A Fused Elastic Net Logistic Regression Model for Multi-Task Binary Classification

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Abstract

Multi-task learning has shown to significantly enhance the performance of multiple related learning tasks in a variety of situations. We present the fused logistic regression, a sparse multi-task learning approach for binary classification. Specifically, we introduce sparsity inducing penalties over parameter differences of related logistic regression models to encode similarity across related tasks. The resulting joint learning task is cast into a form that lends itself to be efficiently optimized with a recursive variant of the alternating direction method of multipliers. We show results on synthetic data and describe the regime of settings where our multi-task approach achieves significant improvements over the single task learning approach and discuss the implications on applying the fused logistic regression in different real world settings.

1 INTRODUCTION

In this paper, we present fused logistic regression, a novel multi-task learning algorithm for solving a set of binary classification tasks that are ordered according to their mutual similarity.

Multi-task learning refers to techniques that jointly address several related learning tasks while leveraging their relatedness. Multi-task learning has been applied in various settings. These include the recognition of spam e-mails in different demographic groups [1]; identification of host-pathogen protein interactions in different infectious diseases [9]; modeling of marketing preferences of similar social groups [6]. A widely adopted mechanism to take advantage of task relatedness is to represent it in the form of a graph with weighted edges in which every node is associated with an individual task or a group of related tasks and to apply the adjacency matrix of this graph as a penalizing multiplier for some metric of the difference between individual model parameters [15] [8]. Multi-task linear regression approaches exploit this (dis-) similarity graph by...
encouraging closely related tasks to share a similar set of relevant input features using structural-sparsity-inducing penalties \[8, 3, 16, 17\].

Sparsity-inducing penalties such as an L1-penalty on the model parameters have been used to perform automatic variable selection \[13, 10, 4\]. \[18\] show that in linear regression, a combination of an L1 and L2 penalty on the model parameters can lead to sparse model fits while preserving or eliminating all parameters associated with groups of strongly correlated predictors. This is particularly useful in the case of small number of training samples, \(n\), compared to the dimensionality of the feature space \(d\) \((n \ll d)\), when the L1 penalty alone has been shown to limit the number of selected features to \(n\) \[18\]. Different studies have shown that sparsity-inducing priors can be applied as well in linear models for classification, such as logistic regression and linear discriminant analysis.

Many real world learning tasks can be cast to the problem of learning multiple related binary classification tasks. For example, in biology, correlation of time series of transcriptome profiles with ordinal phenotypes or in marketing, modeling the buying preferences of different age groups of customers by associating adjacent ranges of age with ordered classification tasks. Therefore, a need is present to develop models for multi-task binary classification, which exploit task relatedness, while performing automatic variable selection in a high-dimensional feature space.

We propose the fused logistic regression to learn binary classifiers in this situation. Specifically, we use (elastic net) logistic regression models for the individual classification tasks. To leverage the similarity across related tasks, we jointly fit all logistic regression models while imposing L1 penalties on the parameter differences of related tasks. The use of an elastic net penalty for each individual model favors sparse estimates of the coefficient vectors while keeping groups of correlated relevant predictor variables.

In section 2 we establish the mathematical notation for the rest of the paper and recapitulate the linear logistic regression model for binary classification. In section 3 we briefly describe the Alternating Direction Method of Multipliers for convex optimization \[2\] of the learning objective induced by the fused logistic regression model. In section 4 we describe the fused elastic net logistic regression model for ordered multi-task binary classification. In section 5 we report experiments of the method conducted on synthetically generated data.

2 LINEAR LOGISTIC REGRESSION FOR BINARY CLASSIFICATION

We start by a brief overview of single-task binary classification. Given is training data which are realizations form \((X_1, Y_1), \ldots, (X_n, Y_n)\) i.i.d., where the predictor or feature vector \(X_i \subset \mathbb{R}^d\), \(i = 1, \ldots, n\), is a random vector and the vector of classes or labels \(Y = (Y_1, \ldots, Y_n) \subset \{-1, 1\}^n\) is a discrete random vector. We denote the training data as the extended matrix \([X|y]\), where \(X = [x_1, \ldots, x_n]^T \in\)
**R**\(^{n \times d}\) is called the design matrix, and the vector \(y = (y_1, ..., y_n) \in \{-1, 1\}^n\) is called the response vector. A classifier is a function, \(C : \mathbb{R}^d \to \{-1, 1\}\), assigning to a predictor vector \(x \in \mathbb{R}^d\) an output label, which is a prediction for the corresponding true label \(y\).

We consider linear logistic regression as our method of choice for finding an estimator of the class probabilities \(\pi_y(x)\), \(y \in \{-1, 1\}\). While it performs comparably to competing methods, such as support vector machines and linear discriminant analysis, logistic regression has some notable advantages in that it provides a direct estimate of \(\pi_y(x)\) and tends to be more robust in the case \(d \gg n\) because it doesn’t make assumptions on the distribution of the predictors \(X\).

The logistic regression model is

\[
\hat{\pi}_y(x; \beta) = \sigma(yx^T \beta) = \frac{1}{1 + \exp(-yx^T \beta)}, \quad y \in \{-1, 1\}
\]  

(1)

where \(\beta = (\beta_0, \beta_1, ..., \beta_d)^T\) is a vector of unknown parameters. The maximum likelihood estimate for the parameters \(\beta\) is found by minimizing the negative conditional log-likelihood function

\[
-\ell(\beta) = \sum \log (1 + \exp(-y \odot X \beta)),
\]  

(2)

where the symbol \(\sum\) without subscript denotes sum over all elements of the underlying vector or matrix, the symbol \(\odot\) denotes the element-wise multiplication between vectors or matrices with the same dimensions and the bold number \(1\) denotes the \(n\)-dimensional real vector having all elements equal to 1.

To improve the generalization performance of the model and to perform automatic variable selection, we consider maximum a-posteriori estimates of \(\beta\) by the use of a product of a Gaussian and Laplacian densities, centered at \(0\) as a regularizing and variable-selecting prior:

\[
p(\beta_\setminus 0) = N\left(0, \frac{1}{\lambda_2} I\right) \times \text{Lap}\left(0, \frac{1}{\lambda_1} I\right),
\]

where \(\lambda_1 > 0\) and \(\lambda_2 > 0\), and by \(\beta_\setminus 0\) we denote the vector \(\beta\) with omitted 0\(^{th}\) element. This results in the elastic net \([18]\) estimate of \(\beta\):

\[
\hat{\beta}_{EN} := \arg \min_{\beta \in \mathbb{R}^{1+d}} -\ell(\beta) + \lambda_1 ||\beta_\setminus 0||_1 + \frac{1}{2} \lambda_2 ||\beta_{(0)}||_2^2
\]  

(3)

### 3 THE ADMM METHOD FOR CONVEX OPTIMIZATION

A major challenge in statistical modeling is to define a model that, on the one hand, is well adaptable to the phenomenon of study, and on the other hand, can

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\footnote{To simplify the notation, we assume that we have added an intercept element equal to 1 as first component of the predictor vector \(x\).}
be fit to the training data in an efficient way. In the case of model-fitting via likelihood or posterior maximization, the fitting procedure reduces to an optimization problem. Linear models like linear regression, logistic regression, linear discriminant analysis and their L2- and L1-regularized variants are expressive and convenient to fit to data since their learning objectives are convex. The Alternating Direction Method of Multipliers (ADMM) is a convex optimization technique that is particularly suited to optimize composite convex objectives, as for instance the objective induced by the fused elastic net logistic regression section 4.

The Alternating Direction Method of Multipliers (ADMM) [2] is an optimization algorithm for solving constrained optimization problems of the form:

\[
\min \; f(\chi) + g(\zeta)
\]

subject to \( P\chi + Q\zeta = s \)

with variables \( \chi \in \mathbb{R}^n \) and \( \zeta \in \mathbb{R}^m \), where \( P \in \mathbb{R}^{p \times n} \), \( Q \in \mathbb{R}^{p \times m} \) and \( s \in \mathbb{R}^p \).

The ADMM algorithm finds a saddle point of the augmented Lagrangian

\[
L_\rho(\chi, \zeta, w) := f(\chi) + g(\zeta) + \omega^T (P\chi + Q\zeta - s) + \frac{1}{2\rho} ||P\chi + Q\zeta - s||^2_2.
\]

by iteratively solving smaller localized optimization tasks. With the scaled dual variable \( \xi := \omega/\rho \), Algorithm 1 lists the general scaled form of ADMM [2].

**Algorithm 1** ADMM (general scaled form)

**Initialization:** \( \chi^0 = \zeta^0 = \xi^0 = 0; \; k = 0 \)

**do**{

\( \chi \)-update:

\[ \chi^{k+1} := \arg \min_{\chi} \left( f(\chi) + \frac{1}{2\rho} ||P\chi + Q\zeta^k - s + \xi^k||^2_2 \right) \]

\( \zeta \)-update:

\[ \zeta^{k+1} := \arg \min_{\zeta} \left( g(\zeta) + \frac{1}{2\rho} ||P\chi^{k+1} + Q\zeta - s + \xi^k||^2_2 \right) \]

\( \xi \)-update:

\[ \xi^{k+1} := \xi^k + P\chi^{k+1} + Q\zeta^{k+1} - s \]

\( k = k + 1 \)

} **while**\((k < MAXITER \) and not converged\)

ADMM doesn’t require strict convexity of its objective. This property makes it a good candidate for solving problems with L1-norm terms on the parameters, which have been shown to be not strictly convex in the case \( d \gg n \) [12].

\[ ^2 \text{Here we use a slightly modified notation from the original paper [2] with the Greek analogs of the letters ‘x’ and ‘z’ in order to avoid the conflict with the name ‘x’ for predictor variables.} \]
4 THE FUSED ELASTIC NET LOGISTIC REGRESSION MODEL

We consider a set of \( t \) ordered binary classification tasks, \( 1, \ldots, t \), on a set of \( n \) \( d \)-dimensional labeled training observations \( \{x_1, \ldots, x_n\} \subset \mathbb{R}^{1+d} \), with \( x_{i1} = 1, \ i = 1, \ldots, n \). The order of the tasks reflects their similarity. For instance, neighboring tasks should be more likely to assign the same label to a test observation, compared to tasks that are ordered far from each other. The training data for all tasks is encoded in the matrix \( [X|Y] \), where \( X = [x_1, \ldots, x_n]^T \in \mathbb{R}^{n \times (1+d)} \) is the common design matrix shared by all tasks and the response vector for task \( j \) is written as the column-vector \( y_j \) of the matrix \( Y = [y_1, \ldots, y_t] \in \{-1, 1\}^{n \times t}, \ j = 1, \ldots, t \). We define a single-task logistic regression model for task \( j = 1, \ldots, t \) with training data \( [X|y_j] \) as:

\[
\hat{\pi}_j(x; \beta^{(j)}) = \sigma(y x^T \beta^{(j)}) = \frac{1}{1 + \exp(-y x^T \beta^{(j)})},
\]

for \( j = 1, \ldots, t \). As we saw in the introduction section, minimizing an L1-L2-penalized version of the negative log-likelihood leads to sparse solutions keeping non-zero parameters for the relevant sets of correlated feature-vectors. This idea reduces to a single-task fitting procedure, in which we find the L1-L2 penalized estimate of the parameters by consecutively solving the optimization problems

\[
\beta^{(j)*} := \arg \min_{\beta^{(j)} \in \mathbb{R}^{1+d}} \left\{ -\ell^{(j)}(\beta^{(j)}; [X|y_j]) + ||\lambda_1 \odot \beta^{(j)}||_1 + \frac{1}{2}||\lambda_2 \odot \beta^{(j)}||_2^2 \right\}
\]

for \( j = 1, \ldots, t \). A small detail of this formulation is that we have presented the regularizing parameters \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) as real vectors of the form \( \lambda_1 = (0, \lambda_1, \ldots, \lambda_1) \in \mathbb{R}^{1+d} \) and \( \lambda_2 = (0, \lambda_2, \ldots, \lambda_2) \in \mathbb{R}^{1+d} \), in order to account for the usually unpenalized intercept \( \beta_0^{(j)} \).

Now we wish to incorporate the prior knowledge about the similarity between neighboring tasks into the model-fitting procedure. An important observation, which directly follows from the continuity of the modeling function in (5), is that two logistic regression models operating on the same data would produce similar output if their parameters were close. Therefore, similarity between neighboring logistic regression models for neighboring tasks can be encoded by penalizing the difference between their parameters. Let \( B := [\beta^{(1)}, \ldots, \beta^{(t)}] \in \mathbb{R}^{(1+d) \times t} \) be

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3 Assume that the first column of the design matrix \( X \) is the constant vector \( 1 \).
the coefficient matrix for all tasks and let \( R \in \mathbb{R}^{t \times t} \) be a matrix defined in the following way:

\[
R_{ij} := \begin{cases} 
1 & \text{if } j = i - 1 \\
0 & \text{otherwise} \\
\text{Circ} & \text{if } (i, j) = (1, t) 
\end{cases}, \quad i, j = 1, \ldots, t.
\]

The variable \( \text{Circ} \) is equal to 0 if no relatedness between task \( t \) and task 1 should be modeled and to 1, otherwise. We call \( R \) the column-rotating matrix for \( B \) because the columns of the \((1 + d) \times t\)-matrix \( BR \) are the same as the columns of \( B \), but rotated by one column to the left.

Let \( \nu \geq 0 \) be a penalizing parameter. Denote by \([\cdot]\) the \((1 + d) \times t\)-matrix with all columns equal to a vector \( \cdot \), by \([\nu]\) the \((1 + d) \times t\)-matrix, each element of which is equal to \( \nu \), and by \( I \) the \( t \times t \)-dimensional identity matrix. We define the multi-task fused L1-L2-penalized negative log-likelihood as the function:

\[
\ell_{MT}(B) := -\sum_{j=1}^{t} \ell(j) \left( \beta^{(j)} \right) + \sum_{j=1}^{t} \|\lambda_1 \odot \beta^{(j)}\|_1 \\
+ \frac{1}{2} \|\lambda_2 \odot \beta^{(j)}\|_F^2 + \|\nu \odot B(I - R)\|_1 \\
= \sum \log \left( [1] + \exp(-Y \odot XB) \right) + \|\lambda_1 \odot B\|_1 \\
+ \frac{1}{2} \|\lambda_2 \odot B\|_F^2 + \|\nu \odot B(I - R)\|_1
\]  

(8)

The first equality shows that if the penalizing parameter \( \nu \) is set to 0, the optimization can be split across the columns of \( B \), and is equivalent to the single-task optimization with elastic net penalty (3). The fusing L1 penalty (8) represents a scaled sum of absolute differences between each pair of consecutive columns of \( B \) and cannot be decomposed column-wise. The MAP fit of the parameters \( B \) to the training data \([X|Y] \) is found by solving the optimization problem

\[
B^* = \arg \min_{B \in \mathbb{R}^{(1+d) \times t}} \ell_{MT}(B).
\]

(9)

As a sum of convex functions, the function \( \ell_{MT} \) is also convex. Through the rest of this section, we show one way to solve this problem (9) reformulating it in an ADMM compliant form.

To begin, we convert problem (9) to the canonical ADMM-form (4) by introducing the variable matrices \( \chi \in \mathbb{R}^{(1+d) \times t} \) and \( \zeta \in \mathbb{R}^{(1+d) \times t} \), and separating the differentiable from the non-differentiable terms as follows:

\[
f(\chi) := \sum \log \left( [1] + \exp(-Y \odot X\chi) \right) + \frac{1}{2} \|\lambda_2 \odot \chi\|_F^2,
\]

\[
g(\zeta) := \|\lambda_1 \odot \zeta\|_1 + \|\nu \odot \zeta(I - R)\|_1.
\]

With this split, the canonical ADMM form for problem (9) is:
min \quad f(\chi) + g(\zeta) \quad \quad (10)
subject to \quad \chi - \zeta = [0].

The scaled form of the ADMM algorithm for problem (10) is given in Algorithm 2:

**Algorithm 2** ADMM for $\ell^{MT}$
 Initialization: $\chi^0 = \zeta^0 = \xi^0 = [0]_{(1+d) \times t}; k := 0$
 do { 
 \chi-update: $\chi^{k+1} := \text{arg min}_{\chi} ( f(\chi) + \frac{1}{2} \rho || \chi - \zeta^k + \xi^k ||_2^2 )$
 \zeta-update: $\zeta^{k+1} := \text{arg min}_{\zeta} ( g(\zeta) + \frac{1}{2} \rho || \chi^{k+1} - \zeta + \xi^k ||_2^2 )$
 \xi-update: $\xi^{k+1} := \xi^k + \chi^{k+1} - \zeta^{k+1}$
 $k := k + 1$
 } while($k < MAXITER$ and not converged )

The convergence criterion is straightforward to implement, following the instructions in [2, p. 16-17].

In the next two subsections, we describe the $\chi$-update and the $\zeta$-update.

**4.1 Newton-Raphson Gradient Descent Procedure for the $\chi$-update**

For the $\chi$-update, we notice that there is no coupling between the columns of the variable matrix $\chi$. Therefore, it is computationally more convenient to obtain $\chi^{k+1}$ by solving separately for $j = 1, ..., t$:

$$\chi^{k+1}_j := \text{arg min}_{\chi_j} \tilde{f}^{k(j)}(\chi_j), \quad (11)$$

where we denoted

$$\tilde{f}^{k(j)}(\chi_j) := \sum \log \left( 1 + \exp(-Y_j \odot X\chi_j) \right)$$

$$+ \frac{1}{2} || \lambda_j \odot \chi_j ||_2^2 + \frac{1}{2} \rho || \chi_j - (\zeta_j - \xi_j^k) ||_2^2.$$

The function $\tilde{f}^{k(j)}(\chi_j)$ is twice differentiable and convex and, therefore, can be optimized efficiently using the Newton-Raphson’s method.

The Newton-Raphson method involves the evaluation of the inverse Hessian of the objective. Due to its possibly large dimensionality, $(d \times d)$, this step can
become prohibitively expensive. Using the conventional method “solve” in R, on a computer with 64-bit 3.1 GHz Intel™ (Core™ i7)-processor, the inversion of the Hessian matrix for \( d = 5000 \) takes \( \sim 160s \), compared to \( \sim 1s \) for \( d = 1000 \), and \( \sim 0.15s \) for \( d = 500 \). If we ignore the costs for calculating the gradient and the hessian, with \( d = 5000 \), a full ten iteration Newton-Raphson’s execution would take approximately 26 minutes. This computational cost is prohibitive, considering that this procedure will be repeated for each ADMM iteration.

It turns out, that we can solve problem (11) by only considering tractable inversions of \( n \)-dimensional matrices [13, 7]. To simplify the notation, we denote \( \Omega^k_j := (\zeta^k_j - \xi^k_j) \). The gradient of \( \tilde{f}^{k(j)} \) is

\[
\nabla \tilde{f}^{k(j)}(\chi_j) = X^T \delta(\chi_j) + \eta(\chi_j),
\]

where we denoted \( \delta(\chi_j) := [-Y_j \odot \exp(-Y_j \odot X\chi_j)] + [1 + \exp(-Y_j \odot X\chi_j)] \), \( \eta(\chi_j) := (\lambda_2 + \rho) \odot \chi_j - \rho \Omega^k_j \) and the symbol ‘\( \odot \)’ denotes element-wise division between its vector or matrix operands and \( I \) denotes the identity matrix.

We know that at the global minimum \( \chi^*_j \) of \( \tilde{f}^{k(j)} \) the gradient \( \tilde{f}^{k(j)} \) should vanish. Setting the gradient to the vector \( 0 \) reveals that there exists an \( n \)-dimensional real vector \( \gamma^*_j := -\delta(\chi^*_j) \), such that

\[
X^T \gamma^*_j = \eta(\chi^*_j) = (\lambda_2 + \rho) \odot \chi^*_j - \rho \Omega^k_j.
\]

The two equations below follow directly from (13):

\[
\chi^*_j = (X^T \gamma^* + \rho \Omega^k_j) \div (\lambda_2 + \rho)
\]

\[
\gamma^*_j = (X X^T)^{-1} X ((\lambda_2 + \rho) \odot \chi^*_j - \rho \Omega^k_j)
\]

Equation (14) shows that \( \chi^*_j \) lies in an \( n \)-dimensional space. Let \( h : \mathbb{R}^n \to \mathbb{R}^d \) and \( h^{-1} : \mathbb{R}^d \to \mathbb{R}^n \) be the following two (mutually inverse) functions:

\[
h(\gamma) : = (X^T \gamma + \rho \Omega^k_j) \div (\lambda_2 + \rho)
\]

\[
h^{-1}(\chi) : = (X X^T)^{-1} X ((\lambda_2 + \rho) \odot \chi - \rho \Omega^k_j)
\]

The following theorem will form the basis of defining an optimization problem over an \( n \)-dimensional variable whose optimum can be used to unambiguously reconstruct the \( d \)-dimensional solution of the initial problem.

**Theorem 1** Let the function \( \phi^{k(j)} : \mathbb{R}^n \to \mathbb{R} \) be defined as:

\[
\phi^{k(j)}(\gamma) := \tilde{f}^{k(j)}(h(\gamma)).
\]

\( \chi^* \) is the global minimum of \( \tilde{f}^{k(j)} \) if and only if \( \gamma^* := h^{-1}(\chi^*) \) is the global minimum of \( \phi^{k(j)} \).
It follows from Theorem 1 that the minimization problem (11) can be solved by minimizing the $n$-dimensional function $\phi^{k(j)}$, and setting

$$\chi^{k+1}_j := h\left(\arg\min_{\gamma \in \mathbb{R}^n} \phi^{k(j)}(\gamma)\right). \quad (18)$$

Minimizing the function $\phi^{k(j)}$ is done again by the Newton-Raphson’s method without suffering from the costly inversion of a $d \times d$-matrix. Analytical expressions for the gradient and hessian of $\phi^{k(j)}$ are provided in the appendix.

4.2 Second-level ADMM for the $\zeta$-update

In the sequel of this section we will rely on a fact, known from subdifferential calculus [11]. For $\kappa \in \mathbb{R}$, $\kappa \geq 0$ and any real number $a$, the soft thresholding operator, $S_{\kappa}$, is defined as:

$$S_{\kappa}(a) := \begin{cases} a - \kappa & a > \kappa \\ 0 & |a| \leq \kappa \\ a + \kappa & a < -\kappa. \end{cases}$$

For the $\chi$-update we use the following result [2]:

**Soft thresholding:** Let $\lambda, \rho > 0$, $x$ is a real variable and $v$ is some real constant. The optimization problem

$$x^* := \arg\min_x \left\{ \lambda |x| + \left( \rho/2 \right) (x - v)^2 \right\}$$

has the closed-form solution

$$x = S_{\lambda/\rho}(v).$$

The $\zeta$-update is:

$$\zeta^{k+1} := \arg\min_\zeta \left\{ \| [\lambda_1] \odot \zeta \|_1 + \| [\nu] \odot \zeta (I - R) \|_1 + \frac{1}{2} \rho \| \zeta - \Omega \|_2^2 \right\}, \quad (19)$$

where $\lambda_1 \in \mathbb{R}^{1+d}$, $\nu \in \mathbb{R}^{1+d}$, $\zeta, \chi^{k+1}, \xi^k \in \mathbb{R}^{(1+d) \times t}$, $(I - R) \in \mathbb{R}^{t \times t}$, $[\cdot] \in \mathbb{R}^{(1+d) \times t}$ denotes the matrix with $t$ columns, equal to the $(1 + d)$-dimensional vector $\cdot$, and $\Omega := \chi^{k+1} + \xi^k$. Due to the two L1-norms, the objective function, unlike the case in the $\chi$-update, is not column-wise decomposable. Again, we use ADMM, to solve problem (19). Because the objective function remains invariant with respect to transposition, we can write problem (19) as:

$$\left(\zeta^{k+1}\right)^T := \arg\min_\zeta \left\{ \| [\lambda_1]^T \odot \zeta^T \|_1 + \| [\nu]^T \odot (I - R)^T \zeta^T \|_1 + \frac{1}{2} \rho \| \zeta^T - \Omega^T \|_2^2 \right\} \quad (20)$$
Defining the two variables $\tilde{\chi} := \zeta^T$ and $\tilde{\zeta} := (I - R)^T \zeta^T$, we present (20) in the canonical ADMM form as:

$$
\min \tilde{f}(\tilde{\chi}) + \tilde{g}(\tilde{\zeta}) 
$$
subject to $(I - R)^T \tilde{\chi} - \tilde{\zeta} = [0]_{t \times (1 + d)}$, where $\tilde{f}(\tilde{\chi}) := ||[\lambda_1]^T \odot \tilde{\chi}||_1 + \frac{1}{2} \rho ||\tilde{\chi} - \Omega^T||_2^2$, and $\tilde{g}(\tilde{\zeta}) := ||[\nu]^T \odot \tilde{\zeta}||_1$. The scaled-form ADMM for problem (21) is given in Algorithm 3.

**Algorithm 3** ADMM for the $\zeta$-update

**Initialization:** $\chi^0 = \zeta^0 = \tilde{\chi}^0 = \tilde{\zeta}^0 = [0]_{t \times (1 + d)}; k = 0$

**do**

**$\tilde{\chi}$-update:**

$\tilde{\chi}^{k+1} := \arg \min_{\tilde{\chi}} \left( \tilde{f}(\tilde{\chi}) + \frac{1}{2} \tilde{\rho} ||(I - R)^T \tilde{\chi} - \zeta^k + \xi^k||_2^2 \right)$

**$\tilde{\zeta}$-update:**

$\tilde{\zeta}^{k+1} := \arg \min_{\tilde{\zeta}} \left( \tilde{g}(\tilde{\zeta}) + \frac{1}{2} \tilde{\rho} ||(I - R)^T \tilde{\chi}^{k+1} - \tilde{\zeta} + \xi^k||_2^2 \right)$

**$\xi$-update:**

$\xi^{k+1} := \xi^k + (I - R)^T \tilde{\chi}^{k+1} - \zeta^{k+1}$

$k = k + 1$

**while**($k < \text{MAXITER}$ and not converged)

**Iterative Soft Thresholding for the $\tilde{\chi}$-update**

The $\tilde{\chi}$-update is:

$$
\tilde{\chi}^{k+1} := \arg \min_{\tilde{\chi}} \left\{ ||[\lambda_1]^T \odot \tilde{\chi}||_1 + \frac{1}{2} \rho ||\tilde{\chi} - \Omega^T||_2^2 
+ \frac{1}{2} \rho ||(I - R)^T \tilde{\chi} - \zeta^k + \xi^k||_2^2 \right\} 
$$

(22)

We notice that the problem (22) is column-wise decomposable, meaning that we can split it into subproblems of the form

$$
\tilde{\chi}_l^{k+1} := \arg \min_{\tilde{\chi}_l} \left\{ ||[\lambda_1]^T \odot \tilde{\chi}_l||_1 + \frac{1}{2} \rho ||\tilde{\chi}_l - \Omega^T||_2^2 
+ \frac{1}{2} \rho ||(I - R)^T \tilde{\chi}_l - \zeta^k + \xi_{l, k}||_2^2 \right\} 
$$

(23)
The term \((I - R)^T \tilde{\chi}\) represents the \(t \times (1 + d)\)-dimensional matrix with \(j^\text{th}\) row representing the row-vector difference \(\tilde{\chi}_j - \tilde{\chi}_{(j+1)}\). Because of this coupling between consecutive rows of \(\tilde{\chi}\), problem \([22]\) cannot be row-decomposed. Therefore, we use a coordinate descent approach for solving problem \([23]\) for \(l = 1, ..., 1 + d\).

Let \(\tilde{\chi}_j\) be the current estimate of \(\tilde{\chi}_k\) from \([23]\), and let \(j \in \{1, ..., t\}\).

Denote \(\Omega_{(j-1)l} := (\tilde{\chi}_{(j-1)l} - \zeta_{(j-1)l}) + \xi_{(j-1)l})\) and \(\Omega_{jl} := (\tilde{\chi}_{(j+1)l} + \zeta_{jl} - \xi_{jl})\).

A coordinate descent step for the \(j^\text{th}\) element of \(\tilde{\chi}_l\) consists in solving

\[
\tilde{\chi}^+_j := \arg \min_{\tilde{\chi}_j} \left\{ \| \lambda_{1l} \tilde{\chi}_j \|_1 + \frac{1}{2} \rho \| \tilde{\chi}_j - \Omega_{jl} \|_2^2 \right\}
\]

\[
= \arg \min_{\tilde{\chi}_j} \left\{ \lambda_{1l} \| \tilde{\chi}_j \|_1 + \frac{\rho + 2\tilde{\rho}}{2} \left( \tilde{\chi}_j - \frac{\rho \Omega_{jl} + \tilde{\rho} \Omega_{(j-1)l}}{\rho + 2\tilde{\rho}} \right)^2 \right\}.
\]

By denoting \(\kappa_{jl} := \frac{\lambda_{1l}}{\rho + 2\tilde{\rho}}\) and \(a_{jl} := \frac{\rho \Omega_{jl} + \tilde{\rho} \Omega_{(j-1)l}}{\rho + 2\tilde{\rho}}\), and using soft thresholding, we find:

\[
\tilde{\chi}_j^+ = S_{\kappa_{jl}}(a_{jl}).
\]

To find \(\tilde{\chi}^{k+1}_j\), we repeat the same step, letting the index \(j\) to iterate cyclically over the \(\{1, ..., t\}\) until satisfying a convergence criterion for the difference in the objective function between two complete cycles.

**Soft Thresholding for the \(\zeta\)-update**

The \(\zeta\)-update is:

\[
\tilde{\zeta}^{k+1} := \arg \min_{\zeta} \left\{ \| \nu^T \odot \zeta \|_1 \right\}
\]

\[
= \arg \min_{\zeta} \left\{ \| \nu^T \odot \zeta \|_1 \right\}
\]

This problem is column- and row-decomposable and can easily be solved for each element of \(\tilde{\zeta}^{k+1}\) by soft thresholding:

\[
\tilde{\zeta}_{jl}^{k+1} = S_{\nu \tilde{\rho}} \left( \tilde{\zeta}_{jl}^{k+1} - \tilde{\chi}_{jl}^{k+1} + \tilde{\chi}_{(j+1)l}^{k+1} \right),
\]

where \(j \in \{1, ..., t\}, l \in \{1, ..., 1 + d\}\).

---

In the case \(j = t\), \(j + 1\) should be thought of as \(1\).
5 EXPERIMENTS

We designed comparative benchmarks with synthetic data-sets in order to evaluate the following models:

(i) **Fused Elastic Net Logistic Regression**: the most general model in which all regularizing parameters are allowed to be non-zero: \( \lambda_1 \geq 0, \lambda_2 \geq 0, \nu \geq 0 \);

(ii) **Fused L1 Logistic Regression**: \( \lambda_1 \geq 0, \lambda_2 = 0, \nu \geq 0 \);

(iii) **Elastic Net Logistic Regression**: \( \lambda_1 \geq 0, \lambda_2 \geq 0, \nu = 0 \);

(iv) **Unpenalized Logistic Regression**: \( \lambda_1 = 0, \lambda_2 = 0, \nu = 0 \);

(v) **Discrete AdaBoost**: Default implementation from the R-package “ada” \[5\] with exponential loss functions and 200 iterations.

Our benchmarks on synthetic data simulate different conditions with respect to the correlation between feature vectors, the sparsity of the model parameters and the degree of similarity between neighboring tasks. Specifically, we simulated an “Independent Features” and a “Correlated Features” scenario and for each of these we generated quartets of 100-dimensional logistic regression coefficient vectors with associated predictor data-sets simulating four sparsity and similarity conditions as shown in Table 1. We measure sparsity in terms of number of non-zero model parameters and similarity in terms of number of matching non-zero parameters between two tasks.

Table 1: Simulated Feature Correlation, Sparsity and Similarity Conditions

| 1. Independent | Low similarity | High similarity |
|----------------|----------------|-----------------|
| Non-sparse     | a) [60, 12]    | b) [60, 48]     |
| Sparse         | c) [10, 2]     | d) [10, 8]      |
| 2. Correlated  | Low similarity | High similarity |
| Non-sparse     | e) [60, 12]    | f) [60, 48]     |
| Sparse         | g) [10, 2]     | h) [10, 8]      |

In brackets are denoted the number of non-zero parameters per task and the number of matching non-zero parameters between neighboring tasks.

For each case (a-h) we generated 20 independent quartet/data-set instances resulting in a total of 80 instances. Each quartet represents the true logistic regression model parameters for four linearly ordered binary classification tasks. The non-zero parameters are sampled from the set \( \{-4, -2, 2, 4\} \), while ensuring the case specific degrees of sparsity and similarity. We use the term “relevant feature” to distinguish a feature for which at least one coefficient in at least one task is non-zero. In scenario 1, “Independent Features”, the feature vectors are drawn from a standard normal distribution: \( \mathbf{x} \sim \mathcal{N}(0, J_{100}) \).
and are classified randomly according to the logistic model (1). In scenario 2, “Correlated Features”, the parameters and the feature vectors are first drawn like in scenario 1. Then, the second $d/2$ parameters are assigned the same values as their corresponding parameters in the first half, while the relevant feature vectors in the second half are re-sampled from a normal distribution $X_{(d/2+j)} \sim \mathcal{N}(X_{(j)}, 0.4I_d)$ so that they represent noisy copies of their corresponding relevant feature vectors in the first half. This procedure guarantees positive correlation in the order of 0.9 for couples of relevant feature vectors corresponding to equal parameters. For each instance of each case (a-h) we trained the five models (i-v) on data-sets ranging from 25 to 400 training observations. To tune the penalizing parameters $\lambda_1$, $\lambda_2$, and $\nu$, we fitted the model-instances using parameter combinations from the Cartesian product of $\Lambda_1 = \{0, 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2, 4, 6, 8\}$, $\Lambda_2 = \{0, 0.05, 0.1, 0.2, 0.4, 1, 2\}$, $\nu = \{0, 0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2, 4, 6, 8\}$ and we used validation data-sets of 1400 observations to estimate and compare the expected prediction errors. Finally, we evaluated the expected $L_{01}$ error of all five models on separate data-sets of 1400 observations that haven’t been used either for the training nor for the parameter tuning.

5.1 Comparison of Model Predictive Performance

The box-plots in Figure 1 depict the estimated expected $L_{01}$ errors, each box representing the empirical distribution of the error obtained from the corresponding 20 quartet/data-set instances for a specific case (a-h) and specific amount of training examples. The fused elastic net and the fused L1-penalized logistic regression models dominate the three single-task models in the majority of cases (cases b, c, d, f and h). This effect becomes significant in the case of high similarity (b, d, f, h), particularly with slightly under-sampled training data-sets (100 to 200 training samples). The fusing L1 penalty seems to be less beneficial for the predictive performance in cases of low task similarity with small training data-sets and/or non-sparse true coefficient profiles (cases a, b with $\leq 50$ training samples, c with $\leq 50$ training samples, e, f with $\leq 50$ training samples and g) as well as when the training data-set is big enough for the single-task models to approach the Bayes risk (cases g and h with 400 training samples). The fused logistic regression model (i) can still be used in these cases with a meta-parameter tuning procedure such as cross validation which would automatically set the fusing parameter $\nu$ to zero.

The benchmarks show only a slight predictive advantage of the fused elastic net model (i) compared to the fused lasso model (ii), particularly in case b and f. A thorough look of the simulation results revealed that for the majority of data-set instances the tuning of $\lambda_2$ has led to very low or zero values.

5.2 Comparison of Model Parameter Recovery

In Figure 2, we plot the fitted models for the “Correlated features” scenario according to their normalized Euclidean distances with respect to the original
Figure 1: Comparison between the expected $L_{01}$ errors of the five tested models. The yellow horizontal lines represent the median of estimated Bayes risks for the 20 quartet/data-set instances.
model parameters which are zero (x-axis) and non-zero (y-axis). In the sparse cases, we notice that the distributions of the fused elastic net and fused L1 models nearly overlap. The elastic net model favors some higher discrepancy for the zero parameters, particularly in the cases f with \( \geq 200 \) training samples, g with \( \geq 200 \) training samples and h with \( \geq 100 \) training samples.

6 DISCUSSION

This work introduced the fused elastic net logistic regression for multi-task binary classification. By means of sparsity inducing priors, learning in this model enables to control sparsity for the individual classifiers as well shared parameter patterns across related classification tasks. Our results suggest that learning performance is enhanced if a small but yet informative amount of data is available for the related classification tasks. We observed no significant differences between the performance between the fused L1 and elastic net logistic regression. We hypothesize that this result will turn out in favor for the elastic net variant if the sets of correlated variables also cover sets of cardinality larger than
two. The good overall classification performance achieved in the considered
simulation setting are encouraging for real world applications, like in genome
wide association studies in biology where the data acquisition typically is costly
and therefore volume of data for individual classification tasks is notoriously
low. The ability to effectively leverage information across different classification
tasks will enable researchers to make progress in this situation.

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APPENDIX

Analytical expressions for the Hessian of $\tilde{f}^{k(j)}$

$$\nabla \nabla \tilde{f}^{(j)}(x_j) = ([w] \odot X)^T ([w] \odot X) + (\lambda_2 + \rho)^T I,$$

where we denoted

$$w := \sqrt{\exp(-Y_j \odot X x_j) \div [exp(-Y_j \odot X x_j)].}$$

The symbol $'\div'$ denotes element-wise division between its vector or matrix operands and $I$ denotes the identity matrix.

Analytical expressions for the gradient and hessian of the transformed objective function $\phi^{k(j)}$ from Section 4.1

$$\tilde{X} := X \div [\sqrt{\lambda_2 + \rho}]^T,$$

$$\tilde{\Omega}^k := \Omega^k \div [\sqrt{\lambda_2 + \rho}],$$

$$\Psi^{k(j)}(\gamma) := \exp \left(-Y_j \odot \tilde{X} \tilde{X}^T \gamma - \rho Y_j \odot \tilde{X} \tilde{\Omega}_j \gamma \right),$$

$$w_j(\gamma) := \left( Y_j \odot \sqrt{\Psi^{k(j)}(\gamma)} \right) \div \left( 1 + \Psi^{k(j)}(\gamma) \right),$$

$$\nabla \phi^{k(j)}(\gamma) = \tilde{X} \tilde{X}^T \left( -Y_j \odot \Psi^{k(j)}(\gamma) \div (1 + \Psi^{k(j)}(\gamma)) \right) + \tilde{X} \tilde{X}^T \gamma$$

$$\nabla \nabla \phi^{k(j)}(\gamma) = \left( w_j(\gamma) \odot \tilde{X} \tilde{X}^T \right)^T \left( w_j(\gamma) \odot \tilde{X} \tilde{X}^T \right) + \tilde{X} \tilde{X}^T \gamma$$

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