SOME REMARKS ON STABLE ALMOST COMPLEX STRUCTURES ON MANIFOLDS

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Abstract. Let $X$ be an $(8k + i)$-dimensional pathwise connected CW-complex with $i = 1$ or 2 and $k \geq 0$, $\xi$ be a real vector bundle over $X$. Suppose that $\xi$ admits a stable complex structure over the $8k$-skeleton of $X$. Then we get that $\xi$ admits a stable complex structure over $X$ if the Steenrod square

$$\text{Sq}^2 : H^{8k-1}(X; \mathbb{Z}/2) \rightarrow H^{8k+1}(X; \mathbb{Z}/2)$$

is surjective. As an application, let $M$ be a 10-dimensional manifold with no 2-torsion in $H^i(M; \mathbb{Z})$ for $i = 1, 2, 3$, and no 3-torsion in $H_1(M; \mathbb{Z})$. Suppose that the Steenrod square

$$\text{Sq}^2 : H^7(M; \mathbb{Z}/2) \rightarrow H^9(M; \mathbb{Z}/2)$$

is surjective. Then the necessary and sufficient conditions for the existence of a stable almost complex structure on $M$ are given in terms of the cohomology ring and characteristic classes of $M$.

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1. Introduction

First we introduce some notations. For a topological space $X$, let $\text{Vect}_\mathbb{C}(X)$ (resp. $\text{Vect}_\mathbb{R}(X)$) be the set of isomorphic classes of complex (resp. real) vector bundles on $X$, and let $\widetilde{K}(X)$ (resp. $\widetilde{KO}(X)$) be the reduced $KU$-group (resp. reduced $KO$-group) of $X$, which is the set of stable equivalent classes of complex (resp. real) vector bundles over $X$. For a map $f : X \rightarrow Y$ between topological spaces $X$ and $Y$, denote by

$$f^*: \widetilde{K}(Y) \rightarrow \widetilde{K}(X) \quad \text{and} \quad f^*: \widetilde{KO}(Y) \rightarrow \widetilde{KO}(X)$$

the induced homomorphisms. For $\xi \in \text{Vect}_\mathbb{R}(X)$ (resp. $\eta \in \text{Vect}_\mathbb{C}(X)$), we will denote by $\xi$ (resp. $\eta$) the stable class of $\xi$ (resp. $\eta$). Moreover, we will denote by

$$w_i(\xi) = w_i(\xi) \quad \text{the } i\text{-th Stiefel-Whitney class of } \xi,$$
$$p_i(\xi) = p_i(\xi) \quad \text{the } i\text{-th Pontrjagin class of } \xi,$$
$$c_i(\eta) = c_i(\eta) \quad \text{the } i\text{-th Chern class of } \eta,$$
$$ch(\eta) \quad \text{the Chern character of } \eta.$$
In particular, if $X$ is a smooth manifold, we denote by
\[
\begin{align*}
  w_i(X) &= w_i(TX) & \text{the } i\text{-th Stiefel-Whitney class of } X,\\
p_i(X) &= p_i(TX) & \text{the } i\text{-th Pontrjagin class of } X,
\end{align*}
\]
where $TX$ is the tangent bundle of $X$.

It is known that there are natural homomorphisms
\[
\begin{align*}
  \tilde{r}_X &
  : \tilde{K}(X) \to \tilde{KO}(X) & \text{the real reduction},\\
  \tilde{c}_X &
  : \tilde{KO}(X) \to \tilde{K}(X) & \text{the complexification},
\end{align*}
\]
which induced from
\[
\begin{align*}
  r_X &
  : \text{Vect}_C(X) \to \text{Vect}_R(X) & \text{the real reduction},\\
  c_X &
  : \text{Vect}_R(X) \to \text{Vect}_C(X) & \text{the complexification},
\end{align*}
\]
respectively. Let $\xi \in \text{Vect}_R(X)$ be a real vector bundle over $X$. We say that $\xi$ admits a stable complex structure over $X$ if there exists a complex vector bundle $\eta$ over $X$ such that $\tilde{r}_X(\tilde{\eta}) = \tilde{\xi}$, that is $\tilde{\xi} \in \text{Im } \tilde{r}_X$. In particular, if $X$ is a smooth manifold, we say that $X$ admits a stable almost complex structure if $T_X$ admits a stable complex structure.

Let $U = \lim_{n \to \infty} U(n)$ (resp. $SO = \lim_{n \to \infty} SO(n)$) be the stable unitary (resp. special orthogonal) group. Denote by $\Gamma = SO/U$. Let $X^q$ be the $q$-skeleton of $X$, and denote by $i: X^q \to X$ the inclusion map of $q$-skeleton of $X$. Suppose that $\xi$ admits a stable complex structure over $X^q$, that is there exists a complex vector bundle $\eta$ over $X^q$ such that
\[
i^*(\tilde{\xi}) = \tilde{r}_{X^q}(\tilde{\eta}).
\]
Then the obstruction to extending $\eta$ over the $(q + 1)$-skeleton of $X$ is denoted by
\[
o_{q+1}(\eta) \in H^{q+1}(X, \pi_q(\Gamma))
\]
where
\[
\pi_q(\Gamma) = \begin{cases} 
\mathbb{Z}, & q \equiv 2 \mod 4, \\
\mathbb{Z}/2, & q \equiv 0, -1 \mod 8, \\
0, & \text{otherwise}.
\end{cases}
\]
(cf. Bott [4] or Massey [12] p.560]).

If $q \equiv 2 \mod 4$, that is the coefficient group $\pi_q(\Gamma) = \mathbb{Z}$, the obstructions $o_{q+1}(\eta)$ have been investigated by Massey [12] Theorem I. For example,
\[
o_3(\eta) = \beta w_2(\xi), \quad o_7(\eta) = \beta w_6(\xi),
\]
where $\beta: H^i(M; \mathbb{Z}/2) \to H^{i+1}(M; \mathbb{Z})$ is the Bockstein homomorphism. Moreover, if $q \equiv -1 \mod 8$, hence $\pi_q(\Gamma) = \mathbb{Z}/2$, the obstruction $o_8(\eta)$ has been studied by Massey [12] Theorem III, Thomas [16] Theorem 1.2, Heaps [11], M. Čadek, M. Crabb and J. Vanžura [6, Proposition 4.1 (a)], Dessai [8] Theorem 1.9 and Yang[18] Corollary 1.6, etc. Furthermore, when $X$ is a $n$-dimensional closed oriented smooth manifold with $n \equiv 0 \mod 8$, the obstruction $o_n(\eta)$ is determined...
by Yang [18, Theorem 1.1]. However, in the case \( q \equiv 0 \pmod{8} \), we only known that \( o_{q+1}(\eta) = 0 \) if \( H^{8q+1}(X; \mathbb{Z}/2) = 0 \).

In this paper, the obstructions \( o_{q+1}(\eta) \) for \( q \equiv 0 \pmod{8} \) are investigated and our results can be state as follows.

Denote by \( Sq^2 : H^i(X; \mathbb{Z}/2) \to H^{i+2}(X; \mathbb{Z}) \) the Steenrod square.

**Theorem 1.** Let \( X \) be a \((8k + i)\)-dimensional pathwise connected CW-complex with \( i = 1 \) or \( 2 \) and \( k \) the non-negative integral number; \( \xi \) be a real vector bundle over \( X \). Suppose that \( X \) satisfying the condition

\[
(*) \quad Sq^2 : H^i(X; \mathbb{Z}/2) \to H^{i+2}(X; \mathbb{Z}/2) \quad \text{is surjective.}
\]

Then \( \xi \) admits a stable complex structure over \( X \) if and only if it admits a stable complex structure over \( X^{8k} \).

**Remark 1.1.** This theorem tell us that the final obstruction

\[
o_{8k+1}(\eta) = 0
\]

if \( X \) satisfying the condition \((*)\). The following examples tell us that the CW-complexes which satisfying the condition \((*)\) are exists.

**Example 1.2.** Trivial case: the CW-complex \( X \) with \( H^{8k+1}(X; \mathbb{Z}/2) = 0 \).

**Example 1.3.** Denote by \( G_k(\mathbb{R}^{n+k}) \) the Grassmannian of \( k \)-planes in real \( n + k \) dimensional space \( \mathbb{R}^{n+k} \), \( \zeta_{k,n} \) the universal \( k \)-plane bundle over \( G_k(\mathbb{R}^{n+k}) \). It is known that (cf. Borel [3]) \( \dim G_k(\mathbb{R}^{n+k}) = kn \) and

\[
H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \cdots, w_k, \bar{w}_1, \cdots, \bar{w}_n]/I_{k,n}
\]

where \( w_i = w_i(\zeta_{k,n}) \) and \( I_{k,n} \) is the ideal generated by the equations

\[
(1 + w_1 + \cdots + w_k)(1 + \bar{w}_1 + \cdots + \bar{w}_n) = 0.
\]

Hence we get that \( \dim G_3(\mathbb{R}^6) = 9, \dim G_2(\mathbb{R}^7) = 10 \) and

\[
H^*(G_3(\mathbb{R}^6); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, w_3]/(w_1^4 + w_1^2 w_2 + w_2^2, w_1^2 w_2 + w_1^2 w_3, w_1^2 w_3 + w_3^2),
\]

\[
H^*(G_2(\mathbb{R}^7); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2]/(w_1^6 + w_1^4 w_2 + w_1^2 w_3, w_1^3 w_2).
\]

Therefore

\[
H^0(G_3(\mathbb{R}^6); \mathbb{Z}/2) \cong \mathbb{Z}/2, \quad H^0(G_2(\mathbb{R}^7); \mathbb{Z}/2) \cong \mathbb{Z}/2,
\]

are all generated by \( w_1^4 w_2^2 \) and we have

\[
Sq^2(w_1^4 w_2^2) = w_1^5 w_2^2.
\]

**Example 1.4.** Denote by \( \mathbb{C}P^2 \) the 2-dimensional complex projective space, \( S^l \) the \( l \)-dimensional sphere. One may easily verify that the following manifolds are all satisfying the condition \((*)\):

- \( M_1 = \mathbb{C}P^2 \times S^5 \times S^1 \),
- \( M_2 = M_1 \times S^{8k}, k > 0 \),
- \( \#_n M_i \) the connected sum of \( n \) copies of \( M_i \), \( i = 1, 2 \),
- etc.
Let $M$ be an $n$-dimensional closed oriented smooth manifold. It is a classical topic in geometry and topology to determine the necessary and sufficient conditions for $M$ to admit a stable almost complex structure. These are only known in the case $n \leq 8$ (cf. [17], [9], [12], [16], [11], [7], [6], [18], etc.). If $n = 10$, Thomas and Heaps determined these conditions in the case $H_1(M; \mathbb{Z}/2) = 0$ and $w_4(M) = 0$ (cf. [16, Theorem 1.6] and [11, Theorem 2]). Moreover, Dessai [8, Theorems 1.2, 1.9] got them in the case $H_1(M; \mathbb{Z}) = 0$ and $H_i(M; \mathbb{Z})$, $i = 2, 3$ has no 2-torsion, no assumption on $w_4(M)$ is made.

One may see from the proof of [16, Theorem 1.6] and [11, Theorem 2] that the assumption $H_1(M; \mathbb{Z}/2) = 0$ is used just to guarantee that the final obstruction $o_9(\eta) = 0$. So as an application of Theorem 1, the assumption $H_1(M; \mathbb{Z}/2) = 0$ in [16, Theorem 1.6] and [11, Theorem 2] can be replaced by the assumption that the Steenrod square $Sq^2: H^7(M; \mathbb{Z}/2) \to H^9(M; \mathbb{Z}/2)$ is surjective. That is trivial, we will not list them here.

As the second application of Theorem 1, we can get the following result.

From now on, $M$ will be a 10-dimensional closed oriented smooth manifold with no 2-torsion in $H_i(M; \mathbb{Z})$, $i = 1, 2, 3$ and no 3-torsion in $H_1(M; \mathbb{Z})$. We may also suppose that the Steenrod square $Sq^2: H^7(M; \mathbb{Z}/2) \to H^9(M; \mathbb{Z}/2)$ is surjective.

Since $H_2(M; \mathbb{Z})$ has no 2-torsion, it follows from the universal coefficient theorem that $H^8(M; \mathbb{Z})$ has no 2-torsion. Hence the mod 2 reduction homomorphism $\rho_2: H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}/2)$ is surjective by using the long exact Bockstein sequence associated to the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. Therefore $M$ is spin$^c$. We may fix an element $c \in H^2(M; \mathbb{Z})$ satisfying $\rho_2(c) = w_2(M)$.

**Definition 1.5.** Set

$$\mathcal{D}(M) \triangleq \{ x \in H^2(M; \mathbb{Z}) \mid x^2 + cx = 2z_x \text{ for some } z_x \in H^4(M; \mathbb{Z}) \}$$

where (uniquely determined) $z_x$ is depending on $x$. One may find that $\mathcal{D}(M)$ is a subgroup of $H^2(M; \mathbb{Z})$ and it does not depend on the choice of $c$.

**Theorem 2.** Let $M$ be a 10-dimensional closed oriented smooth manifold with no 2-torsion in $H_i(M; \mathbb{Z})$, $i = 1, 2, 3$, no 3-torsion in $H_1(M; \mathbb{Z})$. Suppose that the Steenrod square $Sq^2: H^7(M; \mathbb{Z}/2) \to H^9(M; \mathbb{Z}/2)$ is surjective. Then $M$ admits a stable almost complex structure if and only if

(1.2) $$w_3^2(M) \cdot \rho_2(x) = (Sq^2 \rho_2(z_x)) \cdot w_4(M)$$
holds for every $x \in D(M)$.

**Remark 1.6.** This theorem is a generalization of Dessai [8 Theorem]. In fact, the congruence (1.2) is a simplification of the congruence (1.2) in [8 Theorem 1.2].

**Remark 1.7.** One may find that we only need to check the congruence (1.2) for the generators of $D(M)$.

Obviously, one can get that

**Corollary 1.8.** Let $M$ be as in Theorem 2. Suppose that $w_4(M) = 0$. Then $M$ always admits a stable almost complex structure.

**Remark 1.9.** Compare Thomas [16 Theorem 1.6].

For $M$ which has "nice" cohomology, we have

**Corollary 1.10.** Let $M$ be as in Theorem 2. Assume in addition that $H^2(M; \mathbb{Z})$ is generated by $h$ and $h^2 \equiv 0 \mod 2$. Then $M$ always admits a stable complex structure.

**Remark 1.11.** Compare Dessai [8 Corollary 1.3].

**Example 1.12.** One can deduced easily from Corollary 1.10 that the manifold $\mathbb{C}P^2 \times S^5 \times S^1$ (in Example 1.4) must admits a stable complex structure. In fact, $\mathbb{C}P^2 \times S^5 \times S^1$ is a complex manifold because both $\mathbb{C}P^2$ and $S^5 \times S^1$ are complex manifold. Moreover, it follows from Theorem 2 and the cohomology ring of $\#_n \mathbb{C}P^2 \times S^5 \times S^1, n \geq 1$ that they all admit a stable almost complex structure.

This paper is arranged as follows. Firstly, the Theorem 1 is proved in §2. Then in order to prove the Theorem 2 and the Corollary 1.10 in §4, we investigated the obstruction to extend complex vector bundles over the $(2q-1)$-skeleton of $X$ to $(2q+1)$-skeleton in §3.

2. **The proof of Theorem 1**

Let $X$ be a $(8k+i)$-dimensional pathwise connected CW-complex, $i = 1$ or 2. In this section, combining the Bott exact sequence with the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$, we give the proof of Theorem 1.

Let $BU$ (resp. $BO$) be the classifying space of the stable unitary group $U$ (resp. stable orthogonal group $O$). Since $O/U$ is homotopy equivalent to $\Omega^6 BO$ (cf. [4]), the canonical fibering

$$O/U \hookrightarrow BU \to BO$$

gives rise to a long exact sequence of $K$-groups ( we call it the Bott exact sequence )

$$
\cdots \to \widetilde{KO}^{q-2}(X) \to \widetilde{KO}^q(X) \to \widetilde{KO}^q(X) \to \widetilde{KO}^{q-1}(X) \to \cdots
$$
which is similar to the exact sequence given by Bott in [5, p.75].

According to Switzer [14, pp.336-341], the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is the spectral sequence \( \{E^p_{r}, d_r\} \) with
\[
E^p_{1} \cong H^p(X; KO^r), \quad E^p_{\infty} \cong F^{p,q}/F^{p+1,q-1},
\]
where
\[
F^{p,q} = \text{Ker} [i^*_p: KO^{p+q}(X) \to KO^{p+q}(X_{p-1})],
\]
and the coefficient ring of $KO$-theory is (cf. Bott [5, p. 73])
\[
KO^r = \mathbb{Z}[\alpha, x, \gamma, \gamma^{-1}]/(2\alpha, \alpha^2, \alpha x, x^2 - 4\gamma)
\]
with the degrees \(|a| = -1, |x| = -4\) and \(|\gamma| = -8\).

It is well known that the differentials $d_2$ of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ are given as follows (see M. Fujii [10, Formula (1.3)] for instance):
\[
d^2 = \begin{cases} 
\text{Sq}^2, & q \equiv 0 \mod 8, \\
\text{Sq}^2, & q \equiv -1 \mod 8, \\
0, & \text{otherwise}.
\end{cases}
\]

Denote by $j: X \to (X, X^{8k})$ and $i: X^{8k} \to X$ the inclusions. Then we have the following commutative diagram:
\[
\begin{array}{cccccc}
K(X, X^{8k}) & \xrightarrow{f} & \tilde{K}(X) & \xrightarrow{i^*} & \tilde{K}(X^{8k}) & \xrightarrow{\delta} & K^1(X, X^{8k}) \\
\| & & \| & & \| & & \|
\end{array}
\]
\[
\begin{array}{cccccc}
\tilde{K}(X^{8k}) & \xrightarrow{\gamma} & \tilde{K}(X) & \xrightarrow{i^*} & \tilde{K}(X^{8k}) & \xrightarrow{\delta} & K^1(X, X^{8k}) \\
\| & & \| & & \| & & \|
\end{array}
\]
\[
\begin{array}{cccccc}
KO^{-1}(X, X^{8k}) & \xrightarrow{j_\gamma} & KO^{-1}(X) & \xrightarrow{i^*} & KO^{-1}(X^{8k}) & \xrightarrow{\delta} & KO(X, X^{8k}) \\
\| & & \| & & \| & & \|
\end{array}
\]
where the horizontal sequence is the long exact sequence of $K$ and $KO$ groups, the vertical sequence is the Bott exact sequence (2.1).

**Lemma 2.1.** The homomorphism $\tilde{r}: K^1(X, X^{8k}) \to KO^1(X, X^{8k})$ is injective.

**Proof.** Denote by $i^*: (X^{8k+1}, X^{8k}) \to (X, X^{8k})$, $j^*: (X, X^{8k}) \to (X, X^{8k+1})$ the inclusions. Then by the naturality of the long exact sequence of $K$ theory, we got the following exact ladder
\[
\begin{array}{cccccc}
K^1(X, X^{8k+1}) & \xrightarrow{f^1} & K^1(X, X^{8k}) & \xrightarrow{i^*} & K^1(X^{8k+1}, X^{8k}) \\
\| & & \| & & \|
\end{array}
\]
\[
\begin{array}{cccccc}
KO^1(X, X^{8k+1}) & \xrightarrow{\tilde{r}} & KO^1(X, X^{8k}) & \xrightarrow{i^*} & KO^1(X^{8k+1}, X^{8k}) \\
\| & & \| & & \|
\end{array}
\]
Therefore, this lemma can be deduced easily from the facts that $K^1(X, X^{8k+1}) = 0$ and $\tilde{r} : K^1(X^{8k+1}, X^{8k}) \to KO^1(X^{8k+1}, X^{8k})$ is injective.

\textbf{Lemma 2.2.} If the Steenrod square $Sq^2 : H^{8k-1}(X; \mathbb{Z}/2) \to H^{8k+1}(X; \mathbb{Z}/2)$ is surjective, we must have

$$\text{Im } j^*_o \subseteq \text{Im } \tilde{r}_X.$$ 

\textit{Proof.} In the Atiyah-Hirzebruch spectral sequence, since $KO^{-1}(X, X^{8k+1}) = 0$, it follows that

$$F^{8k+2, 8k-3} = \text{Ker}[i^*_o : KO^{-1}(X) \to KO^{-1}(X^{8k+1})] = 0.$$ 

Hence

$$E_{\infty}^{8k+1, 8k-2} = F^{8k+1, 8k-2} / F^{8k+2, 8k-3} = F^{8k+1, 8k-2}.$$ 

Therefore, by the equation (2.2), the surjectivity of the Steenrod square $Sq^2 : H^{8k-1}(X; \mathbb{Z}/2) \to H^{8k+1}(X; \mathbb{Z}/2)$ implies that

$$F^{8k+1, 8k-2} = E_{\infty}^{8k+1, 8k-2} = 0.$$ 

That is, the homomorphism $i^*_o : KO^{-1}(X) \to KO^{-1}(X^{8k})$ is injective. Then it follows from the exactness of the diagram (2.3) that the homomorphism $j^*_o : KO^{-1}(X, X^{8k}) \to KO^{-1}(X)$ is a zero homomorphism. So the composition homomorphism

$$\gamma_X \circ j^*_o = j^*_o \circ \gamma = 0,$$

and we get that

$$\text{Im } j^*_o \subseteq \text{Im } \tilde{r}_X.$$ 

\textit{□}

\textit{Proof of Theorem 1.} Obviously, $\xi$ admits a stable complex structure over $X$ implies that $\xi$ admits a stable complex structure over $X^{8k}$.

Conversely, suppose that $\xi$ admits a stable complex structure over $X^{8k}$. That is there exists a stable complex vector bundle $\tilde{\eta}' \in \tilde{K}(X^{8k})$ such that

$$\tilde{r}_{X^{8k}}(\tilde{\eta}') = i^*_o(\tilde{\xi}).$$ 

Then it follows from the exactness of the diagram (2.3) and the Lemma 2.1 that there is a stable complex vector bundle $\tilde{\eta}_1$ such that $i^*_o(\tilde{\eta}_1) = \tilde{\eta}'$ and

$$i^*_o(\tilde{r}_X(\tilde{\eta}_1) - \tilde{\xi}) = 0.$$ 

That is,

$$\tilde{r}_X(\tilde{\eta}_1) - \tilde{\xi} \in \text{Im } j^*_o.$$ 

Therefore, by the Lemma 2.2 the surjectivity of the Steenrod Square $Sq^2 : H^{8k-1}(X; \mathbb{Z}/2) \to H^{8k+1}(X; \mathbb{Z}/2)$ implies that $\xi$ admits a stable complex structure over $X$.

\textit{□}
3. The obstruction for an extension of a vector bundle over $X^{2q-1}$ to $X^{2q+1}$

Let $X$ be a pathwise connected CW-complex. In order to prove Theorem 2, in this section, we will investigate the obstruction for an extension of a complex vector bundle over the $(2q-1)$-skeleton $X^{2q-1}$ of $X$ to the $(2q+1)$-skeleton $X^{2q+1}$.

**Theorem 3.1.** Let $X$ be a pathwise connected CW-complex and $\tilde{\eta} \in \tilde{K}(X^{2q-1})$ a stable complex vector bundle over $X^{2q-1}$. Denote by $\theta_{2q+1}(\tilde{\eta}) \in H^{2q+1}(X; \mathbb{Z})$ the obstruction to extend $\tilde{\eta}$ to $X^{2q+1}$. Then

$$(q - 1)! \cdot \theta_{2q+1}(\tilde{\eta}) = 0.$$  

**Proof.** Denote by $i: X^{2q-1} \to X^{2q}$ and $j: X^{2q} \to (X^{2q}, X^{2q-1})$ the inclusions. Let $f: \bigsqcup S^{2q} \to X^{2q}$ be the attaching map such that

$$X^{2q+1} = X^{2q} \cup_f \bigsqcup^{2q+1},$$

where the symbol $\bigsqcup$ means the disjoint union. Then it follows from $\tilde{K}(S^{2q-1}) \cong 0$ that we have the following diagram

$$
\begin{array}{cccc}
\tilde{K}(X^{2q+1}) & \to & \tilde{K}(X^{2q}, X^{2q-1}) & \to & \tilde{K}(X^{2q}) \\
\downarrow & & \downarrow f^* & & \downarrow i^* \\
\tilde{K}(X^{2q}) & \to & \tilde{K}(X^{2q-1}) & \to & 0 \\
\Delta_{2q} & \downarrow & & & \\
\tilde{K}(\bigsqcup S^{2q})
\end{array}
$$

where $\Delta_{2q} = f^* \circ j^*$ and the horizontal and vertical sequences are the long exact sequences of $K$ groups for the pairs $(X^{2q}, X^{2q-1})$ and $(X^{2q+1}, X^{2q})$ respectively. Therefore, for any $\tilde{\eta} \in \tilde{K}(X^{2q-1})$, there must exists a stable complex vector bundle $\tilde{\eta}' \in \tilde{K}(X^{2q})$ such that $i^*(\tilde{\eta}') = \tilde{\eta}$, and $\tilde{\eta}$ can be extended to $X^{2q+1}$ if and only if

$$f^*(\tilde{\eta}') \in \text{Im} \Delta_{2q}. $$

Denote by

$$\Sigma: H^{2q}(\bigsqcup S^{2q}; \mathbb{Z}) \to H^{2q+1}(X^{2q+1}, X^{2q}; \mathbb{Z})$$

the suspension which is a isomorphism,

$$f^*: H^{2q}(X^{2q}; \mathbb{Z}) \to H^{2q}(\bigsqcup S^{2q}; \mathbb{Z})$$

and

$$j^*: H^{2q}(X^{2q}, X^{2q-1}; \mathbb{Z}) \to H^{2q}(X^{2q}; \mathbb{Z})$$

the homomorphisms induced by the maps $f$ and $j$ respectively. It is known that the Chern characters

$$ch: \tilde{K}(X^{2q}, X^{2q-1}) \to H^{2q}(X^{2q}, X^{2q-1}; \mathbb{Z}),$$

$$ch: \tilde{K}(\bigsqcup S^{2q}) \to H^{2q}(\bigsqcup S^{2q}; \mathbb{Z})$$
are all isomorphisms, and the composition homomorphism
\[ \Sigma \circ ch \circ \Delta_{2q} \circ ch^{-1} : H^{2q}(X^{2q}, X^{2q-1}; \mathbb{Z}) \to H^{2q+1}(X^{2q+1}, X^{2q}; \mathbb{Z}) \]
is just the cellular coboundary homomorphism
\[ d_{2q} = \Sigma \circ f^* \circ j^* : H^{2q}(X^{2q}, X^{2q-1}; \mathbb{Z}) \to H^{2q+1}(X^{2q+1}, X^{2q}; \mathbb{Z}) \]
of the cellular cochain complex of \( X \) (cf. Atiyah and Hirzebruch [2, pp. 16-18]).

Hence by the equation (3.1), the obstruction \( \vartheta_{2q+1}(\tilde{\eta}) \in H^{2q+1}(X; \mathbb{Z}) \) is just the cohomology class represented by
\[ \Sigma \circ ch(f_u^*(\tilde{\eta})) = \Sigma \left( \frac{c_q(f_u^*(\tilde{\eta}))}{(q-1)!} \right). \]
Therefore, it follows from the surjectivity of \( j^* \) and \( d_{2q} = \Sigma \circ f^* \circ j^* \) that
\[(q-1)! \cdot o(\tilde{\eta}) = 0. \]
\[ \square \]

**Corollary 3.2.** Let \( X \) be a CW-complex. Suppose that \( H^{2q+1}(X; \mathbb{Z}) \) contains no \((q - 1)\)!-torsion. Then every stable complex vector bundle over \( X^{2q-1} \) can be extended to a complex vector bundle over \( X^{2q+2} \).

**Proof.** Note that the homomorphism \( i_u^*: \tilde{K}(X^{2q+2}) \to \tilde{K}(X^{2q+1}) \) is surjective. \( \square \)

Similarly we can get that

**Theorem 3.3.** Let \( X \) be a pathwise connected CW-complex and \( \tilde{\xi} \in \tilde{KO}(X^{4q-1}) \) (resp. \( \tilde{\gamma} \in \tilde{KS}(X^{4q-1}) \)) a stable real (resp. symplectic) vector bundle over \( X^{4q-1} \). Denote by \( \theta_{4q+1}(\tilde{\xi}) \in H^{4q+1}(X; \mathbb{Z}) \) (resp. \( \theta_{4q+1}(\tilde{\gamma}) \in H^{4q+1}(X; \mathbb{Z}) \)) the obstruction to extend \( \tilde{\xi} \) (resp. \( \tilde{\gamma} \)) to \( X^{4q+1} \). Then
\[ (2q - 1)! \cdot a_q \cdot \theta_{4q+1}(\tilde{\xi}) = 0, \]
\[ (2q - 1)! \cdot b_q \cdot \theta_{4q+1}(\tilde{\xi}) = 0, \]
where \( a_q \cdot b_q = 2 \) and \( a_q = 1 \) for \( q \) even and \( a_q = 2 \) for \( q \) odd.

4. The proof of Theorem 2

In this section, we will take Dessai’s strategy, which was used to prove [8, Theorem 1.2], to prove the Theorem 2.

Recall that \( M \) is a 10-dimensional closed oriented smooth manifold with no 2-torsion in \( H_i(M; \mathbb{Z}), i = 1, 2, 3 \) and no 3-torsion in \( H_1(M; \mathbb{Z}) \). It also satisfying that the Steenrod square
\[ Sq^2 : H^7(M; \mathbb{Z}/2) \to H^8(M; \mathbb{Z}/2) \]
is surjective. Then $M$ is spin', and we have fixed an element $c \in H^2(M; \mathbb{Z})$ satisfying $\rho_2(c) = w_2(M)$ and defined

$$\mathcal{D}(M) \doteq \{ x \in H^2(M; \mathbb{Z}) \mid x^2 + cx = 2z_8 \text{ for some } z_8 \in H^4(M; \mathbb{Z}) \}.$$  

Denote by $i^*_\nu: \tilde{K}(M) \to \tilde{K}(M^7)$ and $j^*_\nu: \tilde{K}(M) \to \tilde{K}(M^8)$ the homomorphisms induced by the inclusions $i: M^7 \to M$ and $j: M^8 \to M$ respectively. Let $p_*: H^i(M; \mathbb{Z}) \to H^i(M; \mathbb{Q})$ be the homomorphism induced by the canonical inclusion $p: \mathbb{Z} \to \mathbb{Q}$. Recall that $\tilde{r}_{\nu p_*}: \tilde{K}(M^9) \to \tilde{KO}(M^9)$ is the real reduction homomorphism. Then we get that

**Lemma 4.1.** $M$ has the following properties:

(a) $\rho_2: H^i(M; \mathbb{Z}) \to H^i(M; \mathbb{Z}/2)$ is surjective for $i \neq 4, 5$.
(b) $\rho_2 \circ p_*^{-1}$ is well defined on $p_*(H^8(M; \mathbb{Z}))$.
(c) The annihilator of $\text{Sq}^2 \rho_2 H^8(M; \mathbb{Z})$ with respect to the cup-product is equal to $\rho_2(\mathcal{D}(M))$.
(d) For any stable complex vector bundle $\eta' \in \tilde{K}(M^7)$, there exists a stable complex vector bundle $\eta \in \tilde{K}(M)$ such that $i^*_\nu(\eta) = \eta'$.
(e) Let $\xi \in \tilde{KO}(M)$ be a stable real vector bundle over $M$. Then there must exists a stable complex vector bundle $\eta \in \tilde{K}(M)$, such that $\tilde{r}_{\nu M^7}j^*_\nu(\eta) = i^*_\nu(\xi)$. Moreover, if $\tilde{r}_{\nu M^7}j^*_\nu(\eta) = i^*_\nu(\xi)$, $\xi$ must admits a stable complex structure.

**Proof.** (a) Since $H^i(M; \mathbb{Z})$ has no 2-torsion for $i = 0, 1, 2, 3, 10$, the same is true for $H^i(M; \mathbb{Z}^2)$, $i \neq 5, 6$ (universal coefficient theorem and Poincaré Duality). Hence the statement is true by using the long exact Bockstein sequence.
(b) That is because the kernel of $p_*: H^8(M; \mathbb{Z}) \to H^8(M; \mathbb{Q})$ is an odd torsion which maps to zero under $\rho_2$.
(c) Let $y \in H^2(M; \mathbb{Z}/2)$. Note that the Wu class $V_2$ is $w_2(M)$. Then it follows from Cartan formula that for any $z \in H^8(M; \mathbb{Z})$

$$y \cdot \text{Sq}^2 \rho_2(z) = 0 \quad \text{iff} \quad (w_2(M) \cdot y + y^2) \cdot \rho_2(z) = 0.$$  

Hence the statement is true by the statement (a) and the definition of $\mathcal{T}(M)$.
(d) Since $H^8(M; \mathbb{Z}) \approx H_1(M; \mathbb{Z})$ has no 2-torsion and 3-torsion, the statement can be deduced easily from Corollary 3.2
(e) The statements can be proved by combining the identity (4.1) and the statement (d) with the Theorem 1.

$\square$

Denote by $[M]$ the fundamental class of $M$, $\langle \cdot, \cdot \rangle$ the Kronecker product and

$$\tilde{\Phi}(M) = 1 - \frac{p_1(M)}{24} + \frac{-4p_2(M) + 7p_1^2(M)}{5760}$$

the $\Phi$ class of $M$. For any $x \in H^2(M; \mathbb{Z})$, we will denote by $l_x$ the complex line bundle over $M$ with

$$c_1(l_x) = x.$$
For any \( x \in \mathcal{D}(M) \), we may choose a class \( v_x \in H^8(M; \mathbb{Z}) \) such that
\[
\rho_2(v_x) = Sq^2 \rho_2 z_x.
\]
Since \( M \) has the properties as in Lemma 4.1, the results below can be deduced easily by applying the methods of Dessai in [8].

**Lemma 4.2.** Let \( \tilde{\xi} \in KO(M) \) be a stable oriented vector bundle over \( M \). Then \( \tilde{\xi} \) admits a stable complex structure if and only if
\[
\rho_2 \circ p_{x}^{-1}(ch_4(\tilde{c}_M(\tilde{\eta}) - \tilde{\xi})) \in Sq^2 \rho_2 H^6(M; \mathbb{Z})
\]
for some stable complex vector bundle \( \tilde{\eta} \in \tilde{K}(M) \) satisfying \( \tilde{r}_M i_\psi^*(\tilde{\eta}) = i^\alpha(\tilde{\xi}) \).

**Lemma 4.3.** Let \( c \in H^2(M; \mathbb{Z}) \) be an integral class satisfying \( \rho_2(c) = w_2(M) \). For any \( x \in \mathcal{D}(M) \) there is a stable complex vector bundle \( \tilde{\eta}_x \in \tilde{K}(M) \) trivial over the 3-skeleton such that
\[
e^{x/2} \cdot ch(\tilde{I}_x - \tilde{\eta}_x) \equiv x + \left( \frac{x^3}{6} - \frac{xc^2}{8} - \frac{v_x}{2} \right) \mod H^{28}(M; \mathbb{Q}).
\]

**Lemma 4.4.** Let \( M \) be a 10-dimensional closed oriented smooth manifold as in Theorem 2. Let \( \tilde{\xi} \in KO(M) \) be a stable real vector bundle over \( M \). Choose \( c \in H^2(M; \mathbb{Z}) \) (resp. \( d \in H^2(M; \mathbb{Z}) \)) satisfying \( \rho_2(c) = w_2(M) \) (resp. \( \rho_2(d) = w_2(\tilde{\xi}) \)). Then \( \tilde{\xi} \) admits a stable complex structure if and only if
\[
\langle \hat{\eta}(M) \cdot e^{x/2} \cdot ch(\tilde{I}_x - \tilde{\eta}_x) \cdot ch(\tilde{c}_M(\tilde{\xi} - \tilde{I}_d)), [M] \rangle \equiv 0 \mod 2
\]
holds for every \( x \in \mathcal{D}(M) \).

**Proof of the Lemmas 4.2, 4.3 and 4.4.** cf. the proves of [8] Lemmas 1.7, 1.8, Theorem 1.9. \( \square \)

**Remark 4.5.** Since \( M \) is spin\(^c\), it follow from the Differentiable Riemann-Roch theorem (cf. Atiyah-Hirzebruch [1, Corollary 1]) that the rational number
\[
\langle \hat{\eta}(M) \cdot e^{x/2} \cdot ch(\tilde{I}_x - \tilde{\eta}_x) \cdot ch(\tilde{c}_M(\tilde{\xi} - \tilde{I}_d)), [M] \rangle
\]
is integral, so it make sense to take congruent classes modulo 2.

In fact, the congruence (4.1) can be simplified, hence Lemma 4.4 can be restated as follows.

**Theorem 4.6.** Let \( M \) be a 10-dimensional closed oriented smooth manifold with no 2-torsion in \( H_i(M; \mathbb{Z}) \), \( i = 1, 2, 3 \), no 3-torsion in \( H_2(M; \mathbb{Z}) \). Suppose that the Steenrod square
\[
Sq^2 : H^7(M; \mathbb{Z}/2) \to H^9(M; \mathbb{Z}/2)
\]
is surjective. Let \( \xi \) be a real vector bundle over \( M \). Choose integral class \( d \in H^2(M; \mathbb{Z}) \) such that \( \rho_2(d) = w_2(\xi) \). Set
\[
A_{\xi,x} = \left( \frac{x}{2}, \frac{p_1(\xi) - d^2}{2}, \frac{p_1(\xi) - d^2}{2} - \frac{p_1(M) - c^2}{2} \right), [M] \).
\]
Then \( \xi \) admits a stable complex structure if and only if
\[
A_{\xi,x} \equiv (w_8(\xi) + w_2(\xi)Sq^2(w_4(\xi)) \cdot \rho_2(x) + Sq^2(z_x)w_4(\xi)) \mod 2
\]
holds for every $x \in \mathcal{D}(M)$.

Remark 4.7. One may find that the rational number $A_{\xi,x}$ is integral (see the proof of this Theorem below), so it make sense to take congruent classes modulo 2.

Proof. Let $F = \tilde{\xi} - \tilde{l}_d$. Then $F$ is a stable spin vector bundle since $w_2(F) = 0$. Therefore, the spin characteristic classes

$$q_i(F) \in H^{d_i}(M; \mathbb{Z}), \quad i = 1, 2,$$

of $F$ are defined, and they satisfy the following relations (cf. Thomas [15])

$$p_1(F) = 2q_1(F), \quad \rho_2(q_1(F)) = w_4(F),$$
$$p_2(F) = 2q_2(F) + q_1^2(F), \quad \rho_2(q_2(F)) = w_8(F).$$

Since we have the equations below

$$x^3 = 2xz_x - 2cz_x + c^2x,$$
$$q_1(F) = (p_1(\xi) - d^2)/2,$$
$$w_4(F) = w_4(\xi),$$
$$w_8(F) = w_8(\xi) + w_2(\xi)w_6(\xi) + w_2^2(\xi)w_4(\xi),$$
$$w_6(\xi) = \text{Sq}^2(w_4(\xi)) + w_2(\xi)w_4(\xi),$$

it follows that

$$\langle \hat{H}(M) \cdot e^{c^2/2} \cdot ch(\tilde{l}_x - \tilde{\eta}_x) \cdot ch(\tilde{c}_M(\tilde{\xi} - \tilde{l}_d)), [M] \rangle \equiv 0 \mod 2$$

iff

$$3\langle \hat{H}(M) \cdot e^{c^2/2} \cdot ch(\tilde{l}_x - \tilde{\eta}_x) \cdot ch(\tilde{c}_M(\tilde{\xi} - \tilde{l}_d)), [M] \rangle \equiv 0 \mod 2$$

iff

$$\frac{x}{2} \cdot q_1(F)[q_1(F) - \frac{p_1(M) - c^2}{2}] - q_2(F)x - v_4q_1(F) \equiv 0 \mod 2$$

iff

$$A_{\xi,x} \equiv (w_8(\xi) + w_2(\xi)\text{Sq}^2(w_4(\xi))) \cdot \rho_2(x) + \text{Sq}^2(z_x)w_4(\xi) \mod 2.$$

$\square$

Proof of Theorem 2. Denote by $V_i \in H^i(M; \mathbb{Z}/2)$ the Wu-class which is the unique class satisfying

$$\text{Sq}^i u = V_i \cdot u$$

for any $u \in H^{10-i}(M; \mathbb{Z})$. It is known that they satisfy (cf. [13, p. 132])

$$w_4(M) = \Sigma_{k=0}^8 \text{Sq}^k V_{k-i}.$$

Hence we get that

$$V_2 = w_2(M),$$
$$V_4 = w_4(M) + w_2^2(M),$$
$$V_5 = 0,$$
$$w_8(M) = w_4^2(M) + w_2^4(M).$$
Note that for any \( x \in \mathfrak{T}(M) \), we have
\[
\rho_2^2(x) = \rho_2(x) \cdot w_2(M).
\]
Therefore, for any \( x \in \mathfrak{T}(M) \), we can get that
\[
\rho_2(x)w_2(M)\text{Sq}^2w_4(M) = \text{Sq}^2(\rho_2(x)w_2(M)w_4(M)) = \rho_2(x)w_2^2(M)w_4(M).
\]
Hence
\[
(4.3) \quad \rho_2(x)(w_2^4(M) + w_2(M)\text{Sq}^2w_4(M)) = \rho_2(x)w_2^2(M)(w_4(M) + w_3^2(M)) = \text{Sq}^4(\rho_2(x)w_2^2(M)) = \rho_2(x)w_4^2(M) = \rho_4^2(x)w_2(M) = \text{Sq}^2(\rho_2(x)) = 0.
\]

Then Theorem 2 can be deduced easily from the Theorem 4.6 and the identity (4.3) by choosing that \( \xi = TM \) and \( d = c \).

**Proof of Corollary 1.10** If \( M \) is spin, \( \mathfrak{T}(M) \) generated by \( 2h \). Hence the congruence (1.2) is always true.

If \( M \) is not spin, \( \mathfrak{T}(M) \) generated by \( h \) and \( \rho_2(h) = w_2(M) \). Therefore, we only need to check the congruence (1.2) for \( x = h \). Note that the Wu class \( V_5 \) is zero. In this case, \( z_x = h^2 \),
\[
w_4^2(M)\rho_2(x) = w_2(M)w_4^2(M) = \text{Sq}^2(w_3^2(M)) = (\text{Sq}^1w_4(M))^2 = \text{Sq}^5\text{Sq}^1w_4(M) = 0,
\]
and
\[
\text{Sq}^2(z_x)w_4(M) = \text{Sq}^2(h^2)w_4(M) = 0.
\]
Hence the congruence (1.2) is always true for this case.

These prove the Corollary 1.10.  \( \square \)

**References**

1. M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281.
2. ______, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 7–38.
3. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de lie compacts*, Ann. of Math. **57** (1953), 115–207.
4. R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337.
5. ______, *Lectures on K(X)*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York-Amsterdam, 1969.
6. M. Čadek, M. Crabb, and J. Vanžura, *Obstruction theory on 8-manifolds*, Manuscripta Math. **127** (2008), no. 3, 167–186.
7. M. Čadek and J. Vanžura, *On complex structures in 8-dimensional vector bundles*, Manuscripta Math. 95 (1998), no. 3, 323–330.
8. A. Dessai, *Some remarks on almost and stable almost complex manifolds*, Math. Nachr. 192 (1998), 159–172.
9. C. Ehresmann, *Sur les variétés presque complexes*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, 1952, pp. 412–419.
10. M. Fujii, *$K_O$-groups of projective spaces*, Osaka J. Math. 4 (1967), 141–149.
11. T. Heaps, *Almost complex structures on eight- and ten-dimensional manifolds*, Topology 9 (1970), 111–119.
12. W. S. Massey, *Obstructions to the existence of almost complex structures*, Bull. Amer. Math. Soc. 67 (1961), 559–564.
13. John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 76.
14. R. M. Switzer, *Algebraic topology—homotopy and homology*, Classics in Mathematics, Springer-Verlag, Berlin, 2002.
15. E. Thomas, *On the cohomology groups of the classifying space for the stable spinor groups*, Bol. Soc. Mex. (1962), 57–69.
16. , *Complex structures on real vector bundles*, Amer. J. Math. 89 (1967), 887–908.
17. W.T. Wu, *Sur les classes caractéristiques des structures fibrées sphériques*, Actualités Sci. Ind., no. 1183, Hermann & Cie, Paris, 1952.
18. Huijun Yang, *A note on stable complex structures on real vector bundles over manifolds*, Topology Appl. 189 (2015), 1–9.

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