Coulomb’s law modification in nonlinear and in noncommutative
electrodynamics

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Abstract

We study the lowest-order modifications of the static potential for Born-Infeld electrodynamics and for the $\theta$-expanded version of the noncommutative $U(1)$ gauge theory, within the framework of the gauge-invariant but path-dependent variables formalism. The calculation shows a long-range correction ($1/r^5$-type) to the Coulomb potential in Born-Infeld electrodynamics. However, the Coulomb nature of the potential (to order $e^2$) is preserved in noncommutative electrodynamics.

PACS number(s): 11.10.Ef, 11.10.Nx

I. INTRODUCTION

Renewed interest in non-linear electrodynamics (Born-Infeld theory) has originated in recent string theory investigations. This is primarily because the low energy dynamics of D-branes have been described by a nonlinear Born-Infeld type action [1,2]. It is worth recalling at this stage that Born and Infeld [3] suggested to modify Maxwell’s electromagnetism so as

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to get rid of the divergencies of the theory such as the infinite self-energy of a point charge. The resulting theory is a nonlinear gauge theory endowed with interesting features, like finite electron self-energy and a regular point charge electric field at the origin. However, due to the nonlinearity, the corresponding field equations are very difficult to solve. In addition to the string interest, the Born-Infeld theory has also attracted considerable attention from different viewpoints. For example, in connection to duality symmetry [4,5], also in magnetic monopoles studies [6], in generation of multipole moments for charged particles [7], and possible experimental determination of the parameter that measures the nonlinearity of the theory [8]. The advent of noncommutative field theories also called attention to nonlinear theories. In fact, recently there have been indications that the $\theta$-expanded version of the noncommutative U(1) gauge theory is equivalent to the expansion of the Born-Infeld action up to order $F^3$ [9], which is something that we intend to check in the present work, for the specific case of the static potential between charges.

On the other hand, one may improve our understanding of gauge theories through a proper study of the concepts of screening and confinement. In this respect the interaction energy of an infinitely heavy quark-antiquark pair is a key tool which plays an important role in the understanding of quark confinement. Moreover, the static potential is an essential concept both in electrodynamics and in gravitation. The importance of the static potential is further shown, for example, in the description of non-relativistic bound systems like quarkonia, as well as in the definition of the lattice coupling. In actual calculations it is obtained most directly when a correct separation between gauge-invariant and gauge dependent degrees of freedom is made. Previously, we proposed a general framework for studying the confining and screening nature of the potential in gauge theories in terms of gauge-invariant but path-dependent field variables [10]. An important feature of this methodology is that it provides a physically-based alternative to the usual Wilson loop approach. In this paper, we will examine another aspect of nonlinear theories, namely, the lowest-order modification of the static potential due to the presence of Born-Infeld type terms, and in noncommutative electrodynamics. We address this issue along the lines of reference [10].
In Sec. II we will calculate the lowest-order correction to the Coulomb energy of a fermion-antifermion system, for both Born-Infeld electrodynamics and for the $\theta$-expanded version of noncommutative U(1) gauge theory. Our calculations show that the static interaction of fermions in Abelian gauge theories is determined by the geometrical condition of gauge invariance.

II. INTERACTION ENERGY

A. Two-dimensional Born-Infeld Electrodynamics

As already stated, our principal purpose is to calculate explicitly the interaction energy between static pointlike sources for Born-Infeld electrodynamics. To this end we will calculate the expectation value of the energy operator $H$ in the physical state $|\Phi\rangle$, which we will denote by $\langle H \rangle_\Phi$. However, before going into the four-dimensional Born-Infeld electrodynamics, we would like to first consider the two-dimensional case. This would not only provide the theoretical setup for our subsequent work, but also fix the notation. The starting point is the two-dimensional space-time Lagrangian:

$$L = \beta^2 \left\{ 1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}} \right\} - A_0 J^0, \quad (1)$$

where $J^0$ is the external current. The parameter $\beta$ measures the nonlinearity of the theory and in the limit $\beta \to \infty$ the Lagrangian (1) reduces to the Maxwell theory. In order to handle the square root in the Lagrangian (1) we introduce an auxiliary field $v$, such that its equation of motion gives back the original theory [1]. This allows us to write the Lagrangian as

$$L = \beta^2 \left\{ 1 - \frac{v}{2} \left( 1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} \right) - \frac{1}{2v} \right\} - A_0 J^0. \quad (2)$$

Once this is done, the canonical quantization of this theory from the Hamiltonian analysis point of view is straightforward and follows closely that of references [4,5]. The canonical
momenta read $\Pi^\mu = -vF^{0\mu}$, and one immediately identifies the two primary constraints $\Pi^0 = 0$ and $p \equiv \frac{\partial F}{\partial v} = 0$. The canonical Hamiltonian is thus

$$H_C = \int dx \left\{ -\beta^2 + \Pi_1 \partial^1 A^0 + \frac{1}{2v} \left( \beta^2 - \Pi_1 \Pi^1 \right) + \frac{\beta^2}{2} v + A_0 J^0 \right\}. \quad (3)$$

The consistency condition $\dot{\Pi}_0 = 0$ leads to the secondary constraint $\Gamma_1 (x) \equiv \partial_1 \Pi^1 - J^0 = 0$. The consistency condition for the $p$ constraint yields no further constraints and just determines the field $v$,

$$v = \sqrt{1 - \frac{1}{\beta^2} \Pi_1 \Pi^1}, \quad (4)$$

which will be used to eliminate $v$. The extended Hamiltonian that generates translations in time then reads $H = H_C + \int dx \left( c_0 (x) \Pi_0 (x) + c_1 (x) \Gamma_1 (x) \right)$, where $c_0 (x)$ and $c_1 (x)$ are the Lagrange multipliers. Since $\Pi_0 = 0$ for all time and $\dot{A}_0 (x) = [A_0 (x), H] = c_0 (x)$, which is completely arbitrary, we discard $A_0 (x)$ and $\Pi_0 (x)$ because they add nothing to the description of the system. Then, the Hamiltonian takes the form

$$H = \int dx \left\{ \beta^2 \left( \sqrt{1 - \frac{1}{\beta^2} \Pi_1 \Pi^1} - 1 \right) - c' (x) \left( \partial_1 \Pi^1 - J^0 \right) \right\}, \quad (5)$$

where $c' (x) = c_1 (x) - A_0 (x)$.

The quantization of the theory requires the removal of non-physical variables, which is done by imposing a gauge condition such that the full set of constraints becomes second class. A convenient choice is found to be [10]

$$\Gamma_2 (x) \equiv \int d^\nu A_\nu (z) \equiv \int_0^1 d\lambda x^1 A_1 (\lambda x) = 0, \quad (6)$$

where $\lambda (0 \leq \lambda \leq 1)$ is the parameter describing the spacelike straight path $x^1 = \xi^1 + \lambda (x - \xi)^1$, and $\xi$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^1 = 0$. With this choice the nontrivial Dirac bracket is given by

$$\left\{ A_1 (x), \Pi^1 (y) \right\}^* = \delta^{(1)} (x - y) - \partial_1^\xi \int_0^1 d\lambda x^1 \delta^{(1)} (\lambda x - y). \quad (7)$$
Having outlined the necessary aspects of quantization, we now turn to the problem of obtaining the interaction energy between pointlike sources in Born-Infeld theory, where a fermion is localized at the origin $0$ and an antifermion at $y$. As we have already indicated, we will calculate the expectation value of the energy operator $H$ in the physical state $|\Phi\rangle$.

From (5) we then get for the expectation value

$$\langle H \rangle_\Phi = \langle \Phi | \int dx \left\{ \beta^2 \left( \sqrt{1 - \frac{1}{\beta^2} \Pi_1 \Pi^1} - 1 \right) \right\} |\Phi\rangle.$$

As remarked by Dirac [11], the physical state can be written as

$$|\Phi\rangle \equiv |\bar{\Psi}(y) \Psi (0)\rangle = \bar{\psi}(y) \exp \left( ie \int^{y}_{0} dz A_i (z) \right) \psi(0) |0\rangle,$$

where $|0\rangle$ is the physical vacuum. As before, the line integral appearing in the above expression is along a spacelike path starting at $0$ and ending at $y$, on a fixed time slice. It is worth noting here that the strings between fermions have been introduced in order to have a gauge-invariant function $|\Phi\rangle$. In other terms, each of these states represents a fermion-antifermion pair surrounded by a cloud of gauge fields sufficient to maintain gauge invariance.

Since we are interested in estimating the lowest-order correction to the Coulomb energy, we will retain only the leading quadratic term in the expression (8). Thus the expectation value simplifies to

$$\langle H \rangle_\Phi = \langle \Phi | \int dx \left\{ \frac{1}{2} (\Pi_1)^2 - \frac{1}{8\beta^2} (\Pi_1)^4 \right\} |\Phi\rangle.$$

It is easy to see that the first term inside the curly bracket comes from the usual Maxwell theory while the second one is a correction which comes from the Born-Infeld modification.

From our above Hamiltonian analysis we observe that

$$\Pi_1 (x) |\bar{\Psi}(y) \Psi (0)\rangle = \bar{\Psi}(y) \Psi (0) \Pi_1 (x) |0\rangle - e \int^{y}_{0} dz_1 \delta^{(1)} (z_1 - x) |\Phi\rangle.$$

Substituting this back into (10), we obtain

$$\langle H \rangle_\Phi = \langle H \rangle_0 + \frac{e^2}{2} \int dx \left( \int^{y}_{0} dz_1 \delta^{(1)} (z_1 - x) \right)^2 - \frac{e^4}{8\beta^2} \int dx \left( \int^{y}_{0} dz_1 \delta^{(1)} (z_1 - x) \right)^4,$$
where $\langle H \rangle_0 = \langle 0 | H | 0 \rangle$. We further note that

$$\frac{e^2}{2} \int dx \left( \int_0^y dz \delta^1 (z_1 - x) \right)^2 = \frac{e^2}{2} L, \quad (13)$$

with $|y| \equiv L$. By employing Eq. (13) we can reduce Eq. (12) to

$$V = \frac{e^2}{2} \left( 1 - \frac{e^2}{4\beta^2} \right) L. \quad (14)$$

Hence we see that the static interaction between fermions arises only because of the requirement that the $\left| \nabla \Psi \right>$ states be gauge invariant. The above result reveals that the effect of adding the Born-Infeld term is to decrease the energy. Nevertheless, the confining nature of the potential is preserved.

Eq. (14) exhibits the same formal structure as the one obtained for the massive Schwinger model. In fact, the same calculation for the massive Schwinger model [10] gives

$$V = \frac{q^2}{2} \left( 1 + \frac{e^2}{4\pi^2 m \Sigma} \right)^{-1} L, \quad (15)$$

where $\Sigma = \left( \frac{e}{2\pi \pi} \right) \exp (\gamma_E)$ with $\gamma_E$ the Euler-Mascheroni constant, $m$ and $e$ are the mass and charge of the dynamical fermions. Here $q$ refers to the probe charges. Considering the limit $m \gg e$ and $q \equiv e$, we get

$$V = \frac{e^2}{2} \left( 1 - \frac{e^2}{4\pi^2 m \Sigma} \right) L. \quad (16)$$

Therefore, for this special case the massive Schwinger model simulates the features of the Born-Infeld theory. This analysis suggests the interesting possibility of identifying the parameter $\beta$ with the mass of the dynamical fermions. This, however, is a separate question and which we do not intend to address here.

Before concluding this subsection we discuss an alternative derivation of the result (14), which highlights certain distinctive features of our methodology. We start by considering

$$V \equiv e \left( A_0 (0) - A_0 (y) \right), \quad (17)$$

where the physical scalar potential is given by
This follows from the vector gauge-invariant field expression [12]

\[ \mathcal{A}_0 (x^0, x^1) = \int_0^1 d\lambda x^1 E_1 (\lambda x^1) \].

(B. Four-dimensional Born-Infeld Electrodynamics)

We now turn our attention to the calculation of the interaction energy between static pointlike sources for the four-dimensional Born-Infeld electrodynamics. In such a case the Lagrangian reads

\[ \mathcal{L} = \beta^2 \left\{ 1 - \frac{1}{\sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F_{\mu\nu} - \frac{1}{16\beta^4} (\mathcal{F}_{\mu\nu} \mathcal{F}_{\mu\nu})^2}} - A_0 J^0 \right\} \]

where \( A_0 J^0 \) is the external current.
Before we proceed to work out explicitly the energy, we shall begin by summarizing the Hamiltonian analysis of the theory (23). Once again, we will introduce an auxiliary field \( v \) to handle the square root in the Lagrangian (23). Expressed in terms of this field, the Lagrangian (23) takes the form
\[
L = \beta^2 \left\{ 1 - \frac{v}{2} \left( 1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F^{\mu\nu} F_{\mu\nu})^2 \right) - \frac{1}{2v} \right\} - A^0 J_0. \tag{24}
\]
With this in hand, the canonical momenta are \( \Pi^\mu = -v \left( F^0^\mu - \frac{1}{4\beta^2} F^\alpha^\beta F^\alpha^\beta F^0^\mu \right) \), and one immediately identifies the two primary constraints \( \Pi^0 = 0 \) and \( p \equiv \frac{\partial L}{\partial v} = 0 \). The canonical Hamiltonian of the model can be worked out as usual and is given by the expression
\[
H_C = \int d^3x \left\{ -\beta^2 + \Pi^i \partial^i A^0 + \frac{1}{2v} \left( \Pi^2 + \beta^2 \right) + \frac{v}{2} \left( B^2 + \beta^2 \right) - \frac{1}{2v\beta^2} \left( \frac{\Pi \cdot B}{1 + \frac{B^2}{\beta^2}} \right)^2 + A^0 J^0 \right\}. \tag{25}
\]
Requiring the primary constraint \( \Pi_0 \) to be preserved in time yields the secondary constraint (Gauss’ law) \( \Gamma_1 (x) \equiv \partial_i \Pi^i - J^0 = 0 \). Similarly for the constraint \( p \), we get the auxiliary field \( v \) as
\[
v = \frac{1}{\beta^2 \left( 1 + \frac{1}{\beta^2} B^2 \right)} \sqrt{\beta^2 (\Pi^2 + \beta^2) \left( 1 + \frac{1}{\beta^2} B^2 \right) - (\Pi \cdot B)^2}. \tag{26}
\]
The extended Hamiltonian that generates translations in time then reads \( H = H_C + \int d^3x \left( c_0(x) \Pi_0(x) + c_1(x) \Gamma_1(x) \right) \), where \( c_0(x) \) and \( c_1(x) \) are Lagrange multipliers. As before, neither \( A_0 (x) \) nor \( \Pi_0 (x) \) are of interest in describing the system and may be discarded from the theory. Thus we are left with the following expression for the Hamiltonian
\[
H = \int d^3x \left\{ \sqrt{\beta^2 (\Pi^2 + \beta^2) \left( 1 + \frac{B^2}{\beta^2} \right) - (\Pi \cdot B)^2} - \beta^2 + c'(x) \left( \partial_i \Pi^i - J^0 \right) \right\}. \tag{27}
\]
where \( c'(x) = c_1(x) - A_0(x) \).

Since our main motivation is to compute the static potential for the Born-Infeld theory, we will adopt the same gauge-fixing condition that was used in the last subsection, that is,
\[
\Gamma_2 (x) \equiv \int_{C_{\xi x}} dz^\nu A_\nu (z) \equiv \int_0^1 d\lambda x^i A_i (\lambda x) = 0. \tag{28}
\]
Here again $\lambda$ ($0 \leq \lambda \leq 1$) is the parameter describing the spacelike straight path $x^i = \xi^i + \lambda (x - \xi)^i$ with $i = 1, 2, 3$, and $\xi$ is a fixed (reference) point. There is no essential loss of generality if we restrict our considerations to $\xi^i = 0$. In this way, the fundamental Dirac bracket can be rewritten as

$$\left\{ A_i(x), \Pi^j(y) \right\}^* = \delta^i_j \delta^{(3)}(x-y) - \partial^i_\lambda \int_0^1 d\lambda x^i \delta^{(3)}(\lambda x - y).$$  (29)

We are now in a position to compute the potential energy for static charges in this theory. To do this, we will use the gauge-invariant scalar potential which is now given by

$$\mathcal{A}_0(t, r) = \int_0^1 d\lambda r E_i(t, \lambda r).$$  (30)

It follows from the above discussion that Gauss’ law takes the form

$$\partial_i \frac{E_i}{\sqrt{1 - \frac{E^2}{\rho^2}}} = J^0.$$  (31)

For $J^0(t, r) = e \delta^{(3)}(r)$, the electric field follows as

$$E^i(r) = \frac{e}{4\pi} \frac{1}{\sqrt{|r|^2 + \rho_0^2}} \hat{r}^i,$$  (32)

where $\rho_0 \equiv \frac{e}{4\pi} \beta^2$ and $\hat{r}^i = \frac{r^i}{|r|}$. From this expression it should be clear that the electric field of a pointlike charge is regular at the origin, in contrast to the usual Maxwell theory. As a consequence, equation (30) becomes

$$\mathcal{A}_0(t, r) = -\frac{e}{4\pi} \int_0^1 d\lambda \frac{r}{\sqrt{(\lambda r)^2 + \rho_0^2}}.$$  (33)

Again, as in the two-dimensional case, it is sufficient to retain the leading quadratic term in Eq. (33). Thus we obtain

$$\mathcal{A}_0(t, r) = -\frac{e}{4\pi r} \int_0^1 d\lambda \left\{ \frac{1}{\lambda^2} + \frac{1}{2} \frac{a^4}{\lambda^6} \right\},$$  (34)

where $a^4 \equiv \frac{\rho_0^2}{\lambda^2} = \frac{e^2}{16\beta^2 \pi^2 r^2}$. In terms of $\mathcal{A}_0(t, r)$, the potential for a pair of static pointlike opposite charges located at $0$ and $L$, that is, $J^0(t, r) = e \left\{ \delta^{(3)}(r) - \delta^{(3)}(r - L) \right\}$, is given by

$$V \equiv e \left( \mathcal{A}_0(0) - \mathcal{A}_0(L) \right) = -\frac{e^2}{4\pi L} \left( 1 + \frac{e^2}{160\pi^2 \beta^2 L^4} \right),$$  (35)

with $L = |L|$.  

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FIGURES

FIG. 1. Shape of \( \log V(L) \) (in units of \( \alpha \equiv \frac{e^2}{4\pi} \)), as a function of the distance \( L \). The dashed line represents the Coulomb potential (in units of \( \alpha \equiv \frac{e^2}{4\pi} \)).

This result shows the usual Coulomb potential with a long-range correction due to the second term of the form

\[
|\Delta V| = \frac{e^4}{640\pi^3\beta^2} \frac{1}{L^5}.
\]  

(36)

To \( \mathcal{O} \left( \frac{1}{\beta^2} \right) \) Born-Infeld electrodynamics displays a marked qualitative departure from the usual Maxwell theory. In Fig.1 we show the effect of the \( \beta \) -correction, for the case \( \beta = 10 \). Note that we have plotted the logarithm of \( V(L) \) as a function of \( L \). It is important to notice that the presence of the second term on the right-hand side of Eq. (35), which dominates for small \( L \) values, causes \( V \) to decrease. Thus, from a physical point of view, this discussion allows us to say that the effect of adding the Born-Infeld term is to generate more stable bound states of charged particles. Let us also mention here that if we had considered the value of \( \beta \) as predicted in [8], the correction to the Coulomb potential would have been negligible. Hence it becomes important to obtain an independent lower bound for \( \beta \) from our long-range correction, which we hope to report elsewhere.
C. Non-commutative electrodynamics

We now want to extend what we have done to non-commutative electrodynamics to leading order in $\theta$. As before, we will concentrate on the effect of including the $\theta$ term in the static potential. In such a case the Lagrangian [13–15] reads

$$L = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{8} g^{\alpha\beta} F_{\alpha\beta}^2 F_{\mu\nu}^2 - \frac{1}{2} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} F_{\mu\nu}^2 - A_0 J^0,$$  

(37)

( $\mu, \nu, \alpha, \beta = 0, 1, 2, 3$ ). Here, the field-strength tensor is expressed in terms of potentials in the usual way, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $J^0$ is an external current. As is well known $\theta^{\alpha\beta}$ is a real constant antisymmetric tensor, and from now on we take $\theta^{0\alpha} = 0$ and $\theta^{ij} = \epsilon^{ijk}\theta^k$.

The above Lagrangian will be the starting point of the Dirac constrained analysis [16]. The canonical momenta following from Eq. (37) are $\Pi^\mu = \left(1 - \frac{1}{2} \theta^{\alpha\beta} F_{\alpha\beta}\right) F^{\mu 0} - \theta^{\alpha\beta} F_{\nu\beta} F^{0\nu} - \theta^{0\alpha} F^{0\alpha} F_{\mu\beta}$, which results in the usual primary constraint $\Pi^0 = 0$ and $\Pi^i = \left(1 - \frac{1}{2} \theta^{kl} F_{kl}\right) F^{i0} - \theta^{ij} F_{j0} F^{0j} - \theta^{kl} F^{0k} F^{0l} (i, j, k, l = 1, 2, 3)$. Defining the electric and magnetic fields by $E^i = F^{i0}$ and $B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}$, respectively, the canonical Hamiltonian assumes the form

$$H_C = \int d^3x \left\{ \frac{1}{2} \left( E^2 + B^2 \right) \left(1 + \theta \cdot B\right) - (\theta \cdot E)(E \cdot B) - A_0 \left( \partial_i \Pi^i - J^0 \right) \right\}.$$  

(38)

Time conservation of the primary constraint leads to the secondary constraint $\Gamma_1(x) \equiv \partial_i \Pi^i - J^0 = 0$, and the time stability of the secondary constraint does not induce more constraints, which are first class. It should be noted that the constrained structure for the gauge field remains identical to the Born-Infeld theory. Thus, the quantization can be done in a similar manner to that in the previous subsection. In view of this situation, we pass now to the calculation of the interaction energy.

Following our earlier procedure, we will compute the expectation value of the noncommutative electrodynamics Hamiltonian in the physical state $|\Phi\rangle$ (Eq. (9)). That is,

$$\langle H \rangle_\Phi = \langle \Phi | \int d^3x \left\{ \frac{1}{2} \left( E^2 + B^2 \right) \left(1 + \theta \cdot B\right) - (\theta \cdot E)(E \cdot B) \right\} |\Phi\rangle.$$  

(39)

From our above Hamiltonian analysis, Eq. (39) can be simplified
⟨H⟩Φ = ⟨Φ| ∫ d³x \left\{ \frac{1}{2} \mathbf{E}^2 (1 + \mathbf{θ} \cdot \mathbf{B}) - (\mathbf{θ} \cdot \mathbf{B}) (\mathbf{E} \cdot \mathbf{B}) \right\} |Φ⟩.  \tag{40}

According to the definition of the canonical momenta Πᵢ, we may also write

\[ E^i = (1 + \theta \cdot B) \Pi^i - (\theta \cdot \Pi) B^i - \theta^i (\Pi \cdot B), \tag{41} \]

to lowest order in θ. Using Eq. (41) we can rewrite Eq. (40) in the following way

⟨H⟩Φ = ⟨Φ| ∫ d³x \left\{ \frac{1}{2} \Pi^2 + \frac{3}{2} (\theta \cdot B) \Pi^2 - 3 (\theta \cdot \Pi) (\Pi \cdot B) \right\} |Φ⟩. \tag{42}

Taking into account the preceding Hamiltonian structure, we first note that

\[ \Pi_i (x) = \overline{\psi} (y) \psi (0) \Pi_i (x) |0⟩ + e ∫_0^y dz_i δ(³)(x - z) |Φ⟩. \tag{43} \]

Using this in (42) we then evaluate the expectation value in the presence of the static charges

⟨H⟩Φ = ⟨H⟩₀ + V₁ + V₂ + V₃, \tag{44} \]

where ⟨H⟩₀ = ⟨0| H |0⟩. The V₁, V₂, V₃ terms are given by

\[ V₁ = \frac{e^2}{2} ∫ d³x (∫_0^y dz_i δ(³)(z - x))^2, \tag{45} \]

\[ V₁^θ = \frac{3}{2} e^2 ∫ d³x (θ \cdot B (x)) (∫_0^y dz_i δ(³)(z - x))^2, \tag{46} \]

\[ V₂^θ = -3e^2 ∫ d³x (θ_i B_j (x)) ∫_0^y dz_i δ(³)(z - x) ∫_0^y dz'_j δ(³)(z' - x). \tag{47} \]

The integrals over zᵢ and z'ᵢ are zero except on the contour of integration.

Here we make the following observations. First, we note that the term (45) may look peculiar, but it is nothing but the familiar Coulomb interaction plus a self-energy term. In effect, as was explained in [12], by using spherical coordinates the integral ∫₀^y dz_i δ(³)(z - x) can also be written as

\[ ∫_0^y dz_i δ³(x - z) = \frac{Y_i}{|y| |y - x|^2} ∑_{l,m} Y_{lm}^* (θ', φ') Y_{lm} (θ, φ). \tag{48} \]
By means of (48) and using usual properties for the spherical harmonics, the term (45) reduces to the Coulomb energy after subtracting the self-energy term. On the other hand, it should be noted that in order to evaluate Eqs. (46) and (47) we need to know the magnetic field $\mathbf{B}(\mathbf{x})$. However, since we are dealing with an external constant field $\theta$, we restrict ourselves to constant magnetic fields, $\mathbf{B}(\mathbf{x}) = \mathbf{B}(0)$. Then, with this assumption and using (48), the interquark potential at lowest order in $\theta$ becomes

$$V = -\frac{e^2}{4\pi} \frac{1}{L} \left[ 1 + 3 (\mathbf{\theta} \cdot \mathbf{B}(0)) - 6 (\mathbf{\theta} \cdot \hat{\mathbf{r}}) (\mathbf{B}(0) \cdot \hat{\mathbf{r}}) \right],$$

(49)

where $L \equiv |\mathbf{y}|$ and $\hat{\mathbf{r}}_i = \frac{\mathbf{y}_i}{|\mathbf{y}|}$.

Accordingly, to lowest order in $\theta$, the nature of the static potential remains unchanged. Nevertheless, the introduction of the noncommutative parameter induces a renormalization of the charge, which is absent in the corresponding ordinary spacetime. In such a case, Eq. (49) may be rewritten as

$$V = -\frac{e^2_{\text{ren}}}{4\pi} \frac{1}{L},$$

(50)

To conclude, the expressions for the corrections to the static energy obtained from both the Born-Infeld and noncommutative electrodynamics (to order $e^2$) are quite different. This means that the two theories are not equivalent. Born-Infeld electrodynamics has a rich structure which is reflected in a long-range correction to the Coulomb potential, which is not present in its noncommutative counterpart. On the other hand, the present investigation reveals the general applicability of our methodology. It seems a challenging work to extend to higher orders the above analysis. We expect to report on progress along these lines soon.

### III. ACKNOWLEDGMENTS

Work supported in part by Fondecyt (Chile) grant 1030355.
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