Sasakian metric as a Ricci soliton and related results

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Abstract: We prove the following results: (i) A Sasakian metric as a non-trivial Ricci soliton is null $\eta$-Einstein, and expanding. Such a characterization permits to identify the Sasakian metric on the Heisenberg group $\mathcal{H}^{2n+1}$ as an explicit example of (non-trivial) Ricci soliton of such type. (ii) If an $\eta$-Einstein contact metric manifold $M$ has a vector field $V$ leaving the structure tensor and the scalar curvature invariant, then either $V$ is an infinitesimal automorphism, or $M$ is $D$-homothetically fixed $K$-contact.

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1 Introduction

A Ricci soliton is a natural generalization of an Einstein metric, and is defined on a Riemannian manifold $(M, g)$ by

\[(\mathcal{L}_V g)(X, Y) + 2\text{Ric}(X, Y) + 2\lambda g(X, Y) = 0\] (1)

where $\mathcal{L}_V g$ denotes the Lie derivative of $g$ along a vector field $V$, $\lambda$ a constant, and arbitrary vector fields $X, Y$ on $M$. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as $\lambda$ is negative, zero, and positive respectively. Actually, a Ricci soliton is a generalized fixed point of Hamilton’s Ricci flow $\overrightarrow{\partial}_t g_{ij} = -2R_{ij}$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. For details, see Chow et al. [4]. The vector field $V$ generates the Ricci soliton viewed as a special solution of the Ricci flow. A Ricci soliton is said to be a
gradient Ricci soliton, if $V = -\nabla f$ (up to a Killing vector field) for a smooth function $f$. Ricci solitons are also of interest to physicists who refer to them as quasi-Einstein metrics (for example, see Friedan [9]).

An odd dimensional analogue of Kaehler geometry is the Sasakian geometry. The Kaehler cone over a Sasakian Einstein manifold is a Calabi-Yau manifold which has application in physics in superstring theory based on a 10-dimensional manifold that is the product of the 4-dimensional space-time and a 6-dimensional Ricci-flat Kaehler (Calabi-Yau) manifold (see Candelas et al. [3]). Sasakian geometry has been extensively studied since its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics, for example, $p$-brane solutions in superstring theory, Maldacena conjecture (AdS/CFS duality) [9]. For details, see Boyer, Galicki and Matzeu [2].

In [12] Sharma showed that if a $K$-contact (in particular, Sasakian) metric is a gradient Ricci soliton, then it is Einstein. This was also shown later independently by He and Zhu [8] for the Sasakian case. Recently, Sharma and Ghosh [13] proved that a 3-dimensional Sasakian metric which is a non-trivial (i.e. non-Einstein) Ricci soliton, is homothetic to the standard Sasakian metric on $\text{nil}^3$. In this paper, we generalize these results and also answer the following question of H.-D. Cao (cited in [8]): “Does there exist a shrinking Ricci soliton on a Sasakian manifold, which is not Einstein?” by proving

**Theorem 1** If the metric of a $(2n + 1)$-dimensional Sasakian manifold $M (\eta, \xi, g, \varphi)$ is a non-trivial (non-Einstein) Ricci soliton, then (i) $M$ is null $\eta$-Einstein (i.e. $D$-homothetically fixed and transverse Calabi-Yau), (ii) the Ricci soliton is expanding, and (iii) the generating vector field $V$ leaves the structure tensor $\varphi$ invariant, and is an infinitesimal contact $D$-homothetic transformation.

Conversely, we consider the following question: “What can we say about an $\eta$-Einstein contact metric manifold $M$ which admits a vector field $V$ that leaves $\varphi$ invariant?” and answer it by assuming the invariance of the scalar curvature under $V$, in the form of the following result.

**Theorem 2** If an $\eta$-Einstein contact metric manifold $M$ admits a vector field $V$ that leaves the structure tensor $\varphi$ and the scalar curvature invariant,
then either \( V \) is an infinitesimal automorphism, or \( M \) is \( D \)-homothetically fixed and \( K \)-contact.

**Remark 1** Note that a Ricci soliton as a Sasakian metric is different from the Sasaki-Ricci soliton in the context of transverse Kaehler structure in a Sasakian manifold, for example see Futaki et al. [5]).

**Remark 2** Boyer et al. [2] have studied \( \eta \)-Einstein geometry as a class of distinguished Riemannian metrics on contact metric manifolds, and proved the existence of \( \eta \)-Einstein metrics on many different compact manifolds. We would also like to point out that Zhang [18] showed that compact Sasakian manifolds with constant scalar curvature and satisfying certain positive curvature condition is \( \eta \)-Einstein.

**Remark 3** Theorem 2 provides a generalization of the infinitesimal version of the following result of Tanno [15] “The group of all diffeomorphisms \( \Phi \) which leave the structure tensor \( \varphi \) of a contact metric manifold \( M \) invariant, is a Lie transformation group, and coincides with the automorphism group \( \mathcal{A} \) if \( M \) is Einstein.” Note that the scalar curvature of an Einstein metric is constant. We also note that the set of all vector fields on a contact metric manifold \( M \), that leave \( \varphi \) and scalar curvature invariant, forms a Lie subalgebra of the Lie algebra of all smooth vector fields on \( M \).

## 2 A Brief Review Of Contact Geometry

A \((2n+1)\)-dimensional smooth manifold is said to be contact if it has a global 1-form \( \eta \) such that \( \eta \wedge (d\eta)^n \neq 0 \) on \( M \). For a contact 1-form \( \eta \) there exists a unique vector field \( \xi \) such that \( d\eta(\xi, X) = 0 \) and \( \eta(\xi) = 1 \). Polarizing \( d\eta \) on the contact subbundle \( \eta = 0 \), we obtain a Riemannian metric \( g \) and a \((1,1)\)-tensor field \( \varphi \) such that

\[
d\eta(X,Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 = -I + \eta \otimes \xi
\]  

\( g \) is called an associated metric of \( \eta \) and \((\varphi, \eta, \xi, g)\) a contact metric structure. Following [1] we recall two self-adjoint operators \( h = \frac{1}{2} \mathcal{L}_\xi \varphi \) and \( l = R(\cdot, \xi) \xi \). The tensors \( h, h\varphi \) are trace-free and \( h\varphi = -\varphi h \). We also have these formulas for a contact metric manifold.

\[
\nabla_X \xi = -\varphi X - \varphi hX
\]  

(3)
\[ l - \varphi l \varphi = -2(h^2 + \varphi^2) \]  \hfill (4)
\[ \nabla_\xi h = \varphi - \varphi l - \varphi h^2 \]  \hfill (5)
\[ Trl = Ric(\xi, \xi) = 2n - Tr h^2 \]  \hfill (6)

where \( \nabla, R, Ric \) and \( Q \) denote respectively, the Riemannian connection, curvature tensor, Ricci tensor and Ricci operator of \( g \). For details see [1].

A vector field \( V \) on a contact metric manifold \( M \) is said to be an infinitesimal contact transformation if \( \mathcal{L}_V \eta = \sigma \eta \) for some smooth function \( \sigma \) on \( M \). \( V \) is said to be an infinitesimal automorphism of the contact metric structure if it leaves all the structure tensors \( \eta, \xi, g, \varphi \) invariant (see Tanno [14]).

A contact metric structure is said to be \( K \)-contact if \( \xi \) is Killing with respect to \( g \), equivalently, \( h = 0 \). The contact metric structure on \( M \) is said to be Sasakian if the almost Kaehler structure on the cone manifold \( (M \times R^+, r^2g + dr^2) \) over \( M \), is Kaehler. Sasakian manifolds are \( K \)-contact and \( K \)-contact 3-manifolds are Sasakian. For a Sasakian manifold,

\[(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \]  \hfill (7)
\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad Q\xi = 2n\xi \]  \hfill (8)

For a Sasakian manifold, the restriction of \( \varphi \) to the contact sub-bundle \( D \) \( (\eta = 0) \) is denoted by \( J \) and \( (D, J, d\eta) \) defines a Kaehler metric on \( D \), with the transverse Kaehler metric \( g^T \) related to the Sasakian metric \( g \) as \( g = g^T + \eta \otimes \eta \). One finds by a direct computation that the transverse Ricci tensor \( Ric^T \) of \( g^T \) is given by

\[ Ric^T(X, Y) = Ric(X, Y) + 2g(X, Y) \]

for arbitrary vector fields \( X, Y \) in \( D \). The Ricci form \( \rho \) and transverse Ricci form \( \rho^T \) are defined by

\[ \rho(X, Y) = Ric(X, \varphi Y), \quad \rho^T(X, Y) = Ric^T(X, \varphi Y) \]

for \( X, Y \in D \). The basic first Chern class \( 2\pi c_1^B \) of \( D \) is represented by \( \rho^T \). In case \( c_1^B = 0 \), the Sasakian structure is said to be null (transverse Calabi-Yau). We refer to [2] for details.
A contact metric manifold $M$ is said to be $\eta$-Einstein in the wider sense, if the Ricci tensor can be written as

$$Ric(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$$

(9)

for some smooth functions $\alpha$ and $\beta$ on $M$. It is well-known (Yano and Kon [17]) that $\alpha$ and $\beta$ are constant if $M$ is $K$-contact, and has dimension greater than 3.

Given a contact metric structure $(\eta, \xi, g, \phi)$, let $\bar{\eta} = a\eta, \bar{\xi} = \frac{1}{a}\xi, \bar{\phi} = \phi, \bar{g} = ag + a(a - 1)\eta \otimes \eta$ for a positive constant $a$. Then $(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is again a contact metric structure. Such a change of structure is called a $D$-homothetic deformation, and preserves many basic properties like being $K$-contact (in particular, Sasakian). It is straightforward to verify that, under a $D$-homothetic deformation, a $K$-contact $\eta$-Einstein manifold transforms to a $K$-contact $\eta$-Einstein manifold such that $\bar{\alpha} = \frac{a+2-2a}{a}$ and $\bar{\beta} = 2n - \bar{\alpha}$. We remark here that the particular value: $\alpha = -2$ remains fixed under a $D$-homothetic deformation, and as $\alpha + \beta = 2n$, $\beta$ also remains fixed. Thus, we state the following definition.

**Definition 1** A $K$-contact $\eta$-Einstein manifold with $\alpha = -2$ is said to be $D$-homothetically fixed.

### 3 Proofs Of The Results

**Proof Of Theorem 1:** Using the Ricci soliton equation (11) in the commutation formula (Yano [16], p.23)

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z Ric)(X, Y)$$

(10)

we derive

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z Ric)(X, Y) - (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z)$$

(11)

As $\xi$ is Killing, we have $\mathcal{L}_\xi Ric = 0$ which, in view of (3), the last equation of (8) and $h = 0$, is equivalent to $\nabla_\xi Q = Q\phi - \phi Q$. But for a Sasakian
manifold, \( Q \) commutes with \( \varphi \), and hence \( \text{Ric} \) is parallel along \( \xi \). Moreover, differentiating the last equation of (8), we have \( (\nabla_X Q)\xi = Q\varphi X - 2n\varphi X \). Substituting \( \xi \) for \( Y \) in (11) and using these consequences we obtain

\[
(\mathcal{L}_Y \nabla)(X, \xi) = -2Q\varphi X + 4n\varphi X \tag{12}
\]

Differentiating this along an arbitrary vector field \( Y \), using (7) and the last equation of (8), we find

\[
(\nabla_Y \mathcal{L}_Y \nabla)(X, \xi) - (\mathcal{L}_Y \nabla)(X, \varphi Y) = -2(\nabla_Y Q)\varphi X + 2n(\varphi X)QY - 4n\eta(X)Y
\]

The use of the foregoing equation in the commutation formula [16]:

\[
(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z) \tag{13}
\]

for a Riemannian manifold, shows that

\[
(\mathcal{L}_V R)(X, Y)\xi - (\mathcal{L}_V \nabla)(Y, \varphi X) + (\mathcal{L}_V \nabla)(X, \varphi Y) = -2(\nabla_X Q)\varphi Y + 2(\nabla_Y Q)\varphi X + 2n(\nabla_Y Q)\varphi X - 2\eta(X)QY + 4n\eta(X)Y - 4n\eta(Y)X
\]

Substituting \( \xi \) for \( Y \) in the foregoing equation, using (12) and the formula \( \nabla_\xi Q = 0 \) noted earlier, we find that

\[
(\mathcal{L}_V R)(X, \xi)\xi = 4(QX - 2nX) \tag{14}
\]

Equation (14) gives \( (\mathcal{L}_V g)(X, \xi) + 2(2n + \lambda)\eta(X) = 0 \), which in turn, gives

\[
(\mathcal{L}_V \eta)(X) - g(\mathcal{L}_V \xi, X) + 2(\lambda + 2n)\eta(X) = 0 \tag{15}
\]

\[
\eta(\mathcal{L}_V \xi) = (2n + \lambda) \tag{16}
\]

where we used the Lie-derivative of \( g(\xi, \xi) = 1 \) along \( V \). Next, Lie-differentiating the formula \( R(X, \xi)\xi = X - \eta(X)\xi \) [a consequence of the first formula in (8)] along \( V \), and using equations (14) and (16) provides

\[
4(QX - 2nX) - g(\mathcal{L}_V \xi, X)\xi + 2(2n + \lambda)X = -((\mathcal{L}_V \eta)(X))\xi
\]

By the direct application of (15) to the the above equation we find

\[
\text{Ric}(X, Y) = (n - \frac{\lambda}{2})g(X, Y) + (n + \frac{\lambda}{2})\eta(X)\eta(Y) \tag{17}
\]
which shows that $M$ is $\eta$-Einstein with scalar curvature

$$r = 2n(n + 1) - n\lambda$$  \hspace{1cm} (18)

At this point, we recall the following integrability formula [12]:

$$\mathcal{L}_V r = -\Delta r + 2\lambda r + 2|Q|^2$$  \hspace{1cm} (19)

for a Ricci soliton, where $\Delta r = -\text{div} Dr$. A straightforward computation using (17) gives the squared norm of the Ricci operator as $|Q|^2 = 2n(n^2 - n\lambda + \frac{n^2}{4} + 4n^2)$. Using this and (18) in (19), we obtain the quadratic equation $(2n + \lambda)(2n + 4 - \lambda) = 0$. As $\lambda = -2n$ corresponds to $g$ becoming Einstein, we must have $\lambda = 2n + 4$ and hence the soliton is expanding, which proves part (ii). Moreover, equation (18) reduces to $r = -2n$. Thus equation (17) assumes the form

$$\text{Ric}(Y, Z) = -2g(Y, Z) + 2(n + 1)\eta(Y)\eta(Z)$$  \hspace{1cm} (20)

Hence, as defined in Section 2, $M$ is a $D$-homothetically fixed null $\eta$-Einstein manifold, proving part (i). Using (20) in (11) provides

$$(\mathcal{L}_V \nabla)(Y, Z) = 4(n + 1)\{\eta(Y)\varphi Z + \eta(Z)\varphi Y\}$$  \hspace{1cm} (21)

Differentiating this along $X$, using equations (3) and (7), incorporating the resulting equation in (13), and finally contracting at $X$ we get

$$(\mathcal{L}_V \text{Ric})(Y, Z) = 8(n + 1)\{g(Y, Z) - (2n + 1)\eta(Y)\eta(Z)\}$$  \hspace{1cm} (22)

Equation (20) reduces the soliton equation (1) to the form

$$(\mathcal{L}_V g)(Y, Z) = -4(n + 1)\{g(Y, Z) + \eta(Y)\eta(Z)\}$$  \hspace{1cm} (23)

Next, Lie-differentiating (20) along $V$, and using (23) shows

$$(\mathcal{L}_V \text{Ric})(Y, Z) = 8(n + 1)\{g(Y, Z) + \eta(Y)\eta(Z)\}$$
$$+ 2(n + 1)\{\eta(Z)(\mathcal{L}_V \eta)(Y) + \eta(Y)(\mathcal{L}_V \eta)(Z)\}$$  \hspace{1cm} (24)

Comparing equations (22) with (21) and substituting $\xi$ for $Z$ leads to

$$\mathcal{L}_V \eta = -4(n + 1)\eta$$  \hspace{1cm} (25)
Therefore, substituting $\xi$ for $Z$ in (23) and using (25) we immediately get

$$\mathcal{L}_V \xi = 4(n+1)\xi.$$ 

Operating (25) by $d$, noting $d$ commutes with $\mathcal{L}_V$ and using the first equation of (2) we find

$$(\mathcal{L}_V d\eta)(X,Y) = -4(n+1)g(X,\varphi Y)$$

Its comparison with the Lie-derivative of the first equation of (2) and the use of (23) yields $\mathcal{L}_V \varphi = 0$, completing the proof.

Before proving Theorem 2, we state and prove the following lemma.

**Lemma 1** If a vector field $V$ leaves the structure tensor $\varphi$ of the contact metric manifold $M$ invariant, then there exists a constant $c$ such that

(i) $\mathcal{L}_V \eta = c\eta$, (ii) $\mathcal{L}_V \xi = -c\xi$, (iii) $\mathcal{L}_V g = c(g + \eta \otimes \eta)$.

Though this lemma was proved by Mizusawa in [10], to make the paper self-contained, we provide a slightly different proof as follows.

**Proof:** Lie-differentiating the formulas $\varphi \xi = 0$ and $\eta(\varphi X) = 0$ and using $\mathcal{L}_V \varphi = 0$, we find $\mathcal{L}_V \xi = -c\xi$, and $\mathcal{L}_V \eta = c\eta$ for a smooth function $c$ on $M$. Next, Lie-derivative of the formula $\eta(X) = g(X,\xi)$ along $V$ gives

$$(\mathcal{L}_V g)(X,\xi) = 2c\eta(X) \tag{26}$$

The Lie-derivative of the first equation of (2) along $V$ provides

$$(\mathcal{L}_V g)(X,\varphi Y) = ((dc) \wedge \eta)(X,Y) + cg(X,\varphi Y) \tag{27}$$

Substituting $\xi$ for $Y$ in the above equation we get $dc = (\xi c)\eta$. Taking its exterior derivative, and then exterior product with $\eta$ shows $(\xi c)(d\eta) \wedge \eta = 0$. By definition of the contact structure, $(d\eta) \wedge \eta$ is nowhere zero on $M$, and so $\xi c = 0$. Hence $dc = 0$, i.e. $c$ is constant. Using this consequence, and equations (26) and (27) we obtain (iii), completing the proof.

**Proof Of Theorem 2**: By virtue of Lemma 1, we have

$$\mathcal{L}_V g)(Y,Z) = c\{g(Y,Z) + \eta(Y)\eta(Z)\} \tag{28}$$

Differentiating this and using (3) we get

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = -c\{\eta(Z)g(Y,\varphi X + \varphi hX) + \eta(Y)g(Z,\varphi X + \varphi hX)\} \tag{29}$$
Equation (10) can be written
\[
(\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y) \tag{30}
\]
A straightforward computation using (29) and (30) shows
\[
(\mathcal{L}_V \nabla)(Y, Z) = -c\{\eta(Y)\varphi Y + \eta(Y)\varphi Z + g(Y, \varphi h Z)\xi\}
\]
Its covariant differentiation and use of (2) provides
\[
(\nabla_X \mathcal{L}_V \nabla)(Y, Z) = -c\{\eta(Y)\varphi X + \eta(Y)\varphi h X \varphi Y - g(Y, \varphi X + \varphi h X)\varphi Z \nonumber \\
- g(\varphi h Y, Z)(\varphi X + \varphi h X) + g((\nabla\varphi h) Y, Z)\xi\}
\]
Using this in the commutation formula (13) for a Riemannian manifold, contracting at \(X\), and using equations (2), (3) and also the well known formula: \((\text{div}\varphi)X = -2n\eta(X)\) for a contact metric (see [1]), we find
\[
(\mathcal{L}_V \text{Ric})(Y, Z) = c\{-2g(Y, Z) + 2g(h Y, Z) + 2(2n + 1)\eta(Y)\eta(Z)\} - cg((\nabla\varphi h) Y, Z) \tag{31}
\]
Also, Lie-differentiating (9) along \(V\) and using Lemma 1 we have
\[
(\mathcal{L}_V \text{Ric})(Y, Z) = (V\alpha + c\alpha)g(Y, Z) + (V\beta + c(\alpha + 2\beta))\eta(Y)\eta(Z) \tag{32}
\]
Comparing the previous two equations shows that
\[
[V\alpha + c(\alpha + 2)]g(Y, Z) + [V\beta + c\{\alpha + 2\beta - 2(2n + 1)\}]\eta(Y)\eta(Z) \nonumber \\
- c[2g(h Y, Z) - g((\nabla\varphi h) Y, Z)] = 0
\]
On one hand, we substitute \(Y = Z = \xi\) in the above equation getting one equation, and on the other hand, we contract the above equation (noting that both \(h\) and \(\varphi h\) are trace-free) getting another equation. Solving the two equations we obtain
\[
V\alpha + c(\alpha + 2) = 0, \quad V\beta + c(\alpha + 2\beta - 4n - 2) = 0 \tag{33}
\]
The \(g\)-trace of equation (34) gives the scalar curvature
\[
r = (2n + 1)\alpha + \beta \tag{34}
\]
The divergence of (9) along with the contracted second Bianchi identity yields $dr = 2d\alpha + 2(\xi \beta)\eta$. Taking its exterior derivative, and then exterior product with $\eta$ we have $(\xi \beta)\eta \wedge d\eta = 0$. As $\eta \wedge d\eta$ vanishes nowhere on $M$, we find $\xi \beta = 0$ whence $dr = 2d\alpha$. Hence $V\alpha = Vr = 0$, by hypothesis. Thus, it follows from (34) that $V\beta = 0$. Consequently, equations (33) reduce to: $c(\alpha + 2) = 0$ and $c(\alpha + 2\beta - 4n - 2) = 0$, and hence imply that, either $c = 0$ in which case $V$ is an infinitesimal automorphism, or $\alpha = -2$ and $\alpha + 2\beta = 4n + 2$. In the second case, adding the two equations gives $\alpha + \beta = 2n$. But, from equation (9) we have $\alpha + \beta = Tr.l$. Therefore, $Tr.l = 2n$, and applying equation (6) we obtain $h = 0$, i.e. $M$ is $K$-contact. As $\alpha = -2$, the $\eta$-Einstein structure is $D$-homothetically fixed, completing the proof.

4 An Explicit Example

An explicit example of non-trivial Ricci soliton as a Sasakian metric is the $(2n+1)$-dimensional Heisenberg group $\mathcal{H}^{2n+1}$ (which arose from quantum mechanics) of matrices of type

$$
\begin{bmatrix}
1 & Y & z \\
O^t & I_0 & X^t \\
0 & O & 1
\end{bmatrix},
$$

where $X = (x_1, ..., x_n), Y = (y_1, ..., y_n), O = (0, ..., 0) \in R^n, z \in R$. As a manifold, this is just $R^{2n+1}$ with coordinates $(x^i, y^i, z)$ where $i = 1, ..., n$, and has the left-invariant Sasakian structure $(\eta, \xi, \varphi, g)$ defined by $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i), \xi = 2\frac{\partial}{\partial z}, \varphi(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial y^i}, \varphi(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, \varphi(\frac{\partial}{\partial z}) = 0$, and the Riemannian metric $g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^n ((dx^i)^2 + (dy^i)^2)$. Its $\varphi$-sectional curvature (i.e. the sectional curvature of plane sections orthogonal to $\xi$) is equal to $-3$, so its Ricci tensor satisfies equation (20), as shown by Okumura [11], and hence $\mathcal{H}^{2n+1}$ is a $D$-homothetically fixed null $\eta$-Einstein manifold. Setting $V = \sum_{i=1}^n (V^i \frac{\partial}{\partial x^i} + \bar{V}^i \frac{\partial}{\partial y^i}) + V^z \frac{\partial}{\partial z}$, using equations: $\mathcal{L}_V \xi = 4(n+1)\xi, \mathcal{L}_V \varphi = 0$ obtained in the proof of Theorem 1, and the aforementioned actions of $\varphi$ on the coordinate basis vectors, shows that $V^i$ and $\bar{V}^i$ do not depend on $z$ and yields the PDEs:

$$
\frac{\partial V^i}{\partial x^j} = \frac{\partial V^i}{\partial y^j}, \quad \frac{\partial V^i}{\partial y^j} = -\frac{\partial V^i}{\partial x^j}, \quad y^i \frac{\partial V^i}{\partial y^j} = \frac{\partial V^z}{\partial y^j}
$$

$$
\bar{V}^j = y^i \frac{\partial V^z}{\partial z} - y^i \frac{\partial \bar{V}^i}{\partial y^j}, \quad \frac{\partial V^z}{\partial z} = -4(n+1)
$$

10
The last equation readily integrates as $V^z = -4(n + 1)z + F(x^i, y^i)$. For a special solution, assuming $F = 0$, $V^i = cx^i$, $V^i = cy^i$ and substituting in the above PDEs, we get $c = -2(n + 1)$, and hence the Ricci soliton vector field $V = -2(n + 1)(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + 2z \frac{\partial}{\partial z})$. For dimension 3, this reduces to $V = -4(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2z \frac{\partial}{\partial z})$ which occurs on p. 37 of [4] without the factor 4, but gets adjusted with our $\lambda = 6$ which is 4 times their $\lambda = 3/2$.

**Remark 4** Another conclusion that we draw for Theorem 1 is the following: The value $-2n$ for the scalar curvature $r$ obtained during the proof, and the equation (17) show that the generalized Tanaka-Webster scalar curvature $W = r - \text{Ric}(\xi, \xi) + 4n$ vanishes.

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