Quaternionic Fundamental Cardinal Splines: Interpolation and Sampling

Jeffrey A. Hogan¹ · Peter R. Massopust²

Received: 28 October 2018 / Accepted: 16 July 2019 / Published online: 12 August 2019
© Springer Nature Switzerland AG 2019

Abstract
B-splines $B_q$ of quaternionic order $q$, for short quaternionic B-splines, are quaternion-valued piecewise Müntz polynomials whose scalar parts interpolate the classical Schoenberg splines $B_n$, $n \in \mathbb{N}$, with respect to degree and smoothness. They in general do not satisfy the interpolation property $B_q(n) = \delta_{n,0}$, $n \in \mathbb{Z}$. However, the application of the interpolation filter $(\sum_{k \in \mathbb{Z}} B_q(\xi + 2\pi k))^{-1}$—if well-defined—in the frequency domain yields a cardinal fundamental spline of quaternionic order that satisfies the interpolation property. We handle the ambiguity of the quaternion-valued exponential function appearing in the denominator of the interpolation filter and relate the filter to interesting properties of a quaternionic Hurwitz zeta function and the existence of complex quaternionic inverses. Finally, we show that the cardinal fundamental splines of quaternionic order fit into the setting of Kramer’s Lemma and allow for a family of sampling, respectively, interpolation series.

Keywords Quaternions · Clifford analysis · Fundamental cardinal spline · Hilbert module · Sampling · Kramer’s lemma

Communicated by Uwe Kaehler.

J. A. Hogan: Research partially supported by Australian Research Council Grant DP160101537.
J. A. Hogan, P. R. Massopust: Research partially supported by Bavarian Research Alliance Grant BayIntAn_UPA_2017_76.
This article is part of the topical collection “Higher Dimensional Geometric Function Theory and Hypercomplex Analysis” edited by Irene Sabadini, Michael Shapiro and Daniele Struppa.

Jeffrey A. Hogan
jeff.hogan@newcastle.edu.au

Peter R. Massopust
massopust@ma.tum.de

¹ School of Mathematical and Physical Sciences, Mathematics Bldg V123, University of Newcastle, University Drive, Callaghan, NSW 2308, Australia
² Centre of Mathematics, Research Unit M15, Technical University of Munich, Boltzmannstr. 3, 85748 Garching, Germany
1 Introduction

In [7], a new class of B-splines with quaternionic order ( quaternionic B-splines for short) were introduced with the intention to obtain interpolants and approximants for data that require a multi-channel description. For example, seismic data has four channels, each associated with a different kind of seismic wave: the so-called P (Compression), S (Shear), L (Love) and R (Rayleigh) waves. Similarly, the colour value of a pixel in a colour image has three components—the red, green and blue channels. In order to perform the tasks of processing multi-channel signals and data, an appropriate set of analyzing basis functions is required. These basis functions should have the same analytic properties as those enjoyed by classical B-splines but should in addition be able to describe multi-channel structures. In [11,12], a set of analyzing functions based on wavelets and Clifford-analytic methodologies were introduced in an effort to process four channel seismic data. A multiresolution structure for the construction of wavelets on the plane for the analysis of four-channel signals was outlined in [8].

The Schoenberg cardinal spline interpolation problem on the real line \( \mathbb{R} \) consists of finding a spline \( L \) with the property \( L(k) = \delta_{k,0} \), for all \( k \in \mathbb{Z} \), and which is a linear combination of shifts of a cardinal B-spline \( B: L = \sum_{k \in \mathbb{Z}} c_k B(\cdot - k) \). It is therefore natural to ask whether the newly introduced quaternionic B-splines \( B_q \) also solve a cardinal interpolation problem. In the classical case, only fundamental cardinal splines of even order exist and one should expect that some restrictions on \( q \) in the quaternionic setting will occur as well.

The solution of the cardinal spline interpolation problem in the quaternionic setting is based on finding zero-free regions of a certain linear combination of quaternion-valued Hurwitz zeta functions. These zero-free regions are found by considering zero-free regions for associated complex-valued Hurwitz zeta functions. In the complex-valued case, such zero-free regions were derived in [5] and they will determine nonempty regions in the real quaternionic algebra \( \mathbb{H}_\mathbb{R} \) for which the quaternionic cardinal spline interpolation problem has a unique solution.

Associated with the cardinal spline interpolation problem is Kramer’s sampling theorem which gives conditions under which a function can be represented as an infinite series involving sampled values and a cardinal spline-type sampling function. We will extend Kramer’s lemma to the complex quaternionic setting by first deriving a Riesz representation theorem for left Hilbert modules.

2 Notation and Preliminaries

The real, associative algebra of real quaternions \( \mathbb{H}_\mathbb{R} \) is given by

\[
\mathbb{H}_\mathbb{R} := \left\{ a + \sum_{i=1}^{3} v_i e_i : a, v_1, v_2, v_3 \in \mathbb{R} \right\},
\]
where the imaginary units $e_1, e_2, e_3$ satisfy $e_1^2 = e_2^2 = e_3^2 = -1$, $e_1 e_2 = e_3$, $e_2 e_3 = e_1$ and $e_3 e_1 = e_2$. Because of these relations, $\mathbb{H}_\mathbb{R}$ is a non-commutative algebra.

Each quaternion $q = a + \sum_{i=1}^3 v_i e_i$ may be decomposed as $q = \text{Sc} q + \text{Vec} q$ where $\text{Sc} q := a$ is the scalar part of $q$ and $\text{Vec} q := v = \sum_{i=1}^3 v_i e_i$ is the vector part of $q$. The conjugate $\bar{q}$ of the real quaternion $q = a + v$ is the quaternion $\bar{q} = a - v$. Note that $q\bar{q} = \bar{q} q = |q|^2 = a^2 + |v|^2 = a^2 + \sum_{i=1}^3 v_i^2$. An element $q$ of $\mathbb{H}_\mathbb{R}$ is called a pure quaternion if $\text{Sc} q = 0$. Note also that if $v = \sum_{j=1}^3 v_j e_j$ and $w = \sum_{j=1}^3 w_j e_j$ are pure quaternions, then

$$vw = -\langle v, w \rangle + v \wedge w,$$

(2.1)

where $\langle v, w \rangle := \sum_{j=1}^3 v_j w_j$ is the scalar product of $v$ and $w$ and $v \wedge w := (v_2 w_3 - v_3 w_2)e_1 + (v_3 w_1 - v_1 w_3)e_2 + (v_1 w_2 - v_2 w_1)e_3$ is the vector (cross) product of $v$ and $w$.

If $q = q_0 + \sum_{i=1}^3 q_i e_i \in \mathbb{H}_\mathbb{C} := \{ a + \sum_{i=1}^3 v_i e_i : a, v_1, v_2, v_3 \in \mathbb{C} \}$ is a complex quaternion, we define the conjugate $\bar{q}$ of $q$ by $\bar{q} := \bar{q}_0 - \sum_{i=1}^3 \bar{q}_i e_i$, where $\bar{q}_i$ is the complex conjugate of the complex number $q_i$. If $p = p_0 + \sum_{i=1}^3 p_i e_i \in \mathbb{H}_\mathbb{C}$, we define the inner product $\langle p, q \rangle$ to be the complex number $\langle p, q \rangle := \sum_{i=0}^3 p_i \bar{q}_i = \text{Sc} p \bar{q}$. We also define $e^q$ by the usual series: $e^q := \sum_{j=0}^\infty \frac{q^j}{j!}$. We require the following bounds on $|e^q|$, which we state without proof.

**Lemma 1** Let $q = a + v \in \mathbb{H}_\mathbb{R}$. Then we have

1. $|e^q| = e^a \leq e^{|q|}$.
2. If $z \in \mathbb{C}$ and $q' = zq \in \mathbb{H}_\mathbb{C}$, then $|e^{q'}| \leq e^{\sqrt{2}|q'|}$.

For $q = a + v = a + \sum_{j=1}^3 v_j e_j \in \mathbb{H}_\mathbb{R}$ and $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, the quaternionic power $z^q \in \mathbb{H}_\mathbb{C}$ is defined by

$$z^q := z^a \left[ \cos(|v| \log z) + \frac{v}{|v|} \sin(|v| \log z) \right].$$

(2.2)

This definition of $z^q$ allows for the usual differentiation and integration rules:

$$\frac{d}{dz} z^q = q z^{q-1}$$

and

$$\int z^q dz = \frac{z^{q+1}}{q+1} \text{ + const.}, \quad q \neq -1.$$  

(2.3)

In general, however, the semigroup property $z^{q_1} z^{q_2} = z^{q_1+q_2}$ fails to hold. (See, for instance, [7].)

If $\mathbb{F} \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H}_\mathbb{R}, \mathbb{H}_\mathbb{C} \}$ and $1 \leq p < \infty$, then $L^p(\mathbb{R}, \mathbb{F})$ denotes the Banach space of measurable functions $f : \mathbb{R} \rightarrow \mathbb{F}$ for which $\int_{-\infty}^{\infty} |f(x)|^p \, dx < \infty$, where
the meaning of \( |f(x)| \) is dependent on \( \mathbb{F} \). The Banach space \( L^\infty(\mathbb{R}, \mathbb{F}) \) is defined similarly.

On \( L^2(\mathbb{R}, \mathbb{C}) \), we define an inner product by \( \langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx \), and on \( L^2(\mathbb{R}, \mathbb{H_C}) \) by

\[
\langle f, g \rangle := \text{Sc} \left( \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx \right) = \int_{-\infty}^{\infty} \langle f(x), g(x) \rangle \, dx.
\]

(2.4)

The Fourier-Plancherel transform \( \mathcal{F} \) is defined on \( L^1(\mathbb{R}, \mathbb{F}) \) by

\[
(\mathcal{F} f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-i\xi x} \, dx
\]

and may be extended to \( L^2(\mathbb{R}, \mathbb{F}) \), on which it becomes a multiple of a unitary mapping:

\[
\langle \mathcal{F} f, \mathcal{F} g \rangle = 2\pi \langle f, g \rangle.
\]

3 Quaternionic B-splines

In this section, we state the definition of B-splines of quaternionic order and briefly review some of their properties. More details and proofs can be found in [7].

The complex B-spline \( B_z \) \((z \in \mathbb{C})\) is defined in the Fourier domain to be the function \( \hat{B_z} : \mathbb{R} \rightarrow \mathbb{C} \) given by

\[
\hat{B_z}(\xi) = \left( \frac{1 - e^{-i\xi}}{i\xi} \right)^z, \quad \text{Re} \, z > 1,
\]

and in the time domain by

\[
B_z(t) = \frac{1}{\Gamma(z)} \sum_{k=0}^{\infty} \binom{z}{k} (t - k)^{z-1}, \quad t \in \mathbb{R}, \quad \text{Re} \, z > 1.
\]

(See [3,7]).

A B-spline \( B_q \) of quaternionic order \( q \) (for short quaternionic B-spline) is defined in the Fourier domain to be the function \( \hat{B_q} : \mathbb{R} \rightarrow \mathbb{H_C} \) given by

\[
\hat{B_q}(\xi) := \left( \frac{1 - e^{-i\xi}}{i\xi} \right)^q, \quad \text{Sc} \, q > 1.
\]

(3.1)

Setting \( \Xi(\xi) := \frac{1 - e^{-i\xi}}{i\xi} \), we obtain the precise meaning of (3.1), namely,

\[
\hat{B_q}(\xi) = \Xi(\xi)^{\text{Sc} q} \left( \cos(|v| \log \Xi(\xi)) + \frac{v}{|v|} \sin(|v| \log \Xi(\xi)) \right).
\]

(3.2)
The function \( \Xi \) has a removable singularity at \( \xi = 0 \) with \( \Xi(0) = 1 \). It follows from \( \text{Re} \, \Xi(\xi) = \xi^{-1} \sin \xi \) and \( \text{Im} \, \Xi(\xi) = \xi^{-1} (1 - \cos \xi) \), that graph \( \Xi \cap (\mathbb{R}^{-} \times \{0\}) = \emptyset \). Hence, \( \Xi \) and therefore \( B_q \) are well-defined when \(-\pi < \arg \Xi \leq \pi\). Equation (3.2) also implies that \( \hat{B}_q \in L^2(\mathbb{R}, \mathbb{H}) \) for a fixed \( q \) with \( Sc \, q > \frac{1}{2} \) and in \( L^1(\mathbb{R}, \mathbb{H}) \) for a fixed \( q \) with \( Sc \, q > 1 \) as the fractional B-splines [16] satisfy these conditions. Note that \( B_q \in L^1(\mathbb{R}, \mathbb{H}) \) implies that \( B_q \) is uniformly continuous on \( \mathbb{R} \).

The following results are a direct consequence of the corresponding properties of fractional B-splines [16]. To this end, for a real number \( s \geq 0 \) and \( 1 \leq p \leq \infty \), the Bessel potential space \( H^{s,p}(\mathbb{R}, \mathbb{H}) \) is given by

\[
H^{s,p}(\mathbb{R}, \mathbb{H}) := \left\{ f \in L^p(\mathbb{R}, \mathbb{H}) : \mathcal{F}^{-1}\left[(1 + |\xi|^2)^{s/2}\mathcal{F} f\right] \in L^p(\mathbb{R}, \mathbb{H}) \right\}.
\]

**Proposition 1** Let \( B_q \) be a quaternionic B-spline with \( Sc \, q > \frac{1}{2} \). Then \( B_q \) enjoys the following properties:

(i) **Decay:** \( B_q \in O(|t|^{-Sc\, q-\frac{1}{2}}) \) as \( |t| \to \infty \).

(ii) **Smoothness:** \( B_q \in H^{s,p}(\mathbb{R}, \mathbb{H}) \) for \( 1 \leq p \leq \infty \) and \( 0 \leq s < Sc \, q + \frac{1}{2} \).

(iii) **Reproduction of Polynomials:** \( B_q \) reproduces polynomials up to order \( \lfloor Sc \, q \rfloor \), where the ceiling function \( \lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z} \) is given by \( r \mapsto \min\{n \in \mathbb{Z} : n \geq r\} \).

Of particular interest in this paper is property (i) of the proposition. A proof can be obtained from [16]. Here, we give a less elaborate argument that gives a weaker decay estimate which is nonetheless sufficient for our purposes.

With \( D := \frac{1}{i} \frac{d}{d\xi} \) and \( m \) a non-negative integer, Leibnitz’s formula gives

\[
D^m \hat{B}_q(\xi) = i^{-m} \sum_{k=0}^{m} \binom{m}{k} c_{q,m,k} (1 - e^{-i\xi})^k \xi^{-q-m+k}
\]

with \( c_{q,m,k} (0 \leq k \leq m) \) quaternionic constants. Therefore, for all integers \( \ell \geq 1 \),

\[
\int_{2\pi(\ell-1/2)}^{2\pi(\ell+1/2)} |D^m \hat{B}_q(\xi)| d\xi \leq \sum_{k=0}^{m} c_{q,m,k} \int_{2\pi(\ell-1/2)}^{2\pi(\ell+1/2)} |1 - e^{-i\xi}|^{Sc\, q-k} |\xi|^{-Sc\, q-m+k} d\xi \\
\leq \sum_{k=0}^{m} c_{q,m,k} (\ell - 1/2)^{-Sc\, q-m+k} \int_{2\pi(\ell-1/2)}^{2\pi(\ell+1/2)} |1 - e^{-i\xi}|^{Sc\, q-k} d\xi \\
\leq \sum_{k=0}^{m} c_{q,m,k} (\ell - 1/2)^{-Sc\, q-m+k} \leq c_{q,m} (\ell - 1/2)^{-Sc\, q}
\]

provided \( Sc \, q - m > -1 \). Therefore,

\[
\int_{-\infty}^{\infty} |D^m \hat{B}_q(\xi)| d\xi \leq 2 c_{q,m} \sum_{\ell=1}^{\infty} (\ell - 1/2)^{-Sc\, q} < \infty
\]
provided \( \text{Sc} \, q > 1 \) and \( m < \text{Sc} \, q + 1 \). Hence, for this range of values of \( q \) and \( m \) we have \( B_q(t) \leq c_m |t|^{-m} \) for all \( t \in \mathbb{R} \). Thus, we have proved the following result.

**Proposition 2** Let \( \text{Sc} \, q > 1 \), then

(i) \( B_q \in L^1(\mathbb{R}) \).

(ii) The Fourier series \( \sum_{k=-\infty}^{\infty} B_q(k) e^{ik\xi} \) is absolutely convergent.

The time domain representation of the quaternionic B-spline \( B_q \) is given by

\[
B_q(t) = \frac{1}{\Gamma(q)} \sum_{k=0}^{\infty} (-1)^k \binom{q}{k} (t-k)^{q-1}, \quad t \in \mathbb{R}, \, \text{Sc} \, q > 1. \tag{3.3}
\]

This equality holds in the sense of distributions and in \( L^2(\mathbb{R}) \). Here, the quaternionic binomial coefficient is defined via the quaternionic Gamma function:

\[
\Gamma(q) := \int_0^{\infty} t^{a-1} \cos(|v| \log t) e^{-t} dt + \frac{v}{|v|} \int_0^{\infty} t^{a-1} \sin(|v| \log t) e^{-t} dt. \tag{3.4}
\]

It was shown in [7] that \( \Gamma(q) \) can be written as

\[
\Gamma(q) = \frac{\Gamma(a + i|v|) + \Gamma(a - i|v|)}{2} + \frac{v}{|v|} \left( \frac{\Gamma(a + i|v|) - \Gamma(a - i|v|)}{2i} \right)
= \text{Re} \, \Gamma(z) + \frac{v}{|v|} \text{Im} \, \Gamma(z), \quad z := a + i|v|. \tag{3.5}
\]

The next result shows that there exists also a structure formula for the quaternionic B-splines.

**Theorem 1** If \( q = a + \nu \in \mathbb{H}_\mathbb{R} \) and \( z = a + i|\nu| \in \mathbb{C} \), then

\[
B_q(t) = \text{Re} \, B_z(t) + \frac{\nu}{|\nu|} \text{Im} \, B_z(t), \quad t \in \mathbb{R} \setminus \{0\}.
\]

**Proof** With \( q \in \mathbb{H}_\mathbb{R} \) and \( z \in \mathbb{C} \) as in the statement of the Theorem, in [7] the quaternionic Pochhammer symbol \( (q)_j = q(q-1) \ldots (q-j+1) \) was decomposed as

\[
(q)_j = \text{Re}(z)_j + \frac{\nu}{|\nu|} \text{Im}(z)_j. \tag{3.6}
\]

By (3.6) we have

\[
\binom{q}{k} = \binom{q}{k} = \frac{\text{Re}(z)_k + \nu}{|\nu|} \text{Im}(z)_k = \text{Re} \binom{z}{k} + \frac{\nu}{|\nu|} \text{Im} \binom{z}{k}.
\]
Furthermore,

\[ t^{q-1} - t^{a-1} \left[ \cos(|v| \log t) + \frac{v}{|v|} \sin(|v| \log t) \right] = t^{a-1} \left[ \Re t^{i|v|} + \frac{v}{|v|} \Im t^{i|v|} \right] = \Re t^{z-1} + \frac{v}{|v|} \Im t^{z-1}. \]

Thus,

\[
(\frac{q}{k})(t - k)_{+}^{q-1} = \left[ \Re \left( \frac{z}{k} \right) + \frac{v}{|v|} \Im \left( \frac{z}{k} \right) \right] \left[ \Re ((t - k)_{+}^{\zeta-1}) + \frac{v}{|v|} \Im ((t - k)_{+}^{\zeta-1}) \right] = \Re \left( \frac{z}{k} \right) \Re (t - k)_{+}^{\zeta-1} - \Im \left( \frac{z}{k} \right) \Im (t - k)_{+}^{\zeta-1} + \frac{v}{|v|} \left[ \Re \left( \frac{z}{k} \right) \Im (t - k)_{+}^{\zeta-1} + \Im \left( \frac{z}{k} \right) \Re (t - k)_{+}^{\zeta-1} \right] = \Re \left[ \left( \frac{z}{k} \right) (t - k)_{+}^{\zeta-1} \right] + \frac{v}{|v|} \Im \left[ \left( \frac{z}{k} \right) (t - k)_{+}^{\zeta-1} \right]. \tag{3.7}
\]

Substituting (3.7) into (3.3) yields

\[
B_q(t) = \frac{1}{\Gamma(q)} \left[ \Re \left( \sum_{k=0}^{\infty} \left( \frac{z}{k} \right) (t - k)_{+}^{\zeta-1} \right) + \frac{v}{|v|} \Im \left( \sum_{k=0}^{\infty} \left( \frac{z}{k} \right) (t - k)_{+}^{\zeta-1} \right) \right] = \frac{1}{\Gamma(q)} [\Re(\Gamma(z)B_z(t)) + \frac{v}{|v|} \Im(\Gamma(z)B_z(t))]. \tag{3.8}
\]

On the other hand, using (3.5), we obtain

\[
\frac{1}{\Gamma(q)} = \frac{\Re \Gamma(z) - \frac{v}{|v|} \Im \Gamma(z)}{\Gamma(z)^2}, \tag{3.9}
\]

and substituting (3.9) into (3.8) gives

\[
B_q(t) = \left[ \frac{\Re \Gamma(z) - \frac{v}{|v|} \Im \Gamma(z)}{\Gamma(z)^2} \right] \left[ \Re(\Gamma(z)B_z(t)) + \frac{v}{|v|} \Im(\Gamma(z)B_z(t)) \right] = \frac{1}{\Gamma(z)^2} \left[ \Re(\Gamma(z) \Re(\Gamma(z)B_z(t)) + \Im(\Gamma(z)B_z(t))) \right] + \frac{v}{|v|} \left[ \Re(\Gamma(z) \Im(\Gamma(z)B_z(t)) - \Im(\Gamma(z) \Re(\Gamma(z)B_z(t))) \right] = \frac{1}{\Gamma(z)^2} \left[ \Re(\Gamma(z)^2B_z(t)) + \frac{v}{|v|} \Im(\Gamma(z)^2B_z(t)) \right] = \Re B_z(t) + \frac{v}{|v|} \Im B_z(t). \]

\[\Box\]
4 Quaternion Inverses

Let \( q = z + \sum_{i=1}^{3} w_i e_i \in \mathbb{H}_{\mathbb{C}} \) (i.e., \( z, w_1, w_2, w_3 \in \mathbb{C} \)). Then with \( \tilde{q} = z - w \) we have \( q \tilde{q} = z^2 + \sum_{i=1}^{3} w_i^2 \), so that \( q \) is invertible if and only if \( z^2 + \sum_{i=1}^{3} w_i^2 \neq 0 \) and in this case \( q^{-1} = \tilde{q} / (z^2 + \sum_{i=1}^{3} w_i^2) \). When \( q = a + v \in \mathbb{H}_{\mathbb{R}} \), \( q \) is invertible provided \( q \neq 0 \) and then \( q^{-1} = \tilde{q} / |q|^2 \).

**Lemma 2** If \( \lambda \in \mathbb{C} \) and \( q \in \mathbb{H}_{\mathbb{R}} \) then \( e^{\lambda q} \) is invertible in \( \mathbb{H}_{\mathbb{C}} \) and

\[
(e^{\lambda q})^{-1} = e^{-\lambda q}.
\]

**Proof** By direct calculation we find that

\[
e^{\lambda q} = e^{\lambda a} e^{\lambda v} = e^{\lambda a} \left( \cos(\lambda |v|) + \frac{v}{|v|} \sin(\lambda |v|) \right)
\]

and therefore,

\[
e^{\lambda q} e^{-\lambda q} = e^{\lambda a} \left( \cos(\lambda |v|) + \frac{v}{|v|} \sin(\lambda |v|) \right) e^{-\lambda a} \left( \cos(\lambda |v|) - \frac{v}{|v|} \sin(\lambda |v|) \right)
\]

\[
= \cos^2(\lambda |v|) + \sin^2(\lambda |v|) = 1.
\]

\[\square\]

**Lemma 3** For all \( t \in \mathbb{R}^+ \) and \( q \in \mathbb{H}_{\mathbb{R}} \), we have

\[
(-t)^q = e^{i\pi q t q}.
\]

**Proof** From the definition of a quaternionic power, we have for \( t > 0 \),

\[
(-t)^q = (-t)^a \left[ \cos(|v| \log(-t)) + \frac{v}{|v|} \sin(|v| \log(-t)) \right]
\]

\[
= (-t)^a \left[ \cos(|v| \log t + i\pi) + \frac{v}{|v|} \sin(|v| \log t + i\pi) \right]
\]

\[
= (-t)^a \left[ \cos(|v| \log t) \cosh(\pi |v|) - i \sin(|v| \log t) \sinh(\pi |v|) \right.
\]

\[
+ \frac{v}{|v|} \left( \sin(|v| \log t) \cosh(\pi |v|) + i \cos(|v| \log t) \sinh(\pi |v|) \right)
\]

\[
= (-t)^a \left[ \cos(|v| \log t) \left[ \cosh(\pi |v|) + \frac{iv}{|v|} \sinh(\pi |v|) \right] \right.
\]

\[
+ \sin(|v| \log t) \left[ -i \sinh(\pi |v|) + \frac{v}{|v|} \cosh(\pi |v|) \right]
\]

\[
= e^{i\pi a} e^{i\pi v t a} \left( \cos(|v| \log t) + \frac{v}{|v|} \sin(|v| \log t) \right) = e^{i\pi q t a + v} = e^{i\pi q t q}.
\]

Here, we used the usual convention that \( t^a := e^{i\pi a |t|^a} \), for \( t < 0 \). \[\square\]
5 A Quaternionic Zeta Function

In this section we study the quaternionic analog $\zeta(q, a)$ of the classical Hurwitz zeta function $\zeta(s, a)$, which is defined by

$$ \zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(a + n)^s}, \quad \text{Re } s > 1 \text{ and Re } a > 0. $$

To do so, we first define functions $\chi_+$ and $\chi_-$ by

$$ \chi_{\pm}(v) := \frac{1}{2} \left( 1 \pm i \frac{v}{|v|} \right), \quad \left( v = \sum_{i=1}^{3} v_i e_i \in \mathbb{H}_\mathbb{R} \right) $$

and note that

$$ \chi_{\pm}(v)^2 = \chi_{\pm}(v); \quad \chi_{+}(v)\chi_{-}(v) = \chi_{-}(v)\chi_{+}(v) = 0. \quad (5.1) $$

Given $a > 1$ and $q = a + v \in \mathbb{H}_\mathbb{R}$, we define the quaternionic zeta function $\zeta(q, \cdot) : \mathbb{R} \to \mathbb{H}_\mathbb{R}$ by

$$ \zeta(q, a) := \sum_{k=0}^{\infty} \frac{1}{(a + k)^q}, \quad a > 0. $$

Lemma 4 Let $q = a + v \in \mathbb{H}_\mathbb{R}$. Then with $w = a + i|v| \in \mathbb{C}$, we have

$$ \zeta(q, a) = \chi_{+}(v)\zeta(w, a) + \chi_{-}(v)\zeta(\overline{w}, a), $$

where $\zeta(w, a)$ and $\zeta(\overline{w}, a)$ denote the complex-valued Hurwitz zeta functions. Further, for all positive integers $n$,

$$ \zeta(q, a)^n = \chi_{+}(v)\zeta(w, a)^n + \chi_{-}(v)\zeta(\overline{w}, a)^n. $$

Proof Note that (2.2) and the assumption $\text{Re } q = a > 1$ imply that the infinite series $\sum_{k=0}^{\infty} \frac{1}{(a + k)^q}, a > 0$, converges absolutely. For each $z \in \mathbb{C}^*$ we have

$$ z^q = z^a \left[ \cos(|v| \log z) + \frac{v}{|v|} \sin(|v| \log z) \right] $$

$$ = z^a \left[ e^{i|v| \log z} + e^{-i|v| \log z} \right] \left[ \frac{e^{i|v| \log z} - e^{-i|v| \log z}}{2i} \right] $$

$$ = z^a \left[ \chi_{-}(v)z^{i|v|} + \chi_{+}(v)z^{-i|v|} \right] = \chi_{-}(v)z^w + \chi_{+}(v)z^{\overline{w}}.
where \( w = a + i|v| \in \mathbb{C} \). Thus,

\[
\zeta(q, a) = \sum_{k=0}^{\infty} (a + k)^{-q} \\
= \sum_{k=0}^{\infty} \left[ \frac{\chi_-(v)}{(a + k)^w} + \frac{\chi_+(v)}{(a + k)^{\overline{w}}} \right] = \chi_-(v)\zeta(w, a) + \chi_+(v)\zeta(\overline{w}, a).
\]

Hence, because of (5.1), we have

\[
\zeta(q, a)^n = \sum_{j=0}^{n} \binom{n}{j} \chi_-(v)^j \zeta(w, a)^j \chi_+(v)^{n-j} \zeta(\overline{w}, a)^{n-j} \\
= \chi_-(v)\zeta(w, a)^n + \chi_+(v)\zeta(\overline{w}, a)^n.
\]

\[ \square \]

6 Interpolation with Quaternionic B-splines

In order to solve the cardinal spline interpolation problem using the classical Curry-Schoenberg splines \([1,15]\), one constructs a fundamental cardinal spline function which is a linear bi-infinite combination of polynomial B-splines \( B_n \) of fixed order \( n \in \mathbb{N} \) that interpolates the data set \( \{\delta_{m,0} : m \in \mathbb{Z}\} \). More precisely, one solves the bi-infinite system

\[
\sum_{k \in \mathbb{Z}} c_k^{(n)} B_n \left( \frac{n}{2} + m - k \right) = \delta_{m,0}, \quad m \in \mathbb{Z},
\]

for the sequence \( \{c_k^{(n)} : k \in \mathbb{Z}\} \) and then defines the fundamental cardinal spline \( L_n : \mathbb{R} \to \mathbb{R} \) of order \( n \in \mathbb{N} \) by \( L_n(x) = \sum_{k \in \mathbb{Z}} c_k^{(n)} B_n \left( \frac{n}{2} + x - k \right) \). The Fourier transform of \( L_n \) is given by

\[
\hat{L}_n(\xi) = \frac{\tau_{-n/2} \hat{B}_n(\xi)}{\sum_{k \in \mathbb{Z}} \tau_{-n/2} \hat{B}_n(\xi + 2\pi k)} \quad (6.1)
\]

where \( \tau_{\alpha} f(x) = f(x - \alpha) \) \((x, \alpha \in \mathbb{R})\). Using the Euler-Frobenius polynomials associated with the B-splines \( B_n \), one can show that the denominator in (6.1) does not vanish on the unit circle. For details, see for instance \([1,15]\).

One of our goals is to construct a fundamental cardinal spline \( L_q : \mathbb{R} \to \mathbb{H}_\mathbb{R} \) of quaternionic order \( q = a + v, a > 1 \), of the form

\[
L_q := \sum_{k \in \mathbb{Z}} c_k^{(q)} B_q (\cdot - k), \quad a > 1,
\]

\[ (6.2) \]
satisfying the interpolation problem

\[ L_q(m) = \delta_{m,0}, \quad m \in \mathbb{Z}, \]  
(6.3)

for an appropriate bi-infinite quaternion-valued sequence \( \{ c_k^{(q)} : k \in \mathbb{Z} \} \) and for appropriate \( q \) belonging to some nonempty open subset of \( \mathbb{H}_\mathbb{R} \). The case \( q \in \mathbb{C} \) reduces to the setting for complex B-splines and the existence of fundamental cardinal splines for complex B-splines was investigated and proven in [4,5].

Taking the Fourier transform of (6.2) and applying (6.3) to eliminate the expression containing the unknowns \( \{ c_k^{(q)} : k \in \mathbb{Z} \} \) gives (formally) a formula for \( L_q \) similar to (6.1):

\[ \hat{L}_q(\xi) = \frac{\hat{B}_q(\xi)}{\sum_{k \in \mathbb{Z}} B_q(k) z^k}, \quad z = e^{i\xi}, \quad \xi \in \mathbb{R}. \]  
(6.4)

Equation (6.4) contains a complex quaternion in the denominator. The next result gives necessary and sufficient conditions for the complex quaternion \( \sum_{k \in \mathbb{Z}} B_q(k) z^k \) to have an inverse. In the following, we write \( \sum\ell \) to mean \( \sum_{k \in \mathbb{Z}} \).

**Theorem 2** Let \( q = a + v \in \mathbb{H}_\mathbb{R} \) and \( w = a + i|v| \in \mathbb{C} \). Further, let \( |z| = 1 \). Then \( \sum\ell B_q(\ell)z^\ell \) is invertible in \( \mathbb{H}_\mathbb{C} \) if and only if \( (\sum\ell B_w(\ell)z^\ell) (\sum_k B_{w(k)} z^k) \neq 0 \). In this case we have

\[ \left( \sum\ell B_q(\ell)z^\ell \right)^{-1} = \frac{\sum\ell \left[ \mathcal{R}e B_w(\ell) - \frac{v}{|v|} \mathcal{I}m B_w(\ell) \right] z^\ell}{(\sum\ell B_w(\ell)z^\ell) (\sum_k B_{w(k)} z^k)}. \]  
(6.5)

**Proof** First note that if \( q = a + v \in \mathbb{H}_\mathbb{R} \) and \((w)_k\) is the Pochhammer symbol of the complex number \( w = a + i|v| \) then an application of Theorem 1 gives

\[ \Gamma(q) \sum\ell B_q(\ell)z^\ell = \sum\ell \sum_k (-1)^k \frac{(q)_k}{k!} (\ell - k)_{+}^q z^\ell \]

\[ = \sum_{\ell,k} \frac{(-1)^k}{k!} \left( \mathcal{R}e (w)_k + \frac{v}{|v|} \mathcal{I}m (w)_k \right) (\ell - k)_{+}^q \left[ \cos(|v| \log(\ell - k)_{+}) \right. \]
\[ + \frac{v}{|v|} \sin(|v| \log(\ell - k)_{+}) \left. \right] z^\ell \]

\[ = \sum_{\ell,k} \frac{(-1)^k}{k!} (\ell - k)_{+}^q \left[ \mathcal{R}e (w)_k \cos(|v| \log(\ell - k)_{+}) - \mathcal{I}m (w)_k \sin(|v| \log(\ell - k)_{+}) \right] \]
\[ + \frac{v}{|v|} \left[ \mathcal{I}m (w)_k \cos(|v| \log(\ell - k)_{+}) + \mathcal{R}e (w)_k \sin(|v| \log(\ell - k)_{+}) \right] z^\ell \]

\[ = \sum_{\ell,k} \frac{(-1)^k}{k!} (\ell - k)_{+}^q \left[ \mathcal{R}e ((w)_k e^{i|v| \log(\ell - k)_{+}}) + \frac{v}{|v|} \mathcal{I}m ((w)_k e^{i|v| \log(\ell - k)_{+}}) \right] z^\ell \]
\[
\begin{align*}
= \sum_{\ell} \left[ \text{Re} \left( \sum_k \frac{(-1)^k}{k!} (\ell - k) \binom{w}{k}(\ell - k)_+^{i[v]} \right) \right] \\
+ \frac{v}{|v|} \text{Im} \left( \sum_k \frac{(-1)^k}{k!} (\ell - k) \binom{w}{k}(\ell - k)_+^{i[v]} \right) z^\ell \\
= \sum_{\ell} \left[ \text{Re}(\Gamma(w)B_w(\ell)) + \frac{v}{|v|} \text{Im}(\Gamma(w)B_w(\ell)) \right] z^\ell \\
= \sum_{\ell} \left[ \text{Re}(\Gamma(w) + \frac{v}{|v|} \text{Im}(\Gamma(w))) \left[ \text{Re}(B_w(\ell)) + \frac{v}{|v|} \text{Im}(B_w(\ell)) \right] z^\ell \\
= \Gamma(q) \sum_{\ell} \left[ \text{Re}(B_w(\ell)) + \frac{v}{|v|} \text{Im}(B_w(\ell)) \right] z^\ell,
\end{align*}
\]

where we used (3.5). Let

\[
Z = \sum_{\ell} \text{Re}(B_w(\ell))z^\ell = \frac{1}{2} \sum_{\ell} \left[ B_w(\ell)z^\ell + \overline{B_w(\ell)}z^\ell \right] = \frac{1}{2}(A + B)
\]

and, for \(i \in \{1, 2, 3\}\),

\[
V_i = \frac{v_i}{|v|} \sum_{\ell} \text{Im}((B_w(\ell))z^\ell = \frac{1}{2i|v|} \left[ \sum_{\ell} B_w(\ell)z^\ell - \sum_{\ell} \overline{B_w(\ell)}z^\ell \right] = \frac{1}{2i|v|}(A - B)
\]

so that \(\sum_{\ell} B_q(\ell)z^\ell = Z + \sum_{i=1}^3 V_i e_i\). Then

\[
Z^2 + \sum_{i=1}^3 V_i^2 = \frac{1}{4}(A + B)^2 - \frac{1}{4}(A - B)^2 = AB = \left( \sum_{\ell} B_w(\ell)z^\ell \right) \left( \sum_{\ell} \overline{B_w(\ell)}z^\ell \right).
\]

However for \(w \in \mathbb{C}\) with Pochhammer symbol \((w)_j\) we have \((\overline{w})_j = (\overline{w})_j\) and if \(t\) is a real number, \(t^{\overline{w}} = \overline{t^w}\). Therefore,

\[
B_w(t) = \frac{1}{\Gamma(w)} \sum_k \frac{(-1)^k}{k!} (w)_j(t - k)u^w = \frac{1}{\Gamma(\overline{w})} \sum_k \frac{(-1)^k}{k!} (\overline{w})_j(t - k)\overline{u}^w = B_{\overline{w}}(t),
\]

so that

\[
Z^2 + \sum_{i=1}^3 V_i^2 = \left( \sum_{\ell} B_w(\ell)z^\ell \right) \left( \sum_{\ell} \overline{B_w(\ell)}z^\ell \right).
\]
Thus, $\sum_\ell B_q(\ell)z^\ell$ is invertible if and only if $\left(\sum_\ell B_w(\ell)z^\ell\right)\left(\sum_\ell B_w(\ell)z^\ell\right)^{-1} \neq 0$. In this case the inverse is given by

$$\left(\sum_\ell B_q(\ell)z^\ell\right)^{-1} = \frac{\left(\sum_\ell B_q(\ell)z^\ell\right)^{-1}}{\left(\sum_\ell B_w(\ell)z^\ell\right)\left(\sum_\ell B_w(\ell)z^\ell\right)} = \sum_\ell \left[\Re B_w(\ell) - \frac{v}{|v|} \Im B_w(\ell)\right]z^\ell$$

$$\left(\sum_\ell B_w(\ell)z^\ell\right)\left(\sum_\ell B_w(\ell)z^\ell\right) = \sum_\ell \left[\Re B_w(\ell) - \frac{v}{|v|} \Im B_w(\ell)\right]z^\ell.$$

\[\square\]

By the Poisson summation formula, we have $\sum_\ell B_w(\ell)z^\ell = \sum_\ell \hat{B}_w(z + 2\pi \ell)$. It was shown in [5] that $\sum_\ell \hat{B}_w(z + 2\pi \ell) \neq 0$ iff

$$\zeta(w, \alpha) + e^{-i\pi w} \zeta(w, 1 - \alpha) \neq 0,$$

where we take the principal value of the multi-valued function $e^{-i\pi(\cdot)}$ and where $\zeta(w, \alpha)$ denotes the Hurwitz zeta function with parameter $\xi = \alpha \in (0, 1)$.

In [5], regions $R \subset \mathbb{C}$ for which $w = a + iv \in R$, $a > 2$, implies that the Hurwitz zeta function (6.6) is zero-free were constructed. These regions satisfy $\overline{R} = R$ (i.e., $R$ is symmetric with respect to the real axis) and $R$ is either a rectangular region centered at points $2n \in \mathbb{C}$, $n \in \mathbb{N}$, of width less than one and strictly positive height, or an annular-shaped region centered at points $2n \in \mathbb{C}$, $n \in \mathbb{N}$, of width less than one and angular extend $0 \neq \theta \in (-\pi, \pi)$. Let

$$R \subset \mathbb{C}$$

Then, since $\overline{R} = R$, the denominator of (6.5) is nonzero provided $q \in Q_R$. Applying the Poisson summation formula to the denominator of (6.4) yields

$$\hat{L}_q(\xi) = \frac{\hat{B}_q(\xi)}{\sum_{k \in \mathbb{Z}} \hat{B}_q(\xi + 2\pi k)}, \quad \xi \in \mathbb{R}.$$

We note that $\hat{L}_q(0) = 1$ and $\hat{L}_q(2\pi) = 0$. As the expression in the denominator is $2\pi$-periodic, it suffices to assume that $0 < \xi < 2\pi$. Inserting (3.1) into the above expression for $\hat{L}_q$ and using the facts that $\left(\frac{z_1}{z_2}\right)^q = \frac{z_1^q}{z_2^q}$, $z_1, z_2 \in \mathbb{C}$ and $(it)^q = i^qt^q$, $t \in \mathbb{R}$, with the convention arg $t = -\pi$ for $t < 0$, gives

$$\hat{L}_q(\xi) = \frac{1 - e^{-i\xi}}{\sum_{k \in \mathbb{Z}} (1 - e^{-i(\xi + 2\pi k)})} = \frac{1/\xi^q}{\sum_{k \in \mathbb{Z}} (1 + \xi + 2\pi k)^q}. $$
Setting $\alpha := \frac{\xi}{2\pi} \in (0, 1)$, the sum in the above denominator can be rewritten in the form

$$\sum_{k \in \mathbb{Z}} \frac{1}{(k + \alpha)^q} = \sum_{k=0}^{\infty} \frac{1}{(k + \alpha)^q} + \sum_{k=0}^{\infty} \frac{1}{(\alpha - 1 - k)^q}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k + \alpha)^q} + e^{-i\pi q} \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \alpha)^q}$$

$$= \zeta(q, \alpha) + e^{-i\pi q} \zeta(q, 1 - \alpha), \quad (6.8)$$

where we have used Lemma 3. With the above observations, we obtain the next result which provides zero-free regions for the quaternionic zeta function.

**Proposition 3** Let $Q_R$ be defined as in (6.7). The combination

$$\zeta(q, \alpha) + e^{-i\pi q} \zeta(q, 1 - \alpha)$$

of quaternionic Hurwitz zeta functions is zero-free for all $\alpha \in (0, 1)$ provided $q \in Q_R$.

Combining the above results and observations yields the following theorem.

**Theorem 3** Suppose that $B_q$ is a quaternionic B-spline with $q \in Q_R$. Then

$$L_q(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\xi^{-q} e^{i\xi x}}{\zeta(q, \xi/2\pi) + e^{-i\pi q} \zeta(q, 1 - \xi/2\pi)} \, d\xi, \quad x \in \mathbb{R}, \quad (6.9)$$

is a fundamental interpolating spline of quaternionic order $q$ in the sense that

$$L_q(m) = \delta_{m,0}, \quad \text{for all } m \in \mathbb{Z}.$$  

The Fourier inverse in (6.9) holds in the $L^1$- and $L^2$-sense.

In Figs. 1, 2 and 3, two fundamental cardinal splines with $q_1 = 6.2 + \frac{1}{2\sqrt{2}} e_1 - \frac{1}{3} e_2 + \frac{1}{4} e_3$ and $q_2 = 2.5 + \frac{1}{4\sqrt{2}} e_1 + \frac{1}{8} e_2 - \frac{\sqrt{13}}{8} e_3$ are depicted. The graphs of these functions were constructed by sampling the integrand in Eq. (6.9) and performing an inverse Fourier transform on this set of sampled values.

We can use (6.4) to obtain an estimate on the decay of the coefficients $c_k^{(q)}$. The arguments are based on those given in [13,14] for the case $q \in \mathbb{N}$. Equations (6.2), (6.3) and (6.4) imply that

$$\sum_{k \in \mathbb{Z}} c_k^{(q)} e^{i\xi k} = \frac{1}{\sum_{k \in \mathbb{Z}} B_q(k) z_k} =: \phi_q(\xi). \quad (6.10)$$
The right-hand side $\phi_q$ of the above equation is a function of $z = e^{i\xi}$ which does not have any poles on the unit circle $|z| = 1$ provided $q \in Q_R$. As $\hat{B}_q \in C^{[\text{Sc}q]+1}(\mathbb{R})$ it follows that the Fourier coefficients of $\phi_q(\xi)$ satisfy
\[ |c_k^{(q)}| \leq M_q |k|^{-\left\lfloor \frac{\text{Sc} \ q}{2} \right\rfloor - 1}, \tag{6.11} \]

for some positive constant \( M_q \). The estimate (6.11) implies the following result.

**Proposition 4** The fundamental cardinal spline \( L_q \) with \( q \in \mathbb{Q}_R \) satisfies the following point-wise estimate:

\[ |L_q(x)| \leq A_q |x|^{-\left\lfloor \text{Sc} \ q \right\rfloor}, \quad x \in \mathbb{R}, \tag{6.12} \]

for some positive constant \( A_q \), provided \( \text{Sc} \ q > 1 \).

**Proof** As \( \hat{L}_q(-\xi) = \hat{L}_q(\xi) \) and therefore \( L_q(-x) = L_q(x) \), it suffices to consider \( x \leq 0 \). From (6.2), we obtain

\[ |L_q(x)| = \left| \sum_{k=-\infty}^{x} c_k^{(q)} B_q(x - k) \right| \leq K_q \sum_{k=-\infty}^{x} |k|^{-\left\lfloor \text{Sc} \ q \right\rfloor - 1} \leq A_q |x|^{-\left\lfloor \text{Sc} \ q \right\rfloor}, \]

where in the inequality we used the fact that the quaternionic B-splines are bounded above by some positive constant \( K_q \). This yields the statement. \( \square \)

As a direct corollary of Theorem 1, we now provide a structure theorem for the fundamental cardinal quaternionic splines \( L_q \), relating them to the fundamental cardinal complex splines \( B_z \).

**Corollary 1** Suppose \( q = a + v \in \mathbb{Q}_R \) and \( z = a + i|v| \in \mathbb{C} \). Then

\[ L_q(t) = \text{Re} \ L_z(t) + \frac{v}{|v|} \text{Im} \ L_z(t), \quad t \in \mathbb{R}. \]

**Proof** Let \( c_j = \text{Re} \ w_j + \frac{v}{|v|} \text{Im} \ w_j \) where \( w_j \) is the solution of \( \sum_{j \in \mathbb{Z}} B_z(i - j)w_j = \delta_{i,0}, \ i \in \mathbb{Z} \). Then, for \( t \in \mathbb{R} \),

\[
L_q(t) = \sum_{j \in \mathbb{Z}} B_q(t - j)c_j \\
= \sum_{j \in \mathbb{Z}} \left[ \text{Re}(B_z(t - j)) + \frac{v}{|v|} \text{Im}(B_z(t - j)) \right] \left[ \text{Re}(w_j) + \frac{v}{|v|} \text{Im}(w_j) \right] \\
= \sum_{j \in \mathbb{Z}} \left[ \text{Re}(B_z(t - j)) \text{Re}(w_j) - \text{Im}(B_z(t - j)) \text{Im}(w_j) \right] \\
+ \frac{v}{|v|} \left[ \text{Im}(B_z(t - j)) \text{Re}(w_j) + \text{Re}(B_z(t - j)) \text{Im}(w_j) \right] \\
= \sum_{j \in \mathbb{Z}} \text{Re}(B_z(t - j)w_j) + \frac{v}{|v|} \sum_{j \in \mathbb{Z}} \text{Im}(B_z(t - j)w_j)
\]

where in the inequality we used the fact that the quaternionic B-splines are bounded above by some positive constant \( K_q \). This yields the statement. \( \square \)
\[
\text{Re} \left( \sum_{j \in \mathbb{Z}} B_z(t - j) w_j \right) + \frac{v}{|v|} \text{Im} \left( \sum_{j \in \mathbb{Z}} B_z(t - j) w_j \right) = \text{Re} \, L_z(t) + \frac{v}{|v|} \text{Im} \, L_z(t). \]

We provide now an alternative computation of the fundamental splines \( L_q \) for which error estimates are available. Starting with (6.10) and applying the Poisson summation formula, we write

\[
c^{(q)}_k = \frac{1}{2\pi} \int_{0}^{2\pi} \phi_q(\xi) e^{-ik\xi} \, d\xi = \int_{0}^{2\pi} \sum_j \hat{B}_q(\xi + 2\pi j) \, d\xi. \tag{6.13}
\]

The sum in (6.13) is approximated by the truncation

\[
F^M_q(\xi) = \sum_{k=-M}^{M} \hat{B}_q(\xi + 2\pi k) = \sum_{k=-M}^{M} \left( 1 - e^{-i\xi} \right)^q.
\]

for suitably large \( M > 0 \). We have chosen \( q \) so that the \( 2\pi \)-periodic function \( F_q(\xi) = \sum_k \hat{B}_q(\xi + 2\pi k) \) has no zeroes and \( M \) must be large enough so \( F^M_q \) also has no zeroes on the real line. Then

\[
|F_q(\xi) - F^M_q(\xi)| \leq |1 - e^{-i\xi}| \sum_{|k| > M} \frac{1}{|\xi + 2\pi k|^a} \leq 2|1 - e^{-i\xi}| \sum_{k=M}^{\infty} k^{-a} \leq \frac{2}{\pi^a(a-1) M^{a-1}}
\]

and we conclude that

\[
\left| \frac{1}{F_q(\xi)} - \frac{1}{F^M_q(\xi)} \right| = \mathcal{O}(M^{1-a}) \quad \text{as} \quad M \to \infty. \tag{6.14}
\]

The coefficients \( c^{(q)}_k \), obtained via (6.13) are to be estimated from the discrete Fourier transform

\[
c^{(q)}_{N,M,k} = \frac{2\pi}{N} \sum_{j=0}^{N-1} \frac{e^{-2\pi i j k/N}}{F_q(2\pi j/N)}
\]

for \( N \) sufficiently large. We then have

\[
|c^{(q)}_k - c^{(q)}_{N,M,k}| = \left| \int_{0}^{2\pi} \frac{e^{-i k \xi}}{F_q(\xi)} \, d\xi - \frac{2\pi}{N} \sum_{j=0}^{N-1} \frac{e^{-2\pi i j k/N}}{F^M_q(2\pi j/N)} \right|.
\]
\[ \begin{align*}
&= \left| \int_{0}^{2\pi} e^{-ik\xi} d\xi - \frac{2\pi}{N} \sum_{j=0}^{N-1} e^{-2\pi ij/N} \right| \\
&+ \left| \frac{2\pi}{N} \sum_{j=0}^{N-1} \frac{e^{-2\pi ij/N}}{F_q(2\pi j/N)} - \frac{2\pi}{N} \sum_{j=0}^{N-1} \frac{e^{-2\pi ij/N}}{F_q^M(2\pi j/N)} \right| =: A + B.
\end{align*} \]

(6.15)

Quantity \( B \) on the right hand side of (6.15) is easily handled with an application of (6.14), which gives
\[ B = \mathcal{O}(M^{1-a}), \quad \text{as} \quad M \to \infty. \]

To estimate \( A \), we employ the following simplification of a result due to Epstein [2]:

**Theorem 4** Suppose \( f \) is a \( 2\pi \)-periodic function on the real line with \( \ell \) continuous derivatives \( (\ell \geq 1) \), \( \hat{f}(k) = \int_{0}^{2\pi} f(\xi) e^{-ik\xi} d\xi \) and \( \hat{f}_{N,k} := \frac{2\pi}{N} \sum_{j=0}^{N-1} f(2\pi j/N) e^{-2\pi ijk/N} \) is the \( N \)-point Riemann sum approximation of the integral defining \( \hat{f}(k) \). Then for \( |k| \leq N/2 \),
\[ |\hat{f}(k) - \hat{f}_{N,k}| \leq (2\pi)^2 \left( \frac{12}{N} \right) \| f^{(\ell)} \|_{\infty}. \]

It is easily checked that for quaternionic functions on the line of the form \( f(\xi) = f_s(\xi) + v |v| f_v(\xi) \) with \( v \neq 0 \) a fixed quaternionic vector and \( f_s, f_v \) complex-valued, we have \( \frac{d}{d\xi} (f(\xi)^{-1}) = -\frac{f'(\xi)}{f(\xi)^2} \) at all \( \xi \) for which \( f(\xi) \neq 0 \). Applying Theorem 4 to the estimate of \( A \) in (6.15) gives
\[ A \leq (2\pi)^2 \left( \frac{12}{N} \right) \frac{\| (F_q')_s \|_{\infty} + \| (F_q')_v \|_{\infty}}{\inf_{\xi} |F_q(\xi)|^2} = \mathcal{O}(N^{-1}) \quad \text{as} \quad N \to \infty. \] (6.16)

Returning now to (6.15), we have
\[ |c_k^{(q)} - c_{N,M,k}^{(q)}| = \mathcal{O}(M^{1-a} + N^{-1}) \quad \text{as} \quad M, N \to \infty. \] (6.17)

It remains then to compute (numerically) estimates for the multiplicative constant
\[ S_q = \frac{\| (F_q')_s \|_{\infty} + \| (F_q')_v \|_{\infty}}{\inf_{\xi} |F_q(\xi)|^2}. \]

By direct calculation, we have
\[ F_q'(\xi) = \begin{cases} 
-\frac{i q/2}{1 - e^{-i\xi}} & \text{if} \ \xi \in \mathbb{Z}; \\
\frac{i q}{1 - e^{-i\xi}} [F_{q+1}(\xi) - i e^{-i\xi} F_q(\xi)] & \text{if} \ \xi \notin \mathbb{Z},
\end{cases} \]
and this is used (with \( q_1 = 6.2 + \frac{1}{2\sqrt{2}} e_1 - \frac{1}{4} e_2 + \frac{1}{4} e_3 \) and \( q_2 = 2.5 + \frac{1}{4\sqrt{2}} e_1 + \frac{1}{8} e_2 - \frac{\sqrt{13}}{8} e_3 \)) in Figs. 4 and 5 below.
These calculations provide numerical estimates of the constants of (6.16). We find \( \min |F_{q_1}| \approx 0.1568, \min |F_{q_2}| \approx 0.7799, \| (F_{q_1})'_s \|_\infty + \| (F_{q_1})'_v \|_\infty \approx 3.7889, \| (F_{q_2})'_s \|_\infty + \| (F_{q_2})'_v \|_\infty \approx 2.1753. \)

In these calculations, we have applied Theorem 4 with \( \ell = 1 \), but the use of higher derivatives (to obtain faster decay of errors as \( N \) increases) is possible, at the expense of larger multiplicative constants in the estimates. This will not be pursued here, but we note that given the calculations of \( F_q \) and \( F'_q \) already obtained, the second derivative may be obtained from

\[
F''_q(\xi) = \begin{cases} 
- \frac{q(q + 1)}{4} & \text{if } \xi = 0; \\
\frac{i q}{(1 - e^{-i\xi})^2} \left( (F'_{q+1}(\xi) - i e^{-i\xi} F_q(\xi)) + e^{-i\xi} (i F_{q+1}(\xi) - F_q(\xi)) \right) & \text{if } \xi \neq 0.
\end{cases}
\]
7 A Sampling Theorem

In this section, we derive a sampling theorem for functions in the principal shift-invariant space

\[
V_q = \operatorname{clos}_{\ell^2} \operatorname{span} \{ B_q (\cdot - n) \}_{n=-\infty}^{\infty} = \left\{ f = \sum_k a_k B_q (\cdot - k) : \{ a_k \}_k \in \ell^2 (\mathbb{Z}), \sum_k |a_k|^2 < \infty \right\}
\]

where \( q \in Q_R \). For this purpose, we employ and generalize the following version of Kramer’s lemma [10] (which appears in [6]) to the complex-quaternionic setting.

**Theorem 5** ([6], p. 501) Let \( \emptyset \neq I \subseteq \mathbb{R} \) and let \( \{ \phi_k : k \in \mathbb{Z} \} \) be an orthonormal basis for \( L^2 (I) \). Suppose that \( \{ S_k : k \in \mathbb{Z} \} \) is a sequence of functions \( S_k : \Omega \to \mathbb{C} \) on a domain \( \Omega \subset \mathbb{R} \) and \( t := \{ t_k : k \in \mathbb{Z} \} \) a numerical sequence in \( \Omega \) satisfying the conditions

C1. \( S_k(t_l) = a_k \delta_{k,l} \) \( (k, l) \in \mathbb{Z} \times \mathbb{Z} \), where \( a_k \neq 0 \);
C2. \( \sum_{k \in \mathbb{Z}} |S_k(t)|^2 < \infty \), for each \( t \in \Omega \).

Define a function \( K : I \times \Omega \to \mathbb{C} \) by \( K(x, t) := \sum_{k \in \mathbb{Z}} S_k(t) \overline{\phi_k(x)} \) and a linear integral transform \( K \) on \( L^2 (I) \) by

\[
(KF)(t) := \int_I F(x) K(x, t) \, dx.
\]

Then \( K \) is well-defined and injective. Furthermore, if the range of \( K \) is denoted by

\[
\mathcal{H} := \left\{ f : \mathbb{R} \to \mathbb{C} : f = KF, \ F \in L^2 (I) \right\},
\]

then

(i) \( (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}) \) is a Hilbert space isometrically isomorphic to \( L^2 (I) \) \( (\mathcal{H} \cong L^2 (I)) \) when endowed with the inner product

\[
\langle f, g \rangle_{\mathcal{H}} := \langle F, G \rangle_{L^2 (I)}
\]

where \( f := KF \) and \( g = KG \).

(ii) \( \{ S_k : k \in \mathbb{Z} \} \) is an orthonormal basis for \( \mathcal{H} \).

(iii) Each function \( f \in \mathcal{H} \) can be recovered from its samples on the sequence \( t \) via the formula

\[
f(t) = \sum_{k \in \mathbb{Z}} f(t_k) \frac{S_k(t)}{a_k},
\]

the series converging absolutely and uniformly on subsets of \( \mathbb{R} \) where \( \| K(\cdot, t) \|_{L^2 (I)} \) is bounded.
For the proof and further details, we refer the reader to [6].

To carry out the extension to complex quaternions, let $\mathbb{C}_n$ be the complexified Clifford algebra

$$\mathbb{C}_n := \{ z_A e_A : z_A \in \mathbb{C}, A \subseteq \{1, 2, \ldots, n\} \}.$$ 

Here, if $A = \{i_1, i_2, \ldots, i_k\}$ with $1 \leq i_1 < i_2 < \cdots < i_k$ then $e_A := e_{i_1} e_{i_2} \cdots e_{i_k}$ where $e_j^2 = -1$ and $e_j e_k = -e_k e_j$ for $j \neq k$. We also insist that $e_\emptyset := 1$.

We define an involution on $\mathbb{C}_n$ as the $\mathbb{C}$-conjugate linear mapping $^* : \mathbb{C}_n \rightarrow \mathbb{C}_n$ by

$$(\sum_A z_A e_A)^* := \sum_A \bar{z}_A \bar{e}_A,$$

where $\bar{z}_A$ is the complex conjugate of the complex number $z_A$ and $\bar{e}_A$ is the Clifford conjugate of the Clifford algebra basis element $e_A$, determined by $e_j^2 = -1$ and $e_j e_k = -e_k e_j$ for $j \neq k$. We also insist that $\bar{e}_\emptyset := 1$.

For $z = \sum_A z_A e_A \in \mathbb{C}_n$, we define $|z|^2 := \sum_A |z_A|^2$. Note that $|z|^2 = [zz^*]_0 = [z^* z]_0 = |z^*|^2$ for all $z \in \mathbb{C}_n$.

Let $\mathcal{H}$ be a set on which there are two operations defined:

- **addition**: $+ : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$; and
- **scalar multiplication**: $\cdot : \mathbb{C}_n \times \mathcal{H} \rightarrow \mathcal{H}$.

The triple $(\mathcal{H}, +, \cdot)$ is called a left module over $\mathbb{C}_n$ if $(\mathcal{H}, +)$ is an abelian group and, for all $\lambda, \mu \in \mathbb{C}_n$ and $x, y \in \mathcal{H}$, the following conditions are satisfied:

1. $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$;
2. $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$;
3. $(\lambda \mu) \cdot x = \lambda \cdot (\mu \cdot x)$;
4. $1 \cdot x = x$.

Let $(\mathcal{H}, +, \cdot)$ be a left module over $\mathbb{C}_n$. Given a mapping $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}_n$, we say $(\mathcal{H}, +, \cdot, \langle \cdot, \cdot \rangle)$ is a left Hilbert module over $\mathbb{C}_n$ if, for all $x, y, z \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}_n$, the following requirements hold:

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;
2. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$;
3. $\langle \lambda x, \mu y \rangle = \lambda \langle x, y \rangle \mu^*$;
4. $\langle y, x \rangle = \langle x, y \rangle^*$;
5. $\langle x, x \rangle = 0 \iff x = 0$;
6. $\|x\| := \langle \langle x, x \rangle \rangle_0$ defines a pseudo-norm on $\mathcal{H}$ in the sense that

   (a) $\|x\| \geq 0$ and ($\|x\| = 0 \iff x = 0$);
   (b) $\|x + y\| \leq \|x\| + \|y\|$.
(c) there exists a constant $C > 0$ such that $\|λx\| \leq C |λ| \|x\|$. 

Now suppose that $X$ is a measure space with positive measure $dx$ and

$$L^2(X, \mathbb{C}_n) := \left\{ f = \sum_A f_A e_A : f_A : X \to \mathbb{C} \text{ measurable and } \|f\|^2 = \sum_A \|f_A\|^2 < \infty \right\}.$$ 

A pairing $\langle \cdot, \cdot \rangle : L^2(X, \mathbb{C}_n) \times L^2(X, \mathbb{C}_n) \to \mathbb{C}_n$ is given by

$$\langle f, g \rangle := \int_X f(x)g(x)^* \, dx.$$ 

With this definition, $L^2(X, \mathbb{C}_n)$ becomes a left Hilbert module over $\mathbb{C}_n$. Note that if $f \in L^2(X, \mathbb{C}_n)$, then $\left[ \int f(x)f(x)^* \, dx \right]_0 = \int |f(x)|^2 \, dx$.

Let $\mathcal{H}$ be a left Hilbert module over $\mathbb{C}_n$. A mapping $T : \mathcal{H} \to \mathbb{C}_n$ is called a bounded left linear functional if, for all $f, g \in \mathcal{H}$ and $λ \in \mathbb{C}_n$,

(i) $T(f + g) = Tf + Tg$;
(ii) $T(λf) = λ(Tf)$;
(iii) there exists a constant $M > 0$ such that $|Tf| \leq M\|f\|$.

We call $\|T\| := \inf\{M > 0 : \text{property (iii) is satisfied}\}$ the norm of the functional $T$.

The next theorem is a generalization of the Riesz Representation Theorem to the complex quaternionic setting.

**Theorem 6** (Riesz Representation Theorem) Let $T$ be a bounded left linear functional on a left Hilbert module $\mathcal{H}$ over $\mathbb{C}_n$ for which there exists an orthonormal basis $\{φ_n\}_{n=1}^\infty$, i.e., $\{φ_n\}_{n=1}^\infty \subset \mathcal{H}$ and for all $f \in \mathcal{H}$, we have

(i) $\langle φ_n, φ_m \rangle = δ_{n,m}$ for all $m, n \in \mathbb{N}$;
(ii) $\sum_{n=1}^\infty |\langle f, φ_n \rangle|^2 = \|f\|^2$;
(iii) $f = \sum_{n=1}^\infty \langle f, φ_n \rangle φ_n$.

Then there exists a unique $g \in \mathcal{H}$ such that $Tf = \langle f, g \rangle$, for all $f \in \mathcal{H}$.

**Proof** Let $\{φ_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. Let $a_j := Tφ_j \in \mathbb{C}_n$, and given $f \in \mathcal{H}$, let $c_j := \langle f, φ_j \rangle \in \mathbb{C}_n$ and $f_n := \sum_{j=1}^n c_j φ_j$. Then $\|f - f_n\|_2 \to 0$ as $n \to \infty$ and since $T$ is left linear, $Tf_n = \sum_{j=1}^n c_j Tφ_j = \sum_{j=1}^n c_j a_j$. Also, again by the linearity and boundedness of $T$,

$$|Tf - Tf_n| = |T(f - f_n)| \leq \|T\| \|f - f_n\| \to 0 \text{ as } n \to \infty.$$ 

Consequently, we have that $Tf = \sum_{j=1}^\infty c_j a_j$. Furthermore,

$$\left| \left( \sum_{j=1}^n c_j a_j \right) \right|_0 \leq \left| \sum_{j=1}^n c_j a_j \right| = |Tf_n| \leq \|T\| \|f_n\|_2 = \|T\| \left( \sum_{j=1}^n |c_j|^2 \right)^{1/2}. \quad (7.2)$$
Putting \( c_j = a^*_j \) in (7.2) yields

\[
\sum_{j=1}^{n} |a_j|^2 \leq \|T\| \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2},
\]

from which we obtain the uniform bound \( \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \leq \|T\| \). Now let \( g := \sum_{j=1}^{\infty} a^*_j \varphi_j \in \mathcal{H} \). Then

\[
\langle f, g \rangle = \left( \sum_{j=1}^{\infty} c_j \varphi_j, \sum_{\ell=1}^{\infty} a^*_\ell \varphi_\ell \right) = \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} c_j \langle \varphi_j, \varphi_\ell \rangle a_\ell = \sum_{j=1}^{\infty} c_j a_j = Tf,
\]

as required.

To establish uniqueness, suppose that \( h \) is another such element of \( \mathcal{H} \) for which \( Tf = \langle f, h \rangle \), for all \( f \in \mathcal{H} \). Then we have

\[
\langle g - h, g - h \rangle = \langle g - h, g \rangle - \langle g - h, h \rangle = T(g - h) - T(g - h) = 0,
\]

so that \( \|g - h\|^2 = [\langle g - h, g - h \rangle]_0 = 0 \), i.e., \( g = h \). \( \square \)

The theorem below is the extension of Theorem 5 to the current setting.

**Theorem 7** (Sampling Theorem for \( \mathbb{C}_n \)-valued Functions) Let \( T \) be a measure space with positive measure \( dt \), \( S_n : T \to \mathbb{C}_n \) \((n \geq 1)\) a sequence of \( \mathbb{C}_n \)-valued functions, and \( \{t_k\}_{k=1}^{\infty} \subset T \) such that

1. \( S_n(t_k) = \delta_{n,k} \) \((n, k \geq 1)\);
2. \( \sum_{n=1}^{\infty} |S_n(t)|^2 \leq M < \infty \), for all \( t \in T \).

Let \( \mathcal{H} := \text{clos}_{\ell^2} \text{span} \{S_n\} = \{f = \sum_{n=1}^{\infty} a_n S_n : a_n \in \mathbb{C}_n \text{ and } \sum_{n=1}^{\infty} |a_n|^2 < \infty\} \).

Then for all \( f \in \mathcal{H} \),

\[
f(t) = \sum_{n=1}^{\infty} f(t_n) S_n(t)
\]

with convergence in the norm on \( \mathcal{H} \).

**Proof** Let \((X, dx)\) be a measure space and let \( \{\phi_n\}_{n=1}^{\infty} \) be an orthonormal basis for \( L^2(X, \mathbb{C}_n) \) in the sense that

1. \( \int_X \phi_n(x) \phi_m(x)^* \, dx = \delta_{nm} \), for all \( n, m \in \mathbb{N} \);
2. \( \sum_{n=1}^{\infty} \left| \int_X F(x) \phi_n(x)^* \, dx \right|^2 = \int_X |F(x)|^2 \, dx \), for all \( F \in L^2(X, \mathbb{C}_n) \);
Let $K : X \times T \to \mathbb{C}_n$ be given by $K(x,t) := \sum_{n \in \mathbb{N}} \phi_n(x) S_n(t)$ and define an integral operator $K$ by

$$KF(t) := \int_X F(x) K(x,t) \, dx, \quad F \in L^2(X, \mathbb{C}_n).$$

Note that

$$\int_X |K(x,t)|^2 \, dx = \int_X \left| \sum_{n \in \mathbb{N}} \phi_n(x) S_n(t) \right|^2 \, dx$$

$$= \left[ \int_X \left( \sum_{n \in \mathbb{N}} \phi_n(x) S_n(t) \right)^* \left( \sum_{m \in \mathbb{N}} \phi_m(x) S_m(t) \right) \, dx \right]_0$$

$$= \sum_{n,m \in \mathbb{N}} S_n^*(t) \int_X \phi_n^*(x) \phi_m(x) \, dx \, S_m(t)_0$$

$$= \sum_{n \in \mathbb{N}} [S_n(t)^* S_n(t)]_0 = \sum_{m \in \mathbb{N}} |S_n(t)|^2 \leq M < \infty.$$

If $f = KF$, for some $F \in L^2(X, \mathbb{C}_n)$, then

$$|f(t)| = \left| \int_X F(x) K(x,t) \, dx \right|$$

$$\leq C \int_X |F(x)||K(x,t)| \, dx$$

$$\leq C \left( \int_X |F(x)|^2 \, dx \right)^{1/2} \left( \int_X |K(x,t)|^2 \, dx \right)^{1/2} \leq C \sqrt{M} \|F\|_{L^2(X)}. \quad (7.3)$$

Hence, the mapping $K : F \to f$ is well-defined. $K$ is also one-to-one since if $KF(t) = 0$ for all $t \in T$, then

$$KF(t_k) = \int_X F(x) K(x,t_k) \, dx = \int_X F(x) \phi_k(x) \, dx = 0 \quad \text{for all } k \in \mathbb{N},$$

which implies that $F \equiv 0$.

Now, let $\mathcal{H} := \text{Ran}(K)$ and define a pairing $(\cdot, \cdot)_\mathcal{H}$ on $\mathcal{H}$ by

$$(f, g)_\mathcal{H} := \int_X F(x) G(x)^* \, dx \in \mathbb{C}_n,$$

where $F, G \in L^2(X, \mathbb{C}_n)$ are the unique elements for which $KF = f$ and $KG = g$.

We claim that with this pairing $\mathcal{H}$ becomes a left Hilbert module over $\mathbb{C}_n$. To prove
this, we need to verify that axioms 1–9 are satisfied. First, notice that the addition and scalar multiplication in $\mathcal{H}$ are defined in the usual way: linearity of the operator $\mathcal{K}$ implies that if $\mathcal{K}F = f$ and $\mathcal{K}G = g$, then $\lambda \cdot f = \mathcal{K}(\lambda \cdot F)$ and $\mathcal{K}(F + G) = f + g$. Thus,

$$\begin{align*}
(\lambda \cdot f)(t) &= \int_X (\lambda \cdot F(x))K(x, t)\,dx = \lambda \int_X F(x)K(x, t)\,dx = \lambda(f(t))
\end{align*}$$

and

$$\begin{align*}
(f + g)(t) &= \int_X (F + G)(x)K(x, t)\,dx = \int_X [F(x) + G(x)]K(x, t)\,dx \\
&= \int_X F(x)K(x, t)\,dx + \int_X G(x)K(x, t)\,dx = f(t) + g(t).
\end{align*}$$

Axioms 1–4 follow immediately. Note that if $f$, $g$ are as above and $h = \mathcal{K}H$ then

$$\langle f + g, h \rangle_{\mathcal{H}} = \int_X (F + G)(x)H(x)^*\,dx = \int_X F(x)H(x)^*\,dx + \int_X G(x)H(x)^*\,dx$$

and hence axiom 5 is satisfied. Axiom 6 follows similarly. Next, observe that if $f$, $g$ are as above and $\lambda, \mu \in \mathbb{C}$, then

$$\langle \lambda f, \mu g \rangle_{\mathcal{H}} = \int_X (\lambda F)(x)(\mu G)(x)^*\,dx = \lambda \int_X F(x)G(x)^*\,dx \mu^* = \lambda \langle f, g \rangle_{\mathcal{H}} \mu^*,$$

thus verifying axiom 7. For axiom 8, note that

$$\langle g, f \rangle_{\mathcal{H}} = \int_X G(x)F(x)^*\,dx = \left(\int_X F(x)G(x)^*\,dx\right)^* = \langle f, g \rangle_{\mathcal{H}}^*.$$

Suppose now that $\langle f, f \rangle_{\mathcal{H}} = 0$. Then

$$\int_X |F(x)|^2\,dx = \left[ \int_X F(x)F(x)^*\,dx \right]_0 = [\langle f, f \rangle_{\mathcal{H}}]_0 = 0,$$

so that $F \equiv 0$. Hence $f = \mathcal{K}F \equiv 0$ and axiom 9 is verified. Axiom 10(a) now follows immediately. For axiom 10(b), note that

$$\|f + g\|_{\mathcal{H}} = [\langle f + g, f + g \rangle_{\mathcal{H}}]^{1/2}_0 = \left[ \int_X (F + G)(x)(F + G)(x)^*\,dx \right]^{1/2}_0.$$
Finally, we have

\[ \| \lambda f \|^2_{\mathcal{H}} = |\langle \lambda f, \lambda f \rangle_{\mathcal{H}}| = \int_X |\lambda f(x)|^2 \, dx \leq C^2 |\lambda|^2 \int_X |f(x)|^2 \, dx = C^2 |\lambda|^2 \| f \|^2_{\mathcal{H}}, \]

so that axiom 10(c) is verified.

We need to show that \( \mathcal{H} := \text{Ran}(\mathcal{K}) = \text{clos}_{L^2} \text{span} \{S_n\} \). Since \( \{\phi_n\}_{n=1}^{\infty} \) is an orthonormal basis for \( L^2(X, \mathbb{C}_n) \), for each \( F \in L^2(X, \mathbb{C}_n) \) we have \( F = \sum_{n=1}^{\infty} a_n \phi_n \) for some sequence \( \{a_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}) \). Then \( \mathcal{K}F = \sum_{n=1}^{\infty} a_n S_n \in \text{clos}_{L^2} \text{span} \{S_n\} \). This gives the equivalence of the definitions of \( \mathcal{H} \). We remark that \( \{S_n\}_{n=1}^{\infty} \) is an orthonormal basis for \( \mathcal{H} \). To see this, note that since \( S_n = \mathcal{K}\phi_n \), we have \( \langle S_n, S_n \rangle_{\mathcal{H}} = \int_X \phi_n(x) \phi_n(x)^* \, dx = \delta_{n,m} \). Also, if \( \langle f, S_n \rangle_{\mathcal{H}} = 0 \), for all \( n \in \mathbb{N} \), then

\[ \int_X F(x) \phi_n(x)^* \, dx = \langle f, S_n \rangle_{\mathcal{H}} = 0 \quad \text{for all} \quad n \quad \implies \quad F \equiv 0 \quad \implies \quad f \equiv 0. \]

Moreover,

\[ \sum_{n \in \mathbb{N}} |\langle f, S_n \rangle_{\mathcal{H}}|^2 = \sum_{n \in \mathbb{N}} \left| \int_X F(x) \phi_n(x)^* \, dx \right|^2 = \int_X |F(x)|^2 \, dx = \| f \|^2_{\mathcal{H}}. \]

Equation (7.3) may be interpreted as giving the boundedness of the evaluation functional \( E_t : \mathcal{H} \to \mathbb{C}_n \) defined by \( E_t(f) := f(t) \). By the Riesz Representation Theorem we conclude that there exists a \( k_t \in \mathcal{H} \) such that

\[ f(t) = E_t(f) = \langle f, k_t \rangle_{\mathcal{H}}. \]

Let \( k(t, s) = \langle k_t, k_s \rangle_{\mathcal{H}} = k_s(t) \). Then

\[ \langle f, k(\cdot, s) \rangle_{\mathcal{H}} = \langle f, k_s \rangle_{\mathcal{H}} = f(s). \]

Hence, \( k(t, s) \) is a reproducing kernel for \( \mathcal{H} \). Suppose \( k' \) is another such kernel. With \( k'_t(t) = k'(s, t) \), we have that

\[ k'_t(t) = \langle k'_t, k_t \rangle_{\mathcal{H}} = \langle k_t, k_s^* \rangle_{\mathcal{H}} = k_t(s)^* = \langle k_t, k_s \rangle^* = \langle k_s, k_t \rangle_{\mathcal{H}} = k_s(t), \]

i.e., \( k'(s, t) = k(s, t) \) for all \( s, t \in T \). As \( \{S_n\}_{n=1}^{\infty} \) is an orthonormal basis for \( \mathcal{H} \), we obtain

\[ k_t = \sum_n \langle k_t, S_n \rangle_{\mathcal{H}} S_n = \sum_n \langle S_n, k_t \rangle_{\mathcal{H}}^* S_n = \sum_n S_n(t)^* S_n, \]
so that

\[ k(t, s) = (k_s, k_t)_\mathcal{H} = \sum_{n \in \mathbb{N}} S_n(s)^*S_n, \sum_{m \in \mathbb{N}} S_m(t)^*S_m \]

\[ = \sum_{n, m \in \mathbb{N}} S_n(s)^*S_m(t)^*S_m = \sum_{n \in \mathbb{N}} S_n(s)^*S_n(t). \]

Note also that

\[ \int_X K(x, s)^*K(x, t) \, dx = \int_X \left( \sum_{n \in \mathbb{N}} \phi_n(x)^*S_n(s) \right)^* \left( \sum_{m \in \mathbb{N}} \phi_m(x)^*S_m(t) \right) \, dx \]

\[ = \sum_{n, m \in \mathbb{N}} S_n(s)^* \int_X \phi_n(x)\phi_m(x)^* \, dx S_m(t) \]

\[ = \sum_{n \in \mathbb{N}} S_n(s)^*S_n(t) = k(t, s). \]

Finally, as \( K(x, t_n) = \phi_n(x)^* \), we obtain

\[ \langle f, S_n \rangle_\mathcal{H} = \int_X F(x)\phi_n(x)^* \, dx = \int_X F(x)K(x, t_n) \, dx = f(t_n), \] (7.4)

and therefore, since \( \{ S_n \}_{n=1}^\infty \) is an orthonormal basis for \( \mathcal{H} \), (7.4) yields

\[ f = \sum_{n \in \mathbb{N}} \langle f, S_n \rangle_\mathcal{H} S_n = \sum_{n \in \mathbb{N}} f(t_n)S_n. \]

\[ \square \]

**Remark 1** The above results continue to hold when the index set \( \mathbb{N} \) is replaced by any countably infinite set.

For our purposes we choose \( T := \mathbb{R} \), \( \{ t_k \}_{k \in \mathbb{Z}} := \mathbb{Z} \), and for the interpolating function, \( S_k := L_q(\cdot - k), q \in \mathbb{Q}_R \). Then Theorem 7 implies the next result.

**Theorem 8** Let \( \mathcal{H} := \text{clos}_{\mathcal{L}^2} \text{span} \{ B_q(\cdot - k) \} \) where \( \text{Sc \ } q > 1 \) and \( q \in \mathbb{Q}_R \). Then, for all \( f \in \mathcal{H} \),

\[ f(t) = \lim_{N \to \infty} \sum_{k=-N}^{N} f(k)L_q(t - k) \]

where the convergence is in the \( L^2(\mathbb{R}) \) norm, pointwise, and uniform on compact sets.
Proof We first show that \( \mathcal{H} = \mathcal{H}' := \text{clos}_\ell \text{span} \{ L_q(\cdot - k) \} \). Let \( f \in \mathcal{H} \), i.e., \( f(t) = \sum_k d_k B_q(t - k) \) with \( \{ d_k \}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \). Then

\[
f(t) = \sum_k d_k \sum_\ell b^\ell_k B_q(\ell) L_q(t - \ell)
= \sum_\ell \left( \sum_k d_k b^\ell_{\ell - k} \right) L_q(t - \ell) = \sum_k (d \ast b^q)_\ell L_q(t - \ell)
\]

where \( b^q_k = B_q(k) \) (\( k \in \mathbb{Z} \)). Since \( \{ d_k \}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \) and \( b^q \in \ell^1(\mathbb{Z}) \), an application of Young’s inequality gives

\[
\| d \ast b^q \|_\ell^2 \leq C \| d \|_\ell^2 \| b^q \|_\ell^1
\]

from which we see that \( f \in \mathcal{H}' \) and consequently \( \mathcal{H} \subset \mathcal{H}' \). Note that the function \( \sum_k \widehat{B}_q(\xi + 2\pi k) \) has absolutely convergent Fourier series \( \sum_\ell \widehat{B}_q(\ell)e^{i\ell\xi} \) (since S\( c \) \( q \)). Also, \( \sum_k \widehat{B}_q(\xi + 2\pi k) \) is zero-free, and we conclude from Wiener’s Tauberian theorem [9] that \( 1/\sum_k \widehat{B}_q(\xi + 2\pi k) \) has absolutely convergent Fourier series, i.e.,

\[
\sum_k c^q_k e^{ik\xi} = \sum_k c^q_k e^{-ik\xi}
\]

with \( \sum_k |c^q_k| < \infty \). Suppose \( f \in \mathcal{H}' \), i.e., \( f(t) = \sum_k d_k L_q(t - k) \) with \( \{ d_k \}_{k=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) \). Then

\[
f(t) = \sum_k d_k \sum_j c^q_j B_q(t - j - k) = \sum_j (d \ast c^q)_j B_q(t - j).
\]

But Young’s theorem gives \( \| d \ast c^q \|_{\ell^2} \leq C \| c^q \|_\ell^1 \| d \|_\ell^2 \), and therefore \( f \in \mathcal{H} \) so that \( \mathcal{H}' \subset \mathcal{H} \). We conclude that \( \mathcal{H} = \mathcal{H}' \).

In the spirit of Theorem 7, let \( I = [-\pi, \pi] \) and \( \phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}} \) (\( n \in \mathbb{Z} \)). Then \( \{ \phi_n \}_{n=-\infty}^{\infty} \) is an orthonormal basis for \( L^2(I) \). Let \( S_m(t) = L_q(t - m) \) and \( t_m = m \) (\( m \in \mathbb{Z} \)). Then

\[
K(x, t) = \sum_{n=-\infty}^{\infty} \phi_n(x) S_n(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-inx} L_q(t - n).
\]

Therefore, if \( F \in L^2(I) \),

\[
K F(t) = \frac{1}{\sqrt{2\pi}} \int_0^1 F(x) \sum_k e^{-ikx} L_q(t - k) \, dx = \sum_k \hat{F}(k) L_q(t - k)
\]
where \( \hat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_0^1 F(x)e^{-ikx} \, dx \) is the \( k \)-th Fourier coefficient of \( F \). Furthermore,

\[
\widehat{K}F(\xi) = \sum_k \hat{F}(k) \hat{L}_q(\xi) e^{ik\xi} = F(\xi) \hat{L}_q(\xi).
\]

By the proof of Theorem 7, \( \text{Ran}(K) \) is a Hilbert space with inner product

\[
\langle f, g \rangle_{\text{Ran}(K)} = \langle KF, KG \rangle_{\text{Ran}(K)} = \langle F, G \rangle_{L^2[-\pi, \pi]}.
\]

Let \( f = KF \) for some \( F \in L^2[-\pi, \pi] \). Then

\[
\|f\|_{L^2(\mathbb{R})}^2 = \|KF\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |F(\xi)|^2 |\hat{L}_q(\xi)|^2 \, d\xi = \int_{-\pi}^{\pi} |F(\xi)|^2 \sum_k |\hat{L}_q(\xi + 2\pi k)|^2 \, d\xi \leq C \|F\|_{L^2[-\pi, \pi]}^2 = \|f\|_{\text{Ran}(K)}^2
\]

where

\[
C = \sup_{|\xi| \leq \pi} \frac{\sum_k |\hat{B}_q(\xi + 2\pi k)|^2}{\inf_{|\xi| \leq \pi} \left( \sum_\ell |\hat{B}_q(\xi + 2\pi \ell)|^2 \right)^{\frac{1}{2}}} < \infty
\]

since \( q \in Q_R \) and \( \text{Sc} \, q > 1 \). On the other hand,

\[
\|f\|_{L^2(\mathbb{R})}^2 = \|KF\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |F(\xi)|^2 |\hat{L}_q(\xi)|^2 \, d\xi \geq c \int_{-\pi}^{\pi} |F(\xi)|^2 \, d\xi = c \|F\|_{L^2[-\pi, \pi]}^2 = c \|f\|_{\text{Ran}(K)}^2
\]

where

\[
c = \frac{\inf_{|\xi| \leq \pi} \sum_k |\hat{B}_q(\xi + 2\pi k)|^2}{\sup_{|\xi| \leq \pi} \left( \sum_\ell |\hat{B}_q(\xi + 2\pi \ell)|^2 \right)^{\frac{1}{2}}} \geq \frac{(2/\pi)^{\text{Sc} \, q}}{\sup_{|\xi| \leq \pi} \left( \sum_\ell |\hat{B}_q(\xi + 2\pi \ell)|^2 \right)^{\frac{1}{2}}} > 0.
\]

We therefore have

\[
\|f\|_{L^2(\mathbb{R})} \simeq \|f\|_{\text{Ran}(K)} \simeq \|f\|_H \quad (7.5)
\]

for all \( f \in H \). By Theorem 7, each \( f \in H \) admits the sampling expansion \( f(t) = \sum_{k=-\infty}^{\infty} f(k) L_q(t-k) \) with convergence in the norm of \( \text{Ran}(K) \), or equivalently [by \( (7.5) \)], in the \( L^2(\mathbb{R}) \) norm. By (7.3), norm convergence implies pointwise convergence
and uniform convergence on compact sets since an application of Young’s inequality gives

$$\sum_{n=-\infty}^{\infty} |L_q(t-n)|^2 = \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} c_k^q B_q(t-n-k) \right|^2$$

$$\leq \sum_{k=-\infty}^{\infty} |c_k^q|^2 \left( \sum_{n=-\infty}^{\infty} |B_q(t-n)| \right)^2$$

$$\leq C \left( \inf_{|\xi|\leq\pi} \left| \sum_{k=-\infty}^{\infty} \widehat{B_q}(\xi + 2\pi k) \right| \right)^{-2} \leq C' < \infty$$

where we have used the decay estimate on $B_q$ which is valid since $\text{Sc} \ q > 1$. □

8 Summary

We constructed fundamental cardinal B-splines $L_q$ of quaternionic orders $q$ where $q$ belongs to a certain nonempty region in $\mathbb{H}_\mathbb{R}$. These quaternionic splines satisfy the interpolation conditions $L_q(m) = \delta_{m,0}$, $m \in \mathbb{Z}$. The construction employs interesting properties of an associated quaternionic Hurwitz zeta function and the existence of complex quaternionic inverses. We showed that the cardinal fundamental splines of quaternionic order fit into the setting of Kramer’s Lemma and allow for a family of sampling (respectively, interpolation) series.

Acknowledgements The authors wish to thank the anonymous referees whose insightful questions have led to significant improvements to this paper.

References

1. Chui, C.K.: An Introduction to Wavelets. Academic Press, Cambridge (1992)
2. Epstein, C.L.: How well does the finite Fourier transform approximate the Fourier transform? Commun. Pure. Appl. Math. 58, 1421–1435 (2005)
3. Forster, B., Blu, Th, Unser, M.: Complex B-splines. Appl. Comput. Harmon. Anal. 20, 261–282 (2006)
4. Forster, B., Massopust, P.: Interpolation with fundamental splines of fractional order. In: Proceedings of SampTA, pp. 1–4 (2011)
5. Forster, B., Garunkstis, R., Massopust, P., Steuding, J.: Complex B-splines and Hurwitz zeta functions. Lond. Math. Soc. J. Comput. Math. 16, 61–77 (2013)
6. Garcia, A.G.: Orthogonal sampling formulas: a unified approach. SIAM Rev. 42(3), 499–512 (2000)
7. Hogan, J.A., Massopust, P.R.: Quaternionic B-splines. J. Approx. Theory 224, 43–65 (2017)
8. Hogan, J.A., Morris, A.J.: Quaternionic wavelets. Numer. Funct. Anal. Optim. 33(7–9), 1095–1111 (2012)
9. Katznelson, Y.: An Introduction to Harmonic Analysis, 3rd edn. Cambridge Mathematical Library, Cambridge (2004)
10. Kramer, H.P.: A generalized sampling theorem. J. Math. Phys. 63, 68–72 (1957)
11. Olhede, S., Metikas, G.: The Hyperanalytic wavelet transform. Statistics Section Technical Report TR-06-02, Imperial College, London, pp. 1–49 (2006)
12. Olhede, S., Metikas, G.: The monogenic wavelet transform. IEEE Trans. Signal Process. 57(9), 3426–3441 (2009)
13. Schoenberg, I.J.: Cardinal interpolation and spline functions. J. Approx. Theory 2, 167–206 (1969)
14. Schoenberg, I.J.: Cardinal interpolation and spline functions: II Interpolation of data of power growth. J. Approx. Theory 6, 404–420 (1972)
15. Schoenberg, I.J.: Cardinal Spline Interpolation, CBMS-NSF 12. SIAM, Philadelphia (1973)
16. Unser, M., Blu, Th: Fractional splines and wavelets. SIAM Rev. 42(1), 43–67 (2001)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.