Dirac Monopole from Lorentz Symmetry in 
N-Dimensions: I. The Generator Extension

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Abstract

It is by now well-known that a Lorentz force law and the homogeneous Maxwell equations can be derived from commutation relations among Euclidean coordinates and velocities, without explicit reference to momentum, gauge potential, action or variational principle. More generally, it has been shown that the specification of commutation relations in the coordinate-velocity basis determines a unique Langrangian, from which the full canonical system follows. This result was extended to the relativistic case and shown to correspond to a Stueckelberg-type quantum theory, in which the underlying gauge symmetry depends on the invariant evolution parameter, such that the associated five-dimensional electromagnetism becomes standard Maxwell theory in the equilibrium limit. Bérard, Grandati, Lages and Mohrbach have studied the Lie algebra associated with the O(3) rotational invariance of the Euclidean coordinate-velocity system, and found an extension of the generators that restores the commutation relations in the presence of a Maxwell field, and renders the extended generator a constant of the classical motion. The algebra imposes conditions on the Maxwell field, leading to a Dirac monopole solution. In this paper, we study the generalization of the Bérard, Grandati, Lages and Mohrbach construction to the Lorentz generators in N-dimensional Minkowski space. We find that that the construction can be maximally satisfied in a three dimensional subspace of the full Minkowski space; this subspace can be chosen to describe either the O(3)-invariant space sector, or an O(2,1)-invariant restriction of spacetime. The field solution reduces to the Dirac monopole field found in the nonrelativistic case when the O(3)-invariant subspace is selected. When an O(2,1)-invariant subspace is chosen, the Maxwell field can be associated with a Coulomb-like potential of the type $A^\mu(x) = n^\mu / \rho$, where $\rho = (x^\mu x_\mu)^{1/2}$, similar to that used by Horwitz and Arshansky to obtain a covariant generalization of the hydrogen-like bound state. In both cases, the extended generator is conserved with respect to the invariant parameter under classical relativistic system evolution.
1 Introduction

Since Dyson published his account [1] of Feynman’s early work on the subject, it has become well known that posing commutation relations of the form

\[ [x^i, x^j] = 0 \quad m [x^i, \dot{x}^j] = i\hbar \delta^{ij}, \]  

among the quantum operators for Euclidean position and velocity, where \( \dot{x}^i = dx^i/dt \) and \( i, j = 1, 2, 3 \), restricts the admissible forces in the classical Newton’s second law

\[ m\ddot{x}^i = F^i(t, x, \dot{x}) \]  

to the form

\[ m\ddot{x}^i = E^i(t, x) + \epsilon^{ijk}\dot{x}_j H_k(t, x) \]  

with fields that must satisfy

\[ \nabla \cdot H = 0 \quad \nabla \times E + \frac{\partial}{\partial t} H = 0. \]  

The velocity-dependent part of the interaction in \( 3 \) enters through

\[ m^2 [\dot{x}^i, \dot{x}^j] = -i\hbar F^{ij}(t, x) = -i\hbar \epsilon^{ijk} H_k(t, x) \]  

which is posed as a naive relaxation of assumptions about the velocity operators, and not intended to presuppose the existence of a canonical momentum. Although Dyson treated the “derivation” as something of a curiosity, his article led to small flurry of new results, in particular the proof [2] that the assumptions [1] are sufficiently strong to establish the self-adjointness of the differential equations [2]. It follows from self-adjointness that this system is equivalent to a unique nonrelativistic Lagrangian mechanics [3] with canonical momenta whose relationship to the velocities leads directly to \( 5 \). Several authors observed [4] that supposing Lorentz covariance in \( 4 \) conflicts with the Euclidean assumptions in \( 1 \), and so \( 3 \) cannot be interpreted as the Lorentz force in Maxwell theory. These results were generalized to the relativistic case [5, 6] in curved \( N \)-dimensional spacetime by taking

\[ [x^\mu, x^\nu] = 0 \quad m[x^\mu, \dot{x}^\nu] = -i\hbar g^{\mu\nu}(x) \quad [\dot{x}^\mu, \dot{x}^\nu] = \left( -\frac{i\hbar}{m^2} \right) F^{\mu\nu} \]  

and

\[ m\ddot{x}^\mu = F^\mu(\tau, x, \dot{x}). \]
where \( \mu, \nu = 0, 1, \ldots, N - 1 \) and \( x^\mu(\tau) \) and its derivatives are function of the Poincaré-invariant evolution parameter \( \tau \). The resulting system

\[
m[x^\mu + \Gamma^{\mu\lambda\nu} \dot{x}_\lambda \dot{x}_\nu] = G^\mu(\tau, x) + F^{\mu\nu}(\tau, x) \dot{x}_\nu
\]

(8)
in which the covariant derivative contains the usual affine connection

\[
\Gamma_{\mu\nu\rho} = \frac{1}{2} (\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho})
\]

(9)
and the fields satisfy

\[
\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \quad \partial_\mu G_\nu - \partial_\nu G_\mu + \frac{\partial}{\partial \tau} F_{\mu\nu} = 0
\]

(10)
is equivalent to the \((N + 1)\)-dimensional gauge theory associated with Stueckelberg’s relativistic mechanics [7, 8]. Formally extending the indices to \((N + 1)\)-dimensions

\[
\mu, \nu, \lambda = 0, 1, \ldots, N - 1 \quad \alpha, \beta, \gamma = 0, \ldots, N
\]

(11)
equations (8) and (10) become

\[
m[x^\mu + \Gamma^{\mu\lambda\nu} \dot{x}_\lambda \dot{x}_\nu] = F^{\mu\beta}(\tau, x) \dot{x}_\beta
\]

(13)
and

\[
\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.
\]

(14)
As discussed in [6], the inhomogeneous source equation in the Stueckelberg theory is

\[
\partial_\beta F^{\alpha\beta} = e_j^\alpha,
\]

(15)
which reduces to standard Maxwell theory in an equilibrium limit (with respect to \( \tau \)) of the \((4 + 1)\)-dimensional gauge theory.

More recently [9], Bérard, Grandati, Lages and Mohrbach have studied the Lie algebra associated with the \(O(3)\) invariance of this system. Calculating commutation relations with the angular momentum

\[
L_i = m\epsilon_{ijk} \dot{x}^i \dot{x}^j
\]

(16)
the noncommutivity of the velocities in (5) leads to field-dependent terms,

\[ [x_i, L_j] = -i\hbar \epsilon_{ijk} x_k \]  
\[ \dot{x}_i, L_j = -i\hbar \epsilon_{ijk} \dot{x}^k + \frac{i\hbar}{m} \delta_{ij} (\mathbf{x} \cdot \mathbf{B}) - \frac{i\hbar}{m} x_i B_j \]  
\[ [L_i, L_j] = -i\hbar \epsilon_{ijk} L^k - i\hbar \epsilon_{ijk} x^k (\mathbf{x} \cdot \mathbf{B}). \]  

The authors argue that extending the angular momentum operator \( L_i \) to include the \( O(3) \) invariance of the total particle-field system should recover the closed Lie algebra. Introducing the extended angular momentum \( \tilde{L}_i \) as the sum of the particle angular momentum \( L_i \) and a field-dependent term \( Q_i \),

\[ \tilde{L}_i = L_i + Q_i, \]  
the extended commutation relations must be

\[ [x_i, \tilde{L}_j] = -i\hbar \epsilon_{ijk} x_k \]  
\[ \dot{x}_i, \tilde{L}_j = -i\hbar \epsilon_{ijk} \dot{x}^k \]  
\[ [\tilde{L}_i, \tilde{L}_j] = -i\hbar \epsilon_{ijk} \tilde{L}^k. \]  

It was shown that equations (21) to (23) may be satisfied with the choice

\[ Q_i = -x_i (\mathbf{x} \cdot \mathbf{B}), \]  
which in turn imposes a structural condition on the field \( \mathbf{B} \) given by

\[ x_j B_i + x_i B_j + x_j x_k \partial_i B^k = 0. \]  

Since (25) admits a solution of the form

\[ B_i = -\frac{x_i}{x^3} \]  
the authors argue that the method has led to a magnetic monopole. Using this solution, it is shown that the total angular momentum \( \tilde{L}_i \) is conserved under the classical motion.

In this paper, we generalize the Bérard, Grandati, Lages and Mohrbach construction to the relativistic case in \( N \)-dimensions and study the Lie algebra of the \( O(N - 1,1) \) generators

\[ M^{\mu\nu} = m \left( x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu \right), \quad \mu, \nu = 0, \cdots, N - 1. \]
On departing the realm of 3-dimensional nonrelativistic mechanics, two immediate difficulties arise: the proliferation of terms and tensor indices, and the conceptual difficulty of defining the magnetic monopole in \( N > 4 \) dimensions. These difficulties are conveniently overcome in the spacetime algebra formalism \([10, 11]\), which significantly reduces the notational complexity, and facilitates a general discussion of the monopole in higher dimensions \([12]\). The resulting relativistic construction generalizes equations \((16)\) to \((26)\) and illuminates important features of the symmetric structure not explicit in the 3-dimensional case.

In section 2 we use the spacetime algebra formalism to derive commutation relations for the Lorentz generators \((27)\), involving an antisymmetric tensor field \( W^{\mu \nu} \) associated with the noncommutivity of the velocities \( \dot{x}^\mu \). In section 3, we seek closed commutation relations for the extended generators

\[
\tilde{M}^{\mu \nu} = M^{\mu \nu} + Q^{\mu \nu}
\]

and propose a choice for the field-dependent tensor operator \( Q^{\mu \nu} \) that generalizes \((24)\). We find the structural conditions on the field \( W^{\mu \nu} \) imposed by this choice, which reduce to \((25)\) in the nonrelativistic limit. In section 4, it is shown that when the field \( W^{\mu \nu} \) satisfies the structural conditions, the extended Lorentz operator \( \tilde{M}^{\mu \nu} \) is conserved under system evolution. In section 5, solutions satisfying the structural conditions are shown to be of the general form

\[
W^{\mu \nu}(x) = \frac{1}{(N-2)!} \epsilon^{\mu \nu \lambda_0 \lambda_1 \cdots \lambda_{N-3}} F_{\lambda_0 \lambda_1 \cdots \lambda_{N-3}} = \frac{1}{(N-3)!} \epsilon^{\mu \nu \lambda_0 \lambda_1 \cdots \lambda_{N-3} x_{\lambda_0} U_{\lambda_1 \cdots \lambda_{N-3}}} R(x)
\]

where \( U_{\lambda_1 \cdots \lambda_{N-3}} \) is a fixed antisymmetric tensor of rank \( N - 3 \) and \( R(x) \) is a scalar radial function. In \( N = 4 \), the field \( F_{\lambda_0 \lambda_1 \cdots \lambda_{N-3}} \) reduces to the Liénard-Wiechert solution for an electric charge moving uniformly with four-velocity \( U^\mu \), and since the Levi-Cevita dual exchanges the electric and magnetic fields in the four-dimensional electromagnetic field tensor, \((29)\) can be interpreted as a generalization of the Dirac monopole found in the nonrelativistic case. This interpretation is explored in a second paper. The structural conditions on the fields \( W^{\mu \nu} \) are shown to imply that

\[
x_{\lambda_i} U^{\lambda_1 \cdots \lambda_{N-3}} = 0,
\]

equivalent to the requirement that \( x^\mu \) be orthogonal to \( N - 3 \) mutually orthogonal vectors in \( N \)-dimensions. Thus, the dynamical evolution \( x^\mu(\tau) \) is restricted to the 3-dimensional
subspace normal to $U$, which we denote

$$x^U = \{ x \mid x_\lambda U^{\lambda_1 \cdots \lambda_{N-3}} = 0 \}, \quad (31)$$

and only the three Lorentz generators that leave $x^U$ invariant can be made to satisfy closed commutation relations. Naturally, this restriction has no consequences in the nonrelativistic case. In $N = 4$, we may take the vector $U = \hat{t}$ along the time axis, thereby recovering the O(3)-invariant solution obtained by Bérard, Grandati, Lages and Mohrbach, with the radial function

$$R(x) = r^3 = (x^2)^{3/2} \quad (32)$$
defined on the three space dimensions. On the other hand, by taking $U = \hat{n}$ to be spacelike, the general solution (29) becomes

$$W^{\mu\nu}(x) = \epsilon^{\mu\nu\lambda\rho} \hat{n}_\lambda x_\rho \left( x^2 \right)^{3/2} \quad (33)$$

whose support is on the O(2,1)-invariant subspace

$$x^{\hat{n}} = \{ x \mid x \cdot \hat{n} = 0 \} \quad (34)$$

and the three Lorentz generators (two boosts and one rotation) that leave this subspace invariant will satisfy the closed Lorentz algebra. The field strength (29) is associated with a potential of the type

$$V(x) \sim \left( x^2 \right)^{-1/2} \quad (35)$$

which may be seen as a relativistic generalization of the nonrelativistic Coulomb potential. A solution to the relativistic bound state problem for the scalar hydrogen atom was found in the context of the Horwitz-Piron formalism, using a potential of the form (35). It was shown that a discrete Schrodinger-like spectrum emerges when the dynamics are restricted to the O(2,1)-invariant subspace

$$\text{RMS} (\hat{n}) = \{ x \mid \hat{n}^2 > 0, \, (x \cdot \hat{n})^2 \geq 0 \}. \quad (36)$$

The connection between these cases is discussed in the subsequent paper.
2 Commutation Relations

2.1 Spacetime algebra

The spacetime algebra formalism [10] achieves a high degree of notational compactness by representing the usual tensorial objects of physics as index-free elements in a Clifford algebra. The product of two vectors separates naturally into a symmetric part and antisymmetric part

\[ ab = \frac{1}{2} (ab + ba) + \frac{1}{2} (ab - ba) = a \cdot b + a \wedge b \]  

(37)

where the symmetric part is identified with the scalar inner product, and the rank 2 antisymmetric part is called a bivector. The general Clifford number is a direct sum of multivectors of rank 0, 1, \ldots, N

\[ A = A_0 + A_1 + A_2 + A_3 + \cdots + A_N \]

(38)

expanded on the basis

\[ \{1, e_i, e_i \wedge e_j, e_i \wedge e_j \wedge e_k, \ldots, e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} \} . \]

(40)

The most important algebraic rules are

\[ aA_r = a (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = a \cdot A_r + a \wedge A_r \]

(41)

\[ a \cdot A_r = \sum_{k=1}^r (-1)^{k+1} (a \cdot a_k) a_1 \wedge \cdots \wedge a_{k-1} \wedge a_{k+1} \wedge \cdots \wedge a_r \]

(42)

\[ a \wedge A_r = a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r \]

(43)

\[ i = e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} \]

(44)

\[ i^2 = (-1)^{\frac{N(N-1)}{2}} g_{00} \cdots g_{N-1,N-1} \]

(45)

\[ i [e_{k_1} \wedge \cdots \wedge e_{k_r}] = g_{k_1 k_1} \cdots g_{k_r k_r} \frac{1}{(N-r)!} e^{k_1 \cdots k_r k_{r+1} \cdots k_N} [e_{k_{r+1}} \wedge \cdots \wedge e_{k_N}] \]

(46)

\[ a \cdot (iA_r) = (-1)^{N-1} i (a \wedge A_r) \]

(47)

\[ a \wedge (iA_r) = (-1)^{N-1} i (a \cdot A_r) \]

(48)
2.2 Representations and notation

We begin with the commutation relations among position and velocity
\[
[x^\mu, x^\nu] = 0 \quad [x^\mu, \dot{x}^\nu] = -i\hbar m^2 g^{\mu\nu} \quad [\dot{x}_\mu, f(x)] = \frac{i\hbar}{m} \partial_\mu f(x)
\] (49)
and the relations among velocities
\[
[\dot{x}^\mu, \dot{x}^\nu] = \left(-\frac{i\hbar}{m^2}\right) W^{\mu\nu}(x)
\] (50)
in flat spacetime, where
\[
g^{\mu\nu} = \text{diag} (-1, 1, \cdots, 1) \quad \mu, \nu = 0, 1, \cdots, N - 1.
\] (51)
The Lorentz generators are \( M^{\mu\nu} = m (x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu) \) as given in (27). We represent the vector operators as
\[
x = x^\mu e_\mu \quad \dot{x} = \dot{x}^\mu e_\mu
\] (52)
and the 2nd rank antisymmetric tensors as bivectors
\[
W(x) = W^{\mu\nu}(x) e_\mu \otimes e_\nu = \frac{1}{2} W^{\mu\nu}(x) e_\mu \wedge e_\nu
\] (53)
The entities \( x \) and \( \dot{x} \) are thus composed of operator-valued components \( x^\mu \) and \( \dot{x}^\mu \) that are noncommuting in the operator space but commute in the Clifford algebra, and basis vectors \( e_\mu \) that are noncommuting in the Clifford algebra but commute with the operator-valued components. Exercising care with operator ordering, the manifest antisymmetry of the Lorentz generator permits us to represent the index-free tensor
\[
M = M^{\mu\nu} e_\mu \otimes e_\nu
\] (54)
as a vector product through
\[
M = m (x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu) e_\mu \otimes e_\nu = m x^\mu \dot{x}^\nu (e_\mu \otimes e_\nu - e_\nu \otimes e_\mu) = m (x \wedge \dot{x}).
\] (55)
We may treat operators as Clifford scalars, by introducing auxiliary constants
\[
D = D^\mu e_\mu \quad D^\lambda = g^{\lambda\mu} e_\mu \quad D_2 = D^{(2)} \wedge D^{(1)} = D^{(2)\mu} D^{(1)\nu} e_\mu \wedge e_\nu.
\] (56)
The commutators (49) may be expressed as
\[
[D \cdot x, x] = D_\mu [x^\mu, x^\nu] e_\nu = 0
\] (57)
and

\[ [D \cdot \dot{x}, x] = D_\mu [\dot{x}^\mu, x^\nu] e_\nu = D_\mu \left( \frac{i\hbar}{m g^{\mu\nu}} \right) e_\nu = \frac{i\hbar}{m} D. \]  

Similarly (50) can be written

\[ [D \cdot \dot{x}, \dot{x}] = D_\mu [\dot{x}^\mu, \dot{x}^\nu] e_\nu = -\frac{i\hbar}{m^2} D_\mu W^{\mu\nu} e_\nu = -\frac{i\hbar}{m^2} D \cdot W. \]  

### 2.3 Commutation relations

Using (57) and (58) the commutation relations among generators and position are found as

\[ [D \cdot x, M] = m [D \cdot x, x \wedge \dot{x}] = m [D \cdot x, \dot{x}] + m x \wedge [D \cdot x, \dot{x}] = -i\hbar x \wedge D. \]  

Similarly, the velocity commutators are

\[ [D \cdot \dot{x}, M] = m [D \cdot \dot{x}, x \wedge \dot{x}] = m [D \cdot \dot{x}, \dot{x}] + m x \wedge [D \cdot \dot{x}, \dot{x}] = -i\hbar \dot{x} \wedge D - \frac{i\hbar}{m} x \wedge (D \cdot W). \]  

The bivector equation (61) expresses the \((N - 1)(N - 2)/2\) commutation relations between the Lorentz generators \(M^{\mu\nu}\) and the component of velocity \(\dot{x}\) in the direction of the arbitrary vector \(D\). To obtain the commutators among the generators, it is convenient to write the scalar

\[ D_2 \cdot M = m D_2 \cdot \left[ D_1 \cdot (x \wedge \dot{x}) \right] = m \left[ (D_1 \cdot x) (D_2 \cdot \dot{x}) - (D_2 \cdot x) (D_1 \cdot \dot{x}) \right], \]  

carefully preserving the order of \(x\) and \(\dot{x}\). Using (60) and (61) and extracting \(M\), we are easily led to

\[ [D_2 \cdot M, M] = i\hbar \left[ D_2 \wedge (D_1 \cdot M) - D_1 \wedge (D_2 \cdot M) \right] 
+ i\hbar x \wedge \left[ (D_2 \cdot x) (D_1 \cdot W) - (D_1 \cdot x) (D_2 \cdot W) \right]. \]  

The bivector equation (63) expresses the \((N - 1)(N - 2)/2\) commutation relations between the Lorentz generators \(M^{\mu\nu}\) and the particular generator selected by the arbitrary vectors \(D^{(1)}\) and \(D^{(2)}\). This expression agrees with the closed commutator when \(W = 0\).
3 Restoring the Operator Algebra

3.1 Extended generators

We seek the extended generator

\[ \tilde{M} = M + Q \]  \hspace{1cm} (64)

that satisfies the closed operator algebra

\[ \left[ D \cdot x, \tilde{M} \right] = -i\hbar x \land D \]  \hspace{1cm} (65)

\[ \left[ D \cdot \dot{x}, \tilde{M} \right] = -i\hbar \dot{x} \land D \]  \hspace{1cm} (66)

\[ \left[ D_2 \cdot \tilde{M}, \tilde{M} \right] = i\hbar \left[ D^{(2)} \land (D^{(1)} \cdot \tilde{M}) - D^{(1)} \land (D^{(2)} \cdot \tilde{M}) \right]. \]  \hspace{1cm} (67)

The actual commutation relations, equations (60), (61) and (63), then impose requirements on the form of the generator \( Q \). Applying (60) to (64) it follows that

\[ \left[ D \cdot x, M + Q \right] = \left[ D \cdot x, M \right] + \left[ D \cdot x, Q \right] = -i\hbar x \land D + \left[ D \cdot x, Q \right] \]  \hspace{1cm} (68)

and comparison with (65) leads to

\[ \left[ D \cdot x, Q \right] = 0 \]  \hspace{1cm} (69)

and so that \( Q \) is independent of \( \dot{x} \) and its components commute among themselves

\[ \frac{\partial}{\partial \dot{x}^\mu} Q = 0 \Rightarrow \left[ D_2 \cdot Q, Q \right] = 0. \]  \hspace{1cm} (70)

Now from (64), (66), and (61) we find

\[ \left[ D \cdot \dot{x}, M + Q \right] = -i\hbar \dot{x} \land D - \frac{i\hbar}{m} x \land (D \cdot W) + \left[ D \cdot \dot{x}, Q \right] = -i\hbar \dot{x} \land D \]  \hspace{1cm} (71)

so the new commutation relation is

\[ \left[ D \cdot \dot{x}, \tilde{M} \right] = \left[ D \cdot \dot{x}, M \right] - \frac{i\hbar}{m} [x \land (D \cdot W) - (D \cdot \partial) Q], \]  \hspace{1cm} (72)

where we have used (49). Comparing (72) with (66) leads to the first condition on \( Q \)

\[ \Delta_1 = -\frac{i\hbar}{m} [x \land (D \cdot W) - (D \cdot \partial) Q] = 0. \]  \hspace{1cm} (73)
To find the second condition on \( Q \), we apply (64) to the LHS of (67)

\[
\begin{bmatrix} D_2 \cdot \tilde{M}, \tilde{M} \end{bmatrix} = [D_2 \cdot M, M] + [D_2 \cdot M, Q] + [D_2 \cdot Q, M] \tag{74}
\]

and to the RHS of (67)

\[
\begin{bmatrix} D_2 \cdot \tilde{M}, \tilde{M} \end{bmatrix} = i\hbar \left[ D^{(2)} \land (D^{(1)} \cdot M) - D^{(1)} \land (D^{(2)} \cdot M) \right] + \left[ D^{(2)} \land (D^{(1)} \cdot Q) - D^{(1)} \land (D^{(2)} \cdot Q) \right]. \tag{75}
\]

Applying (62) to (74) provides

\[
[D_2 \cdot M, Q] + [D_2 \cdot Q, M] = i\hbar \left[ D^{(2)} \land (D^{(1)} \cdot Q) - D^{(1)} \land (D^{(2)} \cdot Q) \right] - i\hbar x \land \left[ (D^{(2)} \cdot x) (D^{(1)} \cdot W) - (D^{(1)} \cdot x) (D^{(2)} \cdot W) \right] \tag{76}
\]

so that combining (67), (63), and (76) leads to

\[
\begin{bmatrix} D_2 \cdot \tilde{M}, \tilde{M} \end{bmatrix} = i\hbar \left[ D^{(2)} \land (D^{(1)} \cdot \tilde{M}) - D^{(1)} \land (D^{(2)} \cdot \tilde{M}) \right] - i\hbar x \land \left[ (D^{(2)} \cdot x) (D^{(1)} \cdot W) - (D^{(1)} \cdot x) (D^{(2)} \cdot W) \right] - i\hbar \left[ D^{(2)} \land (D^{(1)} \cdot Q) - D^{(1)} \land (D^{(2)} \cdot Q) \right], \tag{77}
\]

and we arrive at the second condition on the generator \( Q \),

\[
\Delta_2 = -i\hbar \left\{ x \land \left[ (D^{(2)} \cdot x) (D^{(1)} \cdot W) - (D^{(1)} \cdot x) (D^{(2)} \cdot W) \right] + \left[ D^{(2)} \land (D^{(1)} \cdot Q) - D^{(1)} \land (D^{(2)} \cdot Q) \right] \right\} = 0. \tag{78}
\]

### 3.2 Choice of generator

Since \( Q = Q(x) \) cannot depend on \( \dot{x} \) and since \( D = (D \cdot \partial) x \) allows us to write (63) as

\[
(D \cdot \partial) Q = x \land [(D \cdot \partial) x] \cdot W, \tag{79}
\]

we are led to consider the form

\[
Q = x \land (x \cdot W) - x^2 W. \tag{80}
\]

Using

\[
x^2 W = x \cdot (x \land W) + x \land (x \cdot W) \tag{81}
\]

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equation (80) can be rewritten as

\[ Q = x \wedge (x \cdot W) - [x \cdot (x \wedge W) + x \wedge (x \cdot W)] = -x \cdot (x \wedge W). \]  

(82)

The geometrical meaning of (82) can be seen by re-writing (81) as

\[ W = \frac{1}{x^2} [x \cdot (x \wedge W) + x \wedge (x \cdot W)] = \hat{x} \cdot (\hat{x} \wedge W) + \hat{x} \wedge (\hat{x} \cdot W). \]  

(83)

Since \( \hat{x} \cdot W = 0 \) if the unit vector \( \hat{x} \) is orthogonal to the plane spanned by \( W \), we find that (83) represents the decomposition of \( W \) into

\[ W_u = \hat{x} \wedge (\hat{x} \cdot W) \quad W_\perp = \hat{x} \cdot (\hat{x} \wedge W) \]  

(84)

and so comparing with (82), \( Q \) can be described as the component of \( W \) orthogonal to the observation point \( x \).

### 3.3 Conditions on the field \( W \)

Applying the first condition (73) on the generator \( Q \) to the form (82), we find a corresponding condition on the field \( W \)

\[ \Delta_1 = -\frac{i\hbar}{m} \{ x \wedge (D \cdot W) + D \cdot (x \wedge W) + x \cdot (D \wedge W) + x \cdot [x \wedge (D \cdot \partial) W] \} = 0. \]  

(85)

The second condition (78) on the generator \( Q \) is purely algebraic

\[ \Delta_2 = i\hbar (D^{(1)} \cdot x) \left[ x \wedge (D^{(2)} \cdot W) - D^{(2)} \wedge (x \cdot W) \right] \\
+ i\hbar (D^{(2)} \cdot x) \left[ D^{(1)} \wedge (x \cdot W) - x \wedge (D^{(1)} \cdot W) \right] \\
+ i\hbar x^2 \left[ D^{(2)} \wedge (D^{(1)} \cdot W) - D^{(1)} \wedge (D^{(2)} \cdot W) \right] \\
+ i\hbar (D^{(2)} \wedge x) \left( (D^{(1)} \wedge x) \cdot W \right) - i\hbar (D^{(1)} \wedge x) \left( (D^{(2)} \wedge x) \cdot W \right) = 0. \]  

(86)
4 Conserved Evolution

The derivative of the classical extended Lorentz generator with respect to the invariant time

\[
\frac{d}{d\tau} \tilde{M} = \dot{M} + \dot{Q}
\]

\[
= m \frac{d}{d\tau} (x \wedge \dot{x}) + \frac{d}{d\tau} [-x \cdot (x \wedge W)]
\]

\[
= m (x \wedge \ddot{x} + \dot{x} \wedge \dot{x}) - \dot{x} \cdot (x \wedge W) - x \cdot (\dot{x} \wedge W) - x \cdot (x \wedge \dot{W})
\]

\[
= m (x \wedge \ddot{x}) - \dot{x} \cdot (x \wedge W) - x \cdot (\dot{x} \wedge W) - x \cdot (x \wedge \dot{W}). \quad (87)
\]

Using the equations of motion for \(\tau\)-independent fields

\[
W^{\mu\nu}(x, \tau) = W^{\mu\nu}(x) \quad W^{\mu N}(x, \tau) = 0 \quad (88)
\]

leads to

\[
m \ddot{x}^\mu = W^{\mu\nu}\dot{x}_\nu = -\dot{x}_\nu W^{\nu\mu} \quad \longrightarrow \quad \ddot{x} = -\frac{1}{m} \dot{x} \cdot W \quad (89)
\]

and since \(W\) only depends on \(\tau\) through \(x(\tau)\),

\[
\dot{W}^{\mu\nu} = \partial_\lambda W^{\mu\nu}\dot{x}_\lambda \quad \longrightarrow \quad \dot{W} = (\dot{x} \cdot \partial) W \quad (90)
\]

the equations of motion become

\[
\frac{d}{d\tau} \tilde{M} = -x \wedge (\dot{x} \cdot W) - \dot{x} \cdot (x \wedge W) - x \cdot (\dot{x} \wedge W) - x \cdot (x \wedge (\dot{x} \cdot \partial) W).
\]

Applying with \(D = \dot{x}\),

\[
-x \cdot [x \wedge (\dot{x} \cdot \partial) W] = x \wedge (\dot{x} \cdot W) + \dot{x} \cdot (x \wedge W) + x \cdot (\dot{x} \wedge W) \quad (91)
\]

we find

\[
\frac{d}{d\tau} \tilde{M} = -x \wedge (\dot{x} \cdot W) - \dot{x} \cdot (x \wedge W) - x \cdot (\dot{x} \wedge W)
\]

\[
+ x \wedge (\dot{x} \cdot W) + \dot{x} \cdot (x \wedge W) + x \cdot (\dot{x} \wedge W)
\]

\[
= 0 \quad (92)
\]

so that the classical generator is conserved.
5 Solutions

In the previous sections we have shown that in principle we may define an extended Lorentz generator that is conserved under evolution of the particle-field system and satisfies the closed $O(N - 1, 1)$ commutation relations. The resulting system depends, of course, on the existence of a field $W$ satisfying conditions (85) and (86). It turns out such solutions exist in a restricted regime. We first obtain the Liénard-Wiechert field for a uniformly moving charge, and demonstrate that no generalization of this form can satisfy the condition (85). We then show that under limited circumstances, the dual of this solution can satisfy the conditions.

5.1 Liénard-Wiechert field in 4-dimensions

A charge moving uniformly with four-velocity $U^\mu$ induces a potential $A^\mu(x)$ through the Maxwell Green’s function

$$A^\mu(x) = \frac{1}{2\pi} \int d^4x' J^\mu(x') \delta \left[(x - x')^2\right]$$

where

$$J^\mu(x') = \int d\tau U^\mu \delta^4(x' - U\tau).$$

Expanding the Green’s function using

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|\frac{df}{dx}|_{x=x_i}}$$

we arrive at the Liénard-Wiechert potential for uniform motion

$$A^\mu(x) = \frac{U^\mu}{\left[x^2 + (x \cdot U)^2\right]^{1/2}}$$

from which we derive the field strength tensor as

$$F^{\mu\nu}(x) = \frac{U^\mu x^\nu - U^\nu x^\mu}{\left[x^2 + (x \cdot U)^2\right]^{3/2}}.$$  

Since the four-velocity is timelike $U^2 = -1$, and the observation point $x$ can be resolved as

$$x = -U^2 x = -U (U \cdot x + U \wedge x)$$
with
\[ x_{\parallel} = -U (U \cdot x) \] (99)
\[ x_{\perp} = -U (U \wedge x) = -U^2 x + U (U \cdot x) = x + U (U \cdot x), \] (100)
we recognize
\[ (x_{\perp})^2 = x^2 + U^2 (U \cdot x)^2 + 2 (U \cdot x)^2 = x^2 + (U \cdot x)^2. \] (101)

In the index-free notation of spacetime algebra, we may rewrite (99) and (100) as
\[ A(x) = \frac{U}{(x_{\perp})^{1/2}} \] (102)
\[ F(x) = \partial \wedge A(x) = \frac{U \wedge x_{\perp}}{(x^2)^{3/2}} = \frac{U \wedge x}{(x_{\perp}^2)^{3/2}} \] (103)
where we have taken advantage of the identity \( U \wedge x_{\parallel} = 0 \).

5.2 Electric field in \( N \)-dimensions

We attempt a general solution of the form
\[ W(x) = \frac{U \wedge x}{R(x)} \] (104)
with an arbitrary fixed vector \( U \). The first condition (85) on the field is
\[ x \wedge (D \cdot W) + D \cdot (x \wedge W) + x \cdot (D \wedge W) + x \cdot [x \wedge (D \cdot \partial) W] = 0, \] (105)
in which the derivative term requires
\[ (D \cdot \partial) W = (D \cdot \partial) \frac{U \wedge x}{R(x)} = \frac{U \wedge D}{R(x)} - \frac{(U \wedge x) (D \cdot \partial) R(x)}{R^2(x)} \] (106)
so that
\[ x \cdot [x \wedge (D \cdot \partial) W] = x \cdot \left[ \left( \frac{x \wedge U \wedge D}{R(x)} - \frac{(x \wedge U \wedge x) (D \cdot \partial) R(x)}{R^2(x)} \right) \right] = -x \cdot \left( \frac{D \wedge U \wedge x}{R(x)} \right), \] (107)
where we used $x \wedge x = 0$. Similarly, $D \cdot (x \wedge W)$ vanishes identically for this solution, so (105) becomes

$$0 = x \wedge \left( \frac{D \cdot U \wedge x}{R(x)} \right) + x \cdot \left( \frac{D \wedge U \wedge x}{R(x)} \right) - x \cdot \left( \frac{D \wedge U \wedge x}{R(x)} \right)$$

$$= \frac{x \wedge (D \cdot (U \wedge x))}{R(x)}$$

$$= \frac{x \wedge ((D \cdot U) x - U (D \cdot x))}{R(x)}$$

$$= \frac{(U \wedge x) (D \cdot x)}{R(x)}$$

$$= (D \cdot x) W(x).$$

Notice that $D$ is arbitrary, specifying the component of velocity being commuted with the generator. Since $x \wedge x \equiv 0$ causes the second term of (107) to vanish for algebraic reasons, the trial solution (104) fails for any form of the radial function $R(x)$. Moreover, expression (104) is seen to be an unlikely candidate for the field solution, because it leads to

$$Q = -x \cdot (x \wedge W) = -x \cdot \left( x \wedge \frac{U \wedge x}{R(x)} \right) \equiv 0. \quad (109)$$

### 5.3 Dual field in N-dimensions

Generalizing the 3-dimensional result, we assume that the field $W(x)$ is given as the dual of some field $F(x)$, so that we replace

$$W(x) = iF(x) \quad (110)$$

in the requirement (85), and adapt it in N-dimensions by using (47) and (48). The first condition on the field $F$ then becomes

$$0 = \Delta_1 = -\frac{i \hbar}{m} \left\{ \frac{x \wedge (D \cdot iF) + D \cdot (x \wedge iF) + x \cdot (D \wedge iF) + x \cdot [x \wedge (D \cdot \partial) iF]}{x \wedge (D \cdot \partial) F} \right\}$$

$$= -\frac{i \hbar}{m} \left\{ \frac{x \cdot (D \wedge F) + D \wedge (x \cdot F) + x \wedge (D \cdot F) + x \wedge [x \cdot (D \cdot \partial) F]}{x \cdot (D \cdot \partial) F} \right\}. \quad (111)$$

Since the field $W$ was introduced in (50) as a bivector, the Levi-Cevita dual field $F$ must be an $(N - 2)$-vector in N-dimensions. Generalizing expression (104), we may take the general solution for $F(x)$ to be

$$F(x) = \frac{x \wedge U}{R(x)} \quad (112)$$
where $U$ is a fixed multivector of rank $N - 3$. Now
\[(D \cdot \partial) F = (D \cdot \partial) \frac{x \wedge U}{R(x)} = \frac{D \wedge U}{R(x)} - \frac{(x \wedge U)(D \cdot \partial) R(x)}{R^2(x)} \quad (113)\]
and we have
\[x \wedge [x \cdot (D \cdot \partial) F] = \frac{(x \cdot D)(x \wedge U) - x \wedge D \wedge (x \cdot U)}{R(x)} - \frac{x^2(x \wedge U)(D \cdot \partial) R(x)}{R^2(x)}. \quad (114)\]
The other terms in (111) are
\[x \wedge (D \cdot F) = \frac{(D \cdot x)(x \wedge U)}{R(x)} \quad (115)\]
\[D \wedge (x \cdot F) = \frac{x^2D \wedge U - D \wedge x \wedge (x \cdot U)}{R(x)} \quad (116)\]
\[x \cdot (D \wedge F) = \frac{(x \cdot D)(x \wedge U) - x^2D \wedge U + D \wedge x \wedge (x \cdot U)}{R(x)}. \quad (117)\]
Assembling the terms we find
\[0 = \Delta_1 = -\frac{i\hbar}{m} \iota \left\{ \frac{(x \cdot D)(x \wedge U)}{R(x)} - \frac{x^2(x \wedge U)(D \cdot \partial) R(x)}{R^2(x)} - \frac{x \wedge D \wedge (x \cdot U)}{R(x)} \right\}. \quad (118)\]
Taking the radial function $R(x)$ to be
\[R(x) = (x^2)^{3/2} \quad (119)\]
so that
\[\frac{(D \cdot \partial) R(x)}{R^2(x)} = 3 \frac{D \cdot x}{x^2R(x)} \quad (120)\]
leads to
\[0 = \Delta_1 = -\frac{i\hbar}{m} \iota \left[ \frac{D \wedge x \wedge (x \cdot U)}{R(x)} \right] = -\frac{i\hbar}{m} \iota \frac{D \wedge U_v}{(x^2)^{1/2}} \quad (121)\]
where we used (84) to write
\[x \wedge (x \cdot U) = x^2U_v. \quad (122)\]
Again, since the vector $D$ is arbitrary, the condition imposed by (121) is satisfied when the dynamical evolution $x^\mu(\tau)$ is restricted to the subspace defined by
\[x(\tau) \in x^U = \{ x \mid x \cdot U = 0 \}, \quad (123)\]
equivalent to the requirement that $x^\mu$ be orthogonal to $N - 3$ mutually orthogonal vectors in $N$-dimensions. We conclude that the Bérard, Grandati, Lages and Mohrbach construction...
may be generalized to any 3-dimensional subspace of $N$-dimensional Minkowski space. The generators of the $O(2,1)$ or $O(3)$ subgroup of $O(N-1,1)$ that leave the subspace $x^U$ invariant have an extension that is dynamically conserved and satisfies closed commutation relations.

The second condition on the field $W(x)$ was given in (86). With the replacement (110) this condition becomes

\[
\Delta_2 = i\hbar \left( D^{(1)} \cdot x \right) \left[ i x \cdot \left( D^{(2)} \wedge F \right) - i D^{(2)} \cdot (x \wedge F) \right] \\
+ i\hbar \left( D^{(2)} \cdot x \right) \left[ i D^{(1)} \cdot (x \wedge F) - i x \cdot (D^{(1)} \wedge F) \right] \\
+ i\hbar x^2 \left[ i D^{(2)} \cdot \left( D^{(1)} \wedge F \right) - i D^{(1)} \cdot \left( D^{(2)} \wedge F \right) \right] \\
+ i\hbar \left( D^{(2)} \wedge x \right) \left( i D^{(1)} \wedge (x \wedge F) \right) - i\hbar \left( D^{(1)} \wedge x \right) \left( i D^{(2)} \wedge (x \wedge F) \right)
\]

Applying (112) and using $x \wedge x \equiv 0$, this reduces to

\[
\Delta_2 = i\hbar \frac{i}{R(x)} \left( D^{(1)} \cdot x \right) \left[ x \cdot \left( D^{(2)} \wedge x \wedge U \right) \right] - i\hbar \left( D^{(2)} \cdot x \right) \left[ x \cdot \left( D^{(1)} \wedge x \wedge U \right) \right] \\
+ i\hbar \frac{i x^2}{R(x)} \left[ D^{(2)} \cdot \left( D^{(1)} \wedge x \wedge U \right) - D^{(1)} \cdot \left( D^{(2)} \wedge x \wedge U \right) \right],
\]

so that expanding the inner products and restricting the dynamics to the subspace $x^U = \{x \mid x \cdot U = 0\}$ leads to

\[
\Delta_2 = i\hbar \frac{i x^2}{R(x)} \left[ \left( D^{(1)} \wedge x \right) \wedge \left( D^{(2)} \cdot U \right) - \left( D^{(2)} \wedge x \right) \wedge \left( D^{(1)} \cdot U \right) \right].
\]

The arbitrary vectors $D^{(1)}$ and $D^{(2)}$ specify components of the Lorentz generator that undergo commutation with the index-free Lorentz generator tensor. Since (121) restricts the dynamical components to the subspace $x^U = \{x \mid x \cdot U = 0\}$ we may limit attention to the components for which $D^{(1)} \cdot U = D^{(2)} \cdot U = 0$, and so (126) is automatically satisfied.

### 5.3.1 Dual field in 4-dimensions

In $N = 4$, the multivector $U$ is just a four-vector and we recognize (112) as a field of the Liénard-Wiechert type

\[
F(x) = \frac{x \wedge U}{(x^2)^{3/2}} \quad \rightarrow \quad W(x) = i \frac{x \wedge U}{(x^2)^{3/2}}
\]
valid in the subspace for which
\[ x^U = \{ x \mid x \cdot U = 0 \} . \]  

(128)

We may recover the nonrelativistic case by choosing the unit vector along the time axis \( U = e_0 \), imposing the restriction of the dynamics to
\[ x (\tau) = (0, x) \]  

(129)

so that
\[ F(x) = \frac{e_0 \land x}{r^3} = \frac{x^1}{r^3} e_0 \land e_1 + \frac{x^2}{r^3} e_0 \land e_2 + \frac{x^3}{r^3} e_0 \land e_3 \]  

(130)

and
\[ W(x) = E^i e_0 \land e_i + \frac{1}{2} \epsilon^{ijk} B_i e_j \land e_k = i \frac{e_0 \land x}{r^3} = - \frac{x^1}{r^3} e_2 \land e_3 + \frac{x^2}{r^3} e_1 \land e_3 - \frac{x^3}{r^3} e_1 \land e_2 \]  

(131)

where
\[ r = \left[ (x^1)^2 + (x^2)^2 + (x^3)^2 \right]^{1/2} . \]  

(132)

The field strengths are found to be
\[ E = 0 \quad B = - \frac{1}{r^3} \left( x^1, x^2, x^3 \right) \]  

(133)

as in (26). Since \( x \cdot U \) is a scalar in four dimensions, (121) provides the extra term in the commutation relation for velocity as
\[ \Delta_1 = \frac{i \hbar (x \cdot U)}{m R(x)} i (D \land x) . \]  

(134)

We may split this expression into rotations and boosts as
\[ \Delta_1 = \frac{i \hbar x^0}{m R(x)} i \left[ (D^0 e_0 + D) \land (x^0 e_0 + x) \right] \]  

(135)

\[ = \frac{i \hbar x^0}{m R(x)} \left[ (D^0 x^i - x^0 D^i) i (e_0 \land e_i) + i (D \land x) \right] , \]  

(136)

and using
\[ i (e_\mu \land e_\nu) = \epsilon_{\mu\nu\lambda\rho} g^{\lambda\sigma} g^{\rho\kappa} e_\sigma \land e_\zeta , \]  

(137)

we find
\[ \Delta_1 = \frac{i \hbar x^0}{m R(x)} \left[ \frac{1}{2} \epsilon^{0ijk} (D_0 x_i - D_i x_0) e_j \land e_k + (D_i x_j - D_j x_i) e_0 \land e_k \right] . \]  

(138)
The component commutation relations for velocity with the O(3,1) generators can be read off as

\[
\left[ \dot{x}^i, \tilde{M}^{jk} \right] = i\hbar \left( g^{ij} \dot{x}^k - g^{jk} \dot{x}^i \right) - \frac{i\hbar}{m} \epsilon^{ijk} \frac{x^0 x_0}{R(x)} \quad \left[ \dot{x}^0, \tilde{M}^{jk} \right] = 0
\] (139)

and

\[
\left[ \dot{x}^i, \tilde{M}^{0k} \right] = i\hbar \left( g^{i0} \dot{x}^k - g^{ik} \dot{x}^0 \right) - \frac{i\hbar}{m} \epsilon^{ijk} \frac{x^0 x_j}{R(x)} \quad \left[ \dot{x}^0, \tilde{M}^{0k} \right] = i\hbar g^{00} \dot{x}^k,
\] (140)

which, under the restriction \( x \cdot U = x^0 = 0 \) of \([128]\), become the closed relations. Similarly, applying the restriction \([129]\) to \([126]\), the commutation relations among the generators in component form becomes

\[
\left[ \tilde{M}^{\mu\nu}, \tilde{M}^{\lambda\rho} \right] = i\hbar \left\{ g^{\mu\lambda} \tilde{M}^{\nu\rho} - g^{\mu\rho} \tilde{M}^{\nu\lambda} - g^{\nu\lambda} \tilde{M}^{\mu\rho} + g^{\nu\rho} \tilde{M}^{\mu\lambda} \right\} + \Delta_2^{\mu\nu\lambda\rho}
\] (141)

where

\[
\Delta_2^{\mu\nu\lambda\rho} = 2i\hbar \frac{x^2}{R(x)} \left[ g_0^{\mu} \epsilon^{\nu\lambda\delta\rho} x_\delta - g_0^{\nu} \epsilon^{\mu\delta\lambda\rho} x_\delta \right].
\]

Dividing the generators, we find that the algebra of the boosts is broken

\[
\left[ \tilde{M}^{0j}, \tilde{M}^{\lambda\rho} \right] = i\hbar \left\{ g^{0\lambda} \tilde{M}^{j\rho} - g^{0\rho} \tilde{M}^{j\lambda} - g^{j\lambda} \tilde{M}^{0\rho} + g^{j\rho} \tilde{M}^{0\lambda} \right\} + \Delta_2^{0j\lambda\rho}
\]

\[
= i\hbar \left\{ g^{0\lambda} \tilde{M}^{j\rho} - g^{0\rho} \tilde{M}^{j\lambda} - g^{j\lambda} \tilde{M}^{0\rho} + g^{j\rho} \tilde{M}^{0\lambda} \right\} + \frac{2i\hbar x^2}{R(x)} \epsilon^{j\delta\lambda\rho} x_\delta
\] (142)

while the rotation generators obey closed relations

\[
\left[ \tilde{M}^{ij}, \tilde{M}^{\lambda\rho} \right] = i\hbar \left\{ g^{i\lambda} \tilde{M}^{j\rho} - g^{i\rho} \tilde{M}^{j\lambda} - g^{j\lambda} \tilde{M}^{i\rho} + g^{j\rho} \tilde{M}^{i\lambda} \right\} + \Delta_2^{ij\lambda\rho}
\]

\[
= i\hbar \left\{ g^{i\lambda} \tilde{M}^{j\rho} - g^{i\rho} \tilde{M}^{j\lambda} - g^{j\lambda} \tilde{M}^{i\rho} + g^{j\rho} \tilde{M}^{i\lambda} \right\}
\] (143)

Since only those Lorentz generators that leave the vector \( U = e_0 \) invariant enjoy the closed commutations relations, this choice of vector \( U \) restores the closed algebra for the O(3) rotation generators, but not for the boost generators. Thus, we may understand the nonrelativistic result as equivalent to the maximal relativistic result for this choice of unit vector.

We may construct a different kind of solution by choosing \( U = e_0 \) along the z-axis. Then, from

\[
F(x) = \frac{x \wedge e_3}{\rho^3} = \frac{x^0}{\rho^3} e_0 \wedge e_3 + \frac{x^1}{\rho^3} e_1 \wedge e_3 + \frac{x^2}{\rho^3} e_2 \wedge e_3
\] (144)

we find the field

\[
W(x) = E^0 e_0 \wedge e_i + \frac{1}{2} \epsilon^{ijk} B_i e_j \wedge e_k = \frac{x \wedge e_3}{\rho^3} = \frac{x^0}{\rho^3} e_0 \wedge e_1 + \frac{x^1}{\rho^3} e_0 \wedge e_2 - \frac{x^2}{\rho^3} e_0 \wedge e_1,
\] (145)
where
\[
\rho = \left[ - (x^0)^2 + (x^1)^2 + (x^2)^2 \right]^{1/2}
\] (146)

generalizes the spacial separation in the action-at-a-distance problems of nonrelativistic mechanics in the subspace
\[
x = (x^0, x^1, x^2, 0) \in \mathbb{R}^4 = \{ x \mid x \cdot \mathbf{e}_3 = 0 \}
\] (147)
invariant under the O(2,1) subgroup of the full Lorentz group. The field strengths are
\[
E = \frac{1}{\rho^3} (-x^2, x^1, 0) \quad B = \frac{1}{\rho^3} (0, 0, x^0).
\] (148)

In this case, subject to the restriction (147), the extra terms in the velocity relations are
\[
\Delta_1 = \frac{ih}{m R(x)} x^3 i (D \wedge x) = 0
\] (149)
\[
\Delta_{\mu\nu\lambda\rho}^2 = 2i \hbar x^2 \frac{1}{R(x)} \left[ g_{3\mu\nu\delta\lambda\rho} x^\delta - g_{3\nu\mu\delta\lambda\rho} x^\delta \right],
\] (150)

so that closed commutation relations hold for the velocities and among the O(2,1) generators \( \tilde{M}^{01}, \tilde{M}^{02}, \) and \( \tilde{M}^{12} = \tilde{L}_3 \), while the algebra of the generators is broken by field dependent terms for the boost \( \tilde{M}^{03} \) and the rotations \( \tilde{M}^{31} = \tilde{L}_2 \) and \( \tilde{M}^{23} = \tilde{L}_1 \).

5.3.2 Dual field in higher dimensions

In \( N > 4 \), the multivector \( U \) has rank \( N - 3 > 1 \), and we may gain insight into this case by partitioning the \( N \)-dimensional space into the usual 4-dimensional spacetime and \( N - 3 \) ‘extra dimensions’ through
\[
U = n \wedge \mathbf{e}_4 \wedge \cdots \wedge \mathbf{e}_{N-1} = n \wedge \tilde{U}
\] (151)
where
\[
n = n^\mu \mathbf{e}_\mu, \ \mu = 0, 1, 2, 3 \quad \tilde{U} = \mathbf{e}_4 \wedge \cdots \wedge \mathbf{e}_{N-1}.
\] (152)

Under this partition, (112) becomes
\[
F(x) = \frac{x \wedge n \wedge \tilde{U}}{R(x)}
\] (153)
subject to the restriction

\[ x(\tau) \in x^U = \left\{ x \mid x \cdot U = x \cdot \left( n \wedge \tilde{U} \right) = (x \cdot n) \tilde{U} - n \wedge \left( x \cdot \tilde{U} \right) = 0 \right\}. \] (154)

We choose

\[ x(\tau) = x^\mu e_\mu = (x^0, x^1, x^2, x^3, 0, \ldots, 0) \] (155)

so that (154) becomes

\[ x(\tau) \in x^U = \left\{ x \mid (x \cdot n) = 0 \right\} \] (156)

and from (110) the general solution is

\[ W(x) = i x \wedge n \wedge \tilde{U} \] (157)

By similarly partitioning the unit pseudoscalar into the usual 4-dimensional spacetime and \( N - 3 \) ‘extra dimensions’

\[ i = e_0 \wedge \cdots \wedge e_{N-1} = (e_0 e_1 e_2 e_3) (e_4 \cdots e_{N-1}) = i_4 \tilde{U} \] (158)

the general field becomes

\[ W(x) = i_4 \tilde{U} x \wedge n \wedge \tilde{U} x \wedge n \tilde{U}^2 = \left[ g_{14} \cdots g_{N-1,N-1} (-1) \left( \frac{N-4}{2} \right) \frac{N-5}{2} \right] i_4 \tilde{U} \] (159)

which we recognize as the solution (127), up to a sign. For this solution, the extended generators of \( O(N-1,1) \) satisfy the closed commutation relations, up to extra terms of the form

\[ \Delta_2 = i \hbar i x^2 \left\{ \left( D^{(1)} \wedge x \right) \wedge \left[ D^{(2)} \cdot \left( n \wedge \tilde{U} \right) \right] - (D^{(2)} \wedge x) \wedge \left[ D^{(1)} \cdot \left( n \wedge \tilde{U} \right) \right] \right\} \]

\[ = i \hbar i x^2 \left( D^{(1)} \wedge x \right) \wedge \left[ (D^{(2)} \cdot n ) \tilde{U} - n \wedge \left( D^{(2)} \cdot \tilde{U} \right) \right] \]

\[ - i \hbar i x^2 \left( D^{(2)} \wedge x \right) \wedge \left[ (D^{(1)} \cdot n ) \tilde{U} - n \wedge \left( D^{(1)} \cdot \tilde{U} \right) \right] \]

\[ = i \hbar i x^2 \left[ (D^{(2)} \cdot n ) \left( D^{(1)} \wedge x \right) \wedge \tilde{U} - (D^{(1)} \wedge x) \wedge n \wedge \left( D^{(2)} \cdot \tilde{U} \right) \right] \]

\[ - i \hbar i x^2 \left[ (D^{(1)} \cdot n ) \left( D^{(2)} \wedge x \right) \wedge \tilde{U} - (D^{(2)} \wedge x) \wedge n \wedge \left( D^{(1)} \cdot \tilde{U} \right) \right]. \] (160)

As in \( N = 4 \), the extra terms vanish for three of the \( O(3) \) or \( O(2,1) \) generators — the three are determined by the conditions

\[ D^{(1)} \cdot \tilde{U} = D^{(2)} \cdot \tilde{U} = 0 \quad \Rightarrow \quad D^{(1,2)} = D^{(1,2)\mu} e_\mu, \quad \mu = 0, 1, 2, 3 \] (161)
and
\[ D^{(1)} \cdot n = D^{(2)} \cdot n = 0. \] (162)

The examples in the previous section are recovered by the choices \( n = e_0 \) and \( n = e_3 \).

6 Conclusion

We have seen that in the presence of the field
\[ W^{\mu\nu}(x) = \frac{1}{(N-2)!} \epsilon^{\mu\nu\lambda_0\lambda_1\cdots\lambda_{N-3}} F_{\lambda_0\lambda_1\cdots\lambda_{N-3}} = \frac{1}{(N-3)!} \epsilon^{\mu\nu\lambda_0\lambda_1\cdots\lambda_{N-3}} x_{\lambda_0} U_{\lambda_1\cdots\lambda_{N-3}} R(x) \] (163)
the extended O\((N-1,1)\) generators
\[ \tilde{M}^{\mu\nu} = M^{\mu\nu} + Q^{\mu\nu} = m \left( x^\lambda \dot{x}^\nu - x^\nu \dot{x}^\lambda \right) + x^\lambda x^\sigma W^{\sigma\nu} - x^\nu x^\sigma W^{\sigma\mu} - x^\sigma x^\sigma W^{\mu\nu} \] (164)
are constructed in such a way that the commutation relations of the generators with position \( x^\mu \) and velocity \( \dot{x}^\mu \) satisfy the closed Lie algebra
\[ [x^\mu, \tilde{M}^{\rho\lambda}] = i\hbar \left( x^\lambda g^{\mu\rho} - x^\rho g^{\mu\lambda} \right) \quad [\dot{x}^\mu, \tilde{M}^{\rho\lambda}] = i\hbar \left( g^{\mu\rho} \dot{x}^\lambda - g^{\mu\lambda} \dot{x}^\rho \right) \] (165)
when the dynamical evolution is restricted to
\[ x(\tau) \in x^U = \{ x \mid x^{\lambda_1} U_{\lambda_1\lambda_2\cdots\lambda_{N-3}} = 0 \} \] (166)
and
\[ R(x) = (x^2)^{3/2}. \] (167)
Similarly, and the commutation relations among the generators
\[ [\tilde{M}^{\mu\nu}, \tilde{M}^{\lambda\rho}] = i\hbar \left\{ g^{\mu\lambda} \tilde{M}^{\nu\rho} - g^{\rho\nu} \tilde{M}^{\mu\lambda} - g^{\nu\lambda} \tilde{M}^{\mu\rho} + g^{\mu\rho} \tilde{M}^{\nu\lambda} \right\} + \Delta_2^{\mu\nu\lambda\rho} \] (168)
with
\[ \Delta_2^{\mu\nu\rho\sigma} = i\hbar \frac{x^2}{R(x)} \frac{1}{(N-3)!} \epsilon^{\mu\nu\rho\sigma\lambda_2\cdots\lambda_{N-3}} x_{\lambda_2} g^{\lambda_2\lambda_3} U_{\lambda_3\lambda_4\cdots\lambda_{N-3}} \] (169)
satisfy the closed Lie algebra for the three generators of O(3) or O(2,1) that leave the subspace \( x^U \) invariant. We also showed that by appropriate choice of \( U \) it is possible to recover a given four-dimensional solution in any number of dimensions. In particular, the
O(3)-invariant solution recovers the nonrelativistic case for any $N$. Thus, the solution can be interpreted as a generalization of the Dirac monopole to $N > 4$. The field strength in the O(2,1)-invariant solution is associated with a potential of the type

$$V(x) \sim (-t^2 + x^2)^{-1/2}$$  \hspace{1cm} (170)

which may be seen as a relativistic generalization of the nonrelativistic Coulomb potential. A solution to the relativistic bound state problem for the scalar hydrogen atom was found in the context of the Horwitz-Piron formalism, using a potential of this form. It is noteworthy, that this form of potential does not generally follow from the wave equation in $N$-dimensions. The interpretation of the generalized Dirac monopole and its relationship to the relativistic generalization of the bound state problem is discussed in a second paper.

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