TOPOLOGICAL PROPERTIES OF $q$-ANALOGUES OF MULTIPLE ZETA VALUES

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Abstract. In the space of bounded real-valued functions on the interval $(0, 1)$, we study the convergent sequences of $q$-analogues of multiple zeta values which do not converge to 0. And we obtain the derived sets of the set of some $q$-analogue of multiple zeta values.

1. Introduction and statement of main results

Multiple zeta values are natural generalizations of the Riemann zeta values. Let $\mathbb{N}$ be the set of positive integers. For any $d \in \mathbb{N}$ and any multi-index $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ with $k_1 \geq 2$, the multiple zeta value $\zeta(k)$ is defined by the following infinite series

$$\zeta(k) = \zeta(k_1, \ldots, k_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}.$$ 

The condition $k_1 \geq 2$ ensures the convergence of the above infinite series. And we call such a multi-index $k$ admissible. The quantities $k_1 + \cdots + k_d$ and $d$ are called weight and depth of $k$, respectively. Different from other researchers’ work on multiple zeta values, Kumar studied the order structure and the topological properties of the set $\mathcal{Z}$ of all multiple zeta values in [3]. Taking the usual order and the usual topology of the set $\mathbb{R}$ of real numbers, Kumar computed the derived sets of the topological subspace $\mathcal{Z}$ of $\mathbb{R}$, and showed that the set $\mathcal{Z}$, ordered by $\geq$, is well-ordered with the order type $\omega^3$, where $\omega$ is the smallest infinite ordinal.

In this paper, we study the topological properties of some $q$-analogues of multiple zeta values. Let $q \in \mathbb{R}$ with $0 < q < 1$. For any $m \in \mathbb{N}$, let $[m]$ denote the $q$-integer $[m] = \frac{1 - q^m}{1 - q} = 1 + q + \cdots + q^{m-1}$. Then for any admissible multi-index $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we define the multiple $q$-zeta value $\zeta[k]$ by

$$\zeta[k] = \zeta[k_1, \ldots, k_d] = \sum_{m_1 > \cdots > m_d > 0} \frac{q^{m_1(k_1-1) + \cdots + m_d(k_d-1)}}{[m_1]^{k_1} \cdots [m_d]^{k_d}}.$$ 

(1.1)

This $q$-analogue was first studied by Bradley [2] and independently by Zhao [4]. Here we introduce another $q$-analogue of multiple zeta values. Let $r \in \mathbb{N}$, then we
define
\[ \zeta[k, r] = \zeta[k_1, \ldots, k_d, r] = \sum_{m_1 > \cdots > m_d > m_d + 1 > 0} q^{m_1(k_1 - 1) + \cdots + m_d(k_d - 1)} [m_1]^{k_1} \cdots [m_d]^{k_d} m_d^{r + 1}. \]
(1.2)

Different from multiple zeta values, the multiple q-zeta values have a parameter q. Hence we work in the function space \( B(0, 1) \), which is the set of bounded real-valued functions on the open interval \((0, 1)\). Since the multiple q-zeta values we consider here belong to \( B(0, 1) \) (see Remark 2.4), we just study the following two subspaces of \( B(0, 1) \):

\[ QZ = \{ \zeta[k] \mid k \text{ is admissible} \}, \]
\[ QZZ = \{ \zeta[k, r] \mid k \text{ is admissible}, r \in \mathbb{N} \}. \]

We define an order of \( B(0, 1) \) as follows. Let \( f, g \in B(0, 1) \). The function \( f \) is smaller than \( g \), if \( f(q) < g(q) \) for any \( q \in (0, 1) \). We denote this by \( f < g \). Then we can find the maximum element of \( QZ \).

**Theorem 1.1.** For any admissible multi-index \( k \), we have \( \zeta[k] \leq \zeta[2] \). In other words, \( \zeta[2] \) is the maximum element of \( QZ \).

While for the subspace \( QZZ \), we only obtain an upper bound.

**Theorem 1.2.** For any admissible multi-index \( k \) and any \( r \in \mathbb{N} \), we have \( \zeta[k, r] < \zeta[2] \). In other words, \( \zeta[2] \) is an upper bound of \( QZZ \).

We prove Theorem 1.1 and Theorem 1.2 in Section 2.

As in [3], we want to compute the derived sets of the subspace \( QZZ \). Hence some topology of \( B(0, 1) \) is needed. In fact, \( B(0, 1) \) is a complete normed space with the norm given by
\[ \| f \| = \sup_{q \in (0, 1)} |f(q)|, \quad \forall f \in B(0, 1). \]

Under the topology induced from the above norms, we can determine the sequence \((QZZ^{(n)})_{n \geq 0}\) of the derived sets of the subspace \( QZZ \) of \( B(0, 1) \). Here \( QZZ^{(0)} = QZ \) and for any \( n \in \mathbb{N} \), \( QZZ^{(n)} \) is the set of accumulation points of \( QZZ^{(n-1)} \) in \( B(0, 1) \). To state the result, we have to define the tails of multiple q-zeta values. For an admissible multi-index \( k = (k_1, \ldots, k_d) \in \mathbb{N}^d \) and a nonnegative integer \( n \), we set
\[ \zeta[k]_n = \zeta[k_1, \ldots, k_d]_n = \sum_{m_1 > \cdots > m_d > n} q^{m_1(k_1 - 1) + \cdots + m_d(k_d - 1)} [m_1]^{k_1} \cdots [m_d]^{k_d} m_d^{r + 1}. \]
(1.3)

Obviously, we have \( \zeta[k]_0 = \zeta[k] \). Then we have the following theorem, which is proved in Section 3.

**Theorem 1.3.** We have
\[ QZZ^{(1)} = \{ \zeta[k]_1 \mid k \text{ is admissible} \} \cup \{ 0 \} \]
and \( QZZ^{(2)} = \{ 0 \} \).

To save spaces, throughout the paper, the notation \( \{k\}^n \) stands for \( k, \ldots, k \) \( n \) terms.
2. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we give proofs of Theorem 1.1 and Theorem 1.2. We first prepare some lemmas.

Lemma 2.1. For any \( m \in \mathbb{N} \), the function \( f(q) = \frac{q^m}{m} \) is monotonically increasing on the interval \((0, 1)\). In particular, we have
\[
0 < \frac{q^m}{m} < \frac{1}{m}, \quad \forall q \in (0, 1).
\]

Proof. We have
\[
(1 - q^m)^2 f'(q) = q^{m-1} [m - (m + 1)q + q^{m+1}].
\]
Set \( g(q) = m - (m + 1)q + q^{m+1} \), then one gets
\[
g'(q) = (m + 1)(q^m - 1) < 0.
\]
Hence for any \( q \in (0, 1) \), we have
\[
g(q) > g(1) = 0,
\]
which induces that \( f'(q) > 0 \).

Lemma 2.2. For any \( d, j, r, k_1, \ldots, k_d \in \mathbb{N} \) with \( k_1 \geq 2 \) and \( j \leq d \), and any nonnegative integer \( n \), we have
\[
\zeta[k_1, \ldots, k_{j-1}, k_j + 1, k_{j+1}, \ldots, k_d] < \zeta[k_1, \ldots, k_j, \ldots, k_d],
\]
\[
\zeta[k_1, \ldots, k_{j-1}, k_j + 1, k_{j+1}, \ldots, k_d, n] < \zeta[k_1, \ldots, k_j, \ldots, k_d, n]
\]
and
\[
\zeta[k_1, \ldots, k_{j-1}, k_j + 1, k_{j+1}, \ldots, k_d, r] < \zeta[k_1, \ldots, k_j, \ldots, k_d, r).
\]
We also have
\[
\zeta[k_1, \ldots, k_d, r + 1] < \zeta[k_1, \ldots, k_d, r)
\]
and
\[
\zeta[k_1, \ldots, k_d, r] < \zeta[k_1, \ldots, k_d, 1].
\]

Proof. From Lemma 2.1 for any \( m \in \mathbb{N} \), we have \( q^m < 1 \). Multiplying by \( \frac{q^{(k_j-1)m}}{m^{k_j+1}} \) on both sides, we obtain
\[
\frac{q^{k_jm}}{m^{k_j+1}} < \frac{q^{(k_j-1)m}}{m^{k_j}},
\]
which induces the first three inequalities stated in the lemma. For \( m \in \mathbb{N} \), we have
\[
\frac{1}{m^{r+1}} \leq \frac{1}{m^r}, \quad \frac{1}{m^r} \leq \frac{1}{|m|^r} \leq \frac{1}{|m|},
\]
and the equalities holds only if \( m = 1 \). Then we get the last two inequalities stated in the lemma.

Lemma 2.3. For any nonnegative integer \( d \), we have \( \zeta[2, \{1\}^{d+1}] < \zeta[2, \{1\}^d] \).

Proof. We use the duality formula for multiple \( q \)-zeta values proved by Bradley in [2]: for any nonnegative integers \( n \) and \( m \), one has
\[
\zeta[2 + n, \{1\}^m] = \zeta[2 + m, \{1\}^n]. \tag{2.1}
\]
From (2.1), to prove the lemma, it is enough to show \( \zeta[d + 3] < \zeta[d + 2] \), which is just from Lemma 2.2.
Now we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let \( k = (k_1, \ldots, k_d) \). By Lemma 2.2 and Lemma 2.3, we have
\[
\zeta[k] = \zeta[k_1, \ldots, k_d] \leq \zeta[2, \{1\}^{d-1}] \leq \zeta[2],
\]
as desired. \( \square \)

Proof of Theorem 1.2. Let \( k = (k_1, \ldots, k_d) \). By Lemma 2.2 and Lemma 2.3, we have
\[
\zeta[k, r] = \zeta[k_1, \ldots, k_d, r] < \zeta[2, \{1\}^{d}] < \zeta[2],
\]
as desired. \( \square \)

We end this section with a remark.

Remark 2.4. It is easy to see that \( \zeta[2] \) is bounded on \((0, 1)\). Hence from Theorem 1.1 and Theorem 1.2, we find \( QZ \) and \( QZZ \) are subsets of \( B(0, 1) \).

3. Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3.

3.1. Some preliminary results. We first compute the norms of some multiple \( q \)-zeta values. For this, we prepare some lemmas.

Lemma 3.1. For a fixed \( q \in (0, 1) \), the function \( f(x) = \frac{x^{q-1}}{1-x} \) is monotonically decreasing on the interval \([1, +\infty)\).

Proof. We have
\[
(1 - q^x)^2 f'(x) = q^{x-1}(1 - q^x + x \log q).
\]
Set \( g(x) = 1 - q^x + x \log q \), then we have
\[
g'(x) = (1 - q^x) \log q < 0.
\]
Hence we find
\[
g(x) \leq g(1) = 1 - q + \log q.
\]
Set \( h(x) = 1 - x + \log x \), where \( x \in (0, 1) \), then we have
\[
h'(x) = -1 + \frac{1}{x} = \frac{1-x}{x} > 0.
\]
Finally we get
\[
g(1) = h(q) < h(1) = 0,
\]
which implies that \( f'(x) < 0 \). \( \square \)

Lemma 3.2. Let \( d, m_1, \ldots, m_d, k \in \mathbb{N} \) with \( m_1 \geq \cdots \geq m_d \) and \( k \geq d+1 \). Then the function \( f(q) = \frac{q^{m_1(k-1)}}{[m_1][m_2] \cdots [m_d]} \) is monotonically increasing on the interval \((0, 1)\). In particular, for \( m, k \in \mathbb{N} \) with \( k \geq 2 \), the function \( \frac{q^{m(k-1)}}{[m]^k} \) is monotonically increasing on the interval \((0, 1)\).

Proof. Since
\[
f(q) = \frac{(1 - q)^{k+d-1} q^{m_1(k-1)}}{(1 - q^{m_1})^{k-1}(1 - q^{m_2}) \cdots (1 - q^{m_d})},
\]
taking the logarithmic derivative of \( f(q) \), we get
\[
\frac{f'(q)}{f(q)} = \frac{m_1(k-1)}{q} - \frac{k+d-1}{1-q} + \frac{km_1q^{m_1-1}}{1-q^{m_1}} + \sum_{i=2}^{d} \frac{m_iq^{m_i-1}}{1-q^{m_i}}.
\]
Using Lemma 3.1, we get
\[
\frac{f'(q)}{f(q)} \geq \frac{m_1(k-1)}{q} \frac{k + d - 1}{1 - q} + \frac{(k + d - 1)m_1 q^{m_1-1}}{1 - q^{m_1}},
\]
which is equivalent to
\[
g(1-q)(1-q^{m_1}) \frac{f'(q)}{f(q)} \geq g(q)
\]
with
\[
g(q) = [m_1(k-1)(1-q) - (k + d - 1)q] (1-q^{m_1}) + (k + d - 1)m_1 q^{m_1}(1-q).
\]
Then it is enough to show that
\[
g(q) > 0, \quad \forall q \in (0,1).
\]
In fact, we have
\[
g'(q) = m_1 - m_1 k - k - d + 1 + m_1^2 dq^{m_1-1} + (k + d - 1 - m_1 d)(m_1 + 1)q^{m_1},
\]
\[
g''(q) = q^{m_1-2}m_1 [m_1(m_1 - 1)d + (m_1 + 1)(k + d - 1 - m_1 d)].
\]
Set
\[
h(q) = m_1(m_1 - 1)d + (m_1 + 1)(k + d - 1 - m_1 d),
\]
then we get
\[
h(0) = m_1(m_1 - 1)\geq 0, \quad h(1) = m_1(k - d - 1) + k + d - 1 > 0.
\]
Here we have used the condition that \(k \geq d + 1\). Therefore for any \(q \in (0,1)\), we have \(h(q) > 0\) and then \(g''(q) > 0\). This implies that
\[
g'(q) < g'(1) = 0, \quad \forall q \in (0,1),
\]
and then
\[
g(q) > g(1) = 0, \quad \forall q \in (0,1).
\]
We finish the proof. \(\square\)

From Lemma 3.2, we get the norms of height one multiple \(q\)-zeta values.

**Corollary 3.3.** For any nonnegative integers \(n\) and \(m\), we have
\[
||\zeta[2 + n, \{1\}^m]|| = \zeta(2 + n, \{1\}^m).
\]

**Proof.** If \(n \geq m\), we get the result from Lemma 3.2. If \(n \leq m\), applying the duality formula (2.1) and its multiple zeta values’ version, we get the result from Lemma 3.2. \(\square\)

Then we give upper and lower bounds of tails of multiple \(q\)-zeta values.

**Lemma 3.4.** For any admissible multi-index \(k = (k_1, \ldots, k_d) \in \mathbb{N}^d\) and any \(n \in \mathbb{N}\), we have
\[
\left( \frac{q - 1}{\log q} \right)^d \left( \frac{q^n + d}{n + d} \right)^{k_1 + \cdots + k_d - d} \prod_{i=1}^d \frac{1}{k_1 + \cdots + k_i - i} < \zeta[k]_n
\]
\[
< \left( \frac{q - 1}{\log q} \right)^d \left( \frac{q^n}{n} \right)^{k_1 + \cdots + k_d - d} \prod_{i=1}^d \frac{1}{k_1 + \cdots + k_i - i}.
\] (3.1)
Proof. We prove by induction on $d$. For $d = 1$, we have to show that

$$
\frac{q - 1}{\log q} \left( \frac{q^{n+1}}{n+1} \right)^{k_1 - 1} \frac{1}{k_1 - 1} < \zeta[k_1]_n < \frac{q - 1}{\log q} \left( \frac{q^n}{n} \right)^{k_1 - 1} \frac{1}{k_1 - 1}.
$$

(3.2)

In fact, set $f_k(x) = \frac{q^{x(k_1 - 1)}}{(1 - q^x)^{k_1}}$, then we have

$$
(1 - q^x)^k f_k(x) = q^{x(k_1 - 1)}(k_1 - 1 + q^x) \log q < 0, \quad (x \geq 1).
$$

Hence $f_k(x)$ is monotonically decreasing on the interval $[1, +\infty)$ for any $k_1 \in \mathbb{N}$ and any $q \in (0, 1)$. Then we obtain

$$
(1 - q)^k \int_{n+1}^{\infty} \frac{q^{x(k_1 - 1)}}{(1 - q^x)^{k_1}} dx < \zeta[k_1]_n < (1 - q)^k \int_{n}^{\infty} \frac{q^{x(k_1 - 1)}}{(1 - q^x)^{k_1}} dx.
$$

Now if $k_1 \geq 2$, we get

$$
(1 - q)^k \int_{a}^{b} \frac{q^{x(k_1 - 1)}}{(1 - q^x)^{k_1}} dx = \frac{(1 - q)^k}{\log q} \int_{a}^{b} \left( \frac{y^{k_1 - 1}}{1 - y} \right)^{k - 2} \frac{1}{y} dy
$$

$$
= \frac{(1 - q)^k}{(k_1 - 1) \log q} \left( \frac{y}{1 - y} \right)^{k_1 - 1} \bigg|_{y=a}^{y=b},
$$

from which we get (3.2).

If $d > 1$, we have

$$
\zeta[k]_n = \zeta[k_1, \ldots, k_d]_n = \sum_{m_d > n} q^{m_d(k_d - 1)} [m_d]_{k_d} \zeta[k_1, \ldots, k_d - 1]_{m_d}.
$$

Using the induction hypothesis, we get

$$
\Sigma_l < \zeta[k]_n < \Sigma_r
$$

with

$$
\Sigma_l = \sum_{m_d > n} q^{m_d(k_d - 1)} [m_d]_{k_d} \left( \frac{q - 1}{\log q} \right)^{d - 1} \left( \frac{q^{m_d + d - 1}}{m_d + d - 1} \right)^{k_1 + \cdots + k_d - d + 1}
$$

$$
\times \prod_{i=1}^{d - 1} \frac{1}{k_1 + \cdots + k_i - i}.
$$

$$
\Sigma_r = \sum_{m_d > n} q^{m_d(k_d - 1)} [m_d]_{k_d} \left( \frac{q - 1}{\log q} \right)^{d - 1} \left( \frac{q^{m_d}}{m_d} \right)^{k_1 + \cdots + k_d - d + 1} \prod_{i=1}^{d - 1} \frac{1}{k_1 + \cdots + k_i - i}.
$$

Since

$$
\frac{q^{m_d(k_d - 1)}}{[m_d]_{k_d}} = (1 - q)^k f_k(m_d) > (1 - q)^k f_k(m_d + d - 1) = \frac{q^{m_d(d-1)(d-1)}}{[m_d + d - 1]_{k_d}},
$$

we find

$$
\Sigma_l > \left( \frac{q - 1}{\log q} \right)^{d - 1} \prod_{i=1}^{d - 1} \frac{1}{k_1 + \cdots + k_i - i} \zeta[k_1 + \cdots + k_d - d + 1]_{n+d-1}.
$$
Using the lower bound in the case of $d = 1$, we have
\[ \Sigma_l > \left( \frac{q-1}{\log q} \right)^d \left( \frac{q^n}{n+d} \right)^{k_1+\cdots+k_d-d} \prod_{i=1}^d \frac{1}{k_1+\cdots+k_i-i}, \]
as desired. Similarly, since
\[ \Sigma_r = \left( \frac{q-1}{\log q} \right)^{d-1} \prod_{i=1}^{d-1} \frac{1}{k_1+\cdots+k_i-i} \zeta[k_1+\cdots+k_d-d+1]n, \]
using the upper bound in the case of $d = 1$, we have
\[ \Sigma_r < \left( \frac{q-1}{\log q} \right)^d \left( \frac{q^n}{n} \right)^{k_1+\cdots+k_d-d} \prod_{i=1}^d \frac{1}{k_1+\cdots+k_i-i}, \]
as desired. \qed

3.2. Convergent sequences in $QZZ$. To prove Theorem 3.3 we have to know the behaviour of the convergent sequences in the space $QZZ$. We first introduce some notation. Let $(k(n))_{n \in \mathbb{N}} = ((k_1(n), \ldots, k_d(n)))_{n \in \mathbb{N}}$ be a fixed sequence of admissible multi-indices. Set
\[ N_2 = \{ n \in \mathbb{N} \mid k_1(n) = 2 \} \subset \mathbb{N}. \]
For any $n \in N_2$, we define
\[ l(n) = \begin{cases} i & \text{if } d(n) \geq 2 \text{ and } k_2(n) = \cdots = k_{i-1}(n) = 1, k_i(n) \geq 2, \\ 1 & \text{otherwise}, \end{cases} \]
and $v(n) = d(n) - l(n) + 1$. Then for $n \in N_2$ and $l(n) \geq 2$, we set
\[ k = (2, \{1\}^{l(n)-2}, k_1(n), \ldots, k_d(n)) = (2, \{1\}^{l(n)-2}, s_1(n), \ldots, s_v(n)(n)). \]
Finally, we define some subsets of $\mathbb{N}$ as follows
\[ D = \{ d(n) \mid n \in N_2 \}, \]
\[ V = \{ v(n) \mid n \in N_2 \}, \]
\[ W = \{ k_1(n) + \cdots + k_d(n(n)) \mid n \in N_2 \}, \]
\[ W' = \{ s_1(n) + \cdots + s_v(n)(n) \mid n \in N_2, l(n) \geq 2 \}. \]
Then we have the following theorem.

**Theorem 3.5.** Let $(k(n))_{n \in \mathbb{N}} = ((k_1(n), \ldots, k_d(n)))_{n \in \mathbb{N}}$ be a sequence of admissible multi-indices and $(r(n))_{n \in \mathbb{N}}$ be a sequence of positive integers. Assume that 0 is not an accumulation point of the sequence $(\zeta[k(n), r(n)])_{n \in \mathbb{N}}$ in $B(0, 1)$. The following holds.

(i) If $N_2$ is a finite set, then both the sets $\{ d(n) \mid n \in \mathbb{N} \}$ and $\{ k_1(n) + \cdots + k_d(n) \mid n \in \mathbb{N} \}$ are bounded.
(ii) If $N_2$ is an infinite set,
(ii-1) in the case of $D$ is bounded, we have $W$ is bounded;
(ii-2) in the case of both $D$ and $V$ are unbounded, there are only finitely many $n \in N_2$, such that $l(n) \geq 2$;
(ii-3) in the case of $D$ is unbounded while $V$ is bounded, we have $W'$ is bounded.
Proof. (i) Assume that $N_2$ is a finite set. Without loss of generality, we may assume that $N_2 = \emptyset$. Then for any $n \in \mathbb{N}$, we have $k_1(n) \geq 3$. Using Lemma 2.2 and the duality formula (2.1), we have
\[
\zeta[k_1(n), \ldots, k_{d(n)}(n), r(n)) < \zeta[k_1(n), \ldots, k_{d(n)}(n), 1] \leq \zeta[3, \{1\}^{d(n)}] = \zeta[2 + d(n), 1].
\]
Taking norms, we have
\[
\|\zeta[k_1(n), \ldots, k_{d(n)}(n), r(n))\| \leq \|\zeta[2 + d(n), 1]\| = \zeta(2 + d(n), 1),
\]
where the last equality is from Corollary 3.3. If $d(n)$ is unbounded, then there exists an infinite subset $M$ of $\mathbb{N}$, such that
\[
\lim_{n \in M, n \to \infty} d(n) = \infty.
\]
Since
\[
\lim_{d(n) \to \infty} \zeta(2 + d(n), 1) = 0,
\]
we get
\[
\lim_{n \in M, n \to \infty} \|\zeta[k_1(n), \ldots, k_{d(n)}(n), r(n))\| = 0,
\]
which implies that 0 is an accumulation point of the sequence $(\zeta(k(n), r(n)))_{n \in \mathbb{N}}$, a contradiction. Hence $d(n)$ is bounded.

Let $d$ be the maximal element of $\{d(n) \mid n \in \mathbb{N}\}$. For any $1 \leq p \leq d$, set
\[
M_p = \{n \in \mathbb{N} \mid d(n) = p\}.
\]
For a fixed $p$, we show that for any $1 \leq j \leq p$, the sets
\[
\{k_j(n) \mid n \in M_p\}
\]
are all bounded. If $M_p$ is a finite set, we obviously have the result. Now assume that $M_p$ is an infinite set.

For $j = 1$, as above we have
\[
\|\zeta[k_1(n), \ldots, k_p(n), r(n))\| \leq \|\zeta[k_1(n), \ldots, k_p(n), 1]\|
\leq \|\zeta[k_1(n), \{1\}^p]\| = \zeta(k_1(n), \{1\}^p).
\]
If $k_1(n)$ is unbounded for $n \in M_p$, then without loss of generality, we may assume that
\[
\lim_{n \in M_p, n \to \infty} k_1(n) = \infty.
\]
Therefore we have
\[
\lim_{n \in M_p, n \to \infty} \|\zeta[k_1(n), \ldots, k_p(n), r(n))\| = 0,
\]
a contradiction. Hence $k_1(n)$ is bounded for $n \in M_p$.

Assume $2 \leq j \leq p$. We may assume that for any $n \in M_p$, $k_j(n) \geq 2$. We have
\[
\zeta[k_1(n), \ldots, k_p(n), r(n))
= \sum_{m_1 > \cdots > m_{j-1} > m_j} \prod_{i=1}^{j-1} q^{m_i(k_i(n)-1)}_{m_i} \sum_{m_j > \cdots > m_p+1 > 0} \prod_{i=j}^{p} q^{m_i(k_i(n)-1)}_{m_i} m_{p+1}^{r(n)}
< \zeta[k_1(n), \ldots, k_{j-1}(n)] \zeta(k_j(n), \ldots, k_p(n), r(n)).
\]
Assume that we have shown $k_1(n), \ldots, k_{j-1}(n)$ are all bounded for $n \in M_p$, then
\[
\{\|\zeta[k_1(n), \ldots, k_{j-1}(n)]\| \mid n \in M_p\}
\]
is bounded. If $k_j(n)$ is unbounded for $n \in M_p$, then
then as in the case of \( j = 1 \), there exists an infinite subset of \( M_p \), such that 
\[ \| \zeta(k_j(n), \ldots, k_p(n), r(n)) \| \] 
tends to zero when \( n \) belongs to this infinite subset and goes to infinity. Therefore, we again get a contradiction. Hence \( k_j(n) \) is bounded for \( n \in M_p \).

Finally, we find the weight of \( k(n) \) is bounded for \( n \in N \), and (i) is proved.

(ii) Assume that \( N_2 \) is an infinite set.

(ii-1) If \( D \) is bounded, set \( d = \text{max} \ D \). We only need to prove that for any \( 2 \leq j \leq d \), \( k_j(n) \) is bounded for \( n \in N_2 \). Then one may use a similar argument as in (i) to get the result.

(ii-2) Assume that both \( D \) and \( V \) are unbounded and there are infinitely many \( n \in N_2 \), such that \( l(n) \geq 2 \). Without loss of generality, we may assume that for any \( n \in N_2 \), \( l(n) \geq 2 \). Then for \( n \in N_2 \), we have

\[
\zeta([k(n), r(n))] = \zeta([k_1(n), \ldots, k_{l(n)}+v(n)-1(n), r(n)])
\]

\[
= \sum_{m_1 > \ldots > m_{l(n)} > m_{l(n)} > m_{l(n)} > m_{l(n)} > 0} \prod_{i=l(n)}^{l(n)-1} \frac{q^{m_i(k_i(n)-1)}}{m_i^{k_i(n)}} 
\times \sum_{m_{l(n)} > \ldots > m_{l(n)} > m_{l(n)} > m_{l(n)} > 0} \prod_{i=l(n)}^{l(n)+v(n)-1} \frac{q^{m_i(k_i(n)-1)}}{m_i^{k_i(n)}} \frac{1}{m_{l(n)}^r(n)}. 
\]

For \( m_{l(n)} > \ldots > m_{l(n)} > m_{l(n)} > m_{l(n)} > 0 \), we have \( m_{l(n)} \geq v(n)+1 > v(n) \). Hence

\[
\zeta([k(n), r(n))] < \sum_{m_1 > \ldots > m_{l(n)} > m_{l(n)} > v(n)} \prod_{i=1}^{l(n)-1} \frac{q^{m_i(k_i(n)-1)}}{m_i^{k_i(n)}} 
\]

\[
\times \sum_{m_{l(n)} > \ldots > m_{l(n)} > m_{l(n)} > v(n)} \prod_{i=l(n)}^{l(n)+v(n)-1} \frac{q^{m_i(k_i(n)-1)}}{m_i^{k_i(n)}} \frac{1}{m_{l(n)}^r(n)} = \zeta([k_1(n), \ldots, k_{l(n)}-1(n), v(n)]\zeta([k_{l(n)}(n), \ldots, k_{l(n)}+v(n)-1(n), r(n)]).
\]

Using Lemma 2.2 and the duality formula (2.1), we have

\[
\zeta([k(n), r(n)]) < [\zeta[2, \{1\}]^{l(n)-2}]_{v(n)]} [\zeta[2, \{1\}]^{v(n)}] = \zeta[2, \{1\}]^{l(n)-2}]_{v(n)]} [\zeta[2, \{1\}]^{v(n)} + 2].
\]

By Lemma 3.3 we get

\[
\zeta([k(n), r(n)]) < \left( \frac{q-1}{\log q} \right)^{l(n)-1} \frac{q^{\nu(n)}}{v(n)} [\zeta[v(n)] + 2].
\]

Using Lemma 2.3 Corollary 3.3 and the fact that the function \( \frac{q-1}{\log q} \) is monotonically increasing on \((0, 1)\), we have

\[
\| \zeta([k(n), r(n)]) \| \leq \frac{1}{v(n)} [\zeta[v(n)] + 2].
\]

Then from the unboundedness of \( V \), we get a contradiction.

(ii-3) Assume that \( D \) is unbounded and \( V \) is bounded. Then \( l(n) \) is unbounded for \( n \in N_2 \). Hence

\[
\tilde{N}_2 = \{ n \in N_2 \mid l(n) > 2 \}
\]

is an infinite subset of \( N_2 \). Set \( v = \text{max} \ V \), and for \( 1 \leq p \leq v \), set

\[
\tilde{M}_p = \{ n \in \tilde{N}_2 \mid v(n) = p \}.
\]
For a fixed $p$, we need to show that $s_1(n), \ldots, s_p(n)$ are all bounded for $n \in \tilde{M}_p$. Now
\[
\zeta[k(n), r(n)] < \zeta[2, \{1\}^{l(n)-2}]\zeta[s_1(n), \ldots, s_p(n), r(n)]
= \zeta[l(n)]\zeta[s_1(n), \ldots, s_p(n), r(n)].
\]
Then similarly as in the proof of (i), we get the result.

As a consequence, we get the following result, which is used to compute $QZ\mathcal{Z}^{(1)}$.

**Corollary 3.6.** Let $(k(n))_{n \in \mathbb{N}} = ((k_1(n), \ldots, k_d(n)))_{n \in \mathbb{N}}$ be a sequence of admissible multi-indices and $(r(n))_{n \in \mathbb{N}}$ be a sequence of positive integers. Assume that for any $n_1 \neq n_2$, $\zeta[k(n_1), r(n_1)] \neq \zeta[k(n_2), r(n_2)]$. If 0 is not an accumulation point of the sequence $(\zeta[k(n), r(n)])_{n \in \mathbb{N}}$ in $B(0, 1)$, then $(k(n), r(n))_{n \in \mathbb{N}}$ has a subsequence of one of the following forms:
\[
((k_1, \ldots, k_d, \psi(n) + 2))_{n \in \mathbb{N}}, \quad (3.3)
\]
\[
((2, \{1\}^\psi(n)), r)_{n \in \mathbb{N}}, \quad (3.4)
\]
\[
((2, \{1\}^\psi(n), \psi(n) + 2))_{n \in \mathbb{N}}, \quad (3.5)
\]
\[
((2, \{1\}^\psi(n), k_1, \ldots, k_d, r))_{n \in \mathbb{N}}, \quad (3.6)
\]
\[
((2, \{1\}^\psi(n), k_1, \ldots, k_d, \varphi(n) + 2))_{n \in \mathbb{N}}, \quad (3.7)
\]
where $(k_1, \ldots, k_d)$ is a fixed admissible multi-index, $r$ is a fixed positive integer and $(\varphi(n))_{n \in \mathbb{N}}, (\psi(n))_{n \in \mathbb{N}}$ are strictly increasing sequences in $\mathbb{N}$.

**Proof.** We use the same notation as in Theorem 3.5. If $N_2$ is finite, then by Theorem 3.5, $d(n)$ and $k_1(n) + \cdots + k_d(n)$ are bounded for $n \in \mathbb{N}$. Hence there exists an infinite subset $A$ of $\mathbb{N}$, such that $d(n) = d$ is a constant for any $n \in A$. Since $k_1(n)$ is bounded for $n \in A$, there exists an infinite subset $A_1$ of $A$, such that $k_1(n) = k_1$ is a constant for any $n \in A_1$. Similarly, there exists an infinite subset $A_2$ of $A_1$, such that $k_2(n) = k_2$ is a constant for any $n \in A_2$. And finally, there exists an infinite subset $B$ of $A$, such that $k_1(n) = k_1, \ldots, k_d(n) = k_d$
are all constants for any $n \in B$. Now $(r(n))_{n \in B}$ must be unbounded, hence $(k(n), r(n))_{n \in \mathbb{N}}$ has a subsequence of the form (3.3).

Now assume that $N_2$ is an infinite set. If $D$ is bounded, then by Theorem 3.5, $W$ is bounded. A similar argument as above implies that $(k(n), r(n))_{n \in \mathbb{N}}$ has a subsequence of the form (3.3). If both $D$ and $V$ are unbounded, then by Theorem 3.5, there is an infinite subset $A$ of $N_2$, such that $l(n) = 1$ for all $n \in A$. Then $(k(n), r(n))_{n \in \mathbb{N}}$ has a subsequence of the form (3.4) or of the form (3.5) according to the sequence $(r(n))_{n \in A}$ is bounded or unbounded. Finally, if $D$ is unbounded while $V$ is bounded, then by Theorem 3.5, $(k(n), r(n))_{n \in \mathbb{N}}$ has a subsequence of the form (3.6) or of the form (3.7).

Similarly, to compute $QZ\mathcal{Z}^{(2)}$, we need the following result.

**Theorem 3.7.** Let $(k(n))_{n \in \mathbb{N}} = ((k_1(n), \ldots, k_d(n)))_{n \in \mathbb{N}}$ be a sequence of admissible multi-indices. Assume that for any $n_1 \neq n_2$, $\zeta[k(n_1)]_1 \neq \zeta[k(n_2)]_1$. If 0 is not an accumulation point of the sequence $(\zeta[k(n)])_{n \in \mathbb{N}}$ in $B(0, 1)$, then $(k(n))_{n \in \mathbb{N}}$ has a subsequence of one of the following forms:
\[
((2, \{1\}^{\psi(n)}))_{n \in \mathbb{N}}, \quad (3.8)
\]
Proof of Theorem 1.3. We can prove similarly as in Theorem 3.3 and Corollary 3.6. While since
\[ \zeta[k_1(n), \ldots, k_{d(n)}(n)] < \zeta[k_1(n), \ldots, k_{d(n)}(n)] \zeta[k_{d(n)}(n)], \quad (k_{d(n)}(n) \geq 2), \]
if we have shown \( k_1(n), \ldots, k_{d(n)}(n) \) are all bounded, then \( k_{d(n)}(n) \) is also bounded. \( \square \)

3.3. Proof of Theorem 1.3. To prove Theorem 1.3, we recall the concept of double tails of multiple zeta values of Akhilesh from [1]. Let \( k = (k_1, \ldots, k_d) \in \mathbb{N}^d \) be an admissible multi-index and \( n, p \) be two nonnegative integers. Then we define
\[ \zeta(k)_{p,n} = \zeta(k_1, \ldots, k_d)_{p,n} = \sum_{m_1 > \cdots > m_d > n} \left( \frac{m_1 + p}{p} \right)^{-1} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}. \]

We need the duality formula of double tails of multiple zeta values. Any admissible multi-index has the form
\[ k = (a_1 + 1, \{1\}^{b_1 - 1}, \ldots, a_s + 1, \{1\}^{b_s - 1}), \]
where \( s, a_1, b_1, \ldots, a_s, b_s \in \mathbb{N} \). Then the dual index of \( k \) is defined as
\[ \overline{k} = (b_1 + 1, \{1\}^{a_1 - 1}, \ldots, b_s + 1, \{1\}^{a_s - 1}). \]

Lemma 3.8 ([1]). Let \( k \) be an admissible multi-index and \( \overline{k} \) be its dual. Then for any nonnegative integers \( p \) and \( n \), we have
\[ \zeta(k)_{p,n} = \zeta(\overline{k})_{n,p}. \quad (3.10) \]

Let \( p = n = 0 \) in (3.10), we get the well-known duality formula of multiple zeta values
\[ \zeta(k) = \zeta(\overline{k}). \]

To show some sequence of \( B(0, 1) \) does not converge, we need the following simple result.

Lemma 3.9. Let the sequence \( (f(n))_{n \in \mathbb{N}} \) converge to \( f \) in \( B(0, 1) \) as \( n \) tends to infinity. Then \( \|f(n)\| \) is convergent to \( \|f\| \), and for any \( q \in (0, 1) \), \( f_n(q) \) is convergent to \( f(q) \) in \( \mathbb{R} \).

Proof. We have
\[ \lim_{n \to \infty} \|f(n) - f\| = 0. \]

Then the facts
\[ \|f(n)\| - \|f\| \leq \|f(n) - f\| \]
and
\[ |f_n(q) - f(q)| \leq \|f(n) - f\|, \quad (q \in (0, 1)) \]
imply the results. \( \square \)

Now we prove Theorem 1.3.

Proof of Theorem 1.3. We first compute \( \mathcal{QZ}_2^{(1)} \). Let \( n \in \mathbb{N} \). Using Lemma 2.2 and the duality formula (2.1), we have
\[ \zeta[3, \{1\}^n] < \zeta[3, \{1\}^n] = \zeta[n + 1, 1]. \]
Then by Corollary 3.3, we have
\[ \|\zeta[3, \{1\}^n]\| \leq \|\zeta[n + 2, 1]\| = \zeta(n + 2, 1). \]

Since \( \lim_{n \to \infty} \zeta(n + 2, 1) = 0 \), we get
\[ \lim_{n \to \infty} \|\zeta[3, \{1\}^n]\| = 0, \]
which implies that \( 0 \in QZZ^{(1)}. \)

Similarly, let \( \mathbf{k} = (k_1, \ldots, k_d) \) be an admissible multi-index and \( n \in \mathbb{N} \). We have
\begin{align*}
\zeta[k_1, \ldots, k_d, n + 2] & - \zeta[k_1, \ldots, k_d]_1 \\
= & \sum_{m_d+1=2}^{\infty} \sum_{m_d>\cdots>m_1}^{d} \prod_{i=1}^{d} q^{m_i(k_i-1)} \cdot \frac{1}{m_d^{n+2}} \\
& < \zeta[k_1, \ldots, k_d] \sum_{m=2}^{\infty} \frac{1}{m^{n+2}},
\end{align*}
which implies that
\[ \|\zeta[k_1, \ldots, k_d, n + 2] - \zeta[k_1, \ldots, k_d]_1\| \leq \|\zeta[k_1, \ldots, k_d]\| \sum_{m=2}^{\infty} \frac{1}{m^{n+2}}. \]

Since
\[ \lim_{n \to \infty} \sum_{m=2}^{\infty} \frac{1}{m^{n+2}} = 0, \]
we get
\[ \lim_{n \to \infty} \zeta[k_1, \ldots, k_d, n + 2] = \zeta[k_1, \ldots, k_d]_1. \]

And then \( \zeta[k_1, \ldots, k_d]_1 \in QZZ^{(1)}. \)

Conversely, for any \( f \in QZZ^{(1)} \), there exists a sequence \( (\zeta(\mathbf{k}(n), r(n)))_{n \in \mathbb{N}} \) such that \( \mathbf{k}(n) \) is admissible, \( r(n) \in \mathbb{N} \) and
\[ \lim_{n \to \infty} \zeta(\mathbf{k}(n), r(n)) = f. \]
We may assume that \( f \neq 0 \) and for any \( n_1 \neq n_2, \zeta(\mathbf{k}(n_1), r(n_1)) \neq \zeta(\mathbf{k}(n_2), r(n_2)) \).

By Corollary 3.6, the sequence \( (\zeta(\mathbf{k}(n), r(n)))_{n \in \mathbb{N}} \) has a subsequence of one of the forms \([5.3]-[5.7]\). Without loss of generality, we may assume that the sequence \( (\zeta(\mathbf{k}(n), r(n)))_{n \in \mathbb{N}} \) itself is one of the forms \([5.3]-[5.7]\). Now we discuss case by case.

Let \( (\zeta(\mathbf{k}(n), r(n)))_{n \in \mathbb{N}} \) be of the form \([5.3]\). Similarly as above, we have
\[ \lim_{n \to \infty} \|\zeta[k_1, \ldots, k_d, \varphi(n) + 2] - \zeta[k_1, \ldots, k_d]_1\| = 0. \]

Therefore, \( f = \zeta[k_1, \ldots, k_d]_1. \)

Let \( (\zeta(\mathbf{k}(n), r(n)))_{n \in \mathbb{N}} \) be of the form \([5.4]\). Using Lemma 5.4, we have
\[ \|f\| = \lim_{n \to \infty} \|\zeta[2, \{1\}^{\psi(n)}, r]\|. \]
By the definition of norms, we have
\[ \|\zeta[2, \{1\}^{\psi(n)}, r]\| \geq \zeta[2, \{1\}^{\psi(n)}, r]. \]

While if \( r = 1 \), we find
\[ \zeta[2, \{1\}^{\psi(n)}, r] = \zeta(\psi(n) + 3) \to 1 > 0, \quad (n \to \infty). \]
And if $r > 1$, we have

$$
\zeta(2, \{1\}^{\psi(n)}, r) = \zeta(2, \{1\}^{r-2}, \psi(n) + 2) \to \zeta(2, \{1\}^{r-2}, 0, 1 > 0, \ (n \to \infty).
$$

Therefore we always have $\|f\| > 0$. On the other hand, for a fixed $q \in (0, 1)$, we have

$$
0 < \zeta[2, \{1\}^{\psi(n)}, r] < \zeta[2, \{1\}^{\psi(n)}, 1] = \zeta[\psi(n) + 3]
$$

and

$$
\zeta[\psi(n) + 3] = q^{\psi(n) + 2} + \sum_{m=2}^{\infty} \frac{q^{m^{2}+m}}{m^{\psi(n)+3}} < q^{\psi(n) + 2} + \sum_{m=2}^{\infty} \frac{1}{m^{\psi(n) + 3}} \to 0, \ (n \to \infty).
$$

Therefore from Lemma 3.3, we have $f(q) = 0$ for any $q \in (0, 1)$. Hence we find that

$$
\zeta[2, \{1\}^{\psi(n)}, 0] < \zeta[2, \{1\}^{\psi(n)}, 1] = \zeta[\psi(n) + 3]
$$

and

$$
\zeta[\psi(n) + 3] = q^{\psi(n) + 2} + \sum_{m=2}^{\infty} \frac{q^{m^{2}+m}}{m^{\psi(n)+3}} < q^{\psi(n) + 2} + \sum_{m=2}^{\infty} \frac{1}{m^{\psi(n) + 3}} \to 0, \ (n \to \infty).
$$

Let $(\langle k(n), r(n) \rangle)_{n \in \mathbb{N}}$ be of the form (3.6). Similarly as above, we have

$$
\|f\| = \lim_{n \to \infty} \|\zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2]\| \geq \lim_{n \to \infty} \zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2].
$$

Now since

$$
\zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2] = \zeta[2, \{1\}^{\psi(n)}]_{0,1} + \zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2]_{0,1}
$$

and

$$
\zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2]_{0,1} < \zeta[2, \{1\}^{\psi(n)}] \zeta[\varphi(n) + 2]_{0,1}
$$

we have

$$
\lim_{n \to \infty} \zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2]_{0,1} = 0,
$$

and then

$$
\lim_{n \to \infty} \zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2] = \lim_{n \to \infty} \zeta[2, \{1\}^{\psi(n)}]_{0,1}.
$$

Using the duality formula (3.10), we get

$$
\|f\| \geq \lim_{n \to \infty} \zeta[\psi(n) + 2]_{1,0} = \frac{1}{2} > 0.
$$

One the other hand, for any fixed $q \in (0, 1)$, we have

$$
\zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2] = \zeta[2, \{1\}^{\psi(n)}]_{1} + \sum_{m_{1} > \cdots > m_{\psi(n)+1} > m_{\psi(n)+2} > 1} q^{m_{1}} \frac{q^{m_{2}}}{m_{1}^{2}m_{2} \cdots m_{\psi(n)+1}m_{\psi(n)+2}}
$$

$$
< \zeta[2, \{1\}^{\psi(n)}] + \zeta[2, \{1\}^{\psi(n)}] \zeta[\varphi(n) + 2]_{0,1}
$$

$$
= \zeta[\varphi(n) + 2] + \zeta[\varphi(n) + 2]_{0,1} \to 0, \ (n \to \infty).
$$

Hence $f(q) = 0$ for any $q \in (0, 1)$. And we find that $(\zeta[2, \{1\}^{\psi(n)}, \varphi(n) + 2])_{n \in \mathbb{N}}$ does not converge in $B(0, 1)$.

Let $(\langle k(n), r(n) \rangle)_{n \in \mathbb{N}}$ be of the form (3.6). Similarly as above, we have

$$
\|f\| \geq \lim_{n \to \infty} \zeta[2, \{1\}^{\psi(n)}, k_{1}, \ldots, k_{d}, r] = \lim_{n \to \infty} \zeta[1, \psi(n) + 2] = \zeta[1]_{0,1} > 0,
$$

where $I$ is the dual index of $(k_{1}, \ldots, k_{d}, r)$. On the other hand, for any fixed $q \in (0, 1)$, since

$$
\zeta[2, \{1\}^{\psi(n)}, k_{1}, \ldots, k_{d}, r] < \zeta[2, \{1\}^{\psi(n)}] \zeta[k_{1}, \ldots, k_{d}, r]
$$

$$
= \zeta[\psi(n) + 2] \zeta[k_{1}, \ldots, k_{d}, r] \to 0, \ (n \to \infty),
$$

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we have \( f(q) = 0 \). Therefore \((\zeta[2, \{1\}^n, k_1, \ldots, k_d, r])_{n \in \mathbb{N}}\) does not converge in \( \mathcal{B}(0, 1) \).

Let \(((k(n), r(n)))_{n \in \mathbb{N}}\) be of the form \((3.7)\). Similarly as above, we have

\[
\|f\| \geq \lim_{n \to \infty} \zeta(2, \{1\}^{n}, k_1, \ldots, k_d, \varphi(n) + 2) = \lim_{n \to \infty} \zeta(2, \{1\}^{n}, k_1, \ldots, k_d)_{0,1} = \lim_{n \to \infty} \zeta(k, \varphi(n) + 2)_{1,0} = \zeta(k)_{1,1} > 0,
\]

where \(k\) is the dual index of \((k_1, \ldots, k_d)\), and we have used the duality formula \((3.10)\). On the other hand, for any fixed \(q \in (0, 1)\), we have

\[
f(q) = \lim_{n \to \infty} \zeta(2, \{1\}^{n}, k_1, \ldots, k_d, \varphi(n) + 2) = \lim_{n \to \infty} \zeta(2, \{1\}^{n}, k_1, \ldots, k_d)_{1} \leq \zeta(2, \{1\}^{n}) \zeta(k_1, \ldots, k_d)_{1} = \lim_{n \to \infty} \zeta(\varphi(n) + 2) \zeta(k_1, \ldots, k_d)_{1} = 0.
\]

Hence \((\zeta[2, \{1\}^{n}, k_1, \ldots, k_d, \varphi(n) + 2])_{n \in \mathbb{N}}\) does not converge in \( \mathcal{B}(0, 1) \).

Summarily, we have

\[
\mathcal{QZZ}^{(1)} = \{\zeta[k]_1 \mid k \text{ is admissible}\} \cup \{0\}.
\]

Now we compute \(\mathcal{QZZ}^{(2)}\). Since \(\zeta[3, \{1\}^n]_1 < \zeta[3, \{1\}^n] = \zeta[n + 2, 1]\),

we have

\[
\lim_{n \to \infty} \|\zeta[3, \{1\}^n]_1\| = 0.
\]

And hence \(0 \in \mathcal{QZZ}^{(2)}\).

Conversely, for any \(f \in \mathcal{QZZ}^{(2)}\), there exists a sequence \((\zeta[k(n)]_1)_{n \in \mathbb{N}}\) such that \(k(n)\) is admissible and

\[
\lim_{n \to \infty} \zeta[k(n)]_1 = f.
\]

We may assume that \(f \neq 0\) and for any \(n_1 \neq n_2\), \(\zeta[k(n_1)]_1 \neq \zeta[k(n_2)]_1\). By Theorem \((3.7)\) the sequence \((k(n))_{n \in \mathbb{N}}\) has a subsequence of the form \((3.8)\) or of the form \((3.9)\). Without loss of generality, we may assume that the sequence \((k(n))_{n \in \mathbb{N}}\) itself is of the form \((3.8)\) or of the form \((3.9)\).

If \(k = (2, \{1\}^{n})\), then similarly as above, we have

\[
\|f\| \geq \lim_{n \to \infty} \zeta(2, \{1\}^{n})_{0,1} = \lim_{n \to \infty} \zeta(\varphi(n) + 2)_{1,0} = \frac{1}{2},
\]

and

\[
\zeta[2, \{1\}^{n}]_1 < \zeta[2, \{1\}^{n}] = \zeta[\varphi(n) + 2] \to 0, \quad (n \to \infty)
\]

for any fixed \(q \in (0, 1)\). Hence \((\zeta[2, \{1\}^{n}]_1)_{n \in \mathbb{N}}\) does not converge in \( \mathcal{B}(0, 1) \).

If \(k = (2, \{1\}^{n}, k_1, \ldots, k_d)\), then similarly as above, let \(\overline{k}\) be the dual index of \((k_1, \ldots, k_d)\), we have

\[
\|f\| \geq \lim_{n \to \infty} \zeta(2, \{1\}^{n}, k_1, \ldots, k_d)_{0,1} = \lim_{n \to \infty} \zeta(\overline{k}, \varphi(n) + 2)_{1,0} = \zeta(\overline{k})_{1,1},
\]

and

\[
\zeta[2, \{1\}^{n}, k_1, \ldots, k_d]_1 < \zeta[2, \{1\}^{n}] \zeta[k_1, \ldots, k_d]_1
\]
\[= \zeta[\psi(n)] + 2\zeta[k_1, \ldots, k_d]_1 \rightarrow 0, \quad (n \rightarrow \infty)\]

for any fixed \(q \in (0, 1)\). Hence \((\zeta[2, \{1\}^{\psi(n)}, k_1, \ldots, k_d]_{n \in \mathbb{N}}\) does not converge in \(B(0, 1)\).

Finally, we get \(\text{QZZ}^{(2)} = \{0\}\). And Theorem 1.3 is proved. \(\Box\)

We finish the paper with a remark.

**Remark 3.10.** Similarly as in [4], we can give an iterated Jackson’s \(q\)-integral representation for nonzero elements in \(\text{QZZ}^{(1)}\). In fact, for any admissible multi-index \(k = (k_1, \ldots, k_d)\), we have

\[
\zeta[k]_1 = (1 - q)^{k_1 + \cdots + k_d} R^{k_1 - 1} \left[ P[y(t)] R^{k_2 - 1} P[y(t)] \cdots R^{k_d - 1} P[y(t)] \cdots \right] \bigg|_{t=1},
\]

where \(y(t) = \frac{t}{1-t}\) and

\[
P[f](t) = f(t) + f(qt) + f(q^2 t) + \cdots, \quad R[f](t) = f(qt) + f(q^2 t) + \cdots.
\]

**References**

[1] P. Akhilesh, Double tails of multiple zeta values, *J. Number Theory* 170 (2017), 228-249.
[2] D. M. Bradley, Multiple \(q\)-zeta values, *J. Algebra* 283 (2005), 752-798.
[3] K. S. Kumar, Order structure and topological properties of the set of multiple zeta values, *Int. Math. Res. Not.* 2016 (5) (2016), 1541-1562.
[4] J. Zhao, Uniform approach to double shuffle and duality relations of various \(q\)-analogs of multiple zeta values via Rota-Baxter algebras, arXiv:1412.8044.
[5] J. Zhao, Multiple \(q\)-zeta functions and multiple \(q\)-polylogarithms, *Ramanujan J.* 14 (2007), 189-221.

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