Decoherence as Decay of the Loschmidt Echo in a Lorentz Gas.

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Classical chaotic dynamics is characterized by the exponential sensitivity to initial conditions. Quantum mechanics, however, does not show this feature. We consider instead the sensitivity of quantum evolution to perturbations in the Hamiltonian. This is observed as an attenuation of the Loschmidt Echo, $M(t)$, i.e. the amount of the original state (wave packet of width $\sigma$) which is recovered after a time reversed evolution, in presence of a classically weak perturbation. By considering a Lorentz gas of size $L$, which for large $L$ is a model for an unbounded classically chaotic system, we find numerical evidence that, if the perturbation is within a certain range, $M(t)$ decays exponentially with a rate $1/\tau_0$ determined by the Lyapunov exponent $\lambda$ of the corresponding classical dynamics. This exponential decay extends much beyond the Ehrenfest time $t_E$ and saturates at a time $t_s \approx \lambda^{-1} \ln(N)$, where $N \simeq (L/\sigma)^2$ is the effective dimensionality of the Hilbert space. Since $\tau_0$ quantifies the increasing uncontrollability of the quantum phase (decoherence) its characterization and control has fundamental interest.

Chaos justifies the observed macroscopic irreversibility within the reversible laws of Classical Mechanics. One of its characteristic features is the exponential divergence of trajectories corresponding to nearby initial conditions, which leads to deterministic unpredictability. However, quantum dynamics exhibits insensitivity to initial conditions and hence prevents a dynamical definition of quantum chaos. Therefore, quantum signatures of chaos in systems with a chaotic classical equivalent are searched in their steady-state properties such as spectral rigidity, wave function morphologies and decaying parametric correlations of observables. Early attempts to address unitary quantum dynamics did not clarify the connections with the dynamical classical concepts of chaos, but the inclusion of interactions with a dissipative environment was expected to produce an entropy growth controlled by chaos. In a purely Hamiltonian problem, quantum reversibility can be monitored through the amount of revival of a local density excitation, upon time reversal of its unitary quantum evolution, i.e. the Loschmidt Echo. By considering a Lorentz gas where the reversed evolution is disturbed by a static perturbation, we find a Loschmidt Echo that attenuates exponentially with a rate associated to the chaos of the classical system. Under appropriate conditions, the dynamical instability of the classical system translates into quantum phase unpredictability (decoherence) and the classical Lyapunov exponent becomes the quantum irreversibility rate.

A hypersensitivity of time reversal to perturbations was observed in recent NMR experiments on many-body spin systems. In essence, these experiments involve the creation of a local density excitation represented by a state $|0\rangle$ which evolves under a many-spin Hamiltonian $\mathcal{H}$. The Loschmidt Echo (LE) is the probability to return to the initial state when a Hamiltonian evolution for a time $t$ is followed by an identical period of imperfect reversal of that evolution, achieved by the transformation $\mathcal{H} \rightarrow -(\mathcal{H} + \Sigma)$, i.e.:

$$M(t) = |\langle 0 | \exp[i(\mathcal{H} + \Sigma)t/\hbar] \exp[-i\mathcal{H}t/\hbar] |0\rangle|^2,$$

where $\Sigma$ is a constant (or quasi-static) Hermitian perturbation representing the imperfection in the Hamiltonian reversal. Notice that in these experiments $-\ln|M(t)|$ is a measure of the growing entropy. A striking finding was that, for small $\Sigma$, the decay of $M(t)$ becomes independent of $\Sigma$, being proportional only to the dynamical scales of $\mathcal{H}$. The hints that chaotic systems may be unstable under fluctuating perturbations and that many-body systems could be assumed to be intrinsically chaotic suggested us the hypothesis that the most relevant parameters of the Hamiltonian dynamics, the Lyapunov exponents $\lambda$, might control the decay of $M(t)$. Moreover, a semiclassical analysis, assuming a one-body classically chaotic Hamiltonian with a perturbative potential representing a quenched disorder predicted a LE decaying exponentially with a rate $1/\tau_0 = \lambda$. These predictions, based on various approximations, encouraged us to perform exhaustive numerical experiments.

We consider a Lorentz gas, i.e.: a particle in a square billiard of area $L^2$ where we fix an irregular array of $N$ circular scatterers (impurities) of radius $R$. A particular
realization of such system is represented in Fig. 1-a where the length, in a minimal unit \(a\), is \(L = 200a\). The scatterers concentration is \(c = N \pi R^2 / L^2 = 8\pi \times 10^{-2}\). The minimum distance allowed between two scatterer centers is \(3R\). This requirement, together with the imposed periodic boundary conditions, prevents the geometrical localization.

The Lyapunov exponent of a Lorentz gas should be: 
\[
\lambda = \beta v / \ell, \quad \text{where} \quad \ell \text{ is the collision mean free path,} \quad v \text{ the particle velocity and} \quad \beta \approx 1 \text{ a geometrical factor}\]

The Hamiltonian operator is obtained by a lattice discretization on a small scale \(a\). The evolution is calculated using the Trotter-Suzuki (TS) algorithm \([19]\). Its conceptual virtue of this method is that \(\tilde{U}(\tau)\) is always unitary. For an approximation of order \(n\), the difference between \(\tilde{U}(\tau)\) and \(U(\tau)\) is order \(\tau^{n+1}\). Our choice of \(n = 4\) and \(\tau = 0.1h/V = 0.2m_o a^2 / h\) allowed us to obtain high accuracy even for times an order of magnitude larger than needed.

By considering a typical system with \(R/a = 20\), \(c = 8\pi \times 10^{-2}\) and size \(L/a = 200\). The initial state is a Gaussian wave-packet of width \(\sigma = 3a\) and wave number \(k_x = -0.7/a\), \(k_y = 0\) shown in Fig. 1-a. All further simulations will use \(k_x = k_y = 3\pi / 8a\). Since \(kR \gg 1\), comparison with semiclassical calculations is justified. We took \(2\pi / k < \ell = (96 \pm 2)a\), in order to minimize Anderson’s localization effects. The density resulting from a typical evolution for a time \(t = 30h/V\) is shown in panel b). An evolution with the perturbed Hamiltonian \((\alpha = 0.04)\) for the same time is shown in Fig. 1-c. In both panels the classical trajectories corresponding to three initial positions are shown for reference. While densities look similar to the eye, \(M(t)\) is about 0.09 indicating the relevant role of the quantum phase in the attenuation of \(M(t)\). Panel d) shows the LE formed by the perturbed Hamiltonian: 

\[
\Sigma(\alpha) = -\frac{\alpha}{1 + \alpha} \frac{p_x^2}{2m_o} + \alpha \frac{p_y^2}{2m_o}.
\]
backward evolution of the state in panel b). Analyzing $M(t)$ in different realizations we observe that, after the initial transient, $M(t)$ fluctuates around an exponential decay, with a characteristic time $\tau_\phi$. Notably this exponential decay persists much beyond the Ehrenfest time $t_E \approx \frac{1}{2} \ln[2R/\lambda_F] \approx 40 \hbar/V$, which is fixed by the local scale of the potential fluctuation [2]. In a Lorentz gas, in contrast with the usual case of chaotic systems, $t_E$ is independent of the system size [2]. Finally, $M(t)$ saturates in a time scale related to system size. A typical curve for $L/a = 800$ is shown with a bold line in Fig. 2.

In principle, any given precision of $\tau_\phi$ can then be obtained studying a system large enough, however this easily becomes computationally expensive. Alternatively, one can resort to the ensemble averaging of the observable $M(t)$, which reduces noise and defines $\tau_\phi$ with the same precision in much smaller systems. This is exemplified in Fig. 2, where we present the ensemble average $M(t)$ for billiards of three different sizes of $L/a = 200$, 400 and 800 and fixed $\alpha = 0.024$ and $c = 8\pi \times 10^{-2}$ which show the same exponential behavior with a progressively expanded range. Similar plots are obtained for the other parameters with an exponential $M(t)$, showing that the ensemble average does not introduce any spurious effect in the decay. Previous attempts to characterize quantum chaos [3] considered the limiting value $M_\infty$

for the Loschmidt Echo at asymptotically large times. It was proposed $M_\infty \propto 1/\tilde{N}$, where $\tilde{N}$ is the number of energy levels appreciably represented in the initial state. In our case $\tilde{N} \approx (L/\alpha)^2$. The numerical results verify a complete independence of $M_\infty$ on the perturbation. The calculated $M_\infty$ for various sizes are shown in the inset of Fig. 2 for a fixed $\alpha = 0.024$. The predicted relationship is shown with a straight line obtaining $M_\infty = (5.0 \pm 0.1)(\alpha/L)^2$. For all the sizes we kept $c$ and hence $\ell$, $\lambda$ and $t_E$ fixed and we verified that $\tau_\phi$ does not depend on $L/a$. These results imply that the Lyapunov exponential behavior persists up to a time $t_s \approx \lambda^{-1} \ln[\tilde{N}]$.

A representative dependence of $\tau_\phi$ on the parameter $\alpha$ is obtained considering the smallest sample size compatible with a good observation of the exponential, in this case $L/a = 200$. Fig. 2 shows $M(t)$ averaged over 100 realizations that contain at least a scatterer in the classical trajectory of the wave packet. For $\alpha > \alpha_c \approx 0.02$, all the exponential decays coincide, within the numerical error, with the Lyapunov decay associated with the classical system, shown with a thick line for comparison. The initial perturbative parabolic decay, $M(t) \approx 1 - b(\alpha t)^2$ with $b \approx 0.37(V/h)^2$, has a characteristic scale which does depend quadratically on the perturbation strength and prevents the curves to lie superimposed. We show in the inset of Fig. 2 the numerical values of $\tau_\phi^{-1}$ extracted from the exponential part. It shows an initial quadratic dependence on $\alpha$ and a cross-over at $\alpha_c \approx 0.02$ to a saturation at a value close to the classical Lyapunov exponent.

The semiclassical analysis predicts a universal behavior provided that the perturbation is strong enough to be quantically significant, but weak enough to be classically irrelevant. The first condition implies that the length $\ell = \sqrt{\tau_{\phi}}$ required for an important change in the phase as consequence of the perturbation must be shorter than the distance associated to the Lyapunov exponent $v/\lambda$.

A rough estimation for our perturbation $\delta m_{x,x} = \alpha m_o$ is:

$$\tau \approx \frac{\hbar}{\delta E_k} = \frac{\pi m_o a^2}{\alpha b(1 - \cos[ka])} \lesssim \frac{1}{\lambda}$$

which would predict a critical value $\alpha_c \approx 0.13$. However, we see the universal exponential even for perturbations $\alpha > 0.02$ implying a critical value an order of magnitude smaller than Eq. (5). This suggests that either this estimation or the condition required in Ref. 14 is too strong.

The condition for a classically weak perturbation means that it must not modify the system’s global properties. Otherwise, even in absence of chaos (no collisions) the perturbation would spread out the classical trajectories. In our case, this $M(t)$ fits a Gaussian decay. If at the time $1/\lambda$, required for the chaos to set in, the overlap has already decayed by cause of the perturbation, the exponential decay would not be observed.
FIG. 3. Average $M(t)$ for a system with $L = 200a$, $R = 20a$, which makes $t = (96 \pm 2)a$. The values of $\alpha$ are, from top to bottom: 0.006, 0.009, 0.012, 0.018, 0.024, 0.03, 0.04, 0.05 and 0.06. The thick line corresponds to an exponential with the calculated numerical Lyapunov exponent of the system. For long times $M(t)$ saturates to a finite value, $M_\infty$. In the inset the inverse characteristic decay times are shown, evidencing the regime where the decay of $M$ is given by the intrinsic properties of the system. The dotted is the fit, $\tau^{-1} = (130 \pm 6)\alpha^2V/\hbar$, while the dashed line is the classical Lyapunov exponent.

This sets an upper bound for the perturbation from $M(t = 1/\lambda) = \exp[-b(\alpha/\lambda)^2] \leq M_\infty$. Replacing with the parameters of the system shown in Fig. (3) one gets the condition $\alpha \leq 0.3$, consistent with our results.

Therefore, we have shown that in a wide range of parameters $M(t)$ in any large enough individual system decays exponentially until an asymptotic value is reached. The characteristic decay time does not depend on the perturbation, but rather on the intrinsic properties of the Hamiltonian. The range of perturbation parameters with an exponential decay results to be much broader than hinted by the previous semiclassical analysis. From our numerical results one can infer that infinite systems not showing Anderson’s localization should present an unbounded exponential decay. The Lorentz gas represents a broad class of chaotic systems, where it is expected the same overall behavior. It has the additional advantage over other chaotic cavities that here the saturation time ($t_s \approx 2/\alpha \ln[L/\sigma]$) can be made arbitrarily longer than the Ehrenfest time. This manifests the quantum nature of the observed effect. The simplicity of our perturbation evidences a very remarkable property: there is no need of chaoticity or stochasticity in the “environment” to introduce irreversibility. Fig. 4 manifests that the irreversibility produced by small perturbations results through their effect on the wave function’s phase. In this sense, the definition of classical chaos in terms of hypersensitivity to initial conditions and perturbations translates, in the quantum world, as a sensitivity of phases to perturbations in the Hamiltonian, i.e. decoherence. Since we restricted ourselves to a Hamiltonian problem, its solution clarifies how chaos limits our control of dynamics even within the reversible framework of Quantum Mechanics of closed systems. This concept has deep consequences for the diverse fields where decoherence must be minimized, such as classical waves propagation, mesoscopic and molecular electronics and quantum information. In view of the entropic meaning of $-\ln[M(t)]$ the interest for foundations of Statistical Physics is clear.

In summary, we have presented numerical evidence supporting the fact that decoherence, as measured by the exponential decay of the Loschmidt Echo, is controlled by the same parameters that govern classical chaos. This renders $M(t)$, defined in Eq. 1, as a very practical tool to study dynamical quantum chaos.

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