Locally Accelerated Conditional Gradients

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Abstract

Conditional gradient methods form a class of projection-free first-order algorithms for solving smooth convex optimization problems. Apart from eschewing projections, these methods are attractive because of their simplicity, numerical performance, and the sparsity of the solutions outputted. However, they do not achieve optimal convergence rates. We present the Locally Accelerated Conditional Gradients algorithm that relaxes the projection-freeness requirement to only require projection onto (typically low-dimensional) simplices and mixes accelerated steps with conditional gradient steps to achieve local acceleration. We derive asymptotically optimal convergence rates for this algorithm. Our experimental results demonstrate the practicality of our approach; in particular, the speedup is achieved both in wall-clock time and per-iteration progress compared to standard conditional gradient methods and a Catalyst-accelerated Away-Step Frank-Wolfe algorithm.

1 Introduction

We consider problems of the form:

$$\min_{x \in \mathcal{X}} f(x),$$

where \( f \) is a smooth (gradient Lipschitz) convex function and \( \mathcal{X} \subseteq \mathbb{R}^n \) is a convex polytope. We assume first-order access to the objective function \( f \), i.e., given \( x \in \mathcal{X} \), we can compute \( f(x) \) and \( \nabla f(x) \). A typical approach to solving this problem is by using different variants of projected gradient descent. This approach is effective for simple feasible regions such as, e.g., the probability simplex or \( \ell_1 \)-balls. However, for more complicated feasible regions \( \mathcal{X} \), such as, e.g., polytopes arising from structure to be imposed in the optimization or learning process, such projections are often overly computationally expensive. To eschew projections, conditional gradient methods access \( \mathcal{X} \) solely by means of a linear optimization oracle: given a vector \( c \in \mathbb{R}^n \), the oracle returns

$$v = \arg\min_{u \in \mathcal{X}} \langle c, u \rangle.$$

The resulting iterates are then formed as convex combinations, so that the feasible region is never abandoned and, hence, no projections are needed. Due to their simplicity, good real-world performance, and other favorable properties, conditional gradient methods have been an active area of research in recent years (see, e.g., [1,4,5,13–16,18–21,23,27] and references therein). Unfortunately, while variants of these methods (e.g., Away-Step Frank-Wolfe, Pairwise Conditional Gradients, and

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Fully-Corrective Frank-Wolfe: see [21]) achieve linear convergence for smooth strongly convex functions, they do not achieve optimal accelerated rates; the same is true for the smooth (non-strongly) convex case. This is due to a lower bound that significantly limits globally achievable rates.

Limits to Global Acceleration

Acceleration for conditional gradient methods has been an important topic of interest. Apart from several technical challenges, there is a strong lower bound (see [18, 22]) that significantly limits what type of acceleration is achievable. This lower bound applies to arbitrary methods whose access to the feasible region is limited to a linear optimization oracle. As an illustration, let \( \mathcal{X} = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x \geq 0 \} \) be the probability simplex on \( n \) coordinates and consider the problem

\[
\min_{x \in \mathcal{X}} \|x\|^2. \quad \text{(LB)}
\]

If a first-order method has access to \( \mathcal{X} \) only by means of a linear optimization oracle, then it is guaranteed that after \( k < n \) iterations the primal gap satisfies:

\[
f(x_k) - f(x^*) \geq \frac{1}{k} - \frac{1}{n}. \]

In particular, for \( k = n/2 \) (assuming \( n \) even), the primal gap is lower bounded by \( f(x_{n/2}) - f(x^*) \geq \frac{1}{n} \), and an accelerated rate of \( O(1/k^2) \) cannot be achieved in full generality.

Note that the objective in the problem above is also strongly convex. Thus, if we have a generic algorithm that is linearly convergent, contracting the primal gap as \( f(x_k) - f(x^*) \leq e^{r_k}(f(x_0) - f(x^*)) \) with a global rate \( r \), then it follows that \( r \leq 2 \log \frac{n}{\epsilon} \) via the lower bound from above. The commonly used Away-Step Frank-Wolfe algorithm converges with a rate of roughly \( r = \frac{1}{\log n} \) for problem (LB) [21]; similar rates apply for Pairwise Conditional Gradients, Fully-Corrective Frank-Wolfe, etc. Given the lower bound, it follows that up to logarithmic factors these global rates cannot be further improved and, in particular, acceleration to a rate of \( \sqrt{r} = \sqrt{1/(2n)} \) is not possible.

At the same time, it is known that, e.g., Away-Step Frank-Wolfe can be globally accelerated with Catalyst [26] (see also a discussion in [21]). However, the obtained accelerated rate involves huge dimension-dependent constants to be compatible with the lower bound, making the algorithm impractical. Another form of acceleration in the case of linear optimization based methods is achieved by Conditional Gradient Sliding [24]. In [24], the complexity is separated into calls to a first-order oracle and calls to the linear optimization oracle and, while an optimal rate of \( O(1/\sqrt{\epsilon}) \) is achieved for the number of calls to the first-order oracle, the number of calls to the linear optimization oracle is \( O(1/\epsilon) \), compatible with the lower bound.

Contributions and Related Work

We show that dimension-independent acceleration for conditional gradient methods is possible after a burn-in phase whose length does not depend on the target accuracy \( \epsilon \) (but could potentially depend on the dimension). This allows for local acceleration, achieving an asymptotically optimal rate. Our contributions are summarized as follows.

Locally Accelerated Conditional Gradients. We devise a new class of conditional gradient algorithms, which we dub Locally Accelerated Conditional Gradients (LaCG), by mixing a linearly convergent Frank-Wolfe algorithm for strongly convex functions (such as, e.g., Away-Step Frank-Wolfe or Pairwise Conditional Gradients) with an accelerated sequence. The accelerated sequence is partially restarted depending on the condition of the active set of our Frank-Wolfe type iterations and the algorithm makes monotonic primal progress. LaCG achieves an asymptotically optimal iteration count of \( K + O(\sqrt{\frac{L}{r}} \log \frac{1}{\epsilon}) \) to solve \( \min_{x \in \mathcal{X}} f(x) \) up to accuracy \( \epsilon \), where \( f \) is \( L \)-smooth and \( \mu \)-strongly convex and \( K \) is a constant only depending on \( \mathcal{X} \) and \( f \). Slightly simplifying, we achieve acceleration once we identify the optimal face and are reasonably close to the optimal solution. Via standard arguments, our method also extends to the smooth (non-strongly) convex case, achieving an asymptotic iteration complexity of \( \tilde{O}(1/\sqrt{\epsilon}) \), where \( \tilde{O}(\cdot) \) is hiding log factors. In both cases, our reported complexities depend on the actual smoothness \( L \) and strong convexity constant \( \mu \) independent of the dimension of \( \mathcal{X} \), rather than derived constants that have been adjusted.
to account for the geometry of \( \mathcal{X} \), e.g., via the pyramidal width (see [21]) or smoothness and strong-convexity relative to a polytope [17, 29], which both bring in a dimension dependence. Note that such dimension-dependent terms are unavoidable if global linear rates of convergence are sought after, due to the lower bound from above. However, in contrast, our rate is local (i.e., holds after a constant number of burn-in iterations with possibly weaker rates) and as such is not subject to the lower bound, allowing us to achieve much faster local convergence of \( O(\sqrt{\frac{n}{\mu}} \log \frac{1}{\epsilon}) \) both for first-order access and linear optimization access after the burn-in. We demonstrate this also empirically, comparing our algorithm to a Catalyst-augmented Away-Step Frank-Wolfe algorithm, which also achieves acceleration, however with respect to the functional constants \( L \) and \( \mu \) weakened by a geometric correction leading to a rate equivalent of \( O(\sqrt{\frac{n}{\mu}} \frac{\delta^2}{D^2} \log \frac{1}{\epsilon}) \), where \( D \) is the diameter of \( \mathcal{X} \) and \( \delta \) is the pyramidal width of \( \mathcal{X} \); see [21].

To achieve local acceleration, we assume that we can project relatively efficiently onto simplices spanned by sets of a small number of vertices. However, while we do employ projections onto the convex hulls of maintained active sets, we stress that the feasible region is only accessed via a linear optimization oracle, i.e., our method is an LO-based method and the active sets are typically small, so that projection onto those sets is rather cheap. In this sense of employing projections internally but not over the original feasible region, our algorithm is similar to Conditional Gradient Sliding.

**Generalized Accelerated Method.** While there is an extensive literature on accelerated methods in optimization (see, e.g., [2, 3, 6–8, 10, 28, 30, 34]), none of these approaches directly applies to local acceleration of Conditional Gradients. Most relevant to our work is [7], and, in the process of obtaining our LaCG algorithm, we generalize the algorithm \( \mu \text{AGD}^+ \) from [7] in a few important ways (note that this algorithm is also a generalization of Nesterov’s method). First, we show that \( \mu \text{AGD}^+ \) retains its convergence guarantees when coupled with an arbitrary alternative algorithm, where the coupling chooses the point with the lower function value between the two algorithms, in each iteration. This is crucial for turning \( \mu \text{AGD}^+ \) into a descent method and ensures that it makes at least as much progress per iteration as the Frank-Wolfe type method with which it is coupled. This is also what allows us to achieve acceleration without any explicit knowledge of the parameters of the polytope \( \mathcal{X} \) or the position of the function minimizer \( x^* \). Second, we show that \( \mu \text{AGD}^+ \) can tolerate inexact projections onto simplices. While this is not surprising, as similar results have been shown in the past for proximal methods [32], this generalization of \( \mu \text{AGD}^+ \) is a necessary ingredient to ensure that the practical per-iteration complexity of LaCG does not become too high. Finally, we prove that \( \mu \text{AGD}^+ \) converges to the optimal solution at no computational loss even if the convex set on which the projections are performed changes between the iterations, as long as the convex set in iteration \( k \) is contained in the convex set from the preceding iteration and it contains the minimizer \( x^* \). We are not aware of any other results of this type. Note that this result allows us to update the projection simplex with each update of the active set, and, as vertices are dropped from the active set when the algorithm approaches the minimizer, the iterations become less and less expensive.

**Computational Experiments.** We compare our methods to other conditional gradient methods as well as a Catalyst-augmented Away-Step Frank-Wolfe and provide computational evidence that our algorithms achieve a practical speed-up, both in progress per iteration and in wall-clock time, significantly outperforming other methods.

2 Preliminaries

Let \( \| \cdot \| \) be the Euclidean norm and let \( B(x, r) \) denote the ball around \( x \) with radius \( r \) with respect to \( \| \cdot \| \). We say that \( x \) is \( r \)-deep in a convex set \( \mathcal{X} \subseteq \mathbb{R}^n \) if \( B(x, 2r) \cap \text{aff}(\mathcal{X}) \subseteq \mathcal{X} \). The point \( x \) is contained in the relative interior of \( \mathcal{X} \), written as \( x \in \text{rel.int}(\mathcal{X}) \), if there exists an \( r > 0 \) such that \( x \) is \( r \)-deep in \( \mathcal{X} \); if \( \text{aff}(\mathcal{X}) = \mathbb{R}^n \), then \( x \) is contained in the interior of \( \mathcal{X} \), written as \( x \in \text{int}(\mathcal{X}) \).

Further, given a polytope \( \mathcal{X} \), let \( \text{vert}(\mathcal{X}) \subseteq \mathcal{X} \) denote the (finite) set of vertices of \( \mathcal{X} \) and, given a point \( x \in \mathcal{X} \), let \( \mathcal{F}(x) \) denote the smallest face of \( \mathcal{X} \) containing \( x \), which is defined as a subset of \( \text{vert}(\mathcal{X}) \) of minimal cardinality whose convex hull contains \( x \). Finally, let \( \Delta_n \overset{\text{def}}{=} \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1, x \geq 0 \} \subseteq \mathbb{R}^n \) denote the probability simplex in dimension \( n \).
2.1 Conditional Gradient Descent

We provide a very brief introduction to the Conditional Gradient Descent algorithm [25] (also known as Frank-Wolfe algorithm; see [12]). Assume that $f$ is $L$-smooth with $L < \infty$. The Frank-Wolfe algorithm with step sizes $\eta_k \in [0, 1]$ is defined via the following updates:

$$x_{k+1} = (1 - \eta_k)x_k + \eta_k v_k = x_k + \eta_k (v_k - x_k),$$

where $x_0 \in X$ is an arbitrary initial point from the feasible set $X$ and $v_k$ is computed using the linear programming oracle as:

$$v_k = \arg\min_{u \in X} \langle \nabla f(x_k, u) \rangle.$$

2.2 Approximate Duality Gap Technique (ADGT)

To analyze the algorithms proposed in this paper, we use the Approximate Duality Gap Technique (ADGT) [9]. The core idea behind ADGT is to ensure that $A_k G_k$ is non-increasing with iteration count $k$, where $G_k$ is an upper approximation of the optimality gap (namely, $f(x_k) - f(x^*) \leq G_k$, where $x_k$ is the solution point output by the algorithm at iteration $k$), and $A_k$ is a positive strictly increasing function of $k$. If such a condition is satisfied, we immediately have $A_k G_k \leq A_0 G_0$, which implies $f(x_k) - f(x^*) \leq \frac{A_0 G_0}{A_k}$. Thus, as long as $A_0 G_0$ is bounded (it typically corresponds to some initial distance to the minimizer $x^*$), we have that the algorithm converges at rate $1/A_k$.

This also means that, to obtain the highest rate of convergence, one should always aim to obtain the fastest-growing $A_k$ for which it holds that $A_k G_k \leq A_{k-1} G_{k-1}$, for all $k$.

The approximate gap $G_k$ is defined as the difference of an upper bound $U_k$ on the function value at the output point $x_k$, $U_k \geq f(x_k)$, and a lower bound $L_k$ on the minimum function value, $L_k \leq f(x^*)$. Clearly, this choice ensures that $f(x_k) - f(x^*) \leq G_k$. In all the algorithms analyzed in this paper, we will use $U_k = f(x_k)$. The lower bound requires more effort; however, it is similar to those used in previous work [7, 9], and its detailed construction is provided in Appendix A.

Because of its generality, ADGT is well-suited to our setting, as it allows coupling different types of steps (different variants of conditional gradients and accelerated steps) and performing a more fine-grained and local analysis than typical approaches. Further, it allows accounting for inexact minimization oracles invoked as part of the algorithm subroutines in a generic way.

3 Locally Accelerated Conditional Gradients

In this section, we establish our main result. We first consider the case where the optimal solution $x^* \in \text{int}(X)$ as a warm-up to explain our approach and derive a first Locally Accelerated Frank-Wolfe algorithm specifically for this case (Section 3.1). Then, in Section 3.2, we consider the more general case where $x^* \in \text{rel. int}(F)$ with $F$ being a face of $X$. Together with the case from Section 3.1, this covers all cases of interest (except some degenerate cases). Further, the Locally Accelerated Frank-Wolfe algorithm presented in Section 3.2 works for either of the two cases.

3.1 Warm-up: Optimum in the Interior of $X$

In the case where the optimum is contained in the interior of $X$, we have $\nabla f(x^*) = 0$. As such, the unconstrained optimum and the constrained optimum coincide. Thus, one might be tempted to assert that there is no need for an accelerated Frank-Wolfe algorithm in this case. However, whether the optimum is contained in the interior is not known a priori. The presented algorithm is adaptive, as it accelerates if $x^* \in \text{int}(X)$, and otherwise it converges with the standard $1/k$ rate.

The main idea can be summarized as follows. Suppose that $x^*$ is contained $2r$-deep in the interior of $X$. Due to the function’s strong convexity, any method that contracts the optimality gap $f(x_k) - f(x^*)$ over $k$ also contracts the distance $\|x_k - x^*\|$. In particular, this follows by $\frac{r^2}{2} \|x_k - x^*\|^2 \leq f(x_k) - f(x^*)$. Hence, roughly, after an iterate $x_k$ is guaranteed to be inside the $2r$-ball, we can switch to a faster (accelerated) method for unconstrained minimization. This idea, however, requires a careful formalization, for the following reasons:

1. In general, we cannot assume that the algorithm has knowledge of $r$ and $D$, or access to information on $\|x_k - x^*\|$, as $x^*$ is unknown – it is what the algorithm is trying to determine.
2. The algorithm should be able to converge even if \( r = 0 \); we cannot in general assume that it is a priori known that \( x^* \in \text{int}(\mathcal{X}) \). Because, if that were the case and we knew that \( r \geq \varepsilon \), we would be able to run an accelerated algorithm for \( O(\sqrt{n} \log(1/\varepsilon)) \) iterations, without any need to worry about whether the solution that the algorithm outputs belongs to \( \mathcal{X} \).

We show that both issues can be resolved by implementing a monotonic version of a hybrid algorithm that can choose at each iteration whether to perform an accelerated step or a Frank-Wolfe step. Note that monotonicity will be crucial to ensure contraction of the distance to the optimal solution. In this subsection only, we assume that the algorithm has access to a membership oracle for \( \mathcal{X} \) — namely, that for any point \( x \) it can determine whether \( x \in \mathcal{X} \). This is generally a mild assumption, especially when \( \mathcal{X} \) is a polytope, which is a standard setting for Frank-Wolfe.\(^2\) The convergence of the resulting algorithm is summarized in the following theorem. Full technical details are deferred to Appendix B.

**Theorem 3.1.** Let \( x_k \) be the solution output by Algorithm 2 (Appendix B.1) for \( k \geq 1 \). If:

\[
    k \geq \min \left\{ \frac{2LD^2}{\varepsilon}, \frac{LD^2}{\mu r^2} + \sqrt{\frac{L}{\mu}} \log \left( \frac{2(L + \mu) r^2}{\mu \varepsilon} \right) \right\},
\]

then \( f(x_k) - f(x^*) \leq \varepsilon \).

### 3.2 Optimum in the Relative Interior of a Face of \( \mathcal{X} \)

We now formulate the general case that subsumes the case from above. Due to space constraints, we only state the main ideas, while full technical details are deferred to Appendix B.2.

We assume that, given points \( x_1, ..., x_m \) and a point \( y \), the following problem is easily solvable:

\[
    u = \min_{\frac{1}{2} \sum_{i=1}^{m} \lambda_i x_i, x_i \in \Delta_m} \frac{1}{2} \| u - y \|^2. \tag{3.1}
\]

In other words, we assume that the projection onto the convex hull of a given set of vertices can be implemented efficiently; however, we do not require access to a membership oracle anymore. Moreover, note that this projection problem does neither require access to the first-order oracle nor the linear optimization oracle. Finally, due to Lemma 3.2 stated below, we only need to solve this problem to accuracy of the order \( 1/\sqrt{\mu L} \), where \( \varepsilon \) is the target accuracy of the program.

Our LaCG algorithm is a hybrid version of the Away-Step Frank-Wolfe (AFW) algorithm and an accelerated algorithm over the convex hull of certain active sets. While we perform the analysis for the Away-Step Frank-Wolfe variant, it can also be extended to combine **Pairwise Conditional Gradients** with accelerated steps. The assumption that we only consider polytopes is important as the linear convergence for the AFW algorithm established in [21] relies on a constant, the **pyramidal width**, that is only known to be bounded away from 0 for polytopes. For completeness, we provide the pseudocode for one iteration of AFW (as stated in [21]) in Algorithm 3 in Appendix B.2. Similar to the algorithm from the previous subsection, LaCG is monotonic to ensure that enough progress is made regardless of the setting, and that the distance to the optimal solution is contracting. The particular version of the accelerated algorithm we use here is a modification of \( \mu \text{AGD}^+ \) from [7]. Unlike its original version [7], the version we provide here (Lemma 3.2) allows coupling the method with another optimization method, supports inexact minimization oracles, and supports changes in the convex set (which correspond to active sets from AFW) on which projections are performed.

**Lemma 3.2.** (Convergence of the modified \( \mu \text{AGD}^+ \)) Let \( f : \mathcal{X} \to \mathbb{R} \) be an \( L \)-smooth and \( \mu \)-strongly convex function, and let \( \mathcal{X} \) be a closed convex set. Let \( x^* = \arg\min_{u \in \mathcal{X}} f(u) \), and let \( \{C_i\}_{i=0}^{k} \) be a sequence of convex subsets of \( \mathcal{X} \) such that \( C_i \subseteq C_{i-1} \) for all \( i \) and \( x^* \in \bigcap_{i=0}^{k} C_i \). Let \( \{\tilde{x}_i\}_{i=0}^{k} \) be any (fixed) sequence of points from \( \mathcal{X} \). Let \( a_0 = 1 \), \( a_k \frac{a_i}{A_k} = \theta \) for \( k \geq 1 \), where \( A_k = \sum_{i=0}^{k} a_i \) and

\(^2\) For generic LP solvers applied to polynomially-sized LPs, checking membership amounts to evaluating the linear (in)equality describing \( \mathcal{X} \), which is typically much cheaper than linear optimization over \( \mathcal{X} \). For structured LPs that are solvable in polynomial time but have exponential representation (e.g., matching over non-bipartite graphs [31]), there typically exist membership oracles with running times that are comparable to the running time of the corresponding minimization oracle used in Frank-Wolfe steps (e.g., [11]).
\[ \theta = \sqrt{\frac{2}{\pi}}. \] Let \( y_0 \in \mathcal{X}, x_0 = w_0, \) and \( z_0 = L y_0 - \nabla f(y_0). \) For \( k \geq 1, \) define iterates \( x_k \) by:

\[
\begin{align*}
    y_k &= \frac{1}{1 + \theta} x_{k-1} + \frac{\theta}{1 + \theta} w_{k-1}, \\
    z_k &= z_{k-1} - a_k \nabla f(y_k) + \mu a_k y_k, \\
    \hat{x}_k &= (1 - \theta) x_{k-1} + \theta w_k, \\
    x_k &= \text{argmin}\{f(\hat{x}_k), f(x_k)\}
\end{align*}
\]  

(3.2)

where, for all \( k \geq 0, \) \( w_k \) is defined as an \( \epsilon_k^m \)-approximate solution of:

\[
\min_{u \in C_k} \left\{ -\langle z_k, u \rangle + \frac{\mu L_k + \mu_0}{2} \|u\|^2 \right\},
\]

(3.3)

with \( \mu_0 \overset{\text{def}}{=} L - \mu. \) Then, for all \( k \geq 0, \) \( x_k \in \mathcal{X} \) and:

\[
f(x_k) - f(x^*) \leq (1 - \theta)^k (L - \mu) \|x^* - y_0\|^2 + 2 \sum_{i=0}^{k-1} \epsilon_i^m + \epsilon_k^m.
\]

To obtain our locally accelerated algorithm, we will show that from some iteration onwards, we can apply the accelerated method from Lemma 3.2 with \( C_k \) being the convex hull of the vertices from the active set and the sequence \( \hat{x}_k \) being the sequence of the Away-Step Frank-Wolfe algorithm. The pseudocode for the Locally Accelerated Conditional Gradients algorithm is provided in Algorithm 1.

For completeness, pseudocode for one iteration of the accelerated method (ACC), which is based on Eq. (3.2) is provided in Algorithm 5 (Appendix B.2).

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**Algorithm 1** Locally Accelerated Conditional Gradients

1: Let \( x_0 \in \mathcal{X} \) be an arbitrary point, \( S_0^{\text{AFW}} = \{x_0\}, \lambda_0^{\text{AFW}} = [1] \)
2: Let \( y_0 = \hat{x}_0 = w_0 = x_0, z_0 = -\nabla f(y_0) + L y_0, C_1 = \text{co}(S_0^{\text{AFW}}) \)
3: \( a_0 = A_0 = 1, \theta = \sqrt{\frac{2}{\pi}}, \mu_0 = L - \mu \)
4: \( H = \frac{2}{\theta} \log(1/\theta^2 - 1) \quad \triangleright \) Minimum restart period
5: \( r_f = \text{false}, r_c = 0 \quad \triangleright \) Restart flag and restart counter initialization
6: for \( k = 1 \) to \( K \) do
7: \( x_k^{\text{AFW}}, S_k^{\text{AFW}}, \lambda_k^{\text{AFW}} = \text{AFW}(x_{k-1}^{\text{AFW}}, S_{k-1}^{\text{AFW}}, \lambda_{k-1}^{\text{AFW}}) \quad \triangleright \) Independent AFW update
8: \( A_k = A_{k-1}/(1 - \theta), a_k = \theta A_k \)
9: \( \hat{x}_k, z_k, w_k = \text{ACC}(x_{k-1}, z_{k-1}, w_{k-1}, \mu, a_k, A_k, C_k) \)
10: if \( r_f \) and \( r_c \geq H \) then \( \triangleright \) Restart criterion is met
11: \( y_k = \text{argmin}\{f(x_k^{\text{AFW}}), f(\hat{x}_k)\} \quad \triangleright \) Updating feasible set for the accelerated sequence
12: \( C_{k+1} = \text{co}(S_k^{\text{AFW}}) \quad \triangleright \) Restarting accelerated sequence
13: \( a_k = A_k = 1, z_k = -\nabla f(y_k) + L y_k \)
14: \( \hat{x}_k = w_k = \text{argmin}_{u \in C_{k+1}} \{-\langle z_k, u \rangle + \frac{L}{2} \|u\|^2\} \)
15: \( r_c = 0, r_f = \text{false} \quad \triangleright \) Resetting the restart indicators
16: else
17: if \( S_k^{\text{AFW}} \setminus S_{k-1}^{\text{AFW}} \neq \emptyset \) then \( \triangleright \) If a vertex was added to the active set
18: \( r_f = \text{true} \quad \triangleright \) Raise restart flag
19: if \( r_f = \text{false} \) then \( \triangleright \) If AFW did not add a vertex since last restart
20: \( C_{k+1} = \text{co}(S_k^{\text{AFW}}) \quad \triangleright \) Update the feasible set
21: else
22: \( C_{k+1} = C_k \quad \triangleright \) Freeze the feasible set
23: \( x_k = \text{argmin}\{f(x_k^{\text{AFW}}), f(\hat{x}_k), f(x_{k-1})\} \quad \triangleright \) Choose the better step + monotonicity
24: \( r_c = r_c + 1 \quad \triangleright \) Increment the restart counter

A simple observation, which turns out to be key for the coupling to work is that when running the Away-Step Frank-Wolfe algorithm there exists an iteration \( K_0 \) such that for all \( k \geq K_0 \) it holds \( x^* \in \text{co}(S_k^{\text{AFW}}) \), where \( S_k^{\text{AFW}} \) denote the active sets maintained by AFW. This iteration \( K_0 \) only depends on the feasible region \( \mathcal{X} \) and \( x^* \) and, as such, it is a burn-in period of constant length. Our main theorem is stated below, with a proof sketch. The full proof is deferred to Appendix B.2.
We illustrate the performance of our algorithm with two numerical experiments, in order to compare Wolfe. This immediately implies (by a standard Away-Step Frank-Wolfe guarantee; see [21] and where \( H \)) with an argumentation and convergence analysis analogous to the one in Section 3.1. In particular, the whether (variants of) conditional gradients. The resulting algorithm is then fully projection-free in the sense \( \epsilon/ \).

The rest of the proof invokes Lemma 3.2 and its corollary (Corollary B.5), which ensures accelerated minimization oracle (\( \epsilon^m = 0 \) in Lemma 3.2). Clearly, it suffices to have \( \epsilon^m = \frac{\alpha}{8} \epsilon \) and invoke Theorem 3.3 for target accuracy \( \epsilon/2 \).

Remark 3.5 (Running Algorithm 1 when \( x^* \in \text{int}(\mathcal{X}) \)). Usually we do not know ahead of time whether \( x^* \in \text{int}(\mathcal{X}) \) or whether \( x^* \) is in the relative interior of a face of \( \mathcal{X} \). However, we can simply run Algorithm 1 agnostically, as in the case where \( x^* \in \text{int}(\mathcal{X}) \) we still exhibit local acceleration with an argumentation and convergence analysis analogous to the one in Section 3.1. In particular, the assumptions of Section 3.2 are only needed to establish a bound for the estimation in Proposition B.4.

Remark 3.6 (Variant relying exclusively on a linear optimization oracle). Similar as in the Conditional Gradient Sliding (CGS) algorithm [24] we can also solve the arising projection problems using (variants of) conditional gradients. The resulting algorithm is then fully projection-free in the sense that it is only accessing the feasible region by means of the linear optimization oracle. A variant of CGS would then be recovered if we would ignore the AFW steps and only run the accelerated sequence with such projections realized by conditional gradients.

Remark 3.7 (Extension to the smooth non-strongly convex case). Our results from Theorem 3.3 can also be extended to the general smooth (non-strongly) convex case by using a simple argument, see e.g., [33]. For this, given an \( L \)-smooth convex function \( f \) and a target accuracy of \( \epsilon \), we define an auxiliary function \( f_{\epsilon} \equiv f(x) + \frac{D^2}{2\epsilon} \|x_0 - x\|^2 \), where \( D \) is the diameter of \( \mathcal{X} \), which is \((L + \frac{D^2}{2\epsilon})\) smooth and \( \frac{D^2}{\epsilon} \)-strongly convex and optimizing \( f_{\epsilon} \) with accuracy \( \epsilon/2 \) optimizes \( f \) with accuracy \( \epsilon \).

Now, we run LaCG on \( f_{\epsilon} \) and achieve an accuracy of \( \epsilon \) after at most \( \sqrt{\frac{2LD^2}{\epsilon}} \log \left( \frac{(L+\epsilon)D^2}{\epsilon} \right) \) iterations, ignoring the initial burn-in. This is optimal up to a log factor.

4 Computational Results

We illustrate the performance of our algorithm with two numerical experiments, in order to compare the running time and the progress per iteration of our algorithm with that of two other projection-free
methods. The methods we compare to are the Away-Step Frank-Wolfe algorithm [21] and a Catalyst-augmented Away-Step Frank-Wolfe algorithm, obtained applying [26] to the Away-Step Frank-Wolfe algorithm (see the discussion in [21]). The numerical experiments were implemented in Python 3 using standard libraries such as numpy with no non-trivial optimization applied.

All examples involve a positive definite random quadratic with the global minimum outside of the feasible region of interest. One of the feasible regions is the probability simplex, which can be considered a simple feasible region in terms of projections, while the other is the Birkhoff polytope, where projections would be considerably more involved. Both however allow for fast and simple linear minimization oracles. In the case of the probability simplex, the linear optimization is obvious, while in the case of the Birkhoff polytope we use the Hungarian method. In order to solve the minimization problem stated in Eq. (3.3), we use the Projected Gradient Descent (PGD) algorithm in the case of the simplex, and the Accelerated Projected Gradient Descent (APGD) in the case of the Birkhoff polytope, solving the problem to a tolerance $\frac{\epsilon}{\sqrt{2\mu L}}$ and using the rather weak Frank-Wolfe gap against this tolerance as a stopping criterion. Note that in the case of the simplex, PGD and APGD have the same performance, as in the projection problem from Eq. (3.3) (equivalently, from Eq. (3.1)), the condition number of the objective function (determined by the vertices from the active set) is 1, as the vertices from the active set are standard basis vectors.

All plots depict the evolution of the primal gap (normalized by the initial primal gap) over the iteration count and wall-clock time, with the vertical axes in logarithmic scale. The first instance corresponds to the probability simplex of dimension 2000 for a quadratic with condition number $L/\mu = 1000$ (Figure 1). Figure 2 shows the performance of the algorithms on the Birkhoff polytope for a graph with 40 nodes (ambient dimension 1600), for a quadratic with a condition number $L/\mu = 100$.

As can be seen, LaCG exhibits higher progress per iteration, and better wall-clock performance than the two competing methods, shortening the running time with respect to Away-Step Frank Wolfe in both problem instances. We also remark that as opposed to the Catalyst-augmented Away-Step Frank-Wolfe algorithm, the convergence rate for LaCG does not depend on the dimension of the problem, which is reflected in both problem instances. Moreover, the accuracy to which the subproblems need to be solved is constant, whereas the Catalyst approach requires that the subproblem be solved with increasing accuracy.

Figure 1: LaCG comparison for the probability simplex of dimension 2000. Left: normalized optimality gap over iteration count; right: normalized optimality gap over wall-clock time.

Figure 2: LaCG comparison for the Birkhoff polytope of ambient dimension 1600. Left: normalized optimality gap over iteration count; right: normalized optimality gap over wall-clock time.
5 Discussion

We presented the Locally-Accelerated Conditional Gradients method that achieves asymptotically optimal rate in a local region around the minimum and improves upon the existing conditional gradients methods, both in theory and in experiments. As discussed before, such an accelerated rate cannot be achieved globally. The experiments shown here are for the purpose of illustration; more extensive experiments will be conducted in a future version of the paper.

Some interesting questions for future research remain. For example, it would be interesting to understand whether the version of $\mu$AGD+ from Lemma 3.2, which allows changing the projection set $C_k$, can speed up the practical performance of accelerated methods in other (possibly projection-based) optimization settings.

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References

[1] M. A. Bashiri and X. Zhang. Decomposition-invariant conditional gradient for general polytopes with line search. In Proc. NIPS'17, 2017.
[2] A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM J. Imaging Sci., 2(1):183–202, 2009.
[3] M. Betancourt, M. I. Jordan, and A. C. Wilson. On symplectic optimization. arXiv preprint arXiv:1802.03653, 2018.
[4] G. Braun, S. Pokutta, D. Tu, and S. Wright. Blended Conditional Gradients: the unconditioning of conditional gradients. In Proc. ICML’19, 2019.
[5] G. Braun, S. Pokutta, and D. Zink. Lazifying Conditional Gradient Algorithms. In Proc. ICML’17, 2017.
[6] S. Bubeck, Y. T. Lee, and M. Singh. A geometric alternative to Nesterov’s accelerated gradient descent. arXiv preprint, arXiv:1506.08187, 2015.
[7] M. B. Cohen, J. Diakonikolas, and L. Orecchia. On acceleration with noise-corrupted gradients. In Proc. ICML’18, 2018.
[8] J. Diakonikolas and L. Orecchia. Accelerated extra-gradient descent: A novel, accelerated first-order method. In Proc. ITCS’18, 2018.
[9] J. Diakonikolas and L. Orecchia. The approximate duality gap technique: A unified theory of first-order methods. SIAM J. Optimiz., 29(1):660–689, 2019.
[10] D. Drusvyatskiy, M. Fazel, and S. Roy. An optimal first order method based on optimal quadratic averaging. SIAM J. Optimiz., 28(1):251–271, 2018.
[11] L. K. Fleischer, A. N. Letchford, and A. Lodi. Polynomial-time separation of a superclass of simple comb inequalities. Math. of Oper. Res., 31(4):696–713, 2006.
[12] M. Frank and P. Wolfe. An algorithm for quadratic programming. Naval research logistics quarterly, 3(1-2):95–110, 1956.
[13] R. M. Freund, P. Grigas, and R. Mazumder. An extended Frank-Wolfe method with “in-face” directions, and its application to low-rank matrix completion. SIAM J. Optimiz., 27(1):319–346, 2017.
[14] D. Garber and E. Hazan. A linearly convergent conditional gradient algorithm with applications to online and stochastic optimization. arXiv preprint arXiv:1301.4666, 2013.
[15] D. Garber and O. Meshi. Linear-memory and decomposition-invariant linearly convergent conditional gradient algorithm for structured polytopes. arXiv preprint, arXiv:1605.06492v1, 2016.
[16] D. Garber, S. Sabach, and A. Kaplan. Fast generalized conditional gradient method with applications to matrix recovery problems. *arXiv preprint arXiv:1802.05581*, 2018.

[17] D. H. Gutman and J. F. Pena. The condition number of a function relative to a set. *arXiv preprint arXiv:1901.08359*, 2019.

[18] M. Jaggi. Revisiting Frank-Wolfe: Projection-free sparse convex optimization. In *Proc. ICML'13*, 2013.

[19] T. Kerdreux, A. d’Aspremont, and S. Pokutta. Restarting Frank–Wolfe. In *Proc. AISTATS'18*, 2018.

[20] T. Kerdreux, F. Pedregosa, and A. d’Aspremont. Frank–Wolfe with subsampling oracle. *arXiv preprint arXiv:1803.07348*, 2018.

[21] S. Lacoste-Julien and M. Jaggi. On the global linear convergence of Frank-Wolfe optimization variants. In *Proc. NIPS'15*, 2015.

[22] G. Lan. The complexity of large-scale convex programming under a linear optimization oracle. *arXiv preprint arXiv:1309.5550*, 2013.

[23] G. Lan, S. Pokutta, Y. Zhou, and D. Zink. Conditional accelerated lazy stochastic gradient descent. In *Proc. ICML'17*, 2017.

[24] G. Lan and Y. Zhou. Conditional gradient sliding for convex optimization. *SIAM J. Optimiz.*, 26(2):1379–1409, 2016.

[25] E. S. Levitin and B. T. Polyak. Constrained minimization methods. *USSR Computational mathematics and mathematical physics*, 6(5):1–50, 1966.

[26] H. Lin, J. Mairal, and Z. Harchaoui. A universal catalyst for first-order optimization. In *Proc. NIPS'15*, 2015.

[27] F. Locatello, A. Raj, S. P. Reddy, G. Rätsch, B. Schölkopf, S. U. Stich, and M. Jaggi. Revisiting first-order convex optimization over linear spaces. *arXiv preprint arXiv:1803.09539*, 2018.

[28] Y. Nesterov. *Lectures on Convex Optimization*. Springer, 2018.

[29] J. Pena and D. Rodriguez. Polytope conditioning and linear convergence of the Frank-Wolfe algorithm. *Math. of Oper. Res.*, 44(1):1–18, 2018.

[30] B. T. Polyak. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. & Math. Phys.*, 4(5):1–17, 1964.

[31] T. Rothvoß. The matching polytope has exponential extension complexity. *Journal of the ACM (JACM)*, 64(6):41, 2017.

[32] M. Schmidt, N. L. Roux, and F. R. Bach. Convergence rates of inexact proximal-gradient methods for convex optimization. In *Proc. NIPS'11*, 2011.

[33] D. Scieur, A. d’Aspremont, and F. Bach. Regularized nonlinear acceleration. In *Proc. NIPS’16*, 2016.

[34] P. Tseng. On accelerated proximal gradient methods for convex-concave optimization, 2008.
A Lower Bound in ADGT

In this section, we provide the construction of the lower bounds on the minimum function value $f(x^*)$ that are used in our analysis. By $\mu$-strong convexity of $f$, we have that, $\forall x \in \mathcal{X}$:

$$f(x^*) \geq f(x) + \langle \nabla f(x), x^* - x \rangle + \frac{\mu}{2} \|x - x^*\|^2. \quad (A.1)$$

Further, if $x^*$ belongs to the interior of $\mathcal{X}$, then $\nabla f(x^*) = 0$, and $L$-smoothness of $f$ implies, $\forall x \in \mathcal{X}$:

$$f(x^*) \geq f(x) - \frac{L}{2} \|x - x^*\|^2. \quad (A.2)$$

Let $\{x_i\}_{i=0}^k$ be a sequence of points from some feasible set $\mathcal{X}$ and let $\{a_i\}_{i=0}^k$ be a sequence of positive numbers with $a_0 = 1$. Define $A_k \overset{\text{def}}{=} \sum_{i=0}^k a_i$.

Assume first that $x^*$ belongs to the interior of the feasible set $\mathcal{X}$. Then, taking a convex combination of Eq. (A.2) with $x = x_0$ and Eq. (A.1) with $x = x_i$, $1 \leq i \leq k$, we get:

$$f(x^*) \geq \frac{\sum_{i=0}^k a_i f(x_i) + \sum_{i=1}^k a_i (\langle \nabla f(x_i), x^* - x_i \rangle + \frac{\mu}{2} \|x_i - x^*\|^2) - \frac{L}{2} \|x_0 - x^*\|^2}{A_k}$$

$$+ \frac{\mu}{2A_k} \|x_0 - x^*\|^2 - \frac{\mu}{2A_k} \|x_0 - x^*\|^2$$

$$\geq \frac{\sum_{i=0}^k a_i f(x_i) + \min_{u \in \mathbb{R}^d} \{ \sum_{i=1}^k a_i (\langle \nabla f(x_i), u - x_i \rangle + \frac{\mu}{2} \|x_i - u\|^2) + \frac{\mu}{2} \|x_0 - u\|^2 \}}{A_k}$$

$$- \frac{L + \mu}{2A_k} \|x_0 - x^*\|^2.$$

The last expression corresponds to the lower bound used in the proof of Lemma B.2.

Now assume that $x^*$ is not necessarily from the interior of $\mathcal{X}$. Take a convex combination (with weights $a_i/A_k$) of Eq. (A.1) for $x = x_i$, $0 \leq i \leq k$. Let $\mathcal{C}_k$ be any convex subset of $\mathcal{X}$ that contains $x^*$. Then, we have:

$$f(x^*) \geq \frac{\sum_{i=0}^k a_i f(x_i) + \sum_{i=0}^k a_i (\langle \nabla f(x_i), x^* - x_i \rangle + \frac{\mu}{2} \|x_i - x^*\|^2)}{A_k}$$

$$+ \frac{\mu_0}{2A_k} \|x_0 - x^*\|^2 - \frac{\mu_0}{2A_k} \|x_0 - x^*\|^2$$

$$\geq \frac{\sum_{i=0}^k a_i f(x_i) + \min_{u \in \mathcal{C}_k} \{ \sum_{i=0}^k a_i (\langle \nabla f(x_i), u - x_i \rangle + \frac{\mu}{2} \|x_i - u\|^2) + \frac{\mu}{2} \|x_0 - u\|^2 \}}{A_k}$$

$$- \frac{\mu_0}{2A_k} \|x_0 - x^*\|^2.$$

The last expression corresponds to the lower bound used in the proof of Lemma 3.2.

B Omitted Proofs from Section 3

B.1 Proofs and Results for Warm-up: Optimum in the Interior of $\mathcal{X}$

Starting at point $x_k$, the Frank-Wolfe step $x_{k+1}^{FW}$ is defined via:

$$v_k = \arg\min_{u \in \mathcal{X}} \langle \nabla f(x_k), u \rangle,$$

$$x_{k+1}^{FW} = (1 - \eta_k)x_k + \eta_k v_k,$$

where

$$\eta_k = \arg\min_{\eta \in [0,1]} \left\{ f(x_k) + \langle \nabla f(x_k), \eta (v_k - x_k) \rangle + \frac{L}{2\eta^2} \|x_k - v_k\|^2 \right\}. \quad (B.1)$$
On the other hand, the accelerated step \( \hat{x}_{k+1} \) is defined as:

\[
y_{k+1} = \frac{1}{1 + \theta} x_k + \frac{\theta}{1 + \theta} w_k,
\]

\[
w_{k+1} = (1 - \theta) w_k + \theta \left( y_{k+1} - \frac{1}{\mu} \nabla f(y_{k+1}) \right),
\]

\[
\hat{x}_{k+1} = (1 - \theta) x_k + \theta w_{k+1},
\]

where \( \theta = \sqrt{\frac{L}{\mu}} \) and \( w_k \) and \( x_k \) are appropriately initialized. We now proceed to describe the algorithm.

**Algorithm 2** Preliminary Locally Accelerated Frank-Wolfe for \( x^* \in \text{int}(X) \)

Input: \( x_0 \in X, \mu, L, X \)

Initialization: \( w_0 = x_0, \theta = \sqrt{\mu/L} \)

1: for \( k = 0 \) to \( N - 1 \) do
2: Compute \( x_{k+1}^{FW} \), based on Eq. (B.1) and \( \hat{x}_{k+1} \) based on Eq. (B.2)
3: if \( \hat{x}_{k+1} \in X \) then
4: \( x_{k+1} = \arg\min \{ f(x_{k+1}^{FW}), f(\hat{x}_{k+1}) \} \)
5: else
6: \( x_{k+1} = x_{k+1}^{FW} \)
7: \( w_{k+1} = x_{k+1} \)

Note that the “else” branch in Algorithm 2 effectively restarts the accelerated sequence.

Let us now argue about the convergence of the algorithm. Observe first that the algorithm makes at least as much progress as Frank-Wolfe, since, whatever the step is, \( f(x_{k+1}) \leq f(x_{k+1}^{FW}) \). We thus have the following simple proposition, which bounds the length of the so-called burn-in phase.

**Proposition B.1.** Assume that \( r > 0 \). Then, after at most \( K_0 = \left\lfloor \frac{L D^2}{\mu r^2} \right\rfloor \) steps of Algorithm 2, \( f(x_{K_0}) - f(x^*) \leq 2\mu r^2 \). Further, in every subsequent iteration \( k \geq K_0 \), \( \| x_k - x^* \| \leq 2r \).

**Proof.** As in every iteration the algorithm makes at least as much progress as standard Frank-Wolfe (since \( f(x_{k+1}) \leq f(x_{k+1}^{FW}) \)), by standard Frank-Wolfe guarantees (see e.g., [18]), we have that after \( K_0 \) steps \( f(x_{K_0}) - f(x^*) \leq \frac{2L D^2}{K_0 + 1} \), which gives the first part of the lemma. Since none of the iterations of the algorithm can increase the function value, we have that in every subsequent iteration \( f(x_k) - f(x^*) \leq 2\mu r^2 \). By strong convexity and \( \nabla f(x^*) = 0 \), this implies \( \| x_k - x^* \| \leq 2r \).

We can conclude that if \( r > 0 \), for \( k > K_0 = \left\lfloor \frac{L D^2}{\mu r^2} \right\rfloor \) Algorithm 2 never enters the else branch, as \( B(x^*, 2r) \cap \text{aff}(X) \subseteq X \). This is precisely what allows us to obtain accelerated convergence in the remaining iterations. This is formally established by the following lemma.

**Lemma B.2.** Assume that \( r > 0 \) and let \( K_0 = \left\lfloor \frac{L D^2}{\mu r^2} \right\rfloor \). Then, for all \( k \geq K_0 \):

\[
f(x_k) - f(x^*) \leq 2\frac{L + \mu}{\mu} r^2 \left( 1 - \sqrt{\frac{\mu}{L}} \right)^{k - K_0}.
\]

**Proof.** Let \( k_0 \leq K_0 \) be the last iteration in which Algorithm 2 enters the “else” branch – as already argued, for \( k > K_0 \), this cannot happen. Then, from Algorithm 2, we have that \( w_{k_0} = x_{k_0} \), and for all iterations \( k \geq k_0 + 1 \):

\[
y_k = \frac{1}{1 + \theta} x_{k-1} + \frac{\theta}{1 + \theta} w_{k-1},
\]

\[
w_k = (1 - \theta) w_{k-1} + \theta (y_k - \frac{1}{\mu} \nabla f(y_k)),
\]

\[
\hat{x}_k = (1 - \theta) x_{k-1} + \theta w_k,
\]

\[
x_k = \arg\min \{ f(\hat{x}_k), f(x_k^{FW}) \}.
\]

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To analyze the convergence of (B.3), we use the approximate duality gap technique, as described in Section 2.2. Let $a_{k_0} = A_{k_0} = 1$ and $A_k = \sum_{i=k_0}^{k} a_i$, $\frac{\theta}{A_k} = \theta$ for $k \geq k_0 + 1$. Recall that the approximate duality gap $G_k$ is defined as the difference between a lower bound on $f(x^*)$, $L_k$ and an upper bound on the algorithm output, $U_k$. Define $U_k = f(x_k)$ and $L_k$ via (see Appendix A):

$$L_k \defegal \sum_{i=k_0}^{k} a_i f(y_i) + \min_{u \in \mathbb{R}^d} \left\{ \sum_{i=k_0+1}^{k} a_i \left( \langle \nabla f(y_i), u - y_i \rangle + \frac{\mu}{2} \| u - y_i \|^2 \right) + \frac{\mu}{2} \| u - x_{k_0} \|^2 \right\} / A_k$$

$$- \frac{L + \mu}{2 A_k} \| x^* - x_{k_0} \|^2.$$  

(B.4)

We claim that:

$$w_k = \arg\min_{u \in \mathbb{R}^d} \left\{ \sum_{i=k_0+1}^{k} a_i \left( \langle \nabla f(y_i), u - y_i \rangle + \frac{\mu}{2} \| u - y_i \|^2 \right) + \frac{\mu}{2} \| u - x_{k_0} \|^2 \right\} / A_k$$

$$= x_{k_0} + \sum_{i=k_0+1}^{k} a_i (y_i - \frac{1}{\mu} \nabla f(y_i)).$$  

(B.5)

Indeed, Eq. (B.5) implies that $w_{k_0} = x_{k_0}$, while for $k > k_0$ it gives: $A_k w_k = A_{k-1} w_{k-1} + a_k (y_k - \frac{1}{\mu} \nabla f(y_k))$. As $A_k = A_{k-1} + a_k$ and $\frac{\theta}{A_k} = \theta$, (B.5) implies that $w_k = (1 - \theta) w_{k-1} + \theta (y_k - \frac{1}{\mu} \nabla f(y_k))$, which is equivalent to the definition from Eq. (B.3).

Further, observe from (B.3) that $x_{k-1} = (1 + \theta)y_{k-1} - \theta w_{k-1}$, which, combined with $w_k = (1 - \theta) w_{k-1} + \theta (y_k - \frac{1}{\mu} \nabla f(y_k))$ and $\theta = \sqrt{\mu/L}$, implies

$$\dot{x}_k = y_k - \frac{1}{L} \nabla f(y_k).$$  

(B.6)

The rest of the proof bounds the initial gap $G_{k_0}$ and shows that for $k > k_0$, $G_k \leq (1 - \theta) G_{k-1}$. Note that, by construction, $f(x_k) - f(x^*) \leq G_k$.

The initial gap equals $G_{k_0} = \frac{L + \mu}{2} \| x^* - x_{k_0} \|^2$. This follows by simply evaluating $U_{k_0} - L_{k_0}$.

Now let $k > k_0$. As $f(x_k) \leq f(\dot{x}_k)$ and using (B.6):

$$A_k U_k - A_{k-1} U_{k-1} \leq A_k f(\dot{x}_k) - A_{k-1} f(x_{k-1})$$

$$= a_k f(y_k) + A_k (f(\dot{x}_k) - f(y_k)) + A_{k-1} (f(y_k) - f(x_{k-1}))$$

$$\leq a_k f(y_k) - \frac{A_k}{2L} \| \nabla f(y_k) \|^2 + A_{k-1} (f(y_k) - f(x_{k-1})).$$  

(B.7)

To bound the change in the lower bound, denote by:

$$m_k(u) = \sum_{i=k_0+1}^{k} a_i \left( \langle \nabla f(y_i), u - y_i \rangle + \frac{\mu}{2} \| u - y_i \|^2 \right) + \frac{\mu}{2} \| u - x_{k_0} \|^2$$

the function inside the minimum in the definition of $L_k$. Hence:

$$m_k(w_k) = m_{k-1}(w_k) + a_k \langle \nabla f(x_k), w_k - y_k \rangle + a_k \frac{\mu}{2} \| w_k - y_k \|^2.$$

As $w_{k-1}$ minimizes $m_{k-1}(\cdot)$, expanding $m_{k-1}(w_k)$ around $w_{k-1}$, we have:

$$m_{k-1}(w_k) = m_{k-1}(w_{k-1}) + \langle \nabla m_{k-1}(w_{k-1}), w_k - w_{k-1} \rangle + \frac{A_{k-1} \mu}{2} \| w_k - w_{k-1} \|^2,$$

leading to:

$$m_k(w_k) - m_{k-1}(w_{k-1}) = a_k \langle \nabla f(y_k), w_k - y_k \rangle + a_k \frac{\mu}{2} \| w_k - y_k \|^2 + \frac{A_{k-1} \mu}{2} \| w_k - w_{k-1} \|^2$$

$$\geq a_k \langle \nabla f(y_k), w_k - y_k \rangle + \frac{A_k \mu}{2} \| w_k - \frac{A_{k-1}}{A_k} w_{k-1} - \frac{a_k}{A_k} y_k \|^2.$$
where the second line is by Jensen’s inequality. As \( \frac{A_k}{\lambda_k} = \theta = \sqrt{\mu/L} \), using the definition of \( w_k \), we have:
\[
m_k(w_k) - m_k-1(w_{k-1}) \geq a_k \langle \nabla f(y_k), w_k - y_k \rangle + \frac{A_k}{2L} \| \nabla f(y_k) \|^2.
\]
Combining with the definition of \( L_k \), we thus have:
\[
A_k L_k - A_{k-1} L_{k-1} \geq a_k f(y_k) + a_k \langle \nabla f(y_k), w_k - y_k \rangle + \frac{A_k}{2L} \| \nabla f(y_k) \|^2.
\] (B.8)

Combining (B.7) and (B.8), we have:
\[
A_k G_k - A_{k-1} G_{k-1} \leq A_k (f(y_k) - f(x_{k-1})) - a_k \langle \nabla f(y_k), w_k - y_k \rangle - \frac{A_k}{L} \| \nabla f(y_k) \|^2
\]
\[
\leq \langle \nabla f(y_k), A_k y_k - A_{k-1} x_{k-1} - a_k w_k \rangle - \frac{A_k}{L} \| \nabla f(y_k) \|^2
\]
\[
= A_k \langle \nabla f(y_k), y_k - \hat{x}_k \rangle - \frac{A_k}{L} \| \nabla f(y_k) \|^2
\]
\[
= 0,
\]
where the second line is by convexity of \( f \) (namely, by \( f(y_k) - f(x_k) \leq \langle \nabla f(y_k), y_k - x_k \rangle \)), the third line is by the definition of \( \hat{x}_k \) and \( \theta = \frac{a_k}{\lambda_k} \), and the last line is by (B.6).

As \( \frac{A_{k-1}}{A_k} = 1 - \theta \), we have that \( G_k \leq (1 - \theta)^{k-k_0} G_{k_0} = (1 - \theta)^{k-k_0} \frac{\mu + L}{2} \| x^* - x_{k_0} \|^2 \), and, thus:
\[
f(x_k) - f(x^*) \leq (1 - \sqrt{\frac{\mu}{L}})^{k-k_0} \frac{L + \mu}{2} \| x^* - x_{k_0} \|^2.
\]

By the same arguments as in the proof of Proposition B.1, \( f(x_{k_0}) - f(x^*) \leq \frac{3LD^2}{\eta \mu r^2} \). By strong convexity of \( f \), this implies that also \( \mu \| x_{k_0} - x^* \|^2 \leq \frac{4LD^2}{\eta \mu r^2} \). To complete the proof, it remains to argue that
\[
(1 - \sqrt{\frac{\mu}{L}})^{K_0-k_0} \mu \| x_{k_0} - x^* \|^2 \leq (1 - \sqrt{\frac{\mu}{L}})^{K_0-k_0} \frac{4LD^2}{\eta \mu r^2} \leq 4r^2.
\]
This simply follows by arguing that for the choice of \( k_0 \) from the statement of the lemma and \( k_0 \leq K_0 \), we have
\[
(1 - \sqrt{\frac{\mu}{L}})^{K_0-k_0} \frac{1}{\eta \mu r^2} \leq \frac{1}{K_0 \mu r^2},
\]
while the rest follows from Proposition B.1. This is not hard to show and is omitted. □

Finally, we have the following bound on the convergence of Algorithm 2.

**Theorem 3.1.** Let \( x_k \) be the solution output by Algorithm 2 (Appendix B.1) for \( k \geq 1 \). If:
\[
k \geq \min \left\{ \frac{2LD^2}{\epsilon}, \frac{LD^2}{\eta \mu r^2}, \frac{L}{\mu} \log \left( \frac{2(L + \mu)r^2}{\mu \epsilon} \right) \right\},
\]
then \( f(x_k) - f(x^*) \leq \epsilon \).

**Proof.** Follows directly by applying the standard convergence bound for FW, Proposition B.1, and Lemma B.2. □

Note that in the argument in Proposition B.1 we could have also used the Away-Step Frank-Wolfe algorithm achieving linear convergence for the burn-in phase. However, for the easy of exposition we used the simpler bound for the warm-up; we will use the Away-Step Frank-Wolfe algorithm in Section 3.2.

### B.2 Proofs and Results for Optimum in the Relative Interior of a Face of \( \mathcal{A} \) from Section 3.2

In this section we provide full technical details for the results in Section 3.2 and we also restate material from that section here once again to facilitate reading.

We will now formulate the general case that subsumes the case from above. We assume that, given a set of points \( x_1, \ldots, x_m \) and a point \( y \), the following optimization problem is easily solvable:
\[
u = \min_{\lambda \in \Delta} \max_{i=1}^m \lambda_i x_i, \quad \frac{1}{2} \| u - y \|^2.
\]
In other words, we assume that the projection onto the convex hull of a given set of vertices can be implemented efficiently, however we do not require access to a membership oracle anymore. Moreover, note that this projection problem does neither require access to the first-order oracle nor the linear optimization oracle. Finally, due to Lemma 3.2, we only need to solve this problem to accuracy of the order $\frac{1}{\sqrt{r_{opt}}}$, where $\epsilon$ is the target accuracy of the program.

The assumption that we only consider polytopes is important as the linear convergence for the Away-Step Frank-Wolfe (AFW) algorithm established in [21] relies on a constant, the pyramidal width, that is only known to be bounded away from 0 for polytopes. We briefly recall the pseudocode for one iteration of Away-Step Frank-Wolfe algorithm (as stated in [21]) in Algorithm 3. The AFW algorithm starts with an arbitrary point $x_0$ from the feasible set and active set $S_0 = \{x_0\}$. In the following, the vector $\lambda_k \in \Delta_m$ with $m = |S_k|$ denotes the barycentric coordinates of the current iterate $x_k$ over the active set $S_k$.

**Algorithm 3 Away-Step Frank-Wolfe Iteration: AFW($\lambda, S, x$)**

1: Set FW direction: $s = \arg\min_{u \in X} \langle \nabla f(x), u \rangle$, $d^{FW} = s - x$
2: Set Away direction: $v = \arg\max_{u \in S} \langle \nabla f(x), u \rangle$, $d^A = x - v$
3: if $\langle -\nabla f(x), d^{FW} \rangle \geq \langle -\nabla f(x), d^A \rangle$ then
   4: $d = d^{FW}$, $\gamma_{\max} = 1$
   5: else
   6: $d = d^A$, $\gamma_{\max} = \frac{\lambda(v)}{1 - \lambda(v)}$
   7: $\gamma' = \arg\min_{\gamma \in [0, \gamma_{\max}]} f(x + \gamma d)$
   8: $x' = x + \gamma' d$; update $\lambda$ (to $\lambda'$)
   9: $S' = \{ u \in S \cup \{s\} : \lambda'(u) > 0 \}$
10: return $x'$, $S'$, $\lambda'$

We will need the following fact that establishes the existence of a radius $r$ (and hence iteration $K_r$) from which onwards all active sets $S_k$ maintained by our algorithm ensure that $x^* \in \text{co}(S_k)$ for all $k \geq K_r$.

**Fact B.3** (Active set convergence). There exists $r > 0$ such that for any subset $S \subseteq \text{vert}(X)$ and point $x \in X$ with $x \in \text{co}(S)$ and $||x - x^*|| \leq r$ it follows $x^* \in \text{co}(S)$.

**Proof.** Let $S \subseteq \text{vert}(X)$ be an arbitrary subset of vertices, so that $x^* \not\in \text{co}(S)$. As $S$ is closed there exists $0 < r_s \equiv \min_{x \in S} ||x - x^*||$. Let $2r$ be the minimum over all such $S$, which is bounded away from 0 as there are only finitely many such subsets. It follows that if $||x - x^*|| \leq r$ then $x^* \in \text{co}(S_k)$. \hfill $\square$

Let $r_0$ denote the critical radius from Fact B.3 and $K_0$ the critical iteration so that $||x^* - x_k|| \leq r_0$ is ensured for all $k \geq K_0$. The next proposition bounds the magnitude of $K_0$.

**Proposition B.4** (Finite burn-in with linear rate). Denote by $\delta$ the pyramidal width of the polytope $X$, as defined in [21]. Then for all $k \geq K_0$ it holds $x^* \in \text{co}(S_k)$ and for any algorithm that makes in each iteration at least as much progress as the Away-Step Frank-Wolfe Algorithm, we have the bound

$$K_0 \leq \frac{8L}{\mu} \left( \frac{D}{\delta} \right)^2 \log \left( \frac{2(f(x_0) - f(x^*))}{\mu r_0^2} \right).$$

**Proof.** Since the algorithm makes at least as much progress as the Away-Step Frank-Wolfe algorithm, we can use the convergence rate of [21] to bound the primal gap at step $k$. Using the $\mu$-strong convexity of $f$, we have that $f(x_k) - f(x^*) \geq \mu/2 ||x_k - x^*||$, allowing us to relate the primal gap to the distance to the optimum. \hfill $\square$
and that the distance to the optimal solution is contracting. The particular version of the accelerated algorithm we use here is $\mu$AGD+ from [7]; we start by showing the following generic result for a modification of $\mu$AGD+. Note that, unlike its original version from [7], the version we provide here allows coupling the method with another optimization method, supports inexact minimization oracles, and supports changes in the convex set on which projections from Eq. (3.3) are performed.

**Lemma 3.2. (Convergence of the modified $\mu$AGD+)** Let $f : X \to \mathbb{R}$ be an $L$-smooth and $\mu$-strongly convex function, and let $X$ be a closed convex set. Let $x^* = \arg\min_{u \in X} f(u)$, and let $(C_i)_{i=0}^k$ be a sequence of convex subsets of $X$ such that $C_i \subseteq C_{i-1}$ for all $i$ and $x^* \in \bigcap_{i=0}^k C_i$. Let $(\bar{x}_i)_{i=0}^k$ be any (fixed) sequence of points from $X$. Let $a_0 = 1$, $\frac{a_k}{a_{k-1}} = \theta$ for $k \geq 1$, where $A_k = \sum_{i=0}^{k-1} a_i$ and $\theta = \sqrt{\frac{1}{2\pi}}$. Let $y_0 \in X$, $x_0 = w_0$, and $z_0 = L y_0 - \nabla f(y_0)$. For $k \geq 1$, define iterates $x_k$ by:

$$
\begin{align*}
  y_k &= \frac{1}{1+\theta} x_{k-1} + \frac{\theta}{1+\theta} w_{k-1}, \\
  z_k &= z_{k-1} - a_k \nabla f(y_k) + \mu a_k y_k, \\
  \bar{x}_k &= (1-\theta)x_{k-1} + \theta w_k, \\
  x_k &= \arg\min \{ f(\bar{x}_k), f(x_k) \}
\end{align*}
$$

(3.2)

where, for all $k \geq 0$, $w_k$ is defined as an $\epsilon_k^m$-approximate solution of:

$$
\min_{u \in C_k} \left\{ -\langle z_k, u \rangle + \frac{\mu A_k + \mu_0}{2} \|u\|^2 \right\},
$$

(3.3)

with $\mu_0 \overset{\text{def}}{=} L - \mu$. Then, for all $k \geq 0$, $x_k \in X$ and:

$$
\begin{align*}
  f(x_k) - f(x^*) &\leq (1-\theta)^k \frac{(L - \mu) \|x^* - y_0\|^2}{2} + 2 \sum_{i=0}^{k-1} m_i^m + \epsilon_k^m \\
  &\leq A_k
\end{align*}
$$

Proof. We first show by induction on $k$ that $x_k \in X$. The claim is true initially, by the statement of the lemma. Assume that the claim is true for the iterates up to $k - 1$. Then, $\bar{x}_k$ must be from $X$, as it is a convex combination of $x_{k-1} \in X$ (by the inductive hypothesis) and $w_k \in C_k \subseteq X$. By assumption, $\bar{x}_k \in X$, for all $k$. Thus, it must be $x_k \in X$.

The rest of the proof relies on showing that $A_k G_k \leq A_{k-1} G_{k-1} + \epsilon_k^m + \epsilon_{k-1}^m$ and on bounding $A_0 G_0$, where $G_k$ is an approximate duality gap defined as $G_k = U_k - L_k$. Here, the upper bound is defined as $U_k = f(x_k)$, while the lower bound on $L_k \geq f(x^*)$ can be defined as (see Appendix A):

$$
L_k \overset{\text{def}}{=} \sum_{i=0}^{k} a_i f(y_i) + \min_{u \in C_k} m_k(u) - \frac{\mu_0}{2} \|x^* - y_0\|^2,
$$

where $\mu_0 = L - \mu$ and

$$
\begin{align*}
  m_k(u) &\overset{\text{def}}{=} \sum_{i=0}^{k} a_i (\nabla f(y_i), u - y_i) + \sum_{i=0}^{k} a_i \frac{\mu}{2} \|u - y_i\|^2 + \frac{\mu_0}{2} \|u - y_0\|^2.
\end{align*}
$$

It is not hard to verify that:

$$
\text{argmin}_{u \in C_k} \left\{ -\langle z_k, u \rangle + \frac{\mu A_k + \mu_0}{2} \|u\|^2 \right\} = \text{argmin}_{u \in C_k} m_k(u), \quad \forall k.
$$

Let us start by bounding $A_0 G_0$. Recall that $a_0 = A_0 = 1$ and $x_0 = w_0$. By smoothness of $f$,

$$
U_0 = f(x_0) = f(w_0) \leq f(y_0) + \langle \nabla f(y_0), w_0 - y_0 \rangle + L_0 \|w_0 - y_0\|^2.
$$

(B.9)

On the other hand, as $\mu_0 = L - \mu$ and $w_0$ is an $\epsilon_0^m$-approximate minimizer of $\text{argmin}_{u \in C} m_0(u_0)$, we have:

$$
\min_{u \in C_0} m_0(u) \geq m_0(w_0) - \epsilon_0^m = \langle \nabla f(y_0), w_0 - y_0 \rangle + L_0 \|w_0 - y_0\|^2 - \epsilon_0^m.
$$

(B.10)

Combining Eqs. (B.9) and (B.10) with the definition of $L_k$, we have that:

$$
A_0 G_0 \leq \frac{\mu_0 \|x^* - y_0\|^2}{2} + \epsilon_0^m = (L - \mu) \|x^* - y_0\|^2 + \epsilon_0^m.
$$
To complete the proof, it remains to show that $G_k \leq \frac{A_k - 1}{A_k} G_{k-1} = (1 - \theta)G_{k-1}$. Observe first, as $f(x_k) \leq f(\bar{x}_k)$, that we can bound the change in the upper bound as:

$$A_k U_k - A_{k-1} U_{k-1} = A_k f(x_k) - A_{k-1} f(x_{k-1}) \leq a_k f(y_k) + A_k (f(\bar{x}_k) - f(y_k)) + A_{k-1} (f(y_k) - f(x_{k-1})).$$

Using smoothness and convexity of $f$, we further have:

$$A_k U_k - A_{k-1} U_{k-1} \leq a_k f(y_k) + (\nabla f(y_k), A_k \bar{x}_k - A_{k-1} x_{k-1} - a_k y_k) + \frac{A_k L}{2} \| \bar{x}_k - y_k \|^2. \quad (B.11)$$

By the definition of $L_k$, the change in the lower bound is:

$$A_k L_k - A_{k-1} L_{k-1} = a_k f(y_k) + m_k(w_k^*) - m_{k-1}(w_{k-1}^*), \quad (B.12)$$

where $w_k^* = \arg\min_u \mathcal{C}_k m_k(u)$.

To bound $m_k(w_k^*) - m_{k-1}(w_{k-1}^*)$, observe first that:

$$m_k(w_k^*) - m_{k-1}(w_{k-1}^*) \geq m_k(w_k) - m_{k-1}(w_{k-1}^*) - \epsilon_k^m. \quad (B.13)$$

as $w_k \in \mathcal{C}_k$ is an $\epsilon_k^m$-approximate minimizer of $m_k$. Further, observe that $m_k(u) = m_{k-1}(u) + a_k \langle \nabla f(y_k), u - y_k \rangle + a_k \frac{\mu}{2} \| u - y_k \|^2$. Hence, we have:

$$m_k(w_k) - m_{k-1}(w_{k-1}^*) = a_k \langle \nabla f(y_k), w_k - y_k \rangle + a_k \frac{\mu}{2} \| w_k - y_k \|^2 + m_{k-1}(w_k) - m_{k-1}(w_{k-1}^*). \quad (B.14)$$

As $m_k(u)$ can be expressed as the sum of $\frac{\mu A_k}{2} \| u \|^2$ and terms that are linear in $u$, it is $(\mu_0 + \mu A_k)$-strongly convex. Observe that, as $w_{k-1}^*$ minimizes $m_{k-1}$ over $C_{k-1}$ and $w_k \in \mathcal{C}_k \subseteq \mathcal{C}_{k-1}$, by the first-order optimality condition, we have $\langle \nabla m_{k-1}(w_{k-1}^*), w_k - w_{k-1}^* \rangle \geq 0$. Thus, it further follows that:

$$m_{k-1}(w_k) \geq m_{k-1}(w_{k-1}^*) + \frac{\mu_0 + \mu A_{k-1}}{2} \| w_k - w_{k-1}^* \|^2. \quad (B.15)$$

Next, observe that, as $m_{k-1}$ is $(\mu_0 + \mu A_{k-1})$-strongly convex, $w_{k-1}^*$ minimizes $m_{k-1}$, and $w_{k-1}$ is an approximate minimizer, we have:

$$\frac{\mu_0 + \mu A_{k-1}}{2} \| w_{k-1} - w_{k-1}^* \|^2 \leq m_{k-1}(w_{k-1}) - m_{k-1}(w_{k-1}^*) \leq \epsilon_k^m. \quad (B.16)$$

Using Young’s inequality ($(a + b)^2 \leq 2a^2 + 2b^2$ and so $a^2 \geq \frac{(a + b)^2}{2} - b^2$), we have, using Eq. (B.16), that:

$$\frac{\mu_0 + \mu A_{k-1}}{2} \| w_k - w_{k-1}^* \|^2 \geq \frac{\mu_0 + \mu A_{k-1}}{4} \| w_k - w_{k-1} \|^2 - \frac{\mu_0 + \mu A_{k-1}}{2} \| w_{k-1} - w_{k-1}^* \|^2 \geq \frac{\mu_0 + \mu A_{k-1}}{4} \| w_k - w_{k-1} \|^2 - \epsilon_k^m. \quad (B.17)$$

Combining the last inequality with Eqs. (B.13)–(B.15), we have:

$$m_k(w_k^*) - m_{k-1}(w_{k-1}^*) \geq a_k \langle \nabla f(y_k), w_k - y_k \rangle + a_k \frac{\mu}{2} \| w_k - y_k \|^2 + \frac{\mu_0 + \mu A_{k-1}}{4} \| w_k - w_{k-1} \|^2 - \epsilon_k^m - \epsilon_k^m.$$

Using that $\mu_0 \geq 0, \theta = \frac{\mu_k}{A_k}$, and applying Jensen’s inequality to the last expression,

$$m_k(w_k^*) - m_{k-1}(w_{k-1}^*) \geq a_k \langle \nabla f(y_k), w_k - y_k \rangle + \frac{\mu A_k}{4} \| w_k - (1 - \theta)w_{k-1} - \theta y_k \|^2 - \epsilon_k^m - \epsilon_k^m.$$

It is not hard to verify that $\bar{x}_k - y_k = \theta(w_k - (1 - \theta)w_{k-1} - \theta y_k)$. Hence, combining the last inequality with Eq. (B.12):

$$A_k L_k - A_{k-1} L_{k-1} \leq a_k f(y_k) + a_k \langle \nabla f(y_k), w_k - y_k \rangle + \frac{\mu A_k}{4\theta^2} \| \bar{x}_k - y_k \|^2 - \epsilon_k^m - \epsilon_k^m. \quad (B.17)$$
Finally, combining Eqs. (B.11) and (B.17), we have:

$$A_k G_k - A_{k-1} G_{k-1} \leq \langle \nabla f(y_k), A_k \hat{x}_k - A_{k-1} x_{k-1} - a_k w_k \rangle + A_k \left( L - \frac{\mu}{2 \theta^2} \right) \| \hat{x}_k - y_k \|^2 + \epsilon_m + \epsilon_{m-1}

\leq \epsilon_m + \epsilon_{m-1},$$

as $\hat{x}_k = \frac{A_{k-1}}{A_k} x_{k-1} + \frac{a_k}{A_k} w_k$ and $\theta = \sqrt{\frac{2}{2\theta^2}}$, completing the proof.

A simple corollary of Lemma 3.2 that will be useful for our analysis is as follows. It shows that if the algorithm from Lemma 3.2 is not restarted too often, we do not lose more than a constant factor (two) in the final bound on the iteration count.

**Corollary B.5.** Define a restart of the method from Lemma 3.2 as setting $a_k = A_k = 1$, $y_k = x_{k-1}$, $w_k = y_k$, and $z_k = L y_k - \nabla f(y_k)$. Let $\epsilon_m = \frac{\theta}{2} \epsilon$, for some $\epsilon_m \geq 0$. If the method is restarted no more frequently than every $\frac{\theta}{2} \log(1/(2\theta^2) - 1)$ iterations, where $\theta = \sqrt{\mu/(2L)}$, then:

$$f(x_k) - f(x^*) \leq \frac{L - \mu}{\mu} (1 - \theta)^{k/2} (f(x_0) - f(x^*)) + 2\epsilon.$$

Proof. Denote $H = \frac{\theta}{2} \log(1/(2\theta^2) - 1)$. Let the iterations at which the restarts happen be denoted as $k_0 = 0$, $k_1$, $k_2$, ..., and note that, by assumption, $k_i \geq k_{i-1} + H$, for all $i \geq 1$. Assume w.l.o.g. that each $k_i$ is even. We first claim that we have the following contraction between the successive restarts:

$$f(x_{k_i}) - f(x^*) \leq (1 - \theta)^{(k_i - k_{i-1})/2} (f(x_{k_{i-1}}) - f(x^*)) + \epsilon.$$  \hspace{1cm} (B.18)

To prove the claim, observe first using $k_i - k_{i-1} \geq H$ that:

$$\frac{L - \mu}{\mu} (1 - \theta)^{k_i - k_{i-1}} \leq \left( \frac{1}{2\theta^2} - 1 \right) \left( 1 - \theta \right) \frac{1}{2} \text{log}(\frac{1}{2\theta^2}) \leq (1 - \theta)^{k_i - k_{i-1}}.$$  \hspace{1cm} (B.19)

Applying Lemma 3.2 with $x_{k_{i-1}}$ as an initial point and using strong convexity of $f$ (which implies $f(x_{k_{i-1}}) - f(x^*) \geq \frac{\mu}{2} \| x_{k_{i-1}} - x^* \|^2$), we have:

$$f(x_{k_i}) - f(x^*) \leq \frac{L - \mu}{\mu} (1 - \theta)^{k_i - k_{i-1}} (f(x_{k_{i-1}}) - f(x^*)) + \epsilon.$$  \hspace{1cm} (B.20)

Thus, combining the last inequality with (B.19), inequality (B.18) follows.

Applying Eq. (B.18) recursively and using that $k_i - k_{i-1} \geq H$, we further have:

$$f(x_{k_i}) - f(x^*) \leq (1 - \theta)^{k_i/2} (f(x_0) - f(x^*)) + \epsilon \sum_{j=0}^{i} (1 - \theta)^{jH/2} \leq (1 - \theta)^{k_i/2} (f(x_0) - f(x^*)) + 2\theta^2 \epsilon.$$  \hspace{1cm} (B.20)

To complete the proof, fix an iteration $k$ and let $k_i$ be the last iteration up to $k$ in which a restart happened. Applying Lemma 3.2 with $k_i$ as the initial point, we get:

$$f(x_k) - f(x^*) \leq \frac{L - \mu}{\mu} (1 - \theta)^{k - k_i} (f(x_{k_i}) - f(x^*)) + \epsilon \leq \frac{L - \mu}{\mu} (1 - \theta)^{k/2} (f(x_0) - f(x^*)) + (1 + 2\theta^2) \epsilon.$$  \hspace{1cm} (B.20)

It remains to note that $\theta^2 = \mu/(2L) \leq 1/2$. \hspace{1cm} (B.20)

To obtain our locally accelerated algorithm, we will show that from some iteration onwards, we can apply the accelerated method from Lemma 3.2 with $C_k$ being the convex hull of the vertices from the active set and the sequence $x_k$ being the sequence of the Away-Step Frank-Wolfe algorithm. The pseudocode for the Locally Accelerated Conditional Gradients algorithm is provided in Algorithm 1 (Algorithm 4 in the Appendix).
Algorithm 4: Locally Accelerated Conditional Gradients
1: Let $x_0 \in \mathcal{X}$ be an arbitrary point, $S_0 = \{x_0\}$, $\lambda_0 = 1$
2: Let $y_0 = \bar{x}_0 = w_0 = x_0$, $a_0 = -\nabla f(y_0) + Ly_0$, $C_1 = \text{co}(S_0)$
3: $a_0 = A_0 = 1$, $\theta = \sqrt{\frac{L}{\mu}}$, $\mu = L - \mu$
4: $H = \frac{1}{2} \log((1/\theta^2) - 1)$ \text{\textcolor{red}{Minimum restart period}}
5: $r_f = \text{false}$, $r_c = 0$ \text{\textcolor{red}{Restart flag and restart counter initialization}}
6: for $k = 1$ to $K$ do
7: $x_k^{\text{AFW}}, S_k^{\text{AFW}}, A_k^{\text{AFW}} = \text{AFW}(x_{k-1}^{\text{AFW}}, S_{k-1}^{\text{AFW}}, A_{k-1}^{\text{AFW}})$ \text{\textcolor{red}{Independent AFW update}}
8: $A_k = A_{k-1}/(1 - \theta)$, $a_k = \theta A_k$
9: $\hat{x}_k, z_k, w_k = \text{ACC}(x_{k-1}, z_{k-1}, w_{k-1}, \mu, a_k, A_k, C_k)$
10: if $r_f$ and $r_c \geq H$ then \text{\textcolor{red}{Restart criterion met}}
11: $y_k = \text{argmin}\{f(x_k^{\text{AFW}}), f(\hat{x}_k)\}$ \text{\textcolor{red}{Updating feasible set for the accelerated sequence}}
12: $C_{k+1} = \text{co}(S_{k}^{\text{AFW}})$ \text{\textcolor{red}{Restarting accelerated sequence}}
13: $a_k = A_k = 1$, $z_k = -\nabla f(y_k) + Ly_k$
14: $\hat{x}_k = w_k = \text{argmin}_{u \in C_{k+1}} \{-\langle z_k, u \rangle + \frac{L}{2} ||u||^2\}$
15: $r_c = 0$, $r_f = \text{false}$ \text{\textcolor{red}{Resetting the restart indicators}}
16: else
17: if $S_k^{\text{AFW}} \setminus S_{k-1}^{\text{AFW}} \neq \emptyset$ then \text{\textcolor{red}{If a vertex was added to the active set}}
18: $r_f = \text{true}$ \text{\textcolor{red}{Raise restart flag}}
19: if $r_f = \text{false}$ then \text{\textcolor{red}{If AFW did not add a vertex since last restart}}
20: $C_{k+1} = \text{co}(S_{k}^{\text{AFW}})$ \text{\textcolor{red}{Updating the feasible set}}
21: else
22: $C_{k+1} = C_k$ \text{\textcolor{red}{Freeze the feasible set}}
23: $x_k = \text{argmin}\{f(x_k^{\text{AFW}}), f(\hat{x}_k), f(x_{k-1})\}$ \text{\textcolor{red}{Choose the better step + monotonicity}}
24: $r_c = r_c + 1$ \text{\textcolor{red}{Increment the restart counter}}
5: return $\hat{x}_k, z_k, w_k$

Algorithm 5: Accelerated Step ACC($x_{k-1}, z_{k-1}, w_{k-1}, \mu, a_k, A_k, C_k$)
1: $\theta = a_k/A_k$
2: $y_k = \frac{1}{1+\theta} x_{k-1} + \frac{\theta}{1+\theta} w_{k-1}$
3: $z_k = z_{k-1} - a_k f(y_k) + a_k y_k$, $w_k = \text{argmin}_{u \in C_k} \{-\langle z_k, u \rangle + \frac{a_k + \mu}{2} ||u||^2\}$
4: $\hat{x}_k = (1 - \theta)x_{k-1} + \theta w_k$
5: return $\hat{x}_k, z_k, w_k$

Main Theorem 3.3. (Convergence analysis of Locally Accelerated Frank-Wolfe) Let $x_k$ be the solution output by Algorithm 1 and $r_0$ be the critical radius (see Fact B.3 in Appendix B.2). If:

$$k \geq \min \left\{ \frac{8L}{\mu} \left( \frac{D}{\delta} \right)^2 \log \left( \frac{f(x_0) - f(x^*)}{\epsilon} \right), K_0 + H + 2 \sqrt{\frac{2L}{\mu}} \log \left( \frac{L - \mu}{2 \epsilon} \right) \right\},$$

where $H = 2 \sqrt{2L/\mu} \log(L/\mu - 1)$ and $K_0 = \frac{4L}{\mu} \left( \frac{D}{\delta} \right)^2 \log \left( \frac{2(f(x_0) - f(x^*))}{\mu \rho^2 \mu} \right)$, then:

$$f(x_k) - f(x^*) \leq \epsilon.$$

Proof. The statement of the theorem is a direct consequence of the following observations about Algorithm 1 (Algorithm 4 in the Appendix). First, observe that the Away-Step Frank-Wolfe algorithm is run independently of the accelerated sequence and in particular the accelerated sequence has no effect on the AFW-sequence whatsoever. Further, in any iteration, the set $C_k$ that we project onto is the convex hull of some active set $S_k^{\text{AFW}} \subseteq \mathcal{X}$ for some $0 \leq i \leq k - 1$ implying $\hat{x}_k \in \mathcal{X}$—each $\hat{x}_k$ is hence feasible.

Now, as in any iteration $k$ the solution outputted by the algorithm is $x_k = \text{argmin}\{f(x_k^{\text{AFW}}), f(\hat{x}_k)\}$, the algorithm never makes less progress than the Away-Step Frank-Wolfe. This immediately implies (by a standard Away-Step Frank-Wolfe guarantee; see [21] and Proposition B.4) that for $k \geq \frac{8L}{\mu} \left( \frac{D}{\delta} \right)^2 \log \left( \frac{f(x_0) - f(x^*)}{\mu \rho^2 \mu} \right)$ it must be that $f(x_k) - f(x^*) \leq \epsilon$, which establishes the unaccelerated part of the minimum in the asserted rate.
Further by Proposition B.4, there exists an iteration $K \leq K_0$ such that for all $k \geq K$ it holds $x^* \in \text{co}(S_{k}^{AFW})$. Let $K$ be the first such iteration. Then, the Away-Step Frank-Wolfe algorithm must have added a vertex in iteration $K$ as otherwise $x^* \in \text{co}(S_{k-1}^{AFW})$, contradicting the minimality of $K$. Due to the restarting criterion from Algorithm 1, a restart must happen by iteration $K_0 + H$. Thus, for $k \geq K_0 + H$, it must be $x^* \in C_k$.

Further, the restarting criterion implies that we perform at least $H = \frac{2}{\theta} \log \left(1/(2\theta^2) \right) - 1$ iterations between successive restarts of the accelerated sequence $\{\hat{x}_k\}$ and, unless a restart happens, we also have that $C_k \subseteq C_{k-1}$. Thus, starting from iteration $K_0 + H$, Lemma 3.2 and Corollary B.5 apply and $\{x_k\}$ converges to $x^*$ at an accelerated rate. The remaining $2\sqrt{\frac{2}{\mu} \log \left( (L-\mu)\rho^2 \right)}$ part of the minimum in the asserted rate follows now by Corollary B.5.

Remark B.6 (Inexact projection oracles.). For simplicity, we stated Theorem 3.3 assuming exact minimization oracle ($\epsilon^m = 0$ in Lemma 3.2). Clearly, it suffices to have $\epsilon^m = \frac{a}{\beta} \epsilon$ and invoke Theorem 3.3 for target accuracy $\epsilon/2$. This only affects the constant factor in the convergence bound (it gets increased by a factor of $2$).

Remark B.7 (Running Algorithm 1 when $x^* \in \text{int}(X)$). Usually we do not know ahead of time whether $x^* \in \text{int}(X)$ or whether $x^*$ is in the relative interior of a face of $X$. However, we can simply run Algorithm 1 agnostically, as in the case where $x^* \in \text{int}(X)$ we still exhibit local acceleration with an argumentation and convergence analysis analogous to the one in Section 3.1. In particular, the assumptions of Section 3.2 are only needed to establish a bound for the estimation in Proposition B.4.

Remark B.8 (Variant relying exclusively on a linear optimization oracle). Similar as in the Conditional Gradient Sliding algorithm [24] we can also solve the arising projection problems using (variants of) conditional gradients. The resulting algorithm is then fully projection-free in the sense that it is only accessing the feasible region by means of the linear optimization oracle. A variant of CGS would then be recovered if we would ignore the AFW steps and only run the accelerated sequence with such projections realized by conditional gradients.

Remark B.9 (Extension to the smooth non-strongly convex case). Our results from Theorem 3.3 can also be extended to the general smooth and (non-strongly) case by using a simple argument, see e.g., [33]. For this, given an $L$-smooth convex function $f$ and a target accuracy of $\epsilon$, we define an auxiliary function $f_\epsilon \equiv f(x) + \frac{L}{2\rho^2} \|x_0 - x\|^2$, where $D$ is the diameter of $X$, which is $(L + \frac{\sqrt{L}}{2\rho})$-smooth and $\frac{\sqrt{L}}{2\rho}$-strongly convex and optimizing $f_\epsilon$ with accuracy $\epsilon/2$ optimizes $f$ with accuracy $\epsilon$. Now, we run Locally Accelerated Conditional Gradients on $f_\epsilon$ and achieve an accuracy of $\epsilon$ after at most $\sqrt{\frac{2L\rho^2}{\epsilon}} \log \left( \frac{(L+\epsilon)^2}{\epsilon} \right)$ iterations, ignoring the initial burn-in. This is optimal up to a log factor.