Application of Quasi-Cubic Bézier Curves in the Blending of Tubes with Different Radiiuses

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Abstract. The three shape parameters , of a quasi-cubic Bézier curve can adjust the shape of the curves near its specific control points. In this study, given two pipes of different radiiuses and non-coplanar axes, we first blend the axes of the two pipes using the property of the quasi-cubic Bézier curves and further construct the surface that blends the pipes, which has theoretical significance and application value.

1. Introduction

Surface blending, having a wide range of applications, is a basic problem in computer geometry design and geometry modeling. Previous studies have presented many classical methods, such as [1]-[4].

The smooth blending of tubes with non-coplanar axes is a basic yet difficult problem in the field of computer geometry. So far, little research has been done. Lei Na [5] proposed a method that uses an auxiliary cylinder to obtain a two-piece cubic smooth blending surface for two given cylinders with non-coplanar axes using Wu Wenjun formula. Bai Gen-zhu [6] studied the problem of smoothly blending two tubes with non-coplanar axes along their vertical section with Wu Tieru's method which expands a surface into a standard expression with the cutting plane, meanwhile, obtaining the necessary and sufficient conditions for the existence of a cubic surface. When a cubic surface is smoothly blended to two implicit algebraic surfaces with non-coplanar axes, the conditions for the coefficients of the implicit algebraic surfaces are also given. Furthermore, a smooth blending method for tubes with non-coplanar axis based on the smooth blending of the axes is presented. In [7], two tubes with non-coplanar axes are smoothly blended by a tube with a spatial Bézier curve as its axe, which is called a generalized Bézier tube. A tube with a spatial Bézier curve as its axe can be constructed to smoothly blend two tubes with non-coplanar axes. In [8], a circular pipe with rational Bézier curve as its axe is constructed to smoothly blend two circular pipes with the same radius but non-coplanar axes. There are many advantages using the Bézier curves, but the local shape of the curves cannot be changed. However, the shape of rational Bézier curves, a generalization of Bézier curves, can be fully modified given the premise of keeping the control points unchanged. A rational Bézier curve either gets closer to or further away from its control polygon when proper weight factors which correspond to the control points are chosen. But individual weight factor’s effect on the curves is still unclear. Qin Xinqiang et al [9] introduced a quasi-cubic polynomial basis function with three shape parameters . Without changing the vertices of the control polygon of the cubic Bézier curve, a quasi-cubic Bézier curve with three shape parameters
\( \alpha, \beta, \gamma \) is defined. And it is shown that the parameters play a significant role in adjusting the shape of the curve near the control points. On the basis of smooth blending of the axes of two circular tubes with the same radius but non-coplanar axes using quasi-cubic Bézier curves [10], a blending surface of two circular tubes with the same radius is constructed. We will further study the problem of smooth blending two circular pipes of different radiuses using a circular pipe with a quasi-cubic Bezier curve as its axes.

Definition 1[9] For any \( t \in [0,1] \), \( \alpha, \gamma \in [-3,1], \beta \in [-3,3] \), the following is the polynomial of variable \( t \), which is called the quartic polynomial basis function with shape parameters \( \alpha, \beta, \gamma \).

\[
B_{0,4}(t) = (1-\alpha t)(1-t)^3, \\
B_{1,4}(t) = [3+\alpha(1-t)+\beta t]t(1-t)^2, \\
B_{2,4}(t) = [3-\beta(1-t)+\gamma t]t^2(1-t), \\
B_{3,4}(t) = (1-\gamma + t)t^3.
\]

Definition 2 [9] Given four control points \( V_i \in \mathbb{R}^3 \), for \( t \in [0,1] \), define a quasi-cubic Bézier curve by

\[
P(t; \alpha, \beta, \gamma) = \sum_{i=0}^{3} B_{i,4}(t)V_i,
\]

\( \alpha, \gamma \in [-3,1], \beta \in [-3,3] \).

Where \( \alpha, \beta, \gamma \) are shape parameters. The curve is simply referred to as a cubic CE-Bézier curve. The quasi-cubic Bézier curve passes through the first and last vertices and is tangent to the first and last edges of the characteristic polygon, the same way as a Bézier curve. In addition, by appropriately adjusting the value of the parameters, the shape of the curve adjacent to the individual control vertices is changed accordingly.

2. Constructing a tube that smoothly blends tubes with different radiuses and non-coplanar axes based on the smooth blending of the axes

Let \( \Phi_1, \Phi_2 \) be the parameter representation of two circular tubes with non-coplanar axes, where \( a_i(i=1,2) \) is the radius of the circular tube.

\[
\Phi_1 : \begin{cases} 
  x = x_i + a_1N_{11} \cos \phi + a_1B_{11} \sin \phi, \\
  y = y_i + b_1s + a_1N_{12} \cos \phi + a_1B_{12} \sin \phi, \\
  z = a_1N_{13} \cos \phi + a_1B_{13} \sin \phi.
\end{cases}
\]

And \( \Phi_2 : \begin{cases} 
  x = a_2N_{21} \cos \phi + a_2B_{21} \sin \phi, \\
  y = y_2 + a_2N_{22} \cos \phi + a_2B_{22} \sin \phi, \\
  z = z_2 + c_2s + a_2N_{23} \cos \phi + a_2B_{23} \sin \phi.
\end{cases}
\]

\( N_i = (N_{i1}, N_{i2}, N_{i3}) \) And \( B_i = (B_{i1}, B_{i2}, B_{i3}), i=1,2 \) are the normal vectors and binormal vectors when they are \( s = 1 \) and \( s = 0 \) respectively. \( L_1 \) And \( L_2 \) are the axes of the two circular tubes, which are respectively parallel to \( Y \) in the \( OXY \) plane and intersect with the \( Y \) axis in the \( OYZ \) plane.

\[
\begin{align*}
  &L_1 : \begin{cases} 
    x = x_i + 0 \cdot s, \\
    y = y_i + b_1s, \\
    z = 0 + 0 \cdot s,
  \end{cases} \\
  &L_2 : \begin{cases} 
    x = 0 + 0 \cdot s, \\
    y = y_2 + 0 \cdot s, \\
    z = z_2 + c_2s.
  \end{cases}
\end{align*}
\]

Let \( V_0(x_0,0,0) \), \( V_0(x_1, y_1, 0) \) be two points on \( L_1 \) and \( V_2(0, y_2, 0) \), \( V_3(0, y_2, z_2) \) be two points on \( L_2 \).

With these four points taken as the control points, the characteristic polygons corresponding to the three Bézier curves are constructed. According to the definition of quasi-cubic polynomial basis function with shape parameters \( \alpha, \beta, \gamma \) and the definition of quasi-cubic space Bézier curves with shape parameters \( \alpha, \beta, \gamma \), we can obtain the quartic space Bézier curve that smoothly blends tubes with non-coplanar axes. Its parameters are expressed as
In this way, we can construct a blending tube of tubes with different radiuses and non-coplanar axes. Its parameters are expressed as

\[
x(s, \alpha, \beta, \gamma) = \sum_{i=0}^{3} B_{i,4}(s)x_i, \\
y(s, \alpha, \beta, \gamma) = \sum_{i=0}^{3} B_{i,4}(s)y_i, \\
z(s, \alpha, \beta, \gamma) = \sum_{i=0}^{3} B_{i,4}(s)z_i.
\]

The following is a practical example of blending.

**Example 1.** The parameters of the two tubes with non-coplanar axes are expressed as

\[
\Phi_1: \begin{align*}
x &= 5 + a_1 N_{11} \cos \varphi + a_1 B_{11} \sin \varphi, \\
y &= 2 + s + a_1 N_{12} \cos \varphi + a_1 B_{12} \sin \varphi, \\
z &= a_1 N_{11} \cos \varphi + a_1 B_{11} \sin \varphi. \\
\end{align*}
\]

\[
\Phi_2: \begin{align*}
x &= a_2 N_{23} \cos \varphi + a_2 B_{21} \sin \varphi, \\
y &= 6 + a_2 N_{22} \cos \varphi + a_2 B_{22} \sin \varphi, \\
z &= 3 + a_2 N_{23} \cos \varphi + a_2 B_{23} \sin \varphi. \\
\end{align*}
\]

Where \(a_1\) and \(a_2\) are the radiuses of the two circular tubes, \(N_i = (N_{i1}, N_{i2}, N_{i3})\) and \(B_j = (B_{j1}, B_{j2}, B_{j3}), i = 1, 2 \) are the normal and binormal when E and F, respectively.

The control vertices of the quasi-cubic Bézier curve with three shape parameters \(\alpha, \beta\) and \(\gamma\) are \(V_0(5, 0, 0), V_1(5, 2, 0), V_2(0, 6, 0)\) and \(V_3(0, 6, 3)\), respectively. When \(\alpha = 3, \beta = 2\) and \(\gamma = 1\), the quasi-cubic Bézier curve that smoothly blends the two tubes with non-coplanar axes is

\[
x(s, 3, 2, 1) = 10s^3 - 10s^3 - 5s^2 + 5, \\
y(s, 3, 2, 1) = -14s^4 + 28s^3 - 20s^2 + 12s, \\
z(s, 3, 2, 1) = 3s^4.
\]

The following is the curve that smoothly blends two lines that are generatrixes of the two tubes with non-coplanar axes and different radiuses.

\[
x'(s, 3, 2, 1) = 6s^4 - 6s^3 - 3s^2 + 3, \\
y'(s, 3, 2, 1) = -12s^4 + 26s^3 - 21s^2 + 12s, \\
z'(s, 3, 2, 1) = 3s^4.
\]

The two tubes with non-coplanar axes and different radiuses can then be smoothly blended, and the parameters of the blending tube are expressed as the following.
\[
\begin{align*}
    x(s, 3, 2, 1, \phi) &= 10s^4 - 10s^3 - 5s^2 + 5 + d(s)N_i \cos \phi + d(s)B_i \sin \phi, \\
    y(s, 3, 2, 1, \phi) &= -14s^4 + 28s^3 - 20s^2 + 12s + d(s)N_i \cos \phi + d(s)B_i \sin \phi, \\
    z(s, 3, 2, 1, \phi) &= 3s^4 + d(s)N_i \cos \phi + d(s)B_i \sin \phi,
\end{align*}
\]

Where \( N(s) = (N_i, N_j, N_k) \) and \( B(s) = (B_1(s), B_2(s), B_3(s)) \) are the normal vectors and the binormal vectors of the blending tube at \( s \in [0, 1] \), respectively.

\( a_1 = 2 \) and \( a_2 = 1 \) are the radiuses of the two circular tubes, \( d(s) = \|r'(s) - r(s)\| \cdot \| \) is the Euclidean norm. The blending effect diagram is as follows.

**Figure 1.** Blending surface with quasi-cubic curves as axes, when control vertices \( V_0(5,0,0), V_1(5,2,0), V_2(0,6,0), \) and \( V_3(0,6,3) \)

It could be easily verified that the left and right tangent planes on both sides of the blending line are the same.

**Example 2.** The parameters of the tubes with non-coplanar axes are expressed as

\[
\begin{align*}
    \Phi_1: & \quad x = 5 + a_1 N_{i1} \cos \phi + a_1 B_{i1} \sin \phi, \\
    & \quad y = -4 + s + a_1 N_{i1} \cos \phi + a_1 B_{i1} \sin \phi, \quad \phi \in [0, 2\pi] \\
    & \quad z = a_1 N_{i1} \cos \phi + a_1 B_{i1} \sin \phi.
\end{align*}
\]

\[
\begin{align*}
    \Phi_2: & \quad x = a_2 N_{i2} \cos \phi + a_2 B_{i2} \sin \phi, \\
    & \quad y = 6 + a_2 N_{i2} \cos \phi + a_2 B_{i2} \sin \phi, \quad \phi \in [0, 2\pi] \\
    & \quad z = 5 + s + a_2 N_{i2} \cos \phi + a_2 B_{i2} \sin \phi.
\end{align*}
\]

Where \( a_1 \) and \( a_2 \) are the radiuses of the two circular tubes, \( N_i = (N_{i1}, N_{i2}, N_{i3}) \) and \( B_i = (B_{i1}, B_{i2}, B_{i3}), i = 1, 2 \) are the normal and binormal when \( s = 1 \) and \( s = 0 \), respectively. The control vertices of quasi-cubic Bézier curves with three shape parameters \( \alpha, \beta \) and \( \gamma \) are \( V_0(5,-4,0), V_1(5,0,0), V_2(0,6,0), V_3(0,4,5) \).

Respectively. When \( \alpha = 3, \beta = 2, \gamma = 1 \), the quasi-cubic Bézier curve that smoothly blends the two tubes with non-coplanar axes is

\[
\begin{align*}
    x(s, 3, 2, 1) &= 10s^4 - 10s^3 - 5s^2 + 5, \\
    y(s, 3, 2, 1) &= -26s^4 + 52s^3 - 42s^2 + 24s - 4, \\
    z(s, 3, 2, 1) &= 5s^4.
\end{align*}
\]

The following is the curve that smoothly blends two lines that are generatrixes of the two tubes with non-coplanar axes and different radiuses.
The two tubes with non-coplanar axes and different radiiuses can then be smoothly blended, and the parameters of the blending tube are expressed as the following.

\[
\begin{align*}
\begin{cases}
\frac{d}{ds}x(s,3,2,1) &= 6s^6 - 6s^3 - 3s^2 + 3 \\
\frac{d}{ds}y(s,3,2,1) &= (-2s^2 + 10\sqrt{2}s^2 + (2s - 10\sqrt{2}s^2)^3 + (2s + 5\sqrt{2}s^2)^3 + 25 - 4)
\end{cases}
\end{align*}
\]

The blending effect diagram is as the following.

![Blending surface with quasi-cubic curves as axes](image)

Figure 2. Blending surface with quasi-cubic curves as axes, when control vertices \(V_0(5,-4,0), V_1(5,0,0), V_2(0,6,0), \) and \(V_3(0,4,5)\)

It could be easily verified that the left and right tangent planes on both sides of the blending line are the same.

4. Concluding remarks

By using the property that changing the shape parameters \(\alpha, \beta, \gamma\) of a quasi-cubic Bézier curve can adjust the shape of the curve near its control vertices, we can smoothly blend two tubes with different radiiuses and non-coplanar axes by firstly blending their axes. However, a blending tube with a quasi-cubic curve with shape parameters \(\alpha, \beta, \gamma\) as its axe is just one of the many options to blend tubes with non-coplanar axes. Thus, we can explore more spatial curves to blend tubes with non-coplanar axes for many different applications.

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