On the problem of hidden variables for quantum field theory

Andrei Khrennikov
School of Mathematics and Systems Engineering
University of Växjö, S-35195, Sweden

October 12, 2018

Abstract
We show that QFT (as well as QM) is not a complete physical theory. We constructed a classical statistical model inducing quantum field averages. The phase space consists of square integrable functions, \( f(\phi) \), of the classical bosonic field, \( \phi(x) \). We call our model prequantum classical statistical field-functional theory – PCSFFT. The correspondence between classical averages given by PCSFFT and quantum field averages given by QFT is asymptotic. The QFT-average gives the main term in the expansion of the PCSFFT-average with respect to the small parameter \( \alpha \) – dispersion of fluctuations of “vacuum field functionals.” The Scrödinger equation of QFT is obtained as the Hamilton equation for functionals, \( F(f) \), of classical field functions, \( f(\phi) \). The main experimental prediction of PCSFFT is that QFT gives only approximative statistical predictions that might be violated in future experiments.

PACS: 03.65.Ca, 03.50.-z, 03.70.+k

1 Historical introduction on the problem of completeness of quantum theories

The problem of hidden variables is closely related to the problem of completeness of quantum mechanics that was discussed in the paper
of Einstein, Podolsky, Rosen [1] (see also Bohr’s reply to Einstein in [2]). We note that the views of Einstein and Bohr were in the process of the permanent evolution, see, e.g., [3] for comments. However, it is important to remark that A. Einstein was always sure that quantum mechanics is not complete. And this was in spite of so called “NO-GO” theorems (e.g., von Neumann’s theorem [4]).

A. Einstein did not believe that the wave function provides the complete description of a quantum system. In particular, he was one of the founders of the so called ensemble interpretation of the wave function, see also L. Ballentine [5]. By this interpretation the so called pure quantum state $\psi$ is not pure at all. It describes not the state of an individual quantum system, but statistical properties of a huge ensemble $S_\psi$ of quantum systems. Another important remark is that investigations of A. Einstein in the late part of his life were concentrated on finding a pure field model of physical reality, including quantum reality, see, e.g., [1].

We note to that even the last part of Schrödinger’s life was characterized by comeback to creation of purely field foundation of quantum mechanics [6], [7]. But, in contrast to Einstein, Schrödinger’s attitude was toward quantum field theory (Einstein was more interested in classical field theory).

Since typically N. Bohr did not express his views clearly enough, it is not completely clear how he understood “completeness of quantum mechanics” [8] (see also A. Plotnitsky for detail [9], [10]). My personal impression of Bohr’s writings is that he considered completeness with respect to physical phase space $\Omega_{\text{phys}} = \mathbb{R}^3 \times \mathbb{R}^3$. N. Bohr was sure that it is impossible to provide a finer description of a quantum system based on $\Omega$ than given by the $\psi$-function. However, I am not sure that he would claim that it would be impossible to do this on the basis of a more general model of phase space. In any event in his correspondence with W. Heisenberg he always discussed impossibility to provide a detailed description of quantum phenomena by using classical coordinates and momenta [11].

This long historical introduction was presented to convince the reader that there still exists a possibility (in spite of a rather common opinion) to create a model with hidden variables which would reproduce statistical predictions of quantum mechanics. Such a model was presented in [12]-[14]. This is a classical statistical mechanics with phase-space $\Omega = H \times H$, where $H$ is a real Hilbert space. We emphasize that our phase space $\Omega$ is different from the conventional phase
space $\Omega_{\text{phys}} = \mathbb{R}^3 \times \mathbb{R}^3$, cf. with the previous discussion on views of Bohr and Heisenberg. It is extremely important to remark that the conventional quantum mechanics we obtain through a very special choice of $H$, namely $H = L_2(\mathbb{R}^3)$, the space of square integrable functions $\psi : \mathbb{R}^3 \to \mathbb{R}$. Thus quantum mechanics can be reproduced on the basis of classical statistical mechanics on phase space:

$$
\Omega = L_2(\mathbb{R}^3) \times L_2(\mathbb{R}^3).
$$

This is the space of classical vector fields, $\psi(x) = (q(x), p(x))$. Here the field $q(x)$ plays the role of the (infinite-dimensional) coordinate and the field $p(x)$ plays the role of momentum.

Thus our classical field model for quantum mechanics can be considered as the “Einstein-Schrödinger dream” (at least late Einstein and early Schrödinger). The most important deviation from the traditional ideas on a pre-quantum classical statistical model is that in our approach a pre-quantum model does not reproduce precisely quantum averages $<A>_D$ (where $A$ is a quantum observable represented by a self-adjoint operator and $D$ is a statistical state represented by a density operator).

Quantum mechanics is a statistical approximation of pre-quantum classical statistical field theory (PCSFT). There is a small parameter of the model $\alpha$. Where $\alpha \to 0$, PCSFT is reduced to quantum mechanics. We recall that, when $h \to 0$, quantum mechanics is reduced to ordinary classical statistical mechanics on phase space $\Omega_{\text{phys}}$. In [12]-[14] I identified small parameters $\alpha$ and $h$. It seems that it was not correct. In [15] I proposed to distinguish parameters $\alpha$ and $h$. The parameter $\alpha$ is small in quantum mechanics, but the Planck constant $h$ can be chosen as equal to 1 (for the Planck system of units).

As far as I know, in quantum field theory the problem of hidden variables was never discussed, see e.g., [16], [17]. Roughly speaking it was meaningless to study this problem for quantum field theory, since even for quantum mechanics there were proved various NO-GO theorems. It was commonly believed that quantum field theory is a complete theory. The wave function $f(\psi)$ given by the formalism of second quantization provides the complete description of the quantum field. However, after the publication of papers [12]-[15] on the asymptotic solution of the problem of hidden variables in quantum mechanics it became clear that it is not meaningless to consider the problem of hidden variables for quantum field theory. In particular, the postulate on completeness of quantum field theory can be questioned.
In this paper we apply the method of asymptotic dequantization developed in [12]-[15] for quantum mechanics to quantum field theory. We show that (as well as quantum mechanics) quantum field theory can be considered as a statistical approximation of classical statistical mechanics for a specially chosen phase space $\Omega$. Here $\Omega$ consists of functionals $f(\phi)$ of classical fields $\phi$. Classical physical variables are given by functionals of such functionals: $f \to F(f)$. Classical statistical states are given by Gaussian ensembles of functional $f(\phi)$. In this paper we restrict our considerations to the case of scalar boson field $\phi(x)$. The same procedure of asymptotic dequantization can be applied to other fields, but it needs a lot of technical efforts.

We also remark that our investigations on asymptotic dequantization are closely related to so called contextual probabilistic approach to quantum mechanics, see also [18] (cf. with conditional probabilistic approach of G. Mackey, L. Accardi, L. Ballentine, E. Beltrametti, W. De Muynck, S. Gudder, [19]-[23]). We found a natural realization of the general contextual probabilistic model by representing contexts by Gaussian ensembles of classical fields (for quantum mechanics) or field functionals (for quantum field theory). So called prespace [18] - space preceding both quantum noncommutative space (given by the Heisenberg algebra) and classical phase space $\Omega_{\text{phys}} = \mathbb{R}^3 \times \mathbb{R}^3$ - is given by infinite-dimensional phase space $\Omega = H \times H$.

In our model the phase space of the classical prequantum field model is given by $\Omega = L_2(S'(\mathbb{R}^3), \mu) \times L_2(S'(\mathbb{R}^3), \mu)$, where $S'(\mathbb{R}^3)$ is the space of Schwartz distributions, and $\mu$ is the Gaussian measure on $S'(\mathbb{R}^3)$ corresponding to the free boson field $\phi$. Statistical states are represented by Gaussian measures on $\Omega$. They describe ensembles of functionals $f(\phi)$ of classical fields $\phi \in S'(\mathbb{R}^3)$. Physical variables are given by functionals $F(f(\cdot))$ of field functionals $f : S'(\mathbb{R}^3) \to \mathbb{R}$.

Quantum field operators $A$ are obtained as second derivatives of such functionals $F$ at the zero point: $F \to A = \frac{F''(0)}{2}$.

In our approach classical averages are not equal to quantum field averages. There is only an asymptotic relation between the classical average and the quantum field average. Thus the conventional quantum field theory gives only the first order approximation of the prequantum classical statistical model. Our prequantum field model contains a small parameter $\alpha \to 0$. In fact, we consider a one parameter family $M^\alpha$ of classical statistical models. QFT is obtained as the limit $\lim_{\alpha \to 0}$ of $M^\alpha$:

$$\lim_{\alpha \to 0} M^\alpha = N_{\text{QFT}},$$

(2)
where $N_{QFT}$ is the conventional quantum field model. We point out that the problem of the classical limit of QFT has been discussed both on physical and mathematical levels of rigorousness, see, e.g., [24]. The Planck constant $h$ was considered as a small parameter: $N_{QFT} \equiv N_{QFT}^h, h \to 0$. It was shown (see, e.g., [25], [26] for the rigorous mathematical considerations) that:

$$\lim_{h \to 0} N_{QFT}^h = M_{\text{cl}, \text{inf}},$$

where $M_{\text{cl}, \text{inf}}$ is the classical statistical model with the infinite dimensional phase space.

However, we study the opposite problem: to represent the QFT-model $N_{QFT}$ as the $\lim_{\alpha \to 0}$ of classical statistical models $M^\alpha$. In this framework QFT is just the $\alpha \to 0$ approximation of a special classical statistical model. The latter can be called prequantum classical statistical field-functional theory, PCSFFT.

The small parameter $\alpha$ gives the dispersion of fluctuation of prequantum field functionals, $f(\phi)$:

$$\int_{L_2(S'(\mathbb{R}^3)),\mu} \left( \int_{S'(\mathbb{R}^3)} |f(\phi)|^2 d\mu(\phi) \right) d\rho(f) = \alpha$$

(4)

Here $f(\phi)$ is a “classical field” on the infinite-dimensional configuration space $S'(\mathbb{R}^3)$ – field functional, and $\rho$ is a Gaussian measure representing an ensemble of such field functionals.

2 Gaussian quantization of the scalar boson field

Let us consider the pseudo-differential operator $a = \sqrt{-\Delta + m^2}, m > 0$. We pay attention that the operator $a^{-1}$ is continuous in $S(\mathbb{R}^3)$. Thus the quadratic (positively defined) form $b(\phi, \phi) = (a^{-1}\phi, \phi)$ is also continuous on $S(\mathbb{R}^3)$. By the Minlos-Sazonov theorem the Gaussian measure $\mu$ with zero mean value and the covariation operator $b_\mu = \text{cov} \; \mu = a^{-1}$ is $\sigma$-additive on the $\sigma$-algebra of Borel subsets of the space $S(\mathbb{R}^3)$. Let us consider the Hilbert space $L_2(S'(\mathbb{R}^3), \mu)$, consisting of functionals $f : S'(\mathbb{R}^3) \to \mathbb{R}$ such that

$$||f||^2 = \int_{S'(\mathbb{R}^3)} f^2(\phi) d\mu(\phi) < \infty.$$
The basic operators of QFT, e.g., free Hamiltonian $H_0$ and the operator of the number of particles $N$, are constructed with the aid of the procedure of the second quantization, see, e.g., [24]. There is a natural realization of this procedure within the calculus of infinite-dimensional pseudo-differential operators in $L_2(S'(\mathbb{R}^3, \mu))$, [25], [26].

Let an operator $\lambda : S(\mathbb{R}^3) \to S(\mathbb{R}^3)$ be continuous and let it be symmetric with respect to the scalar product in $L_2(\mathbb{R}^3, dx)$. Its second quantization is defined as an operator $d\Gamma(\lambda) : L_2(S'(\mathbb{R}^3, \mu)) \to L_2(S'(\mathbb{R}^3), \mu)$ which can be defined, for example, with the aid of its symbol:

$$d\Gamma(\lambda)(q, p) = (b_\mu \lambda p, p) + i(q, \lambda p), \ p \in S(\mathbb{R}^3), q \in S'(\mathbb{R}^3). \quad (5)$$

The quantization procedure is performed through the representation of the classical field variables, $p \equiv p(x), q \equiv q(x)$ by the operators:

$$(q, r) \to (q, r)f(\phi) = (\phi, r)f(\phi), r \in S(\mathbb{R}^3); \quad (6)$$

$$(s, p) \to (s, p)f(\phi) = \frac{1}{i} \left( s, \frac{\delta}{\delta \phi} \right) f(\phi), s \in S'(\mathbb{R}^3). \quad (7)$$

Thus

$$d\lambda(q, p) = -(b_\mu \lambda \frac{\delta}{\delta \phi}, \frac{\delta}{\delta \phi}) + (\phi, \lambda \frac{\delta}{\delta \phi}). \quad (8)$$

For example, for $\lambda = a = \sqrt{-\Delta + m^2}$ we get the free field Hamiltonian:

$$H_0 = d\Gamma(\sqrt{-\Delta + m^2}) = \frac{1}{2} \int_{\mathbb{R}^3} \frac{\delta^2}{\delta \phi^2}(x) dx + \int_{\mathbb{R}^3} \phi(x) \sqrt{-\Delta + m^2} \frac{\delta}{\delta \phi(x)} dx. \quad (9)$$

If $\lambda = I$ is the unit operator, then we obtain the operator of the number of particles:

$$N = d\Gamma(1) = -\frac{1}{2} \int_{\mathbb{R}^3} \frac{\delta}{\delta \phi(x)} (-\Delta + m^2)^{-1/2} \frac{\delta}{\delta \phi(x)} dx + \int_{\mathbb{R}^3} \phi(x) \frac{\delta}{\delta \phi(x)} dx. \quad (10)$$

We remark that these operators are not bounded in $L_2(S'(\mathbb{R}^3), \mu)$. But they, of course, can be approximated by bounded operators corresponding to approximation of the kernel of the operator $(-\Delta + m^2)^{\pm 1/2}$ by smooth functions. Therefore in our further considerations we restrict ourselves to the QFT-model with bounded quantum field operators.
The QFT-model is defined as the pair:

\[ N_{\text{QFT}} = (\mathcal{D}(\Omega_c), \mathcal{L}_s(\Omega_c)) \]

where \( \Omega_c = L^2_2(S'({\mathbb R}^3), \mu) \) is the space of square integrable with respect to the Gaussian measure \( \mu \) functionals \( f : S'({\mathbb R}^3) \rightarrow \mathbb{C} \); \( \mathcal{D} \) is the space of density operators \( (D : \Omega_c \rightarrow \Omega_c, D \geq 0, \text{Tr}D = 1) \), \( \mathcal{L}_s \) is the space of self-adjoint continuous operators \( (A : \Omega_c \rightarrow \Omega_c, A^* = A) \).

## 3 A classical statistical model for QFT

We choose the phase space \( \Omega \), consisting of square integrable field functionals, \( \phi \rightarrow f(\phi) \). Thus \( \Omega = Q \times P \), where \( Q = P = L_2(S'({\mathbb R}^3), \mu) \).

We consider on \( \Omega \) the canonical symplectic structure given by the operator

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Denote by \( J \) the one parametric group with the generator \( J \):

\[ J_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \theta \in \mathbb{R}. \]

A function (in fact, functions of functionals of fields \( \phi \in S'({\mathbb R}^3) \)) \( F : \Omega \rightarrow \mathbb{R} \) is called \( J \)-invariant if

\[ F(J_\theta f) = F(f), \ f \in \Omega, \quad (11) \]

for any \( \theta \in [0, 2\pi) \). In our further considerations the following simple mathematical fact will play an important role:

**Lemma 1.** Let \( F : \Omega \rightarrow \mathbb{R} \) be two times Frechet differentiable and \( J \)-invariant. Then

\[ F''(0)J = JF''(0) \quad (12) \]

**Corollary 1.** A quadratic form \( F(f) = (Hf, f) \) is \( J \)-invariant iff \( HJ = JH \).

Let us denote by \( \Omega_c \) the phase-space \( \Omega \) endowed with the canonical complex structure induced by the symplectic structure on it:

\[ \Omega_c = Q \oplus iP \equiv L^2_2(S'({\mathbb R}^3), \mu). \]
By Lemma 1, for any $C^2$-map $F : \Omega \to \mathbb{R}$ which is $J$-invariant, its second derivative defines the $C$-linear operator
\[ f''(0) : \Omega_c \to \Omega_c. \]
In particular, any quadratic $J$-invariant form $F(f) = (Hf, f)$ can be represented in the form $F(f) = \langle Hf, f \rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical complex scalar product on $\Omega_c$:
\[ \langle f, g \rangle = \int_{S'(\mathbb{R}^3)} f(\phi) \overline{g(\phi)} d\mu(\phi). \]

We denote by the symbol $\Omega^C$ the complexification $\Omega \oplus i\Omega$ of the phase space $\Omega$:
\[ \Omega^C = [L_2(S'(\mathbb{R}^3), \mu) \times L_2(S'(\mathbb{R}^3), \mu)] \oplus i[L_2(S'(\mathbb{R}^3), \mu) \times L_2(S'(\mathbb{R}^3), \mu)]. \]
The space of classical physical variables, denoted by $\mathcal{V}(\Omega)$, we choose in the following way: a) $F(0) = 0$; b) $F$ can be continued to the analytic function $F : \Omega^C \to \mathbb{C}$; c) $F$ is $J$-invariant; d) $F$ has the exponential growth on $\Omega^C$:
\[ |F(f)| \leq a_F e^{r_F \|f\|^2}, f \in \Omega^C. \quad (13) \]
The following simple mathematical facts will play important roles in our future considerations.

**Lemma 2.** Let a measure $\rho$ on $\Omega$ be $J$-invariant. Then its covariation operator $B = \text{cov}_{\rho}$ commutes with the symplectic operator $J : [B, J] = 0$.

**Lemma 3.** A Gaussian measure $\rho$ (with the zero mean value) is $J$-invariant iff $[B, J] = 0$.

**Lemma 4.** Let a measure $\rho$ on $\Omega$ be $J$-invariant. Then the "coordinate" $q(\phi)$ and the "momentum" $p(\phi)$ give the equal contributions into its dispersion:
\[ \int_{L_2(S'(\mathbb{R}^3), \mu) \times L_2(S'(\mathbb{R}^3), \mu)} \left( \int_{S'(\mathbb{R}^3)} q^2(\phi) d\mu(\phi) \right) d\rho(q, p) \]
\[ = \int_{L_2(S'(\mathbb{R}^3), \mu) \times L_2(S'(\mathbb{R}^3), \mu)} \left( \int_{\mathbb{S}(\mathbb{R}^3)} p^2(\phi) d\mu(\phi) \right) d\rho(q, p) \quad (14) \]
We choose the space of classical statistical states denoted by the symbol $S^G_\alpha(\Omega)$ – consisting of Gaussian measures on the phase space $\Omega$ (having the zero mean value) such that:

a) the dispersion of $\rho \in S^G_\alpha(\Omega)$ equals to $\alpha$:

$$\sigma^2(\rho) = \int_{L^2(S'(\mathbb{R}^3),\mu) \times L^2(S'(\mathbb{R}^3),\mu)} \left( \int_{S'(\mathbb{R}^3)} (q^2(\phi) + p^2(\phi))d\mu(\phi) \right) d\rho(q,p) = \alpha, \alpha \to 0;$$

b) any $\rho \in S^G_\alpha(\Omega)$ is $J$-invariant:

$$\int_{L^2(S'(\mathbb{R}^3),\mu) \times L^2(S'(\mathbb{R}^3),\mu)} f(\cos \theta q + \sin \theta p, -\sin \theta q + \cos \theta p) d\rho(q,p) = \int_{L^2(S'(\mathbb{R}^3),\mu) \times L^2(S'(\mathbb{R}^3),\mu)} f(q,p) d\rho(q,p).$$

We note that

$$\sigma^2(\rho) = \text{Tr}B,$$

where $B = \text{cov} \rho$ is the covariance operator of $\rho$. We also point out that $\rho \in S^G_\alpha(\Omega)$ implies that $[B, J] = 0$, see Lemma 2, and that by Lemma 4:

$$\int_\Omega ||q||^2 d\rho(q,p) = \int_\Omega ||\rho||^2 d\rho(q,p) = \frac{\alpha}{2}.$$

We shall also use the complex covariation operator of $\rho, B^c = \text{cov}^c \rho$ which is given by

$$<B^c u, v> = \int <u, f> <f, v> d\rho(f(\cdot)),$$

where $f(\phi) = q(\phi) + ip(\phi)$.

**Lemma 5.** Let a measure $\rho$ be $J$-invariant. Then $B^c = 2B$ (in particular, there is one-to-one correspondence between real and complex covariation operators).

**Lemma 6.** There is one-to-one correspondence between between Gaussian $J$-invariant measures and complex covariation operators: $\rho \to B^c = \text{cov}^c \rho = 2\text{cov} \rho$.

We pay attention that by using the trace with respect to the complex Hilbert space $\Omega_c$ we can write

$$\sigma^2(\rho) = \text{Tr}B^c$$

---

1They describe ensembles of physical systems having states belonging to the phase space $\Omega$. 

9
We define one parametric family of classical statistical models:

\[ M^\alpha = (S^\alpha_G(\Omega), \mathcal{V}(\Omega)). \]

In the Gaussian integral \( \int_\Omega F(f) d\rho(f) \) we make the scaling: \( f(\phi) = \sqrt{\alpha} f(\phi) \). By considering \( \alpha \) as a small parameter, \( \alpha \to 0 \), and using the Taylor expansion of analytic functionals, \( F(f) \), on the space of square integrable field-functionals, \( f(\phi) \), we obtain the following asymptotic expansion of Gaussian integrals on the phase space, see appendix for the detailed proof (the proof for QM presented in [1] should be modified in that way to become mathematically correct):

**Lemma 7.** Let \( F \in \mathcal{V}(\Omega) \) and let \( \rho \in S^\alpha_G(\Omega) \). Then

\[ < F >_\rho \equiv \int_\Omega F(q,p) d\rho(q,p) = \frac{\alpha}{2} \text{Tr} D^c F''(0) + O(\alpha^2), \alpha \to 0 \quad (18) \]

where \( D^c = \text{cov}^c \rho/\alpha \) and

\[ |O(\alpha^2)| \leq \alpha^2 K_F \int_\Omega e^{r_F||f||^2} d\rho_D^c(f), \]

and \( \rho_D^c \) is the Gaussian measure (\( \sqrt{\alpha} \)-scaling of \( \rho \)) with the complex covariation operator \( D^c \).

The equality (21) motivates the following definition of the asymptotic projection of the one parametric family of classical statistical models \( M^\alpha \) onto the QFT-model \( N_{QFT} \):

\[ T : \mathcal{V}(\Omega) \to \mathcal{L}_a(\Omega_c), T(F) = F''(0)/2 \quad (19) \]

Thus the classical physical variable \( F : \Omega \to \mathbb{R} \) (functional of functionals \( f(\phi) = (q(\phi), p(\phi)) \) of classical fields \( \phi \in S'(\mathbb{R}^3) \)) is mapped into its second derivative. This is really a projection having the huge degeneration.

\[ T : S^\alpha_G(\Omega) \to \mathcal{D}(\Omega_c), \rho \to D^c = \text{cov}^c \rho/\alpha. \quad (20) \]

By Lemma 3 this map is one-to-one. One can formulate previous considerations in the form of a theorem:

**Theorem 1.** The one parametric family of classical statistical models \( M^\alpha \) provides the asymptotic "dequantization" of QFT for the scalar bosonic field. There exists projections given by (19) and (20) of spaces of classical physical variables and statistical states onto spaces...
of self-adjoint operators (quantum field operators) and density operators such that the asymptotic equality of classical and QFT averages take place:

\[ < F >_{\rho} = \alpha < T(F) >_{T(\rho)} + O(\alpha^2), \alpha \to 0. \]  

(21)

Denote by the symbol \( \mathcal{V}_{\text{quad}} \) the space of quadratic forms \( F : \Omega \times \Omega \to \mathbb{R} \) which are \( \mathcal{J} \)-invariant. Thus \( F(f, f) = (Af, f) \), where \([A, J] = 0\). Let us consider the one parametric family of classical statistical models: \( M^\alpha_{\text{quad}} = (S^1_G(\Omega), \mathcal{V}_{\text{quad}}(\Omega)) \).

**Corollary 1.** The family \( M^\alpha_{\text{quad}} \) provides the "explicit dequantization" of QFT. Both dequantization maps, (19) and (20), are one-to-one and classical and QFT averages coincide:

\[ < F >_{\rho} \equiv \int_\Omega (Af, f) d\rho(f) = \text{Tr} \text{cov}^c \rho A. \]  

(22)

However, we consider the explicit dequantization given by corollary 1 as a purely mathematical construction, cf. [1], [27] for QM. The essence of correspondence between classical and quantum worlds is the asymptotic expansion of classical statistical averages\(^2\).

4 **Interpretation, structure of vacuum**

The point \( f_{\text{vacuum}} = 0 \in \Omega \) we call the classical vacuum state. This is the field functional \( f_{\text{vacuum}}(\phi) \) which equals to zero for any classical bosonic field \( \phi \in S'(\mathbb{R}^3) : f_{\text{vacuum}}(\phi) = (q_{\text{vacuum}}(\phi), p_{\text{vacuum}}(\phi)) \) and \( q_{\text{vacuum}} \equiv 0, p_{\text{vacuum}} \equiv 0 \) on \( S'(\mathbb{R}^3) \).

Thus the vacuum field \( f_{\text{vacuum}} \) is defined not on the conventional physical space \( \mathbb{R}^3 \), but on the infinite dimensional space of classical bosonic fields \( S'(\mathbb{R}^3) \). This is the crucial departure from the conventional picture of vacuum. In any event, in our approach “fluctuations of vacuum” are fluctuations of the vacuum field functional \( f_{\text{vacuum}}(\phi) \), cf. [28]–[34]. Such fluctuations can be described by measures on \( \Omega \) having the very small dispersion.

Such a measure represents a random variable \( f(\lambda, \phi) \in \Omega \) (here \( \lambda \) is a random parameter)\(^3\) and its standard deviation gives the measure\(^2\).
of deviation from the vacuum field functional $f_{\text{vacuum}}(\phi)$:

$$Df(\lambda, \phi) = E[|\langle\lambda, \phi| - f_{\text{vacuum}}(\phi)\rangle^2] = E\left(\int_{S'(\mathbb{R}^3)} |f(\lambda, \phi)|^2 d\mu(\phi)\right).$$

Thus $\sigma(f) = \sqrt{Df(\lambda, \phi)}$ tells us how much the random field functional $f(\lambda, \phi)$ deviates from the vacuum. Therefore we can interpret our statistical states $\rho \in S^\alpha_0(\Omega)$ as Gaussian fluctuations of vacuum. Here $\alpha$ can be interpreted as intensity of vacuum fluctuations.

Let $F$ be a classical physical variable, $F : \Omega \rightarrow \mathbb{R}$. We can consider the relative intensity:

$$F_\alpha(f) = \frac{F(f)}{\alpha} \equiv \frac{F(q, p)}{\alpha}$$

The basic equality of the asymptotic dequantization of QFT, see [21], can be written as

$$< F_\alpha >_\rho = < T(F) >_{T(\rho)} + O(\alpha), \alpha \rightarrow 0. \quad (24)$$

Thus the QFT-average $< T(F) >_{T(\rho)} \equiv \text{Tr}_T(\rho)T(F)$ gives us the main term in the expansion of the classical average of the relative intensity with respect to the vacuum of fluctuations,

$$< F_\alpha >_\rho = \int_\Omega F_\alpha(f) d\rho(f) = \frac{1}{\alpha} \int_\Omega F(f) d\rho(f) \approx < T(F) >_{T(\rho)}. \quad (25)$$

5 Quantum field Schrödinger equation as Hamilton equation for field functionals

We consider the system of Hamilton equations on the phase space of field functionals $\Omega = L_2(S'(\mathbb{R}^3), \mu) \times L_2(S'(\mathbb{R}^3), \mu)$:

$$\dot{q} = \frac{\delta H}{\delta p}, \quad \dot{p} = -\frac{\delta H}{\delta q} \quad (25)$$

where $H : \Omega \rightarrow \mathbb{R}$ is a function of the class $C^1$ (so it is Frechet differentiable with continuous first derivative $\nabla H(q, p) = (\frac{\delta H}{\delta q}(q, p), \frac{\delta H}{\delta p}(q, p)) \in \Omega$).

We introduce the symplectic gradient of the Hamilton function:

$$J\nabla H(q, p) = (\frac{\delta H}{\delta p}(q, p), -\frac{\delta H}{\delta q}(q, p)),$$
and we write the system of Hamilton equations in the vector form:

$$\dot{f}(t, \phi) = J\nabla \mathcal{H}(f(t, \phi))$$  \hspace{1cm} (26)

For example, let \( \mathcal{H}(f) = \frac{1}{4} \int_{\mathbb{R}^3} \left[ \left( \int_{\mathbb{R}^3} \frac{\delta f(\phi(x))}{\delta \phi(x)} dx \right)^2 + 2 \left( \int_{\mathbb{R}^3} \phi(x) \sqrt{-\Delta + \mu^2 \frac{\delta f(\phi(x))}{\delta \phi(x)}} \right) f(\phi) \right. $$

$$\left. + \left( \int_{\mathbb{R}^3} |f(\phi(x))|^2 dx \right)^2 \right] d\mu(\phi).$$

Then the Hamilton equation has the form:

$$\dot{f}(t, \phi) = J\mathbf{H}_0 f(t, \phi) + \int_{\mathbb{R}^3} |f(\phi(x))|^2 dx f(t, \phi),$$  \hspace{1cm} (27)

where the linear operator \( \mathbf{H}_0 \) is the Hamiltonian of the free quantum bosonic field, see [19]. Now let us restrict our consideration by quadratic Hamilton functions \( \mathcal{H} \in \mathcal{V}_{\text{quad}}(\Omega) \). In this case \( \mathbf{H} = \mathcal{H}''(0) \) commutes with the symplectic operator \( J \) and, hence, the equation (26) can be written in the complex form:

$$i\dot{f}(t, \phi) = \mathbf{H} f(t, \phi).$$  \hspace{1cm} (28)

This is nothing else than the Schrödinger equation for QFT [16], [24].

**Theorem 2.** For \( \mathcal{H} \in \mathcal{V}_{\text{quad}}(\Omega) \), the Hamilton equation can be written as the Schrödinger equation for QFT-Hamiltonian \( \mathbf{H} = \mathcal{H}''(0) \).

We note that quadratic Hamilton functions describe Harmonic oscillators in the space of field functionals \( f(\phi) \). If \( \mathbf{H} = \text{diag}(R, R) \), then the Hamilton function \( \mathcal{H}(f) = \frac{1}{2}[(Rp, p) + (Rq, q)] \) and the Hamilton equations have the form:

$$\dot{q}(t, \phi) = Rp(t, \phi), \hspace{0.5cm} \dot{p}(t, \phi) = -Rq(t, \phi).$$  \hspace{1cm} (29)

We can call field functionals \( q(t, \phi) \) and \( p(t, \phi) \) \( \phi \in \mathcal{S}'(\mathbb{R}^3) \), mutually-inducing: the presence of \( p(t, \phi) \) induces change of \( q(t, \phi) \) and vice versa, cf. with classical electromagnetic field \( E(t, x), B(t, x), x \in \mathbb{R}^3 \). The system of the Hamilton equations (29) induces the second order equation:

$$\ddot{q}(t, \phi) + R^2 q(t, \phi) = 0.$$  \hspace{1cm} (30)

**Theorem 3.** QFT (for the scalar bosonic field) can be represented as classical statistical mechanics of Gaussian ensembles of harmonic oscillators in the space of classical field functionals \( f(\phi), \phi \in \mathcal{S}'(\mathbb{R}^3) \).
6 Complex representation for the Hamilton dynamics

As usual, we introduce complex variables
\[ f(\phi) = q(\phi) + ip(\phi), \quad f^*(\phi) = q(\phi) - i\phi, \quad \phi \in S'(\mathbb{R}^3). \]

**Proposition 1.** A map \( F(q, p) \) is \( J \)-invariant iff
\[ F(\lambda f, \lambda^* f^*) = F(f, f^*), |\lambda| = 1. \quad (31) \]

**Proposition 2.** Let \( F(f, f^*) \) be analytic. Then it is \( J \)-invariant iff
\[ F(f, f^*) = \sum_{n=0}^{\infty} \frac{\delta^n F}{\delta f^n \delta f^*} (0)(f, \ldots, f, f^*, \ldots, f^*). \quad (32) \]

In the complex variables \( f \) and \( f^* \) the system of Hamilton equations can be written as
\[ i\dot{f}(t, \phi) = 2\frac{\delta H}{\delta f^*}(f(t, \phi), f^*(t, \phi)). \quad (33) \]

cf. [16], [17], [24], [26].

7 Appendix: Proof of lemma 7 on asymptotic expansion of Gaussian functional integrals

In the Gaussian integral \( \int_{\Omega} F(f)d\rho(f) \) we make the scaling:
\[ f(\phi) = \sqrt{\alpha} f(\phi). \quad (34) \]

We obtain:
\[ < F >_\rho = \int_{\Omega} F(\sqrt{\alpha} f)d\rho_{D^\alpha}(f) = \frac{\alpha}{2} \int_{\Omega} (F''(0)f, f) d\rho_{D^\alpha}(f) + \alpha^2 R(\alpha, F, \rho), \quad (35) \]

where
\[ R(\alpha, F, \rho) = \int_{\Omega} g(\alpha, F; f)d\rho_{D^\alpha}(f), g(\alpha, F; f) = \sum_{n=4}^{\infty} \frac{\alpha^{n/2-2}}{n!} F^{(n)}(0)(f, \ldots, f). \]
We note that
\[
\int_{\Omega} (F'(0), f) d\rho_D = 0, \quad \int_{\Omega} F'''(0)(f, f, f) d\rho_D = 0,
\]
because the mean value of \(\rho\) (and, hence, of \(\rho_D\)) is equal to zero. Since \(\rho \in S_G^\alpha(\Omega)\), we have that the real trace \(\text{Tr} D = 1\). Hence, even the complex trace \(\text{Tr} D^c = 1\).

We now estimate the rest term \(R(\alpha, F, \rho)\). By using exponential growth of the functional \(F(f)\) on the complexification \(\Omega^c\) of the phase space \(\Omega\) we obtain: we have for \(\alpha \leq 1\):
\[
|g(\alpha, F; f)| = \sum_{n=4}^\infty \frac{\|F^{(n)}(0)\|\|f\|^2}{n!} \leq c_F \sum_{n=4}^\infty \frac{r_F^n\|f\|^2}{n!} = C_F e^{r_F\|f\|^2}.
\]
Thus: \(|R(\alpha, F, \rho)| \leq c_F \int_{\Omega} e^{r_F\|f\|^2} d\rho_D = \alpha^2\). We obtain:
\[
<F> = \frac{\alpha}{2} \int_{\Omega} (F'(0), f) d\rho_D + O(\alpha^2), \quad \alpha \to 0. \quad (36)
\]

By performing Gaussian integration we finally come the asymptotic equality \((21)\).

REFERENCES

[1] Einstein A., Podolsky B., Rosen N., Phys. Rev., 47 (1935) 777.
Einstein A., The Collected Papers of Albert Einstein (Princeton Univ. Press, Princeton) 1993.
[2] Bohr N., Phys. Rev., 48 (1935) 696.
[3] Plotnitsky A., Quantum atomicity and quantum information: Bohr, Heisenberg, and quantum mechanics as an information theory, in Quantum theory: reconsideration of foundations, edited by Khrennikov A. Yu. (Växjö Univ. Press) 2002, pp. 309-343.
[4] Von Neumann J., Mathematische Grundlagen der Quantenmechanik (Springer, Berlin) 1932.

\[4\] The change of variables can be considered as scaling of the magnitude of statistical (Gaussian) fluctuations. Negligibly small random fluctuations \(\sigma(\rho) = \sqrt{\alpha}\) (where \(\alpha\) is a small parameter) are considered in the new scale as standard normal fluctuations. If we use the language of probability theory and consider a Gaussian random variables \(\xi(\lambda)\), then the transformation is nothing else than the standard normalization of this random variable (which is used, for example, in the central limit theorem): \(\eta(\lambda) = \frac{\xi(\lambda) - E\xi}{\sqrt{E(\xi(\lambda) - E\xi)^2}}\) (in our case \(E\xi = 0\).
Von Neumann J., *Mathematical foundations of quantum mechanics* (Princeton Univ. Press, Princeton, N.J.) 1955.

[5] Ballentine L. E., *Rev. Mod. Phys.*, **42** (1970) 358.

[6] Schrödinger E., *Philosophy and the Birth of Quantum Mechanics*, edited by Bitbol M., Darrigol O. (Editions Frontières, Gif-sur-Yvette) 1992; especially the paper of D’Agostino S., *Continuity and completeness in physical theory: Schrödinger’s return to the wave interpretation of quantum mechanics in the 1950’s*, pp. 339-360.

[7] Schrödinger E., *E. Schrödinger Gesammelte Abhandlungen* (Wieweg and Son, Wien) 1984; especially the paper *What is an elementary particle?*, pp. 456-463.

[8] Bohr N., *Niels Bohr: Collected Works*, Vol. 1-10 (Elsevier, Amsterdam) 1972-1996.

[9] Plotnitsky A., *The Knowable and Unknowable (Modern Science, Nonclassical Thought, and the “Two Cultures”*) (Univ. Michigan Press) 2002.

[10] Plotnitsky A., *Found. Phys.* **33** (2003) 1649.

[11] Heisenberg W., *Physical Principles of Quantum Theory* (Chicago Univ. Press) 1930.

[12] Khrennikov A. Yu., *Prequantum classical statistical model with infinite dimensional phase-space*, *J. Phys. A: Math. Gen.* **38** (2005) 9051.

[13] Khrennikov A. Yu., *Generalizations of quantum mechanics induced by classical statistical field theory*, *Found. Phys. Letters*, **18** (2005) 637.

[14] Khrennikov A. Yu., *Interpretation of stationary states in prequantum classical statistical field theory*, *Found. Phys. Letters*, (2005)-accepted for publication.

[15] Khrennikov A. Yu., *Quantum mechanics as an asymptotic projection of statistical mechanics of classical fields: derivation of Schrödinger’s, Heisenberg’s and von Neumann’s equations*, [http://www.arxiv.org/abs/quant-ph/0511074](http://www.arxiv.org/abs/quant-ph/0511074).

[16] Bogolubov N. N., *Quantum Field Theory* (Taylor and Francis), 1995.

[17] Segal I., *Mathematical Foundations of Quantum Field Theory* (Benjamin, New York) 1964.

[18] Khrennikov A. Yu., *J. Phys.A: Math. Gen.*, **34** (2001) 9965; *Il Nuovo Cimento, B* **117** (2002) 267; *J. Math. Phys.*, **43** (2002) 789; *J. Math. Phys.*, **44** (2003) 2471; *Phys. Lett. A*, **316** (2003) 279; *Annalen der Physik*, **12** (2003) 575.
[19] Accardi L., “The probabilistic roots of the quantum mechanical paradoxes” in The wave–particle dualism. A tribute to Louis de Broglie on his 90th Birthday, edited by Diner S., Fargue D., Lochak G., and Selleri F. (D. Reidel Publ. Company, Dordrecht) 1984, pp. 297–330.

Accardi L., Urne e Camaleoni: Dialogo sulla realtà, le leggi del caso e la teoria quantistica (Il Saggiatore, Rome) 1997.

[20] Ballentine L. E., Quantum mechanics (Englewood Cliffs, New Jersey) 1989.

[21] Beltrametti E. G., The Logic of Quantum Mechanics (Addison-Wesley) 1981.

[22] De Muynck W. M., Foundations of Quantum Mechanics, an Empiricists Approach (Kluwer, Dordrecht) 2002.

[23] Gudder S. P., Axiomatic Quantum Mechanics and Generalized Probability Theory (Academic Press, New York) 1970.

Gudder S. P., An approach to quantum probability in Foundations of Probability and Physics, edited by A. Yu. Khrennikov, Quantum Prob. White Noise Anal., 13 (WSP, Singapore) 2001, pp. 147-160.

[24] Berezin F. A., Method of Second Quantization (Academic Press) 1966.

[25] Khrennikov A. Yu., Equations with infinite-dimensional pseudo-differential operators, Dissertation for the degree of candidate of phys-math. sc., Dept. Mechanics-Mathematics (Moscow State University, Moscow) 1983.

[26] Khrennikov A.Yu., Superalysis (Nauka, Fizmatlit, Moscow) 1997 (in Russian). English translation: (Kluwer, Dordrecht) 1999.

[27] Bach A., J. Math. Phys., 14 (1981) 125; 21 (1980) 789; Phys. Lett. A, 73 (1979) 287.

[28] De la Pena L. and Cetto A. M., The Quantum Dice: An Introduction to Stochastic Electrodynamics (Kluwer, Dordrecht) 1996; Boyer T. H., A Brief Survey of Stochastic Electrodynamics in Foundations of Radiation Theory and Quantum Electrodynamics, edited by Barut A. O. (Plenum, New York) 1980; Boyer T. H., Timothy H., Scientific American, pp. 70-78, Aug 1985; see also an extended discussion on vacuum fluctuations in: M. Scully O., Zubairy M. S., Quantum Optics (Cambridge University Press, Cambridge) 1997; Louisell W. H., Quantum Statistical Properties of Radiation (J. Wiley, New York) 1973; Mandel L. and Wolf E., Optical Coherence and Quantum Optics (Cambridge University Press, Cambridge) 1995.

[29] Cavalleri G., Nuovo Cimento B, 112 (1997) 1193.

[30] Zecca A. and Cavalleri G., Nuovo Cimento B, 112, (1997)
1; Cavalleri G. and Tonni E., “Discriminating between QM and SED with spin”, in C. Carola and A. Rossi, The Foundations of Quantum Mechanics (Historical Analysis and Open Questions) (World Scientific Publ., Singapore), p.111, 2000.

[31] Nelson E., Quantum fluctuation (Princeton Univ. Press, Princeton) 1985.

[32] Albeverio S., Höegh-Krohn R., Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 40 (1977) 59.

[33] Davidson M., J. Math. Phys. 20 (1979) 1865; Physica A, 96 (1979) 465.

[34] Morgan P., Phys. Lett. A, 338 (2005) 8; 321 (2004) 216.