Can extra dimensional effects replace dark matter?

Supratik Pal *, Somnath Bharadwaj † and Sayan Kar ‡

Department of Physics and Centre for Theoretical Studies
Indian Institute of Technology
Kharagpur 721 302, India

Abstract

In the braneworld scenario, the four dimensional effective Einstein equation has extra terms which arise from the embedding of the 3-brane in the bulk. We show that in this modified theory of gravity, it is possible to model observations of galaxy rotation curves and the X-ray profiles of clusters of galaxies, without the need for dark matter. In this scenario, a traceless tensor field which arises from the projection of the bulk Weyl tensor on the brane, provides the extra gravitational acceleration which is usually explained through dark matter. We also predict that gravitational lensing observations can possibly discriminate between the proposed higher dimensional effects and dark matter, the deflection angles predicted in the proposed scenario being around 75% to 80% of the usual predictions based on dark matter.

* Electronic address : supratik@cts.iitkgp.ernet.in; Permanent address : Department of Physics, Ramakrishna Mission Vidyapith, Purulia 723 147 and Department of Physics, Jadavpur University, Kolkata 700 032, India
† Electronic address : somnath@cts.iitkgp.ernet.in
‡ Electronic address : sayan@cts.iitkgp.ernet.in
**Introduction**: To determine the nature of dark matter and how it is distributed is one of the most important problems currently facing physicists and astro-physicists. The problem arises, over a range of astrophysical length-scales, from a variety of observations which determine the dynamical mass. Observations of galaxy rotation curves and the gravitational lensing by galaxies are some of the most direct probes of dark matter on galactic scales (\(\sim 10 - 100 \text{kpc}\)). The X-ray profiles of clusters of galaxies and the gravitational lensing by these objects probe dark matter on larger scales (\(\sim 0.1 - 10 \text{Mpc}\)).

The usual analysis of these observations is based on two assumptions:

1. Einstein’s theory of gravitation as given by the equations \(G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}\) is valid on the length-scales in question.

2. The matter in galaxies and clusters of galaxies is such that relativistic stresses do not make a significant contribution to the stress-energy tensor \(T_{\mu\nu}\), and \(T_{00} = \rho c^2\) is the only non-zero component.

It follows that the Einstein tensor \(G_{\mu\nu}\) has only one non-zero component \(G_{00}\) which can be determined directly from observations of either the rotation curve or the X-ray profiles, or from gravitational lensing, and this allows the total matter distribution to be mapped out. The mass determined from such dynamical means is always found to be in excess of that which can be attributed to the visible matter. This discrepancy is explained by postulating that every galaxy and cluster of galaxy is embedded in a halo made up of some kind of invisible matter, the dark matter. The exact nature of this dark matter is unknown, with exotic supersymmetric particles currently being accepted as the most favoured candidates. Till date, direct searches for the dark matter particles have not yielded any detection.

It is important to take note of the fact that none of the assumptions used have been independently tested on either the galactic or the cluster length-scales. This raises the possibility that there actually may not be any dark matter and it may be possible to explain these observations using a modified theory of gravity. Another possibility is that the dark matter may have relativistic stresses, which will require a different interpretation of the observations and will result in a different inferred dark matter distribution.
The possibility that we live in a warped five dimensional (or possibly higher) space-time, in which the familiar 4-dimensional space-time is a hyper-surface has, of late, received a considerable amount of attention. In this so-called \textit{brane-world} scenario, the effective Einstein equations pick up extra terms which arise from the embedding of our four dimensional space-time hyper-surface, referred to as the brane, in the five dimensional space-time, the bulk. In this \textit{Letter}, we show that it is possible to model the halos of galaxies and clusters of galaxies using the modified Einstein equation, and thereby explain the observed rotation curves and X-ray profiles without the need for dark matter.

The effective four dimensional Einstein equations is given by:

\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2_4 T_{\mu\nu} + \kappa^4_5 S_{\mu\nu} - E_{\mu\nu} \]  

where \( \Lambda \), the brane cosmological constant, depends on the bulk cosmological constant and the brane tension \( \lambda_b \) both of which can be fine-tuned to make \( \Lambda = 0 \) which we adopt throughout. The constants \( \kappa_4 \) and \( \kappa_5 \) are defined as \( \kappa^2_4 = 8\pi G/c^4 = \kappa^2_5 \lambda_b/6 \), and \( E_{\mu\nu} \) is the limit, of the projection on the brane, of a quantity defined in 5-dimensions, which is related to the bulk Weyl tensor and the bulk matter. In the absence of bulk matter, \( E_{\mu\nu} \) is a traceless symmetric tensor. The term \( S_{\mu\nu} \) is quadratic in the brane energy-momentum tensor, and it can be shown to be small compared to both the usual linear energy-momentum tensor \( T_{\mu\nu} \) and \( E_{\mu\nu} \). In the following, we shall ignore the contribution from the term \( S_{\mu\nu} \).

To summarize, the effective Einstein equation on the brane is

\[ G_{\mu\nu} = -E_{\mu\nu} + \kappa^2_4 T_{\mu\nu} \]  

where the difference from the usual Einstein equation is that we have an extra traceless tensor \( E_{\mu\nu} \) which is a purely geometrical term that arises from embedding of the 3-brane in the bulk. We investigate if it is possible to consistently model observations of galaxies and clusters of galaxies without the need for dark matter in this modified theory of gravity.

\textit{Modelling galaxy and cluster halos:} It is possible to interpret observations of the rotation curves of spiral galaxies and the X-ray profiles of clusters of galaxies without reference to any particular theory of gravity or the existence and nature of dark matter, the only assumption being that gravitation arises from the geometry of space time and the gravitational field can be represented by the space-time metric \( g_{\mu\nu} \) which we choose with signature \((-,+,+,+).\)
In the work presented here we make two further assumptions which, though not crucial for
the discussion, substantially simplify the analysis.

First, we assume that galaxies and clusters of galaxies are embedded inside spherically
symmetric gravitational fields which we refer to as halos. The most general static, spherically
symmetric space-time is completely described by only two unknown functions \( \Phi(r) \) and \( \Psi(r) \)
which we choose so that the proper time inside the halo is

\[
c^2 d\tau^2 = -(1 + 2\Phi)c^2 dt^2 + (1 - 2\Phi + 2\Psi)[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]
\] (3)

Further, the gravitational field is assumed to be weak (\( \Phi, \Psi \ll 1 \)), and we retain terms
only to linear order in the potentials \( \Phi \) and \( \Psi \). This is an assumption which we shall justify
later.

We next briefly discuss how observations of the rotation curves of spiral galaxies and the
X-ray profiles of clusters of galaxies can be used to determine the gravitational potentials
inside the halo.

Observations of the 21 cm line from neutral hydrogen (HI) clouds in spiral galaxies show
these clouds to be distributed in a disk, aligned with the plane of the galaxy. Further, the
observed redshifts of the 21 cm emission determine the velocities of the HI clouds. These
observations show the clouds to be in circular orbits around the center of the galaxy. The
circular velocity \( v_c(r) \) of the HI clouds at different radius \( r \) from the center is referred to
as the “rotation curve”. The geodesic equation for the HI clouds, which we treat as test
particles moving in stable circular orbits under the gravitational influence of the halo,

\[
\Phi'(r) = \frac{1}{r} \frac{v_c^2(r)}{c^2}
\] (4)
determines the potential \( \Phi(r) \) in terms of the observed rotation curve. The rotational velocities
are typically in the range \((100 - 400) \text{km/s}\) and as a consequence \( \Phi \sim (v_c/c)^2 \sim 10^{-6} \),
validating our initial assumption that the gravitational field is weak.

Interpreting the observed X-ray emission from the hot, ionized intra-cluster gas in clusters
of galaxies under the assumption that the gas is isothermal allows the density profile \( \rho_g(r) \),
and the pressure \( P_g(r) = (kT/\mu m_p) \rho_g(r) \) of the gas to be determined. Here \( k \), \( T \), \( \mu \) and
\( m_p \) are the Boltzmann constant, gas temperature, mean atomic weight of the particles in
the gas and the proton mass respectively. The gas temperatures are typically in the range
\( 2 \times 10^7 - 10^8 \text{K} \) which implies that \((kT/\mu m_p c^2) \sim 10^{-5} - 10^{-6} \). Considering the energy-
momentum tensor for the intra-cluster gas \( T[gas]_{\mu\nu} = (P_g + \rho_g c)U_\mu U_\nu - P_g g_{\mu\nu} \), we see that
\[ P_g = \left( \frac{kT}{\mu m_p} \right) \rho_g \ll \rho_g c^2. \]

Assuming the gas to be in hydrostatic equilibrium, the energy momentum conservation \( T[^\text{gas}]_{\nu,\mu} = 0 \) gives

\[
\Phi'(r) = -\frac{kT}{\mu m_p c^2} \frac{d \ln \rho_g}{dr}.
\]

which can be used to determine \( \Phi \). It should be noted that in the energy-momentum conservation equation we have dropped terms of order \( P_g \Phi \) and \( P_g \Psi \), as these are much smaller compared to the terms involving \( \Phi \rho_g c^2 \) and \( P_g \) which we have retained. The factor \((kT/\mu m_p c^2)\) in equation (5) ensures that \( \Phi \ll 1 \), justifying the assumption that the field is weak.

While it requires two potentials \( \Phi \) and \( \Psi \) to completely specify the gravitational field inside the halo, only one of the potentials, namely \( \Phi \), can be determined directly from observations of either the rotation curve or the X-ray profiles. It should also be noted that \( \Phi \) is the only one which matters if we are dealing with the motion of non-relativistic \((v/c \ll 1)\) particles. The potential \( \Psi \) is important when considering the motion of relativistic particles, e.g. photons which we shall consider later.

It is necessary to assume a specific theory for gravity if one is to proceed further in modelling galaxy or cluster halos. We shall first briefly outline the standard procedure which is based on Newtonian physics \( \text{i.e.} \) the two assumptions mentioned in the Introduction. The components of the Einstein tensor (which appear in nearly all geometrical theories for gravity) are listed below.

\[
G^0_0 = -2 \nabla^2 (\Phi - \Psi), \quad G^r_r = 2 \frac{\Psi'}{r}, \quad G^\theta_\theta = G^\phi_\phi = \Psi'' + \frac{\Psi'}{r}.
\]

Under the abovementioned Newtonian assumptions \( \text{i.e.} \) Einstein’s theory is valid on these length-scales, we have

\[
G^\mu_\nu = \frac{8\pi G}{c^4} T^\mu_\nu
\]

and that relativistic stresses are absent in \( T^\nu_\nu \) and \( T^0_0 = -\rho c^2 \) is the only non-zero component, implies \( \Psi = 0 \) and

\[
\nabla^2 \Phi = \frac{4\pi G}{c^2} \rho
\]

The reader has probably already realised that \( c^2 \Phi \) is the familiar gravitational potential which appears in Newtonian gravity, and \( \Psi \) quantifies deviations from the Newtonian theory.

In the Newtonian theory, the dark matter problem arises when one uses the potential \( \Phi \) determined from observations of either the rotation curves (eq. 4) or the X-profiles (eq. 5).
in equation (8) to make estimates of the density. These dynamical estimates of the density and the mass are always found to be substantially in excess of the visible matter. Hence it is required to postulate that around $\sim 80\%$, or more, of the matter in the outer parts of spiral galaxies and in clusters of galaxies is invisible, *i.e.* the dark matter.

We next consider the modified theory of gravity as discussed in equation (2), where the Einstein equation has an extra traceless term $E_{\mu}^{\nu}$ arising from the embedding of the 3-brane in higher dimensions.

$$G_{\nu}^{\mu} + E_{\nu}^{\mu} = \frac{8\pi G}{c^4} T_{\nu}^{\mu}. \quad (9)$$

We proceed by taking the trace of eq. (9) which gives us

$$\nabla^2 (\Phi - 2\Psi) = \frac{4\pi G}{c^2} \rho_v \quad (10)$$

where we have assumed that there is no dark matter, and the visible matter with density $\rho_v$ is all that contributes to the energy-momentum tensor. The solution to equation (10) is

$$\Psi = \frac{1}{2} \Phi - \frac{2\pi G}{c^2} (\nabla^2)^{-1}\rho_v \quad (11)$$

where $\Phi$ is determined from observations of either rotation curves or the X-ray profiles. The tensor $E_{\nu}^{\mu}$ can be calculated using eq. (9) once both $\Phi$ and $\Psi$ are known.

The point to note here is that we have a solution for the gravitational field inside the halo, consistent with observations of rotation curves or X-ray profiles, without the need for dark matter. The extra gravitational acceleration required to explain observation of galaxy rotation curves or the cluster X-ray profiles now arises from $E_{\nu}^{\mu}$ which incorporates the geometrical effects arising from the embedding of the 3-brane in the bulk. The proposal that $E_{\nu}^{\mu}$ can replace dark matter has been made earlier [15], but not substantiated in a general situation. The earlier work imposes an ad hoc conformal symmetry to obtain spherically symmetric vacuum solutions of the modified Einstein’s equations (eq. 2) which are consistent with flat rotation curves. In this *Letter* we have outlined, in general and without any ad hoc assumptions, how rotations curves and X-ray profiles can be interpreted without dark matter in the modified theory of gravity.

We next take up a specific example and explicitly calculate $\Phi$ and $\Psi$. Interpreting observations of spiral galaxies is somewhat complicated as the visible matter is mainly distributed in a disk [16] which breaks spherical symmetry. Further, ambiguities in the mass to light
ratio makes it difficult to uniquely determine the mass corresponding to the visible matter. The situation is simpler for clusters where the X-ray gas is the dominant component of visible matter (i.e. $\rho_v = \rho_g$). The mass density of X-ray gas is usually modeled using the spherically symmetric, isothermal $\beta$ model with

$$\rho_g(r) = \rho_0 [1 + (r/r_c)^2]^{-3\beta/2}$$  \hfill (12)$$

where $\rho_0$ is the central density, $r_c$ the core radius and $\beta$ decides the slope at $r \gg r_c$, and these parameters have values in the range $(7-150) \times 10^{-23}$ kg/m$^3$, $0.1-0.8$ Mpc and $0.5-0.9$ respectively. For simplicity we use $\beta = 2/3$ and restrict our analysis to $r \gg r_c$. Further, we use $\rho_0 = 5 \times 10^{-24}$ kg/m$^3$, $r_c = 0.3$ Mpc, $\mu = 0.6$ and $T = 10^8$ K as representative values when making estimates.

Solving equation (13) gives us

$$\Phi = \frac{2kT}{\mu m_p c^2} \ln \frac{r}{r_c}.$$  \hfill (13)$$

In the usual Newtonian analysis $\Psi = 0$, and $\Phi$ is used in eq. (8) to determine the total matter density $\rho(r) = (kT/2\pi G\mu m_p) r^{-2}$ needed to keep the hot X-ray gas in hydrostatic equilibrium. Comparing $\rho(r)$ with $\rho_g(r)$ we find that $\rho_g(r)/\rho(r) = (2\pi G\rho_0 r_c^2/\mu m_p/kT) \sim 0.2$ i.e. 80% of the total matter has to be in an invisible form, the dark matter.

In the modified theory $\Psi \neq 0$ and we solve eq. (11) to obtain

$$\Psi = \left[ \frac{kT}{\mu m_p c^2} - \frac{2\pi G\rho_0 r_c^2}{c^2} \right] \ln \frac{r}{r_c}.$$  \hfill (14)$$

which is a solution of the modified Einstein's equations (eq. 9) without any dark matter. The non-zero components of $E^\mu_\nu$, $E^0_0 = -E^r_r = [(2kT/\mu m_p c^2) - (4\pi G\rho_0 r_c^2/c^2)] r^{-2}$ now provide the extra gravitational acceleration.

The two different theories for gravity considered here interpret the same X-ray observations to infer different space-time geometries for cluster halos. It is necessary to consider other independent probes of the space-time geometry to discriminate between the two possibilities namely dark matter and higher dimensional effects.

**Gravitational Lensing:** Observations of gravitational lensing provide independent constraints on the gravitational field inside halos. These observations probe both $\Phi$ and $\Psi$, and are sensitive to the full geometry of the space-time inside halos. In the standard Newtonian analysis where $\Psi = 0$, the deflection angle $\hat{\alpha}_N$ of a photon from a distant source $(s)$,
propagating through the halo to a distant observer \((o)\) is given to be

\[
\hat{\alpha}_N = 2 \int_s^o \hat{\nabla}_\perp \Phi \, dl
\]  

(15)

where the integral is to be evaluated along the straight line trajectory between the source and the observer, and \(\hat{\nabla}_\perp\) denotes the derivative in the direction perpendicular to this trajectory. Using eq. (13) we find that a photon passing through the halo of a cluster experiences a constant deflection given by

\[
\alpha_N = \frac{4\pi kT}{\mu m_p c^2}
\]  

(16)

Generalizing eq. (15) to the situation where \(\Psi \neq 0\) gives

\[
\hat{\alpha} = \int_s^o \hat{\nabla}_\perp (2\Phi - \Psi) \, dl .
\]

(17)

Using \(\Phi\) and \(\Psi\) calculated for the modified theory of gravity (eqs. 13 and 14) gives the deflection angle to be

\[
\hat{\alpha} = \hat{\alpha}_N \left[ 0.75 + \frac{\pi G \rho_0 r_c^2 \mu m_p}{2 kT} \right]
\]

(18)

The term \(\frac{\pi G \rho_0 r_c^2 \mu m_p}{2 kT}\) which arises from the contribution of the visible matter to \(\Psi\) (eq. 11) is around 0.05 for our choice of cluster parameter and it falls to less than 0.01 if \(r_c = 0.1\) Mpc.

We find that the modified theory with no dark matter predicts a lensing deflection angle which is smaller than that of the usual Newtonian analysis where there is dark matter. For the cluster parameters adopted here, it is 80% of the Newtonian value, and it is expected to be in the range 75% to 80% of the Newtonian value for typical clusters, depending on the cluster parameters. This should, in principle, allow us to observationally discriminate between the two possibilities and determine which is correct. Carrying this out requires X-ray and gravitational lensing observations of the same cluster. The X-ray profiles can be used to determine the metric which will be different in the two scenarios. These can be used to make lensing predictions which can be compared with observations to test which scenario is correct. There presently exists a substantial volume of such observations which have been interpreted in the Newtonian picture using dark matter. A significant fraction of these observations have been interpreted to conclude that the dark matter masses inferred from X-ray observations are significantly smaller (~ 2 to 4 times) than the masses inferred from gravitational lensing [19], while there also are a significant number of claims that the X-ray and lensing observations are consistent [20]. At a preliminary level it may be
speculated that the present uncertainties (statistical and systematic) in the modeling of X-ray and lensing observations are sufficiently large that the alternate possibility considered here, whose predictions differ by around 20%, would fare equally well as the dark matter scenario in simultaneously fitting X-ray and lensing data.

There currently exists a large body of observations on cosmological scales (1 Mpc to 10 Gpc) like the CMBR anisotropies [21] and the clustering of galaxies [22] all of which are consistent, at a high level of precision, with a cosmological model where one third of the present matter density is in cold dark matter and two-thirds in dark energy which has negative pressure, the sum of the two densities being very close to the critical value $3H_0^2/8\pi G$, where $H_0$ is the present value of the Hubble parameter. The interpretation of these observations requires the analysis of the growth of perturbations in an expanding background cosmological model. It is to be seen if the dynamics of perturbations in the presence of $E_{\mu\nu}$ can explain these observations without the need for dark matter.

[1] L. Bergstrom, Rept. Prog. Phys. 63, 793 (2000); E. Hoyashi and J. F. Navorro, AAS, 201 (2002); F. Combes, New Astron. Rev., 46, 755 (2002)
[2] J. J. Binney and S. Tremaine, Galactic Dynamics, Princeton University Press, Princeton (1987); M. Persic, P. Salucci and F. Stel, Month. Not. R. Acad. Sci. 281, 27 (1996); A. Berriello and P. Saluc, Month. Not. R. Acad. Sci. 323, 285 (2001); Y. Sauf and V. Rubin, Ann. Revs. Astron. Astrophy. 39, 137 (2001)
[3] P. Schneider, J. Ehlers and E. Falco, Gravitational lenses, Springer Verlag, Berlin (1992)
[4] N. Bahcall in Formation of structures in the universe A. Dekel and J. P. Ostriker (Ed), Cambridge University Press (1999); P. Rosati, S. Borgani and C. Norman, Ann. Rev. A & A, 40, 539 (2002)
[5] G. Jungman, M. Kamionkowski and K. Griest, Phys Rep, 267, 195 (1996); W. B. Lin, D. H. Huang and R. H. Brandenberger, Phys. Rev. Lett. 86, 954 (2001); A. M. Green, Phys. Rev. D66, 8300 (2002)
[6] M. Milgrom, ApJ, 270, 365 (1983); J. Bekenstein and M. Milgram, ApJ 286, 7 (1984); M. Milgrom, New Astron. Rev. 46, 741 (2002), M. Milgrom, ApJ 599, L25 (2003)
[7] T. Matos and F. S. Guzman, Class. Quant. Grav, 18, 5055 (2001); T. Matos and L. Arturo
Urena-Lopez, Phys. Rev. D66, 023514 (2002); T. Matos and D. Nunez, astro-ph/0303594 (2003); K. Lake, Phys. Rev. Lett. 92, 051101 (2004)

[8] S. Bharadwaj and S. Kar, Phys. Rev. D68, 023516 (2003), and references therein

[9] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999)

[10] C. Csaki, hep-ph/0404096 and references therein

[11] T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D62, 024012 (2000)

[12] A. Padilla, Braneworld Cosmology and Holography, hep-th/0210217 (2002)

[13] R. Maartens, Phys. Rev. D62, 084023 (2000); R. Marteens, gr-qc/0312059 (2003)

[14] N. Dadhich, R. Marteens, P. Papadopoulos and V. Rezania, Phys. Lett. B487, 1 (2000)

[15] T. Harko and M. K. Mak, Phys. Rev. D69, 064020 (2004); M. K. Mak and T. Harko, Phys. Rev. D70 024010 (2004)

[16] J. J. Binney and M. Merrifield, Galactic Astronomy, Princeton University Press, Princeton (1998);

[17] see for example, W. J. G. de Blok and A. Bosma, A & A, 385, 816 (2000); W. J. G. de Blok, S. S. McGaugh and V. C. Rubin, AJ, 122, 2396 (2001); E. F. Bell and R. S. de Jong, ApJ, 550, 212 (2001)

[18] M. Markevitch, W. R. Forman, C. L. Sarazin and A. Vikhlinin, Astrophys. J. , 503, 77 (1998); C. Jones and W. Forman, Astrophys. J. , 276, 38 (1984)

[19] see Y. Mellier, ARAA, 37, 127 (1999) for a review

[20] eg. S. Ettori and M. Lombardi, A & A, 398, L5 (2003); X. P. Wu, T. Chiueh, L. Z. Fang and Y. J. Xue, Month. Not. R. Acad. Sci., 301, 861 (1998); S. W. Allen, Month. Not. R. Acad. Sci., 296, 392 (1998); S. W. Allen, S. Ettori and A. C. Fabian, Month. Not. R. Acad. Sci., 324, 877 (2001); H. Boehringer, Y. Tanaka, R. F. Mushotzky, Y. Ikebe and M. Hattori, A & A, 334, 789 (1998)

[21] eg. WMAP, D. N. Spergel et al., ApJ, 148, 175 (2003)

[22] eg. M. Tegmark, M. et al., Astrophys. J. , 606, 702 (2004); M. Tegmark et al., Phys. Rev. D69, 103501 (2004)