SPACES OF NORMS, DETERMINANT OF COHOMOLOGY AND FEKETE POINTS IN NON-ARCHIMEDEAN GEOMETRY

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ABSTRACT. We describe the leading asymptotics of the determinant of cohomology of large powers of a metrized ample line bundle, on a projective Berkovich space over any complete non-Archimedean field. As consequences, we obtain the existence of transfinite diameters and equidistribution of Fekete points, as in previous work of the first author with Berman and Witt Nyström in the complex setting. Our approach relies on a version of the Knudsen-Mumford expansion for the determinant of cohomology on models over the (possibly non-Noetherian) valuation ring, as a replacement for the asymptotic expansion of Bergman kernels in the complex case, and on the reduced fiber theorem, as a replacement for the Bernstein-Markov inequalities. Along the way, a systematic study of spaces of norms and the associated Fubini-Study type metrics is undertaken.

CONTENTS

Introduction 2

Part 1. Spaces of norms and determinants 7
1. Spaces of norms 7
2. Determinants and successive minima 18
3. Alternative metric structures on spaces of norms 27

Part 2. Models and metrics 31
4. Analytification and models 31
5. Metrics 37
6. Limits of Fubini-Study metrics 43

Part 3. Asymptotics of relative volumes 51
7. Monge-Ampère measures and Deligne pairings 51
8. Asymptotics of relative volumes 61
9. Transfinite diameter and Fekete points 67
Appendix A. Determinant of cohomology and Deligne pairings 75
References 85

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Fekete points and transfinite diameter are classical notions in logarithmic potential theory in the plane. For each \( m \geq 1 \), the \( m \)-diameter \( \delta_m(K) \) of a compact subset \( K \subset \mathbb{C} \) is defined as the supremum of the geometric mean distance between \( m + 1 \) points in \( K \), maximizers being called Fekete configurations. The \( m \)-diameter of \( K \) admits a limit \( \delta_\infty(K) = \lim_{m \to \infty} \delta_m(K) \), called the transfinite diameter of \( K \), which is proved to coincide with the logarithmic capacity of \( K \). Fekete configurations are also known to become asymptotically unique in the limit, in the sense that they equidistribute to a certain probability measure known as the equilibrium measure of \( K \).

In the higher dimensional case, a similar understanding was only rather recently obtained. The first steps were taken by Leja in the 1950’s, introducing a notion of \( m \)-diameter \( \delta_m(K) \) for a compact subset \( K \subset \mathbb{C}^n \) in terms of the supremum of certain Vandermonde-type determinants. The existence of the transfinite diameter \( \delta_\infty(K) = \lim_{m \to \infty} \delta_m(K) \) was established by Zaharjuta in the 1970’s, and the next key step came with Rumely’s observation in [Rum07] that the general results in arithmetic intersection theory developed in [CLR03] yield in particular an exact formula for \( \delta_\infty(K) \) in terms of pluripotential theory (semipositive/plurisubharmonic envelopes and mixed Monge-Ampère operators in the sense of Bedford-Taylor). This triggered joint work of the first author with Berman and Witt Nyström [BB10, BBW11], which built on Bergman kernel asymptotics to establish a general version of Rumely’s formula in the setting of polarized projective manifolds, and combined it with a variational argument to prove the equidistribution of Fekete configurations in this context.

The main purpose of the present paper is to study versions of these results in non-Archimedean (Berkovich) geometry. While many results hold over an arbitrary non-Archimedean complete valued field \( K \), the full picture relies on more refined non-Archimedean pluripotential theory as developed in [BFJ16, BFJ15, BG+16, BJ18], and hence requires \( K \) to be trivially or discretely valued and of residue characteristic 0.

**Asymptotics of relative volumes.** The Bouche-Catlin-Tian-Zelditch asymptotic expansion of Bergman kernels is a fundamental result in complex geometry describing the asymptotic behavior of the \( L^2 \)-norms associated to large tensor powers of a positive Hermitian line bundle. As noticed in [BB10], it can be reformulated as an asymptotic expansion for the logarithmic volume ratio of such \( L^2 \)-norms. More specifically, define the relative volume of two norms \( \| \cdot \|, \| \cdot \|' \) on an \( N \)-dimensional complex vector space \( V \) as

\[
\text{vol}(\| \cdot \|, \| \cdot \|') := \log \left( \frac{\det \| \cdot \|'}{\det \| \cdot \|} \right),
\]

(0.1)

where \( \det \| \cdot \| \) denotes the induced norm on the determinant line \( \text{det} V \). In terms of the unit balls \( B, B' \subset V \) of the two norms, \( \text{vol}(\| \cdot \|, \| \cdot \|') \) coincides with the logarithmic volume ratio

\[
\frac{1}{2} \log \left( \frac{\text{vol} B}{\text{vol} B'} \right),
\]

up to an error term \( O(N \log N) \) that vanishes when the two norms are Hermitian. Let \( X \) be a smooth, \( n \)-dimensional complex projective variety endowed an ample bundle \( L \). Any two smooth positive metrics \( \phi, \psi \) on \( L \) induce for each \( m \in \mathbb{N} \) \( L^2 \)-norms \( \| \cdot \|_{L^2(m\phi)}, \| \cdot \|_{L^2(m\psi)} \)
on the space of sections $H^0(mL) = H^0(X, L^\otimes m)$, and the asymptotic expansion of Bergman kernels turns out to be equivalent to the existence of a full asymptotic expansion

$$
\frac{1}{mN_m} \text{vol} (\| \cdot \|_{L^2(m\phi)}, \| \cdot \|_{L^2(m\psi)}) = a_0 + m^{-1}a_1 + \cdots + O(m^{-\infty}),
$$

with

$$N_m := \dim H^0(mL) = \frac{m^n}{n!}V + O(m^{n-1}).$$

The leading order term $a_0$ can further be identified with a fundamental functional in Kähler geometry, the relative Monge-Ampère energy

$$E(\phi, \psi) := \frac{1}{n+1} \sum_{j=0}^n V^{-1} \int_X (\phi - \psi)(ddc\phi)^j \wedge (ddc\psi)^{n-j}. \tag{0.2}$$

We use additive notation for metrics on line bundles, so that $\phi - \psi$ is a function on $X$, and $ddc\phi, ddc\psi$ denote the curvature $(1,1)$-forms of $\phi, \psi$.

If $\phi, \psi$ are now arbitrary continuous metrics on $L$, denote by $\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}$ the sup-norms they define on $H^0(mL)$. Combining the previous result with a regularization argument and a growth estimate for the distortion between sup-norms and $L^2$-norms (Bernstein-Markov inequality), it was proved in [BB10] that

$$\lim_{m \to \infty} \frac{1}{mN_m} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \to E(P(\phi), P(\psi)). \tag{0.3}$$

Here $P(\phi), P(\psi)$ are the semipositive envelopes of $\phi, \psi$, whose relative Monge-Ampère energy $E(P(\phi), P(\psi))$ can be defined as in $\text{(0.2)}$ using the Bedford-Taylor theory of mixed Monge-Ampère operators.

The main result of the present paper is the following non-Archimedean analogue, generalizing in particular [BG+16, Theorem A] (which deals with the discretely valued case).

**Theorem A.** Let $X$ be a smooth projective variety over a non-Archimedean complete valued field $K$, and $L$ be an ample line bundle on $X$. For any two continuous metrics $\phi, \psi$ on (the Berkovich analytification of) $L$, the scaled relative volumes of the induced sup-norms $\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}$ on $H^0(mL)$ admit a finite limit

$$\lim_{m \to \infty} \frac{1}{mN_m} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \in \mathbb{R}.$$

If the semipositive envelopes $P(\phi), P(\psi)$ are further continuous, then this limit coincides with the Monge-Ampère energy $E(P(\phi), P(\psi))$.

Continuity of $P(\phi), P(\psi)$ is expected to be always true. Besides the trivial case where $\phi, \psi$ are already semipositive, it is known to hold when $K$ is trivially or discretely valued of residue characteristic 0 [BFJ16, BJ18] (see also [GJKM17] for partial results when $K$ is discretely valued of equal positive characteristic). The relative volume of two norms on a $K$-vector space and the Monge-Ampère energy $E$ are still respectively defined by $\text{(0.1)}$ and $\text{(0.2)}$, the latter being this time understood in the sense of Chamber-Loir and Ducros [CLD12].

**Sketch of the proof.** The main tools involved in the proof of Theorem A can be summarized as follows.
Spaces of norms and determinants. The existence of the limit in Theorem A is closely related to [CMac15, WN14]. We deduce it from the filtered linear series technique introduced in [BC], itself based on Newton-Okounkov bodies. This relies on an appropriate version of Minkowski’s second theorem, expressing the relative volume of two norms as a sum of relative successive minima, up to a negligible error term. This material is included in a general study of spaces of norms and determinants, which forms the first part of the present paper.

Fubini-Study metrics. The semipositive envelope \( P(\phi) \) of a continuous metric \( \phi \) appearing in Theorem A is defined as the limit of the Fubini-Study metrics on \( L \) induced by the sup-norms \( \| \cdot \|_{m\phi} \). This gives rise to a systematic study of Fubini-Study metrics and their limits, partly revisiting with a slightly different perspective [CMor15], and forming the second part of the paper.

The reduced fiber theorem. With these preliminary tools in hand, the proof of Theorem A is easily reduced to the case where each metric is induced by an ample model \((X, \mathcal{L})\) of \((X, L)\), i.e. an ample line bundle \( \mathcal{L} \) extending \( L \) to a projective model \( X \) of \( X \) over the valuation ring \( K^\circ \). Besides the sup-norm \( \| \cdot \|_{m\phi} \) defined by the model metric \( \phi = \phi_{\mathcal{L}} \), the space of sections \( H^0(mL) \) is then also equipped with the lattice norm \( \| \cdot \|_{H^0(m\mathcal{L})} \) induced by the \( K^\circ \)-module \( H^0(m\mathcal{L}) = H^0(X, \mathcal{L}^{\otimes m}) \). This lattice norm, which is to some extent the analogue of the \( L^2 \)-norm in the present non-Archimedean context, coincides with the sup-norm when \( X \) has reduced special fiber, but not in general. Using the Bosch-Lütkebohmert-Raynaud reduced fiber theorem [BLR95], we prove however that the distortion between \( \| \cdot \|_{m\phi} \) and \( \| \cdot \|_{H^0(m\mathcal{L})} \) remains bounded as \( m \to \infty \), which enables us to replace the sup-norms with the lattice norms in proving Theorem A.

Knudsen-Mumford expansion. Our main tool is then the Knudsen-Mumford expansion of the determinant of cohomology [KM76], which plays the role of the asymptotic expansion of Bergman kernels in the complex case and provides for \( m \gg 1 \) a polynomial expansion (in additive notation for \( \mathbb{Q} \)-line bundles over \( \text{Spec} \ K^\circ \))

\[
\det H^0(m\mathcal{L}) = \frac{m^{n+1}}{(n+1)!} \langle \mathcal{L}^{n+1} \rangle + \ldots
\]  

with leading order term the Deligne pairing \( \langle \mathcal{L}^{n+1} \rangle \). Since models over \( K^\circ \) are non-Noetherian in the densely valued case, some care is however required to apply this result, and the relevant explanations are provided in Appendix A, based on Ducrot’s approach [Duc05].

Metrics on Deligne pairings. The Knudsen-Mumford expansion yields an expression of the limit of the scaled relative volume as a difference of model metrics on the Deligne pairing \( \langle L^{n+1} \rangle \). The final ingredient in the proof of Theorem A consists in relating model metrics on Deligne pairings with mixed Monge-Ampère integrals, which is accomplished building on the Poincaré-Lelong formula of [CLD12] and a careful monotone regularization argument.

Transfinite diameter and Fekete points. Following the strategy developed in [BB10, BBW11], we use Theorem A to show the existence of transfinite diameters, and combine it with a differentiability result proved in [BFJ15, BG+16, BJ18] under appropriate assumptions on the ground field \( K \) to infer equidistribution of Fekete points.
For simplicity, we only consider the case of a compact set equal to the whole of $X^{an}$. The $m$-diameter of a continuous metric $\phi$ on $L$ is to be defined as the sup-norm, with respect to the induced metric, of the Vandermonde determinant
\[
\det s \in H^0 \left( X^{N_m}, (mL)^{\boxtimes N_m} \right)
\]
associated to a basis $s = (s_i)$ of $H^0(mL)$, defined by
\[
(\det s)(x_1, \ldots, x_{N_m}) = \det (s_i(x_j)).
\]
This requires however to normalize the basis in some way, which can be done using a reference norm $\|\cdot\|_{ref}$ on $H^0(mL)$. We thus define define the $m$-diameter of $\phi$ normalized by $\|\cdot\|_{ref}$ as
\[
\delta_{m,\|\cdot\|_{ref}}(\phi) := \left( \frac{\| \det s\|_{(m\phi)^{\boxtimes N_m}}}{\det \|s_1 \wedge \cdots \wedge s_{N_m}\|_{ref}} \right)^{1/mN_m}
\]
for any choice of basis $s = (s_i)$ of $H^0(mL)$.

**Theorem B.** Let $\phi, \psi$ be two continuous metrics on $L$, and $(\|\cdot\|_m)$ be any sequence of reference norms on the spaces $H^0(mL)$, with subexponential distortion with respect to the sequence of sup-norms $\|\cdot\|_{m\psi}$. Then
\[
\delta_{\infty,\psi}(\phi) := \lim_{m \to \infty} \delta_{m,\|\cdot\|_m}(\phi)
\]
exists and is independent of the choice of $(\|\cdot\|_m)$. We further have
\[
- \log \delta_{\infty,\psi}(\phi) = \lim_{m \to \infty} \frac{1}{mN_m} \text{vol} (\|\cdot\|_{m\phi}, \|\cdot\|_{m\psi}).
\]

This is inferred from Theorem A via an estimate for the operator norms of the embeddings $\det H^0(mL) \hookrightarrow H^0((mL)^{\boxtimes N_m})$ with respect to the sup-norms induced by $\phi$ on both sides, which is again ultimately deduced from the reduced fiber theorem.

An $m$-Fekete configuration for $\phi$ is a point $P \in (X^{N_m})^{an}$ that achieves the sup-norm of $\det s$ with respect to the induced metric, for some (hence any) choice of basis $s$ of $H^0(mL)$.

**Theorem C.** Assume that $K$ is trivially or discretely valued, of residue characteristic 0. Let $\phi$ be a continuous metric, and pick for each $m$ an $m$-Fekete configuration $P_m \in (X^{N_m})^{an}$ for $\phi$. Then $P_m$ equidistributes to the Monge-Ampère measure $\text{MA}(P(\phi)) = V^{-1}(d\xi P(\phi))^n$ as $m \to \infty$.

Indeed, the assumption on $K$ guarantees that the semipositive envelope $P(\phi)$ of any continuous metric $\phi$ is itself continuous [BFJ16, BJ18], and that $E(P(\phi), P(\psi))$ is differentiable with respect to $\phi$ [BFJ15, BG+16, BJ18]. By Theorems A and B, $\log \delta_{\infty,\psi}(\phi)$ is thus differentiable with respect to $\phi$, and we then simply import the variational argument of [BBW11], itself based on an idea of [SUZ97].

**Organization of the paper.** The paper is organized as follows:

- Section 1 contains background material on norms on finite dimensional vector spaces over a complete valued field. We present the results in the Archimedean and non-Archimedean cases as uniformly as possible.
- In Section 2 we discuss determinants of norms and their relationship to successive minima.
• Section 3, which stands somewhat apart from the rest of the paper, applies the previous result to construct metrics on spaces of norms, following [Ger81].
• Sections 4 and 5 contain background material on Berkovich spaces and metrics on line bundles. We recall the standard constructions of model metrics, and compare them to Fubini-Study metrics. We discuss the relation between the reduced fiber theorem and finiteness of integral closures, and infer our first key tool, to wit boundedness of the distortion between the sup-norms and lattice norms induced by a model.
• In Section 6 we study limits of Fubini-Study metrics, compare them to Zhang’s definition of semipositive metrics, and discuss the related notion of semipositive envelope.
• In Section 7 we review the Bedford-Taylor/Chambert-Loir-Ducros mixed Monge-Ampère operators, and relate them to Deligne pairings.
• Section 8 contains the proof of Theorem A.
• Section 9 shows the existence of transfinite diameters and equidistribution of Fekete points, i.e. Theorem B and Theorem C. We also show how the results can be applied in the case of toric varieties.
• In the appendix we explain how Ducrot [Duc05] approach to the Knudsen-Mumford expansion for the determinant of cohomology and the related notion of Deligne pairings can be extended from the usual noetherian case to schemes of finite presentation over the valuation ring of a complete non-Archimedean field.

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Part 1. Spaces of norms and determinants

1. Spaces of norms

The goal of this section is to review some basic material on finite dimensional normed vector spaces, treating in parallel the Archimedean and non-Archimedean cases (including the trivially valued case). All the results are well-known, but our proof of density of diagonalizable norms among ultrametric norms (Theorem 1.19) appears to be new.

1.1. Complete valued fields. Here and throughout the article, \( K \) denotes a field, complete with respect to a (possibly trivial) absolute value \(| \cdot |: K \to \mathbb{R}\). The value group \(|K^*|\) is a subgroup of \( \mathbb{R}_+^\times \), which is thus either either finite or dense. We sometimes use the additive value group \( \Gamma := \log |K^*| \subset \mathbb{R} \).

Recall that \( K \) is Archimedean if, for each non-zero \( \alpha \in K \), there exists \( n \in \mathbb{Z} \) with \(|n\alpha| > 1\). This implies that \( K \) contains \( \mathbb{Q} \), and that the restriction of \(| \cdot | \) to \( \mathbb{Q} \) is equivalent to the standard absolute \(| \cdot |_\infty \), by Ostrowski’s theorem. As a result, \( K \) is a complete field extension of \( \mathbb{R} \), and hence \( K = \mathbb{R} \) or \( \mathbb{C} \), by the Gelfand-Mazur theorem.

Otherwise, \( K \) is non-Archimedean, which holds if and only if \(| \cdot |\) satisfies the ultrametric inequality \(|\alpha + \alpha'| \leq \max\{|\alpha|, |\alpha'|\}\) for all \( \alpha, \alpha' \in K \). We then have a corresponding real-valued valuation \( v_K := -\log |\cdot| \) on \( K \), whose valuation ring \( K^0 \) is thus the closed unit ball of \( K \), with maximal ideal \( K^{\infty} \) the open unit ball and residue field \( \tilde{K} := K^0/K^{\infty} \). Since the value group is a subgroup of \( \mathbb{R} \), the valuation ring \( K^0 \) is of Krull dimension at most 1, and it is Noetherian if and only if \(|K^*|\) is discrete. In that case, \( K^{\infty} \) is a principal ideal; a generator \( \pi_K \) is called a uniformizing parameter, and is unique up to multiplication by a unit. If \( K \) is algebraically closed, then \( \tilde{K} \) is algebraically closed as well, and \(|K^*|\) is divisible. In particular, \( K \) is then either trivially valued or densely valued. The completion of an algebraic closure of any non-Archimedean field \( K \) is denoted by \( \mathbb{C}_K \). It is the smallest complete algebraically closed extension of \( K \).

A field \( K \) is local if its unit ball \( K^0 \) is compact. This holds if and only if \( K \) is either Archimedean, or non-Archimedean with finite residue field \( \tilde{K} \). In the latter case, \( K \) is discretely valued, and is in fact either finite and trivially valued, or isomorphic to a finite extension of \( \mathbb{Q}_p \) or \( \mathbb{F}_p((t)) \) (up to normalization of the absolute value).

An immediate extension of a non-Archimedean field \( K \) is a complete field extension \( L/K \) such that \(|L^*| = |K^*|\) and \( L = \tilde{K} \), and \( K \) is maximally complete if it admits no nontrivial immediate extension. By [Kap42], maximally complete is equivalent to spherically complete. A discretely valued field is maximally complete.

Example 1.1. Let \( k \) be an algebraically closed field of characteristic 0, and endow the field \( K = k((t)) \) of formal Laurent series with the \( t \)-adic valuation. An algebraic closure of \( K \) is given by the field of Puiseux series \( \bigcup_{n \geq 1} k((t^{1/n})) \), and its completion \( \mathbb{C}_K \) is realized as the field of formal power series \( f = \sum_{\tau \in \mathbb{Q}} a_\tau t^\tau \) with \( a_\tau \in k \) whose support \( \text{Supp} f = \{ \tau \in \mathbb{Q} \mid a_\tau \neq 0 \} \) contains only finitely many elements with a given upper bound. The Malcev-Neumann field \( k((\mathbb{Q})) \) of power series \( f = \sum_{\tau \in \mathbb{Q}} a_\tau t^\tau \) with well-ordered support is an immediate extension of \( \mathbb{C}_K \), with \( f = \sum_{n \geq 1} t^{-1/n} \) in \( k((\mathbb{Q})) \backslash \mathbb{C}_K \). As a consequence, \( \mathbb{C}_K \) is not maximally complete.

Similarly, the completion \( \mathbb{C}_p \) of an algebraic closure of \( \mathbb{Q}_p \) is also not maximally complete. Denoting by \( A \) the Witt ring of \( \mathbb{F}_p \) (i.e. the valuation ring of the completion of the maximal unramified extension of \( \mathbb{Q}_p \)), an immediate maximally complete extension of \( \mathbb{C}_p \) is obtained
as the quotient of the Malcev-Neumann ring $A((t))$ by the ideal of formal power series $f = \sum_{r \in \mathbb{Q}} a_r t^r$ such that $\sum_{n \in \mathbb{Z}} a_{r+n p^n} = 0$ in $\mathbb{Z}_p$ for all $r \in \mathbb{Q}$, cf. [Poo93] §4.

1.2. The space of norms. Let $V$ be a fixed finite dimensional $K$-vector space, and set $N := \dim V.$

**Definition 1.2.** A seminorm on $V$ is a function $\| \cdot \| : V \to \mathbb{R}_+$ such that

(i) $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in K$, $v \in V$;
(ii) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$;

It is a norm if $\|v\| = 0 \iff v = 0$. A seminorm $\| \cdot \|$ is ultrametric if $\|v + w\| \leq \max\{\|v\|, \|w\|\}$.

We denote by $\mathcal{N}(V)$ the set of all norms on $V$, and by $\mathcal{N}^{\text{ultr}}(V) \subset \mathcal{N}(V)$ the set of ultrametric norms (which is non-empty if and only if $K$ is non-Archimedean).

The group $\text{GL}(V)$ acts on $\mathcal{N}(V)$ by composition, and this action preserves $\mathcal{N}^{\text{ultr}}(V)$.

**Remark 1.3.** When $K$ is non-Archimedean, ultrametric norms will of course be the relevant ones, but allowing at first arbitrary norms yields a more uniform treatment of the Archimedean and non-Archimedean cases.

**Example 1.4.** If $K$ is Archimedean, mapping a norm $\| \cdot \|$ to its closed unit ball $B$ (centered at 0) sets up a one-to-one correspondence between $\mathcal{N}(V)$ and the set of all convex bodies of $V$ that are centrally symmetric (resp. $S^1$-invariant when $K = \mathbb{C}$). The inverse map is obtained by setting

$$\|v\| = \inf \{ r \geq 0 \mid v \in rB \}.$$

**Example 1.5.** If $K$ is trivially valued, the closed balls $B_r$, of radius $r$, of an ultrametric norm form an increasing filtration of $V$ by linear subspaces, which is exhaustive ($B_r = V$ for $r > 1$), separating ($B_r = \{0\}$ for $r \ll 1$), and right-continuous ($B_r = \bigcap_{r' > r} B_{r'}$). Conversely, any such filtration defines an ultrametric norm by setting

$$\|v\| = \inf \{ r \geq 0 \mid v \in B_r \}.$$

In other words, the data of an ultrametric norm with respect to the trivial absolute value is equivalent to that of an increasing flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of linear subspaces, together with an increasing sequence $0 = r_0 < r_1 < \cdots < r_n$.

Equivalence of norms over $\mathbb{R}$ and $\mathbb{C}$ is usually established as a consequence of the compactness of the unit cube. Crucially, equivalence of norms still holds over any complete valued field.

**Proposition 1.6.** Any two norms $\| \cdot \|$, $\| \cdot \|'$ on $V$ are equivalent, i.e. there exists $C > 0$ such that $C^{-1}\| \cdot \| \leq \| \cdot \|' \leq C\| \cdot \|$. 

**Proof.** For the convenience of the reader, we repeat the simple standard argument, in order to show that it applies to the trivially valued case as well. Note first that the result implies that $V$ is complete with respect to any norm $\| \cdot \|$. Indeed, after choosing a basis $(e_i)$ of $V$, $\| \cdot \|$ will be equivalent to the $\ell^\infty$-norm $\| \cdot \|_\infty$ associated to $(e_i)$, which is complete since it is isometrically isomorphic to $K^N$. We argue by induction on $N = \dim V$, the desired result being trivial for $N = 1$. We are going to show that any given norm $\| \cdot \|$ on $V$ is equivalent to $\| \cdot \|_\infty$. For each subspace $W \neq V$, the restriction of $\| \cdot \|$ to $W$ is complete, by induction. As
a result, \( W \) is closed with respect to \( \| \cdot \| \), and hence \( \inf_{w \in W} \| v + w \| > 0 \) for each \( v \in V - W \). In particular,
\[
c_i := \inf_{\alpha \in K^N} \left\| e_i + \sum_{j \neq i} \alpha_j e_j \right\| > 0
\]
for all \( i \). For each \( \alpha \in K^N \) and each \( i \) with \( \alpha_i \neq 0 \), we get
\[
\left\| \sum_j \alpha_j e_j \right\| = |\alpha_i| \left\| e_i + \sum_{j \neq i} \frac{\alpha_j}{\alpha_i} e_j \right\| \geq c_i|\alpha_i|,
\]
and hence \( \left\| \sum_j \alpha_j e_j \right\| \geq (\min_j c_j) \max \vert \alpha_j \vert \). By the triangle inequality, we also have
\[
\left\| \sum_j \alpha_j e_i \right\| \leq N \max_j |\alpha_j| \vert e_i \vert ,
\]
which proves that \( \| \cdot \| \) and \( \| \cdot \|_{\infty} \) are indeed equivalent. \( \square \)

As noticed during the proof, each linear subspace \( W \subset V \) is closed with respect to any norm \( \| \cdot \| \), and \( \| \cdot \| \) thus induces a quotient norm \( \| \cdot \|_{V/W} \) on \( V/W \), defined as usual by
\[
\| \bar{v} \|_{V/W} := \inf_{w \in W} \| v + w \|
\]
for each \( v \in V \) with image \( \bar{v} \in V/W \).

By Proposition 1.6, we can endow \( N(V) \) with the Goldman-Iwahori metric \( d_\infty \) (named after [GI]), defined by
\[
d_\infty (\| \cdot \|, \| \cdot \|') := \sup_{v \in V \setminus \{0\}} \log \| v \| - \log \| v \|'. \tag{1.1}
\]
The exponential of \( d_\infty (\| \cdot \|, \| \cdot \|') \) is thus the distortion between the two norms, i.e. the smallest constant \( C \geq 1 \) such that
\[
C^{-1} \| \cdot \| \leq \| \cdot \|' \leq C \| \cdot \|.
\]
The action of \( GL(V) \) on \( N(V) \) preserves \( d_\infty \).

**Example 1.7.** Assume that \( \dim V = 1 \), and pick a nonzero \( v \in V \). Then \( \| \cdot \| \mapsto \log \| v \| \) defines an isometry \( (N(V), d_\infty) \simeq \mathbb{R} \), and the action of \( GL(V) \) on \( N(V) \) is then equivalent to the action of the additive value group \( \Gamma = \log |K^*| \) on \( \mathbb{R} \) by translation.

The basic topological properties of \( N(V) \) are as follows.

**Proposition 1.8.** The metric space \( (N(V), d_\infty) \) is complete, and the subset \( N^{ultr}(V) \) of ultrametric norms is closed. If \( K \) is local, i.e. \( K^0 \) is compact, any closed bounded subset of \( N(V) \) is compact.

**Proof.** If \( (\| \cdot \|_n) \) is a Cauchy sequence in \( N(V) \), then \( \log \| v \|_n \) is a Cauchy sequence for each nonzero \( v \in V \), and we easily conclude that \( \| \cdot \|_n \) converges in \( N(V) \), which proves the first assertion. Assume now that \( K \) is local. After choosing a basis, we may assume that \( V = K^N \), which we equip with the \( \ell^\infty \)-norm \( \| \cdot \|_\infty \). By the triangle inequality, each norm \( \| \cdot \| \) with \( d_\infty (\| \cdot \|, \| \cdot \|_\infty) \leq C \) restricts to a \( C \)-Lipschitz continuous function on the compact set \((K^0)^N \). By the Arzela-Ascoli theorem, the closed balls of \( N(V) \) (and hence any bounded closed subset) are thus compact. \( \square \)

**Remark 1.9.** Conversely, if \( \dim V > 1 \), one can show that \( N(V) \) is locally compact only if \( K^0 \) is compact.
1.3. Diagonalizable norms. In order to treat in parallel the Archimedean and non-Archimedean cases, we will use the following terminology.

Definition 1.10. A norm $\| \cdot \|$ on $V$ is diagonalizable if there exists a basis $(e_i)$ such that we have for all $\alpha \in K^N$:

(i) $\sum_i \alpha_i e_i = \sum_i \|\alpha_i e_i\|^2$ (Archimedean case);

(ii) $\|\sum_i \alpha_i e_i\| = \max_i \|\alpha_i e_i\|$ (non-Archimedean case).

The basis $(e_i)$ is then said to be orthogonal for $\| \cdot \|$, and it is orthonormal if it further satisfies $\|e_i\| = 1$. We say that a diagonalizable norm $\| \cdot \|$ is pure if it admits an orthonormal basis. We denote by

$$\mathcal{N}^{\text{diag}}(V) \subset \mathcal{N}(V)$$

the set of diagonalizable norms.

A diagonalizable norm is pure iff it takes values in $|K|$. In the Archimedean case, a norm is diagonalizable if and only if it derives from a Euclidian/Hermitean scalar product, and every such norm is pure. In the non-Archimedean case, every diagonalizable norm is ultrametric. The converse depends on the specific field $K$ we are dealing with, but diagonalizable norms are always dense in the set of ultrametric norms, see [1.4] below.

Example 1.11. If $K$ is trivially valued, any ultrametric norm $\| \cdot \|$ is diagonalizable. Indeed, $\| \cdot \|$ determines a flag of subspaces (cf. Example [1.5]), and a basis $(e_i)$ of $V$ is orthogonal for $\| \cdot \|$ if and only if $(e_{\sigma(i)})$ is compatible with the flag, for some permutation $\sigma \in \mathfrak{S}_N$.

Example 1.12. By [BGR 2.4.2/3], any ultrametric norm is diagonalizable as soon as $K$ is discretely valued (see also [CMor15 Proposition 1.3]). By [BGR 2.4.4], this is more generally true if $K$ is maximally complete; conversely, $K^2$ admits a non-diagonalizable ultrametric norm as soon as $K$ is not maximally complete (e.g. $K = \mathbb{C}_p$). Indeed, let $L/K$ be a nontrivial immediate extension, pick $\pi \in L - K$, and denote by $\| \cdot \|$ the restriction of $\| \cdot \|_L$ to $K + \pi K \simeq K^2$. If $\| \cdot \|$ were diagonalizable, it would admit an orthonormal basis $(e_1,e_2)$, since $\| \cdot \|$ takes values in $|K| = |L|$. Using $\tilde{L} = \tilde{K}$, we then find a unit $\alpha \in K^\circ$ such that $e_1 - \alpha e_2 \in L^{\circ\circ}$, i.e. $\|e_1 - \alpha e_2\| < 1$, contradicting the orthonormality of $(e_1,e_2)$.

In the Archimedean case, diagonalizable norms are of course preserved by restriction and quotient. This is also true in the non-Archimedean case (cf. [BGR 2.4.1/5]):

Lemma 1.13. Let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of vector spaces. If $\| \cdot \|$ is a diagonalizable norm on $V$, then the induced norms on $V'$ and $V''$ are also diagonalizable.

The following codiagonalization result is crucial for what follows. It will be proved in [1.5] after the basic facts on duality have been discussed.

Proposition 1.14. For any two diagonalizable norms $\| \cdot \|, \| \cdot \|' \in \mathcal{N}^{\text{diag}}(V)$, there exists a basis $(e_i)$ of $V$ that is orthogonal for both $\| \cdot \|$ and $\| \cdot \|$.

This can be used to give a simple description of the restriction of $d_{\infty}$ to $\mathcal{N}^{\text{diag}}(V)$.

Lemma 1.15. If $\| \cdot \|, \| \cdot \|' \in \mathcal{N}^{\text{diag}}(V)$ are codiagonalized in a basis $(e_i)$, then

$$d_{\infty}(\| \cdot \|, \| \cdot \|') = \max_i \left| \log \frac{\|e_i\|}{\|e_i\|'} \right|.$$
If $K$ is non-Archimedean, we have more generally

$$d_\infty(\lVert \cdot \rVert, \lVert \cdot \rVert') = \log \max \left\{ \max_i \frac{\|e_i\|'}{\|e_i\|}, \max_i \frac{\|e'_i\|}{\|e'_i\|'} \right\}$$

whenever $(e_i)$ and $(e'_i)$ are orthogonal bases for $\lVert \cdot \rVert$ and $\lVert \cdot \rVert'$, respectively.

**Proof.** Assume first that $K$ is Archimedean. We trivially have

$$d_\infty(\lVert \cdot \rVert, \lVert \cdot \rVert') = \sup_{v \in V \setminus \{0\}} \log \frac{\|v\|'}{\|v\|} \geq m := \max_i \log \frac{\|e_i\|'}{\|e_i\|}.$$ 

Consider conversely $v = \sum_i \alpha_i e_i$ with $\alpha \in K^N$. Then

$$\|v\|^2 = \sum_i |\alpha_i|^2 \|e_i\|^2 \leq e^{2m} \sum_i |\alpha_i|^2 \|e_i\|^2 = e^{2m} \|v\|^2.$$ 

By symmetry, this shows that $e^{-m} \lVert \cdot \rVert \leq \lVert \cdot \rVert' \leq e^m \lVert \cdot \rVert$, i.e. $d_\infty(\lVert \cdot \rVert, \lVert \cdot \rVert') \leq m$. In the non-Archimedean case the result follows from Lemma 1.16 below.

**Lemma 1.16.** Assume that $K$ is non-Archimedean. Let $\lVert \cdot \rVert$ be a diagonalizable norm with orthogonal basis $(e_i)$, and $\lVert \cdot \rVert'$ be any ultrametric seminorm on $V$. Then

$$\sup_{v \in V \setminus \{0\}} \frac{\|v\|'}{\|v\|} = \max_i \frac{\|e_i\|'}{\|e_i\|}.$$ 

**Proof.** As above, we trivially have $\sup_{v \in V \setminus \{0\}} \|v\|'/\|v\| \geq m := \max_i \|e_i\|'/\|e_i\|$, and $v = \sum_i \alpha_i e_i$ satisfies $\|v\|' \leq \max_i |\alpha_i| \|e_i\|' \leq m \max_i |\alpha_i| \|e_i\| = m \|v\|.$

We now discuss in more detail the structure of the set $N^{\text{diag}}(V)$ of diagonalizable norms. To each basis $e = (e_i)$ of $V$ is associated an injective map

$$\iota_e : \mathbb{R}^N \hookrightarrow N^{\text{diag}}(V),$$

which takes $\lambda \in \mathbb{R}^N$ to the unique norm $\lVert \cdot \rVert_{e,\lambda}$ that is diagonalized in $(e_i)$ and such that $\|e_i\|_{e,\lambda} = e^{-\lambda_i}$. The image

$$A_e := \iota_e(\mathbb{R}^N) \subset N^{\text{diag}}(V)$$

is thus the set of norms that are diagonalized in the given basis $e$, and is called an *apartment* (or *flat*) of $N^{\text{diag}}(V)$. By definition, $N^{\text{diag}}(V) = \bigcup_e A_e$, and the Goldman-Iwahori metric $d_\infty$ can then be characterized as follows.

**Proposition 1.17.** The restriction of $d_\infty$ to $N^{\text{diag}}(V)$ is the unique metric such that each $\iota_e : \mathbb{R}^N \hookrightarrow N^{\text{diag}}(V)$ is an isometric embedding with respect to the $\ell^\infty$-norm on $\mathbb{R}^N$.

**Proof.** Each $\iota_e$ is an isometric embedding by Lemma 1.15, and uniqueness follows from the fact that any two points of $N^{\text{diag}}(V)$ belong to the image of some $\iota_e$, by codiagonalization.

This picture will be generalized to any symmetric norm on $\mathbb{R}^N$ (and in particular to the $\ell^2$-norm) in Section 3, leading to the description of $N^{\text{diag}}(V)$ as a Riemannian symmetric space/Euclidian building. In the present setting, the general construction of retractions onto an apartment in building theory specializes as follows (compare [Ger81]).
Definition 1.18. Let $e = (e_i)$ be a basis of $V$, with apartment $\mathcal{A}_e = \iota_e(\mathbb{R}^N) \subset \mathcal{N}^{\text{diag}}(V)$. The Gram-Schmidt projection $\rho_e : \mathcal{N}(V) \to \mathcal{A}_e$ is defined by sending a norm $\| \cdot \|$ to the unique norm $\| \cdot \|_e$ that is diagonalized in $e$ and such that

$$\|e_i\|_e = \inf_{\alpha \in K_N} \left\| e_i + \sum_{j<i} \alpha_j e_j \right\|$$

for $i = 1, \ldots, N$.

Setting $W_i := \text{Vect}(e_1, \ldots, e_i)$ defines a complete flag $W_\bullet$, and $\| \cdot \|$ induces a subquotient norm on each graded piece $W_i/W_{i-1}$, and hence a diagonalizable norm on the graded object $\text{Gr} V = \bigoplus_{1 \leq i \leq N} W_i/W_{i-1}$. The norm $\| \cdot \|_e$ can then be described as the corresponding norm on $V$ under the isomorphism $V \simeq \text{Gr} V$ defined by $(e_i)$. It is straightforward to see that $\rho_e : \mathcal{N}(V) \to \mathcal{A}_e$ is a retraction, i.e. restricts to the identity on $\mathcal{A}_e$.

The chosen terminology comes from the Archimedean case, where the Gram-Schmidt orthogonalization process associates to a Euclidian/Hermitian norm $\| \cdot \|$ and a basis $(e_i)$ the orthogonal basis $(e'_i)$ obtained by projection of each $e_i$ orthogonal to $W_{i-1}$, which satisfies $\|e_i\|_e = \|e'_i\|$.

1.4. Approximation by diagonalizable norms. The goal of this section is to study the closure in $\mathcal{N}(V)$ of the set $\mathcal{N}^{\text{diag}}(V)$ of diagonalizable norms.

Theorem 1.19. The space of diagonalizable norms $\mathcal{N}^{\text{diag}}(V)$ satisfies the following properties.

(i) Each norm $\| \cdot \| \in \mathcal{N}(V)$ is at distance at most $\log N$ of $\mathcal{N}^{\text{diag}}(V)$.

(ii) If $K$ is Archimedean, $\mathcal{N}^{\text{diag}}(V)$ is closed in $\mathcal{N}(V)$.

(iii) If $K$ is non-Archimedean, the closure of $\mathcal{N}^{\text{diag}}(V)$ in $\mathcal{N}(V)$ coincides with the set $\mathcal{N}^{\text{ultr}}(V)$ of ultrametric norms.

In the Archimedean case, (i) in Theorem 1.19 can be deduced from the John ellipsoid theorem; one can also use the simpler Auerbach lemma, whose proof will be basically repeated below. (ii) follows from the fact that Euclidian/Hermitian norms are characterized by the parallelogram law

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Finally, the first part of (iii) is equivalent to the existence of $\alpha$-cartesian bases in the sense of [BGR] 2.6.1/3, which will be recovered below by imitating the Auerbach argument.

Denote by $V^\vee$ the dual of $V$, and by $\det V^\vee = \bigwedge^N V^\vee$ its determinant line. Viewing an element $\omega \in \det V^\vee$ as a multilinear form on $V$, we define its operator norm as

$$\|\omega\|_{\text{op}} := \sup_{(v_1, \ldots, v_N) \in V^N} \left| \frac{\omega(v_1, \ldots, v_N)}{\|v_1\| \cdots \|v_N\|} \right|.$$

This supremum is indeed finite by equivalence of norms.

Lemma 1.20. Let $\| \cdot \|$ be a norm on $V$, and pick a non-zero $\omega \in \det V^\vee$. For each basis $(e_i)$ of $V$, we then have

$$\max_i \|\alpha_i e_i\| \leq \left( \frac{\|\omega\|_{\text{op}} \|e_1\| \cdots \|e_N\|}{\|\omega(e_1, \ldots, e_N)\|} \right) \|\sum_i \alpha_i e_i\|.$$
for all \( \alpha \in K^N \).

**Proof.** The dual basis \( (e_i^\vee) \) satisfies

\[
\langle e_i^\vee, v \rangle = \frac{\omega(e_1, \ldots, e_{i-1}, v, e_{i+1}, \ldots, e_N)}{\omega(e_1, \ldots, e_N)} ,
\]

and hence

\[
\max_i \|\langle e_i^\vee, v \rangle\|e_i\| \leq \left( \frac{\|\omega\|_{op}\|e_1\| \cdots \|e_N\|}{\omega(e_1, \ldots, e_N)} \right) \|v\|,
\]

which is equivalent to the desired result. \( \square \)

**Proof of Theorem 1.19** Assume first that \( K \) is Archimedean. As noted above, (ii) follows from the characterization of diagonalizable norms in terms of the parallelogram law. Let \( \|\cdot\| \) be any norm on \( V \), and fix a nonzero determinant \( \omega \in \det V^\vee \). By compactness, we may choose a basis \( (e_i) \) of \( V \) with \( \|e_1\| = 1 \) and \( \|\omega\|_{op} = |\omega(e_1, \ldots, e_N)| \). For each \( p \in [1, \infty] \), denote by \( \|\cdot\|_p \) the \( \ell^p \)-norm in the basis \( (e_i) \). Lemma 1.20 and the triangle inequality yield \( \|\cdot\|_{\infty} \leq \|\cdot\| \leq \|\cdot\|_1 \). Since \( N^{-1/2} \|\cdot\|_1 \leq \|\cdot\|_2 \leq N^{1/2} \|\cdot\|_\infty \), it follows that \( N^{-1/2} \|\cdot\|_2 \leq \|\cdot\| \leq N^{1/2} \|\cdot\|_2 \), and hence \( d_\infty(\|\cdot\|, \|\cdot\|_2) \leq \log N \), which proves (i) in the Archimedean case since \( \|\cdot\|_2 \in \mathcal{N}^{\text{diag}}(V) \).

Assume now that \( K \) is non-Archimedean, and pick any norm \( \|\cdot\| \in \mathcal{N}(V) \), not necessarily ultrametric at this point. For any \( \varepsilon > 0 \), there exists a basis \( (e_i) \) such that

\[
\|\omega\|_{op} \leq (1 + \varepsilon) \left| \omega(e_1, \ldots, e_N) \right| \left\| e_1 \right\| \cdots \left\| e_N \right\|,
\]

and Lemma 1.20 yields

\[
\max_i \|\alpha_i e_i\| \leq (1 + \varepsilon) \left\| e_1 \right\| \cdots \left\| e_N \right\| \sum_i \alpha_i e_i \tag{1.2}
\]

for all \( \alpha \in K^N \). Since we trivially have

\[
\left\| \sum_i \alpha_i e_i \right\| \leq \sum_i \|\alpha_i e_i\| \leq N \max_i \|\alpha_i e_i\| ,
\]

it follows that the diagonalizable norm \( \|\cdot\|' \in \mathcal{N}^{\text{diag}}(V) \) defined by \( \|\sum_i \alpha_i e_i\|' := \max_i \|\alpha_i\|\|e_i\| \) satisfies

\[
d_\infty \left( \|\cdot\|, \|\cdot\|' \right) \leq \max\{\log N, \log(1 + \varepsilon)\} ,
\]

which proves the non-Archimedean case of (i) since \( \varepsilon > 0 \) was arbitrary. Finally, if \( \|\cdot\| \) is assumed to be ultrametric, then (1.2) implies \( (1 - \varepsilon)\|\cdot\|' \leq \|\cdot\| \leq \|\cdot\|' \), hence \( d_\infty \left( \|\cdot\|, \|\cdot\|' \right) \leq \log(1 + \varepsilon) \), and \( \|\cdot\| \) is thus in the closure of \( \mathcal{N}^{\text{diag}}(V) \). \( \square \)

1.5. **Duality.** To each norm \( \|\cdot\| \) on \( V \) is associated a dual norm \( \|\cdot\|' \) on the dual vector space \( V^\vee \), defined by the usual formula

\[
\|\mu\|' = \sup_{v \in V \setminus \{0\}} \frac{\langle \mu, v \rangle}{\|v\|} .
\]

Again, the supremum is finite by equivalence of norms, and it is straightforward to see that the duality map \( \mathcal{N}(V) \rightarrow \mathcal{N}(V^\vee) \) so defined is 1-Lipschitz continuous with respect to the \( d_\infty \) metric. Diagonalizable norms are preserved by duality:
**Lemma 1.21.** If $\| \cdot \| \in \mathcal{N}^{\text{diag}}(V)$ is diagonalizable, then so is $\| \cdot \|^\vee$. Further, if $(e_i)$ is an orthogonal basis for $\| \cdot \|$, then the dual basis $(e_i^\vee)$ is orthogonal for $\| \cdot \|^\vee$, and $\| e_i \|^\vee = \| e_i \|^{-1}$.

**Proof.** In the Archimedean case, $\| \cdot \|$ is Euclidian/Hermitian, and the result is well-known. In the non-Archimedean case, the result is a simple consequence of Lemma 1.16. 

**Lemma 1.22.** If $W \subset V$ is a linear subspace, the canonical embedding $(V/W)^\vee \hookrightarrow V^\vee$ identifies the dual of the quotient norm $\| \cdot \|_{V/W}$ with the restriction of $\| \cdot \|^\vee$.

**Proof.** The image of $(V/W)^\vee$ in $V^\vee$ is the space $W^\perp$ of linear forms $\mu \in V^\vee$ that vanish on $W$. Denoting by $\bar{v} \in V/W$ the image of $v \in V$ and by $\tilde{\mu} \in V^\vee$ the image of $\mu \in (V/W)^\vee$, we have by definition

$$\| \tilde{\mu} \|^\vee = \sup_{v \in V \setminus \{0\}} \frac{|\langle \tilde{\mu}, v \rangle|}{\| v \|},$$

and

$$\| \mu \|_{V/W}^\vee = \sup_{v \in V-W} \frac{|\langle \mu, v \rangle|}{\| v \|_{V/W}}.$$

Since $\| \tilde{v} \|_{V/W} = \inf_{w \in W} \| v + w \| \leq \| v \|$ and $\langle \tilde{\mu}, v \rangle = 0$ for $v \in W$, we trivially have $\| \tilde{\mu} \|^\vee \leq \| \mu \|_{V/W}^\vee$. Conversely, we have for each $v \in V - W$ and $w \in W$

$$\frac{|\langle \mu, v \rangle|}{\| v + w \|} = \frac{|\langle \tilde{\mu}, v + w \rangle|}{\| v + w \|} \leq \| \tilde{\mu} \|^\vee,$$

hence

$$\frac{|\langle \tilde{\mu}, v \rangle|}{\| \tilde{v} \|_{V/W}} = \sup_{w \in W} \frac{|\langle \tilde{\mu}, v + w \rangle|}{\| v + w \|} \leq \| \tilde{\mu} \|^\vee,$$

and taking the sup over $v$ yields the inequality in the other direction $\| \mu \|_{V/W}^\vee \leq \| \tilde{\mu} \|^\vee$ and we conclude. 

As is well-known, biduality holds in the Archimedean case, as a consequence of the Hahn-Banach theorem:

**Proposition 1.23.** If $K$ is Archimedean, the duality map $\mathcal{N}(V) \rightarrow \mathcal{N}(V^\vee)$ is an involutive isometry.

In the non-Archimedean case, biduality fails, for the simple reason that the dual of any norm is automatically ultrametric. This is however the only obstruction (compare for instance [CMor15, Corollary 1.7]):

**Proposition 1.24.** Assume that $K$ is non-Archimedean. The bidual $\| \cdot \|^{\vee \vee}$ of a norm $\| \cdot \| \in \mathcal{N}(V)$ is the largest ultrametric norm such that $\| \cdot \|^\vee \leq \| \cdot \|$. In particular, $\| \cdot \|^\vee \vee = \| \cdot \|$ if and only if $\| \cdot \|$ is ultrametric, and the duality map $\| \cdot \| \mapsto \| \cdot \|^\vee$ therefore restricts to an involutive isometry $\mathcal{N}^{\text{ultr}}(V) \simeq \mathcal{N}^{\text{ultr}}(V^\vee)$.

**Proof.** By definition of the dual norm, we have $|\langle \mu, v \rangle| \leq \| \mu \|^\vee \| v \|$ for all $v \in V$, $\mu \in V^\vee$, and hence $(\| \cdot \|^\vee)^\vee \leq \| \cdot \|$. Since $\| \cdot \| \mapsto \| \cdot \|^\vee$ is non-decreasing, it is thus enough to show that any ultrametric norm $\| \cdot \|$ satisfies $\| \cdot \|^\vee = \| \cdot \|$. By density of diagonalizable norms in $\mathcal{N}^{\text{ultr}}(V)$ (Theorem 1.19), we may assume that $\| \cdot \|$ is diagonalizable, in which case the result follows from Lemma 1.21.

We are now in a position to prove the codiagonalization result promised in Proposition 1.14.
Proof of Proposition 1.14. That any two diagonalizable norms \( \| \cdot \|, \| \cdot \|' \in N^{\text{diag}}(V) \) are codiagonalizable is a standard fact in the Archimedean case, and we henceforth assume that \( K \) is non-Archimedean. Our argument extends the classical one of [GI], which treats the case of a local field, following a suggestion of Marco Maculan, whom we warmly thank. Recall that a direct sum decomposition \( V = V_1 \oplus \cdots \oplus V_r \) is orthogonal for \( \| \cdot \| \) if

\[
\left\| \sum_i v_i \right\| = \max_i \|v_i\|
\]

for all \( v_i \in V_i \). Given \( v \in V \) and a linear form \( \mu \in V^\vee \) with \( \langle \mu, v \rangle \neq 0 \), it is straightforward to check that the decomposition \( V = K v \oplus \ker \mu \) is orthogonal for \( \| \cdot \| \) if and only if

\[
\frac{|\langle \mu, w \rangle|}{|\langle \mu, v \rangle|} \leq \frac{\|w\|'}{\|v\|'}
\]

for all \( w \in V \). Arguing by induction on \( \dim V \), we will thus be done if we prove the existence of \( v \in V \) and \( \mu \in V^\vee \) with \( \langle \mu, v \rangle \neq 0 \) such that

\[
|\langle \mu, w \rangle| |\langle \mu, v \rangle| \leq \|w\| \|v\| \leq \|w\|' \|v\|' (1.3)
\]

for all \( w \in V \), since \( V = K v \oplus \ker \mu \) will then be orthogonal for both \( \| \cdot \| \) and \( \| \cdot \|' \). Let \( (e'_i) \) be an orthogonal basis for \( \| \cdot \|' \). By Lemma 1.16 we have

\[
\sup_{w \in W \setminus \{0\}} \frac{\|w\|}{\|w\|'} = \frac{\|e'_i\|}{\|e'_i\|'}
\]

for some \( i \), and \( v := e'_i \) therefore satisfies \( \|w\|/\|v\| \leq \|w\|'/\|v\|' \) for all \( w \in V \). Let now \( (e'_j) \) be an orthogonal basis for \( \| \cdot \|' \). Since the dual basis \( (e^\vee_j) \) is orthogonal for the dual norm \( \| \cdot \|^\vee \), we similarly get

\[
\|v\| = \sup_{\mu \in V^\vee \setminus \{0\}} \frac{|\langle \mu, v \rangle|}{\|\mu\|} = \frac{|\langle e^\vee_j, v \rangle|}{\|e^\vee_j\|}
\]

for some \( j \). It follows that \( \mu := e^\vee_j \) satisfies

\[
|\langle \mu, v \rangle| \|\mu\| \|v\| \geq \frac{|\langle \mu, w \rangle|}{\|w\|} \|v\|
\]

for all \( w \in V \setminus \{0\} \), and (1.3) follows. \( \square \)

1.6. Ground field extension. In this section, \( K \) is assumed to be non-Archimedean. In this context, the tensor product of two ultrametric seminorms \( \| \cdot \|, \| \cdot \|' \) on (possibly infinite dimensional) \( K \)-vector spaces \( U, U' \) is the seminorm \( \| \cdot \|_{U \otimes U'} \) defined by setting for each \( w \in U \otimes U' \)

\[
\|w\|_{U \otimes U'} = \inf_{w = \sum_i u_i \otimes u'_i} \max_i \|u_i\| \|u'_i\|',
\]

the infimum ranging over all finite sum decompositions \( w = \sum_i u_i \otimes u'_i \) with \( u_i \in U, u'_i \in U' \). As a special case of this construction, we have:
Definition 1.25. Assume that $K$ is non-Archimedean, and let $K'/K$ be a complete field extension. The ground field extension of an ultrametric seminorm $\| \cdot \|$ on $V$ is the ultrametric seminorm $\| \cdot \|_{K'}$ on $V_{K'} = V \otimes K'$ defined by setting for each $v' \in V_{K'}$

$$\|v'\|_{K'} = \inf_{v'=\sum \beta_i v_i} \max |\beta_i|\|v_i\|,$$

the infimum ranging over all decompositions $v' = \sum \beta_i v_i$ with $\beta_i \in K'$ and $v_i \in V$.

Our notation identifies for simplicity $V$ with its image in $V_{K'}$ under $v \mapsto v \otimes 1$.

Proposition 1.26. Let $K'/K$ be a complete field extension.

(i) For each ultrametric norm $\| \cdot \|$ on $V$, $\| \cdot \|_{K'}$ is an ultrametric norm on $V_{K'}$ which coincides with $\| \cdot \|$ on $V$.

(ii) The map $\| \cdot \| \mapsto \| \cdot \|_{K'}$ is an isometric embedding $\mathcal{N}^{\text{ultr}}(V) \hookrightarrow \mathcal{N}^{\text{ultr}}(V_{K'})$.

(iii) For all $v' \in V_{K'}$ we have

$$\|v'\|_{K'} = \sup_{\mu \in V\setminus\{0\}} \frac{|\langle \mu \otimes 1, v' \rangle|}{\|\mu\|_{K'}^\vee}.$$

This last equality shows the compatibility of our definition with [CMor15, Definition 1.8].

Lemma 1.27. If $\| \cdot \|$ is a diagonalizable norm with orthogonal basis $(e_i)$, then $\| \cdot \|_{K'}$ is also a diagonalizable norm with orthogonal basis $(e_i)$ (viewed as a basis of $V_{K'}$), and $\|e_i\|_{K'} = \|e_i\|$.

Proof. Write $v' = \sum_j \alpha'_j e_j$ with $\alpha'_j \in K'$. By definition, we have

$$\|v'\|_{K'} \leq \max_i |\alpha'_i|\|e_i\|.$$

Conversely, given a decomposition $v' = \sum_i \beta_i v_i$ with $\beta_i \in K'$ and $v_i \in V$, we have to show that

$$\max_j |\alpha'_j|\|e_j\| \leq \max_i |\beta_i|\|v_i\|.$$

Writing $v_i = \sum_j \alpha_{ij} e_j$ with $\alpha_{ij} \in K$, we have for each $j \alpha'_j = \sum_i \beta_i \alpha_{ij}$, and hence

$$|\alpha'_j|\|e_j\| \leq \max_i (|\beta_i||\alpha_{ij}|)\|e_j\| \leq \max_i (|\beta_i|\max_k |\alpha_{ik}|\|e_k\|) = \max_i |\beta_i|\|v_i\|.$$

Proof of Proposition 1.26. Assume first that $\| \cdot \|$ is diagonalizable. By Lemma 1.27 $\| \cdot \|_{K'}$ is an ultrametric norm, and it coincides with $\| \cdot \|$ on $V$. Denote by $\|v'\|'$ the right-hand supremum in (iii). Pick a nonzero $\mu \in V'$, and consider a decomposition $v' = \sum_i \beta_i v_i$ with $\beta_i \in K'$ and $v_i \in V$. Then

$$\frac{|\langle \mu \otimes 1, v' \rangle|}{\|\mu\|_{V'}^\vee} \leq \max_i |\beta_i| \frac{|\langle \mu, v_i \rangle|}{\|\mu\|_{V'}^\vee} \leq \max_i |\beta_i| \|v_i\|.$$

Taking the infimum over all decompositions and the supremum over $\mu$ yields $\|v'\|' \leq \|v'\|_{K'}$.

Conversely, let $(e_i)$ be an orthogonal basis for $\| \cdot \|$, write $v' = \sum_i \alpha'_i e_i$, and pick an index $i$ achieving $\|v'\|_{K'} = \max_i |\alpha'_i|\|e_i\|$. By Lemma 1.21 $\mu := e_i'$ satisfies

$$\frac{|\langle \mu \otimes 1, v' \rangle|}{\|e_i'\|_{V'}^\vee} = |\alpha'_i|\|e_i\| = \|v'\|_{K'},$$
which proves (iii) for \( \| \cdot \| \). In the general case, it is straightforward to see that \( \| \cdot \| \mapsto \| \cdot \|_{K'} \) and \( \| \cdot \| \mapsto \| \cdot \|' \) are both 1-Lipschitz continuous, and we conclude by density of diagonalizable norms in \( \mathcal{N}^{\text{ultr}}(V) \) (Theorem 1.19). \( \square \)

For later use we finally quote the next result from [CMor15, Lemma 1.12 & p.29].

**Lemma 1.28.** Assume that \( K \) is trivially valued, and let \((V_{\alpha}, \| \cdot \|_{\alpha})_{\alpha \in A} \) be an at most countable family of ultrametric norms on finite dimensional \( K \)-vector spaces. We may then find a nontrivially valued complete field extension \( K'/K \) such that for each \( \alpha \in A \), \( \| \cdot \|_{\alpha,K'} \) is the only ultrametric norm on \( V_{\alpha,K'} \) that coincides with \( \| \cdot \|_{\alpha} \) on \( V_{\alpha} \).

1.7. **Lattice norms.** In this section, \( K \) is non-Archimedean, with associated real-valued valuation \( v_K = -\log |\cdot| \). The valuation ring \( K^\circ \) is a Prüfer domain, i.e. every finitely generated ideal of \( K^\circ \) is principal. This implies that a \( K^\circ \)-module \( M \) is flat if and only if it is torsion-free. If \( M \) is further finitely generated, then it is free (since \( K^\circ \) is local).

**Definition 1.29.** A lattice\(^4\) \( \mathcal{V} \) of \( V \) is a finite \( K^\circ \)-submodule \( \mathcal{V} \) of \( V \) such that \( \mathcal{V} \otimes_{K^\circ} K = V \).

A lattice is thus of the form \( \mathcal{V} = \sum_i K^\circ e_i \) with \((e_i)\) a basis of \( V \), and \( \text{GL}(V) \) therefore acts transitively on the set of lattices. A lattice \( \mathcal{V} \) determines a lattice norm \( \| \cdot \|_{\mathcal{V}} \) on \( V \), by setting
\[
\|v\|_{\mathcal{V}} := \inf \{ |\alpha| \mid \alpha \in K, v \in \alpha \mathcal{V} \}.
\]

**Lemma 1.30.** Let \((e_i)\) be a \( K^\circ \)-basis of a lattice \( \mathcal{V} \) of \( V \). Then \((e_i)\) is an orthonormal basis of \( \| \cdot \|_{\mathcal{V}} \). In particular, \( \mathcal{V} \) coincides with the unit ball of \( \| \cdot \|_{\mathcal{V}} \).

**Proof.** Pick \( v \in V \), and write \( v = \sum_i \alpha_i e_i \) with \( \alpha \in K^N \). Given \( \alpha \in K \), we then have \( v \in \alpha \mathcal{V} \) if and only if \( |\alpha_i| \leq |\alpha| \), and hence \( \|v\|_{\mathcal{V}} = \max_i |\alpha_i| \). This means that \((e_i)\) is orthonormal for \( \| \cdot \|_{\mathcal{V}} \), and also implies that \( \mathcal{V} \) is the unit ball of \( \| \cdot \|_{\mathcal{V}} \). \( \square \)

**Lemma 1.31.** Denote by \( \mathcal{N}^{\text{latt}}(V) \subset \mathcal{N}^{\text{diag}}(V) \) the set of lattice norms.

(i) A norm is a lattice norm if and only if it is a pure diagonalizable norm, i.e. it admits an orthonormal basis.

(ii) If \( K \) is discretely valued, \( \mathcal{N}^{\text{latt}}(V) \) is discrete and closed in \( \mathcal{N}^{\text{ultr}}(V) = \mathcal{N}^{\text{diag}}(V) \). If \( K \) is non-trivially valued with uniformizing parameter \( \pi_K \), the closed unit ball \( B \) of any ultrametric norm \( \| \cdot \| \) is a lattice, and the associated lattice norm \( \| \cdot \|_B \) satisfies
\[
d_{\infty}(\| \cdot \|, \| \cdot \|_B) \leq v_K(\pi_K).
\]

(iii) If \( K \) is densely valued, \( \mathcal{N}^{\text{latt}}(V) \) is dense in \( \mathcal{N}^{\text{diag}}(V) \), and hence also in \( \mathcal{N}(V)^{\text{ultr}} \). Further, the unit ball \( B \) of a norm \( \| \cdot \| \) is a lattice if and only \( \| \cdot \| \) is a lattice norm.

(iv) Let \( \| \cdot \| \) be the lattice norm determined by a lattice \( \mathcal{V} \) of \( V \). For each complete field extension \( K'/K \), the induced norm \( \| \cdot \|_{K'} \) on \( V_{K'} \) is the lattice norm determined by the lattice \( \mathcal{V}_{K^\circ} \).

**Proof.** (i) is a direct consequence of Lemma 1.30. Assume that \( K \) is discretely valued. Let \( \| \cdot \|_{\mathcal{V}} \neq \| \cdot \|'_{\mathcal{V}} \) be two distinct lattice norms, and pick a joint orthogonal basis \((e_i)\) as in

\(^4\)In the densely valued case, the present notion of lattice is more restrictive than the one used in [CMor15, §1.3.3].
Proposition 1.14  We then have $\mathcal{V} = \sum_i K^0 \pi_K^{m_i} e_i$ and $\mathcal{V}' = \sum_i K^0 \pi_K'^{m'_i} e_i$ for some integers $m_i, m'_i \in \mathbb{Z}$, and hence
\[ d_\infty(\| \cdot \|_{\mathcal{V}}, \| \cdot \|_{\mathcal{V}'}) = \max_i \left| \log \frac{\|e_i\|_{\mathcal{V}}}{\|e_i\|_{\mathcal{V}'}} \right| = v_K(\pi_K) \max_i |m_i - m'_i| \geq v_K(\pi_K). \]

This shows that $\mathcal{N}_{\text{latt}}(V)$ is discrete and closed. Next, let $\| \cdot \|$ be any diagonalizable norm, pick an orthogonal basis $(e_i)$ for $\| \cdot \|$, and write a given $v \in V$ as $v = \sum_i u_i \pi_K^{n_i} e_i$ with $n_i \in \mathbb{Z}$ and $u_i$ a unit. Then $\|v\| \leq 1$ if and only if $\|\pi_K^{n_i}\|e_i\| \leq 1$ for all $i$, and we infer $B = \sum_i K^0 \pi_K^{m_i} e_i$ with $m_i := \lfloor \log \|e_i\|/v_K(\pi_K) \rfloor$. In particular, $B$ is a lattice with basis $(\pi_K^{m_i} e_i)$, and hence
\[ d_\infty(\| \cdot \|, \| \cdot \|_B) = \max_i \left| \log \frac{\|e_i\|_B}{\|e_i\|} \right| = \max_i |m_i v_K(\pi_K) - \log \|e_i\|| \leq v_K(\pi_K). \]

Assume finally that $K$ is densely valued. That $\mathcal{N}_{\text{latt}}(V)$ is dense in $\mathcal{N}_{\text{diag}}(V)$ is easily seen by approximating the values of a given diagonalizable norm $\| \cdot \|$ on an orthogonal basis $(e_i)$ by elements of the dense subset $|K|$ of $\mathbb{R}_+$. Similarly, any norm $\| \cdot \|$ is determined by its closed unit ball $B$, via
\[ \|v\| = \inf \{ |\alpha| \mid v \in \alpha B \text{ with } \alpha \in K \}. \]

As a result, $\| \cdot \|$ is a lattice norm if and only if $B$ is a lattice. Finally, (iv) is a direct consequence of Lemma 1.30 and Lemma 1.27. \qed

Example 1.32. Let $L/K$ be a finite extension, with induced absolute value $| \cdot |_L$. When $K$ is discretely valued, the unit ball $L^0$ is always finite over $K^0$. In the densely valued case, (iii) shows that $L^0$ is finite over $K^0$ only if $| \cdot |_L$ admits a $K$-orthonormal basis, which implies in particular that $|L| = |K|$, i.e. $L/K$ is unramified.

2. Determinants and successive minima

The goal of this section is to investigate induced norms on the determinant line, leading to the notion of relative volume of two norms. We relate the latter to successive minima via a Minkowski-type theorem, and then apply the results of [BC] to obtain a general convergence result for the relative volumes of certain sequences of norms on the graded pieces of a graded algebra.

In this section, $V$ still denotes a finite dimensional vector space over a field $K$, complete with respect to an absolute value, and we set as before $N := \dim V$.

2.1. The determinant of a norm. The determinant line of $V$ is $\det V := \Lambda^N V$. We have a natural isomorphism $\det(V^\vee) \simeq (\det V)^\vee$, induced by the pairing
\[ \langle v_1 \wedge \cdots \wedge v_N, \mu_1 \wedge \cdots \wedge \mu_N \rangle = \det \langle (v_i, \mu_j) \rangle. \]

In particular, if $(e_i)$ is basis of $V$ with dual basis $(e_i^\vee)$, then
\[ (e_1 \wedge \cdots \wedge e_N)^\vee = e_1^\vee \wedge \cdots \wedge e_N^\vee. \]

Definition 2.1. To each norm $\| \cdot \|$ on $V$, we associate a norm $\det \| \cdot \|$ on $\det V$ by setting for $\tau \in \det V$
\[ \det \|\tau\| = \inf_{\tau = v_1 \wedge \cdots \wedge v_N} \prod_i \|v_i\|, \]
where the infimum runs over all decompositions $\tau = v_1 \wedge \cdots \wedge v_N$ with $v_i \in V$.

By construction, $\det \| \cdot \|$ is the largest seminorm on $\det V$ with the submultiplicativity property
\[
\det \| v_1 \wedge \cdots \wedge v_N \| \leq \prod_i \| v_i \| \tag{2.1}
\]
for all $v_i \in V$. That it is actually a norm follows from the next result, which is readily checked.

**Lemma 2.2.** If we view the dual $\tau^V \in \det V^\vee$ of a nonzero $\tau \in \det V$ as a multilinear form on $V$, then $(\det \| \tau \|)^{-1}$ coincides with the operator norm
\[
\| \tau^V \|_{\text{op}} := \sup_{v_1, \ldots, v_N \in V} \frac{|\tau^V(v_1, \ldots, v_N)|}{\| v_1 \| \cdots \| v_N \|}.
\]

**Lemma 2.3.** The map $\det : \mathcal{N}(V) \to \mathcal{N}(\det V)$ is $N$-Lipschitz continuous with respect to $d_\infty$-metrics.

**Proof.** Given $\| \cdot \|, \| \cdot \|' \in \mathcal{N}(V)$, $C := \exp d_\infty(\| \cdot \|, \| \cdot \|')$ satisfies
\[
C^{-N} \det \| \cdot \| \leq \det \| \cdot \|' \leq C^N \det \| \cdot \|,
\]
i.e. $d_\infty(\| \cdot \|, \| \cdot \|') \leq N \log C = N d_\infty(\| \cdot \|, \| \cdot \|')$. \hfill \Box

Computing the determinant of a norm is typically a hard problem. As an illustration we show:

**Lemma 2.4.** Assume that $K$ is Archimedean. Pick $p \in [1, \infty]$, and denote by $\| \cdot \|_p$ the $\ell^p$-norm in a given basis $(e_i)$. Then $\det \| e_1 \wedge \cdots \wedge e_N \|_p = 1$ for $p \in [1, 2]$, and $\det \| e_1 \wedge \cdots \wedge e_N \|_p < 1$ for $p > 2$.

**Proof.** Set $\tau := e_1 \wedge \cdots \wedge e_N$. After identifying $V$ with $K^N$, Lemma 2.2 yields
\[
(\det \| e_1 \wedge \cdots \wedge e_N \|_p)^{-1} = \sup_{v_i \neq 0} \frac{|\det(v_1, \ldots, v_N)|}{\| v_1 \|_p \cdots \| v_N \|_p}.
\]

For $p = 2$, the classical Hadamard inequality states that
\[
|\det(v_1, \ldots, v_N)| \leq \prod_i \| v_i \|_2
\]
for all $v_i$, i.e. $\det \| e_1 \wedge \cdots \wedge e_N \|_2 = 1$. For $p \in [1, 2]$, we have $\| \cdot \|_2 \leq \| \cdot \|_p$, and hence $\det \| \cdot \|_2 \leq \det \| \cdot \|_p$, and we infer $\det \| e_1 \wedge \cdots \wedge e_N \|_p = 1$. Assume now $p > 2$. The basis $(v_i)$ defined by $v_1 = e_1 + e_2$, $v_2 = -e_1 + e_2$ and $v_i = e_i$ for $i > 2$ satisfies $\det(v_1, \ldots, v_N) = 2$, $\| v_1 \|_p = \| v_2 \|_p = 2^{1/p}$ and $\| v_i \|_p = 1$ for $i > 2$, which yields
\[
\det \| e_1 \wedge \cdots \wedge e_N \|_p \leq 2^{2/p - 1} < 1.
\]
\hfill \Box
Example 2.5. For $p = \infty$, determining the precise value of $\det \|e_1 \wedge \cdots \wedge e_N\|_{\infty}$ amounts to maximizing the determinant of a $N \times N$-matrix with entries in $\{\pm 1\}$, and is known as the Hadamard maximal determinant problem. By [CL65], we have for instance

$$N^{-\frac{N}{2}} \leq \det \|e_1 \wedge \cdots \wedge e_N\|_{\infty} \leq N^{-\frac{N}{2}} \left(1 - \frac{\log(4/3)}{\log N}\right).$$

The lower bound is achieved if and only if there exists an $\ell^2$-orthogonal basis with entries in $\{\pm 1\}$ (which implies that $N$ is a multiple of 4), but the exact value $\det \|e_1 \wedge \cdots \wedge e_N\|_{\infty}$ is unknown in the general case.

2.2. Determinants of diagonalizable norms. As we shall see in this section, the determinant of a diagonalizable norm is very well-behaved. By density, this will also be the case for all ultrametric norms, in the non-Archimedean case.

Lemma 2.6. If $\| \cdot \|$ is diagonalizable, then a basis $(e_i)$ of $V$ satisfies

$$\det \|e_1 \wedge \cdots \wedge e_N\| = \prod_i \|e_i\|$$

if and only if $(e_i)$ is orthogonal for $\| \cdot \|$.

Corollary 2.7. Assume that $K$ is non-Archimedean, and let $V$ be a lattice of $V$. Then $\det \| \cdot \|_V = \| \cdot \|_{\det V}$ is the norm determined by the lattice $\det V := \bigwedge^N V$ of $\det V$.

Proof of Lemma 2.6. When $K$ is Archimedean, the result is easily seen to be equivalent to the classical Hadamard inequality (and the equality case thereof) recalled in Lemma 2.4. Assume now that $K$ is non-Archimedean, and let $(e_i)$ be an orthogonal basis for $\| \cdot \|$. We need to show that each basis $(v_i)$ such that $v_1 \wedge \cdots \wedge v_N = e_1 \wedge \cdots \wedge e_N$ satisfies $\prod_i \|e_i\| \leq \prod_i \|v_i\|$. If we write $v_i = \sum_j \alpha_{ij} e_j$ with $\alpha_{ij} \in K$, then $\det(\alpha_{ij}) = 1$. Expanding out the determinant and using the ultrametric inequality, we get $\prod_i |\alpha_{i\sigma(i)}| \geq 1$ for some permutation $\sigma$. Since $\| \cdot \|$ is diagonalized in $(e_i)$, we have

$$\|v_i\| = \max_j |\alpha_{ij}| \|e_j\| \geq |\alpha_{i\sigma(i)}| \|e_{\sigma(i)}\|,$$

and we obtain as desired

$$\prod_i \|v_i\| \geq \prod_i |\alpha_{i\sigma(i)}| \|e_{\sigma(i)}\| \geq \prod_i \|e_i\|.$$ 

Conversely, any basis $(e_i)$ satisfying $\det \|e_1 \wedge \cdots \wedge e_N\| = \prod_i \|e_i\|$ is orthogonal for $\| \cdot \|$, as a direct consequence of Lemma 1.20.

Lemma 2.8. If $\| \cdot \|$ is a diagonalizable norm on $V$, then $\det (\| \cdot \|_V) = (\det \| \cdot \|)^V$ under the canonical isomorphism $\det (V^V) \simeq (\det V)^V$.

Proof. Let $(e_i)$ be an orthogonal basis for $\| \cdot \|$. By Lemma 1.21, the dual basis $(e_i^V)$ is orthogonal for $\| \cdot \|_V$, and $\|e_i^V\|_V = \|e_i\|^{-1}$. By Lemma 2.6, we infer

$$\det \|e_1^V \wedge \cdots \wedge e_N^V\|_V = \prod_i \|e_i^V\|_V = \left(\prod_i \|e_i\|\right)^{-1} = (\det \|e_1 \wedge \cdots \wedge e_N\|)^{-1},$$

hence the result.

$\square$
Lemma 2.9. Let \( \| \cdot \| \) be a diagonalizable norm on \( V \), and consider an exact sequence of vector spaces

\[
0 \to V' \to V \to V'' \to 0,
\]

with induced norms \( \| \cdot ' \| , \| \cdot '' \| \) on \( V' , V'' \). Under the canonical isomorphism

\[
det V \simeq det V' \otimes det V'' ,
\]

we then have

\[
det \| \cdot \| = det \| \cdot ' \| \otimes det \| \cdot '' \| .
\]

Corollary 2.10. If \( K \) is non-Archimedean, Lemma 2.8 and 2.9 hold for any ultrametric norm.

Proof of Lemma 2.9. Set \( N' := \dim V' , N'' := \dim V'' \), and denote by \( \pi : V \to V'' \) the given surjection. Pick nonzero \( \tau' \in det V' , \tau'' \in det V'' \) and \( \varepsilon > 0 \). Definition 2.1, we may then find \( v' , \ldots , v'_{N'} \in V' \) and \( v''_1 , \ldots , v''_{N''} \in V'' \) such that \( \tau' = v'_1 \wedge \cdots \wedge v'_{N'} , \tau'' = \pi (v''_1) \wedge \cdots \wedge \pi (v''_{N''}), \)

\[
\prod_i \| v'_i \| \leq (1 + \varepsilon ) \det \| \tau' \|
\]

and

\[
\prod_i \| v''_i \| \leq (1 + \varepsilon ) \det \| \tau'' \|
\]

The isomorphism \( det V' \otimes det V'' \simeq det V \) maps \( \tau' \otimes \tau'' \) to

\[
v'_1 \wedge \cdots \wedge v'_{N'} \wedge v''_1 \wedge \cdots \wedge v''_{N''},
\]

which satisfies

\[
det \| \tau' \otimes \tau'' \| = det \| v'_1 \wedge \cdots \wedge v'_{N'} \wedge v''_1 \wedge \cdots \wedge v''_{N''}\| \\
\leq \prod_i \| v'_i \| \prod_i \| v''_i \| \leq (1 + \varepsilon )^2 (\det \| \tau' \| ) (\det \| \tau'' \| ),
\]

hence \( det \| \cdot \| \leq det \| \cdot ' \| \otimes det \| \cdot '' \| \). By Lemma 1.22 we dually have

\[
det (\| \cdot \| ) \leq det (\| \cdot ' \| ) \otimes det (\| \cdot '' \| ).
\]

Since \( \| \cdot \| \) is diagonalizable, so are \( \| \cdot ' \| \) and \( \| \cdot '' \| \), by Lemma 1.13 By Lemma 2.8 we thus have

\[
det (\| \cdot \| )^{-1} \leq (det (\| \cdot ' \| ))^{-1} \otimes (det (\| \cdot '' \| ))^{-1},
\]

hence the result. \( \square \)

Proof of Corollary 2.10. By Theorem 1.19, diagonalizable norms are dense in the set of ultrametric norms, and we conclude by continuity of det (Lemma 2.3). \( \square \)

In the Archimedean case, both Lemma 2.6 and Lemma 2.8 fail in general for non-diagonalizable norms.

Example 2.11. Assume that \( K \) is Archimedean, and let \( \| \cdot \| \) be the \( \ell^\infty \)-norm on \( K^N \). The dual norm \( \| \cdot \| ' \) is the \( \ell^1 \)-norm, and Lemma 2.4 thus shows that \( \det (\| \cdot \| ) \neq (\det (\| \cdot \| ))' \). Also, the exact sequence \( 0 \to V' \to V \to V'' \to 0 \) with \( V' = Ke_1, V'' = Ke_2 \) shows that \( det \| \cdot \| < (\det (\| \cdot \| )) \otimes (\det (\| \cdot \| ))' \). Recall from Section 1.3 that each basis \( e = (e_i) \) of \( V \) defines an apartment \( A_e = \iota_e (\mathbb{R}^N) \) in \( N^\text{diag} (V) \) and a Gram-Schmidt projection \( \rho_e : N(V) \to A_e \). For later use, we show:

Lemma 2.12. For each diagonalizable norm \( \| \cdot \| \), we have \( \det \| \cdot \| = det \rho_e (\| \cdot \| ) \).
Proof. Denote by $W_i = \text{Vect}(e_1, \ldots, e_i)$ the complete flag defined by $e$. By Lemma 2.9 we have
\[
\det \| \cdot \| = \bigotimes_i \det \| \cdot \|_{W_i/W_{i-1}},
\]
under the identification $\det V \simeq \bigotimes_i \det(W_i/W_{i-1})$. By definition of $\| \cdot \|_e := \rho_e(\| \cdot \|)$, we infer
\[
\det \| e_1 \wedge \cdots \wedge e_N \| = \prod_i \| e_i \|_e = \det \| e_1 \wedge \cdots \wedge e_N \|_e,
\]
where the second equality follows from Lemma 2.6. □

2.3. Relative volume. We introduce the following 'additive version' of [Tem14, 2.4.3].

Definition 2.13. The relative volume of two norms $\| \cdot \|, \| \cdot \|'$ on $V$ is defined as the real number
\[
\text{vol}(\| \cdot \|, \| \cdot \|') := \log \left( \frac{\det \| \cdot \|'}{\det \| \cdot \|} \right).
\]

Proposition 2.14. The relative volume satisfies the following properties.

(i) $\text{vol}(\| \cdot \|, \| \cdot \|') = \text{vol}(\| \cdot \|, \| \cdot \|''') + \text{vol}(\| \cdot \|'', \| \cdot \|')$.
(ii) $\text{vol}(\| \cdot \|, e^c \| \cdot \|') = \text{vol}(\| \cdot \|, \| \cdot \|') + Nc$ for $c \in \mathbb{R}$.
(iii) $\| \cdot \|_1 \leq \| \cdot \|_2 \implies \text{vol}(\| \cdot \|, \| \cdot \|_1) \leq \text{vol}(\| \cdot \|, \| \cdot \|_2)$.
(iv) $\text{vol}(\| \cdot \|, \| \cdot \|')$ is $N$-Lipschitz continuous in each variable.
(v) If $K'/K$ is a non-Archimedean extension and $\| \cdot \|_1, \| \cdot \|_2$ are two ultrametric norms on $V$ with ground field extensions $\| \cdot \|_1', \| \cdot \|_2'$ to $V_{K'}$, then $\text{vol}(\| \cdot \|_1', \| \cdot \|_2') = \text{vol}(\| \cdot \|_1, \| \cdot \|_2)$.

Proof. The first three properties are obvious, and imply the fourth one as a formal consequence. By Lemma 1.27 and Lemma 2.6, the last property holds when $\| \cdot \|_1, \| \cdot \|_2$ are diagonalizable, and the general ultrametric case follows by density. \□

As we next show, in the Archimedean case, the relative volume is equivalent to the (logarithmic) volume ratio. The non-Archimedean case will be analyzed in the next section.

Proposition 2.15. Assume that $K$ is Archimedean (i.e. $K = \mathbb{R}$ or $\mathbb{C}$), and let $\| \cdot \|, \| \cdot \|'$ be two norms on $V$, with unit balls $B, B'$.

(i) If $\| \cdot \|, \| \cdot \|'$ are diagonalizable, then
\[
\text{vol}(\| \cdot \|, \| \cdot \|') = \frac{1}{[K : \mathbb{R}]} \log \left( \frac{\text{vol}(B)}{\text{vol}(B')} \right),
\]
where $\text{vol}$ is any choice of Haar measure on $V$.

(ii) In the general case, we have
\[
\text{vol}(\| \cdot \|, \| \cdot \|') = \frac{1}{[K : \mathbb{R}]} \log \left( \frac{\text{vol}(B)}{\text{vol}(B')} \right) + O(N \log N). \tag{2.2}
\]

The constant in $O(N \log N)$ is purely numerical (10 would do).

Proof. If $\| \cdot \|, \| \cdot \|'$ are diagonalizable, we can pick an orthonormal basis $(e_i)$ for $\| \cdot \|$ in which $\| \cdot \|'$ is diagonalized, and the change-of-variable formula yields
\[
\text{vol}(\| \cdot \|, \| \cdot \|') = \log \prod_i \| e_i \|' = \frac{1}{[K : \mathbb{R}]} \log \left( \frac{\text{vol}(B)}{\text{vol}(B')} \right),
\]
which proves (i). To prove (ii), we may assume that \( \| \cdot \| \) is diagonalizable, by the cocycle property. By Theorem \[1.19\] (or the John ellipsoid theorem), we can find a diagonalizable norm \( \| \cdot \|' \) with \( d_\infty (\| \cdot \|, \| \cdot \|') \leq \log N \). Since \( \text{vol} \) is \( N \)-Lipschitz continuous, we infer \( |\text{vol}(\| \cdot \|', \| \cdot \|)| \leq N \log N \). We similarly have

\[
\frac{1}{|K : \mathbb{R}|} \log \left( \frac{\text{vol}(B')}{\text{vol}(B'')} \right) \leq N \log N,
\]

and we conclude thanks to (i) and the cocycle property. \( \square \)

Here again, the error term is generally nonzero in the non-diagonalizable case.

**Example 2.16.** Let \( \| \cdot \|_p \) be the \( \ell^p \)-norm in a given basis \( (e_i) \). By Lemma \[2.4\], we have \( \text{vol}(\| \cdot \|_1, \| \cdot \|_2) = 0 \) for \( p \leq 2 \), while the volume of the unit ball of \( \| \cdot \|_1 \) is strictly smaller than that of \( \| \cdot \|_2 \).

2.4. **The content of a torsion module.** We assume in this section that \( K \) is non-Archimedean and non-trivially valued, with associated valuation \( v_K = -\log |\cdot| \). The next result was observed for instance in \[Tem14, Corollary 2.3.8\].

**Lemma 2.17.** Every finitely presented torsion \( K^\circ \)-module \( M \) is isomorphic to a finite direct sum of cyclic modules

\[
M \simeq \bigoplus_{i=1}^r K^\circ / \pi_i K^\circ.
\]

with \( \pi_i \in K^{\infty} \). The sequence \( v_K(\pi_1), \ldots, v_K(\pi_r) \) is further uniquely determined by \( M \), up to permutation.

**Proof.** Observe first that \( M \) is necessarily of the form \( M \simeq \mathcal{V}/\mathcal{V}' \) with \( \mathcal{V}' \subset \mathcal{V} \) two lattices in a \( K \)-vector space \( \mathcal{V} \). Indeed, pick a finite free \( K^\circ \)-module \( \mathcal{V} \) with a surjection \( \mathcal{V} \twoheadrightarrow M \). Since \( M \) is finitely presented, the kernel \( \mathcal{V}' \) is finitely generated, and hence a lattice of \( \mathcal{V} := \mathcal{V}_K \) since \( M \) is torsion. We claim that \( \mathcal{V} \) admits a basis \( (e_i) \) such that \( \mathcal{V} = \bigoplus_i K^\circ \pi_i e_i \) and \( \mathcal{V}' = \bigoplus_i K^\circ/\pi_i K^\circ \) for some \( \pi_i \in K^\circ \), which will yield as desired \( M \simeq \bigoplus_{i=1}^r K^\circ / \pi_i K^\circ \) with \( v_K(\pi_i) > 0 \) (since \( K^\circ/\pi_i K^\circ = 0 \) when \( v_K(\pi_i) = 0 \)). Indeed, Proposition \[1.14\] yields a basis \( (e_i) \) that jointly diagonalizes \( \| \cdot \|_\mathcal{V} \) and \( \| \cdot \|_\mathcal{V}' \). As lattice norms take values in \( |K| \), we can arrange that \( |e_i|_{\mathcal{V}'} = 1 \) after multiplying each \( e_i \) by a scalar. Since \( \mathcal{V}' \subset \mathcal{V} \), we then have \( |e_i|_{\mathcal{V}} = |\pi_i|^{-1} \) for some \( \pi_i \in K^\circ \), and we get the claim by Lemma \[1.30\]. Uniqueness is proved as in the usual structure theorem for torsion modules over PID’s. \( \square \)

**Definition 2.18.** The content of a finitely presented torsion module \( M \) is defined as

\[
\text{cont}(M) := \sum_i v_K(\pi_i) \in (0, +\infty).
\]

Note that this \( - \log \) of the content as defined in \[Tem14, 2.6.1\]. Alternatively, \( \text{cont}(M) \) is obtained by applying \( v_K \) to the fractional ideal sheaf \( \{ \alpha \in K \mid \alpha \cdot \det M = 0 \} \).

**Example 2.19.** If \( K \) is discretely valued with uniformizing parameter \( \pi_K \), then \( \text{cont}(M) = v_K(\pi_K) \ell(M) \) with \( \ell(M) \) the length of \( M \).

The content is closely related to the relative volume. Indeed, as we saw during the proof of Lemma \[2.17\], every finitely presented torsion module is the quotient of two lattices in the same vector space, and we have:
Lemma 2.20. If \( \mathcal{V}' \subset \mathcal{V} \) are lattices in a given \( K \)-vector space \( V \), then

\[
\text{cont}(\mathcal{V}/\mathcal{V}') = \text{vol}(\|\cdot\|, \|\cdot\|_{\mathcal{V}'}) .
\]

Assuming now that \( K \) is discretely valued, we conclude this section with an analogue of Proposition 2.15, relating the relative volume to the virtual length used in [BG+16]. Recall that the virtual length \( \ell(\mathcal{V}/\mathcal{V}') \in \mathbb{Z} \) of two lattices \( \mathcal{V}, \mathcal{V}' \) in a \( K \)-vector space is defined as

\[
\ell(\mathcal{V}/\mathcal{V}') := \ell(\mathcal{V}/\mathcal{V}'') - \ell(\mathcal{V}'/\mathcal{V}'')
\]

for any lattice \( \mathcal{V}'' \) contained in both \( \mathcal{V} \) and \( \mathcal{V}' \) (cf. [BG+16, Definition 4.1.1], [Ser, III, §1]).

Proposition 2.21. Assume that \( K \) is discretely valued with uniformizing parameter \( \pi_K \), and let \( \|\cdot\|, \|\cdot\|' \) be two ultrametric norms on \( V \). Denote by \( B, B' \) their unit balls, and by \( \|\cdot\|_B, \|\cdot\|_{B'} \) the associated lattice norms. Then

\[
\text{vol}(\|\cdot\|_B, \|\cdot\|_{B'}) = v_K(\pi_K)\ell(B/B') = \text{vol}(\|\cdot\|, \|\cdot\|') + O(N).
\]

Note that the absolute value of \( K \) is normalized by \( v_K(\pi_K) = 1 \) in [BG+16].

Proof of Proposition 2.21. The first equality follows from Example 2.19. By Lemma 1.30, we further have \( d_{\infty}(\|\cdot\|, \|\cdot\|_B) \leq v_K(\pi_K) \) and \( d_{\infty}(\|\cdot\|', \|\cdot\|'_{B'}) \leq v_K(\pi_K) \), and hence \( \text{vol}(\|\cdot\|, \|\cdot\|') = \text{vol}(\|\cdot\|_B, \|\cdot\|_{B'}) + O(N) \) by \( N \)-Lipschitz continuity of vol. \( \square \)

2.5. Relative successive minima. As mentioned in Example 1.5, an ultrametric norm \( \|\cdot\|_0 \) on \( V \) with respect to the trivial absolute \( |\cdot|_0 \) on \( K \) is equivalent to the data of a filtration of \( V \). Following the convention of [BC], the latter will be understood in this section as a decreasing, left-continuous, separating and exhaustive \( \mathbb{R} \)-filtration \( F^*V \), i.e. a family \( (F^\lambda V)_{\lambda \in \mathbb{R}} \) of \( K \)-subspaces of \( V \) such that

(i) \( F^\lambda V \subset F^{\lambda'} V \) when \( \lambda \geq \lambda' \);

(ii) \( F^\lambda V = \bigcap_{\lambda' < \lambda} F^{\lambda'} V \);

(iii) \( F^\lambda V = 0 \) for \( \lambda \gg 0 \);

(iv) \( F^\lambda V = V \) for \( \lambda \ll 0 \).

The correspondence is given by

\[
-\log \|v\|_0 = \sup \left\{ \lambda \in \mathbb{R} \mid v \in F^\lambda V \right\}, \quad F^\lambda V = \left\{ v \in V \mid \|v\|_0 \leq e^{-\lambda} \right\}.
\]

We define the jumping values \( \lambda_1 \geq \cdots \geq \lambda_N \) of a filtration \( (F^\lambda V)_{\lambda \in \mathbb{R}} \) by setting

\[
\lambda_i := \sup \left\{ \lambda \in \mathbb{R} \mid \dim F^\lambda V \geq i \right\}.
\]

We then have

\[
-\frac{d}{d\lambda} \dim F^\lambda V = \sum_{i=1}^{N} \delta_{\lambda_i}
\]

in the sense of distributions.

Definition 2.22. Let \( \|\cdot\|, \|\cdot\|' \in \mathcal{N}(V) \) be two norms on \( V \) (with respect to the given absolute value of \( K \)). The relative successive minima of \( \|\cdot\| \) with respect to \( \|\cdot\|' \) are defined as the jumping values \( \lambda_i = \lambda_i(\|\cdot\|, \|\cdot\|') \) of the filtration

\[
F^\lambda V := \text{Vect} \left\{ v \in V \mid \|v\| \leq e^{-\lambda}\|v\|' \right\}.
\]
As one immediately sees, the relative successive minima can be alternatively characterized via a min-max principle, as follows:

$$\lambda_i(|| \cdot \parallel, || \cdot ||') = \sup_{W \subset V, \dim W \geq i} \left( \inf_{w \in W \setminus \{0\}} \log \frac{\|w\||'}{\|w\|} \right). \quad (2.3)$$

In particular,

$$\lambda_1(|| \cdot ||, || \cdot ||') = \sup_{v \in V \setminus \{0\}} \log \left( \frac{\|v\'||}{\|v\|} \right),$$

$$\lambda_N(|| \cdot ||, || \cdot ||') = \inf_{v \in V \setminus \{0\}} \log \left( \frac{\|v||'}{\|v\|} \right) = -\lambda_1(|| \cdot ||', || \cdot ||)$$

and

$$d_{\infty}(|| \cdot ||, || \cdot ||') = \max \left\{ \lambda_1(|| \cdot ||, || \cdot ||'), \lambda_1(|| \cdot ||', || \cdot ||) \right\}. \quad (2.4)$$

**Lemma 2.23.** Each $\lambda_i(|| \cdot ||, || \cdot ||')$ is a 1-Lipschitz continuous function of $|| \cdot ||, || \cdot ||' \in N(V)$.

**Proof.** As already observed before, this follows formally from the fact that $|| \cdot ||' \mapsto \lambda_i(|| \cdot ||, || \cdot ||')$ is non-decreasing and satisfies $\lambda_i(|| \cdot ||, e^t|| \cdot ||') = \lambda_i(|| \cdot ||, || \cdot ||') + t$ for $t \in \mathbb{R}$, both of which are straightforward from (2.3).

In the diagonalizable case, we can mimick the usual min-max characterization of the eigenvalues of a Hermitian matrix and prove:

**Proposition 2.24.** Assume that $|| \cdot ||, || \cdot ||' \in N^{\text{diag}}(V)$ are diagonalizable, and choose a basis $(e_i)$ of $V$ in which both norms are diagonalized, as in Proposition 1.14 ordered so that

$$\frac{\|e_1\||'}{\|e_1\|} \geq \cdots \geq \frac{\|e_N\||'}{\|e_N\|}.$$

Then

$$\lambda_i(|| \cdot ||, || \cdot ||') = \log \frac{\|e_i\||'}{\|e_i\|}.$$

**Proof.** Set $W_i := \text{Vect}(e_1, \ldots, e_i)$, $W_i' := \text{Vect}(e_i, \ldots, e_N)$, and observe that

$$\log \frac{\|e_i||'}{\|e_i\|} = \inf_{w \in W_i \setminus \{0\}} \log \frac{\|w||'}{\|w\|} = \sup_{w \in W_i \setminus \{0\}} \log \frac{\|w||'}{\|w\|}.$$

By (2.3), the first equality yields $\log \frac{\|e_i\||'}{\|e_i\|} \leq \lambda_i(|| \cdot ||, || \cdot ||')$. On the other hand, each subspace $W \subset V$ with $\dim W \geq i$ satisfies $W \cap W_i' \neq \{0\}$ for dimension reason, and hence

$$\inf_{w \in W \setminus \{0\}} \log \frac{\|w||'}{\|w\|} \leq \sup_{w \in W_i \setminus \{0\}} \log \frac{\|w||'}{\|w\|} = \log \frac{\|e_i||'}{\|e_i\|},$$

and using (2.3) again yields $\lambda_i(|| \cdot ||, || \cdot ||') \leq \log \frac{\|e_i||'}{\|e_i\|}$.

We are now in a position to prove the following analogue of Minkowski’s second theorem.

**Theorem 2.25.** For any two norms $|| \cdot ||, || \cdot ||'$ on $V$, we have

$$\text{vol}(|| \cdot ||, || \cdot ||') = \sum_i \lambda_i(|| \cdot ||, || \cdot ||') + O(N \log N).$$
If \( \| \cdot \| \) and \( \| \cdot \|' \) are diagonalizable (or merely ultrametric, in the non-Archimedean case), then

\[
\mathrm{vol}(\| \cdot \|, \| \cdot \|') = \sum_i \lambda_i(\| \cdot \|, \| \cdot \|').
\]

For non-diagonalizable norms, the term in \( O(N \log N) \) is nonzero in general:

**Example 2.26.** Assume that \( K \) is Archimedean, and denote by \( \| \cdot \|_p \) the standard \( \ell_p \)-norm on \( K^2 \). Then \( \mathrm{vol}(\| \cdot \|_2, \| \cdot \|_1) = 0 \), while \( \lambda_1(\| \cdot \|_2, \| \cdot \|_1) = \log \sqrt{2} \) and \( \lambda_2(\| \cdot \|_2, \| \cdot \|_1) = 0 \).

**Proof of Theorem 2.25.** Note that \( \mathrm{vol}(\| \cdot \|, \| \cdot \|') \) and \( \sum_i \lambda_i(\| \cdot \|, \| \cdot \|') \) are both \( N \)-Lipschitz continuous in \( \| \cdot \| \) and \( \| \cdot \|' \), by Proposition 2.14 and Lemma 2.23, respectively. When both norms are diagonalizable, Lemma 2.6 and Proposition 2.24 yield

\[
\mathrm{vol}(\| \cdot \|, \| \cdot \|') = \sum_i \lambda_i(\| \cdot \|, \| \cdot \|').
\]

In the general case, any norm lies at distance \( O(\log N) \) of \( N_{\text{diag}}(V) \), by Theorem 1.19, and we conclude by \( N \)-Lipschitz continuity. Finally, if \( K \) is non-Archimedean, \( N_{\text{diag}}(V) \) is dense in \( N_{\text{ultr}}(V) \) by Theorem 1.19, and we infer

\[
\mathrm{vol}(\| \cdot \|, \| \cdot \|') = \sum_i \lambda_i(\| \cdot \|, \| \cdot \|').
\]

for any two ultrametric norms, by continuity. \( \square \)

**Remark 2.27.** When \( K \) is discretely valued, the results of this section can be traced back to Mahler’s paper [Mah41].

2.6. Graded norms. In this final section, we deduce from [BC] a general existence result for limits of relative volumes, which will be crucial later on. Let \( R = \bigoplus_{m \in \mathbb{N}} R_m \) be a graded \( K \)-algebra with \( N_m := \dim_K R_m \) finite for all \( m \).

**Definition 2.28.** A graded norm \( \| \cdot \| \) on \( R \) is defined as a sequence of norms \( \| \cdot \|_m \) on the graded pieces \( R_m \). A graded norm \( \| \cdot \| \) is submultiplicative (resp. multiplicative) if

\[
\| s \cdot s' \|_{m+m'} \leq \| s \|_m \cdot \| s' \|_{m'} \quad (\text{resp. } \| s \cdot s' \|_{m+m'} = \| s \|_m \cdot \| s' \|_{m'})
\]

for all \( m, m' \in \mathbb{N}, s \in R_m, s' \in R_{m'} \).

**Definition 2.29.** We say that two graded norms \( \| \cdot \|, \| \cdot \|' \) on \( R \) are

(i) linearly close if \( d_\infty(\| \cdot \|, \| \cdot \|') = O(m) \);

(ii) asymptotically equal if \( d_\infty(\| \cdot \|, \| \cdot \|') = o(m) \).

In other words, \( \| \cdot \| \) and \( \| \cdot \|' \) are linearly close if the distortion between \( \| \cdot \|_m \) and \( \| \cdot \|'_m \) has at most exponential growth, and they are asymptotically equal if the distortion has subexponential growth. The following lemma is a direct consequence of the Lipschitz continuity property of Proposition 2.14.

**Lemma 2.30.** If two graded norms \( \| \cdot \|, \| \cdot \|' \) on \( R \) are asymptotically equal, then

\[
\frac{1}{mN_m} \mathrm{vol}(\| \cdot \|_m, \| \cdot \|'_m) \to 0.
\]

Relying on the Okounkov body technique developed in [BC], we next show:
Theorem 2.31. Assume that the graded $K$-algebra $R$ is integral and of finite type. Let $\| \cdot \|, \| \cdot \|'$ be two linearly close submultiplicative graded norms on $R$, and assume that they are linearly close to some multiplicative graded norm. Then

$$\frac{1}{mN_m} \text{vol}(\| \cdot \|_m, \| \cdot \|'_m)$$

admits a limit in $\mathbb{R}$.

Proof. Let $\| \cdot \|''$ be a multiplicative graded norm on $R$ linearly close to $\| \cdot \|$ and $\| \cdot \|'$. Since

$$\text{vol}(\| \cdot \|_m, \| \cdot \|'_m) = \text{vol}(\| \cdot \|_m, \| \cdot \|''_m) - \text{vol}(\| \cdot \|'_m, \| \cdot \|''_m),$$

it is enough to prove the result when $\| \cdot \|'$ itself is multiplicative. The relative successive minima $\lambda_i(\| \cdot \|_m, \| \cdot \|'_m)$ are the jumping values of the

$$F^\lambda R_m = \left\{ s \in R_m \mid \|s\|_m \leq e^{-\lambda} \|s\|'_m \right\}, \quad (2.5)$$

and they satisfy

$$\text{vol}(\| \cdot \|_m, \| \cdot \|'_m) = \sum_{i=1}^{N_m} \lambda_i(\| \cdot \|_m, \| \cdot \|'_m) + O(N_m \log N_m) \quad (2.6)$$

by Theorem 2.25. Since $\| \cdot \|$ is submultiplicative and $\| \cdot \|'$ is multiplicative, the filtration $F^\bullet R$ is multiplicative in the sense of [BC], i.e. it satisfies

$$F^\lambda R_m \cdot F^\lambda R_{m'} \subset F^{\lambda+\lambda'} R_{m+m'}.$$

Since $\| \cdot \|, \| \cdot \|'$ are linearly close, there exists $C > 0$ with

$$e^{-Cm} \| \cdot \|_m \leq \| \cdot \|'_m \leq e^{Cm} \| \cdot \|_m,$$

and hence $F^\lambda R_m = \{0\}$ for $\lambda > Cm$, $F^\lambda R_m = R_m$ for $\lambda < -Cm$. The filtration $F^\bullet R$ is thus linearly bounded in the terminology of [BC], and [BC] Theorem A implies that the scaled successive minima $(m^{-1} \lambda_i(\| \cdot \|_m, \| \cdot \|'_m))$ equidistribute. In particular,

$$\frac{1}{mN_m} \sum_{i=1}^{N_m} \lambda_i(\| \cdot \|_m, \| \cdot \|'_m)$$

admits a limit in $\mathbb{R}$. Since $R$ is finitely generated, $N_m$ has at most polynomial growth by Hilbert’s theorem, hence $\log N_m = o(m)$, and we are thus done by (2.6).

□

3. Alternative metric structures on spaces of norms

The goal of this section, which stands somewhat apart from the rest of the paper, is to exploit the properties of determinants of norms to endow the space of diagonalizable norms with natural metric structures, recovering in particular the Bruhat-Tits metric in the non-Archimedean case.
3.1. **The triangle inequality.** The relative successive minima of two norms $\|\cdot\|, \|\cdot\|' \in \mathcal{N}(V)$ define a point $\lambda(\|\cdot\|, \|\cdot\|') \in \mathbb{R}^V$ as the unique metric on $\mathbb{R}^V$, set
\[
d_\lambda(\|\cdot\|, \|\cdot\|') := \lambda(\|\cdot\|, \|\cdot\|').\]
The resulting function $d_\lambda : \mathcal{N}(V) \times \mathcal{N}(V) \to \mathbb{R}$ is symmetric, with $d_\lambda(\|\cdot\|, \|\cdot\|') = 0$ if and only if $\|\cdot\| = \|\cdot\|'$. The following result is inspired by Gerardin’s proof of [Ger81 2.4.7, Corollaire 2].

**Theorem 3.1.** For each $\mathcal{G}_N$-invariant norm $\chi$ on $\mathbb{R}^V$, $d_\chi$ satisfies the triangle inequality on $\mathcal{N}^{\text{diag}}(V)$, and is characterized as the unique metric on $\mathcal{N}^{\text{diag}}(V)$ for which
\[
\iota_e : (\mathbb{R}^V, \chi) \to (\mathcal{N}^{\text{diag}}(V), d_\chi)
\]
is an isometric embedding for each basis $e$ of $V$.

By Proposition 1.17 the Goldman-Iwashiri metric $d_\infty$ on $\mathcal{N}^{\text{diag}}(V)$ corresponds to the $\ell^\infty$-norm on $\mathbb{R}^V$. By equivalence of norms on $\mathbb{R}^V$, any metric $d_\chi$ produced by Theorem 3.1 is Lipschitz equivalent to $d_\infty$. Besides the latter, the most important case is the Euclidean metric $d_2$ induced by the $\ell^2$-norm:

**Example 3.2.** When $K$ is Archimedean, $d_2$ coincides with the Riemannian metric of the symmetric space $\mathcal{N}^{\text{diag}}(V) \simeq \text{GL}_N(K)/U_N(K)$ (see Theorem 3.7 below). When $K$ is non-Archimedean, $(\mathcal{N}^{\text{diag}}(V), d_2)$ is a realization of the Bruhat-Tits building of $\text{GL}_N(K)$ with its Euclidean metric, see for instance [Par00, Chapter III]. In both cases, $(\mathcal{N}^{\text{diag}}(V), d_2)$ is a CAT(0) metric space.

**Corollary 3.3.** If $K$ is non-Archimedean, the space $\mathcal{N}^{\text{ultr}}(V)$ of ultrametric norms is complete metric with respect to $d_\chi$, which is characterized as the unique compatible metric such that $\iota_e : (\mathbb{R}^V, \chi) \to (\mathcal{N}^{\text{ultr}}(V), d_\chi)$ is an isometric embedding for all bases $e$. For $\chi = \ell^2$, $(\mathcal{N}^{\text{ultr}}(V), d_2)$ is a CAT(0) metric space.

**Proof.** On $\mathcal{N}^{\text{diag}}(V)$, $d_\chi$ is equivalent to $d_\infty$ and satisfies the triangle inequality. This is also the case on $\mathcal{N}^{\text{ultr}}(V)$, by density of $\mathcal{N}^{\text{diag}}(V)$. Since $\mathcal{N}^{\text{ultr}}(V)$ is complete for $d_\infty$, it is also complete for $d_\chi$, and is thus the completion of $(\mathcal{N}^{\text{diag}}(V), d_\chi)$. Conversely, any metric on $\mathcal{N}^{\text{ultr}}(V)$ with the stated property must coincide with $d_\chi$ on the dense subset $\mathcal{N}^{\text{diag}}(V)$, hence everywhere. For $\chi = \ell^2$, the Bruhat-Tits building $(\mathcal{N}^{\text{diag}}(V), d_2)$ is a CAT(0) metric space, and this property is preserved under completion.

As in the construction of the Euclidean metric on any Euclidean building, the key to the proof of Theorem 3.1 is to show that the Gram-Schmidt projections introduced in Definition 1.18 are distance-decreasing.

**Lemma 3.4.** For each basis $e$ of $V$, the Gram-Schmidt projection $\rho_e : \mathcal{N}^{\text{diag}}(V) \to k_e$ satisfies
\[
d_\chi(\rho_e(\|\cdot\|), \rho_e(\|\cdot\|')) \leq d_\chi(\|\cdot\|, \|\cdot\|')
\]
for all $\|\cdot\|, \|\cdot\|' \in \mathcal{N}^{\text{diag}}(V)$.

We first recall some elementary facts.
**Definition 3.5.** Given \( \lambda, \lambda' \in \mathcal{C} \), one says that \( \lambda \) is majorized by \( \lambda' \), written \( \lambda \preceq \lambda' \), if \( \lambda_1 + \cdots + \lambda_i \preceq \lambda'_1 + \cdots + \lambda'_i \) for all \( i \), with equality for \( i = \mathcal{N} \).

**Lemma 3.6.** We have \( \lambda \preceq \lambda' \) if and only \( \lambda \) belongs to the convex envelope of the \( \mathcal{G}_{\mathcal{N}} \)-orbit of \( \lambda' \), and then \( \chi(\lambda') \leq \chi(\lambda) \) for any \( \mathcal{G}_{\mathcal{N}} \)-invariant norm \( \chi \) on \( \mathbb{R}^\mathcal{N} \).

**Proof.** It is straightforward to see that any \( \lambda \) in the convex envelope of the \( \mathcal{G}_{\mathcal{N}} \)-orbit of \( \lambda' \) satisfies \( \lambda \preceq \lambda' \). As observed in [Rad52], the converse is a simple consequence of the Hahn-Banach theorem. Assuming indeed that \( \lambda \preceq \lambda' \), it is enough to show that for each \( \mu \in \mathbb{R}^\mathcal{N} \) there exists \( \sigma \in \mathcal{G}_{\mathcal{N}} \) with

\[
\sum_i \mu_i \lambda_i \leq \sum_i \mu_i \lambda'_i.
\]

Choose \( \sigma \) such that \( \mu_{\sigma(1)} \geq \cdots \geq \mu_{\sigma(\mathcal{N})} \). Then

\[
\sum_i \mu_i \lambda_i = \sum_i \mu_{\sigma(i)} \lambda_{\sigma(i)} = \sum_{i < N} \left( \mu_{\sigma(1)} - \mu_{\sigma(i+1)} \right) \left( \lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(i)} \right) + \mu_{\sigma(N)} \left( \lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(\mathcal{N})} \right)
\]

\[
\leq \sum_{i < N} \left( \mu_{\sigma(1)} - \mu_{\sigma(i+1)} \right) \left( \lambda'_1 + \cdots + \lambda'_i \right) + \mu_{\sigma(N)} \left( \lambda'_1 + \cdots + \lambda'_\mathcal{N} \right) = \sum_i \mu_{\sigma(i)} \lambda'_i = \sum_i \mu_i \lambda'_{\sigma^{-1}(i)}.
\]

Assume finally that \( \lambda \preceq \lambda' \), and hence \( \lambda = \sum_j t_j \mu_j \) with \( \mu_i \mathcal{G}_{\mathcal{N}} \)-equivalent to \( \lambda' \), \( t_j \in \mathbb{R}_+ \) and \( \sum_j t_j = 1 \). By \( \mathcal{G}_{\mathcal{N}} \)-invariance of \( \chi \), we infer

\[
\chi(\lambda) \leq \sum_j t_j \chi(\mu_i) = \sum_j t_j \chi(\lambda') = \chi(\lambda').
\]

**Proof of Lemma 3.4.** By Lemma 3.6, it will be enough to show that the relative successive minima \( \lambda \) (resp. \( \mu \)) of \( \rho_e(\| \cdot \|') \) with respect to \( \rho_e(\| \cdot \|) \) (resp. \( \| \cdot \|' \) with respect to \( \| \cdot \| \)) satisfy \( \lambda_1 + \cdots + \lambda_i \preceq \mu_1 + \cdots + \mu_i \) for all \( i \), with equality for \( i = \mathcal{N} \). Set \( W := \text{Vect}(e_1, \ldots, e_i) \), and observe that the restriction \( \rho_e(\| \cdot \|)_W \) of \( \rho_e(\| \cdot \|) \) to \( W \) satisfies by definition

\[
\rho_e(\| \cdot \|)_W = \rho_{e_W}(\| \cdot \|_W)
\]

with \( e_W = (e_1, \ldots, e_i) \). By Lemma 2.12 we thus have \( \det(\rho_e(\| \cdot \|)_W) = \det(\| \cdot \|_W) \), and Theorem 2.25 yields

\[
\lambda_1 + \cdots + \lambda_i = \det(\rho_e(\| \cdot \|)_W, \rho_e(\| \cdot \|')_W) = \det(\| \cdot \|_W, \| \cdot \|'_W) = \sum_{j \leq i} \lambda_j(\| \cdot \|_W) = \sum_{j \leq i} \lambda_j(\| \cdot \|'_W) = \mu_1 + \cdots + \mu_i,
\]

where the last inequality follows directly from the min-max principle (2.3), and is an equality when \( i = \mathcal{N} \), i.e. \( W = V \).

**Proof of Theorem 3.7.** By construction, \( \iota_e \) is an isometric embedding with respect to \( \chi \) and \( d_\chi \), and the latter therefore satisfies the triangle inequality on each apartment \( \mathcal{A}_e = \iota_e(\mathbb{R}^\mathcal{N}) \). Pick three diagonalizable norms \( \| \cdot \|_i \in \mathcal{N}_{\text{diag}}(V) \), \( i = 1, 2, 3 \). We may then choose a basis \( e \) with \( \| \cdot \|_1, \| \cdot \|_2 \in \mathcal{A}_e \). Since \( d_\chi \) satisfies the triangle inequality on \( \mathcal{A}_e \), we have

\[
d_\chi(\| \cdot \|_1, \| \cdot \|_2) = d_\chi(\rho_e(\| \cdot \|_1), \rho_e(\| \cdot \|_2))
\]
\[ \leq d_\chi (\rho_e(\| \cdot \|_1), \rho_e(\| \cdot \|_3)) + d_\chi (\rho_e(\| \cdot \|_3), \rho_e(\| \cdot \|_2)), \]

which shows that \( d_\chi \) satisfies the triangle inequality on \( N^{\text{diag}}(V) \), by (3.1). \( \square \)

3.2. The Archimedean case: Finsler metrics. Assume that \( K \) is Archimedean, i.e. \( K = \mathbb{R} \) or \( \mathbb{C} \), and denote by \( H(V) \) the real vector space of quadratic/Hermitian forms \( h \) on \( V \). Each diagonalizable norm \( \| \cdot \| \in N^{\text{diag}}(V) \) is then associated to a positive definite form \( \chi(h) := \|v\|^2 \), thereby defining an embedding of \( N^{\text{diag}}(V) \) as an open convex subset of \( H(V) \). Further, \( N^{\text{diag}}(V) \) is diffeomorphic to the symmetric space \( \text{GL}(N, K)/U(N, K) \), with \( U(N, K) \) the unitary/orthogonal group.

As we now show, the metric \( d_\chi \) constructed above is induced by a natural Finsler metric on \( N^{\text{diag}}(V) \). Recall first that for each \( h, \gamma \in H(V) \) with \( \gamma \) positive definite, one can find a basis \( e = (e_i) \) of \( V \) which is orthonormal for \( \gamma \) and orthogonal for \( h \), i.e.

\[ \gamma \left( \sum_i \alpha_i e_i \right) = \sum_i |\alpha_i|^2 \]

and

\[ h \left( \sum_i \alpha_i e_i \right) = \sum_i \lambda_i |\alpha_i|^2 \]

for all \( \alpha_i \in K \). The spectrum \( (\lambda_i) \) is independent of the choice of \( e \) up to ordering, hence defines a point \( \lambda_\gamma(h) \in \mathcal{C} \), and we then have for any two \( h, h' \in H(V) \)

\[ \lambda_\gamma(h + h') \leq \lambda_\gamma(h) + \lambda_\gamma(h'). \]

This is indeed a simple consequence of the min-max principle, known as the Ky Fan inequality. Given a symmetric norm \( \chi \) on \( \mathbb{R}^N \), it follows as in Lemma 3.6 that setting for each \( \gamma \in N^{\text{diag}}(V) \)

\[ |h|_{\chi, \gamma} := \chi (\lambda_\gamma(h)) \]

defines a norm on \( H(V) \), and we thus get a continuous Finsler norm \( | \cdot |_{\chi} \) on the tangent bundle of \( N \).

**Theorem 3.7.** The metric \( d_\chi \) on \( N^{\text{diag}}(V) \) in Theorem 3.1 coincides with the length metric defined by the Finsler norm \( | \cdot |_{\chi} \). In other words, for any two \( \gamma, \gamma' \in N^{\text{diag}}(V) \), \( d_\chi(\gamma, \gamma') \) is the infimum over all smooth paths \( (\gamma_t)_{t \in [0,1]} \) in \( N \) joining \( \gamma \) to \( \gamma' \) of the corresponding length

\[ \ell_{\chi}(\gamma) := \int_0^1 |\dot{\gamma}_t|_{\chi, \gamma_t} dt. \]

**Lemma 3.8.** For each basis \( e \) of \( V \), the Gram-Schmidt projection \( \rho_e : N^{\text{diag}}(V) \to A_e \) is a smooth map, and it satisfies \( \rho^*_e|_\chi \leq | \cdot |_{\chi} \).

**Proof.** The smoothness of \( \rho_e \) follows from its description in terms of the Gram-Schmidt orthogonalization process. Pick \( \gamma \in N^{\text{diag}}(V) \), \( h \in H(V) \). As in the proof of Lemma 3.4, it will be enough to show that \( \lambda := \lambda_{\rho_e(\gamma)}(d_\gamma \rho_e(h)) \) is majorized by \( \mu := \lambda_\gamma(h) \). Differentiating the identity

\[ \text{vol}(\gamma, \gamma') = \text{vol}(\rho_e(\gamma), \rho_e(\gamma')) \]

with respect to \( \gamma' \) shows that the trace \( \text{Tr}_\gamma(h) \) satisfies

\[ \text{Tr}_\gamma(h) = \text{Tr}_{\rho_e(\gamma)}(d_\gamma \rho_e(h)), \]
i.e. $\lambda_1 + \cdots + \lambda_N = \mu_1 + \cdots + \mu_N$. Arguing as in Lemma 3.4 we apply this fact to the restrictions of $\gamma$ and $h$ to the span $W$ of $(e_1, \ldots, e_i)$ for a given $i$, and get

$$\lambda_1 + \cdots + \lambda_i = \text{Tr}_{\gamma|W}(h|W) \leq \mu_1 \cdots + \mu_i$$

thanks to the min-max principle.

Proof of Theorem 3.4. Pick a basis $e$, and observe that the differential $d\lambda e : \mathbb{R}^N \to H(V)$ at $\lambda \in \mathbb{R}^N$ satisfies for all $\mu \in \mathbb{R}^N$

$$\lambda_{e}(\lambda)(d\lambda e(\mu)) = \mu \pmod{\mathcal{S}_N}.$$

As a result, $\lambda e : |\chi|$ is the constant Finsler norm $\chi$ on $\mathbb{R}^N$, and the $\chi$-length of any smooth path $\gamma : [0, 1] \to \Lambda e$ joining $\gamma = \iota e(\lambda)$ to $\gamma' = \iota e(\lambda')$ thus satisfies

$$\ell_\chi(\gamma) \geq \chi(\lambda' - \lambda) = d_\chi(\gamma, \gamma'),$$

with equality when $\gamma$ is the image of the line segment $[\lambda, \lambda']$. If $\gamma : [0, 1] \to \mathcal{N}$ is now a smooth path joining $\gamma$ to $\gamma'$ in $\mathcal{N}^{\text{diag}}(V)$ only, the previous case applies to $\rho e \circ \gamma : [0, 1] \to \Lambda e$, which combines with (ii) of Lemma 3.8 to give $\ell_\chi(\gamma) \geq \ell_\chi(\rho e \circ \gamma) \geq d_\chi(\gamma, \gamma').$  

Part 2. Models and metrics

4. Analytification and models

This section reviews some well-known facts on Berkovich analytifications and models, with an emphasis on the reduced fiber condition. We provide in particular a direct proof of the relevant version of the Bosch-Lütkebohmert-Raynaud reduced fiber theorem.

As before, $K$ denotes a field equipped with an arbitrary (Archimedean or non-Archimedean) complete absolute value. All schemes over $K$ (or $K^\circ$) considered below are separated, unless otherwise specified.

4.1. Analytification. To each scheme $X$ of finite type over $K$, Berkovich associates in [Ber, §3.4] an analytification $X^{\text{an}}$. In the present paper, we only need the underlying topological space, which is Hausdorff and locally compact. As such, the analytifications of $X$ and of the reduced scheme $X_{\text{red}}$ coincide, but it will nevertheless be useful for inductive arguments to allow $X$ to be non-reduced.

Assume first that $X$ is affine, i.e. $X = \text{Spec} A$ with $A$ a finite type $K$-algebra. The topological space $X^{\text{an}}$ is defined as the set of all multiplicative seminorms $\lambda$ extending the given absolute value on $K$, endowed with the topology of pointwise convergence. The multiplicative seminorm associated to $x \in X^{\text{an}}$ is denoted by $f \mapsto |f(x)|$. The set of $f \in A$ with $|f(x)| = 0$ is a prime ideal, thereby defining a natural continuous map $\rho : X^{\text{an}} \to X$. We thus have $|f(x)| = 0$ if and only if $f$ vanishes at $\xi = \rho(x)$, and $f \mapsto |f(x)|$ defines a norm on the residue field $\kappa(\xi)$.

Lemma 4.1. A function $f \in A$ satisfies $|f| \equiv 0$ on $X^{\text{an}}$ if and only if $f$ is nilpotent.

Proof. For each closed point $\xi \in X$, the absolute value on $K$ (uniquely) extends to the finite field extension $\kappa(\xi)$ of $K$, and $\rho(X^{\text{an}})$ therefore contains the set of closed points of $X$ (in fact, $\rho$ injects $X^{\text{an}}$ as the set of closed points of $X$ if $K$ is Archimedean, while $\rho$ maps $X^{\text{an}}$ onto $X$ when $K$ is non-Archimedean). As a consequence, a function $f \in A$ with $|f| \equiv 0$ vanishes at all closed points of $X$, and hence is nilpotent. □
Consider now an arbitrary $K$-scheme of finite type $X$, and cover it with finitely many affine open subschemes $U_i$. Since $X$ is separated, each $U_{ij} = U_i \cap U_j$ is affine, and $U_{ij}^\an$ is homeomorphic to the inverse image of $U_{ij}$ in both $U_i^\an$ and $U_j^\an$. We can thus glue $U_{ij}^\an$ and $U_{ij}^\an$ together along their common open subset $U_{ij}^\an$ to define the topological space $X^\an$ with its continuous map $\rho : X^\an \to X$. The GAGA theorem [Ber] 3.4.8, 3.5.3 guarantees that $X^\an$ is Hausdorff (since we always assume $X$ separated), locally compact, and $X^\an$ is compact if and only if $X$ is proper.

**Example 4.2.** If $K$ is Archimedean, the Gelfand-Mazur theorem yields the following description of $X^\an$. When $K = \mathbb{C}$, $X^\an$ is the usual analytification of $X$, i.e. $X^\an = X(\mathbb{C})$ with its Euclidian topology. When $K = \mathbb{R}$, $X^\an$ is identified with the set of closed points of $X$, i.e. the quotient of $(X \otimes \mathbb{C})^\an = X(\mathbb{C})$ by complex conjugation.

**Example 4.3.** When $K$ is non-Archimedean with valuation $v_K = -\log |.|$, $X^\an$ can be seen as a space of semivaluations on $X$, i.e. real valuations on the residue fields of points of $X$. More precisely, the bijective map $x \mapsto (\rho(x), v_x)$ with $v_x(f) := -\log |f(x)|$ describes $X^\an$ as the set of pairs $(\xi, v)$ where $\xi \in X$ is a scheme point and $v : \kappa(\xi)^* \to \mathbb{R}$ is a rank 1 valuation on the residue field $\kappa(\xi)$ extending $v_K$.

### 4.2. Models and reduction.

In what follows, $K$ is non-Archimedean (possibly trivially valued). Recall that the valuation ring $K^\circ$ is then noetherian if and only if $K$ is discretely valued. Let $X$ be a $K$-scheme of finite type.

**Definition 4.4.** A model of $X$ is a (separated) flat, finite type $K^\circ$-scheme $\mathcal{X}$ together with an identification of $K$-schemes $\mathcal{X}_K \simeq X$.

If $\mathcal{X}'$, $\mathcal{X}$ are two models of $X$, we say that $\mathcal{X}'$ dominates $\mathcal{X}$ if there is a proper morphism $\mathcal{X}' \to \mathcal{X}$ compatible with the identifications with the general fiber.

The *special fiber* of a model $\mathcal{X}$ is the $\bar{K}$-scheme of finite type $\mathcal{X}_s := \mathcal{X}_K$. In the trivially valued case, $\mathcal{X} = X$ is the only model, and $\mathcal{X}_s = X$.

**Lemma 4.5.** A model $\mathcal{X}$ is automatically finitely presented over $K^\circ$.

This follows from a general result of Raynaud-Gruson [RG71] Théorème 3.4.6, and goes back to Nagata [Nag66] Theorem 3. For the sake of completeness, we reproduce here a simple argument due to Antoine Ducros [Duc], which is basically equivalent to that of Nagata.

**Proof.** We claim that it is enough to prove the result when $\mathcal{X}$ is projective over $K^\circ$. Indeed, arguing locally, we may first assume that $\mathcal{X}$ is affine, and we get the claim by choosing a closed embedding in an affine space and passing to the schematic closure in the corresponding projective space. Pick a closed embedding $\mathcal{X} \hookrightarrow \mathbb{P}_{K^\circ}^N$, and denote by $I$ the corresponding homogeneous ideal of $R := K^\circ[t_0, \ldots, t_N]$. Since both $(R/I) \otimes K$ and $(R/I) \otimes \bar{K}$ are noetherian, we may choose a finitely generated homogeneous ideal $I' \subset I$ such that $R/I' \to R/I$ becomes an isomorphism after tensoring with either $K$ or $\bar{K}$. This means that the finitely presented closed subscheme $\mathcal{X}' \subset \mathbb{P}_{K^\circ}^N$ defined by $I'$ has the same special fiber and generic fiber as $\mathcal{X}$. If we can show that $\mathcal{X}'$ is flat over $K^\circ$, it will coincide with the schematic closure of its generic fiber, which will prove that $\mathcal{X} = \mathcal{X}'$ is finitely presented. But $\mathcal{X}'$ is flat over $K^\circ$ if and only if the finite type $K^\circ$-module $\mathcal{V}_m := (R/I')^m$ is free for all $m \in \mathbb{N}$ large enough, which is indeed the case since $\dim_K \mathcal{V}_m \otimes \bar{K} = \dim_{\bar{K}} \mathcal{V}_m \otimes \bar{K}$, by choice of $I'$. \qed
A model $\mathcal{X}$ of $X$ determines a compact subset $\mathcal{X}^2$ of $X^\text{an}$ and a reduction map
\[ \text{red}_\mathcal{X} : \mathcal{X}^2 \to \mathcal{X}_s, \]
as follows. If $\mathcal{X}$ is affine, i.e. $\mathcal{X} = \text{Spec}(\mathcal{A})$ with $\mathcal{A}$ a flat (i.e. torsion-free) finite type $K^\circ$-algebra, then
\[ \mathcal{X}^2 = \{ x \in X^\text{an} | |f(x)| \leq 1 \text{ for all } f \in A \}, \]
and the reduction $\text{red}_\mathcal{X}(x)$ of $x \in \mathcal{X}^2$ is the point of $\mathcal{X}_s$ induced by the prime ideal $\{ f \in A | |f(x)| < 1 \}$. In the general case, $\mathcal{X}$ is covered by finitely many affine open subschemes $U_i$, whose generic fibers $U_i$ give an affine open cover of $X$, and
\[ \mathcal{X}^2 = \bigcup_i U_i^2 \subset \bigcup_i U_i^\text{an} = X^\text{an}. \]

In the language of Example 4.3, $\mathcal{X}^2$ consists of those semivaluations on $X$ that admit a center on $\mathcal{X}$, and $\text{red}_\mathcal{X}$ maps a semivaluation to its center, which is necessarily on $\mathcal{X}_s$. By the valuative criterion of properness, we thus have:

**Lemma 4.6.** For any model $\mathcal{X}'$ dominating $\mathcal{X}$, we have $(\mathcal{X}')^2 = \mathcal{X}^2$, and $\mathcal{X}^2 = X^\text{an}$ if $\mathcal{X}$ is proper over $K^\circ$ (and hence $X$ proper over $K$).

**Example 4.7.** In the trivially valued case, $X$ is the only model, and $\mathcal{X}^2$ coincides with the construction of \cite{Thu07}.

The compact set $\mathcal{X}^2$ associated to a model $\mathcal{X}$ can also be understood as (the underlying topological space of) the generic fiber in the sense of \cite{Ber94} \S 1 of the formal completion $\hat{\mathcal{X}}$ of $\mathcal{X}$. As above, it is enough to consider the case where $X = \text{Spec}(A)$ and $\mathcal{X} = \text{Spec}(\mathcal{A})$ are affine. For any two nonzero $\pi, \pi' \in K^{\circ\circ}$, there exists $n \gg 1$ with $\pi^n \in K^\circ \pi'$, and the formal completion $\hat{\mathcal{A}}$ of $\mathcal{A}$ with respect to $\pi$ is thus independent of the choice of $\pi$ (set $\hat{\mathcal{A}} = \hat{\mathcal{A}} = A$ in the trivially valued case). The $K^\circ$-algebra $\hat{\mathcal{A}}$ is flat and topologically of finite type, and $\hat{\mathcal{X}} = \text{Spf}(\hat{\mathcal{A}})$ is thus an admissible formal $K^\circ$-scheme, whose generic fiber $\hat{\mathcal{X}}_\eta$ is defined as the set of bounded multiplicative seminorms on the $K$-affinoid algebra $\hat{\mathcal{A}} := \hat{\mathcal{A}} \otimes K$. Composing such a seminorm with the canonical map $A \to \hat{\mathcal{A}}$ defines a continuous map $\hat{\mathcal{X}}_\eta \to \mathcal{X}^2$, which is easily seen to be bijective by density of the image of $A$ in $\hat{\mathcal{A}}$, and hence a homeomorphism.

The following well-known result holds for the reduction map of any admissible formal scheme (see for instance \cite{GRW15} \S 2.13).

**Lemma 4.8.** The reduction map $\text{red}_\mathcal{X} : \mathcal{X}^2 \to \mathcal{X}_s$ of any model $\mathcal{X}$ is anticontinuous and surjective. Further, the preimage $\Gamma(\mathcal{X})$ of the set of generic points of $\mathcal{X}_s$ is a finite set.

We shall call $\Gamma(\mathcal{X})$ the set of Shilov points. When $\mathcal{X}$ (and hence $X$) is affine, $\Gamma(\mathcal{X})$ is exactly the Shilov boundary of the $K$-affinoid domain $\mathcal{X}^2$ in the sense of \cite{Ber} 2.4.4. Covering a general model with affine open subschemes, we get:

**Lemma 4.9.** For any model $\mathcal{X}$ of $X$ and $f \in \mathcal{O}(X)$, the sup-seminorm on $\mathcal{X}^2$ satisfies
\[ \|f\|_{\mathcal{X}^2} := \sup_{\mathcal{X}^2} |f| = \max_{\Gamma(\mathcal{X})} |f|. \]

\footnote{The letter $\beth$ (‘bet’) is the second letter of the Hebrew alphabet. The chosen notation follows the lead of \cite{Thu07}.}
Example 4.10. If $K$ is trivially valued and $\eta$ is a generic point of $X = X_s$, the point of $X^\Sigma$ corresponding to the trivial valuation on $\kappa(\eta) = \mathcal{O}_{X, \eta}$ is the unique Shilov point mapping to $\eta$.

Example 4.11. Assume that $K$ is nontrivially valued and $X$ is normal, i.e. integrally closed in $X$ with $X$ normal. For each generic point $\eta$ of $X_s$, the local ring $\mathcal{O}_{X, \eta}$ is a rank one valuation ring. This is of course well-known when $X$ is noetherian (i.e. $K$ discretely valued), while the general case is proved in [Kna00, Theorem 2.6.1]. As observed in [GST13, Proposition 2.3], one easily checks that the corresponding point of $X^\Sigma$ is the unique Shilov point of $X$ mapping to $\eta$.

Example 4.12. If we merely assume that $X$ is integrally closed in $X$, it is still true that there is a unique Shilov point mapping to any given generic point $\eta$ of $X_s$. In the discretely valued case, this is proved in [CLT09, Lemme 2.1]. In the densely valued case, the assumption implies that $X_s$ is reduced by Theorem 4.19 below, which also implies that red$_X : X^\Sigma \to X_s$ coincides with the affinoid reduction map, and we conclude by [Ber, Proposition 2.4.4].

4.3. Sup-seminorm, integral closure and reduced fiber. The goal of this section is to review the relation between sup-seminorm, integral closure and reduced fiber.

Assume first that $K$ is non-Archimedean and nontrivially valued, $X = \text{Spec}(A)$ is affine and $X' = \text{Spec}(\hat{A})$ is an affine model of $X$. Denote as above by $\widehat{A}$ the formal completion of $A$ with respect to any nonzero $\pi \in K^\circ$, by $\widehat{A} := \widehat{A} \otimes K$ the associated $K$-affinoid algebra, and write $f \mapsto \hat{f}$ for the canonical map $A \to \widehat{A}$ (which is not injective in general, cf. Example 4.14 below). The affinoid algebra $\widehat{A}$ is equipped with the sup-seminorm $\| \cdot \|_{\sup}$, defined by setting for $g \in \widehat{A}$

$$\|g\|_{\sup} := \sup_{X^\Sigma} |g| = \max_{\Gamma_X} |g|,$$

where the second equality holds by [Ber, 2.4.4]. The sup-seminorm on $X^\Sigma$ of $f \in A$ as in Lemma 4.9 can thus be written as

$$\|f\|_{X^\Sigma} = \|\hat{f}\|_{\sup}.$$

By [BGR, 6.2.1/4] and [Ber, 3.4.3], we have:

Lemma 4.13. The sup-seminorm on $\widehat{A}$ is a norm if and only if $\widehat{A}$ is reduced. This holds in particular if $A$ is reduced.

While $A \to \widehat{A}$ has dense image, it is not injective in general, even when $A$ is reduced:

Example 4.14. If $K$ is discretely valued, $A := K$ is of finite type of $K^\circ$, and hence a model of $A = K$, for which $\widehat{A} = \{0\}$ (thanks to Antoine Ducros for this simple example).

Proposition 4.15. The unit ball of $\| \cdot \|_{\sup}$ coincides with the integral closure $\widehat{A}'$ of $\widehat{A}$ in $\widehat{A}$. Similarly, the unit ball of $\| \cdot \|_{X^\Sigma}$ coincides with the integral closure $A'$ of $A$ in $A$, and the induced map $A' \to \widehat{A}'$ further has dense image.

Corollary 4.16. A given $f \in A$ is integral over $A$ if and only if $\hat{f}$ is integral over $\widehat{A}$, and $A$ is integrally closed in $A$ if and only if $\widehat{A}$ is integrally closed in $\widehat{A}$.

We start with a useful observation.
Lemma 4.17. For each \( f \in A \), we have \( f \in A \iff \hat{f} \in \hat{A} \).

**Proof.** We imitate [BLX6 Lemma 1.4]. Pick a non-zero \( \pi \in K^\circ \) such that \( \pi f \in A \), and note that the canonical map \( A \to \hat{A} \) induces an isomorphism \( A/\pi A \simeq \hat{A}/\pi \hat{A} \). If \( \hat{f} \in \hat{A} \), we thus have \( \pi f = \pi g \) for some \( g \in \hat{A} \), and hence \( f = g \in A \) after multiplying by \( \pi^{-1} \) in \( A \).

**Proof of Proposition 4.13.** That the unit ball of \( \| \cdot \|_{\text{sup}} \) is the integral closure of \( \hat{A} \) is a reformulation of [BGR 6.3.4/1] (and the remark that follows). Before dealing with the unit ball of \( \| \cdot \|_{\chi^2} \), we first establish the density of the image of \( A' \) in \( \hat{A}' \).

Let thus \( g \in \hat{A}' \), so that \( g^n + \sum_{i=1}^{n-1} b_i g^{n-i} \in \hat{A} \) for some \( b_i \in \hat{A} \). Since \( A \to \hat{A} \) and \( A \to \hat{A} \) have dense images, we can pick sequences \( f_j \in A \) and \( a_{ij} \in A \) with \( \hat{f}_j \to g \) and \( \hat{a}_{ij} \to b_i \) as \( j \to \infty \). As \( \hat{A} \) is open in \( \hat{A} \) (for instance by Lemma 4.18 below), it follows that \( \hat{f}_j^n + \sum_{i=1}^{n-1} \hat{a}_{ij} \hat{f}_j^{n-i} \in \hat{A} \) for all \( j \gg 1 \), i.e. \( \hat{f}_j^n + \sum_{i=1}^{n-1} a_{ij} f_j^{n-i} \in A \), by Lemma 4.17. As a result, \( f_j \) is integral over \( A \), i.e. \( f_j \in A' \), which proves that \( A' \to \hat{A}' \) has dense image.

It remains to show that an element \( f \in A \) with \( \| f \|_{\chi^2} \leq 1 \) is integral over \( A \). Since \( \| f \|_{\text{sup}} \leq 1 \), we already know that \( \hat{f} \) belongs to \( \hat{A}' \). Since \( A' \to \hat{A}' \) has dense image and \( \hat{A} \) is open in \( \hat{A} \), we find \( f' \in A' \) with \( \hat{f} - \hat{f}' \in \hat{A} \), i.e. \( f - f' \in A \), and hence \( f \in A' \) (see also [CMor15 Theorem 2.10]) for a direct proof.

Besides the sup-seminorm \( \| \cdot \|_{\chi^2} \), \( A \) is also equipped with a 'lattice seminorm' \( \| \cdot \|_A \), defined by

\[
\| f \|_A := \inf \{ |\alpha| \mid \alpha \in K, f \in \alpha A \}.
\]

By Example 4.14 this is again not a norm in general. However, similarly setting for \( g \in \hat{A} \)

\[
\| g \|_{\hat{A}} := \inf \{ |\alpha| \mid \alpha \in K, g \in \alpha \hat{A} \},
\]

does yield a norm on \( \hat{A} \):

**Lemma 4.18.** For each \( g \in \hat{A} \), the infimum defining \( \| g \|_{\hat{A}} \) is achieved. In particular, \( \hat{A} \) is the closed unit ball of \( \| \cdot \|_{\hat{A}} \), and \( K^\circ \hat{A} \) is its open unit ball.

**Proof.** The result is true for the polynomial ring \( B := K^\circ[t_1, \ldots, t_r] \), since \( \| \cdot \|_B \) is then the Gauss norm on the Tate algebra \( B = K[t_1, \ldots, t_r] \). In the general case, choose a surjection \( B := K^\circ[t_1, \ldots, t_r] \to A \) for some \( r \geq 1 \), and observe that \( \| \cdot \|_{\hat{A}} \) is the quotient seminorm of \( \| \cdot \|_{\hat{B}} \) with respect to the induced surjection \( \rho : \hat{B} \to \hat{A} \). The kernel of \( \rho \), being an ideal in a Tate algebra, is strictly closed [BGR 5.2.7/8]. By definition, this means that for each \( g \in \hat{A} \), there exists \( h \in \hat{B} \) such that \( \rho(h) = g \) and \( \| h \|_B = \| g \|_{\hat{A}} \). Since the desired result holds for \( \hat{B} \), we can then find \( \alpha \in K \) with \( |\alpha| = \| h \|_B = \| g \|_{\hat{A}} \), which implies that \( h \in \alpha \hat{B} \), and hence \( g = \rho(h) \in \alpha \hat{A} \).

**Theorem 4.19.** As above, let \( \mathcal{X} = \text{Spec} A \) be an affine model of \( X = \text{Spec} A \). We then have \( \| \cdot \|_{\text{sup}} \leq \| \cdot \|_{\hat{A}} \) on \( \hat{A} \), and hence \( \| \cdot \|_{\chi^2} \leq \| \cdot \|_A \) on \( A \). Consider further the following properties:

(i) \( \mathcal{X}_s \) is reduced;
(ii) \( \| \cdot \|_{\text{sup}} = \| \cdot \|_{\hat{A}} \) on \( \hat{A} \);
(iii) \( \| \cdot \|_{\chi^2} = \| \cdot \|_A \) on \( A \);
(iv) $\hat{A}$ is integrally closed in $\hat{A}$;
(v) $A$ is integrally closed in $\hat{A}$.

Then (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v). If $K$ is densely valued, we also have (v) $\implies$ (i), so that (i)–(v) are then equivalent.

**Proof.** Pick a nonzero $g \in \hat{A}$. By Lemma 4.18 $\|g\|_{\hat{A}}$ is in the value group $|K^*|$, and we may thus assume that $\|g\|_{\hat{A}} = 1$ after multiplying $g$ by a nonzero scalar. We then have $g \in \hat{A}$, hence $|g(x)| \leq 1$ for all $x \in X^2$, which proves that $\|g\|_{\sup} \leq 1$, and hence $\|\cdot\|_{\sup} \leq \|\cdot\|_{\hat{A}}$.

Suppose now that $X_s$ is reduced, and assume by contradiction that $g$ as above satisfies $\|g\|_{\sup} < 1$. By [BGR 6.2.3/2], $g$ is topologically nilpotent, i.e. $g^n \to 0$. For $n \geq 1$, we thus have $\|g^n\|_{\hat{A}} < 1$, i.e. $g^n \in K^{\circ\circ}\hat{A}$; since $A \otimes \hat{K} \simeq \hat{A} \otimes \hat{K}$ is reduced, this implies $g \in K^{\circ\circ}\hat{A}$, which contradicts $\|g\|_{\hat{A}} = 1$. We have thus proved (i)$\implies$(ii), which trivially implies (iii) by composing with $A \to \hat{A}$. If (iii) holds, then $\|\cdot\|_{\hat{A}}$ is power-multiplicative, i.e. $\|f^n\|_{\hat{A}} = \|f\|_{\hat{A}}^n$ for each $f \in A$ and $n \in \mathbb{N}$. In particular, $f^n \in K^{\circ\circ}\hat{A}$ $\iff$ $\|f^n\|_{\hat{A}} < 1$ $\iff$ $f \in K^{\circ\circ}\hat{A}$, which means that $A \otimes \hat{K} \simeq A/K^{\circ\circ}A$ is reduced.

Since $\hat{A}$ (resp. $A$) is the unit ball of $\|\cdot\|_{\hat{A}}$ (resp. $\|\cdot\|_{\hat{A}}$), Proposition 4.15 shows that (ii) and (iii) respectively imply (iv) and (v), while Corollary 4.16 shows that (iv) and (v) are equivalent.

Assume finally that $K$ is densely valued and that (v) holds. To prove (i), we need to show that each $f \in A$ such that $f^n \in \pi A$ for some $n \geq 1$ and $\pi \in K^{\circ\circ}$ actually satisfies $f \in K^{\circ\circ}A$. Since $K$ is densely valued, we can find $\pi' \in K^{\circ\circ}$ with $|\pi'| \leq |\pi| < 1$, and hence $\pi/\pi' \in K^{\circ}$. As a result, $g := \pi'^{-1}f \in A$ satisfies $g^n \in A$, and hence $g \in A$, since $A$ is integrally closed in $A$. We have thus shown as desired that $f = \pi'g \in K^{\circ\circ}A$ (we are grateful to Walter Gubler for his help with this argument).

We conclude this section with the following rather special case of the scheme-theoretic version of the Bosch-Lütkebohmert-Raynaud reduced fiber theorem [BLR95 Theorem 2.1'].

**Theorem 4.20.** Assume that $K$ is non-Archimedean and algebraically closed, and let $X$ be a reduced $K$-scheme of finite type. For each model $X$ of $X$, the integral closure $X'$ of $X$ in $X$ is finite over $X$, and hence a model of $X$ as well. Further, $X'_s$ is reduced.

Note conversely that the existence of a model with reduced special fiber implies that $X$ is reduced, by [EGA IV,12.1.1].

**Proof of Theorem 4.20.** In the trivially valued case, there is nothing to prove, and we thus assume that $K$ is nontrivially valued. Being algebraically closed, it is then densely valued, and Theorem 4.19 thus shows that $X'_s$ will automatically be reduced.

The finiteness of $X'$ over $X$ being local, we assume that $X = \text{Spec}(A)$ is affine and use the above notation. We will reduce the result to the Grauert-Remmert finiteness theorem, basically arguing as in [BLR6 Proposition 1.5] and [Tem10 Theorem 3.5.5, Step 3].

Since $A$ is reduced, $\hat{A}$ is reduced as well by Lemma 4.13 and [BGR 6.4.1/5] thus shows that $\hat{A}'$ is finite over $\hat{A}$, i.e. $\hat{A}' = \sum_i \hat{A}g_i$ for a finite set $g_i \in \hat{A}$, in which we include 1 for convenience. As in the proof of Lemma 4.17 we can find for each $i$ some $f_i \in A$ with $g_i - \hat{f}_i \in \hat{A}$, and hence $\hat{A}' = \sum_i \hat{A}f_i$. By Corollary 4.16 an element $f \in A$ belongs to the integral closure $A'$ of $A$ in $A$ if and only $\hat{f}$ belongs to $A'$, which is also equivalent to $f \in \sum_i A\hat{f}_i$ by Lemma 4.17. We conclude as desired that $A' = \sum_i A\hat{f}_i$ is finite over $A$. $\square$
Remark 4.21. The analogue of Theorem 4.20 fails in general for \( K \) an arbitrary non-Archimedean, nontrivially valued field \( K \). Let indeed \( X = \text{Spec}(A) \) be an affine reduced \( K \)-scheme and \( \mathcal{X} = \text{Spec}(A) \) be a model with integral closure \( \mathcal{X}' \) in \( X \). If \( K \) is discretely valued, \( \mathcal{X}' \) is finite over \( \mathcal{X} \), by excellence (but \( \mathcal{X}' \) is of course not reduced in general). If \( K \) is densely valued, the finiteness of \( \mathcal{X}' \) over \( \mathcal{X} \) implies that \( \mathcal{X}' \) is reduced, and hence that the sup-seminorm \( \| \cdot \|_{\mathcal{X}} = \| \cdot \|_{\mathcal{X}'} = \| \cdot \|_{A} \) takes values in \( |K| \), a condition that is not satisfied in general when the group \( |K^*| \) is not divisible.

5. Metrics

This section introduces the class of Fubini-Study metrics, and compares it with the class of model metrics. The main result is Theorem 5.14, which compares the sup-norms and lattice norms induced by a model metric, and relies on the reduced fiber theorem.

In what follows, \( X \) denotes a projective \( K \)-scheme, where \( K \) is again a field complete with respect to an arbitrary absolute value.

5.1. Continuous and bounded metrics. A line bundle \( L \) on \( X \) admits an analytification \( L^{an} \), which can for instance be obtained as the analytification of the total space of \( L \). By a continuous metric on a line bundle \( L \) over \( X \), we mean an \( \mathcal{O}_X \)-linear function \( |\cdot| : L^{an} \to \mathbb{R}^+ \) such that for any section \( s \) of \( L \) on an open subscheme \( U \) of \( X \), the induced function \( |s| : U^{an} \to \mathbb{R}^+ \) is non-zero and is continuous. We denote by \( C^0(L) \) the space of continuous metrics on \( L \), for which we prefer to use additive notation. This amounts to the following simple rules:

(i) if \( \phi, \phi' \) are continuous metrics on \( L, L' \), then \( -\phi \) denotes the induced metric on \( -L := L^* \) and \( \phi + \phi' \) is the induced metric on \( L + L' := L \otimes L' \);
(ii) if \( \phi \) is a continuous metric on a line bundle \( L \) and \( s \in H^0(U, L) \) is a section of \( L \) on a Zariski open set \( U \subset X \), then \( |s|_\phi : U^{an} \to \mathbb{R}^+ \) denotes the corresponding length function in the metric \( \phi \);
(iii) a continuous metric \( \phi \) on the trivial line bundle \( \mathcal{O}_X \) is identified with the continuous function \(-\log |1|_\phi \in C^0(X)\).

If \( \phi, \phi' \in C^0(L) \) are continuous metrics on the same line bundle \( L \), \( \phi - \phi' \) is thus a continuous function on \( X^{an} \), which satisfies

\[
|s|_{\phi'} = |s|_\phi e^{\phi - \phi'}
\]

for any local section \( s \) of \( L \). In other words, the space \( C^0(L) \) of continuous metrics on \( L \) is an affine space, modelled on the vector space \( C^0(X^{an}) \) of (real-valued) continuous functions on \( X^{an} \).

A bounded metric on \( L \) is a metric of the form \( \phi + u \) with \( u \) a bounded function on \( X^{an} \). We denote by \( \mathcal{L}^\infty(L) \) the space of such metrics, which we endow with the sup-norm distance

\[
d(\phi, \phi') := \sup_{X^{an}} |\phi - \phi'|.
\]

5.2. Fubini-Study metrics. The usual Fubini-Study metric on the tautological ample line bundle \( \mathcal{O}(1) \) over the projective space (also known as the Weil metric in the non-Archimedean context) generalizes in a natural way as follows.

Definition 5.1. A metric \( \phi \) on a line bundle \( L \) over \( X \) is called a Fubini-Study metric if there exists \( m \geq 1 \) with \( mL \) globally generated, a basis \( (s_i) \) of \( H^0(mL) \) and positive constants \( c_i \) such that
(A) \( \phi = \frac{1}{2m} \log \sum_i (c_i |s_i|)^2 \) (Archimedean case);
(NA) \( \phi = \frac{1}{m} \log \max_i (c_i |s_i|) \) (non-Archimedean case).

When \( c_i = 1 \), we say that \( \phi \) is a pure Fubini-Study metric. We denote by
\[
FS(L) \subset C^0(L)
\]
the set of Fubini-Study metrics on \( L \).

The existence of a Fubini-Study metric of course requires \( L \) to be semiample, i.e. \( mL \) is globally generated for some \( m \geq 1 \). The meaning of the notation is that for any choice of an auxiliary continuous metric \( \psi \) on \( L \), we respectively have
\[
\phi - \psi = \frac{1}{2m} \log \sum_i (c_i |s_i|_{m\psi})^2
\]
and
\[
\phi - \psi = \frac{1}{m} \log \max_i (c_i |s_i|_{m\psi}).
\]
This is indeed independent of the choice of \( \psi \) by (5.1). In particular, if \( \ell \) is a local trivialization of \( L \) around a point \( x \), and if we are given a section \( s \) of \( L \), and write the various sections in the form \( s_i = a_i \ell^m \), \( s = a\ell \), then
\[
|s|_\phi(x) = \frac{|a(x)|}{(\sum_i (c_i |a_i(x)|^2)^{1/2m}}
\]
and
\[
|s|_\phi(x) = \frac{|a(x)|}{\max_i (c_i |a_i(x)|)^{1/m}}.
\]
A Fubini-Study metric as above with \( c_i \in |K^*| \) is pure, and every Fubini-Study is thus pure if \( |K^*| = \mathbb{R}_+^* \) (e.g. when \( K \) is Archimedean). In the general case, we have:

**Proposition 5.2.** If \( K \) is non-trivially valued, every Fubini-Study metric is a uniform limit of pure Fubini-Study metrics.

The result fails in the trivially valued case, since the only pure Fubini-Study metric is then the trivial metric (see Example 5.4 below). The result is obvious in the Archimedean case, and we henceforth assume that \( K \) is non-Archimedean, nontrivially valued.

**Lemma 5.3.** Pick \( m \geq 1 \) with \( mL \) globally generated and a generating family \( (s_i)_{1 \leq i \leq r} \) in \( H^0(mL) \). Then \( \phi := \frac{1}{m} \log \max_i |s_i| \) is a pure Fubini-Study metric.

**Proof.** The \( K^\circ \)-module generated by the \( (s_i) \) is a lattice of \( H^0(mL) \), and hence admits a \( K^\circ \)-basis \( (s'_j) \). Writing each \( s_i \) (resp. \( s'_j \)) as a \( K^\circ \)-linear combination of the \( s'_j \) (resp. \( s_i \)), it is straightforward to see that \( \phi = \frac{1}{m} \log \max_j |s'_j| \), which is thus a pure Fubini-Study metric as \( (s'_j) \) is a basis of \( H^0(mL) \). \( \square \)

**Proof of Proposition 5.2.** Any Fubini-Study metric can be written as \( \phi = \frac{1}{m} \max_i (\log |s_i| + \lambda_i) \) with \( (s_i) \) a basis of \( H^0(mL) \) and \( \lambda_i \in \mathbb{R} \). Since additive value \( \log |K^*| \) is non-trivial, \( \mathbb{Q} \log |K^*| \) is dense in \( \mathbb{R} \). Given \( \varepsilon > 0 \), we may thus find \( \ell \geq 1 \) and \( \alpha_i \in K^* \) such that \( |\frac{1}{\ell} \log |\alpha_i| - \lambda_i| \leq m\varepsilon \) for all \( i \). Fix a basis \( (\sigma_j) \) of \( H^0(\ell mL) \), pick \( \alpha \in K^* \) with \( |\alpha| < 1 \), and set for each \( k \geq 1 \)
\[
\phi_k := \frac{1}{\ell m} \log \max \left\{ \max_i |\alpha_i s_i^{\ell} \sigma_j|, \max_j |\alpha^{k\ell m} \sigma_j| \right\}.
\]
By Lemma 5.3, \( \phi_k \) is a pure Fubini-Study metric, and we have by construction
\[
-\varepsilon \leq \phi_k - \phi \leq \max \left\{ \varepsilon, k \log |\alpha| + \log \max \sup \limits_{X^an} |\sigma_j|_{\ell^mO} \right\} \leq \varepsilon
\]
for \( k \gg 1 \).

5.3. Model metrics. In this section, \( K \) is non-Archimedean (possibly trivially valued). A model of a line bundle \( L \) on our given projective \( K \)-scheme \( X \) is a line bundle \( \mathcal{L} \) on a projective model \( \mathcal{X} \) of \( X \) together with an isomorphism \( \mathcal{L}|_{\mathcal{X}_K} \cong L \) compatible with the given isomorphism \( \mathcal{X}_K \cong X \). When \( L = \mathcal{O}_X \), each model of \( L \) is of the form \( \mathcal{O}_{\mathcal{X}}(D) \) where \( D \) is a vertical Cartier divisor on a projective model \( \mathcal{X} \), i.e. \( \text{Supp} \, D \subset \mathcal{X}_s \).

A model \( \mathcal{L} \) of \( L \) defines a continuous metric \( \phi_{\mathcal{L}} \) on \( L \), as follows. Cover \( \mathcal{X} \) with finitely many open subschemes \( \mathcal{U}_i \) with a trivializing section \( \tau_i \) of \( \mathcal{L} \). Since \( \mathcal{X} \) is assumed projective over \( K^\circ \), we have \( X^an = \mathcal{X}^\mathcal{Z} \), which is thus covered by the compact sets \( \mathcal{U}_i^\mathcal{Z} \). We may thus define a continuous metric \( \phi_{\mathcal{L}} \) on \( L^an \) by requiring that \( |\tau_i|_{\phi_{\mathcal{L}}} = 1 \) on \( \mathcal{U}_i^\mathcal{Z} \). This is indeed well-defined, since any other trivializing section of \( \mathcal{L} \) on \( \mathcal{U}_i \) is isomorphic to the trivialization on \( \mathcal{U}_i^\mathcal{Z} \) on the inverse image \( \mathcal{U}_i^\mathcal{Z} \), which satisfies \( \mathcal{U}_i^\mathcal{Z} = \mathcal{U}_{i'}^\mathcal{Z} \) since \( \mathcal{U}_i^\mathcal{Z} \) is proper over \( \mathcal{U}_i^\mathcal{Z} \).

As a consequence of the first property, we may introduce:

Definition 5.6. A model metric on \( L \) is a metric of the form \( \phi = \phi_{\mathcal{L}} \), where \( \mathcal{L} \) is a \( \mathbb{Q} \)-model of \( L \), i.e. \( m\mathcal{L} \) is a model of \( mL \) for some \( m \geq 1 \).

A model function is a model metric \( \phi \) on \( \mathcal{O}_X \), identified with the function \( \phi = -\log |\cdot|_{\phi} \).

A model function \( f \) is thus determined by a vertical \( \mathbb{Q} \)-Cartier divisor \( D \) on some projective model \( \mathcal{X} \) of \( X \). Every line bundle \( L \) on \( X \) admits a model metric \( \phi \), and any other model metric on \( L \) is then of the form \( \phi + f \) with \( f \) a model function.

While the map \( \mathcal{L} \mapsto \phi_{\mathcal{L}} \) is not injective in general, we have:

Lemma 5.7. Let \( \mathcal{L}, \mathcal{L}' \) be two models of a line bundle \( L \) defined on the same projective model \( \mathcal{X} \) of \( X \), and assume that \( \mathcal{X}_s \) is reduced. Then \( \phi_{\mathcal{L}} = \phi_{\mathcal{L}'} \) if and only if \( \mathcal{L} = \mathcal{L}' \) (as models of \( L \)).

Proof. Let \( \mathcal{U} = \text{Spec}(A) \) be an affine open subscheme of \( \mathcal{X} \) with trivializing sections \( \tau \in H^0(\mathcal{U}, \mathcal{L}), \tau' \in H^0(\mathcal{U}, \mathcal{L}') \). Since \( \mathcal{L}, \mathcal{L}' \) are both models of the same line bundle \( L \), the restrictions of \( \tau, \tau' \) to the generic fiber \( U = \text{Spec}(A_K) \) of \( \mathcal{U} \) satisfy \( \tau'|_U = u\tau|_U \) with \( u \in A_K \) a unit. As \( \phi_{\mathcal{L}} \) and \( \phi_{\mathcal{L}'} \) coincide, the definition of model metrics yields \( |u| = 1 \) on \( \mathcal{U}^\mathcal{Z} \). Since \( \mathcal{U}_s \) is reduced, Theorem 4.19 shows that \( \| \cdot \|_{sup} = \| \cdot \|_A \) on \( A_K \). As \( \|u\|_{sup} = \|u^{-1}\|_{sup} = 1 \), it follows that \( u \) and \( u^{-1} \) belong to \( A \), i.e. \( u \) is a unit in \( A \), and we conclude that \( \mathcal{L} = \mathcal{L}' \).
Remark 5.8. By Theorem 4.19, $X_r$ reduced implies $X$ integrally closed in $X$. In the discretely valued case, the latter condition is weaker in general, but it guarantees that $A$ is the unit ball of $\| \cdot \|_{\sup}$ on $A_K$ by [CMor15, Theorem 2.1], and the above argument thus shows that Lemma 5.7 is more generally valid as soon as $X$ is integrally closed in $X$.

The next result describes the behavior of model metrics under pull-back.

Proposition 5.9. Let $L$ be a line bundle on $X$, and $\phi_L$ be the model metric determined by a $\mathbb{Q}$-model $\mathcal{L}$ of $L$ on a model $\mathcal{X}$ of $X$. For any projective morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$, the induced metric $f^*\phi$ on $f^*L$ is also a model metric. More precisely, the set of projective models $\mathcal{Y}$ of $Y$ such that $f$ extends to a $K^\circ$-morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ is cofinite in the set of all projective models, and we have $f^*\phi_L = \phi_{f^*\mathcal{L}}$ for any such model $\mathcal{Y}$. Moreover, after possibly passing to a model $\mathcal{X}' \rightarrow \mathcal{X}$, if $f : \mathcal{Y} \rightarrow \mathcal{X}$ is flat we can assume that $f : \mathcal{Y} \rightarrow \mathcal{X}$ is flat.

Proof. Pick a projective model $\mathcal{Y}'$ of $Y$, and denote by $\mathcal{Y}$ the schematic (or flat) closure in $\mathcal{Y}' \times_{K^\circ} \mathcal{X}$ of the graph of $f$. Then $\mathcal{Y}$ dominates $\mathcal{Y}'$, and the projection $\mathcal{Y} \rightarrow \mathcal{X}$ extends $f$. To see the last point, we may assume that $\mathcal{L}$ is a line bundle. Let $(U_i)$ be a finite open cover of $\mathcal{Y}$ with trivializing sections $\tau_i \in H^0(U_i, L)$. Then $(f^{-1}(U_i))$ is an open cover of $\mathcal{Y}$ with trivializing sections $f^*\tau_i$ for $f^*\mathcal{L}$, and the result easily follows.

For the last point, by the main result of Raynaud-Gruson in [RG71], we can blow up $\mathcal{X}$ and take the proper transform of $\mathcal{Y}$ to obtain a flat morphism $f : \mathcal{X}' \rightarrow \mathcal{Y}'$ of models. By Lemma 5.5 this does not change the metrics induced by the models. \hfill $\Box$

We are now in a position to compare model metrics and Fubini-Study metrics (see [CMor15, Proposition 3.8] for a related result).

Proposition 5.10. Assume that $X$ is projective, and let $\phi$ be a continuous metric on $L$. The following are equivalent:

(i) $\phi$ is a pure Fubini-Study metric;

(ii) $\phi$ is a model metric determined by a semiample $\mathbb{Q}$-model $\mathcal{L}$ of $L$, i.e. $m\mathcal{L}$ is a globally generated line bundle when $m \in \mathbb{N}$ is sufficiently divisible.

Corollary 5.11. A function $u \in C^0(X^{an})$ is a model function if and only if $u = \phi - \phi'$ with $\phi, \phi'$ pure Fubini-Study metrics on some ample line bundle.

Proof of Proposition 5.10. The key observation is that the Weil metric $\log \max_i |z_i|$ on $\mathbb{P}_K^r$ with homogeneous coordinates $[z_0 : \cdots : z_r]$ coincides with the model metric $\phi_{\mathcal{O}(1)}$ determined by the canonical model of $(\mathbb{P}^r, \mathcal{O}(1))$ over $K^\circ$.

Let first $\phi$ be a pure Fubini-Study metric on $L$. After replacing $L$ by a multiple, we may assume that $L$ is globally generated and $\phi = \log \max_i |s_i|$ with $(s_i)$ a basis of $H^0(L)$. These sections induce a morphism of $K$-schemes $f : X \rightarrow \mathbb{P}_K^r$ with an identification $L = f^*\mathcal{O}(1)$ such that $\phi = f^*\phi_{\mathcal{O}(1)}$. By Proposition 5.9, $X$ admits a projective model $\mathcal{X}$ such that $f$ extends to a morphism $f : \mathcal{X} \rightarrow \mathbb{P}_{K^\circ}^r$, and $\phi$ is the model metric defined by $(\mathcal{X}, f^*\mathcal{O}(1))$. Since $f^*\mathcal{O}(1)$ is globally generated, this proves (i)$\Rightarrow$(ii). Conversely, let $\mathcal{L}$ be a semiample $\mathbb{Q}$-model of $L$. After passing to a multiple, we may assume that $\mathcal{L}$ is a globally generated line bundle. Choosing a $K^\circ$-basis $(s_i)$ of the lattice $H^0(L)$ yields a morphism $f : \mathcal{X} \rightarrow \mathbb{P}_{K^\circ}^r$ with $f^*\mathcal{O}(1) = L$, and hence $\phi_L = f^*\phi_{\mathcal{O}(1)} = \log \max_i |s_i|$, by Proposition 5.9. \hfill $\Box$

Proof of Corollary 5.11. One direction is clear, since a pure Fubini-Study metric is a model metric, and a difference of model metrics is a model metric. Conversely, assume that $u$ is a
model function. After multiplying $u$ by a constant, we may assume that it is determined by a vertical Cartier divisor $D$ on a projective model $X$. Let $\mathcal{L}$ be a relatively ample line bundle on $X$, and denote by $L$ its restriction to $X$. For $m \gg 1$, $D + m \mathcal{L}$ is then (very) ample on $X$, and $u = \phi_{D + m \mathcal{L}} - \phi_{m \mathcal{L}}$ is a difference of pure Fubini-Study metrics on $m \mathcal{L}$.

5.4. The sup-norm of a model metric. The data of a bounded metric $\phi$ on $L$ defines a sup-seminorm $\| \cdot \|_\phi$ on $H^0(X, L)$ by setting

$$\|s\|_\phi := \sup_{X^\text{an}} |s|_\phi$$

for each $s \in H^0(X, L)$. It follows from Lemma 4.1 that $\|s\|_\phi = 0$ if and only if $s$ is nilpotent, and that $\| \cdot \|_\phi$ is a norm if $X$ is reduced.

In the remainder of this section, $K$ is non-Archimedean. We first consider the behavior of sup-norms under ground field extension.

Lemma 5.12. Let $K'/K$ be an arbitrary complete field extension, and denote by $(X', L')$ the base change of $(X, L)$ to $K'$. Let also $\phi$ be a bounded metric on $L$, denote by $\phi'$ its pull-back to $L'$, and by $\| \cdot \|'_\phi$ the ultrametric seminorm on $H^0(X', L') = H^0(X, L) \otimes K'$ induced from $\| \cdot \|_\phi$ by ground field extension as in Definition 1.25. Then $\| \cdot \|_\phi \leq \| \cdot \|'_\phi$, and both coincide with $\| \cdot \|_\phi$ on $H^0(X, L)$ viewed as a subspace of $H^0(X', L')$.

Proof. By Proposition 1.26, the restriction of $\| \cdot \|'_\phi$ to $H^0(X, L)$ coincides with $\| \cdot \|_\phi$. Denoting by $p : X^\text{an} \to X^\text{an}$ the surjective projection map, we plainly have

$$\|s\|'_\phi = \sup_{X^\text{an}} |s|_\phi \circ p = \sup_{X^\text{an}} |s|_\phi = \|s\|_\phi.$$

Now pick $s' \in H^0(X', L')$, and consider a decomposition $s' = \sum \alpha_i s_i$ with $\alpha_i \in K'$ and $s_i \in H^0(X, L)$. Since $\|s_i\|'_\phi = \|s_i\|_\phi$, we have $\|s'\|'_\phi \leq \max \|\alpha_i\|_K \|s_i\|_\phi$, and taking the infimum over all such decompositions yields as desired $\|s'\|'_\phi \leq \|s\|'_\phi$. \hfill \qed

The inequality in Lemma 5.12 is strict in general. One has for instance:

Lemma 5.13. Assume that $K$ is discretely and nontrivially valued, and let $K'/K$ be a finite Galois extension. The following are equivalent:

(i) $K'/K$ is tamely ramified, i.e. $K'/K$ is separable and the ramification order $|K'| : |K^*|$ is prime to the residue characteristic.

(ii) for each bounded metric $\phi$ on a line bundle $L$ over a geometrically reduced projective $K$-scheme $X$, we have $\| \cdot \|'_\phi = \| \cdot \|'_\phi$ in the notation of Lemma 5.12.

Proof. In the situation of (ii), $\| \cdot \|'_\phi$ is a Galois invariant norm on $H^0(X', L') = H^0(X, L) \otimes K'$ (since $X'$ is reduced). If $K'/K$ is tamely ramified, the descent result of [Rou77, Proposition 5.1.1] (see also [Pra01]) states that every Galois invariant norm is obtained by ground field extension, and hence (i)$\implies$(ii).

Conversely, for $X := \text{Spec} K'$, $L = \mathcal{O}_X$ and $\phi$ the trivial metric, $\| \cdot \|'_\phi$ is the spectral norm on $K' \otimes_K K'$, and [RTW13 §5.1] thus says that $\| \cdot \|'_\phi = \| \cdot \|'_\phi$ if and only if $K'$ is tamely ramified. \hfill \qed

The next result will be a key element in the proof of Theorem 8.5 below.
Theorem 5.14. Assume that $X$ is geometrically reduced, and let $\phi$ be the model metric determined by a line bundle $L$ on a projective model $X$ of $X$. The sup-norm $\| \cdot \|_{H^0(X, mL)}$ and the lattice norm $\| \cdot \|_{H^0(X, mL)}$ on $H^0(X, mL)$ then satisfy

$$\| \cdot \|_{m\phi} \leq \| \cdot \|_{H^0(X, mL)} \leq C \| \cdot \|_{m\phi}$$

for a uniform constant $C > 1$.

Lemma 5.15. Let $\phi$ be a model metric determined by a line bundle $L$ on a projective model $X$.

(i) For each $s \in H^0(X, L)$, we have $\sup_{X^{\text{an}}} |s|_\phi = \max_{\Gamma(X)} |s|_\phi$, where $\Gamma(X) \subset X^{\text{an}}$ denotes the (finite) set of Shilov points associated to $X$.

(ii) We have $\| \cdot \|_{\phi} \leq \| \cdot \|_{H^0(X, L)}$, the lattice norm determined by the lattice $H^0(X, L)$ of $H^0(X, L)$.

(iii) If $X_s$ is further reduced, then $\| \cdot \|_{\phi} = \| \cdot \|_{H^0(X, L)}$.

Proof. Cover $X$ with finitely many open subschemes $U_i$ with trivializing sections $\tau_i \in H^0(U_i, L)$. Denoting by $U_i$ the generic fiber of $U_i$, we have $s|_{U_i} = f_i \tau_i$ with $f_i \in \mathcal{O}(U_i)$, and $s|_{\phi} = |f_i|$ on $U_i^2$, by definition of a model metric. By Lemma 4.9, it follows that $\sup_{U_i^2} |s|_{\phi} = \max_{\Gamma(U_i)} |s|_{\phi}$, and (i) follows since $X^{\text{an}} = \bigcup_i U_i^2$ and $\Gamma(X) = \bigcup_i \Gamma(U_i)$.

In order to prove (ii) and (iii), we may assume that $s$ is nonzero. Since $\|s\|_{H^0(X, L)}$ belongs to the value group $|K^*|$, we may then multiply $s$ by a scalar and assume that $\|s\|_{H^0(X, L)} = 1$, i.e., $s \in H^0(X, L)$ but $s \notin K^0 H^0(X, L)$. Choose as above a finite cover $X$ by affine open subscheme $U_i = \text{Spec}(A_i)$ with trivializing section $\tau_i \in L$, and write $s|_{U_i} = f_i \tau_i$ with $f_i \in A_i$. On $U_i^2$, we have $|s|_{\phi} = |f_i| \leq 1$, and hence $\|s\|_{\phi} \leq 1$, which proves (ii).

Finally, suppose that $X_s$ is reduced, and assume by contradiction $\|s\|_{\phi} < 1$. For each $i$, we then have $\sup_{U_i^2} |f_i| < 1$, and hence $\pi_i^{-1} f_i \in A_i$ for some nonzero $\pi_i \in K^0$, by Theorem 4.19. Setting $\pi = \pi_{i_0}$ for an index $i_0$ achieving $\max_i |\pi_i|$, we infer $\pi^{-1} f_i \in A_i$ for all $i$, and hence $\pi^{-1} s \in H^0(X, L)$, a contradiction.

Proof of Theorem 5.14: Fix an algebraically closed complete field extension $K'/K$, and denote by $(X', L')$ the base change of $(X, L)$ to $K'$, by $(X', L')$ the base change of $(X, L)$ to $K^0$. The pull-back $\phi'$ of $\phi$ to $L'$ is the model metric determined by $L'$. By Lemma 5.12, the norms $\| \cdot \|_{m\phi'}$ and $\| \cdot \|_{H^0(X', mL')}$ induced from $\| \cdot \|_{m\phi}$ and $\| \cdot \|_{H^0(X, mL)}$ by ground field extension satisfy

$$\| \cdot \|_{m\phi'} \leq \| \cdot \|_{m\phi} \leq \| \cdot \|_{H^0(X, mL)},$$

and the latter is the lattice norm defined by $H^0(X, mL) \otimes K^0 \simeq H^0(X', mL')$, by Lemma 1.31.

Since $X$ is geometrically reduced, $X'$ is reduced, and Theorem 4.20 thus yields a model $\mathcal{X}'$ on $X'$ with a finite morphism $\mu : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'_{m\mu}$ is reduced. By Lemma 5.5, we have $\phi' = \phi_{\mathcal{X}'_{m\mu}}$, and hence $\| \cdot \|_{H^0(X', mL')} = \| \cdot \|_{m\phi'}$, by Lemma 5.15. All in all, we get

$$\| \cdot \|_{H^0(X', mL')} = \| \cdot \|_{m\phi'} \leq \| \cdot \|_{m\phi} \leq \| \cdot \|_{H^0(X, mL)} = \| \cdot \|_{H^0(X', mL')},$$

and it will thus be enough to show that $\| \cdot \|_{H^0(X, mL)} \leq C \| \cdot \|_{H^0(X', mL')} \otimes H^0(X', mL') \hookrightarrow H^0(X', \mathcal{F}(mL'))$, for a uniform constant $C > 0$. By the projection formula, we have an injection of $K^0$-modules

$$H^0(X', mL') \otimes H^0(X', mL') \hookrightarrow H^0(X', \mathcal{F}(mL')),$$
where $F := (\mu_\ast \mathcal{O}_{X''}) / \mathcal{O}_{X''}$ is a coherent module on $X''$ supported in the special fiber (cf. Theorem A.6 for coherence), and hence $\pi$-torsion for some nonzero $\pi \in K^{\times \infty}$. We thus have $\pi H^0(X'', m\mu^r L') \subset H^0(X', mL')$, and hence $\| \cdot \|_{H^0(X', mL')} \leq |\pi|^{-1} \| \cdot \|_{H^0(X'', m\mu^r L')}$, which yields the desired result. □

For later use, we also establish the following technical generalization of Theorem 5.14. A bounded metric on a line bundle $L$ over $X$ induces for each $r \geq 1$ a bounded metric $\phi^\otimes r$ on the external tensor product $L \otimes^\times$ over $X' := X \times_K \cdots \times_K X$ ($r$ times). If $\phi$ is the model metric determined by a line bundle $L$ over $X$, then $\phi^\otimes r$ is the model metric determined by $L \otimes^\times$ over $X' := X \times_K \cdots \times_K X$.

**Theorem 5.16.** Assume that $X$ is geometrically reduced, and let $\phi_i$ be a finite family of model metrics on line bundles $L_i$, with $\phi_i$ determined by a line bundle $L_i$ on a given model $X'$. We can then find a constant $C > 1$ such that for all integers $r, m_i \geq 1$, we have

$$\| \cdot \|_{H^0(X', \sum_i m_i L_i)} \leq C^r \| \cdot \|_{H^0(X'', \sum_i m_i L'_i)}$$

as norms on $H^0(X', \sum_i m_i L_i)$.

Note that $X'$ is reduced for each $r$, since $X$ is geometrically reduced, so that $\| \cdot \|_{H^0(X', \sum_i m_i L_i)}$ is indeed a norm.

**Proof.** Use the notation in the proof of Theorem 5.14. Since $K'$ is algebraically closed, so is its residue field $\tilde{K}'$; the $\tilde{K}'$-scheme $X''$ is thus geometrically reduced, and

$$(X''_{s'})_s = (X'' \times_{K'^o} \cdots \times_{K'^o} X''')_s = X'' \times_{\tilde{K}'} \cdots \times_{\tilde{K}'} X''$$

is therefore reduced. As above, we have

$$\| \cdot \|_{H^0(X'', \sum_i m_i L'_i)} = \| \cdot \|_{H^0(X'', \sum_i m_i L'_i)} \leq \| \cdot \|_{H^0(X', \sum_i m_i L_i)}$$

and

$$H^0 \left( (X''_{s'})_s (\sum_i m_i L'_i) \right) \to H^0 (X', (\sum_i m_i L_i) \otimes^\times, \mathcal{F}_r (\sum_i m_i L'_i) \otimes^\times)$$

with $\mathcal{F}_r := (\mu_{s'} \ast \mathcal{O}_{X''}) / \mathcal{O}_{X''}$. It will thus be enough to show that $\mathcal{F}_r$ is $\pi^r$-torsion, where $\pi \in K^{\times \infty}$ annihilates $\mathcal{F} = (\mu_\ast \mathcal{O}_{X''}) / \mathcal{O}_{X''}$ as above. Let $U' = \text{Spec}(A')$ be an affine open subset of $X'$. Since $\mu$ is finite, $\mu^{-1}(U') = \text{Spec}(A''')$ is also affine. Since $\pi A'''' \subset A$, we get $\pi^r A''' \otimes_{K'} \cdots \otimes_{K'} A'''' \subset A \otimes_{K'} \cdots \otimes_{K'} A'$, which implies as desired $\pi^r (\mu_{s'} \ast \mathcal{O}_{X''}) \subset \mathcal{O}_{X''}$. □

6. LIMITS OF FUBINI-STUDY METRICS

We introduce here uniform limits of Fubini-Study metrics, and a notion of semipositive envelope of a continuous metric that arises naturally in this context, which we compare to existing notions when an appropriate pluripotential theory is available. In what follows, $X$ still denotes a projective scheme over a complete valued field $K$. 

6.1. Asymptotically Fubini-Study metrics. We introduce the following terminology, and discuss its relation to known results on semipositive metrics.

**Definition 6.1.** We say that a continuous metric $\phi$ on a semiample line bundle $L$ over $X$ is asymptotically Fubini-Study if it is a uniform limit of Fubini-Study metrics. We denote by 
\[
\mathcal{FS}(L) \subset C^0(L)
\]
the space of continuous metrics so defined.

**Remark 6.2.** A continuous metric $\phi$ as in Definition 6.1 is simply called semipositive in \cite{CMor15}*§3.3], but we prefer to stick with the above, more precise, terminology to avoid potential confusion with other notions of semipositivity.

Assume first that $K$ is Archimedean (the case $K = \mathbb{R}$ merely consisting in working with conjugation-invariant functions). In that case, both local and global pluripotential theory is fully developed, and a (possibly singular) metric $\phi$ on $L$ is unambiguously called semipositive (or psh) if the function $-\log |\tau|_\phi$ is plurisubharmonic for each local trivialization $\tau$ of $L$. An asymptotically Fubini-Study metric is obviously continuous and semipositive. Conversely:

**Theorem 6.3.** If $K$ is Archimedean and $L$ is ample, every singular semipositive metric $\phi$ on $L$ is the limit of a decreasing sequence of Fubini-Study metrics $\phi_j$. If $\phi$ is further continuous, then $\phi_j$ converges uniformly to $\phi$, and $\phi$ is thus asymptotically Fubini-Study.

**Proof.** The second part of the statement follows directly from Dini’s lemma. When $X$ is smooth, the result was proved by Demailly, based on the deep Ohsawa-Takegoshi $L^2$-extension theorem (see for instance \cite{GZ05}*Theorem 8.1] and its proof). In the general case, we may assume that $L$ is very ample, yielding a closed embedding $X \hookrightarrow \mathbb{P}V$ with $V = H^0(X, L)$. By \cite{CGZ13}*Theorem B'], $\phi$ is the restriction of a singular semipositive metric $\psi$ on $\mathcal{O}(1)$ over $\mathbb{P}V$. Since $\mathbb{P}V$ is smooth, Demailly’s result implies that $\psi$ is the decreasing limit of a sequence $\psi_j$ of Fubini-Study metrics associated to sections of $H^0(\mathbb{P}V, \mathcal{O}(m_j))$ with $m_j \to \infty$. We claim that the restriction of $\psi_j$ to $X$ is a Fubini-Study metric on $L$ for $j \gg 1$, which will conclude the proof. For each $j$, we can find a basis $(s_{ji})$ of $H^0(\mathbb{P}V, \mathcal{O}(m_j))$ such that
\[
\psi_j = \frac{1}{2m_j} \log \sum_i |s_{ji}|^2
\]
If we denote by $\| \cdot \|_j$ the Hermitian norm on $H^0(\mathbb{P}V, \mathcal{O}(m_j))$ that has $(s_{ji})$ as an orthonormal basis, it is well-known that
\[
\log \sum_i |s_{ji}|^2 = \log \max_{s \in H^0(\mathbb{P}V, \mathcal{O}(m_j)) \setminus \{0\}} \frac{|s|}{\|s\|_j},
\]
cf. Lemma 6.13 below. For $j \gg 1$, the restriction map $H^0(\mathbb{P}V, \mathcal{O}(m_j)) \to H^0(X, m_jL)$ is surjective, and $\| \cdot \|_j$ thus induces a quotient norm $\| \cdot \|'_j$ on $H^0(X, m_jL)$, which is also Hermitian and satisfies
\[
\left( \max_{s \in H^0(\mathbb{P}V, \mathcal{O}(m_j)) \setminus \{0\}} \frac{|s|}{\|s\|_j} \right) |X = \max_{\sigma \in H^0(X, m_jL) \setminus \{0\}} \frac{|\sigma|}{\|\sigma\|'_j}.\]
If we choose an orthonormal basis $(\sigma_{jk})$ of $H^0(X, m_jL)$, we thus have
\[
\psi_j|_X = \frac{1}{2m_j} \log \sum_k |\sigma_{jk}|^2,
\]
which proves as desired that $\psi_j^L$ is Fubini-Study. \hfill \Box

We now turn to the non-Archimedean case. The first notion of semipositive metric introduced in that context goes back to the work of Shou-Wu Zhang: a continuous metric $\phi$ on $L$ is **semipositive in the sense of Zhang** if it is a uniform limit of model metrics $\phi_j$ associated to $\mathbb{Q}$-models $(X_j, L_j)$ with $L_j$ nef on the special fiber of $X_j$ (we refer to [GM16] for a thorough discussion). Note that this notion of semipositivity is not appropriate when $K$ is trivially valued, since it is then only satisfied by the trivial metric.

On the other hand, Chambert-Loir and Ducros introduced in [CLD12] a general notion of **smooth psh** function on non-Archimedean analytic spaces, a typical example of which is as follows: given invertible function $u_1, \ldots, u_r$ on a Zariski open set $U \subset X$ and a smooth convex function $\chi : \mathbb{R}^r \to \mathbb{R}$, the function $\chi(\log |u_1|, \ldots, \log |u_r|)$ is smooth psh on $U^{\text{an}}$. Naturally, a continuous metric $\phi$ on a line bundle $L$ is said to be smooth psh if $-\log |\tau|^\phi$ is smooth psh on $U^{\text{an}}$ for each trivializing section $\tau$ of $L$ on a Zariski open $U \subset X$.

**Example 6.4.** Let $s_1, \ldots, s_r \in H^0(X, L)$ be sections without common zeroes, $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, and let $\max_{\mathbb{R}} : \mathbb{R}^r \to \mathbb{R}$ be a regularized max function. Then

$$
\phi := \max_{\mathbb{R}} \{ \log |s_1| + \lambda_1, \ldots, \log |s_r| + \lambda_r \}
$$

defines a smooth psh metric on $L$. This is indeed checked exactly as in [CLD12, 6.3.2].

Slightly generalizing [CLD12, §6.3.1], we introduce the following terminology.

**Definition 6.5.** A (possibly unbounded) metric $\phi$ on a line bundle $L$ is **psh-regularizable** if $\phi$ can be written as the decreasing limit of a sequence of smooth psh metrics $\phi_j$ on $L$.

Note that $\phi$ is automatically usc, and hence bounded above. If $\phi$ is continuous, the convergence $\phi_j \to \phi$ is uniform by Dini’s lemma. Conversely, for any uniformly convergent sequence $(\phi_j)$ of continuous metrics, we can find a sequence of constants $c_j \to 0$ such that the new sequence $(\phi_j + c_j)$ is decreasing, and it follows that a continuous, psh-regularizable metric is the same as a **globally psh-approachable** metric in the sense of [CLD12, §6.3.1].

**Theorem 6.6.** Let $\phi$ be a continuous metric on a line bundle $L$.

(i) If $\phi$ is asymptotically Fubini-Study, then $\phi$ is psh-regularizable.

(ii) If $K$ is non-trivially valued and $L$ is ample, then $\phi$ is asymptotically Fubini-Study if and only if $\phi$ is semipositive in the sense of Zhang.

The proof of (ii) is also sketched in [CMor15, Remark 3.18].

**Proof of Theorem 6.6.** By definition, continuous, psh-regularizable metrics form a closed set under uniform limits, and we may thus assume that $\phi$ is a Fubini-Study metric in proving (i). After replacing $L$ with a multiple, there exists a basis $(s_1, \ldots, s_N)$ of $H^0(X, L)$ without common zeroes and constants $\lambda_i \in \mathbb{R}$ such that $\phi = \max \{ \log |s_1| + \lambda_1 \ldots, \log |s_N| + \lambda_N \}$, and $\phi$ is thus the uniform limit of the smooth psh metrics obtained by replacing max with a regularized max $\max_{\mathbb{R}}$ (compare [CLD12, 6.8.3]).

We turn to (ii). Since $K$ is nontrivially valued, every Fubini-Study metric is a uniform limit of pure Fubini-Study metrics, by Proposition 5.2 By Proposition 5.10 the latter are precisely the model metrics associated to $\mathbb{Q}$-models $(X, L)$ with $L$ semiample, and hence nef on $X$, and it follows that every asymptotically Fubini-Study metric is semipositive in the sense of Zhang.
To prove the converse, it is enough to show that a model metric \( \phi = \phi_L \) associated to a \( \mathbb{Q} \)-model \((\mathcal{X}, \mathcal{L})\) with \( \mathcal{L} \) nef on \( \mathcal{X} \) is asymptotically Fubini-Study. Since \( \mathcal{L} \) is ample, it extends to an ample line bundle \( \mathcal{H} \) on some projective model \( \mathcal{X}' \) dominating \( \mathcal{X} \) (cf. [GM16 Lemma 4.12]). The pull-back \( L' \) of \( \mathcal{L} \) is nef on \( \mathcal{X}' \), and \((1 - \varepsilon)L' + \varepsilon\mathcal{H}\) is thus ample on \( \mathcal{X} \), for every \( \varepsilon \in (0,1) \cap \mathbb{Q} \). By [EGA] IV.9.6.4, \((1 - \varepsilon)L' + \varepsilon\mathcal{H}\) is then ample on \( \mathcal{X} \), and hence also semiample. The model metric \( \phi_\varepsilon \) it defines is thus a (pure) Fubini-Study metric, and one immediately sees that \( \phi_\varepsilon \to \phi_{L'} = \phi_L \) uniformly. \( \square \)

For later use, we also prove:

**Lemma 6.7.** Let \( L \) be a line bundle, \( s \in H^0(X, L) \) a regular section, and denote by \( \log|s| \) the corresponding singular metric on \( L \). We can then find two ample line bundles \( A, B \), an (unbounded) psh-regularizable metric \( \phi_A \) on \( A \) and a smooth psh metric \( \phi_B \) on \( B \) such that \( L = A - B \) and \( \log|s| = \phi_A - \phi_B \).

**Proof.** Pick a very ample line bundle \( B \) such that \( A := L + B \) is also very ample. Let \((s_1, \ldots, s_r)\) be a basis of \( H^0(X, B) \), and set

\[
\phi_A := \max_\varepsilon\{\log|s \cdot s_1|, \ldots, \log|s \cdot s_r|\}, \quad \phi_B := \max_\varepsilon\{\log|s_1|, \ldots, \log|s_r|\},
\]

where \( \max_\varepsilon \) denotes as above a regularized max function for any fixed value of \( \varepsilon \). We clearly have \( \log|s| = \phi_A - \phi_B \), and \( \phi_B \) is smooth psh by Example 6.4. To see that \( \phi_A \) is psh-regularizable, pick a basis \((\sigma_1, \ldots, \sigma_p)\) of \( H^0(X, A) \), and observe that \( \phi_A \) is the decreasing limit of the sequence of smooth psh metrics

\[
\phi_{A,j} := \max_\varepsilon\{\log|s \cdot s_1|, \ldots, \log|s \cdot s_r|, \log|\sigma_1| - j, \ldots, \log|\sigma_p| - j\}.
\]

\( \square \)

**Remark 6.8.** Still assuming that \( K \) is non-Archimedean, a general notion of singular semipositive metric has so far only been introduced when \( X \) is smooth, \( L \) is ample, and one the following holds:

\begin{itemize}
  \item [(ii)] \( K \) is nontrivially valued and \( \dim X = 1 \) [Thu05];
  \item [(iii)] \( K \) is discretely or trivially valued, of residue characteristic 0 [BFJ16 BJT18].
\end{itemize}

In both cases, every singular semipositive metric \( \phi \) is known to be the pointwise limit of a decreasing sequence of Fubini-Study metrics, and hence is psh-regularizable. Note however that it is not at all clear at this point that a smooth psh metric in the sense of [CLD12] is semipositive in the sense of [BFJ16].

### 6.2. From norms to metrics

In this section, \( X \) is assumed to be reduced. The following discussion is closely related to the point of view developed by Chen and Moriwaki in [CMor15 §3.3]. We adopt an "operational" approach, which can be seen as an "\( L^\infty \)-version" of the one used by Donaldson in his study of balanced metrics [Don01, Don2]. We denote by

\[
\mathcal{N}_m := \mathcal{N}(H^0(mL))
\]

the space of norms on \( H^0(mL) \). Since \( X \) is reduced, the sup-seminorm \( \| \cdot \|_{m, \phi} \) defined by a bounded metric \( \phi \) on \( L \) is a norm, and we may therefore introduce:

**Definition 6.9.** The sup-norm operator \( L_m^\infty : \mathcal{L}^\infty(L) \to \mathcal{N}_m \) is defined by sending a bounded metric \( \phi \) to the induced sup-norm \( \| \cdot \|_{m, \phi} \) on \( H^0(mL) \).
Clearly, the sup-norms \( \| \cdot \|_{m\phi} \) induced by a given metric \( \phi \) define a submultiplicative graded norm on the section ring \( R(L) \), in the terminology of Section 2.6. The following properties are straightforward.

**Lemma 6.10.** The sup-norm operator \( L^\infty_m \) is non-increasing and \( m \)-Lipschitz, i.e. we have
\[
d^\infty(\| \cdot \|_{m\phi}, \| \cdot \|_{m\phi'}) \leq m \sup_{X_{an}} |\phi - \phi'| \]
for any two bounded metrics \( \phi, \phi' \) on \( L \). If \( K \) is non-Archimedean, the image of \( L^\infty_m \) is contained in the set \( N^\text{ultr}_m \subset N_m \) of ultrametric norms on \( H^0(mL) \).

**Definition 6.11.** Pick \( m \geq 1 \) with \( mL \) globally generated. To each norm \( \| \cdot \| \in N_m \) we associate a metric \( FS_m(\| \cdot \|) \) on \( L \) by setting
\[
FS_m(\| \cdot \|) := \frac{1}{m} \log \sup_{s \in H^0(mL) \setminus \{0\}} \frac{|s|}{\|s\|}. \quad (6.1)
\]

**Remark 6.12.** In the terminology of [CMor15, §3.2], \( FS_m(\| \cdot \|) \) is the quotient norm induced by the surjective morphism \( H^0(mL) \otimes \mathcal{O}_X \to \mathcal{O}_X(mL) \).

We shall soon see that the metric \( FS_m(\| \cdot \|) \) is in fact asymptotically Fubini-Study (and hence continuous). At any rate, it is clear that
\[
\sup_{X_{an}} |FS_m(\| \cdot \|) - FS_m(\| \cdot \|')| \leq \frac{1}{m} d^\infty(\| \cdot \|, \| \cdot \|') \quad (6.2)
\]
for any two \( \| \cdot \|, \| \cdot \|' \in N_m \).

**Lemma 6.13.** Assume that \( mL \) is globally generated, let \( \| \cdot \| \) be a diagonalizable norm on \( H^0(mL) \), and pick an orthogonal basis \( (s_i) \) for \( \| \cdot \| \). Then
\[
(A) \quad FS_m(\| \cdot \|) = \frac{1}{2m} \log \sum_i \frac{|s_i|^2}{\|s_i\|^2} \quad \text{(Archimedean case)};

(NA) \quad FS_m(\| \cdot \|) = \frac{1}{m} \log \max_i \frac{|s_i|}{\|s_i\|} \quad \text{(non-Archimedean case)}.
\]
In particular, a metric \( \phi \) on \( L \) is Fubini-Study if and only if \( \phi = FS_m(\| \cdot \|) \) with \( \| \cdot \| \in N^\text{diag}_m \) for some \( m \).

**Proof.** The result is well-known in the Archimedean case, and follows directly from Lemma 1.16 in the non-Archimedean case.

**Example 6.14.** Assume that \( K \) is non-Archimedean, and let \( L \) be a semiample model of \( L \). The model metric \( \phi_L \) and the lattice norm \( \| \cdot \|_{H^0(mL)} \) on \( H^0(mL) \) are then related by
\[
\phi_L = FS_m(\| \cdot \|_{H^0(mL)})
\]
for any \( m \geq 1 \) such that \( mL \) is globally generated. This follows indeed from Proposition 5.10 (and its proof).

Since by Proposition 1.6 each norm \( \| \cdot \| \) on \( H^0(mL) \) is equivalent to a diagonalizable one, Lemma 6.13 and (6.2) imply that \( FS_m(\| \cdot \|) \) is a bounded metric, and we thus get an operator
\[
FS_m : N_m \to \mathcal{L}^\infty(L).
\]

**Theorem 6.15.** Fix \( m \geq 1 \) with \( mL \) globally generated.

(i) The image \( FS_m(N_m) \) is closed in \( \mathcal{L}^\infty(L) \), and contained in \( \overline{FS}(L) \). In other words, \( FS_m(\| \cdot \|) \) is asymptotically Fubini-Study for each norm \( \| \cdot \| \) on \( H^0(mL) \).
(ii) If $K$ is non-Archimedean, we further have
\[ \text{FS}_m(\mathcal{N}_m) = \text{FS}_m(\mathcal{N}_m^{\text{ultr}}) = \text{FS}_m(\mathcal{N}_m^{\text{diag}}). \]

(iii) The composition $P_m := \text{FS}_m \circ L_m^\infty$ projects $\mathcal{L}_\infty(L)$ onto $\text{FS}_m(\mathcal{N}_m)$, i.e. $P_m(\phi) = \phi$ if and only if $\phi \in \text{FS}_m(\mathcal{N}_m)$.

(iv) $P_m$ is 1-Lipschitz, and satisfies $P_m(\phi) \leq \phi$ and $\| \cdot \|_{P_m(\phi)} = \| \cdot \|_{m(\phi)}$ for each bounded metric $\phi$.

(v) If $m$ divides $m'$, then $P_m \leq P_{m'}$ and $\text{FS}_m(\mathcal{N}_m) \subset \text{FS}_{m'}(\mathcal{N}_{m'})$.

Proof. We first observe that $\text{FS}_m$ is non-increasing and $1/m$-Lipschitz, by (6.2). By Lemma 6.10, $P_m := \text{FS}_m \circ L^\infty$ is thus non-decreasing and 1-Lipschitz. Next, we claim that
\[ \text{FS}_m \circ L^\infty \leq \text{id}, \quad L^\infty \circ \text{FS}_m \leq \text{id}. \]

For each bounded metric $\phi$ and each section $s \in H^0(mL)$, we obviously have $|s|_{m(\phi)} \leq \|s\|_{m(\phi)}$.

From the definition of $\text{FS}_m$ (cf. 6.1) we conclude $\text{FS}_m(\| \cdot \|_{m(\phi)}) = 0$, and hence $P_m = \text{FS}_m \circ L^\infty \leq \text{id}$ on $L^\infty(L)$. Now pick a norm $\| \cdot \| \in \mathcal{N}_m$ on $H^0(mL)$, and set $\phi := \text{FS}_m(\| \cdot \|)$. By (6.1), each $s \in H^0(mL)$ satisfies $|s|_{m(\phi)} = \sup_X |s|_{m(\phi)} \leq \|s\|$, and hence $L^\infty \circ \text{FS}_m \leq \text{id}$ on $\mathcal{N}_m$. Pre-composing the previous inequalities with $\text{FS}_m$ and $L^\infty$ yields
\[ \text{FS}_m \circ L^\infty \circ \text{FS}_m \leq \text{FS}_m, \quad L^\infty \circ \text{FS}_m \circ L^\infty \leq L^\infty. \]

On the other hand, post-composing with the non-increasing operators $L^\infty$ and $\text{FS}_m$ yields
\[ L^\infty \circ \text{FS}_m \circ L^\infty \geq L^\infty, \quad \text{FS}_m \circ L^\infty \circ \text{FS}_m \geq \text{FS}_m, \]

In terms of $P_m = \text{FS}_m \circ L^\infty$, we have thus proved
\[ P_m \circ \text{FS}_m = \text{FS}_m, \quad L^\infty \circ P_m = L^\infty, \]

which establishes (iii), (iv) as well as the closedness of $\text{FS}_m(\mathcal{N}_m)$ in (i).

By submultiplicativity of the sup-norms, it is easy to see that $m \mapsto mP_m(\phi)$ is a subadditive sequence for each bounded metric $\phi$. As a result, $P_m(\phi) \leq P_{m'}(\phi) \leq \phi$ if $m$ divides $m'$, and hence $\text{FS}_m(\mathcal{N}_m) \subset \text{FS}_{m'}(\mathcal{N}_{m'})$ by the projection property of $P_m$, which settles (v).

By Theorem 1.19, each norm $\| \cdot \| \in \mathcal{N}_m$ lies at metric at most $\log \mathcal{N}_m = O(\log m)$ of $\mathcal{N}_m^{\text{diag}}$. Since $\text{FS}_m$ is 1/m-Lipschitz, it follows that $\text{FS}_m(\| \cdot \|)$ is at distance $O(m^{-1} \log m)$ to $\text{FS}_m(\mathcal{N}_m^{\text{diag}})$, which is contained in $\mathcal{FS}(L)$ by Lemma 6.13. Now pick $\phi \in \text{FS}_m(\mathcal{N}_m)$. By (iii), $\phi$ is also in $\mathcal{FS}_{km}(\mathcal{N}_{km})$ for each $k \geq 1$. It therefore lies at distance $O((km)^{-1} \log(km))$ of $\mathcal{FS}(L)$ for all $k$, which yields as desired $\phi \in \mathcal{FS}(L)$, i.e. the second half of (i).

Assume finally that $K$ is furthermore non-Archimedean. As observed above, the sup-norm $\| \cdot \|_{m(\phi)}$ associated to any bounded metric $\phi$ is then ultrametric. If $\phi$ is in $\text{FS}_m(\mathcal{N}_m)$, then $\phi = P_m(\phi) = \text{FS}_m(\| \cdot \|_{m(\phi)})$ thus belongs to $\text{FS}_m(\mathcal{N}_m^{\text{ultr}})$. By Theorem 1.19, $\mathcal{N}_m^{\text{diag}}$ is dense in $\mathcal{N}_m^{\text{ultr}}$, and $\text{FS}_m(\mathcal{N}_m^{\text{diag}})$ is thus dense in $\text{FS}_m(\mathcal{N}_m) = \text{FS}_m(\mathcal{N}_m^{\text{ultr}})$, which concludes the proof of (ii).

As an application of the previous result, we prove that asymptotically Fubini-Study metrics are preserved under max.

**Proposition 6.16.** Suppose that $\phi, \psi$ are asymptotically Fubini-Study metrics on $L$. Then $\max\{\phi, \psi\}$ is also asymptotically Fubini-Study.
Proof. In the Archimedean case, this follows from Theorem 6.3. Assume \( K \) is non-Archimedean. By Theorem 6.15 and continuity of max, it is enough to show that \( \text{FS}_m(\mathcal{N}_m^{\text{diag}}) \) is preserved under max. Given \( \| \cdot \|, \| \cdot ' \| \in \mathcal{N}_m^{\text{diag}} \), pick a basis \( (s_i) \) of \( H^0(mL) \) that jointly diagonalizes them. Then

\[
\text{FS}_m(\| \cdot \|) = \frac{1}{m} \max_i \{ \log |s_i| - \log \|s_i\| \},
\]

\[
\text{FS}_m(\| \cdot \|') = \frac{1}{m} \max_i \{ \log |s_i| - \log \|s_i\|' \},
\]

and hence

\[
\max \{ \text{FS}_m(\| \cdot \|), \text{FS}_m(\| \cdot \|') \} = \frac{1}{m} \max_i \{ \log |s_i| + \lambda_i \}
\]

with \( \lambda_i = \max \{ -\log \|s_i\|, -\log \|s_i\|' \} \), which shows that \( \max \{ \text{FS}_m(\| \cdot \|), \text{FS}_m(\| \cdot \|') \} \in \text{FS}_m(\mathcal{N}_m^{\text{diag}}) \). \( \square \)

We conclude this section with some remarks on the injectivity of the Fubini-Study operators \( \text{FS}_m \). Pick \( m \geq 1 \) such that \( mL \) is very ample. Since any two norms in \( \mathcal{N}_m^{\text{diag}} \) are jointly diagonalizable, \( \text{FS}_m \) is injective on \( \mathcal{N}_m^{\text{diag}} \) if and only if \( \text{FS}_m \circ \iota_s \) is injective for any basis \( s = (s_i) \) of \( H^0(mL) \). Here \( \iota_s : \mathbb{R}^N_m \hookrightarrow \mathcal{N}_m^{\text{diag}} \) denotes as before the map sending \( \lambda \in \mathbb{R}^N_m \) to the norm \( \| \cdot \| \) diagonalized in \( (s_i) \) and such that \( \|s_i\| = e^{-\lambda_i} \).

**Example 6.17.** Assume that \( K \) is trivially valued. Then \( (\text{FS}_m \circ \iota_s)(0) = \frac{1}{m} \max_i \log |s_i| \) is the trivial metric on \( L \). If \( \max_i \lambda_i = 0 \), [BHJ16, Proposition 2.12] implies that \( \text{FS}_m(\| \cdot \|) = (\text{FS}_m \circ \iota_s)(0) \) if and only if the sections \( s_i \) with \( \lambda_i = 0 \) have no common zeroes, and \( \text{FS}_m \) is therefore not injective on \( \mathcal{N}_m^{\text{diag}} \) for any \( m \gg 1 \) in general.

This is in contrast with the following result, which was very recently proved by Y. Hashimoto [Has17].

**Proposition 6.18.** If \( K = \mathbb{C} \), \( X \) is smooth and \( L \) is ample, the operator \( \text{FS}_m \) is injective on the set \( \mathcal{N}_m^{\text{diag}} \) of Hermitian norms on \( H^0(mL) \) for any \( m \) such that \( mL \) is very ample.

Proof. Using the above notation, pick \( \lambda \in \mathbb{R}^N_m \) and set

\[
\phi_\lambda := (\text{FS}_m \circ \iota)(\lambda) = \frac{1}{2m} \log \sum_i e^{-2\lambda_i} |s_i|^2.
\]

For each continuous metric \( \phi \) on \( L \) and volume form \( \mu \) on \( X \), the corresponding \( L^2 \)-norm \( \| \cdot \|_{\mu, \phi} \) satisfies

\[
\int_X e^{2m(\phi_\lambda - \phi)} d\mu = \sum_i e^{-2\lambda_i} \|s_i\|_{\mu, \phi}^2.
\]

(6.3)

By [Has17, Lemma 2.2], the map \( (\phi, \mu) \mapsto \| \cdot \|_{\mu, \phi} \in \mathcal{N}_m^{\text{diag}} \) is onto. In particular, the values of \( \|s_1\|_{\mu, \phi}, \ldots, \|s_N\|_{\mu, \phi} \) can be arbitrarily prescribed, and (6.3) thus shows as desired that \( \lambda \) is determined by \( \phi_\lambda \). \( \square \)
6.3. Semipositive envelopes. We now consider the limit as $m \to \infty$ of the projection operators $P_m$.

Proposition 6.19. Let $\phi \in \mathcal{L}^\infty(L)$ be a bounded metric on $L$.

(i) The sequence $P_m(\phi)$ converges pointwise to the bounded metric

$$P(\phi) := \sup_m P_m(\phi),$$

where the supremum ranges over all $m$ such that $mL$ is globally generated.

(ii) The resulting operator $P : \mathcal{L}^\infty(L) \to \mathcal{L}^\infty(L)$ is non-decreasing and 1-Lipschitz.

(iii) For all $m \in \mathbb{N}$, the sup-norms induced by $\phi$ and $P(\phi)$ on $H^0(mL)$ coincide.

(iv) If $\phi$ is continuous, then $\phi$ is asymptotically Fubini-Study if and only if $P(\phi) = \phi$, and the convergence $P_m(\phi) \to \phi$ is then uniform.

Proof. It was already noted in the proof of Theorem 6.15 that $m \mapsto mP_m(\phi)$ is subadditive. By Fekete’s subadditivity lemma, this implies (i), and (ii) follows from the analogous properties for $P_m$. By (ii) of Theorem 6.15, $\phi$ and $P_m(\phi)$ induce the same sup-norms on $H^0(mL)$. Since $P_m(\phi) \leq P(\phi) \leq \phi$, we infer (iii).

Assume next that $\phi$ is a Fubini-Study metric. By Lemma 6.13, we have $\phi \in \text{FS}_m(N_m)$ for some $m$. By Theorem 6.15, it follows that $P_m(\phi) = \phi$ for all $k \geq 1$, and hence $P(\phi) = \phi$. By continuity of $P$, the identity $P(\phi) = \phi$ propagates to the closure $\mathcal{FS}(L)$. Conversely, if $P(\phi) = \phi$ is continuous, the convergence $P_m(\phi) \to P(\phi) = \phi$ is uniform by Dini’s lemma, and $\phi$ is thus asymptotically Fubini-Study, which settles (iv). □

Conjecture 6.20. Assume that $L$ is ample. For any continuous metric $\phi$ on $L$, $P(\phi)$ is continuous. Equivalently (by Dini’s lemma), $P_m(\phi)$ converges uniformly as $m \to \infty$.

As we show next, Conjecture 6.20 is equivalent the following basic regularization property, which is known to hold in the cases discussed in Remark 6.8.

Conjecture 6.21. Let $(\phi_\alpha)$ be a family of asymptotically Fubini-Study metrics, uniformly bounded above. Then the usc upper envelope $(\sup_\alpha \phi_\alpha)^*$ can be written as a decreasing net of asymptotically Fubini-Study metrics $\psi_j$.

Lemma 6.22. Conjecture 6.21 and Conjecture 6.20 are equivalent.

Proof. Suppose first that Conjecture 6.21 holds, and let $\phi$ be a continuous metric on $L$. Since each $P_m(\phi)$ is asymptotically Fubini-Study, $P(\phi) = \sup_m P_m(\phi)$ is lsc, and its usc regularization $P(\phi)^*$ is the decreasing limit of a net of asymptotically Fubini-Study metrics $\psi_j$, by Conjecture 6.21. Since $P(\phi) \leq \phi$ and $\phi$ is continuous, $\lim_j \psi_j = P(\phi)^* \leq \phi$, and (a small variant of) Dini’s lemma therefore yields $\psi_j \leq \phi + \varepsilon_j$ for some constants $\varepsilon_j \to 0$. It follows that $\psi_j = P(\psi_j) \leq P(\phi) + \varepsilon_j$, and hence $P(\phi)^* \leq P(\phi)$ in the limit, which proves that $P(\phi)$ is usc.

Conversely, assume that $P(\phi)$ is continuous for each continuous metric $\phi$, and let $(\phi_\alpha)$ be a family of asymptotically Fubini-Study metrics, uniformly bounded above. The metric $\psi := (\sup_\alpha \phi_\alpha)^*$, being usc, can be written as the limit of a decreasing net of continuous metrics $\tau_j$. For each $\alpha, j$, we have $\phi_\alpha \leq \tau_j$, and hence $\phi_\alpha = P(\phi_\alpha) \leq P(\tau_j)$, which in turn yields $\psi \leq P(\tau_j) \leq \tau_j$. We have thus written $\psi$ as the limit of the decreasing net of asymptotically Fubini-Study metrics $P(\tau_j)$. □
Example 6.23. Assume that $K = \mathbb{C}$. By Theorem 6.3 any continuous semipositive metric $\phi$ on $L$ is asymptotically Fubini-Study, and hence satisfies $P(\phi) = \phi$. If $\phi$ is merely locally bounded and semipositive, it can however happen that the usc envelope $Q(\phi)$ of all continuous semipositive metrics $\psi$ with $\psi \leq \phi$ satisfies $Q(\phi) < \phi$ at some point $x \in X$, and hence $P(\phi)^*(x) \leq Q(\phi)(x) < \phi(x)$ as well.

To get an example, it is enough to produce two semipositive metrics $\phi_1 \neq \phi_2$ on $L$ with $\phi_1$ continuous, $\phi_2$ locally bounded, $\phi_1 \leq \phi_2$ everywhere and $\phi_1 = \phi_2$ on a dense subset of $X$. Indeed, we trivially have $\psi \leq \phi_2$ satisfies $\psi \leq \phi_1$ on a dense set, and hence everywhere by lower semicontinuity of $\psi - \phi_1$. Thus $\phi_1(x) = Q(\phi_1) = Q(\phi_2) \neq \phi_2$.

Now pick a dense sequence $(z_n)$ in the unit disc $D \subseteq \mathbb{C}$ and set $u(z) := \sum_n 2^{-n} \log |z - z_n|$. This function is subharmonic, bounded above on $D$, and has a dense polar set. The functions $v_1 := 0$ and $v_2 := e^u$ are both subharmonic on $D$, and satisfy $v_1 \leq v_2 \leq C$ for some constant $C > 0$ and $v_1 = v_2$ on a dense subset of $D$.

Consider the potential $\varphi(z) = \frac{1}{2} \log(1 + |z|^2)$ of the Fubini-Study metric on $O(1)$ over $\mathbb{P}^1$. Set $\alpha := \varphi(1/2)$, $\delta := \varphi(2/3) - \alpha > 0$, $d := [C/d]$ and set $w_i := \max\{d(\varphi - \alpha), v_i\}$ on $D$. For $|z| \leq 1/2$, we have $d(\varphi(z) - \alpha) \leq 0 \leq v_i(z)$, and hence $w_i(z) = v_i(z)$. For $2/3 \leq |z| \leq 1$, $d(\varphi(z) - \alpha) \geq d \geq C \geq v_i(z)$, and hence $w_i(z) = d(\varphi(z) - \alpha)$. We can thus extend each $w_i$ to a subharmonic function on $\mathbb{C}$ by setting $w_i(z) := d(\varphi(z) - \alpha)$ for $|z| \geq 1$.

Since $w_i(z) = d \log |z| + O(1)$ near $\infty$, $w_i$ defines a locally bounded psh metric $\phi_i$ on $O(d)$ over $\mathbb{P}^1$, and $\phi_i$ is even continuous. By construction, we have $\phi_1 \leq \phi_2$, with equality on a dense set of $\mathbb{P}^1$, and we are done.

Part 3. Asymptotics of relative volumes

7. Monge-Ampère measures and Deligne pairings

After reviewing some basic properties of Monge-Ampère operators in the Archimedean ad non-Archimedean cases, we relate them to naturally defined metrics on Deligne pairings (Theorem 7.9).

7.1. Mixed Monge-Ampère measures. Let $X$ be an $n$-dimensional projective $K$-scheme, and $L_1, \ldots, L_n$ be line bundles on $X$. The fundamental class of $X$ is the $n$-dimensional cycle

$$[X] = \sum_i m_i [X_i],$$

where the $X_i$ denote the $n$-dimensional irreducible components of $X$ (with their reduced structure) and $m_i$ is the length of the local ring of $X$ at the generic point of $X_i$. The intersection number of the $L_i$ is then defined as

$$(L_1 \cdots L_n) := \deg \pi_* (c_1(L_1) \cdots c_1(L_n) \cdot [X]),$$

with $\pi : X \to \text{Spec} K$ the structure morphism.

Assume first that $K = \mathbb{C}$, and let $\phi_1, \ldots, \phi_n$ be continuous semipositive metrics on $L_1, \ldots, L_n$. The fundamental work of Bedford and Taylor [BT76] enables to define a mixed Monge-Ampère measure

$$dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n = dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n \wedge \delta_X$$
on $X^\an$, with $\delta_X = \sum_i m_i \delta_{X_i}$ denoting the integration current. Indeed, each point of $X^\an$ admits an (analytic) neighborhood $U$ on which each $L_i$ admits a trivializing section $\tau_i$, and that embeds as a closed complex subspace of a polydisc $\mathbb{D}^N$. After perhaps shrinking the latter, the continuous psh functions $u_i := -\log |\tau_i|$ extend to continuous psh functions on $\mathbb{D}^N$, and \cite{BT76} constructs a positive measure
\[ dd^c u_1 \wedge \cdots \wedge dd^c u_n \wedge \delta_U \]
on $\mathbb{D}^N$, with support in $U$. Viewed as a positive measure on $U$, it does not depend on the choices made, and yields a globally defined positive measure $dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n \wedge \delta_X$ on $X^\an$.

The operator
\[ (\phi_1, \ldots, \phi_n) \mapsto dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n \]
is multi-additive, symmetric, and continuous with respect to uniform convergence of metrics and weak convergence of measures, and satisfies
\[ \int_{X^\an} dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n = (L_1 \cdots \cdot L_n). \]
When $K = \mathbb{R}$, mixed Monge-Ampère measures on $X^\an$ can be similarly defined by pulling-back everything to conjugation invariant objects on $(X_C)^\an$.

Assume now that $K$ is non-Archimedean. Recall from Definition 6.5 that a (possibly unbounded) metric on a line bundle $L$ is psh-regularizable if it can be written as the decreasing limit of a sequence of smooth psh metrics on $L$. By \cite{CLD12} 5.6.5, to any $n$-tuple $\phi_1, \ldots, \phi_n$ of continuous psh-regularizable metrics on $L_1, \ldots, L_n$ is associated a mixed Monge-Ampère measure
\[ dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n = dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n \wedge \delta_X, \]
with the same properties as above. By Theorem 6.6 this applies in particular to asymptotically Fubini-Study metrics.

**Example 7.1.** Assume that $n = 0$, i.e. $X = \text{Spec} A$ with $A$ a finite $K$-algebra. The previous construction produces a positive measure $\delta_X$ on the finite set $X^\an$, which is described as follows. We have a product decomposition $A = \prod_i A_i$ into local finite $K$-algebras $A_i$ corresponding to the connected components of $X$. The (reduced) irreducible components of $X$ are given by $X_i = \text{Spec} K_i$, where the residue field $K_i$ of $A_i$ is a finite extension of $K$, and the unique extension of the absolute value of $K$ to $K_i$ defines a point $x_i \in X^\an$. The current $\delta_X = \sum_i m_i \delta_{X_i}$ is a measure on $X^\an$, and the requirement that $\delta_X$ has total mass deg $\pi*[X_i] = [K_i : K]$ yields $\delta_{X_i} = [K_i : K] \delta_{x_i}$. As $m_i[K_i : K] = \dim_K A_i$, we conclude that
\[ \delta_X = \sum_i (\dim_K A_i) \delta_{x_i}. \]

The theory developed by Chambert-Loir and Ducros in general, and mixed Monge-Ampère measures in particular, have the virtue of being invariant under ground field extension:

**Lemma 7.2.** Let $K'/K$ be a non-Archimedean field extension. Pick continuous, psh-regularizable metrics $\phi_1, \ldots, \phi_n$ on line bundles $L_1, \ldots, L_n$ over $X$, and denote by $\phi'_1, \ldots, \phi'_n$ the pulled-back metrics on $X^\an_{K'}$. Then
\[ p_*(dd^c \phi'_1 \wedge \cdots \wedge dd^c \phi'_n) = dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n, \]
with $p : X^\an_{K'} \to X^\an$ the canonical projection map.
Proof. After regularization, we may assume that the $\phi_i$ are smooth psh. Pick a smooth, compactly supported function $f$ on $X^{an}$, and consider the smooth compactly supported $(n,n)$-form $\alpha := f \ddc \psi_1 \wedge \cdots \wedge \ddc \psi_n$. According to [Gub13 Proposition 5.13], there exists a Zariski open subset $U \subset X$ with a closed embedding $U \hookrightarrow \mathbb{C}^r_{an,K}$ and a compactly supported smooth $(n,n)$-superform on $\mathbb{R}^r$ such that $\alpha$ is obtained by pulling-back $\eta$ by the tropicalization map $\text{Trop}: U^{an} \to \mathbb{R}^r$. The integral $\int_{X^{an}} \alpha$ is then defined as the integral of $\eta$ on the tropical cycle $\text{Trop}(U)$. Unravelling the definitions, it is clear that the pull-back of $\alpha$ to $X^{an}_K$ is simply the pull-back of $\eta$ by the tropicalization map $\text{Trop}': U^{an}_{K'} \to \mathbb{R}^r$. The construction of the tropical cycle of an algebraic variety being invariant under ground field extension, we have $\text{Trop}'(U_{K'}) = \text{Trop}(U)$ as tropical cycles, and we conclude as desired that $\int_{X^{an}} \alpha = \int_{X^{an}_K} \alpha'$.

We next introduce an extra line bundle $L_0$ on $X$. Given continuous psh-regularizable metrics $\phi_i, \psi_i$ on each $L_i$, we then have the basic integration-by-parts formula

$$\int_{X^{an}} (\phi_0 - \psi_0) \ddc(\phi_1 - \psi_1) \wedge \ddc \phi_2 \wedge \cdots \wedge \ddc \phi_n = \int_{X^{an}} (\phi_1 - \psi_1) \ddc(\phi_0 - \psi_0) \wedge \ddc \phi_2 \wedge \cdots \wedge \ddc \phi_n. \tag{7.1}$$

This is indeed a consequence of the Stokes/Green formula in the smooth case [CLD12 Theorem 3.12.2], and the general case follows by approximation.

We next turn to the following version of the Poincaré-Lelong formula, which will be instrumental in relating mixed Monge-Ampère operators to Deligne pairings.

**Theorem 7.3.** Let $\phi_0, \psi_0$ and $\phi_1, \ldots, \phi_n$ be continuous psh-regularizable metrics on $L_0$ and $L_1, \ldots, L_n$, and assume also given a regular section $s \in H^0(X, L_n)$ with divisor $Z$. Then

$$\int_{X^{an}} \log |s|_{\phi_0} \ddc(\phi_0 - \psi_0) \wedge \ddc \phi_1 \wedge \cdots \wedge \ddc \phi_{n-1}$$

$$= \int_{Z^{an}} (\phi_0 - \psi_0) \ddc \phi_1 \wedge \cdots \wedge \ddc \phi_{n-1} - \int_{X^{an}} (\phi_0 - \psi_0) \ddc \phi_1 \wedge \cdots \wedge \ddc \phi_n.$$

The proof is based on the following monotone continuity property of mixed Monge-Ampère measures.

**Lemma 7.4.** Let $\phi_0, \psi_0$ be psr-regularizable metrics on $L_0$ with $\psi_0$ continuous but $\phi_0$ possibly unbounded, and for $i = 1, \ldots, n$ let $(\phi_{ij})_{j \in \mathbb{N}}$ be a decreasing sequence of continuous psr-regularizable metrics converging to a continuous metric $\phi_i$. Then

$$\lim_{j \to \infty} \int_{X^{an}} (\phi_0 - \psi_0) \ddc \phi_{1j} \wedge \cdots \wedge \ddc \phi_{nj} = \int_{X^{an}} (\phi_0 - \psi_0) \ddc \phi_1 \wedge \cdots \wedge \ddc \phi_n.$$

The function $\phi_0 - \psi_0$ is usc, hence bounded above, and the above integrals are understood in $[-\infty, +\infty)$.

**Proof.** The function $\phi_0 - \psi_0$ being usc, the weak convergence of measures

$$\ddc \phi_{1j} \wedge \cdots \wedge \ddc \phi_{nj} \to \ddc \phi_1 \wedge \cdots \wedge \ddc \phi_n$$

already implies

$$\limsup_{j \to \infty} \int_{X^{an}} (\phi_0 - \psi_0) \ddc \phi_{1j} \wedge \cdots \wedge \ddc \phi_{nj} \leq \int_{X^{an}} (\phi_0 - \psi_0) \ddc \phi_1 \wedge \cdots \wedge \ddc \phi_n.$$
By assumption, we can pick a decreasing sequence \((\phi_{0k})\) of smooth psh metrics on \(L_0\) with \(\phi_{0k} \to \phi_0\). By monotone convergence,

\[
\int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \lim_{k \to \infty} \int_{X_{\text{an}}} (\phi_{0k} - \psi_0) dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n
\]

and

\[
\int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_{1j} \wedge \cdots \wedge dd^c\phi_{nj} = \lim_{k \to \infty} \int_{X_{\text{an}}} (\phi_{0k} - \psi_0) dd^c\phi_{1j} \wedge \cdots \wedge dd^c\phi_{nj}
\]

for each \(j\), and it will thus suffice to prove an estimate

\[
\int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_{1j} \wedge \cdots \wedge dd^c\phi_{nj} \geq \int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n - \varepsilon_j \tag{7.2}
\]

with \(\varepsilon_j \to 0\) independently of \(k\). Pick smooth psh metrics \(\psi_1, \ldots, \psi_n\) on \(L_1, \ldots, L_n\). By integration-by-parts (7.1), we have

\[
\int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{nj} - \int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj}
\]

\[
= \int_{X_{\text{an}}} (\phi_1 - \psi_1) dd^c\phi_{0k} \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj} - \int_{X_{\text{an}}} (\phi_1 - \psi_1) dd^c\phi_{0} \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj}
\]

\[
\geq \int_{X_{\text{an}}} (\phi_1 - \psi_1) dd^c\phi_{0k} \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj} - \int_{X_{\text{an}}} (\phi_1 - \psi_1) dd^c\phi_{0} \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj},
\]

since \(\phi_{1j}\) decreases to \(\phi_1\). Another integration-by-parts gives

\[
\int_{X_{\text{an}}} (\phi_1 - \phi_1j) dd^c\phi_{0k} \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj} - \int_{X_{\text{an}}} (\phi_1 - \phi_1j) dd^c\phi_{0} \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj}
\]

\[
= \int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj} - \int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj},
\]

which sums up with the previous inequality to yield

\[
\int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_{1j} \wedge \cdots \wedge dd^c\phi_{nj} \geq \int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj}
\]

\[
+ \int_{X_{\text{an}}} (\phi_1 - \phi_{1j}) dd^c\phi_0 \wedge dd^c\phi_{2j} \wedge \cdots \wedge dd^c\phi_{nj}.
\]

Iterating this argument, we get

\[
\int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_{1j} \wedge \cdots \wedge dd^c\phi_{nj} \geq \int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge dd^c\phi_{2} \wedge \cdots \wedge dd^c\phi_{n}
\]

\[
+ \sum_{i=1}^{n} \int_{X_{\text{an}}} (\phi_i - \phi_{ij}) dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{i-1} \wedge dd^c\phi_0 \wedge dd^c\phi_{i+1,j} \wedge \cdots \wedge dd^c\phi_{nj}.
\]

For each \(i = 1, \ldots, n\), \(dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{i-1} \wedge dd^c\psi_0 \wedge dd^c\phi_{i+1,j} \wedge \cdots \wedge dd^c\phi_{nj}\) has total mass \((L_0 \cdots \widehat{L_i} \cdots L_n)\), and

\[
\left| \int_{X_{\text{an}}} (\phi_0 - \psi_0) dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{i-1} \wedge dd^c\psi_0 \wedge dd^c\phi_{i+1,j} \wedge \cdots \wedge dd^c\phi_{nj} \right|
\]

\[
\leq \sup_{X_{\text{an}}} |\phi_i - \phi_{ij}| (L_0 \cdots \widehat{L_i} \cdots L_n),
\]

thus tends to 0, providing the desired estimate (7.2). \qed
Proof of Theorem 7.3. Assume first that all the metrics are smooth psh. By the Poincaré-Lelong formula [CLD12, Theorem 4.6.5], we have $dd^c \log |s|_{\phi_n} = \delta_Z - dd^c \phi_n$ in the sense of currents, and hence

$$
\int_{X^{an}} \log |s|_{\phi_n} \, dd^c (\phi_0 - \psi_0) \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1} = \int_{X^{an}} (\phi_0 - \psi_0) \, dd^c \log |s|_{\phi_n} \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1}
$$

$$
= \int_{Z^{an}} (\phi_0 - \psi_0) \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1} - \int_{X^{an}} (\phi_0 - \psi_0) \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n.
$$

Consider now the general case, and pick a smooth psh metric $\phi'_n$ on $L_n$. Then

$$
\int_{X^{an}} \log |s|_{\phi_n} \, dd^c (\phi_0 - \psi_0) \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1} = \int_{X^{an}} \log |s|_{\phi'_n} \, dd^c (\phi_0 - \psi_0) \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1}
$$

$$
= \int_{X^{an}} (\phi'_n - \phi_n) \, dd^c (\phi_0 - \psi_0) \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1}
$$

by integration-by-parts. The desired formula for $\phi_n$ is thus equivalent to that for $\phi'_n$, and we may hence assume wlog that $\phi_n$ is smooth psh.

Now pick decreasing sequences of smooth psh metrics $\phi_{0j}, \psi_{0j}, \phi_{1j}, \ldots, \phi_{n-1,j}$ converging to $\phi_0, \psi_0, \phi_1, \ldots, \phi_{n-1}$, so that

$$
\int_{X^{an}} \log |s|_{\phi_n} \, dd^c (\phi_{0j} - \psi_{0j}) \wedge dd^c \phi_{1j} \wedge \cdots \wedge dd^c \phi_{n-1,j}
$$

$$
= \int_{Z^{an}} (\phi_{0j} - \psi_{0j}) \, dd^c \phi_{1j} \wedge \cdots \wedge dd^c \phi_{n-1,j} - \int_{X^{an}} (\phi_{0j} - \psi_{0j}) \, dd^c \phi_{1j} \wedge \cdots \wedge dd^c \phi_{nj}
$$

by the smooth case just treated. On the one hand, the right-hand side of the equality converges to

$$
\int_{Z^{an}} (\phi_0 - \psi_0) \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1} - \int_{X^{an}} (\phi_0 - \psi_0) \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n.
$$

On the other hand, Lemma 6.7 gives a decomposition $\log |s| = \phi_A - \phi_B$ with $\phi_A$ psh-regularizable and $\phi_B$ smooth psh. It follows that $\log |s|_{\phi_n} = \phi_A - (\phi_B + \phi_n)$, and

$$
\int_{X^{an}} \log |s|_{\phi_n} \, dd^c (\phi_{0j} - \psi_{0j}) \wedge dd^c \phi_{1j} \wedge \cdots \wedge dd^c \phi_{n-1,j}
$$

thus converges to

$$
\int_{X^{an}} \log |s|_{\phi_n} \, dd^c (\phi_0 - \psi_0) \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_{n-1},
$$

by Lemma 7.4 which concludes the proof. \(\square\)

Let finally $\phi_1, \ldots, \phi_n$ be model metrics on $L_1, \ldots, L_n$. By Proposition 5.10 each $\phi_i$ can be written as a difference of Fubini-Study metrics, and we can thus make sense of $dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n$ as a signed measure on $X^{an}$. Its support is in fact a finite set of Shilov points; more precisely, [CLD12, Proposition 6.9.2] implies:

Lemma 7.5. Let $L_1, \ldots, L_n$ be $\mathbb{Q}$-models of $L_1, \ldots, L_n$ defined on the same projective model $X$ of $X$, and denote by $\phi_i := \phi_{L_i}$ the corresponding model metrics. The mixed Monge-Ampère measure $dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n$ is then supported in the finite set $\Gamma(X)$ of Shilov points of $X$. 
7.2. Metrics on Deligne pairings. As a special case of a general construction discussed extensively in Appendix [A], the Deligne pairing associates to \( n + 1 \) line bundles \( L_0, \ldots, L_n \) on the \( n \)-dimensional projective \( K \)-scheme \( X \) a line bundle \( \langle L_0, \ldots, L_n \rangle \) on \( \text{Spec} \, K \), i.e. a one-dimensional \( K \)-vector space. This pairing is symmetric, multi-additive, and the data of a regular section \( s_0 \in H^0(X, L_0) \) with zero divisor \( Z \) defines a canonical isomorphism

\[
\langle L_0, \ldots, L_n \rangle \simeq \langle L_1|_Z, \ldots, L_n|_Z \rangle.
\]  

(7.3)

In particular, \( \langle O_X, L_1, \ldots, L_n \rangle \) is canonically identified with the trivial line bundle on \( \text{Spec} \, K \).

Example 7.6. When \( n = 0 \), the functor \( L_0 \mapsto \langle L_0 \rangle \) coincides with the norm functor. More precisely, we then have \( X = \text{Spec} \, A \) with \( A \) a finite \( K \)-algebra, and every line bundle \( L_0 \) on \( X \) admits a trivializing section \( \tau \in H^0(X, L_0) \) (which is then the same as a regular section). By (7.3), \( \tau \) defines an isomorphism of \( K \)-vector spaces \( \langle L_0 \rangle \simeq K \), i.e. a generator \( \langle \tau \rangle \) of \( \langle L_0 \rangle \). Any other trivializing section is of the form \( u \tau \) with \( u \in A \) a unit, and we have

\[
\langle u \tau \rangle = N_{A/K}(u) \langle \tau \rangle
\]  

(7.4)

where the norm \( N_{A/K}(u) \in K^* \) is defined as the determinant of the endomorphism of the \( K \)-vector space \( A \) given by multiplication by \( u \).

Given suitably regular metrics \( \phi_0, \ldots, \phi_n \) on line bundles \( L_0, \ldots, L_n \), our goal is to equip \( \langle L_0, \ldots, L_n \rangle \) with a metric \( \langle \phi_0, \ldots, \phi_n \rangle \) (i.e. a norm, but written in additive notation), required to be at least compatible with the symmetry and multiadditivity isomorphisms.

For \( n = 0 \), this is done by setting for any trivializing section \( \tau \in H^0(X, L_0) \)

\[
\log |\langle \tau \rangle|_{\langle \phi_0 \rangle} = \int_{X_{an}} \log |\tau|_{\phi_0} \delta_X
\]

where \( \delta_X \) is the analytic fundamental class discussed in Example 7.1. In view of (7.4), the next result shows that this defines a metric on \( \langle L_0 \rangle \).

Lemma 7.7. If \( X = \text{Spec} \, A \) is a finite \( K \)-scheme, then

\[
\int_{X_{an}} (\log |u|) \delta_X = \log |N_{A/K}(u)|
\]

for all units \( u \in A \).

Proof. By Example 7.1 we may assume that \( A \) is local, and we then have

\[
\int_{X_{an}} (\log |u|) \delta_X = (\dim_K A) \log |u(x)|,
\]

with \( x \) is the unique point of \( X_{an} \), corresponding to the absolute value on the residue field \( K' \) of \( A \). A standard computation gives \( N_{A/K}(u) = N_{K'/K}(u|_{K'})^m \) with \( m = \dim_{K'} A \) the length of \( A \) (see for instance [Mor, Lemma 1.16.2]). It is also well-known that \( |N_{K'/K}(u)| = |u|_{K'/K} = |u(x)|_{K'^*:K} \), and the result follows since \( \dim_K A = m[K':K] \).

The above construction can be generalized to the setting of a finite and flat morphism of projective \( K \)-schemes \( f : X \rightarrow Y \). Given a line bundle \( L \) on \( X \), we can as before consider the line bundle \( N(L) = N_{X/Y}(L) \) on \( Y \). The above lemma indicates that if \( \phi \) is a metric on \( L \),
$N(L)$ inherits a natural metric $N(\phi)$, defined for any local trivialization $\tau \in H^0(f^{-1}(U), L)$ by

$$|\tau|_{N(\phi)}(y) := \prod_i |\tau|_{\phi}(x_i)^{m_i}. \quad (7.5)$$

Here $y \in Y^{an}$ and $\sum m_i[x_i]$ denotes the fundamental cycle of $f^{-1}(y)$.

**Lemma 7.8.** If $\phi$ is a continuous metric on $L$, then so is $N(\phi)$ on $N(L)$. In the case $\phi$ is asymptotically Fubini-Study, so is $N(\phi)$.

*Proof.* Arguing on each component of $Y$, we may assume wlog that $f$ has constant degree $e$. Assume first $K$ is Archimedean. The finite flat morphism $X^{an} \to Y^{an}$ of constant degree $e$ induces a natural map $Y^{an} \to \text{Sym}^e(X^{an})$, where the latter is the symmetric product. The fact that the norm is continuous now results from the fact that for a continuous function $h: X^{an} \to \mathbb{R}$, the function $\text{Sym}^e(X^{an}) \to \mathbb{R}$ given by $\sum n_i[x_i] \mapsto \prod h(x_i)^{m_i}$ is continuous. This in turn follows from the continuity of the corresponding map $(X^{an})^e \to \mathbb{R}$ and the definition of the quotient topology on $\text{Sym}^e(X^{an})$. The second claim is standard in the Archimedean setting since it is straightforward to verify that the curvature of $N(L)$ is the direct image under $f$ of the curvature of $L$. It hence follows from Theorem 6.3.

Suppose now that $K$ is non-trivially valued and non-Archimedean. It follows from definition (cf. (7.5)) that the norm is an $e$-Lipschitz continuous map from normed line bundles on $X^{an}$ to $Y^{an}$. By density of model functions in the space of continuous functions, it suffices to verify that the norm of a model function is a model function. By Lemma 7.13 and Proposition 5.9 (and its proof) we can assume $f$ extends to a proper and flat morphism $X \to Y$ of models, and that $m\mathcal{L}$ is a model of $mL$. Flatness together with properness implies that the fiber dimension is locally constant, and hence $f$ is quasi-finite. As $f$ is automatically finitely presented these facts taken together implies it is also finite by EGA IV.8.11.1]. Now, for any point $y \in Y$ there is an open neighborhood $U$ of $y$, and a trivialization $\tau$ of $\mathcal{L}$ on $f^{-1}(U)$. Since $|\tau|_{m\phi}(x') = 1$ for $x' \in f^{-1}(y')$ for $y' \in U^{\mathcal{L}}$, it follows from definition that $mN(\phi)$ is the model metric induced by $N(m\mathcal{L})$.

Since any asymptotically Fubini-Study metric is a uniform limit of metrics induced by ample models, it is enough to show that the norm of such a metric is of the same type, which follows from the well-known fact that the norm of an ample line bundle is ample.

In the trivially valued case, let $K'/K$ be a non-trivially valued non-Archimedean field extension. Then the base change $(Y')^{an} \to Y^{an}$ is proper so continuity of any function $Y^{an} \to \mathbb{R}$ follows from continuity after base change. The diagram

$$
\begin{array}{ccc}
(X')^{an} & \longrightarrow & X^{an} \\
\downarrow & & \downarrow \\
(Y')^{an} & \longrightarrow & Y^{an}
\end{array}
$$

is Cartesian in the category of Berkovich spaces, so that $N(L^{an})' = N((L^{an})')$, and hence it follows from the non-trivially valued case.

For any $m$ such that $L$ is globally generated, so is $mN(L)$ by the above discussion. For such $m$, by Lemma 1.28 and the argument in the proof of Proposition 6.16, we find a non-Archimedean complete non-trivially valued extension $K'/K$ such that $\| \cdot \|_{m\phi} = \| \cdot \|_{m\phi'}$ and $\| \cdot \|_{mN(\phi)} = \| \cdot \|_{mN(\phi')} = \| \cdot \|_{mN(\phi')}. The last equality follows from the equality...
\( N(\phi)' = N(\phi) \), which follows from functoriality. It follows that \( P_m(\phi)' = P_m(\phi') \), so that \( \phi' \) is also asymptotically Fubini-Study. By the same reasoning, \( P_m(N(\phi))' = P_m(N(\phi')) \).

We conclude by the above considerations and the non-trivially valued case that \( P_m(N(\phi'))' \) converges uniformly to \( N(\phi)' \). Since \( (Y')^{an} \to Y^{an} \) is surjective this proves that \( P_m(N(\phi)) \) converges uniformly to \( N(\phi) \) so \( N(\phi) \) is necessarily asymptotically Fubini-Study.

Let now \( n \) be arbitrary. When \( K = \mathbb{C} \), Elkik [Elk90] associates to any \((n+1)\)-tuple \( \phi_0, \ldots, \phi_n \) of smooth metrics on line bundles \( L_0, \ldots, L_n \) over an \( n \)-dimensional smooth projective \( K \)-scheme \( X \) a metric \( \langle \phi_0, \ldots, \phi_n \rangle \) on \( \langle L_0, \ldots, L_n \rangle \), compatible with the symmetry and multiadditivity isomorphisms and satisfying the change-of-metric formula

\[
\langle \phi_0, \ldots, \phi_n \rangle - \langle \phi'_0, \phi_1, \ldots, \phi_n \rangle = \int_{X^{an}} (\phi_0 - \phi'_0) \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n
\]

for any other smooth metric \( \phi'_0 \) on \( L_0 \).

From now on, we assume that \( K \) is non-Archimedean. Let \( L_0, \ldots, L_n \) be line bundles on the \( n \)-dimensional projective \( K \)-scheme \( X \), and let \( \mathcal{L}_0, \ldots, \mathcal{L}_n \) be models of \( L_0, \ldots, L_n \) defined on the same projective model \( X' \) of \( X \). By Appendix [A] their Deligne pairing \( \langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle \) is a line bundle over \( \text{Spec} \, k \), compatible with the previous one on the generic fiber, and hence a model of \( \langle L_0, \ldots, L_n \rangle \) (i.e. a lattice in the one-dimensional \( K \)-vector space \( \langle L_0, \ldots, L_n \rangle \)).

This pairing is again symmetric and multi-additive, the corresponding isomorphisms being compatible with the previous ones on the generic fiber. If the \( \mathcal{L}_i \) are merely \( \mathbb{Q} \)-models of the \( L_i \), we can thus make sense of \( \langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle \) as a \( \mathbb{Q} \)-model of \( \langle L_0, \ldots, L_n \rangle \), by multilinearity. The following result will be established in the next section.

**Theorem 7.9.** To any \((n+1)\)-tuple of continuous, psh regularizable metrics \( \phi_0, \ldots, \phi_n \) on line bundles \( L_0, \ldots, L_n \) over an \( n \)-dimensional projective \( K \)-scheme \( X \) is canonically associated a Deligne metric \( \langle \phi_0, \ldots, \phi_n \rangle \) on \( \langle L_0, \ldots, L_n \rangle \), with the following properties:

(i) these metrics are compatible with the multiadditivity and symmetry isomorphisms, and commute with ground field extension;

(ii) for any other continuous, psh-regularizable metric \( \phi'_0 \) on \( L_0 \), we have the change-of-metric formula

\[
\langle \phi_0, \ldots, \phi_n \rangle - \langle \phi'_0, \phi_1, \ldots, \phi_n \rangle = \int_{X^{an}} (\phi_0 - \phi'_0) \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n;
\]

(iii) if \( \phi_0, \ldots, \phi_n \) are model metrics determined by \( \mathbb{Q} \)-models \( \mathcal{L}_0, \ldots, \mathcal{L}_n \) of \( L_0, \ldots, L_n \) on a projective model \( X \) of \( X \), then \( \langle \phi_0, \ldots, \phi_n \rangle = \phi_{\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle} \) is the model metric on \( \langle L_0, \ldots, L_n \rangle \) defined by the Deligne pairing \( \langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle \).

Note that Deligne metrics are uniquely determined by (ii) and (iii).

### 7.3. Proof of Theorem 7.9

Consider first an \((n+1)\)-tuple of model metrics \( \phi_0, \ldots, \phi_n \) on line bundles \( L_0, \ldots, L_n \) over an \( n \)-dimensional projective scheme \( X \). By Lemma [Lemma 5.3] we can find a projective model \( X' \) of \( X \) and \( \mathbb{Q} \)-models \( \mathcal{L}_0, \ldots, \mathcal{L}_n \) of \( L_0, \ldots, L_n \) on \( X' \) such that \( \phi_i = \phi_{\mathcal{L}_i} \) for all \( i \). The main step towards the proof of Theorem 7.9 is the following claim:

**Lemma 7.10.** The model metric \( \langle \phi_0, \ldots, \phi_n \rangle := \phi_{\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle} \) on \( \langle L_0, \ldots, L_n \rangle \) only depends on the metrics \( \phi_i = \phi_{\mathcal{L}_i} \).

This will be established below by induction on \( n \), and we start with:
Lemma 7.11. When \( n = 0 \), the metric \( \langle \phi_0 \rangle \) as defined above coincides with the model metric determined by \( \langle L_0 \rangle \).

Proof. We have \( \mathcal{X} = \text{Spec } \mathcal{A} \) with \( \mathcal{A} \) finite free over \( K^\circ \). As recalled in Lemma [A.12] any line bundle on \( \mathcal{X} \) is trivial in a neighborhood of the special fiber of \( \mathcal{X} \), and hence trivial on \( \mathcal{X} \). A trivializing section \( \tau \in H^0(\mathcal{X}, L_0) \) induces a trivializing section \( \langle \tau \rangle = N_{\mathcal{X}/K^\circ}(\tau) \) of \( \langle L_0 \rangle \), as well as a trivializing section \( \tau_K \in H^0(X, L_0) \), such that \( |\tau_K|_{L_0} = 1 \) on \( X^{an} \). We infer

\[
\log(\tau_K)\langle \phi_0 \rangle = \int_{X^{an}} \log |\tau_K|_{L_0} \delta_X = 0,
\]

which precisely means that \( \langle \phi_0 \rangle \) is the model metric defined by \( \langle L_0 \rangle \).

Recall that a section \( s \in H^0(\mathcal{X}, L) \) of a line bundle \( L \) on \( \mathcal{X} \) is relatively regular if \( s \) is locally a nonzerodivisor and the corresponding Cartier divisor \( Z : = \text{div}(s) \) is flat over \( K^\circ \). This holds if and only if \( s \) does not vanish at the finite set of associated (i.e. generic and embedded) points of \( \mathcal{X} \), and \( s \) then defines a canonical isomorphism

\[
\langle L_0, \ldots, L_n \rangle \simeq \langle L_1|_Z, \ldots, L_n|_Z \rangle,
\]

(7.6)

compatible with (7.3) on the generic fiber. Recall also from Appendix [A] that Deligne pairings are compatible with base change, and hence with arbitrary ground field extension.

Lemma 7.12. Let \( L \) be a relatively ample line bundle on a projective model \( \mathcal{X} \) of \( X \), and let \( \mu : \mathcal{X}' \to \mathcal{X} \) be a higher projective model. For each \( m \gg 1 \) we can then find a relatively regular section \( s \in H^0(\mathcal{X}, mL) \) such that \( \mu^*s \in H^0(\mathcal{X}', mL) \) is also relatively regular.

Proof. It suffices to pick \( s \) that doesn’t vanish on the finite set of \( \mathcal{X} \) made of the associated points of \( \mathcal{X}' \), and by the images of the associated points of \( \mathcal{X}' \). Compare with Proposition [A.9] \( \Box \)

In the case of curves fibered over a discrete valuation ring, the next statement appears in [Eri, p. 117] and the same method generalizes, but we offer instead a direct proof:

Lemma 7.13. If \( \mu : \mathcal{X}' \to \mathcal{X} \) is a higher projective model, then

\[
\phi_\langle L_0, \ldots, L_n \rangle = \phi_{\mu^*L_0, \ldots, \mu^*L_n}
\]

as model metrics on \( \langle L_0, \ldots, L_n \rangle \).

Proof. For \( n = 0 \), this follows from Lemma [7.11]. In the general case, we may assume that all \( L_i \) are relatively ample line bundles, by multiadditivity of Deligne pairings. After replacing \( L_0 \) with a multiple, Lemma [7.12] yields the existence of \( s \in H^0(\mathcal{X}, L_0) \) such that \( s \) and \( \mu^*s \) are both relatively regular. Denote by \( Z \) the zero divisor of \( s \), and by \( Z \) its generic fiber. Since \( Z' : = \mu^{-1}(Z) \) is flat, it coincides with the schematic closure of \( Z \) in \( \mathcal{X}' \), and hence is a model of \( Z \) dominating \( Z \). By the restriction property of Deligne pairings (7.6), the section \( s \) induces isomorphisms

\[
\langle L_0, L_1, \ldots, L_n \rangle \simeq \langle L_1|_Z, \ldots, L_n|_Z \rangle
\]

and

\[
\langle \mu^*L_0, \mu^*L_1, \ldots, \mu^*L_n \rangle \simeq \langle (\mu|_Z)^*(L_1|_Z), \ldots, (\mu|_Z)^*(L_n|_Z) \rangle,
\]

both inducing the same isomorphism \( \langle L_0, \ldots, L_n \rangle \simeq \langle L_1|_Z, \ldots, L_n|_Z \rangle \). We conclude by induction on \( n \). \( \Box \)
Proof of Lemma 7.10. Consider another choice of ℚ-models \( L_i', \ldots, L_n' \) of \( L_0, \ldots, L_n \), defined on a projective model \( X' \) of \( X \), such that \( \phi_{\mathcal{L}_i} = \phi_{\mathcal{L}_i'} \) for all \( i \). Our goal is to show that

\[
\phi_{(\mathcal{L}_0, \ldots, \mathcal{L}_n)} = \phi_{(\mathcal{L}_0', \ldots, \mathcal{L}_n')}.
\]

By multilinearity, we may assume that all \( \mathcal{L}_i, \mathcal{L}_i' \) are line bundles. We will be done if we can reduce to the case where \( X = X' \) with \( X \) reduced, as \( \phi_{\mathcal{L}_i} = \phi_{\mathcal{L}_i'} \) then implies \( \mathcal{L}_i = \mathcal{L}_i' \) as ℚ-models of \( L_i \), by Lemma 5.7. First, pick a projective model \( X'' \) of \( X \) dominating both \( X \) and \( X' \) via \( \mu : X'' \to X \) and \( \mu' : X'' \to X' \). Lemma 5.5 and Lemma 7.13 respectively show that

\[
\phi_{\mu^* \mathcal{L}_i} = \phi_{\mathcal{L}_i} = \phi_{\mu' \mathcal{L}_i'}
\]

and

\[
\phi_{(\mu^* \mathcal{L}_0, \ldots, \mu^* \mathcal{L}_n)} = \phi_{(\mu'^* \mathcal{L}_0', \ldots, \mu'^* \mathcal{L}_n')}.
\]

Replacing the \( \mathcal{L}_i \) and \( \mathcal{L}_i' \) with their pull-backs to \( X'' \), we can thus assume that \( X = X' \). Since all the objects considered are compatible with ground field extension, we may also assume that \( K \) is algebraically closed. By the reduced fiber theorem (Theorem 4.20), we can again pull-back everything to a higher model and arrange as desired that \( X \) is reduced. □

At this point, we have associated to any \((n+1)\)-tuple of model metrics \( \phi_0, \ldots, \phi_n \) on line bundles \( L_0, \ldots, L_n \) a model metric \( \langle \phi_0, \ldots, \phi_n \rangle \) on \( \langle L_0, \ldots, L_n \rangle \). Since Deligne pairings of model line bundles are multi-additive, symmetric, and compatible with ground field extension, the same holds for the Deligne metrics \( \langle \phi_0, \ldots, \phi_n \rangle \). We claim that they also satisfy the change-of-metric formula, which boils down to the next result:

Lemma 7.14. Assume that \( L_0 = \mathcal{O}_X \), and view \( \phi_0 \) as a model function on \( X^{an} \). Then

\[
\langle \phi_0, \ldots, \phi_n \rangle = \int_{X^{an}} \phi_0 \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n
\]

Proof. By multiadditivity, we may assume that \( \mathcal{L}_n \) is (relatively) ample on \( X \). After replacing \( \mathcal{L}_n \) with a multiple, Lemma 7.12 guarantees the existence of \( s \in H^0(X, \mathcal{L}_n) \) such that \( s \) is relatively regular. Denote by \( Z \) the divisor of \( s \), and by \( Z \) its generic fiber. The section \( s \) induces isomorphisms

\[
\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle \simeq \langle \mathcal{L}_0|_Z, \ldots, \mathcal{L}_n-1|_Z \rangle.
\]

Arguing by induction on \( n \), we infer

\[
\langle \phi_0, \ldots, \phi_n \rangle = \langle \phi_0|_Z, \ldots, \phi_n-1|_Z \rangle = \int_{Z^{an}} \phi_0 \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n-1.
\]

On the other hand, the Poincaré-Lelong formula of Theorem 7.3 gives

\[
\int_{X^{an}} \log |s|_{\phi_n} \, dd^c \phi_0 \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n-1 = \int_{Z^{an}} \phi_0 \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n-1 - \int_{X^{an}} \phi_0 \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n.
\]

By Lemma 7.5 the measure \( dd^c \phi_0 \wedge dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n-1 \) is supported in \( \Gamma(X) \). On the other hand, \( \log |s|_{\phi_n} \) vanishes on \( \Gamma(X) \). Hence, by Lemma 7.15 below. We thus have

\[
\int_{Z^{an}} \phi_0 \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n-1 = \int_{X^{an}} \phi_0 \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n,
\]

which concludes the proof of Lemma 7.14. □
Lemma 7.15. Let \( s \in H^0(\mathcal{X}, \mathcal{L}) \) be a relatively regular section of a line bundle \( \mathcal{L} \) on a model \( \mathcal{X} \). Then \( |s|_{\phi_L} = 1 \) on \( \Gamma(\mathcal{X}) \).

Proof. Since \( s \) doesn’t vanish at the associated points of \( \mathcal{X} \), \( s \) is nonzero on the reduction \( \xi := \text{red}_\mathcal{X}(x) \) of any Shilov point \( x \in \mathcal{X}^\text{an} \) defined by \( \mathcal{X} \). In other words, \( s \) defines a trivializing section of \( \mathcal{L} \) on an open neighborhood \( U \) of \( \xi \), and we infer \( |s|_{\phi_L} \equiv 1 \) on \( U \), by definition of the model metric \( \phi_L \). □

Consider finally the case of continuous, psh-regularizable metrics \( \phi_0, \ldots, \phi_n \) on \( L_0, \ldots, L_n \). We then define the metric \( \langle \phi_0, \ldots, \phi_n \rangle \) by forcing the change-of-metric formula, i.e. we set

\[
\langle \phi_0, \ldots, \phi_n \rangle := \langle \psi_0, \ldots, \psi_n \rangle + \frac{1}{n+1} \int_{X^\text{an}} (\phi_i - \psi_i) \, dd^c\psi_0 \wedge \cdots \wedge dd^c\psi_{i-1} \wedge dd^c\phi_{i+1} \wedge \cdots \wedge dd^c\phi_n.
\]

Using the properties already established in the case of model metrics, this definition is easily seen to be independent of the choice of model metrics \( \psi_i \), multi-additive, symmetric, as well as compatible with ground field extension, thanks to Lemma 7.2. This finishes the proof of Theorem 7.9.

8. Asymptotics of relative volumes

In this section, \( X \) is a smooth projective scheme over a field \( K \) complete with respect to an absolute value, \( L \) is an ample line bundle on \( X \), and we set \( n := \dim X, V := (L^n) \), and \( N_m := \dim_K H^0(X, mL) \).

8.1. Monge-Ampère energy. The Monge-Ampère measure of a continuous psh-regularizable metric \( \phi \) on \( L \) is the probability measure

\[ \text{MA}(\phi) := V^{-1} (dd^c\phi)^n. \]

Definition 8.1. The relative Monge-Ampère energy \( E(\phi, \psi) \) of two continuous psh-regularizable metrics \( \phi, \psi \) on \( L \) is defined as

\[
E(\phi, \psi) := \frac{1}{n+1} \sum_{j=0}^n V^{-1} \int_{X^\text{an}} (\phi - \psi)(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}.
\]

(8.1)

Recall that \( \phi - \psi \) is a continuous function on \( X^\text{an} \), which may thus be integrated against the Radon measures \( (dd^c\phi)^j \wedge (dd^c\psi)^{n-j} \). The point of normalizing by the factor \( (n+1)V \) is that we then have

\[ E(\phi + c, \psi) = E(\phi, \psi) + c \]

for all \( c \in \mathbb{R} \). Note also that the Monge-Ampère energy is invariant under ground field extension, by Lemma 7.2. As in [BB10, Proposition 4.7], we further have the following restriction property.

Lemma 8.2. If \( s \in H^0(X, L) \) is a regular section with divisor \( Z \), then

\[
nE_Z(\phi|_Z, \psi|_Z) = (n+1)E_X(\phi, \psi) + \int_{X^\text{an}} \log |s|_{\phi} \text{MA}(\phi) - \int_{X^\text{an}} \log |s|_{\psi} \text{MA}(\psi).
\]
Proof. By Theorem [7.3] we have for $j = 0, \ldots, n - 1$
\[
\int_{X^{an}} \log |s|_\psi (dd^c \phi)^{j+1} \wedge (dd^c \psi)^{n-j} - \int_{X^{an}} \log |s|_\psi (dd^c \phi)^j \wedge (dd^c \psi)^{n-j} = \int_{Z^{an}} (\phi - \psi) (dd^c \phi)^j \wedge (dd^c \psi)^{n-j} - \int_{X^{an}} (\phi - \psi) (dd^c \phi)^{j+1} \wedge (dd^c \psi)^{n-j}.
\]
Summing up these equalities, we get
\[
nVE_\Sigma(\phi, \psi) = (n+1)VEX(\phi, \psi) - \int_{X^{an}} (\phi - \psi) (dd^c \phi)^n + \int_{X^{an}} \log |s|_\psi (dd^c \phi)^n - \int_{X^{an}} \log |s|_\psi (dd^c \psi)^n.
\]
Noting that $\log |s|_\phi = \log |s|_\psi + (\psi - \phi)$ concludes the proof. □

Proposition 8.3. Let $\phi, \phi', \phi''$ be continuous psh-regularizable metrics on $L$. Then
\[
\frac{d}{dt} E(t \phi + (1-t)\phi', \phi'') = V^{-1} \int_{X^{an}} (\phi - \phi') (dd^c \phi)^n = \int_{X^{an}} (\phi - \phi') MA(\phi), \tag{8.2}
\]
and we have the cocycle formula
\[
E(\phi, \phi'') = E(\phi, \phi') + E(\phi', \phi''). \tag{8.3}
\]
Proof. A formal computation based on integration-by-parts (cf. [7.1]) gives [8.2], cf. [BB10] Proposition 4.1. As a consequence, the function $f : [0, 1] \to \mathbb{R}$ given by
\[
f(t) := E(t \phi + (1-t)\phi', \phi'') - E(t \phi + (1-t)\phi', \phi') - E(\phi', \phi'')
\]
satisfies $f'(t) = 0$ for all $t$. Since $f(0) = 0$, we also have $f(1) = 0$, which yields the cocycle formula. □

The cocycle formula suggests that $E(\phi, \psi)$ can be realized as a difference of metric on a line bundle over Spec $K$. We saw in Section 7.2 that any continuous psh-regularizable metric $\phi$ on $K$ induces a metric $\langle \phi^{n+1} \rangle$ on the Deligne pairing $\langle L^{n+1} \rangle$, and the change-of-metric formula indeed yields:

Lemma 8.4. The relative Monge-Ampère energy of two continuous psh-regularizable metrics $\phi, \psi$ on $L$ satisfies
\[
(n + 1)VE(\phi, \psi) = \langle \phi^{n+1} \rangle - \langle \psi^{n+1} \rangle.
\]

8.2. Limits of relative volumes. The goal of the following sections will be to establish the following result, which corresponds to Theorem A of the introduction.

Theorem 8.5. For any pair of continuous metrics $\phi, \psi \in C^0(L)$, the scaled limit
\[
\lim_{m \to \infty} \frac{1}{mN_m} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi})
\]
of the relative volumes of the induced sup-norms $\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}$ on $H^0(mL)$ exists in $\mathbb{R}$. If we further assume that the semipositive envelopes $P(\phi), P(\psi)$ are continuous, then this limit coincides with the relative Monge-Ampère energy $E(P(\phi), P(\psi))$.

The continuity assumption holds in particular if $P(\phi) = \phi$ and $P(\psi) = \psi$, i.e. when $\phi, \psi$ are asymptotically Fubini-Study. Recall that we expect this assumption to be always satisfied cf. Conjecture 6.20.

Theorem 8.5 was first settled in [BB10] when $K$ is Archimedean, and in [BG+16] when $K$ is discretely valued and $\phi, \psi$ are asymptotically Fubini-Study.
We start with a few simple observations, reducing the general case of Theorem 8.5 to that of Fubini-Study metrics.

Lemma 8.6. Assume that \( \frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \rightarrow E(\phi, \psi) \) for any two \( \phi, \psi \) in a dense subset of the set \( \mathcal{F}S(L) \) of Fubini-Study metrics. Then the second part of Theorem 8.5 holds.

Proof. Let \( \phi, \psi \in C^0(L) \), and assume that \( P(\phi), P(\psi) \) are continuous. By Proposition 6.19, \( \phi \) and \( P(\phi) \) induce the same sup-norms on \( H^0(mL) \), and hence

\[
\frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) = \frac{1}{mN} \text{vol}(\| \cdot \|_{mP(\phi)}, \| \cdot \|_{mP(\psi)}).
\]

We may thus assume wlog that \( \phi \) and \( \psi \) are asymptotically Fubini-Study. By assumption, we may then find two sequences \( \phi_j, \psi_j \) of Fubini-Study metrics for which the theorem holds and such that \( \phi_j \rightarrow \phi \) and \( \psi_j \rightarrow \psi \) uniformly. Also by assumption, we have

\[
\lim_{m \rightarrow \infty} \frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi_j}, \| \cdot \|_{m\psi_j}) = E(\phi_j, \psi_j)
\]

for each fixed \( j \). Now Lemma 8.7 below yields

\[
\lim_{j \rightarrow \infty} \frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi_j}, \| \cdot \|_{m\psi_j}) = \frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi})
\]

uniformly with respect to \( m \), while \( E(\phi_j, \psi_j) \rightarrow E(\phi, \psi) \) by continuity of \( E \) with respect to the sup-norm. The result follows. \( \square \)

Lemma 8.7. The relative volume \( \frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \) is 1-Lipschitz continuous in each variable \( \phi, \psi \in C^0(L) \) with respect to the sup-norm.

Proof. By the cocyle formula

\[
\text{vol}(\| \cdot \|_{m\phi_1}, \| \cdot \|_{m\phi_2}) = \text{vol}(\| \cdot \|_{m\phi_1}, \| \cdot \|_{m\phi_3}) + \text{vol}(\| \cdot \|_{m\phi_3}, \| \cdot \|_{m\phi_2}),
\]

it is enough to show that

\[
\left| \frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \right| \leq \sup |\phi - \psi|.
\]

But Proposition 2.14 yields

\[
\left| \frac{1}{N} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \right| \leq d_\infty(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}),
\]

while

\[
d_\infty(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \leq m \sup_{X} |\phi - \psi|
\]

by Lemma 6.10. \( \square \)

8.3. Review of the Archimedean case. In this section, we assume that \( K \) is Archimedean. Theorem 8.5 is then a special case of [BB10, Theorem A]. We review the argument here, taking into account the simplifications owing to the fact that \( L \) is assumed to be ample, as opposed to the general big case (cf. [BB10, §5.3]). By Lemma 8.6, it is enough to show that

\[
\frac{1}{mN} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \rightarrow E(\phi, \psi)
\]
when $\phi, \psi$ are smooth metrics with positive curvature. Let $\mu$ be a smooth, positive probability measure on $X$, and introduce the $L^2$-norm $\| \cdot \|_{\mu,m\phi}$ on $H^0(mL)$ defined by

$$\|s\|_{\mu,m\phi}^2 = \int_X |s|^2_{m\phi} \, d\mu.$$ 

We trivially have $\| \cdot \|_{\mu,m\phi} \leq \| \cdot \|_{m\phi}$, and the Fubini-Study metric

$$P_{\mu,m}(\phi) := \text{FS}_m(\| \cdot \|_{\mu,m\phi})$$

thus satisfies

$$m \sup_X (P_{\mu,m}(\phi) - \phi) = \log \sup_{s \in H^0(mL) \setminus \{0\}} \frac{\|s\|_{m\phi}}{\|s\|_{\mu,m\phi}} = d_{\infty}(\| \cdot \|_{m\phi}, \| \cdot \|_{\mu,m\phi}).$$

By the Bouche-Catlin-Tian-Zelditch theorem on asymptotic expansions of Bergman kernels, we have

$$\lim_{m \to \infty} \frac{1}{N_m} e^{2m(P_{\mu,m}(\phi) - \phi)} \, d\mu = V^{-1}(dd^c\phi)^n = \text{MA}(\phi),$$

(8.4)

in $C^\infty$-topology. In particular,

$$d_{\infty}(\| \cdot \|_{m\phi}, \| \cdot \|_{\mu,m\phi}) = m \sup_X (P_{\mu,m}(\phi) - \phi) = O(\log N_m) = O(\log m),$$

and the Lipschitz continuity and cocycle property of vol (cf. Proposition 2.14) thus shows that the desired result is equivalent to

$$\frac{1}{mN_m} \text{vol}(\| \cdot \|_{\mu,m\phi}, \| \cdot \|_{\mu,m\psi}) \to E(\phi, \psi).$$

(8.5)

By (8.2), the derivative of $\phi \mapsto E(\phi, \psi)$ is given by integration against $\text{MA}(\phi)$, and a simple computation [BB10, Lemma 5.1] shows that the derivative of $\phi \mapsto \frac{1}{mN_m} \text{vol}(\| \cdot \|_{\mu,m\phi}, \| \cdot \|_{\mu,m\psi})$ is given by integration against $\frac{1}{N_m} e^{2m(P_{\mu,m}(\phi) - \phi)} \, d\mu$. Integrating (8.4) along the line segment between $\phi$ and $\psi$ therefore yields (8.5), and we are done.

8.4. Existence of the limit. We assume from now on that $K$ is non-Archimedean (possibly trivially valued until further notice). We first establish:

Lemma 8.8. For any two bounded metrics $\phi, \psi$ on $L$, $\frac{1}{mN_m} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi})$ admits a limit in $\mathbb{R}$.

This result is closely related to the main result [CMac15], and could probably also be obtained by adapting Witt Nyström’s notion of Chebyshev transform [WNL14].

Proof. According to the general convergence result of Theorem 2.31, it is enough to show that the graded norm on $R = \bigoplus_{m \in \mathbb{N}} H^0(mL)$ induced by the sup-norms $\| \cdot \|_{m\phi}$ is linearly close to a multiplicative graded norm on $R$. Equivalently, we seek to produce a sequence of norms $\| \cdot \|_m$ on each $H^0(mL)$ such that

(i) $\|s \cdot s'\|_{m+m'} = \|s\|_m \|s'\|_{m'}$ for all $s \in H^0(mL), s' \in H^0(m'L)$;

(ii) $C^{-m} \| \cdot \|_{m\phi} \leq \| \cdot \|_m \leq C^m \| \cdot \|_{m\phi}$ for some constant $C > 1$.

This is accomplished by Lemma 8.9 below. 

□
Lemma 8.9. There exists $x \in X^{an}$ and a constant $C > 1$ such that
\[ |s(x)|_{m \phi} \leq \| s \|_{m \phi} \leq C^m |s(x)|_{m \phi} \]
for all $m \in \mathbb{N}$ and all $s \in H^0(mL)$. In particular, for this $x$, $\| s \|_m := |s(x)|_{m \phi}$ defines a norm satisfying (i) and (ii) above.

Proof. For any other choice of bounded metric $\phi'$, (5.1) shows that
\[ e^{-m \sup |\phi - \phi'|} |s(x)|_{m \phi'} \leq |s(x)|_{m \phi} \leq e^{m \sup |\phi - \phi'|} |s(x)|_{m \phi'} . \]
It is thus enough to show the result for one bounded metric $\phi$ on $L$, which we may thus assume to be the model metric determined by a $\mathbb{Q}$-model $(\mathcal{X}, \mathcal{L})$ of $(X, L)$ over $K^\circ$ (i.e. the trivial metric, in the trivially valued case). Denoting by $\Gamma(\mathcal{X}) \subset X^{an}$ the associated (finite) set of Shilov points, we then have
\[ \| s \|_{m \phi} = \max_{x \in \Gamma(\mathcal{X})} |s(x)|_{m \phi} \]
for all $m \in \mathbb{N}$ and $s \in H^0(mL)$, by Lemma 5.15. For each $x \in \Gamma(\mathcal{X})$, setting $\| s \|_{x, m} := |s(x)|_{m \phi}$ defines a multiplicative graded norm on $\mathcal{R}$, and it will thus be enough to show that there exists $x \in \Gamma(\mathcal{X})$ such that
\[ \lambda_1(\| \cdot \|_{m \phi}, \| \cdot \|_{x, m}) = \sup_{s \in H^0(mL) \setminus \{0\}} \log \frac{\| s \|_{m \phi}}{\| s \|_{x, m}} \]
is $O(m)$. Since $\| \cdot \|_{m \phi}$ is submultiplicative and $\| \cdot \|_{x, m}$ is multiplicative, $m \mapsto \lambda_1(\| \cdot \|_{m}, \| \cdot \|_{x, m})$ is subadditive, and $m^{-1} \lambda_1(\| \cdot \|_{m \phi}, \| \cdot \|_{x, m})$ admits, by Fekete’s subadditivity lemma, a limit $c_x \in (-\infty, +\infty]$ (compare [BC, Lemma 1.4]). We thus need to show that there is an $x \in \Gamma(\mathcal{X})$ such that $c_x < +\infty$. But since $\max_{x \in \Gamma(\mathcal{X})} \| \cdot \|_{x, m} = \| \cdot \|_{m \phi}$, we have
\[ \min_{x \in \Gamma(\mathcal{X})} m^{-1} \lambda_1(\| \cdot \|_{m \phi}, \| \cdot \|_{x, m}) = 0, \]
and hence $\min_{x \in \Gamma(\mathcal{X})} c_x = 0$. \qed

Remark 8.10. The generalized Izumi inequality proved in [B18, Theorem 2.21] implies that Lemma 8.9 holds true for any choice of quasimonomial/Abhyankar point $x \in X^{an}$.

8.5. Proof of Theorem 8.5. Assume first that $K$ is nontrivially valued.

Lemma 8.11. It is enough to prove Theorem 8.5 when $\phi, \psi$ are model metrics determined by ample line bundles $\mathcal{L}, \mathcal{M}$ on some model $\mathcal{X}$.

Proof. By Proposition 5.2 the set of pure Fubini-Study metrics is dense in $\mathcal{FS}(L)$, and we may thus assume that $\phi, \psi$ are pure Fubini-Study metrics, by Lemma 8.6. According to Proposition 5.10, this means that $\phi, \psi$ are model metrics determined by semialle $\mathbb{Q}$-line bundles $\mathcal{L}, \mathcal{M}$ on some models $\mathcal{X}, \mathcal{X}'$. After replacing $\mathcal{X}, \mathcal{X}'$ with a common higher model, we may assume that $\mathcal{X} = \mathcal{X}'$, and that $\mathcal{L}$ extends to an ample $\mathbb{Q}$-line bundle $\mathcal{H}$ on $\mathcal{X}$ (cf. [GM16, Lemma 4.12]). For each $\varepsilon \in \mathbb{Q} \cap (0, 1)$, the $\mathbb{Q}$-line bundles
\[ \mathcal{L}_\varepsilon := (1 - \varepsilon)\mathcal{L} + \varepsilon \mathcal{H}, \quad \mathcal{M}_\varepsilon := (1 - \varepsilon)\mathcal{M} + \varepsilon \mathcal{H} \]
are ample, and it is immediate to check that the model metrics $\phi_\varepsilon, \psi_\varepsilon$ they determine converge uniformly to $\phi, \psi$ respectively as $\varepsilon \to 0$. Using Lemma 8.7 as in the proof of Lemma 8.6 we are thus reduced to the the case where $\phi, \psi$ are model metrics determined by two ample $\mathbb{Q}$-line
bundies $\mathcal{L}, \mathcal{M}$ on $\mathcal{X}$. Finally, we already know by Lemma 8.4 that $\frac{1}{mN_m} \vol(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi})$ converges; we may thus replace $L$, $\mathcal{L}$ and $\mathcal{M}$ by a multiple and assume that $\mathcal{L}$ and $\mathcal{M}$ are ample line bundles.

Assuming that $\phi = \phi_\mathcal{L}$ and $\psi = \phi_\mathcal{M}$ are as in Lemma 8.11 Theorem 5.14 shows that $d_\infty (\| \cdot \|_{m\phi_\mathcal{L}}, \| \cdot \|_{H^0(\mathcal{L})})$ is bounded, and hence

$$\frac{1}{mN_m} \vol(\| \cdot \|_{H^0(\mathcal{L})}, \| \cdot \|_{H^0(\mathcal{M})}) = O(N_m)$$

by the Lipschitz property of $\vol$. Similarly, $\vol(\| \cdot \|_{m\phi_\mathcal{M}}, \| \cdot \|_{H^0(\mathcal{M})}) = O(N_m)$, and it will thus suffice to show that

$$\frac{1}{mN_m} \vol(\| \cdot \|_{H^0(\mathcal{L})}, \| \cdot \|_{H^0(\mathcal{M})}) \to E(\phi_\mathcal{L}, \phi_\mathcal{M}),$$

(8.6)

thanks to the cocycle property of $\vol$ and $E$. By Corollary 2.7 we have

$$\det \| \cdot \|_{H^0(\mathcal{L})} = \| \cdot \|_{\det H^0(\mathcal{L})}, \quad \det \| \cdot \|_{H^0(\mathcal{M})} = \| \cdot \|_{\det H^0(\mathcal{M})},$$

and hence

$$\vol(\| \cdot \|_{H^0(\mathcal{L})}, \| \cdot \|_{H^0(\mathcal{M})}) = \phi_{\det H^0(\mathcal{L})} - \det H^0(\mathcal{M}),$$

(8.7)

where the right-hand side is value on the point $\spec K^\an$ of the model metric on $\det H^0(\mathcal{L}) - \det H^0(\mathcal{M}) = O_{\spec K} \phi$.

By Serre vanishing, the higher cohomology of $m\mathcal{L}$ and $m\mathcal{M}$ vanishes for $m \gg 1$, and Corollary A.16 yields Knudsen-Mumford expansions

$$\det H^0(\mathcal{L}) = \frac{m^{n+1}}{(n+1)!} \langle \mathcal{L}^n+1 \rangle + O(m^n)$$

and

$$\det H^0(\mathcal{M}) = \frac{m^{n+1}}{(n+1)!} \langle \mathcal{M}^{n+1} \rangle + O(m^n),$$

as $\mathbb{Q}$-line bundles on $\spec K^\circ$, which yields

$$\phi_{\det H^0(\mathcal{L})} - \det H^0(\mathcal{M}) = \frac{m^{n+1}}{(n+1)!} \langle \mathcal{L}^{n+1} \rangle - \langle \mathcal{M}^{n+1} \rangle + O(m^n).$$

Combining this with (8.7), and using $N_m = \frac{m^n}{m} V + O(m^{n-1})$ (by Riemann-Roch) and

$$\phi_{\langle \mathcal{L}^{n+1} \rangle - \langle \mathcal{M}^{n+1} \rangle} = \langle \phi^{n+1} \rangle - \langle \phi^{n+1} \rangle = (n+1) V E(\phi_\mathcal{L}, \phi_\mathcal{M})$$

(by Lemma 8.4), we get as desired (8.6).

Assume finally that $K$ is trivially valued, and pick two Fubini-Study metrics $\phi, \psi$ on $L$. By the Chen-Moriwaki result recalled in Lemma 1.28 we can find a complete field extension $K'/K$ with $K'$ nontrivially valued, and such that for each $m$ the ground field extensions $\| \cdot \|_{m\phi'}, \| \cdot \|_{m\psi'}$ of the sup-norms $\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}$ to $H^0(m\mathcal{L})_{K'}$ are the unique norms on $H^0(m\mathcal{L})_{K'}$ that coincide with $\| \cdot \|_{m\phi'}, \| \cdot \|_{m\psi'}$ on $H^0(m\mathcal{L})$. Since the sup-norms $\| \cdot \|_{m\phi'}, \| \cdot \|_{m\psi'}$ with respect to the pulled-back metrics $\phi', \psi'$ satisfy the latter property, we infer $\| \cdot \|_{m\phi'} = \| \cdot \|_{m\phi'}$ and $\| \cdot \|_{m\psi'} = \| \cdot \|_{m\psi'}$, and hence

$$\vol(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) = \vol(\| \cdot \|_{m\phi'}, \| \cdot \|_{m\psi'}).$$
by Proposition 2.14 and . Since \( K' \) is non-trivially valued, the first part of the proof yields
\[
\frac{1}{mN_m} \text{vol}(\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}) \to E'(\phi', \psi'),
\]
where the relative energy \( E' \) is computed on the base change \((X', L')\) of \((X, L)\) to \( K' \). As already observed, we have \( E(\phi, \psi) = E'(\phi', \psi') \) by Lemma 7.2 and this concludes the proof of Theorem 8.5 in the trivially valued case.

9. Transfinite diameter and Fekete points

Following the strategy developed in [BB10, BBW11] in the complex case, we use Theorem 8.5 to show the existence of transfinite diameters, and then apply the differentiability result of [BFJ15, BG+16, BJ18] to prove a non-Archimedean version of the equidistribution of Fekete points established in [BBW11]. As in the previous section, \( X \) is a smooth projective scheme over a field \( K \), complete with respect to an absolute value, and \( L \) is an ample line bundle on \( X \).

9.1. Existence of transfinite diameters. Set \( N := \dim H^0(X, L) \), and define the Vandermonde embedding
\[
\Psi : \det H^0(X, L) \to H^0(X^N, L^{\otimes N})
\]
as the composition of the antisymmetrization operator \( \det H^0(X, L) \to H^0(X, L)^\otimes N \)
\[
s_1 \wedge \cdots \wedge s_N \mapsto \sum_{\sigma \in S_N} (-1)^{\text{sgn} \sigma} s_{\sigma(1)} \otimes \cdots \otimes s_{\sigma(N)}
\]
with the canonical isomorphism \( H^0(X, L)^\otimes N \cong H^0(X^N, L^{\otimes N}) \). For each \( N \)-tuple \( (s_1, \ldots, s_N) \) of sections of \( L \), \( \Psi(s_1 \wedge \cdots \wedge s_N) \) can be more informally written as the Vandermonde (or Slater) determinant
\[
\Psi(s_1 \wedge \cdots \wedge s_N)(x_1, \ldots, x_N) = \det (s_i(x_j))_{1 \leq i, j \leq N}.
\]
Given a reference norm \( \| \cdot \|_{\text{ref}} \) on \( H^0(L) \) and a continuous metric \( \phi \) on \( L \), the normalized length function
\[
V_{\| \cdot \|_{\text{ref}}} (\phi) := \frac{\abs{\Psi(\omega)}_{\varphi^{\otimes N}}}{\det \| \cdot \|_{\text{ref}}} \in C^0((X^N)_{\text{an}})
\]
is independent of the choice of generator \( \omega \in \det H^0(L) \). Note that the canonical map \( p : (X^N)_{\text{an}} \to (X_{\text{an}})^N \) is a homeomorphism when \( K \) is Archimedean, but is merely continuous and surjective in the general non-Archimedean case.

**Definition 9.1.** The \( m \)-diameter of \( \phi \) normalized by a norm \( \| \cdot \|_{\text{ref}} \) on \( H^0(mL) \) is defined as
\[
d_{m, \| \cdot \|_{\text{ref}}} (\phi) := \left( \sup_{(X^{N_m})_{\text{an}}} V_{\| \cdot \|_{\text{ref}}} (m\phi) \right)^{1/mN_m}.
\]

We will infer from Theorem 8.5 the following existence result for transfinite diameters.

**Theorem 9.2.** Let \( \phi, \psi \) be two continuous metrics on \( L \), and let \( (\| \cdot \|_m) \) be a sequence of norms on \( H^0(mL) \) asymptotically equal to \( (\| \cdot \|_{m\psi}) \), i.e. \( d_\infty(\| \cdot \|_m, \| \cdot \|_{m\psi}) = o(m) \). Then the limit
\[
d_{\infty, \psi}(\phi) := \lim_{m \to \infty} d_{m, \| \cdot \|_m} (\phi)
\]
exists in $\mathbb{R}_+$ and only depends on $\phi, \psi$. In fact, we have

$$-\log \delta_{\infty, \psi}(\phi) = \lim_{m \to \infty} \frac{1}{mN_m} \text{vol} (\| \cdot \|_{m\phi}, \| \cdot \|_{m\psi}),$$

and hence

$$-\log \delta_{\infty, \psi}(\phi) = E (P(\phi), P(\psi))$$

if the semipositive envelopes $P(\phi), P(\psi)$ are assumed to be continuous.

**Definition 9.3.** The above limit $\delta_{\infty, \psi}(\phi)$ is called the transfinite diameter of $(X, \phi)$ with respect to $\psi$.

**Remark 9.4.** As in the complex case, it is more generally possible to introduce the transfinite diameter of $(A, \phi)$ where $A$ is a compact subset of $X^{\text{an}}$, but for simplicity we will stick to the case $A = X^{\text{an}}$ in the present paper.

**Remark 9.5.** The above relation between the transfinite diameter and the energy $E$ led us to choose a normalization that differs from the classical one by an exponent $n/(n+1)$.

Together with Lemma 8.2, Theorem 9.2 yields the following Robin-Rumely formula for the transfinite diameter (compare [Rum07, DMR, BB10]).

**Corollary 9.6.** Suppose as above that $P(\phi), P(\psi)$ are continuous, and let $s_0, \ldots, s_n \in H^0(X, L)$ be a regular sequence of sections. Then

$$\log \delta_{\infty, \psi}(\phi) = \frac{1}{n+1} \sum_{i=0}^{n} \left( \int_{Z_i^\text{an}} \text{log} |s_i| P(\phi)(dd^c P(\phi))^{n-i} - \int_{Z_i^\text{an}} \text{log} |s_i| P(\psi)(dd^c P(\psi))^{n-i} \right)$$

with $Z_i = \{ s_0 = \cdots = s_{i-1} = 0 \}$.

A continuous metric $\phi \in C^0(L)$ defines for each $m \in \mathbb{N}$ sup-norms $\| \cdot \|_{m\phi}$ and $\| \cdot \|_{(m\phi)\oplus N_m}$ on $H^0(mL)$ and $H^0((mL)^{\oplus N_m})$, respectively, which induce in turn two norms $\| \cdot \|_{m\phi}$ and $\Psi_m^* \| \cdot \|_{(m\phi)\oplus N_m}$ on the determinant line $\det H^0(mL)$, with

$$\Psi_m : \det H^0(mL) \to H^0((mL)^{\oplus N_m})$$

the Vandermonde embedding as defined above. The next result compares these two norms on $\det H^0(mL)$, and will easily imply that Theorem 9.2 is in fact equivalent to Theorem 8.5.

**Lemma 9.7.** For each $\phi \in C^0(L)$ we have

$$d_\infty \left( \det \| \cdot \|_{m\phi}, \Psi_m^* \| \cdot \|_{(m\phi)\oplus N_m} \right) = o(mN_m).$$

Observe first that $d_\infty \left( \det \| \cdot \|_{m\phi}, \Psi_m^* \| \cdot \|_{(m\phi)\oplus N_m} \right)$ is $2mN_m$-Lipschitz continuous with respect to $\phi \in C^0(L)$, so that it is enough to prove the result for $\phi$ in a dense subset of $C^0(L)$. The proof will be by comparison with certain pure diagonalizable norms, an $L^2$-norm in the Archimedean case, and a lattice norm in the non-Archimedean.

**Lemma 9.8.** Assume $K$ is Archimedean, pick a continuous metric $\phi$ on $L$ and a smooth volume form $\mu$ on $X$, and denote by $\| \cdot \|_{\mu, \phi}$ and $\| \cdot \|_{\mu^N, \phi^{\oplus N}}$ the induced $L^2$-norms on $H^0(L)$ and $H^0(L^{\oplus N})$. Then

$$\Psi^* \| \cdot \|_{\mu^N, \phi^{\oplus N}} = \sqrt{N!} \det \| \cdot \|_{\mu, \phi}$$

as norms on $\det H^0(mL)$. 
Proof. The statement is equivalent to [BB10, Lemma 5.3], and goes as follows. By Fubini, the \( L^2 \)-norm \( \| - \|_{H^0(\mathcal{L}_{\mathbb{Z}_N})} \) on \( H^0(\mathcal{L}_{\mathbb{Z}_N}) \) corresponds to the tensor norm \( \| - \|_{\mu, \phi} \) under the isomorphism \( H^0(\mathcal{L}_{\mathbb{Z}_N}) \cong H^0(\mathcal{L})^{\otimes N} \). If \( (s_i) \) is an orthonormal basis of \( H^0(\mathcal{L}) \) with respect to \( \| - \|_{\mu, \phi} \), then the tensors \( s_1 \otimes \cdots \otimes s_N \) form an orthonormal basis of \( H^0(\mathcal{L})^{\otimes N} \) with respect to \( \| - \|_{\mu, \phi} \). This implies that the norm of \( s_1 \wedge \cdots \wedge s_N \) under the anti-symmetrization operator \( \det H^0(\mathcal{L}) \hookrightarrow H^0(\mathcal{L})^{\otimes N} \) has squared-norm equal to \( N! \), and the result follows. \( \square \)

Lemma 9.9. Assume \( K \) is non-Archimedean. Let \( \mathcal{L} \) be a model of \( L \), and \( \| - \|_{H^0(\mathcal{L})} \), \( \| - \|_{H^0(\mathcal{L}_{\mathbb{Z}_N})} \) be the induced lattice norms on \( H^0(\mathcal{L}) \) and \( H^0(\mathcal{L}_{\mathbb{Z}_N}) \). Then

\[
\Psi^* \| - \|_{H^0(\mathcal{L}_{\mathbb{Z}_N})} = \det \| - \|_{H^0(\mathcal{L})}.
\]

Proof. The isomorphism \( H^0(\mathcal{X}^N, \mathcal{L}_{\mathbb{Z}_N}) \cong H^0(\mathcal{X}, \mathcal{L})^{\otimes N} \) shows that the lattice norm \( \| - \|_{H^0(\mathcal{L}_{\mathbb{Z}_N})} \) corresponds to the tensor norm \( \| - \|_{H^0(\mathcal{L})} \) under the isomorphism \( H^0(\mathcal{X}^N, \mathcal{L}_{\mathbb{Z}_N}) \cong H^0(\mathcal{X}, \mathcal{L})^{\otimes N} \). On the other hand, if \( (s_i) \) is an orthonormal basis of \( H^0(\mathcal{L}) \) with respect to \( \| - \|_{H^0(\mathcal{L})} \), then the tensors \( s_1 \otimes \cdots \otimes s_N \) form an orthonormal basis of \( H^0(\mathcal{L})^{\otimes N} \) with respect to \( \| - \|_{H^0(\mathcal{L})} \), which means in the ultrametric case that the anti-symmetrization operator \( \det H^0(\mathcal{L}) \hookrightarrow H^0(\mathcal{L})^{\otimes N} \) is an isometric embedding with respect to \( \| - \|_{H^0(\mathcal{L})} \) and \( \| - \|_{H^0(\mathcal{L})} \). \( \square \)

Proof of Lemma 9.7. Assume first that \( K \) is Archimedean, and pick a smooth volume form \( \mu \). By the Bernstein-Markov inequality, the sup-norm \( \| - \|_{m_\phi} \) and \( L^2 \)-norm \( \| - \|_{m_\phi, m_\phi} \) on \( H^0(mL) \) satisfy

\[
d_\infty (\| - \|_{m_\phi}, \| - \|_{m_\phi, m_\phi}) = o(m),
\]

and hence

\[
d_\infty (\text{det} \| - \|_{m_\phi}, \text{det} \| - \|_{m_\phi, m_\phi}) = o(mN_m). \tag{9.2}
\]

As in [BB10, Step 2, p.378], a successive application of the Bernstein-Markov inequality in each variable similarly shows that the induced sup-norm \( \| - \|_{(m_\phi)^{\otimes N_m}} \) and \( L^2 \)-norm \( \| - \|_{m_\phi, (m_\phi)^{\otimes N_m}} \) on \( H^0((mL)^{\otimes N_m}) \) satisfy

\[
d_\infty (\| - \|_{(m_\phi)^{\otimes N_m}}, \| - \|_{m_\phi, (m_\phi)^{\otimes N_m}}) = o(mN_m),
\]

and hence

\[
d_\infty (\Psi^*_m \| - \|_{(m_\phi)^{\otimes N_m}}, \Psi^*_m \| - \|_{m_\phi, (m_\phi)^{\otimes N_m}}) = o(mN_m) \tag{9.3}
\]
as well. Finally, since \( \log (N_m!) = O(m^2 \log m) = o(mN_m) \), Lemma 9.8 yields

\[
d_\infty (\text{det} \| - \|_{m_\phi, m_\phi}, \Psi^*_m \| - \|_{m_\phi, (m_\phi)^{\otimes N_m}}) = o(mN_m),
\]

which combines with (9.2) and (9.3) to yield the desired estimate

\[
d_\infty (\text{det} \| - \|_{m_\phi, (m_\phi)^{\otimes N_m}}) = o(mN_m).
\]

Assume now that \( K \) is non-Archimedean. Arguing as in [8], we may assume after ground field extension that \( K \) is nontrivially valued, so that model metrics are dense in \( C^0(\mathcal{L}) \). As already noted, \( d_\infty (\Psi^*_m \| - \|_{(m_\phi)^{\otimes N_m}}, \text{det} \| - \|_{m_\phi}) \) is plainly \( 2mN_m \)-Lipschitz continuous with respect to \( \phi \in C^0(\mathcal{L}) \); by density, it is thus enough to prove the result when \( \phi = \phi_\mathcal{L} \) is a
model metric, determined by a \( \mathbb{Q} \)-line bundle \( L \) extending \( L \) on some projective model \( X \) of \( X \). After replacing \( X \) with a higher model, we may assume that \( L \) also extends to an (ample) line bundle \( H \) on \( X \) (cf. [GM16, Lemma 4.12]). Fix \( a \geq 1 \) such that \( aL \) is a line bundle, and write \( m = ka + r \) with \( k, r \in \mathbb{N} \) and \( r < a \). Since \( aL \) and \( H \) are line bundles, Theorem 5.16 shows the existence of \( C > 0 \) independent of \( m \) such that

\[
d_{\infty} \left( \| \cdot \|_{(ka\phi + r\phi_H)^{2N_m}}, \| \cdot \|_{H^0((ka\mathcal{L} + r\mathcal{H})^{2N_m})} \right) = O(N_m).
\]

As \( \phi - \phi_H \) and \( r \) are bounded, it follows that

\[
d_{\infty} \left( \| \cdot \|_{(m\phi)^{2N_m}}, \| \cdot \|_{H^0((ka\mathcal{L} + r\mathcal{H})^{2N_m})} \right) = O(N_m),
\]

and hence

\[
d_{\infty} \left( \Psi_{m} \| \cdot \|_{(m\phi)^{2N_m}}, \Psi_{m} \| \cdot \|_{H^0((ka\mathcal{L} + r\mathcal{H})^{2N_m})} \right) = O(N_m) \tag{9.4}
\]

By Lemma 9.9, we have

\[
\Psi_{m} \| \cdot \|_{H^0((ka\mathcal{L} + r\mathcal{H})^{2N_m})} = \det \| \cdot \|_{H^0((ka\mathcal{L} + r\mathcal{H})}. \]

On the other hand, Theorem 5.14 yields

\[
d_{\infty} \left( \| \cdot \|_{ka\phi + r\phi_H}, \| \cdot \|_{H^0((ka\mathcal{L} + r\mathcal{H})} \right) = O(N_m),
\]

hence

\[
d_{\infty} \left( \det \| \cdot \|_{m\phi}, \det \| \cdot \|_{H^0((ka\mathcal{L} + r\mathcal{H})} \right) = O(N_m)
\]

by boundedness of \( \phi - \phi_H \), and we conclude that

\[
d_{\infty} \left( \Psi_{m} \| \cdot \|_{(m\phi)^{2N_m}}, \det \| \cdot \|_{m\phi} \right) = O(N_m) \tag{9.5}
\]

when \( \phi \) is a model metric.

\[\square\]

**Proof of Theorem 9.2.** By definition, we have

\[
\sup_{V_{\| \cdot \|_{m}}} V_{\| \cdot \|_{m}}(m\phi) = \frac{\| \Psi_{m}(\omega_m) \|_{m\phi}^{2N_m}}{\det \| \omega_m \|_{m}}
\]

for any choice of generator \( \omega_m \in \det H^0(mL) \). By Lemma 9.7, it follows that

\[
\log \delta_{m,\psi}(\phi) = \frac{1}{mN_m} \cdot \| \cdot \|_{m}, \| \cdot \|_{m\phi} + o(1) = \frac{1}{mN_m} \cdot \| \cdot \|_{m\psi}, \| \cdot \|_{m\phi} + o(1)
\]

and we thus see that Theorem 9.2 is indeed equivalent to Theorem 8.5. \[\square\]

### 9.2. Equidistribution of Fekete points

We assume in this section that one of the following holds:

(a) \( K \) is Archimedean;

(b) \( K \) is non-Archimedean, either trivially or discretely valued, and of residue characteristic zero.

By [BB10, BFJ15, BG+16, BJ18], the semipositive envelope \( P(\phi) \) of each continuous metric \( \phi \in C^0(L) \) is then continuous, and for any auxiliary continuous semipositive metric \( \psi \), the operator \( \phi \mapsto E(P(\phi), \psi) \) is differentiable on \( C^0(L) \), with directional derivatives given by

\[
\frac{d}{dt} t=0 E(P(\phi + tu), \psi) = \int_{X^{an}} u \cdot MA(P(\phi)). \tag{9.5}
\]
Following the strategy of [BBW11], itself inspired by a variational argument due to Szpiro-Ullmo-Zhang [SUZ97], we will then use (9.5) to infer from Theorem 9.2 an equidistribution property for Fekete configurations.

**Definition 9.10.** Let $\phi \in C^0(L)$ be a continuous metric on $L$. A Fekete configuration for $\phi$ is a point $P \in (X^N)^{an}$ such that

$$\sup_{(X^N)^{an}} |\Psi(\omega)|_{\phi \otimes N} = |\Psi(\omega)|_{\phi \otimes N}(P)$$

for some, hence any, generator $\omega \in \det H^0(L)$.

**Theorem 9.11.** Assume that the ground field $K$ satisfies (a) or (b) above. For each $m \gg 1$, pick a Fekete configuration $P_m \in (X^{N_m})^{an}$ for $m\phi$. Then $P_m$ equidistributes to $MA(P(\phi))$ as $m \to \infty$.

The statement means that

$$\int_{X^{an}} u \delta_{P_m} \to \int_{X^{an}} u \, MA(P(\phi))$$

for each $u \in C^0(X^{an})$, where we associate to $P \in (X^N)^{an}$ the measure $\delta_P$ on $X^{an}$ obtained by averaging over the image of $P$ in $(X^{an})^N$.

**Proof.** Fix an auxiliary continuous semipositive metric $\psi$ on $L$, and pick for each $m$ a generator $\omega_m \in \det H^0(mL)$. To say that $P_m$ is a Fekete configuration for $m\phi$ means that

$$|\Psi_m(\omega_m)|_{(m\phi) \otimes N_m} \in C^0((X^{N_m})^{an})$$

is maximized (and, in particular, nonzero) at $P_m$. In terms of

$$F_m(\phi) := - \frac{1}{mN_m} \log \left( \frac{|\Psi_m(\omega_m)|_{(m\phi) \otimes N_m}(P_m)}{\det \|\omega_m\|_{m\psi}} \right),$$

$P_m$ is Fekete for $m\phi$ iff $F_m(\phi) = - \log \delta_{m,\psi}(\phi)$. For each $u \in C^0(X^{an})$, observe that

$$F_m(\phi + u) = F_m(\phi) + \int_{X^{an}} u \delta_{P_m}. $$

Since $F_m(\phi + u) \geq - \log \delta_{m,\psi}(\phi)$, Theorem 9.2 yields

$$\liminf_{m \to \infty} F_m(\phi + u) \geq E(P(\phi + u), \psi)$$

and

$$\lim_{m \to \infty} F_m(\phi) = E(P(\phi), \psi).$$

Replacing $u$ with $tu$ for $0 < t \ll 1$, we infer

$$t \liminf_m \int_{X^{an}} u \delta_{P_m} = \liminf_m (F_m(\phi + tu) - F_m(\phi + tu)) \geq E(P(\phi + tu), \psi) - E(P(\phi), \psi),$$

and hence

$$\liminf_m \int_{X^{an}} u \delta_{P_m} \geq d \left. \frac{d}{dt} E(P(\phi + tu), \psi) \right|_{t=0} = \int_{X^{an}} u \, MA(P(\phi)).$$

Applying this to $-u$, we conclude as desired

$$\lim_m \int_{X^{an}} u \delta_{P_m} = \int_{X^{an}} u \, MA(P(\phi)).$$
9.3. The pullback formula. We consider in this section a polarized endomorphism $f$ of $(X, L)$, i.e. a morphism $f : X \to X$ together with the data of an isomorphism $f^*L \simeq dL$ for some positive integer $d > 1$. Since $f^*L$ is ample and $f$ is necessarily proper, $(f^*L)^n = d^n(L^n)$ implies that $f$ is finite (and flat, as $X$ is assumed to be smooth), of degree $d^n$. By Theorem A.15, we thus have a canonical isomorphism

$$\langle f^*L^{n+1} \rangle \simeq d^n \langle L^{n+1} \rangle,$$

which combines with the given isomorphism $f^*L \simeq dL$ to yield

$$d^{n+1} \langle L^{n+1} \rangle \simeq d^n \langle L^{n+1} \rangle.$$

This defines a canonical section

$$R_f \in d^n(d-1)\langle L^{n+1} \rangle,$$

which we call the resultant section (see Corollary 9.17 below for the choice of terminology).

On the other hand, the map $C^0(L) \to C^0(L)$ defined by $\phi \mapsto d^{-1}f^*\phi$, being $1/d$-Lipschitz continuous, admits a unique fixed point $\phi_f$, the equilibrium metric of $f$. For any choice of Fubini-Study metric $\phi$, the metrics $d^{-j}(f^j)^*\phi$ are Fubini-Study as well, and they converge uniformly to $\phi_f$, which is thus asymptotically Fubini-Study.

Example 9.12. For any $d \geq 2$, the equilibrium metric of the polarized endomorphism $f$ of $(\mathbb{P}^n, \mathcal{O}(1))$ induced by $(x_0, \ldots, x_n) \mapsto (x_0^d : \cdots : x_n^d)$ is $\phi_f = \max_i \log |x_i|$.

Lemma 9.13. The resultant $R_f \in d^n(d-1)\langle L^{n+1} \rangle$ has norm 1 with respect to the induced metric $d^n(d-1)\langle \phi_f^{n+1} \rangle$.

Proof. For any continuous, psh-regularizable metric $\phi$ on $L$, the isomorphism $\langle f^*L^{n+1} \rangle \simeq d^n \langle L^{n+1} \rangle$ is an isometry with respect to $\langle f^*\phi^{n+1} \rangle$ and $\langle \phi^{n+1} \rangle$. By definition of $\phi_f$, the isomorphism $f^*L \simeq dL$ is an isometry with respect to $f^*\phi_f$ and $d\phi_f$. It follows that the induced isomorphism $d^n(d-1)\langle L^{n+1} \rangle \simeq K$ is an isometry with respect to $d^n(d-1)\langle \phi_f^{n+1} \rangle$ and the canonical metric on $K$, hence the result. \hfill \Box

We now get the following pull-back formula for the transfinite diameter, which generalizes [DMR], [BB10, §6.3] in view of Corollary 9.17 below.

Theorem 9.14. Let $\phi, \psi$ be continuous metrics on $L$, and assume that $P(\phi), P(\psi)$ are continuous. Then

$$\delta_{\infty, \psi}(d^{-1}f^*\phi) = c_f(\psi)\delta_{\infty, \psi}(\phi)^{1/d}$$

with $c_f(\psi) > 0$ such that

$$\log c_f(\psi) = -\frac{1}{V(n+1)d^{n+1}} \log |R_f|_{d^n(d-1)\langle P(\psi)^{n+1} \rangle}.$$

Proof. By Lemma 9.15 we have $P(d^{-1}f^*\phi) = d^{-1}f^*P(\phi)$. Since $d^{-1}f^*\phi_f = \phi_f$, Theorem 9.2 and the cocycle formula for $E$ give
\[
\log \delta_{\infty, \psi}(d^{-1} f^* \phi) = E(P(\psi), P(d^{-1} f^* \phi)) \\
= E(P(\psi), \phi_f) + E(d^{-1} f^* \phi_f, d^{-1} f^* P(\phi)) \\
= E(P(\psi), \phi_f) + d^{-1} E(\phi_f, P(\phi)) \\
= (1 - d^{-1}) E(\psi, \phi_f) + d^{-1} E(P(\psi), P(\phi)) \\
= (1 - d^{-1}) E(\psi, \phi_f) + \log \delta_{\infty, \psi}(\phi)^{1/d}.
\]

On the other hand, Lemma 9.13 yields
\[
\log |R_f|_{d^n(d-1)(P(\psi)^{n+1})} = d^n(d-1) \left( (\phi_f^{n+1}) - (P(\psi)^{n+1}) \right),
\]
and hence
\[
\frac{1}{V d^{n+1}} \log |R_f|_{d^n(d-1)(P(\psi)^{n+1})} = (n+1)(1 - d^{-1}) E(\phi_f, P(\psi)).
\]
The result follows. \(\square\)

**Lemma 9.15.** Let \(f : Y \to X\) be a finite, flat morphism of reduced projective \(K\)-schemes. Suppose \(L\) is a line bundle on \(X\), and \(\phi\) any bounded metric on \(L^n\). Then \(P(f^* \phi) = f^* P(\phi)\), and in particular if \(P(\phi)\) is continuous, so is \(P(f^* \phi)\).

In the Archimedean case, this is a special case of [BB10, Proposition 2.9].

**Proof.** Denote the degree of \(f\) by \(e\). We will show that any asymptotically Fubini-Study metric \(\psi\) on \(f^* L\) with \(\psi \leq f^* \phi\) admits a majoration \(\psi \leq f^* \varphi \leq f^* \phi\), where \(\varphi\) is asymptotically Fubini-Study.

If \(K\) is non-Archimedean non-trivially valued, it follows from Proposition 5.9 and 1-Lipschitz continuity of \(f^*\) that \(f^* \overline{\mathcal{F}\mathcal{S}(L)} \subseteq \overline{\mathcal{F}\mathcal{S}(f^* L)}\). The corresponding statement in the trivially valued case follows from this one and considerations analogues to the proof in Proposition 6.16.

We thus have the inequality \(f^* P(\phi) \leq P(f^* \phi)\) and since \(P\) is non-decreasing we find that
\[
\psi \leq f^* \varphi \leq f^* P(\phi) \leq P(f^* \phi)
\]
which implies the statement by taking supremum over all \(\psi = P_m(f^* \phi)\).

Recall that there is a natural metric \(N(\varphi)\) on the norm of a line bundle with metric \(\varphi\) (cf. (7.5)). If we are given an asymptotically Fubini-Study metric \(\psi \leq f^* \phi\), then \(N(\psi) \leq N(f^* \phi) = e \phi\), and \(f^* \frac{1}{e} N(\psi) \leq f^* \phi\). By Lemma 7.8, \(f^* \frac{1}{e} N(\psi)\) is also asymptotically Fubini-Study. For any positive integer \(k\), define inductively \(A_k \psi = \sup\{f^* \frac{1}{e} N(A_{k-1} \psi), \psi\}\), where we set \(A_0(\psi) = \psi\). It is an asymptotically Fubini-Study metric by Proposition 6.16.

By using an auxiliary bounded reference metric on \(L\), e.g. \(\phi\), we may identify our metrics with functions. Consider the fiberwise supremum
\[
v(y) = \sup_{x' \in f^{-1}(y)} \psi(x').
\]
We want to show that \(A_k \psi \to f^* v\) uniformly as \(k \to \infty\). By the construction
\[
0 \leq (f^* v - A_k \psi(x)) \leq v(f(x)) - \left[ \frac{1}{e} \sum_{x' \in f^{-1}(f(x))} -m_{x'} A_{k-1} \psi(x') \right].
\]
Here \( [f^{-1}(y)] = \sum m_{x'} [x'] \) denotes the fundamental cycle of \( f^{-1}(y) \). Since for any \( y \in \mathbb{Y}^n \), there is always at least one \( x \in f^{-1}(y) \) such that \( A^k \psi(x) = \psi(y) \), it follows that any bound \( M_k \) of the above can be choosen so that \( M_k \leq (1 - \frac{1}{k}) M_{k-1} \leq (1 - \frac{1}{k})^k M_0 \). We conclude that \( A^k \psi \to f^* \psi \) uniformly so that \( f^* \psi \) is asymptotically Fubini-Study and bounded below by \( \psi \), above by \( f^* \phi \). This construction is independent of the implicit auxiliary metric used. \( \square \)

9.4. The case toric varieties. We now illustrate the previous pull-back formula in the toric case. We thus assume that \((X, L)\) is a smooth projective polarized toric variety with respect to a split torus \( T \cong (\mathbb{G}_m)^n \) with character lattice \( M = \text{Hom}(T, \mathbb{G}_m) \), which thus corresponds to a Delzant polytope \( \Delta \subset M_{\mathbb{R}} \).

For each integer \( d \geq 2 \), multiplication by \( d \) on \( M \) induces a polarized endomorphism \( m_d \) of \((X, L)\), defining an equilibrium metric \( \phi_d \) on \( L \) and a resultant section \( R_d \in d^n (d-1)(L^{n+1}) \). The next lemma describes the moduli space of polarized endomorphisms of degree \( d \) of \((X, L)\).

**Lemma 9.16.** Set \( N = \dim H^0(X, L) = \# (M \cap \Delta) \). The space of polarized morphisms of degree \( d \) of \((X, L)\) is parametrized by a Zariski open subset of \( \mathbb{P} (\text{Hom} (H^0(X, L), H^0(X, dL))) \cong \mathbb{P} (H^0(X, dL)^N) \) whose complement \( Z \) has codimension \( N - n \). In particular, \( Z \) has codimension greater than \( 1 \) unless \((X, L) \cong (\mathbb{P}^n, \mathcal{O}(1)) \).

**Proof.** Since \( X \) is smooth, \( R(X, L) \) is generated in degree one, the data of a polarized endomorphism of \((X, L)\) of degree \( d \) is equivalent to that of a linear map \( H^0(X, L) \to H^0(X, dL) \) whose image is basepoint free. As a result, the space of polarized endomorphisms of degree \( d \) is isomorphic to the complement in \( \mathbb{P} (H^0(X, dL)^N) \) of the projection \( Z \) of the incidence variety

\[
I = \{ ([s_1 : \cdots : s_N], x) \in \mathbb{P} (H^0(X, dL)^N) \times X, s_1(x) = \cdots = s_N(x) = 0 \}
\]

Since \( dL \) is basepoint free, the elements of \( H^0(X, dL) \) vanishing at a given closed point \( x \in X \) is a hyperplane, and it follows that \( \dim I = n - 1 + N(N_d - 1) \). We claim that the restriction to \( I \) of the first projection \( H^0(X, dL)^N \times X \to H^0(X, dL)^N \) is generically finite, which will imply \( \text{codim} Z = N N_d - 1 - \dim I = N - n \). Indeed, if \( s_1, \ldots, s_n \in H^0(X, dL) \) is a regular sequence of sections, the fiber of \( I \) over the \( N \)-tuple \( (s_1, \ldots, s_1, s_2, \ldots, s_n) \in H^0(X, dL)^N \) is finite.

The last point of the lemma follows from the embedding \( X \to \mathbb{P} H^0(X, L) \), which is an isomorphism iff \( N = n - 1 \). \( \square \)

In the case \( N = n - 1 \), i.e. \((X, L) \cong (\mathbb{P}^n, \mathcal{O}(1)) \), any polarized morphism of degree \( d \) is given by \( z \mapsto [f_0(z) : \cdots : f_n(z)] \) for homogenous polynomials \( f_0, \ldots, f_n \), of degree \( d \), without common zeros, and the locus \( Z \subset \mathbb{P} (H^0(X, L)^{n+1}) \) is an irreducible divisor of degree \((n+1) d^n \). As a result, it is defined by a unique polynomial \( \text{Res}(f_0, \ldots, f_n) \) of degree \((n+1) d^n \) in the coefficients of the \( f_i \) and normalized by \( \text{Res}(x_0^d, \ldots, x_n^d) = 1 \), cf. [GKZ94 Ch. 13].

**Corollary 9.17.** Let \( f \) be a polarized morphism of degree \( d \) of the smooth polarized toric variety \((X, L)\).

- If \((X, L) \cong (\mathbb{P}^n, \mathcal{O}(1)) \), then \( R_f = \text{Res}(f) R_d \);
- if not, then \( R_f = R_d \).
Proof. If \((X, L) \simeq (\mathbb{P}^n, \mathcal{O}(1))\), we can restrict along the hyperplane determined by \(x_n = 0\) and inductively compare how the Deligne products and the resultants change. In the case of the resultant, the transformation is described by the Poisson formula \([GKZ94, \text{Ch. 13, Theorem 1.2}]\), and the Deligne products transform accordingly. This shows they are equal up to some constant, and the constant is equal to 1 by evaluating at the polarized endomorphism \(m_d\).

If \((X, L)\) is not isomorphic to \((\mathbb{P}^n, \mathcal{O}(1))\), the previous lemma shows that the space of polarized degree \(d\) endomorphisms of \((X, L)\) is isomorphic to an open subset \(U \subset \mathbb{P}(H^0(X, dL)^N)\) whose complement \(Z\) has codimension at least 2. The map \(f \mapsto R_f/R_d\) defines a morphism \(U \to \mathbb{G}_m\), which is thus constant by normality and properness of \(\mathbb{P}(H^0(X, dL)^N)\), and hence equal to 1 by evaluating at \(f = m_d\). \(\square\)

Remark 9.18. It follows from Lemma 9.13 that the constant \(c_f(\psi)\) in Theorem 9.14 can be expressed through \(E(\phi, \phi_{\text{can}})\). This is also the case for the arithmetic height of a toric variety over the integers and one finds that

\[
\log |R_f|_\phi = \log |R_f/R_d| + d^n(d-1)h(X_\Delta, \mathbb{Z}_\phi).
\]

In the case \(\phi\) is given by the usual Fubini-Study metric on \(\mathbb{P}^n\), we have \(h(\mathbb{P}^n, \mathcal{O}(1)) = \sum_{i=1}^n \sum_{j=1}^i \frac{1}{j^2}\). For an exhaustive discussion on computations of arithmetic heights of toric varieties we refer the reader to \([BPS14]\).

Appendix A. Determinant of cohomology and Deligne pairings

The goal of this Appendix is to discuss (a slight generalization of) results of Knudsen-Mumford \([KM76]\), Deligne \([Del]\), Elkik \([Elk89]\), Munoz-Garcia \([MG00]\) and Ducrot \([Duc05]\), which provide a rough Riemann-Roch theorem for the determinant of cohomology.

A.1. Discussion of the results. For a projective scheme \(X\) over a field \(K\), the determinant of cohomology of a line bundle \(L\) is the line (i.e. one-dimensional \(K\)-vector space)

\[
\lambda(L) := \sum_{i=0}^n (-1)^i \det H^i(X, L),
\]

where we use additive notation for tensor products of lines. If \(\pi : X \to Y\) is now a flat projective morphism of locally noetherian schemes, it was shown by Knudsen and Mumford in \([KM76]\) that the fiberwise determinant of cohomology of a line bundle \(L\) on \(X\) glues together to define a line bundle \(\lambda_{X/Y}(L)\) on \(Y\). Indeed, the derived direct image \(R\pi_*L\) is a perfect complex, i.e. there exists a bounded complex \(E^\bullet\) of vector bundles on \(Y\) with \(R^q\pi_*L\) as \(q\)-th cohomology sheaf, and the determinant of cohomology of \(L\) can then be described as

\[
\lambda_{X/Y}(L) = \sum_{i} (-1)^i \det E_i.
\]

Denoting \(n\) the relative dimension of \(\pi\), the main result in F. Ducrot’s paper \([Duc05]\) implies that the functor \(\lambda_{X/Y} : \mathcal{P}(X) \to \mathcal{P}(Y)\) so defined between the Picard categories of line bundles on \(X\) and \(Y\) admits a unique polynomial structure of degree \(n+1\) compatible with base change and restriction to a relative Cartier divisor (see \([A.8]\) for a precise statement). This result recovers in one stroke the construction of Deligne pairings \([Elk90, MG00]\) and
the Knudsen-Mumford expansion [KM76]. Indeed, it implies that the $(n+1)$-st iterated difference

\[ \langle L_0, \ldots, L_n \rangle_{X/Y} := \sum_{I \subset \{0, \ldots, n\}} (-1)^{|I|} \lambda_{X/Y} \left( \sum_{i \in I} L_i \right) \]

defines a multi-additive symmetric functor \( P(X)^{n+1} \to P(Y) \), the Deligne pairing, and that we have for each \( L \in P(X) \) an expansion

\[ \lambda_{X/Y}(mL) = \sum_{i=0}^{n+1} \binom{m+i}{i} M_i \]

with line bundles \( M_i \) on \( Y \) such that \( M_{n+1} = \langle L_{n+1} \rangle_{X/Y} \). Whenever a Grothendieck-Riemann-Roch theorem is available, we infer

\[ c_1(\langle L_0, \ldots, L_n \rangle_{X/Y}) = \pi_*(c_1(L_0) \cdots c_1(L_n)), \]

so that Deligne pairings lift the natural push-forward operation on the right-hand side to the level of line bundles.

In the main body of the present paper, a version of these results in the possibly non-noetherian setting of models over the valuation ring of a complete non-Archimedean field is required, and the purpose of this appendix is to summarize the results leading to this generalization. In order to keep the treatment uniform, we shall say in this appendix that a scheme \( X \) is admissible if either

(i) \( X \) is locally noetherian, or
(ii) \( X \) is locally finitely presented over the valuation ring \( K^0 \) of some complete non-Archimedean field \( K \).

A.2. Polynomial maps. In order to motivate the definitions in [A.3] we briefly recall some background on polynomial maps and difference calculus. It is well-known that a map \( f: \mathbb{Z} \to \mathbb{Z} \) is polynomial of degree (at most) \( n \) if and only if it admits an expansion

\[ f(m) = \sum_{i=0}^{n} \binom{m+i}{i} b_i \]

with coefficients \( b_i \in \mathbb{Z} \). More generally, a map \( f: A \to B \) between commutative groups is said to be polynomial of degree \( n \) if for any given \( x_1, \ldots, x_r \in A \) we have an expansion

\[ f(m_1 x_1 + \cdots + m_r x_r) = \sum_{0 \leq i_1, \ldots, i_r \leq n} \binom{m_1 + i_1}{i_1} \cdots \binom{m_r + i_r}{i_r} b_{i_1 \ldots i_r} \]

for all \( m_i \in \mathbb{Z} \), with coefficients \( b_{i_1 \ldots i_r} \in B \).

Polynomiality can be characterized in terms of difference calculus. For a map \( f: \mathbb{Z} \to B \), define the difference \( \Delta f: \mathbb{Z} \to B \) by

\[ (\Delta f)(m) := f(m+1) - f(m). \]

Then

\[ \Delta \binom{m+i}{i} = \binom{m+i-1}{i-1}, \]

which can be used to show by induction on \( n \) that \( f: \mathbb{Z} \to B \) is a polynomial map of degree \( n \) if and only if \( \Delta^{n+1} f = 0 \). For a map \( f: \mathbb{Z}^r \to B \), one can introduce partial difference
operators $\Delta_i$ and $f$ is polynomial of degree $n$ if and only if $\Delta^\alpha f = 0$ for all $\alpha \in \mathbb{N}^r$ with $|\alpha| = n + 1$, where we have set $\Delta^\alpha = \Delta_1^{\alpha_1} \cdots \Delta_r^{\alpha_r}$ for each multiindex $\alpha$.

Consider now a map $f : A \to B$ between commutative groups. Mimicking differential calculus, one defines the difference at $x$ $\delta_x f : A \to B$ of a map $f : A \to B$ by setting

$$ (\delta_x f)(y) = f(x + y) - f(x), $$

and the $k$-th iterated difference $\delta^k_x f : A^k \to B$ by

$$ (\delta^k_x f)(x_1, \ldots, x_k) := \delta_x \left( y \mapsto (\delta^k_x f)(x_2, \ldots, x_k) \right)(x_1) $$

$$ = (\delta^{k-1}_{x+x_1} f)(x_2, \ldots, x_k) - (\delta^{k-1}_x f)(x_2, \ldots, x_k). $$

The map $\delta^k_x f : A^k \to B$ so defined is symmetric, as follows from the explicit expression

$$ (\delta^k_x f)(x_1, \ldots, x_k) = \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} f\left( x + \sum_{i \in I} x_i \right). \quad (A.1) $$

Given $x_1, \ldots, x_r \in A$, the map $g : \mathbb{Z}^r \to B$ defined by $g(m_1, \ldots, m_r) = f(\sum_i m_i x_i)$ satisfies

$$ (\Delta^\alpha g)(m_1, \ldots, m_r) = (\delta^{|\alpha|} \sum_{i} m_i x_i) f(x_1^{\alpha_1}, \ldots, x_r^{\alpha_r}) \quad (A.2) $$

for all $\alpha \in \mathbb{N}^r$. Using this, we conclude that $f : A \to B$ is polynomial of degree $n$ if and only if $\delta^{|\alpha|+1}_x f = 0$ for all $x \in A$.

It is in fact enough to check this condition for $x = 0$. Indeed, the operator $\delta^k := \delta^k_0$ determines all $\delta^k_x$ by

$$ (\delta^k f)(x_1, \ldots) = (\delta^k f)(x + x_1, \ldots) - (\delta^k f)(x, \ldots). \quad (A.3) $$

It further satisfies

$$ (\delta^{k+1} f)(x_1, y_1, \ldots) = (\delta^k f)(x_1 + y_1, \ldots) - (\delta^k f)(x_1, \ldots) - (\delta^k f)(y_1, \ldots), $$

and we thus see that $f$ is polynomial of degree $n$ if and only if $\delta^n f$ is multi-additive. Note also that $\delta^k$ can be understood as a polarization operator, in the sense that $\delta^k(L(x^k)) = k!L$ for any symmetric multi-additive map $L : A^k \to B$; we can thus view the multi-additive map $\delta^n f : A^n \to B$ associated to a polynomial map $f : A \to B$ of degree $n$ as the polarization of its degree $n$ part.

**Example A.1.** Let $X$ be an $n$-dimensional projective scheme over a field $K$, with structure morphism $\pi : X \to \text{Spec} K$, and pick a line bundle $L$ on $X$. The Euler characteristic

$$ \chi(L) := \sum_{i=0}^n (-1)^i \dim_K H^i(X, L) $$

only depends on the class of $L$ in the Picard group $\text{Pic}(X)$, and Snapper’s theorem implies that $\chi : \text{Pic}(X) \to \mathbb{Z}$ is a polynomial map of degree $n$, with degree $n$ polarization given by the intersection pairing, i.e.

$$ (\delta^n \chi)(L_1, \ldots, L_n) = (L_1 \cdots L_n) := \deg \pi_*(c_1(L_1) \cdots c_1(L_n) \cdot [X]). $$

For later use, we finally note:
Lemma A.2. Let $f : A \to B$ is a polynomial map of degree $n$, and set for all $0 \leq i \leq n$ 

$$f_{n,i}(x) := (\delta^i f)(x^i) - (\delta^{i+1} f)(x^{i+1}) + \ldots + (-1)^{n-i}(\delta^n f)(x^n).$$

Then $f(mx) = \sum_{i=0}^{n}(\binom{m+i}{i})f_{n,i}(x)$ for all $x \in A$ and $m \in \mathbb{Z}$.

Proof. Since $f$ is polynomial of degree $n$, we have

$$g(m) := f(mx) = \sum_{i=0}^{n}(\binom{m+i}{i})b_i$$

for some $b_i \in B$. Since $\Delta^k(\binom{m+i}{i}) = 1$ for $k \leq i$ and $0$ otherwise, (A.2) yields

$$(\delta^k f)(x^k) = \Delta^k_0 g = \sum_{i \geq k} b_i,$$

and hence $b_i = f_{n,i}(x)$.

\[\square\]

A.3. Polynomial functors. A commutative Picard category is a 'stacky version' of a commutative group. It is defined as a category $\mathcal{P}$ in which all arrows are isomorphisms, together with an additivity functor and functorial associativity and commutativity isomorphisms satisfying the expected compatibility conditions, and such that for any object $x$ the endofunctors $y \mapsto x + y$ and $y \mapsto y + x$ are autoequivalences.

These axioms imply the existence of a neutral object $0$ and of an inverse $-x$ for each object $x$, both unique up to unique isomorphism. The sum $\sum_{i \in I} x_i$ of a finite family $(x_i)_{i \in I}$ of objects in $\mathcal{P}$ is well-defined up to unique isomorphism, and satisfies the expected associativity rules. A commutative Picard category $\mathcal{P}$ is strictly commutative if the commutativity isomorphism induces the identity on $x + x$ for each $x$, in which case $x$ and $-x$ can be contracted within a sum without raising any sign issue.

In practice for us, $\mathcal{P}$ will be the category of line bundles or $\mathbb{Q}$-line bundles on a given scheme, and isomorphisms between them, both of which are strictly commutative Picard categories. Note that a commutative group can also be viewed as a strictly commutative Picard category.

In what follows, $\mathcal{P}$ and $\mathcal{Q}$ are strictly commutative Picard categories. An additive functor $F : \mathcal{P} \to \mathcal{Q}$ is a functor equipped with a functorial additivity isomorphism $F(x + y) \simeq F(x) + F(y)$ which is commutative, expressed by the commutativity of the induced diagram

$$F(x + y) \longrightarrow F(x) + F(y),$$

and associative, i.e. the commutativity of the diagram

$$F((x + y) + z) \longrightarrow F(x + y) + F(z) \longrightarrow (F(x) + F(y)) + F(z)$$

and

$$F(x + (y + z)) \longrightarrow F(x) + F(y + z) \longrightarrow F(x) + (F(y) + F(z)).$$

These conditions then yield a consistent system of functorial additivity isomorphisms $F(\sum_{i \in I} x_i) \simeq \sum_{i \in I} F(x_i)$ for all finite families $(x_i)_{i \in I}$ in $\mathcal{P}$. 

A multi-additive functor $F : \mathcal{P}^n \to \mathcal{Q}$ is defined as a functor equipped with functorial commutative and associative additivity data in each variable, such that expanding out sums in the variables does not depend on the order the operation is performed. A symmetric functor $F : \mathcal{P}^n \to \mathcal{Q}$ is a functor equipped with symmetry isomorphisms

$$F(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \simeq F(x_1, \ldots, x_n)$$

for each permutation $\sigma \in \mathfrak{S}_r$, compatible with the group law on $\mathfrak{S}_r$, and a symmetric, multi-additive functor has both structures, with the expected compatibility condition.

Define the $k$-th iterated difference at an object $x$ in $\mathcal{P}$ of a functor $F : \mathcal{P} \to \mathcal{Q}$ as the symmetric functor $\delta^k_x F : \mathcal{P}^k \to \mathcal{Q}$ defined by setting

$$(\delta^k_x F)(x_1, \ldots, x_k) := \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} F\left(x + \sum_{j \in I} x_j\right),$$

For $x = 0$, we simply set $\delta^k := \delta^k_0$. Recalling from [A.2] that a map $f$ between commutative groups is polynomial of degree $n$ if and only if $\delta^n f$ is multi-additive, we introduce:

**Definition A.3.** A polynomial structure of degree $n$ on $F$ is defined as a structure of multi-additive functor on $\delta^n F$, compatible with its canonical symmetry.

Ducrot introduces in [Duc05, Definition 1.6.1] the notion of $k$-cube structure on $F$. By [Duc05, Proposition 1.9], an $(n+1)$-cube structure on $F$ induces a polynomial structure of degree $n$ on $F$ (and the converse is probably true as well, by [Duc05, 1.5.1, (d)]). Ducrot’s terminology comes from the following well-known result.

**Example A.4.** If $L$ is a line bundle on an abelian variety $A$, the theorem of the cube asserts that for any variety $S$, the functor $F_L : A(S) \to \mathcal{P}(S)$ defined by $F_L(x) := x^* L$ admits a 3-cube structure. It is thus quadratic in our sense, i.e. $(x, y) \mapsto (x + y)^* L - x^* L - y^* L + 0^* L$ is biadditive. Further, the whole structure is compatible with base change.

In analogy with Lemma [A.2], we have:

**Lemma A.5.** Suppose that $F : \mathcal{P} \to \mathcal{Q}$ admits a polynomial structure of degree $n$, and define for $0 \leq i \leq n$ a functor $F_{n,i} : \mathcal{P}^i \to \mathcal{Q}$ by setting

$$F_{n,i}(x) = (\delta f)(x^i) - (\delta f^{i+1})(x^{i+1}) + \ldots + (-1)^{n-i}(\delta f)(x^n).$$

For all $x \in \mathcal{P}$ and $m \in \mathbb{Z}$, we then have canonical functorial isomorphisms

$$F(mx) \simeq \sum_{i=0}^{n} \binom{m+i}{i} F_{n,i}(x).$$

**Proof.** Define $g : \mathbb{Z} \to \mathcal{Q}$ by $g(m) := F(mx) - \sum_{i=0}^{n} \binom{m+i}{i} F_{n,i}(x)$. Since

$$\Delta^k_{m} F(mx) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} F((m + ix) = (\delta^k_{mx} F)(x^k),$$

and $\Delta^k_{m} (\binom{m+i}{i}) = 1$ for $k \leq i$ and 0 otherwise, we have canonical isomorphisms $(\Delta^k g)(0) \simeq 0$ for $k = 0, \ldots, n$. Further,

$$(\Delta^{n-1} g)(m + 1) - (\Delta^{n-1} g)(m) = (\Delta^n g)(m) = (\delta^m_{mx} F)(x^n) - (\delta^m F)(x^n)$$
\( \simeq (\delta^n F)((m + 1)x, x, \ldots, x) - (\delta^n F)(mx, x, \ldots, x) - (\delta^n F)(x, \ldots, x) \simeq 0, \)

for all \( m \in \mathbb{Z} \), by multadditivity of \( \delta^n F \). Summing up these relations, we get

\( (\Delta^{n-1} g)(m) \simeq (\Delta^{n-1} g)(0) \simeq 0, \)

and iterating the argument finally yields \( g(m) \simeq 0 \) for all \( m \). \( \square \)

A.4. The determinant of a perfect complex. Let \( X \) be a scheme, and \( E \) be a vector bundle on \( X \), i.e. a finite locally free \( \mathcal{O}_X \)-module. One denotes by \( \det E = \Lambda^{\text{rk} E} E \) the determinant line. If

\[
0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \tag{A.4}
\]

is an exact sequence of vector bundles, then there is a canonical isomorphism

\[
\det E \simeq \det E' + \det E'', \tag{A.5}
\]

where + denotes the tensor product of line bundles, in additive notation. However, given two vector bundles \( E, F \), the isomorphism

\[
\det E + \det F \simeq \det(E \oplus F) \simeq \det(F \oplus E) \simeq \det F + \det E
\]

induced by the canonical isomorphism \( E \oplus F \simeq F \oplus E \) coincides with the canonical commutativity isomorphism only up to a factor \((-1)^{\text{rk} E \text{rk} F}\).

To deal with this sign issue, one introduces the graded determinant functor \( E \mapsto (\det E, \text{rk} E) \) with values in the Picard category of graded line bundles and modifies the commutativity isomorphism as to satisfy the Koszul rule of signs indicated. For the purpose of the present paper, it will however be enough to view \( \det E \) as an object in the Picard category \( \mathcal{P}(X)_\mathbb{Q} \) of \( \mathbb{Q} \)-line bundles on \( X \), and we can thus ignore the previous sign issue.

Next, recall that a complex \( F^\bullet \) of \( \mathcal{O}_X \)-modules is

(i) pseudo-coherent if it is locally quasi-isomorphic to a bounded-above complex of vector bundles;

(ii) of locally finite Tor-dimension if it is locally quasi-isomorphic to a bounded complex of flat \( \mathcal{O}_X \)-modules;

(iii) perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles.

By [Stacks Tag 08CQ], \( F^\bullet \) is perfect if and only if it is pseudo-coherent and of locally finite Tor-dimension. In [KM76, Theorem 2], Knudsen and Mumford showed that setting

\[
\det F^\bullet := \sum_i (-1)^i \det E^i
\]

for each bounded complex of vector bundles \( E^\bullet \) gives rise to a functor \( F^\bullet \mapsto \det F^\bullet \) from the category of perfect complexes \( F^\bullet \) on \( X \) and quasi-isomorphisms between them to \( \mathcal{P}(X) \). This functor commutes with base change (i.e. we have a canonical functorial isomorphism \( \det Lf^* F^\bullet \simeq g^* F^\bullet \) for any morphism \( f : X' \rightarrow X \)), it is additive with respect to short exact sequences of complexes in the sense of [A.5], and satisfies a couple of other properties which then uniquely characterize it up to unique isomorphism.
A.5. The direct image theorem. As mentioned in §A.1 we shall say for convenience that a scheme $X$ is admissible if it is either locally noetherian (‘noetherian case’) or locally of finite presentation over the valuation ring $K^0$ of some complete non-Archimedean field $K$ (‘valuative case’). Obviously, any scheme $X$ locally finitely presented over an admissible scheme is admissible. Further, any scheme $X$ which is flat and locally of finite type (in particular, proper) over a valuation ring $K^0$ as above is locally finitely presented over $K^0$ (by [Nag66, Theorem 3] or [RG71 Théorème 3.4.6], cf. Lemma 4.5), and hence is admissible.

Recall that an $O_X$-module $F$ on a scheme $X$ is coherent if it is locally finitely presented, and for each open $U \subseteq X$, each locally finite submodule of $F|_U$ is locally finitely presented as well. Crucially, the structure sheaf $O_X$ of any admissible scheme $X$ is coherent, by [Ull Corollary 1.8]. The following direct image theorem is of course well-known in the noetherian case, and follows from the work of Kiehl [Kie72] in the valuative case, as explained in [Ull, Theorem 3.5, Proposition 3.6] (note that an admissible scheme admits an open cover by stably coherent rings, by [Ull Example 3.3]).

**Theorem A.6.** Let $Y$ be an admissible scheme and $\pi : X \to Y$ be a proper, finitely presented morphism. For each coherent $O_X$-module $F$, $R^q\pi_*F$ is coherent for all $q \in \mathbb{N}$ (and zero for $q$ large enough, locally on $Y$).

If we further assume that $\pi$ is projective, $Y$ is quasicompact and $L$ is a $\pi$-ample line bundle on $X$, then the usual Serre vanishing theorem is satisfied, i.e. for all $m \gg 1$ $F(mL) := F \otimes L^m$ is $\pi$-globally generated and satisfies $R^q\pi_*F(mL) = 0$ for $q \geq 1$.

**Corollary A.7.** If $\pi : X \to Y$ is a flat, proper, finitely presented morphism with $Y$ admissible, then the derived direct image $R\pi_*E$ is perfect for each vector bundle $E$ on $X$.

**Proof.** Since $R^q\pi_*E = 0$ for $q \gg 1$, $R\pi_*E$ is quasi-isomorphic to a bounded-above complex. As $O_Y$ is coherent, [Ull Lemma 3.2] thus says that $R\pi_*E$ is pseudo-coherent if and only it has coherent cohomology, which is indeed the case by Theorem A.6. It thus remains to see that $R\pi_*E$ has locally finite Tor-dimension, which can be done verbatim as in [Stacks, Tag 08EV] (whose proof only uses the noetherianity assumption to get pseudo-coherence).

A.6. Regular sequences. Let $\pi : X \to Y$ be a flat, locally finitely presented morphism of schemes, and let $s$ be a global section of a line bundle $L$ on $X$, defining a closed subscheme $D \subseteq X$. Recall that $s$ is $\pi$-regular at $x \in X$ if $D$ is a relative Cartier divisor at $x$, i.e. $s$ is a nonzerodivisor in $O_{X,x}$, and $D$ is $\pi$-flat at $x$. If this holds for all $x \in X$, then $s$ is simply called $\pi$-regular. By [EGA IV.11.3.7], $s$ is $\pi$-regular at $x$ if and only the restriction of $s$ to the fiber through $x$ is not a zerodivisor at $x$, and the set of $x \in X$ at which this holds is open.

More generally, a sequence of sections $(s_1, \ldots, s_p)$ of line bundles $L_1, \ldots, L_p$ with zero schemes $D_1, \ldots, D_p$ is $\pi$-regular if

$$s_1, s_2|_{D_1 \cap \cdots \cap D_{p-1}}$$

are $\pi$-regular.

**Lemma A.8.** Let $\pi : X \to Y$ be flat, proper, finitely presented morphism of schemes, and $s$ be a global section of a line bundle $L$ on $X$. Pick $y \in Y$, and assume that $s$ is nonzero at each associated point of $X_y$. Then $s$ is relatively regular over an open neighborhood of $y$.

**Proof.** Denote by $U \subseteq X$ the open set of points at which $s$ is relatively regular. Since $X_y$ is noetherian (being of finite type over a field), the assumption implies that $s|_{X_y}$ is a
nonzerodivisor at each \( x \in X_y \), and hence that \( X_y \subset U \) by the above results. As \( \pi \) is closed, it follows that \( \pi^{-1}(V) \subset U \) for some open neighborhood \( V \) of \( y \).

As a consequence, we then have the following useful existence result for relatively regular sections.

**Proposition A.9.** Let \( \pi : X \to Y \) be a flat, projective, finitely presented morphism with \( Y \) admissible, and let \( L \) be a \( \pi \)-ample line bundle on \( X \). Then \( mL \) admits a relatively regular section locally over \( Y \) for all \( m \gg 1 \).

**Proof.** By coherence of \( \pi_* (mL) \) and Lemma [A.8] it is enough to show that for each closed point \( y \in Y \) and all \( m \gg 1 \), there exists \( s \in \pi_* (mL)_y \) such that \( s|_{X_y} \) is nonzero at all associated points of \( X_y \). By ampleness of \( L|_{X_y} \), for each \( m \gg 1 \) we may find a section in \( H^0(X_y, mL_y) \) not vanishing at the associated points of \( X_y \) (finitely many, since \( X_y \) is noetherian). If we denote by \( a \subset \mathcal{O}_X \) the ideal sheaf of \( X_y \), Theorem [A.6] implies that \( R^1 \pi_* (a(mL)) = 0 \) for all \( m \gg 1 \). The restriction map \( \pi_* (mL)_y \to H^0(X_y, mL_y) \) is thus surjective for \( m \gg 1 \), hence the result.

### A.7. The determinant of cohomology

Let \( \pi : X \to Y \) be a flat, proper, finitely presented morphism with \( Y \) admissible. Thanks to Corollary [A.7] we can introduce:

**Definition A.10.** The determinant of cohomology is the functor \( \lambda_{X/Y} : \mathcal{P}(X) \to \mathcal{P}(Y)_\mathbb{Q} \) that takes a line bundle \( L \) on \( X \) to the \( \mathbb{Q} \)-line bundle

\[
\lambda_{X/Y}(L) := \det R\pi_* L.
\]

If \( L \) is \( \pi \)-acyclic, i.e. \( R^q \pi_* L = 0 \) for \( q > 0 \), then \( \pi_* L \) is a vector bundle, and \( \lambda_{X/Y}(L) \) is simply given as the ‘naive’ determinant \( \lambda_{X/Y}(L) = \det \pi_* L \). This holds in particular when \( n = 0 \) (i.e. \( \pi \) finite flat), or for large enough multiples of a \( \pi \)-ample line bundle.

**Proposition A.11.** The determinant of cohomology satisfies the following two compatibility properties.

(i) It commutes with base change: for any morphism \( f : Y' \to Y \) with \( Y' \) admissible, let

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

be the corresponding Cartesian square. Then we have a canonical functorial isomorphism

\[
\lambda_{X'/Y'}(g^* L) \simeq f^* \lambda_{X/Y}(L).
\]

(ii) If \( D \) is an effective relative Cartier divisor \( D \) on \( X \), then we have a canonical functorial isomorphism

\[
\lambda_{X/Y}(L) - \lambda_{X/Y}(L - D) \simeq \lambda_{D/Y}(L|_D).
\]  
(A.6)

**Proof.** Since \( L \) is flat over \( Y \), we have \( R\pi_* L' = L^* R\pi_* L \), cf. [Stacks, Tag 0A1D]. On the other hand, the determinant functor satisfies \( \det L^* = f^* \det \), hence (i). If \( D \) is an effective relative Cartier divisor, the restriction exact sequence \( 0 \to L(-D) \to L \to L|_D \to 0 \) induces an exact sequence of perfect complexes \( 0 \to R\pi_* L(-D) \to R\pi_* L \to R(\pi|_D)_* L|_D \to 0 \), and (ii) follows by additivity of \( \det \) in exact sequences. \( \square \)
A.8. Deligne pairings and Knudsen-Mumford expansion. In this section, we fix a flat, projective, finitely presented morphism $\pi : X \to Y$ of constant relative dimension $n$, with $Y$ admissible.

When $n = 0$, $\pi : X \to Y$ is a finite flat morphism, and the determinant of cohomology $\lambda_{X/Y}$ provides a canonical and functorial construction of the norm of a line bundle [EGA II.6.5] (compare for instance [Fer98 Proposition 3.3]). To see this, recall first that the norm $N_{X/Y}(f) \in \mathcal{O}_Y$ of $f \in \pi_*\mathcal{O}_X$ is defined as the determinant of the endomorphism of $\pi_*\mathcal{O}_X$ defined by multiplication by $f$, yielding a multiplicative map

$$N_{X/Y} : \pi_*\mathcal{O}_X \to \mathcal{O}_Y.$$  \hspace{1cm} (A.7)

Now define a functor $N_{X/Y} : \mathcal{P}(X) \to \mathcal{P}(Y)$ by setting

$$N_{X/Y}(L) = (\det \pi_*L) - (\det \pi_*\mathcal{O}_X) = \lambda_{X/Y}(L) - \lambda_{X/Y}(\mathcal{O}_X) = (\delta\lambda_{X/Y})(L),$$

Lemma A.12. For each line bundle $L$ on $X$, $N_{X/Y}(L)$ coincides with the norm of $L$ as defined in [EGA II.6.5].

Proof. Observe that if $u \in H^0(X, \mathcal{O}_X^*)$ is a unit and $L$ is a line bundle on $X$, multiplication by $u$ defines an isomorphism $L \simeq L'$, whose induced isomorphism $\det \pi_*L \simeq \det \pi_*L'$, and hence also $N_{X/Y}(L) \simeq N_{X/Y}(L')$, are both given by multiplication by $N_{X/Y}(u)$.

By [EGA II.6.12.1], $L$ is trivial in a neighborhood of each fiber of $\pi$, and $Y$ therefore admits an open cover $(Y_i)$ with $L|_{X_i} \simeq \mathcal{O}_{X_i}$ on $X_i := \pi^{-1}(Y_i)$. Set $Y_{ij} = Y_i \cap Y_j$, $X_{ij} = X_i \cap X_j = \pi^{-1}(Y_{ij})$, and denote by $u_{ij} \in H^0(X_{ij}, \mathcal{O}_{X_{ij}}^*)$ the corresponding cocycle. The transition isomorphism $\mathcal{O}_{X_{ij}} \simeq L|_{X_{ij}} \simeq \mathcal{O}_{X_{ij}}$, is given by multiplication by $u_{ij}$. By the above observation, applying the functor $N_{X/Y}$ yields an isomorphism $\mathcal{O}_{Y_{ij}} \simeq N_{X/Y}(L)|_{Y_{ij}} \simeq \mathcal{O}_{Y_{ij}}$, given by multiplication by $N_{X_{ij}/Y_{ij}}(u_{ij})$, which precisely means that $N_{X/Y}(L)$ coincides with the norm of $L$ as defined in [EGA II.6.5].

Arguing as in [Duc05 4.1.1], we next prove:

Proposition A.13. There is a unique way to assign to each finite flat morphism $\pi : X \to Y$ with $Y$ admissible an additivity structure on the norm functor $N_{X/Y} : \mathcal{P}(X) \to \mathcal{P}(Y)$ that is compatible with base change and such that the additivity isomorphisms

$$N_{X/Y}(L + \mathcal{O}_X) \simeq N_{X/Y}(L) + N_{X/Y}(\mathcal{O}_X)$$

are the canonical ones given by $N_{X/Y}(\mathcal{O}_X) = \mathcal{O}_Y$.

Proof. Pick two line bundles $L, L'$ on $X$ and a cover $(Y_i)$ of $Y$ with trivializations $L|_{X_i} \simeq \mathcal{O}_{X_i}$, $L'|_{X_i} \simeq \mathcal{O}_{X_i}$. Given an additivity isomorphism

$$N_{X/Y}(L + L') \simeq N_{X/Y}(L) + N_{X/Y}(L')$$

with the desired properties, the induced isomorphisms

$$N_{X_{i}/Y_{i}}(\mathcal{O}_{X_{i}} + \mathcal{O}_{X_{i}}) \simeq N_{X_{i}/Y_{i}}(\mathcal{O}_{X_{i}}) + N_{X_{i}/Y_{i}}(\mathcal{O}_{X_{i}})$$

are necessarily equal to the canonical ones obtained by multiplicativity of $[A.7]$, which proves uniqueness. To establish existence, it is then enough to argue locally on $Y$, since compatibility on overlaps will follow from uniqueness, and the result is then straightforward, using again that any line bundle on $X$ is trivial locally over $Y$. \hspace{1cm} \Box
In the terminology of §A.3, Proposition A.13 says that the functor \( \lambda_{X/Y} \) is polynomial of degree 1 when \( n = 0 \). This is generalized by the next result due to F. Ducrot [Duc05, Theorem 4.2].

**Theorem A.14.** [Duc05, Theorem 4.2] There exists a unique way to assign to each flat, projective, finitely presented morphism \( \pi: X \to Y \) of relative dimension \( n \) with \( Y \) admissible a polynomial structure of degree \( n + 1 \) on \( \lambda_{X/Y} : \mathcal{P}(X) \to \mathcal{P}(Y) \mathbb{Q} \) with the following properties:

(i) it commutes with base change;
(ii) it coincides with the above one when \( n = 0 \);
(iii) for any relative effective divisor \( D \) on \( X \), the polynomial structures on \( \lambda_{X/Y} \) and \( \lambda_{D/Y} \) are compatible with the canonical restriction isomorphism

\[
\lambda_{X/Y}(L) - \lambda_{X/Y}(L - D) \simeq \lambda_{D/Y}(L|_D).
\]

More precisely, [Duc05, Theorem 4.2] proves the existence of a canonical \((n + 2)\)-cube structure on \( \lambda_{X/Y} \), which yields (and is probably equivalent to) a polynomial structure of degree \( n + 1 \) on \( \lambda_{X/Y} \), as discussed in §A.3. Strictly speaking, the proof of Theorem A.14 assumes \( X \) and \( Y \) to be locally noetherian, but all the arguments apply to the valuative case as well, once Corollary A.7 is available. Since a polynomial structure of degree \( n + 1 \) on \( \lambda_{X/Y} \) is by definition a multi-additive structure on the \((n + 1)\)-st difference

\[
(\delta^{n+1} \lambda_{X/Y})(L_0, \ldots, L_n) = \sum_{I \subseteq \{0, \ldots, n\}} (-1)^{n+1-|I|} \lambda_{X/Y} \left( \sum_{i \in I} L_i \right),
\]

and we can thus define the Deligne pairing as the functor \( \mathcal{P}(X)^{n+1} \to \mathcal{P}(Y) \mathbb{Q} \) that takes line bundles \( L_0, \ldots, L_n \) on \( X \) to the \( \mathbb{Q} \)-line bundle

\[
\langle L_0, \ldots, L_n \rangle_{X/Y} := (\delta^{n+1} \lambda_{X/Y})(L_0, \ldots, L_n).
\]  

**Theorem A.15.** The Deligne pairing satisfies the following properties.

(i) it is multi-additive, symmetric, and commutes with base change;
(ii) for each relative effective Cartier divisor \( D \) on \( X \), we have canonical multi-additive functorial isomorphisms

\[
\langle \mathcal{O}_X(D), L_1, \ldots, L_n \rangle_{X/Y} \simeq \langle L_1|_D, \ldots, L_n|_D \rangle_{D/Y};
\]

(iii) if \( f: X' \to X \) is a finite flat of degree \( e \), then we have canonical functorial isomorphisms

\[
\langle f^* L_0, \ldots, f^* L_n \rangle_{X'/Y} \simeq e \langle L_0, \ldots, L_n \rangle_{X/Y}.
\]

**Proof.** (i) follows directly from Theorem A.14. Given \( D \) as in (ii), taking the \( n \)-th iterated difference of the restriction isomorphism

\[
\lambda_{D/Y}(L|_D) \simeq \lambda_{X/Y}(L) - \lambda_{X/Y}(L - D)
\]

yields

\[
\langle L_1|_D, \ldots, L_n|_D \rangle_{D/Y} = (\delta^n \lambda_{D/Y})(L_1|_D, \ldots, L_n|_D)
\]

\[
\simeq (\delta^n \lambda_{X/Y})(L_1, \ldots, L_n) - (\delta^n \lambda_{X/Y})(L_1, \ldots, L_n) \simeq - (\delta^{n+1} \lambda_{X/Y})(-D, L_1, \ldots, L_n),
\]

which is isomorphic to

\[
(\delta^{n+1} \lambda_{X/Y})(D, L_1, \ldots, L_n) = \langle D, L_1, \ldots, L_n \rangle_{X/Y},
\]
by multiadditivity of \( \delta^{n+1} \lambda_{X/Y} \). Finally, let \( f : X' \to X \) be finite and flat of degree \( e \), so that \( E := f_* \mathcal{O}_{X'} \) is a rank \( e \) vector bundle. By the projection formula we have

\[
R(\pi \circ f)_*(f^*L) = R\pi_*(Rf_*(f^*L)) = R\pi_*(L \otimes E),
\]

and we get (iii) thanks to [Duc05, Proposition 4.7.1], which yields a canonical isomorphism between the \((n+1)\)-st difference of \( L \mapsto \det R\pi_*(L \otimes E) \) and \( e \delta^{n+1} \lambda_{X/Y} \).

By Lemma A.5 we finally get the following generalization of [KM76, Theorem 4].

**Corollary A.16.** For each \( 0 \leq i \leq n+1 \), define a functor \( F_{n+1,i} : \mathcal{P}(X) \to \mathcal{P}(Y)_Q \) by

\[
F_{n+1,i}(L) := \sum_{j=i}^{n+1} (-1)^{j-i} (\delta^j \lambda_{X/Y})(L^j).
\]

For each line bundle \( L \) on \( X \) and \( m \in \mathbb{Z} \), we then have functorial isomorphisms

\[
\lambda_{X/Y}(mL) \simeq \sum_{i=0}^{n+1} \binom{m+i}{i} F_{n+1,i}(L) = \frac{m^{n+1}}{(n+1)!} \langle L^{n+1} \rangle_{X/Y} + O(m^n),
\]

compatible with base change.

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