Weights of mod $p$ automorphic forms
and partial Hasse invariants

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(with an appendix by Wushi Goldring)

Abstract

For a connected, reductive group $G$ over a finite field endowed with a cocharacter $\mu$, we define the zip cone of $(G, \mu)$ as the cone of all possible weights of mod $p$ automorphic forms on the stack of $G$-zips. This cone is conjectured to coincide with the cone of weights of characteristic $p$ automorphic forms for Hodge-type Shimura varieties of good reduction. We prove in full generality that the cone of weights of characteristic 0 automorphic forms is contained in the zip cone, which gives further evidence to this conjecture. Furthermore, we determine exactly when the zip cone is generated by the weights of partial Hasse invariants, which is a group-theoretical generalization of a result of Diamond–Kassaei and Goldring–Koskivirta.

1 Introduction

This paper is aimed at understanding automorphic forms in characteristic $p$. They are sections of certain automorphic vector bundles over Shimura varieties. The second-named author and W. Goldring have illustrated in several papers (e.g. [GK19a, GK18]) that Shimura varieties share many geometric properties with the stack of $G$-zips of Moonen–Wedhorn and Pink–Wedhorn–Ziegler ([MW04, PWZ11]). In this paper, we study various cones generated by weights of some classes of automorphic forms coming from this stack.

Let $(G, X)$ be a Shimura datum and $Sh_K(G, X)$ the corresponding Shimura variety with level $K$ over a number field $E$ (the reflex field). Let $\mu: \mathbb{G}_m, \mathbb{C} \to G_\mathbb{C}$ be a cocharacter attached to $X$, and $L \subset G_\mathbb{C}$ the Levi subgroup centralizing $\mu$. Choose a Borel pair $(B, T)$ such that $B$ is contained in the parabolic $P$ with Levi $L$ defined by $\mu$. Write $\Phi$ for the set of $T$-roots and $\Phi^+$ for the positive roots (with respect to the opposite Borel $B^+$). Denote by $\Delta$ the set of simple roots and let $I := \Delta_L$ be the simple roots of $L$. For any $L$-dominant character $\lambda \in X^*(T)$, we can attach a vector bundle $V_I(\lambda)$ (called automorphic vector bundle) on $Sh_K(G, X)$, modeled on the $L$-representation $V_I(\lambda) := \text{Ind}_{B^+}^B(\lambda)$ induced from $\lambda$. When $(G, X)$ is of Hodge-type and $p$ is a prime of good reduction, we have an integral model $\mathcal{S}_K$ over $\mathbb{O}_{E_p}$ (where $p | p$) by works of Kisin and Vasiu. Furthermore, $V_I(\lambda)$ extends to a vector bundle over $\mathcal{S}_K$. In this paper, we are interested in the question: For which $\lambda \in X^*(T)$ does $V_I(\lambda)$ admit nonzero global sections?

Set $S_K := \mathcal{S}_K \otimes_{\mathbb{O}_{E_p}} \overline{\mathbb{F}}_p$. When $F = \mathbb{C}$ (resp. $F = \overline{\mathbb{F}}_p$), denote by $C_K(F)$ the cone of $\lambda \in X^*(T)$ such that $V_I(\lambda)$ admits nonzero sections on $Sh_K(G, X) \otimes_{\mathbb{C}} \mathbb{C}$ (resp. $S_K$). For a cone $C \subset X^*(T)$, define the saturation (or saturated cone) of $C$ as the set of $\lambda \in X^*(T)$ such that some positive multiple of $\lambda$ lies in $C$. We always denote the saturation with a calligraphic letter $\mathcal{C}$. For example, write $\mathcal{C}_K(F)$ for the saturation of $C_K(F)$. The set $C_K(F)$ depends on the level $K$, but one can show that the saturated cone $\mathcal{C}_K(F)$ does not ([Kos19, Corollary 1.5.3]). Therefore, we may denote it simply by $\mathcal{C}(F)$.
We first consider the case $F = \mathbb{C}$. Griffiths–Schmid introduced in [GS69] the set:

$$\mathcal{C}_{GS} = \left\{ \lambda \in X^*(T) \ \middle| \ \begin{array}{l}
\langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for } \alpha \in I, \\
\langle \lambda, \alpha^\vee \rangle \leq 0 \text{ for } \alpha \in \Phi^+ \setminus \Phi^+_L
\end{array} \right\}.$$ 

The following conjecture is expected, but we could not find a reference for it.

**Conjecture 1.** One has $\mathcal{C}(\mathbb{C}) = \mathcal{C}_{GS}$.

The inclusion $\mathcal{C}(\mathbb{C}) \subset \mathcal{C}_{GS}$ is proved for general Hodge-type Shimura varieties in [GK22b, Theorem 2.6.4]. The opposite inclusion should follow by studying the Lie algebra cohomology appearing in the cohomology of Shimura varieties.

Regarding $\mathcal{C}(\overline{\mathbb{F}}_p)$, very little is known. Diamond–Kassaei ([DKT17, DK23]) and Goldring–Koskivirta ([GK18]) have shown in the case of Hilbert–Blumenthal Shimura varieties that $\mathcal{C}(\overline{\mathbb{F}}_p) = \mathcal{C}_{pHa}$, the cone generated by the weights of partial Hasse invariants on $S_K$. One goal of this paper is to discuss possible generalizations of this result to other cases. For general groups, we seek a description or an approximation of the cone $\mathcal{C}(\overline{\mathbb{F}}_p)$. Our approach uses the stack of $G$-zips of Moonen–Wedhorn and Pink–Wedhorn–Ziegler. Let $G$ be a reductive group over a finite field $\mathbb{F}_q$ and $\mu: \mathbb{G}_{m,k} \rightarrow G_k$ a cocharacter over $k = \overline{\mathbb{F}}_q$ (in the context of Shimura varieties, we always take $q = p$). The stack of $G$-zips of type $\mu$ is denoted by $G$-Zip$^\mu$. After possibly conjugating $\mu$, we may choose a Borel pair $(B, T)$ over $\mathbb{F}_q$ such that $B$ is contained in the parabolic subgroup $P$ defined by $\mu$ (see §2.2). Write $L \subset G_k$ for the centralizer of $\mu$ and define $I := \Delta_L$. The vector bundles $V_I(\lambda)$ for $\lambda \in X^*(T)$ can also be defined on $G$-Zip$^\mu$. We attach to $(G, \mu)$ a cone $C_{zip} \subset X^*(T)$, defined as the set of $\lambda$ such that $V_I(\lambda)$ admits nonzero sections on $G$-Zip$^\mu$. It is a group-theoretical version of $C_K(\overline{\mathbb{F}}_p)$ and can be interpreted in terms of representation theory of reductive groups (see §2.4). When $(G, \mu)$ arises by reduction from a Hodge-type Shimura datum, Zhang showed ([Zha18]) that there is a smooth map $\zeta: S_K \rightarrow G$-Zip$^\mu$ which is known to be surjective. The map $\zeta$ induces by pullback of sections an inclusions $C_{zip} \subset C_K(\overline{\mathbb{F}}_p)$ and $C_{zip} \subset \mathcal{C}(\overline{\mathbb{F}}_p)$. Goldring and the second-named author have conjectured

**Conjecture 2 ([GK18 Conjecture 2.1.6]).** One has $\mathcal{C}(\overline{\mathbb{F}}_p) = C_{zip}$.

In the case of Hilbert–Blumenthal Shimura varieties one has $C_{zip} = C_{pHa}$, hence Conjecture 1 is compatible with the result of Diamond–Kassaei mentioned above. Aside from this case, Goldring and the second-named author showed this conjecture for Picard modular surfaces at a split prime and Siegel threefolds ([GK18, Theorem D]). They also treat the case of Siegel modular varieties attached to $GSp(6)$ and unitary Shimura varieties of signature $(r, s)$ with $r + s \leq 4$ at split or inert primes (with the exception of $r = s = 2$ and $p$ inert) in the paper [GK22a].

We now describe our results more precisely. We defined in [GK19a] the stack of $G$-zip flags, denoted by $G$-ZipFlag$^\mu$, which is a group-theoretical analogue of the flag space of Ekedahl–van der Geer ([EdvG09]). There is a natural projection $\pi: G$-ZipFlag$^\mu \rightarrow G$-Zip$^\mu$ whose fibers are flag varieties isomorphic to $P/B$. The stack $G$-ZipFlag$^\mu$ carries a family of line bundles $V_{flag}(\lambda)$ for $\lambda \in X^*(T)$ such that $\pi_* V_{flag}(\lambda) = V_I(\lambda)$. In particular, we can identify $H^0(G$-Zip$^\mu, V_I(\lambda))$ and $H^0(G$-ZipFlag$^\mu, V_{flag}(\lambda))$. Moreover, $G$-ZipFlag$^\mu$ admits a stratification $(\mathcal{F}_w)_{w \in W}$ analogous to the Bruhat decomposition, where $W = W(G, T)$ is the Weyl group of $G$. By [IK21b], there exists a family of partial Hasse invariants $\{h_\alpha\}_{\alpha \in \Delta}$ (where $\Delta$ is the set of simple roots). Specifically, $h_\alpha$ is a section of $V_{flag}(\lambda_\alpha)$ (for some $\lambda_\alpha \in X^*(T)$) whose vanishing locus is the closure of a single codimension one stratum in $G$-ZipFlag$^\mu$ (and each such stratum is cut out by exactly one of the $h_\alpha$). The cone generated by the $\langle \lambda_\alpha \rangle_{\alpha \in \Delta}$ is called the partial Hasse invariant cone $C_{pHa}$ (Definition 3.6.1).
One has by construction $C_{\text{ph,}\lambda} \subseteq C_{\text{zip}}$. As an analogue of [DK23, Corollary 8.3], we ask whether $C_{\text{ph,}\lambda} = C_{\text{zip}}$ holds in general. Let $w_{0,L}$ be the longest element in the Weyl group $W_L = W(L,T)$. Let $\sigma$ denote the action of Frobenius on the based root datum of $(G,B,T)$. We show:

**Theorem 1** (Theorem 4.3.1). The following are equivalent:
(i) One has $C_{\text{ph,}\lambda} = C_{\text{zip}}$.
(ii) One has $C_{\text{GS}} \subseteq C_{\text{ph,}\lambda}$.
(iii) $L$ is defined over $\mathbb{F}_q$ and $\sigma$ acts on $\Delta_L$ by $-w_{0,L}$.

Pairs $(G,\mu)$ satisfying condition (iii) are called of Hasse-type. For a Shimura variety $S_K$ as above, we always have $C_{\text{ph,}\lambda} \subseteq C_{\text{zip}} \subseteq \mathcal{C}(\mathbb{F}_p)$. We deduce that a necessary condition for $\mathcal{C}(\mathbb{F}_p)$ to be generated by partial Hasse invariants is that $(G,\mu)$ is of Hasse-type. A classification of Hasse-type cases is given in an appendix by Wushi Goldring (see §A). For example, orthogonal Shimura varieties give rise to pairs $(G,\mu)$ of Hasse-type (see §7.2). Condition (ii) has also an interpretation for Shimura varieties. One can show in general that $C_K(\mathbb{C}) \subseteq C_K(\mathbb{F}_p)$ ([Kos19, Proposition 1.8.3]) and hence $\mathcal{C}(\mathbb{C}) \subseteq \mathcal{C}(\mathbb{F}_p)$. Since it is expected that $\mathcal{C}(\mathbb{C}) = \mathcal{C}_{\text{GS}}$, Condition (ii) is necessary for $C_{\text{ph,}\lambda} = \mathcal{C}(\mathbb{F}_p)$ to hold. From Conjecture 1 and Conjecture 2, we expect that the containment $\mathcal{C}_{\text{GS}} \subseteq \mathcal{C}_{\text{zip}}$ should hold in general, which is now a purely group-theoretical statement. We confirm this expectation:

**Theorem 2** (Theorem 6.4.3). For general $(G,\mu)$, we have $\mathcal{C}_{\text{GS}} \subseteq \mathcal{C}_{\text{zip}}$.

This theorem gives further evidence for Conjecture 2. In [Kos19, Corollary 3.5.6], Theorem 2 was proved only when $P$ is defined over $\mathbb{F}_q$. We now explain the proof of Theorem 2. The proof uses a general technique that makes it possible to reduce questions pertaining to $\mathcal{C}_{\text{zip}}$ to the case of a split group. In the split case, Theorem 2 is already known by [Kos19, Corollary 3.5.6]. We explain how we can reduce to the case of a split group. Denote by $L_0 \subseteq L$ the largest algebraic subgroup defined over $\mathbb{F}_q$. It is a Levi subgroup of $L$ containing $T$. There is a cocharacter $\mu_0$ with centralizer $L_0$, and we consider the pair $(G_{\mathbb{F}_{q_r}},\mu_0)$, where $r \geq 1$ is such that $G_{\mathbb{F}_{q_r}}$ is split. Denote by $C_{\text{zip}}(G_{\mathbb{F}_{q_r}},\mu_0)$ the zip cone of $(G_{\mathbb{F}_{q_r}},\mu_0)$ and $\mathcal{C}_{\text{zip}}(G_{\mathbb{F}_{q_r}},\mu_0)$ for its saturation. Let $w_{0,L}$ and $w_{0,L_0}$ be the longest elements in the Weyl groups of $L$ and $L_0$ respectively. Write $X^*_+(T)$ for the set of $L$-dominant characters. We show the following:

**Theorem 3** (Theorem 6.4.1). We have

$$X^*_+(L)(T) \cap \left( w_{0,L}w_{0,L_0}C_{\text{zip}}(G_{\mathbb{F}_{q_r}},\mu_0) \right) \subseteq C_{\text{zip}}.$$

This theorem is useful in general to reduce questions on $C_{\text{zip}}$ to the case of a split group, as explained in Remark 6.4.2. In particular, Theorem 3 reduces Theorem 2 to the case of a split group, for which it is already known. The proof of Theorem 3 relies on a closer study of the case when $G$ is a Weil restriction (see §3). Our final result is the construction of natural mod $p$ automorphic forms attached to the highest weight vectors of the representations $V_\ell(\lambda)$. Let $\lambda$ be an $L$-dominant character and let $f_\lambda \in V_\ell(\lambda)$ denote the highest weight vector of $V_\ell(\lambda)$. There is a natural way of defining the norm $f_\lambda := \text{Norm}_{L,\varphi}(f_\lambda)$ of $f_\lambda$. Here $L_\varphi$ is a certain finite (generally non-smooth) subgroup of $L$ containing $L_0(\mathbb{F}_q)$. There is an integer $m \geq 0$ determined by $L_\varphi$, such that the norm $\text{Norm}_{L_\varphi}(f_\lambda)$ is a section of $V_\ell(d\lambda)$ (where $d = q^m[L_0(\mathbb{F}_q)]$) over the $\mu$-ordinary locus $U_\mu$ of $G$-Zip $\mu$ (see §3.5 for details). For $\alpha \in \Delta$, let $r_\alpha$ be the smallest integer $r \geq 1$ such that $\sigma^r(\alpha) = \alpha$. 

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Theorem 4 (Proposition 3.5.1). The section $f_\lambda$ extends to $G$-Zip$^\mu$ if and only if for all $\alpha \in \Delta \setminus \Delta_L$ one has
\[ \sum_{w \in W_{\lambda}(\mathbb{F}_q)} \sum_{i=0}^{r_\alpha-1} q^{i+\ell(w)} \langle w\lambda, \sigma^i(\alpha^\vee) \rangle \leq 0. \quad (1) \]

Let $C_{hw}$ be the set of $L$-dominant characters $\lambda$ satisfying the above inequality (1). Theorem 4 shows that $C_{hw} \subset C_{zip}$, which provides another natural subcone of $C_{zip}$. We obtain a family of interesting automorphic forms $(f_\lambda)_{\lambda \in C_{hw}}$ in characteristic $p$ of weight $d_\lambda$ (by pullback via $\zeta$). There is also an analogue of Theorem 4 for the lowest weight vector (§5.2), and we define the lowest weight cone $C_{lw}$ similarly. When $P$ is defined over $\mathbb{F}_q$, one has $C_{lw} = C_{hw}$ but in general $C_{hw} \subset C_{lw}$.

The motivation for introducing the family $(f_\lambda)_{\lambda}$ is the following. As mentioned above, Diamond–Kassaei showed in [DK17] that the weight of any Hilbert modular form in characteristic $p$ is spanned by the weights of partial Hasse invariants. This is also true for the Siegel-type Shimura variety $A_2$, but it fails for $A_n$ when $n \geq 3$. In the case $n = 3$, Goldring and the second-named author showed that the weight of any automorphic form for $A_3$ is spanned by the weights of partial Hasse invariants and of the forms $(f_\lambda)_{\lambda \in C_{hw}}$. Therefore, these forms seem to have some significance for more general groups. Moreover, the vanishing locus of $f_\lambda$ is an interesting subvariety stable by Hecke operators, that we plan to investigate in future papers.

We briefly explain the content of each section. In §2 we review the stack of $G$-zips, vector bundles thereon and the connection with Shimura varieties. Section 3 is dedicated to the study of the cone $C_{zip}$, called the zip cone. We explain the motivation for introducing this set. We define several related subcones which arise naturally. We define automorphic forms on $G$-Zip$^\mu$ attached to highest weight vectors. In section 4, we consider pairs $(G, \mu)$ of Hasse-type and we give a complete characterization in terms of $C_{zip}$. In section 5, similarly to the highest weight vectors, we show that the lowest weight vectors give rise naturally to certain automorphic forms on $G$-Zip$^\mu$. In section 6, we study pairs $(G, \mu)$ where $G$ is the Weil restriction of a reductive group defined over an extension. This machinery makes it possible to reduce several questions to the case of a split group. Using this, we can check in full generality the expectation that $C_{GS} \subset C_{zip}$. Finally, in the last section, we illustrate the results in the case of a unitary group $U(2,1)$ and for odd orthogonal groups. In the appendix by Wushi Goldring, we give an exhaustive classification of pairs $(G, \mu)$ of Hasse-type.

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2 Preliminaries and reminders on the stack of $G$-zips

2.1 Notation

Throughout the paper, $p$ is a prime number, $q$ is a power of $p$ and $\mathbb{F}_q$ is the finite field with $q$ elements. We write $k = \overline{\mathbb{F}}_q$ for an algebraic closure of $\mathbb{F}_q$. The notation $G$ will always denote a connected reductive group over $\mathbb{F}_q$. For a $k$-scheme $X$, we denote by $X^{(q)}$ its $q$-th power Frobenius twist and by $\varphi : X \rightarrow X^{(q)}$ its relative Frobenius morphism. Write
\( \sigma \in \text{Gal}(k/F_\ell) \) for the \( \ell \)-power Frobenius. We will always write \((B, T)\) for a Borel pair of \( G \), i.e. \( T \subset B \subset G \) are a maximal torus and a Borel subgroup in \( G \). We do not assume that \( T \) is split over \( F_\ell \). Let \( B^+ \) be the Borel subgroup of \( G \) opposite to \( B \) with respect to \( T \) (i.e. the unique Borel subgroup \( B^+ \) of \( G \) such that \( B^+ \cap B = T \)). We will use the following notations:

- As usual, \( X^*(T) \) (resp. \( X_*(T) \)) denotes the group of characters (resp. cocharacters) of \( T \). The group \( \text{Gal}(k/F_\ell) \) acts naturally on these groups. Let \( W = W(G_k, T) \) be the Weyl group of \( G_k \). Similarly, \( \text{Gal}(k/F_\ell) \) acts on \( W \). Furthermore, the actions of \( \text{Gal}(k/F_\ell) \) and \( W \) on \( X^*(T) \) and \( X_*(T) \) are compatible in a natural sense.

- \( \Phi \subset X^*(T) \): the set of \( T \)-roots of \( G \).

- \( \Phi^+ \subset \Phi \): the system of positive roots with respect to \( B^+ \) (i.e. \( \alpha \in \Phi^+ \) when the \( \alpha \)-root group \( U_\alpha \) is contained in \( B^+ \)). This convention may differ from other authors. We use it to match the conventions of previous publications \([\text{GK}19\alpha] \), \([\text{Kos}19] \).

- \( \Delta \subset \Phi^+ \): the set of simple roots.

- For \( \alpha \in \Phi \), let \( s_\alpha \in W \) be the corresponding reflection. The system \((W, \{s_\alpha \mid \alpha \in \Delta\})\) is a Coxeter system. We write \( \ell : W \to \mathbb{N} \) for the length function. Hence \( \ell(s_\alpha) = 1 \) for all \( \alpha \in \Phi \). Let \( w_0 \) denote the longest element of \( W \).

- For a subset \( K \subset \Delta \), let \( W_K \) denote the subgroup of \( W \) generated by \( \{s_\alpha \mid \alpha \in K\} \). Write \( w_{0,K} \) for the longest element in \( W_K \).

- Let \( K \) (resp. \( K^{\pm} \)) denote the subset of elements \( w \in W \) which have minimal length in the coset \( W_Kw \) (resp. \( wW_K \)). Then \( K \) (resp. \( K^{\pm} \)) is a set of representatives of \( W_K \setminus W \) (resp. \( W \setminus W_K \)). The map \( g \to g^{-1} \) induces a bijection \( K \to K^{\pm} \). The longest element in the set \( K \) is \( w_{0,K} \).

- \( X_*(T) \) denotes the set of dominant characters, i.e. characters \( \lambda \in X_*(T) \) such that \( \langle \lambda, \alpha \rangle \geq 0 \) for all \( \alpha \in \Delta \).

- For a subset \( I \subset \Delta \), let \( X_*(I) \) denote the set of characters \( \lambda \in X_*(T) \) such that \( \langle \lambda, \alpha \rangle \geq 0 \) for all \( \alpha \in I \). We call them \( I \)-dominant characters.

- Let \( P \subset G_k \) be a parabolic subgroup containing \( B \) and let \( L \subset P \) be the unique Levi subgroup of \( P \) containing \( T \). Then we define a subset \( I_P \subset \Delta \) as the unique subset such that \( W(L, T) = W_{I_P} \). For an arbitrary parabolic subgroup \( P \subset G_k \) containing \( T \), we define \( I_P \subset \Delta \) as \( I_P : = I_{P'} \) where \( P' \) is the unique conjugate of \( P \) containing \( B \).

- For a parabolic \( P \subset G_k \), write \( \Delta^P : = \Delta \setminus I_P \).

- For all \( \alpha \in \Phi \), choose an isomorphism \( u_\alpha : G_a \to U_\alpha \) so that \( (u_\alpha)_{\alpha \in \Phi} \) is a realization in the sense of \([\text{Spr}98] \) 8.1.4]. In particular, we have

\[
tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad \forall x \in G_a, \quad \forall t \in T.
\]

- Let \( \phi_\alpha : SL_2 \to G \) denote the map attached to \( \alpha \), as in \([\text{Spr}98] \) 9.2.2]. It satisfies

\[
\phi_\alpha \left( \begin{array}{c} 1 & x \\ 0 & 1 \end{array} \right) = u_\alpha(x), \quad \phi_\alpha \left( \begin{array}{c} 0 & 1 \\ 1 & 0 \end{array} \right) = u_{-\alpha}(x).
\]

- Fix a \( B \)-representation \((V, \rho)\). For \( j \in \mathbb{Z} \) and \( \alpha \in \Phi \), we define a map \( E^{(j)}_\alpha : V \to V \) as follows. Let \( V = \bigoplus_{\nu \in X^*(T)} V_{\nu} \) be the weight decomposition of \( V \). For \( v \in V_{\nu} \), we can write uniquely

\[
u_0(x)v = \sum_{j \geq 0} x^j E^{(j)}_\alpha(v), \quad \forall x \in G_a,
\]

for elements \( E^{(j)}_\alpha(v) \in V_{\nu + j\alpha} \) ([IK21\alpha] Lemma 3.3.1]). Extend \( E^{(j)}_\alpha \) by additivity to a map \( V \to V \). For \( j < 0 \), put \( E^{(j)}_\alpha = 0 \).
2.2 The stack of $G$-zips

We recall some facts about the stack of $G$-zips of Pink–Wedhorn–Ziegler in [PWZ11].

2.2.1 Definitions

Let $G$ be a connected reductive group over $\mathbb{F}_q$. In this paper, a zip datum is a tuple $\mathcal{Z} := (G, P, L, Q, M, \varphi)$ consisting of the following objects:

(i) $P \subset G_k$ and $Q \subset G_k$ are parabolic subgroups of $G_k$.
(ii) $L \subset P$ and $M \subset Q$ are Levi subgroups such that $L^{(q)} = M$.

For an algebraic group $H$, denote by $R_u(H)$ the unipotent radical of $H$. If $P' \subset G_k$ is a parabolic subgroup with Levi subgroup $L' \subset P'$, any $x \in P'$ can be written uniquely as $x = \overline{x} u$ with $\overline{x} \in L'$ and $u \in R_u(P')$. We denote by $\theta_{L'}^P : P' \to L'$ the map $x \mapsto \overline{x}$. Since $M = L^{(q)}$, we have a Frobenius isogeny $\varphi : L \to M$. Put

$$E := \{ (x, y) \in P \times Q \mid \varphi(\theta_{L'}^{P}(x)) = \theta_{M}^{Q}(y) \}.$$

Equivalently, $E$ is the subgroup of $P \times Q$ generated by $R_u(P) \times R_u(Q)$ and elements of the form $(a, \varphi(a))$ with $a \in L$. Let $G \times G$ act on $G$ by $(a, b) \cdot g := agb^{-1}$, and let $E$ act on $G$ by restricting this action to $E$. The stack of $G$-zips of type $\mathcal{Z}$ ([PWZ11], [PWZ15]) can be defined as the quotient stack

$$G\text{-Zip}^{\mathcal{Z}} = [E\backslash G_k].$$

There is an equivalent definition of $G\text{-Zip}^{\mathcal{Z}}$ in terms of torsors. The stack $G\text{-Zip}^{\mathcal{Z}}$ is the stack over $k$ such that for all $k$-scheme $S$, the groupoid $G\text{-Zip}^{\mathcal{Z}}(S)$ is the category of tuples $I = (I, I_P, I_Q, \iota)$, where $I$ is a $G$-torsor over $S$, $I_P \subset I$ and $I_Q \subset I$ are respectively a $P$-subtorsor and a $Q$-subtorsor of $I$, and $\iota : (I_P/R_u(P))^{(p)} \to I_Q/R_u(Q)$ is an isomorphism of $M$-torsors.

2.2.2 Cocharacter datum

A **cocharacter datum** is a pair $(G, \mu)$ where $G$ is a reductive connected group over $\mathbb{F}_q$ and $\mu : \mathbb{G}_{m,k} \to G_k$ is a cocharacter. One can attach to $(G, \mu)$ a zip datum $\mathcal{Z}_\mu$, defined as follows. First, denote by $P_+(\mu)$ (resp. $P_-(-\mu)$) the unique parabolic subgroup of $G_k$ such that $P_+(\mu)(k)$ (resp. $P_-(\mu)(k)$) consists of the elements $g \in G(k)$ satisfying that the map

$$G_{m,k} \to G_k; \ t \mapsto \mu(t)g\mu(t)^{-1} \quad (\text{resp. } t \mapsto \mu(t)^{-1}g\mu(t))$$

extends to a morphism of varieties $\mathbb{A}^1_k \to G_k$. We obtain a pair of parabolics $(P_+(\mu), P_-(\mu))$ in $G_k$ whose intersection $P_+(\mu) \cap P_-(\mu) = L(\mu)$ is the centralizer of $\mu$ (it is a common Levi subgroup of $P_+(\mu)$ and $P_-(\mu)$). Set $P := P_-(\mu)$, $Q := (P_+(\mu))^{(q)}$, $L := L(\mu)$ and $M := (L(\mu))^{(q)}$. The tuple $\mathcal{Z}_\mu := (G, P, L, Q, M, \varphi)$ is a zip datum, which we call the zip datum attached to the cocharacter datum $(G, \mu)$. We write simply $G\text{-Zip}^{\mu}$ for $G\text{-Zip}^{\mathcal{Z}_\mu}$. We always consider zip data of this form.

**Remark 2.2.1.** A general zip datum $(G, P, L, Q, M, \varphi)$ is of the form $\mathcal{Z}_\mu$ for a cocharacter $\mu : \mathbb{G}_{m,k} \to G_k$ if and only if $\sigma(P)$ and $Q$ are opposite parabolic subgroups with common Levi $M = \sigma(L)$.

**Remark 2.2.2.** If $\mu$ is defined over $\mathbb{F}_q$, then so are $P$ and $Q$. In this case, we have $L = M$ and $P, Q$ are opposite parabolic subgroups with common Levi subgroup $L$. 

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2.2.3 Frames

Let \( Z = (G, P, Q, L, M) \) be a zip datum. In this paper, a frame for \( Z \) is a triple \( (B, T, z) \) where \( (B, T) \) is a Borel pair of \( G_k \) defined over \( \mathbb{F}_q \) satisfying
(i) One has the inclusion \( B \subset P \).
(ii) \( z \in W \) is an element satisfying the conditions
\[
^G B \subset Q \quad \text{and} \quad B \cap M = \text{ad}(z) B \cap M.
\]

We put \( B_M := B \cap M \). Other papers (PWZ11, PWZ15, KW18) use the convention \( B \subset Q \) instead of \( B \subset P \). A frame (as defined here) may not always exist. However, if \( (G, \mu) \) is a cocharacter datum and \( Z_\mu \) is the associated zip datum by 2.2.2, then there exists a \( G_k \)-conjugate \( \mu' = \text{ad}(g) \circ \mu \) (with \( g \in G(k) \)) such that \( Z_{\mu'} \) admits a frame. Hence, it is harmless to assume that a frame exists, and we only consider zip data that admit frames.

With respect to the Borel pair \( (B, T) \), we define subsets \( I, J, \Delta P \) of \( \Delta \) as follows:
\[
I := I_P, \quad J := I_Q, \quad \Delta^P = \Delta \setminus I.
\]

Lemma 2.2.3 ([GK19b, Lemma 2.3.4]). Let \( \mu : G_{m, k} \to G_k \) be a cocharacter, and let \( Z_\mu \) be the attached zip datum. Assume that \( (B, T) \) is a Borel pair defined over \( \mathbb{F}_q \) such that \( B \subset P \). Define the element
\[
z := w_0 w_0, J = \sigma(w_0, I) w_0.
\]
Then \( (B, T, z) \) is a frame for \( Z_\mu \).

2.2.4 Parametrization of the \( E \)-orbits in \( G \)

By [PWZ11, Proposition 7.1], there are finitely many \( E \)-orbits in \( G \). The \( E \)-orbits are smooth and locally closed in \( G \), and the Zariski closure of an \( E \)-orbit is a union of \( E \)-orbits. We review the parametrization of \( E \)-orbits following [PWZ11]. For \( w \in W \), fix a representative \( \dot{w} \in N_{G}(T) \), such that \( (w_1 w_2) = \dot{w}_1 \dot{w}_2 \) whenever \( \ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \) (this is possible by choosing a Chevalley system, [ABD+16, XXIII, §6]). For \( w \in W \), define \( G_w \) as the \( E \)-orbit of \( \dot{w} \dot{w}^{-1} \). If no confusion occurs, we write \( w \) instead of \( \dot{w} \). For \( w, w' \in I^W \), write \( w' \leq w \) if there exists \( w_1 \in W_I \) such that \( w' \leq w_1 w \sigma(w_1)^{-1} \). This defines a partial order on \( I^W \) ([PWZ11, Corollary 6.3]).

Theorem 2.2.4 ([PWZ11, Theorem 7.5, Theorem 11.2, Theorem 11.3, Theorem 11.5]). We have two bijections:
\[
I^W \to \{ E \text{-orbits in } G_k \}, \quad w \mapsto G_w \quad (2.2.1)
\]
\[
W^J \to \{ E \text{-orbits in } G_k \}, \quad w \mapsto G_w \quad (2.2.2)
\]
For \( w \in I^W \cup W^J \), one has \( \dim(G_w) = \ell(w) + \dim(P) \) and the Zariski closure of \( G_w \) is
\[
\overline{G_w} = \bigsqcup_{w' \in I^W, \ w' \leq w} G_{w'} \quad (2.2.3)
\]
for \( w \in I^W \), and
\[
\overline{G_w} = \bigsqcup_{w' \in W^J, \ w' \leq w} G_{w'} \quad (2.2.3)
\]
for \( w \in W^J \).
In particular, there is a unique open $E$-orbit $U_\mathcal{Z} \subset G$ corresponding to the longest elements $w_0, w_0 \in ^l W$ via (2.2.1) and to $w_0 w_0, j \in W^j$ via (2.2.2). If $\mathcal{Z} = \mathcal{Z}_\mu$ (see §2.2.2), write $U_\mu = U_{\mathcal{Z}_\mu}$. In this case, we can choose $z = w_0 w_0, j = \sigma(w_0) w_0$ (Lemma (2.2.3), hence (2.2.2) shows that $1 \in U_\mu$. Using the terminology pertaining to Shimura varieties, we call $U_\mu$ the $\mu$-ordinary stratum of $G$ and the substack $U_\mu := [E \setminus U_\mu]$ the $\mu$-ordinary locus. It corresponds to the $\mu$-ordinary locus in the good reduction of Shimura varieties, studied for example in [Moo04] and [Wor13]. For more details about Shimura varieties, see §2.6 below. For $w \in ^l W$ or $w \in W^j$, write $\mathcal{X}_w := [E \setminus G_w]$ for the corresponding locally closed substack of $G$-Zip$^\circ = [E \setminus G_k]$. We obtain similarly a stratification

$$G$$-Zip$^\circ = \bigsqcup_{w \in ^l W} \mathcal{X}_w$

and one has closure relations between strata similar to (2.2.3).

2.3 Vector bundles on the stack of $G$-zips

2.3.1 Representation theory

For an algebraic group $G$ over a field $K$, denote by $\text{Rep}(G)$ the category of algebraic representations of $G$ on finite-dimensional $K$-vector spaces. We denote a representation $\rho: G \to \text{GL}_K(V)$ by $(V, \rho)$, or sometimes simply $\rho$ or $V$. For an algebraic group $G$ over $\mathbb{F}_q$, a $G_k$-representation $(V, \rho)$ and an integer $m$, we denote by $(V^{[m]}, \rho^{[m]})$ the representation such that $V^{[m]} = V$ and

$$\rho^{[m]}: G_k \xrightarrow{\varphi^m} G_k \xrightarrow{\rho} \text{GL}(V).$$

Let $H$ be a split connected reductive $K$-group and choose a Borel pair $(B_H, T)$ defined over $K$. If $K$ has characteristic zero, $\text{Rep}(H)$ is semisimple. In characteristic $p$ however, this is no longer true in general. For $\lambda \in X^+_w(T)$, let $L_\lambda$ be the line bundle attached to $\lambda$ on the flag variety $H/B_H$ by the usual associated sheaf construction ([Jan03] §5.8]). Define an $H$-representation $V_H(\lambda)$ by:

$$V_H(\lambda) := H^0(H/B_H, L_\lambda)$$

(2.3.1)

In other words, one has $V_H(\lambda) = \text{Ind}^H_{B_H} \lambda$. The representation $V_H(\lambda)$ is of highest weight $\lambda$. If char($K$) = 0, the representation $V_H(\lambda)$ is irreducible. We view elements of $V_H(\lambda)$ as regular maps $f: H \to \mathbb{A}^1$ satisfying

$$f(hb) = \lambda(b^{-1}) f(h), \quad \forall h \in H, \forall b \in B_H.$$  

(2.3.2)

For dominant characters $\lambda, \lambda'$, there is a natural surjective map

$$V_H(\lambda) \otimes V_H(\lambda') \to V_H(\lambda + \lambda').$$

(2.3.3)

In the description given by (2.3.2), this map is $f \otimes f' \mapsto f f'$ (for $f \in V_H(\lambda)$, $f' \in V_H(\lambda)$). Denote by $W_H := W(H, T)$ the Weyl group and $w_{0,H} \in W_H$ the longest element. Then $V_H(\lambda)$ has a unique $B_H$-stable line, which is a weight space for the weight $w_{0,H}\lambda$.

2.3.2 Vector bundles on quotient stacks

For an algebraic stack $\mathcal{X}$, write $\mathcal{VB}(\mathcal{X})$ for the category of vector bundles on $\mathcal{X}$. Let $X$ be a $k$-scheme and $H$ an affine $k$-group scheme acting on $X$. If $\rho: H \to \text{GL}(V)$ is an algebraic representation of $H$, it gives rise to a vector bundle $V_{H,X}(\rho)$ on the stack $[H \setminus X]$. This
vector bundle can be defined geometrically as \([H \setminus (X \times_k V)]\) where \(H\) acts diagonally on \(X \times_k V\). We obtain a functor

\[
\mathcal{V}_{H,X}: \text{Rep}(H) \to \mathfrak{VB}(H \setminus X).
\]  

Similarly to the usual associated sheaf construction [Jan03 §5.8, equation (1)], the global sections of \(\mathcal{V}_{H,X}(\rho)\) are given by

\[
H^0([H \setminus X], \mathcal{V}_{H,X}(\rho)) = \{ f: X \to V \mid f(h \cdot x) = \rho(h)f(x), \forall h \in H, \forall x \in X \}.
\]

2.3.3 Vector bundles on \(G\)-Zip\(^\mu\)

Fix a cocharater datum \((G, \mu)\), let \(\mathcal{Z} = (G, P, L, Q, M, \varphi)\) be the attached zip datum. Fix a frame \((B, T)\) as in 2.2.3 By (2.3.4), we have a functor \(\mathcal{V}_{E,G}: \text{Rep}(E) \to \mathfrak{VB}(G\text{-Zip}\(^\mu\))\), that we simply denote by \(\mathcal{V}\). For \((V, \rho) \in \text{Rep}(E)\), the global sections of \(\mathcal{V}(\rho)\) are

\[
H^0(G\text{-Zip}\(^\mu\), \mathcal{V}(\rho)) = \{ f: G_k \to V \mid f(\epsilon \cdot g) = \rho(\epsilon)f(g), \forall \epsilon \in E, \forall g \in G_k \}.
\]

Since \(G\) admits an open dense \(E\)-orbit (see discussion below Theorem 2.2.4), the space \(H^0(G\text{-Zip}\(^\mu\), \mathcal{V}(\rho))\) is finite-dimensional ([Kos19 Lemma 1.2.1]). The first projection \(p_1: E \to P\) induces a functor \(p_1^*: \text{Rep}(P) \to \text{Rep}(E)\). If \((V, \rho) \in \text{Rep}(P)\), we write again \(\mathcal{V}(\rho)\) for \(\mathcal{V}(p_1^*(\rho))\). In this paper, we only consider \(E\)-representations coming from \(P\) in this way. Let \(\theta_L^P: P \to L\) be the natural projection modulo \(R_u(P)\), as in §2.2.1 It induces a fully faithful functor

\[
(\theta_L^P)^*: \text{Rep}(L) \to \text{Rep}(P)
\]

whose image is the full subcategory of \(\text{Rep}(P)\) of \(P\)-representations trivial on \(R_u(P)\). Hence, we view \(\text{Rep}(L)\) as a full subcategory of \(\text{Rep}(P)\). If \((V, \rho) \in \text{Rep}(L)\), write again \(\mathcal{V}(\rho)\) for \(\mathcal{V}(\theta_L^P(\rho))\). For \(\lambda \in X^*(T)\), write \(B_L := B \cap L\) and define an \(L\)-representation \((V_I(\lambda), \rho_{I,\lambda})\) as follows

\[
V_I(\lambda) = \text{Ind}_B^L \lambda, \quad \rho_{I,\lambda}: L \to \text{GL}(V_I(\lambda)).
\]

This is the representation defined in (2.3.1) for \(H = L\) and \(B_H = B_L\). Let \(V_I(\lambda)\) be the vector bundle on \(G\text{-Zip}\(^\mu\)\) attached to \(V_I(\lambda)\), and call it an \textit{automorphic vector bundle} on \(G\text{-Zip}\(^\mu\)\) associated to \(\lambda\). This terminology stems from Shimura varieties (see §2.6 below for further details). For \(\lambda \in X^*(L)\), viewing \(\lambda\) as an element of \(X^*(T)\) by restriction, the vector bundle \(V_I(\lambda)\) is a line bundle. Note that if \(\lambda \in X^*(T)\) is not \(I\)-dominant, then \(V_I(\lambda) = 0\) and thus \(V_I(\lambda) = 0\).

2.4 Global sections over \(G\text{-Zip}\(^\mu\))

We review some results of [IK21a] regarding the global sections of \(\mathcal{V}(\rho)\) for a \(P\)-representation \(\rho\). We start with sections over the open substack \(U_\mu \subset G\text{-Zip}\(^\mu\). Recall that \(U_\mu = [E \setminus U_\rho]\) and \(1 \in U_\mu\) (see (2.2.4)). By (2.3.5), an element of \(H^0(U_\mu, \mathcal{V}(\rho))\) can be viewed a map \(h: G \to V\) satisfying \(h(ab^{-1}) = \rho(a)h(g)\) for all \((a, b) \in E\) and all \(g \in G\). Since the \(E\)-orbit of 1 is open dense in \(G\) (see paragraph after Theorem 2.2.4), the map \(h \mapsto h(1)\) is an injection

\[
\text{ev}_1: H^0(U_\mu, \mathcal{V}(\rho)) \to V.
\]

We give the image of this map. Let \(L_\varphi\) be the scheme-theoretical stabilizer subgroup of 1 in \(E\). By definition, one has

\[
L_\varphi = E \cap \{(x, x) \mid x \in G_k\}.
\]
which is a 0-dimensional algebraic group (in general non-smooth). The first projection \( E \to P \) induces a closed immersion \( L_\varphi \to P \). Identify \( L_\varphi \) with its image and view it as a subgroup of \( P \). Denote by \( L_0 \subset L \) the largest algebraic subgroup defined over \( \mathbb{F}_q \). In other words,

\[
L_0 = \bigcap_{n \geq 0} L^{(n)}.
\]

(2.4.3)

**Lemma 2.4.1 ([IK21a, Corollary 3.2.3]).** The map \((2.4.1)\) induces an identification

\[
H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L_\varphi}.
\]

Here, the notation \( V^{L_\varphi} \) denotes the space of scheme-theoretical invariants, i.e. the set of \( v \in V \) such that for any \( k \)-algebra \( R \), one has \( \rho(x)v = v \) in \( V \otimes_k R \) for all \( x \in L_\varphi(R) \). We now consider the space of global sections over \( G\text{-}\text{Zip}^\mu \). Restriction of sections to \( \mathcal{U}_\mu \subset G\text{-}\text{Zip}^\mu \) induces an injective map \( H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)) \to H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L_\varphi} \). For simplicity, we assume here that \( P \) is defined over \( \mathbb{F}_q \) (for the general result, see [IK21a, Theorem 3.4.1]).

We will need the general version in the proof of Proposition 3.4.1 but in the simple setting when \( \rho \) is a character \( L \to \mathbb{G}_m \). For \( \alpha \in \Phi \), choose a realization \((u_\alpha)_{\alpha \in \Phi} \) (see §2.1). Fix a \( P \)-representation \((V, \rho)\) and let \( V = \bigoplus_{\nu \in X^*(T)^\sigma} V_\nu \) be its \( T \)-weight decomposition. Define the Brylinski–Kostant filtration (cf. [XZ19] (3.3.2)) indexed by \( c \in \mathbb{R} \) on \( V_\nu \) by:

\[
\text{Fil}^c_{\nu} V_\nu = \bigcap_{j > c} \ker \left( E_{\alpha}^{(j)} : V_\nu \to V_{\nu+j_\alpha} \right)
\]

where the map \( E_{\alpha} \) was defined in §2.1. For \( \chi \in X^*(T)^\sigma \) and \( \nu \in X^*(T) \), set also

\[
\text{Fil}^c_{\chi} V_\nu = \bigcap_{\alpha \in \Delta^\nu} \text{Fil}^c_{(\chi, \alpha \nu)} V_\nu.
\]

The Lang torsor morphism \( \varphi : T \to T, \ g \mapsto g\varphi(g)^{-1} \) induces isomorphisms:

\[
\begin{align*}
\varphi^* : X^*(T)_\mathbb{R} & \xrightarrow{\sim} X^*(T)_\mathbb{R}; \ \lambda \mapsto \lambda \circ \varphi = \lambda - q\sigma^{-1}(\lambda) \\
\varphi_* : X_*(T)_\mathbb{R} & \xrightarrow{\sim} X_*(T)_\mathbb{R}; \ \delta \mapsto \varphi \circ \delta = \delta - q\sigma(\delta).
\end{align*}
\]

(2.4.4)

**Theorem 2.4.3 ([IK21a, Corollary 3.4.2]).** Assume that \( P \) is defined over \( \mathbb{F}_q \). For all \((V, \rho) \in \text{Rep}(P)\), the map \( \text{ev}_{V_1} \) induces an identification

\[
H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(P)} \cap \bigoplus_{\nu \in X^*(T)} \text{Fil}_{\rho^{-1}(\nu)}^c V_\nu.
\]

In the general case of an arbitrary parabolic \( P \), \( V^{L(P)} \) is replaced by \( V^{L_\varphi} \) and \( \text{Fil}^c_{\nu} V_\nu \) is replaced by a generalized Brylinski–Kostant filtration (see [IK21a, Theorem 3.4.1]). In the special case when \( \rho \) is trivial on \( R_\alpha(P) \), Theorem 2.4.3 simplifies greatly. Set \( \delta_\alpha := \varphi^{-1}(\alpha^\vee) \) and define a subspace \( V^\Delta_{\geq 0} \subset V \) by
\[ V_{\geq 0}^{\Delta^P} = \bigoplus_{(\nu, \delta\nu) \geq 0, \forall \alpha \in \Delta^P} V_{\nu}. \]  

(2.4.5)

If \( T \) is split over \( \mathbb{F}_q \), then \( \delta_{\alpha} = -\alpha^\vee/(q - 1) \), and \( V_{\geq 0}^{\Delta^P} \) is the direct sum of the weight spaces \( V_{\nu} \) for those \( \nu \in X^*(T) \) satisfying \( \langle \nu, \alpha^\vee \rangle \leq 0 \) for all \( \alpha \in \Delta^P \).

**Corollary 2.4.4.** Assume that \( P \) is defined over \( \mathbb{F}_q \) and furthermore that \( (V, \rho) \in \text{Rep}(P) \) is trivial on \( R_u(P) \). Then one has

\[ H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) = V^{L(F_q)} \cap V_{\geq 0}^{\Delta^P}. \]

2.5 The stack of \( G \)-zip flags

2.5.1 Definition

Let \((G, \mu)\) be a cocharacter datum with attached zip datum \( Z_{\mu} = (G, P, L, Q, M, \varphi) \) (§2.2.2). Fix a frame \((B, T, z)\) with \( z = \sigma(w_0, i)w_0 = w_0w_0, j \) (Lemma 2.2.3). The stack of zip flags ([GK19a Definition 2.1.1]) is defined as

\[ G\text{-ZipFlag}^\mu = [E \backslash (G_k \times P/B)] \]

where the group \( E \) acts on the variety \( G_k \times (P/B) \) by the rule \((a, b) \cdot (g, hB) := (agb^{-1}, ahB)\) for all \((a, b) \in E\) and all \((g, hB) \in G_k \times P/B\). The first projection \( G_k \times P/B \to G_k \) is \( E \)-equivariant, and yields a natural morphism of stacks

\[ \pi: G\text{-ZipFlag}^\mu \to G\text{-Zip}^\mu. \]  

(2.5.1)

Similarly to the stack \( G\text{-Zip}^\mu \), there is an interpretation of this stack in terms of torsors ([GK19a Definition 2.1.1]). For any \( k \)-scheme \( S \), the \( S \)-points of \( G\text{-ZipFlag}^\mu \) are pairs \((J, L)\) where \( L = (I, I_P, I_Q, \iota) \) is a \( G \)-zip over \( S \) (see the end of §2.2.1) and \( J \subset I_P \) is a \( B \)-torsor. The map (2.5.1) is given by forgetting the \( B \)-torsor \( J \).

Set \( E' := E \cap (B \times G_k) \). Then the injective map \( G_k \to G_k \times P/B; g \mapsto (g, B) \) yields an isomorphism of stacks \([E' \backslash G_k] \cong G\text{-ZipFlag}^\mu \) (see [GK19a (2.1.5)]). We recall the stratification of \( G\text{-ZipFlag}^\mu \). First, define the Schubert stack as the quotient stack

\[ \text{Sbt} := [B \backslash G_k/B]. \]

This stack is finite and smooth. Its topological space is isomorphic to \( W \), endowed with the topology induced by the Bruhat order on \( W \). This follows easily from the Bruhat decomposition of \( G \). One can show that \( E' \subset B \times zB \). In particular, there is a natural projection map \([E' \backslash G_k] \to [B \backslash G_k/zB]\). Composing with the isomorphism \([B \backslash G_k/zB] \to [B \backslash G_k/B]\) induced by \( G_k \to G_k; g \mapsto gz \), we obtain a smooth, surjective map

\[ \psi: G\text{-ZipFlag}^\mu \to \text{Sbt}. \]

For \( w \in W \), put \( \text{Sbt}_w := [B \backslash BwB/B] \); it is a locally closed substack of \( \text{Sbt} \). The flag strata of \( G\text{-ZipFlag}^\mu \) are defined as fibers of the map \( \psi \). They are locally closed substacks (endowed with the reduced structure). Concretely, let \( w \in W \) and put

\[ F_w := B(wz^{-1})zB = BwBz^{-1}, \]
which is the $B \times zB$-orbit of $wz^{-1}$. The set $F_w$ is locally closed in $G_k$, and one has \( \dim(F_w) = \ell(w) + \dim(B) \). Then, via the isomorphism $G\text{-ZipFlag}^\mu \simeq [E'\setminus G_k]$, the flag strata of $G\text{-ZipFlag}^\mu$ are the locally closed substacks

$$\mathcal{F}_w := [E'\setminus F_w], \quad w \in W.$$  \hfill (2.5.2)

The set $F_{w_0} \subset G_k$ is open in $G_k$ and similarly the stratum $\mathcal{F}_{w_0}$ is open in $G\text{-ZipFlag}^\mu$. The $B \times zB$-orbits of codimension 1 are $F_{w,\alpha}$ for $\alpha \in \Delta$. The Zariski closure $\overline{\mathcal{F}}_w$ is normal ([RRS5 Theorem 3]) and coincides with $\bigcup_{w' \leq w} F_{w'}$, where $\leq$ is the Bruhat order of $W$.

### 2.5.2 Vector bundles on $G\text{-ZipFlag}^\mu$

Let $\rho : B \to \text{GL}(V)$ be an algebraic representation, and view $\rho$ as a representation of $E'$ via the first projection $E' \to B$. Via the isomorphism $G\text{-ZipFlag}^\mu \simeq [E'\setminus G_k]$, we obtain a vector bundle $\mathcal{V}_{\text{flag}}$ on $G\text{-ZipFlag}^\mu$. Let $(V, \rho) \in \text{Rep}(P)$ and let $\mathcal{V}(\rho)$ be the attached vector bundle on $G\text{-Zip}^\mu$. Then one has

$$\pi^*(\mathcal{V}(\rho)) = \mathcal{V}_{\text{flag}}(\rho|B).$$

Note that the rank of $\mathcal{V}_{\text{flag}}(\rho)$ is the dimension of $\rho$. In particular, if $\lambda \in X^*(B)$, then $\mathcal{V}_{\text{flag}}(\lambda)$ is a line bundle. For $(V, \rho) \in \text{Rep}(B)$, consider the $P$-representation $\text{Ind}_B^P(\rho)$ defined by

$$\text{Ind}_B^P(\rho) = \{ f : P \to V \mid f(bx) = \rho(b^{-1})f(x), \forall b \in B, \forall x \in P \}.$$  

For $y \in P$ and $f \in \text{Ind}_B^P(\rho)$, the element $y \cdot f$ is the function $x \mapsto f(y^{-1}x)$.

**Proposition 2.5.1** ([IK21b, Proposition 3.2.1]). For $(V, \rho) \in \text{Rep}(B)$, we have the identification $\pi_*(\mathcal{V}_{\text{flag}}(\rho)) = \mathcal{V}(\text{Ind}_B^P(\rho))$. In particular $\pi_*(\mathcal{V}_{\text{flag}}(\rho))$ is a vector bundle on $G\text{-Zip}^\mu$.

In particular, if $\rho$ is a character $\lambda \in X^*(T)$, then $\mathcal{V}_{\text{flag}}(\lambda)$ is a line bundle and one has:

$$\pi_*(\mathcal{V}_{\text{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$$

where the vector bundle $\mathcal{V}_I(\lambda)$ was defined in §2.3.3. Hence, we have

$$H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda)) = H^0(G\text{-ZipFlag}^\mu, \mathcal{V}_{\text{flag}}(\lambda)).$$  \hfill (2.5.3)

If $f : G_k \to k$ is a section of the right hand side of (2.5.3), then the corresponding function $f_I : G_k \to V_I(\lambda)$ on the left hand side of (2.5.3) is given by

$$f_I(g)(x) = f((x^{-1}, \varphi(x)^{-1}) \cdot g) = f(x^{-1}g\varphi(x))$$  \hfill (2.5.4)

for all $g \in G_k$ and $x \in L$, by the construction of the identification. Note also that the line bundles $\mathcal{V}_{\text{flag}}(\lambda)$ satisfy the following identity:

$$\mathcal{V}_{\text{flag}}(\lambda + \lambda') = \mathcal{V}_{\text{flag}}(\lambda) \otimes \mathcal{V}_{\text{flag}}(\lambda'), \quad \forall \lambda, \lambda' \in X^*(T).$$  \hfill (2.5.5)

We can also define vector bundles on the stack Sbt as in [IK21b] §4. For our purpose, it is enough to define line bundles on Sbt. Using (2.5.4), we can attach to each $(\chi_1, \chi_2) \in X^*(T) \times X^*(T)$ a line bundle $\mathcal{V}_{\text{Sbt}}(\chi_1, \chi_2)$ on Sbt. One has

$$\psi^*\mathcal{V}_{\text{Sbt}}(\chi_1, \chi_2) = \mathcal{V}_{\text{flag}}(\chi_1 + (z\chi_2) \circ \varphi) = \mathcal{V}_{\text{flag}}(\chi_1 + q\sigma^{-1}(z\chi_2)).$$  \hfill (2.5.6)
2.5.3 Partial Hasse invariants

We recall some results of [IK21b]. By [GK19a Theorem 2.2.1(a)], the line bundle $\mathcal{V}_{\text{Sbt}}(\chi_1, \chi_2)$ admits a nonzero section over $\text{Sbt}_{w_0}$ if and only if $\chi_1 = -w_0 \chi_2$. If this condition is satisfied, $H^0(\text{Sbt}_{w_0}, \mathcal{V}_{\text{Sbt}}(\chi_1, \chi_2))$ is one-dimensional.

For $\chi \in X^*(T)$, let $h_\chi$ be any nonzero element

$$h_\chi \in H^0(\text{Sbt}_{w_0}, \mathcal{V}_{\text{Sbt}}(-w_0 \chi, \chi)).$$

The divisor of $h_\chi$ is given by Chevalley’s formula (loc. cit., Theorem 2.2.1(c)):

$$\text{div}(h_\chi) = \sum_{\alpha \in \Delta} \langle \chi, \alpha^\vee \rangle B_{w_0 s_\alpha} B,$$

where we view $h_\chi$ as a rational map on $G$. In particular, $h_\chi$ extends to $\text{Sbt}$ if and only if $\chi$ is a dominant character. Using (2.5.6) and $z = \sigma(w_{0,f})w_0$, we obtain a section

$$\text{Ha}_\chi := \psi^*(h_\chi) \in H^0(\mathcal{F}_{w_0}, \mathcal{V}_{\text{flag}}(-w_0 \chi + qw_{0,f}w_0(\sigma^{-1} \chi))).$$

and for $\chi \in X^+_+(T)$ the section $\text{Ha}_\chi$ extends to $G\text{-ZipFlag}^\text{\alpha}$. In particular, let $\alpha \in \Delta$ and suppose $\chi_\alpha$ is a character satisfying

$$\begin{cases} 
\langle \chi_\alpha, \alpha^\vee \rangle > 0 \\
\langle \chi_\alpha, \beta^\vee \rangle = 0 & \text{for all } \beta \in \Delta \setminus \{\alpha\}. 
\end{cases}$$

(2.5.7)

In this case, the section $h_{\chi_\alpha}$ vanishes exactly on the codimension one stratum $\overline{\text{Sbt}}_{w_0 s_\alpha}$. Similarly, the section $\text{Ha}_{\chi_\alpha}$ cuts out the Zariski closure of the codimension one stratum $\mathcal{F}_{w_0 s_\alpha}$.

**Definition 2.5.2.** For $\alpha \in \Delta$ and $\chi_\alpha$ satisfying (2.5.7), we call the section $\text{Ha}_{\chi_\alpha}$ a partial Hasse invariant for the stratum $\mathcal{F}_{w_0 s_\alpha}$.

2.6 Shimura varieties and G-zips

We explain the connection between the stack of G-zips and Shimura varieties. Let $(G, X)$ be a Shimura datum [Del79 2.1.1]. In particular, $G$ is a connected reductive group over $\mathbb{Q}$. Furthermore, $X$ provides a well-defined $G(\mathbb{Q})$-conjugacy class of cocharacters $\{\mu\}$ of $G_{\mathbb{Q}}$. Write $E = E(G, X)$ for the reflex field of $(G, X)$ (i.e. the field of definition of $\{\mu\}$) and $\mathcal{O}_E$ for its ring of integers. Given an open compact subgroup $K \subset G(A_f)$, write $\text{Sh}(G, X)_K$ for Deligne’s canonical model at level $K$ over $E$ (see [Del79]). For $K \subset G(A_f)$ small enough, $\text{Sh}(G, X)_K$ is a smooth, quasi-projective scheme over $E$. Every inclusion $K' \subset K$ induces a finite étale projection $\pi_{K'/K} : \text{Sh}(G, X)_{K'} \to \text{Sh}(G, X)_K$.

Let $g \geq 1$ and let $(V, \psi)$ be a $2g$-dimensional, non-degenerate symplectic space over $\mathbb{Q}$. Write $\text{GSp}(2g) = \text{GSp}(V, \psi)$ for the group of symplectic similitudes of $(V, \psi)$. Write $X_g$ for the double Siegel half-space [Del79 1.3.1]. The pair $(\text{GSp}(2g), X_g)$ is called the Siegel Shimura datum and has reflex field $\mathbb{Q}$. Recall that $(G, X)$ is called of Hodge-type if there exists an embedding of Shimura data $\iota : (G, X) \hookrightarrow (\text{GSp}(2g), X_g)$ for some $g \geq 1$. Henceforth, assume $(G, X)$ is of Hodge-type.

Fix a prime number $p$, and assume that the level $K$ is of the form $K = K_p K^p$ where $K_p \subset G(\mathbb{Q}_p)$ is a hyperspecial subgroup and $K^p \subset G(A_f^p)$ is an open compact subgroup. Recall that a hyperspecial subgroup of $G(\mathbb{Q}_p)$ exists if and only if $G_{\mathbb{Q}_p}$ is unramified, and one has $K_p = \mathcal{G}(\mathbb{Z}_p)$ where $\mathcal{G}$ is a reductive group over $\mathbb{Z}_p$ such that $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq G_{\mathbb{Q}_p}$ and $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is connected.
For any place $v$ above $p$ in $E$, Kisin ([Kis10]) and Vasiu ([Vas99]) constructed a smooth, canonical model $\mathcal{S}_\lambda$ over $\mathcal{O}_{E_v}$ of $Sh(G, X)_K$. Let $\kappa(v)$ denote the residue field of $\mathcal{O}_{E_v}$ and let $\bar{F}_p$ be an algebraic closure of $\kappa(v)$. Set $S_K := \mathcal{S}_\lambda \otimes_{\mathcal{O}_{E_v}} \bar{F}_p$. We can take a representative $\mu \in \{\mu\}$ defined over $E_v$ by [Kot84] (1.1.3) Lemma (a). We can also assume that $\mu$ extends to $\mu : \mathcal{G}_{m, \mathcal{O}_{E_v}} \to \mathcal{G}_{\mathcal{O}_{E_v}}$ ([Kim18] Corollary 3.3.11]). It gives rise to a parabolic subgroup $\mathcal{P} \subset \mathcal{G}_{\mathcal{O}_{E_v}}$, with Levi subgroup $\mathcal{L}$ equal to the centralizer of $\mu$. As explained in [IK21a, §2.5], we can assume (after possibly twisting $\mu$) that there is a Borel pair $(\mathcal{B}, \mathcal{F})$ over $\mathcal{S}$ such that $\mathcal{P}_{\mathcal{O}_{E_v}} \subset \mathcal{P}$. Let $G, P, B, T$ denote the special fibers of $\mathcal{G}, \mathcal{P}, \mathcal{B}, \mathcal{F}$ respectively. By slight abuse of notation, we denote again by $\mu$ its mod $p$ reduction $\mu : \mathcal{G}_{m, k} \to G_k$. Then $(G, \mu)$ is a cocharacter datum, and it yields a zip datum $(G, P, L, Q, M, \varphi)$ as in [2.2.2] (in the context of Shimura varieties, $G$ is an $\mathbb{F}_p^\times$-group, so we always take $q = p$; in particular $\varphi$ is the $p$-th power Frobenius homomorphism).

By a result of Zhang ([Zha18, 4.1]), there exists a natural smooth morphism

$$\zeta : S_K \to G^{Zip}_\mu. \quad (2.6.1)$$

This map is also surjective by [SYZ21, Corollary 3.5.3(1)]. The map $\zeta$ amounts to the existence of a universal $G$-zip $\mathcal{Z} = (I_G, I_P, I_Q, \iota)$ over $S_K$, using the description of $G^{Zip}_\mu$ provided at the end of 2.2.1. In the construction of Zhang, the $G$-torsor $I_G$ and the $P$-torsor $I_P$ over $S_K$ are the reduction of a $\mathcal{G}$-torsor and a $\mathcal{P}$-torsor over $\mathcal{S}_\lambda$, that we denote by $\mathcal{J}_G$ and $\mathcal{J}_P$ respectively. We have a commutative diagram of functors

$$\begin{align*}
\text{Rep}_{\mathcal{S}_\mu}(\mathcal{P}) & \xrightarrow{\psi} \text{VB}(\mathcal{J}_G) \\
\text{Rep}_{\mathcal{S}_\mu}(P) & \xrightarrow{\psi} \text{VB}(S_K)
\end{align*}$$

where the vertical arrows are reduction modulo $p$ and the horizontal arrows are obtained by applying the $\mathcal{P}$-torsor $\mathcal{J}_P$ and the $P$-torsor $I_P$ respectively. The vector bundles obtained in this way on $\mathcal{S}_\lambda$ and $S_K$ are called automorphic vector bundles following [Mih90, III, Remark 2.3].

For each $\mathbf{L}$-dominant character $\lambda \in X^*_+ (T)$, we have the unique irreducible representation $V_I(\lambda)$ of $\mathcal{P}$ over $\mathbb{F}_p$ of highest weight $\lambda$. Since we are in characteristic zero, $V_I(\lambda)$ coincides with $H^0(P/B, \mathcal{L}_\lambda)$, as defined in 2.3.1 in §2.3.1. It admits a natural model over $\mathbb{Z}_p$, namely

$$V_I(\lambda)_{\mathbb{Z}_p} := H^0(\mathcal{P}/\mathcal{B}, \mathcal{L}_\lambda),$$

where $\mathcal{L}_\lambda$ is the line bundle attached to $\lambda$. Its reduction modulo $p$ is the $P$-representation $V_I(\lambda) = H^0(P/B, \mathcal{L}_\lambda)$ over $k = \mathbb{F}_p$. By the above discussion, we obtain a vector bundle $V_I(\lambda)$ on $\mathcal{S}_\lambda$ attached to the $\mathcal{P}$-representation $V_I(\lambda)_{\mathbb{Z}_p}$. The vector bundle $V_I(\lambda)$ for $\lambda \in X^*_+ (T)$ is called the automorphic vector bundles of weight $\lambda$. We denote again by $V_I(\lambda)$ the vector bundle obtained on the special fiber $S_K$. For an $\mathcal{O}_{E_v}$-algebra $R$, we will call the space $H^0(\mathcal{S}_\lambda \otimes_{\mathcal{O}_{E_v}} R, V_I(\lambda))$ the space of automorphic forms of level $K$ and weight $\lambda$ with coefficients in $R$.

The flag space of the Siegel modular variety $A_n$ was first introduced by Ekedahl–van der Geer in [EvG09]. It parametrizes pairs $(\mathcal{A}, \mathcal{F})$ where $\mathcal{A} = (A, \lambda) \in A_n$ is a principally polarized abelian variety of rank $n$ and $\mathcal{F} \subset H^1_{\text{dR}}(A)$ is a full symplectic flag refining the Hodge filtration of $H^1_{\text{dR}}(A)$. In general, we defined in [GK19a, §9.1] the flag space $\text{Flag}(S_K)$.
of $S_K$ as the fiber product
\[
\begin{array}{ccc}
\text{Flag}(S_K) & \xrightarrow{\zeta_{\text{flag}}} & G\text{-ZipFlag}^\mu \\
\pi_K \downarrow & & \downarrow \pi \\
S_K & \xrightarrow{\zeta} & G\text{-Zip}^\mu
\end{array}
\]

The stratification $(F_w)_{w \in W}$ on $G\text{-ZipFlag}^\mu$ induces by pullback via $\zeta_{\text{flag}}$ a stratification $(\text{Flag}(S_K)_w)_{w \in W}$ of $\text{Flag}(S_K)$ by locally closed, smooth subschemes. Moreover, we obtain a line bundle $V_\text{flag}(\lambda)$ on $\text{Flag}(S_K)$. Since $\zeta$ is smooth and $\pi$ is proper, pullback and pushforward commute, so we have $\pi_{K, *} (V_\text{flag}(\lambda)) = V_\mathcal{I}(\lambda)$. In particular, the space of automorphic forms $H^0(S_K, V_\mathcal{I}(\lambda))$ identifies with $H^0(\text{Flag}(S_K), V_\text{flag}(\lambda))$.

3 The zip cone

In this section, we will consider several subsets of $X^*(T)$. A cone in $X^*(T)$ will be an additive submonoid containing 0. If $C \subset X^*(T)$ is a cone, we define its saturation (or saturated cone) as follows
\[
C = \{ \lambda \in X^*(T) \mid \exists N \geq 1, \ N\lambda \in C \}.
\]
We say that $C$ is saturated if $C = \mathcal{C}$. Define also $C_{Q, 0}$ as follows
\[
C_{Q, 0} = \left\{ \sum_{i=1}^{N} a_i \lambda_i \mid N \geq 1, \ a_i \in \mathbb{Q}_{\geq 0}, \ \lambda_i \in C \right\}.
\]
There is a bijection between saturated cones of $X^*(T)$ and additive submonoids of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ stable by $\mathbb{Q}_{\geq 0}$. The bijection is given by the maps $C \mapsto C_{Q, 0}$ and $C' \mapsto C' \cap X^*(T)$.

3.1 Example: Hilbert–Blumenthal Shimura varieties

We recall some results of Diamond–Kassaei in [DK17] and extended in [DK23] that motivate this paper. We give a short explanation of [DK17] Corollary 5.4. The authors study Hilbert automorphic forms in characteristic $p$. Specifically, let $F/\mathbb{Q}$ be a totally real extension of degree $d = [F : \mathbb{Q}]$ and let $G$ be the subgroup of $\text{Res}_{F/\mathbb{Q}}(\text{GL}_2(F))$ defined by
\[
G(R) = \{ g \in \text{GL}_2(R \otimes_{\mathbb{Q}} F) \mid \det(g) \in R^\times \}.
\]
Let $p$ be a prime number unramified in $F$ (in [DK23], $p$ is allowed to be ramified in $F$). The lattice $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F \subset \mathbb{Q}_p \otimes_{\mathbb{Q}} F$ gives rise to a reductive model $\mathcal{G}$ over $\mathbb{Z}_p$. Fix a small enough level $K^p \subset \mathcal{G}(A^p_f)$ outside $p$ and set $K_p := \mathcal{G}(\mathbb{Z}_p)$ and $K = K_p K^p$. Let $S_K$ be the (geometric) special fiber of the corresponding Hilbert–Blumenthal Shimura variety of level $K$. The scheme $S_K$ is smooth of dimension $d$ over $\mathbb{F}_p$. It parametrizes tuples $(A, \lambda, \iota, \eta)$ of abelian schemes over $\mathbb{F}_p$ of dimension $d$ endowed with a principal polarization $\lambda$, an action $\iota$ of $\mathcal{O}_F$ on $A$ and a $K^p$-level structure $\eta$.

Let $\Sigma := \text{Hom}(F, \mathbb{Q}_p)$ be the set of field embeddings $F \rightarrow \mathbb{Q}_p$. Write $(e_\tau)_\tau$ for the canonical basis of $\mathbb{Z}^\Sigma$. Let $\sigma$ denote the action of Frobenius on $\Sigma$. For each $\tau \in \Sigma$, there is an associated line bundle $\omega_\tau$ on $S_K$. For $k = \sum_{\tau} k_\tau e_\tau \in \mathbb{Z}^\Sigma$, define
\[
\omega^k := \bigotimes_{\tau \in \Sigma} \omega^{k_\tau}_\tau.
\]
Elements of $H^0(X_{\mathbb{F}_p}, \omega^k)$ are called mod $p$ Hilbert modular forms of weight $k$. There is an Ekedahl–Oort stratification on $S_K$ given by the isomorphism class of the $p$-torsion $A[p]$ (with its additional structure given by $\lambda$ and $\iota$). There is a unique open stratum (on which $A$ is an ordinary abelian variety). The codimension one strata can be labeled as $(S_{K, r})_{r \in \Sigma}$. Andreattà–Goren ([AG05]) constructed partial Hasse invariants $\mathcal{H}_r$ for each $r \in \Sigma$. The weight of $\mathcal{H}_r$ is given by

$$\mathbf{h}_r := e_r - p e_{r-1, r}.$$ 

Note that the sign of $\mathbf{h}_r$ is different in [AG05], due to a different convention of positivity. The main property of $\mathcal{H}_r$ is that it vanishes exactly on the Zariski closure of the codimension one stratum $S_{K, r}$. It is a special case of the sections $\mathcal{H}_\alpha$ defined in Definition 2.5.2.

Define the partial Hasse invariant cone $\mathcal{C}_{\mathfrak{ph}, \alpha} \subset \mathbb{Z}^\Sigma$ as the cone of $k \in \mathbb{Z}^\Sigma$ which are spanned (over $\mathbb{Q}_{\geq 0}$) by the weights $(\mathbf{h}_r)_{r \in \Sigma}$ defined above.

**Theorem 3.1.1** (Diamond–Kassaei, [DK17] Theorem 5.1, Corollary 5.4).

1. Let $f \in H^0(S_K, \omega^k)$ and assume that $pk_r > k_{\sigma-1, r}$. Then $f$ is divisible by $\mathcal{H}_r$.
2. If $H^0(S_K, \omega^k) \neq 0$, then $k \in \mathcal{C}_{\mathfrak{ph}, \alpha}$.

The authors define a minimal cone $\mathcal{C}_{\min} \subset \mathcal{C}_{\mathfrak{ph}, \alpha}$ as follows:

$$\mathcal{C}_{\min} = \{ k \in \mathbb{Z}^\Sigma \mid pk_r \leq k_{\sigma-1, r} \text{ for all } r \in \Sigma \}.$$ 

Theorem 3.1.1 shows that any Hilbert modular form $f$ of weight $k$ can be written as a product $f = f_{\min} h$, where $f_{\min}$ has weight $k_{\min} \in \mathcal{C}_{\min}$ and $h$ is a product of partial Hasse invariants. In particular (2) is a direct consequence of (1). One motivation of this paper is to understand the natural setting in which one might expect a generalization to other Shimura varieties of Theorem 3.1.1. In [GK22a], Goldring and the second-named author show that (1) also admits a similar generalization for several Hodge-type Shimura varieties.

### 3.2 General setting

We attempt to give an abstract setting in which Theorem 3.1.1 may generalize. First, by observing the example of Hilbert–Blumenthal varieties, we extract the essential properties of the objects we want to study. Specifically, we consider a stack $Y$ over $k = \mathbb{F}_p$ endowed with the following structure:

(a) There is a locally closed stratification $Y = \bigsqcup_{i=1}^N Y_i$ such that the Zariski closure of a stratum is a union of strata.

(b) There is a free, finite-type $\mathbb{Z}$-module $\Lambda$ and a family of line bundles $(\omega(\lambda))_{\lambda \in \Lambda}$ on $Y$, such that $\omega(\lambda + \lambda') = \omega(\lambda) \otimes \omega(\lambda')$ for all $\lambda, \lambda' \in \Lambda$.

(c) For each codimension one stratum $Y_i \subset Y$, there is $\lambda_i \in \Lambda$ and $\mathcal{H}_i \in H^0(X, \omega(\lambda_i))$ such that the support of $\operatorname{div}(\mathcal{H}_i)$ is $\overline{Y}_i$. By analogy, we call $\mathcal{H}_i$ a partial Hasse invariant for $Y_i$.

Denote by $I_1 \subset I$ the indices such that $Y_i$ has codimension one. Let $C_{\mathfrak{ph}} \subset \Lambda$ denote the cone generated by the elements $\{ \lambda_i \mid i \in I_1 \}$, and call it the partial Hasse invariant cone. Put

$$C_Y := \{ \lambda \in \Lambda \mid H^0(Y, \omega(\lambda)) \neq 0 \}.$$ 

By definition, one has $C_{\mathfrak{ph}} \subset C_Y$. If $Y$ is connected, then $C_Y$ is a cone (i.e. an additive submonoid) of $\Lambda$. Indeed, if $\lambda, \lambda' \in \Lambda$ and $f, f'$ are nonzero sections of $\omega(\lambda)$ and $\omega(\lambda')$ respectively, then $ff'$ is a section of $\omega(\lambda + \lambda')$. Since $Y$ is connected, $ff'$ is nonzero. Write $\mathcal{C}_{\mathfrak{ph}}$ and $\mathcal{C}_Y$ for the saturation of $C_{\mathfrak{ph}}$ and $C_Y$ inside $\Lambda$, respectively.
Definition 3.2.1. Let \( Y \) be a stack satisfying (a), (b) and (c). We say that \( Y \) has the Hasse property if \( \mathcal{C}_{\text{pHa}} = \mathcal{C}_Y \).

For example, Theorem [3.1.1 (2)] shows that the geometric special fiber of the Hilbert–Blumenthal Shimura variety at a place of good reduction satisfies the Hasse property. Let \((G, \mu)\) be a cocharacter datum and let \(G\text{-Zip}^\mu\) be the attached stack of \(G\)-zips. Fix a frame \((B, T, z)\) as in §2.2.3. Then, the stack of zip flags \(G\text{-ZipFlag}^\mu(3.2.1)\) satisfies all requirements (a), (b) and (c) above. First, we have the flag stratification \(G\text{-ZipFlag}^\mu = \bigsqcup_{w \in W} \mathcal{F}_w\) as in §2.5.2. Setting \( \Lambda := X^*(T), \) we have the family of line bundles \((\mathcal{V}_I(\lambda))_{\lambda \in \mathcal{X}^*(T)}\) satisfying (b) by (2.5.3). Finally, we have partial Hasse invariants (2.5.3). In this paper, we give a full answer as to whether \(G\text{-ZipFlag}^\mu\) satisfies the Hasse property.

Similarly, let \( Y \) be a scheme endowed with a smooth, surjective morphism \( \zeta : Y \to G\text{-Zip}^\mu. \) Then \( Y \) inherits naturally by pullback all the structure from \(G\text{-ZipFlag}^\mu\), and hence satisfies all required properties (a), (b) and (c) above. In particular, if we start with a scheme \( X \) and a smooth, surjective morphism \( \zeta : X \to G\text{-Zip}^\mu \), then we can consider the flag space \( Y := \text{Flag}(X) \) (similarly to the flag space of \( S_K \) defined at the end of §2.5). It is defined as the fiber product

\[
Y := X \times_{G\text{-Zip}^\mu} G\text{-ZipFlag}^\mu. \tag{3.2.1}
\]

The induced map \( \zeta_{\text{flag}} : Y \to G\text{-ZipFlag}^\mu \) is again smooth and surjective. Hence, \( Y \) inherits the structure as above and satisfies (a), (b) and (c). Denote by \( \pi : Y \to X \) and \( \pi : G\text{-ZipFlag}^\mu \to G\text{-Zip}^\mu \) the natural projections. In both cases, we have \( \pi_* (\mathcal{V}_I(\lambda)) = \mathcal{V}_I(\lambda) \) because \( \zeta \) is smooth and \( \pi \) is proper. Therefore, the cones \( \mathcal{C}_Y \) and \( \mathcal{C}_{G\text{-ZipFlag}^\mu} \) can also be written as follows:

\[
\mathcal{C}_Y = \{ \lambda \in \mathcal{X}^*(T) \mid H^0(X, \mathcal{V}_I(\lambda)) \neq 0 \}, \tag{3.2.2}
\]

\[
\mathcal{C}_{\text{zip}} := \mathcal{C}_{G\text{-ZipFlag}^\mu} = \{ \lambda \in \mathcal{X}^*(T) \mid H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0 \}. \tag{3.2.3}
\]

We will use the notation \( \mathcal{C}_{\text{zip}} \) (introduced in [Kos19]), instead of \( \mathcal{C}_{G\text{-ZipFlag}^\mu} \). When \( Y \) is given as (3.2.1) above, we call \( Y \) the flag space of \((X, \zeta)\). Furthermore, we make the slight abuse of saying that \((X, \zeta)\) satisfies the Hasse property if \((Y, \zeta_{\text{flag}})\) does. In particular, let \( X = S_K \) be the geometric special fiber modulo \( p \) of a Hodge-type Shimura variety with good reduction at \( p \). By Zhang’s result, there is a smooth, surjective morphism \( \zeta : X \to G\text{-Zip}^\mu \), and so we obtain \((Y, \zeta_{\text{flag}})\) as above. Our goal is to investigate which Hodge-type Shimura varieties satisfy the Hasse property.

We now return to a general pair \((X, \zeta)\). Since \( \zeta \) is surjective, pullback by \( \zeta_{\text{flag}} \) induces an inclusion

\[
H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda)) \subset H^0(X, \mathcal{V}_I(\lambda)).
\]

Hence we have inclusions \( \mathcal{C}_{\text{zip}} \subset \mathcal{C}_Y \) and \( \mathcal{C}_{\text{zip}} \subset \mathcal{C}_Y \). Furthermore, Hasse invariants exist already on \(G\text{-ZipFlag}^\mu\) by section 2.5.3, hence the cone \( \mathcal{C}_{\text{pHa}} \) generated by their weights satisfies \( \mathcal{C}_{\text{pHa}} \subset \mathcal{C}_{\text{zip}} \). Therefore, we have in general

\[
\mathcal{C}_{\text{pHa}} \subset \mathcal{C}_{\text{zip}} \subset \mathcal{C}_Y.
\]

In particular, if the pair \((X, \zeta)\) satisfies the Hasse property, then all three cones above coincide. In other words, a necessary condition for \( X \) to satisfy the Hasse property is that \(G\text{-Zip}^\mu\) itself satisfies this property, which is equivalent to the condition \( \mathcal{C}_{\text{zip}} = \mathcal{C}_{\text{pHa}} \). This is an obstruction for a potential generalization of Theorem [3.1.1 (2)] to other Shimura varieties.
When we start with a pair \((X, \zeta)\) and construct \((Y, \zeta_{\text{flag}})\) by fiber product as in (3.2.1), formula (3.2.2) shows immediately that
\[
C_Y \subset X^*_+(T). \tag{3.2.4}
\]
Indeed, this follows simply from the fact that if \(\lambda\) is not \(I\)-dominant, then \(\mathcal{V}_I(\lambda) = 0\). Thus, in the example of Shimura varieties, we have the inclusion (3.2.4).

### 3.3 Previous results

We review previous results from [GK18]. Let \((X, \zeta)\) be a pair consisting of a \(k\)-scheme \(X\) and a smooth, surjective morphism of stacks \(\zeta: X \rightarrow G\text{-Zip}^\mu\), and let \((Y, \zeta_{\text{flag}})\) be the flag space of \(X\). We make the following assumption:

**Assumption 3.3.1.**

(A) For any \(w \in W\) with \(\ell(w) = 1\), the closed stratum \(\overline{Y}_w = \zeta_{\text{flag}}^{-1}(\overline{\mathcal{I}}_w)\) is pseudo-complete (i.e. any element of \(H^0(\overline{Y}_w, \mathcal{O}_{\overline{Y}_w})\) is locally constant on \(\overline{Y}_w\) for the Zariski-topology).

(B) The restriction \(\zeta\) to any connected component \(X^0 \subset X\) is smooth and surjective.

For example, Condition (A) is satisfied if \(X\) is a proper \(k\)-scheme. In general, it can happen that the inclusion \(C_{\text{flag}} \subset C_{\text{zip}}\) is strict. In this case, it is impossible for \(Y\) to satisfy the Hasse property. However, Goldring and the second-named author conjectured in general:

**Conjecture 3.3.2.** Under Assumption 3.3.1, we have \(C_Y = C_{\text{zip}}\).

Let \(S_K\) be the special fiber of a Hodge-type Shimura variety at a prime \(p\) of good reduction. In this case, we write \(C_K(\overline{\mathcal{F}}_p)\) for the cone \(C_Y\), i.e.
\[
C_K(\overline{\mathcal{F}}_p) := \{\lambda \in X^*_+(T) \mid H^0(S_K, \mathcal{V}_I(\lambda)) \neq 0\}. \tag{3.3.1}
\]

By [Kos19, Corollary 1.5.3], the saturation of \(C_K(\overline{\mathcal{F}}_p)\) is independent of \(K\), so we simply denote it by \(C(\overline{\mathcal{F}}_p)\). Let \(\zeta: S_K \rightarrow G\text{-Zip}^\mu\) be the map (2.6.1). We do not know whether the pair \((X, \zeta)\) always satisfies condition (A) of Assumption 3.3.1. However, by [GK19a, Theorem 6.2.1], Zhang’s map \(\zeta: S_K \rightarrow G\text{-Zip}^\mu\) admits an extension to a toroidal compactification
\[
\zeta^\Sigma: S_K^\Sigma \rightarrow G\text{-Zip}^\mu
\]
where \(\Sigma\) is a sufficiently fine cone decomposition. By construction, the pullback \(\mathcal{V}_I^\Sigma(\lambda) := \zeta^{\Sigma*}(\mathcal{V}_I(\lambda))\) is the canonical extension of \(\mathcal{V}_I(\lambda)\) to \(S_K^\Sigma\). Furthermore, by [And23, Theorem 1.2], the map \(\zeta^\Sigma\) is smooth. Since \(\zeta\) is surjective, \(\zeta^\Sigma\) is also surjective. By [WZ23, Proposition 6.20], any connected component \(S^\circ \subset S_K^\Sigma\) intersects the unique zero-dimensional stratum. Since \(\zeta^\Sigma: S^\circ \rightarrow G\text{-Zip}^\mu\) is smooth, it has an open image, therefore it must be surjective. In particular, the pair \((S_K^\Sigma, \zeta^\Sigma)\) satisfies Conditions (A) and (B). Furthermore, in most cases Koecher’s principle holds by [LS18, Theorem 2.5.11], i.e. we have an equality
\[
H^0(S_K^\Sigma, \mathcal{V}_I^\Sigma(\lambda)) = H^0(S_K, \mathcal{V}_I(\lambda)).
\]

In particular, the cone attached to the pair \((S_K^\Sigma, \zeta^\Sigma)\) is the same as the cone attached to \((S_K, \zeta)\), namely \(C_K(\overline{\mathcal{F}}_p)\). Therefore, by the above discussion, we deduce that Conjecture 3.3.2 applies to Shimura varieties and predicts the following:

**Conjecture 3.3.3.** If \(S_K\) is the special fiber of a Hodge-type Shimura variety at a prime \(p\) of good reduction, we have \(C(\overline{\mathcal{F}}_p) = C_{\text{zip}}\).
In [GK18 Theorem D], the authors proved that certain Shimura varieties satisfy the Hasse property. Specifically, they showed the following:

**Theorem 3.3.4 ([GK18 Theorem D]).** Let \((X, \zeta)\) be a pair which satisfies Assumption 3.3.1 and let \((Y, \zeta_{\text{flag}})\) be the flag space of \(X\). Suppose that \((G, \mu)\) is one of the following three pairs:

1. \(G \) is an \(F_p\)-form of \(GL_n^\alpha\) for some \(n \geq 1\), and \(\mu\) is non-trivial on each factor;
2. \(G = GL_{3, F_p}\), and \(\mu : z \mapsto \text{diag}(z, z, 1)\);
3. \(G = \text{GSp}(4, F_p)\), and \(\mu : z \mapsto \text{diag}(z, z, 1, 1)\).

Then \((X, \zeta)\) satisfies the Hasse property. In other words, we have \(C_Y = C_{\text{zip}} = C_{\text{pHa}}\).

The above theorem also holds if we change the group \(G\) to a group with the same adjoint group. By the above discussion, Theorem 3.3.4 applies to Hilbert–Blumenthal Shimura varieties, Picard surfaces at a split prime, Siegel modular threefolds and shows that Conjecture 3.3.2 holds in each case. Goldring and the second-named author proved Conjecture 3.3.2 in [GK22a] for certain Shimura varieties for which the inclusion \(C_{\text{pHa}} \subset C_{\text{zip}}\) is strict. Namely, they showed Conjecture 3.3.2 for the Siegel modular variety \(A_3\) as well as unitary Shimura varieties of signature \((r, s)\) with \(r + s \leq 4\) at split or inert primes, except when \(r = s = 2\) and \(p\) is inert. With the exception of the case \(r = s = 2\) and \(p\) split, the inclusion \(C_{\text{pHa}} \subset C_{\text{zip}}\) is strict in each of these cases.

### 3.4 First properties of \(C_{\text{zip}}\)

Let \((G, \mu)\) be a cocharacter datum over \(F_q\) and \(Z_\mu = (G, P, L, Q, M, \varphi)\) the attached zip datum (§2.2.2). Fix a frame \((B, T, z)\) with \(z = \sigma(w_0, 1)w_0\) (see §2.2.2). Let \(C_{\text{zip}} \subset X^*(T)\) be the zip cone, defined in (3.2.3). We start with some elementary properties of \(C_{\text{zip}}\). As we already noted, we have \(C_{\text{zip}} \subset X_{\text{flag}}^*(T)\). Furthermore, the cone \(C_{\text{zip}}\) has maximal rank in \(X^*(T)\), in the sense that \(\text{Span}_Q(C_{\text{zip}}) = X^*(T) \otimes_Q Q\). This was shown in [GK19a Lemma 3.4.2] (with the notation of loc. cit., \(C_{\text{reg}} \subset C_{\text{zip}}\) and \(C_{\text{reg}}\) has maximal rank). Note that the cocharacter datum is assumed to be Hodge-type in [GK19a §3.4], but this assumption is unnecessary for [GK19a Lemma 3.4.2].

Next, we consider line bundles on \(G\text{-Zip}^\alpha\). Recall that \(V_I(\lambda)\) is a line bundle if and only if \(\lambda \in X^*(L)\) (viewed as a subgroup of \(X^*(T)\)). Define the following set:

\[
X^*_\text{reg}(L) = \{\lambda \in X^*(L) \mid \langle \lambda, \alpha^\vee \rangle < 0, \forall \alpha \in \Delta^P\}.
\] (3.4.1)

These characters were termed \(L\)-ample in [GK19a Definition N.5.1]. The notation used in (3.4.1) is more enlightening, since these characters are in particular in \(X^*_\text{reg}(T)\) (the cone of anti-dominant characters). An immediate consequence of [KW18 Theorem 5.1.4] is the inclusion \(X^*_\text{reg}(L) \subset C_{\text{zip}}\). Set \(X^*_L := X^*(T) \cap X^*(L)\). The stronger inclusion \(X^*_L \subset C_{\text{zip}}\) is claimed in [Kos19 Proposition 1.6.1] with an incomplete proof, so we give one below:

**Proposition 3.4.1.** We have \(X^*_L \subset C_{\text{zip}}\).

**Proof.** Let \(\lambda \in X^*_L \cap X^*(L)\). Applying [IK21a Theorem 3.4.1] to the one-dimensional \(L\)-representation \(V_I(\lambda)\), we obtain:

\[
H^0(G\text{-Zip}^\alpha, V_I(\lambda)) = V_I(\lambda)^L_{\varphi} \cap \bigcap_{\alpha \in \Delta^P} \text{Fil}_{\Delta^P, \alpha} V_\lambda.
\]

Furthermore, \(\text{Fil}_{\Delta^P, \alpha} V_\lambda = V_\lambda = V_I(\lambda)\) if \(\langle \lambda, \delta_\alpha \rangle \geq 0\) and is 0 otherwise. Let \(d_\alpha \geq 1\) be an integer such that \(\alpha\) is defined over \(F_{q^{d_\alpha}}\). We find that \(\delta_\alpha = -\frac{1}{q_{d_\alpha} - 1} \sum_{i=0}^{d_\alpha-1} q^i \sigma^i(\alpha^\vee)\).
Since $\lambda \in X^* (T)$, we have $\langle \lambda , \sigma^r (\alpha^\vee) \rangle \leq 0$ for all $i$, hence $\langle \lambda , \delta_\alpha \rangle \geq 0$. We deduce $H^0 (G \cdot \text{Zip}^\mu , \mathcal{V}_I (\lambda)) = V_I (\lambda)^L$. Finally, if we change $\lambda$ to $N\lambda$ where $N$ divides the order of the finite group scheme $L_\varphi$, we obtain $H^0 (G \cdot \text{Zip}^\mu , \mathcal{V}_I (\lambda)) = V_I (\lambda)$. In particular, this space is nonzero, and this proves the result.

\section{Norm of the highest weight vector}

Recall that we always have $1 \in U_\mu$, where $U_\mu \subset G_k$ is the unique open $E$-orbit. Recall the definition of the finite subgroup $L_\varphi \subset L$ given in (2.4.2). Put $N_\varphi = |L_0 (\mathbb{F}_q)|q^m$ where $L_0$ is the Levi subgroup defined in (2.4.3) and $m \geq 0$ is the smallest integer such that the finite unipotent group $L_\varphi^0$ is annihilated by $\varphi^m$. For $\lambda , \lambda' \in X^* (T)$ and $f \in V_I (\lambda)$, $f' \in V_I (\lambda')$, let $ff' \in V_I (\lambda + \lambda')$ be the image of $f \otimes f'$ by the map (2.3.3). For $\lambda \in X^* (T)$ and $f \in V_I (\lambda)$ define

$$\text{Norm}_{L_\varphi} (f) := \left( \prod_{s \in L_0 (\mathbb{F}_q)} s \cdot f \right)^{q^m} \in V_I (N_\varphi \lambda). \quad (3.5.1)$$

It is clear that $\text{Norm}_{L_\varphi} (f)$ is $L_\varphi$-invariant. In particular, it gives rise to an element in $H^0 (U_\mu , \mathcal{V}(N_\varphi \lambda))$ by Lemma 2.4.2. In general, it is difficult to determine whether $\text{Norm}_{L_\varphi} (f)$ extends to a global section. However, this is possible when $f$ is a highest weight vector, as we now explain.

Let $f_x \in V_I (\lambda)$ be a nonzero element in the highest weight line of $V_I (\lambda)$. The following result generalizes [Kos19, Theorem 2] (where $P$ was assumed to be defined over $\mathbb{F}_p$, here we do not make this assumption). For $\alpha \in \Delta^P$, denote by $r_\alpha$ the smallest integer $r \geq 1$ such that $\sigma^r (\alpha) = \alpha$.

\textbf{Proposition 3.5.1.} The section $\text{Norm}_{L_\varphi} (f_x)$ extends to a global section over $G \cdot \text{Zip}^\mu$ if and only if for all $\alpha \in \Delta^P$, the following holds:

$$\sum_{w \in W_{E_\alpha} (\mathbb{F}_q)} \sum_{i=0}^{r_\alpha - 1} q^{i+\ell(w)} \langle w\lambda , \sigma^i (\alpha^\vee) \rangle \leq 0. \quad (3.5.2)$$

Before giving the proof, we need to recall some facts from [IK21a §3.1]. First, we have

$$G_k \setminus U_\mu = \bigcup_{\alpha \in \Delta^P} Z_\alpha , \quad Z_\alpha = E \cdot s_\alpha$$

where $E \cdot s_\alpha$ denotes the $E$-orbit of $s_\alpha$ and the bar denotes the Zariski closure. This follows easily from Theorem 2.2.4. For any $\alpha \in \Delta^P$, define an open subset

$$X_\alpha := G_k \setminus \bigcup_{\beta \in \Delta^P , \beta \neq \alpha} Z_\beta .$$

Then $U_\mu \subset X_\alpha$ and $X_\alpha \setminus U_\mu$ is irreducible. Choose a realization $(u_\alpha)_{\alpha \in \Phi}$ and let $\phi_\alpha : SL_2 \rightarrow G$ be the map attached to $\alpha$ (see 2.2.1). Set $Y := E \times \mathbb{A}^1$ and $Y_0 := E \times \mathbb{G}_m$. For $\alpha \in \Delta^P$, define $\psi_\alpha : Y \rightarrow G$ by

$$\psi_\alpha : ((x , y) , t) \mapsto x\phi_\alpha (A (t)) y^{-1} \quad \text{where} \quad A (t) = \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}_{2,k}.$$

It satisfies $\psi_\alpha ((x , y) , t) \in X_\alpha$ for all $((x , y) , t) \in Y$ and $\psi_\alpha ((x , y) , t) \in U_\mu$ if and only if $t \neq 0$ (see [IK21a Proposition 3.1.4]).
We now prove Proposition \ref{prop:main}. We use a similar argument as in [Kos19] Theorem 3.5.3. Set $U'_\mu := \pi^{-1}(U_\mu)$, where $\pi \colon G$-ZipFlag$^\mu \to G$-Zip$^\mu$ is the natural projection. One has clearly $U'_\mu \cong [E' \setminus U_\mu]$ via the isomorphism $G$-ZipFlag$^\mu \cong [E' \setminus G]$ explained in §2.5. We have an identification

$$H^0(U_\mu, V_I(N_{\varphi} \lambda)) = H^0(U'_\mu, V_{\text{flag}}(N_{\varphi} \lambda))$$

similarly to \eqref{eq:ident}. In particular, we can view $\text{Norm}_{L_\varphi}(f_\lambda)$ as a function $h \colon U_\mu \to \mathbb{A}^1$ satisfying the relation $h(a x b^{-1}) = \lambda(a)h(x)$ for all $(a, b) \in E'$ and $x \in U_\mu$ (using \eqref{eq:ident}). Specifically, the function $h$ is given by

$$h(x_1 x_2^{-1}) = \text{Norm}_{L_\varphi}(f_\lambda)(\theta^P_L(x_1)^{-1})$$

\hfill \eqref{eq:ident}

for all $(x_1, x_2) \in E$ using \eqref{eq:ident}. The function $h$ is well-defined because $\text{Norm}_{L_\varphi}(f_\lambda)$ is $L_\varphi$-invariant. Furthermore, $\text{Norm}_{L_\varphi}(f_\lambda)$ extends to $G$-Zip$^\mu$ if and only if $h$ extends to a function $G \to \mathbb{A}^1$. By the strategy explained in [Kos19] §3.2 and in [IK21a] §3.1, the function $h$ extends to $G$ if and only if for each $\alpha \in \Delta^P$, the function $h \circ \psi_\alpha \colon Y_0 \to \mathbb{A}^1$ extends to a function $Y \to \mathbb{A}^1$. It remains to compute the $t$-valuation of the function $h \circ \psi_\alpha$, viewed as an element of $R[t, \frac{1}{t}]$ where $R = k[E]$ is the ring of functions of $E$. Put

$$m_\alpha = \min \{m \geq 1 \mid \sigma^{-m}(\alpha) \notin I\}, \quad \alpha \in \Delta^P$$

and $t_\alpha = t^{-1}a(\varphi(\delta_\alpha(t)))^{-1} = t\alpha(\delta_\alpha(t))^{-1} \in t^Q$, where $t$ is an indeterminate and $\delta_\alpha = \varphi^{-1}(\alpha^Q)$ as defined in §2.4. Set

$$u_{t,\alpha} = \prod_{i=1}^{m_\alpha-1} \varphi_{\sigma^{-i}(\alpha)} \left( \begin{array}{cc} 1 & 1 - t_\alpha^{2i} \\ 0 & 1 \end{array} \right)$$

where the product is taken in increasing order of indices. By the proof of [IK21a] Proposition 3.1.4, for all $(x, y) \in E$ and $t \in \mathbb{G}_m$, we can write $\psi_\alpha((x, y), t) = x_1 x_2^{-1}$ with $(x_1, x_2) \in E$ and

$$x_1 = x \varphi_{\alpha} \left( \begin{array}{cc} 1 & 0 \\ -t^{-1} & 1 \end{array} \right) \delta_\alpha(t) u_{t,\alpha}.$$

By definition of $m_\alpha$, all the roots $\sigma^{-i}(\alpha)$ (for $1 \leq i \leq m_\alpha - 1$) appearing in the formula of $u_{t,\alpha}$ lie in $I$. Using \eqref{eq:ident}, we deduce:

$$h \circ \psi_\alpha((x, y), t) = \text{Norm}_{L_\varphi}(f_\lambda)(u_{t,\alpha}^{-1} \delta_\alpha(t)^{-1} \theta^P_L(x)^{-1})$$

$$= \left( \prod_{s \in L_0(\mathbb{F}_q)} f_\lambda(su_{t,\alpha}^{-1} \delta_\alpha(t)^{-1} \theta^P_L(x)^{-1})^{-1} \right)^{q^m}.$$

Consider the element $f_\lambda(su_{t,\alpha}^{-1} \delta_\alpha(t)^{-1} \theta^P_L(x)^{-1})$, which lies in $R[t^Q]$. We can still speak of the $t$-valuation of this element, which is a rational number. Equivalently, to simplify notation, we change $\theta^P_L(x)^{-1}$ to a generic element $g \in L$ and we compute the $t$-valuation of $F_\alpha(t, g) := f_\lambda(su_{t,\alpha}^{-1} \delta_\alpha(t)^{-1} g)$, viewed as an element of $k[L][t^Q]$. Let $v_\alpha(s)$ be this valuation. We put $B^+_L = B^+ \cap L$. Define a parabolic subgroup of $L$ by $Q_0 := L_0B^+_L$. It is clear that $u_{t,\alpha}$ lies in $R_0(Q_0)$, thus for all $s \in L_0(\mathbb{F}_q)$, we have $su_{t,\alpha}^{-1} s^{-1} \in R_0(Q_0)$. Since $f_\lambda$ is invariant by $R_0(B^+_L)$, we obtain $F_\alpha(t, g) = f_\lambda(s \delta_\alpha(t)^{-1} g)$. Now, the rest of the proof is completely similar to [Kos19] Theorem 3.5.3. We recall it briefly.

Let $B_{L_0} := B \cap L_0$ and $B^+_{L_0} := B^+ \cap L_0$. If we change $s$ to $bs$ with $b \in B^+_{L_0}(\mathbb{F}_q)$, then $v_\alpha(bs) = v_\alpha(s)$. Indeed, this follows from $f_\lambda(bs \delta_\alpha(t)^{-1} g) = \lambda(b)^{-1} f_\lambda(s \delta_\alpha(t)^{-1} g)$ since $f_\lambda$
We denote by 

\[ L_0(F_q) = \bigcup_{w \in W_{L_0}(F_q)} B^+_L(F_q)wB^+_L(F_q) \]

as in [Kos19] Lemma 3.4.4. By [Kos19] Lemma 3.4.5, one has

\[ |B^+_L(F_q)wB^+_L(F_q)| = |T(F_q)|q^{\dim(R_u(B_L))} q^w. \]

Thus, we can determine completely \( v_\alpha \) from the values \( v_\alpha(w) \) for \( w \in W_{L_0}(F_q) \). Similarly to [Kos19] Proposition 3.5.2, we have \( v_\alpha(w) = \langle w\lambda, \delta_\alpha \rangle \). We deduce that the \( t \)-valuation of \( h \circ \psi_\alpha((x, y), t) \) is

\[ q^n \sum_{s \in L_0(F_q)} v_\alpha(s) = q^m |T(F_q)|q^{\dim(R_u(B_L))} \sum_{w \in W_{L_0}(F_q)} q^w \langle w\lambda, \delta_\alpha \rangle. \]

The statement of Proposition 3.5.1 then follows by replacing \( \delta_\alpha \) by the expression in (3.5.4).

**Definition 3.5.2.** We denote by \( \mathcal{C}_{hw} \subset X^*_+ (T) \) the subset of characters \( \lambda \) satisfying the inequalities (3.5.2) and call \( \mathcal{C}_{hw} \) the highest weight cone.

By construction, for all \( \lambda \in \mathcal{C}_{hw} \), the section \( f_\lambda := \text{Norm}_{L_s}(f_\lambda) \) is a nonzero section of \( \mathcal{V}_f(N_{\psi_{\lambda}}) \) over \( G\text{-Zip}^w \). In particular, we deduce \( N_{\psi_{\lambda}} \in \mathcal{C}_{zip} \) and hence \( \lambda \in \mathcal{C}_{zip} \). We deduce that \( \mathcal{C}_{hw} \subset \mathcal{C}_{zip} \). If \( S_K \) is the good reduction special fiber of a Hodge-type Shimura variety and \( \zeta : S_K \rightarrow G\text{-Zip}^w \) is the map (2.6.1), we obtain a family of mod \( p \) automorphic forms \( \zeta^*(f_\lambda)_{\chi \in \mathcal{C}_{hw}} \). We also have by (2.5.3) the family \( \zeta^*(H_{\alpha})_{\chi \in X^*_+(T)} \). The vanishing locus of \( H_{\alpha} \) is a union of Ekedahl–Oort strata of codimension one. On the other hand, the vanishing locus of \( f_\lambda \) is highly nontrivial. It is an interesting closed subvariety stable by Hecke operators.

### 3.6 Partial Hasse invariant cone, Griffiths–Schmid cone

We defined in §3.2 the partial Hasse invariant cone \( C_{ph\alpha} \) as the cone generated by the weights of partial Hasse invariants on \( G\text{-ZipFlag}^w \) (see 2.5.3). There is an ambiguity in this definition, because if \( f \) is a partial Hasse invariant for \( \mathcal{T}_{u_0s_0} \) (Definition 2.5.2), then \( \chi f^n \) for \( \chi \in X^*(G) \) and \( n \geq 1 \) is also a partial Hasse invariant for \( \mathcal{T}_{u_0s_0} \). Therefore, we give below an unambiguous definition of \( C_{ph\alpha} \).
**Definition 3.6.1 ([Kos19 Definition 1.7.1])**. Define $C_{pHa}$ as the image of $X^*_+ (T)$ by

$$h_2 : X^* (T) \to X^* (T); \quad \lambda \mapsto \lambda - qw_{0,l} (\sigma^{-1} \lambda).$$

We write $C_{pHa}$ for the saturation of $C_{pHa}$. One has $C_{pHa} \subset X^*_+ (T)$ since $-w_{0,l} \sigma^{-1} (\lambda) \in X^*_+ (T)$ for $\lambda \in X^*_+ (T)$. If $G$ is split over $\mathbb{F}_q$, we have an equivalence

$$\lambda \in C_{pHa} \iff qw_{0,l} \lambda + \lambda \in X^*_+ (T).$$

**Definition 3.6.2.** Let $C_{GS}$ denote the set of characters $\lambda \in X^* (T)$ satisfying

$$\langle \lambda, \alpha^\vee \rangle \geq 0 \quad \text{for } \alpha \in I,$$

$$\langle \lambda, \alpha^\vee \rangle \leq 0 \quad \text{for } \alpha \in \Phi^+ \setminus \Phi^+_L.$$

One sees easily that $\lambda \in C_{GS}$ if and only if $-w_{0,l} \lambda$ is dominant. Clearly $C_{GS}$ is a saturated subcone of $X^* (T)$ and contains $X^*_+ (L)$. We explain the significance of $C_{GS}$. Consider a Hodge-type Shimura variety $Sh(G, X)_K$ over the reflex field $E$, with good reduction at the prime $p$, as in §2.6. Similarly to (3.3.1), we define a cone $C_K (\mathbb{C})$ by

$$C_K (\mathbb{C}) = \{ \lambda \in X^* (T) \mid H^0 (Sh(G, X)_K \otimes_E \mathbb{C}, \mathcal{V}_I (\lambda)) \neq 0 \}.$$

Again, the saturation of $C_K (\mathbb{C})$ is independent of $K$, so we denote it by $C(\mathbb{C})$. Based on the results of [GS69], it is expected that $C(\mathbb{C}) = C_{GS}$, but we could not find a reference for this result. The inclusion $C(\mathbb{C}) \subset C_{GS}$ is proved for general Hodge-type Shimura varieties in [GK22b Theorem 2.6.4].

By reduction modulo $p$, one can show that $C(\mathbb{C}) \subset C(\mathbb{F}_p)$ (see [Kos19 Proposition 1.8.3]). Combining the expectation $C(\mathbb{C}) = C_{GS}$ with Conjecture 3.3.3, one should expect an inclusion $C_{GS} \subset C_{zip}$ (at least for groups attached to Shimura varieties). In Theorem 6.1.3 we confirm this expectation and prove $C_{GS} \subset C_{zip}$ in general (this was previously shown in [Kos19 only in the case when $P$ is defined over $\mathbb{F}_p$). This result gives evidence for Conjecture 3.3.3.

### 3.7 Inclusion relations of cones

Let us briefly summarize in a diagram the cones that appear in our construction:

![Diagram](image)

All arrows of this diagram are inclusions. The lowest weight cone $C_{hw}$ is defined in §5.2. The inclusion $C_{hw} \subset C_{zip}$ is shown only under Condition 5.1.1 (hence the dotted arrow in the above diagram).

**Lemma 3.7.1.** One has $X^*_+ (L) \subset C_{hw}$.

**Proof.** For $\lambda \in X^*_+ (L)$, we have $w \lambda = \lambda$ for all $w \in W_L$. Hence $\langle w \lambda, \sigma^i \alpha^\vee \rangle \leq 0$ for all $i \in \mathbb{Z}$, $w \in W_{\lambda_0} (\mathbb{F}_q)$ and $\alpha \in \Delta^P$. Thus $\lambda \in C_{hw}$. \qed
We postpone the proof of $C_{GS} \subset C_{zip}$ in the general case, which is quite involved. The following was proved in [Kos19 Corollary 3.5.6):

**Lemma 3.7.2.** Assume that $P$ is defined over $\mathbb{F}_q$. Then one has $C_{GS} \subset C_{hw}$.

This shows $C_{GS} \subset C_{zip}$ in the case when $P$ is defined over $\mathbb{F}_q$. However, the inclusion $C_{GS} \subset C_{hw}$ is false in general. This happens for example in the case of Picard modular surfaces of signature $(2, 1)$ at an inert prime, where the group $G$ is a unitary group of rank $3$ over $\mathbb{F}_p$. In this example, all cones $C_{ph\alpha}$, $C_{GS}$, $C_{hw}$ and $C_{zip}$ are distinct and there is no inclusion relation between the first three. These four cones are also distinct for $G = Sp(6)$ ([Kos19 §5.5]), and more generally for $G = Sp(2n)$, $n \geq 3$. In particular, in those cases the inclusion $C_{ph\alpha} \subset C_{zip}$ is strict, hence $G$-$ZipFlag^\mu$ does not satisfy the Hasse property. As a consequence, the Siegel-type Shimura variety $A_n$ does not satisfy the Hasse property for $n \geq 3$.

## 4 Hasse-type zip data

### 4.1 Topology of $C_{zip, \mathbb{R}_{\geq 0}}$

Let $(G, \mu)$ be a cocharacter datum. We showed $X^*_+(L) \subset C_{zip}$ in Proposition 3.4.1. For $X^*_+(L)_{reg}$ (see (3.4.11)), we have a more precise result ([KW18 Theorem 5.1.4]):

**Theorem 4.1.1.** For all $\lambda \in X^*_+(L)_{reg}$, there is a section $h \in H^0(G-Zip^\mu, \mathcal{V}_I(N_c\lambda))$ whose non-vanishing locus is exactly $U_\mu$.

Here $N_c \geq 1$ is the integer defined in §3.5. Since $\lambda \in X^*(L)$, the vector bundle $\mathcal{V}_I(\lambda)$ is a line bundle, and thus $V_I(N_c\lambda) = V_I(\lambda)^{\otimes N_c}$. A subset of an $\mathbb{R}$-vector space stable under linear combination with coefficients in $\mathbb{R}_{\geq 0}$ will be called an $\mathbb{R}_{\geq 0}$-subcone. We endow $X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$ with the subspace topology of $X^*(T)_{\mathbb{R}}$.

**Lemma 4.1.2.** Let $C \subset X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$ be an $\mathbb{R}_{\geq 0}$-subcone and let $\lambda \in C$. Then $C$ is a neighborhood of $\lambda$ in $X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$ if and only if for all $\lambda' \in X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$, there exists $r \in \mathbb{R}_{\geq 0}$ such that $\lambda + r\lambda' \in C$.

**Proof.** First, assume that $C$ is a neighborhood of $\lambda$ in $X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$. Then there is an open subset $V$ of $X^*(T)_{\mathbb{R}}$ such that $\lambda \in V \cap X^*_+,I(T)_{\mathbb{R}_{\geq 0}} \subset C$. Fix $\lambda' \in X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$. For large $r \in \mathbb{R}_{\geq 0}$, we have $\lambda + \frac{\lambda'}{r} \in V$, and this element is also in $X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$. Thus $\lambda + r\lambda' \in C$.

We prove the converse. We claim that for all $\lambda' \in X^*_+,I(T)_{\mathbb{R}_{\geq 0}}$, there exists $r > 1$ such that $\lambda + \frac{\lambda'}{r} \in C$. Indeed, let $r \in \mathbb{R}_{\geq 0}$ such that $\lambda + \frac{\lambda'}{r} \in C$. Then for all $\gamma > 0$, we have $\gamma\lambda + \frac{\gamma\lambda'}{r} = \lambda + \frac{\gamma(\lambda' - \lambda)}{r} + (\gamma - 1) + \frac{\lambda'}{r} \lambda' \in C$. For $\gamma = \frac{r}{r + 1}$, we have $\gamma - 1 + \frac{\lambda'}{r} = 0$ hence $\lambda + \frac{\lambda'}{r + 1} \in C$. Hence, by taking $\lambda$ as the origin, we are reduced to the following:

Let $X \subset \mathbb{R}^n$ be an intersection of closed half-spaces containing $0$, and $0 \in Y \subset X$ a convex subset satisfying: for all $x \in X$, $\exists r \in \mathbb{R}_{\geq 0}$, $\frac{x}{r} \in Y$. Then $Y$ is a neighborhood of $0$ in $X$.

Taking intersections with a neighborhood of $0$ in $\mathbb{R}^n$ which is a convex polytope, we may assume that $X$ is a convex polytope. Since $X$ is the convex hull of finitely many points, there exists $r > 1$ such that $\frac{1}{r}X = \{\frac{x}{r} \mid x \in X\} \subset Y$. Hence, it suffices to show that $\frac{1}{r}X$ is a neighborhood of $0$ in $X$. There are linear forms $u_1, \ldots, u_d$ on $\mathbb{R}^n$ and $m_1, \ldots, m_d \in \mathbb{R}_{\geq 0}$ such that $x \in X$ if and only if $u_i(x) \leq m_i$ for all $i = 1, \ldots, d$. Hence $u = (u_1, \ldots, u_d)$ maps $X$ to $Z = \bigcap_{i=1}^d - \infty, m_i$. For $r > 1$, $\frac{1}{r}Z$ is clearly a neighborhood of $0$ in $Z$, hence $\frac{1}{r}X = u^{-1}(\frac{1}{r}Z)$ is a neighborhood of $0$ in $X$. 

\[\square\]
The following Lemma was proved in a slightly restricted setting in [Kos19, Proposition 2.2.1], so we restate it below.

**Lemma 4.1.3.** The cone $C_{\text{zip},R_{>0}}$ is a neighborhood of $X^*_-(L)_{\text{reg}}$ in $X^*_{+,I}(T)_{R_{>0}}$.

**Proof.** For $\lambda \in X^*_-(L)_{\text{reg}}$, we show that $C_{\text{zip},R_{>0}}$ is a neighborhood of $\lambda$ in $X^*_{+,I}(T)_{R_{>0}}$. By Lemma 4.1.2, it suffices to show that for all $\lambda' \in X^*_{+,I}(T)_{R_{>0}}$, there is $r \in R_{>0}$ such that $\lambda' + r\lambda \in C_{\text{zip},R_{>0}}$. We may assume $\lambda' \in X^*_{+,I}(T)$ by scaling. Let $h \in H^0(G,\text{Zip}^\alpha, V_I(N_\varphi\lambda))$ be the section provided by Theorem 4.1.1. By Lemma 2.4.2, we have $H^0(U_\mu, V_I(N_\varphi\lambda))$ is nonzero; let $h'$ be a nonzero element therein. This section may have poles on the complement of $U_\mu$. However, since $h$ vanishes on the complement of $U_\mu$, there exists $d \geq 1$ such that $h^d h'$ has no poles. Hence $h^d h' \in H^0(G,\text{Zip}^\alpha, V_I(N_\varphi(\lambda'+dN_\varphi\lambda)))$, and thus $N_\varphi(\lambda'+d\lambda) \in C_{\text{zip}}$, hence $\lambda' + d\lambda \in \tilde{C}_{\text{zip}}$. The result follows.

**Lemma 4.1.4.** $C_{\text{GS},R_{>0}}$ and $C_{\text{hw},R_{>0}}$ are neighborhoods of $X^*_-(L)_{\text{reg}}$ in $X^*_{+,I}(T)_{R_{>0}}$.

**Proof.** The open subset of $X^*_{+,I}(T)_{R_{>0}}$ defined by the equations $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_L$ is contained in $C_{\text{GS},R_{>0}}$ and contains $X^*_-(L)_{\text{reg}}$, which proves the first part of the assertion. Replacing $\leq$ by $<$ in the inequalities (3.5.2), we get an open subset of $X^*_{+,I}(T)_{R_{>0}}$ containing $X^*_-(L)_{\text{reg}}$ (same proof as Lemma 3.7.1), which proves the second part.

We may ask whether $C_{\text{ph},R_{>0}}$ is also a neighborhood of $X^*_-(L)_{\text{reg}}$. The proof of the following result is similar to [Kos19, Lemma 2.3.1], where the cocharacter datum $(G, \mu)$ was assumed to be of Hodge-type, but this assumption is superfluous. We reproduce partly the proof to explain the appropriate changes (we replace the character $\eta_\omega$ in [Kos19, Lemma 2.3.1] by the set $X^*_-(L)_{\text{reg}}$). The following holds for an arbitrary cocharacter datum $(G, \mu)$:

**Proposition 4.1.5.** The following are equivalent:

1. The cone $C_{\text{ph},R_{>0}}$ is a neighborhood of $X^*_-(L)_{\text{reg}}$ in $X^*_{+,I}(T)_{R_{>0}}$.
2. One has $C_{\text{GS}} \subseteq \tilde{C}_{\text{ph}}$.
3. $P$ is defined over $F_q$ and the Frobenius $\sigma$ acts on $I$ by $\sigma(\alpha) = -w_{0,I}\alpha$ for all $\alpha \in I$.

**Proof.** Since $C_{\text{GS},R_{>0}}$ is a neighborhood of $X^*_-(L)_{\text{reg}}$ in $X^*_{+,I}(T)_{R_{>0}}$, we have (ii) $\Rightarrow$ (i). Assume that (i) holds. In particular, $X^*_-(L)_{\text{reg}} \subseteq C_{\text{ph},R_{>0}}$, hence $h_{z}^{-1}(X^*_-(L)_{\text{reg}}) \subseteq X^*_{+,I}(T)_{R_{>0}}$. Let $\lambda \in X^*_-(L)_{\text{reg}}$ and write $\lambda = h_{z}(\chi)$ for $\chi \in X^*_{+,I}(T)_{R_{>0}}$. Hence for all $\alpha \in I$, we have $\langle h_{z}(\chi), \alpha^\vee \rangle = 0$, which amounts to $\langle \chi, \alpha^\vee \rangle = q(\chi, \sigma(w_{0,I}\alpha^\vee))$. Since $\alpha \in I$, $w_{0,I}\alpha$ is a negative root, and so is $\sigma(w_{0,I}\alpha^\vee)$. We deduce that $\langle \chi, \alpha^\vee \rangle = \langle \chi, \sigma(w_{0,I}\alpha^\vee) \rangle = 0$ (in particular $\chi \in X^*(L)$). Since $X^*_-(L)_{\text{reg}}$ generates $X^*(L)$, this shows that $h_{z}^{-1}$ maps $X^*(L)_{\mathbb{R}}$ to itself, and all elements in the image satisfy $\langle \chi, \sigma(w_{0,I}\alpha^\vee) \rangle = 0$ for all $\alpha \in I$. For dimension reasons, $h_{z}^{-1}(X^*(L)_{\mathbb{R}}) = X^*(L)_{\mathbb{R}}$, hence any character $\chi \in X^*(L)$ is orthogonal to $\sigma(\alpha^\vee)$ for all $\alpha \in I$. Hence we must have $\sigma(I) = I$, thus $P$ is defined over $F_q$. Next, for $\alpha \in I$, let $\lambda_\alpha \in X^*_{+,I}(T)$ such that $\langle \lambda_\alpha, \beta^\vee \rangle = 0$ for all $\beta \in \Delta \setminus \{\alpha\}$ and $\langle \lambda_\alpha, \alpha^\vee \rangle > 0$. Let $\lambda \in X^*_-(L)_{\text{reg}}$. Then there exist $r \in R_{>0}$ and $\chi_\alpha \in X^*_{+,I}(T)_{R_{>0}}$ such that $h_{z}(\chi_\alpha) = r\lambda + \lambda_\alpha$. As before, we deduce $\langle \chi_\alpha, \beta^\vee \rangle = \langle \chi_\alpha, \sigma(w_{0,I}\beta^\vee) \rangle = 0$ for all $\beta \in I \setminus \{\alpha\}$. The character $\chi_\alpha$ cannot be orthogonal to all $\beta^\vee$ for $\beta \in I$, hence $\langle \chi_\alpha, \alpha^\vee \rangle \neq 0$. Furthermore, since the map $I \to I, \beta \mapsto -\sigma(w_{0,I}\beta)$ is a bijection, we must have $-\sigma(w_{0,I}\alpha) = \alpha$. This shows (i) $\Rightarrow$ (iii). Finally, the implication (iii) $\Rightarrow$ (ii) is completely similar to (3) $\Rightarrow$ (4) in the proof of [Kos19, Lemma 2.3.1] (after changing $p$ to $q$).

**Definition 4.1.6.** We say that a cocharacter datum $(G, \mu)$ is of Hasse-type if the equivalent conditions of Proposition 4.1.5 are satisfied.
The main result of this section is that (i), (ii), (iii) above are also equivalent to the equality $\mathcal{E}_{\text{pH}} = \mathcal{E}_{\text{zip}}$. For the time being, the following is an immediate consequence of Lemma 4.1.3.

**Corollary 4.1.7.** Assume that $\mathcal{E}_{\text{pH}} = \mathcal{E}_{\text{zip}}$ holds. Then $(G, \mu)$ is of Hasse-type.

Recall that $\mathcal{E}_{\text{pH}} = \mathcal{E}_{\text{zip}}$ means by definition that $G$-ZipFlag$^\mu$ satisfies the Hasse property (Definition 3.2.1). This shows that Theorem 3.1.2(2) can only potentially generalize to Hodge-type Shimura varieties $S_K$ such that the associated zip datum $(G, \mu)$ is of Hasse-type. Indeed, if the flag space of $S_K$ satisfies the Hasse property, then so does $G$-ZipFlag$^\mu$, and hence $(G, \mu)$ must be of Hasse-type by Corollary 4.1.7. In Theorem 3.3.4, all three cases (1), (2) and (3) are of Hasse-type.

### 4.2 Maximal flag stratum

We prove some technical results used in the proof of Theorem 4.3.1. Let $(G, \mu)$ be an arbitrary cocharacter datum, and let $(B, T, z)$ be a frame with $z = \sigma(w_{0,1})w_0$ (Remark 4.2.2). Recall that $H^0(G$-ZipFlag$^\mu, V_1(\lambda))$ identifies with $H^0(G$-ZipFlag$^\mu, V_{\text{flag}}(\lambda))$ by (2.5.3). Via the isomorphism $G$-ZipFlag$^\mu \cong [E'/G]$ (see §2.5 and (2.3.5), an element of the space $H^0(G$-ZipFlag$^\mu, V_{\text{flag}}(\lambda))$ can be viewed as a function $f : G \to A^1$ satisfying

$$f(abg^{-1}) = \lambda(a)f(g), \quad \forall (a, b) \in E', \forall g \in G.$$  \hspace{1cm} (4.2.1)

Recall that $G$-ZipFlag$^\mu$ admits a stratification $(\mathcal{F}_w)_{w \in W}$ (2.5) where $\mathcal{F}_w := [E'\setminus F_w]$ and $F_w = BwBz^{-1}$ is the $B \times B^\perp$-orbit of $wz^{-1}$. The unique open stratum is $U_{\text{max}} = \mathcal{F}_{w_0}$. Write also $U_{\text{max}} := F_{w_0} = Bw_0Bz^{-1}$ (the $B \times B^\perp$-orbit of $w_0z^{-1} = \sigma(w_{0,1})^{-1}$). The codimension one $B \times B^\perp$-orbits are the $F_{sa,w_0}$ for $\alpha \in \Delta$. Define $U'_{\mu} := \pi^{-1}(U_{\mu}) \cong [E'\setminus U_{\mu}]$.

**Lemma 4.2.1.**

1. The stabilizer of $\sigma(w_{0,1})^{-1}$ in $B \times B^\perp$ is $S := \{(t, \sigma(w_{0,1})t\sigma(w_{0,1})^{-1}) \mid t \in T\}$.
2. The map $B_M \to U_{\text{max}}, b \mapsto \sigma(w_{0,1})b^{-1}$ induces an isomorphism $[B_M/T] \cong U_{\text{max}},$ where $T$ acts on $B_M$ on the right by the action $B_M \times T \to B_M, (b, t) \mapsto \varphi(t)^{-1}b\sigma(w_{0,1})t\sigma(w_{0,1})^{-1}$.
3. Assume that $P$ is defined over $\mathbb{F}_q$. Then $U_{\text{max}} \subset U_{\mu}$, and $U_{\text{max}} \subset U'_{\mu}$.

**Proof.** We prove (1). Let $(x, y) \in B \times B$ such that $x \sigma(w_{0,1})^{-1}y^{-1} = \sigma(w_{0,1})^{-1}$. Write $y = y'y^{-1}$ with $y' \in B$. Since $z = \sigma(w_{0,1})w_0$, we obtain $xw_0y'^{-1}w_0^{-1}\sigma(w_{0,1})^{-1} = \sigma(w_{0,1})^{-1}$, hence $x = w_0y'^{-1}w_0^{-1}$. It follows that $x \in B\sigma(w_0Bw_0^{-1} = T$. We can write $y = \sigma(w_{0,1})xx\sigma(w_{0,1})^{-1}$, which proves (1). To show (2), note that the map $B \times B \to U_{\text{max}}; (x, y) \mapsto x \sigma(w_{0,1})y^{-1}$ induces an isomorphism $(B \times B)/S \to U_{\text{max}}$, where $S$ is as in (1). Hence $U_{\text{max}}$ is isomorphic to $[E'\setminus B \times B]/S$. We have an isomorphism

$$E'\setminus (B \times B) \to B_M, \quad E' \ni (x, y) \mapsto \varphi(\theta^P_E(x))^{-1}\theta^Q_M(y)$$  \hspace{1cm} (4.2.2)

whose inverse is $B_M \to E'\setminus B \times B; b \mapsto E' \cdot (1, b)$. Identify $T$ and $S$ via the isomorphism $T \to S; t \mapsto (t, \sigma(w_{0,1})t\sigma(w_{0,1})^{-1})$. The action of $S$ on $E'\setminus B \times B$ by multiplication on the right transforms via the isomorphism (4.2.2) to the right action of $T$ defined by $B_M \times T \to B_M; (b, t) \mapsto \varphi(t)^{-1}b\sigma(w_{0,1})t\sigma(w_{0,1})^{-1}$. This proves (2). Finally, we show (3). Assume that $P$ is defined over $\mathbb{F}_q$. Then $U_{\mu}$ coincides with the unique open $P \times Q$-orbit by [Wed14, Corollary 2.15]. Since $B \times B \subset P \times Q$, the set $U_{\mu}$ is a union of $B \times B$-orbits or, since $B \times B \subset U'_{\mu}$, we have $U_{\text{max}} \subset U'_{\mu}$. □

For $\lambda \in X^*(T)$, let $S(\lambda)$ denote the space of functions $h : B_M \to A^1$ satisfying

$$h(\varphi(t)^{-1}b\sigma(w_{0,1})t\sigma(w_{0,1})^{-1}) = \lambda(t)^{-1}h(b), \quad \forall t \in T, \forall b \in B_M.$$
Corollary 4.2.2. The isomorphism from Lemma 4.2.1(2) induces an isomorphism
\[ \vartheta: H^0(U_{max}, V_{flag}(\lambda)) \to S(\lambda). \]

We describe explicitly this isomorphism. Let \( f \in H^0(U_{max}, V_{flag}(\lambda)) \), viewed as a function \( f: U_{max} \to \mathbb{A}^1 \) satisfying (4.2.1). The corresponding element \( \vartheta(f) \in S(\lambda) \) is the function \( B_M \to \mathbb{A}^1; b \mapsto f(\sigma(w_{0,I})b^{-1}) \). Conversely, if \( h: B_M \to \mathbb{A}^1 \) is an element of \( S(\lambda) \), the function \( f = \vartheta^{-1}(h) \) is given by
\[
\vartheta(f)(b_1, b_2) = \lambda(b_1)h(\varphi(\theta_f^e(b_1))^{-1}\theta_M^e(b_2)), \quad (b_1, b_2) \in B \times \mathbb{A}^1.
\]

By the property of \( h \), the function \( f \) is well-defined.

In particular, for a section of \( V_{flag}(\lambda) \) over \( G\text{-ZipFlag}^\mu \), we can restrict it to the open substack \( U_{max} \), and then apply \( \vartheta \) to obtain an element of \( S(\lambda) \). Assume now that \( P \) is defined over \( F_q \). In particular, we have \( \sigma(w_{0,I}) = w_{0,I} \) and \( z = w_{0,I}w_0 \). We also have \( U_{max} \subset U'_\mu \) (cf. Lemma 4.2.1(3)) and inclusions
\[
H^0(G\text{-ZipFlag}^\mu, V_{flag}(\lambda)) \subset H^0(U'_\mu, V_{flag}(\lambda)) \subset H^0(U_{max}, V_{flag}(\lambda)).
\]

Write \( S_{flag}(\lambda) \subset S_\mu(\lambda) \subset S(\lambda) \) respectively for the images under \( \vartheta \) of these three spaces. Choose a realization \( (u_\alpha)_{\alpha \in \Phi} \) (see §2.1). For \( \alpha \in \Delta \), define a map \( \Gamma_\alpha: B_L \times \mathbb{A}^1 \to G \) by
\[
\Gamma_\alpha: (b,t) \mapsto b\phi_\alpha(A(t))w_{0,I}, \quad \text{where } A(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \in SL_2
\]
and \( \phi_\alpha: SL_2 \to G \) is the map attached to \( \alpha \). For \( \alpha \in \Delta \), define an open subset
\[
G_\alpha := G \setminus \bigcup_{\beta \in \Delta, \beta \neq \alpha} V_{s_{\beta}w_0} = U_{max} \cup F_{s_{\alpha}w_0}.
\]

Since \( U_\mu \) coincides with the open \( P \times Q \)-orbit, one sees that \( G_\alpha \subset U_\mu \) if and only if \( \alpha \in I \). In this setting, one has an analogue of [IK21a, Proposition 3.1.4]:

Proposition 4.2.3. The following properties hold:
1. The image of \( \Gamma_\alpha \) is contained in \( G_\alpha \).
2. For all \( b \in B_L \) and \( t \in \mathbb{A}^1 \), one has \( \Gamma_\alpha(b,t) \in U_{max} \iff t \neq 0. \)

Proof. We have \( U_{max} = Bw_0Bz^{-1} = BB^+w_{0,I} \). As in [IK21a, (3.1.3)], one has a decomposition
\[
A(t) = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus for \( t \neq 0 \), we have \( \phi_\alpha(A(t)) = BB^+ \), hence \( \Gamma_\alpha(b,t) \in U_{max} \). For \( t = 0 \), we have \( \phi_\alpha(A(0)) = s_{\alpha} \) and \( \Gamma_\alpha(b,t) \in BS_{\alpha}w_{0,I} \subset BS_{\alpha}w_{0,I}B = F_{s_{\alpha}w_0} \). This shows (1) and (2).

Let \( f \in H^0(U_{max}, V_{flag}(\lambda)) \), viewed as a function \( f: U_{max} \to \mathbb{A}^1 \) satisfying (4.2.1). Let \( h := \vartheta(f) \) be the corresponding element of \( S(\lambda) \). Using (4.2.3), we have for \( \alpha \in \Delta^P \) and \( (b,t) \in B_L \times \mathbb{G}_m \):
\[
f \circ \Gamma_\alpha(b,t) = f \left( b\phi_\alpha \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} w_{0,I} \phi_\alpha \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix} w_{0,I} \right) = \lambda(b) h(\varphi(\phi_\alpha^{-1}w_{0,I}\phi_\alpha^{-1}w_{0,I}))
\]
Similarly, for \( \alpha \in I \) and \((b, t) \in B_L \times \mathbb{G}_m\), one can show the following (we leave out the computation, since we will only need the case \( \alpha \in \Delta_P \) in §4.3):

\[
f \circ \Gamma_\alpha(b, t) = \lambda(b) h \left( \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} \varphi(b)^{-1} \phi_{-w_0, t} \left( \begin{pmatrix} t & 0 \\ -1 & t^{-1} \end{pmatrix} \right) \right).
\]

For \( \alpha \in \Delta \), define a function \( F_{h, \alpha} : B_L \times \mathbb{G}_m \to \mathbb{A}_1 \) by

\[
F_{h, \alpha}(b, t) := h \left( \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} b \phi_{-w_0, t} \left( \begin{pmatrix} t & 0 \\ -1 & t^{-1} \end{pmatrix} \right) \right) \quad \text{if} \quad \alpha \in I,
\]

\[
F_{h, \alpha}(b, t) := h (b w_0, t \alpha^\vee(t)^{-1} w_0, t) \quad \text{if} \quad \alpha \in \Delta_P.
\]

The function \( F_{h, \alpha}(b, t) \) lies in \( k[B_L][t, \frac{1}{t}] \), where \( k[B_L] \) denotes the ring of functions of \( B_L \). Moreover, \( F_{h, \alpha}(b, t) \in k[B_L][t] \) if and only if \( f \circ \Gamma_\alpha(b, t) \) extends to a map \( B_L \times \mathbb{A}_1 \to G \).

**Proposition 4.2.4.** Let \( h \in S(\lambda) \).

1. \( h \in S_{flag}(\lambda) \) if and only if \( F_{h, \alpha} \in k[B_L][t] \) for all \( \alpha \in \Delta \).
2. \( h \in S_{\mu}(\lambda) \) if and only if \( F_{h, \alpha} \in k[B_L][t] \) for all \( \alpha \in I \).

**Proof.** Let \( f = \varphi^{-1}(h) \in H^0(U_{\max}, \mathbb{V}_{\text{flag}}(\lambda)) \). In the terminology of [Kos19, Definition 3.2.1], the map \( \Gamma_\alpha \) is adapted to \( f \) by [Kos19, Lemma 3.2.4], because \( f \) is an eigenfunction for the action of \( E_i \) and we have \( E_i \cdot \Gamma_\alpha(B_L \times \{0\}) = F_{h, w_0} \) using \( B \times \mathbb{A}_1 = E_i(B_L \times \{1\}) \). By [Kos19, Lemma 3.2.2], \( f \) extends to \( G \) if and only if \( f \circ \Gamma_\alpha \) extends to \( B_L \times \mathbb{A}_1 \) for all \( \alpha \in \Delta \), which shows (1). Assertion (2) is proved similarly. \( \square \)

### 4.3 Main result

We state the main result of this section, which is the reciprocal of Corollary 4.1.7.

**Theorem 4.3.1.** Let \((G, \mu)\) be a cocharacter datum of Hasse-type. Then \( G \)-ZipFlag\(^u \) satisfies the Hasse property. Combining with Corollary 4.1.7, we have:

\[
(G, \mu) \text{ is of Hasse-type } \iff E_{\text{zip}} = E_{\text{pHa}}.
\]

We prove Theorem 4.3.1 in the rest of this section. Fix a cocharacter datum \((G, \mu)\), with zip datum \( \mathcal{Z} = \mathcal{Z}_\mu = (G, P, L, Q, M, \varphi) \). For now, we only assume that \( P \) is defined over \( \mathbb{F}_q \) (hence \( L = M \)). Fix also a frame \((B, T, z)\) with \( z = w_0, t w_0 \).

**Proposition 4.3.2** ([ABD+66, XXII, Proposition 5.5.1]). Let \( G \) be a reductive group over \( k \) and let \((B, T)\) be a Borel pair. Choose a total order on \( \Phi^- \). The \( k \)-morphism

\[
\gamma : T \times \prod_{\alpha \in \Phi^-} U_\alpha \to G
\]

defined by taking the product with respect to the chosen order is a closed immersion with image \( B \).

We apply this proposition to \((L, B_L)\). Choose an order on \( \Phi^-_L \) and consider the corresponding map \( \gamma \) as in (4.3.1), with image \( B_L \). For a function \( h : B_L \to \mathbb{A}_1 \), put \( P_h := h \circ \gamma \). Via the isomorphism \( u_\alpha : \mathbb{G}_a \to U_\alpha \), we can view \( P_h \) as a polynomial \( P_h \in k[T]\langle \{x_\alpha\}_{\alpha \in \Phi^-_L} \rangle \), where the \( x_\alpha \) are indeterminates indexed by \( \Phi^-_L \). For \( m = (m_\alpha)_\alpha \in \mathbb{N}^{\Phi^-_L} \) and \( \lambda \in X^+(T) \), denote by \( P_{m, \lambda} \) the monomial

\[
P_{m, \lambda} = \lambda(t) \prod_{\alpha \in \Phi^-_L} x_\alpha^{m_\alpha} \in k[T]\langle \{x_\alpha\}_{\alpha \in \Phi^-_L} \rangle.
\]

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We can write any element \( P \) of \( k[T][\langle x_\alpha \rangle_{\alpha \in \Phi_L}] \) as a sum of monomials

\[
P = \sum_{i=1}^{N} c_i P_{m_i, \lambda_i} \tag{4.3.2}
\]

where for all \( 1 \leq i \leq N \), we have \( m_i \in \mathbb{N}^{\Phi_L} \), \( \lambda_i \in X^*(T) \) and \( c_i \in k \). Furthermore, we may assume that the \((m_i, \lambda_i)\) are pairwise distinct. Under this assumption, the expression \([4.3.2]\) is uniquely determined (up to permutation of the indices). For \( P \in k[T][\langle x_\alpha \rangle_{\alpha \in \Phi_L}] \), write \( h_P : B_L \to \mathbb{A}^1 \) for the function \( P \circ \gamma^{-1} \). For \( m = (m_\alpha)_\alpha \in \mathbb{N}^{\Phi_L} \) and \( \lambda \in X^*(T) \), write \( h_{m, \lambda} := h_{P_{m, \lambda}} \).

**Lemma 4.3.3.** Let \((m, \lambda) \in \mathbb{N}^{\Phi_L} \times X^*(T)\). For all \( a \in T \) and \( b \in B_L \), we have

\[
h_{m, \lambda}(ab) = \lambda(a)h_{m, \lambda}(b), \quad \text{and} \quad h_{m, \lambda}(ba) = \left(\lambda(a) \prod_{\alpha \in \Phi_L} \alpha(a)^{-m_\alpha}\right)h_{m, \lambda}(b).
\]

**Proof.** The first formula is an immediate computation. For the second, let \( b = \gamma(t, (u_\alpha(x_\alpha))_\alpha) \) with \( t \in T \) and \((x_\alpha)_\alpha \in \mathbb{G}_a^{\Phi_L} \). Then

\[
h_{m, \lambda}(ba) = h_{m, \lambda}\left( ta \prod_{\alpha \in \Phi_L} a^{-1} u_\alpha(x_\alpha)a \right) = h_{m, \lambda}\left( ta \prod_{\alpha \in \Phi_L} u_\alpha(\alpha(a)^{-1}x_\alpha) \right) \]

\[
= \lambda(ta) \prod_{\alpha \in \Phi_L} (\alpha(a)^{-1}x_\alpha)^{m_\alpha} = \left(\lambda(a) \prod_{\alpha \in \Phi_L} \alpha(a)^{-m_\alpha}\right)h_{m, \lambda}(b),
\]

where we used the formula \( a^{-1}u_\alpha(x)a = u_\alpha(\alpha(a)^{-1}x) \) for all \( x \in \mathbb{A}^1 \) and all \( a \in T \).

For \((m, \lambda) \in \mathbb{N}^{\Phi_L} \times X^*(T)\) as above, define the weight \( \omega(m, \lambda) \) as

\[
\omega(m, \lambda) := q^{-1}(\lambda) - w_{0,1} \lambda + \sum_{\beta \in \Phi_L} m_\beta(w_{0,1}\beta) \in X^*(T). \tag{4.3.3}
\]

It follows immediately from Lemma \([4.3.3]\) that \( h_{m, \lambda} : B_L \to \mathbb{A}^1 \) lies in \( S(\omega(m, \lambda)) \).

**Lemma 4.3.4.** Let \( \lambda \in X^*(T) \) and \( h \in k[B_L] \) be nonzero. Write \( P_h = \sum_{i=1}^{N} c_i P_{m_i, \lambda_i} \) as in \([4.3.2]\), with \((m_i, \lambda_i)\) pairwise disjoint and \( c_i \neq 0 \) for all \( 1 \leq i \leq N \). Then we have

\[
h \in S(\lambda) \iff \omega(m_i, \lambda_i) = \lambda \quad \text{for all} \quad i = 1, \ldots, N.
\]

**Proof.** The implication "\( \Longleftarrow \)" is obvious. Conversely, if \( h \in S(\lambda) \), then for all \( t \in T \), \( b \in B \), we have \( \lambda(t)h(b) = h(\varphi(t)bw_{0,1}t^{-1}w_{0,1}) = \sum_{i=1}^{N} \omega(m_i, \lambda_i)(t)c_i h_{m_i, \lambda_i}(b) \). The result follows by linear independence of characters.

For \( m \in \mathbb{N}^{\Phi_L} \), \( \lambda \in X^*(T) \) and \( \alpha \in \Delta^P \), we write \( F_{m, \lambda, \alpha} := F_{m, \lambda, \alpha} \) (see \([4.1.2]\)). For all \( \alpha \in \Delta^P \), and all \((b, t) \in B_L \times \mathbb{G}_m \), we find:

\[
F_{m, \lambda, \alpha}(b, t) = t^{-q(\lambda, \sigma^\alpha) + \langle \omega(m, \lambda), \alpha^\vee \rangle} h_{m, \lambda}(b). \tag{4.3.4}
\]

In particular, \( F_{m, \lambda, \alpha} \) is in \( k[B_L][t] \) if and only if \( -q(\lambda, \sigma^\alpha) + \langle \omega(m, \lambda), \alpha^\vee \rangle \geq 0 \). Using \([4.3.3]\), this inequality can also be written as

\[
\langle w_{0,1}\lambda, \alpha^\vee \rangle \leq \sum_{\beta \in \Phi_L} m_\beta \langle w_{0,1}\beta, \alpha^\vee \rangle. \tag{4.3.5}
\]

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Corollary 4.3.5. Let \( \lambda \in X^*(T) \) and \( h \in S(\lambda) \) be nonzero. Write \( P_h = \sum_{i=1}^m c_i P_{m_i, \lambda_i} \), with the \((m_i, \lambda_i)\) pairwise distinct and \( c_i \neq 0 \) for \( 1 \leq i \leq N \). Let \( \alpha \in \Delta^P \).

1. We have
   \[
   F_{h, \alpha} \in k[B_L][t] \iff \forall i = 1, \ldots, N, \ F_{m_i, \lambda_i, \alpha} \in k[B_L][t].
   \]
   \[
   \iff \forall i = 1, \ldots, N, \ -q(\lambda_i, \sigma \alpha^\vee) + \langle \lambda, \alpha^\vee \rangle \geq 0.
   \]

2. Moreover, if \( F_{h, \alpha} \in k[B_L][t] \) then \( \langle w_{0, I} \lambda_i, \alpha^\vee \rangle \leq 0 \) for all \( 1 \leq i \leq N \).

Proof. By (4.3.4), we have \( F_{m_i, \lambda_i, \alpha}(b, t) = t^{d_i} F_{m_i, \lambda_i}(b) \) for some integer \( d_i \in \mathbb{Z} \). Hence, the first equivalence of (1) follows from the assumption that \((m_i, \lambda_i)\) for \( 1 \leq i \leq N \) are pairwise distinct. The second equivalence follows from the previous discussion, using \( \omega(m_i, \lambda_i) = \lambda \) (Lemma 4.3.4). Assertion (2) follows from the inequality (4.3.5) and the fact that \( \langle w_{0, I} \beta, \alpha^\vee \rangle \leq 0 \) for all \( \beta \in \Phi^+_L \). Indeed, recall that \( \langle \beta, \alpha^\vee \rangle \leq 0 \) for any two distinct simple roots \( \alpha, \beta \in \Delta \). Since \( \beta \in \Phi^+_L \), we have \( w_{0, I} \beta \in \Phi^+_L \), hence \( w_{0, I} \beta \) is a sum of simple roots in \( I \). Since \( \alpha \in \Delta^P \), the result follows.

We now study the partial Hasse invariant cone \( C_{pHa} \) (Definition 3.6.1). Fix a positive integer \( n \) such that \( G \) is split over \( \mathbb{F}_{q^n} \). By inverting the map \( h_2 : \lambda \mapsto \lambda - qw_{0, I}(\sigma^{-1}\lambda) \), we can write \( C_{pHa} \) as the set of \( \lambda \in X^*(T) \) such that

\[
\sum_{i=0}^{2n-1} q^i \langle (w_{0, I})^i \sigma^{-i} \lambda, \alpha^\vee \rangle \leq 0, \quad \forall \alpha \in \Delta.
\]

For \( \alpha \in \Delta \) and \( \lambda \in X^*(T) \), define \( K_{\alpha}(\lambda) := \sum_{i=0}^{2n-1} q^i \langle (w_{0, I})^i \sigma^{-i} \lambda, \alpha^\vee \rangle \).

**Lemma 4.3.6.** Assume that \((G, \mu)\) is of Hasse-type. For all \( \lambda \in X_{+I}^*(T) \) and \( \alpha \in I \), we have \( K_{\alpha}(\lambda) \leq 0 \). In particular, we have

\[
C_{pHa} = \{ \lambda \in X_{+I}^*(T) | \forall \alpha \in \Delta^P, \ K_{\alpha}(\lambda) \leq 0 \}.
\]

**Proof.** For all \( \alpha \in I \), we have

\[
\sum_{i=0}^{2n-1} q^i \langle (w_{0, I})^i \sigma^{-i} \lambda, \alpha^\vee \rangle = \sum_{i=0}^{2n-1} q^i \langle \lambda, \sigma^i((w_{0, I})^i \alpha^\vee) \rangle = \sum_{i=0}^{2n-1} (-1)^i q^i \langle \lambda, \alpha^\vee \rangle = \langle \lambda, \alpha^\vee \rangle \left( \frac{q^{2n} - 1}{q + 1} \right) \leq 0,
\]

where we used that \((G, \mu)\) is of Hasse-type in the second equality and the fact that \( \lambda \) is \( I \)-dominant in the last inequality. This shows the result.

For example, if \( P \) is a maximal parabolic, we have \( |\Delta^P| = 1 \), hence \( C_{pHa} \) is given inside \( X_{+I}^*(T) \) by a single inequality. This is in contrast to cases which are not of Hasse-type. For example, if \( G = \text{Sp}(6) \) as explained in [Kos19 §5.5], the cone \( C_{pHa} \) is defined by \( |\Delta| = 3 \) inequalities inside \( X_{+I}^*(T) \).

From now on, assume that \((G, \mu)\) is of Hasse-type. We prove Theorem 4.3.1 by showing that if \( H^0(G, \text{Zip}^\mu, V_I(\lambda)) \neq 0 \), then \( \lambda \in C_{pHa} \). First, recall that \( H^0(G, \text{Zip}^\mu, V_I(\lambda)) \) identifies with \( H^0(G, \text{ZipFlag}^\mu, V_{\text{flag}}(\lambda)) \), and also with \( S_{\text{flag}}(\lambda) \subset S(\lambda) \). Let \( h \in S_{\text{flag}}(\lambda) \) be nonzero. By Proposition 1.2.1 (2), \( F_{h, \alpha} \in k[B_L][t] \) for all \( \alpha \in \Delta \). We will only need this information for \( \alpha \in \Delta^P \). Write again \( P_h = \sum_{i=1}^N c_i P_{m_i, \lambda_i} \) as in (4.3.2), with the \((m_i, \lambda_i)\)
pairwise distinct and \( c_i \neq 0 \) for \( 1 \leq i \leq N \). By Lemma \[4.3.4\] and formula \[4.3.3\], we have in particular
\[
\lambda = q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1 + \sum_{\beta \in \Phi_L^+} m_{1,\beta}(w_{0,I}\beta).
\]

We want to show \( \lambda \in \mathcal{C}_{ph^a} \), which amounts to \( K_\alpha(\lambda) \leq 0 \) for all \( \alpha \in \Delta^P \) by Lemma \[4.3.6\]. We first compute \( K_\alpha(\beta) \) for any \( \beta \in \Phi_L \). We find:
\[
K_\alpha(\beta) = \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^i\sigma^{-i}(\alpha,\alpha^\vee) = \sum_{i=0}^{2n-1} (-1)^i q^i \langle \beta,\alpha^\vee \rangle = -\langle \beta,\alpha^\vee \rangle \left( \frac{q^{2n} - 1}{q + 1} \right).
\]

On the other hand, we have
\[
K_\alpha(q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1) = \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^i\sigma^{-i}(q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1),\alpha^\vee \rangle
\]
\[
= \sum_{i=0}^{2n-1} q^{i+1} \langle (w_{0,I})^{i+1}\sigma^{-i}(\lambda_1),\alpha^\vee \rangle - \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^{i+1}\sigma^{-i}(\lambda_1),\alpha^\vee \rangle
\]
\[
= (q^{2n} - 1) \langle w_{0,I}\lambda_1,\alpha^\vee \rangle,
\]
where we used that \( \sigma^{2n}\lambda_1 = \lambda_1 \). Hence, we find for all \( \alpha \in \Delta^P \):
\[
K_\alpha(\lambda) = K_\alpha(q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1) + \sum_{\beta \in \Phi_L^+} m_{1,\beta} K_\alpha(w_{0,I}\beta)
\]
\[
= \frac{q^{2n} - 1}{q + 1} \left( q + 1 \langle w_{0,I}\lambda_1,\alpha^\vee \rangle - \sum_{\beta \in \Phi_L^+} m_{1,\beta} \langle w_{0,I}\beta,\alpha^\vee \rangle \right).
\]

One has \( K_\alpha(\lambda) \leq 0 \) using the fact that \( \langle w_{0,I}\lambda_1,\alpha^\vee \rangle \leq 0 \) (Corollary \[4.3.5(2)\]) and equation \[4.3.3\] applied to \( F_{m,\lambda_1} \). This terminates the proof of Theorem \[4.3.1\].

5 \( R_u(P_0) \)-invariant subspace

Let \((G,\mu)\) be an arbitrary cocharacter datum, and let \( \mathcal{Z}_\mu = (G, P, L, Q, M, \varphi) \) the attached zip datum. Fix a frame \((B,T,z)\) with \( z = \sigma(w_{0,I})w_0 \). Let \((V,\rho)\) be an \( L \)-representation. For \( f \in V^{L^\vee} \), we can view \( f \) as a section of \( \mathcal{V}_f(\lambda) \) over \( \mathcal{U}_\mu \), by Lemma \[2.4.2\]. When \( P \) is defined over \( \mathbb{F}_q \), this section extends to \( G \)-Zip\(^\mu \) if and only if \( f \in V(\lambda)_{L^\vee}^{>0} \), by Corollary \[2.4.4\]. For general \( P \), the condition on \( f \) is given by the Brylinski–Kostant filtration on \( \mathcal{V}_f(\lambda) \) (see \[IK21\] Theorem 3.4.1)]. Unfortunately, this condition is too complex to understand explicitly. However, let \( P_0 \) be the parabolic \( L_0B \) with \( L_0 \) as in \[2.4.3\], and assume further that \( f \in V(\lambda)^{R_u(P_0)} \). In this case, we can give a more explicit condition for when \( f \) extends. In particular, the lowest weight vector of \( \mathcal{V}_f(\lambda) \) satisfies this condition. This makes it possible to define a "lowest weight cone" \( \mathcal{C}_{hw^a} \) similar to the highest weight cone \( \mathcal{C}_{hw^a} \). When \( P \) is not defined over \( \mathbb{F}_q \), one sees on examples that \( \mathcal{C}_{hw^a} \) is usually very small. On the other hand, the lowest weight cone will be quite large.

5.1 Statement

As in the proof of Proposition \[3.5.1\] define for \( \alpha \in \Delta^P \):
\[
m_\alpha = \min\{m \geq 1 \mid \sigma^{-m}(\alpha) \notin I\}
\]
and \( t_\alpha = t^{-1}\alpha(\varphi(\delta_\alpha(t)))^{-1} = t\alpha(\delta_\alpha(t))^{-1} \in \mathbb{D} \), where \( t \) is an indeterminate. Set also

\[
ut_{t,\alpha} = \prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{i-1}(\alpha)} \left( \begin{pmatrix} 1 & -\frac{1}{t_\alpha^i} \\ 0 & 1 \end{pmatrix} \right). \tag{5.1.1} \]

For \( \alpha \in \Phi \), write \( G_\alpha \subset G \) for the image of the map \( \phi_\alpha : \text{SL}_2 \to G \). For simplicity, we consider the following condition:

**Condition 5.1.1.** For all \( 1 \leq i, j \leq m_\alpha - 1 \) with \( i \neq j \) we have \( \langle \sigma^{i-1}(\alpha), \sigma^{j-1}(\alpha^\vee) \rangle = 0 \) and the subgroups \( G_{\sigma^{j-1}(\alpha)} \) and \( G_{\sigma^{j-1}(\alpha)} \) commute with each other.

**Remark 5.1.2.** Condition 5.1.1 is satisfied in many cases. For example, if \( G \) splits over \( \mathbb{F}_q \), then \( m_\alpha \in \{1, 2\} \) and the condition is trivially satisfied. In particular, all absolutely simple unitary groups satisfy it. The condition also holds for \( G = \text{Res}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(G_0, \mathbb{F}_{q^n}) \) where \( G_0 \) is a split reductive over \( \mathbb{F}_q \).

Let \( (V, \rho) \) be an \( L \)-representation and let \( V = \bigoplus_{V \in X^\bullet(T)} V_\nu \) be its \( T \)-weight decomposition. For \( \alpha \in \Delta \), set \( \delta_\alpha = \rho^{-1}(\alpha^\vee) \) (where \( \varphi_\alpha \) was defined in (2.4.4)). Put \( P_1 := \sigma^{-(m_\alpha-1)}(P) \). We have \( \Delta^P_1 = \sigma^{-(m_\alpha-1)}(\Delta^P) \). Since \( P_0 \subset P_1 \), we have \( \Delta^P_1 \subset \Delta^P_0 \). Define \( V_\Delta^P_1 \) similarly to (2.4.5) by

\[
V_{\Delta^P_1}^{\geq 0} = \bigoplus_{\langle \nu, \delta_\alpha \rangle \geq 0, \forall \beta \in \Delta^P_1} V_\nu.
\]

**Proposition 5.1.3.** Assume that Condition 5.1.1 holds. Then we have

\[
V^{R_\alpha(P)} \cap V^{L_\sigma} \cap V_{\Delta^P_1}^{\geq 0} \subset H^0(G, \text{Zip}^\mu, V(\rho)).
\]

**Proof.** Let \( f \in V^{R_\alpha(P)} \cap V^{L_\sigma} \) and let \( \tilde{f} : U_\mu \to V \) be the function corresponding to \( f \) by Lemma 2.4.2. It suffices to check that \( \tilde{f} \) extends to \( G \). By the proof of [IK21a Theorem 3.4.1], it is enough to show that for all \( \alpha \in \Delta^P \), the function

\[
F_\alpha : t \mapsto \rho(\phi_\alpha \left( \begin{pmatrix} 1 & 0 \\ \frac{1}{t_\alpha} & 1 \end{pmatrix} \right) \delta_\alpha(t) \nu_{t,\alpha} \right) f
\]

lies in \( k[\mathbf{t}] \otimes V \). Since it lies in \( \mathbf{k}[\mathbf{t}, t^{-1}] \otimes V \) by the proof of [IK21a Theorem 3.4.1], it suffices to show that it also lies in \( k[\mathbf{t}] \otimes V \) by the proof of [IK21a Theorem 3.4.1]. Since \( \rho \) is trivial on \( R_\alpha(P) \) and \( \alpha \in \Delta^P \), one has simply \( F_\alpha(t) = \rho(\delta_\alpha(t) \nu_{t,\alpha}) \). Using (5.1.1), we can write

\[
F_\alpha(t) = \rho \left( \delta_\alpha(t) \prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{i-1}(\alpha)} \left( \begin{pmatrix} 1 & 0 \\ \frac{1}{t_\alpha^i} & 1 \end{pmatrix} \right) \right) f = \rho \left( \prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{i-1}(\alpha)} \left( \begin{pmatrix} 1 & \gamma_i \\ 0 & 1 \end{pmatrix} \right) \right) \delta_\alpha(t) \nu_{t,\alpha} \right) f
\]

where \( \gamma_i = -t^{\sigma^{i-1}(\alpha) \delta_\alpha + \frac{1}{t_\alpha^i}} \). We have \( q^{-1}\sigma^{-1}(\delta_\alpha) = \delta_\alpha + q^{-1}\sigma^{-1}(\alpha^\vee) \) and hence by induction \( q^{-i}\sigma^{-1}(\delta_\alpha) = \delta_\alpha + (q^{-1}\sigma^{-1}(\alpha^\vee) + \cdots + q^{-i}\sigma^{-1}(\alpha^\vee)) \). Let \( 1 \leq i \leq m_\alpha - 1 \). By Condition 5.1.1, we deduce \( \langle \sigma^{-i}(\alpha), \delta_\alpha \rangle = q^{-i}(0, \delta_\alpha) = 2 \). Thus

\[
\gamma_i = -t^{(\sigma^{i-1}(\alpha) \delta_\alpha) + q^{-1}(0, \delta_\alpha) = -t^{-1/q}.
\]
Let $f = \sum_{\nu} f_{\nu}$ be the $T$-weight decomposition of $f$. By assumption, we have:

\[
F_\alpha(t) = \sum_{\nu} \rho \left( \prod_{i=1}^{m_{\alpha} - 1} \phi_{\sigma^{-1}(\nu)} \left( \begin{array}{cc} 1 & -1 \nu/\nu \alpha \\
0 & 1 \nu/\nu \alpha \end{array} \right) \right) f_{\nu}
= \sum_{\nu} t^{(\nu, \delta_\alpha)} \rho \left( \prod_{i=1}^{m_{\alpha} - 1} \phi_{\sigma^{-1}(\nu)} \left( \begin{array}{cc} t^{-1/q_\alpha} & -1 \nu/\nu \alpha \\
0 & t^{1/q_\alpha} \nu/\nu \alpha \end{array} \right) \right) \sigma^{-i}(\alpha') \sigma^{-i}(\nu/\nu \alpha) f_{\nu}
= \sum_{\nu} t^{(\nu, \delta_\alpha + \sum_{i=1}^{m_{\alpha} - 1} \nu/\nu \alpha)} \rho \left( \prod_{i=1}^{m_{\alpha} - 1} \phi_{\sigma^{-1}(\nu)} \left( \begin{array}{cc} t^{-1/q_\alpha} & -1 \nu/\nu \alpha \\
0 & t^{1/q_\alpha} \nu/\nu \alpha \end{array} \right) \right) \sigma^{-i}(\alpha') \sigma^{-i}(\nu/\nu \alpha) f_{\nu}.
\]

As before, we have $\delta_\alpha + \sum_{i=1}^{m_{\alpha} - 1} \nu/\nu \alpha = q^{-i}(m_{\alpha} - 1) \sigma^{-i}(m_{\alpha} - 1) (\delta_\alpha)$. Furthermore, we have

\[
\left( t^{-1/q_\alpha} & -1 \\
0 & t^{1/q_\alpha} \right) = \left( 0 & -1 \\
1 & t^{1/q_\alpha} \right) \left( 1 & 0 \\
-t^{-1/q_\alpha} & 1 \right).
\]

Since $P_0$ is defined over $\mathbb{F}_q$, we have $\sigma^{-i}(\alpha) \notin I_{P_0}$ for all $i \in \mathbb{Z}$. By invariance of $f$ under $R_\nu(P_0)$, we deduce

\[
F_\alpha(t) = \sum_{\nu} t^{(\nu, \sigma^{-i}(\delta_\alpha))} \rho \left( \prod_{i=1}^{m_{\alpha} - 1} \phi_{\sigma^{-1}(\nu)} \left( \begin{array}{cc} 0 & -1 \\
1 & t^{1/q_\alpha} \nu/\nu \alpha \end{array} \right) \right) \sigma^{-i}(\alpha') \sigma^{-i}(\nu/\nu \alpha) f_{\nu}.
\]

Since $f \in V_{\Delta^{P_0}_1}$, we have $\langle \nu, \sigma^{-i}(\delta_\alpha) \rangle = \langle \nu, \delta_\alpha \rangle \geq 0$. Hence, the $t$-valuation of $F_\alpha(t)$ is $\geq 0$. The result follows.

5.2 Lowest weight cone

We examine the case $V = V_\lambda(T)$ for $\lambda \in X^*_+ (T)$. The $L$-representation $V_\lambda(T)\gamma^{R_\nu(P_0)}$ is isomorphic to $V_{\nu_0}(w_0, w_0, T)$ by [IK21]. Proposition 6.3.1. Put $\lambda_0 = w_0, w_0, T$.

Let $f_{\text{low}, \lambda} \in V_\lambda(T)$ be a nonzero element in the lowest weight line of $V_\lambda(T)$. Consider the element $\text{Norm}_{L_\nu}(f_{\text{low}, \lambda}) \in V_\lambda(N_\nu \lambda)$, defined in (3.5.1), where $N_\nu = \left| L_0(\mathbb{F}_q) \right| q^n$. By construction, this element lies in $V_\lambda(N_\nu \lambda)$. For $\alpha \in \Delta$, write $r_\alpha$ for the smallest integer $r \geq 1$ such that $\sigma^r(\alpha) = \alpha$.

**Theorem 5.2.1.** Assume Condition 5.1.1. Suppose that for all $\alpha \in \Delta^{P_0}$, one has

\[
\sum_{w \in W_{\nu_0}(\mathbb{F}_q)} \sum_{t=0}^{r_\alpha-1} q^{1+\ell(w)} \langle w\nu_0, \sigma^t(\alpha) \rangle \leq 0.
\]  

Then $\text{Norm}_{L_\nu}(f_{\text{low}, \lambda})$ extends to $G$-Zip$^\nu$.

**Remark 5.2.2.** Formulas (5.2.1) and (3.5.2) (in the case of $f_{\text{high}, \lambda}$) differ in two aspects: $\lambda$ changes to $\lambda_0 = w_0, w_0, T$ and $w_0, w_0, T$ for all $\lambda \in \Delta^{P_0}$ changes to $w_0, w_0, T$.

**Proof.** The lowest weight vector $f_{\text{low}, \lambda}$ is contained in the $L_0$-subrepresentation $V_\lambda(T)\gamma^{R_\nu(P_0)} \cong V_{\nu_0}(\lambda_0)$, which has highest weight $\lambda_0$, lowest weight vector $f_{\text{low}, \lambda}$ and highest weight vector $f_{\text{high}, \lambda_0} := w_0, w_0, f_{\text{low}, \lambda}$. Since $w_0, w_0, \in W_{\nu_0}(\mathbb{F}_q)$, we have

\[
\text{Norm}_{L_\nu}(f_{\text{low}, \lambda}) = \text{Norm}_{L_\nu}(f_{\text{high}, \lambda_0}) = \text{Norm}_{L_\nu(\mathbb{F}_q)}(f_{\text{high}, \lambda_0}) q^n.
\]

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Consider the zip datum $\mathcal{Z}_0 = (G, P_0, L_0, Q_0, L_0, \varphi)$, where $Q_0$ is the opposite parabolic to $P_0$ with Levi subgroup $L_0$. By Remark 2.2.2 we have $\mathcal{Z}_0 = \mathcal{Z}_{\mu_0}$ for some cocharacter $\mu_0 : \mathbb{G}_{m,k} \to G_k$. Since $P_0$ is defined over $\mathbb{F}_q$, we have by Corollary 2.4.1

$$H^0(G\text{-Zip}^{\mu_0}, \mathcal{V}_{l_0} (\lambda_0)) = V_{l_0}(\lambda_0)^{L_0(\mathbb{F}_q)} \cap V_{l_0}(\lambda_0)^{\Delta_{l_0}^+}.$$  

Applying Proposition 3.5.1 to $G\text{-Zip}^{\mu_0}$ and the $L_0$-representation $V_{l_0}(\lambda_0)$, we deduce

$$\text{Norm}_{L_0(\mathbb{F}_q)}(f_{\text{high}, \lambda_0}) \in V_{l_0}(N_0 \lambda_0)^{\Delta_{l_0}^+}$$

where $N_0 = |L_0(\mathbb{F}_q)|$. Combining this with (5.2.2) and using that $\Delta_{l_0}^+ \subset \Delta_{l_0}^\circ$, we find

$$\text{Norm}_{L_0}(f_{\text{low}, \lambda_0}) \in V_{l_1}(N_0 \lambda)^{\Delta_{l_1}^+} \subset V_{l_1}(N_0 \lambda)^{\Delta_{l_1}^+}.$$  

The result follows from Proposition 5.1.3 applied to $V_{l_1}(N_0 \lambda)$.

**Definition 5.2.3.** Define $\mathcal{C}_{lw}$ as the set of $\lambda \in X_{+, l}(T)$ satisfying the inequalities (5.2.1).

We call $\mathcal{C}_{lw}$ the lowest weight cone. Under Condition 5.1.1, one has $\mathcal{C}_{lw} \subset \mathcal{C}_{\text{zip}}$ by Theorem 5.2.1. We do not know if this inclusion holds in general. When $P$ is defined over $\mathbb{F}_q$, one has $P_0 = P$ and hence $\mathcal{C}_{lw} = \mathcal{C}_{lw}$.

**Lemma 5.2.4.** One has $\mathcal{C}_{GS} \subset \mathcal{C}_{lw}$.

**Proof.** For $\lambda \in \mathcal{C}_{GS}$, the character $w_{0,l} \lambda$ is anti-dominant. For all $w \in W_{L_0}(\mathbb{F}_q)$, we have $(w \lambda_0, \sigma^i(\alpha^v)) = (w_{0,l} \lambda, w_{0,l} w^{-1} \sigma^i(\alpha^v))$. Since $w_{0,l}$ and $\alpha \in \Delta_{l_0}^\circ$, the root $w_{0,l} w^{-1} \sigma^i(\alpha)$ is positive. Hence $(w \lambda_0, \sigma^i(\alpha^v)) \leq 0$ for all $w \in W_{L_0}(\mathbb{F}_q)$, and the result follows.

In particular, if Condition 5.1.1 holds, we deduce $\mathcal{C}_{GS} \subset \mathcal{C}_{\text{zip}}$ from Lemma 5.2.4. We will prove this inclusion in the next section in the general case.

### 6 Weil restriction

When Condition 5.1.1 does not hold, we cannot use Proposition 5.1.3 to show $\mathcal{C}_{GS} \subset \mathcal{C}_{\text{zip}}$. We show here that a version of Proposition 5.1.3 holds in general (see Theorem 6.3.1 below). To eliminate the need for Condition 5.1.1, we first study the case of a Weil restriction. More generally, we will prove a useful result that makes it possible to reduce certain questions pertaining to the cone $\mathcal{C}_{\text{zip}}$ to the case of a split group.

#### 6.1 Zip strata of a Weil restriction

We recall some results from [KW18, §4]. Note that loc. cit. uses the convention $B \subset Q$, whereas we assume $B \subset P$. We make the appropriate changes in this section. Let $r \geq 1$ and let $G_1$ be a connected, reductive group over $\mathbb{F}_{q^r}$. Put $G = \text{Res}_{\mathbb{F}_q/\mathbb{F}_{q^r}}G_1$. Over $k$, we can decompose

$$G_k = G_{1,k} \times G_{2,k} \times \cdots \times G_{r,k}$$

where $G_i = \sigma^{-1}(G_1)$. The Frobenius homomorphism $\varphi : G \to G$ maps $(x_1, \ldots, x_r) \bbracket{G_k}$ to $(\varphi(x_r), \varphi(x_1), \ldots, \varphi(x_{r-1}))$. We choose a cocharacter $\mu : \mathbb{G}_{m,k} \to G_k$ written as $(\mu_1, \ldots, \mu_r)$ with $\mu_i : \mathbb{G}_{m,k} \to G_{i,k}$. Consider the attached zip datum $(G, P, L, Q, M, \varphi)$. Assume that there is a Borel pair $(B, T)$ defined over $\mathbb{F}_q$ and $B \subset P$. For all $i = P, L, Q, M, B, T$, one can decompose $i = \prod_{i=1}^{r} i$. Note that $\sigma(B_i) = B_{i+1}$ and $\sigma(T_i) = T_{i+1}$ and $\sigma(L_i) = M_{i+1}$.
where indices are taken modulo $r$. Moreover, $\sigma(P_{i})$ and $Q_{i+1}$ are opposite in $G_{i+1}$. Write $\Delta_{i}$ for the set of simple roots of $G_{i}$. The Weyl group $W := W(G, T)$ decomposes as also $W = W_{1} \times \cdots \times W_{r}$ where $W_{i} := W(G_{i}, T_{i})$. Let $w_{0,i}$ be the longest element in $W_{i}$. The Frobenius induces an automorphism of $W$ again denoted by $\sigma$, and we have $\sigma(W_{i}) = W_{i+1}$. Similarly, we have $tW_{1} = tW_{1} \times \cdots \times tW_{r}$ and $W_{i} = W_{i}^{1,1} \times \cdots \times W_{i}^{r,1}$, where $I_{1}, I_{2} \subset \Delta_{i}$ are the types of the parabolic subgroups $P_{i}$ and $Q_{i}$ respectively.

We obtain a frame $(B, T, z)$ by setting $z := \sigma(w_{0,1})w_{0} = w_{0,0}w_{j}$ (Lemma 2.2.3). Thus $z = (z_{1}, \ldots, z_{r})$ with $z_{i} = w_{0,i}w_{0,i}$ for all $i = 1, \ldots, r$. By the dual parametrization (2.2.2) and the dimension formula for $E$-orbits in Theorem 2.2.4 the $E$-orbits of codimension one in $G$ are

$$C_{i,\alpha} := E \cdot (1, \ldots, 1, w_{0,i}, 1, \ldots, 1), \quad 1 \leq i \leq r, \quad \alpha \in \Delta_{i} \setminus J_{i}. \quad (6.1.1)$$

For each $1 \leq j \leq r$, define parabolic subgroups in $G_{j,k}$ by

$$P_{j}^{i} = \bigcap_{i=0}^{r-1} \sigma^{-i}(P_{i+j}) \quad \text{and} \quad Q_{j}^{i} = \bigcap_{i=0}^{r-1} \sigma^{-i}(Q_{j-i})$$

where the indices are taken modulo $r$. The unique Levi subgroups of $P_{j}^{i}$ and $Q_{j}^{i}$ containing $T_{j}$ are respectively

$$L_{j}^{i} = \bigcap_{i=0}^{r-1} \sigma^{-i}(L_{i+j}) \quad \text{and} \quad M_{j}^{i} = \bigcap_{i=0}^{r-1} \sigma^{-i}(M_{j-i}).$$

By [KW18] Lemma 4.2.1, the tuple $Z_{j} := (G_{j}, P_{j}^{i}, L_{j}^{i}, Q_{j}^{i}, M_{j}^{i}, \varphi^{\sigma})$ is a zip datum over $\mathbb{F}_{q^{r}}$. Clearly $B_{j} \subset P_{j}^{i}$ and $B_{j}^{\circ} \subset Q_{j}^{i}$, since $B$ is defined over $\mathbb{F}_{q}$. It follows that $\sigma^{r}(P_{j}^{i})$ and $Q_{j}^{i}$ are opposite parabolics of $G_{j}$. By Remark 2.2.1 $Z_{j}$ is of cocharacter-type. We denote the zip group of $Z_{j}$ by $E_{j} \subset P_{j}^{i} \times Q_{j}^{i}$ (in [KW18] this group is denoted by $E_{j}$, but we want to avoid confusion with the group $E'$ defined in (2.5)).

Write $\iota_{j}: G_{j} \to G$ for the natural embedding $x \mapsto (1, \ldots, x, \ldots, 1)$. Denote by $X$ the set of $E$-orbits in $G$, and by $X_{j} \subset X$ the set of $E$-orbits which intersect $G_{j}$ (viewed as a subset of $G$ via $\iota_{j}$). We have the following result ([KW18] Theorem 4.3.1):

**Theorem 6.1.1.** The map $C \mapsto C \cap G_{j}$ defines a bijection between $X_{j}$ and the set of $E_{j}$-orbits in $G_{j}$. Furthermore one has $\text{codim}_{G}(C) = \text{codim}_{G_{j}}(C \cap G_{j})$ for all $C \in X_{j}$.

Note that $X_{j}$ always contains the open $E$-orbit, since this orbit contains 1 in $G$. Furthermore, by equation (6.1.1), any $E$-orbit of codimension 1 lies in at least one of the $X_{j}$. There is a natural group homomorphism $\gamma_{j}: E_{j} \to E$, defined as follows. For $(x, y) \in E_{j}$ write $\overline{x} := \theta_{j}^{P_{j}^{i}}(x)$ and set

$$u_{j}(x, y) := (\varphi^{-r+j}(\overline{x}), \ldots, \varphi^{-1}(\overline{x}), x, \varphi(\overline{x}), \ldots, \varphi^{-r-j}(\overline{x})) \in P$$

$$v_{j}(x, y) := (\varphi^{-r+j}(\overline{x}), \ldots, \varphi^{-1}(\overline{x}), y, \varphi(\overline{x}), \ldots, \varphi^{-r-j}(\overline{x})) \in Q$$

$$\gamma_{j}(x, y) := (u_{j}(x, y), v_{j}(x, y)) \in E.$$
We have a commutative diagram

\[
\begin{array}{ccc}
E_j & \xrightarrow{\gamma_j} & E \\
\downarrow{pr_1} & & \downarrow{pr_1} \\
P_j' & \xrightarrow{\gamma} & P
\end{array}
\]

For \(x \in L_j\), we have \(\gamma_j(x) \in L\). Hence, we also have a map \(\gamma_j : L'_j \to L\).

### 6.2 Space of global sections

For each \(1 \leq i \leq r\), let \((V_i, \rho_i)\) be an \(L_i\)-representation and let \((V, \rho)\) be the \(L\)-representation \(\bigotimes_{i=1}^r \rho_i\). For example, if \(\lambda = (\lambda_1, \ldots, \lambda_r)\) is in \(X^*(T) = X^*(T_1) \times \cdots \times X^*(T_r)\), then we have \(V_j(\lambda) = \bigotimes_{i=1}^r V_i(\lambda_i)\). View \(\rho_i\) as a map \(P_i \to \text{GL}(V_i)\) trivial on \(R_u(P_i)\). Using the maps \(\gamma_j : P'_j \to P_j\), we have

\[
\theta_j^*(\mathcal{V}(\rho)) = \bigotimes_{i=1}^r \mathcal{V}(\rho_j^{[i]}_j)
\]

where \(\rho_j^{[i]}_j\) denotes the \(P_j\)'-representation \(P'_j \xrightarrow{\varphi_j^i} P_{j+i} \xrightarrow{\rho_{j+i}} \text{GL}(V_{j+i})\) (indices modulo \(r\)). By definition of \(P'_j\), this composition is well-defined. Note that \(\rho_j^{[i]}_j\) may not be trivial on the unipotent radical of \(P'_j\). Let \(L_\varphi\) be the stabilizer of \(1 \in G\) in \(E\), as defined in \(\S 2.4\) and fix \(f \in V^L_\varphi\). By Lemma \(2.1.2\), we may view \(f\) as a section of \(\mathcal{V}(\rho)\) over the open substack \(U_{\mu} \subset G\text{-Zip}^\mu\). Similarly, since \(\theta_j\) maps \(U_{\mu_j}\) into \(U_{\mu}\) (Theorem \(0.1.1\)), we have \(\theta_j^*(f) \in H^0(U_{\mu_j}, \theta_j^*(\mathcal{V}(\rho)))\).

**Lemma 6.2.1.** The section \(f\) extends to \(G\text{-Zip}^\mu\) if and only if \(\theta_j^*(f)\) extends to \(G_j\text{-Zip}^{2j}\) for all \(1 \leq j \leq r\).

**Proof.** One implication is clear. Conversely, assume that \(\theta_j^*(f) \in H^0(G\text{-Zip}^{2j}, \theta_j^*(\mathcal{V}(\rho)))\) for all \(1 \leq j \leq r\). Viewing \(f\) as a section over \(U_{\mu}\), consider the unique regular map \(\tilde{f} : U_{\mu} \to V\) satisfying \(\tilde{f}(1) = f\) and \(\tilde{f}(axb^{-1}) = \rho(a)f(x)\) for all \(x \in U_{\mu}\) and all \((a, b) \in E\). It suffices to show that \(\tilde{f}\) extends to a regular map \(\tilde{f} : G \to V\) (by density, this regular map will automatically satisfy the \(E\)-equivariance condition).

Consider a codimension one \(E\)-orbit \(C_{i,\alpha}\) for some \(1 \leq i \leq r\) and \(\alpha \in \Delta_i \setminus J_i\) (where \(C_{i,\alpha}\) was defined in equation \((6.1.1)\)). Set \(Y := U_{\mu} \cup C_{i,\alpha}\). It is the complement in \(G\) of the union of the Zariski closures of all other codimension one \(E\)-orbits. In particular \(Y\) is open in \(G\). Define \(X := \iota^{-1}_i(Y)\) and consider the map \(\iota_i : X \to Y\). This map satisfies conditions (1) and (2) of Lemma \(6.2.2\) below (for the group \(H = E\)). By assumption, the function \(\iota_i^*(\tilde{f}) = \tilde{f} \circ \iota_i : U_{\mu} \to V\) extends to a function \(G_i \to V\) (in particular to a map \(X \to V\)). Therefore, we can apply Lemma \(6.2.2\) to deduce that \(\tilde{f}\) extends to a regular map \(Y \to V\). To show that \(\tilde{f}\) extends to \(G\), let \(f_0 : U_{\mu} \to \mathbb{A}^1\) be a coordinate function of \(f\) in some basis of \(V\). By the above discussion, \(f_0\) cannot have a pole along any codimension one \(E\)-orbit of \(G\), hence extends to \(G\) by normality. Hence \(\tilde{f}\) itself extends to \(G\) and the result follows. \(\square\)

**Lemma 6.2.2.** Let \(Y, X\) be irreducible normal \(k\)-varieties, and assume that \(Y\) is endowed with an action of an algebraic group \(H\). Suppose that \(Y\) has an open subset \(U_Y \subset Y\) stable by \(H\). Set \(Z_Y := Y \setminus U_Y\). Let \((V, \rho)\) be an \(H\)-representation and let \(f : U_Y \to V\) be an \(H\)-equivariant regular map on \(U_Y\). Let \(\iota : X \to Y\) be a regular map satisfying the following:

- \(\iota \subseteq U_Y\)
- \(\rho \circ \iota = f\)
- \(\text{codim}_X \iota^{-1}(Z_Y) > 0\)
- \(\dim X < \dim Y\)

Then \(f\) extends to \(\tilde{f} : \iota(X) \to V\).

**Proof.** By induction on \(\dim X\). If \(\dim X = 0\) then \(X\) is an \(H\)-orbit in \(Y\) and \(\tilde{f}\) is well-defined. Suppose \(\dim X > 0\). We may assume that \(X\) is a point. By the above conditions, \(\tilde{f}\) is well-defined on a neighborhood of \(X\) in \(Y\). Since \(\tilde{f}\) is regular, it extends to a regular map \(\tilde{f} : \iota(X) \to V\). \(\square\)
Corollary 6.2.3. Let \( \iota(X) \cap U_Y \neq \emptyset \),
(2) \( H \cdot (\iota(X) \cap Z_Y) \) is Zariski dense in \( Z_Y \).

Define \( U_X := \iota^{-1}(U_Y) \). Then the following holds: The morphism \( f \) extends to an \( H \)-equivariant regular map \( Y \to V \) if and only if \( \iota^*(f) : U_X \to V \) extends to a regular map \( X \to V \).

Proof. One direction is obvious. Conversely, assume that \( \iota^*(f) : U_X \to V \) extends to a regular map \( X \to V \). Consider the map

\[
\phi : H \times X \to Y, \quad (h, x) \mapsto h \cdot \iota(x).
\]

We have \( \phi^{-1}(U_Y) = H \times U_X \). Then \( f \) extends to a regular map \( Y \to V \) if and only if \( \phi^*(f) : H \times U_X \to V \) extends to a regular map \( H \times X \to V \). Indeed, choose a basis of \( V \).

Let \( f_i : U_Y \to \mathbb{A}^1_k \) for \( 1 \leq i \leq \dim V \) be coordinate maps of \( f \) with respect to that basis. Since the image of \( \phi \) is dense in \( Z_Y \) by assumption, \( f_i \) cannot have a pole along \( Z_Y \), hence extends to \( Y \) by normality. Thus, it suffices to show that if \( \iota^*(f) \) extends, then so does \( \phi^*(f) \). But since \( f \) is \( H \)-equivariant, we have for all \( h \in H, x \in U_X \):

\[
\phi^*(f)(h, x) = f(h \cdot \iota(x)) = h \cdot (\iota^*(f))(x).
\]

Hence if \( \iota^*(f) \) extends to \( X \), we can define a function \( H \times X \to V \) using the above formula, and it must coincide with \( \phi^*(f) \) on the open subset \( H \times U_X \). The result follows.

Now, assume that for all \( 1 \leq j \leq r \), \( P_j \) is defined over \( \mathbb{F}_{q^r} \) (for example, this is the case if \( T_1 \) is split over \( \mathbb{F}_{q^r} \)). It is clear that \( P_j \) is then also defined over \( \mathbb{F}_{q^r} \). We apply Corollary 2.4.3 to the \( \mathbb{F}_{q^r} \)-zip datum \( \mathbb{Z}_j \). We deduce that for any \( L_j \)-representation \( (W, \rho_W) \), we have

\[
H^0(G_j \text{-Zip}^\gamma_j, \mathcal{V}(\rho_W)) = W^{L_j(\mathbb{F}_{q^r} \cap W^\Delta_{j, 0})}. \tag{6.2.1}
\]

However, since \( \gamma_j^* (\rho) = \rho \circ \gamma_j \in \text{Rep}(P_j) \) may be non-trivial on \( R_u(P_j) \), we cannot apply this formula directly to \( \gamma_j^*(\rho) \). Denote by \( V^* \subset V \) the subspace of \( f \in V \) which are invariant under \( \gamma_j(R_u(P_j)) \) for all \( 1 \leq j \leq r \). We deduce from (6.2.1) and Lemma 6.2.1.

**Corollary 6.2.3.** Let \( f \in V^{L_j \cap V^*} \). Then \( f \) extends to \( G \text{-Zip}^\rho \) if and only if \( f \in (V|_{L_j})_{\Delta_{j, 0}}^{L_j \cap V^*} \) for all \( 1 \leq j \leq r \), where \( V|_{L_j} \) denotes the \( L_j \)-representation \( \gamma_j^*(\rho) : L_j \rightarrow L \rightarrow GL(V) \).

Write \( V = \bigoplus_{\chi \in X^*(T)} V_\chi \) for the \( T \)-weight space decomposition of \( V \), and write \( \chi = (\chi_1, \ldots, \chi_r) \) where \( \chi_i \in X^*(T_i) \). Similarly, let \( f = \sum f_\chi \) be the decomposition of \( f \). We determine the \( T_j \)-weight decomposition of \( V|_{L_j} \). For \( \chi \in X^*(T) \), define

\[
S_j(\chi) := \sum_{i=0}^{r-1} q^i \sigma^{-i}(\chi_{j+i}) \in X^*(T_j)
\]

(indices taken modulo \( r \)). Then, the \( T_j \)-weight decomposition of \( V|_{L_j} \) is given by

\[
V|_{L_j} = \bigoplus_{\eta \in X^*(T_j)} V_\eta, \quad \text{where} \quad V_\eta = \bigoplus_{\chi \in X^*(T) \atop S_j(\chi) = \eta} V_\chi.
\]

Define \( V_{\geq 0} \subset V \) as the intersection of all \( (V|_{L_j})_{\Delta_{j, 0}}^{L_j \cap V^*} \) for \( 1 \leq j \leq r \) inside \( V \). Put

\[
\varphi_{j, \ast}^{(r)} : X_*(T_j)_{\mathbb{R}} \to X_*(T_j)_{\mathbb{R}}, \quad \delta \mapsto \delta - q^r \sigma^r(\delta)
\]

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as in (2.4.4) (but changing \( \varphi \) to \( \varphi^\vee \)). For \( \alpha \in \Delta_j \), define \( \delta^{(r)}_{j,\alpha} := (\varphi^{(r)}_{\ast})^{-1}(\alpha^\vee) \in X_\ast(T_j)_\mathbb{R} \). By definition, \( (V|_{L_j})^\Delta_{P_j} \geq 0 \) is the direct sum of \( V_\eta \) for \( \eta \in X_\ast(T_j) \) satisfying \( (\eta, \delta^{(r)}_{j,\alpha}) \geq 0 \) for all \( \alpha \in \Delta_{P_j} \). Hence \( V_{\geq 0} \subset V \) is the direct sum of weight spaces \( V_\chi \) satisfying \( (S_j(\chi), \delta^{(r)}_{j,\alpha}) \geq 0 \) for all \( \alpha \in \Delta_{P_j} \) and all \( 1 \leq j \leq r \). We have shown that \( f \) extends to \( G\text{-Zip}^\mu \) if and only if \( f \in V_{\geq 0} \). In other words:

**Proposition 6.2.4.** Let \( \Gamma(\rho) \) be the set of all \( \chi \in X_\ast(T) \) such that \( \langle S_j(\chi), \delta^{(r)}_{j,\alpha} \rangle \geq 0 \) for all \( 1 \leq j \leq r \) and all \( \alpha \in \Delta_{P_j} \). For \( f \in V^{L_\varphi} \cap V^\# \), \( f \) extends to \( G\text{-Zip}^\mu \) if and only if \( f \in V_{\geq 0} = \bigoplus_{\chi \in \Gamma(\rho)} V_\chi \).

Now, assume that \( T_1 \) is split over \( \mathbb{F}_{q'} \). Then for all \( 1 \leq j \leq r \), \( T_j \) is split over \( \mathbb{F}_{q'} \), hence \( \delta^{(r)}_{j,\alpha} = -q^{-1} \alpha^\vee \) for all \( \alpha \in \Delta_j \). Therefore, in this case \( \Gamma(\rho) \) is the set of \( \chi \in X_\ast(T) \) satisfying \( \langle S_j(\chi), \alpha^\vee \rangle \leq 0 \) for all \( \alpha \in \Delta_{P_j} \) and all \( 1 \leq j \leq r \).

### 6.3 Consequence for \( H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) \)

We derive consequences from the above considerations. Let \( G \) be a connected, reductive group over \( \mathbb{F}_q \); \( \mu : G_{\mathbb{A}_{m,k}} \to G_k \) a cocharacter, and \( \mathcal{Z} = (G, P, L, Q, M, \varphi) \) the associated zip datum over \( \mathbb{F}_q \). Choose a frame \( (B, T, z) \) as in §2.2.3. For \( r \geq 1 \), consider the diagonal embedding

\[
\Delta : G \to \tilde{G} := \text{Res}_{\mathbb{F}_{q'}/\mathbb{F}_q}(G_{\mathbb{F}_{q'}}).
\]

The cocharacter \( \tilde{\mu} := \Delta \circ \mu \) induces a zip datum \( \tilde{\mathcal{Z}} = (\tilde{G}, \tilde{P}, \tilde{L}, \tilde{Q}, \tilde{M}, \tilde{\varphi}) \), where for each \( \square = G, P, L, Q, M \) we have \( \tilde{\square}_k = \square_k \times \cdots \times \square_k \). Write \( \tilde{E} \) for the zip group of \( \tilde{\mathcal{Z}} \). We obtain a morphism of stacks

\[
\Delta : G\text{-Zip}^\mu \to \tilde{G}\text{-Zip}^{\tilde{\mu}}.
\]

For all \( 1 \leq i \leq r \), let \( (V_i, \rho_i) \) be an \( L \)-representation, and write \( \tilde{\rho} := \bigotimes_{i=1}^r \rho_i \), viewed as an \( \tilde{L} \)-representation. We have

\[
\Delta^*(\mathcal{V}(\tilde{\rho})) = \bigotimes_{i=1}^r \mathcal{V}(\rho_i).
\]

Since \( \Delta : G \to \tilde{G} \) is a group homomorphism, it satisfies \( \Delta(1) = 1 \), hence the induced map \( \Delta : G\text{-Zip}^\mu \to \tilde{G}\text{-Zip}^{\tilde{\mu}} \) is dominant (1 lies in the open zip stratum). Therefore, pullback via \( \Delta \) induces an injection on the spaces of global sections:

\[
\Delta^* : H^0(\tilde{G}\text{-Zip}^{\tilde{\mu}}, \mathcal{V}(\tilde{\rho})) \to H^0(G\text{-Zip}^\mu, \bigotimes_{i=1}^r \mathcal{V}(\rho_i)).
\]

In particular, let \( (V, \rho) \) be an \( L \)-representation and let \( \rho_0 : L \to \{1\} \) be the trivial character of \( L \). Put \( \rho_1 = \rho \) and \( \rho_i = \rho_0 \) for all \( 2 \leq i \leq r \). We obtain an injection

\[
\Delta^* : H^0(\tilde{G}\text{-Zip}^{\tilde{\mu}}, \mathcal{V}(\text{pr}_1^\ast \rho)) \to H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho))
\]  

(6.3.1)

where \( \text{pr}_1 : \tilde{L} \to L \) is the first projection and \( \text{pr}_1^\ast \rho \) is the \( \tilde{L} \)-representation \( \rho \circ \text{pr}_1 \). Fix \( r \geq 1 \) such that \( P \) is defined over \( \mathbb{F}_{q'} \). We apply Proposition 6.2.4 to \( \text{pr}_1^\ast \rho \). In this case, for each \( 1 \leq j \leq r \), the parabolic subgroup \( P_j^* \) is equal to \( P_0 = \bigcap_{i \in \mathbb{Z}} \sigma^i(P) \), the largest parabolic subgroup defined over \( \mathbb{F}_q \) contained in \( P \). Let \( L_0 \subset P_0 \) be the Levi subgroup containing \( T \), as in (2.4.3). The space \( V^\# \) is clearly \( V^{R_{L_0}(P_0)} \). Any weight of the \( \tilde{T} \)-representation \( \text{pr}_1^\ast \rho \)
is of the form \( \tilde{\chi} = (\chi, 0, \ldots, 0) \) where \( \chi \) is a \( T \)-weight of \( V \). Hence, for each \( 1 \leq j \leq r \), we have \( S_j(\tilde{\chi}) = q^{r-j+1} \sigma^{-(r-j+1)} \chi \). Thus, \( V^0 \) is the direct sum of \( T \)-weight spaces \( V_\chi \) satisfying \( \langle \sigma^{-(r-j+1)} \chi, \delta_\alpha^{(r)} \rangle \leq 0 \) for all \( \alpha \in \Delta^P \) and all \( 1 \leq j \leq r \) (here \( \delta_\alpha^{(r)} \) is independent of \( j \), so we denote it simply by \( \delta_\alpha^{(r)} \)). But since \( P_0 \) is defined over \( \mathbb{F}_q \), this condition is also equivalent to \( \langle \chi, \delta_\alpha^{(r)} \rangle \leq 0 \) for all \( \alpha \in \Delta^P \). Note that \( V^0 \geq 0 \) is very close to the space \( V^0 \), the only difference being that \( \delta_\alpha \) is replaced by \( \delta_\alpha^{(r)} \) in the definition. In other words, we could say that \( V^0 = V^0 \otimes \mathbb{F}_q^* \), where we changed \( P_0 \) to \( P_0 \otimes \mathbb{F}_q^* \). To simplify notation, for any \( L \)-representation \( (V, \rho) \) define
\[
V^0 := \bigoplus_{\langle \chi, \delta_\alpha^{(r)} \rangle \geq 0, \forall \alpha \in \Delta^P} V_\chi.
\]
We showed that \( V^0 = V_0 \otimes \mathbb{F}_q^* \). Denote by \( L_0^{(r)} \) the image of \( \text{Stab}_E(1) \) via the composition of the projection \( E \to \bar{P} \) and the first projection \( \text{pr}_1 : \bar{P} \to P \). By Lemma 2.4.1 we have \( L_0^{(r)} \subset L \). We deduce from Proposition 6.2.4:
\[
V^{L_0^{(r)}}(r) \cap V^{0}(r) \cap V^{R_0}(P) \subset H^0(\tilde{G} \cdot \text{Zip} \tilde{\rho}, \mathcal{V}(\text{pr}_1(\rho))).
\] (6.3.2)

The largest Levi subgroup of \( \tilde{G} \) defined over \( \mathbb{F}_q \) contained in \( \tilde{L} \) is \( \tilde{L}_0 := \text{Res}_{G}^{\tilde{G}, \mathbb{F}_q} L_0 \). Since \( \tilde{L}_0(\mathbb{F}_q) = L_0(\mathbb{F}_q) \), we have \( L_0^{(r)} \subset L_0^{(r)} \times L_0(\mathbb{F}_q) \) by Lemma 2.4.1 Furthermore, \( \Delta \) induces an injection \( \Delta : L_0^{(r)} \to L_0^{(r)} \). Combining (6.3.2) with (6.3.1), we deduce:

**Theorem 6.3.1.** Let \( r \geq 1 \) such that \( P \) is defined over \( \mathbb{F}_q \). One has
\[
V^{L_0^{(r)}}(r) \cap V^{0}(r) \cap V^{R_0}(P) \subset H^0(G \cdot \text{Zip} \rho, \mathcal{V}(\rho)).
\] (6.3.3)

This theorem is slightly weaker than Proposition 5.1.3 but holds in general, independently of Condition 5.1.1. Put \( V^{(r)} \mathcal{V}_{\text{Weil}} := V^{L_0^{(r)}} \cap V^{0}(r) \cap V^{R_0}(P) \).

### 6.4 Applications to \( C_{\text{Zip}} \)

Consider the \( L \)-representation \( V = V^1(\lambda) \) for \( \lambda \in X^+_1(T) \). Let \( r \geq 1 \) such that \( P \) is defined over \( \mathbb{F}_q \). Consider the sub-\( L_0 \)-representation \( V^1_{\lambda}(\lambda) \subset V^1(\lambda) \) with \( \lambda_0 := w_0 \lambda w_0 \lambda \). Then, we have \( V^{R_0}(P) = V_{\lambda}(\lambda_0) \). Let \( Q_0 \) be the opposite parabolic to \( P_0 \) with Levi subgroup \( L_0 \). Let \( \mu_0 : \mathbb{G}_{m,k} \to G_k \) be any dominant cocharacter with centralizer \( L_0 \) (hence \( \mu_0 \) defines the parabolics \( P_0, Q_0 \)). If we base-change \( G \) to \( \mathbb{F}_q^* \), we have by Corollary 2.4.3:
\[
H^0(G_s \cdot \text{Zip} \mu_0, \mathcal{V}_{\lambda}(\lambda_0)) = V_{\lambda}(\lambda_0) = V^{R_0}(P) \subset H^0(G \cdot \text{Zip} \rho, \mathcal{V}(\rho)).
\] (6.4.1)

Hence, the space \( V^{(r)} \mathcal{V}_{\text{Weil}} \) given in (6.3.3) is very close to the space (6.4.1). The only difference is that we take invariants under \( L_0^{(r)} = L^{(r)} \times L_0(\mathbb{F}_q) \) instead of \( L_0(\mathbb{F}_q) \).

Fix \( m \geq 1 \) such that the finite unipotent group \( L^{(r)} \times L_0(\mathbb{F}_q) \) is annihilated by \( \varphi^m \). If \( f \in H^0(G \cdot \text{Zip} \mu_0, \mathcal{V}_{\lambda}(\lambda_0)) \), then \( f^{\varphi^m} \) is stable by \( L^{(r)} \), and hence lies in \( V^1(\varphi^m \lambda) \). We deduce the following: Assume that \( \lambda \in X^+_1(T) \) satisfies \( \lambda_0 \in C_{\text{Zip}}(G_{\mathbb{F}_q^*}, \mu_0) \), where \( C_{\text{Zip}}(G_{\mathbb{F}_q^*}, \mu_0) \) is the zip cone of the zip datum \((G_{\mathbb{F}_q^*}, \mu_0)\). Then \( \lambda \in C_{\text{Zip}} \). We have shown

**Theorem 6.4.1.** Assume that \( P \) is defined over \( \mathbb{F}_q \). Then
\[
X^+_1(T) \cap \left\{ w_0 \lambda w_0 \lambda \epsilon_{\text{zip}}(G_{\mathbb{F}_q^*}, \mu_0) \right\} \subset C_{\text{Zip}}.
\]
Remark 6.4.2. We can apply all results and constructions about the zip cone to \((G_{\mathbb{F}_q^\ast}, \mu_0)\). For example, consider the highest weight cone of \((G_{\mathbb{F}_q^\ast}, \mu_0)\). We deduce from Theorem 6.4.1 and Proposition 3.5.1 that if \(\lambda \in X^+_\pm(T)\) satisfies
\[
\sum_{w \in W_{I_0}(F_q)} q^{r(w)} \langle w\lambda_0, \alpha \rangle \leq 0, \quad \forall \alpha \in \Delta^{po},
\]
then \(\lambda \in \mathcal{C}_{zip}\). This is slightly weaker than Theorem 5.2.1, but holds without any assumption on \((G, \mu)\).

We can finally prove in general:

**Theorem 6.4.3.** One has \(\mathcal{C}_{GS} \subset \mathcal{C}_{zip}\).

**Proof.** Write \(\mathcal{C}_{GS,I} = \mathcal{C}_{GS} \cap \mathcal{C}_{GS,I_0}\) for the Griffiths–Schmid cones of \(I\) and \(I_0\) respectively. By Lemma 3.7.2, we have \(\mathcal{C}_{GS,I_0} \subset \mathcal{C}_{zip}(G_{\mathbb{F}_q^\ast}, \mu_0)\). Since \(w_0,Iw_0,I_0 \mathcal{C}_{GS,I_0} = \mathcal{C}_{GS,I}\), the result follows from Theorem 6.4.1.

### 7 Examples

#### 7.1 The case \(G = U(2, 1)\) with \(p\) inert

We consider the example of Picard modular surfaces. More precisely, let \(E/\mathbb{Q}\) be a quadratic totally imaginary extension and \((V, \psi)\) a hermitian space over \(E\) of dimension 3 such that \(\psi\) has signature \((2, 1)\). There is a Shimura variety of dimension 2 of PEL-type attached to \(G = GU(V, \psi)\). It parametrizes abelian varieties of dimension 3 with a polarization, an action of \(\mathbb{Q}_E\) and a level structure. Let \(p\) be a prime of good reduction, and let \(X\) be the special fiber of the Kisin–Vasiu (canonical) integral model of the Shimura variety. By (2.6.1), we have a smooth, surjective morphism \(\zeta : X \to G\text{-}Zip^p\), where \(G\) is the special fiber of a reductive \(\mathbb{Z}_p\)-model of \(G_{\mathbb{Q}_p}\). In this section, we study the cones attached to \(G\text{-}Zip^p\) when \(p\) is inert in \(E\). To simplify, we consider the case of a unitary group \(G = U(V, \psi)\) (the case of \(G = GU(V, \psi)\) is very similar).

Let \((V, \psi)\) be a 3-dimensional vector space over \(\mathbb{F}_q^2\) endowed with a non-degenerate hermitian form \(\psi : V \times V \to \mathbb{F}_q^2\) (in the context of Shimura varieties, take \(q = p\)). Write \(\text{Gal}(\mathbb{F}_q^2/\mathbb{F}_q) = \{\text{Id}, \sigma\}\). Choose a basis \(B = (v_1, v_2, v_3)\) of \(V\) where \(\psi\) is given by the matrix
\[
J = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

We define a reductive group \(G\) by
\[
G(R) = \{ f \in \text{GL}_{\mathbb{F}_q^2}(V \otimes_{\mathbb{F}_q} R) \mid \psi_R(f(x), f(y)) = \psi_R(x, y), \forall x, y \in V \otimes_{\mathbb{F}_q} R \}
\]
for any \(\mathbb{F}_q\)-algebra \(R\). There is an isomorphism \(G_{\mathbb{F}_q^2} \simeq \text{GL}(V) \simeq \text{GL}_{3, \mathbb{F}_q^2}\). It is induced by the \(\mathbb{F}_q\)-algebra isomorphism \(\mathbb{F}_q^2 \otimes_{\mathbb{F}_q} R \to R \times R, a \otimes x \mapsto (ax, \sigma(a)x)\) (where \(\text{Gal}(\mathbb{F}_q^2/\mathbb{F}_q) = \{\text{Id}, \sigma\}\)). The corresponding action of \(\sigma\) on \(\text{GL}_3(k)\) is given by \(\sigma \cdot A = J\sigma(tA)^{-1}J\). Let \(T\) denote the diagonal torus and \(B\) the lower-triangular Borel subgroup of \(G_k\) (note that \(B\) and \(T\) are defined over \(\mathbb{F}_q\)). Identify \(X^+(T) = \mathbb{Z}^3\) such that \((a_1, a_2, a_3) \in \mathbb{Z}^3\) corresponds to the character \(\text{diag}(x_1, x_2, x_3) \mapsto \prod_{i=1}^3 x_i^{a_i}\). The simple roots are \(\Delta = \{e_1 - e_2, e_2 - e_3\}\), where \((e_1, e_2, e_3)\) is the canonical basis of \(\mathbb{Z}^3\). Define a cocharacter \(\mu : \mathbb{G}_{m,k} \to G_k\) by \(x \mapsto \text{diag}(x, x, 1)\) via the identification \(G_k \simeq \text{GL}_{3,k}\). Let \(\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)\) be the
associated zip datum. We have $\Delta^P = \{e_2 - e_3\}$. Note that the determinant $\det: GL_{3,k} \to \mathbb{C}_{m,k}$ is an invertible section of the line bundle $V_t(p + 1, p + 1, p + 1)$ on $G$-$\text{Zip}^\mu$. Set $D := \mathbb{Z}(1, 1, 1) = X^*(G)$. We have $D \subset C_{\text{zip}}$. Identify

$$
\mathbb{Z}^3/D \simeq \mathbb{Z}^2, \quad (a_1, a_2, a_3) \mapsto (a_1 - a_3, a_2 - a_3).
$$

(7.1.1)

Hence, subcones of $\mathbb{Z}^3$ containing $D$ correspond bijectively to subcones of $\mathbb{Z}^2$ via (7.1.1). For a subcone $C$ of $\mathbb{Z}^3$ containing $D$ and a subcone $C' \subset \mathbb{Z}^2$, we write $C \leftrightarrow C'$ if they correspond via the bijection (7.1.1).

**Proposition 7.1.1.** Via this identification, we have

\[
X^*_{+} (T) \leftrightarrow \{(a_1, a_2) \in \mathbb{Z}^2 \mid a_1 \geq a_2 \}
\]

\[
X^*_{-} (L) \leftrightarrow \mathbb{N}(-1, -1)
\]

\[
C_{\text{GS}} \leftrightarrow \{(a_1, a_2) \in X^*_{+}(T) \mid 0 \geq a_1 \}
\]

\[
C_{\text{zip}} \leftrightarrow \{(a_1, a_2) \in X^*_{+}(T) \mid (q - 1)a_1 + a_2 \leq 0 \}
\]

\[
C_{\text{ph}} \leftrightarrow \{(a_1, a_2) \in X^*_{+}(T) \mid qa_1 - (q - 1)a_2 \geq 0 \text{ and } (q - 1)a_1 + a_2 \leq 0 \}
\]

\[
C_{\text{hw}} \leftrightarrow \{(a_1, a_2) \in X^*_{+}(T) \mid qa_1 - (q - 1)a_2 \leq 0 \}
\]

\[
C_{\text{ph}} = C_{\text{zip}}.
\]

**Proof.** The cone $C_{\text{zip}}$ was determined in [IK21a, Corollary 6.3.3]. The rest is a straightforward computation.

This example is not of Hasse-type since $P$ is not defined over $\mathbb{F}_q$. As predicted by Proposition 4.1.5, $C_{\text{ph}, R_{>0}}$ is not a neighborhood of $X^*(L)_{\text{reg}}$ in $X^*(T)_{R_{>0}}$. Condition 5.1.1 is satisfied, and we have indeed $C_{\text{GS}} \subset C_{\text{hw}}$ (Lemma 5.2.4). However, $C_{\text{GS}} \subset C_{\text{ph}}$ does not hold. For this group, Conjecture 3.3.3 holds by [GK22a, Theorem 4.3.3], i.e. we have $C(\overline{\mathbb{F}}_p) = C_{\text{zip}}$.

### 7.2 The orthogonal group $SO(2n + 1)$

We consider the case of odd orthogonal groups. This example arises in the theory of Shimura varieties of Hodge-type attached to general spin groups $GSpin(2n - 1, 2)$ $(n \geq 1)$. This furnishes an interesting infinite family of examples of zip data of Hasse-type (Definition 4.1.6). To simplify, we only consider the case of odd special orthogonal groups $SO(2n + 1)$, which is completely similar. Assume $p > 2$. Let $J$ be the symmetric square matrix of size $2n + 1$ defined by

$$
J := \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix}.
$$

Let $n \geq 1$ and let $G$ be the reductive, connected, algebraic group over $\mathbb{F}_q$ defined by

$$
G(R) := \{ A \in \text{SL}_{2n+1}(R) \mid ^tAJA = J \}
$$

for all $\mathbb{F}_q$-algebra $R$. Let $T$ be the maximal diagonal torus, given by matrices of the form $t = \text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1})$. Identify $X^*(T) \simeq \mathbb{Z}^n$ such that $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ corresponds to $t \mapsto t_1^{a_1} \ldots t_n^{a_n}$. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{Z}^n$. Fix the Borel
subgroup of lower-triangular matrices in $G$. The positive roots $\Phi^+$ and the simple roots $\Delta$ are respectively

$$\Phi^+ := \{e_i \pm e_j, \ 1 \leq i < j \leq n\} \cup \{e_i, \ 1 \leq i \leq n\},$$

$$\Delta := \{e_1 - e_2, \ldots, e_{n-1} - e_n\}.$$  

The Weyl group identifies as the group of permutations $\sigma$ of $\{1, \ldots , 2n+1\}$ satisfying $\sigma(i) + \sigma(2(n+2) - i) = 2n+2$. In particular, we have $\sigma(n+1) = n+1$. Moreover, $\sigma$ is entirely determined by $\sigma(1), \ldots , \sigma(n)$. For $\sigma \in W$ such that $\sigma(i) = a_i$ for $i = 1, \ldots , n$, write $\sigma = [a_1 \ldots a_n]$. Hence, the identity element is $[1 \ 2 \ldots \ n]$ and the longest element is $w_0 = [2n+1 \ 2n \ldots \ n+2]$. The action of $w_0$ on $X^*(T)$ is given by $w_0\lambda = -\lambda$. Consider the cocharacter

$$\mu : z \mapsto \text{diag}(z,1, \ldots, 1, z^{-1}).$$

Let $Z_\mu := (G, P, L, Q, M, \varphi)$ be the zip datum attached to $\mu$ (since $\mu$ is defined over $\mathbb{F}_q$ we have $M = L$). For $n \geq 2$, one has:

$$I = \Delta \setminus \{e_1 - e_2\}, \quad \Delta^P = \{e_1 - e_2\}$$

(for $n = 1$, one has $I = \emptyset$, $\Delta^P = \Delta = \{e_1\}$). The Levi $L$ is isomorphic to $\text{SO}(2n-1) \times \mathbb{G}_m$.

In particular, $w_{0,I}$ acts on $I$ by $w_{0,I} \alpha = -\alpha$. Since $T$ is $\mathbb{F}_q$-split, one has $\sigma(\alpha) = \alpha = -w_{0,I} \alpha$ for all $\alpha \in I$. This shows that $(G, \mu)$ is of Hasse-type. Put $z := w_{0,I} w_0 = [2n+1 \ 2 \ldots \ n]$. Then $(B,T,I,z)$ is a frame for $Z_\mu$ (Lemma 2.2.3). We determine the cones appearing in Diagram (5.7.1).

**Proposition 7.2.1.** For $n \geq 2$, we have

$$X^*_+, I(T) = \{(a_1, \ldots , a_n) \in \mathbb{Z}^n \mid a_2 \geq \cdots \geq a_n \geq 0\}.$$  

$$X^*(L)_- = \mathbb{Z}_{\leq 0}(1,0, \ldots, 0)$$

$$C_{GS} = \{(a_1, \ldots , a_n) \in \mathbb{Z}^n \mid X^*_+ I(T) \mid a_1 + a_2 \leq 0\}$$

$$C_{\phi}\lambda = \{(a_1, \ldots , a_n) \in X^*_+ I(T) \mid (q+1)a_1 + (q-1)a_2 \leq 0\}$$

$$C_{\text{zip}} = C_{\phi}\lambda$$

$$C_\text{hw} = C_\text{hw} = \{(a_1, \ldots , a_n) \in X^*_+ I(T) \mid (q^{2n-2} - 1)a_1 \leq (q-1) \sum_{i=2}^{n} (q^{2n-1-i}a_i)\}.$$  

**Proof.** The equality $C_{\text{zip}} = C_{\phi}\lambda$ follows from Theorem 4.3.1. Since $P$ is defined over $\mathbb{F}_q$, we have $C_{\text{hw}} = C_{\text{hw}}$. The only nontrivial computation is $C_{\text{hw}}$. Since $T$ is split over $\mathbb{F}_q$, we can use [Kos19] §3.6] (changing $p$ to $q$). Put $\alpha = e_1 - e_2$. Denote by $L_\alpha \subset L$ the centralizer in $L$ of $\alpha^\vee$, and $I_\alpha \subset I$ its type. Then $C_{\text{hw}}$ is the set of $\lambda \in X^*_+ I(T)$ satisfying

$$\sum_{w \in I_\alpha W_I} q^{(w)(\lambda, \alpha^\vee)} \leq 0. \quad (7.2.1)$$

We only carry out the case $n \geq 3$. The set $I_\alpha W_I$ has cardinality $2(n-1)$. Any permutation $w \in I_\alpha W_I$ is entirely determined by $w^{-1}(2)$, and it can be any integer $2 \leq w^{-1}(2) \leq 2n$ different from $n+1$. Writing $w^{-1}(2) = i$, there are two cases to consider: $2 \leq i \leq n$ and $n+2 \leq i \leq 2n$. In the first case, the length of $w$ is $i-2$ and one has $\langle \lambda, w^{-1} \alpha^\vee \rangle = a_i - a_i$ (where $\lambda = (a_1, \ldots , a_n)$). In the second case, the length of $w$ is $i-3$ and $\langle \lambda, w^{-1} \alpha^\vee \rangle = a_1 + a_{2n+2-i}$. Hence we find that the sum in (7.2.1) is equal to

$$\sum_{i=2}^{n} q^{i-2}(a_1 - a_i) + \sum_{i=n+2}^{2n} q^{i-3}(a_1 + a_{2n+2-i}) = \frac{q^{2n-2} - 1}{q-1} a_1 - \sum_{i=2}^{n} (q^{i-2} - q^{2n-1-i}) a_i.$$  

The result follows.

\[\square\]
As predicted by Theorem 4.3.1, one sees that $C_{pHa}$ contains all cones of Proposition 7.2.1 (except of course $X^*_+(T)$). For example, assume that $\lambda \in C_{hw}$. We find $\frac{q^{2n-2}-1}{q-1}a_1 \leq \sum_{i=2}^{n}(q^{i-2} - q^{2n-1-i})a_i \leq (1 - q^{2n-3})a_2$, and hence $\frac{q^{2n-2}-1}{q-1}a_1 + (q - 1)a_2 \leq 0$. In particular, this implies $a_1 \leq 0$. Since $q + 1 \geq \frac{q^{2n-2}-1}{q^{2n-3}-1}$, we have $(q + 1)a_1 + (q - 1)a_2 \leq 0$. This shows $C_{hw} \subset C_{pHa}$ (for $n = 2$ one has actually $C_{hw} = C_{pHa}$). Here is a representation of the cones for $n = 3$. We represent the intersections with the affine hyperplane $a_1 = -(q - 1)$. In other words, the weight $-((q - 1), x, y)$ appears as the point $(x, y)$. Set $a := \frac{q^4-1}{q^3-q-1}$ and $b := \frac{q^4-1}{q^3-q-1}$ (hence we have $q - 1 < a < b < q + 1$).

A Appendix: Classification of Hasse-type zip data

In this appendix, we classify the pairs $(G, \mu)$ which are of Hasse-type (Definition 4.1.6).

A.1 Hasse-type Dynkin triples

Let $D$ be a Dynkin diagram, let $\sigma \in \text{Aut}(D)$ be a diagram automorphism and let $I \subset D$ be a $\sigma$-stable sub-diagram. In this appendix, we classify such Dynkin triples $(D, I, \sigma)$ satisfying the following:

**Condition A.1.1.** The actions of $\sigma$ and the opposition involution $-w_0, I$ of $I$ on $I$ coincide.

The case $I = D$ is allowed; if $I = D$, then the conditions [A.1.1] hold precisely when the opposition involution of $D$ is trivial: $-w_0 := -w_{0,D} = 1$.

A.2 Translation

In the setting of [4.1.5] let $D$ denote the Dynkin diagram of the simple roots $\Delta$ associated to $(G, B, T)$ and let $I$ denote the Dynkin sub-diagram of the type $I \subset \Delta$ of the parabolic $P \supset B$. Then the triples $(D, I, \sigma)$ satisfying [A.1.1] are precisely those which arise from Hasse-type zip data, as characterized by the root-data-theoretic condition [4.1.5].
A.3 Classification

The classification is broken up into two cases, according to whether or not \( \sigma \) is trivial, see \( \text{A.3.1} \) and \( \text{A.3.4} \). The special cases \( (\mathcal{D}, \mathcal{J}, \sigma) \) is maximal and of Hodge-type are singled out in \( \text{A.4} \) and \( \text{A.5} \) respectively. Proofs are given in \( \text{A.6} \).

Isolated vertices of \( \mathcal{J} \) We call a vertex \( v \in \mathcal{J} \) isolated if its connected component in \( \mathcal{J} \) is \( \{v\} \). Assume that \( (\mathcal{D}, \mathcal{J}, \sigma) \) satisfies \( \text{A.1.1} \). If \( \sigma \) fixes an isolated vertex \( v \) of \( \mathcal{J} \), then it is clear that \( (\mathcal{D}, \mathcal{J} \setminus \{v\}, \sigma) \) also satisfies \( \text{A.1.1} \). More generally, if \( v \in \mathcal{J} \) is isolated but not necessarily fixed by \( \sigma \), then the orbit \( \langle \sigma \rangle v \) consists of isolated vertices of \( \mathcal{J} \) and \( (\mathcal{D}, \mathcal{J}, \sigma) \) satisfies \( \text{A.1.1} \) if and only if \( (\mathcal{D}, \mathcal{J} \setminus \langle \sigma \rangle v, \sigma) \) does. For this reason, it suffices to consider triples \( (\mathcal{D}, \mathcal{J}, \sigma) \) where \( \mathcal{J} \) contains no isolated vertices.

Let \( \mathcal{J}^2 \subset \mathcal{J} \) denote the sub-diagram consisting of all connected components with at least two vertices (i.e., the sub-diagram obtained by throwing out all the isolated vertices). When \( \mathcal{J} \) consists entirely of isolated vertices, write \( \mathcal{J}^2 = \emptyset \).

**Theorem A.3.1.** Assume that \( \mathcal{D} \) is connected, that \( \sigma = 1 \) and that \( \mathcal{J}^2 \neq \emptyset \). Then \( \text{A.1.1} \) holds precisely in the following cases:

1. \( \mathcal{D} \) is of type \( B_n \) or \( C_n \) \( (n \geq 2) \) and \( \mathcal{J}^2 \) is of type \( B_m \) or \( C_m \) respectively, for some \( 2 \leq m \leq n \); in other words \( \mathcal{J}^2 \) is connected and contains the multi-laced vertices.
2. \( \mathcal{D} \) is of type \( D_4 \) or \( G_2 \) and \( \mathcal{D} = \mathcal{J}^2 \).
3. \( \mathcal{D} \) is of type \( D_n \) \( (n \geq 5) \) and \( \mathcal{J}^2 \) is the unique sub-diagram of type \( D_{2m} \) for some \( 2 \leq m \leq n/2 \); in other words \( \mathcal{J}^2 \) is connected of even size and contains the two extremal vertices which are permuted by \( \operatorname{Aut}(\mathcal{D}) \).
4. \( \mathcal{D} \) is of type \( F_4 \) and \( \mathcal{J}^2 \) is the unique sub-diagram of type \( B_2 \cong C_2 \ B_3 , \ C_3 \) or \( F_4 \).
5. \( \mathcal{D} \) is of type \( E_6 \) and \( \mathcal{J}^2 \cong D_4 \).
6. \( \mathcal{D} \) is of type \( E_7 \) and the type of \( \mathcal{J}^2 \) is either \( D_4, D_6 \) or \( E_7 \).
7. \( \mathcal{D} \) is of type \( E_8 \) and the type of \( \mathcal{J}^2 \) is either \( D_4, D_6, E_7 \) or \( E_8 \).

The following two remarks explain how \( \text{A.1.1} \) behaves when \( \mathcal{D} \) is disconnected:

**Remark A.3.2.** If \( \mathcal{D} = \bigsqcup_{i=1}^{m} \mathcal{D}_i \) has multiple connected components \( \mathcal{D}_i \) and \( \sigma = 1 \), then \( (\mathcal{D}, \mathcal{J}, \sigma = 1) \) satisfies \( \text{A.1.1} \) if and only if every component \( (\mathcal{D}_i, \mathcal{J} \cap \mathcal{D}_i, \sigma = 1) \) does.

**Remark A.3.3.** Assume that \( \sigma \) permutes the connected components of \( \mathcal{D} \) non-trivially, i.e., that there exists distinct components \( \mathcal{D}_i \) and \( \mathcal{D}_j \) of \( \mathcal{D} \) with \( \sigma \mathcal{D}_i = \mathcal{D}_j \). Then a triple \( (\mathcal{D}, \mathcal{J}, \sigma) \) satisfies \( \text{A.1.1} \) if and only if the type \( \mathcal{J} = \emptyset \) (i.e., if and only if the parabolic \( P = B \) is a Borel). Indeed, the opposition involution \( -w_{0,3} \) leaves stable each component \( \mathcal{D}_i \) (and acts on \( \mathcal{D}_i \) by the \( i \)th component \( -w_{0,\mathcal{D}_i \cap \mathcal{D}_j} \) of \( -w_{0,3} \)).

**Theorem A.3.4.** Assume that \( \sigma \neq 1 \), that \( \mathcal{J}^2 \neq \emptyset \) and that \( \text{A.1.1} \) holds. Then \( \mathcal{D} \) is necessarily connected, of type \( A_n \) \( (n \geq 2) \), \( D_n \) \( (n \geq 4) \) or \( E_6 \). More precisely, the conditions \( \sigma \neq 1, \mathcal{J}^2 \neq \emptyset \) and \( \text{A.1.1} \) hold jointly precisely in the following cases:

1. \( \mathcal{D} \) is of type \( A_n \) for some \( n \geq 2 \), the automorphism \( \sigma = -w_0 \) is the opposition involution of \( \mathcal{D} \) and there is some \( m \) satisfying \( 2 \leq m \leq n \) and \( m \equiv n \pmod{2} \) such that \( \mathcal{J}^2 \) is the unique \( \sigma \)-stable sub-diagram of type \( A_m \).
2. \( \mathcal{D} \) is of type \( D_4 \), the automorphism \( \sigma \) is one of the three transpositions in \( \operatorname{Aut}(\mathcal{D}) \cong S_4 \) and \( \mathcal{J}^2 = \mathcal{J} \cong D_3 \cong A_3 \) is obtained by removing any one of the three extremal vertices.
3. \( \mathcal{D} \) is of type \( D_n \) for some \( n \geq 5 \), \( \sigma \in \operatorname{Aut}(\mathcal{D}) \) is the unique nontrivial element and \( \mathcal{J}^2 \) is the unique sub-diagram of type \( D_{2m+1} \), for some \( 1 \leq m \leq (n-1)/2 \).
4. \( \mathcal{D} \) is of type \( E_6 \), \( \sigma = -w_0 \) is the opposition involution and \( \mathcal{J}^2 \) is the unique \( -w_0 \)-stable sub-diagram of type \( A_3, A_5 \) or \( E_6 \).
A.4 Special cases I: Maximal Dynkin triples

Definition A.4.1. We say that a Dynkin pair \((\mathfrak{D}, \mathfrak{I})\) is maximal if, for every connected component \(\mathfrak{D}_i\) of \(\mathfrak{D}\), either \(\mathfrak{D}_i \cap \mathfrak{I} = \emptyset\) or \(\mathfrak{D}_i \cap \mathfrak{I}\) is a proper, maximal sub-diagram of \(\mathfrak{D}\), i.e., \(\text{Card}(\mathfrak{D}_i \cap \mathfrak{I}) = \text{Card}(\mathfrak{D}_i) - 1\). We say that a Dynkin triple \((\mathfrak{D}, \mathfrak{I}, \sigma)\) is maximal if the underlying pair \((\mathfrak{D}, \mathfrak{I})\) is.

Remark A.4.2. By definition, a Dynkin pair \((\mathfrak{D}, \mathfrak{I})\) is maximal if and only if this is true of every component \((\mathfrak{D}_i, \mathfrak{D}_i \cap \mathfrak{I})\).

Remark A.4.3. If the Dynkin triple \((\mathfrak{D}, \mathfrak{I}, \sigma)\) arises from \((G, \mu, P, B, T)\) as in \[A.2\] then \((\mathfrak{D}, \mathfrak{I})\) is maximal if and only if, for every nontrivial, minimal, normal, connected \(k\)-subgroup \(G_i\) of \(G\), the Levi subgroup \(L = \text{Cent}(\mu) \cap G_i\) of \(G_i\) is either all of \(G_i\) or a proper, maximal Levi of \(G_i\).

A \(\sigma\)-orbit of a Dynkin triple \((\mathfrak{D}, \mathfrak{I}, \sigma)\) is a Dynkin triple \((\mathfrak{D}', \mathfrak{I}', \sigma')\) such that \(\mathfrak{D}'\) is a \(\sigma\)-orbit of connected components of \(\mathfrak{D}, \mathfrak{I}' := \mathfrak{D}' \cap \mathfrak{I}\) and \(\sigma' = \sigma|_{\mathfrak{D}'} \in \text{Aut}(\mathfrak{D}')\).

Corollary A.4.4. A maximal Dynkin triple \((\mathfrak{D}, \mathfrak{I}, \sigma)\) satisfies \[A.1.1\] if and only if every \(\sigma\)-orbit \((\mathfrak{D}', \mathfrak{I}', \sigma')\) is one of the following:

1. \(\mathfrak{D}'\) is of type \(A_1^m\) for some \(m \geq 1\), \(\mathfrak{I}' = \emptyset\) and necessarily \(\sigma'\) permutes the \(m\) components of \(\mathfrak{D}'\) cyclically.
2. \(\mathfrak{D}'\) is of type \(A_2\), \(\mathfrak{I}'\) is of type \(A_1\) and necessarily \(\sigma' = 1\).
3. \(\mathfrak{D}'\) is of type \(B_n\) (resp. \(C_n\)) for some \(n \geq 2\), necessarily \(\sigma' = 1\) and \(\mathfrak{I}'\) is the unique sub-diagram of type \(B_{n-1}\) (resp. \(C_{n-1}\)).
4. \(\mathfrak{D}'\) is of type \(D_4\), \(\sigma'\) has order 2 and \(\mathfrak{I}' \cong D_3 \cong A_3\) is any one of the three sub-diagrams obtained by removing an extremal vertex.
5. \(\mathfrak{D}'\) is of type \(D_{2m}\) for some \(m \geq 3\), \(\sigma' \neq 1\) is the unique nontrivial element and \(\mathfrak{I}'\) is the unique sub-diagram of type \(D_{2m-1}\).
6. \(\mathfrak{D}'\) is of type \(D_{2m+1}\) for some \(m \geq 2\), \(\sigma' = 1\) and \(\mathfrak{I}'\) is the unique sub-diagram of type \(D_{2m}\).
7. \(\mathfrak{D}'\) is of type \(G_2\) (resp. \(F_4\)), necessarily \(\sigma' = 1\) and \(\mathfrak{I}'\) is obtained by removing an extremal vertex, so \(\mathfrak{I}'\) is of type \(A_1\) (resp. \(B_3\) or \(C_3\)).
8. \(\mathfrak{D}'\) is of type \(E_6\), \(\sigma' = -w_0\) and \(\mathfrak{I}'\) is the unique sub-diagram of type \(A_5\).
9. \(\mathfrak{D}'\) is of type \(E_7\), necessarily \(\sigma' = 1\) and \(\mathfrak{I}'\) is the unique sub-diagram of type \(D_6\).
10. \(\mathfrak{D}'\) is of type \(E_8\), necessarily \(\sigma' = 1\) and \(\mathfrak{I}'\) is the unique sub-diagram of type \(E_7\).

A.5 Special cases II: Dynkin triples of Hodge-type

Let \((G, X)\) be a Hodge-type Shimura datum, i.e., a Shimura datum which admits a symplectic embedding \((G, X) \hookrightarrow (\text{GSp}(2g), X_g)\) into a Siegel-type Shimura datum for some \(g \geq 1\). For every prime \(p\) such that \(G_{Q_p}\) is unramified, the process recalled in \[A.2\] produces a connected, reductive \(\mathbb{F}_p\)-group \(G\) from \(G\) and a cocharacter \(\mu \in X_{\mu}(G)\) from \(X\). Then \[A.2\] associates a Dynkin triple \((\mathfrak{D}, \mathfrak{I}, \sigma)\) to \((G, \mu)\) (which may depend on \(p\)).

Definition A.5.1. We say that a Dynkin triple \((\mathfrak{D}, \mathfrak{I}, \sigma)\) is of Hodge-type if it arises from some Hodge-type Shimura datum \((G, X)\) and some prime \(p\) by the process described above.

Combining Deligne’s classification of symplectic embeddings \[\text{[Del79, 1.3.9, 2.3.4-2.3.10]}\] with \[A.4.4\] gives:

Corollary A.5.2. Assume \((\mathfrak{D}, \mathfrak{I}, \sigma)\) is of Hodge-type. Then \((\mathfrak{D}, \mathfrak{I}, \sigma)\) is maximal and \[A.1.1\] holds precisely in cases \((\mathfrak{D}, \mathfrak{I}, \sigma) = (\mathfrak{D}_1, \mathfrak{I}_1, \sigma)\) of \[A.4.4\].
In other words the only maximal triples \((\mathfrak{D}, \mathfrak{J}, \sigma)\) with \(\mathfrak{D}\) classical and connected which satisfy \([A.1.1]\) but are not of Hodge-type are those where \(\mathfrak{D}\) is of type \(C_n\) for some \(n \geq 3\) or of type \(D_{2m}\) for some \(m \geq 2\).

### A.6 Proofs

The proofs are exercises in reading the Planches of Bourbaki [Bou68, Chap. 6, Planches I-IX]. Most importantly, consulting loc. cit., one finds:

**Lemma A.6.1.** The connected Dynkin diagrams \(\mathfrak{D}\) which have trivial opposition involution
\(-w_0 = 1\) in \(\text{Aut}(\mathfrak{D})\) are precisely those of type \(A_1, B_n\ (n \geq 2), C_n\ (n \geq 3), D_{2n}\ (n \geq 2), G_2, F_4, E_7\) and \(E_8\).

**Proof of [A.3.4]** Assume \(\sigma = 1\). Then the problem is to identify the sub-diagrams \(\mathfrak{J}\) which satisfy \(-w_{0,\mathfrak{J}} = 1\). Using [A.6.1] one sees that \(\mathfrak{J}\) should contain none of the following
(1) A connected component of type \(A_m\) with \(m \geq 2\),
(2) a connected component of type \(D_{2k+1}\) with \(k \geq 2\),
(3) a sub-diagram of type \(E_6\).

The first two restrictions (1)-(2) establish [A.3.1] when \(\mathfrak{D}\) is not of type \(E\). Type \(E\) is handled the same way, except that in addition one disqualifies the unique sub-diagram of type \(E_6\).

The connected Dynkin diagrams \(\mathfrak{D}\) which have trivial opposition involution
\(-w_0 = 1\) in \(\text{Aut}(\mathfrak{D})\) are precisely those of type \(A_1, B_n\ (n \geq 2), C_n\ (n \geq 3), D_{2n}\ (n \geq 2), G_2, F_4, E_7\) and \(E_8\).

**Proof of [A.3.4]** Since \(\sigma \neq 1\) and \(\mathfrak{J} \neq \emptyset\), \(\mathfrak{D}\) is connected by [A.3.3] Thus \(\mathfrak{D}\) must be of type \(A_n\ (n \geq 2), D_n\ (n \geq 3)\) or \(E_6\).

Consider type \(E_6\). Since \(\sigma \neq 1\), it must be the opposition involution \(\sigma = -w_0\). There are precisely six \(-w_0\)-stable sub-diagrams without isolated points, of types \(A_2, A_3, A_2 \times A_2, D_4, A_5\) and \(E_6\). The action of \(\sigma\) on the unique sub-diagram of type \(D_4\) is nontrivial, while \(-w_0 = 1\) for \(D_4\). On the other hand, the action of \(\sigma\) on the \(\sigma\)-stable \(A_2\) is trivial, while \(-w_0 \neq 1\) for \(A_2\). Thus both of these sub-diagrams fail to satisfy [A.1.1] The sub-diagram of type \(A_2 \times A_2\) also fails to satisfy [A.1.1] because the Weyl group preserves connected components, hence so does the opposition involution \(-w_0\). The remaining three sub-diagrams \(A_3, A_5\) and \(E_6\) do satisfy [A.1.1] This proves [A.3.4].

In type \(A\), if \(\sigma \neq 1\) then again \(\sigma = -w_0\) is the opposition involution. We then conclude the same way as was argued for \(A_2 \times A_2\) in \(E_6\), that the opposition involution of a diagram preserves its connected components.

In type \(D\), \(\sigma\) will act trivially on \(\sigma\)-stable sub-diagrams of type \(A\) with more than one point, while these have \(-w_0 \neq 1\). On the other hand, \(\sigma \neq 1\) will act non-trivially on a sub-diagram of type \(D\), so the latter must be of type \(D_{2k+1}\) rather than \(D_{2k}\).

**Proof of [A.4.2]** As in [A.4.2] \((\mathfrak{D}, \mathfrak{J}, \sigma)\) is maximal if and only if all its \(\sigma\)-orbits are maximal. The cases where \((\mathfrak{J}') \geq 2 \neq \emptyset\) follow directly from [A.3.1] and [A.3.4] Assume that \((\mathfrak{D}', \mathfrak{J}', \sigma')\) is maximal and \((\mathfrak{J}') \geq 2 = \emptyset\). If \(\mathfrak{D}'\) is disconnected, then \(\mathfrak{J}' = \emptyset\) and \(\mathfrak{D}'\) is of type \(A_1^m\) by [A.3.3]. If \(\mathfrak{D}'\) is connected and \(\mathfrak{J}' \geq 2 = \emptyset\), then \(\mathfrak{D}'\) must have size one or two.

Recall from [Del79, 1.2.5] that a vertex \(v \in \mathfrak{D}\) is special if the corresponding simple root \(\alpha_v \in \Delta\) has multiplicity one in the decomposition of the highest root of the connected component \(\mathfrak{D}_i\) of \(\mathfrak{D}\) containing \(v\). Equivalently, \(v\) is special if and only if the corresponding fundamental coweight is minuscule.

**Proof of [A.5.2]** Assume that \((\mathfrak{D}, \mathfrak{J}, \sigma)\) is of Hodge-type. By [Del79, 1.3.9], every connected component \(\mathfrak{D}_i\) of \(\mathfrak{D}\) is of classical type \((A_n, B_n, C_n)\) or \(D_n\) and for every component satisfying \(\mathfrak{J} \cap \mathfrak{D}_i \subseteq \mathfrak{D}_i\), the complement \(\mathfrak{D}_i \setminus (\mathfrak{J} \cap \mathfrak{D}_i) = \{v_i\}\) for some special vertex
Let \( v_i \in \mathfrak{D}_i \). Let \((\mathfrak{D}', \mathfrak{Y}', \sigma)\) be a \( \sigma \)-orbit and assume that \( \mathfrak{Y}' \neq \emptyset \). Then \( \mathfrak{D}' \) is connected. The cases (4) and (5) of type \( D_{2m} \) with nontrivial \( \sigma \) in A.4.4 are excluded as follows: Since \((G, X)\) is a Shimura datum, the adjoint group \( G^{ad}_{\mathbb{R}} \) over \( \mathbb{R} \) is an inner form of its compact form. The compact form of the adjoint split group of type \( D_{2m} \) is inner to the split form. Therefore, if \( G^{ad} \) is \( \mathbb{Q} \)-simple of type \( D_{2m} \), then the \( \mathbb{F}_p \)-group \( G^{ad} \) is \( \mathbb{F}_p \)-split, because every connected, reductive group over a finite field is quasi-split (Lang-Steinberg). So \( \sigma = 1 \) if \( G^{ad} \) is \( \mathbb{Q} \)-simple of type \( D_{2m} \).

We have excluded all the cases of A.4.4 which don’t appear in A.5.2. Conversely, it follows from Deligne’s classification \([\text{Del79}]\) 2.3.4-2.3.10 that the cases listed in A.4.4 all arise from Shimura data of Hodge-type. Concretely, A.4.4(1) is realized by Hilbert modular varieties, (2) is realized by Picard modular surfaces associated to unitary similitude groups of signature \((2, 1)\) at infinity and \( p \) split in the reflex field, and the remaining cases are realized with \( G \) a spin similitude group of signature \((2, j)\) at infinity.

\[ \text{Remark A.6.2.} \] In \([\text{GK18}]\) it was shown that the cone conjecture (Conjecture 2 of the introduction, 2.1.6 in \([\text{GK18}]\)) holds when \( G = GSp(4) \) (or equivalently when \( G = Sp(4) \)). Note that \( GSp(4) \cong GSpin(2, 3) \) and \( Sp(4)/\{\pm 1\} \cong SO(2, 3) \), so this example is part of the \( B_n \) sub-case of A.4.4(3) listed in the Hodge cases A.5.2.

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