QUANTIZED HEISENBERG SPACE

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ABSTRACT. We investigate the algebra $F_q(N)$ introduced by Faddeev, Reshetikhin and Takhadjian [3]. In case $q$ is a primitive root of unity the degree, the center, and the set of irreducible representations are found. The Poisson structure is determined and the De Concini-Kac-Procesi Conjecture is proved for this case. In the case of $q$ generic, the primitive ideals are described. A related algebra studied by Oh is also treated.

1. INTRODUCTION

Among the candidates for a quantized phase space is the algebra $F_q(N)$ introduced by Faddeev, Reshetikhin and Takhadjian [3]. It is promising since it both contain generators satisfying relations

$$A_1B_1 = qB_1A_1$$

and generators satisfying

$$A_2B_2 - qB_2A_2 = I.$$  \(1.1\)

The higher dimensional analogues of the former are often known as “the quantum affine space” (c.f. S. P. Smith, [13]) whereas the latter is among the candidates for “the quantized Heisenberg algebra”. We shall give the precise defining relations of $F_q(N)$ below in Section 2. In [3] it was called the quantized function algebra of Hermitian space - a name we hesitate to adopt. One reason is, as proved by Korogodsky and Vaksman [7], that the algebra is connected with $SU_q(n, 1)$ and hence not directly connected with a hermitian symmetric space. Thus the name in the title.

The algebra $F_q(N)$ is very closely related to the high dimensional quantum Heisenberg algebra ([4]) as well as to the so-called quantum symplectic space $O_q(spC^{2n})$. It is thus natural to expect that many features of those algebras will show up in $F_q(N)$ as indeed they do. We shall return to this connection when appropriate.

In the present article we analyze algebraic properties (center, degree) as well as the representation theory of $F_q(N)$ when $q$ is a primitive root of unity. The algebra fits into the general framework of Procesi and De Concini ([1]). This first gives a tool for computing the center and the degree. Moreover, and more importantly, it yields a Poisson structure which, hypothetically is connected to the representation theory through the so-called De Concini-Kac-Procesi Conjecture ([2]). We prove that this conjecture is true in the present setting.

Section 2 gives the defining relations for the algebra and its associated quasipolynomial algebra. The relation to other algebras, especially the one considered by Oh ([11]) is given.

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Section 3 describes the quite intricate arguments needed for obtaining explicitly the Poisson structure. The DKP Conjecture is then stated. Finally, the symplectic leaves are described, including their dimensions. In Section 4 the theory of De Concini and Procesi is briefly described and is then applied in Section 5 to finding the degree and more generally the canonical form of a certain number of quasipolynomial algebras. In Section 6, the irreducible representations are found and the conjecture is verified. It is made precise how the algebra in [11] also is covered by these results. Finally, in Section 7 we return briefly to the \( q \) generic case. In [11], Oh described completely the set of primitive ideals for \( \mathcal{O}_q(s\mathfrak{p}\mathbb{C}^2) \). We show briefly that in our case, the primitive ideals can be described in a similar way.

2. Definitions and Basic Properties

The quantum function algebra of the quantum space \( \mathbb{C}^N \) is an associative algebra generated by \( z_0, z_1, \cdots, z_{N-1}, z_0^*, z_1^*, \cdots, z_{N-1}^* \) subject to the following relations:

\[
\begin{align*}
    z_i z_j &= q^{-1} z_j z_i \quad \text{for } i < j, \\
    z_i^* z_j^* &= q z_j^* z_i^* \quad \text{for } i < j, \\
    z_i z_j^* &= q^{-1} z_j^* z_i \quad \text{for } i \neq j, \text{ and} \\
    z_i z_i^* - z_i^* z_i &= (q^2 - 1) \sum_{k > i} z_k z_k^*,
\end{align*}
\]

where \( q \in \mathbb{C}^* \) is the quantum parameter.

**Theorem 2.1.** The quantum function algebra \( F_q(N) \) is an iterated Ore extension and hence Noetherian and an integral domain with the same Hilbert series of the commutative polynomial algebra in \( 2N \) variables. There is a \( \mathbb{C} \)-basis for \( F_q(N) \) consisting of

\[
\mathcal{A} = \{ z_0^{r_0} \cdots z_{N-1}^{r_{N-1}} z_0^* s_0^* \cdots z_{N-1}^* s_{N-1}^* | r_i, s_i \in \mathbb{Z}_+ \}.
\]

**Proof:** Observe that \( F_q(N) \) is an iterated Ore extension

\[
F_q(N) = \mathbb{C}[z_{N-1}] [z_{N-1}^*, \sigma_{N-1}^*, \delta_{N-1}^*] \cdots [z_0, \sigma_0, \delta_0] [z_0^*, \sigma_0^*, \delta_0^*]
\]
for automorphisms \( \sigma_i, \sigma_i^* \) and \( \sigma \)-derivations \( \delta_i, \delta_i^* \) defined as follows:

\[
\begin{align*}
\sigma_i : \mathbb{C}[z_{N-1}^*, z_{N-1}^*, \cdots, z_{i+1}^*, z_{i+1}] & \longrightarrow \mathbb{C}[z_{N-1}, z_{N-1}^*, \cdots, z_{i+1}, z_{i+1}], \\
\sigma_i(z_j) &= qz_j \quad \text{for } j = i + 1, \cdots, N - 1, \\
\sigma_i(z_j^*) &= qz_j^* \quad \text{for } j = i + 1, \cdots, N - 1, \\
\delta_i : \mathbb{C}[z_{N-1}^*, z_{N-1}^*, \cdots, z_{i+1}, z_{i+1}] & \longrightarrow \mathbb{C}[z_{N-1}, z_{N-1}^*, \cdots, z_{i+1}, z_{i+1}], \\
\delta_i(z_j) &= \delta_i(z_j^*) = 0 \quad \text{for } j = i + 1, \cdots, N - 1, \\
\sigma_i^* : \mathbb{C}[z_{N-1}, z_{N-1}^*, \cdots, z_{i+1}, z_{i+1}, z_i] & \longrightarrow \mathbb{C}[z_{N-1}, z_{N-1}^*, \cdots, z_{i+1}, z_i], \\
\sigma_i^*(z_j) &= q^{-1}z_j \quad \text{for } j = i + 1, \cdots, N - 1, \\
\sigma_i^*(z_j^*) &= q^{-1}z_j^* \quad \text{for } j = i + 1, \cdots, N - 1, \\
\delta_i^*(z_j) &= \delta_i^*(z_j^*) = 0 \quad \text{for } j = i + 1, \cdots, N - 1, \\
\delta_i^*(z_i) &= (1 - q^2) \sum_{k>i} z_kz_k^*.
\end{align*}
\]

This completes the proof. \(\Box\)

Let \( \overline{F_q(N)} \) be the associated quasipolynomial algebra of \( F_q(N) \), which is an associative algebra with generators \( \bar{z}_0, \bar{z}_1, \cdots, \bar{z}_{N-1}, \bar{z}_0^*, \bar{z}_1^*, \cdots, \bar{z}_{N-1}^* \) and the following defining relations:

\[
\begin{align*}
\bar{z}_i\bar{z}_j &= q^{-1}\bar{z}_j\bar{z}_i \quad \text{for } i < j, \\
\bar{z}_i^*\bar{z}_j^* &= q\bar{z}_j^*\bar{z}_i^* \quad \text{for } i < j, \\
\bar{z}_i\bar{z}_j^* &= q^{-1}\bar{z}_j^*\bar{z}_i \quad \text{for } i \neq j, \quad \text{and} \\
\bar{z}_i^*\bar{z}_i &= \bar{z}_i^*\bar{z}_i \quad \text{for all } i.
\end{align*}
\]

We finish this section by constructing an irreducible module for \( \overline{F_q(N)} \) of a “high” dimension:

Let \( q^m = 1 \). Let \( \sigma, D \in \text{End}(\mathbb{C}^m) \) be the operators which with respect to the standard basis \( v_0, v_1, \cdots, v_{m-1} \) are given as

\[
\sigma(v_i) = v_{i+1}, \quad D(v_i) = q^iv_i \quad \text{for all } i,
\]

where the index \( i \) in \( v_i \) is \( i \mod m \). We denote by \( \sigma_i \) and similarly \( D_i \) the operators \( 1 \otimes 1 \otimes \cdots \otimes \sigma \otimes 1 \cdots \otimes 1 \) and \( 1 \otimes 1 \otimes \cdots \otimes D \otimes 1 \cdots \otimes 1 \) on \( (\mathbb{C}^m)^{N-1} \) with \( \sigma \) and \( D \) in the \( i \)th positions for \( i = 1, \cdots, N - 1 \).

\[
\begin{align*}
\bar{z}_i &= D_i\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{N-1} \quad \text{for } i = 1, \cdots, N - 2, \\
\bar{z}_0 &= D_1\sigma_1\sigma_2\cdots\sigma_{N-1}, \\
\bar{z}_{N-1} &= D_{N-1}, \\
\bar{z}_{N-1}^* &= D_{N-1}, \\
\bar{z}_i^* &= D_i\sigma_{i+1}^{-1}\cdots\sigma_{N-1}^{-1} \quad \text{for } i = 1, 2, \cdots, N - 2, \quad \text{and} \\
\bar{z}_0^* &= D_1^{-1}\sigma_1^{-1}\cdots\sigma_{N-1}^{-1}.
\end{align*}
\]
Utilizing the relation \( D\sigma = q\sigma D \), elementary computations now yield that the above formulas define a representation of the algebra \( F_q(N) \) in an \( m^{N-1} \) dimensional space. If the quantum parameter \( q \) is a primitive \( m \)th root of unity where \( m \) is an odd positive integer the representation is irreducible. In fact,

\[
\bar{z}_i\bar{z}_i^* = D_i^2 \quad \text{for} \quad i = 1, 2, \ldots , N - 1,
\]

so we get every \( D_i = (D_i^2)^{\frac{m-1}{m}} \) for \( i = 1, 2, \ldots , N - 1 \). Then we can get every \( \sigma_i \) for all \( i = 1, 2, \ldots , N - 1 \). Since \( D, \sigma \) generate \( \text{End}(\mathbb{C}^m) \) the representation is irreducible.

**Remark 2.2.** The algebra considered in [11] is given as follows (changing the notation in an inessential way): It is generated by \( x_1, x_2, \ldots , x_N, x_1^*, x_2^*, \ldots , x_N^* \) subject to the following relations:

\[
x_i x_j = q^{-1} x_j x_i \quad \text{for} \quad i < j,
\]

\[
x_i^* x_j^* = q x_j^* x_i^* \quad \text{for} \quad i < j,
\]

\[
x_i x_j^* = q^{-1} x_j^* x_i \quad \text{for} \quad i \neq j, \quad \text{and}
\]

\[
x_i^* x_i - q^2 x_i^* x_i^* = (q^2 - 1) \sum_{k<i} q^{k-i} x_k x_k^*.
\]

Here, the factor \( q^{k-i} \) may be gotten away with by absorbing \( q^i \) into, say, \( x_i \). After changing \( q \mapsto q^{-1} \) and setting \( z_i = x_{N-i}, z_i^* = x_{N-i}^* \) we get the following equivalent algebra generated by \( z_0, z_1, \ldots , z_{N-1}, z_0^*, z_1^*, \ldots , z_{N-1}^* \) and with relations:

\[
z_i z_j = q^{-1} z_j z_i \quad \text{for} \quad i < j,
\]

\[
z_i^* z_j^* = q z_j^* z_i^* \quad \text{for} \quad i < j,
\]

\[
z_i z_j^* = q z_j^* z_i \quad \text{for} \quad i \neq j, \quad \text{and}
\]

\[
z_i z_i^* - q^2 z_i^* z_i = (q^2 - 1) \sum_{k>i} z_k z_k^*.
\]

We return to this algebra in Remark 6.2.

### 3. The Poisson Structure and Its Symplectic Leaves

#### 3.1. The DKP Conjecture

Let \( A \) be an associative algebra with generators \( x_1, \ldots , x_n \) and defining relations

\[
x_i x_j = q^{h_{ij}} x_j x_i + p_{ij}, \quad \text{where} \quad p_{ij} \in \mathbb{C}[x_1, \ldots , x_{i-1}] \quad \text{if} \quad i > j,
\]

and where \((h_{ij})\) is an anti-symmetric integral matrix. Let \( \varepsilon \) be a primitive \( m \)th root of unity. Assume that every \( x_i^{m\varepsilon} \) is central. Let \( \eta_1, \eta_2, \ldots , \eta_n \) be coordinate functions of \( \mathbb{C}^n \) so that \( \eta_i \) corresponds to \( x_i^{m\varepsilon} \) for each \( i = 1, \ldots , n \). As explained in [11] p. 84-85, there is then a Poisson structure on \( \mathbb{C}^n \) induced from the defining relations of the algebra \( A \) given by

\[
\{\eta_i, \eta_j\} = \left( \lim_{q \to \varepsilon} \frac{[x_i^{m\varepsilon}, x_j^{m\varepsilon}]}{m(q^m - 1)} \right).
\]
Remark 3.1. Observe that up to a constant, the lowest order of \( q^m - 1 \), when expanded around \( \varepsilon = q - \varepsilon \). Indeed, when \( m \) is an odd prime and \( \varepsilon \neq 1 \) is an \( m \)-th root of unity, then \( (q^m - 1) = (q - 1)(q - \varepsilon) \cdots (q - \varepsilon^{m-1}) \). Moreover,

\[
\frac{q^m - 1}{q - 1} = 1 + q + \cdots + q^{m-1},
\]

hence \( (1 - \varepsilon)(1 - \varepsilon^2) \cdots (1 - \varepsilon^{m-1}) = m \). Hence, \( (q^m - 1) \approx \frac{m}{\varepsilon}(q - \varepsilon) \). It is also possible to let \( q \to 1 \), but in this case, the Poisson structure is rather degenerate (c.f. below).

A connected submanifold which is a maximal integral manifold of the distribution defined by the Hamiltonian vector fields corresponding to the Poisson structure, is called a symplectic leaf. These define a foliation of \( \mathbb{C}^n \).

Let \( \pi \) be an irreducible representation of the algebra \( A \). Then there exist a \( p = (p_1, p_2, \cdots, p_n) \) in \( \mathbb{C}^n \) such that \( x_i^m = p_i \) on \( \pi \). Assume that \( O_\pi \) is a symplectic leaf containing the point \( p \).

The DKP Conjecture: For "good" \( m \),

\[
\dim \pi = \frac{1}{2} \dim O_\pi.
\]

Remark 3.2. An algebra with defining relations as in (3.1) and where \( p_{ij} = 0 \) for all \( i > j \) is called a quasipolynomial algebra. It is easy to see that the DKP conjecture holds in this case provided that \( m \) is prime to all the numbers \( h_{ij} \). Indeed, one may, without loss of generality, assume that all the generators in an irreducible module are invertible. But then the dimension of the irreducible module is given as \( \frac{1}{2} \dim r \), where \( r \) is the rank of the matrix \( \{h_{ij}\} \) (c.f. Section 2), whereas the dimension of the symplectic leaf is equal to the rank of the matrix \( \{b_{ij} = \{a_i, a_j\}\} \) where \( b_{ij} = h_{ij}a_ia_j \) and where for all \( i \): \( a_i \equiv x_i^m \). Clearly the two matrices have the same rank.

In the conjecture, the point only enters through the leaf to which it belongs. To understand this, and also for further use, we need some observations from [1]:

Assume that we have a manifold \( M \) and a vector bundle \( V \) of algebras with 1 (i.e. 1 and the multiplication map are smooth sections). We identify the functions on \( M \) with the sections of \( V \) which are multiples of 1. Let \( D \) be a derivation of \( V \), i.e. a derivation of the algebra of sections which maps the algebra of functions on \( M \) into itself. Let \( X \) be the corresponding vector field on \( M \) and let \( (t, p) \mapsto X_t(p) \) denote the local 1-parameter group determined by \( X \).

Proposition 3.3. For each point \( p \in M \) there exists a neighborhood \( U_p \) and a map \( (t, v_p) \mapsto \phi(t, v_p) \) defined for \( |t| \) sufficiently small and \( v_p \in V_{U_p} \) which for each \( t \) is a morphism of vector bundles and which covers the \( t \)-parameter group generated by \( X \). Indeed, for each such \( t \) and each \( p \in U_p \), \( \phi_t \) induces an algebra isomorphism between \( V_p \) and \( V_{X_t(p)} \).

Now, suppose \( M \) is a Poisson manifold. Assume furthermore that the Poisson structure lifts to \( V \) i.e. that each local function \( f \) induces a derivation on sections extending the given Hamiltonian vector field \( X_f \) determined by \( f \).

Proposition 3.4. Under the above hypotheses, the fibers of \( V \) over the points of a given symplectic leaf \( M \) are all isomorphic as algebras.

In the present general set-up, \( M = \mathbb{C}^n \) and for each \( y = (y_1, \cdots, y_n) \in \mathbb{C}^n \), the fiber \( V_y \) of \( V \) over \( y \) is given as

\[
V_y = \frac{A}{< x_i^m - y_i >} ; i = 1, \cdots, n.
\]
It was proved in [1] that the Poisson structure does lift in this situation.

On a symplectic leaf $O$ the fibers of the algebra are isomorphic. Later on, when studying the representations of the algebra $F_q(N)$ connected with some leaf, our strategy will be to choose a good point $p \in O$.

3.2. The Poisson structure defined by $F_q(N)$. Let $a_i, a_i^* \leftrightarrow z_i^m, z_i^{*m}$ for $I = 0, 1, \cdots, N-1$.

**Proposition 3.5.** The Poisson structure is given by

\[
\{a_i, a_j\} = -a_i a_j \quad \text{if } i < j,
\]

\[
\{a_i^*, a_j^*\} = a_i^* a_j^* \quad \text{if } i < j,
\]

\[
\{a_i, a_j^*\} = -a_i a_j^* \quad \text{if } i \neq j, \quad \text{and}
\]

\[
\{a_i, a_i^*\} = 2 \sum_{k > i} a_k a_k^*.
\]

In order to prove this proposition we now state and prove a number of Lemmas.

**Lemma 3.6.** For any positive integer $s$

\[
z_i (z_i^*)^s = (z_i^*)^s z_i + (q^2s - 1) \sum_{k > i} z_k z_k^* (z_i^*)^{s-1}
\]

\[
z_i^* z_i^s = z_i^s z_i^* + (q^{-2s} - 1) \sum_{k > i} z_k^* z_k z_i^{s-1}.
\]

In particular, if $q$ is an $m$th root of unity then the elements $z_i^m, (z_i^*)^m,$ and $z_{N-1}^a (z_{N-1}^*)^{m-a}$ are central for all $i = 0, 1, \cdots, N-2$ and all $a = 0, 1, \cdots, m$.

**Proof:** This follows easily by induction. \hfill \Box

**Lemma 3.7.** Let

\[
\Omega_i = \sum_{k=0}^{N-1} z_k z_k^*, \quad \Omega_i^* = \sum_{k=0}^{N-1} z_k^* z_k^*, \quad \text{and} \quad \Omega = \Omega_0.
\]

We have the following formulas:

\[
\Omega_i z_k = z_k \Omega_i \quad \text{if } i \leq k \leq N-1,
\]

\[
\Omega_i z_k = q^2 z_k \Omega_i \quad \text{if } 0 \leq k \leq i-1,
\]

\[
\Omega_i z_k^* = z_k^* \Omega_i \quad \text{if } i \leq k \leq N-1,
\]

\[
\Omega_i z_k^* = q^{-2} z_k^* \Omega_i \quad \text{if } 0 \leq k \leq i-1,
\]

\[
z_i z_i^* - z_i^* z_i = (q^2 - 1) \Omega_{i+1}, \quad \text{and}
\]

\[
q^2 z_i z_i^* - z_i^* z_i = (q^2 - 1) \Omega_i.
\]

In particular, if $q$ is an $m$th root of unity, then $\Omega$ is central.
Proof: This follows directly from the defining relations. \[ \square \]

Let \( p_i = (q^{-2i} - 1) \). For \( i, j \in \mathbb{N} \) set \( S_{i,j} = \{ a \in \mathbb{N} \mid 1 \leq a \leq j, \text{ and } a \neq i \} \). Let

\[
F_{i,j}(n) = \frac{p_i^{n-1}}{\prod_{a \in S_{i,j}} (p_i - p_a)}.
\] (3.8)

Lemma 3.8. Let

\[
(z_i z_i^*)^n = \sum_{t=0}^{n} a_t(n) z_i^t (z_i^*)^t \Omega_i^{n-t}.
\] (3.9)

Then

\[
a_t(n) = \sum_{b=1}^{t} F_{b,t}(n).
\] (3.10)

Moreover, since the right hand side makes sense for all \( n \), we can define the left hand side by this formula. Doing this, we get that \( a_t(s) = 0 \) for \( s = 1, \cdots, t - 1 \).

Proof: We have

\[
(z_i z_i^*)^{n+1} = \sum_{t=0}^{n} a_t(n) z_i^t (z_i^*)^t \Omega_i^{n-t} z_i z_i^*
\] (3.11)

\[ = \sum_{t=0}^{n} a_t(n) z_i^t (z_i^*)^t z_i z_i^* \Omega_i^{n-t} + \sum_{t=0}^{n} a_t(n) (q^{-2t} - 1) z_i^t (z_i^*)^{t+1} \Omega_i^{n-t} + \sum_{t=0}^{n} a_t(n) (q^{-2t} - 1) z_i^t (z_i^*)^t \Omega_i^{n-t}.
\]

Hence, if we set \( a_{-1}(n) = a_{n+1}(n) = 0 \), we get

\[
\forall t = 0, \cdots, n + 1 : \quad a_t(n + 1) = (q^{-2t} - 1) a_t(n) + a_{t-1}(n).
\] (3.12)

In particular,

\[
\forall n : a_n(n) = 1 \quad \text{and} \quad \forall n : a_0(n) = 0.
\] (3.13)

To check that (3.12) is satisfied by (3.10), observe that \( p_t = (q^{-2t} - 1) \) and that

\[
a_{t-1}(n) = \sum_{b=1}^{t-1} \left( \frac{p_b^{n-1}}{\prod_{a \in S_{b,t-1}} (p_b - p_a)} \right) p_b - p_t = \sum_{b=1}^{t-1} \left( \frac{p_b^{n-1}(p_b - p_t)}{\prod_{a \in S_{b,t}} (p_b - p_a)} \right).
\] (3.14)

As for (3.13) as well as the remaining assertion, observe that

\[
\sum_{b=1}^{t} \frac{p_b^{s-1}}{\prod_{a \in S_{b,t}} (p_b - p_a)}
\] (3.15)

is the \((t, s)\)th entry of the matrix product \( A^{-1} \cdot A \) where \( A \) is the (Vandermonde) matrix.
\[
A = \begin{pmatrix}
1 & p_1 & p_1^2 & \cdots & p_1^{t-1} \\
1 & p_2 & p_2^2 & \cdots & p_2^{t-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & p_t & p_t^2 & \cdots & p_t^{t-1}
\end{pmatrix},
\]  
(3.16)

Lemma 3.9.

\[
\Omega_i^m = \sum_{k=1}^{N-1} z_k^m z_k^{*m}.
\]  
(3.17)

Proof: Observe that

\[
\Omega_i^m = (z_i z_i^* + \Omega_{i+1})^m.
\]  
(3.18)

Since \(z_i z_i^*\) and \(\Omega_{i+1}\) commute we have

\[
\Omega_i^m = \sum_{n=0}^{m} \binom{m}{n} (z_i z_i^*)^n \Omega_{i+1}^{m-n}.
\]  
(3.19)

Therefore

\[
\Omega_i^m = \sum_{n=0}^{m} \binom{m}{n} \sum_{t=0}^{n} a_t(n) z_t z_t^* \Omega_{i+1}^{m-t}.
\]  
(3.20)

The coefficient of \(z_i z_i^* \Omega_{i+1}^{m-t}\) is

\[
\sum_{n=1}^{m} \binom{m}{n} a_t(n).
\]  
(3.21)

By Lemma 3.8 this may be written

\[
\sum_{n=1}^{m} \binom{m}{n} \sum_{b=1}^{t} \frac{p_b^{n-1}}{\prod_{a \in S_{i,j}} (p_b - p_a)} = \sum_{b=1}^{t} \sum_{n=1}^{m} \binom{m}{n} \frac{p_b^{n-1}}{\prod_{a \in S_{i,j}} (p_b - p_a)}.
\]  
(3.22)

The coefficient is clearly 1 if \(t = m\). For all other values of \(t\) it is zero since then \(p_b \neq 0\) and thus

\[
\sum_{n=1}^{m} \binom{m}{n} p_b^{n-1} = \frac{1}{p_b}((p_b + 1)^m - 1) = 0.
\]  
(3.23)

If \(f\) is a polynomial in \(q, q^{-1}\), denote by \([f]\) the value

\[
[f] = \lim_{q \to \epsilon} \frac{f(q)}{m \cdot (q^m - 1)}.
\]  
(3.24)

When computing \(z_k^m z_k^{*m} - z_k^m z_k^{*m}\) we need more than the above \(\Omega_i^m\), namely we need to show that all other terms vanish at the root of unity.

To ascertain this, consider more generally (with \(z \leftrightarrow z_k\) and \(\Omega \leftrightarrow \Omega_{k+1}\)).
Lemma 3.10. Assume that $m$ is odd and that $q$ is a primitive $m$th root of unity. Let

$$z^i z^* = z^* z^i + \sum_{j=1}^i d_{i,j}(s) \Omega^j z^s z^{i-j} \quad (i \leq s). \quad (3.25)$$

Then $[d_{m,j}(m)] = 0$ for $j = 1, \ldots, m - 1$ and $[d_{m,m}(m)] = 2$.

Proof: Let

$$d_{i,j}(s) = (q^{2s} - 1) \cdots (q^{2(s+1-j)} - 1) c_{i,j}(s). \quad (3.26)$$

The recursion relation is: $c_{i+1,j} = q^{-2j} c_{i,j} + q^{-2(j-1)} c_{i,j-1}$ with $c_{i,0} = 1 (= d_{i,0})$, and, for consistency, $c_{i,j} = 0$ if $j > i$. It follows that $c_{i,i} = q^{i^2}$. If we set $f_{i,j} = q^{2(i-1)j} c_{i,j}$ the recursion relation becomes

$$f_{i+1,j} = f_{i,j} + q^{2i} f_{i,j-1}, \quad f_{i,0} = 1, \text{ and } f_{i,i} = q^{2^2-i}. \quad (3.27)$$

The solution is

$$f_{i,j} = \sum_{\delta=-1}^{i-1} \sum_{\gamma=2}^{\delta} \sum_{\alpha=1}^{\gamma-1} \sum_{\beta=0}^{\alpha-1} q^{2(\alpha+\beta+\gamma+\delta)}. \quad (3.28)$$

Another way of expressing $f_{i,j}$ is as follows: Let $Y_{i,j}$ denote the set of “Young diagrams” with $j$ rows with $n_k$ boxes in row $k$ and such that $i - 1 \geq n_1 > \cdots > n_j \geq 0$. If $y \in Y_{i,j}$ let $A(y) = n_1 + \cdots + n_j$. Then

$$f_{i,j} = \sum_{y \in Y_{i,j}} q^{2A(y)}. \quad (3.29)$$

It is easy to verify that (3.29) yields the solution to (3.27).

Now observe that the function $q^{n_1+n_2+\cdots+n_j}$ is invariant under permutations. Also note that since $1 + q + \cdots + q^{m-1} = 0$, the sum of all $q^{n_1+n_2+\cdots+n_j}$ over the set $P^+ = \{ (n_1, n_2, \ldots, n_j) \mid \forall k = 1, \ldots, j : 0 \leq n_k \leq m - 1 \}$ is zero.

But, by the same reason, the sum of $q^{n_1+n_2+\cdots+n_j}$ over sets of the form $P_\omega = \{ (n_1, n_2, \ldots, n_j) \mid \forall k = 1, \ldots, j : 0 \leq n_k \leq m - 1 \text{ and } n_{\omega_1} = \cdots = n_{\omega_j} \}$ is zero.

It follows easily from this that $[d_{m,j}(m)] = 0$ for $j = 1, \ldots, m - 1$. Finally observe that since $m$ is odd (see also Remark 3.1),

$$[d_{m,m}(m)] = \lim_{q \to 1} \frac{(q^{2m} - 1) \cdots (q^2 - 1) \cdot q^{m^2-m}}{m \cdot (q^m - 1)} = 2. \quad (3.30)$$

\[ \square \]

Proof of Proposition 3.3: It should now be clear from Lemma 3.8-3.10 (see also Remark 3.1) that it only remains to consider $[z_i^m, z_j^m]$ where $z_i z_j = q z_j z_i$. Indeed, this is a prototype for the remaining cases. Since

$$[z_i^m, z_j^m] = (q^{m^2} - 1) z_j^m z_i^m, \quad (3.31)$$

this case follows since clearly $[q^{m^2} - 1] = 1$. \[ \square \]
\[ \{\cdot, \cdot\} \{d, \cdot\} \{b, \cdot\} \{a, \cdot\} \{c, \cdot\} \{a^*, \cdot\} \{b^*, \cdot\} \]

\[
\begin{array}{cccccc}
\{\cdot, \cdot\} & \{d, \cdot\} & \{b, \cdot\} & \{a, \cdot\} & \{c, \cdot\} & \{a^*, \cdot\} \\
\hline
d & 0 & bd & ad & cd & a^*d & b^*d \\
\hline
b & -bd & 0 & ab & bc & ba^* & -2aa^* \\
\hline
a & -ad & -ab & 0 & ac & 0 & ab^* \\
\hline
c & -cd & -bc & -ac & 0 & a^*c & b^*c \\
\hline
a^* & -a^*d & -ba^* & 0 & -a^*c & 0 & a^*b^* \\
\hline
b^* & -b^*d & 2aa^* & -ab^* & -b^*c & -a^*b^* & 0 \\
\end{array}
\]

**Figure 1.** For convenience, we list the Poisson brackets for the configuration above

### 3.3. Symplectic leaves.

Consider a point

\[ a = (a_0, a_1, \ldots, a_{N-1}, a^*_0, \ldots, a^*_1, a^*_0) \in \mathbb{C}^{2N}. \]  (3.32)

Let \( i_0 \) denote the biggest \( i \) such that \( a_i \cdot a^*_i \neq 0 \). Suppose that \( a^*_i \neq 0 \) for some \( i < i_0 \). Let \( k_1 \) be the biggest such number (below \( i_0 \)). The Hamiltonian vector field corresponding to \( a_{k_1} \) has the form

\[ \sum_j \left( -a_{k_1} a_j \frac{\partial}{\partial a_j} - a_{k_1} a^*_j \frac{\partial}{\partial a^*_j} \right) + 2a_{i_0} \cdot a^*_{i_0} \frac{\partial}{\partial a^*_{k_1}} \]  (3.33)

The integral curves \((t \text{ is complex})\) of this have the form

\[
\begin{align*}
  a_j(t) &= a_j(0) e^{-a_{k_1} t} \\
  a^*_j(t) &= a^*_j(0) e^{-a_{k_1} t} \quad (j \neq k_1) \quad (3.34) \\
  a^*_{k_1}(t) &= \begin{cases} 
  a^*_k(0) - \frac{a_{i_1}(0) a^*_{i_0}(0)}{a^*_{k_1}} (e^{-2a_{k_1} t} - 1) & \text{for } a_{k_1} \neq 0 \\
  a^*_k(0) + 2a_{i_0}(0) a^*_{i_0}(0) t & \text{for } a_{k_1} = 0
  \end{cases} \quad (3.35)
\end{align*}
\]

So, we can flow to a point where \( a^*_{k_1} = 0 \) unless we are in the exceptional case where

\[ a_{k_1}(0) a^*_{k_1}(0) + a_{i_0}(0) a^*_{i_0}(0) = 0. \]  (3.36)

Consider again the point

\[ a = (a_0, a_1, \ldots, a_{N-1}, a^*_0, \ldots, a^*_1, a^*_0) \in \mathbb{C}^{2N}. \]

Let, as before, \( i_0 \) be the biggest index such that \( a_i a^*_i \neq 0 \). Let \( r_0 \) denote the number of non-zero coordinates (starrred or unstarred) having an index greater than that \( i_0 \). If \( a_i a^*_i = 0 \) for all \( i \), let \( r_0 \) denote the number of non-zero coordinates. Let \( i_1 \) denote the biggest index among those \( i < i_0 \) for which \( a_i a^*_i + a_{i_0} a^*_{i_0} = 0 \). Let \( i_2 \) denote the biggest index among those \( i < i_1 \) for which \( a_i a^*_i + a_{i_2} a^*_i + a_{i_0} a^*_{i_0} \neq 0 \), let \( r_2 \) denote the number of non-zero coordinates \( a_i \) and \( a^*_i \) with \( i_2 < i < i_1 \). Continuing like this we obtain a sequence of indices \( 0 \leq i_s < i_{s-1} < \cdots < i_1 < i_0 \leq N - 1 \). And we set more generally \( r_{2j} \) equal to the total number of non-zero coordinates \( a_i \) and \( a^*_i \) with \( i_{2j} < i < i_{2j-1} \).
Observe first that these quantities are invariant under Hamiltonian flow. Also, we may, according to the preceding, use Hamiltonian flow to move to a point in the same leaf where all coordinates \( a_i \) and \( a_i^* \), respectively, with \( i_{2j+1} < i < i_{2j} \) are zero. We do that. Let us then take a closer look at the \( 2N \times 2N \) matrix whose \( ij \)th entry is \( \{ x_j, x_i \} \) where \( x_i = a_{i-1} \) for \( i = 1, \ldots, N \), and \( x_i = a_{2N-i}^* \) for \( i = N + 1, \ldots, 2N \). The rank of this matrix is (of course) equal to the dimension of the symplectic leaf. The coordinates \( a_i \) and \( a_i^* \) with \( i_{2j+1} < i < i_{2j} \) now have non-zero Poisson bracket with each other and zero with any other coordinate. Thus, they contribute with a total of \( 2(i_{2j} - i_{2j+1} - 1) \) to the rank.

Next observe that \( \{ a_{i_{1}}, b \} \), where \( b \) is some other coordinate different from \( a_{i_0} \) and \( a_{i_0}^* \), is proportional to \( \{ a_{i_{0}}, b \} \). The same observation holds for \( a_{i_1}^* \) and \( a_{i_0}^* \) and even \( a_{i_0} \) and \( a_{i_0}^* \) have similar Poisson brackets. Using this, it is easy to use row and column moves to decouple the matrix into the direct sum of three submatrices - one of rank 2, one involving the non-zero coordinates with \( i > i_0 \) together with \( a_{i_0}^* \) and with a standard quasi-polynomial coupling, and one involving the remaining coordinates, i.e. those \( a_i \) and \( a_i^* \) with \( i < i_1 \). The second summand involves \( r_0 + 1 \) points, and thus contributes with \( 2 \cdot \left[ \frac{r_0+1}{2} \right] \) to the total rank. The last summand may then be attacked with the same strategy, and after a finite number of steps the result is obtained.

The total contribution to the rank from the first round is

\[
2(i_0 - i_1) + 2 \cdot \left[ \frac{r_0+1}{2} \right]. \tag{3.38}
\]

If \( s = 2l \) let \( i_{2l+1} = 0 \) and if \( s = 2l + 1 \) set \( r_{2l+2} + 1 \) equal to the total number of points having an index smaller than \( i_{2l+1} \).

We thus get the following result:

**Proposition 3.11.** Expressed in terms of the introduced data, the dimension of the symplectic leaf \( \mathcal{O}_a \) containing \( a \) is given by

\[
\dim \mathcal{O}_a = \begin{cases} 
2 \cdot \left[ \frac{r_0}{2} \right] & \text{if } \forall i : a_i a_i^* = 0, \\
\sum_{k=0}^{i_0} 2(i_{2k} - i_{2k+1}) + 2 \cdot \left[ \frac{r_{2k+1}}{2} \right] & \text{if } s = 2l, \\
\sum_{k=0}^{i_0} 2(i_{2k} - i_{2k+1}) + \sum_{k=0}^{i_1} \left( 2 \cdot \left[ \frac{r_{2k+1}}{2} \right] \right) & \text{if } s = 2l + 1.
\end{cases}
\]

4. **The theory of De Concini and Procesi**

The main tool used to compute the degrees and centers is the theory developed in [5] by De Concini and Procesi. Indeed, it is straightforward to verify that \( F_q(N) \) and \( F_q(N) \) satisfy the hypotheses of that article and hence by [5, 6.4 Theorem] have the same degree.

Furthermore, there is a bijective correspondence between their centers by means of the highest order term (c.f. [3]).

Given an \( n \times n \) skew-symmetric matrix \( H = (h_{ij}) \) over \( \mathbb{Z} \) one constructs the twisted polynomial algebra \( \mathbb{C}_H[x_1, x_2, \ldots, x_n] \) as follows: It is the algebra generated by elements \( x_1, x_2, \ldots, x_n \) with the following defining relations:

\[
x_ix_j = q^{h_{ij}} x_j x_i \quad \text{for } i, j = 1, 2, \ldots, n. \tag{4.1}
\]
It can be viewed as an iterated twisted polynomial algebra with respect to any ordering of the indeterminates $x_i$. Given $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ we write $x^a = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$.

The degree and center of such an algebra is then given by [1, 7.1 Proposition].

It is well known that a skew-symmetric matrix over $\mathbb{Z}$ such as the matrix $H$ can be brought into a block diagonal form by an element $W \in SL(\mathbb{Z})$. Specifically, there is a $W \in SL(\mathbb{Z})$ and a sequence of $2 \times 2$ matrices $S(m_i) = \begin{pmatrix} 0 & -m_i \\ m_i & 0 \end{pmatrix}$, $i = 1, \ldots, N$, with $m_i \in \mathbb{Z}$ for each $i = 1, \ldots, N$, such that

$$W \cdot H \cdot W^t = \begin{cases} \text{Diag} \left( S(m_1), \ldots, S(m_N), 0 \right) & \text{if } N = \frac{n^2-1}{2}, \text{ if } n \text{ is odd.} \\ \text{Diag} \left( S(1 m_1), \ldots, S(m_N) \right) & \text{if } N = \frac{n^2}{2}, \text{ if } n \text{ is even.} \end{cases}$$

(4.2)

**Definition 4.1.** We call the matrix $H$ the defining matrix. Any matrix of the form of the right-hand-side in (4.2) will be called a canonical form of $H$ and will occasionally be denoted by $J_H$.

Thus, a canonical form of $H$ reduces the algebra to the tensor product of twisted Laurent polynomial algebras in two variables with commutation relation $xy = q^{xy}$. By [1, 7.1 Proposition] it follows in particular that the degree of a twisted Laurent polynomial algebra in two variables is equal to $m/(m, r)$, where $(m, r)$ is the greatest common divisor of $m$ and $r$.

5. The center and the degree of the algebra $F_q(N)$

For use in determining the degree of $F_q(N)$ as well as for later, we now consider a number of quasipolynomial algebras.

As usual, we let $\Omega_i = \sum_{k=1}^{N-1} z_k z_k^*$. Observe that $\Omega_0, \Omega_1, \ldots, \Omega_{N-1}$ form a commutative family.

Consider $n$ points: $z_{i_1}, \ldots, z_{i_n}, \ldots, z_{i_n}$, with $0 \leq i_1 < i_2 < \cdots < i_n \leq N - 1$ together with $\Omega_{j_1}, \Omega_{j_2}, \ldots, \Omega_{j_r}$, with $0 \leq j_1 < j_2 < \cdots < j_r \leq N - 1$. We wish to compute the degree of the quasipolynomial algebra generated by these elements.

For $\ell = 2, \ldots, r$, let $s_\ell$ denote the number of elements from our family $z_{i_1}, \ldots, z_{i_n}$ that have an index $i_\ell$ satisfying $j_{\ell-1} \leq i_\ell < j_\ell$, let $s_1$ denote the number of elements with an index $i_\ell$ satisfying $i_\ell < j_1$, and let $s_{r+1}$ denote the number of elements with an index $i_\ell$ satisfying $i_\ell \geq j_r$. To avoid redundancy, we will from now on assume that $s_\ell \neq 0$ for $\ell = 1, 2, \ldots, r$. (Indeed, if $s_\ell = 0$ then $\Omega_{j_\ell}$ can be removed from the algebra, c.f. below). If $s_{r+1} \neq 0$ we indicate this with an $\downarrow$ and if $s_{r+1} = 0$ we indicate this with an $\uparrow$. More precisely, we denote the algebras corresponding to these two cases by $\mathcal{L}_t(s_1, s_2, \ldots, s_r, s_{r+1})$ and $\mathcal{L}_t(s_1, s_2, \ldots, s_r)$, respectively. Observe that the algebra is completely determined by the listed data. Finally, let $\mathcal{L}_t^T = \mathcal{L}_{t \uparrow}^{\downarrow}(1, 1, \ldots, 1)$ for $T \in \mathbb{N}$.

**Proposition 5.1.** The non-trivial blocks in a canonical form of the defining matrix of $\mathcal{L}_t^T$ consists of $\left\lfloor \frac{T-2}{2} \right\rfloor$ blocks of

$$\begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}, \left\lceil \frac{T}{2} \right\rceil \text{ blocks of } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and (only if } T \text{ is odd) one } \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}. \text{ In particular, the rank of } \mathcal{L}_t^T \text{ is equal to } 2T - 2. \text{ The non-trivial blocks in a canonical form of}$$
the defining matrix of $\mathcal{L}_T^T$ consists of $\left[ \frac{r}{2} \right]$ blocks of \( \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} \), $\left[ \frac{T}{2} \right]$ blocks of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and (only if $T$ is odd) one $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$. In particular, the rank of $\mathcal{L}_T^T$ is equal to $2T$.

**Proof:** When we in the sequel say “by subtracting $a$ from $b$” we mean that we perform row and column operations corresponding to subtracting the row and column of $a$ from the corresponding of $b$. The operation of “clearing” is defined as follows: Suppose that we have a row $r_1$ with a non-zero entry $w$ in the $c_1$th position and such that the entries in the $c_1$th column are integer multiples of $w$. Clearing then means subtracting appropriate multiples of this row from the other rows, thereby creating a matrix whose $c_1$th column consists of zeros except at the $r_1$th place. Of course, these operations should be accompanied by corresponding column operations. In our situations the row we start from will have just one or two non-zero entries.

By subtracting $z_{T-1}$ from $z_T$ and clearing it is easy to see that one can move from $\mathcal{L}_T^T$ to $\mathcal{L}_{T-2}^T$ while picking up one $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the process. This reduces the problem to that of $\mathcal{L}_T^T$. Using the same kind of moves one can here move from $\mathcal{L}_1^T$ to $\mathcal{L}_{T-2}^T$ while picking up both a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and a $\begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}$. The result then follows by induction by paying special attention to the configuration of $\mathcal{L}_1^T$. Indeed, the latter is just $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$. □

For later use, observe that

**Lemma 5.2.** Let $M_x = \sum_{x \geq i > j \geq 1} (E_{i,j} - E_{j,i})$. Let $S_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A canonical form for $M_x$ is diag($S_1, \ldots, S_1$) if $x$ is even, and diag($S_1, \ldots, S_1, 0$) if $x$ is odd.

**Proof:** Starting by subtracting the second row from the first, this follows easily by simple row and column operations. □

A canonical form of the defining matrix of the associated quasipolynomial algebra follows from Proposition 5.3 via the following, easily established result.

**Lemma 5.3.** A canonical form of the defining matrix of the associated quasipolynomial algebra is equal to that of $\mathcal{L}_1^N$. If $m$ is odd, the degree of $F_q(N)$ is equal to $m^{N-1}$.

**Proposition 5.4.** The rank of the defining matrix of $\mathcal{L}_1^T(s_1, s_2, \ldots, s_r, s_{r+1})$ is $R_1 = 2 \left[ \frac{s_{r+1}}{2} \right] + 2 \left[ \frac{s_r+1}{2} \right] + \cdots + 2 \left[ \frac{s_2+1}{2} \right]$. Let $x$ denote the number of odd $s_i$. The defining matrix of $\mathcal{L}_1^T(s_1, s_2, \ldots, s_r, s_{r+1})$ is equivalent to

$$\sum_{i=1}^{r+1} \left[ \frac{s_i}{2} \right] \text{ copies of } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \left\{ \begin{array}{l}
\mathcal{L}_1^T(s_{r+1} \text{ even }) \\
\mathcal{L}_1^T(s_{r+1} \text{ odd })
\end{array} \right. \quad (5.1)
$$

The rank of the defining matrix of $\mathcal{L}_1(s_1, s_2, \ldots, s_r)$ is $R_1 = 2 \left[ \frac{s_r+1}{2} \right] + \cdots + 2 \left[ \frac{s_2+1}{2} \right]$. Let $y$ denote the number of odd $s_i$ $(i = 1, \ldots, r)$. The defining matrix of $\mathcal{L}_1(s_1, s_2, \ldots, s_r)$ is...
equivalent to

\[ \sum_{i=1}^{r} \left[ \frac{S_i}{2} \right] \text{ copies of } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \mathcal{L}_q. \] (5.2)

(This result remains true even if \( s_\ell = 0 \) for one or more \( \ell = 1, 2, \ldots, r. \))

Proof: First of all, if one is just interested in the rank (or the cases \( m \) odd), one may replace the \( \Omega \)'s by \( \Omega_{j_u}^{-1} \) for each pair \( u = 1, \ldots, r-1 \) together with \( \Omega_{j_1} \) and then subtract the resulting rows and columns from everything to decompose the matrix into a direct sum of matrices of the form of Lemma 5.2.

In the general case, let us assume that we have an algebra \( \mathcal{L} \) with \( \Omega_j \). By subtracting \( \Omega_j \) for each pair \( j \), one sees that one may move from \( \mathcal{L} \) to \( \mathcal{L} \) while picking up a \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). The result follows easily from this.

We shall be content to treat the center in the case where \( m \) is odd:

**Proposition 5.5.** Let \( q \) be a primitive \( m \)th root of 1 with \( m \) odd. The center of \( F_q(N) \) is generated by the following elements

\[ \Omega, z_i^m \text{ and } z_i^{+m} \text{ with } i = 0, 1, \ldots, N-1, \quad \text{and} \]

\[ (z_{N-1})^a(z_{N-1}^*)^b \text{ with } a + b = m \text{ and } a, b \in \{1, 2, \ldots, m-1\}. \] (5.3)

Proof: The analogous result for the quasipolynomial algebra follows immediately from the construction of a canonical form. But by comparing the highest order terms, it is easy to see that the centers of \( F_q(N) \) and \( \overline{F_q(N)} \), respectively, have the same magnitude. \( \square \)

### 6. The irreducible representations of the algebra \( F_q(N) \)

In this section, the De Concini-Kac-Procesi Conjecture is proved to be true for the algebra \( F_q(N) \).

If for some element \( w \) in some algebra, \( uw = q^{aw}wu \) for all generators \( u \), then \( w \) is said to be covariant. Observe than if in an irreducible module, \( w \) is covariant, then either \( w = 0 \) or \( w \) is invertible.

Consider an irreducible module \( V \). Let \( z_i^m = a_i \) and \( z_i^{+m} = a_i^m \) for all \( i = 0, 1, \ldots, N-1 \). Construct from a point

\[ a = (a_0, a_1, \ldots, a_{N-1}, a_{N-1}^*, \ldots, a_1^*, a_0^*) \in \mathbb{C}^{2N} \]

the sequence of indices \( 0 \leq i_s < i_{s-1} < \cdots < i_1 < i_0 \leq N-1 \) as previously. Assume furthermore that we, possibly after using Hamiltonian flow, are in a situation where all coordinates \( a_i \) and \( a_i^* \), respectively, with \( i_{2j+1} < i < i_{2j} \) are zero.

Let us first observe (c.f. (3.9)) that

\[ \forall i : \Omega_i^m = \sum_{j>i} a_j a_j^*. \]

Moreover, each \( \Omega_i \) is covariant, hence if \( \Omega_i^m = 0 \) in \( V \), then also \( \Omega_i = 0 \) on \( V \).

Let us then begin by considering points \( z_i, z_i^* \) with \( i > i_0 \). It follows that at most one of \( a_i, a_i^* \) is non-zero and that \( z_i \) commutes with \( z_i \). Suppose \( a_i \neq 0 \). Then \( z_i \) is invertible and \( z_i \)
is nilpotent and \((q-)\)commutes with everything. Hence \(z_i^* = 0\). Of course, if \(a_i = 0\) then also \(z_i = 0\).

The same kind of reasoning gives that \(z_{i_0}\) and \(z_{i_0}^*\) are two commuting invertible operators. Here we send \(z_{i_0}, \Omega_{i_0}\) to a set \(\mathcal{A}\) and discard \(z_{i_0}^*\).

The operators \(z_i\) with \(i_1 < i < i_0\) form a \((q-)\)commuting family of nilpotent operators as do the operators \(z_i^*\) with the same restriction on the index. Moreover, there is a common null space for these operators \(z_i\) which is invariant under all other operators except the corresponding \(z_i^*\). Finally observe that for any \(i_1 < i < i_0\),

\[
z_i z_i^* - z_i^* z_i = (q^2 - i)z_{i_0} z_{i_0}^*
\]

and the right hand side is invertible. Hence the operators \(z_i, z_i^*\) in this region are non-zero.

Let us now move to the operators \(z_{i_1}, z_{i_1}^*\). By assumption and the above remarks concerning \(\Omega_i\) it follows that

\[
z_{i_1} z_{i_1}^* + z_{i_0} z_{i_0}^* = 0.
\]

The defining relations easily give that \(z_{i_1} z_{i_1}^* = q^{-2} z_{i_1}^* z_{i_1}\). Since all operators clearly are invertible we may solve for \(z_{i_1}^*\). The treatment of this operator is thus completed and we will not consider it any more.

Moving in the direction of decreasing indices on the way to \(i_2\) we may pick up some terms \(z_i\) or \(z_j^*\) but never both \(z_i, z_j\). These are sent to \(\mathcal{A}\) and so are \(z_{i_2}, \Omega_{i_2}\), but \(z_{i_2}^*\) is discarded. After this, the picture repeats itself periodically. As a result, it is clear that we obtain the following:

- A family \(N^+\) of non-zero \((q-)\)commuting nilpotent operators \(z_i\) with \(i_2j_1 < i < i_2j_2\).
- A family \(N^-\) of non-zero \((q-)\)commuting nilpotent operators \(z_i^*\) with \(i_2j_1 < i < i_2j_2\).
- A family \(\mathcal{A} \subseteq \text{Alg}_{\mathbb{C}}\{z_0, \ldots , z_{N-1}, z_{N-1}^*, \ldots , z_0^*\}\) of \((q-)\) commuting invertible operators.

These operators satisfy

1. The operators from \(N^+, \mathcal{A}\), and \(N^-\) generate all operators in the representation.
2. For each \(i_2j_1 < i < i_2j_2\): \(z_i z_i^* - z_i^* z_i = z_{i_2j_1} z_{i_2j_2}^*\).
3. \(\mathcal{A}\) leaves the common null spaces of \(N^+\) and \(N^-\), respectively, invariant.
4. The only pairs of commuting operators from \(\mathcal{A}\) are \((z_{i_2j_1}, z_{i_2j_2}^*)\).

Let \(V_0\) denote the common null space of \(N^+\) and let \(\pi\) denote the representation of \(\mathcal{A}\) on \(V_0\). Then, clearly,

- \(\pi\) is irreducible.
- \(V = N^- V_0\).

By \(\text{[B, §7]}\), the representation \(\pi\) is essentially unique. Furthermore, if we set \(N^- = \{w_1^*, \ldots , w_s^*\}\) then any element \(v \in V\) can be written as a sum of elements

\[
w_1^{d_1} \cdots w_s^{d_s} v_d
\]

where \(\forall j = 1, \ldots , s : 0 \leq d_j \leq m - 1\) and \(v_d \in V_0\). Using the action of \(N^+\) it is easy to see that if \(v_d\) is taken from a fixed basis of \(V_0\) then the set of all elements \(\text{[B.1]}\) constitutes a basis of \(V\). Observe that \(s = (i_0 - i_1 - 1) + \cdots + (i_2j_1 - i_2j_1 - 1) + \cdots\).

Conversely, given an irreducible \(\mathcal{A}\) representation \(\pi\) on a space \(V_0\) promote this to a a representation of the algebra \(\text{Alg}_{\mathbb{C}}\{\mathcal{A}, N^+\}\) by letting \(N^+\) act trivially. Then it is easy to see that the induced representation

\[
\text{Alg}_{\mathbb{C}}\{N^-, \mathcal{A}, N^+\} \otimes_{\text{Alg}_{\mathbb{C}}\{\mathcal{A}, N^+\}} V_0
\]
is irreducible.

**Theorem 6.1.** Let $a$ and $V$ be as above. Let $m$ be an odd integer. Then the DKP Conjecture is true in this case. Specifically,

$$\dim(V) = m^{\frac{1}{2}} \dim O_a.$$  \hfill (6.2)

**Proof:** A good deal of the proof has been done in the paragraphs proceeding the statement of the theorem. What remains is to determine $\dim(V_0)$. This is done by the results in Section 4 and by translating the notation of Section 3 (and implicitly, the present) to that situation.

Let us first consider the case where $\forall i : a_i a_i^* = 0$. Our algebra is then a quasipolynomial algebra and (possibly after a renumbering of e.g. the $z_i^*$) the dimension is given by Lemma 5.2.

In the remaining cases it suffices, due to the assumptions on $m$ to use the rank of the defining matrix as given by Proposition 6.4. Indeed, due to the way the algebra is constructed, $s_{r+1} = r_0 + 1$ and thus we are always in the “↓ case”. If we are in the $s = 2\ell$ case of Section 4 then we get the following translation between the remaining notation: $r = \ell + 1$, $\forall i = 1, \ldots, r - 1 : s_{r+1-i} = 2 + r_2i$, and $s_1 = 0$. In this case the contribution from $N^-$ to the dimension is

$$(i_0 - i_1 - 1) + \cdots + (i_{2\ell-2} - i_{2\ell-1} - 1) + (i_{2\ell} - i_{2\ell+1}),$$  \hfill (6.3)

where, moreover, $i_{2\ell+1} = 0$. Thus, the two contributions to $\dim(V)$ do not match up precisely with the two contributions to $\dim O_a$. But the discrepancies clearly cancel. In the remaining case, $s = 2\ell + 1$, the only differences are that $s_1 = r_{2\ell+2} + 2$ and that the contribution to the dimension from $N^-$ now is

$$(i_0 - i_1 - 1) + \cdots + (i_{2\ell-2} - i_{2\ell-1} - 1) + (i_{2\ell} - i_{2\ell+1} - 1).$$  \hfill (6.4)

The claim then follows as above. \hfill □

**Remark 6.2.** As mentioned in the introduction, the algebra in [11] is very closely related to ours. Specifically, when this algebra is given in a form as in Remark 2.2 it is easy to see how the representation theory and symplectic leaves of this algebra are connected with the ones given in the previous sections. Specifically, consider an irreducible representation of the algebra in Remark 2.2. Without loss of generality we may assume that $z_{N-1}^m \neq 0$ or $z_{N-1}^m \neq 0$. Since both $z_{N-1}, z_{N-1}^*$ are covariant we may in fact assume that at least one of them is non-zero (hence invertible). The two cases are equivalent, so let us just assume that $z_{N-1} \neq 0$. Let $w_i = z_i$ and $w_i^* = z_i^* z_{N-1}^2$ for all $i = 0, \ldots, N - 1$. The resulting relations are then, in terms of $w_0, w_1, \ldots, w_{N-1}, w_0^* w_1^*, \ldots, w_{N-1}^*$, the following:

\begin{align*}
w_i w_j &= q^{-1} w_j w_i \text{ for } i < j, \\
w_i^* w_j^* &= q w_j^* w_i^* \text{ for } i < j < N - 1, \\
w_i w_j^* &= q^{-1} w_j^* w_i \text{ for } i \neq j \text{ and } i \neq N - 1, \\
w_i w_i^* - w_i^* w_i = (q^2 - 1) \sum_{k > i} w_k w_k^* \text{ for } i = 0, \ldots, N - 2, \\
w_{N-1} w_j^* &= q w_j^* w_{N-1}, \\
w_i^* w_{N-1} &= q^2 w_{N-1}^* w_i^* \text{ for } i \neq N - 1, \text{ and} \\
w_{N-1} w_{N-1}^* &= q^2 w_{N-1}^* w_{N-1}.  \hfill (6.6)
\end{align*}
In particular, if we let \( \Omega_i = \sum_{k \geq i} w_k w_k^* \) (as before), then
\[
\forall i, j : \Omega_i \Omega_j = \Omega_j \Omega_i, \\
\forall i : w_{N-1} \Omega_i = q^2 \Omega_i w_{N-1}, \\ 
\forall i : w_{N-1}^* \Omega_i = q^{-2} \Omega_i w_{N-1}^*.
\]
(6.7)

This maybe looks rather complicated, but it is in fact not. In dealing with the representation theory, one can proceed with the same strategy as for the previous case. Even more so, after a small adjustment, the same results can be used. Since the Poisson structure (connected with \( \xi \)) behaves in the same way as far as the terms \( \{a_i, a_j^*\} \) are concerned - the fact that we have \( \{a_{N-1}, a_{N-1}^*\} = 2a_{N-1}a_{N-1}^* \) is of no importance here - again allows one to move to a good point where the representation theory breaks up in two parts. The part corresponding to \( N^\pm \) is identical whereas for the algebra corresponding to \( \mathcal{A} \) we get again a family consisting of some elements \( w_i, \Omega_j \) with \( i, j \leq N - 2 \) together with \( w_{N-1}, w_{N-1} w_{N-1}^* \). Here we may set \( \Omega_{N-1} = w_{N-1} w_{N-1}^* \) whereas to deal with the remaining generator we add a new variable \( w_{-1} = w_{N-1}^* \). (Formally, we should then change the numbering by one unit on all variables, but we shall not do so.) We can then take over directly the results of Section 3. All algebras will be of the form \( \mathcal{L}_1 \). The addition of the extra variable of course means a shift in various quantities. For instance, since \( \text{rank} \mathcal{L}_1(1, 1, \ldots, 1) = 2N \), the degree is \( m^N \).

As for the dimensions of the symplectic leaves, observe (this we could also have done for the original algebra) that at a good point, the Poisson structure decouples corresponding to the algebra breaking into \( N^\pm \) and \( \mathcal{A} \). The part corresponding to \( N^\pm \) is completely trivial to deal with. For the remaining part we have a collection of functions \( a_i, a_j^* \). Among the members of this family are \( a_{N-1}, a_{N-1}^* \) and \( a_{N-1} a_{N-1}^* \) at the point. After this there may be variables \( a_{i_1}, a_{i_1}^* \) such that \( a_{i_1} a_{i_1}^* + a_{N-1} a_{N-1}^* = 0 \), and so on, together possibly with some extra functions \( a_k \) and/or \( a_k^* \). The crucial point is that the functions \( a_i, a_k, \) and \( a_j^* \) correspond exactly to the variables \( w_i, w_k \), and \( w_j^* \) in \( \mathcal{A} \). At the same time, for \( i = i_0, i_1, \ldots \), the product \( a_i a_j^* \) corresponds exactly to \( \Omega_j \). Moreover, the original functions can be recovered from these. But this turns the problem into comparing dimensions of representations and symplectic leaves for quasipolynomial algebras, and here the result is well known (see Remark 3.3).

In particular, The DKP conjecture holds for this algebra.

7. Annihilators

In this section we study annihilators of simple modules in the case where \( q \) is generic. If \( u \) is invertible, \( \text{Ad}(u) : w \mapsto uwu^{-1} \) denotes as usual the adjoint representation of \( u \). If \( w \) is covariant we shall occasionally call \( q^{\alpha_u} \) (or simply \( \alpha_u \)) the weight of \( w \) w.r.t. \( u \).

We proceed by observing

**Lemma 7.1.** Let \( \mathcal{C} \) be a quasipolynomial algebra with center \( Z_\mathcal{C} \) and \( V_1 \) an irreducible module in which all generators are invertible. Let \( \xi : Z_\mathcal{C} \mapsto \mathbb{C} \) be the corresponding central character. Let \( Z_\mathcal{C}^\xi \) denote the subalgebra of \( Z_\mathcal{C} \) generated by \( \{c \mapsto \xi(c) \mid c \in Z_\mathcal{C}\} \). Then
\[
\text{Ann}_\mathcal{C} = F_\mathcal{q}(N) \cdot Z_\mathcal{C}^\xi.
\]
(7.1)
Proof: Let $C = \{w_1, \ldots, w_r\}$ and assume $w_i w_j = q^{a_{i,j}} w_j w_i$ all $i, j = 1, \ldots, r$. (It is a standing assumption that the coefficients $a_{i,j}$ are all integers.) The moves which brings the skew-symmetric matrix $\{a_{i,j}\}$ into block diagonal form may simply be interpreted as a series of replacements of the generators by monomials in the generators and their inverses: $w_i \mapsto u_i = w_1^{a_{i,1}} \cdots w_r^{a_{i,r}}$ for all $i = 1, \ldots, r$ and with each $a_{i,j} \in \mathbb{Z}$. Here we may assume that $u_1 u_2 = q^{b_{1,2}} u_2 u_1, \ldots, u_{2s+1} u_{2s+2} = q^{b_{2s+1,2s+2}} u_{2s+2} u_{2s+1}$ $(2s + 2 \leq r)$ and with all other relations trivial (commutative). Now suppose that $u$ is an element in the annihilator. Clearly the new generators are invertible and each $\text{Ad}(u_i)$ leaves the annihilator invariant. Hence we may assume that $u$ has a fixed weight w.r.t. each generator $u_i$. But then evidently $u$ must be a polynomial in the generators $u_{2s+3}, \ldots, u_r$. These generators clearly generate the center and thus the proof is complete. \hfill \Box

From now on, let $V$ be an irreducible module of the algebra $F_q(N)$ and let $\text{Ann}$ denote the annihilator of $V$ in $F_q(N)$.

Observe that if for some index $i$, $\Omega_{i+1} \neq 0$, then $z_i, z_i^*$ do not commute and are not identically 0. If $\Omega_i = 0$ we get that $[z_i, z_i^*] = (\Omega_i - z_i z_i^*)$ and hence $z_i z_i^* = q^{-2} z_i^* z_i$.

Lemma 7.2. Suppose that $\Omega_{i+1} \neq 0$, $\Omega_i \neq 0$, and that

$$\sum_{\alpha} z_i^\alpha p_\alpha + \sum_{\beta} z_i^{*\beta} r_\beta \in \text{Ann}$$

where the $p_\alpha, r_\beta$ may contain powers of $\Omega_i$ but otherwise only contain generators with index $j \neq i$. Then each $z_i^\alpha p_\alpha$ and $z_i^{*\beta} r_\beta$ belongs to $\text{Ann}$. Moreover, we may assume that each $p_\alpha$ and each $r_\beta$ is covariant w.r.t. $\Omega_1$ and $\Omega_{i-1}$.

Proof: Notice that the element considered is the most general element containing $z_i, z_i^*$. Observe that $\text{Ad}(\Omega_i)$ and $\text{Ad}(\Omega_{i+1})$ are identical on $p_\alpha, r_\beta$. Hence the claims follows by weight considerations. \hfill \Box

We now introduce several subalgebras. Let $\mathcal{B}$ denote the set of generators $z_i, z_i^*$ for which $\Omega_{i+1} \neq 0, \Omega_i \neq 0$, and let $\mathcal{B}_0$ denote the subset of $\mathcal{B}$ consisting of generators $z_i, z_i^*$ such that one of them is not injective. Let $\mathfrak{b}$ and $\mathfrak{b}_0$ denote the corresponding sets of indices $i$ of the elements in $\mathcal{B}$ and $\mathcal{B}_0$, respectively. Let $\mathcal{D}$ denote the subalgebra generated by the elements of an index $i \notin \mathfrak{b}_0$ together with the operators $\Omega_i$ with $i \in \mathfrak{b}_0$. Let $\mathcal{C}_1$ denote the subalgebra generated by the elements of an index $i \notin \mathfrak{b}$ together with the operators $\Omega_i$ with $i \in \mathfrak{b}$. The latter is a quasipolynomial algebra and hence its generators are either invertible or identically zero. Let finally $\mathcal{C}$ denote the algebra generated by the invertible elements in $z_0, z_1, \ldots, z_1^*, z_0^*$ having an index $i \notin \mathfrak{b}$ together with the operators $\Omega_i$ with $i \in \mathfrak{b}$. Thus, $\mathcal{C}$ is a subalgebra of $\mathcal{C}_1$.

We now prove some lemmas about the operators in these algebras.

Lemma 7.3. If $i \in \mathfrak{b}_0$ then at most one of the pair $z_i, z_i^*$ is not invertible.

Proof: Let us for simplicity assume that $\forall i \in \mathfrak{b}_0 : z_i$ not injective. Let $V_0$ denote the common null space. This is invariant under all generators whose index is not in $\mathfrak{b}_0$. Hence the space $V$ is of the form

$$V = \text{Span}\{z_i^{*j_1} \cdots z_i^{*j_d} v_{j_1, \ldots, j_d} \mid i_1, \ldots, i_d \in \mathfrak{b}_0, \text{ and } v_{j_1, \ldots, j_d} \in V_0\}.$$ 

Suppose that, say, $z_i^{*j_1}$ annihilates a sum $\omega$ of such elements and that at least one of the summands is non-zero. Let $k$ be the biggest power of $z_i^{*j_1}$ occurring in the expression (and
is an irreducible subspace for (ultimately) $D$

Moreover, since $z_i^k z_i^k = \Omega_i^k + \ell z_i$ this leads to a new sum with fewer and still non-trivial summands (recall that $\Omega_{i+1}$ is invertible and covariant).

Actually, the former proof also yields

**Lemma 7.4.** $V_0$ is an irreducible $D$ module.

**Lemma 7.5.** $\text{Ann}$ is generated by those elements in the annihilator $\text{Ann}_D$ of $V_0$ in $D$ that are covariant with respect to the operators $z_i, z_i^*$ with $i \in \mathfrak{b}_0$.

**Proof:** Let $i \in \mathfrak{b}_0$ and assume as before that $z_i$ is not injective. Assume that $z_i^* r_{\beta} \in \text{Ann}$. Then by injectivity of $z_i^*$, $r_{\beta} \in \text{Ann}$. In particular, $z_i r_{\beta} \in \text{Ann}$. This case then becomes a special case of the following: Suppose that $z_i^* p \in \text{Ann}$. Then $z_i^k p z_i^k \in \text{Ann}$ for all $k = \ell + 1, \ldots$. Notice that if $p$ is a sum of monomials then each monomial is covariant w.r.t. $z_i$. Define $(p)_k$ by $z_i^k p = (p)_k z_i^k$. Observe that $p z_i^k = z_i^k (p)_k$. Then $(p)_k (z_i^k z_i^k - z_i^* z_i^k) \in \text{Ann} \subset \text{Ann}(V_0)$.

Since $z_i^k z_i^k - z_i^* z_i^k = f(q) \Omega_i^k$ on $V_0$ with $f(q) \neq 0$ (c.f. Lemma 3.6), each $(p)_k = 0$. Hence we may assume that $p$ annihilates $V_0$ and has a fixed weight w.r.t. $z_i$ (it already has a fixed weight w.r.t. $\Omega_i$ and $\Omega_{i+1}$). Hence, since $z_i z_i^* = \Omega_i - \Omega_{i+1}$, it has a fixed weight w.r.t. $z_i^*$. According to Lemma 7.2, we are now done with this index. Moving through $\mathfrak{b}_0$ the conclusion is easily reached.

**Lemma 7.6.** $\text{Ann}_D$ is generated by those elements in $\text{Ann}_D \cap \mathcal{C}_1$ that are covariant with respect to the generators of $D$.

**Proof:** This follows again from Lemma 7.2 as in the proof of Lemma 7.5.

**Proposition 7.7.** The annihilator $\text{Ann}$ is generated by those generators $z_i$ and $z_i^*$ of $F_q(N)$ that have an index $i \notin \mathfrak{b}$ and for which $z_i$ and/or $z_i^*$ is identically zero on $V$, together with $\Omega - c$ for some $c \in \mathbb{C}$.

**Proof:** By Lemma 7.5 and Lemma 7.6, an element $a$ in $\text{Ann}$ may be assumed to belong to $\mathcal{C}_1$ and to have a specific weight with respect to all generators (c.f. the proof of Lemma 7.1). Thus, if $a$ is a sum of monomials in the generators, e.g. $a = a_1 + a_2$, it may be assumed that $a_1$ and $a_2$ have the same weight. By covariance, $a_1$ and $a_2$ are either identically zero or invertible. We may for now assume that they are both invertible. But then $a_2 \cdot a_1^{-1}$ is in the center of $F_q(N)$ and hence $a_2 = a_1 \cdot w$ for some $w$ in the center of $F_q(N)$.

We finish by some observations concerning the converse of Proposition 7.4.

Let us then consider an irreducible representation $\hat{V}_0$ of $\mathcal{C}_1$ (or, equivalently, an irreducible representation of $C$ in which all generators are invertible). Let $i \in \mathfrak{b} \setminus \mathfrak{b}_0$. We want to have that

$$\text{Span}\{z_i^r v_r, z_i^s v_s \mid v_r, v_s \in \hat{V}_0\}$$

is an irreducible subspace for (ultimately) $D$, that $z_i, z_i^*$ are injective, etc.

Utilizing the fact that $\Omega_i$ commutes with $z_i, z_i^*$ whereas their weight w.r.t. $\Omega_{i+1}$ is $q^2$, it follows that if $z_i(z_i^* v_s) = 0$ or $z_i^*(z_i^* v_r) \neq 0$ for some $r, s$ then $\Omega_{i+1} = q^n \Omega_i$ for some $n \in \mathbb{Z}$.

We will from now on assume that $\forall n \in \mathbb{Z} : \Omega_{i+1} \neq q^n \Omega_i$.

Elements of the form

$$z_i^r v_r + z_i^{r-1} v_{r-1} + \cdots + v_0$$
are by definition non-zero (polynomials) and hence non-zero if just one \( v_i \neq 0 \). By applying \( z_i \) (and \( z_i^* \)) appropriately, it follows that any invariant subspace must contain such an element. Let \( r \) be minimal \((r \geq 1)\). Observe that \( \Omega_{i+1} = c^* \Omega_i \) for some non-zero complex constant \( c^* \).

Applying \( \Omega_i \) and \( \Omega_{i+1} \) to the element we must still be in the invariant subspace. But then there is an element of the form \( z_i^r v_0 \) in the subspace, and then an element \( v_0 \) in the subspace. Hence, \( V_0 \) is contained in any invariant subspace.

Observe that if \( \Omega_{r+1} = q^n \Omega_i \) for some \( n \in \mathbb{Z} \), then \( z_i \) and \( z_i^* \) satisfy a covariance relation (which generically is not homogeneous w.r.t. \( q \)).

But it follows easily that under the assumptions, \( \Omega_i, \Omega_{i+1} \) are invertible on the space

\[
z_i^r v_r + z_i^{r-1} v_{r-1} + \cdots + v_0 + \cdots + z_i^s v_s.
\]

Indeed, we obtain the following

**Proposition 7.8.** Suppose that \( \pi \) is an irreducible representation of \( \mathcal{C} \) for which \( \forall i \in b, \forall n \in \mathbb{Z} : \Omega_i \neq q^n \Omega_{i+1} \). Then there is an irreducible representation of \( F_q(N) \) for which the annihilator is generated by \( C_1 \setminus \mathcal{C} \) together with \( \Omega - c \). The complex constant \( c \) is determined by \( \pi \) if \( 0 \in b \). Otherwise, \( c = 0 \).

Notice that this description is very similar to Oh’s description by means of admissible sets.

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