Semi-discrete Grüss-Voronovskaya-type and Grüss-type estimates for Bernstein-Kantorovich polynomials

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Abstract. The aim of this note is to prove a semi-discrete Grüss-Voronovskaya-type estimate for Bernstein-Kantorovich polynomials. Also, as a consequence, a perturbed Grüss-type estimate is obtained.

Keywords. Bernstein-Kantorovich polynomials, semi-discrete Grüss-Voronovskaya-type estimate, perturbed Grüss-type estimate, modulus of continuity.

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1 Introduction

A classical result in approximation theory is the asymptotic qualitative result of Voronovskaya for Bernstein polynomials in [11]. It was generalized by Bernstein in [3] and then it was extended to positive and linear operators by Mamedov in [10]. Also, quantitative estimates of Mamedov’s result were obtained in terms of the least concave majorant and a $K$-functional by Gonska in [7] and by Gavrea-Ivan in [6].

Another classical result is the well-known Grüss inequality for positive linear functionals $L : C[0, 1] \to \mathbb{R}$. This inequality gives an upper bound for the generalized Chebyshev functional

$$T(f, g) := L(f \cdot g) - L(f) \cdot L(g), \quad f, g \in C[0, 1].$$

For positive and linear operators $H : C[0, 1] \to C[0, 1]$ reproducing constant functions, this was investigated for the first time in [2], then obtaining in [8] the estimate

$$|H(fg; x) - H(f; x) \cdot H(g; x)| \leq \frac{1}{4} \cdot \tilde{\omega}_1(f; 2 \cdot \sqrt{H(e_2; x) - H(e_1; x)^2}) \cdot \tilde{\omega}_1(g; 2 \cdot \sqrt{H(e_2; x) - H(e_1; x)^2})$$
where \( \omega_1 \) is the least concave majorant of \( \omega_1 \) and \( e_i(x) = x^i \) for \( x \in [0, 1] \).
A mixture between the above two classical results are the so-called Grüss-Voronovskaya-type results obtained for the first time in the paper [5] for Bernstein and Păltănea operators.
On the other hand, in the very recent paper [4], we generalized the asymptotic quantitative Voronovskaya-type results, by obtaining semi-discrete quantitative Voronovskaya-type results for general positive and linear operators.
The main goal of this short note is to use the result in [4] to obtain in Section 2 a semi-discrete Grüss-Voronovskaya-type results for the Bernstein-Kantorovich polynomials. Also, as a consequence, we easily obtain a perturbed Grüss-type estimate for the same polynomials.

## 2 Semi-discrete Grüss-Voronovskaya-type estimate

The main result is the following semi-discrete Grüss-Voronovskaya-type estimate, for the Bernstein-Kantorovich polynomials given by the formula (see [9])

\[
K_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt,
\]

where \( p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) and \( f : [0, 1] \rightarrow \mathbb{R} \) is Riemann (or Lebesgue) integrable on 

\[ [0, 1]. \]
Also, for \( n \in \mathbb{N} \) and \( x, y \in [0, 1] \), let us denote

\[
E_n(x, y) = \frac{1}{(n+1)^2} \cdot \left( x(1-x)(n-1) + \frac{1}{3} \right) + (x-y) \frac{1-2x}{2(n+1)}
\]

\[
F_n(x) = \frac{1}{(n+1)^2} \cdot \left( x(1-x)(n-1) + \frac{1}{3} \right).
\]

Notice that clearly we have \( |E_n(x, y)| = O \left( \frac{1}{n} \right) \) and \( |F_n(x)| = O \left( \frac{1}{n} \right) \), uniformly with respect to \( x, y \in [0, 1] \).

**Theorem 2.1.** For all \( f, g \in C^2[0, 1] \), \( n \in \mathbb{N} \) and \( x, y \in [0, 1] \), \( x \neq y \) we have

\[
\left| K_n(fg)(x) - K_n(f)(x) \cdot K_n(g)(x) + (x-y) \cdot \frac{1-2x}{2(n+1)} (\|f\| \omega_1(x, y) + \|g\| \omega_1(x, y) - f'(x)g'(x)) \right|
\]

\[
\leq \left[ \frac{1}{(n+1)^2} (x(1-x)(n-1) + 1/3) + |x-y| \cdot \frac{1}{\sqrt{3} \sqrt{n+1}} \right]
\]

\[
\cdot \omega_1 \left( (fg)''; |x-y| + \frac{2\sqrt{6}}{\sqrt{n+1}} \right) + \|g\| \omega_1 \left( f''; |x-y| + \frac{2\sqrt{6}}{\sqrt{n+1}} \right)
\]

\[
+ \|f\| \omega_1 \left( g''; |x-y| + \frac{2\sqrt{6}}{\sqrt{n+1}} \right) + |K_n(f)(x) - f(x)| \cdot |K_n(g)(x) - g(x)|.
\]

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where \( [x, y; f] = \frac{f(x) - f(y)}{x - y}, \) \( \omega_1(f; \delta) := \sup\{ |f(x) - f(y)|; x, y \in [0, 1], |x - y| \leq \delta \} \) and \( \|f\| \) denotes the uniform norm of \( f \).

**Proof.** Supposing that \( f, g \in C^2[0, 1] \) and using Corollary 4.1 in [4], we obtain

\[
\left| K_n(fg)(x) - K_n(f)(x) \cdot K_n(g)(x) + (x - y) \cdot \frac{1 - 2x}{2(n + 1)} ([x, y; f] \cdot [x, y; g] - f'(x)g'(x)) - F_n(x) \cdot f'(x) \cdot g'(x) \right|
\]

\[
= \left| \left( K_n(fg)(x) - f(x)g(x) - \frac{1 - 2x}{2(n + 1)} \cdot [x, y; fg] \right) - E_n(x, y) \cdot \frac{(f(x)g(x))''}{2} \right|
\]

\[
- g(x) \left( \left( K_n(f)(x) - f(x) - \frac{1 - 2x}{2(n + 1)} \cdot [x, y; f] \right) - E_n(x, y) \cdot \frac{f''(x)}{2} \right)
\]

\[
- f(x) \left( \left( K_n(g)(x) - g(x) - \frac{1 - 2x}{2(n + 1)} \cdot [x, y; g] \right) - E_n(x, y) \cdot \frac{g''(x)}{2} \right)
\]

\[
+ [K_n(f)(x) - f(x)] \cdot [g(x) - K_n(g)(x)]
\]

\[
\leq \left[ \frac{1}{(n + 1)^2} (x(1 - x)(n - 1) + 1/3) + |x - y| \cdot \frac{1}{\sqrt{3} \sqrt{n + 1}} \right]
\]

\[
\cdot \left( \omega_1 \left( (fg)'', |x - y| + \frac{2\sqrt{6}}{\sqrt{n + 1}} \right) + \|g\| \cdot \omega_1 \left( f''; |x - y| + \frac{2\sqrt{6}}{\sqrt{n + 1}} \right) \right)
\]

\[
+ \|f\| \cdot \omega_1 \left( g''; |x - y| + \frac{2\sqrt{6}}{\sqrt{n + 1}} \right) \right) + |K_n(f)(x) - f(x)| \cdot |K_n(g)(x) - g(x)|,
\]

which is exactly the estimate in the statement. \( \square \)

**Remark 2.2.** Let \( f, g \in C^3[0, 1]. \) Firstly, take \( y \to x, \) multiply by \( n \) both members in the estimate in Theorem 2.1 and use the estimate in [7], page 849, line 7 from below

\[
|K_n(h)(x) - h(x)| \leq \frac{1}{2n} \|h''\| + \frac{8}{9n} \|h''\|, x \in [0, 1], n \in \mathbb{N}, h \in C^2[0, 1].
\]

Then, since

\[
n \cdot F_n(x) = \frac{n(n - 1)}{(n + 1)^2} x(1 - x) + \frac{n}{3(n + 1)^2},
\]

by using the estimate in Theorem 2.1, we easily obtain

\[
\|n[K_n(fg) - K_n(f) \cdot K_n(g)] - e_1(1 - e_1)f'g'\| = O \left( \frac{1}{\sqrt{n}} \right),
\]

thus recapturing the order of approximation in the classical Grüss-Voronovskaya-type estimate given by Theorem 5.1 in [7].

**Remark 2.3.** Since obviously

\[
\left| K_n(fg)(x) - K_n(f)(x) \cdot K_n(g)(x) + (x - y) \cdot \frac{1 - 2x}{2(n + 1)} ([x, y; f] \cdot [x, y; g] - f'(x)g'(x)) \right|
\]
\[
K_n(fg)(x) - K_n(f)(x) \cdot K_n(g)(x) + (x - y) \cdot \frac{1 - 2x}{2(n + 1)} ([x, y; f] \cdot [x, y; g] - f'(x)g'(x))
\]

by Theorem 2.1, for all \( f, g \in C^2[0,1], n \in \mathbb{N}, x, y \in [0,1], x \neq y \), we immediately get the following estimate

\[
\left| K_n(fg)(x) - K_n(f)(x) \cdot K_n(g)(x) + (x - y) \cdot \frac{1 - 2x}{2(n + 1)} ([x, y; f] \cdot [x, y; g] - f'(x)g'(x)) \right|
\]

\[
\leq \frac{1}{(n+1)^2} (x(1-x)(n-1) + 1/3) + |x - y| \cdot \frac{1}{\sqrt{3\sqrt{n+1}}}
\]

\[
= \left[ \omega_1 \left( (fg)'', |x - y| + \frac{2\sqrt{6}}{\sqrt{n+1}} \right) + \|g\| \cdot \omega_1 \left( f'', |x - y| + \frac{2\sqrt{6}}{\sqrt{n+1}} \right) + \|f\| \cdot \omega_1 \left( g'', |x - y| + \frac{2\sqrt{6}}{\sqrt{n+1}} \right) + |K_n(f)(x) - f(x)| \cdot |K_n(g)(x) - g(x)| + |F_n(x)| \cdot |f'(x)g'(x)| \right],
\]

which can be considered as a "perturbed" (discrete) Grüss-type estimate since for \( y \) sufficiently close to \( x \), the left-hand side of the above inequality, becomes sufficiently close to

\[
|K_n(fg)(x) - K_n(f)(x) \cdot K_n(g)(x)|.
\]

Now, if above we take \( y \to x \), then since \( |F_n(x)| = O \left( \frac{1}{n} \right) \), we immediately get

\[
\|K_n(fg) - K_n(f) \cdot K_n(g)\| = O \left( \frac{1}{n} \right),
\]

which is the same order which can be obtained for the classical Grüss-type estimate in terms of the least concave majorant of the modulus of continuity expressed by Theorem 4.2 in \([7]\) for \( f, g \in C^2[0,1] \).

**Remark 2.4.** The results in this note suggest that based on other semi-discrete Voronovskaya-type results in \([4]\), to get for the Bernstein-Kantorovich polynomials other semi-discrete estimates of Grüss-Voronovskaya-type and of Grüss-type. Also, similar results can be obtained for other positive and linear operators too.

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