Abstract

Presented here are algorithms for converting between (decimal) scientific-notation and (binary) IEEE-754 double-precision floating-point numbers. These algorithms are much simpler than those previously published. The values are stable under repeated conversions between the formats. The scientific representations generated have only the minimum number of mantissa digits needed to convert back to the original binary values.

Also presented is an algorithm for printing IEEE-754 double-precision floating-point numbers with a specified number of decimal digits of precision. For the specified number of digits, the decimal numbers produced are the closest possible to the binary values.

Introduction

Articles from Steele and White[SW90, Clinger[Cli90], and Burger and Dybvig[BD96] establish that binary floating-point numbers can be converted into and out of decimal representations without losing accuracy using a minimum number of (decimal) significant digits.

The lossless algorithms from these papers all require high-precision integer calculations, although not for every conversion.

In How to Read Floating-Point Numbers Accurately[Cli90] Clinger astutely observes that successive rounding operations do not have the same effect as a single rounding operation. This is the crux of the difficulty with both reading and writing floating-point numbers. But instead of constructing his algorithm to do a single rounding operation, Clinger and the other authors follow Matula[Mat68, Mat70] in doing successive integer divisions and remainders.

Both Steel and White[SW90] and Clinger[Cli90] claim that the input and output problems are fundamentally different from each other because the floating-point format has a fixed precision while the decimal representation does not. Yet, bignum rounding divisions accomplish accurate conversions in both directions.

The algorithms from How to Print Floating-point Numbers Accurately[SW90] and Printing floating-point numbers quickly and accurately[BD96] are iterative and complicated. The read and write algorithms presented here do at most 2 and 4 bignum divisions, respectively. Over the range of IEEE-754[IEE85] double-precision numbers, the largest intermediate bignum used by presented algorithms is 339 decimal digits (1126 bits). According to Steele and White[SW90], the largest bignum used by their algorithm is 1050 bits. These are not large for bignums, being orders of magnitude smaller than the smallest precisions which get speed benefits from FFT multiplication.
Bignums

Both reading and writing of floating-point numbers can involve division of numbers larger than can be stored in the floating-point registers, causing rounding at undesired points of the conversion.

Bignums (arbitrary precision integers) can perform division of large integers without rounding. What is needed is a bignum division-with-rounding operator, called \texttt{round-quotient} here. It can be implementated in Scheme\cite{KR98} as follows:

\begin{verbatim}
(define (round-quotient num den)
  (define quo (quotient num den))
  (if ((if (even? quo) > >=) (* (abs (remainder num den)) 2) (abs den))
      (+ quo (if (eqv? (negative? num) (negative? den)) 1 -1))
      quo))
\end{verbatim}

If the remainder is more than half of the denominator, then it rounds up; if it is less, then it rounds down; if it is equal, then it rounds to even. These are the same rounding rules as IEEE Standard for Binary Floating-Point Arithmetic\cite{IEEE85} and the Scheme procedure \texttt{round}.

The algorithms presented here use the \texttt{integer-length} function from Common-Lisp and the SLIB Scheme Library:

- Function: \texttt{integer-length} $n$
  
  Returns the number of bits necessary to represent $n$.

  Examples:

  \begin{verbatim}
  (integer-length #b10101010)
  --> 8
  (integer-length 0)
  --> 0
  (integer-length #b1111)
  --> 4
  \end{verbatim}

For positive argument $n$, \texttt{integer-length} returns $\lceil \log_2 n \rceil$.
The algorithms also use \texttt{ldexp} (from C). Here is Scheme code for \texttt{ldexp}:

\begin{verbatim}
(define (ldexp bmant bexp)
  (define DBL_MIN_10_EXP -1074)
  (if (> DBL_MIN_10_EXP (* -2 (abs bexp)))
      (* (* 1. bmant (expt 2 (quotient bexp 2)))
          (expt 2 (- bexp (quotient bexp 2))))
      (* 1. bmant (expt 2 bexp))))
\end{verbatim}

The conditional in \texttt{ldexp} is in order to work with \texttt{bexp} values where $2^{\text{bexp}}$ is out of floating-point range.
The algorithm to find the closest binary floating-point number to a positive integer mantissa \texttt{mant} and integer exponent (of 10) \texttt{point} is straightforward.

Constant \texttt{dbl-mant-dig} is the number of bits in the mantissa field of the floating-point format.

\begin{verbatim}
(define dbl-mant-dig 53)

(define (big->dbl n)
  (define bex (- (integer-length (abs n)) dbl-mant-dig))
  (ldexp (exact->inexact (if (positive? bex)
                           (round-quotient n (expt 2 bex))
                           n))
          bex))

(define (mantexp->dbl mant point)
  (if (>= point 0)
      (big->dbl (* mant (expt 10 point)))
      (let* ((scl (expt 10 (- point)))
              ;; BEX is the binary shift for QUO; BEX is negative.
              (bex (- (integer-length mant) (integer-length scl) dbl-mant-dig))
              (num (* mant (expt 2 (- bex))))
              (quo (round-quotient num scl)))
        (cond
         ((> (integer-length quo) dbl-mant-dig) ;too many bits of quotient
          (set! bex (+ 1 bex))
          (set! quo (round-quotient num (* scl 2))))
         (ldexp (exact->inexact quo) bex)))))
\end{verbatim}

When \texttt{point} is non-negative, the mantissa is multiplied by $10^{\texttt{point}}$ and the product is rounded to fit in \texttt{dbl-mant-dig} bits by \texttt{big->dbl}.

With a negative \texttt{point}, the mantissa will be multiplied by a power of 2, then divided by \texttt{scl} = $10^{-\texttt{point}}$.

Over the floating-point range, the longest a rounded quotient of a $n$ bit number and a $m$ bit power-of-10 can be is $1+n-m$ bits; the shortest is $n-m$ bits.

$2^{-\texttt{bex}}$ is the power-of-two multiplier. The initial value of \texttt{bex} corresponds to the $n-m$ case above. If the number returned by the call to \texttt{round-quotient} is more than \texttt{dbl-mant-dig} bits long, then call \texttt{round-quotient} with double the denominator \texttt{scl}. In either case, the final step is to convert to floating-point using \texttt{ldexp}. 
Writing

The algorithm for writing a floating-point number is more complicated because it must generate the shortest decimal mantissa which reads as the original floating-point input. The inputs are the positive integer mantissa \( \text{mant} \) and the integer exponent (of 2) \( \text{e2} \).

\[
\text{(define (exact-ceiling x) (inexact->exact (ceiling x)))}
\]

\[
\text{(define log_10of2 (/ (log 2) (log 10)))}
\]

\[
\text{(define (dbl->string mant e2)}
\]

\[
\text{(define f (ldexp mant e2))}
\]

\[
\text{(define quo 0)}
\]

\[
\text{(define point 0)}
\]

\[
\text{(if (> e2 0)}
\]

\[
\text{(let ((num (* mant (expt 2 e2))))}
\]

\[
\text{(set! point}
\]

\[
\text{(max 0 (exact-ceiling (* (- (integer-length num) dbl-mant-dig) log_10of2)))}
\]

\[
\text{(let ((den (expt 10 point)))}
\]

\[
\text{(set! quo (round-quotient num den))}
\]

\[
\text{(set! maxw (max maxw num))}
\]

\[
\text{(cond ((not (= (mantexp->dbl quo point) f))}
\]

\[
\text{(set! point (+ -1 point))}
\]

\[
\text{(set! quo (round-quotient num (quotient den 10))))}
\]

\[
\text{(let ((den (expt 2 (- e2))))}
\]

\[
\text{(set! point (exact-ceiling (* e2 log_10of2)))}
\]

\[
\text{(let ((num (* mant (expt 10 (- point)))))}
\]

\[
\text{(set! quo (round-quotient num den))}
\]

\[
\text{(set! maxw (max maxw num))}
\]

\[
\text{(cond ((and (positive? f) (not (= (mantexp->dbl quo point) f)))}
\]

\[
\text{(set! point (+ -1 point))}
\]

\[
\text{(set! maxw (max maxw (* 10 num)))}
\]

\[
\text{(set! quo (round-quotient (* num 10) den))})}
\]

\[
\text{(let* ((dman (number->string quo))}
\]

\[
\text{(lman (string-length dman)))}
\]

\[
\text{(do ((idx (+ -1 lman) (+ -1 idx)))}
\]

\[
\text{(or (zero? idx)) (not (eqv? #\0 (string-ref dman idx))}}
\]

\[
\text{(string-append "." (substring dman 0 (+ 1 idx))
\]

\[
\text{"e" (number->string (+ point lman))))})}
\]

When \( e2 \) is positive, \( \text{num} \) is bound to the product of \( \text{mant} \) and \( 2^{e2} \). \( \text{point} \) is set to the upper-bound of the number of decimal digits of \( \text{num} \) in excess of the floating-point mantissa’s precision. The \textit{round-quotient} of \( \text{num} \) and \( 10^{\text{point}} \) produces the integer \( \text{quo} \). If \( \text{mantexp->dbl} \) of \( \text{quo} \) and \( \text{point} \) is not equal to the original floating-point value \( f \), then the \textit{round-quotient} is computed again with the divisor divided by 10 yielding one more digit of precision.

When \( e2 \) is negative, \( \text{den} \) is bound to \( 2^{-e2} \) and \( \text{point} \) is set to the negation of the upper-bound of the number of decimal digits in \( 2^{-e2} \). \( \text{num} \) is bound to the product of \( \text{mant} \) and \( 10^{\text{point}} \). The \textit{round-quotient} of \( \text{num} \) and \( \text{den} \) produces the integer \( \text{quo} \). If \( \text{mantexp->dbl} \) of \( \text{quo} \) and \( \text{point} \) is not equal to the original floating-point value \( f \), then the \textit{round-quotient} is computed again with \( \text{num} \) multiplied by 10, yielding one more digit of precision.

The last part of \( \text{dbl->string} \) produces a string using Scheme’s \texttt{number->string} to convert the integer mantissa and exponent components to decimal strings. Mantissa trailing zeros are eliminated by scanning the \( \text{dman} \) string in reverse for non-zero digits.
Performance

IEEE-754 floating-point numbers have a finite range. And the bulk of floating-point usage tends to have magnitudes within the range $1 \times 10^{-30}$ to $1 \times 10^{30}$. Thus the asymptotic running time of floating-point conversion operations is of limited practical interest. Instead, this article looks at measured running-times of several Scheme implementations, of which SCM version 5f2 employs the algorithms presented here. These measurements were performed on a 2.30GHz Intel Core i7-3610QM CPU with 16 GB of RAM running Ubuntu GNU/Linux 3.5.0-49.

A Scheme program was written which generates a vector of 100,000 numbers, $10^X$ where $X$ is a normally distributed random variable. Then for each integer $-322 \leq n \leq 307$, the vector of numbers is scaled by $10^n$, written to a file, read back in, and checked against the scaled vector. The CPU time for writing and the time for reading are measured and plotted in Figure 1. An expanded view of the data for $-15 \leq n \leq 45$ is plotted in Figure 2.

There are 4 regions of the write and read curves of Figure 1. In the range around $n = 14$, little or no scaling is performed. In the range $-15 \leq n \leq 45$, the intermediate (bignum) integers are small, fitting in a few cache lines. For $n < -300$ the mantissa is unnormalized and requires smaller bignums than for $n = -300$. In the remaining regions the running time growth is roughly linear in the length of the intermediate bignums.

The measurement program was ported to run on Racket v5.1.3.

SCM (running the algorithms presented here) is faster than Racket for $n > -40$, but slower for $n \leq -40$. 
The measurement program was ported to do runs of 10,000 samples on Petite-Chez-Scheme version 8.4.

Except for writes of floating-point numbers with decimal exponents between 0 and 14, SCM has faster conversions than Chez.

Some numbers between $10^{-311}$ and $10^{-308}$ fail to be write-read invariant in Chez 8.4 (for example 8.8808138989051e-310).

The measurement program was ported to do runs of 10,000 samples on Gambit-Scheme version gambc 4.2.8-1.1.

Gambit’s writes take roughly twice as long as Chez’s. Its positive exponent reads are comparable to SCM’s. But its read performance suffers as the exponent becomes more negative. Figure 9 shows the full range of Gambit’s performance.
Fixed Precision Output

dbl-prec->string takes an additional positive integer argument prec specifying the number of decimal digits of precision to return. 15 is the maximum number of digits for IEEE-754 double-precision numbers.

(define logof10 (log 10))

(define (dbl-prec->string mant e2 prec)
  ;; POINT is the number of decimal digits to scale the binary-scaled
  ;; matissa so that its has PREC > 0 decimal digits; positive scales up.
  (define point 0)
  (define quo 0)
  (cond
   ((> e2 0)
    (let ((num (* mant (expt 2 e2))))
      (set! point (- (exact-ceiling (/ (log num) logof10)) (min prec 15)))
      (let ((den (expt 10 point)))
        (set! quo (round-quotient num den))
        (cond ((>= quo (expt 10 prec))
                   (set! point (+ 1 point))
                   (set! quo (round-quotient num (* 10 den))))))
    (else ; (<= e2 0)
     (set! point (+ (exact-ceiling (* e2 log_10of2))
                        (max 0 (- (exact-floor (* (integer-length mant) log_10of2))
                                 prec))))
    (if (negative? point)
      (let ((den (expt 2 (- -1 point))))
        (set! quo (round-quotient (* 10 num/10) den))
        (cond ((>= quo (expt 10 prec))
                   (set! point (+ 1 point))
                   (set! quo (round-quotient num/10 den))))
      (let ((den (* (expt 2 (- e2)) (expt 10 point))))
        (set! quo (round-quotient mant den))
        (cond ((>= quo (expt 10 prec))
                   (set! point (+ 1 point))
                   (set! quo (round-quotient mant (* 10 den))))))
    (let* ((dman (number->string quo))
            (lman (string-length dman)))
      (string-append "." dman "e" (number->string (+ point lman))))))

For positive e2, the conversion is the same as dbl->string except that prec plays the role of dbl-mant-dig. For negative e2, the calculation of point is similarly modified, but point is then not guaranteed to be non-negative. So the computation treats positive and negative point separately.

Praxis

For all the conversion algorithms, the storage for bignums generated in the course floating-point conversions can be freed when the conversion is complete.
Conclusion
The introduction of an integer \texttt{round-quotient} procedure facilitates algorithms for lossless (and minimal) conversions between (decimal) scientific-notation and (binary) IEEE-754 double-precision floating-point numbers which are much simpler than those previously published.

A variant of the output algorithm prints IEEE-754 double-precision floating-point numbers with a specified number of decimal digits of precision.

Measurements of conversion times versus floating-point exponent were conducted. Implemented in SCM, these conversion algorithms compare favorably over the IEEE-754 range with three other Scheme implementations having floating-point write-read invariance. Thus the simplicity of this approach does not compromise performance compared with previous algorithms.

References

[BD96] Robert G. Burger and R. Kent Dybvig. Printing floating-point numbers quickly and accurately. \textit{SIGPLAN Not.}, 31(5):108–116, May 1996.

[Cli90] William D. Clinger. How to read floating point numbers accurately. \textit{SIGPLAN Not.}, 25(6):92–101, June 1990.

[IEE85] IEEE Task P754. \textit{ANSI/IEEE 754-1985, Standard for Binary Floating-Point Arithmetic}. IEEE, New York, NY, USA, August 1985.

[KR98] W. Kelsey, R. Clinger and J. (Eds) Rees. Revised\textsuperscript{5} report on the algorithmic language scheme. \textit{ACM SIGPLAN Notices}, 33(9):26–76, 1998.

[Mat68] David W. Matula. In-and-out conversions. \textit{Commun. ACM}, 11(1):47–50, January 1968.

[Mat70] D.W. Matula. A formalization of floating-point numeric base conversion. \textit{Computers, IEEE Transactions on}, C-19(8):681–692, Aug 1970.

[SW90] Guy L. Steele, Jr. and Jon L. White. How to print floating-point numbers accurately. \textit{SIGPLAN Not.}, 25(6):112–126, June 1990.