On Matrix Strings, the Large $N$ Limit and Discretized Light-Cone Quantization

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Abstract

We consider the 1 + 1 dimensional supersymmetric matrix field theory obtained from a dimensional reduction of ten dimensional $\mathcal{N} = 1$ super Yang-Mills, which is a matrix model candidate for non-perturbative Type IIA string theory. The gauge group here is $U(N)$, where $N$ is sent to infinity. We adopt light-cone coordinates to parametrize the string world sheet, and choose to work in the light-cone gauge. Quantizing this theory via Discretized Light-Cone Quantization (DLCQ) introduces an integer, $K$, which restricts the light-cone momentum-fraction of constituent quanta to be integer multiples of $1/K$. We show how a double scaling limit involving the integers $K$ and $N$ implies the existence of an extra (free) parameter in the Yang-Mills theory, which plays the role of an effective string coupling constant. The formulation here provides a natural framework for studying quantitatively string dynamics and conventional Yang-Mills in a unified setting.

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1 Introduction

Much of the recent excitement in string theory stems from a conjecture about the essential degrees of freedom underlying $M$-theory; namely, eleven dimensional $M$-theory is given by a $U(N)$ gauge invariant supersymmetric matrix model, which is formally obtained by reducing $9 + 1$ dimensional super Yang-Mills to $0 + 1$ dimensions via classical dimensional reduction [1]. Motivated by the work of Witten [2], we may interpret the eigenvalues of the (Hermitian matrix-valued) Yang-Mills fields as space-time coordinates of D-particles, where the world line trajectories are given by the $0 + 1$ dimensional matrix model Lagrangian. Deviations from classical space-time are therefore seen from the non-commutative properties of these matrix coordinates. Of course, until the correspondence between $M$-Theory and the matrix model is rigorously established, we have to content ourselves with the label “Matrix Theory” for the matrix model, in order to distinguish it (and its consequences) from the formal definition of $M$-theory [3].

Now the underlying Yang-Mills theory of the matrix model is manifestly ten dimensional, and so in order to establish any connection with (eleven dimensional) $M$-Theory, a necessary (and highly non-trivial) ingredient in the conjecture is the assertion that the large $N$ limit of the matrix model effectively gives rise to an additional space-like dimension. The resulting theory is then interpreted as $M$-Theory in the light-cone frame. An attempt to strengthen the plausibility of this assumption by associating the integer $N$ with the harmonic resolution $K$ of Discretized Light-Cone Quantization (DLCQ) [4] has recently been made by Susskind [5].

Evidently, these developments suggest that another (closer) look at Yang-Mills theory in the large $N$ limit is in order if we wish to understand its true underlying dynamics, and, ironically, obtain possible insights into a new kind of ‘string’ theory. The crucial issue here is the (conjectured) existence of a double scaling limit involving the integer $N$, which we send to infinity, and an additional parameter – such as a lattice spacing, $\epsilon$ – which must eventually be sent to zero [6]. Simply put: whether one takes these limits independently or not determines whether one ends up with conventional Yang-Mills, or not. The result is that we have an extra free parameter, in addition to the usual coupling constant, and so we are forced to generalize our usual concept of Yang-Mills theory. Moreover, this new parameter can be shown to be related to the string coupling constant of Type IIA or Type IIB string theory [3, 7]. Conventional Yang-Mills is then recovered when this extra parameter is set to zero.
In summary, developments in non-perturbative string theory have shed new light on the interpretation of large $N$ Yang-Mills theory, giving rise to ‘generalized Yang-Mills’\(\text{[7]}\). We will discuss in this article how DLCQ enables one to clarify these ideas in a quantitative manner.

Figure 1 outlines how a notion of generalized Yang-Mills may be viewed as emerging from string theory related investigations\(\text{[8]}\); the arrows are labeled by key concepts involved in the progress from one development to the next, starting with ordinary Yang-Mills, and moving anti-clockwise around the figure (the broken arrow relating formal $M$-Theory to Matrix Theory reflects the conjectured equivalence between the two). The connection between ordinary and generalized Yang-Mills via DLCQ is also implied, and will be the subject of this paper.

Our main goal therefore is to outline a framework which (we believe) naturally incorporates string dynamics and conventional Yang-Mills in a unified setting. The contents of this paper are organized as follows; in Section 2 we consider the two dimensional super Yang-Mills theory which is a candidate for non-perturbative Type IIA string theory\(\text{[8, 9, 10]}\). We formulate the theory using light-cone coordinates for the string world sheet, and quantize the theory in the light-cone gauge. In Section 3, we discuss Discretized

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\(\text{[8]}\) Evidently, space limitations prevent a more detailed representation than the one given here.
Light-Cone Quantization. This discretization procedure admits a natural formulation of ‘bit strings’, which will represent our string states, and which can be shown to end on D-particle-like configurations. This is the subject matter of Section 4. Subtleties associated with the large $N$ limit, and the implications of an additional dimensionless parameter in Yang-Mills due to a double scaling limit will be addressed in Section 5. We conclude with a comment on possible future directions in Section 6.

2 Matrix String Theory on the Light-Cone

By definition, compactifying eleven dimensional $M$-theory on a circle $S^1$ gives Type IIA string theory. This suggests how one may obtain a possible representation of non-perturbative Type IIA string theory via Matrix Theory; namely, we compactify one of the spatial dimensions of the matrix model on a circle $S^1$ \cite{1,11}. The theory we end up with is called Matrix String Theory, although there is now considerable evidence that it exhibits the known properties of Type IIA string theory \cite{8,9,10}.

Actually, it turns out that Matrix String Theory may be formally obtained by dimensionally reducing 9 + 1 dimensional $\mathcal{N} = 1$ super Yang-Mills to 1 + 1 dimensions. For the sake of completeness, we review the underlying ten dimensional light-cone Yang-Mills theory in Appendix A – in perhaps more detail than is customary – although the ideas should be familiar to many readers.

In order to dimensionally reduce the ten dimensional Yang-Mills action (50) (see Appendix A) to a 1 + 1 dimensional field theory, we simply specify that all fields are independent of the (eight) transverse coordinates $x^I$, $I = 1, \ldots, 8$ (points in Minkowski space are specified by the ten space-time coordinates $(x^0, x^1, \ldots, x^9)$). We may therefore assume that the fields depend only on the light-cone variables $\sigma^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^9)$. The resulting two dimensional theory has $\mathcal{N} = 8$ supersymmetry, and may be described by the action

$$S_{1+1}^{LC} = \int d\sigma^+ d\sigma^- \, \text{tr} \left( \frac{1}{2} D_\alpha X_I D^\alpha X_I + \frac{g^2}{4} [X_I, X_J]^2 - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \\
+ i\theta_R^T D_+ \theta_R + i\theta_L^T D_- \theta_L - \sqrt{2} g \theta_L^T \gamma^I [X_I, \theta_R] \right),$$

where the repeated indices $\alpha, \beta$ are summed over light-cone labels $\pm$, and $I, J$ are summed over $1, \ldots, 8$. The eight scalar fields $X_I(\sigma^+, \sigma^-)$ represent $N \times N$ Hermitian matrix-valued fields. The radius of this circle is identified with the string coupling constant, and so $M$-theory is just the strong coupling limit of ten dimensional Type IIA string theory.
fields, and are remnants of the transverse components of the ten dimensional gauge field $A_\mu$, while $A_\pm(\sigma^+, \sigma^-)$ are the light-cone gauge field components of the residual two dimensional $U(N)$ gauge symmetry. The spinors $\theta_L$ and $\theta_R$ are the remnants of the right-moving and left-moving projections of a sixteen component real spinor in the ten dimensional theory. The components of $\theta_R$ and $\theta_L$ are also $N \times N$ Hermitian matrix-valued fields. $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + ig[A_\alpha, A_\beta]$ is just the two dimensional gauge field tensor, while $D_\alpha = \partial_\alpha + ig[A_\alpha, \cdot]$ is the covariant derivative corresponding to the adjoint representation of the gauge group $U(N)$. The eight $16 \times 16$ real symmetric matrices $\gamma^I$ are defined in Appendix A.

In order to make a connection with string theory, we identify the 1 + 1 dimensional space parametrized by the light-cone coordinates ($\sigma^+, \sigma^-$) as the string world sheet. The eigenvalues of the matrices $X_I$ are then identified with the target space coordinates of the Type IIA fundamental string [10]. Of course, the $X_I$’s are non-commuting in general, and so we cannot simultaneously diagonalize them to obtain a classical description of a propagating string. It is in this sense that Matrix (String) Theory forces us to revise our notion of space-time as an approximately derived concept, and deviations from the classical formulation may be measured in terms of the non-commuting properties of these matrix coordinates.

Since we are working in the light-cone frame, it is natural to adopt the light-cone gauge $A_- = 0$. With this gauge choice, the action (1) becomes

$$\tilde{S}_{1+1}^{LC} = \int d\sigma^+ d\sigma^- \text{tr} \left( \partial_+ X_I \partial_- X_I + i\theta^T_R \partial_+ \theta_R + i\theta^T_L \partial_- \theta_L \right. + \frac{1}{2} (\partial_- A_+)^2 + g A_+ J^+ - \sqrt{2}g \theta^T_L \gamma^I [X_I, \theta_R] + \frac{g^2}{4} [X_I, X_J]^2 \left. \right), \quad (2)$$

where $J^+ = i[X_I, \partial_- X_I] + 2 \theta^T_R \theta_R$ is the longitudinal momentum current. The (Euler-Lagrange) equations of motion for the $A_+$ and $\theta_L$ fields are now

$$\partial^2_- A_+ = g J^+, \quad (3)$$
$$\sqrt{2}i \partial_- \theta_L = g \gamma^I [X_I, \theta_R]. \quad (4)$$

These are evidently constraint equations, since they are independent of the light-cone time $\sigma^+$. The “zero mode” of the constraints above provide us with the conditions

$$\int d\sigma^- J^+ = 0, \text{ and } \int d\sigma^- \gamma^I [X_I, \theta_R] = 0, \quad (5)$$
which will be imposed on the Fock space to select the physical states in the quantum
theory. The first constraint above is well known in the literature, and projects out the
colourless states in the quantized theory\[13\]. The second (fermionic) constraint is per-
haps lesser well known, but certainly provides non-trivial relations governing the small-x
behaviour of light-cone wavefunctions\[4 \, [14]\].

At any rate, equations (3),(4) permit one to eliminate the non-dyna-
mical fields
and
in the theory, which is a particular feature of light-cone gauge theo-
ries. There are
no ghosts. We may therefore write down explicit expressions for the light-cone momentum
and Hamiltonian
in terms of the physical degrees of freedom of the theory, which
are denoted by the eight scalars
, and right-moving spinor
:

\[
P^+ = \int d\sigma^- \, \text{tr} \left( \partial_- X_I \partial_- X_I + i \theta_R^T \partial_- \theta_R \right),
\]
\[
P^- = g^2 \int d\sigma^- \, \text{tr} \left( -\frac{1}{2} J^+ \frac{1}{\partial^2} J^+ - \frac{1}{4} [X_I, X_J]^2 
+ \frac{i}{2} (\gamma^I [X_I, \theta_R])^T \frac{1}{\partial_-} \gamma^J [X_J, \theta_R] \right).
\]

The light-cone Hamiltonian propagates a given field configuration in light-cone time
, and contains all the non-trivial dynamics of the interacting field theory.

In the representation for the \( \gamma^I \) matrices specified by \( [10] \) in Appendix A, we may
write

\[
\theta_R = \begin{pmatrix} u \\ 0 \end{pmatrix},
\]

where \( u \) is an eight component real spinor. The commutation relations at equal light-cone
time \( \sigma^+ = \rho^+ \) take the following form for \( I, J, \alpha, \beta = 1, \ldots, 8 \):

\[
[X^I_{pq}(\sigma^+, \sigma^-), \partial_- X^J_{rs}(\rho^+, \rho^-)] = \frac{i}{2} \delta(\sigma^- - \rho^-) \delta^{IJ} \delta_{ps} \delta_{qr},
\]
\[
\{u^\alpha_{pq}(\sigma^+, \sigma^-), u^\beta_{rs}(\rho^+, \rho^-)\} = \frac{1}{2} \delta(\sigma^- - \rho^-) \delta^{\alpha\beta} \delta_{ps} \delta_{qr},
\]

where the lower indices of the fields label the components of an \( N \times N \) (Hermitian)
matrix. In terms of their Fourier modes, the fields may be expanded at light-cone time
\( \sigma^+ = 0 \) to give\[6\]

\[
X^I_{pq}(\sigma^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left( a^I_{pq}(k^+) e^{-ik^+\sigma^-} + a^I_{qp} \dagger (k^+) e^{ik^+\sigma^-} \right), \quad I = 1, \ldots, 8; \quad (11)
\]

\[4\] If we introduce a mass term, such relations become crucial in establishing finiteness conditions. See
\[4\], for example.

\[5\] The symbol \( \dagger \) denotes quantum conjugation, and does not transpose matrix indices.
\[
\alpha pq \left( \sigma^- \right) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dk^+ \sqrt{2} \left( b^\alpha_{pq}(k^+) e^{-ik^+ \sigma^-} + b_{qp}^\dagger(k^+) e^{ik^+ \sigma^-} \right), \quad \alpha = 1, \ldots, 8, \quad (12)
\]

with

\[
\begin{align*}
[a^I_{pq}(k^+), a^J_{rs}(k'^+)] &= \delta^{IJ} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+), \\
\{b^\alpha_{pq}(k^+), b^\beta_{rs}(k'^+)\} &= \delta^{\alpha\beta} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+). \quad (14)
\end{align*}
\]

An important simplification of the light-cone quantization is that the light-cone vacuum is the Fock vacuum \(|0\rangle\), defined by

\[
a^I_{pq}(k^+)|0\rangle = b^\alpha_{pq}(k^+)|0\rangle = 0, \quad (15)
\]

for all positive longitudinal momenta \(k^+ > 0\). We therefore have \(P^+|0\rangle = P^-|0\rangle = 0\) if we formulate the theory in the continuum, since the (zero measure) point \(k^+ = 0\) may be neglected, and the issue of “zero modes” does not arise.\(^6\)

The “charge-neutrality” condition (first integral constraint from (5)) requires that all the colour indices must be contracted for physical states. Thus the physical states are formed by colour traces of the boson and fermion creation operators \(a^I\dagger, b^\alpha\dagger\) acting on the light-cone vacuum. A single trace of these creation operators may be identified as a single closed string, where each operator (or ‘parton’), carrying some longitudinal momentum \(k^+\), represents a ‘bit’ of the string. A product of traced operators is then a multiple string state. A general superposition of single closed strings with total longitudinal momentum \(P^+\) takes the form

\[
|\Psi(P^+)\rangle = \sum_{q=1}^{\infty} \int_0^\infty dk_1^+ \cdots dk_q^+ \frac{1}{\sqrt{Nq}} \text{tr}[\Gamma_{\alpha_1}(k_1^+) \cdots \Gamma_{\alpha_q}(k_q^+)] |0\rangle, \quad (16)
\]

where the repeated indices \(\alpha_i\) are summed over the eight boson and eight fermion degrees of freedom such that \(\Gamma_{\alpha_i}\) may represent any boson operator \(a^I\), or fermion operator \(b^\alpha\). The wavefunctions \(f_{\alpha_1 \cdots \alpha_q}(k_1^+, \ldots, k_q^+)|0\rangle\) are normalized such that the orthonormality condition

\(^6\)In the continuum formulation, subtleties associated with the singular point \(k^+ = 0\) still arise, but may be handled in the context of certain “cancelation conditions” at vanishing longitudinal momentum \(k^+ \rightarrow 0\). See [14, 15], and references therein.

\(^7\)If we discretize the momenta, however, such that the \(k^+\) integrations are replaced by finite sums, then the point \(k^+ = 0\) can no longer be ignored, and the “zero mode” problem must be addressed [10]. In some cases, neglecting zero modes is legitimate even after discretizing momenta, and we expect that to be the case here. For recent work, see [16], and references therein.

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\[ \langle \Psi(P^+) | \Psi(Q^+) \rangle = \delta(P^+ - Q^+) \] holds. A simple diagrammatic representation of the expansion (16) is shown in Figure 2. Each solid disk represents an \( N \times N \) bosonic or fermionic matrix operator \( \Gamma_{\alpha_i}^\dagger \), and the lines connecting them denote contraction of the matrix indices in the trace. These disks are the ‘string bits’ which each carry some fraction of the total light-cone momentum \( P^+ \). States which are dominated by an infinite number of such partons are evidently candidates for our string states. We will elaborate on this remark in Section 4. States involving multiple closed string states correspond in this formalism to a product of two or more traces in the Fock space representation. A Fock state with two closed strings, for example, would have the general form

\[
\left| \Psi_2 \right> = \frac{1}{\sqrt{N_q}} \text{tr}[\Gamma_{\alpha_1}^\dagger (k_1^+) \cdots \Gamma_{\alpha_q}^\dagger (k_q^+)] \cdot \frac{1}{\sqrt{N_s}} \text{tr}[\Gamma_{\beta_1}^\dagger (\tilde{k}_1^+) \cdots \Gamma_{\beta_s}^\dagger (\tilde{k}_s^+)] |0\rangle,
\]

Figure 2: The single closed string expansion. The wavefunctions \( f_i \) depend on the light-cone momenta of the string bits, and summation over all possible momenta is implied. The total momentum of each closed string Fock state is \( P^+ \), and is conserved in all interactions.

Figure 3: Splitting interactions, such as the one illustrated above, are suppressed by the factor \( 1/N \), which acts as a string coupling constant. One would expect the absence of such processes in the large \( N \) limit, but the existence of a double scaling limit may give rise to an effectively non-zero string coupling even in the \( N \to \infty \) limit.

Formalism to a product of two or more traces in the Fock space representation. A Fock state with two closed strings, for example, would have the general form
where $\sum_{i=1}^q k^+_i + \sum_{j=1}^s \bar{k}^+_j = P^+$. Finally, it should be stressed that the space of states generated by the single closed string states (i.e. all states which may be written as in (10)) forms an invariant subspace of the light-cone Hamiltonian in the large $N$ limit. The reasoning here is that the splitting or joining interactions (see, for example, Figure 3) involving multiple string states is suppressed by a factor $1/N$, which represents (in this scenario) the string coupling constant. This is the conventional interpretation of large $N$ Yang-Mills: all non-planar diagrams are suppressed. However, we shall discuss in Section 4 how a double scaling limit involving $N$, and an ultraviolet cutoff $K$, enables one to have an effectively non-zero string coupling even in the large $N$ limit. But first we need to introduce the concept of Discretized Light-Cone Quantization in order to define the integer $K$.

3 Discretized Light-Cone Quantization (DLCQ)

If we substitute the mode expansions (11) and (12) for the bosonic and fermion fields into expressions (6) and (7), we may explicitly derive the quantized light-cone momentum and Hamiltonian operators $P^\pm$ in terms of the (momentum dependent) creation and annihilation operators $a^I, a^I\dagger$ (bosons) and $b^\alpha, b^\alpha\dagger$ (fermions)\cite{21}. One may then extract boundstates and masses by solving the eigen-equation

$$2P^+P^-|\Psi\rangle = M^2|\Psi\rangle,$$

where $|\Psi\rangle$ is some appropriate superposition of single and multiple closed string Fock states. One can show that $P^+$ commutes with the Hamiltonian $P^-$ (i.e. it is conserved in all interactions), and is already diagonal on the space of closed string Fock states. The problem, therefore, is to diagonalize the light-cone Hamiltonian $P^-$ with respect to the given Fock basis. If we substitute the most general closed string expansion (involving single and multiple strings) for $|\Psi\rangle$ into the eigen-equation (18), we obtain an infinitely-coupled set of integral equations relating wavefunctions from different Fock sectors. Finding analytical solutions to these integral equations is in general a formidable task, and it is here that the DLCQ method has proven to be extremely useful in extracting numerical solutions. In the context of supersymmetric field theories, additional simplifications can be made by noting that in certain cases we may write $P^+P^- \sim (Q^+Q^-)^2$ in terms of the light-cone supercharges $Q^+, Q^-$, and so the eigen-problem in this case is reduced to diagonalizing (via DLCQ) the square root $Q^+Q^-$ \cite{18,19}. Application of
this technique to calculate boundstates in Matrix String Theory is the subject of current investigation \cite{21}.

Let us now briefly review the DLCQ method in the context of Matrix String Theory (for a more detailed treatment, see \cite{4}). The essential idea is surprisingly simple: we discretize the light-cone momenta of constituent partons, or string bits, keeping in mind that the total light-cone momentum $P^+$ is always conserved. In practice, this means we introduce an (ideally large) positive integer $K$ such that $P^+/K$ defines the smallest unit of momentum. The light-cone momentum $k^+$ of any constituent parton is then some positive integer multiple of this smallest unit:

$$k^+ = \frac{n}{K} P^+, \quad n = 1, 2, \ldots$$  \hspace{1cm} (19)

Of course, we recover the continuum formulation in the limit $K \to \infty$. Note that for a Fock state with $q$ partons, with each parton carrying momentum $k_i^+$, this prescription gives

$$k_1^+ + \cdots + k_q^+ = P^+; \quad k_i^+ = \frac{n_i}{K} P^+, \quad (i = 1, \ldots, q),$$  \hspace{1cm} (20)

where the integers $n_i$ lie in the range $1 \leq n_i \leq K$, and satisfy the constraint

$$n_1 + \cdots + n_q = K.$$  \hspace{1cm} (21)

Evidently, if $K$ is fixed and finite, then (21) is satisfied in only a finite number of ways, and so the space of Fock states is also finitely enumerated. In this case, the Hamiltonian is just a finite matrix, which, in principle, can always be diagonalized by some numerical routine. All of this depends on the crucial assumption that one may neglect the “zero modes”; i.e. $n_i > 0$. In the continuum formulation, we may certainly assume $k^+ > 0$, since the point $k^+ = 0$ is a zero measure set, and cannot affect the evaluation of an integral. However, this is no longer the case in the discretized theory \cite{17}. The crucial issue here is that for finite $K$, integrals over light-cone momenta are replaced by finite sums, and so if there are any contributions arising from integrations around singularities at $k^+ = 0^+$ in the continuum theory, then they can only appear in the discretized version of the same theory via the zero mode $k^+ = 0$. Even so, such zero modes may at best only provide a ‘mean field’ picture of the true dynamics at vanishingly small $k^+$ (since $K$ is finite) and so we are not guaranteed of a faithful representation of the theory even after we introduce these zero mode degrees of freedom.

One way to avoid the issue of zero modes in the DLCQ formulation of Matrix String Theory is to always assume the continuum limit $K \to \infty$ in all expressions involving $K$. 
The quantity $1/K$ is then to be interpreted as vanishingly small, and so cannot represent some finite cutoff in the theory. Rather, any singularities that arise at vanishing $k^+$ are to be taken care of by introducing a set of cancelation conditions as in [14, 15].

Admittedly, this formal approach is not always useful, since in most practical implementations of DLCQ one works with finite values of $K$, and the continuum limit $K \to \infty$ is obtained by performing a suitable extrapolation. However, it should be emphasized that neglecting the zero mode in the discretized version of a number of two dimensional theories does not affect the spectrum of massive boundstates, and so, in this case, the strategy of working with finite values of $K$, and then extrapolating to the continuum limit $K \to \infty$ proves to be a remarkably effective way of numerically extracting boundstates and mass spectra. Perhaps the best candidates for such special theories are two dimensional supersymmetric models with unbroken supersymmetry. Matrix String Theory is therefore expected to admit a numerical boundstate analysis via the DLCQ method with finite $K$.

4 D-Partons, Wee-Partons, and Strings

Consider a Fock state representing a single closed string of $q$ partons with total light-cone momentum $P^+ = \sum_{i=1}^{q} k_i^+$:

$$\frac{1}{\sqrt{N_q}} \text{tr}[\Gamma_{\alpha_1}^+(k_1^+) \cdots \Gamma_{\alpha_q}^+(k_q^+)] |0\rangle$$

(22)

where $\Gamma_{\alpha_i}^+(k_i^+)$ is a creation operator for a fermion or boson carrying light-cone momentum $k_i^+$. If we perform DLCQ, then the light-cone momenta must be integer multiples of the smallest unit of momentum, $P^+/K$, and may be identified with the $q$ integers $n_1, \ldots, n_q$. These integers need only satisfy the constraints

$$n_i \geq 1, \quad i = 1, \ldots, q$$

(23)

$$n_1 + \cdots + n_q = K,$$

(24)

For given (finite) values of $K$, enumerating all such integer sets $(n_1, \ldots, n_q)$ satisfying the above constraints is, in principle, accomplished by a suitable computer algorithm, although the processing time would increase at least exponentially as $K$ is increased. What will be of interest to us are the general physical properties that may be ascribed to such integer solution sets in the continuum limit $K \to \infty$. 
The first case we consider is when the total number of string bits, \( q \), is finite. Since \( K \to \infty \), we must have at least one of the integers in the constraint (24) tend to infinity as well. We will call the corresponding partons in this case “D-partons”; all remaining partons will be called “wee-partons”:

"D-Partons" : \( 0 < \lim_{K \to \infty} \frac{n_i}{K} \leq 1 \) for some \( i \in \{1, \ldots, q\} \); \hspace{1cm} (25)

"Wee-Partons" : \( \lim_{K \to \infty} \frac{n_i}{K} = 0 \), for some \( i \in \{1, \ldots, q\} \). \hspace{1cm} (26)

Of course, this is just a formal way of saying that D-partons each have (positive) non-zero light-cone momentum, while the momentum carried by each wee-parton is vanishingly small. Pictorially (see Figure 4), we will distinguish a D-parton from a wee-parton by size.

Now consider what happens if we allow the total number of partons, \( q \), tend to infinity.

![Figure 4: “D-partons” and “wee-partons” will be distinguished diagrammatically according to size.](image)

It is now possible to have an infinite number of wee-partons, even though as a whole they may carry only a finite fraction of the total light-cone momentum \( P^+ \). For example, let us consider the Fock state (22) where each parton carries the smallest possible unit of momentum \( (P^+/K) \). In this case, \( n_i = 1 \) for all \( i = 1, \ldots, q \), and constraint (24) implies that the total number of such partons is \( K \) (i.e. \( q = K \)). As we let \( K \to \infty \), we end up with a closed string consisting of an infinite number of wee-partons (Figure 3(a)).

Now let us consider a state with a single D-parton. This may be accomplished by making the assignments

\[ n_i = \begin{cases} 
1 & \text{for } i = 1, \ldots, q - 1; \\
K/2 & \text{for } i = q 
\end{cases} \]  

(27)

where \( K \) is an even integer here, and \( q = K/2 + 1 \). The total number of wee-partons is thus \( K/2 \), which tends to infinity in the \( K \to \infty \) limit. Half of the total light-cone momentum is carried by this string of wee-partons, while the single D-parton carries the
Figure 5: (a) A closed string made up from an infinite number of wee-partons. (b) A “long” open string of wee-partons ending on a single D-parton. (c) A long string of wee-partons ending on different D-partons. (d) Two D-partons separated by two long strings of wee-partons.

remaining half. This state resembles an open string of wee-partons with both ends fixed to the same D-parton (Figure 5(b)).

For an example of an open string of wee-partons ending on different D-partons (Figure 5(c)), we make the assignments

\[
n_i = \begin{cases} 
1 & \text{for } i = 1, \ldots, q - 2; \\
K/4 & \text{for } i = q - 1 \text{ and } i = q 
\end{cases}
\]  

(28)

where \( q = K/2 + 2 \), and \( K \) is divisible by four. In the continuum limit \( K \to \infty \), we have two adjacent D-partons carrying half the total light-cone momentum, while the “long string” of wee-partons (there being an infinite number of them) carries the other half. We will use the phrase “short string” of wee-partons if there are only a finite number of them.

As a final example, we may construct two open strings of wee-partons ending on two D-partons (Figure 5(d)) by making the assignments

\[
n_i = \begin{cases} 
1 & \text{for } i = 1, \ldots, q - K/2, \text{ and } i = q - K/2 + 2, \ldots, q - 1; \\
K/4 & \text{for } i = q - K/2 + 1 \text{ and } i = q 
\end{cases}
\]  

(29)

where \( q = K/2 + 2 \) and \( K \) is divisible by four. In this case, the two D-partons are
separated by long strings of wee-partons, and collectively share half of the total light-cone momentum.

Evidently, these examples only provide a glimpse at the totality of all possible configurations, and whether a particular configuration dominates in a low energy boundstate or not depends on the dynamical properties of the light-cone Hamiltonian. Actually, in a recent study of QCD coupled to adjoint fermions, low energy boundstates made from only wee partons were found to exist[20]. In any case, one can see from these simple observations that the existence of such “long strings” of wee-partons, along with the D-partons, naturally incorporates the dynamics of open and closed strings, together with D-particle-like objects on which these open strings may end. However, the theory we have constructed may exhibit structures which do not admit an obvious classification from such string and/or D-particle-like configurations, and more detailed studies of the Hamiltonian will be required if we wish to investigate the structure of the low energy boundstates in the theory.

5 The Large $N$ Limit and Double Scaling

In a fully interacting string theory, one has “splitting” and “joining” interactions which can change the number of strings. In the present context, if we take the large $N$ limit while keeping $g^2 N$ fixed then by keeping track of normalizations one can easily show that such interactions are suppressed by a factor $1/N$ (Figure 3). However, in the DLCQ formulation, we have an additional parameter $K$, which is also sent to infinity to recover the continuum, and so there is now a possibility of double scaling between these two diverging integers. In particular, we would like to know whether one can still have an effectively non-zero string coupling constant even in the limit $N \to \infty$.

To investigate these ideas further, we begin by considering the mass term operator in any massive two dimensional bosonic matrix field theory (on the light-cone):

$$P^\text{mass} = \frac{1}{2} m^2 \int_0^\infty \frac{dk^+}{k^+} a^\dagger_{ij}(k^+) a_{ij}(k^+).$$

Such an operator acts on the closed string of $q$ partons $|\Psi_q\rangle = \frac{1}{\sqrt{N^q}}\text{tr} \left[ a^\dagger_{ij}(k_1^+) \cdots a^\dagger_{ij}(q^+) \right] |0\rangle$ as follows:

$$P^\text{mass} \cdot |\Psi_q\rangle = \frac{1}{2} m^2 \left( \frac{1}{k_1^+} + \cdots + \frac{1}{k_q^+} \right) \cdot |\Psi_q\rangle,$$

\(^{8}\)The constant $g^2 N$ has the dimension of mass squared, which acts as the string tension.
where \( k_1^+ + \cdots + k_q^+ = P^+ \). In the DLCQ formulation, the last identity becomes

\[
P_{mass}^- \cdot \left| \Psi_q \rightangle = \frac{K}{P^+} \times \frac{1}{2} m^2 \left( \frac{1}{n_1} + \cdots + \frac{1}{n_q} \right) \cdot \left| \Psi_q \rightangle,
\]

(32)

where the integers \( n_i \in \{1, \ldots, K\} \) are related to the original light-cone momenta \( k_i^+ \) in the usual way: \( k_i^+ = \frac{n_i}{K} P^+ \), and \( n_1 + \cdots + n_q = K \). If \( \left| \Psi_q \right\rangle \) consists only of D-partons (i.e. if \( n_i/K \) is non-zero in the continuum limit \( K \to \infty \)) then \( \sum_{i=1}^{q} \frac{1}{n_i} \sim \frac{1}{K} \), and therefore the mass operator is simply multiplication by a finite constant which is independent of \( K \):

\[
P_{mass}^- \left| \Psi_q \right\rangle \sim \text{const.} \times \left| \Psi_q \right\rangle \text{ as } K \to \infty.
\]

Now consider the case where \( n_i = 1 \) for some finite number of partons, i.e. we have a finite number of wee-partons. In this case, \( P_{mass}^- \left| \Psi_q \right\rangle \sim K \times \left| \Psi_q \right\rangle \text{ as } K \to \infty \), and we therefore have a different scaling behaviour for the mass operator.

Finally, if we set all the integers \( n_i \) to unity, so that we end up with a closed string of (only) wee-partons, then \( P_{mass}^- \left| \Psi_q \right\rangle \sim K^2 \times \left| \Psi_q \right\rangle \text{ as } K \to \infty \), and we thus arrive at another scaling behaviour for the mass operator when the number of wee-partons is allowed to grow to infinity.

In summary, we have observed in a simple case that operators may scale differently with respect to \( K \) depending on the Fock state configuration, and in particular, long strings of wee-partons give rise to the largest scaling exponent for \( K \).

For Matrix String Theory (which has no explicit mass terms), the scaling behaviour is perhaps not immediately calculable without solving the full Hamiltonian – a task which is under current investigation [21] – and so at this stage our presentation will have to be schematic. At any rate, we know the theory is supersymmetric, and so, schematically, we may write\(^9\) \( P^- \sim \{Q^-, Q^-\} \), where \( Q^- \) is a light-cone supercharge operator (in fact, there are eight such operators in all – see Appendix [3]). The supercharge \( Q^- \) contains only cubic interactions\(^{10}\), and in the DLCQ formulation has the (very) schematic representation

\[
Q^- \sim \sqrt{g^2N} \times \frac{K}{\sqrt{N}} \times \sum_{n,l=1}^\infty \left( \frac{1}{n} + \frac{1}{l} \right) \times \left( \Gamma^\dagger(l+n)\Gamma(l)\Gamma(n) + \Gamma^\dagger(n)\Gamma^\dagger(l)\Gamma(n+l) \right),
\]

(33)

where the operators \( \Gamma, \Gamma^\dagger \) annihilate and create partons in a closed string of partons. Note that \( g^2N \) is held fixed as we take the large \( N \) limit. Splitting interactions – such as the one illustrated in Figure [3] – are generated by the terms \( \Gamma^\dagger\Gamma \), and introduce an

---

*Central charges in the theory will be addressed in the forthcoming work [21].

*For more details, the reader is referred to [3] for a related treatment of a light-cone supersymmetric matrix model.
overall factor of $\frac{1}{\sqrt{N}}$, so the amplitude for a single splitting interaction is roughly

$$\frac{K}{N} \times \frac{1}{n} \quad \text{(splitting amplitude)} \quad (34)$$

For a D-parton, the integer $n$ (corresponding to the light-cone momentum $\frac{n}{K}P^+$) must scale like $n \sim K$, and so the amplitude (34) is $1/N$, which vanishes in the large $N$ limit. This suggests that Fock states of D-partons do not give rise to interacting string dynamics in the large $N$ limit. This is the usual interpretation of large $N$ Yang-Mills: all non-planar diagrams are suppressed – which, in this case, is equivalent to restricting to the Fock space of single closed loops of partons.

Now consider a long closed string of wee-partons. To make things simple, assume each parton has the smallest possible light-cone momentum, $P^+/K$, so that we have a closed string of $K$ wee-partons. The integer $n$ appearing in (34) is now equal to unity, and the splitting amplitude is now $K/N$. If all the wee-partons are identical, there may be an additional factor of $K$ which arises from cyclic symmetry, but for now we restrict ourselves to the more general case. It is now clear that with the amplitude $K/N$ one has the possibility of taking a double scaling limit; i.e. we let $N \to \infty$ and $K \to \infty$ while keeping the ratio

$$\frac{K}{N} \equiv \text{“string coupling”} \quad (35)$$

fixed. Although the analysis here is very naive, it is nevertheless intriguing that $K$ must scale in proportion to $N$ if we wish to incorporate interacting string dynamics. It is of course well established that the limit $K \to \infty$ in DLCQ effects a decompactification of a (light-cone) space dimension, and so identifying $K$ with $N$ in the present context suggests that the large $N$ limit is indeed associated with a decompactification of an additional space-like dimension. Related ideas have been proposed in a recent work by Susskind [5].

At any rate, the resulting theory can no longer be identified with conventional Yang-Mills, since we have an additional parameter specified by the ratio $K/N$, which acts as an effective string coupling. Of course, for small values of this ratio, we recover conventional large $N$ Yang-Mills weakly coupled to a string-like theory.

6 Discussion

Our investigations have shown that there is scope for quantitative investigations of field theories that are generalizations of conventional large $N$ Yang-Mills theory. In particular,
we suggest that Discretized Light-Cone Quantization offers a promising approach to unify
in a natural framework open and closed string dynamics coupled to conventional large
$N$ Yang-Mills theory.

A key ingredient in this proposal is the identification of a double scaling limit involving
the $U(N)$ gauge group parameter $N$, and the DLCQ harmonic resolution $K$, which are
both sent to infinity. We provided a crude argument as to why the ratio $K/N$ must be
constant for an interacting string theory, and how such a ratio is related to the effective
string coupling constant. This observation appears to be consistent with the recent
suggestion that Matrix Theory for finite $N$ corresponds to the DLCQ of $M$-Theory, with
harmonic resolution $K = N$.

In any case, it is clear that further numerical studies of two (and possibly higher)
dimensional Yang-Mills theories in the large $N$ limit would help clarify the physical
consequences that follow from taking this limit. As we have seen, the emergence of
an additional parameter via a double scaling limit, and the proposed connection with
conventional string theory, suggests (rather ironically) that we are perhaps understanding
the full implications of Yang-Mills theory for the first time.

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**A Appendix: Yang-Mills in Ten Dimensions**

Let’s start with $\mathcal{N} = 1$ super Yang-Mills theory in 9+1 dimensions with gauge group
$U(N)$, where $\theta = 0$ in the instanton contribution:

$$S_{9+1} = \int d^{10}x \text{tr}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi \right),$$  \hspace{1cm} (36)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu],$$  \hspace{1cm} (37)

$$D_\mu \Psi = \partial_\mu \Psi + ig[A_\mu, \Psi],$$  \hspace{1cm} (38)

and $\mu, \nu = 0, \ldots, 9$. The Majorana spinor $\Psi$ transforms in the adjoint representation of
$U(N)$. The (flat) space-time metric $g_{\mu\nu}$ has signature $(+,-,\ldots,-)$, and we adopt the
normalization $\text{tr}(T^a T^b) = \delta^{ab}$ for the generators of the $U(N)$ gauge group.
In order to realize the ten dimensional Dirac algebra \( \{ \Gamma_{\mu}, \Gamma_{\nu} \} = 2g_{\mu\nu} \) in terms of Majorana matrices, we use as building blocks the reducible \( 8_s + 8_c \) representation of the spin(8) Clifford Algebra. In block form, we have

\[
\gamma^I = \begin{pmatrix}
0 & \beta_I \\
\beta_I^T & 0
\end{pmatrix}, \quad I = 1, \ldots, 8,
\]

(39)

where the \( 8 \times 8 \) real matrices, \( \beta_I \), satisfy \( \{ \beta_I, \beta_J^T \} = 2\delta_{IJ} \). This automatically ensures the spin(8) algebra \( \{ \gamma^I, \gamma^J \} = 2\delta_{IJ} \) for the \( 16 \times 16 \) real-symmetric matrices \( \gamma^I \). An explicit representation for the \( \beta_I \) algebra may be given in terms of a tensor product of Pauli matrices \(^{12}\). In the present context, we may choose a representation such that a ninth matrix, \( \gamma^9 = \gamma^1\gamma^2 \cdots \gamma^8 \), which anti-commutes with the other eight \( \gamma^I \)'s, takes the explicit form

\[
\gamma^9 = \begin{pmatrix}
1_8 & 0 \\
0 & -1_8
\end{pmatrix}.
\]

(40)

We may now construct \( 32 \times 32 \) pure imaginary (or Majorana) matrices \( \Gamma^\mu \) which realize the Dirac algebra for the Lorentz group \( \text{SO}(9,1) \):

\[
\Gamma^0 = \sigma_2 \otimes 1_{16}, \quad (41)
\]

\[
\Gamma^I = i\sigma_1 \otimes \gamma^I, \quad I = 1, \ldots, 8; \quad (42)
\]

\[
\Gamma^9 = i\sigma_1 \otimes \gamma^9. \quad (43)
\]

The Majorana spinor therefore has 32 real components, and since it transforms in the adjoint representation of \( \text{U}(N) \), each of these components may be viewed as an \( N \times N \) Hermitian matrix.

An additional matrix \( \Gamma_{11} = \Gamma^0 \cdots \Gamma^9 \), which is equal to \( \sigma_3 \otimes 1_{16} \) in the representation specified by (40), is easily seen to anti-commute with all other gamma matrices, and satisfies \( (\Gamma_{11})^2 = 1 \). It is also real, and so the Majorana spinor field \( \Psi \) admits a chiral decomposition via the projection operators \( \Lambda_\pm \equiv \frac{1}{2}(1 \pm \Gamma_{11}) \):

\[
\Psi = \Psi_+ + \Psi_-, \quad \Psi_\pm = \Lambda_\pm \Psi. \quad (44)
\]

We will therefore consider only spinors with positive chirality \( \Gamma_{11}\Psi = +\Psi \) (Majorana-Weyl):

\[
\Psi = 2^{1/4} \begin{pmatrix}
\psi \\
0
\end{pmatrix}, \quad (45)
\]

where \( \psi \) is a sixteen component real spinor, and the numerical factor \( 2^{1/4} \) is introduced for later convenience.
Since $\gamma^9$ anti-commutes with the other eight $\gamma^I$’s, and satisfies $(\gamma^9)^2 = 1$, we may construct further projection operators $P_R \equiv \frac{1}{2}(1 + \gamma^9)$ and $P_L \equiv \frac{1}{2}(1 - \gamma^9)$ which project out, respectively, the right-moving and left-moving components of the sixteen component spinor $\psi$ defined in (45):

$$\psi = \psi_R + \psi_L, \quad \psi_R = P_R \psi, \quad \psi_L = P_L \psi. \quad (46)$$

This decomposition is particularly useful when working with light-cone coordinates, since in the light-cone gauge one can express the left-moving component $\psi_L$ in terms of the right-moving component $\psi_R$ by virtue of the fermion constraint equation. We will derive this result shortly. In terms of the usual ten dimensional Minkowski space-time coordinates, the light-cone coordinates are given by

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^9), \quad \text{“time coordinate”} \quad (47)$$
$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^9), \quad \text{“longitudinal space coordinate”} \quad (48)$$
$$x^\perp = (x^1, \ldots, x^8), \quad \text{“transverse coordinates”} \quad (49)$$

Note that the ‘raising’ and ‘lowering’ of the $\pm$ indices is given by the rule $x^\pm = x_\mp$, while $x^I = -x_I$ for $I = 1, \ldots, 8$, as usual. It is now a routine task to demonstrate that the Yang-Mills action (36) for the positive chirality spinor (45) is equivalent to

$$S_{b+1}^{LC} = \int dx^+ dx^- dx^\perp \text{tr} \left( \frac{1}{2} F_{+-}^2 + F_{+I} F_{-I} - \frac{1}{4} F_{IJ}^2 ight. + i \psi_R^T D_+ \psi_R + i \psi_L^T D_- \psi_L + i \sqrt{2} \psi_L^T \gamma^I D_I \psi_R \left. \right), \quad (50)$$

where the repeated indices $I, J$ are summed over $(1, \ldots, 8)$. Some surprising simplifications follow if we now choose to work in the light-cone gauge $A^+ = A^- = 0$. In this gauge $D_- \equiv \partial_-$, and so the (Euler-Lagrange) equation of motion for the left-moving field $\psi_L$ is simply

$$\partial_- \psi_L = -\frac{1}{\sqrt{2}} \gamma^I D_I \psi_R, \quad (51)$$

which is evidently a non-dynamical constraint equation, since it is independent of the light-cone time. We may therefore eliminate any dependence on $\psi_L$ (representing unphysical degrees of freedom) in favour of $\psi_R$, which carries the eight physical fermionic degrees of freedom in the theory. In addition, the equation of motion for the $A_+$ field yields Gauss’ law:

$$\partial_+^2 A_+ = \partial_- \partial_I A_I + g J^+ \quad (52)$$
where \( J^+ = i[A_I, \partial_- A_I] + 2\psi_R^T \psi_R \), and so the \( A_+ \) field may also be eliminated to leave the eight bosonic degrees of freedom \( A_I, I = 1, \ldots, 8 \). Note that the eight fermionic degrees of freedom exactly match the eight bosonic degrees of freedom associated with the transverse polarization of a ten dimensional gauge field, which is of course consistent with the supersymmetry. We should emphasize that unlike the usual covariant formulation of Yang-Mills, the light-cone formulation here permits one to remove explicitly any unphysical degrees of freedom in the Lagrangian (or Hamiltonian); there are no ghosts.

## B  Light-Cone Supersymmetry

The supercharges of \( \mathcal{N} = 8 \) Matrix String Theory can be obtained by the dimensional reduction of the supercharge of \( \mathcal{N} = 1 \) super Yang-Mills in ten dimension. The time component of the reduced ten dimensional supercurrent may be decomposed as follows:

\[
J^+ = \frac{1 - \gamma^9}{2} J^+ + \frac{1 + \gamma^9}{2} J^+, \tag{53}
\]

where

\[
\frac{1 - \gamma^9}{2} J^+ = 2^{\frac{3}{4}} \partial_+ X_I \gamma^I \theta_R, \tag{54}
\]

\[
\frac{1 + \gamma^9}{2} J^+ = 2^{\frac{3}{4}} \partial_- A_\perp \theta_R + i 2^{-\frac{3}{4}} g [X_I, X_J] \gamma^{IJ} \theta_R, \tag{55}
\]

and \( \gamma^{IJ} = [\gamma_I, \gamma_J]/2 \). After eliminating the non-dynamical variables and introducing the eight-component real spinor \( u \), the supercharges of \( \mathcal{N} = 8 \) Matrix String Theory on the light-cone are given by (\( \alpha = 1, \ldots, 8 \)):

\[
Q_\alpha^+ = \int dx - 2^{\frac{3}{4}} (\partial_- X_I \beta_I^T u_\alpha), \tag{56}
\]

\[
Q_\alpha^- = g \int dx \left( 2^{\frac{3}{4}} \partial_-^{-1} J^+ u_\alpha + i 2^{-\frac{3}{4}} [X_I, X_J] (\beta_I^T \beta_J^T - \beta_J^T \beta_I^T)_{\alpha \beta} \cdot u_\beta \right). \tag{57}
\]

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