A classification of nullity classes in the derived category of a ring

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version November 1, 2016

Abstract

For a commutative Noetherian ring \( R \) with finite Krull dimension, we study the nullity classes in \( D^c_{fg}(R) \), the full triangulated subcategory \( D^c_{fg}(R) \) of the derived category \( D(R) \) consisting of objects which can be represented by cofibrant objects with each degree finitely generated. In the light of perversity functions over the prime spectrum \( \text{Spec} R \), we prove that there is a complete invariant of nullity classes thus that of aisles (or equivalently, \( t \)-structures) in \( D^c_{fg}(R) \).

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1 Introduction

The classification of various types of algebraic or geometric objects, such as finite simple groups and differentiable manifolds of dimension 4, is a very fundamental problem in many different areas of mathematics. Finding invariants is considered efficient to distinguish them, which in fact establishes a bijection from the collection of target objects to that of well-studied objects (such as finitely generated abelian groups or prime spectrum of a ring). Our research were concerned with the classification of subcategories. These problems naturally arose in homotopy theory (see Hopkins [12]) and have heavily influenced algebraic geometry (see Thomason [20] and Neeman [15]) and modular representation theory (see Benson, Iyengar and Krause [14]) since then. Explicitly, Gabriel [9] for example, showed that in the category of finitely generated modules over a commutative noetherian ring $R$, the Serre subcategories are one-to-one correspondent to the specialization closed subsets of the prime spectrum Spec$R$. Neeman in [15] and [16] showed that the (co)localizing subcategories of the derived category $D(R)$ are bijectively correspondent to the subsets of Spec$R$. More recently in the context of tensor triangulated category, the radical thick tensor ideals are classified by the closed subsets (thanks to Hochster dual [11]) of Balmer’s spectrum of the tensor triangulated category, see Balmer [2].

In this paper, we are particularly interested in the classification of $t$-structures. The notion of $t$-structures was introduced in the work of Beilinson, Bernstein and Deligne [3], which commenced the theory from a simple idea of truncation of complexes and connected an abelian category and its derived category in a homological way. Later on, Keller and Vossieck [14] pointed out that $t$-structures are bijectively correspondent to the subcategories aisles (see Theorem 3.4). Nullity classes a generalization of aisles, were introduced by Stanley [18], who gave a classification of nullity classes (particularly, aisles thus $t$-structures) in the derived categories $D_{\text{perf}}(R)$ of perfect complexes and also in $D^c_{\text{fg}}(R)$ of bounded derived category whose objects have finitely generated homologies. We generalized his result in the current paper. Recently, translated into our language, Tarrio, Lopez and Saorin [19] gave a classification of compactly generated aisles in $D^+_{\text{fg}}(R)$ for a commutative Noetherian ring $R$, also via the perversity functions over Spec$R$.

In our classification of nullity classes, the cellular tower obstruction of objects in $D(R)$ and their homology supports were used. We strengthened the assumption on the category $D_{\text{fg}}(R)$ to $D^c_{\text{fg}}(R)$ in which every object has a cofibrant representative with each degree finitely generated, thanks to the model category theory (see Dwyer and Spalinski [7] and Hovey [13], for example). The paper is organized as follows. In Section 2 and Section 3 we introduce the basic notions such as nullity classes and support of modules. Then in Section 4 we show as Lemma 4.7 that to construct an object $M$ in $D(R)$ it is sufficient to use the quotient $R/p$ by primes $p$ in the support of $M$. In Section 5 a technical lemma is proved as Lemma 5.10 which roughly shows the converse statement of Lemma 4.7 provided that the ring $R$ has finite Krull dimension. Finally, the complete invariant of nullity classes in $D^c_{\text{fg}}(R)$ is given.
in the last section, also it gives a complete invariant of aisles or \( t \)-structures in the same category.

Throughout the paper, we assume that the ring \( R \) is commutative Noetherian with identity.

## 2 Support and associated primes

Let \( \text{D}(R) \) be the derived category of a ring \( R \). For the definitions and various properties of \( \text{D}(R) \) one can see [21] for example. We will adopt the following notations for various full triangulated subcategories of \( \text{D}(R) \):

1. The category \( \text{D}^*_\text{fg}(R) \) (\( \ast = b, +, - \), blank, resp.) consists of (bounded, bounded below, bounded above, blank, resp.) chain complexes of \( R \)-modules whose homologies are finitely generated;
2. The category \( \text{D}_{\text{perf}}(R) \) consists of bounded complexes of finitely generated projective \( R \)-modules, i.e. compact objects, the Hom functors represented by such objects commute with coproducts;
3. The category \( \text{D}^*_\text{c}(R) \) consists of objects which can be represented by cofibrant objects finitely generated in each degree. See Proposition 2.4 for detail.

We denote by \( \text{Spec} R \) the prime spectrum of \( R \), i.e. the set of all prime ideals of \( R \). For each \( R \)-module \( M \), we can consider two useful subsets of \( \text{Spec}(R) \). Let \( \text{ann}^R(x) = \{ r \in R \mid rx = 0 \} \) be the annihilator of \( x \in M \) in \( R \) and write \( M_p = M \otimes R_p \) for abbreviation. The set of associated primes of \( M \) is defined as \( \text{Ass} M = \{ p \in \text{Spec}(R) \mid p = \text{ann}^R(x) \text{ for some } x \in M \} \), and the support of \( M \) is defined as \( \text{Supp} M = \{ p \in \text{Spec}(R) \mid M_p \neq 0 \} \).

Similarly, we can define the set of associated primes and the support of an object \( M \in \text{D}(R) \) by setting \( \text{Ass} M = \bigcup_i \text{Ass} H_i(M) \) and \( \text{Supp} M = \{ p \in \text{Spec} R \mid M \otimes R_p \neq 0 \} = \bigcup_i \text{Supp} H_i(M) \). Then the following properties about \( R \)-modules can be generalized to the case of chain complexes of \( R \)-modules without difficulty.

Let \( U \subseteq \text{Spec}(R) \) be a subset. Define \( \overline{U} = \{ p \in \text{Spec}(R) \mid p \supseteq q \text{ for some } q \in U \} \) to be the closure of \( U \) under specialization. Particularly, we also denote by \( V(p) \) for \( \overline{U} \) when \( U = \{ p \} \).

**Lemma 2.1.** Let \( A_i, A, B \) and \( C \) be \( R \)-modules and \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) a short exact sequence. Then

1. \( \text{Ass} A \subset \text{Ass} B \subset \text{Ass} A \cup \text{Ass} C \);
2. \( \text{Supp} B = \text{Supp} A \cup \text{Supp} C \);
3. \( \text{Ass} C \subset \overline{\text{Ass} B} = \text{Supp} B \);
4. \( \text{Supp} \bigoplus_i A_i = \bigcup_i \text{Supp} A_i \).

**Proof.** These results can be found in Chapter 3 of [1], Chapter II and IV of [6]. \( \square \)
Example 2.2. Let $p$ be a prime ideal and $k(p)$ the corresponding residue field. Take $q = \text{ann}^R(\frac{r}{s} + pR_p) \in \text{Ass} k(p)$, where $r \in R$ and $s \notin p$. Then for every $x \in q$ we have $x(\frac{r}{s} + pR_p) = pR_p$ thus $\frac{r}{s} \in pR_p$ since $k(p)$ is a field. In particular, $x \in p$. Hence $\text{Ass} k(p) = \{p\}$. Furthermore, Lemma 2.1 implies that $\text{Supp} k(p) = \text{Ass} k(p) = V(p)$. It is clear that $\text{Supp} R/p = V(p)$, also see Exercise 3.19 in [1].

We recall the Nakayama’s Lemma for convenience, which usually allows us to deduce contradictions when considering the annihilation properties of homologies.

Lemma 2.3. (Nakayama) Let $R$ be a commutative ring with identity. Suppose $M$ is a finitely generated $R$-module and $a$ is an ideal of $R$ contained in the Jacobson radical of $R$. Then $aM = M$ implies $M = 0$.

Proof. See Proposition 2.6 in [1]. □

The model category structure we are using here on the category $\text{Ch}(R)$ of chain complexes comes from [13], in which the cofibrant objects are exactly the DG-projective complexes, i.e. $K$-projective complexes such that each degree is projective. Here a complex $P$ is $K$-projective if and only if the functor $\text{Hom}^R(P, -)$ sends acyclic complexes to acyclic ones, or equivalently, $\text{Hom}^R(P, X) \cong \text{Hom}^K(P, X)$ for all $X$. See Proposition 1.4 in [17]. One can consult the definition of total Hom chain complexes $\text{Hom}^R(-, -)$ in Chapter V of [10] and properties of Hom functors in $\text{D}(R)$ in Chapter 10 of [21].

We denote by $k(p)$ the residue field of a prime ideal $p$, defined as $k(p) = R_p/pR_p$.

Proposition 2.4. Suppose $X \in D^c_{fg}(R)$. Then
(1) $\text{Hom}(X, \bigoplus Y_\alpha^i) \cong \bigoplus \text{Hom}(X, Y_\alpha^i)$ for all $R$-modules $Y_\alpha$;
(2) $X \otimes k(p) \cong \bigoplus \Sigma^i \otimes_{a \in I(i)} k(p)$ for finite $I(i)$;
(3) $\text{Hom}(X, Y) \otimes R_p \cong \text{Hom}(X_p, Y_p)$ for every $Y \in D^b(R)$;
(4) $X$ has a projective representative which is finitely generated in each degree.

Proof. For Property (1), since $X$ can be represented by a cofibrant object which is in particular $K$-projective, Proposition 1.4 in [17] implies that it is sufficient to show the functor $\text{Hom}(X_i, -)$ commutes with direct sums in the category of $R$-modules, which is clear since each $X_i$ is finitely generated. Property (2) comes from Lemma 2.17 in [5]. Property (3) is part of our Proposition 2.5, and Property (4) is exactly the first part of Lemma 2.3.6 in [13], since the lifting property of cofibration implies the projectivity. □

We end this section by showing that in $D(R)$, under some sufficient conditions on the two variables of the Hom functor, with which the localization functor commutes. This is usually a key step when considering generalization from local to global. Nevertheless, this proposition is not always true in general.
Proposition 2.5. Suppose one of the following conditions holds,
(1) \( M \in D_{\text{perf}}(R) \) and \( N \in D(R) \),
(2) \( M \in D^+_{fg}(R) \) and \( N \in D^-(R) \) or
(3) \( M \in D_{fg}^c(R) \) and \( N \in D^b(R) \).
Then for every \( p \in \text{Spec} \, R \) we have an isomorphism
\[
\text{Hom}_{D(R)}(M, N) \otimes R_p \cong \text{Hom}_{D(R_p)}(M_p, N_p)
\]
as \( R_p \)-modules.

**Proof.** The key step is to observe that each degree \( D_n = \prod_{p+q=n} \text{Hom}_R(M_{-p}, N_q) \) of the total Hom chain complex \( \text{Hom}_R(M, N) \) becomes a coproduct. Take condition (1) for instance, other cases are similar except (3) which needs Property (4) of Proposition 2.4. Since \( M \) is bounded, the product is finite, hence \( D_n \cong \bigoplus_{p+q=n} \text{Hom}_R(M_{-p}, N_q) \). Note that each \( M_i \) is finitely generated and \( R_p \) is flat, \( \text{Hom}_R(M_{-p}, N_q) \otimes R_p \cong \text{Hom}_{R_p}(M_{-p} \otimes R_p, N_q \otimes R_p) \) by Proposition 2.10 in [8]. Thus \( D_n \otimes R_p \cong \bigoplus_{p+q=n} \text{Hom}_{R_p}(M_{-p} \otimes R_p, N_q \otimes R_p) \), which is the degree \( n \) part of the chain complex \( \text{Hom}_R(M, N) \otimes R_p \), or \( \text{Hom}_{R_p}(M_p, N_p) \) by the definition of total Hom chain complex. Since \( M \) is bounded below and projective in each degree, then we have \( \text{Hom}_{D(R)}(M, N) \cong H_0(\text{Hom}_R(M, N)) \). Therefore,
\[
\text{Hom}_{D(R)}(M, N) \otimes R_p \cong H_0(\text{Hom}_R(M, N)) \otimes R_p \cong H_0(\text{Hom}_R(M, N) \otimes R_p) \cong H_0(\text{Hom}_{R_p}(M_p, N_p)) \cong \text{Hom}_{D(R_p)}(M_p, N_p),
\]
where the second isomorphism comes from Proposition 2.5 in [8]. \( \square \)

3 Aisles, \( t \)-structures and nullity classes

In this section, we introduce the basic notions properties of nullity classes and related topics such as aisles and \( t \)-structures, and establish their fundamental relations. Denote by \( \mathcal{T} \) a triangulated category whose suspension functor is \( \Sigma \). The basic properties of triangulated categories can be found in [3] and [21], for instance.

**Definition 3.1.** A non-empty full subcategory \( \mathcal{A} \) of \( \mathcal{T} \) is a preaisle if:
(1) for every \( X \in \mathcal{A} \), we have \( \Sigma X \in \mathcal{A} \);
(2) for every distinguished triangle \( X \rightarrow Y \rightarrow Z \rightarrow \Sigma X \), if \( X, Z \in \mathcal{A} \) then \( Y \in \mathcal{A} \).

A preaisle \( \mathcal{A} \) is called cocomplete if \( \mathcal{A} \) is closed under coproducts. A preaisle \( \mathcal{A} \) is called an aisle if the inclusion functor \( \mathcal{A} \hookrightarrow \mathcal{T} \) admits a right adjoint. As an easy consequence of the definition of preaisles, we obtain the following proposition.
Proposition 3.2. Suppose $\mathcal{A}$ is a preaisle of $\mathcal{T}$ then:

1. $\mathcal{A}$ contains the zero object, i.e. $0 \in \mathcal{A}$.
2. If $X \in \mathcal{A}$ and $Y \in \mathcal{T}$ is isomorphic to $X$, then $Y \in \mathcal{A}$.

Proof. Since $\mathcal{A} \neq \emptyset$, we can take $X \in \mathcal{A}$. Then it suffices to consider the following two distinguished triangles $X \to 0 \to \Sigma X \to \Sigma X$ and $X \to Y \to 0 \to \Sigma X$.

The concept of $t$-structures traces back to the work of A.A. Beilinson, J. Bernstein and P. Deligne in [3], and that of aisles was introduced in the work of B. Keller and D. Vossieck in [14]. The Theorem 3.4 implies that these two concepts are essentially the same thing.

Definition 3.3. A $t$-structure on a triangulated category $\mathcal{T}$ is a pair of two subcategories $\mathcal{A}, \mathcal{A}' \subset \mathcal{T}$ such that:

1. $\Sigma \mathcal{A} \subset \mathcal{A}$ and $\mathcal{A}' \subset \Sigma \mathcal{A}'$;
2. for every $A \in \mathcal{A}$ and every $B \in \Sigma^{-1} \mathcal{A}'$, we have $\text{Hom}(A, B) = 0$;
3. for every $X \in \mathcal{T}$ there is a distinguished triangle $A \to X \to B \to \Sigma A$ such that $A \in \mathcal{A}$ and $B \in \Sigma^{-1} \mathcal{A}'$.

Let $\mathcal{A} \subset \mathcal{T}$ be a subcategory. We define $\mathcal{A}^\perp = \{X \in \mathcal{T} \mid \text{Hom}(A, X) = 0 \text{ for every } A \in \mathcal{A}\}$. Notice that $\mathcal{A}^\perp$ still lies in $\mathcal{T}$. The following theorem is essentially the first proposition in [14].

Theorem 3.4. A preaisle $\mathcal{A}$ is an aisle, that is the inclusion $\mathcal{A} \hookrightarrow \mathcal{T}$ admits a right adjoint, if and only if $(\mathcal{A}, \Sigma \mathcal{A}^\perp)$ is a $t$-structure on $\mathcal{T}$.

Associated to the $t$-structure $(\mathcal{A}, \Sigma \mathcal{A}^\perp)$, we denote the right adjoint of the inclusion $\mathcal{A} \hookrightarrow \mathcal{T}$ by $(\cdot)(\mathcal{A}) : \mathcal{T} \to \mathcal{A}$, and the left adjoint of the inclusion $\mathcal{A}^\perp \hookrightarrow \mathcal{T}$ by $P_\mathcal{A} : \mathcal{T} \to \mathcal{A}^\perp$. These adjoints are called truncation functors, and we often refer to $P_\mathcal{A}$ the nullification functor as well. Due to Theorem 3.4 for every $X \in \mathcal{T}$ we obtain a natural triangle,

$$X(\mathcal{A}) \longrightarrow X \longrightarrow P_\mathcal{A}(X) \longrightarrow \Sigma X(\mathcal{A}).$$

Proof of Theorem 3.4. Suppose $(\mathcal{A}, \Sigma \mathcal{A}^\perp)$ is a $t$-structure and $X \in \mathcal{T}$. Then we have a distinguished triangle $A \to X \to B \to \Sigma A$ with $A \in \mathcal{A}$ and $B \in \mathcal{A}^\perp$. Applying the functor $\text{Hom}_\mathcal{T}(\mathcal{A}', -)$ to this triangle for any $A' \in \mathcal{A}$ and noticing that $\mathcal{A} \subseteq \mathcal{T}$ is full, we have an exact sequence

$$\text{Hom}_\mathcal{T}(\mathcal{A}', \Sigma^{-1} B) \longrightarrow \text{Hom}_\mathcal{A}(\mathcal{A}', A) \longrightarrow \text{Hom}_\mathcal{T}(\mathcal{A}', X) \longrightarrow \text{Hom}_\mathcal{T}(\mathcal{A}', B),$$

where both ends are trivial since $B \in \mathcal{A}^\perp$ and $\mathcal{A}^\perp$ is closed under desuspension by definition. Hence $\text{Hom}_\mathcal{A}(\mathcal{A}', A) \to \text{Hom}_\mathcal{T}(\mathcal{A}', X)$ is an isomorphism, which is induced by $u : A = X(\mathcal{A}) \to X$. 


Conversely, suppose $\mathcal{A}$ is an aisle. The adjointness $\Sigma \dashv \Sigma^{-1}$ of suspension functors implies $\mathcal{A}' = \Sigma \mathcal{A}^\perp$ is closed under desuspension. Now suppose $$X \langle \mathcal{A} \rangle \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{w} \Sigma X \langle \mathcal{A} \rangle$$ is a distinguished triangle, it remains to show that $Y \in \mathcal{A}^\perp$. Indeed, let $A \in \mathcal{A}$ and $f \in \text{Hom}_T(A,Y)$. Then we have a morphism between two distinguished triangles by definition $$X \langle \mathcal{A} \rangle \xrightarrow{h} B \xrightarrow{g} A \xrightarrow{w} \Sigma X \langle \mathcal{A} \rangle$$ $$X \langle \mathcal{A} \rangle \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{w} \Sigma X \langle \mathcal{A} \rangle.$$ It follows from the adjunction $u_*$ and the commutativity of the first square that $u = gh = u_* (g') h = u g' h$, i.e. $g' h = \text{id}_{X \langle \mathcal{A} \rangle}$, for some $g' : B \to X \langle \mathcal{A} \rangle$. Thus $w f = 0$. Hence $f$ factors through $X$ as $f = v f'$ for some $f' : A \to X$. Therefore $f = v f' = v u_* (f'') = v u f'' = 0$ for some $f'' : A \to X \langle \mathcal{A} \rangle$, by using the adjunction $u_*$ again. □

As an immediate consequence we have

**Corollary 3.5.** Suppose $\mathcal{A}$ is an aisle, then $P_\mathcal{A}(X) = 0$ implies $X \in \mathcal{A}$.

Next we move into the study of the derived category $\text{D}(R)$ of a ring $R$ and its subcategories, nullity classes, basically following the work of Stanley, see his paper [18].

**Definition 3.6.** Let $\mathcal{D} \subseteq \text{D}(R)$ be a full triangulated subcategory and $\mathcal{A}$ be a cocomplete pre-aisle in $\text{D}(R)$. A nullity class in $\mathcal{D}$ is a full subcategory of the form $\mathcal{A} \cap \mathcal{D}$.

We let $E \in \text{D}(R)$, and we denote by $\mathcal{A} = \overline{C}(E)$ the smallest nullity class containing $E$ in $\text{D}(R)$. We denote by $P_E = P_\mathcal{A}$ the nullification functor. Let $F \in \text{D}(R)$ be another object, we denote by $E < F$ if $P_E(F) = 0$.

**Proposition 3.7.** Let $\mathcal{D} \subseteq \text{D}(R)$ be a full triangulated subcategory. Then every aisle in $\mathcal{D}$ is a nullity class and every nullity class is a pre-aisle.

Proposition 3.7 was proved as Proposition 2.14 in [18], which tells us that nullity classes are slightly generalized aisles from preaisles.

**Proposition 3.8.** Let $E,F \in \text{D}(R)$. Then $E < F$ if and only if $\overline{C}(F) \subseteq \overline{C}(E)$. Moreover, $\overline{C}(E) = \{ X \in \text{D}(R) \mid P_E(X) = 0 \}$.

**Proof.** See Proposition 3.3 in [18]. □

Clearly, Proposition 3.8 implies that the killing relation “$<$” is transitive. Also, it is easy to show by the Eilenberg Swindle that nullity classes are closed under retracts, since they are cocomplete and closed under suspensions and extensions.
Lemma 3.9. Let $\overline{C(E)}$ be a nullity class for some $E \in D(R)$. Then $\overline{C(E)}$ is closed under retracts.

Proof. See Lemma 3.2 in [18]. □

Finally, we can deduce the following lemma from the advantage of being a nullity class.

Lemma 3.10. Suppose $E \in D(R)$. If $p \notin \text{Supp}H_i(E)$ for $i \leq n$, then for every $X \in \overline{C(E)}$, we have $p \notin \text{Supp}H_i(X)$ for $i \leq n$.

Proof. Notice that the condition $H_i(X) \otimes R_p = 0$ for $i \leq k$ is closed under suspensions, direct sums and extensions. Indeed, the conclusion follows from the properties that $H_i(\Sigma X) \cong H_{i-1}(X)$, the homology functors commute with direct sums and the localization functor at $p$ is exact, see Proposition 3.3 in [1]. □

4 Koszul complexes and annihilation

As typical objects in the derived category $D_{\text{perf}}(R)$, the Koszul complexes play a crucial role in this paper. Their various properties can be found in [8] and [21], for example. We prove in this section as Lemma 4.7 that in the light of the annihilation property of Koszul complexes, the quotients $R/p$ by $p$ which comes from the support of an object $M \in D(R)$ are sufficient to generate $M$.

Notation 4.1. Let $p = (x_1, ..., x_k)$ be an ideal in $R$. Denote by $K(p) = K(x_1, ..., x_k)$ the Koszul complex associated to the sequence $x_1, ..., x_k$ and $K' = K(x_1^{n_1}, ..., x_k^{n_k})$ for some powers $n_1, ..., n_k \in \mathbb{N}$. Also we define $K(0) = R$, and inductively define

$$K(i) = \text{Cone}(K(i-1) \xrightarrow{x_i} K(i-1)),$$

as the mapping cone of the multiplication map by $x_i$. Moreover, for every $M \in D(R)$ we define $M(i) = M \otimes K(i)$. In particular, $M(k) = M \otimes K(p)$. Similarly, we define $K'(0) = R$ and $K'(i) = \text{Cone}(K'(i-1) \xrightarrow{x_i^{n_i}} K'(i-1))$ for $n_i \in \mathbb{N}$. In particular, $H_0(K(p)) = R/p$, see Section 17.2 in [8].

Recall the annihilation property of Koszul complexes proved as Proposition 17.14 in [8].

Proposition 4.2. Suppose $M$ is an $R$-module and $p$ is an ideal of $R$ having $k$ generators. Then for each $i = 0, 1, ..., k$, the homology $H_i(M \otimes_R K(p))$ is annihilated by every $x \in p$. In particular, such $x$ annihilates all the homologies of $K(p)$.

Example 4.3. Let $q = \text{ann}^R(a) \in \text{Ass}H_i(K(p))$ with $a \in H_i(K(p))$. Then Proposition 4.2 implies that $a \cdot p = 0$ thus $q \in V(p)$. Hence $\text{Supp}K(p) \subseteq V(p)$. Also notice that $\text{Supp}H_0(K(p)) = V(p)$ by Example 2.2. Therefore, $\text{Supp}K(p) = V(p)$. 


Corollary 4.4. Let $m$ be a maximal ideal of $R$ having $k$ generators. Then $H_i(K(m)) \cong \bigoplus_{\Lambda_i} k(m)$ for some index set $\Lambda_i$. In particular, $H_0(K(m)) \cong H_k(K(m)) \cong k(m)$.

Proof. Notice that the differentials of Koszul complex $K(m)$ is defined by multiplication by the generators of $m$, the computation follows immediately from Proposition 4.2 since $m$ is maximal and $R/m \cong k(m)$.

The next technical result of annihilation roughly says that for a fixed element $x \in R$, every map with domain a perfect complex $X$ and arbitrary codomain $Y$ can be annihilated by some power of $x$ as soon as the homologies of $Y$ can be annihilated by some power of $x$. Therefore, in particular, it allows us to obtain a nonzero map from a Koszul complex to an appropriate object in $D(R)$, as Lemma 4.6 shows, which is crucial in the proof of Lemma 4.7.

Lemma 4.5. Let $X \in D_{\text{perf}}(R)$ and $x \in R$. Let $Y \in D(R)$ and $f : X \to Y$ be a morphism in $D(R)$. Suppose for every $\ast$ and every $\alpha \in H_\ast(Y)$, there exists $n \in \mathbb{N}$ such that $x^n\alpha = 0$. Then $x^l f = 0$ for some $l \in \mathbb{N}$.

Proof. This is proved by induction using the filtration $X(i) = \bigoplus_{j=0}^i X_j$ on $X$, assuming that $X$ is bounded between dimension 0 and $k \geq 0$ with each degree free on finitely many generators.

First we show the case for $k = 0$, i.e. $X$ has nontrivial homology for only degree zero. Then that $X$ is perfect implies $\text{Hom}_{D(R)}(X, Y) \cong \bigoplus \text{Hom}_{D(R)}(R, Y) \cong \bigoplus H_0(Y)$, which has only finitely many summands by assumption. Hence any $f \in \text{Hom}_{D(R)}(X, Y)$ is annihilated by some power of $x$.

In general, for $k \geq 1$ we set $f_{k-1}^{(k-1)} = f^{(k-1)}i_{k-1}, f_k^{(k-1)} = f^{(k-1)}i_k$ and $f^{(k-1)} = x^{n_{k-1}}f^{(k-2)}$ such that the following diagram commutes

$$
\begin{array}{ccc}
X(k-1) & \overset{i_{k-1},k}{\longrightarrow} & X(k) & \overset{i_k}{\longrightarrow} & X \\
\downarrow f_k^{(k-1)} & & \downarrow f_k^{(k-1)} & & \downarrow f^{(k-1)} \\
Y & = & Y & = & Y
\end{array}
$$

where $i_{k-1} = i_k \circ i_{k-1}$ and $f^{(0)} = f$. We replace the three vertical maps by $f_k^{(k)} = x^{n_{k-1}}f_{k-1}^{(k-1)} = 0, f_k^{(k)} = x^{n_{k-1}}f_k^{(k-1)}$ and $f^{(k)} = x^{n_{k-1}}f^{(k-1)}$, still preserving the commutativity, where $f_{k-1}^{(k)} = 0$ for some power $n_{k-1}$ by induction. Hence there exists a map $\rho_k$ making the following diagram commute

$$
\begin{array}{ccc}
X(k-1) & \overset{i_{k-1},k}{\longrightarrow} & X(k) & \overset{\pi_k}{\longrightarrow} & X(k)/X(k-1) & \longrightarrow & \Sigma X(k-1) \\
\downarrow f_k^{(k)} & & \downarrow f_k^{(k)} & & \downarrow & & \downarrow \rho_k \\
Y & = & Y & = & Y
\end{array}
$$
so that \( x^{n_k} \rho_k = 0 \) for some \( n_k \) implies \( x^{n_k} f^{(k)} = 0 \). Therefore, since \( X(k) = X \) thus \( i_k = id_X \) and \( f^{(k-1)} = x^{\sum_{i=0}^{k-2} n_i} f \) is defined recursively, we deduce that

\[
0 = x^{n_k} f^{(k)} = x^{n_k} x^{n_k-1} f^{(k-1)} = x^{n_k} x^{n_k-1} f^{(k-1)} i_k = x^{\sum_{i=0}^{k} n_i} f,
\]

as we required. □

**Lemma 4.6.** Let \( M \in D(R) \). Suppose \( \mathfrak{p} = (x_1, \ldots, x_k) \in \text{Supp} M \) is minimal such that \( \mathfrak{p} \in \text{Supp} H_n(M) \) for some \( n \). Then \( \text{Hom}_{D(R)}(\Sigma^n K'(\mathfrak{p}), M) \neq 0 \), for appropriate powers.

**Proof.** Notice that \( K'(\mathfrak{p}) \) is a perfect complex, by Proposition 2.5 it suffices to show that \( \text{Hom}_{D(\mathfrak{p})}(\Sigma^n K'(\mathfrak{p}), \mathfrak{M}_p) \neq 0 \). Indeed, for every \( * \) for which \( H_* (M) \neq 0 \) and, for a decomposition of a cyclic module generated by a nontrivial element \( \alpha \in H_* (M) \), we can deduce that

\[
\mathfrak{p}_i \mathfrak{R}_p \in \text{Ass} \mathfrak{R}_p / \mathfrak{p} \mathfrak{R}_p \subseteq \text{Ass} (\mathfrak{R} \alpha) \mathfrak{p} \subseteq \text{Ass} H_*(\mathfrak{M}_p) \subseteq V(\mathfrak{p} \mathfrak{R}_p),
\]

where the last inclusion holds because the localization at \( \mathfrak{p} \) kills all the primes containing \( \mathfrak{p} \) and the minimality of \( \mathfrak{p} \in \text{Supp} M \) implies that all such primes \( \mathfrak{p}_i \mathfrak{R}_p \) must be identified with \( \mathfrak{p} \mathfrak{R}_p \). Inductively for each \( j = 1, \ldots, k \) there is some \( n_j \) such that \( \alpha \) is annihilated by \( x^{n_j}_j \).

For simplicity, we now assume that \( R \) is local. Suppose \( 0 \neq f_0 \in \text{Hom}(K'(0), M) \cong H_0(M) \). Since for any \( \alpha \in H_* (M) \) some power of \( x_1 \) annihilates \( \alpha \), there is by Lemma 4.2 some \( n_1 \in \mathbb{N} \) such that \( x_1^{n_1} \) annihilates \( f_0 \), hence the map \( f_0 \) extends to a map \( f_1 : K'(1) \to M \) by considering the distinguished triangle

\[
K'(0) \xrightarrow{x_1^{n_1}} K'(0) \to K'(1) \to \Sigma K'(0).
\]

Observe that the fact \( H_0(f_0) \neq 0 \) implies that \( H_0(f_1) \neq 0 \) since \( f_0 \) factors through \( f_1 \). Now suppose the maps \( f_j : K'(j) \to M \) are constructed such that \( H_0(f_j) \neq 0 \) for \( j \leq k - 1 \). By Lemma 4.2 again, that some power of \( x_k \) annihilates \( H_* (M) \) implies the existence of some \( n_k \in \mathbb{N} \) such that \( x_k^{n_k} f_{k-1} = 0 \). Now consider the distinguished triangle

\[
K'(k-1) \xrightarrow{x_k^{n_k}} K'(k-1) \to K'(k) = K' \to \Sigma K'(k-1),
\]

the annihilation implies there is an extension \( f = f_k : K' \to M \). Our inductive assumption implies that \( H_0(f) \neq 0 \), as we required. □

In particular, if \( M \in D(R) \) satisfies \( \text{Supp} M = V(\mathfrak{p}) = \text{Supp} H_n(M) \) for some prime \( \mathfrak{p} \), then Lemma 4.6 applies. We end this section by proving a generalization of Lemma 5.3 in [18], which shows the same thing but in the case when the object \( M \) is bounded.

**Lemma 4.7.** Let \( M \in D(R) \). Then \( \bigoplus_{i \in \mathbb{Z}} \bigoplus_{\mathfrak{p} \in \text{Supp} H_*(M)} \Sigma^i R/\mathfrak{p} < M \).
Proof. Denote by $E = E(M) = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{p \in \text{Supp}H_i(M)} \Sigma^i R/p$. We prove it by contradiction, assuming that $P_EM \neq 0$. Notice that $E < M(E)$ so that $\text{Ass}H_{i-1}M(E) \subseteq \text{Supp}H_{i-1}(E) \subseteq \text{Supp}H_{i-1}(M)$. Thus $\text{Ass}H_iP_EM \subseteq \text{Supp}H_i(M) \cup \text{Ass}H_{i-1}M(E) \subseteq \text{Supp}H_i(M) \cup \text{Supp}H_{i-1}(M)$ by Lemma 5.4 in [18]. Since every Noetherian local ring has finite dimension (see Corollary 11.11 in [11]), one can pick up a minimal prime in $\bigcup_p \text{Ass}H_iP_EM$ such that $p \in \text{Supp}H_n(M)$ for some $n$. The natural distinguished triangle

$$M(E) \rightarrow M \rightarrow P_EM \rightarrow \Sigma M(E)$$

allows us to identity $p$ to be minimal also in $\text{Supp}M$. Hence by Lemma 4.6 there is a map $f : \Sigma^n K(p) \rightarrow P_EM$ such that $H_n(f) \neq 0$. Therefore, the fact $E < \Sigma^n K(p)$ by Lemma 5.3 in [18] implies $\text{Hom}(\Sigma^i E, P_EM) \neq 0$ for some $i \geq 0$ by Proposition 3.4 in [18], a contradiction. \qed

5 The key lemma

This section presents the key ingredient, Lemma 5.10, in order to produce an invariant of nullity classes. In fact, this lemma implies the injectivity of the map $\phi$ defined in the next section, which maps the class of aisles in the derived category $D_{fg}(R)$ into the class of perversity functions on the power set $\mathcal{P}(\text{Spec}R)$.

We first recall two facts which are Lemma 6.1 and Lemma 6.2 respectively in [18].

Lemma 5.1. If $E < F$, then $E \otimes M < F \otimes M$ for every $M \in D(R)$.

Lemma 5.2. If $M \in D(R)$ such that $H_i(M) = 0$ for $i < 0$, then $R < M$.

In Notation 4.1 we define the Koszul complex $K(i)$ as the mapping cone of $x_i : K(i-1) \rightarrow K(i-1)$ with $K(0) = R$ and $M(i) = K(i) \otimes M$.

Lemma 5.3. Let $M \in D_{fg}(R)$ and $H_n(M) \neq 0$ for some $n$. Then

1. if $(R, m)$ is local, then $H_n(M \otimes K(m)) \neq 0$ and,
2. in general, there is a $p \in \text{Supp}H_n(M)$ such that $p \in \text{Supp}H_n(M \otimes K(p))$.

Proof. Suppose $R$ is local with $m = (x_1, ..., x_k)$. Then we have distinguished triangles

$$M(i) \xrightarrow{x_{i+1}} M(i) \longrightarrow M(i + 1)$$

which induce long exact sequences on homologies

$$H_n(M(i)) \xrightarrow{x_{i+1}} H_n(M(i)) \longrightarrow H_n(M(i + 1)).$$

It turns out inductively on $i$, the homology $H_n(M(i + 1)) \neq 0$ because otherwise, the induced map $x_{i+1}$ on the finitely generated module $H_n(M(i))$ is surjective, so that Lemma 2.3 implies $H_n(M(i)) = 0$, contradicting our assumption.
In general, since $H_n(M) \neq 0$, there is a prime ideal $p$ of $R$ such that $H_n(M_p) \cong H_n(M) \otimes R_p \neq 0$ by Proposition 3.8 in [1]. Then $H_n(M \otimes K(p)) \otimes R_p \cong H_n(M_p \otimes K(p)_p) \cong H_n(M_p \otimes K(pR_p)) \neq 0$ by (1) since $(R_p, pR_p)$ is local and $M_p \in D_{fg}(R_p)$. □

It is well-known that associated to each standard truncation $\tau_{\geq n}$ of an object $M$ in $D(R)$ there is a distinguished triangle

$$\tau_{\geq n}M \to M \to \tau_{<n}M$$

and also that $\tau_{\geq n} \tau_{\leq n} = \tau_{\leq n} \tau_{\geq n} = \Sigma^n H_n(M)$, where $\tau_{\leq n} = \tau_{<n+1}$. In particular, we have a distinguished triangle

$$\tau_{\geq n+1}M \to \tau_{\geq n}M \to \Sigma^n H_n(M)$$

for each $n$. See for example [3] and [21].

**Lemma 5.4.** Let $(R, m)$ be a local ring whose maximal ideal has $k$ generators and $M \in D_{fg}(R)$. Suppose $H_i(M \otimes k(m)) = 0$ for $n - k \leq i \leq n$. Then $\tau_{\leq n}M \otimes K(m) = 0$.

**Proof.** Truncating $K(m)$ each step a homology $\Sigma^i H_i(K(m))$, we have a filtration

$$0 = K_{k+1} \to K_k \to K_{k-1} \to \cdots \to K_1 \to K_0 = K(m)$$

for $K(m)$ with $K_i = \tau_{\geq i}K(m)$ such that $K_{i+1} \to K_i \to \Sigma^i H_i(K(m))$ is a distinguished triangle for each $i = 0, 1, \ldots, k$. Since tensor product preserves distinguished triangles, the sequence

$$0 = M \otimes K_{k+1} \to M \otimes K_k \to M \otimes K_{k-1} \to \cdots \to M \otimes K_1 \to M \otimes K_0 = M \otimes K(m)$$

remains a collection of distinguished triangles. Hence by taking the $n$-th homology and the long exact sequences, $H_i(M \otimes k(m)) = 0$ for $n - k \leq i \leq n$ implies $H_n(M \otimes \Sigma^i H_i(K(m))) \cong H_{n-i}(M \otimes (\oplus k(m))) = 0$ for $i = 0, 1, \ldots, k$ by Corollary [14]. Consequently, $H_n(M \otimes K_i) = 0$ for each $i$. In particular, $H_n(M \otimes K(m)) = 0$ as required. □

**Corollary 5.5.** Suppose $M \in D_{fg}(R)$ and $p$ is a prime ideal having $k$ generators, such that $H_i(M \otimes k(p)) = 0$ for $n - k \leq i \leq n$. Then $H_n(M_p) = 0$.

**Proof.** Notice that $M \otimes k(p) \cong M_p \otimes k(pR_p)$. Thus $H_n(M_p \otimes K(pR_p)) = 0$ by Lemma [5.4]. Hence $H_n(M_p) = 0$ thanks to Lemma [5.3]. □

The following lemma is a generalization of Lemma 6.4 in [18] in which the object is assumed to be bounded.

**Lemma 5.6.** Suppose $M \in D_{fg}(R)$ and $p \in \text{Supp} H_n(M)$ has $k$ generators. Then $M < \Sigma^{n-i} k(p)$ for some $0 \leq i \leq k$. In particular, $M < \Sigma^{n-i} k(q)$ for every $q \in V(p)$.
Proof. Observe that $R < k(p)$ implies $M < M \otimes k(p)$ by Lemma 5.1 and 5.2 which is a direct sum of suspensions of finite copies of $k(p)$ by Lemma 2.17 in [1]. Thus $p \in \text{Supp} R_n(M)$ implies $H_{n-i}(M \otimes k(p)) \neq 0$ for some $0 \leq i \leq k$ by Lemma 5.5. Hence $M < \Sigma^{n-k}(p)$, since nullity class is closed under retracts. The second statement follows from the fact that the support sets are specialization closed. \( \square \)

The proof of the next lemma follows precisely that of Proposition 6.6 in [18]. For convenience, we denote by $\bar{k}(p) = \bigoplus_{q \in V(p)} k(q)$.

Lemma 5.7. Suppose $\dim R/\mathfrak{p}$ is finite. Then $\bar{k}(p) < \Sigma^{\dim R/\mathfrak{p}} R/\mathfrak{q}$ for every $\mathfrak{q} \in V(p)$.

Proof. Prove by induction on $\dim R/\mathfrak{p}$. Suppose $\dim R/\mathfrak{p} = 0$, i.e. $\mathfrak{p} = \mathfrak{m}$ is maximal, then the inequality holds since $k(\mathfrak{m}) \cong R/\mathfrak{m}$. Now for a fixed $\mathfrak{q} \in V(p)$, assume $\bar{k}(p) < \Sigma^{\dim R/\mathfrak{q}'} R/\mathfrak{q}'$ for every $\mathfrak{q}' \in V(q) - \{\mathfrak{q}\}$. Consider the short exact sequence of $R$-modules

$$0 \to R/\mathfrak{q} \xrightarrow{f} k(q) \to M \to 0$$

where $f$ is the natural map and $M$ its quotient. Since $f \otimes R/\mathfrak{q}$ becomes an isomorphism, $\text{Supp} M \subseteq V(q) - \{\mathfrak{q}\}$. Hence by induction together with Lemma 4.7, we can deduce that $\bar{k}(p) < \Sigma^{\dim R/\mathfrak{q}} R/\mathfrak{q} < \Sigma^{\dim R/\mathfrak{q} - 1} M$. Furthermore, the distinguished triangle

$$\Sigma^{\dim R/\mathfrak{q} - 1} M \to \Sigma^{\dim R/\mathfrak{q}} R/\mathfrak{q} \to \Sigma^{\dim R/\mathfrak{q}} k(q)$$

allows us to conclude that $\bar{k}(p) < \Sigma^{\dim R/\mathfrak{q}} R/\mathfrak{q}$, since nullity classes are closed under retracts and extensions. \( \square \)

As an easy consequence we can show the following statement.

Corollary 5.8. Suppose $\dim R/\mathfrak{p}$ is finite. Then $\bar{k}(p) < \Sigma^{\dim R/\mathfrak{p}} K'(p)$ for every $K'$ with arbitrary finite powers on the generators of $p$. In particular, $k(\mathfrak{m}) < K'(\mathfrak{m})$ if $\mathfrak{m}$ is maximal.

Proof. By Lemma 4.7, $\bigoplus_{i \geq 0} \bigoplus_{q \in \text{Supp} H_i(K'(p))} \Sigma^i R/\mathfrak{q} < K'(p)$, thus by Lemma 5.7 we have $\bar{k}(p) < \Sigma^{\dim R/\mathfrak{p}} \bigoplus_{i \geq 0} \bigoplus_{q \in \text{Supp} H_i(K'(p))} \Sigma^i R/\mathfrak{q} < \Sigma^{\dim R/\mathfrak{p}} K'(p)$, since nullity class is closed under coproducts and suspensions. \( \square \)

As a crucial step in proving our key lemma, we show that

Lemma 5.9. Suppose $M \in D^b_{lg}(R)$ and $p \in \text{Supp} H_n(M)$, $p \notin \text{Supp} H_i(M)$ for $i < n$. Then there is a map $f : M \to \Sigma^n H_n(M)$ such that $H_n(f_p)$ is an isomorphism.

Proof. Notice that $H_i((\tau_{<n} M)_p) = H_i(\tau_{<n} M)_p = 0$ for all $i < n$. Thus the natural map $\tau_{\geq n} M_p \to M_p$ associated to the standard truncation of $M$ at $n$ then localized at $p$ is a quasi-isomorphism. Hence

$$\text{Hom}(\tau_{\geq n} M, \Sigma^n H_n(M)_p) = \text{Hom}(M_p, \Sigma^n H_n(M)_p) = \text{Hom}(M, \Sigma^n H_n(M)) \otimes R_p$$
by Proposition \[2.5\]
Therefore, there is a morphism \( f : M \to \Sigma^n H_n(M) \) corresponding to the natural map \( \pi : \tau_{\geq n} M \to \Sigma^n H_n(M) \) via the above isomorphisms. Since \( H_n(\pi_p) \) is an isomorphism, so is \( H_n(f_p) \). \( \square \)

Finally we are well-prepared to prove the key lemma.

**Lemma 5.10.** Suppose \( \dim R \) is finite and \( M \in D_{fg}(R) \). Then \( M < \Sigma^n R/p \) for every \( p \in \text{Supp} H_n(M) \).

**Proof.** It is not hard to show that for every degree \( m \) where \( p \) stands, we have \( M < \Sigma^{m+\dim R/p} R/p \) by Lemma \[5.6\] and Lemma \[5.7\] together with the transitivity of nullity classes. Define

\[
  l(p) = \inf \{ i \in \mathbb{Z} \mid p \in \text{Supp} H_i(M) \}.
\]

If \( l(p) = -\infty \), then for each degree \( n \) where \( p \) stands, \( M < \Sigma^{m+\dim R/p} R/p < \Sigma^n R/p \) since \( m \) can be chosen such that \( n - m - \dim R/p \geq 0 \). So fix a prime \( p \) and suppose \( l = l(p) \) is finite, i.e. \( p \in \text{Supp} H_i(M) \) and \( p \notin \text{Supp} H_i(M) \) for \( i < l \). Also, suppose for every \( n \) and every \( q \in \text{Supp} H_n(M) \), \( M < \Sigma^{n+k} R/q \) for \( 1 \leq k \leq \dim R/q \). We need to show that \( M < \Sigma^l R/p \) by downward induction.

The finiteness of \( l(p) \) implies that there is a map \( f : M \to \Sigma^l H_l(M) \) by Lemma \[5.9\]. Now consider the distinguished triangle

\[
  M \xrightarrow{f} \Sigma^l H_l(M) \rightarrow M'
\]

and the associated long exact sequence which splits into short ones as

\[
  0 \to H_{l+1}(M') \to H_l(M) \xrightarrow{H_l(f)} H_l(M) \to H_l(M') \to H_{l-1}(M) \to 0
\]

and \( H_{l+i+1}(M') \cong H_{l+i}(M) \) for either \( i \geq 1 \) or \( i \leq -2 \). Since \( H_l(f_p) \) is an isomorphism, thus \( H_l(M')_p = 0 \) so that \( \text{Supp} H_l(M') \subseteq \text{Supp} H_l(M) \cup \text{Supp} H_{l-1}(M) - \{p\} \) by Lemma \[2.1\]. Next we show that \( M < \Sigma^l R/q \) for every \( q \in V(p) - \{p\} \). Since \( l \) is the minimal number at which \( p \) stands, Lemma 6.7 in \[18\] implies that \( M < N = M \otimes K(q) \) such that \( q \in \text{Supp} H_l(N) \) and \( \text{Supp} N \subseteq V(q) \). Therefore, it is sufficient to show \( N < \Sigma^l R/q \) for every \( q \in V(p) - \{p\} \), which is proved by induction on the dimension \( \dim R/q \), similarly to the proof of Lemma \[5.7\].

In fact, if \( q = m \) is maximal then \( N < \Sigma^l R/m = \Sigma^l k(m) \) by Lemma \[5.6\]. Now suppose it is true for every \( q \in V(p) \) such that \( \dim R/q \leq \dim R/p - 2 \), we need to show it is true for every \( q \in V(p) - \{p\} \), i.e. \( \dim R/q \leq \dim R/p - 1 \). Consider the short exact sequence of \( R \)-modules

\[
  0 \to R/q \xrightarrow{\iota} k(q) \to A \to 0
\]

where \( \iota \) is the natural map and \( A \) its quotient. Since \( \iota_q \) becomes an isomorphism, \( \text{Supp} A \subseteq V(q) - \{q\} \). Hence by induction together with Lemma \[4.7\] we can deduce that \( N < \Sigma^l R/q \) for every \( q \in V(p) - \{p\} \).
\[ \Sigma^l \bigoplus_{q' \in \text{Supp} A} R/q' < \Sigma^l A. \] Furthermore, \( N < \Sigma^l k(q) \) by Lemma 5.6. Hence the distinguished triangle
\[ \Sigma^l R/q \rightarrow \Sigma^l k(q) \rightarrow \Sigma^l A \]
allows us to conclude that \( N < \Sigma^l R/q \) since nullity classes are closed under extensions.

Combine what we have shown that \( M < \Sigma^l R/q \) for every \( q \in V(p) - \{ p \} \) and the fact that the support of \( M' \) is shifted down by one into the support of \( M \) for every degree, we deduce by the downward induction and Lemma 4.7 that \( M < \bigoplus_i \bigoplus_{p \in \text{Supp} H_i(M')} \Sigma^l R/p < M' \). Hence \( M < \Sigma^l H_l(M) \) since nullity classes are closed under extensions. Therefore, \( M < \Sigma^l R/p \) by Lemma 6.8 in [18], and the proof is complete. □

Therefore, together with Lemma 4.7 our key lemma implies that in fact, each object in \( D(R) \) is completely determined by its support in the sense of construction of nullity classes, i.e. they generate the same category.

### 6 A complete invariant

Now we are ready to produce an invariant of nullity classes as well as aisles in the derived category \( D_{\text{fg}}^c(R) \), as soon as we establish the involved maps \( \phi \) and \( N \) as follows.

We call a function \( f : \mathbb{Z} \rightarrow \mathcal{P}(\text{Spec } R) \) a perversity function if \( f(n) = \overline{f(n)} \), i.e. specialization closed, and \( f(n) \subseteq f(n + 1) \) for each \( n \in \mathbb{Z} \). Then for every nullity class \( A \) in a full triangulated subcategory \( D \subseteq D(R) \) we define \( \phi(A) \) by setting \( \phi(A)(n) = \{ p \in \text{Spec } R \mid p \in \text{Supp} H_n(M) \text{ for some } M \in A \} \). Clearly such a function is increasing since nullity class is closed under suspensions and also \( \phi(A)(n) \) is closed under specialization since the support set is. Hence the map

\[ \phi : \{ \text{nullity classes in } D \} \rightarrow \{ \text{perversity functions} \} \]
is well-defined. On the other hand, let \( S(f) = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p \in f(n)} \Sigma^n R/p \) and \( \overline{C(S(f))} \) be the smallest nullity class containing \( S(f) \). We define a function

\[ N : \{ \text{perversity functions} \} \rightarrow \{ \text{nullity classes in } D \} \]
which associates each perversity function \( f \) a nullity class \( N(f) = \overline{C(S(f))} \cap D \).

**Lemma 6.1.** The maps \( N \) and \( \phi \) are order preserving.

**Proof.** Clearly \( \phi \) is order preserving by definition. For \( N \) we consider \( f, g \) any two perversity functions such that \( f \leq g \), i.e. \( f(n) \subseteq g(n) \) for all \( n \). Then \( S(f) \) is a retract of \( S(g) \) by definition, which implies that \( S(f) \in \overline{C(S(g))} \) hence \( \overline{C(S(f))} \subseteq \overline{C(S(g))} \). □
Theorem 6.2. Suppose \( \dim R \) is finite and \( \mathcal{D} = D^c_{fg}(R) \). Then \( \phi : \{ \text{nullity classes in } \mathcal{D} \} \to \{ \text{perversity functions} \} \) is a bijection with inverse given by \( N \).

**Proof.** Let \( \mathcal{A} \) be a nullity class in \( \mathcal{D} \). Take \( M \in \mathcal{A} \). Then by Lemma 5.9 we have \( S(\phi(\mathcal{A})) = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p \in \phi(\mathcal{A})(n)} \Sigma^n R/p < \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p \in \text{Supp} H_n(M)} \Sigma^n R/p \) since the latter coproduct is a retract of the former one. Thus \( S(\phi(\mathcal{A})) < M \) by Lemma 4.7, so that \( M \in \overline{C(S(\phi(\mathcal{A})))} \). Hence \( M \in N\phi(\mathcal{A}) \). Conversely, for every \( p \in \phi(\mathcal{A})(n) \) there is \( M(p,n) \in \mathcal{A} \) such that \( p \in \text{Supp} H_n(M(p,n)) \) by definition. Thus by Lemma 5.10 we have \( M(p,n) < \Sigma^n R/p \), so that \( M = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p \in \phi(\mathcal{A})(n)} M(p,n) < \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p \in \phi(\mathcal{A})(n)} \Sigma^n R/p = S(\phi(\mathcal{A})) \). Hence \( S(\phi(\mathcal{A})) \in \overline{C(M)} \subseteq \mathcal{A} \).

On the other hand, we show that \( \phi N(f)(n) = f(n) \) for every perversity function \( f : \mathbb{Z} \to \mathcal{P}(\text{Spec} R) \) and every \( n \in \mathbb{Z} \). Take \( p \in f(n) \). Then \( S(f) < \Sigma^n R/p \) since \( \Sigma^n R/p \) is a retract of \( S(f) \), so that \( \Sigma^n R/p \in N(f) \). It follows that \( p \in \text{Ass} H_n(\Sigma^n R/p) = \text{Ass} R/p \), thus \( p \in \phi(N(f))(n) \). Conversely, suppose \( p \in \phi(N(f))(n) \). Then there is some \( M \in N(f) = \overline{S(f)} \) such that \( p \in \text{Supp} H_n(M) \). Lemma 3.10 implies that \( p \in \text{Supp} H_i(S(f)) \) for some \( i \leq n \). Hence \( p \in f(i) \subseteq f(n) \), as required. \( \square \)

This theorem has immediate consequences as the following corollaries.

**Corollary 6.3.** Every aisle in \( \mathcal{D} = D^c_{fg}(R) \) with finite \( \dim R \) is of the form \( \overline{C(E)} \cap \mathcal{D} \) for some \( E \in D(R) \).

**Corollary 6.4.** Let \( \mathcal{D} = D^c_{fg}(R) \) with finite \( \dim R \). Suppose \( \mathcal{A} \) and \( \mathcal{A}' \) are aisles in \( \mathcal{D} \). Then \( \mathcal{A} \subseteq \mathcal{A}' \) if and only if \( \phi(\mathcal{A}) \subseteq \phi(\mathcal{A}') \). Thus \( \phi : \{ \text{aisles in } \mathcal{D} \} \to \{ \text{perversity functions} \} \) is injective.

Since aisles are nullity classes, the map \( \phi \) gives a complete invariant for aisles (or \( t \)-structures) thanks to Theorem 6.2.

**References**

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Company, 1969.

[2] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *Journal für die reine und angewandte Mathematik*, (588):149–168, 2005.

[3] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. *Asterisque*, (100):5–171, 1982.

[4] D. Benson, S. B. Iyengar, and H. Krause. Statifying modular representations of finite groups. *Annals of Mathematics*, (174):1643–1684, 2011.
[5] M. Bökstedt and A. Neeman. Homotopy limits in triangulated categories. *Compositio Mathematica*, (86):209–234, 1993.

[6] N. Bourbaki. *Commutative Algebra, Chapters 1-7*. Springer-Verlag, 1988.

[7] W. Dwyer and J. Spalinski. Homotopy theories and model categories. In I. M. James, editor, *Handbook of Algebraic Topology*. 1995.

[8] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, 1994.

[9] P. Gabriel. Des catégories abéliennes. *Bulletin de la Société Mathématique de France*, (90):323–448, 1962.

[10] P. J. Hilton and U. Stammbach. *A Course in Homological Algebra*, volume 4 of *Graduate Text in Mathematics*. Springer-Verlag, 1970.

[11] M. Hochster. Prime ideal structure in commutative rings. *Transactions of the American Mathematical Society*, (142):43–60, 1969.

[12] M. J. Hopkins. Global methods in homotopy theory. In *Homotopy Theory: Proceedings of the Durham Symposium 1985*. Cambridge University Press, 1987.

[13] M. Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2007.

[14] B. Keller and D. Vossieck. Aisles in derived categories. *Bulletin of the Belgian Mathematical Society*, (40):239–253, 1988.

[15] A. Neeman. The chromatic tower for $D(R)$. *Topology*, (3):519–532, 1992.

[16] A. Neeman. Colocalizing subcategory of $D(R)$. *Journal für die reine und angewandte Mathematik*, (653):221–243, 2011.

[17] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Mathematica*, (65):121–154, 1988.

[18] D. Stanley. Invariants of $t$-structures and classification of nullity classes. *Advances in Mathematics*, (224):2662–2689, 2010.

[19] L. Alonso Tarrio, A. Jeremias Lopez, and M. Saorin. Compactly generated $t$-structures on the derived category of a noetherian ring. *Journal of Algebra*, (324):313–346, 2010.

[20] R. W. Thomason. The classification of triangulated subcategories. *Compositio Mathematica*, (105):1–27, 1997.
[21] C. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1993.