Maximizing the encoded information via freezing the estimated parameters of a pulsed driven qubit

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Abstract—We use a rectangular pulse to freeze the possibility of estimating the coherent parameters (θ, φ) of a single qubit and the encoded information. It is shown that, as the possibility of estimating the parameters increases, the amount of encoded information decreases. The pulse strength and the detuning between the qubit and the pulse have a different effect on the estimation degree and the encoded information. We show that if the weight parameter, θ, is estimated, the encoded information depends on the initial state settings. Meanwhile, the encoded information doesn’t depend on the estimated phase parameter, φ. These results may be useful in the context of quantum cryptography, teleportation and secure communication.

I. INTRODUCTION

It is well known that quantum information tasks, e. g. quantum cryptography, quantum encoding [2], [3] and quantum computation [4], require pure states to be implemented with high efficiencies. However, decoherence is an inevitable process due to the interaction with the surroundings. There are different techniques that have been introduced to protect these states’ decoherence. Among of these methods are quantum purification [5], weak measurement [6], and quantum filtering [7].

Recently, it was shown that different pulse shapes can keep the quantum correlation survival and consequently, the phenomena of the longed lived entanglements is depicted [8]. Very recently, Metwally and Hassan [9] investigated the initial parameters which describe the pulsed driven state that maximize/ minimize the Fisher information which contained in the driven state. However, in our previous work, we showed that the possibility of estimating these parameters is very small during the pulse duration and for some cases it is frozen. This means that, one may estimate the these parameters within a certain constant value during the pulsed time and consequently, if this state is captured by any Eavesdropper, may he/she get a minimum information or nothing at all. Theses observations motivated us to investigate the possibility of freezing [10] the pulsed qubits from a sender to a receiver by using the rectangular pulse.

The paper is organized as following. In Sec.(2), we describe the initial system and its driving by the rectangular pulse. In Sec.(3), we evaluate the encoded information of the driven qubit. Finally, we summarize our result in Sec.(4).

II. THE SUGGESTED MODEL

Here, we consider a single qubit taken as 2-level atomic transition of frequency ω\textsubscript{q} and driven by a short laser pulse of arbitrary shape and of circular frequency ω\textsubscript{c} in the absence of any dissipation process. The quantized Hamiltonian of the system (in units of ħ = 1) in the dipole and rotating wave approximation and in a rotating frame of ω\textsubscript{c} is given by[9],

\[ \hat{H} = \Delta \hat{σ}_z + \frac{Ω(t)}{2}(σ_+ + σ_-) \]  

where, the spin-\( \frac{1}{2} \) operators \( \hat{S}_{\pm, z} \) obey the \( SU(2) \) algebra,

\[ [\hat{σ}^\prime_+, \hat{σ}^\prime_-] = 2\hat{σ}^\prime_z, \quad [\hat{σ}^\prime_z, \hat{σ}^\prime_{\pm}] = ±\hat{σ}^\prime_{\pm} \]

and \( Δ = ω_q - ω_c \) is the atomic detuning and \( Ω(t) = Ω_0 f(t) \), is the real laser Rabi frequency with \( f(t) \) is the pulse shape. Heisenberg equation of motion for the spin operators \( \hat{σ}_x = \frac{1}{2}(\hat{σ}^\prime_+ + \hat{σ}^\prime_-) \), \( \hat{σ}_y = \frac{1}{2i}(\hat{σ}^\prime_+ - \hat{σ}^\prime_-) \) and \( \hat{σ}_z \) according to (1), (2) are of the form,

\[ \hat{σ}^\prime_x = -Δ\hat{σ}_y \]
\[ \hat{σ}^\prime_y = Δ\hat{σ}_x - Ω(t)\hat{σ}_z \]
\[ \hat{σ}^\prime_z = = Ω(t)\hat{σ}_y \]

In the case of a rectangular pulse of a short duration \( T \) (much smaller than the life time of the qubit), we have \( Ω(t) = Ω_0; f(t) = 1, t \in [0, T] \) and zero otherwise. In this case, the exact solution of the average Bloch vector components \( s_{x,y,z}(t) = \langle \hat{σ}_{x,y,z}(t) \rangle \) is the matrix form (cf[9], [11]),

\[ \tilde{σ} (t) = A(t)\tilde{σ} (0) \]  

where \( \tilde{S} = (\hat{σ}_x, \hat{σ}_y, \hat{σ}_z) \) and the matrix \( A = [a_{ij}] ; i, j = 1..3 \) with coefficient \( a_{ij} \) are given in the appendix (A).

Initially, we assume that the information is encoded in the single qubit which is prepared in the coherent state, 

\[ |ψ_q\rangle = \cos(θ/2)|0\rangle + e^{-iφ}\sin(θ/2)|1\rangle, \]

where \( 0 ≤ φ ≤ 2π, 0 ≤ θ ≤ π \) and \( |0\rangle, |1\rangle \) are the lower and upper states, respectively. The initial Bloch vector \( \tilde{s} (0) \) with the state (5) has the components,

\[ s_x(0) = \sin θ \cos φ, \quad s_y(0) = \sin θ \sin φ, \quad s_z(0) = -\cos θ \]  

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III. DYNAMICS OF INFORMATION

A. Mathematical Forms

- Fisher Information:
  It is known that, the density operator for 2-level atomic system is given by,
  \[
  \rho_q = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma})
  \]  
  (7)

  where, \( \vec{s} = (s_x(0), s_y(0), s_z(0)) \) is the Bloch vector and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) are the spin Pauli operators.

  In terms of Bloch vector \( \vec{s}(\beta) \), the quantum Fisher information (QFI) with respect to the parameter \( \beta \) is defined as [9], [13],

  \[
  \mathcal{F}_\beta = \begin{cases} 
  \left( \frac{\partial \vec{s}(\beta)}{\partial \beta} \right)^2 & |\vec{s}(\beta)| < 1, \\
  \left( \frac{\partial \vec{s}(\beta)}{\partial \beta} \right)^2 & |\vec{s}(\beta)| = 1 
  \end{cases}
  \]

  where \( \beta \) is the parameter to be estimated. From Eq.(7), it is clear that the final solution depends on the initial parameters \( (\theta, \phi) \) in addition to the system parameters \( \delta, \Omega_0 \).

- The encoded information
  let us assume that Alice has encoded a given information to be used in the context of quantum cryptography, for example. She will use the Bennett and Wiesner protocol [2]. If the final state is given by

  \[
  \rho(t) = \frac{1}{2}(1 + s_x(t)\sigma_x + s_y(t)\sigma_y + s_z(t)\sigma_z)
  \]  
  (8)

  The amount of the coded information is given by

  \[
  I_{\text{cod}} = -\lambda_1\log\lambda_1 - \lambda_2\log\lambda_2
  \]  
  (9)

  where \( \lambda_i, i = 1, 2 \) are the eigenvalues of the state \( |s \rangle \).

B. Numerical results

In the following subsections, we estimate these parameters by calculating their corresponding QFI, \( \mathcal{F}_\beta \). The larger QFI is the higher degree of estimation for the parameter \( \beta \).

Fig.(1a) describes the behavior of the quantum Fisher information with respect to the weight parameter \( \theta \) as a function of the frequency \( \Omega_0 \) at small value of the detuning parameter, \( \Delta \). It is clear that, the quantum Fisher information \( \mathcal{F}_\theta \) is almost zero for any value of \( \Omega_0 < 0.1 \) and any initial value \( \theta \in [0, \pi] \). This means that, in this interval one can not estimate the weight parameter. However, for larger values of \( \Omega_0 \), \( \mathcal{F}_\theta \) increases gradually to reach its maximum values at \( \Omega_0 = 1 \). Also note that for the range \( 0.4 < \Omega_0 < 1 \), the quantum Fisher information decreases for \( \theta \in [0, \pi/4] \) and increases for \( \theta \in [\pi/4, \pi] \). This behavior is displayed in Fig.(1b), as a contour plot, where it is divided into different regions that have the same degree of brightness/darkness. This means that, in these regions, the quantum Fisher information \( \mathcal{F}_\theta \) is frozen. In the more brightened regions, the possibility of estimating the weight parameter \( \theta \) increases, while it decreases as the darkness increases.

In Fig.(2a), we plot the amount of the encoded information in the pulsed state at \( \Delta = 0.2 \). It is clear that, as soon as the pulse is switched on, the encoded information \( I_{\text{cod}} \) is maximum at small values of \( \Omega_0 \) and for any initial values of the weight parameter, \( \theta \). For larger values of \( \Omega_0 \), the quantum encoded information \( I_{\text{cod}} \) gradually decreases with the minimum values of the estimation degree around \( \pi = \pi/2 \). The contour plot, Fig.(2b), displays the regions in which the encoded information is large and decreases as the initial weight parameter \( (\theta) \) decreases. On the other hand, there are no dark regions depicted which means that the encoded information cannot vanishes.

For larger value of the detuning parameter \( (\Delta = 0.9) \) the contour of \( \mathcal{F}_\theta \) in the \( (\theta, \Omega_0) \)-plane is shown in , Fig.(3), where it shows the areas where the quantum fisher information may be frozen. It is clear that, the dark regions are wider than those displayed for small values of the detuning parameter (see Fig.(1b)). This means that the possibility of estimation \( \theta \) decreases as one increases \( \Delta \).

The contour plot of the encoded information, \( I_{\text{cod}} \) in Fig.(4) shows that the size of the bright regions is much larger than that displayed in Fig.(2b). However, the degree of brightness decreases as \( \Omega_0 \) increases which means that there is a leakage of the pulsed information.
From Figs.(1-4), one may conclude that, it is possible to freeze the coherence of the estimation degree of the weight parameter (θ) by controlling the strength of the pulse and the detuning between the qubit and the pulse. For larger values of the detuning and smaller values of the strength one can increase the possibility of freezing the estimation degree of the weight parameter. The amount of the coded information may be maximized as the estimation degree of the weight parameter is minimized.

Figs.(5) and (6) display the contour behavior of the quantum Fisher information and the encoded information, respectively, in the (θ, Δ)-plane. It is clear that, the detuning parameter has a decoherence effect on the Fisher information, with a coherence effect on the encoded information. Fig.(5) shows the size of regions in which one may estimate the weight parameter (θ), where the possibility of freezing the pulsed Fisher information increases as the detuning parameter increases.

The dynamics of the pulsed encoded information, I_{cod} is depicted in Fig.(6), where it reaches its maximum values at Δ = θ = 0 and decreases suddenly as the initial weight parameter increases and vanish completely at θ ≃ π/16. However, at any θ ∈ [π/16, 15π/16] and Δ < 0.1, the encoded information is almost zero. For larger values of Δ and arbitrary value of the weight parameter, the encoded infor-
Fig. 6. The same as Fig.(5) but for the encoded information $I_{\text{cod}}$.

Fig. 7. The contour plot of $\mathcal{F}_\phi$ in the $(\phi, \Delta)$-plane with $\Omega_0 = 0.5, \theta = \pi$.

In Figs.(7) and (8), we investigate the behavior of the Fisher information $\mathcal{F}_\phi$ and the encoded information when the phase parameter $(\phi)$ is estimated, such that the driven qubit is initially prepared in the state $e^{-i\delta}|1\rangle$, namely, we set the weight parameter $\theta = \pi$. It is clear that, the larger values of the detuning has a decoherence effect on the Fisher information $\mathcal{F}_\phi$, where it decreases as $\Delta$ increases. Fig.(7) displays the area in which the Fisher information is frozen, where the degree of the darkness indicates the estimation. As $\Delta$ increases, the darkness increases which means that, the possibility of estimating the phase parameter $(\phi)$ decreases.

On the other hand, Fig.(8), for the encoded information shows that the brightness increases as $\Delta$ increases and the maximum bounds are displayed around $\phi = \pi/2$.

Fig.(9) describes the contour behavior of the encoded information $I_{\text{cod}}$ for a different initial state setting, where it is assumed that the qubit is initially prepared in the state $|\psi(0)\rangle = |0\rangle$, namely, $\theta = 0$. This means that, the initial state doesn’t depend on the phase $\phi$ and may be taken arbitrary. On the other hand, the freezing phenomena of the pulsed encoded information, $I_{\text{cod}}$ is depicted at small values of the detuning and the degree of freezing decreases as the detuning increases.

IV. CONCLUSIONS

In this contribution, we investigate the relation between the pulsed Fisher information of the qubit’s parameters and the encoded information. The suggested system consists of a single qubit driven by a rectangular pulse. These physical quantities, the Fisher and the encoded information, are discussed for different values of the pulse strength and the detuning between the qubit and the pulse.

In case of estimating the weight parameter $(\theta)$, it is shown that, large values of the pulse strength increase the possibility of estimating the weight parameter and decreases the capacity of encoded information in the qubit. Large values
of the detuning increase the size of the frozen areas for the two physical quantities; estimation degree and the channel capacity. However, for increased detuning, the estimation degree of the weight parameter increases, while the channel capacity decreases. The behavior of the Fisher information and the encoded information as functions of the detuning parameter is discussed for small values of the pulse strength. It is shown that, it is possible to maximize the channel capacity at the expense of the estimation degree. Moreover, one can always freeze both quantities for any initial state setting of the weight parameter.

The behavior of Fisher information and the coded information is discussed when the phase parameter \( \phi \) is estimated. In this case, the initial phase plays an important role on the decoherence/ coherence effect of the pulse. The results, show that the encoded information doesn’t depend on \( \phi \), while the Fisher information depend on it.

In conclusion, it is possible to freeze the Fisher information and the amount of the encoded information for both qubit parameters \((\theta, \phi)\). One can increase the size of the frozen area of the encoded information at the expense of Fisher information. We show that, the encoded information doesn’t depend on the phase \( \phi \). We expect that, these results may be useful in the context of cryptography and secure communications.

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V. APPENDIX(A)

The coefficients \( a_{ij} \) of the matrix \( A(t) \) in Eq.(4) are as follows:

\[
\begin{align*}
a_{11} &= \frac{1}{\eta} + \delta^2 \cos(\tau \sqrt{\eta}) - \delta \lambda_1 \\
a_{12} &= \frac{1}{2} (1 + \frac{\lambda_2}{\eta} + \delta \lambda_1) \\
a_{13} &= \frac{\delta}{\eta} \lambda_3 + \lambda_1 \\
a_{21} &= \frac{\lambda_3}{2\eta} + \lambda_1 \\
a_{22} &= \cos(\tau \sqrt{\eta}) - \delta \lambda_1 \\
a_{23} &= \frac{\delta}{\eta} \lambda_3 - \delta \lambda_1 \\
a_{31} &= \frac{\delta}{\eta} \lambda_3, \quad a_{32} = \lambda_1, \quad a_{33} = \frac{\lambda_2}{\eta}
\end{align*}
\]

where,

\[
\begin{align*}
\lambda_1 &= \frac{1}{\sqrt{\eta}} \\
\lambda_2 &= \delta^2 + \cos(\tau \sqrt{\eta}), \\
\lambda_3 &= \frac{1}{2} \eta \lambda_1^2, \quad \lambda_4 = 1 + (\eta + \delta^2) \cos(\tau \sqrt{\eta})
\end{align*}
\]

and \( \delta = \frac{\Delta}{\Omega}, \eta = 1 + \delta^2, \eta = \Omega_0 t \)