An isoperimetric inequality for fundamental tones of free plates with nonzero Poisson’s ratio

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We establish a partial generalization of a prior isoperimetric inequality for the fundamental tone (first nonzero eigenvalue) of the free plate to that of plates of nonzero Poisson’s ratio. Given a tension $\tau > 0$ and a Poisson’s ratio $\sigma$, the free plate eigenvalues $\omega$ and eigenfunctions $u$ are determined by the equation $\Delta^2 u - \tau \Delta u = \omega u$ together with certain natural boundary conditions which involve both $\tau$ and $\sigma$. The boundary conditions are complicated but arise naturally from the plate Rayleigh quotient. We prove the free plate isoperimetric inequality, previously shown in the $\sigma = 0$ case, holds for certain nonzero $\sigma$ and positive $\tau$. We conjecture that the inequality holds for all dimensions, $\tau > 0$, and relevant values of $\sigma$, and discuss numerical and analytic support of this conjecture.

Keywords: isoperimetric; free plate; bi-Laplace; bi-Laplace eigenvalues

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1. Introduction

The eigenvalues of the Laplacian operator and its fourth-order analog the bi-Laplacian appear in many models of physical situations, representing quantities such as frequency or energy. One classic example is that the eigenvalues $\mu$ of the Neumann Laplacian on a bounded region $\Omega$ determine the frequencies of vibration of a free membrane with that shape. If $\Omega^*$ is the ball of same volume as $\Omega$, then we have

$$\mu_1(\Omega) \leq \mu_1(\Omega^*)$$

with equality if and only if $\Omega$ is a ball.

First conjectured by Kornhauser and Stakgold [1], this isoperimetric inequality was proved for simply connected domains in $\mathbb{R}^2$ by Szegö [2,3] and extended to all domains and dimensions by Weinberger [4].

While Laplacian eigenvalue problems represent vibrations of membranes, there are corresponding bi-Laplace problems represent vibrations and buckling energies of plates. Fourth-order plate problems are frequently more difficult than their second-order analogs – the theory of the bi-Laplace operator is not as well understood, and because
the order is higher, exact solutions (such as those used for trial function methods) can require more complicated linear combinations of special functions.

The isoperimetric inequality for the fundamental tone of the Dirichlet Laplacian (drum) was proved by Faber [5] and Krahn [6,7] in the 1920s with the ball as the minimizer. It was not until the 1980s and 90s that it was proved that the ball also the lower bound for the first clamped plate eigenvalue [8–11]. The methods used by Talenti, Nadirashvilli, Ashbaugh and Benguria to prove the clamped plate isoperimetric inequality are quite different than those for the free plate and membrane and only establish the bound in dimensions two and three. The problem remains open for dimensions four and higher, with a partial result by Ashbaugh and Laugesen [12].

Other boundary conditions exist for fourth-order problems, such as the hinged plate investigated by Nazarov and Sweers [13]. Other generalizations of the Szegő–Weinberger inequality include its analog in other spaces. In spaces of constant curvature, the spherical cap (analog of the ball) maximizes the first Neumann eigenvalue, as seen by Ashbaugh and Benguria [14]. In Gauss space, the problem was considered by Chiacchio and Di Blasio [15]. As in Euclidean space, this gives an upper bound on the fundamental tone of the Neumann Gaussian Laplacian (Hermitian); one can also consider lower bounds on the Neumann eigenvalues, e.g. [16,17].

In [18], we adapted Weinberger’s trial function argument to prove the free plate analog of the Szegő–Weinberger inequality for plates with positive tension and assuming the Poisson’s ratio (a property of the material) of the plate was zero. Taking $\omega_1$ to be the fundamental tone, we had that:

$$\omega_1(\Omega) \leq \omega_1(\Omega^*) \quad \text{with equality if and only if } \Omega \text{ is a ball.}$$ (1)

In this paper, we present a generalization of this result to some plates under tension with nonzero values of Poisson’s ratio. We prove that if the dimension, tension, and Poisson’s ratio are such that fundamental mode of the ball has simple angular dependence, we again have the bound (1). Our proof relies on the variational characterization of eigenvalues with suitable trial functions, taken to be extensions of the fundamental mode of the unit ball. This follows both Weinberger’s approach for the free membrane and our prior work for the free plate with zero Poisson’s ratio in [18]. However, because the plate equation is fourth order, finding the trial functions and establishing the appropriate monotonicities is significantly more complicated than in the membrane case. The inclusion of $\sigma$ further complicates matters and prevents us from applying some of our tools from [18], including identifying the fundamental mode of the ball.

Based on numerical evidence and analytic reasoning, we conjecture that the fundamental mode of the ball has simple angular dependence (i.e. in dimension two, the angular part can be written as $\sin(\theta)$ or $\cos(\theta)$) for all dimensions, positive tension and mathematically suitable values of Poisson’s ratio, and so the isoperimetric inequality (1) holds for all plates.

Poisson’s ratio, which we will denote by $\sigma$, is a property of the material of the plate. If a material is stretched in one direction, it usually contracts in the orthogonal directions; the value $\sigma$ is a ratio of the strains. In some materials, the material expands in the orthogonal directions rather than contracting; these materials have $\sigma < 0$ and are called auxetic.

Considering nonzero Poisson’s ratio is a natural generalization of the free plate problem because $\sigma$ appears in the Rayleigh quotient for the plate, even though it does not appear in the eigenvalue equation itself; instead, we see explicit dependence on $\sigma$ in the
natural boundary conditions. Verchota recently established the solvability of the biharmonic Neumann problem \[19\], the boundary conditions for which arise from the zero-tension plate with nonzero Poisson’s ratio. Interestingly, the clamped plate problem is independent of \(\sigma\).

Although the clamped plate problem begins from the plate Rayleigh quotient, integration by parts and the imposed boundary conditions allows the clamped plate quotient to be written in its more familiar form, which is independent of \(\sigma\).

This paper proceeds as follows: we begin by formulating the problem and stating our main theorem, a partial result towards the conjectured isoperimetric inequality. We then prove existence of the discrete spectrum and regularity of the eigenfunctions for specific values of \(\sigma\) in Section 4 and use the Rayleigh quotient to establish bounds on the fundamental tone as a function of \(\sigma\) and \(\tau\) in Section 5.

To prepare to prove the theorem, we establish crucial properties of ultraspherical Bessel functions in Section 6, derive the form of the natural boundary conditions in Section 7. We use these in Section 8 to find the eigenfunctions of the ball, where we also state and discuss our conjecture that the fundamental mode has simple angular dependence. We use the fundamental mode to construct our trial functions and establish some properties of these in Section 9. From there, we proceed to prove our main theorem in Sections 10 and 11.

2. Formulating the problem

Let \(\Omega \subseteq \mathbb{R}^d\) be a smoothly bounded region. We will write \(\Omega^*\) for the ball in \(\mathbb{R}^d\) with the same volume as \(\Omega\).

The generalized plate Rayleigh quotient has the form

\[
Q_\Omega[u] := \frac{\int_\Omega (1 - \sigma)|D^2u|^2 + \sigma(\Delta u)^2 + \tau |Du|^2 \, dx}{\int_\Omega u^2 \, dx}.
\]

(2)

Here the parameter \(\tau\) measures tension over flexural rigidity, and \(\sigma\) is Poisson’s ratio. A positive \(\tau\) represents a plate under tension; taking \(\tau < 0\) gives us a plate under compression. The limiting case as \(\tau \to \infty\) is more naturally understood as the limit as rigidity goes to zero; in other words, the plate should behave like a membrane for larger \(\tau\). For Poisson’s ratio, typically \(\sigma \in [0, 0.5]\) for real-world materials, although a class of materials known as auxetics have negative Poisson’s ratio. We will take \(\sigma\) to be in \((-1/(d-1), 1)\) in order to be assured of coercivity of our form.

From the Rayleigh quotient (2), we derive the partial differential equation and boundary conditions governing the vibrational modes of a free plate. The critical points of the quotient are the eigenstates for the plate satisfying the free boundary conditions and the critical values are the corresponding eigenvalues. We shall show in Section 7 that the differential eigenvalue equation is

\[
\Delta \Delta u - \tau \Delta u = \omega u,
\]

(3)

where \(\omega\) is the eigenvalue, with the natural (i.e. unconstrained or “free”) boundary conditions on \(\partial \Omega\):

\[
Mu := (1 - \sigma) \frac{\partial^2 u}{\partial n^2} + \sigma \Delta u = 0,
\]

(4)

\[
Vu := \tau \frac{\partial u}{\partial n} - (1 - \sigma) \text{div}_{\partial \Omega} \left( P_{\partial \Omega} \left( (D^2 u) n \right) \right) - \frac{\partial \Delta u}{\partial n} = 0.
\]

(5)
Here \( \partial/\partial n \) denotes the normal derivative and \( \text{div}_{\partial \Omega} \) is the surface divergence, and \( P_{\partial \Omega} \) projects a vector into the tangent space of \( \partial \Omega \).

The fundamental tone (lowest nonzero eigenvalue) of the plate with shape \( \Omega \) can then be written with the Rayleigh–Ritz characterization as follows:

\[
\omega_1(\Omega) = \inf \left\{ Q_{\Omega}[u] : u \in H^2(\Omega), \int_{\Omega} u \, dx = 0 \right\}
\]

We conjecture the following isoperimetric inequality:

**Conjecture** Let \( \Omega \subset \mathbb{R}^d \) be a smoothly bounded region, and supposed we have \( \tau > 0 \) and \( \sigma \in (-1/(d - 1), 1) \) fixed. Then

\[
\omega_1(\Omega) \leq \omega_1(\Omega^*),
\]

with equality if and only if \( \Omega = \Omega^* \).

This conjecture has previously been proved true in the case of \( \sigma = 0 \) in [18]. It is supported by numerical evidence and analytic arguments made in Section 8 and our weaker result.

### 3. Main result

In this paper, we will prove the following result:

**Theorem 3.1** Suppose we have \( \tau > 0 \) and \( \sigma \in (-1/(d - 1), 1) \) fixed so that the fundamental mode of the ball \( \Omega^* \) has simple angular dependence. Suppose also that one of the following hold:

- \( d = 2 \) and \( \sigma > -51/97 \) or \( \tau \geq 3(1 - \sigma)/(1 + \sigma) \),
- \( d = 3 \),
- \( d \geq 4 \) and \( \sigma \leq 0 \) or \( \tau \geq (d + 2)/2 \).

Then

\[
\omega_1(\Omega) \leq \omega_1(\Omega^*), \tag{6}
\]

with equality if and only if \( \Omega = \Omega^* \).

The restrictions \( \sigma > -51/97 \) for \( d = 2 \) and the lower bounds on \( \tau \) are specific to our method of proof and do not seem to be inherent to the problem. Furthermore, numerical and analytic evidences suggest that the fundamental mode of the ball \( \Omega^* \) has simple angular dependence for choices of dimension \( d \geq 2 \), Poisson’s ratio \( \sigma \in (-1/(d - 1), 1) \) and tension \( \tau > 0 \). We will argue this more thoroughly in Section 8.

The proof of Theorem 3.1 is a trial function argument like that of Weinberger [4] and our own proof of the \( \sigma = 0 \) case in [18]. It proceeds from a sequence of lemmas, organized into the following sections:

- Section 9: Define the trial functions and prove crucial properties about concavity of the radial part.
Section 10: Prove partial monotonicity of the Rayleigh quotient, treating the cases of positive and negative $\sigma$ separately.

Section 11: Complete the proof using the partial monotonicity and rescaling and rearrangement arguments.

4. The existence of the spectrum

The weak eigenvalue problem is given by the sesquilinear form

$$a(u, v) = \int_{\Omega} (1 - \sigma) \sum_{i,j=1}^{d} u_{x_i x_j} v_{x_i x_j} + \sigma(\Delta u \Delta v) + \tau(Du \cdot Dv) \, dx$$

with form domain $H^2(\Omega)$. Note the plate Rayleigh quotient $Q$ can be written in terms of $a(\cdot, \cdot)$, with $Q[u] = a(u, u)/\|u\|_{L^2}^2$.

**Proposition 4.1** Fix $\tau \in \mathbb{R}$ and $\sigma \in (-\frac{1}{d-1}, 1)$. Then the spectrum of the operator $\Delta^2 - \tau \Delta$ associated with the form $a(\cdot, \cdot)$ consists entirely of isolated eigenvalues of finite multiplicity

$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n \to \infty \quad \text{as} \ n \to \infty.$$ (7)

Furthermore, there exists an associated set of weak eigenfunctions which is an orthonormal basis for $L^2(\Omega)$.

Because the quadratic form involves a convex combination of second-order terms $|D^2 u|^2$ and $|\Delta u|^2$, we will find the following inequality useful in proving Proposition 4.1:

**Fact 1** For any function $u \in H^2(\Omega)$, we have the sharp bound $|\Delta u|^2 \leq d|D^2 u|^2$.

We are now ready to prove the existence of the spectrum of our Rayleigh quotient.

**Proof** (Proposition 4.1) We will prove that the quadratic form $a(\cdot, \cdot)$ is bounded and coercive; that is, we will show the existence of positive constants $c_1$ and $c_2$ such that

$$a(u, u) + c_1\|u\|^2 \geq c_2\|u\|_{H^2(\Omega)}^2.$$  

Once we have this, then by a standard result (see e.g. Corollary 7.7 [20, p.88]), the form $a(\cdot, \cdot)$ has a set of weak eigenfunctions which is an orthonormal basis for $L^2(\Omega)$, and the corresponding eigenvalues are of finite multiplicity and satisfy (7).

To prove boundedness of the form when $\sigma \geq 0$, notice that by Fact 1, we have $(\Delta u)^2 \leq d|D^2 u|^2$; thus

$$a(u, u) \leq \int_{\Omega} (1 - \sigma + d\sigma)|D^2 u|^2 + \tau|Du|^2 \, dx,$$

and so $a(u, u) \leq \max(1 + (d-1)\sigma, \tau)\|u\|_{H^2(\Omega)}^2$. That is, $a(\cdot, \cdot)$ is bounded.

When $\sigma < 0$, we note that $\sigma(\Delta u)^2 \leq 0$ and so

$$a(u, u) \leq \int_{\Omega} (1 - \sigma)|D^2 u|^2 + \tau|Du|^2 \, dx \leq \max(1 - \sigma, \tau)\|u\|_{H^2(\Omega)}^2.$$
To establish coercivity, it is enough to show our form $a(\cdot, \cdot)$ is bounded below by a coercive quadratic form, in our case by a positive constant multiple of the quadratic form for the free plate when $\sigma = 0$. This form was proved to be coercive for all $\tau$ in [18, Proposition 2].

For $0 \leq \sigma < 1$, note that

$$a(u, u) \geq (1 - \sigma)\|D^2 u\|^2 + \tau\|Du\|^2$$

$$= (1 - \sigma)\left(\|D^2 u\|^2 + \frac{\tau}{1 - \sigma}\|Du\|^2\right).$$

The lower bound is $(1 - \sigma)$ times the quadratic form associated with the free plate with zero Poisson’s ratio and positive tension $\tau/(1 - \sigma)$. Since we assumed $\sigma < 1$, this establishes coercivity of $a(\cdot, \cdot)$ for $\sigma \in [0, 1)$.

When instead we have a negative Poisson’s ratio, in particular $0 > \sigma > -1/(d - 1)$, we use Fact 1 to obtain:

$$a(u, u) \geq (1 - \sigma)\|D^2 u\|^2 + d\sigma\|D^2 u\|^2 + \tau\|Du\|^2$$

$$\geq (1 + (d - 1)\sigma)\|D^2 u\|^2 + \tau\|Du\|^2.$$

Again, this is a constant multiple of the quadratic form associated with a free plate under tension and with zero Poisson’s ratio. Because we assumed $\sigma > -1/(d - 1)$, this constant is positive and hence the form $a(\cdot, \cdot)$ is coercive. □

**Proposition 4.2** For any $\tau \in \mathbb{R}$ and smoothly bounded $\Omega$, the weak eigenfunctions associated with our form $a(\cdot, \cdot)$ are real valued and smooth on $\Omega$.

**Proof** Let $u$ be a weak eigenfunction of $a(\cdot, \cdot)$ with associated eigenvalue $\omega$; by Proposition 4.1 we have $u \in H^2(\Omega)$. Then by a theorem in [21, p.668], we have $u \in H^k(\Omega)$ for every positive integer $k$. Thus we have $u \in H^k(\Omega)$ for all $k \in \mathbb{Z}^+$, and so $u \in C^\infty(\Omega)$.

Regularity on the boundary follows from global interior regularity and the Trace Theorem (see e.g. [22, Proposition 4.3, p.286 and Proposition 4.5, p.287]). Thus, we have $u \in C^\infty(\Omega)$, as desired.

Because our eigenvalues are all real valued, if $u$ is an eigenfunction with associated eigenvalue $\omega$, we may take the complex conjugate of the eigenvalue equation and see that $\bar{u}$ is also a eigenfunction with eigenvalue $\omega$. Then, the real and imaginary parts of $u$ are also eigenfunctions, and we may choose real-valued eigenfunctions for our eigenbasis. □

**Remark** It may be possible that the form $a(\cdot, \cdot)$ is coercive for values of $\sigma$ outside the given range if we impose restrictions on $\tau$, such as requiring $\tau > 0$. However, note that in the case $\tau = 0$ and $\sigma = 1$, the form is not coercive. In this case, all $H^2(\Omega)$ harmonic functions are eigenfunctions with eigenvalue zero, and so we have an eigenvalue of infinite multiplicity.

Furthermore, the lower bound on $\sigma$ arises from applying a sharp inequality bounding the Laplacian by the Hessian. This suggests coercivity might fail for $\sigma \leq -1/(d - 1)$. 


5. The fundamental tone as a function of $\tau$ and $\sigma$

The Rayleigh quotient depends on both $\tau$ and $\sigma$, so we can view the fundamental tone $\omega_1$ as a function in either parameter. Because $\omega_1$ is found by taking the infimum of Rayleigh quotients, and the quotients are linear in each of $\tau$ and $\sigma$, the fundamental tone is concave in each parameter. Additionally, by nonnegativity of $|Du|^2$, we have that the quotient and hence $\omega_1$ are increasing in $\tau$.

Fix $\sigma \in (-1/(d - 1), 1)$ and view $\omega_1$ as a function of $\tau$. Then, we can prove the same linear bounds on $\omega_1(\tau)$ that were established for the $\sigma = 0$ case in [18].

**Lemma 5.1** For all $\sigma \in (-1/(d - 1), 1)$ and $\tau > 0$, we have

$$\tau \mu_1 \leq \omega_1(\tau, \sigma) \leq \tau \frac{\Omega d}{\int_{\Omega} |x - \bar{x}|^2 \, dx}$$

where $\bar{x} = \int_{\Omega} x \, dx/|\Omega|$ is the center of mass of $\Omega$. In particular, when $\Omega$ is the unit ball, we have

$$\tau \mu_1 \leq \omega_1(\tau, \sigma) \leq \tau (d + 2).$$

Furthermore, the upper bounds in (8) and (9) hold for all $\tau \in \mathbb{R}$.

**Proof** To prove the upper bound, our argument is virtually the same as that of [18, Lemma 8]: we use as trial functions the linear functions $u_k = x_k - \bar{x}_k$.

To establish the lower bound, we note that for $\sigma \geq 0$, both $(1 - \sigma)|D^2 u|^2$ and $\sigma (\Delta u)^2$ are nonnegative and so

$$Q[u] \geq \frac{\int_{\Omega} \tau |Du| \, dx}{\int_{\Omega} u^2 \, dx}.$$  

If $\sigma < 0$, then we apply Fact 1, and since $\sigma > -1/(d - 1)$, we see

$$Q[u] \geq \frac{\int_{\Omega} (1 + \sigma (d - 1))|D^2 u|^2 + \tau |Du|^2 \, dx}{\int_{\Omega} u^2 \, dx} \geq \frac{\int_{\Omega} \tau |Du| \, dx}{\int_{\Omega} u^2 \, dx}.$$  

In both cases, we’ve bounded $Q[u]$ below by $\tau$ times the free membrane Rayleigh quotient. The lower bound on $\omega_1$ is then obtained by taking the infimum of both sides over all $u \in H^2(\Omega)$ orthogonal to a constant. \qed

**Lemma 5.2** For all $\tau \in \mathbb{R}$,

$$\omega_1 \leq C(\Omega) + \tau \mu_1,$$

where the value

$$C(\Omega) = \frac{\int_{\Omega} (1 - \sigma)|D^2 v|^2 + \sigma (\Delta v)^2 \, dx}{\int_{\Omega} v^2 \, dx}$$

is given explicitly in terms of the fundamental mode $v$ of the free membrane on $\Omega$.

The proof is essentially the same as in [18, Lemma 9] (the $\sigma = 0$ case) and so is omitted. These two lemmas give us the limiting behavior of $\omega_1$ as $\tau \to \infty$:

**Corollary 5.3** For all values of our Poisson’s ratio $-1/(d - 1) < \sigma < 1$, we have

$$\frac{\omega_1(\tau, \sigma)}{\tau} \to \mu_1 \quad \text{as} \quad \tau \to \infty.$$
This tells us that for sufficiently large $\tau$, we expect the free plate to behave much like the free membrane. This matches the physical interpretation of $\tau$ as the reciprocal of rigidity – large values of $\tau$ mean less rigidity.

6. Properties of Bessel functions

In this section, we will define the ultraspherical Bessel functions and summarize or prove properties that we will need to prove Theorem 3.1. Ultraspherical Bessel functions are the generalization of spherical Bessel functions to an arbitrary dimension $d$ and can be defined in terms of the usual Bessel functions $J_\nu(z)$ and $I_\nu(z)$. For more information on Bessel functions and their properties, see, e.g. [23].

We define the $d$-dimensional ultraspherical Bessel function of the first kind of order $l$, written $j_l(z)$, as follows:

$$ j_l(z) := z^{-(d-2)/2} J_{l+d/2-1}(z). $$

This function solves the $d$-dimensional ultraspherical Bessel equation

$$ z^2 w'' + (d-1)zw' + (z^2 - l(l + d - 2))w = 0. $$

Analogously, we define the $d$-dimensional ultraspherical modified Bessel function of the first kind of order $l$, written $i_l(z)$, as follows:

$$ i_l(z) := z^{-(d-2)/2} I_{l+d/2-1}(z). $$

This function solves the $d$-dimensional ultraspherical modified Bessel equation

$$ z^2 w'' + (d-1)zw' - (z^2 + l(l + d - 2))w = 0. $$

Each of the Bessel and modified Bessel equations are second-order and have a second, linearly independent solution; these are ultraspherical Bessel functions of the second kind $N_l(z)$ and ultraspherical modified Bessel functions of the second kind $K_l(z)$. Both of these are singular at $z = 0$ of different orders.

The ultraspherical Bessel functions inherit a family of recurrence relations from their two-dimensional analogues.

**Lemma 6.1** [24] We have the following properties of ultraspherical Bessel functions:

1. $\frac{d-2+2l}{z} j_l(z) = j_{l-1}(z) + j_{l+1}(z)$ and $\frac{d-2+2l}{z} i_l(z) = i_{l-1}(z) - i_{l+1}(z)$.
2. $j_l'(z) = \frac{z}{l} j_l(z) - j_{l+1}(z)$ and $i_l'(z) = \frac{l}{z} i_l(z) + i_{l+1}(z)$.
3. $j_l''(z) = \left(\frac{l^2-1}{z^2} - 1\right) j_l(z) + \frac{d+1}{z} j_{l+1}(z)$ and $i_l''(z) = \left(\frac{l^2-1}{z^2} + 1\right) i_l(z) + \frac{d+1}{z} i_{l+1}(z)$.

We will also need a bound on the roots of the $j_l'(z)$ functions:

**Proposition 6.2** (Lorch and Szego [25]) Let $p_{l,k}$ denote the $k$th positive zero of $j_l'(z)$. Then for $d \geq 3$ and $l \geq 1$,

$$ \frac{l(d+2l)(d+2l+2)}{d+4l+2} < p_{l,1}^2 < l(d+2l). $$
In particular, for \( p_{1,1} \) the first zero of \( j_1' \), we deduce \( d < (p_{1,1})^2 < d + 2 \) for all \( d \geq 2 \).

We will also find the following properties of signs of Bessel functions and their derivatives to be useful:

**Lemma 6.3** [24, Lemmas 5 through 9] We have the following:

1. For \( l = 1, \ldots, 5 \) and \( d \geq 2 \), we have \( j_l > 0 \) on \((0, p_{1,1})\).
2. We have \( j_1^d > 0 \) on \((0, p_{1,1})\).
3. We have \( j_2^d > 0, j_1'' < 0, \) and \( j_1^{(4)} > 0 \) on \((0, p_{1,1})\).

We may also write a power series for the ultraspherical Bessel functions \( j_l(z) \) and \( i_l(z) \) using the series for the corresponding \( J_{l+(d-2)/2} \) and \( I_{l+(d-2)/2} \) functions:

\[
\begin{align*}
    j_l(z) &= \sum_{k=0}^{\infty} (-1)^k c_{l,k} z^{2k+l} \\
    i_l(z) &= \sum_{k=0}^{\infty} c_{l,k} z^{2k+l}
\end{align*}
\]

where \( c_{l,k} := \frac{2^{1-d/2-2k-l}}{k! \Gamma(k + \frac{d}{2} + l)} \).

By examining the power series, it is immediate that \( i_l(z) \) and its derivatives are all positive on \((0, \infty)\). Since the terms of the power series for \( j_l \) and \( i_l \) are the same up to a sign, we also have that the derivatives of \( j_l \) are dominated by those of \( i_l \):

\[
|j_l^{(m)}(z)| \leq i_l^{(m)}(z) \quad \text{for } m \geq 0, \ z \geq 0, \quad (10)
\]

with equality only at \( z = 0 \).

These power series are particularly useful, and we use them to prove some crucial bounds on the Bessel functions with which we work:

**Lemma 6.4** (Bessel bounds) For all dimensions \( d \geq 2 \), we have the following bounds:

\[
\begin{align*}
    c_{1,0} z - c_{1,1} z^3 &\leq j_1(z) \leq c_{1,0} z \\
    c_{1,0} z &\leq c_{1,0} z + c_{1,1} z^3 \leq i_1(z) \\
    j_1''(z) &\leq -d_1 z + d_2 z^3 \\
    i_1'(z) &\leq d_1 z + K_d(M) d_2 z^3 \\
    j_2(z) &\geq n_0 z - n_1 z^3 \\
    i_2'(z) &\leq n_0 z + n_1 K_n(M) z^3
\end{align*}
\]

for all \( z \in [0, \sqrt{d+2}] \), \( z \geq 0 \), \( z \in [0, \sqrt{d+2}] \), \( z \in [0, \sqrt{d+2}] \), \( z \in [0, \sqrt{d+2}] \), \( z^2 \leq M \), \( z^2 \leq M \), \( z \leq M \).

Here \( c_{k,1} \) is the \( k \)th coefficient of \( i_l \) as before, while \( d_k \) is the \( k \)th coefficient of \( i_1'' \) and \( n_k \) is the \( k \)th coefficient of \( i_2' \), given by

\[
\begin{align*}
    d_k &= \frac{2^{1-2k-d/2}(2k+1)}{(k-1)! \Gamma(k+1+d/2)} \\
    n_k &= \frac{2^{2k-d/2}(k+1)}{k! \Gamma(k+2+d/2)}
\end{align*}
\]

The functions \( K_d \) and \( K_n \) are given by

\[
\begin{align*}
    K_d(M) &= \frac{7}{5} + \frac{8}{5M} \left( e^{M/4} - 1 \right) \\
    K_n(M) &= \frac{1}{2} + \frac{2}{M} \left( e^{M/4} - 1 \right)
\end{align*}
\]
We omit the proof as most bounds follow directly from properties of series and straightforward computation and all can be easily confirmed numerically for any dimension using a program such as Mathematica. The most complicated, the proof of the upper bounds on $i''_1(z)$ and $i'_2(z)$, is quite similar to the proof of a similar result in our earlier work in [24, Lemma 10].

7. The natural boundary conditions

In this section, our goal is to derive the form of the natural boundary conditions necessarily satisfied by all eigenfunctions. Consider the weak eigenvalue equation for eigenfunction $u$ with eigenvalue $\omega$ and some test function $\phi \in C^\infty_c(\Omega)$:

$$\int_\Omega \left( (1 - \sigma) \sum_{i,j} u_{x_i x_j} \phi_{x_i x_j} + \sigma \Delta u \Delta \phi + \tau D u \cdot D \phi - \omega u \phi \right) d x = 0.$$ (13)

Because the eigenfunction $u$ is smooth, we may use integration by parts to move most of the derivatives on $\phi$ to $u$; this gives us a volume integral and two surface integrals that must vanish for all $\phi$.

We first state the natural boundary conditions for a smoothly bounded region in arbitrary dimension:

**Proposition 7.1** For any smoothly bounded $\Omega$, the natural boundary conditions for eigenfunctions of the free plate under tension have the form

$$M u := (1 - \sigma) \frac{\partial^2 u}{\partial n^2} + \sigma \Delta u = 0 \quad \text{on } \partial \Omega,$$

$$V u := \tau \frac{\partial u}{\partial n} - (1 - \sigma) \text{div}_\partial \Omega \left( P_{\partial \Omega} \left( (D^2 u) n \right) \right) - \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial \Omega,$$

where $\partial / \partial n$ denotes the normal derivative and $\text{div}_\partial \Omega$ is the surface divergence. The projection $P_{\partial \Omega}$ projects a vector $v$ at a point $x$ on $\partial \Omega$ into the tangent space of $\partial \Omega$ at $x$.

When $\Omega$ is a ball, we can simplify the general boundary conditions.

**Proposition 7.2 (Ball)** In the case, $\Omega$ is the ball of radius $R$, the natural boundary conditions may be written as

$$M u := (1 - \sigma) u_{rr} + \sigma \Delta u = 0 \quad \text{at } r = R,$$ (11)

$$V u := \tau u_r - (1 - \sigma) \frac{1}{r^2} \Delta_S \left( u_r - \frac{u}{r} \right) - (\Delta u)_r = 0 \quad \text{at } r = R.$$ (12)

**Proof of Proposition 7.1** Our eigenfunctions $u$ are smooth on $\overline{\Omega}$ by regularity and satisfy the weak eigenvalue equation $a(u, \phi) - \omega(u, \phi)_{L^2(\Omega)} = 0$ for all $\phi \in H^2(\Omega)$. That is,

$$\int_\Omega \left( (1 - \sigma) \sum_{i,j=1}^d u_{x_i x_j} \phi_{x_i x_j} + \sigma \Delta u \Delta \phi + \tau D \phi \cdot D u - \omega u \phi \right) d x = 0.$$ (13)

Much of the work is already done for us in the proof of the boundary conditions for $\sigma = 0$, in [18].
Let \( n \) denote the outward unit normal to the surface \( \partial \Omega \). We can rewrite the Laplacian term by applying the Divergence theorem twice:

\[
\int_{\Omega} \Delta u \Delta \phi \, dx = \int_{\partial \Omega} (\Delta u) \frac{\partial \phi}{\partial n} \, dS - \int_{\Omega} D(\Delta u) \cdot D\phi \, dx
\]

\[
= \int_{\partial \Omega} (\Delta u) \frac{\partial \phi}{\partial n} - \phi \frac{\partial (\Delta u)}{\partial n} \, dS + \int_{\Omega} (\Delta^2 u) \phi \, dx.
\]

Combining this with the form of the Hessian term found in the \( \sigma = 0 \) case in [18, Proposition 6], we obtain

\[
\int_{\Omega} (1 - \sigma) \sum_{i,j} u_{i,x_j} \phi_{x_i x_j} + \sigma (\Delta u) (\Delta \phi) \, dx
\]

\[
= \int_{\Omega} (\Delta^2 u) \phi \, dx + \int_{\partial \Omega} \frac{\partial \phi}{\partial n} \left( (1 - \sigma) \frac{\partial^2 u}{\partial n^2} + \sigma \Delta u \right) \, dS
\]

\[
- \int_{\partial \Omega} \phi \left( \frac{\partial (\Delta u)}{\partial n} + (1 - \sigma) \text{div}_{\partial \Omega} \left( P_{\partial \Omega} \left[ (\Delta^2 u) n \right] \right) \right) \, dS.
\]

Thus, for \( u \) an eigenfunction associated with eigenvalue \( \omega \), we see (13) can be written as

\[
0 = \int_{\Omega} \phi \left( \Delta^2 u - \tau \Delta u - \omega u \right) \, dx + \int_{\partial \Omega} \frac{\partial \phi}{\partial n} \left( (1 - \sigma) \frac{\partial^2 u}{\partial n^2} + \sigma \Delta u \right) \, dS
\]

\[
+ \int_{\partial \Omega} \phi \left( \tau \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} - (1 - \sigma) \text{div}_{\partial \Omega} \left( P_{\partial \Omega} \left[ (\Delta^2 u) n \right] \right) \right) \, dS.
\]

As in the membrane case, this identity must hold for all \( \phi \in H^2(\Omega) \). If we take any compactly supported \( \phi \), then the volume integral must vanish; because \( \phi \) is arbitrary, we must therefore have \( \Delta^2 u - \tau \Delta u - \omega u = 0 \) everywhere. Similarly, the terms multiplied by \( \phi \) and \( \partial \phi / \partial n \) must vanish on the boundary. Collecting these results, we obtain the eigenvalue Equation (6) and natural boundary conditions of Proposition 7.1.

\[\square\]

Once we have the general form of the boundary conditions, we can find their expression in spherical coordinates when \( \Omega \) is a ball, from which Proposition 7.2 follows directly. The necessary computations were performed in the proof of [18, Proposition 7]; we do not repeat them here.

8. The eigenmodes of the ball

The ball is a rare case in which we can find exact solutions for the vibrating plate. As in the \( \sigma = 0 \) case treated in [24], we factor the eigenvalue equation:

\[
(\Delta + a^2)(\Delta - b^2)u = 0, \quad \text{where } b^2 = a^2 + \tau \quad \text{and } a^2 b^2 = \omega.
\]

After writing the factors \( (\Delta + a^2) \) and \( (\Delta - b^2) \) in spherical coordinates, we are able to write the eigenfunctions \( u \) as

\[
u(r, \hat{\theta}) = \left( j_l(ar) + \gamma_l i_l(br) \right) Y_l(\hat{\theta})
\]
where \( Y_l \) is an \( l \)-th order spherical harmonic and \( \gamma \) is a coefficient determined by the boundary conditions. There are two boundary conditions, so we could use either to express \( \gamma \); we will find it more convenient to write \( \gamma_l = -Mj_l(a)/Mi_l(b) \).

We are able to exclude Bessel functions of the second kind \( n_l \) and modified Bessel functions \( k_l \) of the second kind because they are singular at the origin but of different order for a fixed \( l \), and so no linear combination of \( n_l \) and \( k_l \) will be continuous at the origin. See, e.g. [24,26].

**Proposition 8.1 (Eigenfunctions in spherical coordinates)** Let \( \tau > 0 \) and \( \omega \) be any positive eigenvalue of the free ball \( \mathbb{B}(R) \); that is, \( \omega \) is an eigenvalue of \( \Delta \Delta u - \tau \Delta u = \omega u \) under boundary conditions (11) and (12). Then the corresponding eigenfunctions can be written in the form \( R_l(r)Y_l(\hat{\theta}) \), where \( Y_l \) is a spherical harmonic of some integer order \( l \) and \( R_l \) is a linear combination of Bessel and modified Bessel functions, \( R_l(r) = j_l(ar/R) + \gamma i_l(br/R) \).

Here the positive numbers \( a \) and \( b \) depend on \( \tau \) and \( \omega \) by \( b^2 - a^2 = R^2\tau \) and \( a^2b^2 = R^4\omega \), and \( \gamma \) is a real constant given by \( \gamma := -Mj_l(a)/Mi_l(b) \).

The proof of this proposition is almost identical to the proof of [24, Proposition 8] and so is not repeated here. It proceeds roughly as follows: We argue that \( \Delta^2 - \tau \Delta \) and \( \Delta \gamma \) are simultaneously diagonalizable to justify factoring the eigenvalue equation and writing solutions as a product of radial and angular parts, with the radial part being a linear combination of Bessel and Modified Bessel functions of the first and second kind. The regularity of eigenfunctions is used to conclude the coefficients of the singular second-kind Bessel functions must be zero, and we use the boundary condition \( Mu = 0 \) when \( r = 1 \) to find \( \gamma \).

Since the eigenfunctions must also satisfy the boundary condition \( Vu = 0 \) when \( r = 1 \), we can determine the values of \( a \) from the nontrivial roots of

\[
W_l(a) := Mj_l(a)Vl_i(b) - Ml_i(ab)Vj_l(a) = \left( -(1 - \sigma)a^2j_l'(a) - \sigma a^2j_l(a) \right) + \left( -(1 - \sigma)b^2i_l'(b) + \sigma b^2i_l(a) \right).
\]

For notational simplicity we will sometimes write \( K_l := l(l + d - 2) \).

Based on the \( \sigma = 0 \) case and other supporting evidence, we make the following

**Conjecture** For \( \tau > 0 \) and \( \sigma \in [0, 1) \), the fundamental modes of the ball \( \mathbb{B}(R) \) can be written as linear combinations of

\[
u_1(r, \hat{\theta}) = \left( j_1(ar/R) + \gamma i_1(br/R) \right) Y_1(\hat{\theta}),
\]

with \( a, b, \gamma \) real constants, with \( a \) and \( b \) positive and depending on \( \tau, \sigma \) and \( \omega_1 \) as follows: \( b^2 - a^2 = R^2\tau \) and \( a^2b^2 = R^4\omega_1(\tau, \sigma) \), and \( \gamma \) given by \( \gamma = -Mj_1(a)/Mi_1(b) \).

In this section, we will prove a weaker result and an ancillary lemma, and conclude with a more thorough discussion of the evidence supporting the conjecture. This conjecture was proved for \( \sigma = 0 \) in [24, Theorem 3], treating cases of small and larger index \( l \) separately.
First, we showed that for \( l \geq 1 \), the quotient \( Q[RY_i] \) was an increasing function of \( l \) for any radial function \( R \); this approach fails for \( \sigma > 0 \) because of the inclusion of the term \((\Delta R)^2\), which cannot be rewritten to be monotone in \( l \) for all admissible \( R \). This may be due to the lack of coercivity of the quotient \( \int_{\Omega} (\Delta u)^2 \, dx / \int_{\Omega} u^2 \, dx \).

The second part of the proof, showing that the first eigenvalue for modes with index \( l = 1 \) is lower than the first nonzero eigenvalue for radially symmetric modes (index \( l = 0 \)), can be adapted to nonzero values of \( \sigma \). As a result, we have the following

**Proposition 8.2** For \( \tau > 0 \) and \( \sigma \in [0, 1) \), the fundamental modes of the ball \( B(R) \) can be written as linear combinations of

\[
 u_i(r, \hat{\theta}) = \left( j_i(ar/R) + \gamma i_i(br/R) \right) Y_i(\hat{\theta}),
\]

where the index \( l \geq 1 \) and with \( a, b, \gamma \) real constants, with \( a \) and \( b \) positive and depending on \( \tau, \sigma \) and \( \omega_1 \) as follows: \( b^2 - a^2 = R^2 \tau \) and \( a^2 b^2 = R^4 \omega_1(\tau, \sigma) \), and \( \gamma \) given by

\[
 \gamma = \frac{-M j_1(a)}{M i_1(b)} = \frac{-(1 - \sigma)a^2 j_1''(a) + \sigma a^2 j_1(a)}{(1 - \sigma)b^2 i_1'' + \sigma b^2 i_1(b)}, \tag{14}
\]

**Proof** Let \( p_{1,1} \) denote the first positive zero of \( j_1(a) \).

Recall from Proposition 8.1 and the following text that if \( u_i = (j_i(ar) + \gamma i_i(br))Y_i(\hat{\theta}) \) is an eigenfunction, then \( \omega = a^2(a^2 + \tau) \) where \( a \) satisfies \( W_i(a) = 0 \). The parameter \( \tau \) is positive, so \( \omega \) increases with \( a \). Therefore, to show that the lowest nonzero eigenvalue corresponds to \( l = 1 \) and not \( l = 0 \), we show that the first nonzero root of \( W_i(a) \) is less than the first nonzero root of \( W_0(a) \).

First we consider \( l = 1 \). We will show that \( W_i(a) \) changes sign on the interval \((0, p_{1,1})\). Note first that by Lemma 8.3, the function \( W_i(a) \) is negative as \( a \to 0^+ \).

We next show that \( W_1(p_{1,1}) > 0 \); then by continuity we will have shown \( W_i(a) \) contains a root in the interval \((0, p_{1,1})\). From its definition and Lemma 6.4, we can write:

\[
 W_i(a) = \left((1 - \sigma)(d - 1) - a^2\right) j_i(a) - (1 - \sigma)(d - 1) a j'_i(a) \right) \cdot V i_1(b) - M i_1(b) \cdot \left(ab^2 j'_i(a) + (1 - \sigma)(d - 1)(aj'_i(a) - j_i(a)) \right).
\]

We have \( j'_i(p_{1,1}) = 0 \) by definition of \( p_{1,1} \), so \( W_1(p_{1,1}) \) simplifies considerably. Factoring out \( j_1(p_{1,1}) \) and expanding out \( V i_1(b) \) in terms of \( i_1 \) and \( i'_1 \), we find that

\[
 \frac{1}{j_1(p_{1,1})} W_1(p_{1,1}) = (1 - \sigma)(d - 1) M i_1(b) + \left(p^2_{1,1} - (1 - \sigma)(d - 1) \right)^2 b i'_1(b) + \left(p^2_{1,1} - (1 - \sigma)(d - 1) \right) (1 - \sigma)(d - 1)i_1(b).
\]

Note that \( 0 < (1 - \sigma)(d - 1) < d \), so

\[
 p^2_{1,1} - (1 - \sigma)(d - 1) > p^2_{1,1} - d > 0
\]

since \( p^2_{1,1} > d \) by Proposition 6.2. Thus every summand in \( W_1(p_{1,1})/j_1(p_{1,1}) \) is positive. Since \( j_1(p_{1,1}) > 0 \), this means that \( W_1(p_{1,1}) > 0 \), as desired.
Now we consider \( l = 0 \). Here \( K_l = l(l + d - 2) \) vanishes, so we look for \( a \) solving \( 0 = W_0(a) \). We may simplify \( W_0 \) using hyperspherical Bessel recurrence relations, and so we are interested in solutions to

\[
0 = W_0(a) = \left((1 - \sigma)a^2 j'_1(a) + \sigma a^2 j_0(a)\right)a^2 b i_1(b) + \left((1 - \sigma)b^2 i'_1(b) + \sigma b^2 i_0(b)\right)ab^2 j_1(a),
\]

The functions \( i_0(b) \) and \( i'_1(b) \) are positive for \( b > 0 \) by their power series. Similarly, \( j_1(a) \) and \( j'_1(a) \) are positive on \((0, p_{1,1})\) by Lemma 6.3, and so for \( \sigma \geq 0 \), we have shown \( W_0(a) > 0 \) on \((0, p_{1,1})\).

**Lemma 8.3** For any dimension \( d \geq 2 \), the function \( W_l(a) \) is convex as a function of \( \sigma \) for any \( l \geq 2 \) and linear as a function of \( \sigma \) for \( l = 0, 1 \).

For any dimension \( d \geq 2 \), index \( l \geq 1, \sigma \in [-1/(d-1), 1] \), and positive \( \tau \), the function \( W_l(a) \) is negative as \( a \to 0^+ \).

**Proof** First, we shall establish convexity or linearity of \( W_l(a) \) as a function of \( \sigma \). Treating \( a, \tau, \) and \( \sigma \) all as independent variables, we differentiate twice with respect to \( \sigma \) and simplify using Bessel recurrence relations and Lemma 6.4, obtaining:

\[
\frac{\partial^2}{\partial \sigma^2} W_l(a) = 2K_l \left(a^2 j''_1(a) + a^2 j'_1(a)\right) \left(b i_1(b) - i_1(b)\right) + 2K_l \left(-b^2 i''_1(b) + b^2 i_1(b)\right) \left(a j_1(a) - j_1(a)\right) = K_l(K_l - (d - 1)) \left(j_1(a) b i_{l+1}(b) + i_1(b) a j_{l+1}(a)\right).
\]

Since \( K_l = l(l + d - 2) \), we have that \( K_0 = 0 \) and \( K_1 = (d - 1) \), so this quantity vanishes for those two indices. Otherwise, \( K_l > (d - 1) \) and so by our knowledge of signs of Bessel functions from Lemma 6.3, the above is positive for indices \( l \geq 2 \). Hence \( W_l(a) \) is linear in \( \sigma \) for \( l = 0, 1 \) and convex in \( \sigma \) for \( l \geq 2 \), as desired.

Next, we establish the negativity of \( W_l(a) \) as \( a \to 0^+ \). By the series expansions of Bessel functions, we have that as \( a \to 0^+ \),

\[
j_l(a) = c_{l,0} a^l - c_{l,1} a^{l+2} + O(a^{l+4})
\]

\[
j_l(a) - a j'_1(a) = -(l - 1)c_{l,0} a^l + (l + 1)c_{l,1} a^{l+2} + O(a^{l+4})
\]

\[
a^2 j''_1(a) = (l - 1)c_{l,0} a^l - (l + 2)(l + 1)c_{l,1} a^{l+2} + O(a^{l+4}),
\]

and since \( b = \sqrt{a^2 + \tau} \), we have that as \( a \to 0^+ \),

\[
i_l(b) = i_l(\sqrt{\tau}) + O(a^2),
\]

\[
b i'_l(b) = \sqrt{\tau} i'_l(\sqrt{\tau}) + O(a^2),
\]

\[
b^2 i''_l(b) = \tau i''_l(\sqrt{\tau}) + O(a^2).
\]

Thus for small \( a \) values,
The limiting case of zero tension. By Lemma 5.1, we have for positive tension that
\[ M_j^0(a) = (1 - \sigma)l(l-1)c_{l,0}a^l + O(a^{l+2}), \]
\[ V_j^0(a) = (l\tau + (1 - \sigma)K_l(l-1))c_{l,0}a^l + O(a^{l+2}), \]
\[ Mi_j^0(b) = Mi_j^0(\sqrt{\tau}) + O(a^2), \]
\[ Vi_j^0(b) = (1 - \sigma)K_l\left(\sqrt{\tau}i'_l(\sqrt{\tau}) - i_l(\sqrt{\tau})\right) + O(a^2), \]
and so as \( a \to 0^+ \)
\[ W_l(a) = (1 - \sigma)l(l-1)(1 - \sigma)K_l\left(\sqrt{\tau}i'_l(\sqrt{\tau}) - i_l(\sqrt{\tau})\right)c_{l,0}a^l, \]
\[ - Mi_j^0(\sqrt{\tau})(l\tau + (1 - \sigma)K_l(l-1))c_{l,0}a^l + O(a^{l+2}). \] (15)
Since \( W_l(a) \) is either linear or convex in \( \sigma \), it will be maximized at one of the two extreme values of \( \sigma \). Hence it suffices to show \( W_l(a) < 0 \) as \( a \to 0^+ \) for both \( \sigma = 1 \) and \( \sigma = -1/(d-1) \).

If \( \sigma = 1 \), then \( Mi_j^0(z) = -z^2i_l(z) \) and so (15) simplifies to:
\[ W_l(a) = -\tau^2i_l(\sqrt{\tau})l^2c_{l,0}a^l + O(a^{l+2}). \]
Since \( a^l \) and \( c_{l,0} \) are both nonnegative, by positivity of the Bessel \( i_l \) functions, \( W_l(a) < 0 \) for sufficiently small \( a \).

If \( \sigma = -1/(d-1) \), then \( 1 - \sigma = d/(d-1) \) and we can write \((d-1)Mi_j^0(z) = dz^2i''_l(z) - z^2i_l(z)\), and (15) simplifies to:
\[ (d-1)^2W_l(a) = d^2l(l - 1)K_l\left(\sqrt{\tau}i'_l(\sqrt{\tau}) - i_l(\sqrt{\tau})\right)c_{l,0}a^l \]
\[ - \left(d\tau i''_l(\sqrt{\tau}) - \tau i'_l(\sqrt{\tau})\right)(l\tau(d - 1) + dK_l(l-1))c_{l,0}a^l + O(a^{l+2}). \]
Note \( c_{l,0} \) is positive and so does not affect the sign. The coefficient of the \( c_{l,0}a^l \) terms above can be rewritten using Bessel identities, yielding:
\[ - \frac{l(l - 1)(d - 1)^2\tau^2 + (d - 1)d(l - 1)^2(l + d - 1)\tau}{l + d - 1}i_l(\sqrt{\tau}) \]
\[ - \frac{dl(d - 1)^2\tau^2 + d^2(l - 1)K_l\tau}{l + d - 1}i_{l+1}(\sqrt{\tau}), \]
which is negative for all \( l, d, \) and \( \tau \) under consideration. Thus for sufficiently small values of \( a \), we have \( W_l(a) < 0 \) when \( \sigma = -1/(d-1) \), as desired. \( \square \)

8.1. Evidence for the conjecture

We now discuss the body of evidence for our conjecture. When \( \sigma = 0 \), it has already been proved that the fundamental mode of the ball corresponds to index \( l = 1 \) and has simple angular dependence \[24, Theorem 3\] and so the conjecture is true in this case.

8.1.1. Limiting cases

The limiting case of zero tension. By Lemma 5.1, we have for positive tension that \( \tau \mu_1 \leq \omega_1(\tau, \sigma) \leq \tau(d + 2) \); taking the limit as \( \tau \to 0^+ \) gives us that \( \omega_1(\tau, \sigma) \to 0 \) in this limit.
When \( \tau = 0 \), the differential eigenvalue equation becomes \( \Delta^2 u = \omega u \) and constant and linear functions are all eigenfunctions, and so the eigenvalue \( \omega = 0 \) has \( d + 1 \) multiplicity. By taking the spanning set to be \( u = \text{const} \) and \( u = x_k, k = 1, \ldots, d \), we see the nonconstant eigenfunctions have simple angular dependence, which would be the limiting case of eigenfunctions of the form \( R(r)Y_1(\hat{\theta}) \).

The limiting case of infinite tension (zero rigidity). We saw in Corollary 5.3 that \( \omega_1/\tau \) approaches the fundamental tone of the free membrane as \( \tau \to \infty \); the fundamental modes of the free membrane are of the form \( u(r, \hat{\theta}) = J_1(\rho_1 r/R)Y_1(\hat{\theta}) \).

From Proposition 8.2, the radial part of the fundamental mode can be written as \( J_1(\rho_1 r) + \gamma J_1(\rho_2 r) \) with \( \gamma = -M_1/M_2 \); then in the limit as \( \tau \to \infty \), \( \gamma \to 0 \) and so we obtain eigenfunctions of the form \( J_1(\rho_1 r)Y_1(\hat{\theta}) \). Taking \( l = 1 \) would give us agreement with the free membrane.

8.1.2. Numerical evidence

One consequence of Lemma 8.3 is that it allows us to reduce the number of parameters we need to consider when numerically verifying that \( l = 1 \) gives us the fundamental mode. The convexity in \( \sigma \) means that for any fixed \( a \), the value of \( W_l(a) \) is maximal at either \( \sigma = -1/(d-1) \) or \( \sigma = 1 \). Because \( W_l(0) = 0 \) and is decreasing and negative for small \( a \), we may conclude that for a fixed \( l \), the smallest first root \( a^*_l \) of \( W_l \) occurs at either \( \sigma = -1/(d-1) \) or \( \sigma = 1 \). On the other hand, because \( W_l(a) \) is linear in \( \sigma \), the largest value of the first root \( a^*_l \) occurs at either \( \sigma = -1/(d-1) \) or \( \sigma = 1 \).

Thus if we can show that the for any fixed dimension \( d \), any \( \tau > 0 \), and any index \( l \geq 2 \), the first roots of \( W_l(a, \sigma = 1) \) and \( W_l(a, \sigma = -1/(d-1)) \) are smaller than those of \( W_l(a, \sigma = 1) \) and \( W_l(a, \sigma = -1/(d-1)) \), then we will have proved that the lowest positive eigenvalue does correspond to the index \( l = 1 \).

Because of the sheer complexity of the \( W_l(a) \) functions, it does not seem to be possible to prove this directly. However, it is easy to verify this numerically for any choice of \( d, l \), and \( \tau \). Numerical investigations suggest that for any dimension \( d \geq 2 \), any tension \( \tau > 0 \) and any \( l \geq 2 \), the function \( W_l(a, \sigma = -1/(d-1)) < 0 \) for \( a \in (0, p_{1,1}) \) and that it suffices to consider the \( \sigma = 1 \) case for \( l \geq 2 \). This is demonstrated in Figure 1. The graphs were generated using LogLogPlot in Mathematica; we provide a source file on the ArXiv.

Figure 2 shows the roots \( a^*_l \) as functions of \( \tau \) for dimension \( d = 2 \) and indices \( l \). These images were produced in Mathematica using ContourPlot and LogLogPlot; we provide a source file on the ArXiv, along with images in the \( d = 100 \) case (which are visually quite similar and hence not included here). The thick lines correspond to the roots \( a \) of \( W_l(a, \sigma = 1) \), while the thinner lines are those of higher indices. Log-log plots are used for small values of \( \tau \) so that the separation between the curves is more apparent.

We end with a final useful inequality relating \( \tau \) and \( a \):

**Lemma 8.4** For all dimensions \( d \geq 2 \) and all values \( \tau > 0 \) and \( \sigma \in (-1/(d-1), 1) \), if \( a \) is as in 8.2, we have

\[
\frac{a^4}{d + 2 - a^2} \leq \tau.
\]

**Proof** This follows directly from writing \( \omega_1 = a^2(a^2 + \tau) \) in the lower bound from Lemma 5.1 and solving the inequality for \( \tau \). \( \square \)
Figure 1. Log-log plots of $-W_l(a)$ for indices $l = 2, 3, 4, 5$ and $\tau = 1$ for dimensions $d = 2$ (left) and $d = 100$ (right) for $a \in [10^{-20}, \sqrt{d} + 2]$ and $\tau = 10^{-10}$. Solid curves are $l = 2$; dashed are $l = 3$, dotted are $l = 4$, and dash-dotted are $l = 5$. From these images, we see that $-W_l(a)$ remains positive even for very small $\tau$ and $a$ values.

Figure 2. Graph of the first nontrivial root $a$ of $W_l(a)$ as functions of $\tau$ for extremal $\sigma$ values for dimension $d = 2$. In each image, the solid curve corresponds to $l = 1$ with $\sigma = 0$; the dashed curves are $l = 2, 3, 4$ with $\sigma = 1$.

9. Trial functions

Because we are using a trial function argument to prove Theorem 3.1, we will need to define these functions and establish some properties that will be useful for proving our main theorem.

Proceeding from our assumption that $l = 1$ corresponds to the fundamental mode for the unit ball, we write $R(r) = j_1(ar) + \gamma i_1(br)$ for the radial portion of any fundamental mode eigenfunction. The constants $a, b, \text{ and } \gamma$ are as in Proposition 8.2 and depend on $\tau$ and $\sigma$.

While the value of $a$ is also determined by the boundary conditions and hence dependent on both $\tau$ and $\sigma$, it will occasionally be useful to consider $\gamma$ as a function of independent
variables $\sigma$ and $a$. In these cases, we will define $\gamma$ according to (14) rather than its role as a linear combination constant.

**Lemma 9.1** (Trial functions) Let the radial function $\rho$ be given by the function $R$, extended linearly. That is,

$$
\rho(r) = \begin{cases} 
R(r) & \text{when } 0 \leq r \leq 1, \\
R(1) + (r - 1)R'(1) & \text{when } r \geq 1.
\end{cases}
$$

After translating $\Omega$ suitably, the functions $u_k = x_k \rho(r)/r$, for $k = 1, \ldots, d$, are valid trial functions for the fundamental tone.

**Proof** By construction, we have $x_k \rho(r)$ is continuous; then the functions $u_k$ are in $H^2(\Omega)$ provided there is no singularity introduced at the origin when we divide by $r$. By series expansions of $j_1(ar)$ and $i_1(br)$, we have $\rho(r) c_{1,0}(a+\gamma b)r$ as $r \to 0^+$, and so $R(r)/r \to 0$ as $r \to 0^+$.

The $u_k$ must also be orthogonal to a constant function in order to be admissible trial functions. To achieve this, we use “center of mass” coordinates so that $\int_\Omega u_k \, dx = 0$ for $k = 1, \ldots, d$. This argument relies on the Brouwer Fixed Point Theorem and is identical to that in the proof of [18, Lemma 13].

In the remainder of this section, we will establish several facts about the behavior of $\rho(r)$ along with bounds on the constant $\gamma$. For convenience, we write $\Delta_r \rho := \rho'' - \frac{d-1}{r^2}(\rho - r \rho')$. Note that $\Delta(\rho Y_1) = \Delta_r \rho Y_1$, so $\Delta_r \rho$ can be thought of as the radial part of the Laplacian.

**Lemma 9.2** Fix dimension $d \geq 2$ and constants $a \in (0, p_{1,1})$ and $\tau > 0$. Then $\gamma$ as defined in (14) is an increasing function of $\sigma$.

Furthermore, for $\sigma \in [-1/(d - 1), 1]$, the constant $\gamma$ satisfies the bounds

$$0 \leq \frac{\sigma^2 j_1''(a)}{b^2 i_1''(b)} \leq \gamma \leq \frac{\sigma^3}{b^3} < 1,$$

and when $\sigma \geq 0$ we also have

$$-\frac{\sigma^2 j_1''(a)}{b^2 i_1''(b)} \leq \gamma.$$

**Proof** We compute the derivative directly and simplify:

$$\frac{\partial \gamma}{\partial \sigma} = \frac{a^2 j_1(a)b^2 i_1''(b) + a^2 j_1''(a)b^2 i_1(b)}{(1 - \sigma)b^2 i_1''(b) + \sigma b^2 i_1(b))^2}.$$  

The denominator is always nonnegative, so the sign of $\partial \gamma/\partial \sigma$ is determined by the numerator. Using Lemma 6.1 to rewrite $j_1''$ and $i_1''$, and then using Bessel recurrence relations to rewrite $j_2$ and $i_2$ in terms of $j_1$, $j_3$, $i_1$, and $i_3$, we obtain

$$a^2 j_1(a)b^2 i_1''(b) + a^2 j_1''(a)b^2 i_1(b) = \frac{d - 1}{d + 2} a^2 b^2 \left(j_3(a)i_1(b) + j_1(a)i_3(b)\right),$$

which is nonnegative for $a \in [0, p_{1,1}]$ by our knowledge of signs of Bessel functions from Lemma 6.3. Thus $\gamma$ is increasing in $\sigma$ for fixed $a$ and $\tau$, as desired.
Now we note that for \( \sigma \in (-1/(d-1), 1) \), we have
\[
\gamma \leq \gamma \bigg|_{\sigma=1} = \frac{a^2 j_1(a)}{b^2 i_1(b)} \leq \frac{a^3}{b^3}
\]
with the last inequality by Lemma 6.4. Similarly, we obtain
\[
\gamma \geq \gamma \bigg|_{\sigma=-1/(d-1)} = \frac{-da^2 j''_1(a) - a^2 j_1(a)}{db^2 i'_1(b) - b^2 i_1(b)} = \frac{a^2 j''_2(a)}{b^2 i'_2(b)},
\]
again using Bessel identities from Lemma 6.1 to simplify. This quotient is positive for \( a \in [0, p_{1.1}] \), and hence \( \gamma \geq 0 \) as desired.

Finally, when \( \sigma \geq 0 \), we observe
\[
\gamma \geq \gamma \bigg|_{\sigma=0} = \frac{-a^2 j''_1(a)}{b^2 i''_1(b)}.
\]

Now that we have bounds on \( \gamma \), we may use these to establish some useful properties of our trial function \( \rho \) and its derivatives.

**Lemma 9.3** For all \( \tau > 0 \) and \( \sigma \in (-1/(d-1), 1) \), there exists a unique \( r^* \in (0, 1) \) such that \( \rho'' < 0 \) on \( (0, r^*) \) and \( \rho'' \geq 0 \) on \( [r^*, 1] \).

Additionally, we have that \( \rho - r \rho' \geq 0 \), \( \tau \rho'' - (\Delta \rho) = 0 \) and \( \Delta \rho \leq 0 \) for all \( r \in [0, 1] \).

**Proof** First note that the function \( \rho'' \) is convex in \( r \) on \( [0, 1] \), since
\[
\frac{d^2}{dr^2} \rho''(r) = a^4 j''_1(4)(ar) + \gamma b^4 i''_1(4)(br),
\]
which is positive by Lemmas 9.2 and 6.3.

By differentiating the series expansions for \( j_1 \) and \( i_1 \) from Lemma 6.4, we see that as \( r \to 0 \), we have
\[
\rho''(r) \sim (b^3 \gamma - a^3)c_{1.1} r,
\]
which is negative by Lemma 9.2. So by convexity, \( \rho'' \) is either negative on all of \( (0, 1) \) or has a single, simple root in that interval. Let \( r^* \) denote the root if it exists; otherwise set \( r^* = 1 \). Then \( \rho'' \leq 0 \) on \( [0, r^*) \) and \( \rho'' \geq 0 \) on \( (r^*, 1] \) provided the interval is nonempty.

When \( \sigma = 0 \), the boundary condition \( (4) \) simplifies to \( \rho''(1) = 0 \), and so \( r^* = 1 \).

When \( \sigma < 0 \), we have by Lemma 9.2 that \( \gamma \leq -a^2 j''_1(a)/b^2 i''_1(b) \) and so
\[
\rho''(1) = a^2 j''_1(a) + \gamma b^2 i''_1(b) \leq 0.
\]
We then take \( r^* = 1 \) once again.

When \( \sigma > 0 \), we have \( \gamma \geq -a^2 j''_1(a)/b^2 i''_1(b) \), and so \( \rho''(1) \geq 0 \) in this case. We then have some root \( r^* \in (0, 1] \).

Now we consider \( (\rho - r \rho')(r) \). Note that
\[
(\rho - r \rho')(0) = 0 \quad \text{and} \quad \frac{d}{dr}(\rho - r \rho') = -r \rho''.
\]
Thus the function \((\rho - r \rho')\) is increasing and hence positive on \([0, r^*]\). On \((r^*, 1]\), \((\rho - r \rho')\) is decreasing, and so on this interval
\[
(\rho - r \rho') \geq (\rho - r \rho') \bigg|_{r=1} = aj_2(a) - \gamma b i_2(b)
\]
\[
\geq \frac{1}{b^2i_1(b)} \left( aj_2(a)b^2i_1(b) - a^2 j_1(a)bi_2(b) \right)
\]
by Lemma 9.2. Then by Bessel recurrence relations,
\[
= \frac{a^2 j_3(a)b^2i_1(b) + a^2 j_1(a)b^2i_3(b)}{(d + 2)b^2i_1(b)},
\]
which is positive by Lemma 6.3. This \(\rho - r \rho' \geq 0\) on \([0,1]\) as desired.

Next, we consider the function \(\Delta_r \rho\). Like \(\rho''\), this function is also convex on \([0, 1]\), since
\[
\frac{d^2}{dr^2} \Delta_r \rho(r) = -a^4 j_1''(ar) + \gamma b^2 i_1''(br)
\]
and is positive by Lemmas 6.3 and 9.2. Note that \(\Delta_r \rho(0) = -a^2 j_1(0) + \gamma b^2 i_1(0) = 0\), and recall that we can write
\[
\Delta_r \rho = \rho'' - \frac{d - 1}{r^2}(\rho - r \rho').
\]
We then must have \(\Delta_r \rho \leq \rho''\) on \([0,1]\), and hence \(\Delta_r \rho \leq -\) on \([0, r^*]\).

By the boundary condition (4), we have \((1 - \sigma)\rho'' + \sigma (\Delta_r \rho) = 0\) at \(r = 1\), so either \(\rho''(1) = \Delta_r \rho(1) = 0\) or \(\rho''(1)\) and \(\Delta_r \rho(1)\) have opposite signs. If both are zero, then we have \(r^* = 1\), and \(\Delta_r \rho \leq \rho'' \leq 0\) on \([0,1]\) as desired.

Suppose \(\rho''(1) \neq 0\). Then because \(\Delta_r \rho \leq \rho''\) and \(\Delta_r \rho(1)\) and \(\rho''(1)\) have opposite signs, we must have \(\rho''(1) > 0\), and so \(\Delta_r \rho(1) < 0\). By convexity of \(\Delta_r \rho(r)\), we then conclude \(\Delta_r \rho(r) \leq 0\) on \([0,1]\).

Finally, returning to considering all \(\sigma \in [-1/(d - 1), 1]\), we investigate the sign of \(\tau \rho' - (\Delta_r \rho)_r\). Differentiating, we see
\[
\frac{d}{dr} \left( \tau \rho' - (\Delta_r \rho)_r \right) = \tau \rho'' - (\Delta_r \rho)_r = a^2 b^2 (j_1''(a) - \gamma i_1''(b)),
\]
which is negative by sign properties of \(j_1''\), \(i_1''\), and \(\gamma\). So the function \(\tau \rho' - (\Delta_r \rho)_r\) will be minimal when \(r = 1\). However, at \(r = 1\), we may apply the boundary condition \(Vu = 0\):
\[
Vu = \tau \rho' - (\Delta_r \rho)_r - \frac{(1 - \sigma)(d - 1)}{r^3}(\rho - r \rho') = 0
\]
and so
\[
\tau \rho' - (\Delta_r \rho)_r = \frac{(1 - \sigma)(d - 1)}{r^3}(\rho - r \rho').
\]
Since \(\rho - r \rho' \geq 0\) on \([0,1]\), and \(1 - \sigma \geq 0\) for all \(\sigma\) under consideration we have that \(\tau \rho' - (\Delta_r \rho)_r \geq 0\) at \(r = 1\) and hence on all of \([0,1]\).

\[\square\]

10. **Proof of the isoperimetric inequality**

In this section, we establish the lemmas needed to prove the free plate isoperimetric inequality for nonzero \(\sigma\). Some of the work from the proof of the inequality for the \(\sigma = 0\) case, found in [18], can be applied to our more general case of \(\sigma \in [0,1]\).
Using our trial functions from Section 9, the proof proceeds as follows:

- Evaluating the Rayleigh Quotient for these trial functions for regions \( \Omega \) with volume equal to that of the unit ball.
- Establishing partial monotonicity of the integrand in the numerator and denominator.
- Proving the theorem using scaling and rearrangement arguments.

We first bound our fundamental tone above by a quotient of integrals whose integrands are radial functions. The numerator will be quite complicated, so we write

\[
N[\rho] := (1 - \sigma) \left( \rho''^2 + \frac{3(d - 1)}{r^4} (\rho - r \rho')^2 \right) + \sigma \left( \rho'' - (d - 1) \frac{\rho - r \rho'}{r^2} \right)^2 + \tau (\rho')^2 + \frac{\tau (d - 1)}{r^2} \rho^2
\]

We will also need the following calculus facts:

**Fact 2** [26, Appendix]  We have the sums

\[
\sum_{k=1}^{d} |u_k|^2 = \rho^2 \quad \text{and} \quad \sum_{k=1}^{d} |D^2 u_k|^2 = (\rho''^2 + \frac{3(d - 1)}{r^4} (\rho - r \rho')^2)
\]

\[
\sum_{k=1}^{d} |D u_k|^2 = \frac{d - 1}{r^2} \rho^2 + (\rho')^2 \quad \text{and} \quad \sum_{k=1}^{d} (\Delta u_k)^2 = (\Delta_r \rho)^2.
\]

We may now use the trial functions to bound our fundamental tone by a quotient of integrals.

**Lemma 10.1** (Using the trial functions)  For any \( \Omega \), translated as in Lemma 9.1, we have

\[
\omega \leq \frac{\int_{\Omega} N[\rho] \, dx}{\int_{\Omega} \rho^2 \, dx}
\]

with equality if \( \Omega = \Omega^* \).

**Proof**  For \( u_k \) defined as in Lemma 9.1, we have

\[
\omega \leq Q[u_k] = \frac{\int_{\Omega} (1 - \sigma)|D^2 u_k|^2 + \sigma (\Delta u_k)^2 \tau |D u_k|^2 \, dx}{\int_{\Omega} |u_k|^2 \, dx},
\]

from the Rayleigh–Ritz characterization. We have equality when \( \Omega = \Omega^* \) because the \( u_k \) are the eigenfunctions for the ball associated with the fundamental tone, by our choice of trial functions and hypothesis in Theorem 3.1. Multiplying both sides by \( \int_{\Omega} |u_k|^2 \, dx \) and summing over all \( k \), we obtain
\[
\omega \int_{\Omega} \sum_{k=1}^{d} |u_k|^2 \, dx \leq \int_{\Omega} (1 - \sigma) \sum_{k=1}^{d} |D^2 u_k|^2 + \sigma \sum_{k=1}^{d} (\Delta u_k)^2 + \tau \sum_{k=1}^{d} |Du_k|^2 \, dx
\]  
(17)

again with equality if \( \Omega = \Omega^* \).

By these sums in Fact 2, we see inequality (17) becomes

\[
\omega \int_{\Omega} \rho^2 \, dx \leq \sigma \int_{\Omega} \left( (d-1) \frac{\rho - r \rho'}{r^2} - \rho'' \right)^2 + \tau (\rho')^2 + \frac{(d-1)}{r^2} \rho^2 \, dx + (1 - \sigma) \int_{\Omega} \left( \rho''^2 + \frac{3(d-1)}{r^4} (\rho - r \rho')^2 + \tau (\rho')^2 + \frac{(d-1)}{r^2} \rho^2 \right) \, dx,
\]

once more with equality if \( \Omega \) is the ball \( \Omega^* \). Dividing both sides by \( \int_{\Omega} \rho^2 \, dx \), we obtain (16).

\[\square\]

We now wish to show the quotient (16) in Lemma 10.1 has a sort of monotonicity with respect to the region \( \Omega \), and so we examine the integrands of the numerator and denominator separately. The case of the denominator is much simpler; the partial monotonicity of the integrand of the numerator is much more difficult, and requires several lemmas.

We begin with the denominator.

**Lemma 10.2 (Monotonicity in the denominator)** The function \( \rho(r)^2 \) is increasing.

**Proof** Differentiating, we see

\[
\rho'(r) = \begin{cases} 
 j'_1(ar) + \gamma i'_1(br) & \text{when } 0 \leq r \leq 1, \\
 R'(1) & \text{when } r \geq 1.
\end{cases}
\]

Obviously \( i'_1(br) \geq 0 \). Because we have \( a < p_{1,1} \) from the proof of Proposition 8.2, the function \( j'_1(ar) \) is positive on \([0,1]\). Thus \( \rho'(r) \) is positive everywhere, and \( \rho \) (and therefore \( \rho^2 \)) is an increasing function. \[\square\]

We do not need to prove the integrand of the numerator is strictly decreasing; a weaker “partial monotonicity” condition is sufficient. We will say a function \( F \) is **partially monotonic for \( \Omega \)** if it satisfies

\[
F(x) > F(y) \quad \text{for all } x \in \Omega \text{ and } y \notin \Omega.
\]  
(18)

Our approach to proving partial monotonicity of the numerator will depend on the sign of \( \sigma \). We will also now assume that \( \Omega \) has volume equal to that of the unit ball, so that \( \Omega^* = \mathbb{B}(1) \); we will recover the general case by a scaling argument at the end of the proof.

**10.1. Positive \( \sigma \)**

When \( \sigma > 0 \), we will wish to group terms in \( N[\rho] \) in order to address them separately. So we write

\[
N[\rho] = (1 - \sigma)(\rho'')^2 + (1 - \sigma)h(r) + \sigma g(r),
\]
where we define $h$ and $g$ as
\[
\begin{align*}
    h(r) := & \frac{3(d-1)}{r^4} \left( \rho - r \rho' \right)^2 + \tau \left( \left( \rho' \right)^2 + \frac{(d-1)}{r^2} \left( \rho^2 - \rho' \right) \right), \\
    g(r) := & \left( \Delta_r \rho \right)^2 + \tau \left( \left( \rho' \right)^2 + \frac{(d-1)}{r^2} \rho^2 \right).
\end{align*}
\]

**Lemma 10.3** (Partial monotonicity in the numerator when $\sigma \geq 0$)

Suppose $\sigma \in [0, 1)$ and one of the following is true:

- we have $d = 2$ or $d = 3$ and $\tau > 0$, or
- we have $d \geq 4$ and $\tau \geq (d + 2)/2 > a^2$.

then the function $N[\rho]$ satisfies the partial monotonicity condition (18) for the unit ball.

**Proof** We consider each term of $N[\rho]$ separately.

Because $\rho$ is linear for $r > 1$, we have $\rho'' = 0$ for $r > 1$. So the function $(\rho'')^2$ is nonnegative on $[0, 1]$ and zero otherwise, and hence satisfies condition (18) for the unit ball. Since $\sigma < 1$, we conclude that $(1 - \sigma)(\rho'')^2$ satisfies (18).

We are considering only $\sigma \in [0, 1)$. So we will have partial monotonicity in the other terms if we can show that $h$ and $g$ are also decreasing functions of $r$. This is established in Lemmas 10.4 and 10.5, below.

**Remark** The requirement that $\tau \geq (d + 2)/2$ when dimension $d \geq 4$ comes from the inequality in Lemma 8.4. If we solve this inequality for $a^2$, we obtain
\[
2a^2 \leq -\tau + \sqrt{\tau^2 + 4(d + 2)\tau}.
\]

The inequality $\tau > a^2$ is certainly true when $\tau$ exceeds this upper bound on $a^2$; this occurs when $\tau \geq (d + 2)/2$.

**Lemma 10.4** For dimensions $d = 2, 3$, when $\tau > 0$ the function $g(r)$ is decreasing for $r \in (0, 1)$.

For all dimensions $d \geq 4$, when $\tau \geq a^2$, the function $g(r)$ is decreasing for $r \in (0, 1)$.

Numerical computations in Mathematica strongly suggest that $g(r)$ is decreasing in $r$ for any choice of dimension and all positive $\tau$. However, our method of proof in the case of $\tau < (d + 2)/2$ relies on an upper bound of the Bessel function $i''_1(z)$ on the interval $[0, d + 2]$, and this bound is increasingly poor for high dimensions and proves to be too large for dimensions $d \geq 4$. Hence we restrict ourselves to small dimensions.

**Proof** To show $g(r)$ is decreasing on $(0, 1)$, we compute $g'(r)$ directly and simplify using the relationship between $\Delta_r \rho$ and $\rho''$:
\[
\begin{align*}
    g'(r) = & \left[ 2(\Delta_r \rho)(\Delta_r \rho)_r + 2\tau \left( \rho' \rho'' - \frac{(d - 1)}{r^3} \rho (\rho - r \rho') \right) \right] \\
    = & 2(\Delta_r \rho) \left( (\Delta_r \rho)_r + \tau \rho' \right) - 2\tau \frac{(d - 1)}{r^3} \rho (\rho - r \rho')^2.
\end{align*}
\]
The second term in the final line is clearly negative when \( r > 0 \). We have from Lemma 9.3 that \( \Delta_r \rho \leq 0 \), so it remains only to prove that

\[
(\Delta_r \rho)_r + \tau \rho' \geq 0 \quad \text{on } [0, 1).
\]

(19)

Our general approach will be as follows: write \( \rho \) and its derivatives explicitly in terms of Bessel functions and the linear combination constant \( \gamma \) from (14). For large values of \( \tau \), the positivity will be immediate from properties of Bessel functions. For small values of \( \tau \), we solve for \( \gamma \), transforming (19) into proving an inequality of the form \( \gamma \geq \gamma^* \). We then use (14) and bounds on Bessel functions to obtain a rational function which is lower bound on \( \gamma \). After some algebra, this reduces the problem of proving \( \gamma \geq \gamma^* \) to proving nonnegativity of a particular polynomial expression.

On the interval \([0, 1)\), we can write \( \rho \) as a linear combination of Bessel functions, and so we have

\[
(\Delta_r \rho)_r + \tau \rho' = a(\tau - a^2) j'_1(ar) + \gamma b(\tau + b^2)i'_1(br).
\]

Since \( a < p_{1,1} \), this is clearly positive for all \( \tau \geq a^2 \).

Now suppose \( \tau < a^2 \). Then \( \tau - a^2 \) is negative, and by bounds 6.4 on Bessel functions, we have

\[
a(\tau - a^2) j'_1(ar) + \gamma b(\tau + b^2)i'_1(br) \geq a(\tau - a^2)c_{1,0} + \gamma b(\tau + b^2)c_{1,0}.
\]

So it suffices to show

\[
a(\tau - a^2) + \gamma b(\tau + b^2) \geq 0,
\]

or equivalently,

\[
\gamma \geq \frac{a(a^2 - \tau)}{b(b^2 + \tau)} =: \gamma^*.
\]

(20)

To prove this, we will obtain a rational lower bound on gamma, and show it remains greater than \( \gamma^* \) for all values of \( a \) and \( \tau \) under consideration, and the dimensions \( d = 2, 3 \).

Note that because \( \tau \geq a^4/(d + 2 - a^2) \) by Lemma 8.4, we have in this case that

\[
a^2 > \frac{a^4}{d + 2 - a^2} \quad \Rightarrow \quad a^2 < \frac{d + 2}{2},
\]

and since \( b^2 = \tau + a^2 \), we see

\[
b^2 \leq 2a^2 = d + 2 =: M.
\]

Recall that by Lemma 9.2 for nonnegative \( \sigma \) we have the lower bound \( \gamma \geq -a^2 j''_1(a)/b^2 i''_1(b) \). We may now use our bounds on \(-j''_1\) and \(i''_1\) from Lemma 6.4; then for any \( a^2 \leq \sqrt{d + 2} \) and \( b^2 \leq M \), we have:

\[
\gamma \geq \frac{a^3 d_1 - a^5 d_2}{b^3 d_1 + k d_2 b^5},
\]

where \( k = 7/5 + 8(e^{M/4} - 1)/5M \). We took \( M = d + 2 \), so we will treat \( k \) as a function of dimension \( d \). Note also that \( d_1/d_2 = 6(d + 4)/5 \) depends only on \( d \). Thus to satisfy (20), it suffices to show

\[
\frac{6(d + 4)a^2 - 5a^4}{6(d + 4)b^2 + 5kb^4} - \frac{a^2 - \tau}{b^2 + \tau} \geq 0.
\]
Since both denominators are positive, this is equivalent to proving

\[(6(d + 4)a^2 - 5a^4)(b^2 + \tau) - (a^2 - \tau)(6(d + 4)b^2 + 5kd_2b^4) \geq 0.\]

The left-hand side is a polynomial in \(\tau\) and \(a^2\) (recall that \(b^2 = a^2 + \tau\)). Writing \(x = a^2\), we define

\[p(x, \tau) := (6(d + 4)x - 5x^2)(x + 2\tau) - (x - \tau)(6(d + 4)(x + \tau) + 5k(x + \tau)^2)\]

\[= 5k\tau^3 + (6(d + 4) + 5kx)^2\tau + (12(d + 4) - 5x(k + 2))x\tau - 5(1 + k)x^3.\]

Thus, proving nonnegativity of \(p\) for \(0 < \tau < x < (d + 2)/2\) is sufficient to establish (20) in this case and complete our proof.

First note that in \(p\), the coefficients of \(\tau^3\) and \(x^2\) are positive for all \(d, x,\) and \(k\) under consideration. We wish to see when the coefficient of \(\tau\) is positive for our values of \(x\) under consideration. Since \(x \leq (d + 2)/2\), taking \(k\) as defined above, we have

\[12(d + 4) - 5x(k + 2) = 3.5d + 35 - 4e^{(d+2)/4}.\]

By direct numerical computation, we see that this last expression is positive for \(d = 2, \ldots, 9\) and \(x\) in \([0, (d + 2)/2]\), and so the coefficient of \(\tau\) in \(p(x, \tau)\) is positive in this case. Hence \(p\) is increasing in \(\tau\) for these values of \(x\) and these dimensions, although we only needed this for \(d = 2, 3\).

For \(d = 2\), the constant \(k < 2.09\), and so we have for \(x \in [0, 2]\) that

\[p(x, \tau) \geq p\left(x, \frac{x^3}{4 - x}\right) = \frac{x^3}{5(4 - x)^3} \left(816 - 88x + 280x^2 - 25x^3\right)\]

\[\geq \frac{x^3}{5(4 - x)^3} \left(230x^2 + 640\right) \geq 0\]

with this last by noting that since \(0 \leq x \leq 2\), we have \(-88x \geq -176\) and \(-25x^3 \geq -50x^2\).

For \(d = 3\), our constant \(k < 2.2\). Then for \(x \in [0, 5/2]\) we note \(-5x^3 \geq -25x^2/2\) and by a similar argument to the above, we may conclude \(p(x, \tau) \geq 0.\)

\[\square\]

Remark The function \(p(x, \tau)\) in the above proof will not be nonnegative for all \(x \in [0, (d + 2)/2]\) when our dimension \(d \geq 4\); our bounds for \(i_{1\tau}^r(z)\) on \([0, d + 2]\) are too large since \(k\) grows exponentially in dimension. Numerical investigations support the conjecture that we still have \(\gamma \geq \gamma^*\) for small \(\tau\) in higher dimensions, but we would need a lower bound on \(\gamma\) in order to prove this.

Lemma 10.5 For all dimensions \(d \geq 2\) and values \(0 < \sigma < 1\) and \(\tau > 0\), the function \(h(r)\) is decreasing on \([0, 1]\).

Proof Recall \(h(r) = \frac{3(d-1)}{r^4}(\rho - r\rho')^2 + \tau\left((\rho')^2 + \frac{(d-1)}{r^2}\rho^2\right)\). Then differentiating and simplifying, we obtain:

\[h'(r) = \frac{-2(d-1)}{r^3}(\rho - r\rho') \left(\frac{6}{r^2}(\rho - r\rho') + 3\rho'' + \tau r\right) + 2\tau r\rho'''.\] (21)
Writing \((d - 1)(\rho - r \rho')/r^2 = \rho'' - \Delta_r \rho\), we can also rewrite this as
\[
h'(r) = \frac{2}{r} \Delta_r \rho \left( \frac{6}{r^2} (\rho - r \rho') + 3 \rho'' + \tau \rho \right) \\
+ \frac{2}{r} \rho'' \left( \frac{6}{r^2} (\rho - r \rho') + 3 \rho'' + \tau (\rho - r \rho') \right).
\] (22)

We know from Lemma 9.3 that \(\rho'' < 0\) on \((0, r^*)\) and \(\rho'' > 0\) on \((r^*, 1]\). We consider each case separately.

When \(r \in (0, r^*)\), we write \(h'(r)\) as in (21). That \(h'(r) < 0\) in this case follows from the proof of the free plate isoperimetric inequality for \(\sigma = 0\) in [18, Lemmas 18 through 22]. These lemmas rely on properties of ultraspherical Bessel functions from [24] and the following properties of the function \(\rho\):

1. \(\rho(r) = j_1(ar) + \gamma i_1(br)\) with \(b = \sqrt{a^2 + \tau}, 0 < a < p_{1,1}\), and \(\gamma\) determined by the natural boundary conditions.
2. \((d + 2)\tau > \omega^* > \tau d\).
3. \(\rho'' \leq 0\) for all \(r\) under consideration.
4. \(\gamma \geq -a^2 j''_1(a)b^2 i''_1(b)\) (the proof for \(\sigma = 0\) assumes equality and establishes a lower bound).
5. \(\rho - r \rho' \geq 0\) for all \(r\) under consideration.

Because \(r^* \leq 1\), we meet condition (1) by our choice of trial functions. The bound on \(\omega^*, (2)\), is guaranteed by Lemma 5.1 and Proposition 6.2. The bounds (3), (4), and (5) hold on \([0, r^*]\) by Lemmas 9.2 and 9.3. Thus we have met all the hypotheses of the lemmas from [18], and so \(h'(r) \leq 0\) for \(r \in [0, r^*]\) as desired.

When \(r \in (r^*, 1]\), we write \(h'(r)\) as in (22). We know \(\rho - r \rho' \geq 0\) and \(\Delta_r \rho \leq 0\) here while \(\rho'' \geq 0\), so both terms in (22) are negative. Thus \(h'(r) \leq 0\) for all \(r \in (0, 1]\), completing the proof.

\[\square\]

10.2. **Negative \(\sigma\) (Auxetic case)**

Throughout this section, we will use the notation \(\alpha = |\sigma|\) to reduce risk of confusion over signs; note now that the range of values we consider is \(0 < \alpha < 1/(d - 1)\).

Again, we will treat large and small values of \(\tau\) separately. For this section, the “small” values of \(\tau\) will be any which satisfy the inequality

\[
0 < \tau \leq \frac{3(1 + \alpha)a^2}{(d + 2)(1 - \alpha)} =: \tau_{\text{max}}.
\]

In the case of \(d = 2, 3\) we will further subdivide the range of “small” \(\tau\) values into \(0 < \tau \leq \tau_{\text{mid}}\) and \(\tau_{\text{mid}} < \tau \leq \tau_{\text{max}}\), where

\[
\tau_{\text{mid}} := \frac{(3 - \alpha(d - 1) + \alpha a^2)d^2}{d + 2}.
\]

Let us collect some results on bounds of \(a\) and \(b\) when \(\tau\) is small.
When $d = 3$, we have $\tau_{\text{max}} \leq 9a^2/5$ and $x_{\text{max}} \leq 45/14$. If we additionally assume $0 < \tau \leq \tau_{\text{mid}}$, then we have $a^2 \leq x_{\text{mid}} \leq 15/8$ and $b^2 \leq b_{\text{mid}}^2 \leq 3$.

When $d = 2$, $0 < \tau \leq \tau_{\text{mid}}$, and $0 < \alpha \leq 51/97$, then we have $a^2 \leq x_{\text{mid}} \leq 1.852$ and $b^2 \leq b_{\text{mid}}^2 \leq 3.5$.

Proof First, we will use $\tau \leq \tau_{\text{max}}$ to restrict the values of $a$ we need to consider. Since we have that $\tau > a^4/(d + 2 - a^2)$ by Lemma 8.4, the values of $a$ must satisfy

$$\frac{3(1 + \alpha)a^2}{(d + 2)(1 - \alpha)} - \frac{a^4}{d + 2 - a^2} \geq 0.$$ 

Solving this inequality for $a^2$ gives us the desired bound $a^2 \leq x_{\text{max}}$.

Write $x_{\text{max}} = x_{\text{max}}(d, \alpha)$. By inspection, $x_{\text{max}}$ is increasing in $\alpha$, and so for any fixed $d$ is maximized when $\alpha = 1/(d - 1)$. Evaluating $x_{\text{max}}(d, \alpha)$ at this value and applying a standard calculus maximization argument, we obtain

$$x_{\text{max}}(d, \alpha) \leq \frac{3d(d + 2)}{(d - 1)(d - 4)} \leq \max\{x_{\text{max}}(2, 1), 3\} = \frac{45}{14}.$$ 

We can also use the bounds on $\tau$ and $a^2$ to find an upper bound on $b^2$:

$$b^2 = a^2 + \tau \leq x_{\text{max}} + \tau_{\text{max}} \leq \frac{3(1 + \alpha)}{1 - \alpha} = : b_{\text{max}}^2.$$ 

By inspection, this upper bound $b_{\text{max}}^2$ is increasing in $\alpha$. If we take $d \geq 4$ and $\alpha = 1/(d - 1)$, we obtain the upper bound $b_{\text{max}}^2 \leq 6$.

Now suppose $\tau \leq \tau_{\text{mid}}$. We again apply our lower bound on $\tau$ from Lemma 8.4 and solve for $a^2$, obtaining the bound $a^2 \leq x_{\text{mid}}$, where

$$x_{\text{mid}} = \begin{cases} 7\alpha - 8 + \sqrt{64 - \alpha(52 - 9\alpha)} & \text{when } d = 3 \\ 5\alpha - 7 + \sqrt{49 - \alpha(22 - 9\alpha)} & \text{when } d = 2. \end{cases}$$ 

Similarly, we set bound $b_{\text{mid}}^2 := \tau_{\text{mid}} + x_{\text{mid}}$, which gives us an upper bound on the values of $b^2$ for which $A$ is negative:

$$b_{\text{mid}}^2 := \begin{cases} \frac{3\alpha - 2 + \sqrt{64 - \alpha(52 - 9\alpha)}}{2} & \text{when } d = 3 \\ \frac{3\alpha - 2 + \sqrt{49 - \alpha(22 - 9\alpha)}}{2} & \text{when } d = 2. \end{cases}$$ 

When $d = 2$, standard calculus arguments show that $x_{\text{mid}}$ and $b_{\text{mid}}^2$ are increasing in $\alpha$ and hence maximized at $\alpha = 51/97$, taking values $x_{\text{mid}} \leq 1.852$ and $b_{\text{mid}}^2 \leq 3.5$. When $d = 3$, we similarly obtain $x_{\text{mid}} \leq 1.852$ and $b_{\text{mid}}^2 \leq 3.5$.
Proposition 10.7 Suppose one of the following holds:

(1) The dimension $d \geq 3$ with any $0 < \alpha < 1/(d - 1)$ and any $\tau > 0$
(2) The dimension $d = 2$ with $\tau > \frac{3(1+\alpha)a^2}{(d+2)(1-\alpha)}$ and $0 < \alpha < 1$.
(3) The dimension $d = 2$ with $\tau < \frac{3(1+\alpha)a^2}{(d+2)(1-\alpha)}$ and $0 < \alpha < 51/97$.

Then we have partial monotonicity on the interval $[0,1]$ of the Rayleigh quotient numerator

$$N[r] = (1 + \alpha)\left((\rho''r) + \frac{2d-1}{r^4}(\rho - r\rho')^2 - \alpha(D_1\rho)^2 + \tau(\rho')^2 + \frac{\tau(d - 1)}{r^2}\rho^2\right).$$

Remark The proof of Proposition 10.7 involves showing individual terms of $N[r]$ are either partially monotonic themselves, or in fact decreasing. The latter involves expressing the trial function $\rho$ and its derivatives in terms of Bessel functions. The technical details of this part of the proof are handled in auxiliary Lemmas 10.8 and 10.9, which appear after the proof of Proposition 10.7.

Proof Because we have defined our trial function so that $\rho'' = 0$ for $r > 1$, we already have partial monotonicity of $(\rho'')^2$. We then focus on the remaining terms:

$$\tilde{N}(r) := \frac{1}{2}N[r] - (1 + \alpha)(\rho'')^2 \quad = \frac{1}{2}\left((1 + \alpha)\frac{2d-1}{r^4}(\rho - r\rho')^2 - \alpha(D_1\rho)^2 + \tau(\rho')^2 + \frac{\tau(d - 1)}{r^2}\rho^2\right).$$

Differentiating and regrouping the above yields

$$\tilde{N}'(r) = -\frac{(d - 1)}{r^3}(\rho - r\rho')\left(\frac{6(1 + \alpha)}{r^2}(\rho - r\rho') + 3(1 + \alpha)\rho'' + \tau\rho\right) + \tau\rho\rho'' - \alpha(D_1\rho)(D_1\rho)\cdot$$

We will want to handle the term $\tau\rho\rho'' - \alpha(D_1\rho)(D_1\rho)$, above differently depending on our choice of $\tau$.

We will consider the “large” $\tau$ case first. For this, we rewrite the $\tau\rho\rho''$ term of (23) using $\rho'' = D_1\rho + \frac{d-1}{r}(\rho - r\rho')$. We then obtain:

$$\tilde{N}'(r) = -\frac{(d - 1)}{r^3}(\rho - r\rho')l(r) + k(r),$$

where the functions $k$ and $l$ are given by

$$l(r) = \frac{6(1 + \alpha)}{r^2}(\rho - r\rho') + 3(1 + \alpha)\rho'' + \tau\rho - \alpha\tau r\rho',$$

$$k(r) = (1 - \alpha)\tau\rho\rho'' + \alpha D_1\rho\left(\tau\rho' - (D_1\rho)_r\right).$$

We shall first show $k$ is nonpositive. Because $\rho'' \leq 0$ and $0 < \alpha \leq 1$, the first term in $k(r)$ is nonpositive. The second term will be nonpositive for any values of $r$ such that $(D_1\rho)_r \leq 0$, since $\rho' \geq 0$ and $D_1\rho \leq 0$ for all $r \in [0, 1]$. 
We will thus assume \((\Delta, \rho)_r > 0\) and show that \(k(r) \leq 0\) for these values of \(r\). In this case, the sign of the second term in \(k(r)\) depends on the sign of \(\tau \rho' - (\Delta, \rho)_r\). By Lemma 9.3 this is nonnegative, and so \(k(r) \leq 0\) as desired.

Returning to (23), we note that by positivity of \(\rho - r \rho'\) on \([0,1]\) and our above work with \(k\), we will have that \(\hat{N}(r) \leq 0\) if we can show \(l(r)\) is nonnegative.

As in the \(\sigma > 0\) case, we write \(\rho''\) in terms of \(\rho, \rho', \Delta, \rho\) and express these in terms of \(j_1, i_1, j_3, i_3\) using properties of Bessel functions. We obtain:

\[
l(r) = \left(1 - \alpha\right)\tau - \frac{3(1 + \alpha)a^2}{d + 2} j_1(\alpha r) + \gamma \left(1 - \alpha\right)\tau + \frac{3(1 + \alpha)b^2}{d + 2} i_1(b r)
+ \left(\frac{3(d + 1)(1 + \alpha)}{d + 2}\right) \left(a^2 j_3(\alpha r) + \gamma b^2 i_3(b r)\right) + \alpha \tau (\rho - r \rho').
\]

We now have four terms to consider. Because \(\rho - r \rho' \geq 0\) for all \(r \in [0,1]\), our parameter \(0 < \alpha \leq 1\) regardless of dimension, and properties of \(a\) and \(j_3, i_3\), both terms in the last line are nonnegative. By inspection, the coefficient of the \(i_1(b r)\) term will be positive.

Thus the sign of \(l(r)\) (and hence partial monotonicity of \(N[\rho]\)) hinges on the term involving \(j_1(\alpha r)\). When this term is positive, all terms in \(l(r)\) are nonnegative and so \(l(r) \geq 0\) on \([0,1]\). The \(j_1(\alpha r)\) term’s coefficient is positive when

\[
\tau \geq \frac{3(1 + \alpha)a^2}{(d + 2)(1 - \alpha)} = \tau_{\text{max}},
\]

which is precisely our “large” \(\tau\) condition. Thus we’ve shown \(l(r) \geq 0\) for these values of \(\tau\), and hence \(N[\rho]\) has the desired partial monotonicity.

If \(\tau < \tau_{\text{max}}\) and \(d \geq 4\), then by Lemma 10.8 we again have that \(l(r) \geq 0\).

This leaves the \(\tau < \tau_{\text{max}}\) case for dimensions \(d = 2, 3\). In this case, one can choose \(a, \tau,\) and \(r\) so that the function \(l(r)\) will be negative, and so we need to change how we group the terms in \(\hat{N}'(r)\) in order to achieve partial monotonicity. This time we’ll rewrite the \(-\alpha \Delta, \rho(\Delta, \rho)_r\) term in (23) by expressing it in terms of derivatives of \(\rho\), obtaining

\[
\hat{N}'(r) = -(d - 1) r^{-3} (\rho - r \rho') \tilde{l}(r) + (1 - \alpha) \tau \rho' \rho'' + \rho'' \left(\tau \rho' - (\Delta, \rho)_r\right)
\]

where the function \(\tilde{l}\) is given by

\[
\tilde{l}(r) := \frac{6(1 + \alpha)}{r^2} (\rho - r \rho') + 3(1 + \alpha) \rho'' + \tau \rho - \alpha r (\Delta, \rho)_r.
\]

As before, note that \((1 - \alpha) \tau \rho' \rho''\) and \(\rho'' \left(\tau \rho' - (\Delta, \rho)_r\right)\) will both be nonpositive. Thus to show \(\hat{N}(r) \leq 0\), we only need nonnegativity of \(\tilde{l}\). But if \(\tau < \tau_{\text{max}}\) and \(d = 3\) with \(0 < \alpha < 1/(d - 1)\) or \(d = 2\) with \(\alpha < 51/97\), then by Lemma 10.9 our function \(\tilde{l}\) is nonnegative for any \(r \in [0,1]\), completing our proof. \(\square\)

**Lemma 10.8** For dimensions \(d \geq 4\) and all \(0 < \alpha < 1/(d - 1)\), if \(\tau\) and \(a\) are such that

\[
\tau < \frac{3(1 + \alpha)a^2}{1 - \alpha},
\]

then...
then the function
\[
l(r) = \frac{6(1 + \alpha)}{r^2} (\rho - r \rho') + 3(1 + \alpha)\rho'' + \tau \rho - \alpha \tau r \rho'
\]
is nonnegative for all \( r \in [0, 1] \).

**Proof** We will rewrite \( l(r) \) in terms of \( j_1, j_3, i_1, \) and \( i_3 \) as we did in the proof of Proposition 10.7. Then for these small \( \tau \) values, the coefficient of \( j_1(\alpha r) \) in \( l(r) \) is negative while all others are positive, so by Lemma 6.4 and our work in the early part of the proof of Proposition 10.7, we have
\[
l(r) \geq \left( (1 - \alpha)\tau - \frac{3(1 + \alpha)a^2}{d + 2} \right) j_1(\alpha r) + \gamma \left( (1 - \alpha)\tau + \frac{3(1 + \alpha)b^2}{d + 2} \right) i_1(br)
\]
\[
\geq \left( (1 - \alpha)\tau - \frac{3(1 + \alpha)a^2}{d + 2} \right) c_0 ar + \gamma \left( (1 - \alpha)\tau + \frac{3(1 + \alpha)b^2}{d + 2} \right) c_0 br.
\]
Thus for small \( \tau \) values and dimensions \( d \geq 4 \), it suffices to show that
\[
\left( (1 - \alpha)\tau - \frac{3(1 + \alpha)a^2}{d + 2} \right) a + \gamma \left( (1 - \alpha)\tau + \frac{3(1 + \alpha)b^2}{d + 2} \right) b \geq 0. \tag{24}
\]
Our approach is now similar to the one we took in the case of positive \( \sigma \): we solve for \( \gamma \), bound below by a rational function, and reduce the problem to proving positivity of a particular polynomial.

By solving (24) for \( \gamma \), we see that the inequality holds if and only if
\[
\gamma \geq \frac{a}{b} \frac{3a^2(1 + \alpha) - (d + 2)(1 - \alpha)\tau}{3b^2(1 + \alpha) + (d + 2)(1 - \alpha)\tau} =: \gamma^*.
\]
Viewing \( \gamma^* \) as a function of \( \alpha \) with the variables \( a, \tau, \) and \( d \) seen as independent, then straightforward calculus shows that \( \gamma^* \) is increasing in \( \alpha \), with
\[
\gamma^* \leq \frac{a}{b} \frac{3a^2d - (d + 2)(d - 2)\tau}{3b^2d + (d + 2)(d - 2)\tau}.
\]
To obtain a rational lower bound on \( \gamma \), we recall from Lemma 9.2 that similarly we may view \( \gamma \) as increasing in \( \sigma \), and hence decreasing in \( \alpha = -\sigma \). We then use this value of \( \alpha \), Bessel function identities, and Lemma 6.4 to obtain the lower bound
\[
\gamma \geq \frac{a^2 j_2(a)}{b^2 i_2''(b)} \geq \frac{a^3((d + 4) - a^2)}{b^3((d + 4) + \frac{5}{3}b^2)}.
\]
Note that in our application of Lemma 6.4, we are restricting ourselves to \( d \geq 4 \), \( M = b_{\text{max}}^2 = 3d/(d - 2) \) (from Lemma 10.6), and hence \( K < 5/3 \). We now have that
\[
\gamma - \gamma^* \geq \frac{a}{b} \left( \frac{a^2((d + 4) - a^2)}{b^2((d + 4) + \frac{5}{3}b^2)} - \frac{3da^2 - (d^2 - 4)\tau}{3db^2 + (d^2 - 4)\tau} \right).
\]
We wish the right-hand side to be positive. Then following our approach in Lemma 10.4, we set \( x = a^2 \) and use algebra to obtain a polynomial \( p(x, \tau) \) whose nonnegativity guarantees \( \gamma - \gamma^* \geq 0 \). In this case, \( p \) is given by
\[ p(x, \tau) := \frac{5}{3}(d^2 - 4)\tau^3 + \frac{1}{3}(5(2d^2 - 3d - 8)x + 3(d^2 - 4)(d + 4))\tau^2 \]
\[ + \frac{1}{3}(2d^2 - 39d - 8)x + 6(d^2 - 4)(d + 4))x\tau - 8dx^3. \]

It now suffices to show that \( p(x, \tau) \) is positive for all dimensions \( d \geq 4 \) and all \( \tau \) and \( x \) such that
\[ 0 \leq x \leq \frac{3d(d + 2)}{(d + 4)(d - 1)} := x_{\text{max}}, \quad \frac{x^2}{d + 2 - x} < \tau < \tau_{\text{max}}. \]

As in the positive \( \sigma \) case, we investigate the signs of the coefficients in \( p \) individually. The nonnegativity of the coefficients of \( \tau^3 \) and \( \tau^2 \) is easy; the linear coefficient requires slightly more work, treating dimensions \( 4 \leq d \leq 10 \) and \( d \geq 20 \) as separate cases. The details are purely algebraic computation and hence omitted.

Thus we may conclude \( p(x, \tau) \) is increasing in \( \tau \), and so minimized when \( \tau \) is. Evaluating \( p(x, \tau) \) at the minimal value \( \tau = x^2/(d + 2 - x) \) yields a rational expression whose nonnegativity can be shown algebraically. A slightly different argument is needed for the cases of \( d = 4 \) and \( d \geq 5 \).

Thus we have show that for all values \( d \geq 4 \) and \( \tau, x = a^2 \) under consideration, the function \( p(x, \tau) \), and hence \( \gamma - \gamma'' \), is nonnegative.

Remark This method of proof cannot be extended to the physical case \( d = 2, 3 \); numerical investigations show that there exist values of \( \tau, a, \alpha, \) and \( r \) in those dimensions for which \( \hat{l}(r) < 0 \). We will need to group the terms in \( \hat{N}'(r) \) differently to obtain a proof for these small dimensions and small \( \tau \). This alternate grouping can also be used to prove the small \( \tau \) case for higher dimensions, however it is more cumbersome.

**Lemma 10.9 (Small \( \tau \) for \( d = 2, 3 \))** Suppose \( d = 3 \) and \( 0 < \alpha < 1/2 \) or \( d = 2 \) and \( \alpha \leq 51/97 \), and \( \tau < \tau_{\text{max}} \). Then we have that
\[ \hat{l}(r) = \frac{6(1 + \alpha)}{r^2}(\rho - r\rho') + 3(1 + \alpha)\rho'' + \tau\rho - \alpha r\Delta r\rho(\Delta r\rho)_r \geq 0. \]

Remark The upper bound on \( \alpha \leq 51/97 \) in the \( d = 2 \) case is specific to our method of proof and not a strict upper bound on \( \alpha \) values for which \( \hat{l}(r) \geq 0 \) holds. Numerical estimates suggest \( \hat{l}(r) \geq 0 \) in fact holds for all \( 0 < \alpha < 1 \).

**Proof** Using the notation of Lemma 10.6, since \( \tau \leq \tau_{\text{max}} \), we also have \( 0 \leq a^2 \leq x_{\text{max}} \) and \( 0 \leq b \leq b_{\text{max}} \).

As in our previous proofs, we use Bessel identities to write \( \rho - r\rho' \) and \( \rho' \) in terms of Bessel \( j_1, i_1, j_3, \) and \( i_3 \). Then \( \hat{l}(r) \) can be rewritten as
\[ \hat{l}(r) = Aj_1(ar) + \gamma Bi_1(br) + Cj_3(ar) + \gamma Di_3(br), \]
where the coefficients are
\[ A = \tau - \frac{3(1 + \alpha)}{d + 2}a^2 + \frac{\alpha a^2}{d + 2}(d + 2 - a^2r^2) \]
\[ B = \tau + \frac{3(1 + \alpha)}{d + 2}b^2 - \frac{\alpha b^2}{d + 2}(d + 2 + b^2r^2) \]
\[
C = \frac{a^2}{d+2} \left(3(1+\alpha)(d+1) - \alpha a^2 r^2 \right) \quad D = \frac{b^2}{d+2} \left(3(1+\alpha)(d+1) + \alpha b^2 r^2 \right).
\]

The coefficient \(D\) of \(i_3(br)\) is nonnegative by inspection. Since \(r \in [0, 1]\) and \(a^2 \in [0, d+2]\), the coefficient \(C\) of \(j_3(ar)\) can be bounded below by a nonnegative quotient through simple algebra.

It now remains to show \(A j_1(ar) + \gamma B i_1(br)\) is nonnegative. Because \(A\) will be negative for some values of \(\tau\) and positive for others, this will require two separate cases based on our range of \(\tau\) values.

First, we will show \(B\) is nonnegative for all values of \(\tau\) and \(b\) satisfying \(\tau \leq \tau_{\max}\) and \(b \leq b_{\max}\). We’ll first rewrite \(B\) as

\[
B = \frac{(d+2)\tau + b^2(3 - \alpha(d-1) - \alpha r^2 b^2)}{d + 2}.
\]

Note that \(B\) is decreasing in both \(r\) and \(\alpha\), and \(\alpha\) is constrained by our dimension. Thus \(B\) can be bounded below as follows:

\[
B \geq \frac{1}{10} \left(-\tau^2 + 2(7 - \alpha^2)\tau + 4a^2 - \alpha^2 \right) =: B_3(\tau) \quad \text{when } d = 3
\]

\[
B \geq \frac{1}{388} \left(-51\tau^2 + (628 - 102\alpha^2)\tau + \alpha^2(80 - 17\alpha^2) \right) =: B_2(\tau) \quad \text{when } d = 2.
\]

Both functions \(B_2, B_3\) are concave in \(\tau\), and so minimized at the extreme values of \(\tau\). For \(d = 3\), we have \(0 \leq \tau \leq \tau_{\max} = 9a^2/5\) and \(0 \leq a^2 \leq x_{\max} = 1.875\). When \(d = 2\) and \(\alpha \leq 51/97\), we have \(0 \leq \tau \leq \tau_{\max} < 2.41a^2\) and \(0 \leq a^2 \leq x_{\max} = 1.72\). Direct computation of \(B_2\) and \(B_3\) at these extreme values yields nonnegative quantities, as desired.

Finally, we consider the sign of \(A\). We Note it is minimized when \(r = 1\), and hence

\[
A \geq \frac{(d+2)\tau - a^2(3 - \alpha(d-1) + \alpha a^2)}{d + 2}.
\]

Note also that if \(\tau\) is large enough relative to \(a\), then \(A\) will be nonnegative. However, if \(\tau\) is too small, specifically if

\[
\tau < \tau_{\text{mid}} = \frac{(3 - \alpha(d-1) + \alpha a^2)a^2}{d + 2},
\]

then we have \(A < 0\) when \(r = 1\). We will then need to consider the \(j_1\) and \(i_1\) terms together in order to prove nonnegativity of \(\hat{l}(r)\).

We now show that \(\hat{l}(r)\) is positive even when \(\tau \leq \tau_{\text{mid}}\). Since we’ve already established the nonnegativity of \(C\) and \(D\), we only need to show \(A j_1(ar) + \gamma B i_1(br) \geq 0\). Since we assume \(\tau < \tau_{\text{mid}}\), we have \(A < 0 < B\) and will apply the bounds from Lemma 6.4

\[
-j_1(z) \geq -c_{1,0}z \quad i_1(z) \geq c_{1,0}z,
\]
where $c_{1,0}$ is a positive constant coming from the series expansion and depends only on the dimension. Then
\[
l(r) \geq c_0 r (Aa + \gamma Bb) \\
= c_0 r \left( a \left( (d+2)\tau - a^2 (3 - \alpha(d-1) + \alpha a^2) \right) + \gamma b \left( (d+2)\tau + b^2 (3 - \alpha(d-1) - ab^2) \right) \right).
\]

Solving for $\gamma$ as usual, we see the above is positive if and only if
\[
\gamma \geq \frac{a}{b} \left( \frac{a^2 (3 - \alpha(d-1) + \alpha a^2) - (d+2)\tau}{b^2 (3 - \alpha(d-1) - ab^2) + (d+2)\tau} \right) =: \gamma^*.
\]

As before, we will find a rational lower bound on $\gamma$ so that we can prove nonnegativity of a polynomial rather than a transcendental quantity involving Bessel functions. We can bound $\gamma$ below by setting $\alpha = 1/(d-1)$, and then apply Lemma 6.4. This yields the bound
\[
\gamma \geq \frac{a^2 j_2'(a)}{b^2 i_2'(b)} = \frac{a^3 ((d+4) - a^2)}{b^3 ((d+4) + k(b_{mid}^2)b^2)},
\]
where the constant $k(M)$ is given in Lemma 6.4 as
\[
k(M) = \frac{1}{2} + \frac{2}{M} \left( e^{M/4} - 1 \right).
\]

For $d = 2, 3$, recall that our bounds on $b_{mid}^2$ from Lemma 10.6 both give us $k(b_{mid}^2) < 1.3$. Then once again, we write $x = a^2$ and obtain a polynomial $p$ so that $p(x, \tau) \geq 0$ implies $\gamma - \gamma^* \geq 0$. In this case, $p$ is given by
\[
p(x, \tau) = \frac{13(d+2)}{10} \tau^3 - \frac{x^3}{10} \left( 3x\alpha + 69 + 103\alpha - 3d\alpha \right) \tau^2 + \frac{1}{10} \left( -3ax^2 + (13(2d+1) + \alpha(3d - 53))x + 10(d+2)(d+4) \right) \tau + \frac{x}{10} \left( -6ax^2 - 3(34 - d + 2\alpha(26 - d))x + 20(d+2)(d+4) \right).
\]

Once again, we can prove this quantity is nonnegative for $0 < \tau \leq \tau_{mid}$, $0 \leq x < x_{mid}$, $d = 2, 3$ and appropriate $\alpha$ using familiar arguments: show $p$ is minimized when $\tau = \tau_{min}$, and then show $p(x, \tau_{min}) \geq 0$. We use our bounds on $x_{mid}$ from Lemma 10.6. The details are straightforward calculus and algebra, and omitted for brevity.

Thus we have that $l(r) \geq 0$ for all $r \in [0, 1]$, as desired.

**Remark**  In the case of $d = 2$, our polynomial $p(x, \tau_{min})$ becomes
\[
p \left( x, \frac{x^2}{4 - x} \right) = \frac{4x^3}{5(4-x)^3} \left( -(4-x)(194 - 19x)\alpha - 5x^3 + 55x^2 - 138x + 408 \right).
\]

This function will *not* be nonnegative for all $\alpha \in [0, 1]$; this is where our restriction of $0 \leq \alpha \leq 51/97$ became necessary for completing the proof.

□
11. Completing the proof

Now that we have established the desired monotonicity of our quotient, we need two more lemmas before we can prove the isoperimetric inequality for the free plate under tension. Our first of these is a special case of more general rearrangement inequalities:

**Lemma 11.1** [18, Lemma 14] For any radial function function $F(r)$ that satisfies the partial monotonicity condition (18) for $\Omega^*$,

\[
\int_{\Omega} F \, dx \leq \int_{\Omega^*} F \, dx \quad \text{with equality if and only if } \Omega = \Omega^*.
\]

For any strictly increasing radial function $F(r)$,

\[
\int_{\Omega} F \, dx \geq \int_{\Omega^*} F \, dx \quad \text{with equality if and only if } \Omega = \Omega^*.
\]

The final lemma describes how the eigenvalues change with the dilation of the region, and is used in the proof of the theorem to show we need only consider $\Omega$ with volume equal to that of the unit ball. We will use the notation $s\Omega := \{x \in \mathbb{R}^d : x/s \in \Omega\}$ for $s > 0$.

**Lemma 11.2** (Scaling) For all $s > 0$, we have

\[
\omega(\tau, \sigma, \Omega) = s^4 \omega(s^{-2}\tau, \sigma, s\Omega).
\]

The proof is straightforward and nearly identical to that of [18, Lemma 15], and so not repeated here.

We can now prove our main result.

**Proof of Theorem 3.1** Once we have established inequality (3) for all regions $\Omega$ of volume equal to that of the unit ball and all $\tau > 0$, we obtain (3) for regions of arbitrary volume, since for all $s > 0$, by Lemma 11.2,

\[
\omega(\tau, \sigma, \Omega) = s^4 \omega(s^{-2}\tau, \sigma, s\Omega) \leq s^4 \omega(s^{-2}\tau, \sigma, s\Omega^*) = \omega(\tau, \Omega^*).
\]

Thus it suffices to prove the theorem for $\Omega$ with volume equal to that of the unit ball, so that $\Omega^*$ is the unit ball. We may also translate $\Omega$ as in Lemma 9.1, which leaves the fundamental tone unchanged. Then, applying Lemma 10.1 and then Lemmas 10.2, 10.3, and 11.1,

\[
\omega \leq \frac{\int_{\Omega} N[\rho] \, dx}{\int_{\Omega} \rho^2 \, dx} \leq \frac{\int_{\Omega^*} N[\rho] \, dx}{\int_{\Omega^*} \rho^2 \, dx} = \omega^*,
\]

with this last by applying the equality condition in Lemma 10.1. Finally, if equality holds, then $\Omega$ must be a ball, by the equality statement in Lemma 11.1. □
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