Ion Sizes and Finite-Size Corrections for Ionic-Solvation Free Energies

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Free energies of ionic solvation calculated from computer simulations exhibit a strong system size dependence. We perform a finite-size analysis based on a dielectric-continuum model with periodic boundary conditions. That analysis results in an estimate of the Born ion size. Remarkably, the finite-size correction applies to systems with only eight water molecules hydrating a sodium ion and results in an estimate of the Born radius of sodium that agrees with the experimental value.

Calculation of ionic-hydration free energies from computer simulations require great care to avoid artifacts due to the long-range electrostatic interactions. We have recently shown that Ewald summation can give results for single-ion free energies that are essentially system-size independent for as few as about 16 water molecules. That has been achieved by including the self-interactions 0.5$q^2ξ_{Ew}$ of the ion ($ξ_{Ew} ≈ −2.837297/L$ with $L$ the length of the cubic box). Ewald summation determines the electrostatic interactions using lattice sums for a periodically replicated simulation box. This provides a natural description of the electrostatics in the periodic space resulting from the periodic boundary conditions commonly used in computer simulations. Deviations from the approximate finite-size correction are expected for solvents with finite dielectric constant $\epsilon < \infty$ (i.e., in the infinite-dilution limit of an ion in a non-conducting solvent) and if the ion size is comparable to the dimensions of the simulation box.

The self-interaction of an ion is introduced in the Ewald summation through interactions with the compensating background charge with density $−q/L^3$. The infinite system is obtained by taking the limit ($L → ∞$) and if the ion size is comparable to the dimensions of the simulation box.

The self-interaction ($ξ$) is important for small system sizes. We also show that the analysis of the system-size dependence of ionic free energies leads to the definition of an ion size in dielectric media in excellent agreement with the Born radius.

We approximate the free energy of charging an ion as the sum of the explicit simulation contribution $F_{\text{sim}}$ and a self-interaction $0.5q^2ξ_{Ew}$, as in Ref. 3. To account for further effects of finite system size, we add the difference of charging an ion in an infinite and finite system, $F_∞ − F_L$,

$$F_0 = F_{\text{sim}} + \frac{1}{2}q^2ξ_{Ew} + (F_∞ − F_L) \tag{1}$$

The finite-size correction $F_∞ − F_L$ is evaluated for a simplified model of a point charge $q$ in a spherical cavity with radius $R$ inside a dielectric continuum with dielectric constant $ε$, as schematically shown in Fig. 1. The finite system is formed by an ion in a periodically replicated box of length $L$, where the box is charge neutral through addition of a homogeneous background charge with density $−q/L^3$. The infinite system is obtained by taking the limit $L → ∞$ and corresponds to the Born model. If the model used to calculate $F_∞ − F_L$ were exact, $F_L$ would cancel $F_{\text{sim}} + 0.5q^2ξ_{Ew}$. However, the dielectric model is only an approximation to the simulation system with explicit solvent, making the finite-size correction only approximate.

The finite-size correction is the difference of charging the ion in an infinite ($L → ∞$) and finite system,

$$F_∞ − F_L = \frac{1}{2}q^2 [ξ(ε, R, L → ∞) − ξ(ε, R, L)] \tag{2}$$

$ξ(ε, R, L)$ is the electrostatic potential minus the bare self-interaction at the position of a unit charge,

$$ξ(ε, R, L) = \lim_{r \rightarrow 0} [φ(r) − 1/r] \tag{3}$$

We determine the electrostatic potential $φ(r)$ for the dielectric model described above by solving the corresponding Poisson equation,

$$\nabla [ε(\mathbf{r})\nabla φ(\mathbf{r})] = −4π [δ(\mathbf{r}) − 1/L^3] \tag{4}$$

We define potentials $φ_−(\mathbf{r})$ and $φ_+(\mathbf{r})$ inside and outside the sphere. For $R < L/2$, the boundary conditions are

$$\nabla [ε(\mathbf{r})\nabla φ_+(\mathbf{r})] = −4π δ(\mathbf{r}) \tag{5}$$

$$\nabla [ε(\mathbf{r})\nabla φ_−(\mathbf{r})] = 0 \tag{6}$$

$$φ_+(\mathbf{r}) = φ_−(\mathbf{r}) \quad \text{for} \quad 0 < r < R \tag{7}$$

$$φ_+(\mathbf{r}) = φ_+(\mathbf{r}) + ξ(ε, R, L) \tag{8}$$

$$φ_+(\mathbf{r}) = φ_+(\mathbf{r}) + ξ(ε, R, L) \tag{9}$$
\[
\phi_- (r) = \phi_+ (r) \quad \text{for} \quad |r| = R ,
\]
\[
\frac{\partial \phi_- (r)}{\partial r} = \epsilon \frac{\partial \phi_+ (r)}{\partial r} \quad \text{for} \quad |r| = R ,
\]
\[
\frac{\partial \phi_+ (r)}{\partial x} = 0 \quad \text{for} \quad |x| = L/2 .
\]

The last condition reflects the periodicity of the potential and applies analogously for \( y \) and \( z \).

To solve this electrostatic problem, we expand \( \phi_- \) and \( \phi_+ \) into a complete set of functions that satisfy Poisson’s equation Eq. (4) and then choose the expansion coefficients to satisfy the boundary conditions Eq. (5). Such functions are the kubic-harmonic polynomials \( K_n \) introduced by von der Lage and Bethe. We define

\[
\phi_- (r) = \frac{1}{r} + \frac{\pi r^2}{3L^3} + \sum_{n=2}^{N/2} a_{2n} K_{2n} (r) + C ,
\]

\[
\phi_+ (r) = \frac{b_{-1}}{r} + b_0 + \frac{\pi r^2}{3L^3} + \sum_{n=2}^{N/2} b_{2n} K_{2n} (r) + C .
\]

For a finite set of functions \( K_{2n} \) up to polynomial order \( 2n \leq N \), the solution is only approximate. We choose the coefficients in the least-square sense, defining a \( \chi^2 \) functional,

\[
\chi^2 = R^2 \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta \ d\theta \left\{ [\phi_- (r) - \phi_+ (r)]^2 + L \left[ \frac{\partial \phi_- (r)}{\partial r} - \epsilon \frac{\partial \phi_+ (r)}{\partial r} \right]^2 \right\} |_{r=R} + 6 \int \int_{y^2 + z^2 < L^2/4} dy \ dz \left[ \frac{\partial \phi_+ (r)}{\partial x} \right]_{x=L/2}^2 ,
\]

with spherical polar coordinates \( r, \varphi \) and \( \theta \). The last integral extends only over a circular region because this results in an analytical solution of that integral. Also, the deviations from Eq. (5c) of truncated kubic-harmonic expansions of the Ewald potential are largest in the corners of the box. Fitting only a circular region therefore produces smaller distortions of the potential at the center of the box.

Minimizing \( \chi^2 \) with respect to the coefficients \( b_{-1}, b_0, a_4, b_4 \) etc. results in a set of linear equations for those coefficients. Using the computational algebra package REDUCE we solve for the unknown expansion coefficients for different orders \( N \) of the kubic-harmonic expansion. We then apply Taylor expansion with respect to \( 1/\epsilon \) and the size parameter \( R/L \). This results in an asymptotic expansion of \( \xi (\epsilon, R, L) \) as

\[
\xi (\epsilon, R, L) = \frac{c_0}{\epsilon L} - \frac{2\pi R^2 \epsilon - 1}{3L^3} + \frac{c_2 R^2}{L^3 \epsilon} + \frac{c_4 R^4}{L^5 \epsilon^3} + O \left( \frac{R^5}{L^6} \epsilon^{-5/2} \right) .
\]

Notice that because of the finite system size, \( \phi (r) \) is normalized by adding a constant \( C \) such that the average potential in the simulation box is zero.

\[
\int_{\text{box}} d\mathbf{r} \phi (\mathbf{r}) = 0 ,
\]

rather than \( \phi (r) \to 0 \) for \( r \to \infty \). The familiar Born term \( 0.5(1 - \epsilon^{-1})/R \) is thus contained in the normalization constant \( C \).

The coefficients \( c_0, c_2 \) and \( c_4 \) are listed in Table I. As the order \( N \) of expansion functions increases, the coefficient \( c_0 \) converges towards the self-energy of a point charge in a cubic Wigner lattice \( c_0 \to L \xi_\infty \approx -2.837297 \). The coefficients \( c_2 \) and \( c_4 \) appear to converge to values close to zero. The most interesting term in Eq. (5) is the lowest order correction for the finite size of an ion, \( -2\pi (\epsilon - 1) R^2/(3L^3) \). Eq. (5) converges to the correct limit for \( \epsilon \to \infty \), which can be found independently using the continuity of \( \phi (r) \) at the dielectric interface \( |r| = R \) and the spherical geometry for \( \epsilon \to \infty \),
\[ \xi(\epsilon \to \infty, R, L) = -\frac{2\pi R^2}{3L^3} + \frac{16\pi^2 R^6}{45L^6}. \]

The last term is contained in the \( \mathcal{O}\left(R^6/L^6, \epsilon^{-5}\right) \) term of Eq. (8).

We can now calculate the finite-size correction to the free energy of an ion with radius \( R \) in a dielectric medium with Ewald-summation electrostatics,

\[ F_\infty - F_L = -\frac{1}{2}q^2 \xi(\epsilon, R, L), \]

using Eq. (9) and \( \xi(\epsilon, R, L \to \infty) = 0 \) for the particular choice of \( C \) [Eq. (8)]. This results in an approximation to the free energy \( F_0 \) of charging the ion including finite-size effects:

\[ F_0 \approx F_{\text{sim}} + \frac{1}{2}q^2 \epsilon - \frac{1}{\epsilon} \left( \xi_{\text{Ew}} + \frac{2\pi R^2}{3L^3} \right). \]

In the limit \( \epsilon \to \infty \), the finite-size correction \( F_\infty - F_L \) does not contain terms of order \( L^{-1} \) and \( L^{-2} \). This explains the success of using \( F_{\text{sim}} + 0.5q^2 \xi_{\text{Ew}} \) alone for the free energy of charging an ion in a conducting or highly polar environment, without further finite-size correction that takes the ion size or the dielectric constant of the solvent into account.

The validity of the approximate finite-size correction Eq. (12) is illustrated in Fig. 2 for a sodium ion in water. We fit the sodium-ion data \( F_{\text{sim}} \) for electrostatic free energies calculated from simulations with \( M = 8 \) to \( M = 256 \) water molecules \( L \) to Eq. (12) with \( F_0 \) and \( R \) as parameters. For the dielectric constant, \( \epsilon \to \infty \) is used, but values of \( \epsilon \approx 80 \) have little effect on the result. That fit yields \( F_0 = -402.5 \pm 1 \) kJ mol\(^{-1}\) for the electrostatic contribution to the solvation free energy of sodium. The radius \( R \) of the sodium ion is found to be \( R = 0.18 \) nm, in excellent agreement with the effective Born radius 0.180 nm of that ion, as determined by Latimer, Pitzer and Slansky[5] or, more recently, 0.188 nm, as compiled by Marcus[6]. In the fit, the ion radius \( R \) is determined mostly by the data for small system sizes. Using only the \( M = 8 \) and 16 data and fixing \( F_0 \) at -402.5 kJ mol\(^{-1}\), a fit of \( R \) yields 0.159 nm (\( M = 16 \), 0.190 nm (\( M = 8 \)) and 0.184 nm (\( M = 8 \) and 16).

The simulation result for the Born radius is 0.172 nm. That is, \( F_0 \) and \( R \) are self-consistent assuming an uncertainty greater than 0.01 nm in \( R \). We can therefore fit the free-energy data with the solvation free energy \( F_0 \) as the only parameter by substituting the Born expression \( F_0 \approx -0.5q^2(1 - \epsilon^{-1})/R \) for \( R \),

\[ F_0 \approx F_{\text{sim}} + \frac{1}{2}q^2 \epsilon - \frac{1}{\epsilon} \left( \xi_{\text{Ew}} + \frac{\pi q^2(1 - \epsilon^{-1})^2}{6F_0^2L^3} \right). \]

That fit yields \( F_0 = -403 \pm 1 \) kJ mol\(^{-1}\) (for \( \epsilon \to \infty \)). These results show that the system-size effects on the hydration of sodium can be described accurately for \( M \geq 8 \) water molecules by the dielectric continuum model with periodic boundary conditions shown in Fig. 1. It is remarkable that continuum-electrostatics approximations apply even for as few as eight water molecules solvating the sodium ion provided the periodic boundary conditions are considered. Of course, determination of the value of \( F_0 \) and the Born radius requires further molecular considerations. To treat solutes with more complicated shapes will require additional computational effort, as was suggested previously.

In summary, we have found an approximate finite-size correction for ions in water that takes into account the size of the ions as well as the dielectric constant of the solvent. These findings extend the previous work by Hummer et al.[3] and Figueirido et al.[4] The results of Refs. 3 and 11 are recovered in the limit of a conducting solvent (\( \epsilon \to \infty \)) and a point ion \( R/L \to 0 \). In the limit of a point ion but for a finite dielectric constant \( \epsilon < \infty \), we obtain the finite-size correction of Ref. 3. For strongly polar solvents, such as water (\( \epsilon \approx 80 \)), and typical system sizes of hundreds of more solvent molecules with ions of comparable size, the difference to the finite-size correction obtained previously[3,4] for \( \epsilon \to \infty \) and \( R/L \to 0 \) will be small (\( \lesssim 1\text{-}2\% \)). We have found that the finite-size correction derived in this paper gives qualitatively and quantitatively correct behavior. This finite-size analysis results in an estimate of the sodium-ion size that is in agreement with the experimental data.

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TABLE I. Coefficients of the approximation Eq. (8) to $\xi$ for different levels $N$ of kubic-harmonic expansions. ($N = 2$ means that no kubic harmonics have been used.)

| $N$ | $c_0$    | $c_2$    | $c_4$    |
|-----|---------|---------|---------|
| 2   | -2.9037 | 0.9751  | -0.8581 |
| 4   | -2.8309 | -0.1638 | 0.1558  |
| 6   | -2.8362 | -0.0728 | 0.0711  |
| 8   | -2.8398 | 0.0371  | -0.0366 |
| 10  | -2.8370 | 0.0022  | -0.0023 |

FIG. 1. Schematic representation of the contributions to the electrostatic solvation free energy of an ion, $F_{\text{ion}} + 0.5q^2\xi_{\text{Ew}} + (F_{\infty} - F_L)$.

FIG. 2. Finite-size correction for the hydration free energy of a sodium ion. Shown is the free energy of charging a sodium ion in water from charge zero to $e$ as a function of the inverse simulation box length $1/L$. The symbols are simulation data for Ewald-summation electrostatics from Ref. 3 for $M = 8, 16, 32, 64, 128$, and $256$ water molecules. The dashed line is the fit to the correction formula of Ref. 3 (for $M \geq 16$), corresponding to Eq. (12) with $\epsilon \rightarrow \infty$ and $R = 0$. That fit yields an extrapolated free energy $F_0$ of $-405 \text{ kJ mol}^{-1}$. The solid line shows the fit to the correction formula Eq. (12) with $\epsilon \rightarrow \infty$, where the sodium-ion radius $R$ and the free energy of charging $F_0$ are estimated to be $0.18 \text{ nm}$ and $-402.5 \text{ kJ mol}^{-1}$, respectively.
The graph shows the relationship between $F_{\text{sim}}$ (kJ mol$^{-1}$) and $L^{-1}$ (nm$^{-1}$). The data points are represented by squares, and the solid line indicates the trend. The dashed line is a theoretical fit to the data.