WHITNEY NUMBERS FOR POSET CONES

GALEN DORPALEN-BARRY AND VICTOR REINER

Abstract. Hyperplane arrangements dissect \( \mathbb{R}^n \) into connected components called chambers, and a well-known theorem of Zaslavsky counts chambers as a sum of nonnegative integers called Whitney numbers of the first kind. His theorem generalizes to count chambers within any cone defined as the intersection of a collection of halfspaces from the arrangement, leading to a notion of Whitney numbers for each cone. This paper focuses on cones within the braid arrangement, consisting of the reflecting hyperplanes \( x_i = x_j \) inside \( \mathbb{R}^n \) for the symmetric group, thought of as the type \( A_{n-1} \) reflection group. Here,

- cones correspond to posets,
- chambers within the cone correspond to linear extensions of the poset,
- the Whitney numbers of the cone interestingly refine the number of linear extensions of the poset.

We interpret this refinement explicitly for two families of posets: width two posets, and disjoint unions of chains. In the latter case, this gives a geometric re-interpretation to Foata’s theory of cycle decomposition for multiset permutations, and leads to a simple generating function compiling these Whitney numbers.

1. Introduction

This paper concerns arrangements \( \mathcal{A} = \{H_1, \ldots, H_m\} \) of hyperplanes \( H_i \), which are affine-linear codimension one subspaces of a real vector space \( V = \mathbb{R}^n \). Each such arrangement dissects \( V \) into the connected components of its complement \( V \setminus \bigcup_{i=1}^m H_i \), called chambers. We denote by \( \mathcal{C}(\mathcal{A}) \) the collection of all such chambers.

The theory of hyperplane arrangements is rich and well-explored, with connections to reflection groups, braid groups, random walks and card-shuffling, and discrete geometry of polytopes and oriented matroids; see \cite{10,14}. In particular, the number \( \#\mathcal{C}(\mathcal{A}) \) of chambers has a famous interpretation due to Zaslavsky, expressed in terms of the intersection poset \( \mathcal{L}(\mathcal{A}) \), consisting of all intersection subspaces \( X = H_{i_1} \cap H_{i_2} \cdots \cap H_{i_k} \), ordered via reverse inclusion. This poset is known to have the property that every lower interval \([V, X] := \{Y \in \mathcal{L}(\mathcal{A}) : V \leq Y \leq X\}\) from its unique bottom element \( V \) to any intersection space \( X \) forms a geometric lattice. In particular, each such \([V, X]\) is a ranked poset, with rank function given by the codim \( (X) := n - \dim(X) \). Zaslavsky’s result asserts that

\[
\#\mathcal{C}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)| = \sum_{k=0}^{n} c_k(\mathcal{A}) = |\text{Poin}(\mathcal{A}, t)|_{t=1},
\]

where \( \mu(-,-) \) denotes the Möbius function of \( \mathcal{L}(\mathcal{A}) \), while the nonnegative integers

\[
c_k(\mathcal{A}) := \sum_{X \in \mathcal{L}(\mathcal{A}) : \text{codim}(X) = k} |\mu(V, X)|,
\]

are often called the (signless) Whitney numbers of the first kind for \( \mathcal{A} \), and their generating function

\[
\text{Poin}(\mathcal{A}, t) := \sum_{k} c_k(\mathcal{A}) t^k
\]

is called the Poincaré polynomial.\[1\]

\[1\]This is because it is the generating function for the Betti numbers of the complexified complement \( \mathbb{C}^n \setminus \mathcal{A} \); see \cite{10} Chap. 5.

5
Our starting point is a less widely-known generalization of equation (1), already proven by Zaslavsky [16]. It applies more generally to count the chambers of $\mathcal{A}$ that lie within a cone $\mathcal{K}$, defined to be the intersection of any collection of open halfspaces for hyperplanes of $\mathcal{A}$; said differently, a cone $\mathcal{K}$ of $\mathcal{A}$ is a chamber in $\mathcal{C}(\mathcal{A}')$ for some subarrangement $\mathcal{A}' \subseteq \mathcal{A}$. Results on the set $\mathcal{C}(\mathcal{K})$ of all chambers of $\mathcal{A}$ inside a cone $\mathcal{K}$ have appeared more recently in work of Brown on random walks [3], and independently Aguiar and Mahajan [1, Theorem 8.22]. Define the poset of interior intersections for $\mathcal{K}$ to be the following order ideal in $\mathcal{L}(\mathcal{A})$:

$$\mathcal{L}^{\text{int}}(\mathcal{K}) = \{ X \in \mathcal{L}(\mathcal{A}) \mid X \cap \mathcal{K} \neq \emptyset \}.$$ 

Zaslavsky observed in [16, Example A, p. 275] that (1) generalizes to cones $\mathcal{K}$, asserting

$$\#\mathcal{C}(\mathcal{K}) = \sum_{X \in \mathcal{L}^{\text{int}}(\mathcal{K})} |\mu(V, X)| = \sum_{k=0}^{\dim(X)} c_k(\mathcal{K}) = |\text{Poin}\,(\mathcal{K}, t)|_{t=1}. \tag{2}$$

Here we again define nonnegative integers, the (signless) Whitney numbers of the first kind for the cone $\mathcal{K}$

$$c_k(\mathcal{K}) := \sum_{X \in \mathcal{L}^{\text{int}}(\mathcal{K}) : \text{codim}(X) = k} |\mu(V, X)|,$$

with generating function $\text{Poin}(\mathcal{K}, t) := \sum_k c_k(\mathcal{K}) t^k$, which we call the the Poincaré polynomial of $\mathcal{K}$.

For example, inside $\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\}$ in $\mathbb{R}^2$ shown below on the left, we have shaded one of four possible cones $\mathcal{K}$ defined by the subarrangement $\mathcal{A}' = \{H_4, H_5\}$, containing $\#\mathcal{C}(\mathcal{K}) = 5$ chambers of $\mathcal{A}$:

Zaslavsky’s formula (2) computes this as follows. The poset of interior intersections $\mathcal{L}^{\text{int}}(\mathcal{K})$ has Hasse diagram:

$$\mathcal{K} = \{ \mathcal{K}, \mathcal{K} \cap H_3, H_2 \cap H_3, H_1, H_2, H_3 \}$$

Here $\mu(V, X) = (-1)^{\text{codim}(X)}$ for all $X$, so that $(c_0(\mathcal{K}), c_1(\mathcal{K}), c_2(\mathcal{K})) = (1, 3, 1)$, and

$$\#\mathcal{C}(\mathcal{K}) = |\text{Poin}(\mathcal{K}, t)|_{t=1} = \frac{c_0(\mathcal{K}) + c_1(\mathcal{K}) t + c_2(\mathcal{K}) t^2}{c_0(\mathcal{K}) + c_1(\mathcal{K}) + c_2(\mathcal{K})} = 5.$$

The object of this paper is to understand the distribution of the signless Whitney numbers as a refinement of $\#\mathcal{C}(\mathcal{K})$ as in equation (2), for cones $\mathcal{K}$ in the braid arrangement. The braid arrangement $A_{n-1} = \{H_{ij}\}_{1 \leq i < j \leq n}$ is the set of $\binom{n}{2}$ reflecting hyperplanes

$$H_{ij} = \{ (x_1, \ldots, x_n) \in V = \mathbb{R}^n \mid x_i - x_j = 0 \}$$

for the symmetric group $S_n$ on $n$ letters, though of as the reflection group of type $A_{n-1}$. There is a well-known and easy bijection between the chambers $\mathcal{C}(A_{n-1})$ and the permutations $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ in $S_n$, sending $\sigma$ to the chamber:

$$K_\sigma := \{ x \in V = \mathbb{R}^n : x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_n} \}. \tag{3}$$
More generally, one has an easy bijection, reviewed in Section 2, between posets on the set \([n] := \{1, 2, \ldots, n\}\) and cones in the braid arrangement \(A_{n-1}\), sending a poset \(P\) to the cone
\[
\mathcal{K}_P := \{x \in V = \mathbb{R}^n : x_i < x_j \text{ for } i < j \text{ and } j < n\}.
\]
It is readily checked that the chamber \(\mathcal{K}_\sigma\) lies in the cone \(C(\mathcal{K}_P)\) if and only \(\sigma\) is a linear extension of \(P\), meaning that \(i < j\) implies \(i < \sigma_j\), regarding \(\sigma\) as a total order \(\sigma_1 < \sigma_2 < \cdots < \sigma_n\) on \([n]\). Letting \(\text{LinExt}(P)\) denote the set of all linear extensions \(\sigma\), this shows that \(\#C(\mathcal{K}_P) = \#\text{LinExt}(P)\), and hence (2) becomes
\[
\#\text{LinExt}(P) = \sum_{k \geq 0} c_k(P) = [\text{Poin}(P, t)]_t = 1
\]
abbreviating \(c_k(P) := c_k(\mathcal{K}_P)\) and \(\text{Poin}(P, t) := \text{Poin}(\mathcal{K}_P, t)\) from now on. Our primary goal is to understand the following.

**Main Problem.** Given a poset \(P\) on \([n]\), find a statistic \(\text{LinExt}(P) \xrightarrow{\text{stat}} \{0, 1, 2, \ldots\}\) interpreting (4) as follows:
\[
\sum_{\sigma \in \text{LinExt}(P)} t^{\text{stat}(\sigma)} = \sum_{k \geq 0} c_k(P) t^k = \text{Poin}(P, t).
\]

A motivating special case occurs when \(P\) is the antichain poset on \([n]\) that has no order relations, so that \(\#\text{LinExt}(P) = \mathcal{S}_n\) itself, and the signless Whitney number \(c_{n-k}(\mathcal{K}_P)\) of the first kind is well-known [15] Prop. 1.3.7 to be the signless Stirling number of the first kind \(c(n, k)\) that counts permutations in \(\mathcal{S}_n\) having \(k\) cycles. Consequently, (4) becomes the easy summation formula
\[
(5) \quad n! = |\mathcal{S}_n| = \sum_k c(n, k),
\]
which is the \(t = 1\) specialization of the generating function
\[
(6) \quad 1(1 + t)(1 + 2t) \cdots (1 + (n - 1)t) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{fcyc}(\sigma)} = \sum_k c(n, k) t^{n-k} = \text{Poin}(P, t).
\]

The remainder of this paper provides similar interpretations of \(\text{Poin}(P, t)\) for two other families of posets.

Section 2 gives preliminaries on the intersection lattice and cones in braid arrangements. In particular, it gives an explicit combinatorial description of the interior intersections \(\mathcal{C}(\mathcal{K}_P)\) for a poset cone \(\mathcal{K}_P\).

Section 3 then examines posets of width two, that is, posets \(P\) decomposable as \(P = P_1 \cup P_2\) where the subposets \(P_1, P_2\) are chains (i.e. totally ordered subsets) inside \(P\). Here the Whitney numbers \(c_k(P)\) are interpreted by a descent-like statistic on \(\sigma\) in \(\text{LinExt}(P)\):
\[
\text{des}_{P_1, P_2}(\sigma) := \#\{i \in [n-1] : \sigma_i \in P_2, \sigma_{i+1} \in P_1, \text{ with } \sigma_i, \sigma_{i+1} \text{ incomparable in } P\}.
\]

**Theorem 1.1.** For a width two poset decomposed into two chains as \(P = P_1 \cup P_2\), one has
\[
\text{Poin}(P, t) = \sum_{\sigma \in \text{LinExt}(P)} t^{\text{des}_{P_1, P_2}(\sigma)}.
\]

In Example 3.5 below, Theorem 1.1 is applied to show that for \(P = 2 \times n\), the Cartesian product of chains having sizes 2 and \(n\), the Whitney numbers \(c_k(2 \times n)\) are Narayana numbers, counting \(2 \times n\) standard tableaux according to their number of descents.

Section 4 examines posets which are disjoints unions of chains. Given any composition \(\pi = (a_1, \ldots, a_\ell)\) of \(n\), meaning that \(\pi \in \{0, 1, 2, \ldots\}\) and \(|\pi| := \sum_{i=1}^\ell a_i = n\), let \(a_i\) denote an \(a_i\)-element chain poset, and then
\[
\bigcup_{i=1}^\ell a_i := a_1 \cup a_2 \cup \cdots \cup a_\ell
\]
is a disjoint union of incomparable chains of sizes \(a_1, a_2, \ldots, a_\ell\). Here one can identify \(\text{LinExt}(P_\pi)\) with multiset permutations of \(1^{a_1}2^{a_2}\cdots \ell^{a_\ell}\). Section 4 reviews the beautiful theory of cycle decompositions for such multiset permutations due to Foata [4]. Letting \(\text{fcyc}(\sigma)\) denote the number of prime cycles in Foata’s unique decomposition for \(\sigma\), one has this remarkable generalization of (5), (6) above.

**Theorem 1.2.** For any composition \(\pi\) of \(n\), the disjoint union poset of chains poset \(P_\pi\) has
\[
\text{Poin}(P_\pi, t) = \sum_{\sigma \in \text{LinExt}(P_\pi)} t^{n-\text{fcyc}(\sigma)}.
\]
Foata’s theory is then used to prove the following generating function.

**Theorem 1.3.** For ℓ = 1, 2, . . ., one has

\[
\sum_{\pi \in S_{\ell}} \mathrm{Poin}(\mathbb{P}_n, t) \cdot x^{\pi} = \frac{1}{1 - \sum_{j=1}^{\ell} e_j(x) \cdot (t-1)(2t-1) \cdots ((j-1)t-1)}
\]

where \(x^{\pi} := x_1^{a_1} \cdots x_\ell^{a_\ell}\) and \(e_j(x) := \sum_{1 \leq i_1 < \cdots < i_j \leq \ell} x_{i_1} \cdots x_{i_j}\) is the \(j\)th elementary symmetric function.

2. The braid arrangement, its intersection lattice, and its cones

### 2.1. Preliminaries on arrangements.

We begin with some preliminaries on hyperplane arrangements, focusing on braid arrangements. Good references include [10], [14], [15] §3.11, [11] §3.3.

**Definition 2.1.** A hyperplane in \(V = \mathbb{R}^n\) is an affine linear subspace of codimension one. An arrangement of hyperplanes in \(\mathbb{R}^n\) is a finite collection \(\mathcal{A} = \{H_1, \ldots, H_m\}\) of distinct hyperplanes. A chamber of \(\mathcal{A}\) is an open, connected component of \(\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H\). The set of all chambers of \(\mathcal{A}\) is denoted by \(C(\mathcal{A})\).

**Example 2.2.** The type A reflection arrangement, \(A_{n-1}\), also called the braid arrangement, consists of the \(\binom{n}{2}\) hyperplanes of the form

\[H_{ij} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i - x_j = 0\}\]

for integers 0 ≤ i < j ≤ n. There are n! chambers \(\mathcal{K}_{\sigma} = \{x \in \mathbb{R}^n : x_{\sigma_1} < \cdots < x_{\sigma_n}\}\) of \(A_{n-1}\), naturally indexed by the permutations \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_n\) of [n], that give the strict inequalities ordering the coordinates within the chamber, as in [3]. For example, when \(n = 4\),

\[
\mathcal{K}_{1243} = \{x \in \mathbb{R}^4 : x_1 < x_2 < x_4 < x_3\},
\]

\[
\mathcal{K}_{4213} = \{x \in \mathbb{R}^4 : x_4 < x_2 < x_1 < x_3\}
\]

are two out of the 4! = 24 chambers of \(C(A_{4-1})\).

**Definition 2.3.** Let \(\mathcal{A}\) be a hyperplane arrangement in \(\mathbb{R}^n\). An intersection of \(\mathcal{A}\) is a nonempty subspace of the form \(X = H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}\) where \(\{H_{i_1}, H_{i_2}, \ldots, H_{i_k}\} \subseteq \mathcal{A}\). Here the ambient vector space \(V = \mathbb{R}^n\) is considered to the intersection \(\bigcap_{H \in \mathcal{A}} H\) of the empty set of hyperplanes. We denote the set of intersections of \(\mathcal{A}\) by \(\mathcal{L}(\mathcal{A})\).

**Example 2.4.** The intersections of \(A_{n-1}\) are described by equalities between the variables.

- For all \(n \geq 1\), the line \(x_1 = x_2 = \cdots = x_k\) is the intersection of all the hyperplanes of \(A_{n-1}\).
- When \(n = 4\) the intersection of \(H_{12}\) and \(H_{34}\) is the subspace of \(\mathbb{R}^4\) in which \(x_1 = x_2\) and \(x_3 = x_4\). On the other hand, the intersection of \(H_{12}\) and \(H_{13}\) is the subspace of \(\mathbb{R}^4\) in which \(x_1 = x_2 = x_3\).

More generally, there is a bijection \(\pi \mapsto X_\pi\) between the collection \(\Pi_n\) of all set partitions \(\pi = \{B_1, \ldots, B_k\}\) of [n] = \{1, 2, ..., n\} and the set of all intersections of \(A_{n-1}\). The bijection sends the set partition \(\pi\) to the subspace \(X_\pi\) where one has equal coordinates \(x_i = x_j\) whenever \(i, j\) lie in a common block \(B_k\) of \(\pi\). We sometimes denote the set partition \(\pi = \{B_1, \ldots, B_k\}\) with the notation \(\pi = B_1|B_2|\cdots|B_k\), and may or may not include commas and set braces around the elements of each block \(B_i\). E.g., \(1 \mid 23 \mid 456\) and \(\{\{1\}, \{2, 3\}, \{4, 5, 6\}\}\) represent the same set partition of [6].

- For example, the set partition \(12|3\cdots|n\) in which all elements appear as singletons corresponds to \(X_{12|3\cdots|n} = V = \mathbb{R}^n\), the empty intersection, which is the ambient space.
- For all \(n \geq 1\), the set partition \(123\cdots n\) having all the elements in the same block corresponds to the line \(X_{123\cdots n}\) defined by \(x_1 = x_2 = \cdots = x_n\).
- When \(n = 4\), one has \(X_{12|34} = H_{12} \cap H_{34}\) and \(X_{13|24} = H_{12} \cap H_{13}\).

The collection \(\mathcal{L}(\mathcal{A})\) of all intersections of an arrangement \(\mathcal{A}\) will be partially ordered by reverse inclusion, and called the intersection poset of \(\mathcal{A}\). It has unique minimal element, namely the intersection

\[\bigcap_{H \in \mathcal{A}} H = V = \mathbb{R}^n\]

For the braid arrangement \(A_{n-1}\), the intersection poset \(\mathcal{L}(A_{n-1})\) is easy to describe.
Proposition 2.5 ([14 pp. 26-27]). The bijection $\pi \mapsto X_\pi$ from Example 2.4 gives a poset isomorphism

$$\Pi_n \cong \mathcal{L}(A_{n-1})$$

where $\Pi_n$ denotes the lattice of set partitions on $[n]$, ordered via refinement: $\pi_1 \leq \pi_2$ if $\pi_1$ refines $\pi_2$.

For any hyperplane arrangement $\mathcal{A}$, each of the lower intervals $[V, X] := \{ Y \in \mathcal{L}(\mathcal{A}) : V \leq Y \leq X \}$ forms a geometric lattice [15 Prop. 3.11.2]. In particular, this implies that each such lower interval is a ranked poset, with rank function given by the codimension $\text{codim}(X) = \dim(V) - \dim(X)$. Furthermore, this implies that its Möbius function values $\mu(V, X)$, defined recursively by $\mu(V, V) := 1$ and $\mu(V, X) := -\sum_{Y : V \leq Y < X} \mu(V, Y)$, will alternate in sign in the sense that $(-1)^{\text{codim}(X)} \mu(V, X) \geq 0$.

For the braid arrangement, these Möbius function values have a simple expression.

Proposition 2.6 ([15 Example 3.10.4]). For any set partition $\pi = \{ B_1, \ldots, B_k \} \in \Pi_n$, one has

$$\mu(V, X_\pi) = (-1)^{n-k} \prod_{i=1}^{k} (\# B_i - 1)! \quad ( = \mu(1|2|\cdots|n|, \pi) )$$

with the convention $0! := 1$. Here $\mu(V, X_\pi), \mu(1|2|\cdots|n|, \pi)$ are $\mu(-,-)$ values in $\mathcal{L}(A_{n-1}), \Pi_n$, respectively.

Definition 2.7. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^n$. For $0 \leq k \leq n$, the $k$th signless Whitney number of $\mathcal{A}$ is

$$c_k(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A}) : \text{codim}(X) = k} |\mu(V, X)| = (-1)^{k} \sum_{X \in \mathcal{L}(\mathcal{A}) : \text{codim}(X) = k} \mu(V, X).$$

Henceforth, we call $\{c_k(\mathcal{A})\}_{k=0}^n$ the Whitney numbers of $\mathcal{A}$. One of the standard ways to compile them into a generating function is their Poincaré polynomial $\text{Poin}(\mathcal{A}, t) := \sum_{k=0}^{n} c_k(\mathcal{A}) t^k$; see [10 §2.3].

As mentioned in the Introduction, we aim to understand the chambers, intersections, and Whitney numbers for cones in $\mathcal{A}$, of which the chambers, intersections, and Whitney numbers for $\mathcal{A}$ are a special case.

Definition 2.8. Let $\mathcal{A}$ be an arrangement of hyperplanes in $V = \mathbb{R}^n$. A cone $\mathcal{K}$ of $\mathcal{A}$ is any nonempty intersection $\mathcal{A} \neq \mathcal{K} \subseteq V = \mathbb{R}^n$ of (open) halfspaces defined by a subset $\mathcal{A}'$ of the hyperplanes from $\mathcal{A}$. That is, a cone $\mathcal{K}$ is any one of the (open) chambers from the set of all chambers $\mathcal{C}(\mathcal{A}')$ for some subarrangement $\mathcal{A}' \subset \mathcal{A}$. For example, in the following arrangement in $\mathbb{R}^2$ there are four cones defined by the dashed hyperplanes. One such cone $\mathcal{K}$ is shaded below.

Each cone $\mathcal{K}$ of $\mathcal{A}$ has its collection of chambers, namely those chambers in $\mathcal{C}(\mathcal{A})$ that lie inside $\mathcal{K}$:

$$\mathcal{C}(\mathcal{K}) = \{ C \in \mathcal{C}(\mathcal{A}) : C \subset \mathcal{K} \}.$$

The poset of interior intersections of the cone $\mathcal{K}$ is the following order ideal within the poset $\mathcal{L}(\mathcal{A})$:

$$\mathcal{L}_{\text{int}}(\mathcal{K}) = \{ X \in \mathcal{L}(\mathcal{A}) : X \cap \mathcal{K} \neq \emptyset \}.$$

For each $X$ in $\mathcal{L}_{\text{int}}(\mathcal{K})$, its lower interval $[\mathcal{V}, X]$ is still a geometric lattice, with same rank function $\text{codim}(X)$, so that one can define the $k$th (signless) Whitney number of $\mathcal{K}$ by

$$c_k(\mathcal{K}) = \sum_{X \in \mathcal{L}_{\text{int}}(\mathcal{K}) : \text{codim}(X) = k} |\mu(V, X)| = (-1)^{k} \sum_{X \in \mathcal{L}_{\text{int}}(\mathcal{K}) : \text{codim}(X) = k} \mu(V, X),$$

along with their generating function $\text{Poin}(\mathcal{K}, t) := \sum_{k=0}^{n} c_k(\mathcal{K}) t^k$, the Poincaré polynomial for $\mathcal{K}$.

\textsuperscript{2}Aguiar and Mahajan [1] call these objects top-cones.
The starting point for our study is the following result of Zaslavsky [16] counting the number \( \#C(\mathcal{K}) \) of chambers of an arrangement \( \mathcal{A} \) lying inside one of its cones \( \mathcal{K} \).

**Theorem 2.9** ([16, Example A, p. 275]). Let \( \mathcal{K} \) be a cone of an arrangement \( \mathcal{A} \) in \( V = \mathbb{R}^n \). Then

\[
\#C(\mathcal{K}) = \sum_{X \in \mathcal{L}^\text{int}(\mathcal{K})} |\mu(V, K)| = c_0(\mathcal{K}) + c_1(\mathcal{K}) + \cdots + c_n(\mathcal{K}) = \left| \text{Poin}(\mathcal{K}, t) \right|_{t=1}.
\]

Zaslavsky proved in his doctoral thesis the better-known special case of Theorem 2.9 for the full arrangement, that is, where \( \mathcal{K} = V = \mathbb{R}^n \).

The following two examples illustrate Theorem 2.9 for two cones in \( A_3 \).

**Example 2.10.** Consider the braid arrangement \( A = A_4 = \{H_{12}, H_{13}, H_{14}, H_{23}, H_{24}, H_{34}\} \) inside \( V = \mathbb{R}^4 \). On the left below we have drawn a linearly equivalent picture of its intersection with the hyperplane where \( x_1 + x_2 + x_3 + x_4 = 0 \), isomorphic to \( \mathbb{R}^3 \), and depicted the intersection of the hyperplanes with the unit 2-sphere in this 3-dimensional space. Here we pick the cone \( \mathcal{K} \) to be the one defined by the halfspace \( x_3 < x_4 \) for the hyperplane \( H_{34} \), and draw the intersection of \( H_{34} \) with the unit sphere as the equatorial circle, with the other five hyperplanes \( H_{ij} \) depicted as great circles intersecting the hemisphere where \( x_3 < x_4 \). On the right below the non-hyperplane interior intersection subspaces \( X_\pi \) are labeled.

Therefore the intersection poset \( \mathcal{L}^\text{int}(\mathcal{K}) \) of this cone is

\[
\begin{align*}
H_{23} & \quad H_{24} \\
H_{13} & \quad H_{14} \\
V = \mathbb{R}^4 & \\
X_{123} & \quad X_{124} \\
X_{1324} & \quad X_{1423}
\end{align*}
\]

We have \( (c_0(\mathcal{K}), c_1(\mathcal{K}), c_2(\mathcal{K})) = (1, 5, 6) \). Summing these gives \( 1 + 5 + 6 = 12 \), and a quick visual verification assures that there are 12 chambers in this cone.

**Example 2.11.** Consider the cone \( \mathcal{K} \) of \( A_3 \) in which \( x_3 < x_4 \) and \( x_1 < x_2 \). On the left below we have drawn the same picture as Example 2.10 with the cone corresponding to \( \mathcal{K} \) shaded. We depict \( \mathcal{L}^\text{int}(\mathcal{K}) \) on the right.

Therefore the intersection poset \( \mathcal{L}^\text{int}(\mathcal{K}) \) of this cone is

\[
\begin{align*}
H_{23} & \quad H_{24} \\
H_{13} & \quad H_{14} \\
V = \mathbb{R}^4 & \\
X_{123} & \quad X_{124} \\
X_{1324} & \quad X_{1423}
\end{align*}
\]

We have \( c_0(\mathcal{K}) = 1, c_1(\mathcal{K}) = 4, \) and \( c_2(\mathcal{K}) = 1 \). Summing these gives \( 1 + 4 + 1 = 6 = \#C(\mathcal{K}) \).

For the remainder of this paper, we focus on cones \( \mathcal{K} \) inside braid arrangements \( A_{n-1} \). It is well-known (see, e.g., [11, §3.3]) and easy to see that such cones correspond bijectively with posets \( P \) on \([n]\) via this rule:
one has \( x_i < x_j \) for all points in the cone \( K \) if and only if \( i <_P j \). We will denote the cone associated to \( P \) by \( K_P \), and abbreviate \( \mathcal{L}^{\text{int}}(P) := \mathcal{L}^{\text{int}}(K_P) \), along with \( c_k(P) := c_k(K_P) \) and \( \text{Poin}(P,t) := \text{Poin}(K_P,t) \).

**Example 2.12.** The cone inside \( A_3 \) in Example 2.10 given by the inequality \( x_3 < x_4 \) on \( V = \mathbb{R}^4 \) has defining poset \( P_1 \) with order relation \( 3 <_{P_1} 4 \) on \( [4] = \{1,2,3,4\} \), while the cone in Example 2.11 given by the inequalities \( x_1 < x_2 \) and \( x_3 < x_4 \) has defining poset \( P_2 \) with order relations \( 1 <_{P_2} 2 \) and \( 3 <_{P_2} 4 \). These posets \( P_1, P_2 \) are shown here:

\[
P_1 = \begin{array}{ccc}
1 & 2 & 3 \\
4 & & \\
\end{array}
\quad P_2 = \begin{array}{ccc}
2 & 4 & \\
1 & 3 & \\
\end{array}
\]

2.2. **Preposets, posets, cones and a characterization of \( \mathcal{L}^{\text{int}}(P) \).** By Theorem 2.16, the intersection poset \( \mathcal{L}(A_{n-1}) \) is isomorphic to the set partition lattice \( \Pi_n \), and hence for each cone \( K_P \) in \( A_{n-1} \), one should be able to identify the interior intersection poset as some order ideal \( \mathcal{L}^{\text{int}}(P) := \mathcal{L}^{\text{int}}(K_P) \) inside \( \Pi_n \). This is our next goal, which will be aided by recalling some facts about preposets, posets, binary relations, and cones.

**Definition 2.13.** Recall that a preposet \( Q \) on \([n]\) is a binary relation \( Q \subseteq [n] \times [n] \), which is both reflexive \( ((i,i) \in Q) \) and transitive \( ((i,j), (j,k) \in Q \Rightarrow (i,k) \in Q) \). If in addition \( Q \) is antisymmetric \( ((i,j), (j,i) \in Q \Rightarrow i = j) \) then \( Q \) is called a poset on \([n]\); in this case, we sometimes write \( i \leq_Q j \) when \( (i,j) \in Q \).

A set partition \( \pi \in \Pi_n \) is identified with an equivalence relation \( \pi \subseteq [n] \times [n] \) having \( (i,j) \in \pi \) when \( i, j \) appear in the same block of \( \pi \). That is, \( \pi \) is reflexive, transitive, and symmetric \( ((i,j) \in \pi \Rightarrow (j,i) \in \pi) \). We will sometimes write this binary relation as \( i \equiv \pi j \) when \( (i,j) \in \pi \).

The union \( Q_1 \cup Q_2 \subseteq [n] \times [n] \) of two reflexive binary relations will be reflexive, but possibly not transitive, so not a poset. However, the transitive closure operation \( Q \mapsto \overline{Q} \) lets one complete it to a preposet \( Q_1 \cup Q_2 \).

We will use a slight rephrasing of the folklore cone-preposet dictionary, as discussed by Postnikov, Reiner, and Williams in [11, Section 3.3]. This dictionary is a bijection between preposets \( Q \) on \([n]\) and closed cones of any dimension that are intersections in \( V = \mathbb{R}^n \) of closed halfspaces of the form \( \{x_i \leq x_j\} \). Under this bijection, any such closed cone \( C \) corresponds to a preposet \( Q_C \) via

\[
C \mapsto Q_C := \{(i,j) \mid x_i \leq x_j \text{ for all } x \in C\}.
\]

Conversely, any preposet \( Q \) on \([n]\) corresponds to a closed cone \( C_Q \) via

\[
Q \mapsto C_Q := \bigcap_{(i,j) \in Q} \{x_i \leq x_j \} = \{x \in \mathbb{R}^n \mid x_i \leq x_j \text{ for all } (i,j) \in Q\}.
\]

For a subset \( A \subseteq \mathbb{R}^n \), denote its interior and relative interior by \( \text{int}(A) \), \( \text{relin}(A) \). Then for a preposet \( Q \),

\[
\text{relin}(C_Q) = \left\{ x \in \mathbb{R}^n : x_i < x_j \text{ if } (i,j) \in Q \text{ but } (j,i) \not\in Q \right\}.
\]

Also, one has the following assertions, using the notation of this dictionary:

- for \( \pi \in \Pi_n \), the subspace denoted \( X_\pi \) is the (non-pointed) cone \( C_\pi \), regarding \( \pi \) as a preposet, and
- for any poset \( P \) on \([n]\), the open \( n \)-dimensional cone denoted \( K_P \) earlier is \( \text{relin}(C_P)(= \text{int}(C_P)) \).

We will need one further dictionary fact.

**Proposition 2.14 ([11 Proposition 3.5]).** For preposets \( Q, Q' \), one has \( C_Q \cap C_{Q'} = C_{Q \cup Q'} \).

The following definition will help to characterize the set partitions \( \pi \) having \( X_\pi \) in \( \mathcal{L}^{\text{int}}(P) \).

**Definition 2.15.** Given a poset \( P \) on \([n]\) and a set partition \( \pi = \{B_1, \ldots, B_k\} \) in \( \Pi_n \), define a preposet \( P/\pi \) on the set \( \{B_1, \ldots, B_k\} \) as the transitive closure of the (reflexive) binary relation having \( (B_i, B_j) \in P/\pi \) whenever there exists \( p \in B_i \) and \( q \in B_j \) with \( p <_P q \).

**Proposition 2.16.** For \( P \) a poset on \([n]\) and \( \pi = \{B_1, \ldots, B_k\} \) a set partition in \( \Pi_n \), the following are equivalent, and define \( \pi \) being a \( P \)-transverse partition.

(i) The intersection space \( X_\pi \) has \( X_\pi \cap K_P = \emptyset \), that is, \( X_\pi \in \mathcal{L}^{\text{int}}(P) \).

(ii) If \( i <_P j \) and \( i \neq j \), then \( (j,i) \notin P/\pi \).
(iii) Every block $B_i \in \pi$ is an antichain of $P$, and the preposet $P/\pi$ is actually a poset.

Remark 2.17. Aguiar and Mahajan [1, p.230] have a similar concept, which they call a prelinear extension of $P$. A prelinear extension of $P$ is equivalent to a $P$-transverse partition $\pi$ together with a linear ordering on the blocks of $\pi$ that extends the partial order $P/\pi$ from Proposition 2.16(iii).

Before giving a proof of Proposition 2.16, we consider a few examples.

Example 2.18. Let $P := P_2$ be the second poset on $[4]$ from Example 2.12, with $x_1 <_P x_2$ and $x_3 <_P x_4$. Then

- $\pi = 13|24$ is $P$-transverse.
- $\pi = 12|34$ is not $P$-transverse as it fails condition (ii): $1 <_P 2$, but $(2, 1) \in P \cup \pi$.
- $\pi = 14|23$ is not $P$-transverse, failing condition (ii): $1 <_P 2$, but $(2, 1) \in P \cup \pi \setminus \pi$.

The six $P$-transverse partitions give a subposet of $\Pi_4$ isomorphic to $\mathcal{L}^{\text{int}}(P)$, as in Example 2.11:

Example 2.19. If $P$ is the following poset

then $\pi = \{\{1, 4, 9\}, \{2, 5\}, \{3, 6, 8\}, \{7, 10\}\}$ in $\Pi_{10}$ is $P$-transverse, represented here by shading the blocks:

Viewed in this way, Proposition 2.16(iii), roughly speaking, states that $\pi$ is $P$-transverse if and only if one can “stack its blocks without crossings” with respect to the Hasse diagram for $P$.

Proof of Proposition 2.16. We will show a cycle of implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) implies (ii):
Assume (i), so that there exists some $x$ in $\mathbb{R}^n$ lying in the nonempty set

$$X_\pi \cap K_P = X_\pi \cap \text{int}(C_P) = \text{relint}(X_\pi \cap C_P) = \text{relint}(C_P \cup \pi),$$

where the first equality comes from the definition of $K_P$ and $C_P$, the second from the fact that $K_P, C_P$ are full $n$-dimensional, the third from (7) above, and the fourth from Proposition 2.14. Now to see that (ii) holds, given any pair $i, j$ with $i <_P j$, then $x_i < x_j$ since $x \in K_P$, but then since $(i, j) \in P \subseteq P \cup \pi$, the conditions above imply $(j, i) \notin P \cup \pi$, as desired for (ii).

(ii) implies (iii):
Assume (ii) holds. Then every block $B$ of $\pi$ must be an antichain in $P$, else there exists $i \neq j$ in $B$ with $i <_P j$, and then $(j, i) \in \pi \subseteq P \cup \pi$, contradicting (ii).

Now suppose for the sake of contradiction that $P/\pi$ is not a poset. Since $P/\pi$ is a preposet, it can only fail to be antisymmetric, that is, there are blocks $B \neq B'$ of $\pi$ having both $(B, B'), (B', B)$ in $P/\pi$. Since
both $P, \pi$ are transitive binary relations, this means there must exist a (periodic) sequence of elements of the form
\[ \cdots \equiv_\pi p_1 \prec_p p_2 \equiv_\pi p_3 \prec_p p_4 \equiv_\pi \cdots \prec_p p_{m-2} \equiv_\pi p_{m-1} \prec_p p_m \equiv_\pi p_1 \prec_p p_2 \equiv_\pi \cdots \]
alternating relations $(p_i, p_{i+1})$ lying in $P$ and in $\pi$. Then $p_1 \prec_p p_2$ and $(p_2, p_1) \in P \cup \pi$, contradicting (ii).

(iii) implies (i):
Assume (iii), that is, the blocks of $\pi$ are antichains of $P$, and $P/\pi$ is a poset. One can then reindex the blocks of $\pi$ such that $(B_1, B_2, \ldots, B_k)$ is a linear extension of $P/\pi$. Use this indexing to define a point $x \in \mathbb{R}^n$ whose $p^{th}$ coordinate $x_p = i$ if $p$ lies in block $B_i$ of $\pi$.

We claim $x$ lies in $X_\pi \cap K_P$, verifying (i). By construction $x$ lies in $X_\pi$, since its coordinates are constant within the blocks of $\pi$. To verify $x \in K_P$, given $p \prec_q q$, one must check that $x_p < x_q$. Assume that $p, q$ lie in blocks $B_i, B_j$ of $\pi$, so that $x_p = i$ and $x_q = j$. Since the blocks of $\pi$ are antichains in $P$ and $p \prec_q q$, one has $i \neq j$, and since $(B_1, B_2, \ldots, B_k)$ is a linear extension of $P/\pi$, one must have $i < j$, that is, $x_p < x_q$. □

In the remainder of this paper, we will often identify the interior intersection poset $L_{\text{int}}(P)$ with the subposet of $P$-transverse partitions inside the partition lattice $\Pi_n$.

2.3. Linear extensions and a refinement of the Main Problem. We recall here the bijection between the the chambers of braid arrangement $A_{n-1}$ inside a cone $K_P$ and the linear extensions of $P$, in order to give a more detailed version of the Main Problem from the Introduction.

Definition 2.20. Given two posets $P, Q$ on $[n]$, say that $Q$ extends $P$ if $i \leq_Q j$ implies $i \leq_P j$, that is, $P \subseteq Q$ as binary relations on $[n]$, or equivalently, the cone $K_Q \subseteq K_P$. When $Q$ is a total or linear order $\sigma_1 < \cdots < \sigma_n$ on $[n]$, we identify it with a permutation $\sigma = \sigma_1 \ldots \sigma_n$, and call $\sigma$ a linear extension $\sigma = \sigma_1 \ldots \sigma_n$ of $P$. Let $\text{LinExt}(P)$ denote the set of all linear extensions of $P$.

Example 2.21 noted that chambers of the braid arrangement $A_{n-1}$ are of the form $K_\sigma$ for permutations $\sigma$. Then $K_\sigma$ is a chamber lying in the cone $C(K_P)$ if and only if $\sigma$ lies in $\text{LinExt}(P)$, giving a bijection
\[ \text{LinExt}(P) \quad \text{σ} \quad \mapsto \quad C(K_P) \quad \sigma \quad \mapsto \quad K_\sigma. \]

See also [14 Example 1.3].

Example 2.21. The poset defined by $1 \prec_P 2$ and $3 \prec_P 4$ from Example 2.11 has six linear extensions, shown here labeling the chambers in $C(K_P)$:

Recall that the Main Problem from the Introduction was stated as follows: Given a poset $P$ on $[n]$, define a statistic $\text{LinExt}(P) \xrightarrow{\text{stat}} \{0, 1, 2, \ldots\}$ that refines $\#\text{LinExt}(P) = \sum_{k=0}^{n} c_k(P) = [\text{Poin}(P, t)]_{t=1}$ as follows:
\[ \sum_{\sigma \in \text{LinExt}(P)}^\text{stat}(\sigma) = \text{Poin}(P, t). \]
However, now we have re-interpreted the elements $X_\pi$ in $\mathcal{L}^{\text{int}}(K_P)$ as being indexed by the $P$-transverse partitions $\pi = \{B_1, B_2, \ldots\}$, and $\text{codim}(X_\pi) = n - \# \text{blocks}(\pi)$, so that

$$c_k(P) = \sum_{X_\pi \in \mathcal{L}^{\text{int}}(K_P) : \text{codim}(\pi) = k} |\mu(V, X)| = \sum_{\pi = \{B_1, B_2, \ldots, B_{n-k}\} \text{transverse}} \prod_{i} (\#B_i - 1)!$$

Therefore, one way to solve the Main Problem is by providing a map

$$\#\text{LinExt}(P) \to \{P\text{-transverse partitions}\}$$

such that $\pi = \{B_1, B_2, \ldots\}$ has $\#f^{-1}(\pi) = \prod_{i} (\#B_i - 1)!$, and then define $\text{stat}(\sigma) = n - \#\text{blocks}(f(\sigma))$. In the following sections, this is how we will solve the Main Problem for two families of posets.

Example 2.22. Given a poset $P$ on $[n]$, its dual or opposite poset $P^{\text{opp}}$ has the same underlying set $[n]$, but with opposite order relation: $i \leq_P j$ if and only if $j \leq_{P^{\text{opp}}} i$. One can readily check that conditions (ii) and (iii) in Proposition 2.16 are self-dual in the sense that $\pi$ in $\Pi_n$ is $P$-transverse if and only if it is $P^{\text{opp}}$-transverse. Consequently, one has

$$\text{Poin}(P^{\text{opp}}, t) = \text{Poin}(P, t).$$

Example 2.23. Given posets $P_1, P_2$, respectively, their ordinal sum $P_1 \oplus P_2$ is the poset whose underlying set is the disjoint union $P_1 \sqcup P_2$, and having order relations $x \leq_{P_1 \oplus P_2} y$ if either

- $x, y$ in $P_i$ and $x \leq_P y$ for some $i = 1, 2$, or
- $x \in P_1$ and $y \in P_2$.

If the underlying sets for $P_1, P_2$ are $[n_1], [n_2]$, one can readily check from either of Proposition 2.16(ii) or (iii) that a partition $\pi$ of $[n_1] \sqcup [n_2]$ is $P_1 \oplus P_2$-transverse if and only if it is of form $\pi = \{A_1, \ldots, A_k, B_1, \ldots, B_t\}$ where $\pi_1 = \{A_i\}_{i=1}^k$ and $\pi_2 = \{B_j\}_{j=1}^t$ are $P_1$-transverse and $P_2$-transverse partitions of $[n_1]$ and $[n_2]$, respectively. Bearing in mind that $V = \mathbb{R}^{n_1+n_2} = V_1 \oplus V_2$ where $V_i = \mathbb{R}^{n_i}$ for $i = 1, 2$, one has

$$[V, X_\pi] \cong [V_1, X_{\pi_1}] \times [V_2, X_{\pi_2}]$$

$$\mu(V, X_\pi) = \mu(V_1, X_{\pi_1}) \cdot \mu(V_2, X_{\pi_2})$$

and therefore also

$$\text{Poin}(P_1 \oplus P_2, t) = \text{Poin}(P_1, t) \cdot \text{Poin}(P_2, t).$$

Remark 2.24. There is a motivation for trying to answer the Main Problem by such a map $f$ above, and more generally, for any cone $K$ in an arrangement $\mathcal{A}$, to seek a map $f : \mathcal{C}(K) \to \mathcal{L}^{\text{int}}(K)$ having $\#f^{-1}(X) = |\mu(V, X)|$ for all $X \in \mathcal{L}^{\text{int}}(K)$. Brown [3] Section 4.2] considered random walks on $\mathcal{C}(K)$ that generalize the Bidigare-Hanlon-Rockmore random walks on $\mathcal{L}(A)$. He completely analyzed the spectrum of their transition matrices in [3] Theorem 2], showing that for each $X$ in $\mathcal{L}^{\text{int}}(K)$ one has an easily computable eigenvalue $\lambda(X)$ whose multiplicity is $|\mu(V, X)|$.

Remark 2.25. There is another well-studied generating function for $\text{LinExt}(P)$, the $P$-Eulerian polynomial,

$$\sum_{\sigma \in \text{LinExt}(P)} f^{\text{des}(\sigma)}$$

which counts linear extensions $\sigma$ of $P$ according to their number of descents $\text{des}(\sigma)$, assuming that $P$ has been naturally labeled in the sense that the identity permutation $\sigma = 12 \cdot \cdot \cdot n$ lies in $\text{LinExt}(P)$. The $P$-Eulerian polynomial can be interpreted as the $h$-polynomial of the order complex for the distributive lattice $J(P)$ of order ideals in $P$, or of the $P$-partition triangulation of the order polytope for $P$; see [12] Proposition 2.1, Proposition 2.2] and [14] Sections 3.4, 3.8, 3.13] for more on this.

The $P$-Eulerian polynomial differs, in general, from the Poincaré polynomial $\text{Poin}(P, t)$ considered here. For example, when $P$ is an antichain with three elements, the $P$-Eulerian polynomial is $1 + 4t + t^2$, while $\text{Poin}(P, t) = 1 + 3t + 2t^2$. Nevertheless, Corollary 2.24 below describes a situation where the two coincide.
3. Posets of width two and proof of Theorem 1.1

The width of a poset $P$ is the maximum size of an antichain in $P$. A famous result of Dilworth from 1950 (see [15, Ch 3, Exer 77(d)]), asserts that the width $d$ of $P$ is the minimum number of chains required in a chain decomposition $P = P_1 \cup P_2 \cup \cdots \cup P_d$, that is, where each $P_i$ is a totally ordered subset $P_i \subseteq P$. This section answers the Main Problem for posets of width two, starting with the following observation.

**Corollary 3.1.** For posets $P$ of width two, $\#\text{LinExt}(P) = \#\mathcal{L}^{\text{int}}(P)$, the number of $P$-transverse partitions, and more generally,

$$\text{Poin}(P,t) = \sum_{X_\pi \in \mathcal{L}^{\text{int}}(P)} t^{\text{pairs}(\pi)}$$

where pairs$(\pi)$ is the number of two-element blocks $B_i$ in $\pi$.

**Proof.** Every $X_\pi$ in $\mathcal{L}^{\text{int}}(P)$ has $\pi = \{B_1, \ldots, B_k\}$ a $P$-transverse partition, with $\#B_i \leq 2$ as the $B_i$ are antichains of $P$ by Proposition 2.16(iii). All such $\pi$ have $|\mu(V, X_\pi)| = \prod_i (\#B_i - 1)! = 1$, and also codim$(X_\pi) = n - \#\text{blocks}(\pi) = \text{pairs}(\pi)$. This proves the second equation; setting $t = 1$ gives the first. \hfill $\square$

**Example 3.2.** Let $P = a \sqcup b$ be a poset which is a disjoint union of two chains $a, b$ having $a, b$ elements respectively. One can check that an $P$-transverse partition having pairs$(\pi) = k$ is completely determined by the choice of a $k$ element subset $x_1 <_P \cdots <_P x_k$ from $a$ and a $k$ element subset $y_1 <_P \cdots <_P y_k$ from $b$ to constitute the two-element blocks, as follows: $\{x_1, y_1\}, \ldots, \{x_k, y_k\}$. This implies

$$\text{Poin}(a \sqcup b, t) = \sum_{k=0}^{\min(a,b)} \binom{a}{k} \binom{b}{k} t^k.$$  

This is consistent with $\#\text{LinExt}(a \sqcup b) = \binom{a+b}{a}$, since setting $t = 1$ in the equation above gives

$$\binom{a+b}{a} = \sum_{k=0}^{\min(a,b)} \binom{a}{k} \binom{b}{k}$$

which is an instance of the Chu-Vandermonde summation.

Corollary 3.1 suggests that, for a poset $P$ of width two, there should be an explicit bijection from its linear extensions $\sigma$ to its $P$-transverse partitions $\pi$. We give such a bijection by first choosing a chain decomposition of $P = P_1 \sqcup P_2$ into disjoint chains. This bijection will then have the property that the non-singleton (two-element) blocks $B$ of $\pi$ correspond to the indices $i$ in this set:

$$\text{Des}_{(P_1, P_2)}(\sigma) := \left\{ i \in [n-1] : \sigma_i \in P_2, \sigma_{i+1} \in P_1, \text{ and } \sigma_i, \sigma_{i+1} \text{ are incomparable in } P \right\}.$$  

Denoting $\text{des}_{(P_1, P_2)}(\sigma) := \#\text{Des}_{(P_1, P_2)}(\sigma)$, the following gives a more precise version of Theorem 1.1.

**Theorem 3.3.** For a poset $P$ of width two and choice of decomposition $P = P_1 \sqcup P_2$ into two chains $P_1, P_2$, there is a bijection

$$\text{LinExt}(P) \xrightarrow{f} \{P\text{-transverse partitions}\}$$

such that the non-singleton blocks of $f(\sigma)$ are the blocks $\{\sigma_i, \sigma_{i+1}\}$ for $i \in \text{Des}_{(P_1, P_2)}(\sigma)$. Consequently,

$$\pi(P, t) = \sum_{\sigma \in \text{LinExt}(P)} t^{\text{des}_{(P_1, P_2)}(\sigma)}.$$  

**Proof.** We describe $f$ and $f^{-1}$ recursively, via induction on $n := \#P$. There are two cases, based on whether $P$ has one or two minimal elements.

**Case 1.** There is a unique minimum element $p_0 \in P$.

In this case, given $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $\text{LinExt}(P)$, we must have $\sigma_1 = p_0$, so that $\{p_0\}$ should be a singleton block of $\pi = f(\sigma)$, and one produces the remaining blocks of $\pi$ by applying $f$ recursively to $(\sigma_2, \ldots, \sigma_n)$. This is depicted schematically here:

---

3So we assume here that $P_1 \cap P_2 = \varnothing$, but there may be order relations between elements of $P_1$ and $P_2$. 

When defining \( f \), the unique minimum element \( p_0 \) of \( P \) must lie in a singleton block \( \{p_0\} \) in \( \pi \). So make \( f^{-1}(\pi) = \sigma \) have \( \sigma_1 = p_0 \), and construct \( \sigma_2 \cdots \sigma_n \) by applying \( f^{-1} \) recursively to the \((P - \{p_0\})\)-transverse partition obtained from \( \pi \) by removing the block \( \{p_0\} \).

**Case 2.** There are two minimal elements of \( P \).

Label these two minimal elements \( p_1, p_2 \) of \( P \) so that \( p_i \in P \) for \( i = 1, 2 \). Note that this implies that every \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) in \( \text{LinExt}(P) \) has either \( \sigma_1 = p_1 \) or \( \sigma_1 = p_2 \). Note also that any \( P \)-transverse partition \( \pi \) only has blocks of cardinality 1 or 2, which yields two subcases for defining \( f \) and \( f^{-1} \):

- The Subcase 2a for
  - defining \( f \) occurs when \( \sigma_1 = p_1 \),
  - defining \( f^{-1} \) occurs when \( \{p_1\} \) appears as a singleton block within \( \pi \).
- The Subcase 2b for
  - defining \( f \) occurs when \( \sigma_1 = p_2 \),
  - defining \( f^{-1} \) occurs when \( p_1 \) appears in a two-element block within \( \pi \).

**Subcase 2a.**

When defining \( f \), if \( \sigma_1 = p_1 \), then make \( \{p_1\} \) a singleton block of \( \pi = f(\sigma) \), and produce the remaining blocks of \( \pi \) by applying \( f \) recursively to \( \{\sigma_2, \ldots, \sigma_n\} \).

To define \( f^{-1} \), if \( \{p_1\} \) is a singleton block of \( \pi \), make \( f^{-1}(\pi) = \sigma \) have \( \sigma_1 = p_1 \), and construct \( \sigma_2 \cdots \sigma_n \) by applying \( f^{-1} \) recursively to the \((P - \{p_1\})\)-transverse partition obtained from \( \pi \) by removing the block \( \{p_1\} \).

**Subcase 2b.**

When defining \( f \), if \( \sigma_1 = p_2 \), then \( p_1 \) appears elsewhere in \( \sigma \), say \( p_1 = \sigma_{i+1} \) where \( i \geq 1 \). Because \( \sigma \) lies in \( \text{LinExt}(P) \) and \( \sigma_{i+1} = p_1 \) is the minimum element of \( P \), this forces \( \sigma_1, \sigma_2, \ldots, \sigma_i \) to all be elements of \( P \). In this case, add to \( \pi \) the singleton blocks \( \{\sigma_1\}, \{\sigma_2\}, \ldots, \{\sigma_{i-1}\} \) along with the two-element block \( \{\sigma_i, \sigma_{i+1}\} = \{p_2, p_1\} \), and compute the rest of \( f(\sigma) = \pi \) recursively by replacing \((P, \sigma)\) with \((P - \{\sigma_1, \sigma_2, \ldots, \sigma_{i+1}\}, \{\sigma_{i+2}, \sigma_{i+3}, \ldots, \sigma_n\}) \). Here is the schematic picture:
When defining \( f^{-1}(\pi) \), if \( p_1 \) appears in some two-element block of \( \pi \), then it appears in some block \( \{p_1, p_2\} \) for some \( p'_2 \) in \( P_2 \). We claim that \( \pi \) being \( P \)-transverse then forces any elements \( p <_{\mu} p'_2 \) in \( P_2 \) to lie in singleton blocks \( \{p\} \) of \( \pi \). To see this claim, assume not, so that some such \( p \) lies in a two-element block of \( \pi \), necessarily of the form \( \{p'_1, p\} \) for some \( p'_1 \) in \( P_1 \) with \( p_1 <_{\mu} p'_1 \). This leads to a contradiction of Proposition \( \text{2.10\ii} \), since \( (p'_1, p_1) \) would then be a relation in \( P \cup \pi \) via this transitive chain of relations: \( p'_1 \equiv_{\pi} p <_{\mu} p'_2 \equiv_{\pi} p_1 \).

In this subcase, list the totally ordered (and possibly empty) collection of all elements \( p \) in \( P_2 \) with \( p <_{\mu} p'_2 \) at the beginning of \( \sigma \) as \( \sigma_1, \sigma_2, \ldots, \sigma_{i-1} \), followed by \( \sigma_i \sigma_{i+1} = p_2 p_1 \). Then compute the rest of \( f^{-1}(\pi) = \sigma \) recursively, by applying \( f^{-1} \) to the \((P - \{\sigma_1, \sigma_2, \ldots, \sigma_{i+1}\})\)-transverse partition obtained from \( \pi \) by removing the singleton blocks \( \{\sigma_1\}, \{\sigma_2\}, \ldots, \{\sigma_{i-1}\} \) and the two-element block \( \{\sigma_i, \sigma_{i+1}\} = \{p'_2, p_1\} \).

It is not hard to check that the two maps \( f, f^{-1} \) defined recursively in this way are actually mutually inverse bijections. By construction, \( f \) has the property that the two-element blocks of \( \pi = f(\sigma) \) are exactly those containing \( P \)-incomparable pairs \( \{\sigma_i, \sigma_{i+1}\} \) for which \( \sigma_i \in P_1 \) and \( \sigma_{i+1} \in P_2 \), as claimed. \( \square \)

The following corollary tells us when \( \text{Des}_{(P_1, P_2)}(\sigma) \) corresponds to the usual descent set of \( \sigma \), that is,

\[
\text{Des}(\sigma) := \{ i \in [n-1] \mid \sigma_i > \sigma_{i+1} \}.
\]

Denote its cardinality by \( \text{des}(\sigma) = \#\text{Des}(\sigma) \).

**Corollary 3.4.** When \( P \) is a width two poset with a chain decomposition \( P_1 \cup P_2 \) where \( P_1 \) is an order ideal of \( P \), then the Poincaré polynomial for \( P \) coincides with the \( P \)-Eulerian polynomial from Remark \( \text{2.23} \)

\[
Poin(P, t) = \sum_{\sigma \in \text{LinExt}(P)} t^{\text{des}(\sigma)}.
\]

**Proof.** Let \( \#P_1 = n_i \) for \( i = 1, 2 \), so that \( n = \#P = n_1 + n_2 \). One can then choose a natural labeling for \( P \) by \([n]\) such that the elements of the order ideal \( P_1 \) are labeled by the initial segment \([n_1] = \{1, 2, \ldots, n_1\} \), and \( P_2 \) is labeled by \( \{n_1 + 1, n_1 + 2, \ldots, n\} \). In this situation, one of the conditions for an index \( i \) to lie in \( \text{Des}_{(P_1, P_2)} \) becomes vacuous: if \( \sigma_i \in P_2 \) and \( \sigma_{i+1} \in P_1 \), then this already implies \( \sigma_i, \sigma_{i+1} \) are incomparable in \( P \), because \( P_1 \) is an order ideal. On the other hand, since \( P_1, P_2 \) are both totally ordered in \( P \), and \( \sigma \) lies in \( \text{LinExt}(P) \), one has \( \sigma_i \in P_2 \) and \( \sigma_{i+1} \in P_1 \) if and only if \( \sigma_i > \sigma_{i+1} \), that is, if and only if \( i \) lies in \( \text{Des}(\sigma) \). \( \square \)

**Example 3.5.** An interesting family of posets to which Corollary \( \text{3.3} \) applies are the posets \( P_{\lambda/\mu} \) associated with two-row skew Ferrers diagrams \( \lambda/\mu \). A Ferrers diagram associated to a partition (of a number) \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) has \( \lambda_i \) square cells drawn left-justified in row \( i \). A skew Ferrers diagram \( \lambda/\mu \) for two partitions \( \lambda, \mu \) having \( \lambda_i \geq \mu_i \) is the diagram for \( \lambda \) with the cells occupied by the diagram for \( \mu \) removed. There is a poset structure \( P_{\lambda/\mu} \) on the cells of \( \lambda/\mu \) in which a cell \((i, j)\) in row \( i \) and column \( j \) has \((i, j) \leq_{P_{\lambda/\mu}} (i', j') \) if \( i \leq i' \) and \( j \leq j' \).

When \( \lambda/\mu \) has only two parts, we will call it a two-row skew Ferrers diagram. Three examples of such \( \lambda/\mu \) and their associated \( P_{\lambda/\mu} \) are shown below.
The decomposition $P_{\lambda/\mu} = P_1 \cup P_2$ where $P_i$ correspond to the cells in row $i$ of $\lambda/\mu$ shows that $P_{\lambda/\mu}$ has width two, and furthermore $P_1$ forms an order ideal. Therefore Corollary 3.4 implies that for any two-row skew Ferrers diagram $\lambda/\mu$ one has

$$Poin(P_{\lambda/\mu}, t) = \sum_{\sigma \in \text{LinExt}(P_{\lambda/\mu})} t^{\text{des}(\sigma)}$$

On the other hand, there is a well-known bijection between linear extensions $\sigma$ of $P_{\lambda/\mu}$ and the standard Young tableaux $Q$ of shape $\lambda/\mu$, which are (bijective) labelings of the cells of the diagram by $[n]$ where $n = \sum_i \lambda_i - \sum_i \mu_i$, with the numbers increasing left-to-right in rows and top-to-bottom in columns; see [13, §7.10]. There is also a notion of descent set $\text{Des}(Q)$ for such tableaux, having $i \in \text{Des}(Q)$ whenever $i + 1$ labels a cell in a lower row of $Q$ than $i$. However, in general when $\sigma$ corresponds to $Q$, one does not have $\text{des}(\sigma) = \text{des}(Q)$, so that $Poin(P_{\lambda/\mu}, t)$ differs from the generating function $\sum_Q t^{\text{des}(Q)}$ of standard tableaux $Q$ shape $\lambda/\mu$. For example, there are two standard tableaux of shape $\lambda/\mu = (2, 1)/(0, 0)$

$Q_1 = \begin{array}{ccc} 1 & 2 \\ 3 & 2 \end{array}$

$Q_2 = \begin{array}{ccc} 1 & 3 \\ 2 & 2 \end{array}$

both having $\text{des}(Q_1) = 1$, however $Poin(P_{\lambda/\mu}, t) = 1 + t$.

In two special cases, however, they (essentially) coincide.

- When $P_{\lambda/\mu} = a \sqcup b$ is a disjoint union of two chains, as in Example 3.2 one can check that, if one (naturally) labels $a \sqcup b$ so that the elements of the order ideal $b$ are labeled $1, 2, \ldots, b$ while $a$ is labeled $b + 1, b + 2, \ldots, b + a$, then one does have $\text{des}(\sigma) = \text{des}(Q)$, and hence

$$\sum_Q t^{\text{des}(Q)} = Poin(P_{a \sqcup b}, t) = \sum_k \binom{a}{k} \binom{b}{k} t^k.$$  

- When $\lambda/\mu$ is a $2 \times n$ rectangle, so that $P_{\lambda/\mu} = 2 \times n$ is a Cartesian product poset, then $\sigma$ in $\text{LinExt}(P)$ and standard Young tableaux $Q$ of shape $2 \times n$ can both be identified with Dyck paths of semilength $n$, that is, lattice paths from $(0, 0)$ to $(2n, 0)$ in $\mathbb{Z}^2$ taking steps northeast or southwest and staying weakly above the $x$-axis. One can check that

- $\text{Des}(\sigma)$ corresponds to valleys (i.e. southwest steps followed by a northeast step), while
- $\text{Des}(Q)$ correspond to peaks (i.e. northeast steps followed by a southwest step).

In general, such a Dyck path has one more peak than valley [13, Exercises 6.19(i, ww, aaa)], and hence

$$Poin(2 \times n, t) = \frac{1}{t} \sum_Q t^{\text{des}(Q)} = \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n}{k} t^k,$$

which is the generating function for the Narayana numbers $N(n, k) := \frac{1}{n} \binom{n}{k} \binom{2n}{k}$ (see [2, p.2] and [13, Exer. 6.36(a)]), and which upon setting $t = 1$ sums to the Catalan number

$$\#\text{LinExt}(2 \times n) = \frac{1}{n+1} \binom{2n}{n}.$$  

14
Note that for any (non-skew) partition \( \lambda \), the celebrated hook-length formula of Frame, Robinson and Thrall [14] gives a product expression for the number \( f^\lambda \) of standard Young tableaux of shape \( \lambda \), and hence for \( \# \text{LinExt}(P_\lambda) = |\text{Poin}(P_\lambda, t)|_{t=1} \).

**Open Problem 3.6.** Combinatorially interpret \( \text{Poin}(P_\lambda, t) \) for other partitions \( \lambda \), and in particular, for \( m \times n \) rectangular partitions, where \( P_\lambda = m \times n \) is a Cartesian product of chains.

Here is a tiny bit of data on \( \text{Poin}(3 \times n, t) \):

\[
Poin(3 \times 2, t) = 1 + 3t + t^2, \\
Poin(3 \times 3, t) = 1 + 11t + 26t^2 + 16t^3 + 2t^4. \\
Poin(3 \times 4, t) = 1 + 18t + 92t^2 + 174t^3 + 133t^4 + 40t^5 + 4t^6.
\]

4. **Disjoint Unions of Chains and proofs of Theorems 1.2 and 1.3**

The goal of this section is to resolve the Main Problem from the Introduction for posets \( P \) which are disjoint unions of chains. In this case, the Whitney numbers turn out have an elegant expression utilizing Foata’s theory of multiset permutations, generalizing the answer (6) for the antichain poset \( P \).

In Subsection 4.1 we review Foata’s theory of multiset permutations, in particular his work with the intercalation product. Subsection 4.2 reviews its relation to partial commutation monoids. Subsection 4.3 employs Foata’s theory to give a generalization of MacMahon’s Master Theorem which specializes to Theorem 1.3, a generating function compiling the Poincaré polynomials for disjoint unions of chains.

4.1. **Multiset Permutations.** This subsection gives background on the theory of multiset permutations as introduced by Foata in his PhD thesis [4, Section 3.2], and extended in later publications [5, Chapters 3-5]. It also appears in Knuth [8, Section 5.1.2].

**Definition 4.1.** Recall that a (weak) composition \( \mathbf{a} = (a_1, \ldots, a_\ell) \) of \( n \) is a sequence of nonnegative integers having sum \( |\mathbf{a}| := \sum_i a_i = n \). We will regard \( \mathbf{a} \) as specifying the multiplicities in a multiset \( M := 1^{a_1}2^{a_2} \cdots \ell^{a_\ell} \), that is, a set with repetitions

\[
M = \left( \overset{a_1 \text{ times}}{1, 1, \ldots, 1}, \overset{a_2 \text{ times}}{2, 2, \ldots, 2}, \ldots, \overset{a_\ell \text{ times}}{\ell, \ell, \ldots, \ell} \right)
\]

A multiset permutation \( \sigma = \sigma_1 \cdots \sigma_n \) is a rearrangement of the elements of \( M \), which we will often write in a two-line notation that generalizes that of permutations:

\[
\sigma = \left( \begin{array}{ccccccc}
1 & \cdots & 1 & 2 & \cdots & 2 & \cdots \\
\sigma_1 & \cdots & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+\cdots+a_{\ell-1}+1} & \sigma_{a_1+\cdots+a_{\ell-1}+2} & \cdots & \sigma_n
\end{array} \right)
\]

We denote the set of all multiset permutations of \( M \) by \( \mathcal{S}_M \). For any \( \sigma \in \mathcal{S}_M \), we call \( M \) the support of \( \sigma \), and write \( M = \text{supp}(\sigma) \).

**Example 4.2.** The composition \( \mathbf{a} = (3, 3, 2, 2) \) gives the multiplicities of the multiset

\[
M = 1^32^33^24^2 = (1, 1, 1, 2, 2, 2, 3, 3, 4, 4).
\]

Then the following multiset permutation \( \sigma \) is an element of \( \mathcal{S}_M \):

\[
\sigma = \left( \begin{array}{ccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\
2 & 2 & 2 & 4 & 1 & 3 & 4 & 1 & 1 & 3
\end{array} \right).
\]

Foata [4, §3.2] defined an associative intercalation product operation on multiset permutations \( (\sigma, \rho) \mapsto \sigma \triangleright \rho \). Knuth [8, §5.1.2] describes it algorithmically: think of \( \sigma, \rho \) in two-line notation as a sequence of columns \( \left( \begin{array}{c} i \\ j \end{array} \right) \), and juxtapose these sequences of columns. Then perform swaps to sort the columns according to their top entries, never swapping two with the same top entry. For example,

\[
\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \end{pmatrix} \triangleright \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 2 & 2 & 4 \\ 2 & 3 & 1 & 1 & 2 & 4 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 1 & 1 & 2 & 1 \end{pmatrix}.
\]
Definition 4.3. For each $\ell$, the intercalation monoid $\Int_\ell$ is the submonoid of all multiset permutations $\sigma$ whose support $M = 1^{a_1}2^{a_2}\ldots\ell^{a_\ell}$ involves only the letters $\{1,2,\ldots,\ell\}$. The empty permutation () is the identity element for $\tau$, since $(()) \tau \sigma = \sigma = \sigma \tau ()$, so we will denote it by $1 := ()$ in $\Int_\ell$ when we want to emphasize the monoid structure.

Note that, just as permutations in the symmetric group $\mathfrak{S}_n$ do not commute in general, the monoid $\Int_\ell$ is not commutative. For example

\[
\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tau \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \tau \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \tau \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.
\]

However, one can check that $\sigma \tau \rho = \rho \tau \sigma$ when $\sigma, \rho$ are disjoint, that is, $\text{supp}(\sigma) \cap \text{supp}(\rho) = \emptyset$.

Definition 4.4. Say $\sigma$ in $\Int_\ell$ is prime if the only factorizations $\sigma = \rho \tau \tau$ have either $\rho = ()$ or $\tau = ()$.

Example 4.5. The permutation $(2\ 4\ 5\ 7) 5\ 7\ 4\ 2$ is prime. However, $(1\ 1\ 2\ 3) 2\ 3\ 1\ 1$ is not prime, not since $(1\ 2\ 3\ 1) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$.

On the other hand $(2\ 4\ 5\ 7) 5\ 7\ 2\ 4$ is not prime, even though its support is multiplicity free, since $(2\ 5\ 2) \tau (4\ 7\ 4) = (4\ 7\ 2) \tau (2\ 5\ 4)$. It is not obvious, but turns out to be true that $\sigma$ is prime if and only if both

- $\text{supp}(\sigma) = M$ is multiplicity free, that is, $M$ is a set not a multiset, and
- $\sigma$ consists of a single $\#M$-cycle permuting this set $M$.

We therefore call prime elements prime cycles. More generally, one has the following.

Theorem 4.6 (Foata, 1969 [5 8]). Let $\sigma$ be a multiset permutation. Then $\sigma$ has a decomposition into a product of prime cycles. That is, there exist $k \geq 0$ cycles $\sigma^{(1)},\ldots,\sigma^{(k)}$ such that

\[
\sigma = \sigma^{(1)} \tau \sigma^{(2)} \tau \cdots \tau \sigma^{(k)}.
\]

Further, this cycle decomposition of $\sigma$ is unique up to successively interchanging pairs of adjacent prime cycles with disjoint support. In particular $k$ is unique.

Definition 4.7. Call $\text{fcyc}(\sigma) := k$ the number of prime cycles in the decomposition of $\sigma$ from Theorem 4.6.

Example 4.8. The element $\sigma$ from Example 4.4 has $\text{fcyc}(\sigma) = 4$ and two prime cycle decompositions

\[
\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 2 & 2 & 2 & 4 & 1 & 3 & 4 & 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \tau \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tau \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \tau \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \tau \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \tau \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tau \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.
\]

4.2. Partial Commutation Monoids. It will be helpful to view the intercalation monoid $\Int_\ell$ as a partial commutation monoid. We briefly review some relevant facts about partial commutation monoids.

Definition 4.9. Given a set $\mathcal{A}$, which we call an alphabet and a subset of its pairs $C \subseteq \binom{\mathcal{A}}{2}$, the associated partial commutation monoid $\mathcal{M}$ is defined to be the set of equivalence classes on words $\alpha_1\alpha_2\ldots\alpha_k$ in the alphabet $\mathcal{A}$ under the equivalence relation

\[
\alpha_1\alpha_2\ldots\alpha_i\alpha_{i+1}\ldots\alpha_k \equiv \alpha_1\alpha_2\ldots\alpha_{i+1}\alpha_i\ldots\alpha_k
\]

if $\{\alpha_i, \alpha_{i+1}\} \in C$.

From this perspective, Foata’s Theorem 4.6 asserts that $\Int_\ell$ is a partial commutation monoid, whose associated alphabet $\mathcal{A}$ is the set of all prime cycles, and with $C$ being the pairs of prime cycles having disjoint supports.

For later use, we point out the following (nontrivial) proposition, see [8 §5.1.2, Exercise 11] and [15 Exercise 3.123]. Given a factorization of an element $\alpha = \alpha_1\alpha_2\ldots\alpha_k$ in $\mathcal{M}$ a partial commutation monoid.
define a poset $\mathcal{P}_\alpha$ on $[k]$ as the transitive closure of the binary relation containing $(i,j) \in \mathcal{P}_\alpha$ when $i <_{\mathbb{Z}} j$ and either $\alpha_i = \alpha_j$ or $\alpha_i \alpha_j \neq \alpha_i \alpha_j$.

**Proposition 4.10.** Given a factorization of $\alpha = \alpha_1 \alpha_2 \ldots \alpha_k \in M$ a partial commutation monoid,

1. $\mathcal{P}_\alpha$ does not depend on the choice of factorization of $\alpha$, and
2. there is a bijection between $\text{LinExt}(\mathcal{P}_\alpha)$ and the factorizations of $\alpha$ given by

$$(i_1, \ldots, i_k) \mapsto \alpha_{i_1} \ldots \alpha_{i_k}.$$ 

**Example 4.11.** The multiset permutation $\sigma$ from Example 4.8 had two prime cycle factorizations

$$\sigma = \sigma^{(1)} \cdot \sigma^{(2)} \cdot \sigma^{(3)} \cdot \sigma^{(4)}$$

$$= \sigma^{(1)} \cdot \sigma^{(3)} \cdot \sigma^{(2)} \cdot \sigma^{(4)}$$

corresponding to the two linear extensions of the poset $\mathcal{P}_\sigma$ on $[4]$ with this Hasse diagram:

```
4
挖掘
2
挖掘
1
3
```

4.3. **Proof of Theorem 4.12.** Our goal in this subsection is to use Foata’s Theorem 4.6 to prove Theorem 4.18 below, which is a more precise version of Theorem 1.2.

**Definition 4.12.** For a composition $\vec{n} = (a_1, \ldots, a_\ell)$ of $n$, let $|\vec{n}| = a_1 + \cdots + a_\ell = n$ denote its sum, and let $P_{\vec{n}} = a_1 \sqcup a_2 \sqcup \cdots \sqcup a_\ell$ denote the poset which is the disjoint union of chains having $a_1, \ldots, a_\ell$ elements. Also, let $M(\vec{n}) = 1^{a_1}2^{a_2} \cdots \ell^{a_\ell}$ denote the multiset with multiplicities specified by $\vec{n}$.

We wish to interpret linear extensions $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ of $P_{\vec{n}}$ as multiset permutations and use $\text{fcyc}(\sigma)$ to interpret its Poincaré polynomial. In order to do this, we introduce two labelings of $P_{\vec{n}}$.

- The **multiset labeling** of $P_{\vec{n}}$ gives all elements of the $k$th chain $a_k$ the same label $k$.
- The **[n]-labeling** labels the element of the $k$th chain $a_k$ chain in increasing order from bottom to top with the distinct labels $(\Sigma_k + 1, \Sigma_k + 2, \ldots, \Sigma_k + a_k)$ where $\Sigma_{k-1} := a_1 + a_2 + \cdots + a_{k-1}$.

Via this relabeling, linear extensions $\sigma$ in $\text{LinExt}(P_{\vec{n}})$ biject with multiset permutations $\mathfrak{S}_{M(\vec{n})}$.

**Example 4.13.** Suppose $\vec{n} = (3, 3, 2, 2)$. Then $P = P_{\vec{n}} = 3 \sqcup 3 \sqcup 2 \sqcup 2$, with its two labelings shown below:

```
1
4
2
3
4

3
2
1
4
3

1
2
3
4
```

Using the [n]-labeling, $\sigma = (4, 5, 6, 9, 1, 7, 10, 2, 3, 8)$ lies in $\text{LinExt}(P_{\vec{n}})$, and corresponds under the multiset labeling to $\sigma = (2, 2, 2, 4, 1, 3, 4, 1, 1, 3)$ in $\mathfrak{S}_{M(\vec{n})}$, or in two-line notation

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 2 & 2 & 2 & 4 & 1 & 3 & 4 & 1 & 1 & 3 \end{pmatrix}.$$ 

The two labelings of $P_{\vec{n}}$ lead to a notion of relabeled support for each prime cycle $\sigma^{(i)}$ in the decomposition of any $\sigma = \sigma^{(1)} \cdot \sigma^{(2)} \cdots$ in $\mathfrak{S}_{M(\vec{n})}$. To compute this relabeled support, first decorate the entries in the top row of the two-line notation for $\sigma$ with subscripts $1, 2, \ldots, n := |\vec{n}|$ from left-to-right. Then simply preserve the subscripts in the top row as one decomposes $\sigma$ into prime cycles; the relabeled support of $\sigma^{(i)}$ is precisely its set of top row subscripts.

**Example 4.14.** Let $a = (3, 3, 2, 2)$. Consider the multiset permutation $\sigma \in \mathfrak{S}_{M(\vec{n})}$ from Example 4.11. The factorization of $\sigma$ with its top row decorated looks like this:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 2 & 2 & 2 & 4 & 1 & 3 & 4 & 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 2 & 4 & 4 & 9 & 37 & 410 \\ 1 & 2 & 2 & 4 & 1 & 3 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 2 & 4 & 4 & 9 & 37 & 410 \\ 1 & 2 & 2 & 4 & 1 & 3 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 2 & 4 & 4 & 9 & 37 & 410 \\ 1 & 2 & 2 & 4 & 1 & 3 & 3 & 1 \end{pmatrix}.$$
From here, the relabeled supports can be read off from the subscripts on the top row:

\[
\text{relsupp}_\sigma \left( \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 4 & 1 \end{array} \right) = \{1, 4, 9\},
\]

\[
\text{relsupp}_\sigma \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right) = \{2, 5\},
\]

\[
\text{relsupp}_\sigma \left( \begin{array}{cc} 3 & 4 \\ 4 & 3 \end{array} \right) = \{7, 10\},
\]

\[
\text{relsupp}_\sigma \left( \begin{array}{cc} 1 & 2 \\ 3 & 1 \end{array} \right) = \{3, 6, 8\}.
\]

**Definition 4.15.** Given a composition \( \overline{\pi} \) of \( n \), define the *Foata map*

\[
\mathcal{S}_{M(\overline{\pi})} \xrightarrow{f} \Pi_n \\
\sigma \mapsto f(\sigma) = \{B_1, B_2, \ldots, B_k\}
\]

where \( B_k = \text{relsupp}_\sigma (\sigma^{(i)}) \) in the unique prime cycle decomposition \( \sigma = \sigma^{(1)} \cdot \cdots \cdot \sigma^{(k)} \) from Theorem 4.6.

**Example 4.16.** Let \( a = (3, 3, 2, 2) \). Then \( \sigma \in \mathcal{S}_{\overline{\pi}} \) as in Example 4.14 has

\[
f(\sigma) = \{\{1, 4, 9\}, \{2, 5\}, \{3, 6, 8\}, \{7, 10\}\},
\]

which can be represented as in Example 2.19 by coloring the blocks of \( f(\sigma) \) on the Hasse diagram of \( P_{\overline{\pi}} \):

![Hasse diagram](image)

**Example 4.17.** Consider the poset \( P \) from Example 2.21 and its two labelings:

\[
\begin{array}{cc}
1 & 2 \\
1 & 2
\end{array} \quad \begin{array}{cc}
2 & 4 \\
1 & 3
\end{array}
\]

We apply \( f \) to each of the six \( \sigma \) in \( \mathcal{S}_{M(\overline{\pi})} \) with \( \overline{\pi} = (2, 2) \):

\[
\begin{align*}
f \left( \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{array} \right) &= f \left( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \right) = 1 | 2 | 3 | 4 \\
f \left( \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 \end{array} \right) &= f \left( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \right) = 1 | 23 | 4 \\
f \left( \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 \end{array} \right) &= f \left( \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \right) = 13 | 2 | 4 \\
f \left( \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 2 \end{array} \right) &= f \left( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \right) = 1 | 24 | 3 \\
f \left( \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 2 \end{array} \right) &= f \left( \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right) = 14 | 2 | 3 \\
f \left( \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 1 \end{array} \right) &= f \left( \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right) = 13 | 24
\end{align*}
\]

Compare this to the elements of \( \mathcal{L}^{\text{int}}(P) \), which are
We can now state the result which is the goal of this subsection, generalizing Theorem 1.2:

**Theorem 4.18.** Fix a composition \( \pi = (a_1, \ldots, a_\ell) \) of \( n \). Then the image of the Foata map \( f : \mathfrak{S}_\mathfrak{M}(\pi) \to \Pi_n \) is exactly the set of \( P \)-transverse partitions \( \mathcal{L}^{\text{int}}(P) \). Furthermore, for each such \( P \)-transverse partition \( \pi \), one has \( \#f^{-1}(\pi) = |\mu(V, X_\pi)| \), so that

\[
Poin(P_{\pi}, t) = \sum_{\sigma \in \mathfrak{S}_\mathfrak{M}(\pi)} t^{\text{n-fcycle}(\sigma)}.
\]

**Proof.** Let \( \sigma \in M(\pi) \subseteq \text{Int}_{\ell} \) have prime decomposition \( \sigma = \sigma^{(1)} \cdots \sigma^{(k)} \), so that

\[
\pi = f(\sigma) = \{B_1, \ldots, B_k\}.
\]

To show \( \pi \in \mathcal{L}^{\text{int}}(P) \), we will use Proposition 2.10(iii). Note that each \( B_i \) is an antichain of \( P \) since prime cycles always have support which is multiplicity free. We are left to show that the preorder \( P/\pi \) on \( \{B_1, \ldots, B_k\} \) given in Proposition 2.10(iii) is a poset; let us view this as a preorder on \( \{1, 2, \ldots, k\} \) by relabeling. Viewing \( \sigma \) as an element of the partial commutation monoid \( \text{Int}_{\ell} \), recall that the discussion preceding Proposition 4.10 defined a poset \( P_\sigma \) on \( \{1, 2, \ldots, k\} \): we will check that \( P/\pi = P_\sigma \) as binary relations, showing \( P/\pi \) is a poset.

Note \( P_\sigma \) is the transitive closure of the relations \((i, j) \in P_\sigma \) if \( i <_\pi j \) and \( \supp(\sigma^{(i)}) \cap \supp(\sigma^{(j)}) \neq \emptyset \). Also \( P/\pi \) is the transitive closure of the relations \((i, j) \in P/\pi \) if \( p_i < p_j \) for some \( p_i \in B_i, \ p_j \in B_j \). Since \( p_i, p_j \) are comparable in \( P \) if and only if they have the same label in \( P \) (in the \( M(\pi) \) labeling, not the \([n]\)-labeling), we leave it to the reader to verify that Proposition 4.10 implies that these two binary relations are the same. This completes the proof that the image of \( f \) is contained in the subposet \( \mathcal{L}^{\text{int}}(P) \) of \( \Pi_n \).

To see that the image of \( f \) equals \( \mathcal{L}^{\text{int}}(P) \) and simultaneously prove the formula \( \#f^{-1}(\pi) = |\mu(V, X_\pi)| \), assume we are given some \( \pi = \{B_1, \ldots, B_k\} \) in \( \mathcal{L}^{\text{int}}(P) \). By Proposition 2.10, the preorder \( P/\pi \) defined in part (iii) of the proposition is a poset. We reindex the blocks of \( P \) so that \( (B_1, \ldots, B_k) \) is a linear extension of \( P/\pi \). Let \( \text{unlabel} : [n] \to [\ell] \) be the map (depending upon \( M(\pi) \)) that sends an element labeled \( i \) in \([n]\) during the \([n]\)-labeling of \( P = f_{\pi} \) to its original label \( j \) in \([\ell] \) from the \( M(\pi) \)-labeling; that is \( \text{unlabel}(i) = j \) if \( i \) labels an element of the \( j \)-th chain \( a_j \) in the decomposition \( f_{\pi} = a_1 \sqcup \cdots \sqcup a_\ell \). Then the first part of this proof shows that \( \sigma \) in \( \text{Int}_{\ell} \) has \( f(\sigma) = \pi \) if and only if \( \sigma = \sigma^{(1)} \cdots \sigma^{(k)} \) where \( \sigma^{(i)} \) is a prime cycle with \( \supp(\sigma^{(i)}) = \text{unlabel}(B_i) \) for \( i = 1, 2, \ldots, k \). Since there exactly \((#B_i - 1)! \) prime cycles with support set \( \text{unlabel}(B_i) \), this shows

\[
\#f^{-1}(\pi) = \prod_{i=1}^{k} (#B_i - 1)! = |\mu(V, X_\pi)|
\]

where the last equality used Proposition 2.6. \( \Box \)

**Example 4.19.** Let \( a = (3, 3, 2, 2) \) and consider \( \pi = \{\{1, 4, 9\}, \{2, 5\}, \{3, 6, 8\}, \{7, 10\}\} \) from Example 4.10.
The preimages $f^{-1}(π)$ of $π$ under the Foata map $f$ are as follows:

$$f^{-1}(\{\{1,4,9\}, \{2,5\}, \{3,6,8\}, \{7,10\}\}) = \begin{cases}
\left\{ \begin{array}{c}
1 & 2 & 4 \\
a & b & c \\
\end{array} \right\} \tau \left\{ \begin{array}{c}
1 & 2 \\
a & b \\
\end{array} \right\} \tau \left\{ \begin{array}{c}
3 & 4 \\
1 & 2 \\
\end{array} \right\} \tau \left\{ \begin{array}{c}
1 & d \\
d & e \\
\end{array} \right\} & (a,b,c) = (2,4,1) \text{ or } (4,1,2), \\
\end{cases}$$

Thus the size of the fiber of $π$ under the map $f$ is

$$\#f^{-1}(124 | 25 | 368 | 7(10)) = \left| \mu(1|2|3|4|5|6|7|8 \cup 148 | 25 | 368 | 7(10)) \right| = (3 − 1)! (2 − 1)! (2 − 1)! (3 − 1)! = 4.$$

4.4. **Proof of Theorem 4.3.** Our goal here is a generating function compiling the Poincaré polynomials $P(\pi, t)$ for all compositions $\pi$ of length $\ell$. This uses more of Foata’s theory for the intercalation monoid $\text{Int}_\ell$, similar to his deduction of MacMahon’s Master Theorem.

Since each multiset permutation $σ$ has only finitely many intercalation factorizations $σ = ρ ⊕ τ$, one can define a convolution algebra on the set of functions $φ : \text{Int}_\ell → \mathbb{Z}$ with pointwise addition:

$$(φ_1 * φ_2)(σ) := \sum_{ρ ⊕ τ = σ} φ_1(ρ) * φ_2(τ).$$

Let $ζ : \text{Int}_\ell → \mathbb{Z}$ denote the zeta function defined by $ζ(σ) = 1$ for all $σ$ in $\text{Int}_\ell$. The zeta function has a unique convolutional inverse $μ$, called the Möbius function. Foata proved that the Möbius function can be expressed by the following explicit formula

$$μ(σ) = \begin{cases}
(-1)^{\text{fcyc}(σ)} & \text{if } σ \text{ is simple}, \\
0 & \text{else},
\end{cases}$$

where $σ ∈ \text{Int}_\ell$ is simple if all the letters of $σ$ are distinct, that is $\text{supp}(σ)$ is a set, not a multiset. This may be formulated as an identity in a completion $\mathbb{Z}[[\text{Int}_\ell]] := \{ \sum_{σ ∈ \text{Int}_\ell} z_σ σ : z_σ ∈ \mathbb{Z} \}$ of the monoid algebra $\mathbb{Z}[[\text{Int}_\ell]]$, allowing infinite $\mathbb{Z}$-linear combinations of elements of $\text{Int}_\ell$ (see [5, Théorème 2.4]):

$$1 = \left( \sum_{σ ∈ \text{Int}_\ell} σ \right) \left( \sum_{σ ∈ \text{Int}_\ell} μ(σ) σ \right) = \left( \sum_{σ ∈ \text{Int}_\ell} σ \right) \left( \sum_{σ ∈ \text{Int}_\ell} \sum_{\text{simple } σ ∈ \text{Int}_\ell} (-1)^{\text{fcyc}(σ)} σ \right)$$

Now introduce an $ℓ × ℓ$ matrix $B := (b_{ij})_{i,j=1,2,...,ℓ}$ of indeterminates, and let $\mathbb{Z}[[b_{ij}, t]]$ be the (usual, commutative) power series ring in $\{b_{ij}\}_{i,j=1}^\ell$ along with one further indeterminate $t$. One can then define a ring homomorphism

$$\mathbb{Z}[[\text{Int}_\ell]] \xrightarrow{u_t} \mathbb{Z}[[b_{ij}, t]] : σ \mapsto t^{\text{fcyc}(σ)} b_σ,$$

where if $σ = (i_1 \sigma_1 i_2 \sigma_2 \cdots i_n \sigma_n)$ then $b_σ := \prod_{k=1}^n b_{i_k \sigma_k}$.

Applying the homomorphism $u_t$ to both sides of (11) gives a $t$-version of MacMahon’s Master Theorem.

**Theorem 4.20.** In $\mathbb{Z}[[b_{ij}, t]]$ one has the identity

$$\sum_{σ ∈ \text{Int}_\ell} t^{\text{fcyc}(σ)} b_σ = \left( \sum_{\text{simple } σ ∈ \text{Int}_\ell} (-t)^{\text{fcyc}(σ)} b_σ \right)^{-1} = \left( \sum_{H ⊆ [\ell]} \sum_{σ ∈ \mathcal{G}_H} (-t)^{\text{fcyc}(σ)} b_σ \right)^{-1}.$$

**Remark 4.21.** Setting $t = 1$ in Theorem 4.20 gives an identity in $\mathbb{Z}[[b_{ij}]]$:

$$\sum_{σ ∈ \text{Int}_\ell} b_σ = \left( \sum_{H ⊆ [\ell]} \sum_{σ ∈ \mathcal{G}_H} (-1)^{\text{fcyc}(σ)} b_σ \right)^{-1}.$$
which is equivalent to Foata’s proof of the (commutative) MacMahon Master Theorem, as we recall here. Introduce two sets of ℓ variables \( x = (x_1, \ldots, x_\ell), y = (y_1, \ldots, y_\ell) \) related by the matrix of indeterminates \( B \) as follows: \( y = Bx \), that is, \( y_i = \sum_j b_{ij}x_j \). Then MacMahon’s Master Theorem is this identity in \( \mathbb{Z}[[b_{ij}]] \):

\[
\sum_{\pi \subseteq \{0,1,2,\ldots\}^\ell} \text{(coefficient of } x^\pi\text{ in } y^\pi\text{)} = \det(I_\ell - B)^{-1}.
\]  

(13)

It is not hard to check that the left sides and right sides of (13) and (12) are the same: the left side of (12) needs to be grouped according to the multiplicity vector \( \pi \) giving the support \( \text{supp}(\sigma) \), and the right side must be reinterpreted in terms of the permutation expansion of a determinant.

**Remark 4.22.** Theorem 4.20 is similar in spirit to Garoufalidis-Lê-Zeilberger’s quantum MacMahon Master Theorem [6, Theorem 1] (see also Konvalinka-Pak [9, Theorem 1.2]). Their quantum version inserts a \((q)^{-\text{inv}}\) in order to produce a \( q \)-determinant, but \( \text{inv}(\sigma) \neq \text{fcyc}(\sigma) \).

We now specialize \( b_{ij} = x_j \) in Theorem 4.20 to deduce Theorem 1.3 whose statement we recall here.

**Theorem 1.3.** For \( \ell = 1, 2, \ldots, \) one has

\[
\sum_{\pi \subseteq \{0,1,2,\ldots\}^\ell} \text{Poin}(\ell; t) \cdot x^\pi = \frac{1}{1 - \sum_{j=1}^\ell e_j(x) \cdot (t - 1)(2t - 1) \cdot \cdots \cdot ((j - 1)t - 1)}.
\]

(14)

where \( x^\pi := x_1^{i_1} \cdots x_\ell^{i_\ell} \) and \( e_j(x) := \sum_{1 \leq i_1 < \cdots < i_j \leq \ell} x_{i_1} \cdots x_{i_j} \) is the \( j \)th elementary symmetric function.

**Proof.** Setting \( b_{ij} = x_j \) in Theorem 4.20 gives

\[
\sum_{\sigma \in \text{Int}_\ell} t^{\text{fcyc}(\sigma)} \prod_k x_{\sigma_k} = \left( \sum_{H \subseteq \ell} \sum_{\sigma \in \mathfrak{S}_H} (-t)^{\text{fcyc}(\sigma)} \prod_{k \in H} x_k \right)^{-1}.
\]

Let us manipulate both sides of equation (14). On the left, grouping terms according to \( \text{supp}(\sigma) \) gives

\[
\sum_{\pi \subseteq \{0,1,2,\ldots\}^\ell} x^\pi \sum_{\sigma \in \mathfrak{S}_M(\pi)} t^{\text{fcyc}(\sigma)}.
\]

On the right of (14), note that any subset \( H \subseteq \ell \) of cardinality \( j \geq 1 \) has \( \mathfrak{S}_H \cong \mathfrak{S}_j \), and hence same sum

\[
\sum_{\sigma \in \mathfrak{S}_H} (-t)^{\text{fcyc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_j} (-t)^{\text{fcyc}(\sigma)} = (-t)^{(j - 1)t}(2 - t) \cdot \cdots \cdot (j - 1 - t)
\]

using \([6]\). Therefore grouping according to \( j = \# H \), and noting \( \sum_{H \subseteq \ell} \prod_{k \in H} x_k = e_j(x) \) lets one rewrite the sum inside the parentheses on the right of (14) as this:

\[
1 + \sum_{j=1}^\ell (-t)(1 - t)(2 - t) \cdot \cdots \cdot (j - 1 - t) \cdot e_j(x).
\]

So far this gives

\[
\sum_{\pi \subseteq \{0,1,2,\ldots\}^\ell} x^\pi \sum_{\sigma \in \mathfrak{S}_M(\pi)} t^{\text{fcyc}(\sigma)} = \left( 1 + \sum_{j=1}^\ell (-t)(1 - t)(2 - t) \cdot \cdots \cdot (j - 1 - t) \cdot e_j(x) \right)^{-1}.
\]

Now perform two more substitutions: first replace \( t \) by \( t^{-1} \), giving this

\[
\sum_{\pi \subseteq \{0,1,2,\ldots\}^\ell} x^\pi \sum_{\sigma \in \mathfrak{S}_M(\pi)} t^{-\text{fcyc}(\sigma)} = \left( 1 + \sum_{j=1}^\ell (-t^{-1})(1 - t^{-1})(2 - t^{-1}) \cdot \cdots \cdot (j - 1 - t^{-1}) \cdot e_j(x) \right)^{-1},
\]

and then replace \( x_i \) by \( tx_i \) for \( i = 1, 2, \ldots, \ell \), so that \( x^\pi \to t^\pi x^\pi \) and \( e_j(x) \to t^j e_j(x) \), giving this

\[
\sum_{\pi \subseteq \{0,1,2,\ldots\}^\ell} x^\pi \sum_{\sigma \in \mathfrak{S}_M(\pi)} t^{\text{fcyc}(\sigma)} = \left( 1 - \sum_{j=1}^\ell (t - 1)(2t - 1) \cdot \cdots \cdot ((j - 1)t - 1) \cdot e_j(x) \right)^{-1}.
\]
Comparison of the left side with Theorem 1.2 shows that this last equation is Theorem 1.3.

5. Real-rootedness

At the 2019 Mid-Atlantic Algebra, Geometry, and Combinatorics (MAAGC) Workshop, Phillip Zhang observed that the three polynomials listed in equation (9) are real-rooted. For a partition \( \lambda \), an exhaustive search determines that \( \text{Poin}(P_\lambda, t) \) has real roots when \( \lambda \) has most 6 cells. This does not, however, extend to arbitrary skew shapes. For example, when \( \lambda = (6, 4, 2) \) and \( \mu = (4, 2) \) we have \( P_{\lambda/\mu} = 2 \sqcup 2 \sqcup 2 \) and

\[
\text{Poin}(P_{\lambda/\mu}, t) = \text{Poin}(2 \sqcup 2 \sqcup 2, t) = 4x^4 + 30x^3 + 43x^2 + 12x + 1,
\]

which has a pair of complex roots. Even when the skew shape is connected one can encounter complex roots. For example, the ribbon skew shape \( \lambda/\mu = (4, 4, 3, 2, 1)/(3, 2, 1, 0, 0) = \)

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

has \( \text{Poin}(P_{\lambda/\mu}, t) = 36t^6 + 246t^5 + 507t^4 + 424t^3 + 150t^2 + 21t + 1 \). On the other hand, computations show that \( \text{Poin}(P, t) \) is real-rooted for all posets \( P \) of width two having at most 8 elements. This leads to the following questions.

**Question 5.1.** Is \( \text{Poin}(P, t) \) real-rooted when the poset \( P \) has width two, or \( P = P_\lambda \) for a partition \( \lambda \)?

Acknowledgements

The authors gratefully acknowledges Dennis Stanton for conversations about MacMahon’s Master Theorem, as well as Jesus DeLoera, Chi-Ho Yuen, and Philip Zhang for enlightening discussions.

References

[1] Marcelo Aguiar and Swapneel Mahajan. *Topics in Hyperplane Arrangements*. American Mathematics Society, 2017.
[2] Petter Brändén. \( q \)-Narayana numbers and the flag \( h \)-vector of \( J(2 \times n) \). *Discrete Math.*, 281(1-3):67–81, 2004.
[3] Kenneth Brown. Semigroups, Rings, and Markov Chains. *Journal of Theoretical Probability*, 13(3):342–351, 2000.
[4] Dominique Foata. *Etude algébrique de certains problèmes d’analyse combinatoire et du calcul des probabilités*. PhD thesis, Publ. Inst. Statist. Univ. Paris, 1965.
[5] Dominique Foata and Pierre Cartier. Problèmes combinatoires de commutation et réarrangements. Springer Verlag, Lecture Notes in Mathematics, 1969.
[6] Stavros Garoufalidis, Thang TQ Lê, and Doron Zeilberger. The Quantum MacMahon Master Theorem. *J. Algebra*, March 2003.
[7] Regina Gente. *The Varchenko Matrix for Cones*. PhD thesis, Universität Marburg, 2013.
[8] Donald Ervin Knuth. *The Art of Computer Programming*. Addison-Wesley, 2015.
[9] Matjaz Konvalinka and Igor Pak. Non-commutative Extensions of MacMahon’s Master Theorem. *Advances in Mathematics*, 2007.
[10] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
[11] Alexander Postnikov, Victor Reiner, and Lauren Williams. Faces of Generalized Permutohedra. *Documenta Mathematica*, 13:207–273, 2008.
[12] Victor Reiner and Volkmar Welker. On the Charney-Davis and Neggers-Stanley conjectures. *Journal of Combinatorial Theory, Series A*, 109(2):247–280, February 2005.
[13] Richard Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, UK, 1999.
[14] Richard Stanley. *An Introduction to Hyperplane Arrangements*. *Geometric Combinatorics IAS/Park City Mathematics Series*, pages 389–496, 2007.
[15] Richard Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, New York, NY, USA, 2 edition, 2012.
[16] Thomas Zaslavsky. A Combinatorial Analysis of Topological Dissections. *Advances in Mathematics*, 25(3):267–285, 1977.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

E-mail address: dorpa003@umn.edu, reiner@umn.edu