Correlations in noisy Landau-Zener transitions

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We analyze the influence of classical Gaussian noise on Landau-Zener transitions during a two-level crossing in a time-dependent regular external field. Transition probabilities and coherence factors become random due to the noise. We calculate their two-time correlation functions, which describe the response of this two-level system to a weak external pulse signal. The spectrum and intensity of the magnetic response are derived. Although fluctuations are of the same order of magnitude as averages, the results is obtained in an analytic form.

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I. INTRODUCTION

The Landau-Zener (LZ) theory [1, 2] (see also [3]) plays an important role in many different physical problems ranging from chemistry, biology, theory of collisions to the tunneling of Bose-condensates, dynamics of glasses and spin reversal in magnetic wires and nanomagnets, as well as quantum computing. LZ theory treats a rather simple and general situation of two-level crossing. Near the point of (avoided) level crossing, adiabaticity is violated and transitions occur. The LZ Hamiltonian can be represented by a 2×2 matrix acting in a two-dimensional space of vectors spanned by the basis vectors |↑⟩ and |↓⟩,

\[ H_{LZ} = \begin{pmatrix} E_↑ & \Delta \\ \Delta^* & E_↓ \end{pmatrix}, \]

where \( E_↑ = -E_↓ = -\frac{\hbar}{2}\nu t \) and the transition matrix element \( \Delta \) does not depend on time. Without loss of generality one can assume \( \Delta \) to be a positive constant. Landau and Zener have found the evolution matrix

\[ U_{LZ} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \]

which connects initial amplitudes in the diabatic basis \( |\psi_0\rangle \) with their final values. The entries

\[ a = e^{-\pi \gamma^2}, \quad b = -\frac{\sqrt{2\pi}}{\gamma \Gamma(i\gamma^2)} e^{-\pi \gamma^2/2-i\pi/4} \]

depend only on the dimensionless parameter

\[ \gamma = \frac{\Delta}{\hbar \nu} \]

which we call the LZ parameter.

In many applications the LZ theory must be modified to take into account thermal noise or noise of a different nature. In a pioneering work Kayanuma [4] has solved such a problem on the basis of three simplifying assumptions: i) the regular transition matrix element is equal to zero (i.e., \( \Delta = 0 \)), the noise is completely responsible for transitions; ii) the correlation function of noise has a simple exponential form; iii) the noise is fast. The first two limitations were lifted in the work by Pokrovsky and Sinitsyn [4]. Their analysis of time scales for different processes distinguishes the correlation time of noise \( \tau_n \), the accumulation time \( \tau_{acc} = (\nu \tau_n)^{-1} \) during which the noise effectively produces transitions, and the LZ time \( \tau_{LZ} = \Delta/(\hbar \nu) \) during which the standard LZ transition (without noise) proceeds. The noise is called fast if \( \tau_n \ll \tau_{LZ} \ll \tau_{acc} \). In this case the separation of times allows to neglect the regular matrix element \( \Delta \) at \( |t| \gg \tau_{LZ} \) and to neglect the noise in the interval \( |t| \ll \tau_{acc} \). Since these intervals overlap, one can solve first the reduced problem with the noise producing transitions and then match this solution with the LZ solution. In this way the authors of Ref. [4] have found the average value of the density matrix.

The two-level problem is as usual equivalent to the problem of spin 1/2 rotating in the time-dependent field containing regular and stochastic parts. In their second work [5] the same authors represented the density matrix in terms of the Bloch vector \( \mathbf{g} \),

\[ \hat{\rho}(t) = \frac{1}{2} [\mathbf{1} + \mathbf{g}(t) \cdot \hat{\sigma}] , \]

where \( \hat{\sigma} \) is the triplet of Pauli matrices for spin 1/2. It is a well known fact that the square of the Bloch vector is an integral of motion. In Ref. [5] this fact was used to calculate the square fluctuation

\[ \langle |\delta \mathbf{g}(t)|^2 \rangle = \mathbf{g}^2 - \langle \mathbf{g}(t) \rangle^2 . \]

However, the fluctuations of separate components, \( \langle |\delta g_i(t)|^2 \rangle, (i = x, y, z) \) were not calculated, neither two-time correlators \( \langle g_i(t)g_j(t') \rangle \) were found.

Such fluctuations are observable, for example, in a system of magnetic molecules subject to the same magnetic field (with identical regular and random contributions). The measurement of the z component of
In two-level systems the fluctuation will be suppressed if the system feel different random fields. Therefore, by an environment, for example, by thermal phonons or magnetic field is much larger than the linear size of the system.

In two-level systems the random field can be realized by an environment, for example, by thermal phonons or magnetic field of nuclei. In such a situation different two-level systems feel different random fields. Therefore, in the ensemble consisting of a large number of members there will be fluctuations proportional to $1/\sqrt{N}$. However, the two-time correlation function describes the linear response of such a system to a small perturbation $\delta h(t)$ (identical for all members) according to a standard linear-response equation:

$$\delta \langle s_i \rangle = \int_{-\infty}^{t} dt' \langle s_i(t) s_j(t') \rangle \delta h_j(t'). \quad (6)$$

Such a perturbation can be realized as a pulse of electric magnetic or acoustic field. In this work we determine all these correlations under the same assumption of fast noise.

The outline of the paper is as follows. In Sec. II we formulate the model for LZ transitions in the presence of noise we focus on. In Sec. III we calculate in a general form the time evolution of averages and autocorrelations of the Bloch vector in the absence of the regular transition matrix element $\Delta$. These general findings are evaluated to obtain the averages and fluctuations of transition probabilities first for $\Delta = 0$ in Sec. IV and then also for $\Delta > 0$ in Sec. V. Implications for the spectral width and intensity of fluctuations for transitions in a gas of colliding atoms or molecules are discussed in Sec. VI. We close with concluding remarks in Sec. VII. Some calculational details are collected in an appendix.

II. FUNDAMENTAL

We consider the dynamics of a quantum spin $\hat{S}$ in the presence of a time dependent effective magnetic field $\mathbf{B}(t)$. Absorbing the Landé factor and the Bohr magneton into this field, the Hamiltonian reads

$$\hat{H}(t) = -\mathbf{B}(t) \cdot \hat{S} \quad \text{. (7)}$$

The effective field $\mathbf{B}(t) = \mathbf{b}(t) + \eta(t)$ is composed of a regular part $\mathbf{b}$ and a noisy part $\eta$ with zero average, $\langle \eta(t) \rangle = 0$. In the LZ theory, an avoided level crossing is described by the regular field which may be chosen as

$$\mathbf{b}(t) = b_z(t) \mathbf{e}_z + b_x \mathbf{e}_x \quad \text{. (8)}$$

with a $z$ component approximately linear in time,

$$b_z(t) \approx vt, \quad \nu = b_z(0), \quad \text{. (9)}$$

and an approximately time independent perpendicular component $b_x = -(2/\hbar)\Delta$. The noise is considered as Gaussian distributed and uncorrelated in the longitudinal and transverse directions with variances $(i, j = x, y, z)$

$$\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} f_i(t - t') \quad \text{. (10)}$$

The correlation functions $f_i$ naturally are even functions. A further planar symmetry $f_x = f_y \equiv f_\perp$ is assumed.

Under the action of the Hamiltonian $\hat{H}$, the dynamics of the density matrix is equivalent to the classical equation of motion (Bloch equation)

$$\dot{\mathbf{g}}(t) = -\nabla(t) \times \mathbf{g}(t) \quad \text{. (11)}$$

For the analysis of the Bloch equation it is convenient to introduce

$$g \equiv \frac{1}{\sqrt{2}} (g_x + ig_y) \quad \text{. (12)}$$

as well as $\eta_\perp, b_\perp$, and $B_\perp$ by analogous definitions. In these terms the Bloch equation can be rewritten as

$$\dot{\eta}_\perp = \mp i B_\perp g_\perp \pm i B_\perp g_z \quad \text{. (13a)}$$

$$\dot{g}_\perp = i (B_- g_+ - B_+ g_-) \quad \text{. (13b)}$$

From these equations the longitudinal field component $B_z(t)$ can be eliminated by going over to the interaction representation with respect to the Hamiltonian

$$\hat{H}_0(t) = -B_z(t) \hat{S}_z \quad \text{. (14)}$$

Using the time evolution operator

$$\hat{U}_0(t) = \exp \left\{ \frac{1}{\hbar} \int_{t_0}^{t} dt' \hat{H}_0(t') \right\} \quad \text{. (15)}$$

we rewrite the density matrix $\hat{\rho}$ as

$$\dot{\hat{\rho}}(t) = \frac{1}{2} \left[ \hat{1} + \hat{U}_0(t) \hat{g}(t) \hat{\sigma} \hat{U}^\dagger_0(t) \right] \quad \text{. (16)}$$

in terms of the transformed Bloch vector

$$\hat{g}_\pm(t) = g_\pm(t) \exp \left\{ \pm i \int_{t_0}^{t} dt' B_z(t') \right\} \quad \text{. (17)}$$

The transformation does not affect the longitudinal component, $\hat{g}_z(t) = g_z(t)$. Using analogous transformations for $\hat{\eta}$, $\hat{b}$ and $\hat{B}$, the Bloch equation simplifies to

$$\dot{\hat{g}}_\pm = \pm i \hat{B}_\pm \hat{g}_\pm \quad \text{. (18a)}$$

$$\dot{\hat{g}}_z = i [\hat{B}_- \hat{g}_+ - \hat{B}_+ \hat{g}_-] \quad \text{. (18b)}$$
In the limit of short noise correlation time $\tau_n$, the correlations for the transformed noise read

\[
\langle \tilde{\eta}_+(t)\tilde{\eta}_+(t') \rangle = \langle \tilde{\eta}_-(t)\tilde{\eta}_-(t') \rangle = 0, \tag{19a}
\]

\[
\langle \tilde{\eta}_+(t)\tilde{\eta}_-(t') \rangle = \tilde{f}_\perp(t, t'), \tag{19b}
\]

with the correlator

\[
\tilde{f}_\perp(t, t') \equiv f_\perp(t - t') \exp \left\{ i \int_{t'}^t dt'' B_z(t'') \right\} 
\approx f_\perp(t - t') e^{i\omega(t^2 - t'^2)/2}. \tag{19c}
\]

Since $f_\perp(t - t')$ decays for $|t - t'| \gg \tau_n$ one may neglect the contribution of $\tilde{\eta}_z(t'')$ to the time integral provided

\[
\tau_n^2(\tilde{\eta}_z^2) \ll 1. \tag{20}
\]

This condition is assumed in the following. It can be considered as a further limitation for the noise correlation time or as a limitation for the noise amplitude. However, it is not crucial and is introduced only for simplification. If the longitudinal noise is independent of the transverse one, its contribution to the averaged exponent in equation (19) can be reduced to the Debye-Waller factor $W(t, t') = \exp \left\{ -\frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 (\eta(t_1)\eta(t_2)) \right\}$. Theory becomes less transparent and more cumbersome, but in essence remains the same except for the range of very strong longitudinal noise.

Although the longitudinal noise is not effective in $\tilde{f}_\perp$, it cannot be neglected in general. From Eq. (19) one recognizes that $\eta_z$ leads to a diffusive precession of the transverse components of $\mathbf{g}$. Therefore, the expectation values $\langle g_\pm \rangle$ decay exponentially on the dephasing time

\[
\tau_\phi = \frac{2}{F_{z0}} \tag{21}
\]

with $F_{z0} = \int_{-\infty}^{\infty} dt \ (\langle \tilde{\eta}_z(t)\tilde{\eta}_z(0) \rangle)$. For short noise correlations, $\tau_\phi \sim 1/\tau_n(\tilde{\eta}_z^2)$. Thus, longitudinal dephasing can be neglected during the accumulation period $|t| \ll \tau_{acc} = 1/\nu\tau_n$ of transverse noise for $\langle \tilde{\eta}_z^2 \rangle \ll \nu$. On the other hand, on sufficiently large time scales $|t| \gg \tau_\phi$ dephasing always sets in. If the dynamics of the system is followed over that long times, the averages of $g_x$ and $g_y$ vanish because of the random precession around the $z$ axis. Nevertheless, in this case the transverse amplitude

\[
g_\perp \equiv (g_x^2 + g_y^2)^{1/2} = (2g_x) = (2g_y) = (2\tilde{g}_x) = (2\tilde{g}_y)^{1/2}, \tag{22}
\]

which is identical to the absolute value of the off-diagonal element of the density matrix, can have a finite expectation value.

### III. NOISY DYNAMICS

In this section, we temporarily neglect regular transitions (we set $b_z = 0$) and focus on transitions solely due to noise. To solve the equations of motion for the propagation from an initial time $t_0$ to $t > t_0$, we integrate the equations of motion (18) to

\[
\tilde{g}_\pm(t) = \tilde{g}_\pm(t_0) + i \int_{t_0}^t dt' \tilde{g}_\pm(t') \tilde{g}_z(t') \tag{23a}
\]

\[
\tilde{g}_z(t) = \tilde{g}_z(t_0) - \int_{t_0}^t dt_1 \int_{t_1}^{t_t} dt_2 w(t_1, t_2) \tilde{g}_z(t_2) 
- \int_{t_0}^t dt' [i\tilde{\eta}_+(t') \tilde{g}_-(t_0) + cc] \tag{23b}
\]

with the vertex pair function

\[
w(t_1, t_2) = \tilde{\eta}_+(t_1) \tilde{\eta}_-(t_2) + cc. \tag{24}
\]

Wherever “cc” occurs, it stands for the complex conjugation of the preceding term.

Equation (23b) may be fed back iteratively into itself and into Eq. (23a), to express $\tilde{g}(t)$ entirely in terms of $\tilde{g}(t_0)$. Each term arising from this iteration can be visualized by a diagram. Defining the integral series

\[
W_{t_0}^t(t') = \delta(t_0 - t') - \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 w(t_1, t_2) \delta(t_2 - t') 
+ \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \int_{t_0}^{t_3} dt_4 w(t_1, t_2, t_3, t_4) \delta(t_4 - t') - \ldots \tag{25}
\]

one obtains

\[
\tilde{g}_z(t) = \int_{-\infty}^{\infty} dt' W_{t_0}^t(t') \left\{ \tilde{g}_z(t_0) - \int_{t_0}^{t'} dt'' [i\tilde{\eta}_+(t'') \tilde{g}_-(t_0) + cc] \right\}, \tag{26a}
\]

\[
\tilde{g}_\pm(t) = \tilde{g}_\pm(t_0) + i \int_{t_0}^t dt' \tilde{\eta}_\pm(t') \int_{-\infty}^{\infty} dt'' W_{t_0}^t(t') \left\{ \tilde{g}_z(t_0) - \int_{t_0}^{t'} dt'' [i\tilde{\eta}_+(t'') \tilde{g}_-(t_0) + cc] \right\}. \tag{26b}
\]
In the diagrammatic representation, every factor $W$ corresponds to an arbitrary number of vertex pairs $w$. Corresponding to the definition (24), two “polarities” have to be considered per pair (cf. Fig. 1). During the time evolution (26) of $\tilde{g}$, besides such paired vertices, “excess” vertices can appear as first or last vertices during the evolution period.

![Diagram](image)

**Fig. 1:** Diagrammatic representation of $w$ and contributions to $G_{i,n}(t, t_0)$. The time axis is chosen to point to the left. An encircled $+$ or $-$ represents a noise vertex $\tilde{\eta}_\pm$. (a) A pair of vertices connected by a gray bond represents $-w(t_1, t_2)$. A pair of bold circles contains the sum over both possible polarities. (b) To leading order in small $\tau_n$, only diagrams with noncrossing vertex contractions contribute. $G_{i,n}(t, t_0)$ consists only of noise contraction (wiggly lines) within vertex pairs. (c) $G_{i,+}(t, t_0)$ consists on a negative excess vertex as first vertex (closest to $t_0$) and a positive excess vertex as last vertex. Contractions can be performed only from pair to pair.

According to the Wick theorem for Gaussian fields, the noise averaging corresponds to all possible contractions between noise vertices. The correlations (19) imply that contractions can be performed only between vertices with opposite charge (in a pictorial language, we associate a “charge” $\pm 1$ with every vertex $\tilde{\eta}_\pm$). Therefore, only neutral diagrams (where the number of vertices $\tilde{\eta}_+$ equals the number of vertices $\tilde{\eta}_-$) do not vanish.

### A. Dynamics of averages

Before addressing correlations, we analyze which diagrams contribute to the average $\langle \tilde{g}(t) \rangle$. Because of the neutrality constraint, this average reduces to

$$\langle \tilde{g}_+(t) \rangle = \int_{-\infty}^{\infty} dt' \langle W_{i0}(t') \rangle \langle \tilde{g}(t) \rangle,$$  \hspace{1cm} (27a)

$$\langle \tilde{g}_+(t) \rangle = \left\{ 1 - \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{t''} dt''' \times \langle \tilde{\eta}_\pm(t') W_{i0}(t'') \tilde{\eta}_\mp(t''') \rangle \right\} \langle \tilde{g}(t) \rangle$$  \hspace{1cm} (27b)

In factorizing the expectation values we assume that noise before and after $t_0$ is statistically independent.

To leading order in $\tau_n \ll \tau_{acc}$, only contractions between neighboring vertices (in time order) contribute. In $\langle \tilde{g}_+(t) \rangle$, all contractions are performed within the pair functions $w(t, t')$ (cf. Fig. 1b), which (for $t - t_0 \gg \tau_n$) give rise to factors

$$\int_{t_0}^{t} dt' (w(t, t')) = F_\pm(\nu t)$$

with the Fourier transform

$$F_\pm(\Omega) = \int_{-\infty}^{\infty} d\tau \cos(\Omega \tau) f_\pm(\tau).$$  \hspace{1cm} (28)

Returning to Eq. (27a), we find that the series can be easily summed up to

$$\langle \tilde{g}_+(t) \rangle = G_{i,+}(t, t_0) \langle \tilde{g}(t) \rangle$$  \hspace{1cm} (29a)

with

$$G_{i,+}(t, t_0) = \int_{-\infty}^{\infty} dt' \langle W_{i0}(t') \rangle = \exp \left\{ - \int_{t_0}^{t} dt' F_\pm(\nu t') \right\}.$$  \hspace{1cm} (29b)

This result was obtained in the Ref. [1] via the differential equation

$$\frac{d\langle \tilde{g}_+(t) \rangle}{dt} = -F_\pm(\nu t) \langle \tilde{g}_+(t) \rangle$$  \hspace{1cm} (30)

which immediately follows from the integral formula (24) and confirms the statistical independence of the events before and after some moment of time $t$ on a time scale much larger than $\tau_n$.

Likewise, to leading order in $\tau_n \ll \tau_{acc}$, $\langle \tilde{g}_-(t) \rangle$ consists only of contractions linking neighboring pairs due to the presence of excess vertices (cf. Fig. 1c). Such a contraction leads to a factor

$$\int_{t_0}^{t} dt' \langle \tilde{g}_\pm(t') \tilde{g}_\mp(t) \rangle = F_\pm^+(\nu t)$$

with

$$F_\pm^+(\Omega) = \int_{0}^{\infty} d\tau e^{i\Omega \tau} f_\pm(\tau).$$

Apparently, $F_\pm = F_\pm^+ + F_\pm^-$ and $F_\pm^+ = F_\pm^-$ because of the even parity of the noise correlator $f_\pm(t)$. Hence, resummation of diagrams gives

$$\langle \tilde{g}_-(t) \rangle = G_{i,-}(t, t_0) \langle \tilde{g}(t) \rangle$$  \hspace{1cm} (31)

with

$$G_{i,-}(t, t_0) = \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{t''} dt''' \times \langle \tilde{\eta}_\pm(t') W_{i0}(t'') \tilde{\eta}_\mp(t''') \rangle$$

$$= \exp \left\{ - \int_{t_0}^{t} dt' F_\pm^+(\nu t') \right\}.$$  \hspace{1cm} (32)
In summary, the dynamics of averages can brought into the compact form ($\alpha = z, \pm$)

\[
\langle \tilde{g}_\alpha(t) \rangle = G_{\alpha\alpha}(t, t_0) \langle \tilde{g}_\alpha(t_0) \rangle , \quad (33a)
\]

\[
G_{\alpha\alpha}(t, t_0) \equiv e^{-|\theta_\alpha(t) - \theta_\alpha(t_0)|} \quad (33b)
\]

by means of noise integrals

\[
\theta_\pm(t) \equiv \int_{-\infty}^{t} dt' F_{\pm}^r(\nu t') \quad (34a)
\]

\[
\theta_z(t) \equiv \int_{-\infty}^{t} dt' F_{\perp}^r(\nu t') \quad (34b)
\]

Note that the subscripts of $\theta$ relate to the components of the Bloch vector rather than to the noise components.

B. Dynamics of pair correlations

Using the diagrammatic approach established above, we now proceed to calculate pair correlations of the Bloch vector for $b_x = 0$.

1. Reduction to equal-time correlators

We start by realizing that

\[
\langle \tilde{g}_\alpha(t) \tilde{g}_\beta(t) \rangle = e^{-[\theta_\alpha(t) - \theta_\alpha(t)]} \langle \tilde{g}_\alpha(t) \tilde{g}_\beta(t) \rangle
\]

(35)

since noise is uncorrelated before and after $t$, assuming (without loss of generality) that $t > \tilde{t}$. Thus we need to calculate only equal-time correlators

\[
\tilde{C}_{\alpha\beta}(t) \equiv \langle \tilde{g}_\alpha(t) \tilde{g}_\beta(t) \rangle .
\]

The correlations of principal interest are

\[
\langle g_z(t) g_z(t) \rangle = \langle \tilde{g}_z(t) \tilde{g}_z(t) \rangle , \quad (36a)
\]

\[
\langle g_+(t) g_-(t) \rangle = \langle \tilde{g}_+(t) \tilde{g}_-(t) \rangle . \quad (36b)
\]

They are invariant under the transformation to the interaction representation.

We describe the evolution of the equal-time correlators

\[
\tilde{C}_{\alpha\beta}(t) = \sum_{\gamma\delta} G_{\alpha\beta,\gamma\delta}(t, t_0) \tilde{C}_{\gamma\delta}(t_0)
\]

in terms of propagators $G_{\alpha\beta,\gamma\delta}(t, t_0)$. In the diagrammatic representation, these propagators consist only of ladder-like diagrams: The times axes $t$ and $\tilde{t}$ represent the legs of the ladder. Noise contractions between the sides are the rungs. A contribution of any diagram with intersecting or overlapping lines is smaller than that of the main sequence by a factor of the order $T_{\alpha}/\tau_{\text{acc}}$. Since only “neutral” diagrams survive averaging, many propagators vanish. The surviving propagators satisfy “charge conservation”

\[
\chi(\alpha) + \chi(\beta) = \chi(\gamma) + \chi(\delta)
\]

with the charge function

\[
\chi(\alpha) \equiv \begin{cases} 1 & \text{for } \alpha = +, \\ 0 & \text{for } \alpha = z, \\ -1 & \text{for } \alpha = - \end{cases}
\]

which now refers to propagator indices, not to noise indices. At the initial time $t_0$, the propagator charge is opposite to the charge of the first excess vertex. At the final time, the propagator charge coincides with the charge of the last excess vertex.

Among the nonvanishing propagators, several propagators are mutually dependent. In particular, there is a trivial symmetry $G_{\alpha\beta,\gamma\delta} = G_{\beta\alpha,\delta\gamma}$ corresponding to the exchange of the legs of the ladder. Additional relations follow from the time-reversal invariance or the complex conjugation, such as $G_{z-,z-} = G_{z+,z+}^*.$

We therefore find the time evolution of the equal-time correlators captured by the relations

\[
\tilde{C}_{z\pm}(t) = G_{zz,zz}(t, t_0) \tilde{C}_{z\pm}(t_0) + 2G_{z-,zz}(t, t_0) \tilde{C}_{z\mp}(t_0) , \quad (37a)
\]

\[
\tilde{C}_{+,\mp}(t) = G_{+,-,zz}(t, t_0) \tilde{C}_{zz}(t_0) + [G_{+,-,zz}(t, t_0) \tilde{C}_{+,\mp}(t_0) , \quad (37b)
\]

\[
\tilde{C}_{+,+}(t) = G_{+,+,zz}(t, t_0) \tilde{C}_{zz}(t_0) \quad (37c)
\]

\[
\tilde{C}_{z\mp}(t) = [G_{z-,z\pm}(t, t_0) \tilde{C}_{z\mp}(t_0) + G_{z+,z\mp}(t, t_0) \tilde{C}_{z\pm}(t_0) . \quad (37d)
\]

It is important to realize that the dynamics of the “neutral” correlations $\tilde{C}_{z\pm}$ and $\tilde{C}_{+,\mp}$ is decoupled from the “non-neutral” $\tilde{C}_{z\mp}$ and $\tilde{C}_{+,\pm}$. This implies that the latter correlations will not be generated by noise if they are absent initially. However, such correlations can be generated in general by the transverse components of the magnetic field.

For the convenience of the reader we anticipate the results of our subsequent explicit calculation of the propagators:

\[
G_{zz,zz}(t, t_0) = \frac{1}{3} (1 + 2e^{-3\theta_z}) , \quad (38a)
\]

\[
G_{+,-,zz}(t, t_0) = \frac{1}{3} (1 - e^{-3\theta_z}) , \quad (38b)
\]

\[
G_{zz,-}(t, t_0) = G_{+,-,zz}(t, t_0) , \quad (38c)
\]

\[
G_{+,-,-}(t, t_0) = \frac{1}{6} (2 + 3e^{-\theta_z} + e^{-3\theta_z}) , \quad (38d)
\]

\[
G_{+,+,zz}(t, t_0) = \frac{1}{6} (2 - 3e^{-\theta_z} + e^{-3\theta_z}) , \quad (38e)
\]

introducing the abbreviation

\[
\theta_\alpha \equiv \theta_\alpha(t) - \theta_\alpha(t_0) . \quad (39)
\]

These explicit calculations are based on integral equations which we now derive and solve for the propagators $G_{zz,zz}$ and $G_{zz,-}$ as two representative examples. The
diagrammatic derivation of the other propagators is deferred to the appendix. We refrain from presenting the expressions for $G_{zz,zz}$, $G_{zz,+z}$, and $G_{zz,-z}$ since they are not relevant for our purposes.

2. Propagator $G_{zz,zz}$

We start with the propagator for the longitudinal correlator

$$G_{zz,zz}(t,t_0) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \langle \bar{W}_{t_0}^i(t')W_{t_0}^i(t'') \rangle.$$  

This expression follows directly from the contribution to the autocorrelation of Eq. (42a) stemming from the initial value $\bar{g}_z(t_0)$.

To leading order in $\tau_n/\tau_{acc}$, the contributing diagrams have the ladder structure mentioned previously (cf. Fig. 2). On leg segments between two neighboring rungs of the ladder, an arbitrary number of noncrossing vertex contractions can be performed. Since there are no final excess vertices close to time $t$, all contractions between the last rung (at time $t_1$) and the final time $t$ are intra-pair contractions. Between the last rung and the last but one rung, all contractions are inter-pair contraction, forming a ring structure limited by time $t_1$ and $t_2$. Then, the next contractions on the legs prior to $t_2$ can be intra-pair contractions again.

Since all possible diagrams prior to $t_2$ have the same structure as the diagrams prior to $t$, the propagator satisfies a Bethe-Salpeter integral equation:

$$G_{zz,zz}(t,t_0) = G_{zz,zz}(t_0) + 2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 G_{zz,zz}^2(t,t_1) \times f(\nu_1)G_{zz,zz}(t_1,t_2)G_{zz,zz}(t_2,t_0).$$  

The first term represents the diagram without rungs. In the second term, the last pair of rungs at times $t_1$ and $t_2$ is separated from possible additional pairs prior to $t_2$. On the ring between $t_1$ and $t_2$, the intra-pair noise contractions are “oriented”. The explicit factor 2 arises from the two possible orientations.

To simplify the integral equation, it is convenient to substitute the time variables by variables $y \equiv \vartheta_z(t) - \vartheta_z(t_0)$ and $y_1 \equiv \vartheta_z(t_1) - \vartheta_z(t_0)$. This leads to

$$G_{zz,zz}(y,0) = e^{-2y} + 2 \int_{0}^{y} dy_1 \int_{0}^{y_1} dy_2 \times e^{-2(y-y_1)}e^{-(y_1-y_2)}G_{zz,zz}(y_2,0).$$  

This integral equation can now be transformed into a differential equation in two steps:

$$\frac{d}{dy}[e^{2y}G_{zz,zz}(y,0)] = 2 \int_{0}^{y} dy_2 e^{y+y_2}G_{zz,zz}(y_2,0)$$  

$$\frac{d}{dy}[e^{-y}G_{zz,zz}(y,0)] = 2e^yG_{zz,zz}(y,0)$$

Performing the derivatives, the last equation simplifies to

$$\frac{d^2}{dy^2}G_{zz,zz}(y,0) + 3 \frac{d}{dy}G_{zz,zz}(y,0) = 0.$$  

This differential equation must be solved with the initial conditions

$$G_{zz,zz}(0,0) = 1, \quad (43a)$$

$$\frac{d}{dy}G_{zz,zz}(y,0)|_{y=0} + 2G_{zz,zz}(0,0) = 0. \quad (43b)$$

The first one expresses the trivial fact that there is no time evolution over a vanishing time interval and follows from equation (10). The second one stems directly from Eq. (12a). Both determine a surprisingly simple form of the final result (33) for $G_{zz,zz}(t,t_0)$.

Considering an initial state with $\bar{g}(t_0) = \tilde{g}_z(t_0)e_z$, the conservation of $g^2$ implies the normalization

$$G_{zz,zz} + 2G_{+-,zz} = 1.$$  

This implies the result (33) for $G_{+-,zz}(t,t_0)$ which is derived also diagrammatically in the appendix.

3. Propagator $G_{zz,+-}$

As an example for the diagrammatic calculation of the other propagators, we consider here

$$G_{zz,+-}(t,t_0) = \int_{-\infty}^{\infty} dt' \int_{t_0}^{t'} dt'' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \langle \bar{W}_{t_0}^i(t')W_{t_0}^i(t'')\tilde{\eta}_-(t''')\tilde{\eta}_+(t''') \rangle.$$  

FIG. 2: Diagrams contributing to $G_{zz,zz}(t,t_0)$. First, there is the “ladder” without rungs. The diagrams with rungs can be represented as propagators $G_{zz,zz}(t,t_1)$ on both legs of the ladder, following the ring contraction between the last rungs as times $t_1$ and $t_2$. Prior to $t_2$ additional leg propagators and rings may occur. In the Bethe-Salpeter equation, these additional contraction sum up to $G_{zz,zz}(t_2,t_0)$. The diagrams with rungs can be represented as propagators $G_{zz,zz}(t,t_1)$ on both legs of the ladder, following the ring contraction between the last rungs as times $t_1$ and $t_2$. Prior to $t_2$ additional leg propagators and rings may occur. In the Bethe-Salpeter equation, these additional contraction sum up to $G_{zz,zz}(t_2,t_0).$
This expression follows directly from the contribution to the autocorrelation of Eq. (26a) stemming from the initial value \( \hat{g}(t_0) \). Its diagrams (cf. Fig. 3) have no excess vertices close to the final time \( t \) but opposite excess vertices close to the initial time \( t_0 \). Therefore, close to \( t_0 \) only inter-pair contractions are possible which are connected through the first rung at the time \( t_1 \). All possible contractions between \( t_1 \) and \( t \) have the same structure as the diagrams of \( G_{zz,zz} \). Therefore, the diagrams can be summed implicitly to

\[
G_{zz,+}(t,t_0) = \int_{t_0}^{t} dt_1 G_{zz,zz}(t,t_1) f(\nu t_1) \times G_{+,+}(t_1,t_0) G_{-,+}(t_1,t_0) . \tag{46}
\]

After a transition to variables \( y \) the integral can be performed explicitly and yields \( G_{zz,+-} = G_{+,-zz} \).

The normalization condition

\[
G_{zz,-+}(t,t_0) + G_{+,+-}(t,t_0) + G_{+,-+}(t,t_0) = 1 ,
\]

which expresses the conservation of \( g^2 \) for an initial transverse state, determines already the sum of the propagators \( G_{+,+-}(t,t_0) \) and \( G_{+,-+}(t,t_0) \) which enter Eqs. (47) only in this combination. Nevertheless, both propagators \( G_{+,+-}(t,t_0) \) and \( G_{+,-+}(t,t_0) \) are calculated separately in the appendix.

### IV. TRANSITIONS PRODUCED BY NOISE ONLY

On the basis of the propagators derived in the previous section we now evaluate transition probabilities and their fluctuations, still focusing on the subcase \( b_x = 0 \). We consider an arbitrary initial state at \( t_0 = -\infty \) with arbitrary Bloch vector \( \mathbf{g}(-\infty) \). We are interested in the final time \( t = \infty \), where the average is given by \( \hat{g} \)

\[
\langle g_z(\infty) \rangle = e^{-\Theta} g_z(-\infty) , \tag{47a}
\]

\[
\langle \hat{g}_\pm(\infty) \rangle = e^{-\Theta/2} \hat{g}_\pm(-\infty) . \tag{47b}
\]

According to Eqs. (43). Since the noise correlators are even functions of time differences, only the real parts of \( \vartheta_\alpha \) contribute to the difference \( \vartheta_\alpha(\infty) - \vartheta_\alpha(-\infty) \). We have introduced

\[
\Theta_\alpha \equiv \vartheta_\pm(\infty) = \frac{\pi}{\nu} (\gamma_2^2 + \gamma_0^2)
\]

for abbreviation. It is important to realize that the decay of the averaged Bloch vector depends only on the instantaneous noise correlation.

Although the final slow amplitudes \( \hat{g}_\pm(\infty) \) will be finite if the initial amplitudes \( \hat{g}_\pm(-\infty) \) were finite, the original amplitudes vanish,

\[
\langle g_\pm(\infty) \rangle = 0 \tag{48}
\]

due to the diffusive random precession leading to a decay on the dephasing time \( \tau_\vartheta \) given in Eq. (21). However, the transverse amplitude \( g_\perp \) decays like \( \hat{g}_\perp \),

\[
\langle g_\perp(\infty) \rangle = e^{-\Theta/2} g_\perp(-\infty) . \tag{49}
\]

For the variances we obtain – suppressing the obvious time arguments \( (\infty, -\infty) \) of the propagators –

\[
\langle g_z^2(\infty) \rangle = G_{zz,zz} g_z^2(-\infty) + G_{zz,+} g_\perp^2 (\infty) \tag{50a}
\]

\[
\langle g_\perp^2(\infty) \rangle = 2 G_{+,-zz} g_\perp^2 \tag{50b}
\]

and for the fluctuations

\[
\langle [\delta g_z(\infty)]^2 \rangle = \langle g_z^2(\infty) \rangle - \langle g_z(\infty) \rangle^2 = \frac{1}{3} \{ g_\perp^2 + 2 [e^{-3\Theta} - 3 e^{-2\Theta}] g_z^2(\infty) - e^{-3\Theta} g_z^2(-\infty) \} . \tag{51a}
\]

\[
\langle [\delta g_\perp(\infty)]^2 \rangle = \langle g_\perp^2(\infty) \rangle - \langle g_\perp(\infty) \rangle^2 = \frac{1}{3} \{ 2 g_\perp^2 - 2 e^{-3\Theta} g_\perp^2(\infty) \} + [e^{-3\Theta} - 3 e^{-2\Theta}] g_z^2(-\infty) \} . \tag{51b}
\]

For systems prepared initially in a state with a well defined \( g \), the correlations \( C_{zz} \) and \( C_{zz} \) do not necessarily vanish. They are not explicitly considered here since they do not couple to \( C_{zz} \) and \( C_{zz} \) according to Eqs. (47). Furthermore, an long time scales, phase diffusion due to \( \eta_\varphi \) annihilates these non-neutral correlations anyway.

To discuss the statistics of transitions between the two levels, we consider the system to be located initially in the state \( |\uparrow\rangle \), i.e. \( \mathbf{g}(-\infty) = \mathbf{e}_z \). A measurement which detects at time \( t \) whether the system is in the state \( |\uparrow\rangle \) or \( |\downarrow\rangle \) yields the probabilities

\[
\Pi_{\uparrow\downarrow} (t) = \text{Tr} \left\{ \frac{1}{2} (1 \pm \hat{\sigma}_z) \hat{\rho}(t) \right\} = \frac{1}{2} [1 \pm g_z(t)] . \tag{52}
\]
The trace accounts for the average over quantum fluctuations for a given realization of noise. The additional averaging over the noise leads to the final expectation values

\[
\langle \Pi_{t\uparrow\downarrow}(\infty) \rangle = \frac{1}{2} [1 + e^{-\Theta}], \quad \langle \Pi_{t\downarrow\uparrow}(\infty) \rangle = \frac{1}{2} [1 - e^{-\Theta}] \quad (53a)
\]

Since the probabilities \(\Pi_{t\uparrow\downarrow}/\Pi_{t\downarrow\uparrow}\) depend on the noise, they are fluctuating quantities themselves. Their fluctuations are

\[
\langle [\delta \Pi_{t\downarrow\uparrow}(\infty)]^2 \rangle = \frac{1}{12} (1 - 3e^{-2\Theta} + 2e^{-3\Theta}) . \quad (53b)
\]

Remarkably, the fluctuations of probabilities in general have the same order of magnitude as average values. They are weak only if the noise is weak, i.e., \(\langle [\delta \Pi_{t\downarrow\uparrow}(\infty)]^2 \rangle \approx \Theta^2/4\) for \(\Theta \ll 1\). In the opposite limiting case \(\Theta \gg 1\) of strong noise the two levels become equipopulated, \(\langle \Pi_{t\uparrow\downarrow}(\infty) \rangle = \langle \Pi_{t\downarrow\uparrow}(\infty) \rangle = \frac{1}{2}\), and the fluctuation is equal to \(\langle [\delta \Pi_{t\downarrow\uparrow}(\infty)]^2 \rangle = \frac{1}{12}\).

V. LZ SYSTEM WITH NOISE

In this section we assume that not only the noise, but also the regular part of the Hamiltonian has a finite nondiagonal matrix element \(\Delta\) corresponding to a finite transverse component of the regular magnetic field. As it was shown in Ref. [4], if the noise is fast, there exists a well-defined time separation for the transition due to the noise and due to the regular part of the Hamiltonian. Consider matching at times \(\pm t_x\) with \(t_{1\text{LZ}} \ll t_x \ll \tau_{\text{acc}}\). The time evolution may be decomposed into three intervals. (i) From \(t = -\infty\) to \(t = -t_x\) one may neglect \(\Delta\). The transition is only under the influence of noise. Since \(t_x \ll \tau_{\text{acc}}\), this time evolution is approximately the same as from \(t = -\infty\) to \(t = 0\) in the absence of \(\Delta\). (ii) From \(t = -t_x\) to \(t = t_x\) the LZ transition occurs. Since \(t_x \gg \tau_{\text{LZ}}\), time evolution is approximately the same as from \(t = \infty\) to \(t = \infty\) in the absence of noise. (iii) From \(t = t_x\) to \(t = \infty\) the situation is again analogous to the regime (i).

We represent the time evolution operator as

\[
\hat{U}(t, t_0) = \hat{U}_0(t) \hat{\omega}(t, t_0) \hat{U}_0^\dagger(t_0) \quad (54)
\]

and approximate

\[
\hat{\omega}(\infty, -\infty) = \hat{\omega}_{\text{iii}} \hat{U}_{\text{iii}} \hat{U}_i , \quad (55a)
\]

\[
\hat{U}_i = \hat{U}(0, -\infty) \quad \text{for} \quad \Delta = 0 , \quad (55b)
\]

\[
\hat{\omega}_{\text{ii}} = \hat{\omega}(\infty, -\infty) \quad \text{for} \quad \eta = 0 , \quad (55c)
\]

\[
\hat{\omega}_{\text{iii}} = \hat{\omega}(\infty, 0) \quad \text{for} \quad \Delta = 0 \quad (55d)
\]

for the time evolution operator acting on the slow vector.

We assume that at \(t = -\infty\) the density matrix is diagonal (complete decoherence), i.e.,

\[
g(\infty) = g_z(\infty) \mathbf{e}_z .
\]

Then

\[
\langle g_z(\infty) g_z(\infty) \rangle = g_z^2(\infty)
\]

is the only non-vanishing correlator at \(t = -\infty\). Since \(\tau_0 \ll t_x\), the noise in intervals (i) and (iii) is statistically independent and the averaging can be performed separately for both intervals.

A. Interval (i)

Neglecting \(\Delta\) in the first time interval, we find at \(t = -t_x\) the averages

\[
\langle \hat{g}_z(t_x) \rangle = e^{-\Theta/2} g_z(-\infty) , \quad (56a)
\]

\[
\langle \hat{g}_{\pm}(t_x) \rangle = 0 , \quad (56b)
\]

and correlations

\[
\hat{C}_{z z}(-t_x) = G_{zz,zz}(0, -\infty) g_z^2(-\infty) = \frac{1}{3} [1 + 2e^{-3\Theta/2}] g_z^2(-\infty) , \quad (57a)
\]

\[
\hat{C}_{+ -}(-t_x) = G_{+,zz}(0, -\infty) g_z^2(-\infty) = \frac{1}{3} (1 - e^{-3\Theta/2}) g_z^2(-\infty) . \quad (57b)
\]

According to Eqs. [57] all other independent pair correlations vanish at \(t = -t_x\).

B. Interval (ii)

Time evolution at intermediate times from \(-t_x\) to \(t_x\) is given by

\[
\hat{U}_i = \hat{U}_{\text{LZ}} .
\]

Hence the density matrix evolves according to

\[
\hat{\rho}(t_x) = \hat{U}_{\text{LZ}} \hat{\rho}(-t_x) \hat{U}_{\text{LZ}}^\dagger
\]

which implies the transformation

\[
\hat{\mathbf{g}}(t_x) \cdot \mathbf{\sigma} = \hat{U}_{\text{LZ}} \hat{\mathbf{g}}(-t_x) \cdot \mathbf{\sigma} \hat{U}_{\text{LZ}}^\dagger
\]

for the transformed Bloch vector \(\hat{\mathbf{g}}\). The explicit transformation of its components reads

\[
\begin{pmatrix}
\hat{g}_+(t_x) \\
\hat{g}_z(t_x) \\
\hat{g}_-(t_x)
\end{pmatrix} = \mathbf{U}^\text{LZ} \cdot
\begin{pmatrix}
\hat{g}_+(-t_x) \\
\hat{g}_z(-t_x) \\
\hat{g}_-(-t_x)
\end{pmatrix} \quad (58a)
\]

with the rotation matrix

\[
\mathbf{U}^\text{LZ} =
\begin{pmatrix}
a^2 & -\sqrt{2}a^* b & -b^2 \\
\sqrt{2}a^* b \ |a|^2 - |b|^2 & \sqrt{2}ab^* & -b^2 \\
-b^2 & -\sqrt{2}ab & a^2
\end{pmatrix} . \quad (58b)
\]
From these relations we obtain

\[
\langle \tilde{g}_z(t_x) \rangle = U_{zz}^{LZ} \langle \tilde{g}_z(t) \rangle \\
= (|a|^2 - |b|^2)e^{-\Theta/2}g_z(-\infty), \quad (59a)
\]

\[
\langle \tilde{g}_+^L(t_x) \rangle = U_{zz}^{LZ} \langle \tilde{g}_z(t) \rangle \\
= -\sqrt{2}a^*b^*e^{-\Theta/2}g_z(-\infty) \quad (59b)
\]

for the averages. Analogously, correlations are transformed according to

\[
\tilde{C}_{\alpha\beta}(t_x) = \sum_{\gamma\delta} U_{\gamma\alpha}^{LZ} U_{\delta\beta}^{LZ} \tilde{C}_{\gamma\delta}(-t_x). \quad (60)
\]

Since we are ultimately interested only in the correlations \( \tilde{C}_{zz}(\infty) \) and \( \tilde{C}_{+\cdot}(\infty) \) and since the time evolution in interval (iii) again follows Eqs. (61), we need to evaluate at time \( t = t_x \) only the two correlations

\[
\tilde{C}_{zz}(t_x) = (U_{zz}^{LZ})^2 \tilde{C}_{zz}(-t_x) \\
+ 2U_{zz}^{LZ} U_{zz}^{LZ} \tilde{C}_{zz}(-t_x) \\
= \frac{1}{3}[1 + 2(1 - 6|a|^2|b|^2)e^{-3\Theta/2}]g_z^2(-\infty) \quad (61a)
\]

\[
\tilde{C}_{+\cdot}(t_x) = U_{zz}^{LZ} U_{zz}^{LZ} \tilde{C}_{zz}(-t_x) \\
+ [U_{zz}^{LZ} U_{zz}^{LZ} + U_{zz}^{LZ} U_{zz}^{LZ}] \tilde{C}_{zz}(-t_x) \\
= \frac{1}{3}[1 - (1 - 6|a|^2|b|^2)e^{-3\Theta/2}]g_z^2(-\infty) \quad (61b)
\]

As a check of these results, we verify the normalization relation \( \tilde{C}_{zz}(t_x) + 2\tilde{C}_{+\cdot}(t_x) = g_z^2(-\infty) \).

C. Interval (iii)

As already mentioned before, the time evolution in interval (iii) is absolutely analogous to the evolution in interval (i). We therefore obtain

\[
\langle \tilde{g}_z(\infty) \rangle = G_{zz}(\infty, 0) \langle \tilde{g}_z(t_x) \rangle \\
= (|a|^2 - |b|^2)e^{-\Theta}g_z(-\infty), \quad (62a)
\]

\[
\langle \tilde{g}_+^L(\infty) \rangle = G_{+\cdot}(\infty, 0) \langle \tilde{g}_z(t_x) \rangle \\
= -\sqrt{2}a^*b^*e^{-\Theta}g_z(-\infty) \quad (62b)
\]

These results were obtained already in Ref. [7]. In terms of \( \textbf{g} \) they mean

\[
\langle g_z(\infty) \rangle = (|a|^2 - |b|^2)e^{-\Theta}g_z(-\infty), \quad (63a)
\]

\[
\langle g_{x\cdot}(\infty) \rangle = \sqrt{2}|a||b|e^{-3\Theta/4}g_z(-\infty) \quad (63b)
\]

We recall that the averages \( \langle g_z(\infty) \rangle = \langle g_{y\cdot}(\infty) \rangle = 0 \) vanish due to the presence of the longitudinal noise component. The correlations follow from Eqs. (63) with \( t = \infty \) and \( t_0 = t_x \):

\[
\tilde{C}_{zz}(\infty) = \frac{1}{3}[1 + 2(1 - 6|a|^2|b|^2)e^{-3\Theta}]g_z^2(-\infty) \quad (64a)
\]

\[
\tilde{C}_{+\cdot}(\infty) = \frac{1}{3}[1 - (1 - 6|a|^2|b|^2)e^{-3\Theta}]g_z^2(-\infty) \quad (64b)
\]

In the presence of regular transitions, the transition probabilities and their fluctuations – given in Eqs. (56) in the absence of \( \Delta \) – finally become

\[
\langle \Pi_{\uparrow\uparrow/\uparrow\downarrow}(\infty) \rangle = \frac{1}{2}[1 + (|a|^2 - |b|^2)e^{-\Theta}] \quad (65a)
\]

\[
\langle [\delta\Pi_{\uparrow\downarrow/\uparrow\downarrow}(\infty)]^2 \rangle = \frac{1}{12}[1 - 3(1 - 4|a|^2|b|^2)e^{-2\Theta} + 2(1 - 6|a|^2|b|^2)e^{-3\Theta}] \quad (65b)
\]

To obtain the explicit dependence on the LZ parameter one may use \( |a|^2 = e^{-2\pi\gamma^2} \) and the unitarity relation \( |b|^2 = 1 - |a|^2 \).

VI. RESPONSE SPECTRUM AND INTENSITY

Let a short pulse of an effective field directed along axis \( i \) act on the LZ system at some moment of time \( t \). Its response at a later moment of time \( t' = t + \tau \) is determined by the correlation function, \( K_{i,j}(t, \tau) = \langle s_i(t)s_j(t + \tau) \rangle \). It is reasonable to measure the spectral content of the response at a fixed time of initial pulse \( t \). It is given by the Fourier-transform

\[
K_{i,j}(t; \omega) = \int_0^{\infty} K_{i,j}(t, \tau)e^{i\omega\tau}d\tau. \quad (66)
\]

According to general relations (65a) and (65b), in the limit \( \tau \to \infty \) such a function vanishes or saturates to a finite value. For example, in the case \( i = j = z \) and \( t > t_x \) it is equal to \( \tilde{C}_{zz}(t) \exp(-\int_0^{\infty} F_{\perp}(\nu \tau')d\tau') \). Therefore, the spectral density of the response contains a \( 1/\omega \) component at small frequency with an intensity that depends on the time of excitation. This is not surprising since the LZ process violates the homogeneity of time. At infinite time \( t \) this intensity reaches a finite limit \( \tilde{C}_{zz}(\infty) \). It happens since the noise becomes ineffective at sufficiently large time and does not randomize the quantum amplitudes any more. Therefore, any quantum amplitude or a component of the Bloch vector \( \textbf{g} \) changes after this time in an almost deterministic way. On the other hand, the spectral intensity remains finite even at \( t \to \infty \) since the value of \( \textbf{g} \) reached at the moment, when the noise becomes ineffective, is random.

Another (incoherent) contribution to the spectral density arises from the variation of the correlation function at a finite time:

\[
K_{zz}(\tau, t) - K_{zz}(\infty, t) = K_{zz}(\infty, t) \times \left[ \exp \left( -\int_{t+|\tau|}^{\infty} F_{\perp}(\nu \tau')d\tau' \right) - 1 \right]. \quad (67)
\]

To estimate the spectral width of this contribution we assume that either time \( t \) is large enough or the noise is weak. Then the exponent in equation (67) can
be expanded to the linear term. Its spectral width is easily estimated as \( \Delta \omega_1 = 1/\tau_{acc} \). The spectral intensity of this line at \( t \sim \tau_{acc} \) can be estimated as \( I_1 \propto C_{zz}(\infty) (\tilde{\eta}_x^2 + \tilde{\eta}_y^2) / \nu \). It decreases rapidly with growing time \( t \).

If the LZ processes are repeated randomly with the average time \( \tau_{coll} \) between the events (as it happens in a gas of colliding atoms or molecules), the \( 1/\omega \)-line is smeared out to the width \( \Delta \omega_0 \sim 1/\tau_{coll} \ll \Delta \omega_1 \), whereas its average intensity \( \bar{I}_0 \) is proportional to \( C_{zz}(\infty) \). The width of the incoherent line remains the same \( \Delta \omega_1 = 1/\tau_{acc} \), but its intensity is stabilized at a value \( I_1(\tau_{coll}) \propto I_1 \varphi(\tau_{coll}/\tau_{acc}) \) the smaller the larger is \( \tau_{coll} \) [here \( \varphi(x) \) is a function describing the decay of the noise correlations, which depends on details].

Thus, the main contribution to the spectral width of the population or induced field fluctuations consists of two narrow lines. The first has the width determined by collisions and a permanent intensity per particle. The second has a permanent width, but its intensity is determined by collision time. In the rarefied gas, or more generally if \( \tau_{coll} \gg \tau_{acc} \), the first line is narrower and stronger than the second one.

VII. CONCLUSIONS

We have calculated the correlation functions of the Bloch vector components in the LZ process subject to a fast Gaussian noise. These correlators determine the linear response of the LZ system to a weak, time dependent probe signal. Two-time correlation functions are factorizable at the time scale much larger than the noise correlation time \( \tau_0 \) due to statistical independence of random processes. Thus the main problem is the calculation of simultaneous averages. The condition of the fast noise allows to use for these averages the ladder graphs only. The resulting Bethe-Salpeter equation can be reduced to differential equations which are exactly solvable. In general, fluctuations are strong, of the same order of magnitude as the average values.

Since the noise is ineffective for transitions at sufficiently large time \( t \gg \tau_{acc} \), the transition probabilities or the components of the Bloch vector remain deterministically coherent after this time. Therefore, the spectrum of the fluctuations contains a narrow \( 1/\omega \) line, whose intensity is determined by the distribution of these values at the time when the noise became ineffective. Besides of this line, there exists another narrow line with a width of about \( 1/\tau_{acc} \) and an intensity depending on the pulse time and going to zero when this time goes to infinity.

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APPENDIX A: DETAILS FOR CORRELATORS

Here we provide some details on the calculation of correlation propagators.

1. Propagator \( G_{+-,zz} \)

The calculation of the propagator \( G_{+-,zz} \) can be performed largely in parallel to the calculation of \( G_{zz,+-} \) in Sec. III B. Starting point is the equation

\[
G_{+-,zz}(t, t_0) = \int_{t_0}^{t} \int_{t_0}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt'' dt''' dt''' dt'''' \times \langle \tilde{\eta}_-(t') \tilde{\eta}_+(t') W_{t_0}^{t'}(t'') W_{t_0}^{t'}(t''') \rangle .
\]

Using the Wick theorem, we perform the contractions

\[
G_{+-,zz}(t, t_0) = \int_{t_0}^{t} dt_1 G_{+,-,zz}(t, t_1) G_{-,+,zz}(t_1, t_0),
\]

where \( t_1 \) is the time of the last rung of the ladder (last means closest to \( t \), see Fig. IV). After a substitution from \( t_1 \) to \( y_1 \) the integral is elementary with the result

\[
G_{+-,zz}(t, t_0) = \frac{1}{3} \left( 1 - e^{-3[\theta_+(t) - \vartheta_-(t_0)]} \right) .
\]

2. Propagator \( G_{+,+,zz} \)

From the equations of motion (26) one extracts

\[
G_{zz,zz}(t_1, t_0).
\]

FIG: The diagrams of \( G_{+-,zz} \) are analogous to the diagrams of \( G_{zz,+-} \) with the only difference that now the excess vertices are the last vertices.
Eq. (38d)

After averaging, the sum over the ladder diagrams can be captured by a Bethe-Salpeter equation

\[
G_{-,-+}(t, t_0) = 1 - \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \langle \bar{n}_{-}(t') W_{t_0}^{l'}(t'') \bar{n}_{+}(t''') \rangle
\]

\[
- \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \langle \bar{n}_{+}(t') W_{t_0}^{l'}(t'') \bar{n}_{-}(t''') \rangle
\]

\[
+ \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \int_{-\infty}^{\infty} dt'''' \int_{t_0}^{t''''} d\bar{t}'''' \langle \bar{n}_{-}(t') W_{t_0}^{l'}(t'') \bar{n}_{+}(t''') \bar{n}_{+}(\bar{t}) W_{t_0}^{l'''}(\bar{t}''') \bar{n}_{-}(t''''') \rangle .
\]

After averaging, the sum over the ladder diagrams can be captured by a Bethe-Salpeter equation

\[
G_{-,-+}(t, t_0) = G_{++,}(t, t_0) G_{-,-}(t, t_0)
\]

\[
+ \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 G_{++,}(t, t_1) G_{-,,-}(t_1, t) f(\nu t_1) G_{zz,zz}(t_1, t_2) f(\nu t_2) G_{++,}(t_2, t_0) G_{-,,-}(t_2, t_0)
\]

where \( t_1 \) is the time of the last rung and \( t_2 \) the time of the first rung (see Fig. 5). All contractions in between sum up to \( G_{zz,zz}(t_1, t_2) \). Since the latter propagator is explicitly known, the integrals can be performed directly, yielding Eq. (38d).

![Diagram](image)

**FIG. 5:** \( G_{-,-+} \) consists of diagrams with excess vertices as first and last vertices on both time legs. There are disconnected diagrams (where the “ladder” has no rungs) and connected diagrams (where the ladder has rungs). In the second case, the excess vertices are connected by rungs as \( t_1 \) and \( t_2 \) and between \( t_1 \) and \( t_2 \) additional pairs can be contracted as in \( G_{zz,zz}(t_1, t_2) \).

### 3. Propagator \( G_{--++} \)

Similar to the previous propagator, one now has

\[
G_{--++}(t, t_0) = \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \int_{t_0}^{t''} dt'''' \int_{t_0}^{t''''} d\bar{t}'''' \langle \bar{n}_{-}(t') W_{t_0}^{l'}(t'') \bar{n}_{-}(t''') \bar{n}_{+}(t''') \bar{n}_{+}(\bar{t}) W_{t_0}^{l''} \rangle .
\]

The configuration of the excess vertices necessitates at least two rungs at times \( t_1 \) and \( t_2 \) (see Fig. 6). The contractions in between again sum up to \( G_{zz,zz}(t_1, t_2) \), leading to

\[
G_{--++}(t, t_0) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 G_{++,}(t, t_1) G_{--+}(t_1, t) f(\nu t_1) G_{zz,zz}(t_1, t_2) f(\nu t_2) G_{++,}(t_2, t_0) G_{--+}(t_2, t_0)
\]
FIG. 6: The diagrams of $G_{+,-,+-}$ as similar to these of $G_{+,-,-+}$ with the difference that the polarity of the first excess vertices is inverted. Therefore, no disconnected diagrams can be formed.

and eventually to Eq. \[1\].

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