Extraction of Product and Higher Moment Weak Values: Applications in Quantum State Reconstruction and Entanglement Detection

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Weak measurements introduced by Aharonov, Albert and Vaidman (AAV) can provide informations about the system with minimal back action. Weak values of product observables (commuting) or higher moments of an observable are informationally important in the sense that they are useful to resolve some paradoxes, realize strange quantum effects, reconstruct density matrices, etc. In this work, we show that it is possible to access the higher moment weak values of an observable using weak values of that observable with pairwise orthogonal post-selections. Although the higher moment weak values of an observable are inaccessible with Gaussian pointer states, our method allows any pointer state. We have calculated product weak values in a bipartite system for any given pure and mixed pre selected states. Such product weak values can be obtained using only the measurements of local weak values (which are defined as single system weak values in a multi-partite system). As an application, we use higher moment weak values and product weak values to reconstruct unknown quantum states of single and bipartite systems, respectively. Further, we give a necessary separability criteria for finite dimensional systems using product weak values and certain class of entangled states violate this inequality by cleverly choosing the product observables and the post selections. By such choices, positive partial transpose (PPT) criteria can be achieved for these classes of entangled states. Robustness of our method which occurs due to inappropriate choices of quantum observables and noisy post-selections is also discussed here. Our method can easily be generalized to the multi-partite systems.

I. INTRODUCTION

In quantum systems, measurement of product observables of two or more can contain information about quantum correlations and quantum dynamics. In the recent past, many theoretical and experimental works have been performed regarding the measurements of product observables using weak measurements. They have been used to resolve the Hardy’s Paradox [1] with experimental verification [2]. EPR-Bohm experiment [3], direct measurement of a density matrix [4], reconstruction of entangled quantum states [5]. It has also been reported that a strange quantum effect namely the “Quantum Cheshire Cats” where the properties of a quantum particle can be disembodied (e.g., photon’s position and polarization degrees of freedoms can be separated from each other) can be realized using product weak values [6]. See also the references [7] and [8] for the experimental test of the existence of Quantum Cheshire Cats and the exchange of grins between two such Quantum Cheshire Cats, respectively. Weak measurements of product observables also play an important role in understanding the quantum mechanics such as Bell tests [9-11], nonlocality via post-selection [12].

Higher moment weak values are useful to obtain the weak-valued probability distribution [13] of an observable in the pre and post-selected systems (which will be shown in this work later). Some applications of weak-valued probability distribution are: (a) Ozawa’s measurement-disturbance relation has experimentally been verified using weak-valued probability distribution [14], (b) to obtain all of the values of the observables relevant to a Bell test experimentally, weak-valued probabilities have been used [15], (c) the authors of [16] have shown that there is a connection between weak-valued joint probabilities and incompatibility. There are some other applications of weak-valued probabilities e.g., (d) experimental realization of the Quantum Box Problem [17], (e) to solve Hardy’s paradox [18], (f) justification of Scully et al.’s claim [19] that the momentum disturbance associated with which-way measurement in Young’s double-slit experiment can be avoided has been shown by the negativity of the weak-valued probabilities corresponding to the momentum disturbance, which consequently have zero variance [19, 20], (g) to control the probe wave packet of the target system by pre and post-selections, one can use the higher moment weak values [21]. (h) to obtain the modular value of an observable in pre and post-selected systems, higher moment weak values can be used as there is an exact connection between them [22, 23].

Weak measurements in particular weak values are known to provide useful informations with simple experimental setups. The “weak value” was first introduced by Aharonov, Albert, and Vaidman (AAV) [24]. It was inspired by the two-time formulation of the quantum-mechanical system [25]. The mechanism of AAV method is that they took the von Neumann measurement scheme [26] one step forward by considering the coupling coefficient very small i.e., weak followed by a strong measurement in succession. This formulation is characterized by the pre- and postselected states of the system. By preparing a system initially in the state $|\psi\rangle$ and post-selecting...
in the state $|\phi\rangle$, as a result we obtain the weak value of any observable $A$ which is defined as

$$\langle A_w \rangle_{\phi} = \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}. \quad (1)$$

This is a complex number and the spatial and momentum displacements of the pointer state give the real and imaginary parts of that weak value $^{27,28}$, respectively. By this way, we get the full knowledge of the complex weak value. One of the exciting and interesting features of weak value is that it can lie outside the max-min range of the eigenvalues of the operator of interest. Simultaneously, measurement disturbance is quite small which gives one to perform further measurements or simultaneous measurement of multiple observables.

Weak measurements have been proven useful in understanding quantum systems such as for the direct measurement of the wave function of a quantum system $^{29}$, to calculate slow- and fast-light effects in birefringent photonic crystals $^{30}$, the confirmation of the Heisenberg-Ozawa uncertainty relationship $^{31}$, for detecting tiny spatial shifts $^{32,33}$. Weak value can also be used to measure non-Hermitian operators $^{34,35}$, in hot thermometry $^{36}$, to detect entanglement $^{37}$ and it’s higher orders i.e., $\mathcal{D}$-values $^{34,35}$, in slow- and fast-light effects in birefringent photonic crystals $^{37}$. Also, to reconstruct the states of single and bipartite systems, we have generalized the measurement of projection operators to arbitrary observables. Our method can be generalized to the multipartite systems. Further, we give a necessary separability inequality for finite dimensional systems using product weak values. This inequality is violated by certain class of entangled states by cleverly choosing the product observables and the post selections. By such choices, The PPT criteria can be achieved for these class of entangled states. In particular, we give several examples namely (i) two-qubit Werner state (noisy singlet), (ii) mixture of two-Bell states, (iii) mixture of arbitrary pure entangled and maximally mixed states, (iv) mixture of two arbitrary entangled states, (v) mixture of four-Bell states, (vi) two qudit Werner states, (vii) higher dimensional isotropic states. The criteria can potentially detect more classes of entangled states with suitably choosing product observables and post-selections. Finally we show that our methods of “extraction of product and higher moment weak values” are robust against the errors which occur due to the inappropriate choices of system observables and unsharp postselections.

This paper is organized as follows. In sec. III we provide the formulation of our method. In sec. IV we apply our method to reconstruct quantum states of single as well as bipartite systems separately. Entanglement detection criteria is shown in sec. V. We show the robustness of our method in sec. VI and finally conclude in sec. VII.

II. FORMULATION

The following identity which is sometimes referred as Vaidman’s formula $^{40}$ will be used to derive the main results of this paper

$$A|\phi\rangle = \langle A\rangle_{\phi} |\phi\rangle + \langle \Delta A\rangle_{\phi} |\phi_{\perp}^A\rangle, \quad (2)$$

where $A$ is an Hermitian operator and $|\phi\rangle$ is any quantum state vector in the Hilbert space $\mathcal{H}$. The state vector $|\phi_{\perp}^A\rangle$ is orthogonal to $|\phi\rangle$, $\langle A\rangle_{\phi} = \langle \phi|A|\phi\rangle$ and $\langle \Delta A\rangle_{\phi} = \langle \phi_{\perp}^A|A|\phi\rangle$. For the derivation see Appendix A.
A. Higher moment weak values

If $|\psi\rangle$, $A$ and $|\phi\rangle$ are the pre-selected state, the observable and the post-selected state, respectively, then the weak value of the observable $A$ is given by

$$\langle A_w \rangle_\psi = \left(\frac{\langle \psi | A | \phi \rangle}{\langle \psi | \phi \rangle}\right)^* = \langle A \rangle_\phi + \langle \Delta A \rangle_\phi \frac{\langle \phi^\perp | A | \psi \rangle}{\langle \phi | \psi \rangle} \frac{\langle \phi | \psi \rangle}{\langle \phi^\perp | \psi \rangle},$$

(3)

where we have used Eq. (2). A similar expression was considered in Ref. [41] to explain the origin of the complex and anomalous nature of a weak value. The Eq. (3) for the expression of the weak value will be useful for deriving the following result.

Result 1.- The weak value of the operator $A^2$ which we call the "second moment weak value" has the following expression

$$\langle (A^2)_w \rangle_\psi = \langle A \rangle_\phi \left(\langle A_w \rangle_\psi - \langle A_w \rangle_\phi^\perp\right) + \langle A_w \rangle_\phi \langle A_w \rangle_\psi^\perp,$$

(4)

where $\langle A_w \rangle_\psi$ and $\langle A_w \rangle_\phi^\perp$ are the weak values of the operator $A$ for the given pre selection $|\psi\rangle$ with two post-selections $|\phi\rangle$ and $|\phi^\perp\rangle$, respectively.

Proof.

$$\langle (A^2)_w \rangle_\psi = \left(\frac{\langle \psi | A^2 | \phi \rangle}{\langle \psi | \phi \rangle}\right)^* = \left(\frac{1}{\langle \psi | \phi \rangle} \langle \psi | A | [A]_\phi \rangle + \langle \Delta A \rangle_\phi \langle \phi^\perp | \phi \rangle\right)^* = \langle A \rangle_\phi \frac{\langle \psi | A | \phi \rangle}{\langle \psi | \phi \rangle} + \langle \Delta A \rangle_\phi \frac{\langle \psi | A | \phi \rangle}{\langle \phi | \phi \rangle}^* = \langle A \rangle_\phi \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} + \langle \Delta A \rangle_\phi \frac{\langle \phi^\perp | A | \psi \rangle}{\langle \phi | \psi \rangle} \frac{\langle \phi^\perp | \psi \rangle}{\langle \phi | \psi \rangle}.$$  (5)

From Eq. (3), using $\langle \Delta A \rangle_\phi \frac{\langle \phi^\perp | \psi \rangle}{\langle \phi | \psi \rangle} = \langle A_w \rangle_\psi - \langle A \rangle_\phi$ in Eq. (5), we will obtain Eq. (4).

In the similar way we obtain all the higher moment weak values which take the general form as

$$\langle A_n^w \rangle_\psi = \langle A \rangle_\phi \left(\langle A_w \rangle_\psi^{n-1} - \langle A_w \rangle_\psi^{n-1} \langle A_w \rangle_\phi^\perp\right) + \langle A_w \rangle_\phi \langle A_w \rangle_\psi^\perp \langle A_w \rangle_\phi,$$

(6)

for $n = 1, 2, \cdots$.

Now consider the second moment weak value i.e., Eq. (4), where $\langle A_w \rangle_\psi$ and $\langle A_w \rangle_\phi^\perp$ are extractable from the one and the same experimental set-up for the post-selection of $|\phi\rangle$ and $|\phi^\perp\rangle$, respectively as these two states are orthogonal to each other. Note that, although the post-selection can be realized here in one and the same measurement set-up, nevertheless, in order to actually find out the weak values $\langle A_w \rangle_\psi$ and $\langle A_w \rangle_\phi^\perp$, measurements of phase-space displacements for the two post-selected states $|\phi\rangle$ and $|\phi^\perp\rangle$ need to be performed. $\langle A \rangle_\phi$ is the average value of $A$ for the post selected state $|\phi\rangle$. Extraction of higher moment weak values becomes extremely simple in two dimensional Hilbert space discussed in the following.

Two dimensional case: In two dimensional Hilbert space, there are only two pairwise orthogonal post-selections which occur at the same time and hence both the weak values with orthogonal post-selections can be extracted simultaneously. Particularly in this dimension, we find that only by knowing the weak values $\langle A_w \rangle_\psi$ and $\langle A_w \rangle_\phi^\perp$, we are able to obtain all the higher moment weak values without any further complications in comparison to Ref. [42]. So the number of measurements is being reduced considerably than the earlier proposal [42].

Higher dimensional case: In higher dimensional Hilbert space, to extract second moment weak value of the observable $A$, one can perform the projective measurements $\{\phi\} \cup \{\phi^\perp\}$, $I - |\phi\rangle \langle \phi| - |\phi^\perp\rangle \langle \phi^\perp|$, for post-selections.

It can be shown that the $n$-th moment weak value i.e., $\langle A_n^w \rangle_\psi$, consists $n$ number of different weak values, namely $\langle A_w \rangle_\psi$, $\langle A_w \rangle_\phi^\perp$, \ldots $\langle A_w \rangle_\phi^{n-1}$. Here $\langle (\phi^\perp)^n \rangle = \langle ((\phi^\perp)^{n-1}) \phi | (\phi^\perp)^{n-1} \rangle$. Now, if $n$ is even, then there exist pairwise orthogonal post-selected states i.e., $(|\phi|, |\phi^\perp|)$, $(|\phi^\perp|^2 |\phi|, |\phi^\perp|^2 |\phi^\perp|), \cdots (|\phi^\perp|^{n-1} |\phi|, |\phi^\perp|^{n-1} |\phi^\perp|)$. So, it is possible to obtain weak values $\langle A_w \rangle_\psi$ and $\langle A_w \rangle_\phi^\perp$ simultaneously for the first pair of post-selected states, so on and so forth. So, effectively the total number of measurements to be performed according to the AAV method to extract the $n$-th moment weak value is $n/2$. For odd $n$, the number of measurements is $(n + 1)/2$. Note that, all the measurements are here to be done by only changing the post-selections while keeping the observable $A$ fixed in the AAV method. Once we extract the $n$-th moment weak value, then all the lower moment weak values can be calculated from the data of $n$-th moment weak value.

The higher moment weak values of an observable are inaccessible with Gaussian pointer states. The reason is that, in the RS method [38], the higher moment weak value terms will vanish due to the properties of the Gaussian pointer state. The whole expression can be found in Ref. [42] (equation 4). More specifically, it can be seen in the above mentioned expression that when the orbital angular momentum (OAM) of the pointer state is zero (which corresponds to the two dimensional Gaussian pointer state, i.e., OAM state with zero orbital angular momentum), the higher moment weak value terms vanish. To retrieve higher moment weak values, we need to use OAM states with non-zero orbital angular momentum. The key factor for such cases is that the two dimensional OAM states are not factorizable in two different directions for non-zero orbital angular momentums. In doing that one needs to engineer OAM states with higher winding numbers, or superpositions of OAM states to obtain the higher moment weak values. This procedure can become difficult as the moments increase. One needs to prepare pointer states with different
combination of orbital angular momentum \[^{[42]}\]. Moreover, there are several disadvantages of the RS method from experimental perspective which we will discuss later in this section. See \[^{[43]}\] for a comment regarding the extraction of higher moment weak values.

As an application, we will use the higher moment weak values to reconstruct an unknown pure state of a single system. See Appendix \[^{[13]}\] for the derivation of the second or higher moment weak values for the mixed pre selected state case.

### B. Product weak values

**Product weak values for pure pre-selected state:** The product weak value of the observable \(A \otimes B\) in a bipartite system is given by

\[
\langle (A \otimes B)_w \rangle_{AB}^{\phi_A \phi_B} = \frac{\langle \phi_A \phi_B | (A \otimes B) | \psi_{AB} \rangle}{\langle \phi_A \phi_B | \psi_{AB} \rangle},
\]

(7)

**Result 2:** The product weak value in Eq. (7) can be realized via local weak values as

\[
\langle (A \otimes B)_w \rangle_{AB}^{\phi_A \phi_B} = \langle A \rangle_{\phi_A} \left( \langle B^{\text{local}}_w \rangle_{\psi_{AB}}^{\phi_A \phi_B} - \langle B^{\text{local}}_w \rangle_{\psi_{AB}}^{\phi_A \phi_B} \right) + \langle A^{\text{local}}_w \rangle_{\psi_{AB}}^{\phi_A \phi_B} \langle B^{\text{local}}_w \rangle_{\psi_{AB}}^{\phi_A \phi_B} \quad \text{(9)}
\]

where \(\langle A^{\text{local}}_w \rangle_{\psi_{AB}}^{\phi_A \phi_B} \cdot \langle B^{\text{local}}_w \rangle_{\psi_{AB}}^{\phi_A \phi_B}\) and \(\langle B^{\text{local}}_w \rangle_{\psi_{AB}}^{\phi_A \phi_B}\) are the “local” weak values and \(\langle \phi_A \rangle = \frac{1}{\langle \Delta \phi_A \rangle} \left( A - \langle A \rangle_{\phi_A} \right) \langle \phi_A \rangle\) is given by (2) for the subsystem A.

**Proof.**

\[
\langle (A \otimes B)_w \rangle_{AB}^{\phi_A \phi_B} = \left( \frac{\langle \psi_{AB} | (A \otimes B) | \phi_A \phi_B \rangle}{\langle \psi_{AB} | \phi_A \phi_B \rangle} \right)^* \quad \text{(8)}
\]

\[
= \left( \frac{\langle \psi_{AB} | (A)_{\phi_A} | \phi_A \rangle + \langle \Delta \phi_A | \phi_A \rangle \otimes B | \phi_B \rangle}{\langle \psi_{AB} | \phi_A \phi_B \rangle} \right)^*
\]

\[
= \langle A \rangle_{\phi_A} \langle \psi_{AB} | (A)_{\phi_A} | \phi_A \rangle + \langle \Delta \phi_A | \phi_A \rangle \otimes B | \phi_B \rangle
\]

\[
= \langle A \rangle_{\phi_A} \langle B^{\text{local}}_w | \phi_A \phi_B \rangle + \langle B^{\text{local}}_w | \phi_A \phi_B \rangle \left( \langle A^{\text{local}}_w | \phi_A \phi_B \rangle - \langle A \rangle_{\phi_A} \right),
\]

where we have used Eq. (2) in the second line for the subsystem A and Eq. (8) in the fourth line. After the manipulation, we have Eq. (9).

**Product weak values in terms of local weak values:** We have obtained a product weak value using only local weak values. Note that the local weak values \(\langle B^{\text{local}}_w | \phi_A \phi_B \rangle\) and \(\langle B^{\text{local}}_w | \phi_A \phi_B \rangle\) can be measured in the same experimental setup as the post-selected states \(| \phi_A \rangle\) and \(| \phi_A \rangle\) are orthogonal to each other. We need another measurement setup for \(\langle A^{\text{local}}_w | \phi_A \phi_B \rangle\). So, effectively the total number of measurements to be performed according to the AA V method to extract the product weak value \(\langle (A \otimes B)_w \rangle_{\psi_{AB}}^{\phi_A \phi_B}\) is only two. In experiment, local weak values like \(\langle B^{\text{local}}_w | \phi_A \phi_B \rangle\) can be realized as

\[
\langle B^{\text{local}}_w | \phi_A \phi_B \rangle = \frac{\langle \phi_B | (B \otimes I) | \psi_{AB} \rangle}{\langle \phi_B | \psi_{AB} \rangle},
\]

(10)

\[
= \frac{\langle \phi_B | B_{\psi_{BA}}^{\phi_A} \rangle}{\langle B_{\psi_{BA}}^{\phi_A} \rangle},
\]

(11)

where \(| \psi_{AB}^{\phi_A} \rangle = \langle \phi_A | \psi_{AB} \rangle\) is an unnormalized state for the subsystem B. That is, we first measure the projection operator \(\Pi_{\phi_A} = | \phi_A \rangle \langle \phi_A |\) on the subsystem A, then the state of the subsystem B becomes \(\langle \phi_A | \psi_{AB} \rangle / \langle \psi_{AB} | \psi_{AB} \rangle\) which
we consider to be pre-selected state for the subsystem B. This pre-selected state is unknown as \( |\psi_{AB}\rangle\) is unknown. So, for each given projector related to the subsystem A, there exists an unknown pre-selected state of the subsystem B. The observable is B and the post-selection is \( |\phi_B\rangle\). So from Eq. (11), we see that the local weak value \( \langle B_{local} \rangle A_{\phi_B} \) can be realized according to the AAV method related to the subsystem B.

**Product weak values for mixed pre-selected state:** If the pre-selected state is unknown, the product weak value can be realized via local weak values as

\[
\langle (A \otimes B)_{w} \rangle_{\rho_{AB}} = \langle A \rangle_{\phi_A} \langle B_{local} \rangle_{\rho_{AB}} + \frac{(\Delta A)_{\phi_A}}{2p(\rho_{AB}, \phi_{AB})} \sum_{i=1}^{m} \lambda^{i}_{A} \langle B_{local} \rangle_{\rho_{AB}} p(\rho_{AB}, i_{A} \phi_{B}) + \lambda^{i}_{A} \langle B_{local} \rangle_{\rho_{AB}} p(\rho_{AB}, i_{A} \phi_{B}),
\]

(13)

where \( \{\lambda^{i}_{A}, |i_{A}\rangle\} \) and \( \{\lambda^{i}_{A}', |i'_{A}\rangle\} \) satisfy the spectral decomposition for the normal operators \(|\phi_{A}\rangle \langle \phi_{A}\rangle + |\phi_{A}'\rangle \langle \phi_{A}'\rangle \) and \(|\phi_{A}\rangle \langle \phi_{A}\rangle - |\phi_{A}'\rangle \langle \phi_{A}'\rangle \), respectively. \( p(\rho_{AB}, \phi_{AB}) = \langle \phi_{AB}\rangle |\rho_{AB}\rangle \phi_{AB} \rangle \) is the probability of obtaining the post-selected state \(|\phi_{A}\rangle |\phi_{B}\rangle\) for the given pre-selected state \(\rho_{AB}\) and ‘m’ is the dimension of the subsystem A.

\[
\langle (A \otimes B)_{w} \rangle_{\rho_{AB}} = \langle A \rangle_{\phi_A} \langle B_{local} \rangle_{\rho_{AB}} + \frac{(\Delta A)_{\phi_A}}{2p(\rho_{AB}, \phi_{AB})} \sum_{i=1}^{m} \lambda^{i}_{A} \langle B_{local} \rangle_{\rho_{AB}} p(\rho_{AB}, i_{A} \phi_{B})
\]

(14)

where we have used Eq. (2). Now \( \langle \phi_{A}' \rangle |\phi_{B}\rangle |I_{A} \otimes B\rangle \rho_{AB} |\phi_{A} \phi_{B}\rangle = \text{Tr} \left[ (|\phi_{A}\rangle \langle \phi_{A}'\rangle \otimes |\phi_{B}\rangle \langle \phi_{B}\rangle) |I_{A} \otimes B\rangle \rho_{AB} \right] \) can be calculated as

\[
\langle \phi_{A}' \rangle |\phi_{B}\rangle |I_{A} \otimes B\rangle \rho_{AB} |\phi_{A} \phi_{B}\rangle + \langle \phi_{A} \phi_{B}\rangle |I_{A} \otimes B\rangle \rho_{AB} |\phi_{A}' \phi_{B}\rangle = \text{Tr} \left[ (|\phi_{A}\rangle \langle \phi_{A}'\rangle + |\phi_{A}'\rangle \langle \phi_{A}\rangle) \otimes |\phi_{B}\rangle \langle \phi_{B}\rangle) |I_{A} \otimes B\rangle \rho_{AB} \right],
\]

(15)

where \(|\phi_{A}\rangle \langle \phi_{A}'\rangle + |\phi_{A}'\rangle \langle \phi_{A}\rangle \) is a normal operator satisfying \( XX^\dagger = X^\dagger X \) (where X is any operator) and hence can be written in spectral decomposition

\[
|\phi_{A}\rangle \langle \phi_{A}'\rangle + |\phi_{A}'\rangle \langle \phi_{A}\rangle = \sum_{i} \lambda^{i}_{A} |i_{A}\rangle \langle i_{A}|,
\]

(16)

where \( \{|i_{A}\rangle\} \) is the set of eigenvectors with eigenvalues \( \{\lambda^{i}_{A}\} \). Using Eq. (16) in (15), we have

\[
\langle \phi_{A}' \rangle |\phi_{B}\rangle |I_{A} \otimes B\rangle \rho_{AB} |\phi_{A} \phi_{B}\rangle + \langle \phi_{A} \phi_{B}\rangle |I_{A} \otimes B\rangle \rho_{AB} |\phi_{A}' \phi_{B}\rangle = \sum_{i} \lambda^{i}_{A} \langle i_{A} \phi_{B}\rangle |I_{B} \otimes B\rangle \rho_{AB} |i_{A} \phi_{B}\rangle
\]

(17)

where we have used the definition of the weak value \( \langle B_{local} \rangle_{\rho_{AB}} \) of the observable B for the given pre and post-selections \( \rho_{AB} \) and \( |i_{A} \phi_{B}\rangle \), respectively. \( p(\rho_{AB}, i_{A} \phi_{B}) = \langle i_{A} \phi_{B}\rangle |\rho_{AB}\rangle i_{A} \phi_{B}\rangle \) is the probability of obtaining the product state \(|i_{A} \phi_{B}\rangle\). Now similarly,

\[
\langle \phi_{A}' \rangle |\phi_{B}\rangle |I_{A} \otimes B\rangle \rho_{AB} |\phi_{A} \phi_{B}\rangle - \langle \phi_{A} \phi_{B}\rangle |I_{A} \otimes B\rangle \rho_{AB} |\phi_{A}' \phi_{B}\rangle = \sum_{i} \lambda^{i}_{A} \langle B_{local} \rangle_{\rho_{AB}} p(\rho_{AB}, i_{A} \phi_{B}),
\]

(18)

where we have used the fact that \(|\phi_{A}\rangle \langle \phi_{A}'\rangle - |\phi_{A}'\rangle \langle \phi_{A}\rangle \) is also a normal operator with the spectral decomposition

\[
|\phi_{A}\rangle \langle \phi_{A}'\rangle - |\phi_{A}'\rangle \langle \phi_{A}\rangle = \sum_{i} \lambda^{i}_{A} |i'_{A}\rangle \langle i'_{A}|.
\]

(19)
Now adding two equations \(17\) and \(18\), we have

\[
\langle \phi_A \phi_B | (I_A \otimes B) \rho_{AB} | \phi_A \phi_B \rangle = \frac{1}{2} \sum_{i}^{m} \lambda_A^i \langle B_{\text{w}}^{(i)} \rho_{AB} \rangle \left( p(\rho_{AB}, i_A \phi_B) + \lambda_A^i \langle B_{\text{w}}^{(i)} \rho_{AB} \rangle \right). \tag{20}
\]

Using Eq. \(20\) in \(14\), we obtain Eq. \(13\).

**III. QUANTUM STATE TOMOGRAPHY**

In the following, we discuss some methods of quantum state tomography of a single and bipartite system using higher moments weak values and product weak values.

**A. State reconstruction of a single system**

1. **Pure state**

As an application of higher moment weak values we reconstruct a pure state. The method of quantum state reconstruction using weak values was introduced by Lundeen et al. \([7]\) as follows. Any state can be written in computational basis \(\{i\}\) as

\[
|\psi\rangle = \sum_{i=0}^{\lambda} \alpha_i |i\rangle, \tag{21}
\]

where \(\alpha_i = \langle i | \psi \rangle\). Now the weak value of a projection operator \(\Pi_i = |i\rangle \langle i|\) is given by

\[
\langle \langle \Pi_i \rangle \rangle_{\psi}^b = \frac{\langle b | i \rangle \langle i | \psi \rangle}{\langle b | \psi \rangle}, \tag{22}
\]

with \(\langle b | \psi \rangle \neq 0\). where \(|b\rangle\) is a post-selection. So using Eq. \(22\), we finally construct the pure state Eq. \(21\)

\[
|\psi\rangle = \sum_{i} \frac{\langle \langle \Pi_i \rangle \rangle_{\psi}^b}{\langle b | i \rangle \langle i | b\rangle} |i\rangle. \tag{23}
\]

The complex number \(\langle b | \psi \rangle\) is not taken into account as it corresponds to the global phase factor after normalization. To measure a pure state, we need to obtain weak values of the projection operators \(\Pi_i\) with pre selection \(|\psi\rangle\) and post-selection \(|b\rangle\), respectively.

Instead of measuring weak values of the projection operators individually, we want to use the higher moment weak values of the observable which satisfies spectral decomposition with those projection operators and using those higher moment weak values we will obtain weak values of the projection operators. Let the observable be

\[
A = \sum_i a_i \Pi_i, \tag{24}
\]

where \(a_i\) are the eigenvalues of the observable \(A\). Now the weak value and higher moment weak values of the observable...
are
\begin{align}
(A_w^b)_{i\psi} &= \sum_i a_i \langle \Pi_i w \rangle_{i\psi}, \quad (25) \\
(A_w^n)_{i\psi} &= \sum_i a_i^n \langle \Pi_i w \rangle_{i\psi}, \quad (26)
\end{align}

where ‘n’ is any positive integer. Eqs. (25) and (26) can be solved to obtain the weak values of projection operators for different ‘n’. For example in three dimensional Hilbert Space, we require only up to second moment weak values because one can use the completeness relation for the projection operators with pre selection \(|\psi\rangle\) and post-selection \(|b\rangle\)

\begin{equation}
1 = \sum_i \langle \Pi_i \rangle_{i\psi}. \quad (27)
\end{equation}

In Appendix C we explicitly show how to solve the above equations to obtain the weak values of the projection operators.

From (21), the highest moment weak value is \(A_w^{d-1}\) and as we have discussed in II A that, extracting the highest moment weak value is enough to calculate all the lower moments weak values. Hence the total number of measurements needed to reconstruct a pure state is \(d/2\) if the dimension \(d\) is even and \((d-1)/2\) if the dimension \(d\) is odd (see II A for detail discussion).

To compare with Lundeen et al. [29] and Wu [47], the number of measurement operators which is the complete set of projection operators with a fixed post-selection is \((d-1)\). Moreover, we measure only one system operator \(A\), but post-selections are to be changed, while, in their method, there are \((d-1)\) system operators (projection operators) to be measured according to the AAV method.

Note that the weak values of projection operators in Eq. (22) are exactly the weak-valued probabilities which were mentioned in the introduction section.

Alternative:- The weak value of an observable \(C\) with pre and post-selections \(|\psi\rangle\) and \(|0\rangle\), respectively is

\begin{equation}
\langle C_w \rangle_0 = \frac{\langle 0 | C | \psi \rangle}{\langle 0 | \psi \rangle}. \quad (28)
\end{equation}

Inserting the identity operator \(I = \sum_i |i\rangle \langle i|\) in numerator of the right hand side of Eq. (28), we have

\begin{equation}
\langle C_w \rangle_0 = C_{00} = \sum_i C_{0i} \frac{\alpha_i}{\alpha_0}, \quad (29)
\end{equation}

where \(C_{0i} = \langle 0 | C | i \rangle\). Like Eq. (29), we have to measure a set of observables to obtain the values of \(\alpha_1/\alpha_0, \alpha_2/\alpha_0, \ldots, \alpha_{(d-1)}/\alpha_0\) (see Appendix C). The value of \(\alpha_0\) can be obtained from the normalization condition. Here, the number of measurement operators is \((d-1)\). This method will be used to reconstruct an unknown quantum pure state of a bipartite system.

2. Mixed state

Measurement of a mixed state of a quantum system was also introduced by Lundeen et al. [4] using product weak values of two non commuting projection operators. It was later simplified by Shengjun Wu [47] where weak values of complete set of projection operators with complete set of post-selected states have been used. Here, we develop another method by using the weak values of arbitrary observables which can be thought as the generalization of the reference [47]. This part is added here because we will use the same procedure while dealing with bipartite mixed state reconstruction using product weak values.

Let the unknown mixed state be the pre-selected state then the weak value of the observable \(C\) with post-selection \(|j\rangle\) is

\begin{equation}
\langle C_w \rangle_\rho = \frac{\langle j | C \rho | j \rangle}{\langle j | \rho | j \rangle}. \quad (30)
\end{equation}

Inserting the identity operator \(I = \sum_i |i\rangle \langle i|\), we have

\begin{equation}
p(\rho, j) \langle C_w \rangle_\rho = \sum_i C_{ji} \rho_{ij}, \quad (31)
\end{equation}

where \(p(\rho, j) = \langle j | \rho | j \rangle\) is the probability of obtaining the basis state \(|j\rangle\) as a post-selection and \(C_{ji} = \langle j | C | i \rangle\), \(\rho_{ij} = \langle i | \rho | j \rangle\). To obtain the \(j\)-th column of the density matrix \(\rho\) from Eq. (31), we need to measure a set of arbitrary observables according to the AAV method to get a set of equations like (31) (see Appendix D).

To compare with the work by Lundeen et al. [4] where each matrix element is directly obtainable via sequential measurements of two non-commuting projection operators in the AAV method. Different combinations of position and momentum correlations are to be measured where the correlations are of second order in terms of the interaction coefficient. Our method is more efficient as we only need to measure \((d-1)\) arbitrary single observables according to AAV method. We do not discard any post-measurements data and thus reduces the number of experimental runs (see Appendix D).

Weak measurement methods have several key advantages for state reconstruction of a quantum system over the standard schemes [4]. For example, the state disturbance is minimum and thus it is possible for characterization of the state of a system during a physical process in an experiment. Unlike standard schemes, global reconstruction is not required by our methods as states can be determined locally i.e., each matrix element can directly be accessed. Standard scheme typically requires \(O(d^2)\) measurements, while our method require \(O(d-1)\) measurements for mixed state reconstruction.

Recently Vallone et al. [8] have shown: “Strong Measurements Give a Better Direct Measurement of the Quantum Wave Function”. Namely, they have considered the von Neumann Hamiltonian with basis projection operator (of the system) and Pauli operators (of the two dimensional pointer) and coupling coefficient (without approximation). After the evolution of the system and the pointer, the system is projected in the uniform superposition of the basis states. After that, each wavefunctions or basis coefficients of the concerned system state are calculated using the experimental probabilities obtained from the measurement of the two dimensional pointer’s different observables. To compare, (i) In our method, the state disturbance is minimum. While in
the work of Vallone et al. [48], the system will be disturbed strongly.

(ii) In the method of Vallone et al. [48], the post-selection of the system has to be of particular forms to make the scheme successful otherwise (a) the systematic error (trace distance) will be independent of the interaction coefficient which is one of their main concerns in the scheme (b) wavefunctions for each computational basis will diverge and hence the scheme will fail. Dimension of the pointer’s Hilbert space is considered to be two dimensional (finite dimensional). By such restrictions, the method can only be used for limited number of quantum systems (e.g., optical systems). While in our methods, there are no such restrictions on pointer states. The most suitable ways can be applied to obtain single weak values.

(iii) (a) In the method of Vallone et al. [48], for pure state reconstruction of a single system, there are effectively ‘d−1’ number of projection operators and for each projection operator, three different measurement observables are needed. So, the total number of measurement settings is ‘3(d−1)’. While in our method, the measurement settings are nearly ‘d/2’ using higher moment weak values. (b) For the mixed state reconstruction, they need two independent pointers, three different joint pointer operators and ‘d−1’ number of projection operators and one post-selection in the system [49]. Using such combinations, one need to calculate mean values of different combination of tripartite observables. In our method, ’d−1’ number of weak values are required and there are no such joint operations.

(iv) Vallone et al. [48] have shown that strong measurements outperform weak measurements in both the “precision and accuracy” for arbitrary quantum states in most cases. In our case, by performing the experiment many times on identically prepared systems, it is possible to reduce the uncertainty in the mean pointer displacement to any arbitrary precision [50] (in order to obtain the real and imaginary part of the weak values)

(v) For the given finite ensemble size, our scheme can’t give better performance than the methods of [48, 49] in terms of precision and accuracy [51, 52].

B. State reconstruction of a bipartite system

1. Pure state

In the above, we introduced reconstruction of an unknown pure state of a given system using single observable (or projection) weak values. For the reconstruction of a bipartite state, one needs to measure product weak values namely the weak values of the tensor product observables. In standard scheme i.e., von Neumann measurement scheme, measurement of product observables can not be realized directly as it requires the interaction Hamiltonian of the two distant subsystems to be of the form \( H \propto (A \otimes B) \) which implies an instantaneous interaction between the two distant subsystems (a relativistic constraint). In this section, we use our version of product weak values \((\text{iv})\) in a bipartite system to reconstruct a pure state following the same method of Eq. (29) as we saw for the pure state case in a single quantum system.

The pure state of a bipartite system can be written in computational basis as

\[
|\psi_{AB}\rangle = \sum_{ij} \alpha_{ij} |i_A\rangle \otimes |j_B\rangle,
\]

where \(\alpha_{ij} = \langle ij | \psi_{AB}\rangle\) and \(|ij\rangle = |i_A\rangle \otimes |j_B\rangle\). The product weak value of the observable \(C_A \otimes C_B\) in a bipartite system is given by

\[
\langle (C_A \otimes C_B)_{w|\psi_{AB}} \rangle = \frac{\langle 00 | (C_A \otimes C_B) | \psi_{AB}\rangle}{\langle 00 | \psi_{AB}\rangle},
\]

where \(|\psi_{AB}\rangle\) is the bipartite pre selected state which is to be reconstructed. \(|00\rangle = |0_A\rangle \otimes |0_B\rangle\) is the post-selected state. Now inserting identity operator \(I = \sum_{ij} |ij\rangle \langle ij|\) of the joint Hilbert space in Eq. (33), we have

\[
\langle (C_A \otimes C_B)_{w|\psi_{AB}} \rangle_{00} - \langle [C_A]_{00} [C_B]_{00} \rangle = \sum_{ij \neq (0,0)} \langle [C_A]_{0i} [C_B]_{0j} \rangle \alpha_{ij},
\]

where \(|C_A]_{0i} [C_B]_{0j}\rangle = \langle 00 | (C_A \otimes C_B) | ij\rangle\) and \(\alpha_{00} \neq 0\).

Again we have to solve a matrix equation using Eq. (34) for a set of product operators to obtain the values \(\alpha_{ij}\) (see Appendix E).

We have found using the matrix equation (E1) that we do not require to measure all the product weak values. For example, consider a two-qubit system where

\[
C_A^{(1)} \otimes C_B^{(1)} = I_A \otimes \sigma_x^B, \quad C_A^{(2)} \otimes C_B^{(2)} = \sigma_x^A \otimes I_B
\]

\[
C_A^{(3)} \otimes C_B^{(3)} = \sigma_x^A \otimes \sigma_x^B,
\]

then the square matrix in (E1) becomes an identity matrix having nonzero determinant and hence all the coefficients can be determined. This way, the number of measurements of product weak values can be reduced. In this particular case, we need only one product weak value i.e., \(\langle \sigma_x^A \otimes \sigma_x^B \rangle_{w|\psi_{AB}}\) to be measured and according to Eq. (9), it can be calculated using local weak values \(\langle \sigma_x^A \rangle_{w|\psi_{AB}}\) and \(\langle \sigma_x^B \rangle_{w|\psi_{AB}}\). So the total number of local weak values is only three to reconstruct two-qubit pure state. Note that, the local weak value \(\langle \sigma_x^B \rangle_{w|\psi_{AB}}\) can be calculated by using \(\langle \sigma_x^B \rangle_{w|\psi_{AB}}\) with the completeness relation for \(|0\rangle\) and \(|1\rangle\).

To compare with the method of Pan et al. [5], our method is experimentally simple because it does not depend on the nature of the pointer’s state (entangled or product) and locally measurable (using local weak values) only while in their method, the use of entangled pointer’s states are necessary (which might not be an easy task to perform) and local modular values as well as modular values of the sum of the local operators are required. For certain cases, some of the probability amplitudes with the entangled pointer’s states are considered to be sufficiently small. Complicated situations may arise for higher dimensions and multi-partite systems because of the entangled pointer states. Our method can be generalized both in higher dimensions and multi-partite systems with
local weak value measurements only. In the method of Pan et al. \[5\], the number of product weak values is \((m - 1)(n - 1)\) and each product weak value consists of one modular value of the sum of the two local projectors as well as two local projector modular values. Here ‘m’ and ‘n’ are the number of dimensions of the subsystems A and B, respectively. In our method, there are \((d - 1)\) (where \(d = mn\)) numbers of product weak values and each product weak value can be extracted with only two numbers of AA V type weak measurements. But as we have seen for the case of two-qubit system \((55)\), we do not need to calculate \((d - 1)\) numbers of product weak values all the time. For example, effectively we need only two local weak values to reconstruct the pure state of the two-qubit system. Note that, in Ref. \[5\] for two-qubit system, the total number of measurements is three in which one is the modular value of the sum of the two local projectors and two local projector modular values. So, in most of the cases, it is possible to reduce the number of product weak values considerably in our method of state reconstruction.

2. Mixed state

To reconstruct a mixed state of a bipartite system, we will use the method of Eq. \((31)\). Now, let the pre-selection of the system be \(\rho_{AB}\) which is unknown and post-selection be any computational basis state \(|kl\rangle\). Then the product weak value of the operator \(C_A \otimes C_B\) is given by

\[
\langle (C_A \otimes C_B)_{\psi}^{kl}_{\rho_{AB}} \rangle = \frac{\langle kl \mid (C_A \otimes C_B) \rho_{AB} \mid kl \rangle}{\langle kl \mid \rho_{AB} \mid kl \rangle}.
\]

(36)

Now inserting the identity operator \(I = \sum_{i,j} |ij\rangle \langle ij|\) in Eq. \((36)\), we have

\[
p(\rho_{AB}, kl)\langle (C_A \otimes C_B)_{\psi}^{kl}_{\rho_{AB}} \rangle = \sum_{i,j} [C_A]_{ki}[C_B]_{lj} \langle \rho_{AB} \rangle_{ij,kl}.
\]

(37)

where \([C_A]_{ki}[C_B]_{lj} = \langle kl \mid C_A \otimes C_B \mid ij \rangle\), \(\langle \rho_{AB} \rangle_{ij,kl} = \langle ij \mid \rho_{AB} \mid kl \rangle\) and \(p(\rho_{AB}, kl) = \langle kl \mid \rho_{AB} \mid kl \rangle\) is the probability of the successful post-selection \(\mid kl \rangle\). So to obtain the \(kl\)-th column of the density matrix \(\rho_{AB}\), we have to form a matrix equation using Eq. \((37)\) for a set of product operators (see Appendix F).

Clearly, mixed state reconstruction is more resource intensive than the pure state case in a bipartite system. The number of product weak values to be calculated here is \((d - 1)\) and each product weak value can be extracted with only three numbers of AAV type weak measurements (see II B). Here \(d = mn\), where ‘m’ and ‘n’ are dimensions of the subsystems A and B, respectively. We will get advantage of using matrix equation \((31)\) where for some cases we do not require to calculate all the product weak values as we have seen for the case of bipartite pure state reconstruction.

It is important to note that, our method of calculating product weak values for pure and mixed states in a bipartite system can also be applied for projection operators and hence one can reconstruct pure state using the state reconstruction method of Ref. \[5\]. For mixed state reconstruction, one should look to the Ref. \[47\] by considering bipartite system conditions.

Full knowledge of the state of a quantum system is always crucial to understand a system better and for controlling quantum technologies. In particular, the measurement of bipartite (multi-partite) states are useful for information transfer, cryptography protocols, etc. They are also used to study nonlocality, quantum discord, entanglement entropy, etc. We have shown the application of product and higher moment weak values as quantum state reconstruction of a single and bipartite systems only. The calculations of product weak values of a bipartite system are even more fascinating because of their local realizations. We can have applications of product weak values to extract informations about multi-partite systems for future technologies. Product weak values with local realization can find its applications in quantum steering, to perform some nonlocal tasks, etc.

IV. ENTANGLEMENT DETECTION

Due to an immense application of entangled systems \[53\], \[54\], it is, by default, an important task in the field of quantum information to detect whether the shared states are entangled or not. Here we show that product weak values (introduced in sec. II) can be used to detect entanglement of a bipartite system’s state. Product weak values are experimentally accessible quantities and we have discussed in sec. II how one can do that. We have found a necessary separability criteria for finite-dimensional systems. By clever choices of product observable and post-selections, it is possible to achieve the PPT criteria for entanglement detection of several important class of entangled bipartite states. Our method of entanglement detection can definitely be used for more class of entangled states.

There are some existing necessary separability criteria \[53\], \[54\] for detection of entanglement for finite-dimensional systems based on local uncertainty relation (standard deviation based) \[55\], entropic uncertainty relations \[56\], separability inequalities on Bell correlations \[57\] (which are exponentially stronger than the corresponding local reality inequalities), etc. It is worth mentioning here that Uffink and Seevinck provided a single separability inequality \[58\], (although the choice of the observables being state-dependent) quadratic in nature is used to detect separability / entanglement of all two-qubit states.

The separable states are considered to be of the following form

\[
\rho = \sum_{i} p_i \rho^i_A \otimes \rho^i_B,
\]

(38)

where \(\rho^i_A = |\psi^i_A\rangle \langle \psi^i_A|\), \(\rho^i_B = |\psi^i_B\rangle \langle \psi^i_B|\) and \(\sum_i p_i = 1\). We will consider the following quantity which is directly connected to the product weak value \((12)\) for mixed states
\[ |\langle \phi_A \phi_B | (A \otimes B) | \phi_A \phi_B \rangle |^2 \]
\[ = \sum_i p_i \langle \phi_A | A \rho_A | \phi_A \rangle \langle \phi_B | B \rho_B | \phi_B \rangle \]
\[ = \sum_i \left\{ \sqrt{p_i} \sqrt{\frac{\langle \phi_A | A \rho_A^i | \phi_A \rangle}{\langle \phi_A | A \rho_A | \phi_A \rangle}} \langle \phi_B | B \rho_B^i | \phi_B \rangle \right\} \right)^2 \]
\[ \leq \left( \sum_i \frac{\langle \phi_A | A \rho_A^i | \phi_A \rangle}{\langle \phi_A | A \rho_A | \phi_A \rangle} \right) \left( \sum_i \frac{\langle \phi_B | B \rho_B^i | \phi_B \rangle}{\langle \phi_B | B \rho_B | \phi_B \rangle} \right) \]
\[ = \left( \sum_i p_i \langle \phi_A | A \rho_A^i | \phi_A \rangle \langle \phi_B | B \rho_B^i | \phi_B \rangle \right) \left( \sum_i p_i \langle \phi_A | A \rho_A^i | \phi_A \rangle \langle \phi_B | B \rho_B^i | \phi_B \rangle \right) \]
\[ = \langle \phi_A \phi_B | (A \otimes I) \rho(A \otimes I) | \phi_A \phi_B \rangle + \langle \phi_A \phi_B | (I \otimes B) \rho(I \otimes B) | \phi_A \phi_B \rangle \]
\[ = \langle \phi_A | A | \phi_A \rangle \langle \phi_B | B \rangle + \langle \phi_A | B | \phi_B \rangle \langle \phi_B | A \rangle \]
\[ = \langle \phi_A | A^2 | \phi_A \rangle \langle \phi_B | B^2 \rangle + \langle \phi_A | A | \phi_B \rangle \langle \phi_B | A \rangle \]
\[ = \langle \phi_A | A^2 | \phi_A \rangle \langle \phi_B | B^2 \rangle + \langle \phi_A | A | \phi_B \rangle \langle \phi_B | A \rangle \]
\[ \text{for } p > 1/3 \text{ (PPT criterion).} \]

The violation of the above inequality will imply entanglement of the given bipartite state. Now, the following examples will show the potential of the above inequality to detect entanglement of certain class of entangled states.

(i) **Two-qubit Werner state (noisy singlet):**
\[ \rho = p |\psi^-_{AB}\rangle \langle \psi^-_{AB}| + (1-p) \frac{I_A \otimes I_B}{4}, \]
where \( |\psi^-_{AB}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \). By choosing \( A = \sigma_A^+ \), \( B = \sigma_B^-, |\phi_A\rangle = |1\rangle \) and \( |\phi_B\rangle = |0\rangle \), it can be shown that the inequality is violated for \( p > 1/3 \) (PPT criterion).

(ii) **Mixture of two Bell states:**
\[ \rho = p |\phi_{AB}^+\rangle \langle \phi_{AB}^+| + (1-p) |\phi_{AB}^-\rangle \langle \phi_{AB}^-|, \]
where \( |\phi_{AB}^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) and \( |\phi_{AB}^-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \).
Consider \( A = \sigma_A^+ \), \( B = \sigma_B^-, |\phi_A\rangle = |1\rangle \) and \( |\phi_B\rangle = |1\rangle \). The inequality is violated for \( p \neq 1/2 \) (PPT criterion).

(iii) **The following density operator**
\[ \rho = p |\psi_{AB}\rangle \langle \psi_{AB}| + (1-p) \frac{I_A \otimes I_B}{4}, \]
where \( |\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) and \( |a|^2 + |b|^2 = 1 \), is entangled if and only if \( p > 1/(1 + 4|ab|) \) (PPT criterion).

Using the above separability criterion with the choices \( A = \sigma_A^+ \), \( B = \sigma_B^- \), \( |\phi_A\rangle = |1\rangle \) and \( |\phi_B\rangle = |1\rangle \). The inequality is violated for \( p > 1/(1 + 4|ab|) \).

(iv) **The density operator**
\[ \rho = p |\psi_{AB}^{(1)}\rangle \langle \psi_{AB}^{(1)}| + (1-p) |\psi_{AB}^{(2)}\rangle \langle \psi_{AB}^{(2)}| \]
where \( |\psi_{AB}^{(1)}\rangle = b_1 |01\rangle + c_1 |10\rangle \), \( |\psi_{AB}^{(2)}\rangle = b_2 |01\rangle + c_2 |10\rangle \), \( |b_1|^2 + |c_1|^2 = 1 \) and \( |b_2|^2 + |c_2|^2 = 1 \), is entangled if and only if \( p \) (PPT criterion) \( |b_2|^2 c_1 + (1-p) |b_2|^2 c_2| > 0 \).

Using the separability criterion with the choices \( A = \sigma_A^+ \), \( B = \sigma_B^-, |\phi_A\rangle = |0\rangle \) and \( |\phi_B\rangle = |1\rangle \). The inequality is violated for \( |b_2|^2 c_1 + (1-p) |b_2|^2 c_2| > 0 \).

(v) **Mixture of 4-Bell states:**
\[ \rho = p_1 |\psi_{AB}^+\rangle \langle \psi_{AB}^+| + p_2 |\psi_{AB}^-\rangle \langle \psi_{AB}^-| + p_3 |\phi_{AB}^+\rangle \langle \phi_{AB}^+| + p_4 |\phi_{AB}^-\rangle \langle \phi_{AB}^-| \]
where \( |\psi_{AB}^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \), \( |\psi_{AB}^-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \) and \( p_1 + p_2 + p_3 + p_4 = 1 \). This density matrix is entangled if and only if \( p_i > 1/2 \), \( j < 1/2 \), \( i \neq j \) and \( j = 1, 2, 3, 4 \) (PPT criterion).

Consider (a) \( A = \sigma_A^+ \), \( B = \sigma_B^-, |\phi_A\rangle = |0\rangle \) and \( |\phi_B\rangle = |1\rangle \). The inequality is violated for \( p_1 > 1/2 \) or \( p_2 > 1/2 \), (b) \( A = \sigma_A^-, B = \sigma_B^+, |\phi_A\rangle = |0\rangle \) and \( |\phi_B\rangle = |1\rangle \). The inequality is violated for \( p_3 > 1/2 \) or \( p_4 > 1/2 \).

(vii) **Two qubit Werner states [5](5)**
\[ \rho = (1-p) \frac{2}{d^2 + d} P^+(i) + p \frac{2}{d^2 - d} P^-(i), \quad 0 \leq p \leq 1, \]
where the projectors \( P^+(i) = (I + V)/2 \), \( P^-(i) = (I - V)/2 \) with identity \( I \) and flip operation \( V = \sum_{i,j=0}^{d-1} i j | j \rangle \otimes | i \rangle. \)
and \(\{|i\}\) is the basis states. The state \(\rho\) is entangled if and only if \(p > 1/2\) (PPT criterion).

To see, which values of ‘\(p\)’ are achievable via the separability inequality Eq. \(59\), we first calculate the entanglement condition and then will see some physically implementable systems. Consider \(\langle i_A' j_B' \rangle | C_A \otimes C_B = | j_A' j_B' \rangle\), where \(\langle i_A' | j_A' \rangle = 0\) or \(\langle i_B' | j_B' \rangle = 0\) or both. Then from Eq. \(59\), the LHS - RHS becomes

\[
|\lambda^{(+) -} (j_A' | j_B' \rangle | j_B' | j_A' \rangle)^2 - \left[|\lambda^{(+) +} (j_A' | j_B' \rangle | j_B' | j_A' \rangle)^2\right] \times \left[|\lambda^{(+) -} (j_B' | j_A' \rangle | j_A' | j_B' \rangle)^2\right]
\]

where \(\lambda^{(+) +} = \frac{1 - p}{d^2} + \frac{p}{d^2} - \frac{1}{d^2}\) and \(\lambda^{(+) -} = \frac{1 - p}{d^2} - \frac{p}{d^2}\). Now by making the choices \(|j_B' | j_A' \rangle = |j_B' | j_A' \rangle = 1\), it is easy to show that LHS-RHS \(40\) is always positive for \(p > \frac{3(d-1)}{2(d^2-1)}\) which is the entanglement condition for the Werner state in \(d \otimes d\). It known that for \(\frac{1}{2} < p \leq \frac{d}{d^2}\) and \(p > \frac{3(d-1)}{2(d^2-1)}\), the Werner state is bound entangled (conjectured) and distillable respectively. That is, our separability criterion \(59\) is able to detect the distillability of the Werner state but not the bound entanglement (if any). In Appendix \(41\) we give the examples of how to fulfill the choices we made here in the physical systems.

(vi) Higher dimensional isotropic states \(64\):

\[
\rho = p |\psi_{AB}^+\rangle \langle \psi_{AB}^+| + (1 - p) I_A \otimes I_B/d^2,
\]

where \(|\psi_{AB}^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i_A i_B\rangle\) and ‘\(d\)’ is the dimension of the subsystems.

By choosing the spin flip operators \(A = \sigma_A^x, B = \sigma_B^x\) such that \((\sigma_A^x \otimes \sigma_B^x) |i_A i_B\rangle = |j_A j_B\rangle\), \(j_A \neq i_A, j_B \neq i_B\), and \(|\phi_A\rangle = |i_A\rangle, |\phi_B\rangle = |i_B\rangle\), it can be shown that the inequality is violated for \(p > 1/(d + 1)\) (PPT criterion).

In comparison with most of the existing works, the above separability inequality is easier to implement in experiments due to the simple realization of weak measurement and less number of measurement settings. In particular, compared to the case of universal (i.e., state-independent) detection of two-qubit entanglement using two copies of the state at a time and using the notion of weak values \(37\), the aforesaid inequality \(39\) (involving product weak values) uses only a single copy of the bipartite state \(\rho\) at a time. Moreover, the criterion is resource-wise better than tomography, based on local realization and dependent on one type of measurement set-up.

We do believe that the separability inequality \(39\) is of universal nature at least for the set of all two-qubit states. Needless to say that the choice of the local observables \(A, B\) as well as the post-selected state \(|\phi_A \phi_B\rangle\) do depend upon the choice of the input bi-partite state \(\rho\).

V. ROBUSTNESS

In AAV method, the coupling between the system and the pointer is extremely small and hence the state collapse is avoided. During the process, any resolution is insufficient to distinguish the different eigenvalues of the observable. Nevertheless, by performing the experiment many times on identically prepared systems, it is possible to reduce the uncertainty in the mean pointer displacement to any arbitrary precision \(50\).

There are other type of errors which are inevitable due to the inappropriate choices of system observables and unsharp post-selections. Here, we show that our methods of “extrac-

(i) Error in choice of observable: In experiment, let’s say, we want to measure a spin-1/2 observable according to the AAV method but due to some technical difficulties, we are unable to measure the actual spin-1/2 observable (slightly changed \(\theta\) and \(\phi\), where \(\theta\) and \(\phi\) define a point on the bloch sphere). Now let ‘\(A\)’ be the correct observable while ‘\(A’\) be the erroneous one such that \(|A - A’| \leq \delta\), where \(|X| = Tr \sqrt{X^\dagger X}\) is the trace norm of a square matrix \(X\). So the error occurring in the weak value is given by

\[
\Delta(\rho, A, \phi) = \left| \langle A_w \rangle ^\phi _\rho - \langle A_w \rangle ^{\phi'} _\rho \right| = \frac{|\langle \phi | (A - A') | \phi \rangle|}{\langle \phi | | \phi \rangle},
\]

where ‘\(m\)’ is the minimum of the probabilities for all the possible choices of rank-one post-selections with a given pre-selection. Consider the spectral decomposition \(A - A’ = \sum_{i} \lambda_i |i\rangle \langle i|\) where \(|\{i\}\rangle\) is the complete set of orthogonal basis. Then

\[
\Delta(\rho, A, \phi) \leq \frac{1}{m} \sum_{i} |\lambda_i| = \frac{|A - A'|}{m}\leq \frac{\delta}{m}.\]

(ii) Noisy post-selection: Now we consider another type of error which is common in experiments is due to the unsharp post-selections. Let us assume that the unsharp post-selection is a mixture of the true post-selection \(|\phi\rangle\) with probability \((1 - \epsilon)\) and noise state \(\sigma\) with probability \(\epsilon\)

\[
\Phi' = (1 - \epsilon) |\phi\rangle \langle \phi| + \epsilon \sigma.
\]

where \(\epsilon\) is a sufficiently small positive quantity. Then the difference between the perturbed and true weak values is

\[
\langle A_w \rangle ^{\phi'} _\rho - \langle A_w \rangle ^\phi _\rho \approx \epsilon \left[ Tr(\sigma A \rho) - \langle A_w \rangle ^\phi _\rho Tr(\sigma \rho) \right].\]  

So in both the cases (Eqs. \((43\) and \((45)\), the weak values are robust. Now it is not hard to realize that product and higher moment weak values are also robust. The only thing
we need to do is to replace the observable $A$ by $A^2$ for a single system and $C_A \otimes C_B$ for a bipartite system in Eqs. (43) and (45). Hence the weak values which we have used to reconstruct the state of a single and bipartite systems are also robust.

VI. CONCLUSION

We have derived the methods of extracting higher moment weak values and product weak values using Vaidman’s formula. Such higher moment weak values are calculated using only the weak values of that observable with pairwise orthogonal post-selections. Two dimensional Hilbert space becomes the simplest case for extracting the higher moment weak values. Our methods turn out to be simple from experimental perspective as we don’t need to measure the N pointer’s correlations as required in the previous works. Previously, it was thought that with Gaussian pointers’ states, it is not possible to obtain the higher moment weak values but we have shown that instead of looking for different pointer states (e.g., OAM states) to obtain the higher moment weak values, we can use Vaidman’s formula. To extract the product weak values in a bipartite system, we have again used Vaidman’s formula in one of the subsystems. The product weak values can be calculated using only local weak values. The key factor for such local realization is that the action of the local operator on the local post-selected state is equivalent to the superposition of that post-selected state and a unique orthogonal state to that given post-selected state. Our method can be used to verify Hardy’s Paradox, to confirm the existence of quantum Cheshire cats, to perform EPR-Bohm experiment, to realize non-locality via post-selections, etc.

As an application, we have shown how to reconstruct quantum states of a single and bipartite systems separately. We have used higher moment weak values to reconstruct an unknown pure state of a single system. The number of measurements are nearly half of the measurements required in previous works. Mixed state reconstruction has been shown using arbitrary observables. We have used product weak values for reconstruction of pure and mixed states in a bipartite system. Such reconstructions become simply feasible in experiment using only the measurements of local weak values. In the previous works, projection measurement operators were the central for direct quantum state tomography. But we have generalized it to any arbitrary observables for both single and bipartite systems. Comparisons between the previous works and our work have been considered from various perspective (e.g., number of measurements according to the AAV method and experimental feasibility). A necessary separability criteria (in terms of an inequality) for finite dimensional bi-partite systems using product weak values has been derived. This inequality is turned out to be strong as the PPT criteria can be achieved for certain class of entangled states by cleverly choosing the product observables and the post-selections. The criteria can, in principle detect more classes of entangled states with suitably choosing product observables and post-selections. Finally, we have shown that our methods are robust against the errors which are inevitable due to the inappropriate choices of system observables and unsharp post-selections. Our method can easily be extended to multi-partite systems.

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[1] Y. Aharonov, et al., Phys. Lett. A 301 (2001) 130
[2] J. S. Lundeen and A. M. Steinberg, Phys. Rev. Lett. 102, 020404 (2009)
[3] Y. Aharonov, D. Rohrlich, Quantum Paradoxes: Quantum Theory for the Perplexed, Wiley–VCH; Y. Aharonov et al., Phys. Rev. A 87 (2012) 014105
[4] Jeff S. Lundeen and Charles Bamber, Phys. Rev. Lett. 108, 070402 (2012)
[5] Wei-Wei Pan et al., Phys. Rev. Lett. 123, 150402 (2019)
[6] Yakir Aharonov et al., 2013 New J. Phys. 15 13015
[7] Tobias Denkmayr et al., 10.1038/ncomms5492
[8] Debmalya Das and Arun Kumar Pati 2020 J. Phys. Commun. 4 075007
[9] B. L. Higgins, et al., Phys. Rev. A 91, 012113 (2015)
[10] S. Marcovitch and B. Reznik, arXiv:1005.236
[11] Y. Aharonov et al., Annals of Physics 355(2015)258–268
[12] Y. Aharonov, and E. Cohen, in Quantum Nonlocality and Reality, arXiv:1504.03797
[13] Abraham G. Kofman, Sahel Ashhab, Franco Nori, Physics Reports 520 (2012) 43–133
[14] A P Lund and H M Wiseman, New J. Phys. 12 093011 (2010)
[15] B. L. Higgins et al., Physical Review A 91, 012113 (2015)
[16] Michael J. W. Hall, Arun Kumar Pati, and Junde Wu, Phys. Rev. A 93, 052118 (2016)
[17] K.J. Resch, J.S. Lundeen, and A.M. Steinberg, Physics Letters A 324 (2004) 125–131
[18] M. O. Scully, B.-G. Englert, and H. Walther, Nature 351, 111–116 (1991)
[19] H. M. Wiseman, Directly observing momentum transfer in twin-slit “which-way” experiments, Phys. Lett. A 311, 285 (2003)
[20] R. Mir, J. S. Lundeen, M. W. Mitchell, A. M. Steinberg, J. L. Garretson, and H. M. Wiseman, New J. Phys. 9, 287 (2007)
[21] Kazuhiro Ogawa, Natsuki Abe, Hirokazu Kobayashi, and Akihisa Tomita, Phys. Rev. Research 3, 033077 (2021)
[22] Le Bin Ho, Nobuyuki Imoto, Physics Letters A 380 (2016) 2129–2135
[23] Y. Kedem and L. Vaidman, PRL. 105, 230401 (2010)
[24] Y. Aharonov, D. Z. Albert, and L. Vaidman,
Here $|\phi\rangle$ is normalized. So the normalized orthogonal state vector to $|\phi\rangle$ is

$$|\phi_A\rangle = \frac{|\phi_A\rangle_{un}}{\sqrt{\langle \phi_A | \phi_A \rangle_{un}}}$$  \hspace{1cm} (A3)

From Eqs. (A2) and (A3) we find $\alpha = \langle \phi | A | \phi \rangle = \langle A | \phi \rangle$ and $\beta = \langle \phi_A | A | \phi \rangle = \langle \Delta A | \phi \rangle$. Where $\langle A | \phi \rangle$ and $\langle \Delta A | \phi \rangle$ are the average and standard deviation of the observable $A$ in the state $|\phi\rangle$, respectively. So the Eq. (A1) becomes

$$A |\phi\rangle = \langle A | \phi \rangle |\phi\rangle + \langle \Delta A | \phi \rangle |\phi_A\rangle.$$  \hspace{1cm} (A4)

Eq. (A4) is sometimes referred as Vaidman’s formula.

### Appendix B

For a given mixed pre selected state $\rho$ and post-selected state $|\phi\rangle$, the second moment weak value of the observable $A$ is given by

$$\langle (A^2)_\rho \rangle^\phi = \frac{\langle \phi | A^2 | \phi \rangle^\rho}{\langle \phi | \phi \rangle^\rho} = \langle A | \phi \rangle \frac{\langle \phi | A \rho | \phi \rangle}{\langle \phi | \rho | \phi \rangle} + \langle \Delta A | \phi \rangle \frac{\langle \phi_A \rho | A | \phi \rangle}{\langle \phi_A | \rho | \phi \rangle},$$  \hspace{1cm} (B1)

Now $\langle \phi_A | A \rho | \phi \rangle = Tr \left( |\phi\rangle \langle \phi_A | A \rho \right)$ can be calculated as

$$\langle \phi_A | A \rho | \phi \rangle + \langle \phi | A \rho | \phi \rangle = Tr \left( [\{|\phi\rangle \langle \phi_A | + |\phi_A\rangle \langle \phi |\} A \rho \right),$$  \hspace{1cm} (B2)

where $|\phi\rangle |\phi_A\rangle + |\phi_A\rangle |\phi\rangle$ is a normal operator satisfying $X X^\dagger = X^\dagger X$ (where $X$ is any operator) and hence can be written in spectral decomposition

$$|\phi\rangle |\phi_A\rangle + |\phi_A\rangle |\phi\rangle = \sum_i \lambda_i |i\rangle \langle i|.$$  \hspace{1cm} (B3)
where $|i\rangle$ is the eigenvector with eigenvalue $\lambda_i$ and ‘$d$’ is the dimension of the Hilbert space. Using Eq. (B3) in (B2), we have

$$
\langle \phi_A^+ | A \rho | \phi \rangle + \langle \phi | A \rho | \phi_A^+ \rangle = \sum_i^d \lambda_i \langle i | A \rho | i \rangle \\
= \sum_i^d \lambda_i \langle A \rho | i \rangle p(\rho, i), \quad (B4)
$$

where we have used the fact that $|\phi \rangle \langle \phi_A^+| - |\phi_A^+ \rangle \langle \phi| \rho$ is also a normal operator with the spectral decomposition $|\phi \rangle \langle \phi_A^+| = \sum_i^d \lambda_i |i \rangle \langle i| \rho$.

Now adding two equations (B4) and (B5), we have

$$
\langle \phi_A^+ | A \rho | \phi \rangle = \frac{1}{2} \sum_i^d \left( \lambda_i \langle A \rho | i \rangle p(\rho, i) + \lambda_i' \langle A \rho | i' \rangle p(\rho, i') \right). \quad (B7)
$$

So the final expression of Eq. (B1) is

$$
\langle (A^2) \rangle_{\rho, \phi} = \langle A \phi | A \phi \rangle + \frac{\langle \Delta A \phi \rangle}{2p(\rho, \phi)} \sum_i^d \left( \lambda_i \langle A \rho | i \rangle p(\rho, i) + \lambda_i' \langle A \rho | i' \rangle p(\rho, i') \right). \quad (B8)
$$

where $p(\rho, \phi) = \langle \phi | \rho | \phi \rangle$. The $n$-th moment weak value is

$$
\langle (A^n) \rangle_{\rho, \phi} = \langle A \phi | A^{n-1} \phi \rangle + \frac{\langle \Delta A \phi \rangle}{2p(\rho, \phi)} \sum_i^d \left( \lambda_i \langle A^{n-1} \rho | i \rangle p(\rho, i) + \lambda_i' \langle A^{n-1} \rho | i' \rangle p(\rho, i') \right). \quad (B9)
$$

Note that, $\{\langle A \rho | i \rangle \}_{i=1}^d$ can be measured in the same experimental setup according to the AAV method as the post-selections $\{ |i\rangle \}$ form a complete set of orthogonal basis. Similarly, $\{\langle A \rho | i \rangle \}_{i=1}^d$ can also be measured within the same experiment with the complete set of post-selections $\{ |i'\rangle \}$ according to the AAV method. So, to extract the second moment weak value, one needs only three AAV type measurements and these are $\langle A \rho | i \rangle$, $\langle A \rho | i \rangle^d_{i=1}$ and $\langle A \rho | i \rangle^d_{i=1}$. Number of measurements will increase for mixed state case than when the system is prepared in the pure state.

**Appendix C**

For the measurement of a pure state of a single system, we have followed a method slightly different from Lundeen et al. [29]. Namely we consider higher moment weak values as discussed in the main text. Here we show how to complete the process. $A$ is an observable having spectral decomposition [24], is measured according to AAV method. Now consider Eq. (25) and (26) with $n=1, \ldots, d-1$, where $d$ is the dimension of the Hilbert space. So, for different ‘$n$’, we obtain a matrix equation

$$
\begin{pmatrix}
1 \\
\langle A \rho | b \rangle \\
\langle A^{d-1} \rho | b \rangle
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
a_0 & a_1 & \cdots & a_{d-1} \\
a_0^{d-1} & a_1^{d-1} & \cdots & a_{d-1}^{d-1}
\end{pmatrix}
\begin{pmatrix}
\langle (\Pi_0) \rho | b \rangle \\
\langle (\Pi_1) \rho | b \rangle \\
\langle (\Pi_{d-1}) \rho | b \rangle
\end{pmatrix}. \quad (C1)
$$

The solution of the above equation exists if the determinant of the square matrix is non-zero. The weak values of the projection operators are then extracted from (C1) using the higher moment weak values of $A$.

**Alternative:** To obtain the values of $\alpha_i/\alpha_0$, we have to solve the matrix equation using the Eq. (29) for a set of arbitrary observables $C^{(n)}$, $n=1, \ldots, (d-1)$. So

$$
\begin{pmatrix}
C^{(1)}_{10} \langle C^{(1)}_0 | b \rangle - C^{(1)}_{00} \\
C^{(2)}_{10} \langle C^{(2)}_0 | b \rangle - C^{(2)}_{00} \\
\vdots \\
C^{(d-1)}_{10} \langle C^{(d-1)}_0 | b \rangle - C^{(d-1)}_{00}
\end{pmatrix} =
\begin{pmatrix}
C^{(1)}_{01} & C^{(1)}_{02} & \cdots & C^{(1)}_{0(d-1)} \\
C^{(2)}_{01} & C^{(2)}_{02} & \cdots & C^{(2)}_{0(d-1)} \\
\vdots \\
C^{(d-1)}_{01} & C^{(d-1)}_{02} & \cdots & C^{(d-1)}_{0(d-1)}
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{d-1}
\end{pmatrix}. \quad (C2)
$$

The solution exists if the determinant of the square matrix
above is non-zero. It may be noted here that there will always exist at least one set of observables \( \{C^{(n)}\}_{n=1}^{d-1} \) for which the determinant of the coefficient matrix on the right hand side of (2) is non-zero. Now we obtain the values of \( \alpha_i/\alpha_0 \) and then from Eq. (2), we have

\[
|\psi\rangle = \alpha_0 \left( |0\rangle + \sum_{i=1}^{d} \frac{\alpha_i}{\alpha_0} |i\rangle \right).
\]  

(C3)

The value of \( \alpha_0 \) can be obtained from the normalization condition. If the determinant of the square matrix (2) is zero with the post-selection \( |0\rangle \), then one should look for the other post-selections such as \( |1\rangle \), \( |2\rangle \) and so on and hence solve the corresponding matrix equations.

**Appendix D**

The \( j \)-th column of the density matrix with post-selection \( |j\rangle \) can be obtained using Eq. (31) for a set of arbitrary observables \( C^{(n)} \), \( n = 1, \ldots, d \)

\[
\begin{pmatrix}
(C^{(1)}_{A^0})_0 \otimes (C^{(1)}_{B^0})_0 |0\rangle_{AB} - (C^{(1)}_{A^0})_0 \otimes (C^{(1)}_{B^0})_0 |0\rangle_{AB} \\
(C^{(2)}_{A^0})_0 \otimes (C^{(2)}_{B^0})_0 |0\rangle_{AB} - (C^{(2)}_{A^0})_0 \otimes (C^{(2)}_{B^0})_0 |0\rangle_{AB} \\
\vdots \\
(C^{(d-1)}_{A^0})_0 \otimes (C^{(d-1)}_{B^0})_0 |0\rangle_{AB} - (C^{(d-1)}_{A^0})_0 \otimes (C^{(d-1)}_{B^0})_0 |0\rangle_{AB}
\end{pmatrix}
= \begin{pmatrix}
(C^{(1)}_{A^0})_0 \otimes (C^{(1)}_{B^0})_0 \\
(C^{(2)}_{A^0})_0 \otimes (C^{(2)}_{B^0})_0 \\
\vdots \\
(C^{(d-1)}_{A^0})_0 \otimes (C^{(d-1)}_{B^0})_0
\end{pmatrix} \begin{pmatrix}
(C^{(1)}_{A^0})_0 \otimes (C^{(1)}_{B^0})_0 \\
(C^{(2)}_{A^0})_0 \otimes (C^{(2)}_{B^0})_0 \\
\vdots \\
(C^{(d-1)}_{A^0})_0 \otimes (C^{(d-1)}_{B^0})_0
\end{pmatrix}
\]

with the post selection \( |00\rangle \) then one should look for the other post-selections such as \( |01\rangle \), \( |10\rangle \) and so on and hence solve the corresponding matrix equations.

**Appendix F**

The \( kl \)-th column of the density matrix with post-selection \( |kl\rangle \) can be obtained using Eq. (37) for a set of arbitrary product observables \( C^{(n)}_A \otimes C^{(n)}_B \), \( n = 1, \ldots, d \)
With $C_A^{(1)} \otimes C_B^{(1)} = I_A \otimes I_B$, an identity operator in the
joint Hilbert space. Solution exists for non zero determinant of the
above square matrix. The $k'\ell'$-th column can be calculated measuring
the same set of observables with post-selection $|k\ell\rangle$ and by solving
the corresponding matrix equations. Remember that post-selections are
complete set of projection operators and hence can be measured simultane-
ously for each product observables $C_A^{(n)} \otimes C_B^{(n)}$. So measuring an
observable $C_A^{(n)} \otimes C_B^{(n)}$ according to AAV method, we ob-
tain all the weak values for a complete set of post selections
\{\{k\ell\}\}. One does not need to perform a measurement for the
last column as it can be obtained from the normalization and
Hermiticity condition of the density matrix.

\textbf{Appendix G}

The choices for which the quantity LHS - RHS is positive
when $p > \frac{3(d-1)}{2d^2-1}$ are
\begin{equation}
|j_B'||j_A'| = 1, \quad |j_A'||j_B'| = 1. \tag{G1}
\end{equation}

To realize these conditions physically, we consider spin sys-
tems of spin-1 ($d=3$) and spin-3/2 ($d=4$) only (one can extend
such realizations for higher dimensions also).

(i) Spin-1 system: the dimension is $d=3$ with the basis states
\{0\} = (1,0,0)^T, \{1\} = (0,1,0)^T, \{2\} = (0,0,1)^T. The spin
operators in x, y and z directions are $S_x, S_y, S_z$ respectively
and the ladder operators are $S_{\pm} = S_x \pm iS_y$

\begin{align*}
S_x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
S_{\pm} &= \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_{-} = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}

We consider
\begin{align*}
\langle i_A'|C_A = \langle 2|S_x = \langle 1| = |j_A'|, \\
\langle i_B'|C_B = \langle 1|S_+ = \langle 2| = |j_B'| \tag{G2}
\end{align*}

and hence the condition (G1) is fulfilled. The op-
cator $C_A \otimes C_B = S_x \otimes S_+ = S_x \otimes S_x + iS_x \otimes S_y$
in the LHS of Eq. (39) can be realized via product weak
values as $\langle \phi_A|\phi_B|(S_x \otimes S_x + iS_x \otimes S_y)|\phi_A\phi_B\rangle =
\langle \phi_A\phi_B|(S_x \otimes S_y)|\phi_A\phi_B\rangle + i \langle \phi_A\phi_B|(S_x \otimes S_y)|\phi_A\phi_B\rangle$.

Alternative: Consider
\begin{align*}
\langle i_A'|C_A = \langle 0| + \langle 2| = \langle 1\rangle = |j_A'|, \\
\langle i_B'|C_B = \langle 1| = \langle 0| + \langle 2| = \langle 0| \tag{G3}
\end{align*}

and the condition (G1) is satisfied.

(ii) Spin-3/2 system: the dimension is $d=4$ with the ba-
sis states \{0\} = (1,0,0,0)^T, \{1\} = (0,1,0,0)^T, \{2\} =
(0,0,1,0)^T, \{3\} = (0,0,0,1)^T. The spin operators in x, y
and z directions are $J_x, J_y, J_z$ respectively and the ladder op-
erators are $J_{+}$ and $J_{-}$

\begin{align*}
J_x &= \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \\
J_y &= \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, \\
J_z &= \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad J_{+} = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
J_{-} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}

Consider the following
\begin{align*}
\langle i_A'|C_A = \langle 3|J_x = \langle 2| = |j_A'|, \\
\langle i_B'|C_B = \langle 2|J_+ = \langle 3| = |j_B'| \tag{G4}
\end{align*}

and the condition (G1) is fulfilled. Note that, normalizing fac-
tors in the calculations (G2), (G3) and (G4) are not important
as these factors will get canceled out in Eq. (39).