ON WEIGHTED POINCARÉ INEQUALITIES

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Abstract. The aim of this note is to show that Poincaré inequalities imply corresponding weighted versions in a quite general setting. Fractional Poincaré inequalities are considered, too. The proof is short and does not involve covering arguments.

1. Introduction

Let \((X, \rho)\) be a metric space with a positive \(\sigma\)-finite Borel measure \(dx\), we will write \(|E| = \int_E dx\) for the measure of a Borel set \(E \subset X\). We fix some point \(x_0 \in X\) and set \(B_r = \{x \in X : \rho(x, x_0) < r\}\), \(\overline{B}_r = \{x \in X : \rho(x, x_0) \leq r\}\).

We call a function \(\phi : B_1 \to [0, \infty)\) a radially decreasing weight, if \(\phi\) is a radial function, i.e. \(\phi = \Phi(\rho(\cdot, x_0))\) and its profile \(\Phi\) is nonincreasing and right-continuous with left-limits. We assume that \(\phi\) is not identically zero on \(B_1 \setminus \overline{B}_{1/2}\).

For any such a weight \(\phi\) there exists a positive, non-zero \(\sigma\)-finite Borel measure \(\nu\) on \((\frac{1}{2}, 1]\), such that

\[
\phi(x) = \int^1_{\rho(x, x_0)\vee 1/2} \nu(dt) = \int^1_{1/2} \chi_{B_t}(x) \nu(dt), \quad x \in B_1 \setminus \overline{B}_{1/2}.
\]

(Note that we put \(\int^b_a f(t) \nu(dt) = \int_{[a,b]} f(t) \nu(dt).\))

For a function \(u\) we denote by

\[
u_E = \frac{1}{|E|} \int_E u(x) \, dx
\]

the mean of \(u\) over the set \(E\), and by

\[
\nu^\phi_E = \frac{\int_E u(x) \phi(x) \, dx}{\int_E \phi(x) \, dx}
\]

the mean of \(u\) over the set \(E \subset B_1\) with respect to the weight function \(\phi\).

Our main result is the following:

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Theorem 1. Let \(1 \leq p < \infty\) and let \(\phi\) be a radially decreasing weight with \(\phi = \Phi(\rho(\cdot, x_0))\). Let \(F : L^p(X) \times (\frac{1}{2}, 1] \to [0, \infty]\) be a functional satisfying
\[
F(u + a, r) = F(u, r), \quad a \in \mathbb{R},
\]
for every \(r \in (\frac{1}{2}, 1]\) and every \(u \in L^p(X)\). Then for \(M = \frac{8^p|B_1|}{|B_{1/2}|} \Phi(0)\)
\[
\int_{B_1} |u(x) - u_{B_1}|^p \, dx \leq F(u, r),
\]
and
\[
\hat{B}_r |u(x) - u_{B_r}|^p \, dx \leq M \int_{1/2}^1 F(u, t) \, \nu(dt)
\]
for every \(u \in L^p(B_1)\), where \(\nu\) is as in (1).

By choosing the functional \(F\) appropriately, (4) becomes a Poincaré inequality with weight \(\phi\), see Section 3. Such inequalities have been studied extensively because of their importance for the regularity theory of partial differential equations, see the exposition in [5].

2. Proof

Lemma 2. Let \(\Omega\) be a finite measure space and \(p \geq 1\). Assume \(f \in L^p(\Omega)\) with \(\int_{\Omega} f = 0\). Then
\[
\|f + a\|_{L^p(\Omega)} \geq \frac{1}{2} \|f\|_{L^p(\Omega)}
\]
for every \(a \in \mathbb{R}\).

Proof. We may assume \(a > 0\). Then
\[
\int_{\Omega \cap \{f > 0\}} |f + a|^p \geq \int_{\Omega \cap \{f > 0\}} |f|^p \quad \text{and} \quad \int_{\Omega \cap \{f < -2a\}} |f + a|^p \geq 2^{-p} \int_{\Omega \cap \{f < -2a\}} |f|^p.
\]
Furthermore, since \(\int_{\Omega \cap \{f \leq 0\}} |f| = \int_{\Omega \cap \{f > 0\}} |f|\), we obtain
\[
\int_{\Omega \cap \{-2a \leq f \leq 0\}} |f|^p \leq (2a)^{p-1} \int_{\Omega \cap \{-2a \leq f \leq 0\}} |f| \leq (2a)^{p-1} \int_{\Omega \cap \{f > 0\}} |f| \leq 2^{p-1} \int_{\Omega \cap \{f > 0\}} |f + a|^p,
\]
where we use \((a^p)b \leq (b + a)^p(a + b)\) for positive \(a, b\). Combining these observations we obtain the result.

Proof of Theorem 1. First we observe that it is enough to prove that
\[
\int_{B_1} |u(x) - u_{B_1}|^p \hat{\phi}(x) \, dx \leq \frac{2^{2p}|B_1|}{|B_{1/2}|} \int_{1/2}^1 F(u, t) \, \nu(dt),
\]
where \(\hat{\phi}(x) = \phi(x) \land \Phi(\frac{1}{2})\). Indeed, we have
\[
\frac{\Phi(\frac{1}{2})}{\Phi(0)} \phi(x) \leq \phi(x) \land \Phi(\frac{1}{2}) \leq \phi(x).
\]
Hence if \([5]\) holds, then
\[
\int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx \geq \frac{\Phi(\frac{1}{2})}{\Phi(0)} \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx
\]
\[
\geq \frac{\Phi(\frac{1}{2})}{\Phi(0)} 2^{-p} \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx,
\]
where in the last line we have used Lemma \([2]\).

Now we prove \([5]\). To simplify the notation, we assume that \(\phi(x) = \Phi(\frac{1}{2})\) for \(x \in B_{1/2}\), so that \(\hat{\phi} = \phi\).

Because of \([2]\), by subtracting a constant from \(u\), we may and do assume that \(u_{B_1}^\phi = 0\), which means that
\[
(6) \quad 0 = \int_{B_1} u(x) \phi(x) \, dx = \int_{1/2}^1 \int_{B_1} u(x) \, dx \, \nu(dt) = \int_{1/2}^1 u_{B_1}|B_1| \, \nu(dt).
\]

We start from the integral on the right hand side of \((4)\) and use \((3)\)

\[
R := \int_{1/2}^1 F(u, t) \, \nu(dt) \geq \int_{1/2}^1 \int_{B_1} |u(x) - u_{B_1}|^p \, dx \, \nu(dt)
\]
\[
= \frac{1}{2} \int_{1/2}^1 \int_{B_1} |u(x) - u_{B_1}|^p \, dx \, \nu(dt) + \frac{1}{2} \int_{1/2}^1 \int_{B_1} |u(x) - u_{B_1} \chi_{B_1}(x)|^p \, dx \, \nu(dt)
\]
\[
=: I_1 + I_2.
\]

(In fact \(I_1 = I_2\), but we treat them differently.) We now deal with the inner integral in \(I_2\). For \(x \in B_{1/2}\) we have
\[
\int_{1/2}^1 |u(x) - u_{B_1}|^p \chi_{B_1}(x) \, \nu(dt) \geq \frac{1}{|B_1|} \int_{1/2}^1 |u(x) - u_{B_1}|^p |B_1| \, \nu(dt).
\]

Since \(\int_{1/2}^1 u_{B_1}|B_1| \, \nu(dt) = 0\), by Lemma \([2]\) we obtain
\[
\int_{1/2}^1 |u(x) - u_{B_1}|^p |B_1| \, \nu(dt) \geq 2^{-p} \int_{1/2}^1 |u_{B_1}|^p |B_1| \, \nu(dt).
\]

Therefore
\[
I_2 \geq \frac{2^{-p}}{2|B_1|} \int_{B_{1/2}} \int_{1/2}^1 |u_{B_1}|^p |B_1| \, \nu(dt) \, dx = \frac{2^{-p}|B_{1/2}|}{2|B_1|} \int_{1/2}^1 |u_{B_1}|^p |B_1| \, \nu(dt).
\]

Using the inequality \(|a|^p + |b|^p \geq 2^{1-p}|a + b|^p\) we obtain
\[
I_1 + I_2 \geq \frac{1}{2} \int_{1/2}^1 \int_{B_1} \left( |u(x) - u_{B_1}|^p + \frac{2^{-p}|B_{1/2}|}{|B_1|} |u_{B_1}|^p \right) \, dx \, \nu(dt)
\]
\[
\geq \frac{2^{-p}|B_{1/2}|}{2|B_1|} 2^{1-p} \int_{1/2}^1 \int_{B_1} |u(x)|^p \, dx \, \nu(dt)
\]
\[
= \frac{|B_{1/2}|}{|B_1|} 2^{-2p} \int_{B_1} |u(x)|^p \phi(x) \, dx
\]
and the proof is finished. \(\Box\)
3. Applications

Let us discuss some corollaries. Corollary 3 is well-known [5]. Our approach allows for more general weights. Proposition 4 allows to deduce a weighted Poincaré inequality for fractional Sobolev norms from an unweighted version. Corollaries 5 and 6 give a more concrete result for fractional Sobolev norms. The first allows for more general kernels and exponents $p$. Corollary 6 improves [2, Theorem 5.1] because the result is robust for $s \to 1$ and allows for general weights and exponents $p$.

**Corollary 3.** Let $p \geq 1$ and $\phi$ be a radially decreasing weight. Consider $X = \mathbb{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. There exists a positive constant $C$ depending on $p, d$ and $\phi$ such that

$$
\int_{B_1} |u(x) - u_{B_1}\phi|^p dx \leq C \int_{B_1} |\nabla u(x)|^p \phi(x) dx,
$$

for every $u \in W^{1,p}(B_1)$.

**Proposition 4.** Let $p \geq 1$ and let $\phi$ be a radially decreasing weight of the form $\phi = \Phi(\rho(\cdot, x_0))$. Assume that for some kernel $k : B_1 \times B_1 \to [0, \infty)$ and some positive constant $C$ the following inequality holds

$$
\int_{B_r} |u(x) - u_{B_1}|^p dx \leq C \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) dy dx,
$$

whenever $r \in (\frac{1}{2}, 1]$ and $u \in L^p(X)$. Then with $M = \frac{\Phi(1/2)}{\Phi(1/2)}$

$$
\int_{B_1} |u(x) - u_{B_1}\phi|^p \phi(x) dx \leq CM \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y)(\phi(y) \wedge \phi(x)) dy dx
$$

for $u \in L^p(X)$.

**Corollary 5.** Let $\phi$ be a radially decreasing weight of the form $\phi = \Phi(\rho(\cdot, x_0))$ and $p \geq 1$. Let $k : B_1 \times B_1 \to [0, \infty)$ be a kernel satisfying $k \geq c$ for some constant $c > 0$. There is a positive constant $M$ depending on $d, p$ and $\Phi$ such that for $u \in L^p(X)$

$$
\int_{B_1} |u(x) - u_{B_1}\phi|^p \phi(x) dx \leq \frac{M}{c} \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y)(\phi(y) \wedge \phi(x)) dy dx
$$

for $u \in L^p(X)$.

**Corollary 6.** Let $p \geq 1$, $R \geq 1$ and $0 < s_0 \leq s < 1$. Consider $X = \mathbb{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. Let $\phi$ be a radially decreasing weight of the form $\phi = \Phi(|\cdot|)$. Then there exists a positive constant $C$ depending on $p, d, s_0$ and $\Phi$ such that

$$
\int_{B_1} |u(x) - u_{B_1}\phi|^p \phi(x) dx \leq C(1-s) \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \chi_{\{|x-y| \leq \frac{1}{2}\}}(\phi(y) \wedge \phi(x)) dy dx
$$

for all $u \in L^p(B_1)$.

**Proof of Corollary 6.** It is well-known that the following Poincaré inequality holds

$$
\int_{B_r} |u(x) - u_{B_r}|^p dx \leq r^p \int_{B_r} |\nabla u(x)|^p dx
$$
for every \( u \in W^{1,p}(B_r) \) and \( r > 0 \) where \( c > 0 \) depends on \( p \) and \( d \). Set
\[
F(u, r) = c \int_{B_r} |\nabla u(x)|^p \, dx,
\]
for \( u \in W^{1,p}(B_1) \) and \( F(u, r) = \infty \) otherwise. Then for \( u \in W^{1,p}(B_1) \)
\[
\int_{1/2}^1 F(u, t) \, dt = c \int_{1/2}^1 t^p \int_{B_1} |\nabla u(x)|^p \chi_{B_1}(x) \, dx \, dt
\]
\[
\leq c \int_{B_1} |\nabla u(x)|^p \int_{1/2}^1 \chi_{B_1}(x) \, dt \, dx = c \int_{B_1} |\nabla u(x)|^p \phi(x) \, dx.
\]
By Theorem 4 the assertion follows with \( C = 2^{3p+d} \frac{\Phi(0)}{\Phi(1/2)} c. \)

**Proof of Proposition 4.** Let
\[
F(u, r) = c \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) \, dy \, dx.
\]
Then
\[
\int_{1/2}^1 F(u, t) \, dt = c \int_{1/2}^1 \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \chi_{B_1}(y) \chi_{B_1}(x) \, dy \, dx \, dt
\]
\[
= c \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \int_{1/2}^1 \chi_{B_1}(y) \chi_{B_1}(x) \, dt \, dy \, dx
\]
\[
= c \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) (\phi(y) \wedge \phi(x)) \, dy \, dx.
\]
The assertion now follows from Theorem 1.

**Proof of Corollary 5.** First we use a well-known argument to obtain a nonweighted Poincaré inequality. By calculus and convexity of the function \( x \mapsto |x|^p \) we conclude that \( |a + b|^p \geq |a|^p + |b|^p |a|^{p-1} \text{sgn}(a) \). Thus
\[
\int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) \, dy \, dx \geq c \int_{B_r} \int_{B_r} |(u(x) - u_{B_r}) + (u_{B_r} - u(y))|^p \, dy \, dx
\]
\[
\geq c |B_r| \int_{B_r} |u(x) - u_{B_r}|^p \, dx
\]
\[
\geq c |B_{1/2}| \int_{B_r} |u(x) - u_{B_r}|^p \, dx,
\]
whenever \( u \in L^p(B_r) \) and \( \frac{1}{2} < r \leq 1 \).
The assertion follows now from Proposition 4.

In the proof of Corollary 6 we use the following auxiliary result.

**Lemma 7.** Let \( R \geq 1, p \geq 1 \) and \( 0 < s < 1 \). Then
\[
(13) \quad \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} \, dy \, dx \leq (3R)^{p(1-s)} \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} \chi_{\{|x-y| \leq \frac{R}{2}\}} \, dy \, dx
\]
for all $u \in L^p(B_1)$.

Proof. Let $n$ be a natural number such that $n \geq 2R > n - 1$. We introduce

$$A_k = A_k(x,y) = \frac{k}{n}y + \frac{n-k}{n}x, \quad k = 0, 1, \ldots n.$$  

Then

$$I = \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} dy \, dx = \int_{B_1} \int_{B_1} \frac{\sum_{k=1}^n (u(A_{k-1}) - u(A_k))^p}{|x-y|^{d+ps}} dy \, dx$$

$$\leq n^{p-1} \sum_{k=1}^n \int_{B_1} \int_{B_1} \frac{|u(A_{k-1}) - u(A_k)|^p}{|x-y|^{d+ps}} dy \, dx.$$ 

Note that $|A_{k-1} - A_k| = \frac{1}{n}|x-y|$. If we substitute $\hat{x} = A_{k-1}, \hat{y} = A_k$, then $d\hat{y} \, d\hat{x} = n^{-d} dy \, dx$ (which follows by an elementary calculation, see also [3, page 570]). Moreover, $\hat{x}, \hat{y} \in B_1$ with $|\hat{x} - \hat{y}| \leq 2nR$. Hence

$$I \leq n^{p-ps} \int_{B_1} \int_{B_1} \frac{|u(\hat{x}) - u(\hat{y})|^p}{|\hat{x} - \hat{y}|^{d+ps}} \chi_{|\hat{x} - \hat{y}| \leq \frac{1}{n}} \, d\hat{y} \, d\hat{x}.$$ 

Since $n < 2R + 1 \leq 3R$, the assertion follows. □

Proof of Corollary [7]. From [4] and [1, page 80] we know that there exists a constant $C = C(p,d,s_0)$, such that for $s_0 \leq s < 1$

$$\int_{B_r} |u(x) - u_{B_r}|^p \, dx \leq C(1-s)^{p\lambda} \int_{B_r} \int_{B_r} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} dy \, dx,$$

for all $u \in L^p(B_1)$. The assertion now follows from (14), Proposition 4 and Lemma 7. □

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