Analyzing the Structure of the Non-examples in the Instructional Example Space for Function in Abstract Algebra

Rosaura Uscanga1 · John Paul Cook2

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Abstract
The concept of function is critical in mathematics in general and abstract algebra in particular. We observe, however, that much of the research on functions in abstract algebra (1) reports widespread student difficulties, and (2) focuses on specific types of functions, including binary operation, homomorphism, and isomorphism. Direct, detailed examinations of the function concept itself—and such fundamental properties as well-definedness and everywhere-definedness—are scarce. To this end, in this paper we examine non-examples of function in abstract algebra by conducting a textbook analysis and semi-structured interviews with abstract algebra instructors. In doing so, we propose four key categories based upon the definitive function properties of well-definedness and everywhere-definedness. These categories identify specific characteristics of the kinds of non-examples of function that abstract algebra instruction should emphasize, enabling us to hypothesize how students might be able to develop a robust view of function and explain in greater detail the nature of the reported difficulties that students experience.

Keyword Function · Abstract algebra · Example space · Non-examples · Well-definedness · Everywhere-definedness

Introduction
The function concept is critical in mathematics and is a core topic in the secondary and undergraduate mathematics curriculum (Bagley et al., 2015; Dubinsky & Wilson, 2013; Even & Tirosh, 1995; Hitt, 1998; Oehrtman et al., 2008). In abstract
algebra, a nationally representative sample of abstract algebra experts recently concluded that topics like homomorphism, isomorphism, and binary operation—all of which are specific types of functions—are some of the most important concepts in the course (Melhuish, 2019). Indeed, nearly all of the research on functions in abstract algebra has examined key aspects of these various types of functions (e.g., Brown et al., 1997; Hausberger, 2017; Larsen, 2009; Leron et al., 1995; Melhuish et al., 2020b; Rupnow, 2019). One theme that emerges from this literature is that students experience considerable challenges reasoning about these types of functions. The function concept itself, however, has received considerably less attention in these advanced settings. We also note that much of the functions literature has focused on examples of functions and has largely overlooked non-examples. To this end, in this paper we investigate the contents and structure of the non-examples in the instructional example space (Watson & Mason, 2005; Zazkis & Leikin, 2008) for function in abstract algebra. Our research question is: what non-examples of function do students encounter in introductory abstract algebra, and what are the key characteristics by which these non-examples might be productively classified?

**Literature Review**

**Characterizations of the Function Concept**

Much of the functions literature focuses on a covariational (e.g., Carlson, 1998; Carlson et al., 2002; Oehrtman et al., 2008) approach to functions, in which a function is viewed primarily as a relationship between two quantities that are changing in tandem. Although a covariational perspective is a very useful way to conceive of functions in courses like algebra and calculus, it is not useful in an abstract algebra setting because it “superimposes an ordinal system on function, which does not underlie many of the discrete structures in abstract algebra” (Melhuish & Fagan, 2018, p. 22). Thus, a significant portion of the research on functions in the mathematics education literature is not able to account for the ways in which students must reason about functions in abstract algebra. Instead, we take a relational (Slavit, 1997) view of function in order to focus on “relationships between input–output pairs” (p. 262). This includes relationships between “individual inputs and outputs” (Slavit, 1997, p. 262) as well as relationships between sets of inputs and sets of outputs.

Our relational focus highlights a need to specify in greater detail how we define the relationship between the inputs and outputs of a function. Weber and colleagues (2020) pointed out that there are two common ways to do so. The first defines a function in terms of “a domain, a codomain, and a correspondence between the domain and the codomain such that each member of the domain is assigned exactly one element of the codomain” (Weber et al., 2020, p. 2). The correspondence mentioned here is often referred to in the literature as the *rule*. 
From this perspective, the phrase ‘exactly one element’ is conventionally interpreted in terms of two conditions: (a) each element of the domain maps to at most one element of the codomain (i.e., the proposed mapping must be well-defined\(^1\)), and (b) each element of the domain maps to at least one element of the codomain (i.e., the proposed mapping must be everywhere-defined). The second characterization involves viewing a function \(f\) as a set of ordered pairs such that, if \((x_1, y_1)\) and \((x_2, y_2)\) are in \(f\), if \(x_1 = x_2\) then \(y_1 = y_2\). Here, the domain of \(f\) is defined as the set of all of the first coordinates of these ordered pairs and the codomain (equivalent in this case to the range) is the set of all of the second coordinates. A subtle but critical difference between these two characterizations is that correspondences defined using this second characterization are automatically everywhere-defined, and thus the only condition that must be satisfied for a proposed correspondence to be a function is well-definedness.

We adopt the first characterization of function because it is commonly used in abstract algebra. For example, functions are often used to define a relationship between a familiar, well-understood algebraic structure and one that is unfamiliar in order to familiarize oneself with the latter (this is one of the many uses of the First Isomorphism Theorem). This choice shaped the study in important ways, particularly the way we operationalized the study’s central notion of a non-example of function (see “The Importance of Non-examples” section).

**Literature on Everywhere-definedness and Well-definedness**

Research on functions generally emphasizes the importance of attending to well-and everywhere-definedness. In the abstract algebra literature, this includes studies that examine students’ reasoning about binary operation (e.g., Brown et al., 1997; Melhuish & Fagan, 2018; Melhuish et al., 2020a), homomorphism (e.g., Hausberger, 2017; Rupnow, 2019), and isomorphism (e.g., Larsen, 2009; Leron et al., 1995; Nardi, 2000). Collectively, these studies point out that developing a robust understanding of function is a key precursor to reasoning about these specific types of functions. For example, closure—one of the definitive characteristics of a binary operation—can be framed as a specific manifestation of everywhere-definedness. Additionally, with respect to reasoning coherently about homomorphisms, Melhuish and colleagues (2020b) noted that “a fractured or rich understanding of function may serve as a hindrance or support, respectively” (p. 14). In short, the abstract algebra literature very clearly illustrates the implications of well-definedness and everywhere-definedness for reasoning with subsequent function-related ideas. Studies that involve direct, detailed examinations of these notions in their own right, however, are scarce.

The core function concept has also received attention in the broader literature base on functions. We note two themes from these studies. First, well-definedness has received considerably more attention than everywhere-definedness, which

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\(^1\) Well-definedness is also referred to as univalence in the literature.
remains a critical but oft-overlooked concept. Second, the nuance of well-definedness creates some difficulties for students, who find it difficult to articulate what it means and why it is important (e.g., Even, 1993; Even & Tirosh, 1995). Students might also associate it primarily with procedural conceptions of the vertical line test (e.g., Clement, 2001; Kabael, 2011; Thomas, 2003). Third, students have difficulties adapting well-definedness (and the vertical line test) to functions whose domains are not the real numbers (e.g., Dorko, 2017; Even & Tirosh, 1995). We note that the vertical line test is of limited use in abstract algebra as (1) many functions do not lend themselves to a useful graphical illustration (which is required for the vertical line test), and (2) as it is typically stated, the vertical line test does not address everywhere-definedness. Thus, though it has been more than two decades since this observation was originally made by Even and Bruckheimer (1998), we believe it is still very much true that well-definedness “deserves more careful attention than it receives” (p. 30).

The difficulties students experience with well- and everywhere-definedness can be explained in part by an overreliance on the proposed rule used to define the correspondence between the domain and codomain (e.g., Bailey et al., 2019; Breidenbach et al., 1992; Carlson, 1998; Clement, 2001). Thompson (1994), for example, noted that the “predominant image evoked in students by the word ‘function’ is of two written expressions separated by an equal sign” (p. 68). Indeed, a rule-only view of function can “mask definitional properties such as well-definedness” (Melhuish et al., 2020b, p. 4) and, we propose, everywhere-definedness. To help students overcome these difficulties, researchers have suggested that it is critical for students to consider the rule in conjunction with the domain and codomain when determining when a proposed correspondence is or is not a function (e.g., Dorko, 2017; Kabael, 2011; Oehrmtman et al., 2008; Zandieh & Knapp, 2006). How to emphasize the importance of the domain and codomain, however – such as the characteristics of non-examples that an instructor might use to encourage students to attend to these features – has not been explored.

Theoretical Perspective

The Importance of Non-examples

Examples and non-examples are critical in mathematical reasoning because they can provide concrete illustrations of abstract ideas (e.g., Goldenberg & Mason, 2008; Tsamir et al., 2008; Zaslavsky, 2019). In particular, non-examples of a concept can illuminate insights that are not always apparent when considering examples of that same concept. As noted by Watson and Mason (2005), non-examples have the potential to “demonstrate the boundaries or necessary conditions of a concept” (p. 65) and, in turn, showcase the essential aspects and features of definitions (such as the features of everywhere- and well-definedness in the definition of function). In particular, non-examples can make these key conceptual features more apparent by illustrating what happens when they are not satisfied (e.g., Tsamir et al., 2008).
Our characterization of a non-example of function is shaped by the characterization of function we adopted in the “Characterizations of the Function Concept” section: a function is a proposed mapping $f : A \rightarrow B$ that is both well-defined and everywhere-defined. We view a non-example of function, therefore, as a proposed correspondence that fails to satisfy either the well-definedness condition or the everywhere-definedness condition (or both). This is a key distinction: with the other characterization, functions that are defined in terms of sets of ordered pairs are automatically everywhere-defined (and thus a non-example would simply be a relation that is not well-defined). Our choice here reflects the literature’s emphasis on the importance of (yet notable lack of attention afforded to) everywhere-definedness. Another related consequence of this choice is that changing the domain or codomain changes the nature of the proposed correspondence, even if the rule remains the same. Weber and colleagues (2020) offered the example of the squaring function from $\mathbb{R}$ to $\mathbb{R}$ and note that it is a different function than the squaring function from $\mathbb{R}$ to $[0, \infty)$. Extrapolating this point, we note that changing the domain or codomain of a proposed correspondence could, for instance, change a non-example of function into a function (a process that we call repairing a non-example).

Generally, we note that, while example-based research has received a fair amount of attention in abstract algebra (e.g., Cook & Fukawa-Connelly, 2015; Fukawa-Connelly & Newton, 2014), research that leverages non-examples in this domain is scarce. Thus, non-examples are a potentially valuable but currently underutilized tool. Indeed, the literature suggests that such an analysis could be particularly productive at the advanced undergraduate level. Melhuish and colleagues (2020b), for instance, noted that “a lack of unification between the general function [concept] and specific AA functions was pervasive” (p. 15, emphasis added). We infer, then, that examining specific non-examples of function could yield similar insights into the nature of the general function concept. Similarly, Even (1993) and Even and Tirosh (1995) called attention to the importance of being able to distinguish between functions and non-functions and illustrated that having students consider well-chosen non-examples of function could be particularly beneficial in helping them develop a clearer image of this distinction. What it means for a collection of non-examples of function to be ‘well-chosen,’ however, is currently unclear. To address this issue, in this paper we examine the non-examples contained in the instructional example space for function in abstract algebra.

The Instructional Example Space

We employ Watson and Mason’s (2005) notion of example space—that is, the collections of examples that are associated with a particular concept. We interpret the term ‘example’ in a holistic way to refer to any specific, concrete manifestation of an abstract mathematical principle, concept, or idea. This can include exercises, illustrations, or, importantly for this study, non-examples. Watson and Mason (2005) distinguished between different kinds of example spaces, two of which are relevant for our objectives here. A personal example space is the collection and organization of examples and non-examples that an individual associates with a particular
mathematical topic. The conventional example space is the collection of examples “as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students” (Watson & Mason, 2005, p. 76). Zazkis and Leikin (2008) proposed a useful refinement of the conventional example space, distinguishing between expert example spaces and instructional example spaces. Expert example spaces are the personal example spaces of experts and display the “rich variety of expert knowledge” whereas instructional example spaces involve what examples are “displayed in textbooks” and are used in instruction (Zazkis & Leikin, 2008, p. 132). In this paper, in order to investigate what it means for a collection of non-examples of function to be ‘well-chosen,’ we examine the non-examples contained in the instructional example space.

Example spaces are not only characterized by lists of examples and non-examples; they also include the means by which these examples and non-examples might be organized and structured (Sinclair et al., 2011). We therefore distinguish between the contents and the structure of the non-examples in the instructional example space. For our purposes, the contents of the instructional example space are the union of the instructional non-examples that specific, individual experts consider to be useful in their instruction. Thus, to say that an example is in the instructional example space for function in abstract algebra is to say that there is a specific individual (in this case, an abstract algebra instructor or abstract algebra textbook author) who (1) views the proposed correspondence as a non-example of function, and (2) considers it to be useful in their instruction. We consider the structure to be the characteristics that we infer (from instructors’ descriptions and explanations) about what certain non-examples illustrate and why they are important. Inferences about the structuring of the non-examples in the instructional example space might involve, for instance, (1) researchers’ own perceptions of what conceptual aspect a non-example can (or is intended to) illustrate, (2) researchers’ interpretations of why an expert believes a particular characteristic to be important, or (3) researchers’ conjectures about the key distinctions between non-examples in a given collection.

Methods

We employed two methodologies: an analysis of introductory abstract algebra textbooks and semi-structured interviews with algebraists. First, we conducted a textbook analysis because (1) the instructional example space, by definition, contains the examples in textbooks, and (2) textbook analyses can provide insight into “how experts in a field […] define and frame foundational concepts” (Lockwood et al., 2017, p. 389). Accordingly, while the primary purpose of the textbook analysis was to identify the non-examples in the instructional example space (the contents), we were also attentive to insights in the textbooks regarding how experts might organize these non-examples (the structure). Second, upon completion of the textbook analysis, we conducted a series of semi-structured interviews (Fylan, 2005) with abstract algebra instructors. Semi-structured interviews were important for our objectives because they allowed us to “address aspects that are important to individual participants” (Fylan, 2005, p. 66) and thus provided a means by which to flexibly pursue
emerging themes we inferred related to the structure of the instructional example space. Indeed, the primary purpose of these interviews was to gain insight into the structure of the instructional example space (though we remained open to identifying additional contents as well).

Two considerations guided our selection of textbooks: relevance (to select textbooks that are currently in use in current undergraduate abstract algebra courses in the United States) and depth (to select a sample that is large enough to saturate any categories that emerge in our analysis). In total, we collected data from 13 textbooks, 9 of which we had verified were being used ubiquitously (Melhuish, 2019) or at prominent universities (National University Rankings, n.d.) (to ensure relevance), and 4 of which we introduced ourselves (following Lockwood et al., 2017) (to ensure depth) – see Table 1. Certainly this sample was relevant: according to Melhuish (2019), the 4 textbooks in row 1 of Table 1 alone were in use at a combined total of 60% of the 1244 institutions surveyed (nearly 750 institutions); supplementing with 5 textbooks in use at the top Research-1 institutions increases the percentage of market share (and therefore relevance) of our sample. In the event that we determined this initial sample to be insufficient for achieving saturation, we had planned to incorporate more textbooks as needed using similar criteria. Post-analysis, however, we concluded that, even though the 4 textbooks included for depth certainly helped us

Table 1 Description of the textbook sample

| Description                                                                 | Textbooks                                                                 |
|-----------------------------------------------------------------------------|---------------------------------------------------------------------------|
| 4 textbooks that, according to Melhuish’s (2019) national study, are the most popular in the United States | Fraleigh (2002), Gallian (2017), Gilbert and Gilbert (2015), Hungerford (2014) |
| 5 additional textbooks* (not already accounted for in Melhuish’s study) that were used in the 2019–2020 academic year at the top 25 Research-1 universities in the United States (National University Rankings, n.d.) | Artin (2011), Beachy and Blair (2019), Dummit and Foote (2004), Herstein (1975), Pinter (2010) |
| Following the approach of Lockwood et al. (2017), we included 4 textbooks from our personal textbook libraries | Davidson and Gulick (1976), Herstein (1996), Hodge et al. (2014), Rotman (2006) |

*We obtained this information by either (a) consulting the information provided by the relevant campus bookstores, or (b) finding websites from instructors who had taught the introductory abstract algebra course in the 2019–2020 academic year. We were able to include the textbooks in use at 22 of the Top 25 Research-1 universities in the United States. The five textbooks listed in the second row of Table 1 are not a complete listing of the textbooks used in these institutions. In particular, several institutions used Gallian’s (2017), Fraleigh’s (2002), and Hungerford’s (2014) textbooks, all of which were already included in the first row of Table 1 on account of Melhuish’s (2019) study. Additionally, we were unable to obtain copies of the textbooks from three of these institutions, which either used proprietary course materials or textbooks that at the time were otherwise difficult to obtain.

2 For each textbook in the sample, we obtained and analyzed the most recent edition available to us. The two citations for Herstein (1975; 1996) correspond to two different books (and not two editions of the same book). For more information, see the Textbooks in Our Sample.
to illustrate the categories of our framework, the 9 textbooks selected for relevance would have been sufficient on their own for achieving saturation; as such, additional selection measures were not necessary.

We created a list of terms (informed by the literature and our own knowledge of abstract algebra) related to functions in abstract algebra: function, relation, map, correspondence, well-definedness, everywhere-definedness, domain, codomain, rule, binary operation, homomorphism, and isomorphism. The first author then identified the sections in each textbook that corresponded to these terms by using the table of contents and the index. Next, the first author collected the sections in each textbook related to these terms by obtaining a digital PDF file of the textbook (when available) or by scanning the desired sections from a hard copy of the textbook (we included section-ending exercises as part of each section). Then the first author read the relevant sections of each textbook, highlighting any excerpts that contained (1) non-examples of function (to identify the contents), and (2) any associated descriptions and explanations related to a given non-example (to infer the structure). The textbooks with the greatest market share (row 1 of Table 1) were analyzed first; the first author then used theoretical sampling techniques (Creswell, 2012) to select the textbooks that, based upon her initial examination of the textbooks in data collection, she believed would enable her to elaborate and refine codes and emerging themes most effectively. We identified a total of 71 non-examples of function (this number includes non-examples that emerged in the semi-structured interviews, described below). To analyze the data, we followed Creswell’s (2012) method for thematic analysis. This analysis was exploratory but did involve the use of some a priori codes: once a non-example of function was identified, we initially coded it (and any associated descriptions or explanations) as either (a) a well-definedness issue, (b) an everywhere-definedness issue, or (c) both. To enable us to focus more clearly on specific well- or everywhere-definedness issues, in this paper we focus only on those non-examples that satisfy either (a) or (b) but not both. Once the first author had assigned one of these three codes to each excerpt and trimmed those coded as both (a) and (b), she re-read each remaining excerpt, creating and assigning secondary codes based upon her interpretations of what the textbook authors were identifying as key characteristics of these non-examples. All codes were continually refined, revised, and reorganized as coding progressed. During this process, the second author reviewed all coded excerpts and proposed different ways by which they might be plausibly interpreted and organized; each code was then discussed and negotiated until agreement was reached.

Five mathematicians (whom we refer to as Professors A, B, C, D, and E) participated in the semi-structured interviews. All were tenured or tenure-track faculty members at a midwestern Research 1 university who had taught an undergraduate abstract algebra course in the last five years. All interviews were conducted and recorded on Zoom (on account of the COVID-19 pandemic). We began data collection with a semi-structured group interview with all 5 abstract algebra instructors because we hypothesized that a group setting would be more conducive to generating non-examples than an individual interview—that is, we anticipated that in such a setting the group would “become more than the sum of its parts [and] exhibit a synergy that individuals alone don’t possess” (Krueger & Casey, 2009, p. 19). A central question of the group interview was, “what
Individual semi-structured interviews followed. The prompts for these interviews were informed by the results of the textbook analysis and group interview; though the nature of semi-structured interviews prevents us from providing a comprehensive listing of all questions asked, a representative sample is included in Fig. 1.

All mathematicians participated in at least one individual interview in addition to the group interviews; each mathematician was invited to participate in multiple individual interview sessions, but some were unable to do so due to varying availability. Professors A and E participated in a total of three individual interviews, Professor B participated in two, and Professors C and D participated in one. Each individual interview session lasted approximately 1 to 1.5 h. To analyze the data from the (group and individual) interviews, we transcribed each session in its entirety and employed the same procedures for thematic analysis (Creswell, 2012) that we employed for the textbook analysis (the one distinction being that we began this phase of the analysis with the codes that resulted from the textbook analysis). This iterative process resulted in four key themes: well-definedness – domain choice, well-definedness – codomain choice, everywhere-definedness – domain restriction, and everywhere-definedness – codomain expansion. These codes correspond to the four categories by which we structure the instructional example space; the characteristics by which we assigned these codes are included as part of the results.

**Results**

We now characterize and illustrate four categories that, we propose, can be used to productively organize the non-examples in the instructional example space for function in abstract algebra. We wish to call attention to three points before proceeding. First, as mentioned in the “Methods” section, to enhance the clarity of our analysis, we restrict ourselves here to non-examples that either have (a) a well-definedness issue or (b) an everywhere-definedness issue (but not both). Second, as such, we do not claim that these categories partition the entire space of non-examples (that is, we do not consider these categories to be exhaustive or disjoint). Finally, we focus

| Targeted aspect of instructional example space | Sample Interview Questions |
|-----------------------------------------------|----------------------------|
| The contents                                  | What non-examples of function do you use in your instruction, if any? Why? |
| The structure                                 | What do you see as key aspects of the following non-examples of function? Why? |
|                                               | $f: \mathbb{Q} \rightarrow \mathbb{Z}$ given by $f\left(\frac{a}{b}\right) = a + b$ |
|                                               | $g: \mathbb{Z} \rightarrow \mathbb{N}$ given by $g(x) = x^3$ |
|                                               | $\phi: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$ given by $\phi(a, b) = \frac{a}{b}$ |
|                                               | $p: (0, \infty) \rightarrow \mathbb{R}$ given by $p(x) = \pm \sqrt{x}$ |

Fig. 1 Typical questions asked of the abstract algebra instructors in the semi-structured interviews

are 3–4 non-examples that you like to use to illustrate the function concept, and why?”
our analysis on a small number of what we considered to be vivid, prototypical non-examples in each category.

**Well-definedness**

We classify a non-example in the *well-definedness* category if there exists at least one element of the proposed domain with at least two corresponding images contained in the proposed codomain. For example, consider \( \phi : \mathbb{Q} \to \mathbb{Z} \) given by \( \phi \left( \frac{a}{b} \right) = a + b \) (non-example 2.1 in Fig. 2). Gallian (2017) explained that \( \phi \) “does not define a function since \( \frac{1}{2} = \frac{2}{4} \) but \( \phi(1/2) \neq \phi(2/4) \)” (p. 21). That is, as noted by Professor D, for “one half and two fourths, you get different answers. So if you get different answers for the same input, it’s not a function.” Consider also \( f : \mathbb{R}^+ \to \mathbb{R} \) given by \( f(a) = \pm \sqrt{a} \) (non-example 2.4). Here, as noted by Rotman (2006), “there are two candidates for \( \sqrt{9} \), namely 3 and -3” (p. 83) and, thus, “\( f(a) = \pm \sqrt{a} \) is not single-valued, and hence it is not a function” (Rotman, 2006, p. 88). Similarly, Professor C noted that the input 9 maps to “plus or minus 3 […]. But plus or minus three isn’t a number, it’s two numbers.” Other non-examples that we classified in this category are displayed in Fig. 2.

We further refine these non-examples into categories based upon a distinction we inferred from the way the experts in our study discussed them. A key element of this distinction involved the nature of the *choices* one makes when evaluating a proposed correspondence at a particular input value. Professor A, for example, proposed that certain non-examples with well-definedness issues “demand a different treatment.” He then proposed that this ‘different treatment’ could be framed

| Non-example | Source |
|-------------|--------|
| (2.1) \( \phi : \mathbb{Q} \to \mathbb{Z} \) given by \( \phi \left( \frac{a}{b} \right) = a + b \) | Gallian (2017, p. 21) |
| (2.2) \( A \) is the union of two subsets \( A_1 \) and \( A_2 \) \( f \) from \( A \) to the set \( \{0,1\} \) where \( f \) maps elements of \( A_1 \) to 0 and elements of \( A_2 \) to 1 | Dummit & Foote (2004, p. 1-2) |
| (2.3) \( f : \mathbb{Z}_4 \to \mathbb{Z}_6 \) given by \( f([x]_4) = [x]_6 \) | Beachy & Blair (2019, p. 57) |
| (2.4) \( f : \mathbb{R}^+ \to \mathbb{R} \) given by \( f(a) = \pm \sqrt{a} \) | Rotman (2006, p. 88) |

Fig. 2 Non-examples with well-definedness issues

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3 When needed, for clarity we occasionally modified or reformulated some of non-examples throughout this paper – usually by inferring reasonable domains and codomains or introducing clear notation – without altering the underlying structure of the proposed correspondence. Additionally, many non-examples appeared in multiple textbooks; throughout this paper we typically note only one.
in terms of the following question: “Where is the choice taking place? Is it in your input? Or is it, uh, in the execution of the rule?” Professor B similarly framed this distinction – which he described as “two different types of problems” – in terms of the same choice:

Your function could be, um, not well-defined because the value in the domain is not well-defined, or that you have to make a choice in the value of the domain. Or they could be not well-defined because the value of the output is not well-defined and you have to make a choice of that value of the output.

Broadly, then, we infer that this distinction centers primarily on whether one is making a choice in the domain or the codomain. We account for this distinction by introducing two subcategories, which we elaborate below.

**Well-definedness – Domain Choice**

The aforementioned choice in the domain refers to the choice of different yet equivalent representations for a given domain element. This issue was explicitly attended to in both the textbooks and interviews. For example:

- “Problems arise when the element $x$ can be described in more than one way, and the rule or formula for $f(x)$ depends on how $x$ is written” (Beachy & Blair, 2019, p. 56).
- If “there are multiple ways to represent elements in the domain (like in $\mathbb{Z}_n$ or $\mathbb{Q}$), then we need to know whether our mapping is well-defined before we worry about any other properties the mapping might possess” (Hodge et al., 2014, p. 129).
- “If the defining rule for a possible binary operation is stated in terms of a certain type of representation of the elements, then the rule does not define a binary operation unless the result is independent of the representation for the elements” (Gilbert & Gilbert, 2015, p. 305).
- “The function is deliberately taking […] a particular presentation of the rationals … That’s the issue … that’s a problem. Like if you’re going to, if you’re gonna use a representative … then you have to be extra careful.” (Professor E)

We identify two elements common to these excerpts: each mentions the importance of attending to multiple representations of a domain element as well as the image of these representations under the rule. These features correspond to the two definitive characteristics of the non-examples in the well-definedness – domain choice category:

1. elements in the domain can be represented in different yet equivalent ways, and
2. these equivalent representations are mapped to different outputs by the rule.

In light of these characteristics, we propose that the previously discussed non-example 2.1 belongs in this category. Notice that (1) each rational number (such
as $\frac{1}{2}$) admits different yet equivalent representations, and (2) the rule maps each representation to a different element of the codomain. Consider also non-example 2.3. Beachy and Blair (2019) pointed out that, “in defining functions on $\mathbb{Z}_n$ it is

| $\mathbb{Z}/(3) \rightarrow \mathbb{Z}_6$ | $f: \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$ given by $f \left( \frac{a}{b} \right) = \frac{a+c}{b+d}$ |
|----------------------------------------|--------------------------------------------------|
| ($x + \langle 3 \rangle \mapsto 3x$    | (Rotman, 2006, p. 105)                           |
| (Gallian, 2017, p. 195)                |                                                 |
| $f: \mathbb{Q}^{\geq 0} \rightarrow \mathbb{Z}$ given by $f \left( \frac{m}{n} \right) = 2^m 3^n$ | $\phi: \mathbb{Z}_6 \rightarrow \mathbb{Z}_5$ given by $\phi(a_6) = a_5$ |
| (Herstein, 1996, p. 14)                | (Davidson & Gulick, 1976, p. 94)                |
| * defined on $\mathbb{Z}_n$ as follows:   | * on $\mathbb{Z}_n$ given by:                        |
| $[a] \ast [b] = \begin{cases} [1] & \text{if } a \text{ and } b \text{ have the same parity} \\ [0] & \text{if } a \text{ and } b \text{ have opposite parity} \end{cases}$ | $[a] \ast [b] = \begin{cases} [1] & \text{if } a = b \pmod{5} \\ [0] & \text{if } a \neq b \pmod{5} \end{cases}$ |
| (parity refers to whether an integer is even or odd) | (Hodge, Schlicker, & Sundstrom, 2014, p. 61) |
| (Hodge, Schlicker, & Sundstrom, 2014, p. 53) |                                                 |
| $f: \mathbb{Q} \rightarrow \mathbb{Z}$ given by $f \left( \frac{a}{b} \right) = a$ | $f: \mathbb{Q} \rightarrow \mathbb{Q}$ given by $f \left( \frac{m}{n} \right) = \frac{m+1}{n+1}$ |
| (Dummit & Foote, 2004, p. 4)           | (Beachy & Blair, 2019, p. 64)                   |
| $f: \mathbb{Z}_4 \rightarrow \mathbb{Z}_8$ given by $f([a]_4) = [3a]_8$ | $p: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_5$ given by $p([x]_{12}) = [2x]_5$ |
| (Hodge, Schlicker, & Sundstrom, 2014, p. 134) | (Beachy & Blair, 2019, p. 65)                   |

Fig. 3 Illustrations of non-examples 2.1 and 2.3 (well-definedness – domain choice)

Fig. 4 Additional non-examples in the well-definedness – domain choice category
necessary to be very careful that the given formula is independent of the numbers chosen to represent each congruence class” (p. 53). Referring to the same non-example, Professor B’s comment illustrates why such caution is indeed necessary: “if I take \( x \) equal to the equivalence class of 1 mod 4, well that’s equivalent to 5. And, if I were to choose 5, it would map to 5 and if I were to choose 1, it would map to 1. And 1 and 5 are not equivalent in the codomain.” Regarding the characteristics of this category, this comment illustrates that (1) the elements of the domain \( \mathbb{Z}_4 \) can be represented in multiple, equivalent ways, highlighting the aforementioned notion of ‘choice’ in the domain, and (2) the rule maps these equivalent representations to outputs that are \textit{not} equivalent in the codomain. See Fig. 3 for an illustration of non-examples 2.1 and 2.3; for other non-examples in this category, see Fig. 4.

**Well-definedness – Codomain Choice**

The \textit{choice in the codomain} to which we refer above involves choosing amongst multiple outputs that are associated with a single, unambiguously represented input. This issue was explicitly attended to in the interviews. For example:

- Professor A: “We’ve got to make a choice. […] There is not a choice in the domain, […] there is a choice of things that satisfy the statement in your rule.”
- Professor B: “I wouldn’t say equivalence is at the heart of [it]. […] The definition has two possible values. […] You have to clarify which value you’re going to choose. That’s a problem with multiple values.”
- Professor E: “You’re not taking advantage of any strange representation. The problem is just, like, with the function itself.”

We identify two features common to these excerpts. First, each mentions that the well-definedness issue is not attributed to equivalent representations in the domain. Second, we infer that the well-definedness issue is instead attributed to a choice in the codomain caused by multiple values of the rule. These two features correspond to the two definitive characteristics of the non-examples in the \textit{well-definedness – codomain choice} category:

1. the proposed correspondence does \textit{not} invoke different yet equivalent representations of elements in the domain, and
2. despite the lack of equivalent representations of elements in the domain, the rule still forces a choice to be made amongst outputs in the codomain.

The aforementioned non-example 2.4 exemplifies these characteristics: (1) the domain \((\mathbb{R}^+\)) causes no issues with respect to representation, yet (2) there is still an input that the rule maps to two outputs. Non-example 2.2 can also, we propose, be classified in the \textit{codomain choice} category. Dummit and Foote (2004) explained that “this unambiguously defines \( f \) unless \( A_1 \) and \( A_2 \) have elements in common (in which case it is not clear whether these elements should map to 0 or to 1)” (pp.
1–2). For example, if $A_1 = 2\mathbb{Z}$ and $A_2 = 3\mathbb{Z}$, then it is not clear whether the domain element 6 maps to 0 or 1; to use the language of the algebraists, a choice must be made in the codomain. Additionally, Professor E pointed out that, in this non-example, “there’s no issue of representative, you know … you’re not invoking a presentation of elements of $[A_1]$ or $[A_2]$.” Through the lens of the characteristics of this category, these comments collectively call attention to the fact that (1) the elements of the domain do not admit multiple representations, and (2) the rule is ambiguous and possibly maps at least one input to two outputs. See Fig. 5 for an illustration of non-examples 2.2 and 2.4; other non-examples that we classified in this category appear in Fig. 6.

![Fig. 5 Illustrations of 2.2 and 2.4 (well-definedness – codomain choice)](image)

![Fig. 6 Non-examples included in the codomain choice category](image)
**Everywhere definedness**

We classify a non-example in the everywhere definedness category if there exists at least one element of the proposed domain for which there is no corresponding image in the proposed codomain. For instance, consider \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(a, b) = a/b \) (Pinter, 2010). Professor D pointed out that \( f \) is not a function because, generally, “you have to know that you can’t divide by zero.” Pinter (2010) specifically pointed out that “there are ordered pairs such as (3,0) whose quotient 3/0 is undefined” (p. 19). Fraleigh (2002), commenting on a similar non-example, noted that there is no element of the codomain that “is assigned by this rule to the pair (2,0)” (p. 25). Additionally, consider the proposed correspondence \( g : \mathbb{Z} \rightarrow \mathbb{N} \) given by \( g(x) = x^3 \) (non-example 7.5). Professor C concluded that \( g \) is not a function, rhetorically asking “where does -1 go?” Professor E similarly specified that “-1 cubed is not a natural number, so it doesn’t go anywhere in its codomain.” Figure 7 displays other non-examples with everywhere-definedness issues.

We now introduce a distinction the mathematicians attended to related to the specific ways in which the input–output correspondence fails. Professor B, for instance, mentioned that “there’s no division by zero, ever.” In contrast, when examining non-example 7.5, he pointed out that \(-1\) (the output associated with the input \( x = -1 \)) exists but is “not contained in the codomain.” We therefore infer that he is distinguishing between instances in which the output does not exist at all and those in which it does exist but not in the specified codomain. Other algebraists made this distinction as well. Professor A, for example, suggested that, amongst non-examples with everywhere-definedness issues, “there are situations where you just can’t execute the instructions and there are situations where you could execute the instructions but [miss] only by a margin

| (7.1) | (7.4) |
| --- | --- |
| + on the set of all odd integers defined as: the usual addition on \( \mathbb{R} \) | + on the set \( M(\mathbb{R}) \) of all matrices with real entries defined as: the usual matrix addition + |
| (Davidson & Gulick, 1976, p. 10) | (Fraleigh, 2002, p. 21) |

| (7.2) | (7.5) |
| --- | --- |
| Division on the set \( \mathbb{R} \) of the real numbers, i.e. \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(a, b) = \frac{a}{b} \) | \( g : \mathbb{Z} \rightarrow \mathbb{N} \) given by \( g(x) = x^3 \) |
| (Pinter, 2010, p. 19) | (inspired by Gallian (2017), p. 23) |

| (7.3) | (7.6) |
| --- | --- |
| \( f(x) = \sqrt{x} \) | \( g(x) = \frac{1}{x} \) |
| (Beachy & Blair, 2019, p. 56) | (Professor E) |

**Fig. 7** Non-examples with everywhere-definedness issues
target.” We therefore introduce two subcategories, which are characterized and elaborated below.

**Everywhere-definedness – Codomain Expansion**

The *everywhere-definedness – codomain expansion* subcategory includes the everywhere-definedness non-examples for which the output exists in some natural, accessible superset of the proposed codomain. We observed that these kinds of non-examples were often discussed by the mathematicians in the context of expanding the proposed codomain (hence the category name):

- **Professor A:** “The codomain should generally be some space that’s large enough.”
- **Professor D:** “It’s not a function because that formula is not defined on every element of the domain. So [we’re] having to adjust the codomain.”
- **Professor E:** These kinds of non-examples “are basically not functions in the same way. There is a way to extend the codomain to make them functions.”

For example, Hungerford (2014) considered the rule $f(x) = x^2$ in which the proposed domain and codomain are both $\mathbb{Z}$, pointing out that that “the rule of $f$ makes sense for odd integers” (p. 513). We interpret this to mean that the rule can, in fact, be evaluated for odd integers (such as 9) to obtain some number (9/2). However, they go on to note that “$f(9) = 9/2$, which is not in $\mathbb{Z}$” (p. 513), the codomain. In light of our comments above, we observe that replacing the proposed codomain $\mathbb{Z}$ with, say, $\mathbb{Q}$, repairs this non-example and resolves the issue. We note that these kinds of non-examples can also be repaired by restricting the domain–for instance, restricting the domain of $f$ to $2\mathbb{Z}$ also resolves the issue – but our focus here is on the fact that they *can* be repaired by broadening the codomain. This serves to distinguish this subcategory from the *everywhere-definedness – domain restriction* subcategory (which, as we will discuss in the “Everywhere-definedness – Domain Restriction” section, *must* be repaired by restricting the domain because it *cannot* be easily or naturally repaired by broadening the codomain). Thus, we propose the following characteristics of *everywhere-definedness – codomain expansion*:

1. There exists at least one input for which the corresponding output is not an element of the proposed codomain.
2. The proposed correspondence *can* be repaired by broadening the proposed codomain in a natural way.

Consider, for instance, the aforementioned non-example 7.5 in Fig. 7. Notice that (1) the cube of each negative integer exists, but many of these outputs are not contained in $\mathbb{N}$, the proposed codomain, and (2) this non-example can be repaired by broadening the codomain from $\mathbb{N}$ to $\mathbb{Z}$. We also classify non-example 7.3 in the *everywhere-definedness – codomain expansion* category. Beachy and Blair (2019), for instance, called attention to the fact that “we immediately run into a problem: the
square root of a negative number cannot exist in the set of real numbers” (Beachy & Blair, 2019, p. 52). The mathematicians identified the same issue. Professor B, for example, noted that the proposed correspondence “is not defined for negative real numbers, so therefore it’s not a function.” Professor A specified that, with the proposed codomain, “there is no square root negative one or something, so this process is no good.” Professor A later clarified, however, that \(\sqrt{-1}\) does exist, noting that “we’re going to have to deal with complex roots, which means we need to modify the codomain.” Along these lines, Beachy and Blair (2019) pointed out that “we can enlarge the codomain to the set \(\mathbb{C}\) of all complex numbers, in which case the formula \(f(x) = \sqrt{x}\) yields a function \(f : \mathbb{R} \to \mathbb{C}\)” (p. 52). Non-example 7.3 belongs to the everywhere-definedness – codomain expansion category, then, because (1) there is indeed at least one input for which the corresponding output is not an element of the codomain, and (2) the non-example can be repaired by broadening the proposed codomain from \(\mathbb{R}\) to \(\mathbb{C}\). See Fig. 8 for an illustration of non-examples 7.1 and 7.3; other non-examples that we classified in this category appear in Fig. 9.

| Subtraction – on \(\mathbb{Z}^+\) | Addition defined on \(\{\pm n^2 : n \in \mathbb{Z}\}\) |
|---------------------------------|--------------------------------------------------|
| (Dummit & Foote, 2004, p. 16)   | (Davidson & Gulick, 1976, p. 10)                 |
| \(S\) and \(T\) are both the set of positive integers | Addition defined on \(\{0\} \cup \{\pm 2^n : n \in \mathbb{Z}^+\}\) |
| \(f : S \to T\) given by \(f(s) = s - 1\) | (Davidson & Gulick, 1976, p. 10)                 |
| (Herstein, 1996, p. 13)         | Division on the set \(\mathbb{Z}^+\) of positive integers, i.e. \(f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+\) given by \(f(a, b) = \frac{a}{b}\). |
| \(\star\) defined on \(\mathbb{Z}^+\) by: | (Professor E) |
| \(a \star b = c\) where \(c\) is the largest integer less than the product of \(a\) and \(b\) | |
| (Fraleigh, 2002, p. 27)         | |

**Fig. 8** Illustrations of non-examples 7.3 and 7.5 (everywhere-definedness – codomain expansion)

**Fig. 9** Additional non-examples in the everywhere-definedness – codomain expansion category
Everywhere-definedness – Domain Restriction

The *everywhere-definedness – domain restriction* category refers to those everywhere-definedness non-examples for which the image of a given input does not exist in any set that is accessible to an introductory abstract algebra student. This idea was often discussed in terms of restricting the domain to repair the non-example in question:

- Professor B: “You just *have to* change the domain. […] If there’s no sensible definition at some point in your domain, then you *have to* change the domain.”
- Professor C: “You restrict the domain to make that a function.”
- Professor E: “It’s a domain problem, a domain error. […] Modifying the domain, you know, you can always make it smaller”.

The key theme we observe from these excerpts is that the mathematicians viewed the issue as domain-related. In particular, we note Professor B’s use of the phrase “*have to*,” which highlights the fact that repairing such non-examples by replacing the proposed codomain is perhaps not sensible (we illustrate this point using non-examples 7.2 and 7.4 below). This theme forms the basis for the characteristics of this category of non-examples:

1. There exists at least one input in the proposed domain for which the corresponding output is not an element of the proposed codomain.
2. The proposed correspondence can only be repaired by restricting the domain.

Consider again non-example 7.2. Characteristic 1 is satisfied because, as previously noted, there exist inputs in the domain (such as (2,0)) for which there is no corresponding image in the codomain. Regarding Characteristic 2, the mathematicians generally framed ‘divide by 0’ as an issue that could be resolved by repairing the *domain*. Professor E, for instance, noted that he “would fix this by changing the domain.” Illustrating one possible way to do this, Fraleigh (2002) restricted the domain to pairs of positive rational numbers (i.e., the set $\mathbb{Q}^+ \times \mathbb{Q}^+$) and noted that, as a result of this modification, the conditions for function “are satisfied” (p. 25). Professor B made an even stronger statement, noting that “you just *have to* fix the domain by saying that $b$ has to not be zero. And then it makes sense.” Put another

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*We have chosen to describe this subcategory in this way because some of its non-examples can indeed be repaired by broadening the codomain, but it is usually simpler (and more coherent in the given context) to restrict the domain. For example, Professor A pointed out that, when repairing non-example 7.6, “you absolutely can pass to the Riemann sphere or something and have it make sense” by defining 1/0 to be the point at infinity. Our use of the term “accessible” is intended to acknowledge that, while instructors and experienced abstract algebra students might be able to repair such non-examples by broadening the codomain in this way, it is arguably simpler and more accessible in the early stages of an introductory abstract algebra course to address such issues by restricting the domain.*
way, there is no (accessible) superset containing the proposed codomain that can be used to repair this non-example in a natural way.

We also note that non-example 7.4 can be classified in the everywhere-definedness – domain restriction category. Regarding characteristic 1, Fraleigh (2002) explained that “the usual matrix addition is not a binary operation on $M_{m\times n}(\mathbb{R})$ since $A + B$ is not defined for an ordered pair $(A,B)$ of matrices having different numbers of rows or of columns” (p. 21). So, for example, the input (ordered pair) consisting of, say, a $2 \times 3$ matrix $A$ and a $2 \times 2$ matrix $B$ does not have a sum in the proposed codomain because $A$ and $B$ have a different number of columns (3 vs. 2, respectively). For the related case of matrix multiplication, Professor B pointed out that “matrix multiplication is only defined if the number of columns in $A$ equals the number of rows in $B$.” We note that the ordered pair consisting of the aforementioned $2 \times 3$ matrix $A$ and the $2 \times 2$ matrix $B$ also has no image in the proposed codomain with respect to matrix multiplication. Regarding characteristic 2, the mathematicians suggested that these non-examples could be repaired by restricting the domain. For instance, Professor E’s proposed restricting the domain to $M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R})$: “you could fix this by fixing the size, you know? You could say, like, square matrices of $n$ by $n$... $n$ by $n$ matrices would be fine.” Professor B also commented that “in this case, you have to restrict that set [the domain] to the pairs of matrices that are consistent.” We again interpreted the use of the phrase ‘have to’ to refer to the fact that repairing this non-example by expanding the codomain is neither natural nor sensible—put another way, there is no accessible superset of $M_{m\times n}(\mathbb{R})$ that contains the matrices $A + B$ and $AB$ (as stipulated above). In such cases, Professor B noted, “we just say it’s undefined.” See Fig. 10 for an illustration of non-examples 7.2 and 7.4; for other non-examples that we classified in this category, see Fig. 11.
Using a textbook analysis and semi-structured interviews with mathematicians, we have identified categories that highlight key characteristics of the non-examples in the instructional example space. These categories, we propose, offer a clearer image of what it means for a set of non-examples of function in introductory abstract algebra to be ‘well chosen’ (see Fig. 12 for a summary of these categories and their characteristics).

In this section, we outline our conjectures pertaining to the importance and implications of these categories, identify the contributions of this work, and discuss limitations and future research.

**Importance and Implications of These Categories**

We examined the instructional example space to identify the characteristics of non-examples that would be advantageous for students to have in their personal example spaces. Here we discuss the four categories in light of this objective.

Recall the theme in the literature that students experience considerable challenges with functions–even in advanced mathematics–in part because they have a view of function that focuses predominantly on the rule (to the exclusion of the domain and codomain). We propose that these categories provide a meaningful way to parse the instructional example space because they can each be viewed and characterized in terms of meaningful relationships with the domain or codomain and therefore hold

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5 Recall that we do not intend for these categories to partition the non-examples in the instructional example space, but rather to simply point out the key characteristics of these non-examples.
potential for supporting students in developing a more comprehensive view of function as a coordination of the rule, domain, and codomain. The two subcategories in the well-definedness category, for instance, can be characterized by either a choice in the domain (amongst equivalent representations of the same element) or the codomain (amongst multiple output values assigned by the rule). The two subcategories in the everywhere-definedness category can be characterized by determining whether the non-example can be repaired by expanding the codomain (if the targeted output exists in some accessible superset) or must be repaired by restricting the domain (if the targeted output does not exist in an accessible superset). We therefore hypothesize that non-examples chosen according to these four categories do indeed offer potential opportunities for students to move beyond rule-only reasoning to develop a more comprehensive view of function that explicitly attends to the domain and codomain.

We consider the well-definedness – domain choice and everywhere-definedness – codomain expansion categories to be particularly important to include in introductory abstract algebra instruction (and, accordingly, for students to incorporate into their own example spaces) because these categories include non-examples not included in a typical introductory student’s personal example space. Put another way, we suspect that many introductory abstract algebra students at the beginning of the course are more familiar with non-examples in the well-definedness – codomain choice and everywhere-definedness – domain restriction categories. This is notable for two reasons. First, experience with well-definedness – domain choice and everywhere-definedness – codomain expansion is critical for subsequent reasoning with functions in abstract algebra. Well-definedness – domain choice is critical

| Category name                  | Well-definedness | Everywhere-definedness |
|-------------------------------|------------------|------------------------|
|                               | Domain choice    | Codomain choice        | Domain restriction | Codomain expansion |
| Prototypical non-example       | \( \phi: \mathbb{Q} \rightarrow \mathbb{Q} \) given by \( \phi(\frac{a}{b}) = a + b \) | \( f: \mathbb{R}^+ \rightarrow \mathbb{R} \) given by \( f(x) = \pm \sqrt{x} \) |                             | \( g: \mathbb{Z} \rightarrow \mathbb{N} \) given by \( g(x) = x^3 \) |
| Illustration                  | ![Illustration](image) | ![Illustration](image) | ![Illustration](image) | ![Illustration](image) |
| Relationship to domain/codomain | Involves a choice in the domain | Involves a choice in the codomain | Must be repaired by restricting the domain | Can be repaired by expanding the codomain |
| Relative familiarity for introductory AA students | Relatively unfamiliar | Somewhat familiar from previous courses | Somewhat familiar from previous courses | Relatively unfamiliar |

**Fig. 12** Summary of categories of non-examples in the instructional example space
for reasoning with functions defined on sets of equivalence classes or quotient structures (as in the First Isomorphism Theorem or results related to the formal construction of the rational numbers). In particular, students need to check well-definedness issues related to equivalence every time they define a function whose domain involves a quotient structure, a common task. Everywhere-definedness – codomain expansion is important in abstract algebra when reasoning, for example, about the closure of a proposed binary operation or when attempting to define a mapping between two algebraic structures. While students likely have some prior exposure to domain restriction (e.g., inverse functions), the notion of expanding the codomain is likely to be less familiar.

Second, we suspect that rule-only reasoning is insufficient to determine that many of the non-examples in well-definedness – domain choice and everywhere-definedness – codomain expansion are, in fact, non-examples. Consider, for instance, non-examples 2.1 and 2.3 from the well-definedness – domain choice category. While the issue with these non-examples resides in the representation of elements in the domain, they both feature simple, familiar formulas (addition and the identity) that are commonly associated in previous courses with functions on the real numbers. There are no obvious pitfalls (such as the prototypical division by zero, see non-examples 7.2 and 7.6) or multiple outputs (such as the vertical line test, see non-example 2.4) that are observable simply by examining the rule. That is, we suspect that rule-only reasoners would likely overlook the existence of equivalent representations (well-definedness – domain choice characteristic 1) and thus there would be no means of perceiving that the rule maps the same input to different outputs (characteristic 2). We also observe the same feature in the everywhere-definedness – codomain expansion category. Non-examples 7.1 and 7.5, for instance, also have familiar rules that typically have been associated with functions in students’ experiences. A student only attending to the rule is therefore unlikely to notice that there is an input for which the corresponding output is not an element of the proposed codomain (everywhere-definedness – codomain expansion characteristic 1) and thus, in their view, there is no need to repair anything (characteristic 2). Thus, non-examples in well-definedness – domain choice and everywhere-definedness – codomain expansion are exactly the kinds of non-examples for which rule-only reasoning is the least well-suited. This underscores the need to deliberately incorporate these categories into instruction to provide students with opportunities to incorporate the domain and codomain into their views of function.

Contributions

In addition to providing conjectures about how we might support students’ learning about function in abstract algebra, this paper makes two primary contributions. First, it contributes to the literature on examples and example spaces. We consider our methodology—a textbook analysis paired with semi-structured interviews with experts—to be a particularly helpful way to gain insight into the instructional example space (and the conceptual structure of mathematical ideas more generally). This paper is also one of only a few analyses of non-examples, which are a key element
of example spaces that have not received much attention in the literature. We believe our analysis emphasizes the potential that the non-examples in the instructional example space hold for affording insight into the key aspects of a concept, a tool that is currently underutilized.

Second, as previously noted, the literature on functions in abstract algebra is substantial but largely focused on binary operation, homomorphism, and isomorphism. This paper provides one of the few direct, detailed analyses of well- and everywhere-definedness. Our analysis illustrates (1) the various ways in which well- and everywhere-definedness can manifest in various non-examples, and (2) how these manifestations relate to the key notions of the domain and codomain. We note that our structuring of the instructional example space, in addition to providing hypotheses about supporting students’ learning about function in advanced contexts, extends findings from this body of literature. For instance, the framework provides a frame of reference for why students might see functions in advanced mathematics as different from functions in secondary mathematics (e.g., Zandieh et al., 2017). The framework we propose in this study enables us to build upon this idea by proposing a refined conjecture: introductory abstract algebra students experience considerable challenges with function in abstract algebra because their personal example spaces are likely to involve only the well-definedness – codomain choice and everywhere-definedness – domain restriction categories, but much of abstract algebra involves the unfamiliar well-definedness – domain choice and everywhere-definedness – codomain expansion categories. Equivalently, we hypothesize that instructors can support students in overcoming these difficulties by providing them with myriad opportunities to reason about non-examples from each category in the framework.

Future Research

In this paper, we have outlined our investigation of the ways in which experts view the function concept. The implications for students’ learning that we set forth above are therefore empirically-based hypotheses that provide clear direction for testing and refinement in future research. Relatedly, while the literature has generally identified that a function should be understood as a coordination of the rule, domain, and codomain, such a view has not been explicated or directly examined in any detail. We believe that the structuring of the instructional example space reported here can inform such efforts. Though students’ conceptual structures can not be adequately captured by descriptions of behaviors, we note that doing so can be a useful first step because it then enables the research to ask, “how might the student be thinking about this idea that might explain their behaviors?” To this end, we propose that it is a useful first (though by no means last) step to initially characterize a productive view of function as one that enables students to reason successfully about non-examples from all 4 categories outlined in this paper. This hypothesis could be pursued in future research via task-based clinical interviews (Goldin, 2000) or teaching experiments (Steffe & Thompson, 2000).
Our structuring of non-examples in this study also motivates us to consider whether there might be a similar structuring for examples. This could be pursued directly using a similar design in which examples (instead of non-examples) are the primary focus. Alternatively, the notion of ‘repairing’ a non-example—that is, modifying a given non-example in some way so that it becomes an example—might provide some insight into this issue. For instance, we propose that repairing could be used to extend these categories of non-examples so that they also include the associated repaired examples (see Fig. 13). Given that the instances of repairing that we observed in this study involved explicit attention to the domain or codomain, we further suspect that instructional tasks that prompt students to repair non-examples could support students in moving beyond a rule-only view of function to develop a robust, coordinated view of function as a coordination of the rule, domain, and codomain. We therefore hypothesize that repairing could have potential advantages for researchers (as a productive way to bridge between the non-examples and examples in an example space) and students (as a productive way to develop a robust view of function).

Another fruitful path for future research could involve examining specific types of functions—such as binary operations, homomorphisms, and isomorphisms—through this new lens. One potential option in this vein would be to use the framework to
further explore the aforementioned connection between everywhere-definedness and the closure of a binary operation. Another relates to the notions of injectivity and surjectivity, as the injectivity and surjectivity of a function \( f : A \to B \) is equivalent to the well-definedness and everywhere-definedness (respectively) of its inverse \( f^{-1} : B \to A \). In particular, future research could explore how students whose personal example spaces are structured according to our framework might reason with these subsequent topics.

**Declarations**

**Conflict of Interests** On behalf of all authors, the corresponding author states that there are no conflicts of interest or competing interests. Additionally, the work described in this paper has not been published before and is not under consideration for publication anywhere else.

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