Surjectivity of the completion map for rings of $C^\infty$-functions, necessary conditions and sufficient conditions

Genrich Belitskii, Alberto F. Boix and Dmitry Kerner

1. Introduction

1.1. Let $R$ be a commutative ring, filtered by a decreasing sequence of ideals $R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$, set $I_\infty := \cap_{j \geq 0} I_j$, and consider the corresponding completion map $R \to \hat{R}^{(\ast)} := \lim_{\to} R/I_j$. Its kernel is $I_\infty$ and thus for many rings/filtrations this map is injective. On the other hand, for many traditional (non-complete) rings of Commutative Algebra, this map is far from being surjective.

Let $R$ be a ring of smooth functions, e.g. one of

\[ C^\infty(\mathbb{R}^m, Z), \quad C^\infty(\mathcal{U})/J, \quad C^\infty((\mathbb{R}^m, Z) \times \mathcal{U})/J. \]

(Here $\mathcal{U} \subset \mathbb{R}^m$ is an open subset. $(\mathbb{R}^m, Z)$ denotes the germ of $\mathbb{R}^m$ along a closed subset $Z \subset \mathbb{R}^m$.)

Take a filtration, $\{I_j\}$ and the completion, $R \to \hat{R}^{(\ast)}$. When is this map surjective?

Borel’s lemma ensures the surjectivity of the completion $C^\infty(\mathbb{R}^m, o) \to \hat{R}^{(\ast)}$, for the filtration $\{(\mathbb{R}^m)^{\ast}\}$. In this case, specifying an element of completion is the same as specifying all the derivatives at $o$. More generally, Whitney’s extension theorem gives the necessary and sufficient conditions to extend a function with prescribed derivatives on $Z \subset \mathcal{U}$ to a smooth function on $\mathcal{U}$. In the particular case, $Z$ is a manifold, and the filtration is $\{I_j = I(Z)^{\ast}\}$, specifying derivatives on $Z$ is equivalent to specifying an element of $\hat{R}^{(\ast)}$. In this case the surjectivity follows by Whitney theorem.

For more general subsets and filtrations the data of derivatives/elements of completion are essentially different.

1.2. In this short note we address the surjectivity of the completion map for the rings of (1). In (2) we reduce the considerations to the case $R = C^\infty(\mathcal{U})$, for an open $\mathcal{U} \subset \mathbb{R}^m$. In (3) we obtain a (non-trivial) necessary condition. Our main result is Theorem 4.1 for a rather general class of filtrations the completion map is surjective, and moreover, the preimage of $\hat{f} \in \hat{R}^{(\ast)}$ can be chosen real-analytic off the prescribed (closed) set.

This surjectivity is the necessary starting point for various questions, e.g.:

i. Artin approximation type for $C^\infty$-rings, see e.g. [Bel.Boi.Ker];
ii. The study of determinacy/algebraizability of non-isolated singularities of maps and schemes, [B.K.16b], [Boi.Gre.Ker], [Bel.Ker].

It will be interesting to extend these surjectivity results to various subclasses of smooth functions.

1.3. Notations. For any ideal $I \subset C^\infty(\mathcal{U})$ we take its (reduced) set of zeros, $Z = V(I) \subset \mathcal{U}$. For any subset $Z \subset \mathcal{U}$ denote by $I(Z) \subset C^\infty(\mathcal{U})$ the set of functions vanishing on $Z$. Thus $I(V(I)) \supseteq I$. Not much can be said about the converse inclusion, because of the flat functions.

We denote the derivatives by multi-indices, $g^{(\underline{k})}$. We abbreviate the condition

\[ \text{“any derivative } \partial^{i_1}_{x_1} \ldots \partial^{i_m}_{x_m} g, \text{ with } \sum_{i_j} |i_j| = |\underline{k}| \text{ satisfies } |\partial^{i_1}_{x_1} \ldots \partial^{i_m}_{x_m} g| \leq \ldots” \]

by writing: \[ |g^{(\underline{k})}| < \ldots. \]

For a closed subset $Z \subset \mathbb{R}^m$ we denote by $C^\infty(\mathbb{R}^m, Z)$ the ring of germs of smooth functions at $Z$. These are functions defined on (small) neighborhoods of $Z$, with equivalence relation: $f_1 \sim f_2$ if for some open neighborhood $Z \subset \mathcal{U}$ holds $f_1|\mathcal{U} = f_2|\mathcal{U}$. 

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2. Preparations

Let \( S \) be one of the rings \( C^\infty(\mathbb{R}^m, Z), C^\infty(\mathcal{U}), C^\infty([\mathbb{R}^m, Z] \times \mathcal{U}) \), here \( \mathcal{U} \subseteq \mathbb{R}^n \) is an open subset. For an ideal \( J \subseteq S \) take the quotient, \( R := S/J \). The ring \( R \) can be thought as the ring of functions on the subscheme \( \text{Spec}(R) \subseteq \text{Spec}(S) \).

Fix a filtration by ideals, \( R = I_0 \supsetneq I_1 \supsetneq \cdots \). Take the corresponding completion, \( R \to \hat{R}^{(t_\ast)} := \lim_{\leftarrow} R/I_j \). Its elements are Cauchy sequences of function-germs, \( \left\{ f_j \right\} \in R \), such that \( f_{j+i} - f_j \in I_j \), for all \( i, j > 0 \). Equivalently, the elements can be expressed as (formal) sums \( \sum_{j=0} g_j \), for \( g_j \in I_j \).

2.1. Surjectivity for \( S \) vs surjectivity for \( R \). Fix a filtration \( I_\ast \) of \( S \), we assume \( I_j \supsetneq I_0 \) for any \( j \). This induces the filtration \( \tilde{I}_j := I_j/I_0 \) of \( R \) and the diagram on the right. The maps \( \pi_\ast \), \( \pi_0 \) are surjective. Thus the surjectivity of \( \phi_\ast \) implies that of \( \phi_0 \). Vice versa, assume that \( \phi_0 \) is surjective. Fix an element \( \hat{g} \in \hat{S}^{(t_\ast)} \) and take any \( g \in \pi_\ast^{-1} \phi_\ast \pi_0(\hat{g}) \subseteq S \). Then \( \phi_\ast(g) - \hat{g} \in \phi_\ast(S)(J) = 0 \in \hat{S}^{(t_\ast)} \). Thus \( \phi_\ast \) is surjective.

Therefore it is enough to verify the surjectivity of \( S \to \hat{S} \), where \( S \) is one of the rings

\[
C^\infty(\mathbb{R}^m, Z), \quad C^\infty(\mathcal{U}), \quad C^\infty([\mathbb{R}^m, Z] \times \mathcal{U}).
\]

2.2. Reduction to the ring \( C^\infty(\mathcal{U}) \). For any \( f \in C^\infty(\mathbb{R}^m, Z) \) and any open neighborhood \( Z \subseteq \mathcal{U} \) one can choose a representative \( \hat{f} \in C^\infty(\mathcal{U}) \) of \( f \). For any ideal \( I \subseteq C^\infty(\mathbb{R}^m, Z) \) fix some generators, \( I = \left\{ q_\alpha \right\} \). These are germs of smooth functions, choose their representatives, \( \hat{q}_\alpha \in C^\infty(\mathcal{U}) \). These define the ideal \( \hat{I} \subseteq C^\infty(\mathcal{U}) \), a representative of \( I \). For any \( g \in \hat{I} \) there exists a representative \( \tilde{g} \in \hat{I} \). Indeed, expand \( g = \sum c_\alpha q_\alpha \), choose some representatives \( \hat{c}_\alpha \), and define \( \tilde{g} := \sum \hat{c}_\alpha q_\alpha \).

Take a filtration \( \{ I_j \} \subseteq C^\infty(\mathbb{R}^m, Z) \) and some element \( g \in C^\infty(\mathbb{R}^m, Z) \). These are germs of smooth functions, choose their representatives, \( \hat{q}_\alpha \in C^\infty(\mathcal{U}) \). These define the ideal \( \hat{I} \subseteq C^\infty(\mathcal{U}) \), a representative of \( I \). For any \( g \in \hat{I} \) there exists a representative \( \tilde{g} \in \hat{I} \). Indeed, expand \( g = \sum c_\alpha q_\alpha \), choose some representatives \( \hat{c}_\alpha \), and define \( \tilde{g} := \sum \hat{c}_\alpha q_\alpha \).

Thus we have the diagram with surjective horizontal maps. And the surjectivity of \( \phi_{(\mathbb{R}^m, Z)} \) is implied by that of \( \phi_{\mathcal{U}} \). Similar argument apply to the ring \( C^\infty(\mathbb{R}^m, Z) \times \mathcal{U} \). Therefore it suffices to establish the surjectivity \( C^\infty(\mathcal{U}) \to \hat{C}^\infty(\mathcal{U}) \).

3. The necessary condition for the surjectivity of \( C^\infty(\mathcal{U}) \to \hat{C}^\infty(\mathcal{U}) \)

Let \( R = C^\infty(\mathcal{U}) \) for some open \( \mathcal{U} \subseteq \mathbb{R}^m \). For a filtration \( R = I_0 \supsetneq I_1 \supsetneq \cdots \) define the loci of \( i \)’th multiplicity: \( V_i(I_j) = \left\{ x \in \mathcal{U} \mid I_j \subseteq m_x^i \right\} \). Thus

\[
V_i(I_j) \subseteq V_i(I_{j+1}) \subseteq \cdots \subseteq V_i(I_j) =: Z_i.
\]

Here the loci \( \{ Z_i \} \) satisfy \( \mathcal{U} \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \), they are not necessarily closed.

**Example 3.1.** i. In the simplest case take the filtration \( \{ I_j = I^j \} \). Then \( Z_1 = Z_2 = \cdots = V(I) \subseteq \mathcal{U} \).

ii. Let \( \{ L_1 \} \) be a collection of linear subspaces (of whichever dimensions) through the origin \( 0 \in \mathbb{R}^m \), finite or infinite. Take their defining ideals, \( a_i := I(L_i) \). Take the filtration \( I_j = (\cap_{i \geq 1} a_i) \cap (\cap_{i \geq 2} a_i^2) \cap \cdots \cap (\cap_{i \geq j} a_i^j) \). Then

\[
Z_i = \bigcup_{i \geq 1} L_i \supseteq Z_2 = \bigcup_{i \geq 2} L_i \supseteq \cdots \supseteq Z_j = \bigcup_{i \geq j} L_i \supseteq \cdots
\]

Take a compactly embedded subset, \( U_0 \subseteq \overline{U_0} \subseteq \mathcal{U} \), where \( \overline{U_0} \) is compact in \( \mathbb{R}^m \). Take the restrictions \( Z_i|_{U_0}, I_j|_{U_0} \subseteq C^\infty(\mathcal{U}) \).

**Lemma 3.2.** Suppose the completion is surjective, \( C^\infty(\mathcal{U}) = R \to \hat{R}^{(t_\ast)} \). For any compactly embedded open subset \( U_0 \subseteq \mathcal{U} \) the restricted filtration \( I_{\ast|_{U_0}} \) is equivalent to the filtration \( \{ I_j \cap I(Z_1) \cap I(Z_2)^2 \cap \cdots \cap I(Z_j)^i \}|_{U_0} \). In particular, the restrictions of chains in equation (4) stabilize, \( Z_1|_{U_0} = V_i(I_j)|_{U_0} \) for \( j > 1 \). Therefore all \( \{ Z_i \} \subseteq \mathcal{U} \) are closed.

**Proof.** To check the equivalence we should show: for any \( j \) exists \( n_j < \infty \) such that

\[
I_{n_j}|_{U_0} \subseteq I_j \cap I(Z_1) \cap I(Z_2)^2 \cap \cdots \cap I(Z_j)^i|_{U_0} \subseteq C^\infty(\mathcal{U}).
\]

For this it is enough to show: for any \( i \) the chain \( \{ V_i(I_j)|_{U_0} \} \) in equation (4) stabilizes. We prove this by induction on \( i \).

**Case** \( i = 1 \), for \( Z_1 \). Suppose the loci \( V_i(I_j)|_{U_0} \) do not stabilize. Replace \( \{ I_j \} \) by its (equivalent) subsequence that satisfies: \( V_i(I_j)|_{U_0} \subseteq V_i(I_{j+1})|_{U_0} \subseteq \cdots \). Fix a sequence of points, \( x_j \in V_i(I_{j+1}) \setminus V_i(I_j)|_{U_0} \), and small balls,
First we reduce the proof to the ring $C(10)$.

\[ g_j(x) = \begin{cases} 
0, & x \not\in \text{Ball}_{\epsilon_j}(x_j) \\
> 0, & x \in \text{Ball}_{\epsilon_j}(x_j) 
\end{cases} \quad j!, \ x = x_j \]

(These are constructed from elements of $I_j$ by using the standard bump functions.)

Suppose the element $\sum_{j=0}^\infty g_j \in \hat{R}(\star)$ is presented by some $f \in C^\infty(U)$, i.e. $f = \sum_{j=0}^N g_j \in I_{N+1}$ for any $N$. Then $f$ is bounded on the compact set $\mathcal{U}$, contradicting the construction: $f(x_N) = \sum_{j=0}^N g_j(x_N) \geq N$.

**The general case** is similar. Assuming the statement for $i$, i.e. for $Z_1, \ldots, Z_i$, we pass to the (equivalent) sub-filtration satisfying:

\[ Z_1|_{\mathcal{U}} = V_1(I_1)|_{\mathcal{U}}, \quad Z_2|_{\mathcal{U}} = V_2(I_1)|_{\mathcal{U}}, \quad \ldots \quad Z_i|_{\mathcal{U}} = V_i(I_i)|_{\mathcal{U}}. \]

Suppose the loci \(\{V_{i+1}(I_j)|_{\mathcal{U}}\}_j\) do not stabilize. We can assume $V_{i+1}(I_1)|_{\mathcal{U}} \subseteq V_{i+1}(I_2)|_{\mathcal{U}} \subseteq \ldots$ Fix a sequence of points and the small balls, \(\{x_j \in \text{Ball}_{\epsilon_j}(x_j)\}\), before. Fix a sequence of functions \(\{g_j \in I_j\}\) satisfying:

\[ \partial_j^i g_j(x) = \begin{cases} 
0, & x \not\in \text{Ball}_{\epsilon_j}(x_j) \\
> 0, & x \in \text{Ball}_{\epsilon_j}(x_j) 
\end{cases} \quad j!, \ x = x_j \]

(Here $\partial_j^i$ is the $i$'th partial derivative with respect to the first coordinate.)

Then we get a contradiction as before: the function $f$ must have bounded $i$'th derivatives on $\mathcal{U}$, but $\partial_j^i f|_{\mathcal{U}} = \sum_{j=0}^N \partial_j^i g_j(x_N) \geq N$.

**Remark 3.3.** The restriction to the compactly embedded subsets, \(\ldots|_{\mathcal{U}}\), is important. For example, let $U = (0, 1) \subset \mathbb{R}^1$ and $I_j = \{f|_U : f = 0 \text{ on } [\frac{1}{j}, 1 - \frac{1}{j}]\} \subset C^\infty(0, 1)$. Then $Z_1 = \cup_j V_j(I_j)$ does not stabilize, for any $i$. But the completion is surjective, $C^\infty(0, 1) \to C^\infty(0, 1)$ (\(\star\)). Indeed, for any sequence $\{g_j \in I_j\}$ we have $\sum g_j \in C^\infty(0, 1)$, as the sum is finite on small neighborhoods of each point of $(0, 1)$.

4. **The condition ensuring surjectivity of $R \to \hat{R}$**

As was shown in [2], the surjectivity question is reduced to the ring $C^\infty(U)$.

**Theorem 4.1.** Let $R = C^\infty(U)$, for some open $U \subseteq \mathbb{R}^m$. Suppose there exists an open cover $U = \cup U_a$ such that, when restricted to each $U_a$, the filtration $\{I_{j_a}\}$ is equivalent to the filtration $\{a_0 + \sum_k a_k \cdot b_{k,j}\}$, where (all the ideals depend on $U_a$)

- The ideals $a_0$, $\{a_k\}$ do not depend on $j$: the collection $\{a_k\}$ is finite and they are all finitely generated.
- The zero loci satisfy: $V(b_{k,j}) = V(b_{k,1})$ for any $k, j$.
- The ideals $\{b_{k,j}\}$ satisfy: $b_{k,j} \subseteq I(V(b_{k,1}))^n$, for a sequence $n_j \to \infty$.

1. The $\{I_j\}$-completion map is surjective, $R \to \hat{R}^\uparrow$.
2. Moreover, if a closed subset $Z \subset U$ satisfies $I_\infty \supseteq I(Z)^\infty$ then any element $\hat{f} \in \hat{R}^\uparrow$ admits a preimage which is real analytic off $Z$, i.e. $f \in C^\infty(U) \cap C^\infty(U \setminus Z)$.

**Proof.** Given $\sum g_j$, with $g_j \in I_j \subset C^\infty(U)$, we should construct a function $f \in C^\infty(U)$, satisfying:

\[ \forall N : f - \sum_{j=0}^N g_j \in I_N. \]

First we reduce the proof to the ring $C^\infty(\text{Ball}_1(0))$ and a very particular filtration. Then we estimate the growth of derivatives of $g_j$. Then we construct $f$ using the cutoff functions with controlled growth. Finally, in Step 5, we use Whitney approximation theorem to achieve a function real-analytic off $Z$.

**Step 1.** (Simplifying the filtration $\{I_j\}$) We reduce the question to the particular case of the filtration $I_{\star}$ of $C^\infty(\text{Ball}_1(0))$ satisfying:

\[ \{V(I_j) = V(I_1)\}_j \quad \text{ and } \quad \{I_j \subseteq I(V(I_1))^j\}_j \]

i. Take an open cover by small balls, $U = \cup U_a$, such that on each ball $\{I_j\}$ is equivalent to the corresponding $\{a_0 + \sum_k a_k \cdot b_{k,j}\}$. We can assume that this covering is locally finite (by shrinking the balls if needed). Take the corresponding partition of unity,

\[ \{u_\alpha \in C^\infty(U)\}_\alpha : \quad 0 < u_\alpha|_{\text{Ball}_a} \leq 1, \quad u_\alpha|_{U \setminus \text{Ball}_a} = 0, \quad \sum u_\alpha = 1_U. \]

Suppose we have proved the surjectivity on each ball. Thus for any $\sum g_j \in \hat{R}^\uparrow$ and each $\text{Ball}_a$ the element $\sum_j u_\alpha g_j$ is realized, i.e. we have $f_\alpha \in C^\infty(\text{Ball}_a)$ satisfying:

\[ \forall N : f_\alpha = \sum_{j=0}^N u_\alpha g_j \in I_{N+1} \cdot C^\infty(\text{Ball}_a). \]
Then \( f := \sum u_\alpha f_\alpha \) is the needed function. Indeed, \( f \in C^\infty(\mathcal{U}) \), as the sum is locally finite, and

\[
(14) \quad f - \sum_j g_j = \sum_{\alpha} u_\alpha f_\alpha - \sum_j 1_{ij} \cdot g_j = \sum_{\alpha} u_\alpha (f_\alpha - \sum_j u_\alpha g_j) \in I_\infty.
\]

Thus we restrict to \( C^\infty(Ball_1(0)) \) and replace \( \{ I_j \} \) by the equivalent filtration as in the assumptions. Thus \( I_j = a_0 + \sum_k a_k \cdot b_{k,j} \).

ii. An element of \( \hat{R} \) is \( c_0 + \sum_{j \geq 1} (g_j^{(0)} + g_j^{(>0)}) \), where \( g_j^{(0)} \in a_0 \) and \( g_j^{(>0)} \in \sum_k a_k \cdot b_{k,j} \). We should construct \( f \in R \) that satisfies: \( f - c_0 = -\sum_{j=1}^N g_j^{(0)} - g_j^{(>0)} \in I_{N+1} \) for any \( N \). As \( d^0 \subseteq I_j \), for each \( j \), one can omit \( g_j^{(0)} \). This reduces the statement to the filtration \( \{ \sum_k a_k \cdot b_{k,j} \} \).

iii. Suppose \( \{ I_j = \sum_k a_k \cdot b_{k,j} \} \). For each \( a_k \) fix a (finite) set of generators, \( \{ a^{(k)} \}_i \). Then an element \( g_j \in I_j \) is written as \( \sum_{i,k} a^{(k)} \cdot b_{i,k} \), with \( b_{i,k} \in b_{k,j} \). Therefore \( g_j \in \hat{R} \) is presentable as \( \sum_{i,k} a^{(k)} (\sum_{j=0}^\infty b_{i,k,j}) \). (Here the sum over \( i, k \) is finite.) It is enough to find some \( C^\infty \)-representatives \( \{ b_{i,k} \} \) of \( \{ g_j \} \), i.e. \( b_{i,k} - \sum_j b_{i,k,j} \in \cap_j b_{k,j} \). Indeed, for such representatives we get:

\[
(15) \quad \sum_{i,k} a^{(k)} \cdot b_{i,k} - \sum_{k,j} a^{(k)} (\sum_{j=0}^\infty b_{i,k,j}) \in \cap_j a_k \cdot b_{k,j} = I_\infty.
\]

Thus it is enough to consider just the filtration \( \{ b_j \} \), with \( b_j \subseteq I(V(b_1))^{n_j} \), for a sequence \( n_j \to \infty \). Furthermore, we pass to an equivalent filtration satisfying \( b_j \subseteq I(V(b_1)\{ j \} \).

Therefore it is enough to establish the surjectivity \( R \to \hat{R}^{(\ast)} \) for the filtration of the particular type as in equation (11).

**Step 2.** We have \( \{ g_j \in I_j \} \) for the specific filtration of the ring \( C^\infty(Ball_1(0)) \) as in (11). By slightly shrinking the ball we can assume \( g_j \in C^\infty(Ball_1(0)) \), in particular each derivative of each \( g_j \) is bounded.

We claim: for any \( j \) and any \( k \) with \( |k| < j \), and any \( x \in Ball_1(0) \) holds

\[
(16) \quad |g_j^{(k)}(x)| < C_{g_j} \cdot dist(x, Z)^{j-|k|}.
\]

(Here \( \{ C_{g_j} \} \) are some constants that depend on \( g_j \) only.)

Indeed, fix some \( x \in Ball_1(0) \setminus Z \) and some \( z \in Z \) for which \( dist(x, z) - dist(x, Z) \leq dist(x, Z) \). By the assumption \( g_j \in m_1 \), thus \( g_j^{(k)} |z| = 0 \) for \( |k| < j \). Therefore the Taylor expansion with remainder (in \( Ball_{dist(x, z)}(z) \)) gives:

\[
(17) \quad g_j(x) = \sum_{|k| \geq j} \frac{|k|!}{k!} \left( \int_0^{1-t} |z|^{k-1} g_j^{(k)}(z + t(x-z)) dt \right)(x-z)^k.
\]

(Here \( x, z \) are the coordinates of \( x, z, k! = k_1! \cdots k_m! \), and \( g^{(k)}(\ldots) \) is a multi-linear form.)

Note that \( |z - z'|^k \leq (dist(x, Z) + \epsilon)^{j} \) and the derivatives \( g_j^{(k)} \) are bounded on \( Ball_1(0) \). Thus \( |g_j(x)| \leq C_{g_j} \cdot dist(x, Z)^j \), for a constant \( C_{g_j} \).

The bounds on the derivatives, \( |g^{(k)}(\ldots)| \leq \ldots \), are obtained in the same way, by Taylor expanding \( g^{(k)} \) at \( z \).

**Step 3.** We use a particular cutoff function with controlled growth of derivatives:

**Theorem 1.4.2 of [Hor'mander], pg. 25** For any compact set with its neighborhood, \( Z \subset U \subset \mathbb{R}^n \), and a positive decreasing sequence \( \{ d_j \} \), satisfying \( \sum d_j < dist(Z, \partial U) \), there exists a smaller neighborhood, \( Z \subset V \subset \mathcal{U} \), and a function \( \tau \in C^\infty(\mathbb{R}^n) \) satisfying

(a) \( \tau|_{\mathbb{R}^n \setminus U} = 0 \), \( \tau|_{V} = 1 \);

(b) for any \( k \) and \( x, y_1, \ldots, y_k \in \mathbb{R}^n \) holds: \( |\tau^{(k)}(x)(y_1, \ldots, y_k)| \leq \frac{C_{\tau(k)}}{d_1 \cdots d_k} \).

(Here the constant \( C \) depends only on the dimension \( n \).)

In our case the set \( Z \subset Ball_1(0) \) is closed and we can assume it is compact by shrinking the ball. Define the \( \epsilon \)-neighborhood, \( U_\epsilon(Z) := \{ x \mid dist(x, Z) < \epsilon \} \subset \mathbb{R}^n \). Fix a decreasing sequence of positive numbers \( \{ \epsilon_j \} \), \( \epsilon_j \to 0 \). Assume it decreases fast, so that for each \( j \) exists a cutoff function satisfying:

\[
(18) \quad \tau_j|_{\mathcal{U} \epsilon_{j+1}} = 1, \quad \tau_j|_{Ball_1(0) \setminus \mathcal{U} \epsilon_j} = 0, \quad \text{and } |\tau_j^{(k)}| \text{ is bounded as above, for any } k.
\]

Define \( f(x) := \sum_j \tau_j(x) \cdot g_j(x) \). We claim that \( f \in C^\infty(\mathbb{R}^n) \), when \( \{ \epsilon_j \} \) decrease fast.
The statement $f \in C^\infty(\mathbb{R}^n \setminus Z)$ is obvious, as for any $x \in \mathbb{R}^n \setminus Z$ the summation is finite. To check the behaviour on/near $Z$ we bound the derivatives:

$$\left| (\tau_j(x) \cdot g_j(x)) (\mathbf{y}) \right| = \left| \sum_{0 \leq |\mathbf{L}| \leq k} \left( \frac{\partial^{|\mathbf{L}|}}{\partial x_1^{j_1} \cdots \partial x_{i}^{j_i}} \right) \tau_j(x) \cdot g_j^{\mathbf{L}}(x) \right| < \sum_{0 \leq |\mathbf{L}| \leq k} \left( \frac{\partial^{|\mathbf{L}|}}{\partial x_1^{j_1} \cdots \partial x_{i}^{j_i}} \right) \cdot C_{g_j} \cdot \text{dist}(x, Z)^{1-|\mathbf{L}|} \cdot \left| \mathbf{y} \right|^{k+|\mathbf{L}|}$$

$$< \sum_{0 \leq |\mathbf{L}| \leq k} \left( \frac{\partial^{|\mathbf{L}|}}{\partial x_1^{j_1} \cdots \partial x_{i}^{j_i}} \right) C^{\mathbf{L}} \cdot C_{g_j} \cdot \text{dist}(x, Z)^{1-|\mathbf{L}|} \cdot \frac{1}{d_1 \cdots d_l} < \text{dist}(x, Z) \cdot \sum_{0 \leq |\mathbf{L}| \leq k} \left( \frac{\partial^{|\mathbf{L}|}}{\partial x_1^{j_1} \cdots \partial x_{i}^{j_i}} \right) C^{\mathbf{L}} \cdot C_{g_j} \cdot \frac{1}{d_1 \cdots d_l} \cdot \left| \mathbf{y} \right|^{k+|\mathbf{L}|-1}.$$

We assume the sequence $\{\epsilon_j\}$ decrease fast to ensure, for $j > |\mathbf{L}| + 1$:

$$\sum_{0 \leq |\mathbf{L}| \leq k} \left( \frac{\partial^{|\mathbf{L}|}}{\partial x_1^{j_1} \cdots \partial x_{i}^{j_i}} \right) C^{\mathbf{L}} \cdot C_{g_j} \cdot \frac{1}{d_1 \cdots d_l} \cdot \left| \mathbf{y} \right|^{k+|\mathbf{L}|-1} < \frac{1}{j!}.$$

Present $f(\mathbf{y}) = \sum_{j=0}^{\infty} \left( (\tau_j(x) \cdot g_j(x)) (\mathbf{y}) \right) + \sum_{j>|\mathbf{L}|+1} \ldots$. And our bounds ensure that the infinite tail converges uniformly on the whole $\mathbb{R}^n$.

**Step 4.** We claim: $\tau_j \cdot g_j - g_j \in I_{\infty}$, for any $j$. For this, we construct a function $q \in I_{\infty}$, satisfying $Z = q^{-1}(0)$. For any $x_0 \in \text{Ball}(0) \setminus Z$ fix some $q_0 \in I_{\infty}$ such that $q_0(x_0) \neq 0$. (This exists as $V(I_{\infty}) = Z$.) By compactness considerations we get a finite subset $\{q_{n_0}\}$ such that the function $q(x) := \sum q_{n_0}(x) \in I_{\infty}$ does not vanish at any point of $\text{Ball}(0) \setminus Z$.

Finally, $\tau_j \cdot g_j - g_j$ vanishes on $U_{\epsilon_j}$, thus $\frac{\tau_j \cdot g_j - g_j}{q}$ extends to a smooth function on $\text{Ball}(0)$. Therefore $\tau_j \cdot g_j - g_j \in (q) \in I_{\infty}$.

Hence $f - \sum_{j=0}^{N} g_j \in I_{\infty}$, for any $N$, and the completion map sends $f$ to $\sum g_j$.

**Step 5.** We prove part 2 of the theorem.

Let $\{I_j\}$ be a filtration of $C^\infty(U)$, as in the assumption. Take an element of the completion, $\sum g_j \in C^\infty(U)/I_*$, and construct a representative $f \in C^\infty(U)$ of $\sum g_j$. Take a closed set $Z \subset U$ and assume $I_{\infty} \supseteq Z(\infty)$. Apply the Whitney extension theorem, see [Whitney pg. 65], to the restriction of $f$ and all of its derivatives onto $Z$. We have the continuous functions $\{f(\mathbf{y})|Z\}$ which satisfy the compatibility conditions of Whitney. (Because they are all restrictions of the derivatives of $f \in C^\infty(U)$.) Then we get a function $f_{\text{ann}} \in C^\infty(U) \cap C^\infty(U\setminus Z)$, whose derivatives (of all orders) on $Z$ coincide with the derivatives of $f$. Which means: $f_{\text{ann}} - f \in I(Z)\infty \subseteq I_{\infty}$. Thus $f_{\text{ann}}$ is also a representative of $\sum g_j$.

**Example 4.2.** The class of filtrations of the theorem, locally equivalent to $\left\{a_0 + \sum_{k} a_k \cdot b_{k,j}\right\}$, is rather large.

i. As the simplest case suppose $V(I_j) = V(I_{j+1}) = V(m)$. For $I_j = m^j \subset C^\infty(\mathbb{R}^m, o)/j$, or more generally when the filtration $\{I_j\}$ is equivalent to $\{m^j\}$, we get the Borel lemma.

ii. Suppose $V(I_j) = V(m)$ and $I_j \subseteq m^{n_j}$, with $n_j \to \infty$, but $I_j \not\supseteq m^{N_j}$, for any $N_j < \infty$. (This happens, e.g. when $I_j$ is generated by flat functions.) We still get the surjectivity of completion, though not implied by Borel’s lemma: for $\sum g_j \in C^\infty(\mathbb{R}^m, o)/j$ we have a representative $f \in C^\infty(\mathbb{R}^m, o)/j$, with $f - \sum g_j \in I_{\infty}$.

However, as in this case $I_{\infty} \supseteq m^{\infty}$, we cannot use part 2 of theorem [4.1] to ensure analyticity off the origin. One can pass to the filtration $m^{\infty}$, to ensure (by Borel) an analytic representative

$$\sum g_j \in C^\infty(\mathbb{R}^m, o/j) \cap C^\infty(\mathbb{R}^m \setminus \{o\}/j).$$

But this satisfies only $f_{\text{ann}} - \sum g_j \in m^{\infty}$ rather than $f_{\text{ann}} - \sum g_j \in I_{\infty}$.

iii. More generally, suppose $V(I_j) = V(I_{j+1}) = Z$ and $I_j \subseteq (I(Z))^{n_j}$, for $n_j \to \infty$. Again, theorem [4.1] implies the surjectivity of completion. Note that we do not assume any regularity/subanalytic conditions on the closed set $Z$.

If $Z \subset U$ is a discrete subset then we get a “multi-Borel” lemma.

iv. Take the ring $C^\infty(\mathbb{R}^m \times \mathbb{R}^n, o)$ with coordinates $x, y$, and the filtration $\{(y)^j\}$. The completion map is the Taylor map in $y$-coordinates, and theorem [4.1] ensures its surjectivity:

$$C^\infty(\mathbb{R}^m \times \mathbb{R}^n, o) \to C^\infty(\mathbb{R}^m \times \mathbb{R}^n, o)[[y]] = C^\infty(\mathbb{R}^m, o)[[y]].$$

(And, moreover, the preimage can be chosen $y$-analytic for $y \neq 0$.) This recovers the classical Borel’s theorem, see [Hörmander Theorem 1.2.6, pg. 16], and [Moerdijk-Reyes Theorem 1.3. pg. 18].

v. Many important filtrations are not of the type $\{I^j\}$, and not equivalent to this, see example [3.1]. In Singularity Theory when studying the germ (at the origin) of a non-isolated hypersurface singularity with singular locus is $\{x_1 = 0 = x_2\}$ one often considers the filtration

$$I_j = (x_1, x_2)^j \cdot (x_1, \ldots, x_m)^j \subset C^\infty(\mathbb{R}^m, o).$$

More generally, $I_j = x_1^j(x_1, y_1)^{n_1} + x_2^j(x_2, y_2)^{n_2} + \cdots + x_m^j(x_m, y_m)^{n_m} \subset C^\infty(\mathbb{R}^m \times \mathbb{R}^n, o)$ is a typical filtration for complete intersections with non-isolated singularities.
Using the surjectivity of completion, one pulls-back various formal results, over $\mathbb{R}[[x,y]]$, to the $C^\infty$-statements.

vi. For the ring $R = C^\infty((\mathbb{R}^m,o) \times (0,1))/I_j$, we can interpret the elements as the families of function germs. Then we get the surjectivity of completion in families, $R \to \hat{R}$. For example, for $I_j = (x^j)$ we get a particular version of Borel lemma on families: any power series $\sum a_m(t) x^m$, with $a_m(t) \in C^\infty([0,1])$, is the $x^j$-Taylor expansion of some function germ $f_j(x) \in C^\infty((\mathbb{R}^m,o) \times (0,1))$.

**Remark 4.3.** For some rings/filtrations one can apply the following surjectivity argument. Assume $(R,m)$ is local and $I_j \subseteq m^n$, with $n_j \to \infty$. Then the completion $R \to \hat{R}^m(n_j)$ factorizes through $R \to \hat{R}^m(I_j) \to \hat{R}^m(n_j)$. Thus, if the map $R \to \hat{R}^m(I_j)$ is surjective, the map $R \to \hat{R}^m(n_j)$ is injective. However, a necessary condition for the injectivity $\hat{R}^m(I_j) \to \hat{R}^m(n_j)$ is $I_{\infty} \supseteq m^\infty$. And this does not hold for many filtrations.

**Remark 4.4.** The surjectivity question is naturally related to the classical Whitney extension problem:

\[ f(\varphi)|_Z = h(\varphi)|_Z \]

Whitney’s theorem ensures the existence of $f$ under the “compatibility” conditions (e.g. §1.5.5 and §1.5.6 of Narasimhan):

\[ \forall k, \forall x, y \in Z : h_k(x) - \sum_{k \in \mathbb{N}^m} \frac{h_k+1(y)}{k!}(x-y)^k \in I(Z)^\infty. \]

(24) Whitney’s theorem ensures the existence of $f$ under the “compatibility” conditions (e.g. §1.5.5 and §1.5.6 of Narasimhan):

\[ \forall k, \forall x, y \in Z : h_k(x) - \sum_{k \in \mathbb{N}^m} \frac{h_k+1(y)}{k!}(x-y)^k \in I(Z)^\infty. \]

(23) Given a closed subset $Z \subseteq U$ and a collection of $C^\infty$-functions, $\{h_k\}_k$, on $Z$, does there exist $f \in C^\infty(U)$ with the prescribed derivatives $\{f(\varphi)|_Z = h_k(\varphi)|_Z\}$.

(25) Whitney’s theorem ensures the existence of $f$ under the “compatibility” conditions (e.g. §1.5.5 and §1.5.6 of Narasimhan):

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**Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be’er Sheva 84105, Israel.**

E-mail address: gernich@math.bgu.ac.il

E-mail address: fernanaly@post.bgu.ac.il

E-mail address: dmitry.kerner@gmail.com