A Central Limit Theorem for incomplete U-statistics over triangular arrays

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Abstract. We analyze the fluctuations of incomplete U-statistics over a triangular array of independent random variables. We give criteria for a Central Limit Theorem (CLT, for short) to hold in the sense that we prove that an appropriately scaled and centered version of the U-statistic converges to a normal random variable. Our method of proof relies on a martingale CLT. An application, a CLT for the hitting time for random walks on random graphs, will be presented in Löwe and Terveer (2020).

1 Introduction

U-statistics constitute a general method to construct unbiased minimum variance estimators in the theory of statistics. A thorough investigation of their properties can be found, e.g., in the monographs by Denker (1985) or Lee (1990). U-statistics also naturally appear in other contexts, like in the theory of random graphs where they count occurrences of certain subgraphs (cf. Janson (1990)). In the latter case the U-statistics are incomplete, by which we mean that not all possible combinations of the random variables are taken into account. Such a ”dilution” can also be random, as considered in Janson (1984). After having established a law of large numbers for U-statistics (cf. Christofides (1992)) the most obvious next question is to analyze their asymptotic distribution. This was already investigated in a seminal paper by Hoeffding (1948). In general, whether a U-statistic is asymptotically normal or not, may depend on whether its kernel function is degenerate (Denker (1985)), i.e. on whether the conditional expectation of the kernel function given some the variables is zero. Berry-Esseen theorems and Edgeworth expansions around this CLT were analyzed e.g. in Bentkus and Götze (1995), Bickel et al. (1986), and Tan (2013). Fluctuation results for U-statistics on the level of large or moderate deviations were obtained

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in Eichelsbacher and Löwe (1995), Eichelsbacher and Löwe (1998), and Eichelsbacher (1998).

In this note we will study a situation where the random variables in the U-statistic stem from a triangular array as in Malevich and Abdurakhmanov (1987). Additionally, we consider incomplete U-statistics, where a random variable determines whether a certain summand is taken into account or not. Finally, also the kernel function $h$ may change with $n$, the line number of the triangular array. This situation is motivated by our analysis of hitting times for random walks on random graphs in Löwe and Terveer (2020). However, as this situation also is a generalization of the settings in Janson (1984), Jammalamadaka and Janson (1986), and Malevich and Abdurakhmanov (1987), we think it might be also interesting in its own rights.

More precisely let $X_{n1}, \ldots, X_{nn}, n \in \mathbb{N}$ be a triangular array of random variables with values in $\mathbb{R}$, independent of each other and having the same distribution function $F_n(x)$ in each row. Let $h_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a real-valued, symmetric Borel function. For $i, j = 1, \ldots, n$, let

$$\Phi_n(i, j) = Z_{ij} \cdot h_n(X_{ni}, X_{nj})$$

(of course, $\Phi_n$ is a function of $Z_{ij}, X_{ni}$, and $X_{nj}$ rather than just of $i, j$; however, for the sake of brevity we will omit the variables here and in the following definitions). Here the $Z_{ij} = Z_{ji}$ are assumed to be i.i.d. $\text{Ber}(p_n)$ random variables (apart from the constraint that $Z_{ij} = Z_{ji}$) that are independent of the $(X_{ni})$. We assume $p = p_n$ may depend on $n$, but that $np \to \infty$. Moreover, assume that for all $n \in \mathbb{N}$ and $1 \leq i \neq j \leq n$

$$\mathbb{E}[h_n(X_{ni}, X_{nj})] = 0 \quad \text{and} \quad \mathbb{E}[h_n^2(X_{ni}, X_{nj})] < \infty. \quad (1.1)$$

Let us construct the following U-statistic

$$U_n = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \Phi_n(i, j) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} Z_{ij} \cdot h_n(X_{ni}, X_{nj}).$$

To construct a Hoeffding-type decomposition (see Hoeffding (1948)) we introduce

$$\Psi_n(i) := \mathbb{E}[\Phi_n(i, j) \mid X_i, Z_{ij}] = Z_{ij} \mathbb{E}[h_n(X_{ni}, X_{nj}) \mid X_{ni}] \quad (1.2)$$

$$\beta_n^2 := \mathbb{E}[\Phi_n^2(1, 2)], \quad \gamma_n^2 := \mathbb{E}\left[\left(\Psi_n(1)\right)^2\right], \quad \text{and} \quad \theta_n^2 := np_n \gamma_n^2 + \beta_n^2/2. \quad (1.3)$$

Then obviously, $\Phi_n(i, j)$ and $\Psi_n(i)$ are centered. Next, put

$$\tilde{\Phi}_n(i, j) = \Phi_n(i, j) - \Psi_n(i) - \Psi_n(j),$$

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\( \tilde{h}_n(X_{ni}, X_{nj}) = h_n(X_{ni}, X_{nj}) - \mathbb{E}[h_n(X_{ni}, X_{nj}) | X_{ni}] - \mathbb{E}[h_n(X_{ni}, X_{nj}) | X_{nj}] \).

Then \( \tilde{\Phi}_n(i, j) = Z_{i,j} \tilde{h}_n(X_{ni}, X_{nj}) \) is again centered – we even have for every \( i \neq j \) and every \( k \)

\[
\mathbb{E} \left[ \tilde{h}_n(X_{ni}, X_{nj}) | X_{nk} \right] = 0, \tag{1.4}
\]
even for \( k = i, j \) (as can be seen by applying the definition of \( \tilde{h}_n \)).

In the following we omit the index \( n \) whenever suitable. We will also frequently write \( h(i, j) \) and \( \tilde{h}(i, j) \) as shorthand notations for \( h_n(X_{ni}, X_{nj}) \) and \( \tilde{h}_n(X_{ni}, X_{nj}) \). Let us collect some properties of the above quantities in the following lemma, whose proof is deferred to the appendix:

**Lemma 1.1.** For any \( i \neq j \) we have:

1. \( \mathbb{E} \left[ \tilde{\Phi}^2(i, j) \right] = \beta_n^2 - 2\gamma_n^2 \).
2. \( \mathbb{E} \left[ \mathbb{E} \left[ h(i, j) | X_i \right]^2 \right] = \frac{\gamma_n^2}{p}, \mathbb{E} \left[ h^2(i, j) \right] = \frac{\beta_n^2}{p} \) and \( \mathbb{E} \left[ \tilde{h}(i, j)^2 \right] = \frac{\beta_n^2 - 2\gamma_n^2}{p} \).

We can now compute the Hoeffding decomposition of \( U_n \) (again the proof is given in the appendix):

**Lemma 1.2.** We can rewrite \( U_n \) in the following way:

\[
U_n = \sum_{i=1}^{n} \left( \left( \frac{n}{2} \right)^{-1} \sum_{j=1}^{i-1} \tilde{\Phi}(i, j) + \left( \frac{n}{2} \right)^{-1} \sum_{j=1}^{n} \sum_{j \neq i} \Psi_j(i) \right) \tag{1.5}
\]

This allows to compute the asymptotic variance of \( U_n \) (the proof is given in the appendix):

**Lemma 1.3.** For the variance of \( U_n \) we have the following asymptotic identities

\[
\forall U_n \sim \left( \frac{n}{2} \right)^{-1} (\beta_n^2 + 2np\gamma_n^2) = \left( \frac{n}{2} \right)^{-1} 2\theta_n^2 \text{ and } \sqrt{\mathbb{V} U_n} \left( \frac{n}{2} \right) \sim n\theta_n.
\]

Here and below, for two sequences \( (a_n) \) and \( (b_n) \) we write \( a_n \sim b_n \), if \( a_n/b_n \to 1 \). To prove a CLT for \( U_n \) we will consider

\[
\frac{U_n}{\sqrt{\mathbb{V} U_n}} \sim \sum_{i=1}^{n} \xi_i
\]

with \( \xi_{i,n} = \xi_i = \xi^{(1)}_i + \xi^{(2)}_i \) for \( i = 1, \ldots, n \), where

\[
\xi^{(1)}_i = \xi^{(1)}_{i,n} = \frac{1}{n\theta_n} \sum_{j=1}^{n} \Psi_j(i), \quad \xi^{(2)}_i = \xi^{(2)}_{i,n} = \frac{1}{n\theta_n} \sum_{j=1}^{n} \tilde{\Phi}(i, j).
\]
We are aiming to prove the following results:

**Theorem 1.4.** Assume that for all \( \varepsilon > 0 \)

\[
\eta_1 = \sum_{i=1}^{n} \mathbb{E} \left[ \xi_i^2 \mathbb{1}_{\{|\xi_i| \geq \varepsilon\}} \right] \left( X_k \right)_{k=1,\ldots,i-1}, \left( Z_{l,m} \right)_{l=1,\ldots,i-1}, \frac{1}{n} \xrightarrow{\mathbb{P}} 0, \quad (1.6)
\]

\[
\eta_2 = \sum_{i=1}^{n} \mathbb{E} \left[ \xi_i^2 \mathbb{1}_{\{|\xi_i| \geq \varepsilon\}} \right] \left( X_k \right)_{k=1,\ldots,i-1}, \left( Z_{l,m} \right)_{l=1,\ldots,i-1}, \frac{1}{n} \xrightarrow{\mathbb{P}} 1. \quad (1.7)
\]

Then \( \frac{U_n}{\sqrt{\nu_n}} \) converges in distribution to a standard normal random variable.

For an alternative conditions introduce

\[
G_k(i,j) = \mathbb{E} \left[ \Phi(i,k) \Phi(j,k) \mid X_i, X_j, Z_{ik}, Z_{jk} \right] =: Z_{ik} Z_{jk} H(i,j) \quad (1.8)
\]

\[
\tilde{G}_k(i,j) = \mathbb{E} \left[ \tilde{\Phi}(i,k) \tilde{\Phi}(j,k) \mid X_i, X_j, Z_{ik}, Z_{jk} \right] =: Z_{ik} Z_{jk} \tilde{H}(i,j). \quad (1.9)
\]

Then,

**Theorem 1.5.** Assume that for all \( \varepsilon > 0 \)

\[
\frac{1}{n \theta_n^2} \mathbb{E} \left[ \left( \sum_{j=2}^{n} \Psi_j(1) \right)^2 \mathbb{1}_{\{|\sum_{j=2}^{n} \Psi_j(1)| \geq \varepsilon \theta_n n\}} \right] \xrightarrow{n \to \infty} 0 \quad (1.10)
\]

\[
\theta_n^{-2} \mathbb{E} \left[ \tilde{\Phi}^2(1,2) \mathbb{1}_{\{|\tilde{\Phi}(1,2)| \geq \varepsilon \theta_n n\}} \right] \xrightarrow{n \to \infty} 0 \quad (1.11)
\]

\[
p \theta_n^{-2} \mathbb{E} \left[ \tilde{H}(1,1) \mathbb{1}_{\{|\tilde{H}(1,1)| \geq \varepsilon \theta_n n\}} \right] \xrightarrow{n \to \infty} 0 \quad (1.12)
\]

\[
\theta_n^{-4} \mathbb{E} \left[ G_1^2(2,3) \right] \xrightarrow{n \to \infty} 0 \quad (1.13)
\]

Then \( \frac{U_n}{\sqrt{\nu_n}} \) converges in distribution to a standard normal random variable.

**Remark 1.6.** It is well known that there are situations where \( U_n \) does not obey a CLT. E.g. when the \( Z_{ij} \equiv 1 \) with probability 1, and if the kernel function \( h = h_n \) satisfies \( \mathbb{E} h(X_i, Y) = 0 \) for independent \( X, Y \) with the same distribution as \( X_1 \) (for the time being we assume that the distribution of \( X_{ni} \) does not depend on \( n \) and \( i \)). For a typical situation consider \( \mathbb{P}(Z_{ij} = 1) = 1 \) and \( h(X_1, X_2) = X_1 X_2 \). If then, \( \mathbb{E} X_1 = 0 \) and \( \mathbb{E} X_1^2 = 1 \), by the CLT together with an application of the continuous mapping theorem, the rescaled \( U \)-statistic \( n U_n \) does not converge to a normally distributed random variable but to a \( \chi^2 \)-random variable. But then also condition (1.13) breaks down. Indeed, one checks that \( \theta_n^2 = \frac{1}{2} \), since \( \beta_n^2 = 1 \), and that \( \gamma_n^2 = 0 \). Moreover,

\[
G_1(2,3) = H(2,3) = \mathbb{E} \left[ h(1,2) h(1,3) \mid X_2, X_3 \right] = X_2 X_3 \mathbb{E} \left[ X_1^2 \right] = X_2 X_3
\]
(recall that \( Z_{i,j} \equiv 1 \)). This means for (1.13) that
\[
\theta_n^{-1} \mathbb{E} \left[ G_n^2(2, 3) \right] = 4 \cdot \mathbb{E} \left[ X_2^2 X_3^2 \right] = 4 \mathbb{E} \left[ X_2^2 \right] \mathbb{E} \left[ X_3^2 \right] = 4,
\]
which does not go to 0. Hence (1.13) is violated.

The rest of this note is organized in the following way. In Section 2 we will prove Theorem 1.4. A main ingredient to this end will be a martingale CLT due to Girko (see Theorem 2.1 below). In Section 3 we will prepare for the proof of Theorem 1.5 by giving an alternative condition for (1.13) (see condition (3.1) below). The core of the paper is the proof of Theorem 1.5 in Section 4. We will see that (1.10)-(1.13) (resp. (3.1)) imply the conditions of Theorem 1.4. Finally, in Section 5, we will give some alternative conditions for (1.10)-(1.12) that are easier to check in some applications. The appendix contains the proofs of our technical results.

2 Proof of Theorem 1.4

As mentioned in the Introduction we will base our arguments on the following theorem (Girko, 1990, Theorem 5.4.11):

**Theorem 2.1.** Consider a triangular array of martingale differences \( (Y_{i,n}) \), \( i = 1, \ldots, n \) with respect to a sequence of filtrations \( (F_{i,n})_{i=1,\ldots,n} \) (that is, for fixed \( n \in \mathbb{N} \) and all \( i = 1, \ldots, n \), \( Y_{i,n} \) is \( F_{i,n} \)-measurable, integrable and satisfies \( \mathbb{E} [Y_{i,n} \mid F_{i-1,n}] = 0 \) almost surely). If for any \( \varepsilon > 0 \)
\[
\sum_{i=1}^{n} \mathbb{E} \left[ Y_{i,n}^2 \mathbb{1}_{\{ |Y_{i,n}| \geq \varepsilon \}} \right] \mathbb{P} \xrightarrow{n \to \infty} 0 \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{E} \left[ Y_{i,n}^2 \mid F_{i-1,n} \right] \mathbb{P} \xrightarrow{n \to \infty} 1
\]
hold, then \( \sum_{i=1}^{n} Y_{i,n} \) converges in distribution to a standard normal random variable.

To apply this result let \( F_i = F_{i,n} = \sigma\left( (X_k)_{k=1,\ldots,i}, (Z_{l,m})_{l=1,\ldots,i, m=1,\ldots,n, m \neq l} \right) \).

Using the notation from Section 1, \( \Psi_j(i) \) is \( F_i \)-measurable, hence \( \xi_i^{(1)} \) is \( F_i \)-adapted. For \( j < i \), \( X_j \) is also \( F_i \)-measurable, so that \( \Phi(i, j) \) and therefore \( \xi_i^{(2)} \) are also \( F_i \)-adapted. Hence \( \xi_i \) is \( F_i \)-adapted. Now,
\[
\mathbb{E} [\xi_i \mid F_{i-1}] = \frac{1}{n \theta_n} \sum_{j < i} \mathbb{E} [Z_{ij} \mathbb{E} [h(i, j) \mid X_i] \mid F_{i-1}]
+ \sum_{j > i} \mathbb{E} [Z_{ij} \mathbb{E} [h(i, j) \mid X_i] \mid F_{i-1}] + \sum_{j < i} \mathbb{E} [Z_{ij} \tilde{h}(i, j) \mid F_{i-1}] \).
\]
By definition, for \( j < i \), \( Z_{ij} = Z_{ji} \) are \( F_{i-1} \)-measurable, while \( X_i \) is independent of \( F_{i-1} \). For the second term notice that both, \( X_i \) and \( Z_{ij} \), are independent of \( F_{i-1} \), if \( j > i \). In the third sum, the \( Z_{i,j} \) is measurable again. This leaves only \( \tilde{h}(i,j) \), which only depends on \( X_j \). Therefore

\[
E[\xi_i | F_{i-1}] = \frac{1}{n\theta_n} \left( \sum_{j<i} Z_{ij}E[h(i,j)] + \sum_{j>i} E[Z_{ij}]E[h(i,j)] + \sum_{j<i} Z_{ij}E[h(i,j) | X_j] \right) = 0
\]

by (1.1) and (1.4). Thus \( \xi_i \) is a martingale difference. Setting \( Y_i = Y_{i,n} = \xi_{i,n} \) in Theorem 2.1 we can rewrite conditions a) and b) in this theorem as

\[
\eta_1 = \sum_{i=1}^{n} \mathbb{E} \left[ \xi_i^2 1_{\{|\xi_i| \geq \epsilon\}} \bigg| (X_k)_{k=1,...,i-1}, (Z_{l,m})_{l=1,...,i-1}, m=1,...,n, m \neq l \right] \xrightarrow{p} 0, \quad \eta_2 = \sum_{i=1}^{n} \mathbb{E} \left[ \xi_i^2 1_{\{|X_k| \leq \epsilon\}} \bigg| (X_k)_{k=1,...,i-1}, (Z_{l,m})_{l=1,...,i-1}, m=1,...,n, m \neq l \right] \xrightarrow{p} 1.
\]

This proves Theorem 1.4.

### 3 An alternative condition for (1.13)

The purpose of this section is to prove

**Proposition 3.1.** Condition (1.13) implies

\[
\theta_n^{-4} \mathbb{E} \left[ \tilde{G}_1^2(2,3) \right] \xrightarrow{n \to \infty} 0.
\]  

**Proof.** For \( i, j, k \) pairwise different, we have by definition of \( \tilde{G}_k \)

\[
\tilde{G}_k(i,j) = \mathbb{E} \left[ \Phi(i,k)\Phi(j,k) - \Psi_k(i)\Phi(j,k) - \Psi_k(j)\Phi(i,k) \right] \\
- \Psi_k(i)\Psi_k(j) + \Psi_k(i)\Psi_k(j) + \Psi_k(j)\Psi_k(i) + \Psi_k(i)\Psi_k(j) \\
+ \Psi_k(i)\Psi_k(j) | X_i, X_j, Z_{i,k}, Z_{j,k} \\
= G_k(i,j) - \Psi_k(i)\Psi_k(j) - \mathbb{E} \left[ \Psi_k(i)\Phi(j,k) | X_i, X_j, Z_{i,k}, Z_{j,k} \right] \\
- \Psi_k(i)\Psi_k(j) - \mathbb{E} \left[ \Psi_k(j)\Phi(i,k) | X_i, X_j, Z_{i,k}, Z_{j,k} \right] + \Psi_k(i)\Psi_k(j) \\
+ \Psi_k(i)\mathbb{E} \left[ \Psi_j(k) | X_i, X_j, Z_{i,k}, Z_{j,k} \right] + \Psi_k(j)\mathbb{E} \left[ \Psi_i(k) | X_i, X_j, Z_{i,k}, Z_{j,k} \right] \\
+ \mathbb{E} \left[ \Psi_i(k)\Psi_j(k) | X_i, X_j, Z_{i,k}, Z_{j,k} \right].
\]
Now $\mathbb{E}[\Psi_j(k) \mid X_i, X_j, Z_{i,k}, Z_{j,k}] = 0$ and three of the above terms only differ by their sign. Thus

$$\tilde{G}_k(i,j) = G_k(i,j) - \Psi_k(i)\Psi_k(j) - \mathbb{E}[\Psi_i(k)\Phi(j,k) \mid X_i, X_j, Z_{i,k}, Z_{j,k}]$$

$$- \mathbb{E}[\Psi_j(k)\Phi(i,k) \mid X_i, X_j, Z_{i,k}, Z_{j,k}] + \mathbb{E}[\Psi_i(k)\Psi_j(k) \mid X_i, X_j, Z_{i,k}, Z_{j,k}]$$

$$= G_k(i,j) - \Psi_k(i)\Psi_k(j) - A + B + C.$$  

(3.2)

By independence, $\mathbb{E}\left[(\Psi_k(i)\Psi_k(j))^2\right] = \mathbb{E}\left[\Psi_k^2(i)\right]\mathbb{E}\left[\Psi_k^2(j)\right] = \gamma_n^4$, and for $A$, we have again by independence and Cauchy-Schwarz

$$\mathbb{E}\left[A^2\right] = \mathbb{E}\left[Z_{i,k}Z_{j,k}\left(\mathbb{E}[h(i,k) \mid X_k]h(j,k) \mid X_j]\right)^2\right]$$

$$\leq p^2\mathbb{E}\left[\mathbb{E}[h(i,k) \mid X_k]^2 \mathbb{E}[h(j,k) \mid X_j]\right]$$

$$= p^2\mathbb{E}\left[\mathbb{E}[h(i,k) \mid X_k]^2 \mathbb{E}[h^2(j,k) \mid X_j]\right]$$

and by Lemma 1.1 this equals

$$= p^2\mathbb{E}\left[\frac{\gamma_n^2}{p}\mathbb{E}[h^2(j,k) \mid X_j]\right] = p^2\frac{\gamma_n^2}{p} = \beta_n^2\gamma_n^2.$$

$\mathbb{E}[B^2]$ has the same upper bound, which is seen analogously. As for $C$, we again use measurability and independence to obtain

$$\mathbb{E}\left[C^2\right] = \mathbb{E}\left[\mathbb{E}[\Psi_i(k)\Psi_j(k) \mid X_i, X_j, Z_{i,k}, Z_{j,k}]^2\right]$$

$$= \mathbb{E}\left[Z_{i,k}Z_{j,k}\mathbb{E}[h(i,k) \mid X_k]\mathbb{E}[h(j,k) \mid X_j]\right]^2$$

$$\leq p^2\mathbb{E}\left[\mathbb{E}[h(i,k) \mid X_k]^2\right]^2 = p^2\left(\frac{\gamma_n^2}{p}\right)^2 = \gamma_n^4.$$

by Cauchy-Schwarz and Lemma 1.1.

We can combine all this to conclude

$$\frac{1}{\theta_n^4}\mathbb{E}\left[\tilde{G}_1^2(2,3)\right] \leq \frac{1}{\theta_n^4}\mathbb{E}\left[(G_1(2,3) - \Psi_1(2)\Psi_1(3) - A - B + C)^2\right]$$

$$\leq \frac{25}{\theta_n^4}\mathbb{E}[G_1^2(2,3)] + \mathbb{E}[\Psi_1^2(2)\Psi_1^2(3)] + \mathbb{E}[A^2] + \mathbb{E}[B^2] + \mathbb{E}[C^2]]$$

$$\leq \frac{25}{\theta_n^4}\mathbb{E}[G_1^2(2,3)] + \frac{25}{\theta_n^4}2\gamma_n^4 + \frac{25}{\theta_n^4}2\beta_n^2\gamma_n^2 \leq \frac{25}{\theta_n^4}\mathbb{E}[G_1^2(2,3)] + \frac{50}{(np)^2} + \frac{50}{np}$$

due to (1.3), $\theta_n^4 \geq np\gamma_n^2$ and $\theta_n^2 \geq \beta_n^2$. The second and third term go to 0 as $np \to \infty$. The first term is the term from (1.13), and hence converges to 0. This completes the proof. \qed
4 Proof of Theorem 1.5

The proof of Theorem 1.5 immediately follows from

Proposition 4.1. Conditions (1.6) and (1.7) follow from (1.10), (1.11), (1.12) and (1.13).

Remark 4.2. Conditions (1.10)–(1.13) may be tricky to check. In fact, in many settings it may be unreasonable to prove conditions for \( \tilde{\Phi} \) instead of \( \Phi \) etc. In Proposition 5.1 below we will give alternative, more straightforward conditions for (1.10)–(1.12).

We split the proof of Proposition 4.1 into two Lemmas:

Lemma 4.3. (1.6) follows from (1.10), (1.11), (1.12), (1.13) and (3.1).

Proof. Since \( \eta_1 \geq 0 \), it suffices to show \( \mathbb{E} [\eta_1] \xrightarrow{n \to \infty} 0 \) to obtain \( \eta_1 \xrightarrow{p} 0 \).

Using Lemma A.8 with \( k = 2 \), \( a_1 = \xi_1^{(1)} \), and \( a_2 = \xi_1^{(2)} \) we get

\[
\eta_1 \leq 4 \sum_{i=1}^{n} \left( \mathbb{E} \left[ \left( \xi_1^{(1)} \right)^2 \mathbb{1}_{\left\{ |\xi_1^{(1)}| \geq \frac{\varepsilon}{2} \right\}} \mid (X_k)_{k=1,\ldots,i-1}, (Z_{l,m})_{l=1,\ldots,i-1, m=1,\ldots,n, m \neq l} \right] \right.
\]

\[
+ \mathbb{E} \left[ \left( \xi_1^{(2)} \right)^2 \mathbb{1}_{\left\{ |\xi_1^{(2)}| \geq \frac{\varepsilon}{2} \right\}} \mid (X_k)_{k=1,\ldots,i-1}, (Z_{l,m})_{l=1,\ldots,i-1, m=1,\ldots,n, m \neq l} \right] \right),
\]

and consequently,

\[
\mathbb{E} [\eta_1] \leq 4 \sum_{i=1}^{n} \mathbb{E} \left[ \left( \frac{1}{n \theta n} \sum_{j=1}^{n} \Psi_j(i) \right)^2 \mathbb{1}_{\left\{ \left| \frac{1}{n \theta n} \sum_{j=1, j \neq i}^{n} \Psi_j(i) \right| \geq \frac{\varepsilon}{2} \right\}} \right]
\]

\[
+ 4 \sum_{i=1}^{n} \mathbb{E} \left[ \left( \frac{1}{n \theta n} \sum_{j=1}^{i-1} \tilde{\Phi}(i, j) \right)^2 \mathbb{1}_{\left\{ \left| \frac{1}{n \theta n} \sum_{j=1, j \neq i}^{i-1} \tilde{\Phi}(i, j) \right| \geq \frac{\varepsilon}{2} \right\}} \right] =: S_1 + T_1.
\]

Now,

\[
S_1 = \frac{4}{n^2 \theta^2} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \sum_{j=1, j \neq i}^{n} \Psi_j(i) \right)^2 \mathbb{1}_{\left\{ \left| \frac{1}{n \theta n} \sum_{j=1, j \neq i}^{n} \Psi_j(i) \right| \geq \frac{\varepsilon \theta n}{2} \right\}} \right]
\]

\[
= \frac{4}{n \theta^2} \mathbb{E} \left[ \left( \sum_{j=2}^{n} \Psi_j(1) \right)^2 \mathbb{1}_{\left\{ \left| \sum_{j=2}^{n} \Psi_j(1) \right| \geq \frac{\varepsilon \theta n}{2} \right\}} \right]
\]

by identical distribution. Hence, by (1.10), we have \( S_1 \xrightarrow{n \to \infty} 0 \).
The estimate for $T_1$ is slightly longer:

$$T_1 = \frac{4}{n^2 \theta_n^2} \sum_{i=1}^n \sum_{j=1}^{i-1} \mathbb{E} \left[ \left( \sum_{j=1}^{i-1} \tilde{\Phi}(i, j) \right)^2 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} \right]$$

$$= \frac{4}{n^2 \theta_n^2} \sum_{i=1}^n \sum_{j=1}^{i-1} \mathbb{E} \left[ \tilde{\Phi}^2(i, j) 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} 1 \left\{ \left| \phi(i, j) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} \right]$$

$$+ \frac{4}{n^2 \theta_n^2} \sum_{i=1}^n \sum_{j=1}^{i-1} \mathbb{E} \left[ \tilde{\Phi}(i, j) 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} 1 \left\{ \left| \phi(i, j) \right| < \frac{\varepsilon \theta_n n}{4} \right\} \right]$$

$$+ \frac{4}{n^2 \theta_n^2} \sum_{i=1}^n \left[ \sum_{1 \leq j < k \leq i-1} \tilde{\Phi}(i, j) \tilde{\Phi}(i, k) 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} \right]$$

$$\leq \frac{4}{\theta_n^2} \mathbb{E} \left[ \tilde{\Phi}^2(1, 2) 1 \left\{ \left| \phi(1,2) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} \right]$$

By (1.11), we have $S_2 \to 0$. For $S_3$, we manipulate the indicators to see that

$$1 \left\{ \left| \sum_{k=1}^{i-1} \phi(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} 1 \left\{ \left| \phi(i, j) \right| < \frac{\varepsilon \theta_n n}{4} \right\} \leq 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\}$$

and use (1.9) to obtain

$$S_3 \leq \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \tilde{\Phi}^2(i, j) 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} 1 \left\{ (X_k)_{k=1}^{1,\ldots,i}, (Z_i, k)_{k=1,\ldots,i-1} \right\} \right]$$

$$= \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \tilde{\Phi}^2(i, j) \mid X_i, Z_{i,j} \right] 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\}$$

$$= \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \tilde{G}(i, j) 1 \left\{ \left| \sum_{k=1}^{i-1} \tilde{\Phi}(i, k) \right| \geq \frac{\varepsilon \theta_n n}{4} \right\} \right]$$
By adding another indicator, for any \( \tilde{\varepsilon} > 0 \), this can be rewritten as

\[
S_3 \leq \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \tilde{G}_j(i, i) \mathbb{1} \left\{ \left| \sum_{k=1}^{i-1} \Phi(i, k) \right| \geq \frac{\theta_n}{4} \right\} \mathbb{1} \left\{ |\tilde{H}(i, i)| \geq \frac{\theta_n^2}{16} \right\} \right] 
+ \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \tilde{G}_j(i, i) \mathbb{1} \left\{ \left| \sum_{k=1}^{i-1} \Phi(i, k) \right| \geq \frac{\theta_n}{4} \right\} \mathbb{1} \left\{ |\tilde{H}(i, i)| < \frac{\theta_n^2}{16} \right\} \right]
\]

\[
\leq \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \tilde{G}_j(i, i) \mathbb{1} \left\{ |\tilde{H}(i, i)| \geq \frac{\theta_n^2}{16} \right\} \right] 
+ \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \tilde{G}_j(i, i) \mathbb{1} \left\{ \left| \sum_{1 \leq k \leq i-1 \atop k \neq j} \Phi(i, k) \right| \geq \frac{\theta_n}{4} \right\} \mathbb{1} \left\{ |\tilde{H}(i, i)| < \frac{\theta_n^2}{16} \right\} \right]
\]

\[=: S_{31} + S_{32}. \]

For \( S_{31} \) we have by independence, (1.9) and (1.12)

\[
S_{31} = \frac{4}{n^2 \theta_n^2} \binom{n}{2} \mathbb{P} \left[ H(1, 1) \mathbb{1} \left\{ |H(1, 1)| \geq \frac{\theta_n^2}{16} \right\} \right] \xrightarrow{n \to \infty} 0. \tag{4.2}
\]

On the other hand, by applying the two indicators

\[
S_{32} \leq \frac{4}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ Z_{i,j} \frac{\tilde{\varepsilon} \theta_n^2 n}{16 \theta_n^2} \mathbb{1} \left\{ \left| \sum_{1 \leq k \leq i-1 \atop k \neq j} \Phi(i, k) \right| \geq \frac{\theta_n}{4} \right\} \right]
\]

\[
\leq \frac{4 \tilde{\varepsilon}}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \frac{\tilde{\varepsilon} \theta_n^2 n^2}{16} \mathbb{1} \left\{ \left| \sum_{1 \leq k \leq i-1 \atop k \neq j} \Phi(i, k) \right| \geq \frac{\theta_n}{4} \right\} \right]
\]

\[
\leq \frac{4 \tilde{\varepsilon}}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \left( \left| \sum_{1 \leq k \leq i-1 \atop k \neq j} \Phi(i, k) \right| \right)^2 \mathbb{1} \left\{ \left| \sum_{1 \leq k \leq i-1 \atop k \neq j} \Phi(i, k) \right| \geq \frac{\theta_n}{4} \right\} \right]
\]

\[
\leq \frac{4 \tilde{\varepsilon}}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \sum_{1 \leq k \leq i-1 \atop k \neq j} \Phi^2(i, k) + \sum_{1 \leq k \leq i-1 \atop k \neq j} \sum_{1 \leq l \leq i-1 \atop l \neq j} \Phi(i, k) \Phi(i, l) \right]
\]

\[
\leq \frac{4 \tilde{\varepsilon}}{n^2 \theta_n^2} n(n-1) \left( (n-1) \mathbb{E} \left[ \Phi^2(1, 2) \right] + (n-1)^2 \mathbb{E} \left[ \Phi(1, 2) \Phi(1, 3) \right] \right)
\]

By Lemma 1.1, the first expectation is smaller than \( \beta_n^2 \). By Lemma A.2, the second expectation is 0. Thus

\[
S_{32} \leq \frac{4(n-1)^2 \tilde{\varepsilon}}{n^2 \theta_n^2} \beta_n^2 \leq \frac{8 \tilde{\varepsilon} (n-1)^2}{n^2}.
\]
by (1.3). As we may chose $\varepsilon > 0$ arbitrarily, this shows that $S_{32} \to 0$, and together with (4.2) we obtain that $S_3 \xrightarrow{n \to \infty} 0$.

Finally, for $S_4$, we compute

$$S_4 \leq \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \sum_{1 \leq j \leq i-1}^{i-1} \tilde{\Phi}(i,j) \tilde{\Phi}(i,k) \mathbb{1}_{\{|\sum_{j=1}^{i-1} \tilde{\phi}(i,j)| \geq \frac{c_n}{2n} \}} \right]$$

$$+ \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \leq \frac{c_n^2}{2p} \}} \sum_{1 \leq j \neq k}^{i-1} \tilde{\Phi}(i,j) \tilde{\Phi}(i,k) \mathbb{1}_{\{|\sum_{j=1}^{i-1} \tilde{\phi}(i,j)| \geq \frac{c_n}{2n} \}} \right]$$

$$=: S_{41} + S_{42}$$

Consider $S_{41}$ first. Because of $\left| \sum_{j \neq k} x_j x_k \right| \leq \left( \sum_{j=1}^{n} x_j \right)^2 + \sum_{j=1}^{n} x_j^2$, we obtain

$$S_{41} \leq \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \left( \sum_{j=1}^{i-1} \tilde{\Phi}(i,j) \right)^2 \mathbb{1}_{\{|\sum_{j=1}^{i-1} \tilde{\phi}(i,j)| \geq \frac{c_n}{2n} \}} \right]$$

$$+ \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \sum_{j=1}^{i-1} \tilde{\Phi}^2(i,j) \mathbb{1}_{\{|\sum_{j=1}^{i-1} \tilde{\phi}(i,j)| \geq \frac{c_n}{2n} \}} \right]$$

$$\leq \frac{8}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \sum_{j=1}^{i-1} \tilde{\Phi}^2(i,j) \right]$$

$$+ \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \sum_{1 \leq j \neq k \leq i-1} \tilde{\Phi}(i,j) \tilde{\Phi}(i,k) \right]$$

$$=: S'_{41} + S''_{41}$$

For the first of these summands we obtain

$$S'_{41} = \frac{8}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \tilde{\Phi}^2(i,j) \right]$$

$$= \frac{8}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \mathbb{E} \left[ \tilde{\Phi}^2(i,j) \mid X_i, Z_{ij} \right] \right]$$

$$= \frac{8}{n^2 \theta_n^2} \sum_{1 \leq j < i \leq n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i,i)| \geq \frac{c_n^2}{2p} \}} \tilde{G}_j(i,i) \right]$$

$$= \frac{8}{n^2 \theta_n^2} \binom{n}{2} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(1,1)| \geq \frac{c_n^2}{2p} \}} \tilde{H}(1,1) Z_{1,1} \right]$$
\[ S_{41}' = \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \sum_{1 \leq j, k \leq i-1 \atop j \neq k} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| \geq \frac{\varepsilon \theta_n n}{2p}\}} \tilde{H}(i, k) \right] \]

We estimate \( S_{41}' \) by

\[
S_{41}' = \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \sum_{1 \leq j, k \leq i-1 \atop j \neq k} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| \geq \frac{\varepsilon \theta_n n}{2p}\}} \tilde{H}(i, k) \right] \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| \geq \frac{\varepsilon \theta_n n}{2p}\}} \tilde{H}(i, j) \right] \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| \geq \frac{\varepsilon \theta_n n}{2p}\}} \tilde{H}(i, k) \right] = 0,
\]

which follows from (1.4). Altogether this gives \( S_{41}' \xrightarrow{n \to \infty} 0 \).

Considering now \( S_{42} \) we see that (using Cauchy-Schwarz for the second inequality)

\[
S_{42} \leq \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| < \frac{\varepsilon \theta_n n}{2p}\}} \sum_{1 \leq j, k \leq i-1 \atop j \neq k} \tilde{H}(i, j) \tilde{H}(i, k) \mathbb{1}_{\{|\sum_{l=1}^{i-1} \tilde{H}(i, l)| \geq \frac{\varepsilon \theta_n n}{2}\}} \right]
\]

\[
\leq \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| < \frac{\varepsilon \theta_n n}{2p}\}} \left( \sum_{j, k=1}^{i-1} \tilde{H}(i, j) \tilde{H}(i, k) \right)^2 \right] \frac{1}{2} \mathbb{P} \left( \left| \sum_{l=1}^{i-1} \tilde{H}(i, l) \right| \geq \frac{\varepsilon \theta_n n}{2} \right) \]

\[
= \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} A_i B_i. \tag{4.3}
\]

We estimate \( A_i \) by

\[
A_i^2 = \sum_{j, k=1}^{i-1} \mathbb{1}_{\{j \neq k\}} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| < \frac{\varepsilon \theta_n n}{2p}\}} \tilde{H}^2(i, j) \tilde{H}^2(i, k) \right]
\]

\[
+ \sum_{j, k=1}^{i-1} \sum_{l, m=1}^{i-1} \mathbb{1}_{\{j \neq k\}} \mathbb{1}_{\{l \neq m\}} \mathbb{E} \left[ \mathbb{1}_{\{|\tilde{H}(i, i)| < \frac{\varepsilon \theta_n n}{2p}\}} \tilde{H}(i, j) \tilde{H}(i, k) \tilde{H}(i, l) \tilde{H}(i, m) \right]
\]

\[
= A_{i1} + A_{i2}
\]
Using the properties of (conditional) expectation for $A_{i1}$ we see that
\[
E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{\Phi}^2(i, j) \tilde{\Phi}^2(i, k) \right] \\
= E\left[ E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{\Phi}^2(i, j) \tilde{\Phi}^2(i, k) \mid X_i, X_j, Z_{i,j}, Z_{i,k} \right] \mid X_i, Z_{i,j}, Z_{i,k} \right] \\
= E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{\Phi}^2(i, j) \mid X_i, Z_{i,j} \right] E\left[ \tilde{\Phi}^2(i, k) \mid X_i, Z_{i,k} \right]
\]
and by (1.9) this is equal to
\[
E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{G}_j(i, i) \tilde{G}_k(i, i) \right] \\
= E\left[ Z_{i,j} \right] E\left[ Z_{i,k} \right] E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{H}(i, i) \tilde{H}(i, i) \right] \\
< p^2 \cdot \frac{\epsilon \theta_n^2 n}{2p} \cdot E\left[ \tilde{H}(i, i) \right] \leq p \cdot \frac{\epsilon \theta_n^2 n}{2} \cdot \frac{\beta_n^2}{p} = \frac{\epsilon \theta_n^2 n \beta_n^2}{2},
\]
where in the last inequality we applied Corollary A.1.

On the other hand, for each of the terms in $A_{i2}$, we know that at least one of the values $j, k, l, m$ is different from the others. Without loss of generality, this is $m$. Then,
\[
E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{\Phi}(i, j) \tilde{\Phi}(i, k) \tilde{\Phi}(i, l) \tilde{\Phi}(i, m) \right] \\
= E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{\Phi}(i, j) \tilde{\Phi}(i, k) \tilde{\Phi}(i, l) \tilde{\Phi}(i, m) \mid Z_{i,j}, Z_{i,k}, Z_{i,l}, Z_{i,m}, X_i, X_j, X_k, X_l \right] \\
= E\left[ \mathbb{1}_{\{h(i,i) < \frac{\epsilon \theta_n n}{2p} \}} \tilde{\Phi}(i, j) \tilde{\Phi}(i, k) \tilde{\Phi}(i, l) Z_{i,m} \right] E\left[ \tilde{h}(i, m) \mid X_i \right] = 0,
\]
due to (1.4). Altogether, $A_i^2 = A_{i1} \leq n^2 \cdot \frac{\epsilon \theta_n^2 n}{2} \cdot \beta_n^2 \leq n^2 \cdot \epsilon n \theta_n^2 \cdot \beta_n^2$, hence
\[
A_i \leq n \beta_n \theta_n \sqrt{\epsilon n}, \tag{4.4}
\]
To give a bound for $B_i$, we use the fact that
\[
E\left[ \left( \sum_{l=1}^{i-1} \tilde{\Phi}(i, l) \right)^2 \right] = E\left[ \sum_{l=1}^{i-1} \tilde{\Phi}^2(i, l) \right] + E\left[ \sum_{l \neq m}^{i-1} \tilde{\Phi}(i, l) \tilde{\Phi}(i, m) \right] = \sum_{l=1}^{i-1} E\left[ \tilde{\Phi}^2(i, l) \right] \leq n \beta_n^2,
\]
by Lemma A.2 and Lemma 1.1. By Markov’s inequality
\[
B_i = P\left( \left| \sum_{l=1}^{i-1} \tilde{\Phi}(i, l) \right| \geq \frac{\epsilon \theta_n n}{2} \right)^{1/2} \leq \left( \frac{4}{\epsilon^2 \theta_n^2 n^2} E\left[ \left( \sum_{l=1}^{i-1} \tilde{\Phi}(i, l) \right)^2 \right] \right)^{1/2} \leq \frac{2 \beta_n}{\epsilon \theta_n \sqrt{n}}, \tag{4.5}
\]
Then (4.3), (4.4), (4.5) and (1.3) (this, in particular implies $\theta_n^2 \geq \beta_n^2 / 2$) give

$$S_{42} \leq \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} A_i B_i \leq \frac{4}{n^2 \theta_n^2} \sum_{i=1}^{n} (n \beta_n \theta_n \sqrt{\varepsilon n} \cdot \frac{2 \beta_n}{\varepsilon \theta_n \sqrt{n}}) \leq \frac{4}{n^2 \theta_n^2} n \cdot n \cdot 2 \sqrt{\varepsilon} \leq 16 \sqrt{\varepsilon}$$

As $\varepsilon$ was chosen arbitrarily, we obtain, $S_{42} \xrightarrow{n \to \infty} 0$, and hence $S_{4} \xrightarrow{n \to \infty} 0$.

Wrapping things up, this tells us that $E[\eta_1] \leq S_1 + T_1 \leq S_1 + S_2 + S_3 + S_4 \to 0$ and thus, (1.6) holds. \hfill $\square$

**Lemma 4.4.** Condition (1.7) follows from (1.10), (1.11), (1.12), (1.13) and (3.1).

**Proof.** We start by calculating $\xi_i^2$:

$$\xi_i^2 = (\xi_i^{(1)} + \xi_i^{(2)})^2 = \left( \frac{1}{n \theta_n} \sum_{j=1}^{n} \Psi_j(i) + \frac{1}{n \theta_n} \sum_{j=1}^{i-1} \Phi(i, j) \right)^2$$

$$= \frac{1}{n^2 \theta_n^2} \sum_{j=1}^{n} \Psi_j^2(i) + \frac{1}{n^2 \theta_n^2} \sum_{j \neq i}^{n} \Psi_j(i) \Psi_k(i) + 2 \frac{1}{n^2 \theta_n^2} \sum_{j=1}^{n} \sum_{k=1}^{i-1} \Psi_j(i) \Phi(i, k)$$

$$+ \frac{1}{n^2 \theta_n^2} \sum_{j=1}^{i-1} \Phi^2(i, j) + \frac{1}{n^2 \theta_n^2} \sum_{j \neq k, j \neq i}^{i-1} \Phi(i, j) \Phi(i, k) \quad (4.6)$$

We will compute the sum (in $i$) of the conditional expectations for of each of these summands. For the second one: observe that for any choice of $i \neq j, k$ we have $\Psi_k(i) = \Psi_j(i) Z_{ik} / Z_{ij}$. Hence, using that $Z_{ij}^2 = Z_{ij}$ we get

$$\sum_{i=1}^{n} \frac{1}{n^2 \theta_n^2} \sum_{j, k=1}^{n} \sum_{j \neq i, j \neq k} \mathbb{E} \left[ \Psi_j(i) \Psi_k(i) \mid (X_k)_{k=1,...,i-1}, (Z_{l,m})_{l=1,...,i-1}, m=1,...,n, m \neq l \right]$$

$$= \sum_{i=1}^{n} \frac{1}{n^2 \theta_n^2} \sum_{j, k=1}^{n} \sum_{j \neq i, k \neq i} \mathbb{E} \left[ Z_{i,k} \Psi_j^2(i) \mid (X_k)_{k=1,...,i-1}, (Z_{l,m})_{l=1,...,i-1}, m=1,...,n, m \neq l \right]$$

$$= \sum_{i=1}^{n} \frac{1}{n^2 \theta_n^2} \sum_{j=1}^{n} \sum_{k=1}^{i-1} \sum_{k \neq j}^{i-1} \sum_{k \neq j}^{i-1} \sum_{m=1}^{n} \mathbb{E} \left[ \Psi_j^2(i) \mid (X_k)_{k=1,...,i-1}, (Z_{l,m})_{l=1,...,i-1}, m=1,...,n, m \neq l \right]$$
where we applied measurability of $Z_{i,k}$ for $k < i$ with respect to the condition and independence of the condition for $Z_{i,k}$, $k > i$. The conditional expectation of the first two summands in (4.6) together then is

$$
\eta_{21} := \sum_{i=1}^{n} \frac{1}{n^2 \theta^2_n} \sum_{j=1}^{n} \sum_{j \neq i} E \left[ \Psi^2_j(i) \mid (X_k)_{k=1, \ldots, i-1}, (Z_{l,m})_{l=1, \ldots, i-1, m=1, \ldots, n, m \neq l} \right] \cdot \left( \sum_{k=1}^{i-1} Z_{i,k} + \sum_{k=i+1}^{n} p + 1 \right)
$$

$$
= \sum_{i=1}^{n} \frac{1}{n^2 \theta^2_n} \sum_{j=1}^{n} \sum_{j \neq i} E \left[ Z^2_i \mathbb{E}[h(i, j) \mid X_i] \right] \cdot \left( \sum_{k=1}^{i-1} Z_{i,k} + \sum_{k=i+1}^{n} p + 1 \right)
$$

$$
= \frac{\gamma_n^2}{n^2 p \theta^2_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j \neq i} E \left[ Z_{i,j} \cdot \left( \sum_{k=1}^{i-1} Z_{i,k} + \sum_{k=i+1}^{n} p + 1 \right) \right] \cdot \left( Z_{i,i} \right)_{l=1, \ldots, i-1}
$$

since $E \left[ \mathbb{E}[h(i, j) \mid X_i] \right]$ is given by Lemma 1.1, and $Z^2_{i,j} = Z_{i,j}$. It is easily seen that for this term, the relation

$$
E \left[ \eta_{21} \right] = \frac{\gamma_n^2}{n^2 p \theta^2_n} n(n-1) (n-2) p + 1) p \sim \frac{n p \gamma_n^2}{\theta^2_n} \quad (4.7)
$$

holds. Next, the conditional expectation of the fourth summand in (4.6) can be computed as:

$$
\eta_{22} := \frac{1}{n^2 \theta^2_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j \neq i} E \left[ \hat{\Phi}^2(i, j) \mid (X_k)_{k=1, \ldots, i-1}, (Z_{l,m})_{l=1, \ldots, i-1, m=1, \ldots, n, m \neq l} \right]
$$

$$
= \frac{1}{n^2 \theta^2_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j \neq i} E \left[ \hat{\Phi}^2(i, j) \mid X_j, Z_{i,j} \right] = \frac{1}{n^2 \theta^2_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{G}_i(j, j)
$$

by (1.9). By Corollary A.1 the expectation for this term satisfies

$$
E \left[ \eta_{22} \right] = \left( \frac{n}{2} \right) \frac{1}{n^2 \theta^2_n} (\beta_n^2 - 2 \gamma_n^2) \sim \frac{\beta_n^2 - 2 \gamma_n^2}{2 \theta^2_n} \quad (4.8)
$$

Furthermore, the sum of the conditional expectations of 5'th summand in (4.6) by (1.9) is

$$
\eta_{23} := \frac{1}{n^2 \theta^2_n} \sum_{i=1}^{n} \sum_{j,k=1}^{n} \tilde{G}_i(j, k).
$$

For the 3'rd summand in (4.6)
we obtain: \( \eta_{24} := \frac{2}{n^2 \theta_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{i-1} \mathbb{E}[\Psi_j(i) \tilde{\Phi}(i, m) \mid X_m, (Z_{l,i})_{l=1,\ldots,i-1}] \).

Then \( \eta_2 \sim \eta_{21} + \eta_{22} + \eta_{23} + \eta_{24} \). One can immediately conclude from (4.7) and (4.8) that \( \mathbb{E}[\eta_{21} + \eta_{22}] \sim \frac{\eta \gamma_n^2}{\theta_n^4} + \frac{\beta_n^2 - 2\gamma_n^2}{2\theta_n^2} \sim \frac{1}{2} \frac{\beta_n^2 + \eta \gamma_n^2}{\theta_n^2} \) and by definition of \( \theta_n^2 \) (cf. (1.3))

\[
\mathbb{E}[\eta_{21} + \eta_{22}] \xrightarrow{n \to \infty} 1.
\] (4.9)

Let us split up \( \eta_{22} \), by choosing \( \varepsilon > 0 \) arbitrarily:

\[
\eta_{22} = \frac{1}{n^2 \theta_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} i^{-1} \tilde{G}_i(j, j) \{ |\tilde{H}(j, j)| < \varepsilon \theta_n^2 \} + \frac{1}{n^2 \theta_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{G}_i(j, j) \{ |\tilde{H}(j, j)| \geq \varepsilon \theta_n^2 \}
\]

\[
=: \eta_{22}' + \eta_{22}''.
\]

Consider the second summand first. By definition

\[
\mathbb{E}[\eta_{22}''] = \frac{1}{n^2 \theta_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j, j} \tilde{G}_i(j, j) \mathbb{I}\{ |\tilde{H}(j, j)| \geq \varepsilon \theta_n^2 \}.
\]

Then (1.12) yields for any choice of \( \varepsilon \) that \( \mathbb{E}[\eta_{22}'] \xrightarrow{n \to \infty} 0 \) and hence \( \eta_{22}'' \xrightarrow{n \to \infty} 0 \) and by (4.9)

\[
\mathbb{E}[\eta_{21} + \eta_{22}] \xrightarrow{n \to \infty} 1.
\] (4.10)

For \( \mathbb{E}[(\eta_{21} + \eta_{22})^2] \) define \( \Lambda_j(i) := \sum_{k=1}^{i-1} Z_{i,k} + \sum_{k=i+1}^{n} p + 1 \) and compute

\[
\mathbb{E}[\eta_{21}^2] = \frac{\gamma_n^4}{n^4 \theta_n^4} \sum_{i, i'=1}^{n} \sum_{j=1}^{n} \sum_{j'=1}^{n} \sum_{j \neq i}^{j} \mathbb{E}[\Lambda_j(i) Z_{i,j} \mid (Z_{l,i})_{l=1,\ldots,i-1}] \cdot \mathbb{E}[\Lambda_j(i') Z_{i',j'} \mid (Z_{l,i'})_{l=1,\ldots,i'-1}].
\]

If \( i = i', j = j' \), by Jensen’s inequality we find

\[
\mathbb{E}[\Lambda_j(i) Z_{i,j} \mid (Z_{l,i})_{l=1,\ldots,i-1}] \cdot \mathbb{E}[\Lambda_j(i') Z_{i',j'} \mid (Z_{l,i'})_{l=1,\ldots,i'-1}]
\]

\[
\leq \mathbb{E} \left[ Z_{i,j}^2 \Lambda_j^2(i) \right] = \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \Lambda_j(i) \Lambda_j(i') \right]
\]

If \( i = i', j \neq j' \), \( \Lambda_j(i) \) and \( \Lambda_j(i') \) are \((Z_{l,i})_{l=1,\ldots,i-1}\)-measurable. Then, dragging the second conditional into the first one and additionally conditioning on \( Z_{i,j} \) in it (which is possible due to \( j \neq j' \)),

\[
\mathbb{E}[\Lambda_j(i) Z_{i,j} \mid (Z_{l,i})_{l=1,\ldots,i-1}] \cdot \mathbb{E}[\Lambda_j(i') Z_{i',j'} \mid (Z_{l,i'})_{l=1,\ldots,i'-1}]
\]
Similarly to the previous step we get by (1.9)

\[
\Lambda_j(i) \Lambda_j'(i) \cdot \mathbb{E}[Z_{i,j} \mid (Z_{i,i})_{l=1,\ldots,i-1}] = \mathbb{E}[Z_{i,j} \mid (Z_{i,i})_{l=1,\ldots,i-1}]
\]

Finally, if \( i \neq i' \), by independence and law of total expectation we have

\[
\mathbb{E}\left[ \mathbb{E}\left[ \Lambda_j(i) Z_{i,j} \mid (Z_{i,i})_{l=1,\ldots,i-1} \right] \cdot \mathbb{E}\left[ \Lambda_j'(i') Z_{i',j'} \mid (Z_{i,i})_{l=1,\ldots,i'-1} \right] \right]
\]

Thus

\[
\mathbb{E}[\eta_{21}^2] \leq \gamma_n^2 \frac{n^2 p^2}{n^4 p^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j' \neq i}^{n} \sum_{j'' \neq i'}^{n} \mathbb{E}\left[ Z_{i,j} Z_{i',j'} \Lambda_j(i) \Lambda_j'(i') \right]
\]

By Lemma A.4 this immediately leads to \( \mathbb{E}[\eta_{21}^2] \lesssim \frac{(np)^2 \gamma_n^4}{\theta_n^2} \). Moreover,

\[
\mathbb{E}[\eta_{21} \eta_{22}']
\]

\[
\leq \gamma_n^2 \frac{n^2 p^2}{n^4 p^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j' \neq i}^{n} \mathbb{E}\left[ Z_{i,j} \Lambda_j(i) \mid (Z_{i,i})_{l=1,\ldots,i-1} \right] \cdot \frac{1}{n^2 \theta_n^2} \sum_{i'=1}^{n} \sum_{j' \neq i'}^{n} \tilde{G}_{i'}(j', j')
\]

Similarly to the previous step we get by (1.9)

\[
\mathbb{E}[\eta_{21} \eta_{22}'] \leq \gamma_n^2 \frac{n^2 p^2}{n^4 p^2} \sum_{i'=1}^{n} \sum_{j' \neq i}^{n} \sum_{j \neq i'}^{n} \mathbb{E}\left[ \Lambda_j(i) Z_{i,j} Z_{i',j'} \tilde{h}^2(i', j') \right]
\]

Now the \( \tilde{h} \) term only depends on the \( X_{i'} \) and as such is independent of \( \Lambda_j(i) Z_{i,j} \). By Lemma 1.1, \( \mathbb{E}\left[ \tilde{h}^2(i, j) \right] = \frac{\beta^2}{p} \leq \frac{\beta^2}{p} \) such that

\[
\mathbb{E}[\eta_{21} \eta_{22}'] \leq \gamma_n^2 \frac{n^2 p^2}{n^4 p^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j' = 1}^{n} \sum_{j'' = 1}^{n} \mathbb{E}\left[ \Lambda_j(i) Z_{i,j} Z_{i',j'} \right]
\]
Applying Lemma A.5 yields $\mathbb{E} [\eta_{21} \eta_{22}^2] \lesssim \frac{np}{4^n} \frac{\beta^2_n}{\gamma_n^2}$. Finally,

$$
\mathbb{E} \left[ (\eta_{22}')^2 \right] = \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \mathbb{E} \left[ \hat{G}_i(j,j) \mathbb{1} \{ |\hat{H}(j,j)| < \frac{\epsilon \theta_n^2}{p} \} \hat{G}'_{i'}(j',j') \mathbb{1} \{ |\hat{H}(j',j')| < \frac{\epsilon \theta_n^2}{p} \} \right] 
$$

and applying (1.9) to both sums gives

$$
\leq \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \hat{H}^2(j,j) \mathbb{1} \{ |\hat{H}(j,j)| < \frac{\epsilon \theta_n^2}{p} \} \right] + \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \sum_{j' \neq j} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \hat{H}(j,j) \hat{H}(j',j') \right]
$$

By independence we arrive at

$$
\mathbb{E} \left[ (\eta_{22}')^2 \right] \leq \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \right] \mathbb{E} \left[ \hat{H}^2(j,j) \right] \mathbb{1} \{ |\hat{H}(j,j)| < \frac{\epsilon \theta_n^2}{p} \} + \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \sum_{j' \neq j} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \hat{H}(j,j) \hat{H}(j',j') \right]
$$

$$
\leq \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \right] \frac{\epsilon \theta_n^2}{p} \mathbb{E} \left[ \hat{H}(j,j) \right] + \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \sum_{j' \neq j} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \hat{H}(j,j) \hat{H}(j',j') \right]
$$

$$
\leq \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \right] \frac{\epsilon \theta_n^2}{p} + \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i'=1}^{n} \sum_{j'=1}^{n} \sum_{j' \neq j} \mathbb{E} \left[ Z_{i,j} Z_{i',j'} \right] \frac{\beta_n^2}{p^2}
$$
where we applied Corollary A.1 and $\beta_n^2 - 2\gamma_n^2 \leq \beta_n^2$. By $Z_{i,j}^2 = Z_{i,j}$

$$
\mathbb{E} \left[ (\eta_{22})^2 \right] \leq \frac{1}{n^4 \theta_n^4} \frac{\xi^2 n \beta_n^2}{p} \left( \binom{n}{2} p + \binom{n}{2} (n-1)p^2 \right) + \frac{1}{n^4 \theta_n^4} \left( \binom{n}{2} p^2 \frac{\beta_n^2}{p^2} \right)^2 \\
\sim \frac{1}{n^4 \theta_n^4} \frac{\xi^2 n \beta_n^2}{p} \left( \binom{n}{2} (n-1)p^2 + \frac{1}{n^4 \theta_n^4} \left( \frac{\beta_n^2}{p} \right)^2 \right)^2 \sim \tilde{\epsilon} + \frac{\beta_n^4}{4 \theta_n^4}
$$

by $\frac{\beta_n^2}{2} \leq \theta_n^2$. Thus, after a quick calculation

$$
\mathbb{E} \left[ (\eta_{21} + \eta'_{22})^2 \right] \lesssim \frac{(np)^2 \gamma_n^4}{\theta_n^4} + \frac{2}{2 \theta_n^4} \gamma_n^2 \beta_n^2 + \frac{\beta_n^4}{4 \theta_n^4} + \tilde{\epsilon}
$$

$$
= \frac{1}{\theta_n^4} \left( np \gamma_n^2 + \frac{1}{2} \beta_n^2 \right)^2 + \tilde{\epsilon} = 1 + \tilde{\epsilon}
$$

Putting this together with (4.10), we obtain $\mathbb{V}(\eta_{21} + \eta'_{22}) \lesssim \tilde{\epsilon} + o(1)$. Hence $\eta_{21} + \eta'_{22}$ converges in probability to the limit of its expectation, which is 1.

It remains to show that $\eta_{23}, \eta_{24} \xrightarrow{p} 0$, then $\eta_{21} + \eta_{22} + \eta_{23} + \eta_{24} \xrightarrow{p} 1$. We start with $\eta_{23}$. By similar calculations as above:

$$
\mathbb{E} \left[ \eta_{23}^2 \right] = \frac{1}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j,k=1}^{n} \sum_{j' \neq k}^{n-1} \sum_{i'=1}^{n-1} \sum_{j' \neq k'}^{n-1} \mathbb{E} \left[ \tilde{\Phi}(i', j') \tilde{\Phi}(i', k') \tilde{\Phi}(i,j) \tilde{\Phi}(i,k) \right]
$$

which converges to 0 by Lemma A.6. Thus $\eta_{23}^2 \xrightarrow{p} 0$ follows.

Finally for $\eta_{24}$,

$$
\mathbb{E} \left[ \eta_{24}^2 \right] = \frac{4}{n^4 \theta_n^4} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{i-1} \sum_{m'=1}^{i-1} \sum_{j' \neq i} \sum_{j' \neq i'} \mathbb{E} \left[ \mathbb{E} \left[ \Psi_j(i') \tilde{\Phi}(i', m) \mid X_m, (Z_{l,i})_{l=1,\ldots,i-1} \right] \mathbb{E} \left[ \Psi_{j'}(i') \tilde{\Phi}(i', m') \mid X_{m'}, (Z_{l,i'})_{l=1,\ldots,i'-1} \right] \right]
$$

which converges to 0 by Lemma A.7 and consequently $\eta_{24} \xrightarrow{p} 0$. By the above estimates for the $\eta_{2i}$, $i = 1, \ldots, 4$ we conclude

$$
\eta_{2i} \xrightarrow{p} 1
$$

which completes the proof. \qed
5 Alternative conditions for the Central Limit Theorem

As mentioned above, it may be cumbersome to check (1.10)–(1.12) in Theorem 1.5. We now give an alternative.

**Proposition 5.1.** The conditions (1.10)-(1.12) follow from: For any $\varepsilon > 0$

\[ n^2 \theta_n^{-2} \mathbb{E} \left[ \Psi_2^2(1) \mathbb{1}_{\{|\Psi_2(1)| \geq \varepsilon \theta_n\}} \right] \xrightarrow{n \to \infty} 0 \]  
\[ \theta_n^{-2} \mathbb{E} \left[ \Phi^2(1, 2) \mathbb{1}_{\{|\Phi(1, 2)| \geq \varepsilon \theta_n\}} \right] \xrightarrow{n \to \infty} 0 \]  
\[ p \theta_n^{-2} \mathbb{E} \left[ H(1, 1) \mathbb{1}_{\{|H(1, 1)| \geq \varepsilon \theta_n\}} \right] \xrightarrow{n \to \infty} 0, \]  

**Proof.** We use the definitions of $\tilde{\Phi}$ and $\tilde{G}$:

(1.10): We use Lemma A.8 for $l = 2, \ldots, n$ and $a_l = \Psi_l(1)$. Then

\[ \frac{1}{n \theta_n^2} \mathbb{E} \left[ \left( \sum_{j=2}^{n} \Psi_j(1) \right)^2 \mathbb{1}_{\{|\sum_{j=2}^{n} \Psi_j(1)| \geq \varepsilon \theta_n\}} \right] \leq \frac{1}{n \theta_n^2} \mathbb{E} \left[ n^2 \sum_{j=2}^{n} \Psi_j^2(1) \mathbb{1}_{\{|\Psi_j(1)| \geq \varepsilon \theta_n\}} \right] \leq \frac{n^2}{\theta_n^2} \mathbb{E} \left[ \Psi_2^2(1) \mathbb{1}_{\{|\Psi_2(1)| \geq \varepsilon \theta_n\}} \right] \to 0 \]

by identical distribution and (5.1). Therefore, (1.10) is true.

(1.11): By Lemma A.8 for $k = 3$,

\[ \theta_n^{-2} \mathbb{E} \left[ \tilde{\Phi}^2(1, 2) \mathbb{1}_{\{|\tilde{\Phi}(1, 2)| \geq \varepsilon \theta_n\}} \right] \leq 9 \theta_n^{-2} \mathbb{E} \left[ \Phi^2(1, 2) \mathbb{1}_{\{|\Phi(1, 2)| \geq \varepsilon \theta_n\}} \right] + 9 \theta_n^{-2} \mathbb{E} \left[ \Psi_2^2(1) \mathbb{1}_{\{|\Psi_2(1)| \geq \varepsilon \theta_n\}} \right] + 9 \theta_n^{-2} \mathbb{E} \left[ \Psi_1^2(2) \mathbb{1}_{\{|\Psi_1(2)| \geq \varepsilon \theta_n\}} \right] \]

By (5.2), the first term converges to 0. By (5.1), so do the other two. Therefore, (1.11) is true.

(1.12): For $i \neq k$, we have

\[ \tilde{H}(i, i) = \mathbb{E} \left[ \tilde{h}(i, k) \tilde{h}(i, k) \mid X_i \right] \]

\[ = \mathbb{E} [h(i, k) h(i, k) - \mathbb{E} [h(i, k) \mid X_i] h(i, k) - \mathbb{E} [h(i, k) \mid X_k] h(i, k) ]

\[ - \mathbb{E} [h(i, k) \mid X_i] h(i, k) - \mathbb{E} [h(i, k) \mid X_k] h(i, k) ]

\[ + \mathbb{E} [h(i, k) \mid X_i] \mathbb{E} [h(i, k) \mid X_i] + \mathbb{E} [h(i, k) \mid X_i] \mathbb{E} [h(i, k) \mid X_k] \]
By measurability and independence we obtain

\[ \tilde{H}(i, i) = H(i, i) - \mathbb{E} [h(i, k) \mid X_i] - \mathbb{E} [\mathbb{E} [h(i, k) \mid X_k] h(i, k) \mid X_i] \]

\[ - \mathbb{E} [h(i, k) \mid X_i^2 - \mathbb{E} [\mathbb{E} [h(i, k) \mid X_k] h(i, k) \mid X_i] \]

\[ + \mathbb{E} [h(i, k) \mid X_i^2 + \mathbb{E} [h(i, k) \mid X_i] \mathbb{E} [\mathbb{E} [h(i, k) \mid X_k]]] \]

\[ + \mathbb{E} [h(i, k) \mid X_i] \mathbb{E} [\mathbb{E} [h(i, k) \mid X_k]] + \mathbb{E} \left[ \mathbb{E} [h(i, k) \mid X_k^2] \right] \]

As \( h(i, j) \) is centered, and by Lemma 1.1

\[ = H(i, i) - \mathbb{E} [h(i, k) \mid X_i] - 2\mathbb{E} [\mathbb{E} [h(i, k) \mid X_k] h(i, k) \mid X_i] + \frac{\gamma_n^2}{p} \]

\[ =: H(i, i) - A - 2B + \frac{\gamma_n^2}{p}. \quad (5.4) \]

Firstly, by Lemma 1.1 \( \mathbb{E} [A] = \mathbb{E} \left[ \mathbb{E} [h(i, k) \mid X_k]^2 \right] = \frac{\gamma_n^2}{p} \).

By Cauchy-Schwarz and Lemma 1.1, we obtain

\[ \mathbb{E} \left[ 2B \mathbb{I}_{\{|2B| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] \leq 2 \sqrt{\mathbb{E} \left[ \mathbb{E} [h(i, k) \mid X_k]^2 \right] \mathbb{E} [h(i, k)^2 \mathbb{I}_{\{|2B| \geq \frac{\epsilon \theta_n^2}{p} \}}]} \]

\[ \leq 2 \sqrt{\mathbb{E} \left[ \mathbb{E} [h(i, k) \mid X_k]^2 \right] \mathbb{E} [h(i, k)^2]} \leq 2 \frac{\gamma_n \beta_n}{p} \]

We obtain by similar arguments as in Lemma A.8

\[ p \theta_n^2 \mathbb{E} \left[ H(1, 1) \mathbb{I}_{\{|H(1, 1)\| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] \]

\[ \leq 4p \theta_n^2 \left( \mathbb{E} \left[ H(1, 1) \mathbb{I}_{\{|H(1, 1)\| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] + \mathbb{E} \left[ A \mathbb{I}_{\{|A| \| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] \right) \]

\[ + \mathbb{E} \left[ 2B \mathbb{I}_{\{|2B| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] + \frac{\gamma_n^2}{p} \]

\[ \leq 4p \theta_n^2 \left( \mathbb{E} \left[ H(1, 1) \mathbb{I}_{\{|H(1, 1)\| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] + \mathbb{E} [A] + 2 \frac{\gamma_n \beta_n}{p} + \frac{\gamma_n^2}{p} \right) \]

\[ \leq 4p \theta_n^2 \left( \mathbb{E} \left[ H(1, 1) \mathbb{I}_{\{|H(1, 1)\| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] + 2 \frac{\gamma_n^2}{p} + 2 \frac{\gamma_n \beta_n}{p} \right) \]

\[ = 4p \theta_n^2 \mathbb{E} \left[ H(1, 1) \mathbb{I}_{\{|H(1, 1)\| \geq \frac{\epsilon \theta_n^2}{p} \}} \right] + 8p \theta_n^2 \frac{\gamma_n^2}{p} + 8p \theta_n^2 \frac{\beta_n \gamma_n}{p} \]
Since \( \theta_n^2 \geq np\gamma_n^2 \) and \( \theta_n^2 \geq \frac{1}{2}\beta_n^2 \), the last two terms immediately converge to 0. By (5.3), so does the first one. Therefore, (1.12) is true.

This completes the proof. \( \square \)

Appendix A

We start the appendix by proving the lemmas in the introduction.

Proof of Lemma 1.1. We begin with (1)

\[
\mathbb{E}\left[\tilde{\Phi}^2(i, j)\right] = \mathbb{E}\left[(\Phi(i, j) - \Psi_j(i) - \Psi_i(j))^2\right] \\
= \mathbb{E}\left[\Phi^2(i, j) - 2\Psi_j(i)\Phi(i, j) - 2\Psi_i(j)\Phi(i, j) + \Psi_j^2(i) + 2\Psi_j(i)\Psi_i(j) + \Psi_i^2(j)\right] \\
= \mathbb{E}\left[\Phi^2(i, j)\right] - 4\mathbb{E}\left[\Psi_j(i)\Phi(i, j)\right] + 2\mathbb{E}\left[\Psi_j^2(i)\right] + 2\mathbb{E}\left[\Psi_j(i)\Psi_i(j)\right]
\]
due to identical distribution. The last term is 0, since \( \mathbb{E}\left[h(i, j) \mid X_i\right] \) and \( \mathbb{E}\left[h(i, j) \mid X_j\right] \) are independent of each other and of \( Z_{ij} \), and centered. The first and third term are known from (1.3). As for the second term,

\[
\mathbb{E}\left[\Psi_j(i)\Phi(i, j)\right] = \mathbb{E}\left[\mathbb{E}\left[\Phi(i, j) \mid X_i, Z_{ij}\right]\Phi(i, j)\right] \\
= \mathbb{E}\left[\mathbb{E}\left[\Phi(i, j) \mid X_i, Z_{ij}\right]\Phi(i, j) \mid X_i, Z_{ij}\right] \\
= \mathbb{E}\left[\mathbb{E}\left[\Phi(i, j) \mid X_i, Z_{ij}\right]^2\right] = \mathbb{E}\left[\Psi_j^2(i)\right] = \gamma_n^2,
\]

by measurability, so that \( \mathbb{E}\left[\tilde{\Phi}^2(i, j)\right] = \beta_n^2 - 4\gamma_n^2 + 2\gamma_n^2 = \beta_n^2 - 2\gamma_n^2 \).

As for the statements on \( h \) and \( \tilde{h} \), i.e. (2): We have

\[
\gamma_n^2 = \mathbb{E}\left[\Psi_j^2(i)\right] = \mathbb{E}\left[Z_{i,j}\mathbb{E}\left[\mathbb{E}\left[h(i, j) \mid X_i\right]^2\right]\right] = p\mathbb{E}\left[\mathbb{E}\left[h(i, j) \mid X_i\right]^2\right]
\]

by definition of \( \Psi_j(i) \) and independence. Moreover,

\[
\beta_n^2 = \mathbb{E}\left[\Phi^2(i, j)\right] = \mathbb{E}\left[Z_{i,j}\mathbb{E}\left[h^2(i, j)\right]\right] = \mathbb{E}\left[Z_{i,j}\mathbb{E}\left[h^2(i, j)\right]\right] = p\mathbb{E}\left[\mathbb{E}\left[h^2(i, j)\right]\right]
\]

by definition of \( \Phi(i, j) \) and independence. Finally,

\[
\beta_n^2 - 2\gamma_n^2 = \mathbb{E}\left[\tilde{\Phi}^2(i, j)\right] = \mathbb{E}\left[Z_{i,j}\tilde{h}^2(i, j)\right] = \mathbb{E}\left[Z_{i,j}\mathbb{E}\left[\tilde{h}^2(i, j)\right]\right] = p\mathbb{E}\left[\mathbb{E}\left[\tilde{h}^2(i, j)\right]\right]
\]

by (1) and independence. This proves the claim. \( \square \)
Proof of Lemma 1.2. We have
\[ \mathcal{U}_n = \left( \frac{n}{2} \right)^{-1} \sum_{1 \leq i < j \leq n} \tilde{\Phi}(i, j) + \left( \frac{n}{2} \right)^{-1} \sum_{1 \leq i < j \leq n} \Psi_j(i) + \left( \frac{n}{2} \right)^{-1} \sum_{1 \leq i < j \leq n} \Psi_i(j) \]

\[ = \left( \frac{n}{2} \right)^{-1} \sum_{1 \leq i < j \leq n} \tilde{\Phi}(i, j) + \frac{n}{2} \sum_{i=1}^{n} \sum_{j=1 \atop j \neq i}^{n} \Psi_j(i) \]

\[ = \left( \frac{n}{2} \right)^{-1} \left( \sum_{1 \leq i < j \leq n} \tilde{\Phi}(i, j) + \sum_{i=1}^{n} \sum_{j=1 \atop j \neq i}^{n} \Psi_j(i) \right). \]

□

Proof of Lemma 1.3. Since \( \mathcal{U}_n \) is centered, we obtain from (1.5)
\[ \mathbb{V}[\mathcal{U}_n] = \mathbb{E}[\mathcal{U}_n^2] \]
\[ = \left( \frac{n}{2} \right)^{-2} \mathbb{E}\left[ \sum_{i < j} \tilde{\Phi}^2(i, j) \right] + \left( \frac{n}{2} \right)^{-2} \mathbb{E}\left[ \sum_{i < j} \sum_{k < l \atop \{i,j\} \neq \{k,l\}} \tilde{\Phi}(i, j)\tilde{\Phi}(k, l) \right] \]
\[ + \left( \frac{n}{2} \right)^{-2} \mathbb{E}\left[ \sum_{i < j} \tilde{\Phi}(i, j) \sum_{k \neq i} \Psi_i(k) \right] \]
\[ + \left( \frac{n}{2} \right)^{-2} \mathbb{E}\left[ \sum_{j \neq i} \Psi_j^2(i) \right] + \left( \frac{n}{2} \right)^{-2} \mathbb{E}\left[ \sum_{j \neq i \atop \{i,j\} \neq \{k,l\}} \Psi_j(i)\Psi_l(k) \right] \]

\[ =: A + B + C + D + E \]

Let us consider the summands separately: Note that from Lemma 1.1
\[ A = \left( \frac{n}{2} \right)^{-1} \mathbb{E}[\tilde{\Phi}^2(1, 2)] = \left( \frac{n}{2} \right)^{-1} \left( \beta_n^2 - 2\gamma_n^2 \right). \quad (A.1) \]

Moreover, \( B = C = 0 \) as follows from Lemma A.2. For \( D \) notice that
\[ D = \left( \frac{n}{2} \right)^{-1} 2\gamma_n^2 \quad (A.2) \]
Finally, consider $E$. For $k \neq i$, the expectation is 0 (see the arguments given in the proof of Lemma A.2). For $k = i$, we have that $j \neq l$ and therefore

$$
\mathbb{E}[\Psi_j(i)\Psi_l(k)] = \mathbb{E}[Z_{ij}Z_{il}\mathbb{E}[h(i, j) \mid X_i] \mathbb{E}[h(i, l) \mid X_i]] = \mathbb{E}[Z_{il}\mathbb{E}[\Psi_j^2(i)]] = \mathbb{E}[Z_{il}p^2] = p\gamma_n^2
$$

Thus

$$
E = \left(\frac{n}{2}\right)^{-1} \cdot 2(n - 2)p\gamma_n^2
$$

(A.3)

and

$$
\mathbb{V}_n = \left(\frac{n}{2}\right)^{-1} (\beta_n^2 - 2(n - 2)p\gamma_n^2) \sim \left(\frac{n}{2}\right)^{-1} (\beta_n^2 + 2np\gamma_n^2) = \left(\frac{n}{2}\right)^{-1} 2\theta_n^2,
$$

from which we conclude the assertion. \hfill \Box

We now prove a couple of lemmas that were used in the proof of Theorem 1.5 in Section 4.

**Corollary A.1.** For $i \neq j$, $\mathbb{E}[\tilde{G}_j(i, i)] = \beta_n^2 - 2\gamma_n^2$ and $\mathbb{E}[\tilde{H}(i, i)] = \frac{\beta_n^2 - 2\gamma_n^2}{p}$.

**Proof.** The claim follows immediately from the tower property, the definition of $\tilde{G}$ and Lemma 1.1:

$$
\mathbb{E}[\tilde{G}_j(i, i)] = \mathbb{E}[\mathbb{E}[\tilde{\Phi}(i, j)\tilde{\Phi}(i, j) \mid X_i, Z_{i,j}]] = \mathbb{E}[\tilde{\Phi}^2(i, j)] = \beta_n^2 - 2\gamma_n^2.
$$

For $\tilde{H}$, one can use

$$
p\mathbb{E}[\tilde{H}(i, i)] = \mathbb{E}[Z_{i,j}]\mathbb{E}[\tilde{H}(i, i)] = \mathbb{E}[Z_{i,j}\tilde{H}(i, i)] = \mathbb{E}[\tilde{G}_j(i, i)]
$$

and apply the above result. \hfill \Box

**Lemma A.2.** For $\{i, j\} \neq \{k, l\}$ we have $\mathbb{E}[\tilde{\Phi}(i, j)\tilde{\Phi}(k, l)] = 0$.

For any $\{i, j\}, \{k, l\}$ we have $\mathbb{E}[\tilde{\Phi}(i, j)\Psi_l(k)] = 0$.

**Proof.** Consider two cases:

If $\{i, j\} \cap \{k, l\} = 1$, then by identical distributions and (1.4)

$$
\mathbb{E}[\tilde{\Phi}(1, 2)\tilde{\Phi}(1, 3)] = \mathbb{E}[Z_{12}Z_{13}]\mathbb{E}[\tilde{h}(1, 2)\tilde{h}(1, 3)] = \mathbb{E}[Z_{12}Z_{13}]\mathbb{E}[\mathbb{E}[\tilde{h}(1, 2)\tilde{h}(1, 3) \mid X_1, X_2]]
$$
\[
\begin{align*}
&= \mathbb{E}[Z_{12}Z_{13}] \mathbb{E}[\hat{h}(1, 2)\mathbb{E}[\hat{h}(1, 3) \mid X_1, X_2]] \\
&= \mathbb{E}[Z_{12}Z_{13}] \mathbb{E}[\hat{h}(1, 2)\mathbb{E}[\hat{h}(1, 3) \mid X_1]] = 0.
\end{align*}
\]

If \( \{i, j\} \cap \{k, l\} = \emptyset \), then by similar reasoning

\[
\mathbb{E}[\hat{h}(1, 2)\hat{h}(3, 4)] = 0.
\]

This shows the first claim. The second can be shown in the same fashion. □

Lemma A.3. For \( \Lambda_j(i) \) the following relation holds

\[
\mathbb{E}[\Lambda_j^2(i)] \leq O((np)^2)
\]

Proof. With \( \Lambda_j(i) := \sum_{k=1}^{i-1} Z_{i,k} + \sum_{k=i+1}^{n} p + 1 \) we can immediately conclude

\[
\Lambda_j^2(i) \leq 9 \left[ \left( \sum_{k=1}^{i-1} Z_{i,k} \right)^2 + ((n - i)p)^2 + 1 \right]
\]

\[
= 9 \left[ \sum_{k=1}^{i-1} Z_{i,k} + \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} Z_{i,k}Z_{i,l} + ((n - i)p)^2 + 1 \right]
\]

Therefore, using the independence of the \( Z_{i,j} \)

\[
\mathbb{E}[\Lambda_j^2(i)] \leq 9 \left[ \sum_{k=1}^{i-1} p + \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} p^2 + ((n - i)p)^2 + 1 \right]
\]

\[
\leq 9 \left[ np + n^2 p^2 + n^2 p^2 + 1 \right],
\]

which, due to \( np \to \infty \), confirms \( \mathbb{E}[\Lambda_j^2(i)] \leq O((np)^2) \). □

Lemma A.4. With \( \Lambda_j(i) \) as in Section 4 we have:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j'=1}^{n} \sum_{j''=1}^{n} \sum_{j \neq j'} \sum_{j' \neq j''} \mathbb{E}[Z_{i,j}Z_{i,j'}\Lambda_j(i)\Lambda_{j'}(i')] \leq n^4 p^2 (np)^2
\]
Proof. Recall that since \( np \to \infty \)

\[
\Lambda_j(i) := \sum_{k=1}^{i-1} Z_{i,k} + \sum_{\substack{k=i+1 \atop k \neq j}}^{n} p + 1 \quad \text{and} \quad E[\Lambda_j(i)] = (n-2)p + 1 \sim np. \quad (A.4)
\]

Let us differentiate cases.

If \( i \neq i' \) (and we’re not in the case \( i = j', \ j = i' \), which will be considered later), then \( (Z_{i,k})_{k=1,..,i-1}, Z_{i,j}, (Z'_{i',k})_{k=1,..,i'-1} \) and \( Z_{i',j'} \) are independent (regardless of \( j \) and \( j' \)). Then by independence the expectation can be reduced to \( E[\Lambda_j(i)]^2 E[Z_{i,j}]^2 \sim (np)^2 p^2 \) by \( (A.4) \). There are \( n \cdot (n-1) \cdot ((n-1) \cdot (n-1) - 1) \) possibilities for this case.

If \( i = j' \) and \( j = i' \), the independence between \( \Lambda_j(i), \Lambda_i(j) \) and \( Z_{i,j} = Z_{j,i} = Z_{i',j'} \) still holds, as well as the independence between \( \Lambda_j(i) \) and \( \Lambda_i(j) \) and we obtain

\[
E[\Lambda_j(i)\Lambda_j(i')] E[Z_{i,j}Z_{i',j'}] = E[\Lambda_j(i)] E[\Lambda_i(j)] E[Z^2_{i,j}] \sim (np)^2 p
\]

There are \( n(n-1) \) possibilities for this case.

If \( i = i' \) but \( j \neq j' \), then \( Z_{i,j} \) may appear in the random sum in \( \Lambda_j(i') \) (and correspondingly, if we interchange \( j, j' \)). We introduce

\[
\Lambda_{j,j'}(i) := \sum_{k=1}^{i-1} Z_{i,k} + \sum_{\substack{k=i+1 \atop k \neq j,j'}}^{n} p + 1, \quad \text{with} \quad E[\Lambda_{j,j'}(i)] = (n-3)p + 1 \sim np \quad (A.5)
\]

and \( \Lambda_j(i) = \Lambda_{j,j'}(i) + Z_{i,j} \). Then

\[
E[\Lambda_j(i)\Lambda_j(i') Z_{i,j} Z_{i',j'}] = E[\Lambda_j(i)\Lambda_j(i') Z_{i,j} Z_{i',j'}]
\]

\[
= E[\Lambda_j^2(i) Z_{i,j} Z_{i,j'}] + E[Z_{i,j} Z_{i,j'}] + 2E[\Lambda_j(i) Z_{i,j} Z_{i,j'}]
\]

\[
= E[\Lambda^2_{j,j'}(i)] p^2 + p^2 + 2E[\Lambda_{j,j'}(i)] p^2.
\]

After some considerations one finds

\[
E[\Lambda^2_{j,j'}(i)] = O((np)^2),
\]

so that combining this and \( (A.5) \) gives

\[
E[\Lambda_j(i)\Lambda_j(i') Z_{i,j} Z_{i',j'}] \leq O((np)^2) p^2 + p^2 + 2O(np) p^2 = O((np)^2) \cdot p^2
\]

by \( np \to \infty \). There are \( n(n-1)(n-2) \) possibilities for the case \( i = i', j \neq j' \).

If \( i = i' \) and \( j = j' \), we may again use independence to arrive at

\[
E[\Lambda_j(i)\Lambda_j(i') Z_{i,j} Z_{i',j'}] = E[\Lambda^2_j(i) Z^2_{i,j}] \leq O((np)^2) p,
\]
by Lemma A.3. Again, there are \( n(n - 1) \) possibilities for this case. Putting this together, we see that the sum of all expectations is asymptotically bounded from above by \( n^4(np)^2p^2 \).

\[ \square \]

**Lemma A.5.** We have:

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i' \neq i}^{n} \mathbb{E} \left[ \Lambda_j(i)Z_{i,j}Z_{i',j'} \right] \leq \frac{1}{2}n^4p^2(np) \]

**Proof.** The strategy of proof is exactly the same as in Lemma A.4, just the expectations and the number of summands differ. We therefore leave the proof to the reader. \[ \square \]

**Lemma A.6.** Under the assumptions of Theorem 1.5 as \( n \to \infty \) we have:

\[ \frac{1}{n^4 \theta^4_n} \sum_{i=1}^{n} \sum_{j,k=1}^{n} \sum_{i' \neq i}^{n} \sum_{j' \neq j'}^{n} \mathbb{E} \left[ \Phi(i', j')\Phi(i', k')\Phi(i, j)\Phi(i, k) \right] \xrightarrow{n \to \infty} 0. \]

**Proof.** Without loss of generality, assume \( i' \geq i \). We let

\[ Q := \mathbb{E} \left[ \tilde{G}_i(j, k)\tilde{G}_j(j', k') \right], \quad \tilde{Q} := \mathbb{E} \left[ \tilde{\Phi}(i, j)\tilde{\Phi}(i, k)\tilde{\Phi}(i', j')\tilde{\Phi}(i', k') \right] \]

and \( Q_1 \cdot Q_2 := \mathbb{E} \left[ Z_{i,j}Z_{i,k}Z_{i',j'}Z_{i',k'} \right] \mathbb{E} \left[ \tilde{h}(i, j)\tilde{h}(i, k)\tilde{h}(i', j')\tilde{h}(i', k') \right] \). Note that all three notations denote the same object. However, we will use all these notations throughout the proof.

Now, let us go through all possible cases for \( i, j, k, i', j', k' \).

1. If \( i = i' \) and \( |\{j, k\} \cap \{j', k'\}| = 2 \) by independence \( Q = \mathbb{E} \left[ \tilde{G}_i^2(j, k) \right] \).
2. The cases \( i = i' \) and \( |\{j, k\} \cap \{j', k'\}| = 1 \) and \( i = i' \) and \( |\{j, k\} \cap \{j', k'\}| = 0 \) are almost identical. Consider the first: without loss of generality take \( j = j' \). Then, by total expectation, the tower property, and independence

\[ Q_2 := \mathbb{E} \left[ \tilde{h}(i, j)^2\tilde{h}(i, k)\tilde{h}(i, k') \right] = \mathbb{E} \left[ \mathbb{E} \left[ \tilde{h}(i, j)^2\tilde{h}(i, k)\tilde{h}(i, k') \mid X_i, X_j, X_k \right] \right] \\
= \mathbb{E} \left[ \tilde{h}(i, j)^2\tilde{h}(i, k)\mathbb{E} \left[ \tilde{h}(i, k') \mid X_i, X_j, X_k \right] \right] \\
= \mathbb{E} \left[ \tilde{h}(i, j)^2\tilde{h}(i, k)\mathbb{E} \left[ \tilde{h}(i, k') \mid X_i \right] \right] = 0, \]

since by (1.4) \( \mathbb{E} \left[ \tilde{h}(l, m) \mid X_r \right] = 0 \) if \( l \neq m \) for every \( r \). Thus, \( Q = 0 \).
3. Again the cases \( i < i' \), \( i \in \{j', k'\} \), and \( |\{j, k\} \cap \{j', k'\}| = 1 \) and \( i < i' \), \( i \in \{j', k'\} \), and \( |\{j, k\} \cap \{j', k'\}| = 0 \) are very similar. Consider the first: Without loss of generality \( i = j', k = k' \), and along the lines of the previous cases we get \( Q_2 = 0 \).
4. Next consider the case \( i < i', \ i \notin \{j', k'\}, \) and \(|\{j, k\} \cap \{j', k'\}| = 2\).
Without loss of generality, \( j = j', k = k' \) and by the definition of \( \tilde{G} \) and independence we compute

\[
Q = \mathbb{E}[\tilde{G}(j, k) \tilde{G}(j, k)]
= \mathbb{E}\left[\mathbb{E}[Z_{i,j}Z_{i,k}\tilde{h}(i, j)\tilde{h}(i, k) \mid X_j, X_k, Z_{i,j}Z_{i,k}] \cdot \mathbb{E}[Z_{i',j}Z_{i',k}\tilde{h}(i', j)\tilde{h}(i', k) \mid X_j, X_k, Z_{i',j}Z_{i',k}]\right]
= \mathbb{E}[Z_{i,j}Z_{i,k}Z_{i',j}Z_{i',k}\mathbb{E}[\tilde{h}(i, j)\tilde{h}(i, k) \mid X_j, X_k]^2]
= \mathbb{E}[Z_{i,j}Z_{i,k}(Z_{i,j}Z_{i,k}\mathbb{E}[\tilde{h}(i, j)\tilde{h}(i, k) \mid X_j, X_k, Z_{i,j}Z_{i,k}])^2]
= \mathbb{E}[Z_{i',j}Z_{i',k}(\tilde{G}(j, k))^2] = p^2\mathbb{E}[\tilde{G}(j, k)\tilde{G}(j, k)]
\]

5. Finally, the cases the case \( i < i', \ i \notin \{j', k'\}, \) and \(|\{j, k\} \cap \{j', k'\}| = 0, 1\) follow the arguments in cases (2) and (3) to give \( Q_2 = 0 \).

Altogether, we get the following: The only situation where the given expectation is non-zero is when \(|\{j, k\} \cap \{j', k'\}| = 2\). If \( i = i' \), there are at most \( n(n-1)^2 \) possibilities for this (\( n \) for \( i \), and since \( j \) and \( k \) are smaller than \( i \) and different, at most \( n-1 \) for each of those). If \( i \neq i' \), there are additional \( n-1 \) possibilities for \( i' \), which makes at most \( n(n-1)^3 \) possibilities. Thus, we see that the given sum of expectations is bounded by

\[
n(n-1)^2\mathbb{E}[(\tilde{G}(j, k))^2] + n(n-1)^3p^2\mathbb{E}[(\tilde{G}(j, k))^2] \leq n^4\mathbb{E}[\tilde{G}^2(j, k)].
\]

Then for the sum of the considered expectations we have

\[
\frac{1}{n^4\theta^4} \sum_{i=1}^{n} \sum_{j \neq k}^{n-1} \sum_{j' \neq k'}^{n-1} \sum_{j' \neq i}^{n-1} \mathbb{E}[\Phi(i', j')\Phi(i', k')\Phi(i, j)\Phi(i, k)] \leq \frac{\mathbb{E}[\tilde{G}^2(j, k)]}{\theta^4_n}
\]

By (3.1), this converges to 0. \( \square \)

**Lemma A.7.** Under the assumptions of Theorem 1.5 as \( n \to \infty \) we have

\[
\frac{1}{n^4\theta^4} \sum_{i=1}^{n} \sum_{j \neq i}^{n-1} \sum_{m=1}^{i-1} \sum_{m' = 1}^{m'} \mathbb{E}[\Psi_j(i)\Phi(i, m) \mid X_m, (Z_{i,t})_{t=1,\ldots,i-1}] \cdot \mathbb{E}[\Psi_{j'}(i')\Phi(i', m') \mid X_{m'}, (Z_{i',t'})_{t'=1,\ldots,i'-1}] \xrightarrow{n \to \infty} 0.
\]
Proof. We denote
\[
Q := \mathbb{E} \left[ \Psi_j(i) \tilde{\Phi}(i, m) \mid X_m, (Z_{l,i})_{l=1,...,i-1} \right] \mathbb{E} \left[ \Psi_j(i') \tilde{\Phi}(i', m') \mid X_{m'}, (Z_{l,i'})_{l=1,...,i'-1} \right]
\]
\[
= Q_1 \cdot Q_2,
\]
where
\[
Q_1 = \mathbb{E} \left[ Z_{i,j} Z_{i,m} \mid (Z_{l,i})_{l=1,...,i-1} \right] \cdot \mathbb{E} \left[ Z_{i',j'} Z_{i',m'} \mid (Z_{l,i'})_{l=1,...,i'-1} \right]
\]
\[
Q_2 = \mathbb{E} \left[ \mathbb{E} [h(i, j) \mid X_i] \tilde{h}(i, m) \mid X_m \right] \mathbb{E} \left[ \mathbb{E} [h(i', j') \mid X_{i'}] \tilde{h}(i', m') \mid X_{m'} \right].
\]
By independence between the $Z$- and $X$-terms, \( \mathbb{E} [Q] = \mathbb{E} [Q_1] \mathbb{E} [Q_2] \).

In the case \( m \neq m' \), we have \( \mathbb{E} [Q_2] = 0 \) by independence of \( X_m \) and \( X_{m'} \) and \( \mathbb{E} \left[ \mathbb{E} [h(i, j) \mid X_i] \tilde{h}(i, m) \mid X_m \right] = 0 \), which we find by adding a conditional expectation on \( X_i \).

For \( m = m' \), note that the conditional expectations in \( Q_2 \) do not depend on the choice of \( i \) and \( i' \), hence we choose \( i = 1, i' = 2, \) and \( m = 3 \). Then:
\[
\mathbb{E} [Q_2] = \mathbb{E} \left[ \mathbb{E} [h(1, j) \mid X_1] \tilde{h}(1, 3) \mid X_3 \right] \mathbb{E} \left[ \mathbb{E} [h(2, j') \mid X_2] \tilde{h}(2, 3) \mid X_3 \right]
\]
\[
= \mathbb{E} \left[ \mathbb{E} [h(1, j) \mid X_1] \mathbb{E} [h(2, j') \mid X_2] \tilde{H}_3(1, 2) \right].
\]
By Cauchy-Schwarz, independence and Lemma 1.1
\[
\mathbb{E} [Q_2] \leq \left( \mathbb{E} \left[ h(1, j) \mid X_1 \right]^2 \right) \mathbb{E} \left[ h(2, j') \mid X_2 \right]^2 \mathbb{E} \left[ \tilde{H}_3^2(1, 2) \right]^{1/2} = \frac{\gamma_n^2}{p} \mathbb{E} \left[ \tilde{H}_3^2(1, 2) \right]^{1/2}
\]

Furthermore, if \( i = i' \) and \( |\{j, j', m\}| \leq 2 \), \( \mathbb{E} [Q_1] \leq p \) and we have at most \( n^3 \) possibilities to choose \( i, j, m, i', j', m' \).

If \( i = i' \) and \( |\{j, j', m\}| = 3 \), \( \mathbb{E} [Q_1] \leq p^2 \) and we have at most \( n^4 \) possibilities to choose.

If \( i \neq i' \) and \( |\{j, j', m\}| \leq 2 \), \( \mathbb{E} [Q_1] \leq p^2 \) and we have at most \( n^4 \) possibilities to choose.

If \( i \neq i' \) and \( |\{j, j', m\}| = 3 \), \( \mathbb{E} [Q_1] \leq p^4 \) and we have at most \( n^5 \) possibilities to choose.

Combining all this and keeping in mind that we assume that \( np \to \infty \) yields
\[
\frac{1}{n^4 \theta_4^4} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{i-1} \sum_{i'=1}^{n} \sum_{j'=1}^{i'-1} \sum_{m'=1}^{i'-1} \mathbb{E} \left[ \Psi_j(i) \tilde{\Phi}(i, m) \mid X_m, (Z_{l,i})_{l=1,...,i-1} \right]
\]
\[ \cdot E \left[ \Psi_{j'}(i', \Phi(i', m') \mid X_{m'}, (Z_{t,i'})_{t=1,...,l'-1}) \right] \]

\[
\leq \frac{1}{n^4 \theta_n^4} \frac{\gamma_n^2}{p} \sqrt{E \left[ \hat{H}_1^2(2,3) \right]} (pn^3 + p^3 n^4 + p^2 n^4 + p^4 n^5) \\
\leq \frac{2}{n^4 \theta_n^4} \frac{\gamma_n^2}{p} \sqrt{E \left[ \hat{H}_1^2(2,3) \right]} (p^2 n^4 + p^4 n^5) \\
\leq \frac{2}{np} \sqrt{\frac{1}{\theta_n^4} p^2 E \left[ \hat{H}_1^2(2,3) \right]} + 8p \sqrt{\frac{1}{\theta_n^4} p^2 E \left[ \hat{H}_1^2(2,3) \right]} \\
\leq \frac{2}{np} \sqrt{\frac{1}{\theta_n^4} E \left[ \hat{G}_1^2(2,3) \right]} + 8p \sqrt{\frac{1}{\theta_n^4} E \left[ \hat{G}_1^2(2,3) \right]}.
\]

where the last two inequalities follow from (1.3) and (1.9). By \( np \to \infty \) and (3.1), this converges to 0. \( \square \)

A very general lemma for the purpose of reminding us of a basic fact is

**Lemma A.8.** For any \( k \in \mathbb{N} \) and any sequence \((a_l)_{l=1,...,k} \), the following relation holds

\[
\left( \sum_{l=1}^{k} a_l \right)^2 1_{ \left\{ \sum_{l=1}^{k} |a_l| \geq \varepsilon \right\} } \leq k^2 \sum_{l=1}^{k} a_l^2 1_{ \left\{ |a_l| \geq \frac{\varepsilon}{k} \right\} }.
\]

The proof of this lemma is elementary.

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