Maximum dissociation sets in subcubic trees

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Abstract
A subset of vertices in a graph $G$ is called a maximum dissociation set if it induces a subgraph with vertex degree at most 1 and the subset has maximum cardinality. The dissociation number of $G$, denoted by $\psi(G)$, is the cardinality of a maximum dissociation set. A subcubic tree is a tree of maximum degree at most 3. In this paper, we give the lower and upper bounds on the dissociation number in a subcubic tree of order $n$ and show that the number of maximum dissociation sets of a subcubic tree of order $n$ and dissociation number $\psi$ is at most $1.466^{4n - 5\psi + 2}$.

Keywords Extremal graph theory · Enumeration in graph theory · Maximum dissociation set · Trees

1 Introduction

We consider only finite, simple, and undirected labeled graphs, and use (Bondy and Murty 2008) for terminology and notations not defined here.

In a graph $G$, an independent set is a set of pairwise non-adjacent vertices of $G$. An independent set is maximal if it is not a proper subset of any other independent set,
and maximum if it has maximum cardinality. The independence number of a graph \( G \), denoted by \( \alpha(G) \), is the cardinality of a maximum independent set of \( G \).

A dissociation set in a graph \( G \) is a vertex subset \( F \) such that the subgraph \( G[F] \) induced by \( F \) has vertex degree at most 1. A maximum dissociation set of \( G \) is a dissociation set of maximum cardinality. The dissociation number of a graph \( G \), denoted by \( \psi(G) \), is the cardinality of a maximum dissociation set of \( G \). The problem of finding a maximum dissociation set in a given graph has been introduced by Yannakakis (1981) and is known to be NP-hard for bipartite graphs. The complexity of the problem for some classes of graphs has been studied (Alekseev et al. 2007; Cameron and Hell 2006; Orlovich et al. 2011; Xiao and Kou 2017; Yannakakis 1981). Note that a set \( F \) of vertices of a graph \( G \) is a dissociation set if and only if its complement \( V(G) \setminus F \) is a so-called 3-path vertex cover, that is, a set of vertices of \( G \) intersecting every path of order 3 in \( G \). The 3-path vertex cover problem is to find a minimum 3-path vertex cover in a graph \( G \) and has received considerable attention in the literature (Brešar et al. 2011; Kardoš et al. 2011; Katrenič 2016; Tu and Zhou 2011; Xiao and Kou 2017).

In 1960s, Erdős and Moser raised the problem of determining the maximum number of maximal independent sets for a general graph \( G \) of order \( n \). This problem was solved by Erdős, and later Moon and Moser (1965). Since then, the problem was extensively studied for various classes of graphs, including trees (Wilf 1986; Sagan 1988), connected graphs (Füredi 1987; Griggs et al. 1988), bipartite graphs (Liu 1993), unicyclic connected graphs (Koh et al. 2008), connected graphs with at most \( r \) cycles (Sagan and Vatter 2006). A number of authors have also studied the problem of determining the maximum number of maximum independent sets for various classes of graphs (Zito 1991; Jou and Chang 2000; Sagan and Vatter 2006; Mohr and Rautenbach 2018, 2021; Derikvand and Oboudi 2014). On the other hand, the problem of determining the maximum number of minimal (or minimum) dominating sets was also studied in the literature (Alvarado et al. 2019; Bród and Skupień 2006; Connolly et al. 2016; Fomin et al. 2005).

The study of dissociation sets has many applications in theory and practice. For example, a lower bound for the 1-improper chromatic number of a graph could be found by computing the dissociation number (Havet et al. 2009) and the maximum dissociation set problem has played a role in telecommunications, scheduling, wireless sensor networks (Acharya et al. 2012; Brešar et al. 2011). Furthermore, the problem of enumerating and counting all subgraphs that satisfy a given property is a fundamental problem in computer science and the core of data analysis, and it is also interesting in terms of computational complexity theory (Conte et al. 2022; Tsukiyama et al. 1977).

A subcubic tree is a tree of maximum degree at most 3. Recently, Mohr and Rautenbach (2018) considered subcubic trees and proved the following.

**Theorem 1.1** (Mohr and Rautenbach 2018) If \( T \) is a subcubic tree of order \( n \) and independence number \( \alpha \), then the number of maximum independent sets in \( T \) is at most \( 1.618^{2n-3\alpha+1} \).

Inspired by these aforementioned problems, we consider the analogous problem of determining the maximum number of maximum dissociation sets in a graph \( G \). In Tu et al. (2021), we determined the maximum number of maximum dissociation
sets in a tree of order $n$ and characterized the extremal trees. In the present paper, we consider the dissociation number and the number of the maximum dissociation sets of a subcubic tree. In the next section, we will give the lower and upper bounds on the dissociation number of a subcubic tree of order $n$. In Sect. 3, we will show that the number of maximum dissociation sets of a subcubic tree of order $n$ and dissociation number $\psi$ is at most $1.4664^n - 5\psi + 2$.

2 The bounds on the dissociation number of a subcubic tree

Let $G$ be a graph and $v$ be a vertex in $G$. The neighborhood $N_G(v)$ is the set of vertices adjacent to $v$ and the closed neighborhood $N_G[v] = N_G(v) \cup \{v\}$. For a subset $X$ of vertices, the induced subgraph $G[X]$ is the subgraph of $G$ whose vertex set is $X$ and whose edge set consists of all edges of $G$ which have both ends in $X$. If $U$ is the set of vertices deleted, the resulting subgraph is denoted by $G - U$. If $U = \{v\}$, we write $G - v$ for $G - \{v\}$. A vertex in a tree $T$ is called a leaf if it has degree exactly one. The neighbor of a leaf is called a support vertex of $T$. If a support vertex of $T$ is adjacent to at least two leaves, then it is a strong support vertex of $T$.

Let $k$ be a positive integer. A set $S$ of vertices in $G$ is called a $k$-path vertex cover if every path of order $k$ in $G$ contains at least one vertex from $S$. The $k$-path vertex cover number $\tau_k(G)$ of $G$ is the cardinality of a minimum $k$-path vertex cover in $G$.

**Theorem 2.1** (Brešar et al. 2011) Let $T$ be a tree of order $n$ and $k$ a positive integer. Then, $\tau_k(G) \leq n/k$.

A set $S$ of vertices of a graph $G$ is a 3-path vertex cover if and only if its complement $V(G) \setminus S$ is a dissociation set. Therefore, we can obtain a lower bound on the dissociation number of a tree of order $n$.

**Theorem 2.2** If $T$ is a tree of order $n$, then $\psi(T) \geq \frac{2n}{3}$.

For a positive integer $\ell$, let $T_\ell$ arise by attaching a pendant edge to every vertex of a path of order $\ell$. Hence, $T_\ell$ is a subcubic tree of order $3\ell$ and $\psi(T_\ell) = 2\ell$. It follows that for a subcubic tree the lower bound on the dissociation number given by Theorem 2.2 is tight.

Next, we give the upper bound on the dissociation number of a subcubic tree of order $n$. If $T$ is a tree, then $T'$ arises from $T$ by attaching a $P_5$ such that $V(T')$ is the disjoint union of $V(T)$ and $\{x, y, z, j, k\}$, and $E(T') = E(T) \cup \{uz, zy, zj, yx, jk\}$, where $u$ is some vertex of $T$.

**Theorem 2.3** If $T$ is a subcubic tree of order $n$, then,

$$\psi(T) \leq \frac{4n + 2}{5}. \quad (1)$$

Furthermore, equality holds in (1) if and only if $T$ arises from $K_2$ by iteratively attaching $P_5$s.
Proof Suppose, for a contradiction, that the theorem is false, and let \( n \) be the smallest order for which it fails. Let \( T \) be a subcubic tree of order \( n \) and dissociation number \( \psi \) such that either \( \psi > \frac{4n+2}{5} \) or \( \psi = \frac{4n+2}{5} \) and \( T \) doesn’t arise from \( K_2 \) by iteratively attaching \( P_5 \)s. It is easy to see that \( T \) has diameter at least 3. We root \( T \) at an endvertex of a longest path in \( T \). Let \( u \) be a leaf of maximum depth in \( T \) and \( uwxv \) be a path of \( T \).

Claim 1 \( d_T(v) = 2 \).

Proof of Claim 1 Suppose, for a contradiction, that \( d_T(v) = 3 \). Let \( T' = T - (N_T[v] \ \{w\}) \), and then \( T' \) has order \( n - 3 \) and dissociation number \( \psi - 2 \). By the choice of \( n \), we have

\[
\psi = \psi(T') + 2 \leq \frac{4|V(T')|+2}{5} + 2 = \frac{4(n-3)+2}{5} + 2 < \frac{4n+2}{5},
\]

which contradicts the choice of \( T \). \( \square \)

Claim 2 \( d_T(w) = 3 \) and \( w \) is not a support vertex.

Proof of Claim 2 If \( d_T(w) = 2 \), then \( T' = T - \{u, v, w\} \) has order \( n - 3 \) and dissociation number \( \psi - 2 \). Similarly, we obtain \( \psi < \frac{4n+2}{5} \), which contradicts the choice of \( T \). Therefore, \( d_T(w) = 3 \).

Suppose, for a contradiction, that \( w \) is a support vertex. Let \( v' \) be the child of \( w \) distinct from \( v \) and \( v' \) is a leaf. Let \( T' = T - \{u, v, w, v'\} \), and then \( T' \) has order \( n - 4 \) and dissociation number \( \psi - 3 \). By the choice of \( n \), we obtain

\[
\psi = \psi(T') + 3 \leq \frac{4|V(T')|+2}{5} + 3 = \frac{4(n-4)+2}{5} + 3 < \frac{4n+2}{5},
\]

which contradicts the choice of \( T \). \( \square \)

By the above two claims, \( w \) has a child \( v' \) distinct from \( v \) and \( v' \) has exactly one child \( u' \) in \( T \). Let \( T' = T - \{u, u', v, v', w\} \), and then \( T' \) has order \( n - 5 \) and dissociation number \( \psi - 4 \). By the choice of \( n \), we obtain

\[
\psi = \psi(T') + 4 \leq \frac{4|V(T')|+2}{5} + 4 = \frac{4(n-5)+2}{5} + 4 = \frac{4n+2}{5},
\]

which implies that \( \psi = \frac{4n+2}{5} \) and the tree \( T' \) arises from \( K_2 \) by iteratively attaching \( P_5 \)s. Since \( w \) has degree 3 and each of children of \( w \) has exactly a child that is a leaf, \( T \) also arises from \( K_2 \) by iteratively attaching \( P_5 \)s. This contradiction completes the proof. \( \square \)

Theorem 2.4 If \( T \) is a subcubic tree of order \( n \), then \( \frac{2n}{5} \leq \psi(T) \leq \frac{4n+2}{5} \). Moreover, both bounds are tight.
3 The number of maximum dissociation sets in a subcubic tree

In this section, our main result concerns the largest possible value of the number of maximum dissociation sets in a subcubic tree of order $n$ and dissociation number $\psi$.

Let $G$ be a graph and $v$ be a vertex of $G$. The set of all maximum dissociation sets of $G$ is denoted by $MD(G)$ and its cardinality by $\Phi(G)$. Let

$$
\Phi_v(G) = |\{F \in MD(G) : v \in F\}|,
$$

$$
\Phi_{\overline{v}}(G) = |\{F \in MD(G) : v \not\in F\}|,
$$

$$
\Phi_v^0(G) = |\{F \in MD(G) : v \in F \text{ and } d_G(F)(v) = 0\}|,
$$

$$
\Phi_v^1(G) = |\{F \in MD(G) : v \in F \text{ and } d_G(F)(v) = 1\}|,
$$

It follows that $\Phi(G) = \Phi_v(G) + \Phi_{\overline{v}}(G)$ and $\Phi_v(G) = \Phi_v^1(G) + \Phi_v^0(G)$.

**Lemma 3.1** Let $T$ be a tree of order $n$ and dissociation number $\psi$. If $v$ is a leaf of $T$ such that $\psi(T - v) = \psi(T)$, then $\Phi_v(T) \leq \min\{\Phi_v^0(T), \Phi_v^1(T)\}$ and $\Phi_v(T) \leq \frac{1}{3}\Phi(T)$.

**Proof** Let $u$ be the neighbour of $v$ in $T$ and $T' = T - v$. Since $\psi(T') = \psi(T)$, for every maximum dissociation set $F$ of $T'$ the vertex $u$ is contained by the set $F$ and $d_{T'[F]}(u) = 1$. Let $N_{T'}(u) = \{u_1, \ldots, u_k\}$. Next, we will prove that there exists a vertex in $N_{T'}(u)$ such that it is in all maximum dissociation sets of $T'$.

Without loss of generality, suppose to the contrary that there are two maximum dissociation sets $F_1$ and $F_2$ of $T'$ such that $u_1 \in F_1$ and $u_2 \in F_2$. Let $D_1, \ldots, D_k$ be the connected components of $T' - u$ such that $u_i \in V(D_i)$. See Fig. 1. It is easy to see that $F_1 \cap V(D_1)$ is a maximum dissociation set of $D_1$ and every maximum dissociation set of $D_1$ contains $u_1$. Since $F_2 \cap V(D_1)$ is a dissociation set of $D_1$ not containing $u_1$, $|F_2 \cap V(D_1)| < |F_1 \cap V(D_1)|$. Now, $[F_1 \cap V(D_1)] \cup [\bigcup_{i=2}^{k}(F_2 \cap V(D_i))]$ is a maximum dissociation set of $T'$ that does not contain $u$, this is a contradiction. Thus, we verify that there exists a vertex in $N_{T'}(u)$, say $u_1$, such that it is in all maximum dissociation sets of $T'$.

If $F$ is a maximum dissociation set in $T'$, then $F_1 = (F \cup \{v\}) \setminus \{u\}$ is a maximum dissociation set of $T$ such that $d_{T[F_1]}(v) = 0$, and $F_2 = (F \cup \{v\}) \setminus \{u_1\}$ is a maximum dissociation set of $T$. Therefore, $\Phi_v(T) \leq \min\{\Phi_v^0(T), \Phi_v^1(T)\}$ and $\Phi_v(T) \leq \frac{1}{3}\Phi(T)$.
dissociation set in $T$ such that $d_{T[F_2]}(v) = 1$. On the other hand, $F$ is also a maximum dissociation set in $T$ that does not contain $v$. Thus, we have

$$\Phi(T') = \Phi(T) \leq \min\{\Phi_v^0(T), \Phi_v^1(T)\}.$$  

Since $\Phi(T) = \Phi(T) + \Phi_v^0(T) + \Phi_v^1(T)$, it follows that $\Phi(T) \leq \frac{1}{3} \Phi(T)$. □

**Theorem 3.2** If $T$ is a subcubic tree of order $n$ and dissociation number $\psi$, then

$$\Phi(T) \leq 1.466^{4n-5\psi+2}.$$  

**Proof** Suppose, for a contradiction, that the theorem is false, and let $n$ be the smallest order for which it fails. Let $T$ be a subcubic tree of order $n$ and dissociation number $\psi$ such that $\Phi(T) > 1.466^{4n-5\psi+2}$. It is easy to see that $T$ has diameter at least 3. We can assume that $n \geq 4$ and $\psi \geq 3$.

Let $\alpha$ be the largest solution of the equation $\alpha^3 - \alpha^2 - 2\alpha - 1 = 0$, that is, $\alpha \approx 2.148$. Let $\lambda = \sqrt[3]{\alpha} \approx 1.466$.

**Claim 1** Let $T_1$ be a subcubic tree of order $n_1$ and dissociation number $\psi_1$. If $n_1 \leq n - 3$, then for any vertex $v \in V(T_1)$ with $d_{T_1}(v) < 3$, we have $\Phi(T_1) \leq \lambda^{4n_1-5\psi_1}$.

**Proof of Claim 1** Let $v \in V(T_1)$ and $d_{T_1}(v) < 3$. If $v$ is in all maximum dissociation sets of $T_1$, then $\Phi(T_1) = 0$. Otherwise, let $T_2$ arise from $T_1$ by attaching a pendant edge $e$ to $v$. Then $T_2$ is a subcubic tree of order $n_1 + 2$ and dissociation number $\psi_1 + 2$. If $F$ is a maximum dissociation set of $T_1$ such that $v \notin F$, $F \cup V(e)$ is a maximum dissociation set of $T_2$. By the choice of $n$, this implies

$$\Phi(T_1) \leq \Phi(T_2) \leq \lambda^{4(n_1+2)-5(\psi_1+2)+2} = \lambda^{4n_1-5\psi_1}.$$  

We complete the proof of the claim. □

We root $T$ at an endvertex of a longest path in $T$. Let $u$ be a leaf of maximum depth in $T'$ and $uvwxy$ be a path of $T$. For a vertex $z \in V(T)$, let $T_z$ be the subtree consisting of $z$ and all of its descendants in $T$.

**Claim 2** $d_T(v) = 2$.

**Proof of Claim 2** Suppose, for a contradiction, that $d_T(v) = 3$. Let $T' = T - V(T_v)$, and then $T'$ has order $n - 3$ and dissociation number $\psi - 2$. Since $d_{T'}(w) < 3$, by Claim 1, we have $\Phi(T') \leq \lambda^{4(n-3)-5(\psi-2)} = \lambda^{4n-5\psi-2}$. A maximum dissociation set in $T$ not containing $w$ can be extended in three ways to a maximum dissociation set in $T'$, while a maximum dissociation set in $T'$ containing $w$ can only be extended in a unique way to a maximum dissociation set in $T$. Since all maximum dissociation sets in $T$ are of one of these types, we obtain

$$\Phi(T) = 3 \cdot \Phi(T') + \Phi_w(T')$$  

$$= \Phi(T') + 2 \cdot \Phi(T')$$

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\[ \lambda^4(n-3) - 5(\psi - 2) - 2 + 2\lambda^{4n-5}\psi - 2 \]
\[ = \left( \frac{1}{\lambda^2} + \frac{2}{\lambda^4} \right)\lambda^{4n-5}\psi + 2 \]
\[ < \lambda^{4n-5}\psi + 2, \]

where we use \( \lambda^2 + 2 < \lambda^4 \). This contradiction completes the proof of the claim.

\[ \square \]

**Claim 3** \( w \) is not a support vertex.

**Proof of Claim 3** Suppose, for a contradiction, that \( w \) is a support vertex. Let \( v' \) be a leaf of \( T \) that is adjacent to \( w \). Let \( T^{(1)} = T - V(T_w) \), and then \( T^{(1)} \) has order \( n - 4 \) and dissociation number \( \psi - 3 \). Let \( T^{(2)} = T - \{u, v\} \), and then \( T^{(2)} \) has order \( n - 2 \) and dissociation number \( \psi - 1 \) or \( \psi - 2 \). Now, we consider the following two cases.

**Case 1.** \( \psi(T^{(2)}) = \psi - 1 \).

In this case, \( w \) is in all maximum dissociation sets in \( T^{(2)} \). Furthermore, a set \( F \) is a maximum dissociation set of \( T \) if and only if

- either \( F = F' \cup \{u, v, v'\} \), where \( F' \) is a maximum dissociation set of cardinality \( \psi - 3 \) of \( T^{(1)} \),
- or \( F = F' \cup \{u\} \), where \( F' \) is a maximum dissociation set of cardinality \( \psi - 1 \) of \( T^{(2)} \).

By the choice of \( n \), this implies

\[ \Phi(T) = \Phi(T^{(1)}) + \Phi(T^{(2)}) \]
\[ \leq \lambda^{4(n-4)} - 5(\psi - 3) + 2 + \lambda^{4(n-2)} - 5(\psi - 1) + 2 \]
\[ = \left( \frac{1}{\lambda^2} + \frac{1}{\lambda^4} \right)\lambda^{4n-5}\psi + 2 \]
\[ = \lambda^{4n-5}\psi + 2, \]

where we use \( \lambda^2 + 1 = \lambda^3 \). This contradiction completes the proof of the case.

**Case 2.** \( \psi(T^{(2)}) = \psi - 2 \).

In this case, every maximum dissociation set of \( T \) contains both \( u \) and \( v \). So we have \( \Phi(T) = \Phi(T^{(1)}) \leq \lambda^{4(n-4)} - 5(\psi - 3) + 2 < \lambda^{4n-5}\psi + 2 \). This contradiction completes the proof of the case.

We complete the proof of the claim.

\[ \square \]

**Claim 4** \( d_T(w) = 2 \)

**Proof of Claim 4** Suppose, for a contradiction, that \( w \) has a child \( v' \) distinct from \( v \). By Claims 2 and 3, the vertex \( v' \) has exactly one child \( u' \) and \( u' \) is a leaf. Let \( T' = T - V(T_w) \). Then \( T' \) has order \( n - 5 \) and dissociation number \( \psi - 4 \). Since every maximum dissociation set of \( T \) contains \( u, u', v \) and \( v' \), we have \( \Phi(T) = \Phi(T') \leq \lambda^{4(n-5)} - 5(\psi - 4) + 2 = \lambda^{4n-5}\psi + 2 \), which contradicts the choice of \( T \). This contradiction completes the proof of the claim.

\[ \square \]
Since we root $T$ at an endvertex of a longest path in $T$, if the vertex $x$ is the root of $T$, then $T \cong P_4$ and $\Phi(P_4) = 2 < \lambda^{4 \times 4 - 5 \times 3 + 2}$. Thus, we can assume that $x$ is a child of a vertex $y$ in $T$.

**Claim 5** $x$ is not a support vertex.

**Proof of Claim 5** Suppose, for a contradiction, that $x$ has a child $w'$ that is a leaf. Let $T^{(1)} = T - V(T_x)$, and then $T^{(1)}$ has order $n - 5$ and dissociation number $\psi - 3$ or $\psi - 4$. Now, we consider the following two cases.

**Case 1.** $\psi(T^{(1)}) = \psi - 3$.

In this case, every maximum dissociation set of $T^{(1)}$ contains $y$. Let $T^{(2)} = T - \{u, v, w\}$, and then $T^{(2)}$ has order $n - 3$ and dissociation number $\psi - 2$. A set $F$ is a maximum dissociation set of $T$ if and only if

- either $F \in \{F' \cup \{u, w, w'\}, F' \cup \{v, w, w'\}\}$, where $F'$ is a maximum dissociation set of cardinality $\psi - 3$ of $T^{(1)}$,
- or $F = F' \cup \{u, v\}$, where $F'$ is a maximum dissociation set of cardinality $\psi - 2$ of $T^{(2)}$.

By the choice of $n$, this implies

$$\Phi(T) = 2 \cdot \Phi(T^{(1)}) + \Phi(T^{(2)})$$

$$\leq 2\lambda^{4(n-5)-5(\psi-3)+2} + \lambda^{4(n-3)-5(\psi-2)+2}$$

$$= \left(\frac{2}{\lambda^5} + \frac{1}{\lambda^4}\right)\lambda^{4n-5\psi+2}$$

$$< \lambda^{4n-5\psi+2},$$

where we use $\lambda^3 + 2 < \lambda^5$. This contradiction completes the proof of the case.

**Case 2.** $\psi(T^{(1)}) = \psi - 4$.

In this case, there is at least one maximum dissociation set $F$ in $T^{(1)}$ such that $y \notin F$. On the other hand, every maximum dissociation set in $T$ contains $u, v, x$ and $w'$. So we have $\Phi(T) = \Phi(T^{(1)}) \leq \Phi(T^{(1)}) \leq \lambda^{4n-5\psi+2}$, which contradicts the choice of $T$.

We complete the proof of the claim. \qed

**Claim 6** $x$ has no child that is a support vertex.

**Proof of Claim 6** Suppose, for a contradiction, that $x$ has a child $w'$ that is a support vertex. By Claims 2 and 3, the vertex $w'$ has exactly one child $v'$ that is a leaf. Let $T^{(1)} = T - \{u, v\}$. Because it is easy to see that there exists at least one maximum dissociation set $F$ of $T$ such that $v \notin F$, $T^{(1)}$ has order $n - 2$ and dissociation number $\psi - 1$. Let $T^{(2)} = T - \{u, v, w\}$, and then $T^{(2)}$ has order $n - 3$ and dissociation number $\psi - 2$. Let $T^{(3)} = T - V(T_x)$, and then $T^{(3)}$ has order $n - 6$ and dissociation number $\psi - 4$.

A set $F$ is a maximum dissociation set of $T$ if and only if

- either $F = F' \cup \{u\}$, where $F'$ is a maximum dissociation set of cardinality $\psi - 1$ of $T^{(1)}$,  

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• or $F = F' \cup \{u, v\}$, where $F'$ is a maximum dissociation set of cardinality $\psi - 2$ of $T^{(2)}$,

• or $F = F' \cup \{v, w, v', w'\}$, where $F'$ is a maximum dissociation set of cardinality $\psi - 4$ of $T^{(3)}$.

By the choice of $n$, this implies

$$
\Phi(T) = \Phi(T^{(1)}) + \Phi(T^{(2)}) + \Phi(T^{(3)}) \\
\leq \lambda^{4(n-2)-5(\psi-1)+2} + \lambda^{4(n-3)-5(\psi-2)+2} + \lambda^{4(n-6)-5(\psi-4)+2} \\
= \left(\frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2}\right) \lambda^{4n-5\psi+2} \\
= \lambda^{4n-5\psi+2},
$$

where we use $\lambda^2 + 2\lambda + 1 = \lambda^4$. This contradiction completes the proof of the claim. □

**Claim 7** $d_T(x) = 2$

**Proof of Claim 7** Suppose, for a contradiction, that $x$ has a child $w'$ distinct from $w$. By Claims 5 and 6, the vertex $w'$ has a child $v'$ that has a child $u'$. By Claims 2 and 3, $d_T(w') = d_T(v') = 2$. Let $T^{(1)} = T - V(T_x)$, and then $T^{(1)}$ has order $n - 7$ and dissociation number $\psi - 4$ or $\psi - 5$. Now, we consider the following two cases.

**Case 1.** $\psi(T^{(1)}) = \psi - 4$.

Let $T^{(2)} = T - (V(T_x) \setminus \{x\})$, and then $T^{(2)}$ has order $n - 6$ and dissociation number $\psi - 4$. In this case, $\psi(T^{(2)}) = \psi(T^{(1)})$, by Lemma 3.1, we have

$$
\Phi_{\bar{T}}(T^{(2)}) \leq \frac{1}{3} \Phi(T^{(2)}),
$$

(2)

$$
2 \cdot \Phi_{\bar{T}}(T^{(2)}) + \Phi^0_x(T^{(2)}) \leq \Phi(T^{(2)}).
$$

(3)

A maximum dissociation set in $T^{(2)}$ not containing $x$ can be extended in nine ways to a maximum dissociation set in $T$, a maximum dissociation set $F$ in $T^{(2)}$ with $x \in F$ and $d_{T^{(2)}[F]}(x) = 0$ can be extended in three ways to a maximum dissociation set in $T$, and a maximum dissociation set $F$ in $T^{(2)}$ with $x \in F$ and $d_{T^{(2)}[F]}(x) = 1$ can only be extended in a unique way to a maximum dissociation set in $T$. Since all maximum dissociation sets of $T$ are of such forms, we obtain

$$
\begin{align*}
\Phi(T) &= 9 \cdot \Phi_{\bar{T}}(T^{(2)}) + 3 \cdot \Phi^0_x(T^{(2)}) + \Phi^1_x(T^{(2)}) \\
&= \Phi(T^{(2)}) + 8 \cdot \Phi_{\bar{T}}(T^{(2)}) + 2 \cdot \Phi^0_x(T^{(2)}). 
\end{align*}
$$

By inequalities (2) and (3), consider the following linear programming:

$$
\begin{align*}
\max \ & 8 \cdot \Phi_{\bar{T}}(T^{(2)}) + 2 \cdot \Phi^0_x(T^{(2)}) \\
\text{s. t.} \ & \Phi_{\bar{T}}(T^{(2)}) \leq \frac{1}{3} \Phi(T^{(2)}) \\
& 2 \Phi_{\bar{T}}(T^{(2)}) + \Phi^0_x(T^{(2)}) \leq \Phi(T^{(2)}).
\end{align*}
$$
\[ \Phi_{\mathcal{T}}(T^{(2)}) \geq 0, \quad \Phi_{\mathcal{X}}^{0}(T^{(2)}) \geq 0. \]

The linear programming has an unique optimal solution
\[ (\Phi_{\mathcal{T}}(T^{(2)}), \Phi_{\mathcal{X}}^{0}(T^{(2)})) = (\frac{1}{3} \Phi(T^{(2)}), \frac{1}{3} \Phi(T^{(2)})) . \]

Thus,
\[
\Phi(T_0) \leq \Phi(T^{(2)}) + \frac{8}{3} \cdot \Phi(T^{(2)}) + \frac{2}{3} \cdot \Phi(T^{(2)}) \\
= \frac{13}{3} \cdot \Phi(T^{(2)}) \\
\leq \frac{13}{3} \lambda^{4(n-6)-5(\psi-4)+2} \\
< \lambda^{4n-5\psi+2},
\]

where we use \( \frac{13}{3} < \lambda^4 \). This contradiction completes the proof of the case.

**Case 2.** \( \psi(T^{(1)}) = \psi - 5 \).

In this case, there is at least one maximum dissociation set \( F \) in \( T^{(1)} \) such that either \( y \notin F \) or \( y \in F \) and \( d_{T^{(1)}}[F](y) = 0 \). Furthermore, a maximum dissociation set of \( T^{(1)} \) can be extended at most three ways to a maximum dissociation set of \( T \). Thus,
\[
\Phi(T) \leq 3 \cdot \Phi(T^{(1)}) \\
\leq 3 \lambda^{4(n-7)-5(\psi-5)+2} \\
< \lambda^{4n-5\psi+2},
\]

where we use \( 3 < \lambda^3 \). This contradiction completes the proof of the case.

We complete the proof of the claim. \( \square \)

We are now in a position to derive a final contradiction. By the above claims, \( d_T(v) = d_T(w) = d_T(x) = 2 \). Let \( T^{(1)} = T - V(T_x) \). Then \( T^{(1)} \) has order \( n - 4 \) and dissociation number \( \psi - 2 \) or \( \psi - 3 \). We consider the following two cases.

**Case 1.** \( \psi(T^{(1)}) = \psi - 2 \).

In this case, for every maximum dissociation set \( F \) of \( T^{(1)} \), we have \( y \in F \) and \( d_{T^{(1)}}[F](y) = 1 \). Let \( T^{(2)} = T - \{ u, v \} \), and then \( T^{(2)} \) has order \( n - 2 \) and dissociation number \( \psi - 1 \); let \( T^{(3)} = T - \{ u, v, w \} \), and then \( T^{(3)} \) has order \( n - 3 \) and dissociation number \( \psi - 2 \). A set \( F \) is a maximum dissociation set of \( T \) if and only if
- either \( F = F' \cup \{ v, w \} \), where \( F' \) is a maximum dissociation set of cardinality \( \psi - 2 \) of \( T^{(1)} \),
- or \( F = F' \cup \{ u \} \), where \( F' \) is a maximum dissociation set of cardinality \( \psi - 1 \) of \( T^{(2)} \),
- or \( F = F' \cup \{ u, v \} \), where \( F' \) is a maximum dissociation set of cardinality \( \psi - 2 \) of \( T^{(3)} \).

By the choice of \( n \), this implies
\[
\Phi(T) = \Phi(T^{(1)}) + \Phi(T^{(2)}) + \Phi(T^{(3)})
\]
Thus, we have

\[ L(\Phi) = \Phi(1) + \Phi(2) = 2 \cdot \Phi(1) + \Phi(1) = 3 \cdot \Phi(1). \]

This final contradiction completes the proof. \( \square \)

**Corollary 3.3** Let \( T \) be a subcubic tree of order \( n \) and dissociation number \( \psi \). If

\[ \psi = \frac{4n+2}{5}, \]

then \( T \) has exactly one maximum dissociation set, i.e., \( \Phi(T) = 1 \).

**Proof** The result can be easily obtained from Theorem 3.2. In fact, if \( \psi = \frac{4n+2}{5} \), by Theorem 2.3, the structure of \( T \) is known and it can be seen that \( \Phi(T) = 1 \). \( \square \)

**Corollary 3.4** Let \( T \) be a subcubic tree of order \( n \). Then

\[ \Phi(T) \leq 2.418 \cdot 1.29^n. \]

**Proof** By Theorems 2.4 and 3.2, we have

\[ \Phi(T) \leq \lambda^{4n-5\psi+2} = \lambda^{\frac{2n}{3}+2} \leq 2.418 \cdot 1.29^n. \]

\( \square \)
**Corollary 3.5** Let \( T \) be a subcubic tree of dissociation number \( \psi \). Then

\[
\Phi(T) \leq 1.466^{\psi+2}.
\]

**Proof** By Theorems 2.4 and 3.2, we have

\[
\Phi(T) \leq \lambda^{\frac{3\psi}{2}} - 5\psi + 2 = \lambda^{\psi+2} = 1.466^{\psi+2}.
\]

\( \Box \)

4 Concluding remarks

In this paper, we give the lower and upper bounds on the dissociation number in a subcubic tree of order \( n \) and show that the number of maximum dissociation sets of a subcubic tree of order \( n \) and dissociation number \( \psi \) is at most \( 1.466^{4n-5\psi+2} \).

An edge \( e \) of a graph \( G \) is called critical if \( \psi(G-e) > \psi(G) \), and a subgraph of \( G \) is called critical if all its edges are critical in \( G \). The second author with two others (Tu et al. 2021) proved that the larger the number of critical paths of order 3 in a tree, the larger the maximum number of maximum dissociation sets of the tree. And they showed that the number of maximum dissociation sets of a tree of order \( n \) is at most \( 1.442^{n-3} + n/3 + 1 \). Compared the result with Corollary 3.4, one can find that the bound on the number of maximum dissociation sets in a subcubic tree is tighter than in a tree, because the number of critical paths of order 3 in a subcubic tree is fewer than in a tree.

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**Data availability** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** The authors have no relevant financial or non-financial interests to disclose.

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