Critical behavior of the contact process on small-world networks

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Abstract. We investigate the role of clustering on the critical behavior of the contact process (CP) on small-world networks using the Watts-Strogatz (WS) network model with an edge rewiring probability \( p \). The critical point is well predicted by a homogeneous cluster-approximation for the limit of vanishing clustering \( (p \to 1) \). The critical exponents and dimensionless moment ratios of the CP are in agreement with those predicted by the mean-field theory for any \( p > 0 \). This independence on the network clustering shows that the small-world property is a sufficient condition for the mean-field theory to correctly predict the universality of the model. Moreover, we compare the CP dynamics on WS networks with rewiring probability \( p = 1 \) and random regular networks and show that the weak heterogeneity of the WS network slightly changes the critical point but does not alter other critical quantities of the model.

PACS. 89.75.Hc Networks and genealogical trees – 05.70.Jk Critical point phenomena – 64.60.an Finite-size systems – 05.70.Ln Nonequilibrium and irreversible thermodynamics

1 Introduction

The Watts-Strogatz (WS) model is recognized as a typical example of a small-world network [1]. The small-world property in a network constituted by vertices connected by edges means that the average shortest path between vertices, \( \langle l \rangle \), increases logarithmically or slower with the number of vertices \( N \). This property is a central feature shared by many complex networks [2]. Differently from scale-free (SF) networks, which do not present a characteristic scale for the fluctuations around the average number of connections (the vertex degree) \( \langle k \rangle \), the WS networks have a narrow degree distribution \( P(k) \), defined as the probability that a randomly chosen vertex has degree \( k \), which decays exponentially fast for large \( k \). These networks are commonly treated as homogeneous [3,4] even being heterogeneous in a strict sense.

In addition to the small-world property, a high average clustering coefficient \( \langle c \rangle \) is another important topological property exhibited by many complex systems [5]. This quantity is defined by [6]

\[
\langle c \rangle = \frac{1}{N} \sum_i \frac{e_i}{k_i(k_i - 1)/2},
\]

where \( e_i \) is the number of edges between the neighbors of a vertex \( i \) and the denominator contains the maximum number of these edges in a vertex with \( k_i \) connections.

In the WS model, the clustering is controlled by the rewiring probability \( p \) such that this model interpolates from a regular lattice for \( p = 0 \) to an almost random network for \( p = 1 \). Despite of the high clustering coefficient observed in many real-world networks presenting a long-tailed degree distribution \( P(k) \sim k^{-\gamma} \) \( (2 < \gamma < 3) \), random SF networks have a vanishing clustering in the thermodynamic limit [7,8]. Contrastingly, both small-world and high clustering properties can coexist in WS networks, providing an interesting substrate to study dynamical processes [9,10,11].

Simple dynamical processes with absorbing states exhibit rich features when running on the top of SF networks [12,13,14,15]. The simplest example is the contact process (CP) [16], a paradigmatic interacting particle system involving spontaneous annihilation of particles at unitary rate and catalytic creation in a pair of occupied/empty nearest neighbors at a rate \( \lambda \). The configuration devoid from particles is called absorbing: once the system visited an absorbing configuration the dynamics remains permanently trapped within this state. The model undergoes an absorbing state phase transition for a critical value \( \lambda_c \) of the control parameter [17].

The CP dynamics has been investigated in networks using the heterogeneous mean-field (HMF) theory [18,19,20], in which all dynamical correlations are disregarded and quantities of interest depend only on the vertex degree [21]. According to the HMF theory for the CP, the critical density of particles \( \rho \) and characteristic time \( \tau \) obey the finite-size scaling (FSS) relations [19]

\[
\rho \sim (gN)^{-1/2} \quad \text{and} \quad \tau \sim (N/g)^{1/2},
\]

where the factor \( g = (k^2)/\langle k \rangle^2 \) depends on the form of \( P(k) \) and introduces an anomalous dependence on the cutoff \( k_c \) of the distribution \( P(k) \) for the case of SF networks [19]. Even for quenched SF networks having an asymptotically null clustering coefficient, the HMF the-
ory is able to predict the correct FSS exponents of the CP dynamics independently of the presence of dynamical correlations [14]. However, dynamical correlations lead to wrong predictions of the critical points.

While the CP dynamics on nonclustered SF networks has been subject of several recent studies [12, 13, 14, 18], the role of the clustering has not been carefully investigated up to this moment. In this work, we address this issue by investigating the criticality of the CP model on the top of WS networks with different rewiring probabilities. Our results show that the critical exponents agree with those of the HMF theory even for highly clustered networks with \( p > 0 \). On the other hand, the position of the critical point strongly depends on the parameter \( p \). In addition, we compare the WS results with those for non-clustered homogeneous networks obtained with the random regular (RR) networks [15].

We have organized this paper as follows. The model is presented in section 2. Section 3 is devoted to a brief review of the critical properties of the CP in a mean-field level. Simulations are presented and discussed in section 4. Our conclusions and remarks are drawn in section 5. Two appendixes with pair and three-vertex approximations for the CP complement the paper. These approximations, particularly that for three-vertices, can be useful in the investigation of other similar problems.

2 The model

The CP is a reaction-diffusion system including self-annihilation and catalytic creation of particles in pairs of occupied and empty vertices [16]. The CP rules in arbitrary graphs are defined as follows [17]: A vertex \( i \) can be occupied (\( \sigma_i = 1 \)) or empty (\( \sigma_i = 0 \)). At a rate \( \lambda \), occupied vertices try to create an offspring in one of their nearest neighbors selected at random. Creation events successes only on empty vertices. An occupied vertex becomes spontaneously empty at unitary rate while an empty vertex \( i \) is occupied at a rate \( \lambda \sum_j \sigma_j/k_j \) where the sum runs over all neighbors of \( i \) and \( k_j \) is the respective vertex degree. This dynamics can drive the system to a frozen phase devoid from particles, the absorbing state. However, if the control parameter \( \lambda \) is sufficiently large the active phase becomes stable and a finite fraction of the network is occupied for \( t \to \infty \). The stationary density of occupied vertices \( \rho \) is the order parameter. The critical value \( \lambda_c \) separates the active (\( \rho > 0 \) for \( \lambda > \lambda_c \)) and absorbing (\( \rho = 0 \) for \( \lambda \leq \lambda_c \)) phases.

We used the original WS model [11] as the underlying substrate for the CP dynamics. This model allows to interpolate from a regular lattice to a random network combining two important topological features: a high clustering and a low average shortest path between vertices. According to the WS model [11], a small-world network can be built from \( N \) vertices initially ordered in a one-dimensional lattice with periodic boundary conditions, in which each vertex has \( K \) connections with the nearest neighbors. Vertices are clockwise visited and for each vertex the clockwise edges are rewired with probability \( p \), also in the clockwise sense. The rewiring rules generate connected networks and conserve the number of edges implying that \( \langle k \rangle = K \). We used \( K = 6 \) in our simulations.

The rewiring of edges introduces \( pNK/2 \) long-range connections, reducing the average shortest path \( \langle l \rangle \) between vertices. For \( p = 0 \), the network is a regular lattice with \( \langle l \rangle \approx N/2K \) and a high average clustering coefficient \( \langle c \rangle \approx 3/5 \) [3]. On the other hand, for \( p \to 1 \), the network converges to both vanishing clustering \( \langle c \rangle \sim K/N \) and small average shortest path \( \langle l \rangle \sim \ln(N)/\ln(K) \), typical of random networks [3]. Interestingly, there is a broad range of \( p \) values where a large \( \langle c \rangle \) and a short \( \langle l \rangle \) are concomitant. However, for any finite \( p \) and sufficiently large sizes, the presence of shortcuts in the networks renders the small-world property [9, 22].

3 Mean field approximation for the CP on networks

Analytical insights for the critical behavior of the CP on random networks can be obtained from the HMF theory [12, 15, 19, 20]. Assuming networks with degree distribution \( P(k) \) and the absence of dynamical and degree correlations, the HMF equation for \( \rho_k \) is [12]:

\[
\frac{d\rho_k}{dt} = -\rho_k + \frac{\lambda k}{\langle k \rangle} (1 - \rho_k) \rho, \tag{2}
\]

where

\[
\rho = \sum_k P(k) \rho_k
\]

is the overall density of particles in the networks. The WS network is asymptotically free from degree correlations due to the fast decay of the \( P(k) \) for large \( k \). It was shown that the critical point for Eq. (2) is \( \lambda_c = 1 \) independently of the degree distribution [19].

In the stationary state, \( d\rho(t)/dt = 0 \), we have

\[
\rho_k = \frac{\lambda k \rho / \langle k \rangle}{1 + \lambda k \rho / \langle k \rangle} \tag{3}
\]

Near the critical point, such that \( k \rho \ll 1 \), Eq. (3) yields

\[
\rho_k \sim \lambda k \rho / \langle k \rangle. \tag{4}
\]

Plugging this result in Eq. (2) we obtain

\[
\frac{d\rho}{dt} = -\rho + \lambda (1 - g \rho) \rho. \tag{4}
\]

where \( g = \langle k^2 \rangle / \langle k \rangle^2 \) carries the dependence on the degree distribution \( P(k) \). For the WS networks the factor \( g \) is practically independent of \( N \). Therefore, we found the usual mean-field result for CP where a continuous phase transition \( \rho \sim (\lambda - \lambda_c)^\beta \) with \( \lambda_c = 1 \) and \( \beta = 1 \) are found.

In a mean-field level, equation (4) is the macroscopic equation of the one-step process [18]:

\[
w_{n-1,n} = n \]

\[
w_{n+1,n} = n(1 - gn/N)^\gamma \]

where \( w_{m,n} \) is the transition rate from the state \( m \) to the state \( n \). For finite size systems, one must consider the
quasistationary (QS) analysis where only surviving samples are considered in the averages [17]. The QS analysis of this one-step processes was performed in Ref. [20]. The central result is the probability \( P_n \) that \( n \) vertices are occupied in the QS regime is given by a scaling form

\[
\bar{P}_n = \frac{1}{\sqrt{n}} f \left( \frac{n}{\sqrt{n}} \right),
\]

(6)

where \( \Omega = N/g \) and \( f(x) \) is a scaling function. One can directly derive the scaling of the basic critical properties of the system as

\[
\bar{\rho} = \frac{1}{N} \sum_n n P_n \sim (gN)^{-1/2} \sim N^{-1/2}
\]

(7)

and

\[
\tau = \frac{1}{N} \sim (N/g)^{1/2} \sim N^{1/2}.
\]

(8)

The network heterogeneity still plays a role in the supercritical phase. The density of particles presents a dependence on the degree distribution given by [19]

\[
\bar{\rho} \sim \bar{\rho}_{\text{hom}} / g,
\]

(9)

where \( \bar{\rho}_{\text{hom}} \) is the solution for the strictly homogeneous networks with \( g \equiv 1 \). Equation (9) can be directly obtained either from Eq. (4) of this paper or Eq. (37) of Ref. [19].

### 4 Results

The CP simulations were performed with the usual scheme [17]: An occupied vertex \( i \) is chosen at random and the time incremented by \( 1/(1+\lambda)n(t) \), where \( n(t) \) is the number of particles at time \( t \). With probability \( p = 1/(1+\lambda) \), the particle \( i \) is eliminated. With the complementary probability \( 1 - p \), one of the \( k_i \) nearest neighbors of \( i \) is randomly chosen and, if empty, occupied. The finite size inherent to simulations will force the system to visit the absorbing configuration even for \( \lambda > \lambda_c \) at some sufficiently long time due to the stochastic fluctuations [17]. Therefore, a suitable simulation method is necessary to circumvent this problem.

We adopted a QS simulation method [23] (see Refs. [14] [15] [20] [24] for applications of the QS method in dynamical process on networks), in which the system history is used to replace the absorbing state. The method is implemented by keeping and constantly updating a list of \( M = 400 \) active configurations taken from the evolution of the system. Every time the system visits an absorbing state, this state is replaced by a configuration picked up at random from the list. An update of the list consists in replacing a stored configuration, chosen at random, by the current one with probability \( p_{\text{rep}} \). In our simulations we used \( p_{\text{rep}} = 10^{-2} \) to \( 10^{-4} \) (The large the network size the smaller the value of \( p_{\text{rep}} \)). After a relaxation time \( t_r = 10^5 \), the probability \( \bar{P}_n \) is determined during an averaging time \( t_a = 10^7 \). All relevant quantities derive from the QS distribution \( \bar{P}_n \). The QS density of particles given by Eq. (7) and the characteristic time given by Eq. (6), are the basic quantities that we use to compute critical exponents.

Since HMF theory disregards dynamical correlations between nearest neighbors, one does not expect that the mean-field critical point \( \lambda_c = 1 \) is the correct value for quenched networks [14] [25]. The critical point of the CP on networks was determined using moment ratios of the order parameter [14] [26] that are independent of the network size when \( \bar{P}_n \) has a scaling form as in Eq. (6). This method was successfully applied to find the thresholds of the dynamical processes taking place on SF networks [14] [24]. The moment ratios in the form

\[
M^q_r = \frac{\langle \rho^q \rangle}{\langle \rho \rangle^q} \text{ with } q = r + s,
\]

(10)

intersect at \( \lambda_c \) for different network sizes [14].

We investigate the CP dynamics on networks with different clustering coefficients built with rewiring probabilities \( p = 0.01, p = 0.10, \) and \( p = 1.00. \) The determination of the critical points for \( p = 0.01 \) and \( p = 0.1 \) are exemplified in Fig. 1 in which we show the moment ratio \( M^1_1 = \langle \rho^2 \rangle / \langle \rho \rangle^2 \) against creation rate. The corresponding curves for \( p = 1 \) are shown in Fig. 2. The same crossing points were obtained for higher order moment ratios. The critical points and critical moment ratios up to fourth order are reported in Table 1.

![Fig. 1. Second order moment ratio for the CP dynamics on WS networks with \( \langle k \rangle = 6 \). The main plot shows the curves for the rewiring probability \( p = 0.01 \) and the inset for \( p = 0.10 \).](image-url)
in relation to the HMF prediction \( \lambda_c = 1 \). Notice that the displacement increases for highly clustered networks. Moreover, the threshold \( \lambda_{\text{PA}} \) obtained in a standard pair-approximation for nonclustered networks (see Appendix A), which fits remarkably well critical points of CP and related models in random SF networks [20, 24, 25], is a good approximation only in the limit of \( p \rightarrow 1 \) when the network exhibits an asymptotically vanishing clustering coefficient [2]. So, a necessary condition for the accuracy of the thresholds in a pair-approximation is a low clustering. The accuracy is substantially enhanced when compared with a three-vertices approximation \( \lambda_{3}\text{VA} \) (see Appendix B) for which the difference is of only 0.2%. A simple modification of the pair-approximation including the possibility of loops, yields \( \lambda_c = k/(k - 1 - c/2) \) where \( c \) is the clustering coefficient. Details of the theory are given in Appendix A. This result qualitatively explains the decay of \( \lambda_c \) for increasing \( p \). However, this approach is still quantitatively inaccurate for small values of \( p \), as shown in Appendix A.

FSS analyses were performed for networks sizes ranging from \( 10^4 \) to \( 10^7 \). Figure 3 shows the scaling of \( \bar{\rho} \) and \( \bar{\tau} \) as functions of \( N \) at the critical point. The results for highly clustered structures \( (p = 0.01 \text{ and } p = 0.10) \) show that both quantities obey the scaling exponents of HMF theory (see Table 2). An interesting remark is that both FSS exponents as well as moment ratios of CP on WS networks agree with the corresponding values on annealed network. For highly heterogeneous SF networks, HMF theory yields exponents in good agreement with simulations but fails in predicting moment ratios dependent of \( P(k) \) [14]. Finally, corrections to the scaling, which play a central role for critical CP in highly heterogeneous substrates [14, 19, 20], are absent in WS networks.

![Fig. 2.](image)

**Table 1.** Critical points \( \lambda \) and moment critical points for CP on WS networks with different rewiring probabilities and random regular (RR) networks with \( \langle k \rangle = 6 \). The values corresponding to annealed networks (ANN) taken from Ref. [14] are also included for comparison.

| Network | \( \lambda \) | \( M_1^2 \) | \( M_2^2 \) | \( M_4^2 \) |
|---------|-------------|-------------|-------------|-------------|
| \( p = 0.01 \) | 1.7691(1)   | 1.64(4)     | 2.1(1)      | 3.2(2)      |
| \( p = 0.10 \) | 1.4498(1)   | 1.65(3)     | 2.1(1)      | 3.4(1)      |
| \( p = 1.00 \) | 1.2130(1)   | 1.65(3)     | 2.1(1)      | 3.4(1)      |
| RR      | 1.2155(1)   | 1.664(6)    | 2.18(1)     | 3.41(3)     |
| ANN     | 1.00        | 1.667(3)    | 2.190(4)    | 3.452(3)    |

![Fig. 3.](image)

**Table 2.** FSS exponents \( \hat{\nu} \sim N^{-\hat{\nu}} \) and \( \hat{\sigma} \sim N^{\hat{\sigma}} \) for the CP on WS and random regular (RR) networks with \( \langle k \rangle = 6 \). The mean-field result is also shown.

| Network | \( \hat{\nu} \) | \( \hat{\sigma} \) | \( \alpha \) | \( \lambda \) |
|---------|---------------|---------------|-----------|-----------|
| \( p = 0.01 \) | 0.50(1)       | 0.49(2)       | 1/2       |
| \( p = 0.10 \) | 0.50(1)       | 0.49(1)       | 1/2       |
| \( p = 1.00 \) | 0.50(1)       | 0.50(2)       | 1/2       |
| RR      | 0.50(1)       | 0.50(2)       | 1/2       |

The role of the clustering in the evolution of the CP on WS networks is qualitatively illustrated in Fig. 4 in which we show space-time patterns consisting of configurations taken between time intervals \( \Delta t = 1 \), sequentially ordered in time. For highly clustered networks \( (p = 0.01) \), one can neatly see localized activity lasting for a finite time. The few shortcuts are responsible by the emergence of isolated patches of activity that are forbidden in regular lattices [27]. For \( p = 0.1 \), the clustered patterns are highly reduced while the low clustered network \( p = 1.0 \) does not exhibit space-time structures due to high number of shortcuts.
Fig. 4. Space-time patterns generated by the CP dynamics on WS networks starting with all vertices occupied. Bottompanels correspond to a highly clustered network with rewiring probability \( p = 0.01 \), middle to \( p = 0.1 \) while the top considers a nonclustered network with \( p = 1.0 \). Subcritical (left), critical (center), and supercritical (right) configurations are shown.

The weak heterogeneity of WS network does not affect the universality of neither critical moment ratios nor exponents, which agree with the HMF theory. However, WS network does not have a strictly homogeneous degree distribution. In order to investigated the role of this weak heterogeneity, we compare the CP dynamics on WS networks with \( p = 1 \) (the most randomized case) with that on RR networks. In the latter, all vertices have exactly the same degree but connections are done at random excluding multiple and self-connections [15]. The critical points are slightly different as one can see in Table 1. Curiously, the critical point for the strictly homogeneous RR networks is further from mean-field critical point (\( \lambda_c \approx 1.20103 \) in a three-site approximation) than WS networks with \( p = 1 \). We expect that higher order mean-field theories must converge to the critical point observed for the homogeneous RR networks since heterogeneity is expected to play some role in dynamical correlations and consequently in the critical point. Figure 5 shows the FSS of the critical quantities \( \bar{\rho} \) and \( \tau \). As expected, the critical exponents and moment ratios for CP on RR networks agree very well with the mean-field theory as shown in tables 1 and 2.

Further comparisons between CP dynamics on WS and RR networks are given in Fig. 6, where QS density is shown as a function of the creation rate. The critical exponent \( \beta = 1.05(4) \) agrees with the mean-field exponent \( \beta = 1 \). The density predicted by the mean-field theory, Eqs. (14) or (24) suitably shifted to vanish at the real critical point, describes the simulations for RR networks much better than for WS networks. If, nevertheless, the numerically calculated factor \( g = 1.0832 \) is explicitly included in the heterogeneous mean-field theory, Eq. (9), a very good agreement is also found for WS networks.

Fig. 5. Double logarithmic plots of the QS quantities of the critical CP on WS and RR networks with \( \langle k \rangle = 6 \). Dashed line has a slope 1/2.

Fig. 6. QS density against the creation rate for CP dynamics on WS networks with \( p = 1 \) and RR networks, both with \( N = 10^7 \) vertices. Solid lines represent the mean-field solution (See appendix B) shifted to vanish at the actual critical points given in Table 1. Dashed line shows the solution predicted by heterogeneous mean-field theory, Eq. (9). Inset shows the supercritical density against the distance to critical point.

5 Conclusions

The Watts-Strogatz model [11] generates networks that interpolate between regular lattices and random graphs allowing to adjust the clustering of the network with a rewiring probability \( p \). The small-world property is always present on large networks for any \( p > 0 \) while the clustering vanishes as \( p \to 1 \). We investigated the role of these structural properties on the dynamics of the contact processes, focusing on the critical properties of the ensuing absorbing state phase transition. We show that the critical exponents do not change when the rewiring probability and, consequently, the clustering coefficient are varied in a wide range. Thus, our results corroborate previous evidences that mean-field approximations yield the correct critical exponents of statistical physics models running on...
the top of WS networks for any finite rewiring probability [0,28].

We also analyzed dimensionless moment ratios and show that universal values are found independently of the rewiring probability p > 0. The ratios are equal to those obtained for annealed networks [14], for which they are independent of the degree distribution $P(k)$, differently from the distribution dependence observed in scale-free quenched networks [14]. Moreover, our results support the conjecture that the moment ratios converge to the annealed values when $\gamma \rightarrow \infty$ [17].

Differently from the exponents and moment ratios, the critical points strongly depend on the rewiring probability. Mean-field cluster approximations (see appendixes A and B) yield a very good prediction for critical point in the limit $p \rightarrow 1$, but fails for highly clustered networks. In fact, the existence of loops is essential to the critical point determination as qualitatively clarified in a modified pair mean-field theory (Appendix A).

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A Homogeneous pair-approximation

In a homogeneous pair-approximation, the states of the vertices do not depend on their degrees. So, let us introduce the notation $[A]$ to represent the probability that a vertex is in the state $A$, $[AB]$ that a pair of nearest neighbors are in states $A$ and $B$, and $[ABC]$ the generalization for a cluster of three vertices. Representing an occupied vertex by 1 and an empty one by 0, we introduce the following notation: $[1] = \rho$, $[0] = 1 - \rho$, $[10] = [01] = \phi$, $[11] = \psi$, $[00] = \omega$. The normalization conditions $\psi + \phi = \rho$ and $\psi + 2\phi + \omega = 1$ are derived straightforwardly. In addition, we assume that the chance of two vertices to have one or more common neighbors is negligible, which is an incorrect premise for highly clustered networks. This approach is equivalent to the one used in Ref. 29 for the susceptible-infected-recovered-susceptible epidemic model. The following dynamical equations are obtained for CP dynamics in a homogeneous network with a fixed vertex degree $k$

$$\frac{d\rho}{dt} = -\rho + \lambda\phi$$

and

$$\frac{d\phi}{dt} = -\phi - \lambda\phi\frac{c}{k} + \psi + \lambda \left(\frac{k-1}{k}\left[001\right] - \frac{k}{k}\left[010\right]\right).$$

The first three terms of Eq. (12) reckon the reactions inside the pair that contribute to change the density $[01]$. The fourth term increases the density $[01]$ due to the creation in the right vertex of an empty pair 00 due to its other $k-1$ neighbors. Finally, the last one represents the creation in the empty vertex of a pair 01.

We now truncate triplets in Eq. (12) using the pair-approximation 30:

$$[ABC] \approx \frac{[AB][BC]}{[B]},$$

resulting

$$\frac{d\phi}{dt} = -\phi - \lambda\phi\frac{c}{k} + \lambda\left(\frac{k-1}{k}\left[001\right] - \frac{k}{k}\left[010\right]\right).$$

Solving equations (11) and (13) in the stationary state $(d\rho/dt = d\phi/dt = 0)$ and using normalization conditions, we obtain

$$\tilde{\rho} = \frac{\lambda - \lambda_c}{1 + \lambda - \lambda_c}.$$ (14)

with the critical point

$$\lambda_c = \frac{k}{k-1}. \quad (15)$$

A simple approximation to clustered networks is derived assuming that the triplets form a loop with probability $c$, the clustering coefficient. In the case of loops, only $k-2$ neighbors contribute to the creation/annihilation of 01’s in the last terms of Eq. (12). We then rewrite Eq. (12) as

$$\frac{d\rho}{dt} = -\phi - \lambda\phi\frac{c}{k} + \lambda\left(\frac{k-1}{k}\left(1 - c\right) + \frac{k-2}{k}\right) - \psi \right)$$

$$\times ([001] - [101]).$$

Performing the pair-approximation we have

$$\lambda_c = \frac{k}{k-1 - c/2}, \quad (17)$$

which qualitatively explains the critical points increasing for larger clustering.

The clustering coefficient of WS model as a function of $p$ is well described by $9$

$$c(p) = \frac{3}{4}(K-1)(1-p)^3. \quad (18)$$

Using the clustering coefficient given by Eq. (18), one obtains from the Eq. (17) $\lambda_c = 1.2742$ and 1.2548 for $p = 0.01$ and $p = 0.10$, respectively, which are quite far from the transition points observed in simulations of highly clustered networks given in Table [11].

B Homogeneous three-site approximation

Let us introduce the notation for triplets: $a = [001] = [100]$, $b = [011] = [110]$, $c = [111]$, $d = [000]$, $e = [010]$, and $f = [101]$. We have that the following independent relations hold: $a + f = \phi$, $b + c = \psi$, $e + b = \phi$, $d + a = \omega$. We need two more independent equations. We have chosen to
write them for $a$ and $e$. The resulting dynamical equations are

$$\frac{da}{dt} = -\left(1 + \frac{\lambda}{k}\right)a + b + f + \lambda \frac{k - 1}{k}[1001] - \lambda \frac{k - 2}{k}[1001]$$

and

$$\frac{de}{dt} = -\left(1 + \frac{2\lambda}{k}\right)e + 2b + \lambda \frac{k - 2}{k}[1000] - 2\lambda \frac{k - 1}{k}[1010].$$

The first three terms of Eq. (19) represent the reactions inside the triplet 001, the forth term represents the events caused by the creation due to the $k - 1$ nearest neighbors of the rightmost and leftmost vertices of the triplets 000 and 001, respectively, and the last one reckons the creation due to the $k - 2$ nearest neighbors of the middle vertex of the triplet 001. Similar interpretations hold for Eq. (20).

We now replace quadruplets using the cluster approximation [30]

$$[ABCD] \approx \left[\frac{ABC}{BC}\right]$$

to obtain

$$\frac{da}{dt} = -\left(1 + \frac{\lambda}{k}\right)a + b + f + \lambda \frac{k - 1}{k}\omega - 2\lambda \frac{2k}{k + 1} - 3\frac{a^2}{\omega}$$

and

$$\frac{de}{dt} = -\left(1 + \frac{2\lambda}{k}\right)e + 2b + \lambda \frac{k - 2}{k}\omega - 2\lambda \frac{k - 1}{k}\phi$$

Solving Eqs. (19), (21), (22) and (23) with the normalization conditions, the stationary density is

$$\bar{\rho} = \frac{(k - 1)(3k - 4)(\lambda - \lambda_c)(\lambda + \lambda_c - \frac{2k}{2k - 4})}{\lambda(k - 1)(3k - 4)\lambda + (k - 4) - k},$$

where the critical point is given by

$$\lambda_c = \frac{k + 2\sqrt{k^2 - k}}{3k - 4}.$$

It is worth mentioning that independently of the apparently very different forms of Eqs. (19) and (24) they are almost indistinguishable in plots $\bar{\rho}$ versus $\lambda - \lambda_c$.

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