Methods for measuring energy dissipation rate in anisotropic turbulence

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Abstract

Energy dissipation rate, \( \varepsilon \), is an important parameter for nearly every experiment on turbulent flow. Mathematically precise relationships between \( \varepsilon \) and other measurable statistics for the case of anisotropic turbulence are useful to experimentalists. Such relationships are obtained for which the measurable statistics are the 3rd-order and 2nd-order velocity structure functions as well as the acceleration-velocity structure function. The relationships are derived using the Navier-Stokes equation without approximation. Approximate versions are obtained on the basis of local stationarity and local homogeneity. The latter are valid for arbitrary Reynolds numbers for the case of stationary, homogeneous turbulence. Precise use of the mathematics requires care noted in the Discussion section.
I. INTRODUCTION

Recently, many experiments have produced turbulence by novel means and/or measured energy dissipation rate $\varepsilon$ by novel means. Luthi et al. (2005), use a magnetically driven flow that is nearly isotropic, and obtain $\varepsilon$ from velocity derivatives including $2\nu \langle s_{ij} s_{ij} \rangle$, where $s_{ij}$ is the rate of strain, $\nu$ is kinematic viscosity, and repeated indices denote summation over coordinate directions. Mann et al. (1999) and Ott & Mann (2005) use an oscillating grid, particle tracking, and obtain $\varepsilon$ from $\langle (a_i - a'_i) (u_i - u'_i) \rangle$, where $a_i$ and $u_i$ are acceleration and velocity. Similarly, Ott & Mann (2000) compare those methods with $\varepsilon$ determined from the 3rd- and 2nd-order structure function. Voth et al. (2002) and La Porta et al. (2001) generate turbulence between counter-rotating blades in a cylindrical enclosure; using particle tracking, they obtain $\varepsilon$ from 2nd-order structure functions. Ouellette et al. (2006) create the same turbulence and include evaluation of $\varepsilon$ from $\langle (a_i - a'_i) (u_i - u'_i) \rangle$. Berg et al. (2006) use multiple propellers to create turbulence, use particle tracking, and obtain $\varepsilon$ from 2nd-order velocity structure functions. Tsinober et al. (1992), Kolmyansky et al. (2001), and Gulitski et al. (2007) measure in the atmospheric surface layer using multi-wire probes. Without use of Taylor’s frozen flow hypothesis they obtain $\varepsilon$ from velocity derivatives including $2\nu \langle s_{ij} s_{ij} \rangle$.

A common means of obtaining $\varepsilon$ is from the inertial range of the energy spectrum or that of the 2nd-order velocity structure function. That relationship is based on dimensional analysis under the assumption of local isotropy, and based on empirical validation, and on empirical evaluation of the Kolmogorov constant. For anisotropic turbulence, that empirical basis must be reevaluated for each anisotropic flow. Those relationships have not been derived from the Navier-Stokes equation, and are therefore not considered here.

The usefulness of asymptotic relationships for an experimenter’s purpose of quantifying $\varepsilon$ using measurements of other statistics is limited. For every different type of flow and every Reynolds number, the experimenter must demonstrate to what accuracy the asymptotic limit permits determination of $\varepsilon$. Duchon & Robert (2000) and Eyink (2003) use time and space averages of arbitrary extent and $\nu = 0$ such that the Reynolds number is infinite and use an orientation average to remove the effect of anisotropy. Both the time and space averages must be of nonvanishing extent (Eyink (2003) and §6.2 in Hill (2006)). The requirement $\nu = 0$ is not applicable to experiments. Eyink (2003) considers the possi-
bility of experimental tests of their results, concluding that “a slow approach to asymptopia makes a direct test of the local results, especially a verification of the numerical prefactor, rather more difficult.” A likely cause of inaccuracy in experiments of the results of Duchon & Robert (2000) and Eyink (2003) is an effect of random sweeping described in §6.1 in Hill (2006).

The time and space averages used by Nie and Tanveer (1999) are not applicable to experiments. Nie & Tanveer (1999) base their relationship between $\varepsilon$ and the 3rd-order velocity structure function on time averaging over an infinite duration to remove effects of nonstationarity, on space averaging over either all of space without boundary conditions or space averaging over an entire spatial period of a spatially periodic flow to remove effects of inhomogeneity, and on a sufficiently large Reynolds number. They use an orientation averaging to remove the effect of anisotropy.

Danaila et al. (2002, 2004, and references therein) present structure function equations in which terms that describe particular effects of large-scale inhomogeneity are retained. They use data to demonstrate that those retained large-scale terms are the dominant inhomogeneous terms for several types of flows. Their equations are approximate because the large-scale inhomogeneity terms are approximated, and the other terms are approximated using local isotropy.

In contrast, the method in this paper avoids approximation. On the basis of algebra, calculus, incompressibility, the Navier-Stokes equation, and use of no approximations whatsoever, the structure function equation that contains energy dissipation rate is given in §4. The new, quantifiable definitions of local homogeneity and local stationarity are illustrated in §3. In §4, terms that are negligible for local homogeneity and local stationarity are identified. For simplicity, but not for necessity, those terms are not carried forward in (11)–(23). Those terms are considered in §7. For simplicity, an ensemble average is used herein because the slight complications from spatial and temporal averages were thoroughly documented in Hill (2002a,b). With attention to those complications, spatial and temporal averages may be substituted for the ensemble average used here. An average conditioned on the value of some hydrodynamic quantity cannot be used because of its unknown commutation with respect to spatial and temporal derivatives; that topic must await future study.

Several methods of obtaining energy dissipation rate $\varepsilon$ without use of local isotropy are given here. These methods derive from the Navier-Stokes equation with no restriction
on the flow symmetry; they differ only in the type of average employed. These methods are expected to be of use to experimenters in interpretation of their data. Our purpose here is to provide pragmatic means for measuring $\varepsilon$ using no assumptions and with the greatest mathematical precision possible. Cautions to experimenters on precise use of the relationships are in §7.

II. NOTATION

The velocities, accelerations, and energy dissipation rates at spatial points $x$ and $x'$ and times $t$ and $t'$ are denoted by

$$u_i \equiv u_i(x, t), u'_i \equiv u_i(x', t'), a_i \equiv a_i(x, t), a'_i \equiv a_i(x', t'), \varepsilon \equiv \varepsilon(x, t), \varepsilon' \equiv \varepsilon(x', t'),$$

(1)
etc.; $x$, $t$, $x'$, $t'$ are independent variables. For particle tracking measurements, $u_i$ could be the velocity of one particle at position $x$ at time $t$, and $u'_i$ could be the velocity of another particle at position $x'$ at time $t'$, where $t = t'$ or $t \neq t'$, but if $u_i$ and $u'_i$ are the velocities of the same particle at different $x$ and $x'$, then clearly $t \neq t'$. A sequence of point pairs may come from particle trajectories, but it is useful to consider them as pairs of coordinate locations. Define a new set of independent variables:

$$X \equiv (x + x')/2 \quad \text{and} \quad r \equiv x - x', \quad \text{and} \quad r \equiv |r|; \quad T \equiv (t + t')/2 \quad \text{and} \quad t_\parallel \equiv t - t'.$$

(2)
The significance of variables $X$ and $T$ is that they are the location and time of the measurement, respectively. Define the following:

$$d_{ij} \equiv (u_i - u'_i) (u_j - u'_j)$$

(3)

$$d_{ijk} \equiv (u_i - u'_i) (u_j - u'_j) (u_k - u'_k)$$

(4)

$$A_{ij} \equiv (a_i - a'_i) (u_j - u'_j) + (a_j - a'_j) (u_i - u'_i)$$

(5)

$$F_{iik} \equiv (u_i - u'_i) (u_i - u'_i) \frac{u_k + u'_k}{2}$$

(6)

Below, we use numerical subscript 1 to denote a component in the direction of $r$; and 2 and 3 to denote components transverse to $r$.  

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III. WHAT ARE LOCAL HOMOGENEITY AND LOCAL STATIONARITY?

Local homogeneity and local stationarity only apply for very large Reynolds numbers and sufficiently small $r$. Require that the definition of local homogeneity produces the same results as homogeneity, and that the approximation be quantifiable. The simplest case is the incompressibility relationship on the 2nd-order structure function, namely the divergence vanishes: $\partial_{rj} \langle d_{ij} \rangle = 0$, where we denote an ensemble average by angle brackets. Familiar examples are obtained by substitution of the isotropic formula for $\langle d_{ij} \rangle$ into $\partial_{rj} \langle d_{ij} \rangle = 0$; that gives the well-known incompressibility relationship that $r \partial_r \langle d_{11} \rangle + 2 [\langle d_{11} \rangle - \langle d_{22} \rangle] = 0$; subsequent substitution of the inertial-range 2/3 power-law formula for $\langle d_{ij} \rangle$ gives the familiar inertial-range result that $\langle d_{22} \rangle = (4/3) \langle d_{11} \rangle$, and similarly substituting the viscous-range formula for $\langle d_{ij} \rangle$ gives the familiar relationship $\langle (\partial u_2 / \partial x_1)^2 \rangle = 2 \langle (\partial u_1 / \partial x_1)^2 \rangle$. Use of algebra, calculus, and incompressibility, i.e., $\partial_x u_i = 0$ and $\partial_x' u_i' = 0$, but no approximations and no average, gives (Hill 2002a,b)

$$\partial_{rj} [\left((u_i - u'_i)(u_j - u'_j)\right)] = \partial_{X_i} [\left((u_i + u'_i)(u_j - u'_j)\right)] / 2.$$  \hspace{1cm} (7)

Apply an ensemble average and the definition (3) in (7). Then, to obtain $\partial_{rj} \langle d_{ij} \rangle = 0$ from (7), local homogeneity must be the approximation that

$$\partial_{rj} \langle d_{ij} \rangle \gg \partial_{X_i} \langle (u_i + u'_i)(u_j - u'_j) \rangle / 2.$$  \hspace{1cm} (8)

This is a quantifiable approximation because $\langle (u_i + u'_i)(u_j - u'_j) \rangle$ in (8) can be measured at several locations $X$ such that the derivative on the right-hand side of (8) can be calculated numerically. Similarly, local stationarity is the approximation that the derivatives of statistics with respect to $T$ are negligible. The cases of time and space averages applied to (7) is given in Hill (2002a,b).

Kolmogorov (1941) introduced a formalism of local homogeneity that uses the joint probability distribution function (JPDF) of velocity differences. The moment $\langle (u_i + u'_i)(u_j - u'_j) \rangle$ in (8) cannot be calculated from that JPDF. Because of (7)–(8), Kolmogorov’s formalism cannot be used to obtain the incompressibility relation $\partial_{rj} \langle d_{ij} \rangle = 0$, nor is it applicable to simplifying the structure-function equations deduced from the Navier-Stokes equation (Hill, 2001, 2002a,b, 2006). Kolmogorov’s (1941) formalism invokes a region of vague size for use of the JPDF. In contrast, (8) is truly local because it is a derivative;
experimentalists need only displace $X$ sufficiently to use the 3-point numerical derivative formula.

**IV. APPLICATION OF THE NAVIER-STOKES EQUATION FOR ANISOTROPIC TURBULENCE**

From the Navier-Stokes equations, we obtain an exact equation relating 3rd- and 2nd-order velocity structure functions and other statistics (Hill, 2002a,b, 2006). “Exact” means that no approximations were used; calculus and algebra were used. The trace is performed because it greatly simplifies the term involving the pressure-gradient difference. We obtain

$$A_{ii} = \partial_T d_{ii} + \partial_X F_{iik} + \partial_r d_{iik} = 2\nu \partial_r \partial_r d_{ii} - 2(\varepsilon + \varepsilon') + W,$$

where

$$W \equiv -2\partial_X [(p-p')(u_i - u_i')] + \frac{\nu}{2} \partial_X \partial_X d_{ii} - 2\nu \partial_X \partial_X (p + p').$$

(10)

No average exists in (9)–(10).

After performing the ensemble average, use of the approximation of local homogeneity as in §3, i.e., derivatives with respect to $X$ are negibible, causes the average of all terms in (10) as well as the term $\partial_X \langle F_{iik} \rangle$ to be negligible because they are all the rate of change of a statistic with respect to where the measurement is performed, i.e., $X$. Likewise, the approximation of local stationarity in §3 causes the one term $\partial_T \langle d_{ii} \rangle$ to be negligible because it is the rate of change of $\langle d_{ii} \rangle$ with respect to when the measurement is performed, i.e., $T$. We will return to the evaluation of those neglected terms in the discussion §7. The result is the approximate structure-function equation (Hill, 2006).

$$\langle A_{ii} \rangle = \partial_r \langle d_{iik} \rangle = 2\nu \partial_r \partial_r \langle d_{ii} \rangle - 2 \langle \varepsilon + \varepsilon' \rangle.$$

(11)

Note the two equality signs in both (9) and (11). In (11) $\partial_r \langle d_{iik} \rangle$ is the divergence of the vector $\langle d_{iik} \rangle$; e.g., in Cartesian coordinates $\partial_r \langle d_{iik} \rangle \equiv \partial_{r_1} \langle d_{i1k} \rangle + \partial_{r_2} \langle d_{i2k} \rangle + \partial_{r_3} \langle d_{i3k} \rangle$. The energy dissipation rate in (9) and (11) is defined by

$$\varepsilon \equiv 2\nu s_{ij} s_{ij}.$$ 

(12)

The calculation of exact averages is described elsewhere (Hill 2001, 2002a,b, 2006). Structure functions that contain the two-point pressure difference present formidable experimental
difficulties; it is therefore significant that the pressure does not appear in (11) on the basis of local homogeneity (Hill 2002a,b, 2006). Local isotropy was not used to obtain (11); therefore, (11) can provide methods for measuring the energy dissipation rate in anisotropic turbulence.

For simplicity of notation, let

$$\bar{\varepsilon}(r) \equiv \langle \varepsilon + \varepsilon' \rangle / 2.$$  \hspace{1cm} (13)

Dependence on $r$ in (13) is because $\bar{\varepsilon}(r)$ depends on energy dissipation rates at two points separated by $r$. Anisotropic turbulence causes dependence of $\bar{\varepsilon}(r)$ on the direction of $r$; hence the emphasis on $r$ in (13). In (13) the argument list could be $(r, t)$, which assumes that the ensemble contains events for fixed $r$ and $t$. However, if an experimenter chooses an ensemble of events having fixed $r$ but various $t$, then there is an implicit average over $t$ such that $t$ should be deleted from the argument list. Henceforth, $t$ is deleted from the argument list.

For simplicity, consider the case for which $r$ is much greater than dissipation scales; for that case we neglect $2\nu \partial_{r_i} \partial_{r_j} \langle d_{ij} \rangle$ in (11), but we later include $2\nu \partial_{r_i} \partial_{r_k} \langle d_{ij} \rangle$. Then, because $A_{ii} \equiv 2(a_i - a_i') (u_i - u_i')$, (11) gives

$$\langle (a_i - a_i') (u_i - u_i') \rangle = -2\bar{\varepsilon}(r).$$  \hspace{1cm} (14)

For anisotropic turbulence, (14) makes it clear that $\langle (a_i - a_i') (u_i - u_i') \rangle$ depends on the direction of $r$. Mann et al. (1999) obtained a relationship similar to (14), and they (Mann et al., 1999, Ott & Mann, 2000, 2005) used their relationship to obtain energy dissipation rate from their measurements of acceleration and velocity. For our purpose of mathematical precision, it is necessary to note the distinctions that (14) is obtained here and in Hill (2006) without the assumption by Mann et al. (1999) that a certain derivative moment may be neglected and that both $\varepsilon$ and $\varepsilon'$ define $\bar{\varepsilon}(r)$ in (13), i.e., two space-time points appear here.

Average (14) over orientations of $r$ to obtain:

$$\frac{1}{4\pi} \int \int \langle (a_i - a_i') (u_i - u_i') \rangle d\Omega = -2\bar{\varepsilon}_{\text{orientation}} (r)$$ \hspace{1cm} (15)

$$\bar{\varepsilon}_{\text{orientation}} (r) \equiv \frac{1}{4\pi} \int \int \bar{\varepsilon} d\Omega$$  \hspace{1cm} (16)

which is a function of $r$; $d\Omega$ is the differential of solid angle, and the double integral is understood to be over $4\pi$ steradians.
Use of particle tracking data allows calculation of the average of (14) over a sphere in \( r \)-space to obtain energy dissipation averaged within the sphere as follows:

\[
\frac{3}{4\pi r_S^3} \int_{|r| \leq r_S} \left( \frac{1}{4\pi} \int_0^{r_S} \left( \frac{1}{4\pi} \int_0^{r_S} \left< a_i - a_i' \right> \left( u_i - u_i' \right) d\Omega \right) r^2 dr \right) = -2 \bar{\varepsilon}_{sphere} (r_S). \tag{17}
\]

\[
\bar{\varepsilon}_{sphere} (r_S) \equiv \frac{3}{4\pi r_S^3} \int_{|r| \leq r_S} \bar{\varepsilon} dr. \tag{18}
\]

The sphere has radius \( r_S \) such that \( \bar{\varepsilon}_{sphere} \) depends on \( r_S \), not on \( r \). The average over orientations of \( r \) is performed first, resulting in a function of \( r \); the \( r \)-integration is performed second. In (18), the average produces the same result as that given in the first description of intermittency theory in Obukhov (1962) which was used by Kolmogorov (1962).

Another method is to use the equality in (11) that contains \( \partial_r \left< d_{iik} \right> \). To avoid substituting data into the divergence of the 3rd-order structure function, the \( r \)-sphere average is performed and the divergence theorem is used to express the result as an integral over the surface of the outward normal of the vector \( \left< d_{iik} \right> \); then

\[
\frac{3}{4\pi r_S^3} \int_{|r| \leq r_S} \partial_r \left< d_{iik} \right> dr = \frac{3}{4\pi r_S^3} \int_{|r| = r_S} \frac{r_k}{r} \left< d_{iik} \right> ds = \frac{1}{4\pi} \int \left< d_{i11} \right>_{|r| = r_S} d\Omega
\]

\[
= -\frac{4}{3} r_S \bar{\varepsilon}_{sphere} (r_S). \tag{19}
\]

The differential of surface area on the sphere is \( ds \); \( \hat{r} \) is the unit vector in the direction of \( r \) such that \( \hat{r} \left< d_{iik} \right> = \left< d_{i11} \right> \). Specifically, \( \left< d_{i11} \right> \equiv \left< d_{111} + d_{221} + d_{331} \right> \). The subscript notation \( \left< (d_{i11}) \right>_{|r| = r_S} \) means evaluate the quantity within the square brackets at \( |r| = r_S \).

In (19), we have the orientation average of \( \left< d_{i11} \right> \). Taylor et al. (2003) use DNS data to demonstrate the efficacy of the orientation average of the 3rd-order structure function in its relationship to \( \varepsilon \).

Neglect of the term \( \nu \partial_r \partial_r \left< d_{i11} \right> \) in (11) is unnecessary. The Laplacian is the divergence of the gradient such that the divergence theorem gives

\[
\int_{|r| \leq r_S} \partial_r \left< d_{i11} \right> dr = \int_{|r| = r_S} \frac{r_k}{r} \partial_r \left< d_{i11} \right> ds = r_S^2 \int \left[ \partial_r \left< d_{i11} \right> \right]_{|r| = r_S} d\Omega,
\]
where $\partial_{r_1}$ is the gradient in the direction of $r$. Thus, more generally, for anisotropic turbulence

$$\langle (a_i - a_i') (u_i - u_i') \rangle - \nu \partial_{r_k} \partial_{r_k} \langle d_{ii} \rangle = -2\varepsilon (r). \quad (20)$$

$$\frac{1}{4\pi} \int_0^r \left( \frac{1}{4\pi} \int_0^r \langle (a_i - a_i') (u_i - u_i') \rangle \, d\Omega \right) r^2 \, dr - \nu \partial_{r_1} \partial_{r_1} \langle d_{ii} \rangle = -\frac{2}{3} r^3 \varepsilon_{\text{sphere}} (r). \quad (22)$$

$$\frac{1}{4\pi} \int_0^r \left( \frac{1}{4\pi} \int_0^r \langle d_{ii1} \rangle - 2\nu \partial_{r_1} \langle d_{ii} \rangle \right) r^2 \, dr - \nu \partial_{r_1} \langle d_{ii} \rangle = -\frac{4}{3} r^2 \varepsilon. \quad (23)$$

For stationary, homogeneous turbulence, (20)–(23) are valid for all $r$ and all Reynolds numbers. We began with the hydrodynamic definition of energy dissipation rate, $\varepsilon$, in (12); (13) is only simplified notation. We obtain in (20)–(23) three different averages of $\varepsilon$. The three values differ only because the averaging operations differ. There is no distinction as to which value to prefer. Any choice of averaging operation is left to the judgment of the experimenter.

V. THE LIMIT OF LOCAL ISOTROPY

We can determine whether or not the above formulas give the corresponding classic results for locally isotropic turbulence. All of the energy dissipation rates in (20)–(23) become the same value for local isotropy, so here we denote them all by $\langle \varepsilon \rangle$. For local isotropy, both (20) and (21) become

$$\langle (a_i - a_i') (u_i - u_i') \rangle = -2 \langle \varepsilon \rangle,$$

where $\partial_{r_k} \partial_{r_k} = r^{-2} \partial_r (r^2 \partial_r)$ was used, and (22)–(23) become

$$\int_0^r \langle (a_i - a_i') (u_i - u_i') \rangle r^2 \, dr - \nu r^2 \partial_r \langle d_{ii} \rangle = -\frac{2}{3} r^3 \langle \varepsilon \rangle. \quad (25)$$

$$\langle d_{ii1} \rangle - 2\nu \partial_r \langle d_{ii} \rangle = -\frac{4}{3} r \langle \varepsilon \rangle. \quad (26)$$

The inertial range formulas obtained from (24)–(26) are the well-known formulas

$$\langle (a_i - a_i') (u_i - u_i') \rangle = -2 \langle \varepsilon \rangle \quad \text{and} \quad \langle d_{ii1} \rangle = -\frac{4}{3} r \langle \varepsilon \rangle$$
The viscous-range formulas from (24)–(26) are $\nu r^{-2} \partial_r (r^2 \partial_r) \langle d_{ii} \rangle = 2 \langle \varepsilon \rangle$ and $\nu \partial_r \langle d_{ii} \rangle = \frac{2}{3} r \langle \varepsilon \rangle$. In the viscous range, Taylor series expansion, local isotropy, and incompressibility give

$$\langle d_{ii} \rangle = \left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^2 + 2 \left( \frac{\partial u_2}{\partial x_1} \right)^2 \right\rangle r^2 = 5 \left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle r^2 = \frac{5}{2} \left\langle \left( \frac{\partial u_2}{\partial x_1} \right)^2 \right\rangle r^2. \quad (27)$$

Note that $r^{-2} \partial_r (r^2 \partial_r) r^2 = 6$. Then, from (24)–(26), both $\nu r^{-2} \partial_r (r^2 \partial_r) \langle d_{ii} \rangle = 2 \langle \varepsilon \rangle$ and $\nu \partial_r \langle d_{ii} \rangle = \frac{2}{3} r \langle \varepsilon \rangle$ give the classic formulas

$$\left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle = \frac{1}{15} \nu \langle \varepsilon \rangle, \quad \text{and} \quad \left\langle \left( \frac{\partial u_2}{\partial x_1} \right)^2 \right\rangle = \frac{2}{15} \nu \langle \varepsilon \rangle. \quad (28)$$

VI. DISCUSSION

The mathematics is precise. Experimenters must be careful to follow with precision when using the equations. In particular, precise evaluation of $\varepsilon$ requires evaluation of not only (20)–(23), but also the terms in (9) that describe inhomogeneity, i.e., $\partial X_k F_{ik}$ and $W$, as well as the term that describes nonstationarity, i.e., $\partial T d_{ii}$. Those terms must be operated upon with the same averages that appear in (20)–(23). Danaila et al. (2002, 2004, and references therein) give approximate evaluations of some of those terms for several flows. The pressure that appears in $W$ in (10) presents a future challenge, but techniques of pressure measurement combined with hot-wire velocity measurement have advanced (Tsuji et al., 2007).

The same velocity and acceleration that appear in the Navier-Stokes equation also appear above. That is, the velocity fluctuation and acceleration fluctuation do not appear above. If the experimenter performs a Reynolds decomposition, e.g. $u = \langle u \rangle + \tilde{u}$ where $\tilde{u}$ is the fluctuation of velocity, then many more terms must appear. Those terms must be evaluated quantitatively to obtain quantitative energy dissipation rate. The energy dissipation rate may be expressed as being caused by the average flow and the fluctuation flow and terms descriptive of the interaction of the two. The subject of Reynolds decomposition for the structure-function equations, and the many resultant terms that must be quantified or neglected is discussed in Hill (2002b). If those many terms are neglected, the resultant approximate equations for structure functions of fluctuations are described fully in Hill (2002b).
An important advantage of using \( \langle A_{ii} \rangle \) over \( \partial_{r_k} \langle d_{ii} \rangle \) in (11) is the necessity to neglect \( \partial_T \langle d_{ii} \rangle + \partial_X \langle f_{iik} \rangle \) in (9) if \( \partial_{r_k} \langle d_{iik} \rangle \) is to be used. For the case of a nonzero mean velocity as well as for random sweeping by the large-scale flow, the term \( \partial_X \langle f_{iik} \rangle \) can be a significant effect [see §7 of Hill (2006) and experimental evaluation in Danaila et al. (2002, 2004, and references therein)].

The definition of local homogeneity used above holds at the center of symmetry of a flow. For example, in the center of the cylinder containing the flow between rotating blades, the rate of change with respect to where the measurement is performed (i.e., \( X \)) is zero because any statistic evaluated both above and below the center of symmetry has the same value at both points. The same is true for any statistic evaluated at two points equally spaced in any direction from the center of symmetry.

For use of Taylor’s hypothesis of frozen flow, the correction for fluctuating convection velocity is given in Hill (1996) for any statistic on the basis of local isotropy; that makes that correction inapplicable to anisotropic turbulence. However, the correction is qualitatively useful. An advantage of the 3rd-order structure function \( \langle d_{ijk} \rangle \) is that the correction to Taylor’s hypothesis caused by fluctuating velocity vanishes in the inertial range of \( \langle d_{ijk} \rangle \) (Hill, 1996); the same can be shown to be true for \( \langle A_{ij} \rangle \).

In (20)–(21) the calculation of the three-dimensional Laplacian of data might pose problems. Danaila et al. (2002, 2004, and references therein) have approximated the Laplacian by only its \( r \)-derivatives on the basis that local isotropy should be adequate at the dissipation and viscous-range scales where \( \nu \partial_{r_k} \partial_{r_k} \langle d_{ii} \rangle \) is an important term within (20)–(21). For axisymmetric turbulence one may express the Laplacian in terms of only two independent variables using the cylindrical coordinate system used by Lindborg (1995) or the coordinates used by Batchelor (1946). The axisymmetric analogues of (28) are given by George and Hussein (1991).

The results (20)–(23) relate the energy dissipation rates on the right-hand sides to quantities that are measurable on the left-hand sides. The results are valid for anisotropic turbulence. The left-hand sides of (20)–(23) are measurable with particle tracking technology (Voth et al. 2002, La Porta et al. 2001, Mann et al. 1999, Ott & Mann 2000, 2005, Luthi et al., 2005, Berg et al. 2006, Ouellette et al. 2006). With the exception of the acceleration-velocity structure function, the left-hand sides of (20)–(23) are also measurable by multi-wire probes (Tsinober et al. 1992, Kolmyansky et al. 2001, Gulitski et al. 2007).
We expect that (20)–(23) will be of significant use to experimenters.

The author thanks A. Pumir, B. Luthi, H. Xu & E. Bodenschatz for valuable comments.

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