A weak pseudo-Hermitian two band model, artificial Hawking radiation and tunneling

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We examine the possibility of artificial Hawking radiation by proposing a non-$\mathcal{PT}$-symmetric weakly pseudo-Hermitian two band model containing a tilting parameter. We also determine the tunneling probability using our Hamiltonian through the event horizon that acts as a classically forbidden barrier.

Keywords: Hawking radiation, weak pseudo-Hermiticity, tunneling, WKB approximation

1 Introduction

Non-Hermitian quantum mechanics is an emerging field of interest with a wide range of applications [1, 2]. In particular, the sub-class embodying $\mathcal{PT}$-symmetry has proved to be an area of continuous activity [3, 4]. In fact, over the past two decades a large family of exactly solvable $\mathcal{PT}$-symmetric systems has been discovered reflecting their intriguing spectral properties. Briefly, $\mathcal{PT}$-symmetry addresses a complex extension of quantum mechanics that is controlled by the combined action of parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) transformations [3] namely, $\mathcal{P}: x \to -x, \ p \to -p, \ \mathcal{T}: x \to x, \ p \to -p, \ i \to -i$. Non-Hermitian Hamiltonians undergo non-unitary evolution and generally describe open quantum systems in the presence of gain and loss of particles.

Hamiltonians respecting $\mathcal{PT}$-symmetry may exhibit, under certain condition related to $\mathcal{PT}$ being exact, appearance of real spectra of eigenvalues, implying balanced loss and gain. However, $\mathcal{PT}$-symmetry is neither necessary nor sufficient for the reality of the spectrum. An exceptional point appears where symmetry breaking takes place [5, 6]. In such a situation one finds the eigenvalues corresponding to two states to coalesce and the accompanying eigenfunctions become linearly dependent with respect to each other. However, in approaching the exceptional point, the phases of the eigenfunctions are not rigid and hence information from outside may get through to the system [7]. Exceptional points play an important role in the characterization of non-Hermitian Hamiltonians.

The idea of $\mathcal{PT}$-symmetry has found extension in the formulation of pseudo-Hermiticity. For the pseudo-Hermitian operators one takes recourse to the concept bi-orthogonality

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of wavefunctions [8]. The Hamiltonian $H$ is called pseudo-Hermitian if there exists a Hermitian and invertible operator $\eta$ satisfying

$$H^\dagger = \eta H \eta^{-1}$$  \hspace{1cm} (1.1)$$

where the Hermitian conjugation is taken in the Hilbert space that is endowed with a specific inner product. Imposing pseudo-Hermiticity serves as a necessary condition for the reality of the energy spectrum [9]. Like for unbroken $\mathcal{PT}$ systems, pseudo-Hermitian systems can be constructed where one encounters full balance of loss and gain (see, for example, [10] and references therein). However, in what follows, we will focus on a weak pseudo-Hermitian operator $\eta$ by not restricting it to be Hermitian [11]. Such a relaxation opens up the possibility of connecting to a wider class of non-Hermitian systems [12–15].

Lately, much interest has been focused on the issue of topological phases that are special for non-Hermitian systems and do not appear in the Hermitian setups [16–18]. In particular, $\mathcal{PT}$-symmetry is observed to have a subtle role to play in stable topologically protected nodal points for gapped and gapless semimetals [19, 20] where Bloch bands signalize distinct topological invariants. Of course, non-Hermitian support for stable topological phase has been in the news for sometime [21,22]. Topological photonics is gradually becoming a rich area to explore. In particular, mention may be made of the topological phases in the non-Hermitian SSH model [23]. While band crossings are prevalent in three-dimensions, because of the role of $\mathcal{PT}$-symmetry, stable nodal points occur in lesser dimensions which are topologically preserved. Specifically, these consist of topologically stable exceptional points in two-dimensions [24].

The aim of this note is to view the recent work of [25] in the perspective of a weak pseudo-Hermitian model. It is to be noted that the guiding Hamiltonian used in it although invariant under a special type of transformation, is not conventionally $\mathcal{PT}$-symmetric. However we can re-interpret it as a composition of two parts, one of which is $\mathcal{PT}$-weak pseudo-Hermitian by suitably identifying $\eta$, while the other is trivially Hermitian. In section 2, we review briefly the background of two-band structure and subsequently propose a weak pseudo-Hermitian Hamiltonian as a replacement of the previously used non-Hermitian form. In section 3, we calculate the contribution of such a Hamiltonian to the tunneling and show that the results of both approaches coincide because the $\gamma$-contribution is ignored in the earlier work. Finally, in section 4, some concluding remarks are presented.

## 2 Pseudo-Hermitian Hamiltonian

We begin with a tilted Weyl Hamiltonian distorted in the $x$-direction [25, 26]

$$H = \xi p_x I + \vec{p} \cdot \vec{\sigma}$$  \hspace{1cm} (2.1)$$

where $\xi \in \mathbb{R}$ is the tilting parameter, $I$ is the three-dimensional identity matrix, $\vec{p}$ is the three-dimensional momentum with components $(p_x, p_y, p_z)$, $\vec{\sigma}$’s are a set of Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ which are Hermitian and unitary and we have set the fermi velocity to be 1. The accompanying energy eigenvalues are

$$E_{\pm} = \xi p_x \pm \sqrt{p_x^2 + p_y^2 + p_z^2}$$  \hspace{1cm} (2.2)$$

where the two signs reflect two zones of a cone, the upper and lower, in the energy-momentum space. The study of the Hamiltonian $H$ reveals that the Weyl cones touch when the two energies become equal and actually cross the Fermi level as Weyl node is
overtilted ($|\xi| > 1$). The existence of strongly tilted Weyl cones has been proposed to exist in layered transition metals [20].

Adopting the tetrad representation

$$H = e^i_a p_i \sigma^a + e_i^\alpha p_{\alpha}$$

(2.3)

where $e^i_a$ are the vielbiens satisfying the orthnormality condition $e^a_\mu e^{\mu}_b = \delta^a_b$. $\mu, \alpha = (0, x, y, z)$ and $i, a = (x, y, z)$ subject to the inner product-signature constraint

$$g^{\mu\nu} = e^\mu_\alpha e^{\nu}_\alpha \eta^{\alpha\beta}$$

(2.4)

where $\eta^{\alpha\beta} = diag(-1, 1, 1, 1)$ is the Minkowski metric of flat spacetime, and comparing with (2.1) yields for the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(1 - \xi^2) dt^2 + 2\xi dx dt + dx^2$$

(2.5)

There is no singularity at $|\xi| = 1$ which represents the event horizon. Stalhammar et al [25] tried to explore the topology of an exceptional nodal structure by invoking $\mathcal{PT}$-symmetry. They found a connection between the exceptional cone and the light cone of an observer placed around a Schwarzschild black hole\(^1\). By enacting a scenario of critical-tilting of the cone, where $\mathcal{PT}$ phase transition takes place, emission of Hawking-like radiation is envisaged through pair production involving light-like particle-anti-particle pair. By exploiting the results of the Hermitian counterpart, they could derive a relation between the exceptional cone and the light cone of a radially infalling observer. Note that Hawking radiation [27] refers to thermal radiation\(^2\), emitted by a black hole off its event horizon if the quantum effects are taken into account. The contention is that pair production leads to one of the particles escaping the boundary of the black hole to infinite space leaving the other of negative energy returning into it. Incessant transitions of negative energy particles back into the black hole inevitably reduces its mass until the whole black hole disappears leaving a cloud of radiation. In the literature, an analogy has been drawn between Weyl semimetals with inhomogeneous tilting and spacetime conforming to black holes. This triggers off the idea of an artificial Hawking radiation in Weyl semimetals [28–31]. Very recently, a study of the reasonableness of connecting (over-)tilted Weyl nodes with the manifestation of black holes, along with an overall experimental viability, has received further attention [26]

Consider the following non-Hermitian Hamiltonian of a topologically insulating two band model

$$\mathcal{H} = p_x \sigma^x + p_y \sigma^y + \iota(p_z - \lambda p_x) \sigma^z$$

(2.6)

where the third term in the right specifies the presence of non-Hermiticity which contains tilting in the x-direction, and $\lambda$ signifies the coupling strength taken to be real. (2.6) has been studied in relation to an exceptional cone tilted in the x-direction. $\mathcal{H}$ was claimed to be $\mathcal{PT}$-symmetric according to the prescription

$$\mathcal{H} = (\mathcal{PT}) \mathcal{H}^* (\mathcal{PT})^{-1}$$

(2.7)

under the representations $\mathcal{P} = \sigma^z$, $\mathcal{T} = \sigma^y$ but note that while $\mathcal{P}^2$ is unity, $\mathcal{T}^2$ is not -1. One can easily check that $\mathcal{H}$ does not commute with the $\mathcal{PT}$ operator. Actually, the explicit presence of the $p_y$ term in (2.6) spoils the $\mathcal{PT}$ character of $\mathcal{H}$.

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\(^1\)The (3+1)-dimensional spherically symmetric Schwarzschild metric is an elegant platform to addresses curved spacetime around a black hole singularity.

\(^2\)Although approximately so because with radiation continuously taking place, the mass of the black hole varies.
Note that for the Hamiltonian $H$ we can solve for $p_z$ from the indicial equation to get

$$p_z^\pm = \lambda p_x \pm \sqrt{p_x^2 + p_y^2} \tag{2.8}$$

On comparing with its Hermitian counterpart, namely

$$H = \lambda p_x \sigma^x + p_y \sigma^y \tag{2.9}$$

whose energy eigenvalues read

$$E_{\pm} = \lambda p_x \pm \sqrt{p_x^2 + p_y^2} \tag{2.10}$$

one easily observes that $E_{\pm}$ and $p_z^\pm$ are interchangeable entities affording $p_z^\pm$ to be interpreted as a Hamiltonian-like operator.

We offer an interesting interpretation here. Although $H$ is not entirely $\mathcal{PT}$-symmetric by itself, we can treat it as a combination$^3$ of two Hamiltonians one of which is Hermitian ($H_h$) while the other is weak pseudo-Hermitian ($H_w$) with respect to $\mathcal{PT}$. Indeed we can write it as

$$H = H_h + H_w \tag{2.11}$$

where

$$H_h = p_y \sigma^y \tag{2.12}$$

$$H_w = p_x \sigma^x + \imath (p_z - \lambda p_x) \sigma^z \tag{2.13}$$

While the Hermiticity of $H_h$ is trivial, the weak pseudo-Hermiticity of $H_w$ follows from the condition

$$H_w^\dag = \rho H_w \rho^{-1} \tag{2.14}$$

as is evident on employing $\rho = -\imath \sigma^x (\neq \rho^\dag)$. However, it should be borne in mind that $H_w$ is not $\rho$-symmetric. The operator $\rho$ is not invariant under the conventional transformation of $\mathcal{PT}$. Examples of Hamiltonians endowed with the property of weak pseudo-Hermiticity but not being $\mathcal{PT}$-symmetric have been explicitly constructed before [12].

The secular equation for (2.13) furnishes the simpler form

$$p_x^2 - (p_z - \lambda p_x)^2 = 0 \tag{2.15}$$

implying

$$p_x^\pm = p_z \pm \lambda p_x \tag{2.16}$$

For $H_w$ the corresponding Hermitian Hamiltonian is

$$H_h = p_x \sigma^x + \lambda p_x \sigma^z \tag{2.17}$$

which supports the energy eigenvalues

$$E_{\pm} = p_z \pm \lambda p_z \tag{2.18}$$

$^3$Interfacing the Hermitian and non-Hermitian systems has been a topic of interest in the literature [32,33].
The forms of $E_\pm$ and $p^\pm_z$ are similar. Check that (2.16) and (2.18) are respectively the reduced versions of (2.8) and (2.10) when $p_y$ is absent.

Employing the same arguments leading to (2.5), here too with the Hamiltonian $H$ results in a similar metric containing the coupling parameter $\lambda$ and with an additional presence of a term $dy^2$. If we consider a slice of $y = \text{constant}$, then the latter contribution drops out and the metric transforms to the Schwarzschild black hole in Painlevé-Gullstrand coordinates [34, 35]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\tau^2 + 2\sqrt{\frac{2M}{r}}drd\tau + dr^2$$ (2.19)

with evident identifications of $x$ with $r$, $t$ with $\tau$, the Painlevé time and fixing $\lambda$ as the quantity $\sqrt{\frac{2M}{r}}$, $M$ denoting the mass of the black hole. We now turn to the process of tunneling concerning the Hawking radiation across the black hole horizon.

### 3 Tunneling probability

The tunneling probability of the Hawking radiation is simple to compute. Before that, we want to note that in [25], the spatial dependence was assumed to be carried entirely by the $x$ or the radial $r$ coordinate: in other words, the influence of the $y$-coordinate was suppressed. Hence the contribution comes from the first and third terms of the right side of $H$. In our case, we evaluate the tunneling from the weak pseudo-Hermitian part (2.13) which is our guiding Hamiltonian. In fact, the $y$-component cannot be present in $H_w$ to preserve its weak pseudo-Hermiticity.

The particle escaping from the black hole has an energy $\omega$ and so the mass of black hole is reduced from $M \to M - \omega$. Likewise, for the antiparticle, the mass of the black hole is enhanced from $M \to M + \omega'$, for energy $\omega'$. More precisely, when the pair production is happening inside the event horizon, the positive energy particle will tunnel out and when the pair production takes place outside the event horizon, the negative energy particle tunnels in [36, 37]. For our calculation of the tunneling corresponding to the weak pseudo-Hermitian component $H_w$ of $H$, the procedure is standard. The situation that we encounter resembles a contrived scenario of Hawking radiation, where the pair production of particles occurs near the event horizon of the black hole, which is given by the metric (2.19), the event horizon playing a potential barrier for the outgoing particle. Here the action $\zeta$ is imaginary and we can profitably use the semi-classical WKB approximation to estimate the tunneling probability.

Using the Legendre transformation, the Lagrangian for $H_w$ reads

$$L_w = (1 + \iota) \ p_r \cdot \dot{r} - H_w$$ (3.1)

It implies for the action the form

$$\zeta = \int L_w dt = \int (1 + \iota) \ \left(\frac{-\sqrt{2Mrp_z \pm rp_z}}{r - 2M}\right) \cdot dr - \int H_w \ dt$$ (3.2)

where because of the exceptional points coalescing at the origin the second term in the right side has no role to play. The action eventually takes the form

$$\zeta = (1 + \iota) \left[ \int_0^{2\omega} dr_* \left(\frac{-4Mp_z}{r_*}\right) + \int_0^{2\omega'} dr_* \left(\frac{4Mp_z}{r_*}\right) \right]$$ (3.3)
where \( r^* = r - 2M \), whose imaginary part contributes in the tunneling. In arriving at (3.3) we assumed \( \omega = \omega' \). Applying Plemelj-Sokhotski formulae for the Cauchy principal value, the result for the tunneling probability to leading order in \( \omega \) turns out to be

\[
\Gamma \approx e^{-2\Im(\zeta)} \approx e^{-8\pi M|p_z|}
\]

(3.4)

This result is in standard form\(^4\) and in tune with the description of Hawking radiation in terms of quantum tunneling across a black hole horizon and analogous to the Boltzmann factor for a particle of energy \( \omega \) corresponding to the inverse value of the Hawking temperature \( 8\pi M \) \([36]\).

### 4 Concluding remarks

Against the background of some recent works seeking analogues of black holes in Weyl semimetals coming from the mapping of the Weyl Hamiltonian to the Schwarzschild metric in Painlevé-Gullstrand coordinates, we have set up, in this paper, a Hamiltonian relevant for a tilted two band model much in the spirit of modeling a Weyl semimetal. The Hamiltonian is constructed to be weakly pseudo-Hermitian. The notion of weak pseudo-Hermiticity extends the definition of pseudo-Hermitian operators to exclude the constraint of the operator \( \eta \) to be Hermitian. However, our Hamiltonian is not \( \mathcal{PT} \)-symmetric and different in spirit from a recent proposal of a toy model for the same. Our scheme reflects similar features of a Schwarzschild black hole when translated to Painlevé-Gullstrand coordinates. By writing down the action whose imaginary part contributes to the tunneling effect, we provide an estimate of the tunneling probability that matches with the expected result.

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\(^4\)It also justified the neglect of the \( y \)-contribution on enforcing the radial coordinate to coincide with \( x \) \([25]\).
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