First order ODEs, Symmetries and Linear Transformations

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Abstract

An algorithm for solving first order ODEs, by systematically determining symmetries of the form $\xi = F(x)$, $\eta = P(x)y + Q(x)$ - where $\xi \partial/\partial x + \eta \partial/\partial y$ is the symmetry generator - is presented. To these linear symmetries one can associate an ODE class which embraces all first order ODEs mappable into separable through linear transformations $\{t = f(x), u = p(x)y + q(x)\}$. This single ODE class includes as members, for instance, 78\% of the 552 solvable first order examples of Kamke’s book. Concerning the solving of this class, a restriction on the algorithm being presented exists only in the case of Riccati type ODEs, for which linear symmetries always exist but the algorithm will succeed in finding them only partially.

1 Introduction

One of the most attractive aspects of Lie’s method of symmetries is its generality: roughly speaking, all solving methods for DEs can be correlated to particular forms of the symmetry generators [1, 14]. However, for first order ODEs, Lie’s method seems to be, in principle, not as useful as in the higher order case. The problem is that the determining PDE - whose solution gives the infinitesimals of the symmetry group - has the original first order ODE in its characteristic strip. Hence, finding these infinitesimals requires solving the original ODE, which in turn is what we want to solve using these infinitesimals, thus invalidating the approach.

For higher order ODEs, the strategy consists of restricting the cases handled to the universe of ODEs having point symmetries, so that the infinitesimals depend on just two variables, and then the determining PDE is overdetermined. Although few second or higher order ODEs have point symmetries, and the solving of such a PDE system for the infinitesimals may be a major problem in itself [4], the hope is that one will be able to solve the system by taking advantage of the fact that it is overdetermined.

One basic motivation in this approach is also that the finite transformations associated to point symmetries are pointlike and these transformations form a group by themselves - not just with respect to the Lie group parameter. Actually, the composition of any two point transformations is also a point transformation. Consequently any two point symmetries can be obtained from each other through a point transformation. Lie point symmetries can then be used to tackle the ODE class all of whose members can be obtained from (are equivalent to) each other through point transformations, and which includes a member we know how to solve (missing the dependent variable).

Such a powerful approach however is not useful in the case of first order ODEs - the subject of this paper - for which “point symmetries” are already the most general ones. The alternatives left then, roughly
speaking, consist of: looking for particular solutions to the determining PDE (3), or restricting the form of the infinitesimals trying to emulate what is done in the higher order case, such that the problem can be formulated in terms of an *overdetermined* linear PDE (4, 9). The question in this latter approach, however, is what would be an “appropriate restriction” on the symmetries such that:

- The related invariant ODE family includes a reasonable variety of non-trivial cases typically arising in mathematical physics;
- The determination of these symmetries, when they exist, can be performed systematically, preferably without solving any differential equations;
- The related finite transformations form a group by themselves - not just with respect to the Lie group parameter - so that the method applies to a whole ODE class.

Bearing this in mind, this paper is concerned with first order ODEs and *linear symmetries* of the form

\[ \xi = F(x), \quad \eta = P(x) y + Q(x) \]  

where \( \{\xi, \eta\} \) are the infinitesimals, the symmetry generator is \( \xi \partial / \partial x + \eta \partial / \partial y \) and \( x \) and \( y \equiv y(x) \) are respectively the independent and dependent variables. Concerning the arbitrary functions \( \{F, P, Q\} \), the requirements are those implied by the fact that (1) generates a Lie group of transformations. The linear symmetries (1) have interesting features; for instance, the related finite transformations are also *linear*, of the form

\[ t = f(x), \quad u = p(x) y + q(x) \]  

where \( t \) and \( u \equiv u(t) \) are respectively the new independent and dependent variables, and \( \{f, p, q\} \) are arbitrary functions of \( x \). Linear transformations (2) form a group by themselves too - not just with respect to the Lie parameter. So, as it happens with point symmetries in the higher order case, *any* two linear symmetries (1) can be *transformed into each other* by means of a linear transformation (2), and hence to the symmetries (1) we can associate an *ODE class*. Since separable ODEs have symmetries of this form, the class of ODEs admitting linear symmetries (1) actually includes all first order ODEs which can be mapped into separable by means of (2).

We note that in the particular case of polynomial ODEs, e.g. of Abel type,

\[ y' = f_3(x) y^3 + f_2(x) y^2 + f_1(x) y + f_0(x) \]  

(3) actually defines their respective classes. Since a separable Abel ODE can be obtained by just taking the coefficients \( f_i \) all equal, this means that there are complete Abel ODE *classes* all of whose members can be transformed into separable by means of (3). Such a case was discussed and solved at the end of the nineteenth century by Liouville, Appell and others and is presented in textbooks such as [10, 11].

More generally, for polynomial ODEs of the form

\[ y' = f_n(x) y^n + f_1(x) y + f_0(x) \]  

(4) Chini [6, 10] presented at the beginning of the twentieth century a method similar to this mapping into separable but through transformations (2) with \( q = 0 \). Chini’s method is equivalent to solving (4) by determining, when they exist, symmetries (1) with \( Q = 0 \).

In connection with the above, this work presents a generalization of these methods as an algorithm for determining whether or not a first order ODE of *arbitrary form*:

\[ y' = \Psi(x, y), \]

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1 The Abel ODE members of a given class can be mapped between themselves through (3). There are infinitely many non-intersecting such Abel ODE classes but only some of them (still infinitely many) admit symmetries of the form (1).

2 Riccati ODEs are partially excluded from the discussion - see sec. 3.
belongs to this class admitting linear symmetries \([\xi, \eta]\), and, if so, for explicitly finding the symmetry, without restrictions to the form of \([F, P, Q]\). Both the determination of the existence of a solution as well as of the symmetry itself are performed without solving any auxiliary differential equations.

The exposition is organized as follows. In sec. 2, the connection between the symmetries of the form \([\xi, \eta]\) and linear transformations of the form \([t, \nu]\) is analyzed and a solving algorithm for the related ODE class is presented. Some examples illustrating the type of ODE problem which can be solved using this method are shown in sec. 2.2. In sec. 3, a discussion of Riccati ODEs in terms of their symmetries is made and a variant of the method of sec. 2 to solve a subset of the Riccati problem, is shown. In sec. 4, a discussion and some statistics related to the classification of Kamke’s first order ODE examples is presented. Finally, the conclusions contain general remarks about this work.

## 2 Linear transformations and symmetries

To determine whether or not a given first order ODE has symmetries of the form \([\xi, \eta]\), following \([4]\), we take advantage of the fact that for such a symmetry, the related invariant ODE family can be computed in closed form. With the invariant family in hands we then show how one can algorithmically compute infinitesimals \([\xi, \eta]\) entering \([t, \nu]\) as

\[
\xi = \frac{1}{f'}, \quad \eta = \frac{p' y + q'}{f' p}
\]

A direct computation of the finite transformations generated by \([\xi, \eta]\) shows that they are linear, of the form

\[
t = X, \quad u = \frac{p(x) y + q(x) - q(X)}{p(X)}
\]

where \(X\) is a solution of \(f(X) - f(x) - \alpha = 0\) and \(\alpha\) is the (Lie) group parameter.

Regarding the invariant ODE family related to \([\xi, \eta]\), this family can be obtained, for instance, by computing differential invariants \([I_0, I_1]\) of order 0 and 1 related to \([\xi, \eta]\):

\[
I_0 = py + q, \quad I_1 = \frac{f'}{p' y + py' + q'}
\]

and hence, the invariant ODE, given by \(\Lambda(I_0, I_1) = 0\), where \(\Lambda\) is arbitrary, can be conveniently written as \(I_1 G(I_0) = 1\), with arbitrary \(G\), resulting in

\[
y' = \frac{f'}{p} G(py + q) - \frac{q'}{p} - \frac{p'}{p} y
\]

Due to this connection between linear transformations and symmetries \([\xi, \eta]\), the same invariant family \([I_0, I_1]\) can be obtained directly from an autonomous ODE,

\[
u' = G(u)
\]

by just changing variables in it using \([t, \nu]\). The solution to \([8]\) can be obtained in the same way, by changing variables in the solution to \([7]\):

\[
\int \frac{f' y + q'}{G(z)} dz + C_1 = 0
\]

In fact \([3]\) too can be obtained from the symmetry \([\xi = 1, \eta = 0]\) of \([\xi, \eta]\) by changing variables using \([t, \nu]\). We recall that the knowledge of \([\xi, \eta]\) entering \([t, \nu]\) suffices to express the solution of any member of the class \([\xi, \eta]\) by quadratures and so it is equivalent to the determination of \([f, p, q]\) entering \([10]\).

\[\text{In this section we assume } f' \neq 0, \text{ since, otherwise, } \text{[8]} \text{ would be a first order linear ODE. The limiting case where } f' \to \infty, \xi = 0, \text{ however, is not excluded.}\]
Theorem 1 Consider an ODE $y' = \Psi(x, y)$ with $\Psi_{yy} \neq 0$. Both determining whether or not this ODE belongs to the class (8), and, in the positive case, computing the infinitesimals (5) themselves, can be performed algorithmically from $\Psi$ by quadratures.

Proof. We start by noting an intrinsic feature of ODEs that are members of the class (8): the determination of $p(x)$ up to a constant factor - say $\kappa$ - suffices to map any member of the class into another one having a symmetry of the form

$$\xi = F(x), \quad \eta = Q(x)$$

(11)

This is easily verified by changing variables

$$y = \frac{u}{\kappa p}$$

(12)
in (8), arriving at $[\xi = 1/f', \eta = -q\kappa/f']$, which is of the form (11). In turn, symmetries of the form (11), when they exist, can be systematically determined as shown in [4]. In what follows we develop the proof by first showing how to map any ODE member of (8) into one having a symmetry of the form (11), and then, for completeness, briefly reviewing how such a symmetry is determined when it exists.

Now, in view of (8), for all ODEs $y' = \Psi$ members of that class we have that

$$\Psi_y = f'G'(py + q) - \frac{\xi}{p},$$

$$\Psi_{yy} = f'pG''(py + q),$$

$$\Psi_{yyy} = f'p^2G'''(py + q).$$

(13)

Thus let

$$A \equiv \frac{\Psi_{yy}}{\Psi_{yyy}} = \frac{1}{p} K(py + q) \quad \text{where} \quad K = \frac{G''}{G'''}.$$\hspace{1cm}

(14)

Three cases now arise, respectively related to whether $A_y = 0$, $A_{yy} = 0$ or $A_{yy} \neq 0$.

Case $A_y = 0$

In this case, $K' = 0$, so that $K = \kappa$ for some non-zero constant $\kappa$, and hence $A = \kappa/p$. So, from (12), when $y' = \Psi$ is indeed a member of this class, by changing variables using

$$y = A u$$

(15)

the resulting ODE family will have a symmetry of the form (11).

Case $A_{yy} = 0$, $A_y \neq 0$

In this case, $K'' = 0$ so that from (14)

$$A = \frac{\kappa_1(py + q) + \kappa_0}{p}$$

(16)

for some constant $\kappa_0$ and some non-zero constant $\kappa_1$. Here the necessary condition for $y' = \Psi$ to be a member of (8) is that the ratio above be linear in $y$. In such a case, when the ODE is indeed a member of this class, by introducing

$$u = \ln(A)$$

(17)

the resulting ODE in $u$ will have a symmetry of the form (11); this can be verified straightforwardly by performing the change of variables directly in (8).
Case $A_{yy} \neq 0$

In this case let
\[ I \equiv \frac{A_{yx}}{A_{yy}} = \frac{p'y + q'}{p} \quad (18) \]

The necessary condition for $y' = \Psi$ to be a member of (8) is then that $I$ be linear in $y$, from where
\[ p(x) = \exp \left( \int I_y dx \right) \quad (19) \]

So, when the ODE belongs to this class, from (12), changing variables $y = u/p$ will lead to an ODE having a symmetry of the form (11).

Once we have shown how a member of (8) can be mapped into another one having a symmetry of the form (11), what remains to be done in the proof of Theorem 1 is to review how that symmetry can be obtained by quadratures.

2.1 Symmetries of the form $[\xi = F(x), \eta = Q(x)]$

By computing differential invariants, as we did to arrive at (8), the invariant ODE family associated to $[\xi = F(x), \eta = Q(x)]$ can be written as
\[ y' = \Phi(x, y) \equiv \frac{1}{F(x)} \left( Q(x) + G \left( y - \int \frac{Q(x)}{F(x)} dx \right) \right) \quad (20) \]

where $F$, $Q$ and $G$ are arbitrary functions of their arguments. So far we have shown that if an ODE belongs to (8), then after changing variables as shown for the cases $A_y = 0$, $A_{yy} = 0$, $A_{yy} \neq 0$, the resulting ODE will be a member of this family (20).

Now, to determine $F$ and $Q$, following [4], we first build an expression depending on $x$ and $y$ only through $G$
\[ K \equiv \frac{\oplus \oplus}{\oplus \oplus} = \frac{G^2}{G^2} \quad (21) \]

where we assume $\Phi_{yy} \neq 0$. As explained in [4], the problem then splits into two cases.

Case $K \neq 1$

In this case, we can obtain the ratio $Q(x)/F(x)$ - only depending on $x$ - by taking
\[ \Upsilon \equiv \frac{K_1}{K_1} = -\frac{Q(x)}{F(x)} \quad (22) \]

The knowledge of this ratio in turn permits the elimination of $Q$ from the determining PDE for the infinitesimals, leading to
\[ F(x) = C_1 e^{-\int \left( \frac{\Upsilon \Phi_y - \Upsilon x - \Phi_x}{\Phi + \Upsilon} \right) dx} \quad (23) \]

which together with (22) gives the solution we are looking for. The necessary and sufficient conditions for the existence of such a symmetry are:
\[ \frac{\partial}{\partial y} \left( \frac{K_1}{K_1} \right) = 0, \quad \frac{\partial}{\partial y} \left( -\frac{\Upsilon \Phi_y - \Upsilon x - \Phi_x}{\Phi + \Upsilon} \right) = 0 \quad (24) \]

4If $Q = 0$ or $F = 0$ then the invariant ODE is separable or linear, so in [20] and henceforth we assume $F \neq 0$, $Q \neq 0$.  
5If $\Phi_{yy} = 0$, then (11) is already a first order linear ODE solvable in terms of quadratures.  
6For more details see [4]
Case $K_1 = t$

Since $K_1 = -K_1 \mathcal{F}/Q$, then when $K_1 = t$, $K_1 = t$ too, so $K = \kappa$ for some non-zero constant $\kappa$. Hence, the right-hand-side of (20) satisfies:

\[ \frac{\Phi_y}{\Phi_{yy}} = \kappa \]  
(25)

and so (20) - the invariant ODE family - is of the form

\[ y' = \Phi = A(x) + B(x) e^{y/\kappa} \]  
(26)

where $A$ and $B$ are arbitrary functions. For a given ODE of this type, $A$ and $B$ can be determined by inspection, and the determining PDE for the infinitesimals can be solved directly in terms of $A$ and $B$ as

\[ F(x) = \frac{-e}{B} \int \frac{A}{\kappa} \, dx, \quad Q(x) = A F(x) \]  
(27)

2.2 Examples

1. Consider the first order ODE example number 128 from Kamke's book:

\[ xy' + ay - f(x)g(x^a y) = 0 \]  
(28)

For this ODE, Kamke shows a change of variables mapping the ODE into separable, derived for this particular ODE family in [7]. Using the algorithm presented in this paper, we tackle this ODE by computing $A$ in (14)

\[ A = \frac{g''}{x^a g'''} \]  
(29)

so we are in “Case $A_{yy} \neq 0$”. We then proceed with computing $I$ (see (18)) arriving at

\[ I = \frac{a y}{x} \]  
(30)

The existence condition that $I$ be linear in $y$ is satisfied, hence, according to (19),

\[ p(x) = x^a \]  
(31)

Now, changing variables $y = u/p$ as indicated in (12), (28) becomes

\[ u' = g(u) \frac{f(x)x^a}{x} \]  
(32)

which is already separable (and thus naturally has a symmetry of the form (11)). The solution to (28) is then obtained by changing variables back in the solution to (32), leading to:

\[ \int x^{a-1} f(x) \, dx - \int x^{a y} \frac{1}{g(z)} \, dz - C_1 = 0 \]  
(33)

2. Let’s now discuss an example for which $A_{yy} = 0$,

\[ y' = (x^3 y^4 + 4 x^4 y^3 + 6 x^5 y^2 + 4 x^6 y + x^7) (x^n + 1) - \frac{y}{x} - 2 \]  
(34)

where $n$ is an arbitrary constant. For this ODE, from (14),

\[ A = \frac{y + x}{2} \]  
(35)
so the change of variables here is \( u = \ln(A) \) (see (17)), mapping (34) into

\[
u' = t^{7+n}(t^n + 1)e^6u + \frac{7}{t}
\] (36)

For this ODE, a symmetry of the form (11) is computed algorithmically (see sec. 2.1)

\[
\xi = \frac{1}{8(t^n + 1)}, \quad \eta = -\frac{1}{8t(t^n + 1)}
\] (37)

Changing variables back directly in the above we arrive at a symmetry for (34)

\[
\xi = \frac{1}{8(x^n + 1)}, \quad \eta = -\frac{y + 2x}{8x(x^n + 1)}
\] (38)

from where an implicit solution to (34) follows as

\[
x + \frac{1}{3x^3(y + x)^3} + \frac{x^{(1+n)}}{1 + n} = C_1
\] (39)

3. As an example of the case in which \( A_y = 0 \), consider

\[
y' = b e^{ax}y x^a + \frac{(x^2 - 1)y}{x} - \frac{1}{x^2} + \ln(x) + c
\] (40)

where \( a, b \) and \( c \) are arbitrary constants. From (14), \( A = 1/(ax) \), so that by changing variables as indicated in (15), (33) becomes

\[
u' = ut + abt^{(a+1)}e^u - \frac{a}{t} + at(\ln(t) + c)
\] (41)

and this ODE has a symmetry of the form (11)

\[
\xi = \frac{1}{t}, \quad \eta = -\frac{a}{t^2}
\] (42)

from where, by changing variables back, (40) admits the symmetry

\[
\xi = \frac{1}{x}, \quad \eta = -\frac{xy + 1}{x^3}
\] (43)

which suffices to integrate (40) by either using canonical coordinates or computing an integrating factor.

4. The algorithm presented is applicable to higher degree ODEs too (see Table 1. in sec. 4), provided that it is possible to solve the given ODE for \( y' \). Consider for instance Kamke’s example 394

\[
(y')^2 + 2fyy' + gy^2 + (f^2 - g) e^{-2 \int_a^y f(z)dz} = 0
\] (44)

where \( f \equiv f(x) \) and \( g \equiv g(x) \) are arbitrary functions. For this problem Kamke presents a particular change of variables derived in [12]. To tackle this example using our algorithm we first solve the ODE for \( y' \)

\[
y' = -fy \pm \sqrt{(y^2 - e^{-2 \int_a^y f(z)dz})(f^2 - g)}
\] (45)

Now, by taking any of the two branches of (13), for instance the “+” one, and computing the second derivative of (14), we find

\[
A_{yy} = \frac{2}{3y^3(e^{\int_a^y f(z)dz})^2}
\] (46)
so that $I$ in (18) is given by

$$ I = f y $$

(47)

from where we compute $p$ using (19), finally arriving at the symmetry

$$ \xi = \frac{1}{\sqrt{f^2 - g}}, \quad \eta = -\frac{f y}{\sqrt{f^2 - g}} $$

(48)

actually admitted by both branches of (45).

It is worth mentioning that if on one hand these four examples are straightforward problems for the single solving algorithm presented, on the other hand we are not aware of any other algorithm for tackling examples like 2 or 3; also, for examples 1 and 4 the changes of variables presented in Kamke are non-obvious and presented in connection with different problems [7, 12]. Despite the presence of arbitrary functions and parameters, both Kamke’s examples 128 and 394 are actually particular cases of the class represented by (8).

3 Riccati equations

The case of Riccati type ODEs

$$ y' = f_2 y^2 + f_1 y + f_0 $$

(49)

where $f_i = f_i(x)$, $f_2 \neq 0$ and $f_0 \neq 0$, deserves a separate discussion. All Riccati ODEs admit symmetries of the form (3), and so all of them can be mapped into separable using transformations of the form (2). However, it is easy to verify that to find such a transformation requires solving the Riccati ODE itself. The algorithm of the previous section - which does not rely in solving auxiliary differential equations - only works when $\Psi_{yyy} \neq 0$ - see Theorem 1. The usual approach for solving (49) then consists of converting it to a linear second order ODE and using the various methods available for this other problem.

Nonetheless, there are entire subfamilies of (49) for which symmetries of the form (3) can be found following an approach such as the one presented in the previous section, without using techniques for linear second order ODEs. Such an approach is interesting since it enriches the algorithms available for tackling the problem and could be of use for solving some linear ODEs by mapping them into Riccati ones as well. For the purpose of discussing these cases and without loss of generality we rewrite (3) - the general form of the symmetries admitted by Riccati ODEs - redefining $f' \rightarrow f/p$

$$ \xi = \frac{p}{f}, \quad \eta = -\frac{p' y + q'}{f} $$

(50)

If now, in (3), we redefine $f'$ in the same way and take $G$ as a square mapping depending on two arbitrary constants $a$ and $b$,

$$ G = u \mapsto u^2 + a u + b $$

(51)

we arrive at the form of an arbitrary Riccati ODE - as general as (49) - but expressed in terms of these two constants $a$ and $b$ and the functions $\{f, p, q\}$ appearing in its symmetry generator (50):

$$ y' = f y^2 + \frac{(a + 2 q) f - p'}{p} y + \frac{(a + q) q + b f - q' p}{p^2} $$

(52)

Case $p' = 0$

A first solvable case happens when $p' = 0$, so that in (50) both infinitesimals depend only on $x$ and hence the symmetry can be systematically determined - also for Riccati ODEs - as explained in sec. 2.1.

Case $q' = 0$

A second solvable case happens when, in (52), $q' = 0$, so that the infinitesimals (50) are of the form

$$ \xi = F(x), \quad \eta = \mathcal{P}(x) y $$

(53)
An algorithm for solving such an ODE was presented by Chini [6]. In the case of Riccati ODEs, Chini’s algorithm can be summarized as “to check for the constant character” of the expression

\[ I \equiv \left( f_0' f_2 - f_0 f_2' - 2 f_0 f_1 f_2 \right)^2 \left( f_0 f_2 \right)^3 \]  

(54)

where \( f_i \) are the coefficients of \( y \) in (49). Whenever \( I \) is constant, the problem is systematically solvable in terms of quadratures (see for instance [10] - p.303). Concerning (52) at \( q' = 0 \), a direct computation of \( I \) confirms that in such a case \( I \) is constant. Conversely, another direct computation shows that whenever \( I \) is constant, the ODE will have a symmetry of the form (53). To check that, it suffices to solve (54) for \( f_1 \) and substitute the result into (49); the resulting Riccati ODE will admit the symmetry

\[ \xi = \frac{1}{f_2} \sqrt{\frac{f_2}{f_0}}, \quad \eta = \frac{\left( f_0' f_2 - f_0 f_2' \right)}{2 f_0^2 f_2} \sqrt{\frac{f_2}{f_0}} y \]

(55)

which is of the form (53).

We can also see what is the ODE class solved by Chini’s algorithm, as well as explain the previous results, by noticing that (54) is an absolute invariant for (49) under transformations of the form

\[ t = \bar{f}(x), \quad u = \bar{p}(x) y \]

(56)

that is, of the form (3) with \( q = 0 \). In fact, (54) can be written as

\[ I = \frac{s_3^2}{s_2^3} \]

(57)

where

\[ s_2 = f_0 f_2, \quad s_3 = f_0' f_2 - f_2' f_0 - 2 f_0 f_1 f_2 \]

(58)

are relative invariants of weight 2 and 3 with respect to transformations (56).

In summary: Chini’s algorithm solves the ODE class generated by changing variables (56) in the general Riccati ODE (52) at \( q' = 0 \), all of whose members have \( I = \text{constant} \).

Three additional solvable Riccati families, where the invariant \( I \) is non-constant, are obtained by equating in (50) any two of the three arbitrary functions \( \{ f, p, q \} \) with \( p' \neq 0 \) and \( q' \neq 0 \).

Case \( f = p \)

When a given Riccati ODE belongs to this family, then by changing variables

\[ y = \frac{u}{f} \]

(59)

in the given ODE and in the general form of its symmetry (51), we see that the resulting ODE in \( u \) will admit the symmetry

\[ \xi = 1, \quad \eta = -q' \]

(60)

which can be determined as explained in sec. 2.1.

Case \( q = p \)

When a given Riccati ODE belongs to this family, then by changing variables

\[ y = u - 1 \]

(61)

\footnote{This connection between the constant character of an expression like (54), built with the coefficients of a polynomial ODE of the form (4), and symmetries of the form (53), is valid not only for Riccati ODEs but for ODEs of the form (4) in general.}

\footnote{\( f \) is the coefficient of \( y^2 \) in the given ODE.
in the given ODE and in (50) we see that the resulting ODE in \( u \) will admit a symmetry of the form

\[
\xi = \frac{p}{f}, \quad \eta = -\frac{p'}{f}u \tag{62}
\]

that is, infinitesimals of the form (53), and hence the ODE will be solvable using Chini’s method.

Case \( f = q \)

From (52), the Riccati family corresponding to this case is given by

\[
y' = fy^2 + \frac{(f(a + 2f) - p')}{p} y + \frac{((a + f)f + b)f - f'p}{p^2} \tag{63}
\]

From (51), this ODE family admits the symmetry

\[
\xi = \frac{p}{f}, \quad \eta = -\frac{p' y + f'}{f} \tag{64}
\]

We haven’t found an obvious transformation of the form \( y = Pu + Q \) to map this symmetry to one of the forms (11) or (53). A possible approach would then be to directly set up the determining PDE

\[
\eta_x + (\eta y - \xi') \left( f_2 y^2 + f_1 y + f_0 \right) - \xi \left( f_2' y^2 + f_1' y + f_0' \right) - \eta \left( 2f_2 y + f_1 \right) = 0
\]

for the coefficients \( \xi \) and \( \eta \) of the infinitesimal symmetry generator \( \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \) of an arbitrary Riccati ODE (49). Then take \( \xi \) and \( \eta \) as given by (64) and run a differential elimination process solving for \( p \). That approach works but the resulting symbolic expressions are large enough to become untractable even with simple examples.

An alternative approach leading to more tractable expressions is based on using the information we have in (63) concerning the existence of two constants \( a \) and \( b \). So, by equating the coefficients of (63) with those of (49) we arrive at the system

\[
f_2 - f = 0,
\]

\[
f_1 - \frac{f_2(a + 2f_2) - p'}{p} = 0, \quad f_0 - \frac{f_2(f_2a + f_2^2 + b) - f_2'p}{p^2} = 0 \tag{65}
\]

This system can be solved for \( a \) and \( b \), so that when a Riccati equation belongs to this family the following two expressions formed with its coefficients \( f_i \) will be constants:

\[
a = \frac{f_1p - 2f_2^2 + p'}{f_2}
\]

\[
b = \frac{f_0p^2 + f_2(f_2^{2} - f_1p - p') + f_2'p}{f_2} \tag{66}
\]

At this point, however, we cannot verify the constant character of these expressions because \( p \) is still unknown. An expression for \( p \) can be obtained by computing the integrability conditions for \( f_2 \) and \( p \) implied by (65):

\[
f_2'' = f_1f_2' - 2f_0p' - pf_0' + \frac{(f_2')^2 + pff_2'}{f_2},
\]

\[
p'' = 2f_2f_2' - p'f_1 - pf_1' + \frac{f_2'p' + pf_1f_2'}{f_2} \tag{67}
\]

\[
10
\]
Using (67) to eliminate $p'$ from (68) we arrive at a solution for $p$:

$$p = \frac{f_2 \left( 3 s_2 f_2'' s_2' - 2 s_2^2 f_2''' + \left( (s_2'' - s_4) s_2^2 - 8 s_2^3 - 2 \left( s_2' \right)^2 + 2 s_3^2 \right) f'_2 \right)}{s_2 \left( 2 s_2 s_3 - 3 s_3 s_2' \right)}$$

(69)

where

$$s_4 = \frac{2 s_2 s_3' - 3 s_3 s_2' + 3 s_3^2}{2 s_2}$$

(70)

is the relative invariant of weight 4 for Riccati ODEs with respect to transformations of the form (56) [2]. We note that, from (57), when the denominator of (69) is zero the ODE has a constant invariant $I$ (54) and so it is already solvable using Chini’s algorithm.

In summary, a strategy - not relying on solving second order linear ODEs - for finding linear symmetries of the form (1) for Riccati equations could consist of

1. Check if the ODE has a symmetry of the form $[\xi = F(x), \eta = Q(x)]$ (algorithm of [1] - see sec. 2.1); or $[\xi = F(x), \eta = P(x) u]$ (Chini’s algorithm);
2. Check if the ODE belongs to one of the two families “$f = p$” or “$p = q$” by using the transformations (59) and (61) and re-entering the previous step;
3. Check if the ODE belongs to the family “$f = q$” (63); for that purpose:
   (a) compute the invariants (58);
   (b) use these invariants to compute $p$ using (69);
   (c) plug the resulting $p$ into (66) and verify if the two right-hand-sides are constant - if so, the ODE admits the symmetry (64).

It is our belief that with the development of computer algebra software and faster computers, these type of algorithms for Riccati subclasses (there are of course many other possibilities) will become each day more relevant, also as an alternative for tackling second order linear ODEs.

4 Classification of Kamke’s examples and discussion

We have prepared a computer algebra prototype of the algorithms presented in sec. 2 and sec. 3 using the Maple system. We then used this prototype to analyze the set of Kamke’s 576 first order ODE examples. This type of computational activity interesting. In the first place, Kamke’s book contains a fair collection of examples arising in applications. In the second place, this book collects many, perhaps most, of the solving algorithms available in the literature for first order ODEs. So, generally speaking, such an analysis conveys a reasonable evaluation of the range of application and novelty of a new ODE solving algorithm.

In order to perform the classification, we first excluded from the 576 Kamke examples all those for which a solution is not shown and we were not able to determine it by other means - most of them just because the ODE is too general. So our testing arena starts with 552 ODEs.

Then, we know that all ODEs of type separable, linear, homogeneous, Bernoulli, Riccati and Abel with constant invariant [10], that is, 372 of Kamke’s examples, have symmetries of the form (1) and then belong to the ODE class discussed in this paper. So the first thing we wanted to know is how many of the remaining 552 - 372 = 180 examples admit linear symmetries of the form (1)? Is there any other classification known for these 180 examples? The information for answering these and related questions is summarized in this table:

---

9The numbers of the Kamke examples we excluded in this way are: 47, 48, 50, 55, 56, 74, 79, 82, 202, 205, 206, 219, 234, 235, 237, 265, 250, 253, 269, 331, 370, 461, 503 and 576.
10For an enumeration of Kamke’s examples of Abel type with constant invariant see [1] and concerning other classes see [4].
Class First degree in $y'$: 88 Higher degree in $y'$: 92 Total: 180 ODEs

|                      | $[\xi = F, \eta = P_y + Q]$ | $\xi = F, \eta = P$ | “Two of $\{f, p, q\}$ are equal” | Total of ODEs |
|----------------------|-------------------------------|----------------------|----------------------------------|---------------|
| Abel (non-constant invariant) | 20                            | 37                   | 57                               |               |
| Clairaut             | 15                            | 0                    | 15                               |               |
| d’Alembert           | 0                             | 15                   | 15                               |               |
| Unknown              | 33                            | 21                   | 54                               |               |

Table 1. Classification of 180 non “linear, separable, Bernoulli, Riccati or Abel c.i.” Kamke’s examples.

From these numbers, some first conclusions can be drawn. First, 372 + 57 = 429 ODEs, that is, 78 % of Kamke’s 552 solvable examples, have linear symmetries of the form (1). Also Table 1. shows that in Kamke - even among these particular 180 ODEs which exclude the easy ones - there are more examples having symmetries of the form (1) which can be systematically determined as explained in sec. 2, than examples of Abel (non-constant invariant), Clairaut and d’Alembert types all together.

In the second place, if we discard the 61 examples of Riccati type found in Kamke, the solvable set is reduced to 552 - 61 = 491 ODEs, and from this set, 429 - 61 = 368 ODEs, that is, 75 % of these 491 can be solved algorithmically as shown in sec. 2.

Moreover, a classification of Kamke’s examples of Riccati type according to sec. 3 shows:

| Class       | $[\xi = F, \eta = Q]$ | $[\xi = F, \eta = P_y]$ | “Two of $\{f, p, q\}$ are equal” | Total of ODEs |
|-------------|------------------------|--------------------------|----------------------------------|---------------|
| Riccati     | 7/61                   | 22/61                    | 2/61                             | 31/61         |

Table 2. Classification according to sec. 3 of the 61 Riccati ODE examples of Kamke’s book.

So one half of these Riccati examples is still solvable using the algorithms described in sec. 3, and so without mapping the problem into a second order linear ODE nor having to solve auxiliary differential equations.

5 Conclusions

In this work we presented an algorithm for solving first order ODEs, consisting of determining symmetries of the form (1). From the discussions of sec. 2 to these symmetries one can associate an ODE class - represented by (5) - which embraces all first order ODEs mappable into separable ones through linear transformations. From the numbers of sec. 4, this class appears to us as the widest first order ODE class we are aware of, all of whose members are algorithmically solvable as shown in sec. 2 or sec. 3, or mappable into second order linear ODEs when the ODE is of Riccati type and the methods in sec. 3 don’t cover the case.

The algorithms presented also do not require solving additional differential equations nor do they rely on the ODE member of the class being algebraic (i.e.: rational in $y$ and its derivatives) or on restrictions to the function fields - the only requirement on the functions entering the infinitesimals (1) are those implied by the fact that these infinitesimals do generate a Lie group of transformations.

Concerning other related works we are aware of, the method presented in [13] for solving Abel ODEs with constant invariant is a particular case of the one presented here in that those ODEs are the subclass of (5) of type Abel. In the same line, the method by Chini [6] - a generalization of the method for Abel ODEs with constant invariant which solves more general ODEs of the form (1) - is also a particular case in that (5) is a very restricted subclass of (6). Also, the class solvable through Chini’s method is equivalent to a separable ODE only through transformations (2) with $q = 0$. In this sense, the algorithm presented in sec. 3 generalizes both the one discussed in [13] and the one presented in [6].

The fact that this class (5) is algorithmically solvable turns this classification relevant for modern computer algebra implementations too. As shown in sec. 4, taking as framework for instance Kamke’s examples,
78% belong to this ODE-class. Even after discarding Riccati ODEs, 75% of the remaining Kamke examples belong to this class (8) and so are solvable by the algorithm presented in sec. 2. This algorithm actually solves many ODE families not solved in the presently available computer algebra systems (CAS). For instance, from the 4 examples shown in sec. 2.2, three of them cannot be solved by Maple 6 or Mathematica 4 - the last versions of these major CAS. Even concerning the Riccati families presented in sec. 3, two of them (cases “f = q” and “p = q”) are also not solved by these two CAS, which base their strategy in mapping Riccati ODEs into linear second order ones.

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