On distributional spectrum of piecewise monotonic maps

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Abstract. We study a certain class of piecewise monotonic maps of an interval. These maps are strictly monotone on finite interval partitions, satisfy the Markov condition, and have generator property. We show that for a function from this class distributional chaos is always present and we study its basic properties. The main result states that the distributional spectrum, as well as the weak spectrum, is always finite. This is a generalization of a similar result for continuous maps on the interval, circle, and tree. An example is given showing that conditions on the mentioned class can not be weakened.

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1. Introduction

Distributional chaos and the structure of spectrum for continuous maps of one-dimensional spaces such as an interval, a circle or a tree are well known, see [7, 13, 15] for details. The class of piecewise monotonic, not necessarily continuous, interval maps is a natural generalization of these maps. In their papers [2–4, 6, 9, 10] F. Hofbauer and P. Raith provided basic tools and properties of this class of dynamical systems and showed that in general the dynamics of a map with discontinuity points can have significantly different behavior [5]. Therefore we focus on piecewise monotonic maps which fulfil the Markov condition and have a generator. In this case, we have obtained that the spectrum of distributional functions is finite.

We also provide an example that shows that without a generator, a piecewise monotonic map can have an infinite spectrum. This can also happen when we consider a continuous map on more general spaces like dendroids [15].

Let $I$ be an interval. For a map $f : I \to I$ and any $n \geq 0$ let $f^n$ denote the $n$-th iteration of $f$, it is defined recursively by $f^0 = id_I$ and $f^n = f \circ f^{n-1}$.
The trajectory of a point $x \in I$ is a sequence of its iterations $(f^i(x))_{i=0}^\infty$ and the orbit of a point $x \in I$ is a set of its iterations $\{x, f(x), f^2(x), \ldots\}$. The $\omega$-limit set of a point $x \in I$ is a set of accumulation points of its trajectory and it is denoted by $\omega_f(x)$.

Based on the work of A.N. Sharkovsky in [11,12], maximal $\omega$-limit sets were characterized and their properties widely studied, see e.g. [1,8]. Results on the finiteness of the spectrum in the case of continuous maps are based on the properties of these $\omega$-limit sets. We adopt this characterization, but we drop the assumption of maximality because in the case of discontinuous maps a maximal $\omega$-limit may not exist [5]. If an $\omega$-limit set is finite then it is a cycle. If it is infinite and does not contain a periodic point then it is called a solenoid. The $\omega$-limit set is called a basic set if it is infinite and contains a periodic point. The last case of $\omega$-limit sets plays a crucial role in distributional chaos.

We say that $\omega_f(x)$ is maximal if there is no $y \in I$ such that $\omega_f(x) \subset \omega_f(y)$. We denote with $\Omega$ as the set of maximal basic $\omega$-limit sets.

In [13], Schweizer and Smítal defined the notion of distributional chaos. The definition is based on a distributional function of pairs of points. For any $x, y \in I$, a positive integer $n$ and any real $t$, let

$$\xi(x, y, t, n) = \# \{i; 0 \leq i \leq n - 1 \text{ and } \delta_{xy}(i) < t\}$$

(1)

where $\#S$ is the cardinality of the set $S$ and $\delta_{xy}(i) := |f^i(x) - f^i(y)|$ is the distance of the $i$-th iteration of points $x$ and $y$. Put

$$F^*_{xy}(t) = \limsup_{n \to \infty} \frac{1}{n} \xi(x, y, t, n),$$

(2)

$$F_{xy}(t) = \liminf_{n \to \infty} \frac{1}{n} \xi(x, y, t, n).$$

(3)

The function $F^*_{xy}$ is called an upper distributional function and $F_{xy}$ is called a lower distributional function. Both $F^*_{xy}$ and $F_{xy}$ are nondecreasing functions with $F^*_{xy}(t) = F_{xy}(t) = 0$ for every $t < 0$ and $F^*_{xy}(t) = F_{xy}(t) = 1$ for every $t > |I|$. Let $f$ be a map on an interval $I$ and $x, y \in I$. Then the pair $(x, y)$ is isotectic (with respect to $f$) if, for every positive integer $n$, the $\omega$-limit sets $\omega_{f^n}(x)$ and $\omega_{f^n}(y)$ are subsets of the same maximal $\omega$-limit set of $f^n$.

Put $Iso(f) = \{(x, y) \in I^2; (x, y)$ is isotectic$\}$ and $D(f) = \{F_{xy}; (x, y) \in Iso(f)\}$. Then the spectrum of $f$ is the set of minimal elements of $D(f)$, denote such a set by $\Sigma(f)$. The weak spectrum of $f$, denoted by $\Sigma_w(f)$, is the set of minimal elements of the set $D_w(f) = \{F_{xy} | \liminf_{n \to \infty} \delta_{xy}(n) = 0\}$. See [13, 14] for more information on Distributional spectrum.

For any subset $K \subset I$ let $\text{int}(K)$ be its interior, $\partial(K)$ be its border and $K^c$ be its complement. A map $f : I \to I$ is called piecewise monotone if there is a family $J$ of pairwise disjoint nondegenerate intervals $\{J_1, \ldots, J_N\}$ such that
$f|_{\text{int}(J_i)}$ is continuous and strictly monotone, and $f(c)$ coincides with one of the one-sided limits at $c \in \partial(J_i)$.

Let $C_0(f)$ be the set of endpoints of $J_i \in \mathcal{J}$, $C_0(f) = \{c | c \in \partial(J_i)\}$, and we name it the set of critical points. For a nonnegative integer $n$ let $C_n(f)$ be the set of points which will map to the set of critical points in $n$ iterations, or less, $C_n(f) = \{x \in I | f^i(x) \in C_0(f) \text{ for some } 0 \leq i \leq n\}$, and let $C(f)$ be the limit case of $C_n(f), C(f) = \{x \in I | f^i(x) \in C_0 \text{ for some } i \in \mathbb{N}\}$.

We say that a piecewise monotonic map $f$ satisfies the Markov condition if for any $c \in C_0$ the one-sided limits are in $C_0$. The family of intervals $\mathcal{J}$ is a generator if for every sequence $\{k_i\}_{i=0}^{\infty}$ of natural numbers $k_i \in \{1, 2, \ldots, N\}$ the set $\bigcap_{j=0}^{\infty} f^{-j}(J_{k_j})$ contains at most one point. We assume that the family of intervals from the definition of a piecewise monotone map is a generator.

We denote the class of piecewise monotone maps that satisfy the Markov condition and have a generator by $\mathcal{M}(I)$.

From the definitions of lower and upper distribution functions (see (2), (3)) we get that for every $x, y \in I$ and positive integer $k$ $F_{f^k(x)f^k(y)}(t) = F_{xy}(t)$ and $F^*_{f^k(x)f^k(y)}(t) = F^*_{xy}(t)$. If we have points $x, y \in C(f)$, then there are positive integers $k, l$ such that $f^k(x) \in C_0(f)$ and $f^l(y) \in C_0(f)$. We take $m$ as the maximum of these two positive integers and we know that $f_m(x), f_m(y) \in C_0(f)$. We label $u = f^m(x)$ and $v = f^m(y)$ and from the previous we know that $F_{xy}(t) = F_{uv}(t)$ and $F^*_{xy}(t) = F^*_{uv}(t)$. In view of this fact, it is sufficient to consider $C_0(f)$ instead of $C(f)$ in theorems and lemmas, when we deal with $F^*, F$ of pair of points.

We end this section with notations of needed properties. First of them is the property of irreducibility, which describes a subset of generator which is in some sense maximal and any member in this subset can access any member in this subset. The second one replaces the classic definition of a periodic set, which does not work very well in the case of a discontinuous map.

We say that a set $\mathcal{G} \subseteq \mathcal{J}$ has the property of irreducibility if for all pairs of sets $J_i, J_j$ from $\mathcal{G}$ there is a positive integer $n_{i,j}$ such that $f^{n_{i,j}}(\text{int}(J_i))$ covers $\text{int}(J_j)$ and if for all pairs of sets $J_i, J_k$, where $J_i$ is from $\mathcal{G}$ and $J_k$ is from $\mathcal{J} \setminus \mathcal{G}$, no iterations of $\text{int}(J_k)$ intersect $\text{int}(J_i)$ or no iterations of $\text{int}(J_i)$ intersect $\text{int}(J_k)$. Let $A$ be a set with $\text{int}(A) \neq \emptyset$, we say that $A$ is an $f^*$-periodic set if there is $m \geq 1$ with $\text{int}(f^m(A)) = \text{int}(A)$. For $\mathcal{L} \subseteq \mathcal{J}$ we define the set of all admissible sequences $S(f, \mathcal{L}) = \{(\alpha_i) | J_{\alpha_i} \in \mathcal{L} \text{ and } f(J_{\alpha_i}) \cap J_{\alpha_{i+1}} \neq \emptyset\}$.

The following theorem is our main result. Its proof can be found in Sect. 4.

**Theorem 1.** Let $f$ be a piecewise monotonic map with the Markov condition and a generator, then both the Spectrum $\Sigma(f)$ and the Weak Spectrum $\Sigma_w(f)$ are nonempty and finite.
2. Properties of \( \omega \)-limit set

**Lemma 2.** Let \( f \) be a piecewise monotone map with the Markov condition and let \( u, v \in I \). Let \( \{U_i\}_{i=1}^{\infty}, \{V_i\}_{i=1}^{\infty} \) be compact intervals with \( \lim_{i \to \infty} U_i = u \), \( \lim_{i \to \infty} V_i = v \), and such that for all positive integers \( i, j \) there are positive integers \( u(i, j) \) and \( v(i, j) \) with \( V_j \subset f^{u(i,j)}(U_i) \) and \( U_j \subset f^{v(i,j)}(V_i) \). Then \( \{u, v\} \subset \omega(y) \) for some \( y \in I \).

**Proof.** Define a decreasing sequence \( \{J_i\}_{i=1}^{\infty} \) of compact intervals and an increasing sequence \( \{n(i)\}_{i=1}^{\infty} \) of positive integers as follows: \( J_1 = U_1 \) and \( n(1) = u(1, 2) \). Then \( f^{n(1)}(J_1) \supset V_2 \), choose \( J_2 \subset J_1 \) such that \( f^{n(1)}(J_2) = V_2 \). Take \( n(2) = n(1) + v(2, 3) \). Then \( f^{n(2)}(J_2) = f^{n(2, 3)}(V_2) \supset U_3 \) and there is \( J_3 \subset J_2 \) such that \( f^{n(2)}(J_3) = U_3 \). Then there are \( J_4 \) and \( n(3) \) such that \( f^{n(3)}(J_4) = V_4 \), etc. Let \( y \in \bigcap_{i=1}^{\infty} J_i \). Since the trajectory of \( y \) visits every neighborhood of \( u \) and every neighborhood of \( v \), the result follows. \( \square \)

**Lemma 3.** Let \( f \in \mathcal{M}(I) \) and let \( J \) be its generator. Let there be a sequence of integers \( \{\beta_i\}_{i=0}^{\infty} \) such that \( \bigcap_{i=0}^{\infty} f^{-i}(J_{\beta_i}) \neq \emptyset \), then \( \bigcap_{i=0}^{\infty} f^{-i}(J_{\beta_i}) \setminus \bigcup_{i=0}^{\infty} f^{-i}(J_{\beta_i}) \subset C(f) \).

**Proof.** Note that \( f^{-i}(J_{\alpha}) = f^{-i}(J_{\alpha}) \cup C^{i, \alpha} \), where \( C^{i, \alpha} = \{c_1, c_2 \mid f^i(c_1), f^i(c_2) \in \partial(J_{\alpha})\} \subset C(f) \). Therefore it is easy to see that \( \bigcap_{i=0}^{\infty} f^{-i}(J_{\beta_i}) \setminus \bigcup_{i=0}^{\infty} f^{-i}(J_{\beta_i}) \) is either empty, or is equal to \( \{c\} \), where \( c \in C(f) \). \( \square \)

The following lemma is probably the most important lemma of this section, it shows us a strong connection between a maximal \( \omega \)-limit set and a subset of the generator with the property of irreducibility and we use this lemma to prove the lemmas which follow. An alternative version of this lemma can be found in the paper \[4\], where the author used Symbolic dynamics. We do not use these tools in our paper and therefore we give our own proof.

**Lemma 4.** Let \( f \in \mathcal{M}(I) \) and let \( J \) be its generator. Then there is a \( \mathcal{G} \subset J \) with the property of irreducibility if and only if there is \( x \in I \) such that \( \omega(x) \subset \bigcup_{J_i \in \mathcal{G}} \bar{I}_i \) is a maximal basic \( \omega \)-limit set.

**Proof.** Let \( \mathcal{G} \subset J \) have the property of irreducibility. To show that there is \( x \in I \) with maximal basic \( \omega \)-limit set \( \omega(x) \subset \bigcup_{J_i \in \mathcal{G}} \bar{I}_i \) we follow the well-known procedure from Symbolic Dynamics (for details see Theorem 2 in \[4\]), where we construct an infinite sequence of indices of intervals such that any finite admissible sequence from \( S(f, \mathcal{G}) \) will be there and this constructed sequence \( \{\beta_i\}_{i=0}^{\infty} \) is also from \( S(f, \mathcal{G}) \).

By Lemma 3 and the properties of a generator we can see that the intersection \( \bigcap_{i=0}^{\infty} f^{-i}(J_{\beta_i}) \) contains exactly one point \( x \). The trajectory of this point \( x \) visits any neighborhood of any point \( y \) for which its iterations will remain in \( \mathcal{G} \). To see that \( \omega(x) \) is maximal assume that there is some other maximal \( \omega(z) \)
on the irreducible \( \mathcal{G} \), then any point \( y \in \bigcup_{J_i \in \mathcal{G}} \text{int}(J_i) \) for which its iterations will remain in \( \mathcal{G} \) has to be in \( \omega(z) \) and \( \omega(x) \subset \omega(z) \). If there is some some point \( p \in \omega(z) \setminus \omega(x) \), then \( p \in \text{int}(J_k) \) where \( J_k \in \mathcal{J} \setminus \mathcal{G} \) which contradicts the irreducibility of \( \mathcal{G} \).

Conversely, let there is \( x \in \bigcup_{J_i \in \mathcal{G}} J_i \) with maximal basic \( \omega \)-limit set \( \omega(x) \). If there are some \( J_k, J_i \in \mathcal{G} \) with \( f^n(\text{int}(J_k)) \cap \text{int}(J_i) = \emptyset \) for any positive integer \( n \), then it would be impossible for an iteration of the point \( x \) to be infinitely many times in any neighbourhood of any point from the intervals \( J_k \) and \( J_i \), therefore we get that for any two \( J_k, J_i \) from \( \mathcal{G} \) there has to be some positive integer \( n \) with \( f^n(\text{int}(J_k)) \cap \text{int}(J_i) \neq \emptyset \). If there is \( J_k \in \mathcal{J} \setminus \mathcal{G} \) with positive integers \( n, m \) such that \( f^n(\text{int}(J_k)) \cap \text{int}(J_i) \neq \emptyset \) and \( f^m(\text{int}(J_i)) \cap \text{int}(J_k) \neq \emptyset \) for some \( J_i, J_j \in \mathcal{G} \), then \( \mathcal{G} \cup \{J_k\} \) is an irreducible subset of the generator and we can find \( z \) such that \( \omega(z) \) is a maximal basic set with \( \omega(z) \cap \text{int}(J_k) \neq \emptyset \) and \( \omega(x) \subset \omega(z) \), which contradicts the maximality of \( \omega(x) \).

\( \square \)

Remark 5. If \( f(\text{int}(J_j)) \subset \bigcup_{J_i \in \mathcal{G}} J_i \) hold for any \( J_j \in \mathcal{G} \), then there is \( x \in I \) such that \( \omega(x) = \bigcup_{J_i \in \mathcal{G}} J_i \).

Lemma 6. For any interval \( I_p \) from the generator \( \mathcal{J} \) there is at most one \( \omega \in \Omega \) with \( \text{int}(I_p) \cap \omega \neq \emptyset \).

Proof. Assume that \( I_p \in \mathcal{J} \) and \( \omega(x) \) is a maximal basic \( \omega \)-limit set with \( \omega(x) \cap \text{int}(I_p) \neq \emptyset \). From Lemma 4 we can see that \( I_p \in \mathcal{G} \) where \( \mathcal{G} \subset \mathcal{J} \) has the property of irreducibility. If there is some other maximal basic \( \omega \)-limit set \( \omega(y) \neq \omega(x) \) with \( \omega(y) \cap \text{int}(I_p) \neq \emptyset \), then by Lemma 4 there is \( \mathcal{H} \subset \mathcal{J} \) with the property of irreducibility such that \( \omega(y) \subset \bigcup_{J_j \in \mathcal{H}} J_j \) and \( I_p \in \mathcal{H} \). Therefore, \( \mathcal{G} \cup \mathcal{H} \) forms a set with the property of irreducibility and by Lemma 4 there is some maximal basic \( \omega \)-limit set \( \omega(z) \) such that \( \omega(z) \subset \omega(z) \) which contradicts the maximality of \( \omega(x) \).

\( \square \)

The previous lemma tells us that the interior of a set from the generator can intersect only with one maximal basic set, which immediately gives us a finite number of maximal basic sets and also that the intersection of two different maximal basic sets is always finite and under \( C_0(f) \). The following lemma is standard and follows from Lemma 4.

Lemma 7. Let \( f \in \mathcal{M}(I) \) and \( \omega \in \Omega \), then the periodic points of \( \omega \) form a dense subset of \( \omega \).

Lemma 8. Let \( f \in \mathcal{M}(I) \) and \( \mathcal{J} = \{J_1, J_2, \ldots, J_n\} \) be its generator. Let \( \omega \in \Omega \), then \( \omega \) is a perfect set.

Proof. By definition \( \omega \) is a closed set. Now, assume that \( x \in \omega \) is an isolated point. If \( x \) was a nonperiodic point then we would get a contradiction with Lemma 7. Therefore we may assume that \( x \) is a periodic point and since \( \omega \) is
infinite \( x \) is an attracting periodic point, but that is in contradiction to the assumption of existence of a generator and the Markov condition.

\[ \square \]

**Lemma 9.** Let \( f \in M(I) \), let \( x \in I \) such that \( \omega(x) \subset \tilde{\omega} \), where \( \tilde{\omega} \in \Omega \). Then there is a positive integer \( N \) such that \( f^n(x) \in \tilde{\omega} \) for any \( n \geq N \).

**Proof.** From Lemma 4, \( \tilde{\omega} \subset \bigcup_{J_i \in G} \overline{J_i} \) where \( G \) is a subset of \( J \) with the property of irreducibility. If \( f^n(x) \notin \omega(x) \) for any positive integer \( n \), then in \( \omega(x) \) there has to be some attracting periodic point, which is in contradiction to the existence of a generator. Therefore, there is a positive integer \( N \) such that \( f^N(x) \in \omega(x) \subset \tilde{\omega} \) and \( f^N(x) \in J_i \) where \( J_i \in G \). If \( f^N(x) \in J_i \cap C(f) \), then the \( \omega \)-limit set of \( f^N(x) \) is a cycle which cannot be attracting and therefore \( f^{N+j} \in \omega(x) \) for any positive integer \( j \). If \( f^N(x) \in J_i \cap C(f)^\ast \) then from the irreducibility of \( G \) we see that \( f^{N+j}(x) \in J_{\alpha_j} \) where \( J_{\alpha_j} \in G \) for any \( j \), therefore \( f^{N+j}(x) \in \tilde{\omega} \), since \( \tilde{\omega} \) contains any point of \( J_{\alpha_j} \in G \) whose iterations stay in \( \bigcup_{J_{\alpha_j} \in G} \) (see proof of Lemma 4 for details). \( \square \)

The rest of this section deals with results about properties of \( f^\ast \)-periodic sets and are used to prove Lemma 14, which is irreplacable for our Main Theorem. In the rest of this section we assume that our function \( f \) is from \( M(I) \). The following lemma is due to the presence of a generator.

**Lemma 10.** \( C(f) \) is a dense subset of \( I \).

**Lemma 11.** Let \( J = \{J_1, \ldots, J_n\} \) be the generator. For any interval \( (a, b) \subset J_i \), \( J_i \in J \), there is an interval \( (c, d) \subset (a, b) \), a positive integer \( k \) and a set \( L \subset \{1, \ldots, n\} \) such that \( \operatorname{int}(f^k((c, d))) = \operatorname{int}(\bigcup_{l \in L} J_l) \).

**Proof.** By Lemma 10 there are points \( c, d \in (a, b) \), \( c < d \), and positive integers \( j_c, j_d \) such that \( f^{j_c}(c) \in C_0(f) \), \( f^{j_d}(d) \in C_0(f) \) and \( f^{j_{\max}j_{c}, j_{d}}(c) \neq f^{j_{\max}j_{c}, j_{d}}(d) \). The lemma follows from that. \( \square \)

**Corollary 12.** If \( K = (a, b) \subset \operatorname{int}(J_i) \), where \( J_i \in J \), then \( K \) is not an \( f^\ast \)-periodic set.

This corollary tells us that the only possible \( f^\ast \)-periodic sets are sets from the generator, or some union of sets from the generator. To see that there is always an \( f^\ast \)-periodic set it is sufficient to look at the interior of iterations of the whole set \( I \). Let \( \operatorname{int}(f^i(I)) = \operatorname{int}(\bigcup_{l \in L_i} J_l) \) where \( L_i \subset \{1, \ldots, n\} \) and \( J_l \in J \). Either \( L_{i+1} \subset L_i \), or \( L_{i+1} = L_i \). Since we have a finite number of sets in the generator, equality is inevitable. Any \( f^\ast \)-periodic set \( U \) of period \( m \geq 1 \) is either \( U = \operatorname{int}(J_l) \) or \( U = \operatorname{int}(\bigcup_{l \in L} J_l) \), where \( J_l \in J \) and \( L \subset \{1, \ldots, n\} \).

**Corollary 13.** Let \( \omega \in \Omega \). Then there is a minimal \( f^\ast \)-periodic set \( U \) such that \( \operatorname{Orb}(U) \supset \omega \).
Lemma 14. Let $U$ be a minimal $f^*$-periodic set (of period $m \geq 1$) such that $\text{Orb}(U) \supset \omega \in \Omega$. Then if $K_1, K_2$ are intervals such that $K_1 \cap \omega$ is infinite and $K_2 \subset \text{int}(U)$ then $f^{a+bn}(K_1) \supset K_2$ for some $a$ and for all sufficiently large $b$.

Proof. By Lemma 4 $\omega \subset \bigcup_{J_i \in \mathcal{G}} J_i$ where $\mathcal{G}$ is an irreducible subset of $\mathcal{J}$. We can choose $J_l \in \mathcal{G}$ such that $K_1 \cap J_l \cap \omega$ is infinite, $\text{int}(J_l \cap K_1)$ is an interval under $J_l$, by Lemma 11 we can find $k$ such that $f^k(\text{int}(J_l \cap K_1)) \supset \text{int}(\bigcup_{J_g \in \mathcal{G}_1} J_g)$, where $\mathcal{G}_1 \subset \mathcal{G}$. The proof follows from Corollary 13 and the property of irreducibility.

We will describe a situation as above by saying that $f|_\omega$ is strongly transitive in $U$.

3. Distributional functions

Lemma 15. Let $f \in \mathcal{M}(I)$ and $x \in C(f)^c$. Then for any $n \in \mathbb{N}$ there is $\delta > 0$ such that $f|_{f^i(x-\delta,x+\delta)}$ is continuous for $0 \leq i \leq n$.

Proof. For any positive integer $n$ the set $C_n(f)$ is finite, therefore there is $\delta > 0$ such that $(x - \delta, x + \delta) \cap C_n(f) = \emptyset$. □

The following lemmas are modified versions of lemmas from [13] for our case. From now on let $F_{xy}(t)$, $F^*_{xy}(t)$ and $\xi(x,y,k,t)$ be as defined in the Introduction.

Lemma 16. Let $f \in \mathcal{M}(I)$, $c \in C(f)$ and $x, y \in C(f)^c$.

Case a): Let $F^*_{xy}(t)$ and $F_{xy}(t)$ be continuous at $t$, then, for any $\varepsilon > 0$, there are positive integers $k, q$, arbitrarily large, and $\delta > 0$ such that

$$\frac{1}{k} \xi(u,v,k,t) < F_{xy}(t) + \varepsilon$$

(4)

and

$$\frac{1}{q} \xi(u,v,q,t) > F^*_{xy}(t) - \varepsilon$$

(5)

whenever $|x - u| < \delta$ and $|y - v| < \delta$.

Case b): Let $F^*_{xc}(t)$ and $F_{xc}(t)$ be continuous at $t$, then for any $\varepsilon > 0$, there are positive integers $k, q$, arbitrarily large, and $\delta > 0$ such that

$$\frac{1}{k} \xi(u,c,k,t) < F_{xc}(t) + \varepsilon$$

(6)

and

$$\frac{1}{q} \xi(u,c,q,t) > F^*_{xc}(t) - \varepsilon$$

(7)

whenever $|x - u| < \delta$. 
Lemma 17. Let $f \in \mathcal{M}(I)$, $c \in C_0(f)$ and let $\omega_1$ and $\omega_2$ be maximal basic sets. Assume that there are $f^*$-periodic sets $U$, $V$ and countable sets $Q \subset I^2$ and $P \subset I$, where $Q$ is the set of pairs $(u, v)$ such that $u \in \omega_1 \cap \text{int}(U) \cap C(f)^c$ and $v \in \omega_2 \cap \text{int}(V) \cap C(f)^c$, and $P$ is the set of points $p \in \omega_1 \cap \text{int}(U) \cap C(f)^c$, and, furthermore, that $f|_{\omega_1}$ is strongly transitive in $\text{int}(U)$ and $f|_{\omega_2}$ is strongly transitive in $\text{int}(V)$.

Case a): Then there are points $x \in \omega_1 \cap U$ and $y \in \omega_2 \cap V$ such that for any $t > 0$:

$$F_{xy}(t) \leq \inf \{F_{uv}(t); (u, v) \in Q\}$$  \hspace{1cm} (8)

and

$$F_{xy}^*(t) \geq \sup \{F_{uv}^*(t); (u, v) \in Q\}.$$  \hspace{1cm} (9)

Case b): Then there is a point $x \in \omega_1 \cap U$ such that for any $t > 0$

$$F_{xc}(t) \leq \inf \{F_{uc}(t); u \in P\}$$  \hspace{1cm} (10)

and

$$F_{xc}^*(t) \geq \sup \{F_{xc}^*(t); u \in P\}.$$  \hspace{1cm} (11)

Proof. First we prove Case a) Let $T$ be a countable set, dense in $I$, and such that, for any $(u, v) \in Q$ and any $t \in T$, both $F_{uv}$ and $F_{uv}^*$ are continuous at $t$. Let $\{t_j\}_{j=1}^{\infty}$ and $\{u_j, v_j\}_{j=1}^{\infty}$ be sequences of points from $T$ and $Q$, respectively, such that for any $t \in T$ and any pair $(u, v) \in Q$, $t = t_j$, $u = u_j$ and $v = v_j$ for infinitely many $j$.

Next, using induction, we define positive integers

$$k_1 < q_1 < k_2 < q_2 \cdots < k_i < q_i < \cdots$$

and decreasing sequences $\{U_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ of compact intervals with

$$\lim_{i \to \infty} \text{diam}(U_i) = \lim_{i \to \infty} \text{diam}(V_i) = 0,$$

and such that for any $u \in U_n$ and $v \in V_n$ and any $j \leq n$ we get,

$$\frac{1}{k_j} \xi(u, v, k_j, t_j) \leq F_{u_j, v_j}(t_j) + \frac{1}{j}.$$  \hspace{1cm} (12)
and
\[
\frac{1}{q_j} \xi(u, v, q_j, t_j) \geq F_{u_j, v_j}(t_j) - \frac{1}{j}.
\] (13)

Take \( U_1 = U, V_1 = V, k_1 = 1, q_1 = 2 \) and assume that \( U_n, V_n, k_n \) and \( q_n \) have been defined so that \( f^j(U_n) \cap \omega_1 \) and \( f^j(V_n) \cap \omega_2 \) are infinite whenever \( j \) is sufficiently large. Since \( f|_{\omega_1} \) is strongly transitive on \( U \) and \( f|_{\omega_2} \) is strongly transitive on \( V \), there is \( s > q_n \) such that \( u_{n+1} \in f^s(U_n) \) and \( v_{n+1} \in f^s(V_n) \). Let \( a \in U_n \) and \( b \in V_n \) be such that \( f^s(a) = u_{n+1} \) and \( f^s(b) = v_{n+1} \). Then it is easy to see that \( F_{ab} = F_{u_{n+1}v_{n+1}} \) and \( F_{ab}^* = F_{u_{n+1}v_{n+1}}^* \). And the existence of \( U_{n+1} \subset U_n, V_{n+1} \subset V_n, k_{n+1} > q_n \) and \( q_{n+1} > k_{n+1} \) follows from Lemma 16. Note that \( a, b, U_{n+1}, V_{n+1} \) can be chosen so that \( f^s(U_{n+1}) \cap \omega_1 \) resp. \( f^s(V_{n+1}) \cap \omega_2 \) are infinite, see Lemma 8.

Take \( x \in \bigcap_{j=1}^\infty U_j \) and \( y \in \bigcap_{j=1}^\infty V_j \). Then for any \( t \in T \) and any \( (u, v) \in Q \) take \( j \) such that \( t = t_j, u = u_j \) and \( v = v_j \). Since \( x \in U_j \) and \( y \in V_j \), (12) holds with \( u = x \) and \( v = y \). From the fact that \( j \) can be arbitrarily large we have \( F_{xy}(t) \leq F_{uv}(t) \). Therefore (8) is true for any \( t \in T \), and since \( T \) is dense in \( I \), for any \( t \). The argument for (9) is analogous.

It remains to show that \( x \) can be chosen in \( \omega_1 \) and \( y \) in \( \omega_2 \). Let \( w \in \omega_1 \cap \omega \) be such that \( \omega_f(w) = \omega_1 \), and let \( \{W_i\}_{i=1}^\infty \) be a decreasing sequence of compact neighborhoods of \( w \) with \( \lim_{i \to \infty} W_i = w \). Now we apply Lemma 2 to obtain that \( \omega_f(x) \subset \omega_1 \) and similarly \( \omega_f(y) \subset \omega_2 \). By Lemma 9 there is \( n \in \mathbb{N} \) such that \( f^n(x) \in \omega_1 \) and \( f^n(y) \in \omega_2 \). It is easy to see that \( n \) can be chosen so that \( f^n(x) \in U \) and \( f^n(y) \in V \) and it is straightforward that (8) and (9) remain valid.

The proof of Case b) is analogous to the previous proof, with the exception that for any \( j, v_j = v = y = c \).

**Theorem 18.** Let \( f \in \mathcal{M}(I) \) and let \( \Omega = \{\omega_i\}_{i=1}^M \) be the set of maximal basic \( \omega \)-limit sets. For any \( i, j \) from \( \{1, \ldots, M\} \), set \( G_{ij} = \inf \{F_{uv} | u \in \omega_i \text{ and } v \in \omega_j\} \).

Then

1. Each \( G_{ij} \) is zero on an interval \([0, \epsilon(i, j)]\), where \( \epsilon(i, j) \) is a positive number.
2. The set \( \{G_{ij} | \omega_i \cap \omega_j \neq \emptyset\} \) has a finite number of minimal elements.

**Proof.** (1) By Lemma 7 there are periodic points \( p, q \) such that \( p \in \omega_i, q \in \omega_j \) and \( \min \delta_{pq}(i) = \epsilon > 0 \) and therefore \( F_{pq}(t) = 0 \) for \( t \in [0, \epsilon] \). Let \( \epsilon(i, j) = \epsilon, G_{ij} \leq F_{pq} \) gives us the result.

2. We will solve the problem by dividing it into three possible situations.

(a) \( u \in C(f) \cap \omega_i, v \in C(f) \cap \omega_j \) without loss of generality we can assume that \( u \in C_0(f) \cap \omega_i \) and \( v \in C_0(f) \cap \omega_j \). Since we have a finite number of critical points the set \( \{F_{uv} | u \in C_0(f) \cap \omega_i \text{ and } v \in C_0(f) \cap \omega_j\} \) is also finite.
(b) \( u \in C(f) \cap \omega_i, v \in C(f)^c \cap \omega_j \), we can assume that \( u \in C_0(f) \cap \omega_i \). Since \( \omega_j \) is a maximal basic set we have an \( f^* \)-periodic set \( U \) with \( \omega_j \subset \text{Orb}(U) \) by Corollary 13 and Lemma 14 \( f|_{\omega_j} \) is strongly transitive in \( U \). By Lemma 17 for \( u \in C(f) \cap \omega_i \) and any \( v \in C(f)^c \cap \omega_j \) there is \( x \in \omega_j \) with \( F_{ux} \leq F_{uv} \), together with a finite number of critical points we get that the set of minimal elements of the set of distributional functions of such points is finite.

(c) \( u \in C(f)^c \cap \omega_i, v \in C(f)^c \cap \omega_j \), both \( \omega_i \) and \( \omega_j \) are maximal basic \( \omega \)-limit sets, we can find \( f^* \)-periodic sets \( U, V \) with \( \omega_i \subset \text{Orb}(U) \) \( \omega_j \subset \text{Orb}(V) \), \( f|_{\omega_i} \) is strongly transitive in \( U \) and \( f|_{\omega_j} \) is strongly transitive in \( V \). We use Lemma 17 and we have points \( x \in \omega_i \cap U \) and \( y \in \omega_j \cap V \) such that \( F_{xy} \leq F_{uv} \). Therefore for one pair of maximal basic omega-limit sets we have a minimal distributional function, together with a finite number of maximal basic omega-limit set we have a finite number of minimal elements. \( \square \)

**Corollary 19.** The set \( G_{ii} \) has a finite number of minimal elements.

4. Main result

**Theorem 20.** Let \( f \in \mathcal{M}(I) \), then both the Spectrum \( \Sigma(f) \) and the Weak Spectrum \( \Sigma_w(f) \) are nonempty and finite.

**Proof.** To prove the first part, let \( D = \{ F_{uv}|u \text{ and } v \text{ are isotetic} \} \) and \( E = \{ F_{uv}|u, v \in \omega_i, \omega_i \in \Omega \} \). Clearly \( E \subset D \), let \( F_{uv} \in D \). Both \( u \) and \( v \) are isotetic, therefore there is some maximal basic omega-limit set \( \omega \in \Omega \) such that \( \omega_{f^n}(u) \subset \omega \) and \( \omega_{f^n}(v) \subset \omega \) for every positive integer \( n \). From this we immediately get \( \omega(u) \subset \omega \) and \( \omega(v) \subset \omega \). We apply Lemma 9 and we find \( M \in \mathbb{N} \) such that \( f^m(u), f^m(v) \in \omega \) for any \( m \geq M \), let \( u_1 = f^M(u) \) and \( v_1 = f^M(v) \). From the definition of the Distribution function we can see that \( F_{uv} = F_{u_1v_1} \), therefore \( F_{uv} \in E \) and \( D = E \). The Corollary of Theorem 18 gives us the Finite Spectrum \( \Sigma(f) \).

To prove the second part, let \( D_w = \{ F_{uw}|u \in C(f)^c, v \in I \text{ such that } \lim \inf_{n \to \infty} \delta_{uw}(n) = 0 \} \) and \( E_w = \{ F_{uv}|u \in (\omega_i \cap C(f)^c), v \in \omega_j, \omega_i \cap \omega_j \neq \emptyset \} \), where \( \omega_i \) and \( \omega_j \) are maximal \( \omega \)-limit sets. Let \( F_{uv} \in D_w \), \( \omega(u) \subset \omega_i \), which \( u \in C(f)^c \), where \( \omega_i \in \Omega \). If \( v \in C(f) \), then \( \omega(v) \in \Omega \). From \( \lim \inf_{n \to \infty} \delta_{uw}(n) = 0 \) we get that \( \omega(v) \cap \omega_i \neq \emptyset \), see Lemma 4, and therefore \( F_{uv} \in E_w \). If \( v \in C(f)^c \), then \( \omega(v) \subset \omega_j \), where \( \omega_j \in \Omega \). If \( \omega_i \neq \omega_j \), then, since \( \lim \inf_{n \to \infty} \delta_{uw}(n) = 0 \), there is a critical point \( c \) in the intersection of \( \omega_i \) and \( \omega_j \), see Lemma 4, and \( F_{uv} \in E_w \). Therefore \( D_w \subset E_w \). Conversely, let \( F_{uv} \in E_w \). If \( u \in \omega_i \), let \( w \in \omega_i \cap \omega_j \).

Case 1). Let \( u \in C(f)^c \) and \( v \in C_0(f) \). Let us fix \( v \) and let \( u \in \omega_i \in \Omega \). We apply Lemma 17 and we find \( x \in \omega_i \) such that \( F_{xy} \leq F_{uv} \) and \( F^*_{xy} = \chi_{(0,\infty)} \). Therefore \( F_{xy} \in D_w \).
Case 2). If both $u$ and $v$ are in $C(f)^C$, then we can take $Q = \{(u,v), (w,w)\}$ and apply Lemma 17 to get $x, y$ such that $F_{xy} \leq F_{uv}$ and $F^*_{xy} = \chi_{(0,\infty)}$. Hence $\liminf_{n \to \infty} \delta_{xy}(n) = 0$, which implies $F_{xy} \in D_w$.

Thus we proved $D_w \subset E_w$ and if at least one of $u, v$ is not in $C_0(f)$, then $E_w$ has a lower bound in $D_w$, if not (i.e. both $u, v$ are in $C_0(f)$), then we have only finitely many $F_{uv}$. This argument together with Theorem 18 gives us a finite Weak Spectrum $\Sigma_w(f)$.

\[ \square \]

5. Case with infinite spectrum

In this section we present an example of a piecewise monotone map without generator and with infinite spectrum (Fig. 1).

Let $f : I \to I$, where $I = [0,1]$, $J_1 = [0,1/3]$, $J_2 = (1/3, 2/3]$, $J_3 = (2/3, 1]$, and $f(x) = \begin{cases} f_1(x) = 2/3 - x & x \in J_1 \\ f_2(x) = 4/3 - x & x \in J_2 \\ f_3(x) = x - 2/3 & x \in J_3 \end{cases}$

We can see that for any point $x$ in $\text{int}(J_i)$ we have $f^3(x) = x$.

Let $n$ be a positive integer, let $x_n = \frac{4n-1}{18n}$, $y_n = f_1(x_n) = \frac{8n+1}{18n}$, $z_n = f_2(y_n) = \frac{16n-1}{18n}$.

Let additionally $a_n = y_n - x_n = \frac{2n+1}{9n}$ and $b_n = z_n - y_n = \frac{4n-1}{9n}$. We can see that for any $n > 1$ $a_n < b_n$.

The distributional function of points $x_n$ and $y_n$ is $F_{x_n,y_n}(t) = \begin{cases} 0 & 0 \leq t < a_n \\ 1/3 & a_n \leq t < b_n \\ 2/3 b_n \leq t < a_n + b_n \\ 1 & a_n + b_n \leq t \end{cases}$

Since $a_n$ is a decreasing sequence and $b_n$ is increasing we can see that for any $n < m$: $a_m < a_n$ and $b_n < b_m$, $F_{x_n,y_n}(t) < F_{x_m,y_m}(t)$ for $t \in (a_m, a_n)$ and
Figure 2. Lower distributional functions $F_{x_2,y_2}, F_{x_{2000},y_{2000}}$

$F_{x_n,y_n}(t) < F_{x_n,y_n}(t)$ for $t \in (b_n,b_m)$ and therefore they are incomparable. In the Fig. 2 we can see situation for $F_{x_2,y_2}$ and $F_{x_{2000},y_{2000}}$

To see that $f(x)$ does not have a generator let $K = [a,b] \subset \text{int}(J_1)$, then $f^{3m}(K) = K$ for any positive integer $m$.

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