The speciality index as invariant indicator in the BKL mixmaster dynamics*

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Received 13 August 2004, in final form 11 February 2005
Published 12 April 2005
Online at stacks.iop.org/CQG/22/1763

Abstract
The long-standing difficulty in general relativity of classifying the dynamics of cosmological models, e.g. as chaotic, is directly related to the gauge freedom intrinsic to relativistic spacetime theories: in general the invariance under diffeomorphisms makes any analysis of dynamical evolution dependent on the particular choice of time slicing one uses. We show here that the speciality index, a scalar dimensionless curvature invariant that has been mainly used in numerical relativity as an indicator of the special or non-special Petrov-type character of a spacetime, is a time-independent quantity (a pure number) at each Kasner step of the Belinski–Khalatnikov–Lifshitz (BKL) map approximating the mixmaster cosmology. Thus the BKL dynamics can be characterized in terms of the speciality index, i.e. in terms of curvature invariants directly related to observables. Possible applications for the associated mixmaster dynamics are discussed.

PACS number: 04.20.Cv

1. Introduction
The speciality index (SI) \cite{1} is a curvature dimensionless invariant indicator of the special or non-special Petrov algebraic character of the Weyl curvature of a given spacetime. This

* During the revision of this paper, we were saddened by the news of the death of our coauthor Zoltan Perjes. We would like to remember him here as a wonderful human being with undiminished enthusiasm for engaging younger colleagues in discussing a broad range of problems in gravitational physics.
quantity has been introduced and used in numerical relativity, for instance in the study of a black hole radiating gravitational waves, or to analyse black holes merging [1, 2]. It has recently found application also in more conceptual problems, such as that of the Petrov classification of perturbed spacetimes [3, 4] and wave extraction in numerical relativity [5].

In general relativity and other spacetime theories based on general covariance, i.e. invariance under diffeomorphisms, the choice of time slicing is arbitrary. A consequence of this gauge freedom is that in general it is intrinsically problematic to characterize the dynamics—by definition the evolution in some time—in an invariant manner. This problem has been particularly relevant in the study of the cosmological dynamics in the approach to singularities, a context where the mixmaster universe has a paradigmatic role. Using fractal methods that are supposedly observer independent, it was shown in [6] that the mixmaster dynamics may be classified as chaotic. These powerful methods are directly related to the nature and characteristics of the phase space, but do not connect explicitly to observable quantities. It would then be interesting to find other invariant ways of characterizing the dynamics, using quantities related to observables such as curvature invariants, as indicators of the possible chaotic behaviour. In this paper we use the SI $S$ to study the Belinski–Khalatnikov–Lifshitz (BKL) map [7–9] associated with the mixmaster dynamics, defining in this way a curvature dimensionless invariant indicator of the Petrov-type ‘changes’ during the evolution. As is well known, the BKL map essentially consists of jumps between different Kasner models, each characterized by a triad of exponents $p_i$ ($i = 1, 2, 3$). The peculiarity of $S$ is that its value for a given Kasner model is time independent, being a number depending only on the $p_i$. Thus, while $S$ assumes values on the segment $0 \leq S \leq 1$ during the BKL evolution, its value at each step is a pure number invariantly characterizing the specific Kasner step.

2. Petrov classification: speciality index

The algebraic properties of curvature are very useful to obtain insight into the character of a given spacetime metric. In particular the properties of the Weyl tensor, which is the trace-free part of the Riemann tensor and in vacuum coincides with it, play a central role in general relativity. The Penrose–Debever equation $l_{\mu} C_{\rho q r l} l^l l^l = 0$ states the existence of four distinct null eigenvectors for the most general spacetime: these are known as ‘principal null directions’ (PND) [10]. If the PND are all distinct one has the algebraically general case (type I). When some of them coincide, this gives rise to the algebraically special cases summarized as follows: type II (one pair of PND coincides), type D (two pairs of PND coincide), type III (three PND coincide), type N (all four PND coincide) and type O (no PND, because of conformal flatness).

Defining the complex tensor $\tilde{C}_{abcd} = C_{abcd} - i^* C_{abcd}$, one can introduce the two complex curvature invariants

$$I = \frac{1}{32} \tilde{C}_{abcd} \bar{\tilde{C}}^{abcd}$$
$$J = \frac{1}{384} \tilde{C}_{abcd} \bar{\tilde{C}}^{cd} m_n \bar{\tilde{C}}^{mnab}.$$  \hspace{1cm} (2.1)

These can be used to define the speciality index [1]

$$S = 27 J^2 / I^3.$$ \hspace{1cm} (2.2)

For some spacetimes this quantity might be not well defined [2] because of the possible vanishing of $I$ and/or $J$. However, for the vacuum Kasner spacetimes [11], as shown in the following, this is not the case and $S$ marks, in an invariant way, the transition from certain algebraically special solutions ($S = 1$) and the general Petrov type I ($S \neq 1$) [10].
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The Kasner vacuum solutions of Einstein equations form a one-parameter family (see also (3.1)) with metric
\[ ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2, \] (2.3)

where
\[ p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \] (2.4)
and the \( p_i \) indices can take values in the interval \([-\frac{1}{3}, 1]\) only. The Kasner metric admits two special subcases when two of the \( p_i \) indices are equal: it then follows from (2.4) that either \( p_1 = p_2 = 0 \), \( p_3 = 1 \) (and permutations) and the spacetime is flat in this case, or \( p_1 = -\frac{1}{3}, p_2 = p_3 = \frac{2}{3} \) (and permutations) and one has the Kasner locally rotational symmetric type D solution, with a spindle-like singularity [10, 12]. For other choices of the parameter the Kasner spacetime is of Petrov type I. A simple calculation, using constraints (2.4), shows [3] that
\[ I = \frac{p_3^2(1 - p_3)}{t^4}, \quad J = \frac{I^2 J^2}{-2}. \] (2.5)

Hence the SI for any Kasner spacetime is
\[ S = \frac{27}{4} t^4 I = \frac{27}{4} p_1 p_2 p_3. \] (2.6)

Thus for the Kasner case \( S \) does not even depend on the coordinates: each Kasner model is characterized in an invariant way by a single pure number, a constant, the value of \( S \).

By direct inspection \( S \) ranges from the value \( S = 1 \) (the type D case) to \( S = 0 \) (the flat case), and is well defined with continuity for any value of the parameters \( p_i \).

The Petrov O case, given by any pair of the \( p_i \)s vanishing\(^7\), corresponds to \( I = J = 0 \) and requires a clarification. From the general definition (2.2) it would appear that in general \( S \) is ill-defined whenever \( I = 0 \). However it is clear from (2.5) that this is not the case for Kasner models, thanks to the proportionality of \( I \) and \( J \), resulting in expression (2.6), which is clearly well defined for any value of the \( p_i \)s. Thus for the Petrov-type O Kasner model, although both \( I \) and \( J \) vanish in this case, the SI \( S \) is perfectly well defined and actually vanishes. In the light of the fact that this specific model just represents Minkowski in anisotropically expanding coordinates this is not surprising, as one would expect that any reasonably defined curvature invariant vanishes for a flat spacetime.

3. Applications to BKL dynamics

The class of algebraically general Kasner spacetimes contains only algebraically special subcases of either type D and type O, and its SI has the simple time independent form (2.6). This makes it convenient to use the latter to describe the mixmaster dynamics as approximated by BKL Kasner epochs [7–9, 11]. By using the standard parametrization [11]:
\[ p_1 = -\frac{u}{u^2 + u + 1}, \quad p_2 = \frac{u + 1}{u^2 + u + 1}, \quad p_3 = \frac{u(u + 1)}{u^2 + u + 1}, \quad u \in [1, +\infty) \] (3.1)
satisfying the ordering
\[ -\frac{1}{3} \leq p_1 \leq 0, \quad 0 \leq p_2 \leq \frac{2}{3}, \quad \frac{2}{3} \leq p_3 \leq 1, \] (3.2)
with the sequence of Kasner epochs given by the rule (Gauss map):
\[ u_{n+1} = u_n - 1 \quad \text{if} \quad 2 \leq u_n < \infty, \] (3.3)
\(^7\) Since metric (2.3) is a vacuum solution, conformal flatness is equivalent to flatness, i.e. the case \( p_1 = p_2 = 0 \) and permutations.
Figure 1. Speciality index for the vacuum BKL map with $u_1 = 7.2328$ in terms of epochs $n$. Interpolation for visual clarification only. The flattening on the right is due to the map sensibility, which generates extremely long monotonic behaviour close to zero until an oscillating regime starts again.

Figure 2. Speciality index for the vacuum BKL map with $u_1 = 1.164\,398\,987$ in terms of epochs $n$.

\begin{equation}
    u_{n+1} = \frac{1}{u_n - 1} \quad \text{if} \quad 1 \leq u_n \leq 2, \quad (3.4)
\end{equation}

we obtain from (2.6) an $n$-dependent speciality index labelled by each epoch

\begin{equation}
    S_n = \frac{27}{4} \frac{u_n^2 (u_n + 1)^2}{(u_n^2 + u_n + 1)^3}. \quad (3.5)
\end{equation}

Although irrational initial values $u_1$ only should be considered [11], in numerical simulations this requirement is clearly an abstraction, because only truncated rational numbers can be handled. Using as an example the sequence given by Berger [13] which starts ‘close’ to the flat spacetime configuration with $u_1 = 7.2328$, we get the speciality index evolution shown in figure 1.

In figure 2 we show instead the evolution of $S$ corresponding to a different initial value, $u_1 = 1.164\,398\,987$.

We point out that as soon as the system gets close to the type D configuration, it moves rapidly towards to the type O region, and then gently evolves until it gets close to the type D case again, and so on. Clearly the overall behaviour is absolutely not regular, as one would
expect from the fact that the BKL dynamics as described by the Gauss map (3.3)–(3.4) is chaotic [18, 15]. The flattening of the trend in figure 1 is due to the well-known sensibility of the Gauss map which can generate very long periods of monotonic behaviour until erratic oscillations start again. Given that $S$ is a dimensionless scalar curvature invariant, its evolution gives gauge-invariant information on the BKL dynamics. Furthermore, at each Kasner step its value is a time-independent pure number associated with that step (Kasner model), even if the single curvature invariants $I$ and $J$ depend on time. The behaviour of the SI $S$ in the BKL dynamics and the fact that it only takes value on the segment $[0, 1]$ thus strongly suggest that it can be fruitfully used in numerical experiments on the general mixmater evolution, to have information on the chaotic nature of this dynamics in terms of a curvature invariant quantity, e.g. studying the mostly explored regions of the segment $[0, 1]$ during the evolution of $S$.

Pictorially, using the representation of the mixmater as the motion on a contracting triangular potential well in the time direction moving away from the initial singularity, the motion close to the type D case corresponds to an almost perpendicular bounce on the middle of the side of the triangle, with the incoming free-motion phase before the bounce corresponding to the flat space Kasner indices and the outgoing free-motion phase after the bounce corresponding to the Kasner indices [14].

The bounce is equivalent to a transition from the one set of Kasner indices to the other, for the asymptotic behaviour away from the straight wall but still far from its time spent at the opposite end in the ‘channel’ corner of the potential where the exact Taub solution originates and then returns, and space curvature effects remain important since the system point is always close to the potential walls of the channel.

We point out that the BKL parametrization has cancelled any information concerning the specific direction in which the motion is happening, leaving in our case an invariant dynamics in an ‘abstract Petrov space’.

By using the probability distribution associated on $u$, given by [7, 15]

$$\rho(u) = \frac{1}{\ln 2 \ u(u + 1)}$$

(3.6)

we can evaluate the mean value of each $p_i$, namely $\bar{p}_1 = -0.2287$, $\bar{p}_2 = 0.3564$, $\bar{p}_3 = 0.8723$. Furthermore, we can perform a probabilistic study of the BKL SI too. The relation $S = S(u)$ is monotonic in the interval $[1, \infty)$, consequently results invertible, i.e. $u = u(S)$. The distribution function for $S$ results then

$$\hat{\rho}(S) = \rho \left[ u(S) \right] \left| \frac{d u(S)}{d S} \right|,$$

(3.7)

leading to the mean value of the SI

$$\bar{S} = \int_0^1 \hat{\rho}(S) S \, dS = \frac{3}{8 \ln 2} \approx 0.541.$$ 

(3.8)

As a final remark we point out that if one wants to visualize concretely the dynamical merging or splitting of the four PNDs in order to have a clear geometrical understanding of the Petrov-type ‘changes’, the use of a different formalism, such as, for example, the one presented in [16], is required. In the context of Kasner spacetime and its BKL implementation, however, one must face the problem of the time coordinate $t$ present in the PNDs, in contrast to what happens for the SI which does not require this information. For this reason this extension will not be addressed here.
4. Conclusions

We have introduced the speciality index in mixmaster BKL dynamics, in analogy with its use in the numerical treatment of gravitational wave sources. Because of its gauge invariant nature, time independence and adimensionality, the Kasner SI and its derived BKL version can be used, as additional tool with respect to the standard ones [17], to compare the discrete map approximation with the real mixmaster dynamics. In fact, as pointed out in a recent review on the subject [15] (see also references therein), ‘a remaining open question is how closely an actual mixmaster evolution is approximated by a single BKL sequence’. A direct comparison of the BKL SI (which approximates the mixmaster dynamics) with the corresponding exact Bianchi IX one will be an appropriate possible way for approaching the problem numerically. In analogy with black-hole physics, this useful geometrical tool could be used for more complicated numerical experiments of theoretical cosmology, in order to obtain, possibly, invariant information concerning chaos [15, 18, 6].

Acknowledgments

The authors thank Robert T Jantzen and Giovanni Montani for useful suggestions and comments. CC would like to acknowledge Bruce Bassett and Beverly K Berger for stimulating discussions.

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