A PERSPECTIVE ON THE CMB ACOUSTIC PEAK

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Draft version January 16, 2022

ABSTRACT

CMB angular spectrum measurements suggest a flat universe. This paper clarifies the relation between geometry and the spherical harmonic index of the first acoustic peak ($\ell_{\text{peak}}$). Numerical and analytic calculations show that $\ell_{\text{peak}}$ is approximately a function of $\Omega_K/\Omega_M$ where $\Omega_K$ and $\Omega_M$ are the curvature ($\Omega_K > 0$ implies an open geometry) and mass density today in units of critical density. Assuming $\Omega_K/\Omega_M \ll 1$, one obtains $\ell_{\text{peak}} \approx \frac{11 \sqrt{3}}{9(\sqrt{a_0 + a_{eq}} - \sqrt{a_0})} \left(2 + \frac{\Omega_K}{\Omega_M}\right)$ where $a_0$ and $a_{eq}$ are the scale factor at decoupling and radiation-matter equality. The derivation of $\ell_{\text{peak}}$ gives another perspective on the widely-recognized $\Omega_M-\Omega_L$ degeneracy in flat models. This formula for near-flat cosmogonies together with current angular spectrum data yields familiar parameter constraints.

Subject headings: cosmic microwave background — cosmological parameters

1. INTRODUCTION

It has long been recognized that the first peak in the CMB angular spectrum provides information about the curvature of the universe (Doroshkevich, Zel’dovich, & Syunyaev 1978; Bond et al. 1994; Kamionkowski, Spergel, & Sugiyama 1994; Efstathiou & Bond 1999; Cornish 2000; Weinberg 2000). The data are now in. Analyses of the data strongly suggest a flat universe (Hu et al. 2001; Lee et al. 2001; Halverson et al. 2001). Analyses of the data strongly suggest a flat universe (Hu et al. 2001; Jaffe et al. 2000; Stompor et al. 2001; Pryke et al. 2001; Dodelson & Knox 2000). The MAP satellite, TOCO, BOOMERanG, MAXIMA, and DASI have measured the peak position (Müller et al. 1999; Netterfield et al. 2001; Page 2000). The data are now in.

As new data resolve higher order peaks, attention shifts to the physics at angular scales beyond that of the first maximum. The object of the present analysis is to clarify the physics derived from the position of the first peak, hereafter called “the peak index” or $\ell_{\text{peak}}$. The size of the sound horizon $r_{\text{ss}}$ at an angular diameter distance $D_a$ to decoupling determines the peak index.\textsuperscript{3} This is widely recognized and serves as a starting point. In Sections 2, 3, and 4, numerical and analytic calculations yield the peak index as a function of $\Omega_M$ and $\Omega_K$. All results are checked with cmbfast (Seljak & Zaldarriaga 1996). In Section 5, a simple formula for $\ell_{\text{peak}}$, applicable to low $\Omega_K/\Omega_M$ universes, is developed alongside geometric and classical interpretations of the formal results. It is found that the peak index approximates a function of $\Omega_K/\Omega_M$. Although our analysis grounds itself in the familiar concepts of $D_a$ and $r_{\text{ss}}$, the results are not widely recognized. While the physical effects responsible for $\ell_{\text{peak}}$ are understood, the interplay that gives the $\Omega_K/\Omega_M$ dependence is not intuitive and holds a number of surprises. We end the investigation by using the functional form of $\ell_{\text{peak}}$ and current angular spectrum data to obtain parameter constraints resembling those of more sophisticated treatments.

Throughout this work, the $(\Omega_M, \Omega_K)$ plane serves as the parameter space. Other quantities enter, though the relation between $\ell_{\text{peak}}$ and $\Omega_K/\Omega_M$ is relatively independent of them. To be concrete, we take $\Omega_B h^2 = 0.02$, consistent with nucleosynthesis (Burles, Nollett, & Turner 2001). The redshift of decoupling, $z_{\text{ss}}$, is taken as 1400 from the Saha equation. The Hubble constant assumes 72 km/s/Mpc in accord with the HST Key Project results (Freedman et al. 2001).

2. ANGULAR DIAMETER DISTANCE

The comoving Friedmann-Robertson-Walker (FRW) metric for a closed universe ($\Omega_K < 0$) may be written as

$$ds^2 = d\eta^2 - R^2 (d\chi^2 + \sin^2 \chi d\Sigma^2)$$

where $\eta$ denotes conformal time, $R^2 = -1/\Omega_K$, and $d\Sigma^2$ is the line element of a unit two-sphere. All spacetime intervals are given in units of $H_0^{-1}$. The angular diameter distance $D_a$ to an object of proper length $d\ell$ at comoving distance $R\chi$ ($d\eta = d\chi = 0$) is defined so that $d\Sigma = |d\ell|/D_a$.

$$D_a = R \sin \chi(a)$$

where $\chi^2$ is the solution to the Friedmann equation:

$$d\chi^2 = \frac{-da^2}{\Omega_M(a/\Omega_K + a_{eq}/\Omega_K) + \Omega_\Lambda a^4/\Omega_K + a^2}$$

Flat and open geometries are treated analogously.

Numerical Result $D_a$ can be evaluated numerically. The distance to redshift 1400 is computed for models across the $(\Omega_M, \Omega_K)$ plane using cosmography routines from David Hogg (Hogg 1999).

Analytic Result To complement the numerical result, $D_a$ can be estimated analytically by setting $\Omega_B, a_{eq}, a_0$ to zero in equation (3), and then using Mattig’s solution (cf.

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\textsuperscript{3} An expression subscripted by ‘*’ or ‘eq’ is evaluated at decoupling or matter-radiation equality respectively.
Fig. 1.— The angle subtended by the sound horizon at decoupling, \(\Delta \Theta_s = r_{ss}/D_{as}\). Data favors \(\Delta \Theta_s = 0.60^\circ\) (Knox, Christensen, & Skordis 2001). In both numerical and analytic plots, equation (7) gives the sound horizon \(r_{ss}\). The numerical plot, the angular diameter distance to last scatter \(D_{as}\), is computed using the cosmography code (Hogg 1999). In the analytic result, the classical equations for \(D_{as}\) are integrated with \(\Omega_b, a_{eq}, \alpha_s = 0\), and the resulting formulae are used without assuming \(\Omega_M + \Omega_K = 1\) (see eq. [4]). \(\Delta \Theta_S\) from either method approximates a function of \(\Omega_K/\Omega_M\). The bending of the contours in the analytic result arises from radiation-dominated dynamics before last scatter. In the numerical result, explicit inclusion of \(\Omega_k\) in the Friedmann equation balances the radiation effect: \(\Delta \Theta_S\) contours are straight lines converging on zero in the full calculation. At far right is illustrated the comoving coordinates of a closed FRW universe with time and azimuthal angular coordinates suppressed. Inspection of the drawing shows \(\Delta \Theta_s\) is computed using the cosmography code (Hogg 1999). In the analytic result, the classical equations for \(D_{as}\) are plotted in Figure 1.

Sound travels through a tightly coupled photon-baryon system with speed \(c_s\):

\[
c_s = \frac{1}{\sqrt{3(1 + Q)}} \tag{6}
\]

where \(Q = 3\rho_B/4\rho_\gamma\), and \(\rho_B\) and \(\rho_\gamma\) are the energy densities of baryons and radiation, respectively. The sound horizon at decoupling \(r_{ss}\) in comoving coordinates is then

\[
r_{ss} = \frac{2}{\sqrt{3}H_0} \int_{a=0}^{a_s} \frac{\bar{\Omega}_q}{\bar{Q}_q} \ln \left( \sqrt{\frac{1 + Q_s}{Q_s} + \frac{Q_s + Q_{eq}}{1 + Q_{eq}}} \right) \tag{7}
\]

\((\text{Hu} \& \text{Sugiyama} 1996)\). Note that curvature does not affect sound dynamics before decoupling. In any geometry, \(r_{ss}\) will be the same.

4. THE HORIZON ANGLE AND THE PEAK INDEX

\(D_{as}\) and \(r_{ss}\) give the angular size of the horizon:

\[
\Delta \Theta_s = \frac{r_{ss}}{D_{as}}. \tag{8}
\]

Values for \(\Delta \Theta_s\) corresponding to the numerical and analytic results for \(D_{as}\) (see Section 2) are plotted in Figure 1. Curves of constant \(\Delta \Theta_s\) approximate straight lines, which intersect the origin of the \((\Omega_M, \Omega_K)\) plane. The angle subtended by the sound horizon, and so the position of the first peak, approximates a function of \(\Omega_K/\Omega_M\). A well known corollary to this general statement is that a peak corresponding to \(\Omega_K = 0\) should be insensitive to variations in \(\Omega_M\) (Bond et al. 1994; Hu et al. 2001).

To check whether this simple analysis agrees with the standard model, we compare the above results to those from CMBFAST (Seljak \& Zaldarriaga 1996). CMBFAST calculates the CMB angular spectrum.

Then, \(\hat{\ell}_{\text{peak}}\) from the spectrum multiplies \(\Delta \Theta_s\) from Figure 1 to give the constant of proportionality \(\alpha = \hat{\ell}_{\text{peak}} \times \Delta \Theta_s\). As shown in Figure 2, the numerically derived \(\alpha\) increases from 110 \((\Omega_M = 0.14)\) to 125 \((\Omega_M = 0.74)\) and has weak if any dependence on curvature. The near constant graphs of \(\alpha\) suggests a well-defined agreement between the peak indices from CMBFAST and those derived from this work's numerical and analytic calculations (see middle frame of Figure 2).

5. DISCUSSION

To gain intuition about \(\Delta \Theta_s\), consider the closed comoving universe (eq. [1]) illustrated in Figure 1, where time and azimuthal angular coordinates have been suppressed to produce the familiar two-sphere geometry. In this picture, CMB photons follow great circles, and, assuming a small angle, \(\Delta \Theta_s\) derives from inspection: \(\Delta \Theta_s = r_{ss}/R\sin \chi_s\), which is exactly equation (8).

To better understand the parameter dependence of \(r_{ss}\), take the sound speed \(c_s\) (eq. [6]) to be constant at \(9/10\sqrt{3}\). (With \(\Omega_Bh^2 = 0.02\) and radiation density derived from the COBE FIRAS measurement, the sound speed decreases at a near constant rate \((dc_s/da \approx \text{constant})\) from \(c/\sqrt{3}\) at \(a = 0\) to four-fifths that value at decoupling (Fixen et al. 1997).) The sound horizon is then

\[
r_{ss} = c_s \int_0^{a_s} \frac{da}{\bar{Q}_{eq}} = \frac{9}{5\sqrt{3}H_0} \left( \sqrt{a_s + a_{eq}} - \sqrt{a_{eq}} \right) \tag{9}
\]

\(4\) CMBFAST inputs are \((\Omega_B = 0.04, \Omega_\nu = 0, \Omega_M = 72, T_{cmb} = 2.726, Y_{He} = 0.24, N_{\nu(\text{massless})} = 3.04, N_{\nu(\text{massive})} = 0, \text{revisited, no reion, scalar only, primordial index 1, adiabatic})\).
where the final equality follows from the Friedmann equation (3) with $\Omega_K = \Omega_\Lambda = 0$.

To obtain a simple formula for $\ell_{\text{peak}}$, expand $D_{\alpha s}$ to first order in $\gamma = 2 \mid \Omega_K \mid / \Omega_M$:

$$D_{\alpha s} = \frac{1}{\sqrt{\Omega_M}} \left( 2 + \frac{\Omega_K}{\Omega_M} \right).$$  \hspace{1cm} (10)

The expansion applies to models with a low curvature-to-matter ratio. Coincidentally, such models are favored by experiment, so that the resulting formula for $\ell_{\text{peak}}$ is useful. Given equations (9) and (10) and a value of $\alpha$ from Figure 2,

$$\ell_{\text{peak}} = \frac{\alpha D_{\alpha s}}{r_{ss}} \approx \frac{11 \sqrt{3}}{9(\sqrt{a_s} + a_{eq} - \sqrt{a_{eq}})} \left( 2 + \frac{\Omega_K}{\Omega_M} \right).$$  \hspace{1cm} (11)

This near-flat approximation is plotted in Figure 2 where $a_{eq} \approx 2.4 \times 10^{-5}(\Omega_M h^2)^{-1}$. For flat models, the distance to last scatter (eq. [10]) scales as $\Omega_M^{1/2}$. One may intuit that self-gravitation leads to smaller cosmological separations. This same “gravitational” effect decreases the distance between the big bang and last scatter, and therefore $r_{ss}$ (eq. [9]) also scales with an overall factor of $\Omega_M^{-1/2}$. In equation (11), the $\Omega_M^{-1/2}$ dependence of $D_{\alpha s}$ cancels that of $r_{ss}$. This cancellation helps explain the $\Omega_M^2\Omega_\Lambda$ degeneracy of flat cosmogonies. Furthermore, the $\Omega_K/\Omega_M$ term in equation (11) is proportional to the curvature times the area between light rays in a two-dimensional representation of an $\Omega_K \approx 0$ universe (e.g. Figure 1). This suggests that one may think of $(\Omega_M, \Omega_K)$ dependence of $\ell_{\text{peak}}$ in near-flat spacetimes as resulting from light curving like the geodesics of a two-dimensional space of constant curvature. Finally, the effect of radiation on early universe dynamics and, in particular, on $r_{ss}$, is made explicit by the appearance of $a_{eq}$ in equation (9). Radiation-dominated cosmological growth per expansion scale $(d\eta/da)$ is less than that of matter-dominated dynamics. In low $\Omega_M$ universes, radiation brings last scattering even closer to the big bang and so shortens $r_{ss}$. Thus, the effect of radiation in the early universe is to spoil the pure $\Omega_K/\Omega_M$ dependence of $\ell_{\text{peak}}$ as manifest by the curved contours in the plot of equation (11) in Figure 2. The straightness of the $\ell_{\text{peak}}$ contours is restored in the numerical result shown in Figure 1. This suggests that explicit inclusion of $\Omega_\Lambda$ in the computation of $D_{\alpha s}$ balances the $a_{eq}$ dependence of $r_{ss}$.

6. CONCLUSION

The peak index has long been recognized as an indicator of geometry. It is hoped that the present analysis sheds light on $\ell_{\text{peak}}$. The peak index does not determine the magnitude of curvature, but rather the ratio of curvature to matter. A measurement of the peak’s angular scale gives the precise geometry only if $\Omega_K \approx 0$, otherwise $\ell_{\text{peak}}$ is a function of $\Omega_K/\Omega_M$. Furthermore, in deriving the $\Omega_K/\Omega_M$ dependence of $\ell_{\text{peak}}$, unexpected cosmological cancellations were discovered. Particularly useful is the balance of overall matter dependencies in $r_{ss}$ and $D_{\alpha s}$ which helps account for the $\Omega_M\Omega_\Lambda$ degeneracy in flat models. At the same time, however, it is remarkable that the admittedly simple arguments of this work yield such a decisive cosmological indicator. Within the next few years, NASA’s MAP satellite data should give $\ell_{\text{peak}}$ to cosmic-variance levels. This measurement will burn a sharp line of possible worlds across the $(\Omega_M, \Omega_K)$ plane.

This letter was begun at Cambridge as part of DAMTP’s tripos. TM thanks Ofer Lahav and Daniel Wesley for early guidance, David Hogg for cosmography computer code, and Lyman Page and James Peebles for suggestions regarding the final draft. TM is an NSF Graduate Research Fellow and is supported by NSF grant PHY-0099493.
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