Abstract

We start the study of the enumeration complexity of different satisfiability problems in first-order team logics. Since many of our problems go beyond DelP, we use a framework for hard enumeration analogous to the polynomial hierarchy, which was recently introduced by Creignou et al. (Discret. Appl. Math. 2019). We show that the problem to enumerate all satisfying teams of a fixed formula in a given first-order structure is DelNP-complete for certain formulas of dependence logic and independence logic. For inclusion logic formulas, this problem is even in DelP. Furthermore, we study the variants of this problem where only maximal, minimal, maximum and minimum solutions, respectively, are considered. For the most part these share the same complexity as the original problem. An exception is the minimum-variant for inclusion logic, which is DelNP-complete.

1 Introduction

Decision problems in general ask for the existence of a solution to some problem instance. In contrast, for enumeration problems we aim at generating all solutions. For many—or maybe most—real-world tasks, enumeration is therefore more natural or practical to study; we only have to think of the domain of databases where the user is interested in all answer tuples to a database query. Other application areas include web search engines, data mining, web mining, bioinformatics and computational linguistics. From a theoretical point of view, maybe the most important problem is that of enumerating all satisfying assignments of a given propositional formula.

Clearly, even simple enumeration problems may produce a big output. The number of satisfying assignments of a formula can be exponential in the length of the formula. In [14], different notions of efficiency for enumeration problems were first proposed, the most important probably being DelP (“polynomial delay”), consisting of those enumeration problems where, for a given instance $x$, the time between outputting any two consecutive solutions as well as pre- and postcomputation times (see [18]) are polynomially bounded in $|x|$. Another notion of tractability is captured by the class IncP where the delay and post-computation time can also depend on the number of solutions that were already output. The separation DelP $\subset$ IncP was mentioned in [20], although one should note that slightly
different definitions were used there. Several examples of membership results for tractable classes can be found in [17, 16, 13, 10, 7, 6]. As a notion of higher complexity, recently an analogue of the polynomial hierarchy for enumeration problems has been introduced [2]. Lower bounds for enumeration problems are obtained by proving hardness (under a suitable reducibility notion) in a level $\Sigma^p_k$ of that hierarchy for some $k \geq 1$ and are regarded as evidence for intractability.

Here, we consider enumeration tasks for so-called team-based logics, where first-order formulas with free variables are evaluated in a given structure not for a single assignment to these variables but for sets of such assignments; these sets are called teams. The logical language is extended by so-called generalised dependency atoms (sometimes referred to as team atoms) that allow to specify properties of teams, e.g., that the value of a variable functionally depends on some other variable(s) (the dependence atom $=\ldots$ [21]), that a variable is independent of some other variable(s) (the independence atom $\perp$ [11]), or that the values of a variable occur as values of some other variable(s) (the inclusion atom $\subseteq$ [8]).

Team-based logics were introduced by Jouko Väänänen [21] and have been used for the study of various dependence and independence concepts important in many areas such as database theory and Bayesian networks (see, e.g., the articles in the textbook by Abramsky et al. [1]).

For a fixed first-order formula and a given input structure, the complexity of the problem of counting all satisfying teams has been studied by Haak et al. [12], where completeness for classes such as $\# \cdot \text{P}$ and $\# \cdot \text{NP}$ was obtained. In the enumeration context, and in analogy to the case of classical propositional logic as above, it is now natural to ask for algorithms to enumerate all satisfying teams of a fixed formula in a given input structure. Enumerating teams for formulas with the above mentioned dependency atom thus means enumerating all sets of tuples in a relational database that fulfil the given Boolean combination of FO-statements and functional dependencies. In this paper, we consider this problem and initiate the study of enumeration complexity for team based logics. Notice that, the task of enumerating teams has been considered before in the propositional setting by Meier and Reinbold [18]. We consider team-based logics with the inclusion, the dependence and the independence atom, and study the problems of enumerating all satisfying teams or certain optimal satisfying teams, where optimal can mean maximal or minimal with respect to inclusion or cardinality. Our results are summarised in Table 1 on p. 12. It is known that in terms of expressive power dependence logic corresponds to the class $\text{NP}$. Hence one cannot expect efficient algorithms for enumerating teams, and in fact, we prove that the problem is DelNP-complete (i.e., Del$\Sigma^p_1$-complete) in all variants (enumerating all or optimal satisfying teams). Analogous results hold for independence logic. Inclusion logic, however, in a model-theoretic sense is equal to the class $\text{P}$ (at least in so-called lax semantics [9]). Consequently, inclusion logic is less expressive than dependence logic (under the assumption $\text{P} \neq \text{NP}$), and the picture in the enumeration context reflects this: We prove that for each inclusion logic formula, there is a polynomial-delay algorithm for enumerating all satisfying teams in a given structure. This is also true when we want to enumerate all maximal, minimal, or maximum satisfying teams. Interestingly, enumerating minimum satisfying teams is DelNP-complete, as for the other logics we consider.

In the next section, we introduce team semantics and the relevant logics. There, we also introduce algorithmic enumeration and the needed complexity classes, and we formally define the enumeration problems we want to classify in this paper. In Sect. 3 we present an efficient enumeration algorithms for inclusion logic, while Sect. 4 is devoted to the presentation of our completeness proofs for the class DelNP. Finally, we summarise our results and conclude with some open questions. Due to space restrictions, most proofs are only sketched in the
paper, but all full details can be found in the appendix.

2 Definitions and Preliminaries

We assume familiarity with basic notations from complexity theory [19]. We will make use of the complexity classes P and NP.

2.1 Team logic

A vocabulary \( \sigma \) is a finite set of relations with corresponding arities. For each relation \( R \in \sigma \) denote by \( \text{ar}(R) \in \mathbb{N}_+ \) the arity of \( R \). A \( \sigma \)-structure \( A = (A, (R_i)_{R_i \in \sigma}) \) consists of a universe \( A \) that is a set, and an interpretation of the relations of \( \sigma \) in \( A \), i.e., \( R_i^A \subseteq A^{\text{ar}(R)} \) for each \( R_i \in \sigma \). Let \( D \) be a finite set of first-order variables and \( A \) be some set. An assignment \( s: D \rightarrow A \) is a function over domain \( D \) and codomain \( A \). The algorithms that we construct later assume an arbitrary order on assignments and thereby on singleton teams. For our purposes a lexicographically ordered assignment suffices. Moreover, if \( s \leq t \) and there exists a \( 1 \leq j \leq n \) such that \( s(x_j) < t(x_j) \) then we write \( s < t \).

Given an assignment \( s \), a variable \( x \) and an element \( a \) from \( A \), the assignment \( s(a/x): D \cup \{x\} \rightarrow A \) is defined by \( s(a/x)(y) =_{\text{def}} a \) and \( s(a/x)(y) =_{\text{def}} s(y) \) for \( y \neq x \). We call \( s(a/x) \) a supplemetning function. A team is a finite set of assignments with common domain and codomain. For a team \( X \), let \( \max(X) \) be the largest assignment contained in \( X \) with respect to the lexicographical order on assignments defined before.

Considering a team \( X \), a finite set \( A \), and a function \( F: X \rightarrow \mathcal{P}(A) \setminus \{\emptyset\} \), we then define \( X[A/x] \) as the modified team \{ \( s(a/x) \mid s \in X, a \in A \) \}. Furthermore, we denote by \( X[F/x] \) the team \{ \( s(a/x) \mid s \in X, a \in F(s) \) \}. If \( X \) is a team whose codomain is the universe of a \( \sigma \)-structure \( A \), we say \( X \) is a team of \( A \).

Now, we proceed with the definition of syntax and semantics of first-order team logic. Let \( \sigma \) be a vocabulary. Then, the syntax of first-order team logic, FO[\( \sigma \)], is defined by the following grammar:

\[
\varphi ::= x = y \mid x \neq y \mid R(\bar{x}) \mid \neg R(\bar{x}) \mid (\varphi \land \psi) \mid (\varphi \lor \psi) \mid \exists x.\varphi \mid \forall x.\varphi,
\]

where \( \bar{x} \) is a tuple of first-order variables, \( x, y \) are first-order variables, and \( R \in \sigma \). Notice that we restricted the syntax to atomic negation. The reason for that restriction is the high complexity of problems on formulas with arbitrary negation symbols both in first-order as well as propositional logic [21, 13].

\begin{definition}[Team semantics] Let \( \sigma \) be a vocabulary, \( A \) be a \( \sigma \)-structure, \( X \) be a team of \( A \), \( x, y \) be first-order variables, \( \bar{\psi} \) be a tuple of first-order variables, \( R \) be a relation symbol, and \( \varphi, \psi \in \text{FO}(\sigma) \). The satisfaction relation \( \models_X \) for FO[\( \sigma \)]-formulas is defined as:

\[
\begin{align*}
A \models_X x = y & \iff \forall s \in X \text{ we have that } s(x) = s(y), \\
A \models_X x \neq y & \iff \forall s \in X \text{ we have that } s(x) \neq s(y), \\
A \models_X R(\bar{x}) & \iff \forall s \in X \text{ we have that } s(\bar{x}) \in R^A, \\
A \models_X \neg R(\bar{x}) & \iff \forall s \in X \text{ we have that } s(\bar{x}) \notin R^A, \\
A \models_X (\varphi \land \psi) & \iff A \models_X \varphi \text{ and } A \models_X \psi, \\
A \models_X (\varphi \lor \psi) & \iff \exists Y, Z \subseteq X \text{ with } Y \cup Z = X \text{ and } A \models_Y \varphi \text{ and } A \models_Z \psi, \\
A \models_X \exists x.\varphi & \iff A \models_{X[A/x]} \varphi, \\
A \models_X \exists x.\varphi & \iff A \models_{X[F/x]} \varphi \text{ for some } F: X \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}.
\end{align*}
\end{definition}
4 Enumerating Teams in First-Order Team Logics

If the underlying vocabulary is clear from the context or not relevant, we usually omit $\sigma$ the expression $\text{FO}[\sigma]$ and write $\text{FO}$ instead. Let $\varphi \in \text{FO}$ be a first-order team logic formula. We denote by $\text{free}(\varphi)$ the set of free variables in $\varphi$. Observe that on singletons, the semantics of $\varphi_1 \lor \varphi_2$ resemble that of the classical disjunction. On teams, however, this generalises to the so-called split junction operator which literally splits the team into (not necessarily disjoint) parts where each of the formulas $\varphi_1$ and $\varphi_2$ has to be satisfied by one of the parts. Notice that the previously defined semantics are called lax semantics. Furthermore, observe that the empty team satisfies any formula. This yields the desirable flatness property (a team satisfies a formula if and only if every assignment/singleton from the team satisfies the formula).

Example 2. Consider the formula $\varphi \overset{\text{def}}{=} R(x,y) \lor \neg R(x,y)$, the structure $\mathcal{A}$ with $R^\mathcal{A} = \{(0,1), (1,0)\}$ and the team $X = \{s_1, s_2\}$ defined with $s_1(x) = 0, s_1(y) = 1$, and $s_2(x) = 1 = s_2(y)$. Then $\mathcal{A} \models_X \varphi$ as we can split $X$ into $X_1 = \{s_1\}$ and $X_2 = \{s_2\}$ such that $\mathcal{A} \models_X R(x,y)$ and $\mathcal{A} \models_X \neg R(x,y)$.

Additionally to the connectives defined in the FO-syntax above, we will make use of so-called generalised dependency atoms. We will use the dependence atom $=_{\mathcal{A}}(\mathcal{A}, y)$, the inclusion atom $\subseteq_{\mathcal{A}}(\mathcal{A}, \mathcal{B})$ and the independence atom $\perp_{\mathcal{A}}(\mathcal{A}, \mathcal{C})$ where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are tuples of first-order variables and $y$ is a first-order variable. Now for any subset $\mathcal{A} \subseteq \{=, (\ldots), \subseteq, \perp\}$, we define $\text{FO}(\mathcal{A})$ as first-order logic extended by the respective atoms. More precisely, we extend the grammar $\{E\}$ by adding a rule for each atom in $\mathcal{A}$. For example, for $\text{FO}(\{\subseteq\})$ we add the rule $\varphi := :_{\subseteq}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ for any tuples $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of FO-variables. For convenience, we often omit the curly brackets and write for example $\text{FO}(\subseteq)$ instead of $\text{FO}(\{\subseteq\})$. The logics $\text{FO}(=)$, $\text{FO}(\subseteq)$ and $\text{FO}(\perp)$ are called dependence logic, inclusion logic and independence logic, respectively.

Intuitively, an independence atom expresses that two tuples are independent with respect to a third tuple. A tuple $\mathcal{A}$ depends on another tuple $\mathcal{C}$, so $=_{\mathcal{A}}$, if for every pair of assignments from the team that agree on $\mathcal{A}$ also agree on $\mathcal{C}$. This is the idea of functional dependency in the database setting. A tuple $\mathcal{A}$ is included in a tuple $\mathcal{C}$, that is $\subseteq_{\mathcal{A}}$, if for every assignment $t_1$ in the team there exists another one $t_2$ such that $\mathcal{A}$ under $t_1$ coincides with $\mathcal{C}$ under $t_2$.

Before we formally define the semantics for these three atoms, we need to introduce a little bit of notation. If $\mathcal{A} = (x_1, \ldots, x_n)$ is a tuple of first-order variables for $n \in \mathbb{N}$, and $s$ is an assignment, then $\mathcal{A}(s) = \overset{\text{def}}{=} (s(x_1), \ldots, s(x_n))$.

Definition 3 (Generalised dependency atoms semantics). Let $\sigma$ be a vocabulary, $\mathcal{A}$ be a $\sigma$-structure, $X$ be a team of $\mathcal{A}$, and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be tuples of first-order variables. The satisfaction relation $\models_X$ for $\text{FO}(\sigma)$-formulas then is extended as follows:

- $\mathcal{A} \models_X \mathcal{A} \subseteq_{\mathcal{A}} \mathcal{B}$ $\iff$ $\forall s, t \in X \text{ with } \mathcal{A}(s) = \mathcal{B}(t)$ $\exists u \in X \text{ such that } \mathcal{A}(u) = \mathcal{A}(s)$.

- $\mathcal{A} \models_X \mathcal{A} \subseteq_{\mathcal{A}} \mathcal{B}$ $\iff$ $\forall s \in X \exists t \in X \text{ such that } \mathcal{A}(s) = \mathcal{A}(t)$.

- $\mathcal{A} \models_X \mathcal{A} \subseteq_{\mathcal{A}} \mathcal{B}$ $\iff$ $\forall s, t \in X \text{ we have that } \mathcal{A}(s) = \mathcal{A}(t) \text{ implies } \mathcal{A}(s) = \mathcal{A}(t)$.

In the following, we define the model checking problem on the level of first-order team logic formulas in the setting of data complexity (fixed formula).

| Problem: | VERIFYTEAM$\varphi$ |
|----------|---------------------|
| Input:   | Structure $\mathcal{A}$, team $X$ |
| Question:| $\mathcal{A} \models_X \varphi \wedge X \neq \emptyset$ |

Lemma 4. Let $\mathcal{A} \subseteq \{\perp, \subseteq, =\}$, $\varphi \in \text{FO}(\mathcal{A})$. Then $\text{VERIFYTEAM}_\varphi \in \text{NP}$.
Proof. Every fixed formula is of bounded width (width is the maximal number of free
variables in subformulas of a given formula). As all of the generalised dependency atoms
in \( A \) can be evaluated in polynomial time, a result from Grädel [10] Theorem 5.1 applies,
yielding \( \text{VerifyTeam}_\varphi \in \text{NP} \).

Our algorithms often start with either \( \emptyset \) or \( \text{dom}(A)^{\text{free}(\varphi)} \) (the full team) as one of their
inputs, for a fixed formula \( \varphi \) and a structure \( A \). Instead of \( \text{dom}(A)^{\text{free}(\varphi)} \) we will write \( X \).

The following proposition summarises important results from literature that are referenced
later in proofs. It mainly states key connection between team logics and predicate logic, also
mentioning descriptive complexity results that are consequences of these connections.

\[\text{Proposition 5 (}[8,16,9] ).\]
1. Over sentences both \( \text{FO}(\bot) \) and \( \text{FO}(=\ldots) \) are expressively equivalent to \( \Sigma_1^{\text{free}} \).
   Every \( \sigma \)-sentence of \( \text{FO}(\bot) \) (or \( \text{FO}(=\ldots) \)) is equivalent to a \( \sigma \)-sentence \( \psi \) of \( \Sigma_1^{\text{free}} \), i.e., for
   any \( \sigma \)-structure \( A \), \( A \models \varphi \iff A \models \psi \), and vice versa. As a consequence of Fagin’s
   Theorem [7], over finite structures both \( \text{FO}(\bot) \) and \( \text{FO}(=\ldots) \) capture \( \text{NP} \).
2. Let \( \varphi(R) \) be a myopic \( \sigma \)-formula, that is, \( \varphi(R) = \forall \varphi(R(\overline{x})) \rightarrow \psi(R(\overline{x})) \), where \( \psi \) is a
   first order \( \sigma \)-formula with only positive occurrences of \( R \). Then there exists a \( \sigma \)-formula
   \( \chi \in \text{FO}(\subseteq) \) such that for all \( \sigma \)-structures \( A \) and all teams \( X \) we have \( A \models \chi(\overline{x}) \iff A, \text{rel}(X) \models \varphi(R) \).

2.2 Enumeration

For the basics of enumeration complexity theory, we follow Creignou et al. [4].

In contrast to decision problems where one gets an input and often has to answer whether
there is a “solution” to the input, for enumeration problems one has to compute the set of
all solutions to the input. As an example see the difference between the decision problem
\( \text{SAT}^\text{team}_\varphi \) and the enumeration problem \( \text{E-SAT}^\text{team}_\varphi \).

| Problem: \( \text{SAT}^\text{team}_\varphi \) | Problem: \( \text{E-SAT}^\text{team}_\varphi \) |
|----------|----------|
| Input: Structure \( A \) | Input: Structure \( A \) |
| Question: \( \{ X \mid A \models \chi \varphi \wedge X \neq \emptyset \} \neq \emptyset \) | Output: \( \{ X \mid A \models \chi \varphi \wedge X \neq \emptyset \} \) |

Note that for all our problem definitions, if not otherwise stated, \( \varphi \) is a formula from \( \text{FO}(A) \)
for some \( A \subseteq \{=\ldots,\subseteq,\bot\} \).

As these sets can get exponentially large compared to the input our, classical measures
(like runtime of the machine/algorithm) will not suffice. To be able to talk about tractability
and intractability of problems in the enumeration setting we need to define new classes.

The idea is that we will not bound the time of the whole computation, but the time of the
computations between the outputs of two consecutive solutions, which we will call \( \text{delay} \).
Instead of Turing machines we will use random access machines (RAMs), to be able to access
the (potentially) exponential “memory” in polynomial time.

\[\text{Definition 6 (}[4] ).\] Let \( C \) be a decision complexity class and \( p \) be a polynomial. The
enumeration class \( \text{DelC} \) consists of all enumeration problems \( E \), for which there exists a
\( \text{RAM} \) \( M \) with oracle \( L \in C \) such that for all inputs \( x \), \( M \) enumerates the output set of \( E \)
with \( p(|x|) \) delay and all oracle queries are bounded by \( p(|x|) \).

To be able to show hardness for our new classes we need a suitable definition of reducibility.
The reduction we use is quite similar to a turing reduction in the decision case. For this we
give a machine access to an enumeration oracle to solve another enumeration problem. The
kind of machine we use here is called \( \text{enumeration oracle machine} \) (EOM) which is a RAM
with some new special registers: an infinite number of registers for the oracle questions and one register for the answer. The machine can write an oracle question into the respective registers (one bit per register) and in one step the answer appears in the register for the answer. If there are further solutions to the question that were not given before, the answer is a solution. Otherwise, the answer is a special symbol, meaning that all solutions have been given. The machines that we use are also oracle-bounded, that is, all oracle questions are polynomial in the size of the input.

Definition 7 ([4]). Let \( E_1, E_2 \) be enumeration problems. We say that \( E_1 \) reduces to \( E_2 \) via \( D \)-reductions, \( E_1 \leq_D E_2 \), if there is an oracle-bounded EOM \( M \) that enumerates \( E_1 \) using oracle \( E_2 \) with polynomial delay and independently of the order in which the \( E_2 \)-oracle enumerates it answers.

Proposition 8 ([4]). The class \( \text{Del} \Sigma^p_k \) is closed under \( D \)-reductions for any \( k \in \mathbb{N} \).

Let \( E \) be the enumeration problem, given input \( x \), to output the set of solutions \( S(x) \). We denote by \( \text{Exist-E} \) the problem to decide, given \( x \), whether \( |S(x)| \geq 1 \).

Proposition 9 ([4]). Let \( E \) be an enumeration problem and \( k \geq 1 \) such that \( \text{Exist-E} \) is \( \Sigma^p_k \)-hard. Then we have that \( E \) is \( \text{Del} \Sigma^p_k \)-hard under \( D \)-reductions.

We slightly generalise this theorem:

Theorem 10. Let \( A \) be an \( \Sigma^p_k \)-hard decision problem and let \( E \) be an enumeration problem such that \( A \) can be decided in polynomial time by an algorithm that has access to oracle \( E \). Then it holds that \( E \) is \( \text{Del} \Sigma^p_k \)-hard under \( D \)-reductions.

Proof. The proof is essentially the same as the one for Prop. 9. Let \( B \in \text{Del} \Sigma^p_k \) and \( L \in \Sigma^p_k \) be a witness for \( B \in \text{Del} \Sigma^p_k \), that is, there is an algorithm with access to oracle \( L \) that enumerates \( B \) with polynomial delay. Since \( A \) is \( \Sigma^p_k \)-hard and by the precondition of the theorem (\( A \) can be decided in polynomial time by an algorithm with an \( E \)-oracle), we can answer the oracle questions to \( L \) by asking \( E \) instead. It follows that \( B \) can be enumerated by an algorithm with an \( E \)-oracle with polynomial delay.

We will close this subsection defining four more enumeration problems. In the following two sections we analyse the complexity of the defined problems for our different logics.

---

### Efficient Enumeration

In this section, we study the class \( \text{DelP} \). All the results are for inclusion logic and rely on the fact that \( \text{MaxSubTeam} \)—the problem to compute the maximal subteam of a given team satisfying a given inclusion logic formula in a given structure—is computable in polynomial time [9]. Note that for inclusion logic the maximal satisfying team is unambiguous.
Problem: MaxSubTeam

Input: Structure $A$, formula $\varphi \in \text{FO}(\subseteq)$, team $X$

Output: $X'$ with $A \models_X \varphi \land X' \subseteq X \land \forall X'' \subseteq X: |X''| > |X'| \Rightarrow A \not\models_X \varphi$

**Theorem 11.** For any formula $\varphi \in \text{FO}(\subseteq)$ it holds that $\text{E-Sat}_{\text{team}}^\varphi \in \text{DelP}$.

**Proof.** We construct a recursive algorithm with access to a MaxSubTeam oracle that on input $(A, X, Y)$ enumerates all satisfying subteams $X' \neq \emptyset$ of $X$ with $Y \subseteq X'$. To compute for a given $A$ all satisfying subteams, we then need to run this algorithm on input $(A, X, \emptyset)$.

**Algorithm 1:** Algorithm used to show $\text{E-Sat}_{\text{team}}^\varphi \in \text{DelP}$ for $\varphi \in \text{FO}(\subseteq)$

1. **Function** EnumerateSubteams(structure $A$, teams $X, Y$)
2. $X = \text{MaxSubTeam}(A, X)$
3. if $X \neq \emptyset \land Y \subseteq X$ then
4. output $X$
5. for $s \in X$ do
6. $Y = \{s' \mid s' < s \land s' \in X\}$
7. EnumerateSubteams($A, X \setminus \{s\}, Y$)

The algorithm does not output any solution more than once. In the recursive calls, it only outputs solutions where at least one assignment is omitted from the maximal solution, which is the only solution output before. Also, when the assignment $s$ is chosen in the for-loop, the next recursive call only outputs solutions that omit $s$, but contain all assignments $s' < s$ that were present in $X$. In contrast, in every solution found in previous recursive calls, at least one of the assignments $s' < s$ from $X$ was omitted. On the other hand, the algorithm outputs every solution at least once. Every solution is a subset of the maximal satisfying subteam of $X$ and the algorithm starts with that maximal solution and then recursively looks for all strict subsets of it. This can be seen by noticing that when choosing the assignment $s$ in the for-loop, the next recursive call outputs all satisfying subteams of $X$ that exclude $s$, except for those that also exclude some $s' < s$ from $X$ and were hence output before.

**Theorem 12.** Let $\varphi \in \text{FO}(\subseteq)$. Then $\text{E-MinSat}_{\text{team}}^\varphi \in \text{DelP}$.

**Proof.** This can be proven similar to Theorem 11 by slightly modifying Algorithm 1 such that it takes input $(A, X, Y)$ and computes all inclusion minimal satisfying subteams $X' \neq \emptyset$ of $X$ with $Y \subseteq X'$. The only change needed for this is that it only outputs a team $X$, if MaxSubTeam answers $\emptyset$ for all $X \setminus \{s\}$, where $s \in X$.

The next result follows from the fact, that MaxSubTeam can be computed in polynomial time, since the solution set only consists of the maximal satisfying team for both problems.

**Theorem 13.** For $\varphi \in \text{FO}(\subseteq)$ the problems $\text{E-MaxSat}_{\text{team}}^\varphi, \text{E-CMaxSat}_{\text{team}}^\varphi$ are included in DelP.

Note that there is an enumeration problem we did not mention in this section, which is $\text{E-CMinSat}_{\text{team}}^\varphi$. This is due to the fact, that this problem is actually DelNP-complete as we will see in the next section.
### 4 A Characterisation of DelNP

We show that for certain formulas the problem $E$-$Sat_{\varphi}$ captures the class DelNP. Moreover, we will extend this result to all remaining cases, that is, all combinations of logics and problems we did not classify already in Section 3.

**Theorem 14.** Let $A \subseteq \{=,\ldots,\bot\}$. There exists a formula $\varphi \in FO(A)$ such that the problem $Sat_{\varphi}$ is NP-hard.

**Proof.** We show the result for $A = \{\bot\}$. The proof for $A = \{=,\ldots\}$ works analogously by reducing from the NP-complete problem $\Sigma_1$-$CNF^-$, that is, given a propositional formula $\varphi \in \Sigma_1$-$CNF^-$, decide whether $\varphi$ is satisfiable. Here, $\Sigma_1$-$CNF$ is the class of propositional formulas with existential quantifiers in prenex normal form and where the quantifier-free part is in conjunctive normal form. The negative fragment $\Sigma_1$-$CNF^-$ further restricts formulas by allowing free variables to only occur negatively.

We reduce from the NP-complete problem $CNF$-$Sat$ to the problems $Sat_{rel}^{\varphi}$ and $Sat_{rel}^{\varphi}$ for some $\varphi \in \Sigma_1$, see below for formal definitions. By Proposition 5.1 we get that $Sat_{\varphi}$ is NP-hard, for a formula $\varphi' \in FO(\bot)$. Let $\varphi$ be a $\Sigma_1^2$-formula.

| Problem: $Sat_{\varphi}^{rel}$ | Problem: $Sat_{\varphi}^{rel}$ |
|-------------------------------|-------------------------------|
| Input: Structure $A$ | Input: Structure $A$ |
| Question: $\{R \mid A, R \models \varphi\} \neq \emptyset$ | Question: $\{R \mid R \neq \emptyset \land A, R \models \varphi\} \neq \emptyset$ |

Let $\psi(x_1,\ldots,x_n) = \bigwedge_i^m C_i$ be a propositional formula in conjunctive normal form, with $C_i = \bigvee_j l_{i,j}$. We encode $\psi$ via the structure $A(\psi) = \{x_1,\ldots,x_n,C_1,\ldots,C_m\}$, $P^2,N^2\}$, where $(C,x) \in P$ if and only if variable $x$ occurs positively (negatively) in clause $C$. We define the following $\Sigma_1^2$-formula $\chi(R)$ over vocabulary $(P^2,N^2)$:

$$\chi(R) = \forall C \exists x P(C,x) \land R(x) \lor \exists x N(C,x) \land \lnot R(x).$$

Now, we have that $\exists R : A(\psi), R \models \chi \iff \psi$ is satisfiable, showing $CNF-Sat \leq_p Sat_{rel}^{\varphi}$.

Next, we will show NP-hardness for $Sat_{rel}^{\varphi}$. This follows from an easy reduction from $Sat_{rel}^{\varphi}$ to $Sat_{rel}^{\varphi}$ which holds for all $\varphi \in \Sigma_1$. Let $\varphi'(R) = \varphi(R) \lor \varphi(\emptyset)$. Now, for all structures $A$ we claim that $\exists R : A, R \models \varphi \iff \exists R' \neq \emptyset A, R' \models \varphi'$.

“$\Rightarrow$”: If $A, R \models \varphi$ only holds for $R = \emptyset$, then $A, R' \models \varphi'$ holds for any $R'$, in particular for any $R' \neq \emptyset$. If $A, R \models \varphi$ for any $R \neq \emptyset$, then $A, R \models \varphi'$ also holds.

“$\Leftarrow$”: Since $A, R \not\models \varphi$ for all $R$, in particular we have $A, \emptyset \not\models \varphi$. This immediately shows $A, R \not\models \varphi'$ for all $R$.

**Corollary 15.** For $A \subseteq \{=,\ldots,\bot\}$ there exists a formula $\varphi \in FO(A)$ such that the problems $E$-$Sat_{\varphi}$, $E$-$MaxSat_{\varphi}$, $E$-$CMaxSat_{\varphi}$, $E$-$MinSat_{\varphi}$, $E$-$CMinSat_{\varphi}$ are DelNP-hard.

**Proof.** By Theorem 14, there is a formula $\varphi \in FO(A)$ (with $A \subseteq \{=,\ldots,\bot\}$) such that $Sat_{\varphi}$ is NP-hard. Since $Sat_{\varphi}$ can be decided in polynomial time by an algorithm with oracle access to any of the problems mentioned in this corollary (simply ask the oracle and return “no” if and only if the output is $\bot$), by Theorem 10 it follows that all of these problems are DelNP-hard.

**Theorem 16.** For $A = \{\bot,=,\ldots\} \subseteq \emptyset$ and $\varphi \in FO(A)$, we have that $E$-$Sat_{\varphi}$ is DelNP-hard.

**Proof.** We give a recursive algorithm enumerating $E$-$Sat_{\varphi}$ with polynomial delay, when given oracle access to $ExtEndTeam_{\varphi}$ (for definition see below) and $VerifyTeam_{\varphi}$.
The maximum cardinality

The algorithm searches these teams

We describe a recursive algorithm that enumerates the solutions with polynomial

Proof.

▶ Theorem 18.

Proof.

There is a recursive algorithm that on input \( (A, s_{\min}) \) to get all satisfying teams, where \( s_{\min} \) is the smallest assignment.

Algorithm 2: Algorithm used to show \( \text{E-Sat}_{\varphi}^{\text{team}} \in \text{DelNP} \) for \( \varphi \in \text{FO}(A) \)

\[
\begin{align*}
\text{Function} & \quad \text{EnumerateSuperteams}(\text{structure } A, \text{ team } X) \\
Y & = \bigcup_{s < \text{max}(X)} A \wedge s \wedge X
\end{align*}
\]

1. if \( \text{VerifyTeam}_{\varphi}(A, X) \) then output \( X \)
2. if \( \text{ExtendTeam}_{\varphi}(A, X, Y) \) then
3. \quad forall \( s > \text{max}(X) \) do
4. \quad EnumerateSuperteams(\( A, X \cup \{s\} \))

▶ Theorem 17. For \( A = \{\bot, =, (\ldots), \subseteq\}, \varphi \in \text{FO}(A) \), we have that \( \text{E-CMaxSat}_{\varphi}^{\text{team}} \in \text{DelNP} \).

Proof. There is a recursive algorithm that on input \( (A, X, k) \) enumerates all satisfying superteams of \( X \) having cardinality \( k \) with polynomial delay. The algorithm is very similar to the one used for Theorem 16. The only differences are that \( |X| = k \) is checked before a team \( X \) is output and that \( \text{ExtendCMaxTeam}_{\varphi} \) is used as the oracle instead of \( \text{ExtendTeam}_{\varphi} \).

The maximum cardinality \( k \) can be computed by asking the \( \text{ExtendCMaxTeam}_{\varphi} \) oracle on input \( (A, \emptyset, \emptyset, i) \) for \( i = |\text{dom}(A)|^{|\text{free}(\varphi)|}, \ldots, 1 \).

▶ Theorem 18. Let \( A = \{\bot, =, (\ldots), \subseteq\}, \varphi \in \text{FO}(A) \). Then \( \text{E-MaxSat}_{\varphi}^{\text{team}} \in \text{DelNP} \).

Proof. We describe a recursive algorithm that enumerates the solutions with polynomial delay, given oracle access to \( \text{VerifyTeam}_{\varphi} \) and the problem \( \text{ExtendMaxTeam}_{\varphi} \) defined below.

Problem: \( \text{ExtendMaxTeam}_{\varphi} \)

Input: Structure \( A \), team \( X \)

Output: \( \{X' \mid A \models_{X'} \varphi \land X \subseteq X' \land X' \cap Y = \emptyset \land |X'| = k \} \neq \emptyset \)
Note that \( \text{ExtendMaxTeam}_\varphi \in \text{NP} \): we do not need to check the maximality condition, since if there exists a satisfying superteam then there is also an inclusion maximal one. Also, the extend oracle (\( \text{ExtendMaxTeam} \)) we use here does not require an additional set of assignments \( Y \) and the rule \( X' \cap Y = \emptyset \) as above. This is due to the fact, that we would output teams that are inclusion maximal with respect to the set \( X \setminus Y \), but not with respect to \( X \). In Theorem \ref{thm:extend_max} this was not a problem since we had the maximum cardinality \( k \) to check whether a team is a maximum one. The challenge here is to compute a suitable replacement for \( Y \) to make sure not to output any team twice.

The algorithm takes as input a structure \( A \), a team \( X \) and a set of assignments \( Y \) and outputs all inclusion maximal satisfying superteams of \( X \) that are a subset of \( Y \). One can then run this algorithm on input \( (A, \emptyset, X) \) to enumerate all maximal satisfying teams.

The algorithm consists of three phases and a subroutine that—given input \( (A, X, Y, s) \)—computes the maximal satisfying team \( X' \) with \( X \cup \{s\} \subseteq X' \subseteq Y \). The combination of \( Y \) and \( s \) is our replacement for the missing \( Y' \) in the oracle question: We start with \( Y = X \) and cut assignments \( s' \) from it during the run of the computation when we are sure that we have computed all solutions containing \( s' \). We use the input \( s \) to force it to be in the next output, to compute different solutions with the same \( Y \). The subroutine works similar to Algorithm \ref{algo:extend_max}, that is, it alternates between asking the extend oracle and adding assignments (from \( Y \)), checking with the verify oracle in the end.

In the first phase, we simply compute the first solution by calling the subroutine with the input values \( (A, X, Y, \bot) \). Since there is no chance of duplicate outputs, we do not need to force one certain assignment to be in this solution.

In the second phase, we repeatedly compute solutions that contain at least one assignment \( s \) that has not been in any solution before. If there is no such solution left, we go to phase 3.

In the third phase, we go through all pairs of teams \( A, B \) that have been output before. Note that the set of these teams is only polynomial, since it contains at most one team per assignment (by the design of phase 2). For each pair we start a recursive call of the algorithm with input \( (A, X \cup \{s, s'\}, Y') \) for all pairs \( s \in A \setminus B, s' \in B \setminus A \). Here, \( Y' \) contains all assignments that were contained in solutions we output before in the current recursive call that are larger than both \( s \) and \( s' \). With this approach, for two pairs of teams \( (A, B), (C, D) \) there could be a common pair \( (s, s') \), that is, \( s \in A \setminus B, s' \in B \setminus A \) and \( s \in C \setminus D, s' \in D \setminus C \). To make sure that we do not start a second recursive call for \( s, s' \) we check—every time before we start a recursive call—if there was a call for the current pair before. Note that the set of such pairs is again polynomial, since the set of the teams we analyse in this phase is polynomial. The choice of \( Y' \) ensures that different recursive calls do not compute any common solutions. This is essentially the same idea as in Algorithm \ref{algo:extend_max} where we only add larger assignments to the current team. Note that in phase three, it suffices to work with the assignments and teams that were used before in other solutions, since after phase two every assignment occurring in any solution will be present in one of the computed solutions. ▶

\textbf{Theorem 19.} For \( A \subseteq \{\bot, =, \ldots, \subseteq\} \) and \( \varphi \in \text{FO}(A) \) the problems \( \text{E-MinSat}_\varphi^\text{team} \), \( \text{E-CMinSat}_\varphi^\text{team} \) are included in \( \text{DelNP} \).

\textbf{Proof.} For \( \text{E-MinSat}_\varphi^\text{team} \) we can run a slightly modified version of Algorithm \ref{algo:extend_max} on input \( (A, \emptyset) \), which was originally used for \( \text{E-Sat}_\varphi^\text{team} \). The only modification needed is that the new algorithm terminates after outputting a solution.

We can solve \( \text{E-CMinSat}_\varphi^\text{team} \) similarly, but this time adjust the algorithm we described in Theorem \ref{thm:e-minsat-team}. We compute the minimal \( k \) (instead of the maximal) for which \( \text{ExtendE-MaxTeam}_\varphi \) is true before starting the Algorithm with that \( k \). Also, the new
algorithm again terminates after outputting a solution.

In the next result, we show NP-hardness for the decision problem \( \text{CMinSat}_\varphi^\text{team} \), for an inclusion logic formula \( \varphi \). By this and Theorem \( \text{11} \) we can conclude DelNP-hardness for \( \text{E-CMinSat}_\varphi^\text{team} \). We reduce from the NP-complete problem IS (INDEPENDENTSET) to \( \text{CMinSat}_\varphi^\text{team} \) with two intermediate steps. The problems we need for this reduction are defined next. For this, we represent propositional assignments \( \beta \) by the set of variables it maps to \( 1 \). Also, we call \( |\beta| \) the weight of \( \beta \).

| Problem: \( \text{CMIN SAT}_\varphi^\text{team} \) | Problem: \( \text{CMIN SAT}_\varphi^\text{rel} \) for \( \varphi \in \Sigma_1^1 \) | Problem: IS |
| --- | --- | --- |
| Input: Structure \( A, k \in \mathbb{N} \) | Input: \( A, k \in \mathbb{N} \) | Input: Graph \( G = (V, E), k \in \mathbb{N} \) |
| Question: \( \{ X \mid A \models X \land |X| \leq k \} \neq \emptyset \) | Question: \( \{ R \mid A, R \models \varphi \land |R| \leq k \} \neq \emptyset \) | Question: \( \{ V' \mid \forall u, v \in V'; \{ i, j \} \notin E \land |V'| \geq k \land V' \subseteq V \} \neq \emptyset \) |
| Problem: \( \text{MAX ZEROS DualHorn} \) |   |   |
| Input: Propositional dual-horn formula \( \varphi, k \in \mathbb{N} \) |   |   |
| Question: \( \{ \beta \mid \beta \models \varphi \land |\beta| \leq k \} \neq \emptyset \) |   |   |

\[ \text{Theorem 20.} \quad \text{There is a formula } \varphi \in \text{FO}(\subseteq) \text{ such that } \text{CMinSat}_\varphi^\text{team} \text{ is NP-hard.} \]

**Proof.** We reduce from the NP-complete problem IS, showing that there are a myopic formula \( \varphi' \in \Sigma_1^1 \) and a formula \( \varphi \in \text{FO}(\subseteq) \) such that

\[ \text{IS} \leq_P \text{MAX ZEROS DualHorn} \leq_P \text{CMIN SAT}_\varphi^\text{rel} \leq_P \text{CMIN SAT}_\varphi^\text{team}. \]

For (1) an arbitrary \((G = (V, E), k)\) is mapped to \((\varphi = \bigwedge_{\{i,j\} \in E} x_i \lor x_j, |V| - k)\). Intuitively, assigning a variable \( x_i \) to 0 in \( \varphi \) corresponds to picking the vertex \( i \) in \( G \) for an independent set. The formula \( \varphi \) expresses that at most one of the variables in any clause may be set to 0, corresponding to the condition that at most one of the endpoints of an edge can be in an independent set. From this it can easily be seen that there is a 1-1 correspondence between independent sets \( V' \) of \( G \) of size at least \( k \) and satisfying assignments of \( \varphi \) of weight at most \( k \). Note that \( \varphi \) is obviously a DualHorn formula.

Let \( \sigma = (P^2, N^2) \) be a vocabulary. A propositional CNF-formula \( \chi \) can be encoded as a \( \sigma \)-structure \( A_\chi \) as follows: The universe contains the variables and clauses of \( \chi \). The relation \( P^{A_\chi} (N^{A_\chi}) \) contains a pair \((C, x)\), if \( C \) is a clause in \( \chi \), \( x \) is a variable and \( x \) occurs positively (negatively) in \( C \) in the formula \( \chi \).

For (2), define the myopic second-order formula \( \varphi' \) over \( \sigma \) as follows:

\[
\varphi'(R) = \forall x (R(x) \rightarrow (\forall C ((\neg \exists z N(C, z)) \rightarrow (\exists y P(C, y) \land R(y)))) \\
\land (N(C, x) \rightarrow (\exists y P(C, y) \land R(y))))
\]

Now for all assignments \( \beta \) it holds that \( \beta \models \chi \iff A_\chi, \beta \models \varphi' \).

Finally, (3) follows from Proposition \( \text{5.2} \) since \( \varphi' \) is a myopic formula.

The second and third reductions are essentially the same that were used to show \( \#\text{DualHorn} \subseteq \#\text{FO}(\subseteq) \) \( \text{12} \). The difference is that in the counting case, the number of solutions to the DualHorn-formula must be equal to number of solutions to the \( \text{FO}(\subseteq) \)-formula, and in our case the size of maximal and minimal solutions must preserved. Fortunately the given formula in the second reduction delivers both, as the solutions are exactly the same for both formulas.

Note that this reduction also works if we use positive 2CNF-formulas (propositional formula in conjunctive normal form, where each clause has two positive literals) instead of DualHorn-formulas, since the given formula \( \varphi = \bigwedge_{\{i,j\} \in E} x_i \lor x_j \) is a positive 2CNF-formula.
Corollary 21. Let $\mathcal{E} = \{\text{E-Sat}, \text{E-MaxSat}, \text{E-CMaxSat}, \text{E-MinSat}, \text{E-CMinSat}\}$.

1. For all $E \in \mathcal{E}$ and $\varphi \in \text{FO}(A)$ with $A \subseteq \{\bot, =, \in\}$ $E_{\varphi}^{\text{team}}$ is in DelNP.

2. There are formulas $\varphi_1 \in \text{FO}(\{\in\}), \varphi_2 \in \text{FO}(\bot), \varphi_3 \in \text{FO}(\subseteq)$ such that for all $E \in \mathcal{E}$ the problems $E_{\varphi_1}^{\text{team}}, E_{\varphi_2}^{\text{team}}$ and $E_{\text{CMinSat}}^{\text{team}}$ are DelNP-complete.

Proof. Statement 1. follows directly from Theorems 16, 18, 17 and 19. For statement 2., the hardness for the case of inclusion logic follows from Theorem 10 together with Theorem 20, as CMinSat is trivially decidable in polynomial time with oracle access to E-CMinSat.$E_{\varphi_3}^{\text{team}}$: Simply get a solution from the oracle, compute its cardinality and compare it to $k$. The other cases follow from Corollary 15.

By Corollary 21, we get a characterization of the class DelNP as the closure of the mentioned problems under the enumeration reducibility notion.

5 Conclusion

In Table 1, we summarise the complexity results we obtained in this paper. We completely classified all here considered enumeration problems and obtained either polynomial-delay algorithms or completeness for DelNP.

There are some open issues that immediately lead to questions for further research. First, all our results are obtained for a certain fixed set of generalised dependency relations. Our selection was motivated by those logics found in the literature, but essentially arbitrary. It will be interesting to see whether other atoms or combinations of atoms lead to different (higher?) complexity.

Also, there is a notion of strict semantics (see, e.g., the work of Galliani [8]). Our results do not immediately transfer to strict semantics, since, for example, Lemma 4 is not true for independence logic with strict semantics. It would be interesting to study the enumeration complexity of team logics in strict semantics.

Maybe even more interesting is the extension of the logical language by the so called strong (or classical) negation. Observe that our logics only allow atomic negation. It is known that with full classical negation, many generalised dependency atoms can be simulated (in modal logic, negation is even complete in the sense that it can simulate any FO-expressible dependency). We consider it likely that enumeration problems for logics with classical negation will lead us out of the class DelNP and potentially even to arbitrary levels of the hierarchy.
References

1. Samson Abramsky,Juha Kontinen,Jouko Väänänen,and Heribert Vollmer, editors. Dependence Logic, Theory and Applications. Springer, 2016.

2. Guillaume Bagan, Arnaud Durand, and Etienne Grandjean. On acyclic conjunctive queries and constant delay enumeration. In CSL, volume 4646 of Lecture Notes in Computer Science, pages 208–222. Springer, 2007.

3. Nofer Carmeli,Batya Kenig, and Benny Kimelfeld. Efficiently enumerating minimal triangulations. In PODS, pages 273–287. ACM, 2017.

4. Nadia Creignou,Markus Kröll, Reinhard Pichler, Sebastian Skritek, and Heribert Vollmer. A complexity theory for hard enumeration problems. Discret. Appl. Math., 268:191–209, 2019.

5. Nadia Creignou and Heribert Vollmer. Parameterized complexity of weighted satisfiability problems: Decision, enumeration, counting. Fundam. Inform., 136(4):297–316, 2015.

6. Arnaud Durand, Nicole Schweikardt, and Luc Segoufin. Enumerating answers to first-order queries over databases of low degree. In PODS, pages 121–131. ACM, 2014.

7. Ronald Fagin. Generalized first-order spectra, and polynomial time recognizable sets. SIAM-AMS Proceedings, 7:43–73, 1974.

8. Pietro Galliani. Inclusion and exclusion dependencies in team semantics - on some logics of imperfect information. Ann. Pure Appl. Logic, 163(1):68–84, 2012.

9. Pietro Galliani and Lauri Hella. Inclusion logic and fixed point logic. In CSL, volume 23 of LIPIcs, pages 281–295. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013.

10. Erich Grädel. Model-checking games for logics of imperfect information. Theor. Comput. Sci., 493:2–14, 2013.

11. Erich Grädel and Jouko A. Väänänen. Dependence and independence. Studia Logica, 101(2):399–410, 2013.

12. Anselm Haak, Juha Kontinen, Fabian Müller, Heribert Vollmer, and Fan Yang. Counting of teams in first-order team logics. In MFCS, volume 138 of LIPIcs, pages 19:1–19:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

13. Miika Hannula, Juha Kontinen, Jonni Virtema, and Heribert Vollmer. Complexity of propositional logics in team semantics. ACM Trans. Comput. Log., 19(1):2:1–2:14, 2018.

14. David S. Johnson, Christos H. Papadimitriou, and Mihalis Yannakakis. On generating all maximal independent sets. Inf. Process. Lett., 27(3):119–123, 1988.

15. Benny Kimelfeld and Phokion G. Kolaitis. The complexity of mining maximal frequent subgraphs. ACM Trans. Database Syst., 39(4):32:1–32:33, 2014.

16. Juha Kontinen and Jouko A. Väänänen. On definability in dependence logic. Journal of Logic, Language and Information, 18(3):317–332, 2009.

17. Claudio L. Lucchesi and Sylvia L. Osborn. Candidate keys for relations. J. Comput. Syst. Sci., 17(2):270–279, 1978.

18. Arne Meier and Christian Reinbold. Enumeration complexity of poor man’s propositional dependence logic. In FoIKS, volume 10833 of Lecture Notes in Computer Science, pages 303–321. Springer, 2018.

19. Nicholas Pippenger. Theories of computability. Cambridge University Press, 1997.

20. Yann Strozecki. Enumeration complexity and matroid decomposition. PhD thesis, 2010.

21. Jouko A. Väänänen. Dependence Logic - A New Approach to Independence Friendly Logic, volume 70 of London Mathematical Society student texts. Cambridge University Press, 2007.
A Appendix

In this section we provide the algorithms we omitted in sections 3 and 4 and analyse the delay of one of these algorithms.

Algorithm 3 belongs to Theorem 12 and shows that \( \text{E-MinSat}^\text{team}_\varphi \in \text{DelP} \) for \( \varphi \in \text{FO}(\subseteq) \).

**Algorithm 3:** Algorithm used to show \( \text{E-MinSat}^\text{team}_\varphi \in \text{DelP} \)

```plaintext
1 Function EnumerateMinSubteams(A, X, Y)
2     X = MAXSUBTEAM(A, X)
3     if X \neq \emptyset \land Y \subseteq X then
4         if \forall s \in X \text{ MAXSUBTEAM}(A, X \setminus \{s\}) = \emptyset then output X
5             for s \in X do
6                 Y = \{s' | s' < s \land s' \in X\}
7                 EnumerateMinSubteams(A, X \setminus \{s\}, Y)
```

Algorithm 4 shows that \( \text{E-MaxSat}^\text{team}_\varphi \in \text{DelNP} \) for \( \varphi \in \text{FO}(\subseteq) \) (see Theorem 18). For this it uses Algorithm 5 as a subroutine.

**Algorithm 4:** Algorithm used to show \( \text{E-MaxSat}^\text{team}_\varphi \in \text{DelNP} \)

```plaintext
1 Function EnumerateMaximalSuperteams(A, X, Y)
2     X' = ComputeNextMaximalSuperteam(A, X, Y, \perp)
3     if X' = \perp then return \perp
4     output X'
5     Y' = X'
6     Sol = \{X'\}
7     for s \in Y \setminus Y' do
8         X' = ComputeNextMaximalSuperteam(A, X, Y, s)
9         if X' \neq \perp then
10            output X'
11            Y' = Y' \cup X'
12            Sol = Sol \cup \{X'\}
13     for A, B \in Sol do
14         C = A \setminus B
15         D = B \setminus A
16         for s_c \in C, s_d \in D do
17             if \{s_c, s_d\} \notin \text{Pairs} then
18                 Pairs = Pairs \cup \{\{s_c, s_d\}\}
19                 Y'' = \{s | s \in Y', s > s_c, s > s_d\}
20                 EnumerateMaximalSuperteams(A, X \cup \{s_c, s_d\}, Y'')
```

Algorithm 6 shows \( \text{E-CMaxSat}^\text{team}_\varphi \in \text{DelNP} \) for \( \varphi \in \text{FO}(\subseteq) \) and \( \varphi \in \text{FO}(\subseteq) \) (see Theorem 17). Since we never explained why our algorithms have polynomial delay we do this exemplary for Algorithm 6.
Algorithm 5: Algorithm computes maximal satisfying team \( X' \) with \( X \subset X' \) by adding assignment \( s \) and assignments from \( Y \)

1. **Function** `ComputeNextMaximalSuperteam(A, X, Y, s)`
   2. if \( s = \bot \) then \( X' = X \)
   3. else \( X' = X \cup s \)
   4. if `ExtendMaxTeam(A, X', Y)` then
      5. for \( s' \in Y \setminus X \) do
         6. if `ExtendMaxTeam(A, X' \cup s')` then
            7. \( X' = X' \cup s' \)
      8. for \( s' \in Y \setminus X \) do
         9. if `VerifyTeam(A, X' \cup s')` then
            10. \( X' = X' \cup s' \)
     11. else if `VerifyTeam(A, X')` then return \( X' \)
     12. else return \( \bot \)

The single steps of the algorithm (computing \( Y \), using the oracles et cetera) obviously take polynomial time. The problem that could occur is, that the algorithm could start an exponential number of recursive calls between two outputs. We argue that by the design of the algorithm this can not happen: In each recursive call the algorithm either starts no new recursive call — when the extend oracle outputs no — or a polynomial number of recursive calls. In the latter case (at least) one of these recursive calls gets us a step closer to the next solution by adding a “right” assignment (because the extend oracle did output yes). Since there are only a polynomial number of assignments this is repeated at most polynomially often until the next solution is output.

Algorithm 6: Algorithm used to show \( \text{E-MaxSat}^\text{team} \in \text{DelNP} \)

1. **Function** `EnumerateCardMaxTeams(A, X, k)`
   2. \( Y = \bigcup_{s < \max(X) \land s \notin X} s \)
   3. if `VerifyTeam(A, X) \land |X| = k` then output \( X \)
   4. else if `ExtendCMaxTeam(A, X, Y, k)` then
      5. for \( s > \max(X) \) do
         6. `EnumerateCardMaxTeams(A, X \cup \{s\}, k)`