Enforcing termination of interprocedural analysis

Stefan Schulze Frielinghaus · Helmut Seidl · Ralf Vogler

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Abstract Interprocedural analysis by means of partial tabulation of summary functions may not terminate when the same procedure is analyzed for infinitely many abstract calling contexts or when the abstract domain has infinite strictly ascending chains. As a remedy, we present a novel local solver for general abstract equation systems, be they monotonic or not, and prove that this solver fails to terminate only when infinitely many variables are encountered. We clarify in which sense the computed results are sound. Moreover, we show that interprocedural analysis performed by this novel local solver, is guaranteed to terminate for all non-recursive programs—irrespective of whether the complete lattice is infinite or has infinite strictly ascending or descending chains.

Keywords Static program analysis · Local solvers · Abstract interpretation · Interprocedural analysis · Partial tabulation

1 Introduction

It is well known that static analysis of run-time properties of programs by means of abstract interpretation can be compiled into systems of equations over complete lattices [10]. Thereby, various interesting properties require complete lattices which may have infinite strictly ascending or descending chains [6,8,16]. In order to determine a (post-) solution of a system of equations over such lattices, Cousot and Cousot propose to perform a first phase of iteration using a *widening* operator to obtain a post-solution which later may be improved by...
a second phase of iteration using a narrowing operator. This strict arrangement into separate phases, though, has the disadvantage that precision unnecessarily may be given up which later is difficult to recover. It has been observed that widening and narrowing need not be organized into separate phases [2,4,5]. Instead various algorithms are proposed which intertwine widening with narrowing in order to compute a (reasonably small) post-fixpoint of the given system of equations. The idea there is to combine widening with narrowing into a single operator and then to iterate according to some fixed ordering over the variables of the system. Still, monotonicity of all right-hand sides is required for the resulting algorithms to be terminating [2,4].

Non-monotonic right-hand sides, however, are introduced by interprocedural analysis in the style of [3] when partial tabulation of summary functions is used. In order to see this, consider an abstract lattice $\mathcal{D}$ of possible program invariants. Then the abstract effect of a procedure call can be formalized as a transformation $f^\# : \mathcal{D} \rightarrow \mathcal{D}$. For rich lattices $\mathcal{D}$ such transformations may be difficult to represent and compute with. As a remedy, each single program procedure may be decomposed into a system of equations with equation variables—one for each possible argument—where each such variable now receives values from $\mathcal{D}$ only. As a result, the difficulty of dealing with elements of $\mathcal{D}$ is replaced with the difficulty of dealing with systems of equations which are infinite when $\mathcal{D}$ is infinite. Moreover, composition of abstract functions is translated into indirect addressing of variables (the outcome of the analysis for one function call determines for which argument another function is queried)—implying non-monotonicity [15]. Thus, termination of interprocedural analysis by means of the solvers from [2,4] cannot be guaranteed. Interestingly, the local solver $\text{SLR}_3$ [2] terminates in many practical cases. Nontermination, though, may arise in two flavors:

- infinitely many variables may be encountered, i.e., some procedure may be analyzed for an ever growing number of calling contexts;
- the algorithm may for some variable switch infinitely often from a narrowing iteration back to a widening iteration.

From a conceptual view, the situation still is unsatisfactory: any solver used as a fixpoint engine within a static analysis tool should reliably terminate under reasonable assumptions. In this paper, we therefore re-examine interprocedural analysis by means of local solvers. First, we extend an ordinary local solver to a two-phase solver which performs widening and subsequently narrowing. The novel point is that both iterations are performed in a demand-driven way so that also during the narrowing phase fresh variables may be encountered for which no sound over-approximation has yet been computed.

In order to enhance precision of this demand-driven two-phase solver, we then design a new local solver which intertwines the two phases. In contrast to the solvers in [2,4], however, we can no longer rely on a fixed combination of a widening and a narrowing operator, but must enhance the solver with extra logic to decide when to apply which operator. For both solvers, we prove that they terminate—whenever only finitely many variables are encountered: irrespective whether the abstract system is monotonic or not. Both solvers are guaranteed to return (partial) post-solutions of the abstract system of equations only if all right-hand sides are monotonic. Therefore, we make clear in which sense the computed results are nonetheless sound—even in the non-monotonic case. For that, we provide a sufficient condition for an abstract variable assignment to be a sound description of a concrete system—given only a (possibly non-monotonic) abstract system of equations. This sufficient condition is formulated by means of the lower monotonization of the abstract system. Also, we elaborate for partial solutions in which sense the domain of the returned variable assignment provides
sound information. Here, the formalization of purity of functions based on computation trees and variable dependencies plays a crucial role. The concept of lower monotonization requires a Galois connection to relate concrete sets of states and abstract values. Later, we lift this restriction enabling us to verify that our solvers work correctly also in a more general setting.

Finally, we prove that interprocedural analysis in the style of [3,12] with partial tabulation using our local solvers terminates for all non-recursive programs and every complete lattice with or without infinite strictly ascending or descending chains. For the presented interprocedural analysis we make use of the same domain for contexts as well as the actual invariants in which one is interested in. However, this is not a requirement, i.e., the analysis can be modified without hassle to include only partial contexts or even a completely different domain. Furthermore, we analyze a procedure separately for each occurring context in order to keep precision. This can also be seen as virtually copying a procedure into its corresponding procedure call. If precision has no high priority, then one could also think of analyzing a procedure precisely only for a certain amount of different contexts.

Parts of this paper have been published at the SAS conference in 2016 [24]. For the journal version, we have provided the following additions:

- we generalized the framework by not requiring a Galois connection but only a concretization function $\gamma$;
- we provided explicit proofs of the Theorems 1, 3, and 4.

The paper is organized as follows. In Sect. 2 we recall the basics of abstract interpretation and introduce the idea of a lower monotonization of an abstract system of equations. The theoretical exposition here requires that we are given a Galois connection between the concrete and abstract domains. In Sect. 3 we recapitulate widening and narrowing. As a warm-up, a terminating variant of round-robin iteration is presented in Sect. 4. In Sect. 5 we formalize the idea of local solvers based on the notion of purity of functions of right-hand sides of abstract equation systems and provide a theorem indicating in which sense local solvers for non-monotonic abstract systems compute sound results for concrete systems. A first local solver is presented in Sect. 6 where widening and narrowing is done in conceptually separated phases. In Sect. 7, we present a local solver where widening and narrowing is intertwined. In Sect. 8 we lift the assumption about Galois connections and indicate that the algorithms are also correct given only a concretization function $\gamma$—if only widening and narrowing operators are used which are sound w.r.t. $\gamma$ [9]. Section. 9 considers the abstract equations systems encountered by interprocedural analysis. A concept of stratification is introduced which is satisfied if the programs to be analyzed are non-recursive. These notions enable us to prove our main result concerning termination of interprocedural analysis with partial tabulation by means of the solvers from Sects. 6 and 7. Finally, Sect. 10 reports about preliminary experiments with an implementation of the new solvers within the static analyzer GOBLINT.

2 Basics on abstract interpretation

In the following we recapitulate the basics of abstract interpretation as introduced by Cousot and Cousot [10,13]. Assume that the concrete semantics of a system is described by a system of equations

$$x = f_x, \quad x \in X$$

where $X$ is a set of variables taking values in some power set lattice $(\mathcal{C}, \subseteq, \cup)$ where $\mathcal{C} = 2^Q$ for some set $Q$ of concrete program states, and for each $x \in X$, $f_x : (X \to \mathcal{C}) \to \mathcal{C}$ is the
defining right-hand side of $x$. For the concrete system of equations, we assume that all right-hand sides $f_x, x \in X$, are monotonic. Accordingly, this system of equations has a unique least solution $\sigma$ which can be obtained as the least upper bound of all assignments $\sigma_\tau, \tau$ an ordinal. The assignments $\sigma_\tau : X \to C$ are defined as follows. If $\tau = 0$, then $\sigma_\tau x = \bot$ for all $x \in X$. If $\tau = \tau' + 1$ is a successor ordinal, then $\sigma_\tau x = f_x \sigma_{\tau'}$, and if $\tau$ is a limit ordinal, then $\sigma_\tau x = \bigcup \{ \sigma_{\tau'} | \tau' < \tau \}$. An abstract system of equations
\[ y = f_y^\sharp, \quad y \in Y \tag{2} \]
specifies an analysis of the concrete system of equations. Here, $Y$ is a set of abstract variables which may not necessarily be in one-to-one correspondence to the concrete variables in the set $X$. The variables in $Y$ take values in some complete lattice $(D, \sqcap, \sqcup)$ of abstract values and for every abstract variable $y \in Y$, $f_y^\sharp : (Y \to D) \to D$ is the abstract defining right-hand side of $y$. The elements $d \in D$ are meant to represent invariants, i.e., properties of states. It is for simplicity that we assume the set $D$ of all possible invariants to form a complete lattice, as any partial order can be embedded into a complete lattice so that all existing least upper and greatest lower bounds are preserved [22]. In order to relate concrete sets of states with abstract values, let us assume for the moment that there is a Galois connection between $C$ and $D$, i.e., there are monotonic functions $\alpha : C \to D$, $\gamma : D \to C$ such that for all $c \in C$ and $d \in D$, $\alpha(c) \sqsubseteq d$ iff $c \sqsubseteq \gamma(d)$. A similar assumption has already been made in [10]. In [13] this assumption is weakened to frameworks where no best abstract descriptions of relations $R$ can be obtained via an abstraction function $\alpha$. In Sect. 8, we therefore later will consider this more general situation.

In the presence or absence of a Galois connection, we assume that there is a description relation $R \subseteq X \times Y$ between the sets of concrete and abstract variables. Via the Galois connection between $C$ and $D$, the description relation $R$ between variables is lifted to a description relation $R^\sharp$ between assignments $\sigma : X \to C$ and $\sigma^\sharp : Y \to D$ by defining $\sigma R^\sharp \sigma^\sharp$ iff for all $x \in X$, $y \in Y$, $\sigma(x) \sqsubseteq \gamma(\sigma^\sharp(y))$ whenever $x R y$ holds. Following [13], we do not assume that the right-hand sides of the abstract equation system are necessarily monotonic. For a sound analysis, we only assume that all right-hand sides respect the description relation, i.e., that for all $x \in X$ and $y \in Y$ with $x R y$,
\[ f_x \sigma \sqsubseteq \gamma(f_y^\sharp \sigma^\sharp) \tag{3} \]
whenever $\sigma R^\sharp \sigma^\sharp$ holds. Our key concept for proving soundness of abstract variable assignments w.r.t. the concrete system of equations is the notion of the lower monotonization of the abstract system. For every function $f^\sharp : (Y \to D) \to D$ we consider the function
\[ f^\sharp \sigma = \bigcap \{ f^\sharp \sigma' | \sigma \sqsubseteq \sigma' \} \tag{4} \]
which we call lower monotonization of $f^\sharp$. By definition, we have:

**Lemma 1** For every function $f^\sharp : (Y \to D) \to D$ the following holds:
1. $f^\sharp$ is monotonic;
2. $f^\sharp \sigma^\sharp \sqsubseteq f^\sharp \sigma^\sharp$ for all $\sigma^\sharp$;
3. $f^\sharp = f^\sharp$ whenever $f^\sharp$ is monotonic. $\hfill \Box$

The lower monotonization of the abstract system (2) then is defined as the system
\[ y = f_y^\sharp, \quad y \in Y \tag{5} \]
Since all right-hand sides of (5) are monotonic, this system has a least solution.
Example 1 Consider the single equation

\[ y_1 = \text{if } y_1 = 0 \text{ then } 1 \text{ else } 0 \]

over the complete lattice \( \mathbb{N}^\infty \) of non-negative integers equipped with the natural ordering and extended by an infimum element. This system is not monotonic. Its lower monotonization is given by \( y_1 = 0 \).

\[ \square \]

Lemma 2 Assume that \( \sigma \) is the least solution of the concrete system (1). Then \( \sigma \mathcal{R}^* \sigma^\sharp \) for every post-solution \( \sigma^\sharp \) of the lower monotonization (5).

Proof For every ordinal \( \tau \), let \( \sigma_\tau \) denote the \( \tau \)th approximation of the least solution of the concrete system and assume that \( \sigma^\sharp \) is a post-solution of the lower monotonization of the abstract system, i.e., \( \sigma^\sharp y \sqsupseteq \bigcup_{\tau'} f^*_y \sigma^\sharp \) holds for all \( y \in Y \). By ordinal induction, we prove that \( \sigma_\tau \mathcal{R}^* \sigma^\sharp \). The claim clearly holds for \( \tau = 0 \). First assume that \( \tau = \tau' + 1 \) is a successor ordinal, and that the claim holds for \( \tau' \), i.e., \( \sigma_\tau \mathcal{R}^* \sigma^\sharp \). Accordingly, \( \sigma_\tau \mathcal{R}^* \sigma^\sharp \) holds for all \( \sigma^\prime \sqsupseteq \sigma^\sharp \). Consider any pair of variables \( x, y \) with \( x \mathcal{R} y \). Then \( \sigma_\tau x = f_x \sigma_\tau' \sqsubseteq \gamma(f^*_y \sigma^\prime) \) for all \( \sigma^\prime \sqsupseteq \sigma^\sharp \). Accordingly, \( \alpha(\sigma_\tau x) \sqsubseteq \bigcup_{\sigma^\prime} \gamma(f^*_y \sigma^\prime) \) for all \( \sigma^\prime \sqsupseteq \sigma^\sharp \), and therefore,

\[ \alpha(\sigma_\tau x) \sqsubseteq \bigcap_{\sigma^\prime} (f^*_y \sigma^\prime | \sigma^\prime \sqsupseteq \sigma^\sharp) = f^*_y \sigma^\sharp \sqsubseteq \sigma^\sharp y \]

since \( \sigma^\sharp \) is a post-solution. From that, the claim follows for the ordinal \( \tau \). Now assume that \( \tau \) is a limit ordinal, and that the claim holds for all ordinals \( \tau' < \tau \). Again consider any pair of variables \( x, y \) with \( x \mathcal{R} y \). Then

\[ \sigma_\tau x = \bigcup_{\sigma_{\tau'}} x | \tau' < \tau \sqsubseteq \bigcup_{\gamma(\sigma^\sharp y) | \tau' < \tau} \gamma(\sigma^\sharp y) \]

and the claim also follows for the limit ordinal \( \tau \).

\[ \square \]

Note that for the proof of Lemma 2 it was crucial to assume that there is a Galois connection between the concrete and abstract domains. From Lemma 2 we conclude that for the abstract system from Example 1 the assignment \( \sigma^\sharp = \{y_1 \mapsto 0\} \) is a sound description of every corresponding concrete system, since \( \sigma^\sharp \) is a post-solution of the lower monotonization \( y_1 = 0 \).

In general, Lemma 2 provides us with a sufficient condition guaranteeing that an abstract assignment \( \sigma^\sharp \) is sound w.r.t. the concrete system (1) and the description relation \( \mathcal{R} \), namely, that \( \sigma^\sharp \) is a post-solution of the system (5). This sufficient condition is remarkable as it is an intrinsic property of the abstract system since it does not refer to the concrete system. As a corollary we obtain:

Corollary Every post-solution \( \sigma^\sharp \) of the abstract system (2) is sound.

Proof For all \( y \in Y \), \( \sigma^\sharp y \sqsupseteq f^*_y \sigma^\sharp \sqsupseteq f^*_y \sigma^\sharp \) holds. Accordingly, \( \sigma^\sharp \) is a post-solution of the lower monotonization of the abstract system and therefore sound.

\[ \square \]

3 Widening and narrowing

It is instructive to recall the basic algorithmic approach to determine non-trivial post-solutions of abstract systems (2) when the set \( Y \) of variables is finite, all right-hand sides are monotonic and the complete lattice \( \mathbb{D} \) has finite strictly increasing chains only. In this case, chaotic
iteration may be applied. This kind of iteration starts with the initial assignment \( \bot \) which assigns \( \bot \) to every variable \( y \in Y \) and then repeatedly evaluates right-hand sides to update the values of variables until the values for all variables have stabilized. This method may also be applied if right-hand sides are non-monotonic: the only modification required is to update the value for each variable not just with the new value provided by the left-hand side, but with some upper bound of the old value for a variable with the new value. As a result, a post-solution of the system is computed which, according to Corollary 2, is sound.

The situation is more intricate, if the complete lattice in question has strictly ascending chains of infinite length. Here, we follow Cousot and Cousot [9, 10, 13] who suggest to accelerate iteration by means of widening and narrowing. A widening operator \( \triangledown : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D} \) takes the old value \( a \in \mathbb{D} \) and a new value \( b \in \mathbb{D} \) and combines them to a value \( a \sqcup b \sqsubseteq a \triangledown b \) with the additional understanding that for any sequence \( b_i, i \geq 0 \), and any value \( a_0 \), the sequence \( a_{i+1} = a_i \triangledown b_i, i \geq 0 \), is ultimately stable. In contrast, a narrowing operator \( \Delta : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D} \) takes the old value \( a \in \mathbb{D} \) and a new value \( b \in \mathbb{D} \) and combines them to a value \( a \sqcap b \sqsubseteq a \Delta b \sqsubseteq a \) with the additional understanding that for any sequence \( b_i, i \geq 0 \), and any value \( a_0 \), the sequence \( a_{i+1} = a_i \Delta b_i, i \geq 0 \), is ultimately stable.

While the widening operator is meant to reach a post-solution after a finite number of updates to each variable of the abstract system, the narrowing operator allows to improve upon a variable assignment once it is known to be sound. In particular, if all right-hand sides are monotonic, the result of a narrowing iteration, if started with a post-solution of the abstract system, again results in a post-solution. Accordingly, the returned variable assignment can easily be verified to be sound. In analyzers which iterate according to the syntactical structure of programs such as ASTRÉE [14], this strict separation into two phases, though, has been given up. There, when iterating over one loop, narrowing for the current loop is triggered as soon as locally a post-solution has been attained. This kind of intertwining widening and narrowing is systematically explored in [2, 4]. There, a widening operator is combined with a narrowing operator into a single derived operator \( \boxslash \) defined by

\[
a \boxslash b = \begin{cases} a \Delta b & \text{if } b \sqsubseteq a \\ a \triangledown b & \text{else} \end{cases}
\]

also called warrowing. Solvers which perform chaotic iteration and use warrowing to combine old values with new contributions, necessarily return post-solutions—whenever they terminate. In [4, 5], termination could only be guaranteed for systems of equations where all right-hand sides are monotonic. For non-monotonic systems as may occur at interprocedural analysis, only practical evidence could be provided for the proposed algorithms to terminate in interesting cases.

Here, our goal is to lift these limitations by providing solvers which terminate for all finite abstract systems of equations and all complete lattices—no matter whether right-hand sides are monotonic or not. For that purpose, we dissolve the operator \( \boxslash \) again into its components. Instead, we equip the solving routines with extra logic to decide when to apply which operator.

### 4 Terminating structured round-robin iteration

Let us consider a finite abstract system as given by:

\[
y_i = f_i^y, \quad i = 1, \ldots, n
\]

(6)
In [4], a variation of round-robin iteration is presented which is guaranteed to terminate for monotonic systems, while it may not terminate for non-monotonic systems. In order to remedy this failure, we re-design this algorithm by additionally maintaining a flag which indicates whether the variable presently under consideration has or has not reached a sound value (Fig. 1). Solving starts with a call `solve(false, n)` where `n` is the highest priority of a variable. A variable `yi` has a higher priority than a variable `yj` whenever `i > j` holds. A call `solve(b, i)` considers variables up to priority `i` only. The Boolean argument `b` indicates whether a sound abstraction (relative to the current values of the higher priority variables) has already been reached. The algorithm first iterates on the lower priority variables (if there are any). Once solving of these is completed, the right-hand side `f^♯_i` of the current variable `yi` is evaluated and stored in the variable `tmp`. Additionally, `b'` is initialized with the Boolean argument `b`. First assume that `b` has already the value `true`. Then the old value `σ[yi]` is combined with the new value in `tmp` by means of the narrowing operator giving the new value of `tmp`. If that is equal to the old value, we are done and `solve` returns. Otherwise, `σ[yi]` is updated to `tmp`, and `solve(true, i)` is called tail-recursively. Next assume that `b` has still the value `false`. By induction hypothesis, all recursive calls eventually terminate. In light of Theorem 1, the resulting algorithm is called terminating structured round-robin iteration or TSRR for short.

**Theorem 1** The algorithm in Fig. 1 terminates for all finite abstract systems of the form (6). Upon termination, it returns a variable assignment `σ` which is sound. If all right-hand sides are monotonic, `σ` is a post-solution.

**Proof** By induction on `i`, we prove that `solve(b, i)` terminates. For `i = 0`, the statement is obviously true. Now assume that `i > 0` and that by induction hypothesis, `solve(b', i - 1)` terminates for `b' ∈ {true, false}`. First consider the case where `b = true`. In this case, the flag `b'` for the tail-recursive call will be equal to `true` as well, and only narrowing will be applied to `yi`. Therefore, the sequence of tail-recursive calls `solve(b, i)` eventually will terminate. Now consider the case where `b = false`. By induction hypothesis, all recursive

```c
void solve(b, i) {
    if (i ≤ 0) return;
    solve(b, i - 1);
    tmp := f^♯_i σ;
    b' := b;
    if (b) tmp := σ[yi] Δ tmp;
    else if (tmp ⊆ σ[yi]) {
        tmp := σ[yi] Δ tmp;
        b' := true;
    } else tmp := σ[yi] ∨ tmp;
    if (σ[yi] = tmp) then return;
    σ[yi] := tmp;
    solve(b', i);
}
```
calls \( \text{solve}(b', i - 1) \) terminate. Consider the sequence of tail recursive calls where the flag \( b' \) is not set to \text{true}. Within this sequence, the new values for \( y_j \) form an ascending chain \( d_0 \sqsupset d_1 \sqsupset d_2 \ldots \) where \( d_{j+1} = d_j \lor a_j \) for suitable values \( a_j \). Due to the properties of a widening operator, this sequence is finite, i.e., there is some \( j \) such that \( d_{j+1} \sqsubseteq d_j \). In this case the call either terminates directly or recursively calls \( \text{solve}(b', i) \) for \( b' = \text{true} \). Therefore, solving terminates also in this case.

It remains to prove that upon termination, a sound variable assignment \( \sigma \) is found. For \( j = 1, \ldots, n \), and a variable assignment \( \rho : \{ y_1, \ldots, y_n \} \rightarrow \mathbb{D} \), we consider the system \( \mathcal{E}_{\rho, j} \) defined by:

\[
y_i = f_{\rho, i}^\sharp \quad (i = 1, \ldots, j)
\]

with \( f_{\rho, i}^\sharp \sigma^\sharp = f_i(\rho \oplus \sigma^\sharp) \) for \( \sigma^\sharp : \{ y_1, \ldots, y_j \} \rightarrow \mathbb{D} \). Here, the operator \( \oplus \) is meant to overwrite the values of \( \rho \) by the corresponding values of \( \sigma^\sharp \) whenever \( \sigma^\sharp \) is defined. Let \( \mathcal{E}_{\rho, j} \) denote the lower monotonization of \( \mathcal{E}_{\rho, j} \). We claim:

1. Assume \( \sigma_1 \) is a post-solution of the system \( \mathcal{E}_{\rho, j} \). Then \( \text{solve}(\text{true}, j) \) when started with \( \rho_1 = \rho \oplus \sigma_1 \), returns with a variable assignment \( \rho_2 = \rho \oplus \sigma_2 \) where \( \sigma_2 \) is still a post-solution of \( \mathcal{E}_{\rho, j} \).
2. \( \text{solve}(\text{false}, j) \) returns a post-solution of \( \mathcal{E}_{\rho, j} \).

We proceed by induction on \( j \). For \( j = 0 \), nothing must be proven. Therefore assume \( j > 0 \). Consider the first claim. As post-solutions of \( \mathcal{E}_{\rho, j} \) are preserved by each update which combines an old value \( \sigma[y_j] \) with the value of the corresponding right-hand side \( f_{\rho, i}^\sharp \) for \( \sigma \) by means of \( \sqcap \) and thus also by \( \Delta \), the claim follows.

For a proof of the second claim, let us consider the sub-sequence of tail-recursive calls \( \text{solve}(b, j) \) where \( b' \) remains \text{false}. Eventually this sequence ends with a last call where \( b' \) is set to \text{true}. Let \( \rho' \) denote the variable assignment before this update occurs. Then \( \rho'(y_j) \sqsupseteq f_j \rho' \sqsupseteq f_j \rho' \). Likewise, by induction hypothesis, \( \rho'[\{ y_1, \ldots, y_{j-1} \} \) is a post-solution of \( \mathcal{E}_{\rho', j-1} \). Altogether therefore, \( \rho' = \rho \oplus \sigma' \) for some variable assignment \( \sigma' : \{ y_1, \ldots, y_j \} \rightarrow \mathbb{D} \) which is a post-solution of \( \mathcal{E}_{\rho', j} \). Accordingly, \( \text{solve}(\text{false}, j) \) either directly terminates with \( \rho' \), and the second claim follows, or \( \text{solve}(\text{true}, j) \) is called, and the second claim follows from the first one. This completes the proof of the two claims. Since the second claim, instantiated with \( j = n \), implies that the variable assignment returned by the algorithm is a post-solution of the lower monotonization of the system, it is sound. And by Lemma 1.3, it is then also a post-solution of the original abstract system whenever all right-hand sides are monotonic.

In fact, for monotonic systems, the new variation of round-robin iteration behaves identical to the algorithm SRR from [4].

## 5 Local solvers

Local solving may gain speed-ups by querying the value of only a small subset of variables which is still sufficient to answer the initial query. Such solvers are at the heart of program analysis frameworks such as the CIAO system [17,18] or GOBLINT. In order to reason about partial variable assignments as computed by local solvers, we can no longer consider right-hand sides in equations as black boxes, but require a notion of variable dependence.

For the concrete system we assume that right-hand sides are mathematical functions of type \( (X \rightarrow \mathbb{C}) \rightarrow \mathbb{C} \) where for any such function \( f \) and variable assignment \( \sigma : X \rightarrow \mathbb{C} \),
we are given a superset $\text{dep}(f, \sigma)$ of variables onto which $f$ possibly depends, i.e.,

$$(\forall x \in \text{dep}(f, \sigma). \sigma[x] = \sigma'[x]) \implies f \sigma = f \sigma'$$  \hspace{1cm} (7)

for all $\sigma' : X \to \mathbb{C}$. Let $\sigma : X \to \mathbb{C}$ denote a solution of the concrete system. Then we call a subset $X' \subseteq X$ of variables $\sigma$-closed, if for all $x \in X'$, $\text{dep}(f_x, \sigma) \subseteq X'$. Then for every $x$ contained in the $\sigma$-closed subset $X'$, $\sigma[x]$ can be determined already if the values of $\sigma$ are known for the variables in $X'$ only.

In [23,25] it is observed that for suitable formulations of interprocedural analysis, the set of all run-time calling contexts of procedures can be extracted from $\sigma$-closed sets of variables.

**Example 2**  Consider the following example program

$$
\text{main} \rightarrow \text{u} \xrightarrow{p()} p(); p() \rightarrow \text{v} \xrightarrow{g}\n$$

consisting of a procedure main which at some program point $u$ inside a loop calls the procedure $p$ twice in a row. The procedure $p$ iterates on some program point $v$ by repeatedly applying the function $g : Q \to 2^Q$ which describes the operational behaviour of the body of the loop at $v$. The following system of equations arises from the program

$$
\langle u, q \rangle = \bigcup \{ \langle v, q_1 \rangle | q_1 \in \bigcup \{ \langle v, q_2 \rangle | q_2 \in \langle u, q \rangle \} \cup \{ q \} \} \\
\langle v, q \rangle = \bigcup \{ g \langle q_1 | q_1 \in \langle v, q \rangle \} \cup \{ q \}
$$

for $q \in Q$. Here, $Q$ is a superset of all possible system states. The set of variables of this system is given by $X = \{ \langle u, q \rangle, \langle v, q \rangle \mid q \in Q \}$ where $\langle u, q \rangle, \langle v, q \rangle$ represent the sets of program states possibly occurring at program points $u$ and $v$, respectively, when the corresponding procedures have been called in context $q$. For any variable assignment $\sigma$, the dependence sets of the right-hand sides are naturally defined by:

$$
\text{dep}(f_{\langle u, q \rangle}, \sigma) = \{ \langle u, q \rangle \} \cup \{ \langle v, q_2 \rangle | q_2 \in \sigma \langle u, q \rangle \} \\
\text{dep}(f_{\langle v, q \rangle}, \sigma) = \{ \langle v, q \rangle \}
$$

where $f_x$ again denotes the right-hand side function for a variable $x$. Assuming that $g(q_0) = \{ q_1 \}$ and $g(q_1) = \emptyset$, the least solution $\sigma$ maps $\langle u, q_0 \rangle, \langle v, q_0 \rangle$ to the set $\{ q_0, q_1 \}$ and $\langle u, q_1 \rangle, \langle v, q_1 \rangle$ to $\{ q_1 \}$. Accordingly, the set $\{ \langle u, q_i \rangle, \langle v, q_i \rangle \mid i = 0, 1 \}$ is $\sigma$-closed. We conclude, given the program is called with initial context $q_0$, that the procedure $p$ is called with contexts $q_0$ and $q_1$ only. \hfill \Box

In concrete systems of equations, right-hand sides may depend on infinitely many variables. Since abstract systems are meant to give rise to effective algorithms, we impose more restrictive assumptions onto their right-hand side functions. For these, we insist that only finitely many variables may be queried. Following the considerations in [19–21], we demand that every right-hand side $f^x$ of the abstract system is pure in the sense of [20]. This means that, operationally, the evaluation of $f^x$ for any abstract variable assignment $\sigma^x$ consists of a finite sequence of variable look-ups before eventually, a value is returned. Technically, $f^x$ can be represented by a computation tree, i.e., is an element of

$$
\text{tree} ::= \text{Answer } \mathbb{D} \mid \text{Query } Y \times (\mathbb{D} \to \text{tree})
$$

Thus, a computation tree either is a leaf immediately containing a value or a query, which consists of a variable together with a continuation which, for every possible value of the
variable returns a tree representing the remaining computation. Each computation tree defines a function \([t] : (Y \rightarrow \mathbb{D}) \rightarrow \mathbb{D}\) by:

\[
[\text{Answer } d] \sigma = d \\
[\text{Query } (y, e)] \sigma = [e (\sigma[y])] \sigma
\]

Following [20], the tree representation is uniquely determined by (the operational semantics of) \(f^\sharp\).

**Example 3** Computation trees can be considered as generalizations of binary decision diagrams to arbitrary sets \(\mathbb{D}\). For example, let \(\mathbb{D} = \mathbb{N}^\infty\) and the function \(f^\sharp : (Y \rightarrow \mathbb{D}) \rightarrow \mathbb{D}\) with \(\{y_1, y_2\} \subseteq Y\), defined by

\[f^\sharp(x) = \begin{cases} 1 + \sigma[y_2] & \text{if } \sigma[y_1] > 5 \\ \sigma[y_1] & \text{otherwise} \end{cases}\]

is then represented by the tree

\[
\text{Query } (y_1, \text{fun } d_1 \rightarrow \text{if } d_1 > 5 \text{ then Query } (y_2, \text{fun } d_2 \rightarrow \text{Answer } (1 + d_2)) \text{ else Query } (y_1, \text{fun } d_1 \rightarrow \text{Answer } d_1))
\]

A set \(\text{dep}(f^\sharp, \sigma^\sharp) \subseteq Y\) with a property analogous to (7) can be explicitly obtained from the tree representation \(t\) of \(f^\sharp\) by defining \(\text{dep}(f^\sharp, \sigma^\sharp) = \text{treedep}(t, \sigma^\sharp)\) where:

\[
\begin{aligned}
\text{treedep}(\text{Answer } d, \sigma^\sharp) &= \emptyset \\
\text{treedep}(\text{Query } (y, e), \sigma^\sharp) &= \{y\} \cup \text{treedep}(c (\sigma^\sharp[y]), \sigma^\sharp)
\end{aligned}
\]

Technically, this means that the value \(f^\sharp(x)\) can be computed already for partial variable assignments \(\sigma' : Y' \rightarrow \mathbb{D}\), whenever \(\text{dep}(f^\sharp, \top \oplus \sigma') = \text{treedep}(t, \top \oplus \sigma') \subseteq Y'\).

Here, \(\top : Y \rightarrow \mathbb{D}\) maps each variable of \(Y\) to \(\top\) and \(\top \oplus \sigma'\) returns the value \(\sigma'[y]\) for every \(y \in Y'\) and \(\top\) otherwise.

**Example 4** Consider the function \(f^\sharp\) from Example 3 together with the partial assignment \(\sigma' = \{y_1 \mapsto 3\}\). Then \(\text{dep}(f^\sharp, \top \oplus \sigma') = \{y_1\}\).

We call a partial assignment \(\sigma' : Y' \rightarrow \mathbb{D}\) closed (w.r.t. an abstract system (2)), if for all \(y \in Y'\), \(\text{dep}(f^\sharp, \top \oplus \sigma') \subseteq Y'\).

In the following, we strengthen the description relation \(\mathcal{R}\) additionally to take variable dependencies into account. We say that the abstract system (2) simulates the concrete system (1) (relative to the description relation \(\mathcal{R}\)) iff for all pairs \(x, y\) of variables with \(x \mathcal{R} y\), such that for the concrete and abstract right-hand sides \(f_x\) and \(f^\sharp_y\), respectively, property (3) holds and additionally \(\text{dep}(f_x, \sigma) \mathcal{R} \text{dep}(f_y^\sharp, \sigma^\sharp)\) whenever \(\sigma \mathcal{R}^* \sigma^\sharp\). Here, a pair of sets \(X', Y'\) of concrete and abstract variables is in relation \(\mathcal{R}\) if for all \(x \in X', x \mathcal{R} y\) for some \(y \in Y'\). Theorem 2 demonstrates the significance of closed abstract assignments which are sound.

**Theorem 2** Assume that the abstract system (2) simulates the concrete system (1) (relative to \(\mathcal{R}\)) where \(\sigma\) is the least solution of the concrete system. Assume that \(\sigma^\sharp : Y' \rightarrow \mathbb{D}\) is a partial assignment with the following properties:

1. \(\sigma^\sharp\) is closed;
2. \(\top \oplus \sigma^\sharp\) is a post-solution of the lower monotonization of the abstract system.

Then the set \(X' = \{x \in X \mid \exists y \in Y', x \mathcal{R} y\}\) is \(\sigma\)-closed.

**Proof** By Lemma 2, \(\sigma \mathcal{R}^* (\top \oplus \sigma^\sharp)\) holds. Now assume that \(x \mathcal{R} y\) for some \(x \in X\) and \(y \in Y'\).

By definition therefore, \(\text{dep}(f_x, \sigma) \mathcal{R} \text{dep}(f_y^\sharp, \top \oplus \sigma^\sharp)\). Since the latter is a subset of \(Y'\), the former must be a subset of \(X'\), and the assertion follows. \(\square\)
6 Terminating structured two-phase solving

We first present a local version of a two-phase algorithm to determine a sound variable assignment for an abstract system of equations. As the algorithm is local, no pre-processing of the equation system is possible. Accordingly, variables where widening or narrowing is to be applied must be determined dynamically (in contrast to solvers based on static variable dependencies where widening points can be statically determined [7]). We solve this problem by assigning priorities to variables in decreasing order in which they are encountered, and consider a variable as a candidate for widening/narrowing whenever it is queried during the evaluation of a lower priority variable. The second issue is that during the narrowing iteration of the second phase, variables may be encountered which have not yet been seen and for which therefore no sound approximation is available. In order to deal with this situation, the algorithm does not maintain a single variable assignment, but two distinct ones. While assignment $\sigma_0$ is used for the widening phase, $\sigma_1$ is used for narrowing with the understanding that, once the widening phase is completed, the value of a variable $y$ from $\sigma_0$ is copied as the initial value of $y$ into $\sigma_1$. This clear distinction allows to continue the widening iteration for every newly encountered variable $y'$ in order to determine an acceptable initial value before continuing with the narrowing iteration. The resulting algorithm can be found in Figs. 2 and 3. Initially, the priority queue $Q$ and the sets $\text{dom}_0$ and $\text{dom}_1$ are empty. Accordingly, the mappings $\sigma_i : \text{dom}_i \rightarrow \mathbb{D}$ and $\text{infl} : \text{dom} \rightarrow 2^Y$ are also empty. Likewise, the set $\text{point}$ is initially empty. Solving for the variable $y_0$ starts with the call $\text{solve}_1(y_0, 0)$.

Let us first consider the functions $\text{solve}_0$, $\text{iterate}_0$, $\text{do-var}_0$. These are meant to realize a local widening iteration. A call $\text{solve}_0(y)$ first checks whether $y \in \text{dom}_0$. If this is the case, solving immediately terminates. Otherwise, $\sigma_0[y]$ is initialized with $\bot$, $y$ is added to $\text{dom}_0$, $\text{infl}[y]$ is set to the empty set, and $y$ receives the next available priority by means of the call $\text{next_prio}()$. Subsequently, $\text{do-var}_0(y)$ is called, followed by a call to $\text{iterate}_0(prio[y])$ to complete the widening phase for $y$. Upon termination, a call $\text{iterate}_0(n)$ for an integer $n$ has removed all variables of priority at most $n$ from the queue $Q$. It proceeds as follows. If $Q$ is empty or contains only variables of priority exceeding $n$, it immediately returns. Otherwise,

```c
void iterate_0(n) {
    if (Q \neq \emptyset \land \text{min_prio}(Q) \leq n) {
        y := \text{extract_min}(Q);
        do_var_0(y);
        iterate_0(n);
    }
}
void solve_0(y) {
    if (y \in \text{dom}_0) return;
    \text{dom}_0 := \text{dom}_0 \cup \{y\};
    prio[y] := \text{next_prio}();
    \sigma_0[y] := \bot;
    \text{infl}[y] := \emptyset;
    do_var_0(y);
    iterate_0(prio[y]);
}
void iterate_1(n) {
    if (Q \neq \emptyset \land \text{min_prio}(Q) \leq n) {
        y := \text{extract_min}(Q);
        solve_1(y, \text{prio}[y] - 1);
        do_var_1(y);
        iterate_1(n);
    }
}
void solve_1(y, n) {
    if (y \in \text{dom}_1) return;
    solve_0(y);
    \text{dom}_1 := \text{dom}_1 \cup \{y\};
    \sigma_1[y] := \sigma_0[y];
    \text{forall} (z \in \{y\} \cup \text{infl}[y]) \text{ insert } z \text{ into } Q;
    \text{infl}[y] := \emptyset;
    iterate_1(n);
}
```

Fig. 2 The solver TSTP, part 1
Fig. 3  The solver TSTP, part 2

the variable $y$ with least priority is extracted from $Q$. Having processed do_var$_0(y)$, the iteration continues with the tail-recursive call iterate$_0(n)$.

It remains to describe the function do_var$_0$. When called with a variable $y$, the algorithm first determines whether or not $y$ is a widening/narrowing point, i.e., contained in the set point. If so, $y$ is removed from point, and the flag isp is set to true. Otherwise, isp is set to false. Then the right-hand side $f^y$ is evaluated and the result stored in the variable $tmp$. For its evaluation, the function $f^y$, however, does not receive the current variable assignment $\sigma_0$ but an auxiliary function eval$_0$ which serves as a wrapper to the assignment $\sigma_0$. The wrapper function eval$_0$, when queried for a variable $z$, first calls solve$_0(z)$ to compute a first non-trivial value for $z$. If the priority of $z$ is greater or equal to the priority of $y$, a potential widening point is detected. Therefore, $z$ is added to the set point. Subsequently, the fact that $z$ was queried during the evaluation of the right-hand side of $y$, is recorded by adding $y$ to the set infl$_[z]$. Finally, $\sigma_0[y] \triangledown tmp$ is returned.

Having evaluated $f^y$ and stored the result in $tmp$, the function do_var$_0$ then applies widening only if isp equals true. In this case, $tmp$ receives the value of $\sigma_0[y] \triangledown tmp$. In the next step, $tmp$ is compared with the current value $\sigma_0[y]$. If both values are equal, the procedure returns. Otherwise, $\sigma_0[y]$ is updated to $tmp$. The variables in infl$_[y]$ are inserted into the queue $Q$, and the set infl$_[y]$ is reset to the empty set. Only then the procedure returns.

The functions solve$_1$, iterate$_1$ and do_var$_1$, on the other hand, are meant to realize the narrowing phase. They essentially work analogously to the corresponding functions solve$_0$, iterate$_0$ and do_var$_0$. In particular, they re-use the mapping infl which records the currently encountered variable dependencies as well as the variable priorities and the priority queue $Q$. Instead of $\sigma_0$, dom$_0$, however, they now refer to $\sigma_1$, dom$_1$, respectively. Moreover, there are the following differences.

First, the function solve$_1$ now receives not only a variable, but a pair of an integer $n$ and a variable $y$. When called, the function first checks whether $y \in$ dom$_1$. If this is the case, solving immediately terminates. Otherwise, solve$_0(y)$ is called first. After that call, the widening phase for $y$ is assumed to have terminated where the resulting value is $\sigma_0[y]$. 

\begin{verbatim}
void do_var$_0(y)$id 
isp := y \in point;
point := point \{ y \};
if \((\text{prio}[z] \geq \text{prio}[y])
point := point \{ z \};
infl[z] := infl[z] \{ y \};
return \sigma_0[z];
}
tmp := f^y eval$_0$;
if (isp) tmp := \sigma_0[y] \triangledown tmp;
if (\sigma_0[y] = tmp) return;
\sigma_0[y] := tmp;
forall (z \in infl$_[y]$) insert z $Q$;
infl$_[y]$ := \emptyset;
return;
}
\end{verbatim}
Accordingly, $\sigma_1[y]$ is initialized with $\sigma_0[y]$, and $y$ is added to $\text{dom}_1$. As the value of $\sigma_1$ for $y$ has been updated, $y$ together with all variables in $\text{infl}[y]$ are added to the queue, whereupon $\text{infl}[y]$ is set to the empty set, and $\text{iterate}_1(n)$ is called to complete the narrowing phase up to the priority $n$. Upon termination, a call $\text{iterate}_1(n)$ for an integer $n$ has removed all variables of priority at most $n$ from the queue $Q$. In contrast to $\text{iterate}_0$, however, it may extract variables $y$ from $Q$ which have not yet been encountered in the present phase of iteration, i.e., are not yet included in $\text{dom}_1$ and thus have not yet received a value in $\sigma_1$. To ensure initialization, $\text{solve}_1(y, n)$ is called for $n = \text{prio}[y] - 1$. This choice of the extra parameter $n$ ensures that all lower priority variables have been removed from $Q$ before $\text{do}_1(y)$ is called.

It remains to explain the function $\text{do}_1(y)$. Again, it essentially behaves like $\text{do}_0(y)$—with the distinction that the narrowing operator is applied instead of the widening operator. Furthermore, the auxiliary local function $\text{eval}_0$ is replaced with $\text{eval}_1$ which now uses a call to $\text{solve}_1$ for the initialization of its argument variable $z$ (instead of $\text{solve}_0$) where the extra integer argument is given by $\text{prio}[y] - 1$, i.e., an iteration is performed to remove all variables from $Q$ with priorities lower than the priority of $y$ (not of $z$).

In light of Theorem 3, we call the algorithm from Figs. 2 and 3 terminating structured two-phase solver.

**Theorem 3** The local solver $\text{TSTP}$ from Figs. 2 and 3 when started with a call $\text{solve}_1(y_0, 0)$ for a variable $y_0$, terminates for every system of equations whenever only finitely many variables are encountered.

Upon termination, assignments $\sigma_i^y : Y_i \rightarrow \mathcal{D}, i = 0, 1$ are obtained for finite sets $Y_0 \supseteq Y_1$ of variables so that the following holds:

1. $y_0 \in Y_1$;
2. $\sigma_0^y$ is a closed partial post-solution of the abstract system (2);
3. $\sigma_1^y$ is a closed partial assignment such that $\mathcal{I} \oplus \sigma_1^y$ is a post-solution of the lower monotonization of the abstract system (2).

**Proof** Assume that only finitely many variables are encountered during the run of the algorithm, i.e., from some point neither $\text{dom}_0$ nor $\text{dom}_1$ receive new elements. Since $\text{solve}_0(y)$ is called before the variable $y$ is added to $\text{dom}_1$, and $\text{solve}_0(y)$ enforces that $y$ is included in $\text{dom}_0$, we have that $\text{dom}_1 \subseteq \text{dom}_0$ throughout the algorithm. Due to the initial call $\text{solve}_1(y_0, 0)$, $y_0$ is contained in $Y_1$ implying the first item in the list.

Variables $y$ are added into sets $\text{infl}[z]$ only during the evaluation of a call to $\text{eval}_i$ and after an appropriate call to $\text{solve}_i$—implying that $y$ is contained in $\text{dom}_i$ whenever $\text{eval}_i$ was called inside a call $\text{do}_1(y)$. Accordingly, all variables added to the priority queue necessarily are contained in $\text{dom}_0$. Thus, all variables for which $\text{do}_0$ is called at a call of $\text{iterate}_0(n)$ are all contained in $\text{dom}_0$, while all variables for which $\text{do}_1$ is called at a call of $\text{iterate}_1(n)$ are already contained in $\text{dom}_1$. Therefore, we define $Y_i = \text{dom}_i$ when the iteration has terminated for $i = 0, 1$. We claim that for every priority $n$, the following holds:

1. Every call $\text{iterate}_0(n)$ during the evaluation of $\text{solve}_0(0, y_0)$ terminates.
2. Every call $\text{iterate}_1(n)$ during the evaluation of $\text{solve}_1(0, y_0)$ terminates as well.

In order to prove the first claim, assume for a contradiction that there is some $n$ such that the call $\text{iterate}_0(n)$ does not terminate. Since $Y_0$ is finite, there must be a variable $y$ of maximal priority $\text{prio}(y) \leq n$ so that $\text{do}_0(y)$ is evaluated infinitely often. This means that from some point on, $y$ is the variable of maximal priority for which $\text{do}_0$ is called.
Let \( d_i, i \geq 0 \) denote the sequence of the new values for \( y \). We claim that for every \( i \geq 0 \), \( d_{i+1} = d_i \lor a_i \) holds for some suitable value \( a_i \). This holds if \( y \in \text{point} \) from the first evaluation onward. Clearly, if this were the case, we arrive at a contradiction, as any such widening sequence is ultimately stable. Accordingly, it remains to prove that from the first evaluation onward, \( y \) is contained in \( \text{point} \) whenever \( \text{do\_var}_0(y) \) is called. Assume for a contradiction that there is a first such call where \( y \) is not contained in \( \text{point} \). Assume that this call provided the \( i \)th value \( d_i \) for \( y \). This means that, since the last evaluation of \( f_y^x \), no query to the value of \( y \) during the evaluation of lower priority variables has occurred. Accordingly, the set \( \text{infl}[y] \) does not contain any lower priority variables, which means that no further variable is evaluated before the next call \( \text{do\_var}_0(y) \). But then this next evaluation of \( f_y^x \) will return the value \( a_i \). Subsequently, the queue \( Q \) does no longer contain variables of priority less than or equal to \( n \), and therefore the iteration would terminate—in contradiction to our assumption.

Now consider the second claim. For a contradiction now assume that there is some \( n \) so that the call \( \text{iterate}_1(n) \) does not terminate. Since every call \( \text{iterate}_0(m) \) encountered during its evaluation is already known to terminate, we conclude that there must be a variable \( y \) of priority less then or equal to \( n \) so that \( \text{do\_var}_1(y) \) is evaluated infinitely often. As before this means that from some point on, \( y \) is the variable of maximal priority for which \( \text{do\_var}_1(y) \) is called. Let \( d_i, i \geq 0 \) denote the sequence of the new values for \( y \). We claim that for every \( i \geq 0 \), \( d_{i+1} = d_i \Delta a_i \) holds for some suitable value \( a_i \). This holds if \( d_i \subseteq d_{i+1} \) and \( y \in \text{point} \) from \( i = 1 \) onward. Again, if this were the case, we arrive at a contradiction, as any such widening sequence is ultimately stable. Accordingly, it remains to prove that from the first evaluation onward, \( y \) is contained in \( \text{point} \) whenever \( \text{do\_var}_1(y) \) is called. This, however, follows by the same argument as for \( \text{iterate}_0(y) \). This completes the proof of the claim.

By the claim which we have just proven, each occurring call \( \text{iterate}_i(n) \) will terminate. From that, the termination of the call \( \text{solve}_1(y_0, 0) \) follows as stated by the theorem.

It remains to prove the remaining two assertions of the enumeration. Again, we assume that only finitely many variables are encountered in a run of the local two-phase solver when started for a variable \( y_0 \), and assume that after some call to \( \text{do\_var}_i \), no further variable is added to \( \text{dom}_0 \), and likewise no further variable is added to \( \text{dom}_1 \). In order to prove the second assertion, we prove that the following invariants hold before every call \( \text{do\_var}_i(y_1) \):

1. For every variable \( y \) in the current domain \( \text{dom}_0 \), \( \text{infl}[y] \) contains (at least) all variables \( z \notin Q \cup \{y_1\} \) whose last evaluation of \( f_y^x \) has called \( \text{eval}_i(y) \);
2. If \( y \in \text{dom}_0 \setminus (Q \cup \{y_1\}) \), then \( \sigma_0[y] \ni f_y^x(
abla \sigma_0) \);
3. If \( y \in \text{dom}_1 \setminus (Q \cup \{y_1\}) \), then \( \sigma_1[y] \ni f_y^x(T \sigma_1) \).

Here, \( \nabla \) is the variable assignment which maps each variable in \( Y_0 \) to \( \nabla \). Here, we only prove the third invariant. For that, consider a call to \( \text{do\_var}_1(y_1) \). If this is the very first call of \( \text{do\_var}_1 \) for \( y_1 \), then this occurs inside a call \( \text{solve}_1(y_1) \). Accordingly, the value \( \sigma_1[y_1] \) has been initialized to \( \sigma_0[y_1] \). Then we have, by the second invariant:

\[
\sigma_1[y_1] = \sigma_0[y_1] \ni f_y^x(T \sigma_0) \ni f_y^x(T \sigma_1)
\]

At that moment, the priority queue does not contain any variable \( y \) with priority less or equal the priority of \( y_1 \), implying that for all these \( y \), \( \sigma_1[y] \ni f_y^x(T \sigma_0) \) holds. This property is preserved by updating \( \sigma_1[y_1] \) with a value exceeding \( f_y^x(T \sigma_1) \). Accordingly, all variables \( z \) from \( \text{infl}[y_1] \) with priority less or equal to \( y_1 \) will subsequently be iterated upon with \( \text{iterate}_1 \). But since these variables \( z \) satisfy \( \sigma_1[z] \ni f_y^x(T \sigma_1) \), the invariant holds for the calls of \( \text{do\_var}_1 \) therein.
By construction, 0 is the maximal priority of any variable. \( y_0 \) receives the greatest priority. Therefore, \( \text{solve}_1(0, y_0) \) returns with an empty queue \( Q \). By the third invariant the second assertion of the theorem follows. \( \square \)

7 Terminating structured mixed-phase solving

The draw-back of the two-phase solver TSTP from the last section is that it may lose precision already in very simple situations.

Example 5 Consider the system:

\[
\begin{align*}
    y_1 &= \max(y_1, y_2) \\
    y_2 &= \min(y_3, 2) \\
    y_3 &= y_2 + 1
\end{align*}
\]

over the complete lattice \( \mathbb{N}^\infty \) and the following widening and narrowing operators:

\[
\begin{align*}
    a \triangledown b &= \begin{cases} 
    \infty & \text{if } a < b \\
    a & \text{else}
    \end{cases} \\
    a \triangledown b &= \begin{cases} 
    b & \text{if } a = \infty \\
    a & \text{else}
    \end{cases}
\end{align*}
\]

Then \( \text{solve}_0(y_1) \) detects \( y_2 \) as the only widening point resulting in

\[
\sigma_0 = \{ y_1 \mapsto \infty, y_2 \mapsto \infty, y_3 \mapsto \infty \}
\]

A call to \( \text{solve}_1(y_1, 0) \) therefore initializes \( y_1 \) with \( \infty \) implying that \( \sigma_1[y_1] = \infty \) irrespective of the fact that \( \sigma_1[y_2] = 2 \). \( \square \)

We may therefore aim at intertwining the two phases into one—without sacrificing the termination guarantee. The idea is to operate on a single variable assignment only and iterate on each variable first in widening and then in narrowing mode. In order to keep soundness, after every update of a variable \( y \) in the widening phase, all possibly influenced lower priority variables are iterated upon until all stabilize with widening and narrowing. Only then the widening iteration on \( y \) continues. If on the other hand an update for \( y \) occurs during narrowing, the iteration on possibly influenced lower priority variables is with narrowing only. The distinction between the two modes of the iteration is maintained by a flag where \text{false} and \text{true} correspond to the widening and narrowing phases, respectively. The algorithm is provided in Figs. 4 and 5.

Initially, the priority queue \( Q \) and the set \( \text{dom} \) are empty. Accordingly, the mappings \( \sigma : \text{dom} \to \mathbb{D} \) and \( \text{infl} : \text{dom} \to Y \) are also empty. Likewise, the set \( \text{point} \) is initially empty.

```c
void iterate(b, n) {
    if (Q \neq \emptyset \land \min_{prio}(Q) \leq n) {
        y := extract_{min}(Q);
        b' := do_var(b, y);
        n' := prio[y];
        if (b \neq b' \land n > n') {
            iterate(b', n');
            iterate(b, n);
        } else iterate(b', n);
    }
}
```

```c
void solve(y) {
    if (y \in \text{dom}) \text{return};
    \text{dom} := \text{dom} \cup \{y\};
    \text{prio}[y] := \text{next}_{prio}();
    \sigma[y] := \text{false};
    \text{infl}[y] := \emptyset;
    b' := do_var(\text{false}, y);
    iterate(b', \text{prio}[y]);
}
```

Fig. 4 The solver TSMP, part 1
Solving for the variable $y_0$ starts with the call `solve($y_0$). Solving for some variable $y$ first checks whether $y \in \text{dom}$. If this is the case, solving immediately terminates. Otherwise, $y$ is added to dom and receives the next available priority by means of a call to next_prio. That call should provide a value which is less than any priority of a variable in dom. Subsequently, the entries $\sigma[y]$ and $\text{infl}[y]$ are initialized to $\perp$ and the empty set, respectively, and do_var is called for the pair ($\text{false}$, $y$). The return value of this call is stored in the Boolean variable $b'$. During its execution, this call may have inserted further variables into the queue $Q$. These are dealt with by the call `iterate($b'$, $\text{prio}[y]$).

Upon termination, a call `iterate($b$, $n$) has removed all variables of priority at most $n$ from the queue $Q$. It proceeds as follows. If $Q$ is empty or contains only variables of priority exceeding $n$, it immediately returns. Otherwise, the variable $y$ with least priority $n'$ is extracted from $Q$. For ($b$, $y$), `do_var` is called and the return value of this call is stored in $b'$.

Now we distinguish several cases. If $b = \text{true}$, then the value $b'$ returned by `do_var` will necessarily be true as well. In that case, iteration proceeds by tail-recursively calling again `iterate(true, n)`. If on the other hand $b = \text{false}$, then the value $b'$ returned by `do_var` can be either true or false. If $b' = \text{false}$ or $b' = \text{true}$ and $n' = n$, then `iterate($b'$, $n$) is tail-recursively called. If, however, $b' = \text{true}$ and $n > n'$, then first a sub-iteration is triggered for ($true$, $n'$) before the main loop proceeds with the call `iterate(false, $n$).

It remains to describe the function `do_var`. When called for a pair ($b$, $y$) consisting of a Boolean value $b$ and variable $y$, the algorithm first determines whether or not $y$ is a widening/narrowing point, i.e., contained in the set point. If so, $y$ is removed from point, and the flag isp is set to true. Otherwise, isp is just set to false. Then the right-hand side $f_y^\sigma$ is evaluated and the result stored in the variable temp. For its evaluation, the function $f_y^\sigma$, however, does not receive the current variable assignment $\sigma$, but an auxiliary function eval which serves as a wrapper to $\sigma$. The wrapper function eval, when queried for a variable $z$, first calls `solve(z` to compute a first non-trivial value for $z$. If the priority of $z$ exceeds or is equal to the priority of $y$, a potential widening/narrowing point is detected. Therefore, $z$ is added to the set point. Subsequently, the fact that the value of $z$ was queried during the evaluation of the right-hand side of $y$, is recorded by adding $y$ to the set infl[z]. Finally, the value $\sigma[z]$ is returned.

Having evaluated $f_y^\sigma$ eval and stored the result in temp, the function `do_var` then decides whether to apply widening or narrowing or none of them according to the following scheme. If isp has not been set to true, no widening or narrowing is applied. In this case, the flag

```plaintext
bool do_var(b, y) {
    isp := y \in \text{point};
    point := point \setminus \{y\};
    if (solve(z) {
        point := point \cup \{z\};
        \text{infl}[z] := \text{infl}[z] \cup \{y\};
        return \sigma[z];
    }
    \text{eval}()
    \text{if (isp)}{
        if (b) temp := \sigma[y] \Delta \text{temp;}
        else if (\text{temp }\subseteq \sigma[y]) {
            temp := \sigma[y] \Delta \text{temp;}
            b' := \text{true;}
        } else temp := \sigma[y] \nabla \text{temp;}
        if (\sigma[y] = \text{temp}) \text{return true;}
        \sigma[y] := \text{temp;}
        forall (z \in \text{infl}[y]) \text{ insert } z \text{ Q;}
        \text{infl}[y] := \emptyset;
        return b';
    }...
}
```

Fig. 5 The solver TSMP, part 2
$b'$ receives the value $b$. Therefore now consider the case $isp = \text{true}$. Again, the algorithm distinguishes three cases. If $b = \text{true}$, then necessarily narrowing is applied, i.e., $tmp$ is updated to the value of $\sigma[y] \setminus tmp$, and $b'$ still equals $b$, i.e., $\text{true}$. If $b = \text{false}$ then narrowing is applied whenever $tmp \subseteq \sigma[y]$ holds. In that case, $tmp$ is set to $\sigma[y] \setminus tmp$, and $b'$ to $\text{true}$. Otherwise, i.e., if $b = \text{false}$ and $tmp \not\subseteq \sigma[y]$, then widening is applied by setting $tmp$ to $\sigma[y] \cup tmp$, and $b'$ obtains the value $\text{false}$.

In the next step, $tmp$ is compared with the current value $\sigma[y]$. If their values are equal, the value of $b'$ is returned. Otherwise, $\sigma[y]$ is updated to $tmp$. The variables in $\text{infl}[y]$ are inserted into the queue $Q$, and the set $\text{infl}[y]$ is reset to the empty set. Only then the value of $b'$ is returned.

**Example 6** Consider the system of equations from Example 5. Calling $\text{solve}$ for variable $y_1$ will assign the priorities $0, -1, -2$ to the variables $y_1, y_2$ and $y_3$, respectively. Evaluation of the right-hand side of $y_1$ proceeds only after $\text{solve}(y_2)$ has terminated. During the first update of $y_2$, $y_2$ is inserted into the set $\text{point}$, implying that at the subsequent evaluation the widening operator is applied resulting in the value $\infty$ for $y_2$ and $y_3$. The subsequent narrowing iteration on $y_2$ and $y_3$ improves these values to 2 and 3, respectively. Only then the value for $y_1$ is determined which is 2. During that evaluation, $y_1$ has also been added to the set $\text{point}$. The repeated evaluation of its right-hand side, will however, again produce the value 2 implying that the iteration terminates with the assignment  

$$\sigma = \{ y_1 \mapsto 2, y_2 \mapsto 2, y_3 \mapsto 3 \}$$

In light of Theorem 4, we call the algorithm from Figs. 4 and 5, terminating structured mixed-phase solver or TSMP for short.

**Theorem 4** The local solver TSMP from Figs. 4 and 5 when started for a variable $y_0$, terminates for every system of equations whenever only finitely many variables are encountered.

Upon termination, an assignment $\sigma^\sharp : Y_0 \rightarrow \mathbb{D}$ is returned where $Y_0$ is the set of variables encountered during $\text{solve}(\text{false}, y_0)$ such that the following holds:

- $y_0 \in Y_0$,
- $\sigma^\sharp$ is a closed partial assignment such that $\top \oplus \sigma^\sharp$ is a post-solution of the lower monotonization of the abstract system (2).

**Proof** By Theorem 4 the only condition for TSMP to terminate is that only finitely many variables are encountered. No further assumptions, e.g., w.r.t. monotonicity of right-hand sides must be made as in [2, 4]. Upon termination, the algorithm is guaranteed to return sound results. The returned variable assignment is a (partial) post-solution of the lower monotonization of the system, which means it may not necessarily be a post-solution of the original system—given that some right-hand sides are not monotonic.

Assume that only variables from the finite set $Y_0$ are encountered during the run of the algorithm. We claim that for every priority $i$, the following holds:

1. Every call $\text{iterate}(\text{true}, i)$ during the evaluation of $\text{solve}(y_0)$ terminates.
2. Every call $\text{iterate}(\text{false}, i)$ during the evaluation of $\text{solve}(y_0)$ terminates as well.

In order to prove the first claim, assume for a contradiction that there is some $i$ such that the call $\text{iterate}(\text{true}, i)$ does not terminate. Note that then any subsequent call to $\text{do\_var}$ as well as $\text{iterate}$ will always be evaluated for the Boolean value $\text{true}$. Since $Y_0$ is finite, there is a variable $y$ of maximal priority $\text{prio}(y) \leq i$ so that $\text{do\_var}(\text{true}, y)$ is evaluated infinitely often. This means that from some point on, $y$ is the variable of maximal priority for which
**do_var** is called. Let \( d_i, i \geq 0 \) denote the sequence of the new values for \( y \). We claim that for every \( i \geq 0, d_{i+1} = d_i \Delta a_i \) holds for some suitable value \( a_i \). This holds if \( y \in \text{point} \) from the first evaluation onward. Clearly, if this were the case, we arrive at a contradiction, as any such narrowing sequence is ultimately stable. Accordingly, it remains to prove that from the first evaluation onward, \( y \) is contained in \( \text{point} \)—whenever \( \text{do_var} (\text{true}, y) \) is called. Assume for a contradiction that there is a first such call where \( y \) is not contained in \( \text{point} \). Assume that this call provided the \( i \)th value \( d_i \) for \( y \). This means in particular that, since the last evaluation of \( f^y_x \), no query to the value of \( y \) during the evaluation of lower priority variables has occurred. Accordingly, the set \( \text{infl} [y] \) does not contain any lower priority variables, which means that no further variable is evaluated before the next call \( \text{do_var} (\text{true}, y) \). But then this next evaluation of \( f^y_x \) will return the value \( a_i \). Subsequently, the queue \( Q \) no longer contains variables of priority \( \leq i \), and therefore the iteration would terminate—in contradiction to our assumption.

Let us therefore now consider the second claim. For a contradiction now assume that there is some \( i \) so that the call \( \text{iterate} (\text{false}, i) \) does not terminate. Since every call \( \text{iterate} (\text{true}, j) \) encountered during its evaluation is already known to terminate, we conclude that there must be a variable \( y \) of maximal priority \( \leq i \) so that \( \text{do_var} (\text{false}, y) \) is evaluated infinitely often. As before this means that from some point on, \( y \) is the variable of maximal priority for which \( \text{do_var} (\text{false}, y) \) is called. Let \( d_i, i \geq 0 \) denote the sequence of the new values for \( y \). We claim that for every \( i \geq 0, d_{i+1} = d_i \lor a_i \) holds for some suitable value \( a_i \) where \( y \in \text{point} \) from \( i = 1 \) onward. Again, if this were the case, we arrive at a contradiction, as any such widening sequence is ultimately stable. Accordingly, it remains to prove that from the first evaluation onward, \( y \) is contained in \( \text{point} \)—whenever \( \text{do_var} (\text{false}, y) \) is called. This, however, follows by the same argument as for \( \text{iterate} (\text{true}, y) \). This completes the proof of the claim.

Now assume that only finitely many variables are encountered in a run of \( \text{TSMP} \) when started for a variable \( x \), and assume that after some call to \( \text{do_var} \), no further variable is encountered. Let \( Y_0 \) denote this set of variables. By the claim which we have just proven, each subsequent call to the function \( \text{iterate} \) will terminate. From that, the termination of the call \( \text{solve} (Y_0) \) follows as stated by the theorem.

It remains to prove the second assertion. We remark that, whenever a new variable is encountered, it is added into the set \( \text{dom} \) and never removed. Let \( Y_0 \) again denote the finite set of variables encountered during \( \text{solve} (Y_0) \), i.e., the final value of \( \text{dom} \). In particular, \( y_0 \) is contained in \( Y_0 \). In order to prove the second assertion, we note that the following invariants hold before every call \( \text{do_var} (b, y_1) \):

1. For every variable \( y \) in the current domain \( \text{dom} \), \( \text{infl} [y] \) contains (at least) all variables \( z \notin Q \cup \{ y_1 \} \) whose last evaluation of \( f^z_x \) has called \( \text{eval} (y) \);
2. If \( y \in \text{dom} \setminus (Q \cup \{ y_1 \}) \), then \( \sigma [y] \subseteq f^y_x (\top \oplus \sigma) \);
3. If \( b = \text{true} \), then \( \sigma [y_1] \subseteq f^y_{y_1} (\top \oplus \sigma) \).

Here, \( \top \) is the variable assignment which maps each variable in \( Y_0 \) to \( \top \). In order to see the second statement, we observe that iteration on a variable always starts with the flag \( \text{false} \). Now consider a call to \( \text{do_var} (\text{false}, y_1) \) where \( b' \) is set to \( \text{true} \). This is the case when \( \sigma [y_1] \supseteq \text{tmp} \) where \( \text{tmp} \) is the value of the last evaluation of the right-hand side of \( y_1 \). Accordingly,

\[
\sigma [y_1] \supseteq f^y_{y_1} (\top \oplus \sigma) \supseteq f^y_{y_1} (\top \oplus \sigma)
\]

At that moment, the priority queue does not contain any variable \( y \) with priority less or equal the priority of \( y_1 \), implying that for \( y, \sigma [y] \supseteq f^y_{y_1} (\top \oplus \sigma) \) holds. This property is preserved by updating \( \sigma [y_1] \) with a value exceeding \( f^y_{y_1} (\top \oplus \sigma) \). Accordingly, all variables \( y \) from \( \text{infl} [y_1] \)

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with priority less or equal to $y_1$ will subsequently be iterated upon with $b = \text{true}$. But since these satisfy $\sigma[z] \supseteq f^*_1(\top \oplus \sigma)$, the invariant holds for the calls of $\text{do\_var}$ therein.

By construction, $y_0$ receives the greatest priority. Therefore, $\text{solve}(y_0)$ returns with an empty queue $Q$. By the second invariant the second assertion of the theorem follows. \hfill $\square$

8 Concretization only

So far, we have assumed that we are given a Galois connection to relate concrete sets of states and abstract values. In some applications, though, such a Galois connection may not exist (see [13] for a detailed discussion). In this case, a best abstract description of a set of states may not always exist. Accordingly, the meet of two sound descriptions need no longer be a sound description. This implies that the concept of lower monotization as presented in Sect. 2 can no longer be used for proving the correctness of fixpoint algorithms. In the following we indicate how we can deal with this more general setting.

Assume thus that we are given a monotonic concretization function $\gamma : D \rightarrow C$ with $\gamma(\top) = Q$ only which provides for each element $d \in D$ the set $\gamma(d)$ described by $d$. For a non-empty subset $D \subseteq D$, we define the $\gamma$-closure $\uparrow_\gamma D$ by

$$\uparrow_\gamma D = \{d \in D | Q_D \subseteq \gamma(d)\}$$

where

$$Q_D = \bigcap\{\gamma(d') | d' \in D\}$$

Note that, according to our assumption on the value $\gamma(\top)$, $\top$ is always contained in $\uparrow_\gamma D$. We call a non-empty subset $D \subseteq D$ $\gamma$-closed, if $D = \uparrow_\gamma D$. For $D$ consider the set $\overline{D}_\gamma$ consisting of all $\gamma$-closed subsets $D \subseteq D$, i.e., $\overline{D}_\gamma = \{D \subseteq D | D = \uparrow_\gamma D\}$. We establish functions $\tilde{\alpha} : C \rightarrow \overline{D}_\gamma$ and $\tilde{\gamma} : \overline{D}_\gamma \rightarrow C$ by

$$\tilde{\alpha}(Q) := \{d \in D | Q \subseteq \gamma(d)\} \quad \text{and} \quad \tilde{\gamma}(D) := Q_D$$

With the ordering $\subseteq_\gamma$ given by $D_1 \subseteq_\gamma D_2$ iff $D_1 \supseteq D_2$, the set $\overline{D}_\gamma$ is a complete lattice where the least upper bound is given by $\bigcup_\gamma D = \bigcap D$. For the latter, we need to convince ourselves that $\bigcap D$ is indeed $\gamma$-closed:

$$\bigcap D = \{d \in D | \forall D \in D. d \in D\} = \{d \in D | \bigcup\{Q_D | D \in D\} \subseteq \gamma(d)\} = \tilde{\alpha}(\bigcup\{Q_D | D \in D\}) \in \overline{D}_\gamma$$

Accordingly, the pair $(\tilde{\alpha}, \tilde{\gamma})$ forms a Galois connection between $C$ and $\overline{D}_\gamma$.

The greatest lower bound operation $\bigcap_\gamma$ on $\overline{D}_\gamma$ exists (as $\overline{D}_\gamma$ is a complete lattice) but is more complicated than plain union of sets:

$$\bigcap_\gamma D = \uparrow_\gamma(\bigcup D)$$

as $\gamma$-closedness is not preserved under union.

Furthermore, we consider the mapping $\iota_\gamma : D \rightarrow \overline{D}_\gamma$ defined by

$$\iota_\gamma(d) = \{d' \in D | \gamma(d) \subseteq \gamma(d')\} = \uparrow_\gamma\{d\}$$
This mapping is monotonic, but not necessarily injective. Still, the following two properties hold:

1. \( y(d) = \tilde{y}(\iota_y(d)) \), and
2. \( \tilde{y}(D) \subseteq y(d) \) for all \( d \in D \).

For a function \( f : (Y \rightarrow \mathbb{D}) \rightarrow \mathbb{D} \), we introduce the \( \gamma \)-monotization \( f_\gamma : (Y \rightarrow \overline{\mathbb{D}}_\gamma) \rightarrow \overline{\mathbb{D}}_\gamma \) as the substitute of the lower monotization in case that we are given a concretization function \( \gamma \) only. The function \( f_\gamma \) is defined by

\[
\begin{align*}
  f_\gamma \sigma &= \bigcap \{ \iota_y(f(\sigma')) | \forall y \in Y. \sigma'(y) \in \sigma(y) \} \\
  &= \uparrow_y(\bigcup \{ \iota_y(f(\sigma')) | \forall y \in Y. \sigma'(y) \in \sigma(y) \}) \\
  &= \bar{\alpha}(\bigcap \{ \gamma(f(\sigma')) | \forall y \in Y. \sigma'(y) \in \sigma(y) \})
\end{align*}
\]

We notice that the function \( f_\gamma \) is monotonic where for every \( \sigma' : Y \rightarrow \mathbb{D} \),

\[
  f_\gamma(\iota_y \circ \sigma') \subseteq f_\gamma(f(\sigma'))
\]

Thus, if \( f \) is sound (w.r.t. \( R \) and \( \gamma \)) then so is \( f_\gamma \) (w.r.t. \( R \) and \( \tilde{y} \)). As the \( \gamma \)-monotization of the system (2), we therefore define the system

\[
y = f_{\gamma,Y}, \quad (y \in Y)
\]

As all right-hand sides of equations in the system (8) are monotonic, this system has a least solution \( \tilde{\sigma} : Y \rightarrow \overline{\mathbb{D}}_\gamma \) which is sound. Accordingly, any assignment \( \sigma : Y \rightarrow \mathbb{D} \) so that \( \iota_\gamma \circ \sigma \) is a post-solution of the \( \gamma \)-monotization, i.e., with \( \tilde{\sigma} \subseteq \gamma_\gamma(\iota_\gamma \circ \sigma) \), is sound as well.

Assume that the binary operators \( \triangledown \) and \( \Delta \) on \( \mathbb{D} \) satisfy the properties

\[
\begin{align*}
  \gamma(a) \cup \gamma(b) &\subseteq \gamma(a \triangledown b) \\
  \gamma(a) \cap \gamma(b) &\subseteq \gamma(a \Delta b) \subseteq \gamma(a)
\end{align*}
\]

In particular, for any \( \gamma \)-closed set \( D, a, b \in D \) implies that

\[
D \subseteq \iota_\gamma(\sigma(a) \cap \gamma(b)) \subseteq \iota_\gamma(\sigma(a \Delta b)) \subseteq \iota_\gamma(\sigma(a))
\]

holds. We additionally assume that for all sequences \( b_i, i \geq 0 \) the sequences \( a_i, i \geq 0 \) and \( a'_i, i \geq 0 \), defined by:

\[
a_{i+1} = a_i \triangledown b_i \quad \text{and} \quad a'_{i+1} = a'_i \Delta b_i
\]

both are ultimately stable for all \( a_0, a'_0 \in \mathbb{D} \). Note that the stability notion is defined relative to \( \mathbb{D} \) (and not relative to \( \overline{\mathbb{D}}_\gamma \)). Operators with these properties have already been proposed in [9] where they are called sound (w.r.t. the concretization \( \gamma \)).

**Theorem 5** Assume that the local solvers TSTP and TSMP from Sects. 6 and 7 are equipped with sound widening and narrowing operators. Upon termination, each of them returns a closed partial assignment \( \sigma \subseteq Y \rightarrow \mathbb{D} \) so that \( \iota_\gamma \circ (\bigcup \Theta \sigma) \) is a post-solution of system (8), i.e., the \( \gamma \)-lower monotization of system (2). Consequently, the mapping \( \bigcup \Theta \sigma \) is sound (w.r.t. the description relation \( R \) and the concretization \( \gamma \)).

We remark that at no point during the whole construction we had to refer to the original ordering of \( \mathbb{D} \), but instead based all our considerations onto the function \( \gamma \), the element \( \top \) and the operators \( \triangledown \) and \( \Delta \). Therefore, we may drop the assumption for the algorithms that \( \mathbb{D} \) is a complete lattice and work with ordinary partial orders, possibly extended with a maximal element \( \top \) where the start value \( \bot \) may be any element in \( \mathbb{D} \).
9 Interprocedural analysis

As seen in Example 2, the concrete semantics of programs with procedures can be formalized by a system of equations over a set of variables \( X = \{ (u, q) \mid u \in U, q \in Q \} \) where \( U \) is a finite set of program points and \( Q \) is the set of possible system states. A corresponding abstract system of equations for interprocedural analysis can be formalized using abstract variables from the set \( Y = \{ (u, a) \mid u \in U, a \in D \} \) where the complete lattice \( D \) of abstract values may also serve as the set of abstract calling contexts for which each program point \( u \) may be analyzed. The description relation \( R \) between concrete and abstract variables is then given by \( \langle u, q \rangle R \langle u, a \rangle \iff q \in \gamma(a) \) for all \( (u, q) \in X \) and \( (u, a) \in Y \) and program points \( u \in U \). Moreover, we require that for all right-hand sides \( f_x \) of the concrete system and \( f^x_Y \) of the abstract system that \( f_x q \subseteq \gamma(f^x_Y a) \) holds, whenever \( x R y \) and \( q \in \gamma(a) \). Right-hand sides for abstract variables are given by expressions \( e \) according to the following grammar:

\[
e::=d \mid \alpha \mid g^x e_1 \ldots e_k \mid \langle u, e \rangle
\]  

(9)

where \( d \in D \) denotes arbitrary constants, \( \alpha \) is a dedicated variable representing the current calling context, \( g^x : D \to \cdots \to D \) is a \( k \)-ary function, and \( \langle u, e \rangle \) with \( u \in U \) refers to a variable of the equation system. Each expression \( e \) describes a function \( [e]^x : D \to (Y \to D) \to D \) which is defined by:

\[
\begin{align*}
[d]^x a \alpha &= d \\
[g^x e_1 \ldots e_k]^x a \alpha &= g^x ([e_1]^x a \alpha) \ldots ([e_k]^x a \alpha) \\
[\alpha]^x a \alpha &= a \\
[\langle u, e \rangle]^x a \alpha &= \sigma \langle u, [e]^x a \alpha \rangle
\end{align*}
\]

A finite representation of the abstract system of equations is then given by the finite set of schematic equations

\[
\langle u, \alpha \rangle = e_u, \quad u \in U
\]  

(10)

for expressions \( e_u \). Each schematic equation \( \langle u, \alpha \rangle = e_u \) denotes the (possibly infinite) family of equations for the variables \( \langle u, a \rangle, a \in D \). For each \( a \in D \), the right-hand side function of \( \langle u, a \rangle \) is given by the function \( [e_u]^x a \). This function is indeed pure for every expression \( e_u \) and every \( a \in D \). Such systems of equations have been used, e.g., in [3,11] to specify interprocedural analyses.

Example 7 Consider the schematic system:

\[
\langle u, \alpha \rangle = \langle v, \langle v, \langle u, \alpha \rangle \rangle \rangle \sqcup \alpha \quad \langle v, \alpha \rangle = g^x \langle v, \alpha \rangle \sqcup \alpha
\]

for some unary function \( g^x : D \to D \). The resulting abstract system simulates the concrete system from Example 2, if \( g(q) \subseteq \gamma(g^x(a)) \) holds whenever \( q \in \gamma(a) \). 

As we have seen in the example, function calls result in indirect addressing via nesting of variables. In case that the program does not have recursive procedures, there is a mapping \( \lambda : U \to \mathbb{N} \) so that for every \( u \) with current calling context \( \alpha \), right-hand side \( e_u \) and every subexpression \( \langle u', \epsilon \rangle \) of \( e_u \) the following holds:

- If \( \lambda(u') = \lambda(u) \), then \( \epsilon' = \epsilon \);
- If \( \lambda(u') \neq \lambda(u) \), then \( \lambda(u') < \lambda(u) \).

If this property is satisfied, we call the equation scheme stratified where \( \lambda(u) \) is the level of \( u \). Intuitively, stratification means that a new context is created only for some point \( u' \) of a strictly lower level. For the interprocedural analysis as formalized, e.g., in [3], all program points of a given procedure may receive the same level while the level decreases whenever...
another procedure is called. The system from Example 7 is stratified: we may, e.g., define \( \lambda(u) = 2 \) and \( \lambda(v) = 1 \).

**Theorem 6** The solver TSTP as well as the solver TSMP terminate for stratified equation schemes.

**Proof** We only consider the statement of the theorem for solver TSMP. Assume we run the solver TSMP on an abstract system specified by a stratified equation scheme. In light of Theorem 4, it suffices to prove that for every \( u \in U \), only finitely many contexts \( a \in D \) are encountered during fixpoint computation. First, we note that variables \( \langle v, a \rangle \) may not be influenced by variables \( \langle u, a' \rangle \) with \( \lambda(u) > \lambda(v) \). Second, let \( D_{v,a} \) denote the set of variables \( \langle u, a' \rangle \) with \( \lambda(u) = \lambda(v) \) onto which \( \langle v, a \rangle \) may depend. Then all these variables share the same context. We conclude that new contexts for a point \( v \) at some level \( k \) are created only by the evaluation of right-hand sides of variables of smaller levels. For each level \( k \), let \( U_k \subseteq U \) denote the set of all \( u \) with \( \lambda(u) \leq k \). We proceed by induction on \( k \).

Assume that we have proven termination for all calls \( \text{solve} \langle u, a' \rangle, \lambda(u) < k \) for any subset of variables \( \langle u', a'' \rangle \) which have already been solved. Then evaluating a call \( \text{solve} \langle v, a \rangle \) with \( \lambda(v) = k \) will either query the values of other variables \( \langle v', a' \rangle \) where \( \lambda(v') = k \). In this case, \( a' = a \). Therefore, only finitely many of these are encountered. Or variables \( \langle v', a' \rangle \) are queried with \( \lambda(v') < k \). For those which have not yet been encountered \( \text{solve} \langle v', a' \rangle \) is called. By induction hypothesis, all these calls terminate and therefore query only finitely many variables. As the evaluation of \( \text{solve}(\langle v, a \rangle) \) encounters only finitely many variables, it terminates.

\[ \Box \]

A similar argument explains why interprocedural analyzers based on the functional approach of Sharir/Pnueli [1,26] terminate not only for finite domains but also for full constant propagation—if only the programs are non-recursive.

### 10 Experimental evaluation

We implemented the solvers TSTP and TSMP presented in Sects. 6 and 7 within the analysis framework GOBLINT.\(^1\) For that, these solvers have been extended to deal with side-effects (see [3] for a detailed discussion of this mechanism) to jointly deal with flow- and context-sensitive and flow-insensitive analyses. In order to perform a fair comparison of the new solvers with warrowing-based local solving as proposed in [2,4], we provided a simplified version of TSMP. This simplified solver performs priority based iteration in the same way as TSMP but uses the warrowing operator instead of selecting operators according to extra flags. These three solvers were evaluated on the SPECint benchmark suite\(^2\) consisting of not too small real-world C programs (1600 to 34,000 LOC). Furthermore, the following C programs where analyzed: ent,\(^3\) figlet,\(^4\) maradns,\(^5\) wget,\(^6\) and some programs from the coreutils\(^7\) package. The analyzed program wget is the largest one with around 77,000 LOC@.

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1. http://goblint.in.tum.de/.
2. https://www.spec.org/cpu2006/CINT2006/.
3. http://www.fourmilab.ch/random/ (version 28.01.2008).
4. http://www.figlet.org/.
5. http://www.maradns.org/.
6. https://www.gnu.org/s/wget/.
7. https://www.gnu.org/s/coreutils/.

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The analyses which we performed are put on top of a basic analysis of pointers, strings and enums. For `enum` variables, `sets` of possible values are maintained. The benchmark programs were analyzed with full context-sensitivity of local data while globals were treated flow-insensitively.

The experimental setting is a fully context-sensitive interval analysis of `int` variables. However, the programs 482.sphinx and `wget` recur over `int` arguments. Note that for the conference version [24] of this paper `wget` was analyzed even with intervals in contexts. However, since then, we improved the precision of `Goblint` rendering the analysis with intervals in contexts non-terminating. Therefore, we had to exclude intervals in contexts for `wget`, and for 482.sphinx also the `enum` values.

Interestingly, the `warrowing` solver behaves exactly the same as `TSMP` on all of our benchmark programs. We interpret this by the fact that for the given analysis the right-hand sides are effectively monotonic. Accordingly, Fig. 7 only reports the relative precision of `TSTP` compared to `TSMP`.

Figure 6 compares the solvers `TSMP` and `TSTP` in terms of space and time. For a reasonable metric for space we choose the total number of variables (i.e., occurring pairs of program points and contexts) and for time the total number of evaluations of right-hand sides of a corresponding variable. The table indicates that the solver `TSMP` requires only around half of the variables of the solver `TSTP`. Interestingly, the percentage of evaluations of right-hand sides in a run of `TSMP` is still around 60–70% of the solver `TSTP`.

In the second experiment, as depicted by Fig. 7, we compare the precision of the two solvers. For a reasonable metric we only compare the values of variables which occur in both solver runs. As a result, between 80 and 95% of the variables receive the same value. It is remarkable that the mixed phase solver is not necessarily more precise. In many cases, an increase in precision is observed, yes—there are, however, also input programs where the two-phase solver excels.

In practice also the wall-clock time and the memory footprint are of interest. Benchmarking was done on an Intel Xeon CPU E3-1270 v3 (3.50 GHz) with 32 GB of RAM. The run-times of the analyses varied from less than ten seconds up to almost two hours.
Fig. 7 Precision of TSMP versus TSTP. Percentage of variables occurring during a run of the solvers. a For which TSMP and TSTP compute the same value; b for which TSMP computes more precise results then TSTP; c for which TSTP computes more precise results then TSMP; d for which the results computed by TSMP and TSTP are incomparable. (Color figure online)

benchmark 401.bzip2 had the maximal wall-clock time and was analyzed in 113 minutes by the solver TSTP and in 58 minutes by the solver TSMP. The maximal memory footprint also varied from less then 100 MB up to 2.4 GB.

11 Conclusion

We have presented local solvers which are guaranteed to terminate for all abstract systems of equations given that only finitely many variables are encountered—irrespective of whether right-hand sides of the equations are monotonic or not or whether the complete lattice has infinite strictly ascending/descending chains or not. Furthermore, we showed that interprocedural analysis with partial tabulation of procedure summaries based on these solvers is guaranteed to terminate with the only assumption that the program has no recursive procedures. Clearly, theoretical termination proofs may only give an indication that the proposed algorithms are well-suited as fixpoint engines within a practical analysis tool. Termination within reasonable time and space bounds is another issue. The numbers provided by our preliminary practical experiments within the analysis framework GOBLINT seem encouraging. Interestingly, a direct comparison of the two-phase versus mixed-phase solver for full context-sensitive interprocedural analysis, indicated that TSMP was virtually always faster, while the picture w.r.t. precision is not so clear. Also, the new solvers always returned post-solutions of the abstract systems—although they are not bound to do so.

In case that widening and narrowing were defined relative to the ordering on \( \mathbb{D} \), our correctness proofs required a Galois connection to relate the concrete with the abstract domain. In Sect. 8, however, we showed that our algorithms remain correct when widening and narrowing operators are used which are sound w.r.t. the concretization. In this case, the requirement of a Galois connection can be dropped. What still remains to be explored is to extend two phased approaches to a many-phase iteration as suggested in [9].
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