Above-well, Stark, and potential-barrier resonances of an open square well in a static external electric field

A. Emmanouilidou\textsuperscript{1}, N. Moiseyev\textsuperscript{2}

\textsuperscript{1}Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany and

\textsuperscript{2}Department of Chemistry and Minerva Center of Nonlinear Physics in Complex Systems Technion-Israel Institute of Technology Haifa 32000, Israel

(Dated: October 12, 2018)

Abstract

Besides the well known Stark resonances, which are localized in the potential well and tunnel through the potential barrier created by the dc-field, “strange” long and short-lived resonances are analytically obtained. These resonances are not localized inside the potential well. We show that the narrow ones are localized above the potential well. These narrow resonances give rise to a peak structure in a 1D scattering experiment. We also show that the broad overlapping resonances are associated with the static electric field potential barrier. These “strange” overlapping resonances do not give rise to a peak structure in a 1D scattering experiment. We propose a 2D experimental set-up where in principle these short-lived states should be observed as peaks. Broad overlapping resonances, associated only with the static electric field potential barrier, could also have observable effects in a $N > 1$ array of quantum wells in the presence of a truncated static electric field. This last problem is associated with the resonance tunnelling phenomena which are used in the construction of resonance-tunnelling diodes and transistors.
PACS number(s): 32.60.+i, 34.50.-s

I. INTRODUCTION

Elucidating the mechanisms underlying the decay processes in mesoscopic as well as atomic systems is of fundamental importance. Valuable insight into the decay mechanisms of these systems is obtained when one expresses their dynamical properties in terms of the resonance states of the system (complex eigenvalues in the lower half energy plane). Resonances in the presence of a static electric field that are directly associated with the bound or resonance states of the field-free Hamiltonian, known as Stark resonances, are well understood and studied for many years [1]. Less known and understood are resonances in the presence of a static electric field that can not be traced back to the bound, excited or resonance states of the field-free Hamiltonian.

A simple model where the later type of resonances appear is the one-dimensional $\delta$-potential well in the presence of a static electric field. For this simple model, Ludvikson has shown the existence of two families of an infinite number of resonances that can not be traced back to the bound state of the field-free Hamiltonian [2]. The effect of these families of resonance states in physically measurable quantities was studied by Emmanouilidou and Reichl [3]. In this latter study, the authors, using a complex spectral decomposition [4] for Ludvikson’s one-dimensional model, were able to qualitatively reproduce the main features of the experimentally obtained photodetachment cross section of the $H^{-}$ ion in the presence of a static electric field when driven by a weak time-periodic electric field [5, 6] and directly associate these features to the two-families of infinite resonance states. More recently, Ludvikson’s model was studied in ref[7], and using two-$\delta$ potentials in ref[8] in the context of avoided crossings.

Open questions yet remain regarding the “nature” of these resonance states which do not stem from the spectrum of the field-free Hamiltonian. In the current work we address some of these questions using as our model system a one-dimensional square well in the presence of a static electric field. We show that this model potential supports Stark resonances that are localized inside the square well and tunnel through the potential barrier created by the dc-field. We also find two families of infinite resonances that can not be traced to the spectrum of the field-free Hamiltonian. One family consists of long-lived resonance
states that are localized above the potential well, while the other family consists of short-lived overlapping resonances that are localized below the potential well. Resonance states similar to the long-lived ones that are not associated with the field-free Hamiltonian have been probed as peaks in experiments measuring the photodetachment rate of negative ions driven by a weak time periodic field in the presence of a static field [5, 6]. However, little is known about the “nature” of the short-lived states that are not associated with the field-free Hamiltonian. Since these resonance states do not give rise to a peak structure in a 1D scattering experiment, it is an open question what is an appropriate experimental set up where these short-lived overlapping resonances could be probed as peaks.

We shed light into the nature of these resonance states, by showing first that they are associated only with the static electric field potential barrier. Next, by introducing a cut-off to the static electric field we find overlapping resonances similar to the broad overlapping resonances that are present in the case of the square well plus static electric field. In both cases, with and without the cut-off, the square well plus static electric field supports overlapping resonances that are only associated with the static electric field potential barrier. Using exterior complex scaling, we show that the overlapping resonances of the truncated potential have a nodal structure. Future work might involve studying the effect of the long and short-lived resonances on the transmission properties through an $N > 1$ array of 1D quantum wells in the presence of a static electric field. This latter problem is associated with the resonance tunnelling phenomena which are used in the construction of resonance-tunnelling diodes and transistors. Our results can be perhaps used in the design of a new type of controlled electronic switches. Finally, we also propose a two-dimensional scattering set-up where in principle the short-lived states of the square well plus static electric field should give rise to a peak structure. The proposed 2D set-up is based on an idea proposed by Narevicius and Moiseyev [10] involving a light atom getting trapped among two heavy ones.

II. MODEL

The model we use describes the behaviour of a single particle, of mass $m$ and charge $q$, in one space dimension, in the presence of a static electric field and a square well potential.
The one dimensional Hamiltonian is:

\[ H(\xi) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \xi^2} - F\xi - V(\xi), \]  

(1)

where \( V(\xi) \) is given by:

\[
V(\xi) = \begin{cases} 
-\xi, & \xi < -\alpha' \\
-V'_0 - \xi, & -\alpha' < \xi < \alpha' \\
-\xi, & \xi > \alpha' 
\end{cases}
\]

(2)

with \( F/q \) the strength of the static electric field and \( V'_0 \) the depth of the square well. If we introduce the dimensionless variables

\[
x = \xi/\xi_0, \quad E = E'/\epsilon_0, \quad t = t'/\epsilon_0, \quad V = V'/\xi_0\epsilon_0, \quad z = z'/\epsilon_0,
\]

(3)

where

\[
\xi_0 = \left(\frac{\hbar^2}{2mF}\right)^{1/3}, \quad \epsilon_0 = F\xi_0
\]

(4)

the Hamiltonian is given by

\[ H(x) = -\frac{\partial^2}{\partial x^2} - x - V(x) \]

(5)

with

\[
V(x) = \begin{cases} 
-x, & x < -\alpha \\
-V_0 - x, & -\alpha < x < \alpha \\
-x, & x > \alpha 
\end{cases}
\]

(6)

Note that \( t', z' \) are the time, and the energy on the complex plane variables, respectively, and \( t, z \) are the corresponding dimensionless variables. The above Hamiltonian has a continuous energy spectrum in the Hilbert space. One way to obtain the resonance states of the system, which are given by complex eigenvalues in the lower energy plane, is to obtain the poles of the analytically continued energy Green’s function in the lower half energy plane. In the current work, we employ a different approach that allows us to find the resonance states of the system in a very simple way. In this simple approach, the wavefunction satisfies Gamov-Siegert boundary conditions, that is, it satisfies outgoing wave boundary conditions:

\[
\Psi(x, z) = \begin{cases} 
NAi(-x - z), & x < -\alpha \\
N (a_1(z)Ai(-x - V_0 - z) + a_2(z)Bi(-x - V_0 - z)), & -\alpha < x < \alpha \\
Na_3(z)Ci^+(-x - z), & x > \alpha
\end{cases}
\]

(7)
where \( C_i^+ \) describes an outgoing wave in the positive \( x \) direction in the presence of a static electric field and \( C_i^+ = iA_i + B_i \). For \( z \) in the lower half energy plane, the wavefunction \( \Psi(x, z) \) goes to zero as \( x \to -\infty \) but is not bounded as \( x \to \infty \). Thus, due to the outgoing wave boundary conditions we imposed, \( \Psi(x, z) \) is not a square integrable wavefunction. From the continuity of the wavefunction and its first derivative at \( x = -\alpha \) and \( x = \alpha \) we find \( a_1(z), a_2(z), a_3(z) \):

\[
a_1 = \pi (A_i(\alpha - z)B_i'(\alpha - V_0 - z) - A_i'(\alpha - z)B_i(\alpha - V_0 - z)),
\]

\[
a_2 = \pi (A_i'(\alpha - z)A_i(\alpha - V_0 - z) - A_i(\alpha - z)A_i'(\alpha - V_0 - z)),
\]

\[
a_3 = \frac{(a_1A_i(-\alpha - V_0 - z) + a_2B_i(-\alpha - V_0 - z))/C_i^+(-\alpha - z)},
\]

as well as the condition for obtaining the resonances, \( z_n \), of the system:

\[
(A_i(-\alpha - V_0 - z_n)iA_i'(-\alpha - z_n) + A_i(-\alpha - V_0 - z_n)B_i'(-\alpha - z_n) -
A_i'(-\alpha - V_0 - z_n)iA_i(-\alpha - z_n) - A_i'(-\alpha - V_0 - z_n)B_i(-\alpha - z_n))

(8)

\[
(A_i(-\alpha - z_n)B_i'(\alpha - V_0 - z_n) - A_i'(\alpha - z_n)B_i(\alpha - V_0 - z_n) +
B_i(-\alpha - V_0 - z_n)B_i'(-\alpha - z_n) + B_i(-\alpha - V_0 - z_n)iA_i'(-\alpha - z_n) -
B_i'(-\alpha - V_0 - z_n)iA_i(-\alpha - z_n) - B_i'(-\alpha - V_0 - z_n)B_i(-\alpha - z_n))

(4)

(4)

\[
(A_i'(-\alpha - z_n)A_i(\alpha - V_0 - z_n) - A_i(\alpha - z_n)A_i'(\alpha - V_0 - z_n)) = 0.
\]

Using Eq.(8) we find the resonances of the square well plus static electric field for \( V_0 = 5 \) and \( \alpha = 1 \), shown in Table I, and their corresponding wavefunctions satisfy:

\[
H(x)\Psi(x, z_n) = z_n\Psi(x, z_n)
\]

with \( \Psi(x, z_n) \) not an element of the Hilbert space and \( H(x) \) being non Hermitian.

### III. STARK RESONANCES

The square well plus static electric field supports resonance states that stem from the bound as well as the resonance states of the field-free square well. In Table I, the resonances denoted by 0 and 1 stem from the bound states of the field-free square well and in what
follows we refer to them as type I Stark resonances. Indeed, one can easily show that the
conditions for bound states of the square well in dimensionless units are:

\[-\sqrt{E} + \sqrt{V_0 - E \tan(\sqrt{V_0 - E})}\alpha = 0\]

\[\sqrt{E} + \sqrt{V_0 - E \cot(\sqrt{V_0 - E})}\alpha = 0.\]

From the above conditions one finds that for \(V_0 = 5\) the square well supports two bound
states, one with even symmetry and value \(E = -3.8525\) and one with odd symmetry and
value \(E = -0.9314\). These energies are very close to the real part of the resonances 0 and
1, respectively, in Table I.

In addition to the type I Stark resonances, the square well plus static electric field supports
resonance states (type II Stark resonances) that stem from the resonance states of the field-
free square well. When computing the transmission probability for the one-dimensional
field-free square well one finds peaks for energies equal to the real part of the resonance
states. The field-free square well supports resonance states \[9\] whose real part is given by:

\[E_{R,l} = \left(\frac{l\pi}{2\alpha}\right)^2 - V_0,\]

where \(\left(\frac{l\pi}{2\alpha}\right)^2\) with \(l = 1, 2, \ldots\) are the energy levels of an infinite square well that extends from
\(-\alpha\) to \(\alpha\). Using Eq.\[10\], we find that for \(V_0 = 5\) the lowest resonance state of the field-free
square well has a real part that is given by \(E_{R,2} = 4.8696\). Note, that this energy is close to
the real part of the resonance state denoted by \(n = 3\) in Table I.

In Fig.\[11\], we plot the absolute value of the resonance wavefunctions \(\Psi_n(x)\) as a function
of the one-dimension \(x\). The normalization constant in Eq.\[7\] is evaluated so that

\[\int_{-\infty}^{-\alpha} \Psi^2(x, z_n)dx + \int_{\alpha}^{\infty} \Psi^2(x, z_n)dx - \left[\int \Psi^2(x, z_n)dx\right]_{x=\alpha} = 1.\]

Using this definition of norm the wavefunctions corresponding to different resonances are
orthogonal, for more details on using this norm see ref.\[11\]. The Stark resonances of the
square well plus static electric field are localized inside the potential well and exhibit a
nodal structure. Note that in order for one to see the antinode-node structure one would
have to plot separately the real and the imaginary part of the wavefunction. In the current
work we plot the absolute value of the wavefunction and that is the reason why only the
maxima of the wavefunction are exposed. For brevity we refer to this structure as nodal.
This can be clearly seen for the type I Stark resonances that are plotted in Fig.\[11\]. Less clear
is the nodal pattern of the type II Stark resonance $z_3$ due to its interference with the long-lived resonance states stemming from the continuum that are localized above the barrier, as we discuss in the following section. A comparison with Fig. (2) reveals that the resonance state $z_3$ has a nodal pattern with two maxima inside the square well since $z_3$ seems to be associated with the $E_{R,2}$ resonance state of the field-free square well. A more detailed study of the resonance dependence with the strength of the static electric field would determine beyond doubt the resonance states in Table I that are associated with the resonances in the field-free square well.

IV. “NEW-BORN” RESONANCE STATES THAT ARE NOT ASSOCIATED WITH THE POLES OF THE FIELD-FREE POTENTIAL WELL.

In addition to the Stark resonances, the square well plus static electric field supports resonance states that can not be traced back to the bound or resonance state poles of the field-free square well. We refer to these resonance states as “new-born” states. There are two families of an infinite number of these resonance states with one family consisting of long-lived states and the other one consisting of short-lived ones.

A. Long-lived “new-born” above-well resonance states

For $V_0 = 5$ and $\alpha = 1$, the family of long-lived states stemming from the continuum are the $z_2, z_4, z_5, \ldots$ resonances in Table I. Using Eq. (8), one can show that when $V_0 \to \infty$ the family of the long-lived resonance states stemming from the continuum are given by $Ai(\alpha - E) = 0 \ [2]$. That is, to the lowest order in $1/V$, the real part of these long-lived states is given by $\alpha + (-3\pi/8 + 3l\pi/2)^{2/3}$, with $l = 1, 2, \ldots$. In Table II we show the real part of the first eight long-lived states when $V_0 \to \infty$. The value of $V_0 = 5$ and $\alpha = 1$ we use to find the resonance states corresponds to a deep potential well. A comparison of Table I and Table II reveals that the real part of the resonance states $z_n$ with $n \geq 6$ is very similar to the energies in Table II with $l > 4$. The resonance states $z_2, z_4$ and $z_5$, in Table I, have a real part not as close to the energies corresponding to $l = 1, 2$ and $l = 3$ in Table II, due to the interference of the family of the long-lived resonance states with the $z_3$ Stark resonance.

When the square well becomes very deep the family of the long-lived resonance states are
very close to the eigenstates of the triangular well that is formed by the static electric field in the region \( x < -\alpha \) and a wall at \( x = -\alpha \). In Fig. 1 we see that the long-lived states, denoted by the dashed curves in Fig. 1, are localized above the barrier and have a nodal structure inside the triangular well. Due to the interference with the \( z_3 \) Stark resonance, the nodal pattern of the family of long-lived states has also a nodal structure consisting of two maxima in the region where the square well extends. Long-lived “new born” resonance states similar to the ones currently under consideration have been known to give rise to a peak structure in photodetachment experiments of negative ions driven by a weak-time periodic field in the presence of a static electric field.

In support of our statement that there is interference between the family of long-lived resonance states and the type II Stark resonances we include Figs. 2 and 3. In these Figures, we plot the absolute value of the wavefunction \( z_1 \) and \( z_6 \) shown in Table III, as a function of the one-dimension \( x \) for \( V_0 = 30 \). Using Eq. (10) one finds that the lowest type II Stark resonance of the field-free square well is \( E_{R,4} = 9.478 \), for \( V_0 = 30 \). The Stark resonance \( z_6 \) in Table III stems from the \( E_{R,4} \) state. In Fig. 2 we see that \( |\Psi_{z_6}(x)| \) has a nodal structure with four pronounced maxima in the region where the square well extends, since it stems from the \( E_{R,4} \) resonance state of the field-free square well. In addition, the nodal pattern of \( |\Psi_{z_6}(x)| \) involves 6 maxima localized in the triangular region above the square well indicating interference with the family of long-lived states. In Fig. 3 we plot \( |\Psi_{z_1}(x)| \) and we see that it has one pronounced maximum inside the triangular region above the square well as well as four less pronounced ones for \( -\alpha < x < \alpha \) due to the interference with the \( z_6 \) Stark resonance.

### B. Short-lived “new-born” potential-barrier resonance states

The square well potential plus static electric field supports in addition to the resonances studied in the previous sections, a family of an infinite number of short-lived resonance states that are stemming from the continuum. For \( V_0 = 5 \), these resonance states are denoted as \( z_{-1}, z_{-2}, \ldots \) in Table I. These short-lived states are overlapping resonances, that is, the spacing of the real energy part is less than the width of these states. When \( V_0 \to \infty \) one can show that the condition for obtaining these states is \( Ci(-z_n - \alpha) = 0 \). That is, to the lowest order in \( 1/V \), these short-lived states are given by \((-3\pi/8 + 3l\pi/2)^{2/3}e^{-i2\pi/3} - \alpha, \ldots\)
Understanding the “nature” of these broad resonances is harder than understanding the “nature” of the long-lived ones for the following reasons. First, in the limit $V_0$ becomes very large, the real part of the long-lived states are the eigenenergies of the triangular well formed by the static electric field for $x < -\alpha$ and a wall at $x = -\alpha$, as discussed in section IVA, see also ref. [3] and references there in. The fact that the long-lived “new-born” states are localized above the square well and primarily in the above mentioned triangular region is also corroborated by the nodal structure of these states shown in Fig. (1). It is not known, however, what part of the potential are the short-lived states associated with. The fact that these resonance states do not have a nodal structure, see Fig. (1), does not make it evident what part of the potential these states are associated with. Second, in a 1D scattering experiment, that one measures, for example, the wigner delay time as a function of the incident energy one finds peaks for energies equal to the real energy of the long-lived states but one does not find a peak structure for the broad overlapping resonances [3]. In photodetachment experiments of negative ions driven by a week time periodic field in the presence of a static electric field [5, 6], the long-lived states are known to give rise to a peak structure [5, 6] while the short-lived ones do not. These latter ones however have been shown to have observable effects in the photodetachment rate of the negative ions when driven by a weak time periodic field in the presence of a static field [3]. Mainly, the short-lived resonance states stemming from the continuum of the $\delta$-potential plus static electric field [2] have been shown to be responsible for the asymmetry of the peaks as well as the considerable photodetachment rate for photon energies smaller than the binding energy of the loosely bound electron of the $H^-$ ion, that is experimentally observed as a “shoulder” structure.

It is thus of interest to elucidate the “nature” of the short-lived resonances and suggest a possible experimental set up where they could give rise to peaks as is the case for the long-lived “new-born” resonances. To do so, we first show that the short-lived states are associated only with the static electric field potential barrier. Then, by introducing a cut-off and thus truncating the static electric field, we find broad overlapping resonances, that are again associated only with the static electric field potential barrier. Using complex scaling, we show that the nodal structure is hidden under the natural exponential divergent nature of these broad resonance states. Lastly, we discuss a 2D scattering set-up where the broad
resonances of the square well plus static field should in principle give rise to *peaks*.

1. **Potential barrier resonances**

In what follows, we show that the short-lived overlapping resonance states stemming from the continuum are *associated only with the potential barrier formed by the static electric field for* \( x > \alpha \), see Fig. 4. We take the potential barrier to be

\[
V(x) = \begin{cases} 
-V_0 - \alpha, & x < \alpha \\
-x, & x > \alpha 
\end{cases}
\]  

(12)

To find the resonance states of the potential barrier we follow the same approach as in section II, that is, we find the wavefunction satisfying outgoing wave boundary conditions which is found to be:

\[
\Psi(x, z) = \begin{cases} 
Ne^{-ikx}, & x < \alpha \\
Nb_1(z)C\text{i}^{+}(-x - z), & x > \alpha 
\end{cases}
\]  

(13)

where \( k = \sqrt{V_0 + \alpha + z} \). From the continuity of the wavefunction and its first derivative at \( x = \alpha \) we obtain the condition for resonance states:

\[
C\text{i}^{+}(-\alpha - z_n)ik - C\text{i}'(-\alpha - z_n) = 0
\]  

(14)

where \( C\text{i}'(-x - z_n) \) is the derivative with respect to \( x \). We choose the \( \text{Im}(k) > 0 \) so that \( \Psi(x, z) \to 0 \) as \( x \to -\infty \). As \( x \to \infty \), \( \Psi(x, z_n) \to \infty \). We find that the potential barrier, supports a family of an infinite number of short-lived resonance states. *These states are given in Table IV, and a comparison with Table I reveals that they are almost identical with the short-lived states of the square well plus static electric field.*

The one-dimensional symmetric and nonsymmetric Eckart potential barrier are smooth barriers that support short-lived overlapping states, as has been shown by Ryaboy and Moiseyev [12]. In ref [12], it was shown that after complex scaling, the wavefunctions of these short-lived states are localized inside the potential barrier and have a nodal structure. It is also interesting to note that in contrast to the sharp edges of the potential barrier currently under consideration the Eckart barrier is a smooth potential. We believe that this latter fact suggests that the existence of the short-lived resonance states of the square well plus static electric field is not due to the sharp edges of this potential.
So far, we have shown, that the short-lived overlapping resonance states of the one-dimensional square well plus static electric field are eigenstates of the static electric field potential barrier, as is the case for the Eckart potential-barrier resonances. However, unlike the Eckart potential-barrier resonances the short-lived overlapping resonances of the square well plus static electric field do not have a nodal structure, as we have shown. In the next section, we show that by truncating the static electric field one finds a nodal structure for overlapping resonances associated only with the static field potential barrier.

2. Square well plus static electric field with cut-off

In what follows, we truncate the static electric field by introducing a cut-off. We find that the square well plus the truncated static electric field supports, in addition to other resonances, broad overlapping ones which are again associated only with the static electric field potential barrier and they are thus similar in nature with the broad overlapping resonances of the square well plus static electric field. Using, complex scaling we show that the overlapping resonances that are associated with the truncated potential barrier have a nodal structure. This suggests that it can be the case that the broad overlapping resonances can have observable effects in the transmission properties of this 1D truncated potential. It would be of interest in the future to study the effect of the potential barrier resonances on the transmission properties of the truncated 1D system with $N > 1$ quantum wells.

The one dimension potential is given by:

$$V(x) = \begin{cases} 
-x, & x < -\alpha \\
-V_0 - x, & -\alpha < x < \alpha \\
-x, & \alpha < x < x_0 \\
-x_0, & x > x_0 
\end{cases}$$

(15)

while the wavefunction, after applying outgoing boundary conditions, is given by:

$$\Psi(x, z_n) = \begin{cases} 
NAi(-x - z_n), & x < -\alpha \\
N(c_1 Ai(-x - V_0 - z_n) + c_2 Bi(-x - V_0 - z_n)), & -\alpha < x < \alpha \\
N(c_3 Ai(-x - z_n) + c_4 Bi(-x - z_n)), & \alpha < x < x_0 \\
Nc_5 e^{ikx}, & x > x_0 
\end{cases}$$

(16)
where \( k = \sqrt{x_0 + z_n} \). Note that \( \Psi(x, z_n) \to 0 \) as \( x \to -\infty \), while for \( Im(k) < 0 \) as \( x \to \infty \)
\( \Psi(x, z_n) \to \infty \). The resonance states of the square well plus static electric field with a cut-off
are shown in Table V for \( V = 5 \), \( \alpha = 1 \) and \( x_0 = 20 \).

In the same way as in the previous section, one can show that the resonance states denoted
by \( b \) in Table V are almost identical to the resonance states supported by the potential barrier
formed by the static electric field with a cut-off. By introducing a cut-off the lifetime of the
short-lived states of the square well plus static electric field increases but they are still
overlapping resonances, see states \( z_{-1}, z_{-2}, \ldots \) in Table V. We next apply exterior complex
scaling to the wavefunction in Eq.(16) to obtain:

\[
\Psi(x, z_n) = \begin{cases}
NAi(-x - z_n), & x < -\alpha \\
N(c_1Ai(-x - V_0 - z_n) + c_2Bi(-x - V_0 - z_n)), & -\alpha < x < \alpha \\
N(c_3Ai(-x - z_n) + c_4Bi(-x - z_n)), & \alpha < x < x_0 \\
Nc_5e^{i((x-x_0)e^{i\theta}+x_0)}, & x > x_0
\end{cases}
\]  \hspace{1cm} (17)

where \( k = \sqrt{x_0 + z_n} \). The normalization is chosen so that \( \int_{-\infty}^{x_0} \Psi(x, z_n)^2 dx + e^{i\theta} \int_{x_0}^{\infty} e^{2i((x-x_0)e^{i\theta}+x_0)} dx = 1 \). Note that \( \Psi(x, z_n) \to 0 \) as \( x \to \infty \) and in addition the wave-
functions of different resonance states are orthogonal, that is, \( \int_{-\infty}^{x_0} \Psi(x, z_n)\Psi(x, z_n') dx + e^{i\theta} \int_{x_0}^{\infty} \Psi(x, z_n)\Psi(x, z_n') dx = \delta_{n,n'} \). \( \theta \) is chosen so that \( \theta > \phi \) with \( x_0 + z_n = k^2 = |k|^2 e^{2i\phi} \).

In Fig.[6] we show that the complex scaled wavefunctions of the overlapping resonances
\( z_{-1}, z_{-2}, \ldots \) have a nodal structure which is explored due to the use of the exterior scaling
transformation. For an explanation of the complex scaling transformations including the
reason we do not take the complex conjugate of the “bra” state when we calculate expecta-
tion values, as is usually done when the conventional scalar product is applied, see Ref.[13].

As it is shown in Fig.[6], the complex scaled wavefunction of the states \( z_{-17}, z_{-16}, \ldots, z_{-9} \)
have a nodal structure with 1,2,\ldots,10 maxima respectively. In Fig.[5] we can see the nodal
structure of the states \( z_{-1}, z_{-2}, z_{-3}, z_{-4}, z_{-5} \) with 18,17,16,15,14 maxima respectively. This
nodal pattern, if \( |\Psi(x)e^{-ikx}| \) is shifted by the real part of the resonance, is localized in the
region formed by the static electric field for \( x > \alpha \) and a wall at \( x = x_0 \). Therefore, the
states are not localized inside the potential barrier as in the Eckart potential case but over
the barrier. Note, however, that these resonances are still considered as potential-barrier
resonances since they “pop up” only when the potential barrier appears and even in the
absence of the field-free square potential well. In Fig.[7] we plot the states \( z_{26}, z_{36}, z_{46}, z_{66}, z_{86} \).
which also have a nodal pattern with 19, 20, 21, 22, 23 maxima and when shifted by the real part of the energy seem to be localized between $\alpha$ and $x_0$ above the well.

We have thus shown in this section that the states indicated by b in Table V are eigenstates of the potential barrier with a cut-off, are localized over the barrier, and definitely not inside the potential well, and have a nodal structure.

As we have already mentioned, it might be interesting to study in the future the effect of the long-lived as well as of the overlapping “new born” resonances in a 1D scattering experiment with $N > 1$ quantum wells in the presence of a static field with a cut-off. If for example one considers two quantum wells plus a static field with a cut-off, see Fig. (8), one could consider an initial wavefunction localized in one of the quantum wells and compute the probability for this wavefunction to tunnel to the region with constant potential, that is, to the region with $x > x_0$. The long-lived “new born” resonances should be observable as “peaks” in the transmission probability when plotted as a function of energy but it also might be the case that the overlapping “new born” resonances affect, in some observable way, the transmission probability as well.

C. 2D experimental setup

Finally, we propose a two-dimensional (2D) experimental set up where the short-lived overlapping resonances of the square well plus static electric field (without a cut-off), that were studied in section IVB1, should in principle give rise to a peak structure. Our proposed 2D set-up is based on an idea by Narevicius and Moiseyev [10]. In ref. [10] the short-lived resonance states of the 1D Eckart potential [12], which are localized in a classically inaccessible region have been considered. It has been shown that these resonances become a very good approximation for 2D resonance states that do yield a peak structure and are localized in a classically accessible region. That idea is based on the trapping of a light atom between two heavy ones. The motion of the light atom along the coordinate $x$ perpendicular to the distance $y(x)$ between the heavy atoms is treated as the slow one. This is a reasonable approximation if it is for the light particle to get temporarily trapped between the heavy atoms. In this adiabatic approximation the eigenenergies along the $y(x)$ direction become the adiabatic potentials along the $x$ direction, which in the case of ref. [10] is the 1D Eckart potential.
Given the above idea we propose the following 2D potential to observe the short lived resonance states of the square well plus static electric field: The electrons move “freely” in a 2D quantum well/dot that has the shape shown in Fig. (9). This 2D potential is associated with a $y(x)$ which is given by,

$$y(x) = \begin{cases} 
\frac{\pi}{2} \times \frac{1}{\sqrt{|x + \Delta_0|}}, & -\Delta_0 < x < -\alpha \\
\frac{\pi}{2} \times \frac{1}{\sqrt{|x + \Delta_0 + V_0|}}, & -\alpha < x < \alpha \\
\frac{\pi}{2} \times \frac{1}{\sqrt{|x + \Delta_0|}}, & x > \alpha.
\end{cases}$$  

(18)

The motion of the electrons along the $y(x)$ direction is much faster than the motion along the $x$ direction. Therefore the adiabatic approach is applicable here as in the case studied before when a light atom is scattered from two heavy ones. The eigenenergy of the light particle along the $y(x)$ direction is the ground state of a particle in a box. This eigenenergy is also the adiabatic potential along the $x$ direction which we take it to be the square well plus static electric field. For a particle in a box the ground state energy is given by

$$E_0 = -\frac{\pi^2}{(4y(x))^2}.$$  

(19)

We design our proposed experiment such that $y(x)$ is given by $y(x) = \pi/(2 \times \sqrt{|E_0|})$ with $E_0$ being the ground state of the square well which in our case is given by the potential in Eq.(6) shifted downwards by $\Delta_0$. The downward shift introduces a cut-off for $x < -\Delta_0$ such that on one hand it does not affect the short-lived states and on the other it allows for a finite potential for $x < -\Delta_0$. 

V. CONCLUSIONS

In conclusion, using a simple model, a square well in the presence of a static electric field we have shown that, besides the Stark resonances, there are two-families of resonances that are not associated with the spectrum of the field-free potential. One family consists of long-lived states that are localized above the square well. The other family consists of short-lived overlapping resonances that in a 1D scattering experiment do not give rise to a peak structure. We have shown that these short-lived resonance states are associated with the potential barrier. By introducing a cut-off in the square well plus static electric field we have found again overlapping resonances which are associated only with the static field potential barrier and that have nodal structure. Future studies could involve studying the
effects of these potential barrier resonances on the transmission properties of the truncated 1D potential with $N > 1$ wells. Finally, we have proposed a 2D scattering experiment where in principle the short-lived resonance states of the square well plus static electric field should yield a peak structure.
Table I: Poles $z_n$ of the square well plus static electric field in the energy range $-5 < \text{Re}(z_n) < 12$, for $V_0 = 5$.

| n | $z_n$                  |
|---|------------------------|
| -5 | -4.8582935-7.3068739*i  |
| -4 | -4.2903538-6.3150215*i  |
| -3 | -3.6721261-5.2295407*i  |
| -2 | -2.9761088-3.9997772*i  |
| -1 | -2.1214686-2.5051412*i  |
|  0 | -3.970613-0.0004836*i   |
|  1 | -1.0511476-0.3388278*i  |
|  2 |  3.0676534-0.3872741*i  |
|  3 |  4.5571629-0.7205034*i  |
|  4 |  5.4849666-0.5994673*i  |
|  5 |  6.7267609-0.3737202*i  |
|  6 |  7.9088730-0.2989037*i  |
|  7 |  9.0068094-0.2684895*i  |
|  8 | 10.0385628-0.2575935*i  |
|  9 | 11.0173509-0.2578512*i  |
| 10 | 11.9525012-0.2658891*i  |

Table II: Eigenenergies of the triangular well formed by the static electric field in the region $x < -\alpha$ and a wall at $x = -\alpha$.

| n | $\lambda_n$ |
|---|-------------|
| 16 | -1.0511476-0.3388278*i |
Table III: Poles $z_n$ of the square well plus static electric field in the energy range $0 < Re(z_n) < 10$ for $V_0 = 30$.

| $n$ | $z_n$                     |
|-----|---------------------------|
| 1   | 3.2499347-0.0765441*i     |
| 2   | 4.9503494-0.1167936*i     |
| 3   | 6.3426275-0.1714228*i     |
| 4   | 7.5729956-0.2513833*i     |
| 5   | 8.6823136-0.3926178*i     |
| 6   | 9.4906152-0.57571047*i    |

Table IV: Poles $z_n$ of the barrier shown in Fig. 4 in the energy range $-5 < Re(z_n) < 0$.

| $n$ | $z_n$                     |
|-----|---------------------------|
| -5  | -4.8574370-7.2999738*i    |
| -4  | -4.2885652-6.3071804*i    |
| -3  | -3.6688695-5.2205217*i    |
| -2  | -2.9713127-3.9888788*i    |
| -1  | -2.1234444-2.47845124*i   |

Table V: Poles $z_n$ of the square well plus static electric field with cut-off for $-20 < Re(z_n) < 7$ with $x_0 = 20$ and $V_0 = 5$.
| n   | $z_n$                                      |
|-----|-------------------------------------------|
| -17b| -17.2097016-1.0943095*i                   |
| -16b| -15.5425549-1.0265055*i                  |
| -15b| -14.1537578-0.9791738*i                 |
| -14b| -12.9162592-0.9431633*i                 |
| -13b| -11.7796093-0.9142266*i                 |
| -12b| -9.7125118-0.8694805*i                  |
| -11b| -8.1548697-0.8514889*i                  |
| -10b| -7.8364663-0.8355591*i                  |
| -9 b | -6.9515568-0.8212841*i                   |
| -8 b | -6.0957260-0.8083655*i                   |
| -7 b | -5.2654870-0.7965786*i                   |
| -6 b | -4.4580053-0.7857675*i                   |
| -5 b | -3.6708589-0.7761792*i                   |
| -4 b | -2.9016305-0.7689446*i                   |
| -3 b | -2.1417117-0.7637019*i                   |
| -2 b | -1.3432008-0.7307476*i                   |
| -1 b | -0.40709318-0.8295943*i                  |
|  0  |  -3.97061424-0.0004822*i                 |
|  1  |  -1.0386799-0.3352285*i                  |
|  2 b |   0.4649898-0.9253466*i                  |
|  3 b |   1.3855199-0.9779873*i                  |
|  4 b |   2.3564670-0.9696458*i                  |
|  5  |   3.0590434-0.3809258*i                  |
|  6 b |   3.4826672-0.9145274*i                  |
|  7  |   4.4107251-0.6948808*i                  |
|  8 b |   4.8838984-0.7687683*i                  |
|  9  |   5.5023958-0.5534036*i                  |
| 10 b|   6.2087394-0.9667513*i                  |
| 11  |   6.7318740-0.3683771*i                  |
FIG. 1: Absolute value of the wavefunction $|\Psi_n(x)|$. The plot for each wavefunction is shifted by the real part of the resonance state $z_n$. The solid lines indicate the type I Stark resonances, the dashed lines indicate the long-lived “new-born” resonance states, while the dashed-dot lines indicate the short-lived “new-born” resonance states. The normalization constant in Eq.(4) is evaluated so that $\int_{-\infty}^{-\alpha} \Psi^2(x, z_n)dx + \int_{-\alpha}^{\alpha} \Psi^2(x, z_n)dx - [\int \Psi^2(x, z_n)dx]_{x=\alpha} = 1$. 
FIG. 2: Absolute value of the wavefunction $|\Psi_{z_6}(x)|$. The plot is shifted by the real part of the resonance state $z_6$.

FIG. 3: Absolute value of the wavefunction $|\Psi_{z_1}(x)|$. The plot is shifted by the real part of the resonance state $z_1$. 
FIG. 4: The potential barrier

FIG. 5: $|\Psi_n(x)e^{-ikx}|$ for the $z_{-1}, z_{-2}, z_{-3}, z_{-4}, z_{-5}$ states in Table V. Their nodal pattern consists of $18,17,16,15,14$ maxima respectively. This nodal pattern is localized inside the region defined by the static electric field for $x > \alpha$ and a wall at $x = x_0$. 
FIG. 6: $|\Psi_n(x)e^{-ikx}|$ for the $z_{-17}, z_{-16}, \ldots, z_{-9}$ states in Table V with a nodal pattern of 1,2,...,10 maxima respectively. This nodal pattern is localized inside the region defined by the static electric field for $x > \alpha$ and a wall at $x = x_0$.

FIG. 7: $|\Psi_n(x)e^{-ikx}|$ for the $z_2, z_3, z_4, z_6, z_8$ in Table V with a nodal pattern of 19,20,21,22,23 maxima respectively. This nodal pattern is localized in the region between $\alpha$ and $x_0$. 
FIG. 8: The wavefunction initially localized at the left well is tunnelling to the region with \( x > x_0 \) where \( x_0 \) is taken equal to 10 in this plot.

FIG. 9: 2D experimental set up
[1] A. Maquet, Shih-I Chu and W.P. Reinhardt, Phys. Rev. A 27, 2946 (1983).
[2] A. Ludviksson, J. Phys. A 20, 4733 (1987).
[3] A. Emmanouilidou and L. E. Reichl, Phys. Rev. A 62, 022709 (2000); A. Emmanouilidou and L. E. Reichl, Chaos Solitons and Fractals, 12 (14-15), 2613 (2001).
[4] J. C. Nickel and L. E. Reichl, Phys. Rev. A 58, 4210 (1998).
[5] J. E. Stewart, H. C. Bryant, P. G. Harris, A. H. Mohagheghi, J. B. Donahue, C. R. Quick, R. A. Reeder, V. Yuan, C. R. Hummer, W. W. Smith, and S. Cohen, Phys. Rev. A 38, 5628 (1988).
[6] P. G. Harris, H. C. Bryant, A. H. Mohagheghi, C. Tang, J. B. Donahue, C. R. Quick, R. A. Reeder, S. Cohen, W. W. Smith, J. E. Stewart, and C. Johnstone, Phys. Rev. A 41, 5968 (1990).
[7] G. Alvarez and B. Sundaram, Phys. Rev. A 68, 013407 (2003).
[8] H. J. Korsch and S. Mossmann, J. Phys. A 36, 2139 (2003).
[9] Franz Schwabl Quantum Mechanics Springer-Verlay Berlin Heidelberg 1995.
[10] E. Narevicius and N. Moiseyev, Molecular Physics 94, 897, (1998).
[11] Ya. B. Zeldovich, Soviet Phys. JETP 12 (1961) 542. R. Zavin and N. Moiseyev, J. Phys. A (to appear April 2004).
[12] V. Ryaboy and N. Moiseyev, J. Chem Phys., 98, 9619 (1993).
[13] N. Moiseyev, Phys. Rep. 302, 211, (1998).