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MATRIX MODELS AS INTEGRABLE SYSTEMS

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The theory of matrix models is reviewed from the point of view of its relation to integrable hierarchies. Determinantal formulas, relation to conformal field models and the theory of Generalized Kontsevich model are discussed in some detail. Attention is also paid to the group-theoretical interpretation of τ-functions which allows to go beyond the restricted set of the (multicomponent) KP and Toda integrable hierarchies.
1 Introduction

The purpose of these notes is to review one of the branches of modern string theory: the theory of matrix models. We put emphasize on their intrinsic integrable structure and almost ignore direct physical applications which are broadly discussed in the literature. Also most of technical details and references are omitted: they can be found in a recent review [1].

Both “matrix models” and “integrability” are somewhat misleading names for the field to be discussed, they refer more to the history of the subject than to its real content. In fact the problem which is actually addressed is that of description of non-perturbative partition functions in quantum theory. The term “non-perturbative partition function” is now widely used to denote the generating functional of all the exact correlation functions in a given quantum model. Such quantity is given by a functional integral where the weight in the sum over trajectories is defined by effective action, which contains either all the possible (local or non-local) counterterms or generic coupling to external fields, so that any correlator can be obtained as derivative with respect to appropriate coupling constants or background fields. These exact generating functionals possess new peculiar properties, resulting from the possibility to perform arbitrary change of integration variables in the functional integral. Such properties are never studied in the orthodox quantum field theory because there the freedom to change integration variables is severely restricted by requirements of locality and renormalizability, which at last lost their role as fundamental principles of physics with creation of string theory.

Every change of integration variables can be alternatively described as some change of parameters of non-perturbative partition function (i.e. the coupling constants or background fields in effective action). Thus invariance of the integral implies certain relations (Ward identities) between partition functions at different values of parameters. Since all the fields of the model are integrated over, the set of relations is actually exhaustively large: more or less any two sets of parameters are related. Exact formulation of this property is yet unknown. The natural first step in these investigations is to look at the finite-dimensional integrals and then proceed to functional integrals by increasing the number of integrations. In turn, the natural way to do it is to make use of the well studied “matrix models”, which deal with $N \times N$ matrix integrals and their behaviour in the large-$N$ limit. It appears that non-perturbative partition functions of matrix models, at least when they can be handled with the presently available techniques, are closely related to “$\tau$-functions”, introduced originally in the study of integrable hierarchies. It looks very probable that some crucial characteristics of such partition functions are in fact not so peculiar for these simple examples, but remain true in the absolutely general setting. Extraction of such properties and construction of the adequate notion of “generalized $\tau$-function” is the main task of further work in the theory of “matrix models” and “integrable systems”.

One of the most straightforward and still promising approaches is based on interpretation of Ward identities for non-perturbative partition functions as Hirota-like equations (supplemented by a much smaller set of “string equations”), while generalized $\tau$-functions, which are solutions to these equations, are interpreted as group-theoretical objects (generating functionals of all the matrix elements of a group element in particular representation). If successful, this approach can provide description of exact correlation functions in terms of some (originally hidden) symmetry of the given class of theories, thus raising to the new height the relation between physical theories and symmetries, which was the guiding line for development of theoretical physics during the
last decades.

Analysis of non-perturbative partition functions is very important for one more reason. Construction of a generating functional is essentially “exponentiation of perturbations”, i.e. it deformes original (bare) action of the model. When perturbation parameters (extra coupling constants or background fields) are non-infinitesimal, one in fact obtains entire set of models instead of original one. Moreover, original model is no longer distinguished within this class non-perturbative partition functions are associated with classes of models, not with a single model. One can easily recognize here realization of the main idea of the string programm (see for example [2]). The study of non-perturbative partition functions even for such a simple class as ordinary matrix models can lead to much better understanding of the general idea. As usual, this can help to figure out what the adequate questions are and to develop effective technique to answer these questions.

The purpose of these notes is to briefly illustrate the general ideas with very simple examples. A lot of work is still required to obtain applications to the really interesting problems. Still, simple examples are enough to understand the ideas, and often conceptual level is no less important than that of technical effectiveness.

We begin in the following sections from consideration of the simplest matrix models, to be referred to as discrete and Kontsevich (continuous) models. Again, these names reflect more the history than the real content of the subject (continuous models were originally described non-explicitly as specific (multiscaling) large-$N$ limits of the discrete ones). In the context of non-perturbative partition functions the difference is that discrete models possess effective actions with all the possible counterterms added, while in Kontsevich model an external field (source) is introduced. Analysis of these theories includes their characterization as eigenvalue models and derivation of related determinant representations. We also consider description of discrete models in the language of conformal field theory. It is important for connecting these matrix models to the physically relevant Liouville theory of $2d$ gravity and - more essential for our notes - to the concept of KP and Toda $\tau$-functions.

Then we turn to the free-fermion (Grassmannian) description of KP/Toda $\tau$-functions and list some results from the theory of Generalized Kontsevich model [3]-[8]. Some other interesting models, which look a priori very different, are in fact just particular examples of Kontsevich model, which actually describes a big family of theories. This example should be very instructive for the future understanding of the interplay between perturbative and non-perturbative information contained in the non-perturbative partition function.

The last topic of these notes concerns group-theoretical interpretation of $\tau$-functions. The natural object, which arises in this way, while possessing the most important properties of conventional $\tau$-functions, is in fact much more general. In this framework the KP/Toda $\tau$-functions are associated with fundamental representations of $SL(N)$ and the closely related theory of the simply-laced Kac-Moody algebras of level $k = 1$. In fact $\tau$-function can be easily defined for any representation of any group (including also quantum groups - this can be important for the future construction of string field theory, where the idea of “third quantization” requires consideration of operator-valued $\tau$-functions; since it takes our non-perturbative partition function as input - effective action of the full string field theory).
2 The basic example: discrete 1-matrix model

The sample example of matrix model is that of 1-matrix integral

\[ Z_N\{t\} \equiv c_N \int_{N \times N} dH e^{\sum_{k=0}^{\infty} t_k \text{Tr} H^k}, \quad (2.1) \]

where the integral is over \( N \times N \) matrix \( H \) and \( dH = \prod_{i,j} dH_{ij} \). This measure is invariant under the conjugation \( H \to UHU^\dagger \) with any unitary \( N \times N \) matrix \( U \), and the “action” \( \sum_{k=0}^{\infty} t_k \text{Tr} H^k \) in (2.1) is the most general one consistent with this invariance. Thus \( Z_N\{t\} \) is indeed an example of the non-perturbative partition function in the sense, described in the Introduction. All observables in the theory are given by algebraic combinations of \( \text{Tr} H^k \) and their correlation functions can be obtained by action of \( t_k \)-derivatives on \( Z_N\{t\} \). Our goal now is to find some more invariant description of this quantity, not so specific as the matrix integral (2.1).

2.1 Ward identities

Such description is provided by Ward identities. The integral is invariant under any change of integration matrix-variable \( H \to f(H) \). It is convenient to choose the special basis in the space of such transformations:

\[ \delta H = \epsilon_n H^{n+1}. \quad (2.2) \]

This integral is often referred to as Hermitean. In most of our considerations we do not need to specify integration contours in matrix integrals, in particular eigenvalues of \( H_{ij} \) do not need to be real. What is indeed important is the “flatness” of the measure \( dH = \prod_{i,j} dH_{ij} \). Below “Hermitean” (as opposed, for example, to “unitary”) will imply just this choice of the measure and not any reality condition.

Here \( \epsilon_n \) is some infinitesimal matrix and, of course, \( n \geq -1 \). Invariance of the integral implies the following identity:

\[ \int_{N \times N} dH e^{\sum_{k=0}^{\infty} t_k \text{Tr} H^k} = \int d(H + \epsilon_n H^{n+1}) e^{\sum_{k=0}^{\infty} t_k \text{Tr}(H + \epsilon_n H^{n+1})^k}, \quad (2.3) \]

i.e.

\[ \int dH e^{\sum_{k=0}^{\infty} t_k \text{Tr} H^k} \left( \sum_{k=0}^{\infty} k t_k \text{Tr} H^{k+n} + \text{Tr} \frac{\delta H^{n+1}}{\delta H} \right) \equiv 0. \quad (2.4) \]

In order to evaluate the Jacobian \( \text{Tr} \frac{\delta H^{n+1}}{\delta H} \) let us restore the matrix indices:

\[ (\delta H^{n+1})_{ij} = \sum_{k=0}^{n} (H^k \delta H H^{n-k})_{ij} = \sum_{k=0}^{n} (H^k)_{il}(\delta H)_{lm}(H^{n-k})_{mj}, \quad (2.5) \]

and to obtain \( \text{Tr} \frac{\delta H^{n+1}}{\delta H} \) put \( l = i \) and \( m = j \), so that

\[ \text{Tr} \frac{\delta H^{n+1}}{\delta H} = \sum_{k=0}^{n} \text{Tr} H^k \text{Tr} H^{n-k}. \quad (2.6) \]

Any correlation function can be obtained as variation of the coupling constants:

\[ < \text{Tr} H^{a_1}...\text{Tr} H^{a_n} > = \int dH e^{\sum_{k=0}^{\infty} t_k \text{Tr} H^k} \text{Tr} H^{a_1}...\text{Tr} H^{a_n} = \]

\[ = \frac{\partial^n}{\partial t_{a_1}...\partial t_{a_n}} Z_N\{t\}. \quad (2.7) \]

This relation together with (2.6) can be used to rewrite (2.4) as:

\[ L_n Z_N\{t\} = 0, \quad n \geq -1 \quad (2.8) \]

with

\[ L_n = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_k \partial t_{n-k}}. \quad (2.9) \]

Note that according to the definition (2.1)

\[ \frac{\partial}{\partial t_0} Z_N = N Z_N. \quad (2.10) \]
2.1.1 Details and comments

Several remarks are now in order.

First of all, expression in brackets in (2.4) represents just all the equations of motion for the model (2.1), and (2.8) is nothing but another way to represent the same set of equations. This example illustrates what “exhaustively large” set of Ward identities is: it should be essentially the same as the set of all equations of motion.

Second, commutator of any two operators $L_n$ appearing in (2.8) should also annihilate $Z_N\{t\}$. Another indication that we already got a complete set of constraints, is that $L_n$’s form a closed algebra:

$$[L_n, L_m] = (n - m)L_{n+m}, \quad n, m \geq -1.$$  \hspace{1cm} (2.11)

Its particular representation (2.9) is referred to as “discrete Virasoro algebra” (to emphasize the difference with “continuous Virasoro” constraints, see eq.(3.64) below).

Third, (2.8) can be considered as invariant formulation of what is $Z_N$: it is a solution of this set of compatible differential equations. From this point of view eq.(2.1) is rather a particular representation of $Z_N$ and it is sensible to look for other representations as well (we shall later discuss two of them: one in terms of CFT, another in terms of Kontsevich integrals).

Fourth, one can try to analyze the uniqueness of the solutions to (2.8). If there are not too many solutions, the set of constraints can be considered complete. A natural approach to classification of solutions to the algebra of constraints is in terms of the orbits of the corresponding group [9]. Let us consider an oversimplified example, which can still be useful to understand implications of the complete set of WI as well as to clarify the meaning of classes of universality and of integrability.

Imagine, that instead of (2.8) with $L_n$’s defined in (2.9) we would obtain somewhat simpler equations:

$$l_nZ = 0, \quad n \geq 0 \quad \text{with} \quad l_n = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+n}}.$$  \hspace{1cm} (2.12)

Then operator $l_1$ can be interpreted as generating the shifts

$$t_2 \rightarrow t_2 + \epsilon_1 t_1,$$
$$t_3 \rightarrow t_3 + 2\epsilon_1 t_2,$$
$$\ldots$$  \hspace{1cm} (2.13)

We can use it to shift $t_2$ to zero, and eq. $l_1Z = 0$ then implies that

$$Z(t_1, t_2, t_3, \ldots) = Z(t_1, 0, \tilde{t}_3, \ldots)$$  \hspace{1cm} (2.14)

($\tilde{t}_k = t_k - \frac{(k-1)\tilde{t}_{k+1}}{t_1}, \quad k \geq 3$).

Next, operator $l_2$ generates the shifts

$$t_3 \rightarrow t_3 + \epsilon_2 t_1,$$
$$t_4 \rightarrow t_4 + 2\epsilon_2 t_2,$$
$$\ldots$$  \hspace{1cm} (2.15)

and does not affect $t_2$. We can now use eq. $l_2Z = 0$ to argue that

$$Z(t_1, t_2, t_3, t_4, \ldots) = Z(t_1, 0, \tilde{t}_3, \tilde{t}_4, \ldots) = Z(t_1, 0, 0, \tilde{t}_4, \ldots)$$  \hspace{1cm} (2.16)

\footnote{One can call them "classical" approximation to (2.8), since they would arise if the variation of measure (i.e. a "quantum effect") was not taken into account in the derivation of (2.8). Though this concept is often used in physics it does not have much sense in the present context, when we are analyzing exact properties of functional (matrix) integrals.}
etc. Assuming that $Z$ is not very much dependent on $t_k$ with $k \to \infty$, \(^3\) we can conclude, that

$$Z(t_1, t_2, t_3, ...) = Z(t_1, 0, 0, ...) = Z(1, 0, 0, ...) \quad (2.17)$$

(at the last step we also used the equation $l_0 Z = 0$ to rescale $t_1$ to unity).

All this reasoning had sense provided $t_1 \neq 0$. Otherwise we would get $Z(0, 1, 0, 0, ...)$, if $t_1 = 0$, $t_2 \neq 0$, or $Z(0, 0, 1, 0, ...)$, if $t_1 = t_2 = 0$, $t_3 \neq 0$ etc. In other words, we obtain classes of universality (such that the value of partition function is just the same in the whole class), which in this oversimplified example are labeled just by the first non-vanishing time-variable. Analysis of the orbit structure for the actually important realizations of groups, like that connected to eq.(2.9) has never been performed in the context of matrix model theory.

In this oversimplified case the constraints actually allow one to eliminate all the dependence on the time-variables, i.e. to solve equations for $Z$ exactly. In realistic examples one deals with less trivial representations of the constraint algebra, like (2.11). It appears that in this general framework constraints somehow imply the integrability structure of the model, what can thus be considered as a slightly more complicated version of the same solvability phenomenon.

### 2.2 CFT interpretation of 1-matrix model

Given a complete set of the constraints on partition function of infinitely many variables which form some closed algebra we can now ask an inverse question: how these equations can be solved or what is the integral representation of partition function. One approach to this problem is analysis of orbits, briefly mentioned at the end of the previous subsection. Now we turn to another technique [10], which makes use of the knowledge from conformal field theory. This constructions can have some meaning from the “physical” point of view, which implies certain duality between the 2-dimensional world surfaces and the spectral surfaces, associated to conﬁguration space of the string theory. However, our present goal is more formal than discussion of this duality: we are going to use the methods of CFT for solving the constraint equations.

This is especially natural when the algebra of constraints is Virasoro algebra, as is the case with the 1-matrix model, or some other algebra known to arise as a chiral algebra in some simple conformal models. In fact the approach to be discussed is rather general and can be applied to construction of matrix models, associated with many different algebraic structures: the only requirement is existence of the (massless) free-field representation.

We begin from the set of ”discrete Virasoro constraints” (2.8). The CFT formulation of interest should provide the solution to these equations in the form of some correlation function in some conformal field theory. Of course, it becomes natural if we somehow identify the operators $L_n$ (2.9) with the harmonics of a stress-tensor $T_n$, which satisfy the same algebra, and manage to relate the constraint that $L_n$ annihilate the correlator to the statement that $T_n$ annihilate the vacuum state. Thus the procedure is naturally split into two steps. First, we should find a $t$-dependent operator (”Hamiltonian”) $H(t)$, such that

$$L_n(t) \langle e^{H(t)} | T_n \rangle = \langle e^{H(t)} | T_n \rangle \quad (2.18)$$

This will relate differential operators $L_n$ to $T_n$’s expressed through the fields of conformal model. Second, we need to enumerate the states, that are annihilated

\(^3\)This, by the way, is hardly correct in this particular example, when the group has no compact orbits.
by the operators $T_n$ with $n \geq -1$, i.e. solve equation
\begin{equation}
T_n | G \rangle = 0 \tag{2.19}
\end{equation}
for the ket-states, what is an internal problem of conformal field theory. If both ingredients $H(t)$ and $| G \rangle$ are found, solution to the problem is given by
\begin{equation}
\langle e^{H(t)} | G \rangle. \tag{2.20}
\end{equation}

To be more explicit, for the case of the discrete Virasoro constraints we can just look for solutions in terms of the simplest possible conformal model: that of a one holomorphic scalar field
\begin{equation}
\phi(z) = \hat{q} + \hat{p} \log z + \sum_{k \neq 0} \frac{J_{-k}}{k} z^k \tag{2.21}
\end{equation}
\begin{equation}
[J_n, J_m] = n \delta_{n+m,0}, \quad [\hat{q}, \hat{p}] = 1.
\end{equation}
Then the procedure is as follows. Define vacuum states
\begin{equation}
J_k|0\rangle = 0, \quad \langle N | J_{-k} = 0, \quad k > 0
\end{equation}
\begin{equation}
\hat{p}|0\rangle = 0, \quad \langle N | \hat{p} = N \langle N |,
\end{equation}
the stress-tensor
\begin{equation}
T(z) = \frac{1}{2} [\partial \phi(z)]^2 = \sum_{k \geq 0} T_n z^{-n-2}, \quad T_n = \sum_{k \geq 0} J_{-k} J_{k+n} + \sum_{a+b \geq 0} J_a J_b, \tag{2.23}
\end{equation}
and the Hamiltonian
\begin{equation}
H(t) = \frac{1}{\sqrt{2}} \sum_{k \geq 0} t_k J_k = \frac{1}{\sqrt{2}} \oint C_0 U(z) J(z) \tag{2.24}
\end{equation}
\begin{equation}
U(z) = \sum_{k \geq 0} t_k z^k, \quad J(z) = \partial \phi(z).
\end{equation}
It is easy to check that
\begin{equation}
L_n \langle N | e^{H(t)} = \langle N | e^{H(t)} T_n \tag{2.25}
\end{equation}
and
\begin{equation}
T_n|0\rangle = 0, \quad n \geq -1. \tag{2.26}
\end{equation}
As an immediate consequence, any correlator of the form
\begin{equation}
Z_N \{ t | G \} = \langle N | e^{H(t)} G | 0 \rangle \tag{2.27}
\end{equation}
gives a solution to (2.8) provided
\begin{equation}
[T_n, G] = 0, \quad n \geq -1. \tag{2.28}
\end{equation}
In fact operators $G$ that commute with the stress tensor are well known: these are just any functions of the ”screening charges” \footnote{For notational simplicity we omit the normal ordering signs, in fact the relevant operators are $: e^H :$ and $: e^{\pm \sqrt{2} \phi} :$}
\begin{equation}
Q_\pm = \oint J_\pm = \oint e^{\pm \sqrt{2} \phi}. \tag{2.29}
\end{equation}
The correlator (2.27) will be non-vanishing only if the matching condition for zero-modes of $\phi$ is satisfied. If we demand the operator to depend only on $Q_+$, this implies that only one term of the expansion in powers of $Q_+$ will contribute to (2.27), so that the result is essentially independent on the choice of the function $G(Q_+)$, we can for example take $G(Q_+) = e^{Q_+}$ and obtain:
\begin{equation}
Z_N \{ t \} \sim \frac{1}{N!} \langle N | e^{H(t)} (Q_+)^N | 0 \rangle. \tag{2.30}
\end{equation}
This correlator is easy to evaluate using Wick theorem and the propagator
\[ \phi(z) \phi(z') \sim \log(z-z'). \] Finally we get

\[
Z_N \{ t \} = \frac{1}{N!} \langle N | : e^{\frac{-1}{N} \oint \phi U(z) \partial \phi(z)} : \prod_{i=1}^N \oint_{C_i} \phi_i : e^{\sqrt{2} \phi(z_i)} : | 0 \rangle =
\]

\[
= \frac{1}{N!} \prod_{i=1}^N \oint_{C_i} \phi_i e^{U(z_i)} \prod_{i<j}^N (z_i-z_j)^2
\]

in the form of a multiple integral. This integral does not yet look like the matrix integral (2.1). However, it is the same: (2.31) is an “eigenvalue representation” of matrix integral, see [11] and eq.(2.53) in the next subsection 2.3.

### 2.2.1 Details and comments

Thus in the simplest case we resolved the inverse problem: reconstructed an integral representation from the set of discrete Virasoro constraints. However, the answer we got seems a little more general than (ref1mamo) and (2.53): the r.h.s. of eq.(2.31) still depends on the contours of integration. Moreover, we can also recall that the operator \( G \) above could depend not only on \( Q_+ \), but also on \( Q_- \). The most general formula is a little more complicated than (2.31):

\[
Z_N \{ t | C_i, C_r \} \sim \frac{1}{(N+M)!M!} \langle N | e^{H(t)} (Q_+)^{N+M} (Q_-)^M | 0 \rangle =
\]

\[
= \frac{1}{(N+M)!M!} \prod_{i=1}^{N+M} \oint_{C_i} \phi_i e^{U(z_i)} \prod_{r=1}^{M+1} \oint_{C_r} \phi_r e^{U(z_r)} \prod_{i<j}^{N+M} (z_i-z_j)^2 \prod_{r<s}^M (z'_r-z'_s)^2 \prod_{i}^{N+M} \prod_{r}^M (z_i-z_r)^2.
\]

We refer to the papers [10] for discussion of the issue of contour-dependence. In certain sense all these different integrals can be considered as branches of the same analytical function \( Z_N \{ t \} \). Dependence on \( M \) is essentially eliminated by Cauchy integration around the poles in denominator in (2.32).

Above construction can be straightforwardly applied to any other algebras of constraints, provided:

(i) The free-field representation of the algebra is known in the CFT-framework, such that the generators are polynomials in the fields \( \phi \) (only in such case it is straightforward to construct a Hamiltonian \( H \), which relates CFT-realization of the algebra to that in terms of differential operators w.r.t. to the \( t \)-variables; in fact under this condition \( H \) is usually linear in \( t \)'s and \( \phi \)'s). There are examples (like Frenkel-Kac representation of level \( k = 1 \) simply-laced Kac-Moody algebras [12] or generic reductions of the WZNW model [13],[14],[17]) when generators are exponents of free fields, then this construction should be slightly modified.

(ii) It is easy to find vacuum, annihilated by the relevant generators (here, for example, is the problem with application of this approach to the case of "continuous" Virasoro and \( W \)-constraints). The resolution to this problem involves consideration of correlates on Riemann surfaces with non-trivial topologies, often - of infinite genus.

(iii) The free-field representation of the "screening charges", i.e. operators that commute with the generators of the group within the conformal model, is explicitly known.

These conditions are fulfilled in many case in CFT, including conventional \( W \)-algebras [18] and \( \mathcal{N} = 1 \) supersymmetric models.

For illustrative purposes we present here several formulas from the last paper.

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\(^5\)In the case of \( \mathcal{N} \leq 2 \) supersymmetry a problem arises because of the lack of reasonable screening charges. At the most naive level the relevant operator to be integrated over superspace (over \( d^2 \theta \)) in order to produce screening charge has dimension \( 1 - \frac{1}{2} \mathcal{N} \), which vanishes when \( \mathcal{N} = 2 \).
of ref.[10] for the case of the $W_{r+1}$-constraints, associated with the simply-laced algebras $G$ of rank $r$.

Partition function in such "conformal multimatrix model" is a function of "time-variables" $t^{(k)}$, $k=0...\infty$, $\lambda = 1...r = \text{rank} G$, and also depends on the integer-valued $r$-vector $\vec{N} = \{N_1...N_r\}$. The $W_{r+1}$-constraints imposed on partition function are:

$$W_n^{(a)}(t)Z_{\vec{N}}^G(t) = 0, \quad n \geq 1 - a, \quad a = 2...r + 1. \quad (2.33)$$

The form of the $W$-operators is somewhat complicated, for example, in the case of $r + 1 = 3$ (i.e. for $G = SL(3)$)

$$W_n^{(2)} = \sum_{k=0}^{\infty} \left( kt_k \frac{\partial}{\partial t_{k+n}} + k\bar{t}_k \frac{\partial}{\partial \bar{t}_{k+n}} \right) +$$
$$+ \sum_{a+b=n} \left( \frac{\partial^2}{\partial t_a \partial \bar{t}_b} + \frac{\partial^2}{\partial \bar{t}_a \partial t_b} \right) \quad (2.34)$$

$$W_n^{(3)} = \sum_{k,l=0}^{\infty} \left( kt_k \bar{t}_l \frac{\partial}{\partial t_{k+n+l}} - k\bar{t}_k t_l \frac{\partial}{\partial \bar{t}_{k+n+l}} - 2kt_k \bar{t}_l \frac{\partial}{\partial k+n+l} \right) +$$
$$+ 2 \sum_{k>0} \left[ \sum_{a+b+c=n+k} \left( kt_k \frac{\partial^2}{\partial t_a \partial \bar{t}_b} - k\bar{t}_k \frac{\partial^2}{\partial \bar{t}_a \partial t_b} - 2kt_k \bar{t}_l \frac{\partial^2}{\partial t_a \partial \bar{t}_b} \right) \right] +$$
$$+ \frac{4}{3} \sum_{a+b+c=n+k} \left( \frac{\partial^3}{\partial t_a \partial \bar{t}_b \partial \bar{t}_c} - \frac{\partial^3}{\partial \bar{t}_a \partial t_b \partial \bar{t}_c} \right), \quad (2.35)$$

and two types of time-variables, denoted through $t_k$ and $\bar{t}_k$, are associated with two orthogonal directions in the Cartan plane of $A_2$: $\mathbf{e} = \frac{\vec{a}}{\sqrt{2}}$, $\mathbf{e} = \frac{\vec{a}}{\sqrt{2}}$. \footnote{Such orthogonal basis is especially convenient for discussion of integrability properties of the model, these $t$ and $\bar{t}$ are linear combinations of time-variables $t^k_\alpha$ appearing in eqs. (2.36) and (2.41).}

All other formulas, however, are very simple: Conformal model is usually that of the $r$ free fields, $S \sim \int \bar{\phi} \partial \bar{\phi} \partial \phi^2 \partial z$, which is used to describe representation of the level one Kac-Moody algebra, associated with $G$. Hamiltonian

$$H(t^{(1)} ... t^{(r+1)}) = \sum_{\lambda=1}^{r+1} \sum_{k>0} t^{(k)}_\lambda \mu_\lambda J_k, \quad (2.36)$$

where $\{\mu_\lambda\}$ are associated with "fundamental weight" vectors $\vec{\nu}_\lambda$ in Cartan hyperplane and in the simplest case of $G = SL(r+1)$ satisfy

$$\mu_\lambda \cdot \bar{\nu}_\lambda = \delta_{\lambda \lambda'} - \frac{1}{r+1}, \quad \sum_{\lambda=1}^{r+1} \mu_\lambda = 0,$$

thus only $r$ of the time variables $t^{(1)} ... t^{(r+1)}$ are linearly independent. Relation between differential operators $W_n^{(a)}(t)$ and operators $W_n^{(a)}$ in the CFT is now defined by

$$W_n^{(a)}(\vec{N}) e^{H(t)} = \langle \vec{N} | e^{H(t)} W_n^{(a)} \rangle, \quad (2.37)$$

where

$$W_n^{(a)}(z) = \sum_{\lambda} [\mu_\lambda \partial \vec{\nu}(z)]^a + \ldots \quad (2.38)$$

are spin-$a$ generators of the $W_{r+1}$ algebra. The screening charges, that commute with all the $W^{(a)}(z)$ are given by

$$Q^{(a)} = \oint j^{(a)} = \oint e^{\vec{a} \vec{\phi}} \quad (2.39)$$

$\{\vec{a}\}$ being roots of finite-dimensional simply laced Lie algebra $G$.

Thus partition function arises in the form:

$$Z_{\vec{N}}^G(t) = \langle \vec{N} | e^{H(t)} G \{Q^{(a)}\} | 0 \rangle \quad (2.40)$$
where $G$ is an exponential function of screening charges. Evaluation of the free-field correlator gives:

$$Z^G_N\{t^\alpha\} \sim \int \prod_{\alpha} \left[ \prod_{i=1}^{N_\alpha} d\zeta_i(\alpha) \exp \left( \sum_{\lambda, \kappa > 0} t^\lambda_k (\mu(\lambda\alpha)) (\zeta_i(\alpha))^k \right) \right] \times$$

$$\times \prod_{(\alpha, \beta) = 1}^{N_\alpha N_\beta} \prod_{i,j = 1}^{(\alpha, \beta)} (\zeta_i(\alpha) - \zeta_j(\beta))^\alpha \beta \tag{2.41}$$

In fact this expression can be rewritten in terms of an $r$-matrix integral – a "conformal multimatrix model":

$$Z^G_N\{t^\alpha\} = c_1^{p-1} \int_{N \times N} dH^{(1)} \ldots dH^{(p-1)} e^{N_\alpha t^\alpha H_{\alpha\beta}} \prod_{(\alpha, \beta)} \text{Det} \left( H^{(\alpha)} \otimes I - I \otimes H^{(\alpha+1)} \right)^{\bar{\alpha}\bar{\beta}} \tag{2.42}$$

In the simplest case of $W_3$ algebra eq.(2.41) with insertion of only two (of the six) screenings $Q_{\alpha_1}$ and $Q_{\alpha_2}$ turns into

$$Z^{SL(3)}_{N_1, N_2}(t, \bar{t}) = \frac{1}{N_1!N_2!} \langle N_1, N_2 | e^{H(t, \bar{t})} (Q^{(\alpha_1)})^{N_1} (Q^{(\alpha_2)})^{N_2} | 0 \rangle =$$

$$= \frac{1}{N_1!N_2!} \prod_i \int dx_i e^{U(x_i)} \prod_j \int dy_j e^{U(y_j)} \Delta(x) \Delta(x, y) \Delta(y), \tag{2.43}$$

where $\Delta(x, y) = \Delta(x) \Delta(y) \prod_{i,j}(x_i - y_j)$. This model is associated with the algebra $G = SL(3)$, while the original 1-matrix model (2.30)-(2.32) - with $G = SL(2)$.

The whole series of models (2.41-2.42) for $G = SL(r+1)$ is distinguished by its relation to the level $k = 1$ simply-laced Kac-Moody algebras. In this particular situation the underlying conformal model has integer central charge $c = r = \text{rank } G$ and can be "fermionized". The main feature of this formulation is that the Kac-Moody currents (which after integration turn into "screening charges" in the above construction) are quadratic in fermionic fields, while they are represented by exponents in the free-boson formulation.

In fact fermionic (spinor) model naturally possesses $GL(r + 1)$ rather than $SL(r + 1)$ symmetry (other simply-laced algebras can be embedded into larger $GL$-algebras and this provides fermionic description for them in the case of $k = 1$). The model contains $r + 1$ spin-1/2 fields $\psi_i$ and their conjugate $\tilde{\psi}_i$ ($b, c$-systems):

$$S = \sum_{j=1}^{r+1} \int \bar{\psi}_j \partial \psi_j d^2 z, \tag{2.44}$$

central charge $c = r + 1$, and operator algebra is

$$\bar{\psi}_j(z) \psi_k(z') = \frac{\delta_{jk}}{z - z'} + : \bar{\psi}_j(z) \psi_k(z') :$$

$$\psi_j(z) \bar{\psi}_k(z') = (z - z') \delta_{jk} : \psi_j(z) \bar{\psi}_k(z') : + (1 - \delta_{jk}) : \bar{\psi}_j(z) \psi_k(z') : \tag{2.45}$$

The Kac-Moody currents of the level-one $GL(r + 1)$ are just $J_{jk} = : \bar{\psi}_j \psi_k :$, $j, k = 1 \ldots r + 1$, and screening charges are $Q^{(\alpha)} = i E^{(\alpha)}_{jk} : \bar{\psi}_j \psi_k :$, where $E^{(\alpha)}_{jk}$ are representatives of the roots $\vec{\alpha}$ in the matrix representation of $GL(r + 1)$. Cartan subalgebra is represented by $J_{jj}$, while positive and negative Borel subalgebras - by $J_{jk}$ with $j < k$ and $j > k$ respectively. In eq.(2.32) $Q_+ = \ldots$
Let us now turn to particular example of the 1-matrix integral (2.1). First of all, this model possesses gauge symmetry, associated with the unitary (angular) rotation of matrices, $H_\alpha \to U_\alpha^\dagger H_\alpha U_\alpha$. This illustrates the general phenomenon: matrix models are usually gauge theories. In the case of eigenvalue models this symmetry is realized without "gauge fields" $V_{\alpha\beta}$, which would depend on pairs of indices $\alpha, \beta$ and transform like $V_{\alpha\beta} \to U_\alpha^\dagger V_{\alpha\beta} U_\beta$. In other words, eigenvalue models are gauge theories without gauge fields, i.e. are pure topological. The case of the 1-matrix model (2.1)

\begin{equation}
Z_N\{t\} \equiv c_N \int_{N \times N} dH e^{\sum_{k=0}^{\infty} t_k \text{Tr} H^k},
\end{equation}

is especially simple, because separation of eigenvalue and angular variables does not involve any information about unitary-matrix integrals. Take

\begin{equation}
H = U^\dagger D U,
\end{equation}

where $U$ is a unitary matrix and diagonal matrix $D = \text{diag}(h_1...h_N)$ has eigenvalues $h_1...h_N$. The case of the 1-matrix model (2.1) in eigenvalue representation

\[ i \oint \tilde{\psi}_1 \psi_2, \quad Q_- = i \oint \tilde{\psi}_2 \psi_1 \quad \text{while in eq.}(2.43) \quad Q^{(a_1)} = i \oint \tilde{\psi}_1 \psi_2, \quad Q^{(a_2)} = i \oint \tilde{\psi}_1 \psi_3 \quad \text{(and} \quad Q^{(a_3)} = i \oint \tilde{\psi}_2 \psi_3, \quad Q^{(a_4)} = i \oint \tilde{\psi}_2 \psi_1, \quad Q^{(a_5)} = i \oint \tilde{\psi}_3 \psi_1, \quad Q^{(a_6)} = i \oint \tilde{\psi}_3 \psi_2). \quad Q^{(a_0)} \text{can be substituted instead of} \quad Q^{(a_2)} \text{in (2.43) without changing the answer. For generic} \quad r \quad \text{the similar choice of "adjacent" (not simple!) roots (such that their scalar products are +1 or 0) leads to selection of the following} \quad r \quad \text{screening operators} \quad Q(1) = i \oint \tilde{\psi}_1 \psi_2 \quad Q(2) = -i \oint \psi_2 \tilde{\psi}_3, \quad Q(3) = i \oint \tilde{\psi}_3 \psi_4, \ldots \quad \text{i.e.} \quad Q(j) = i \oint \tilde{\psi}_j \psi_{j+1} \text{for odd} \quad j \quad \text{and} \quad Q(j) = -i \oint \psi_j \tilde{\psi}_{j+1} \text{for even} \quad j. \]

2.3 1-matrix model in eigenvalue representation

In the last section we found solution of Virasoro constraints (2.8) in the form of the multiple integral (2.31). Now we shall see that this expression is in fact equal to original matrix integral (2.1) and arises after “auxiliary” angular variables are explicitly integrated out. These angular variables are in fact the ones associated with physical vector bosons and the possibility to solve this sector of the model explicitly is peculiar feature of the simplest class of matrix models, naturally named eigenvalue models. The theory of eigenvalue models is in a sense equivalent to the theory of conventional integrable hierarchies and thus is rather straightforward to work on. Inclusion of non-trivial angular integration in the general scheme is still a sophisticated task, with no universal solution found so far. Also unknown is solution of inverse problem: how can be arbitrary eigenvalue model - with integration over eigenvalues of some matrix - “lifted to” the full matrix model where integration goes over entire matrix\(^8\); in other words what is the way to couple vector bosons to the “topological” eigenvalue sector so that the two sectors are interacting only in the “solvable” fashion and angular integrations can be easily performed.

\(^8\)see also ref.[19]
values of $H$ as its entries. Then integration measure\(^9\)

\[
dH = \prod_{i,j=1}^{N} dH_{ij} = \frac{[dU]}{[dU_{\text{Cartan}}]} \prod_{i=1}^{N} dh_{i} \Delta^{2}(h), \tag{2.51}
\]

where ”Van-der-Monde determinant” $\Delta(h) \equiv \det(h_{ij}) = \prod_{i>j}(h_{i} - h_{j})$ and $[dU]$ is Haar measure of integration over unitary matrices.

It remains to note that the ”action” $\text{Tr} \, U(H) \equiv \sum_{k=0}^{\infty} t_{k} \text{Tr} H^{k}$ with $H$ substituted in the form (2.47) is independent of $U$:

\[
\text{Tr} \, U(H) = \sum_{i=1}^{N} U(h_{i}). \tag{2.52}
\]

\(^9\) In order to derive eq.(2.51) one can consider the norm of infinitesimal variation

\[
\| \delta H \|^2 = \sum_{i,j=1}^{N} | \delta H_{ij} |^2 = \sum_{i,j=1}^{N} (\delta h_{ij} \delta h_{ji}) = \text{Tr}(\delta H)^{2} = \text{Tr} \left( -U^{\dagger} \delta U U^{\dagger} DU + U^{\dagger} D \delta U + U^{\dagger} \delta DU \right)^{2} = \text{Tr}(\delta D)^{2} + 2i \text{Tr} \delta u[D, \delta D] + 2i \text{Tr} \left( -\delta u D \delta u D + (\delta u)^{2} D^{2} \right),
\]

where $\delta u \equiv \frac{1}{2} \delta U U^{\dagger} = \delta a^{\dagger}$ and $\delta D = \text{diag}(\delta h_{1}, \ldots, \delta h_{N})$. The second term at the r.h.s. vanishes because both $D$ and $\delta D$ are diagonal and commute. Therefore

\[
\| \delta H \|^2 = \sum_{i=1}^{N} (\delta h_{i})^{2} + \sum_{i,j=1}^{N} (\delta u)_{ij}(\delta u)_{ji}(h_{i} - h_{j})^{2}. \tag{2.49}
\]

Now it remains to recall the basic relation between the infinitesimal norm and the measure: if $\| \delta \|^{2} = G_{ab} \delta a^{\dagger} \delta a^{b}$ then $[d\|] = \sqrt{\det G_{ab}} \prod_{a} da^{a}$, to obtain eq.(2.51) with Haar measure $[dU] = \prod_{i,j=1}^{N} du_{ij}$ being associated with the infinitesimal norm

\[
\| \delta a \|^2 = \text{Tr}(\delta a)^{2} = \sum_{i,j=1}^{N} \delta u_{ij} \delta u_{ji} = \sum_{i,j=1}^{N} | \delta u_{ij} |^{2}, \tag{2.50}
\]

and $[dU_{\text{Cartan}}] = \prod_{i=1}^{N} du_{ii}$.

Thus

\[
Z_{N}\{t_{0}; t_{k} = -\frac{1}{2} \text{tr} \Lambda^{-k} + \frac{1}{2} \delta k_{2} \} = \frac{1}{N!} \prod_{i=1}^{N} \int dh_{i} e^{U(h_{i})} \prod_{i>j}(h_{i} - h_{j})^{2} = \frac{1}{N!} \prod_{i=1}^{N} \int dh_{i} e^{U(h_{i})} \Delta^{2}(h), \tag{2.53}
\]

provided $c_{N}$ in (2.1) and (2.46) is chosen to be

\[
c^{-1}_{N} = N! \frac{\text{Vol}_{U(N)}}{(\text{Vol}_{U(1)})^{N}}, \tag{2.54}
\]

where the volume of unitary group in Haar measure is equal to

\[
\text{Vol}_{U(N)} = \frac{(2\pi)^{N(N+1)/2}}{\prod_{k=1}^{N} k!}. \tag{2.55}
\]

### 2.4 Kontsevich-like representation of 1-matrix model

Matrix-integral representation (2.1) is, however, not the only possible one for the given eigenvalue model (2.31). Expression (2.1) involves the “most general action”, consistent with the symmetry $H \rightarrow U H U^{\dagger}$. As was already mentioned in the introduction, alternative representation of the partition function should instead involve the general coupling to background (source) field. In the theory of matrix models such representations are known under the name of Kontsevich models. They will be the subject of detailed discussion in the next sections of this paper. What we need now is one simple identity, relating original (2.1) and Kontsevich-like representations of the 1-matrix theory:

\[
Z_{N}\{t_{0}; t_{k} = -\frac{1}{2} \text{tr} \Lambda^{-k} + \frac{1}{2} \delta k_{2} \} = \int_{N \times N} dH e^{\sum_{k=0}^{\infty} t_{k} \text{Tr} H^{k}} = \int_{N \times N} dH e^{\sum_{k=0}^{\infty} t_{k} \text{Tr} H^{k}} = \int_{N \times N} dH e^{\frac{1}{2} \text{tr} \Lambda^{2}} \int_{n \times n} dX (\text{det}X)^{N} e^{-\text{tr} \frac{1}{2} \Lambda^{2} + \text{tr} \Lambda X} = Z_{N}^{\Lambda^{2}} \{N, t\},
\]

\(\text{Vol}_{U(N)}\)
where $Z_N \{ t_k = \frac{1}{2} \delta_{k,2} \} = (-2\pi)^{\frac{N^2}{2}} c_N$. This relation follows from another identity:

$$
\int_{N \times N} dH e^{\frac{1}{2} \text{tr} H^2} \text{Det}(\Lambda \otimes I - I \otimes H) = 
\int_{N \times N} dH e^{\frac{1}{2} \text{tr} H^2} = 
\int_{n \times n} dX e^{\frac{1}{2} \text{tr} X^2} \text{det}^N(X + \Lambda),
$$

(2.57)

which is valid for any $\Lambda$ and can be proved by different methods: see [1] and references therein. Note that integrals on the right and left hand sides are of different sizes: $N \times N$ at the l.h.s. and $n \times n$ at the r.h.s. While $N$-dependence is explicit at both sides of the equation, the $n$-dependence at the l.h.s. enters only implicitly: through the allowed domain of variation of variables $t_k = -\frac{1}{2} \text{tr} \Lambda^{-k} + \frac{1}{2} \delta_{k,2}$. This is the usual feature of Kontsevich integrals: explicit $n$-dependence disappears once the integral is expressed through the $t$-like variables.

Eq.(2.6) can be used to perform analytical continuation in $N$ and define what is $Z_N$ for $N$, which are not positive integers. Since $c_N = 0$ for all negative integers (see [1]), the same is true for $Z_N$. This property ($\tau_N = 0$ for all negative integers $N$) is in fact characteristic for $\tau$-functions of forced hierarchies of which the partition function (2.53) is an example.

3 Generalized Kontsevich Model (GKM)

Let us now proceed to investigation of Kontsevich models of a rather general type. Further generalizations, leading directly to theories with physical vector bosons (see for example [20]), are beyond the scope of the present notes. The basic mathematical fact, responsible for solvability (integrability) of Kontsevich models is Duistermaat-Heckmann theorem, which allows to evaluate explicitly the celebrated non-linear Harish-Chandra-Itzykson-Zuber integral over unitary matrices and thus transform matrix integral into an eigenvalue model.

3.1 Kontsevich integral. The first step

Kontsevich integral is defined as

$$
\mathcal{F}_{V,n}\{ L \} = \int_{n \times n} dX e^{-\text{tr} V(X) + N \text{tr} \log X + \text{tr} LX},
$$

(3.1)

In fact it depends only on eigenvalues of the matrix $L$. Indeed, substitute $X = U_X^\dagger D_X U_X$; $L = U_L^\dagger D_L U_L$ in (3.1) and denote $U \equiv U_X U_L^\dagger$. Then

$$
\mathcal{F}_{V,n}\{ L \} = 
\prod_{a=1}^n \int dx_a x_a^N e^{-V(x_a)} \Delta^2(x) \int_{n \times n} \frac{[dU]}{[dU_{\text{Cartan}}]} \exp \left( \sum_{a,b=1}^n x_a t_b \mid U_{ab} \right)^2.
$$

(3.2)

In order to proceed further one needs to evaluate the integral over unitary matrices, which appeared at the r.h.s.

This integral can actually be represented in two different forms:

$$
I_n\{ X, L \} = \int_{n \times n} \frac{[dU]}{[dU_{\text{Cartan}}]} e^{\text{tr} X U L U^\dagger} = 
\int_{n \times n} \frac{[dU]}{[dU_{\text{Cartan}}]} e^{\sum_{a,b=1}^n x_a t_b \mid U_{ab}^2}
$$

(3.3)

(the $U$’s in the two integrals are related by transformation $U \rightarrow U_X U_L^\dagger$ and Haar measure is both left and right invariant). Formula (3.3) implies that $I_n\{ X, L \}$ satisfies a set of simple equations [21]:

$$
\left( \text{tr} \left( \frac{\partial}{\partial \text{tr} X} \right)^k - \text{tr} L^k \right) I_n\{ X, L \} = 0, \quad k \geq 0,
$$

$$
\left( \text{tr} \left( \frac{\partial}{\partial \text{tr} L} \right)^k - \text{tr} X^k \right) I_n\{ X, L \} = 0, \quad k \geq 0,
$$

(3.4)
which by themselves are not very restrictive. However, another formula, (3.3), implies that $I_n \{X, L\}$ in fact depends only on the eigenvalues of $X$ and $L$, and for such $I_n \{X, L\} = \hat{I} \{x_a, l_b\}$ eqs.(3.4) become very restrictive \(^{10}\) and allow to determine $\hat{I} \{x_a, l_b\}$ unambiguously as a formal power seria in positive powers of $x_a$ and $l_b$. The final answer is

$$I_n \{X, L\} = \frac{(2\pi)^{\frac{n(n-1)}{2}}}{n!} \det_{ab} e^{x_a l_b} \Delta(x) \Delta(l),$$  \hspace{1cm} (3.6)

Normalization constant can be defined by taking $L = 0$, when

$$I_n \{X, L = 0\} = \frac{\text{Vol} \{U(n)\}}{\text{Vol} \{U(l)\}} = \frac{(2\pi)^{\frac{n(n-1)}{2}}}{\prod_{k=0}^{n} k!},$$  \hspace{1cm} (3.7)

and using the fact that

$$\frac{\det_{ab} f_a(l_b)}{\Delta(l)} \bigg|_{\{l_b = 0\}} = \left(\prod_{k=0}^{n-1} \frac{1}{k!}\right) \det_{ab} \delta^{-1} f_a(0).$$  \hspace{1cm} (3.8)

### 3.2 Itzykson-Zuber integral and Duistermaat-Heckmann theorem

Eq.(3.6) is usually refered to as the Itzykson-Zuber formula \cite{22}. In mathematical literature it was earlier derived by Kharish-Chandra \cite{23}, and in fact the integral (3.3) is the basic example of the coadjoint orbit integrals \cite{24}-\cite{26}, which can be exactly evaluated with the help of the Duistermaat-Heckmann theorem \cite{27}-\cite{30}. We now interrupt our discussion of Kontsevich model for a brief illustration of this important phenomenon. Duistermaat-Heckmann theorem claims that under certain restrictive conditions (that dynamical flow is consistent with the action of a compact group) some integrals can be expressed in a simple way through extrema of the integrand. This almost sounds like a statement that quasiclassical approximation can be exact, with two correction that all the extrema - not only the deepest minimum of the action - should be taken into account, and that the quantum measure should be adjusted appropriately.

When applicable, the theorem states that

$$\int [d\phi] e^{-S(\phi)} \sim \sum_{\phi: \partial S/\partial \phi = 0} \left(\frac{\partial^2 S}{\partial \phi^2}\right)^{-1/2} e^{-S(\phi)}$$  \hspace{1cm} (3.9)

The simplest example is given by the integral $\int [\sin \theta] e^{i \mu \cos \theta} = \frac{e^{i \mu} - e^{-i \mu}}{\mu}$. We shall not dwell upon the reasons why DH theorem is true in the case of the Itzykson-Zuber integral (the basic requirement: existence of the compact group action - that of unitary group - is obviously fulfilled in this case). Instead we just evaluate the r.h.s. of (3.9) provided the l.h.s. is given by $\int [dU] \exp \left( \text{tr} XULU^\dagger \right)$. Then equation of motion for $U$ looks like

$$[X, ULU^\dagger] = 0$$  \hspace{1cm} (3.10)

We assume that $X$ and $L$ are already diagonal matrices. Then (3.10) has an obvious solution $U = I$ (identity matrix), but it is not unique. For generic diagonal $X$, $L$ the most general solution is given by arbitrary permutation matrix $P$: $U = P$. The “classical action” on such solution is equal to $\text{tr} XULU^\dagger = \sum_a x_a l_{P(a)}$, while the preexponential factors provide Van-der-Monde determinants $\Delta(x) \Delta(l)$ in denominator and the sign factor $(-)^P$. Since

$$\sum_P (-)^P \exp \left( \sum_a x_a l_{P(a)} \right) = \det_{ab} e^{x_a l_b}$$

10 When acting on $\hat{I}$, which depends only on eigenvalues, matrix derivatives turn into:

$$\text{tr} \frac{\partial}{\partial X} \hat{I} = \sum_a \frac{\partial}{\partial x_a} \hat{I};$$  \hspace{1cm} (3.5)

$$\text{tr} \frac{\partial^2}{\partial X^2} \hat{I} = \sum_a \frac{\partial^2}{\partial x_a^2} \hat{I} + \sum_{a \neq b} \frac{1}{x_a - x_b} \left( \frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_b} \right) \hat{I};$$  \hspace{1cm} (3.5)

e tc.
we immediately obtain the IZ formula (3.6). Unfortunately the Duistermaat-Heckmann theory is not yet well developed and even if the vacuum average is exactly calculable, it does not provide immediate prescription for evaluation of correlators (or, what is essentially the same, corrections to DH formula - when it is not exactly true - are not yet described in a universal way). The very important general technique of exact evaluation of non-Gaussian unitary-matrix integrals is now doing its first steps (see [31]-[34]).

3.3 Kontsevich integral. The second step

Now we turn back to the eigenvalue representation of Kontsevich integral (3.1). Substitution of (3.6) into (3.2) gives:

$$
F_{V,n} \{ L \} = \frac{(2\pi)^{n(n-1)}}{\Delta(l)} \prod_{b=1}^{n} \int dx_b e^{-V(x_b)} \Delta(x) \frac{1}{n!} \det_{ab} e^{x_a l_b} =
$$

(3.11)

where we used antisymmetry of $\Delta(x)$ under permutations of $x_a$'s in order to change $\frac{1}{n!} \det_{ab} e^{x_a l_b}$ for $e^{\sum x_a l_b}$ under the sign of the $x_b$ integration.

We can now use the fact that $\Delta(x) = \det_{ab} x_a^{-1}$ in order to rewrite the r.h.s. of (3.11):

$$
F_{V,n} \{ L \} = (2\pi)^{n(n-1)} \frac{\det_{ab} \hat{\varphi}_{a+N}(l_b)}{\Delta(l)},
$$

(3.12)

where

$$
\hat{\varphi}_{a}(l) \equiv \int dx x^{a-1} e^{-V(x)+lx}, \quad a \geq 1.
$$

(3.13)

These functions $\hat{\varphi}(l)$ satisfy a simple recurrent relation:

$$
\hat{\varphi}_{a} = \frac{\partial \hat{\varphi}_{a-1}}{\partial l} = \left( \frac{\partial}{\partial l} \right)^{a-1} \hat{\Phi}
$$

(3.14)

with

$$
\hat{\Phi}(l) \equiv \hat{\varphi}_{1}(l) = \int dx e^{-V(x)+lx}.
$$

(3.15)

This completes the transformation of Kontsevich integral to the form of eigenvalue model.

3.4 “Phases” of Kontsevich integral. GKM as the “quantum piece” of $F_{V} \{ L \}$ in Kontsevich phase

One of the natural things to do in the study of functions, defined in the integral form, is to investigate various asymptotics when the arguments tend to various distinguished limits. In the case of Kontsevich integral the arguments are just eigenvalues $l_a$ of the matrix $L$, while their distinguished values are either zero or positions of singularities of potential $V(x)$. For simplicity we assume that $V(x)$ has only a pole of the order $p+1$ at infinity, i.e. $V(x)$ is a polynomial of degree $p + 1$. Then it remains to consider separately only two different asymptotics: small $l_a$ and large $l_a$. Of course, since there are many different $l_a$ one actually has a vast variety of possibilities: some of $l_a$’s are small and the rest are large.

The two extreme cases are when all the $l_a$’s are either small or large, and they are referred to as the “character phase” and “Kontsevich phase” respectively. The word “phase” is used instead of the more exact “asymptotics” in order to emphasize the relation of these two cases to the strong and weak coupling phases in lattice models of Yang-Mills theories. We refer to [35] for some more discussion of these phases and their properties, here only some basic things will be mentioned.

In the character phase the main observation is that the r.h.s. of the Itzykson-Zuber formula (3.6) is essentially the character of the group element $g = e^{L}$ of
$SL(n)$:
\[
\chi_R(e^L) = \frac{\det e^{L_a x_b}}{\Delta(e^L)} \Delta(x),
\]
(3.16)
where the set \( \{x_a\} \) specifies representation \( R \) of the $SL$ group\(^{11}\). Accordingly the integral (3.1) can be represented as a linear combination of characters with various \( R \), with coefficients depending on the choice of potential \( V(x) \) and \( N \).

This explains the reason why this limit (when everything is assumed to be expandable in positive powers of \( l_a \) or \( e^{L_a} \)) is refered to as the character phase. One should keep in mind, of course, that the most natural, from this point of view, is the situation when integral over \( x_a \)'s is changed for a discrete sum over integer \( x_a \)'s - this is one of the possible directions of the search for “quantum deformations” of GKM.

Let us now turn to the limit of large \( l_a \). Then the natural expansion would be in negative powers of the arguments, but the integral (3.1) does not have such expansion, as it is. One should first extract a “quasiclassical factor” and it will be the remaining “quantum part” that will possess such an expansion. This quantum piece is the most interesting one, and it is for it that the name GKM is usually used. So, in Kontsevich limit, integral (3.1) can be evaluated with the help of the steepest descent method, with the classical solution defined from \( V'(X_0) = l_0 \). (Note that when doing so we include the logarithmic piece\(^{12}\).)

\(^{12}\) Correction factor $\Delta(l)/\Delta(e^L)$ can be restored if one considers appropriate generalization of Kontsevich integral (which describes not just a puncture but a hole on a surface - in terms of the naive string theory), see, for example, ref.\(^{20}\). Actually one needs to substitute
\[
\int [dU] \exp \left( \text{tr} XUULU^\dagger \right)
\]
by the loop integral
\[
\int [dU(s)] \exp \left( \int ds \text{tr} X \left( U(s)\partial_s U^\dagger(s) + U(s)\partial_s U^\dagger(s) \right) \right) = \frac{\chi_R(e^L)}{dR}
\]
When this integral is evaluated for integer \( x_a \)'s one essentially substitutes every item in the product $\Delta(l) = \prod_{a<b}(l_a - l_b)$ by a new infinite product over harmonics, $l_a - l_b \rightarrow \prod_{a}(l_a - l_b + 2\pi ik) \sim \sinh \left( \frac{l_a - l_b}{2} \right) \sim e^{L_a} - e^{L_b}$
in the measure, not in the action.) Let us call solution of this equation \( X_0 = \Lambda \). One could use \( \Lambda \) from the very beginning instead of \( L \) to parametrize Kontsevich integral, writing $\text{tr} V'(\Lambda)X$ instead of $\text{tr} LX$ in the exponent in (3.1).

This is a natural parameter in Kontsevich phase, while \( L = V'(\Lambda) \) plays this role in the character phase. The quasiclassical contribution to (3.1), i.e. exponent of the classical action divided by determinant of quadratic fluctuations, is equal to:
\[
C_V \{\Lambda|N\} = \frac{(2\pi)^{n^2/2} \exp \left[ \text{tr} \left( \frac{1}{2} V'(\Lambda) - V(\Lambda) \right) \right]}{\sqrt{\det V''(\Lambda)}} (\det \Lambda)^N
\]
and partition function of GKM is by definition
\[
Z_{V,n} \{\Lambda|N\} = \frac{f_V \{\Lambda|N\}}{C_V \{\Lambda|N\}}
\]
This function can be expanded in negative integer powers of \( \lambda_a \) (i.e. in fractional negative powers of original \( l_a \), at least as a formal series. Moreover, there is a symmetry between all the eigenvalues \( \lambda_a \), thus $Z^{GKM}$ is in fact a function (formal series) of the “time variables” (the name is historic trace from the theory of integrable hierarchies)
\[
T_k = \frac{1}{k} \text{tr} \Lambda^{-k}
\]
Remarkably, if considered as a function of these \( T_k \)'s \( Z \) becomes independent of \( n! \) We refer to \[4\] and \[1\] for more details (including exact definition of $V''(\Lambda)$ in (3.17)).

What remains to be considered here is the eigenvalue representation of \( Z \). If (3.12) is divided by the quasiclassical factor $C_V \{\Lambda|N\}$, we get:
\[
Z_V \{N, T\} = \frac{1}{(\det \Lambda)^N} \cdot \frac{\det_{ab} \varphi_{a+N}(\Lambda_b)}{\Delta(\Lambda)}
\]
Extraction of the quasiclassical factor converts $\varphi(l)$ into the properly normalized expansions in negative integer powers of \( \lambda \):
\[
\varphi_a(\lambda) = \frac{e^{-\lambda V''(\lambda) + V(\lambda)}}{\sqrt{2\pi}} \frac{V''(\lambda)}{\sqrt{\Delta(\lambda)}} \hat{\varphi}_a(V'(\lambda)) = \lambda^{a-1}(1 + O(\lambda^{-1})),
\]
(3.21)
It also changes $\Delta(l) = \Delta(V'(\lambda))$ in the denominator of (3.12) for $\Delta(\lambda)$ in (3.20).

Note that, as a corollary of normalization condition (3.21), whenever one puts $\lambda_n = \infty$ the $n \times n$ determinant in (3.20) just turns into $(n - 1) \times (n - 1)$ determinant of the same form. One can easily understand that this implies the $n$-independence of $Z$ as a function of $T$-variables (3.19).

Instead of the simple recurrent relations (3.14) for $\varphi$ the normalized functions $\varphi$ satisfy:

$$\varphi_a(\lambda) = A\varphi_{a-1}(\lambda) = A^{a-1}\Phi(\lambda),$$

(3.22)

where $\Phi(\lambda) = \varphi_1(\lambda)$ and operator

$$A = \frac{1}{V''(\lambda)} \frac{\partial}{\partial \lambda} - \frac{1}{2} \frac{V''(\lambda)}{(V''(\lambda))^2} + \lambda$$

(3.23)

now depends on the shape of potential $V(x)$.

### 3.5 Relation between time- and potential-dependencies

Consider the vector space $\mathcal{H}$ of all formal Laurent series in some variable $\lambda$. The set of functions

$$\{\lambda^{-a}, \lambda^{a-1} | a = 1, 2, \ldots\}$$

is one of the many possible basis in this vector space. Let us consider a “half-space” $\mathcal{H}_+$ consisting of all the series in non-negative powers of $\lambda$. Then

$$\{\lambda^a | a = 1, 2, \ldots\}$$

is a possible basis in $\mathcal{H}_+$. Every rotation of the linear subspace $\mathcal{H}_+$ within entire $\mathcal{H}$ can be represented by projection of some basis in rotated subspace onto original one. In other words, any semi-infinite set of functions $\{\phi_a(\lambda)\}$, such that

$$\phi_a(\lambda) = \lambda^{a-1}(1 + O(\lambda^{-1})) = \lambda^{a-1}(1 + \sum_{b > 0} S_{ab} \lambda^{-b})$$

(3.24)

can be considered as describing some particular rotation of $\mathcal{H}_+$ in $\mathcal{H}$. Of course, the same rotation can be represented by different matrices $S_{ab}$, and in fact rotations are in one-to-one correspondence with the factor of the set $\{\phi_a\}$ modulo triangular transformations, which has the natural name of the Universal Grassmannian $\mathcal{GR}$. Eq.(3.20) is obviously invariant under such triangular transformations, thus $Z$, as a function of its argument $V(x)$ can be considered as a function on $\mathcal{GR}$ (if the shape of potential is changed, the set $\{\phi_a\}$ is also changed).

Normalization condition (3.24) is, however, invariant not only under triangular transformations in $\{\phi_a\}$, but also under the changes $\lambda \rightarrow \lambda(1 + O(\lambda^{-1}))$. Such transformations change the point of Grassmannian and they also induce a triangular linear transformation of time-variables: $T_k \rightarrow T_k + \text{lin}(T_{k+1}, T_{k+2}, \ldots)$. In other words, $Z$ depends on the choice of variable $\lambda$ on the “spectral curve” and on the point of $\mathcal{GR}$, i.e. is essentially a function on the tensor product $\mathcal{GR} \times \mathcal{GR}$ of two different Grassmannians. One of them is a space of various models (related to the choice of potential in GKM), another - specifies the basis in the space of observables in a given model (related to the choice of time-variables). As one expects from the physical arguments and as we just saw on a more formal level these two dependencies are in fact interrelated. When consideration is restricted to the set of GKM’s (from that of all the models of string theory) a much more definite statement can be made [36]. Being a priori a function of two distinct types of variables: the times $\{T_k = \frac{1}{k} \text{tr} \Lambda^{-k}\}$ and potential $V(x)$, the GKM partition function in fact depends only on the type of singularities of $V(x)$ (which is a kind of “discrete” information) and on peculiar combination of these variables. If $V(x)$ is a polynomial of degree $p + 1$ (i.e. has only finite order pole at infinity), then

$$Z_V(T_k) \sim \tau_p \left( \frac{1}{k} \text{tr} W(A)^{-k/p} + \frac{p}{k(p-k)} \text{res} \frac{W(x)^{1-k/p}}{dx} \right),$$

(3.25)
where \( W(x) \equiv V'(x) \), and the shape of the function \( \tau_p \) depends only on the value of \( p \). (In order to substitute \( \sim \) by \( \equiv \) in (3.25) one should slightly redefine the quasiclassical factor and thus \( Z \): one should in fact work with the variable \( L^{1/p} = W(\Lambda)^{1/p} \) instead of \( \Lambda \). As often happens, different variables are nice for different purposes.)

If all the arguments \( \hat{T}_k \) with \( k > p \) are put equal to zero, we get a reduced \( \tau \)-function

\[
\tilde{\tau}_p(\hat{T}_1, \ldots, \hat{T}_p) \equiv \tau_p(\hat{T}_k)|_{T_k=0} \text{ for } k>p
\]

It appears to be a solution to "quasiclassical KP-hierarchy", which arises from pure algebraic construction and can be also identified as partition function of topological Landau-Ginsburg model with the superpotential \( W(x) \).

### 3.6 Kac-Schwarz problem

The function \( \tau_p(T_k) \) is of course very far from arbitrary. First of all it possesses peculiar determinant representation of the type (3.20), which in turn implies some restrictive bilinear (Hirota) equations and as result \( \tau_p \) appears to be a KP \( \tau \)-function. Second, \( \tau_p \) is further distinguished even among \( \tau \)-functions by peculiar features of the functions \( \varphi_a(\lambda) \). In particular case of GKM one of the ways to represent these properties is to use the recursive relations (3.22),

\[
\varphi_a = \mathcal{A}^{a-1}\Phi, \quad \text{supplemented by another obvious property } W(\lambda)\Phi(\lambda) = \varphi_{p+1}(\lambda)
\]

(it follows from invariance of the integral \( \Phi(\lambda) \) under the change of integration \( x \)-variable). These two relations together give rise to an equation on \( \varphi_1(\lambda) = \Phi(\lambda): (\mathcal{A}^p - W(\lambda))\Phi(\lambda) = 0 \), which is just a \( p \)-th order differential equation.

In this way one specifies non-perturbative partition function of some string model (in the case of GKM this is in fact a \( (p, 1) \)-minimal model coupled to \( 2d \)-gravity) in terms of invariant points of certain operators acting on the Universal Module space \( \mathcal{GR} \). This reformulation, though far more abstract than the original one, can be very useful for non-trivial generalizations - and be a natural step in the search for generic configuration space of the string theory. It was first introduced by V.Kac and A.Schwarz [37], and is not yet studied as deep as it deserves, even if one deals only with the space of KP \( \tau \)-functions, determinant formulas and ordinary Universal Grassmannian. In this (actually somewhat narrow) context, the general problem is to describe common invariant points in \( \mathcal{GR} \) of two operators, acting on formal Laurent series in \( \lambda \): \( \forall a \)

\[
\mathcal{A}\phi_a \in \text{Span}\{\phi_b\}, \quad \mathcal{K}\phi_a \in \text{Span}\{\phi_b\}
\]

In the case of GKM \( \mathcal{A} \) and \( \mathcal{K} \) are differential operators of the 1-st and 0-th order respectively. Moreover, \( \mathcal{A} \) is a “gap-one” operator (the gap is equal to \( g \) if \( \mathcal{A}\phi_a = \sum_{b=1}^{a+g} \mathcal{A}_{ab}\phi_b \)). See [19] for some more comments on the gap-one case. When both gaps are different from unity, the system (3.26)-(3.27) usually describes a multi-parametric set of invariant points, the simplest example being associated with \( (p, q) \)-minimal models (where \( p \) and \( q \) are in fact the values of gaps for \( \mathcal{K} \) and \( \mathcal{A} \) respectively). In this situation Kontsevich integral describes only duality transformation of \( (p, q) \)-model into the \( (q, p) \)-one [38] (note that these models do not coincide after they are coupled to \( 2d \) gravity).

### 3.7 Ward identities for GKM

Advantage of the Kac-Schwarz reformulation of GKM is that it is very easy to deform, since there are no real restriction imposed on the choice of operators \( \mathcal{A} \) and \( \mathcal{K} \). However, instead, this formulation does not introduce any reach structure and does not immediately provide any valuable information about the
form and properties of solutions to (3.26)-(3.27), i.e. does not explicitly reveal any nice properties which could be common for all the string models. Thus it could serve as a starting, but not the final point of analysis of non-perturbative partition functions. Alternative approach, making use of the Ward identities, provides a better description of GKM, but can appear too restrictive to allow for any interesting deformations. From the point of view of Grassmannian, the Ward identities specify some subset of points in \( \mathcal{GR} \), which are invariant under certain subalgebras of \( U GL(\infty) \) (the symmetry group of entire Grassmannian). The corresponding “homogeneous spaces” are normally not discrete and contain a vast variety of points. By themselves the Ward identities are not enough to specify particular points in \( \mathcal{GR} \) uniquely. They should still be supplemented by some extra conditions, like “reduction constraints” (in fact one should keep one of the Kac-Schwarz constraints, (3.27), and only another one can be usually substituted by the symmetry-like relation, coming from the Ward identities).

### 3.7.1 Gross-Newmann equation

We refer to [1] for a very detailed review of the Ward identities in GKM. They are all corollaries of the simple Gross-Newmann (GN) equation [39], imposed on Kontsevich integral (3.1) as result of its invariance under arbitrary change of the integration variable \( X \):

\[
\int dX e^{-trV(x)+Ntr log X+tr LX} (-V'(X)+NX^{-1}+L) = 0.
\]

or

\[
\left( V' (\partial/\partial L_{tr}) - L - N (\partial/\partial L_{tr})^{-1} \right) F_V = 0. \tag{3.28}
\]

Being just equation of motion for \( \mathcal{F}_V \{ L \} \) eq.(3.28) provides complete information about this function. However, this statement needs to be formulated more carefully. One of the reasons is that (3.28) does not account explicitly for a very important property of \( \mathcal{F}_V \{ L \} \): that it actually depends only on eigenvalues of \( L \). This information should still be taken into account explicitly. If this is kept in mind, it becomes a tedious but straightforward work to substitute \( \mathcal{F}_V = C_V Z_V \) and express \( L \)-derivatives through \( T_k \)-derivatives in order to derive Virasoro and \( W \)-constraints in conventional form [1]. We shall briefly sketch some pieces of this derivation and related problems in the remaining part of this section.

### 3.7.2 \( \tilde{W} \)-operators in Kontsevich models

First of all, the Gross-Newmann equation (3.28) for Kontsevich models can be easily expressed in terms of the so called \( \tilde{W} \)-operators. Namely, we shall prove the following identity [40]:

\[
\frac{\partial}{\partial t} \left\{ \Lambda \right\}^{m+1} Z(T_k) = (\pm)^{m+1} \sum_{l \geq 0} \Lambda^{-t-l} \tilde{W}_{l-m+1}(T) Z(T_k), \tag{3.29}
\]

valid for any function \( Z \) which depends on \( T_k = \mp \frac{1}{k} \text{tr} \Lambda^{-1} \), \( k \geq 1 \) and \( T_0 = \pm \text{tr} \log \Lambda \).

The \( \tilde{W} \)-operators are defined [40] by the following construction. Consider the action of \( \text{Tr} \frac{\partial^m}{\partial L^m} L^n \) on \( e^{\text{tr} U(L)} = e^{\sum_k t_k \text{tr} L^k} \). It gives some linear combination of terms like

\[
\text{tr} L^{a_1} \ldots \text{tr} L^{a_l} e^{\text{tr} U(L)} = \frac{\partial^l}{\partial t_{a_1} \ldots \partial t_{a_l}} e^{-\text{tr} U(L)} \tag{3.30}
\]

i.e. we obtain a combination of differential operators with \( t \)-derivatives, to be denoted \( \tilde{W}(t) \):

\[
\tilde{W}_{m+1}^{(m+1)}(t) e^{\text{tr} U(L)} = \text{Tr} \frac{\partial^m}{\partial L^m} L^n e^{\text{tr} U(L)}, \quad m, n \geq 0. \tag{3.31}
\]
For example,

\[ \tilde{W}_n^{(1)} = \frac{\partial}{\partial n}, \quad n \geq 0; \]

\[ \tilde{W}_n^{(2)} = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_{k} \partial t_{n-k}}, \quad n \geq -1; \]

\[ \tilde{W}_n^{(3)} = \sum_{k,l=1}^{\infty} k t_k U_l \frac{\partial}{\partial t_{k+l+n}} + \sum_{k=1}^{\infty} k t_k \sum_{a+b=k+n} \frac{\partial^2}{\partial t_a \partial t_b} + \sum_{k=1}^{\infty} \sum_{a+b+c=n+1} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c} + \frac{n+1}{2} \frac{(n+2)}{\partial n} \]

\[ \cdots \]

(3.32)

Note, that while \( \tilde{W}_n^{(1)} \) and \( \tilde{W}_n^{(2)} \) are just the ordinary \((U(1))\)-Kac Moody and Virasoro operators respectively, the higher \( \tilde{W}_n^{(m)} \)-operators do not coincide with the generators of the \( \tilde{W} \)-algebras: already

\[ \tilde{W}_n^{(3)} \neq W_n^{(3)} = \sum_{k,l=1}^{\infty} k t_k U_l \frac{\partial}{\partial t_{k+l+n}} + \sum_{k=1}^{\infty} k t_k \sum_{a+b=k+n} \frac{\partial^2}{\partial t_a \partial t_b} + \sum_{a+b+c=n+1} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c} \]

(3.33)

\( \tilde{W} \)-operators (in variance with ordinary \( W \)-operators) satisfy recurrent relation:

\[ \tilde{W}_n^{(m+1)} = \sum_{k=1}^{\infty} k t_k \tilde{W}_{n+k}^{(m)} + \sum_{k=0}^{m+n-1} \frac{\partial^2}{\partial t_k} \cdot \tilde{W}_{n-k}^{(m)}, \quad n \geq -m. \]

(3.34)

Actually not too much is already known about the \( \tilde{W} \) operators and the structure of \( \tilde{W} \)-algebras (in particular it remains unclear whether the negative harmonics \( \tilde{W}_n^{(m+1)} \) with \( n < -m \) can be introduced in any reasonable way), see [40] for some preliminary results.

Now we can come back to the identity (3.29). Its most straightforward application is to the Gaussian Kontsevich model with potential \( V(x) = \frac{x^2}{2} \), see the next subsection. In other cases calculations with the use of identity (3.29), accounting for the quasiclassical factor \( C_V(L) \) and the difference between \( L = V'(\Lambda) \) and \( \Lambda \) become somewhat more involved, though still seem sufficiently straightforward. Also for particular potentials \( V(X) \) partition function \( Z_V(T) \) is actually independent of certain (combinations of) time-variables (for example, if \( V(X) = \frac{x^{p+1}}{p+1} \), it is independent of all the \( T_{pk}, \ k \in \mathbb{Z}_+ \), and this is important for appearance of the constraints in the standard form, i.e. for certain reduction of \( \tilde{W} \)-constraints to the ordinary \( W \)-constraints. This relation between \( \tilde{W} \) and \( W \)-operators deserves further investigation.

The proof of eq.(3.29) is provided by the following trick. Let us make a sort of Fourier transformation:

\[ Z\{T\} = \int dH \ G(H) e^{\sum_{k=0}^{\infty} T_k \text{Tr} H^k}, \]

(3.35)

where integral is over \( N \times N \) Hermitian matrix \( H \).\(^{12}\) Then it is clear that once the identity (3.29) is established for \( Z\{T\} \) substituted by \( e^{\text{Tr} U(H)} \), \( U(H) = \sum_{k=0}^{\infty} T_k \text{Tr} H^k, \) with any matrix \( H \), it is valid for any function \( Z\{T\} \). The advantage of such substitution is that we can now make use of the definition (3.31) of the \( \tilde{W} \) operators in order to rewrite (3.29) in a very explicit form:

\[ \left( \frac{\partial}{\partial \text{Tr} r} \right)^{m+1} e^{\text{Tr} U(H)} = (+)^{m+1} \sum_{l=0}^{\infty} \Lambda^{l-1} \tilde{W}_{-l}^{(m+1)} (T)e^{\text{Tr} U(H)} = \]

\[ = (+)^{m+1} \sum_{l=0}^{\infty} \Lambda^{l-1} \text{Tr} \left( \frac{\partial}{\partial H_{tt}} \right)^m H^l e^{\text{Tr} U(H)} = \]

\[ = (+)^{m+1} \text{Tr} \left( \frac{\partial}{\partial H_{tt}} \right)^m \frac{1}{\Lambda \otimes I - I \otimes H} e^{\text{Tr} U(H)}. \]

\(^{12}\) Here it is for the first time that we encounter an important idea: matrix models - the ordinary 1-matrix model (2.1) in this case - can be considered as defining integral transformations. This view on matrix models can to large extent define their role in the future development of string theory.
Now expression for $T$'s in terms of $\Lambda$ should be used. Then

$$e^{\text{Tr}(U)} = \text{Det}^{\pm 1}(\Lambda \otimes I - I \otimes H)$$

and substituting this into (3.36) we see that (3.29) is equivalent to

$$\left( \left( \frac{\partial}{\partial \Lambda_{tr}} \right)^{m+1} - (\pm)^{m+1} I \cdot \text{Tr} \left( \frac{\partial}{\partial H_{tr}} \right)^m \frac{1}{\Lambda \otimes I - I \otimes H} \right) \cdot \text{Det}^{\pm 1}(\Lambda \otimes I - I \otimes H) = 0.$$

This is already a matrix identity, valid for any $\Lambda$ and $H$ of the sizes $n \times n$ and $N \times N$ respectively. For example, if $m = 0$ ($\tilde{W}^{(1)}$-case), it is obviously satisfied.

If both $n = N = 1$, it is also trivially true, though for different reasons for different choice of signs: for the upper signs, the ratio at the l.h.s. is just unity and all derivatives vanish; for the lower signs we have:

$$\left( \frac{\partial}{\partial \lambda} \right)^m - \left( \frac{\partial}{\partial h} \right)^m = \left( \sum_{a, b \geq 0} \left( \frac{\partial}{\partial \lambda} \right)^a \left( - \frac{\partial}{\partial h} \right)^b \left( \frac{\partial}{\partial \lambda} \right) + \frac{\partial}{\partial \lambda} \right),$$

and this obviously vanishes since $(\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial h})f(\lambda - h) \equiv 0$ for any $f(x)$.

If $m > 0$ and $\Lambda$, $H$ are indeed matrices, direct evaluation becomes much more sophisticated. We present the first two nontrivial examples: $m = 1$ and $m = 2$. The following relations will be useful. Let $Q = \frac{1}{N^2}$. Then

$$\text{Det}^{\pm 1}Q \frac{\partial}{\partial \Lambda_{tr}} \text{Det}^{\mp 1}Q = \pm \{[I \otimes \text{Tr}]Q\};$$

$$\text{Det}^{\pm 1}Q \frac{\partial}{\partial H_{tr}} \text{Det}^{\mp 1}Q = \mp \{[I \otimes \text{Tr}]Q\}.$$  \hspace{1cm} (3.41)

This is already enough for the proof in the case of $m = 1$. Indeed:

$$\text{Det}^{\pm 1}Q \left( \text{Det}^{\mp 1}Q \frac{\partial}{\partial \Lambda_{tr}} \otimes I \pm \frac{\partial}{\partial H_{tr}} \otimes I \right) Q \text{Det}^{\mp 1}Q =$$

$$= \{- [I \otimes \text{Tr}]Q \pm ([I \otimes \text{Tr}]Q)\} \mp$$

$$\mp \{[I \otimes \text{Tr}]Q\} \pm \{[I \otimes \text{Tr}]Q\} = 0.$$  \hspace{1cm} (3.42)

The first two terms at the r.h.s. come from $\Lambda$-, while the last two – from $H$-derivatives.

In the case of $m = 2$ one should take derivatives once again. This is a little more tricky, and the same compact notation are not sufficient. In addition to (3.41) we now need:

$$\left( \frac{\partial}{\partial \Lambda_{tr}} \otimes I \right) [I \otimes \text{Tr}]Q = - \{[I \otimes \text{Tr}]Q\}^2 Q - B.$$  \hspace{1cm} (3.43)

Here

$$[I \otimes \text{Tr}]Q = \{I \otimes \text{Tr}[I \otimes \text{Tr}]Q\},$$

while in order to write $B$ explicitly we need to restore matrix indices (Greek for the $\Lambda$-sector and Latin - for the $H$ one). The $(\alpha i, \gamma k)$-component of (3.43) looks like:

$$\left( \frac{\partial}{\partial \Lambda_{\beta \alpha}} \delta^{im} \right) Q_{\delta \gamma}^{m j} Q_{\beta \gamma}^{j k} = -Q_{\delta \beta}^{i j} Q_{\beta \gamma}^{j k} Q_{\alpha \gamma}^{k} - Q_{\delta \beta}^{i j} Q_{\alpha \delta}^{i j} Q_{\beta \gamma}^{j k}.$$  \hspace{1cm} (3.45)
and appearance of the second term at the r.h.s. implies, that $B^{ik}_{\alpha\gamma} = Q^i_{\alpha\delta} Q^j_{\delta\beta} Q^k_{\beta\gamma}$. Further,

$$\left( \frac{\partial}{\partial H_{tr}} \right) [(I \otimes I) Q] Q =$$

$$-[(I \otimes I) [(tr \otimes I) Q] Q] Q - [(I \otimes I) [(I \otimes I) Q] Q] Q;$$

$$\left( \frac{\partial}{\partial H_{tr}} \right) [(tr \otimes I) Q] Q =$$

$$+(tr \otimes I) [(I \otimes I) Q] Q + (I \otimes I) [(I \otimes I) Q] Q + B.$$

It is important that $B$ that appears in the last relation in the form of $B^{ik}_{\alpha\gamma} = Q^i_{\alpha\delta} Q^j_{\delta\beta} Q^k_{\beta\gamma}$ is exactly the same $B$ as in eq.(3.43).

Now we can prove (3.39) for $m = 2$:

$$Det^{\pm 1} Q \left( \left( \frac{\partial}{\partial L_{tr}} \right)^2 \otimes I - I \otimes \left( \frac{\partial}{\partial H_{tr}} \right)^2 \right) Q Det^{\pm 1} Q =$$

$$\pm [(I \otimes I) Q] Q - [(tr \otimes I) Q] Q - [(I \otimes I) Q] Q -$$

$$- [(tr \otimes I) [(tr \otimes I) Q] Q] Q - B];$$

$$\pm [(tr \otimes I) [(I \otimes I) Q] Q + (I \otimes I) [(I \otimes I) Q] Q +$$

$$+[(tr \otimes I) [(I \otimes I) Q] Q] Q + B]$$

$$\pm [(tr \otimes I) [(I \otimes I) Q] Q] Q + [(I \otimes I) [(I \otimes I) Q] Q] Q$$

where the terms 1,2,3,4,5,6 in the first braces cancel the terms 1,3,2,4,6,5 in the second braces and identity (3.44) and its counterpart with $(tr \otimes I) \rightarrow (I \otimes Tr)$ is used.

Explicit proof of eq.(3.39) for generic $m$ is unknown.

### 3.7.3 Discrete Virasoro constraints for the Gaussian Kontsevich model

As a simplest illustration we derive now the constraints for the Gaussian Kontsevich model [41] with potential $V(X) = \frac{1}{2} X^2$:

$$Z_{\Sigma^2} \{N, T\} = \frac{e^{-\frac{1}{2} L^2}}{(det L)^N} \int dX (det X)^N e^{-\sqrt{\eta} L^2 + LX}.$$  \hspace{1cm} (3.48)

In this case $L = V'(\Lambda) = \Lambda$, and the time-variables are just

$$T_k = \frac{1}{k} \Lambda^{-k} = \frac{1}{k} L^{-k}.$$  \hspace{1cm} (3.49)

The model is non-trivial because of the presence of ”zero-time” variable $N$. The Gross-Neumann equation (3.28) looks like

$$\frac{e^{-\frac{1}{2} L^2}}{(det L)^N} \left( \frac{\partial}{\partial L_{tr}} \right)^{n-1} \cdot \left( \frac{\partial}{\partial L_{tr}} - N \left( \frac{\partial}{\partial L_{tr}} + \Lambda \right) \right);$$

$$\cdot (det L)^N e^{+\frac{1}{2} L^2} Z_{\Sigma^2} \{N, T\} = 0.$$  \hspace{1cm} (3.50)

In order to get rid of the integral operator $(\frac{\partial}{\partial L_{tr}})^{-1}$ one should take here $n \geq 0$ rather than $n \geq -1$. In fact all the equations with $n > 0$ follow from the one with $n = 0$, and we restrict our consideration to the last one. For $n = 0$ we obtain from (3.50):

$$\left( \frac{\partial}{\partial L_{tr}} + N \left( \frac{\partial}{\partial L_{tr}} + \frac{N}{L} + L \right) \right) Z = 0$$

or

$$\left( \frac{\partial}{\partial L_{tr}} \right)^2 + \left( L + \frac{2N}{L} \right) \frac{\partial}{\partial L_{tr}} + \frac{N^2}{L^2} - \frac{N}{L} \frac{L}{L} Z = 0,$$  \hspace{1cm} (3.51)
One can now use eq.(3.29) to obtain:

\[
\sum_{m=-1}^{\infty} \frac{1}{L^{m+2}} \left( \sum_{k=1}^{\infty} \left( \text{tr} \frac{1}{L^{k}} \right) \frac{\partial}{\partial T_{k+m}} + \sum_{k=1}^{m-1} \frac{\partial^{2}}{\partial T_{k} \partial T_{m-k}} \right) - \frac{\partial}{\partial T_{m+2}} - 2N \frac{\partial}{\partial T_{m}} - N^{2} \delta_{m,0} - N \left( \text{tr} \frac{1}{L} \right) \delta_{m,1} Z = 0.
\]

(3.53)

Here \( L_{m}(t) = \bar{W}_{m}^{(2)}(t) \) are just the generators (2.9) of "discrete" Virasoro algebra (2.8):

\[
e^{NT_{0}} L_{m}(t) e^{-NT_{0}} = e^{NT_{0}} \left( \sum_{k=1}^{\infty} \frac{kt_{k}}{L} \frac{\partial}{\partial t_{k+m}} + \sum_{k=0}^{m-1} \frac{\partial^{2}}{\partial t_{k} \partial t_{m-k}} \right) e^{-NT_{0}}.
\]

(3.54)

and at the r.h.s. of (3.53) \( r_{k} = -\frac{1}{2} \delta_{k,2}. \)

Thus we found that the Ward identities for the Gaussian Kontsevich model (3.48) coincide with those for the ordinary 1-matrix model (2.1), moreover the size of the matrix \( N \) in the latter model is associated with the "zero-time" in the former one. This result [41] of course implies, that the two models are identical:

\[
e^{-NT_{0}} Z_{\mathfrak{g},2} \{ N, T_{1}, T_{2}, \ldots \} \sim Z_{N} \{ T_{0}, T_{1}, T_{2}, \ldots \}.
\]

(3.55)

See [42] and [1] for explicit proof of this identity.

\footnote{This small correction is manifestation of a very general phenomenon which was already mentioned in s.3.5 above: from the point of view of symmetries (Ward identities) it is more natural to consider \( Z_{V} \) not as a function of \( T \)-variables, but of some more complicated combination \( \bar{T}_{k} + r_{k} \), depending on the shape of potential \( V \). If \( V \) is a polinomial of degree \( p+1 \), \( \bar{T}_{k} = \frac{1}{k} \text{tr}(V'(\lambda))^{-k/p} \), while \( r_{k} = \frac{p}{(p-k)} \text{Res}(V'(\mu))^{1-\frac{k}{p}} d\mu \). For monomial potentials these expressions become very simple: \( \bar{T}_{k} = T_{k} \) and \( r_{k} = -\frac{p}{p+1} \delta_{k,p+1} \). See [36] for more details. In most places in these notes we prefer to use invariant potential-independent times \( T_{k} \), instead of \( \bar{T}_{k} \), but then Ward identities acquire some extra terms with \( r_{k} \).}

3.7.4 Continuous Virasoro constraints for the \( V = \frac{\lambda^{3}}{3} \) Kontsevich model

This example is a little more complicated than that in the previous subsection, and we do not present calculations in full details (see [43] and [4]). Our goal is to demonstrate that the constraints which arise in this model, though still form (Borel subalgebra of) some Virasoro algebra, are different from (2.8).

From the point of view of the CFT-formulation the relevant model is that of the twisted (in this particular case - antiperiodic) free fields. These so called "continuous Virasoro constraints" give the simplest illustration of the difference between discrete and continuous matrix models: this is essentially the difference between "homogeneous" (Kac-Frenkel) and "principal" (soliton vertex operator) representation of the level \( k = 1 \) Kac-Moody algebra. From the point of view of integrable hierarchies this is the difference between Toda-chain-like and KP-like hierarchies.

Another (historically first) aspect of the same relation also deserves mentioning, since it also illustrates the interrelation between different models. The discrete 1-matrix model arises naturally in description of quantum 2d gravity as sum over 2-geometries in the formalism of random equilateral triangulations. The model, however, describes only lattice approximation to 2d gravity and (double-scaling) continuum limit should be taken in order to obtain the real (continuous) theory of 2d gravity. This limit was originally formulated [44] in terms of the constraint algebra (equations of motion or "loop" or "Schwinger-Dyson" equations - terminology is taste-dependent), leaving open the problem of what is the form of partition function \( Z_{\text{cont}} \) of continuous theory. Since the relevant algebra appeared to be just the set of Ward identities for Kontsevich model (with \( V(X) = \frac{\lambda^{3}}{3} \)), this proves that the latter one is exactly the
continuous theory of pure 2d gravity. At the same time, Kontsevich model itself can be naturally introduced as a theory of topological gravity (in fact this is how the model was originally discovered in [3]). From this point of view the constraint algebra, to be discussed below in this subsection, plays the central role in the proof of equivalence between pure 2d quantum gravity and pure topological gravity (in both cases “pure” means that “matter” fields are not included).

After these introductory remarks we proceed to calculations. Actually they just repeat those from the previous subsection for the Gaussian model, but formulas get somewhat more complicated. This time we do not include zero-time \( N \) and use \( V(X) = \frac{X^3}{3} \). Also, this time it is much more tricky (though possible) to work in matrix notations (because fractional powers of \( L \) will be involved) and we rewrite everything in terms of the eigenvalues of \( L \).

Substitute

\[
C_{X} = \frac{\prod_l e^{\frac{\lambda}{\beta_l}}}{\sqrt{\prod_{a,b} (\lambda_a + \lambda_b)}}
\]

and introduce a special notation for

\[
\left( \frac{\partial^2}{\partial T^2} \right)_{aa} = \frac{\partial^2}{\partial \lambda_a^2} + \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} \left( \frac{\partial}{\partial \lambda_a} - \frac{\partial}{\partial \lambda_b} \right)
\]

(3.56)

and

\[
\sum_{l \neq a} \frac{1}{\lambda_a - \lambda_l} \left( \frac{1}{\sqrt{\lambda_a}} - \frac{1}{\sqrt{\lambda_b}} \right).
\]

(3.57)

Then (3.28) turns into

\[
\left( \left( \frac{D}{D \lambda_a} \right)^2 + \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} \left( \frac{D}{D \lambda_a} - \frac{D}{D \lambda_b} \right) \right) Z_{X^3} \{ T \} = 0.
\]

(3.58)

Now we need explicit expression for \( T \):

\[
T_k = \frac{1}{k} L^{-k},
\]

(3.59)

and - as we already know from the previous subsection - we also need

\[
r_k = -\frac{2}{3} \delta_{k,3}.
\]

(3.60)

\( Z_{X^3} \{ T \} \) is in fact independent of all the time-variables with even numbers (subscripts), see [4], [1] for the explanation. Therefore we can take only \( k = 2l + 1 \) in (3.59),

\[
T_{2l+1} = \frac{1}{2l+1} \sum_b \lambda_b^{l-\frac{3}{2}},
\]

(3.61)

and

\[
\frac{\partial}{\partial \lambda_a} Z_{X^3} \{ T \} = \sum_{l=0}^{\infty} \frac{\partial T_{2l+1}}{\partial \lambda_a} \frac{\partial Z}{\partial T_{2l+1}} = -\frac{1}{2} \sum_{a=0}^{\infty} \lambda_a^{-l-\frac{3}{2}} \frac{\partial Z}{\partial T_{2l+1}} + \frac{1}{2} \sum_{l=0}^{\infty} \left( l + \frac{3}{2} \right) \lambda_a^{-l-\frac{3}{2}} \frac{\partial Z}{\partial T_{2l+1}}.
\]

(3.62)

These expressions should be now substituted into (3.58) and we obtain:

\[
-\frac{1}{4} \sum_{l,m=0}^{\infty} \lambda_a^{-l-m-3} \frac{\partial Z}{\partial T_{2l+1}} \frac{\partial Z}{\partial T_{2m+1}} + \sum_{l=0}^{\infty} \left[ \lambda_a^{-l-\frac{3}{2}} - \frac{1}{2} \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} \left( \lambda_a^{-l-\frac{3}{2}} - \lambda_b^{-l-\frac{3}{2}} \right) \right.
\]

\[
- \sum_{a=0}^{\infty} \left( \lambda_a^{-l-\frac{3}{2}} - \frac{1}{4} \lambda_a^{-l} - \frac{1}{2} \sum_{b \neq a} \frac{1}{\lambda_a (\sqrt{\lambda_a} + \sqrt{\lambda_b})} \right) \lambda_a^{-l-\frac{3}{2}} \frac{\partial Z}{\partial T_{2l+1}} + [\ldots] Z = \sum_{n=-1}^{\infty} \frac{1}{\lambda_a^{n+2}} C_n Z
\]

(3.63)
with

\[
\mathcal{L}_{2n} = \sum_{l=0}^{\infty} \left( l + \frac{1}{2} \right) (T_{2l+1} + r_{2l+1}) \frac{\partial}{\partial T_{2l+1}} + \left( l + 1 \right) \frac{\partial^2}{\partial T_{2l+1} \partial T_{2l+1}} + \frac{1}{4} \sum_{l,m \geq 0} \frac{\partial^2}{\partial T_{2l+1} \partial T_{2m+1}} + \frac{1}{16} \delta_{n,0} + \frac{1}{4} T_1^2 \delta_{n,-1} =
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} k(T_k + r_k) \frac{\partial}{\partial T_{k+2n}} + \frac{1}{4} \sum_{k=1}^{2n-1} \frac{\partial^2}{\partial T_k \partial T_{2n-k}} + \frac{1}{16} \delta_{n,0} + \frac{1}{4} T_1^2 \delta_{n,-1}.
\]

(3.64)

Factor \( \frac{1}{2} \) in front of the first term at the r.h.s. in (3.64) is important for \( \mathcal{L}_{2n} \) to satisfy the properly normalized Virasoro algebra:\footnote{Therefore it could be reasonable to use a different notation: \( \mathcal{L}_n \) instead of \( \mathcal{L}_{2n} \). We prefer \( \mathcal{L}_{2n} \), because it emphasises the property of the model to be 2-reduction of KP hierarchy (to KdV).}

\[
[\mathcal{L}_{2n}, \mathcal{L}_{2m}] = (n - m) \mathcal{L}_{2n+2m}.
\]

(3.65)

Coefficient \( \frac{1}{4} \) in front of the second term can be eliminated by rescaling of time-variables: \( T \rightarrow \frac{1}{2} T \), then the last term turns into \( \frac{1}{16} T_1^2 \delta_{n,-1} \).

We shall not actually discuss evaluation of the coefficient in front of \( Z \) (with no derivatives), which is denoted by \([\ldots]\) in (3.63) (see [43] and [4]). In fact, almost all the terms in original complicated expression cancel, giving finally

\[
[\ldots] = \frac{1}{16} \lambda_a^2 + \frac{T_1^2}{4 \lambda_a},
\]

(3.66)

and this is represented by the terms with \( \delta_{n,0} \) and \( \delta_{n,-1} \) in expressions (3.64) for the Virasoro generators \( \mathcal{L}_{2n} \).

The term with the double \( T \)-derivative in (3.63) is already of the necessary form. Of intermediate complexity is evaluation of the coefficient in front of \( \frac{\partial^2}{\partial T_{2l+1} \partial T_{2l+1}} \) in (3.63), which we shall briefly describe now. First of all, rewrite this coefficient, reordering the items:

\[
\frac{1}{2} \left[ \left( l + \frac{3}{2} \right) \lambda_a^{-l-\frac{3}{2}} - \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} \left( \lambda_a^{-l-\frac{3}{2}} - \lambda_b^{-l-\frac{3}{2}} \right) \right] + \left( \frac{1}{4} - \frac{1}{4} \lambda_a^{-l-\frac{3}{2}} \sum_{b \neq a} \frac{\lambda_b^{-l-2}}{\sqrt[14]{\lambda_a} + \sqrt[14]{\lambda_b}} \right) - \lambda_a^{-l-1}.
\]

(3.67)

The first two terms together are equal to the sum over all \( b \) (including \( b = a \)):

\[
- \frac{1}{2} \sum_{b} \frac{1}{\lambda_a - \lambda_b} \left( \lambda_a^{-l-\frac{3}{2}} - \lambda_b^{-l-\frac{3}{2}} \right) = \frac{1}{2} \sum_{b} \lambda_a^{l+\frac{3}{2}} - \lambda_b^{l+\frac{3}{2}} \lambda_a^{l+\frac{3}{2}} = \frac{1}{2} \lambda_a^{l+\frac{3}{2}} \sum_{b} \frac{\lambda_a^{l+\frac{3}{2}}}{\lambda_a - \lambda_b} . \frac{1}{\lambda_b^{l+\frac{3}{2}}}. \]

(3.68)

Similarly, the next two terms can be rewritten as

\[
\frac{1}{2} \sum_{b} \frac{\lambda_b^{-l-2}}{\sqrt[14]{\lambda_a} + \sqrt[14]{\lambda_b}} = \frac{1}{2} \lambda_a^{l+\frac{3}{2}} \sum_{b} \frac{\sqrt[14]{\lambda_a} - \sqrt[14]{\lambda_b}}{\lambda_a - \lambda_b} = \frac{1}{2} \lambda_a^{l+\frac{3}{2}} \sum_{b} \frac{\lambda_a^{l+\frac{3}{2}} - \lambda_b^{l+\frac{3}{2}}}{\lambda_a - \lambda_b} \lambda_b^{l+\frac{3}{2}}.
\]

(3.69)

The sum of these two expressions is equal to

\[
\frac{1}{2} \lambda_a^{l+\frac{3}{2}} \sum_{b} \frac{\lambda_a^{l+\frac{3}{2}} - \lambda_b^{l+\frac{3}{2}}}{\lambda_a - \lambda_b} \lambda_b^{l+\frac{3}{2}}.
\]

(3.70)

Note that powers \( l + 2 \) are already integer and the remaining ratio can be represented as a sum of \( l + 2 \) terms. Adding also the last term from the l.h.s. of (3.67), we finally obtain:

\[
- \frac{1}{\lambda_a^{l+\frac{3}{2}}} + \frac{1}{2} \sum_{n=-1}^{a} \frac{1}{\lambda_a^n} \sum_{b} \frac{1}{\lambda_b^{l-n+\frac{3}{2}}} = \frac{1}{2} \sum_{n=-1}^{a} \frac{1}{\lambda_a^n} \lambda_a^{n+1}(2a - 2n + 1)(T + r)2l-2n+1
\]

(3.71)
in accordance with (3.63) and (3.64).

Calculations can be repeated for every particular monomial potential \( V(x) = \frac{x^{p+1}}{p+1} \), but they become far more tedious and no general derivation of \( W(p) \)-constraints [44] is yet found on these lines. See [45] for detailed examination of the \( W(3) \)-constraints in the \( \mathbb{A}^4 \)-Kontsevich model.

4 KP/Toda \( \tau \)-function in terms of free fermions

There are several different definitions of \( \tau \)-functions, but all of them are particular realizations of the following idea: \( \tau \)-function is a generating functional of all the matrix elements of some group element in particular representation. Since methods of geometrical quantization allow to express all the group theoretical objects in terms of quantum theory of free fields, generic \( \tau \)-functions can be also considered as non-perturbative partition functions of such models. The basic property of \( \tau \)-function, which can be practically derived in such a general context, is that it always satisfy certain bilinear equations, of which Hirota equation for conventional KP \( \tau \)-function is the simplest example.

KP/Toda \( \tau \)-functions are associated with the free particles of a peculiar type: free fermions in \( 1 + 1 \) dimensions [46]. Existence of fermionization is a very rare property of free field theory (in variance with bosonization which is always available). If existing it leads to dramatic simplification of the formalism and to especially simple determinant formulas (instead of sophisticated and often somewhat abstract objects like chiral determinants \( \text{det} \partial \) in generic case). In the case of Kac-Moody algebras the corresponding \( \tau \)-function is nothing but non-perturbative partition function of the corresponding Wess-Zumino-Novikov-Witten model. Among simply-laced algebras only \( GL(N)_{k=1} \) is straightforwardly fermionized, and the formalism is much simpler in this case than for generic Wess-Zumino-Witten model with arbitrary level \( k \). For \( N = 1 \) we obtain KP/Toda \( \tau \)-functions, while \( N \neq 1 \) are related to the “\( N \)-component KP/Toda systems”. Level-one Kac-Moody algebras \( SL(N)_{k=1} \) are distinguished because their universal enveloping are essentially the same as those of their Cartan subalgebras. This allows to define generation functions with the help of sets of mutually commuting generators and makes evolution, described by commuting Hamiltonian flows, complete (acting transitively on the orbits of the group). This is why such systems are distinguished from the point of view of Hamiltonian integrability – and why they are the usual personages in the theory of integrable hierarchies. In general case \( (k \neq 1) \) one naturally deals with the set of flows that form closed but non-Abelian algebra. In the language of matrix models restriction to \( k = 1 \) and free fermions is essentially equivalent to restriction to eigenvalue models. Serious consideration of non-eigenvalue models, aimed at revealing their integrable (solvable) structure will certainly involve the theory of generic \( \tau \)-functions.

4.1 Explicit definition

Let us introduce two fields (a spin-1/2 \( b,c \)-system) \( \psi(z) \) and \( \bar{\psi}(\bar{z}) \) satisfying canonical commutation relation:

\[
[
\hat{\psi}(\bar{z}), \psi(z)\]_+ = \delta(\bar{z} - z)d\bar{z}^{1/2}dz^{1/2}.
\]

(4.1)

Then

\[
\tau\{A\} \sim \langle 0 | \exp\left(\oint_{dz} \oint_{d\bar{z}} A(z, \bar{z})\psi(z)\bar{\psi}(\bar{z})\right) | 0 \rangle.
\]

(4.2)
Now it is usual to expand in Laurent series around $z = 0$:

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^n dz^{1/2}; \quad \tilde{\psi}(z) = \sum_{n \in \mathbb{Z}} \tilde{\psi}_n z^{-n-1} dz^{1/2};$$

$[\tilde{\psi}_m, \psi_n]_+ = \delta_{m,n};$

$$\psi_m | 0 \rangle = 0 \text{ for } m < 0; \quad \tilde{\psi}_m | 0 \rangle = 0 \text{ for } m \geq 0;$$

$$A(z, \tilde{z}) = \sum_{m,n \in \mathbb{Z}} z^{-m-1} z^n A_{mn} dz^{1/2} d\tilde{z}^{1/2};$$

so that

$$\oint dz \oint d\tilde{z} A(z, \tilde{z}) \psi(z) \tilde{\psi}(\tilde{z}) = \sum_{m,n \in \mathbb{Z}} A_{mn} \psi_m \tilde{\psi}_n.$$  \hspace{1cm} (4.4)

In fact this expansion could be around any pair of points $z_0, z_\infty$ and on a 2-surface of any topology: topological effects can be easily included as specific shifts of the functional $A(z, \tilde{z})$ - by combinations of the "hadle-gluing operators". Analogous shifts can imitate the change of basic functions $z^n$ for $z^{n+\alpha}$ and more complicated expressions (holomorphic 1/2-differentials with various boundary conditions on surfaces of various topologies).

One can now wonder, whether local functionals $A(z, \tilde{z}) = U(z) \delta(\tilde{z} - z) dz^{1/2} d\tilde{z}^{1/2}$ play any special role. The corresponding contribution to the Hamiltonian looks like

$$H_{Cartan} = \oint dz U(z) \psi(z) \tilde{\psi}(z) = \oint dz U(z) J(z),$$  \hspace{1cm} (4.5)

where

$$J(z) = \psi(z) \tilde{\psi}(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1} dz$$  \hspace{1cm} (4.6)

is the $U(1)_{k=1}$ Kac-Moody current;

$$J_n = \sum_{m \in \mathbb{Z}} \psi_m \tilde{\psi}_{m+n}; \quad [J_m, J_n] = m \delta_{m+n,0}.$$  \hspace{1cm} (4.7)

If scalar function (potential) $U(z)$ is expanded as $U(z) = \sum_{k \in \mathbb{Z}} t_k z^k$, then

$$H_{Cartan} = \sum_{k \in \mathbb{Z}} t_k J_k.$$  \hspace{1cm} (4.8)

This contribution to the whole Hamiltonian can be considered distinguished for the following reason. Let us return to original expression (4.2) and try to consider it as a generating functional for all the correlation functions of $\tilde{\psi}$ and $\psi$. Naively, variation w.r.t $A(z, \tilde{z})$ should produce bilinear combination $\psi(z) \tilde{\psi}(\tilde{z})$ and this would solve the problem. However, things are not just so trivial, because operators involved do not commute (and in particular, the exponential operator in (4.2) should still be defined less symbolically, see next subsection). Things would be much simpler, if we can consider commuting set of operators: this is where abelian $U(1)_{k=1}$ subgroup of the entire $GL(\infty)_{k=1}$ (and even its purely commuting Borel subalgebra) enters the game. Remarkably, it is sufficient to deal with this abelian subgroup in order to reproduce all the correlation functions. The crucial point is the identity for free fermions (generalizable to any $b, c$-systems):

$$: \psi(\lambda) \tilde{\psi}(\tilde{\lambda}) : = : \exp \left( \int \lambda \ J \right) :$$  \hspace{1cm} (4.9)

\footnote{We once again emphasize that this trick is specific for the free fermions and for the level $k = 1$ Kac-Moody algebras, which can be expressed entirely in terms of free fields, associated with Cartan generators (modulo some unpleasant details, related to "cocycle factors" in the Frenkel-Kac representations [12], which are in fact reminiscents of free fields associated with the non-Cartan generators (parafermions) [47]; but can, however, be put under the carpet or/and taken into account "by hands" as "unpleasant but non-essential(?)" sophistications).}
which is widely known in the form of bosonization formulas: \(^{16}\) if \(J(z) = \partial \phi(z)\),

\[
\tilde{\psi}(\tilde{\lambda}) \sim e^{\phi(\tilde{\lambda})} : \psi(\infty) \tilde{\psi}(\tilde{\lambda}) : = : e^{(\phi(\tilde{\lambda}) - \phi(\infty))} : ;
\]

\[
\psi(\lambda) \sim e^{-\phi(\lambda)} : \psi(\lambda) \tilde{\psi}(\infty) : = : e^{(\phi(\lambda) - \phi(\infty))} : .
\]

This identity implies that one can generate any bilinear combinations of \(\psi\)-operators by variation of potential \(U(z)\) only, moreover this variation should be of specific form:

\[
\Delta \int UJ = \Delta \left( \sum_{k \in \mathbb{Z}} t_k J_k \right) = \int_z^\infty J = \sum_{k \in \mathbb{Z}} \int_z^\infty z^{-k-1} dz =
\]

\[
= \sum_{k \in \mathbb{Z}} \frac{1}{k} J_k \left( \frac{1}{z_k} - \frac{1}{\tilde{z}_k} \right),
\]

i.e.

\[
\Delta t_k = \frac{1}{k} \left( \frac{1}{z_k} - \frac{1}{\tilde{z}_k} \right) .
\]

Note that this is not an infinitesimal variation and that it has exactly the form, consistent with Miwa parametrization.

Since any bilinear combination can be generated in this way from \(U(z)\), it is clear that the entire Hamiltonian \(\sum A_{mn} \tilde{\psi}_m \psi_n\) can be also considered as resulting from some transformation of \(V\) (i.e. of "time-variables" \(t_k\)). In other words,

\[
\tau\{A\} = \mathcal{O}_A[t] \tau\{A = U\}. \quad (4.14)
\]

These operators \(\mathcal{O}_A\) are naturally interpreted as elements of the group \(GL(\infty)\), acting on the Universal Grassmannian \(GR\) \([48]-[50]\), parametrized by the matrices \(A_{mn}\) modulo changes of coordinates \(z \rightarrow f(z)\). This representation for \(\tau\{A\}\) is, however, not very convenient, and usually one considers infinitesimal version of the transformation, which just shifts \(A\)

\[
\tau\{t \mid A + \delta A\} = \hat{\mathcal{O}}_{\delta A}[t] \tau\{t \mid A\}, \quad (4.15)
\]

note that this transformation clearly distinguishes between the dependencies of \(\tau\) on \(t\) and on all other components of \(A\). The possibility of such representation with the privileged role of Cartan generators is the origin of all simplifications, arising in the case of free-fermion \(\tau\)-functions. Relation (4.15) is the basis of the orbit interpretation of \(\tau\)-functions \([49]\).

### 4.2 Basic determinant formula for the free-fermion correlator

Let us consider the following matrix element:

\[
\tau_N\{t, \bar{t} \mid G\} = \langle N \mid e^H G e^\bar{H} \mid N \rangle \quad (4.16)
\]
where

\[
\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^n dz^{1/2}; \quad \tilde{\psi}(z) = \sum_{n \in \mathbb{Z}} \tilde{\psi}_n z^{-n-1} dz^{1/2};
\]

\[
G = \exp \left( \sum_{m,n \in \mathbb{Z}} A_{mn} \psi_m \tilde{\psi}_n \right);
\]

\[
H = \sum_{k>0} i k J_k, \quad \tilde{H} = \sum_{k>0} i k J_{-k}
\]

\[
J(z) = \psi(z) \tilde{\psi}(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1} dz; \quad J_n = \sum_k \psi_k \tilde{\psi}_{k+n};
\]

\[
[J_m, J_n] = i \delta_{m+n,0}; \quad [J_m, J_n] = m \delta_{m+n,0};
\]

\[
\psi_m | N \rangle = 0, \quad m < N; \quad \langle N | \psi_m = 0, \quad m \geq N;
\]

\[
\tilde{\psi}_m | N \rangle = 0, \quad m \geq N; \quad \langle N | \tilde{\psi}_m = 0, \quad m < N;
\]

\[
J_m | N \rangle = 0, \quad m > 0; \quad \langle N | J_m = 0, \quad m < 0.
\]

The "$N$-th vacuum" $| N \rangle$ is defined as the Dirac sea, filled up to the level $N$:

\[
| N \rangle = \prod_{i=N}^\infty \tilde{\psi}_i | \infty \rangle = \prod_{i=-\infty}^{N-1} \psi_i | -\infty \rangle;
\]

\[
\langle N | = \langle \infty | \prod_{i=N}^\infty \psi_i = \langle -\infty | \prod_{i=-\infty}^{N-1} \tilde{\psi}_i,
\]

where the "empty" (bare) and "completely filled" vacua are defined so that:

\[
\tilde{\psi}_m | -\infty \rangle = 0, \quad \langle -\infty | \tilde{\psi}_m = 0;
\]

\[
\psi_m | \infty \rangle = 0, \quad \langle \infty | \psi_m = 0.
\]

for any $m \in \mathbb{Z}$. For the only reason that operators $J$, $H$, $\tilde{H}$ and $G$ are defined so that they always have $\tilde{\psi}$ at the very right and $\psi$ at the very left, we get also:

\[
J_m | -\infty \rangle = 0, \quad \langle -\infty | J_m = 0,
\]

\[
G^{\pm 1} | -\infty \rangle = | -\infty \rangle; \quad \langle -\infty | G^{\pm 1} = \langle -\infty |;
\]

\[
e^{\pm \tilde{H}} | -\infty \rangle = | -\infty \rangle; \quad \langle -\infty | e^{\pm \tilde{H}} = \langle -\infty |.
\]

Now we can use all these formulas to rewrite our original correlator (4.16) as:

\[
\langle N | e^H G e^\tilde{H} | N \rangle = \langle -\infty | \left( \prod_{i=-\infty}^{N-1} \psi_i \right) e^H G \left( \prod_{i=-\infty}^{N-1} \psi_i \right) | -\infty \rangle = 0
\]

\[
= \langle -\infty | e^{-\tilde{H}} \left( \prod_{i=-\infty}^{N-1} \psi_i \right) e^H G e^{\tilde{H}} \left( \prod_{i=-\infty}^{N-1} \psi_i \right) e^{-\tilde{H}} | -\infty \rangle = \langle -\infty | \prod_{i=-\infty}^{N-1} \tilde{\psi}_i | \prod_{j=-\infty}^{N-1} G_j | -\infty \rangle = 0
\]

\[
= \text{Det}_{-\infty<i,j<N} \langle -\infty | \tilde{\psi}_i | G_j | -\infty \rangle = 0
\]

The last two steps here were introduction of "$GL(\infty)$-rotated" fermions,

\[
\tilde{\Psi}_i[t] \equiv e^{-\tilde{H}} \psi_i e^H; \quad \Psi_j[t] \equiv e^H \psi_j e^{-\tilde{H}}; \quad G_j[t] \equiv G \psi_j e^{-\tilde{H}} G^{-1},
\]

and application of the Wick theorem to express multifermion correlation function through pair correlators

\[
\mathcal{H}_{ij}(t, \tilde{t}) \equiv \langle -\infty | \tilde{\Psi}_i[t] G_j[t] | -\infty \rangle = \langle -\infty | \tilde{\Psi}_i[t] G \Psi_j[t] | -\infty \rangle,
\]

(once again the fact that $G^{-1} | -\infty \rangle = | -\infty \rangle$ was used). The only non-trivial dynamical information entered through applicability of the Wick theorem, and
for that it was crucial that all the operators $e^{iH}, e^{i\hat{H}}, G$ are quadratic exponents, i.e. can only modify the shape of the propagator, but do not destroy the quadratic form of the action (fields remain free). This is exactly equivalent to the statement that "Heisenberg" operators $\Psi[t]$ are just "rotations" of $\psi$, i.e. that transformations (4.22) are linear.

We shall now describe these transformations in a little more explicit form. Namely, their entire time-dependence can be encoded in terms of the ordinary Shur polynomials $P_n(t)$. These are defined to have a very simple generating function (which we already encountered many times in the theory of matrix models):

$$\sum_{n\geq 0} P_n(t) z^n = \exp \left( \sum_{k=1}^{\infty} t_k z^k \right)$$

(4.24)

(i.e. $P_0 = 1$, $P_1 = t_1$, $P_2 = t_1^2 + t_2$ etc.), and satisfy the relation

$$\frac{\partial P_n}{\partial t_k} = P_{n-k}.$$  

(4.25)

Since

$$\exp \left( \sum_{k=1}^{\infty} t_k z^k \right) = \prod_{k=0}^{\infty} \left( \sum_{n_k \geq 0} \frac{1}{n_k!} t_k^{n_k} z^{kn_k} \right),$$

(4.26)

Shur polynomials can be also represented as

$$P_n(t) = \sum_{\sum k_{n_k} = n} \left( \prod_{k=0}^{n} \frac{1}{n_k!} t_k^{n_k} \right).$$

(4.27)

Now, since

$$e^{-B} A e^B = A + [A, B] + \frac{1}{2} [[A, B], B] + \frac{1}{3} [[[A, B], B], B] + \ldots$$

(4.28)

and

$$[\tilde{\psi}_i, J_k] = \tilde{\psi}_{i+k}, \quad [[\tilde{\psi}_i, J_k], J_{k_2}] = \tilde{\psi}_{i+k_1+k_2}, \ldots$$

(4.29)

we have for every fixed $k$:

$$e^{-t_k J_k} \tilde{\psi}_i e^{t_k J_k} = \sum_{n_k \geq 0} \frac{t_k^{n_k}}{n_k!} \tilde{\psi}_{i+k n_k}.$$  

(4.30)

It remains to note that all the harmonics of $J$ in $H = \sum_{k>0} t_k J_k$ commute with each other, to obtain:

$$\tilde{\Psi}(t) = e^{-H} \tilde{\psi}_i e^{H} = \left( \prod_{k>0} e^{-t_k J_k} \right) \tilde{\psi}_i \left( \prod_{k>0} e^{t_k J_k} \right) =$$

$$= \sum_{n \geq 0} \tilde{\psi}_{i+n} \left( \sum_{\sum k_{n_k} = n} \left( \prod_{k=0}^{\infty} \frac{1}{n_k!} t_k^{n_k} \right) \right) = \sum_{n \geq 0} \tilde{\psi}_{i+n} P_n(t) = \sum_{\sum i_{n_k} = n} \tilde{\psi}_i P_{-i}(t).$$

(4.31)

Similarly, relation $[J_k, \psi_j] = \psi_{k+j}$ implies that

$$\Psi_j(t) = e^{H} \psi_j e^{-H} = \sum_{n \geq 0} \psi_{j+n} P_n(t) = \sum_{m \geq j} \psi_m P_{m-j}(t)$$

(4.32)

and finally

$$\mathcal{H}_{ij} = \sum_{i \geq j} \langle -\infty | \tilde{\psi}_i G \psi_m | -\infty \rangle P_{i-m}(t) P_{m-j}(t) =$$

$$= \sum_{i \geq j} T_{im} P_{i-m}(t) P_{m-j}(t),$$

(4.33)

which implies also that

$$\frac{\partial \mathcal{H}_{ij}}{\partial t_k} = \mathcal{H}_{i+k,j};$$

$$\frac{\partial \mathcal{H}_{ij}}{\partial t_k} = \mathcal{H}_{i,j+k}.$$  

(4.34)

Matrix

$$T_{im} \equiv \langle -\infty | \tilde{\psi}_i G \psi_m | -\infty \rangle$$

(4.35)
is the one which defines fermion rotations under the action of $GL(\infty)$-group element $G$:

$$G\psi_m G^{-1} = \sum_{l \in \mathbb{Z}} \psi_l T_{lm};$$

or

$$G^{-1}\psi G = \sum_{m \in \mathbb{Z}} T_{lm} \tilde{\psi}_m,$$

or

$$G^{-1}\tilde{\psi} G = \sum_{m \in \mathbb{Z}} (T^{-1})_{lm} \tilde{\psi}_m.$$ \hspace{1cm} (4.36)

If $G = 1$, $T_{lm} = \delta_{lm}$. If all $t_k = \bar{t}_k = 0$, $H_{ij} = T_{ij}$.

4.3 KP hierarchy and other reductions

In the previous subsection a formula

$$\tau_N\{t, \bar{t} \mid G\} = \text{Det}_{i,j < 0} H_{i+N,j+N}$$ \hspace{1cm} (4.37)

was derived for the basic correlator, which defines "Toda-lattice $\tau$-function". For obvious reasons the variables $\bar{t}$ are often referred to as "negative-times". $\tau$-function can be normalized by division over the same quantity with all the time-variables vanishing, but this is not always convenient. Eq.(4.37) has generalizations - when similar matrix elements in a multifermion system is considered - this leads to "multicomponent Toda" (or AKNS) $\tau$-functions. Generalizations to arbitrary conformal models should be considered as well. It has also particular "reductions", of which the most important are: KP (Kadomtsev-Petviashvili), forced (semi-infinite) and Toda-chain $\tau$-functions. This is the subject to be discussed in this subsection.

Idea of linear reduction is that the form of operator $G$, or, what is the same, of the matrix $T_{lm}$ in eq.(4.33), can be adjusted in such a way, that $\tau_N\{t, \bar{t} \mid G\}$ becomes independent of some variables, i.e. equation(s)

$$\left(\sum_k \alpha \frac{\partial}{\partial t_k} + \sum_k \beta \frac{\partial}{\partial \bar{t}_k} + \sum_k \beta_k D_N(k) + \gamma\right) \tau_N\{t, \bar{t} \mid G\} = 0$$ \hspace{1cm} (4.38)

can be solved as equations for $G$ for all the values of $t, \bar{t}$ and $N$ at once. (In (4.38) $D_N(k)f_N = f_{N+k} - f_N$.) In this case the system of integrable equations (hierarchy), arising from Hirota equation for $\tau$, gets reduced and one usually speaks about "reduced hierarchy". Usually equation (4.38) is imposed directly on matrix $H_{ij}$, of course than (4.38) is just a corollary.

We shall refer to the situation when (4.38) is fulfilled for any $t, \bar{t}, N$ as to "strong reduction". It is often reasonable to consider also "weak reductions", when (4.38) is satisfied on particular infinite-dimensional hyperplanes in the space of time-variables. Weak reduction is usually a property of entire $\tau$-function as well, but not expressible in the form of a local linear equation, satisfied identically for all values of $t, \bar{t}, N$. Now we proceed to concrete examples:

**Toda-chain hierarchy.** This is a strong reduction. The corresponding constraint (4.38) is just

$$\frac{\partial H_{ij}}{\partial t_k} = \frac{\partial H_{ij}}{\partial \bar{t}_k},$$ \hspace{1cm} (4.39)

or, because of (4.34), $H_{i+k,j} = H_{i,j+k}$. It has an obvious solution:

$$H_{i,j} = \hat{H}_{i+j},$$ \hspace{1cm} (4.40)

i.e. $H_{ij}$ is expressed in terms of a one-index quantity $\hat{H}_i$. It is, however, not enough to say, what are restrictions on $H_{ij}$ - they should be fulfilled for all $t$ and $\bar{t}$ at once, i.e. should be resolvable as equations for $T_{lm}$. In the case under consideration this is simple: $T_{lm}$ should be such that

$$T_{lm} = \hat{T}_{l+m}.$$ \hspace{1cm} (4.41)

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Indeed, then
\[
\mathcal{H}_{ij} = \sum_{l,m} T_{lm} P_{l-i}(t) P_{m-j}(\bar{t}) = \sum_{l,m} \hat{T}_{l+m} P_{l-i}(t) P_{m-j}(\bar{t}) = \\
= \sum_{n \geq 0} \hat{T}_{n+i+j} \left( \sum_{k=0}^{n} P_{k}(t) P_{n-k}(\bar{t}) \right),
\]
(4.42)
and
\[
\hat{\mathcal{H}}_{i} = \sum_{n \geq 0} \hat{T}_{n+i} \left( \sum_{k=0}^{n} P_{k}(t) P_{n-k}(\bar{t}) \right).
\]
(4.43)

Volterra hierarchy. Toda-chain \( \tau \)-function can be further weakly reduced to satisfy the identity
\[
\frac{\partial \tau_{2N}}{\partial t_{2k+1}} \bigg|_{\{t_{2l+1}=0\}} = 0, \quad \text{for all } k,
\]
(4.44)
i.e. \( \tau_{2N} \) is requested to be even function of all odd-times \( t_{2l+1} \) (this is an example of "global characterization" of the weak reduction). Note that (4.44) is imposed only on Toda-chain \( \tau \)-function with even values of zero-time. Then (4.44) will hold whenever \( \hat{\mathcal{H}}_{i} \) in (4.43) are even (odd) functions of \( t_{\text{odd}} \) for even (odd) values of \( i \). Since Shur polynomials \( P_{k}(t) \) are even (odd) functions of odd-times for even (odd) \( k \), it is enough that the sum in (4.43) goes over even (odd) \( n \) when \( i \) is even (odd). In other words, the restriction on \( T_{lm} \) is that
\[
T_{lm} = \hat{T}_{l+m}, \quad \text{and} \quad \hat{T}_{2k+1} = 0 \quad \text{for all } k.
\]
(4.45)

Forced hierarchies. This is another important example of strong reduction. It also provides an example of singular \( \tau \)-functions, arising when \( G = \exp \left( \sum A_{mn} \tilde{\psi}_{m} \tilde{\psi}_{n} \right) \) blows up and normal ordered operators should be used to define regularized \( \tau \)-functions. Forced hierarchy appears when \( G \) can be represented in the form \([51] \) \( G = G_{0} P_{+} \), where projection operator \( P_{+} \) is such that
\[
P_{+} \mid N \rangle = | N \rangle \quad \text{for } N \geq N_{0},
\]
\[
P_{+} \mid N \rangle = 0 \quad \text{for } N < N_{0}.
\]
(4.46)

Explicit expression for this operator is\(^{17}\)
\[
P_{+} \psi = \exp \left( - \sum_{l<N_{0}} \tilde{\psi}_{l} \psi_{l} \right) = \prod_{l<N_{0}} (1 - \tilde{\psi}_{l} \psi_{l}) = \prod_{l<N_{0}} \tilde{\psi}_{l} \psi_{l}.
\]
(4.47)

Because of (4.46), \( P_{+} \mid -\infty \rangle = 0 \), and the identity \( G \mid -\infty \rangle = | -\infty \rangle \), which was essentially used in the derivation in (4.27), can be satisfied only if \( G_{0} \) is singular and \( T_{lm} = \infty \). In order to avoid this problem one usually introduces in the vicinity of such singular points in the universal module space a sort of normalized (forced) \( \tau \)-function \( \tau_{N} = \frac{\tau_{N}}{\tau_{N_{0}}} \). One can check that now \( T_{lm}^{f} = \infty \) for all \( l, m < N_{0} \), and \( \tau^{f} \) can be represented as determinant of a final-dimensional matrix \([52],[51]\):
\[
\tau_{N}^{f} = \text{Det}_{N_{0} \leq i, j < N} \mathcal{H}_{ij}^{f} \quad \text{for } N > N_{0};
\]
\[
\tau_{N_{0}}^{f} = 1;
\]
\[
\tau_{N}^{f} = 0 \quad \text{for } N < N_{0}.
\]
(4.48)

For \( N > N_{0} \) we have now determinant of a finite-dimensional \((N-N_{0}) \times (N-N_{0})\) matrix. The choice of \( N_{0} \) is not really essential, therefore it is better to put \( N_{0} = 0 \) in order to simplify formulas, phrasing and relation with the discrete matrix models (\( N_{0} \) is easily restored if everywhere \( N \) is substituted by \( N-N_{0} \)).

For forced hierarchies can also represent \( \tau \) as
\[
\tau_{N}^{f} = \text{Det}_{0 \leq i, j < N} \delta_{ij} \mathcal{H}_{ij}^{f},
\]
(4.49)

\(^{17}\) Normal ordering sign : : means that all operators \( \tilde{\psi} \) stand to the left of all operators \( \psi \). The product at the r.h.s. obviously implies both the property (4.46) and projection property \( P_{+}^{2} = P_{+} \).
where $\mathcal{H}^{t} = \mathcal{H}_{00}^{t}$ and $\partial_{1} = \frac{\partial}{\partial t_{1}}, \bar{\partial}_{1} = \frac{\partial}{\partial \bar{t}_{1}}$. For forced Toda-chain hierarchy this turns into even simpler expression:

$$\tau_{N}^{t} = \text{Det}_{0 \leq i,j < N} \partial_{1}^{i+j} \mathcal{H}^{t},$$

(4.50)

while for the forced Volterra case we get a product of two Toda-chain $\tau$-functions with twice as small value of $N$ [53]:

$$\tau_{2N}^{t} = \left(\text{Det}_{0 \leq i,j < N} \partial_{2}^{i+j} \mathcal{H}^{t}\right) \cdot \left(\text{Det}_{0 \leq i,j < N} \partial_{2}^{i+j} (\partial_{2} \mathcal{H}^{t})\right) = \tau_{N}^{t}[\mathcal{H}^{t}] \cdot \tau_{N}^{t}[\partial_{2} \mathcal{H}^{t}].$$

(4.51)

Forced $\tau_{N}^{t}$ can be always represented in the form of a scalar-product matrix model. Indeed,

$$\hat{\mathcal{H}}_{ij} = \sum_{i,m} T_{im} P_{t-i}(t) P_{m-j}(\bar{t}) = \int \int e^{U(h) + \bar{U}(\bar{h})} h_{i} \bar{h}_{j} T(h, \bar{h}) dhd\bar{h},$$

(4.52)

where $T(h, \bar{h}) = \sum_{i,m} T_{im} h^{i-1} \bar{h}^{-m-1}$, and $e^{U(h)} = e^{\sum_{k \geq 0} t_{k} h_{k}} = \sum_{l \geq 0} h_{l} P_{t}(l)$. Then, since $\text{Det}_{0 \leq i,j < N} h^{i} = \Delta_{N}(h)$ - this is where it is essential that the hierarchy is forced -

$$\text{Det}_{0 \leq i,j < N} \hat{\mathcal{H}}_{ij} = \prod_{i} \int \int e^{U(h_{i}) + \bar{U}(\bar{h}_{i})} T(h_{i}, \bar{h}_{i}) dhd\bar{h} \cdot \Delta_{N}(h) \Delta_{N}(\bar{h}),$$

(4.53)

i.e. we obtain a scalar-product model with

$$d\mu_{h, \bar{h}} = e^{U(h) + \bar{U}(\bar{h})} T(h, \bar{h}) dhd\bar{h}. \quad (4.54)$$

Inverse is also true: partition function of every scalar-product model is a forced Toda-lattice $\tau$-function.

**KP hierarchy.** In this case we just ignore the dependence of $\tau$-function on times $t$. Every Toda-lattice $\tau$-function can be considered also as KP $\tau$-function: just operator $G^{KP} = Ge^{\hat{H}}$ (a point of Grassmannian) becomes $\tilde{t}$-dependent. Usually $N$-dependence is also eliminated - this can be considered as a little more sophisticated change of $G$. When $N$ is fixed, extra changes of field-variables are allowed, including transformation from Ramond to Neveu-Schwarz sector etc. Often KP hierarchy is from the very beginning formulated in terms of Neveu-Schwarz (antiperiodic) fermionic fields (associated with principal representations of Kac-Moody algebras), i.e. expansions in the first line of (4.17) are in semi-integer powers of $z$: $\psi_{NS}(z) = \sum_{n \in Z} \psi_{n} z^{n+\frac{1}{2}} dz^{1/2}$.

Given a KP $\tau$-function one can usually construct a Toda-lattice one with the same $G$, by introducing in appropriate way dependencies on $\tilde{t}$ and $N$. For this purpose $\tau^{KP}$ should be represented in the form of (4.37):

$$\tau^{KP}\{t \mid G\} = \text{Det}_{i,j < 0} \mathcal{H}_{ij}^{KP},$$

(4.55)

where $\mathcal{H}_{ij}^{KP} = \sum_{i} T_{ij} P_{t-i}(t)$. Since $T_{im}$ is a function of $G$ only, it does not change when we build up a Toda-lattice $\tau$-function:

$$\tau_{N}\{t, \tilde{t} \mid G\} = \text{Det}_{i,j < 0} \mathcal{H}_{i+N,j+N}^{KP};$$

$$\mathcal{H}_{ij} = \sum_{i,m} T_{im} P_{t-i}(t) P_{m-j}(\tilde{t}) = \sum_{m} \mathcal{H}_{im}^{KP} P_{m-j}(\tilde{t}).$$

(4.56)

Then

$$\tau^{KP}\{t \mid G\} = \tau_{0}\{t, 0 \mid G\}. \quad (4.57)$$

If we go in the opposite direction, when Toda-lattice $\tau$-function is considered as KP $\tau$-function,

$$\tau_{0}\{t, \tilde{t} \mid G\} = \tau^{KP}\{t \mid \hat{G}(\tilde{t})\};$$

$$\hat{H}_{ij} = \sum_{m} \mathcal{H}_{im} P_{m-j}(\tilde{t}) \quad \text{and} \quad \hat{T}_{ij}\{\hat{G}(\tilde{t})\} = \sum_{m} T_{im}\{G\} P_{m-j}(\tilde{t}).$$

(4.58)
KP reduction in its turn has many further weak reductions (KdV and Boussinesq being the simplest examples).

4.4 Fermion correlator in Miwa coordinates

Let us now return to original correlator (4.16) and discuss in a little more details the implications of bosonization identity (4.9). In order not to write down integrals of \( J \), we introduce scalar field:\(^{18}\)

\[
\phi(z) = \sum_{k \neq 0}^{\infty} \frac{J_k}{k} z_k + \phi_0 + J_0 \log z,
\]

such that \( \partial \phi(z) = J(z) \). Then (4.9) states that:

\[
: \psi(\lambda) \tilde{\psi}(\tilde{\lambda}) : = e^{\phi(\tilde{\lambda}) - \phi(\lambda)} : \tag{4.60}
\]

"Normal ordering" here means nothing more but the requirement to neglect all mutual contractions (or correlators) of operators in between when Wick theorem is applied to evaluate correlation functions. One can also get rid of the normal ordering sign at the l.h.s. of (4.60), then

\[
\psi(\lambda) \tilde{\psi}(\tilde{\lambda}) = : e^{\phi(\tilde{\lambda})} : \tag{4.61}
\]

In distinguished coordinates on a sphere, when the free field propagator is just \( \log(z - \tilde{z}) \), one also has:

\[
\psi(z) \tilde{\psi}(\tilde{z}) = \frac{1}{z - \tilde{z}} : \psi(z) \tilde{\psi}(\tilde{z}) : .
\]

Our task now is to express operators \( e^H \) and \( e^{\tilde{H}} \) through the field \( \phi \). This is simple:

\[
H = \oint_0 U(z)J(z) = \oint_0 U(z)\partial \phi(z) = - \oint_0 \phi(z) \partial U(z). \tag{4.63}
\]

Here as usual \( U(z) = \sum_{k>0} t_k z^k \) and integral is around \( z = 0 \). This is very similar to generic linear functional of \( \phi_-(\lambda) \equiv - \sum_{k>0} k J_k \lambda^{-k} \),

\[
H = \int \phi_-(\lambda) f(\lambda) d\lambda, \tag{4.64}
\]

one should only require that\(^{19}\)

\[
\partial U(z) = \oint \frac{f(\lambda)}{z - \lambda} d\lambda, \tag{4.65}
\]

i.e.

\[
U(z) = \oint \log \left( \frac{1 - z}{\lambda} \right) f(\lambda) d\lambda. \tag{4.66}
\]

In terms of time-variables this means that

\[
t_k = - \frac{1}{k} \int \lambda^{-k} f(\lambda) d\lambda. \tag{4.67}
\]

Here we required that \( U(z = 0) = 0 \), sometimes it can be more natural to introduce also

\[
t_0 = \int \log \lambda f(\lambda) d\lambda. \tag{4.68}
\]

This change from the time-variables to "time density" \( f(\lambda) \) is known as Miwa transformation. In order to establish relation with fermionic representation and also with matrix models we shall need it in "discretized" form:

\[
t_k = \xi \left( \sum_{a} \lambda_a^{-k} - \sum_{a} \tilde{\lambda}_a^{-k} \right), \tag{4.69}
\]

\[
t_0 = -\xi \left( \sum_{a} \log \lambda_a - \sum_{a} \log \tilde{\lambda}_a \right).
\]

\(^{18}\) One can consider \( \phi \) as introduced for simplicity of notation, but it should be kept in mind that the scalar-field representation is in fact more fundamental for generic \( \tau \)-functions, not related to the level \( k = 1 \) Kac-Moody algebras (this phenomenon is well known in conformal field theory, see [13] for more details).

\(^{19}\) The factor \( 2\pi i \) is included into the definition of contour integral \( \oint \).
We changed integral over $\lambda$ for a discrete sum (i.e. the density function $f(\lambda)$ is a combination of $\delta$-functions, picked at some points $\lambda_a$, $\tilde{\lambda}_a$. This is of course just another basis in the space of the linear functionals, but the change from one basis to another one is highly non-trivial. The thing is, that we selected the basis where amplitudes of different $\delta$-functions are the same: parameter $\xi$ in (4.69) is independent of $a$. Thus the real parameters are just positions of the points $\lambda_a$, $\tilde{\lambda}_a$, while the amplitude is defined by the density of these points in the integration (summation) domain. This domain does not need to be a priori specified: it can be real line, any other contour or - better - some Riemann surface.) Parameter $\xi$ is also unnecessary to introduce, because basises with different $\xi$ are essentially equivalent. We shall soon put it equal to one, but not before Miwa transformation will be discussed in a little more detail.

Our next steps will be as follows. Substitution of (4.64) into (4.69), gives:

$$H = -\xi \sum_a \phi_-(\lambda_a) + \xi \sum_a \phi_-(\tilde{\lambda}_a).$$

(4.70)

In fact, what we need is not the operator $H$ itself, but the state which is created by $e^H$ from the vacuum state $\langle N |$. Then, since $\langle N | J_m = 0$ for $m < 0$, $\langle N | e^{-\xi \phi_-(\lambda)}$ is essentially equivalent to $\langle N | e^{-\xi \phi(\lambda)}$ with $\phi_-(\lambda)$ substituted by entire $\phi(\lambda)$. If $\xi = 1$, $e^{-\phi(\lambda)}$ can be further changed for $\psi(\lambda)$ and we obtain an expression for the correlator (4.16) where $e^H$ is substituted by a product of operators $\psi(\lambda_a)$. The same is of course true for $e^H$. Then Wick theorem can be applied and a new type of determinant formulas arises like, for example,

$$\tau \sim \frac{\Delta(\lambda, \tilde{\lambda})}{\Delta^2(\lambda)\Delta^2(\tilde{\lambda})} \text{det}_{ab} \langle N | \psi(\lambda_a)\bar{\psi}(\tilde{\lambda}_b) G | N \rangle$$

(4.71)

It can be also obtained directly from (4.21), (4.23) and (4.33) by Miwa transformation. The rest of this subsection describes this derivation in somewhat more details.

The first task is to substitute $\phi_-$ by $\phi$. For this purpose we introduce operator

$$\sum_{k=-\infty}^{\infty} t_k J_k = H_+ + H_-,$$

(4.72)

where $H_+ = \sum_{k>0} t_k J_k$ is just our old $H$, $H_- = \sum_{k>0} t_{-k} J_k$, and "negative times" $t_{-k}$ are defined by "analytical continuation" of the same formulas (4.67) and (4.69):

$$t_{-k} = \frac{1}{k} \int \lambda^k f(\lambda) \dd \lambda = -\xi \left( \sum_a \phi(\lambda_a) - \sum_a \phi(\tilde{\lambda}_a) \right).$$

(4.73)

Then

$$\sum_{k=-\infty}^{\infty} t_k J_k = H_+ + H_- = -\xi \left( \sum_a \phi(\lambda_a) - \sum_a \phi(\tilde{\lambda}_a) \right).$$

(4.74)

Further

$$e^{H_+ + H_-} = e^{-\frac{1}{2} s(t)} e^{H_+} e^{H_-} = e^{\frac{1}{2} s(t)} e^{H_-} e^{H_+},$$

(4.75)

where

$$s(t) \equiv \sum_{k>0} k t_k t_{-k} = -\xi^2 \sum_{k, k' \geq 0} \frac{1}{k} \left( \sum_a (\lambda_a^{-k} - \tilde{\lambda}_a^{-k}) \sum_b (\lambda_b^k - \tilde{\lambda}_b^k) \right) = \xi^2 \log \left( \prod_{a,b} \frac{(1 - \frac{\lambda_a}{\lambda_b})(1 - \frac{\lambda_b}{\lambda_a})}{(1 - \frac{\lambda_a}{\tilde{\lambda}_b})(1 - \frac{\tilde{\lambda}_b}{\lambda_a})} \right) + \text{const},$$

(4.76)

where prime means that the terms with $a = b$ are excluded from the product in the numerator and accounted for in the infinite "constant", added at the r.h.s. In other words,

$$e^{\frac{1}{2} s(t)} = \text{const} \cdot \left( \prod_{a > b} (\lambda_a - \lambda_b)(\tilde{\lambda}_a - \tilde{\lambda}_b) \right)^{\xi^2} =$$

$$= \text{const} \cdot \left( \frac{\Delta^2(\lambda)\Delta^2(\tilde{\lambda})}{\Delta(\lambda, \tilde{\lambda})} \right)^{\xi^2}.$$
Since $\langle N \mid J_m = 0 \rangle$ for all $m < 0$, we have $\langle N \mid e^{H_+} = \langle N \mid e^{H_-} = \langle N \mid$, and therefore

$$\langle N \mid e^{H} = \langle N \mid e^{H_+} + \langle N \mid e^{H_-} = e^{-\frac{1}{2} s(t)} \langle N \mid e^{H_+ + H_-}.$$  \tag{4.78}$$

From eq.(4.74),

$$e^{H_+ + H_-} = \text{const} \cdot \prod_a : e^{-\xi \phi(\lambda_a)} : \cdot e^{\xi \phi(\tilde{\lambda}_a)} : \tag{4.79}$$

where "const" is exactly the same as in (4.77). If $\xi = 1$, eq.(61) can be used to write:

$$\langle N \mid e^{H} = \frac{\Delta(\lambda, \tilde{\lambda})}{\Delta^2(\lambda) \Delta^2(\tilde{\lambda})} \langle N \mid \prod_a \psi(\lambda_a) \prod_a \tilde{\psi}(\tilde{\lambda}_a)$$ \tag{4.80}$$

Similarly,

$$e^{\tilde{H}} | N \rangle = \prod_b \tilde{\psi}(\tilde{\lambda}_b) \prod_b \tilde{\psi}(\tilde{\lambda}_b) | N \rangle \frac{\Delta(\tilde{\lambda}, \tilde{\lambda})}{\Delta^2(\tilde{\lambda}) \Delta^2(\tilde{\lambda})}; \tag{4.81}$$

where

$$\tilde{t}_k = -\frac{1}{K} \sum_b \left( \tilde{\lambda}_b^k - \tilde{\lambda}_b^k \right) \tag{4.82}$$

and we used the fact that $J_m | N \rangle = 0$ for all $m > 0$. Finally,

$$\tau_N \{ t, \tilde{t} \mid G \} = \langle N \mid e^{H} G e^{\tilde{H}} | N \rangle = \frac{\Delta(\lambda, \tilde{\lambda})}{\Delta^2(\lambda) \Delta^2(\tilde{\lambda})} \frac{\Delta(\tilde{\lambda}, \tilde{\lambda})}{\Delta^2(\tilde{\lambda}) \Delta^2(\tilde{\lambda})}. \tag{4.83}$$

Let us now put $N = 0$ and define normalized $\tau$-function

$$\tilde{\tau}_0 \{ t, \tilde{t} \mid G \} = \frac{\tau_0 \{ t, \tilde{t} \mid G \}}{\tau_0 \{ 0, 0 \mid G \}}, \tag{4.84}$$

i.e. divide r.h.s. of (4.83) by $\langle 0 \mid G \mid 0 \rangle$. Wick theorem now allows to rewrite the correlator at the r.h.s. as a determinant of the block matrix:

$$\begin{vmatrix} \langle 0 \mid \psi(\lambda_a) \tilde{\psi}(\tilde{\lambda}_a) G \mid 0 \rangle & \langle 0 \mid \psi(\lambda_a) G \tilde{\psi}(\tilde{\lambda}_a) \mid 0 \rangle \\ \langle 0 \mid G \mid 0 \rangle & \langle 0 \mid G \mid 0 \rangle \end{vmatrix} \tag{4.85}$$

Special choices of points $\lambda_a, \ldots, \tilde{\lambda}_b$ can lead to simpler formulas. If $\tilde{\lambda}_a \rightarrow \tilde{\lambda}_a$, so that $\tilde{t}_k \rightarrow 0$, the matrix elements at the right lower block in (4.85) blow up, so that the off-diagonal blocks can be neglected. Then

$$\tau \{ t, \tilde{t} \mid G \} \rightarrow \tau^{KP} \{ t \mid G \} = \frac{\langle 0 \mid e^H G \mid 0 \rangle}{\langle 0 \mid G \mid 0 \rangle} = \frac{\Delta(\lambda, \tilde{\lambda})}{\Delta^2(\lambda) \Delta^2(\tilde{\lambda})} \det_{ab} \langle 0 \mid \psi(\lambda_a) \tilde{\psi}(\tilde{\lambda}_b) G \mid 0 \rangle. \tag{4.86}$$

This function no longer depends on $\tilde{t}$-times and is just a KP $\tau$-function.

Matrix element

$$\varphi(\lambda, \tilde{\lambda}) = \frac{\langle 0 \mid \psi(\lambda) \tilde{\psi}(\tilde{\lambda}) G \mid 0 \rangle}{\langle 0 \mid G \mid 0 \rangle} \tag{4.87}$$

is singular, when $\lambda \rightarrow \tilde{\lambda}$: $\varphi(\lambda, \tilde{\lambda}) \rightarrow \frac{1}{\tilde{\lambda} - \lambda}$. If now in (4.86) all $\tilde{\lambda} \rightarrow \infty$,

$$\tau^{KP} \{ t \mid G \} = \frac{\Delta(\lambda)}{\Delta(\lambda)} \frac{\Delta(\lambda, \tilde{\lambda})}{\Delta^2(\lambda) \Delta^2(\tilde{\lambda})}. \tag{4.88}$$

where

$$\varphi(\lambda) \equiv \langle 0 \mid \psi(\lambda) (\partial^{h-1} \tilde{\psi}) (\infty) G \mid 0 \rangle \sim \lambda^{h-1} \left( 1 + \mathcal{O} \left( \frac{1}{\lambda} \right) \right). \tag{4.89}$$

This is the main determinant representation of KP $\tau$-function in Miwa parametrization.
Starting from representation (4.88) one can restore the corresponding matrix \(H_{ij}^{KP}\) in eq.(4.55) [42]:

\[
H_{ij}^{KP}\{t\} = \int z^i\varphi_{-j}(z)e^{\sum_k t_k z_k^k}dz,
\]

(4.90)
i.e.

\[
T_{ij}^{KP} = \int z^i\varphi_{-j}(z).
\]

(4.91)

Then obviously \(\frac{\partial H_{ij}^{KP}}{\partial t_k} = \frac{\partial H_{ij}^{KP}}{\partial \lambda_k}\). Now we need to prove that the \(\tau\)-function is given at once by \(\frac{\text{det} \varphi_a(\lambda)}{\Delta(\lambda)}\) and \(\text{Det}H_{ij}^{KP}\{t\}\). In order to compare these two expressions one should take \(t_k = \frac{1}{k} \sum_a \lambda_a^{-k}\), so that

\[
\exp\left(\sum_{k>0} t_k z_k^k\right) = \prod_{a=1}^{n} \frac{\lambda_a}{\lambda_a - z} = \left(\prod_a \lambda_a\right) \sum_a \frac{(-)^a \Delta_a(\lambda)}{z - \lambda_a},
\]

(4.92)

where

\[
\Delta_a(\lambda) = \prod_{a > \beta, a, \beta \neq a} (\lambda_a - \lambda_\beta) = \frac{\Delta(\lambda)}{\prod_{a \neq a} (\lambda_a - \lambda_\beta)},
\]

(4.93)

and

\[
H_{ij}^{KP}\big|_{t_k = \frac{1}{k} \sum_a \lambda_a^{-k}} = \left(\prod_a \lambda_a\right) \sum_a \frac{(-)^a \Delta_a(\lambda)}{\Delta(\lambda)} \lambda_a \varphi_{-j}(\lambda_a).
\]

(4.94)

As far as \(n\) is kept finite, determinant of the infinite-size matrix (4.94), \(\text{Det}_{i,j<0}H_{ij}^{KP}\big|_{t_k = \frac{1}{k} \sum_a \lambda_a^{-k}} = 0\) since it is obvious from (4.94) that the rank of the matrix is equal to \(n\). Therefore let us consider the maximal non-vanishing determinant,

\[
\text{Det}_{-n \leq i,j < 0}H_{ij}^{KP}\big|_{t_k = \frac{1}{k} \sum_a \lambda_a^{-k}} = \left(\prod_a \lambda_a\right) \det_{ia} \left(\frac{(-)^{a+1} \Delta_a(\lambda)}{\lambda_a^a \Delta(\lambda)} \right) \varphi_{-j}(\lambda_a) =
\]

(4.95)

\[
= \frac{\det_{ia} \varphi_{-j}(\lambda_a)}{\Delta(\lambda)}.
\]

We used here the fact that determinant of a matrix is a product of determinants and reversed the signs of \(i\) and \(j\). Also used were some simple relations:

\[
\prod_{a=1}^{n} \frac{\Delta_a(\lambda)}{\Delta(\lambda)} = \frac{1}{\Delta^2(\lambda)},
\]

\[
\det_{ia} \frac{\lambda_a}{\lambda_a^a \Delta(\lambda)} = (\prod_a \lambda_a)^{-1} \Delta(1/\lambda),
\]

(4.96)

\[
\Delta(1/\lambda) = \prod_{a > \beta} \left(1 - \frac{1}{\lambda_a} \frac{1}{\lambda_\beta}\right) = (-)^{n(n-1)/2} \Delta(\lambda) \left(\prod_a \lambda_a\right)^{-(n-1)},
\]

thus

\[
\left(\prod_{a=1}^{n} \lambda_a\right) \left(\prod_{a=1}^{n} \frac{\Delta_a(\lambda)}{\Delta(\lambda)} \det_{ia} \frac{1}{\lambda_a^a}\right) = \frac{1}{\Delta(\lambda)}.
\]

Since (4.95) is true for any \(n\), one can claim that in the limit \(n \to \infty\) we recover the statement, that \(\tau^{KP}\{t\} = \text{Det}_{i,j<0}H_{ij}^{KP}\) with \(H_{ij}^{KP}\) given by eq.(4.49) (that formula does not refer directly to Miwa parametrization and is defined for any \(t\) and any \(j < 0\) and \(i\)). This relation between \(\varphi_a\)'s and \(H_{ij}^{KP}\) can now be used to introduce negative times \(t_k\) according to the rule (4.58). Especially simple is the prescription for zero-time: \(H_{ij} \to H_{i+N,j+N}\), when expressed in terms of \(\varphi\) just implies that

\[
\frac{\text{det} \varphi_{a+N}(\lambda_b)}{\Delta(\lambda)} \to \frac{\text{det} \varphi_{a}(\lambda_b)}{(\det \lambda)^N \Delta(\lambda)}.
\]

(4.97)

Generalizations of (4.90), like

\[
H_{ij}\{\ell, \bar{\ell}\} = \oint \oint z^i z^j \psi(0) G(z) \psi(\bar{z}) \left| 0\right> e^{\sum_k (t_k z_k^k + \bar{t}_k \bar{z}_k^k)} dz d\bar{z},
\]

(4.98)
can be also considered.
4.5 1-Matrix model versus Toda-chain hierarchy

At the end of this section we use an explicit example of discrete 1-matrix model [54] to demonstrate how a more familiar Lax description of integrable hierarchies arises from determinant formulas. Lax representation appears usually after some coordinate system is chosen in the Grassmannian. In the example which we are now considering this system is introduced by the use of orthogonal polynomials.

Formalism of orthogonal polynomials was intensively used at the early days of the theory of matrix models. It is applicable to scalar-product eigenvalue models (see [1] for details about this notion) and allows to further transform (diagonalize) the remaining determinants into products. In variance with both reduction from original $N^2$-fold matrix integrals to the eigenvalue problem, which (when possible) reflects a physical phenomenon - decoupling of angular (unitary-matrix) degrees of freedom (associated with $d$-dimensional gauge bosons), - and with occurrence of determinant formulas which reflects integrability of the model, orthogonal polynomials appear more as a technical device. Essentially orthogonal polynomials are necessary if wants to explicitly separate dependence on the size $N$ of the matrix in the matrix integral ("zero-time") from dependencies on all other time-variables and to explicitly construct variables, which satisfy Toda-like equations. However, modern description of integrable hierarchies in terms of $\tau$-functions does not require explicit separation of the zero-time and treats it more or less on the equal footing with all other variables, thus making the use of orthogonal polynomials unnecessary. Still this technique remains in the arsenal of the matrix model theory\textsuperscript{20} and we now briefly explain what it is about.

In the context of the theory of scalar-product matrix models orthogonal polynomials naturally arise when one notes that after partition functions appears in a simple determinantal form,

$$Z_N = \frac{1}{N!} \prod_{k=1}^{N} \int d\mu_{h_k,\bar{h}_k} \text{Det} h_k^{i-1} \text{Det} \bar{h}_k^{j-1} = \text{Det}_{ij} \int d\mu_{h,\bar{h}} h^{i-1} \bar{h}^{j-1} = \text{Det}_{ij} \langle h^{i-1} | \bar{h}^{j-1} \rangle,$$

(of which eq.(2.53) is a simple example), any linear change of bases $h^i \to Q_i(h) = \sum_k A_{ik} h_k$, $\bar{h}^j \to \bar{Q}_j(h) = \sum_l B_{jl} \bar{h}_l$ can be easily performed and $Z \to Z \cdot \text{det}A \cdot \text{det}B$. In particular, if $A$ and $B$ are triangular with units at diagonals, their determinants are just unities and $Z$ does not change at all. This freedom is, however, enough, to diagonalize the scalar product and choose polynomials $Q_i$ and $\bar{Q}_j$ so that

$$\langle Q_i(h) | \bar{Q}_j(\bar{h}) \rangle = e^{\phi_i} \delta_{ij}.$$

$Q_i$ and $\bar{Q}_j$ defined in this way up to normalization are called orthogonal polynomials. (Note that $\bar{Q}$ does not need to be a complex conjugate of $Q$: "bar" does not mean complex conjugation.) Because of above restriction on the form of matrices $A$ and $B$ these polynomials are normalized so that

$$Q_i(h) = h^i + \ldots; \quad \bar{Q}_j(h) = \bar{h}^j + \ldots$$

i.e. the leading power enters with the unit coefficient. From (4.99) and (4.100) it follows that

\textsuperscript{20} Of course, one can also use this link just with the aim to put the rich and beautiful mathematical theory of orthogonal polynomials into the general context of string theory. Among

\textsuperscript{21} Interesting problems here is the matrix-model description of $q$-orthogonal polynomials.
This formula is essentially the main outcome of orthogonal polynomials theory for matrix models: it provides complete separation of the $N$-dependence of $Z$ (on the size of the matrix) from that on all other parameters (which specify the shape of potential, i.e. the measure $d\mu_hh$), this information is encoded in a rather complicated fashion in $\phi_i$. As was already mentioned, any feature of matrix model can be examined already at the level of eq.(4.99), which does not refer to orthogonal polynomials and thus they are not really relevant for the subject.

Consider now the case of the local measure, $d\mu_hh = d\mu_h\delta(h, \bar{h})$, when $\bar{Q}_i = Q_i$. The local measure is distinguished by the property that multiplication by (any function of) $h$ is Hermitean operator:

$$\langle hf(h) \mid g(\bar{h})\rangle = \langle f(h) \mid h\bar{g}(\bar{h})\rangle, \text{ if } d\mu_hh \sim \delta(h - \bar{h}).$$  \hspace{1cm} \text{(4.103)}$$

This implies further that the coefficients $c_{ij}$ in the recurrent relation

$$hQ_i(h) = Q_{i+1}(h) + \sum_{j=0}^{i} c_{ij}Q_j(h)$$ \hspace{1cm} \text{(4.104)}$$

are almost all vanishing. Indeed: for $j < i$

$$c_{ij} = \frac{\langle hQ_i(h) \mid Q_j(h)\rangle}{\langle Q_j(h) \mid Q_j(h)\rangle} = \frac{\langle Q_i(h) \mid \bar{h}Q_j(h)\rangle}{\langle Q_j(h) \mid Q_j(h)\rangle} = \delta_{i,j+1} = \delta_{j,i-1}e^{\phi_i - \phi_{i-1}}.$$ \hspace{1cm} \text{(4.105)}$$

In other words, polynomials, orthogonal w.r.to a local measure are obliged to satisfy the "3-term recurrent relation":

$$hQ_n(h) = Q_{n+1}(h) + c_nQ_n(h) + R_nQ_{n-1}(h)$$ \hspace{1cm} \text{(4.106)}$$

(the coefficient in front of $Q_{n+1}$ can be of course changed by the change of normalization). Parameter $c_n$ vanishes if the measure is even (symmetric under the change $h \rightarrow -h$), then polynomials are split into two orthogonal subsets: even and odd in $h$. Partition function (4.102) of the one-component model can be expressed through parameters $R_i = e^{\phi_i - \phi_{i-1}}$ of the 3-term relation:

$$Z_N = Z_1 \prod_{i=1}^{N-1} R_i^{N-i},$$ \hspace{1cm} \text{(4.107)}$$

thus defining a one-component matrix model (i.e. particular shape of potential), associated with any system of orthogonal polynomials.

Coming back to the 1-matrix model (2.53), one can say that all the information is contained in the determinant formula (4.37) together with the rule (4.34), which defines time-dependence of $\mathcal{H}_{ij}^f = \langle h^i \mid h^j \rangle = \mathcal{H}_{i+j}^f$:

$$\frac{\partial \mathcal{H}_{ij}^f}{\partial t_k} = \mathcal{H}_{i+k,j}^f = \mathcal{H}_{i,j+k}^f, \text{ or}$$ \hspace{1cm} \text{(4.108)}$$

The possibility to express everything in terms of $\mathcal{H}_{ij}^f$ with a single matrix index $i$ is the feature of Toda-chain reduction of generic Toda-lattice hierarchy.

However, in order to reveal the standard Lax representation we need to go into somewhat more involved considerations. Namely, we consider representation of two operators in the basis of orthogonal polynomials. First,

$$h^kQ_n(h) = \sum_{m=0}^{n+k} \frac{\langle n \mid h^k \mid m\rangle}{\langle m \mid m\rangle} Q_m(h) = \sum_{m=0}^{n+k} \gamma_{nm}^{(k)} Q_m(h)$$ \hspace{1cm} \text{(4.109)}$$
and \( \gamma^{(k)}_{nm} = \frac{\langle n \mid h^k \mid m \rangle}{\langle m \mid m \rangle} \). Second,

\[
\frac{\partial Q_n(h)}{\partial t_k} = -\sum_{m=0}^{n-1} \frac{\langle n \mid h^k \mid m \rangle}{\langle m \mid m \rangle} Q_m(h) = -\sum_{m=0}^{n-1} \frac{\gamma^{(k)}_{nm}}{\gamma^{(0)}_{nm}} Q_m(h),
\]

(4.110)

\[
\frac{\partial \phi_n}{\partial t_k} = \frac{\langle n \mid h^k \mid n \rangle}{\langle n \mid n \rangle} = \gamma^{(k)}_{nn}.
\]

(These last relations arise from differentiation of orthogonality condition (4.100):

\[
e^{\phi_n} \frac{\partial \phi_n}{\partial t_k} \delta_{nm} = \frac{\partial \langle Q_n \mid Q_m \rangle}{\partial t_k} =
\]

\[
= \langle Q_n \mid Q_m \rangle + \langle Q_n \mid \frac{\partial Q_m}{\partial t_k} \rangle + \langle Q_n \mid h^k \mid Q_m \rangle
\]

(4.111)

by looking at the cases of \( m < n \) and \( m = n \) respectively.)

From these relations one immediately derives the Lax-like formula:

\[
\frac{\partial \gamma^{(k)}_{nm}}{\partial t_q} = -\sum_{l=m-k}^{n-1} \gamma^{(q)}_{nl} \gamma^{(k)}_{lm} + \sum_{l=m+1}^{n+k} \gamma^{(k)}_{nl} \gamma^{(q)}_{lm} \quad (4.112)
\]

or, in a matrix form,

\[
\frac{\partial \gamma^{(k)}}{\partial t_q} = [R \gamma^{(q)}, \gamma^{(k)}],
\]

(4.113)

where

\[
R \gamma^{(k)}_{nm} = \begin{cases} 
-\gamma^{(k)}_{nm} & \text{if } m > n, \\
\gamma^{(k)}_{nm} & \text{if } m < n
\end{cases}
\]

(4.114)

(We remind that usually \( R \)-matrix acts on a function \( f(h) = \sum_{n=-\infty}^{+\infty} f_n h^n \) according to the rule: \( R f(h) = \sum_{n \geq 1} f_n h^n - \sum_{n < l} f_n h^n \) with some "level" \( l \).) These \( \gamma^{(k)} \) are not symmetric matrices, but one can also rewrite all the formulas above in terms of symmetric ones:

\[
L^{(k)}_{mn} = e^{\frac{1}{2}(\phi_n - \phi_m)}, \quad (k)_{mn} = \frac{\langle m \mid h^k \mid n \rangle}{\sqrt{\langle m \mid m \rangle \langle n \mid n \rangle}}
\]

(4.115)

From eqs. (4.112) one can easily deduce Toda-equations for \( \phi_n \):

\[
\frac{\partial^2 \phi_n}{\partial t_k \partial t_l} = \frac{\partial}{\partial t_k} \langle n \mid h^l \mid n \rangle =
\]

\[
= \left( \sum_{m>n} - \sum_{m<n} \right) \langle n \mid h^k \mid m \rangle \langle m \mid h^l \mid n \rangle \langle m \mid m \rangle \langle n \mid n \rangle,
\]

where the r.h.s. can be expressed in terms of \( R_m = e^{\phi_m - \phi_{m-1}} \). In particular,

\[
\frac{\partial^2 \phi_n}{\partial t_1 \partial t_1} = R_{n+1} - R_n = e^{\phi_{n+1} - \phi_n} - e^{\phi_{n} - \phi_{n-1}}.
\]

(4.117)

Let us also mention that in this formalism the Ward identities (Virasoro constraints) follow essentially from the relation

\[
\left( \frac{\partial}{\partial h} \right)^q = -\frac{\partial}{\partial h} - \sum_{k>0} k t_k h^{k-1},
\]

(4.118)

where Hermitean conjugation is w.r.t. the scalar product \( \langle \mid \rangle \). For example, this relation implies, that

\[
\langle Q_n \mid \frac{\partial Q_n}{\partial h} \rangle = -\langle \frac{\partial Q_n}{\partial h} \mid Q_n \rangle - \sum_{k>0} k t_k \langle Q_n \mid h^{k-1} \mid Q_n \rangle.
\]

(4.119)

Now we note that \( \frac{\partial Q_n}{\partial h} \) is a polinomial of degree \( n-1 \), thus \( \langle Q_n \mid \frac{\partial Q_n}{\partial h} \rangle = 0 \). (In fact

\[
\frac{\partial Q_n}{\partial h} = -\sum_{k>0} k t_k \left( \sum_{m=0}^{n-1} \gamma^{(k-1)}_{nm} Q_m \right) = -\sum_{k>0} k t_k \frac{\partial Q_n}{\partial t_{k-1}}.
\]

Also we recall that \( \langle Q_n \mid h^{k-1} \mid Q_n \rangle = \langle Q_n \mid Q_n \rangle \frac{\partial \phi_n}{\partial t_{k-1}} \), and obtain:

\[
\sum_{k>0} k t_k \frac{\partial \phi_n}{\partial t_{k-1}} = 0
\]

(4.120)
for any $n$. This should be supplemented by relation $\frac{\partial \phi_n}{\partial t_n} = \phi_n$. In order to get the lowest Virasoro constraint (string equation), $L_{-1} Z_N = 0$ or $L_{-1} \log Z_N = 0$ it is enough just to sum over $n$ from 0 to $N - 1$.

For more details about 1-matrix model, Toda-chain hierarchy and application of the formalism of orthogonal polynomials in this context see [54].

5 \( \tau \)-function as a group-theoretical quantity

This section contains some remarks about the general notion of \( \tau \)-function on the lines suggested in ref.[1]. Examples below are taken from [55] and [56].

As mentioned in the beginning of the previous section we define the (generalized) \( \tau \)-function as the generating functional of all the matrix elements of a given group element $g \in G$ in a given representation $R$:

$$
\tau_R(t, \bar{t}|g) \equiv \sum_{\{m, \bar{m}\} \in R} s^R_{m, \bar{m}}(t, \bar{t}) < m|g|m > \quad (5.1)
$$

The choice of functions $s^R_{m, \bar{m}}(t, \bar{t})$ is the main ambiguity in the definition of \( \tau \)-function and needs to be fixed in some clever way, not yet known in full generality. The only a priori requirement is that it is indeed a generating functional, i.e. there should be some (\( g \)-independent) operators $\mathcal{M}$, acting on $t, \bar{t}$-variables, which allow to extract all particular matrix elements once $\tau_R(t, \bar{t})$ is known:

$$
< m|g|m > = \langle \mathcal{M}_{R,m,\bar{m}}(t, \bar{t}) | \tau_R(t, \bar{t}|g) >
$$

The ambiguity in the choice of $s^R_{m, \bar{m}}(t, \bar{t})$ can be partly fixed (at least in the case of the highest weight representations $R$) by the requirement that

$$
\tau_R(t, \bar{t}|g) = < \text{vac}_R | U(t) g \bar{U}(\bar{t}) | \text{vac}_R > \quad (5.2)
$$

where operators $U$ and $\bar{U}$ do not depend on $R$. In order to be even more specific one can further request that evolution operators are group elements, i.e.

$$
\Delta U(t) = U(t) \otimes U(t) = (U(t) \otimes I) (I \otimes U(t)) \quad (5.3)
$$

where $\Delta$ denotes group comultiplication law. In the case of Lie algebras $\Delta(T_{2k}) = T_{2k} \otimes I + I \otimes T_{2k}$, and (5.3) is true at least for the evolution operators in KP/Toda systems. Later we shall see that in the case of quantum groups it can be natural to slightly modify the condition (5.3).

Remarkably, the $\tau$-function defined in (5.1), always satisfies a family of nonlinear equations [55], relating $\tau_R$ with different $R$’s, which reflect just the fact that matrix elements of the same group element in different representations are not independent. Conventional bilinear Hirota equation for KP/Toda $\tau$-functions is nothing but particular case of this generic construction\(^{21}\), which has two (essentially identical) interpretations: in terms of fundamental representations of $GL(\infty)$ and in terms of the level $k = 1$ Kac-Moody $U(1)$ algebra.

5.1 From intertwining operators to bilinear equations

The following construction [55] in terms of intertwining operators is the general source of bilinear equations for the $\tau$-function (5.1). One can easily recognize the standard free-fermion derivation of Hirota equations for KP/Toda $\tau$-functions as a particular example (with $G$ being the level $k = 1$ Kac-Moody algebras $\hat{G}_{k=1}$, $V$ a fundamental representation, and $W$ - the simplest fundamental representation corresponding to the very left root of the Dynkin diagram). Construction below involves a lot of arbitrariness. In order to make the consideration more

\(^{21}\) Note, that it is somewhat different from approach, advocated by V.Kac [49] (see also [1]), which makes use of Casimir operators and is less universal than the one to be described below (using intertwining operators).
transparent, we formulate our construction explicitly for finite-dimensional Lie algebras and their $q$-counterparts.

Bilinear equations which we are going to derive are relating $\tau$-functions (5.1) for four different Verma modules $V$, $\hat{V}$, $V'$, $\hat{V}'$. Given $V$, $V'$, every allowed choice of $\hat{V}$, $\hat{V}'$ provides a separate set of bilinear identities. Of course, not all of these sets are actually independent and can be parametrized by source modules $V$ and $V'$ and by a weight of finite-dimensional representation. Also different choices of positive root systems and their ordering in (5.2) provides equations in somewhat different forms. A more invariant description of the minimal set of bilinear equations for given $G$ would be clearly interesting to find.

1. Our starting point is embedding of Verma module $\hat{V}$ into the tensor product $V \otimes W$, where $W$ is some irreducible finite-dimensional representation of $G$ (in the case of Kac-Moody algebra evaluation representation should be used). Once $V$ and $W$ are specified, there is only finite number of choices for $\hat{V}$.

Now we define right vertex operator of the $W$-type as homomorphism of $G$-modules:

$$E_R : \hat{V} \rightarrow V \otimes W.$$ (5.4)

This intertwining operator can be explicitly continued to the whole representation once it is constructed for the vacuum (highest-weight) state:

$$\hat{V} = \left\{ |n_\alpha\rangle_{\hat{V}} = \prod_{\alpha > 0} (\Delta(T_-\alpha)^{n_\alpha} |0\rangle_{\hat{V}}) \right\},$$ (5.5)

where comultiplication $\Delta$ provides the action of $G$ on the tensor product of representations, and

$$|0\rangle_{\hat{V}} = \left( \sum_{\{p_\alpha, i_\alpha\}} A\{p_\alpha, i_\alpha\} \left( \prod_{\alpha > 0} (T_-\alpha)^{p_\alpha} \otimes (T_-\alpha)^{i_\alpha} \right) \right) |0\rangle_V \otimes |0\rangle_W.$$ (5.6)

For finite-dimensional $W$’s, this gives every $|n_\alpha\rangle_{\hat{V}}$ in a form of finite sums of states $|m_\alpha\rangle_V$ with coefficients, taking values in elements of $W$.

2. The next step is to take another triple, defining a left vertex operator,

$$\hat{E}_L : \hat{V}' \rightarrow W^* \otimes V',$$ (5.7)

Note the change of ordering at the r.h.s., this is different from $V' \otimes W^*$ in the case of quantum groups. The product $W \otimes W^*$ of the module $W$ and its conjugate contains unit representation of $G$. The projection to this unit representation

$$\pi : W \otimes W^* \rightarrow I$$ (5.8)

is explicitly provided by multiplication of any element of $W \otimes W^*$ by

$$\pi = \sum_{\{i_\alpha, i'_\alpha\}} \prod_{\alpha > 0} \left( \prod_{\alpha > 0} (T_+\alpha)^{i_\alpha} \otimes (T_+\alpha)^{i'_\alpha} \right).$$ (5.9)

Using this projection, if it is not occasionally orthogonal to the image of $E \otimes E'$, one can build a new intertwining operator

$$\Gamma : \hat{V} \otimes \hat{V}' \rightarrow V \otimes W \otimes W^* \otimes V' \otimes I \otimes \pi \otimes I \rightarrow V \otimes V',$$ (5.10)

which possesses the property

$$\Gamma(g \otimes g) = (g \otimes g)\Gamma$$ (5.11)

for any group element $g$ such that

$$\Delta(g) = g \otimes g.$$ (5.12)
3. It now remains to take a matrix element of (5.11) between four states,
\[ \langle k' | \Gamma | n \rangle \psi | n' \rangle \varphi = \langle k' | \Gamma | g \otimes g | n \rangle \varphi | n' \rangle \varphi, \]
and rewrite this identity in terms of generating functions (5.1).

5.2 The case of KP/Toda \( \tau \)-functions

We do not present here the standard derivation of Hirota equations in the free-fermion formalism, because it is both well known and easily recognizable in the general picture from the previous subsection. Instead we describe here its slight variation — starting from the fundamental representations of \( SL(\infty) \). The reason why this case is the closest one to the standard integrable hierarchies is that, in variance with generic Verma modules for group \( G \neq SL(2) \), the fundamental representations are generated by subset of the \textit{mutually commuting} operators, not by entire set of generators from maximal nilpotent subalgebra. We describe the basic construction for \( G = SL(n) \), since in this case the (finite) Grassmannian construction is the most similar to the conventional infinite-dimensional \( (G = U(1)) \) situation.

The Lie algebra \( SL(n) \) is generated by operators \( T_{\pm \alpha} \) and Cartan operators \( H_{\beta} \), such that \[ [H_{\beta}, T_{\pm \alpha}] = \pm \frac{1}{2}(\beta \alpha)T_{\pm \alpha}. \] All elements of all representations are eigenfunctions of \( H_{\beta} \), \( H_{\beta}|\lambda\rangle = \frac{1}{2}(\beta \lambda)|\lambda\rangle \). The highest weight of representation \( F^{(k)} \) is \( \tilde{\mu}_k \). Vectors \( \tilde{\mu}_k \)'s are “dual” to the simple roots \( \alpha_i \), \( i = 1, \ldots, r \): \( \tilde{\mu}_i \alpha_j \rangle = \delta_{ij} \), and \( \tilde{\rho} = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_i \tilde{\mu}_i \).

There are as many as \( r = \text{rank } G = n - 1 \) fundamental representations of \( SL(n) \). Let us begin with the simplest fundamental representation \( F \equiv F^{(1)} \) — the \( n \)-plet, which consists of the states
\[ \psi_i = T_{-(i-1)} \ldots T_{-2}T_{-1}\psi_1, \quad i = 1, \ldots, n. \] Moreover
\[ T_{-i} \psi_j = \delta_{ij} \psi_{i+1}, \] and the weights are given by
\[ \lambda((\psi_i) = \tilde{\mu}_1 - \alpha_1 - \ldots - \alpha_{i-1}, \] where \( \tilde{\mu}_1 \) is the highest weight of \( F^{(1)} \). Here \( T_{\pm \alpha} \) are generators, associated with the simple roots. Let us denote the corresponding basis in Cartan algebra \( H_i = H_{\alpha_i} \), and \( H_i|\tilde{\lambda}\rangle = \frac{1}{2}(\alpha_i \tilde{\lambda})|\tilde{\lambda}\rangle = \lambda_i|\tilde{\lambda}\rangle \). Then
\[ \lambda((\psi_i) = \frac{1}{2}(\delta_{ij} - \delta_{i,j-1}). \] This formula, together with (5.14) and defining commutation relations between the positive and negative simple-root generators implies that \( ||\psi_i||^2 = 1 \), and, with the help of the classical comultiplication formula, \( \Delta(T) = T \otimes I + I \otimes T \), one immediately observes that the antisymmetric combinations \( \psi_1 \ldots \psi_{k} \) are all the highest weight vectors (i.e. annihilated by all \( \Delta_k(T_{\alpha}) \) and, thus by all \( \Delta_k(T_{\alpha}) \)). These combinations are the highest vectors of all the other fundamental representations \( F^{(k)} \), which are thus skew powers of \( F = F^{(1)} \):
\[ F^{(k)} = \{ \psi^{(k)}_{i_1 \ldots i_k} \sim \psi_{i_1} \ldots \psi_{i_k} \}. \] From this description it is clear that \( k \leq n \), moreover \( F^{(0)} \) and \( F^{(n)} \) are respectively the singlet and dual singlet representations.

According to (5.15) one can also describe all the states of \( F^{(1)} \) in terms of a single generator \( T_- \), which is a sum of those for all the \( r \) simple roots of \( G \), \( T_- = \sum_{i=1}^r T_{-\alpha_i} \):
\[ \psi_i = T_{-i}^{-1}|0\rangle_F, \quad i = 1, \ldots, n. \]
The intertwining operators which are of interest for us are
\[ R_k(T_\perp) = T_\perp \otimes I \otimes \ldots \otimes I + I \otimes T_\perp \otimes \ldots \otimes I + I \otimes I \otimes \ldots \otimes T_\perp. \quad (5.20) \]

These operators obviously commute with each other. For given \( k \) exactly \( k \) of them (with \( i = 1, \ldots, k \)) are independent. However, they are neither (linear combinations of) the generators of Lie algebra acting in \( F^{(k)} \), nor even their algebraic functions (note that \( R_k(T_\perp) \neq ((R_k(T_\perp))^i) \). If one wants to make clear that \( F^{(k)} \) is indeed a representation of \( G \), it is better to say that it is generated by another set of operators,

\[ \Delta^{k-1}(T_\perp), \quad T_\perp = \sum_{\alpha, h(\alpha) = i} T_{-\alpha} \quad (5.21) \]

The “height” \( h(\alpha) \) is the number of items in linear decomposition of the root \( \alpha \) in simple roors \( \alpha_i \) (in \( F^{(1)} \)) where all the generators \( T_{\alpha} \) are represented by \( n \times n \) matrices \( T_i \) are matrices with all zero entries except for units at the \( i \)-th subdiagonal). In particular, \( T_\perp = T_1 \). Operators (5.21) are obviously generators of \( G \), instead their mutual commutativity is somewhat less transparent. Since both sets (5.21) and (5.20) generate \( F^{(k)} \), it is a matter of convenience which of them is used in particular considerations. In dealing with KP/Toda hierarchies the explicitly commuting set (5.20) is more convenient. It is exactly the lack of such equivalence of two sets which makes consideration of KP/Toda hierarchies more subtle in the quantum \((q \neq 1)\) case, see s.5.4 below.

The intertwining operators which are of interest for us are

\[ I_{(k)} : \quad F^{(k+1)} \rightarrow F^{(k)} \otimes F; \]
\[ I^*_{(k)} : \quad F^{(k-1)} \rightarrow F^* \otimes F^{(k)}; \quad \text{and} \]
\[ \Gamma_{k|k'} : \quad F^{(k+1)} \otimes F^{(k'-1)} \rightarrow F^{(k)} \otimes F^{(k')}. \quad (5.22) \]

Here

\[ F^* = F^{(r)} = \{ \psi^i \sim \epsilon^{i_1 \ldots i_r} \psi[i_1 \ldots \psi[i_r] \}, \]
\[ I_{(k)} : \quad \Psi^{(k+1)}_{i_1 \ldots i_{k+1}} = \Psi^{(k)}_{i_1 \ldots i_k} \psi_{i_{k+1}}, \]
\[ I^*_{(k)} : \quad \Psi^{(k-1)}_{i_1 \ldots i_{k-1}} = \Psi^{(k)}_{i_1 \ldots i_{k-1}} \psi^i, \]

and \( \Gamma_{k|k'} \) is constructed with the help of embedding \( I \rightarrow F \otimes F^* \), induced by the pairing \( \psi^i \psi^j \): the basis in linear space \( F^{(k+1)} \otimes F^{(k'-1)} \), induced by \( \Gamma_{k|k'} \) from that in \( F^{(k)} \otimes F^{(k')} \) is:

\[ \Psi^{(k)}_{i_1 \ldots i_k} \Psi^{(k')}_{r_1 \ldots r'_{k'-1}}. \quad (5.24) \]

Operation \( \Gamma \) can be now rewritten in terms of matrix elements

\[ g^{(k)} \begin{pmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{pmatrix} \equiv \langle \Psi_{i_1 \ldots i_k} | g | \Psi_{j_1 \ldots j_k} \rangle = \det_{a,b \leq k} g^{a_b} \quad (5.25) \]
as follows:

\[ g^{(k)} \begin{pmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{pmatrix} g^{(k')} \begin{pmatrix} i'_1 \ldots i'_{k'} \\ j'_1 \ldots j'_{k'-1} \end{pmatrix} \]
\[ = g^{(k+1)} \begin{pmatrix} i_1 \ldots i_k | i'_{k+1} \\ j_1 \ldots j_{k+1} | j'_{k'-1} \end{pmatrix} g^{(k'-1)} \begin{pmatrix} i'_1 \ldots i'_{k'-1} \\ j'_1 \ldots j'_{k'-1} \end{pmatrix} \]

This is the explicit expression for eq.(5.11) in the case of fundamental represen-
tions, and it is certainly identically true for any $g^{(k)}$ of the form (5.25).\(^{22}\)

Let us note that one can use the minors (5.25) to construct local coordinates in the Grassmannian. Bilinear Plucker relations satisfied by these coordinates are nothing but defining equations of the Grassmannian consisting of all the $k$-dimensional vector subspaces of $n$-dimensional vector space. Parametrizing determinants (5.25) by time variables (see (5.33)), one gets a set of bilinear differential equations on the generating function of these Plucker coordinates, which is just a $\tau$-function [58].

Now let us introduce time-variables and rewrite (5.26) in terms of $\tau$-functions. We shall denote time variables through $s_1, \ldots, s_r$, $i = 1, \ldots, r$ in order to emphasize their difference from generic $t_{\bar{\alpha}}, \bar{t}_{\alpha}$ labeled by all the positive roots $\bar{\alpha}$ of $G$. Note that in order to have a closed system of equations we need to introduce all the $r$ times $s_i$ for all $F^{(k)}$ (though $\tau^{(k)}$ actually depends only on $k$ independent combinations of these).

Since the highest weight of representation $F^{(k)}$ is identified as

$$|0\rangle_{F^{(k)}} = |\Psi^{(k)}_{1 \ldots k}\rangle,$$

we have:

$$\tau^{(k)}(t, \bar{\tau} | g) = \langle \Psi^{(k)}_{1 \ldots k} | \exp \left( \sum_i t_i R_k(T_i^+ \bar{t}^-) \right) g \exp \left( \sum_i \bar{t}_i R_k(T_i^-) \right) |\Psi^{(k)}_{1 \ldots k}\rangle. \tag{5.28}$$

\(^{22}\)To see this directly it is enough to rewrite the l.h.s. of (5.26) as

$$g^{(k)} \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_k \end{array} \right) g^{(k')}_{j_{k+1} \ldots j_{k'}} \left( \begin{array}{c} j'_{k+1} \ldots j'_{k-1} \\ j_1 \ldots j_{k-1} \end{array} \right)$$

(expansion of the determinant $g^{(k')}$ in the first column) and now the first two factors can be composed into $g^{(k+1)}$ (expansion of the determinant $g^{(k+1)}$ in the first row), thus giving the r.h.s. of (5.26).

Now,

$$\exp \left( \sum_i t_i R_k(T_i^+ \bar{t}^-) \right) = \exp \left( R_k \left( \sum_i t_i T_i^+ \bar{t}^- \right) \right) = \left( \exp \left( \sum_i t_i T_i^+ \right) \right)^{\otimes k} = \left( \sum_j P_j(t) T_j^+ \right)^{\otimes k}, \tag{5.29}$$

where we used the definition of Schur polynomials

$$\exp \left( \sum_i t_i z_i \right) = \sum_j P_j(t) z^j. \tag{5.30}$$

Essential property of Shur polynomials is that

$$\frac{\partial}{\partial t_i} P_j(t) = (\frac{\partial}{\partial t_1})^i P_j(t) = P_{j-i}(t). \tag{5.31}$$

Because of (5.29), we can rewrite the r.h.s. of (5.28) as

$$\tau^{(k)}(t, \bar{\tau} | g) = \sum_{i_1, \ldots, i_k, j_1, \ldots, j_k} P_{i_1}(t) \cdots P_{i_k}(t) \langle \Psi^{(k)}_{1+i_1,2+i_2,\ldots,k+i_k} | g | \Psi^{(k)}_{1+j_1,2+j_2,\ldots,k+j_k} \rangle P_{j_1}(t) \cdots P_{j_k}(t) = \det_{1 \leq \alpha, \beta \leq k} H^\alpha_\beta(t, \bar{\tau}), \tag{5.32}$$

where

$$H^\alpha_\beta(t, \bar{\tau}) = \sum_{i,j} P_{\alpha-i}(t) g^i_j P_{\beta-j}(\bar{\tau}). \tag{5.33}$$

This formula can be considered as including infinitely many times $s_i$ and $\bar{s}_i$, and it is only due to the finiteness of matrix $g^i_j \in SL(n)$ that $H$-matrix is
additionally constrained

\[ \left( \frac{\partial}{\partial t_1} \right)^n H^\alpha_{\beta} = 0, \]

\[ \cdots \]

\[ \frac{\partial}{\partial t_i} H^\alpha_{\beta} = 0, \quad \text{for } i \geq n. \]

\[ \frac{\partial}{\partial t_i} H^\alpha_{\beta} = H^\alpha_{\beta + i}, \quad \frac{\partial}{\partial \bar{t}_i} H^\alpha_{\beta} = H^\alpha_{\beta + i}. \]

(5.35)

The characteristic property of \( H^\alpha_{\beta} \) is that it satisfies the following “shift” relations (see (5.31)):

\[ \frac{\partial}{\partial t_i} H^\alpha_{\beta} = H^\alpha_{\beta} + i \beta, \quad \frac{\partial}{\partial \bar{t}_i} H^\alpha_{\beta} = H^\alpha_{\beta} + i. \]

(5.35)

Coming back to bilinear relation (5.26), it can be easily rewritten in terms of \( H \)-matrix: it is enough to convolute them with Schur polynomials. For the sake of convenience let us denote \( H^{(\alpha_1 \ldots \alpha_k)}_{(\beta_1 \ldots \beta_k)} = \det_{1 \leq a, b \leq k} H^\alpha_{\beta} \). In accordance with this notation \( \tau^{(k)} = H^{(1 \ldots k)}_{(1 \ldots k)} \), while bilinear equation turns into:

\[ H^{(\alpha_1 \ldots \alpha_k)}_{(\beta_1 \ldots \beta_k)} H^{(\alpha'_1 \ldots \alpha'_{k-1})}_{(\beta_{k+1} \ldots \beta_{k-1})} = H^{(\alpha_1 \ldots \alpha_k|\alpha'_k)}_{(\beta_1 \ldots \beta_k \beta_{k+1})} H^{(\alpha'_1 \ldots \alpha'_{k-1})}_{(\beta'_1 \ldots \beta'_{k-1})}. \]

(5.36)

Just like original (5.26) these are just matrix identities, valid for any \( H^\alpha_{\beta} \). However, after the switch from \( g \) to \( H \) we, first, essentially represented the equations in the \( n \)-independent form and, second, opened the possibility to rewrite them in terms of time-derivatives.

For example, in the simplest case of

\[ \alpha_i = i, \quad i = 1, \ldots, k'; \]

\[ \beta_i = i, \quad i = 1, \ldots, k + 1; \]

\[ \alpha'_i = i, \quad i = 1, \ldots, k - 1, \quad \alpha'_{k} = k + 1; \]

\[ \beta'_i = i, \quad i = 1, \ldots, k - 1 \]

we get:

\[ H^{(1 \ldots k)}_{(1 \ldots k)} H^{(k + 1, 1 \ldots k - 1)}_{(k + 1, 1 \ldots k - 1)} - H^{(1 \ldots k - 1, k)}_{(1 \ldots k - 1, k + 1)} H^{(k + 1, 1 \ldots k - 1)}_{(k, 1 \ldots k - 1)} = \]

\[ = H^{(1 \ldots k + 1)}_{(1 \ldots k + 1)} H^{(1 \ldots k - 1)}_{(1 \ldots k - 1)}. \]

(5.38)

(all other terms arising in the process of symmetrization vanish). This in turn can be represented through \( \tau \)-functions:

\[ \partial_1 \bar{\partial}_1 \tau^{(k)} \cdot \tau^{(k)} - \partial_1 \tau^{(k)} \cdot \partial \tau^{(k)} = \tau^{(k + 1)} \cdot \tau^{(k - 1)}. \]

(5.39)

This is the usual lowest Toda-lattice equation. For finite \( n \) the set of solutions is labeled by \( g \in SL(n) \) (rather than \( SL(\infty) \)) as a result of additional constraints (5.34).

We can now use the chance to illustrate the ambiguity of definition of \( \tau \)-function, or, to put it differently, that in the choice of time-variables. Eq.(5.39) is actually a corollary of two statements: the basic identity (5.26) and the particular choice of evolution operators in eq.(5.2), which in the case of (5.28) implies (5.33) with \( P \)'s being ordinary Schur polynomials (5.30). At least, in this simple situation (of fundamental representations of \( SL(n) \)) one could define \( \tau \)-function not by eq.(5.2), but just by eq.(5.32), with
\[ H^\alpha_\beta(t, \bar{t}) \rightarrow \mathcal{H}^\alpha_\beta(t, \bar{t}) = \sum_{i,j} P_{i-\alpha}(t) g_j^I P_{j-\beta}(\bar{t}) \] (5.40)

with any set of independent functions (not even polynomials) \( P_\alpha \). Such

\[ \tau^{(k)}_P = \det_{1 \leq \alpha, \beta \leq k} \mathcal{H}^\alpha_\beta \] (5.41)

still remains a generating function for all matrix elements of \( G = SL(n) \) in representation \( F^{(k)} \). This freedom should be kept in mind when dealing with “generalized \( \tau \)-functions”. As a simple example, one can take \( P_\alpha(t) \) to be \( q \)-Schur polynomials,

\[ \prod_i e_q(t_i z^i) = \sum_j \hat{P}_j^{(q)}(t) z^j, \] or

\[ \prod_i e_{q'}(t_i z^i) = \sum_j \hat{\hat{P}}_j^{(q)}(t) z^j, \] (5.42)

which satisfy

\[ D_{t_i} P_j^{(q)}(t) = (D_{t_i})^I P_j^{(q)}(t) = P_{j-I}(t). \] (5.43)

where \( D \) are finite-difference operators. Then instead of (5.35) we would have:

\[ D_{t_i} \mathcal{H}^\alpha_\beta = \mathcal{H}^{\alpha + I}_\beta, \quad D_{t_i} \mathcal{H}^\beta_\alpha = \mathcal{H}^{\beta + I}_\alpha \] (5.44)

and

\[ \tau^{(k)}_{P^{(q)}}(t, \bar{t}|g) = \det_{1 \leq \alpha, \beta \leq k} D_{t_i}^{\alpha-I} D_{\bar{t}_i}^{\beta-I} \mathcal{H}_I^1(t, \bar{t}). \] (5.45)

So defined \( \tau \)-function satisfies difference rather than differential equations [59, 60]:

\[ \tau^{(k)} \cdot D_{t_i} D_{\bar{t}_i} \tau^{(k)} - D_{t_i} \tau^{(k)} \cdot D_{\bar{t}_i} \tau^{(k)} = \tau^{(k-1)} \cdot M^+_{t_i} M^+_{\bar{t}_i} \tau^{(k+1)}, \] (5.46)

We emphasize, however, that this is just another description of the \( SL(n) \), not \( SL_q(n) \) \( \tau \)-function, if it is interpreted as a generating function of matrix elements. In particular, this \( \tau \)-function takes \( c \)- rather than \( q \)-number values. Still, as concerns its times-, not \( g \)-dependence, it has something to do with the \( SL_q(n) \) group, in the spirit of relation between \( q \)-hypergeometric functions and quantum groups (see, for example, [61]).

### 5.3 Example of \( SL(2)_q \)

Construction from subsection 5.1 is immediately applicable to the case of quantum groups, the only thing one should keep in mind is that our definition (5.1) gives \( \tau \) as an element of “coordinate ring” \( A(G_q) \), not just a \( c \)-number. If one wants to obtain a \( c \)-number \( \tau \)-function for \( q \neq 1 \) it is necessary to restrict the construction further to particular representation of coordinate ring (this last step will not be discussed in this paper). We present here in full detail the simplest possible example of \( SL(2)_q \) [55].

#### 5.3.1 Bilinear identities

To begin with, fix the notations. We consider generators \( T_+ \), \( T_- \) and \( T_0 \) of \( U_q(SL(2)) \) with commutation relations

\[ \cdots \]
\[ q^{T_0} T_\pm q^{-T_0} = q^{\pm 1} T_\pm, \]
\[ [T_+, T_-] = \frac{q^{2T_0} - q^{-2T_0}}{q - q^{-1}}, \]

and comultiplication

\[ \Delta(T_\pm) = q^{T_0} \otimes T_\pm + T_\pm \otimes q^{-T_0}, \]
\[ \Delta(q^{T_0}) = q^{T_0} \otimes q^{T_0}. \]

Verma module \( V_\lambda \) with highest weight \( \lambda \) (not obligatory half-integer), consists of the elements

\[ |n\rangle_\lambda \equiv T^n |0\rangle_\lambda, \quad n \geq 0, \]

such that

\[ T_- |n\rangle_\lambda = |n + 1\rangle_\lambda, \]
\[ T_0 |n\rangle_\lambda = (\lambda - n) |n\rangle_\lambda, \]
\[ T_+ |n\rangle_\lambda = b_n(\lambda) |n - 1\rangle_\lambda, \]

\[ b_n(\lambda) = [n][2\lambda + 1 - n], \quad [x] = \frac{x^q - q^{-x}}{q - q^{-1}}, \]
\[ ||n||^2_\lambda \equiv \lambda |n\rangle_\lambda = \frac{|n|!}{\Gamma_q(2\lambda + 1 - n)} \frac{[2\lambda][n]!}{[2\lambda - n]!} \]

Now,

\[ (\Delta(T_-))^n = q^{nT_0} \otimes T^n + [n]T_- q^{(n-1)T_0} \otimes T^{n-1} q^{-T_0} + \ldots + \]
\[ + [n]T^{n-1} q^{T_0} \otimes T_- q^{-(n-1)T_0} + T^n \otimes q^{-nT_0}. \]

Let us manifestly derive equations (5.13) taking for \( W \) an irreducible spin \( \frac{1}{2} \) representation of \( U_q(SL(2)) \). Then \( \hat{V} = V_{\lambda \pm \frac{1}{2}}, \) \( V = V_\lambda \) and the highest weights of \( \hat{V} \) in \( W \otimes V \) or \( V \otimes W \) are\(^{23}\):

\[ |0\rangle_\lambda \pm \frac{1}{2} = |+\rangle_\lambda, \quad |+\rangle_\lambda = |0\rangle_\lambda, \]

or \( |0\rangle_\lambda |+\rangle \); \n
\[ |0\rangle_\lambda \pm \frac{1}{2} = |+\rangle_\lambda - q^{(\lambda + \frac{1}{2})}[2\lambda] |0\rangle_\lambda, \quad |+\rangle_\lambda \equiv |1\rangle_\lambda. \]

(5.53)

Entire Verma module is generated by the action of \( \Delta(T_-) \):

\[ |n\rangle_\lambda \pm \frac{1}{2} = (\Delta(T_-))^n |0\rangle_\lambda \pm \frac{1}{2} \rightarrow q^{n/2} \left(|+\rangle_\lambda + q^{-\frac{1}{2}}[2\lambda]|-\rangle_\lambda \right), \]

or \( q^{-n/2} \left(|+\rangle_\lambda + q^{\frac{1}{2}}[2\lambda]|-\rangle_\lambda \right); \)

\[ |n\rangle_\lambda \pm \frac{1}{2} = (\Delta(T_-))^n |0\rangle_\lambda \pm \frac{1}{2} \rightarrow q^{n/2} \left(|+\rangle_\lambda + q^{\frac{1}{2}}[2\lambda]|-\rangle_\lambda \right), \]

or \( q^{-n/2} \left(|+\rangle_\lambda + q^{-\frac{1}{2}}[2\lambda]|-\rangle_\lambda \right); \)

Step 2 to be made in accordance with our general procedure is to project the tensor product of two different \( W \)'s onto singlet state \( S = |+\rangle |-\rangle - q|+\rangle |-\rangle \);\(^{24}\)

\[ (A|+\rangle + B|-\rangle) \otimes (|+\rangle C + |-\rangle D) \rightarrow AD - qBC. \]

(5.56)

With our choice of \( W \) we can now consider two different cases:

\(^{23}\)Hereafter we omit the symbol of tensor product from the notations of the states \( |+\rangle \otimes |0\rangle_\lambda \) etc.

\(^{24}\)This state is a singlet of \( U_q(SL(2)) \). In the case of \( U_q(GL(2)) \) one should account for the \( U(1) \) non-invariance of \( S \). This is the origin of the factor \( \det_q g \) at the r.h.s. of the final equation (5.64).
Let us note that the derivation of these equations can be presented in the form (5.61)

\[ \hat{\lambda}_l \hat{\lambda}'_{l'} \langle l | n' \rangle_{\lambda' \lambda} \rightarrow q^{\frac{n' - n - 1}{2}} \left( [n' - 2\lambda] q^{\lambda'} [n + 1] \langle n' \rangle_{\lambda'} - [n - 2\lambda] q^{-\lambda} [n] \langle n' \rangle_{\lambda} \right). \]  

(5.57)

Case B:

\[ \langle n \rangle_{\lambda + \frac{1}{2}} | n' \rangle_{\lambda' - \frac{1}{2}} \rightarrow q^{\frac{n' - n + 1}{2}} \left( [n' - 2\lambda] q^{\lambda'} [n] \langle n' \rangle_{\lambda} - [n] q^{+\lambda + 1} [n - 1] \langle n' \rangle_{\lambda} \right). \]  

(5.58)

Now we proceed to the step 3. Consider any “group element”, i.e. an element \( g \) from some extension of \( U_q(G) \), which possesses the property:

\[ \Delta(g) = g \otimes g, \]  

(5.59)

and take matrix elements of the formula (5.11):

\[ \lambda \langle k' | \lambda \langle k | (g \otimes g) \Gamma = \langle n \rangle_{\lambda} \langle n' \rangle_{\lambda'}. \]  

(5.60)

The action of operator \( \Gamma \) can be represented as:

\[ \Gamma | n \rangle_{\lambda} \langle n' \rangle_{\lambda'} = \sum_{l, l'} | l \rangle_{\lambda} | l' \rangle_{\lambda'} \Gamma(l, l' | n, n'). \]  

(5.61)

and in these terms (5.60) turns into:

\[ \sum_{m, m'} \Gamma(k, k'|m, m') | \lambda \rangle_{\lambda} | k' \rangle_{\lambda'} \langle m | g | n \rangle \langle m' | g | n' \rangle_{\lambda'} \]  

(5.62)

In order to rewrite this as a difference equation, we use our definition of \( \tau \)-function:

\[ \tau_{\lambda}(t, \tilde{t} | g) \equiv (\lambda | e_q(tT^+ + \tilde{t}T^-) | \lambda) = \sum_{m, n} \langle m | g | n \rangle \lambda t^m \tilde{t}^n. \]  

(5.63)

Then, one can write down the generating formula for the equation (5.62), using the manifest form (5.57)-(5.58) of matrix elements \( \Gamma(l, l' | n, n') \):

Case A:

\[ \sqrt{M_t^+ M_t^-} \left( q^{\lambda \lambda'} D_t^{(2\lambda')} - q^{-\lambda} \tilde{t} D_t^{(2\lambda)} D_t^{(0)} \right) \tau_{\lambda}(t, \tilde{t} | g) \tau_{\lambda'}(t', \tilde{t}' | g) = \]  

\[ = [2\lambda] [2\lambda'] (\det_q^g) \sqrt{M_{-1}^- M_{1}^+} \left( q^{\lambda \lambda'} t' - q^{\lambda \lambda'} t \right) \tau_{\lambda - \frac{1}{2}}(t, \tilde{t} | g) \tau_{\lambda'}(t', \tilde{t}' | g). \]  

(5.64)

Here \( D_t^{(n)} \equiv q^{-n} M_t^+ - q^n M_t^- \) and \( \boxed{M} \) are multiplicative shift operators, \( M_t^\pm f(t) = f(q^{\pm 1} t) \).

Case B:

\[ \sqrt{M_t^+ M_t^-} \left( q^{\lambda \lambda'} D_t^{(2\lambda')} - q^{\lambda \lambda'} t' \right) \tau_{\lambda}(t, \tilde{t} | g) \tau_{\lambda'}(t', \tilde{t}' | g) = \]  

\[ = [2\lambda] [2\lambda + 1] \sqrt{M_{-1}^- M_{1}^+} \left( q^{\lambda \lambda'} t D_t^{(2\lambda + 1)} - q^{\lambda \lambda'} t' D_t^{(0)} \right) \tau_{\lambda + \frac{1}{2}}(t, \tilde{t} | g) \tau_{\lambda'}(t', \tilde{t}' | g). \]  

(5.65)

Let us note that the derivation of these equations can be presented in the form which looks even closer to conventional free-fermion formalism. It is possible to
represent operator $\Gamma$ in component form as $E^R_i \otimes E^R_\bar{j} - qE^R_\bar{i} \otimes E^L_j$, where $E_i$’s are components of the vertex operator (given by fixing different vectors from $W$). Then the equation (5.11) can be rewritten

$$\varphi\langle 0|e_q(tT_\pm) E^R_i |0\rangle\varphi = \varphi\langle 0|e_q(tT_\pm) E^R_i |0\rangle\varphi - q\varphi\langle 0|e_q(tT_\pm) E^R_i |0\rangle\varphi = \varphi\langle 0|e_q(tT_\pm) E^R_i |0\rangle\varphi - q\varphi\langle 0|e_q(tT_\pm) E^R_i |0\rangle\varphi$$

(5.66)

We can easily obtain commutation relations of $E_i$’s with generators of algebra as well as their action on vacuum states. Then, it is straightforward to commute $E_i$’s with $q$-exponentials in the expression (5.66) and represent the result of the commutation by the action of difference operators. Of course, the results (5.64) and (5.65) are reproduced in this way.

### 5.3.2 Solution to bilinear identities

In this particular case (of $SL(2)\times$) one can easily evaluate the $\tau$-function explicitly, and let us use this possibility to show how bilinear equations are satisfied. The fact that our $\tau$-function is operator-valued is of principal importance.

Let us begin from the case of $\lambda = \frac{1}{2}$. Then

$$\tau_{\frac{1}{2}}(t, \bar{t}|g) = \langle +|g|+ \rangle + \bar{t}\langle +|g|- \rangle + t\langle -|g|+ \rangle + t\bar{t}\langle -|g|- \rangle = a + b\bar{t} + ct + dt\bar{t},$$

(5.67)

where $a, b, c, d$ are elements of the matrix

$$\mathcal{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the commutation relations dictated by $\mathcal{T} \mathcal{T} \mathcal{T} = \mathcal{R} \mathcal{T} \mathcal{T}$ equation [62]

$$ab = qba,$$

$$ac = qca,$$

$$bd = qdb,$$

$$cd = qdc,$$

$$bc = cb,$$

(5.69)

$$ad - da = (q - q^{-1})bc.$$

If $b$ or $c$ or both are non-vanishing, $\tau_{\frac{1}{2}}(t, \bar{t}|g)$ with different values of time-variables $t, \bar{t}$ do not commute. Still such $\tau_{\frac{1}{2}}(t, \bar{t}|g)$ does satisfy the same bilinear identity (5.64), moreover, for this to be true it is essential that commutation relations (5.69) are exactly what they are. Indeed, the l.h.s. of the equation (5.64) is equal to

$$-q^{\frac{1}{2}}M_t^+(b + dt)\sqrt{M_{\bar{t}}^+(a + ct)} + q^{-\frac{1}{2}}\sqrt{M_t^-(a + ct)}\sqrt{M_{\bar{t}}^-(b + dt')} =$$

$$= (q^{-\frac{1}{2}}ab - q^{\frac{1}{2}}ba) + (q^{-\frac{1}{2}}cd - q^{\frac{1}{2}}dc)t\bar{t}' + (q^{-\frac{1}{2}}cb - q^{\frac{1}{2}}db)t + (q^{-\frac{1}{2}}da + q^{\frac{1}{2}}bc)t' =$$

$$= (q^{-\frac{1}{2}}t' - q^{\frac{1}{2}}t)\det_qg,$$

(5.70)

which coincides with the r.h.s. of the equation (5.64).

To perform the similar check for any half-integer-spin representation, let us note that the corresponding $\tau$-function can be easily written in terms of $\tau_{\frac{1}{2}}$. 

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Indeed,

\[ |n\rangle_\lambda = (q^{T_0} \otimes T_- + T_+ \otimes q^{-T_0})^n |0\rangle_\lambda \otimes |0\rangle_\lambda = q^{-n/2} \left( |n\rangle_\lambda \otimes |0\rangle_\lambda + [n]|_\lambda^n |n-1\rangle_\lambda \otimes |1\rangle_\lambda \right); \]

\[ \lambda \langle n | = \lambda - \frac{1}{4} \langle 0 | \otimes \frac{1}{4} \langle 0 | = (q^{T_0} \otimes T_+ + T_+ \otimes q^{-T_0})^n = q^{-n/2} \left( \lambda - \frac{1}{4} \langle n | \otimes \frac{1}{4} \langle n | + [n]q^{\lambda} \lambda - \frac{1}{4} \langle n-1 | \otimes \frac{1}{4} \langle 1 | \right). \]

Thus

\[ \lambda (k|g|n)_\lambda = q^{\frac{k}{2} + \frac{1}{4}} \left[ \lambda - \frac{1}{4} \langle k|g|n | \lambda - \frac{1}{4} \langle +|g|+ \rangle + q^{\lambda} \lambda - \frac{1}{4} \langle k|g|n-1 | \lambda - \frac{1}{4} \langle +|g| - \rangle + q^{\lambda} \lambda - \frac{1}{4} \langle k-1|g|n | \lambda - \frac{1}{4} \langle -|g| + \rangle + q^{2\lambda} |k|n | \lambda - \frac{1}{4} \langle k-1|g|n-1 | \lambda - \frac{1}{4} \langle -|g| - \rangle \right] \]

or, in terms of generating (\tau-)functions:

\[ \tau_\lambda(t, \bar{t}|g) = \sqrt{M_t M_{\bar{t}}} \left( \tau_{\lambda-\frac{1}{2}}(t, \bar{t}|g) (a + q^{\lambda}tb + q^{\lambda}tc + q^{2\lambda}t\bar{t}d) \right). \]

Applying this procedure recursively we get:

\[ \tau_\lambda(t, \bar{t}|g) = \tau_{\lambda-\frac{1}{2}}(q^{\frac{1}{2}t}, q^{\frac{1}{2}\bar{t}}|g) \tau_{\frac{1}{2}}(q^{\lambda - \frac{1}{2}t}, q^{\lambda - \frac{1}{2}\bar{t}}|g) \]

if \( \lambda \in \mathbb{Z}/2 \),

\[ \tau_{\frac{1}{2}}(q^{\frac{1}{2}t}, q^{\frac{1}{2}\bar{t}}|g) \tau_{\frac{1}{2}}(q^{\frac{1}{2}t}, q^{\frac{1}{2}\bar{t}}|g) \tau_{\frac{1}{2}}(q^{\frac{1}{2}t}, q^{\frac{1}{2}\bar{t}}|g) \ldots \tau_{\frac{1}{2}}(q^{\frac{1}{2}t}, q^{\frac{1}{2}\bar{t}}|g), \]

i.e. for half-integer \( \lambda \) \( \tau_\lambda \) is a polynomial of degree \( 2\lambda \) in \( a, b, c, d \).

For example,

\[ \tau_1(t, \bar{t}|g) = \tau_{\frac{1}{2}}(q^{\frac{1}{2}t}, q^{\frac{1}{2}\bar{t}}|g) \tau_{\frac{1}{2}}(q^{\frac{1}{2}t}, q^{\frac{1}{2}\bar{t}}|g) = \]

\[ = (a + q^{-\frac{1}{2}}tb + q^{-\frac{1}{2}}tc + q^{-1}\bar{t}d)(a + q^{\frac{1}{2}}tb + q^{\frac{1}{2}}tc + q\bar{t}d) = \]

\[ a^2 + (q^{\frac{3}{2}}ab + q^{-\frac{3}{2}}ba)\bar{t} + (q^{\frac{3}{2}}ac + q^{-\frac{3}{2}}ca)t + b^2t^2 + \]

\[ + (qad + bc + cb + q^{-1}da)\bar{t} + c^2t^2 + (q^{\frac{3}{2}}bd + q^{-\frac{3}{2}}db)t\bar{t} + \]

\[ + (q^{\frac{3}{2}}cd + q^{-\frac{3}{2}}dc)t^2\bar{t} + d^2t^2\bar{t}^2, \]

Using the relations like

\[ q^{\frac{1}{2}}ab + q^{-\frac{1}{2}}ba = [2]q^{\frac{1}{2}}ba = [2]q^{-\frac{1}{2}}ab \quad \text{etc.} \]

one gets for this case

\[ \tau_\lambda(t, \bar{t}|g) = a^2 + [2]q^{-\frac{1}{2}}ab\bar{t} + [2]q^{-\frac{1}{2}}act + b^2t^2 + ([2]qbc + [2]da)t\bar{t} + c^2t^2 + \]

\[ + [2]q^{\frac{3}{2}}db\bar{t}^2 + [2]q^{\frac{3}{2}}dct\bar{t} + d^2t^2\bar{t}^2. \]

With this explicit expression, one can trivially make the calculations similar to (5.70) in order to check manifestly equation (5.64) for \( \lambda = 1, \lambda' = \frac{1}{2}, 1 \) and equation (5.65) for \( \lambda = \lambda' = \frac{1}{2} \).

Thus, we showed explicitly (for the case of \( SL_q(2) \)) that the quantum bilinear identities have as many solutions as the classical ones, provided the \( \tau \)-function is allowed to take values in non-commutative ring \( A(G) \).

### 5.4 Comments on the quantum deformation of KP/Toda \( \tau \)-functions

As we saw in the previous subsection, the generic construction is easily applicable to quantum groups, still the problem of quantum KP/Toda hierarchies deserves separate consideration and is not yet fully resolved. The problem is, that the evolution operator \( U(t) \) in (5.2) is usually constructed from all the operators of the algebra, not just from a commuting set - as it happens in particular case of fundamental representations of \( GL(N) \). As result generic evolution of \( \tau \) with variation of \( t \)'s is not described as a set of commuting flows, rather they form a closed, but non-trivial, algebra. This manifests itself also in the fact that naturally the number of independent time-variables is rather close to dimension than to the rank of the group. The problem of quantum deformation of
KP/Toda hierarchy is to find a deformation which, while dealing with \( \tau \)-function for quantum group, is still describable in terms of few time-variables. If at all resolvable this is the problem of a clever choice of the weight functions \( s^R_{m,n}(t,\bar{t}) \) in (5.1). Following [56] we shall now demonstrate that the problem is resolvable in principle, though at the moment it is a quantum deformation of somewhat non-conventional description of KP/Toda system (with evolution, introduced differently from that in s.5.2, and it is not just a change of time-variables: the transformation is representation \( R \)-dependent).

According to [57] parametrization of group elements which allows the most straightforward quantum deformation involves only simple roots \( \pm \vec{\alpha}_i \), \( i = 1, \ldots, r_G \):

\[
g = g_U g_D g_L, \quad g_U = \prod_s e^{\theta_s T(s)}, \quad g_L = \prod_s e^{\chi_s T_{-s}(i)}, \quad g_D = \prod_{i=1}^{r_G} e^{\bar{\phi}_i H} \tag{5.78}
\]

Every particular simple root \( \vec{\alpha}_i \) can appear several times in the product, and there are different parametrizations of the group elements of such type, depending on the choice of the set \( \{s\} \) and the mapping \( i(s) \). Quantum deformation of such formula is especially simple because comultiplication rule is especially simple for generators, associated with simple roots:

\[
\Delta(T_i) = T_i \otimes \tau^{-2H_i} + I \otimes T_i, \quad \Delta(T_{-i}) = T_{-i} \otimes I + q^{2H_i} \otimes T_{-i} \tag{5.79}
\]

For \( q \neq 1 \) any expression of the form (5.78) remains just the same, provided exponents in \( g_U \) and \( g_L \) are understood as \( q \)-exponents (in the simply-laced case, \( q^{||\vec{\alpha}_i||^2/2} \)-exponents in general), and parameters \( \psi, \chi, \bar{\phi} \) become non-commuting generators of “coordinate ring” \( \mathcal{A}(G_q) \). Actually they form a kind of a very simple Heisenberg-like algebra:

\[
\theta_s \theta_{s'} = q^{-\vec{\alpha}_{i(s)}^T \vec{\alpha}_{i(s')}} \theta_s \theta_{s'}, \quad s < s', \quad \chi_s \chi_{s'} = q^{-\vec{\alpha}_{i(s)}^T \vec{\alpha}_{i(s')}} \chi_s \chi_{s'}, \quad s < s', \tag{5.80}
\]

\[
q^{\vec{\alpha}_i} \theta_s = \theta_s q^{\vec{\alpha}_i \vec{\alpha}_{i(s)}}, \quad q^{\vec{\alpha}_i} \chi_s = \chi_s q^{\vec{\alpha}_i \vec{\alpha}_{i(s)}}
\]

These relations imply that \( \Delta(g) = g \otimes g \).

The simplest possible assumption about evolution operators would be to say that, just as it was in the case of the standard KP/Toda theory (see s.5.2), \( U(t) \) is always an object of the type \( g_U \), while \( \bar{U}(\bar{t}) \) - of the type \( g_L \). However, these are no longer group elements:

\[
\Delta(g_U) \neq g_U \otimes g_U, \quad \Delta(g_L) \neq g_L \otimes g_L,
\]

because of the lack of factors \( g_D \). Still the simplest possibility is to insist on identification of \( U \) and \( \bar{U} \) as objects of the type \( g_U \) and \( g_L \) respectively, and explicitly investigate implications of the failure of (5.3). As result one obtains instead of (5.3)

\[
\Delta(U(\xi)) = U_L^{(2)}(\xi) \cdot U_R^{(2)}(\xi), \tag{5.81}
\]

where

\[
U(\xi) = \prod_s \xi_s, \quad U_L^{(2)} = \prod_s \xi_s \cdot U_R^{(2)} = \prod_s \xi_s \cdot I = I \otimes U(\xi), \tag{5.82}
\]

and this has some simply accountable implications for determinant formulas for quantum \( \tau \)-functions.
In s.5.2 we essentially used an evolution operator of the type

\[ U(\xi) = \prod_{1 \leq i \leq N} \prod_{1 \leq j < i} \exp (\xi_{ij} T_{i-j}) \]  \hspace{1cm} (5.83)

where \( \xi_{ij} \) are certain functions of only \( N \) independent variables \( t \). While (5.83) is trivial to deform in the direction of \( q \neq 1 \), it is a separate (yet unresolved) problem to find such reduction to only \( N \)-variables, consistent with the commutation relations between \( \xi_{ij} \),

\[ \xi_{ij} \xi_{j'k'} = q^{-\tilde{\alpha}_{ij} \tilde{\alpha}_{j'k'}} \xi_{ij} \xi_{j'k'}, \quad \{ i, j \} < \{ i', j' \}. \]

One can instead use a much simpler evolution,

\[ \hat{U}(\xi) = \prod_{i=1}^{r_G} \exp \left( \xi_i T_i \right) \]  \hspace{1cm} (5.84)

This is enough to generate all the states of any fundamental representation from the corresponding vacuum (highest vector) state, but \(<\text{vac}_{F_n}| U(\xi) \) has nothing to do with the usual \(<\text{vac}_{F_n}| U(t) \), where \( U(t) \) is given by (5.83). It can be better to say, that identification \(<\text{vac}_{F_n}| U(\xi) = <\text{vac}_{F_n}| U(t) \) defines a relation \( \xi_i(t) \), which explicitly depends on \( n \).

One can of course build the theory of KP/Toda hierarchies in terms of \( \xi \)-variables instead of conventional \( t \)-variables, but it can not be obtained by just change of time-variables: the whole construction will look different. Instead this new construction is immediately deformed to the case of \( q \neq 1 \): instead of (5.84) we just write

\[ \hat{U}(\xi) = \prod_{i=1}^{r_G} e_\xi (\xi_i T_i) \]  \hspace{1cm} (5.85)

where \( \xi \)'s are non-commuting variables,

\[ \xi_i \xi_j = q^{-\tilde{\alpha}_{ij}} \xi_j \xi_i, \quad i < j, \]  \hspace{1cm} (5.86)

and it is easy to derive quantum counterpart of any statement of the classical \((q=1) \) theory once it is formulated for \( \xi \)-parametrization.

In what follows we first briefly describe the conventional KP/Toda hierarchy in this non-standard parametrization, then consider the corresponding quantum deformation and derive the substitute of determinant formulas for \( \tau_n \equiv \tau_{F_n} \) in the case of \( q \neq 1 \).

5.4.1  On the modified KP/Toda hierarchy

Our first purpose is to demonstrate that all the main ingredients of description of the classical KP/Toda hierarchy, as described in s.5.2, are preserved if evolution (5.85) is used instead of (5.83), in particular, there are determinant formulas and a hierarchy of differential equations.

From now on we denote the \( \tau \)-function associated with the evolution (5.85) through \( \tau(\xi, \xi|g) \). This \( \tau \)-function is linear in each time-variable \( \xi_i \), hence, it satisfies simpler determinant formulas and simpler hierarchy of equations. Indeed, now we have

\[ \hat{\tau}(\xi, \xi|g) = <0_{F_1}| \hat{U}(\xi) g \hat{U}(\xi)|0_{F_1} > = \sum_{k,k \geq 0} s_k \bar{s}_k < k|g|\bar{k} > \]  \hspace{1cm} (5.87)

where \( s_k = \xi_1 \xi_2 \ldots \xi_k \), \( s_0 = 1 \), and

\[ \hat{\tau}_{1m}^{mm}(\xi, \xi|g) = <m_{F_1}| \hat{U}(\xi) g \hat{U}(\xi)|m_{F_1} > = \frac{1}{s_m s_{\bar{m}}} \sum_{k \geq m} s_k \bar{s}_k < k|g|\bar{k} > = \frac{1}{s_m s_{\bar{m}}} \sum_{k \geq m} \frac{\partial}{\partial \log s_k} \frac{\partial}{\partial \log \bar{s}_k} \tau_{1m}(\xi, \xi|g) = \frac{1}{s_m} \frac{\partial}{\partial \xi_m} \frac{\partial}{\partial \bar{\xi}_m} \tau_{1m}(\xi, \xi|g). \]  \hspace{1cm} (5.88)
Thus,\(^{25}\)

\[ \tilde{\tau}_{n+1} = \begin{pmatrix} \prod_{m=1}^{n} s_m \hat{s}_{\hat{m}} \end{pmatrix}^{-1} \det_{m\hat{m}} \left( \sum_{k \geq m}^{n} s_k \hat{s}_{\hat{k}} < k | g | \hat{k} > \right) = \]

\[ = \frac{1}{s_n \hat{s}_n} \sum_{k, \hat{k} \geq n} s_k \hat{s}_{\hat{k}} \prod_{m=1}^{n} s_m \hat{s}_{\hat{m}} \]

\[ = \frac{1}{s_n \hat{s}_n} \sum_{k, \hat{k} \geq n} s_k \hat{s}_{\hat{k}} \left( \begin{pmatrix} g_{m\hat{m}} & g_{m\hat{k}} \\ g_{k\hat{m}} & g_{m\hat{m}} \end{pmatrix} \right) \equiv \frac{1}{s_n \hat{s}_n} \sum_{k, \hat{k} \geq n} s_k \hat{s}_{\hat{k}} D_{kk}^{(n)}. \quad (5.89) \]

### 5.4.2 \(q\)-Determinant-like representation

In this section we demonstrate how the technique developed in the previous sections is deformed to the quantum case and, in particular, obtain \(q\)-determinant-like deformation of (5.89). Our evolution operator (5.85) satisfies the following comultiplication rule:

\[ \Delta^{n-1}(U\{T_i\}) = \prod_{m=1}^{n} U^{(m)} \quad (5.90) \]

where

\[ U^{(m)} = U \left\{ I \otimes \ldots \otimes I \otimes \xi_i T_i \otimes q^{-2H_i} \otimes \ldots \otimes q^{-2H_i} \right\} \quad (5.91) \]

\(^{25}\) One can compare determinant representations (5.41) and (5.89) to find the connection between different coordinates \(t\) and \(\xi\). For every given \(n\) the variables \(s_k\) are some functions of \(P_j(t)\). For example, in the simplest case of the first fundamental representation \(F^{(1)}\) we have \(\tau_1(t|g) = \tau(t|g)\) and \(s_k = P_k(t)\). However, identification of \(\tau_n(t)\) and \(\tilde{\tau}_n(\xi)\) with \(n \neq 1\) will lead to different relations between \(\xi\) and \(t\). Thus the two different evolutions are not related just by a change of time-variables, relation is representation-dependent, and can not be lifted to the actual KP/Toda case when \(n = \infty\). Two evolutions provide two equally nice, but not just equivalent descriptions of the same hierarchy.

and \(T_i\) appears at the \(m\)-th place in the tensor product. Similarly

\[ \tilde{U}^{(m)} = \tilde{U} \left\{ q^{2H_i} \otimes \ldots \otimes q^{2H_i} \otimes T_{-i} \otimes I \otimes \ldots \otimes I \right\}. \quad (5.92) \]

Now let us transform the operator-valued \(q\)-factors into \(c\)-number ones. Let

\[ H_i |j_{F_1} > = h_{i,j} |j_{F_1} >, \quad < j_{F_1} |H_i = h_{i,j} < j_{F_1} | \]

(in fact for \(SL(N)\) \(2h_{i,i-1} = +1, 2h_{i,i} = -1\), all the rest are vanishing). Then

\[ \tilde{\tau}^{j_1 \ldots j_n \hat{j}_1 \ldots \hat{j}_m} (\xi_i, \hat{\xi}_i | g) \equiv \]

\[ \equiv (\otimes_{m=1}^{n} < j_m |) \Delta^{n-1}(U) g^{\otimes n} \Delta^{-1}(\tilde{U}) (\otimes_{m=1}^{n} | j_m >) = \]

\[ = \prod_{m=1}^{n} < j_m | U \left\{ \xi_i T_i q^{-2 \sum_{i=m+1}^{n} h_{i,j_i}} \right\} g \tilde{U} \left\{ \xi_i T_{-i} q^{2 \sum_{i=1}^{n-1} h_{i,j_i}} \right\} | j_m > = \]

\[ = \prod_{m=1}^{n} \tilde{\tau}^{j_1 \ldots \hat{j}_m} (\xi_i q^{-2 \sum_{i=m+1}^{n} h_{i,j_i}}, \xi_i q^{2 \sum_{i=1}^{n-1} h_{i,j_i}}). \quad (5.93) \]

In order to get the analogue of (5.41), one should replace antisymmetrization by \(q\)-antisymmetrization, since, in quantum case, fundamental representations are described by \(q\)-antisymmetrized vectors. We define \(q\)-antisymmetrization as a sum over all permutations,

\[ ([1, \ldots , k]_q) = \sum_{P} (-q)^{\text{deg} P} P(P(1), \ldots , P(k)), \quad (5.94) \]

where

\[ \text{deg} P = \# \text{ of inversions in } P. \quad (5.95) \]

Then, \(q\)-antisymmetrizing (5.93) with \(j_k = k - 1, \hat{j}_k = \hat{k} - 1\), one finally gets
A commutation relations of the matrix elements $j_i$

Note that this is not obligatory the same as $1|q_n$. It is the same only for peculiar $a_i$.$q$ $A_i \sim A_i$ $\det q$ $a_i$($P_1$) $q$ $\sum_{l=0}^{n-1} h_{i(s),P_1 (l)}$)

This would be just a $q$-determinant, be there no $q$-factors which twist the time-variables.

To make this expression more transparent let us consider the simplest example of the second fundamental representation:

$$
\tau_2 = \tau_1^{00} (q\xi_1, q^{-1}\xi_2, \xi_i; \xi_1, \xi_2, \xi_i) \tau_1^{11} (q\xi_1, q^{-1}\xi_2, \xi_i; q\xi_1, q\xi_2, \xi_i) -
- q\tau_1^{01} (q\xi_1, q^{-1}\xi_2, \xi_i; \xi_1, \xi_2, \xi_i) \tau_1^{10} (q\xi_1, q^{-1}\xi_2, \xi_i; q\xi_1, q\xi_2, \xi_i) -
- q^2 \tau_1^{11} (q^{-1}\xi_1, q\xi_2, \xi_i; \xi_1, \xi_2, \xi_i) \tau_1^{00} (q^{-1}\xi_1, q\xi_2, \xi_i; q^{-1}\xi_1, q\xi_2, \xi_i) +
+ q^2 \tau_1^{11} (q^{-1}\xi_1, q\xi_2, \xi_i; \xi_1, \xi_2, \xi_i) \tau_1^{00} (q^{-1}\xi_1, q\xi_2, \xi_i; q^{-1}\xi_1, q\xi_2, \xi_i).
$$

This can be written in a more compact form with the help of operators

$$
\mathbb{D}^L_i \equiv D_i \otimes I, \quad \mathbb{D}^R_i \equiv \prod_j M_j^{-\vec{a}_i} \otimes D_i,
$$

$$
\mathbb{D}^L_i \equiv D_i \otimes \prod_j M_j^{-\vec{a}_i}, \quad \mathbb{D}^R_i \equiv I \otimes D_i.
$$

These operators have simple commutation relations:

$$
\mathbb{D}^L_i \mathbb{D}^R_j = q^{\vec{a}_i \vec{a}_j} \mathbb{D}^R_j \mathbb{D}^L_i,
$$

$$
\mathbb{D}^L_i \mathbb{D}^R_j = q^{\vec{a}_i \vec{a}_j} \mathbb{D}^R_j \mathbb{D}^L_i.
$$

Then,

$$
\tau_2 = (M^+_1 \otimes M^+_1) \left( \mathbb{D}^R_1 - q \mathbb{D}^L_1 \right) \cdot \left( \mathbb{D}^R_1 - q \mathbb{D}^L_1 \right) \tau_1 \otimes \tau_1.
$$

6 Conclusion

These notes combine presentation of some well established facts with that of more recent and sometime disputable speculations. There are all reasons to believe that further developments will prove that the theory of generalized $\tau$-functions and non-perturbative partition functions can become a flourishing branch of mathematical physics with applications well beyond the present modest scope of topological theories and $c < 1$ string models and with profound relations to other fields of the string theory. It is also important that there are plenty of “small problems” at all the levels of this theory, which are enjoyable to think about.

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