BOUNDARY STABILIZATION OF NON-DIAGONAL SYSTEMS
BY PROPORTIONAL FEEDBACK FORMS

IONUȚ MUNTEANU
Alexandru Ioan Cuza University, Department of Mathematics
and Octav Mayer Institute of Mathematics (Romanian Academy)
Carol I No. 11, 700506 Iași, Romania

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Abstract. In this work, we are concerned with the problem of boundary exponential stabilization, in a Hilbert space $H$, of parabolic type equations, namely equations for which their linear parts generate analytic $C_0$-semigroups. We consider the case where the projection of the linear leading operator, on a given Riesz basis of $H$, is non-diagonal. We do not assume that the linear operator has compact resolvent. Therefore, the Riesz basis is not necessarily an eigenbasis. The boundary stabilizer is given in a simple linear feedback form, of finite-dimensional structure, involving only the Riesz basis. To illustrate the results, at the end of the paper, we provide an example of stabilization of a fourth-order evolution equation on the half-axis.

1. Introduction. Here, we consider the following abstract boundary control system

$$
\begin{cases}
\frac{dY}{dt}(t) = AY(t), & t \geq 0, \\
BY(t) = v(t), & t \geq 0, \\
Y(0) = Y_0,
\end{cases}
$$

(1.1)

where $A : \mathcal{D}(A) \subset H \to H$ is a linear (unbounded) operator; $B : \mathcal{D}(B) \subset H \to U$, with $\mathcal{D}(A) \subset \mathcal{D}(B)$, is a linear boundary operator; and $v : [0, \infty) \to U$ is a boundary control. Here, $H$ and $U$ are separable Hilbert spaces over $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. We denote by $\langle \cdot, \cdot \rangle$ and by $\| \cdot \|$ the scalar product and the induced norm in $H$, respectively. And by $\langle \cdot, \cdot \rangle_0$ the scalar product in $U$. (Further details will be given in the sequel.) Below, we shall understand by scalarly multiplying two elements in $H$, taking their inner product in $H$.

In this work, our aim is to design a control $v = v(Y)$, given in a feedback form, such that, once inserted into the system (1.1), it yields that the unique solution of the corresponding closed-loop system (1.1) satisfies

$$
\|Y(t)\| \leq Ce^{-\rho t}\|Y_0\|, \quad \forall t \geq 0,
$$

for some positive constants $C, \rho$.

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For different particular cases of the operators $A$ and $B$, the above problem has been previously considered and solved in many works. We mention the pioneering works of Lasiecka and Triggiani [11, 21], and of Russell [19, 20]. Latter, other types of controls were built, such as: dynamic control [2], collocated control [18], or even noise control [1]. For more references see [4, 16]. But, concerning explicit and easy manageable feedback controllers we distinguish between two main approaches: the backstepping control design technique, see for instance [9] and the references therein; and the direct-proportional control design technique, first introduced in [3] then improved in [16]. The very simple form of the last one, reworded important results on the stabilization problem of the Navier-Stokes equations, MHD equations, Cahn-Hilliard equations or stochastic PDEs, for details see [16]. (These problems, and many more, have been solved also via the backstepping approach.) Unlike the backstepping control design, there are some disadvantages in the use of the control in [16]. First of all, the method requires that the operator $A$ has a countable set of semi-simple eigenvalues. (This implies that the eigenbasis projection of $A$ has a diagonal form.) Moreover, it demands that the corresponding eigenvectors form a Riesz basis in $H$. (This is related to the compact resolvent assumption on $A$.) In many cases, such kind of hypotheses do not hold true, or are hard to be checked. Besides this, the feedback form of the controller relies on the first $N$ eigenvectors of the operator $A$, which, in general, cannot be computed explicitly and it is hard to approximate them. In this work, we shall overcome these issues. More precisely, we drop the assumptions of semi-simple eigenvalues and compact resolvent of $A$, and assume instead the existence of a suitable Riesz basis in $H$ (see Assumption 2 and Assumption 3 below). Consequently, the projection of $A$, on this basis, may have a non-diagonal form. In this work, doing some adjustments to the control design developed in [16], we will construct a direct-proportional stabilizing feedback form $v$ for the non-diagonal case of (1.1), see Theorem 3.1 below. It should be emphasized the recent work of Lasiecka and Triggiani [10], where the non-diagonal Navier-Stokes equations are stabilized by both a boundary and internal-tangential proportional type control.

In order to illustrate the results, at the end of the paper we apply the stabilizing method to a fourth-order evolution equation on the half-axis. Russell [20] established a connection between stabilization and controllability for linear partial differential equations. At the same time, Micu and Zuazua [14] established the absence of the null-controllability for the heat equation on the half-axis. For boundary stabilization the heat equation on the half-axis we refer to [8], while, concerning the problem of boundary stabilization of fourth order PDEs on unbounded domains, as far as we know there are no results in the literature yet. Anyway, there are some studies on the stability of the solutions of fourth order PDEs in unbounded domains, see [15], or for the wave equation, see [7]. Concerning the problem of controllability and optimal control problems associated with equations on unbounded domains, see e.g. [6].

2. The main assumptions. For $z \in \mathbb{K}$, we denote by $\bar{z}$, the complex conjugate of $z$. We denote by $\langle \cdot, \cdot \rangle_N$ the standard euclidean scalar product in $\mathbb{K}^N$. For a matrix $A \in \mathcal{M}_{m \times n}(\mathbb{K})$ we denote by $A^T$ its transpose. It is recalled that $\ker(B)$ is the kernel of $B$ while $R(B)$ stands for the range of $B$. In the following, we shall work under the next hypotheses:
Assumption 1. The operator $A_0$, defined on the domain $\mathcal{D}(A_0) := \mathcal{D}(A) \cap \ker(B)$ by $A_0 := A|_{\mathcal{D}(A_0)}$, is the generator of a $C_0$-semigroup $\{e^{tA_0} : t \geq 0\}$ on $H$.

In the sequel, we are going to apply the well-known projection method. Namely, we assume the existence of a suitable Riesz basis of $H$, then we will project the operator $A$ on this basis, obtaining an infinite system of ordinary differential equations. More precisely, we suppose that

Assumption 2. There exists a countable Riesz basis $\{\varphi_j\}_{j \in \mathbb{N}^*} \subset \mathcal{D}(A_0)$ of the space $H$.

Let us denote by $\{\psi_j\}_{j \in \mathbb{N}^*}$ the associated bi-orthogonal system of the basis $\{\varphi_j\}_{j \in \mathbb{N}^*}$. Namely, we have $\langle \varphi_i, \psi_j \rangle = \delta_{ij}$, $i, j \in \mathbb{N}^*$, where $\delta_{ij}$ is the Kronecker symbol.

Let $N \in \mathbb{N}$. We set

$$X_u := \text{span} \{\varphi_j\}_{j=1}^N$$

and

$$X_s := \text{span} \{\varphi_j\}_{j \geq N+1}.$$ 

Also, we denote the projections $P_N : H \to X_u$ and $Q_N = I - P_N$. Here, $I$ stands for the identity operator.

Assumption 3. We have

$$P_N A_0 = P_N A_0 P_N$$

Moreover, $Q_N A_0 Q_N$ generates an exponentially stable $C_0$-semigroup.

Next, we introduce the matrix $\Lambda = (\lambda_{ij})_{i,j=1}^N$, where

$$\lambda_{ij} := \langle \psi_i, A_0 \varphi_j \rangle, \quad 1 \leq i, j \leq N.$$ 

Assumption 4. For $\gamma > 0$ large enough, and for each $\beta \in R(B)$, there exists a unique solution to the equation

$$-A y + 2 \sum_{i,j=1}^N \lambda_{ij} \langle y, \psi_i \rangle \varphi_j + \gamma y = 0; \quad B y = \beta. \quad (2.1)$$

This way, we may introduce the lifting operator $D_\gamma : R(B) \to H$, $D_\gamma \beta := y$, $y$ solution to (2.1).

Successively scalarly multiplying equation (2.1) by $\psi_1, ..., \psi_N$, we get

$$(\Lambda + \gamma I) \begin{pmatrix} \langle D_\gamma \beta, \psi_1 \rangle \\ \langle D_\gamma \beta, \psi_2 \rangle \\ ... \\ \langle D_\gamma \beta, \psi_N \rangle \end{pmatrix} = \begin{pmatrix} \langle \beta, l_1 \rangle_0 \\ \langle \beta, l_2 \rangle_0 \\ ... \\ \langle \beta, l_N \rangle_0 \end{pmatrix}, \quad (2.2)$$

where $\{l_1, l_2, ..., l_N\}$ are functions in $U$ which do not depend on $\gamma$ or $\beta$. Here, $I$ is the identity matrix of order $N$. We set

$$L := \begin{pmatrix} l_1 \\ l_2 \\ ... \\ l_N \end{pmatrix}.$$ 

Assumption 5. The matrix $[L \Lambda L \Lambda^2 L ... \Lambda^{N-1} L]$ has full rank in a nonzero measure set.

Assumption 5 says, in fact, that the couple $(\Lambda L)$ satisfies the Kalman controllability rank condition.
3. Control design. Let us notice that, for $\gamma > 0$ large enough, the matrix $\Lambda + \gamma I$ is nonsingular. Next, we fix $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_N$, $N$ large enough positive numbers such that for each of them equation (2.1) is well-posed, and such that the matrices $\Lambda + \gamma_k I$, $k = 1, 2, \ldots, N$ are all invertible. We denote by $D_{\gamma_k}$, $k = 1, 2, \ldots, N$, the corresponding lifting operators.

As in [16], we let $B$ the Gram matrix of the system $\{l_j\}_{j=1}^N$ in the Hilbert space $U$. That is

$$B := \begin{pmatrix}
\langle l_1, l_1 \rangle_0 & \langle l_1, l_2 \rangle_0 & \cdots & \langle l_1, l_N \rangle_0 \\
\langle l_2, l_1 \rangle_0 & \langle l_2, l_2 \rangle_0 & \cdots & \langle l_2, l_N \rangle_0 \\
\cdots & \cdots & \cdots & \cdots \\
\langle l_N, l_1 \rangle_0 & \langle l_N, l_2 \rangle_0 & \cdots & \langle l_N, l_N \rangle_0
\end{pmatrix}.$$  (3.1)

We denote by

$$B_k := (\Lambda + \gamma_k I)^{-1} B (\Lambda^T + \gamma_k I)^{-1}, \quad k = 1, \ldots, N.$$  (3.2)

(Note that, in the semi-simple eigenvalues case in [16], i.e., diagonal $\Lambda$, we defined $B_k$, see [16, Eq. (2.21)], in terms of the diagonal matrix $\Lambda_{\gamma_k}$. If $\Lambda$ is diagonal, then $(\Lambda + \gamma_k I)^{-1}$ and $\Lambda_{\gamma_k}$ coincide.)

The following result is essential in the definition of the feedback control. It is the counterpart of [16, Proposition 2.1]. Due to the non-diagonal $\Lambda$, its proof is more elaborate and is given in the Appendix.

Proposition 1. The sum of $B_k$‘s, i.e. $B_1 + B_2 + \ldots + B_N$ is an invertible matrix.

Consequently, we are allowed to define

$$A := (B_1 + B_2 + \ldots + B_N)^{-1}.$$  (3.3)

Now, let us introduce the following proportional-type feedback laws

$$v_k(Y(t)) := -\left(\langle \Lambda^T + \gamma_k I \rangle^{-1} A \begin{pmatrix}
\langle Y(t), \psi_1 \rangle \\
\langle Y(t), \psi_2 \rangle \\
\cdots \\
\langle Y(t), \psi_N \rangle
\end{pmatrix} \begin{pmatrix}
l_1 \\
l_2 \\
\cdots \\
l_N
\end{pmatrix}\right)_N,$$  (3.4)

t $\geq 0$, for $k = 1, 2, \ldots, N$.

Then, define $v$ as

$$v := \sum_{k=1}^N v_k = \left(\sum_{k=1}^N (\langle \Lambda^T + \gamma_k I \rangle^{-1} A \begin{pmatrix}
\langle Y(t), \psi_1 \rangle \\
\langle Y(t), \psi_2 \rangle \\
\cdots \\
\langle Y(t), \psi_N \rangle
\end{pmatrix} \begin{pmatrix}
l_1 \\
l_2 \\
\cdots \\
l_N
\end{pmatrix}\right)_N,$$  (3.5)

for all $t \geq 0$.

Recall the lifting operators $D_{\gamma_k}$, assured by Assumption 4. For latter purpose, we show that

$$\begin{pmatrix}
\langle D_{\gamma_k} v_k, \psi_1 \rangle \\
\langle D_{\gamma_k} v_k, \psi_2 \rangle \\
\cdots \\
\langle D_{\gamma_k} v_k, \psi_N \rangle
\end{pmatrix} = -B_k A \begin{pmatrix}
\langle Y(t), \psi_1 \rangle \\
\langle Y(t), \psi_2 \rangle \\
\cdots \\
\langle Y(t), \psi_N \rangle
\end{pmatrix},$$  (3.6)
for all $k = 1, ..., N$. This is indeed so. We have by (2.2) that
\[
\begin{pmatrix}
\langle D_{\gamma_k}v_k, \psi_1 \rangle \\
\langle D_{\gamma_k}v_k, \psi_2 \rangle \\
\vdots \\
\langle D_{\gamma_k}v_k, \psi_N \rangle
\end{pmatrix}
= (\Lambda + \gamma_k I)^{-1}
\begin{pmatrix}
\langle v_k, l_1 \rangle \\
\langle v_k, l_2 \rangle \\
\vdots \\
\langle v_k, l_N \rangle
\end{pmatrix}.
\]
Which, by virtue of (3.4) gives
\[
\begin{pmatrix}
\langle D_{\gamma_k}v_k, \psi_1 \rangle \\
\langle D_{\gamma_k}v_k, \psi_2 \rangle \\
\vdots \\
\langle D_{\gamma_k}v_k, \psi_N \rangle
\end{pmatrix}
= -(\Lambda + \gamma_k I)^{-1}B(\Lambda^T + \gamma_k I)^{-1}A
\begin{pmatrix}
\langle Y(t), \psi_1 \rangle \\
\langle Y(t), \psi_2 \rangle \\
\vdots \\
\langle Y(t), \psi_N \rangle
\end{pmatrix},
\]
and so, by (3.2), (3.6) is proved.

At this point, we are able to state and prove the main result of this paper concerning the system (1.1).

**Theorem 3.1.** Under Assumption 1 to Assumption 5, once we plug the feedback form (3.5) into the equation (1.1), it yields the exponential stability of the corresponding closed-loop system. More exactly, the unique solution of
\[
\begin{aligned}
\frac{dY}{dt}(t) &= AY(t), \quad t \geq 0, \\
BY(t) &= v_1 + v_2 + \ldots + v_N \\
Y(0) &= Y_0,
\end{aligned}
\]
\[
= -\left(\sum_{k=1}^N (\Lambda^T + \gamma_k I)^{-1}A \begin{pmatrix}
\langle Y(t), \psi_1 \rangle \\
\langle Y(t), \psi_2 \rangle \\
\vdots \\
\langle Y(t), \psi_N \rangle
\end{pmatrix}, \begin{pmatrix}
l_1 \\
l_2 \\
\vdots \\
l_N
\end{pmatrix}\right)_N,
\]
satisfies the exponential decay $\|Y(t)\| \leq Ce^{-\rho t}\|Y_0\|$, $\forall t \geq 0$, for some positive constants $C, \rho$.

**Proof.** Mostly, we shall follow the ideas in [16]. In order to get null boundary conditions in (3.7), let us define $Z := Y - \sum_{j=1}^N D_{\gamma_j}v_j$. We show that the feedback forms (3.4), can be rewritten in terms of $Z$ as
\[
v_k = -\frac{1}{2}\left(\Lambda^T + \gamma_k I\right)^{-1}A \begin{pmatrix}
\langle Z(t), \psi_1 \rangle \\
\langle Z(t), \psi_2 \rangle \\
\vdots \\
\langle Z(t), \psi_N \rangle
\end{pmatrix}, \begin{pmatrix}
l_1 \\
l_2 \\
\vdots \\
l_N
\end{pmatrix}\}_N,
\]
\[
t \geq 0, \text{ for } k = 1, 2, \ldots, N.
\]
This is indeed so. We have

\[- \frac{1}{2} \begin{pmatrix} \sum_{j=1}^{N} \frac{1}{2} \langle (A^T + \gamma_k I)^{-1} A \rangle & \langle Z(t), \psi_1 \rangle \\ \langle Z(t), \psi_2 \rangle & \langle Z(t), \psi_N \rangle \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix} \]

Now we have

\[- \frac{1}{2} \begin{pmatrix} \sum_{j=1}^{N} \frac{1}{2} \langle (A^T + \gamma_k I)^{-1} A \rangle & \langle Y(t), \psi_1 \rangle \\ \langle Y(t), \psi_2 \rangle & \langle Y(t), \psi_N \rangle \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix} \]

(where using (3.6), we get)

\[- \frac{1}{2} \begin{pmatrix} \sum_{j=1}^{N} \frac{1}{2} \langle (A^T + \gamma_k I)^{-1} A \rangle & \langle D\gamma v_j, \psi_1 \rangle \\ \langle D\gamma v_j, \psi_2 \rangle & \langle D\gamma v_j, \psi_N \rangle \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix} \]

by taking into account that \((B_1 + B_2 + \ldots + B_N)^{-1} = A\). Then, similarly as in (3.6), now we have

\[
\begin{pmatrix} \langle D\gamma v_k, \psi_1 \rangle \\ \langle D\gamma v_k, \psi_2 \rangle \\ \vdots \\ \langle D\gamma v_k, \psi_N \rangle \end{pmatrix} = -\frac{1}{2} B_k A \begin{pmatrix} \langle Z(t), \psi_1 \rangle \\ \langle Z(t), \psi_2 \rangle \\ \vdots \\ \langle Z(t), \psi_N \rangle \end{pmatrix} \]

(3.9)

Next, by (3.7), it follows that

\[
\frac{d}{dt} Z(t) = \frac{d}{dt} Y - \sum_{k=1}^{N} \frac{d}{dt} D\gamma v_k = A_0 Z + \sum_{k=1}^{N} \lambda_k A D\gamma v_k - \sum_{k=1}^{N} \frac{d}{dt} D\gamma v_k
\]

Where invoking (2.1), we get

\[
\frac{d}{dt} Z = A_0 Z + 2 \sum_{k=1}^{N} \lambda_k \langle D\gamma v_k, \psi_i \rangle \varphi_j + \sum_{k=1}^{N} \gamma_k D\gamma v_k - \sum_{k=1}^{N} \frac{d}{dt} D\gamma v_k. \quad (3.10)
\]
We apply $P_N$ on (3.10) and take advantage of Assumption 3, to deduce that
\[
\frac{d}{dt} P_N Z(t) = P_N A_0 P_N Z(t) + 2 \sum_{k=1}^{N} \sum_{i,j=1}^{N} \lambda_{ij} (D\gamma_k v_k, \psi_i) \varphi_j
\]
\[
+ \sum_{k=1}^{N} \gamma_k P_N D\gamma_k v_k - P_N \sum_{k=1}^{N} \frac{d}{dt} D\gamma_k v_k.
\]
Likewise in [16, Eqs. (2.38)-(2.39)], successively scalarly multiplying the above equation by $\psi_j$, $j = 1, 2, ..., N$, and using (3.9), we arrive at
\[
\frac{d}{dt} Z = A Z - Z - \frac{1}{2} \sum_{k=1}^{N} \gamma_k B_k A Z + \frac{1}{2} \frac{d}{dt} Z.
\]
Here, $Z$ is the vector of the first $N$ modes of $Z$, i.e. $Z = \begin{pmatrix} \langle Z, \psi_1 \rangle \\ \langle Z, \psi_2 \rangle \\ \vdots \\ \langle Z, \psi_N \rangle \end{pmatrix}$. The above equation is equivalent with
\[
\frac{d}{dt} Z = -\gamma_1 Z + \sum_{k=2}^{N} (\gamma_1 - \gamma_k) B_k A Z, \quad t > 0.
\]
Recalling the definition of $B_k$ in (3.2), it is easy to see that
\[
\langle B_k W, W \rangle_N \geq 0, \quad \forall \ W \in K^N, \quad \forall k = 1, 2, ..., N.
\]
Indeed, we have
\[
\langle B_k W, W \rangle_N = \left\langle B(\Lambda^T + \gamma_k I)^{-1} W, (\Lambda^T + \gamma_k I)^{-1} W \right\rangle_N \geq 0,
\]
since $B$ is a Gram matrix, hence positive semi-definite. Scalarly multiplying the above equation by $A Z$, we get
\[
\frac{1}{2} \frac{d}{dt} \|A \frac{1}{2} Z\|^2_N = -\gamma_1 \|A \frac{1}{2} Z\|^2_N + \sum_{k=2}^{N} (\gamma_1 - \gamma_k) \langle B_k A Z, A Z \rangle_N \leq -\gamma_1 \|A \frac{1}{2} Z\|^2_N,
\]
by the positive semi-definiteness of $B_k$s and the fact that $\gamma_1 - \gamma_k \leq 0$, $k = 2, 3, ..., N$. Here, $\| \cdot \|_N$ stands for the classical norm in $K^N$. The above implies
\[
\|Z(t)\|^2_N \leq C e^{-2\gamma_1 t} \|Z(0)\|^2_N, \quad t \geq 0,
\]
where we took into account the positive-definiteness of the symmetric matrix $A$. (Note that each $B_k$ is symmetric, positive semi-definite, and $A = (B_1 + B_2 + ... + B_N)^{-1}$, thus $A$ is symmetric and positive definite.) Or, equivalently,
\[
\|P_N Z(t)\|^2 \leq C e^{-2\gamma_1 t} \|P_N Z(0)\|^2, \quad t \geq 0. \quad (3.11)
\]
(See relation (2.41) in [16].)

On the other hand, applying the operator $Q_N$ on (3.10) yields
\[
\frac{d}{dt} Q_N Z(t) = Q_N A_0 Q_N Z(t) + Q_N A_0 P_N Z(t), \quad t > 0.
\]
By Assumption 3 we know that $Q_N A_0 Q_N$ generates an exponentially decaying $C_0$-semigroup. This, together with (3.11), implies that
\[
\|Q_N Z(t)\|^2 \leq C e^{-2\mu t} \|Z(0)\|^2, \quad t \geq 0, \quad (3.12)
\]
for some positive constants $C, \mu$.

Making use of the fact that $Z = P_N Z + Q_N Z$, relations (3.11) and (3.12) lead to

$$\|Z(t)\| \leq Ce^{-\rho t}\|Z(0)\|, \quad t \geq 0. \quad (3.13)$$

Where $\rho = \min\{\gamma_1, \mu\}$.

Finally, returning to the notation $Y = Z + \sum_{k=1}^{N} D_{\gamma_k} v_k(Z)$, it is easy to see that (3.13) implies the conclusion of the theorem. Thereby completing the proof.

**Remark 1.** Let $F : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$, $d_1 \in \mathbb{N} \setminus \{0\}$, a nonlinear map such that $F(0) = 0$; and such that:

**Assumption 6.** We have

$$\|\partial_{y_j} F(Y)\|_{\mathbb{R}^{d_1}} \leq C(\|Y\|_{\mathbb{R}^{d_1}}^m + 1), \quad \forall Y \in \mathbb{R}^{d_1}, \quad j = 1, 2, ..., d_1.$$

Based on the results in Theorem 3.1, following the ideas in [3, Theorem 4.1], one may derive a stabilization result for the equation (1.1) perturbed by the nonlinearity $F$. Namely, we have: under Assumption 1 to Assumption 6, once we plug the feedback form (3.5) into the nonlinear version of equation (1.1), it yields the exponential stability of the corresponding closed-loop system in a neighborhood of the origin. More exactly, the unique solution of

$$\begin{cases}
\frac{dY}{dt}(t) = \mathcal{A}Y(t) + F(Y(t)), \quad t \geq 0, \\
BY(t) = v_1 + v_2 + ... + v_N \\
\left. \begin{array}{c}
\langle Y(t), \psi_1 \rangle \\
\langle Y(t), \psi_2 \rangle \\
\vdots \\
\langle Y(t), \psi_N \rangle \\
\end{array} \right\} = -\left( \sum_{k=1}^{N} (\mathcal{A}^T + \gamma_k I)^{-1} \mathcal{A} \left( \begin{array}{c}
\langle Y(t), \psi_1 \rangle \\
\langle Y(t), \psi_2 \rangle \\
\vdots \\
\langle Y(t), \psi_N \rangle \\
\end{array} \right) \right),
\end{cases} \quad (3.14)$$

satisfies the exponential decay $\|Y(t)\| \leq Ce^{-\rho t}\|Y_0\|$, $\forall t \geq 0$, for some positive constants $C, \rho$, provided that $\|Y_0\| \leq \varepsilon$, for some $\varepsilon$ sufficiently small.

4. **Applications.** It is clear that the present control design method can be applied to solve the problem of boundary exponential stabilization for general parabolic type equations, as (1.1), where the leading linear operator $\mathcal{A}$ generates an analytic $C_0$-semigroup and has a compact resolvent, and consequently, via the Fredholm theory, it has a countable set of eigenvalues which accumulates at infinite. Moreover, the corresponding eigenvectors form a Riesz basis in $H$. In this case, the basis claimed to exist in Assumption 2 can be chosen to be exactly the eigenbasis corresponding to $\mathcal{A}$. In the case where the eigenvalues of $\mathcal{A}$ are not semi-simple, its eigenbasis projection is a non-diagonal matrix. Then, provided that Assumption 5 can be verified for $\mathcal{A}$, it clearly follows by Theorem 3.1 that a stabilizing feedback law, as (3.5), can be constructed.

Nevertheless, there are also other important cases which can be solved with the present method. Namely, the case of operators $\mathcal{A}$ which have a countable set of eigenvalues, but the corresponding eigenvectors do not form a Riesz basis in $H$. Sometimes it is possible to construct a Riesz basis in $H$ from the eigenvectors. But, even if the operator $\mathcal{A}$ is self-adjoint, its projection on this Riesz basis may be non-diagonal. As an example, one can see the case of boundary stabilization of PDEs with non-local boundary conditions in [17].
Another important example consists of operators $\mathcal{A}$ for which the proof that they have compact resolvent is not a simple task. Consequently, in such case one cannot guarantee the existence of a countable eigenbasis. But, one can find a basis in $H$ which is invariant under $\mathcal{A}$. More details are given in the following example which treats the case of a fourth-order eigenbasis on the half-axis. The case of boundary stabilization of a second-order evolution equation on the half-axis was solved in [8].

Let us consider the problem of stabilization for the following evolution equation on the half-axis

$$
\begin{align*}
\partial_t y + \partial_{xx} \left[ e^{-x^2/2} \partial_{xx} \left( e^{x^2/2} y \right) \right] + \alpha \partial_{xxx} y + cy &= 0, \ t > 0, \ x \in (0, \infty), \\
y(t, 0) &= v, \ \partial_{xx} y(t, 0) = 0, \ t > 0, \\
y(0, x) &= y_0(x), \ x \in [0, \infty).
\end{align*}
$$

(4.1)

Where $\alpha$ is some constant such that $\alpha \not\in \{2k - 1 : k \in \mathbb{N}\}$ and such that relation (4.8) below holds true; and $c$ is any constant.

In the present case, the operator $\mathcal{A}_0$ is given by

$$
\mathcal{A}_0 y := - \left\{ \partial_{xx} \left[ e^{-x^2/2} \partial_{xx} \left( e^{x^2/2} y \right) \right] + \alpha \partial_{xxx} y + cy \right\}, \ \forall y \in \mathcal{D}(\mathcal{A}_0),
$$

where

$$
\mathcal{D}(\mathcal{A}_0) = \{ y \in H^4(0, \infty) : y(0) = 0, \ \partial_{xx} y(0) = 0 \}.
$$

The operator $\mathcal{A}_0$ can be equivalently written as

$$
\mathcal{A}_0 y = -\partial_{xxxx} y - 2x \partial_{xxx} y - (x^2 + 5 + \alpha) \partial_{xx} y - 2x \partial_x y - (2 + c)y.
$$

We see that $\mathcal{A}_0$ is governed by the elliptic operator

$$
\mathcal{A}_1 y = -\partial_{xxxx} y - (5 + \alpha) \partial_{xx} y - (2 + c)y, \ y \in \mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}_0),
$$

which has continuous spectrum, since for all $\lambda \in [2 + c, \infty)$, we have $\mathcal{A}_1 \varphi_\lambda = -\lambda \varphi_\lambda$, where

$$
\varphi_\lambda = \sin(\mu_\lambda x), \ \mu_\lambda := \sqrt{\frac{5 + \alpha + \sqrt{(5 + \alpha)^2 - 4(2 + c - \lambda)}}{2}}.
$$

Next, $\mathcal{A}_0$ is perturbed by the operator $\mathcal{A}_2 y = -2x \partial_{xxx} y - x^2 \partial_{xx} y - 2x \partial_x y$, which has unbounded coefficients. Then, by virtue of the results in [13], we guess that $\mathcal{A}_0$ has continuous spectrum as well. Even if we would be able to show that $\mathcal{A}$ provides a countable eigenbasis for $L^2(0, \infty)$, the exact form of the eigenfunctions would be hard to determine. Thus, at the end of the day, the explicit form of the feedback controller would not be known. Anyway, in this case we do not care about this since we can find a suitable Riesz basis, of the Hilbert space $H = L^2(0, \infty)$, consisting of the Hermite functions. More precisely, we use Hermite polynomials and functions, which are defined by

$$
H_n := (-1)^n \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x^n} e^{-x^2/2}, \ n \in \mathbb{N},
$$

$$
G_n(x) := \sqrt{\frac{2}{\pi}} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} e^{-x^2/2}, \ n \in \mathbb{N}.
$$

These functions with odd exponents form a biorthogonal system on $\mathbb{R}_+$, i.e.,

$$
\int_0^\infty H_{2n+1}(x) G_{2m+1}(x) dx = \delta_{nm}, \ n, m \in \mathbb{N}.
$$
Moreover, the set of odd Hermite functions forms a Riesz basis in $L^2(0, \infty)$. Therefore, to fulfil Assumption 2 we take

$$\varphi_j := G_{2j-1} \text{ and } \psi_j := H_{2j-1}, \ j \in \mathbb{N}^*.$$  

Now, let us compute $\langle A_0 \varphi_j, \psi_i \rangle$, $i, j \in \mathbb{N}^*$. Taking into account the well-known identities

$$\partial_{xx} H_n = n(n - 1) H_{n-2}, \quad (4.2)$$

and

$$H_{n+1} = x H_n - \partial_x H_n = -e^{x^2/2} \partial_x \left( e^{x^2/2} H_n \right), \ n \in \mathbb{N}, \quad (4.3)$$

and the boundary conditions of $H_n$ and $G_n$, we get that

$$\langle A_0 \varphi_j, \psi_i \rangle = -\int_0^\infty \partial_{xx} \left[ e^{x^2/2} \partial_{xx} \left( e^{x^2/2} \varphi_j \right) \right] \psi_i dx$$

$$- \int_0^\infty \alpha \partial_{xx} \varphi_j \psi_i dx - c \int_0^\infty \varphi_j \psi_i dx$$

$$= - \int_0^\infty e^{-x^2/2} \partial_{xx} \left( e^{x^2/2} \varphi_j \right) \partial_{xx} \psi_i dx$$

$$- \alpha \int_0^\infty \varphi_j \partial_{xx} \psi_i dx - c \delta_{ij}. \quad (4.4)$$

It follows, by relation (4.2), that

$$\langle A_0 \varphi_j, \psi_i \rangle = -(2i - 1)(2i - 2) \int_0^\infty \partial_{xx} \left( e^{x^2/2} \varphi_j \right) e^{-x^2/2} \psi_{i-1} dx$$

$$- \alpha(2i - 1)(2i - 2) \int_0^\infty \varphi_j \psi_{i-1} dx - c \delta_{ij}. \quad (4.4)$$

Invoking relation (4.3), we get that

$$\int_0^\infty \partial_{xx} \left( e^{x^2/2} \varphi_j \right) e^{-x^2/2} \psi_{i-1} dx = - \int_0^\infty \partial_x \left( e^{x^2/2} \varphi_j \right) \partial_x \left( e^{-x^2/2} \psi_{i-1} \right) dx$$

$$= \int_0^\infty \partial_x \left( e^{x^2/2} \varphi_j \right) e^{-x^2/2}(-e^{x^2/2}) \partial_x \left( e^{-x^2/2} \psi_{i-1} \right) dx$$

$$= \int_0^\infty \partial_x \left( e^{x^2/2} \varphi_j \right) e^{-x^2/2} H_{2i-2} dx$$

$$= - \int_0^\infty e^{x^2/2} \varphi_j \partial_x \left( e^{-x^2/2} H_{2i-2} \right) dx$$

$$= \int_0^\infty \varphi_j H_{2i-2} dx = \int_0^\infty \varphi_j \psi_i dx = \delta_{ij}.$$

This, plugged in (4.4), yields

$$\langle A_0 \varphi_j, \psi_i \rangle = [(2i - 1)(2i - 2) - c] \delta_{ij} - \alpha(2i - 1)(2i - 2) \delta_{j(i-1)}, \ \forall i, j \in \mathbb{N}^*. \quad (4.5)$$

It is easy to see that relation (4.5) assures that $P_N A_0 = P_N A_0 P_N$. Furthermore, $Q_N A_0 Q_N$, projected on the basis $\{\varphi_j\}_{j=1}^\infty$, is the infinite matrix

$$\begin{pmatrix}
-(2N + 1)2N - c & 0 & 0 & \cdots & 0 & \cdots \\
-\alpha(2N + 3)(2N + 2) & -(2N + 3)(2N + 2) - c & 0 & \cdots & 0 & \cdots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
\end{pmatrix}.$$
It is clear that $Q_N A_0 Q_N$ generates a $C_0$–semigroup which is exponentially decaying with the decay rate $-(2N+1)2N - c < 0$, for $N$ large enough. Consequently, Assumption 3 holds true for the present case.

The matrix $\Lambda$, in our case, reads as

$$
\Lambda = \begin{pmatrix}
-c & 0 & 0 & \cdots & 0 & 0 \\
-6\alpha & -6 - c & 0 & \cdots & 0 & 0 \\
0 & -20\alpha & -20 - c & \cdots & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -(2N-1)(2N-2)\alpha & -(2N-1)(2N-2) - c
\end{pmatrix}
$$

Thus, operator $A_0$ is non-diagonal in the basis formed by $\{\varphi_j\}_{j \geq 1}$.

We go on and define the lifting operator $D$, which is the solution to

$$
\begin{align*}
\partial_{xx} \left[ e^{-x^2/2} \partial_{xx} \left( e^{x^2/2} D \right) \right] + \alpha \partial_{xx} D + cD &+ 2 \sum_{i,j=1}^{N} \lambda_{ij} \langle D, \psi_i \rangle \varphi_j + \gamma D = 0 \text{ in } (0, \infty), \\
D(0) = 1, \quad \partial_{xx} D(0) = 0.
\end{align*}
$$

(Note that, since we are in the one-dimensional case, $\beta$ from Assumption 4 is a scalar. Hence it suffices to consider the boundary condition $D(0) = 1$.) For $\gamma > 0$ sufficiently large there exists a unique solution $D$ to (4.7), see e.g. [12]. This $D$ assures that Assumption 4 is fulfilled.

Now, let us compute the quantities $\langle D, \psi_i \rangle$, $i = 1, 2, ..., N$. We have, by scalarly multiplying equation (4.7) by $\psi_i$, and doing similar computations as in (4.4)-(4.5) (with $D$ instead of $\varphi_j$), that

$$
0 = [(2i-1)(2i-2) + c] \int_0^\infty D\psi_i dx + \alpha(2i-1)(2i-2) \int_0^\infty D\psi_{i-1} dx + 2 \sum_{k=1}^{N} \lambda_{ki} \langle \psi_k \rangle \psi_i + \gamma \langle \psi_i \rangle - \partial_{xxx} \psi_i(0) - \alpha \partial_x \psi_i(0), \quad i = 1, 2, ..., N.
$$

So, in our case, relation (2.2) reads as

$$
(\Lambda + \gamma I) \begin{pmatrix} 
\langle D, \psi_1 \rangle \\
\langle D, \psi_2 \rangle \\
\cdots \\
\langle D, \psi_N \rangle 
\end{pmatrix} = L := \begin{pmatrix}
1 + \alpha \\
-3!!(\alpha - 1) \\
5!!(\alpha - 3) \\
-7!!(\alpha - 5) \\
\ddots \\
(-1)^{N-1}(2N-1)!!(\alpha - (2N-3))
\end{pmatrix}.
$$

We see that, in our case $l_1 = -(1 + \alpha)$ and

$$
l_i = (-1)^{i-1}(2i - 1)!!(\alpha - (2i - 3)), \quad i = 2, 3, ..., N.
$$

By the choice of $\alpha$, none of $l_i$ equals zero. Above, we understand by $(2n-1)!!$ the product of the first $n$ odd positive integers.

Finally, let us verify that Assumption 5 holds true for our case. As mentioned before, we must verify the Kalman condition for the couple $(\Lambda L)$. Recall that the
Kalman rank condition is equivalent with the Fattorini-Hautus test, i.e.,
\[ \text{ker}(\lambda - \Lambda^T) \cap \text{ker}L^T = \{0\} \].

Let us suppose by contradiction that there exists some \( z \in \mathbb{K}^N \) such that \( z \neq 0 \) and \( z \in \text{ker}(\lambda - \Lambda^T) \cap \text{ker}L^T \). By \( (\lambda - \Lambda^T)z = 0 \), by virtue of the lower triangular form of \( \Lambda^T \), we deduce that
\[ \lambda \in \{\lambda_{ii} : i = 1, 2, \ldots, N\} \].

Assume that, for some \( l \in \{1, 2, \ldots, N\} \) we have \( \lambda = \lambda_{ll} \). In the matrix \( \Lambda^T - \lambda_{ll}I \) we replace the \( l \)th line with \( L^T \). We denote this new matrix by \( \hat{\Lambda} \). We have \( \hat{\Lambda}z = 0 \).

The form of \( \hat{\Lambda} \) involves that the last \( N-l \) unknowns of \( z \) satisfy the system \( \hat{\Lambda}z = 0 \), where
\[
\hat{\Lambda} = \begin{pmatrix}
0 & 0 & \ldots & \lambda_{l+1,l+1} - \lambda_{ll} & \mu_1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & \lambda_{l+2,l+2} - \lambda_{ll} & \mu_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \lambda_{NN} - \lambda_{ll}
\end{pmatrix}.
\]

Therefore, \( z_{l+1} = \ldots = z_N = 0 \).

Next, let us consider \( M_l \) which is the upper left corner minor of order \( l \) of the matrix \( \hat{\Lambda} \). More precisely,
\[
M_l := \begin{pmatrix}
(2l-1)(2l-2) & -6\alpha & \ldots & 0 & 0 \\
0 & -6 + (2l-1)(2l-2) & -20\alpha & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\lambda_1 & \lambda_2 & \ldots & \lambda_l
\end{pmatrix}.
\]

Due to the above observations, we have
\[
M_l \begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_l
\end{pmatrix} = 0.
\]

We assume about \( \alpha \) that
\[ \det M_l \neq 0 \text{ for all } l = 1, 2, \ldots, N. \] (4.8)
(In any case, \( \det M_l = 0 \) is a polynomial of order at most \( l \) in \( \alpha \). So, in fact, we ask that \( \alpha \) is not the root of such polynomials.) Consequently, \( z_1 = \ldots = z_l = 0 \). In conclusion, \( z = 0 \), which is in contradiction with the initial assumption. Thus, the pair \( (\Lambda, L) \) satisfies the Kalman rank condition.

Let \( \gamma_1 < \ldots < \gamma_N \), \( N \) large enough constants. We define
\[
B := \begin{pmatrix}
l_1 \times l_1 & l_1 \times l_2 & \ldots & l_1 \times l_N \\
l_2 \times l_1 & l_2 \times l_2 & \ldots & l_2 \times l_N \\
\vdots & \vdots & \ddots & \vdots \\
l_N \times l_1 & l_N \times l_2 & \ldots & l_N \times l_N
\end{pmatrix}.
\]

We denote by
\[
B_k := (\Lambda + \gamma_k I)^{-1} B (\Lambda^T + \gamma_k I)^{-T}, \ k = 1, \ldots, N.
\] (4.9)

And by
\[
A = (B_1 + B_2 + \ldots + B_N)^{-1}.
\]

Since all the hypotheses are fulfilled, we conclude with the counterpart of Theorem 3.1.
Theorem 4.1. The unique solution of
\[
\begin{aligned}
    \partial_t y + \partial_{xx} \left[ e^{-x^2/2} \partial_{xx} \left( e^{x^2/2} y \right) \right] + \alpha \partial_{xx} y + cy &= 0, 
    t > 0, \ x \in (0, \infty), \\
    y(t, 0) &= -\sum_{k=1}^{N} (\Lambda^T + \gamma_k I)^{-1} A \begin{pmatrix} \langle y(t), \psi_1 \rangle \\ \langle y(t), \psi_2 \rangle \\ \vdots \\ \langle y(t), \psi_N \rangle \end{pmatrix}, \\
    \partial_{xx} y(t, 0) &= 0, \ t > 0, \\
    y(0, x) &= y_0(x), \ x \in [0, \infty). 
\end{aligned}
\]

satisfies the exponential decay
\[\|y(t)\| \leq C e^{-\rho t} \|y_0\|, \ \forall t \geq 0,\]
for some positive constants $C, \rho$.

5. Conclusions. In this paper, we improved the results in [16] in the sense that, we dropped the semi-simple eigenvalues assumption and impose a weaker one related to a Kalman rank controllability condition. Let us notice that the Kalman hypothesis (Assumption 5) is natural in the given context. More exactly, let us consider the system (1.1) in one-dimension.

\[\frac{d}{dt} y = Ay; \ By = v. \tag{5.1}\]

And the lifting operator $D$ solving
\[AD = 0; \ BD = 1.\]

Arguing as in [5], we get that (5.1) can be equivalently rewritten as
\[\frac{d}{dt} y = Ay + ADv. \tag{5.2}\]

(Here $A$ is the extension to the whole $H$ of $\mathcal{A}$, see [5].) Let us assume that we have in hand a suitable Riesz basis of $H$, as in Assumption 2. Then, with the notations in Section 2, projecting (5.2) on $\mathcal{X}_u$, we get
\[\frac{d}{dt} \mathcal{Y} = \Lambda \mathcal{Y} + \begin{pmatrix} v l_1 \\ v l_2 \\ \vdots \\ v l_N \end{pmatrix} = \Lambda \mathcal{Y} + vL, \ t > 0.\]

Here, $\mathcal{Y}$ is the vector of the first $N$ modes of $y$. Then, Assumption 5 says that the pair $(\Lambda L)$ satisfies the Kalman condition. We know that this implies the existence of a matrix $K \in M_{1 \times N}(\mathbb{R})$ such that the matrix $\Lambda +LK$ has the spectrum in the left half-plane. In other words, (5.2) is stabilizable by the feedback control $v = KY$. Matrix $K$ is obtained by solving a Riccati equation. In the present work, for the one-dimensional case, in fact, we give the form of the matrix $K$ directly.

In multi-dimensions, things are totally different. Now, the lifting operator solves $AD = 0; \ BD = \beta$. And the equivalent reformulation of (5.1) projected on $\mathcal{X}_u$ is given by
\[\frac{d}{dt} \mathcal{Y} = \Lambda \mathcal{Y} + \begin{pmatrix} \langle v, l_1 \rangle_0 \\ \langle v, l_2 \rangle_0 \\ \vdots \\ \langle v, l_N \rangle_0 \end{pmatrix}, \ t > 0.\]
Assume that $v$ is of the finite-dimensional structure $v = \sum_{k=1}^{N} u_k(t)l_k$, where $u_k$, scalar functions depending only on $t$, are the controls, $k = 1, 2, \ldots, N$. Then, the above finite-dimensional system can be rewritten as

$$\frac{d}{dt} v = \Lambda v + BV, \quad t > 0,$$

where

$$B := \begin{pmatrix} \langle l_1, l_1 \rangle_0 & \langle l_2, l_1 \rangle_0 & \cdots & \langle l_N, l_1 \rangle_0 \\ \vdots & \ddots & \ddots & \vdots \\ \langle l_1, l_N \rangle_0 & \langle l_2, l_N \rangle_0 & \cdots & \langle l_N, l_N \rangle_0 \end{pmatrix},$$

and $V := \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{pmatrix}$. Assume that $(\Lambda B)$ satisfies the Kalman condition and $B$ is invertible, then, the Kalman matrix is given by $K = -\delta B^T$, $\delta > 0$, and the control $v = -\delta B^T Y$. But, $B$ is invertible if and only if the set $\{l_1, l_2, \ldots, l_N\}$ is linearly independent in $U$. This, usually, does not hold true. So, again, the matrix $K$ is known to exist but its exact form is unknown. In this work, in fact, we give directly its form.

**Appendix. Proof of Proposition 1.** Let $z = (z_1 \ z_2 \ \ldots \ z_N)^T \in \mathbb{K}^N$ such that $(\sum_{k=1}^{N} B_k)z = 0$. Hence,

$$\langle B_1 z, z \rangle_N + \langle B_2 z, z \rangle_N + \cdots + \langle B_N z, z \rangle_N = 0.$$

Or, equivalently by the definition of $B_k$

$$\sum_{k=1}^{N} \left\langle B (\Lambda^T + \gamma_k I)^{-1} z, (\Lambda^T + \gamma_k I)^{-1} z \right\rangle_N = 0.$$

Recall that, $B$, being a Gramian, is positive semi-definite. Thus,

$$\left\langle B (\Lambda^T + \gamma_k I)^{-1} z, (\Lambda^T + \gamma_k I)^{-1} z \right\rangle_N = 0, \ \forall k = 1, 2, \ldots, N.$$

Due to the definition of $B$, the above implies that

$$\left\langle L_k (\Lambda^T + \gamma_k I)^{-1} z, (\Lambda^T + \gamma_k I)^{-1} z \right\rangle_N = 0, \ k = 1, 2, \ldots, N.$$

Which yields

$$\left\langle L_k (\Lambda^T + \gamma_k I)^{-1} z \right\rangle_N = 0 \text{ almost everywhere, } k = 1, 2, \ldots, N.$$

Or, equivalently,

$$\left\langle (\Lambda + \gamma_k I)^{-1} L_k, z \right\rangle_N = 0 \text{ almost everywhere, } k = 1, 2, \ldots, N.$$

Recall that $L := (l_1 \ l_2 \ \ldots \ l_N)^T$. The above can be viewed as a linear system in $U$, of order $N$, with the unknowns $z_1, z_2, \ldots, z_N$, which are constants in $U$. It has only the trivial solution if and only if the determinant of the matrix of the system

$$\det \left[ (\Lambda + \gamma_1 I)^{-1} L \ (\Lambda + \gamma_2 I)^{-1} L \ \ldots \ (\Lambda + \gamma_N I)^{-1} L \right] \neq 0,$$

in a nonzero measure set. Performing elementary transformations in the determinant, namely subtracting from each column $k = 2, 3, \ldots, N$ the first column, i.e.

$$(\Lambda + \gamma_k I)^{-1} L - (\Lambda + \gamma_1 I)^{-1} L = (\gamma_1 - \gamma_k) (\Lambda + \gamma_1 I)^{-1} (\Lambda + \gamma_k I)^{-1} L,$$
k = 2, ..., N, the above is equivalent with
\[
\det \left[ (\Lambda + \gamma I)^{-1} L (\Lambda + \gamma I)^{-1} (\Lambda + \gamma I)^{-1} L \ldots (\Lambda + \gamma N I)^{-1} L \right]
\]
is not equal zero. This holds true if and only if
\[
\det \left[ L (\Lambda + \gamma I)^{-1} L (\Lambda + \gamma I)^{-1} L \ldots (\Lambda + \gamma N I)^{-1} L \right] \neq 0.
\]
Similar actions as above, namely subtracting from each column \( k = 3, 4, ..., N \) the second column, then multiplying the result by \( \Lambda + \gamma I \), lead to the equivalent condition
\[
\det \left[ (\Lambda + \gamma I) L L (\Lambda + \gamma I)^{-1} L \ldots (\Lambda + \gamma N I)^{-1} L \right] \neq 0.
\]
We go on like this and arrive at
\[
\det \left[ \prod_{k=1}^{N} (\Lambda + \gamma I) L L \ldots (\Lambda + \gamma N I)^{-1} L \right] \neq 0.
\]
Again elementary transformations: subtracting from the \((N-1)\)th column the \( N \)th column multiplied by \( \gamma N \), yield
\[
\det \left[ \prod_{k=1}^{N} (\Lambda + \gamma I) \ldots (\Lambda + \gamma I) L \ldots (\Lambda + \gamma N I)^{-1} L \right] \neq 0.
\]
Then, subtracting from the \((N-2)\)th column the \((N-1)\)th column multiplied by \( \gamma N + \gamma N - 1 \) and the \( N \)th column multiplied by \( \gamma N \gamma N - 1 \) we obtain that
\[
\det \left[ \prod_{k=1}^{N} (\Lambda + \gamma I) \ldots (\Lambda + \gamma I) L \ldots (\Lambda^2 L L) \ldots L \right] \neq 0.
\]
The procedure goes in a similar way until we get that the above is equivalent with the fact that the determinant
\[
\det \left[ \Lambda^{N-1} L \Lambda^{N-2} L \ldots L \right] \neq 0.
\]
By virtue of Assumption 5, this holds true. We conclude that, \( \sum_{k=1}^{N} B_k = 0 \) if and only if \( z = 0 \). Or, in other words, the matrix \( \sum_{k=1}^{N} B_k \) is invertible.

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E-mail address: ionut.munteanu@uaic.ro