ERROR ANALYSIS OF DEEP RITZ METHODS FOR ELLIPTIC EQUATIONS

YULING JIAO∗, YANMING LAI †, YISU LO‡, YANG WANG §, AND YUNFEI YANG ¶

Abstract. Using deep neural networks to solve PDEs has attracted a lot of attentions recently. However, why the deep learning method works is falling far behind its empirical success. In this paper, we provide a rigorous numerical analysis on deep Ritz method (DRM) [43] for second order elliptic equations with Dirichlet, Neumann and Robin boundary condition, respectively. We establish the first nonasymptotic convergence rate in $H^1$ norm for DRM using deep networks with smooth activation functions including logistic and hyperbolic tangent functions. Our results show how to set the hyper-parameter of depth and width to achieve the desired convergence rate in terms of number of training samples.

Key words. DRM, Neural Networks, Approximation error, Rademacher complexity, Chaining method, Pseudo dimension, Covering number.

AMS subject classifications. 65N99

1. Introduction. Partial differential equations (PDEs) are one of the fundamental mathematical models in studying a variety of phenomena arising in science and engineering. There have been established many conventional numerical methods successfully for solving PDEs in the case of low dimension ($d \leq 3$), particularly the finite element method [6, 7, 33, 39, 22]. However, one will encounter some difficulties in both of theoretical analysis and numerical implementation when extending conventional numerical schemes to high-dimensional PDEs. The classic analysis of convergence, stability and any other properties will be trapped into troublesome situation due to the complex construction of finite element space [7, 6]. Moreover, in the term of practical computation, the scale of the discrete problem will increase exponentially with respect to the dimension.

Motivated by the well-known fact that deep learning method for high-dimensional data analysis has been achieved great successful applications in discriminative, generative and reinforcement learning [18, 14, 37], solving high dimensional PDEs with deep neural networks becomes an extremely potential approach and has attracted much attentions [3, 38, 27, 34, 43, 45, 5, 17]. Roughly speaking, these works can be divided into three categories. The first category is using deep neural network to improve classical numerical methods, see for example [40, 42, 21, 15]. In the second category, the neural operator is introduced to learn mappings between infinite-dimensional spaces with neural networks [24, 2, 25]. For the last category, one utilizes deep neural networks to approximate the solutions of PDEs directly including physics-informed neural networks (PINNs) [34], deep Ritz method (DRM) [43] and deep Galerkin method (DGM) [45]. PINNs is based on residual minimization for solving PDEs [3, 38, 27, 34]. Proceed from
the variational form, [43, 45, 44] propose neural-network based methods related to classical Ritz and Galerkin method. In [45], weak adversarial networks (WAN) are proposed inspired by Galerkin method. Based on Ritz method, [43] proposes the deep Ritz method (DRM) to solve variational problems corresponding to a class of PDEs.

1.1. Related works and contributions. The idea of using neural networks to solve PDEs goes back to 1990’s [23, 8]. Although there are great empirical achievements in recent several years, a challenging and interesting question is to provide a rigorous error analysis such as finite element method. Several recent efforts have been devoted to making processes along this line, see for example [11, 28, 30, 32, 26, 20, 36, 41, 10]. In [28], least squares minimization method with two-layer neural networks is studied, the optimization error under the assumption of over-parametrization and generalization error without the over-parametrization assumption are analyzed. In [26, 44, 19], the generalization error bounds of two-layer neural networks are derived via assuming that the exact solutions lie in spectral Barron space. Although the studies in [26, 44] can overcome the curse of dimensionality, it should be pointed out that it is difficult to generalize these results to deep neural networks or to the situation where the underlying solutions are living in general Sobolev spaces.

Two important questions have not been addressed in the abovementioned related study are those: Can we provide a convergence result of DRM only requiring the target solution living in $H^2$? How to determine the depth and width to achieve the desired convergence rate? In this paper, we give a firm answers on these questions by providing an error analysis of using DRM with sigmoidal deep neural networks to solve second order elliptic equations with Drichlet, Neumann and Robin boundary condition, respectively.

Let $u_{\phi_A}$ be the solution of a random solver for DRM (i.e., $u_{\phi_A}$ is the solution of (3.6)) and use the notation $N_\rho(D, n_D, B_\theta)$ to refer to the collection of functions implemented by a $\rho$–neural network with depth $D$, total number of nonzero weights $n_D$ and each weight being bounded by $B_\theta$. Set $\rho = \frac{1}{1+e^{-x}}$ or $\frac{e^x-e^{-x}}{e^x+e^{-x}}$. Our main contributions are as follows:

- Let $u_{R}$ be the weak solution of Robin problem (3.1)(3.2c). For any $\epsilon > 0$ and $\mu \in (0, 1)$, set the parameterized function class

$$\mathcal{P} = N_\rho \left( C \log(d+1), C(d, \beta)\epsilon^{-d/(1-\mu)}, C(d, \beta)\epsilon^{-(9d+8)/(2-2\mu)} \right)$$

and number of samples

$$N = M = C(d, \Omega, \text{coe}, \alpha, \beta)\epsilon^{-Cd\log(d+1)/(1-\mu)},$$

if the optimization error of $u_{\phi_A}$ is $\mathcal{E}_{opt} \leq \epsilon$, then

$$\mathbb{E}_{(X_i)_{i=1}^N, (Y_j)_{j=1}^M} \|u_{\phi_A} - u_R\|_{H^1(\Omega)} \leq C(\Omega, \text{coe}, \alpha)\epsilon.$$

- Let $u_{D}$ be the weak solution of Dirichlet problem (3.1)(3.2a). Set $\alpha = 1, g = 0$. For any $\epsilon > 0$, let $\beta = C(\text{coe})\epsilon$ as the penalty parameter, set the parameterized function
\[ P = \mathcal{N}_p \left( C \log(d+1), C(d) \epsilon^{-5d/2(1-\mu)}, C(d) \epsilon^{-(45d+40)/(4-4\mu)} \right) \]

and number of samples

\[ N = M = C(d, \Omega, \text{coe}) \epsilon^{-C(d+1)/(1-\mu)}, \]

if the optimization error \( E_{\text{opt}} \leq \epsilon \), then

\[ E \left\{ X_i \right\}_{i=1}^N, \left\{ Y_j \right\}_{j=1}^M \parallel u_{\phi,A} - u_D \parallel_{H^1(\Omega)} \leq C(\Omega, \text{coe}) \epsilon. \]

We summarize the related works and our results in the following table 1.1.

| Paper | Depth and activation functions | Equation(s) | Regularity Condition |
|-------|-------------------------------|-------------|---------------------|
| [28]  | \( D = 2 \) ReLU\(^3\)       | Second order differential equation | \( u^* \in \text{Barron class} \) |
| [26]  | \( D = 2 \) Softplus         | Poisson equation and Schrödinger equation | \( u^* \in \text{Barron class} \) |
| [19]  | \( D = 2 \) ReLU\(^k\)       | 2\(m\)-th order differential equation | \( u^* \in \text{Barron class} \) |
| [9]   | \( D = O(\log d) \) ReLU\(^2\) | Second order elliptic equation | \( u^* \in C^2 \) |
| This paper | \( D = O(\log d) \) Logistic and Hyperbolic tangent | Second order elliptic equation | \( u^* \in H^2 \) |

The rest of the paper are organized as follows. In Section 2, we give some preliminaries. In Section 3, we present the DRM method and the error decomposition results for analysis of DRM. In Section 4 and 5, we give detail analysis on the approximation error and statistical error. In Section 6, we present our main results. We give conclusion and short discussion in Section 7.

2. Neural Network. Due to its strong expressivity, neural network function class plays an important role in machine learning. A variety of neural networks are chosen as parameter function classes in the training process. We now introduce some notation related to neural network which will simplify our later discussion. Let \( D \in \mathbb{N}^+ \). A function \( f : \mathbb{R}^d \rightarrow \mathbb{R}^{n_D} \) implemented by a neural network is defined by

\[
\begin{align*}
    & f_0(x) = x, \\
    & f_\ell(x) = \rho(A_\ell f_{\ell-1} + b_\ell) \quad \text{for } \ell = 1, \ldots, D - 1, \\
    & f := f_D(x) = A_D f_{D-1} + b_D, \quad \text{(2.1)}
\end{align*}
\]
where $A_\ell = \left(a_{ij}^{(\ell)}\right) \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$ and $b_\ell = \left(b_i^{(\ell)}\right) \in \mathbb{R}^{n_\ell}$. $\rho$ is called the activation function and acts componentwise. $D$ is called the depth of the network and $W := \max\{n_\ell : \ell = 1, \ldots, D\}$ is called the width of the network. $\phi = \{A_\ell, b_\ell\}_\ell$ are called the weight parameters. For convenience, we denote $n_i$, $i = 1, \ldots, D$, as the number of nonzero weights on the first $i$ layers in the representation (2.1). Clearly $n_D$ is the total number of nonzero weights. Sometimes we denote a function implemented by a neural network as $f_\rho$ for short. We use the notation $N_\rho(D, n_D, B_\theta)$ to refer to the collection of functions implemented by a $\rho$-neural network with depth $D$, total number of nonzero weights $n_D$ and each weight being bounded by $B_\theta$.

### 3. Deep Ritz Method and Error Decomposition

Let $\Omega$ be a convex bounded open set in $\mathbb{R}^d$ and assume that $\partial \Omega \in C^\infty$. Without loss of generality we assume that $\Omega \subset [0,1]^d$. We consider the following second order elliptic equation:

$$-\Delta u + wu = f \text{ in } \Omega \quad (3.1)$$

with three kinds of boundary condition:

$$u = 0 \text{ on } \partial \Omega \quad (3.2a)$$

$$\frac{\partial u}{\partial n} = g \text{ on } \partial \Omega \quad (3.2b)$$

$$\alpha u + \beta \frac{\partial u}{\partial n} = g \text{ on } \partial \Omega, \quad \alpha, \beta \in \mathbb{R}, \beta \neq 0 \quad (3.2c)$$

which are called Dirichlet, Neumann and Robin boundary condition, respectively. Note that for Dirichlet problem, we only consider the homogeneous boundary condition here since the inhomogeneous case can be turned into homogeneous case by translation. We also remark that Neumann condition (3.2b) is covered by Robin condition (3.2c). Hence in the following we only consider Dirichlet problem and Robin problem.

We make the following assumption on the known terms in equation:

(A) $f \in L^\infty(\Omega)$, $g \in H^{1/2}(\Omega)$, $w \in L^\infty(\Omega)$, $w \geq c_w$ where $c_w$ is some positive constant. In the following we abbreviate $C\left(\|f\|_{L^\infty(\Omega)}, \|g\|_{H^{1/2}(\Omega)}, \|w\|_{L^\infty(\Omega)}, c_w\right)$, constants depending on the known terms in equation, as $C(\text{coe})$ for simplicity.

For problem (3.1)(3.2a), the variational problems is to find $u \in H^1_0(\Omega)$ such that

$$\langle \nabla u, \nabla v \rangle + (wu, v) = (f, v), \quad \forall v \in H^1_0(\Omega). \quad (3.3a)$$

The corresponding minimization problem is

$$\min_{u \in H^1_0(\Omega)} \frac{1}{2} \int_\Omega \left(\|\nabla u\|^2 + wu^2 - 2fu\right) \, dx. \quad (3.3b)$$

**Lemma 3.1.** Let (A) holds. Let $u_D$ be the solution of problem (3.3a)(also (3.3b)). Then $u_D \in H^2(\Omega)$. 
Proof. See [12]. \[ \Box \]

For problem (3.1)(3.2c), the variational problem is to find \( u \in H^1(\Omega) \) such that

\[
(\nabla u, \nabla v) + (wu, v) + \frac{\alpha}{\beta}(T_0 u, T_0 v)|_{\partial \Omega} = (f, v) + \frac{1}{\beta}(g, T_0 v)|_{\partial \Omega}, \quad \forall v \in H^1(\Omega)
\]  
(3.4a)

where \( T_0 \) is zero order trace operator. The corresponding minimization problem is

\[
\min_{u \in H^1(\Omega)} \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} wu^2 - fu \right) dx + \frac{1}{\beta} \int_{\partial \Omega} \left( \frac{\alpha}{2}(T_0 u)^2 - gT_0 u \right) ds.
\]  
(3.4b)

**Lemma 3.2.** Let (A) holds. Let \( u_R \) be the solution of problem (3.4a)(also (3.4b)). Then \( u_R \in H^2(\Omega) \) and \( \|u_R\|_{H^2(\Omega)} \leq \frac{C(\text{coe})}{\beta} \) for any \( \beta > 0 \).

**Proof.** See [12]. \[ \Box \]

Intuitively, when \( \alpha = 1, g = 0 \) and \( \beta \to 0 \), we expect that the solution of Robin problem converges to the solution of Dirichlet problem. Hence we only need to consider the Robin problem since the Dirichlet problem can be handled through a limit process. Define \( L \) as a functional on \( H^1(\Omega) \):

\[
L(u) := \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} wu^2 - fu \right) dx + \frac{1}{\beta} \int_{\partial \Omega} \left( \frac{\alpha}{2}(T_0 u)^2 - gT_0 u \right) ds.
\]

The next lemma verify the assertion.

**Lemma 3.3.** Let (A) holds. Let \( \alpha = 1, g = 0 \). Let \( u_D \) be the solution of problem (3.3a)(also (3.3b)) and \( u_R \) the solution of problem (3.4a)(also (3.4b)). There holds

\[
\|u_R - u_D\|_{H^1(\Omega)} \leq C(\text{coe})\beta.
\]

**Proof.** We first have

\[
\int_{\Omega} \nabla u_D \cdot \nabla v dx - \int_{\partial \Omega} T_1 u_D v ds + \int_{\Omega} w u_D v dx = \int_{\Omega} f v dx, \quad \forall v \in H^1(\Omega).
\]

with \( T_1 \) being first order trace operator. Hence for any \( u \in H^1(\Omega) \),

\[
L(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} wu^2 - fu \right) dx + \frac{1}{2\beta} \int_{\partial \Omega} (T_0 u)^2 ds
\]

\[
= \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} wu^2 \right) dx + \frac{1}{2\beta} \int_{\partial \Omega} (T_0 u)^2 ds - \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial \Omega} T_1 u_D v ds - \int_{\Omega} w u D v dx,
\]

\[
= \int_{\Omega} \left( \frac{1}{2} |\nabla u - \nabla u_D|^2 + \frac{1}{2} w(u - u_D)^2 \right) dx + \frac{1}{2\beta} \int_{\partial \Omega} (T_0 u + \beta T_1 u_D)^2 ds
\]

\[
- \int_{\Omega} \left( \frac{1}{2} |\nabla u_D|^2 + \frac{1}{2} w u_D^2 \right) dx - \frac{\beta}{2} \int_{\partial \Omega} (T_1 u_D)^2 ds.
\]  
(3.5)

Define

\[
R_\beta(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u - \nabla u_D|^2 + \frac{1}{2} w(u - u_D)^2 \right) dx + \frac{1}{2\beta} \int_{\partial \Omega} (T_0 u + \beta T_1 u_D)^2 ds.
\]
Since \( u_R \) is the minimizer of \( \mathcal{L} \), from (3.5) we conclude that it is also the minimizer of \( \mathcal{R} \).

Note \( u_D \in H^2(\Omega) \)(Lemma 3.1), by trace theorem we know \( T_1 u_D \in H^{1/2}(\partial \Omega) \) and hence there exists \( \phi \in H^1(\Omega) \) such that \( T_0 \phi = -T_1 u_D \). Set \( \bar{u} = \beta \phi + u_D \), then

\[
C(\text{coe}) \| u_R - u_D \|^2_{H^1(\Omega)} \leq \mathcal{R}(u_R) \leq \mathcal{R}(\bar{u}) = \beta^2 \int_{\Omega} \left( \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} w \phi^2 \right) \, dx = C(u_D, \text{coe}) \beta^2.
\]

Note that \( \mathcal{L} \) can be equivalently written as

\[
\mathcal{L}(u) = |\Omega| \mathbb{E}_{X \sim U(\Omega)} \left( \frac{1}{2} |\nabla u(X)|^2 + \frac{1}{2} w(X)u^2(X) - f(X)u(X) \right)
+ \frac{|\partial \Omega|}{\beta} \mathbb{E}_{Y \sim U(\partial \Omega)} \left( \frac{\alpha}{2}(T_0 u)^2(Y) - g(Y)T_0 u(Y) \right)
\]

where \( U(\Omega) \) and \( U(\partial \Omega) \) are uniform distribution on \( \Omega \) and \( \partial \Omega \), respectively. We then introduce a discrete version of \( \mathcal{L} \):

\[
\hat{\mathcal{L}}(u) := \frac{|\Omega|}{N} \sum_{i=1}^{N} \left( \frac{1}{2} |\nabla u(X_i)|^2 + \frac{1}{2} w(X_i)u^2(X_i) - f(X_i)u(X_i) \right)
+ \frac{|\partial \Omega|}{\beta M} \sum_{j=1}^{M} \left( \frac{\alpha}{2}(T_0 u)^2(Y_j) - g(Y_j)T_0 u(Y_j) \right)
\]

where \( \{X_i\}_{i=1}^{N} \) and \( \{Y_j\}_{j=1}^{M} \) are i.i.d. random variables according to \( U(\Omega) \) and \( U(\partial \Omega) \) respectively. We now consider a minimization problem with respect to \( \hat{\mathcal{L}} \):

\[
\min_{u_\phi \in \mathcal{P}} \hat{\mathcal{L}}(u_\phi) \tag{3.6}
\]

where \( \mathcal{P} \) refers to the parameterized function class. We denote by \( \hat{u}_\phi \) the solution of problem (3.6). Finally, we call a (random) solver \( \mathcal{A} \), say SGD, to minimize \( \hat{\mathcal{L}} \) and denote the output of \( \mathcal{A} \), say \( u_{\phi_\mathcal{A}} \), as the final solution.

The following error decomposition enables us to apply different methods to deal with different kinds of error.

**Proposition 3.1.** Let (A) holds. Assume that \( \mathcal{P} \subset H^1(\Omega) \). Let \( u_R \) and \( u_D \) be the solution of problem (3.4a) (also (3.4b)) and (3.3a) (also (3.3b)), respectively. Let \( u_{\phi_\mathcal{A}} \) be the solution of problem (3.6) generated by a random solver.

(1)

\[
\| u_{\phi_\mathcal{A}} - u_R \|^2_{H^1(\Omega)} \leq C(\text{coe}) [\varepsilon_{\text{app}} + \varepsilon_{\text{sta}} + \varepsilon_{\text{opt}}]^{1/2}.
\]

where

\[
\varepsilon_{\text{app}} = \frac{1}{\beta} C(\Omega, \text{coe}, \alpha) \inf_{u \in \mathcal{P}} \| \bar{u} - u_R \|^2_{H^1(\Omega)}
\]

\[
\varepsilon_{\text{sta}} = \sup_{u \in \mathcal{P}} \left[ \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right] + \sup_{u \in \mathcal{P}} \left[ \hat{\mathcal{L}}(u) - \mathcal{L}(u) \right]
\]

\[
\varepsilon_{\text{opt}} = \left[ \frac{\alpha}{2} \mathbb{E}_{Y_0 \sim U(\partial \Omega)} (T_0 u)^2(Y_0) - g(Y_0)T_0 u(Y_0) \right]
\]
\[ \mathcal{E}_{opt} = \hat{\mathcal{E}}(u_{\phi, \alpha}) - \hat{\mathcal{E}}(\bar{u}_\phi). \]

(2) Set \( \alpha = 1, g = 0. \)

\[ \|u_{\phi, \alpha} - u_D\|_{H^1(\Omega)} \leq C(\text{coe}) \left[ \mathcal{E}_{app} + \mathcal{E}_{sta} + \mathcal{E}_{opt} + \|u_R - u_D\|_{H^1(\Omega)}^2 \right]^{1/2}. \]

**Proof.** We only prove (1) since (2) is a direct result from (1) and triangle inequality. For any \( u \in \mathcal{P}, \) set \( v = u - u_R, \) then

\[ \mathcal{L}(u) = \mathcal{L}(u_R + v) \]

\[ = \frac{1}{2} \langle \nabla(u_R + v), \nabla(u_R + v) \rangle_{L^2(\Omega)} + \frac{1}{2} \|u_R + v, u_R + v\|_{L^2(\Omega; w)} - \langle u_R + v, f \rangle_{L^2(\Omega)} \]

\[ + \frac{\alpha}{2\beta} \langle T_0 u_R + T_0 v, T_0 u_R + T_0 v \rangle_{L^2(\partial \Omega)} - \frac{1}{\beta} \langle T_0 u_R + T_0 v, g \rangle_{L^2(\partial \Omega)} \]

\[ = \frac{1}{2} \langle \nabla u_R, \nabla u_R \rangle_{L^2(\Omega)} + \frac{1}{2} \langle u_R, u_R \rangle_{L^2(\Omega; w)} - \langle u_R, f \rangle_{L^2(\Omega)} + \frac{\alpha}{2\beta} \langle T_0 u_R, T_0 v \rangle_{L^2(\partial \Omega)} \]

\[ - \frac{1}{\beta} \langle T_0 u_R, g \rangle_{L^2(\partial \Omega)} + \frac{1}{2} \langle \nabla v, \nabla v \rangle_{L^2(\Omega)} + \frac{1}{2} \|v, v\|_{L^2(\Omega; w)} + \frac{\alpha}{2\beta} \langle T_0 v, T_0 v \rangle_{L^2(\partial \Omega)} \]

\[ = \mathcal{L}(u_R) + \frac{1}{2} \langle \nabla v, \nabla v \rangle_{L^2(\Omega)} + \frac{1}{2} \|v, v\|_{L^2(\Omega; w)} + \frac{\alpha}{2\beta} \langle T_0 v, T_0 v \rangle_{L^2(\partial \Omega)}, \]

where the last equality is due to the fact that \( u_R \) is the solution of equation (3.4a). Hence

\[ C(\text{coe}) \|v\|^2_{H^1(\Omega)} \leq \mathcal{L}(u) - \mathcal{L}(u_R) = \frac{1}{2} \langle \nabla v, \nabla v \rangle_{L^2(\Omega)} + \frac{1}{2} \|v, v\|_{L^2(\Omega; w)} + \frac{\alpha}{2\beta} \langle T_0 v, T_0 v \rangle_{L^2(\partial \Omega)} \]

\[ \leq \frac{1}{\beta} C(\Omega, \text{coe}, \alpha) \|v\|^2_{H^1(\Omega)}, \]

here we apply trace inequality

\[ \|T_0 v\|_{L^2(\partial \Omega)} \leq C(\Omega) \|v\|_{H^1(\Omega)}. \]

See more details in [1]. In other words, we obtain

\[ C(\text{coe}) \|u - u_R\|^2_{H^1(\Omega)} \leq \mathcal{L}(u) - \mathcal{L}(u_R) \leq \frac{1}{\beta} C(\Omega, \text{coe}, \alpha) \|u - u_R\|^2_{H^1(\Omega)}. \]

(3.7)

Now, let \( \bar{u} \) be any element in \( \mathcal{P}, \) we have

\[ \mathcal{L}(u_{\phi, \alpha}) - \mathcal{L}(u_R) \]

\[ = \mathcal{L}(u_{\phi, \alpha}) - \hat{\mathcal{L}}(u_{\phi, \alpha}) + \hat{\mathcal{L}}(u_{\phi, \alpha}) - \hat{\mathcal{L}}(\bar{u}_\phi) + \hat{\mathcal{L}}(\bar{u}_\phi) - \hat{\mathcal{L}}(\bar{u}) + \hat{\mathcal{L}}(\bar{u}) - \mathcal{L}(\bar{u}) + \mathcal{L}(\bar{u}) - \mathcal{L}(u_R) \]

\[ \leq \sup_{u \in \mathcal{P}} \left[ \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right] + \left[ \hat{\mathcal{L}}(u_{\phi, \alpha}) - \hat{\mathcal{L}}(\bar{u}_\phi) \right] + \sup_{u \in \mathcal{P}} \left[ \hat{\mathcal{L}}(u) - \mathcal{L}(u) \right] + \frac{1}{\beta} C(\Omega, \text{coe}, \alpha) \|\bar{u} - u_R\|^2_{H^1(\Omega)}, \]

where the last step is due to inequality (3.7) and the fact that \( \hat{\mathcal{L}}(\bar{u}_\phi) - \hat{\mathcal{L}}(\bar{u}) \leq 0. \) Since \( \bar{u} \) can
Moreover, the weights in the neural network are bounded in absolute value by

\[ \|f\|_{W^{s,p}} \leq \frac{1}{\beta} C(\Omega, \text{cof}, \alpha) \|\hat{u} - u\|_{H^1(\Omega)} + \sup_{u \in \mathcal{P}} \|\mathcal{L}(u) - \hat{\mathcal{L}}(u)\| + \sup_{u \in \mathcal{P}} \|\hat{\mathcal{L}}(u) - \mathcal{L}(u)\| + \|\hat{\mathcal{L}}(u) - \hat{\mathcal{L}}(\hat{u})\|. \]  

(3.8)

Combining (3.7) and (3.8) yields the result. \( \square \)

4. Approximation Error. For neural network approximation in Sobolev spaces, [16] is a comprehensive study concerning a variety of activation functions, including ReLU, sigmoidal type functions, etc. The key idea in [16] to build the upper bound in Sobolev spaces is to construct an approximate partition of unity.

Denote \( F_{s,p,d} := \{ f \in W_{s,p}^d([0,1]^d) : \|f\|_{W_{s,p}^d([0,1]^d)} \leq 1 \} \).

**Theorem 4.1** (Proposition 4.8, [16]). Let \( p \geq 1, s, k, d \in \mathbb{N}^+, s \geq k + 1 \). Let \( \rho \) be logistic function \( \frac{e^x}{1 + e^x} \) or tanh function \( \frac{e^x - e^{-x}}{e^x + e^{-x}} \). For any \( \epsilon > 0 \) and \( f \in F_{s,p,d} \), there exists a neural network \( f_\rho \) with depth \( C \log(d + s) \) and \( C(d, s, p, k)\epsilon^{-d/(s-k-m\mu)} \) non-zero weights such that

\[ \|f - f_\rho\|_{W^{s,p}([0,1]^d)} \leq \epsilon. \]

Moreover, the weights in the neural network are bounded in absolute value by

\[ C(d, s, p, k)\epsilon^{-2\frac{2(d/p + d + s + k + \mu)h}{d/p - k - \mu}} \]

where \( \mu \) is an arbitrarily small positive number.

**Remark 4.1.** The bounds in the theorem can be found in the proof of [16, Proposition 4.8], except that they did not explicitly give the bound on the depth. In their proof, they partition \([0,1]^d\) into small patches, approximate \( f \) by a sum of localized polynomial \( \sum m \phi_m p_m \), and approximately implement \( \sum m \phi_m p_m \) by a neural network, where the bump functions \( \{\phi_m\} \) form an approximately partition of unity and \( p_m = \sum_{|c| \leq s} c_{f,m,a} x^a \) are the averaged Taylor polynomials. As shown in [16], \( \phi_m \) can be approximated by the products of the \( d \)-dimensional output of a neural network with constant layers. And the identity map \( I(x) = x \) and the product function \( \chi(a,b) = ab \) can also be approximated by neural networks with constant layers. In order to approximate \( \phi_m x^a \), we need to implement \( d + s - 1 \) products. Hence, the required depth can be bounded by \( C \log(d + s) \).

Since the region \([0,1]^d\) is larger than the region \( \Omega \) we consider (recall we assume without loss of generality that \( \Omega \subset [0,1]^d \) at the beginning), we need the following extension result.

**Lemma 4.2.** Let \( k \in \mathbb{N}^+, 1 \leq p < \infty \). There exists a linear operator \( E \) from \( W^{k,p}_0(\Omega) \) to \( W^{k,p}_0([0,1]^d) \) and \( Eu = u \) in \( \Omega \).

**Proof.** See Theorem 7.25 in [13]. \( \square \)

From Lemma 3.2 we know that our target function \( u_R \in H^2(\Omega) \). Hence we are able to obtain an approximation result in \( H^1 \)-norm.

**Corollary 4.3.** Let \( \rho \) be logistic function \( \frac{e^x}{1 + e^x} \) or tanh function \( \frac{e^x - e^{-x}}{e^x + e^{-x}} \). For any \( \epsilon > 0 \) and \( f \in H^2(\Omega) \) with \( \|f\|_{H^2(\Omega)} \leq 1 \), there exists a neural network \( f_\rho \) with depth \( C \log(d + 1) \) and
\[ C(d)\varepsilon^{-d/(1-\mu)} \text{ non-zero weights such that} \]
\[ \|f - f_\rho\|_{H^1(\Omega)} \leq \varepsilon. \]
Moreover, the weights in the neural network are bounded by \( C(d)\varepsilon^{-(9d+8)/(2-2\mu)} \), where \( \mu \) is an arbitrarily small positive number.

Proof. Set \( k = 1, s = 2, p = 2 \) in Theorem 4.1 and use the fact \( \|f - f_\rho\|_{H^2(\Omega)} \leq \|\widetilde{E}f - f_\rho\|_{H^2([0,1]d)} \), where \( \widetilde{E} \) is the extension operator in Lemma 4.2. \( \blacksquare \)

5. Statistical Error. In this section we investigate statistical error \( \sup_{u \in \mathcal{P}} \pm \left[ \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right] \).

Lemma 5.1.

\[ \mathbb{E}_{\{X_i\}_{i=1}^N,\{Y_j\}_{j=1}^M} \sup_{u \in \mathcal{P}} \pm \left[ \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right] \leq \sum_{k=1}^5 \mathbb{E}_{\{X_i\}_{i=1}^N,\{Y_j\}_{j=1}^M} \sup_{u \in \mathcal{P}} \pm \left[ \mathcal{L}_k(u) - \hat{\mathcal{L}}_k(u) \right]. \]

where

\[ \mathcal{L}_1(u) = \frac{|\Omega|}{2} \mathbb{E}_{X \sim \mathcal{U}(\Omega)} |\nabla u(X)|^2, \]

\[ \mathcal{L}_2(u) = \frac{|\Omega|}{2} \mathbb{E}_{X \sim \mathcal{U}(\Omega)} w(X)u^2(X), \]

\[ \mathcal{L}_3(u) = -|\Omega| \mathbb{E}_{X \sim \mathcal{U}(\Omega)} f(X)u(X), \]

\[ \mathcal{L}_4(u) = \frac{\alpha |\Omega|}{2\beta} \mathbb{E}_{Y \sim \mathcal{U}(\partial\Omega)} (Tu)^2(Y), \]

\[ \mathcal{L}_5(u) = -\frac{\beta |\Omega|}{\beta} \mathbb{E}_{Y \sim \mathcal{U}(\partial\Omega)} g(Y)Tu(Y). \]

and \( \hat{\mathcal{L}}_k(u) \) is the discrete version of \( \mathcal{L}_k(u) \), for example,

\[ \hat{\mathcal{L}}_1(u) = \frac{|\Omega|}{2N} \sum_{i=1}^N |\nabla u(X_i)|^2. \]

Proof. Direct result from triangle inequality. \( \blacksquare \)

By the technique of symmetrization, we can bound the difference between continuous loss \( \mathcal{L} \) and empirical loss \( \hat{\mathcal{L}} \) by Rademacher complexity.

Definition 5.1. The Rademacher complexity of a set \( A \subseteq \mathbb{R}^N \) is defined as

\[ \mathfrak{R}(A) = \mathbb{E}_{\{\sigma_k\}_{k=1}^N} \left[ \sup_{a \in A} \frac{1}{N} \sum_{k=1}^N \sigma_k a_k \right], \]

where, \( \{\sigma_k\}_{k=1}^N \) are \( N \) i.i.d Rademacher variables with \( \mathbb{P}(\sigma_k = 1) = \mathbb{P}(\sigma_k = -1) = \frac{1}{2} \). The Rademacher complexity of function class \( \mathcal{F} \) associate with random sample \( \{X_k\}_{k=1}^N \) is defined as

\[ \mathfrak{R}(\mathcal{F}) = \mathbb{E}_{\{X_k,\sigma_k\}_{k=1}^N} \left[ \sup_{u \in \mathcal{F}} \frac{1}{N} \sum_{k=1}^N \sigma_k u(X_k) \right]. \]

For Rademacher complexity, we have following structural result.

Lemma 5.2. Assume that \( w : \Omega \to \mathbb{R} \) and \( |w(x)| \leq B \) for all \( x \in \Omega \), then for any function
Lemma 5.3. Now we bound the difference between continuous loss and empirical loss in terms of Rademacher complexity.

Proof.

\[
\mathfrak{R}_N(w \cdot F) \leq \mathcal{B}\mathfrak{R}_N(F),
\]

where \( w \cdot F := \{ \tilde{w} : \tilde{w}(x) = w(x)u(x), u \in F \} \).

Now we bound the difference between continuous loss and empirical loss in terms of Rademacher complexity.
where

\[
\begin{align*}
\mathcal{F}_1 &= \{[\nabla u]^2 : u \in \mathcal{P}\}, \\
\mathcal{F}_2 &= \{u^2 : u \in \mathcal{P}\}, \\
\mathcal{F}_3 &= \{u : u \in \mathcal{P}\}, \\
\mathcal{F}_4 &= \{u^2|_{\Omega} : u \in \mathcal{P}\}, \\
\mathcal{F}_5 &= \{u|_{\Omega} : u \in \mathcal{P}\}.
\end{align*}
\]

Proof. We only present the proof with respect to \(L_2\) since other inequalities can be shown similarly. We take \(\{\tilde{X}_k\}_{k=1}^N\) as an independent copy of \(\{X_k\}_{k=1}^N\), then

\[
L_2(u) - \hat{L}_2(u) = \frac{|\Omega|}{2N} \left[ \mathbb{E}_{X \sim U(\Omega)} w(X)u^2(X) - \frac{1}{N} \sum_{k=1}^N w(X_k)u^2(X_k) \right]
\]

\[
= \frac{|\Omega|}{2N} \mathbb{E}_{X \sim U(\Omega)} \left( \sum_{k=1}^N \left[ w(\tilde{X}_k)u^2(\tilde{X}_k) - w(X_k)u^2(X_k) \right] \right).
\]

Hence

\[
\mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{P}} \left| L_2(u) - \hat{L}_2(u) \right|
\]

\[
\leq \frac{|\Omega|}{2N} \mathbb{E}_{\{X_k, \tilde{X}_k\}_{k=1}^N} \sup_{u \in \mathcal{P}} \sum_{k=1}^N \left[ w(\tilde{X}_k)u^2(\tilde{X}_k) - w(X_k)u^2(X_k) \right]
\]

\[
= \frac{|\Omega|}{2N} \mathbb{E}_{\{X_k, \tilde{X}_k, \sigma_k\}_{k=1}^N} \sup_{u \in \mathcal{P}} \sum_{k=1}^N \sigma_k \left[ w(\tilde{X}_k)u^2(\tilde{X}_k) - w(X_k)u^2(X_k) \right]
\]

\[
\leq \frac{|\Omega|}{2N} \mathbb{E}_{\{X_k, \tilde{X}_k, \sigma_k\}_{k=1}^N} \sup_{u \in \mathcal{P}} \sum_{k=1}^N \sigma_k w(X_k)u^2(X_k) + \frac{|\Omega|}{2N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \sup_{u \in \mathcal{P}} \sum_{k=1}^N -\sigma_k w(X_k)u^2(X_k)
\]

\[
= \frac{|\Omega|}{N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \sup_{u \in \mathcal{P}} \sum_{k=1}^N \sigma_k w(X_k)u^2(X_k)
\]

\[
= |\Omega| \mathcal{G}_N(w \cdot \mathcal{F}_2) \leq C(\Omega, \text{cov}) \mathcal{G}_N(\mathcal{F}_2),
\]

where the second step is due to the fact that the insertion of Rademacher variables doesn’t change the distribution, the fourth step is because \(\sigma_k w(\tilde{X}_k)u^2(\tilde{X}_k)\) and \(-\sigma_k w(X_k)u^2(X_k)\) have the same distribution, and we use Lemma 5.2 in the last step. \(\blacksquare\)

In order to bound Rademacher complexities, we need the concept of covering number.

**Definition 5.4.** An \(\epsilon\)-cover of a set \(T\) in a metric space \((S, \tau)\) is a subset \(T_\epsilon \subset S\) such that for each \(t \in T\), there exists a \(t_\epsilon \in T_\epsilon\) such that \(\tau(t, t_\epsilon) \leq \epsilon\). The \(\epsilon\)-covering number of \(T\), denoted as \(\mathcal{C}(\epsilon, T, \tau)\), is defined to be the minimum cardinality among all \(\epsilon\)-cover of \(T\) with respect to the metric \(\tau\).

In Euclidean space, we can establish an upper bound of covering number for a bounded set easily.
Lemma 5.5. Suppose that $T \subset \mathbb{R}^d$ and $\|t\|_2 \leq B$ for $t \in T$, then

$$C(\epsilon, T, \|\cdot\|_2) \leq \left(\frac{2B\sqrt{d}}{\epsilon}\right)^d.$$ 

Proof. Let $m = \left\lceil \frac{2B\sqrt{d}}{\epsilon} \right\rceil$ and define

$$T_c = \left\{ -B + \frac{\epsilon}{\sqrt{d}}, -B + \frac{2\epsilon}{\sqrt{d}}, \ldots, -B + \frac{m\epsilon}{\sqrt{d}} \right\}^d,$$

then for $t \in T$, there exists $t_c \in T_c$ such that

$$\|t - t_c\|_2 \leq \sqrt{\sum_{i=1}^{d} \left(\frac{\epsilon}{\sqrt{d}}\right)^2} = \epsilon.$$

Hence

$$C(\epsilon, T, \|\cdot\|_2) \leq |T_c| = m^d \leq \left(\frac{2B\sqrt{d}}{\epsilon}\right)^d.$$ 

A Lipschitz parameterization allows us to translate a cover of the function space into a cover of the parameter space. Such a property plays an essential role in our analysis of statistical error.

Lemma 5.6. Let $F$ be a parameterized class of functions: $F = \{f(x; \theta) : \theta \in \Theta\}$. Let $\|\cdot\|_\Theta$ be a norm on $\Theta$ and let $\|\cdot\|_F$ be a norm on $F$. Suppose that the mapping $\theta \mapsto f(x; \theta)$ is $L$-Lipschitz, that is,

$$\|f(x; \theta) - f(x; \tilde{\theta})\|_F \leq L \|\theta - \tilde{\theta}\|_\Theta,$$

then for any $\epsilon > 0$, $C(\epsilon, F, \|\cdot\|_F) \leq C(\epsilon/L, \Theta, \|\cdot\|_\Theta)$.

Proof. Suppose that $C(\epsilon/L, \Theta, \|\cdot\|_\Theta) = n$ and $\{\theta_i\}_{i=1}^n$ is an $\epsilon/L$-cover of $\Theta$. Then for any $\theta \in \Theta$, there exists $1 \leq i \leq n$ such that

$$\|f(x; \theta) - f(x; \theta_i)\|_F \leq L \|\theta - \theta_i\|_\Theta \leq \epsilon.$$

Hence $\{f(x; \theta_i)\}_{i=1}^n$ is an $\epsilon$-cover of $F$, implying that $C(\epsilon, F, \|\cdot\|_F) \leq n$. \qed

To find the relation between Rademacher complexity and covering number, we first need the Massart’s finite class lemma stated below.

Lemma 5.7. For any finite set $A \subset \mathbb{R}^N$ with diameter $D = \sup_{a \in A} \|a\|_2$,

$$\mathcal{R}_N(A) \leq \frac{D}{N} \sqrt{2\log |A|}.$$ 

Proof. See, for example, [35, Lemma 26.8]. \qed
Lemma 5.8. Let $\mathcal{F}$ be a function class and $\|f\|_\infty \leq B$ for any $f \in \mathcal{F}$, we have

$$\mathcal{R}_N(\mathcal{F}) \leq \inf_{0 < \delta < B/2} \left(4\delta + \frac{12}{\sqrt{N}} \int_{\delta}^{B/2} \sqrt{\log C(\epsilon, \mathcal{F}, \|\cdot\|_\infty)} d\epsilon \right).$$

Proof. We apply chaining method. Set $\epsilon_k = 2^{-k+1}B$. We denote by $\mathcal{F}_k$ such that $\mathcal{F}_k$ is an $\epsilon_k$-cover of $\mathcal{F}$ and $|\mathcal{F}_k| = C(\epsilon_k, \mathcal{F}, \|\cdot\|_\infty)$. Hence for any $u \in \mathcal{F}$, there exists $u_k \in \mathcal{F}_k$ such that $\|u - u_k\|_\infty \leq \epsilon_k$. Let $K$ be a positive integer determined later. We have

$$\mathcal{R}_N(\mathcal{F}) = \mathbb{E}_{\{\sigma_i, X_i\}_{i=1}^N} \left[ \sup_{u \in \mathcal{F}} \frac{1}{N} \sum_{i=1}^{N} \sigma_i u(X_i) \right]$$

$$= \mathbb{E}_{\{\sigma_i, X_i\}_{i=1}^N} \left[ \frac{1}{N} \sup_{u \in \mathcal{F}} \sum_{i=1}^{N} \sigma_i (u(X_i) - u_K(X_i)) \right] + \sum_{j=1}^{K-1} \mathbb{E}_{\{\sigma_i, X_i\}_{i=1}^N} \left[ \frac{1}{N} \sum_{i=1}^{N} \sigma_i (u_{j+1}(X_i) - u_j(X_i)) \right] + \mathbb{E}_{\{\sigma_i, X_i\}_{i=1}^N} \left[ \frac{1}{N} \sup_{u \in \mathcal{F}} \sum_{i=1}^{N} \sigma_i u_1(X_i) \right].$$

We can choose $\mathcal{F}_1 = \{0\}$ to eliminate the third term. For the first term,

$$\mathbb{E}_{\{\sigma_i, X_i\}_{i=1}^N} \sup_{u \in \mathcal{F}} \frac{1}{N} \left[ \sum_{i=1}^{N} \sigma_i (u(X_i) - u_K(X_i)) \right] \leq \mathbb{E}_{\{\sigma_i, X_i\}_{i=1}^N} \sup_{u \in \mathcal{F}} \frac{1}{N} \sum_{i=1}^{N} |\sigma_i| \|u - u_K\|_\infty \leq \epsilon_K.$$ 

For the second term, for any fixed samples $\{X_i\}_{i=1}^{N}$, we define

$$V_j := \{(u_{j+1}(X_1) - u_j(X_1), \ldots, u_{j+1}(X_N) - u_j(X_N)) \in \mathbb{R}^N : u \in \mathcal{F}\}.$$ 

Then, for any $v^j \in V_j$,

$$\|v^j\|^2 = \left( \sum_{i=1}^{n} |u_{j+1}(X_i) - u_j(X_i)|^2 \right)^{1/2} \leq \sqrt{n} \|u_{j+1} - u_j\|_\infty$$

$$\leq \sqrt{n} \|u_{j+1} - u\|_\infty + \sqrt{n} \|u_j - u\|_\infty = \sqrt{n} \epsilon_{j+1} + \sqrt{n} \epsilon_j = 3\sqrt{n} \epsilon_{j+1}.$$

Applying Lemma 5.7, we have

$$\sum_{j=1}^{K-1} \mathbb{E}_{\{\sigma_i\}_{i=1}^N} \left[ \sup_{v^j \in V_j} \frac{1}{N} \sum_{i=1}^{N} \sigma_i (u_{j+1}(X_i) - u_j(X_i)) \right]$$

$$= \sum_{j=1}^{K-1} \mathbb{E}_{\{\sigma_i\}_{i=1}^N} \left[ \sup_{v^j \in V_j} \frac{1}{N} \sum_{i=1}^{N} \sigma_i v^j_i \right] \leq \sum_{j=1}^{K-1} \frac{3\epsilon_{j+1}}{\sqrt{N}} \sqrt{2 \log |V_j|}.$$
By the denition of $V_j$, we know that $|V_j| \leq |F_j||F_{j+1}| \leq |F_{j+1}|^2$. Hence

$$\sum_{j=1}^{K-1} \mathbb{E}(\sigma_{j, X_j}) \leq \frac{1}{N} \sum_{i=1}^{N} \sigma_{i} (u_{j+1}(X_i) - u_j(X_i)) \leq \frac{K-1}{\sqrt{N}} \sqrt{\log |F_{j+1}|}.$$  

Now we obtain

$$\mathfrak{R}_N(F) \leq \epsilon_K + \frac{6\epsilon_{j+1}}{\sqrt{N}} \sqrt{\log |F_{j+1}|} = \epsilon_K + \frac{12}{\sqrt{N}} \sum_{j=1}^{K-1} (\epsilon_{j+1} - \epsilon_{j+2}) \sqrt{\log C(\epsilon_{j+1}, F, \| \cdot \|_{\infty})} \leq \epsilon_K + \frac{12}{\sqrt{N}} \int_{\epsilon_{K+1}}^{B/2} \sqrt{\log C(\epsilon, F, \| \cdot \|_{\infty})} d\epsilon.$$  

We conclude the lemma by choosing $K$ such that $\epsilon_{K+2} < \delta \leq \epsilon_{K+1}$ for any $0 < \delta < B/2$. 

From Lemma 5.6 we know that the kep step to bound $C(\epsilon, F_i, \| \cdot \|_{\infty})$ with $F_i$ deined in Lemma 5.3 is to compute the upper bound of Lipschitz constant of class $F_i$, which is done in Lemma 5.9-5.12.

**Lemma 5.9.** Let $\mathcal{D}, n_{\mathcal{D}}, n_i \in \mathbb{N}^+$, $n_{\mathcal{D}} = 1$, $B_\theta \geq 1$, $B_\rho \leq 1$. Set the parameterized function class $\mathcal{P} = \mathcal{N}_\rho(\mathcal{D}, n_{\mathcal{D}}, B_\theta)$. For any $f(x; \theta) \in \mathcal{P}$, $f(x; \theta)$ is $\sqrt{n_{\mathcal{D}}B_\theta^{D-1}} \left( \prod_{i=1}^{D-1} n_i \right)$-Lipschitz连续 with respect to variable $\theta$, i.e.,

$$|f(x; \theta) - f(x; \bar{\theta})| \leq \sqrt{n_{\mathcal{D}}B_\theta^{D-1}} \left( \prod_{i=1}^{D-1} n_i \right) \| \theta - \bar{\theta} \|_2, \forall x \in \Omega.$$

**Proof.** For $\ell = 2, \cdots, \mathcal{D}$ (the argument for the case of $\ell = \mathcal{D}$ is slightly different),

$$|f^{(\ell)}_q - \tilde{f}^{(\ell)}_q| = \rho \left( \sum_{j=1}^{n_{\ell-1}} a^{(\ell)}_{qj} f^{(\ell-1)}_j + b^{(\ell)}_q \right) - \rho \left( \sum_{j=1}^{n_{\ell-1}} a^{(\ell)}_{qj} \tilde{f}^{(\ell-1)}_j + \tilde{b}^{(\ell)}_q \right)$$

$$\leq L_{\rho} \sum_{j=1}^{n_{\ell-1}} \left| a^{(\ell)}_{qj} \right| \left| f^{(\ell-1)}_j - \tilde{f}^{(\ell-1)}_j \right| + L_{\rho} \sum_{j=1}^{n_{\ell-1}} \left| a^{(\ell)}_{qj} - \tilde{a}^{(\ell)}_{qj} \right| \left| f^{(\ell-1)}_j \right| + L_{\rho} \left| b^{(\ell)}_q - \tilde{b}^{(\ell)}_q \right|$$

$$\leq B_{\theta} L_{\rho} \sum_{j=1}^{n_{\ell-1}} \left| f^{(\ell-1)}_j - \tilde{f}^{(\ell-1)}_j \right| + B_{\rho} L_{\rho} \sum_{j=1}^{n_{\ell-1}} \left| a^{(\ell)}_{qj} - \tilde{a}^{(\ell)}_{qj} \right| \left| f^{(\ell-1)}_j \right| + L_{\rho} \left| b^{(\ell)}_q - \tilde{b}^{(\ell)}_q \right|$$

$$\leq B_{\theta} \sum_{j=1}^{n_{\ell-1}} \left| f^{(\ell-1)}_j - \tilde{f}^{(\ell-1)}_j \right| + \sum_{j=1}^{n_{\ell-1}} \left| a^{(\ell)}_{qj} - \tilde{a}^{(\ell)}_{qj} \right| + \left| b^{(\ell)}_q - \tilde{b}^{(\ell)}_q \right|.$$
For \( \ell = 1 \),

\[
|f_q^{(1)} - \tilde{f}_q^{(1)}| = |\rho \left( \sum_{j=1}^{n_0} a_{qj}^{(1)} x_j^{(1)} b_q^{(1)} \right) - \rho \left( \sum_{j=1}^{n_0} \tilde{a}_{qj}^{(1)} x_j^{(1)} + \tilde{b}_q^{(1)} \right)| \\
\leq \sum_{j=1}^{n_0} |a_{qj}^{(1)} - \tilde{a}_{qj}^{(1)}| + |b_q^{(1)} - \tilde{b}_q^{(1)}| = \sum_{j=1}^{n_1} |\theta_j - \tilde{\theta}_j|.
\]

For \( \ell = 2 \),

\[
|f_q^{(2)} - \tilde{f}_q^{(2)}| \leq B_\theta \sum_{j=1}^{n_1} |f_j^{(1)} - \tilde{f}_j^{(1)}| + \sum_{j=1}^{n_1} |a_{qj}^{(2)} - \tilde{a}_{qj}^{(2)}| + |b_q^{(2)} - \tilde{b}_q^{(2)}| \\
\leq B_\theta \sum_{j=1}^{n_1} \sum_{k=1}^{n_1} |\theta_{k} - \tilde{\theta}_{k}| + \sum_{j=1}^{n_1} |a_{qj}^{(2)} - \tilde{a}_{qj}^{(2)}| + |b_q^{(2)} - \tilde{b}_q^{(2)}| \\
\leq n_1 B_\theta \sum_{j=1}^{n_2} |\theta_j - \tilde{\theta}_j|.
\]

Assuming that for \( \ell \geq 2 \),

\[
|f_q^{(\ell)} - \tilde{f}_q^{(\ell)}| \leq \left( \prod_{i=1}^{n_1} \theta_i \right) B_\theta^{\ell-1} \sum_{j=1}^{n_2} |\theta_j - \tilde{\theta}_j|,
\]

we have

\[
|f_q^{(\ell+1)} - \tilde{f}_q^{(\ell+1)}| \leq B_\theta \sum_{j=1}^{n_1} |f_j^{(\ell)} - \tilde{f}_j^{(\ell)}| + \sum_{j=1}^{n_1} |a_{qj}^{(\ell+1)} - \tilde{a}_{qj}^{(\ell+1)}| + |b_q^{(\ell+1)} - \tilde{b}_q^{(\ell+1)}| \\
\leq B_\theta \sum_{j=1}^{n_1} \left( \prod_{i=1}^{n_1} \theta_i \right) B_\theta^{\ell-1} \sum_{j=1}^{n_2} |\theta_{k} - \tilde{\theta}_{k}| + \sum_{j=1}^{n_1} |a_{qj}^{(\ell+1)} - \tilde{a}_{qj}^{(\ell+1)}| + |b_q^{(\ell+1)} - \tilde{b}_q^{(\ell+1)}| \\
\leq \left( \prod_{i=1}^{n_1} \theta_i \right) B_\theta^{\ell} \sum_{j=1}^{n_2} |\theta_j - \tilde{\theta}_j|.
\]

Hence by induction and Hölder inequality we conclude that

\[
|f - \tilde{f}| \leq \left( \prod_{i=1}^{n_1} n_i \right) B_\theta^{D-1} \sum_{j=1}^{n_D} |\theta_j - \tilde{\theta}_j| \leq \sqrt{n_D} B_\theta^{D-1} \left( \prod_{i=1}^{D-1} n_i \right) \| \theta - \tilde{\theta} \|_2.
\]

**Lemma 5.10.** Let \( D, n_D, n_1, \in \mathbb{N}^+ \), \( n_D = 1 \), \( B_\theta \geq 1 \) and \( \rho \) be a function such that \( \rho' \) is bounded by \( B_\rho \). Set the parameterized function class \( \mathcal{P} = \mathcal{N}_\rho (D, n_D, B_\theta) \). Let \( p = 1, \cdots, d \). We have

\[
|\partial_{x_p} f_q^{(\ell)}| \leq \left( \prod_{i=1}^{\ell-1} n_i \right) (B_\theta B_\rho)^\ell, \quad \ell = 1, 2, \cdots, D - 1,
\]

\[
|\partial_{x_p} f| \leq \left( \prod_{i=1}^{D-1} n_i \right) B_\theta^D B_\rho^{D-1}.
\]
Proof. For $\ell = 1, 2, \cdots, D - 1,$

$$\left| \partial_{x_p} f_q^{(\ell)} \right| = \left| \sum_{j=1}^{n_{\ell-1}} a_{qj}^{(\ell)} \partial_{x_p} f_j^{(\ell-1)} \rho' \left( \sum_{j=1}^{n_{\ell-1}} a_{qj}^{(\ell)} f_j^{(\ell-1)} + b_q^{(\ell)} \right) \right| \leq B_\rho B_{\rho'} \sum_{j=1}^{n_{\ell-1}} \left| \partial_{x_p} f_j^{(\ell-1)} \right|$$

$$\leq (B_\rho B_{\rho'})^2 \sum_{k=1}^{n_{\ell-1}} \left( \sum_{j=1}^{n_{\ell-2}} \left| \partial_{x_p} f_j^{(\ell-2)} \right| = n_{\ell-1} (B_\rho B_{\rho'})^2 \sum_{j=1}^{n_{\ell-2}} \left| \partial_{x_p} f_j^{(\ell-2)} \right| \right)$$

$$\leq \cdots \leq \left( \prod_{i=2}^{\ell-1} n_i \right) (B_\rho B_{\rho'})^{\ell-1} \sum_{j=1}^{n_{\ell-1}} \left| \partial_{x_p} f_j^{(\ell-1)} \right|$$

$$\leq \left( \prod_{i=1}^{\ell-1} n_i \right) (B_\rho B_{\rho'})^{\ell-1} \sum_{j=1}^{n_{\ell-1}} B_\rho B_{\rho'} = \left( \prod_{i=1}^{\ell-1} n_i \right) (B_\rho B_{\rho'})^\ell.$$ The bound for $|\partial_{x_p} f|$ can be derived similarly. □

**Lemma 5.11.** Let $D, n_D, n_i \in \mathbb{N}^+, n_D = 1, B_\rho \geq 1$ and $\rho$ be a function such that $\rho, \rho'$ are bounded by $B_\rho, B_{\rho'} \leq 1$ and have Lipschitz constants $L_\rho, L_{\rho'} \leq 1,$ respectively. Set the parameterized function class $\mathcal{P} = \mathcal{N}_p (D, n_D, B_\rho).$ Then, for any $f(x; \theta) \in \mathcal{P}, p = 1, \cdots, d, \partial_{x_p} f(x; \theta)$ is $\sqrt{n_D(D + 1)B_\rho D^2} \left( \prod_{i=1}^{D-1} n_i \right)^2$-Lipschitz continuous with respect to variable $\theta$, i.e.,

$$\left| \partial_{x_p} f(x; \theta) - \partial_{x_p} f(x; \tilde{\theta}) \right| \leq \sqrt{n_D(D + 1)B_\rho D^2} \left( \prod_{i=1}^{D-1} n_i \right)^2 \left\| \theta - \tilde{\theta} \right\|_2, \forall x \in \Omega.$$ 

Proof. For $\ell = 1,$

$$\left| \partial_{x_p} f_q^{(1)} - \partial_{x_p} \tilde{f}_q^{(1)} \right| = \left| a_{q1}^{(1)} \rho \left( \sum_{j=1}^{n_0} a_{qj}^{(1)} x_j + b_q^{(1)} \right) - \tilde{a}_{q1}^{(1)} \rho \left( \sum_{j=1}^{n_0} \tilde{a}_{qj}^{(1)} x_j + \tilde{b}_q^{(1)} \right) \right|$$

$$\leq \left| a_{q1}^{(1)} - \tilde{a}_{q1}^{(1)} \right| \rho \left( \sum_{j=1}^{n_0} a_{qj}^{(1)} x_j + b_q^{(1)} \right) + \left| \tilde{a}_{q1}^{(1)} \right| \rho \left( \sum_{j=1}^{n_0} \tilde{a}_{qj}^{(1)} x_j + \tilde{b}_q^{(1)} \right) - \rho \left( \sum_{j=1}^{n_0} \tilde{a}_{qj}^{(1)} x_j + \tilde{b}_q^{(1)} \right)$$

$$\leq B_{\rho'} \left| a_{q1}^{(1)} - \tilde{a}_{q1}^{(1)} \right| + B_\rho L_{\rho'} \sum_{j=1}^{n_0} \left| a_{qj}^{(1)} - \tilde{a}_{qj}^{(1)} \right| + B_\rho L_{\rho'} \left| b_q^{(1)} - \tilde{b}_q^{(1)} \right| \leq 2B_\rho \sum_{k=1}^{n_1} \left| \theta_k - \tilde{\theta}_k \right|$$

For $\ell \geq 2,$ we establish the Recurrence relation:

$$\left| \partial_{x_p} f_q^{(\ell)} - \partial_{x_p} \tilde{f}_q^{(\ell)} \right|$$

$$\leq \sum_{j=1}^{n_{\ell-1}} \left| a_{qj}^{(\ell)} \right| \left| \partial_{x_p} f_j^{(\ell-1)} \right| \rho' \left( \sum_{j=1}^{n_{\ell-1}} a_{qj}^{(\ell)} f_j^{(\ell-1)} + b_q^{(\ell)} \right) - \rho' \left( \sum_{j=1}^{n_{\ell-1}} \tilde{a}_{qj}^{(\ell)} \tilde{f}_j^{(\ell-1)} + \tilde{b}_q^{(\ell)} \right)$$

$$+ \sum_{j=1}^{n_{\ell-1}} \left| a_{qj}^{(\ell)} \partial_{x_p} f_j^{(\ell-1)} - \tilde{a}_{qj}^{(\ell)} \partial_{x_p} \tilde{f}_j^{(\ell-1)} \right| \rho' \left( \sum_{j=1}^{n_{\ell-1}} \tilde{a}_{qj}^{(\ell)} \tilde{f}_j^{(\ell-1)} + \tilde{b}_q^{(\ell)} \right)$$

$$\leq \left( \prod_{i=2}^{\ell-1} n_i \right) (B_\rho B_{\rho'})^{\ell-1} \sum_{j=1}^{n_{\ell-1}} \left| \partial_{x_p} f_j^{(\ell-1)} \right| (B_\rho B_{\rho'})^\ell.$$
For \(\ell = 2\),

\[
\left| \partial_{x_p} f_q^{(2)} - \partial_{x_p} \tilde{f}_q^{(2)} \right| \leq B_0 \sum_{j=1}^{n_1} \left| \partial_{x_p} f_j^{(1)} - \partial_{x_p} \tilde{f}_j^{(1)} \right| + B_0^2 n_2 \sum_{k=1}^{n_2} \left| \theta_k - \tilde{\theta}_k \right|
\]

\[
\leq 2B_0 n_1 \sum_{k=1}^{n_1} \left| \theta_k - \tilde{\theta}_k \right| + B_0^2 n_2 \sum_{k=1}^{n_2} \left| \theta_k - \tilde{\theta}_k \right| \leq 3B_0^2 n_2 \sum_{k=1}^{n_2} \left| \theta_k - \tilde{\theta}_k \right|
\]

Assuming that for \(\ell \geq 2\),

\[
\left| \partial_{x_p} f_q^{(\ell)} - \partial_{x_p} \tilde{f}_q^{(\ell)} \right| \leq (\ell + 1) B_0^2 \left( \prod_{i=1}^{\ell-1} n_i \right) \sum_{k=1}^{n_\ell} \left| \theta_k - \tilde{\theta}_k \right|
\]

we have

\[
\left| \partial_{x_p} f_q^{(\ell+1)} - \partial_{x_p} \tilde{f}_q^{(\ell+1)} \right|
\]

\[
\leq B_0 \sum_{j=1}^{n_\ell} \left| \partial_{x_p} f_j^{(\ell)} - \partial_{x_p} \tilde{f}_j^{(\ell)} \right| + B_0^2 n_\ell \sum_{k=1}^{n_\ell+1} \left| \theta_k - \tilde{\theta}_k \right|
\]

\[
\leq B_0 (\ell + 1) B_0^2 \left( \prod_{i=1}^{\ell-1} n_i \right) \sum_{k=1}^{n_\ell} \left| \theta_k - \tilde{\theta}_k \right| + B_0^2 n_\ell \sum_{k=1}^{n_\ell+1} \left| \theta_k - \tilde{\theta}_k \right|
\]

17
\[
\leq (\ell + 2)B_0^{2\ell + 2} \left( \prod_{i=1}^{\ell} n_i \right)^2 \sum_{k=1}^{n_{\ell+1}} |\theta_k - \tilde{\theta}_k|
\]

Hence by induction and H"older inequality we conclude that

\[
|\partial_{x_p} f - \partial_{x_p} \tilde{f}| \leq (D + 1)B_0^{2D} \left( \prod_{i=1}^{D-1} n_i \right)^2 \sum_{k=1}^{n_D} |\theta_k - \tilde{\theta}_k| \leq \sqrt{n_D}(D + 1)B_0^{2D} \left( \prod_{i=1}^{D-1} n_i \right)^2 \|\theta - \tilde{\theta}\|_2
\]

\[\blacksquare\]

**Lemma 5.12.** Let \(D, \mathfrak{n}_D, n_i \in \mathbb{N}^+, n_D = 1\), \(B_\theta \geq 1\) and \(\rho\) be a function such that \(\rho, \rho'\) are bounded by \(B_\rho, B_\rho' \leq 1\) and have Lipschitz constants \(L_\rho, L_\rho' \leq 1\), respectively. Set the parameterized function class \(P = \mathcal{N}_\rho(D, n_D, B_\theta)\). Then, for any \(f_i(x; \theta), f_i(x; \tilde{\theta}) \in F_i, i = 1, \ldots, 5\), we have

\[
|f_i(x; \theta)| \leq B_1, \quad \forall x \in \Omega, \\
|f_i(x; \theta) - f_i(x; \tilde{\theta})| \leq L_i\|\theta - \tilde{\theta}\|_2, \quad \forall x \in \Omega,
\]

with \(B_1 = d \left( \prod_{i=1}^{D-1} n_i \right)^2 B_\theta^{2D}, B_2 = B_4 = B_3^2, B_3 = B_5 = (n_D - 1) + 1)B_\theta\), and

\[
L_1 = 2d\sqrt{n_D}(D + 1)B_\theta^{2D} \left( \prod_{i=1}^{D-1} n_i \right)^3, \quad L_2 = 2\sqrt{n_D}B_\theta (n_D - 1) \left( \prod_{i=1}^{D-1} n_i \right), \\
L_3 = \sqrt{n_D}B_\theta^{D-1} \left( \prod_{i=1}^{D-1} n_i \right), \quad L_4 = L_2, \quad L_5 = L_3.
\]

**Proof.** Direct result from Lemma 5.9, 5.11 and some calculation. \[\blacksquare\]

Now we state our main result with respect to statistical error.

**Theorem 5.13.** Let \(D, \mathfrak{n}_D, n_i \in \mathbb{N}^+, n_D = 1\), \(B_\theta \geq 1\) and \(\rho\) be a function such that \(\rho, \rho'\) are bounded by \(B_\rho, B_\rho' \leq 1\) and have Lipschitz constants \(L_\rho, L_\rho' \leq 1\), respectively. Set the parameterized function class \(P = \mathcal{N}_\rho(D, n_D, B_\theta)\). Then, if \(N = M\), we have

\[
\mathbb{E}(X_i) \leq \mathbb{E}(Y_i) \sup_{u \in P} \pm \left[ \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right] \leq C(\Omega, \omega, \alpha) \frac{d\sqrt{\mathfrak{n}_D B_\theta^{2D}}}{\sqrt{N}} \log \left( \frac{d\mathfrak{n}_D B_\theta N}{\delta} \right).
\]

**Proof.** From Lemma 5.5, 5.6 and 5.8, we have

\[
\mathbb{R}_N(F_i) \leq \inf_{0 < \delta < B_i/2} \left( 4\delta + \frac{12}{\sqrt{N}} \int_{\delta}^{B_i/2} \sqrt{\log \mathcal{C}(\epsilon, F_i, \|\cdot\|_\infty)} d\epsilon \right) \\
\leq \inf_{0 < \delta < B_i/2} \left( 4\delta + \frac{12}{\sqrt{N}} \int_{\delta}^{B_i/2} \sqrt{\mathfrak{n}_D \log \left( \frac{2L_i B_\theta \sqrt{\mathfrak{n}_D}}{\epsilon} \right)} d\epsilon \right) \\
\leq \inf_{0 < \delta < B_i/2} \left( 4\delta + \frac{6\sqrt{\mathfrak{n}_D} B_i}{\sqrt{N}} \log \left( \frac{2L_i B_\theta \sqrt{\mathfrak{n}_D}}{\delta} \right) \right).
\]
Choosing $\delta = 1/\sqrt{N} < B_i/2$ and applying Lemma 5.12, we have

$$
\mathcal{R}_N(F_i) \leq \frac{4}{\sqrt{N}} \sqrt{\frac{6\sqrt{B_i}}{\sqrt{N}}} \sqrt{\log \left( \frac{2L_iB_\theta\sqrt{\overline{n}_B\sqrt{N}}}{\sqrt{N}} \right)}
$$

$$
\leq C \frac{d\sqrt{\overline{n}_D} \left( \prod_{i=1}^{D-1} n_i \right)^2 B_\theta^2}{\sqrt{N}} \sqrt{\log \left( \frac{4d\overline{n}_D(D+1)B_\theta^{3D+1} \left( \prod_{i=1}^{D-1} n_i \right)^3 \sqrt{N}}{\sqrt{N}} \right)}
$$

$$
\leq C \frac{d\sqrt{\overline{n}_D} B_\theta^2}{\sqrt{N}} \sqrt{\log (d\overline{n}_DB_\theta N)}
$$

Combining Lemma 5.1, 5.3 and (5.1), we obtain, if $N = M$,

$$
\mathbb{E}_{(X_i)_{i=1}^N; (Y_i)_{j=1}^M} \sup_{u \in \mathcal{P}} \left| \mathcal{L}(u) - \tilde{\mathcal{L}}(u) \right| \leq \frac{C(\Omega, \text{coe}, \alpha)}{\beta} \frac{d\sqrt{\overline{n}_D} B_\theta^2}{\sqrt{N}} \sqrt{\log (d\overline{n}_DB_\theta N)}.
$$

6. Convergence Rate for the Ritz Method. Theorem 6.1. Let (A) holds. Assume that $\mathcal{E}_\text{opt} = 0$. Let $\rho$ be logistic function $\frac{1}{1+e^{-x}}$ or tanh function $\frac{e^x-e^{-x}}{e^x+e^{-x}}$. Let $u_{\phi_A}$ be the solution of problem (3.6) generated by a random solver and $\tilde{u}_\phi$ be an optimal solution of problem (3.6).

(1) Let $u_R$ be the weak solution of Robin problem (3.1)-(3.2c). For any $\varepsilon > 0$ and $\mu \in (0, 1)$, set the parameterized function class

$$
\mathcal{P} = N_P \left( C\log(d+1), C(d, \beta)\varepsilon^{-d/(1-\mu)}, C(d, \beta)\varepsilon^{-(9d+8)/(2-2\mu)} \right)
$$

and number of samples

$$
N = M = C(d, \Omega, \text{coe}, \alpha, \beta)\varepsilon^{-C\log(d+1)/(1-\mu)},
$$

if the optimization error $\mathcal{E}_\text{opt} = \tilde{\mathcal{L}}(u_{\phi_A}) - \tilde{\mathcal{L}}(\tilde{u}_\phi) \leq \varepsilon$, then

$$
\mathbb{E}_{(X_i)_{i=1}^N; (Y_i)_{j=1}^M} \left\| u_{\phi_A} - u_R \right\|_{H^1(\Omega)} \leq C(\Omega, \text{coe}, \alpha)\varepsilon.
$$

(2) Let $u_D$ be the weak solution of Dirichlet problem (3.1)(3.2a). Set $\alpha = 1, g = 0$. For any $\varepsilon > 0$, let $\beta = C(\text{coe})\varepsilon$ as the penalty parameter, set the parameterized function class

$$
\mathcal{P} = N_P \left( C\log(d+1), C(d)\varepsilon^{-5d/2/(1-\mu)}, C(d)\varepsilon^{-(45d+40)/(4-4\mu)} \right)
$$

and number of samples

$$
N = M = C(d, \Omega, \text{coe})\varepsilon^{-C\log(d+1)/(1-\mu)},
$$

if the optimization error $\mathcal{E}_\text{opt} \leq \varepsilon$, then

$$
\mathbb{E}_{(X_i)_{i=1}^N; (Y_i)_{j=1}^M} \left\| u_{\phi_A} - u_D \right\|_{H^1(\Omega)} \leq C(\Omega, \text{coe})\varepsilon.
$$
Remark 6.1. Dirichlet boundary condition corresponding to a constrained minimization problem, which may cause some difficulties in computation. The penalty method has been applied in finite element methods and finite volume method [4, 29]. It is also been used in deep PDEs solvers [43, 34, 44] since it is not easy to construct a network with given values on the boundary. Recently, [32, 31] also study the convergence of DRM with Dirichlet boundary condition via penalty method. However, the analysis in [32, 31] is based on some additional conditions, and we do not need these conditions to obtain the error induced by the penalty. More importantly, we provide the convergence rate analysis involving the statistical error caused by finite samples used in the SGD training, while in [32, 31] they do not consider the statistical error at all.

Proof. We first normalize the solution.

\[
\inf_{\bar{u} \in \mathcal{P}} \| \bar{u} - u_R \|_{H^1(\Omega)} = \| u_R \|_{H^1(\Omega)} \inf_{u \in \mathcal{P}} \left\| \frac{\bar{u}}{\| u_R \|_{H^1(\Omega)}} - \frac{u_R}{\| u_R \|_{H^1(\Omega)}} \right\|_{H^1(\Omega)} = \| u_R \|_{H^1(\Omega)} \inf_{u \in \mathcal{P}} \left\| \bar{u} - \frac{u_R}{\| u_R \|_{H^1(\Omega)}} \right\|_{H^1(\Omega)} \leq \frac{C_c}{\beta} \inf_{u \in \mathcal{P}} \left\| \bar{u} - \frac{u_R}{\| u_R \|_{H^1(\Omega)}} \right\|_{H^1(\Omega)}
\]

where in the third step we apply Lemma 3.2. By Lemma 3.2 and Corollary 4.3, there exists a neural network function

\[
u_{\rho} \in \mathcal{P} = \mathcal{N}_{\rho} \left( \mathcal{C}(d), \mathcal{D} \left( \frac{1}{\beta^3/2}, \epsilon \right), C(d) \left( \frac{1}{\beta^3/2}, \epsilon \right) \right)
\]

such that

\[
\left\| \nu_{\rho} - \frac{u_R}{\| u_R \|_{H^1(\Omega)}} \right\|_{H^1(\Omega)} \leq \beta^{3/2} \epsilon.
\]

Hence,

\[
\mathcal{E}_{app} = \frac{1}{\beta} \mathcal{C}(\Omega, \alpha, \epsilon) \inf_{\bar{u} \in \mathcal{P}} \| \bar{u} - u_R \|^2_{H^1(\Omega)} \leq \frac{1}{\beta^3} \mathcal{C}(\Omega, \alpha, \epsilon) \inf_{\bar{u} \in \mathcal{P}} \left\| \frac{\bar{u}}{\| u_R \|_{H^1(\Omega)}} - \frac{u_R}{\| u_R \|_{H^1(\Omega)}} \right\|^2_{H^1(\Omega)} \leq \mathcal{C}(\Omega, \alpha, \epsilon) \epsilon^2.
\]

(6.1)

Since \( \rho, \rho' \) are bounded and Lipschitz continuous with \( B_\rho, B_\rho', L_\rho, L_\rho' \leq 1 \) for \( \rho = \frac{\epsilon}{\epsilon+1} \) and \( \rho' = \frac{\epsilon}{\epsilon+1} \), we can apply Theorem 5.13 with \( \mathcal{D} = \mathcal{C}(d+1), n_{\mathcal{D}} = \mathcal{C}(d) \left( \frac{1}{\beta^3}, \epsilon \right), B_\theta = \mathcal{C}(d) \left( \frac{1}{\beta^3}, \epsilon \right) \). Now we conclude that by setting

\[
N, M = \mathcal{C}(d, \Omega, \alpha, \epsilon) \left( \frac{1}{\beta^3} \right)^{Cd(d+1)/(1-\mu)}
\]

(6.2)

we have

\[
\mathcal{E}_{sta} = \mathbb{E} \left( X_1^{N}, Y_1^{M} \right) \sup_{u \in \mathcal{P}} \left[ \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right] + \sup_{u \in \mathcal{P}} \left[ \hat{\mathcal{L}}(u) - \mathcal{L}(u) \right] \leq \epsilon^2.
\]

(6.3)

Combining Proposition 3.1(1), (6.1) and (6.3) yields (1).
Setting the penalty parameter

\[ \beta = C(c \varepsilon) \varepsilon \]  

(6.4)

and combining Lemma 3.3, Proposition 3.1(2) and (6.1) – (6.4) yields (2). \( \square \)

7. Conclusions and Extensions. This paper provided an analysis of convergence rate for deep Ritz methods for elliptic equations with Drichlet, Neumann and Robin boundary condition, respectively. Specifically, our study shed light on how to set depth and width of networks and how to set the penalty parameter to achieve the desired convergence rate in terms of number of training samples.

There are several interesting further research directions. First, the approximation and statistical error bounds deriving here can be used for studying the nonasymptotic convergence rate for residual based method, such as PINNs. Second, the similar result may be applicable to deep Ritz methods for optimal control problems and inverse problems.

8. Acknowledgements. We would like to thank Ingo Gühring and Mones Raslan for helpful discussions on approximation error.

The work of Y. Jiao is supported in part by the National Science Foundation of China under Grant 11871474 and by the research fund of KLATA SDSMOE. The work of Y. Wang is supported in part by the Hong Kong Research Grant Council grants 16308518 and 16317416 and HK Innovation Technology Fund ITS/044/18FX, as well as Guangdong-Hong Kong-Macao Joint Laboratory for Data-Driven Fluid Mechanics and Engineering Applications.

REFERENCES

[1] R. A. Adams and J. J. Fournier, Sobolev spaces, Elsevier, 2003.
[2] A. Anandkumar, K. Azizzadenesheli, K. Bhattacharya, N. Kovachki, Z. Li, B. Liu, and A. Stuart, Neural operator: Graph kernel network for partial differential equations, in ICLR 2020 Workshop on Integration of Deep Neural Models and Differential Equations, 2020.
[3] C. Antunes, E. Atroschchenko, N. Alajlan, and T. Rabczuk, Artificial neural network methods for the solution of second order boundary value problems, Cmc-computers Materials & Continua, 59 (2019), pp. 345–359.
[4] I. Babuska, The finite element method with penalty, Mathematics of Computation, 27 (1973), pp. 221–228.
[5] J. Berner, M. Dahlander, and P. Grohs, Numerically solving parametric families of high-dimensional kolmogorov partial differential equations via deep learning, in Advances in Neural Information Processing Systems, vol. 33, Curran Associates, Inc., 2020, pp. 16615–16627.
[6] S. Brenner and R. Scott, The mathematical theory of finite element methods, vol. 15, Springer Science & Business Media, 2007.
[7] P. G. Ciarlet, The finite element method for elliptic problems, SIAM, 2002.
[8] M. Dissanayake and N. Phan-Thien, Neural-network-based approximations for solving partial differential equations, Communications in Numerical Methods in Engineering, 10 (1994), pp. 195–201.
[9] C. Duan, Y. Jiao, Y. Lai, X. Lu, and Z. Yang, Convergence rate analysis for deep ritz method, arXiv preprint arXiv:2103.13330, (2021).
[10] W. E, C. Ma, and L. Wu, The Barron space and the flow-induced function spaces for neural network models, arXiv preprint arXiv:1906.08039, (2021).
[11] W. E and S. Wojtowytsch, Some observations on partial differential equations in Barron and multi-layer spaces, arXiv preprint arXiv:2012.01484, (2020).
[12] L. C. Evans, Partial differential equations, Graduate studies in mathematics, 19 (1998).
Y. Shin, Z. Zhang, and G. Karniadakis, *Error estimates of residual minimization using neural networks for linear pdes*, arXiv preprint arXiv:2010.08019, (2020).

D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre, G. Van Den Driessche, J. Schrittwieser, I. Antonoglou, V. Panneershelvam, M. Lanctot, et al., *Mastering the game of go with deep neural networks and tree search*, nature, 529 (2016), pp. 484–489.

J. A. Sirignano and K. Spiliopoulos, *Dgm: A deep learning algorithm for solving partial differential equations*, Journal of Computational Physics, 375 (2018), pp. 1339–1364.

J. Thomas, *Numerical Partial Differential Equations: Finite Difference Methods*, vol. 22, Springer Science & Business Media, 2013.

K. Um, R. Brand, Y. R. Fei, P. Holl, and N. Thuerey, *Solver-in-the-loop: Learning from differentiable physics to interact with iterative pde-solvers*, in Advances in Neural Information Processing Systems, vol. 33, Curran Associates, Inc., 2020, pp. 6111–6122.

S. Wang, X. Yu, and P. Perdikaris, *When and why pinn fail to train: A neural tangent kernel perspective*, arXiv preprint arXiv:2007.14527, (2020).

Y. Wang, Z. Shen, Z. Long, and B. Dong, *Learning to discretize: Solving 1d scalar conservation laws via deep reinforcement learning*, Communications in Computational Physics, 28 (2020), pp. 2158–2179.

E. Weinan and T. Yu, *The deep ritz method: A deep learning-based numerical algorithm for solving variational problems*, Communications in Mathematics and Statistics, 6 (2017), pp. 1–12.

J. Xu, *Finite neuron method and convergence analysis*, Communications in Computational Physics, 28 (2020), pp. 1707–1745.

Y. Zang, G. Bao, X. Ye, and H. Zhou, *Weak adversarial networks for high-dimensional partial differential equations*, Journal of Computational Physics, 411 (2020), p. 109409.