Abstract. Let $G$ be an algebraic group defined over an algebraically closed field $k$ of characteristic zero. We give a simple proof of the following result: if $H^1(K_0, G) = \{1\}$ for some finitely generated field extension $K_0/k$ of transcendence degree $\geq 3$ then $H^1(K, G) = \{1\}$ for every field extension $K/k$.

ON A PROPERTY OF SPECIAL GROUPS

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1. Introduction

Let $G$ be an algebraic group. J.-P. Serre stated the following conjectures in [Se$_2$] (see also [Se$_3$, Chapter III]).

Conjecture I: If $G$ is connected then $H^1(K, G) = \{1\}$ for every field $K$ of cohomological dimension $\leq 1$.

Conjecture II: If $G$ is semisimple, connected and simply connected then $H^1(K, G) = \{1\}$ for every field $K$ of cohomological dimension $\leq 2$.

Conjecture I was proved by Steinberg [St$_1$]. Conjecture II remains open, though significant progress has been made in recent years; see [BP] and [Gi].

Our main result is a partial converse of Conjectures I and II. Recall that an algebraic group $G$ is called special if $H^1(K, G) = \{1\}$ for every field $K$ of transcendence degree $d$.

Theorem 1. Let $G$ be an algebraic group defined over an algebraically closed field $k$ of characteristic zero. Suppose $H^1(K, G) = \{1\}$ for some finitely generated field extension $K$ of transcendence degree $d$.

(a) If $d \geq 1$ then $G$ is connected.

(b) If $d \geq 2$ then $G$ is simply connected.

(c) If $d \geq 3$ then $G$ is special.

Note that the cohomological dimension of $K$ equals $d$; see [Se$_3$, Section II.4]. Thus, informally speaking, the theorem may be interpreted as saying that Conjectures I and II cannot be extended or strengthened in a meaningful way.

Our proof of Theorem 1 is rather simple; the idea is to use nontoral finite abelian subgroups of $G$ as obstructions to the vanishing of $H^1$. We remark that our argument (and, in particular, the proof of Lemma 3) does not rely on canonical resolution of singularities; cf. [RY, Remark 4.4].

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Ph. Gille recently showed us an alternative proof of Theorem 1, based on case by case analysis and properties of the Rost invariant. We would like to thank him, J.-L. Colliot-Thélène and R. Parimala for informative discussions.

2. Preliminaries

Throughout this note $k$ will denote an algebraically closed base field of characteristic zero. All fields, varieties, morphisms, algebraic groups, etc., will be assumed to be defined over $k$.

Let $G$ be an algebraic group. An abelian subgroup $A$ of $G$ is called toral if $A$ is contained in a torus of $G$ and nontoral otherwise.

Lemma 2. Let $G$ be an algebraic group, $L$ be a Levi subgroup of $G$ and $A$ be a finite abelian subgroup of $L$. If $A$ is nontoral in $L$ then $A$ is nontoral in $G$.

Proof. Assume the contrary: $A \subset T$ for some torus $T$ of $G$. Since $T$ is reductive, it lies in a Levi subgroup $L_1$ of $G$; see [OV, Theorem 6.5]. Denote the unipotent radical of $G$ by $U$; then $L$ and $L_1$ project isomorphically onto $G/U$. Since $A$ is toral in $L_1$, it is toral in $G/U$, and hence, in $L$, as claimed. \qed

Recall that a $G$-variety $X$ is an algebraic variety with a $G$-action; $X$ is generically free if $G$ acts freely on a dense open subset of $X$ and primitive if $k(X)^G$ is a field (note that $X$ is allowed to be reducible). Elements of $H^1(K, G)$ are in 1—1 correspondence with $G$-torsors over $K$, i.e., birational classes of primitive generically free $G$-varieties $X$ such that $k(X)^G = K$; see e.g., [Po, Section 1.3]. If $X$ is a primitive generically free $G$-variety, we shall write $\text{cl}(X)$ for the class in $H^1(k(X)^G, G)$ given by $X$.

Our proof of Theorem 1 is based on the following result.

Lemma 3. ([RY, Lemma 4.3]) Let $G$ be an algebraic group, $A$ be a nontoral finite abelian subgroup of $G$ and $X$ be a generically free primitive $G$-variety. Suppose $A$ fixes a smooth point of $X$. Then $\text{cl}(X) \neq 1$ in $H^1(k(X)^G, G)$.

3. Construction of a nontrivial torsor

Lemma 4. Let $A$ be an abelian group of rank $r$ and let $K$ be a finitely generated field extension of $k$ of transcendence degree $d \geq r$. Then there exists an $A$-variety $Y$ such that (i) $k(Y)^A = K$ and (ii) $Y$ has a smooth $A$-fixed point.

Proof. Since $k$ is algebraically closed, $A$ has a faithful $r$-dimensional representation $V_1$. Let $V_2$ be the trivial $(d-r)$-dimensional representation of $A$, and $V = V_1 \oplus V_2$. Then the (geometric) quotient $V/A$ is isomorphic to the affine space $k^d$. Denote the origin of $V$ by 0, and its image in $V/A$ by $\bar{0}$.

Let $Y_0$ be an affine variety over $k$ such that $k(Y_0) = K; \dim(Y_0) = d$. Let $y_0 \in Y_0$ be a smooth point. Identifying $V/A = k^d$, we can find a dominant projection $f: Y_0 \to V/A$ such that $f(y_0) = \bar{0}$ and $f$ is étale at $y_0$.

Now set $Y = Y_0 \times_{V/A} V$; the $A$-action on $Y$ is induced from $V$. The natural projection $Y \to Y_0$ is a rational quotient map for this action; see, e.g., [R,
Lemma 2.16(a)]. Thus \( Y \) satisfies (i). To prove (ii), set \( y = (y_0, 0) \); \( y \) is fixed by \( A \). The morphism \( Y \to V \) is obtained from \( f \) by a base change, and hence, \( \text{étale} \) at \( y \); the smoothness of \( V \) implies then that \( Y \) is smooth at \( y \). Thus, \( y \in Y \) is a smooth point fixed by \( A \).

**Proposition 5.** Let \( G \) be an algebraic group, \( A \) be a nontoral abelian subgroup of \( G \) of rank \( r \), and \( K/k \) be a field extension of transcendence degree \( d \). If \( d \geq r \) then \( H^1(K, G) \neq \{1\} \).

**Proof.** Choose an \( A \)-variety \( Y \) and a smooth \( A \)-fixed point \( y \in Y \), as in Lemma 4. We claim that the image of \( c_Y \) under the natural map \( H^1(K, A) \to H^1(K, G) \) is nontrivial. Indeed, recall that the image of \( c_Y \) in \( H^1(K, G) \) is \( c_{X} \), where \( X = G \ltimes_A Y = (G \times Y) / A \) is the (geometric) quotient for the \( A \)-action on \( G \times Y \) given by \( a(g, y') = (ga^{-1}, ay') \); see [PV, Section 4.8]. By [PV, Proposition 4.22], \( G \ltimes_A Y \) is smooth at \( x = (1_G, y) \) since \( Y \) is smooth at \( y \). Moreover, \( x \) is an \( A \)-fixed point of \( X \); thus Lemma 3 tells us that \( c_X \neq 1 \) in \( H^1(K, G) \), as claimed.

4. **Proof of Theorem 1**

In view of Proposition 5 it is sufficient to show that \( G \) contains a nontoral finite abelian subgroup \( A \), where

- \( (a') \) rank(\( A \)) = 1, if \( G \) is not connected,
- \( (b') \) rank(\( A \)) \leq 2, if \( G \) is not simply connected and
- \( (c') \) rank(\( A \)) \leq 3, if \( G \) is not special.

Moreover, in view of Lemma 2 we only need to prove \( (a') \), \( (b') \), and \( (c') \) under the assumption that \( G \) is reductive (otherwise we may replace \( G \) by its Levi subgroup).

Proof of \( (a') \): Write \( G = F G_0 \), where \( G_0 \) is the identity component of \( G \) and \( F \) is a finite group; see [V, Proposition 7]. Since \( G \) is disconnected, \( F \) is not contained in \( G_0 \). Choose \( a \in F \setminus G_0 \) and set \( A = \langle a \rangle \). Then \( A \) is cyclic, finite (because \( a \in F \)) and nontoral (because every torus of \( G \) is contained in \( G_0 \)), as desired.

Proof of \( (b') \): In view of \( (a') \) we may assume without loss of generality that \( G \) is connected. Now the desired conclusion follows from [St2, Theorem 2.27].

Proof of \( (c') \): Suppose \( G \) is not special. By [Se4, 1.5.1], \( G \) has a torsion prime \( p \), and by [St2, Theorem 2.28] \( G \) has a nontoral elementary \( p \)-abelian subgroup \( A \) of rank \( \leq 3 \). see also [Se4, 1.3].

**\( \square \)**
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