Differential operators and BV structures in noncommutative geometry

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Abstract We introduce a new formalism of differential operators for a general associative algebra $A$. It replaces Grothendieck’s notion of differential operators on a commutative algebra in such a way that derivations of the commutative algebra are replaced by $\mathbb{D}er(A)$, the bimodule of double derivations. Our differential operators act not on the algebra $A$ itself but rather on $\mathcal{F}(A)$, a certain ‘Fock space’ associated to any noncommutative algebra $A$ in a functorial way. The corresponding algebra $\mathcal{D}(\mathcal{F}(A))$ of differential operators is filtered and $\text{gr} \mathcal{D}(\mathcal{F}(A))$, the associated graded algebra, is commutative in some ‘wheeled’ sense. The resulting ‘wheeled’ Poisson structure on $\text{gr} \mathcal{D}(\mathcal{F}(A))$ is closely related to the double Poisson structure on $T_A \mathbb{D}er(A)$ introduced by Van den Bergh. Specifically, we prove that $\text{gr} \mathcal{D}(\mathcal{F}(A)) \cong \mathcal{F}(T_A (\mathbb{D}er(A)))$, provided the algebra $A$ is smooth. Our construction is based on replacing vector spaces by the new symmetric monoidal category of wheelspaces. The Fock space $\mathcal{F}(A)$ is a commutative algebra in this category (a “commutative wheelgebra”) which is a structure closely related to the notion of wheeled PROP. Similarly, we have Lie, Poisson, etc., wheelgebras. In this language, $\mathcal{D}(\mathcal{F}(A))$ becomes the universal enveloping wheelgebra of a Lie wheelgebroid of double derivations. In the second part of the paper, we show, extending a classical construction of Koszul to the noncommutative setting, that any Ricci-flat, torsion-free bimodule connection on $\mathbb{D}er(A)$ gives rise to a second-order (wheeled) differential operator, a noncommutative analogue of the

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Batalin–Vilkovisky (BV) operator, that makes $\mathcal{F}(T_A(\text{Der}(A)))$ a BV wheelgebra. In the final section, we explain how the wheeled differential operators $\mathcal{D}(\mathcal{F}(A))$ produce ordinary differential operators on the varieties of $n$-dimensional representations of $A$ for all $n \geq 1$.

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### 1 Introduction

#### 1.1 Differential operators on noncommutative algebras

The general definition of differential operator on an abstract commutative algebra was introduced by Grothendieck. For noncommutative algebras, such as tensor algebras, Grothendieck’s definition still makes sense, but does not necessarily lead to a good notion (see Remark 2.1.3, 2.1.9 for details).

In this paper, we introduce a new notion of differential operators on associative algebras. Our approach is based on the observation that the tensor algebra of a vector space may be viewed as a *twisted* commutative algebra. The notion of twisted commutative algebra dates back at least to the 1950’s, appearing in algebraic topology, cf. also [1]. One way to think about twisted commutative algebras is to interpret them as *commutative algebra objects* in the monoidal category of $\mathbb{S}$-modules, where an $\mathbb{S}$-module is a graded vector space equipped with symmetric group actions on its homogeneous components, cf. §2.3.

As a first step of our construction, we introduce a rather general definition of differential operators for a commutative algebra object in an *arbitrary* abelian symmetric monoidal category. Applying our definition in the special case of the monoidal category of $\mathbb{S}$-modules, one obtains quite a reasonable theory of differential operators on a twisted commutative algebra. Thus, for any twisted commutative algebra $A$, our construction produces a filtered algebra $\mathcal{D}(A)$ of differential operators. We show that $\text{gr} \, \mathcal{D}(A)$, the associated graded algebra, is twisted commutative and, moreover, it has a natural structure of a twisted Poisson algebra (Theorem 2.5.1). Twisted Poisson structures on tensor algebras are related to Van den Bergh’s double Poisson structures [23], as explained in [22].