A Information-Theoretic View on Spacetime

Amir Khosravi and Frank Saueressig

Institute for Mathematics, Astrophysics and Particle Physics (IMAPP), Radboud University Nijmegen, Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherlands

Many approaches to quantum gravity predict that spacetime has to be replaced with a more fundamental structure at distances below the Planck length \( l_{Pl} \approx 10^{-35} \) m. In this essay we argue in a quantitative way that the unitarity principle together with the quantum information bound on correlation functions imply: 1) Geometry becomes operationally inaccessible at short distances. 2) Spacetime cannot be discrete.

Keywords: Gravity; Entanglement; Emergent Spacetime.

Essay written for the Gravity Research Foundation 2020 Awards for Essays on Gravitation
I. Introduction

The quest to unravel the ultimate nature of space and time is one of the oldest and most challenging ones in the history of intellectual endeavors. From Democritus’ idea of indivisible atoms floating in an unbounded space [1] to the Aristotelian geocentric finite Universe [1] up to modern ideas originating from the combination of quantum mechanics and general relativity [2, 3], we have gone a long way.

A rather radical proposal [4–9] suggests that space and time themselves are not fundamental but emerge from more abstract thermodynamical/information-theoretic concepts giving rise to an effective geometry at length scales larger than some fundamental scale \( l_0 \). In this work, we elaborate on this idea and prove that this fundamental length scale cannot be a hard cutoff for a discrete spacetime.

In Sects. II A and II B we review the entanglement entropy properties and its relation to the geometry of spacetime. Furthermore, we use the unitarity principle along with the quantum information upper bound on correlation functions to argue that geometry is an emergent concept. In Sect. II C it is shown that if spacetime emerges from a discrete structure, then we have to either discard unitarity or subadditivity of entanglement. Finally, a summary is provided.

II. Quantum Information, Entanglement and Spacetime

A. What Do We Mean By Geometry?

Consider two arbitrary 3d regions in a 4d spacetime, say \( C \) and \( D \), that are disjoint except probably for a shared 2d area between them. Suppose the total information about the geometry of region \( E = C \cup D \) is stored in a density matrix \( \rho_{CD} \). The mutual information \( I_{CD} \) then reads [10]

\[
I_{CD} \equiv S_C + S_D - S_{CD} \geq 0 \tag{1}
\]

where \( S_C \) and \( S_D \) are the entanglement entropy between \( C \) and \( D \) and \( S_{CD} \) is the total entanglement of the \((C, D)\)-system. In a general state, \( I_{CD} \) is a measure of correlation between \( C \) and \( D \). If all the correlation is quantum, i.e. \( S_{CD} = 0 \), then we have a pure state \( S_C = S_D \).

At an arbitrary point \( x \) in a 4d spacetime we can choose a time-like unit vector \( v^\alpha \) and generate a 3d region \( E \) by drawing space-like geodesics out from \( x \) in all directions. Following [11], we can set up a Riemann normal coordinate system using \( v^\alpha \) and 3 other space-like vectors \( x^i = r n^i \) for some unit vectors \( n^i \) and geodesic distance \( r \)

\[
h_{ij} = \delta_{ij} - \frac{1}{3} r^2 R_{ijkl} x^k x^l + O(r^3) \tag{2}
\]

It is not hard to show that for any small subregion \( C \) of \( E \) with a fixed characteristic volume \( V \propto r^3 \), the variation of surface area of \( C \) is proportional to the intrinsic Ricci curvature, \( R = R_{ij} \) [11], evaluated at \( x \)

\[
\delta A_C \propto -r^3 R_C \tag{3}
\]

Now consider a covering of the region \( E \) by a meshing of smaller subregions with typical size \( l \). The change in the boundary area of a subregion \( C \) can be expressed as \([9]\)

\[
\delta A_C = \frac{1}{2\alpha} \sum_i \delta I_{CD_i} \tag{4}
\]

where the sum is over all subregions \( D_i \) whose mutual information with \( C \) has been changed and \( \alpha \) is some normalization constant. Using \([3]\) and \([4]\) we can relate the change in the mutual information of \( C \) with its surrounding to the spatial curvature in \( C \) \([9]\)

\[
R_C \propto -\frac{1}{r^3} \sum_i \delta I_{CD_i} \tag{5}
\]

Then the question rises: "When can we define a metric structure \( g \) that is independent of the size \( l \) of the probed region?" We expect that the metric structure is independent of \( l \) if the size of the probed region is larger than \( l_0 \). In this case the concept of an effective spacetime metric \((g, l_0)_E\) is meaningful. Noting that \( R_C \) is a function of renormalization scale \( \mu \propto 1/l \), we propose:
As long as \( R_C(\mu) \ll \frac{1}{\mu^2} \), one could access the metric information \((g, l_0)_C\) without significantly altering the mutual information between \(C\) and its surrounding.

**B. I-Theorem**

The next interesting question is: “How does \( R_C \) change with the renormalization scale?” To answer the question “how geometry behaves when probed at smaller distances?” we have to calculate \( \partial_\mu I \). It follows from (5) and Statement-1 that if \( \partial_\mu I > 0 \), by moving towards the UV we continue to induce more curvature into the probed region. At some point we will reach \( R \approx \frac{1}{l_0^2} \) and the geometry becomes operationally inaccessible. This argument is a more fundamental version (derived in terms of mutual information’s properties) of the popular “break down of spacetime” argument based on the uncertainty principle. We prove that indeed for generic unitary QFTs we have \( \partial_\mu I > 0 \). To do so we begin by the Callan-Symanzik equation for \( n \)-point correlation function,

\[
\left[ \mu \partial_\mu - \frac{1}{2} \eta (g(\mu)) \right] \Gamma^{(n)}(p_j, g(\mu), \mu) = 0,
\]

in which \( \eta (g(\mu)) \) is the anomalous dimension, \( g(\mu) \) is a set of coupling constants and \( \Gamma^{(n)}(p_j, g(\mu), \mu) \) indicates the \( n \)-point function. The general solution to (6) can be written as:

\[
\Gamma^{(n)}(p_j, g(\mu), \mu) = \exp \left[ \frac{1}{2} \int_{\mu_0}^{\mu} d\mu' \eta (g(\mu')) \right] \Gamma^{(n)}(p_j, g(\mu_0), \mu_0)
\]

The crucial point here is that for a generic unitary QFT it is safe to assume \( \eta (g(\mu)) > 0 \) and \( \langle O_C O_D \rangle_c > 0 \) \citep{20}, where \( \langle O_C O_D \rangle_c \) denotes the connected two-point function. Consequently, using (7), we get \( \partial_\mu \langle O_C O_D \rangle_c > 0 \).

On the other hand, for any two operators \( O_C \) and \( O_D \), the mutual information between two regions \( C \) and \( D \) obeys \citep{10}:

\[
I_{CD} \geq \frac{(\langle O_C O_D \rangle_c)^2}{2|O_C|^2 |O_D|^2}
\]

Now, using \( \partial_\mu \langle O_C O_D \rangle_c > 0 \) and (8) we can claim:

\[
\partial_\mu I > 0
\]

This result hints towards spacetime being a coarse grained entity that originates from something more fundamental, see Fig. 1.

---

FIG. 1: On the left, we see a 3D smooth geometry that is probed at length scale \( l_1 \). On the right, same geometry is probed to smaller length scale \( l_2 < l_0 < l_1 \). According to (5), Statement-1 and (9) this generates the bumpy regions with high curvature.

---

\(^1\) Also see \citep{21} for a general argument showing that positive central charge of a CFT (i.e. unitarity) leads to \( \partial_\mu \langle O_C O_D \rangle_c > 0 \).
C. Spacetime Can Not Be Discrete

According to Van Raamsdonk [7] the entanglement between two regions is proportional to their shared area and the distance between them. The original argument is presented in an AdS/CFT [7, 12] setup; here we present a simpler version using properties of entanglement entropy in quantum field theory (QFT). References [13, 14] suggest that to leading order in the UV cutoff we have

$$S \sim A$$

where $A$ is the shared area (area of entangling surface) between the regions and $S$ indicates their entanglement entropy. In light of this, reducing the entanglement entropy and mutual information is equivalent to reducing the shared area.

Furthermore, for a typical massive field theory with mass $m$ we have

$$\langle O_C(\vec{x}) O_D(\vec{y}) \rangle \propto e^{-mL}$$

where $\vec{x}$ and $\vec{y}$ are spatial positions in $C$ and $D$ and $L$ indicates the geodesic distance between these points. In the light of (8), this means $I \ll 1$ is equivalent to $L \gg \frac{1}{m}$.

Combining last two paragraphs, one can summarize Van Raamsdonk’s result as [7]:

**Statement-2**

Zero entanglement is equivalent to $C$ and $D$ being far away and disconnected (zero shared area) regions of spacetime.

The core of our argument, essentially Sect. [11B], survives the transition to a fundamental theory of nature (whatever that is) as long as this theory admits an effective QFT description that is unitary. If QFT itself survives the transition, we have a scenario like asymptotic safety [22, 23] in which $I$ reaches its maximum at the UV fixed point. If QFT is an effective description, we have something like a string theory effective description of the bulk [3, 12]. Regarding a discrete spacetime, the combination of Statement-2 and (9) is at odds with it: according to Statement-2, when probing a discrete spacetime we reach $I = 0$ below the discreteness scale. Then we have two options: 1) Accept unitarity and consequently (9) which means we have $I < 0$ in the IR limit. As a result, the positivity bound on $I$ and, in turn, subadditivity of entanglement will be violated [10]. 2) Drop the unitarity principle.

III. Summary

In this manuscript we argued that two crucial ingredients of a fundamental description of nature are the following:

(i) Spacetime is not a fundamental concept. It is a result of coarse graining something more fundamental. This follows from [5], Statement-1 and (9).

(ii) Spacetime cannot emerge from coarse graining of a fundamentally discrete structure. This is a consequence of Statement-2, (9) and the positivity of $I$.

The approach followed in this essay is a bottom-up approach in the sense that we assume a geometric structure is already defined on a region $E$ and check whether the concept of effective geometry persists in the UV. One can imagine adopting a top-down approach like the one pursued in [11] and try to construct an emergent geometry starting from an abstract Hilbert space. In fact our arguments can be considered as both a motivation, (i), and a constraint, (ii), for a top-down construction of spacetime.
[1] D. Furley, The Greek Cosmologists, Cambridge University Press (1987).
[2] C. Rovelli, L. Smolin, Spin Networks and Quantum Gravity, Phys.Rev. D52 (1995) 5743-5759, arXiv:9505006
[3] G. T. Horowitz, Spacetime in string theory, New J.Phys. 7 (2005) 201, arXiv:0410049v3; N. Seiberg, Emergent Spacetime, Rapporteur talk at the 23rd Solvay Conference in Physics(2005), arXiv:0601234.
[4] T. Jacobson, Thermodynamics of Spacetime: The Einstein Equation of State, arXiv:gr-qc/9504004
[5] S. Lloyd, A theory of quantum gravity based on quantum computation, arXiv:gr-qc/9504004.
[6] T. Padmanabhan, Thermodynamical Aspects of Gravity: New insights, Rep. Prog. Phys. 73, 046901 (2010), arXiv:0911.5004.
[7] M. Van Raamsdonk, Building up spacetime with quantum entanglement, Gen. Rel. Grav. 42, 2323 (2010), Int. J. Mod. Phys. D 19, 2429 (2010), arXiv:1005.3035.
[8] E. Verlinde, On the Origin of Gravity and the Laws of Newton, JHEP 1104:029 (2011), arXiv:1001.0785.
[9] J. M. Maldacena, The Large N Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)], arXiv:hep-th/9711200.
[10] M. Srednicki, Entropy and Area, Phys.Rev.Lett. 71 (1993) 666-669, arXiv:9303048; Bombelli, L., R. K. Koul, J. Lee, and R. D. Sorkin, Quantum source of entropy for black holes, Phys.Rev. D 34, 373 (1986).
[11] A. Einstein, B. Podolsky and N. Rosen, Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?, Phys. Rev. 47, 777 (1935).
[12] A. Mari, G. De Palma, and V. Giovannetti, Experiments testing macroscopic quantum superpositions must be slow, Sci. Rep. 6, 22777 (2016), arXiv:1509.02408.
[13] A. Belenchia, R. M. Wald, F. Giacomini, E. Castro-Ruiz, C. Brukner, and M. Aspelmeyer, Quantum superposition of massive objects and the quantization of gravity, Phys. Rev. D 98, 126009 (2018), arXiv:1807.07015.
[14] A. Belenchia, R. M. Wald, F. Giacomini, E. Castro-Ruiz, C. Brukner, and M. Aspelmeyer, Information Content of the Gravitational Field of a Quantum Superposition, arXiv:1905.04496.
[15] J. Maldacena, L. Susskind, Cooling black holes, Fortsch. Phys. 61 (2013) 781811, arXiv:1306.0533.
[16] K. Higashijima, E. Itou, Unitarity Bound of the Wave Function Renormalization Constant, Prog.Theor.Phys. 110 (2003) 107-114, arXiv:0304047.
[17] L. Susskind, Dear Qubitzers, GR=QM, arXiv:1708.03040.
[18] I. Papadimitriou and K. Skenderis, AdS/CFT correspondence and Gravity, IRMA Lect. Math. Theor. Phys. 8 (2005) 33, arXiv:0405176.
[19] S. Weinberg, Critical Phenomena for Field Theorists, In: A. Zichichi (eds) Understanding the Fundamental Constituents of Matter, The Subnuclear Series, vol 14 Springer, Boston, MA, (1978).
[20] R. Percacci, An Introduction to Covariant Quantum Gravity and Asymptotic Safety, World Scientific, Singapore, (2017); M. Reuter and F. Saueressig, Quantum Gravity and the Functional Renormalization Group, Cambridge University Press, (2019).