Chern-Simons Duality and the Quantum Hall Effect

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Abstract

In previous work on the quantum Hall effect on an annulus, we used $O(d,d;\mathbb{Z})$ duality transformations on the action describing edge excitations to generate the Haldane hierarchy of Hall conductivities. Here we generate the corresponding hierarchy of “bulk actions” which are associated with Chern-Simons (CS) theories, the connection between the bulk and edge arising from the requirement of anomaly cancellation. We also find a duality transformation for the CS theory exactly analogous to the $R \rightarrow \frac{1}{R}$ duality of the scalar field theory at the edge.
1 Introduction

Chern Simons (CS) gauge theories are known to be particularly appropriate for describing the quantum Hall effect (QHE) \[1\, 2\]. The low energy effective action for the electromagnetic vector potential, obtained by integrating out the electronic degrees of freedom in a Hall system, is known to have a CS term. The coefficient of this term is proportional to the Hall conductivity. This fact is easily shown as follows. Let us assume that the effective action (apart from the usual Maxwell term) is

\[
S_{\text{eff}}[A] = -\frac{1}{2}\sigma_{XY} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda
\]  

(1.1)

Then we get for the expectation value of the current,

\[
\langle j^\mu_{\text{em}} \rangle_A = -\frac{\delta}{\delta A_\mu} S_{\text{eff}}[A] = \sigma_{XY} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho
\]  

(1.2)

We thus see that there is a current in the X- direction when there is an electric field in the Y- direction. This is the Hall effect, the Hall conductivity being \(\sigma_{XY}\).

In studying the Hall effect, we will be interested in CS theories involving several vector potentials (one of which is the electromagnetic field). These comprise the “statistical” gauge fields and the fields describing excitations in the bulk \[1\, 2\, 3\, 4\, 5\]. The former are introduced for the purpose of changing the statistics of the excitation fields in the action while the latter represent collective degrees of freedom such as vortices or other quasiparticles and can describe both bosons and fermions.

Experimentally, the Hall conductivity in certain systems is quantized in integers or in certain definite fractions \[6\] corresponding respectively to integer and fractional QHE (IQHE and FQHE). Several scenarios have been proposed to explain such quantizations. The hierarchy schemes of Haldane \[8\] and Jain \[9\] are perhaps the most attractive, in that they explain most of the observed experimental fractions. CS theories of the type mentioned above, involving several vector potentials, lend themselves naturally to these
schemes \([3, 7]\). The Jain scheme has already been written in this form \([3]\) while we will work out a similar description of the Haldane scheme in this paper.

However, the CS action is not gauge invariant on a manifold with a boundary (like an annulus), such a manifold being the appropriate geometry for a physical Hall sample. One has to include non-trivial dynamical degrees of freedom at the edge \([1, 10, 11, 12, 13]\) to restore gauge invariance. In this way, one predicts the existence of edge states, which nicely corroborates completely different arguments showing the existence of chiral edge currents in a Hall sample \([14]\). These states have been studied in detail by many authors \([4, 10, 11, 12, 13, 15, 16, 17]\). In \([17]\), we described them by a conformal field theory of massless chiral scalar fields taking values on a torus. The most general action for these scalar fields contains a symmetric matrix \(G_{ij}\) and an antisymmetric matrix \(B_{ij}\). In \([17]\), we saw that the Hall conductivity depends on \(G_{ij}\). In this paper, we will show, in detail, how the anomaly cancellation argument enables us to relate this matrix to a corresponding matrix in the bulk CS theories. Thus, once we implement the hierarchy arguments in the CS theory, we have a rationale for particular choices of this matrix made in \([17]\).

Now, as in string theories having torally compactified spatial dimensions, there are certain duality transformations of the edge theory that leave the spectrum invariant \([18]\). These transformations change \(G_{ij}\) and \(B_{ij}\) in well-defined ways and hence also change the Hall conductivity \([17]\). It was shown in \([17]\) that one can reproduce most of the conductivity fractions of the Haldane and Jain schemes by means of these generalized duality transformations. The connection of the edge theory with the CS theory in the bulk then suggests that similar transformations can be implemented in the bulk CS theory also. This conjecture turns out to be at least partly realizable in that one can implement a duality transformation of the type \(R \rightarrow 1/R\) in the bulk. The demonstration of this result is a generalization of the one that has been used in \([19]\) to implement duality in scalar theories. We think that both this proof and the result are interesting and could
have implications in many other areas as well.

This paper is organized as follows: In Section 2 we describe how to implement the Haldane construction using CS theories. In Section 3 we describe the connection between the bulk and edge actions. Finally, in Section 4 we show how to implement duality in the CS theory.

2 The Haldane Hierarchy and CS Theory

In this Section we would like to describe Haldane’s construction using CS gauge fields. Let us first recall the physical arguments. The Haldane approach exploits the superfluid analogy and treats the Hall fluid as a bosonic condensate.

We have a system with $N_e$ electrons per unit area in a magnetic field of strength $B$. The number of flux quanta per unit area is $Be/2\pi$ (in units where $\hbar = c = 1$), which we denote by $N_\phi$. In the usual integer effect with one filled Landau level, we have the equality

$$N_\phi = N_e \quad (2.1)$$

which tells that the degeneracy of the Landau levels is exactly equal to the number of flux quanta piercing the Hall sample. This equation is a consequence of solving the Landau level problem for fermions. The incompressibility follows from the existence of a gap between Landau levels and the fermionic nature of the electrons that fixes the number of electrons that one can place in one Landau level.

There is another way to think of the same system. According to (2.1) the number of flux quanta is equal to the number of electrons. It is therefore like attaching one flux quantum to each electron. The composite object behaves like a boson and can Bose condense. The resulting superfluid is the Hall fluid. The energy gap follows from the usual arguments for superfluidity due to Feynman [20], where he showed, using the
bosonic nature of the condensate, that the only low energy excitations are long wavelength
density fluctuations. However, in the Hall fluid, since the density is tied to the fixed
external magnetic field by (2.1), there are no density fluctuations. Thus we have no
massless excitations (in the bulk) at all, that is, the fluid is incompressible.

A simple generalization of the above arguments can be applied to a system that obeys

\[ N_{\phi} = m N_e, \quad m \in 2\mathbb{Z} + 1. \]  

(2.2)

It can be interpreted as the attachment of an odd number \( m \) of flux tubes to each
electron. Since the composite will be bosonic, there will be bose condensation and one
can again invoke arguments from superfluidity. Thus we have a new incompressible state
at the filling fraction \( 1/m \). These are the Laughlin fractions.

Thus in the Haldane approach, the final dynamical degrees of freedom are bosonic
objects, a circumstance which suggests that we rewrite the original action, which describes
fermions (electrons) in a magnetic field, in terms of a new set of variables that describe
these dynamical excitations. Thus we will re-express the fermionic electron field in terms
of a bosonic field and a statistical gauge field. As we shall see below, one can implement,
in this way, the ideas described in previous paragraphs, in terms of a low energy effective
field theory.

One can also proceed to generalize these ideas to get other filling fractions. The system
admits Nielsen-Olesen vortices \([21]\) as excitations or quasiparticles. As the magnetic field
is changed, it is energetically favourable for the excess or deficit magnetic field to organize
itself as flux tubes threading vortices in the condensate, so that quasiparticles which are
one of these Nielsen-Olesen vortices are formed. At a certain point a large number of
these quasiparticles form and condense, so that we now have a finite number density of
quasiparticles and a new ground state is also created.

If one were to think of these quasiparticles or vortices as carrying a new form of charge,
then the gauge field to which they couple are in fact the duals to the Goldstone (phase)
mode of the condensate \[8, 7\]. The way one shows this point \[8, 7\] is by noticing that if the electron current is represented in a dual representation using a one-form, then the “electric field” associated to this one-form has a behaviour, outside a vortex, identical to that of a usual electric field outside an ordinary electric charge.

Thus the flux quanta, in this dual representation, are the electrons themselves. The quasiparticles are bosons. Clearly, their bosonic nature will be maintained if an even number (say \(2p_1\)) of dual flux quanta (that is, electrons) get attached to each of these quasiparticles. These statements can be summarized by the following two equations:

\[
N_\phi = mN_e + N^{(1)} \tag{2.3}
\]

\[
N_e = 2|p_1 N^{(1)}| \tag{2.4}
\]

Here \(N^{(1)}\) is the number density of quasiparticles. Unlike \(N_e\), it can have either sign depending on whether the associated fluxons point in a direction parallel or antiparallel to the original magnetic field.

If we make \(p_1\) negative whenever \(N^{(1)}\) is, then we can omit the modulus sign in equation (2.4). In that case, we can solve these equations to get the filling fraction:

\[
\frac{N_e}{N_\phi} = \frac{1}{m + \frac{1}{2p_1}} \tag{2.5}
\]

This is the second level of the hierarchy.

We can next imagine that there are new quasiparticle excitations over the ground state as we increase the magnetic field further. These new quasiparticles can have “flux tubes” attached to them and can in turn condense. Now the new flux quanta are dual representations of the quasiparticles of the first level. This process can be iterated as many times as one wants and it generates a series of filling fractions. The equations describing this process are the hierarchy equations of \[8\]:

\[
N_\phi = mN_e + N^{(1)}, \tag{2.6}
\]

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\( N_e = 2|p_1 N^{(1)}| + N^{(2)}, \quad (2.7) \)

\( N^{(1)} = 2|p_2 N^{(2)}| + N^{(3)}, \quad (2.8) \)

......

In equations (2.7), as in (2.3), the quasiparticle density \( N^{(2)} \) can be less than zero, but should still be such that \( N_e \) itself does not become negative.

We can in fact choose to omit the modulus signs in these equations if we allow the \( p_i \)'s also to be less than zero whenever the \( N^i \)'s are, so that their product themselves are always non-negative. We shall assume that this is done in the following, where we implement these ideas using CS fields. The basic techniques are described in [3].

Following [3], we describe the electron by a scalar field coupled to a statistical gauge field, \( a_\mu \). Furthermore if this bosonic order parameter develops an expectation value, then we have a massless Goldstone boson \( \eta \) - the phase of the original scalar field. It fulfills the equation \( \partial_\mu \partial^\mu \eta = 0 \), being massless. The electron current \( \partial^\mu \eta \) can be represented by a dual vector field \( \alpha_\mu \) defined by \( \partial_\mu \eta = \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_\lambda \). The field equation of \( \eta \) turns into an identity in this dual representation. We can also implement a minimal coupling to the external electromagnetic vector potential \( A_\mu \). The action thus far is

\[
\int_{(D\setminus H) \times \mathbb{R}^1} d^3 x \left[ -e J^\mu (A_\mu - a_\mu) - \frac{e^2}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right],
\]

\[
J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_\lambda
\]

\( D\setminus H \equiv \text{Disk D with a hole H removed (or an annulus)} \)

where the last term is an abelian CS term for the statistical gauge field \( a_\mu \) and \( \mathbb{R}^1 \) accounts for time. Its coefficient has been chosen to ensure that it converts the boson to a fermion as may be seen in the following way: On varying (2.9) with respect to \( \alpha_\mu \), we get

\[
\epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda.
\]

(2.10)
On varying with respect to $a_\mu$, we get
\[ \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda. \] (2.11)
so that
\[ \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda. \] (2.12)

This equation relates the number density $J^0 = N_e$ of electrons to the number density $N_\phi = \frac{e}{2\pi} B$ of flux quanta $\frac{2\pi}{e}$. In fact it says that $N_e = N_\phi$. Thus there is one flux quantum per electron which converts the latter to a fermion.

The filling fraction $\nu$ is 1 for (2.9) since $N_\phi = N_e$. It thus describes the IQHE (see (2.1)). We can also eliminate $\alpha$ and $a$ to get an effective action dependent only on the electromagnetic gauge field. Thus the electromagnetic current $-eJ^\mu$ of (2.9) is equal to
\[ -\frac{e^2}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \] by (2.12) and this current is reproduced by
\[ S = -\frac{e^2}{4\pi} \int_M d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \] (2.13)
\[ M = (D\setminus H) \times \mathbb{R}^1. \]

This is the electromagnetic CS term (and a signature of the Hall effect) for the Hall conductivity $\sigma_H = \frac{e^2}{2\pi}$.

One can immediately generalize (2.9) to obtain the Laughlin fractions by changing the coefficient $\frac{e^2}{4\pi}$ to $\frac{e^2}{4\pi m}$ with $m$ odd:
\[ S^{(0)} = \int_M d^3x \left[ -eJ^\mu (A_\mu - a_\mu) - \frac{e^2}{4\pi m} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right], \quad m \in 2\mathbb{Z} + 1. \] (2.14)

This changes (2.11)(2.12) to
\[ \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = \frac{e}{2\pi m} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda, \] (2.15)
\[ \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = \frac{e}{2\pi m} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda. \] (2.16)
Equation (2.15) says that $N_e = \frac{1}{m} N_\phi$. Since $m$ is odd, this is the same as (2.2) and therefore implies that the composite is bosonic, as it should be for this description of the electron to be consistent. The filling fraction now is $\nu = \frac{1}{m}$ while (2.13) is changed to 

$$S^{(0)} = -\frac{e^2}{4\pi m} \int_M d^3x \, e^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

(2.17)

This is the CS action giving the first level of the Haldane hierarchy.

Next, we modify (2.14) by adding a coupling of the quasiparticle current $J^{(1)}\mu$ to the gauge field $\alpha_\mu$. Thus we have the action

$$\int_M d^3x \left[ -e J^{\mu} (A_\mu - a_\mu) - \frac{e^2}{4\pi m} e^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + 2\pi J^{(1)}\mu \alpha_\mu \right]$$

(2.18)

The choice of the coefficient $2\pi$ in the last term can be motivated as follows. Suppose that there is a vortex localised at $z$ so that $J^{(1)0}(x) = \delta^2(x - z)$ while the electron density $J^{(0)}$ is some smooth function. Then since $J^{(0)} = \frac{e}{2\pi m} e^{0ij} \partial_i a_j$ by equations of motion, $e^{0ij} \partial_i a_j$ is also smooth. Now variation of $\alpha$ gives $\frac{2\pi}{e} J^{(1)0} = e^{0ij} (\partial_i A_j + \partial_j a_i)$ so that the magnetic flux attached to the vortex is the flux quantum $\frac{2\pi}{e}$. As this is the unit of magnetic flux we want to attach to the vortex, the choice of $2\pi$ is seen to be correct.

Suppose next that the quasiparticles condense. Then we can write $J^{(1)}\mu = \partial^\mu \eta^{(1)}$ where $\eta^{(1)}$ is the Goldstone boson phase degree of freedom. As before, $\eta^{(1)}$ being massless and hence $\partial_\mu \partial^\mu \eta^{(1)} = 0$, one can write a dual version of the current by defining a field $\beta_\mu$

$$J^{(1)}\mu = \partial^\mu \eta^{(1)} = e^{\mu\nu\lambda} \partial_\nu \beta_\lambda.$$ 

(2.19)

We can also introduce a statistical gauge field $b_\mu$ and attach flux tubes of $b$ to the quasiparticle. Since the quasiparticles correspond to vortices which are assumed to be bosonic, here we attach an even number of the elementary $b$ flux tubes to each vortex to preserve the bosonic nature. Bearing this in mind, we add some more CS terms to (2.18) to get

$$S^{(1)} = \int_M \left[ -e(A - a) d\alpha - \frac{e^2}{4\pi m} ada + 2\pi \alpha d\beta - e b d\beta - \frac{e^2}{4\pi (2p_1)} b db \right], \ m \in 2\mathbb{Z} + 1, \ p_1 \in \mathbb{Z}.$$ 

(2.20)
[Here, we have used the form notation to save writing the antisymmetric symbol repeatedly. A symbol $\xi = A, \alpha, \beta, a$ or $b$ now denotes the one-form $\xi_\mu dx^\mu$.]

The equations of motion from (2.20) are

$$\frac{e}{2\pi} dA = \frac{e}{2\pi} da + d\beta,$$

$$md\alpha = \frac{e}{2\pi} da,$$

$$d\alpha = \frac{e}{2\pi} db,$$

$$d\beta = -\frac{e}{2\pi(2p_1)} db$$

(2.21)

The equations for $\alpha$ and $\beta$ here are seen to be precisely the hierarchy equations (2.7) and (2.8) [with $N^{(2)} = 0$] on eliminating $a$ and $b$.

Now these equations for $\alpha$ and $\beta$ are reproduced also by

$$\mathcal{S}^{(1)} = \int_M \left[ -e A d\alpha + \pi m \alpha d\alpha + 2\pi \alpha d\beta + \pi (2p_1) \beta d\beta \right]$$

(2.22)

$$= \int_M \left[ -e A d\alpha + \pi (\alpha \beta) \left( \begin{array}{c} m \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 2p_1 \end{array} \right) \left( \begin{array}{c} d\alpha \\ d\beta \end{array} \right) \right].$$

(2.23)

We have here used matrix notation to display the form of the “metric” in the CS theory.

The generalization to higher levels is as follows: Introduce $d$ vector fields $\alpha_I; I = 1, \cdots, d$. [In the above example, $d = 2$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$.] Then consider the Lagrangian form

$$\mathcal{L} = -e A d\alpha_1 + \pi \alpha_1 K^{IJ} d\alpha_J$$

(2.24)

with

$$\alpha_I = \alpha_{I\mu} dx^\mu,$$

$$K^{IJ} = \left( \begin{array}{cccccc} m & 1 & 0 & \cdots & \cdots \\
1 & 2p_1 & 1 & 0 & 0 \\
0 & 1 & 2p_2 & 1 & 0 & \cdots \\
\cdots & 0 & 1 & 2p_3 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right).$$

(2.25)

The equation of motion for $\alpha_1$ gives

$$edA = 2\pi K^{1J} d\alpha_J$$

(2.26)
while the equations of motion for the remaining $\alpha_I$’s give

$$K^{IJ}d\alpha_J = 0 \text{ for } I \neq 1.$$  \hfill (2.27)

These are the hierarchy equations. We can solve for the $d\alpha_I$’s:

$$d\alpha_I = \frac{e}{2\pi} (K^{-1})_{II} dA.$$  \hfill (2.28)

Substitute back into (2.25) to get

$$\mathcal{L} = -\frac{e^2}{4\pi} A(K^{-1})_{11} dA.$$  \hfill (2.29)

This is the CS Lagrangian form that gives rise to the Haldane hierarchy. Its filling fraction $\nu$ is just $(K^{-1})_{11}$ where $K^{IJ}$ is given by (2.25). $\nu$ is in fact the continued fraction obtained in the Haldane hierarchy:

$$\nu = \frac{1}{m - \frac{1}{2p_1 - \frac{1}{2p_2 - \frac{1}{2p_3 - \ldots}}}}.$$  \hfill (2.30)

We can prove (2.30) easily. Let

$$\Delta(\xi_1, \xi_2, \ldots, \xi_n) = \det \begin{bmatrix} \xi_1 & 1 & 0 & \cdots \\ 1 & \xi_2 & 1 & 0 \\ 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 & \xi_n \end{bmatrix}.$$  \hfill (2.31)

Then

$$\Delta(\xi_1, \xi_2, \ldots, \xi_n) = \xi_1 \Delta(\xi_2, \ldots, \xi_n) - \Delta(\xi_3, \ldots, \xi_n),$$  \hfill (2.32)

and

$$\nu = \frac{\Delta(2p_1, 2p_2, \ldots, 2p_n)}{\Delta(m, 2p_1, \ldots, 2p_n)}.$$  \hfill (2.33)

We get (2.30) from (2.32) and (2.33).
3 Anomaly Cancellation and the Bulk-Edge Connection

In this section we will show that the requirement of gauge invariance forces the “metric” $K^{IJ}$ introduced in the previous section to be the same as the inverse of the target space “metric” $G_{IJ}$ of the scalar theory describing the edge excitations. In our previous work \cite{17} on edge excitations, we had assumed the form (2.25) for $(G^{-1})^{IJ}$. The results of this and the previous section provide the necessary motivation for this assumption. Let us consider the CS action

$$S = \frac{1}{2} \int_M \alpha d\alpha$$

(3.1)

without any electromagnetic coupling. If $\tilde{M}$ has a closed (compact and boundaryless) spatial slice, this action has the gauge invariance

$$\alpha \rightarrow \alpha + d\Lambda$$

(3.2)

If $\tilde{M}$ is a manifold such as $M$ where the spatial slice $\Sigma$ has a boundary like an annulus $D \setminus H$, then the gauge variation results in a surface term. For $M = D \setminus H \times \mathbb{R}^1$, we have, for the variation of the action,

$$\delta S = \frac{1}{2} \int_{\partial D \times \mathbb{R}^1} \Lambda d\alpha - \frac{1}{2} \int_{\partial H \times \mathbb{R}^1} \Lambda d\alpha,$$

(3.3)

where as usual we assume that $\Lambda$ vanishes in the infinite past and future. One can recover gauge invariance at the boundary by adding to the action the following two-dimensional action containing a new scalar field $\phi$:

$$\Delta S = -\frac{1}{2} \int_{\partial M} d\phi \alpha + \frac{1}{4} \int_{\partial M} d^2 x (\tilde{D}_\mu \phi)(\tilde{D}^\mu \phi)$$

(3.4)

Here the gauge transformation law for $\phi$ is

$$\phi \rightarrow \phi - \Lambda$$

(3.5)
so that $\tilde{D}_\mu \phi$ is $\partial_\mu \phi + \alpha_\mu$. [The coefficient $\frac{1}{4}$ outside the kinetic energy term in (3.4) is determined by requiring that the edge current be chiral, that is, that we can impose the following condition consistently with the equations of motion:

$$\tilde{D}_- \phi \equiv (\tilde{D}_0 - \tilde{D}_\theta) \phi = 0$$

(3.6)

(see [17]).]

The combined action $S + \Delta S$ is gauge invariant.

A more formal way of justifying the above procedure to recover gauge invariance is to first look at the generators of the “edge” gauge transformations in the absence of the scalar field action at the boundary.

The operator that generates the transformation (3.2) at a fixed time with $\Lambda|_{\partial D} \neq 0$ and $\Lambda|_{\partial H} = 0$ ($\Lambda$ being a function on the annulus $D \setminus H$, the choice $\Lambda|_{\partial H} = 0$ being made for simplicity) is

$$Q(\Lambda) := \int_{D \setminus H} d\Lambda \alpha.$$  

(3.7)

The algebra generated by these operators is specified by (22)

$$[Q(\Lambda), Q(\Lambda')] = -i \int_{\partial D} \Lambda d\Lambda'.$$

(3.8)

If one tries to impose the gauge invariance condition $Q(\Lambda)|\cdot\rangle = 0$ on physical states $|\cdot\rangle$, one is led to a contradiction because the commutator of two $Q$’s acting on a (physical) state would also have to vanish, whereas (3.8) specifies the value of this commutator to be a non-zero $c$-number.

However, if we now augment this action by the above action (3.4) describing new degrees of freedom at the boundary, the generators of the “edge” gauge transformations get modified. The modification is by the terms

$$q(\Lambda) = \int_{\partial D} \Lambda (\Pi_\phi - \frac{1}{2} \dot{\phi}').$$

(3.9)
where \( \Pi_{\phi} := \frac{1}{2}(D_0 \phi + A_\theta) \), is the canonical momentum conjugate to \( \phi \) and obeys the usual commutation relations.

\( q(\Lambda) \) generates the transformations

\[
\begin{align*}
\phi & \to \phi - \Lambda \\
\Pi_{\phi} & \to \Pi_{\phi} + \frac{1}{2} \partial_\theta \Lambda
\end{align*}
\]  

(3.10)

The algebra generated by the \( q(\Lambda) \)'s is given by

\[
[q(\Lambda), q'(\Lambda')] = i \int_{\partial D} A d\Lambda'.
\]  

(3.11)

Thus the new generators \( \tilde{Q}(\Lambda) := Q(\Lambda) + q(\Lambda) \) now commute amongst themselves and can be chosen to annihilate the physical states.

Let us now attempt to couple electromagnetism to the action \( S \) in (3.1). \( *d\alpha \) represents a current so that the obvious coupling is

\[
S^1 = -q \int_M A d\alpha
\]  

(3.12)

Here \( A \) is a background electromagnetic field.

However, we run into a problem when we consider the equation of motion implied by \( S + S^1 \). On varying with respect to \( \alpha \), the equation of motion that we get in the bulk is

\[
d\alpha = q dA
\]  

(3.13)

while on the boundary, it is

\[
\frac{1}{2} \alpha = q A.
\]  

(3.14)

(3.13) and (3.14) are incompatible. (3.14) implies a relation between the values of the field strengths of \( \alpha \) and \( A \) on the boundary that differs by a factor of two from that implied by (3.13) in the bulk whereas by continuity they should be equal.
There is, however, the following simple modification of the minimal coupling (3.12) that gives a consistent set of equations. Consider the action

\[ S^2 = -\frac{1}{2} \int q(Ad\alpha + \alpha dA) \]  

(3.15)

With this action, the boundary equation (3.14) is modified to

\[ \alpha = qA. \]  

(3.16)

Thus (3.13) and (3.16) together say that \( \alpha = qA \) everywhere classically, up to gauge transformations that vanish on the boundary. Gauge transformations that do not vanish on the boundary and consistent with the equations of motion have the form

\[ \alpha \to \alpha + qd\Lambda, \quad A \to A + d\Lambda \]  

(3.17)

But while we have achieved consistency of the equations of motion at the edge and in the bulk, the action \( S + S^2 \) given by (3.1) and (3.15) is no longer gauge invariant under (3.17). This is very similar to what happens at the edge \[17\], where gauge invariance and chirality are incompatible with the equations of motion. The cure there (see \[17\] and references therein), was to introduce a coupling to the bulk. Similarly, here, the cure is to couple to degrees of freedom living only at the boundary, just as was done in the beginning of this section for the action (3.1) (see (3.1)-(3.4)). Thus we need to introduce a scalar field \( \phi \) with a boundary action of the form

\[ \Delta S^2 = \frac{q}{2} \int_{\partial M} d\phi A + \frac{1}{4} \int_{\partial M} d^2 x (D_\mu \phi)^2, \]  

(3.18)

\[ D_\mu \phi = \partial_\mu \phi - qA_\mu \]

to maintain invariance under (3.17), namely the electromagnetic gauge transformations. Here \( \phi \) transforms under (3.17) in the following way:

\[ \phi \to \phi + q\Lambda \]  

(3.19)
[Once again, we can justify this addition by noting as before that with this addition, the generators of the “edge” gauge transformations can be required to annihilate the states.]

The full action $S = S + S^2 + \Delta S^2$ is thus gauge invariant under the electromagnetic gauge transformations. It is also easy to see that it gives equations of motion in the bulk and the boundary that are compatible with each other. The final action is thus

$$S = \int_{D \setminus H \times R^1} \left\{ \frac{1}{2} \alpha d\alpha - \frac{q}{2} (A d\alpha + \alpha dA) \right\} + \frac{q}{2} \int_{\partial D \times R^1} d\phi A + \frac{1}{4} \int_{\partial D \times R^1} d^2 x (D_{\mu} \phi)^2$$  \hspace{1cm} (3.20)

Let us summarize what we have done with one CS field before we generalize to the case of $d$ fields. We began with a CS action for a gauge field $\alpha$, where $d\alpha$ represents the current of electrons or quasiparticles. We then introduced a coupling to a background electromagnetic field. Naively, this action has a gauge invariance even without introducing any edge degrees of freedom. However there is an inconsistency between the bulk and the boundary equations. When the naive coupling is modified to restore consistency, the action is no longer gauge invariant. The solution is to introduce a scalar degree of freedom at the edge. The final action is then given by (3.20).

It is now straightforward to extend this to the case with $d$ CS fields and the action

$$S = \pi K^{IJ} \int_M \alpha_I d\alpha_J$$ \hspace{1cm} (3.21)

We have introduced the “metric” $K^{IJ}$ that we had in the last section. This theory has $d$ U(1) gauge invariances:

$$\alpha_I \to \alpha_I + d\Lambda_I$$  \hspace{1cm} (3.22)

We now introduce $d$ background gauge fields $A^I$, one of which represents the physical electromagnetic field and the rest are fictitious. They can be used, for instance, to calculate correlations between the different quasiparticle currents. Thus once we integrate out the quasiparticles from the theory, the resultant action will depend on these gauge fields. Functional differentiation with respect to these fields then gives the correlators of
the currents (the connected Green functions). They are, thus, a means of keeping track of the information in the original action after integrating out the $\alpha$ exactly, much as the “sources” of conventional field theory. Following the earlier procedure as earlier of first introducing a coupling as in (3.13) and then introducing edge scalar fields for restoring gauge invariance, the final action becomes

$$S = \int_M \left\{ -\frac{1}{2} (A^I d\alpha_I + \alpha_I dA^I) + \pi K^{IJ} \alpha_I d\alpha_J \right\} + \int_{\partial M} \frac{1}{4\pi} (K^{-1})_{IJ} \phi^I A^J$$
$$+ \frac{1}{8\pi} \int_{\partial M} (K^{-1})_{IJ} D_\mu \phi^I D^\mu \phi^J$$

As in (3.4), here too the coefficient of the kinetic term is fixed by requiring consistency between the chirality of the edge currents (3.6) and the equations of motion [17].

We can also specialize to the case where only the electromagnetic background is non-zero. Then we can set $A^I = q^I A_{cm}$ and get the expression for the Hall conductivity used in [17]. The expression used in the last section is obtained by further specializing to the case $q^1 = e$ and $q^2 = q^3 = ... q^d = 0$.

### 4 “$T$-Duality” in CS Theory

Let us first review Buscher’s duality argument [19] for the scalar field in 1+1 dimensions. We shall later see that a straightforward generalization works for the CS theory.

Consider the action

$$S = \frac{R^2}{4\pi} \int_{S^1 \times \mathbb{R}} d^2 x \partial_\mu \phi \partial^\mu \phi$$

with $\phi$ identified with $\phi + 2\pi$:

$$\phi \approx \phi + 2\pi.$$  

(4.2)

The translational invariance $\phi \to \phi + \alpha$ of (4.1) can be gauged to arrive at the action

$$\tilde{S} = \frac{R^2}{4\pi} \int d^2 x (\partial_\mu \phi + W_\mu)^2$$

(4.3)
where $W_\mu$ transforms according to

$$W_\mu \rightarrow W_\mu - \partial_\mu \alpha. \quad (4.4)$$

We now introduce a Lagrange multiplier field $\lambda$ which constrains $W$ in the following way:

$$F \equiv dW = 0, \quad (4.5)$$

$$\oint_C W \in 2\pi \mathbb{Z}. \quad (4.6)$$

Here $C$ is any closed loop, space-like or time-like. [This latter possibility arises if we identify $t = -\infty$ with $t = +\infty$ in the functional integral so that the transition amplitude is between an initial state and a final state obtained after transport around a closed path in the configuration space (of fields other than the lagrange multiplier field $\lambda$). In this case our manifold can be thought of as $T^2 = S^1 \times S^1$.] If $W$ satisfies the conditions (4.5) and (4.6), then it has no observable effects on any other fields and so it can be “gauged away” \[19\] from (4.3) to get back $S$.

Let us thus consider

$$S' = \frac{R^2}{4\pi} \int d^2x \left( \partial_\mu \phi + W_\mu \right)^2 + \frac{1}{2\pi} \int d\lambda W, \quad (4.7)$$

$$\left( \partial_\mu \phi + W_\mu \right)^2 \equiv \left( \partial_\mu \phi + W_\mu \right)\left( \partial^\mu \phi + W^\mu \right) \quad (4.8)$$

where $\lambda$ is a function. It follows from the equations of motion of (4.7) that the Lagrange multiplier field $\lambda$ constrains $F(= dW)$ to be zero. The condition (4.6) on the holonomies to be quantized also follows \[20\] if we require that $\lambda$ be identified with $\lambda + 2\pi$ just as $\phi$ itself was. (In the Appendix, we show that this latter condition is in fact necessary for the theory to be consistent).

An alternative derivation of the quantization of the holonomies uses the functional integral approach and is as follows. Consider the path integral

$$Z_\lambda := \int \mathcal{D}\lambda e^{\frac{i}{\hbar} \int d\lambda W} \quad (4.9)$$
(which is the part of the full path integral that involves $\lambda$). Since $\lambda \approx \lambda + 2\pi$, we can expand $d\lambda$ according to

$$d\lambda = \sum_n \alpha_n d\lambda_n^{(0)} + n_x \omega_x + n_t \omega_t, \quad \alpha_n \in \mathbb{R}, \quad n_x, n_t \in \mathbb{Z}. \quad (4.10)$$

Here $\lambda_n^{(0)}$ is a complete set of single-valued functions on $T^2$ while $\omega_x$ and $\omega_t$ are one-forms such that

$$\oint_x \omega_x = \oint_t \omega_t = 2\pi, \quad (4.11)$$

$$\oint_x \omega_x = \oint_x \omega_t = 0 \quad (4.12)$$

where $\oint_x$ and $\oint_t$ refer respectively to the integrals along the circles in $x$ and $t$ directions. Thus

$$Z_\lambda = \sum_{n_x, n_t} \int d\alpha_n e^{\frac{i}{2\pi} [\alpha_n \int d\lambda_n^{(0)} W + n_x \int \omega_x W + n_t \int \omega_t W]}$$

$$\sim \prod_n \delta[\int d\lambda_n^{(0)} W] \sum_{n_x, n_t} e^{\frac{i}{2\pi} [n_x \int \omega_x W + n_t \int \omega_t W]}. \quad (4.13)$$

Now

$$\sum_ne^{inX} = 2\pi \sum_m \delta(X - 2\pi m). \quad (4.14)$$

Therefore

$$Z_\lambda \sim \delta[dW] \sum_{m_1, m_2} \delta(\frac{\omega_x}{2\pi} W - 2\pi m_1) \delta(\frac{\omega_t}{2\pi} W - 2\pi m_2) \quad (4.15)$$

(where, to get the first delta functional, we have done a partial integration of the corresponding term in (4.13) and used the fact that $\lambda_n^{(0)}$ forms a basis).

On using (4.11), we now get

$$Z_\lambda \sim \delta[dW] \sum_{m_1, m_2} \delta(\oint_t W - 2\pi m_1) \delta(\oint_x W - 2\pi m_2). \quad (4.16)$$

Here we have used the fact that $\omega_x$ ($\omega_t$) can be chosen to be independent of $t$ ($x$) by adding an exact form. This addition does not affect the values of the integrals in (4.13) because of the multiplying delta functional $\delta[dW]$. 

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Thus the conditions (4.5) and (4.6) follow.

Under these conditions, we can therefore gauge away $W$ to get back the original action (4.1).

If on the other hand, we decide to integrate out the $W$ field first, then

$$Z_W = \int \mathcal{D}W e^{i \left[ \frac{k^2}{4\pi} \int (\partial_\mu \phi + W_\mu)^2 + \frac{1}{R^2} \int e^{\mu \nu} \partial_\mu \lambda W_\nu \right]}$$

$$\sim e^{i \left[ -\frac{k^2}{4\pi} \int (\partial_\mu \phi - \frac{1}{R^2} e^{\mu \nu} \partial_\nu \lambda)^2 + \frac{1}{4\pi R^2} \int (\partial_\mu \phi)^2 \right]}$$

$$\sim e^{\frac{i}{4\pi R^2} \int (\partial_\mu \lambda)^2}. \quad (4.17)$$

We have used the fact here that $e^{\frac{i}{\pi} \int d\phi d\lambda} = 1$ which is a consequence of the identification $\phi \approx \phi + 2\pi$ and $\lambda \approx \lambda + 2\pi$.

Thus the theory we get now has the “dual” action

$$S_d = \frac{1}{4\pi R^2} \int (\partial_\mu \lambda)^2. \quad (4.18)$$

This completes our review of the duality argument for the scalar field theory.

We will now repeat this argument for the CS case. To begin with we have the action

$$S = \frac{k}{2\pi} \int_M \alpha d\alpha, \quad (4.19)$$

$M$ being an oriented three-manifold with an annulus (say) as its spatial slice, and with time compactified to a circle. This latter condition is equivalent to assuming that the fields at $t = \pm \infty$ take the same values so that the path integral (restricted to the Lagrange multiplier field that will be introduced shortly) leads to the transition amplitude between states after transport around a closed loop in the configuration space (consisting of fields other than the Lagrange multiplier field).

As with the scalar field, here too we need an extra condition on the $\alpha$’s which disallows arbitrary rescalings of the $\alpha$. Without such a condition, $k$ can be changed to $\lambda^2 k$ by
changing $\alpha$ according to the scheme $\alpha \rightarrow \alpha \lambda$, $\lambda$ being a real number. The condition that we impose is

$$\oint_{C \subset \partial M} \alpha \in 2\pi \mathbb{Z}, \quad (4.20)$$

where $C$ is any closed loop on the boundary $\partial M$ of the manifold. This condition is to be thought of as a generalization of the condition $\phi \approx \phi + 2\pi$ on the scalar field.

Under the transformation

$$\alpha \rightarrow \alpha + \omega \quad (4.21)$$
on $\alpha$ where $\omega$ is a closed one-form, the Lagrangean three-form is not invariant, but changes by an exact three-form:

$$\alpha d\alpha \rightarrow \alpha d\alpha - d(\omega \alpha) \quad (4.22)$$

We can make it exactly invariant by introducing a “connection” one-form $A$, transforming according to

$$A \rightarrow A - \omega \quad (4.23)$$

and “gauging” $S$ to obtain

$$\tilde{S} = \frac{k}{2\pi} \int \alpha d\alpha + \frac{k}{2\pi} \int A d\alpha. \quad (4.24)$$

But the action $\tilde{S}$ is obviously not equivalent to the action $S$ because the equations of motion are different. We therefore introduce a Lagrange multiplier one-form $\lambda$ as before to constrain $A$ by the equations

$$dA = 0, \quad (4.25)$$

$$\oint_{C \subset \partial M} A \in 2\pi \mathbb{Z}. \quad (4.26)$$

When $A$ fulfills (4.25) and (4.26), we can redefine $\alpha$ using the transformation (4.21) and get back (4.19) and (4.20).

We thus write

$$S' = \frac{k}{2\pi} \int \alpha d\alpha + \frac{k}{2\pi} \int A d\alpha + \frac{1}{2\pi} \int d\lambda A, \quad (4.27)$$
where
\[ \oint_{C \in \partial M} \lambda \in 2\pi \mathbb{Z}. \quad (4.28) \]

Consider
\[ Z_\lambda = \int D\lambda e^{\frac{\lambda}{2\pi}} \int d\lambda A. \quad (4.29) \]
to see how (4.25) and (4.26) emerge when we integrate out \( \lambda \). If now, each connected component of the boundary \( \partial M \) of \( M \), denoted by \( (\partial M)_a \), contains \( p_a \) cycles \( C_{ai} \) which can serve to define the generators of its first homology group, then there exist also \( p_a \) closed one-forms \( \omega_{ai} \) (for each \( a \)) on \( \partial M \) such that
\[ \oint_{C_{ai}} \omega_{ai}^j = 2\pi \delta_{ij} \delta_{aa'}, \; i, j = 1, 2, \ldots, p_a. \quad (4.30) \]
[We assume that the above homology group is torsion-free.] If \( M \) is compact, as we assume, \( (\partial M)_a \) is compact and has no boundaries. As \( M \) is oriented, \( \partial M \) too is oriented. Hence each connected component \( (\partial M)_a \) is a sphere with handles and its homology group has an even number of generators [27]. [When the spatial slice is an annulus (say), \( \partial M \) is \( T^2 \sqcup T^2 \).] Hence \( p_a \) has to be even. In this case we can order the \( \omega_{ai} \)'s such that [27]
\[ \int_{\partial M} \omega_{a,2l-1} \omega_{a',j} = 4\pi^2 \delta_{2l,j} \delta_{aa'} \; l = 1, 2, \ldots, \frac{p_a}{2}. \quad (4.31) \]
Given any such \( \omega \) on \( \partial M \), we can associate an \( \omega \) on \( M \) by requiring [28]
\[ \nabla^2 \omega = 0. \quad (4.32) \]
Here \( \nabla^2 \) is the Laplacian operator on one-forms on \( M \) defined using some Euclidean metric on \( M \). [The pull-back of this \( \omega \) to \( \partial M \) must of course agree with the \( \omega \) given there.] A choice of \( \omega_\perp \) (the component of \( \omega \) perpendicular to \( \partial M \)) needs to be made for solving (4.32). We can choose it to be zero.

Now, using (4.28), we can write
\[ \lambda = \lambda^{(0)} + \sum_{a,i} n_{ai} \omega_{ai}, \; n_{ai} \in \mathbb{Z} \quad (4.33) \]
where \( \lambda^{(0)} \) is a one-form (on \( M \)) such that

\[
\oint_{C_{a_j}} \lambda^{(0)} = 0.
\] (4.34)

Now, given any three-manifold \( M \), the operator \( \ast d \ast d \) (defined by choosing some Euclidean metric on \( M \)) on the space of one-forms \( \gamma \) admits the following boundary condition compatible with the self-adjointness of \( \ast d \ast d \) (the inner product being defined using the same Euclidean metric) [29]:

\[
\text{Pull-back of } \gamma \text{ to } \partial M \equiv \gamma|_{\partial M} = 0.
\] (4.35)

This means that the one-form \( \lambda^{(0)} \) of equation (4.33) can be expanded in a basis of one-forms \( \gamma_n \) which satisfy the above boundary condition (as in a Fourier expansion so that the convergence is only in the “mean-square” sense). Therefore

\[
\lambda = \sum_n \beta_n \gamma_n + \sum_{a,i} n_{ai} \omega_{ai}, \quad \gamma_n|_{\partial M} = 0.
\] (4.36)

Thus

\[
Z_\lambda = \sum_{n_{ai}} \int \prod_n d\beta_n e^{\frac{i}{2\pi} \left[ \sum_n \beta_n \int_M d\gamma_n A + \sum_{a,i} n_{ai} \int_M d\omega_{ai} A \right]}
\sim \prod_n \delta\left( \int d\gamma_n A \right) \sum_{n_{ai}} e^{\frac{i}{2\pi} \sum_{a,i} n_{ai} \int_M d\omega_{ai} A}
\sim \delta[dA] \sum_{n_{ai}} e^{\frac{i}{2\pi} \sum_{a,i} n_{ai} \int_{\partial M} \omega_{ai} A}.
\] (4.37)

[In arriving at the delta functional here, we have done a partial integration and used the completeness of the \( \gamma_n \)'s while to arrive at the integral in the exponent, we have again done a partial integration and then neglected the bulk term. The latter is justified owing to the multiplying delta functional.]

As before (see (1.14)), this means that

\[
Z_\lambda \sim \delta[dA] \prod_{a,i} \left( \sum_{m_{ai}} \delta\left( \int_{\partial M} \frac{\omega_{ai}}{2\pi} A - 2\pi m_{ai} \right) \right), \quad m_{ai} \in \mathbb{Z}.
\] (4.38)
Since the delta functional above implies that $A$ is a closed one-form, we can expand $A$ on the boundary $\partial M$ as

$$A|_{\partial M} = d\xi + \sum_{a,i} r_{ai} \omega_{ai}$$  \hspace{1cm} (4.39)

where $\xi$ is a function on $\partial M$ and $r_{ai}$ are valued in reals.

Substituting (4.39) in the second delta function in (4.38), and using (4.31) and the fact that $\int_{\partial M} \omega_{ai} d\xi = 0$ (\(\omega_{ai}\)'s being closed one-forms at the boundary), we finally get

$$Z_\lambda \sim \delta [dA] \prod_{a,i} \delta (r_{ai} - m_{ai}).$$  \hspace{1cm} (4.40)

Thus, integrating out $\lambda$ gives exactly the conditions (4.25) and (4.26) that we wanted and shows that $S$ is equivalent to the original action (4.19).

If on the contrary, we choose to integrate out the $A$ field from the action $S'$ in (4.27), we get

$$Z_A = \int D Ae^{i \frac{k}{2\pi} \int \alpha d\alpha + \frac{k}{2\pi} \int A d\alpha + \frac{k}{2\pi} \int d\lambda A} \sim \delta (\frac{k}{2\pi} d\alpha - \frac{1}{2\pi} d\lambda) e^{i \frac{k}{2\pi} \int \alpha d\alpha}.$$  \hspace{1cm} (4.41)

Since the delta functional here implies that

$$d\alpha = \frac{1}{k} d\lambda,$$  \hspace{1cm} (4.42)

we have

$$\alpha = \frac{1}{k} \lambda + \omega^{(1)},$$  \hspace{1cm} (4.43)

$\omega^{(1)}$ being a closed one-form on $M$.

Thus $Z_A$ can be simplified to

$$Z_A \sim \delta (\frac{k}{2\pi} d\alpha - \frac{1}{2\pi} d\lambda) e^{i \frac{k}{2\pi} \int_M \lambda d\lambda + \frac{1}{2\pi} \int_M \omega^{(1)} d\lambda} \hspace{1cm} (4.44)$$

The last term in the exponent above is a surface term because $\omega^{(1)}$ is a closed one-form. Hence the “dual” action obtained by integrating out $A$ is

$$S_d = \frac{1}{2\pi k} \int_M \lambda d\lambda - \frac{1}{2\pi} \int_{\partial M} \omega^{(1)} \lambda$$  \hspace{1cm} (4.45)
where $\lambda$ is subject to the condition

$$\oint_{C \in \partial M} \lambda \in 2\pi \mathbb{Z}, \ C = \text{any cycle on } \partial M. \quad (4.46)$$

Since the second term in $(4.45)$ is a surface term, it does not contribute to the equations of motion. Moreover, on using the equations of motion $d\lambda = 0$ arising from the first term, we see that the second term vanishes.

Although we have worked in this Section first with a single scalar field and then with a single CS field, these considerations generalize (in a sense to be made precise below) to the case with many scalar fields coupled by a matrix $G_{ij}$ (as in [17]) and the case with many CS fields coupled by a matrix $K^{IJ}$ as in the previous Sections.

The procedure to get the dual theory is always as follows [26]:

1. Introduce a gauge field $A$ for some particular transformation that is a “symmetry” of the action (be it the scalar field theory or the CS theory).
2. Introduce a Lagrange multiplier field which constrains $A$ by the conditions $dA = 0$ and $\oint_{C} A \in 2\pi \mathbb{Z}$.
3. Integrate out the original gauge fields to obtain the “dual” theory containing the Lagrange multiplier field.

We get different dual theories, depending on the “symmetries” we choose to gauge. It should however be noted that the duality group that we get using this procedure is still not the full $O(d,d; \mathbb{Z})$ [18] group, because we do not have a method of incorporating antisymmetric matrices (which are needed for the $O(d,d; \mathbb{Z})$ transformations) in this approach.

5 Concluding Remarks

There is a prevalent point of view that CS theories involving several vector potentials are quite effective in reproducing the Hall effect. In this paper we have provided further
evidence in support of this viewpoint by showing that the Haldane hierarchy can be implemented using a sequence of such CS theories.

The connection of these CS theories to chiral scalar field theories at the edge has also been demonstrated. The argument consisted of three stages. The first is that the algebra of observables is the same for these two theories. The second is that both give rise to the same Hall conductivity in the bulk. The third (and perhaps the most important part of the argument) is due to the fundamental requirement of gauge invariance. The CS theory when gauged gives rise to an effective CS theory for the electromagnetic potential. This is not gauge invariant and requires a surface action to restore gauge invariance. The gauged chiral scalar field theory at the boundary serves precisely as this surface action.

Another interesting result we have in this paper has to do with a generalization of the duality transformations for scalar field theories \[18, 19\]. In a previous work \[17\], we showed how such duality transformations relate Hall conductivities at different fractions. There we worked purely with a chiral scalar field theory at the edge to arrive at this result. It is therefore satisfying to note that analogous duality transformations exist also for the CS theory in the bulk. At this point, we have only an analogue of the \(R \rightarrow 1/R\) duality for the CS theory. It would be interesting to check if we can also obtain an \(O(d, d; \mathbb{Z})\) duality for the CS theory with \(d\) CS fields.

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A Appendix

In Section 4, the identification of \( \lambda \) in (4.7) with \( \lambda + 2\pi \) was necessary for the arguments presented there. Here we will give an argument showing that this condition is a particular case of a more general condition that is required for consistency of the theory.

Consider the action (4.7) rewritten as below:

\[
S' = \frac{R^2}{4\pi} \int d^2x [(\dot{\phi} + W_0)^2 - (\phi' + W_1)^2] + \frac{1}{2\pi} \int d^2x (\dot{\lambda}W_1 - \lambda'W_0) \quad (A.1)
\]

Let \( \Pi_\phi \) be the momentum conjugate to \( \phi \). Then we have the equal time commutation relation

\[
[\Pi_\phi(x), \phi(y)] = -i\delta(x - y) \quad (A.2)
\]

Also (A.1) implies that \( W_1 \) is the momentum conjugate to \( \lambda \) so that \([W_1(x), \lambda(y)] = -\frac{i}{2\pi} \delta(x - y)\).

The Hamiltonian that follows from (A.1) is

\[
H = \int dx [\frac{\pi}{R^2} \Pi_\phi^2 + \frac{R^2}{4\pi} (\phi' + W_1)^2], \quad (A.3)
\]

while the Gauss law is

\[
\Pi_\phi(x) - \frac{1}{2\pi} \lambda'(x) \approx 0. \quad (A.4)
\]

The global Gauss law that follows by integrating above expression over the whole of space (here a circle) is

\[
Q = \int dx \Pi_\phi(x) - \frac{1}{2\pi} \Delta \lambda \quad \Delta \lambda = \lambda(2\pi) - \lambda(0). \quad (A.5)
\]

Now, since \( \phi \approx \phi + 2\pi \),

\[
\text{Spectrum of } \int dx \Pi_\phi(x) = Z + c, \quad c \text{ a constant.} \quad (A.6)
\]

As \( Q \) must vanish on states, we end up with the requirement

\[
\Delta \lambda = 2\pi Z + 2\pi c. \quad (A.7)
\]
With these conditions, $Q$ generates $U(1)$ on arbitrary states, and Gauss law picks out singlet.

We thus see that the requirement that $\lambda$ be identified with $\lambda + 2\pi$ is natural provided $c$ in (A.6) vanishes.

Thus for $c = 0$, the $\lambda$ obtained from canonical quantization has an expansion in the form (4.33) showing that the identification of $\lambda$ with $\lambda + 2\pi$ is natural, in the canonical approach, for this particular value of $c$.

Nonzero values of $c$ too can be incorporated in the functional integral as well by identifying $\lambda$ with $\lambda + 2\pi(1 + c)$. It then becomes appropriate for a scalar field canonically quantized with $c \neq 0$. But we will not enter into this generalization of the functional integral here.

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