Willmore Legendrian surfaces in pseudoconformal 5-sphere

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Introduction

For a compact oriented submanifold $X: M^n \hookrightarrow \mathbb{E}^N$ in a Euclidean space, Willmore functional $\mathcal{W}(X)$ is defined by

$$\mathcal{W}(X) = \int_M \frac{1}{n} |B_0|^n \, dX,$$

where $B_0$ is the trace free part of the second fundamental form of $X$. It was introduced by Willmore in a slightly different but equivalent form for surfaces in $\mathbb{E}^3$. The functional is invariant under conformal transformation, and it is natural to extend it for submanifolds in conformal $N$-sphere $S^N = \mathbb{E}^N \cup \{\infty\}$. Willmore conjecture asks if Clifford torus in $S^3$ is the unique minimizer of $\mathcal{W}(X)$ among all immersed tori.

Li and Yau introduced the notion of conformal area of a compact Riemann surface, and showed that Willmore conjecture is true if the induced complex(conformal) structure
of the competing torus lies in a specific neighborhood of the square torus [LY]. Bryant proved a duality theorem, and studied a class of Willmore surfaces for which Willmore dual is constant [Br2][Br3]. He gave a classification of such surfaces in terms of complete minimal surfaces of finite total curvature with embedded ends of zero logarithmic growth. Ros answered the conjecture in the affirmative among tori invariant under antipodal involution using his solution to isoperimetric problem in \( \mathbb{R}P^3 \) [Ro]. Recently Topping gave an integral geometric proof of Ros’s result [To]. Babich and Bobenko constructed examples of Willmore tori in \( \mathbb{R}^3 \) with a planar reflectional symmetry by gluing two minimal surfaces in hyperbolic space [BB]. Helein gave a loop group formulation of constructing all Willmore tori [He]. Montiel and Urbano studied a Willmore type functional for surfaces in \( \mathbb{C}P^2 \) [MU].

The problem has a pseudoconformal(CR) cousin. Let \( \Sigma^n = S^{2n+1} \) be the pseudoconformal sphere with \( \text{SU}(n+1,1) \) as the group of automorphisms [ChM]. Let \( X : M^n \hookrightarrow \Sigma^n \) be a Legendrian submanifold. Then \( M \) inherits a conformal structure, and Willmore functional \( \mathcal{W}(X) \) in this case is a pseudoconformally invariant second order functional defined similarly as above, see (15). A Legendrian submanifold is called Willmore if it is critical for this functional.

The purpose of this paper is to study Willmore Legendrian surfaces in the spirit of [Br2]. There are two main results. The first is a construction of Willmore Legendrian dual for a class of Willmore Legendrian surfaces, **Theorem 3.1**. The second is a characterization of Willmore Legendrian surfaces with constant Willmore dual in terms of immersed meromorphic curves in \( \mathbb{C}^2 \) satisfying an appropriate real period condition, **Theorem 4.1**. We show that every compact Riemann surface admits a generally one to one, conformal, Willmore Legendrian immersion in \( \Sigma^2 = S^5 \) with constant Willmore dual, **Theorem 4.2**. As a corollary, every compact Riemann surface can be conformally immersed in \( \mathbb{C}^2 \) as an exact, algebraic Lagrangian surface.

In Section 1, we give an explicit formulation of pseudoconformal stereographic pro-
jection. The associated pseudoconformal involution will play a role in our ends analysis of Willmore Legendrian surfaces with constant Willmore dual. In Section 2, a basic structure equation for Legendrian surfaces in $\Sigma^2$ is established. A complex quartic differential $\Phi$ arises as a fourth order invariant of the immersion. In case $\Phi = 0$, a complex sextic differential $\Psi$ is defined and holomorphic. In addition, the umbilic loci in this case, the points where the trace free part of second fundamental form vanish, is a closed subset with no interior. In section 3, we define the Willmore functional, and identify its Euler-Lagrange equation; a Legendrian surface is Willmore when $\Phi$ is holomorphic with respect to the induced complex structure. It is a fifth order elliptic equation for the immersion. When $\Phi = 0$, an associated Willmore dual is well defined and can be smoothly extended across the umbilic loci. In section 4, we give a Weierstraß type representation for surfaces with $\Phi = \Psi = 0$, which is equivalent to Willmore dual being constant \(^{(20)}\). By Riemann-Roch, every Willmore Legendrian sphere belongs to this class. We show that there exists Willmore sphere with $\mathcal{W} = 4\pi k$ for each integer $k \geq 0$.

One of our motivation is the question, "What is the best compact Legendrian surface of given genus in $\Sigma^2 = S^5$?" It is perhaps the case that the following is the pseudoconformal analogue of Willmore conjecture; the minimal Legendrian hexagonal torus

$$T = \{ (\xi^1, \xi^2, \xi^3) \mid |\xi^i|^2 = \frac{1}{3}, \Im(\xi^1\xi^2\xi^3) = 0 \} \subset S^5 \subset \mathbb{C}^3$$

uniquely minimizes Willmore functional among all immersed Legendrian tori. A study of this and other related problems for Willmore Legendrian tori will be the subject of the subsequent paper.
1 Pseudoconformal stereographic projection

In this section, we describe $5$-sphere $S^5$ as the standard model of pseudoconformal space. Pseudoconformal analogue of the usual stereographic projection in conformal geometry is introduced, which enables us to compare the geometry of Legendrian surfaces in $S^5$ with the geometry of exact Lagrangian surfaces in $\mathbb{C}^2$.

Let $\mathbb{C}^{3,1}$ be the complex vector space with coordinates $z = (z^0, z^i, z^3)$, $1 \leq i \leq 2$, and a Hermitian scalar product

$$\langle z, \bar{z} \rangle = z^i \bar{z}^i + i(z^0 \bar{z}^3 - z^3 \bar{z}^0).$$

Let $\Sigma$ be the set of equivalence classes up to scale of null vectors with respect to this product. Let $\text{SU}(3,1)$ be the group of unimodular linear transformations that leave the form $\langle z, \bar{z} \rangle$ invariant. $\text{SU}(3,1)$ acts transitively on $\Sigma$, and

$$p : \text{SU}(3,1) \to \Sigma = \text{SU}(3,1)/P$$

for an appropriate subgroup $P$ [ChM].

Explicitly, consider an element $Z = (Z_0, Z_i, Z_3) \in \text{SU}(3,1)$ as an ordered set of four column vectors in $\mathbb{C}^{3,1}$ such that $\det(Z) = 1$, and that

$$\langle Z_i, \bar{Z}_j \rangle = \delta_{ij}, \quad \langle Z_0, \bar{Z}_3 \rangle = -\langle Z_3, \bar{Z}_0 \rangle = i,$$  \hspace{1cm} (2)

while all other scalar products are zero. We define

$$p(Z) = [Z_0],$$

where $[Z_0]$ is the equivalence class of null vectors represented by $Z_0$. The left invariant Maurer-Cartan form $\pi$ of $\text{SU}(3,1)$ is defined by the equation

$$dZ = Z \pi,$$
which is in coordinates

\[
d(Z_0, Z_i, Z_3) = (Z_0, Z_j, Z_3) \begin{pmatrix}
\pi^0_0 & \pi^0_i & \pi^0_3 \\
\pi^j_0 & \pi^j_i & \pi^j_3 \\
\pi^3_0 & \pi^3_i & \pi^3_3
\end{pmatrix}.
\]

Components of \( \pi \) are subject to the relations obtained from differentiating (2) which are

\[
\begin{align*}
\pi^3_0 &= \bar{\pi}^3_0, & \pi^0_3 &= \bar{\pi}^0_3 \\
\pi^i_3 &= -i \bar{\pi}^i_0, & \pi^i_0 &= i \bar{\pi}^0_i \\
\pi^i_j + \bar{\pi}^i_j &= 0, & \pi^0_0 + \bar{\pi}^3_3 &= 0 \\
\text{tr} \, \pi &= 0,
\end{align*}
\]

and \( \pi \) satisfies the structure equation

\[
- \, d\pi = \pi \wedge \pi.
\]

It is well known that the SU(3,1)-invariant CR structure on \( \Sigma \subset CP^3 \) as a real hypersurface is biholomorphically equivalent to the standard CR structure on \( S^5 = \partial B^3 \), where \( B^3 \subset \mathbb{C}^3 \) is the unit ball. The structure equation (3) shows that for any local section \( s : \Sigma \to SU(3,1) \), this CR structure is defined by the hyperplane fields \((s^* \pi^3_0)^\perp = \mathcal{H})\) and the set of \((1,0)\)-forms \( \{s^* \pi^0_0\}\).

Let \( \mathbb{L} \subset \mathbb{C}^{3,1} \) be the cone of nonzero null vectors. SU(3,1) acts transitively on \( \mathbb{L} \), and \( \mathbb{L} \to \Sigma \) is a \( \mathbb{C}^* \)-bundle. Take \( w_3 \in \mathbb{L} \), and let

\[
\Sigma_{w_3} = \{ w \in \mathbb{L} \mid \langle w, \bar{w}_3 \rangle = i \}.
\]

As a submanifold of \( \mathbb{C}^{3,1} \), \( \Sigma_{w_3} \) inherits a real contact form \( i \langle dw, \bar{w} \rangle \) and a degenerate Hermitian metric of signature \((2,0)\), which together form a subHermitian structure. Moreover, the natural projection \( \pi : \Sigma_{w_3} \to \Sigma - \{p_\infty\} \) is easily seen to be a pseudoconformal equivalence, where \( p_\infty = [w_3] \).
Take $w_0 \in \Sigma_{w_3}$, and denote $E = \langle w_0, w_3 \rangle \perp \simeq \mathbb{C}^2 \subset \mathbb{C}^{3,1}$. From the choice of $w_0$, the induced Hermitian metric on $E$ is positive definite. Let $z = (z^1, z^2)$ be the standard coordinate of $E$.

Let $P : \Sigma_{w_3} \to \mathbb{E}$ be the orthogonal projection defined by

$$P(w) = w - w_0 - i \langle w, \bar{w}_0 \rangle w_3. \quad (5)$$

Let $\dot{w}$ be a tangent vector to $\Sigma_{w_3}$ so that $\langle \dot{w}, \bar{w}_3 \rangle = 0$. Then

$$\langle P_*(\dot{w}), \bar{P}_*(\dot{w}) \rangle = \langle \dot{w}, \bar{\dot{w}} \rangle,$$

and $P$ is a Riemannian submersion whose fibers are in the null direction. In fact, since $P$ is linear, it is easy to see that

$$P(w) = P(w') \iff w - w' = s w_3, \quad s \in \mathbb{R},$$

and $\Sigma_{w_3}$ is ruled by real lines parallel to $w_3$.

**Definition 1.1** Let $p_{\infty} \in \Sigma$, and let $w_0, w_3 \in \mathbb{L}$ be such that $p_{\infty} = [w_3]$, and that $\langle w_0, \bar{w}_3 \rangle = i$. The pseudoconformal stereographic projection at $p_{\infty}$ is the (orthogonal) projection defined by (5)

$$P : \Sigma - \{p_{\infty}\} \simeq \Sigma_{w_3} \to \mathbb{E} = \langle w_0, w_3 \rangle \perp \simeq \mathbb{C}^2.$$

**Remark.** Given a point $p_{\infty} \in \Sigma$, the stereographic projection from $p_{\infty}$ depends on the choice null vectors $w_0, w_3 \in \mathbb{L}$. For definiteness, we assume such a choice is made whenever stereographic projection is used.

Let $t$ be the coordinate of $\mathbb{R}$, and let $V = \mathbb{R} \oplus \mathbb{E}$ with the standard subHermitian structure given by the contact form $dt + \frac{i}{2} \left( \langle dz, \bar{z} \rangle - \langle z, d\bar{z} \rangle \right)$, and $\mathbb{E}$-wise subHermitian metric.

**Proposition 1.1** There exists a quadratic subHermitian isomorphism

$$\hat{Q} : V \to \Sigma_{w_3}.$$
Proof. Let $Q : \mathbb{E} \to \Sigma_{w_3}$ be the quadratic map

$$Q(z) = z + w_0 - \frac{i}{2} \langle z, \bar{z} \rangle w_3$$

for $z \in \mathbb{E}$. Then $\langle Q(z), \bar{w}_3 \rangle = i$, $\langle Q(z), \bar{Q}(z) \rangle = 0$, and $Q$ is well defined.

Note

$$P(Q(z)) = z$$

$$\langle Q_*(\dot{z}), \bar{Q}_*(\dot{z}) \rangle = \langle \dot{z}, \bar{z} \rangle,$$

for $\dot{z} \in \mathbb{E}$. Now define

$$\hat{Q}(t, z) = Q(z) + tw_3.$$  (7)

From (6), $Q$ is one to one. Since fibers of $P$ are real lines parallel to $w_3$, $\hat{Q}$ is also one to one, and obviously onto, and hence a diffeomorphism. The rest follows by direct computation. The inverse of $\hat{Q}$ is given by

$$\hat{P}(w) = \hat{Q}^{-1}(w) = \left( \frac{i}{2} (\langle w, \bar{w}_0 \rangle - \langle w_0, \bar{w} \rangle), P(w) \right). \quad \square$$

$\Sigma = V \cup \{p_\infty\}$ can thus be identified with the one point pseudoconformal compactification of $V$, and Legendrian surfaces in $\Sigma - \{p_\infty\}$ are in one to one correspondence with exact Lagrangian surfaces in $\mathbb{C}^2$ via pseudoconformal stereographic projection. A Lagrangian surface $M \subset \mathbb{C}^2$ with symplectic form $\varpi$ is exact if an(any) antiderivative $\eta$ of $\varpi$ in $\mathbb{C}^2$, $d\eta = \varpi$, is an exact 1-form when restricted to $M$.

For later purpose, we describe the pseudoconformal involution. Suppose we swap $(w_3, w_0)$ to $(w_0, -w_3)$ in the above construction. Under the isomorphism (7), this induces the pseudoconformal involution of $\Sigma = V \cup \{p_\infty\}$ given by

$$(t, z) \to (-\text{Re} \lambda, \lambda z), \quad \lambda = \frac{1}{t - \frac{i}{2} \langle z, \bar{z} \rangle}.\quad \text{(8)}$$

It interchanges $p_\infty = [w_3]$ with $(0, 0) = [w_0] \in V$. 

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Let $M$ be a compact surface with a divisor $D \subset M$. Let $f : M - D \hookrightarrow E$ be an exact Lagrangian surface which is complete and with embedded ends $D$. If $f$ satisfies an appropriate asymptotic regularity condition on $D$, the natural Legendrian lift $X_f$ of $f$ in $V = \mathbb{R} \oplus E \simeq \Sigma_{w_3}$ closes up across $D$ to be a smooth Legendrian immersion $X_f : M \hookrightarrow \Sigma$. One such occasion will be described in Section 4.

2 Legendrian surfaces

To study Legendrian surfaces in $\Sigma$, we employ the method of moving frames and construct an adapted frame bundle, on which the rest of our analysis is based. Willmore functional for Legendrian surfaces we are interested in is nothing but the well known Willmore functional restricted to exact Lagrangian surfaces in $\mathbb{C}^2$ under pseudoconformal stereographic projection.

The projection map (1) makes $\text{SU}(3, 1)$ into a right principal bundle over pseudoconformal 5-sphere $\Sigma$ with fiber

$$
G_0 = \left\{ \begin{pmatrix} a^{-1} & i a^{-1} \bar{v}^t A & c \\ 0 & A & v \\ 0 & 0 & \bar{a} \end{pmatrix} \middle| \begin{array}{l}
 a \neq 0, \quad v^t = (v^1, v^2), \quad |v|^2 = i(\bar{a} \bar{c} - ac) \\
 A \in U(2), \quad a^{-1} \bar{a} \det A = 1
\end{array} \right\}.
$$

This verifies that the 1-form $\pi_3^3$ from (3) is well defined up to scale on $\Sigma$, and defines a contact structure $\mathcal{H} = (\pi_3^3)^\perp$. The 1-forms $\{ \pi_0^1, \pi_0^2 \}$ are well defined mod $\pi_0^3$ up to conformal unitary transformations, and defines a conformal Hermitian structure on $\mathcal{H}$.

A Legendrian surface consists of an oriented surface $M$ and a Legendrian immersion

$$ X : M \hookrightarrow \Sigma, \quad X_*(TM) \subset \mathcal{H}. $$

We wish to study its properties invariant under the group action (1).
The 0-adapted frame bundle $F_0$ of $X$ is by definition

$$F_0 = X^* \text{SU}(3,1).$$

(9)

Since $X$ is Legendrian, $\pi_0^3 = 0$, $\pi_0^i \wedge \bar{\pi}_0^i = 0$ on $F_0$, and $\{\pi_0^1, \pi_0^2\}$ are semi-basic 1-forms on $M$. An element $g \in G_0$ acts on $\{\pi_0^1, \pi_0^2\}$ by

$$R_g^* \left( \begin{array}{c} \pi_0^1 \\ \pi_0^2 \end{array} \right) = a^{-1} A^{-1} \left( \begin{array}{c} \pi_0^1 \\ \pi_0^2 \end{array} \right),$$

where $R_g$ is the right action by $g$.

We may thus define the 1-adapted frame bundle $F_1$ by

$$F_1 = \{ \pi_0^i = \omega^i \text{ is real, } \omega^1 \wedge \omega^2 > 0 \} \subset F_0,$$

with the structure group $G_1 = \{ g \in G_0 | aA = a\bar{A}, a^4 > 0 \}$. Note that we have used the fact $M$ is oriented. An oriented Legendrian surface inherits a conformal(complex) structure, for $(aA) (aA)^t = a\bar{a}, \text{det} (aA) = \frac{a^4}{a\bar{a}} > 0$.

The structure equation (3) with this relation gives

$$-d\omega^i = (\pi_j^i - \delta_{ij} \pi_0^0) \wedge \omega^j.$$

Set $\pi_j^i - \delta_{ij} \pi_0^0 = \alpha_j^i + i \beta_j^i$. Then

$$-d\omega^i = \alpha_j^i \wedge \omega^j, \quad 0 = \beta_j^i \wedge \omega^j.$$

From the relations (4), $\pi_j^i + \bar{\pi}_i^j = 0$ implies

$$\alpha_j^i + \alpha_i^j = -\delta_{ij} (\pi_0^0 + \bar{\pi}_0^0) = -\delta_{ij} 2 \phi, \quad \phi \text{ is real},$$

$$\beta_j^i - \beta_i^j = 0,$$

so that in particular

$$\text{tr} (\alpha_j^i) = -2 \phi,$$

$$i \text{tr} (\beta_j^i) = -2 (\pi_0^0 - \bar{\pi}_0^0).$$
By Cartan’s lemma, \( \beta_j^i = h_{ijk} \omega^k \) for a coefficient \( h_{ijk} \) that is fully symmetric in its indices. A computation shows that an element \( g \in G_1 \) now acts by

\[
R_g^* i (\pi_0^0 - \bar{\pi}_0^0) = i (\pi_0^0 - \bar{\pi}_0^0) + (a^{-1} \bar{v}^1 + \bar{a}^{-1} v^1) \omega^1 + (a^{-1} \bar{v}^2 + \bar{a}^{-1} v^2) \omega^2.
\]

We may thus define the 2-adapted frame bundle \( F_2 \) by

\[
F_2 = \{ \text{tr} (\beta_j^i) = 0 \} \subset F_1,
\]

with the structure group \( G_2 = \{ g \in G_1 | av + \bar{a} \bar{v} = 0 \} \).

The geometric meaning of this normalization is clear. Let \( p_\infty(m) = [Z_3(m)] \in \Sigma \) for \( m \in M \), and let \( P = \Sigma - \{ p_\infty(m) \} \rightarrow \mathbb{C}^2 \) be the stereographic projection at \( p_\infty(m) \). For any 2-adapted frame \( Z = (Z_0, Z_i, Z_3) \), the Lagrangian surface \( P \circ X : M - X^{-1}(p_\infty(m)) \hookrightarrow \mathbb{C}^2 \) has zero mean curvature vector at \( m \).

Set \( \pi_3^i = \mu^i + i \nu^i \). Differentiating \( \text{tr} (\beta_j^i) = 0 \), we get \( \mu^k \wedge \omega_k = 0 \). By Cartan’s lemma, \( \mu^i = p_{ij} \omega^j \) for a coefficient \( p_{ij} = p_{ji} \). Using the group action by \( G_2 \) as before, we may translate \( \text{tr} (p_{ij}) = 0 \). We may thus define the 3-adapted frame bundle \( F \) by

\[
F = \{ \text{tr} (p_{ij}) = 0 \} \subset F_2.
\]

Differentiating \( \text{tr} (p_{ij}) = 0 \), we get \( \pi_3^0 = 2 q_i \omega^i \) for a coefficient \( q_i \).

Since we’ll utilize complex geometry later on, we introduce complex notation here.

Set \( Z_\pm = Z_1 \pm i Z_2 \), and denote

\[
\omega = \omega^1 + i \omega^2, \quad \rho = \alpha_2^1, \quad \mu = \mu^1 - i \mu^2, \quad \nu = \nu^1 + i \nu^2,
\]

\[
h = h_{111} - i h_{112}, \quad p = p_{11} - i p_{12}, \quad q = q_1 - i q_2,
\]

so that

\[
\beta_1^1 - i \beta_2^1 = h \omega, \quad \mu = p \omega, \quad \pi_3^0 = q \omega + \bar{q} \bar{\omega}.
\]

We summarize the results of the analysis as follows.
Proposition 2.1 Let \( X : M \hookrightarrow \Sigma \) be an oriented Legendrian surface. Then there exists a 3-adapted frame bundle \( F \subset X^*SU(3,1) \) on which the following structure equations hold.

\[
d(Z_0, Z_+, Z_-, Z_3) = (Z_0, Z_+, Z_-, Z_3) \begin{pmatrix}
\phi & \nu + i\bar{p}\bar{\omega} & -i\bar{\nu} + i\bar{p}\omega & q\omega + \bar{q}\bar{\omega} \\
\frac{1}{2}\bar{\omega} & i\rho & i\bar{h}\omega & \frac{1}{2}(i\bar{\nu} + p\omega) \\
\frac{1}{2}\omega & i\bar{h}\bar{\omega} & -i\rho & \frac{1}{2}(i\nu + \bar{p}\bar{\omega}) \\
0 & -i\omega & -i\bar{\omega} & -\phi
\end{pmatrix}. \tag{11}
\]

The structure coefficients \( h, p, q \) satisfy the following equations.

\[
\begin{align*}
\frac{dh}{\omega} + (3i\rho + \phi) h &= -p\bar{\omega} + z\omega, \\
\frac{dp}{\omega} + (2i\rho + 2\phi) p &= -h\nu - q\bar{\omega} + y\omega, \\
\frac{dq}{\omega} + (i\rho + 3\phi) q &= -p\nu - r\bar{\omega} + x\omega, \\
\frac{dz}{\omega} + (4i\rho + 2\phi) z &= -2h\bar{\nu} - z_{-1}\bar{\omega} \mod \omega, \\
\frac{dy}{\omega} + (3i\rho + 3\phi) y &= -2p\bar{\nu} - z\nu - y_{-1}\bar{\omega} \mod \omega, \\
\frac{dx}{\omega} + (2i\rho + 4\phi) x &= -2q\bar{\nu} - y\nu - x_{-1}\bar{\omega} \mod \omega,
\end{align*}
\]

where \( z, y, x, x_{-1} \) are complex coefficients, \( r \) is real, and

\[
z_{-1} = y + 3h|h|^2, \; y_{-1} = x + 3|h|^2 p.
\]

Remark. When \( r = 0, \; x_{-1} = |h|^2 q + p^2 \bar{h}. \)

Set

\[
\Phi = (hq - \frac{1}{2}p^2)\omega^4. \tag{13}
\]

A computation with the structure equation shows that \( \Phi \) is a well defined complex quartic differential on \( M \). Suppose \( \Phi \equiv 0 \) on \( M \), and consider

\[
\Psi = (zx - \frac{1}{2}y^2)\omega^6. \tag{14}
\]
The structure equation shows that \( r \equiv 0, hz - py + qz \equiv 0 \), and that \( \Psi \) is a well defined complex sextic differential on \( M \) that is holomorphic under the induced complex(conformal) structure. This observation will be important in our characterization of a class of Legendrian surfaces in Section 4.

We close this section with an observation on the umbilic locus \( U_X = \{ m \in M \mid h(m) = 0 \} \).

**Proposition 2.2** Let \( X : M \hookrightarrow \Sigma \) be a connected Legendrian surface with \( \Phi \equiv 0 \). Then either \( h \equiv 0 \), or the umbilic locus \( U_X \) is a closed subset with no interior.

**Proof.** Let \( m \in U_X \), and let \( \xi \) be a local coordinate of \( M \) centered at \( m \). Since every conformal structure on a surface is locally trivial, take a section of the 3-adapted bundle \( F \) such that \( \rho = \phi = 0 \) [Br2, p41].

First note that \( \partial \bar{\xi} (hq - \frac{1}{2}p^2) \equiv 0 \) implies \( hr \equiv 0 \). Suppose \( r(m) \neq 0 \). Then \( h = 0 \) on a neighborhood of \( m \), which implies \( r = 0 \) by (12), a contradiction. Hence \( r \equiv 0 \), and from (12) again, we compute

\[
\partial \bar{\xi} \begin{pmatrix} h \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\nu(\partial \bar{\xi}) & 0 & -1 \\ 0 & -\nu(\partial \bar{\xi}) & 0 \end{pmatrix} \begin{pmatrix} h \\ p \\ q \end{pmatrix}.
\]

By a well known theorem [Br2, p42], there exists an integer \( k(m) \), \( 0 \leq k(m) \leq \infty \), such that

\[
\begin{pmatrix} h \\ p \\ q \end{pmatrix} = \xi^k \begin{pmatrix} h_1 \\ p_1 \\ q_1 \end{pmatrix},
\]

for some smooth coefficients \( h_1, p_1, q_1 \) such that \( (h_1(m), p_1(m), q_1(m)) \neq (0, 0, 0) \). Assume \( k < \infty \) first. Note \( h_1q_1 - \frac{1}{2}p_1^2 = 0 \), and these coefficients also satisfy

\[
\partial \bar{\xi} \begin{pmatrix} h_1 \\ p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\nu(\partial \bar{\xi}) & 0 & -1 \\ 0 & -\nu(\partial \bar{\xi}) & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ p_1 \\ q_1 \end{pmatrix}.
\]
Suppose $h_1(m) \neq 0$, then $m$ is obviously an isolated zero of $h$.

Suppose $h_1(m) = 0$, $p_1(m) \neq 0$. Then $\partial_\xi h_1(m) \neq 0$, and $m$ is not an interior point of the zero set of $h_1$, hence of $h = \xi^k h_1$.

Suppose $h_1(m) = 0$, $p_1(m) = 0$, $q_1(m) \neq 0$. Then as above, $\partial_\xi p_1(m) = -q_1(m) \neq 0$, and $m$ is not an interior point of the zero set of $p_1$. Since $h_1 = \frac{1}{2n} p_1^2$ in a neighborhood of $m$, $m$ is not an interior point of the zero set of $h_1$, hence of $h = \xi^k h_1$.

The argument above shows that each of the sets $M_0 = \{ m \mid k(m) < \infty \}$, $M_\infty = \{ m \mid k(m) = \infty \}$, is open. The proposition follows for $M$ is connected. □

3 A duality theorem

We continue to use the notations in Section 1 and Section 2.

Let

$$\Omega_X = i |h|^2 \omega \wedge \bar{\omega}. \tag{15}$$

The structure equations (11), (12) shows this 2-form is well defined on $M$.

**Definition 3.1** Willmore functional for a compact Legendrian surface $X : M \hookrightarrow \Sigma$ is the second order functional

$$\mathcal{W}(X) = \int_M \Omega_X.$$

Let $p_\infty \in \Sigma - X(M)$, and $P : \Sigma - \{p_\infty\} \to \mathbb{C}^2$ be the stereographic projection from $p_\infty$. Then it is easy to check

$$\mathcal{W}(X) = \int_M \frac{1}{2} |B_0|^2 dX, \tag{16}$$

where $B_0$ is the trace free part of the second fundamental form of the Lagrangian surface $P \circ X(M)$.

A problem of interest of course is to find, if exists, the oriented Legendrian surface that minimizes the functional among the set of compact Legendrian surfaces of fixed
For example in genus 0 case, the absolute minimum value 0 is attained by totally geodesic Legendrian 2-spheres, i.e., Legendrian surfaces for which $h \equiv 0$.

Remark. Minicozzi in his thesis studied the problem of minimizing a functional equivalent to (16) among Lagrangian tori in $\mathbb{C}^2$, and proved the existence and regularity of the embedded minimizer [Min]. Note however an embedded Lagrangian surface in $\mathbb{C}^2$ cannot be exact due to the existence of holomorphic disk spanning the Lagrangian surface [Gro], and the present geometric variational problem is, although related, essentially distinct from Minicozzi’s work.

A Legendrian surface $X : M \hookrightarrow \Sigma$ is called Willmore if it is critical for $W$ for any compactly supported Legendrian variation. The purpose of this section is to identify the Euler-Lagrange equation for this variational problem, and analyze its geometric consequences.

**Proposition 3.1** A Legendrian surface $X : M \hookrightarrow \Sigma$ is Willmore if the complex quartic form $\Phi$, (13), is holomorphic.

We postpone the proof to the appendix. From the structure equation (12), this is equivalent to $r = 0$, which we assume from now on.

**Example 3.1** Let $X : M \hookrightarrow S^5 = U_3/U_2$ be an oriented Legendrian surface. From the general theory moving frames, the second fundamental form of $X$ can be expressed as a symmetric cubic differential $\beta \in C^\infty(S^3 T^*M)$. Let $\beta_0$ be the trace free part of $\beta$, and let $2\eta = \frac{1}{2}\text{tr} \beta$ be the mean curvature 1-form of $X$. For a local oriented orthonormal coframe $\{\omega^1, \omega^2\}$, we have

$$\beta_0 = \text{Re} \left( h_1 - i h_2 \right) (\omega^1 + i \omega^2)^3$$

$$\eta = \delta_1 \omega^1 + \delta_2 \omega^2,$$

for coefficients $h_i, \delta_i$. The Gauß curvature of the induced metric of $X$ for instance is
\[ K = 1 - 2(h_1^2 + h_2^2) + 2(\delta_1^2 + \delta_2^2). \] The Willmore functional is written as
\[ W(X) = \int_M 2(h_1^2 + h_2^2) \omega^1 \wedge \omega^2. \]

A computation shows that \( X \) is Willmore when
\[ * d * \left( \frac{1}{2} \Delta \eta + \eta \right) + 2 \nabla \eta |\eta|^2 + \frac{2}{3} \langle \eta^* \wedge \beta_0, \nabla \eta \rangle = 0. \] (17)

Here \( * \) is the Hodge dual operator, \( \eta^* \in C^\infty(TM) \) is the metric dual of the 1-form \( \eta \), and \( \nabla \eta \in C^\infty(S^2 T^* M) \) is the covariant derivative of \( \eta \) (note \( d\eta = 0 \)). Let \( D\delta_i = \delta_{ij} \omega^j \) be the covariant derivatives of \( \delta_i \). Then
\[ \frac{1}{3} \langle \eta^* \wedge \beta_0, \nabla \eta \rangle = (\delta_{11} - \delta_{22})(\delta_1 h_1 + \delta_2 h_2) + 2\delta_{12}(\delta_1 h_2 - \delta_2 h_1). \]

When \( \eta = 0 \) and \( X \) is minimal, the quartic differential (13) vanishes.

From the arguments in (13), (14), Willmore Legendrian surfaces can be naturally divided into three disjoint classes.

A. \( \Phi \not\equiv 0 \): general case

B. \( \Phi \equiv 0, \Psi \not\equiv 0 \): Willmore dual (to be explained below) is defined

C. \( \Phi \equiv 0, \Psi \equiv 0 \): Willmore dual, when it is defined, is constant

Note when \( M = S^2 \), every Willmore Legendrian immersion is of type C due to Riemann-Roch theorem. The type C surfaces will be completely characterized in terms of Weierstraß type holomorphic data in Section 4.

The main result of this section is the following duality theorem for type B surfaces analogous to Bryant’s duality theorem [Br2, Theorem C].

**Theorem 3.1** Let \( X : M \hookrightarrow \Sigma \) be a Willmore Legendrian surface of type B. Then there exists a smooth Willmore dual \( \hat{X} : M \hookrightarrow \Sigma \), which is a weakly conformal, generally singular, Willmore Legendrian immersion of type B. The singular locus of \( \hat{X} \), the points where \( \hat{X} \) is not an immersion, is a closed set with no interior. \( \hat{X} = X \) whenever it is defined.
Proof. From the structure equations (11), (12), consider the map \( Y : F \to \mathbb{L} \) defined by
\[
Y = \frac{i}{2} (|p|^2 Z_0 + h \bar{p} Z_+ + \bar{h} p Z_-) + |h|^2 Z_3,
\]
so that \( \langle Y, \bar{Y} \rangle = 0 \).

- Assume first \( Y \neq 0 \). One computes from (11), (12), that
\[
dY + 3 \phi Y = Y_+ \frac{1}{2} \bar{\omega} + Y_- \frac{1}{2} \omega,
\]
where
\[
Y_+ = i(p\bar{y} - q\bar{p}) Z_0 + i(h\bar{y} - \frac{1}{2} p\bar{p}) Z_+ + i(p\bar{z} - q\bar{h}) Z_- + (2h\bar{z} - p\bar{h}) Z_3
\]
\[
Y_- = i(y\bar{p} - p\bar{q}) Z_0 + i(z\bar{p} - h\bar{q}) Z_+ + i(y\bar{h} - \frac{1}{2} p\bar{p}) Z_- + (2z\bar{h} - h\bar{p}) Z_3,
\]
and \( \hat{X} = [Y] : M \hookrightarrow \Sigma \) is well defined. A computation shows
\[
\langle Y, \bar{Y}_\pm \rangle = 0, \quad \langle Y_+, \bar{Y}_- \rangle = 0,
\]
\[
\langle Y_\pm, \bar{Y}_\pm \rangle = 2|pz - hy|^2,
\]
and, in particular, \( \hat{X} \) is weakly conformal Legendrian. Moreover, it is easy to check \( \hat{X} \)
is in fact Willmore of type \( B \) whenever it is an immersion by direct computation.

It thus suffices to show that the zero locus of \( pz - hy \) is a closed set with no interior.
Differentiating \( hhq - \frac{1}{2} p^2 = 0, \) we get \( hz - py + qz = 0, \) which implies that \( (pz - hy)(qy - px) - \frac{1}{2}(hx - qz)^2 = 0. \) As in the proof of Proposition 2.2
\[
\partial_\xi \begin{pmatrix} p(z - hy) \\ hx - qz \\ qy - px \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \nu(\partial_\xi) & 0 & 1 \\ 0 & \nu(\partial_\xi) & 0 \end{pmatrix} \begin{pmatrix} p(z - hy) \\ hx - qz \\ qy - px \end{pmatrix}.
\]
By the same argument as before, either the zeros of \( pz - hy \) is a closed set with no interior, or it vanishes identically. But the latter case forces \( \Psi \equiv 0, \) contrary to our assumption that the surface is of type \( B. \)
• Assume $Y(m) = 0$ for $m \in M$, i.e., $h(m) = p(m) = 0$. Following the notation of Proposition 2.2 let $(h, p, q) = \xi^k(h_1, p_1, q_1)$ for a local coordinate $\xi$ centered at $m$.

There are two cases.

Case $(h_1(m), p_1(m)) \neq (0, 0)(k \geq 1)$: Let $\lambda = (\xi \bar{\xi})^{-k}$, and extend $\hat{X}$ across $m$ smoothly by $\hat{X} = [Y_0] = [\lambda Y]$ by using a local cut off function. We have

$$dY_0 \equiv \lambda(Y_+ \frac{1}{2} \omega + Y_- \frac{1}{2} \omega), \mod Y_0$$

A computation shows

$$pz - hy \equiv k \xi^{2k-1}(p_1 h_1 - h_1 p_1) \equiv 0 \mod \xi^k.$$

Since $k \geq 1$ by assumption, $\hat{X}$ is smooth across $m$. A similar argument as above shows that the singular locus of $\hat{X}$ is a closed set with no interior.

Case $(h_1(m), p_1(m)) = (0, 0)$: As before, from the choice of $k$, $q_1(m) \neq 0$, and $h_1 = \frac{1}{2q_1} p_1^2$. Set $\lambda = (p \bar{p})^{-1}$, and define $\hat{X} = [Y_0] = [\lambda Y]$ by using a local cut off function. From [18], this gives a well defined extension of $\hat{X}$ across $m$. Moreover, since $h = \frac{1}{2q} p^2$, it is easy to check that $pz - hy \equiv 0 \mod p^2$. Thus $\hat{X}$ is smooth across $m$. A similar argument as above shows that the singular locus of $\hat{X}$ is a closed set with no interior. □

4 Willmore Legendrian surfaces with constant dual

Willmore Legendrian surfaces of type $C$ can be completely characterized in terms of the following set of holomorphic data.

$$\begin{cases}
    \text{A compact Riemann surface } M \text{ with a divisor } D. \\
    \text{A meromorphic immersion } f = (f^1, f^2) : M \hookrightarrow \mathbb{C}^2 \text{ with simple poles on } D. \quad (20) \\
    \text{Re} (f^1 df^2 - f^2 df^1) \text{ is exact on } M - D \text{ with zero logarithmic term on } D.
\end{cases}$$
For the definition of zero logarithmic term, see the proof of Theorem 4.1 below.

Suppose a Willmore Legendrian surface $M$ of type $C$ is not totally geodesic. Form the structure equations (12), we may translate $p = 0$ on the dense open subset $M^*$ where $h \neq 0$. Then $q = 0$, $x = 0$, $y = 0$, and from (15), (19), the associated Willmore dual is constant on $M^*$, and hence constant on $M$.

**Theorem 4.1** Let $X : M \hookrightarrow \Sigma$ be a compact, connected Willmore Legendrian surface of type $C$. Assume $X$ is not totally geodesic, and denote the constant Willmore dual $\hat{X}(M) = p_\infty$. Let $D = X^{-1}(p_\infty)$, which is a divisor on $M$. Let $P : \Sigma - \{p_\infty\} \to \mathbb{C}^2$ be a stereographic projection from $p_\infty$. Then $P \circ X : M - D \hookrightarrow \mathbb{C}^2$ is a complete, exact, minimal Lagrangian surface of finite total curvature with embedded ends of zero logarithmic growth. Hence, upon a linear change of coordinates of $\mathbb{C}^2$, $P \circ X : M - D \hookrightarrow \mathbb{C}^2$ is holomorphic, and completes across $D$ as a holomorphic immersion $f_X : M \hookrightarrow \mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P_\infty^1$ with transversal intersection with $\mathbb{C}P_\infty^1$ on $D$.

Conversely, suppose a Weierstraß type data (20) is given. Then there is an associated conformal Legendrian lift $X_f : M - D \hookrightarrow \Sigma$ that completes across $D$ as a Willmore Legendrian immersion of type $C$.

In this case, the value of the Willmore functional $W(X) = 4\pi(|D| + g - 1)$, where $g$ is the genus of $M$.

**Proof.** From the geometric description of Willmore dual, $P \circ X : M - D \hookrightarrow \mathbb{C}^2$ is obviously an exact, minimal Lagrangian surface. But minimal Lagrangian surfaces in $\mathbb{C}^2$ are holomorphic curves under a linear change of coordinates of $\mathbb{C}^2$. Let $f : M - D \hookrightarrow \mathbb{C}^2$ denote this holomorphic curve. It is of finite total curvature for

$$\int_{M - D} -K \, dA = \int_{M - D} \frac{1}{2} |B_0|^2 \, dA = \int_{M - D} \Omega_X = W(X) < \infty,$$

where $K$ is the Gauß curvature of $f(M - D)$. It is complete with simple poles and embedded ends because, from the construction of pseudoconformal stereographic projection in Section 1, $E = \langle w_0, w_3 \rangle^\perp$ can be identified with the contact hyperplane at
infinity $p_\infty$. That each end is of zero logarithmic growth follows from the argument for the converse below.

By the classical theorem of Chern-Osserman [ChO], the Gauß map of $f_X : M - D \hookrightarrow \mathbb{C}^2$ extends holomorphically across $D$. Since $f_X$ is itself holomorphic, this implies the holomorphic extension $f_X : M \hookrightarrow \mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1_\infty$ is well defined. It is easy to see that $f_X$ is an immersion on $D$, for the original $X : M \hookrightarrow \Sigma$ is an immersion.

Conversely, suppose such a holomorphic curve $f : M - D \hookrightarrow \mathbb{C}^2$ is given. Upon an appropriate linear change of coordinates of $\mathbb{C}^2$, $\text{Re} \left( f^1 \left( df^2 - f^2 \, df^1 \right) \right)$ becomes the standard symplectic form of $\mathbb{C}^2$, and $f$ is exact, minimal, and Lagrangian. Let

$$X_f : M - D \hookrightarrow \Sigma = (\mathbb{R} \oplus \mathbb{C}^2) \cup \{p_\infty\} \quad (22)$$

be an associated Legendrian lift. It suffices to show that $X_f$ extends across $D$ as a smooth Legendrian immersion (then it will be necessarily of type C).

Take $m \in D$, and let $\xi$ be a local coordinate of $M$ centered at $m$. From the given description, $f$ has a local expansion

$$f = f_{-1} \frac{1}{\xi} + f_0 + f_1 \xi + ...$$

Since $\text{Re} \left( f^1 \left( df^2 - f^2 \, df^1 \right) \right)$ is exact without logarithmic terms, it cannot have any $\text{Re} \frac{1}{\xi} \, d\xi$ term (this is the definition of zero logarithmic terms), and hence $(f^1 \, df^2 - f^2 \, df^1)$ itself cannot have any $\frac{1}{\xi} \, d\xi$ term. Note that the associated Legendrian lift is not smooth if there is a nonzero $\text{Re} \frac{1}{\xi} \, d\xi$ term. Thus for meromorphic curves in $\mathbb{C}^2$ with simple poles on $D$, each ends has zero logarithmic growth when the meromorphic 1-form $(f^1 \, df^2 - f^2 \, df^1)$ is locally exact. We may thus write

$$dt = \text{Re} \left( (c_{-1} \xi^{-2} + c_0 + ...) \, d\xi \right).$$

for a local smooth function $t$ in a neighborhood of $m$. It now follows easily from the inversion formula (7) that the Legendrian lift $X_f$ extends smoothly across $m$. 

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A complete minimal surface of finite total curvature in a Euclidean space with $d$ embedded ends is conformally equivalent to a compact Riemann surface with $d$ points removed, and with the total Gauß curvature $-4\pi(d+g-1)$ [ChO]. $W(X) = 4\pi(|D| + g - 1)$ follows from [21]. □

**Remark.** $D \neq \emptyset$.

**Corollary 4.1** Let $X : S^2 \hookrightarrow \Sigma$ be a Willmore Legendrian immersion. Then it is of type $C$, and $W(X) = 4\pi(d - 1)$ for some positive integer $d$.

**Proof.** By Riemann-Roch, $S^2$ does not support any nonzero holomorphic differential. Hence both $\Phi$ and $\Psi$ are zero. □

The definition of Legendrian surface can be suitably modified to include nonorientable surfaces. In this regard, we have the following analogue of the well known theorem that $\mathbb{R}P^2$ cannot occur as a minimal surface in $S^3$, nor as a minimal Legendrian surface in $S^5$. The difference is that in our case it follows from the global geometry of Willmore Legendrian spheres, whereas in minimal surface case it follows from the rigidity of such minimal spheres.

**Corollary 4.2** $\mathbb{R}P^2$ cannot occur as a Willmore Legendrian surface in $\Sigma$.

**Proof.** Let $X_0 : S^2 \hookrightarrow \Sigma$ be a Willmore Legendrian immersion invariant under antipodal involution $\tau$ of $S^2$, so that it factors through $X = X_0/\tau : \mathbb{R}P^2 = S^2/\tau \hookrightarrow \Sigma$. By **Theorem 4.1** there exists an associated holomorphic immersion $\mathbb{R}P^2 \hookrightarrow \mathbb{C}P^2$, a contradiction. □

**Example 4.1** Consider a meromorphic map $f = (f^1, f^2) : \mathbb{C}P^1 \rightarrow \mathbb{C}^2$ defined by

$$
\begin{pmatrix}
  f^1(z) \\
  f^2(z)
\end{pmatrix} = \begin{pmatrix}
  P(z) \\
  Q(z) \\
  A(z) \\
  B(z)
\end{pmatrix}
$$

where $P, Q$ are polynomials of degree $k$, and $A, B$ are polynomials of degree $l$. By taking generic polynomials, we may assume $f$ is a meromorphic immersion with $k + l$
simple poles on $D$. From

$$f^1 df^2 - f^2 df^1 = \frac{PQ(BA' - AB') - AB(QP' - PQ')}{Q^2B^2} dz,$$

further generic choice of $P, Q, A, B$ implies that the meromorphic 1-form $f^1 df^2 - f^2 df^1$ has no residue, and is exact on $\mathbb{C}P^1 - D$. By Theorem 4.1 we have a Willmore Legendrian sphere $X_f : \mathbb{C}P^1 \rightarrow \Sigma$ with $\mathcal{W}(X_f) = 4\pi(k + l - 1)$. This is in contrast with the case of Willmore spheres in conformal 3-sphere, where certain values $4\pi k$ are prohibited [Br3]. It is not difficult to construct $f$ so that $X_f$ is an embedding on $\mathbb{C}P^1 - D$, [LY].

A question naturally arises as to how many compact Riemann surfaces support the Weierstraß type data (20).

**Theorem 4.2** Every compact Riemann surface $M$ admits a generally one to one, conformal Legendrian immersion $X : M \hookrightarrow \Sigma$ as a Willmore Legendrian surface of type $C$.

**Proof.** We follow the construction of Bryant [Br1], and the notations therein. Let $(z^0, z^1, z^2)$ be a coordinate of $\mathbb{C}^3$. Bryant showed that for any compact Riemann surface $M$, there exists a meromorphic embedding $F = (F^0, F^1, F^2) : M \hookrightarrow \mathbb{C}^3$ with simple poles on a divisor $D \subset M$ as an integral curve of the holomorphic contact form $dz^0 - z^1 dz^2 + z^2 dz^1$. Let $f = (F^1, F^2) : M \hookrightarrow \mathbb{C}^2$, which is necessarily an exact meromorphic immersion with simple poles on $D$. The construction in [Br1, Theorem G] implies $f$ is generally one to one. The Legendrian lift of $f$, (22), is a graph over $f$, and hence a generally one to one immersion on $M - D$. □

The argument in the proof above shows that for every holomorphic Legendrian immersion $F : M \hookrightarrow \mathbb{C}P^3$, there exist associated (branched) Willmore Legendrian surfaces of type $C$. For a discussion of the moduli of holomorphic Legendrian curves, [ChMo].
Corollary 4.3  Every compact Riemann surface can be conformally immersed in $\mathbb{C}^2$ as an exact, algebraic Lagrangian surface.

It is known that every compact Riemann surface can be conformally embedded in Euclidean 3-space as an algebraic surface [Ga].

Appendix. Euler-Lagrange equation

Proof of Proposition 3.1  Let $X_t : (-\delta, \delta) \times M \hookrightarrow \Sigma$ be one parameter family of Legendrian immersions with the associated 3-adapted bundle $F_t \rightarrow (-\delta, \delta) \times M$, and the induced Maurer-Cartan form $\pi$ such that

$$
\pi = (\pi^A_B) = \begin{pmatrix}
\phi + (x_0 + iy_0)dt & \nu_j + i\mu_j + (v_j + iv_j)dt & q_i\omega^i + wdt \\
\omega^i + i\lambda_i dt & \pi^i_j & \mu_j + iv_j + (u_j + iv_j)dt \\
\lambda dt & -i(\omega^j - i\lambda_j dt) & -\phi + (-x_0 + iy_0)dt
\end{pmatrix},
$$

where

$$
\pi^i_j - \delta^i_j \pi^0_0 = \alpha^i_j + i\beta^i_j + (x^i_j + iy^i_j)dt, \quad \pi^i_j + \bar{\pi}^j_i = 0,
$$

and $\phi, \omega^i, \alpha, \beta, \mu, \nu$ are without $dt$-terms.

Let $\Omega = \text{Im} (\pi^1_1 - \pi^2_2) \wedge \text{Im} \pi^3_2$, and set

$$
\Omega_t = \Omega - dt \wedge (\frac{\partial}{\partial t} \Omega).
$$

Then $\frac{\partial}{\partial t} \Omega_t = 0$, and

$$
\mathcal{L}_{\frac{\partial}{\partial t}} \Omega_t|_{t=0} \equiv \frac{\partial}{\partial t} d\Omega|_{t=0} \mod \text{exact form}.
$$

It thus suffices to compute $\frac{\partial}{\partial t} d\Omega|_{t=0}$.

From $-d\pi = \pi \wedge \pi$, we find

$$
d\lambda \equiv \lambda (\pi^0_0 - \pi^3_3) - 2\lambda_k \omega^k, \quad d\lambda_i \equiv -\lambda_j \omega^i_j + y^i_j \omega^j - \lambda \nu_i, \quad \mod dt.
$$
A direct computation with these equations gives

\[
\frac{i}{2} \frac{\partial}{\partial t} \int d\Omega |_{t=0} = y^i p_{ij} \omega^j \wedge \omega^2 + (\lambda_1 \nu_1 - \lambda_2 \nu_2) \wedge \beta_2^1 - \beta_1^1 \wedge (\lambda_1 \nu_2 + \lambda_2 \nu_1),
\]

where \( \mu_i = p_{ij} \omega^j \). We wish to put this expression into a form involving \( \lambda \) only by integration by parts. After a short computation,

\[
\frac{i}{2} \frac{\partial}{\partial t} \int d\Omega |_{t=0} + d(\lambda_1 \mu_2 - \lambda_2 \mu_1) = (\lambda_1 q_1 + \lambda_2 q_2) - \lambda(\nu_1 \wedge \mu_2 - \nu_2 \wedge \mu_1).
\]

Adding \( \frac{1}{2} d(\lambda(q_1 \omega^2 - q_2 \omega^1)) \), we get

\[
\frac{i}{2} \frac{\partial}{\partial t} \int d\Omega |_{t=0} + d(\lambda_1 \mu_2 - \lambda_2 \mu_1) + \frac{1}{2} d(\lambda(q_1 \omega^2 - q_2 \omega^1)) = \lambda r \omega^1 \wedge \omega^2.
\]

Since \( \lambda \) is an arbitrary function of compact support, this forces \( r = 0 \). □

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