The Lamm–Rivière system I: $L^p$ regularity theory

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Abstract
Motivated by the heat flow and bubble analysis of biharmonic mappings, we study further regularity issues of the fourth order Lamm–Rivière system

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \text{div}(w \nabla u) + (\nabla \omega + F) \cdot \nabla u + f$$

in dimension four, with an inhomogeneous term $f$ which belongs to some natural function space. We obtain optimal higher order regularity and sharp Hölder continuity of weak solutions. Among several applications, we derive weak compactness for sequences of weak solutions with uniformly bounded energy, which generalizes the weak convergence theory of approximate biharmonic mappings.

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1 Introduction

1.1 Background and motivation

In the calculus of variations, finding regular critical points of the variational functional

\[ u \mapsto \int F(x, u(x), Du(x))\,dx \]

with quadratic growth has been one of the most attractive topics. Among many other interesting geometric models, those related to conformally invariant variational problems, are of particular interests.

A fundamental work of Morrey [22] shows that minimizers in \( W^{1,2}(B^2, N) \), \( N \subset \mathbb{R}^m \), of the standard Dirichlet energy are locally Hölder continuous and thus are as smooth as the (embedded) Riemannian manifold \( N \). However, when the domain has higher dimensions (greater than or equal to three), one loses conformal/scaling invariance of minimizers and in this case, there exist discontinuous minimizers, much less to say about general critical points. A well-known conjecture along this direction was formulated by Hildbrandt [12]:

**Conjecture** Critical points of coercive conformally invariant Lagrangian with quadratic growth are regular.

In his pioneer work [11], Helein confirmed this conjecture in the case of weakly harmonic mappings: every weakly harmonic mappings from the two dimensional disk \( B^2 \subset \mathbb{R}^2 \) into any closed manifold are smooth via the nowadays well-known moving frame method.

In 2007, this conjecture was fully settled down by Rivière in his remarkable work [25]. More precisely, in [25], he proposed the general second order linear elliptic system

\[
-\Delta u = \Omega \cdot \nabla u \quad \text{in } B^2
\]

(1.1)

where \( u \in W^{1,2}(B^2, \mathbb{R}^m) \) and \( \Omega = (\Omega_{ij}) \in L^2(B^2, so_m \otimes \Lambda^1 \mathbb{R}^2) \). As was verified in [25], (1.1) includes the Euler-Lagrange equations of critical points of all second order conformally invariant variational functionals which act on mappings \( u \in W^{1,2}(B^2, N) \) from \( B^2 \subset \mathbb{R}^2 \) into a closed Riemannian manifold \( N \subset \mathbb{R}^m \). The approach of Rivière involves finding a map \( A \in L^\infty \cap W^{1,2}(B^2, Gl(m)) \) and \( B \in W^{1,2}(B^2, M_m) \) satisfying \( \nabla A - A\Omega = \nabla^\perp B \), such that system (1.1) can be written equivalently as the conservation law

\[
\text{div} \left( A \nabla u + B \nabla^\perp u \right) = 0.
\]

(1.2)

Then the continuity of weak solutions of system (1.1) follows rather easily from the conservation law (1.2). As (1.1) includes the equations of weakly harmonic mappings from \( B^2 \) into \( N \), this recovered the regularity result of Hélein [11]. In fact, Rivière’s work has far more applications beyond conformally invariant problems; see [26,27] for a comprehensive overview.

Starting from the celebrated work of Eells and Sampson [6], there have been great attempts to find regular solutions for the heat flow of harmonic mappings (see for instance [21,23,31,32] and the references therein). This leads to the general consideration of the inhomogeneous Rivière system

\[
-\Delta u = \Omega \cdot \nabla u + f \quad \text{in } B^2,
\]

(1.3)

where the drift term \( f : B^2 \rightarrow \mathbb{R}^m \) belongs to certain natural function space. In case of heat flow of harmonic mappings, \( f \in L^p(B^2, \mathbb{R}^m) \) shall denote the first order partial derivative
of the flow with respect to time. Two basic topics, with large mathematical interest, related to (1.3) are the weak compactness of Palais-Smale sequences and the energy identity (or bubble analysis). The energy identity quantifies the limiting behaviour of certain energy of a sequence of weak solutions. To be more precise, let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence of weak solutions (say, to the harmonic mapping system) with uniformly bounded energy, for which we denoted by \( E(u_n) \). In general, a subsequence of \( \{u_n\} \) shall converge weakly to some limiting map \( u \), but the convergence does not necessarily have to be strong. One can show, with some effort, that away from a finite set \( \Sigma = \{x_1, \cdots, x_k\} \), the convergence \( u_n \to u \) will be strong. Moreover, the loss of energy during the limiting process happens exactly because of energy concentration at these finite points \( x_i, i = 1, \cdots, k \).

The study of energy identity for harmonic mappings was initiated by Sacks and Uhlenbeck in the seminal work [29] and then attracted great attention in geometric analysis of various mappings and equations. Based on the new method of Rivière [25], energy identity for the general system (1.3) was obtained very recently in [18,20]. An important intermediate step towards these results is to establish a higher order \( L^p \)-regularity theory for weak solutions of the system (1.3), which was done in the very interesting work of Sharp and Topping [30]; see also [23] for applications of higher \( L^p \)-regularity theory in the study of heat flow of harmonic mappings.

Moving to four dimensions and taking into account of the conformal invariance in \( \mathbb{R}^4 \), in order to obtain smooth mappings \( u: B^4 \to N \hookrightarrow \mathbb{R}^m \), it is natural to consider critical points of the bi-energy (or intrinsic bi-energy), that is, critical points of the \( L^2 \)-norm of \( \Delta u \) (or \( (\Delta u)^T \), respectively). These critical points are called extrinsic (intrinsic, respectively) biharmonic mappings and they form a natural generalization of harmonic mappings. Chang et al. [5] initiated the study of regularity theory of extrinsic biharmonic mappings from the \( n \)-dimensional Euclidean ball \( B^n \) into Euclidean spheres and proved smoothness of these mappings. Shortly after that, Wang developed a regularity theory of both extrinsic and intrinsic biharmonic mappings into general closed Riemannian manifolds in a series of pioneer works [37–39] via the method of Coulomb frames. For more results on biharmonic mappings, see e.g. [4,28,33,34] and the references therein.

Similar to harmonic mappings, there has been great interest to find regular solutions for the heat flow of biharmonic mappings (see for instance [13,15,16,40]). This leads to the general consideration of the inhomogeneous Lamm–Rivière system

\[
\Delta^2 u = \Delta (V \cdot \nabla u) + \text{div}(w \nabla u) + (\nabla \omega + F) \cdot \nabla u + f \quad \text{in} \ B^4,
\]

where all the involved coefficients belong to some natural function spaces. When \( f = 0 \), the homogeneous system (1.4) was first introduced by Lamm and Rivière [17]. It includes both Euler-Lagrange equations of the extrinsic and intrinsic biharmonic mappings from Euclidean balls into Riemannian manifolds as well as their variants such as approximate biharmonic mappings. One particular motivation for Lamm and Rivière to consider system (1.4) is to extend the new powerful method of Rivière [25] to fourth order system and to give a unified treatment of the regularity theory for the above mentioned mapping classes. Based on the fundamental work of Lamm and Rivière [17], the first two authors of the present paper established the weak compactness of Palais-Smale sequences in [9] and it remains a natural problem to study the energy identity (or bubble analysis) for sequences of weak solutions of (1.4), extending the corresponding results for (approximate) biharmonic mappings obtained in [14,19,41].

\(^1\) Here \((\Delta u)^T\) denotes the projection of \(\Delta u\) into \(T_u N\).
In the present paper, we aim at establishing a higher order $L^p$-regularity theory for weak solutions of the inhomogeneous Lamm–Rivière system (1.4), akin to that of Sharp and Topping [30] for the inhomogeneous Rivière system (1.3). In a following-up work, we shall apply the $L^p$-regularity theorems to establish the energy identity for system (1.4).

### 1.2 Main results

Let $B_r \subset \mathbb{R}^4$ be an open ball with radius $r$, $m \in \mathbb{N}$ and $u \in W^{2,2}(B_{10}, \mathbb{R}^m)$ a weak solution of the inhomogeneous Lamm–Rivière system

$$
\Delta^2 u = \Delta(V \cdot \nabla u) + \text{div}(w \nabla u) + (\nabla \omega + F) \cdot \nabla u + f \quad \text{in } B_{10},
$$

(1.5)

where

$$
V \in W^{1,2}(B_{10}, M_m \otimes \Lambda^1 \mathbb{R}^4), \quad w \in L^2(B_{10}, M_m),
$$

(1.6)

$$
\omega \in L^2(B_{10}, \text{so}_m), \quad F \in L^{\frac{4}{3}:1}(B_{10}, M_m \otimes \Lambda^1 \mathbb{R}^4)
$$

and $f \in L \log L(B_{10}, \mathbb{R}^m)$. For the definitions of the Lorentz function spaces $L^{\frac{4}{3}:1}$ and $L \log L$, see Sect. 2.

We begin our discussion by recording the following fundamental result of Lamm and Rivière [17].

**Theorem** (Lamm and Rivière [17]) For any $m \in \mathbb{N}$, there exist constants $C_m > 0$ and $\epsilon_m > 0$ such that if

$$
\|V\|_{W^{1,2}(B_{10})} + \|w\|_{L^2(B_{10})} + \|\omega\|_{L^2(B_{10})} + \|F\|_{L^{4/3,1}(B_{10})} < \epsilon_m,
$$

(1.7)

then there exist $A \in W^{2,2} \cap L^\infty(B_8, M(m))$ and $B \in W^{1,4/3}(B_8, M(m) \otimes \Lambda^2 \mathbb{R}^4)$ satisfying

$$
\nabla \Delta A + \Delta AV - \nabla Aw + AW = \text{curl}(B) \quad \text{in } B_8,
$$

where $\text{curl}(B) = \sum_k \partial_{s_k} B_{s_k} \partial_{t_k}$. Moreover,

$$
\|A\|_{W^{1,2}(B_8)} + \|\text{dist}(A, SO_m\|_{L^\infty(B_8)} + \|B\|_{W^{1,4/3}(B_8)} \leq C_m \left( \|V\|_{W^{1,2}(B_{10})} + \|w\|_{L^2(B_{10})} + \|\omega\|_{L^2(B_{10})} + \|F\|_{L^{4/3,1}(B_{10})} \right).
$$

(1.8)

Consequently, $u$ solves (1.5) if and only if it satisfies the conservation law

$$
\Delta(A \Delta u) = \text{div}(K) + Af \quad \text{in } B_8,
$$

(1.9)

where

$$
K = 2\nabla A \Delta u - \Delta A \nabla u - Aw \nabla u - \nabla AV \cdot \nabla u + A \nabla(V \cdot \nabla u) + B \cdot \nabla u.
$$

(1.10)

Combining (1.6) with the regularity of $A$ and $B$, one easily verifies that $K \in L^{\frac{4}{3}:1}(B_8)$. Thus $A \Delta u \in W^{1,\frac{4}{3}:1}(B_8)$, which in turn gives $\Delta u \in W^{1,\frac{4}{3}:1}(B_8)$. By elliptic regularity theory, this implies $u \in W^{3,\frac{4}{3}:1}(B_8)$ and thus $u \in C(B_8)$ by Lorentz-Sobolev embedding theorems. For details, see the proof of Theorem 1 of [17] or Theorem 1.6 below.

In a recent very work [9], the first two authors of the present paper further established H"{o}lder continuity of weak solutions of (1.5) (with $f = 0$) by deriving a decay estimate via the conservation law (1.9). We mention that it is also possible to obtain the Hölder continuity without using conservation law, for details see [10].

Our first theorem deals with the optimal Hölder continuity of weak solutions to (1.5).
Theorem 1.1 (Hölder continuity) Let \( u \in W^{2,2}(B_{10}, \mathbb{R}^m) \) be a weak solution of (1.5) and assume \( f \in L^p(B_{10}) \) for \( p \in (1, \frac{4}{3}) \). Then \( u \) is locally \( \alpha \)-Hölder continuous with exponent \( \alpha = 4(1 - \frac{1}{p}) \).

Moreover, there exists \( C = C(p, m) > 0 \) such that for all \( 0 < r < 1 \), there holds

\[
\|\nabla u\|_{L^{4,2}(B_r)} + \|\Delta u\|_{L^2(B_r)} \leq C r^\alpha \left( \|\nabla u\|_{L^{4,2}(B_1)} + \|\Delta u\|_{L^2(B_1)} + \|f\|_{L^p(B_1)} \right). \tag{1.11}
\]

The Hölder continuity is optimal, as one can see from the simplest case \( \Delta^2 u = f \). Moreover, as one can easily notice, applying Theorem 1.1 to the case \( f \equiv 0 \) yields that every weak solution \( u \in W^{2,2}(B_{10}) \) of the Lamm–Rivière system (1.5) is locally \( \alpha \)-Hölder continuous for all \( \alpha \in (0, 1) \). On the other hand, the Hölder continuity is the best possible regularity that one can expect for weak solutions of (1.5) (even when \( f \equiv 0 \)). Indeed, we shall construct a weak solution \( u : B \to \mathbb{R}, B \subset \mathbb{R}^4 \), which belongs to \( C^{0,\alpha}(B) \cap W^{2,2}(B) \) for any \( \alpha \in (0, 1) \), of the system

\[
\Delta^2 u = \Delta(V \cdot \nabla u)
\]

for some \( V \in W^{1,2}(B, \mathbb{R}^4) \). But \( u \) fails to be (locally) Lipschitz continuous in \( B \). For details, see Remark 4.2 below.

In our second theorem, we derive optimal higher order regularity of weak solutions.

Theorem 1.2 (local \( L^p \) estimates) Let \( u \in W^{2,2}(B_{10}, \mathbb{R}^m) \) be a weak solution of (1.5) with \( f \in L^p(B_{10}) \) for \( p \in (1, \frac{4}{3}) \). Then

\[
u \in W_{\text{loc}}^{3, \frac{4p}{p-2}}(B_{10}).
\]

Moreover, there exist \( \varepsilon = \varepsilon(p, m) > 0 \) and \( C = C(p, m) > 0 \) such that if the smallness condition (1.7) is satisfied with \( \varepsilon_m = \varepsilon \), then

\[
\|u\|_{W^{3, \frac{4p}{p-2}}(B_{1/2})} \leq C \left( \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)} \right). \tag{1.12}
\]

If, in addition, we assume \( V \in W^{2, \frac{4}{3}}(B_{10}) \) and \( w \in W^{1, \frac{4}{3}}(B_{10}) \), then

\[
u \in W_{\text{loc}}^{4, p}(B_{10}),
\]

and

\[
\|u\|_{W^{4, p}(B_{1/2})} \leq C \left( \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)} \right). \tag{1.13}
\]

Theorem 1.2 can be regarded as a counterpart of Sharp-Topping [30, Theorem 1.1] to the fourth order system (1.5). Some special cases of Theorem 1.2 can be found in the literature.

In order to study the global existence of extrinsic biharmonic map flow, Lamm and Rivière [17, Lemma 3.1] proved a regularity result for \( f \in L^2 \) under some special conditions on \( V, w, \omega, F \). Wang and Zheng [41, Lemma 2.3] proved \( W^{4, p} \) regularity for approximate extrinsic biharmonic mappings with \( f \in L^p \) for some \( p > 1 \). Laurain and Rivière [19, Theorem 3.3] also obtained \( W^{4, p} \) regularity for (1.5), but under special growth conditions

\[
|V| \leq C|\nabla u|,
\]

\[
|F| \leq C|\nabla u| \left( |\nabla^2 u| + |\nabla u|^2 \right) \text{ almost everywhere}
\]

\[
|w| + |\omega| \leq C \left( |\nabla^2 u| + |\nabla u|^2 \right).
\]
Theorem 1.6 Let \( u \in W^{2,2}(B_{10}, \mathbb{R}^m) \) be a weak solution of (1.5) with \( f \in L \log L(B_{10}, \mathbb{R}^m) \). Then, there exist some \( \epsilon = \epsilon(m) > 0 \) and \( C = C(m) > 0 \) such that if the smallness condition (1.7) is satisfied with \( \epsilon_m = \epsilon \), then
\[
u \in W^{3, \frac{4}{3}, 1}_{\text{loc}}(B_{10})
\]
with
\[
\| \nu \|_{W^{3, \frac{4}{3}, 1}(B_{1/2})} \leq C \left( \| f \|_{L \log L(B_{10})} + \| \nu \|_{L^1(B_{1})} \right). \tag{1.14}
\]
In particular, \( u \) is continuous by the Sobolev embedding \( W^{3, \frac{4}{3}, 1}(B_{10}) \subset C(B_{10}) \).

As an application of Theorem 1.6, we obtain the following compactness result; compare it with [9, Theorem 1.3].

Theorem 1.7 Let \( \{ u_n \} \subset W^{2,2}(B_{10}, \mathbb{R}^m) \) be a sequence of weak solutions of
\[
\Delta^2 u_n = \Delta(V_n \cdot \nabla u_n) + \text{div}(w_n \nabla u_n) + (\nabla \omega_n + F_n) \cdot \nabla u_n + f_n.
\]
Suppose there exists a constant $\Lambda > 0$ such that
$$\sup_n \left( \| f_n \|_{L \log L(B_{10})} + \| u_n \|_{L^1(B_{10})} \right) \leq \Lambda.$$  

Then, there exist some $\epsilon = \epsilon(m) > 0$ and a mapping $u \in W^{2,2}(B_{10}, \mathbb{R}^m)$ such that, if the sequences $\{ V_n, w_n, \omega_n, F_n \} \in \mathbb{N}$ satisfy (1.7) with a common $\epsilon_m = \epsilon$, then after passing by to a subsequence,
$$u_n \to u \text{ in } W^{2,2}_{\text{loc}}(B_{10}, \mathbb{R}^m).$$

### 1.3 Strategy of the proof

Before ending this section, we would like to make some comments on the techniques that we shall use in the proofs of Theorems 1.1 and 1.2. We follow the scheme of Sharp and Topping [30] and divide the proofs into three steps.

1. In the first step, we derive the decay estimates in Theorem 1.1 via the conservation law and Hodge decomposition, from which Hölder continuity follows. The decay estimates show that $\nabla^2 u$ and $\nabla u$ belong to some Morrey spaces.

2. In the second step, we combine the above fact, together with the Riesz potential theory of Adams [1] (see Lemma 2.5), to deduce an almost optimal higher order Sobolev regularity. A bootstrapping argument is also applied.

3. In the last step, we show that all the concerned local estimates are uniform with respect to the parameters so that we can pass to the limit to conclude the optimal higher order Sobolev regularity with the critical exponent.

As we are dealing with fourth order equation, the situation becomes more complicated than that of second order equation, especially in the first and the last step, which are quite different from that of Sharp and Topping [30]. The first step was the key step in their proofs. They adopted a very delicate iteration argument to derive sharp decay estimates. In order to run the iteration procedure, a very precise control on some coefficients of related estimates is needed so that the key coefficients are sufficiently small. A fact that plays a crucial role in their proof is the (nondecreasing) monotonicity of the average function
$$r \mapsto \frac{1}{r^2} \int_{B_r(x)} f$$
whenever $f$ is a subharmonic function. Such a monotonicity property seems unknown in higher order cases. In particular, for a biharmonic function $h$ in $\mathbb{R}^4$, we do not know whether the average function $r \mapsto \frac{1}{r^2} \int_{B_r(x)} |h|^k (k \geq 1)$ is monotone or not. Furthermore, we have to consider not only the energy of $\nabla u$, but also the energy of $\Delta u$. Thus, it seems impossible to gain optimal coefficients simultaneously in front of the decay of these two energies. Consequently, the iteration procedure of Sharp and Topping [30, Proof of Lemma 7.3] fails to apply here. To overcome this difficulty, we borrow some ideas from [9,41], and use Hodge decomposition together with Lorentz-Sobolev embedding to derive the optimal decay. To emphasize the differences between these two approaches, we included an alternative proof of the decay estimate of Sharp-Topping for the inhomogeneous system (1.5) in Sect. 3.

Another severe technical difficulty occurs in the last step. In the case of Sharp-Topping, they run a similar scaling and iteration argument as in the first step due to the well control of coefficient of decay of energy of $\nabla u$. Moreover, since in the second order case they have proved that every solution belongs to $W^{2,p}_{\text{loc}}$, the equation $-\Delta u = \Omega \cdot \nabla u$ can be used as a
pointwise identity to deduce estimate for $\nabla^2 u$ directly. In the fourth order case, we cannot run a similar iteration argument as we do not have good enough control on the coefficients of the decay of energies of $\nabla u$ and $\nabla^2 u$. Furthermore, we have no fourth order regularity for the solutions, and thus equation only has the meaning of distributions. To overcome these difficulties, we apply a duality trick to obtain good control on the decay of energy of $\nabla^2 u$, and then apply the similar scaling and iteration procedure as Sharp and Topping. During the process, we will need several different uniform estimates, in which the usually used elliptic regularity theory has to be refined. These difficulty makes the problem become more interesting.

This paper is organized as follows. Section 2 contain some preliminaries and auxiliary results for later proofs. In Sect. 3, we present an alternative proof of a key result of Sharp-Topping [30], which leads to [30, Theorem 1.1]. Our main theorems are proved in Sects. 4, 5 and 6. In the final section, Sect. 7, we prove the compactness theorem. We also add three appendices to include certain auxiliary results that were used in the proofs of our main theorems.

Our notations are standard. By $A \lesssim B$ we mean there exists a universal constant $C > 0$ such that $A \leq CB$.

2 Preliminaries and auxiliary results

2.1 Function spaces and related

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $1 \leq p < \infty$ and $0 \leq s < n$. The Morrey space $M^{p,s}(\Omega)$ consists of functions $f \in L^p(\Omega)$ such that

$$
\|f\|_{M^{p,s}(\Omega)} \equiv \left( \sup_{x \in \Omega, r > 0} \frac{1}{r^{-s}} \int_{B_r(x) \cap \Omega} |f|^p \right)^{1/p} < \infty.
$$

Denote by $L^p_*$ the weak $L^p$ space and define the weak Morrey space $M^{p,s}_*(\Omega)$ as the space of functions $f \in L^p_*(\Omega)$ such that

$$
\|f\|_{M^{p,s}_*(\Omega)} \equiv \left( \sup_{x \in \Omega, r > 0} \frac{1}{r^{-s}} \|f\|_{L^p_*(B_r(x) \cap \Omega)}^p \right)^{1/p} < \infty,
$$

where

$$
\|f\|_{L^p_*(B_r(x) \cap \Omega)}^p \equiv \sup_{t > 0} t^p \left| \{ x \in B_r(x) \cap \Omega : |f(x)| > t \} \right|.
$$

For a measurable function $f : \Omega \to \mathbb{R}$, denote by $\delta f(t) = |\{ x \in \Omega : |f(x)| > t \}|$ its distributional function and by $f^*(t) = \inf\{s > 0 : \delta f(s) \leq t, t \geq 0\}$, the nonincreasing rearrangement of $|f|$. Define

$$
f^{**}(t) \equiv \frac{1}{t} \int_0^t f^*(s) \, ds, \quad t > 0.
$$

The Lorentz space $L^{p,q}(\Omega)$ ($1 < p < \infty, 1 \leq q \leq \infty$) is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$
\|f\|_{L^{p,q}(\Omega)} \equiv \begin{cases} 
\left( \int_0^{\infty} (t^{1/p} f^{**}(t)) q \frac{dt}{t} \right)^{1/q}, & \text{if } 1 \leq q < \infty, \\
\sup_{t > 0} t^{1/p} f^{**}(t), & \text{if } q = \infty.
\end{cases}
$$
is finite.

It is well-known that \( L^{p,p} = L^p \) and \( L^{p,\infty} = L^p _\infty \). We will need the following Hölder’s inequality in Lorentz spaces.

**Proposition 2.1** ([24]) Let \( 1 < p_1, p_2 < \infty \) and \( 1 \leq q_1, q_2 \leq \infty \) be such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \leq 1.
\]

Then, \( f \in L^{p_1,q_1}(\Omega) \) and \( g \in L^{p_2,q_2}(\Omega) \) implies \( fg \in L^{p,q}(\Omega) \). Moreover,

\[
\| fg \|_{L^{p,q}(\Omega)} \leq \| f \|_{L^{p_1,q_1}(\Omega)} \| g \|_{L^{p_2,q_2}(\Omega)}.
\]

The first order Lorentz-Sobolev space \( W^{1,p,q}(\Omega) \) for \( 1 < p < \infty \), \( 1 \leq q \leq \infty \) consists of functions \( f \in L^{p,q}(\Omega) \) with weak gradient \( \nabla f \in L^{p,q}(\Omega) \). A natural norm for a Lorentz-Sobolev function \( f \in W^{1,p,q}(\Omega) \) is defined by

\[
\| f \|_{W^{1,p,q}(\Omega)} = \left( \| f \|_{L^{p,q}(\Omega)}^p + \| \nabla f \|_{L^{p,q}(\Omega)}^p \right)^{1/p}.
\]

Higher order Lorentz-Sobolev spaces can be defined analogously.

**Proposition 2.2** ([36]) Let \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \). Then

1. \( W^{1,p}(\Omega) = W^{1,p,p}(\Omega) \);
2. If \( \Omega \) is bounded and smooth, then \( W^{1,p,q}(\Omega) \) embeds continuously into \( L^{p',q}(\Omega) \) for \( 1 < p < n \), where \( 1/p' = 1/p - 1/n \).
3. \( W^{1,n,1}(\Omega) \subset C(\Omega) \).

We shall need the space \( L \log L \) as well. Recall that \( L \log L(\Omega) \) consists of all functions \( f : \Omega \to \mathbb{R} \) such that

\[
\| f \|_{L \log L(\Omega)} := \int_0^\infty f^*(t) \log \left(2 + \frac{1}{t}\right) dt < \infty.
\]

The following elementary fact on functions in \( L \log L \) can be found in [30, Lemma 2.1].

**Lemma 2.3** Suppose \( f \in L \log L(B_r) \) and \( r \in (0, \frac{1}{2}) \). Then there exists a constant \( C > 0 \), independent of \( r \), such that

\[
\| f \|_{L^1(B_r)} \leq C \left[ \log \left( \frac{1}{r} \right) \right]^{-1} \| f \|_{L \log L(B_r)}.
\]

### 2.2 Fractional Riesz operators

Let \( 0 < \alpha < n \) and \( I_\alpha = c_{n,\alpha} |x|^{\alpha-n}, x \in \mathbb{R}^n \), be the usual fractional Riesz operators, where \( c_{n,\alpha} \) is a positive normalization constant. The following well-known estimates on fractional Riesz operators in Lorentz spaces can be found in Adams and Fournier [2].

**Proposition 2.4** For \( 0 < \alpha < n \), \( 1 < p < n/\alpha \), \( 1 \leq q \leq q' \leq \infty \), the fractional Riesz operators

\[
I_\alpha : L^{p,q}(\mathbb{R}^n) \to L^{\frac{np}{n-\alpha p},q'}(\mathbb{R}^n)
\]

and

\[
I_\alpha : L^1(\mathbb{R}^n) \to L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)
\]

are bounded.
The following lemma can be viewed as an improved version of the classical Riesz potential theory. It plays a key role in the proofs of Sharp-Topping [30], and also in our approach.

**Lemma 2.5** ([1], Proposition 3.1) Let $0 < \alpha < \beta \leq n$ and $f \in M^{1,n-\beta}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then, $I_\alpha f \in L^{\frac{p \beta}{\beta - \alpha}}(\mathbb{R}^n)$ with

$$
\|I_\alpha f\|_{L^{\frac{p \beta}{\beta - \alpha}}(\mathbb{R}^n)} \leq C_{\alpha, \beta, n, p} \|f\|_{M^{1,n-\beta}(\mathbb{R}^n)} \|\|^{\frac{\beta - n}{p}}_{L^\infty(\mathbb{R}^n)}.
$$

In particular, in the case $1 < p < n/\beta$, we have

$$
\frac{\beta p}{\beta - \alpha} > \frac{np}{n - \alpha p},
$$

which implies that $I_\alpha f$ has better integrability than the typical one from $L^p$ boundedness. This improved Riesz potential estimate comes from the fact that $f$ has additional fine property, that is, $f$ also belongs to some Morrey space. Lemma A.3 of [30] gives a local version of Lemma 2.5.

2.3 Scaling invariance of (1.5)

We shall use a scaling argument in our later proofs. Let $u$ be a weak solution of (1.5). For any $B_{2R}(x_0) \subset B_{10}$ and $x \in B_2 = B_2(0)$, set

$$
u_R(x) = u(x_0 + Rx), \quad V_R(x) = RV(x_0 + Rx), \quad w_R(x) = R^2 w(x_0 + Rx),$$

$$\omega_R(x) = R^2 \omega(x_0 + Rx), \quad F_R(x) = R^3 F(x_0 + Rx), \quad f_R(x) = R^4 f(x_0 + Rx).$$

It is straightforward to verify that $u_R$ satisfies

$$
\Delta^2 u_R = \Delta(V_R \cdot \nabla u_R) + \text{div}(w_R \nabla u_R) + (\nabla \omega_R + F_R) \cdot \nabla u_R + f_R \quad \text{in} \ B_2.
$$

Moreover, for any $0 < r < 1$ and $1 \leq q \leq \infty$, there holds

$$
\|\nabla u_R\|_{L^4,q(B_r(0))} = \|\nabla u\|_{L^4,q(B_r(x_0))}, \quad \|\Delta u_R\|_{L^2(B_r(0))} = \|\Delta u\|_{L^2(B_r(x_0))},$$

$$
\|V_R\|_{L^4,q(B_r(0))} = \|V\|_{L^4,q(B_r(x_0))}, \quad \|\nabla V_R\|_{L^2(B_r(0))} = \|\nabla V\|_{L^2(B_r(x_0))},$$

$$
\|w_R\|_{L^2(B_r(0))} = \|w\|_{L^2(B_r(x_0))}, \quad \|\omega_R\|_{L^2(B_r(0))} = \|\omega\|_{L^2(B_r(x_0))},$$

$$
\|F_R\|_{L^4/3,q(B_r(0))} = \|F\|_{L^4/3,q(B_r(x_0))}, \quad \|f_R\|_{L^p(B_r(0))} = R^4(1-1/p)\|f\|_{L^p(B_r(x_0))},$$

and

$$
\|f(x_0 + R \cdot)\|_{L^\infty L(B_1)} \leq C \|f\|_{L^\infty L(B_R(x_0))}.
$$

by [30, Lemma 2.2] whenever $B_R(x_0) \subset B_{10}$.

3 Warm up

Let $u \in W^{1,2}(B_{10}, \mathbb{R}^m), B_{10} \subset \mathbb{R}^2$, be a weak solution of system (1.3) and $f \in L^p(B_{10}, \mathbb{R}^m)$. Sharp and Topping [30, Lemma 7.3] proved that if $p \in (1, 2)$, then

$$
\|\nabla u\|_{L^2(B_r)} \leq C r^\alpha \left(\|\nabla u\|_{L^2(B_1)} + \|f\|_{L^p(B_1)}\right)
$$
for $0 < r < 1$ under a smallness assumption on $\Omega$, where $\alpha = 2(1 - \frac{1}{p})$, from which the local $\alpha$-Hölder continuity follows. The method there is quite tricky and requires a very delicate control on coefficients of various inequalities throughout their arguments.

The aim of this section is to reproduce the above decay estimate by refining the technique of [30, Lemma 7.3]. The refined technique will be applied to the fourth order system (1.5) in the next section, but in a more complexed way.

**Theorem 3.1** Let $u \in W^{1,2}(B_{10}, \mathbb{R}^m)$ be a weak solution to (1.3). Set $\alpha = 2\left(1 - \frac{1}{p}\right)$ if $1 < p < 2$ and $\alpha$ to be any number in $(0, 1)$ if $p \geq 2$. Then there exist constants $\epsilon = \epsilon(p, m) > 0$ and $C = C(p, m, \alpha)$, such that if $\|\Omega\|_{L^{2}(B_{10})} \leq \epsilon$, then

$$
\|\nabla u\|_{L^{2,1}(B_r)} \leq C\gamma^{\alpha} \left(\|\nabla u\|_{L^{2,1}(B_1)} + \|f\|_{L^{p}(B_1)}\right).
$$

**Proof** Choose $\epsilon$ so small that there exist $A \in W^{1,2} \cap L^{\infty}$ and $B \in W^{1,2}$ such that (1.3) can be written as

$$
-\text{div}(A \nabla u) = \nabla^{\perp} B \cdot \nabla u + Af.
$$

Next extend all the functions from $B_1$ to the whole space $\mathbb{R}^2$ in such a way that their norms in $\mathbb{R}^2$ are bounded by a constant multiply of the corresponding norms in $B_1$. With no confuse of notations, we use the same symbols for all the extended functions.

Applying the Hodge decomposition for $A \nabla u$, we obtain

$$
Adu = dr + *dg \quad \text{in} \ \mathbb{R}^2.
$$

Let $\Gamma(x) = -\frac{1}{2\pi} \log|x|$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^2$. Set $r_1 = \Gamma * (\nabla^{\perp} B \cdot \nabla u)$ and $r_2 = \Gamma * (Af)$ in $\mathbb{R}^2$. It follows that

$$
-\Delta(r - r_1 - r_2) = 0 \quad \text{in} \ B_1.
$$

Thus we obtain

$$
Adu = dr_1 + dr_2 + *dg + h \quad \text{in} \ B_1
$$

for some harmonic 1-form in $B_1$.

We estimate each term in the above decomposition as follows: First note that $\nabla^{\perp} B \cdot \nabla u$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$. So

$$
\|\nabla r_1\|_{L^{2,1}(\mathbb{R}^2)} \lesssim \|\nabla \Gamma * (\nabla^{\perp} B \cdot \nabla u)\|_{L^{2,1}(\mathbb{R}^2)} \lesssim \|\nabla^{\perp} B \cdot \nabla u\|_{\mathcal{H}^1(\mathbb{R}^2)}
$$

$$
\lesssim \|B\|_{L^2(\mathbb{R}^2)} \|\nabla u\|_{L^2(\mathbb{R}^2)} \lesssim \epsilon \|\nabla u\|_{L^2(B_1)},
$$

where in the second inequality above we used the fact that $\nabla \Gamma : \mathcal{H}^1(\mathbb{R}^2) \rightarrow L^{2,1}(\mathbb{R}^2)$ is bounded (see e.g. [30, Section A.4]). Similarly, we have

$$
\|\nabla r_2\|_{L^{2,1}(\mathbb{R}^2)} \lesssim \|\nabla A\|_{L^2(\mathbb{R}^2)} \|\nabla u\|_{L^2(\mathbb{R}^2)} \lesssim \epsilon \|\nabla u\|_{L^2(B_1)}.
$$

Since $\nabla^2 \Gamma$ is a singular operator,

$$
\|\nabla^2 r_2\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(B_1)}.
$$

Set $\tilde{p} = \frac{2p}{2-p}$ for $1 < p < 2$ and any finite number if $p \geq 2$. When $1 < p < 2$, applying the Lorentz-Sobolev embedding (see Proposition 2.2), we have

$$
\|\nabla r_2\|_{L^{\tilde{p},p}(\mathbb{R}^2)} \lesssim \|f\|_{L^p(B_1)}.
$$
In this case, Hölder’s inequality gives
\[ \| \nabla r_2 \|_{L^2,1(R^r)} \lesssim \gamma^{2(1-\frac{1}{p})} \| \nabla r_2 \|_{L^p,1(R^r)} \lesssim \gamma^{2(1-\frac{1}{p})} \| f \|_{L^p(B_1)} \]
for any \( \gamma \in (0, 1) \). If \( p \geq 2 \), we may use Hölder’s inequality and the Sobolev embedding for \( p \geq 2 \) to get
\[ \| \nabla v \|_{L^2,1(R^r)} \lesssim \gamma \alpha \| f \|_{L^p(B_1)} \]
for any \( \gamma \in (0, 1) \).

Combining the above estimates together, it follows that \( \nabla u \in L^{2,1}(B_1) \). Moreover, when \( 1 < p < 2 \), we have
\[ \| \nabla u \|_{L^{2,1}(B_1)} \leq \| h \|_{L^{2,1}(B_1)} + \| \nabla r_1 \|_{L^{2,1}(B_1)} + \| \nabla g \|_{L^{2,1}(B_1)} + \| \nabla r_2 \|_{L^{2,1}(B_1)} \]
\[ \lesssim \gamma \| h \|_{L^{2,1}(B_1)} + \| \nabla r_1 \|_{L^{2,1}(B_1)} + \| \nabla g \|_{L^{2,1}(B_1)} + \gamma^{2(1-\frac{1}{p})} \| f \|_{L^p(B_1)} \]
\[ \lesssim \gamma \| \nabla u \|_{L^2(B_1)} + \epsilon_m \| \nabla u \|_{L^{2,1}(B_1)} + \gamma^{2(1-\frac{1}{p})} \| f \|_{L^p(B_1)} \]
\[ \lesssim (\gamma + \epsilon_m) \| \nabla u \|_{L^{2,1}(B_1)} + \gamma^{2(1-\frac{1}{p})} \| f \|_{L^p(B_1)} , \]
and similarly, when \( p \geq 2 \),
\[ \| \nabla u \|_{L^{2,1}(B_1)} \lesssim (\gamma + \epsilon_m) \| \nabla u \|_{L^{2,1}(B_1)} + \gamma^\alpha \| f \|_{L^p(B_1)} \] (3.3)
for any \( \alpha \in (0, 1) \).

The last step is to iterate. Let \( \gamma > 0 \) to be determined and choose \( \gamma \geq \epsilon_m \) so that
\[ \| \nabla u \|_{L^{2,1}(B_1)} \leq C \left( \gamma \| \nabla u \|_{L^{2,1}(B_1)} + \gamma^\alpha \| f \|_{L^p(B_1)} \right) \).

Note that the equation is scaling invariant: for any \( \tau > 0 \), the functions \( u_\tau(x) = u(x_0 + \tau x) \), \( f_\tau(x) = \tau^2 f(x_0 + \tau x) \) and \( \Omega_\tau = \tau \Omega(x_0 + \tau x) \) satisfy
\[-\Delta u_\tau = \Omega_\tau \cdot \nabla u_\tau + f_\tau .\]

Hence the sequence \( \{ a_n \}_{n \in \mathbb{N}} \), with \( a_n = \| \nabla u \|_{L^{2,1}(B_{\tau n})} \), satisfies
\[ a_n \leq C \gamma a_{n-1} + C \gamma^{\alpha n} \| f \|_{L^p(B_1)} . \]

Iteration gives
\[ a_n \leq (C \gamma)^n a_0 + \gamma^{\alpha n} \left( \sum_{i=0}^{n-1} (C \gamma^{1-\alpha})^i \right) \| f \|_{L^p(B_1)} . \]

Choose \( \gamma \) such that \( C \gamma^{1-\alpha} < 1 \) and we achieve
\[ a_n \leq C \gamma^{\alpha n} (a_0 + \| f \|_{L^p(B_1)}) . \]

The proof is complete. \( \square \)

4 Hölder regularity via decay estimates

In this section, we prove Theorem 1.1, which is a fourth order analog of Theorem 3.1. The idea of the proof is quite similar to that used in Theorem 3.1, but more complicated.
Proof of Theorem 1.1 We begin with the conservation law of Lamm and Riviére [17]. Take $\epsilon_m > 0$ sufficiently small such that the smallness condition (1.7) holds. Then there exist $A \in W^{2,2} \cap L^\infty(B_8, M(m))$ and $B \in L^2(B_8, M(m) \otimes \wedge^2 \mathbb{R}^4)$ such that

$$
\Delta(A \Delta u) = \text{div} K + Af \quad \text{in} \ B_8,
$$

(4.1)

with $K$ given by (1.10). As in the previous section, we extend all the relevant functions from $B_1$ to $\mathbb{R}^4$ such that their norms in $\mathbb{R}^4$ are bounded by a constant multiple of the corresponding norms in $B_1$. To simplify the notation, we use the same symbols for all the extended functions.

Below we first estimate the decay of $\| \Delta u \|_{L^2(B_r)}$, and then the decay of $\| \nabla u \|_{L^{4,2}(B_r)}$; finally we combine the two decay estimates together to conclude the proof.

1. Decay of $\| \Delta u \|_{L^2(B_r)}$.

By an elementary computation using Proposition 2.1 and (1.8), we obtain

$$
\| K \|_{L^{\frac{4}{3},1}(B_1)} \lesssim \epsilon_m (\| \nabla u \|_{L^{4,2}(B_1)} + \| \nabla^2 u \|_{L^2(B_1)}).
$$

(4.2)

For instance, for the first term $\nabla A \Delta u$, we have

$$
\| \nabla A \Delta u \|_{L^{\frac{4}{3},1}(B_1)} \leq \| \nabla A \|_{L^{4,2}(B_1)} \| \Delta u \|_{L^2(B_1)} \lesssim \epsilon_m \| \nabla^2 u \|_{L^2(B_1)}.
$$

The rest terms are estimated similarly. For details, see e.g. [9, proof of Lemma 3.1].

Next let $I_2$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^4$ and set $u_1 = I_2 * \text{div} K, u_2 = I_2 * Af$ in $\mathbb{R}^4$. The theory of singular integrals implies that

$$
\| \nabla u_1 \|_{L^{4,3,1}(\mathbb{R}^4)} \lesssim \| K \|_{L^{4,3,1}(\mathbb{R}^4)} \lesssim \| K \|_{L^{\frac{4}{3},1}(B_1)}
$$

and

$$
\| \nabla^2 u_2 \|_{L^p(\mathbb{R}^4)} \lesssim \| f \|_{L^p(\mathbb{R}^4)} \lesssim \| f \|_{L^p(B_1)}.
$$

(4.3)

Combining the embedding $W^{1,\frac{4}{3},1}(\mathbb{R}^4) \subset L^{2,1}(\mathbb{R}^4)$ and (4.2), we deduce

$$
\| u_1 \|_{L^2(B_1)} \lesssim \| u_1 \|_{L^{2,1}(B_1)} \leq \| u_1 \|_{L^{2,1}(\mathbb{R}^4)} \lesssim \| \nabla u_1 \|_{L^{4,3,1}(\mathbb{R}^4)}
$$

$$
\lesssim \| K \|_{L^{\frac{4}{3},1}(B_1)} \lesssim \epsilon_m (\| \nabla u \|_{L^{4,2}(B_1)} + \| \nabla^2 u \|_{L^2(B_1)}).
$$

(4.4)

Now it is easy to see that the function $v = A \Delta u - u_1 - u_2$ is harmonic in $B_1$. Therefore, for any $0 < \tau < 1$,

$$
\int_{B_\tau} |v|^2 \leq C \tau^4 \int_{B_1} |v|^2.
$$

As a consequence, for any $0 < \tau < 1$, it holds

$$
\int_{B_\tau} |\Delta u|^2 \lesssim \int_{B_1} |v|^2 + \int_{B_1} |u_1|^2 + \int_{B_\tau} |u_2|^2
$$

$$
\lesssim \tau^4 \int_{B_1} |v|^2 + \int_{B_1} |u_1|^2 + \int_{B_\tau} |u_2|^2
$$

$$
\lesssim \tau^4 \int_{B_1} |\Delta u|^2 + (1 + \tau^4) \int_{B_1} |u_1|^2 + \int_{B_\tau} |u_2|^2 + \tau^4 \int_{B_1} |u_2|^2
$$

(4.5)

$$
\lesssim (\tau^4 + \epsilon_m^2) \int_{B_1} |\nabla^2 u|^2 + \epsilon_m^2 \| \nabla u \|_{L^{4,2}(B_1)}^2 + \int_{B_\tau} |u_2|^2 + \tau^4 \int_{B_1} |u_2|^2
$$

$$
\lesssim (\tau^4 + \epsilon_m^2) \int_{B_2} |\Delta u|^2 + \epsilon_m^2 \| \nabla u \|_{L^{4,2}(B_2)}^2 + \int_{B_\tau} |u_2|^2 + \tau^4 \int_{B_1} |u_2|^2.
$$
In the first line above we applied the fact that \(|\Delta u| \approx |A \Delta u|\) holds since \(|A - \text{Id}| \leq \epsilon_m\).

In the last second line we applied the estimate (4.4) of \(u_1\). Continuing from the last line, we apply the interior \(L^2\) estimate to derive

\[
\|\nabla^2 u\|_{L^2(B_1)} \lesssim \|\Delta u\|^2_{L^2(B_2)} + \|\nabla u\|^2_{L^4(B_2)}, \tag{4.6}
\]

by assuming in a priori that \(\int_{B_2} u = 0\) so that \(\|u\|^2_{L^4(B_2)} \lesssim \|\nabla u\|^2_{L^4(B_2)}\).

Finally, combining Hölder’s inequality and (4.3) yields

\[
\int_{B_1} |u_2|^2 \lesssim \tau^{8(1 - \frac{1}{p})} \left(\int_{B_1} |u_2|^{2\frac{p}{2 - p}}\right)^{\frac{2 - p}{p}} \lesssim \tau^{8(1 - \frac{1}{p})} \|f\|^2_{L^p(B_1)},
\]

which together with (4.5) leads to the decay estimate of \(\Delta u\) for \(\tau < 1\):

\[
\int_{B_1} |\Delta u|^2 \lesssim \left(\tau^4 + \epsilon_m^2\right) \int_{B_2} |\Delta u|^2 + \epsilon_m^2 \|\nabla u\|^2_{L^4(B_2)} + \left(\tau^4 + \tau^{8(1 - \frac{1}{p})}\right) \|u_2\|^2_{L^{2\frac{p}{2 - p}}(B_1)}
\]

\[
\lesssim \left(\tau^4 + \epsilon_m^2\right) \int_{B_2} |\Delta u|^2 + \epsilon_m^2 \|\nabla u\|^2_{L^4(B_2)} + \tau^{8(1 - \frac{1}{p})} \|u_2\|^2_{L^{2\frac{p}{2 - p}}(B_1)} \tag{4.7}
\]

\[
\lesssim \left(\tau^4 + \epsilon_m^2\right) \int_{B_2} |\Delta u|^2 + \epsilon_m^2 \|\nabla u\|^2_{L^4(B_2)} + \tau^{8(1 - \frac{1}{p})} \|f\|^2_{L^p(B_1)}.
\]

We used in the second line the fact that \(\tau^4 \leq \tau^{8(1 - \frac{1}{p})}\) since \(p < 2\) and \(\tau < 1\).

To continue, we have to estimate the decay of \(\|\nabla u\|_{L^4(B_1)}\).

2. Decay of \(\|\nabla u\|_{L^4(B_1)}\).

First use (4.1) to rewrite our system as

\[
\Delta \text{div}(A \nabla u) = \text{div}(\hat{K}) + Af \quad \text{in } B_8,
\]

where \(\hat{K} = K + \nabla^2 A \cdot \nabla u + \nabla A \cdot \nabla^2 u \in L^{\frac{4}{3}, 1}(B_{10})\) satisfies the similar estimate:

\[
\|\hat{K}\|_{L^{\frac{4}{3}, 1}(B_R)} \lesssim \epsilon_m(\|\nabla u\|_{L^4(B_R)} + \|\nabla^2 u\|_{L^2(B_R)}). \tag{4.8}
\]

for any \(0 < R \leq 8\).

Keep in mind that we have extended all the related functions from \(B_1\) into \(\mathbb{R}^4\) with controlled norms. By the Hodge decomposition, we have

\[
Adu = dr + *dg \quad \text{in } \mathbb{R}^4
\]

where

\[
\Delta^2 r = \Delta \text{div}(A \nabla u) = \text{div}(\hat{K}) + Af \quad \text{and} \quad \Delta g = *(dA \wedge du).
\]

Denote by \(\Gamma = c \log(\cdot)\) the fundamental solution of \(\Delta^2\) in \(\mathbb{R}^4\), and let \(\tilde{r} = \Gamma \ast \text{div}(\hat{K})\) and \(v = \Gamma \ast (Ah)\). Then \(\Delta^2 (r - \tilde{r} - v) = 0\) in \(B_1\). Thus there exists a biharmonic 1-form \(h\) in \(B_1\) such that

\[
Adu = d\tilde{r} + dv + *dg + h \quad \text{in } B_1.
\]

We estimate the terms above as follows. Applying the Riesz potential estimates in Proposition 2.4 with \(\alpha = 2\), \(n = 4\), \(p = \frac{4}{3}\), \(q = 1\) and \(\theta' = 2\) (noticing \(\frac{np}{n-\alpha}p = 4\)), we infer

\[
\|\nabla \tilde{r}\|_{L^{4, 2}(\mathbb{R}^4)} \lesssim \|I_2(\hat{K})\|_{L^{4, 2}(\mathbb{R}^4)} \lesssim \|\hat{K}\|_{L^{\frac{4}{3}, 1}(\mathbb{R}^4)} \lesssim \|\hat{K}\|_{L^{\frac{4}{3}, 1}(B_1)} \lesssim \epsilon_m(\|\nabla u\|_{L^4(B_1)} + \|\nabla^2 u\|_{L^2(B_1)}). \tag{4.9}
\]
Note that $|\nabla g| = |\nabla I_2(dA \wedge du)| \approx |\nabla^2 I_2(A \nabla u)|$. The singular integral theory implies

$$\|\nabla g\|_{L^{4,2}(\mathbb{R}^d)} \lesssim \|A\nabla u\|_{L^{4,2}(\mathbb{R}^d)} \lesssim \epsilon_m \|\nabla u\|_{L^{4,2}(B_1)} \quad (4.10)$$

Since $\nabla^4 \Gamma$ is a singular operator,

$$\|\nabla^4 v\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(B_1)}.$$

Using the Lorentz-Sobolev embedding $W^{3,p}(\mathbb{R}^d) \subset L^{\bar{p},p}(\mathbb{R}^d)$, where $\bar{p} = \frac{4p}{4-3p}$, we derive

$$\|\nabla v\|_{L^{\bar{p},p}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(B_1)}.$$

Hence, for any $\gamma \in (0, 1)$, Hölder’s inequality gives

$$\|\nabla v\|_{L^{4,2}(B_2)} \lesssim \gamma^{4(1-\frac{1}{p})} \|\nabla v\|_{L^{\bar{p},p}(B_2)} \lesssim \gamma^4 \|f\|_{L^p(B_1)}.$$

Now we can conclude that for any $\gamma \in (0, 1)$, there holds

$$\|\nabla u\|_{L^{4,2}(B_2)} \leq \|h\|_{L^{4,2}(B_2)} + \|\nabla f\|_{L^{4,2}(B_2)} + \|\nabla g\|_{L^{4,2}(B_2)} + \|\nabla u\|_{L^{4,2}(B_2)} \lesssim \gamma \|h\|_{L^{4,2}(B_1)} + \|\nabla f\|_{L^{4,2}(B_1)} + \|\nabla g\|_{L^{4,2}(B_1)} + \gamma^{4(1-\frac{1}{p})} \|f\|_{L^p(B_1)}$$

$$\lesssim \gamma \|\nabla u\|_{L^{4,2}(B_1)} + \epsilon_m (\|\nabla u\|_{L^{4,2}(B_1)} + \|\nabla^2 u\|_{L^2(B_1)}) + \gamma^{4(1-\frac{1}{p})} \|f\|_{L^p(B_1)} \quad (4.11)$$

$$\leq C (\gamma + \epsilon_m) (\|\nabla u\|_{L^{4,2}(B_2)} + \|\Delta u\|_{L^2(B_2)}) + \gamma^{4(1-\frac{1}{p})} \|f\|_{L^p(B_2)}.$$

where we have used (4.9) and (4.10) in the last second line, and (4.6) in the last line.

3. Finally, let $\alpha = 4(1 - 1/p)$ and $\beta = (\alpha + 1)/2$, and choose $\epsilon_m \leq \gamma$ and then choose $\gamma \in (0, 1)$ such that $2C \gamma \leq \gamma^\beta$. Combining (4.7) and (4.11) together, we infer that

$$\|\nabla u\|_{L^{4,2}(B_2)} \leq \|\nabla u\|_{L^{4,2}(B_1)} + \|\Delta u\|_{L^2(B_2)} \leq \gamma^\beta (\|\nabla u\|_{L^{4,2}(B_2)} + \|\Delta u\|_{L^2(B_2)}) + C \gamma^{4(1-\frac{1}{p})} \|f\|_{L^p(B_2)}.$$

The proof is complete after a scaling (see Sect. 2) and an iteration argument as that of Theorem 3.1. We omit the details. \[\square\]

**Remark 4.1** Similarly as in the planar case, we may replace all the $L^{4,2}$-norms by the corresponding $L^4$-norms in the above proof to arrive at the following decay estimate in $L^4$-scale,

$$\|\nabla u\|_{L^4(B_2)} \leq \|h\|_{L^4(B_2)} + \|\Delta u\|_{L^2(B_2)} \leq C \gamma^\alpha (\|\nabla u\|_{L^4(B_1)} + \|\Delta u\|_{L^2(B_1)} + \|f\|_{L^p(B_1)}), \quad (4.12)$$

where $\alpha = 4(1 - 1/p)$ if $1 < p < \frac{4}{3}$ and $\alpha$ can be any number in $(0, 1)$ if $p \geq \frac{4}{3}$.

The following example (with $n = 4$) shows that the Hölder continuity is the best possible regularity that one can expect for the Lamm–Rivièr system (1.5) even $f \equiv 0$.

**Remark 4.2** (A non-Lipschitz continuous example) For any $n \geq 2$, let $B = B_{1/2}(0) \subset \mathbb{R}^n$ be the ball centered at the origin with radius $\frac{1}{2}$ and define $v : B \to \mathbb{R}$ as

$$v(x) = (x_1^2 - x_2^2)(-\log|\lambda|)^{1/2}.$$

Direct computation shows

$$\Delta v(x) = \frac{x_2^2 - x_1^2}{2|x|^2} \left\{ \frac{n + 2}{(-\log|\lambda|)^{1/2}} + \frac{1}{2(-\log|\lambda|)^{3/2}} \right\} =: f.$$
Set $V = \left( \frac{f_1}{x_1}, 0, \cdots, 0 \right)$ and consider $u = v_{x_1} : B \to \mathbb{R}$. Then $u \in C^{0, \alpha}(B) \cap W^{2,2}(B)$ for any $\alpha \in (0, 1)$. It is straightforward to verify that $V \in W^{1,2}(B, \mathbb{R}^n)$ and $u$ is a weak solution of

$$\Delta u = V \cdot \nabla u \quad \text{in } B,$$

which is of the form $\Delta^2 u = \Delta(V \cdot \nabla u)$. However, note that $u \in C^\infty(B \setminus \{0\})$ and

$$\lim_{x \to 0} u_{x_1}(x) = \infty.$$

We thus infer that $u$ is not Lipschitz continuous in $B$.

**Decay estimate for the borderline case**

It is natural to ask whether one can obtain any decay estimate for the borderline case $p = 1$. For later use in Theorem 1.7, we deduce in below a decay estimate for the case $f \in L \log L(B_{10})$. Since the proof is rather similar to that used in Theorem 1.1, we only sketch it for simplicity.

It would be interesting to know whether the assumption $f \in L \log L(B_{10})$ can be replaced with $f \in h^1(B_{10})$, where $h^1$ is the local Hardy space, for definitions see [30, Appendix A.2].

**Proposition 4.3** Let $u \in W^{2,2}(B_{10}, \mathbb{R}^m)$ be a weak solution of (1.5) and assume $f \in L \log L(B_{10})$. Then there exist $0 < \gamma < 1$ and $C > 0$ such that

$$\|\Delta u\|_{L^2(B_{\gamma r})} + \|\nabla u\|_{L^2(B_{\gamma r})} \leq \frac{1}{2} \left( \|\Delta u\|_{L^2(B_r)} + \|\nabla u\|_{L^2(B_r)} \right) + C \|f\|_{L^1(B_{\gamma r})}\|f\|_{L^1(B_{\gamma r})}.$$

(4.13)

**Proof** As in the proof of Theorem 1.1, we first take $\epsilon_m$ sufficiently small so that the conservation law holds, and then extend all functions from $B_1$ to $\mathbb{R}^4$ with controlled norms.

1. Decay estimate of $\int_{B_1} |\Delta u|^2$.

We shall use the same notations as in Step 1 of the proof of Theorem 1.1. Note that $f \in L \log L(B_{10})$ implies $Af \in L \log L(B_{10})$ and $\|Af\|_{L \log L(B_{10})} \lesssim \|f\|_{L \log L(B_{10})}$. So $u_2 \in W^{2,1}(\mathbb{R}^4) \subset W^{1,4/3,1}(\mathbb{R}^4) \subset L^{2,1}(\mathbb{R}^4)$ and

$$\|\nabla^2 u_2\|_{L^1(\mathbb{R}^4)} \lesssim \|Af\|_{L \log L(\mathbb{R}^4)} \lesssim \|f\|_{L \log L(B_{10})},$$

and

$$\|\nabla^2 u_2\|_{L^{1,\infty}(\mathbb{R}^4)} \lesssim \|Af\|_{L^1(\mathbb{R}^4)} \lesssim \|f\|_{L^1(B_{10})},$$

from which it follows

$$\|u_2\|_{L^{2,1}(B_1)} \lesssim \|f\|_{L \log L(B_{10})}$$

and

$$\|u_2\|_{L^{1,\infty}(B_1)} \lesssim \|f\|_{L^1(B_{10})}.$$
As a result,

\[
\begin{align*}
\int_{B_r} |\Delta u|^2 & \lesssim (\tau^4 + \epsilon_m^2) \int_{B_2} |\Delta u|^2 + \epsilon_m^2 \|\nabla u\|_{L^4(B_2)}^2 + \int_{B_1} |u_2|^2 \\
& \lesssim (\tau^4 + \epsilon_m^2) \int_{B_2} |\Delta u|^2 + \epsilon_m^2 \|\nabla u\|_{L^4(B_2)}^2 + \|f\|_{L^1(B_1)}^{1/2} \|f\|_{L^2 \log L(B_1)}^{1/2}.
\end{align*}
\] (4.14)

Consequently, we obtain

\[
\|\nabla u\|_{L^4(B_2)}^2 \lesssim (\gamma + \epsilon_m) \left( \|\Delta u\|_{L^2(B_2)}^2 + \|\nabla u\|_{L^4(B_2)}^2 \right) + \|\nabla v\|_{L^4(B_2)}^2
\lesssim (\gamma + \epsilon_m) \left( \|\Delta u\|_{L^2(B_2)}^2 + \|\nabla u\|_{L^4(B_2)}^2 \right) + \|f\|_{L^1(B_1)}^{1/2} \|f\|_{L^2 \log L(B_1)}^{1/2}.
\] (4.15)

Finally, combining (4.14) and (4.15), we conclude

\[
\begin{align*}
\|\Delta u\|_{L^2(B_r)} + \|\Delta u\|_{L^4(B_r)} + \|\nabla u\|_{L^4(B_r)} + \|\nabla v\|_{L^4(B_r)}
& \lesssim (\gamma + \epsilon_m) \left( \|\Delta u\|_{L^2(B_2)} + \|\nabla u\|_{L^4(B_2)}^2 \right) + \|f\|_{L^1(B_1)}^{1/2} \|f\|_{L^2 \log L(B_1)}^{1/2}.
\end{align*}
\]

Choosing \(\gamma, \epsilon_m\) small to obtain the desired estimate. \(\square\)

**Remark 4.4** Similarly, one can show that for any \(1 \leq s < \infty\),

\[
\|\Delta u\|_{L^2(B_r)} + \|\nabla u\|_{L^4(B_r)}^s 
\leq \frac{1}{2} \left( \|\Delta u\|_{L^2(B_2)}^s + \|\nabla u\|_{L^4(B_2)}^s \right) + C \|f\|_{L^1(B_1)}^{1-s} \|f\|_{L^2 \log L(B_1)}^{s}.
\]

**5 Higher order regularity**

In this section, we shall prove the higher order regularity asserted in Theorem 1.2. The key to derive the improved regularity is to use Lemma 2.5. To illustrate the scheme clearly, we begin with the simple first order case. Throughout this section, we assume

\[1 < p < 4/3\]
\[ \alpha = 4(1 - 1/p) \quad \text{and} \quad M \equiv \|u\|_{W^{2.2}(B_1)} + \|f\|_{L^p(B_1)}. \]

### 5.1 \( W^{1,q} \)-estimate with some \( q > 4 \)

The decay estimate in Theorem 1.1 together with Hölder’s inequality imply that \( \Delta u \in M^{1.2+\alpha}(B_1) \), that is,

\[
\sup_{x \in B_1/2, 0 < r < \frac{1}{2}} r^{-(2+\alpha)} \int_{B_r(x)} |\Delta u| \leq CM.
\]

We claim that \( \nabla u \in L^q(B_{1/4}) \) with

\[
\|\nabla u\|_{L^q(B_{1/4})} \leq C_\alpha M,
\]

where

\[
q = \frac{2(2 - \alpha)}{1 - \alpha} = \frac{2(4 - 2p)}{4 - 3p} > 4.
\]

To prove this claim, take a cut-off function \( \eta \in C_0^\infty(B_{1/2}) \) such that \( 0 \leq \eta \leq 1 \) in \( B_{1/2} \), \( \eta \equiv 1 \) in \( B_{1/4} \). Then \( \eta u \in W^{2.2}(\mathbb{R}^4) \). Thus \( \eta u = I_2(-\Delta(\eta u)) \), where \( I_2 \) is the fundamental solution of \( -\Delta \) in \( \mathbb{R}^4 \). As a consequence,

\[
|\nabla(\eta u)| \lesssim I_1(\eta)|\Delta u| + |\nabla \eta||\nabla u| + |\Delta \eta||u|).
\]

Easy to verify that \( \eta|\Delta u| \in M^{1.2+\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4) \). Hence by applying Lemma 2.5 (with \( \alpha = 1, \beta = 2 - 4(1 - 1/p), p = 1 \)) we find that \( I_1(\eta|\Delta u|) \in L^q(\mathbb{R}^4) \) with

\[
\|I_1(\eta|\Delta u|)\|_{L^q(\mathbb{R}^4)} \leq C\alpha M \frac{\alpha}{2\pi^2} \|\eta \Delta u\|_{L^2(\mathbb{R}^4)}^{1 - \frac{\alpha}{2\pi}} \leq C_\alpha M.
\]

Note that the lower order term \( |\nabla \eta||\nabla u| + |\Delta \eta||u| \) belongs to \( L^4(\mathbb{R}^4) \). Thus the usual Riesz potential theory implies that

\[
\|I_1(|\nabla \eta||\nabla u| + |\Delta \eta||u|)\|_{L^q(\mathbb{R}^4)} \leq C_\alpha M.
\]

The claim follows easily from the above two estimates since \( \nabla u = \nabla(\eta u) \) in \( B_{1/4} \).

Note that we have improved the Lebesgue integrability of \( \nabla u \) from 4 to \( q \), even though \( q \) is not the final optimal exponent.

### 5.2 \( W^{2,q} \)-estimate with any \( q < \frac{2p}{2-p} \)

We now derive the second order regularity. More precisely, we shall prove the following result.

**Proposition 5.1** Let \( u \in W^{2,2}(B_{10}, \mathbb{R}^m) \) be a weak solution of the inhomogeneous system (1.5) with \( f \in L^p(B_{10}) \) for \( p \in (1, \frac{4}{7}) \). Then \( u \in W^{2,q}_{\text{loc}}(B_{10}) \) for any \( q < \frac{2p}{2-p} \).

**Proof** By the definition (1.10) of \( K \) and the decay estimate in Theorem 1.1, we can easily verify that

\[
\sup_{x \in B_{1/2}, 0 < r < 1/2} r^{-\frac{4}{7} \alpha} \int_{B_r(x)} |K|^{4/3} \leq CM.
\]
By Hölder’s inequality, this implies that \( K \in M^{1,1+\alpha}(B_{1/2}) \), that is,
\[
\sup_{x \in B_{1/2}, 0 < r < 1/2} r^{-(1+\alpha)} \int_{B_r(x)} |K| \leq CM.
\]

Now we extend \( K \) from \( B_{1/2} \) into \( \mathbb{R}^4 \) such that
\[
\|K\|_{M^{1,1+\alpha}(\mathbb{R}^4)} \lesssim \|K\|_{M^{1,1+\alpha}(B_{1/2})} \lesssim M
\]
and \( \|K\|_{L^{4/3}(\mathbb{R}^4)} \lesssim \|K\|_{L^{4/3}(B_{1/2})} \). Then it follows from Lemma 2.5 (with \( \alpha = 1, \beta = 3 - \alpha, n = 4, p = 4/3 \)) that
\[
I_1(K) \in L^{\frac{4 \cdot 3 - \alpha}{3 - \alpha}}(\mathbb{R}^4).
\]
Write
\[
q_0 = \frac{4 \cdot 3 - \alpha}{3 - \alpha}.
\]
As a result, \( I_2(\mathrm{div}K) \approx I_1(K) \in L^{q_0}(\mathbb{R}^4) \).

Define \( v_1 \) and \( v_2 \) in \( \mathbb{R}^4 \) as
\[
v_1 = I_2(\mathrm{div}K) \approx I_1(K), \quad v_2 = I_2(Af).
\]
Here we also extend \( A, f \) from \( B_{1/2} \) into \( \mathbb{R}^4 \) with controlled norms. Then, our previous estimate shows that \( v_1 \in L^{q_0}(\mathbb{R}^4) \) and
\[
v_2 \in W^{2,p}(\mathbb{R}^4) \subset L^{2p/(2-p)}(\mathbb{R}^4).
\]
Since \( p > 1 \), we have \( q_0 < \frac{2p}{2-p} \). Thus, using the fact that \( A \Delta u - v_1 - v_2 \) is a harmonic function in \( B_{1/2} \), we infer that \( \Delta u \in L^{q_0}(B_{1/4}) \). In other words, we obtain
\[
u \in W^{2,q_0}(B_{1})\).
\]
Note that \( \alpha > 0 \) implies \( q_0 > 2 \). Thus we have improved the regularity of \( u \) from \( W^{2,2} \) to \( W^{2,q_0} \).

Next we use a bootstrapping argument to repeatedly improve the second order regularity of \( u \). We claim that
\[
u \in W^{2,q}_{\text{loc}}\text{ with } q < \frac{2p}{2-p} \implies \nu \in W^{2,\frac{4q}{4+q} \cdot \frac{3-q}{2-q}}_{\text{loc}}.
\]

This is true because if \( u \in W^{2,q}_{\text{loc}} \) with \( q < \frac{2p}{2-p} \), then the definition (1.10) of \( K \) implies that \( K \in L^{4} \cdot L^{q} \subset L^{q_0} \) with \( 1/q_0 = 1/4 + 1/q \). Since \( K \in M^{1,1+\alpha}(B_{1/2}) \), Lemma 2.5 implies that
\[
v_1 \approx I_1(K) \in L^{q_0 \cdot \frac{3-q}{2-q}} = L^{\frac{4q}{4+q} \cdot \frac{3-q}{2-q}}.
\]
Notice that
\[
\frac{4q}{4+q} \cdot \frac{3-q}{2-q} < \frac{2p}{2-p} \iff q < \frac{2p}{2-p}
\]
and that when \( q \searrow \frac{2p}{2-p} \), we have \( \frac{4q}{4+q} \cdot \frac{3-q}{2-q} \searrow \frac{2p}{2-p} \). Also recall that \( v_2 \in W^{2,p} \subset L^{2p/(2-p)} \).

Thus the same argument as the above implies that \( \Delta u \in L^{\frac{4q}{4+q} \cdot \frac{3-q}{2-q}}_{\text{loc}}(B_{1}) \). That is,
\[
u \in W^{2,\frac{4q}{4+q} \cdot \frac{3-q}{2-q}}_{\text{loc}}(B_{1}).
\]
Thus, by iterating the bootstrapping claim (5.1), we find that
\[ u \in W^{2,q}_\text{loc} \quad \text{for all } q < \frac{2p}{2 - p}. \]

The proof of Proposition 5.1 is complete. \( \square \)

**Remark 5.2** As in the second order case of Sharp-Topping [30], the classical Calderón–Zygmund theory does not give additional improvement on the second order Sobolev exponent. Indeed, by the previous step, we have \( \nabla^2 u \in L^{q_0} \) for some \( q_0 > 2 \) and \( \nabla u \in L^3 \). Then, this implies that \( K \in L^4 \cdot L^{q_0} \subset L^{\tilde{q}_0} \) with \( 1/\tilde{q}_0 = 1/4 + 1/q_0 \). As a result, \( v = I_2(\text{div} K) \in W^{1,\tilde{q}_0} \subset L^{\tilde{q}_0}, \) and \( w = I_2(Af) \in W^{2,p} \). Since \( A\Delta u - v - w \) is harmonic, this gives \( u \in W^{3,\tilde{q}_0} \subset W^{2,q_0} \). Note that we do **not** obtain any improvement for the integrability of \( \nabla^2 u \). This reflects the importance of Lemma 2.5 in obtaining higher Sobolev regularity.

### 5.3 \( W^{3,q} \)-estimate with \( q > \frac{4}{3} \)

With Proposition 5.1 at hand, we immediately obtain the third order regularity.

**Proposition 5.3** Let \( u \in W^{2,2}(B_{10}, \mathbb{R}^m) \) be a weak solution of the inhomogeneous system (1.5) with \( f \in L^p(B_{10}) \) for \( p \in (1, \frac{4}{3}) \). Then \( u \in W^{3,q}_\text{loc}(B_1) \) for any \( q < \frac{4p}{4 - p} \).

**Proof** By Proposition 5.1, we know that \( u \in W^{2,q}_\text{loc} \) for all \( q < \frac{2p}{2 - p} \). As a consequence, the definition (1.10) of \( K \) implies that \( K \in L^4 \cdot L^{q_0} \subset L^{\tilde{q}_0} \) with \( 1/\tilde{q}_0 = 1/4 + 1/q_0 \). This implies \( v_1 := I_2(\text{div} K) \in W^{1,\tilde{q}_0} \). On the other hand, the classical Calderón–Zygmund estimate implies that \( v_2 = I_2(Af) \in W^{2,p} \subset W^{1,\frac{4p}{4 - p}} \). Note that \( q < \frac{2p}{2 - p} \) if and only if \( \tilde{q} < \frac{4p}{4 - p} \). Therefore, by the same argument as that used in the proof of Proposition 5.1, we infer that \( u \in W^{3,\tilde{q}}_\text{loc} \) for all \( \tilde{q} < \frac{4p}{4 - p} \). \( \Box \)

**Remark 5.4** In the next section, we will show that \( u \in W^{2,\frac{2p}{2 - p}}_\text{loc} \). Then, by the same argument as the above, we conclude that \( u \in W^{3,\frac{4p}{4 - p}}_\text{loc} \).

We would like to point out that the third order regularity as obtained in Theorem 1.1 is the best possible, and in general there is no hope to obtain fourth order regularity.

**Example 5.5** (Solutions without \( W^{4,p} \)-regularity) Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function with the following properties:
- \( g \in W^{3,2}((-1, 1)) \) but \( g \notin W^{4,1}((-1, 1)) \);
- \( g \geq 1 \) on \((-1, 1)\).

Consider the map \( u : B_1 \to \mathbb{R}, B_1 \subset \mathbb{R}^4 \), defined by
\[ u(x) = x_1 g(x_2). \]

Set
\[ V_1(x) = x_1 \frac{g''(x_2)}{g(x_2)} \quad \text{and} \quad V(x) = (V_1(x), 0, 0, 0). \]

It is straightforward to verify that \( V \in W^{1,2}(B_1) \) and
\[ \Delta^2 u = \Delta (V \cdot \nabla u) \quad \text{in } B_1. \]

However, the regularity of \( g \) implies that \( u \notin W^{4,1}(B_1) \).
6 Optimal local estimates

In this section we complete the proof of Theorem 1.2. First we give the following lemma for later usage.

**Lemma 6.1** There exists a constant \( C = C(p) > 0 \) satisfying the following property. Let \( B_1 = B_1(0) \subset \mathbb{R}^4 \). For any \( x_0 \in B_{1/2}(0) \) and \( 0 < R < 1/2 \), if \( h \) satisfies the equation

\[
\begin{aligned}
\Delta h &= 0 \quad \text{in } B_R(x_0), \\
h &= A \Delta u \quad \text{on } \partial B_R(x_0),
\end{aligned}
\]  

(6.1)

then

\[
\| h \|_{L^p(B_{R/2}(x_0))} \leq C \left( \| u \|_{W^{2,2}(B_1)} + \| f \|_{L^p(B_1)} \right).
\]  

(6.2)

**Proof** Since \( u \in W^{3,4/3}(B_1) \), the existence of \( h \) for equation (6.1) can be easily deduced from Lemma C.1. Take a scaling transform \( u_R(x) = u(x_0 + Rx) \), \( A_R = A(x_0 + Rx) \) and \( h_R(x) = R^2 h(x_0 + Rx) \) for \( x \in B_1 \) such that

\[
\begin{aligned}
\Delta h_R &= 0 \quad \text{in } B_1, \\
h_R &= A_R \Delta u_R \quad \text{on } \partial B_1.
\end{aligned}
\]

Applying Lemma C.1 (with \( n = 4, p = 4/3 \)), we have \( h_R \in W^{1,4/3}(B_1) \) and

\[
\| \nabla h_R \|_{L^{4/3}(B_1)} \leq \| \nabla (A_R \Delta u_R) \|_{L^{4/3}(B_1)}
\]

\[
\leq C \left( \| \nabla \Delta u \|_{L^{4/3}(B_1)} + \| \nabla A_R \|_{L^4(B_1)} \| \Delta u_R \|_{L^2(B_1)} \right)
\]

for some constant \( C > 0 \) by (C.2). Since \( \nabla A \in L^4(B_1) \), \( \| \nabla A \|_{L^4(B_1)} = \| \nabla A \|_{L^4(B_R(x_0))} \) is uniformly bounded with respect to \( x_0 \) and \( R \). Hence,

\[
\| \nabla h_R \|_{L^{4/3}(B_1)} \leq C \left( \| \nabla \Delta u \|_{L^{4/3}(B_R(x_0))} + \| \Delta u_R \|_{L^2(B_R(x_0))} \right).
\]

Then, applying Theorem 1.1, we obtain

\[
\| \nabla h_R \|_{L^{4/3}(B_1)} \leq C(p) \left( \| u \|_{W^{2,2}(B_1)} + \| f \|_{L^p(B_1)} \right) R^4(1-1/p).
\]

Here we used a simple fact that

\[
\| \nabla \Delta u \|_{L^{4/3}(B_R(x_0))} \leq C(p) \left( \| u \|_{W^{2,2}(B_1)} + \| f \|_{L^p(B_1)} \right) R^4(1-1/p).
\]

We leave the proof for interested readers. As a consequence,

\[
\| h_R \|_{L^2(B_1)} \leq \| h_R - A_R \Delta u_R \|_{L^2(B_1)} + \| A_R \Delta u_R \|_{L^2(B_1)}
\]

\[
\leq C_p \left( \| u \|_{W^{2,2}(B_1)} + \| f \|_{L^p(B_1)} \right) R^4(1-1/p).
\]

In particular, this implies that

\[
\| h_R \|_{L^p(B_{1/2})} \leq C \| h_R \|_{L^2(B_1)} \leq C_p \left( \| u \|_{W^{2,2}(B_1)} + \| f \|_{L^p(B_1)} \right) R^4(1-1/p),
\]

which is equivalent to (6.2). The proof is complete.

Now we can prove Theorem 1.2.
Proof of Theorem 1.2 Set \( q = \frac{4p}{4-p} \). The idea is to establish a uniform estimate for \( \| \nabla^3 u \|_{L^q(B_{1/2})} \) in terms of \( \left( \| f \|_{L^p(B_1)} + \| u \|_{L^1(B_1)} \right) \). The proof consists of two steps. In the first step, we prove \( u \in W^{3,q}_{\text{loc}} \). By Remark 5.4, it suffices to show \( u \in W^{2,p}_{\text{loc}} \) for \( \tilde{p} = \frac{2p}{2-p} \).

In the second step we deduce the desired estimate.

We have proved that \( u \in W^{2,\gamma}_{\text{loc}}(B_1) \) for any \( \gamma < \frac{2p}{2-p} \) whenever \( 1 < p < \frac{4}{3} / \). The idea is to show that \( \| \nabla^2 u \|_{L^\gamma(B_{1/4})} \) is uniformly bounded from above with respect to \( \gamma < \frac{2p}{2-p} \).

In the below, let

\[
\tilde{p} = \frac{2p}{2-p} \quad \text{and} \quad \gamma \in (\frac{\tilde{p}}{2}, \tilde{p}).
\]

By (B.2), there exists a constant \( C > 0 \) depending only on \( p \), such that

\[
\| \nabla^2 u \|_{L^\gamma(B_{1/4})} \leq C \left( \| \Delta u \|_{L^\gamma(B_{1/2})} + \| u \|_{L^1(B_{1/2})} \right)
\]

holds for all \( \gamma \in (\frac{\tilde{p}}{2}, \tilde{p}) \).

Decompose \( A \Delta u = v + h \) in \( B_1 \) such that \( h \) is a harmonic function in \( B_1 \) and \( v = 0 \) on \( \partial B_1 \). From (1.9) we have

\[
\Delta v = \text{div}(K) + Af
\]

in \( B_1 \), with \( K \) being given by

\[
K = 2 \nabla A \cdot \Delta u - A \Delta A \nabla u + A w \nabla u - \nabla A (V \cdot \nabla u) + A \nabla (V \cdot \nabla u) + B \cdot \nabla u.
\]

We first estimate \( \| v \|_{L^\gamma(B_1)} \). To this end, notice by duality that

\[
\| v \|_{L^\gamma(B_1)} = \sup_{\varphi \in C_0^\\infty(B_1), \| \varphi \|_{L^{\gamma'}(B_1)} \leq 1} \int_{B_1} v \varphi dx,
\]

where \( \gamma' = \frac{\gamma}{\gamma - 1} \) is the conjugate exponent of \( \gamma \). Let \( \psi \) be the solution to the Dirichlet problem \( \Delta \psi = \varphi \) on \( B_1 \) with \( \psi = 0 \) on \( \partial B_1 \). Since \( \tilde{p}/2 < \gamma < \tilde{p} \), we have

\[
\frac{3}{2} + \frac{4}{3(3p-2)} = \frac{4}{3p-2} < \gamma' < \left( \frac{\tilde{p}}{2} \right)' = \frac{1}{2} + \frac{1}{p-1}.
\]

Combining this bound together with the Calderón–Zygmund theory (see Section A), there exists a constant \( C = C(p) > 0 \) independent of \( \gamma \) such that

\[
\| \psi \|_{W^{2,\gamma'}(B_1)} \leq C \| \varphi \|_{L^{\gamma'}(B_1)} \leq C.
\]

Let \( \gamma^* = (\gamma')^* \) be defined by

\[
\frac{1}{\gamma^*} = 1/\gamma - 1/4.
\]

Then we infer that

\[
\| v \|_{L^\gamma(B_1)} \leq C_p \sup_{\psi \in W^{2,\gamma'} \cap W^{1,\gamma^*}_{0}(B_1), \| \psi \|_{W^{2,\gamma'}(B_1)} \leq 1} \int_{B_1} v \Delta \psi dx.
\]

Now we estimate the above supremum as follows. Since \( v = 0, \psi = 0 \) on \( \partial B_1 \), we have

\[
\int_{B_1} v \Delta \psi dx = \int_{B_1} \Delta v \psi dx = \int_{B_1} K \cdot \nabla \psi + Af \psi dx.
\]
Note that $\gamma' > \tilde{p}'$ since $\gamma < \tilde{p}$, where $\tilde{p}'$ is the conjugate exponent of $\tilde{p}$. Thus, by the Sobolev embedding $W^{2,\tilde{p}'}(B_1) \subset L^{\tilde{p}'}(B_1)$ and Hölder’s inequality, we get
\[
\|\psi\|_{L^{\tilde{p}'}(B_1)} \leq C_p \|\psi\|_{W^{2,\tilde{p}'}(B_1)} \leq C_p \|\psi\|_{W^{2,\gamma'}(B_1)} \leq C_p.
\]
Hence
\[
\int_{B_1} Af \psi \, dx \lesssim \|f\|_{L^p(B_1)} \|\psi\|_{L^{\gamma'}(B_1)} \leq C_p \|f\|_{L^p(B_1)}. \tag{6.5}
\]
For the integral $\int_{B_1} K \cdot \nabla \psi \, dx$, we estimate term by term by Hölder’s inequality and the smallness assumption. For the first term $\nabla A \Delta u$ of $K$, we have
\[
\int_{B_1} |\nabla A \Delta u \nabla \psi| \leq \|\nabla A\|_{L^{\gamma'}(B_1)} \|\Delta u\|_{L^{\gamma'}(B_1)} \|\psi\|_{L^{\gamma'}(B_1)} \lesssim \epsilon_m \|\Delta u\|_{L^{\gamma'}(B_1)}.
\]
The rest terms can be estimated similarly. This finally leads us to
\[
\int_{B_1} K \cdot \nabla \psi \, dx \lesssim \epsilon_m \left( \|\nabla^2 u\|_{L^{\gamma'}(B_1)} + \|\nabla u\|_{L^{\gamma'}(B_1)} \right). \tag{6.6}
\]
Therefore, for any $\psi \in W^{2,\gamma'} \cap W^{1,\gamma^*}_0(B_1)$ with $\|\psi\|_{W^{2,\gamma'}(B_1)} \leq 1$, (6.4)–(6.6) implies
\[
\int_{B_1} v \Delta \psi \, dx \leq C_p \epsilon_m \left( \|\nabla^2 u\|_{L^{\gamma'}(B_1)} + \|u\|_{L^{\gamma'}(B_1)} \right) + C_p \|f\|_{L^p(B_1)}. \tag{6.7}
\]
It remains to estimate $\|\nabla u\|_{L^{\gamma'}(B_1)}$. Using the Sobolev embedding theorem, similar to the estimate (30) of [30], we may find a constant $C > 0$, independent of $t$, such that for any $t \in (1, 4)$,
\[
\|\nabla u\|_{L^{\frac{\gamma'}{2}}(B_1)} \leq \frac{C}{4-t} \|u\|_{W^{2,\frac{\gamma'}{2}}(B_1)} \leq \frac{C}{4-t} \left( \|\nabla^2 u\|_{L^{\gamma'}(B_1)} + \|u\|_{L^{\gamma'}(B_1)} \right).
\]
Applying this estimate with $t = \gamma$, and taking supremum with respect to $\psi$ in (6.7), we achieve
\[
\|v\|_{L^{\gamma'}(B_1)} \leq C_p \epsilon_m \left( \|\nabla^2 u\|_{L^{\gamma'}(B_1)} + \|u\|_{L^{\gamma'}(B_1)} \right) + C_p \|f\|_{L^p(B_1)}.
\]
Thus
\[
\|A \Delta u\|_{L^{\gamma'}(B_1/2)} \leq \|v\|_{L^{\gamma'}(B_1/2)} + \|h\|_{L^{\gamma'}(B_1/2)} \leq C_p \epsilon_m \left( \|\nabla^2 u\|_{L^{\gamma'}(B_1)} + \|u\|_{L^{\gamma'}(B_1)} \right) + C_p \left( \|f\|_{L^p(B_1)} + \|h\|_{L^{\tilde{p}}(B_1/2)} \right), \tag{6.8}
\]
Finally, we infer from (6.3) and (6.8) that
\[
\|\nabla^2 u\|_{L^{\gamma'}(B_1/4)} \leq C \epsilon_m \|\nabla^2 u\|_{L^{\gamma'}(B_1)} + C \|f\|_{L^p(B_1)} + \|u\|_{L^{\gamma'}(B_1)} + \|h\|_{L^{\tilde{p}}(B_1/2)} \tag{6.9}
\]
holds for some $C = C(p, m) > 0$ which is independent of $\gamma$.

With (6.9) at hand, the remaining step is to use a standard scaling technique as that of [30, Proof of Lemma 7.2]. Namely, we first use scaling to deduce, for any $B_R(z) \subset B_1$,
\[
\|\nabla^2 u\|_{L^{\gamma'}(B_{R}(\frac{z}{4}))} \leq C \epsilon \|\nabla^2 u\|_{L^{\gamma'}(B_R(z))} + C R^{-6} \left( \|f\|_{L^p(B_R(z))} + \|u\|_{L^{\gamma'}(B_R(z))} + \|h\|_{L^{\tilde{p}}(B_R(z))} \right),
\]
for $\beta = 6 \tilde{p} > 0$ (independent of $\gamma$). At this moment, (6.2) implies that we have
\[
\|\nabla^2 u\|_{L^{\gamma'}(B_{R}(\frac{z}{4}))} \leq C \left( \|u\|_{W^{2,\gamma'}(B_1)} + \|f\|_{L^p(B_1)} \right)
\]

for all $B_R(z) \subset B_1$. Then, we use an iteration lemma of Simon (see e.g. [30, Lemma A.7]) to derive the uniform estimate with respect to $\gamma$:
\[
\|\nabla^2 u\|_{L^p(B_{1/4})} \leq C \left( \|u\|_{W^{2,2}(B_1)} + \|f\|_{L^p(B_1)} \right) 
\]  
(6.10)
with a constant $C = C(p, m)$ independent of $\gamma$. Letting $\gamma \to \widetilde{\gamma}$ yields $\nabla^2 u \in L^{4}(B_{1/4})$. Consequently, $u \in W^{3,4}(B_{1/4})$.

In the second step, we want to refine estimate (6.10) to obtain the following quantitative estimate:
\[
\|u\|_{W^{3,4}(B_{1/2})} \leq C \left( \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)} \right). 
\]  
(6.11)
We use an interpolation argument.

By the conservation law, we have
\[
div (\nabla (A\Delta u)) = div (K) + Af.
\]
Thus elliptic regularity theory implies that
\[
\|\nabla (A\Delta u)\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} \lesssim \|K\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} + \|f\|_{L^p(B_1)} + \|A\Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})},
\]
from which it follows
\[
\|\Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} \lesssim \epsilon m \left( \|\Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} + \|\Delta u\|_{L^{p_1}(B_{\frac{1}{2}})} \right) + \|\Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} + \|f\|_{L^p(B_1)},
\]
where $p_1 = 4p/(4 - 3p)$ is the Sobolev exponent of $W^{2,4}(\mathbb{R}^4)$ embedding into $L^{p_1}(\mathbb{R}^4)$. On the other hand, by the Sobolev embedding and interpolation inequality, there holds
\[
\|\Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} \lesssim ||\Delta u||_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} + ||\Delta u||_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} \lesssim \|\nabla \Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} + \|u\|_{L^1(B_1)}
\]
and
\[
\|\Delta u\|_{L^{\frac{4}{3}}} \leq \|\nabla \Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} + C \epsilon \|u\|_{L^1(B_1)}.
\]
Combining all these estimates, we thus conclude that
\[
\|\nabla \Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} \lesssim \epsilon \|\nabla \Delta u\|_{L^{\frac{4}{3}}(B_{\frac{1}{2}})} + C \left( \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)} \right). 
\]  
(6.12)
From (6.12), a scaling argument as that of (6.10) gives (6.11). The proof is complete.

Now we can prove Corollary 1.5.

**Proof of Corollary 1.5** By the proof of previous proposition, we know there exists a constant $C = C(p, m) > 0$ such that
\[
\|\nabla^2 u\|_{L^{4}(B_{1/2})} + \|\nabla u\|_{L^{\frac{4p}{4-p}}(B_{1/2})} \leq C(p, m)\|u\|_{L^1(B_1)}.
\]
Using a simple scaling, we then deduce
\[
\|\nabla^2 u\|_{L^{4}(B_{R})} + \|\nabla u\|_{L^{\frac{4p}{4-p}}(B_{R})} \leq C(p, m)R^{-4(1-1/p)}\|u\|_{W^{2,2}(\mathbb{R}^4)},
\]  
(6.13)
Sending $R \to \infty$ gives $\nabla u = 0$ in $\mathbb{R}^4$, and so $u$ is a constant. Since $u \in L^2(\mathbb{R}^4)$, $u \equiv 0$ in $\mathbb{R}^4$. 

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6.1 Optimal $W^{k,p}$ estimate in a special case

In this section, we deduce $W^{k,p}$ estimate under the additional assumption $V \in W^{2,\frac{4}{3}}$ and $w \in W^{1,\frac{4}{3}}$. Note that the system of biharmonic mappings is included in this case.

**Proposition 6.2** Under the assumptions of Theorem 1.2, if in addition $V \in W^{2,\frac{4}{3}}(B_{10})$ and $w \in W^{1,\frac{4}{3}}(B_{10})$, then $u \in W^{k,p}_{\text{loc}}(B_{1})$ and

$$\|u\|_{W^{k,p}(B_{1/2})} \leq C \left( \|f\|_{L^p(B_{1})} + \|u\|_{L^1(B_{1})} \right).$$

**Proof** By the conservation law, we have

$$\Delta^2 u + 2 A^{-1} \nabla A \cdot \nabla \Delta u + A^{-1} \Delta A \Delta u = A^{-1} \text{div} K + f.$$  

Equivalently, we have

$$\Delta^2 u = \text{div}(A^{-1} K) + \tilde{f},$$

where $\tilde{f} = f + \nabla A^{-1} \cdot K - 2A^{-1} \nabla A \cdot \nabla \Delta u - A^{-1} \Delta A \Delta u \in L^p(B_{2})$ with

$$\|\tilde{f}\|_{L^p(B_{1})} \lesssim \|f\|_{L^p(B_{1})} + \epsilon m \left( \|\nabla^3 u\|_{L^p(B_{1})} + \|\nabla^2 u\|_{L^p(B_{1})} + \|\nabla u\|_{L^p(B_{1})} \right).$$

We use (6.15) to prove the desired result. Since we have proved $u \in W^{3,\frac{4p}{3p-4}}$, under the additional assumptions $V \in W^{2,\frac{4}{3}}(B_{10})$, $w \in W^{1,\frac{4}{3}}(B_{10})$, it is straightforward to verify that $K \in W^{1,p}_{\text{loc}}(B_{1})$, with

$$\|\nabla K\|_{L^p(B_{2/3})} \lesssim \|u\|_{W^{3,\frac{4p}{3p-4}}(B_{3/4})} \lesssim (\|f\|_{L^p(B_{1})} + \|u\|_{L^1(B_{1})}).$$

As a consequence, both $\tilde{f}$ and $\text{div} A^{-1} K$ belong to $L^p_{\text{loc}}(B_{1})$ with the estimate

$$\|\text{div}(A^{-1} K)\|_{L^p(B_{2/3})} + \|\tilde{f}\|_{L^p(B_{2/3})} \lesssim \epsilon \|u\|_{W^{3,\frac{4p}{3p-4}}(B_{3/4})} \lesssim \epsilon (\|f\|_{L^p(B_{1})} + \|u\|_{L^1(B_{1})}).$$

Hence, combining (6.15) with the standard elliptic regularity theory, we achieve the desired estimate (6.14). The proof is complete.

7 Borderline case and the compactness result

In this section we prove Theorems 1.6 and 1.7.

**Proof of Theorem 1.6** The argument is quite similar as that used in Theorem 1.2, so we only sketch the proof.

We shall use the conservation law (1.9) as there. First extend all relevant functions from $B_{1}$ to $\mathbb{R}^4$ with controlled norms. Then let $v_1 = I_2(\text{div} (K))$, $v_2 = I_2(A f)$ and $h := A \Delta u - v_1 - v_2$. As before, $h$ is a harmonic function in $B_{1}$. The condition $K \in L^{\frac{4}{3},1}$ implies $v_1 \in W^{1,\frac{4}{3},1}(B_{1})$ with the estimate

$$\|\nabla v_1\|_{L^{4/3,1}(B_{1})} \lesssim \|K\|_{L^{4/3,1}(B_{1})} \lesssim \epsilon m (\|\nabla^2 u\|_{L^{2,1}(B_{1})} + \|\nabla u\|_{L^{4,1}(B_{1})}).$$
Since \( f \in L \log L(B_1) \), we have \( Af \in L \log L(B_1) \subset h^1(\mathbb{R}^4) \), where \( h^1(\mathbb{R}^4) \) is again the local Hardy space (see [30, Appendix A.2]). Then the singular integral theory implies that \( v_2 \in W^{2,1}(B_1) \subset W^{1,\frac{4}{3},1}(B_1) \) together with the estimate
\[
\| \nabla^2 v_2 \|_{L^1(B_1)} + \| \nabla v_2 \|_{L^{4/3,1}(B_1)} \lesssim \| f \|_{L \log L(B_1)}.
\]
Hence \( A \Delta u = v_1 + v_2 + h \in W^{1,\frac{4}{3},1}(B_{\frac{2}{3}}) \). In particular, this implies that \( u \in W^{3,\frac{4}{3},1}(B_{\frac{2}{3}}) \).

Next using the same arguments as in the proof of Proposition 4.3 (see also Remark 4.4), we obtain
\[
\| \Delta u \|_{L^{2,1}(B_{\frac{7}{8}})} + \| \nabla u \|_{L^{4,1}(B_{\frac{7}{8}})} \lesssim \| f \|_{L \log L(B_1)} + \| u \|_{L^1(B_1)}.
\]
Consequently,
\[
\| K \|_{L^{4/3,1}(B_{\frac{7}{8}})} \lesssim \| \nabla^2 u \|_{L^{2,1}(B_{\frac{7}{8}})} + \| \nabla u \|_{L^{4/3,1}(B_{\frac{7}{8}})} \lesssim \| f \|_{L^{4/3,1}(B_1)} + \| u \|_{L^1(B_1)}.
\]
Returning to system (1.9), the elliptic regularity theory yields
\[
\| \nabla (A \Delta u) \|_{L^{4/3,1}(B_{\frac{3}{4}})} \lesssim \| K \|_{L^{4/3,1}(B_{\frac{7}{8}})} + \| Af \|_{L \log L(B_{\frac{7}{8}})} + \| A \Delta u \|_{L^{2,1}(B_{\frac{7}{8}})} \lesssim \| f \|_{L^{4/3,1}(B_1)} + \| u \|_{L^1(B_1)}.
\]
Hence, combining the interior \( L^2 \)-theory and the above estimates, we obtain
\[
\| \nabla^3 u \|_{L^{4/3,1}(B_{\frac{1}{2}})} \lesssim \| \Delta \nabla u \|_{L^{4/3,1}(B_{\frac{3}{4}})} + \| \nabla u \|_{L^{4,1}(B_{\frac{3}{4}})} \lesssim \| \nabla (A \Delta u) \|_{L^{4/3,1}(B_{\frac{3}{4}})} + \| \nabla u \|_{L^{4,1}(B_{\frac{3}{4}})} \lesssim \| f \|_{L^{4/3,1}(B_1)} + \| u \|_{L^1(B_1)}.
\]
The proof is complete. \( \square \)

Next we follow the idea of Sharp and Topping [30] to apply Theorem 1.6 to prove Theorem 1.7.

Proof Fix a ball \( B_R(x) \subset B_1 \). By Theorem 1.6, we know \( \{u_n\} \) is uniformly bounded in \( W^{3,\frac{4}{3},1}(B_R) \) and hence also bounded in \( W^{2,2,1}(B_R) \). Since \( u_n \to u \) in \( W^{2,2}(B_1) \), we only need to show that both \( \nabla u_n \to \nabla u \) and \( \nabla^2 u_n \to \nabla^2 u \) strongly in \( L^2(B_R) \). The first strong convergence is clear and we are left to show the second strong convergence.

Applying [30, Lemma A.6] with \( V_n = \nabla^2 u_n \), it suffices to show that
\[
\lim_{r \to 0} \limsup_{n \to \infty} \| \nabla^2 u_n \|_{L^2(B_r(x))} = 0. \tag{7.1}
\]

Applying (4.13) with \( \tau < 1 \) and scaling we obtain
\[
\| \Delta u \|_{L^2(B_{\tau r})} + \| \nabla u \|_{L^{4,2}(B_{\tau r})} \leq \frac{1}{2} \left( \| \Delta u \|_{L^2(B_r)} + \| \nabla u \|_{L^{4,2}(B_r)} \right) + C \| f \|_{L^1(B_r)} \| f \|^{1/2}_{L \log L(B_r)}.
\]
Applying Lemma 2.3 to the last term yields
\[
\| \Delta u \|_{L^2(B_{\tau r})} + \| \nabla u \|_{L^{4,2}(B_{\tau r})} \leq \frac{1}{2} \left( \| \Delta u \|_{L^2(B_r)} + \| \nabla u \|_{L^{4,2}(B_r)} \right) + C \left( \log \frac{1}{r} \right)^{-\frac{1}{2}} \| f \|_{L \log L(B_r)}.
\]
Hence,
\[
\lim_{r \to 0} \limsup_{n \to \infty} \| \Delta u_n \|_{L^2(B_{\tau r})}^2 \leq \frac{1}{2} \lim_{r \to 0} \limsup_{n \to \infty} \| \Delta u_n \|_{L^2(B_{\tau r})}^2.
\]
from which we conclude
\[
\lim_{r \to 0} \lim_{n \to \infty} \| \Delta u_n \|_{L^2(B_r)} = 0.
\]
This together with the standard $L^2$ theory for elliptic equations gives (7.1).

**Remark 7.1** In view of Remark 4.4, the above arguments also imply that $u_n \to u$ strongly in $W^{2,s}_{loc}$ for all $1 < s \leq \infty$. But we cannot conclude a strong convergence in $W^{2,1}_{loc}$. This seems to be a case on the borderline. Indeed, slightly strengthen the assumption by assuming $f \in L^{\log p} L(B_{10})$ for some $p > 1$, then for any $0 < r < 1$, there holds
\[
\int_{B_r} |f| \log(2 + |f|) \leq \left( \int_{B_r} |f|^p \log^p (2 + |f|) \right)^{1/p} \left( \int_{B_r} |f| \right)^{1 - \frac{1}{p}} 
\lesssim \log \left( \frac{1}{r} \right)^{\frac{p - 1}{p}} \int_{B_r} |f| \log^p (2 + |f|).
\]
Combining this inequality together with the decay estimate for $s = 1$ in Remark 4.4, the same arguments yield the strong convergence in $W^{2,1}_{loc}$.

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**Appendix A: A note on Calderón–Zygmund estimate**

The aim of this section is to prove that the Calderón–Zygmund estimate is locally uniform with respect to $p$.

**Proposition A.1** For each $\delta \in (0, \frac{1}{2})$, there exists $C = C_{\delta,n}$ such that for any $p \in [1 + \delta, \frac{1 + \delta}{2}]$, the following Calderón–Zygmund estimate holds
\[
\| \nabla^2 u \|_{p,B_1/2} \leq C_{\delta,n} \left( \| \Delta u \|_{p,B_1} + \| u \|_{p,B_1} \right) \tag{A.1}
\]
for all $u \in W^{2,p}(B_1)$.

**Proof** We first prove a global version. That is, for each $\delta \in (0, \frac{1}{2})$, there exists $C = C_{\delta,n}$ such that for any $p \in (1 + \delta, \frac{1 + \delta}{2})$,
\[
\| \nabla^2 u \|_{L^p(\mathbb{R}^n)} \leq C_{\delta,n} \left( \| \Delta u \|_{L^p(\mathbb{R}^n)} + \| u \|_{L^p(\mathbb{R}^n)} \right) \tag{A.2}
\]
for all $u \in W^{2,p}(\mathbb{R}^n)$.

By the well-known estimates for Calderón–Zygmund operators (see e.g. [8]), we have
\[
\| \nabla^2 u \|_{L^{1,\infty}(\mathbb{R}^n)} \leq C_{1,n} \| \Delta u \|_{L^1(\mathbb{R}^n)}
\]
and
\[
\| \nabla^2 u \|_{L^2(\mathbb{R}^n)} \leq C_{2,n} \| \Delta u \|_{L^2(\mathbb{R}^n)}.
\]
By [7, Corollary 9.10], we can take $C_{2,n} = 1$. For $1 < p < 2$ and $u \in W^{2,p}(\mathbb{R}^n)$, the Marcinkiewicz interpolation theorem (see e.g. [7, Theorem 9.8] or [8, Theorem 1.3.2]) implies
\[
\| \nabla^2 u \|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \| \Delta u \|_{L^p(\mathbb{R}^n)},
\]
where
\[ C_{p,n} = 2 \left( \frac{p}{(p-1)(2-p)} \right)^{1/p} (C_{1,n})^\theta (C_{2,n})^{1-\theta} \]
and \( \theta = \frac{2}{p} - 1 \). Thus, for any given \( \delta \in (0, \frac{1}{2}) \) and \( 1 < p \leq 1 + \delta \), we have \( 2 - p > 1/2 \) and \( \theta \in (\frac{1}{3}, 1) \). Consequently, we infer that
\[ C_{p,n} \leq \frac{C(n)}{p-1}. \]

Fix \( \delta \in (0, 1/2) \). We apply the Riesz-Thorin interpolation theorem (see e.g. [8, Theorem 1.3.4] with \( p_0 = q_0 = 1 + \delta, p_1 = q_1 = 2 \)), to obtain, for any \( 1 + \delta \leq p \leq 2 \),
\[ \|\nabla^2 u\|_{L^p(\mathbb{R}^n)} \leq C_{1+\delta,n}^\theta C_{2,n}^{1-\theta} \|\Delta u\|_{L^p(\mathbb{R}^n)} \leq C_{\delta,n} \|\Delta u\|_{L^p(\mathbb{R}^n)}, \]
where in the last inequality we used the fact that \( \theta = \frac{2(p-1-\delta)}{p(1-\delta)} \leq \frac{2}{1+\delta} \).

For \( 2 \leq p \leq \frac{1}{\frac{1}{1+\delta} - 1} = \frac{1+\delta}{\delta} \), we conclude by duality that
\[ \|\nabla^2 u\|_{L^p(\mathbb{R}^n)} \leq C_{\delta,n} \|\Delta u\|_{L^p(\mathbb{R}^n)} \]
holds for all \( u \in W^{2,p}(\mathbb{R}^n) \). This proves (A.2).

Now we can prove the local Calderón–Zygmund estimate (A.1).

For any given \( u \in W^{2,p}(B_1) \), we extend \( u \) to \( \mathbb{R}^n \) as zero outside \( B_1 \) and choose \( \eta \in C_0^\infty(B_1) \) such that \( \eta \equiv 1 \) on \( B_{1/3} \), \( 0 \leq \eta \leq 1 \) on \( \mathbb{R}^n \) and \( \max\{\|\nabla \eta\|_{L^\infty(\mathbb{R}^n)}, \|\Delta \eta\|_{L^\infty(\mathbb{R}^n)}\} \leq C_n \). Then for any \( p \in [1+\delta, \frac{1+\delta}{\delta}] \), we apply the previous global estimate to find a constant \( C_{\delta,n} > 0 \) such that
\[ \|\nabla^2(\eta u)\|_{L^p(\mathbb{R}^n)} \leq C_{\delta,n} \|\Delta(\eta u)\|_{L^p(\mathbb{R}^n)}. \]

As a consequence, we have
\[ \|\nabla^2 u\|_{p,B_{1/2}} \leq C_{\delta,n} \left( \|\eta \Delta u\|_{L^p(\mathbb{R}^n)} + 2\|\nabla \eta \cdot \nabla u\|_{L^p(\mathbb{R}^n)} + \|\Delta \eta u\|_{L^p(\mathbb{R}^n)} \right) \]
\[ \leq 2C_{\delta,n} \left( \|\Delta u\|_{p,B_1} + \|\nabla u\|_{p,B_1} + \|u\|_{p,B_1} \right). \] (A.3)

On the other hand, by the interpolation inequality for Sobolev spaces (see e.g. [7, Theorem 7.28]), there exists \( C = C(n) \) such that for any \( \epsilon > 0 \),
\[ \|\nabla u\|_{p,B_1} \leq \epsilon \|\Delta u\|_{p,B_1} + \frac{C(n)}{\epsilon} \|u\|_{p,B_1}. \]

Substituting the above inequality with \( \epsilon = 1 \) into (A.3), we finally obtain
\[ \|\nabla^2 u\|_{p,B_{1/2}} \leq C_{\delta,n} \left( \|\Delta u\|_{p,B_1} + \|u\|_{p,B_1} \right). \]

The proof is complete. \( \square \)

**Appendix B: A slightly improved Calderón–Zygmund estimate**

In this section, we prove the following proposition which states a slightly improved Calderón–Zygmund estimate. It seems very possible that this proposition was already established in some literature. As we did not find a precise reference at hand, we present a detailed proof here for the reader’s convenience.
Proposition B.1 For each $\delta \in (0, \frac{1}{4})$, there exists $C = C_{\delta,n}$ such that for any $p \in [1+\delta, n-\delta]$, the following Calderón–Zygmund estimate holds
\[
\|\nabla^2 u\|_{p,B_{1/2}} \leq C_{\delta,n} \left( \|\Delta u\|_{p,B_1} + \|u\|_{1,B_1} \right)
\] (B.1)
for all $u \in W^{2,p}(B_1)$.

In our case, $n = 4$ and $1 < p < 4/3$, so $1 < p/(2-p) < \gamma < 2p/(2-p) < 4$. Thus, there exists a constant $C$ independent of $\gamma \in (\frac{p}{2-p}, \frac{2p}{2-p})$ such that
\[
\|\nabla^2 u\|_{\gamma,B_{1/2}} \leq C \left( \|\Delta u\|_{\gamma,B_1} + \|u\|_{1,B_1} \right).
\] (B.2)

Proof We first recall the following results from [2, Chapter 5].

- **P1.** There exists $C_n > 0$ depending only on $n$ such that there exists an extension operator $E : W^{1,p}(B_1) \to W_0^{1,p}(B_2)$ for all $1 \leq p < \infty$ satisfying
\[
\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C_n \|u\|_{W^{1,p}(B_1)}.
\]

- **P2.** Let $1 < p < n$ and $u \in W^{1,p}(B_1)$ for $B_1 \subset \mathbb{R}^n$. Then
\[
\|u\|_p \leq \|u\|_\theta \|u\|_p^{1-\theta},
\]
where $\frac{1}{p} = 1 + (1-\theta)(\frac{1}{p^*} - 1)$ or equivalently $\theta = \frac{p}{np-n+p}$.

By the Sobolev embedding theorem and property **P1**, we have
\[
\|u\|_{p^*,B_1} \leq \|Eu\|_{p^*,B_2} \leq C_p \|\nabla (Eu)\|_{p,B_2} \leq C_p C_n (\|u\|_{p,B_1} + \|\nabla u\|_{p,B_1}).
\]

Here $C_p$ is the best Sobolev constant satisfying
\[
C_p \leq \frac{n-1}{\sqrt{n}} \frac{p}{n-p}.
\]

Thus,
\[
\|u\|_{p,B_1} \leq \|u\|_{1,B_1} \left( C_p C_n (\|u\|_{p,B_1} + \|\nabla u\|_{p,B_1}) \right)^{1-\theta}.
\]

Next we apply the following interpolation theorem (see e.g. [2, Theorem 5.2]): there exists $C_n > 0$ such that for all $1 \leq p < \infty$ and $\epsilon > 0$,
\[
\|\nabla u\|_{p,B_1} \leq \epsilon \|\Delta u\|_{p,B_1} + \frac{C(n)}{\epsilon} \|u\|_{p,B_1},
\]
Taking $\epsilon = 1$, we obtain
\[
\|u\|_{p,B_1} \leq \|u\|_{1,B_1} \left( C_p C_n (\|u\|_{p,B_1} + \|\nabla u\|_{p,B_1}) \right)^{1-\theta}.
\]

Since $a^\theta b^{1-\theta} \leq e^{-\frac{1-a}{\theta a}} + \epsilon b$, we have
\[
\|u\|_{p,B_1} \leq e^{-\frac{1}{\theta a}} \|u\|_{1,B_1} + \epsilon C_p C_n (\|u\|_{p,B_1} + \|\Delta u\|_{p,B_1}).
\]

Take $\epsilon = 1/(2C_p C_n)$ yields
\[
\|u\|_p \leq (2C_p C_n)^{\frac{1}{\theta a}} \|u\|_1 + \frac{1}{2} \|\Delta u\|_{p,B_1}.
\]
Note that $1/n \leq \theta \leq 1$ for all $1 < p < n$, and $\theta \to 1$ as $p \to 1$, $\theta \to 1/n$ as $p \to n$. Thus, $0 \leq 1 - \theta \leq \frac{1-\theta}{p} \leq n(1-\theta) \leq n$ and so

$$(2C\mu C_n)^{1-p} \leq 2^n C_n \left( \frac{p}{n-p} \right)^{1-p}$$

is locally uniformly bounded for $p \in [1, n)$.

Finally, combining the above estimate with Proposition A.1, we obtain

$$\|\nabla^2 u\|_{p,B_1/2} \leq C_{\delta,n} C_p \left( \|\Delta u\|_{p,B_1} + \|u\|_{1,B_1} \right)$$

with a constant $C_{\delta,n} C_p$ uniformly bounded by $C(\delta, n)$ for all $p \in [1 + \delta, n - \delta]$. \hfill \Box

### Appendix C: A regularity result

The following lemma is a special case of Theorem 3.31 of [3]. We sketch the proof for readers’ convenience.

**Lemma C.1** Let $f \in L^p(B_1)$ and $v \in W^{1,p}(B_1)$ for some $1 < p < 2$. Then there exists a unique $h \in W^{1,p}(B_1)$ solving equation

$$\begin{cases}
-\Delta h = \text{div } f & \text{in } B_1, \\
h = v & \text{on } \partial B_1.
\end{cases} \tag{C.1}
$$

Moreover, there exists a constant $C = C(n, p) > 0$, such that

$$\|h\|_{W^{1,p}(B_1)} \leq C \left( \|f\|_{L^p(B_1)} + \|v\|_{W^{1,p}(B_1)} \right).$$

**Proof** First assume $f, v \in C^2(\overline{B}_1)$ such that there exists a solution $h$ of equation (C.1). Let $h = h - v \in W^{1,p}_0(B_1)$. Then

$$-\Delta \tilde{h} = \text{div } (f + \nabla v)$$

in $B_1$. Put $F = |\nabla \tilde{h}|^{p-2} \nabla \tilde{h} \in L^{p'}(B_1)$. Note $p' > 2$. Hodge decomposition gives a function $\varphi \in W^{0,p'}_0(B_1)$ and a function $G \in L^{p'}(B_1)$ satisfying $\text{div } G = 0$ in $B_1$, and

$$\|\nabla \varphi\|_{L^{p'}(B_1)} + \|G\|_{L^{p'}(B_1)} \leq C(n, p) \|F\|_{L^{p'}(B_1)} = C(n, p) \|\nabla \tilde{h}\|_{L^{p'}(B_1)}^{p-1}.$$

Thus, using this Hodge decomposition and Hölder’s inequality, we obtain

$$\|\nabla \tilde{h}\|_{L^p(B_1)} \leq \int_{B_1} \nabla \tilde{h} \cdot F = \int_{B_1} \nabla \tilde{h} \cdot \nabla \varphi = -\int_{B_1} (f + \nabla v) \cdot \nabla \varphi \leq (\|f\|_{L^p(B_1)} + \|\nabla v\|_{L^p(B_1)}) \|\nabla \varphi\|_{L^{p'}(B_1)} \leq C(n, p) \left( \|f\|_{L^p(B_1)} + \|\nabla v\|_{L^p(B_1)} \right) \|\nabla \tilde{h}\|_{L^{p'}(B_1)}^{p-1}.$$

This implies $\|\nabla \tilde{h}\|_{L^p(B_1)} \leq C(n, p) \left( \|f\|_{L^p(B_1)} + \|\nabla v\|_{L^p(B_1)} \right)$, which in turn gives

$$\|\nabla h\|_{L^p(B_1)} \leq C \left( \|f\|_{L^p(B_1)} + \|\nabla v\|_{L^p(B_1)} \right)$$ \tag{C.2}

for some $C > 0$ depending only on $n$ and $p$. \hfill \Box
Then, using Poincaré’s inequality, we obtain
\[\|h\|_{L^p(B_1)} \leq \|h - v\|_{L^p(B_1)} + \|v\|_{L^p(B_1)} \leq C_{n,p} \|\nabla h - \nabla v\|_{L^p(B_1)} + \|v\|_{L^p(B_1)}.
\]
Combining the estimate (C.2) together with the above estimate yields
\[\|h\|_{L^p(B_1)} \leq C \left(\|f\|_{L^p(B_1)} + \|v\|_{W^{1,p}(B_1)}\right).
\]
This completes the proof in the case \(f, v \in C^2(\bar{B}_1)\).

The general case follows from a standard approximation argument. We omit the details. The uniqueness proof is also standard. \(\square\)

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