Higher geometric sheaf theories

Raffael Stenzel

May 19, 2022

Abstract

We introduce an ∞-category of geometric ∞-categories, whose objects are defined by dropping the effectivity condition on colimits in Anel and Joyal’s definition of ∞-logos. We propose a theory of higher geometric sheaves on geometric ∞-categories C, which will be shown to differ from the ordinary geometric sheaf theory on C by a cotopological fragment. We prove that this fragment is crucial: for instance every ∞-topos is the theory of higher geometric sheaves over itself, but the corresponding cotopological localization of its ordinary geometric sheaf theory is generally non-trivial. The notion of higher geometric sheaves over geometric ∞-categories hence faithfully generalizes Lurie’s definition of sheaves over ∞-toposes.

We define this class of sheaf theories by way of an adaption of Anel and Leena Subramaniam’s ∆-modulators. The according sheaves are characterized by a limit preservation property that is generally not captured by the classical sheaf condition as known from ordinary sheaf theory over topological spaces (or over geometric categories more generally). The latter arises as a special case instead. To motivate this class of limits, we first introduce and discuss simpler but analogous sheaf theories for extensive and regular ∞-categories.

1 Introduction

Notation. As is often custom, in all of the following the prefix “(∞, 1)” will be abbreviated to “∞”. The ∞-category of spaces will be denoted by S, the ∞-category of functors between two ∞-categories C, D will be denoted by Fun(C, D) or D^C depending on the context. The ∞-category of presheaves Fun(C^{op}, S) over a small ∞-category C will be denoted by Ș.

A motivation While the motivation for this paper is ultimately multifold, the underlying principle in spirit is a faithful realisation of the universal duality between topology and logic, a paradigm that emerged from the work of numerous people over many years (see [13], [5], [2]). Particularly, it is about the consequently specific nature of a topology given the context of a specific logic. The idea of this duality goes back to the Stone dualities between various categories of topological spaces, and associated categories of categorical models for various first-order theories. The

The author acknowledges the support of the Grant Agency of the Czech Republic under the grant 19-00902S.
arguably most popular of such is given by the adjunction

$$\text{Top} \xleftrightarrow{\text{pt}} \text{Frm}^{\text{op}},$$

between the category Top of topological spaces and the opposite of the category Frm of frames and frame morphisms – that is, the category of categorical models of “propositional theories” ([5, 1.2.5.(f)]). Here, $\mathcal{O}(X)$ is the frame of opens on a topological space $X$, and $\text{pt}(F)$ is the space of points associated to a frame $F$. This adjunction restricts to a well-known equivalence of full subcategories, spanned by the sober topological spaces on the one side, and the frames with enough points on the other side ([9, Corollary II.1.7]). The identification of $X$ with $\mathcal{O}(X)$ holds up especially when it comes to sheaf theory, since sheaves on a space $X$ are by definition sheaves on the frame $\mathcal{O}(X)$ equipped with its canonical Grothendieck topology. Generally, the canonical topology on a frame $F$ is generated by covers of the form $(U_i \to U | i \in I)$ such that $\bigcup_{i \in I} U_i = U$. In other words, a cover of $U$ is a set-indexed diagram $U_i : I \to F/U$ such that the associated map $\text{colim} U_i \to U$ is an isomorphism in $F$.

A faithful categorification of the theory of frames is the theory of geometric categories. These are the categorical models for general geometric theories, where propositional theories are exactly the geometric theories with no sorts and 0-ary relation symbols only. Classifying toposes for coherent and, more generally, geometric theories play a central role both in geometry and categorical logic. In the following, let “geometric” be an umbrella term for coherent, geometric and \(\kappa\)-geometric for some regular cardinal \(\kappa\).

Every geometric category comes canonically equipped with the structure of a site whose generating covers are the jointly epic families of arrows. From a logical standpoint, these notions of covers arise naturally in the characterization of the categorical model theory of geometric theories (see [5, 1]), and hence ultimately are based on the semantics of formulas in the associated internal language of $C$. Or in other words, on colimits and finite limits of diagrams of subobjects in $C$. The same holds for other fragments of first-order logic, such as extensive or regular theories. Indeed, the topologies considered in these contexts are always colimit covers of some form: a family $X_i$ covers an object $C$ if $C$ is the colimit of a diagram associated to the family $X_i$, where the shape of the diagram depends on the specific sort of theory of which $C$ is considered to be a model. This is obviously the case for the extensive topology on an extensive category, and less obviously so for the regular and geometric covers on a regular or geometric category (see Example 5.1 for the latter). The sheaves for each of these topologies are exactly the presheaves which take the corresponding class of colimits to limits of sets. Equivalently, they are the presheaves local for the covering sieves

$$\prod_{i \in I} yC_i \xrightarrow{(p_i)_{i \in I}} yC$$

generated by the basis of associated covers of the form $p = (C_i \xrightarrow{p_i} C | i \in I)$.

Now, the circumstance that formulas are modelled as subobjects is an immediate consequence of proof-irrelevance (and consequently extensionality) of first-order
logic. In the ∞-categorical context of cartesian ∞-categories with possibly more logical structure however, the internal languages are proof-relevant and non-extensional type theories ([12], [11], [18]), and hence require the modelling of formulas as type families which are generally far from being monic. On the flipside, this poses the question what notion of topology ought to be considered to reflect this.

We are therefore motivated to study “structured” topologies on ∞-categories, generated by arrows of the form

$$\text{colim}_y F \to y(\text{colim} F)$$

for certain classes $K$ of diagrams $F : I \to \mathcal{C}$. The according sheaves, defined as the presheaves local for the class

$$\text{Cov}_K = \{\text{colim}_y F \to y(\text{colim} F) | F \in K\}$$

are exactly the presheaves $\mathcal{C}^{op} \to \mathcal{S}$ which take the chosen colimits to limits. One may think of such covers as structured analogues to their ordinary counterparts obtained after propositional truncation in $\hat{\mathcal{C}}$. Generally, an ordinary cover is given by a family $p = (C_i \to C | i \in I)$ indexed by a set $I$, and an element of its associated sieve $S_p \hookrightarrow yC$ over some $d : D \to C$ is exactly some lift of $d$ to a component $F_i$. The structure of a sieve does not record the explicit lift, in particular it does not have the space to distinguish between different lifts. In fact, this is by design, as the notion of a covering sieve explicitly forgets data. This is the reason why sets are the only sensible choice to index the covering diagrams with in the first place, as any further structure would be forgotten by virtue of the eventual propositional truncation anyway. In contrast, the elements over an arrow $d : D \to \text{colim} F$ of a cover of the form (1) is an explicit lift of $d$ to a component $F_i$, and given two such lifts $l_1 : D \to F_i$, $l_2 : D \to F_j$, the space of identifications between $l_1$ and $l_2$ – computed as the equalizer of $l_i : yD \to \text{colim}_y F$ in $\hat{\mathcal{C}} / (y(\text{colim} F))$ – can a priori be of any homotopy type.

Thus, independently of the above raised question of logical interpretation, it is an interesting question in itself to ask for which classes of diagrams $K$ the ∞-category $\text{Sh}_K(\mathcal{C})$ of Cov$_K$-local presheaves forms an ∞-subtopos of $\hat{\mathcal{C}}$.

**Example.** If for instance, ignoring size issues for the moment being, $\mathcal{C}$ is locally presentable, we may consider the class $K$ of all small diagrams in $\mathcal{C}$. Then the Yoneda-embedding induces an equivalence $y : \mathcal{C} \to \text{Sh}_K(\mathcal{C})$. In particular, the sheaf theory $\text{Sh}_K(\mathcal{C})$ is an ∞-topos if and only if $\mathcal{C}$ was an ∞-topos in the first place. This in fact is exactly the definition of the ∞-category of sheaves on an ∞-topos in [14].

**Example.** If instead we pick a very simple class $K$ of diagrams, say, finite discrete diagrams in an extensive ∞-category $\mathcal{C}$, then we obtain nothing new: The sheaves for this class are exactly the ordinary extensive sheaves (Lemma 3.3).

The main part of this paper is concerned with a middle ground: a collection $K_\infty$ of higher covering diagrams in suitably defined geometric ∞-categories $\mathcal{C}$. We will see that the sheaf theories for the associated bases Cov$_{K_\infty}$ yield a new class of ∞-toposes in general, but recover Lurie’s definition of sheaves over an ∞-topos whenever $\mathcal{C}$ is an ∞-topos. In particular, we will see that every ∞-topos is the ∞-category of Cov$_{K_\infty}$-sheaves over itself, and show that this is a distinguishing feature from the ordinary geometric Grothendieck topology on an ∞-topos.

Technically, one of the perks of these structured sheaf theories is that their development avoids the common rather artificial decomposition of arrows into epimorphisms and monomorphisms, the isolated argumentation for monomorphisms
first, and then the deduction of a statement about general arrows second, which is prevalent in ordinary topos theory, as well as in the construction of \(\infty\)-toposes in two steps from a concrete topological followed by an abstract cotopological localization.

**Summary of results** In this paper, we study \(\infty\)-categorical instances of three classical ordinary categorical sites, but do so essentially by replacing property by structure in the way described above. The questions we will pursue are when does this generate an \(\infty\)-topos, and what can we say about it? For instance, how do these constructions relate to their ordinary counterparts?

To answer these questions, we will make use of the theory of modulators defined in [4], which serve the purpose of bases for generalized topologies on small \(\infty\)-categories. We will introduce the notion of an Id-modulator and discuss its relation to the various notions of modulators from [4] in Section 2. The other three sections each study a canonical Id-modulator over \(\kappa\)-extensive, regular, and \(\kappa\)-geometric \(\infty\)-categories, respectively, where each of the three notions will be defined accordingly.

In Section 3, we define the \(\kappa\)-extensive Id-modulator

\[
\text{Cov}_{\text{Ext}} = \left\{ \prod_{i \in I} y(C_i) \to y(\prod_{i \in I} C_i) \mid I \in \text{Set}_\kappa, C_i \in \mathcal{C} \right\}
\]

over small \(\kappa\)-extensive \(\infty\)-categories \(\mathcal{C}\), and show that their associated \(\infty\)-category \(\text{Sh}_{\text{Ext}}(\mathcal{C})\) of sheaves coincides with the \(\infty\)-category \(\text{Sh}_{\text{Ext}}(\mathcal{C})\) of ordinary \(\kappa\)-extensive sheaves on \(\mathcal{C}\) (Lemma 3.3). In particular, the localization \(\mathcal{C} \to \text{Sh}_{\text{Ext}}(\mathcal{C})\) is topological and sub-canonical. Furthermore, we will show that in the finite case \(\kappa = \aleph_0\) the resulting \(\infty\)-topos is hypercomplete (Corollary 3.7), and hence has enough points whenever \(\mathcal{C}\) is lextensive (Corollary 3.8).

In Section 4, we will discuss regular \(\infty\)-categories and the notion of an associated canonical Id-modulator. This example conceptually is an odd one out, but yields a natural environment to introduce notions relevant for the rest of the paper. We will look at the largest Id-modulator contained in the modulator generated by the effective epimorphisms in \(\mathcal{C}\) – that is, the modulator generated by the \(\infty\)-connected maps in \(\mathcal{C}\) – and see that the associated sheaf theory \(\text{Sh}_{\text{Eff}}(\mathcal{C})\) is generally not sub-canonical and non-topological. In fact, it is generally not a localization of the \(\infty\)-category \(\text{Sh}_{\text{Eff}}(\mathcal{C})\) of ordinary regular sheaves, or vice versa (Proposition 4.4). Yet, we obtain an inclusion of points (Remark 4.5). We further show that \(\text{Sh}_{\text{Eff}}(\mathcal{C})\) is generally not hypercomplete, and hence generally does not have enough points, while \(\text{Sh}_{\text{Eff}}(\mathcal{C})\) is hypercomplete, and does have enough points (Proposition 4.6).

Section 5 is the main section of this paper. Here, we define a notion of \(\kappa\)-small higher covering diagrams in \(\infty\)-categories with pullbacks and \(\kappa\)-small colimits, and show that for \(\kappa\)-geometric \(\infty\)-categories \(\mathcal{C}\) (i.e. \(\infty\)-categories with pullbacks and universal \(\kappa\)-small colimits), the collection \(\text{hcd}_\kappa\) of \(\kappa\)-small higher covering diagrams yields an Id-modulator \(\text{Cov}_{\text{hcd}_\kappa}\) (Theorem 5.11). In a nutshell, a diagram \(F: I \to \mathcal{C}\) is higher covering if \(I\) has pullbacks, \(F\) preserves pullbacks, and furthermore all higher homotopical structure of the object \(\text{colim} F \in \mathcal{C}\) can be computed by colimits of associated higher homotopical structures in \(I\) after pushforward with \(F\). We show that the topological part of the left exact localization \(\mathcal{C} \to \text{Sh}_{\text{hcd}_\kappa}(\mathcal{C})\) is exactly the \(\infty\)-topos \(\text{Sh}_{\text{geo}}(\mathcal{C})\) of ordinary \(\kappa\)-geometric sheaves on \(\mathcal{C}\) (Proposition 5.13). We will see that whenever \(\mathcal{C}\) has descent for \(\kappa\)-small diagrams (i.e. \(\mathcal{C}\) is a \(\kappa\)-logos”), then the higher covering diagrams in \(\mathcal{C}\) are exactly the pullback preserving diagrams from small \(\infty\)-categories with pullbacks (Lemma 5.17). We will deduce that every \(\infty\)-topos is the \(\infty\)-topos of such higher geometric sheaves over itself (Theorem 5.19), and that the same is not true for the ordinary geometric sheaves over itself. This
will yield examples where the cotopological localization \( \text{Sh}_{\text{geo}}(\mathcal{C}) \to \text{Sh}_{\text{hcd}}(\mathcal{C}) \) is non-trivial (Proposition 5.22). Yet, we will see that \( \text{Sh}_{\text{hcd}}(\mathcal{C}) \) is generally not the hypercompletion of \( \text{Sh}_{\text{geo}}(\mathcal{C}) \) as it is not always hypercomplete itself (Corollary 5.16).

In this sense, the higher geometric sheaf theories faithfully generalize Lurie’s definition of sheaves on \( \infty \)-toposes to geometric \( \infty \)-categories, and as such yield a counterpart to the set-valued geometric sheaves on geometric 1-categories. This raises the question whether these sheaf theories may be shown to arise as classifying \( \infty \)-toposes of something that may be referred to as geometric homotopy type theories in the future.

**Acknowledgments.** The author would like to thank Nathanael Arkor, John Bourke, Jonas Frey and Nima Rasekh for much appreciated comments and conversations.

### 2 Bases via modulators

We recall the various notions of modulators and their associated plus-construction from [4] applied to our basic case of interest. That is, we fix a small \( \infty \)-category \( \mathcal{C} \) and consider modulators for the \( \infty \)-category \( \hat{\mathcal{C}} \) locally presented by the representables on \( \mathcal{C} \).

A **pre-modulator** \( M \) on \( \mathcal{C} \) is a collection of sets of objects \( M(\mathcal{C}) \subset \hat{\mathcal{C}}_{/yC} \) such that each fiber \( M(\mathcal{C}) \subset \hat{\mathcal{C}}_{/yC} \) is a set and contains the identity 1 on \( yC \). A **modulator** \( M \) on \( \mathcal{C} \) is a full subfibration \( M \to \hat{\mathcal{C}} \) such that each fiber \( M(\mathcal{C}) \subset \hat{\mathcal{C}}_{/yC} \) is a set and contains the identity 1 on \( yC \). A **lex modulator** \( M \) on \( \mathcal{C} \) is a modulator whose fibers \( M(\mathcal{C}) \subset \hat{\mathcal{C}}_{/yC} \) are co-filtered ([4, Definition 3.4.1]). In particular, a modulator which is fiberwise closed under finite limits is a lex modulator.

It has been shown in [4] that lex modulators \( M \) on \( \hat{\mathcal{C}} \) are generators of left exact modalities, and hence of accessible left exact localizations of \( \hat{\mathcal{C}} \) in the following sense. Anel and Leena Subramaniam show that, first, every accessible left exact localization of \( \hat{\mathcal{C}} \) is generated by some lex modulator on \( \hat{\mathcal{C}} \), and, second, that the localization of \( \hat{\mathcal{C}} \) at a lex modulator \( M \) is always accessible and left exact. Furthermore, they show that the class \( \mathcal{L} \) of arrows in \( \hat{\mathcal{C}} \) which are inverted by such a localization – that is, the left class of the associated left exact modality – is not only the strong saturation of \( M \) (as is always the case), but the saturation already ([4, Theorem 3.4.2]). General modulators generate general modalities in analogous fashion ([4, Theorem 3.3.8]).

Next we recall the plus-construction associated to a pre-modulator \( M \) on \( \mathcal{C} \). Given a pre-modulator \( M \) on \( \mathcal{C} \) and a map \( f : X \to Z \) in \( \hat{\mathcal{C}} \), we obtain a factorization \( X \xrightarrow{f^-} X^+_Z \xrightarrow{f^+} Z \) of \( f \) (that is, a 2-cell with according boundaries \( (f^+, f, f^-) \)) given via the colimit

\[
M^- = \text{colim} \left( (M \downarrow f) \to M \leftarrow \text{Fun}(\Delta^1, \hat{\mathcal{C}}) \right) .
\]

The right map \( f^+ \) is the canonical map out of the colimit into \( Z \) as in [4, Theorem 2.4.8]. The left map is induced by restriction to \( \mathcal{C} \downarrow f \subset M \downarrow f \) using that \( M \) is a pre-modulator. If \( M \) is a modulator, the object \( X^+_Z \) can be computed pointwise by
the formula
\[
(X^+_\varphi)(C) = \colim_{w \in M(C)^{op}} \Fun(\Delta^1, \hat{\mathcal{C}})(w, f)
\] (2)
as in [4, Remark 2.4.4]. For a presheaf \( X \), this in particular yields the presheaf \( X^+ \in \hat{\mathcal{C}} \) pointwise by
\[
X^+(C) = \colim_{w \in M(C)^{op}} \hat{\mathcal{C}}(\text{dom}(w), X).
\]

Transfinite iteration of the plus-construction is an explicit way to compute the (relative) sheafification of objects and arrows in terms of a given lex modulator \( M \) ([4, Theorem 3.4.2]).

Recall that every accessible left exact localization of an \( \infty \)-topos factors through an essentially unique topological localization followed by a cotopological localization ([14, Proposition 6.5.2.19, Remark 6.5.2.20]).

Given an \( \infty \)-category \( \mathcal{C} \) with pullbacks, the Čech-nerve \( \check{\mathcal{C}}(f) \) of a map \( f: E \to B \) in \( \mathcal{C} \) (if it exists) is given by the right Kan extension of the edge \( \{ f \}: (\Delta^1)^{op} \to \mathcal{C} \) to the opposite of the category \( \Delta_i \) of augmented simplicial sets along the (opposite of the) fully faithful inclusion \( \Delta_i \hookrightarrow \Delta^+ \), \( i \mapsto i-1 \) (see [14, 6.1.2]). Thus, \( \check{\mathcal{C}}(f) \) is an augmented simplicial object \( \check{\mathcal{C}}(f): \Delta_i^{op} \to \mathcal{C} \) which restricts to \( f \) on degree \( \leq 0 \), together with equivalences
\[
\check{\mathcal{C}}(f)_n \overset{\sim}{\to} E \times_B E \cdots \times_B E
\]
induced by the points \([0] \to [n] \) for all \( n \geq 1 \). The Čech-nerve \( \check{\mathcal{C}}(f) \) of a map \( f \) in an \( \infty \)-category \( \mathcal{C} \) plays the role of the kernel pair associated to a map in a 1-category. Whenever \( \mathcal{C} \) is an \( \infty \)-topos for example, it will be used to compute the \((1)\)-truncation \( f_{-1}: |\check{\mathcal{C}}(f)| \hookrightarrow B \) of \( f \) as the natural map from the geometric realization (that is, the colimit) of the underlying simplicial object of \( \check{\mathcal{C}}(f) \) which corresponds to the diagram \( \check{\mathcal{C}}(f) \) understood as a cocone.

**Lemma 2.1.** Let \( \mathcal{C} \) be a small \( \infty \)-category and \( M \) be a modulator on \( \mathcal{C} \).

1. The collection
\[
M_{-1}(C) := \{ f_{-1}: |\check{\mathcal{C}}(f)| \hookrightarrow yC \mid f \in M(C) \}
\]
of sieves obtained by \((-1)\)-truncation of the maps in \( M \) generates a Grothendieck topology \( J \) whose sheaves are exactly the \((M_{-1})\)-local objects.

2. The Grothendieck topology \( J \) consists exactly of those monomorphisms with representable codomain which are contained in the modality \( \mathcal{L} \) generated by \( M \).

**Proof.** For Part 1, since \((-1)\)-truncation in \( \check{\mathcal{C}} \) is pullback- stable, the class \( M_{-1} \) is a modulator which consists of monomorphisms. Hence, by [4, Corollary 3.4.14], the (topological) modality \( \mathcal{L}_{-1} \) it generates is left exact. We therefore obtain a Grothendieck topology \( J \) which consists of the maps in \( \mathcal{L}_{-1} \) with representable codomain. Being another generating set of \( \mathcal{L}_{-1} \), has the desired property.

For Part 2, let \( \mathcal{L} \) be the modality generated by \( M \) and \( \mathcal{M} \) be the class of monomorphisms in \( \check{\mathcal{C}} \). First, we note that \( M_{-1} \subset \mathcal{L} \) because \( \mathcal{L} \) is pullback-stable and
right cancellable ([3, Proposition 3.1.7.4]) and hence closed under \((-1)\)-truncation (via the proof of [8, Lemma 2.4]). Since \(J\) is generated from \(M_{-1}\) by its associated plus-construction ([4, Theorem 2.4.8]) and \(L\) is closed under colimits, we see that \(J \subseteq L \cap M\). In the following we show the other direction, i.e. that every monomorphism of the form \(a: A \rightarrow yC\) which is contained in \(L\) is already contained in the Grothendieck topology \(J\). Therefore, let \(a: A \rightarrow yC\) be a monomorphism in \(\hat{C}\). We obtain a canonical functor of over-categories

\[
\begin{align*}
M \downarrow a &\rightarrow J \downarrow a,
\end{align*}
\]

over the target functor into \(\hat{C}_{/yC}\) which maps a square with domain in \(M\) to its associated pullback square (whose domain is contained in \(J\) because \(a\) is monic and \(J\) is upwards closed). By definition, the plus-construction of both modulators \(M\) and \(J\) applied to \(a\) is the colimit of the target functor indexed by the respective of the two over-categories above. We therefore obtain a map of the form

\[
\iota_1: (a)^+_M \rightarrow (a)^+_J
\]

over \(yC\). Since \(J\) is a Grothendieck topology and \(a\) is monic, the plus-construction associated to \(J\) converges after one step ([4, Proposition 3.4.22]). Sheafification for \(J\) is left exact, and so \((a)^+_J\) is again a subobject of \(yC\). Let \(\lambda\) be a regular cardinal such that \((\cdot)^+_M\) converges (via [4, Theorem 2.3.4, Theorem 2.4.8]). Given \(\mu \leq \lambda\) and, recursively, maps of the form

\[
\iota_\mu: ((a)^+_M)^{(\mu)} \rightarrow (a)^+_J
\]

in \(\hat{C}_{/yC}\), we obtain associated functors

\[
\begin{align*}
M \downarrow ((a)^+_M)^{(\mu)} &\rightarrow M \downarrow (a)^+_J \rightarrow J \downarrow (a)^+_J
\end{align*}
\]

over the target fibration into \(\hat{C}_{/yC}\). Here, the right map is again obtained by mapping a square to its associated pullback square, using again that the arrow \((a)^+_J\) is monic. We obtain a map between the colimits over \(yC\) as follows.

\[
((a)^+_M)^{(\mu+1)} \rightarrow (a)^+_J \simeq (a)^+_J
\]

By ordinal recursion, we eventually obtain a map between the relative sheafifications at stage \(\lambda\). Now, whenever \(a\) is contained in \(L\), the map \((a)^+_J^{(\lambda)}\) is an equivalence over \(yC\). It follows that the monomorphism \((a)^+_J\) exhibits a section and hence is an equivalence as well. Thus, \(a \in J\).

**Corollary 2.2.** Let \(C\) be a small \(\infty\)-category, \(M\) be a modulator on \(C\) such that the localization \(\hat{C} \rightarrow \text{Sh}_M(C)\) at \(M\) is left exact, and \(J\) be the associated Grothendieck topology from Lemma 2.1.1. Then the factorization of the left exact localization \(\hat{C} \rightarrow \text{Sh}_M(C)\) into a topological localization followed by a cotopological localization is given by

\[\hat{C} \rightarrow \text{Sh}_J(C) \rightarrow \text{Sh}_M(C)\]

**Proof.** This follows from Lemma 2.1.2 along the lines of the proof of [14, Proposition 6.5.2.19]. Indeed, the only part left to show is that the latter localization is cotopological. Therefore we have to show that whenever \(f: X \rightarrow Y\) is an inclusion of \(J\)-sheaves which is mapped to an equivalence in \(\text{Sh}_M(C)\), then \(f\) was an equivalence in \(\text{Sh}_J(C)\) already. Therefore, given such an inclusion \(f\) between \(J\)-sheaves, it follows that all pullbacks to representables of \(f\) in \(\hat{C}\) are inclusions which are each
mapped to equivalences in $\text{Sh}_M(\mathcal{C})$. That means, they are all elements of the left exact modality $\mathcal{L}$ and as such, by Lemma 2.1.2, contained in $J$. It follows that all pullbacks to representables of $f$ are mapped to equivalences in $\text{Sh}_J(\mathcal{C})$, and hence so is $f$.

While lex modulators serve well the development of general theory, they yet appear somewhat impractical for the computation of concrete examples. Fortunately the condition of co-filteredness is not strictly necessary for the generated modality to be left exact. The authors of [4] introduce the notion of $\Delta$-modulators, a notion of modulator which is weaker than lexness and yet generates left exact localization by way of transfinite iteration of the associated plus-construction ([4, Theorem 3.4.16]). By definition, a modulator $M$ on $\mathcal{C}$ is a $\Delta$-modulator whenever for all $m \in M$, all higher diagonals of $m$ are contained in the modality $\mathcal{L}_M$ generated by $M$ ([4, Definition 3.4.10]). Yet, this weaker condition appears somewhat inconvenient to work with as well since the modality $\mathcal{L}_M$ is in practice a less tangible notion than $M$ itself. We therefore introduce the following intermediate variation, essentially replacing the condition for the higher diagonals to be contained in $\mathcal{L}_M$ by the stronger condition to be locally contained in $M$ itself.

**Definition 2.3.** A modulator $M$ on $\mathcal{C}$ is an Id-modulator if for every $m: X \to yC$ in $M(\mathcal{C})$ and every pair of sections $s_1, s_2$ to $m$, there is an equalizer $\text{Equ}_{yC}(s_1, s_2) \to yC$ in $\hat{\mathcal{C}}/yC$ again contained in $M(\mathcal{C})$. Elements of an Id-modulator $M$ will be referred to as M-covers.

**Example 2.4.** Grothendieck topologies on $\mathcal{C}$ are Id-modulators in trivial fashion. Indeed, the diagonal $\Delta_m: X \to X \times_{yC} X$ of a monomorphism $m: X \to yC$ is an equivalence, and as the equalizer $\text{Equ}_{yC}(s_1, s_2) \to yC$ of two sections $s_1, s_2$ to $m$ is computed by the pullback of the diagonal $\Delta_m$ along the pair $(s_1, s_2): yC \to X \times_{yC} X$, it is an equivalence itself. Thus, it is a $J$-cover trivially.

The covers $m \in M(\mathcal{C})$ of a modulator $M$ are generally not monic and thus may exhibit pairwise distinct sections if any. To be an Id-modulator for $M$ however means that any two given sections of an $M$-cover, although potentially distinct, yet agree on an $M$-cover worth of data, in the sense that the object of witnesses of their equivalence is $M$-covering.

In the following, whenever we state that a given construction $X \to yC$ over a representable, which is defined only up to equivalence, is contained in an Id-modulator $M$, we implicitly mean that there is a representative of the equivalence class of $K \to yC$ which is contained in $M$.

**Lemma 2.5.** Let $\mathcal{C}$ be a small $\infty$-category.

1. Every modulator on $\mathcal{C}$ which is fiberwise closed under finite limits (and hence lex) is an Id-modulator.

2. Every Id-modulator on $\mathcal{C}$ is a $\Delta$-modulator.

**Proof.** The first part is immediate since the equalizer of two sections of a map in $M$ over $yC$ is a finite limit of elements in $M(\mathcal{C})$ over $yC$.

For the second part, first note that, given a span $f: Y \to Z$, $g: X \to Z$ in $\hat{\mathcal{C}}$ and lifts $d_1, d_2: Y \to X$ of $f$ along $g$, we have an equivalence $\text{Equ}_Z(d_1, d_2) \simeq \text{Equ}_{yZ}(s_1, s_2) \to yZ$ again contained in $M(\mathcal{C})$. Elements of an Id-modulator $M$ are essentially the lifts $d_1, d_2: Y \to X$ of $f$ along $g$.
Equ}_Y(f^*d_1,f^*d_2)$ of equalizers computed over their respective bases.

\[ \begin{array}{ccc}
  f^*X & \longrightarrow & X \\
  \downarrow^p & & \downarrow^g \\
  f^*d_1 \downarrow^d & \longrightarrow & f^*d_2 \downarrow^d \\
  \downarrow_f & & \downarrow_g \\
  Y & \longrightarrow & Z
\end{array} \]

This follows directly from the fact that the diagonal of $f^*g$ over $Y$ is the pullback of the diagonal of $g$ over $Z$ along the canonical map $f^*X \times_Y f^*X \to X \times_Z X$.

We now show that whenever $m \in M$, then the higher diagonals $\Delta^n(m)$ are locally contained in $M$ by induction on $n$. It follows that they are contained in the associated modality $\mathcal{L}_M$ since modalities are closed under equivalences and (cartesian) colimits. For $n = 0$, the map $\Delta^0(m) = m$ is contained in $M$ by assumption, and hence so are its pullbacks to representables since $M$ is a modulator. For $n \geq 0$, assume that $\Delta^k(m)$ is locally contained in $M$ for all $k \leq n$. Then $\Delta^{n+1}(m) : X \to X \times_{t(\Delta^n(m))} X$ is by definition the diagonal $\Delta(\Delta^n(m))$. Given a map $(d_1,d_2) : yC \to X \times_{t(\Delta^n(m))} X$, we have $(d_1,d_2)^*\Delta^{n+1}(m) \simeq \text{Eq}_{yC}(d_1^*\Delta^n(m),d_2^*\Delta^n(m))$. Since both $d_1^*\Delta^n(m)$ are contained in $M(C)$ by inductive assumption, we have $(d_1,d_2)^*\Delta^{n+1}(m) \to yD$ in $M(C)$ since $M$ is an Id-modulator.

**Remark 2.6.** Say a modulator $M$ on $\mathcal{C}$ is transitive if, whenever $n : X \to Y$ is a map locally contained in $M$ (up to equivalence), and $m : Y \to yC$ is a map contained in $M(C)$, then $mn \in M(C)$ (up to equivalence). Then one can show by similar means that Lemma 2.5.1 holds in both directions, i.e. a transitive modulator $M$ on $\mathcal{C}$ is an Id-modulator if and only if it is fiberwise closed under finite limits.

We end this section with one more definition for deliberate use in the next three sections.

**Definition 2.7.** Let $\mathcal{C}$ be a locally small $\infty$-category. A left exact accessible reflective localization $\mathcal{C} \to \mathcal{E}$ is sub-canonical if the Yoneda-embedding $y : \mathcal{C} \to \mathcal{C}$ factors through the associated right-adjoint inclusion $\mathcal{E} \hookrightarrow \mathcal{C}$.

## 3 The extensive modulator

We formulate the definitions in this section for the finite case only, but everything up to Proposition 3.6 can be phrased for arbitrary regular cardinals $\kappa$ in straightforward fashion.

**Definition 3.1.** Let $\mathcal{C}$ be a small $\infty$-category with finite coproducts and pullbacks along coproduct inclusions. Given a finite collection of objects $\{C_i \in \mathcal{C} \mid i \leq n\}$, for a pair $i,j \leq n$ consider the pullback

\[ \begin{array}{ccc}
P_{i,j} & \longrightarrow & C_i \\
\downarrow & & \downarrow_{i} \\
C_j & \longrightarrow & \coprod_{i \leq n} C_i.
\end{array} \]

Coproducts in $\mathcal{C}$ are disjoint if for every such finite collection of objects and every pair $i,j \leq n$, we have $P_{i,j} \simeq \emptyset$ whenever $i \neq j$. Coproducts in $\mathcal{C}$ are universal if for any such finite collection of objects and every map $D \to \coprod_{i \leq n} C_i$, the induced map $\prod_{i \in I}(C_i \times_C D) \to D$ is an equivalence. The $\infty$-category $\mathcal{C}$ is extensive if pullbacks along coproduct inclusions exist, and coproducts in $\mathcal{C}$ are both disjoint and universal.
Corollary 3.4. Let \( \mathcal{C} \) be a small extensive \( \infty \)-category. Then the localization \( \text{Sh}_{\text{Ext}}(\mathcal{C}) \) at \( \text{Cov}_{\text{Ext}} \) is topological (and left exact). \( \square \)
Lemma 3.3, one can also directly show that the pre-modulator \( \text{Cov}_{\text{Ext}} \) is an \( \text{Id} \)-modulator whenever \( \mathcal{C} \) is extensive. Indeed, the maps \( \prod_{i \leq n} y(C_i) \to y(\prod_{i \leq n} C_i) \) in \( \text{Cov}_{\text{Ext}} \) are coproducts of monomorphisms, and hence 0-truncated each. Therefore, their diagonals are monomorphisms themselves, which when pulled back to a representable \( y(C_i \times_C C_j) \) are equivalent to the identity whenever \( i = j \) and empty otherwise. As such they are locally contained in \( \text{Cov}_{\text{Ext}} \). Furthermore, being monomorphisms, their higher diagonals vanish. In fact, \( \text{Cov}_{\text{Ext}} \) is transitive as well and hence fiberwise closed under finite limits as noted in Remark 2.6.

Proposition 3.6. Let \( \mathcal{C} \) be a small extensive \( \infty \)-category. Then the geometric inclusion \( \iota : \text{Sh}_{\text{Ext}}(\mathcal{C}) \hookrightarrow \hat{\mathcal{C}} \) preserves sifted colimits. In particular, it preserves effective epimorphisms.

Proof. We want to show that a sifted colimit of \( \text{Cov}_{\text{Ext}} \)-sheaves in \( \hat{\mathcal{C}} \) is a \( \text{Cov}_{\text{Ext}} \)-sheaf again. Therefore, recall that sheafification can be computed by a transfinite (in this case in fact an \( \omega \)-long) iteration of the plus-construction associated to the modulator \( \text{Cov}_{\text{Ext}} \) ([4, Theorem 2.4.8, Theorem 3.4.11]). Furthermore, a presheaf \( X \in \hat{\mathcal{C}} \) is a sheaf if and only if \( X^{+} \simeq X \). Thus, given a sifted simplicial set \( S \) and a diagram \( F : S \to \text{Sh}_{\text{Ext}}(\mathcal{C}) \), we want to show that the natural map

\[
\text{colim}(\iota \circ F) \to (\text{colim}(\iota \circ F))^{+}
\]

is an equivalence in \( \hat{\mathcal{C}} \). This can be verified pointwise, and so given an object \( C \in \mathcal{C} \), we have the following sequence of equivalences.

\[
(\text{colim}(\iota \circ F))^{+}(C) \simeq \text{colim}_{(X \to yC) \in \text{Cov}_{\text{Ext}}(C)} \hat{\mathcal{C}}(X, \text{colim}(\iota \circ F)) \tag{3}
\]

\[
\simeq \text{colim}_{\prod_{i \leq n} yC_i \in \text{Cov}_{\text{Ext}}(C)} \hat{\mathcal{C}}(\prod_{i \leq n} yC_i, \text{colim}(\iota \circ F)) \tag{4}
\]

\[
\simeq \text{colim}_{\prod_{i \leq n} y(C_i) \in \text{Cov}_{\text{Ext}}(C)} \prod_{i \leq n} \hat{\mathcal{C}}(y(C_i), \text{colim}(\iota \circ F)) \tag{5}
\]

Here, equivalence (3) holds because \( \text{Cov}_{\text{Ext}} \) is a modulator ([4, Remark 2.4.4]), and equivalence (4) holds because sifted colimits commute both with finite products and arbitrary other colimits ([14, Remark 5.5.8.12]). Equivalence (5) holds because we assumed \( \iota F \) to be a diagram of sheaves.

Consequently, the inclusion \( \iota : \text{Sh}_{\text{Ext}}(\mathcal{C}) \hookrightarrow \hat{\mathcal{C}} \) preserves effective epimorphisms, because by definition a map \( f \in \text{Sh}_{\text{Ext}}(\mathcal{C}) \) is an effective epimorphism if and only if it is the colimit of its Čech-nerve \( \hat{\mathcal{C}}(f) : \Delta^{\text{op}} \to \text{Sh}_{\text{Ext}}(\mathcal{C}) \). I.e., the simplicial category \( \Delta^{\text{op}} \) is sifted ([14, Lemma 5.5.8.4]), and the inclusion \( \iota \) preserves both finite limits and sifted colimits. Hence, given an effective epimorphism \( f \in \text{Sh}_{\text{Ext}}(\mathcal{C}) \), we have

\[
\iota(f) \simeq \iota(\text{colim}(\hat{\mathcal{C}}(f))) \simeq \text{colim}(\hat{\mathcal{C}}(f)) \simeq \text{colim}(\hat{\mathcal{C}}(\iota(f)))
\]
and so \( \iota(f) \in \hat{C} \) is an effective epimorphism as well.

**Corollary 3.7.** Let \( \mathcal{C} \) be a small extensive \( \infty \)-category. Then the \( \infty \)-topos \( \text{Sh}_{\text{Ext}}(\mathcal{C}) \) is hypercomplete.

**Proof.** Recall that a map in an \( \infty \)-topos is \( \infty \)-connected if and only if all its higher diagonals are effective epimorphisms (this follows from [14, Proposition 6.5.1.19]). The inclusion \( \iota : \text{Sh}_{\text{Ext}}(\mathcal{C}) \hookrightarrow \hat{\mathcal{C}} \) preserves finite limits, and we have seen in Proposition 3.6 that it preserves effective epimorphisms as well. Thus, if \( f \in \text{Sh}_{\text{Ext}}(\mathcal{C}) \) is \( \infty \)-connected, then so is \( \iota(f) \in \hat{\mathcal{C}} \). But presheaf \( \infty \)-toposes are hypercomplete (since an \( \infty \)-connected map in a presheaf \( \infty \)-category is pointwise \( \infty \)-connected and hence a (pointwise) equivalence by Whitehead’s Theorem, see [14, Remark 6.5.4.7]), and so \( \iota(f) \) is an equivalence. Thus, \( f \in \text{Sh}_{\text{Ext}}(\mathcal{C}) \) is an equivalence as well.

**Corollary 3.8.** Let \( \mathcal{C} \) be a small lextensive \( \infty \)-category, i.e. \( \mathcal{C} \) is extensive and left exact. Then the \( \infty \)-topos \( \text{Sh}_{\text{Ext}}(\mathcal{C}) \) has enough points. These are up to equivalence exactly the left exact and finite coproduct preserving functors of type \( F : \mathcal{C} \to \mathcal{S} \).

**Proof.** The first statement follows immediately from Corollary 3.7 together with [15, Corollary 3.22] and [15, Theorem 4.1]. The second statement is a standard argument via left Kan extension along the Yoneda embedding (see [14, Lemma 5.1.5.5, Proposition 5.5.4.20 and Proposition 6.1.5.2]).

### 4 The \( \infty \)-regular modulator

In ordinary category theory, a classical example of a Grothendieck topology generated by covers of single arrows is the regular topology on a small regular category ([10]). Here, an arrow \( f : E \to B \) is a cover if the natural map \( \hat{\mathcal{C}}(f) \to B \) is an equivalence. Accordingly, it is a colimit-topology as discussed in the Introduction. In Section 5 we will study this condition in a more general setting as every regular cover is a coherent cover. In this section instead, we consider a naive way to realize the ideas of proof-relevance laid out in the Introduction, implemented by simply not propositionally truncating the covers in \( \hat{\mathcal{C}} \) without contemplating their implicit diagrammatic structure. It gives a simple illustration of the fact that mere non-truncation can lead to a somewhat awkward situation, and also gives rise to the occasion to make definitions required in Section 5 either way.

Let \( \mathcal{C} \) be a small \( \infty \)-category with pullbacks. Say that a modulator \( M \) on \( \mathcal{C} \) is **representable** if there is a pullback-stable class of maps \( S \subseteq \text{Fun}(\Delta^1, \mathcal{C}) \) in \( \mathcal{C} \) which contains all identities, such that

\[
M(C) = \gamma[S(C)]
\]

for all \( C \in \mathcal{C} \). Here, \( S(C) \subseteq \mathcal{C}_{/C} \) denotes the set of objects in \( S \) with codomain \( C \). So \( M \) is representable if and only if the inclusion \( M \subseteq \sum_{C \in \mathcal{C}} \mathcal{O}_C(C) \) factors as follows.

\[
\begin{tikzcd}
M \arrow{r} & \sum_{C \in \mathcal{C}} \mathcal{C}_{/C} \arrow{r} & \sum_{C \in \mathcal{C}} \hat{\mathcal{C}}_{/yC} \arrow{d}
\end{tikzcd}
\]

Whenever \( S \subseteq \text{Fun}(\Delta^1, \mathcal{C}) \) is closed under finite limits, the modulator \( M \) represented by \( S \) is fiberwise closed under finite limits as well. An \( M \)-sheaf \( X \) is a presheaf such that \( X(f) \) is an equivalence of spaces for all \( f \in S \), and so a representable \( \gamma C \) is
an $M$-sheaf if and only if $C$ is $S$-local in $C$. In particular, if we take $S$ to consist of all arrows in $C$, then the corresponding sheaves are exactly the locally constant presheaves on $C$.

**Remark 4.1.** Whenever a given Id-modulator $M$ on $C$ is representable, the inclusion $\text{Sh}_M(C) \hookrightarrow \hat{C}$ of its associated localization preserves all colimits. This can be shown exactly along the lines of the proof of Proposition 3.6.

Pre-modulators of the form $M_S \subseteq C$ for classes of arrows $S$ in $C$ which are closed under compositions and contain the identities are considered in [4, Remark 2.4.16]. Lex modulators arising from classes $S \subseteq C$ which are closed under finite limits are considered in [4, Definition 3.4.1].

**Definition 4.2.** An $\infty$-category $C$ is regular if it is finitely complete, and the Čech nerve $\hat{C}(f)$ of every morphism $f : E \to B$ in $C$ is effective ([14, Definition 6.1.2.14]) and universal. The latter means that for every map $b : B' \to B$, the natural map $|\hat{C}(b \cdot f)| \to b^*|\hat{C}(f)|$ is an equivalence.

A map $f : E \to B$ in $C$ is an effective epimorphism if $|\hat{C}(f)| \to B$ is an equivalence. Due to effectivity of Čech nerves in a regular $\infty$-category $C$, the map $E \to |\hat{C}(f)|$ is an effective epimorphism for every $f : E \to B$ in $C$. Following the proof of [14, Proposition 6.2.3.4], one sees that the natural map $|\hat{C}(f)| \to B$ is always $(-1)$-truncated.

It follows that the class $\text{Eff} \subseteq C^{\Delta^1}$ of effective epimorphisms in a regular $\infty$-category $C$ is pullback-stable and contains all equivalences. Furthermore, the pair of effective epimorphisms and $(-1)$-truncated maps form a factorization system on $C$.

Although the class $\text{Eff} \subseteq C^{\Delta^1}$ is closed under pullbacks and compositions ([14, Proposition 6.2.3.15]), it is generally not left exact. For example, assuming that $\hat{C}$ has disjoint coproducts, one can show that a presheaf $X \in \hat{C}$ is local for all finite limits of the modulator $M_{\text{Eff}}$ represented by $\text{Eff}$ if and only if $X$ is constant (since for every object $C \in C$ the map $\emptyset \to C$ is the limit of a span of effective epimorphisms over $C$). In particular, whenever $\hat{C}$ has disjoint coproducts, the smallest left exact localization of $\hat{C}$ which inverts $M_{\text{Eff}}$ is the localization at the set of all representable maps. Thus, the notion of such “structured” regular sheaves is rather trivial.

Yet, the classical regular Grothendieck topology $J_{\text{Eff}}$ on $C$ is obtained by localizing at the $(-1)$-truncations of the maps contained in $M_{\text{Eff}}$ itself. This is due to an identification of finite limits and finite products under $(-1)$-truncation, which manifests in the fact that every modulator which consists of monomorphisms already is an Id-modulator.

Against the background that the smallest left exact localization generated by $M_{\text{Eff}}$ over a regular $\infty$-category $C$ is often trivial, we may ask what the largest (transitive) Id-modulator contained in $M_{\text{Eff}}$ is. One easily computes that this is the class of effective epimorphisms $f \in C$ such that all higher diagonals of $f$ are again effective epimorphisms.

**Definition 4.3.** A map $f$ in a regular $\infty$-category $C$ is called $\infty$-connected if all its higher diagonals (including the 0-th) are effective epimorphisms. Let $\text{Eff}_\infty \subseteq C^{\Delta^1}$ be the class of $\infty$-connected maps in $C$. An object $C$ in $\hat{C}$ is hypercomplete if $C$ is $\text{Eff}_\infty$-local.

Since effective epimorphisms are pullback-stable, and the pullback of a span of maps $f \to h \leftarrow g$ in $C$ can be computed as a composition of a pullback of $f$, $g$ and $\Delta(h)$ each (see the proof of [4, Theorem 3.4.11]), it follows that the modulator
\( M_{\text{Eff}} \) represented by \( \text{Eff}_{\infty} \) is a transitive modulator which is fiberwise left exact. By construction, it is the largest Id-modulator contained in \( M_{\text{Eff}} \).

Let \( \text{Sh}_{\text{Eff}}(\mathcal{C}) := \hat{\mathcal{C}}[\mathcal{J}_{\text{Eff}}^{-1}] \) be the \( \infty \)-topos of regular sheaves on \( \mathcal{C} \), and let \( \text{Sh}_{\text{Eff}, \infty}(\mathcal{C}) := \hat{\mathcal{C}}[(M_{\text{Eff}, \infty})^{-1}] \).

**Proposition 4.4.** Let \( \mathcal{C} \) be small and regular, and let \( J_{\infty} \) be the Grothendieck topology associated to the modulator \( M_{\text{Eff}, \infty} \) of \( \infty \)-connected maps in \( \mathcal{C} \) (via Lemma 2.1.1). Then there is a diagram of left exact localizations of the following form.

\[
\begin{array}{ccc}
\text{Sh}_{\text{Eff}}(\mathcal{C}) & \xrightarrow{(1)} & \text{Sh}_{J_{\infty}}(\mathcal{C}) \\
\hat{\mathcal{C}} & \xrightarrow{\text{top'}} & \hat{\mathcal{C}}
\end{array}
\]

Furthermore, there are small regular \( \infty \)-categories \( \mathcal{C} \) such that \( \text{Sh}_{\text{Eff}, \infty}(\mathcal{C}) \) is not sub-canonical. In particular, in this case the cotopological localization (3) is non-trivial and \( \text{Sh}_{\text{Eff}, \infty}(\mathcal{C}) \) is not a localization of \( \text{Sh}_{\text{Eff}}(\mathcal{C}) \). Furthermore, there are small regular \( \infty \)-categories such that the localizations (1) and (3) are trivial and (2) is non-trivial. Thus, in general, the two leaves on the right hand side of the diagram are incomparable localizations.

**Proof.** The fact that (1) is topological and (3) is cotopological follows directly from Corollary 2.2. The fact that every regular sheaf \( X \in \text{Sh}_{\text{Eff}}(\mathcal{C}) \) is a \( J_{\infty} \)-sheaf follows from the fact that \( J_{\infty} \) is generated by the sieves \((yf)^{-1} : \text{im}(yf) \rightarrow y\mathcal{C}\) for \( \infty \)-connected maps \( f \). Each of those is a covering sieve for the regular Grothendieck topology since \( \infty \)-connected maps are effective epimorphisms in particular. We thus obtain the localization (2). Thus, since the regular Grothendieck topology is sub-canonical, so is \( J_{\infty} \).

However, a representable \( y\mathcal{C} \in \hat{\mathcal{C}} \) is a sheaf for \( M_{\text{Eff}, \infty} \) if and only if \( \mathcal{C} \) is hyper-complete in \( \mathcal{C} \). Thus, to give an example of a small regular \( \mathcal{C} \) such that \( \text{Sh}_{\text{Eff}, \infty}(\mathcal{C}) \) is non sub-canonical, it suffices to give a small regular \( \mathcal{C} \) which exhibits a non hyper-complete object. Such is given for instance by any non-hypercomplete \( \infty \)-topos when trimmed down to small size. E.g. let \( E \) be the non-hypercomplete Dugger-Hollander-Isaksen \( \infty \)-topos ([16, 11.3]), and let \( \kappa \) be a suitable regular cardinal large enough so the full \( \infty \)-subcategory \( \mathcal{E}_{\kappa} \) of \( \kappa \)-compact objects contains both an object \( E \) and an \( \infty \)-connected map \( f \) such that \( E \) is not \( f \)-local. Then \( \mathcal{E}_{\kappa} \) is small and regular and contains a non-hypercomplete object.

However, whenever \( \mathcal{C} \) is a small regular \( n \)-category for some \( n < \infty \), the class \( \text{Eff}_{\infty} \) is trivial in \( \mathcal{C} \), and hence so are the localizations (1) and (3). Since there are many examples of ordinary regular categories such that the regular topology is non-trivial, so is (2) in those cases.

**Remark 4.5.** The points of \( \text{Sh}_{\text{Eff}}(\mathcal{C}) \) are (up to equivalence) exactly the left exact functors \( \mathcal{C} \rightarrow S \) which preserve regular epimorphisms. The points of \( \text{Sh}_{\text{Eff}, \infty}(\mathcal{C}) \) are (up to equivalence) exactly the left exact functors \( \mathcal{C} \rightarrow S \) which preserve \( \infty \)-connected maps. Thus, despite the general incompatibility of the two sheaf theories themselves, it follows that \( \text{pt}(\text{Sh}_{\text{Eff}}(\mathcal{C})) \subseteq \text{pt}(\text{Sh}_{\text{Eff}, \infty}(\mathcal{C})) \) for all small regular \( \infty \)-categories \( \mathcal{C} \).

In fact, we have the following proposition.
Proposition 4.6. 1. The topological \(\infty\)-topos \(\text{Sh}_{\text{Eff}}(C)\) of regular sheaves on a small regular \(\infty\)-category \(C\) is generally not hypercomplete. Hence, in general, it does not have enough points.

2. The \(\infty\)-topos \(\text{Sh}_{\text{Eff}}(C)\) on a small regular \(\infty\)-category \(C\) is hypercomplete, and has enough points.

Proof. For Part 1, we note that the Yoneda embedding \(y: C \to \text{Sh}_{\text{Eff}}(C)\) preserves both pullbacks and effective epimorphisms. Indeed, for an effective epimorphism \(f: E \to B\) in \(C\), the sequence \(yE \to |\hat{C}(yf)| \to yB\) factors \(yf\) in \(\hat{C}\) into an effective epimorphism followed by a \(J_{\text{Eff}}\)-local monomorphism. Since the localization \(\hat{C} \to \text{Sh}_{\text{Eff}}(C)\) preserves pullbacks and colimits, it preserves effective epimorphisms, and so the map \(yf\) is equivalent to an effective epimorphism in \(\text{Sh}_{\text{Eff}}(C)\). Hence, it preserves \(\infty\)-connected maps. Furthermore, recall that \(\text{Sh}_{\text{Eff}}(C)\) is sub-canonical. Thus, whenever \(C\) exhibits a non-hypercomplete object \(E\), then the representable \(yE\) is non-hypercomplete in \(\text{Sh}_{\text{Eff}}(C)\). As hypercompleteness is a necessary condition for an \(\infty\)-topos to have enough points ([14, Remark 6.5.4.7]), this proves Part 1.

For Part 2, hypercompleteness of \(\text{Sh}_{\text{Eff}}(C)\) follows exactly along the lines of the corresponding proof for the extensive case (via Remark 4.1, Corollary 3.7). That means, that \(\text{Sh}_{\text{Eff}}(C)\) is the hypercompletion of the topological localization \(\text{Sh}_{J_\infty}(C)\). As \(J_\infty\) is generated by single arrow covers, it is finitary, and so it follows again from [15, Corollary 3.22] and [15, Theorem 4.1] that \(\text{Sh}_{\text{Eff}}(C)\) has enough points.

\[\square\]

5 The \(\infty\)-geometric modulator

In this section we define a notion of higher covering diagram in an \(\infty\)-category with universal colimits (of restricted size \(\kappa\)), such that the \(\infty\)-category of presheaves on \(C\) which take colimits of such higher covering diagrams to limits yields a new \(\infty\)-topos \(\text{Sh}_{\text{hcd}}(C)\). We will see that every ordinary \(\kappa\)-geometric cover in \(C\) can be expressed as a higher covering diagram, and that whenever \(C\) has descent for \(\kappa\)-small diagrams, that for a diagram \(F: I \to C\) to be covering only requires \(I\) to be an \(\infty\)-category with pullbacks and \(F\) to preserve them. Whenever \(C\) is an \(\infty\)-topos, we re-derive Lurie’s definition of sheaves on an \(\infty\)-topos. We will show that there are \(\infty\)-toposes \(\mathcal{E}\) which admit ordinary geometric sheaves over themselves which are not in \(\text{Sh}_{\text{hcd}}(C)\). This indicates that hcd-sheaves play a role in \(\infty\)-category theory similar to that which geometric sheaves play in ordinary category theory, but that the two notions when considered \(\infty\)-categorically are generally different.

5.1 Motivation

Recall that an ordinary category \(C\) is \(\kappa\)-geometric if it is regular and its subobject-posets \(\text{Sub}(C)\) for \(C \in C\) have pullback-stable \(\kappa\)-small unions. A category \(C\) is geometric if it is \(\kappa\)-geometric for all cardinals \(\kappa\). Given an ordinary \((\kappa\)-geometric category \(C\), the \((\kappa\)-geometric covering sieves on \(C\) are generated by \((\kappa\)-small jointly epimorphic families \(\{F(i) \mid i < \kappa\}\) of objects over a given object \(C\). That means, a sieve \(S\) on \(C/C\) is covering whenever every map \(f: D \to C\) contained in \(S\) factors
through one of the components of \( F \).

\[
\begin{array}{ccc}
\exists ? & \rightarrow & F(i) \\
\downarrow & & \downarrow \\
D & \overset{f}{\rightarrow} & C
\end{array}
\]

The sieves \( S \) themselves record only the mere existence of the lifts, not the explicit lifts themselves. This justifies the discrete indexing of the generating covering families. If one is to record the lifts explicitly however, the diagrammatic shape of \( F \) indeed is relevant for the higher homotopical structure of the associated presheaf \((\mathcal{C}_I)^{op} \rightarrow \mathcal{S}\) which maps an arrow to the space of such lifts.

With this in mind, in this section we study an \( \text{Id} \)-modulator which consists of comparison maps

\[
\eta_F : \text{colim} yF \rightarrow y(\text{colim} F)
\]

for a class “hcd” of not necessarily discrete diagrams \( F : I \rightarrow \mathcal{C} \). Here, mere existence of a lift of a map \( yf : yD \rightarrow y(\text{colim} F) \) to \( \text{colim} yF \) corresponds exactly to the existence of a lift of \( yf \) to the sieve generated by the components \( F(i) \). But the higher homotopical structure of such lifts is generally non-trivial.

The collection of such formal colimit-diagrams will always form a modulator whenever hcd contains all diagrams indexed by some contractible simplicial set, and \( \mathcal{C} \) has the necessary pullbacks together with the necessary universal colimits. For this collection to be furthermore an \( \text{Id} \)-modulator, it suffices to assure that for a diagram \( F : I \rightarrow \mathcal{C} \) in hcd and vertices \( i, j \in I \), the pullback

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\eta} & \text{colim} F(i) \\
\downarrow & & \downarrow \Delta \\
\text{colim}(i,j) \in I \times I \left( y(F(i)) \times_{y(\text{colim} F)} y(F(j)) \right) & \xrightarrow{\eta_{ij}} & \text{colim} \end{array}
\]

is contained in the modulator as well. Thus the pullback \( F_i \times_{\text{colim} F} F_j \) is to be expressed as the colimit of a suitable diagram \( F_{ij} : I_{ij} \rightarrow \mathcal{C}_{/(F_i \times_{\text{colim} F} F_j)} \) such that equivalence of the common restrictions of \( F_i \xrightarrow{\eta_i} \text{colim} F \) and \( F_j \xrightarrow{\eta_j} \text{colim} F \) to \( F_{ij}(k) \) is not only trivially witnessed in \( \text{colim} F \), but within a component \( F(l(i,j,k)) \) together with an actual zig-zag

\[
\begin{array}{ccc}
& & F(l(i,j,k)) \\
& F_{ij}(k) \overset{-}{\searrow} & \searrow F(i) \\
\swarrow F(i) \times_{\text{colim} F} F(j) & & \swarrow F(j) \\
& F(j) \xrightarrow{\eta_j} & \text{colim} F
\end{array}
\]

in the image of \( F \). Intuitively, we want to cover each parametrized path-object \( F(i) \times_{\text{colim} F} F(j) \) of \( \text{colim} F \) by pieces so small that the homotopy of the canonical pair of maps into \( \text{colim} F \) restricted to each piece \( F_{ij}(k) \) lifts to an actual component \( F(l(i,j,k)) \) through a zig-zag between \( F(i) \) and \( F(j) \).

**Example 5.1** (Ordinarily covering diagrams). Suppose \( \mathcal{C} \) is a small \( \infty \)-category with pullbacks such that the subobject-posets \( \text{Sub}(C) \) for objects \( C \in \mathcal{C} \) have
pullback-stable $\kappa$-small unions for some regular cardinal $\kappa$. Thus, $\mathcal{C}$ is ordinarily $\kappa$-geometric if $\mathcal{C}$ is furthermore regular. The classic definition of $\kappa$-geometric covers on $\mathcal{C}$ yields a class ocd of “ordinarily covering diagrams” which consists of diagrams given by set-indexed jointly epic collections $\{F(i)\mid i < \kappa\}$ of objects in $\mathcal{C}/\mathcal{C}$ after formally closing them diagrammatically under all pullbacks over $\mathcal{C}$. Here, the zig-zags as in (7) exist trivially, given by the span $F(i) \dashv F(i) \times_C F(j) \to F(j)$ itself. We thus obtain an according Id-modulator $\text{Cov}_{\text{ocd}}$.

More precisely, consider the simplicial $\kappa$-small set
$$
\kappa^{[r]} : \Delta^{op} \to \text{Set}_\kappa
$$
and its Grothendieck construction $\sum_{[n] \in \Delta^{op}} \kappa^{[n]}$ discretely fibered over $\Delta$. Furthermore, consider functors of type
$$
F : \left( \sum_{[n] \in \Delta^{op}} \kappa^{[n]} \right) \to \mathcal{C}/\mathcal{C}
$$
(8)
such that

- the natural map
  $$
  F([n], \vec{i}) \to F([0], i_0) \times_C \cdots \times_C F([0], i_n)
  $$
  (9)
induced by the points $\{j\} : [0] \to [n]$ for $j \leq n$ is an equivalence for all $n \geq 0$ and $\vec{i} \in \kappa^{[n]}$;

- $F$ is a colimit cocone over $\mathcal{C}$.

Then the required lifts to zig-zags as in (7) exist globally on the pullbacks $F([n], \vec{i}) \times_{\text{colim}} F([m], \vec{j})$, because the whole span associated to this pullback is contained in the image of $F$.

Indeed, the Id-modulator

$$
\text{Cov}_{\text{ocd}}(\mathcal{C}) := \{ \text{colim} F \to y\mathcal{C} \mid F \text{ is a diagram in } \mathcal{C}/\mathcal{C} \text{ as above}\}
$$
consists of monomorphisms, and is the usual set of generating covering sieves for the ordinary $\kappa$-geometric Grothendieck topology on $\mathcal{C}$. This can be seen using that the colimit of the pushforward $y \circ F : \left( \sum_{[n] \in \Delta^{op}} \kappa^{[n]} \right) \to \hat{\mathcal{C}}/\mathcal{y}(\mathcal{C})$ for a functor $F$ as in (8) can be computed by the colimit of its (global) left Kan extension along the cocartesian fibration $p : \left( \sum_{[n] \in \Delta^{op}} \kappa^{[n]} \right) \to \Delta^{op}$. We thus obtain that the colimit $\text{colim} F \to y\mathcal{C}$ is the colimit of the simplicial diagram

$$
\text{Lan}_p(yF)_0 \leftarrow \text{Lan}_p(yF)_1 \leftarrow \text{Lan}_p(yF)_2 \cdots
$$
(10)
over $y\mathcal{C}$. By [14, Proposition 4.3.3.10], each $\text{Lan}_p(yF)_n$ is the colimit of the restriction of $F$ to the fiber $p^{-1}([n]) = \kappa^{[n]}$. I.e., $\text{Lan}_p(yF)_n \simeq \coprod_{\vec{i} \in \kappa^{[n]}} F([n], \vec{i})$. Using Condition (9), we see that the simplicial object (10) is equivalent to the Čech nerve of $\text{Lan}_p(yF)_0 \simeq \coprod_{i \in \kappa} yF([0], i)$ over $y\mathcal{C}$. Thus, $\text{colim} yF \simeq (\coprod_{i \in \kappa} yF([0], i))_{-1}$ over $y\mathcal{C}$, which is the covering sieve for the ordinary $\kappa$-geometric Grothendieck topology on $\mathcal{C}$ generated by the cover $\{ F_i \to C \mid i \in \kappa \}$ via [14, Lemma 6.2.3.18].
5.2 Higher covering diagrams

Recall that an ∞-category $C$ is said to be $\kappa$-closed for some regular cardinal $\kappa$ if $C$ has colimits of diagrams of size $\kappa$ or less ([14, Definition 5.3.3.1]). We follow Anel’s definition of a logos ([2], [1]) for small $\infty$-categories in the following definition. Therefore, transcribing the notation of [2], given a $\kappa$-closed $\infty$-category $C$ with pullbacks, and a $\kappa$-small diagram $F: I \to C$, furthermore recall that there is a canonical adjoint pair

$$
\begin{array}{ccc}
\mathcal{C}_{/\lim_{i \in I} F_i} & \xrightarrow{\text{rest}_F} & \mathcal{C}_{/\colim_{i \in I} F_i} \\
\downarrow & \uparrow & \downarrow \\
\mathcal{C}_{/\colim_{i \in I} F_i} & \xleftarrow{\text{glue}_F} & \lim_{i \in I} \mathcal{C}_{/F_i}.
\end{array}
$$

The colimit of $F$ is universal if the counit of (11) is a natural equivalence. The colimit of $F$ is effective if the unit of (11) is a natural equivalence. The colimit of $F$ satisfies descent if it is both universal and effective.

**Definition 5.2.** Let $\kappa$ be a regular cardinal. A $\kappa$-closed $\infty$-category $C$ is (higher) $\kappa$-geometric if it has finite limits, and $\kappa$-small colimits in $C$ are universal. The $\infty$-category $C$ is a $\kappa$-logos if furthermore $\kappa$-small colimits in $C$ are effective.

Thus $\kappa$-logoi are $\kappa$-closed $\infty$-categories with pullbacks which satisfy descent for $\kappa$-small diagrams.

**Remark 5.3.** For the vast majority of the following constructions it suffices to assume that $C$ has pullbacks rather than all finite limits in Definition 5.2. We make the additional assumption mainly just to stay within the conventions of the literature, as a terminal object is assumed both in the ordinary categorical context of geometric categories, as well as in Anel’s notion of $\infty$-logoi.

The following two lemmata and their corollary formally introduce canonical maps of the form (7) associated to suitable diagrams $F: I \to C$ which will give rise to the notion of a higher covering diagram. Lemma 5.5 introduces a natural transformation $\phi_F$ over $F \times F: I \times I \to C$ such that for every pair $i, j \in I$, the component $\phi_F(i, j) \in \mathcal{C}_{/F_i \times F_j}$ represents the cocone that we will require to be colimiting in Definition 5.8 (when computed for $F: I \to \mathcal{C}_{/\colim F}$). Lemma 5.6 shows that under suitable assumptions this natural transformation $\phi_F$ is cartesian. In Corollary 5.7 we will use this to show that under additional assumptions (which will apply to the $\infty$-category $\mathcal{C}_{/y(\colim F)}$), the components $\phi_F(i, j)$ indeed compute the pullbacks in (6). We first introduce some notation.

**Notation 5.4.** Given an $\infty$-category $\mathcal{C}$, a cofibration of $\infty$-categories $\iota: I \hookrightarrow J$ and a map $f: I \to \mathcal{C}$, let

$$
\begin{array}{ccc}
\text{Fun}_f(J, \mathcal{C}) & \xrightarrow{\iota^*} & \text{Fun}(f, \mathcal{C}) \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{\iota^*} & \text{Fun}(I, \mathcal{C})
\end{array}
$$

be the $\infty$-category of $f$-lifts to $J$.

Let $I$ be a $\kappa$-small $\infty$-category and consider the free span $\Lambda_0^2 = (1 \leftarrow 0 \to 2)$ in the category of simplicial sets. The according $\infty$-category $I^{\Lambda_0^2}$ of spans in $I$ fits into
a pullback square of the following form.

\[
\begin{array}{ccc}
I^\Lambda^2_0 & \xrightarrow{\lambda_I} & (I^\partial \Delta^1)^\Delta^1 \\
\downarrow (ev_0,ev_{(1,2)}) & \downarrow \Uparrow & \downarrow (s,t) \\
I \times I^\partial \Delta^1 & \rightarrow & I^\partial \Delta^1 \times I^\partial \Delta^1
\end{array}
\]

Thus, the associated 2-cell

\[
\begin{array}{ccc}
I^\Lambda^2_0 & \xrightarrow{ev_{(1,2)}} & I^\partial \Delta^1 \\
ev_0 & \Rightarrow & \lambda_I \\
\downarrow & \downarrow & \downarrow \\
I & \rightarrow & I^\partial \Delta^1
\end{array}
\]

is a comma square in the cartesian closed \(\infty\)-cosmos \(\text{Cat}_\infty\) ([17, 2.3.1]). It follows from [17, Proposition 5.3.9] that the induced 2-cell

\[
\begin{array}{ccc}
\text{Fun}(I^\partial \Delta^1, C) & \xrightarrow{\Delta^*} & \text{Fun}(I, C) \\
\downarrow & \downarrow & \downarrow \\
\text{Fun}(I^\partial \Delta^1, C) & \xrightarrow{ev_0^*} & \text{Fun}(I^\Lambda^2_0, C)
\end{array}
\]

(12)

satisfies the Beck-Chevalley condition with respect to the associated (global) left Kan extensions (which exist because \(C\) has all \(\kappa\)-small colimits). Explicitly, the 2-cell \(\mu\) is given by the composite natural transformation \(\mu\) in the square

\[
\begin{array}{ccc}
\text{Fun}((I^\partial \Delta^1)^\Delta^1, C^\Delta^1) & \xrightarrow{\lambda^*_I} & \text{Fun}(I^\Lambda^2_0, C^\Delta^1) \\
\downarrow (\Delta^1) & \downarrow \mu & \downarrow (s,t) \\
\text{Fun}(I^\partial \Delta^1, C) & \xrightarrow{(ev_0^*\Delta^*, ev_{(1,2)}^*)} & \text{Fun}(I^\Lambda^2_0, C) \times \text{Fun}(I^\Lambda^2_0, C).
\end{array}
\]

Additionally, the 2-cell \(\lambda_I\) is natural in \(I\), i.e. for a functor \(F: I \rightarrow C\), the cube

\[
\begin{array}{ccc}
F^\Lambda^2_0 & \xrightarrow{\lambda_F} & (F^\partial \Delta^1)^\Delta^1 \\
\downarrow (ev_0,ev_{(1,2)}) & \downarrow \Uparrow & \downarrow (s,t) \\
C \times C^\partial \Delta^1 & \rightarrow & C^\partial \Delta^1 \times C^\partial \Delta^1
\end{array}
\]

commutes.

Furthermore, if \(C\) is an \(\infty\)-category with finite products, we obtain a product functor \(\times: C \times C \rightarrow C\) together with an additional homotopy-cartesian square

\[
\begin{array}{ccc}
C^\Lambda^2_0 & \xrightarrow{\pi} & C^\Delta^1 \\
\downarrow (ev_0,ev_{(1,2)}) & \downarrow \Uparrow & \downarrow (s,t) \\
C \times C^\partial \Delta^1 & \rightarrow & (1,\times) \rightarrow C
\end{array}
\]
where the top arrow $\pi$ maps a span of the form $c \leftarrow a \rightarrow b$ to the universal map $a \rightarrow b \times c$. The composite map

$\text{diag}_C: C \xrightarrow{\xi} C^{\Delta^2_0} \xrightarrow{\pi} C^{\Delta^1}$

takes an object $C \in C$ to its diagonal $\Delta^1_C: C \rightarrow C \times C$. The concatenation of homotopies $(\text{diag}_C \circ \text{ev}_0) \ast (\times \Delta^1_0 \circ \lambda_C): C^{\Delta^2_0} \rightarrow C^{\Delta^2}$ is pointwise at a span $b \leftarrow a \rightarrow c$ given by (the essentially unique) 2-cell

$\lambda_C(f,g) = f \times g \quad \Rightarrow \quad b \times c$ \quad (14)

Its 1-boundary computes to be equivalent to $\pi$ as it comes together with the determining 2-cells to $f$ and $g$, respectively, when postcomposed with the respective projections.

Lastly, given a functor $F: I \rightarrow C$, the diagonal $\text{diag}_C$ induces a natural transformation $\pi \circ (\Lambda^2_0)^* \circ \text{ev}_0^*\circ F \circ \Delta^0_0: I \rightarrow C^{\Delta^1}$ from $F$ to the composition

$I \xrightarrow{\Delta} I \partial \Delta^0 \xrightarrow{\dim F \partial \Delta^0} C \partial \Delta^0 \xrightarrow{\times \dim F} C.$

Whenever $C$ has both $I$- and $I \times I$-indexed colimits, the corresponding composite cocone

$I \times \Delta^0 \xrightarrow{\text{dim} F \partial \Delta^0} (I \times I) \ast \Delta^0 \xrightarrow{\text{colim}(F \times F)} C$

induces a natural map $\gamma_\Delta: \text{colim}_I \Delta^* (F \times F) \rightarrow \text{colim}_{I \times I} (F \times F)$ in $C$.

**Lemma 5.5.** Let $\kappa$ be a regular cardinal. Let $C$ be a $\kappa$-closed $\infty$-category with finite products. Let $I$ be a $\kappa$-small $\infty$-category and $F: I \rightarrow C$ a diagram. Then there is a natural transformation $\phi_F: \text{Lan}_{\text{ev}_{(1,2)}}(F \circ \text{ev}_0) \rightarrow F \times F$ which can be computed pointwise as the natural map

$\phi_F(i,j): \text{colim}_k F_k \xrightarrow{\text{colim}_k (F_{\alpha}, F_{\beta})} F_i \times F_j$

that represents the cocone

$\text{Fun}_{(i,j)}(\Lambda^2_0, I) \xrightarrow{\text{ev}^*_{(i,j)}} \text{Fun}_{(F_i, F_j)}(\Lambda^2_0, C) \xrightarrow{\pi_{(F_i, F_j)}} \text{C}_{/F_i \times F_j}.$

It comes together with a 2-cell in $C$ of the form

$\text{colim}_{I \times I} (\text{Lan}_{\text{ev}_{(1,2)}}(F \circ \text{ev}_0)) \xrightarrow{\varnothing} \text{colim}_I F \xrightarrow{\text{colim}_I (\text{dim}_F)} \text{colim}_I \Delta^* (F \times F) \xrightarrow{\gamma_\Delta} \text{colim}_{I \times I} (F \times F)$. 

20
Proof. Consider the following pasting diagram in the $\infty$-cosmos $\text{Cat}_\infty$ of small $\infty$-categories.

\[
\begin{array}{cccccc}
\text{Fun}(I^{\partial\Delta^1}, C) & \xrightarrow{\epsilon \circ \mu \circ \eta} & \text{Fun}(\Delta^*, \text{Lan}_{\Delta}(F)) & \xrightarrow{\epsilon \circ \mu \circ \eta} & \text{Fun}(\Delta^*, \text{Lan}_{\Delta}(F)) \\
\text{Fun}(I, C) & \xrightarrow{\infty} & \text{Fun}(I^{\partial\Delta^1}, C) & \xrightarrow{\infty} & \text{Fun}(I^{\partial\Delta^1}, C) & \xrightarrow{\epsilon \circ \mu \circ \eta} & \text{Fun}(I^{\partial\Delta^1}, C) \\
\text{Fun}(I, C) & \xrightarrow{\infty} & \text{Fun}(I, C) & \xrightarrow{\infty} & \text{Fun}(I, C) & \xrightarrow{\infty} & \text{Fun}(I, C) \\
\text{Fun}(I, C) & \xrightarrow{\infty} & \text{Fun}(I, C) & \xrightarrow{\infty} & \text{Fun}(I, C) & \xrightarrow{\infty} & \text{Fun}(I, C) \\
\end{array}
\]

The left half of the diagram is filled by the 3-cell given by the triangle identity of the adjunction $\text{Lan}_{\Delta} \circ \Delta^*$. The right half of the diagram is filled by the obvious degenerate 3-cell associated to the 2-cell $\epsilon_{(1,2)} \circ \mu$.

The composition of the front square 2-cells $\kappa := \epsilon_{(1,2)} \circ \mu \circ \eta \Delta$ is the mate of $\mu$ in (12), and hence a natural equivalence in virtue of the associated Beck-Chevalley condition. We obtain two homotopy-commutative squares in $\text{Fun}(I^{\partial\Delta^1}, C)$ as follows.

\[
\begin{array}{cccc}
\text{Lan}_{\text{ev}(1,2)} \circ \text{ev}_0^*(F) & \xrightarrow{\pi} & \text{Lan}_\Delta(F) & \\
\text{Lan}_{\text{ev}(1,2)} \circ \text{ev}_0^*(\Delta^*(F \times F)) & \xrightarrow{\pi} & \text{Lan}_\Delta(\Delta^*(F \times F)) & \\
\text{Lan}_{\text{ev}(1,2)} \circ \text{ev}_0^*(\Delta^*(F \times F)) & \xrightarrow{\pi} & \text{Lan}_\Delta(\Delta^*(F \times F)) & \\
\end{array}
\]

The upper square commutes, because $\kappa : \text{Fun}(I, C) \to \text{Fun}(I^{\partial\Delta^1}, C)$ is a natural equivalence. The 2-cell making the bottom square commute comes from the composite 3-cell (18) applied to $F \times F \in \text{Fun}(I^{\partial\Delta^1}, C)$. The bottom-left composite defines the natural transformation $\phi_F$.

Let us first construct the associated 2-cell of colimits in $C$. Thus, pushforward along the colimit functor $\text{colim} : \text{Fun}(I^{\partial\Delta^1}, C) \to C$ yields the following diagram in $C$.

\[
\begin{array}{cccc}
\text{colim}(\text{Lan}_{\text{ev}(1,2)} \circ \text{ev}_0^*(F)) & \xrightarrow{\text{colim}(\kappa_F)} & \text{colim}(\text{Lan}_\Delta F) & \xrightarrow{\text{colim}(\epsilon \circ \mu)} & \text{colim}F \\
\text{colim}(\text{Lan}_\Delta \circ \Delta^*(F \times F)) & \xrightarrow{\text{colim}(\epsilon \circ \mu \circ \eta \Delta)} & \text{colim}(\Delta^*(F \times F)) & \\
\text{colim}(F \times F) & \xrightarrow{\text{colim}(\epsilon \circ \mu \circ \eta \Delta)} & \text{colim}(\Delta^*(F \times F)) & \\
\end{array}
\]

Here, the upper right square comes from the fact that global left Kan extensions commute with the respective colimit functors. The lower right triangle commutes by definition of $\gamma \Delta$ since the vertical arrow $\text{colim}(\epsilon \Delta(F \times F))$ represents exactly to the cocone (16).

Second, to compute $\phi_F$ pointwise, for $(i, j) \in I \times I$, consider the following
proof.

Let \( \mathcal{C} \) be \( \kappa \)-geometric. Let \( I \) be a \( \kappa \)-small \( \infty \)-category with pullbacks. Whenever \( F : I \to \mathcal{C} \) is a pullback preserving functor, the natural transformation \( \phi_F : \text{Lan}_{ev \circ \mu}(F) \to F \times F \) from Lemma 5.5 is a cartesian natural transformation.

**Proof.** Given an arrow \((\alpha, \beta) : (i', j') \to (i, j)\) in \( I \times I\), the induced pushforward

\[
\Sigma_{(\alpha, \beta)} : \text{Fun}(i', j'; \Lambda^2_0, I) \to \text{Fun}(i, j; \Lambda^2_0, I)
\]

has a right adjoint \((\alpha, \beta)^*\) which maps a span \(i \leftarrow k \to j\) to the limit \((k \times_i i') \times_k (k \times_j j')\). Since \(F : I \to \mathcal{C}\) preserves pullbacks, the square

\[
\begin{array}{ccc}
\text{Fun}(i, j; \Lambda^2_0, I) & \xrightarrow{F^\Lambda^2_0(i,j)} & \text{Fun}(F_i, F_j; \Lambda^2_0, \mathcal{C}) \\
(\alpha, \beta)^* & \downarrow \cong_F & (F(\alpha), F(\beta))^* \\
\text{Fun}(i', j'; \Lambda^2_0, I) & \xrightarrow{F^\Lambda^2_0(i',j')} & \text{Fun}(F_{i'}, F_{j'}; \Lambda^2_0, \mathcal{C})
\end{array}
\]

commutes up to equivalence. As \( \kappa \)-small colimits in \( \mathcal{C} \) are universal and right adjoints are cofinal (in the sense of [14, 4.1.1], classically called “final” instead), via

Diagram of \( \infty \)-categories.

The 2-cell \( \phi_F : \text{Lan}_{(1,2)}(F \circ ev_0) \to F \times F \) is by definition the transpose of the pasting \( \mu_{F \times F \star \text{diag}_F} : F \circ ev_0 \to F \times F \circ ev_{(1,2)}^* \). This pasting, as a natural transformation, is equivalent to the composition \( \pi \circ F^{\Lambda^2_0} \) on the upper half of Diagram (19) by (14) and (15), (18). Since

\[
\text{Lan}_{(1,2)}(F \circ ev_0) \circ (i, j) \simeq \text{Lan}_{(i,j)^*\circ ev_{(1,2)}}(F \circ ev_0 \circ \iota) \simeq \text{colim}(F \circ ev_0 \circ \iota),
\]

it follows that the precomposition

\[
\phi_F(i, j) \simeq \phi_F \circ (i, j) : \text{colim}(F \circ ev_0 \circ \iota) \to F_i \times F_j
\]

is the transpose of the cocone \( (\mu_{F \times F \star \text{diag}_F}) \circ \iota : F \circ ev_0 \circ \iota \to F_i \times F_j \). Hence, this cocone is equivalent to the restriction

\[
\pi_{i(F_i, F_j)} \circ F^{\Lambda^2_0(i,j)} : \text{Fun}(i, j; \Lambda^2_0, I) \to \mathcal{C}_{i \times j}.
\]

\[\square\]

**Lemma 5.6.** Let \( \mathcal{C} \) be \( \kappa \)-geometric. Let \( I \) be a \( \kappa \)-small \( \infty \)-category with pullbacks. Whenever \( F : I \to \mathcal{C} \) is a pullback preserving functor, the natural transformation \( \phi_F : \text{Lan}_{ev_{(1,2)}} \circ ev_0^*(F) \to F \times F \) from Lemma 5.5 is a cartesian natural transformation.

**Proof.** Given an arrow \((\alpha, \beta) : (i', j') \to (i, j)\) in \( I \times I\), the induced pushforward

\[
\Sigma_{(\alpha, \beta)} : \text{Fun}(i', j'; \Lambda^2_0, I) \to \text{Fun}(i, j; \Lambda^2_0, I)
\]

has a right adjoint \((\alpha, \beta)^*\) which maps a span \(i \leftarrow k \to j\) to the limit \((k \times_i i') \times_k (k \times_j j')\). Since \(F : I \to \mathcal{C}\) preserves pullbacks, the square

\[
\begin{array}{ccc}
\text{Fun}(i, j; \Lambda^2_0, I) & \xrightarrow{F^\Lambda^2_0(i,j)} & \text{Fun}(F_i, F_j; \Lambda^2_0, \mathcal{C}) \\
(\alpha, \beta)^* & \downarrow \cong_F & (F(\alpha), F(\beta))^* \\
\text{Fun}(i', j'; \Lambda^2_0, I) & \xrightarrow{F^\Lambda^2_0(i',j')} & \text{Fun}(F_{i'}, F_{j'}; \Lambda^2_0, \mathcal{C})
\end{array}
\]

commutes up to equivalence. As \( \kappa \)-small colimits in \( \mathcal{C} \) are universal and right adjoints are cofinal (in the sense of [14, 4.1.1], classically called “final” instead), via
Lemma 5.5 we obtain a cartesian square of the form

\[
\begin{array}{c}
\text{colim}_{i \leftarrow k \rightarrow j} F_k' \\
F_{i'} \times F_{j'} \rightarrow F_i \times F_j
\end{array}
\]

\[\phi_F(i',j') \quad \phi_F(i,j)
\]

\[
\begin{array}{c}
\Delta_{\text{colim} F} \\
\pi_1 \rightarrow \pi_2
\end{array}
\]

Corollary 5.7. Let \( \mathcal{C} \) be a \( \kappa \)-logos. Let \( I \) be a \( \kappa \)-small \( \infty \)-category with pullbacks and \( F : I \to \mathcal{C} \) be a pullback preserving functor. Then for all \( i, j \in I \), there is a cartesian square of the form

\[
\begin{array}{c}
\text{colim}_{i \leftarrow k \rightarrow j} F_k \\
F_i \times F_j \rightarrow \text{colim}_{I} F \times \text{colim}_{I} F.
\end{array}
\]

Proof. The natural transformation \( \phi_F : \text{Lan}_{\text{ev}}(1,2) F \circ \text{ev}_0 \to F \times F \) in \( \text{Fun}(I^{0\Delta^1}, \mathcal{C}) \) is cartesian by Lemma 5.6. Furthermore, Lemma 5.5 shows that the pointwise colimit of \( \phi_F \) is the vertical map \( \gamma_\Delta \circ \text{colim}_I (\text{diag}_F) : \text{colim}_I F \to \text{colim}_{I \times I}(F \times F) \).

Whenever \( \mathcal{C} \) has universal \( \kappa \)-small colimits, the latter is equivalent to the diagonal of \( \text{colim}_I F \), since both

\[
\text{colim}_I F \xrightarrow{\gamma_\Delta \circ \text{colim}_I (\text{diag}_F)} \text{colim}_{I \times I}(F \times F) \xrightarrow{\pi_1 \pi_2} \text{colim}_I F
\]

compose to the identity. By descent, it follows that the squares (20) are cartesian.

Given a small \( \kappa \)-geometric \( \infty \)-category \( \mathcal{C} \), the Yoneda embedding \( y : \mathcal{C} \to \hat{\mathcal{C}} \) preserves finite limits and \( \hat{\mathcal{C}} \) has descent (as do its slices). Thus, by Corollary 5.7 we see that pullback preserving functors \( F : I \to \mathcal{C} \) from small \( \infty \)-categories with pullbacks yield an \( \text{Id} \)-modulator on \( \mathcal{C} \) as described in Section 5.1 whenever the natural map

\[
\phi_F(i, j) : \text{colim}_{i \leftarrow k \rightarrow j} F_k \to F_i \times \text{colim}_F F_j
\]

associated to the colimit cocone \( F : I \to \mathcal{C}/\text{colim}F \) is an equivalence, and whenever this condition is furthermore stable under diagonals. That means, we have to require this condition for iterated spans and iterated pullbacks of components over \( \text{colim}F \) as well. We therefore make the following definitions.

Let \( S^\infty \) be the poset generated by the diagram

\[
\begin{array}{c}
x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \\
y_0 \leftarrow y_1 \leftarrow y_2 \leftarrow \cdots
\end{array}
\]

Let \( S^n \) be the truncation of \( S^\infty \) at stage \( n \), and \( D^{n+1} \) be the join \( \Delta^0 \ast S^n \); that is, the poset given as follows.

\[
\begin{array}{c}
x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n \leftarrow x_{n+1} \\
y_0 \leftarrow y_1 \leftarrow y_2 \leftarrow \cdots \leftarrow y_n
\end{array}
\]
Note that for all \( n \geq 0 \),
\[
S^{n+1} \cong S^0 \ast S^n \cong S^n \ast S^0 \cong D^{n+1} \cup S^n \ D^{n+1},
\]
(21)
\[
D^{n+1} \cong D^n \ast S^0.
\]

We obtain canonical inclusions \( \iota: S^n \hookrightarrow S^m \) for \( n \leq m \), \( \iota_n: S^n \hookrightarrow D^{n+1} \) given by the obvious inclusions \( S^n \hookrightarrow S^m \ast S^n \), \( S^n \hookrightarrow \Delta^0 \ast S^n \). We furthermore obtain inclusions \( \iota_1: S^n \rightarrow S^{n+1} \) given by the obvious inclusions \( S^n \hookrightarrow S^n \ast S^0 \).

Given an \( n \)-dimensional path \( p: S^n \rightarrow I \) in an \( \infty \)-category \( I \), we will make extensive use of the \( \infty \)-category of disc-extensions \( \text{Fun}_p(D^{n+1}, I) \) defined as in Notation 5.4 with respect to the sphere-inclusion \( \iota_n: S^n \hookrightarrow D^{n+1} \). Equivalently, \( \text{Fun}_p(D^{n+1}, I) \) is the over-category \( I/p \) defined in [14, 1.2.9].

For every such path \( p: S^n \rightarrow I \) in a \( \kappa \)-small \( \infty \)-category \( I \), and every diagram \( F: I \rightarrow C \) into an \( \infty \)-category with \( \kappa \)-small colimits and pullbacks, we make the following two observations. First, the limit of the composition \( F \circ \iota_n: S^n \rightarrow C/\text{colim}F \) exists and can be computed by the iterated pullback
\[
F(p(x_n)) \times \left( F(p(x_{n-1})) \times \cdots \times F(p(y_{n-1})) \right)
\]
(22)
This can be seen by a proof by induction along the dimension \( n \). Second, the diagram
\[
\begin{CD}
\text{Fun}_p(D^{n+1}, I) @>{\text{ev}}_{x_{n+1}}>> I @>>> C/\text{colim}F \\
\downarrow {\text{Fun}_Fp(\Delta^0 \ast S^n, C/\text{colim}F)} @>{\text{ev}}_{x_{n+1}}>> F @>>> C/\text{colim}F
\end{CD}
\]
commutes, where the bottom vertex \( \text{Fun}_Fp(\Delta^0 \ast S^n, C/\text{colim}F) \) is the \( \infty \)-category of cones over the composition \( F \circ \iota \), and as such is equivalent to the overcategory \( (C/\text{colim}F)/\text{lim}F \simeq C/\text{lim}F \). Since this \( \infty \)-category has \( \kappa \)-small colimits as well and \( \text{ev}_{x_{n+1}} \) both preserves and reflects them, we obtain
\[
\text{colim}F \ast \in C/\text{lim}F
\]
(23)
with \( \text{ev}_{x_{n+1}}(\text{colim}F \ast) \simeq \text{colim}(F \circ \text{ev}_{x_{n+1}}) \). Note that in the case \( n = 0 \), for any \( p = (i, j): S^0 \rightarrow I \) the element \( \text{colim}F \ast \in C/F_i \ast \text{colim}F_j \) is given by \( \phi_F(i, j) \) from Lemma 5.5 applied to \( F: I \rightarrow C/\text{colim}F \).

**Definition 5.8.** Let \( C \) be a \( \kappa \)-closed \( \infty \)-category with pullbacks, and \( I \) be a \( \kappa \)-small \( \infty \)-category with pullbacks. Let \( F: I \rightarrow C \) be a functor which preserves pullbacks. Say that \( F \) is a higher covering diagram if for all \( n \geq 0 \), and all paths \( p: S^n \rightarrow I \), the vertex in (23) is a terminal object.

Thus, equivalently, a pullback preserving diagram \( F: I \rightarrow C \) from a \( \kappa \)-small \( \infty \)-category \( I \) with pullbacks is higher covering whenever the natural map
\[
\text{colim} \left( \text{Fun}_p(D^{n+1}, I) \xrightarrow{\text{ev}_{x_{n+1}}} I \xrightarrow{F} C \right) \rightarrow (22)
\]
given by (23) is an equivalence in \( C \).

**Remark 5.9.** The case \( n = -1 \) in Definition 5.8 is trivial (here note that the limit of \( F \circ \iota \) is computed for \( F: I \rightarrow C/\text{colim}F \) in the slice \( C/\text{colim}F \)), so without loss of generality one may require the condition on the map (24) to be an equivalence for all \( n \geq -1 \).
Examples 5.10. Suppose \( \mathcal{C} \) is a \( \kappa \)-closed \( \infty \)-category with pullbacks for some regular cardinal \( \kappa \).

1. Whenever a \( \kappa \)-small \( \infty \)-category \( \mathcal{I} \) has both pullbacks and non-empty finite products, and \( F: \mathcal{I} \to \mathcal{C} \) is a pullback preserving functor, then \( F \) is a higher covering diagram if and only if \( F: \mathcal{I} \to \mathcal{C}/\text{colim}F \) preserves products. That is, because in this case for every \( p: S^n \to \mathcal{I} \), the \( \infty \)-category \( \text{Fun}_p(D^{n+1}, I) \) has a terminal object, given by the iterated pullback
   \[
   p(x_n) \times_{p(x_0) \times p(y_0)} p(y_n).
   \]
   Hence, to be higher covering means exactly to preserve these limits. In particular, if \( \mathcal{I} \) has all finite limits and hence a terminal object \( t \), then \( \text{colim}F \simeq F(t) \) and so every pullback preserving functor \( F: \mathcal{I} \to \mathcal{C} \) is higher covering.

2. The identity functor on \( \mathcal{C} \) admits a colimit if and only if \( \mathcal{C} \) has a terminal object. In this case, \( \mathcal{C} \) has all finite limits, and hence by Example 1, the identity on \( \mathcal{C} \) is a higher covering diagram (apart from the size restrictions).

3. Every \( \kappa \)-small discrete diagram \( F: \mathcal{I} \to \mathcal{C} \) is higher covering if and only if \( \mathcal{C} \) has disjoint \( \kappa \)-small coproducts and coproduct inclusions in \( \mathcal{C} \) are monic. Thus, \( \kappa \)-small discrete diagrams are higher covering in \( \kappa \)-extensive \( \infty \)-categories.

4. Whenever \( \mathcal{I} \) is a \( \kappa \)-small poset with finite meets, a functor \( F: \mathcal{I} \to \mathcal{C} \) is higher covering if and only if it factors through a meet-preserving morphism of posets
   \[
   \begin{array}{ccc}
   \text{Sub}(\text{colim}F) & \to & \mathcal{C}/\text{colim}F \\
   \downarrow & & \downarrow \\
   \mathcal{I} & \to & \mathcal{C}/\text{colim}F.
   \end{array}
   \]
   This follows from Example 1 and the fact that for all \( i \in \mathcal{I} \) the degenerate triple \( i = i = i \) is terminal in the category of spans \( i \leftarrow k \to i \) whenever \( \mathcal{I} \) is a poset, so \( F_i \cong F_i \times_{\text{colim}F} F_i \) if \( F \) is higher covering.

5. Whenever \( \mathcal{C} \) is an \( \infty \)-groupoid, every diagram \( F: \mathcal{I} \to \mathcal{C} \) is pullback preserving. As \( \mathcal{C} \) is \( \kappa \)-closed, it has an initial object and hence is contractible. It follows that every diagram in \( \mathcal{C} \) whose domain has pullbacks is higher covering. Even if \( \mathcal{C} \) is not assumed to be \( \kappa \)-closed however, but rather to have enough coproducts (and hence, having all pushouts, to have all colimits necessary) to define what it means for a diagram \( F: \mathcal{I} \to \mathcal{C} \) to be higher covering, the same is true, since its slices are contractible and so the object (23) is always terminal trivially.

Theorem 5.11. Let \( \kappa \) be a regular cardinal and \( \mathcal{C} \) be a small \( \kappa \)-geometric \( \infty \)-category. Let

\[
\text{Cov}_{\text{hcd}}(\mathcal{C}) := \{ y^*c_F: \text{colim}yF \to y\mathcal{C} | I \text{ is } \kappa\text{-small with pullbacks}, \]
\[
F: I \to \mathcal{C} \text{ is a higher covering diagram,} \\
\text{c}_F: F \to \mathcal{C} \text{ is a colimiting cocone}, \}
\]

where each \( y^*c_F \) denotes a fixed representative for its corresponding equivalence class in \( \hat{\mathcal{C}}/y\mathcal{C} \). Then \( \text{Cov}_{\text{hcd}}(\mathcal{C}) \) is an Id-modulator on \( \mathcal{C} \).

Proof. \( \text{Cov}_{\text{hcd}}(\mathcal{C}) \) is a set by virtue of the cardinal boundaries. It is a pre-modulator since the constant functors of type \( \Delta^0 \to \mathcal{C} \) with any value \( C \in \mathcal{C} \) are higher covering.
diagrams. To show that it is a modulator one easily verifies, given a higher covering diagram $F: I \to C$ together with a colimit cocone $c_F: I \to C/I$ of $F$, that for all $f: D \to C$ the composition $\pi_D \circ f^* \circ c_F: I \to C/I \to C/D \to C$ given by pulling back pointwise is higher covering again because $C$ has the necessary universal colimits. The composition $f^* \circ c_F: I \to C/I \to C/D$ is a colimit cocone of $\pi_D \circ f^* \circ c_F$, again because $C$ has the necessary universal colimits.

To show that it is an Id-modulator, we have to show that the diagonal of a diagram $y \circ c_F: I \to \hat{C}_{/yC}$; this holds whenever for all $i,j \in I$ the composition

$$\text{Fun}_{(i,j)}(D^1, I) \xrightarrow{\text{ev}_{x_1}} I \xrightarrow{F \circ } C$$

is higher covering again. Thus, first note that $\text{Fun}_{(i,j)}(D^1, I)$ has pullbacks whenever $I$ does, and that $\text{ev}_{x_1}$ preserves pullbacks. In particular, so does the composition (25). For $p \in \text{Fun}(S^n, \text{Fun}_{(i,j)}(D^1, I))$ consider the natural map from the colimit of the diagram (26). For $p \in \text{Fun}(S^n, \text{Fun}_{(i,j)}(D^1, I))$ consider the natural map from the colimit of the diagram

$$\text{Fun}_p(D^{n+1}, \text{Fun}_{(i,j)}(D^1, I)) \xrightarrow{\text{ev}_{x_{n+1}}} \text{Fun}_{(i,j)}(D^1, I) \xrightarrow{\text{ev}_{x_1}} I \xrightarrow{F \circ } C$$

into the iterated pullback

$$F(\text{ev}_{x_1}(p(x_n))) \times \left( F(\text{ev}_{x_1}(p(x_{n-1}))) \times \cdots \times \text{colim}(F \circ \text{ev}_{x_1}) \right)$$

(27)

There is an isomorphism of the form

$$\text{Fun}_p(D^{n+1}, \text{Fun}_{(i,j)}(D^1, I)) \xrightarrow{\text{ev}_{x_{n+1}}} \text{Fun}_{(i,j)}(D^1, I) \xrightarrow{\text{ev}_{x_1}} I$$

$$\downarrow \cong$$

$$\text{Fun}_q(D^{n+2}, I) \xrightarrow{\text{ev}_{x_{n+2}}}$$

for $q := p(i,j): S^{n+1} \to I$ a path which restricts to $(i,j): S^0 \to I$ on $i: S^0 \hookrightarrow S^{n+1}$, and such that the square

$$\begin{array}{ccc}
S^n & \xrightarrow{P} & \text{Fun}_{(i,j)}(D^1, I) \\
\downarrow_{i+1} & & \downarrow_{\text{ev}_{x_1}} \\
S^{n+1} & \xrightarrow{q} & I
\end{array}$$

(28)

commutes. This can be seen to exist via a series of transpositions and the calculations in (21), using that the join $P \ast S^0$ (computed via the alternative join, [14, 4.2.1], which is the same on posets $P$) can be constructed as the coequalizer of the pair

$$\begin{array}{ccc}
P \times S^0 & \xrightarrow{1 \times i_{t_0}} & P \times D^1 \\
\downarrow_{((a,b) \circ 1_P) \times t_0} & & \downarrow_{P \times D^1}
\end{array}$$

for any given pair of points $a,b \in P$. We apply this to $P = D^{n+1}, S^n$ and $a,b = x_0, y_0$. It follows that the colimit of (26) is equivalent to

$$\text{colim} \left( \text{Fun}_q(D^{n+2}, I) \xrightarrow{\text{ev}_{x_{n+2}}} I \xrightarrow{F \circ } C \right).$$

(29)
Second, the iterated pullback (27) is equivalent to
\[ F(q(x_{n+1})) \times \left( F(q(x_n)) \times F(q(y_n)) \right) F(q(y_{n+1})) \]
via the square (28) and the fact that \( F: I \to C \) is higher covering, so \( \text{colim}(F \circ ev_{x_1}) \simeq F(i) \times \text{colim}_F F(j) \). It follows that the canonical map from (26) to (27) is equivalent to the canonical map from (29) to (30), and hence is an equivalence again because \( F \) is higher covering.

Given a small \( \kappa \)-geometric \( \infty \)-category \( C \), we thus obtain the \( \infty \)-topos
\[ \text{Sh}_{\text{hcd}}(\kappa, C) := \hat{C}[\text{Cov}_{\text{hcd}}^{-1}] \]
of sheaves for \( \kappa \)-small higher covering diagrams in \( C \). We will refer to its elements as the higher \( \kappa \)-geometric sheaves on \( C \). By construction, a presheaf \( X: C^{\text{op}} \to S \) is higher \( \kappa \)-geometric if and only if it takes colimits of \( \kappa \)-small higher covering diagrams to limits in \( S \).

**Remark 5.12.** Let \( C \) be a small \( \kappa \)-geometric \( \infty \)-category. We make the following list of observations.

1. The localization \( \hat{C} \to \text{Sh}_{\text{hcd}}(\kappa, C) \) is sub-canonical.
2. The canonical indexing \( (C/ \_ )^\sim: C^{\text{op}} \to S \) is higher \( \kappa \)-geometric whenever \( C \) is a \( \kappa \)-logos. This is analogous to the fact that the presheaf \( C/ \_ : C^{\text{op}} \to \text{Cat} \) is a stack for the \( \kappa \)-geometric Grothendieck topology on a \( \kappa \)-geometric category \( C \) whenever \( C \) is a \( \kappa \)-pretopos (i.e. whenever \( C \) is furthermore \( \kappa \)-extensive and exact).
3. Whenever \( C \) has disjoint coproducts, it is \( \kappa \)-extensive, and so every \( \text{hcd}_{\kappa} \)-sheaf is a \( \kappa \)-extensive sheaf by Example 5.10.3.

For uncountable regular cardinals \( \kappa \), a small \( \kappa \)-geometric \( \infty \)-category \( C \) automatically satisfies the conditions of Example 5.1. In particular, it is ordinarily \( \kappa \)-geometric wheneverČech nerves in \( C \) are effective. We may hence equip any such \( C \) with the \( \kappa \)-geometric Grothendieck topology given by the modulator \( \text{Cov}_{\text{geo}} \) considered in Example 5.1.

**Proposition 5.13.** Let \( \kappa \) be an uncountable regular cardinal and let \( C \) be a small \( \kappa \)-geometric \( \infty \)-category. Then we obtain a sequence
\[ \hat{C} \to \text{Sh}_{\text{ocd}}(\kappa, C) \to \text{Sh}_{\text{hcd}}(\kappa, C) \]
of left exact localizations where the first localization is topological and the second localization is cotopological.

**Proof.** We show that \( (\text{Cov}_{\text{hcd}}^{-1}) = \text{Cov}_{\text{ocd}} \) and \( \text{Cov}_{\text{ocd}} \subseteq \text{Cov}_{\text{hcd}} \). It then follows from Lemma 2.1 that the Grothendieck topology generated by the set of \((-1)\text{-truncations} \ (\text{Cov}_{\text{hcd}}^{-1}) \) is exactly the \( \kappa \)-geometric Grothendieck topology on \( C \), and so the statement follows from Corollary 2.2.

On the one hand, the inclusion \( (\text{Cov}_{\text{hcd}}^{-1}) \subseteq \text{Cov}_{\text{ocd}} \) is easy to verify. Indeed, given a \( \kappa \)-small higher covering functor of the form \( F: I \to C \) together with a colimit cocone \( c: F \Rightarrow C \) so that \( y_{C, c}: \text{colim}_F F \to yC \) is in \( \text{Cov}_{\text{hcd}}(\kappa, C) \), we note that the \((-1)\text{-truncation} \ (\text{colim}_F)_{-1} \to yC \) is the \( \kappa \)-geometric covering sieve generated by the family \( \{c_i: F(i) \to C[i \in I]\} \). This follows from [14, Lemma 6.2.3.13]
(whose proof applies to $\kappa$-geometric $\infty$-categories without change) and [14, Lemma 6.2.3.18].

On the other hand, justifying the motivation from Example 5.1, proof of the inclusion $\text{Cov}_{\text{octd}} \subseteq \text{Cov}_{\text{hcd}}$ would be a straight-forward matter if the simplex category $\Delta^{op}$ had pullbacks. This apparent design flaw can be resolved by replacing $\Delta$ with the category $\text{FinSet}_+$ of non-empty finite sets. Indeed, given a $\kappa$-small set $I$, the simplicial object $I[[n]]: \Delta^{op} \to \text{Set}_{<\kappa}$ admits an extension to a symmetric simplicial set

$$I[[n]]: \text{FinSet}^{op}_+ \to \text{Set}_{\kappa}.$$  

This can be seen via [7, Theorem 4.2], mapping the main transpositions of a non-empty finite set $[n]$ to the according permutations of components of tuples in $I[[n]]$. We obtain the following pullback of discretely fibered Grothendieck constructions.

$$\begin{array}{ccc}
\sum_{[n] \in \Delta^{op}} I[[n]] & \sum_{[n] \in \text{FinSet}^{op}_+} & I[[n]] \\
\downarrow & \downarrow & \downarrow \\
\Delta^{op} & \to & \text{FinSet}^{op}_+
\end{array}$$

In the following we denote the top inclusion by $j^+: \Delta(I) \hookrightarrow \text{FS}(I)$. The canonical inclusion $\Delta^{op} \hookrightarrow \text{FinSet}^{op}_+$ is cofinal (which can be shown by the same proof of [14, Lemma 6.5.3.7]), and Kan fibrations are smooth maps ([14, Proposition 4.1.2.15]). It follows that the pullback $j^+: \Delta(I) \hookrightarrow \text{FS}(I)$ is cofinal as well by [14, Remark 4.1.2.10]. The presheaf $I[[n]]: \text{FinSet}^{op}_+ \to \text{Set} \subseteq \text{Cat}_{\text{hcd}}$ is in particular and indexed left exact category whose domain $\text{FinSet}^{op}_+$ has finite non-empty products and pullbacks. It follows that the associated total category $\text{FS}(I)$ has finite products and pullbacks, too.

Lastly, we note that the simplicial object $F: \Delta(I) \to \mathcal{C}/_C$ associated to an ordinary $\kappa$-geometric cover $\{F([0], i) \to C \mid i \in I\}$ in $\mathcal{C}$ can be extended to a higher covering diagram

$$F^+: \text{FS}(I) \to \mathcal{C}/_C$$

along the cofinal inclusion $j^+$. Therefore, consider the composite fully faithful inclusion

$$\iota: \{[0]\} \times I \hookrightarrow \sum_{[n] \in \Delta^{op}} I[[n]] \hookrightarrow \sum_{[n] \in \text{FinSet}^{op}_+} I[[n]].$$

As $I$ is discrete, so is the undercategory $([n], \vec{i})_{/\iota} := \{([0]\} \times I \times \text{FS}(I)_{(\{0\}, \vec{i})/}$ for every object $([n], \vec{i}) \in \text{FS}(I)$. This undercategory is furthermore finite, because the tuple $\vec{i}$ has finite length. Since $\mathcal{C}$ has pullbacks, the slice $\mathcal{C}/_C$ has products. It follows that for every $([n], \vec{i}) \in \text{FS}(I)$, the functor

$$(\{[0]\} \times I \times \text{FS}(I)_{(\{0\}, \vec{i})/}$$

has a limit in $\mathcal{C}/_C$. By [14, Lemma 4.3.2.13] it follows that $F$ admits a pointwise right Kan extension $F^+: \text{FS}(I) \to \mathcal{C}/_C$ along $\iota$. By [14, Definition 4.3.2.2], for all tuples $([n], \vec{i})$ we have equivalences

$$F^+([n], \vec{i}) = F^+((\{[0]\} \times I \times \text{FS}(I))$$

$$\simeq \lim\left(\{[0]\} \times I \times \text{FS}(I) \xrightarrow{F^+} \mathcal{C}/_C\right)$$

$$\simeq F([0], i_0) \times_C \cdots \times_C F([0], i_n)$$

(32)
Hence, by assumption that the maps (9) in Example 5.1 are equivalences, the diagram $F^+$ extends $F$ (up to equivalence).

We are left to show that $F^+$ is higher covering. Therefore, we use that all objects in the discrete full subcategory $\{[0]\} \times I$ are small-injective in $FS(I)$ with respect to products and pullbacks. That means, that for every accordingly indexed diagram $G: I \to FS(I)$, the natural map

$$\text{colim}(G/\iota) \to (\lim G)/\iota,$$

for $G/\iota$ the composite $I^{op} \xrightarrow{G^{op}} FS(I)^{op} \xrightarrow{\dashv/\iota} \text{Set}$, is an equivalence (of sets). Via the formula (32), it follows that the right Kan extension $F^+$ preserves all such limits. Furthermore, $C \simeq \text{colim} F$ is a colimit of $F^+$ by cofinality of $j^+$. By Example 5.10.1 together with the earlier observation that $FS(I)$ has products and pullbacks, it follows that $F^+$ is a higher covering diagram.

Eventually, again by virtue of cofinality of $j^+$, the sieves $\text{colim} y F \to y C$ and $\text{colim} y F^+ \to y C$ are mutually equivalent in $\hat{C}$. Thus, we have shown that every covering sieve for the $\kappa$-geometric Grothendieck topology generated by some covering family $(F([0], i) \to C)_{i \in I}$ is contained in $\text{Cov}_{\text{hcd}_n}$ by way of the higher covering diagram $F^+$. This finishes the proof. \qed

**Remark 5.14.** In Proposition 5.13 we assumed the cardinal $\kappa$ to be uncountable so that the cardinal featured in both $\text{Sh}_{\text{hcd}_n}(C)$ and $\text{Sh}_{\text{ocd}_n}(C)$ is the same. In the case $\kappa = \aleph_0$, the proof of Proposition 5.13 only generates factorizations of the form

$$\begin{array}{ccc}
\hat{C} & \xrightarrow{\text{Sh}_{\text{hcd}_0} -1} & \text{Sh}_{\text{hcd}_0}(C) \\
\downarrow & & \downarrow \\
\text{Sh}_{\text{ocd}_0}(C) & \xrightarrow{\text{Sh}_{\text{hcd}_0}} & \text{Sh}_{\text{hcd}_1}(C).
\end{array}$$

The increase in cardinality occurs here because the higher covering diagram $F^+$ associated to a finite cover $\{F_0(i) \to C \mid i \in I\}$ in $C$ has countably infinite domain $\sum_{\{i\} \in \text{FinSet}_+} F[[i]]$. Although it still has finite “width”, it invariably has countably infinite “length”. In this sense, the finite case is somewhat singular.

**Corollary 5.15.** Let $\kappa$ be an uncountable regular cardinal, and let $C$ be a small $\kappa$-geometric $\infty$-category (with a terminal object). Then the $\infty$-toposes $\text{Sh}_{\text{ocd}_n}(C)$ and $\text{Sh}_{\text{hcd}_n}(C)$ have the same class of points. By construction, these are the left exact functors $M: C \to S$ which preserve colimits of $\kappa$-small higher covering diagrams.

**Proof.** This follows immediately from the fact that the localization $\text{Sh}_{\text{hcd}_n}(C) \to \text{Sh}_{\text{ocd}_n}(C)$ is cotopological, together with the general observations that (the left adjoint part of) points preserve $\infty$-connected maps, and that $S$ is hypercomplete. \qed

Although $\kappa$-geometric $\infty$-categories $C$ are not necessarily regular as Čech-nerves in $C$ need not be effective, the notion of effective epimorphisms in $C$ is still well-defined and pullback-stable. It hence gives rise to a basis for the regular Grothendieck topology on $C$ whose generating covering sieves are the $\kappa$-geometric covering sieves of cardinality 1. It hence is contained in the ordinary $\kappa$-geometric Grothendieck topology on $C$.

**Corollary 5.16.** Let $C$ be a small $\kappa$-geometric $\infty$-category. Whenever $C$ contains a non-trivial $\infty$-connected map $f$, the $\infty$-topos $\text{Sh}_{\text{hcd}_n}(C)$ is not hypercomplete. In particular, it generally does not have enough points.
Proof. The proof is essentially the same as the proof of Proposition 4.6.1. Given an effective epimorphism \( f : E \to B \) in \( C \), the representable map \( yf \in \hat{C} \) factors into an effective epimorphism followed by the regular (and hence ordinarily \( \kappa \)-geometric) covering sieve \( |y(C(f))| \to yB \), which is contained in \( \text{Cov}_{\text{hcd}} \) by Proposition 5.13. It follows that the composite Yoneda embedding \( y : C \to \text{Sh}_{\text{hcd}}(C) \) preserves effective epimorphisms. In particular, since the higher diagonals are preserved as well, i.e. \( \Delta^n(yf) \simeq y(\Delta^n(f)) \) for all \( n \geq 0 \), we see that \( C \xrightarrow{y} \text{Sh}_{\text{hcd}}(C) \) preserves \( \infty \)-connected maps. Furthermore, \( yf \in \text{Sh}_{\text{hcd}}(C) \) is an equivalence if and only if \( f \in C \) is such, because \( \text{Sh}_{\text{hcd}}(C) \) is sub-canonical as noted in Remark 5.12.1.

While it may appear complicated to determine whether a given diagram in a \( \kappa \)-geometric \( \infty \)-category \( C \) is higher covering, it is a perfectly redundant matter in the case that \( C \) is a \( \kappa \)-logos.

Lemma 5.17. Suppose \( C \) is a small \( \kappa \)-logos and let \( I \) be a \( \kappa \)-small \( \infty \)-category with pullbacks. Then every pullback preserving functor \( F : I \to C \) is a higher covering diagram.

Proof. Suppose \( F : I \to C \) preserves pullbacks. The slice \( C_{/\text{colim} F} \) has descent for \( \kappa \)-small diagrams as well, and so we can apply Corollary 5.7 to the diagram \( F : I \to C_{/\text{colim} F} \). Thus, the squares

\[
\begin{array}{ccc}
\text{colim}_{i \to k \to j} F_k & \to & \text{colim}_{i \in I} F_i \\
\phi_F(i,j) \downarrow & & \downarrow \Delta \\
F_i \times_{\text{colim} F} F_j & \to & \text{colim}_{i,j \in I} (F_i \times_{\text{colim} F} F_j)
\end{array}
\]

are pullback squares. As \( \kappa \)-small colimits in \( C \) are universal, the right vertical map is an equivalence, and hence so are the left vertical maps. But these are exactly the maps in Definition 5.8 for \( n = 0 \). The same argument applies to all precompositions of \( F \) with \( \text{ev}_{x_{n+1}} : \text{Fun}_p(D^{n+1}, I) \to I \) for \( n \geq 0 \) and \( p : S^n \to I \) as in Definition 5.8. The recursive combinatorial structure of the spheres then again allows a proof by induction over \( n \geq 0 \) showing that the maps in Definition 5.8 are equivalences for all \( n \geq 0 \).

Indeed, we just verified that the map \((24) \to (22)\) in Definition 5.8 is an equivalence for \( n = 0 \). Assuming that it is an equivalence for all \( m \leq n \), one computes that in the case \( n + 1 \) the map \((24) \to (22)\) is equivalent to the natural map from the colimit of the sequence

\[
\text{Fun}(p|_{\text{ev}_{x_n}(D^n)}) \circ \text{ev}_{x_n} : \text{Fun}_p(D^n, I) \to C
\]

into the pullback

\[
\text{ev}_{x_n}(p|_{\text{ev}_{x_n}(D^n)}) \times_{\text{colim}(\text{Fun}_p)} \text{ev}_{x_n}(p|_{\text{ev}_{x_n}(D^n)}),
\]

where \( \text{ev}_{x_n}, \text{ev}_{y_n} : D^n \to S^n \) are the canonical embeddings with \( x_n \mapsto x_n \), and \( x_n \mapsto y_n \), respectively. So we can apply Corollary 5.7 to the composition

\[
\text{Fun}_p(D^n, I) \xrightarrow{\text{ev}_{x_n}} I \to C
\]

as stated to show that \((24) \to (22)\) is an equivalence in the case \( n + 1 \) as well. \( \square \)
Recall that every 1-topos is equivalent to the category of sheaves for the geometric site over itself ([10, Proposition C.2.2.7]). Accordingly, we have the following.

Say a (possibly large) ∞-category C is (higher) geometric if it is κ-geometric for all cardinals κ. Accordingly, a presheaf C^{op} → S is higher geometric if it is higher κ-geometric for all cardinals κ. I.e. if it takes colimits of all small higher covering diagrams in C to limits in S. Let Sh_{hcd}(C) ⊂ Fun(C^{op}, S) denote the full ∞-subcategory of higher geometric sheaves on C.

Furthermore, whenever C is an ∞-topos, recall the notation Sh_{P}(C) for the ∞-category of small limit-preserving functors C^{op} → D for an ∞-category D from [14, Notation 6.3.5.16]. Lurie refers to such functors as D-valued sheaves on the ∞-topos C. We recover this sheaf condition over ∞-toposes as follows.

**Proposition 5.18.** Let E be an ∞-topos and D be an ∞-category which admits all small limits. Then a functor E^{op} → D preserves all small limits if and only if it takes colimits of small higher covering diagrams in E to limits in D.

**Proof.** One direction is trivial. We show the other direction in two steps. First, let C be a small ∞-category with pullbacks, and suppose E ≃ C. For every X ∈ C, the canonical diagram

\[ C_{/X} \to C \xrightarrow{y} \hat{C} \]

comes with a colimiting cocone to X. It is a higher covering diagram by Lemma 5.17, because C_{/X} has pullbacks, both functors C_{/X} → C and y: C → \hat{C} preserve pullbacks, and \hat{C} has descent. Thus, whenever F: \hat{C}^{op} → D takes colimits of small higher covering diagrams in \hat{C} to limits in D, it follows that F is the pointwise right Kan extension of its restriction along y: C^{op} → \hat{C}^{op}. By [14, Lemma 5.1.5.5] it follows that F preserves all small limits.

Second, suppose E is a general ∞-topos. By [14, Proposition 6.1.5.3] there is a small ∞-category \hat{C} with pullbacks together with a left exact accessible localization functor L: \hat{C} → E. Suppose F: E^{op} → D takes colimits of small higher covering diagrams in E to limits in D. Since L: \hat{C} → E preserves both pullbacks and colimits, every higher covering diagram G: I → \hat{C} yields a higher covering diagram L \circ G: I → E by push forward along L. Thus, the composition

\[ F \circ L: \hat{C}^{op} → D \]

takes colimits of higher covering diagrams in \hat{C} to limits in D. By the first part of the proof it follows that F \circ L: \hat{C}^{op} → D preserves all small limits. By [14, Proposition 5.5.4.20] and fully faithfulness of the right adjoint E ≃ \hat{C}, it follows that F: E^{op} → D is small limit preserving.

**Theorem 5.19.** Every ∞-topos is the ∞-category of higher geometric sheaves over itself. More precisely, whenever E is an ∞-topos, we have the following.

1. A presheaf E^{op} → S (of small spaces) is higher geometric if and only if it is representable. In particular, the Yoneda embedding

\[ y: E → Sh_{hcd}(E) = Sh_{S}(E) \]

is essentially surjective and hence an equivalence.

2. Suppose E is contained in some Grothendieck universe U, and let S^{+} be the ∞-category of large spaces. Then a presheaf E^{op} → S^{+} (of large spaces) is higher geometric if and only if it preserves all U-small limits. I.e. Sh_{S^{+}}(E) is the ∞-category of large higher geometric sheaves on E.

31
Proof. Both statements follow from Proposition 5.18 for $\mathcal{D} = \mathcal{S}$ in the first case and $\mathcal{D} = \mathcal{S}^+$ in the second. 

We can use Lemma 5.17 to construct examples of small $\infty$-categories whose ordinary geometric and higher geometric sheaf theories provably differ. We do so in the next proposition, which in particular will show that Theorem 5.19 does not hold for the ordinary geometric sheaf theory over an $\infty$-topos $\mathcal{E}$. In that sense, it follows that the ordinary geometric Grothendieck topology on an $\infty$-topos $\mathcal{E}$ is insufficient to recover $\mathcal{E}$ as a sheaf theory over itself. Therefore, we first state and prove one more general lemma.

**Lemma 5.20.** Let $\mathcal{E}$ be an $\infty$-topos. Then the hypercompletion endofunctor $\tau_\infty: \mathcal{E} \to \mathcal{E}$ associated to the left exact localization $\mathcal{E} \to \tau_\infty(\mathcal{E})$ ([14, 6.5.2]) preserves effective epimorphisms and coproducts.

**Proof.** First, to see that hypercompletion in an $\infty$-topos $\mathcal{E}$ always preserves effective epimorphisms, let $f: E \to B$ be an effective epimorphism in $\mathcal{E}$. We obtain the following map of hypercompletions in $\mathcal{E}$.

\[
\begin{array}{ccc}
E & \xrightarrow{\eta_E} & \tau_\infty(E) \\
\downarrow f & & \downarrow \tau_\infty(f) \\
B & \xrightarrow{\eta_B} & \tau_\infty(B)
\end{array}
\]

The map $f$ is an effective epimorphism by assumption, the two vertical maps are $\infty$-connected and as such in particular effective epimorphisms as well. It follows that $\tau_\infty(f)$ is an effective epimorphism by compositionality and right cancellability of effective epimorphisms ([14, Corollary 6.2.3.12]).

To see that $\tau_\infty$ preserves coproducts, it suffices to show that the class of hypercomplete objects in $\mathcal{E}$ is closed under coproducts. Therefore suppose that $I$ is a set and that we are given a collection \( \{X_i|i \in I\} \) of hypercomplete objects in $\mathcal{E}$. Let $f: A \to B$ be $\infty$-connected and $g: A \to X$ for $X \simeq \coprod_{i \in I} X_i$ be a map. We are to show that $g$ lifts along $f$ in essentially unique fashion. Therefore, note that since $\mathcal{E}$ is extensive, for $A_i \simeq A \times X_i$ we obtain a collection of maps $g_i: A_i \to X_i$ for $i \in I$ together with an equivalence

\[
\begin{array}{ccc}
A & \xrightarrow{g} & \coprod_{i \in I} X_i \\
\downarrow \cong & & \downarrow \\
\coprod_{i \in I} A_i & \xrightarrow{\coprod_{i \in I} g_i} & \coprod_{i \in I} X_i.
\end{array}
\]

Furthermore, we obtain maps $f_i: A_i \to B$ such that $f \simeq (f_i)_{i \in I}$. Since $f$ is $\infty$-connected, its 0-truncation

\[
\begin{array}{ccc}
\coprod_{i \in I} A_i & \xrightarrow{\eta_{A_i}} & \tau_0(\coprod_{i \in I} A_i) \\
\downarrow (f_i)_{i \in I} & & \downarrow \tau_0((f_i)_{i \in I}) \\
B & \xrightarrow{\eta_B} & \tau_0(B)
\end{array}
\]

is an equivalence. Now, $n$-truncation $\tau_n$ for $n \geq 0$ preserves coproducts, because the localization $\mathcal{E} \to \tau_n \mathcal{E}$ is generated by the tensors $E \otimes \partial \Delta^{n+1} \to E \otimes \Delta^n$ for $E \in \mathcal{E}$.
and the \((n+1)\)-sphere for \(n \geq 0\) is connected ([14, Proposition 5.5.6.18]). Again using that \(E\) is extensive, for \(B_i \simeq B \times_{\tau_0(B)} \tau_0(A_i)\) the map \(f : A \to B\) is equivalent to the coproduct \(\coprod_{i \in I} f_i : \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i\). Since \(f\) is \(\infty\)-connected and the class of \(\infty\)-connected maps is closed under pullback, each \(f_i\) is \(\infty\)-connected as well. We thus are given a lifting problem of the form

\[
\begin{array}{ccc}
\coprod_{i \in I} A_i & \overset{\coprod_{i \in I} g_i}{\longrightarrow} & \coprod_{i \in I} X_i \\
\coprod_{i \in I} f_i \downarrow & & \downarrow \\
\coprod_{i \in I} B_i & & \\
\end{array}
\]

As each \(X_i\) is hypercomplete and each \(f_i\) is \(\infty\)-connected, this admits a solution. This solution is essentially unique whenever every map of type \(B_i \to X\) extending \(g_i : A_i \to X_i\) factors through the component \(X_i \to X\). This indeed is satisfied, since the inclusion \(X_i \to \coprod_{i \in I} X_i\) is \((-1)\)-truncated and \(f_i : A_i \to B_i\) is \((-1)\)-connected, and so the square

\[
\begin{array}{ccc}
A_i & \overset{g_i}{\longrightarrow} & X_i \\
\downarrow & & \downarrow \\
B_i & \longrightarrow & \coprod_{i \in I} X_i \\
\end{array}
\]

exhibits a lift.

Remark 5.21. Although not needed here, the proof of Lemma 5.20 applies not only to the hypercompletion endofunctor \(\tau_\infty : E \to E\), but as well to the finite truncation functors \(\tau_{\leq n} : E \to E\) for every \(n \geq -1\) in the case of effective epimorphisms, and for every \(n \geq 0\) in the case of coproducts. For such natural numbers \(n < \infty\), we only need that \(E\) is presentable and regular for the first case, and furthermore extensive for the second case.

Proposition 5.22. There are \(\infty\)-toposes \(E\) such that the cotopological localization

\[
\text{Sh}_{\text{ocd}}(\mathcal{C}) \to \text{Sh}_{\text{hcd}}(\mathcal{C})
\]

is non-trivial. Accordingly, there are small \(\kappa\)-logoi \(\mathcal{C}\) for some \(\kappa\) such that the cotopological localization

\[
\text{Sh}_{\text{ocd},\kappa}(\mathcal{C}) \to \text{Sh}_{\text{hcd},\kappa}(\mathcal{C})
\]

is non-trivial.

Proof. Let \(E\) be an \(\infty\)-topos with the following two properties.

1. \(E\) is generated by a set \(G\) of objects which is closed under fibre products and such that each \(g \in G\) is hypercomplete.

2. \(E\) is not hypercomplete itself.

Let \(\kappa \geq |G|\) be a regular cardinal such that there is a \(\kappa\)-compact non-hypercomplete object \(E \in E\), and such that the \(\infty\)-category \(G \downarrow E\) is \(\kappa\)-small. Let \(\mu\) be a regular cardinal sharply larger than \(\kappa\) ([14, Definition 5.4.2.8]) such that the accessible endofunctor \(T_\infty : E \to E\) takes \(\mu\)-small objects to \(\mu\)-small objects ([19, Lemma 8.3.4]). Let \(\mathcal{C} \subset E\) be the full \(\infty\)-subcategory of \(\mu\)-compact objects.

Then \(\mathcal{C}\) is small, \(\kappa\)-closed, left exact, and has descent for \(\kappa\)-small diagrams. Hence, \(\mathcal{C}\) is a \(\kappa\)-logos. Now, for every ordinary \(\kappa\)-geometric sheaf \(X\) on \(\mathcal{C}\), the precomposition \(X \circ T^{op}_\infty : \mathcal{C}^{op} \to S\) with the endofunctor \(T_\infty : \mathcal{C} \to \mathcal{C}\) is an ordinary
κ-geometric sheaf again by Lemma 5.20 and by the fact that \( T_\infty : \mathcal{C} \to \mathcal{C} \) preserves finite limits. In the following we show that the composition \( yE \circ T_\infty^\text{op} \) is not higher κ-geometric. Since the representable \( yE \) however is ordinary κ-geometric, this proves the statement (assuming that the \( \infty \)-topos \( \mathcal{E} \) exists).

Therefore, we use that \( E \) is the canonical colimit of the generators \( \pi_E : G/E \to \mathcal{C} \), and consider the induced map

\[
\text{colim}_{g \in G/E} T_\infty(g) \to T_\infty(\text{colim}_{g \in G/E}(g)). \tag{33}
\]

As \( T_\infty(g) \simeq g \) for all \( g \in G \) by property 1., the domain of (33) is equivalent to \( E \) itself, while its codomain is the hypercompletion of \( E \) in \( \mathcal{E} \) by construction.

If the representable \( yE \) applied to the map (33) was an equivalence of spaces, we would obtain a retract to the map (33) in \( \mathcal{C} \). Since the collection of hypercomplete objects is closed under retracts, that would imply that \( E \) is hypercomplete as well, which is contrary to our assumption.

Since \( yE \) preserves colimits itself, it follows that the presheaf \( yE \circ T_\infty : \mathcal{C}^\text{op} \to \mathcal{S} \) does not preserve the colimit of the canonical functor \( \pi_E : G/E \to \mathcal{C} \). But the \( \infty \)-category \( G/E \) has pullbacks by property (1) which furthermore are preserved by \( \pi_E \). Since \( \mathcal{C} \) has descent for κ-small diagrams and \( G/E \) is κ-small, it follows that \( \pi_E \) is a higher covering diagram by Lemma 5.17. It hence follows that the presheaf \( yE \circ T_\infty \) is not a higher κ-geometric sheaf. It yet is ordinarliy κ-geometric by the observations put forward at the beginning of the proof.

In order to finish the proof, we are therefore left to present an \( \infty \)-topos \( \mathcal{E} \) which has the properties listed in 1 and 2. Therefore, we simply note that the \( \infty \)-topos of sheaves \( \text{Sh}_\tau(\mathcal{C}) \) on any sub-canonical 1-site \( (\mathcal{C}, \tau) \) where \( \mathcal{C} \) has pullbacks satisfies property 1. Indeed, since it is sub-canonical, it is generated by the representables \( \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}} \to \text{Sh}_\tau(\mathcal{C}) \). As \( \mathcal{C} \) is of finite homotopy type, each representable is of finite homotopy type and thus is in particular hypercomplete. An example of such an \( \infty \)-topos which is not hypercomplete is the localic Dugger-Hollander-Isaksen-topos we used for other examples as well ([16, 11.3]).

**Remark 5.23.** It follows from Proposition 5.22 that the ordinary geometric Grothendieck topology on an \( \infty \)-topos \( \mathcal{E} \) is in general not canonical, in the sense that its associated sheaf theory \( \text{Sh}_{\text{hcd}}(\mathcal{E}) \) is not the smallest sub-canonical left exact accessible localization of \( \hat{\mathcal{E}} \). Indeed, it follows from Theorem 5.19 that \( \text{Sh}_{\text{hcd}}(\mathcal{E}) \) is canonical instead: if \( \hat{\mathcal{E}} \to \mathcal{X} \) is a left exact accessible localization such that \( y : \mathcal{E} \to \hat{\mathcal{E}} \) factors through \( \mathcal{X} \xrightarrow{\eta} \hat{\mathcal{E}} \), then the essential image \( \text{Sh}_{\text{hcd}}(\mathcal{E}) \) of \( y \) is clearly contained in \( \mathcal{X} \).

### 5.3 The \( \infty \)-category of κ-geometric \( \infty \)-categories

In this section we define the \( \infty \)-category of κ-geometric \( \infty \)-categories and relate it to the \( \infty \)-categories of \( \infty \)-toposes. Following the 1-categorical tradition captured by [5, Proposition 1.4.8], we define κ-geometric functors to be the left exact functors which preserves κ-geometric covers.

**Definition 5.24.** A functor \( F : \mathcal{C} \to \mathcal{D} \) between κ-geometric \( \infty \)-categories is κ-geometric if it is left exact and preserves colimits of κ-small higher covering diagrams. The \( \infty \)-category \( \text{GeoCat}_\kappa \subset \text{Cat} \) is the \( \infty \)-subcategory of κ-geometric \( \infty \)-categories, κ-geometric functors and all higher cells. The same applies for the \( \infty \)-category \( \text{GeoCat} \) of geometric \( \infty \)-categories and geometric functors.
Let \( \text{LTop} \) denote the (superlarge) \( \infty \)-category of \( \infty \)-toposes and left exact left adjoints ([14, Definition 6.3.1.5]). Basically by construction we obtain the following right adjoint embedding.

**Proposition 5.25.** For every regular \( \kappa \) (including \( \kappa = \vert \text{Ord} \vert \)), the forgetful functor

\[
U : \text{LTop} \rightarrow \text{GeoCat}_\kappa,
\]

has a left adjoint which is given on a \( \kappa \)-geometric \( \infty \)-category \( \mathcal{C} \) by \( \text{Sh}_{hcd}(\mathcal{C}) \). Furthermore, the forgetful functor

\[
U : \text{LTop} \rightarrow \text{GeoCat}
\]

is fully faithful, i.e. the notion of geometric morphism between \( \infty \)-toposes is unambiguous.

**Proof.** Given a \( \kappa \)-geometric \( \infty \)-category \( \mathcal{C} \) and an \( \infty \)-topos \( \mathcal{E} \), the equivalence \( \text{Fun}(\mathcal{C}, \mathcal{E}) \simeq \text{LTop}(\hat{\mathcal{C}}, \mathcal{E}) \) given by left Kan extension along \( y : \mathcal{C} \rightarrow \hat{\mathcal{C}} \) restricts to an equivalence between \( \text{Geom}_\kappa(\mathcal{C}, \mathcal{E}) \) and the full \( \infty \)-subcategory of \( \text{LTop}(\hat{\mathcal{C}}, \mathcal{E}) \) given by those functors which map each arrow in the class \( \text{Cov}_{hcd} \) to an equivalence in \( \mathcal{E} \). This in turn is up to equivalence the space \( \text{LTop}(\text{Sh}_{hcd}(\mathcal{C}), \mathcal{E}) \). This shows that the embedding \( y : \mathcal{C} \rightarrow \text{Sh}_{hcd}(\mathcal{C}) \) is initial in \( \mathcal{C} \downarrow \text{LTop} \) and hence a unit for an adjunction with right adjoint \( U \).

The fact that \( U : \text{LTop} \rightarrow \text{GeoCat} \) is fully faithful can be shown in the same way as Proposition 5.18: one shows that a left exact functor \( f : \mathcal{E} \rightarrow \mathcal{D} \) from an \( \infty \)-topos \( \mathcal{E} \) into a cocomplete \( \infty \)-category \( \mathcal{D} \) preserves all small colimits if and only if it preserves colimits of small higher covering diagrams.

In this sense, \( \text{Sh}_{hcd}(\mathcal{C}) \) is the free \( \infty \)-topos generated by a \( \kappa \)-geometric \( \infty \)-category \( \mathcal{C} \). Proposition 5.25 also implies Theorem 5.19.1 directly.

**Remark 5.26.** It may appear awkward to obtain a definition of \( \text{GeoCat} \) which requires all \( \kappa \)-small colimits to exist in each of its objects, but whose functors are required only to be preserve a chosen few of them (although, as we have seen, this discrepancy vanishes at least in the case of \( \infty \)-toposes). However, the definition of a geometric functor is imposed by the covers for the associated sheaf theories, and it follows from the first Example in the Motivation of the Introduction that a notion of cover which enfolds all small colimits in a geometric \( \infty \)-category cannot yield a sheaf theory for geometric \( \infty \)-categories which are not \( \infty \)-toposes themselves.

In this context it may be worth to point out that throughout this section we could get away with the assumption of the existence of colimits of \( \kappa \)-small higher covering diagrams in \( \kappa \)-geometric \( \infty \)-categories only, if it was not the very definition of higher covering diagram already using the existence of colimits in the first place.

**References**

[1] M. Anel, *Descent & univalence*, https://www.uwo.ca/math/faculty/kapulkin/seminars/hottestfiles/Anel-2019-05-2-HoTTesT.pdf, 2nd May 2019, Slides from the talk at the HoTTesT-Series.

[2] M. Anel and A. Joyal, *Topo-logie*, New Spaces in Mathematics: Formal and Conceptual Reflections (G. Catren M. Anel, ed.), Cambridge University Press, 2021, pp. 155–257.
[3] M. Anel, A. Joyal, E. Finster, and G. Biedermann, *Higher sheaves and left-exact localizations of $\infty$-topoi*, https://arxiv.org/abs/2101.02791.

[4] M. Anel and C. Leema Subramaniam, *Small object arguments, plus-construction and left-exact localizations*, https://arxiv.org/abs/2004.00731, Last update 12 April 2020.

[5] O. Caramello, *Theories, sites, toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'*; Universitext, Oxford University Press, 2018.

[6] A. Carboni, S. Lack, and R.F.C. Walters, *Introduction to extensive and distributive categories*, Journal of Pure and Applied Algebra 84 (1993), 145–158.

[7] M. Grandis, *Finite sets and symmetric simplicial sets*, Theory and Applications of Categories 8 (2001), no. 8, 244–252.

[8] G. Bin Im and G. M. Kelly, *On classes of morphisms closed under limits*, Journal of the Korean Mathematical Society 23 (1986), no. 1, 1–18.

[9] P.T. Johnstone, *Stone spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, 1982.

[10] ______, *Sketches of an elephant: A topos theory compendium*, Oxford Logic Guides, vol. 43, Clarendon Press, 2003.

[11] C. Kapulkin, *Locally cartesian closed quasicategories from type theory*, Journal of Topology 10 (2015), no. 4.

[12] C. Kapulkin and K. Szumiło, *Internal language of finitely complete $(\infty, 1)$-categories*, https://arxiv.org/abs/1709.09519, 2017, [Online, v1 accessed 27 Sep 2017].

[13] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, Universitext, Springer-Verlag New York, Inc., 1992.

[14] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, no. 170, Princeton University Press, 2009.

[15] ______, *Derived algebraic geometry vii: Spectral schemes*, http://people.math.harvard.edu/~lurie/papers/DAG-VII.pdf, 2011, [Version Nov 5th 2011].

[16] C. Rezk, *Toposes and homotopy toposes (version 0.15)*, https://www.researchgate.net/publication/255654755_Toposes_and_homotopy_toposes_version_015, 2010.

[17] E. Riehl and D. Verity, *Kan extensions and the calculus of modules for $\infty$-categories*, Algebraic & Geometric Topology 17 (2017), 189–271.

[18] M. Shulman, *All $(\infty, 1)$-toposes have strict univalent universes*, https://arxiv.org/abs/1904.07004, 2019, [Online, last revised 26 Apr 2019].

[19] R. Stenzel, *On univalence, Rezk completeness and presentable quasi-categories*, Ph.D. thesis, University of Leeds, Leeds LS2 9JT, 3 2019.