On a Characterization of Geodesic Trajectories and Gravitational Motions

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Abstract: We shall here discuss a characterization of geodesics trajectories. We shall show that the action of the gravitational field on mass particles can be essentially identified with the force that cannot be absolutely eliminated. This leads to an alternative formulation of equivalence principle.

1. Introduction

It has been noted that gravitational force can be eliminated at a point in spacetime by a class of observers, called free fall observers. Some authors hence noted that gravitational field is properly described by tidal forces which in fact cannot be eliminated by the choice of the observer; see [1], [2], [3].

These properties are consequence of equivalence principle which claims in fact that mass particles falls along geodesic worldlines. In particular one usually claims that there are free fall observers for which mass particles behave nearly as in special relativity; see [4], [5]. From a foundational viewpoint there are good reasons to try to characterize worldlines of mass particles without resorting to special relativity. In fact general relativity (GR) aims to be a fundamental theory and special relativity should be obtained as a weak field approximation. Hence, in principle, one should not use special relativity essentially in the definition of GR.

We shall hereafter show that mass particle worldlines can be characterized completely in terms of few basic properties.

First, we know that worldlines are uniquely determinated once one specifies the initial point and the initial velocity (i.e. the direction in spacetime). Of course only timelike direction can be used for mass particle worldlines. In any event we expect worldlines to be solutions of ordinary differential equations, of order 2 and Cauchy theorem should hold for it, i.e. the equation should be in normal form. In other words we shall start from an equation in the general form

$$\ddot{q}^\mu = f^\mu(s, q, \dot{q})$$  \hspace{1cm} (1.1)

where dots denote the derivatives with respect to the affine parameter $s$ along the curve. Of course, one should discuss reasonable regularity conditions for the function $f^\mu(s, q, \dot{q})$.

Second, as any equation in GR we expect the equation to be covariant, i.e. coordinate independent or, equivalently, diffeomorphism covariant. At least one should describe what happens to the equation when a spacetime diffeomorphism is applied.

Third, since the parametrization of the worldline is unphysical (in the sense that two different parametrizations of the same trajectory represent the same motion of the particle), the equation should be covariant with respect to arbitrary reparametrizations. At least one should describe what happens to the equation when a reprarametrization is applied.

We shall show that there are still many equations obeying these axioms. This is quite an expected result since one would require these properties to hold for any mass particle subjected...
to an arbitrary force field as well, not when only gravitational field acts. Hence one needs to characterize pure gravitational motions.

Fourth property will be such characterization. Pure gravitational motions cannot be eliminated absolutely from the general equation. Here by absolutely we mean that one cannot set gravitational field to disappear for any observer and independently of the parametrization used for worldlines at the same time. We shall below show how to mathematically require this property. From a physical viewpoint, for example electromagnetic fields acts on charges but it does not act on neutral particles. As a consequence one could expect there is an equation for charges in which coupling to electromagnetic field is active and an equation for neutral particles in which there is no interaction with electromagnetic field. When a neutral particle is described the coupling terms to the electromagnetic field is not there in an absolute sense since the particle is neutral for any observer and regardless the parametrization one can use along the worldlines. On the contrary the gravitational interaction is universal. It can be switched off by specific observers but there are always observers for which the gravitational field is there.

We shall show that these four properties in fact characterize completely gravitational interaction and that the equation for mass particles which obeys these properties is necessarily the equation for geodesic trajectories of some connection. We shall also briefly review the issue concerning the observability of such a connection; see [5] and [6]. In particular, we shall show that geodesic trajectories are associated to a class of projectively equivalent connections.

From a fundamental viewpoint one can use this procedure to introduce geodesics in GR to describe mass particle worldlines, without resorting to special relativity.

2. The Equation for Geodesic Trajectories

Let us first fix notation. Let us denote by $M$ the spacetime manifold. It is a connected $m$-dimensional smooth manifold on which global Lorentzian metrics exist. Lightrays determine on $M$ a conformal structure $[g]$, i.e. a class of conformally equivalent Lorentzian metrics. The conformal structure is equivalent to the sheaf of lightcones and it defines lightlike, timelike and spacelike directions in spacetime; see [5], [7], [8].

A curve is a parametrized curve in $M$, i.e. a map $\gamma : I \rightarrow M$ for some interval $I \subset \mathbb{R}$. It is not to be confused with its image $\gamma(I) \subset M$ which is a 1-dimensional submanifold which we shall called a path.

Physically speaking, when two curves are two different parametrizations of the same path then they represent the same motion. For this reason we define such two curves to be equivalent. An equivalence class $[\gamma]$ with respect to this relation is called a trajectory in $M$. A trajectory is characterized by a the path which is shared by all representatives of the trajectory.

Let us now expand the right hand side of equation (1.1) as a formal series of the velocities, i.e.

$$\ddot{q}^\alpha = A^\alpha(s, q) + A^\alpha_{\alpha\mu}(s, q)\dot{q}^\mu + A^\alpha_{\alpha\beta\gamma}(s, q)\dot{q}^\beta\dot{q}^\gamma + \ldots$$

This can be done whenever the function $f(s, q, \dot{q})$ is smooth.

Let now $\Phi : M \rightarrow q \rightarrow q' = Q(q)$ be a spacetime diffeomorphism. Then one has

$$\begin{cases} \dot{q}^\alpha = J^\alpha_\mu \dot{q}^\mu \\ \ddot{q}^\alpha = J^\alpha_\mu \dot{q}^\mu + J^\alpha_\nu \ddot{q}^\nu \end{cases}$$

(2.2)
where \( J_\alpha^\nu \) and \( J_\alpha^\nu \) denotes the Jacobian and Hessian of the diffeomorphism \( q' = Q(q) \).

Hence the coefficients of equation (2.1) under the diffeomorphism tranform as

\[
\begin{align*}
A^\mu &\rightarrow J_\mu^\nu A^\nu \\
A_\alpha^\mu &\rightarrow J_\mu^\nu A_\nu^\alpha \bar{J}_\alpha^\lambda \\
A_{\alpha\beta}^\mu &\rightarrow J_\mu^\nu (A_\nu^\rho \bar{J}_\rho^\alpha \bar{J}_\beta^\beta - \bar{J}_\nu^\alpha) \\
A_{\alpha\beta\gamma}^\mu &\rightarrow J_\mu^\nu A_\nu^\rho \bar{J}_\rho^\alpha \bar{J}_\beta^\beta \bar{J}_\gamma^\gamma
\end{align*}
\]

(2.3)

where the bar denotes Jacobians and Hessians of the inverse map \( q = \bar{Q}(q') \). Hence to ensure covariance one only needs all coefficients to be tensor fields except the coefficient of order 2. The transformation rules of the coefficient \( A_{\rho\sigma}^\nu \) can be easily compared to the transformation rules of a connection \( \Gamma_\nu^\rho_\sigma \) to conclude that necessary there exists a connection such that \( A_{\rho\sigma}^\nu = -\Gamma_\nu^\rho_\sigma \).

Hence the most general covariant equation of the form (2.1) is

\[
\bar{q}^\mu + \Gamma_\alpha^\mu q^\alpha \bar{q}^\beta = A^\mu + A_\alpha^\mu q^\alpha + A_{\alpha\beta}^\mu q^\alpha q^\beta + \ldots
\]

(2.4)

understanding that all coefficients \( (A^\mu, A_\alpha^\mu, A_{\alpha\beta}^\mu, \ldots) \) are tensorial and \( \Gamma_\alpha^\mu_\beta \) is a connection. Since the equation is insensitive to torsion of \( \Gamma \) one can restrict without loss of generality to torsionless connections.

Now let us consider an arbitrary reparametrization \( q'(s') = q(\phi(s')) \). One has

\[
\begin{align*}
\dot{q}' &= \dot{q}\dot{\phi} \\
\ddot{q}' &= \ddot{q}\dot{\phi}^2 + \dot{q}\ddot{\phi}
\end{align*}
\]

(2.5)

and the trasformed equation has new coefficients

\[
\begin{align*}
A^\mu &= \dot{\phi}^2 A^\mu \\
A_\alpha^\mu &= \dot{\phi} A_\alpha^\mu + \ddot{\phi} \delta_\alpha^\mu \\
\Gamma_\rho^\alpha_\beta &= \Gamma_\rho^\mu_\beta \\
A_{\alpha\beta\gamma}^\mu &= \frac{1}{\phi} A_{\alpha\beta\gamma}^\mu \\
\ldots
\end{align*}
\]

(2.6)

Hence while the connection is invariant with respect to reparametrizations, the other tensorial coefficients are not. Notice however that \( (A^\mu, A_\alpha^\mu, A_{\alpha\beta\gamma}, \ldots) \) transform linearly and if they are set to zero in one parametrisation they are zero in any parametrization. Not the same can be said for the coefficient \( A_\alpha^\mu \) which if zero in one parametrization is in general non-zero in the others parametrizations. According to our definition of gravitational field, the coefficients \( (A^\mu, A_\alpha^\mu, A_{\alpha\beta\gamma}, \ldots) \) can be set to zero absolutely and they represent non-gravitational interactions. The equation for pure gravitational interactions thence simplifies to

\[
\bar{q}^\mu + \Gamma_\alpha^\mu q^\alpha \bar{q}^\beta = A_{\alpha}^\mu \bar{q}^\alpha
\]

(2.7)

Also part of the last coefficient \( A_\alpha^\mu \) can be set to zero absolutely. In fact one can decompose \( A_\alpha^\mu \) as a traceless part \( a_\alpha^\mu \) and a pure trace part, i.e. \( A_\alpha^\mu = a_\alpha^\mu + \lambda \delta_\alpha^\mu \). One can easily show that

\[
\begin{align*}
X &= \dot{\phi} \lambda + \frac{\bar{q}}{\phi} \\
\alpha_\alpha^\mu &= \dot{\phi} a_\alpha^\mu
\end{align*}
\]

(2.8)
and the traceless part $a^\mu_\alpha$ can be also be set to zero absolutely. The equation for gravitational motions further reduces to

$$\ddot{q}^\mu + \Gamma^\mu_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta = \lambda \dot{q}^\mu$$ (2.9)

### 3. Projective Structures on Spacetime

The equation (2.9) coincides with the equation for geodesic trajectories of $\Gamma$. Let us in fact consider a (torsionless) connection $\Gamma^\alpha_{\mu\nu}$ on $M$. A geodesic motion is a curve locally expressed by $\gamma : I \to M : s \mapsto q^\mu(s)$ which satisfies the following equation

$$\ddot{q}^\alpha + \Gamma^\alpha_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0$$ (3.1)

A trajectory $[\gamma]$ is called a geodesic trajectory for $\Gamma$ if one of its representatives is a geodesic motion. One can easily show that for any representative of a geodesic trajectory $\gamma' \in [\gamma]$ there exists a function $\lambda(s)$ such that locally $\gamma' : I \to M : s \mapsto q'^\mu(s)$ and

$$\ddot{q}'^\alpha + \Gamma^\alpha_{\mu\nu} q'^\mu q'^\nu = \lambda \dot{q}'^\mu$$ (3.2)

We have hence shown that under very mild regularity conditions and the axioms discussed above the equation (1.1) reduces necessarily in the form of this equation.

Let us now define an equivalence relation on the space of all (torsionless) connections. Let us say that two connections are projectively equivalent when they have the same geodesics trajectories. Two connections $\Gamma^\mu_{\alpha\beta}$ and $\Gamma'^\mu_{\alpha\beta}$ are projectively equivalent if and only if locally there exists a covector $A$ for which

$$\Gamma'^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \delta^\mu_{\alpha} A_{\beta}$$ (3.3)

If a motion $\gamma$ is a representative for a geodesics trajectory for $\Gamma$ it is also a geodesic trajectory for $\Gamma'$. Hence it obeys the equations

$$\ddot{q}^\mu + \Gamma^\mu_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta = \lambda \dot{q}^\mu \quad \ddot{q}'^\mu + \Gamma'^\mu_{\alpha\beta} \dot{q}'^\alpha \dot{q}'^\beta = \lambda' \dot{q}'^\mu$$ (3.4)

where we set $\lambda' = \lambda + A_\mu \dot{q}^\mu$. A class of projectively equivalent connections is called a projective structure on spacetime; see [6]. The equation of geodesic trajectories is compatible with the quotient onto projective structures and hence it is attached to a projective structure on spacetime, not to a specific connection. In principle one cannot observe a representative of a projective structure just observing worldlines of mass particles.

### 4. Conclusions

We showed that gravitational free fall of mass particles is uniquely characterized by few axioms. We know that free fall worldlines are described by a second order normal ordinary differential equation, which has to be covariant with respect to spacetime diffeomorphisms and reparametrizations. Then gravitational interaction is characterized by the fact that gravitational field cannot be switched off absolutely. Under these only mild assumptions one can obtain equations of geodesic trajectories for a projective structure on spacetime.
In EPS paper (see [5]) one could argue that particle worldlines select a projective structure on spacetime. This is done without resorting to special relativity. In particular there is no reason to assume that the connection describing free fall is Levi-Civita connection of the metric describing lightcones. Extended theories of gravitation, namely f(R) models, in its metric-affine formulation do in fact provide examples in which free fall turns out to be described by a Weyl connection which is metric but not for the metric originally used to describe lightcones. On the contrary free fall is associated to a conformal metric so that there is a representative of conformal structure which also describe free fall; see [7], [8], [9]. Although it is well-known that conformal transformations (acting on the metric and leaving the connection unchanged) maps a formalism into an equivalent one, still there is one representative (known as the Einstein frame) which can be canonically selected; [10].

It may be worth noticing that this situation is paradigmatic about how absolute knowledge arises in relativistic theories. An observer chooses its coordinates and parametrizations to describe worldlines. To write the equation a pair (Γ, λ) must be chosen. Equivalently any new pair (Γαβμ = Γαβμ + δαμ(α Aβ), ̃λ = λ + Aμ(λ)) is equally good for that observer. This observer dependent setting contains some absolute knowledge about real world together with some convention which is only a characteristic of the observer, namely a gauge. One absolutely knows that another observers can use another coordinate system, other parametrizations and other pairs (Γ′, λ′). However, the new observer is not completely free in its choices if it has to describe the same reality of the previous observer. The connections must be projectively equivalent and the functions λ and λ′ are accordingly constrained. An absolute description of reality in fact emerges from a set of compatible relative descriptions. As usual absolute knowledge is encoded in transformation rules as an absolute description of a manifold is encoded in transition functions of an atlas.

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