Infinite families of (non)-Hermitian Hamiltonians associated with exceptional $X_m$ Jacobi polynomials

Bikashkali Midya and Barnana Roy

Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata 700108, India

E-mail: bikash.midya@gmail.com and barnana@isical.ac.in

Received 12 November 2012, in final form 14 March 2013
Published 9 April 2013
Online at stacks.iop.org/JPhysA/46/175201

Abstract

Using an appropriate change of variable, the Schrödinger equation is transformed into a second-order differential equation satisfied by recently discovered Jacobi-type $X_m$ exceptional orthogonal polynomials. This facilitates the derivation of infinite families of exactly solvable Hermitian as well as non-Hermitian trigonometric Scarf potentials and a finite number of Hermitian and an infinite number of non-Hermitian $\mathcal{PT}$-symmetric hyperbolic Scarf potentials. The bound state solutions of all these potentials are associated with the aforesaid exceptional orthogonal polynomials. These infinite families of potentials are shown to be extensions of the conventional trigonometric and hyperbolic Scarf potentials by the addition of some rational terms characterized by the presence of classical Jacobi polynomials. All the members of a particular family of these 'rationally extended polynomial-dependent' potentials have the same energy spectrum and possess translational shape-invariant symmetry. The obtained non-Hermitian trigonometric Scarf potentials are shown to be quasi-Hermitian in nature ensuring the reality of the associated energy spectra.

PACS numbers: 03.65.Ge, 03.65.Fd, 11.30.Pb

1. Introduction

In recent years, there has been a surge of interest in the study of exactly solvable quantum systems associated with exceptional orthogonal polynomials (EOPs) introduced in the seminal paper by Gomez-Ullate et al [1, 2]. The $X_m$ EOPs are the solutions of the second-order Sturm–Liouville eigenvalue problem with rational coefficients. A distinguishing property of these polynomials is that the lowest eigenpolynomial of the sequence need not be of degree zero, even though the full set of eigenfunctions still forms a basis of the weighted $L^2$ space.

Exceptional $X_1$ Laguerre- or Jacobi-type polynomials were shown to be eigenfunctions of the rationally extended radial oscillator or Scarf I potentials by the point canonical transformation method [3] and by the supersymmetric quantum-mechanical method
The latter is based on a reparametrization of the starting Hamiltonian and redefining the conventional superpotential in terms of modified couplings, while allowing for the presence of extra rational contribution expressed in terms of some polynomial function. Depending on the degree of the latter, the wavefunctions of the extended potentials are shown to be $X_1$ or $X_2$ Laguerre, Jacobi EOPs. Employing the similar approach in the context of second-order supersymmetry, rational extensions of the generalized Poschl–Teller, Scarf I potentials with the corresponding bound state wavefunctions given in terms of EOPs have been constructed [5]. Two distinct families of Laguerre- and Jacobi-type $X_m$, $m = 1, 2, 3 \ldots$, EOPs as eigenfunctions of infinitely many shape-invariant potentials by deforming the radial oscillator and the hyperbolic/trigonometric Poschl–Teller potentials were constructed in [6, 7], and the properties of these $X_m$ EOPs have been thoroughly studied subsequently [8–12]. The multi-indexed version of exceptional $X_m$ orthogonal polynomials, $X_{m_1, m_2, \ldots, m_k}$, is constructed by using multi-step Darboux transformation [13], Crum–Adler mechanism [14], higher order SUSYQM [15] and multi-step Darboux–Backlund transformation [16].

$X_m$ EOPs were shown to be obtainable through several other approaches such as Darboux–Crum transformation [17–19], Darboux–Backlund transformation [16, 20, 21] and prepotential approach [22, 23]. In fact, $X_m$ Laguerre polynomials have been characterized in terms of an isospectral Darboux transformation. The shape invariance of these polynomial families is shown to be a direct consequence of the permutability property of the Darboux transformation [17]. It is shown to be possible to obtain rational extensions of every translationally shape-invariant potentials via Darboux–Backlund transformations based on negative eigenfunctions built from excited states of the initial Hamiltonian. The potentials obtained by this method are either strictly isospectral or quasi-isospectral extensions of the initial ones and have enlarged shape-invariant property. All the quantal systems related to the exceptional Laguerre and Jacobi polynomials can be constructed by the prepotential approach [22, 23] which does not need the concept of shape invariance and the Darboux–Crum transformation. The prepotential, the deforming function, the potential, the eigenfunctions and eigenvalues can be derived within the same framework. The exceptional polynomials obtained in this way are expressible as a bilinear combination of the deforming functions and its derivatives.

Recently, EOPs have been studied in diverse scenarios. For instance, these new polynomials were shown to be associated with the solutions of some conditionally exactly solvable potentials [24, 25], solutions for position-dependent mass systems [26], and main part of the eigenfunctions of the Dirac equations coupled minimally and non-minimally with some external fields and the Fokker–Planck equations [27]. Four types of infinitely many exactly solvable Fokker–Planck equations which are the generalized or deformed versions of the Rayleigh process and the Jacobi process are also shown to be related to the EOP [28]. The structure of the $X_m$ Laguerre polynomials was considered within the quantum Hamilton–Jacobi formalism [29], within $N$-fold supersymmetry [30] and its dynamical breaking in the context of position-dependent mass scenario [31].

On the other hand, the relaxation of hermiticity for the reality of the discrete spectrum of a quantum-mechanical Hamiltonian has given rise to some very interesting investigations over the last few years stepped up by a conjecture of Bender and Boettcher [32] that $PT$-symmetric non-Hermitian Hamiltonians could possess a real bound state spectrum. The complex $PT$-symmetric Hamiltonians possess a real discrete spectrum if the energy eigenstates are also the eigenstates of $PT$; otherwise, the $PT$ symmetry is said to be spontaneously broken and the energy eigenvalues occur in complex conjugate pairs [33]. Here, $P$ stands for parity transformation and $T$ stands for time reversal. Subsequently, Mostafazadeh [34] showed that the reality of the spectrum of a $PT$-symmetric Hamiltonian is ensured if the Hamiltonian $\hat{H}$ is Hermitian with respect to a positive definite inner product $\langle \cdot | \cdot \rangle$ on the Hilbert space $\mathcal{H}$ in
which \( \tilde{H} \) is acting. This renders the Hamiltonian \( \tilde{H} \) to be pseudo-Hermitian [35] \( \tilde{H}^\dagger = \eta \tilde{H} \eta^{-1} \), where the Hermitian linear automorphism \( \eta : \mathcal{H} \rightarrow \mathcal{H} \) is bounded and positive definite. A word of caution is due here if the operator \( \eta \) is not a bounded operator. An unbounded metric operator \( \eta \) cannot be used to define a consistent Hilbert space structure. In that case, an alternative construction, which is not based on the introduction of an \( \eta \) operator, has been proposed in [37]. Another equivalent condition for the reality of the energy spectrum of \( \tilde{H} \) is the quasi-hermiticity [36–38], i.e. the existence of an invertible operator \( \rho \) such that \( H = \rho \tilde{H} \rho^{-1} \) is Hermitian with respect to the usual inner product \( \langle .. , . \rangle \). A quasi-Hermitian Hamiltonian shares the same energy spectrum of the equivalent Hermitian Hamiltonian \( H \) and the wavefunctions are obtained by operating \( \rho^{-1} \) on those of \( H \). Most of the analytically solvable non-Hermitian Hamiltonians are constructed by either making the coupling constant of the potential present in the relevant Schrödinger Hamiltonian imaginary [39, 40] or by shifting the coordinate with an imaginary constant [41, 42]. Several of these classes of Hamiltonians are argued to be pseudo-Hermitian under \( \eta = e^{-i\theta} \eta \) [43, 44]. For a real \( \alpha \) and \( p = -i\frac{d}{dx} \), the operator \( \eta \) shifts the coordinate \( x \) to \( x + i\alpha \).

In this paper, we have found infinite families of the exactly solvable Hermitian as well as non-Hermitian Hamiltonians whose bound state wavefunctions are given in terms of exceptional \( X_m \) Jacobi orthogonal polynomials. The non-Hermitian Hamiltonians, obtained here, are shown to be \( \mathcal{PT} \)-symmetric and quasi-Hermitian in nature rendering these Hamiltonians to have real eigenvalues. Also, using SUSYQM, we have shown that all the infinite families of Hamiltonians have translational shape-invariant property. The motivation for doing this comes from the fact that, compared to the exactly solvable real potentials associated with \( X_m \) EOPs, little attention [45, 4] has been paid to looking for complex potentials having these polynomials as their bound state eigenfunctions. In fact, a few complex potentials have been found involving only \( X_1 \)-type Jacobi, Laguerre EOPs. Considering the recent flurry of interest in the non-Hermitian Hamiltonians with \( \mathcal{PT} \) symmetry and/or pseudo-hermiticity, this study would be worthwhile. The organization of this paper is as follows. In section 2, some properties of \( X_m \)-type Jacobi EOPs, which are necessary to analyze the results, are given. The construction of infinite families of Hermitian and non-Hermitian Hamiltonians by transforming the relevant Schrödinger equation into the differential equation satisfied by the \( X_m \) Jacobi polynomials is carried out in section 3. In section 4, the shape-invariant properties of the obtained potentials are discussed within the framework of supersymmetric quantum mechanics. Finally, section 5 is devoted to the summary and outlook.

2. Exceptional \( X_m \) Jacobi orthogonal polynomials

We present here some properties of \( X_m \) Jacobi polynomials which are studied in detail in [19]. For a fixed integer \( m \geq 1 \) and real \( a, b > -1 \), the exceptional \( X_m \) Jacobi orthogonal polynomials \( P_{n(a,b,m)}(x) \), \( n = m, m + 1, m + 2, \ldots \), satisfy the differential equation

\[
(1 - x^2) \frac{d^2 y}{dx^2} + Q_1(x) \frac{dy}{dx} + R_1(x)y = 0, \tag{2.1}
\]

where \( Q_1(x) \) and \( R_1(x) \) can be written in terms of classical Jacobi orthogonal polynomial \( P_{n(a,b)}(x) \) as

\[
Q_1(x) = (a - b - m + 1)(1 - x^2)P_{n-1(a,b)}(x)P_{n-1(a,b-1)}(x) \tag{2.2}
\]

\[
R_1(x) = b(a - b - m + 1)(1 - x)P_{n-1(a,b)}(x)^2 + n^2 + n(a + b - 2m + 1) - 2bm.
\]
For n, l ≥ m, this new family of polynomials is orthonormal with respect to the weight function \( W_{a,b,m}(x) = \frac{(1-x)^a(1+x)^b}{P_m^{(-a-1,b-1)}(x)} \). Evidently, for \( L^2 \)-orthonormality to hold, the denominator of this weight function should be nonzero in \(-1 \leq x \leq 1\). According to [46, 19], the polynomial \( P_m^{(a,b-1)}(x) \) has no zeros in \([-1, 1]\) if and only if the following conditions hold simultaneously:

\[
\begin{align*}
(i) & \quad b \neq 0, a, a - b - m + 1 \neq 0, 1, \ldots, m - 1, \\
(ii) & \quad a > m - 2, \text{sgn}(a - m + 1) = \text{sgn}(b),
\end{align*}
\]

where \( \text{sgn}(x) \) is the signum function. With these restrictions, the \( L^2 \)-norms of the exceptional \( X_m \) Jacobi polynomials are given by

\[
\int_{-1}^{1} \frac{(1-x)^a(1+x)^b}{P_m^{(-a-1,b-1)}(x)} \left( \sum_{j=0}^{m} \sum_{l=0}^{m} \frac{P_j(a,b,m)(x)\hat{P}_l(a,b,m)(x)dx}{m!} \right)^2 dx
\]

\[
= \frac{2^{a+b+1}(n+b)(n-2m+a+1)\Gamma(n-m+a+2)\Gamma(n+b)}{(2n-2m+a+b+1)(n-m+a+1)^2(n-m)!\Gamma(n-m+a+b+1)} \delta_{nl},
\]

where \( \delta_{nl} \) is the Kronecker delta function. Moreover, the \( X_m \) Jacobi orthogonal polynomials \( \hat{P}_n(a,b,m)(x) \) are related to the classical Jacobi polynomials \( P_n^{(a,b)}(x) \) by the following relations:

\[
\hat{P}_n(a,b,0)(x) = P_n^{(a,b)}(x)
\]

\[
\hat{P}_n(a,b,m)(x) = (-1)^n \left[ \frac{a+b+j+1}{2(a+j+1)} (x-1) P_j^{(-a-1,-b-1)} P_{m-j}^{(a+2,b)} + \frac{a-m+1}{a+j+1} P_j^{(-a-2,b)} P_{m-j}^{(a+1,b-1)} \right], \quad j = n-m \geq 0.
\]

It is worth mentioning here that following identities of the classical Jacobi polynomials are extensively used to simplify the results presented in the later part of the paper:

\[
(1-x^2)(a+b+m+2) P_{m-2}^{(a+2,b+2)} + 2[b-a-(a+b+2)x] P_{m-1}^{(a+1,b+1)} + 4m P_m^{(a,b)} = 0
\]

\[(x-1)(a+b+m+1) P_{m-1}^{(a+1,b+1)} = 2(a+m) P_{m-1}^{(a-1,b+1)} - 2a P_m^{(a,b)}
\]

\[
P_m^{(a,b-1)} - P_{m-1}^{(a+1,b+1)} = 2(a+m) P_{m-1}^{(a-1,b+1)} - 2a P_m^{(a,b)}
\]

\[
\frac{d'}{dx'} P_m^{(a,b)} = \frac{\Gamma(a+b+m+r+1)}{2\Gamma(a+b+n+1)} P_{m-r}^{(a+r,b+r)}
\]

Identity (2.6a), which is absent in the handbook [47], directly follows from the differential equation of the classical Jacobi polynomials \( P_m^{(a,b)}(x) \).

3. Infinite family of Hermitian and non-Hermitian Hamiltonians associated with Jacobi-type \( X_m \) EOPs

Here, we shall derive some exactly solvable Hermitian as well as non-Hermitian potentials whose bound state solutions are connected to the Jacobi-type \( X_m \) EOP. For this, we have used an appropriate coordinate transformation which transforms the Schrödinger equation into a differential equation of the \( X_m \) Jacobi EOP. In the following, we first briefly describe this method.
3.1. Construction method

Following the discussion of [48, 45], we consider the transformation of the Schrödinger equation (with $h = 2m = 1$)

$$H\psi(x) = -\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

(3.1)

into a second-order differential equation satisfied by a special function $F(g)$. For this, the solution $\psi(x)$ of equation (3.1) can be assumed as

$$\psi(x) \sim f(x)F(g(x)).$$

(3.2)

Substituting this form of $\psi(x)$ into (3.1), we obtain the following second-order ordinary differential equation of the special function $F(g)$:

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F = 0,$$

(3.3)

where $Q(g(x))$ and $R(g(x))$ are given by

$$Q(g) = \frac{g''}{g'^2} + \frac{2f'}{fg'},$$

(3.4a)

$$R(g) = \frac{E - V(x)}{g'^2} + \frac{f''}{fg'^2},$$

(3.4b)

respectively. After performing some algebraic manipulations, equations (3.4a) and (3.4b) reduce to

$$f(x) \approx g'(x)^{-1/2} e^{\frac{i}{2}Q(g)}.$$

(3.5a)

$$E - V(x) = \frac{g''}{2g} - \frac{3}{4} \left(\frac{g''}{g}\right)^2 + g^2 \left(R - \frac{1}{2} \frac{dQ}{dg} - \frac{1}{4} Q^2\right).$$

(3.5b)

The above expression of $E - V(x)$ contains the unknown functions $Q(g)$, $R(g)$ and the change of variable $g(x)$. A particular choice of the special function $F(g)$ provides the complete functional forms of the first two unknowns, whereas a simple way of finding the unknown function $g(x)$ has been first proposed by Bhattacharjee and Sudarshan [49]. According to them, if there is a constant $(E)$ on the left-hand side of (3.5b), there must be one on the right-hand side too. Hence, a specific special function $F(g)$ and a reasonable choice of $g(x)$ make the Schrödinger equation (3.1) an exactly solvable problem for the potential $V(x)$ (which is different for different choices of the special function). The solution of the corresponding Schrödinger equation can be obtained using equations (3.2) and (3.5a) as

$$\psi(x) \sim g'(x)^{-1/2} e^{\frac{i}{2}Q(g)}F(g(x)).$$

(3.6)

This procedure has been exploited in [48], and a systematic list of shape-invariant real potentials has been obtained whose solutions are related to the classical orthogonal polynomials. Analogous study, corresponding to $X_l$ EOPs, has been carried out in [3] for Hermitian systems and in [45] for quasi-Hermitian systems. In the following, we use the above-mentioned method to obtain infinitely many exactly solvable Hermitian as well as non-Hermitian Hamiltonians whose bound state wavefunctions are connected to the exceptional $X_m$ Jacobi orthogonal polynomials for arbitrary $m = 1, 2, 3 \ldots$.  

Here, we choose the special function $F(g)$ to be Jacobi-type $X_m$ EOP e.g. $F(g) \sim \hat{P}_{\alpha,\beta}^{(m)}(g)$. In this case, we have, from equations (2.1) and (3.3),

$$Q(g) = \frac{Q_1(g)}{1 - g^2}, \quad R(g) = \frac{R_1(g)}{1 - g^2},$$

(3.7)
where \( Q_1(g) \) and \( R_1(g) \) are given in equation (2.2). Using equations (2.2) and (3.7), equation (3.5b) reduces to

\[
E - V(x) = \frac{g''}{4} - \frac{3}{4} \left( \frac{g''}{g} \right)^2 + \frac{1 - a^2}{4} \frac{g^2}{(1 - g^2)^2} + \frac{1 - b^2}{4} \frac{g^2}{(1 + g^2)^2} + \frac{2n^2 + 2m(a + b - 2m + 1) + 2m(a - 3b - m + 1) + (a + 1)(b + 1)}{2} g^2
\]

\[
+ \frac{(a - b - m + 1)[a + b + (a - b + 1)g]g^2}{1 - g^2} \frac{P_{m-1}^{(-a,b)}(g)}{P_{m-1}^{(-a-1,b-1)}(g)},
\]

\[
- \frac{(a - b - m + 1)^2 g^2}{2} \left[ \frac{P_{m-1}^{(-a,b)}(g)}{P_{m-1}^{(-a-1,b-1)}(g)} \right]^2.
\]

(3.8)

At this point, we choose \( g^2/(1 - g^2) = \text{constant} = c \in \mathbb{R} - \{0\} \), which is satisfied by

\[
g(x) = \sin(\sqrt{c}x + d).
\]

(3.9)

Without loss of generality, we will first choose \( d = 0 \). The nonzero real values of \( d \) do not make any significant difference in the potential and its solutions. The nonzero complex values of \( d \) give rise to complex potentials with real spectra as will be shown later. The choice (3.9) of \( g(x) \) helps to generate a constant term on the right-hand side corresponding to the energy on the left-hand side of equation (3.8). Here two cases may arise depending on the sign of the chosen constant \( c \). In the following, we consider two cases \( c > 0 \) and \( c < 0 \) separately.

**c > 0** gives the infinite numbers of rationally extended Hermitian as well as quasi-Hermitian trigonometric Scarf potentials, while \( c < 0 \) corresponds to the finite number of Hermitian and infinite number of non-Hermitian \( PT \)-symmetric hyperbolic Scarf potential family.

### 3.2. Infinite family of Hermitian Hamiltonians

**Case I: c > 0, d = 0; an extended real trigonometric Scarf potential family.** Let \( c = k^2 \), \( k \neq 0 \). In this case, we have \( g(x) = \sin kx \). Substituting \( g(x) = \sin kx \) into equation (3.8) and separating out the potential \( V(x) \) and energy \( E \), we have

\[
V^{(m)}(x) = \frac{k^2(2a^2 + 2b^2 - 1)}{4} \sec^2 kx - \frac{k^2(b^2 - a^2)}{2} \sec kx \tan kx - 2k^2m(a - b - m + 1)
\]

\[
- k^2(a - b - m + 1)[a + b + (a - b + 1) \sin kx] \frac{P_{m-1}^{(-a,b)}(\sin kx)}{P_{m-1}^{(-a-1,b-1)}(\sin kx)}
\]

\[
+ \frac{k^2(a - b - m + 1)^2 \cos^2 kx}{2} \left[ \frac{P_{m-1}^{(-a,b)}(\sin kx)}{P_{m-1}^{(-a-1,b-1)}(\sin kx)} \right]^2, \quad \frac{\pi}{2k} < x < \frac{\pi}{2k},
\]

\[(3.10a)\]

\[
E_n^{(m)} = \frac{k^2}{4} (2n - 2m + a + b + 1)^2, n = m, m + 1, m + 2, \ldots.
\]

(3.10b)

Also, the expression of the wavefunction \( \psi(x) \) given in equation (3.6) reduces to

\[
\psi^{(m)}(x) = N_n^{(m)} \frac{(1 - \sin kx)^{\frac{1}{2} + \frac{1}{2}m} (1 + \sin kx)^{\frac{1}{2} + \frac{1}{2}m} P_n^{(a,b,m)}(\sin kx)}{P_n^{(a-1,b-1)}(\sin kx)}, n \geq m.
\]

(3.11)

The normalization constant \( N_n^{(m)} \) can be obtained using equation (2.4) in the following form:

\[
N_n^{(m)} = \left[ \frac{k(2n - 2m + a + b + 1)(n - m + a + 1)^2(n - m)!(n - m + a + 2)!\Gamma(n - m + a + 1)}{2^{a+b+1} (n + b) (n - 2m + a + 1) \Gamma(n - m + a + 2) \Gamma(n + b)} \right]^2.
\]

(3.12)
At this point, a few remarks on the obtained potentials \( V^{(m)}(x) \) are worth mentioning.

- The new potentials \( V^{(m)}(x) \), \( m = 0, 1, 2, \ldots \), become singular at the zeros of the Jacobi polynomial \( P_{m}^{(a-1,b-1)}(x) \) in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\). These singularities can be avoided by restricting the potential parameters \( a, b \) according to the conditions mentioned in equation (2.3).
- The potentials \( V^{(m)}(x) \) given in (3.10a) are infinite in number because each integer value of \( m \geq 0 \) gives rise to a new exactly solvable potential.
- For \( m = 0 \), the last three terms in the expression of \( V^{(m)}(x) \) vanish and one is left with the well-known trigonometric Scarf potential

\[
V^{(0)}(x) = \frac{k^2(2a^2 + 2b^2 - 1)}{4} \sec^2 kx - \frac{k^2(b^2 - a^2)}{2} \sec kx \tan kx.
\]

By redefining the parameters \( a = \alpha - \beta - 1/2 \), \( b = \alpha + \beta - 1/2 \), the above equation (3.13) reduces to the well-known trigonometric Scarf potential [48, 50] and its bound state energy spectrum given by

\[
E_n^{(0)} = k^2(n + \alpha + \beta)^2, \quad n = 0, 1, 2, \ldots
\] (3.14)

respectively. Corresponding wavefunctions can be written in terms of classical Jacobi polynomials

\[
\psi_n^{(0)}(x) = N_n^{(0)}(1 - \sin kx)^{\frac{\alpha -1}{2}}(1 + \sin kx)^{\frac{\beta -1}{2}} P_{\frac{n}{2}}^{(\alpha - \frac{1}{2}, \beta - \frac{1}{2})}(\sin kx).
\] (3.15)

(b) \( m = 1 \). For \( m = 1 \) and the same choices of parameters mentioned in the case of \( m = 0 \), equation (3.8) reduces to

\[
V^{(1)}(x) = k^2[\alpha(\alpha - 1) + \beta^2] \sec^2 kx - k^2 \beta(2\alpha - 1) \sec kx \tan kx + \frac{2k^2(2\alpha - 1)}{2\alpha - 1 - 2\beta \sin kx} \frac{2k^2(2\alpha - 1)^2 - 4\beta^2}{(2\alpha - 1 - 2\beta \sin kx)^2}
\]

\[
E_n^{(1)} = k^2(n + \alpha - 1)^2, \quad n = 1, 2, 3, \ldots
\] (3.16)

The potential \( V^{(1)}(x) \) is the rationally extended trigonometric Scarf potential, studied earlier in [3]. The bound state solutions can be written in terms of exceptional \( X_1 \) Jacobi
polynomials [1, 2]
\[ \psi_n^{(1)}(x) = N_n^{(1)} \left( \frac{1 - \sin kx}{2} \right)^{x = \alpha} \left( \frac{1 + \sin kx}{2} \right)^{x = \beta} P_n^{(\alpha - \beta, \frac{1}{2}, \alpha + \beta, \frac{1}{2})}(\sin kx), \quad n = 1, 2, \ldots \]
(3.17)

(c) \( m = 2 \). In this case, we have the following potential, energies and bound state wavefunctions involving \( X_2 \) exceptional Jacobi orthogonal polynomials:
\[ V^{(2)}(x) = k^2 \left[ \alpha(\alpha - 1) + \beta^2 \right] \sec^2 kx - k^2 \beta(2\alpha - 1) \sec kx \tan kx \]
\[ + \frac{4k^2}{3(2\alpha - 1)(2\beta + 1) \sin kx - 2\beta(2\beta + 1) - 8\alpha(\alpha - 1)} \]
\[ + \frac{2(\beta + 1)(2\beta + 1) \sin^2 kx + 2(2\beta + 1)(2\alpha - 1) \sin kx + 4\alpha(\alpha - 1) - 2\beta - 1}{8(2\beta + 1)^2 k^2 \cos^2 kx[2(1 + \beta) \sin kx - 2\alpha + 1]^2} - 8k^2 \]
\[ E_n^{(2)} = \frac{k^2}{4} (n + \alpha - 2)^2, \quad n = 2, 3, 4 \ldots \] (3.18)
and
\[ \psi_n^{(2)}(x) = N_n^{(2)} \left( \frac{1 - \sin kx}{2} \right)^{x = \alpha} \left( \frac{1 + \sin kx}{2} \right)^{x = \beta} \]
\[ \times P_n^{(\alpha - \beta, \frac{1}{2}, \alpha + \beta, \frac{1}{2})}(\sin kx), \quad n = 2, 3, \ldots , \] (3.19)
respectively.

3.3. Infinite family of non-Hermitian Hamiltonians

Case II: \( c > 0, d = ie \); an extended quasi-Hermitian trigonometric Scarf potential family. In section 3.2, we have considered the constant of integration \( d \) to be zero. The nonzero value of \( d \) corresponds to the shift of the coordinate \( x \). In the Hermitian case, this coordinate shift is not so important as it does not influence the energy eigenfunctions and eigenvalues. However, in \( \mathcal{PT} \)-symmetric quantum mechanics, the imaginary coordinate shift plays a significant role. This gives rise to complex potentials with an entirely real spectrum. Here, we shall examine how the potentials \( V^{(m)}(x) \) and associated energy spectra behave if we allow a purely imaginary value of \( d \). For this, we set \( d = ie \) and \( c = k^2 \), in which case equations (3.9) and (3.10a) reduce to
\[ g(x) = \sin(ke + e) \] (3.20)
and
\[ \tilde{V}^{(m)}(x) = \frac{k^2(2a^2 + 2b^2) - 1}{4} \sec^2(kx + ie) - \frac{k^2(b^2 - a^2)}{2} \sec(kx + ie) \tan(kx + ie) \]
\[ - 2k^2 m(a - b - m + 1) \]
\[ - k^2(a - b - m + 1)[a + b + (a - b + 1) \sin(kx + ie)] \frac{P_{m-1}^{(-a,b)}(\sin(kx + ie))}{P_{m-1}^{(-a-1,b-1)}(\sin(kx + ie))} \]
\[ + \frac{k^2(a - b - m + 1)^2 \cos^2(kx + ie)}{2} \left[ \frac{P_{m-1}^{(-a,b)}(\sin(kx + ie))}{P_{m-1}^{(-a-1,b-1)}(\sin(kx + ie))} \right]^2, \]
\[ - \infty < x < \infty \]
\[ \tilde{E}_n^{(m)} = \frac{k^2}{4} (2n - 2m + a + b + 1)^2, \quad n = m, m + 1, m + 2, \ldots \] (3.21)
Similar to the Hermitian case, the potentials $\tilde{V}^{(m)}(x)$, $m = 0, 1, 2, \ldots$, are infinite in number and all the members are isospectral. Here, $a$ and $b$ are chosen to be real; otherwise, the solution of this potential will contain Jacobi polynomials with complex indices and complex arguments. Such complex polynomials are not suitable for physical applications because of their non-trivial orthogonality properties which depend on the interplay between integration contour and parameter values. It is to be noted here that the non-Hermitian Hamiltonians $\tilde{H}^{(m)}$ with the potentials $\tilde{V}^{(m)}(x)$ have entirely real bound state energies $\tilde{E}^{(m)}_n$. This can be proved in the following way. Let us define a Hermitian, positive definite operator $\rho$ as

$$\rho = e^{i\tilde{F} p}, \quad p = -\frac{d}{dx},$$

which has the following properties [43]:

$$\rho x \rho^{-1} = x - \frac{i\epsilon}{k}, \quad \rho \rho \rho^{-1} = p, \quad \rho \tilde{f}(x) \rho^{-1} = \tilde{f}\left(x - \frac{i\epsilon}{k}\right).$$

With the help of the operator $\rho$, we have the following similarity transformation:

$$\rho \tilde{V}^{(m)}(x) \rho^{-1} = V^{(m)}(x).$$

(3.24)

This shows that the non-Hermitian Hamiltonians $\tilde{H}^{(m)}$ are quasi-Hermitian with respect to the Hermitian, positive definite operator $\rho$. The equivalent Hermitian counterparts are the Hamiltonians with the potentials $V^{(m)}(x)$ (3.10a) corresponding to $d = \epsilon = 0$. The energy eigenfunctions of the potentials $\tilde{V}^{(m)}(x)$ can be obtained using $\tilde{\psi}^{(m)}_n(x) = \rho^{-1} \tilde{\psi}^{(m)}_n(x) = \psi^{(m)}_n(x + i\epsilon)$, where $\psi^{(m)}_n(x)$ are given by equation (3.11). It will not be difficult to show that the Hamiltonian $\tilde{H}^{(m)}$ is also pseudo-Hermitian, i.e.

$$\eta \tilde{H}^{(m)} \eta^{-1} = \tilde{H}^{(m)}.$$  

(3.25)

with respect to the positive definite operator

$$\eta = \rho^2 = e^{i\tilde{F} p}.$$  

(3.26)

$PT$ symmetry of the potential $\tilde{V}^{(m)}(x)$ cannot be achieved, in general, if $d$ has a nonzero real component because the finite shift along the coordinate $x$ makes the potential different from its $PT$ counterpart. Now, we shall try to determine under what condition the above potential remains $PT$ invariant. A necessary requirement for a Hamiltonian to be $PT$-symmetric is that the potential should satisfy the following condition:

$$\tilde{V}^{(m)}(x) = -\tilde{V}^{(m)}(x).$$

(3.27)

This implies that for $m = 0, 1$, the potentials $\tilde{V}^{(m)}(x)$ are $PT$-symmetric iff $a = \pm b$. For $m \geq 2$, they are $PT$-symmetric if $a = -b$. Hence, we have derived infinitely many non-Hermitian rationally extended trigonometric Scarf potentials with entirely real spectra and whose solutions are associated with an exceptional $X_m$ Jacobi EOP.

Case III: $c < 0, d = 0$; a non-Hermitian $PT$-symmetric hyperbolic Scarf potential family. Let $c = -k^2, k \neq 0$ and $d = 0$. In this case, we have from (3.9), $g(x) = i \sinh kx$. The corresponding potential in equation (3.8) (we rename them as $U^{(m)}(x)$) reduces to

$$U^{(m)}(x) = -\frac{k^2(2a^2 + b^2 - 1)}{4} \text{sech}^2 k x + \frac{k^2(b^2 - a^2)}{2} \text{sech} k x \tanh k x + 2k^2 m(a - b - m + 1)$$

$$+ k^2(a - b - m + 1)(a + b + (a - b + 1)i \sinh k x) \frac{P^{-a, b}_{m-1}(i \sinh k x)}{P^{-a, -1, b-1}_{m-1}(i \sinh k x)}$$

$$= \frac{k^2(a - b - m + 1)^2 \cosh^2 k x}{2} \left[ \frac{P^{-a, b}_{m-1}(i \sinh k x)}{P^{-a, -1, b-1}_{m-1}(i \sinh k x)} \right]^2, -\infty < x < \infty.$$  

(3.28)
The bound state wavefunctions and energy spectrum of the Schrödinger equation for this potential \( U^{(m)}(x) \) are given by equations (3.8) and (3.6) as

\[
\psi_{m}^{(m)}(x) = N_{n}^{(m)} \frac{(1 - i \sinh kx)^{\frac{n}{2}} (1 + i \sinh kx)^{\frac{n}{2}}}{P_{m}^{(a,b,m)}(i \sinh kx)} \tilde{P}_{n}^{(a,b,m)}(i \sinh kx),
\]

\[
E_{n}^{(m)} = -\frac{k^2}{4} (2n - 2m + a + b + 1)^2, \quad n = m, m + 1, m + 2, \ldots < \left( m - \frac{a + b + 1}{2} \right),
\]

respectively. A few comments on the potential family \( U^{(m)}(x) \) are as follows.

- Since the potential is free from singularity in \(-\infty < x < \infty\), so the wavefunction should vanish asymptotically at \pm \infty. This imposes an upper bound on the quantum number \( n < (m - \frac{a + b + 1}{2}) \).
- The potentials \( U^{(m)}(x) \), \( m = 0, 1, 2, \ldots \) are in general non-Hermitian with entirely real energy spectrum. Moreover, they are \( \mathcal{PT} \)-symmetric for all the real values of \( a, b \). The potentials \( U^{(m)}(x) \) are also rationally extended version of the conventional non-Hermitian hyperbolic Scarf potentials [52] by the addition of some polynomial-dependent terms.
- For \( m = 0, 1 \), the corresponding potentials \( U^{(m)}(x) \) are real provided \( a = b \). For \( m \geq 2 \), they become real if \( a = -b \). But in the latter case, the corresponding energy spectrum becomes empty. So the real potential corresponding to \( m \geq 2 \) is not physically interesting.
- All the rationally extended non-Hermitian hyperbolic Scarf potentials are isospectral with the energy spectrum \( E_{n}^{(m)} = -\frac{k^2}{4} (2n' + a + b + 1)^2, n' = 0, 1, 2, \ldots \).

Hence, the potentials \( U^{(m)}(x), m = 0, 1, 2, \ldots \), can be interpreted as an infinite family of non-Hermitian \( \mathcal{PT} \)-symmetric extended hyperbolic Scarf potentials with entirely real energy spectrum. In particular, for the case \( a = b \) the potentials \( U^{(0)} \) and \( U^{(1)}(x) \) become Hermitian.

4. Supersymmetric shape-invariant approach

Here, we combine the results, obtained in section 3, with supersymmetric quantum mechanics. Supersymmetry makes use of two linear first-order differential operators \( A^{(m)\pm} = \mp \frac{d}{dx} + W^{(m)}(x) \), where \( W^{(m)}(x) \) is the superpotential and generally defined [50] in terms of the ground state wavefunction \( \psi_{m}^{(m)}(x) \) as

\[
\psi_{m}^{(m)}(x) \sim \exp \left(-\int^{x} W^{(m)}(r) \, dr \right).
\]

The two operators \( A^{(m)\pm} \) give rise to two partner Hamiltonians \( H^{(m)\pm} \):

\[
H^{(m)\mp} = A^{(m)\pm} A^{(m)\mp} - \frac{d^2}{dx^2} + V^{(m)\mp}(x) = \varepsilon,
\]

where \( \varepsilon \) is the factorization energy and \( V^{(m)\pm} \) are the two partner potentials given in terms of superpotential

\[
V^{(m)\pm}(x) = W^{(m)}(x)^2 \pm W^{(m)'}(x).
\]

The partner Hamiltonians have the same spectrum except the zero energy state, i.e. \( E_{n+1}^{(m)-} = E_{n}^{(m)+} \) and \( E_{0}^{(m)-} = 0 \). If the zero energy state of one of the Hamiltonians \( H^{(m)\pm} \) is known, then the eigenfunctions of the other Hamiltonian can be determined using \( \psi_{n+1}^{(m)+} \sim A^{-1} \psi_{n+1}^{(m)-} \).
and \( \psi_m^{(m)-} \sim A^+ \psi_{m-1}^{(m)+} \). The two partner potentials \( V^{(m)+}(x) \) are said to be shape invariant if they satisfy \[ V^{(m)+}(x, a_0) = V^{(m)-}(x, a_1) + R(a_0), \] where \( a_1 = f(a_0) \) and \( R(a_0) \) is independent of \( x \). The beauty of the shape-invariant property is that whenever two supersymmetric partner potentials are related by the shape-invariant condition, the energy eigenvalues and the eigenvectors can be determined algebraically [50].

For convenience, we identify the potential \( V^{(m)}(x) \), given in equation (3.10a), with \( V^{(m)-}(x) \). The ground state solution (follow the equation (3.11)) of this potential can be written as

\[
\psi_m^{(m)-} \sim (1 - \sin kx) \frac{2}{1 + \sin kx} \left[ 1 - \frac{P_{m-1}^{(-a-1,b)}(\sin kx)}{P_m^{(-a-1,b-1)}(\sin kx)} \right]. \tag{4.5}
\]

Using equations (4.1) and (4.5) and after some algebraic manipulations with the help of the recurrence relations (2.6), we obtain the superpotential as

\[
W^{(m)}(x) = \frac{k(a - b)}{2} \sec kx + \frac{k(a + b + 1)}{2} \tan kx - \frac{k(a - b - m + 1) \cos kx}{2} \\
\times \left[ \frac{P_{m-1}^{(-a-1,b)}(\sin kx)}{P_m^{(-a-1,b-1)}(\sin kx)} - \frac{P_{m-1}^{(-a-1,b+1)}(\sin kx)}{P_m^{(-a-2,b)}(\sin kx)} \right]. \tag{4.6}
\]

From equations (4.3) and (4.6), we obtain the simplified expressions of the partner potentials \( V^{(m)+} \) as

\[
V^{(m)+}(a, b, x) = \frac{k^2 (2a^2 + 2b^2 - 1)}{4} \sec^2 kx - \frac{k^2 (b^2 - a^2)}{2} \sec kx \tan kx \\
- 2k^2 m(a - b - m + 1) \\
- k^2 (a - b - m + 1)[a + b + (a - b + 1) \sin kx] \frac{P_{m-1}^{(-a,b)}(\sin kx)}{P_m^{(-a-1,b-1)}(\sin kx)} \\
+ \frac{k^2 (a - b - m + 1)^2 \cos^2 kx}{2} \left( \frac{P_{m-1}^{(-a,b)}(\sin kx)}{P_m^{(-a-1,b-1)}(\sin kx)} \right)^2 - \frac{k^2 (a + b + 1)^2}{4} \tag{4.7}
\]

and

\[
V^{(m)-}(a, b, x) = \frac{k^2 [2(a + 1)^2 + 2(b + 1)^2 - 1]}{4} \sec^2 kx \\
- \frac{k^2 [(b + 1)^2 - (a + 1)^2]}{2} \sec kx \tan kx \\
- k^2 (a - b - m + 1)[a + b + 2 + (a - b + 1) \sin kx] \frac{P_{m-1}^{(-a-1,b+1)}(\sin kx)}{P_m^{(-a-2,b)}(\sin kx)} \\
+ \frac{k^2 (a - b - m + 1)^2 \cos^2 kx}{2} \left( \frac{P_{m-1}^{(-a-1,b+1)}(\sin kx)}{P_m^{(-a-2,b)}(\sin kx)} \right)^2 - \frac{k^2 (a + b + 1)^2}{4} \tag{4.8}
\]

The potential \( V^{(m)-} \) matches, apart from an additive factorization energy \( \varepsilon = \frac{k^2 (a+b+1)^2}{4} \), with the potential (3.10a) obtained in section 3. The potential \( V^{(m)+}(x) \) is another infinite set of exactly solvable potential. It is not very difficult to show that the bound state solutions of these potentials are also associated with \( X_m \) Jacobi polynomials. Also, it is very easy to check that...
the two partner potentials $V^{(m)_+}$ given in equations (4.7) and (4.8) are connected to each other by

$$V^{(m)_+}(a, b, x) = V^{(m)_-}(a + 1, b + 1, x) + k^2(a + b + 2),$$  \hspace{1cm} (4.9)

i.e. they have the translational shape-invariant symmetry. Analogously, one can study other generalized potential families, obtained in section 3, in the framework of supersymmetric quantum mechanics to show that they have the shape-invariant symmetry.

5. Summary

In summary, we have obtained infinitely many exactly solvable Hermitian as well as non-Hermitian trigonometric Scarf potentials $V^{(m)}(x)$ and $\tilde{V}^{(m)}(x)$, $m = 0, 1, 2, \ldots$, respectively. We have also obtained a finite number of Hermitian and an infinite number of non-Hermitian $\mathcal{PT}$-symmetric hyperbolic Scarf potentials $U^{(m)}(x)$ with entirely real energy spectra. The bound state wavefunctions of all these potentials are associated with the exceptional $X_m$ Jacobi polynomials. All the potentials belonging to a particular family of potentials $V^{(m)}(x)$, $U^{(m)}(x)$ or $\tilde{V}^{(m)}(x)$ are isospectral to each other. The supersymmetric partners of the potentials, obtained here, possess the shape-invariant symmetry. The non-Hermitian Hamiltonians involving the complex trigonometric Scarf potentials $\tilde{V}^{(m)}(x)$ are shown to be quasi-Hermitian with respect to an invertible operator $\rho = e^{i\gamma}$. This implies that corresponding energy spectra are all real. It has also been shown that for $a = -b$, the potentials $\tilde{V}^{(m)}(x)$ are $\mathcal{PT}$-symmetric, whereas the potentials $U^{(m)}(x)$ are $\mathcal{PT}$-symmetric for all $a$, $b$.

Acknowledgment

One of the authors (BM) thanks Robert Milson for communication regarding exceptional $X_m$ Jacobi polynomials.

References

[1] Gomez-Ullate D, Kamran N and Milson R 2010 J. Approx. Theory 162 987
[2] Gomez-Ullate D, Kamran N and Milson R 2010 J. Math. Anal. Appl. 359 352
[3] Quesne C 2008 J. Phys. A: Math. Theor. 41 392001
[4] Bagchi B, Quesne C and Roychoudhury R 2009 Pramana J. Phys. 73 337
[5] Quesne C 2009 SIGMA 5 084
[6] Odake S and Sasaki R 2009 Phys. Lett. B 679 414
[7] Odake S and Sasaki R 2010 Phys. Lett. B 684 173
[8] Ho C.L, Odake S and Sasaki R 2011 SIGMA 7 107
[9] Ho C-L and Sasaki R 2011 Zeros of the exceptional Laguerre and Jacobi polynomials arXiv:1102.5669
[10] Gomez-Ullate D, Kamran N and Milson R 2012 Asymptotic behaviour of zeros of exceptional Jacobi and Laguerre polynomials arXiv:1204.2282
[11] Odake S and Sasaki R 2010 J. Math. Phys. 51 053513
[12] Quesne C 2011 Int. J. Mod. Phys. A 26 5337
[13] Gomez-Ullate D, Kamran N and Milson R 2012 J. Math. Anal. Appl. 387 410
[14] Odake S and Sasaki R 2011 Phys. Lett. B 702 164
[15] Quesne C 2011 Mod. Phys. Lett. A 26 1843
[16] Grandati Y 2012 Ann. Phys. 327 2411
[17] Gomez-Ullate D, Kamran N and Milson R 2010 J. Phys. A: Math. Theor. 43 434016
[18] Sasaki R, Tsujimoto S and Zhedanov A 2010 J. Phys. A: Math. Theor. 43 315205
[19] Gomez-Ullate D, Kamran N and Milson R 2012 Contemp. Math. 563 51
[20] Grandati Y 2011 J. Math. Phys. 52 103505
[21] Grandati Y 2011 Ann. Phys. 326 2074
[22] Ho C-L 2011 J. Math. Phys. 52 122107
[23] Ho C-L 2011 Prog. Theor. Phys. 126 185
[24] Dutta D and Roy P 2011 J. Math. Phys. 52 032104
[25] Junker G and Roy P 1998 Ann. Phys. 270 155
[26] Midya B and Roy B 2009 Phys. Lett. A 373 4117
[27] Ho C-L 2011 Ann. Phys. 326 797
[28] Chou C-I and Ho C-L 2012 Fokker–Planck equation which are the generalized, or deformed versions of the Rayleigh process and the Jacobi process are arXiv:1207.6001
[29] Ranjani S Sree et al 2012 J. Phys. A: Math. Theor. 45 055210
[30] Tanaka T 2010 J. Math. Phys. 51 032101
[31] Midya B, Roy B and Tanaka T 2012 J. Phys. A: Math. Theor. 45 205303
[32] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
[33] Bender C M 2005 Contemp. Phys. 46 277
Bender C M 2007 Rep. Prog. Phys. 70 947
[34] Mostafazadeh A 2002 J. Math. Phys. 43 205
Mostafazadeh A 2002 J. Math. Phys. 43 2814
Mostafazadeh A 2002 J. Math. Phys. 43 3944
[35] Mostafazadeh A 2010 Int. J. Geom. Methods Mod. Phys. 7 1191
[36] Mostafazadeh A and Batal A 2004 J. Phys. A: Math. Gen. 37 11645
[37] Kretschmer R and Szymański L 2004 Phys. Lett. A 325 112
[38] Scholtz F G, Geyer H B and Hahne F J W 1992 Ann. Phys. 213 74
[39] Ahmed Z 2001 Phys. Lett. A 282 343
[40] Bagchi B and Roychoudhury R 2000 J. Phys. A: Math. Gen. 33 L1
[41] Levai G and Znojil M 2000 J. Phys. A: Math. Gen. 33 7165
[42] Levai G and Znojil M 2001 Mod. Phys. Lett. A 16 1973
[43] Ahmed Z 2001 Phys. Lett. A 290 19
[44] Midya B, Roy B and Roychoudhury R 2010 Phys. Lett. A 374 2605
[45] Midya B 2012 Phys. Lett. A 376 2851
[46] Szegő G 1975 Orthogonal Polynomials (American Mathematical Society Colloquium Publications vol 23)
(Providence, RI: American Mathematical Society)
[47] Gradshteyn I S and Ryzhik I M 1996 Tables of Integrals, Series, and Products 5th edn (New York: Academic)
[48] Levai G 1989 J. Phys. A: Math. Gen. 22 689
[49] Bhattacharjee A and Sudarshan E C G 1962 Nuovo Cimento 25 864
[50] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 257 267
[51] Gendenshtein L 1983 JETP Lett. 38 356
[52] Levai G, Cannata F and Ventura A 2002 Phys. Lett. A 300 271