SHADOW SYSTEM APPROACH TO A PLANKTON MODEL
GENERATING HARMFUL ALGAL BLOOM

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Abstract. Spatially localized blooms of toxic plankton species have negative
impacts on other organisms via the production of toxins, mechanical damage,
or by other means. Such blooms are nowadays a worldwide spread environmental
issue. To understand the mechanism behind this phenomenon, a two-
prey (toxic and nontoxic phytoplankton)-one-predator (zooplankton) Lotka-
Volterra system with diffusion has been considered in a previous paper. Nu-
merical results suggest the occurrence of stable non-constant equilibrium solu-
tions, that is, spatially localized blooms of the toxic prey. Such blooms appear
for intermediate values of the rate of toxicity $\mu$ when the ratio $D$ of the dif-
fusion rates of the predator and the two prey is rather large. In this paper,
we consider a one-dimensional limiting system (we call it a shadow system) in
$(0, L)$ as $D \to \infty$ and discuss the existence and stability of non-constant equi-
librium solutions with large amplitude when $\mu$ is globally varied. We also show
that the structure of non-constant equilibrium solutions sensitively depends on
L as well as $\mu$.

1. Introduction. The term harmful algal bloom (in short, HAB) indicates an algal
bloom that has negative impacts on other organisms via the production of toxins,
mechanical damage, or by other means. HAB includes different types of taxa such as
dinoflagellates, diatoms, and cyanobacteria (commonly known as blue-green algae).
The latter are of special importance because of their potential impact on drinking
or recreational waters. In fact, they can produce a variety of potent toxins called
cyanotoxins (e.g. Falconer and Humpage [3]). These compounds have been found
to be hepatotoxic or neurotoxic for a wide range of organisms, including humans,
and several intoxication cases have been reported worldwide (Jochimsen et al. [2]).
Therefore, in the recent years, the formation of toxic blooms of cyanobacteria in
lakes and rivers has been causing more and more concern. Ecological evidence sug-
gest that toxic and nontoxic species of freshwater phytoplankton hardly coexist in
absence of other species. In particular, competition experiments have shown that

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the toxic strain of Microcystis is a very poor competitor for light (Kardinaal et al. [9]). In these experiments the toxic strain always lost the competition against the nontoxic one, even when given a strong initial advantage. Then a natural question is: how can these species survive and actually bloom? Regarding this question, it is ecologically discussed that toxin-producing Microcystis has overall an inhibitory effect on the growth of most herbivore taxa. Nevertheless, zooplankton usually grazes on both toxic and nontoxic species (Fulton and Pearl [5]). This is interesting, since the toxic or noxious chemicals produced by blue green algae may inhibit feeding and, over long term, cause mortality of zooplankton (Porter and Orcutt [13], Lampert [10], [11], Fulton and Paerl [4]). In particular, while a few species like the rotifer Brachionus calyciorus and the cladoceran Bosmina longirostris apparently make no great distinction between toxic and non toxic prey, the feeding rates of other small-bodied cladocerans, rotifers, and copepods seem to be strongly related to the toxicity of Microcystis (Fulton and Paerl [4]). These observations suggest that predator and toxic prey have an inhibitory effect on each other. Then another naive question is: can the existence of such interaction promote the spatial pattern formation and local algal blooms? This ecological question motivates us to theoretically understand the mechanism behind the formation of spatial blooms of toxic plankton. For this purpose, we proposed a two-prey (toxic and nontoxic phytoplankton)-one-predator (zooplankton) Lotka-Volterra system with diffusion in a previous paper Scotti et al. [14], under the assumptions that (i) in absence of the predator, the toxic prey is a weaker competitor for common resources than the nontoxic one, and (ii) depending on the toxicity and another parameter, the predator is more or less inclined to avoid the toxic prey. The dynamics discussed above are described by the following reaction-diffusion system:

\[
\begin{align*}
\frac{\partial P}{\partial t} &= r_1 P \left(1 - \frac{P + aX}{K_1} - Z\right) + D_1 \Delta P \\
\frac{\partial X}{\partial t} &= r_2 X \left(1 - \frac{X + bP}{K_2} - d\mu Z\right) + D_2 \Delta X \\
\frac{\partial Z}{\partial t} &= r_3 Z (P - \mu X - 1) + D_3 \Delta Z,
\end{align*}
\]

where \(P(t, x), X(t, x)\) and \(Z(t, x)\) are respectively the population densities of the nontoxic and toxic phytoplankton and of the zooplankton, and the parameters \(r_i, D_i (i = 1, 2, 3), K_1, K_2, a, b, d\) and \(\mu\) are all positive constants. Ecologically, an important parameter in (1) is \(\mu\), which is the rate of toxicity. We assume that the predation rate of the toxic prey \(d\) decreases in \(\mu\). Here we simply specify \(d = d(\mu) = 1/(1 + \mu)\). The ecological explanation of (1) is discussed in [14].

By using suitable transformations of time and space, (1) can be rewritten as

\[
\begin{align*}
\frac{\partial P}{\partial t} &= P \left(1 - \frac{P + aX}{K_1} - Z\right) + \Delta P \\
\frac{\partial X}{\partial t} &= rX \left(1 - \frac{X + bP}{K_2} - d(\mu)Z\right) + \sigma \Delta X \\
\frac{\partial Z}{\partial t} &= RZ(P - \mu X - 1) + D \Delta Z,
\end{align*}
\]
where \( r = r_2/r_1 \), \( R = r_3/r_1 \), \( \sigma = D_2/D_1 \) and \( D = D_3/D_1 \) are respectively the ratios of the growth rates and the diffusion rates of the nontoxic phytoplankton and the toxic phytoplankton and/or the zooplankton.

2. **A mathematical model.** In this paper, simply assuming that \( r = 1 \), \( K_1 = K_2 = K \) and \( \sigma = 1 \) in (2), we consider the one dimensional system of (2) in a finite interval \( 0 < x < L \), that is,

\[
\begin{align*}
\frac{dP}{dt} &= P \left( 1 - \frac{P + aX}{K} - Z \right) + P_{xx} \\
\frac{dX}{dt} &= X \left( 1 - \frac{X + bP}{K} - d(\mu)Z \right) + X_{xx} \quad t > 0, \ 0 < x < L \\
\frac{dZ}{dt} &= RZ(P - \mu X - 1) + DZ_{xx},
\end{align*}
\]

where \( a, b, K, \mu, R \) and \( D \) are positive constants. We treat (3) under the Neumann boundary conditions

\[
P_x = X_x = Z_x = 0, \quad t > 0, \ x = 0, L
\]

and the initial conditions

\[
P(0, x) = P_0(x) \geq 0, \ X(0, x) = X_0(x) \geq 0, \ Z(0, x) = Z_0(x) \geq 0, \ 0 \leq x \leq L.
\]

For the system (3), we impose the following assumptions:

(A1) \( K > 1 \),

which implies that predator and nontoxic prey coexist in the absence of toxic prey.

(A2) \( a < 1 < b \),

which implies that in the absence of predator, the nontoxic prey is a superior competitor for resources with respect to the toxic one. Assumption (A1) corresponds to the coexistence of two species of (nontoxic) phytoplankton and zooplankton (e.g. Hutchinson [6]). Assumption (A2) is based on the idea that toxic strains are eventually outcompeted by nontoxic ones (e.g. Kardinaal et al. [9], Lampert [10]).

For (3) with (4), we easily find that the spatially constant equilibrium solution \( E_3 = (1, 0, (K - 1)/K) \) exists for any \( \mu > 0 \). Instead of (A1), we assume \( K \) to satisfy

(A3) \( K > b \).

Putting \( \mu_c = (b - 1)/(K - b) \), we know that \( E_3 \) is stable for \( 0 < \mu < \mu_c \), while it is unstable for \( \mu_c < \mu \). Furthermore, when \( \mu \) increases, a spatially positive constant equilibrium solution of (3) with (4), say, \( E_4 = (P_\mu, X_\mu, Z_\mu) \) super-critically bifurcates from \( E_3 \) at \( \mu = \mu_c \), that is, \( E_4 \) exists for \( \mu > \mu_c \). Here we assume \( R \) suitably large to require that \( E_4 \) is stable for \( \mu > \mu_c \) in the ODEs corresponding to (3) in the absence of diffusion. In addition to (A2) - (A3), we assume \( \mu \) to satisfy

(A4) \( \mu > \mu_c \).

Here we note that \( E_4 \) is not necessarily stable in (3) with (4), that is, the stability of \( E_4 \) depends on \( \mu, R, D \) and \( L \). Figures (1a) and (b) show respectively the linearized stability of \( E_4 \) in the \((D, \mu)\) plane \((L = 30)\) and \((L, \mu)\) plane \((D = 2500)\) for suitably fixed \( a, b, K \) and \( R \). When \( \mu(> \mu_c) \) is small, \( E_4 \) is stable for any fixed \( D \) (or \( L \)), while when \( \mu \) is suitably large, \( E_4 \) loses its stability as \( D \) (or \( L \)) increases. This
Figure 1. Bifurcation curves of $E_4$ in the $(D, \mu)$-plane ($L = 30$) (a) and the $(L, \mu)$-plane ($D = 2500$) (b), where $a = 0.95$, $b = 1.2$, $K = 2.9$ and $R = 0.43$. The curve $n$ corresponds to the $n$-mode bifurcations, where the zero solution of the linearized problem of (3) with (4) around $E_4$ destabilizes under the $n$th eigenmode $\cos(n\pi x/L)$ perturbation.

Figure 2. Global structure of equilibrium solutions of (3) with (4) when $\mu$ is varied, where $L = 30$, $D = 2500$. The other parameters are the same as the ones in Figure 1. Solid (resp. dashed) lines represent stable (resp. unstable) equilibrium solutions of (3) with (4). The right figure is a magnification of the left one where $\mu$ is close to $\mu_{c1}$.

Destabilization is called diffusion-induced instability as stated by Turing ([15]). In fact, suppose that $D$ is suitably large (for instance, $D = 2500$). By using AUTO, we can show the global structure of equilibrium solutions of (3) with (4) when $\mu$ is varied (see Figure 2). This figure shows the occurrence of two bifurcation values of...
Figure 3. 1-mode equilibrium solutions \( (\bar{P}_1^+(x), \bar{X}_1^+(x), \bar{Z}_1^+(x)) \) of (3) with (4) for (a) \( \mu = 0.15 \), (b) \( \mu = 0.5 \) and (c) \( \mu = 3.1 \). The other parameters are the same as the ones in Figure 1 and \( D = 2500 \). Here \( P_1^+, X_1^+ \) and \( Z_1^+ \) are drawn in blue, green and red colors, respectively.

Figure 4. Dependency of \( D \) on the global structures of equilibrium solutions of the system (3) with (4) when \( L = 30 \). (a) \( D = 800 \), (b) \( D = 1500 \), (c) \( D = 2500 \), (d) \( D = 5000 \), (e) \( D = 10000 \) and (e’) is a magnification of (e) around \( \mu = \mu_c1 \). The other parameters are the same as the ones in Figure 1. Solid (resp. dashed) lines represent stable (resp. unstable) equilibrium solutions of (3) with (4).

\( \mu \), say \( \mu_{c1} \) and \( \mu_{c2} \) \((\mu_{c1} < \mu_{c2})\), at which spatially non-constant 1-mode equilibrium solutions, say \((\bar{P}_1^+(x), \bar{X}_1^+(x), \bar{Z}_1^+(x))\) super-critically bifurcate from \( E_1 \), and exist and are stable for \( \mu_{c1} < \mu < \mu_{c2} \), where \((\bar{P}_1^+(x), \bar{X}_1^+(x), \bar{Z}_1^+(x))\) is shown in Figure 3 and \((\bar{P}_1^-(x), \bar{X}_1^-(x), \bar{Z}_1^-(x))\) is given by \((\bar{P}_1^+(L-x), \bar{X}_1^+(L-x), \bar{Z}_1^+(L-x))\). Here a 1-mode equilibrium solution stands for a new non-constant equilibrium solution which appears via the 1-mode bifurcation. These numerical results indicate that
a HAB (stable non-constant equilibrium solutions with large amplitude) does not appear for either small or large \( \mu \), while it appears for intermediate \( \mu \).

We now address the following question: can we show the existence and stability of such non-constant equilibrium solutions of (3) and (4) analytically? One of the ways is to assume that \( D \) is rather large in (3) so that the problem (3) with (4) and (5) becomes simpler, because one can expect \( Z(t,x) \) to be spatially homogeneous.

In Figure 4, we show the dependency of \( D \) on the structures of equilibrium solutions of (3) with (4) when \( \mu \) is globally varied. We notice that they do not qualitatively change if \( D \) becomes larger and larger, as shown in Figures 4(c) - (e). This result motivates us to study the limiting system of (3) with (4) as \( D \to \infty \) in order to discuss the existence and stability of non-constant equilibrium solutions.

3. The shadow system. We formally derive the limiting system from (3) with (4) as \( D \to +\infty \). We first start by dividing the third equation of (3) by \( D \) so that we obtain

\[
\frac{1}{D} Z_t = \frac{R}{D} Z(P - \mu X - 1) + Z_{xx}. \tag{6}
\]

If we assume that \( Z_t, P, X \) and \( Z \) remain bounded as \( D \to +\infty \), then (6) implies that the limit of \( Z \), say, \( \xi \) satisfies

\[
\xi_{xx} = 0. \tag{7}
\]

It follows from (7) and the boundary conditions (4) that \( \xi \) must be spatially constant. On the other hand, integrating the equation of \( Z \) in (3) on the interval \([0, L]\) with respect to \( x \), we get

\[
\frac{\partial}{\partial t} \int_0^L Z \, dx = \int_0^L RZ(P - \mu X - 1) \, dx \tag{8}
\]

by the boundary conditions (4) and then

\[
L \xi_t = R \xi \left( \int_0^L P \, dx - \mu \int_0^L X \, dx - L \right),
\]

that is,

\[
\xi_t = R \xi \left( \frac{1}{L} \int_0^L P \, dx - \mu \frac{1}{L} \int_0^L X \, dx - 1 \right)
\]

by (7) and (5). Consequently when \( D \to +\infty \), we formally obtain the following limiting system, which is called a shadow system, of (3) with (4) for the unknowns \((P(t,x), X(t,x), \xi(t))\) :

\[
\begin{cases}
    P_t = P \left( 1 - \frac{P + aX}{K} - \xi \right) + P_{xx} \\
    X_t = X \left( 1 - \frac{X + bP}{K} - d(\mu)\xi \right) + X_{xx}, \quad t > 0, \ 0 < x < L \\
    \xi_t = R \xi \left( \frac{1}{L} \int_0^L P \, dx - \mu \frac{1}{L} \int_0^L X \, dx - 1 \right)
\end{cases} \tag{9}
\]

with

\[
(P_x, X_x)(t, 0) = (0, 0) = (P_x, X_x)(t, L), \quad t > 0. \tag{10}
\]

In order to obtain equilibrium solutions of (9) with (10), we first assume \( \xi \) to be suitably fixed and consider the first two equations of (9) with (10), which are the well known competition-diffusion system for two species. If we could find an
equilibrium solution \((\bar{P}(x; \xi), \bar{X}(x; \xi))\) of the system and substitute it into the third equation of (9), we obtain the equation of \(\xi\) only

\[ H(\xi) \equiv \frac{1}{L} \int_0^L \bar{P} dx - \frac{\mu}{L} \int_0^L \bar{X} dx - 1 = 0. \] (11)

If we can find \(\bar{\xi}\) to satisfy (11), we get an equilibrium solution \((\bar{P}(x; \bar{\xi}), \bar{X}(x; \xi), \bar{\xi})\) of the shadow system (9) with (10).

In order to obtain equilibrium solutions of the competition-diffusion system for two species, we can apply the results by Kan-on [8]. For this purpose, we introduce a new parameter \(\varepsilon = 1/\sqrt{L^2}\) into (9) with (10) to normalize the interval \([0, L]\) to the unit one \([0, 1]\). Then, the shadow system (9) with (10) is rewritten to the rescaled normal system with a parameter \(\varepsilon\) as follows:

\[
\begin{align*}
  P_t &= P \left( 1 - \frac{P + aX}{K} - \xi \right) + \varepsilon P_{xx} \\
  X_t &= X \left( 1 - \frac{X + bP}{K} - d(\mu)\xi \right) + \varepsilon X_{xx}, \quad t > 0, \; 0 < x < 1 \\
  \xi_t &= R\xi \left( \int_0^1 P dx - \mu \int_0^1 X dx - 1 \right)
\end{align*}
\] (12)

with

\[(P_x, X_x)(t, 0) = (0, 0) = (P_x, X_x)(t, 1), \quad t > 0.\] (13)

In this section, we fix \(\mu\) arbitrarily to satisfy \(\mu > \mu_c\) and ignore the \(\mu\)-dependency of equilibrium solutions of the shadow system (9) with (10).

3.1. 2-component competition system of (12) with (13) for fixed \(\xi\). In this subsection, we look for equilibrium solutions of the following 2-component competition system of (12) with (13) for a fixed \(\xi\) when \(\varepsilon\) is a free parameter:

\[
\begin{align*}
  P_t &= P \left( 1 - \frac{P + aX}{K} - \xi \right) + \varepsilon P_{xx} \\
  X_t &= X \left( 1 - \frac{X + bP}{K} - d(\mu)\xi \right) + \varepsilon X_{xx}, \quad t > 0, \; 0 < x < 1 \\
  (P_x, X_x)(t, 0) &= (0, 0) = (P_x, X_x)(t, 1), \quad t > 0.
\end{align*}
\] (14)

Here we note that, depending on the parameter values \(a, b, K, \mu\) and \(\xi\), the dynamics of solutions of (14) possesses three different cases, (a) mono stability, (b) bistability and (c) coexistence, as shown in Figure 5. Since our concern is to find non-constant equilibrium solutions of (14), we restrict values of the parameters \(a, b, K, \mu\) and \(\xi\) to satisfy the case (b) in Figure 5 because this is the only case for which such solutions exist (Zhou and Pao [17], Kan-On [8]). Figure 5(b) is equivalent to assume

\((A5)\) \(ab > 1\),

which is identical to \(0 < \frac{1 - a}{1 - ad(\mu)} < \frac{b - 1}{b - d(\mu)} < 1\). We define the interval \(I\) by

\[ I = (\xi, \bar{\xi}) \text{ with } \xi = \frac{1 - a}{1 - ad(\mu)} \text{ and } \bar{\xi} = \frac{b - 1}{b - d(\mu)} \] (15)
and assume $\xi$ to satisfy $\xi \in I$. Then the system (14) has a positive constant equilibrium solution $(\bar{P}(\xi), \bar{X}(\xi))$ which is explicitly represented as

$$
\bar{P}(\xi) = \frac{(1 - a - (1 - ad(\mu))\xi)K}{1 - ab}, \quad \bar{X}(\xi) = \frac{(1 - b + (b - d(\mu))\xi)K}{1 - ab}.
$$

(16)

**Remark 1.** We easily find that

$$
\lim_{\xi \to \xi} (\bar{P}(\xi), \bar{X}(\xi)) = \left(0, \frac{(1 - d(\mu))K}{1 - ad(\mu)}\right) \quad \text{and} \quad \lim_{\xi \to \xi} (\bar{P}(\xi), \bar{X}(\xi)) = \left(\frac{(1 - d(\mu))K}{b - d(\mu)}, 0\right).
$$

Taking $\xi$ as a bifurcation parameter, we look for non-constant equilibrium solutions of (14), which are bifurcated from the constant solution $(\bar{P}(\xi), \bar{X}(\xi))$. By simple calculation, it can be easily checked that for arbitrarily fixed $\xi \in I$, the linearized operator of (14) around $(\bar{P}(\xi), \bar{X}(\xi))$ has the zero eigenvalue at $\xi = \varepsilon_0(\xi) =

![Figure 5. Three different structures of the nullclines of (14) with $d = d(\mu) = 1/(1 + \mu)$. (a-1) P-monostability, (a-2) X-monostability, (b) bistability and (c) coexistence. The red and white circles stand for stable and unstable equilibrium solutions of (14), respectively.](image-url)
and the corresponding eigenfunction is an eigenvector of the matrix

\[ A(\xi) = \begin{bmatrix} \pi^2 \varepsilon_0^2(\xi) + \bar{P}(\xi)/K & a\bar{P}(\xi)/K \\ b\bar{X}(\xi)/K & \pi^2 \varepsilon_0^2(\xi) + \bar{X}(\xi)/K \end{bmatrix} \]

(18)
corresponding to the zero eigenvalue. Here we used the relation \( \varepsilon_0^0(\xi) = \varepsilon_0(\xi)/n^2 \) for each \( n \in \mathbb{N} \) and normalized \( u_1 \) to 1 so that \( u_2 = -(K\pi^2 \varepsilon_0^0(\xi) + \bar{P}(\xi))/ (a\bar{P}(\xi)) \).

Then the following results on equilibrium solutions of (14) are given, as shown in Figure 6:

**Proposition 1.** (Kan-on [8]). Assume (A2) - (A5) and consider the problem (14) with arbitrarily fixed \( \xi \in I \). For each \( n \in \mathbb{N} \), we have the followings:

(i) For any \( \varepsilon \in (0, \varepsilon_0^0(\xi)) \), there exists a pair of spatially non-constant positive equilibrium solutions \( (\bar{P}_n(\xi; \varepsilon), \bar{X}_n^2(\xi; \varepsilon)) \) of (14) and (17). (ii) (14) has no positive equilibrium solutions other than \( (\bar{P}(\xi), \bar{X}(\xi)) \) and \( (\bar{P}_j(\xi; \varepsilon), \bar{X}_j^2(\xi; \varepsilon)) \) \( (j = 1, 2, \ldots, n) \) for any \( \varepsilon \in [\varepsilon_0^{n-1}(\xi), \varepsilon_0^n(\xi)) \).

(iii) \( \lim_{\varepsilon \to \varepsilon_0^n(\xi)} (\bar{P}_n(\xi; \varepsilon), \bar{X}_n^2(\xi; \varepsilon)) = (\bar{P}(\xi), \bar{X}(\xi)) \).

So far we arbitrarily fixed \( \xi \in I \) and used \( \varepsilon = 1/L^2 \) as a free parameter to obtain non-constant equilibrium solutions of (14). However, in order to obtain non-constant equilibrium solutions of the shadow system (12) with (13), we have to find \( \xi \in I \) for which (14) has non-constant equilibrium solutions for a fixed \( \varepsilon \). In order to do it, we use the relation \( \varepsilon = \varepsilon_0(\xi) = Q_0(\xi)/(n\pi)^2 \). If \( \varepsilon = Q_0(\xi)/(n\pi)^2 \) has a root \( \xi_n(\varepsilon) \in I \) for a given \( \varepsilon \), this \( \xi_n(\varepsilon) \) is a bifurcation value corresponding to

![Figure 6. Schematic global structure of the constant and non-constant equilibrium solutions](image)
\( \varepsilon_0^n(\xi) \) for each \( n \in \mathbb{N} \), at which \( n \)-mode equilibrium solutions of (14) bifurcate. Note that the function \( Q_0(\xi) \) in (17) is positive and convex upward on \( I \), as shown in Figure 7. Let \( Q^* = \max_{\xi \in I} Q_0(\xi) \). Then if a natural number \( n \) satisfies \( n^2 \pi^2 \varepsilon < Q^* \) for a given \( \varepsilon \), \( \varepsilon = Q_0(\xi)/(n^2 \pi^2) \) has two real roots, say \( \xi_n^l(\varepsilon) < \xi_n^r(\varepsilon) \), which satisfy \( \xi < \xi_n^l(\varepsilon) < \xi_n^r(\varepsilon) < \xi \). This implies that \( n \)-mode equilibrium solutions \( (P_n^\pm(x; \xi, \varepsilon), \bar{X}_n^\pm(x; \xi, \varepsilon)) \) bifurcate from \( (\bar{P}(\xi), \bar{X}(\xi)) \) at \( \xi = \xi_n^l(\varepsilon) \) and \( \xi = \xi_n^r(\varepsilon) \) and exist on the interval \( (\xi_n^l(\varepsilon), \xi_n^r(\varepsilon)) \), as shown in Figure 8. Furthermore, \( \xi_n^l(\varepsilon) \) is strictly increasing while \( \xi_n^r(\varepsilon) \) is strictly decreasing with respect to \( n \). On the other hand, if \( n \) satisfies \( n^2 \pi^2 \varepsilon > Q^* \), there is no \( n \)-mode equilibrium solutions of (14).
3.2. Existence of non-constant equilibrium solutions of the shadow system. Under the assumptions (A2) - (A5), we first note that (12) with (13) has a unique positive constant equilibrium solution \((\bar{P}^*, \bar{X}^*, \bar{\xi}^*)\) which is given by

\[
\begin{align*}
\bar{P}^* &= \bar{P}(\bar{\xi}^*) = \frac{1 - a(\mu) + K\mu - Kd(\mu)\mu}{1 - a(\mu) + b\mu - d(\mu)\mu} = \frac{1 - a + \mu + K\mu^2}{1 - a + b\mu + b\mu^2}, \\
\bar{X}^* &= \bar{X}(\bar{\xi}^*) = \frac{K - b + d(\mu) - Kd(\mu)}{1 - ad(\mu) + b\mu - d(\mu)\mu} = \frac{(K - b)(\mu - \mu_*)}{1 - a + b\mu + b\mu^2}, \\
\bar{\xi}^* &= \frac{ab - 1 + K - aK - K\mu + bK\mu}{K(1 - ad(\mu) + b\mu - d(\mu)\mu)} = 1 + \frac{1}{K\mu} - \frac{(a + \mu)(1 - a + \mu + K\mu^2)}{K\mu(1 - a + b\mu + b\mu^2)} \quad (\in I),
\end{align*}
\]

where \(d(\mu) = 1/(1 + \mu)\). We now look for \(n\)-mode non-constant equilibrium solutions of (12) with (13), when \(\varepsilon\) is suitably fixed, by using the \(n\)-mode equilibrium solutions \((\bar{P}^+\varepsilon(x; \xi, \varepsilon), \bar{X}^+\varepsilon(x; \xi, \varepsilon))\) of (14) which exist for \(\xi \in (\xi^l\varepsilon, \xi^u\varepsilon) \subset I\) (see Proposition 1). From now on, we restrict \((\bar{P}^+\varepsilon(x; \xi, \varepsilon), \bar{X}^+\varepsilon(x; \xi, \varepsilon))\) only, since the case of \((\bar{P}^\varepsilon_n(x; \xi, \varepsilon), \bar{X}^\varepsilon_n(x; \xi, \varepsilon))\) is similarly treated. Now, we define \(H^+_n(\xi; \varepsilon)\) by

\[
H^+_n(\xi; \varepsilon) = \int_0^1 \bar{P}^+\varepsilon(x; \xi, \varepsilon)dx - \mu \int_0^1 \bar{X}^+\varepsilon(x; \xi, \varepsilon)dx - 1
\]

for \(\xi \in (\xi^l\varepsilon, \xi^u\varepsilon)\), as shown in Figure 8. We note that \(H^+_n(\xi; \varepsilon)\) is a continuous function of \(\xi\). If there exists a solution \(\xi^+_n(\varepsilon)\) of \(H^+_n(\xi; \varepsilon) = 0\), then \((\bar{P}^+\varepsilon(x; \xi^+_n\varepsilon, \varepsilon), \bar{X}^+\varepsilon(x; \xi^+_n\varepsilon, \varepsilon), \xi^+_n\varepsilon(\varepsilon))\) gives an \(n\)-mode positive equilibrium solution of (12) with (13).

Before solving \(H^+_n(\xi; \varepsilon) = 0\), we remark that the positive constant equilibrium solution \((\bar{P}^*, \bar{X}^*, \bar{\xi}^*)\) can be obtained by using

\[
H_0(\xi) = \int_0^1 \bar{P}(\xi)dx - \mu \int_0^1 \bar{X}(\xi)dx - 1,
\]

where \((\bar{P}(\xi), \bar{X}(\xi))(\xi \in I)\) is the positive constant equilibrium of (14). Then \(H_0(\xi)\) is explicitly represented as

\[
H_0(\xi) = K\frac{1 - ad(\mu) + b(b - d(\mu))}{ab - 1}(\xi - 1) = \frac{1 - ab + \mu K(1 - b) - K(1 - a)}{ab - 1}.
\]

It is obvious that \(H = H_0(\xi)\) is a straight line with positive slope, satisfying \(H_0(\xi) > 0\) by (A3) and (A4) \((H_0(\xi) < 0\) is obvious). Therefore one easily knows that \(H_0(\xi) = 0\) has the unique root \(\xi^*\), which gives the positive constant solution \((P^*, X^*, \xi^*)\) in (19) (see Figure 9).

We now consider (20) in order to find non-constant equilibrium solutions of (12) with (13), that is, using the \(n\)-mode equilibrium solution \((\bar{P}^+_n(x; \xi, \varepsilon), \bar{X}^+_n(x; \xi, \varepsilon))\) of (14) for \(\xi \in (\xi^l\varepsilon, \xi^u\varepsilon)\), we find \(\xi \in (\xi^l\varepsilon_n, \xi^u\varepsilon_n)\) so as to satisfy \(H^+_n(\xi; \varepsilon) = 0\).

Before doing this, we show the following general result on a general shape of \(H^+_n(\xi; \varepsilon)\):

**Theorem 3.1.** Assume (A2) - (A5). For each \(n \in \mathbb{N}\), the following results hold:

(i) For any \(\varepsilon \in (0, \varepsilon^*_0(\xi^*))\), there exists at least a pair of spatially non-constant \(n\)-mode equilibrium solutions \((\bar{P}^+_n(x; \xi, \varepsilon), \bar{X}^+_n(x; \xi, \varepsilon))\) of (12) with (13), where \(\bar{P}^+_n(x; \varepsilon) \equiv P^+_n(x; \xi^l\varepsilon_n, \varepsilon)\) and \(\bar{X}^+_n(x; \varepsilon) \equiv X^+_n(x; \xi^u\varepsilon_n, \varepsilon)\).

(ii) \(\lim_{\varepsilon \to \varepsilon^*_0(\xi^*)} (\bar{P}^+_n(x; \varepsilon), \bar{X}^+_n(x; \varepsilon)) = (P^*, X^*, \xi^*)\).
The proof will be stated in Section 5.1.

Unfortunately, this theorem does not give any information on the number of non-constant equilibrium solutions of (12) with (13). It really depends on the functional form of $H_n^\pm(\xi;\varepsilon)$ when $\varepsilon$ is suitably fixed. Consider two cases where (a) $\xi^*_n(\varepsilon) \leq \bar\xi^*$ and (b) $\xi^*_n(\varepsilon) > \bar\xi^*$. If $H_n^+(\xi;\varepsilon)$ is a monotone increasing function of $\xi \in (\xi^*_n(\varepsilon), \xi^*_n(\varepsilon))$ as shown in Figure 9, we easily find that for the case (a), $H_n^+(\xi^*_n(\varepsilon);\varepsilon) < 0$ holds so that (12) with (13) have no $n$-mode equilibrium solutions, while for the case (b), $H_n^+(\xi^*_n(\varepsilon);\varepsilon) > 0$ holds so that only one root $\bar\xi^*_n(\varepsilon)$ of $H_n^+(\xi;\varepsilon) = 0$ exists, that is, (12) with (13) has an $n$-mode equilibrium solution $\bar\Pi_n^+(x;\varepsilon) \equiv (\bar P_n^+(x;\varepsilon), \bar X_n^+(x;\varepsilon), \bar \xi^*_n(\varepsilon))$. However, we have not yet understood whether $H_n^+(\xi;\varepsilon)$ is monotone increasing in $\xi$ or not. Therefore we have to compute $H_n^+(\xi;\varepsilon)$ by using $(\bar P_n^+(x;\xi,\varepsilon), \bar X_n^+(x;\xi,\varepsilon))$.

\[
\begin{align*}
\text{Figure 9. Dependency of } H_n^\pm(\xi;\varepsilon) \text{ on } \varepsilon. \quad &\text{(a) } \xi^*_n(\varepsilon) \leq \bar\xi^* \quad \text{and (b) } \xi^*_n(\varepsilon) > \bar\xi^*. \\
\end{align*}
\]

In the previous sections, we fixed $\mu(>\mu_c)$ arbitrarily and considered the dependency of $\varepsilon (=1/L^2)$ on the existence of non-constant equilibrium solutions of (12) with (13). However, as it was shown in Figure 4(e), we demonstrated the global structures of equilibrium solutions of (3) with $D$ sufficiently large ($D = 10000$) and (4) when $\mu$ is varied. The reason why we take $\mu$ as a free parameter is that it is an important parameter of toxicity in the system. We therefore discuss the $\mu$-dependency of equilibrium solutions of (9) and (10) when $\varepsilon = 1/L^2$ is suitably fixed. Then we write $(\bar P_n^+(x;\xi,\varepsilon), \bar X_n^+(x;\xi,\varepsilon))$, $\bar\Pi_n^+(x;\varepsilon) \equiv (\bar P_n^+(x;\varepsilon), \bar X_n^+(x;\varepsilon), \bar \xi^*_n(\varepsilon), \bar \xi^*_n(\varepsilon))$ and $H_n^+(\xi;\varepsilon)$ as $(\bar P_n^+(x;\xi,\varepsilon,\mu), \bar X_n^+(x;\xi,\varepsilon,\mu), \bar \Pi_n^+(x;\xi,\varepsilon,\mu) \equiv (\bar P_n^+(x;\xi,\varepsilon,\mu), \bar X_n^+(x;\xi,\varepsilon,\mu), \bar \xi^*_n(\varepsilon,\mu), \bar \xi^*_n(\varepsilon,\mu))$, respectively.

4. Dependency of equilibrium solutions of the shadow system on the toxicity $\mu$. In this section, by suitably fixing the interval length $L$ and using $\mu$ as a free parameter, we consider (9) with (10). Since Figure 4(e) suggests that when $D$ is sufficiently large, 1-mode equilibrium solutions of (9) with (10) are exist and are stable, we restrict the dependency of $\mu$ to 1-mode equilibrium solutions $\bar\Pi_n^+(x;1/L^2,\mu)$ of (9) with (10) for different values of $L$.

We first consider the case $L = 30$ and compute $H_n^+(\xi;1/L^2,\mu)$ when $\mu$ is a free parameter. As numerically shown in Figure 10, we can expect that the forms of
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Figure 10. The functional forms of $H_1^+(ξ;1/L^2,µ)$ for (a) $µ = 0.1$, $(α,β) = (0.404069,0.668669)$, (b) $µ = 0.2$, $(α,β) = (0.264145,0.531191)$, (c) $µ = 1$, $(α,β) = (0.104722,0.2785)$, (d) $µ = 6$, $(α,β) = (0.0636,0.184446)$, (e) $µ = 12$, $(α,β) = (0.0593018,0.173621)$ and (f) $µ = 13$, $(α,β) = (0.0589696,0.172776)$, where other parameters are fixed as $a = 0.95$, $b = 1.2$, $K = 2.9$, $R = 0.43$ and $L = 30$. The horizontal (resp. vertical) axes are $ξ$ (resp. $H_1^+(ξ;1/L^2,µ)$).

$H_1^+(ξ;1/L^2,µ)$ are monotone increasing in $ξ ∈ (α,β)$ where $α = ξ_1^+(1/L^2,µ)$ and
\[ \beta = \xi^{1}_{c}(1/L^2, \mu) \] and that Figures (10)b)-(d) clearly show that there is a unique \( \xi \) satisfying \( H^{+}_{1}(\xi; 1/L^2, \mu) = 0 \) for suitable range of \( \mu \).

Next consider the case \( L = 35 \). Figure (11) shows the functional forms of \( H^{+}_{1}(\xi; 1/L^2, \mu) \) from which we can notice that one big difference from the case of \( L = 30 \) is that these are not necessarily monotone in such a way that one maximal and one minimal points exist in \((\alpha, \beta)\) for suitable values of \( \mu \) (see Figures (11)b)-(d)). Therefore, the number of \( \xi \) satisfying \( H^{+}_{1}(\xi; 1/L^2, \mu) = 0 \) is not unique, depending on values of \( \mu \). For \( \mu = 0.1275 \) (Figure (11)b)), there are two solutions of \( H^{+}_{1}(\xi; 1/L^2, \mu) = 0 \), for \( \mu = 0.1281 \) (Figure (11)c)), there are three, and for \( \mu = 0.13 \) (Figure (11)d)) there is one. The results above indicate that we need to study more precisely the dependency of non-constant equilibrium solutions of (9) with (10) on \( \mu \).

We therefore employ AUTO to obtain the global structures of the equilibrium solutions of (9) with (10) when \( \mu \) is globally varied. The case of \( L = 30 \) is shown in Figure (12) which shows that there exist 1- and 2- modes equilibrium solutions when \( \mu \) is varied. The 1-mode equilibrium solutions \( \Pi^{+}_{1}(x; 1/L^2, \mu) \) primarily and super-critically bifurcate from the constant equilibrium solution \( E_{1} = (\bar{P}^{*}, \bar{X}^{*}, \bar{z}^{*}) \) at \( \mu = \mu_{c1} \) and \( \mu_{c2} \) so that they exist and are stable for any \( \mu \in (\mu_{c1}, \mu_{c2}) \), while the 2-mode equilibrium solutions exist but are unstable. The bifurcation values \( \mu_{c1} \) and \( \mu_{c2} \) can be examined by using the key relation between \( \varepsilon \) and \( \mu \), which is \( \varepsilon = \varepsilon_{0}^{1}(\xi^{c}) = Q_{0}(\xi^{c})/(\pi^{2}) \), where \( Q_{0}(\xi) \) is defined in (17) which depends on \( \mu \). Figure (13) shows the occurrence of two 1-mode bifurcation values \( \mu = \mu_{c1} = 0.13 \cdots \) and \( \mu = \mu_{c2} = 12.51 \cdots \). Theorem 3.1 says that the 1-mode equilibrium solutions exist for arbitrary \( \mu \in (\mu_{c1}, \mu_{c2}) \). Similarly there appear two 2-mode bifurcation values \( \mu = \mu_{c1} \) and \( \mu = \mu_{c2} \) and the 2-mode equilibrium solutions exist for \( \mu \in (\mu_{c1}, \mu_{c2}) \), but no 3-mode equilibrium solutions exist in this parameter setting. Moreover, there exist no saddle-node bifurcation points on the 1-mode branch in Figure (12). This corresponds to the result that \( H^{+}_{1}(\xi; 1/L^2, \mu) = 0 \) has a unique solution (see Figure (17)).

For the case of \( L = 35 \), AUTO exhibits the structure of 1-mode equilibrium solutions which is almost similar to the one for the case of \( L = 30 \) in a sense that the 1-mode equilibrium solutions super-critically bifurcate, as shown in Figure (13). This result is verified exactly in Section 4.2. However, it exhibits an S-shaped structure with respect to \( \mu \) so that there exist three 1-mode equilibrium solutions for suitable range of \( \mu \). The difference on global structures of equilibrium solutions for \( L = 30 \) and \( L = 35 \) is well-consistent with the ones obtained from the information on \( H^{+}_{1}(\xi; 1/L^2, \mu) \) in Figures (10) and (11) respectively.

Let us show one more important example. It is the case \( L = 60 \) where \( H^{+}_{1}(\xi; 1/L^2, \mu) \) is shown in Figure (15). We can observe one hump in \( H^{+}_{1}(\xi; 1/L^2, \mu) \) so that there are two solutions \( \xi \) of \( H^{+}_{1}(\xi; 1/L^2, \mu) = 0 \) in a suitable range of \( \mu \). AUTO exhibits the corresponding global structure of equilibrium solutions in Figure (16). It is emphasized that 1-mode equilibrium solutions sub-critically bifurcate from the constant one \( E_{1} \) at \( \mu = \mu_{c1} \) so that there are two 1-mode equilibrium solutions for suitable range for \( \mu \).

Consequently, the numerical results above assert the followings on equilibrium solutions of the shadow system:

1. 1-mode equilibrium solutions \( \Pi^{+}_{1}(x; 1/L^2, \mu) \) primarily bifurcate from \((\bar{P}^{*}, \bar{X}^{*}, \bar{z}^{*})\) when \( \mu \) increases.
2. Whether the bifurcation of the 1-mode equilibrium solutions from \((\bar{P}^{*}, \bar{X}^{*}, \bar{z}^{*})\)
Figure 11. The functional forms of $H_1^+(\xi; 1/L^2, \mu)$ for (a) $\mu = 0.12$, $(\alpha, \beta) = (0.353533, 0.638713)$, (b) $\mu = 0.1275$, $(\alpha, \beta) = (0.340841, 0.626635)$, (c) $\mu = 0.1281$, $(\alpha, \beta) = (0.339872, 0.625696)$, (d) $\mu = 0.13$, $(\alpha, \beta) = (0.336846, 0.622744)$, (e) $\mu = 17$, $(\alpha, \beta) = (0.0565952, 0.171585)$ and (f) $\mu = 18$, $(\alpha, \beta) = (0.0564297, 0.171148)$, where other parameters are fixed as $a = 0.95$, $b = 1.2$, $K = 2.9$, $R = 0.43$ and $L = 35$. The horizontal (resp. vertical) axes are $\xi$ (resp. $H_1^+(\xi; 1/L^2, \mu)$).

is supercritical or subcritical depends on the interval length $L$. 
Figure 12. Bifurcation diagram of the shadow system (9) with (10) when \( \mu \) is a free parameter. Other parameters are fixed at \( a = 0.95, b = 1.2, K = 2.9, R = 0.43 \) and \( L = 30 \). Solid (resp. dashed) lines represent stable (resp. unstable) equilibrium solutions of (9) with (10).

Figure 13. Relations between \( \varepsilon \) and \( \mu \) with respect to the bifurcation curves of \( n \)-mode equilibrium solutions \((n = 1, 2, 3)\) of (9) and (10) with \( \varepsilon = 1/L^2 \), where \( a = 0.95, b = 1.2, K = 2.9, R = 0.43 \), and \( L = 30 \). (a) \( \mu \in (\mu_c, 20.0) \), (b) \( \mu \in (\mu_c, 2.0) \), which is a magnification of (a), where \( \mu_c = 0.11 \cdots, \mu_{c1} = 0.13 \cdots\) and \( \mu_{c2} = 12.51 \cdots \).

(3) The number and stability of 1-mode equilibrium solutions also depends on the interval length \( L \).

The assertion (1) is proved in Theorem 3.1. In the next subsection, the assertion (2) will be discussed by using the local bifurcation theory (II).

4.1. Direction of the pitchfork bifurcation. In the above, we have numerically shown the existence of the \( n \)-mode equilibrium solutions of the shadow system (12) with (13) when \( L \) is varied. Here we are interested in showing the local stability of the 1-mode equilibrium solutions \( \Pi^+_1(x; 1/L^2, \mu) \) of (12) with (13). In order to do it, we can apply the local bifurcation theory (II) to the stationary problem of
Figure 14. Bifurcation diagram of the shadow system (9) with (10) when $\mu$ is a free parameter. Other parameters are fixed at $a = 0.95$, $b = 1.2$, $K = 2.9$, $R = 0.43$ and $L = 35$. Solid (resp. dashed) lines represent stable (resp. unstable) equilibrium solutions of (9) with (10). In the right corner, it is shown a magnification around the primary bifurcation point ($n = 1$).

Our shadow system (12) with (13) to examine whether $n$-mode equilibrium solutions bifurcate super-critically or sub-critically from $(\bar{P}^*, \bar{X}^*, \bar{\xi}^*)$. First, we define some function spaces. $Y = \{w(x) \in H^2(0, 1) \mid w_x(0) = 0 = w_x(1)\}$ and $Z(n) = \{(P(x), X(x))^T \in Y \times Y \mid \int_0^1 P(x) \cos(n\pi x)dx + \int_0^1 u_2 X(x) \cos(n\pi x)dx = 0\}$

with the usual $L^2$ inner product $<f(x), g(x)> = \int_0^1 f(x)g(x)dx$. Then by Theorem 1.7 of [1], we have

**Proposition 2.** Under the assumptions (A2) - (A5), there exists a positive constant $\delta$ such that non-constant equilibrium solutions of (12) and (13) near $(\bar{P}, \bar{X}, \bar{\xi}, \varepsilon) = (P^*, X^*, \xi^*, \varepsilon_0(\xi^*)) \in Y \times Y \times R^2$ can be represented by

\[
\begin{align*}
\bar{P}_n(x; s) &= P^* + s \cos(n\pi x) + s^2 \tilde{P}_n(x; s) \\
\bar{X}_n(x; s) &= X^* + s u_2 \cos(n\pi x) + s^2 \tilde{X}_n(x; s) \\
\bar{\xi}_n(s) &= \tilde{\xi}^* + s^2 \tilde{\xi}_n(s) \\
\varepsilon_n(s) &= \varepsilon_0(\xi^*) + s \tilde{\varepsilon}_n(s)
\end{align*}
\]

for any $s \in (-\delta, \delta)$ and each $n \in \mathbb{N}$, where $(\bar{P}_n(x; s), \bar{X}_n(x; s)) \in Z(n)$ and $\bar{\xi}_n(s)$, $\tilde{\varepsilon}_n(s) \in R$.

Though Theorem 3.1 is identical to this proposition limited to a neighborhood of the bifurcation points, the representation of the solution is different. For example, $(\bar{P}_n(x; \varepsilon), \bar{X}_n(x; \varepsilon), \bar{\xi}_n^+(\varepsilon))$ in Theorem 3.1 corresponds to $(\bar{P}_n(x; s), \bar{X}_n(x; s), \bar{\xi}_n(s))$ for $s \in (0, \delta)$ in Proposition 2. For the proof of Proposition 2 we refer to the paper by Q. Wang, C. Gai and J. Yan [16], so we omit it.
Figure 15. The functional forms of $H_1^+(\xi; 1/L^2, \mu)$ for (a) $\mu = 0.1$, $(\alpha, \beta) = (0.375538, 0.683272)$, (b) $\mu = 0.11$, $(\alpha, \beta) = (0.355264, 0.66457)$, (c) $\mu = 0.12$, $(\alpha, \beta) = (0.337375, 0.647172)$, (d) $\mu = 0.13$, $(\alpha, \beta) = (0.321474, 0.630946)$, (e) $\mu = 50$, $(\alpha, \beta) = (0.0521753, 0.168414)$ and (f) $\mu = 58$, $(\alpha, \beta) = (0.0520414, 0.168035)$, where other parameters are fixed at $a = 0.95$, $b = 1.2$, $K = 2.9$, $R = 0.43$ and $L = 60$. The horizontal (resp. vertical) axes are $\xi$ (resp. $H_1^+(\xi; 1/L^2, \mu)$).
Figure 16. Bifurcation diagram of the shadow system (9) with (10) when $\mu$ is a free parameter. Other parameters are fixed at $a = 0.95$, $b = 1.2$, $K = 2.9$, $R = 0.43$ and $L = 60$. Solid (resp. dashed) lines represent stable (resp. unstable) equilibrium solutions of (9) with (10). In the right corner, it is shown a magnification around the primary bifurcation point ($n = 1$).

Noting that $(\tilde{P}_n(x; s), \tilde{X}_n(x; s), \tilde{\xi}_n(s))$ are $C^3$-class functions of $s$, we can rewrite (23) in the following form:

\[
\begin{align*}
\tilde{P}_n(x; s) &= \tilde{P}^* + s \cos(n \pi x) + s^2 \phi_2(x) + s^3 \phi_3(x) + o(s^3) \\
\tilde{X}_n(x; s) &= \tilde{X}^* + su_2 \cos(n \pi x) + s^2 \psi_2(x) + s^3 \psi_3(x) + o(s^3) \\
\tilde{\xi}_n(s) &= \tilde{\xi}^* + s^2 \eta_2 + s^3 \eta_3 + o(s^3) \\
\varepsilon_n(s) &= \varepsilon_n^0(\tilde{\xi}^*) + s\mathcal{K}_1 + s^2\mathcal{K}_2 + o(s^2),
\end{align*}
\]

where $(\phi_i, \psi_i) \in \mathcal{Z}(n)$ ($i = 2, 3$) and $\eta_2, \eta_3, \mathcal{K}_1, \mathcal{K}_2$ are constants. Here, though $\phi_i, \psi_i, \eta_i$ ($i = 2, 3$) and $\mathcal{K}_i$ ($i = 1, 2$) depend on $n \in \mathbb{N}$, we dropped the index $n$ for simplicity. We can show the following theorem:

**Theorem 4.1.** Under the assumptions (A2) - (A5), $\mathcal{K}_1 = 0$ and $\mathcal{K}_2$ is computed by the following three steps:

1. Calculate

\[
\begin{align*}
< \phi_2, 1 > &= -\mu( -b \tilde{P}^* u_2 - \tilde{P}^* u_2^2 + d(\mu) \tilde{X}^* + ad(\mu) u_2 \tilde{X}^*) \\
2(\text{ad}(\mu) + d(\mu) \mu - 1 - b\mu) \tilde{P}^* \tilde{X}^*
\end{align*}
\]

\[
< \psi_2, 1 > = -\mu( -b \tilde{P}^* u_2 - \tilde{P}^* u_2^2 + d(\mu) \tilde{X}^* + ad(\mu) u_2 \tilde{X}^*) \\
2(\text{ad}(\mu) + d(\mu) \mu - 1 - b\mu) \tilde{P}^* \tilde{X}^*
\]

\[
\eta_2 = \frac{1 + av_2}{2KP^*} + \frac{(a + \mu)( -b \tilde{P}^* u_2 - \tilde{P}^* u_2^2 + d(\mu) \tilde{X}^* + ad(\mu) u_2 \tilde{X}^*)}{2K(\text{ad}(\mu) + d(\mu) \mu - 1 - b\mu) \tilde{P}^* \tilde{X}^*},
\]

where $d(\mu) = 1/(1 + \mu)$.

2. Solve the following linear system with respect to $< \phi_2, \cos(2n\pi x) >$ and <
ψ_2, \cos(2n\pi x) >:

\[
\begin{align*}
\{ (2\pi)^2 \varepsilon^1_0(\xi^*) + \frac{P^*}{K} \} < \phi_2, \cos(2n\pi x) > + \frac{aP^*}{K} < \psi_2, \cos(2n\pi x) > &= \frac{(1 + au_2)}{4K} \\
\frac{b\bar{X}^*}{K} < \phi_2, \cos(2n\pi x) > + \left\{ (2\pi)^2 \varepsilon^1_0(\xi^*) + \frac{\bar{X}^*}{K} \right\} < \psi_2, \cos(2n\pi x) > &= \frac{u_2(u_2 + b)}{4K}.
\end{align*}
\]

(26)

By using the results obtained in (1) and (2), \( K_2 \) is explicitly represented as

\[
\begin{align*}
K_2^2 &= \left[ \{ 2 + au_2 + bu_2^2 \} \{ < \phi_2, \cos(2n\pi x) > + < \phi_2, 1 > \} / K \\
&+ \{ a + bu_2 + 2u_2^2 \} \{ < \psi_2, \cos(2n\pi x) > + < \psi_2, 1 > \} / K \right] / ((n\pi)^2(1 + u_2^2)).
\end{align*}
\]

(27)

The proof will be stated in Section 5.2.

Remark 2. If \( K_2(n) < 0 \), the pitchfork bifurcation of the \( n \)-mode solutions is supercritical, on the other hand, if \( K_2(n) > 0 \), it is subcritical.

Remark 3. Since \( < \phi_2, \cos(2n\pi x) >, < \psi_2, \cos(2n\pi x) > \) and \( u_2 = - (K\pi^2 \varepsilon^1_0(\xi^*) + \bar{P}(\xi^*)) / (a\bar{P}(\xi^*)) \) do not depend on \( n \), the numerator of the right hand side of (27) also does not depend on \( n \). Thus, the sign of \( K_2(n) \) does not depend on the mode number \( n \).

4.2. Bifurcation of the \( n \)-mode equilibrium solutions from \((\bar{P}^*, \bar{X}^*, \bar{\xi}^*)\). In this subsection, we study the relation between the sign of the principal eigenvalue of the linearized problem (see [23]) and that of \( K_2(n) \) of the \( n \)-mode equilibrium solutions \((\bar{P}_n(x; s), \bar{X}_n(x; s), \bar{\xi}_n(s))\) of the following time dependent system:

\[
\begin{align*}
P_t &= P \left( 1 - \frac{P + aX}{K} - \xi \right) + \varepsilon P_{xx} \\
X_t &= X \left( 1 - \frac{X + bP}{K} - d(\mu)\xi \right) + \varepsilon X_{xx}, \quad t > 0, \ 0 < x < 1 \\
\xi_t &= R\xi \left( \int_0^1 Pdx - \mu \int_0^1 Xdx - 1 \right) \\
(P_x, X_x)(t, 0) &= (0, 0) = (P_x, X_x)(t, 1), \quad t > 0.
\end{align*}
\]

(28)

For this purpose, we apply the classical results by Crandall and Rabinowitz [2] on the linearized stability to (28). Then we consider the following linearized eigenvalue
problem around \((\bar{P}_n(x; s), \bar{X}_n(x; s), \bar{\xi}_n(s))\):

\[
\begin{align*}
\lambda P &= \left(1 - \frac{2\bar{P}_n(x; s) + a\bar{X}_n(x; s)}{K} - \bar{\xi}_n(s)\right) P \\
&\quad - \frac{a}{K} \bar{P}_n(x; s)X - \bar{P}_n(x; s)\xi + \varepsilon P_{xx} \\
\lambda X &= -\frac{b}{K} \bar{X}_n(x; s)P + \left(1 - \frac{2\bar{X}_n(x; s) + b\bar{P}_n(x; s)}{K}\right) X - d(\mu)\bar{X}_n(x; s)\xi + \varepsilon X_{xx}, \\
\lambda \xi &= R\bar{\xi}_n(s) \left( \int_0^L Pdx - \mu \int_0^1 Xdx \right) \\
(P_x, X_x)(0) &= (0, 0) = (P_x, X_x)(1),
\end{align*}
\]  

(29)

where \(\lambda(s; n)\) is an eigenvalue and \((P(x; s; n), X(x; s; n), \xi(s; n))\) is the corresponding eigenfunction of the linearized operator.

On the other hand, the linearized eigenvalue problem (28) around \((\bar{P}^*, \bar{X}^*, \bar{\xi}^*)\) at \(\varepsilon = \varepsilon_0^*(\xi^*)\) satisfies the following equation:

\[
\begin{align*}
0 &= -\frac{\bar{P}^*}{K} P - \frac{a}{K} \bar{P}^* X - \bar{P}^* \xi + \varepsilon_0^*(\xi^*) P_{xx} \\
0 &= -\frac{b}{K} \bar{X}^* P - \frac{\bar{X}^*}{K} X - d(\mu)\bar{X}^* \xi + \varepsilon_0^*(\xi^*) X_{xx}, \\
0 &= R\bar{\xi}^* \left( \int_0^1 Pdx - \mu \int_0^1 Xdx \right) \\
(P_x, X_x)(0) &= (0, 0) = (P_x, X_x)(1),
\end{align*}
\]  

(30)

where \((P(x; n), X(x; n), \xi(n)) = (\cos(n\pi x), u_2 \cos(n\pi x), 0)\). That is, zero is a simple eigenvalue of the linearized problem (30) with the eigenfunction \(\varepsilon_0^*(\xi^*)\) in \(J\) and continuously differentiable functions \(\nu(\varepsilon; n)\) defined on \(J\) and \(\lambda(s; n)\) defined on \((-\delta, \delta)\) with \(\nu(\varepsilon_0^*(\xi^*); n) = 0\) and \(\lambda(0; n) = 0\) such that \(\lambda(s; n)\) is an eigenvalue of (29) and \(\nu(\varepsilon; n)\) is an eigenvalue of the following eigenvalue problem:

\[
\begin{align*}
\nu P &= -\frac{\bar{P}^*}{K} P - \frac{a}{K} \bar{P}^* X - \bar{P}^* \xi + \varepsilon P_{xx} \\
\nu X &= -\frac{b}{K} \bar{X}^* P - \frac{\bar{X}^*}{K} X - d(\mu)\bar{X}^* \xi + \varepsilon X_{xx}, \\
\nu \xi &= R\bar{\xi}^* \left( \int_0^1 Pdx - \mu \int_0^1 Xdx \right) \\
(P_x, X_x)(0) &= (0, 0) = (P_x, X_x)(1),
\end{align*}
\]  

(31)
Theorem 4.2. Under the assumptions (A2) - (A5), there exists $\delta > 0$ such that $sgn\{\lambda(s; n)\} = sgn\{X_2(n)\}$ for $s \in (-\delta, 0) \cup (0, \delta)$ and $n \in \mathbb{N}$.

The proof will be stated in Section 5.3.

Noting the order relation of the bifurcation values $\varepsilon = \varepsilon_0^L(\xi^*)$ for each $n \in \mathbb{N}$, we easily find that the 1-mode equilibrium solutions $(\bar{P}_1(x; s), \bar{X}_1(x; s), \xi_1(s))$ of (28) bifurcate primarily from the constant equilibrium solution $(\bar{P}^*, \bar{X}^*, \xi^*)$. Then we have

Corollary 1. Assume (A2) - (A5). In a neighborhood of the 1-mode bifurcation value $\varepsilon = \varepsilon_0^L(\xi^*)$, the 1-mode equilibrium solutions $(\bar{P}_1(x; s), \bar{X}_1(x; s), \xi_1(s))$ of (28) are asymptotically stable (resp. unstable) if their bifurcation is supercritical (resp. subcritical). On the other hand, for $n \geq 2$ the $n$-mode equilibrium solutions $(\bar{P}_n(x; s), \bar{X}_n(x; s), \xi_n(s))$ of (28) are unstable in a neighborhood of the $n$-mode bifurcation value $\varepsilon = \varepsilon_0^L(\xi^*)$.

Finally, we consider the type of bifurcation and the stability of the 1-mode equilibrium solutions in a neighborhood of the bifurcation points. For these purpose, we have to calculate the sign of $X_2(1)$ (see Theorem 4.1). In Figure 17 the solid curves represent the relation between $L$ and $\mu_0$ on which 1-mode equilibrium solutions bifurcate from the constant equilibrium solution $E_4$. The dashed curves represent $X_2 = X_2(1)$ on which 1-mode equilibrium solutions bifurcate depending on $\mu$. First, for a given $L$ we can get the bifurcation values of $\mu_0$ by using the solid curves and, next, we find the sign of $X_2(1)$ corresponding to the bifurcation values $\mu_0$ by using the dashed curves. Figure 17 shows the following: when $L = 30$, 1-mode equilibrium solutions bifurcate at $\mu = \mu_1$ and $\mu = \mu_2$. Both of them super-critically bifurcate and are stable near the bifurcation points. When $L = 35$, a similar situation occurs. On the other hand, when $L = 60$, 1-mode equilibrium solutions bifurcate at $\mu = \mu_1^*$ and $\mu = \mu_2^*$. Figure 17(c) says that $X_2(1) > 0$ at $\mu = \mu_1^*$, which means that it sub-critically bifurcates and is unstable near the bifurcation point, while Figure 17(a) says that $X_2(1) < 0$ at $\mu = \mu_2^*$, which means that it super-critically bifurcates and is stable near the bifurcation point. Of course, these results agree with the ones obtained by AUTO (see Figures 12, 14, and 16).

5. Proofs.

5.1. Proof of Theorem 3.1. To show (ii) is easy from Proposition 1 so we show only (i). We consider $(\bar{P}_n^+(x; \xi, \varepsilon), \bar{X}_n^+(x; \xi, \varepsilon))$ only since the case with $(\bar{P}_n^-(x; \xi, \varepsilon), \bar{X}_n^-(x; \xi, \varepsilon))$ is treated similarly. Since $(\bar{P}(\xi^*), \bar{X}(\xi^*), \xi^*) = (\bar{P}^*, \bar{X}^*, \xi^*)$ is a constant equilibrium solution of (12) with (13), we find easily that $\varepsilon_0^L(\xi^*)$ is a bifurcation point of an $n$-mode non-constant equilibrium solution by using the linearized analysis for each $n \in \mathbb{N}$. Here we consider the simple case when $H_n^+(\xi; \varepsilon)$ is monotone increasing with respect to $\xi$. If $H_n^+(\xi; \varepsilon)$ is monotone increasing for $n \geq 2$, the proof is the same as the case $n = 1$. On the other hand, if $H_n^+(\xi; \varepsilon)$ is not
monotone, it is clear that (12) with (13) has at least one $n$-mode equilibrium solution. Therefore, we assume $H_{1}^{+}(\xi;\varepsilon)$ is monotone increasing and prove the assertion (i).

First let us consider the case (i) $\varepsilon > \varepsilon_{0}^{1}(\xi^{*})$ in Figure 18. For the case (i-1), $H_{1}^{+}(\xi;\varepsilon)$ is not defined because $\xi_{1}^{1}(\varepsilon)$ and $\xi_{1}(\varepsilon)$ are not defined while for the case (i-2), $H_{1}^{+}(\xi;\varepsilon) = 0$ has no real roots, which is shown in Figure 19 (i-2). In both cases, $H_{0}(\xi) = 0$ has only one solution $\bar{\xi}$, which corresponds to the constant equilibrium solution. Next, we consider the case (ii) $\varepsilon = \varepsilon_{0}^{1}(\xi^{*})$ in Figure 18. The graph of $H_{1}^{+}(\xi;\varepsilon)$ is indicated in Figure 19 (ii). $H_{1}^{+}(\xi;\varepsilon) = 0$ has only one root $\bar{\xi}$, which is the same as that of $H_{0}(\xi) = 0$. But this shows the beginning of the appearance of a 1-mode equilibrium solution. Then, $\varepsilon$ becomes smaller, which corresponds to the case (iii) in Figure 18 and the graph of $H_{1}^{+}(\xi;\varepsilon)$ is shown in Figure 19 (iii). Clearly, $H_{1}^{+}(\xi;\varepsilon) = 0$ has a new root $\xi_{1}^{+}(\varepsilon)$, which corresponds to a 1-mode equilibrium solution of (12) with (13). This completes the proof.

5.2. Proof of Theorem 4.1. Substituting (24) into the stationary problem of (12) and equating the same power of $O(s^{2})$ and $O(s^{3})$, we have respectively
The bifurcation curve of $\varepsilon = \varepsilon_0^1(\xi) = Q_0(\xi)/\pi^2$ for $\xi \in I$. For given $\xi^*$, $\varepsilon_0^1(\xi^*)$ is determined.

\[
\begin{cases}
\varepsilon_0^1(\xi^*) \phi_2 - \mathcal{K}_1 (n\pi)^2 \cos(n\pi x) - \frac{\bar{P}^*}{K} (\phi_2 + a\psi_2) \\
- \frac{1}{K} (1 + au_2)(\cos(n\pi x))^2 - \bar{P}^* \eta_2 = 0 \\
\varepsilon_0^1(\xi^*) \psi_2 - u_2\mathcal{K}_1 (n\pi)^2 \cos(n\pi x) - \frac{\bar{X}^*}{K} (\psi_2 + b\phi_2) \\
- \frac{u_2}{K} (u_2 + b)(\cos(n\pi x))^2 - d(\mu)\bar{X}^* \eta_2 = 0 \\
\int_0^1 \phi_2 dx - \mu \int_0^1 \psi_2 dx = 0
\end{cases}
\]

(32)

and

\[
\begin{cases}
\varepsilon_0^1(\xi^*) \phi_3 + \mathcal{K}_1 \phi_2 - \mathcal{K}_2 (n\pi)^2 \cos(n\pi x) - \frac{\bar{P}^*}{K} (\phi_3 + a\psi_3) \\
- \frac{1}{K} (\phi_2 + a\psi_2) \cos(n\pi x) - \frac{1}{K} \phi_2 (1 + au_2) \cos(n\pi x) \\
- \bar{P}^* \eta_3 - \eta_2 \cos(n\pi x) = 0 \\
\varepsilon_0^1(\xi^*) \psi_3 + \mathcal{K}_1 \psi_2 - u_2\mathcal{K}_2 (n\pi)^2 \cos(n\pi x) - \frac{\bar{X}^*}{K} (\psi_3 + b\phi_3) \\
- \frac{u_2}{K} (\psi_2 + b\phi_2) \cos(n\pi x) - \frac{1}{K} \psi_2 (u_2 + b) \cos(n\pi x) \\
- d(\mu)\bar{X}^* \eta_3 - d(\mu)u_2\eta_2 \cos(n\pi x) = 0 \\
\int_0^1 \phi_3 dx - \mu \int_0^1 \psi_3 dx = 0,
\end{cases}
\]

(33)
where \((\phi_i, \psi_i)^T \in \mathbb{Z}(n)\) \((i = 2, 3)\). Multiplying the first equation of (32) by \(\cos(n\pi x)\) and then integrating on \([0, 1]\) by parts, we have

\[
\bar{P}^*(\phi_2, \cos(n\pi x)) > -K_1(n\pi)^2 < \cos(n\pi x), \cos(n\pi x) >
-\frac{\bar{P}^*}{K} < (\phi_2 + a\psi_2), \cos(n\pi x) > \frac{1}{K}(1 + au_2) < \cos(n\pi x))^2, \cos(n\pi x) >
\]

\[ -\bar{P}^*\eta_2 < 1, \cos(n\pi x) >= 0. \tag{34} \]

Similarly, multiplying the second equation of (32) by \(\cos(n\pi x)\) and then integrating on \([0, 1]\) by parts, we have

\[
\bar{X}^*(\psi_2, \cos(n\pi x)) > -u_2K_1(n\pi)^2 < \cos(n\pi x), \cos(n\pi x) >
-\frac{\bar{X}^*}{K} < (\psi_2 + b\phi_2), \cos(n\pi x) > -\frac{u_2}{K}(u_2 + b) < (\cos(n\pi x))^2, \cos(n\pi x) >
\]

\[ -d(\mu)\bar{X}^*\eta_2 < 1, \cos(n\pi x) >= 0. \tag{35} \]
By using the relations

\[
\begin{align*}
<\dot{\phi}_2, \cos(n\pi x)> &= -(n\pi)^2 <\phi_2, \cos(n\pi x)>, \quad <1, \cos(n\pi x)> = 0, \\
<\cos(n\pi x), \cos(n\pi x)> &= \frac{1}{2}, \quad <(\cos(n\pi x))^2, \cos(n\pi x)> = 0, \\
<\dot{\psi}_2, \cos(n\pi x)> &= -(n\pi)^2 <\psi_2, \cos(n\pi x)>,
\end{align*}
\]

(36)

lead to the following linear system:

\[
A(\xi^*) \begin{bmatrix} <\phi_2, \cos(n\pi x)> \\ <\psi_2, \cos(n\pi x)> \end{bmatrix} = -K_1 \cdot \frac{(n\pi)^2}{2} \begin{bmatrix} 1 \\ u_2 \end{bmatrix},
\]

(37)

where \(A(\xi^*)\) is defined in (18). Note that \(A(\xi^*)\) is singular and the eigenvector corresponding to the zero eigenvalue is \((1, u_2)^T\). Then the solvability condition of (37) is \(K_1 = 0\). Furthermore (37) and the relation that \((\phi_2, \psi_2)^T \in Z(n)\) lead to

\[
<\phi_2, \cos(n\pi x)> = 0 = <\psi_2, \cos(n\pi x)>.
\]

(38)

Here we note that \(K_1 = 0\) and integrate (32) on \([0, 1]\), then we have

\[
\begin{bmatrix} 1 & a & K \\ b & 1 & d(\mu)K \\ 1 & -\mu & 0 \end{bmatrix} \begin{bmatrix} <\phi_2, 1> \\ <\psi_2, 1> \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\frac{(1 + au_2)}{2P^*} \\ \frac{u_2(u_2 + b)}{2X^*} \\ 0 \end{bmatrix}.
\]

(39)

We can solve this equation explicitly and obtain (25).

Next, let us consider (33) with \(K_1 = 0\). Multiplying the first and the second equations of (33) by \(\cos(n\pi x)\) and then integrating on \([0, 1]\) by parts, we have

\[
\begin{align*}
\varepsilon^0_0(\xi^*) <\dot{\phi}_3, \cos(n\pi x)> &= -K_2 (n\pi)^2 <\cos(n\pi x), \cos(n\pi x)> \\
&= -\frac{P^*}{K} <\phi_3 + \psi_3, \cos(n\pi x)> \\
&= -\frac{1}{K} <\phi_2 + a\psi_2, \cos(n\pi x), \cos(n\pi x)> \\
&= -\frac{1}{K} (1 + au_2) <\phi_2, \cos(n\pi x), \cos(n\pi x)> -\dot{P}^* \eta_3 <1, \cos(n\pi x)> \\
&= -\eta_2 <\cos(n\pi x), \cos(n\pi x)> = 0,
\end{align*}
\]

\[
\varepsilon^0_0(\xi^*) <\dot{\psi}_3, \cos(n\pi x)> = -u_2 K_2 (n\pi)^2 <\cos(n\pi x), \cos(n\pi x)> \\
&= -\frac{X^*}{K} <\psi_3 + b\phi_3, \cos(n\pi x)> \\
&= -\frac{u_2}{K} <\psi_2 + b\phi_2, \cos(n\pi x), \cos(n\pi x)> \\
&= -\frac{1}{K} (u_2 + b) <\psi_2, \cos(n\pi x), \cos(n\pi x)> -d(\mu)X^* \eta_3 <1, \cos(n\pi x)> \\
&= -d(\mu)u_2 \eta_2 <\cos(n\pi x), \cos(n\pi x)> = 0.
\]

(40)
By using the relations (36) and

\[
\begin{align*}
< \dot{\psi}, \cos(n\pi x) > &= -(n\pi)^2 < \phi, \cos(n\pi x) >, \\
< \dot{\phi}, \cos(n\pi x) > &= -(n\pi)^2 < \psi, \cos(n\pi x) >, \\
< (\phi + a\psi) \cos(n\pi x), \cos(n\pi x) > &= \frac{1}{2} \{ < \phi, \cos(2n\pi) > + a < \psi, \cos(2n\pi) > \\
&\quad + < \phi, 1 > + a < \psi, 1 > \}, \\
< \phi^2 \cos(n\pi x), \cos(n\pi x) > &= \frac{1}{2} \{ < \phi, \cos(2n\pi) > + < \phi, 1 > \}, \\
< (\psi + b\phi) \cos(n\pi x), \cos(n\pi x) > &= \frac{1}{2} \{ b < \phi, \cos(2n\pi) > + < \psi, \cos(2n\pi) > \\
&\quad + b < \phi, 1 > + < \psi, 1 > \}, \\
< \psi^2 \cos(n\pi x), \cos(n\pi x) > &= \frac{1}{2} \{ < \psi, \cos(2n\pi) > + < \psi, 1 > \},
\end{align*}
\]

leads to the linear system

\[
A(\xi^*) \begin{bmatrix} < \phi, \cos(n\pi x) > \\ < \psi, \cos(n\pi x) > \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},
\]

where

\[
p_1 = \frac{(n\pi)^2}{2} \mathcal{K} - \frac{(2 + au)u_2}{2K} < \phi, \cos(2n\pi) > - \frac{a}{2K} < \psi, \cos(2n\pi) > \\
&\quad - \frac{(2 + au)u_2}{2K} < \phi, 1 > - \frac{a}{2K} < \psi, 1 > - \frac{1}{2} \eta_2,
\]

\[
p_2 = -u_2 \frac{(n\pi)^2}{2} \mathcal{K} - \frac{bu_2}{2K} < \phi, \cos(2n\pi) > - \frac{(2u_2 + b)}{2K} < \psi, \cos(2n\pi) > \\
&\quad - \frac{bu_2}{2K} < \phi, 1 > - \frac{(2u_2 + b)}{2K} < \psi, 1 > - \frac{1}{2} d(\mu)u_2 \eta_2.
\]

Similarly to the case $O(s^3)$, we find that the solvability condition of (41) is

\[
(1 + u_2^2)(n\pi)^2 \mathcal{K} + (2 + au_2 + bu_2^2) < \phi, \cos(2n\pi) > + < \phi, 1 > /K \\
+ (a + bu_2 + 2u_2^2) < \psi, \cos(2n\pi) > + < \psi, 1 > /K
\]

\[
+ \eta_2 (1 + d(\mu)u_2^2) = 0,
\]

from which we have (27). Also we have

\[
< \phi, \cos(n\pi x) > = 0 = < \psi, \cos(n\pi x) >.
\]

Finally, multiplying the first and the second equations of (32) with $\mathcal{K} = 0$ by $\cos(2n\pi x)$ and then integrating on $[0, 1]$ by parts, we have (26). Here we used the relations (36), $n^2 \varepsilon_0^2(\xi^*) = \varepsilon_0^2(\xi^*)$ and $< \cos(n\pi x) >, \cos(2n\pi x) >= 1/4$. Then the proof is completed.
5.3. **Proof of Theorem 4.2.** Differentiating (31) with respect to $\varepsilon$, setting $\varepsilon = \varepsilon_0^*(\xi^*)$ and using $\nu(\varepsilon_0^*(\xi^*); n) = 0$, we have

\[
\begin{align*}
\dot{\nu}(\varepsilon_0^*(\xi^*); n)P &= \frac{P^*}{K} \dot{P} - \frac{a}{K} P^* \dot{X} \\
&\quad - \frac{P^* \dot{\xi} + \varepsilon_0^n(\xi^*)P_{xx}}{\varepsilon_0^n(\xi^*)}P_{xx} + P_{xx} \\
\dot{\nu}(\varepsilon_0^*(\xi^*); n)X &= -\frac{b}{K} \dot{X} + \frac{\dot{X}^*}{K} \dot{X} \\
&\quad - d(\mu)X^* \dot{\xi} + \frac{\varepsilon_0^n(\xi^*)}{\nu(\varepsilon_0^*(\xi^*); n)}X_{xx} + X_{xx},
\end{align*}
\]

(45)

where $(P(x; \varepsilon_0^*(\xi^*), n), X(x; \varepsilon_0^*(\xi^*); n), \xi(\varepsilon_0^*(\xi^*); n)) = (\cos(n\pi x), u_2 \cos(n\pi x), 0)$ and $\bar{P}(x; \varepsilon_0^*(\xi^*); n) = \frac{\partial}{\partial \varepsilon} P(x; \varepsilon; n)\bigg|_{\varepsilon = \varepsilon_0^*(\xi^*)}$, $\dot{X}(x; \varepsilon_0^*(\xi^*); n) = \frac{\partial}{\partial \varepsilon} X(x; \varepsilon; n)\bigg|_{\varepsilon = \varepsilon_0^*(\xi^*)}$, $\dot{\xi}(\varepsilon_0^*(\xi^*); n) = \frac{\partial}{\partial \varepsilon} \delta_\varepsilon(\varepsilon; n)|_{\varepsilon = \varepsilon_0^*(\xi^*)}$.

Multiplying the first and second equations of (45) by $\cos(n\pi x)$ and integrating them on $[0, 1]$ by parts, we have

\[
A(\xi^*) \begin{bmatrix}
\langle \dot{P}(x; \varepsilon_0^*(\xi^*); n), \cos(n\pi x) \rangle \\
\langle \dot{X}(x; \varepsilon_0^*(\xi^*); n), \cos(n\pi x) \rangle
\end{bmatrix}
= -\frac{1}{2} \begin{bmatrix}
\{\nu(\varepsilon_0^*(\xi^*); n) + n^2 \pi^2\} \\
u_2 \{\nu(\varepsilon_0^*(\xi^*); n) + n^2 \pi^2\}
\end{bmatrix}.
\]

(46)

Since the coefficient matrix is singular and its kernel is spanned by $(1, u_2)^T$, (46) has a unique solution $(\langle \dot{P}(x; \varepsilon_0^*(\xi^*); n), \cos(n\pi x) \rangle, \langle \dot{X}(x; \varepsilon_0^*(\xi^*); n), \cos(n\pi x) \rangle)^T$ if and only if the vector of the right-hand of (46) is perpendicular to $(1, u_2)^T$. That is,

\[
\dot{\nu}(\varepsilon_0^*(\xi^*); n) = -n^2 \pi^2 < 0
\]

(47)

holds. On the other hand, by virtue of Theorem 1.16 in [2] we find that for any $s \in (-\delta, \delta)$, $\lambda(s; n)$ and $-s\dot{\nu}(\varepsilon_0^*(\xi^*); n)\dot{\varepsilon}_n(s)$ have the same sign and the same zeros. Furthermore

\[
\lim_{s \to 0, \lambda(s; n) \neq 0} -s\dot{\nu}(\varepsilon_0^*(\xi^*); n)\dot{\varepsilon}_n(s) / \lambda(s; n) = 1.
\]

(48)

Noting that $\dot{\varepsilon}_n(s) = \frac{d}{ds}\varepsilon_n(s) = 2s\kappa_2(n) + o(s)$ (see [24]), we have Theorem 4.2.

This completes the proof.

6. **Concluding remarks.** In this paper, we have considered the one dimensional shadow system given in (12). Such system is derived from the three component reaction-diffusion system of Lotka-Volterra type (3) with sufficiently large $D$. Under the zero-flux boundary conditions, we have shown, in Theorem 3.1 that (12)
with (13) has at least a pair of non-constant equilibrium solutions. Among these solutions, an essential one is given by a pair of 1-mode equilibrium solutions. Furthermore, by using the local bifurcation theory and complementarily the numerical continuation software AUTO, we have suggested that the number of 1-mode equilibrium solutions and their stability depend not only on the interval length $L$ but also on the toxicity $\mu$, as shown in Figures 12, 14 and 16. This result gives one of the most important information to understand the occurrence of harmful algal bloom arising in ecosystems from a theoretical point of view. This is also interesting in the following mathematical sense: though $(\bar{P}_1^\pm(x; \xi_1^\pm(\varepsilon), \varepsilon), \bar{X}_1^\pm(x; \xi_1^\pm(\varepsilon), \varepsilon))$ are unstable equilibrium solutions of the two component competition system (14) (8), $(\bar{P}_1^\pm(x; \xi_1^\pm(\varepsilon), \varepsilon), \bar{X}_1^\pm(x; \xi_1^\pm(\varepsilon), \varepsilon), \bar{\xi}_1^\pm(\varepsilon))$ become stable equilibrium solutions of the shadow system (12) with (13). That is, the presence of a third component $\xi$ changes the stability properties.

In this work, we restricted our discussion to the shadow system as $D \to \infty$. However, we address the following natural question: what is the dynamics of the full system $\mathcal{S}$ with relative large $D$? Regarding this question, we could numerically answer that the global structure of non-constant equilibrium solutions of the full system converges to the one of the shadow system, as $D \to \infty$. Along this direction, we refer the paper by Nishiura [12] that discusses a stationary problem of a two component activator-inhibitor reaction diffusion system. The author shows that bifurcating branches of equilibrium solutions of a shadow system are nice approximations to the ones of the corresponding full system with a suitable norm when one of the diffusion coefficients becomes large. This gives a partial answer to the above question for the two component-system. The convergence of the time dependent problem of the two systems is also very important. Such problem has not yet been solved completely and will be the object of future work.

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