Dedekind sums take each value infinitely many times

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Abstract

For \( a \in \mathbb{Z} \) and \( b \in \mathbb{N} \), \( (a, b) = 1 \), let \( s(a, b) \) denote the classical Dedekind sum. We show that Dedekind sums take this value infinitely many times in the following sense. There are pairs \( (a_i, b_i), \ i \in \mathbb{N} \), with \( b_i \) tending to infinity as \( i \) grows, such that \( s(a_i, b_i) = s(a, b) \) for all \( i \in \mathbb{N} \).

1. Introduction and result

Let \( a \) be an integer, \( b \) a natural number, and \( (a, b) = 1 \). The classical Dedekind sum \( s(a, b) \) is defined by

\[
s(a, b) = \sum_{k=1}^{b} ((k/b))((ak/b)).
\]

Here

\[
((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}; \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}
\]

(see [9, p. 1]).

Originally, Dedekind sums appeared in the theory of modular forms (see [1]). But these sums have also interesting applications in a number of other fields, so in connection with class numbers, lattice point problems, topology, and algebraic geometry (see [2, 7, 9, 10]). Starting with Rademacher [8], several authors have studied the distribution of Dedekind sums (for instance, [3, 5, 11]).

It is often more convenient to work with

\[
S(a, b) = 12s(a, b)
\]

instead. We call \( S(a, b) \) a normalized Dedekind sum.

Let \( r \) be a rational number such that there exist \( a \in \mathbb{Z} \) and \( b \in \mathbb{N} \), \( (a, b) = 1 \) with \( S(a, b) = r \). Then

\[
S(a + jb, b) = r \text{ for all } j \in \mathbb{Z}.
\]  (1)

Accordingly, the value \( r \) is taken infinitely many times in a trivial sense (\( b \) fixed, \( a \) running through a congruence class mod \( b \)).

In the present paper, however, we say that the value \( r \) is taken infinitely many times if, and only if, there exists a sequence \( (a_i, b_i), \ i \in \mathbb{N} \), such that \( b_i \to \infty \) as \( i \to \infty \) and \( S(a_i, b_i) = r \) for all \( i \in \mathbb{N} \).
The only possible $r \in \mathbb{Z}$ that can be the value of a normalized Dedekind sum is $r = 0$. The value 0 is taken infinitely many times since $S(a, a^2 + 1) = 0$ for all $a \in \mathbb{N}$. This is well-known (see [9, p. 28]). Our main result is the following theorem.

**Theorem 1** Let $r \in \mathbb{Q}$ be the value of a normalized Dedekind sum. Then the value $r$ is taken infinitely many times.

2. The proof

Let $x$ be a real number, $0 < x < 1$. We consider the regular continued fraction expansion

$$x = \lfloor 0, c_1, c_2, \ldots \rfloor,$$

where the $c_i$ are natural numbers. This expansion is finite, if, and only if, $x \in \mathbb{Q}$. In this case it has the form

$$x = \lfloor 0, c_1, \ldots, c_n \rfloor$$

with $n \geq 1$ and $c_n \geq 2$. In the present setting the only irrational numbers of interest are quadratic irrationals. A number $x$ is a quadratic irrational if, and only if, its continued fraction expansion is infinite and periodic. We need only quadratic irrationals that are nearly purely periodic, i.e.,

$$x = \lfloor 0, c_1, \ldots, c_L \rfloor = \lfloor 0, c_1, \ldots, c_L, c_1, \ldots, c_L, \ldots \rfloor.$$

Let $p_k/q_k$, $k \geq 0$, be the $k$th convergent of $x$. The convergents are defined recursively in a well-known way (see [6, p. 250]). In particular, $p_k \in \mathbb{Z}$, $q_k \in \mathbb{N}$, and $(p_k, q_k) = 1$ for all $k \geq 0$. Hence $S(p_k, q_k)$ is the value of a normalized Dedekind sum. If $x = a/b = \lfloor 0, c_1, \ldots, c_n \rfloor$ is rational, $a, b \in \mathbb{N}$, $(a, b) = 1$, then $p_n = a$, $q_n = b$ (and so $p_n/q_n = x$). Otherwise, $q_k$ tends to infinity for $k \to \infty$.

The core of the proof of Theorem 1 is the following lemma, which is one of the main results of [4].

**Lemma 1** Let $x = \lfloor 0, c_1, \ldots, c_L \rfloor$ be a quadratic irrational with odd period length $L$. If $k \geq 0$, $k \equiv L - 1 \mod 2L$, then

$$S(p_k, q_k) = S(p_{L-1}, q_{L-1}).$$

Remark. The constant value $S(p_k, q_k)$, $k \equiv L - 1 \mod 2L$, takes the form

$$S(p_k, q_k) = \sum_{j=1}^{L} (-1)^{j-1} c_j + x + x',$$  \hspace{1cm} (2)

where $x'$ is the algebraic conjugate of $x$ (see [4]).

**Proof of Theorem 1** Let $S(a, b)$ be the value of a normalized Dedekind sum. By (1), we may assume that $0 \leq a < b$, so $0 \leq a/b < 1$. Due to the remark that precedes Theorem 1 we suppose that $S(a, b) \neq 0$. Then $a/b \neq 0$. Let

$$a/b = \lfloor 0, c_1, \ldots, c_n \rfloor$$
be the continued fraction expansion of $a/b$. We have $n \geq 1$ and, in particular, $c_n \geq 2$.

**Case 1:** $n$ is even. Choose an arbitrary natural number $c$ and define
\[
x = [0, c_1, \ldots, c_n, c].
\]
So this quadratic irrational has the odd number $L = n + 1$ as a period length. By Lemma 1 the convergents of $x$ satisfy
\[
S(p_k, q_k) = S(p_{L-1}, q_{L-1}) = S(p_n, q_n) = S(a, b)
\]
for $k \geq 0$, $k \equiv L - 1 \mod 2L$. Since $q_k \to \infty$ for $k \to \infty$, the value $S(a, b)$ is taken infinitely many times.

**Case 2:** $n$ is odd. We write $a/b = [0, c_1, \ldots, c_{n-1}, c_n - 1, 1]$ and put
\[
x = [0, c_1, \ldots, c_{n-1}, c_n - 1, 1, 1].
\]
So the odd number $L = n + 2$ is a period length of $x$. Observe that $p_{n+1} = a, q_{n+1} = b$. We obtain
\[
S(p_k, q_k) = S(p_{L-1}, q_{L-1}) = S(p_{n+1}, q_{n+1}) = S(a, b)
\]
for $k \geq 0$, $k \equiv L - 1 \mod 2L$. This gives the same result as in Case 1. \qed

*Example.* Let $a = 5$, $b = 14$. Then $S(a, b) = 18/7$ and $a/b = [0, 2, 1, 3, 1, 4]$. Here $n = 3$, so Case 2 of the proof applies. Accordingly, we define $x = [0, 2, 1, 3, 1, 1] = -5/7 + \sqrt{226}/14$ and have $L = 5$. We obtain $p_4 = a, q_4 = b$ and, for instance, $p_{14} = 4535, q_{14} = 12614, p_{24} = 4090565, q_{24} = 11377814, p_{34} = 3689685095, q_{34} = 10262775614$, where 14, 24 and 34 are $\equiv L - 1 \mod 2L$. Indeed, $S(p_{34}, q_{34}) = S(p_{24}, q_{24}) = S(p_{14}, q_{14}) = S(a, b) = 18/7$. From (2) we also obtain $S(a, b) = 2 - 1 + 3 - 1 - 10/7 = 18/7$.

*Remarks.* 1. The proof of Theorem 1 has exhibited a sequence $(a_i, b_i)$ such that $b_i \to \infty$ and $S(a_i, b_i) = S(a, b)$ for all $i \in \mathbb{N}$. The sequence $b_i$, however, grows exponentially in $i$. This is a consequence of the exponential growth of the denominators $q_k$ of the convergents of $x$. Accordingly, the set of the numbers $b_i$ is rather thin within the set $\mathbb{N}$. In a number of special cases the author could establish a sequence $(a_i, b_i)$ of this kind such that $b_i$ is a polynomial of degree 4 in $i$ — and so the set of the numbers $b_i$ is considerably denser in $\mathbb{N}$.

2. It would be interesting to know more about the density of the set of all suitable numbers $b_i$. In our case, the following values of $(a_i, b_i)$ with $b_i < 1000$ yield $S(a_i, b_i) = 18/7$: $(5, 14), (27, 70), (13, 119), (31, 259), (157, 406), (47, 455), (293, 707), (111, 854)$. This suggests that the set of all suitable $b_i$ could be relatively dense in $\mathbb{N}$.

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