Bäcklund transformation of Painlevé III\((D_8)\) \(\tau\) function

M A Bershtein\(^{1,2,3,4,5}\) and A I Shchechkin\(^{2,3,6}\)

1 Landau Institute for Theoretical Physics, Chernogolovka, Russia
2 Skolkovo Institute of Science and Technology, Moscow, Russia
3 National Research University Higher School of Economics, Moscow, Russia
4 Institute for Information Transmission Problems, Moscow, Russia
5 Independent University of Moscow, Moscow, Russia
6 Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine

E-mail: mbersht@gmail.com and shch145@gmail.com

Received 16 August 2016, revised 11 January 2017
Accepted for publication 17 January 2017
Published 20 February 2017

Abstract
We study the explicit formula (suggested by Gamayun, Iorgov and Lisovyy) for the Painlevé III\((D_8)\) \(\tau\) function in terms of Virasoro conformal blocks with a central charge of 1. The Painlevé equation has two types of bilinear forms, which we call Toda-like and Okamoto-like. We obtain these equations from the representation theory using an embedding of the direct sum of two Virasoro algebras in a certain superalgebra. These two types of bilinear forms correspond to the Neveu–Schwarz sector and the Ramond sector of this algebra. We also obtain the \(\tau\) functions of the algebraic solutions of the Painlevé III\((D_8)\) from the special representations of the Virasoro algebra of the highest weight \((n + 1/4)^2\).

Keywords: Painlevé equations, Virasoro algebra, Bäcklund transformations, bilinear equations

(Some figures may appear in colour only in the online journal)

1. Introduction
This paper is a sequel to [4]. We continue our study of the relation between the Painlevé equations and conformal field theory. In this paper we restrict ourselves to the most degenerate case of the Painlevé III equation. This equation has different names: it is called the Painlevé III\((D_8)\) equation in the geometric approach (see e.g. [26]), it is also called the Painlevé III\(_3\) equation (see e.g. [11]) and, furthermore, it is equivalent to the radial sine-Gordon equation (see e.g. [9]).
In the paper [11] (following their previous work [10]), Gamayun, Iorgov and Lisovyy suggested that the \( \tau \) function of this equation had the form

\[
\tau(\sigma, s|z) = \sum_{n \in \mathbb{Z}} C(\sigma + n)s^n \mathcal{F}((\sigma + n)^2|z),
\]

(1.1)

where \( s, \sigma \) are integration constants, the function \( \mathcal{F}(\Delta|z) \) denotes the Whittaker limit of the Virasoro conformal block in the module with the highest weight \( \Delta \) and central charge \( c = 1 \), and the function \( C(\sigma) = 1/(G(1 - 2\sigma)G(1 + 2\sigma)) \), where \( G \) is a Barnes \( G \)-function. This formula was proven in [15] and [4] by different methods.

On the other hand, it is known that this Painlevé equation has a Bäcklund transformation \( \pi \) of order two. The main topic of this paper is the relation between the decomposition (1.1) and this Bäcklund transformation. Our initial motivation was a \( q \)-deformation of the formula (1.1), and the corresponding results are reported in a separate paper [5].

Let us discuss the content of the paper. In section 2 we recall the necessary definitions and notations of the Painlevé III\( (D_8) \) equation, its \( \tau \) function and Bäcklund transformation. The relation between \( \tau \) and the transformed \( \tau' = \pi(\tau) \) is written in the form of bilinear relations.

We have two types of bilinear relations, namely Okamoto-like and Toda-like

\[
\begin{align*}
D_{\log z}^2(\tau, \eta) &- \frac{1}{2} \left( \frac{d}{dz} - \frac{1}{8} \right) D_{\log z}^1(\tau, \eta) = 0, \\
D_{\log z}^3(\tau, \eta) &- \frac{1}{2} \left( \frac{d}{dz} - \frac{1}{8} \right) D_{\log z}^1(\tau, \eta) = 0,
\end{align*}
\]

(1.2)

where \( D_{\log z}^k \) are Hirota differential operators (2.8) (see propositions 2.2 and 2.3). We obtain these equations using a standard method from the Painlevé theory, the only difference being the fact that \( \pi \) has order two instead of the infinite order Bäcklund transformations usually used. We also prove the converse statement, i.e. to what extent equations (1.2) determine the Painlevé III\( (D_8) \) \( \tau \) function.

In section 3 we recall the necessary notations from the representation theory of the Virasoro algebra in order to state formula (1.1). Then we show that

\[
\eta(\sigma, s|z) \propto \tau(\sigma - 1/2, s|z),
\]

(1.3)

where \( \propto \) stands for constant (with respect to \( z \)) proportionality. In the section 4, we discuss the interpretation of the bilinear equations (1.2) (on the \( \tau \) function (1.1)) in the framework of the representation theory of the Virasoro algebra. As in the paper [4], the main tool is the embedding \( \text{Vir} \oplus \text{Vir} \subset \text{F} \oplus \text{NSR} \) of a direct sum of two Virasoro algebras into a sum of the Majorana fermion and super Virasoro algebra. Actually, using the Neveu–Schwarz sector of the algebra \( \text{F} \oplus \text{NSR} \) we already proved (up to some details discussed in section 4.3) in [4] that the right-hand side of (1.1) satisfies the Toda-like equations. This proves the formula (1.1), although it is a slight simplification of the proof in [4], where we used another bilinear equation of order four. In section 4.4 we show that the Ramond sector of the algebra \( \text{F} \oplus \text{NSR} \) gives Okamoto-like equations for the right-hand side of (1.1).

The Painlevé III\( (D_8) \) equation has two algebraic solutions. The corresponding \( \tau \) functions have the form \( \tau(z) \propto z^{1/16} e^{\pi i z} \). In section 3.3 we give an interpretation of these \( \tau \) functions in the framework of the representation theory of the Virasoro algebra. Namely, these \( \tau \) functions correspond to the special representations of the highest weights \( (n + 1/4)^2, n \in \mathbb{Z} \), studied by Zamolodchikov in [29]. Also note that in this case, the \( \tau \) function coincides with the special case of the dual partition function introduced by Nekrasov and Okounkov in [23].
In this paper, all continuous variables are considered to belong to the field \( \mathbb{C} \) unless otherwise stated. All representations and algebras are considered over the field \( \mathbb{C} \).

2. Painlevé III(D_8) equation

2.1. Hamiltonian and \( \tau \) form of the Painlevé III(D_8) equation

We recall several facts about one of the simplest Painlevé equations: the Painlevé III(D_8) (or Painlevé III_3) following [11, 26].

The Painlevé III(D_8) equation on function \( w(z) \) has the form

\[
\frac{d^2 w}{dz^2} = w^3 \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{2w^2}{z^2} - \frac{2}{z}.
\]

(2.1)

Note that in the work [26], rescaled \( w \) and \( z \) are used.

We now proceed to the Hamiltonian (or \( \zeta \)) form of the Painlevé III(D_8). The Painlevé equations can be rewritten as non-autonomous Hamiltonian systems. This means that they can be obtained by eliminating an auxiliary momentum \( p(z) \) from the equations

\[
\frac{dw}{dz} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H}{\partial w}
\]

where the Hamiltonian \( H(z) \) for the Painlevé III(D_8) equation has the form

\[
\zeta = zH = p^2w^2 - w - zw.
\]

(2.2)

It is also convenient to use the function \( \zeta(z) = zH(z) \), which is just a Hamiltonian with respect to the time \( \log z \). Below, we will denote the dot differentiation by \( z \) and the prime differentiation by \( \log z \). The Hamilton equations, in terms of \( p \) and \( w \), read

\[
w = 2pw^2/z, \quad \dot{p} = -2p^3w/z + 1/z - 1/w^2.
\]

(2.3)

Note that if we know function \( \zeta(z) \) on the trajectories of motion, then we can find \( w(z) \) and \( p(z) \). Differentiating (2.2) once and twice, and using the Hamilton equations to differentiate \( p(z) \) and \( w(z) \), we can express these functions by the formulas

\[
w(z) = -\frac{1}{\zeta(z)}, \quad p(z) = \frac{z\dot{\zeta}(z)}{2}.
\]

(2.4)

Substituting these expressions into (2.2) we get the Hamiltonian (or \( \zeta \)) form of the Painlevé III(D_8) equation

\[
(z\ddot{\zeta}(z))^2 = 4\dot{\zeta}(z)^2(\zeta(z) - z\dot{\zeta}(z)) - 4\dot{\zeta}(z).
\]

(2.5)

In this paper, we will consider the solutions of this equation except the constant \( \zeta(z) \) and \( \zeta(z) = z + 1 \) (these are only solutions such that \( \ddot{\zeta} = 0 \)).

Then it can be checked directly that each solution of (2.5) corresponds to the solution of (2.1) by the first formula of (2.4). Inversely, the solution \( w(z) \) of (2.1) gives us \( p(z) \) by the first formula of (2.3) and then \( \zeta(z) \) given by (2.2) satisfies (2.5). So, we have one-to-one correspondence between the solutions \( w(z) \) of (2.1) and \( \zeta(z) \) of (2.5).

Remark 2.1. The Painlevé III(D_8) equation appears in a physical framework, for instance, as the radial sine-Gordon equation on function \( r(r) \) (see e.g. [9, chapter 3]).
\[ v_r + \frac{v}{r} = 1/2 \sin 2v. \]  
\[ (2.6) \]

One can proceed to this equation from (2.1) by substituting
\[ w(z)/\sqrt{z} = e^{2iv}, \quad z = r^4/4096. \]

Let us introduce the \( \tau \) function by the formula
\[ \zeta(z) = z \frac{d \log \tau(z)}{dz} \quad \text{and inverse} \quad \tau = \exp \left( \int \zeta(z) d \log z \right). \]  
\[ (2.7) \]

Note that the \( \tau \) function is defined up to a multiplication by a constant factor.

One can obtain the equation on the \( \tau \) function from (2.5). Differentiate (2.5) by \( z \) and divide the result by \( \zeta \). Substituting the first formula of (2.7) and multiplying by \( \tau^2 \), we obtain a bilinear equation on the \( \tau \) function. It is convenient to write this equation by use of the Hirota differential operators \( D_k \). In our paper, we only use Hirota derivatives with respect to the logarithm of a variable. These operators on the functions \( f(z) \), \( g(z) \) are defined by the formula
\[ f(e^{\sigma}z)g(e^{-\sigma}z) = \sum_{k=0}^{\infty} D_{\log z}^k f(z), g(z) \frac{\alpha^k}{k!}. \]  
\[ (2.8) \]

The first examples of the Hirota operators are
\[ D_{\log z}^0(f(z), g(z)) = f(z)g(z), \quad D_{\log z}^1(f(z), g(z)) = z \frac{d}{dz} (f(z)g(z)) - f(z)z g(z). \]

Then, the \( \tau \) form of the Painlevé III(\( D_8 \)) equation can be written as
\[ D^{III}(\tau(z), \zeta(z)) = 0, \quad \text{where} \quad D^{III} = \frac{1}{2} D_{\log z}^4 - z \frac{d}{dz} D_{\log z}^2 + \frac{1}{2} D_{\log z}^2 + 2z D_{\log z}^0. \]  
\[ (2.9) \]

Because we differentiate (2.5) to obtain (2.9) we have extra solutions of (2.9). More precisely, (2.9) is equivalent to the so-called Painlevé III(\( D_7 \)) equation
\[ (\zeta^2(z))^2 = 4(\zeta(z))^2(\zeta(z) - \zeta(z)) - 4\zeta(z) + 1/\theta. \]

In this work, we will only consider solutions of (2.9), which correspond to the case \( \theta = \infty \), i.e. the Painlevé III(\( D_8 \)) in the form (2.5). These solutions can be distinguished by the asymptotic behavior of the \( \tau \) function.

The following proposition follows from results proven in [24], (see also book [9] and the original papers [17, 22, 25]).

**Proposition 2.1.** There exists a two-parametric family of solutions of the equation (2.5), such that the asymptotic behavior of the corresponding \( w(z) \) and \( \tau(z) \) for \( z \to 0 \) is given by
\[ w(\sigma, \tilde{s} | z) = 4\sigma^2 \tilde{s}z^{2\sigma}(1 + o(1)), \]  
\[ (2.10) \]
\[ \tau(\sigma, \tilde{s} | z) \propto z^{\sigma} \left[ 1 + \frac{z}{2\sigma^2} - \frac{\tilde{s}^{-1}}{(1 - 2\sigma)^2(2\sigma)^2 z^{1-2\sigma} + o(\tilde{s})} \right]. \]  
\[ (2.11) \]

\(^7\)We are grateful to A Its for the explanation of this point and help with the references.
where $\propto$ means constant proportionality (with respect to $z$), and $\sigma, \tilde{s}$ are the integration constants which belong to the domain $0 < \text{Re} \sigma < 1/2, \tilde{s} = 0$. Moreover, any solution of (2.5) with such asymptotics belongs to this family, and for a given $\sigma$ and $\tilde{s}$ it is unique.

It was also proven in [24] that solutions which do not belong to this family can be parametrized by a lower number of parameters (three real numbers). Therefore, one can think of the family from proposition 2.1 as a family of generic solutions.

2.2. The Bäcklund transformation and the Okamoto-like and Toda-like equations

The group of Bäcklund transformations of the Painlevé III($D_8$) equation is $\mathbb{Z}_2$ (see [26, section 2.3]). This group is generated by the transformation $\pi$ which acts on the solutions of the Painlevé III($D_8$) by the formula

$$z \mapsto z, \quad w \mapsto w_1 = z/w, \quad p \mapsto p_1 = -\frac{w(2wp - 1)}{2z}. \quad (2.12)$$

By (2.2), this transformation leads to a transformation of $\zeta(z)$. We will mark the variables after the transformation using the subscript 1. We have two useful formulas for the transformation of the function $\zeta(z)$

$$\zeta_1 = \zeta - pw + 1/4, \quad \zeta_1' = z \quad (2.13)$$

which follow from (2.2) and (2.4) respectively.

In terms of the sine-Gordon equation (see remark 2.1), the Bäcklund transformation is just $v \mapsto -v$.

**Proposition 2.2.**

(i) Consider a solution $\zeta(z)$ of (2.5), its Bäcklund transformation $\zeta_1(z)$, and the functions $\tau(z)$ and $\eta(z)$ corresponding to $\zeta(z)$ and $\zeta_1(z)$ by (2.7). Then the functions $\tau(z)$ and $\eta(z)$ satisfy equations

$$D_{\log z}^2(\tau, \eta) - \frac{1}{2} \left( z \frac{d}{dz} - \frac{1}{8} \right) (\tau \eta) = 0, \quad (2.14)$$

$$D_{\log z}^2(\tau, \eta) - \frac{1}{2} \left( z \frac{d}{dz} - \frac{1}{8} \right) D_{\log z}^2(\tau, \eta) = 0. \quad (2.15)$$

(ii) Conversely, consider functions $\tau(z)$ and $\eta(z)$ satisfying (2.14) and (2.15), and the functions $\zeta(z)$ and $\zeta_1(z)$ corresponding to $\tau(z)$ and $\eta(z)$ by (2.7), $\zeta(z) \neq 0$ and $\zeta_1(z) \neq 0$. Then there exists $D \neq 0$ such that functions $\zeta(Dz), \zeta_1(Dz)$ satisfy the equations (2.5) and $\pi(\zeta(zD)) = \zeta(zD)$. Usually, the bilinear equations in Painlevé theory are obtained using an infinite order Bäcklund transformation (see e.g. [26, 27]). We follow this approach using $\pi$ of order two for the Painlevé III($D_8$); that is why we call equations (2.14) and (2.15) ‘Okamoto-like equations’.

Equation (2.14) is symmetric under the transposition $\tau \leftrightarrow \eta$ and equation (2.15) is skew-symmetric under this transposition. This is natural since $\pi^2 = 1$.

**Remark 2.2.** Okamoto-like equations (2.14) and (2.15) have a symmetry in rescaling $z$, i.e. if $\tau(z)$ and $\eta(z)$ is a solution, then $\tau(Dz), \eta(Dz)$ and $D \neq 0$ is also a solution. In other words, any solution of (2.14) and (2.15) can be obtained from a Painlevé $\tau$ function using such rescaling, and a solution with asymptotic behavior (2.11) corresponds to the value $D = 1$.  

M A Bershtein and A I Shchechkin  
J. Phys. A: Math. Theor. 50 (2017) 115205
Proof.

(i) From (2.4) we have
\[ \frac{d}{dz} \zeta = -\frac{z}{w}, \quad \frac{d}{dz} \zeta' = -w. \] (2.16)

One can check that the simple algebraic identities hold
\[ \zeta - \zeta' = \frac{d}{dz} \log \frac{\tau}{\eta} = \frac{D_{\log z}(\tau, \eta)}{\tau \eta}, \] (2.17)
\[ \frac{d}{dz} (\zeta + \zeta') = \frac{D_{\log z}^2(\tau, \eta)}{\tau \eta} - \left( \frac{D_{\log z}(\tau, \eta)}{\tau \eta} \right)^2. \] (2.18)

\[ \left( \frac{d}{dz} \right)^2 (\zeta - \zeta') = \frac{D_{\log z}^3(\tau, \eta)}{\tau \eta} - \frac{3 D_{\log z}^2(\tau, \eta) D_{\log z}(\tau, \eta)}{\tau \eta} + 2 \left( \frac{D_{\log z}(\tau, \eta)}{\tau \eta} \right)^2. \] (2.19)

Using (2.18), (2.13) and (2.16) we have
\[ \frac{D_{\log z}(\tau, \eta)}{\tau \eta} - (\zeta - \zeta')^2 = \frac{z}{w} (\zeta + \zeta') = -w - z/w = \zeta - p^2 w^2 = \zeta - \left( \zeta - \zeta' + \frac{1}{4} \right)^2, \]
where we used (2.16) in the second equality, (2.2) in the third equality and the first equation of (2.13) in the fourth. Using (2.17) we finally obtain (2.14).

Analogously, we could obtain (2.15). From (2.19) we have
\[ \frac{D_{\log z}^3(\tau, \eta)}{\tau \eta} - \frac{3 D_{\log z}^2(\tau, \eta) D_{\log z}(\tau, \eta)}{\tau \eta} - 2 \left( \frac{D_{\log z}(\tau, \eta)}{\tau \eta} \right)^2 = \frac{z}{w} (\zeta + \zeta') \]
\[ = \frac{d}{dz} \zeta - \left( \frac{D_{\log z}^3(\tau, \eta)}{\tau \eta} - (\zeta - \zeta')^2 \right) \left( 2(\zeta - \zeta') + \frac{1}{2} \right), \]

where first we use (2.16), then (2.3), then the first equation of (2.13) and (2.16) and then (2.18); i.e. we have
\[ \frac{D_{\log z}^3(\tau, \eta)}{\tau \eta} - \frac{3 D_{\log z}^2(\tau, \eta) D_{\log z}(\tau, \eta)}{\tau \eta} + 2 \left( \frac{D_{\log z}(\tau, \eta)}{\tau \eta} \right)^2 = \frac{z}{w} (\zeta + \zeta'). \]

Then, using (2.14) we obtain (2.15).

(ii) From (2.14) and (2.15) we obtain respectively
\[ \zeta' + \zeta' = \zeta - \frac{1}{4} \left( 2\zeta - 2\zeta' + \frac{1}{2} \right)^2, \] (2.20)
\( \zeta'' - \zeta''' = \zeta' - \left(2\zeta - 2\zeta_1 + \frac{1}{2}\right)\zeta' + \zeta_1. \)  \hfill (2.21)

Let us differentiate the first equation and then the sum and subtract it from the second equation. We obtain

\( \zeta'' = -\left(2\zeta - 2\zeta_1 - \frac{1}{2}\right)\zeta', \)  \hfill (2.22)

\( \zeta''' = \left(2\zeta - 2\zeta_1 + \frac{1}{2}\right)\zeta'_1. \)  \hfill (2.23)

From these equations we obtain

\( \zeta''/\zeta'_1 = 1 - \zeta''/\zeta' \Leftrightarrow \zeta''/\zeta'_1 = D\zeta, \)

where \( D \neq 0 \) is the integration constant (see second equation of (2.13)).

Eliminating \( \zeta, \zeta'_1 \) from (2.20) using \( \zeta''/\zeta'_1 = D\zeta \) and (2.22), we obtain

\( (\zeta'' - \zeta')^2 = 4(\zeta^2)(\zeta - \zeta') - 4D\zeta' \Leftrightarrow (z\zeta(z))^2 = 4(\zeta(z))^2(\zeta(z) - z\zeta(z)) - 4D\zeta(z), \)

and evidently we have the same equation on \( \zeta_1 \). From this we see that \( \zeta(z/D) \) and \( \zeta_1(z/D) \) satisfy (2.5).

Let us check that \( \pi(\zeta(z/D)) = \zeta(z/D). \) These functions satisfy the second equation of (2.13), from which it follows that \( \zeta_1(z/D) = \pi(\zeta(z/D)) \) is a constant. Since \( \zeta(z/D) \) and \( \pi(\zeta(z/D)) \) satisfy (2.5) then \( \zeta(z/D) \neq \pi(\zeta(z/D)) \) iff \( \zeta(z/D)' = 0, \) which contradicts (2.13). \[ \Box \]

Now we will present the other equations, which we call 'Toda-like', because they are analogous to similar equations called 'Toda equations' in [26].

**Proposition 2.3.**

(i) Let \( \zeta(z) \) denote a solution of (2.5), \( \zeta(z) \) denotes its Bäcklund transformation and the functions \( \tau(z) \) and \( \eta(z) \) correspond to \( \zeta(z) \) and \( \zeta_1(z) \) by (2.7). Then the functions \( \tau(z) \) and \( \eta(z) \) satisfy equations

\[
D^2_{\log \tau}(\tau, \tau) = 2C \tau^{1/2} \tau_1^2,
\]

\[
D^2_{\log \eta}(\eta, \eta) = 2C^{-1} \tau^{1/2} \tau_1^2.
\]  \hfill (2.24)

(ii) Consider that the functions \( \tau(z) \) and \( \eta(z) \) satisfy (2.24), and the functions \( \zeta(z) \) and \( \zeta_1(z) \) correspond to \( \tau(z) \) and \( \eta(z) \) by (2.7) and \( \zeta(z) \neq 0, \zeta_1(z) \neq 0. \) Then there exists \( K \) such that functions \( \zeta(z) - K, \zeta_1(z) - K \) satisfy the equations (2.5) and \( \pi(\zeta(z) - K) = \zeta_1(z) - K. \)

A constant \( C \) in (2.24) depends on the normalization of \( \tau \) and \( \eta. \)

**Remark 2.3.** Usually, Toda-like equations are written in logarithmic form, i.e. (2.24) can be rewritten as

\[
\left( \frac{d}{dz} \right)^2 \log \tau = C \tau^{1/2} \tau_1^2, \quad \left( \frac{z}{dz} \right)^2 \log \eta = C^{-1} \tau^{1/2} \tau_1^2.
\]  \hfill (2.25)
Remark 2.4. The Toda-like equations (2.24) have symmetry in multiplying by $z^K$, i.e. if $\tau(z)$ and $\eta(z)$ is a solution then $z^K \tau(z)$ and $z^K \eta(z)$ for any $K$ is also a solution. In other words, any solution of (2.24) can be obtained from the Painlevé $\tau$ function by this multiplication, and the solution with asymptotic behavior (2.11) corresponds to the value $K = 0$.

Proof.
(i) Using (2.4), (2.13) and (2.17) we have

$$\frac{d}{dz} \left( \log \frac{\tau}{\eta} \right) = \zeta - \zeta_1 = pw - 1/4 = \frac{z}{2\zeta} - 1/4 = \frac{\zeta''}{2\zeta'} + 1/4 \Leftrightarrow \zeta^{-1/2}$$

$$= C \frac{T_1^2}{T^2} \Leftrightarrow \left( \log \frac{\tau}{\eta} \right)'' = C z^{1/2} \frac{T_1^2}{T^2}$$

from which we have a Toda-like equation with a constant $C$ and its symmetric one with a constant $C_1$

$$D_{\log;}(\tau, \tau) = 2C T_1^2 z^{1/2}, \quad D_{\log;}(\eta, \eta) = 2C T_1^2 z^{1/2}.$$

Multiplying these equations by each other we have $\zeta' \zeta'_1 = CC_1z$. However, from the second equation of (2.13) we have $CC_1 = 1$.

(ii) Multiplying the first and second equations of (2.25) we have $\zeta' \zeta'_1 = z$. Acting by $\left(\frac{d}{dz}\right)^2$ on the second equation of (2.25) we have

$$-\frac{\zeta''}{2\zeta'} + \frac{\zeta'''}{2\zeta''} = \zeta' - \zeta_1.$$

Substituting $\zeta' = \frac{z}{\zeta'}$ we have

$$-\frac{\zeta''}{2\zeta'} + \frac{\zeta'''}{2\zeta''} = 2\zeta'^3 - 2z\zeta'.$$

Let us denote

$$f(z) = z^2\zeta'(z)^2 - 4\zeta'(z)^2(\zeta(z) - z\zeta'(z)) + 4\zeta'(z) = \frac{1}{z}((\zeta'' - \zeta')^2 - 4\zeta'^3(\zeta - \zeta') + 4z\zeta').$$

Differentiating the expression for $f(z)$ we have

$$\frac{z^2}{2(\zeta'' - \zeta')''} = \zeta'' + 2\zeta' + 6\zeta'' - 4\zeta' + 2z.$$

Then equation (2.26) can be rewritten as

$$\frac{z^2}{2(\zeta'' - \zeta')''} = 2\zeta' = \zeta' \Leftrightarrow f = 4K \zeta'^2,$$

where $K$ is some integration constant. Evidently, we have the same equation on $\zeta_1$ with the integration constant $K_1$. Then the functions $\zeta(z) - K$ and $\zeta_1 - K_1$ are solutions of (2.5).
Similar to the proof of proposition 2.2, we get \( \pi(\zeta - K) = \zeta - K_1 \), because \( \zeta' = z \). Similar to the proof of proposition 2.2, we see that \( \zeta(z) = z + 1 + K \). It remains to prove that \( K_1 = K \), and this follows from the fact that both \( \pi(\tau) \) and \( \gamma_1 \) satisfy the first equation of (2.24).

In the section 4 we will see the appearance of Toda-like and Okamoto-like equations in the framework of the representation theory of the Virasoro algebra.

3. The Painlevé \( \tau \) function and the Virasoro algebra

3.1. Virasoro conformal blocks

The Virasoro algebra (which we denote by \( \text{Vir} \)) is generated by \( L_n \) and \( n \in \mathbb{Z} \) with relations

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{n^3 - n}{12}c\delta_{n+m,0}.
\]

(3.1)

Here, \( c \) is an additional central generator, which acts on the representations below as a multiplication by a complex number. Therefore, we consider \( c \) to be a complex number, which we call the central charge.

Denote the Verma module of \( \text{Vir} \) by \( \pi^\Delta \text{Vir} \). This module is generated by a highest weight vector \( |\Delta\rangle \)

\[
L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_n|\Delta\rangle = 0, \quad n > 0,
\]

(3.2)

where \( \Delta \in \mathbb{C} \) is called the weight of \( |\Delta\rangle \). Here and below in the analogous situations, the representation space is freely spanned by vectors obtained by the action of the operators \( L_{-n} \), \( n > 0 \) on \( |\Delta\rangle \)—i.e. the vectors of the form \( L_{-m}L_{-n}\cdots L_{-k}|\Delta\rangle \) for \( m > n > \cdots > k \) form a basis in the Verma module \( \pi^\Delta \text{Vir} \). Actually, the Verma module is determined by the pair of complex numbers \( \Delta \) and \( c \), but we just write \( \pi^\Delta \text{Vir} \) if a fixed value of \( c \) has been chosen.

Everywhere in this work, we use a complex symmetric scalar product, and not a Hermitian. We define the scalar product on \( \pi^\Delta \text{Vir} \) by the conjugation \( L_n^\dagger = L_{-n} \) and the normalization \( \langle \Delta|\Delta\rangle = 1 \). This scalar product is called the Shapovalov form. Note that in this paper we will normalize all the highest weight vectors of the \( \text{Vir} \) Verma modules on 1 unless otherwise stated.

We will say that \( \Delta \) is generic if \( \Delta \neq \Delta_{m,n} = ((b^{-1} + b)^2 - (mb^{-1} + nb)^2)/4 \), where the parameter \( b \) is defined by the equation \( c = 1 + 6(b^{-1} + b)^2 \) and \( m, n \in \mathbb{Z}_{>0} \). For such a generic \( \Delta \) Shapovalov form on the Verma module, \( \pi^\Delta \text{Vir} \) is nondegenerate [13, theorem 4.2]. Therefore, for the generic \( \Delta \) Verma module, \( \pi^\Delta \text{Vir} \) is irreducible.

Let us define the so-called irregular limit of the conformal block. First we define the Whittaker vector \( |W(z)\rangle \) by the formula

\[
|W(z)||\rangle = z^\Delta \sum_{N=0}^{\infty} z^N|N\rangle, \quad |N\rangle \in \pi^\Delta \text{Vir}, \quad L_0|N\rangle = (\Delta + N)|N\rangle,
\]

(3.3)

where

\[
L_0|N\rangle = |N - 1\rangle, \quad N > 0, \quad L_0|N\rangle = 0, n > 1,
\]

(3.4)

or equivalently,

\[
L_0|W(z)\rangle = z|W(z)\rangle, \quad L_k|W(z)\rangle = 0, k > 1.
\]

(3.5)

Using these conditions and the normalization \( |0\rangle = |\Delta\rangle \), one can compute all the \( |N\rangle \) vectors inductively. Moreover, it is easy to see that if a Shapovalov form is nondegenerate, then such
a system of vectors exists and is unique. Therefore, for generic values of \( \Delta \), the Whittaker vector is well defined.

Note that it is enough to impose the \( L_1 \) and \( L_2 \) relations since the action of the other \( L_k \) and \( k > 2 \) follows from the Virasoro commutation relations. Also note that this definition of \( |W(z)\rangle \) slightly differs from the one used in [4], and our \( z \) is \( z^{1/2} \) in loc. cit.

The irregular (or Whittaker or Gaiotto) limit of the conformal block is defined by

\[
\mathcal{F}_\Delta(z) = \langle W(1)|W(z)\rangle = z^\Delta \sum_{N=0}^{\infty} z^N \langle N|N\rangle.
\]  

We are ready to formulate the fact conjectured in [11] (which is based on a result of the work [10]) and proved in different ways in [4] and [15].

**Theorem 3.1.** The expansion of the Painlevé III\((D_8)\) \( \tau \) function near \( z = 0 \) can be written as

\[
\tau(\sigma, s|z) \propto \sum_{n \in \mathbb{Z}} C(\sigma + n)|w^a\mathcal{F}(\sigma + n^2|z), \quad \Re \sigma \in \mathbb{R} \setminus \left\{ \frac{1}{2}, -\frac{1}{2} \right\}, \quad s \in \mathbb{C} \setminus \{0\},
\]  

where \( \mathcal{F}(\sigma^2|z) = \mathcal{F}_\infty(\sigma^2|z) \). The coefficients \( C(\sigma) \) are defined by \( C(\sigma) = 1/(G(1 - 2\sigma)G(1 + 2\sigma)), \) where \( G(\cdot) \) is the Barnes \( G \)-function. The parameters \( s \) and \( \sigma \) in (3.7) are the integration constants of equation (2.5). Notation \( \propto \) means constant proportionality.

**Remark 3.1.** Actually, the parameter \( \sigma \) in the formula (3.7) can be \( \sigma \in \mathbb{C} \setminus \left\{ \frac{1}{2}, -\frac{1}{2} \right\} \). We use a smaller region in order to compare these \( \tau \) functions with the ones from the family given in proposition 2.1.

In fact, the formula (3.7) gives us the full power series, whose first terms one sees in the asymptotic behavior (2.11) (see the next section for details). The parameters \( \sigma \) in these formulas are the same, and the connection between \( s \) and \( \tilde{s} \) is given in the next section.

### 3.2. The Bäcklund transformation in terms of the \( \tau \) functions

The Bäcklund transformation acts on the \( \tau \) function due to the formulas (2.2) and (2.7). We want to understand its relation with (3.7). We will see that the action of the Bäcklund transformation is just an action on the parameters \( \sigma, \tilde{s} \).

The \( \tau \) function (3.7) has obvious symmetries

\[
\tau(\sigma, s|z) = s\tau(\sigma + 1, s|z), \quad \tau(\sigma, s|z) = \tau(-\sigma, s^{-1}|z).
\]  

Therefore, we can move \( \Re \sigma \) into the interval \((0, \frac{1}{2})\).

Let us mention the fact that the coefficients \( C(\tilde{\sigma} + n) \) in (3.7) could be made rational functions. Indeed

\[
\frac{C(\sigma + n)}{C(\sigma)} = \left( \frac{\Gamma(-2\sigma)}{\Gamma(2\sigma)} \right)^{2n} \left\{ \begin{array}{ll}
\frac{(-1)^{[n]}}{(2\sigma)^{2n-1} \prod_{i=1}^{2n-1} (-2\sigma + i)^{2i(n-i)}}, & n \geq 0 \\
\frac{(-1)^{-[n]}}{(2\sigma)^{-2n-1} \prod_{i=1}^{-2n-1} (2\sigma + i)^{-2i(n-i)}}, & n < 0
\end{array} \right.
\]  

\[
\tilde{C}(\sigma, n),
\]  

(3.9)
where $2n \in \mathbb{Z}$. So, we can introduce \( \delta(s, \sigma) = s \left( \frac{\Gamma(1-2\sigma)}{\Gamma(2-2\sigma)} \right)^2 \) and consider
\[
\tau(\sigma, \delta \mid z) \propto \sum_{n \in \mathbb{Z}} \tilde{C}(\sigma, n) \delta^n \mathcal{F}(s + n)^2 \mid z \propto \tau(\sigma, s \mid z).
\]

**Proposition 3.1.** The Bäcklund transformation of the \( \tau \) function is given by the formula
\[
\eta(\sigma, s \mid z) = \pi(\tau(\sigma, s \mid z)) \propto \tau(1/2 - \sigma, s^{-1} \mid z).
\]

Now, we give a straightforward proof based on the proposition 2.1. This proof does not use the formula (3.7), and can actually be given in section 2. Below in section 4.3 we will give another proof using representation theory.

**Proof.** Let us calculate the asymptotic behavior of (3.7) in the case of \( \sigma \ll \Re \in \mathbb{C} \setminus \{0\} \)
\[
\tau(\sigma, s \mid z) \propto z^{\sigma^2} \left( 1 + \frac{z}{2s^2} \right) + \tilde{C}(\sigma, -1) \delta^{1-\sigma -1} + o(|z|^{s^2+1}).
\]

This is just the asymptotic behavior (2.11) with the coinciding parameters \( \sigma \) and \( \delta \). Let us then calculate the asymptotic behavior for \( w(z) \) (see (2.10))
\[
\begin{align*}
\zeta(z) &= \frac{\sigma^2}{\delta} \log z + \frac{z}{2s^2} - \frac{\delta}{\sigma(1-2\sigma)} + \frac{1}{2} \delta \frac{\delta^{-2} - \sigma^2}{(2\sigma)^2(1-2\sigma)^2} + o(|z|); \\
w(z) &= -\frac{1}{\zeta(z)} = -\frac{1}{\frac{\sigma^2}{\delta} - \frac{z}{2s^2} + \frac{1}{2} \delta \frac{\delta^{-2} - \sigma^2}{(2\sigma)^2(1-2\sigma)^2} + o(1)} = 4\sigma^2 \delta z^{2\sigma}(1 + o(1)).
\end{align*}
\]

After the Bäcklund transformation \( w \mapsto z/w \) the asymptotic will be \( z^{1-2\sigma}/(4\sigma^2) (1 + o(1)) \). Therefore, the function \( \pi(\tau(\sigma, s \mid z)) \) belongs to the same two-parametric family due to the proposition 2.1, and the corresponding parameters \( \sigma_1, \delta_1 \) are given by
\[
\sigma_1 = 1/2 - \sigma, \quad \delta_1 = \frac{1}{(2\sigma)^2(1-2\sigma)^2} \delta_1 \Rightarrow s_1 = s^{-1}.
\]

So finally
\[
\tau(\sigma, s \mid z) \mapsto \eta(\sigma, s \mid z) \propto \tau(\sigma, 1/2 - \sigma, s^{-1} \mid z) = \tau(\sigma - 1/2, s \mid z). \tag{3.10}
\]

**Remark 3.2.** The power series decomposition of \( \eta \) can be written as
\[
\eta(\sigma, s \mid z) \propto \sum_{n \in \mathbb{Z} + 1/2} C(\sigma + n)^2 \mathcal{F}(s + n)^2 \mid z, \quad \Re \sigma \in \mathbb{R} \setminus \left\{ \frac{1}{2} \mathbb{Z} \right\}, \quad s \in \mathbb{C} \setminus \{0\}, \tag{3.11}
\]
which only differs from (3.7) in the region of summation.

Using proposition 3.1 and propositions 2.2 and 2.3, we can see that \( \tau(\sigma, s \mid z) \), given by the right-hand side of (3.7), satisfies the equations on the function \( \pi(\sigma \mid z) \), which follows from the Toda-like and Okamoto-like equations as equations in \( \sigma \) and \( z \). These equations will be differential on \( z \) and the difference on \( \sigma \). For instance, the Okamoto-like equation turns to
The analogous form of the Toda-like equation is given in the next remark.

**Remark 3.3.** We can directly determine the constant $C$ in the Toda-like equations (2.24) for $\tau = \tau(\sigma, s|z)$ and $\eta = \tau(\sigma - 1/2, s|z)$, where the normalization of $\tau(\sigma, s|z)$ is given by the equality in (3.7). Indeed, the first equation of (2.24) can be rewritten as

$$\zeta' = Cz \begin{pmatrix} \tau(1/2 - \sigma, s^{-1}|z)^2 \\ \tau(\sigma, s|z)^2 \end{pmatrix}$$

with the asymptotic behavior of the l.h.s. and r.h.s.

$$\zeta' = -\frac{z^{1-2\sigma}}{4\sigma^3} (1 + o(1)), \quad r.h.s. = Cz \begin{pmatrix} (1/2 - \sigma) \\ C(\sigma) \end{pmatrix} (1 + o(1))z^{1/2 - 2\sigma} \Rightarrow C = -s^{-1}.$$ 

So, the $\tau$ function given by the decomposition (3.7) satisfies the differential-difference equation

$$1/2D_{logz}^2(\tau(\sigma|z), \tau(\sigma|z)) = -s^{-1}z^{1/2}\tau(\sigma - 1/2)^2,$$

or using the first relation from (3.8)

$$1/2D_{logz}^2(\tau(\sigma|z), \tau(\sigma|z)) = -z^{1/2}\tau(\sigma + 1/2|z)\tau(\sigma - 1/2|z).$$

Analogously for the normalization of $\eta$ given by the equality in (3.11), we have $C = -1$.

We will use the differential-difference equations (3.12), (3.13) and (3.15) in section 4.

### 3.3. The algebraic solution of the Painlevé III($D_8$) equation

There exist only two rational solutions of the Painlevé III($D_8$) equation: $w(z) = \pm \sqrt{z}$ ([12]). These solutions are only invariant solutions under the Bäcklund transformation $\pi(\pm \sqrt{z}) = \pm \sqrt{z}$. Using (2.4) we obtain

$$\hat{\zeta}(z) = \mp \frac{1}{\sqrt{z}} \Rightarrow \rho(z) = \pm \frac{1}{4\sqrt{z}}.$$ 

Therefore, using (2.2) and (2.7) we find the $\zeta$ and $\tau$ functions

$$\zeta(z) = 1/16 \mp 2\sqrt{z}, \quad \tau(z) \propto z^{1/16}e^{\mp 4\sqrt{z}}.$$ 

This formula is also more or less known, for example, for the upper sign it follows (as well as the next formula (3.17)), from [6, equation (3.52)] after the substitution $\sigma = 1/4$, and probably from other sources too.

On the other hand, from proposition 3.1 it follows that there are two Bäcklund invariant solutions $\tau(1/4, \pm 1|z)$ given by the right-hand side of (3.7). Comparing the first terms of the power series expansion of (3.7) and the expression (3.16) for the $\tau$ function we obtain
\[ \tau(1/4, \pm 1|z) = C(1/4)z^{1/16}e^{\pi i/4}e^z, \]  
(3.17)

So, using (3.9), we obtain the following relation on conformal blocks
\[ \sum_{n \in \mathbb{Z}} (\mp 1)^n B_n \mathcal{F}(1/4 + n)^2|z) = z^{1/16}e^{\pi i/4}e^z, \]  
(3.18)

where the coefficients \( B_n \) are equal to
\[ B_n = \frac{24n^2 + 2n}{\prod_{i=0}^{2n-i-1} (2i + 1)^{2n-i-1}}, \quad n \geq 0, \quad B_n = \frac{24n^2 + 2n}{\prod_{i=0}^{2n-2} (2i + 1)^{2n-1-i}}, \quad n < 0. \]  
(3.19)

In the remaining part of this section, we will prove the relation (3.18) using representation theory.

Introduce the Heisenberg algebra with the generators \( a_r, r \in \mathbb{Z} + 1/2 \) and the relations \([a_r, a_s] = r \delta_{r+s,0}\). Consider the Fock module \( F \) generated by the highest weight vector \(|\emptyset\rangle\), which satisfies \( a_r |\emptyset\rangle = 0, \quad r > 0 \). Then, one can introduce the action of the algebra \( \text{Vir} \) with \( c = 1 \) by the formula
\[ \sum_{\in \mathbb{Z} + 1/2} a_{n-r} a_r : + \frac{1}{16} \delta_{n,0}, \]  
(3.20)

where \( : \ldots : \) is standard Heisenberg normal ordering. Our first goal is to describe \( F \) as a \( \text{Vir} \) module. This module cannot be irreducible since the Heisenberg algebra has half-integer indices, but \( \text{Vir} \) only has integer indices.

On \( F \) we have the scalar product defined by \( a_r^+ a_r \langle \emptyset | \emptyset \rangle = 1 \). This product is nondegenerate. Due to (3.20) we have \( L_n^+ = L_{-n} \). Therefore, this scalar product coincides with the Shapovalov form on the \( \text{Vir} \) submodules on \( F \).

Since \( [L_0, a_r] = -ra_r \) and \( L_0 |\emptyset\rangle = \frac{1}{16} |\emptyset\rangle \) the character of \( F \) equals
\[ \text{ch}(F) = \text{Tr}(z^{L_0}) = \frac{z^{1/16}}{\prod_{r=0}^{\infty} (1 - z^{r+1/2})}. \]

This formula can be rewritten using the Gauss relation
\[ \sum_{k=0}^{\infty} z^{k(k+1)/2} = \prod_{k=1}^{\infty} \frac{1 - z^{2k}}{1 - z^{2k+1}}, \]  
(3.21)

which follows from the Jacobi triple product identity
\[ \prod_{k=1}^{\infty} (1 - z^{2k})(1 + z^{2k-1}y^2)(1 + z^{2k-1}y^{-2}) = \sum_{k = -\infty}^{\infty} z^k y^{2k} \]  
(3.22)
after the substitution \( y \mapsto z \), and then \( z \mapsto \sqrt{z} \).

Now using (3.21) we get
\[ \text{ch}(F) = \sum_{n \in \mathbb{Z}} \frac{z^{(n+1/4)^2}}{\prod_{k=0}^{\infty} (1 - z^k)}. \]  
(3.23)

On the right-hand side of (3.23) we have the sum of the characters of the Verma modules \( \pi_{n}^{1/4}/\text{Vir} \). Moreover, we have the following:
Proposition 3.2. The Fock module $F$ is isomorphic to the direct sum of the Vir $c = 1$ Verma modules with the highest weight $(n + 1/4)^2$, $n \in \mathbb{Z}$

$$F \cong \bigoplus_{n \in \mathbb{Z}} \pi_{\text{Vir}}^{(n+1/4)^2}. \quad (3.24)$$

Proof. It is sufficient to prove the existence of the vectors $|(n + 1/4)^2\rangle$ such that

$$L_k((n + 1/4)^2) = 0, \text{ for } k > 0, \quad L_0((n + 1/4)^2) = (n + 1/4)^2(n + 1/4)^2 \quad (3.25)$$

and the submodules generated by $|(n + 1/4)^2\rangle$ are orthogonal with respect to the Shapovalov form on $F$. Indeed, the Verma modules on the Virasoro algebra with $c = 1$ and $\Delta = (n + 1/4)^2$ are irreducible (since the corresponding $\Delta$ are generic), therefore the vectors $|(n + 1/4)^2\rangle$ generate the Verma modules $\pi_{\text{Vir}}^{(n+1/4)^2}$. These modules are linearly independent due to their orthogonality, and their direct sum is isomorphic to $F$ due to the character identity (3.24).

The existence of vectors $|(n + 1/4)^2\rangle$ follows by induction. As a base we choose $|0 + 1/4)^2\rangle = |\varnothing\rangle$. Assume the existence of the vectors $|(n + 1/4)^2\rangle$ and $-m < n < m$, then we can define $|(m + 1/4)^2\rangle$ as the highest weight vector of the orthogonal complement $(\oplus_{n=-m+1}^{m-1} \pi_{\text{Vir}}^{(n+1/4)^2})^\perp$. The conditions (3.25) will be satisfied due to the orthogonality and character identity (3.23). Then, we define $|(m + 1/4)^2\rangle$ as the highest weight vector of the orthogonal complement $(\oplus_{n=-m+1}^{m-1} \pi_{\text{Vir}}^{(n+1/4)^2})^\perp$ and this finishes the induction step. \hfill \Box

The decomposition (3.24) was stated in [29] and [28] (and probably in other sources too). We will deduce the formula (3.18) from this decomposition. We will need explicit formulas for the vectors $|(n + 1/4)^2\rangle$, and these formulas are given in e.g. [7]. First, we should recall the so-called boson-fermion correspondence (see e.g. [19]).

Let us extend the Heisenberg algebra by the generators $a_n, n \in \mathbb{Z}, [a_n, a_m] = n \delta_{n+m,0}$, $[a_n, a_r] = 0$ and $r \in \mathbb{Z} + 1/2$. Also, consider the corresponding extension $F(k)$ of the Fock module $F$ with $a_0$ acting on this module as a number $k/\sqrt{2}$ and $k \in \mathbb{Z}$. We will denote the corresponding vacua vectors by $|\varnothing, k\rangle$. Consider then the sum $\bigoplus_{k \in \mathbb{Z}} F(k)$ and denote by $S$ the operator $S: F(k) \to F(k + 1)$ determined by the formulas

$$S|\varnothing, k\rangle = |\varnothing, k + 1\rangle, \quad [S, a_n] = 0, \text{ for } n \neq 0.$$ 

Introduce the operators $\tilde{\psi}_r, \tilde{\psi}^+_r$, and $r \in \mathbb{Z} + \frac{1}{4}$ by the formulas

$$\tilde{\psi}(z) = \sum_{r \in \mathbb{Z} + \frac{1}{4}} \tilde{\psi}_r z^{-r-1/4} = S z^w \sqrt{j} \exp \left( \sum_{2d \in \mathbb{Z}_{\geq 0}} \frac{z \exp (2d/\sqrt{2})}{j/\sqrt{2}} \right) \exp \left( \sum_{2d \in \mathbb{Z}_{\geq 0}} \frac{-z^{-1}}{j/\sqrt{2}} \right),$$

$$\tilde{\psi}^+(z) = \sum_{r \in \mathbb{Z} + \frac{1}{4}} \tilde{\psi}^+ r^{-r-1/4} = S^{-1} z^{-w} \sqrt{j} \exp \left( \sum_{2d \in \mathbb{Z}_{\geq 0}} \frac{z^{-1}}{j/\sqrt{2}} \right) \exp \left( \sum_{2d \in \mathbb{Z}_{\geq 0}} \frac{z^{-1}}{j/\sqrt{2}} \right). \quad (3.26)$$

The boson-fermion correspondence can be stated as follows:

Proposition 3.3. The operators $\tilde{\psi}_r, \tilde{\psi}^+_r$ satisfy the Clifford algebra relations:

$$\{\tilde{\psi}_r, \tilde{\psi}_s\} = 0, \quad \{\tilde{\psi}_r, \tilde{\psi}^+_s\} = \delta_{r+s,0}, \quad \{\tilde{\psi}_r, \tilde{\psi}^+_s\} = 0.$$
The space $\bigoplus_{k \in \mathbb{Z}} \mathcal{F}(k)$ is a Fock representation of this Clifford algebra generated by the vector $|\varnothing, 0\rangle$ such that
\[
\bar{\psi}_r |\varnothing, 0\rangle = \bar{\psi}_r^* |\varnothing, 0\rangle = 0, \quad \text{for} \ r > 0.
\]

The Heisenberg algebra generators $a_j$ and $j \in \frac{1}{2} \mathbb{Z}$ in terms of $\bar{\psi}_r, \bar{\psi}_r^*$ are given by the formula
\[
a_j = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{4}} \bar{\psi}_r \bar{\psi}_{j-r}^*,
\]
where the usual fermion normal ordering is used.

Remark 3.4. Our numeration of the indices differs from the standard one where the fermions have half-integer indices and the bosons have integer indices. This difference is nonessential for boson-fermion correspondence.

Now we can write the explicit formula for the vectors $|(n + 1/4)^2\rangle$ from the proposition 3.2.

Proposition 3.4. The highest weight vectors of $\mathcal{F}$ in $\mathcal{F}$ are given by
\[
\begin{align*}
|(n + 1/4)^2\rangle &= \prod_{k=1}^{n} \bar{\psi}_{-k+1/4} \bar{\psi}_{-k+1/4}^* |\varnothing, 0\rangle, \quad \text{for} \ n > 0, \\
|(n + 1/4)^2\rangle &= \prod_{k=1}^{-n} \bar{\psi}_{k-1/4} \bar{\psi}_{k-1/4}^* |\varnothing, 0\rangle, \quad \text{for} \ n < 0.
\end{align*}
\]

Proof. First, we prove that the vectors defined in (3.28) belong to $\mathcal{F} \subset \mathcal{F}(0)$, i.e. they only depend on $a_r$ and $r \in \frac{1}{2} \mathbb{Z}$. Indeed, $a_j |(n + 1/4)^2\rangle = 0$ and $j \in \mathbb{Z}_{n, 0}$ due to the commutation relations (which follow from (3.26))
\[
[a_j, \bar{\psi}_r] = \frac{1}{\sqrt{2}} \bar{\psi}_j \bar{\psi}_{j+r}, \quad [a_j, \bar{\psi}_r^*] = -\frac{1}{\sqrt{2}} \bar{\psi}_j \bar{\psi}_{j+r}, \quad \text{where} \ j \in \frac{1}{2} \mathbb{Z}, r \in \frac{1}{2} \mathbb{Z} + \frac{1}{4}.
\]

Also, $a_0 |(n + 1/4)^2\rangle = 0$. Then we prove that these vectors satisfy (3.25). This can be done by direct calculation using formulas (3.20) and (3.29). We will use another approach, and introduce the full Virasoro algebra generators by the formula
\[
L_{\text{full}}^n = \frac{1}{2} \sum_{j \in \mathbb{Z} + \frac{1}{2}} :a_{n-j}a_j r : + \frac{1}{16} \delta_{n, 0}.
\]

Then we have
\[
[L_{\text{full}}^n, \bar{\psi}_r] = (-\frac{n}{2} - r) \bar{\psi}_{n+r}, \quad [L_{\text{full}}^n, \bar{\psi}_r^*] = (-\frac{n}{2} - r) \bar{\psi}_{n+r}^*,
\]

Hence, we get the relation (3.25) for $L_{\text{full}}^n$. However, the vectors $|(n + 1/4)^2\rangle$ do not depend on $a_j$ or $j \in \mathbb{Z}$, so $L_{\text{full}}^k |(n + 1/4)^2\rangle = L_k |(n + 1/4)^2\rangle$ for $k > 0$. \qed

The natural scalar product on the space $\bigoplus_{k \in \mathbb{Z}} \mathcal{F}(k)$ is defined by the conjugation $\bar{\psi}_r^* = \bar{\psi}_{-r}^*$ and the unit norm of $|\varnothing, 0\rangle$. Then, it follows from (3.28) that the vectors $(n + 1/4)^2)$ have a unit norm. Due to the formula (3.27) this product is consistent with the Shapovalov form on the
Therefore, this product coincides with the Shapovalov form on the Virasoro submodules from (3.24).

Now we return to the proof of (3.18). Introduce the vectors $\ket{\emptyset^\pm_{\epsilon E} w z}$ e za 11 62 2 12. One can take any $\epsilon, \epsilon' \in \{-, +\}$ and calculate the scalar product

$$\langle w(1) | w(z) \rangle = z^{1/16} \sum_{i=0}^{\infty} \langle \epsilon \sigma_{i/2, i/4} \rangle \theta^{i+\epsilon} = z^{1/16} e^{\epsilon+\epsilon'}. $$

The result coincides with the right-hand side of (3.18).

On the other hand, due to (3.24) the vectors $|w(z)\rangle^{\epsilon}$ can be decomposed into orthogonal summands belonging to $\pi^{(a + 1/4)}$, $n \in \mathbb{Z}$. Moreover:

**Proposition 3.5.** The vectors $|w(z)\rangle^{\epsilon}$, $\epsilon = \pm$ decompose into a sum of the Vir Whittaker vectors, namely

$$|w(z)\rangle^{\epsilon} = \sum_{n \in \mathbb{Z}} l_n^{\epsilon} |W_n(z)\rangle, \quad |W_n(z)\rangle \in \pi^{(a + 1/4)}_n $$

with certain coefficients $l_n^{\epsilon}$ which do not depend on $z$.

**Proof.** It is enough to check that

$$L_1 |w(z)\rangle^{\pm} = z |w(z)\rangle^{\pm}, \quad L_2 |w(z)\rangle^{\pm} = 0. $$

This is done by a standard calculation. 

It is clear from the definition of $|w(z)\rangle^{\pm}$ that $l_n^{\epsilon} = (-1)^n l_n^{\epsilon}$. It follows from the definition that $l_n^{\epsilon}$ are given by the formula

$$l_n^{\epsilon} = \langle (n + 1/4)^2 | w(1) \rangle^{\epsilon}, $$

and we calculate them using the formulas (3.33).

**Proposition 3.6.** The coefficients $l_n^{\epsilon}$, $n \in \mathbb{Z}$ are given by the formula $l_n^{\epsilon} = \sqrt{B_n}$.

**Proof.** Recall that under the boson-fermion correspondence, the decomposable fermionic vectors correspond to the Schur polynomials ([19, Lec. 6]). In particular, the vectors on the right-hand side of (3.28) correspond to polynomials with a staircase diagram after the substitution $p_j \to \sqrt{2} a_{j/2}$, where $p_j$ are the power sum polynomials

$$|(n + 1/4)^2\rangle = S_{k,k-1,...,1}(\sqrt{2} a_{-1/2}, \sqrt{2} a_{-1}, \sqrt{2} a_{-3/2}, \ldots) |\emptyset\rangle, \quad k = \begin{cases} 2n, & n > 0 \\ -2n - 1, & n < 0 \end{cases} $$

(3.34)

The Schur polynomials satisfy $\langle p^{|\lambda|} | S_\lambda \rangle = |\lambda| / h(\lambda)$, where $h(\lambda) = \prod_{s \in \lambda} h(s)$ is the product of the hook length (see e.g. [21, section 1.4, example 3]). Therefore,

$$l_n^{\epsilon} = \langle \bigotimes_{k,k-1,...,1}(\sqrt{2} a_{-1/2}) \bigotimes \emptyset \rangle = 2^n \prod_{i=0}^{k-1} (2i + 1)^{-i} = \sqrt{B_n},$$

where $N = k(k + 1)/2$.
Now we calculate \( \langle w(1)|w(z)\rangle' \) using the decomposition (3.32) and proposition 3.6 and get
\[
\langle w(1)|w(z)\rangle' = \sum_{n \in \mathbb{Z}} (ee\gamma^\alpha B_n \mathcal{F}((n + 1/4)^2 z),
\]
which coincides with the left-hand side of (3.18).

**Remark 3.5.** Actually, the space \( \mathcal{F} \) has a natural basic module structure over \( \widehat{s}l_2 \). In particular, one can introduce the operator \( h_0 \) such that
\[
h_0 v = 2nv
\]
if \( \pi \in \pi_{\text{Vir}}^{(n+1/4)^2} \). Then, due to (3.7) and the calculations above one can write the Painlevé III(\( D_8 \)) function for \( \sigma = 14/2 \) as
\[
\sum_{n \in \mathbb{Z}} B_n (-s) \mathcal{F}((n + 1/4)^2 z) = -(w(1)|s^{1/2}|w(z))'.
\]
(3.35)

Taking into account the AGT correspondence [1], the right-hand side of this formula coincides with the dual partition function introduced in [23] (see equation (5.25) in loc.cit.).

4. The bilinear relations from the algebra \( F \oplus NSR \)

4.1. The algebra \( F \oplus NSR \) and its conformal blocks

The \( F \oplus NSR \) algebra is a direct sum of the free-fermion algebra \( F \) with the generators \( f_r \) \( (r \in \mathbb{Z} + \delta) \) and \( NSR \) (Neveu–Schwarz–Ramond or super Virasoro) algebra with the generators \( L_n, G_r \) \( (n \in \mathbb{Z}, r \in \mathbb{Z} + \delta, \delta = 0, 1/2) \). These generators satisfy the commutation relations
\[
\{f_r, f_s\} = \delta_{r+s,0}, \quad \{f_r, G_s\} = 0
\]
\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{(n^3 - n)}{8} c_{NSR} \delta_{n+m,0}
\]
\[
\{G_r, G_s\} = 2L_{r+s} + \frac{1}{2} c_{NSR} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0}
\]
\[
[L_n, G_s] = \left( \frac{1}{n} - r \right) G_{n+r}.
\]
(4.1)

It is convenient to express the central charge by
\[
c_{NSR} = 1 + 2Q^2, \quad Q = b + b^{-1}.
\]
The case of \( \delta = 1/2 \), i.e. the half-integer indices \( r \) of \( G_r \) and \( f_r \), is called the NS sector of the algebras defined above. The case of \( \delta = 0 \), i.e. the integer indices, is called the R sector. Note that \( c_{NSR} \) differs from the Virasoro central charge \( c = \frac{3}{2} c_{NSR} \).

Below, we will use a parametrization of the highest weight \( \Delta \) by
\[
\Delta^\delta = \frac{1 - 2\Delta}{16} + \frac{1}{2} \left( \frac{Q^2}{4} - p_1^2 \right),
\]
(4.2)
where \( \Delta^\delta \equiv \Delta^{NS} \) if \( \delta = 1/2 \), and \( \Delta^\delta \equiv \Delta^R \) if \( \delta = 0 \).

In the NS case, we denote by \( \pi_{F, NSR}^{\Delta^\delta} \) a Verma module of the \( F \oplus NSR \) algebra. This module is isomorphic to a tensor product of the Verma modules \( \pi_{F}^{\Delta^F} \) and \( \pi_{NSR}^{\Delta^R} \) which are generated by the highest weight vectors \( |1\rangle \) and \( |\Delta^{NS}\rangle \) defined by
\[
f_r |1\rangle = 0, \quad r > 0,
\]
\[
G_r |\Delta^{NS}\rangle = 0, \quad r > 0.
\]
and
\[ L_0|\Delta^{\text{NS}}\rangle = \Delta^{\text{NS}}|\Delta^{\text{NS}}\rangle, \quad L_n|\Delta^{\text{NS}}\rangle = 0, \quad G_\ell|\Delta^{\text{NS}}\rangle = 0, \quad n, \ell > 0. \]

We denote the highest weight vector $|1\rangle \otimes (\Delta^{\text{NS}})$ as $|\Delta^{\text{NS}}\rangle$.

In the $R$ sector, we have an analogous construction $\pi^{\text{F\#NSR}} = \pi^{\text{R}} \otimes \pi^{\text{F\#NSR}}$. Here, $\pi^{\text{R}}$ is a Verma module with two highest weight vectors $|\pm\rangle$, defined by
\[ f_r|\pm\rangle = 0, \quad r > 0, \quad f_0|\pm\rangle = \frac{1}{\sqrt{2}}|\pm\rangle. \quad (4.3) \]

By $\pi^{\text{F\#NSR}}$, we denote the Verma module with the highest weight vectors $|\Delta^{\text{R}}, \pm\rangle$ defined by
\[ G_\ell|\Delta^{\text{R}}, \pm\rangle = 0, \quad r > 0, \quad L_n|\Delta^{\text{R}}, \pm\rangle = 0, \quad n > 0, \quad G_0|\Delta^{\text{R}}, \pm\rangle = -\frac{i\nu}{\sqrt{2}}|\Delta^{\text{R}}, +\rangle, \quad L_0|\Delta^{\text{R}}, \pm\rangle = \Delta^{\text{R}}|\Delta^{\text{R}}, \pm\rangle. \quad (4.4) \]

Actually, the formula for the action $L_0$ follows from the relation $G_0^2 = L_0 - c_{\text{NSR}}/16$ and the parametrization (4.2).

Let us denote the highest weight vectors of $\pi^{\text{F\#NSR}}$ as
\[ |\mu\rangle \otimes |\Delta^{R}, \nu\rangle = |\mu; \Delta^{R}, \nu\rangle, \quad \mu, \nu = \pm. \quad (4.5) \]

We define the action of $G_0$ on the highest weight vector $|\mu; \Delta^{R}, \nu\rangle$ by
\[ G_0|\pm; \Delta^{R}, \nu\rangle = |1\rangle \otimes G_0|\Delta^{R}, \nu\rangle, \quad \nu = \pm. \]

Then, due to the anticommutativity of $f_0$ and $G_0$
\[ G_0|\mu; \Delta^{R}, \nu\rangle = \mu|\nu\rangle \otimes G_0|\Delta^{R}, \nu\rangle, \quad \mu, \nu = \pm. \quad (4.6) \]

Now let us define the irregular limit of the conformal block for the NSR algebra. First, we define the NSR Whittaker vectors $|W_\ell(z)\rangle$ for both sectors (see [16] for the $R$ sector, and [3] for the $\text{NS}$ sector)
\[ |W_{\text{NS}}(z)\rangle = z^{\Delta^{\text{NS}}} \sum_{2N = 0}^\infty z^N|N\rangle, \quad |N\rangle^{\text{NS}} \in \pi^{\text{NSR}}, \quad L_0|N\rangle^{\text{NS}} = (\Delta^{\text{NS}} + N)|N\rangle^{\text{NS}}, \quad (4.7) \]
\[ |W_{R, \pm}(z)\rangle = z^{\Delta^{\text{R}}} \sum_{2N = 0}^\infty z^N|N\rangle^{R, \pm}, \quad |N\rangle^{R, \pm} \in \pi^{\text{NSR}}, \quad L_0|N\rangle^{R, \pm} = (\Delta^{\text{R}} + N)|N\rangle^{R, \pm}, \quad (4.8) \]

where the vectors $|N\rangle^{\text{NS}}$ and $|N\rangle^{R, \pm}$ satisfy
\[ G_{1/2}|N\rangle^{\text{NS}} = |N - 1/2\rangle^{\text{NS}}, \quad N > 0, \quad G_{3/2}|N\rangle^{\text{NS}} = 0, \quad (4.9) \]
and
\[ L_4|N\rangle^{R, \pm} = 1/2 |N - 1\rangle^{R, \pm}, \quad N > 0, \quad G_4|N\rangle^{R, \pm} = 0. \quad (4.10) \]

Equivalently, in terms of Whittaker vectors, the last equations can be written as
\[ G_{1/2}|W_{\text{NS}}(z)\rangle = z^{1/2}|W_{\text{NS}}(z)\rangle, \quad G_r|W_{\text{NS}}(z)\rangle = 0, \quad r \geqslant 3/2 \quad (4.11) \]

J. Phys. A: Math. Theor. 50 (2017) 115205

M A Bershtein and A I Shchechkin
for the NS sector and

\[ L_r |W_{R,+}(z)\rangle = \frac{1}{2} z |W_{R,+}(z)\rangle, \quad G_r |W_{R,+}(z)\rangle = 0, r > 0 \] (4.12)

for the R sector. These conditions, coupled to the normalization of \( |0\rangle^\text{NS} = |\Delta^\text{NS}\rangle \), \( |0\rangle^\text{R} = \pm |\Delta^\text{R}\rangle \), completely determine the Whittaker vectors for generic values of \( \Delta \), as in the Virasoro case. To be more precise, we say that \( P \) is generic if \( P \in \mathbb{Z} \) and

\[ \alpha = \alpha \in \mathbb{Z}, \quad \beta = \beta \in \mathbb{Z}, \quad \gamma = \gamma \in \mathbb{Z}, \]

for the corresponding values of \( \Delta \) the Whittaker vector is well defined.

The formulas for the action of the operators \( L_k, G_r \) for \( k > 1, r > 1/2 \) on \( |W_{NS}(z)\rangle, |W_{R,+}(z)\rangle \) follow from the NSR commutation relations.

We define the complex Shapovalov form on

\[ \pi \oplus \Delta^\text{NS}, \quad \pi \oplus \Delta^\text{NS}, \]

and

\[ \pi \oplus \Delta^\text{R}, \quad \pi \oplus \Delta^\text{R}, \]

in which the conjugation of the operators is

\[ \gamma = \gamma \in \mathbb{Z}, \quad \delta = \delta \in \mathbb{Z}, \quad \epsilon = \epsilon \in \mathbb{Z}, \]

(4.13)

In order to define the scalar product completely, we should fix the Shapovalov form on the highest weight vectors. In the NS sector, we only have one highest weight vector \( |\Delta^\text{NS}\rangle \), which we normalize so that \( \langle \Delta^\text{NS} | \Delta^\text{NS} \rangle = 1 \). In the R sector, we have additional scalar products between the different highest weight vectors. Due to the skew-symmetry of the \( f_0 \) operator, we have that \( \langle \Gamma^\ast | \Gamma \rangle = 0 \) and \( \langle \Gamma^\ast | \Gamma' \rangle = -\langle \Gamma | \Gamma' \rangle \). We normalize \( \langle \Gamma^\ast | \Gamma \rangle = -\langle \Gamma | \Gamma^\ast \rangle = 1 \); we also state \( \langle \Delta^\text{R}, + | \Delta^\text{R}, - \rangle = 0 \) and \( \langle \Delta^\text{R}, - | \Delta^\text{R}, + \rangle = 1 \).

The irregular (or Whittaker or Gaiotto) limit of the NSR conformal block is defined by

\[ F^\text{NSR}(\Delta^\text{NS}|z) = \langle W_{NS}(1)|W_{NS}(z)\rangle, \quad F^\text{NSR}(\Delta^\text{R}|z) = \langle W_{R,+}(1)|W_{R,+}(z)\rangle, \] (4.14)

where the R conformal block does not depend on the subscript of the R Whittaker vector. This is because the properties (4.4) have symmetry in the interchange of vectors \( |\Delta^\text{R}\rangle, \pm \).

**Remark 4.1.** The scalar product \( \langle \Delta^\text{R}, + | \Delta^\text{R}, - \rangle \) is not zero in general, but one can choose a different pair of highest weight vectors using the formulas

\[ |\Delta^\text{R}, + \rangle = |\Delta^\text{R}, + \rangle - \alpha |\Delta^\text{R}, - \rangle, \quad |\Delta^\text{R}, - \rangle = |\Delta^\text{R}, - \rangle - \alpha |\Delta^\text{R}, + \rangle \]

where the non-primed vectors are normalized on 1. Indeed, \( |\Delta^\text{R}, \pm \rangle \) satisfies (4.4) just as well as \( |\Delta^\text{R}, \pm \rangle \). Moreover, if \( \alpha \) is a solution of the equation

\[ 0 = (1 + \alpha^2)\langle \Delta^\text{R}, + | \Delta^\text{R}, - \rangle - 2\alpha \]

then we get \( \langle \Delta^\text{R}, - | \Delta^\text{R}, + \rangle = 0 \).

### 4.2. Verma module decomposition

Let us recall the free-field realization of the NSR algebra. Consider the algebra generated by \( c_n, n \in \mathbb{Z} \) and \( \psi_r, r \in \mathbb{Z} + \delta \) (free boson and free fermion) with relations

\[ [c_n, c_m] = n\delta_{n+m,0}, \quad [c_n, \psi_r] = 0, \quad [\psi_r, \psi_s] = \delta_{r+s,0}. \]

Consider two sets of such generators, which we will distinguish by the superscript \( \mp \), and add the zero mode \( c_0^\mp \) to them as \( \mp i\hat{p} \). We will omit the superscript when it is not confusing. Then,
a Fock representation of this algebra is generated by the vacuum vector $|P\rangle$ for the NS sector and the vacuum vectors $|P\rangle^\pm$ for the R sector. The vector $|P\rangle$ satisfies $\psi_0|P\rangle = c_0|P\rangle = 0$ and $\hat{P}|P\rangle = P|P\rangle$ for $r, n > 0$ and analogously for the vectors $|P\rangle^\pm$. For the R case, we should also add $\psi_0^\pm|P\rangle^\mu = \pm(1/\sqrt{2})|P\rangle^\mu$, $\mu = \pm$. On this Fock module, we can define the action of the NSR algebra by the formulas

$$L_n = \frac{1}{2} \sum_{k=0,n} c_k c_{n-k} + \frac{1}{2} \sum_r r_\psi n - r_\psi r + \frac{i}{2} (Qn \mp \hat{P}) c_n, \quad n \neq 0,$$

$$L_0 = \sum_{k \geq 0} c_k c_k + \sum_{r > 0} r_\psi n - r_\psi r + \frac{1 - 2b}{16} + \frac{1}{2} \left\{ Q^2 - \hat{P}^2 \right\},$$

$$G_r = \sum_{n=0} c_n r_{n-r} + i (Qr \mp \hat{P}) \psi_r.$$ (4.15)

The operators $c_n^\pm, \psi_r^\pm$ with two different signs give us two different free-field realizations of the NSR algebra.

Recall that we say that $P$ is generic if $P \notin \{ n(b + nb^{-1}) | m, n \in \mathbb{Z} \}$. For a generic $P$, the NSR module defined by (4.15) is irreducible (see e.g. case V in [14]), and is isomorphic to the Verma module $\pi_{\text{NSR}}$ in the NS sector and to $\pi_{\text{NSR}}^\perp$ in the R sector. The weights $\Delta_{\text{NS}}$ and $\Delta_{\text{R}}$ are defined by (4.2). The operators $c_n^\pm, \psi_n^\pm$ acting on $\pi_{\text{NSR}}^\perp$ are conjugated by the so-called super-Liouville reflection operator.

As the main tool, we will use the action of the algebra $\text{Vir} \oplus \overline{\text{Vir}}$ on the representations $\pi_{\text{NSR}}^\perp, \delta = 0, 1/2$ (following [8, 20]). The generators of the algebra $\text{Vir} \oplus \overline{\text{Vir}}$ are defined by the formulas

$$L_n^{(1)} = \frac{b^{-1}}{b^{-1} - b} L_n^{(1)} - \frac{b^{-1} + 2b}{b^{-1} - b} \left( \frac{1}{2} \sum_{r \in \mathbb{Z} + b} r : f_{n-r} f_r : + \frac{1 - 2b}{16} \delta_{n,0} \right) + \frac{1}{b^{-1} - b} \sum_{r \in \mathbb{Z} + b} f_{n-r} G_r,$$

$$L_n^{(2)} = \frac{b}{b - b^{-1}} L_n^{(1)} - \frac{b + 2b^{-1}}{b - b^{-1}} \left( \frac{1}{2} \sum_{r \in \mathbb{Z} + b} r : f_{n-r} f_r : + \frac{1 - 2b}{16} \delta_{n,0} \right) + \frac{1}{b - b^{-1}} \sum_{r \in \mathbb{Z} + b} f_{n-r} G_r.$$ (4.16)

The expression in parentheses from the second summand is just the expression for $L_n^e$, which defines the representation of the algebra $c = 1/2$ algebra on $\pi_{\overline{\text{Vir}}}$ in the appropriate sector. Note that the expressions for $L_n^{(1)}$ and $\eta = 1, 2$ contain infinite sums and belong to a certain completion of the universal enveloping algebra of $\mathfrak{F} \oplus \text{NSR}$. The conjugation of $L_n^{(1)}$ is standard: $L_n^{(1)\perp} = L_n^{(1)\perp}$ due to (4.13).

It is convenient to express the central charge and the highest weights of the Virasoro algebra by

$$\Delta(P, b) = \frac{Q^2}{4} - P^2, \quad c(b) = 1 + 6Q^2, \quad \text{where} \quad Q = b + b^{-1}. \quad (4.17)$$

Then, the central charges of the $\text{Vir}^{(1)}$ and $\text{Vir}^{(2)}$ subalgebras are equal to

$$c^{(1)} = c(b^{(1)}), \quad \eta = 1, 2, \quad (b^{(1)})^2 = \frac{2b^2}{1 - b^2}, \quad (b^{(2)})^2 = \frac{2b^{-2}}{1 - b^{-2}}.$$ (4.18)

Note that the symmetry $b \leftrightarrow b^{-1}$ permutes $\text{Vir}^{(1)}$ and $\text{Vir}^{(2)}$, here and below $b^2 \neq 0, 1$. 

20
Now consider the space \( \pi^{\Delta_{NS}}_{F\oplus\text{NSR}} \) as a representation of \( \text{Vir} \oplus \text{Vir} \). Clearly, the vector \( |\Delta^{(1)}_{NS}\rangle = |1\rangle \otimes |\Delta^{(0)}_{NS}\rangle \) is the highest weight vector with respect to \( \text{Vir} \oplus \text{Vir} \). This vector generates a Verma module \( \pi^{\Delta^{(1)},\Delta^{(0)}}_{\text{Vir}\oplus\text{Vir}} \). The highest weight \( (\Delta^{(1)}, \Delta^{(2)}) \) can be found from (4.16), namely

\[
\Delta^{(1)} = \frac{b^{-1} - b}{b^{-1} - b^{-1}} \Delta_{NS}, \quad \Delta^{(2)} = \frac{b}{b - b^{-1}} \Delta_{NS}
\]  

(4.19)

The whole space \( \pi^{\Delta_{NS}}_{F\oplus\text{NSR}} \) is larger than \( \pi^{\Delta^{(1)},\Delta^{(0)}}_{\text{Vir}\oplus\text{Vir}} \). The following decomposition was proven in [2].

**Theorem 4.1.** For a generic \( P \), the space \( \pi^{\Delta_{NS}}_{F\oplus\text{NSR}} \) is isomorphic to the sum of the \( \text{Vir} \oplus \text{Vir} \) modules

\[
\pi^{\Delta_{NS}}_{F\oplus\text{NSR}} \cong \bigoplus_{2n \in \mathbb{Z}} \pi^{n}_{\text{Vir}\oplus\text{Vir}}.
\]  

(4.20)

The highest weight \( (\Delta^{(1)}_{n}, \Delta^{(2)}_{n}) \) of the Verma module \( \pi^{n}_{\text{Vir}\oplus\text{Vir}} \) is defined by \( \Delta^{(1)}_{n} = \Delta(P^{(n)}_{\eta}, b^{(n)}) \), \( \eta = 1, 2 \), where

\[
P^{(1)}_{n} = P^{(1)} + nb^{(1)}, \quad P^{(2)}_{n} = P^{(2)} + n(b^{(2)})^{-1}, \quad P^{(1)} = \frac{P}{\sqrt{2 - 2b}}, \quad P^{(2)} = \frac{P}{\sqrt{2 - 2b^{-1}}}
\]  

(4.21)

The highest weight vectors of \( \pi^{n}_{\text{Vir}\oplus\text{Vir}} \) are given by the formula

\[
|P, n\rangle \propto \prod_{r=1/2}^{(4n-1)/2} \chi^{\pm}_{r} \langle \Delta^{(n)}_{NS} \rangle, \quad n > 0, \quad |P, n\rangle \propto \prod_{r=1/2}^{(-4n-1)/2} \chi^{\mp}_{r} \langle \Delta^{(n)}_{NS} \rangle, \quad n < 0, \quad |P, 0\rangle = \langle \Delta^{(0)}_{NS} \rangle.
\]  

(4.22)

where \( \chi^{\pm}_{r} = f_{r} - i\psi^{\pm}_{r} \).

Let us now formulate and prove the analogous decomposition for the Ramond sector.

Introduce the parity operator \( \mathbb{P} : \pi^{\Delta_{NS}}_{F\oplus\text{NSR}} \rightarrow \pi^{\Delta_{NS}}_{F\oplus\text{NSR}} \). The operator \( \mathbb{P} \) is defined by the action \( \mathbb{P}|+\rangle_{\Delta^{(s)}, +} = |+\rangle_{\Delta^{(s)}, +} \), and the property in which \( \mathbb{P} \) commutes with even operators and anticommutes with odd operators of the \( F \oplus \text{NSR} \) algebra.

Evidently, \( \mathbb{P}^{2} = 1 \) and the operators \( \mathbb{P}^{1/2} \) are projectors on the \( \mathbb{P} \) eigenspaces with the eigenvalues 1 and \(-1\), which we will call even and odd subspaces respectively. We will mark the objects related to these subspaces by the index \( \varepsilon \) equals 0 for the even subspace and 1 for the odd subspace. So, \( \pi^{\Delta_{R}}_{F\oplus\text{NSR}} \) decomposes into a direct sum of \( \pi^{\Delta_{R}}_{F\oplus\text{NSR}}^{0} \) and \( \pi^{\Delta_{R}}_{F\oplus\text{NSR}}^{1} \).

The operator \( \mathbb{P} \) can also be defined separately on \( \pi^{\Delta_{R}}_{F\oplus\text{NSR}} \) and \( \pi^{\Delta_{NS}}_{F\oplus\text{NSR}} \) in an obvious way. The operator defined above is just the tensor product of these operators. To make the last definition precise, we need to add the condition \( \mathbb{P}|1^{+}\rangle = |1^{+}\rangle \). This decomposition is represented schematically in figure 1.

It appears that the summands \( \pi^{\Delta_{R}}_{F\oplus\text{NSR}}^{\varepsilon} \) decompose into a direct sum of the \( \text{Vir} \oplus \text{Vir} \) modules. Namely we have:

**Theorem 4.2.** For a generic \( P \), the space \( \pi^{\Delta_{R}}_{F\oplus\text{NSR}} \) is isomorphic to the sum of the \( \text{Vir} \oplus \text{Vir} \) modules

\[
\pi^{\Delta_{R}}_{F\oplus\text{NSR}} \cong \bigoplus_{2n \in \mathbb{Z}} \pi^{n}_{\text{Vir}\oplus\text{Vir}}^{0} \oplus \bigoplus_{2n+1/2 \in \mathbb{Z}} \pi^{n}_{\text{Vir}\oplus\text{Vir}}^{1}.
\]  

(4.23)
where the highest weights of the modules $\pi^{n,\epsilon}_{\text{Vir} \oplus \text{Vir}}$ are $(\Delta^{(1)}_n, \Delta^{(2)}_n)$, defined in theorem 4.1; the superscript $\epsilon = 0, 1$ denotes parity. This decomposition is represented schematically in figure 2.

**Proof.** The arguments are analogous to the proof of the theorem 4.1 (see [2, 4]).

First let us find the highest weight vectors for the $\text{Vir} \oplus \text{Vir}$ algebra in $\pi \oplus \Delta_{FN \text{SR}}$. We will denote the highest weight vectors of $\pi^{\epsilon}_{\text{Vir} \oplus \text{Vir}}$, $\epsilon = 0, 1$, with the weights $\eta_{\text{L}} = \eta_{\text{L}}(0), \eta = 1, 2$.

In $\pi^{\epsilon}_{FN \text{SR}}$, we have four highest weight vectors $\mu \nu_{\text{L}}$, $\mu, \nu = \pm 1, 2$. Clearly, $\mu \nu_{\text{L}} = \eta_{\text{L}}(0), \eta = 1, 2$.

The vectors $\mu_{\text{L}}$ and $\mu = \pm$ are even, and the vectors $\mu_{\text{L}} - \nu = \eta_{\text{L}}(0), \eta = 1, 2$.

The following linear combinations are eigenvectors of $\eta_{\text{L}}(0)$, i.e. the highest weight vectors of the $\text{Vir} \oplus \text{Vir}$ algebra

$$\eta_{\text{L}}(0) \chi_{\text{L}} = \eta_{\text{L}}(0) \chi_{\text{L}}$$

where we used $\chi_{\text{L}} = f_{\epsilon} - i \psi_{\text{L}}^{-\epsilon}$ as for the NS case but with integer indices. Using (4.16) and (4.15), one can calculate the commutation relations between $L_{\text{L}}^{(i)}$, $\eta = 1, 2$ and $\chi_{\text{L}}$

$$[L_{\text{L}}^{(i)} + L_{\text{L}}^{(2)}, \chi_{\text{L}}] = -\left(\frac{n}{2} + r\right) \chi_{r+n},$$

$$[b_{\text{L}}^{(1)} + b_{\text{L}}^{(2)}, \chi_{\text{L}}] = -(n + r) \hat{Q} + \hat{P}) \chi_{r+n} + i \sum_{m=0}^{n} c_m \chi_{r+n-m}.$$  

(See [2, equation (3.23)] for the NS sector.) Then, it is easy to check that for the $2n \in \mathbb{Z}, n \geq 0$ vectors
are the highest weight vectors for the Vir ⊕ Vir algebra with the appropriate highest weight. Given by (4.16), the generators of the Vir ⊕ Vir algebra are even, so the vectors |ε\n, +\rangle generate Verma modules π^{⊕ε}_Vir with parity ε. For a generic P, the Verma modules π^{⊕ε}_Vir with certain parity ε are irreducible and linearly independent. The linear independence between the 0 and 1 modules with the same highest weight follows from the different parity of the modules.

From (4.21), it follows that ∆_R^{(1)} + ∆_R^{(2)} = ∆_R - 1/16 + 2n^2.

In order to finish the proof of the decomposition (4.23), it is sufficient to check the equality of the characters. Indeed,

\[
\text{ch}(\pi^{NSR}_F) = z^{2n+1/16} \frac{\prod_{k=0}^{\infty} (1 + z^k)^2}{\prod_{k=1}^{\infty} (1 - z^k)} = 2 \sum_{2n \in \mathbb{Z} + 1/2} z^{2n+2n^2 - 1/16} \prod_{k=1}^{\infty} \frac{1}{(1 - z^k)^2} = \sum_{2n \in \mathbb{Z} + 1/2} \text{ch}(\pi^{\oplus}_\text{Vir}^{\oplus}_{\text{Vir}}),
\]

where we used the Jacobi triple product identity (3.22) in the case z \mapsto \sqrt{z}, y^2 \mapsto \sqrt{z}. \quad \square
Remark 4.2. We have the isomorphism of modules $\pi_{F_0}^{\oplus_{NSR}} \cong \pi_{F_0}^{\oplus_{NSR}}$, so it is convenient to consider only one sector in the calculations below.

4.3. The Whittaker vector decomposition for the NS sector; Toda-like equations

In [4, remark 4.1] certain bilinear relations on conformal blocks were considered. In the case of the $c = 1$ conformal blocks, these relations do not provide a differential equation on the $\tau$ function given by (3.7). However, these relations do provide the differential-difference equation (3.15) on this $\tau$ function. We first recall these bilinear relations; this part is a brief version of the arguments in [4, section 4.2.]. Then we deduce (3.15) from the bilinear relations on the conformal blocks; this part is not written in [4], but is similar to the arguments in loc.cit.

The $F \oplus_{NSR}$ Whittaker vector $|t \otimes W_{NSR}(z)|$ of the NS sector is defined as a tensor product of the $F$ vacuum, and the $NSR$ Whittaker vector. The following proposition was proven in [4, section 3.2].

Proposition 4.1. The decomposition of the $F \oplus_{NSR}$ Whittaker vector of the NS sector in terms of the subalgebra $\mathfrak{Vir} \oplus \mathfrak{Vir}$ has the form

$$|t \otimes W_{NSR}(z)| = \sum_{2n \in \mathbb{Z}} l_n(P, b) (|W^{(1)}(\beta^{(1)}(z))_n \otimes |W^{(2)}(\beta^{(2)}(z))_n|).$$

(4.27)

Here, $|W^{(1)}(\beta^{(1)}(z))_n \otimes |W^{(2)}(\beta^{(2)}(z))_n|$ denotes the tensor product of the Whittaker vectors in $\pi_{Vir \oplus Vir}$, and the coefficients $l_n(P, b)$ do not depend on $z$. The parameters $\beta^{(\eta)}, \eta = 1, 2$ are defined by the formulas

$$\beta^{(1)} = \left(\frac{b^{-1}}{b - b^{-1}}\right)^2, \quad \beta^{(2)} = \left(\frac{b}{b - b^{-1}}\right)^2.$$  

(4.28)

Introduce the functions

$$s_{\text{even}}(x, n) = \prod_{i,j \geq 0, i+j \leq 2n \atop i+j \equiv 0 \mod 2} (x + ib + jb^{-1}), \quad s_{\text{odd}}(x, n) = 2^{1/8} \prod_{i,j \geq 0, i+j \leq 2n \atop i+j \equiv 1 \mod 2} (x + ib + jb^{-1}),$$

for $n \geq 0$ and

$$s_{\text{even}}(x, n) = (-1)^n s_{\text{even}}(Q - x, -n), \quad s_{\text{odd}}(x, n) = s_{\text{odd}}(Q - x, -n)$$

(4.29)

for $n < 0$. We will use the function $s_{\text{odd}}$ in section 4.4. The coefficients $l_n(P, b)$ in the formula (4.27) were calculated in [18] and [4, section 3.3].

Proposition 4.2. The coefficients $l_n(P, b)$ are given by

$$l_n(P, b) = \frac{\text{det}(\frac{b^{-1}}{b - b^{-1}})(\beta^{(1)})^\Delta_1(\beta^{(2)})^\Delta_2)}{s_{\text{even}}(2P, 2n) s_{\text{even}}(2P + Q, 2n)}.$$  

(4.31)

Note that the normalization $(P, n|P, n) = 1$ determines the vectors $|P, n|$ only up to a sign; therefore, the coefficients $l_n(P, b)$ are also determined up to a sign. However, $l_n(P, b)$ will only appear in the bilinear relation as $l_n^2(P, b)$.

Let us introduce operator $H$:

$$H = bL^{(1)}_0 + b^{-1}L^{(2)}_0$$

(4.32)

and define $\widehat{F}_{\Delta}$ by the formulas
We can calculate $\mathcal{F}_k$ in two ways. The first calculation is in terms of the \textit{Vir} $\oplus$ \textit{Vir} generators and the modules using proposition 4.1, and it gives

$$\mathcal{F}_k = \sum_{2n \in \mathbb{Z}} \tilde{I}_n^2(P, b) \cdot D_{h, b \cdot \log z}^2 (\mathcal{F}_n^{(1)}, \mathcal{F}_n^{(2)}),$$

(4.34)

where we used the shorter notation $\mathcal{F}_n^{(k)}$ for $\mathcal{F}_n(\Omega_{a | b}^{(k)})$ and the generalized Hirota differential operators $D_{e_1, e_2}^{(k)}$ are defined by

$$f(e^\sigma z)g(e^\sigma z) = \sum_{n=0}^{\infty} D_{e_1, e_2}^n(f(z), g(z)) \frac{\Delta^n}{n!},$$

(4.35)

where we take the derivatives with respect to the logarithm of the variable as before.

In another way, using the formulas (4.16) we have

$$\sum_{r \in \mathbb{Z} + 1/2} r \cdot \mathcal{F}_k = -z^{1/2} \mathcal{F}_{\text{NS}},$$

(4.36)

Then we calculate $\mathcal{F}_k$ in terms of the $\mathcal{F} \oplus \text{NSR}$ generators and modules. This gives for $k = 0, 2$

$$\mathcal{F}_0 = \mathcal{F}_{\text{NS}}, \quad \mathcal{F}_2 = -z^{1/2} \mathcal{F}_{\text{NS}}.$$  

(4.37)

where we used the shorter notation $\mathcal{F}_{\text{NS}}$ for $\mathcal{F}_{\text{NS}}(\Delta_{\text{NS}} | z)$. Eliminating $\mathcal{F}_{\text{NS}}$ from this equation and using (4.34) we have

$$\sum_{2n \in \mathbb{Z}} \tilde{I}_n^2(P, b) \cdot D_{h, b \cdot \log z}^2 (\mathcal{F}_n^{(1)}, \mathcal{F}_n^{(2)}) = -z^{1/2} \sum_{2n \in \mathbb{Z}} \tilde{I}_n^2(P, b) \mathcal{F}_n^{(1)} \mathcal{F}_n^{(2)}.$$  

(4.38)

This is just the relation written in [4, remark 4.1].

Now we deduce the Toda-like equation (3.15) on the $\tau$ function (3.7) from the equation (4.38). Let us substitute this $\tau$ function decomposition into (3.15) and collect the terms with the same powers of $s$. The vanishing condition of the $s^m$ coefficient has the form

$$\sum_{n \in \mathbb{Z}} C(\sigma + n + m) C(\sigma - n) \{ 1/2 D_{\log z}^2 (\mathcal{F}(\sigma + n + m)^2 | z), \mathcal{F}(\sigma - n)^2 | z)$$

$$+ z^{1/2} \mathcal{F}((\sigma + 1/2 + n + m)^2 | z) \mathcal{F}((\sigma - 1/2 - n)^2 | z) \} = 0.$$  

Clearly this $s^m$ coefficient coincides with the $s^{m+2}$ coefficient after the shift $\sigma \mapsto \sigma + 1$. Therefore, it is sufficient to prove the vanishing of the $s^0$ and $s^1$ coefficients.

To obtain these bilinear relations we set in (4.38) $\mathcal{F}_{\text{NSR}} = 1 \in b = i$, $c^{(0)} = 1$, $\eta = 1, 2$. We also substitute $P = 2i \sigma, z \mapsto 4z$. Splitting (4.38) into the relations with powers $z^{2\sigma + N}, N \in \mathbb{Z}$ and $z^{2\sigma + N}, N \in \mathbb{Z} + 1/2$ and shifting $n$ and $\sigma$ in an appropriate way we correspondingly obtain

$$\sum_{n \in \mathbb{Z}} \tilde{I}_n^2(\sigma) D_{\log z}^2 (\mathcal{F}(\sigma + n)^2 | z), \mathcal{F}(\sigma - n)^2 | z)$$

$$= 2z^{1/2} \sum_{n \in \mathbb{Z}} \tilde{I}_n^2(\sigma + 1/2) \mathcal{F}(\sigma + 1/2 + n)^2 | z) \mathcal{F}(\sigma - 1/2 - n)^2 | z),$$

$$\sum_{n \in \mathbb{Z}} \tilde{I}_n^2(\sigma + 1/2) D_{\log z}^2 (\mathcal{F}(\sigma + n + 1)^2 | z), \mathcal{F}(\sigma - n)^2 | z)$$

$$= 2z^{1/2} \sum_{n \in \mathbb{Z}} \tilde{I}_n^2(\sigma + 1/2) \mathcal{F}(\sigma + 3/2 + n)^2 | z) \mathcal{F}(\sigma - 1/2 - n)^2 | z),$$
where we used shorter notation \( l_n(\sigma) \) for \( l_n(2i\sigma, i) \). It remains to compare the coefficients of these relations and the \( s^0 \) and \( s^1 \) relations. Using (4.31) and the shift relation on \( G(\sigma) \) we get

\[
C(\sigma + n)C(\sigma - n) = \frac{1}{2^{2|n|-1} \prod_{k=1}^{2|n|} (k^2 - 4\sigma^2)^{2|n| - k} (4\sigma^2)^2 |n|} = 4^{-2\sigma^2}(-1)^{2n}l_n^2(\sigma), \tag{4.39}
\]

where \( 2n \in \mathbb{Z} \). Then we have for \( n \in \mathbb{Z} \)

\[
\frac{\ell_n^2(\sigma)}{\ell_n^2(\sigma)} = \frac{C(\sigma + n)C(\sigma - n)}{C(\sigma)^2}, \quad \frac{\ell_{n+\frac{1}{2}}^2(\sigma)}{\ell_n^2(\sigma)} = \frac{C(\sigma + \frac{1}{2} + n)C(\sigma - n - \frac{1}{2})}{C(\sigma)^2},
\]

\[
\frac{\ell_{n+\frac{1}{2}}^2(\sigma + \frac{1}{2})}{\ell_n^2(\sigma)} = \frac{C(\sigma + n + \frac{1}{2})C(\sigma - n - \frac{1}{2})}{C(\sigma + \frac{1}{2})^2}, \quad \frac{\ell_{n+\frac{1}{2}}^2(\sigma + \frac{1}{2})}{\ell_n^2(\sigma)} = \frac{C(\sigma + n + \frac{1}{2})C(\sigma - n - \frac{1}{2})}{C(\sigma + \frac{1}{2})^2},
\]

which completes the proof.

**Remark 4.3.** From the results of section 2 and the proposition 3.1, it follows that the left-hand side of (3.7) satisfies (3.15), and this determines the function. Here we proved that the right-hand side of (3.7) satisfies (3.15), i.e. as a byproduct we proved (3.7). This simplifies the proof of theorem 3.1 in comparison with [4], because we do not consider equation (2.9) of order four.

On the other hand, in section 2 we proved that the functions \( \tau(z) \) and \( \pi(\tau(z)) \) satisfy the Toda-like equations (2.24). Here we proved that \( \tau(\sigma, z) \) and \( \tau(\sigma - 1/2, z) \) satisfy these Toda-like equations. So, as a byproduct we have another proof of proposition 3.1. Note that the obtained proof is rather difficult, since it uses theorem 3.1.

### 4.4. The Whittaker vector decomposition for the R sector; Okamoto-like equations

In this section we will obtain the bilinear relations on the conformal blocks in the \( R \) sector analogously to the \( NS \) sector. In terms of the \( \tau \) functions, these relations for \( c = 1 \) will have the same form as the Okamoto-like equations (3.12) and (3.13). However, we do not give a representational theoretic proof of these equations since we do not find explicit formulas for the coefficients \( l_n \), see conjecture 4.1 below.

We have four \( F \oplus NSR \) Whittaker vectors \( z^{1/16}| \mu, \nu \rangle_{W_k} \) and the \( \mu, \nu = \pm \) of the \( R \) sector, where \( 1/16 \) is the conformal dimension of the fermionic vacuum. These Whittaker vectors start from the highest weight vectors \( | \mu; \Delta^R, \nu \rangle \) with certain parity \( \mu \nu \) (notation \( \mu \nu \) means the product of the signs in the natural sense). Moreover, we have the following:

**Lemma 4.1.** The Whittaker vectors \( | W_{R, \pm} \rangle \) have a certain parity

\[
| \Psi | W_{R, \pm} \rangle = \pm | W_{R, \pm} \rangle.
\]

**Proof.** The vector \( \Psi | W_{R, \pm} \rangle \) satisfies the properties (4.12) and starts from \( \pm z^{16} | \Delta^R, \pm \rangle \). As was mentioned in section 4.1, these properties and the normalization condition determine the Whittaker vector.

Then, analogously to proposition 4.1, we have the following:
Proposition 4.3. The decomposition of the $F \oplus \text{NSR}$ Whittaker vector of the $R$ sector in terms of the subalgebra $\text{Vir} \oplus \text{Vir}$ has the form

$$z^{1/16} |\mathcal{P} \otimes W_{R,\epsilon}(z)\rangle = \sum_{\mu, \nu \in \mathbb{Z}} l_n^{\mu,\nu}(P, b) (|W^{(1)}(\beta^{(1)}z)_n \otimes |W^{(2)}(\beta^{(2)}z)_n|).$$

(4.40)

Here, $|W^{(1)}_n \otimes |W^{(2)}_n|$ denotes the tensor product of the Whittaker vectors of $\pi_{\text{Vir} \oplus \text{Vir}}$, and the coefficients $l_n^{\mu,\nu}(P, b)$ do not depend on $z$. The parameters $\beta^{(1,2)}$, $\eta = 1, 2$ are defined by the formulas (4.28).

As in NS, the sector coefficients $l_n^{\mu,\nu}$ are determined up to the sign. However, the ratio of $l_n^{\mu,\nu}$ with the same parity is well defined, because the decomposition (4.40) starts with the same highest weight vectors. Below, we will only reach the relations for even subspaces (see remark 4.2). Then:

Proposition 4.4. We have the relations between $l_n^{+\pm}(P, b)$ and $l_n^{-\pm}(P, b)$

$$l_n^{-\pm}(P, b) = (-1)^{2n+1/2} l_n^{+\mp}(P, b).$$

(4.41)

Proof. The coefficients $l_n^{\mu,\nu}(P, b)$ are given by

$$l_n^{\mu,\nu}(P, b) = \langle \mu, n | \mathcal{P} \otimes W_{R,\epsilon}(1) \rangle.$$ 

We have the isomorphism between the module $\pi_{F \oplus \text{NSR}}$ with the highest weight vectors $|\mu; \Delta^R, \nu\rangle$, the momentum $P$, the same module with the transposed highest weight vectors $|\mu; \Delta^R, \nu\rangle = i\epsilon |\mu; \Delta^R, -\nu\rangle$ and the same momentum $P$, $\mu, \nu = \pm$ (we denote this module by $\pi_{F \oplus \text{NSR}}$). This isomorphism follows from the properties (4.3), (4.4) and the normalization condition. Consider then the $F \oplus \text{NSR}$ Whittaker vector $z^{1/16} |\mathcal{P} \otimes W_{R,\epsilon}(z)\rangle$ in the module $\pi_{F \oplus \text{NSR}}$. Due to the properties (4.12), this vector is also a Whittaker vector $z^{1/16} |\mathcal{P} \otimes W_{R,-\epsilon}(z)\rangle$ in $\pi_{F \oplus \text{NSR}}$, and its normalization is standard.

We also have that the $\text{Vir} \oplus \text{Vir}$ highest weight vectors $|P, \pm 1/4\rangle$, $\epsilon = 0, 1$ in $\pi_{F \oplus \text{NSR}}$ given by the equality in (4.24) are connected with the analogously defined $\text{Vir} \oplus \text{Vir}$ highest weight vectors in module $\pi_{F \oplus \text{NSR}}$ by

$$|P, \pm 1/4\rangle' = \epsilon |P, \pm 1/4\rangle,$$

where we used (4.24). Then, using (4.26), we have that

$$|P, n + 1/4\rangle_0 = (-1)^{2n} |P, n + 1/4\rangle_0,$$

where $2n \in \mathbb{Z}$. Then:

$$l_n^{-\pm}(P, b) = \langle \mu, n | \mathcal{P} \otimes W_{R,-\epsilon}(1) \rangle = (-1)^{2n-1/2} \langle \mu, n | \mathcal{P} \otimes W_{R,\epsilon}(1) \rangle = (-1)^{2n+1/2} l_n^{+\mp}(P, b),$$

which completes the proof. □

The coefficients $l_n^{-\pm}(P, b)$ could possibly be calculated in a way which is analogous to that in [4, section 3.3.] or in [18]. It is natural to conjecture the following:
Conjecture 4.1. Coefficient $I_0^{+,-}(P, b)$ is given by the expression

$$I_0^{+,-}(P, b) = \frac{2^{2n-2} \alpha^{1/2} \beta^{1/2}}{s_{\text{odd}}(2P, 2n) s_{\text{odd}}(2P + Q, 2n)}$$

where function $s_{\text{odd}}$ was defined in (4.29).

This was checked by computer calculations for $|n| \leq 9/4$. More precisely, we check the first relation of the relations in (4.47), with these coefficients up to $z$ in the power $\Delta_R + 1/16 + 10$.

Now, we want to calculate $\mathcal{F}_k$ defined by

$$\mathcal{F}_k = z^{1/16}(P' \otimes W_{R, +}(1)) [H^k] \otimes W_{R, +}(z)),$$

where even operator $H$ is given by (4.32), $\mu, \nu = \pm$. We choose the ket Whittaker vector with the signs $+$, because only the relative signs of the bra and ket Whittakers are interesting (this follows from the isomorphism mentioned in the proof of proposition 4.4 and the analogous isomorphism obtained by the transposition $[\mu, \Delta^F, -\nu] = [\mu, \Delta^F, -\nu]$). It follows from lemma 4.1 that only the functions $\mathcal{F}_k^{n, \mu}$ are nonzero, and we will denote them simply by $\mathcal{F}_k^{n, \mu}$.

We can calculate $\mathcal{F}_k$ using the r.h.s. of (4.40)

$$\sum_{k=0}^{\infty} \mathcal{F}_k^{n, \mu} \frac{\alpha^k}{k!} = z^{1/16}(P' \otimes W_{R, +}(1)) [e^{\alpha H}] \otimes W_{R, +}(z))$$

$$= \sum_{2n+1/2 \in \mathbb{Z}} \mathcal{F}_k^{n, \mu}(P, b) \mathcal{F}_k^{n, \nu}(P, b) (W_n^{1}(\beta^{1/2}) | e^{\alpha H} | W_n^{1}(\beta^{1/2})) (W_n^{2}(\beta^{1/2}) | e^{\alpha H} | W_n^{2}(\beta^{1/2}))$$

$$= \sum_{2n+1/2 \in \mathbb{Z}} \mathcal{F}_k^{n, \mu}(P, b) \mathcal{F}_k^{n, \nu}(P, b) D_{\alpha, \beta} k^{n} (\mathcal{F}_k^{1}(\beta^{1/2} z), \mathcal{F}_k^{2}(\beta^{1/2} z)) \frac{\alpha^k}{k!}$$

(4.44)

where we used the definition of the generalized Hirota differential (4.35).

On the other hand, we can rewrite $H$ in terms of the $F \oplus NSR$ generators using (4.16)

$$H = \alpha \left( \sum_{r \in \mathbb{Z}} r : f_{r} f_{r}^{-1} : +1/8 \right) - \sum_{r \in \mathbb{Z}} f_{r} G_{r}$$

(4.45)

We will use the calculations of $(H - Q/8) z^{1/16} [1^+ \otimes W_{R, +}(z))]$ for $k \leq 3$

$$(H - Q/8) z^{1/16} [1^+ \otimes W_{R, +}(z))] = -\frac{z}{2} G_{\Delta^{1/16}} [1^+ \otimes W_{R, +}(z))]$$

$$(H - Q/8) z^{1/16} [1^+ \otimes W_{R, +}(z))] = -\frac{z}{2} G_{\Delta^{1/16}} [1^+ \otimes W_{R, +}(z))]$$

and after the multiplication on $[1^+ \otimes W_{R, +}(z)]$ we get

$$\mathcal{F}_0^{10} = z^{1/16} \mathcal{F}_R,$$

$$\mathcal{F}_1^{10} = -\frac{Q}{2} z^{1/16} \mathcal{F}_R,$$

$$\mathcal{F}_2^{10} = -1/2 z^{1/16} \left( \frac{d}{dz} - c_{NSR/16} (z) \right) \mathcal{F}_R,$$

$$\mathcal{F}_3^{10} = -\frac{Q}{4} z^{1/16} \left( \frac{d}{dz} - c_{NSR/16} (z) \right) \mathcal{F}_R$$

(4.46)
where we denote by $F'_k^\mu$ the modified $F_k^\mu$ with $H \mapsto H - Q/8$ in definition (4.43). We also shorten the notation $F_{\text{NSR}}(\Delta^R|z|$ to $F_R$.

Now we are interested in two relations following from (4.46) by the elimination of $F_R'''' = \frac{-1}{4\pi i\sqrt{\eta}}$. In this specialization $F_k^\mu = F_k^\prime$, and using (4.44), the relations (4.47) (4.44) turn to

$$\sum_{2n+1/2 \in \mathbb{Z}} D_{\log z}^3 \mathcal{F}((\sigma + n)^2|z), \mathcal{F}((\sigma - n)^2|z)) = -\frac{1}{2} \left( \frac{d}{dz} - \frac{1}{8} \right) \sum_{2n+1/2 \in \mathbb{Z}} \rho_n^+ \mathcal{F}((\sigma + n)^2|z), \mathcal{F}((\sigma - n)^2|z))$$

Due to (4.44), these relations can be viewed as bilinear differential relations on $\text{Vir}$ conformal blocks.

As in the $\text{NS}$ sector we set $\eta = \eta_{c,b} = 1,1,1,2$ in (4.47). In this specialization $F_k^\mu = F_k^\prime$, and using (4.44), the relations (4.47) turn to

$$\sum_{2n+1/2 \in \mathbb{Z}} D_{\log z}^3 \mathcal{F}((\sigma + n)^2|z), \mathcal{F}((\sigma - n)^2|z)) = -\frac{1}{2} \left( \frac{d}{dz} - \frac{1}{8} \right) \sum_{2n+1/2 \in \mathbb{Z}} \rho_n^+ \mathcal{F}((\sigma + n)^2|z), \mathcal{F}((\sigma - n)^2|z))$$

where we used (4.41) and shortened the notation $D_{\log z}^3$ to $D_{\log z}^3$. Since $\rho_n^+ = \rho_n^-$ (due to (4.42) in case $Q = 0$), we can divide these sums by two and get

$$\sum_{n \in \mathbb{Z}} D_{\log z}^3 \mathcal{F}((\sigma + n + 1/4)^2|z), \mathcal{F}((\sigma - n - 1/4)^2|z)) = \frac{1}{2} \left( \frac{d}{dz} - \frac{1}{8} \right) \sum_{n \in \mathbb{Z}} \rho_n^+ \mathcal{F}((\sigma + n + 1/4)^2|z), \mathcal{F}((\sigma - n - 1/4)^2|z))$$

Now we deduce the Okamoto-like equations (3.12) and (3.13) on the $\tau$ function (3.7). Let us substitute this $\tau$ function decomposition into (3.12) and (3.13), and collect the terms with the same powers of $s$. The vanishing condition of the $s^0$ coefficient has the form

$$\sum_{n \in \mathbb{Z}} C(\sigma + n + m)C(\sigma - 1/2 - n)D_{\log z}^3 \mathcal{F}((\sigma + n + m)^2|z), \mathcal{F}((\sigma - 1/2 - n)^2|z))$$

$$= \frac{1}{2} \left( \frac{d}{dz} - \frac{1}{8} \right) \sum_{n \in \mathbb{Z}} C(\sigma + n + m)C(\sigma - 1/2 - n)\mathcal{F}((\sigma + n + m)^2|z), \mathcal{F}((\sigma - 1/2 - n)^2|z))$$

29
\[ \sum_{n \in \mathbb{Z}} C(\sigma + n + m)C(\sigma - 1/2 - n)D_{\log z}^{1}(\mathcal{F}(\sigma + n + m)^2|z), \mathcal{F}(\sigma - 1/2 - n)^2|z)) \]
\[ = \frac{1}{2}\left(- \frac{d}{dz} - \frac{1}{8}\right) \sum_{n \in \mathbb{Z}} C(\sigma + n + 1/4)C(\sigma - 1/4 - n)D_{\log z}^{1}(\mathcal{F}(\sigma + n + 1/4)^2|z), \mathcal{F}(\sigma - 1/4 - n)^2|z) \]

Clearly this $s^m$ coefficient coincides with the $s^{m+1}$ coefficient after the shift $\sigma \mapsto \sigma + 1/2$. Therefore, it is sufficient to prove the vanishing of the $s^0$ coefficient.

Let us use the substitution $\sigma \mapsto \sigma + 1/4$ in the $s^0$ relation
\[ \sum_{n \in \mathbb{Z}} C(\sigma + n + 1/4)C(\sigma - 1/4 - n)D_{\log z}^{1}(\mathcal{F}(\sigma + n + 1/4)^2|z), \mathcal{F}(\sigma - 1/4 - n)^2|z) \]
\[ = \frac{1}{2}\left(- \frac{d}{dz} - \frac{1}{8}\right) \sum_{n \in \mathbb{Z}} C(\sigma + n + 1/4)C(\sigma - 1/4 - n)\mathcal{F}(\sigma + n + 1/4)^2|z), \mathcal{F}(\sigma - 1/4 - n)^2|z) \]

These relations coincide with the relations (4.48) and (4.49) up to the coefficients. Using the shift relation on $G(\sigma)$ and (4.42) we obtain
\[ \frac{C(\sigma + n + 1/4)C(\sigma - n - 1/4)}{C(\sigma + 1/4)C(\sigma - 1/4)} = \frac{1}{2^{[n+1/4]-3/2} \prod_{k=0}^{2[n+1/4]-1/2-1} ((1/2 + k)^2 - 4\sigma^2)^{2[n+1/4]-1/2-1}} = \frac{2^{1/8} \cdot 4^{n+1/2}}{i_{n+1/4}(\sigma)}} \]
(4.50)

which completes the proof.

Acknowledgments

We thank P Gavrylenko, A Its, O Lisovyy, A Marshakov, H Sakai and A Sciarappa for their interest in our work and discussions.

This work has been funded by the Russian Academic Excellence Project ‘5–100’. AS was also supported in part by the Young Russian Mathematics award and the joint NASU-CNRS project F14-2016; MB was also supported in part by the Young Russian Mathematics award and RFBR grant mol_a_ved 15-32-20974. The study of the algebraic solutions was performed under a grant from the Russian Science Foundation 14-050-00150 for IITP.

References

[1] Alday L F, Gaiotto D and Tachikawa Y 2010 Liouville correlation functions from four-dimensional gauge theories Lett. Math. Phys. 91 167–97
[2] Belavin A, Bershtein M, Feigin B, Litvinov A and Tarnopolsky G 2013 Instanton moduli spaces and bases in coset conformal field theory Commun. Math. Phys. 319 269–301
[3] Belavin V 2007 $\mathcal{N} = 1$ SUSY conformal block recursive relations Theor. Math. Phys. 152 1275
[4] Bershtein M and Shchechkin A 2015 Bilinear equations on Painlevé $\tau$ functions from CFT Commun. Math. Phys. 339 1021–61
[5] Bershtein M and Shchechkin A 2017 $q$-deformed Painlevé $\tau$ function and $q$-deformed conformal blocks J. Phys. A: Math. Theor. 50 085202
31

[6] Bonelli G, Grassi A and Tanzini A 2017 Seiberg–Witten theory as a Fermi gas Lett. Math. Phys. 107 1–30
[7] Carlsson E 2015 AGT and the Segal–Sugawara construction (arXiv:1509.00075)
[8] Crnkovic C, Painov R, Sotkov G and Stanishkov M 1990 Fusions of conformal models Nucl. Phys. B 336 637
[9] Fokas A, Its A, Kapaev A and Novokshenov V 2006 Painlevé Transcendents: the Riemann–Hilbert Approach (Mathematical Surveys and Monographs vol 128) (Providence, RI: American Mathematical Society)
[10] Gamayun O, Iorgov N and Lisovyy O 2012 Conformal field theory of Painlevé VI J. High Energy Phys. JHEP10(2012)38
[11] Gamayun O, Iorgov N and Lisovyy O 2013 How instanton combinatorics solves Painlevé VI, V and III’s J. Phys. A: Math. Theor. 46 335203
[12] Gromak V 1984 Reducibility of the Painlevé equations Differ. Equ. 20 1191–8
[13] Iohara K and Koga Y 2011 Representation Theory of the Virasoro Algebra (Springer Monographs in Mathematics) (London: Springer)
[14] Iohara K and Koga Y 2003 Representation theory of Neveu–Schwarz and Ramond algebras I: Verma modules Adv. Math. 178 1–65
[15] Iorgov N, Lisovyy O and Teschner J 2015 Isomonodromic τ functions from Liouville conformal blocks Commun. Math. Phys. 336 2 671–94
[16] Ito Y 2012 Ramond sector of super Liouville theory from instantons on an ALE space Nucl. Phys. B 861 387–402
[17] Jimbo M 1982 Monodromy problem and the boundary condition for some Painlevé equations Publ. RIMS, Kyoto Univ. 18 1137–61
[18] Hadasz L and Jaskólski Z 2014 Super-Liouville–Double-Liouville correspondence J. High Energy Phys. JHEP(2014)124
[19] Kac V and Raina A 1987 Highest Weight Representations of Infinite Dimensional Lie Algebras (Advanced Series in Mathematical Physics vol 2) (Singapore: World Scientific)
[20] Lashkevich M 1993 Superconformal 2D minimal models and an unusual coset construction Mod. Phys. Lett. A8 851–60
[21] Macdonald I 1995 Symmetric Functions and Hall Polynomials 2nd edn (New York: Oxford University Press)
[22] McCoy B, Tracy C and Wu T 1977 Painlevé functions of the third kind J. Math. Phys. 18 1058–92
[23] Nekrasov N and Okounkov A 2006 Seiberg–Witten theory and random partitions Prog. Math. 244 525–96
[24] Niles D 2009 The Riemann–Hilbert–Birkhoff inverse monodromy problem and connection formulae for the third Painlevé transcendents PhD Thesis Purdue University
[25] Novokshenov V 1985 On the asymptotics of the general real solution of the Painlevé equation of the third kind Sov. Phys.—Dokl. 30 666–8
[26] Ohyama Y, Kavamuko H, Sakai H and Okamoto K 2006 Studies on the Painlevé equations, V, third Painlevé equations of special type PIII(D7) and PIII(D8) J. Math. Sci. Univ. Tokyo 13 145–204
[27] Okamoto K 1999 The Hamiltonians associated to Painlevé equations The Painlevé Property One Century Later (Berlin: Springer) pp 735–87
[28] Zamolodchikov A and Zamolodchikov AI 1989 Conformal field theory and critical phenomena in two-dimensional systems Sov. Sci. Rev. Sect. A 10 269–433
[29] Zamolodchikov AI 1987 Conformal scalar field on the hyperelliptic curve and critical Ashkin–Teller multipoint correlation functions Nucl. Phys. B 285 481–503