(c, d)$$

$$= \sum_{st} \frac{C_s}{C_{N-t}} A_{us}(a, b) B_{N-t,u}(a, b) \delta_{s,N-t} = \sum_{s} A_{us}(a, b) B_{sv}(a, b) = \delta_{uv}.$$
A straight-forward computation reveals that Theorem 7.2 corresponds to the special case when \((a, b, c, d) \mapsto (cb/a, a/b, b/a/b)\) and

\[
C_s = q^{2\sum\ell_i<\ell_j}s_{\ell_j}} \prod_{i=1}^{n} x_i^{2\ell_i} \left( \frac{b^2c}{qa} \right)^{|s|} \frac{\Delta(x)}{\Delta(xq^s)} \frac{(a, c)_{2|s}}{(aq/b, q^{1|N}|bc)_{|s|}} \times \prod_{i=1}^{n} \frac{(ax_i/b, aq^{1+N}x_i/b)_{|s|}(b/x_i, bc/ax_i)_{|s|-s_i}(x_i, aq^{1-|N|x_i/b^2c})_{s_i}}{(ax_i/b, aqx_i/b)_{|s|+s_i} \prod_{j=1}^{n} (q^x_i/x_j, q^{-N_j}x_i/x_j)_{s_i}}.
\]

Clearly, the same proof can be used to obtain more general, or different, biorthogonal systems.

Finally, we explain how Theorem 7.2 can be obtained from (6.3). Since the details of the computations are of little interest, we will be quite brief. First, we specialize \(w, S\) and \(T\) to (7.8). We also assume that

\[
z = (\zeta_1, \ldots, \zeta_1q^{N_1-1}, \ldots, \zeta_n, \ldots, \zeta_nq^{N_n-1}),
\]
\[
z_U = (\zeta_1q^{N_1-u_1}, \ldots, \zeta_1q^{N_1-1}, \ldots, \zeta_nq^{N_n-u_n}, \ldots, \zeta_nq^{N_n-1}),
\]
\[
z_V = (\zeta_1q^{N_1-u_1}, \ldots, \zeta_1q^{N_1-1}, \ldots, \zeta_nq^{N_n-u_n}, \ldots, \zeta_nq^{N_n-1}).
\]

Consider the symbol \(R_{SU}^{XY}(\lambda; w; z)\) in (6.3). By Corollary 6.5, it vanishes identically unless \(X = [M - x + 1, M]\) and

\[
z_Y = (\zeta_1q^{N_1-y_1}, \ldots, \zeta_1q^{N_1-1}, \ldots, \zeta_nq^{N_n-y_n}, \ldots, \zeta_nq^{N_n-1}).
\]

This means that \(R_{SU}^{XY}\) can be expressed as in (7.11), where \(M, N, L, s, \lambda, \omega\) are unchanged, while the remaining parameters are replaced by \(t \mapsto L - |y|, |U| \mapsto L - s, |V| \mapsto |y|, k_j \mapsto \min(u_j, y_j), l_j \mapsto N_j - \max(u_j, y_j), \eta_j \mapsto q^{1-\max(u_j, y_j)} \xi_j, \xi_j \mapsto q^{1-N_j + \max(u_j, y_j)}\). As for the other generalized 6\(j\)-symbol in (6.3), we first apply Corollary 6.3 to write

\[
R_{XY}^{VT}(\lambda; z; w) = R_{X\setminus Y\setminus C}(\lambda + M + N - 2L; w^{-1}; z^{-1}).
\]

We can then express it as in (7.11), where \(M\) and \(N\) are unchanged, while \(L \mapsto M + N - L, s \mapsto M + |y| - L, t \mapsto M - t, |U| \mapsto N - |y|, |V| \mapsto N + t - L, \lambda \mapsto \lambda + M + N - 2L, \omega \mapsto q^{1-M} \omega, k_j \mapsto N_j - \max(v_j, y_j), l_j \mapsto \min(u_j, y_j), \eta_j \mapsto q^{1+\max(v_j, y_j)} - N_j, \xi_j \mapsto q^{N_j - \min(v_j, y_j)} \xi_j\). Inserting these explicit formulas in (6.3), it reduces to (7.13), where \(a = q^{\lambda+1+M-2L}, b = q^{\lambda+1-L}, c = q^{N-1}, x_j = q^{-N_j} / \xi_j\). Since \(M\) and \(L\) are non-negative integers with \(L \leq M + N\), we only obtain (7.13) under additional discreteness conditions on the parameters. However, these conditions can be removed by analytic continuation, similarly as in the proof of [12], Cor. 5.3. In that sense, we have obtained an algebraic proof of Theorem 7.2.

**Appendix. Algebra symmetries.**

The equations (2.25), (3.7), (3.8) and (3.9) reflect different forms of unitarity for symmetries of cobraised \(b\)-bialgebroids. We will describe how to encompass these
in a general framework, which in particular simplifies the proof of Proposition 3.3. It turns out that there are four types of unitarity, corresponding to a choice of direct or opposite product and coproduct. Moreover, one can incorporate twists by affine automorphisms of $\mathfrak{h}^*$ (e.g. the map $\lambda \mapsto -\lambda - 2$ in Proposition 3.3).

For $A$ an $\mathfrak{h}$-bialgebroid, two opposite $\mathfrak{h}$-bialgebroid structures $A^{\text{op}}$ and $A^{\text{coop}}$ on the complex vector space underlying $A$ were introduced in [KoN]. We will write $A^{\text{coop}} = (A^{\text{op}})^{\text{op}} = (A^{\text{op}})^{\text{coop}}$. The bigradings on the opposite $\mathfrak{h}$-bialgebroids are given by

$$A_{\alpha \beta}^{\text{op}} = A_{-\alpha, -\beta}, \quad A_{\alpha \beta}^{\text{coop}} = A_{\beta \alpha}, \quad A_{\alpha \beta}^{\text{coop}} = A_{-\beta, -\alpha}.$$ 

The moment maps are given by

$$\mu_i^{A^{\text{op}}}(f)x = x\mu_i^A(f), \quad \mu_i^{A^{\text{coop}}}(f)x = \mu_i^A(f)x, \quad \mu_i^{A^{\text{coop}}}(f)x = x\mu_i^A(f),$$

and the same equations with $l$ and $r$ interchanged. The product on $A^{\text{op}}$ is the same as that on $A$, while $A^{\text{op}}$ and $A^{\text{coop}}$ are equipped with the opposite product $m^A \circ \sigma$. The coproduct on $A^{\text{op}}$ is the same as that on $A$, while $A^{\text{cop}}$ and $A^{\text{coop}}$ have the opposite product $\sigma \circ \Delta^A$. The counit on $A^{\text{op}}$ is the same as that on $A$, while $A^{\text{op}}$ and $A^{\text{coop}}$ have counit $S^Dh \circ \varepsilon^A$. Finally, if $A$ is an $\mathfrak{h}$-Hopf algebroid with invertible antipode, then so are the opposite structures, with antipode $S^{A^{\text{cop}}} = S^A$, $S^{A^{\text{op}}} = S^{A^{\text{coop}}} = (S^A)^{-1}$.

Let $\chi^{\mathfrak{h}^*}$ be a linear automorphism of $\mathfrak{h}^*$ and $\chi^{M_{\mathfrak{h}^*}}$ a field automorphism of $M_{\mathfrak{h}^*}$ satisfying

$$\chi^{M_{\mathfrak{h}^*}} \circ T_\alpha = T_{\chi^{\mathfrak{h}^*}(\alpha)} \circ \chi^{M_{\mathfrak{h}^*}}.$$ 

(A.1)

For instance, given an invertible affine map $\lambda \mapsto A\lambda + \lambda_0$ on $\mathfrak{h}^*$ one may define

$$\chi^{\mathfrak{h}^*}(\lambda) = \lambda^{-1}, \quad \chi^{M_{\mathfrak{h}^*}}(f)(\lambda) = f(A\lambda + \lambda_0).$$ 

(A.2)

It follows from (A.1) that

$$\chi^{Dh}(fT_\alpha) = \chi^{M_{\mathfrak{h}^*}}(f)T_{\chi^{\mathfrak{h}^*}(\alpha)}$$

defines an algebra automorphism $\chi^{Dh}$ of $D_h$. From now on, we suppress the upper indices, denoting all three automorphisms by $\chi$. We will also write

$$\chi^{\text{op}} = S^{Dh} \circ \chi^{Dh} = \chi^{Dh} \circ S^{Dh}.$$

Next, we recall some rudiments of the duality theory for $\mathfrak{h}$-bialgebroids [RI §3.1]. It will be convenient to write $\{x, \xi\} = \xi(x)$, where $\xi$ is a $\mathbb{C}$-linear map from an $\mathfrak{h}$-bialgebroid $A$ to $D_h$. Let $A'$ be the space of such maps $\xi$ such that

$$\{\mu_i(f)x, \xi\} = f \circ \{x, \xi\}, \quad \{x\mu_i(f), \xi\} = \{x, \xi\} \circ f.$$

It is an associative algebra with product

$$\{x, \xi \eta\} = \sum_{(x)} \{x', \xi\} T_{\omega(x)} \{x'', \eta\}$$

and unit element $\varepsilon$. 
Lemma A.1. Fix $\chi$ as above, and let $A$ and $B$ be two $\mathfrak{h}$-bialgebroids. Let $\phi : A \to B$ be a $\mathbb{C}$-linear map such that $\phi(A_{\alpha\beta}) \subseteq B_{\chi^{-1}(\alpha),\chi^{-1}(\beta)}$, 

$$\phi(\mu_t^A(f)x) = \mu_t^B(\chi^{-1}(f))\phi(x), \quad \phi(\mu_r^A(f)x) = \mu_r^B(\chi^{-1}(f))\phi(x).$$ 

Then, $\{x, \phi'(|\xi\rangle\} = \chi(\{\phi(|\xi\rangle\}$ defines a map $\phi' : B' \to A'$. Moreover, if 

$$(\phi \otimes \phi) \circ \Delta^A = \Delta^B \circ \phi, \quad \chi \circ \varepsilon^B \circ \phi = \varepsilon^A,$$

then $\phi'$ is an algebra homomorphism.

The proof is straight-forward.

Lemma A.2. Let $A$ be an $\mathfrak{h}$-bialgebroid equipped with a cobrading. For $s \in \{\emptyset, \text{op}, \text{cop}, \text{coop}\}$, there is an algebra homomorphism $i^s : A^s \to (A^s)'$ given by 

$$\{y, i(x)\} = \langle x, y \rangle, \quad \{y, i_{\text{op}}(x)\} = S^{D_h}(\langle y, x \rangle),$$

$$\{y, i_{\text{cop}}(x)\} = \langle y, x \rangle, \quad \{y, i_{\text{coop}}(x)\} = S^{D_h}(\langle x, y \rangle).$$

Again, the proof is straight-forward.

Definition A.3. Let $A$ be an $\mathfrak{h}$-bialgebroid equipped with a cobrading. Fix $\chi$ as above, and let $s \in \{\emptyset, \text{op}, \text{cop}, \text{coop}\}$. Then, a map $\phi : A \to A$ is called $(\chi, s)$-unitary if, when viewed as a map $A \to A^s$, it is an algebra homomorphism, satisfies all conditions of Lemma [A.1], and

$$\phi' \circ i^s \circ \phi = i.$$ 

(A.3)

More explicitly, $\phi$ is $(\chi, \text{id})$-unitary if it is an algebra homomorphism and

$$\phi(\mu_t(f)) = \mu_t(\chi^{-1}(f)), \quad \phi(\mu_r(f)) = \mu_r(\chi^{-1}(f)),$$

$$\phi(A_{\alpha\beta}) \subseteq A_{\chi^{-1}(\alpha),\chi^{-1}(\beta)}, \quad (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi, \quad \chi \circ \varepsilon \circ \phi = \varepsilon,$$

$$\langle x, y \rangle = \chi(\langle \phi(x), \phi(y) \rangle);$$

(A.4a)

it is $(\chi, \text{op})$-unitary if it is an algebra antihomomorphism and

$$\phi(\mu_t(f)) = \mu_t(\chi^{-1}(f)), \quad \phi(\mu_r(f)) = \mu_r(\chi^{-1}(f)),$$

$$\phi(A_{\alpha\beta}) \subseteq A_{-\chi^{-1}(\alpha),-\chi^{-1}(\beta)}, \quad (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi, \quad \chi^{\text{op}} \circ \varepsilon \circ \phi = \varepsilon,$$

$$\langle x, y \rangle = \chi^{\text{op}}(\langle \phi(y), \phi(x) \rangle);$$

(A.4b)

it is $(\chi, \text{cop})$-unitary if it is an algebra homomorphism and

$$\phi(\mu_t(f)) = \mu_r(\chi^{-1}(f)), \quad \phi(\mu_r(f)) = \mu_l(\chi^{-1}(f)),$$

$$\phi(A_{\alpha\beta}) \subseteq A_{\chi^{-1}(\beta),\chi^{-1}(\alpha)}, \quad \sigma \circ (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi, \quad \chi \circ \varepsilon \circ \phi = \varepsilon,$$
\begin{align}
\langle x, y \rangle &= \chi(\langle \phi(y), \phi(x) \rangle); \quad \text{(A.4c)}
\end{align}

and, finally, \( \phi \) is \((\chi, \text{coop})\)-unitary if it is an algebra antihomomorphism and
\[
\phi(\mu_l(f)) = \mu_r(\chi^{-1}(f)), \quad \phi(\mu_r(f)) = \mu_l(\chi^{-1}(f)),
\]
\[
\phi(A_{\alpha\beta}) \subseteq A_{-\chi^{-1}(\beta), -\chi^{-1}(\alpha)},
\]
\[
\sigma \circ (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi, \quad \chi^{\text{op}} \circ \varepsilon \circ \phi = \varepsilon,
\]
\[
\langle x, y \rangle = \chi^{\text{op}}(\langle \phi(x), \phi(y) \rangle). \quad \text{(A.4d)}
\]

If \( A \) is equipped with an invertible antipode, then it is natural to require
\[
\phi \circ S = S^{A^s} \circ \phi, \quad \text{(A.5)}
\]
that is, \( \phi \circ S = S \circ \phi \) for \( s \in \{\emptyset, \text{cop}\} \) and \( \phi \circ S = S^{-1} \circ \phi \) for \( s \in \{\text{op, coop}\} \).

Since \((\chi, s)\) is an equality between compositions of algebra homomorphisms, it is natural in the sense that if the two sides agree on two elements \( x \) and \( y \), they agree on \( xy \). We also need a dual version of this naturality. To this end, we observe that if \( \phi \) satisfies all conditions of Lemma \( (A.1) \) then this is also true when \( \phi \) is viewed as a map from \( A^{\text{cop}} \) to \( B^{\text{cop}} \). It is then easy to check that \((A.3)\) is equivalent to
\[
\phi' \circ (i^s)^{\text{cop}} \circ \phi = i^{\text{cop}},
\]
when \( \phi \) is viewed as a map \( A^{\text{cop}} \to (A^s)^{\text{cop}} \). Together, the naturality properties of these two versions of \((A.3)\) mean that, assuming the other conditions of \( \phi \), if one of the equalities \((A.4)\) hold with \( (x, y) \) replaced by \( (x_1, y) \) and \( (x, y_2) \) it holds also for \( (x_1x_2, y) \), and if it holds for \( (x, y_1) \) and \( (x, y_2) \), it holds also for \( (x, y_1y_2) \). Thus, it is enough to check unitarity on a set of generators. Having made this observation, the proof of Proposition \( (A.3) \) is reduced to straight-forward verification.

If we let \( \chi_0 = \psi_{D_h} \) denote the automorphism constructed as in \((A.2)\) from the affine map \( \lambda \mapsto -\lambda - 2 \) of \( h^* \) = \( \mathbb{C} \), then we can give examples of eight types of \((\chi, s)\)-unitary maps according to the following table (in each case, the additional axiom \((A.3)\) is valid):

|   | id | * | * \circ S | S | \psi | \phi | \phi \circ S | \psi \circ S |
|---|----|---|-----------|---|-----|-----|-------------|-------------|
| \chi | id | id | id | id | \chi_0 | \chi_0 | \chi_0 | \chi_0 |
| s   | op | cop | coop | id | op | coop | coop | coop |

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