Relating entanglement spectra and energy spectra via path-integral on replica manifold

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We develop a new algorithm to overcome the exponential growth of computational complexity in reliably extracting low-lying entanglement spectrum from quantum Monte Carlo simulation combined with stochastic analytic continuation. We test the method on Heisenberg spin ladder with long entangled boundary between two chains and the results support the Haldane's conjecture on entanglement spectrum of topological phase. We then explain the conjecture via the wormhole effect in the path integral framework and show that it can be further generalized for systems beyond gapped topological phases. Our further simulation results on the bilayer antiferromagnetic Heisenberg model with two dimensional entangled boundary across the \((2 + 1)d\) \(O(3)\) quantum phase transition clearly demonstrate the correctness of such understanding of Haldane's conjecture from the wormhole effect.

Introduction.- The fruitful dialogue and fusion between quantum informatics and highly entangled condensed matter systems, have been gradually appreciated and recognized in recent years \cite{1, 2}. Within this trend, quantum entanglement serves as the quintessential quantity to detect and characterize the informational, field-theoretical and topological properties of many-body quantum states \cite{3, 4}. It offers, among many interesting features, the direct connection to the conformal field theory (CFT) and categorical description of the problem at hand \cite{17, 22}. More than a decade ago, Li and Haldane proposed that the entanglement spectrum (ES) is an important, maybe more fundamental, measurement in this regard than the entanglement entropy (EE) \cite{23, 24}. Since then, low-lying ES has been widely employed as a fingerprint of CFT and topology in the investigation in highly entangled quantum matter \cite{26, 42}. Moreover, for topological states (e.g. quantum Hall state), they pointed out a possible deep correspondence between the low-lying ES and the true energy spectra on the edge. This is another famous Haldane's conjecture, other than the one for the gapped spin-1 chain. Later, Qi, Katsura and Ludwig theoretically demonstrated the general relationship between entanglement spectrum of \((2+1)d\) gapped topological states and the spectrum on their \((1+1)d\) edges \cite{13}. However, besides such gapped topological phases, how universal the Li and Haldane's conjecture is remains an interesting and open question to this day.

On the numeric front, most of the ES studies so far have focused on (quasi) 1d systems. Due to the exponentially growth of computation complexity and memory cost, the existing numerical methods such as exact diagonalization (ED) and density matrix renormalization group (DMRG) have obvious limitations for entangling region with long boundaries and higher dimensions. Quantum Monte Carlo (QMC) is a powerful tool for studying large size and higher dimensional open quantum many-body systems \cite{44, 45}, as the important sampling scheme can in principle convert the exponential complexity into polynomial \cite{46, 47}. But in the first appearance, it looks difficult to obtain ES from QMC, as it can’t obtain the quantum (ground state) wavefunction directly. However, it has been successfully shown that the computation of Rényi EE can be cast into the sampling of the partition function in modified manifold with different boundary condition for the entangling region and the rest of the system \cite{2, 16, 54}, and the universal information of many interesting quantum many-body phases and phase transitions in EE have been reliably extracted in QMC simulations \cite{16, 18, 20, 55, 59}.

This paper is a response to these open questions and recent developments. Here we first develop a protocol to overcome the exponential growth of computational complexity in obtaining the ES via QMC combined with stochastic analytic continuation \cite{13, 16, 18, 67, 68}. Then, we test the method on a Heisenberg ladder with long entangling region between two coupled chains. Our QMC results remove the finite size effects in previous ED results \cite{27} and further show the low-lying ES is indeed behave as the energy spectra of one chain, highly consistent with the Haldane’s conjecture.

In the next step, we develop a pedagogical understanding of the conjecture within the path-integral framework, in which we find a wormhole effect on the imaginary time edges of environment to induce the modes of entanglement Hamiltonian. According to such physical picture, the Haldane’s conjecture should not only be true in the gapped topological phase, but can be further generalized to other systems such as the quantum critical point (QCP). To demonstrate this point, we compute the ES of antiferromagnetic (AFM) Heisenberg bilayer model across the \((2 + 1)d\) \(O(3)\) QCP, where the entangling region is between the two coupled layers. The ESs for the bilayer inside the Néel phase, at the QCP and in-
side the dimerized phase are obtained readily. All these ESs are similar with the magnon dispersion of the layer system, that is, Goldstone modes of the AFM Heisenberg model on square lattice [11,13], consistent with the prediction of our wormhole effect in the ES computation in path-integral. As far as we are aware of, these results also serve as the first measurement of ES with 2d entangling region.

As depicted in Fig. 1, $Z_A^{(n)}$ is a partition function in a replica-manifold – the "Qi Ku" geometry [18,20,72] – where the boundaries of area $A$ of the $n$ replicas are connected in imaginary time and the boundaries of the area $\overline{A}$ are independent (for sites in $\overline{A}$ for each replica, the usual periodical boundary condition of $\beta$ is maintained). It can be seen that the effective $\beta_A = n$ of the subsystem $A$ is in the unit of integer numbers whereas the $\beta = 1/T$ of the total system is in the inverse unit of the physical energy scale of the original system, $J$ of the Heisenberg model, for instance.

As shown in Fig. 1, since the imaginary times $\tau_A$ between two nearest replicas differs by $\delta \tau_A = 1$ (an effective imaginary time evolution operator $e^{-\beta \mathcal{H}_A}$ of subsystem acts between them), we can measure the imaginary time correlation function at integer points, to obtain $G(\tau_A)$ of $\tau_A = 0, 1, 2, ..., n$. The correlation function $G(\tau_A) \sim e^{-\Delta \tau_A}$ when $\beta_A \rightarrow \infty$ where $\Delta$ is the lowest energy gap in the ES. When there is large gap, the $G(\tau_A)$ decays very fast and it is very hard to extract the high energy part of spectrum because the distance $\delta \tau_A = 1$ is not small enough. However, since the important information of ES is usually encoded in the long-time imaginary time correlations in QMC simulations, i.e. the more number of replicas $n$, the longer the imaginary time correlation $\tau_A$, and lower "energy" $\omega_{\mathcal{H}_A}$ in ES accessed. With the good quality $G(\tau_A)$ at hand, the stochastic analytic continuation (SAC) scheme [73] can reveal reliable spectral information, $S(\omega_{\mathcal{H}_A}(k))$, as have been widely tested in fermionic and bosonic quantum many-body systems in 1d, 2d and 3d [4,5,7,10,14,18,21,23,73,79]. In the following two examples, we use stochastic series expansion (SSE) QMC for quantum spin systems [13,35,36] combined with SAC to obtain the related ESs. All the imaginary time correlations are computed via spin $S^z$ operators, i.e., $G_k(\tau_A) = \langle S^z_k(\tau_A) S^z_0(0) \rangle$.

**Spin-1/2 Heisenberg Ladder.-** We compute the ES of Heisenberg ladder with $L = 100$ and compare with the ED results in small sizes $L = 10, 12, 14, 25$. The spins on ladder are coupled through the nearest neighbor Heisenberg interactions [73] as shown Fig. 2 (a), with the strength $J$ along the leg and $J'$ on the rung. We first

\[
Z_A^{(n)} = \text{Tr}[e^{-\beta \mathcal{H}_A}] = \text{Tr}[e^{-\beta \mathcal{H}_{A \setminus \overline{A}}}],
\]

**Method.-** The ES of a subsystem $A$ coupled with environment $\overline{A}$ is constructed via the reduced density matrix (RDM), defined as the partial trace of the total density matrix $\rho$ over a complete basis of $\overline{A}$, $\rho_A = \text{Tr}_{\overline{A}} \rho$. The RDM $\rho_A$ can be interpreted as an effective thermodynamic density matrix $e^{-\beta \mathcal{H}_A}$ through an entanglement Hamiltonian $\mathcal{H}_A$. As known in closed system, spectral function $S(\omega)$ for physical observable, represented as $\mathcal{O}$, can be written by the eigenstates $|n\rangle$ with the eigenvalue $E_n$ of the Hamiltonian $\mathcal{H}$,

\[
S(\omega) = \frac{1}{\pi} \sum_{m,n} e^{-\beta E_n} \langle \langle m | \mathcal{O} | n \rangle \rangle^2 \delta(\omega - (E_m - E_n)).
\]

Therefore, there is a relation between energy spectrum $S(\omega)$ and imaginary time correlation $G(\tau)$ as \[G(\tau) = \int_0^{\infty} d\omega K(\omega, \tau) S(\omega),\] The $K(\omega, \tau)$ is a kernel with slightly different expressions for bosonic/fermionic $\mathcal{O}$ [4,7,10,11]. The energy spectrum of the corresponding operator can be analytically continued from the correlation function in imaginary time. However, the relation between RDM and the modular Hamiltonian, \[\rho_A = e^{-\beta \mathcal{H}_A},\] doesn’t contain any information of a effective imaginary time $\beta_A$ of the subsystem. To compute ES, the first task is to construct a "partition function" of $\mathcal{H}_A$ with effective imaginary time $\beta_A$.

The solution comes from the $n$-th order of RDM, $\rho_A^n$, which can be written as $\rho_A^n = e^{-n \mathcal{H}_A}$. In this way, we can readily make use of such effective imaginary time $\beta_A = n$ at $n = 1, 2, 3, \cdots$ points. It’s similar to how the Rényi EE is computed in QMC via the replica partition function [18,20,49,55,58,70,71].

\[
Z_A^{(n)} = \text{Tr}[\rho_A^n] = \text{Tr}[e^{-n \mathcal{H}_A}].
\]
simulate with $J = 1$ and $J' = 1.732$ at $\beta = 100$ and $\beta_A = 200$ (200 replicas).

Our QMC ES in Fig. 2(b) are consistent with the ED results, but our larger system sizes clearly reveal new features at the thermodynamic limit (TDL). First, with $L = 100$, the finite size gaps at $k = 0$ and $\pi$ are much smaller than the ED results with $L = 14$, e.g. $\Delta(\pi) \sim 0.48$ in ED whereas $\Delta(\pi) \sim 0.1$ in QMC, suggesting the gap closes at TDL. Second, the ES here is expected to decay with the central charge $c = 1$ and $v$ the velocity of ES near the gapless point [25], i.e. the Cloizeaux-Pearson spectrum of the quantum spin chain, $v |\sin(k)|$ [5]. The ED fit at $L = 14$ gave $v \sim 2.36$, however, the fitting of QMC reveals the $v \sim 4.58$, as shown by the fitting line in Fig. 2(b). This again reflects ES is greatly affected by finite size effect and it is necessary to access larger system sizes for quantitative information.

Furthermore, we simulate the case with ferromagnetic (FM) $J = -1$ and antiferromagnetic (AFM) $J' = 1.732$ at $\beta = 100$ and $\beta_A = 800$ on the same ladder to compare with the results in Ref. [25], where the ladder is in the gapped rung singlet phase. Since the ED in TDL is expected to show spectrum of FM spin chain, i.e. quadratic dispersion $\sin^2(k/2)$ close to $k = 0$ [80 81], we purposely chose $\beta_A = 800$ such that more low-lying ES can be obtained. As shown in Fig. 2(c), the obtained dispersion of ES indeed resembles that of an edge Hamiltonian of the FM chain. In the ED results with $L = 14$ [25], a linear dispersion is found and it was attributed to the finite size effect. Now that we can access $L = 100$, the ES indeed reveals a quadratic dispersion $7.96 \sin^2(k/2)$ as shown by the white line therein. The results in Fig. 2(b) and (c), clearly demonstrate the correspondence between the ES and the true spectra of the edge Hamiltonian, consistent with the previous ED study, and achieve the TDL readily.

**FIG. 2.** (a) Heisenberg spin ladder. The red dashed line cut it into two entangled constituents, $A$ and $\bar{A}$. (b) The low-lying ES with $L = 100$, $J' = 1.732$, $J = 1$ and $\beta = 100$, $\beta_A = 200$. The white line is fitting to the data with the dispersion $4.58 \sin(k)$. (c) The low-lying ES with $L = 100$, $J' = 1.732$, $J = -1$ and $\beta = 100$, $\beta_A = 800$. The white line is fitting to the data with the dispersion $7.96 \sin^2(k/2)$.

**FIG. 3.** (a) Wormhole effect of worldlines going through a replica. The gray part is the subsystem $A$ and the white part is the environment $\bar{A}$. The arrows go into bulk will decay to zero as $\beta \to \infty$. At the same time, the arrows go through the imaginary-time-edge of environment will reach the other side through "wormhole" without much attenuation. (b) The $G_{k=0}(\tau)$ along imaginary time direction of the replica system for AFM spin ladder in Fig. 2(b) with $L = 100$ and $\beta_A = 100$. The fast mode inside one replica and the slow mode between the replicas are clearly seen. (c) A worldline evolution in a normal path-integral of closed system. (d) A worldline evolution in a replica system as Fig. 1. The real worldline (solid line) locates at the $\tau_A = 0$, 1, 2,... layers and the line goes in/out of a hole means it goes in/out of one replica. Between a pair of holes, we can imagine a virtual process (dashed line) which is equivalent to a real process in normal system.

**Wormholes in the path integral.** We find the geometrical manifold of replicas (Fig. 1) provides a very intuitive picture to understand the Haldane’s conjecture, in particular, one can use a wormhole effect of the worldline in QMC simulation in space-time to explain the tracing of the environment in the path-integral. As shown in Fig. 3(a), the gray part is the subsystem $A$ and the white part is the environment $\bar{A}$, when a worldline goes into one replica, it has many choices for the possible paths. The imaginary time correlations for the worldlines going into bulk will decay to zero as $\beta \to \infty$ for obtaining the ground state. At the same time, the ones go through the imaginary-time-edge of environment will reach the other side through "wormhole" without much attenuation. Thus, tracing $\bar{A}$ actually provides a wormhole-like
escapeway to connect both sides in environment of imaginary time, and it automatically guarantee the imaginary time correlations near the connection between replicas stronger, which contribute to the ES.

This picture is proved in our numerics via the correlation function along imaginary time direction in Fig. 3 (b), where the $G_k=\tau(\tau)$ for AFM Heisenberg ladder [the case of Fig. 2 (b)] is shown. It is obvious that there are two time/energy scales: the slow-decaying one is led by the entanglement Hamiltonian, that is the envelope of upper boundary at the $\beta_A = 0, 1, 2, ..., \tau$ generated by the wormhole effect. The fast-decaying one is led by the original Hamiltonian, which happens in the bulk of one replica between two integer $\beta_A$ points. It goes into the depth of bulk as the "decay" ones in Fig. 3 (a). We note such wormhole effect of the worldline in QMC simulation of ES in the path-integral framework has not been seen before.

![Fig. 4](attachment:image.png)

**FIG. 4.** (a) Antiferromagnetic Heisenberg bilayer. The red dashed line cut it into two entangled constituent layers, $A$ and $\bar{A}$. (b) The low-lying ES of in the Néel phase with $L = 50$, $J' = 1.732, J = 1$ and $\beta = 100, \beta_A = 32$. (c) The low-lying ES at the quantum critical point with $L = 50, J' = 2.522, J = 1$ and $\beta = 100, \beta_A = 32$. (d) The low-lying ES in the dimerized phase with $L = 50, J' = 3, J = 1$ and $\beta = 100, \beta_A = 32$. We show the obtained ES along the high-symmetry path $(0,0) - (\pi,0) - (\pi,\pi) - (0,0)$. The white line is fitting to the data with the dispersion $7.5\sqrt{1 - \frac{1}{4}(\cos(k_x) + \cos(k_y))^2}$, the linear spin wave for antiferromagnetic Heisenberg model on square lattice.

With such picture, it is easy to understand the working of Haldane’s conjecture, i.e. the entanglement spectrum of $A$ is similar with the energy spectra of a real closed $A$ system without coupling to $\bar{A}$. Fig. 3 (c) is an example of the evolution of worldline in a real closed $A$ system and Fig. 3 (d) is that of a replica system such as that in Fig. 1. For a replica system, we can measure the correlation of entanglement Hamiltonian only at the integer $\beta_A$ points. Then the real worldline (solid line) means it is at the $\tau_A = 0, 1, 2, ..., \tau$ layers, i.e., the connected part of two nearest replicas, and the virtual process goes in/out of one replica are denoted as the dashed line. Furthermore, because the entanglement Hamiltonian is independent on imaginary time, the rule of worldline evolution should be same in any time. Therefore, between a pair of replicas (holes), we can imagine such a virtual process (dashed line) which is equivalent to a real process in a normal system [Fig. 2 (c)]. Thus, the entanglement Hamiltonian is likely to be the similar as Hamiltonian of real closed $A$ system.

The above argument can be applied to the case where every sites of subsystem $A$ is coupled with environment $\bar{A}$, such as the ladder case, and extend the previous spatial-cut explanation of the Haldane’s conjecture where the subsystem and environment are coupled only on boundary [43]. To further verify our wormhole picture, we simulate a coupled bilayer system, in which one layer is the subsystem $A$ and the other is the environment $\bar{A}$. According to our picture, the ES should be similar to the energy Hamiltonian of a closed one layer system. The results are shown below.

**Antiferromagnetic Heisenberg Bilayer.** An AFM Heisenberg model on bilayer square lattice [73] as shown Fig. 4 (a), where $J$ and $J'$ are the intra- and inter-layer couplings, are studied. We compute the ES with the bottom layer as $A$ and the top layer as $\bar{A}$. The $(2+1)$D O(3) QCP, separating the Néel phase and inter-layer dimer product state, is found to locate at $J'/J = 2.5220(1)$ from high-precision QMC simulations [18, 21, 88, 89]. The EE of antiferromagnetic Heisenberg bilayer has been studied in Ref. [90], but the ES is still lacking. We simulate three cases to demonstrate our prediction: $J'/J = 1.732$ in the Néel phase [Fig. 4 (b)], $J'/J = 2.522$ at the critical point [Fig. 4 (c)], $J'/J = 3$ in the dimerized phase [Fig. 4 (d)] at $\beta = 100$ and $\beta_A = 32$ with size $L = 50$.

All the three cases strongly support our understanding: the ESs have two gapless modes with a strong one at $(\pi,\pi)$ and a weak one at $(0,0)$, closely resembling those the Goldstone modes in square antiferromagnetic Heisenberg model [11, 13, 91]. We therefore fit all the three ESs in Fig. 4 (b), (c) and (d) with the same linear spin wave [92, 93] dispersion $7.5\sqrt{1 - \frac{1}{4}(\cos(k_x) + \cos(k_y))^2}$ along the high-symmetry-path and find the fitting lines go through the data. As far as we are aware of, these results serve as the first measurement of ES in 2d entanglement region, also consistent with our wormhole picture of worldline in QMC simulation of ES.

**Discussions.** The picture of path-integral in replica-manifold to explain and predict ES is very general. One only needs to compare the different energy cost in different paths along the imaginary time. Rough estimation gives the ratio of the cost between the subsystem and environment as $\beta_\Delta : \Delta_\tau$, where $\Delta_\tau$ and $\Delta_\tau$ are the energy gaps of subsystem and environment, and $\beta$ and $1$ are the length of imaginary time between two close replicas in subsystem and environment. The gap near entangled edge is more important than in deep environment under consideration of the dissipation of real space. Therefore,
the ES is like the edge modes in the gapped topological phases \[\beta \Delta \gg \Delta\]
simply because \[\beta \Delta \gg \Delta\] and entangling region has a gapless edge mode. On the contrary, if the environment and entangling edge are gapped and subsystem bulk is exactly gapless, the ES will be more like the energy spectra of subsystem bulk. In addition, the strength of the coupling between the environment and the subsystem will obviously affect the choice of worldline path. When the coupling is weak, this also increases the cost of the worldline going into the environment. The path integrals in subsystem and environment are competing to determine a less energy-cost path.

Overall, we develop a practical scheme to extract the low-lying ES from QMC simulation. Combined with the unifying picture of wormhole effect in path-integral, our method makes the ES measurement possible for high dimension quantum many-body systems with large entangling region. Our method is not only limited to QMC for spins, as the existed pioneering works in computing the ES for interacting fermion systems [39–41], but can also be extended to other numerical approaches for highly entangled quantum matter, such as the finite temperature tensor-network algorithm [93,96].

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SUPPLEMENTAL MATERIAL

Hamiltonian of ladder and bilayer systems

For the quantum Monte Carlo (QMC) simulations in this paper, we use the stochastic series expansion (SSE) algorithm[1–3] to deal with the Heisenberg ladder [Fig.2 (a) in main text] and antiferromagnetic (AFM) Heisenberg bilayer model [Fig.4 (a) in main text].

The Hamiltonian of the spin-1/2 ladder is

\[ H = J \sum_{\langle ij \rangle} (S_{A,i} S_{A,j} + S_{\overline{A},i} S_{\overline{A},j}) + J' \sum_i S_{A,i} S_{\overline{A},i} \]  

Here, \( S_{A,i} \) and \( S_{\overline{A},i} \) are spin-1/2 operators at site \( i \) of \( A \) and \( \overline{A} \). \( \langle i, j \rangle \) denotes a pair of nearest-neighbor sites on the spin chain of \( L \) sites with periodic boundary conditions. \( J \) is the coupling strength intra-chain and \( J' \) is coupling inter-chains.

Similarly, we can define the spin-1/2 Hamiltonian on a bilayer lattice via the same equation. Where \( S_{A,i} \) and \( S_{\overline{A},i} \) are spin-1/2 operators at site \( i \) of \( A \) and \( \overline{A} \) as Fig.4 (a) in main text. \( \langle i, j \rangle \) denotes a pair of nearest-neighbor sites on the square lattice of \( L \times L \) sites with periodic boundary conditions. \( J \) is the coupling strength intra-layer and \( J' \) is coupling inter-layers.

Stochastic analytical continuation

We employ a stochastic analytical continuation (SAC) [4–8] method to obtain the spectral function \( S(\omega) \) from the imaginary time correlation \( G(\tau) \) measured from QMC, which is generally believed a numerically unstable problem.

The spectral function \( S(\omega) \) is connected to the imaginary time Green’s function \( G(\tau) \) through:

\[ G(\tau) = \int_{-\infty}^{\infty} d\omega S(\omega) K(\tau, \omega) \]  

here \( K(\tau, \omega) \) is the kernel function depending on the temperature and the statistics of the particles. We restrict ourselves to the case of spin systems and with only positive frequencies in the spectral, where \( K(\tau, \omega) = (e^{-\tau \omega} + e^{-(\beta - \tau) \omega})/\pi \). Then, we have

\[ G(\tau) = \int_{0}^{\infty} d\omega \frac{e^{-\tau \omega} + e^{-(\beta - \tau) \omega}}{1 + e^{-\beta \omega}} B(\omega) \]  

here \( B(\omega) = S(\omega)(1 + e^{-\beta \omega}) \) is the renormalized spectral function.

In fact, \( G(\tau) \) for a set of imaginary time \( \tau_i \) is obtained by QMC with the statistical errors. The renormalized spectral function can be set into large number of equal-amplitude \( \delta \)-functions

\[ B(\omega) = \sum_{i=0}^{N_\omega} a_i \delta(\omega - \omega_i) \]
Then the fitted Green’s functions $\tilde{G}_i$ from Eq. [S3] and the measured Greens functions $\bar{G}_i$ are compared by the fitting goodness

$$\chi^2 = \sum_{i,j=1}^{N_r} (\tilde{G}_i - \bar{G}_i)(C^{-1})_{ij}(\tilde{G}_j - \bar{G}_j)$$

(S5)

where the covariance matrix is defined as

$$C_{ij} = \frac{1}{N_B(N_B-1)} \sum_{b=1}^{N_B} (G^b_i - \bar{G}_i)(G^b_j - \bar{G}_j),$$

(S6)

with $N_B$ the number of bins, the measured Green’s functions of each $G^b_i$.

The weight for a given spectrum is taken to follow a Boltzmann distribution via Metropolis sampling

$$W(\{a_i, \omega_i\}) \sim \exp \left( -\frac{\chi^2}{2\Theta} \right)$$

(S7)

with $\Theta$ a virtue temperature to balance the goodness of fitting $\chi^2$ and the smoothness of the spectral function. All the spectral functions of sampled series will be averaged to obtain the final spectrum.

In this paper, we use the correlation of $S^z$ operators to obtain the spectra. For a broad class of spin models, the operator $S^z$ can reveal a lot excitations unless it can not connect the ground state and the excited state (e.g., the excitation changes the total $S^z$). For example, previous works on extracting the spectra of other models even without SU(2) symmetry via $S^z$ correlations: neutron scattering spectrum through transverse field Ising model [9]; interaction between visons in quantum dimer model [10]; and the Higgs mode in a weak Zeeman field Heisenberg model [11], etc.

In addition, we will need to design different operators to probe some special spectra. These can be readily implement in many other systems, for instance, people have measured the vison and dimer correlations and their spectra in quantum dimer models [10, 12], the $S^+S^-$ and bond-bond correlations and their spectra in quantum spin models [11, 13]. The corresponding entanglement spectral measurements, can be carried out in the similar manner.

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