Variational Perturbation Theory for Summing Divergent Non-Borel-Summable Tunneling Amplitudes

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We present a method for evaluating divergent non-Borel-summable series by an analytic continuation of variational perturbation theory. We demonstrate the power of the method by an application to the exactly known partition function of the anharmonic oscillator in zero space-time dimensions. In one space-time dimension we derive the imaginary part of the ground state energy of the anharmonic oscillator for all negative values of the coupling constant $g$, including the non-analytic tunneling regime at small $-g$. As a highlight of the theory we retrieve the divergent perturbation expansion from the action of the critical bubble and the contribution of the higher loop fluctuations around the bubble.

I. INTRODUCTION

None of the presently known resummation schemes \cite{1, 2} is able to deal with non-Borel-summable series. Such series arise in the theoretical description of many important physical phenomena, in particular tunneling processes. In the path integral, these are dominated by non-perturbative contributions coming from nontrivial classical solutions called critical bubbles \cite{3, 4} or bounces \cite{5}, and fluctuations around these.

Any Borel-summable series becomes non-Borel-summable if the expansion parameter, usually some coupling constant $g$, is continued to negative values. Here we show that non-Borel-summable series can be evaluated with any desired accuracy by an analytic continuation of variational perturbation theory \cite{2, 4} in the complex $g$-plane. This implies that variational perturbation theory can give us information on non-perturbative properties of the theory.

Variational perturbation theory has a long history \cite{6, 7, 8, 9}. It is based on the introduction of a dummy variational parameter $\Omega$ on which the full perturbation expansion does not depend, while the truncated expansion does. An optimal $\Omega$ is selected by the principle of minimal sensitivity \cite{10}, requiring the quantity of interest to be stationary as function of the variational parameter. The optimal $\Omega$ is usually taken from a zero of the derivative with respect to $\Omega$. If the first derivative has no zero, a zero of the second derivative is chosen. For Borel-summable series, these zeros are always real, in contrast to statements in the literature \cite{11, 12, 13, 14} which have proposed the use of complex zeros. Complex zeros produce in general wrong results for Borel-summable series, as was recently shown in Ref. \cite{15}.

The purpose of this paper is to show that there does exist a wide range of applications of complex zeros if one wants to resum non-Borel-summable series, which have so far remained intractable. These arise typically in tunneling problems, and we shall see that variational perturbation theory provides us with an efficient method for evaluating these series, rendering their real and imaginary parts with any desirable accuracy, if only enough perturbation coefficients are available. An important problem which had to be solved is the specification of the proper choice the optimal zero from the many possible candidates existing in higher orders. A non-Borel-summable series is associated with a function which has an essential singularity at the origin in the complex $g$-plane, which is the starting point of a left-hand cut. Near the tip of the cut, the imaginary part of the function approaches zero rapidly like $\exp(-\alpha/|g|)$ for $g \to 0^-$. If the variational approximation is plotted against $g$ with an enlargement factor $\exp(\alpha/|g|)$, oscillations become visible near $g = 0$. The choice of the optimal complex zeros of the derivative with respect to the variational parameter is fixed by the requirement of obtaining, in each order, the least oscillating imaginary part when approaching the tip of the cut. We may call this selection rule the principle of minimal sensitivity and oscillations.

In Section \textbf{II} we shall explain and test the new principle on the exactly known partition function $Z(g)$ of the anharmonic oscillator in zero space time dimensions. In Section \textbf{III} we apply the method to the critical-bubble regime of small $-g$ of the anharmonic oscillator and find the action of the critical bubble and the corrections caused by the fluctuations around it. In Section \textbf{IV} we present yet another method of calculating the properties of the critical-bubble regime. This method is restricted to quantum mechanical systems. Its results for the anharmonic oscillator give more evidence for the correctness of the general method of Sections \textbf{II} and \textbf{III}.
II. TEST OF RESUMMATION OF NON-BOREL-SUMMABLE EXPANSIONS

The partition function $Z(g)$ of the anharmonic oscillator in zero space-time dimensions is

$$Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-x^2/2 - g x^4/4\right) dx = \frac{\exp\left(1/8g\right)}{\sqrt{4\pi g}} K_{1/4}(1/8g),$$  \hspace{1cm} (2.1) \hspace{1cm} \{Fokker\}

where $K_{\nu}(z)$ is the modified Bessel function. For small $g$, the function $Z(g)$ has a divergent Taylor series expansion, to be called weak-coupling expansion:

$$Z^{(L)}_{weak}(g) = \sum_{l=0}^{L} a_l g^l, \quad \text{with} \quad a_l = (-1)^l \frac{\Gamma(2l + 1/2)}{l! \sqrt{\pi}},$$  \hspace{1cm} (2.2) \hspace{1cm} \{FP-Weak\}

For $g < 0$, this is non-Borel-summable. For large $|g|$ there exists a convergent strong-coupling expansion:

$$Z^{(L)}_{strong}(g) = g^{-1/4} \sum_{l=0}^{L} b_l g^{-l/2}, \quad \text{with} \quad b_l = (-1)^l \frac{\Gamma(l/2 + 1/4)}{2l! \sqrt{\pi}},$$  \hspace{1cm} (2.3) \hspace{1cm} \{FP-Strong\}

As is obvious from the integral representation (2.1), $Z(g)$ obeys the second-order differential equation

$$16g^2 Z''(g) + 4(1 + 8g) Z'(g) + 3Z(g) = 0,$$  \hspace{1cm} (2.4) \hspace{1cm} \{DGL\}

which has two independent solutions. One of them is $Z(g)$, which is finite for $g > 0$ with $Z(0) = a_0$. The weak-coupling coefficients $a_l$ in (2.2) can be obtained by inserting into (2.4) the Taylor series and comparing coefficients. The result is the recursion relation

$$a_{l+1} = -\frac{16l(l+1) + 3}{4(l+1)} a_l.$$  \hspace{1cm} (2.5) \hspace{1cm} \{Rec\}

A similar recursion relation can be derived for the strong-coupling coefficients $b_l$ in Eq. (2.3). We observe that the two independent solutions $Z(g)$ of (2.4) behave like $Z(g) \propto g^{\alpha}$ for $g \to \infty$ with the powers $\alpha = -1/4$ and $-3/4$. The function (2.1) has $\alpha = -1/4$. It is convenient to remove the leading power from $Z(g)$ and define a function $\zeta(x)$ such that $Z(g) = g^{-1/4} \zeta(g^{-1/2})$. The Taylor coefficients of $\zeta(x)$ are the strong-coupling coefficients $b_l$ in Eq. (2.3). The function $\zeta(x)$ satisfies the differential equation and initial conditions:

$$4\zeta''(x) - 2x\zeta'(x) - \zeta(x) = 0, \quad \text{with} \quad \zeta(0) = b_0 \quad \text{and} \quad \zeta'(0) = b_1.$$  \hspace{1cm} (2.6) \hspace{1cm} \{DGL2\}

The Taylor coefficients $b_l$ of $\zeta(x)$ satisfy the recursion relation

$$b_{l+2} = \frac{2l + 1}{4(l+1)(l+2)} b_l.$$  \hspace{1cm} (2.7) \hspace{1cm} \{Rec2\}

Analytic continuation of $Z(g)$ around $g = \infty$ to the left-hand cut gives:

$$Z(-g) = (-g)^{-1/4} \zeta((-g)^{-1/2})$$  \hspace{1cm} (2.8) \hspace{1cm} \{Cut\}

$$= (-g)^{-1/4} \sum_{l=0}^{\infty} b_l(-g)^{-l/2} \exp\left[-\frac{i\pi}{4}(2l + 1)\right] \quad \text{for} \ g > 0,$$  \hspace{1cm} (2.9)

so that we find an imaginary part

$$\text{Im} \ Z(-g) = -(4g)^{-1/4} \sum_{l=0}^{\infty} b_l(-g)^{-l/2} \sin\left[-\frac{i\pi}{4}(2l + 1)\right]$$  \hspace{1cm} (2.10) \hspace{1cm} \{Cut1\}

$$= -(4g)^{-1/4} \sum_{l=0}^{\infty} \beta_l(-g)^{-l/2},$$  \hspace{1cm} (2.11)

where

$$\beta_0 = b_0, \quad \beta_1 = b_1, \quad \beta_{l+2} = -\frac{2l + 1}{4(l+1)(l+2)} \beta_l.$$  \hspace{1cm} (2.12) \hspace{1cm} \{Cut2\}
It is easy to show that
\[ \sum_{l=0}^{\infty} \beta_l x^l = \zeta(x) \exp(-x^2/4), \] (2.13) \{\Phi\}
so that
\[ \text{Im } Z(-g) = -\frac{1}{\sqrt{2}} g^{-1/4} \exp(-1/4g) \sum_{l=0}^{\infty} b_l g^{-l/2}. \] (2.14) \{\text{CUT-STRONG}\}

From this we may re-obtain the weak-coupling coefficients \( a_l \) by means of the dispersion relation
\[ Z(g) = -\frac{1}{\pi} \int_0^\infty \frac{\text{Im } Z(-z)}{z + g} dz \] (2.15) \{\text{DISP}\}
\[ = \frac{1}{\pi \sqrt{2}} \sum_{j=0}^{\infty} b_j \int_0^\infty \exp(-1/4z) z^{-j/2 - 1/4} / z + g \, dz. \] (2.16) \{\text{DISP-2}\}

Indeed, replacing \( 1/(z + g) \) by \( \int_0^\infty \exp(-x(z + g)) \, dx \), and expanding \( \exp(-x g) \) into a power series, all integrals can be evaluated to yield:
\[ Z(g) = \frac{1}{\pi} \sum_{j=0}^{\infty} 2^j b_j \sum_{l=0}^{\infty} (-g)^l \Gamma(l + j/2 + 1/4). \] (2.17) \{\text{DISP-3}\}

Thus we find for the weak-coupling coefficients \( a_l \) an expansion in terms of the strong-coupling coefficients
\[ a_l = \frac{(-1)^l}{\pi} \sum_{j=0}^{\infty} 2^j b_j \Gamma(l + j/2 + 1/4). \] (2.18) \{\text{DISP-3}\}

Inserting \( b_j \) from Eq. (2.3), this becomes
\[ a_l = \frac{(-1)^l}{2 \pi^{3/2}} \sum_{j=0}^{\infty} \frac{2^j (-1)^j}{j!} \Gamma(j/2 + 1/4) \Gamma(l + j/2 + 1/4) = (-1)^l \frac{\Gamma(2l + 1/2)}{l! \sqrt{\pi}}, \] (2.19) \{\text{DISP-3}\}

coinciding with \{FP-EPS\}.\{\text{DISP-3}\}

Variational perturbation theory is a well-established method for obtaining convergent strong-coupling expansions from divergent weak-coupling expansions in quantum-mechanical systems such as the anharmonic oscillator [4, 16] as well as in quantum field theory [2, 17]. We have seen in Eq. (2.8), that the strong-coupling expansion can easily be continued analytically to negative \( g \). This continuation can, however, be used for an evaluation only for sufficiently large \( |g| \) where the strong-coupling expansion converges. In the tunneling regime near the tip of the left-hand cut, the expansion diverges. In this paper we shall find that an evaluation of the weak-coupling expansion according to the rules of variational perturbation theory continued into the complex plane gives extremely good results on the entire left-hand cut with a fast convergence even near the tip at \( g = 0 \).

The \( L \)th variational approximation to \( Z(g) \) is given by (see [2, 17])
\[ Z_{\text{var}}^{(L)}(g, \Omega) = \Omega^p \sum_{j=0}^{L} \left( \frac{g}{\Omega^q} \right)^{-\sigma} \epsilon_j(\sigma), \] (2.20) \{FP-VAR\}
with
\[ \sigma = \Omega^{q-2}(\Omega^q - 1)/g, \] (2.21) \{FP-s\}
where \( q = 2/\omega = 4, \ p = -1 \) and
\[ \epsilon_j(\sigma) = \sum_{l=0}^{j} a_l \left( \frac{(p - lq)/2}{j - l} \right) (-\sigma)^{j-l}. \] (2.22) \{FP-EPS\}
To apply the principle of minimal sensitivity, the zeros of the derivative of $Z_{\text{var}}^{(L)}(g, \Omega)$ with respect to $\Omega$ are needed. They are given by the zeros of the polynomials in $\sigma$:

$$P^{(L)}(\sigma) = \sum_{l=0}^{L} a_l (p - lq + 2L) \left( \frac{(p - lq)/2}{L - l} \right) (-\sigma)^{L-l} = 0,$$

(2.23)  \{FP-DERIV-II\}

since it can be shown that the derivative depends only on $\sigma$:

$$\frac{dZ_{\text{var}}^{(L)}(g, \Omega)}{d\Omega} = \Omega^{p-1} g \left( \frac{g}{\Omega} \right)^L P^{(L)}(\sigma).$$

(2.24)  \{FP-DERIV-II\}

Consider in more detail the lowest non-trivial order with $L = 1$. From Eq. (2.23) we obtain

\begin{align*}
\sigma &= \frac{5}{2}, \quad \text{corresponding to} \quad \Omega = \frac{1}{2} \left( 1 \pm \sqrt{1 + 10g} \right). \quad (2.25)  \{Zero\}
\end{align*}

In order to ensure that our method reproduces the weak-coupling result for small $g$, we have to take the positive sign in front of the square root. In Fig. 1 we have plotted $Z_{\text{var}}^{(1)}(g)$ (dashed curve) and $Z_{\text{var}}^{(2)}(g)$ (solid curve) compared with the exact result (dotted curve) in the tunneling regime. The agreement is quite good even at these low orders. Next we study the behavior of $Z_{\text{var}}^{(L)}(g)$ to higher orders $L$. For selected coupling values in the non-Borel-summable region, $g = -0.1, -1, -10$, we want to see the error as a function of the order. We want to find from this model system the rule for selecting systematically the best zero of $P^{(L)}(\sigma)$ solving Eq. (2.24), which leads to the optimal value of the variational parameter $\Omega$. For this purpose we plot the variational results of all zeros. This is shown in Fig. 2 where the logarithm of the deviations from the exact value is plotted against the order $L$. The outcome of different zeros cluster strongly near the best value. Therefore, choosing any zero out of the middle of the cluster is reasonable, in particular, because it does not depend on the knowledge of the exact solution, so that this rule may be taken over to realistic cases.

We wish to emphasize, that for the Borel-summable domain with $g > 0$, variational perturbation theory has the usual fast convergence in this model. In fact, for $g = 10$, probing deeply into the strong-coupling domain, we find rapid convergence like $\Delta(L) \simeq 0.02 \exp(-0.73L)$ for $L \to \infty$, where $\Delta(L) = \log |Z_{\text{var}}^{(L)} - Z_{\text{exact}}|$ is the logarithmic error as a function of the order $L$. This is shown in Fig. 3. Furthermore, the strong-coupling coefficients $b_l$ of Eq. (2.23) are reproduced quite satisfactorily. Having solved $P^{(L)}(\sigma) = 0$ for $\sigma$, we obtain $\Omega^{(L)}(g)$ by solving Eq. (2.24). Inserting this and (2.22) into (2.24), we bring $g^{1/4} Z_{\text{var}}^{(L)}(g)$ into a form suitable for expansion in powers of $g^{-1/2}$. The expansion coefficients are the strong-coupling coefficients $b_l^{(L)}$ to order $L$. In Fig. 4 we have plotted the logarithms of their
absolute and relative errors over the order $L$, and find very good convergence, showing that variational perturbation theory works well for our test-model $Z(g)$.

A better selection of the optimal $\Omega$ values comes from the following observation. The imaginary parts of the approximations near the singularity at $g = 0$ show tiny oscillations. The exact imaginary part is known to decrease extremely fast, like $\exp(1/4g)$, for $g \to 0^-$, practically without oscillations. We can make the tiny oscillations more visible by taking this exponential factor out of the imaginary part. This is done in Fig. 3. The oscillations differ strongly for different choices of $\Omega^{(L)}$ from the central region of the cluster. To each order $L$ we see that one of them is smoothest in the sense that the approximation approaches the singularity most closely before oscillations begin. If this $\Omega^{(L)}$ is chosen as the optimal one, we obtain excellent results for the entire non-Borel-summable region $g < 0$. As an example, we pick the best zero for the $L = 16$th order. Fig. 3 shows the normalized imaginary part calculated to this order, but based on different zeros from the central cluster. Curve C appears optimal. Therefore we select the underlying zero as our best choice at order $L = 16$ and calculate with it real and imaginary part for the non-Borel-summable region $-2 < g < -0.008$, to be compared with the exact values. Both are shown in Fig. 3, where we have again renormalized the imaginary part by the exponential factor $\exp(-1/4g)$. The agreement with the exact result (solid curve) is excellent as was to be expected because of the fast convergence observed in Fig. 2. It is indeed much better than the strong-coupling expansion to the same order, shown as a dashed curve. This is the essential improvement of our present theory as compared to previously known methods probing into the tunneling regime [19].

This non-Borel-summable regime will now be investigated for the quantum-mechanical anharmonic oscillator.
FIG. 3: Logarithm of deviation of variational results from exactly known value $\Delta(L) = \log |Z_{\text{var}}(L) - Z_{\text{exact}}|$, plotted against the order $L$ for $g = 10$ in Borel-summable region. The real positive optimal $\Omega$ have been used. There is only one real zero of the first derivative in every odd order $L$ and none for even orders. There is excellent convergence $\Delta(L) \simeq 0.02 \exp(-0.73L)$ for $L \to \infty$.

FIG. 4: Relative logarithmic error $\Delta_r = \log |1 - b_l^{(L)}/b_l^{(\text{exact})}|$ on the left, and the absolute logarithmic error $\Delta_a = \log |b_l^{(L)} - b_l^{(\text{exact})}|$ on the right, plotted for some strong-coupling coefficients $b_l$ with $l = 0, 4, 8, 12, 16, 20$ against the order $L$. 
FIG. 5: Normalized imaginary part \( \text{Im}[Z_{\text{var}}^{(16)}(g) \exp(-1/4g)] \) as a function of \( g \) based on six different complex zeros (thin curves). The fat curve represents the exact value, which is \( Z_{\text{exact}}(g) \approx -0.7071 + 0.524g - 1.78g^2 \). Oscillations of varying strength can be observed near \( g = 0 \). Curves A and C carry most smoothly near up to the origin. Evaluation based on either of them yields equally good results. We have selected the zero belonging to curve C as our best choice to this order \( L = 16 \).

III. TUNNELING REGIME OF QUANTUM-MECHANICAL ANHARMONIC OSCILLATOR

The divergent weak-coupling perturbation expansion for the ground state energy of the anharmonic oscillator in the potential \( V(x) = x^2/2 + gx^4 \) to order \( L \)

\[
E_{0,\text{weak}}^{(L)}(g) = \sum_{l=0}^{L} a_l \ g^l, \tag{3.1 \ WEAK}
\]

where \( a_l = (1/2, 3/4, -21/8, 333/16, -30885/128, \ldots) \), is non-Borel-summable for \( g < 0 \). It may be treated in the same way as \( Z(g) \) of the previous model, making use as before of Eqs. (2.20)–(2.23), provided we set \( p = 1 \) and
\[\omega = 2/3,\text{ so that } q = 3,\] accounting for the correct power behavior \(E_0(g) \propto g^{1/3}\) for \(g \to \infty\). According to the principle of minimal dependence and oscillations, we pick a zero best for the order \(L = 64\) from the cluster of zeros of \(P_L(\sigma)\), and use it to calculate the logarithm of the normalized imaginary part:

\[f(g) := \log \left[ \frac{-\pi g/2}{E_{0,\text{vac}}(g)} \right] - 1/3g. \tag{3.2} \]

This quantity is plotted in Fig. 4 against \(\log(-g)\) close to the tip of the left-hand cut for \(-2 < g < -0.006\). Comparing our result to older values from semi-classical calculations [20],

\[f(g) = b_1g - b_2g^2 + b_3g^3 - b_4g^4 + \ldots, \tag{3.3} \]

with

\[b_1 = 3.95833, \quad b_2 = 19.344, \quad b_3 = 174.21, \quad b_4 = 2177, \tag{3.4}\]

shown in Fig. 4 as a thin curve, we find very good agreement. This expansion contains the information on the fluctuations around the critical bubble. It is divergent and non-Borel-summable for \(g < 0\). In Appendix A we have rederived it in a novel way which allowed us to extend and improve it considerably.

Remarkably, our theory allows us to retrieve the first three terms of this expansion from the perturbation expansion. Since our result provides us with a regular approximation to the essential singularity, the fitting procedure depends somewhat on the interval over which we fit our curve by a power series. A compromise between a sufficiently long interval and the runaway of the divergent critical-bubble expansion is obtained for a lower limit \(g > -0.0229 \pm 0.0003\) and an upper limit \(g = -0.006\). Fitting a polynomial to the data, we extract the following first three coefficients:

\[b_1 = 3.9586 \pm 0.0003, \quad b_2 = 19.4 \pm 12, \quad b_3 = 135 \pm 18. \tag{3.5}\]

The agreement of these numbers with those in [20] demonstrates that our method is capable of probing deeply into the critical-bubble region of the coupling constant.

Further evidence for the quality of our theory comes from a comparison with the analytically continued strong-coupling result plotted to order \(L = 22\) as a fat curve in Fig. 4. This expansion was derived by a procedure of summing non-Borel-summable series developed in Chapter 17 of the textbook [4]. It was based on a two-step process: the derivation of a strong-coupling expansion of the type [23] from the divergent weak-coupling expansion, and an analytic continuation of the strong-coupling expansion to negative \(g\). This method was applicable only for large enough coupling strength where the strong-coupling expansion converges, the so-called sliding regime. It could not invade into the tunneling regime at small \(g\) governed by critical bubbles, which was treated in [4] by a separate variational procedure. The present work fills the missing gap by extending variational perturbation theory to all \(g\) arbitrarily close to zero, without the need for a separate treatment of the tunneling regime.

It is interesting to see, how the correct limit is approached as the order \(L\) increases. This is shown in Fig. 5 based on the optimal zero in each order. For large negative \(g\), even the small orders give excellent results. Close to the singularity the scaling factor \(\exp(-1/3g)\) will always win over the perturbation results. It is surprising, however, how fantastically close to the singularity we can go.

IV. DYNAMIC APPROACH TO THE CRITICAL-BUBBLE REGIME

Regarding the computational challenges connected with the critical-bubble regime of small \(g < 0\), it is worth to develop an independent method to calculate imaginary parts in the tunneling regime. For a quantum-mechanical system with an interaction potential \(gV(x)\), such as a the harmonic oscillator, we may study the effect of an infinitesimal increase in \(g\) upon the system. It induces an infinitesimal unitary transformation of the Hilbert space. The new Hilbert space can be made the starting point for the next infinitesimal increase in \(g\). In this way we derive an infinite set of first order ordinary differential equations for the change of the energy levels and matrix elements (for details see Appendix B):

\[E_n'(g) = V_{nn}(g), \tag{4.1} \]

\[V_{mn}'(g) = \sum_{k \neq n} \frac{V_{mk}(g)V_{kn}(g)}{E_m(g) - E_k(g)} + \sum_{k \neq m} \frac{V_{mk}(g)V_{kn}(g)}{E_n(g) - E_k(g)}. \tag{4.2}\]

This system of equations holds for any one-dimensional Schroedinger problem. Individual differences come from the initial conditions, which are the energy levels \(E_n(0)\) of the unperturbed system and the matrix elements \(V_{nm}(0)\) of the
FIG. 7: Logarithm of the imaginary part of the ground state energy of the anharmonic oscillator with the essential singularity factored out for better visualization, \( l(g) = \log \left( \sqrt{-\pi g/2} E_{g,\text{var}}^{(64)}(g) \right) - 1/3g \), plotted against small negative values of the coupling constant \(-0.2 < g < -0.006\) where the series is non-Borel-summable. The thin curve represents the divergent expansion around a critical bubble of Ref. [20]. The fat curve is the 22nd order approximation of the strong-coupling expansion, analytically continued to negative \( g \) in the sliding regime calculated in Chapter 17 of the textbook [4].

FIG. 8: Logarithm of the normalized imaginary part of the ground state energy \( \log \left( \sqrt{-\pi g/2} E_{0,\text{var}}^{(L)}(g) \right) - 1/3g \), plotted against \( \log (-g) \) for orders \( L = 4, 8, 16, 32 \) (curves). It is compared with the corresponding results for \( L = 64 \) (points). This is shown for small negative values of the coupling constant \(-0.2 < g < -0.006\), i.e. in the non-Borel-summable critical-bubble region. Fast convergence is easily recognized. Lower orders oscillate more heavily. Increasing orders allow closer approach to the singularity at \( g = 0^- \).
interaction $V(x)$ in the unperturbed basis. For a numerical integration of the system a truncation is necessary. The obvious way is to restrict the Hilbert space to the manifold spanned by the lowest $N$ eigenvectors of the unperturbed system. For cases like the anharmonic oscillator, which are even, with even perturbation and with only an even state to be investigated, we may span the Hilbert space by even basis vectors only. Our initial conditions are thus for $n = 0, 1, 2, \ldots, N/2$:

$$E_{2n}(0) = 2n + 1/2$$
$$V_{2n,2m} = 0 \text{ if } m < 0 \text{ or } m > N/2$$
$$V_{2n,2n}(0) = 3(8n^2 + 4n + 1)/4$$
$$V_{2n,2n\pm 2}(0) = (4n + 3)\sqrt{(2n + 1)(2n + 2)/2}$$
$$V_{2n,2n\pm 4}(0) = \sqrt{(2n + 1)(2n + 2)(2n + 3)(2n + 4)/4}$$

For the anharmonic oscillator with a $V(x) = x^4$ potential, all sums in equation (4.1) are finite with at most four terms due to the near-diagonal structure of the perturbation.

In order to find a solution for some $g < 0$, we first integrate the system from 0 to $|g|$, then around a semi-circle $g = |g| \exp(i\varphi)$ from $\varphi = 0$ to $\varphi = \pi$. The imaginary part of $E_0(g)$ obtained from a calculation with $N = 64$ is shown in Fig. refVIx, where it is compared with the variational result for $L = 64$. The agreement is excellent. It must be noted, however, that the necessary truncation of the system of differential equations introduces an error, which cannot be made arbitrarily small by increasing the truncation limit $N$. The approximations are asymptotic sharing this property with the original weak-coupling series. Its divergence is, however, reduced considerably, which is the reason why we obtain accurate results for the critical-bubble regime, where the weak-coupling series fails completely to reproduce the imaginary part.

V. APPENDIX A

We determine the ground state energy function $E_0(g)$ for the anharmonic oscillator on the cut, i.e. for $g < 0$ in the bubble region, from the weak coupling coefficients $a_l$ of equation (3.1). The behavior of the $a_l$ for large $l$ can be cast
into the form

$$a_l/a_{l-1} = -\sum_{j=-1}^l \beta_j \ell^{-j}. \quad (5.1) \{B1\}$$

The $\beta_j$ can be determined by a high precision fit to the data in the large $\ell$ region of $250 < \ell < 300$ to be

$$\beta_{-1, 0, 1, \ldots} = \left\{ \begin{array}{c} 3, \frac{95}{24}, \frac{113}{24}, \frac{391691}{6}, \frac{40783}{48}, \frac{1915121357}{124416}, \frac{10158832895}{71663616}, \\ \frac{267843837757582}{4478976}, \frac{43743}{1423}, \frac{351954117229}{12122186977970425}, \frac{1915121357}{3550} \end{array} \right\}, \quad (5.2) \{B2\}$$

where the rational numbers up to $j = 6$ are found to be exact, whereas the higher ones are approximations.

Equation (5.1) can be read as recurrence relation for the coefficients $a_l$. Now we construct an ordinary differential equation for $E(g) := E_{0,\text{weak}}(g)$ from this recurrence relation and find:

$$\left[ \left( g \frac{d}{dg} \right)^L + g \sum_{j=0}^{L+1} \beta_{L-j} \left( g \frac{d}{dg} + 1 \right)^j \right] E(g) = 0 \cdot \quad (5.3) \{B3\}$$

All coefficients being real, real and imaginary part of $E(g)$ each have to satisfy this equation separately. The point $g = 0$, however, is not a regular point. We are looking for a solution, which is finite when approaching it along the negative real axis. Asymptotically $E(g)$ has to satisfy

$$E(g) \simeq \exp\left( 1/g \beta_{-1} \right) = \exp\left( 1/3g \right). \quad \text{Therefore we solve } (5.3) \text{ with the ansatz}$$

$$E(g) = g^\alpha \exp\left( \frac{1}{3g} - \sum_{k=1} b_k (-g)^k \right) \quad (5.4) \{B4\}$$

to obtain $\alpha = -1/2$ and

$$\left\{ \begin{array}{c} 95, 619, 200689, 2229541, 104587909, 7776055955, 9339313153349, 127213593813181, \\ 24, 32, 1152, 1024, 3072, 12288, 688128, 524288, \\ 139886592, 5439163728720, 143593922458208, \\ 10905182471547987497465746723, 45574017678173074497482074500364087, \\ 348951880031792748, 3780312033677803520 \end{array} \right\}, \quad (5.5) \{B5\}$$

This is in agreement with equation (5.1) and an improvement compared to the WKB results of [20]. Again, the first six rational numbers are exact, followed by approximate ones.

### VI. APPENDIX B

Given a one-dimensional quantum system

$$(H_0 + g V)|n, g\rangle = E_n(g)|n, g\rangle \quad (6.1) \{A1\}$$

with Hamiltonian $H = H_0 + g V$, eigenvalues $E_n(g)$ and eigenstates $|n, g\rangle$ we consider an infinitesimal increase $dg$ in the coupling constant $g$. The eigenvectors will undergo a small change:

$$|n, g + dg\rangle = |n, g\rangle + dg \sum_{k \neq n} u_{nk}|k, g\rangle \quad (6.2) \{A2\}$$

so that

$$\frac{d}{dg} |n, g\rangle = \sum_{k \neq n} u_{nk}|k, g\rangle. \quad (6.3) \{A3\}$$
Given this, we take the derivative of (6.1) with respect to \( g \) and multiply by \( \langle m, g \rangle \) from the left to obtain:

\[
\langle m, g \rangle [V - E'_n(g)] |n, g\rangle = \sum_{k \neq n} u_{nk} \langle m, g \rangle H_0 + g V - E_n(g) |k, g\rangle.
\]  

(6.4) \{44\}

Setting now \( m = n \) and \( m \neq n \) in turn, we find:

\[
E'_n(g) = \langle V'_{nn}(g) \rangle
\]  

(6.5) \{45\}

\[
V'_{mn}(g) = \langle u_{nm} (E_m(g) - E_n(g)) \rangle
\]  

(6.6) \{46\}

where \( V_{mn}(g) = \langle m, g \rangle V |n, g\rangle \).

Equation (6.5) governs the behavior of the eigenvalues as functions of the coupling constant \( g \). In order to have a complete system of differential equations, we must also determine how the \( V_{nm}(g) \) change, when \( g \) changes. With the help of equations (6.3) and (6.6), we obtain:

\[
V'_{mn} = \sum_{k \neq m} u_{mk} V_{kn} + \sum_{k \neq n} V_{mk} V_{kn} + \sum_{k \neq n} V_{mk} V_{kn} - E_k
\]  

Equations (6.5) and (6.8) together describe a complete set of differential equations for the energy eigenvalues \( E_n(g) \) and the matrix-elements \( V_{nm}(g) \). The latter determine via (6.6) the expansion coefficients \( u_{nm}(g) \). Initial conditions are given by the eigenvalues \( E_n(0) \) and the matrix elements \( V_{nm}(0) \) of the unperturbed system.

[1] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Clarendon, Oxford, 1989.
[2] H. Kleinert, V. Schulte-Frohlinde Critical Properties of \( \Phi^4 \)-Theories, World Scientific, Singapore, 2001 [http://www.physik.fu-berlin.de/~kleinert/b8].
[3] J.S. Langer, Ann. Phys. 41, 108 (1967).
[4] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics and Polymer Physics, World Scientific, Singapore, 1995.[http://www.physik.fu-berlin.de/~kleinert/b5].
[5] S. Coleman, Nucl. Phys. B 298, 178 (1988).
[6] R. Sezne and J. Zinn-Justin, J. Math. Phys. 20, 1398 (1979); T. Barnes and G.I. Ghandour, Phys. Rev. D 22, 924 (1980); B.S. Shaverdyan and A.G. Ushveridze, Phys. Lett. B 123, 316 (1983); K. Yamazaki, J. Phys. A 17, 345 (1984); H. Mitter and K. Yamazaki, J. Phys. A 17, 1215 (1984); P.M. Stevenson, Phys. Rev. D 30, 1712 (1985); D 32, 1389 (1985); P.M. Stevenson and R. Tarrach, Phys. Lett. B 176, 436 (1986); A. Okopinska, Phys. Rev. D 35, 1835 (1987); D 36, 2415 (1987); W. Namgung, P.M. Stevenson, and J.F. Reed, Z. Phys. C 45, 47 (1989); U. Ritschel, Phys. Lett. B 227, 44 (1989); Z. Phys. C 51, 469 (1991); M.H. Thoma, Z. Phys. C 44, 343 (1991); I. Stancu and P.M. Stevenson, Phys. Rev. D 42, 2710 (1991); R. Tarrach, Phys. Lett. B 262, 294 (1991); H. Haugerud and F. Raund, Phys. Rev. D 43, 2376 (1991); A.N. Sissakian, I.L. Solovtsov, and O.Y. Shevchenko, Phys. Lett. B 313, 367 (1993).
[7] J.R.C. Buckley, A. Duncan, H.F. Jones, Phys. Rev. D 47, 2554 (1993); C.M. Bender, A. Duncan, H.F. Jones, Phys. Rev. D 49, 4219 (1994).
[8] H. Kleinert and W. Janke, Convergence of variational perturbation expansion, A method for locating Bender-Wu singularities, Phys. Lett. A 206, 283 (1995) [quant-ph/9509005].
[9] R. Guida, K. Konishi, and H. Suzuki, Annals Phys. 249, 190 (1996) [hep-th/9505084].
[10] P.M. Stevenson, Phys. Rev. D 30, 1712 (1985); D 32, 1389 (1985); P.M. Stevenson and R. Tarrach, Phys. Lett. B 176, 436 (1986).
[11] B.Bellet, P.Garcia, A.Neveu, Int. J. of Mod. Phys. A 11, 5587 (1997).
[12] J.-L. Kneur, D. Reynaud, [hep-th/0205133] and references [8–16], quoted therein.
[13] E. Braaten, E. Radescu, (cond-math/0206186v1).
[14] F. F. de Souza Cruz, M. B. Pinto and R. O. Ramos, Phys. Rev. A 65, 053613 (2002) [cond-mat/0112306]; Phys. Rev. B 64, 014515 (2001); J.-L. Kneur, M. B. Pinto, R. O. Ramos, (cond-mat/0207295), (cond-mat/0207295), Phys.Rev.Lett. 89 (2002) 210403.
[15] B. Hamprecht and H. Kleinert, Dependence of Variational Perturbation Expansions on Strong-Coupling Behavior. Inapplicability of \( \delta \)-Expansion to Field Theory, [hep-th/0302116].
[16] W. Janke and H. Kleinert, Convergence Strong-Coupling Expansions from Divergent Weak-Coupling Perturbation Theory Phys. Rev. Lett. 75, 2787 (1995) [quant-ph/9502019].
[17] H. Kleinert, Strong-Coupling Behavior of \( \Phi^4 \)-Theories and Critical Exponents, Phys. Rev. D 57, 2264 (1998); Addendum: Phys. Rev. D 58, 107702 (1998) [cond-mat/9803208]; Seven Loop Critical Exponents from Strong-Coupling \( \phi^4 \)-Theory in Three Dimensions, Phys. Rev. D 60, 085001 (1999) [hep-th/9812197]; H. Kleinert, Strong-Coupling \( \phi^4 \)-Theory in \( 4 - \epsilon \) Dimensions, and Critical Exponent, Phys. Lett. B 434, 74 (1998) [cond-mat/9801167].
This was proved in W. Janke and H. Kleinert, *Scaling property of variational perturbation expansion for general anharmonic oscillator with $x^p$-potential*, Phys. Lett. A **199**, 287 (1995) (quant-ph/9502018) for $p/q = 1$ (see also the textbook [4], Appendix 5A), but can easily be generalized to hold for arbitrary $p$ and $q$.

The low-order results were first obtained by H. Kleinert, Phys. Lett. B **300**, 261 (1993) ([http://www.physik.fu-berlin.de/~kleinert/214](http://www.physik.fu-berlin.de/~kleinert/214)), and extended by R. Karrlein and H. Kleinert, Phys. Lett. A **187**, 133 (1994) (hep-th/9504048).

J. Zinn-Justin, J. Math Phys. **22**(3), 511 (1981). The first 10 coefficients of expansion (3.3) are calculated.