Weighted Inequalities For The Numerical Radius

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Abstract

In this article, we obtain several new weighted bounds for the numerical radius of a Hilbert space operator. The significance of the obtained results is the way they generalize many existing results in the literature; where certain values of the weights imply some known results, or refinements of these results. In the end, we present some numerical examples that show how our results refine the well known results in the literature, related to this topic.

Keywords

Bounded linear operators · Numerical radius · Operator norm · Inequality

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1 Introduction

Let \( \mathcal{B}(\mathcal{H}) \) denote the \( C^* \)-algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \). In the sequel, upper case letters will be used to denote elements of \( \mathcal{B}(\mathcal{H}) \), while lower case letters will denote real numbers. For \( A \in \mathcal{B}(\mathcal{H}) \), the numerical radius and operator norm are defined, respectively, by

\[
\omega(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle| \quad \text{and} \quad \|A\| = \sup_{\|x\|=1} \|Ax\|.
\]

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where \( \| \cdot \| \) is the norm induced by the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H} \). It is well known that both quantities \( \omega(\cdot) \) and \( \| \cdot \| \) define equivalent norms on \( \mathcal{B}(\mathcal{H}) \) via the inequalities
\[
\frac{1}{2} \| A \| \leq \omega(A) \leq \| A \|, \quad A \in \mathcal{B}(\mathcal{H}). \tag{1.1}
\]
The significance of such bounds lies in finding easier lower and upper bounds for the numerical radius; due to the difficulty in computing the exact value of \( \omega(A) \). Thus, sharpening the bounds in (1.1) has received a considerable attention in the literature.

Among the most well established interesting results in this direction are the following inequalities due to Kittaneh [6, 7]
\[
\omega(A) \leq \frac{1}{2} \| |A| + |A^*| \|, \tag{1.2}
\]
\[
\omega(A)^2 \leq \frac{1}{2} \| |A|^2 + |A^*|^2 \|, \tag{1.3}
\]
and
\[
\omega(A) \leq \frac{1}{2} \left( \| A \| + \| A^2 \| \right), \tag{1.4}
\]
where \( A^* \) is the adjoint operator of \( A \) and \( |A| = (A^*A)^{1/2} \). We emphasize that the notation \( \omega(A)^2 \) means \( (\omega(A))^2 \). We refer the reader to [8, 10, 11] for some treatments of the inequalities (1.2), (1.3) and (1.4). In [12], a refinement of (1.3) was shown as follows
\[
\omega(A)^2 \leq \left\| \int_0^1 ((1-t)|A| + t|A^*|^2)^2 dt \right\|. \tag{1.5}
\]

The Aluthge transform \( \tilde{A} \) of \( A \in \mathcal{B}(\mathcal{H}) \) has appeared in many studies treating numerical radius inequalities, and has provided some interesting refinements. Recall that the Aluthge transform \( \tilde{A} \) of \( A \in \mathcal{B}(\mathcal{H}) \) is defined by \( \tilde{A} = |A|^\frac{1}{2} U |A|^\frac{1}{2} \), where \( U \) is the partial isometry appearing in the polar decomposition \( A = U |A| \) of \( A \), [2]. Yamazaki showed the following better estimates of (1.4) in [14]
\[
\omega(A) \leq \frac{1}{2} \left( \| A \| + \omega(\tilde{A}) \right). \tag{1.6}
\]

In this article, we further explore related numerical radius inequalities, by providing weighted versions of a parameter \( t \) which, upon selecting certain values, imply the above inequalities or some sharper bounds.

The following results will be needed in our analysis.

**Lemma 1.1** [5] Let \( A \in \mathcal{B}(\mathcal{H}) \) and let \( x, y \in \mathcal{H} \) be any vectors. If \( 0 \leq t \leq 1 \),
\[
|\langle Ax, y \rangle|^2 \leq \left( |A|^{2(1-t)} x, x \right) \left( |A^*|^2 t y, y \right).
\]

**Lemma 1.2** [9] Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive operator and let \( x \in \mathcal{H} \) be a unit vector. Then
\[
\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle, \quad (r \geq 1),
\]
\[
\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r, \quad (0 \leq r \leq 1).
\]

**Lemma 1.3** [14] Let \( A \in \mathcal{B}(\mathcal{H}) \). Then
\[
\omega(\tilde{A}) \leq \| \tilde{A} \| \leq \| A^2 \|^{\frac{1}{2}}.
\]
Lemma 1.4 [4] Let $A \in \mathcal{B}(\mathcal{H})$ and let $k \in \mathbb{N}$. Then
\[ \omega(A^k) \leq \omega(A)^k. \]

Lemma 1.5 [14] Let $A \in \mathcal{B}(\mathcal{H})$. Then
\[ \omega(A) = \sup_{\theta} \| \Re \left( e^{i\theta} A \right) \|, \]
where $\Re(T)$ is the real part of the operator $T$, defined by $\Re T = \frac{T + T^*}{2}$.

Lemma 1.6 [1] Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then
\[ r(AB + CD) \leq \frac{1}{2} \left( \omega(BA) + \omega(DC) + \sqrt{\omega(BA)^2 + \omega(DC)^2 + 4\|BC\| \cdot \|DA\|} \right), \]
where $r(\cdot)$ is the spectral radius.

Lemma 1.7 [3] Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then
\[ \|A^tB^t\| \leq \|AB\|^t, \quad 0 \leq t \leq 1. \]
Further, the function $f(t) = \|A^tB^t\|$ is log-convex on $[0, 1]$.

2 Main Results

2.1 Results Involving the Aluthge Transform

We begin with the following inequality, which gives a weighted upper bound in terms of the Aluthge transform. The value of this result can be seen in Corollary 2.1 below, where it turns out that this form implies refinements of (1.6). In the sequel, the notation $\tilde{A}_t$ will be used to denote the weighted Aluthge transform defined by
\[ \tilde{A}_t = |A|^{1-t} U |A|^{t}, \quad 0 \leq t \leq 1, \]
where $U$ is the partial isometry in the polar decomposition $A = U |A|$ of $A$. In what follows, $\tilde{A}^* = (\tilde{A})^*$.

Theorem 2.1 Let $A \in \mathcal{B}(\mathcal{H})$ and $0 \leq t \leq 1$. Then
\[ \omega(A) \leq \frac{1}{2} \sqrt{\frac{1}{4} \| |A|^t + |A|^{4(1-t)} \| + \frac{1}{4} \| A \|^2 + \frac{1}{4} \| \tilde{A}_t \|^2 + \| \tilde{A}_t^* \|^2 + \frac{1}{2} \omega(\tilde{A}_t^2) + \| |A|^2 + |A|^{2(1-t)} \| \omega(\tilde{A}_t)}. \]

Proof Let $A = U |A|$ be the polar decomposition of $A$. Noting the identity
\[ \Re(x, y) = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right), \quad x, y \in \mathcal{H}, \]
we have for any unit vector $x$,

$$
\Re \left( e^{i\theta} Ax, x \right) = \Re \left( e^{i\theta} U |A|x, x \right) \\
= \Re \left( e^{i\theta} U |A^t|x, x \right) \\
= \Re \left( e^{i\theta} |A^t|x, |A^t|U^* x \right) \\
= \frac{1}{4} \left\| \left( e^{i\theta} |A|^{1-t} + |A^t|U^* \right)x \right\|^2 - \frac{1}{4} \left\| \left( e^{i\theta} |A|^{1-t} - |A^t|U^* \right)x \right\|^2 \\
\leq \frac{1}{4} \left\| e^{i\theta} |A|^{1-t} + |A^t|U^* \right\|^2.
$$

On the other hand,

$$
\frac{1}{4} \left\| e^{i\theta} |A|^{1-t} + |A^t|U^* \right\|^2 \\
= \frac{1}{4} \left\| \left( e^{i\theta} |A|^{1-t} + |A^t|U^* \right) \left( e^{i\theta} |A|^{1-t} + |A^t|U^* \right)^* \right\| \\
= \frac{1}{4} \left\| |A|^2t + |A|^{2(1-t)} + e^{i\theta} \tilde{A}_t + e^{-i\theta} \tilde{A}_t^* \right\| \\
= \frac{1}{4} \left\| \left( |A|^2t + |A|^{2(1-t)} \right) + \left( e^{i\theta} \tilde{A}_t + e^{-i\theta} \tilde{A}_t^* \right)^2 \right\|^2 \\
+ \left( |A|^2t + |A|^{2(1-t)} \right) \left( e^{i\theta} \tilde{A}_t + e^{-i\theta} \tilde{A}_t^* \right) \left( |A|^2t + |A|^{2(1-t)} \right) \right\|^2 \\
\leq \frac{1}{4} \left( \left\| |A|^2t + |A|^{2(1-t)} \right\|^2 + \left\| e^{i\theta} \tilde{A}_t + e^{-i\theta} \tilde{A}_t^* \right\|^2 \\
+ \left\| \left( |A|^2t + |A|^{2(1-t)} \right) \left( e^{i\theta} \tilde{A}_t + e^{-i\theta} \tilde{A}_t^* \right) \right\| \right)^2,
$$

where the last inequality is obtained using the triangle inequality and the fact that $\|T^2\| \leq \|T\|^2$ for any operator $T$. Then direct calculations show that

$$
\frac{1}{4} \left\| e^{i\theta} |A|^{1-t} + |A^t|U^* \right\|^2 \\
\leq \frac{1}{4} \left( \left\| |A|^4t + |A|^{4(1-t)} + 2|A| \right\|^2 + \left\| \tilde{A}_t \right\|^2 + \left\| \tilde{A}_t^* \right\|^2 + 2 \Re \left( e^{2i\theta} \tilde{A}_t^2 \right) \right) \\
+ \left\| \left( |A|^2t + |A|^{2(1-t)} \right) \left( e^{i\theta} \tilde{A}_t + e^{-i\theta} \tilde{A}_t^* \right) \right\| \\
+ \left\| \left( e^{i\theta} \tilde{A}_t + e^{-i\theta} \tilde{A}_t^* \right) \left( |A|^2t + |A|^{2(1-t)} \right) \right\|^2.
$$
Applying the triangle inequality implies

\[
\frac{1}{4} \left\| e^{i \theta} |A|^{1-t} + |A|^t \right\|^2
\]

\[
\leq \frac{1}{4} \left( \left\| |A|^{4t} + |A|^{4(1-t)} \right\| + 2 \| A \|^2 + \| |\tilde{A}r|^2 + |\tilde{A}t|^2 \| + 2 \left\| \text{Re} \left( e^{2i \theta} \tilde{A}^2_t \right) \right\| 
\right)
\]

\[
+ 4 \left\| |A|^{2t} + |A|^{2(1-t)} \right\| \left\| \text{Re} \left( e^{i \theta} \tilde{A}_t \right) \right\| \frac{1}{2}
\]

\[
= \frac{1}{2} \left( \frac{1}{4} \left\| |A|^{4t} + |A|^{4(1-t)} \right\| + \frac{1}{2} \| A \|^2 + \frac{1}{4} \left\| |\tilde{A}r|^2 + |\tilde{A}t|^2 \| + \frac{1}{2} \left\| \text{Re} \left( e^{2i \theta} \tilde{A}^2_t \right) \right\| 
\right)
\]

\[
+ \left\| |A|^{2t} + |A|^{2(1-t)} \right\| \left\| \text{Re} \left( e^{i \theta} \tilde{A}_t \right) \right\| \frac{1}{2}
\]

\[
\leq \frac{1}{2} \left( \frac{1}{4} \left\| |A|^{4t} + |A|^{4(1-t)} \right\| + \frac{1}{2} \| A \|^2 + \frac{1}{4} \left\| |\tilde{A}r|^2 + |\tilde{A}t|^2 \| + \frac{1}{2} \omega(\tilde{A}^2) + \| |A|^{2t} + |A|^{2(1-t)} \| \omega(\tilde{A}_t) 
\right)
\]

Thus,

\[
\omega(A)
\]

\[
\leq \frac{1}{2} \sqrt{\frac{1}{4} \left\| |A|^{4t} + |A|^{4(1-t)} \right\| + \frac{1}{2} \| A \|^2 + \frac{1}{4} \left\| |\tilde{A}r|^2 + |\tilde{A}t|^2 \| + \frac{1}{2} \omega(\tilde{A}^2) + 2 \| A \| \omega(\tilde{A})}. 
\]

This completes the proof of the theorem. \qed

Letting \( t = \frac{1}{2} \) in Theorem 2.1, we reach the following result, whose significance is explained next.

**Corollary 2.1** Let \( A \in B(H) \). Then

\[
\omega(A) \leq \frac{1}{2} \sqrt{\| A \|^2 + \frac{1}{4} \left\| |\tilde{A}|^2 + |\tilde{A}^*|^2 \| + \frac{1}{2} \omega(\tilde{A}^2) + 2 \| A \| \omega(\tilde{A})}. 
\]

The significance of Corollary 2.1 is shown in the next remark, where multiple refinements of (1.4) are found.

**Remark 2.1** Taking Lemmas 1.3 and 1.4 into account, it follows from Corollary 2.1 that

\[
\omega(A) \leq \frac{1}{2} \sqrt{\| A \|^2 + \frac{1}{4} \left\| |\tilde{A}|^2 + |(\tilde{A})^*|^2 \| + \frac{1}{2} \omega(\tilde{A}^2) + 2 \| A \| \omega(\tilde{A})}
\]

\[
\leq \frac{1}{2} \sqrt{\| A \|^2 + \frac{1}{2} \| \tilde{A}^2 + \frac{1}{2} \omega(\tilde{A}^2) + 2 \| A \| \omega(\tilde{A})}
\]

\[
\leq \frac{1}{2} \sqrt{\| A \|^2 + \frac{1}{2} \| \tilde{A}^2 + \frac{1}{2} \omega^2(\tilde{A}) + 2 \| A \| \omega(\tilde{A})}
\]
\[
\frac{1}{2} \sqrt{\|A\|^2 + \|\widetilde{A}\|^2 + 2\|A\|\|\widetilde{A}\|}
= \frac{1}{2} \sqrt{(\|A\| + \|\widetilde{A}\|)^2}
= \frac{1}{2} (\|A\| + \|\widetilde{A}\|)
\leq \frac{1}{2} (\|A\| + \|A^2\|^{\frac{1}{2}}).
\]

2.2 Other Weighted Forms

Now we move to another type of weighted versions. The following presents the general form of (1.3), which gives (1.3) upon letting \(t = \frac{1}{2}\).

**Theorem 2.2** Let \(A \in \mathcal{B}(\mathcal{H})\). Then

\[
\omega(A)^2 \leq \inf_{0 < t < 1} \left\| (1 - t)|A|^{\frac{1}{1-t}} + t|A^*|^{\frac{1}{t}} \right\|
\]

**Proof** Let \(x \in \mathcal{H}\) be a unit vector. Then

\[
|\langle Ax, x \rangle|^2 \leq |\langle A|x, x \rangle| |\langle A^*|x, x \rangle|
= |\langle |A|^\frac{1}{1-t} x, x \rangle| |\langle A^*|^\frac{1}{t} x, x \rangle|
\leq |\langle |A|^\frac{1}{1-t} x, x \rangle| |\langle A^*|^\frac{1}{t} x, x \rangle|
\leq (1 - t) |\langle |A|^\frac{1}{1-t} x, x \rangle| + t |\langle A^*|^\frac{1}{t} x, x \rangle|
= \left( (1 - t)|A|^{\frac{1}{1-t}} + t|A^*|^{\frac{1}{t}} \right) |x, x|
\leq \left\| (1 - t)|A|^{\frac{1}{1-t}} + t|A^*|^{\frac{1}{t}} \right\|
\]

where the first inequality follows from Lemma 2.4 using \(y = x\) and \(t = \frac{1}{2}\), while the second inequality is obtained using the second inequality in Lemma 1.2 and the third inequality is obtained by the arithmetic-geometric mean inequality. Thus, we have shown

\[
|\langle Ax, x \rangle|^2 \leq \left\| (1 - t)|A|^{\frac{1}{1-t}} + t|A^*|^{\frac{1}{t}} \right\|
\]

By taking supremum over all unit vectors \(x \in \mathcal{H}\), we get

\[
\omega(A)^2 \leq \left\| (1 - t)|A|^{\frac{1}{1-t}} + t|A^*|^{\frac{1}{t}} \right\|
\]

Now, by taking infimum over all \(t \in (0, 1)\), we infer that

\[
\omega(A)^2 \leq \inf_{0 < t < 1} \left\| (1 - t)|A|^{\frac{1}{1-t}} + t|A^*|^{\frac{1}{t}} \right\|
\]

which completes the proof. \(\square\)

On the other hand, a weighted version of (1.2) can be stated as follows. Although the form is different from (1.2), we will show in Remark 2.3 how (2.1) below implies (1.2).
Theorem 2.3 Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq t \leq 1$. Then
\[
\omega(A)^2 \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 - \frac{t(1-t)}{R}(|A| - |A^*|)^2 \right\|, \tag{2.1}
\]
where $R = \max\{t, 1-t\}$.

Proof By Lemma 3.12 in [13],
\[
(1-t)|A|^2 + t|A^*|^2 \leq (|A| + |A^*|)^2, \tag{2.2}
\]
where $0 \leq t \leq 1$ and $R = \max\{t, 1-t\}$.

On the other hand,
\[
(1-t)|A|^2 + t|A^*|^2 - (|A| + |A^*|)^2 = (1-t)^2(|A| - |A^*|)^2
\]
\[
= (1-t)^2(|A|^2 + |A^*|^2 - |A||A^*| + |A^*||A|)
\]
\[
= (1-t)(2t-1)(|A|^2 + |A^*|^2 - |A||A^*| - |A^*||A|)
\]
Hence,
\[
(1-t)|A|^2 + t|A^*|^2 - ((1-t)|A| + t|A^*|)^2 = (1-t)^2(|A| - |A^*|)^2.
\]

Consequently,
\[
(1-t)|A|^2 + t|A^*|^2 - ((1-t)|A| + t|A^*|)^2 = t(1-t)(|A| - |A^*|)^2. \tag{2.3}
\]

By (2.3) and (2.2), we have
\[
\left( \frac{|A| + |A^*|}{2} \right)^2 \leq \frac{|A|^2 + |A^*|^2}{2} - \frac{t(1-t)}{2R}(|A| - |A^*|)^2. \tag{2.4}
\]

On the other hand,
\[
|\langle Ax, x \rangle|^2 \leq \langle |A|x, x \rangle \langle |A^*|x, x \rangle \quad \text{(by Lemma 1.1)}
\]
\[
\leq \left( \frac{\langle |A|x, x \rangle + \langle |A^*|x, x \rangle}{2} \right)^2 \quad \text{(by the arithmetic-geometric mean inequality)}
\]
\[
= \left( \frac{|A| + |A^*|}{2} \right)^2 \langle x, x \rangle
\]
\[
\leq \left( \frac{|A| + |A^*|}{2} \right)^2 \langle x, x \rangle \quad \text{(by Lemma 1.2).} \tag{2.5}
\]

Therefore,
\[
|\langle Ax, x \rangle|^2 \leq \left( \frac{|A|^2 + |A^*|^2}{2} - \frac{t(1-t)}{2R}(|A| - |A^*|)^2 \right) \langle x, x \rangle. \tag{2.6}
\]
Taking the supremum in (2.6) over $x \in \mathcal{H}$, $\|x\| = 1$ we deduce the desired result. \qed
Remark 2.2 It follows from the inequality (2.2) that
\[
\frac{|A||A^*| + |A^*||A|}{2} \leq \frac{|A|^2 + |A^*|^2}{2} - \frac{1}{R} \left( (1-t)|A|^2 + t|A^*|^2 - ((1-t)|A| + t|A^*|)^2 \right).
\]

Remark 2.3 Analyzing the inequality (2.1), we see that the inequality is best attained when \(\frac{t(1-t)}{R}\) attains its maximum value. This occurs when \(t = 1/2\) and the maximum is 1/2. Substituting these values into (2.1) we reach the following
\[
\omega(A)^2 \leq \left\| \frac{|A|^2 + |A^*|^2}{2} - \frac{1}{4} \left( |A| - |A^*| \right)^2 \right\| = \left\| \left( \frac{|A| + |A^*|}{2} \right)^2 \right\| = \left\| \frac{|A| + |A^*|}{2} \right\|^2.
\]
Thus, we have shown that
\[
\omega(A) \leq \frac{1}{2} \| |A| + |A^*| \|
\]
which is (1.2). Thus, Theorem 2.3 provides a new proof of (1.2).

Another weighted inequality that refines (1.4) can be stated as follows.

**Theorem 2.4** Let \(A \in \mathcal{B}(H)\). Then
\[
\omega(A) \leq \frac{1}{2} \left( \| A \| + \sqrt{\| A^t A^* A^t \| \cdot \| |A|^{1-t} |A^*|^{1-t} \|} \right), \quad 0 \leq t \leq 1.
\]

**Proof** Since \(|A| + |A^*|\) is a positive operator, it follows that \(\| |A| + |A^*| \| = r(|A| + |A^*|)\). Therefore, Lemma 1.6 implies
\[
\omega(A) \leq \frac{1}{2} \| |A| + |A^*| \|
= \frac{1}{2} r(|A| + |A^*|)
= \frac{1}{2} \left( |A|^{1-t} + |A^*|^{1-t} |A^t| \right)
\leq \frac{1}{4} \left( \omega(|A|) + \omega(|A^*|) + \sqrt{\omega(|A|) - \omega(|A^*|) + 4 \| A^t A^* A^t \|} \cdot \| |A|^{1-t} |A^*|^{1-t} \| \right)
= \frac{1}{4} \left( \| A \| + \| A^* \| + 2 \sqrt{\| A^t A^* A^t \| \cdot \| |A|^{1-t} |A^*|^{1-t} \|} \right)
= \frac{1}{2} \left( \| A \| + \sqrt{\| A^t A^* A^t \| \cdot \| |A|^{1-t} |A^*|^{1-t} \|} \right).
\]
This completes the proof. \(\square\)
Remark 2.4 To see how Theorem 2.4 refines (1.4), we use Lemma 1.7 to find that
\[
\sqrt{||A|| l ||A^*|| l ||A||^{1-t} ||A^*||^{1-t}} \leq \sqrt{||A|| l ||A^*|| l ||A||^{1-t} ||A^*||^{1-t}}
\]
\[
= \sqrt{||A||} ||A^*||^{1/2}
\]
\[
= ||A^*||^{1/2}.
\]

Corollary 2.2 Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( 0 \leq t \leq 1 \). Then
\[
\omega(A) \leq \frac{1}{2} \left( ||A|| + \|||A||^{1/2} ||A^*||^{1/2} \| \right)
\]
\[
\leq \frac{1}{2} \left( ||A|| + \sqrt{||A||^{1/2} ||A^*||^{1/2} l ||A||^{1-t} ||A^*||^{1-t}} \right)
\]
\[
\leq \frac{1}{2} \left( ||A|| + ||A^2||^{1/2} \right).
\]

Proof By Lemma 1.7, both
\[
||A||^{1-t} ||A^*||^{1-t}
\]
are log-convex functions in \( t, 0 \leq t \leq 1 \). Consequently, the function
\[
f(t) = \sqrt{||A||^{1-t} ||A^*||^{1-t}}
\]
is log-convex, hence is convex. Since \( f \) is convex and symmetric about \( t = \frac{1}{2} \), it follows that \( f \) attains its minimum at \( \frac{1}{2} \). Thus,
\[
f\left( \frac{1}{2} \right) \leq f(t), \quad 0 \leq t \leq 1.
\]
This together with Theorem 2.4 and Remark 2.4 imply the desired result. \( \square \)

In the next result, we present a refinement of (1.5). First, we present the following needed calculations. The inequality (2.2) states that
\[
(1-t)||A||^2 + t||A^*||^2 \leq ((1-t)||A|| + t||A^*||)^2 + 2R \left( \frac{||A||^2 + ||A^*||^2}{2} - \left( \frac{||A|| + ||A^*||}{2} \right)^2 \right),
\]
where \( 0 \leq t \leq 1 \) and \( R = \max\{t, 1-t\} \). So,
\[
\frac{1}{2R} \left( (1-t)||A||^2 + t||A^*||^2 \right) \leq \frac{1}{2R} \left( (1-t)||A|| + t||A^*|| \right)^2 + \left( \frac{||A||^2 + ||A^*||^2}{2} - \left( \frac{||A|| + ||A^*||}{2} \right)^2 \right).
\]
From this, it follows that
\[
\left( \frac{||A|| + ||A^*||}{2} \right)^2 \leq \frac{||A||^2 + ||A^*||^2}{2} - \frac{1}{2R} \left( ((1-t)||A||^2 + t||A^*||^2) - ((1-t)||A|| + t||A^*||)^2 \right).
\]
Then (2.7) and (2.3) imply that
\[
\left( \frac{||A|| + ||A^*||}{2} \right)^2 \leq \frac{||A||^2 + ||A^*||^2}{2} - \frac{t(1-t)}{2R} (||A|| - ||A^*||)^2.
\]
In fact, (2.8), can be proved for any positive operators $A$ and $B$ to obtain
\[
\left( \frac{A + B}{2} \right)^2 \leq \frac{A^2 + B^2}{2} - \frac{t(1-t)}{2R} (A - B)^2. \tag{2.9}
\]
In (2.9), if we replace $A$ and $B$ by $(1-t)|A| + t|A^*|$ and $(1-t)|A^*| + t|A|$, respectively, we reach
\[
\left( \frac{|A| + |A^*|}{2} \right)^2 = \left( \frac{(1-t)|A| + t|A^*| + (1-t)|A^*| + t|A|}{2} \right)^2
\leq \frac{(1-t)|A| + t|A^*|)^2 + ((1-t)|A^*| + t|A|)^2}{2}
- \frac{t(1-t)(1-2t)^2}{2R}(|A| - |A^*|)^2. \tag{2.10}
\]

**Theorem 2.5** Let $A \in \mathcal{B}(\mathcal{H})$. Then
\[
\omega(A)^2 \leq \left\| \int_0^1 ((1-t)|A| + t|A^*|)^2 dt - \frac{1}{48}(|A| - |A^*|)^2 \right\|
\]

**Proof** Since $R = \max\{t, 1-t\} = \frac{1-t+t|1-t-t|}{2} = \frac{1+2|1-2t|}{2}$, we get from (2.10)
\[
\left( \frac{|A| + |A^*|}{2} \right)^2
\leq \frac{(1-t)|A| + t|A^*|)^2 + ((1-t)|A^*| + t|A|)^2}{2}
- \frac{t(1-t)(1-2t)^2}{1+|1-2t|}(|A| - |A^*|)^2.
\]
By taking integral over $0 \leq t \leq 1$, and using the fact that
\[
\int_0^1 ((1-t)|A| + t|A^*|)^2 dt = \int_0^1 ((1-t)|A^*| + t|A|)^2 dt,
\]
we can obtain
\[
\left( \frac{|A| + |A^*|}{2} \right)^2 \leq \int_0^1 ((1-t)|A| + t|A^*|)^2 dt - \frac{1}{48}(|A| - |A^*|)^2.
\]
By the same method used in the proof of inequality (2.5), we have
\[
|\langle Ax, x \rangle|^2 \leq \left( \left( \frac{|A| + |A^*|}{2} \right)^2 x, x \right).
\]
Thus,
\[
|\langle Ax, x \rangle|^2 \leq \left( \left( \int_0^1 ((1-t)|A| + t|A^*|)^2 dt - \frac{1}{48}(|A| - |A^*|)^2 \right) x, x \right),
\]
which implies
\[
\omega(A)^2 \leq \left\| \int_0^1 ((1-t)|A| + t|A^*|)^2 dt - \frac{1}{48}(|A| - |A^*|)^2 \right\|,
\]
completing the proof. \( \square \)
We conclude this subsection with the following weighted inequality, which implies (1.3) upon letting $t = \frac{1}{2}$.

**Theorem 2.6** Let $A \in B(H)$. Then

$$\omega(A)^2 \leq \min_{0 \leq t \leq 1} \left\| \frac{|A|^4(1-t) + |A^*|^{4t}}{4} + \frac{(1-t)|A|^2 + t|A^*|^2}{2} \right\|.$$  

**Proof** For any unit vector $x \in H$ and $0 \leq t \leq 1$, we have

$$|(Ax, x)|^2 \leq \left( |A|^{2(1-t)}x, x \right) \left( |A^*|^{2t}x, x \right) \leq \left( \frac{|A|^{2(1-t)}x, x + |A^*|^{2t}x, x}{2} \right)^2 \leq \frac{|A|^{4(1-t)}x, x + |A^*|^{4t}x, x}{4} + \frac{2|A|^{2(1-t)}x, x}{4} \frac{|A^*|^{2t}x, x}{4} \leq \frac{|A|^{4(1-t)}x, x + |A^*|^{4t}x, x}{4} + \frac{2|A|^2x, x}{4} \left( 1 - t \right) \frac{|A|^2x, x + t|A^*|^2x, x}{4} \leq \left( \frac{|A|^{4(1-t)} + |A^*|^{4t}}{4} + \frac{(1-t)|A|^2 + t|A^*|^2}{2} \right) x, x \right).$$

Taking the supremum over unit vectors $x$ implies

$$\omega(A)^2 \leq \left( \frac{|A|^{4(1-t)} + |A^*|^{4t}}{4} + \frac{(1-t)|A|^2 + t|A^*|^2}{2} \right).$$

This completes the proof. \qed

### 3 Further Results

As an important tool to obtain numerical radius inequalities, we present the following inequality for the inner product of Schwarz type. First, notice Lemmas 1.1 and 1.2 imply

$$|\langle Ax, x \rangle|^2 \leq \langle |Ax, x \rangle \langle |Ax^*|, x, x \rangle \leq \sqrt{\langle |Ax, x \rangle^2 \langle |Ax^*|, x, x \rangle^2} \leq \sqrt{\langle |Ax^2, x \rangle \langle |Ax^*|^2, x, x \rangle}.$$ 

So,

$$|\langle Ax, x \rangle|^2 \leq \sqrt{\langle |Ax^2, x \rangle \langle |Ax^*|^2, x, x \rangle},$$

for any $A \in B(H)$ and $x \in H$. In the next result, we improve the last inequality, to obtain a form that enables us to find a new weighted numerical radius inequality.
Theorem 3.1 Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then for any vector \( x \in \mathcal{H} \),

\[
|\langle B^*Ax, x \rangle - \langle B^*x, x \rangle \langle Ax, x \rangle| \leq \sqrt{\langle A^2x, x \rangle \langle B^2x, x \rangle - \langle (Ax, x) \| (Bx, x) \rangle}.
\]

In particular,

\[
|\langle Ax, x \rangle|^2 + |\langle A^2x, x \rangle - \langle Ax, x \rangle|^2 \leq \langle A^2x, x \rangle \langle A^2x, x \rangle.
\]

(3.1)

Proof First of all, notice that

\[
\| (A - \langle Ax, x \rangle)x \| = \sqrt{\langle A^2x, x \rangle - \langle (Ax, x) \|^2}.
\]

(3.2)

Replacing \( A \) by \( B \) in (3.2), we get

\[
\| (B - \langle Bx, x \rangle)x \| = \sqrt{\langle B^2x, x \rangle - \langle (Bx, x) \|^2}.
\]

On the other hand, by the Schwarz inequality,

\[
|\langle B^*Ax, x \rangle - \langle B^*x, x \rangle \langle Ax, x \rangle| = |\langle (B^* - \langle B^*x, x \rangle)(A - \langle Ax, x \rangle)x, x \rangle| \\
= |\langle A - \langle Ax, x \rangle x, B - \langle Bx, x \rangle x \rangle| \\
\leq \| A - \langle Ax, x \rangle x \| \| B - \langle Bx, x \rangle x \|.
\]

We can conclude from the discussion above that

\[
|\langle B^*Ax, x \rangle - \langle B^*x, x \rangle \langle Ax, x \rangle| \leq \sqrt{\langle A^2x, x \rangle - \langle (Ax, x) \|^2} \sqrt{\langle B^2x, x \rangle - \langle (Bx, x) \|^2} \\
\leq \sqrt{\langle A^2x, x \rangle \langle B^2x, x \rangle - \langle (Ax, x) \|^2 \langle (Bx, x) \|^2},
\]

where the last inequality follows from the simple inequality \((a^2 - b^2)(c^2 - d^2) \leq (ac - bd)^2\) for \( a, b, c, d \in \mathbb{R}^+ \). Consequently,

\[
|\langle B^*Ax, x \rangle - \langle B^*x, x \rangle \langle Ax, x \rangle| \leq \sqrt{\langle A^2x, x \rangle \langle B^2x, x \rangle - \langle (Ax, x) \|^2 \langle (Bx, x) \|^2} \\
\tag{3.3}
\]

which implies the desired inequality. Putting \( B^* = A \) in (3.3), we obtain the inequality (3.1). \( \square \)

Corollary 3.1 Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then for any vector \( x \in \mathcal{H} \),

\[
|\langle B^*x, x \rangle \langle Ax, x \rangle| \leq \frac{\sqrt{\langle A^2x, x \rangle \langle B^2x, x \rangle + \langle B^*Ax, x \rangle}}{2}.
\]

Proof Using the triangle inequality for the modulus,

\[
|\langle Bx, x \rangle \langle Ax, x \rangle| - |\langle B^*Ax, x \rangle| = |\langle B^*x, x \rangle \langle Ax, x \rangle| - |\langle B^*Ax, x \rangle| \\
\leq |\langle B^*Ax, x \rangle - \langle B^*x, x \rangle \langle Ax, x \rangle|.
\]

(3.4)

Combining the inequalities (3.3) and (3.4), we get

\[
|\langle B^*x, x \rangle \langle Ax, x \rangle| \leq \frac{\sqrt{\langle A^2x, x \rangle \langle B^2x, x \rangle + \langle B^*Ax, x \rangle}}{2}.
\]

This completes the proof. \( \square \)
Corollary 3.2 Let $A \in \mathcal{B}(\mathcal{H})$. Then

$$\omega(A)^2 \leq \frac{1}{2} \min_{0 \leq r \leq 1} \left( \left\| \frac{1}{2} (1-t) |A|^\frac{1}{r} + \frac{1}{2} (1-t) |A^*|^\frac{1}{r^*} \right\|^2 + \omega(A^2) \right).$$

Proof From Corollary 3.1, we have

$$|\langle Ax, x \rangle|^2 \leq \frac{\sqrt{\langle |A|^2 x, x \rangle \langle |A^*|^2 x, x \rangle + \langle A^2 x, x \rangle}}{2},$$

for any unit vector $x$. Then for $0 \leq t \leq 1$,\n
$$|\langle Ax, x \rangle|^2 \leq \frac{\sqrt{\langle |A|^2 x, x \rangle \langle |A^*|^2 x, x \rangle + \langle A^2 x, x \rangle}}{2} \leq \frac{\sqrt{\langle \left| A \right|^\frac{2}{r} x, x \rangle \langle \left| A^* \right|^\frac{2}{r^*} x, x \rangle \frac{1}{1-t} + \langle A^2 x, x \rangle}}{2} \leq \frac{\sqrt{\langle \left| t A \right|^\frac{2}{r} + (1-t) \left| A^* \right|^\frac{2}{r^*} x, x \rangle \langle A^2 x, x \rangle}}{2}.$$

Taking the supremum over all unit vectors $x$ implies the desired inequality. \qed

4 Examples

In this section, we present different examples to show that the obtained results provide non-trivial refinements of the well known results; such as (1.2), (1.3) and (1.4).

Example 4.1 In this example, we investigate the inequality

$$\omega(A)^2 \leq \min_{0 \leq t \leq 1} \left\| (1-t) |A|^{\frac{1}{r}} + t |A^*|^{\frac{1}{r^*}} \right\|$$

obtained in Theorem 2.2.

Let $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{bmatrix}$. Then $|A^*| = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and $|A| = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Thus,

$$\min_{0 \leq t \leq 1} \left\| (1-t) |A|^{\frac{1}{r}} + t |A^*|^{\frac{1}{r^*}} \right\| = \min_{0 \leq t \leq 1} \left\| \begin{array}{ccc} (1-t)4^{\frac{1}{r}} + t2^{\frac{1}{r^*}} & 0 & 0 \\ 0 & (1-t)2^{\frac{1}{r}} + t3^{\frac{1}{r^*}} & 0 \\ 0 & 0 & (1-t)3^{\frac{1}{r}} + t4^{\frac{1}{r^*}} \end{array} \right\| = \min_{0 \leq t \leq 1} \max \left\{ (1-t)4^{\frac{1}{r}} + t2^{\frac{1}{r^*}}, (1-t)2^{\frac{1}{r}} + t3^{\frac{1}{r^*}}, (1-t)3^{\frac{1}{r}} + t4^{\frac{1}{r^*}} \right\} = \min_{0 \leq t \leq 1} \left\{ (1-t)3^{\frac{1}{r^*}} + t4^{\frac{1}{r^*}} \text{ when } 0 < t \leq 0.5557, \quad \frac{1}{t} \text{ when } 0.5558 \leq t < 1 \right\} \approx 12.002.$$

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On the other hand,
\[ \frac{1}{2} \left\| A^2 + |A^*|^2 \right\| = 12.5. \]

Consequently, in this case
\[ \min_{0 \leq t \leq 1} \left( (1 - t)|A|^{1-t} + t|A^*|^t \right) \leq \frac{1}{2} \left\| A^2 + |A^*|^2 \right\|. \]

Thus, this example shows that Theorem 2.2 provides a non-trivial refinement of (1.3).

In fact, this example shows also that Theorem 2.2 provides a non-trivial refinement of (1.2), because Theorem 2.2 implies
\[ \omega(A) \leq \min_{0 \leq t \leq 1} \left( (1 - t)|A|^{1-t} + t|A^*|^t \right)^{1/2} \approx \sqrt{12.002} < 3.5 = \frac{1}{2} ||A| + |A^*||. \]

Indeed, the inequality (1.2) is better than (1.3) and (1.4). Thus, showing that our result is better than (1.2) implies that it is better than both (1.3) and (1.4).

**Example 4.2** In this example, we show that the inequality
\[ \omega(A)^2 \leq \min_{0 \leq t \leq 1} \left( \frac{|A|^{4(1-t)} + |A^*|^{4t}}{4} + \frac{(1 - t)|A|^2 + t|A^*|^2}{2} \right) \]

obtained in Theorem 2.6 provides a non-trivial refinement of the three inequalities (1.2), (1.3) and (1.4). To do so, it suffices to show that it is better than (1.2).

Let \( A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix} \). Then \(|A^*| = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 2 \end{bmatrix}\) and \(|A| = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\). Consequently,
\[ \min_{0 \leq t \leq 1} \left( (1 - t)|A|^2 + t|A^*|^2 \right) = \min_{0 \leq t \leq 1} \left( \frac{(16)^{1-t} + (81)^t}{4} + \frac{4 + 5t}{2} \right) \]
\[ = \min_{0 \leq t \leq 1} \left[ \frac{(16)^{1-t} + (81)^t}{4} + \frac{4 + 5t}{2} , \frac{(81)^{1-t} + (256)^t}{4} + \frac{9 - 7t}{2} , \frac{(256)^{1-t} + (16)^t}{4} + \frac{16 - 12t}{2} \right] \]
\[ = \min_{0 \leq t \leq 1} \max \left\{ \frac{(16)^{1-t} + (81)^t}{4} + \frac{4 + 5t}{2} , \frac{(81)^{1-t} + (256)^t}{4} + \frac{9 - 7t}{2} , \frac{(256)^{1-t} + (16)^t}{4} + \frac{16 - 12t}{2} \right\} \]
\[ = \min_{0 \leq t \leq 1} \left\{ \frac{(256)^{1-t} + (16)^t}{4} + \frac{16 - 12t}{2} \right\} \quad \text{for } 0 \leq t \leq 0.5286, \]
\[ \approx 9.32. \]

On the other hand,
\[ \frac{1}{2} ||A| + |A^*|| = 3.5. \]

So, we have shown that, in this example,
\[ \min_{0 \leq t \leq 1} \left( \frac{(1 - t)|A|^2 + t|A^*|^2}{2} + \frac{|A|^{4(1-t)} + |A^*|^{4t}}{4} \right)^{1/2} < \sqrt{9.34} < 3.1 < 3.5 = \frac{1}{2} ||A| + |A^*||. \]
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References

1. Abu Omar, A., Kittaneh, F.: A numerical radius inequality involving the generalized Aluthge transform. Stud. Math. 216, 69–75 (2013)
2. Aluthge, A.: Some generalized theorems on $p$-hyponormal operators. Integr. Equ. Oper. Theory 24, 497–501 (1996)
3. Bhatia, R., Davis, C.: A Cauchy-Schwarz inequality for operators with applications. Linear Algebra Appl. 223–224, 119–129 (1995)
4. Halmos, P.R.: A Hilbert Space Problem Book, 2nd edn. Graduate Texts in Mathematics, vol. 19. Springer, New York (1982)
5. Kato, T.: Notes on some inequalities for linear operators. Math. Ann. 125, 208–212 (1952)
6. Kittaneh, F.: A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. Stud. Math. 158, 11–17 (2003)
7. Kittaneh, F.: Numerical radius inequalities for Hilbert space operators. Stud. Math. 168, 73–80 (2005)
8. Kittaneh, F., Moradi, H.R.: Cauchy–Schwarz type inequalities and applications to numerical radius inequalities. Math. Inequal. Appl. 23, 1117–1125 (2020)
9. McCarthy, C.C.: $C_p$, Israel J. Math. 5, 249–271 (1967)
10. Moradi, H.R., Sababheh, M.: More accurate numerical radius inequalities (II). Linear Multilinear Algebra 69, 921–933 (2021)
11. Omidvar, M.E., Moradi, H.R.: Better bounds on the numerical radii of Hilbert space operators. Linear Algebra Appl. 604, 265–277 (2020)
12. Sababheh, M., Moradi, H.R.: More accurate numerical radius inequalities (i). Linear Multilinear Algebra 69, 1964–1973 (2021)
13. Sababheh, M.: Convexity and matrix means. Linear Algebra Appl. 506, 588–608 (2016)
14. Yamazaki, T.: On upper and lower bounds of the numerical radius and an equality condition. Stud. Math. 178, 83–89 (2007)

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