A weak law of large numbers is established for a sequence of systems of $N$ classical point particles with logarithmic pair potential in $\mathbb{R}^n$, or $S^n$, $n \in \mathbb{N}$, which are distributed according to the configurational microcanonical measure $\delta(E - H)$, or rather some regularization thereof, where $H$ is the configurational Hamiltonian and $E$ the configurational energy. When $N \to \infty$ with non-extensive energy scaling $E = N^2 \varepsilon$, the particle positions become i.i.d. according to a self-consistent Boltzmann distribution, respectively a superposition of such distributions. The self-consistency condition in $n$ dimensions is some nonlinear elliptic PDE of order $n$ (pseudo-PDE if $n$ is odd) with an exponential nonlinearity. When $n = 2$, this PDE is known in statistical mechanics as Poisson-Boltzmann equation, with applications to point vortices, 2D Coulomb and magnetized plasmas and gravitational systems. It is then also known in conformal differential geometry, where it is the central equation in Nirenberg’s problem of prescribed Gaussian curvature. For constant Gauss curvature it becomes Liouville’s equation, which also appears in two-dimensional so-called quantum Liouville gravity. The PDE for $n = 4$ is Paneitz’ equation, and while it is not known in statistical mechanics, it originated from a study of the conformal invariance of Maxwell’s electromagnetism and has made its appearance in some recent model of four-dimensional quantum gravity. In differential geometry, the Paneitz equation and its higher order $n$ generalizations have applications in the conformal geometry of $n$-manifolds, but no physical applications yet for general $n$. Interestingly, though, all the Paneitz equations have an interpretation in terms of statistical mechanics.

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1 INTRODUCTION

Early in his scientific career Joel Lebowitz spent a postdoctoral year or so working with Lars Onsager. There are many scientific similarities between these two towering figures of statistical physics: both in possession of penetrating mathematical powers; both broadly interested in physics; both with a good taste for choosing problems worthy to work on; and both with deep insights into physics that have inspired – and continue to inspire – many other physicists and mathematicians around the world. Of course, Joel has also worked on many problems that were inspired by some work of Onsager, and while it would be impossible for me here even to merely mention them all, one contribution to a problem of Onsager that I was privileged to collaborate on with Joel has recently had some interesting mathematical spin-off, and I am very glad to present these new results in this paper in honor of Joel’s 70th birthday.

In the next section, I will first describe Onsager’s application of statistical mechanics to point vortices in two dimensions, the questions it has raised, and our contribution to it. In the two subsequent sections I will describe Nirenberg’s problem in the conformal differential geometry of two-dimensional manifolds and its recent generalization to n-manifolds. Point vortices in two dimensional flows have logarithmic pair interactions. In the last section I will explain how statistical mechanics of logarithmic interactions, but now in all dimensions, provides solutions to the Nirenberg-type problems in conformal geometry.

2 ONSAGER’S POINT VORTEX DISTRIBUTIONS

In a pioneering paper on statistical fluid dynamics \[57\], Onsager studied, among other things, the microcanonical ensemble of \( N \) classical point vortices in a two-dimensional, incompressible Euler flow. The motion of \( N \) vortices in a connected domain \( \Lambda \subseteq \mathbb{R}^2 \) is governed by Kirchhoff’s Hamiltonian \[43, 44, 47\],

\[
H^{(N)}(x_1, \ldots, x_N) = \frac{1}{2} \sum_{1 \leq i, j \leq N} c_i c_j G^*(x_i, x_j) + \sum_{1 \leq i \leq N} c_i F^{(N)}(x_i),
\]

(1)

where \( x_i \in \Lambda \) is the position of the \( i \)-th vortex; \( c_i \) its circulation, expressed as dimensionless multiple of a suitable reference unit; \( G^*(x, y) = G^*(y, x) \) is a renormalized Green’s function for \(-\Delta\) on \( \Lambda \), i.e. \( G^*(x, y) = G(x, y) \) if \( x \neq y \), with \( G(x, y) \) solving \(-\Delta G(x, y) = 2\pi \delta_y(x)\) in \( \Lambda \), and

\[
G^*(x, x) = \lim_{y \rightarrow x} \left( G(x, y) + \ln |x - y| \right).
\]

(2)

In case of a bounded domain with piecewise regular boundary \( \partial \Lambda \), the physically appropriate 0-Dirichlet boundary conditions for \( G \) are imposed on \( \partial \Lambda \), but in principle other boundary conditions are of interest too. Finally, \( F^{(N)} \) is some externally applied stream function (whose strength may be proportional to \( N \)). Considering a bounded domain with finite area \( |\Lambda| \), and noting that up to a trivial factor \( \sqrt{|c_i|} \) the canonically conjugate variables of the vortices are given by the Cartesian
components of their positions in \( \Lambda \), Onsager observed that the phase space volume of the set \( \{ H^{(N)} < E \} \) in the \( N \) vortices phase space \( \Lambda^N \),

\[
\Phi_{\Lambda}(E, N) = \left| \{ H^{(N)} < E \} \right|,
\]

is a monotonically increasing function of \( E \), bounded by \( |\Lambda|^N \). He noticed that this implies that Boltzmann’s entropy \( S_{\Lambda}(E, N) = \ln \left( |\Lambda|^{-N} \Phi'_{\Lambda}(E, N) \right) \) (where \( \Phi'_{\Lambda}(E, N) = \partial_E \Phi_{\Lambda}(E, N) \)) must reach a maximum at a particular value \( E_m(N; \Lambda) \) of energy, such that [57], pp. 281: “negative “temperatures”...will occur if \( E > E_m \), ... vortices of the same sign will tend to cluster ...[and] large compound vortices [be] formed in this manner.”

Onsager’s insight not only predated Ramsey’s prediction of negative temperatures in spin systems [63] by 7 years, Onsager was (once again) way ahead of his time. His program was not picked up until almost a quarter century later, when Montgomery pointed out [50] that numerical simulations of turbulent fluid flows suggested an explanation in terms of Onsager’s 1949 paper. But Onsager had not made any attempt to extract continuum vorticities from his statistical vortex distributions, and so several authors [36, 51, 61, 46, 38] now came up with the proposal to use mean-field theory for this purpose.

As it is with traditions, and statistical mechanics surely has a long tradition, certain general wisdoms tend to be passed on which sometimes may not be so generally valid. One such general wisdom holds that ‘mean-field theory is wrong.’ This traces back to the failure of the prototype mean-field theories of van der Waals and Curie-Weiss to give the correct data for the critical point of the condensation and magnetization phase transitions, respectively. In particular, they predict incorrect critical exponents. However, as I recall Michael Fisher pointing out in a lecture on Coulombic criticality, mean-field theory is not that bad after all. Another general wisdom states that thermodynamic concepts such as temperature become precise only in the thermodynamic (bulk) limit of an infinitely big system. A third general wisdom says that the statistical ensembles are equivalent. Against this background it is easier to appreciate that it took a while to realize that neither wisdom is correct in the case of Onsager’s statistical theory of vortex clustering.

Indeed, the first attempt [51] to put Onsager’s theory on a more rigorous basis was made in terms of the traditional bulk thermodynamic limit for a neutral two-species vortex system, using formal central limit arguments of Khinchin which where asserted to apply also to the negative temperature regime. Moreover, in that paper [51] also the BBGKY hierarchy was considered, and the mean-field equation for the distributions at negative \( T \), which in [36] had been obtained by the standard van-der-Waals-theory type combinatorics, was now obtained as Vlasov approximation. A subsequent rigorous study by Fröhlich and Ruelle [33] however showed that negative temperatures do, in fact, not exist in the bulk thermodynamic limit of a neutral two-species vortex system.

Inspection of the phenomenon that Onsager predicted reveals, however, that \( O(N^2_k) \) truly long range pair interactions in the \( k \)th cluster of vortices of the same sign are involved [46]. Hence, we are dealing with a strictly nonuniform system with non-extensive energy scaling. Understood from this perspective, the relevant limit \( N \to \infty \) in which Onsager’s prediction of negative vortex temperatures attains a rigorous meaning is not the bulk thermodynamic limit but rather a continuum
(Euler) fluid limit. Moreover, in this limit mean-field theory should become *exact* in the sense of a weak law of large numbers.

Thus, the following picture emerges. For the neutral two-species system, with $|c_i| = 1$, and with $F \equiv 0$ for simplicity, distributed in a bounded domain $\Lambda \subset \mathbb{R}^2$ according to the microcanonical measure

$$
\mu^{(N,E)}(dx_1...dx_N) = \frac{1}{\Phi'_{\Lambda}(E,N)} \delta \left( E - H^{(N)} \right) dx_1...dx_N ,
$$

(4)

where $dx$ denotes Lebesgue measure on $\mathbb{R}^2$, we have to fix $\Lambda \subset \mathbb{R}^2$ and $\varepsilon = E/N^2 > 0$. Then, in the limit $N \to \infty$, the Boltzmann entropy per vortex will converge to a continuous function of $\varepsilon$,

$$
\lim_{N \to \infty} \frac{1}{N} \ln \left( |\Lambda|^{-N} \Phi'_{\Lambda}(N^2\varepsilon, N) \right) = s_{\Lambda}(\varepsilon) ,
$$

(5)

which is given by a variational principle,

$$
s_{\Lambda}(\varepsilon) = \max_{\rho^+,\rho^-} \sum_{\sigma = \pm} - \int_{\Lambda} \rho^\sigma \ln \left( |\Lambda| \rho^\sigma \right) dx ,
$$

(6)

where the maximization is carried out over the probability densities $\rho^\pm$ which satisfy the energy constraint

$$
\frac{1}{2} \int_{\Lambda} \int_{\Lambda} G(x,y) \omega(x) \omega(y) dxdy = \varepsilon ,
$$

(7)

where $\omega = \rho^+ - \rho^-$. Moreover, the microcanonical equilibrium measure itself will converge, for $\varepsilon > 0$ and in a suitable topology, to a convex linear superposition of infinite products of those absolutely continuous one-vortex measures whose densities $\rho^\pm_\varepsilon$ are the maximizers for (6)-(7). In particular, if the superposition measure is a singleton (a unique maximizing pair $\rho^\pm_\varepsilon$ exists), then we have the weak law of large numbers, that for all $f \in C^0(\Lambda)$,

$$
\lim_{N \to \infty} \frac{2}{N} \sum_{j=1}^{N/2} f(x^\pm_j) = \int_{\Lambda} f(x) \rho^\pm_\varepsilon(x) dx
$$

(8)

in probability. In (8), the summation extends over either all positive or all negative vortices. If the superposition is not a singleton, $\rho^\pm_\varepsilon$ in (8) is to be replaced by the corresponding superposition of probabilities. Furthermore, the maximizers satisfy the self-consistency conditions

$$
\rho^\pm_\varepsilon(x) = \exp \left( \beta \left[ \mu^\pm_{ch} \mp \int_{\Lambda} G(x,y) \omega_\varepsilon(y) dy \right] \right) ,
$$

(9)

where $\omega_\varepsilon = \rho^+_\varepsilon - \rho^-_\varepsilon$, and where the constants $\beta$ and $\mu^\pm_{ch}$ are to be chosen so that the constraint $\int_{\Lambda} \rho^\pm dx = 1$ and (8) are satisfied. Not only are the equations (3) clearly mean-field, solutions having $\beta < 0$ are known to exist, and the corresponding continuum vorticities $\omega_\varepsilon$ satisfy the stationary Euler equations for incompressible flows [47]. Onsager’s prediction of negative temperatures in vortex systems indeed attains a precise meaning in this limit.

There can be hardly any doubt that the above picture is correct, and that it will eventually be verified rigorously. An almost complete verification is available for the mildly simpler single species
system. By generalizing a result of Messer and Spohn [49] for Lipschitz continuous interactions to the logarithmically singular point vortex interactions, first the canonical point vortex ensemble was conquered, independently in [9] and [39]. While the microcanonical ensemble in the strict sense described above has not yet been treated, rigorous works exist in which the microcanonical point vortex measure \(\delta(H - E)\) is replaced by a regularized measure, the regularization being removed after the limit \(N \to \infty\) has been taken. Again, the first result, by Eyink and Spohn [31], was for regularized interactions. The singular point vortex interactions where then treated by Caglioti et al. [10]. The limit \(N \to \infty\) in [10] is, however, constructed under the assumption that the microcanonical and canonical ensembles are equivalent, an assumption which is not generally valid for these finite-domain mean-field limits. This last barrier was finally overcome by Joel and myself in [41], where we go beyond equivalence of ensembles for the logarithmically singular (1). Of course, the mean-field theory obtained is precisely the single-species version of the one described above.

Future work should extend these results to the neutral two-species vortex system. In contrast to the many rigorous results obtained for these neutral systems in the traditional thermodynamic limit, see [33] and references therein, only few facts are rigorously known for their Euler fluid limit. Curiously enough, while Fröhlich and Ruelle [33] proved absence of negative temperatures in the standard bulk limit, in [40] I was able to prove absence of positive temperatures in the Euler fluid limit for the neutral two-species system; in fact, \(\beta < 0\) is bounded away from zero. This raises the question as to the whereabouts of Onsager’s \(E_m\), at which temperatures are supposed to switch from positive to negative values in a neutral system? The answer is that there is much room between the low-energy regime \(E = Ne\) of [33] and the high-energy regime \(E = N^2 \varepsilon\) of [11, 10, 31]. In fact, in [11] it was found that for a neutral two-species system \(E_m(N) \sim CN \ln N\), falling precisely inbetween these two regimes. The vicinity of \(E_m\), analyzed in [53, 11], turns out to be a small-entropy regime where \(S\), not \(S/N\), converges to a limit when \(N \to \infty\).

The results in [53, 11] are the only ones obtained directly from \(\delta(H - E)\). The construction of the Euler fluid limit \(N \to \infty\) directly from \(\delta(H - E)\) remains an open problem. For now, [41] is the final word in the construction of the Euler fluid limit for Onsager’s statistical vortex distributions.

Interestingly, [11] also holds the key to some answers to questions in conformal geometry, to which we turn next.

3 NIRENBERG’S PROBLEM

Many years ago, see [53], Louis Nirenberg raised the following question: “Which real functions \(K(x)\) on \(S^2\) are Gauss curvature functions for a surface over \(S^2\) whose metric \(g\) is pointwise conformal to the standard metric \(g_0\) on \(S^2\)?” To see what this question has to do with Onsager’s vortex distributions, we need to recall a few basic concepts of differential geometry as found, for instance, in [3, 66]. As we shall see, the connection with statistical mechanics, or at least thermodynamics, must have been suspected by differential geometers long ago!

Consider first, for simplicity, surfaces over \(R^2\) that are embedded in \(R^3\). Thus, let \(x \in \Lambda \subset R^2\). The two Cartesian coordinates \(x^1, x^2\) of \(x\) provide two independent real parameters for the local
representation of a surface $S \subset \mathbb{R}^3$, given by $S = X(\mathbf{x})$, $\mathbf{x} \in \Lambda$, with $X \in \mathbb{R}^3$. The line element on $S$ is given by $d\sigma^2 = g_{ij}dx^i dx^j$, where we use Einstein’s summation convention, and where $g_{ij} = \langle \partial_i X, \partial_j X \rangle$ (Euclidean inner product) are the components of the metric tensor. If $\nu$ is the unit normal at $S$ induced by the representation $X(\mathbf{x})$, then the Gauss curvature $K(\mathbf{x})$ is defined by ($\times$ means cross product in $\mathbb{R}^3$)

$$\partial_x \nu \times \partial_x \nu = K(\mathbf{x}) \partial_x X \times \partial_x X$$  \hspace{1cm} (10)

Gauss’ Theorema Egregium asserts that $K(\mathbf{x})$ depends only on the $g_{ij}$. Writing $g_{11} = E$, $g_{12} = F$, and $g_{22} = G$, and moreover $x^1 = s$ and $x^2 = t$, we have the Frobenius formula

$$K = -\frac{1}{4(EG - F^2)^2} \det \begin{pmatrix} E & F & G \\ E_s & F_s & G_s \\ E_t & F_t & G_t \end{pmatrix} + \frac{1}{2\sqrt{EG - F^2}} \left[ \partial_s \left( \frac{F_t - G_s}{\sqrt{EG - F^2}} \right) - \partial_t \left( \frac{E_t - F_s}{\sqrt{EG - F^2}} \right) \right]$$  \hspace{1cm} (11)

see [3, 66], from which it follows that $K$ is independent of the parametric representation $X(\mathbf{x})$ of $S$. This freedom can be used to simplify (11). In particular, the representation for which $E = G$ and $F = 0$ is conformal, i.e. $d\sigma^2 = Edx^2$. Interestingly, it is known in differential geometry as isothermic parameter representation. In this representation, (11) reduces to $K = -\frac{1}{2E} \left( \partial_s^2 \ln E + \partial_t^2 \ln E \right)$, or, setting $\ln E = 2u$ and recalling $(s,t) = \mathbf{x}$,

$$K(\mathbf{x}) = -e^{-2u(\mathbf{x})} \Delta u(\mathbf{x})$$  \hspace{1cm} (12)

which gives us $K$ when $u$ (i.e. $E$) is given. Nirenberg’s question, here its analog on $\mathbb{R}^2$, addresses the inverse problem, i.e. to prescribe the putative Gauss curvature function $K$ and study (12) as a non-linear elliptic PDE for the unknown $u(\mathbf{x})$, $\mathbf{x} \in \Lambda \subset \mathbb{R}^2$. In particular, when $K$ is constant, $K = \pm 1$ by scaling, then (12) is known in differential geometry as Liouville’s equation [1]. Clearly, Liouville’s equation is identical to what in statistical physics goes under the name Poisson-Boltzmann equation,

$$-\Delta \psi(\mathbf{x}) = 2\pi e^{\beta[\mu - \psi(\mathbf{x})]}$$  \hspace{1cm} (13)

for the spatial density of a ‘perfect gas’ in $\Lambda \subset \mathbb{R}^2$, in thermal equilibrium at temperature $\beta^{-1}$ and chemical potential $\mu_{ch}$, distributed in its own Coulomb (or Newton) potential field $\psi$. Hence, the notion of ‘isothermic parameters,’ so it seems; however, I have not been able to trace the originator of this terminology. In any event, (13) is of course identical to the one-species specialization of (3), i.e. (3) with $\rho^+ \equiv \rho$ and $\rho^- \equiv 0$, and with $\psi(\mathbf{x}) = \int_{\Lambda} G(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) d\mathbf{y}$ the stream function, so that $-\Delta \psi(\mathbf{x}) = 2\pi \rho(\mathbf{x})$. We have come back full circle to Onsager’s problem.

Liouville [1] himself already showed that the equation derived by him is in a certain sense completely integrable. In particular, all entire solutions on $\mathbb{R}^2$ with finite integral curvature $\kappa = \int_{\mathbb{R}^2} K e^{2u(\mathbf{x})} d\mathbf{x}$ have been identified; namely, no entire solution exists when $K = -1$ [10, 58], while for $K = +1$ Liouville’s equation is solved when $\exp(2u)$ is any point on the conformal orbit of the Jacobian of the stereographic map $S^2 \to \mathbb{R}^2$, and $\kappa = 4\pi$ for all solutions, then; see [23, 12, 27, 21].
For the prescribed Gauss curvature problem in all \( \mathbb{R}^2 \) with \( K \neq \text{const.} \), no general solution of (12) is available, yet over the years a vast knowledge has accumulated, e.g. [65], [54], [48], [4], [25], [24], [20], [26], [22], [62]. Of particular interest are radially symmetric Gauss curvature functions, for then the interesting question arises whether solutions \( u \) exist that break the radial symmetry. In the following, the sign of \( K \) (as a function) is defined as: \( \text{sign} (K) = 1 \) if \( K \equiv 0 \) and \( K(x) \geq 0 \) for all \( x \); \( \text{sign} (K) = -1 \) if \( K \equiv 0 \) and \( K(x) \leq 0 \) for all \( x \); and \( \text{sign} (K) = 0 \) if \( K \equiv 0 \). In all other cases \( \text{sign} (K) \) is not defined. The next theorem is taken from [22].

**Theorem 1**: Assume \( K \in C^\alpha (\mathbb{R}^2) \) is radially symmetric, has well-defined sign, and satisfies

\[
\int_{\mathbb{R}^2} |K(x)| e^{2h(x)} |x|^\lambda \, dx < \infty
\]

for some non-constant, harmonic function \( h : \mathbb{R}^2 \to \mathbb{R} \), and some \( \lambda > 0 \). Assume also that

\[
\int_{B_1(y)} |y-x|^{-\gamma} |K(x)| e^{2H(x)} \, dx \to 0 \quad \text{as} \ |y| \to \infty ,
\]

for all \( 0 < \gamma < 2 \). If \( \text{sign} (K) = -1 \), define

\[
\kappa^*(K; h) = -2\pi \sup \{ \lambda > 0 : (14) \text{ is satisfied} \}
\]

where \( \kappa^* \) might be \( -\infty \) for some \( K \leq 0 \). Then, for any such \( K \) and \( h \), and \( \kappa \) satisfying

\[
\kappa \in \begin{cases} 
(\kappa^*, 0) & \text{if } \text{sign} (K) = -1; \\
[0] & \text{if } \text{sign} (K) = 0; \\
(0, 4\pi) & \text{if } \text{sign} (K) = +1; 
\end{cases}
\]

there exists a classical solution \( u = U_{h,\kappa} \) of (12) with prescribed Gaussian curvature function \( K \), uniquely if \( K \leq 0 \), having integral curvature

\[
\int_{\mathbb{R}^2} K(x) e^{2U_{h,\kappa}(x)} \, dx = \kappa
\]

and asymptotic behavior

\[
U_{h,\kappa}(x) = h(x) - \frac{\kappa}{2\pi} \ln |x| + o(\ln |x|) \quad \text{as} \ |x| \to \infty .
\]

Theorem 1 considerably generalizes an earlier result of this kind (Thm.III in [25]) which is restricted to compactly supported \( K \) in \( \mathbb{R}^2 \). Theorem 1 is proven in [22] as corollary of the construction, also given in [22], of the Euler fluid limit in \( \mathbb{R}^2 \) for the canonical point vortex ensemble, which generalizes the one for finite domains \( \Lambda \subset \mathbb{R}^2 \) [39, 1]. The canonical point vortex measure on \( \mathbb{R}^{2N} \) is given by

\[
\mu^{(N,\beta)}(dx_1 \ldots dx_N) = \frac{1}{Z_{\beta,N}} \exp \left( -\beta N^{-1} H^{(N)} \right) dx_1 \ldots dx_N ,
\]

with \( G(x,y) = -\ln |x-y| \) and \( F^{(N)}(x) = -N\beta^{-1} [\ln |K(x)| + 2h(x)] \) in \( H^{(N)} \), [4]. The reciprocal temperature \( \beta \) in (20) and the integral curvature \( \kappa \) in Theorem 1 are related by \( \kappa = -\beta \pi \), with \( -\beta \in (\kappa^*/\pi, 4) \).
Encouraged by these achievements of the canonical statistical mechanics approach to the prescribed Gauss curvature problem on \( R^2 \), we now return to Nirenberg’s problem in its original setting on the sphere \( S^2 \). Writing the conformal deformation of the metric as \( g = e^{2u(x)} g_0 \), where \( g_0 \) is the standard metric on \( S^2 \), the problem is to find all \( K(x), x \in S^2 \), for which

\[- \Delta u(x) = K(x)e^{2u(x)} - 1 \tag{21}\]

has a solution \( u \) on \( S^2 \). Here, \( \Delta \) is the Laplace-Beltrami operator on \( S^2 \) w.r.t. \( g_0 \). Superficially \( (21) \) is hardly any different from the prescribed Gauss curvature equation on \( R^2 \) \( (12) \). However, Nirenberg’s problem for \( S^2 \) is a hard problem, indeed; see [53], [37], [15], [16], [34]; see also [37], [29] for more general compact Riemann surfaces.

There are many obstructions to finding admissible \( K \) for \( (21) \). The celebrated Kazdan-Warner theorem [37] is one of them. According to [37], a Gauss curvature function \( K \) on \( S^2 \) cannot be axially symmetric and monotonic, unless it is a constant function, say \( K \equiv 1 \). In the latter case the problem is completely solved, [52], [1], [56]. Another, more serious obstruction is the famous Gauss-Bonnet theorem [66], which relates the Gauss curvature \( K_g \) on a general compact 2-manifold \( (M^2, g) \) without boundary to the Euler-Poincaré characteristic \( \chi(M^2) \), a topological invariant, by

\[ \chi(M^2) = \frac{1}{2\pi} \int_{M^2} K_g d\text{vol}_g. \tag{22} \]

Since \( \chi(S^2) > 0 \), it follows from \( (22) \) that \( K(x) \) has to be positive for some \( x \in S^2 \), [53]. Furthermore, we have \( \chi(S^2) = 2 \). Thus, \( \kappa = 4\pi \), i.e.

\[ \int_{S^2} K(x)e^{2u(x)} dx = 4\pi, \tag{23} \]

which of course results also by directly integrating \( (21) \) over \( S^2 \). Contrast \( (23) \) with the range of integral curvatures covered in Theorem 1! In particular, notice that \( \kappa = 4\pi \) is not included in Theorem 1, and this is a matter of principle. Indeed, the restriction to \( -\beta < 4 \) in the canonical ensemble is an obstruction imposed on us by the local integrability requirement of the singularities in \( (21) \), and this will not improve if we put our vortex system on the sphere \( R^2 \). Thus, \( (23) \) dashes any hope to apply the Euler fluid limit of the canonical point vortex ensemble at \( \beta = -4 \) to the prescribed Gauss curvature problem on \( S^2 \).

Not all hopes are dashed, though. Physically speaking, what happens at \( -\beta = 4 \) in the finite \( N \), single-species canonical ensemble is that the vortex system concentrates onto a single point. This is done at the expenses of the ‘heat bath,’ which delivers a positive infinite amount of energy into the vortex system at fixed negative temperature. Taking the limit \( N \to \infty \) of this concentrated singular state gives a Dirac \( \delta \) mass on \( S^2 \), corresponding to \( \varepsilon = +\infty \). However, performing the mean-field limit \( N \to \infty \) first for \( -\beta < 4 \) produces a regular solution of the resulting analog on \( S^2 \) of the Poisson-Boltzmann equation \( (13) \), and subsequently letting \( -\beta \nearrow 4 \) may, or may not result in a regular limiting solution at \( -\beta = 4 \). This can be made more precise with the so-called concentration-compactness alternative of P.L. Lions, compactness of the sequence of solutions as
−β ↗ 4 implying a regular limiting solution at −β = 4. The canonical ensemble may nevertheless not provide enough control to decide the concentration-compactness alternative. However, whenever compactness holds, a regular solution exists which has finite energy ε, and some finite entropy s. We conclude that the study of the mean-field limit for the microcanonical ensemble of point vortices on $S^2$, with external stream function $F^{(N)}$, can be expected to yield valuable new information on Nirenberg’s problem.

As remarked earlier, the microcanonical ensemble in its strict sense has not yet succumbed to rigorous analysis. However, for the prescribed Gauss curvature problem on $S^2$ the construction given in [11] is fully sufficient. As a matter of fact, we shall state our microcanonical results in the more general setting on $S^n$, $n \geq 2$. To see what this now is about, we need to briefly digress into Paneitz theory.

### 4 Paneitz Equations

Initiated by the conformal covariance in Minkowski space-time $\mathbb{R}^{1,3}$ of the Maxwell equations of electromagnetism, S. Paneitz [59] in 1983 discovered a quartic conformally covariant differential operator for arbitrary pseudo-Riemannian 1,3-manifolds, together with an associated new conformal curvature invariant. Recently this theory has received considerable attention, see [4], [8], [17], [7], [13], [14]. The significance of the Paneitz curvature, $Q_g$, becomes apparent through an analog of the Gauss-Bonnet formula for a general compact 4-manifold $(M^4, g)$ without boundary, [18],

$$\chi(M^4) = \frac{1}{8\pi^2} \int_{M^4} \left( \frac{1}{4} |W(x)|^2 + Q_g(x) \right) d\text{vol}_g$$  \hspace{1cm} (24)

Here, $\chi(M^4)$ is the Euler-Poincaré characteristic of $M^4$, and $W$ is the pointwise conformally invariant Weyl tensor. Moreover, $Q_g$ is defined by

$$6Q_g(x) = -\Delta_g R_g(x) + \frac{1}{4} R_g^2(x) - 3 |\tilde{\text{Ric}}_g(x)|^2$$  \hspace{1cm} (25)

where $\Delta_g$ is the Laplace-Beltrami operator on $(M^4, g)$, $R_g$ is the scalar curvature and $\tilde{\text{Ric}}_g$ the traceless Ricci tensor, see [21], [13]. Associated with $Q_g$ is Paneitz’ quartic conformally covariant operator, such that, given $(M^4, g_0)$ and a conformal change of metric written as $g = e^{2u(x)} g_0$, $x \in (M^4, g_0)$, the new curvature $Q_g$ is given by

$$Q_g(x) = e^{-4u(x)} \left( \Delta_{g_0}^2 u(x) + \delta_{g_0} \left( \frac{2}{3} R_{g_0}(x) I - 2 \text{Ric}_{g_0}(x) \right) d_{g_0} u(x) + Q_{g_0}(x) \right)$$  \hspace{1cm} (26)

Here, $d$ is the differential and $\delta$ the divergence.

An analog of Nirenberg’s problem on the 4-manifold $(S^4, g_0)$, with $g_0$ the standard metric, can be formulated thus: “Which real functions $Q(x)$, $x \in S^4$, are Paneitz curvature functions for a 4-manifold whose metric is pointwise conformal to the standard one?” This analog of Nirenberg’s problem can be rephrased in terms of [26], namely to find all functions $Q(x)$ on $S^4$ such that the fourth-order PDE

$$\Delta^2 u(x) - 2\Delta u(x) = Q(x) e^{4u(x)} - 6$$  \hspace{1cm} (27)
has a solution $u$ on $S^4$. Generalizations to $S^n$ (and, hence, to $\mathbb{R}^n$) of the Paneitz equation (27) have been derived as well. On $S^n$, $n \geq 2$, we have

$$P_n u(x) = Q(x) e^{\eta u(x)} - (n - 1)!$$

with

$$P_n = \begin{cases} 
\prod_{k=0}^{n-2} (-\Delta + k(n - k - 1)); & n \text{ even} \\
\sqrt{-\Delta + \left(\frac{n-1}{2}\right)^2} \prod_{k=0}^{n-2} (-\Delta + k(n - k - 1)); & n \text{ odd},
\end{cases}$$

see [12], [4], [19], [14]. On $\mathbb{R}^n$, we simply have

$$(-\Delta)^{n/2} u(x) = Q(x) e^{\eta u(x)}$$

with $x \in \mathbb{R}^n$ now. The operator $P_n$ in (28) is the Paneitzian, and $Q(x)$ the Queervature (pardon the pun) of order $n$. Notice that for $n$ odd, $P_n$ is a pseudo differential operator.

5 LOGARITHMIC INTERACTIONS IN ALL DIMENSIONS

The increased complexity of the operators $P_n$ given in (29) for high dimensions gives the Paneitz equations (28) a formidable appearance. Also (30) is not too familiar when $n > 2$. However, notice that the resolvent kernel of $P_n$ on $S^n (\mathbb{R}^n)$, with $P_n$ restricted to the orthogonal complement of its kernel space, is always $G(x, y) = -\ln |x - y|$, with $x, y \in S^n (x, y \in \mathbb{R}^n)$, and with $| . |$ the chordal distance on $S^n$ (Euclidean distance in $\mathbb{R}^n$), cf. [12]. More precisely, $-P_n \ln |x - y| = (1/2)(n - 1)!|S^n|((\delta y(x) - |S^n|^{-1})$ on $S^n$. Hence, all the equations (28), as well as (30), have a statistical mechanics interpretation whenever $Q$ has a well defined sign.

In the following we discuss the prescribed Paneitz curvature problems on $S^n$, (28), (29), using the mean-field limit of the regularized microcanonical ensemble of [41]. Nirenberg’s problem on $S^2$ is contained in the analysis as special case $n = 2$. We also remark on the equations in $\mathbb{R}^n$, (30), using just the canonical ensemble.

Let us begin with the simpler (30). A brief moment of reflection reveals that Theorem 1, and its proof, have an immediate generalization to the Paneitz equation (30). In (4) we then have to set $G(x, y) = -\ln |x - y|$, $x, y \in \mathbb{R}^n$. We also replace $|K|$ in $F(N)$ by $|Q|$, and $2h$ by $nh$, where instead of a harmonic function as in Theorem 1 now $h$ is a non-constant element of the kernel space of $(-\Delta)^{n/2}$ on $\mathbb{R}^n$, to which we may want to refer as higher harmonic function. Moreover, $Q$ shall have well defined sign and satisfy analogous integrability conditions as (14) and (15) in Theorem 1. Finally, the critical $\beta = -4$ for the canonical ensemble changes to $\beta = -2n$, and the corresponding critical integral Gauss curvature $\kappa = 4\pi$ in Theorem 1 changes to a critical integral Paneitz curvature $q = (n - 1)!|S^n|$, where $q = \int_{\mathbb{R}^n} Q(x) e^{\eta u(x)} dx$. These numbers are also the sharp constants in the Trudinger-Moser type inequalities on $\Lambda \subset \mathbb{R}^n$ and $S^n$, [67, 52, 4, 12].
We now come to the problem on $S^n$. Since $S^n$ is a compact manifold without boundary we do not need any delicate unbounded-domain estimates of the sort needed in [41] or [42]. The technique of [41] carries over to $S^n$ without major changes. In [41] we simply have to set $G(x,y) = -\ln |x-y|$, $x,y \in S^n$. We also let $F^{(N)}(x) = Nf(x)$, with $f$ some continuous function on $S^n$ satisfying $\int_{S^n} f(x)dx = 0$. Notice that not all $f$ will correspond to a Paneitz curvature function.

Our regularized microcanonical probability measure on $(S^n)^N$ is of the form [41]

$$\mu^{(N,\varepsilon,\sigma)}(dx_1...dx_N) = \frac{1}{Z} \exp\left[-N \frac{1}{2\sigma^2} \left(\varepsilon - \frac{1}{N^2}H^{(N)}\right)^2\right] dx_1...dx_N,$$

where $\sigma > 0$ and $\varepsilon$ are fixed real numbers, and

$$Z(N,\varepsilon,\sigma) = \int_{(S^n)^N} \exp\left[-N \frac{1}{2\sigma^2} \left(\varepsilon - \frac{1}{N^2}H^{(N)}\right)^2\right] dx_1...dx_N.$$  \hspace{1cm} (31)

The Hamiltonian is given in [41], with the identifications of $G$ and $F$ mentioned above. The microcanonical ensemble at fixed $N$ is obtained in the limit $\sigma \to 0$ in (31), giving a delta measure concentrated on \{\mathcal{H}^{(N)} = E\}, with $E = N^2\varepsilon$, as can be easily verified using geometric measure theory.

Let $P((S^n)^N)$ denote the probability measures on $(S^n)^N$. For any $N \in \mathbb{N}$, we define the entropy of $\varrho_N \in P((S^n)^N)$ relative to the normalized uniform measure $|S^n|^{-N}dx_1...dx_N$ by

$$S(\varrho_N) = -\int_{(S^n)^N} \rho_N \ln \left(|S^n|^N \rho_N\right) dx_1...dx_N,$$  \hspace{1cm} (32)

if $\varrho_N$ is absolutely continuous w.r.t. uniform measure on $(S^n)^N$, having density $\rho_N$, and provided the integral on the r.h.s. of (32) exists; $S(\varrho_N) = -\infty$ in all other cases. For $\varrho_1 = \varrho \in P(S^n)$, we define a one-particle penalized entropy functional by

$$R_{\varepsilon,\sigma}(\varrho) = S(\varrho) - \frac{1}{2\sigma^2} \left(\varepsilon - \frac{1}{2} \int_{S^n} \int_{S^n} G(x,y)\varrho(dx)\varrho(dy) - \int_{S^n} f(x)\varrho(dx)\right)^2$$  \hspace{1cm} (33)

for those $\varrho(dx) = \rho dx$, with $\rho$ a probability density, for which $S(\varrho) > -\infty$. We set $R_{\varepsilon,\sigma}(\varrho) = -\infty$ in all other cases. Here, $S(\varrho) = S(\varrho_1)$ is the one-particle entropy as defined in (33). We write $M_{\varepsilon,\sigma}$ for the set $\{\varrho_{\varepsilon,\sigma}\} \subset P(S^n)$ of maximizers of $R_{\varepsilon,\sigma}$.

By $\Omega = (S^n)^\mathbb{N}$ we denote the $S^n$-valued infinite exchangeable sequences, by $P_{sym}(\Omega)$ the permutation-invariant probability measures on $\Omega$. According to the theorem of de Finetti – Dynkin [32, 30] every $\mu \in P_{sym}(\Omega)$ is a unique convex linear superposition of product measures, see also [28] and [35]. By a theorem of Hewitt and Savage [35], the product states $\varrho_{\otimes N}$ are also the extreme points of the convex set $P_{sym}(\Omega)$. Hence, we have the extremal decomposition

$$\mu_k(dx_1...dx_k) = \int_{P(S^n)} \nu(\mu|d\varrho) \varrho_{\otimes k}(dx_1...dx_k),$$  \hspace{1cm} (34)

with $\mu_k(dx_1...dx_k) \in P((S^n)^k)$ the $k$-th marginal measure of $\mu$.

As in [41], we first take the limit $N \to \infty$ of (31) for fixed $\varepsilon$ and $\sigma$. This gives us mean-field theory with a two-parameter penalized entropy principle, as follows.

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Theorem 2: For each $\varepsilon \in \mathbb{R}$ and $\sigma > 0$ fixed, one has
\[
\lim_{N \to \infty} \frac{1}{N} \ln [\mathbb{S}^n]^{-N} Z(N, \varepsilon, \sigma) = R_{\varepsilon, \sigma}(\theta_{\varepsilon, \sigma})
\] (36)
with $\theta_{\varepsilon, \sigma} \in M_{\varepsilon, \sigma}$. Moreover, (34) has at least one limit point in the corresponding subset of $P^{sym}(\Omega)$, convergence understood for all the marginals in the sense of Kolmogorov, [5], here weakly in $L^p$, $p < \infty$. The decomposition measure $\nu(\mu(\varepsilon, \sigma)|d\rho)$ of any limit point $\mu(\varepsilon, \sigma)$ is concentrated on $M_{\varepsilon, \sigma}$.

The subsequent limit $\sigma \to 0$ now gives the anticipated mean-field variational principle for the microcanonical entropy. We denote by $L^{1,+}_{1}(\mathbb{S}^n)$ the subset of the positive cone of $L^1(\mathbb{S}^n)$ whose elements $\rho$ satisfy $\int_{\mathbb{S}^n} \rho \, dx = 1$, and by $L^{1,+}_{1;\varepsilon}(\mathbb{S}^n)$ the subset of $L^{1,+}_{1}(\mathbb{S}^n)$ for which
\[
\frac{1}{2} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} G(x, y) \rho(dx) \rho(dy) + \int_{\mathbb{S}^n} f(x) \rho(dx) = \varepsilon,
\] (37)
with $\rho(dx) = \rho(x) \, dx$. Let $\varepsilon_0(f)$ denote the minimum over $P(\mathbb{S}^n)$ of the functional on the left side of (37).

Theorem 3: Fix $\varepsilon \in \mathbb{R}$.

Part 1. Let $\varepsilon \geq \varepsilon_0$. Then the limit
\[
s(\varepsilon) = \lim_{\sigma \to 0} R_{\varepsilon, \sigma}(\theta_{\varepsilon, \sigma})
\] (38)
exists and satisfies the variational principle
\[
s(\varepsilon) = \max \left\{ S(\rho) \mid \rho \, dx = \rho dx; \rho \in L^{1,+}_{1;\varepsilon} \right\}.
\] (39)
If $\varepsilon > \varepsilon_0$, all maximizers $\rho_\varepsilon$ for (39) satisfy the Euler-Lagrange equation
\[
\rho_\varepsilon(x) = \exp \left( \beta \left[ \mu_{ch} - \int_{\mathbb{S}^n} G(x, y) \rho_\varepsilon(y) \, dy - f(x) \right] \right),
\] (40)
where $\beta$ and $\mu_{ch}$ are real Lagrange parameters for the constraints $\rho \in L^{1,+}_{1;\varepsilon}$. For $\varepsilon = \varepsilon_0$, the maximizer(s) solve the free boundary problem obtained from (40) in the limit $\beta \to +\infty$.

Furthermore, let $M_{\varepsilon}$ denote the set of maximizers $\rho_\varepsilon$ for (39). Let $\mu(\varepsilon)$ be a weak limit point, as $\sigma \to 0$, of the measure $\mu(\varepsilon, \sigma)$. Then $\mu(\varepsilon) \in P^{sym}(\Omega)$, and its decomposition measure $\nu(\mu(\varepsilon)|d\rho)$ is concentrated on $M_{\varepsilon}$.

Part 2. Let $\varepsilon < \varepsilon_0$. In this case $\lim_{\sigma \to 0} R_{\varepsilon, \sigma}(\theta_{\varepsilon, \sigma}) = -\infty$.

The proofs of Theorems 2 and 3 are nearly verbatim copies of the proofs for two-dimensional domains given in [11], with a few functional analytical differences. Details will appear elsewhere.

Our (34) is the dual equation to (28) on $\mathbb{S}^n$. The differential geometric problem is to find such $f$ for which (40) has a solution with $\beta = -2n$, in which case $(n-1)!|\mathbb{S}^n| \exp(2nf(x))$ can be identified with a prescribed Paneitz curvature functions $Q(x)$. Clearly, only such $Q$ with $\text{sign}(Q) = +1$ can be found this way. The parameter $\beta \mu_{ch}$, which accounts for the requirement that $\rho$ is a probability density, is simply absorbed in $u(x)$, but the parameter $\beta$ in (40) is implicitly determined by the choice of $\varepsilon$. Hence, the problem becomes: “Find all $f$ for which the map $\varepsilon \mapsto \beta(\varepsilon)$ takes the value $-2n$.”
One obvious such $f$ is $f ≡ 0$, in which case the equation (10) with $β = −2n$ is completely integrable [12, 9]. Namely, (10) is then invariant under the full conformal group on $S^n$, having a unique solution — up to rotations and reflections on $S^n$ — for each $ε ≥ ε_0$.

The problem to find admissible $f ≠ 0$ may not appear any simpler than the original Nirenberg problem and its generalization to higher $n$, but now thermodynamics comes to the rescue. It is straightforward to show that $s(ε)$ is continuous and piecewise differentiable and that $\text{Ran}(∂_ε s)$ is connected. As in ordinary thermodynamics, so also here we have the identification

$$\beta = ∂_ε s(ε)$$

(41)

wherever the derivative is defined. We have $∂_ε s(ε) → +∞$ as $ε → ε_0^+$, except when $f ≡ 0$. Moreover, for all $f ≠ 0$ we have $s(ε_∞) = 0$ and $∂_ε s(ε_∞) = 0$, where $ε_∞ = (1/2) \int_{S^n} \int_{S^n} G(x,y)|S^n|^{-2}dxdy$. For $ε > ε_∞$, we have $∂_ε s(ε) < 0$. Therefore, it suffices to solve the simpler problem of finding those $f$ for which $∂_ε s(ε) < −2n$ for some large enough $ε > ε_∞$. Using physical intuition as guidance, a little reflection reveals that a solution to the generalized Nirenberg problem should exist whenever $f$ allows particles to cluster in at least two spatially separated regions when $β < 0$, or, in technical language:

**Conjecture 4:** Any $f ∈ C^0(S^n)$ satisfying $∫_{S^n} f(x)dx = 0$ which has at least two isolated maxima can be identified with a Paneitz curvature function by $Q(x) = (n − 1)!|S^n| \exp (2nf(x))$.

A variant of Conjecture 4 for $S^2$ with somewhat stronger regularity assumptions on the Gauss curvature has been proven with PDE techniques in [15], see their Theorems I and II. Our micro-canonical mean-field limit now offers a new perspective on proving a result like Conjecture 4 — simultaneously in all dimensions. I hope to report on the details of such an effort in some future publication.

As a final remark, I mention that also the ground state problem $ε = ε_0$, while only indirectly relevant to our conformal geometric problems on $S^n$, is of quite some interest, in other contexts; see [12, 64].

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