Modified Excluded Volume Hadron Resonance Gas Model with Lorentz Contraction

Somenath Pal\textsuperscript{1,*}, Abhijit Bhattacharyya\textsuperscript{1,†} and Rajarshi Ray\textsuperscript{2‡}

\textsuperscript{1}Department of Physics, University of Calcutta, 92, A.P.C. Road, Kolkata-700009, India
\textsuperscript{2}Center for Astroparticle Physics \\& Space Science, Block-EN, Sector-V, Salt Lake, Kolkata 700091, India
\& Department of Physics, Bose Institute, 93/1, A. P. C Road, Kolkata 700009, India

In this work we discuss a modified version of Excluded Volume Hadron Resonance Gas model (MEVHRG) and also study the effect of Lorentz contraction of the excluded volume on scaled pressure and susceptibilities of conserved charges. We take four different sets of hadronic radii and compare their results. We find that a larger variety of radii for hadrons enlarges the difference between the results of MEVHRG model and EVHRG model.

* somenathpal1@gmail.com
† abhattacharyyacu@gmail.com
‡ rajarshi@jcbose.ac.in
I. INTRODUCTION

Studies of strongly interacting matter at high temperatures and/or high densities have been of great interest for some decades. In the early universe, a few microseconds after the Big Bang, strongly interacting matter is expected to have existed in the color charge deconfined quark-gluon phase [1]. On the other hand, dense strongly interacting matter can be found inside neutron stars [2]. There are several ongoing and up-coming experiments with ultra-relativistic heavy ion collision that are recreating such phases of strongly interacting matter. Among the experimental programmes, Large Hadron Collider (LHC) at CERN, Geneva and Relativistic Heavy Ion Collider (RHIC) at Brookhaven, New York, have already enriched lot of our understanding in this direction. The ongoing experiments in these facilities as well as the upcoming facilities at GSI, Darmstadt and at JINR, Dubna, are expected to further expand our knowledge about the various facets of strongly interacting matter at high temperature and density.

The experimental investigations have been very well supported by significant advancements of the theoretical approaches. One of the main objectives of these explorations is to understand the thermodynamic phases as well as the phase diagram of strong interactions at high temperatures and high densities. At high temperature and low density, the phase boundary between hadronic and quark-gluon matter is found to be a crossover [3, 4], while at low temperature and high density there is possibly a first order phase transition [5–10]. Thereby the possibility of observing signatures of a critical end point (CEP) [11–14] has become an extremely active field of research.

Though quantum chromodynamics (QCD) is the theory of strong interactions, the traditional perturbative methods of field theory is inadequate because the strong interaction coupling may not be small for the temperatures and densities concerned. Lattice QCD (LQCD) is the most important non-perturbative tool that can describe strongly interacting matter at high temperatures [3, 15–29]. However, the Monte Carlo techniques of LQCD cannot be applied to a system with finite baryon chemical potential $\mu_B$, as the fermion determinant becomes complex. However, the Taylor expansion of thermodynamic quantities around $\mu_B = 0$, for a given temperature $T$, can be used until $\mu_B$ is close to a phase boundary. For this reason, people build effective models to study properties of strongly interacting matter in nonperturbative domain. Some examples of such models are Polyakov Loop Extended Nambu-Jona-Lasinio (PNJL) Model [30–46], Hadron Resonance Gas (HRG) Model [47–56], Polyakov-Quark-Meson (PQM) model [57–60], Chiral Perturbation Theory [61], etc., which successfully describe some aspects of strongly interacting matter.

HRG model is used to describe a system of dilute gas consisting of hadrons and is based on Dashen, Ma, Bernstein theorem [62]. In this model attractive interactions among hadrons are taken care of by considering the unstable resonance particles as stable particles. HRG model successfully describes the experimental data of a system at freeze out [63]. However, repulsive interaction among the particles in the hadronic system is also important [64]. This is taken into account in the modified version of HRG model, namely, Excluded Volume Hadron Resonance Gas (EVHRG) model where repulsive interaction comes into play due to finite excluded hardcore volume of the particles [65–71].

Fluctuations of conserved charges are useful indicators of phase transition between hadronic and quark gluon plasma phase. Existence of CEP can also be indicated by divergent fluctuations. One can calculate charge susceptibilities which are related to fluctuations via fluctuation-dissipation theorem. If net baryon number of the system is small then
transition from hadronic to QGP phase is continuous and fluctuations are expected not to show singular behavior. On the other hand Lattice QCD calculation shows that at small chemical potentials, susceptibilities show rapid increment near the crossover region. Higher order susceptibilities are considered to be more sensitive to phase transition. Fluctuation of conserved charges has been studied in Ref. [72–75].

In this work we present a modified version of EVHRG model and also include the effect of Lorentz contraction on excluded volume of hadrons since, LQCD inherently takes care of Lorentz contraction. The motivation of this work is as follows. The Polyakov enhanced chiral quark models have satisfactorily described the first principle lattice QCD data above the crossover temperature $T_c$ [30, 39–41]. Below $T_c$, these models in the mean field framework were found to be insufficient especially in describing the susceptibilities. Introduction of a hybrid model, with HRG along with the PNJL model, seemed to replicate the lattice data much better [74]. Similarly a beyond mean field motivated approach with the PNJL model also described satisfactorily the bulk thermodynamic and fluctuation properties [76]. Here we would like to explore the effects of finite size of the hadrons up to a limited range of temperatures and compare with the lattice data.

This paper is organised as follows. In the next section we briefly discuss the Hadron Resonance Gas (HRG) model and its interacting version Excluded Volume Hadron Resonance Gas (EVHRG) model briefly. We also introduce a modified version of EVHRG model, namely, MEVHRG model and its extended version Lorentz contracted MEVHRG model or LMEVHRG model. In section (III) we discuss the pressure of the system within these four models and compare them with LQCD results. In section (IV) we present a brief introduction of fluctuation and susceptibilities of conserved charges and also discuss our findings on these quantities. Finally, in section (V) we conclude our findings.

## II. HADRON RESONANCE GAS MODEL

Here we present a brief discussion of Hadron Resonance Gas (HRG) model and its interacting version namely Excluded Volume Hadron Resonance Gas (EVHRG) Model. More details about these models can be found in Refs. [47–55, 64–71].

### A. Pure HRG

In HRG model, a dilute system of strongly interacting matter is considered as a gas of free resonances. The attractive interactions are taken care of by considering all the resonances as stable particles [62].

The grand canonical partition function is $Z^{id}$.

$$\ln Z^{id} = \sum_i \ln Z^{id}_i$$

Here sum runs over all hadron species and "$id$" in superscript refers to ideal gas.

$$\ln Z^{id}_i = \pm \frac{V g_i}{2\pi^2} \int_0^\infty p^2 \, dp \ln \{1 \pm \exp[-(E_i - \mu_i)/T]\}$$

Here $V$ is the volume of the system, $g_i$ is the degeneracy factor, $p$ is the momentum of a particle, $E_i = \sqrt{p^2 + m_i^2}$ is the energy of a single particle, $m_i$ is the mass of particle species
$i$, $T$ is the temperature, $\mu_i = B_i \mu_B + S_i \mu_S + Q_i \mu_Q$ is the chemical potential of particle species $i$, $B_i, Q_i, S_i$ are baryon number, electric charge and strangeness of $i$'th hadron species respectively and $\mu$'s are respective chemical potentials, + sign is for fermions and - sign is for bosons.

From partition function we can calculate various thermodynamic quantities like pressure $P_i$, number density $n_i$ as follows

$$P_i^{id} = \frac{T}{V} \ln Z_i^{id} = \frac{T g_i}{2 \pi^2} \int_0^{\infty} p^2 dp \ln\{1 \pm \exp[-(E_i - \mu_i)/T]\}$$  \hspace{1cm} (3)\

$$n_i^{id} = \frac{T}{V} \left( \frac{\partial \ln Z_i^{id}}{\partial \mu_i} \right)_{V,T} = \frac{g_i}{2 \pi^2} \int_0^{\infty} \frac{p^2 dp}{\exp[(E_i - \mu_i)/T]} \pm 1$$  \hspace{1cm} (4)

These equations are called equation of state (EOS) of the system.

### B. EVHRG Model

In EVHRG Model, a short range hardcore repulsive hadron-hadron interaction is taken into account by considering excluded volume \[66, 67, 69\] of the hadrons.

In EVHRG model pressure is given as

$$P(T, \mu_1, \mu_2, ...) = \sum_i P_i^{id}(T, \hat{\mu}_1, \hat{\mu}_2, ...)$$  \hspace{1cm} (5)

where $\hat{\mu}_i$ is the effective chemical potential for $i$-th particle species which can be written as

$$\hat{\mu}_i = \mu_i - V_{ev,i} P(T, \mu_1, \mu_2, ...)$$  \hspace{1cm} (6)

where $V_{ev,i} = \frac{4}{3} \pi R_i^3$ is the excluded volume for $i$-th hadron having hard-core radius $R_i$.

Above two equations are iteratively solved to get the pressure. Since effective chemical potential $\hat{\mu}_i$ is smaller than chemical potential $\mu_i$, pressure $P(T, \mu_1, \mu_2, ...)$ is smaller than ideal gas pressure $P^{id}$. From equations (5) and (6), we can calculate various thermodynamic quantities like number density ($n_i^E$) as

$$n_i^E = n_i^{id}(T, \mu_1, \mu_2, ...) = \frac{\partial P}{\partial \mu_i} = \frac{n_i^{id}(T, \hat{\mu}_i)}{1 + \sum_k V_{ev,k} n_k^{id}(T, \hat{\mu}_k)}$$  \hspace{1cm} (7)

### C. Modified EVHRG Model

As mentioned earlier that various authors have discussed a varying size distribution for various hadron species [73]. To analyze the effect of unequal size of different hadron species, we take recourse to virial expansion method. Such an analysis is done in Ref. [77, 78]. In this analysis, excluded volume for a single particle is taken to be half the volume excluded by two touching spheres of unequal radii instead of equal radii as taken in the previous EVHRG analysis. We name this Model as Modified Excluded Volume Hadron Resonance Gas Model (MEVHRG model).
The second virial coefficients of particle \( k \) and \( n \) are

\[
a_{kn} = \frac{2}{3} \pi (R_k + R_n)^3 = \frac{2}{3} \pi (R_k^3 + 3R_k^2R_n + 3R_kR_n^2 + R_n^3)
\] (8)

Then pressure of the system becomes [78]

\[
P^M = T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left[ 1 - \frac{4}{3} \pi R_k^3 \sum_{n=1}^{N} \phi_n e^{\frac{\mu_n}{T}} - 2\pi R_k^2 \sum_{n=1}^{N} R_n \phi_n e^{\frac{\mu_n}{T}} - 2\pi R_k \sum_{n=1}^{N} R_n^2 \phi_n e^{\frac{\mu_n}{T}} \right]
\] (9)

where \( \phi_k = g_k |S_K| \int \frac{d^4p}{(2\pi)^4} \exp\left[-\frac{\sqrt{p^2 + m^2}}{T}\right] \) is thermal density of particles and 'M' stands for MEVHRG model. Here \( |S_K| \) is the strangeness suppression factor and \( |S_K| \) is the number of valence strange quarks and antiquarks in that particular hadron. We make an approximation that surface and curvature terms in the above expression which are proportional to \( R_k^2 \) and \( R_k \) respectively are equal and then we get

\[
P^M \simeq T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left[ 1 - \frac{4}{3} \pi R_k^3 \sum_{n=1}^{N} \phi_n e^{\frac{\mu_n}{T}} - 4\pi R_k^2 \sum_{n=1}^{N} R_n \phi_n e^{\frac{\mu_n}{T}} \right]
\] (10)

To facilitate an iterative algorithm for obtaining \( P^M \), we replace \( \phi_n e^{\frac{\mu_n}{T}} \simeq \frac{P_n}{T} \)

\[
P^M \simeq T \sum_{k=1}^{N} \phi_k e^{\frac{\mu_k}{T}} \left[ 1 - \frac{4}{3} \pi R_k^3 \frac{P^M}{T} - 4\pi R_k^2 \sum_{n=1}^{N} \frac{R_n P_n}{T} \right]
\] (11)

For \( \mu_k/T < 1 \) we further approximate

\[
P^M \simeq T \sum_{k=1}^{N} \phi_k \exp \left[ \frac{\mu_k}{T} - \frac{4}{3} \pi R_k^3 \frac{P^M}{T} - 4\pi R_k^2 \sum_{n=1}^{N} \frac{R_n P_n}{T} \right]
\] (11)

Eqn. (11) can be written as

\[
P^M = \sum_{k=1}^{N} P_k(T, \tilde{\mu}_k^M)
\] (12)

where

\[
\tilde{\mu}_k^M = \mu_k - \frac{4}{3} \pi R_k^3 P^M - 4\pi R_k^2 \sum_{n=1}^{N} R_n P_n
\] (13)

Eqn. (12) and (13) can be solved iteratively to get pressure.

Number density of i-th hadron is given as

\[
n_i^M = \frac{1 + \sum_k 4\pi R_k^3 n_{ki}(T, \tilde{\mu}_k^M)) n_{ij}(T, \tilde{\mu}_j^M) - \sum_j 4\pi R_j^3 n_{ij}(T, \tilde{\mu}_j^M) R_i n_{ij}(T, \tilde{\mu}_i^M)}{(1 + \sum_k 4\pi R_k^3 n_{ki}(T, \tilde{\mu}_k^M)))(1 + \sum_j 4\pi R_j^3 n_{ij}(T, \tilde{\mu}_j^M) - \sum_j 4\pi R_j^3 n_{ij}(T, \tilde{\mu}_j^M))}
\] (14)
D. Effect of Lorentz Contraction (LMEVHRG Model)

In a hadron gas, the constituting particles can have large kinetic energies and hence they can have large velocities. So the excluded volume of a particular particle in the rest frame of the thermal medium is Lorentz contracted. Consideration of Lorentz contraction has also been discussed in Ref. [77]. This weakens the excluded volume repulsive interaction compared to the case where the effect of Lorentz contraction is not taken into account. In this paper we take into account the effect of Lorentz contraction on the excluded volume of the particles and compare the results with Lattice QCD results. If a particle has momentum $p$, its size is contracted in the direction of motion according to the formula $R' = R\sqrt{1 - \frac{v^2}{c^2}}$ where $v$ is the velocity of a particle. Now, $p = mv = \frac{m_0}{\sqrt{1 - v^2}}v$ where $m_0$ is the rest mass of a particle. From this we get $1 - v^2 = \frac{m_0^2}{p^2 + m_0^2}$

So, the effective chemical potential with Lorentz contraction becomes

$$\hat{\mu}_k^{LM} = \mu_k - \frac{4}{3}\pi R_k^3 \frac{m_0}{\sqrt{p^2 + m_0^2}} P^{LM} - 4\pi R_k^2 \frac{m_0}{\sqrt{p^2 + m_0^2}} \sum_{n=1}^{N} R_n P_n^{LM}$$

(15)

Here 'LM' in the superscript stands for Lorentz Contracted Modified EVHRG Model.

We shall now explore the effects of these various possible modifications of the HRG model to take into account the effects of finite size of the hadrons.

III. PRESSURE

In this work we have used 4 different sets of radii of hadrons and have compared their results. The first set corresponds to those in Ref. [79] where the freeze-out data was reproduced. The other three sets were chosen generally to check which radii affects which quantity the most. The radii used in these sets are mentioned in the following table. Here $R_\pi$=radius of pions, $R_m$=radius of mesons, $R_b$=radius of baryons, $R_k$=radius of kaons, $R_\Lambda$=radius of lambda particles.

| Set | Radii used |
|-----|------------|
| (a) | $R_\pi = 0.15 fm, R_m = 0.42 fm, R_b = 0.365 fm, R_k = 0.395 fm, R_\Lambda = 0.085 fm$ |
| (b) | $R_\pi = 0.3 fm, R_m = 0.2 fm, R_b = 0.3 fm$ |
| (c) | $R_\pi = 0.4 fm, R_m = 0.2 fm, R_b = 0.3 fm$ |
| (d) | $R_\pi = 0.5 fm, R_m = 0.2 fm, R_b = 0.3 fm$ |
In Fig 1, we have shown the scaled pressure \((P/T^4)\) as a function of temperature for HRG, EVHRG, MEVHRG and LMEVHRG models for four sets of hadron radii. We take radii set (a) because this set fits AGS, SPS and RHIC hadron yield ratios the best [79]. We also choose some simple set of radii namely (b), (c) and (d) with radii for mesons \(R_m=0.2\) fm and that for baryons \(R_b=0.3\) fm but various choices of pion radii. We have taken into account all particles listed in particle data book up to 3 GeV mass. The pressure corresponding to pure HRG is maximum. When we include hardcore repulsive interaction, there is a reduction in scaled pressure. Effect of repulsive interaction is prominent after \(T=0.11\) GeV for set (a), \(T=0.06\) GeV for sets (b), (c) and (d).

Scaled pressure for MEVHRG model is greater than that for EVHRG model since the effective chemical potential in MEVHRG case is larger than the EVHRG case. Scaled pressure for LMEVHRG model is even higher because in this case repulsive interaction is weaker. Scaled pressure for EVHRG and MEVHRG differ only slightly from each other. This difference is somewhat more prominent for set (a) and negligible for sets (b), (c) and
(d). This shows that the effect of unequal radii becomes more and more significant as one takes different values of radii for each different hadron species instead of using the same radius for all the mesons and the same radius for all the baryons only.

In this section we have found that the HRG model seems to give the best fit for pressure. However, we would like to look at other thermodynamic quantities also, like fluctuations and correlations, in the light of different variants of HRG which we will do in the next section.

IV. FLUCTUATIONS OF CONSERVED CHARGES

Fluctuations of conserved charges like net baryon number, electric charge, strangeness are useful indicators of thermalisation and hadronization of matter produced in ultra relativistic heavy ion collision [80–82]. Large fluctuations in various thermodynamic quantities are important signatures of existence of Critical End Point (CEP) in the phase diagram.

Susceptibilities are defined as derivatives of grand canonical partition function $Z$. The $n$th order susceptibility is defined as

$$\chi^n_x = \frac{1}{VT^3} \frac{\partial^n(ln Z)}{\partial (\mu_x T)^n}$$

where $\mu_x$ is the chemical potential for conserved charge $x$, where $x$ may be B(baryon number), Q(electric charge) or S(strangeness). We calculate pressure at various chemical potentials around $\mu_x = 0$ and find out the charge susceptibilities by fitting the data into Taylor expansion series.

We have shown in Figs. (2)-(7) second and fourth order susceptibilities of the conserved charges around zero chemical potentials ($\mu_B = \mu_Q = \mu_S=0$) for different radii combinations of hadrons and compare them with LQCD results. From the plots one can see that susceptibilities for all the four models are almost same at low temperatures. As temperature increases, deviations among various models become prominent. For HRG model where no repulsive interaction is included, susceptibilities of all orders increase rapidly with temperature. Susceptibility for HRG Model is greater than corresponding EVHRG (where hardcore repulsion is included) case. Susceptibilities for MEVHRG model are little bit higher or lower compared to EVHRG case depending on the choice of radii of hadrons and also on the type of conserved charge. It is seen that $\chi^n_Q > \chi^n_S > \chi^n_B$. This is expected since susceptibilities at a certain temperature is governed by the dominating hadrons at that temperature carrying that charge. Since pions (lightest electrically charged hadrons) are lighter than kaons (lightest strange hadrons) which is lighter than protons (lightest baryons), hence the result.
FIG. 2. Second order susceptibility for baryon number

In Fig. 2 we have shown second order baryon susceptibility for different radii combinations of hadrons at zero chemical potentials ($\mu_B = \mu_Q = \mu_S = 0$) and compare them with LQCD results given in Ref. [83]. It is seen that results of all the four model for $\chi_B^2$ coincide up to $T = 0.12$ GeV. The EVHRG, MEVHRG and LMEVHRG results don't show significant difference up to somewhat higher temperature. Difference between EVHRG and MEVHRG models is most prominent for Fig. 2(a) and least prominent for Fig. 2(d). It is seen that a larger variety of radii shifts the results of MEVHRG model up to a greater extent than EVHRG model. Here figure 2(b) seems to have the best match with LQCD data among all four plots. The dependence on pion radius comes through equations (5) and (6) where it is seen that an indirect dependence of meson radii on $\chi_B^2$ is present through the effective
In Fig. 3 we have shown second order electric charge susceptibility for different radii combinations of hadrons at zero chemical potentials ($\mu_B = \mu_Q = \mu_S=0$) and compare them with LQCD results given in Ref. [83]. It is seen that in Fig. 3(a), results of all the four models for $\chi_Q^2$ coincide up to $T=0.1$ GeV. Other three figures show different results for different models from sufficiently low temperature. For Figs. 3(b), 3(c) and 3(d), EVHRG and MEVHRG values deviate significantly from each other at medium range of temperature. The LMEVHRG data in sets (b) and (c) has the best match with LQCD data among all
four plots. The difference in the radius of pions plays the most significant role to determine $\chi_Q^2$.

![Graphs showing second order strangeness susceptibility](image)

**FIG. 4.** Second order susceptibility for strangeness.

In Fig. 4 we have shown second order strangeness susceptibility for different radii combinations of hadrons at zero chemical potentials ($\mu_B = \mu_Q = \mu_S = 0$) and compare them with LQCD results given in Ref. [83]. It is seen that results of all the four model for $\chi_S^2$ coincide upto $T=0.11$ GeV. Difference between EVHRG and MEVHRG models is most prominent for Fig. 4(a) and almost same for other three figures. As we see that $\chi_S^2$ deviates from LQCD data much more than that of $\chi_B^2$ and $\chi_Q^2$. For $\chi_S^2$, the variation of pion radius is...
Insignificant. Rather the kaon is the most dominant contributor to this susceptibility. The difference between the results of different radii sets seem to favour a smaller kaon radius.

FIG. 5. Fourth order susceptibility for baryon number.

In Fig. 5 we have shown the fourth order baryon susceptibility for different radii combinations of hadrons at zero chemical potentials (μ_B = μ_Q = μ_S=0) and compare them with LQCD results given in Ref. [83]. It is seen that all the four model results for χ_B^4 coincide upto T=0.12 GeV. The EVHRG, MEVHRG and LMEVHRG results don't show significant difference upto somewhat higher temperature. Difference between EVHRG and MEVHRG models is most prominent for Fig. 5(a) but not so much for other three figures. It is seen that a larger variety of radii shifts the results of MEVHRG model upto a greater extent than
EVHRG model. Fig. 5(a) has the best match with LQCD data among all the four plots, up to temperature 0.16 GeV. For all the four sets, $\chi^4_B$ increases gradually with temperature, reaches a maximum and then starts to decrease. Thus the interacting HRG models have an excellent qualitative behavior when compared to LQCD data.

In Fig. 6 we have shown fourth order electric charge susceptibility for different radii combinations of hadrons at zero chemical potentials ($\mu_B = \mu_Q = \mu_S=0$) and compare them with LQCD results given in Ref. [83]. It is seen that all the four model results for $\chi^4_Q$ coincide up to $T=0.1$ GeV. The pion is the most significant contributor to $\chi^4_Q$. Therefore the other three sets show different results from sufficiently low temperature. For Figs. 6(b), 6(c) and
6(d), EVHRG and MEVHRG plots deviate significantly from each other at medium range of temperature. Here again the qualitative behavior of the results obtained in interacting HRG models have the same features as that from LQCD.

From the study of fluctuations we note that the quantitative results of pure HRG model is too far from the LQCD data even for low temperature unlike pressure. It is only the interacting HRG models that can approximately reproduce the Lattice data for $\chi^4_Q$.

In Fig. 7 we have shown fourth order strangeness susceptibility for different radii combinations of hadrons at zero chemical potentials ($\mu_B = \mu_Q = \mu_S = 0$) and compare them with LQCD results given in Ref. [83]. It is seen that all the four model results for $\chi^4_S$ coincide upto
T=0.11 GeV. The difference between set (a) and the others is clearly due to the different radii of kaon. The change of pion radius has almost no effect.

Though LQCD data for scaled pressure shows good agreement with HRG results, second order baryon susceptibility and electric charge susceptibility are significantly less than HRG data at high temperatures. This indicates that excluded volume correction is required to explain such behavior.

Fourth order baryon susceptibility and electric charge susceptibility show a sign of saturation at high temperatures much like LQCD results though they do not match exactly. Such behaviour is indeed expected if one takes into account excluded volume correction.

V. CONCLUSION AND DISCUSSION:

In this work we have presented a modified version of EVHRG model where we have taken into account two additional things namely, (1) effect of unequal radii for excluded volume of different particle species and (2) Lorentz contraction of excluded volume, which have not been investigated within this framework before. We conclude the following important things:

(i) Strangeness susceptibilities in LQCD are higher than those in HRG because there are some strange baryons undiscovered experimentally which are predicted by quark model and also seen in LQCD spectrum.

(ii) We find that a larger variety of particle radii gives better qualitative agreement with LQCD data for most of the quantities shown in the plots. So, to improve the result even more, one can try with different radii for all the particle species.

(iii) Including Lorentz contraction has significant impact on pressure and susceptibilities as can be seen from our work and this is also important since LQCD inherently takes Lorentz contraction into account.

(iv) The aim of this work is not to get exact agreement with LQCD results but to see what important features of different thermodynamic quantities can be found if we do some important upgradation in the EVHRG model. That is why we have compared our results for four different sets of hadron radii.

(v) Better agreement with LQCD data can be achieved by taking into account other modifications like temperature dependent particle mass instead of constant mass as used in this work and also some hybrid model with both HRG and quarks. We plan to do these in our future work.

Recently there is a renewed interest in studying multiplicity data \cite{84, 85}. However, a more prudent approach has been followed in Ref. \cite{86} where HRG model has been studied in the light of both multiplicity and fluctuation data. The authors have treated temperature and chemical potential as parameters and tried to fit those by fitting the data using non-interacting HRG model. On the other hand, we have studied different variants of HRG model and tried to find out which model fits the data best.

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1. E. W. Kolb and M. S. Turner, *The Early Universe*, in Frontiers in Physics, D. Pines (Ed.), Perseus Books Group (Pub.) (1994).
2. K. Rajagopal and F. Wilczek, in *The frontier of particle physics / Handbook of QCD*, M. Shifman (Ed.), 3, 2061 (2000).
3. Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, Nature (London) **443**, 675 (2006).
4. F. R. Brown et al., Phys. Rev. Lett. **65**, 2491 (1990).
5. S. Ejiri, Phys. Rev. D **78**, 074507 (2008).
6. R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).
7. M. Asakawa and K. Yazaki, Nucl. Phys. A **504**, 668 (1989).
8. M. A. Halasz, A. D. Jackson, R. E. Shrock, M. A. Stephanov, and J. J. M. Verbaarschot, Phys. Rev. D **58**, 096007 (1998).
9. Y. Hatta and T. Ikeda, Phys. Rev. D **67**, 014028 (2003).
10. E. S. Bowman and J. I. Kapusta, Phys. Rev. C **79**, 015202 (2009).
11. Z. Fodor and S. D. Katz, J. High Energy Phys. **04** 050 (2004).
12. M. A. Stephanov, Prog. Theor. Phys. Suppl. **153**, 139 (2004).
13. R. V. Gavai and S. Gupta, Phys. Rev. D **71**, 114014 (2005).
14. B. J. Schaefer and J. Wambach, Phys. Rev. D **75**, 085015 (2007).
15. R. V. Gavai and S. Gupta, Phys. Rev. D **78**, 114503 (2008).
16. M. Cheng et al., Phys. Rev. D **79**, 074505 (2009).
17. G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lugermeier, and B. Peterson, Nucl. Phys. B **469**, 419 (1996).
18. J. Engels, O. Kaczmarek, F. Karsch, and E. Laermann, Nucl. Phys. B **558**, 307 (1999).
19. Z. Fodor and S. D. Katz, Phys. Lett. B **534**, 87 (2002).
20. Z. Fodor, S. D. Katz, and K. K. Szabo, Phys. Lett. B **568**, 73 (2003).
21. C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, Ch. Schmidt, and L. Scorzato, Phys. Rev. D **66**, 074507 (2002).
22. C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and Ch. Schmidt, Phys. Rev. D **68**, 014507 (2003).
23. C. R. Allton, M. Doring, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and K. Redlich, Phys. Rev. D **71**, 054508 (2005).
24. P. de Forcrand and O. Philipsen, Nucl. Phys. B **642**, 290 (2002); **673**, 170 (2003).
25. Y. Aoki, Z. Fodor, S. D. Katz, and K. K. Szabo, Phys. Lett. B **643**, 46 (2006).
26. A. Bazavov et al., Phys. Rev. D **86**, 034509 (2012).
27. S. Borsnyi et al., J. High Energy Phys. **11** 077 (2010).
28. M. Cheng et al., Phys. Rev. D **77**, 014511 (2008).
29. S. Borsnyi et al., J. High Energy Phys. **1201** 138 (2012).
30. C. Ratti, M. A. Thaler, and W. Weise, Phys. Rev. D **73**, 014019 (2006).
31. P. N. Meisinger and M. C. Ogilvie, Phys. Lett. B **379**, 163 (1996).
32. P. N. Meisinger, M. C. Ogilvie Nucl. Phys. B (Proc. Suppl.) **47**, 519 (1996).
33. K. Fukushima, Phys. Lett. B **591**, 277 (2004).
[34] E. Megas, E. R. Arriola, and L. L. Salcedo, Phys. Rev. D 74, 065005 (2006); E. Megias, E. R. Arriola, and L. L. Salcedo, Phys. Rev. D 74, 114014 (2006).
[35] E. Megas, E. R. Arriola and L. L. Salcedo, High Energy Phys. 01 073 (2006).
[36] S. K. Ghosh, T. K. Mukherjee, M. G. Mustafa and R. Ray, Phys. Rev. D 77, 094024 (2008).
[37] S. K. Ghosh, T. K. Mukherjee, M. G. Mustafa, and R. Ray, Phys. Rev. D 73, 114007 (2006).
[38] S. Mukherjee, M. G. Mustafa, and R. Ray, Phys. Rev. D 75, 094015 (2007).
[39] A. Bhattacharyya, P. Deb, S. K. Ghosh and R. Ray, Phys. Rev. D 82, 014021 (2010).
[40] A. Bhattacharyya, P. Deb, A. Lahiri and R. Ray, Phys. Rev. D 82 114028 (2010).
[41] A. Bhattacharyya, P. Deb, A. Lahiri and R. Ray, Phys. Rev. D 83 014011 (2011).
[42] A. Bhattacharyya, P. Deb, S. K. Ghosh, R. Ray and S. Sur, Phys. Rev. D 87, 054009 (2013).
[43] A. Bhattacharyya, S. K. Ghosh, S. Maity, S. Raha, R. Ray, K. Saha and S. Upadhaya, Phys. Rev. D 95, 054005 (2017).
[44] A. Bhattacharyya, S. K. Ghosh, A. Lahiri, S. Majumder, S. Raha and R. Ray, Phys. Rev. C 89 064905 (2014).
[45] A. Bhattacharyya, S. K. Ghosh, S. Majumder and R. Ray, Phys. Rev. D 86 096006 (2012).
[46] P. Deb, A. Bhattacharyya, S. Datta and S. K. Ghosh, Phys. Rev. C 79 055208 (2009).
[47] P. Braun-Munzinger, K. Redlich, and J. Stachel, in Quark Gluon Plasma 3, edited by R. C. Hwa and X. N. Wang (World Scientific, Singapore, 2004).
[48] P. Braun-Munzinger, J. Stachel, J. P. Wessels, and N. Xu, Phys. Lett. B 344, 43 (1995).
[49] J. Cleymans, D. Elliott, H. Satz, and R. L. Thews, Z. Phys. C 74, 319 (1997).
[50] P. Braun-Munzinger, I. Heppe, and J. Stachel, Phys. Lett. B 465, 15 (1999).
[51] J. Cleymans and K. Redlich, Phys. Rev. C 60, 054908 (1999).
[52] F. Becattini, J. Manninen, M. Gadzicki, Phys. Rev. C 73, 044905 (2006).
[53] P. Braun-Munzinger, D. Magestro, K. Redlich, and J. Stachel, Phys. Lett. B 518, 41 (2001).
[54] A. Andronic, P. Braun-Munzinger, and J. Stachel, Nucl. Phys. A 772, 167 (2006).
[55] A. Andronic, P. Braun-Munzinger, and J. Stachel, Phys. Lett. B 673, 142 (2009).
[56] A. Bhattacharyya, S. K. Ghosh, R. Ray and S. Samanta, Euro. Phys. Lett. 115, 62003 (2016).
[57] B. J. Schaefer, J. M. Pawlowski and J. Wambach, Phys. Rev. D 76 074023 (2007).
[58] T. K. Herbst, J. M. Pawlowski and B. J. Schaefer, Phys. Lett. B 696 58 (2011).
[59] J. Wambach, B. J. Schaefer and M. Wagner, Acta Phys. Polon. Supp. 3 691 (2010).
[60] S. Chatterjee and K. A. Mohan, Phys. Rev. D 86 114021 (2012).
[61] G. Ecker, Chiral perturbation theory, Progress in Particle and Nuclear Physics, 35, 1-80 (1995).
[62] R. Dashen and S. K. Ma, Phys. Rev. A 4, 700 (1971).
[63] F. Karsch and K. Redlich, Phys. Lett. B 695 136 (2011).
[64] V. V. Begun, M. Gadzicki, and M. I. Gorenstein, Phys. Rev. C 88, 024902 (2013).
[65] R. Hagedorn and J. Rafelski, Phys. Lett. B 97, 136 (1980).
[66] D. H. Rischke, M. I. Gorenstein, H. Stöcker, and W. Greiner, Z. Phys. C 51, 485 (1991).
[67] J. Cleymans, M. I. Gorenstein, J. Stalnacke, and E. Suhonen, Phys. Scr. 48, 277 (1993).
[68] C. P. Singh, B. K. Patra, and K. K. Singh, Phys. Lett. B 387, 680 (1996).
[69] G. D. Yen, M. I. Gorenstein, W. Greiner, and S. N. Yang, Phys. Rev. C 56, 2210 (1997).
[70] A. Andronic, P. Braun-Munzinger, J. Stachel, and M. Winn, Phys. Lett. B 718, 80 (2012).
[71] J. Fu, Phys. Rev. C 85, 064905 (2012).
[72] A. Bhattacharyya, R. Ray and S. Sur, Phys. Rev. D 91 051501 (2015).
[73] A. Bhattacharyya, S. Das, S. K. Ghosh, R. Ray and S. Samanta, Phys. Rev. C 90 034909 (2014).
[74] A. Bhattacharyya, S. K. Ghosh, S. Maity, S. Raha, R. Ray, K. Saha, S. Samanta and S. Upadhaya, Phys. Rev. C 99, 045207 (2019).
[75] A. Bhattacharyya, R. Ray, S. Samanta and S. Sur, Phys. Rev. C 91 041901 (2015).
[76] S. Roessner, T. Hell, C. Ratti and W. Weise, Nucl. Phys. A 814 118 (2008).
[77] K. A. Bugaev, M. I. Gorenstein, H. Stoecker and W. Greiner, Phys. Lett. B 485, 121 (2000).
[78] V. V. Sagun, A. I. Ivanytskyi, K. A. Bugaev and I. N. Mishustin, Nucl. Phys. A 924, 24 (2014).
[79] V. V. Sagun et al., Eur. Phys. J. A 54, 100 (2018).
[80] M. Asakawa, U. Heinz, and B. Müller, Phys. Rev. Lett. 85, 2072 (2000).
[81] D. Bower and S. Gavin, Phys. Rev. C 64, 051902 (2001).
[82] M. Abdel-Aziz and S. Gavin, Phys. Rev. C 70, 034905 (2004).
[83] A. Bazavov et al., Phys. Rev. Lett. 109, 192302 (2012); C. Schmidt (for the BNL-Bielefeld Collaboration), PoS ConfinementX, 187 (2012); A. Bazavov et al., Phys. Rev. Lett. 111, 082301 (2013); A. Bazavov et al., Phys.Rev.Lett. 113, 072001 (2014).
[84] S. Bhattacharyya, D. Biswas, S. K. Ghosh, R. Ray and P. Singha, Phys. Rev. D 100 054037 (2019).
[85] S. Bhattacharyya, D. Biswas, S. K. Ghosh, R. Ray and P. Singha, Phys. Rev. D 101 054002 (2020).
[86] S. Gupta, D. Mallick, D. K. Mishra, B. Mohanty and N. Xu, arXiv:2004.04681 [hep-ph].