Research article

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Concentration results for a magnetic Schrödinger-Poisson system with critical growth

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Abstract: This paper is concerned with the following nonlinear magnetic Schrödinger-Poisson type equation
\[
\begin{aligned}
\left(\frac{\sqrt{\varepsilon}}{i} \nabla - A(x)\right)^2 u + V(x)u + e^{-2(|x|^{-1} * |u|^2)}u &= f(|u|^2)u + |u|^4u \quad \text{in } \mathbb{R}^3, \\
u &\in H^1(\mathbb{R}^3, \mathbb{C}),
\end{aligned}
\]

where \( \varepsilon > 0, V : \mathbb{R}^3 \to \mathbb{R} \) and \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) are continuous potentials, \( f : \mathbb{R} \to \mathbb{R} \) is a subcritical nonlinear term and is only continuous. Under a local assumption on the potential \( V \), we use variational methods, penalization technique and Ljusternick-Schnirelmann theory to prove multiplicity and concentration of nontrivial solutions for \( \varepsilon > 0 \) small.

Keywords: Schrödinger-Poisson system, Magnetic field, Critical growth, Variational methods

MSC: 35J60, 35J25

1 Introduction and main results

In this paper, we study multiplicity and concentration of the nontrivial solutions of the following Schrödinger-Poisson type equations with critical growth
\[
\begin{aligned}
\left(\frac{\sqrt{\varepsilon}}{i} \nabla - A(x)\right)^2 u + V(x)u + e^{-2(|x|^{-1} * |u|^2)}u &= f(|u|^2)u + |u|^4u \quad \text{in } \mathbb{R}^3, \\
u &\in H^1(\mathbb{R}^3, \mathbb{C}),
\end{aligned}
\]

where \( u \in H^1(\mathbb{R}^3, \mathbb{C}), \varepsilon > 0 \) is a parameter, \( V : \mathbb{R}^3 \to \mathbb{R} \) is a continuous function, \( f \in C(\mathbb{R}, \mathbb{R}) \) has a subcritical growth, the magnetic potential \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) is Hölder continuous with exponent \( \alpha \in (0, 1] \), and the convolution potential is defined by \( |x|^{-1} * |u|^2 = \int_{\mathbb{R}^3} |x - y|^{-1}u(y)^2dy \).

In recent years a considerable amount of work has been devoted to investigating the existence and multiplicity of solutions for nonlinear Schrödinger-Poisson system without magnetic field. We notice that, by using minimax theorema and the Ljusternik-Schnirelmann theory, He [25] gave multiplicity and concentration of positive solutions of the following problem
\[
\begin{aligned}
- e^2 \Delta u + V(x)u + \phi(x)u &= f(u), \quad \text{in } \mathbb{R}^3, \\
- e^2 \Delta \phi &= u^2, \quad \text{in } \mathbb{R}^3, \\
u &\in H^1(\mathbb{R}^3), \ u(x) > 0, \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]

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where \( f \in C^1(\mathbb{R}) \) has the subcritical growth and the potential \( V \) satisfies a global condition introduced by Rabinowitz [31]. In [26], He and Zou studied the existence and concentration of ground state solutions for the following Schrödinger-Poisson system with the critical growth

\[
\begin{align*}
-\varepsilon^2 \Delta u + V(x)u + \phi(x)u &= f(u) + |u|^4 u, \quad \text{in } \mathbb{R}^3, \\
-\varepsilon^2 \Delta \phi &= u^2, \quad u(x) > 0, \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \( f \in C^1(\mathbb{R}) \) and the potential \( V \) satisfies a global condition. Then, He [27] studied the multiplicity of concentrating positive solutions for Schrödinger-Poisson system (1.2) with nonlinear term \( f \in C^1(\mathbb{R}) \) under a local assumption introduced by del Pino and Felmer [17]. For further results on Schrödinger-Poisson system without magnetic field, we refer to [1, 4, 5, 14, 15, 22, 32, 33, 36, 40] and the references therein (see also [21] for the fractional case).

Concerning the magnetic nonlinear Schrödinger equation (1.1), we refer to [6–8, 10–13, 16, 19, 23, 24, 29, 38, 39] and references therein. It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [19]. They used the concentration-compactness principle and minimization arguments to obtain solutions for \( \varepsilon > 0 \) fixed. In [39], the authors studied multiplicity and concentration of solutions for magnetic relativistic Schrödinger equations, Xia [38] studied a critical fractional Choquard-Kirchhoff problem with magnetic field. In particular, due to our scope, we want to mention [41] where the authors studied a Schrödinger-Poisson type equation with magnetic field by using the method of the Nehari manifold, the penalization method and Ljusternik-Schnirelmann category theory for subcritical nonlinearity \( f \in C^1 \). If \( f \) is only continuous, then the arguments in [41] failed. Recently, by variational methods, penalization technique, and Ljusternik-Schnirelmann theory, for the magnetic Schrödinger-Poisson system with subcritical growth nonlinearity \( f \) which is only continuous, in [29] we proved multiplicity and concentration properties of nontrivial solutions for \( \varepsilon > 0 \) small. For the fractional Schrödinger-Poisson type equations with magnetic field, we refer to [2, 3].

Inspired by [27, 29], we intend to prove multiplicity and concentration of nontrivial solutions for problem (1.1) with critical growth. Since the problem we deal with has the critical growth, we need more refined estimates to overcome the lack of compactness. On the other hand, due to the appearance of magnetic field \( A(x) \) and the nonlocal term \(|x|^{-1} \cdot |u|^2\), problem (1.1) will be more difficult, and some estimates are also more complicated.

In this paper, we make the following assumptions on the potential \( V \):

(V1) There exists \( V_0 > 0 \) such that \( V(x) \geq V_0 \) for all \( x \in \mathbb{R}^3 \);

(V2) There exists a bounded open set \( \Lambda \subset \mathbb{R}^3 \) such that

\[
V_0 = \min_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x).
\]

Observe that

\[
M := \{ x \in \Lambda : V(x) = V_0 \} \neq \emptyset.
\]

On the nonlinearity \( f \in C(\mathbb{R}, \mathbb{R}) \), we require that:

(f1) \( f(t) = 0 \) if \( t \leq 0 \), and \( \lim_{t \to 0^+} \frac{f(t)}{t} = 0 \);

(f2) There exist \( \sigma, q \in (4, 6) \) and \( \mu > 0 \) such that

\[
f(t) \geq \mu t^{\frac{\sigma}{q}} \quad \forall t > 0, \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(t)}{t^{\frac{q}{\sigma}}} = 0;
\]

(f3) there is a positive constant \( \theta \in (4, 6) \) such that

\[
0 < \frac{\theta}{2} F(t) \leq tf(t), \quad \forall t > 0, \quad \text{where} \quad F(t) = \int_0^t f(s)ds;
\]

(f4) \( \frac{f(t)}{t} \) is strictly increasing in \((0, \infty)\).

The main result of this paper is listed as follows:
Theorem 1.1. Assume that $V$ satisfies (V1), (V2) and $f$ satisfies (f1)–(f4). Then, for any $\delta > 0$ such that

$$M_\delta := \{ x \in \mathbb{R}^3 : \text{dist}(x, M) < \delta \} \subset A,$$

there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (1.1) has at least $\text{cat}_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{ \varepsilon_n \}$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, if we denote by $u_{\varepsilon_n}$ one of these solutions of (1.1) for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^3$ the global maximum point of $|u_{\varepsilon_n}|$, then

$$\lim_{n \to +\infty} V(\eta_{\varepsilon_n}) = V_0.$$  

The paper is organized as follows. In Section 2 we indicate the functional setting and give some preliminary results. In Section 3, we study the modified problem, and prove the Palais-Smale condition for the modified functional and provide some tools which are useful to establish a multiplicity result. In Section 4, we study the autonomous limit problem associated. It allows us to show the modified problem has the multiple solutions. Finally, the proof of Theorem 1.1 is derived in Section 5.

### Notation

- $C, C_1, C_2, \ldots$ denote any positive constants, whose exact values are not relevant;
- $B_R(y)$ denotes the open disk centered at $y \in \mathbb{R}^3$ with radius $R > 0$ and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^3$;
- $\| \cdot \|, \| \cdot \|_q$, and $\| \cdot \|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^3, \mathbb{R})$, $L^6(\mathbb{R}^3, \mathbb{R})$, and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^3$, $(\cdot, \cdot)_0$ denotes the inner product of the space $H^1(\mathbb{R}^3, \mathbb{R})$.

## 2 Abstract setting and preliminary results

For $u : \mathbb{R}^3 \to \mathbb{C}$, let us denote by

$$\nabla_A u := (\nabla - A)u,$$

and

$$D_A^1(\mathbb{R}^3, \mathbb{C}) := \{ u \in L^6(\mathbb{R}^3, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^3, \mathbb{R}) \},$$

$$H_A^1(\mathbb{R}^3, \mathbb{C}) := \{ u \in D_A^1(\mathbb{R}^3, \mathbb{C}) : u \in L^2(\mathbb{R}^3, \mathbb{C}) \}.$$  

The space $H_A^1(\mathbb{R}^3, \mathbb{C})$ is an Hilbert space endowed with the scalar product

$$\langle u, v \rangle := \text{Re} \int_{\mathbb{R}^3} (\nabla_A u \nabla_A v + u \overline{v}) \, dx,$$

for any $u, v \in H_A^1(\mathbb{R}^3, \mathbb{C}),$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover we denote by $\| u \|_A$ the norm induced by this inner product.

On $H_A^1(\mathbb{R}^3, \mathbb{C})$, an important tool is the following diamagnetic inequality (see e.g. [28, Theorem 7.21])

$$|\nabla_A u(x)| \geq |\nabla| u(x)|. \tag{2.1}$$

Now, by a simple change of variables, we can see that (1.1) is equivalent to

$$\left( \frac{1}{i} \nabla - A_{\varepsilon}(x) \right)^2 u + V_{\varepsilon}(x)u + (|x|^{-1} \ast |u|^2)u = f(|u|^2)u \quad \text{in} \ \mathbb{R}^3, \tag{2.2}$$

where $A_{\varepsilon}(x) = A(\varepsilon x)$ and $V_{\varepsilon}(x) = V(\varepsilon x)$.

Let $H_{\varepsilon}$ be the Hilbert space obtained as the closure of $C_0^\infty(\mathbb{R}^3, \mathbb{C})$ with respect to the scalar product

$$\langle u, v \rangle_{\varepsilon} := \text{Re} \int_{\mathbb{R}^3} (\nabla_A u \nabla_A v + V_{\varepsilon}(x)u \overline{v}) \, dx.$$
We obtain the following $t$-Riesz formula

$$\phi_{|u|}(x) = c \int_{\mathbb{R}^3} |x - y|^{-1} |u(y)|^2 dy.$$ 

Arguing as in [14, 32, 40], the function $\phi_{|u|}$ possesses the following properties.

**Lemma 2.1.** For any $u \in H_{\varepsilon}$, we have

(i) $\phi_{|u|} : H^1(\mathbb{R}^3, \mathbb{R}) \to D^{1,2}(\mathbb{R}^3, \mathbb{R})$ is continuous and maps bounded sets into bounded sets;

(ii) if $u_n \rightharpoonup u$ in $H_{\varepsilon}$, then $\phi_{|u_n|} \rightharpoonup \phi_{|u|}$ in $D^{1,2}(\mathbb{R}^3, \mathbb{R})$, and

$$\liminf_{n} \int_{\mathbb{R}^3} \phi_{|u_n|} |u_n|^2 dx \leq \int_{\mathbb{R}^3} \phi_{|u|} |u|^2 dx;$$

(iii) $\phi_{|u|} = r^2 \phi_{|u|}$ for all $r \in \mathbb{R}$ and $\phi_{|u(z+y)|} = \phi_{|u|}(x+y)$;

(iv) $\phi_{|u|} \equiv 0$ for all $u \in H_{\varepsilon}$ and we have

$$\|\phi_{|u|}\|_{D^{1,2}} \leq C \|u\|_{L^6(\mathbb{R}^3)}^2 \leq C \|u\|_{C}^2, \text{ and } \int_{\mathbb{R}^3} \phi_{|u|} |u|^2 dx \leq C \|u\|_{L^6(\mathbb{R}^3)}^4 \leq C \|u\|_{C}^8.$$

For compact supported functions in $H^1(\mathbb{R}^3, \mathbb{R})$, the following result will be very useful for some estimates below.

**Lemma 2.2.** If $u \in H^1(\mathbb{R}^3, \mathbb{R})$ and $u$ has compact support, then $\omega := e^{iA(0) - x} u \in H_{\varepsilon}$.

**Proof.** Assume that $\text{supp}(u) \subset B_R(0)$. Since $V$ is continuous, it is clear that

$$\int_{B_R(0)} V_c(x)|\omega|^2 dx = \int_{B_R(0)} V_c(x)|\omega|^2 dx \leq C \|u\|_{C}^2 < +\infty.$$

Moreover, since $V$ and $A$ are continuous, we have

$$\int_{\mathbb{R}^3} |\nabla A_c| \omega|^2 dx = \int_{\mathbb{R}^3} |\nabla \omega|^2 dx + \int_{\mathbb{R}^3} |A_c(x)|^2 |\omega|^2 dx + 2 \text{Re} \int_{\mathbb{R}^3} iA_c(x) \omega \nabla \omega dx$$

$$\leq 2 \int_{\mathbb{R}^3} |\nabla \omega|^2 dx + 2 \int_{\mathbb{R}^3} |A_c(x)|^2 |\omega|^2 dx$$

$$\leq C \left[ \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |u|^2 dx \right] < +\infty$$

and we conclude. \qed

### 3 The modified problem

To study (1.1), we modify suitably the nonlinearity $f$ so that, for $\varepsilon > 0$ small enough, the solutions of such modified problem are also solutions of the original one. More precisely, we choose $K > 2$. By (f4) there exists
a unique number \( a > 0 \) verifying \( f(a) + a^2 = V_0/K \), where \( V_0 \) is given in (V1). Hence we consider the function
\[
\tilde{f}(t) := \begin{cases} 
  f(t) + (t^*)^2, & t \leq a, \\
  V_0/K, & t > a.
\end{cases}
\]

Now we introduce the penalized nonlinearity \( g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \)
\[
g(x, t) := \chi_A(x)f(t) + (t^*)^2 + (1 - \chi_A(x))\tilde{f}(t),
\]
where \( \chi_A \) is the characteristic function on \( A \) and \( G(x, t) := \int_0^t g(x, s)ds \).

From (f1)–(f4), \( g \) is a Carathéodory function satisfying the following properties:

(g1) \( g(x, t) = 0 \) for each \( t \leq 0 \);
(g2) \( \lim_{t \to 0} \frac{g(x, t)}{t^*} = 0 \) uniformly in \( x \in \mathbb{R}^3 \);
(g3) \( g(x, t) \leq f(t) + t^2 \) for all \( t \geq 0 \) and uniformly in \( x \in \mathbb{R}^3 \);
(g4) \( 0 < \theta G(x, t) \leq 2g(x, t)t \), for each \( x \in A, t > 0 \);
(g5) \( 0 < G(x, t) \leq g(x, t)t \leq V_0 t/K \), for each \( x \in A', t > 0 \);
(g6) for each \( x \in A \), the function \( t \mapsto \frac{g(x, t)}{t} \) is strictly increasing in \( t \in (0, +\infty) \) and for each \( x \in A' \), the function \( t \mapsto \frac{g(x, t)}{t} \) is strictly increasing in \( (0, a) \).

Then we consider the modified problem
\[
\left( \frac{1}{t} \nabla - A_c(x) \right)^2 u + V_c(x)u + (|x|^{-1} \ast |u|^2)u = g(ex, |u|^2)u \quad \text{in} \ \mathbb{R}^3.
\]

Note that, if \( u \) is a solution of problem (3.2) with
\[
|u(x)|^2 \leq a \quad \text{for all} \ x \in A_c, \quad A_c := \{ x \in \mathbb{R}^3 : ex \in A \},
\]
then \( u \) is a solution of problem (2.2).

We observe that the weak solutions of the modified problem (3.2) can be found as the critical points of the \( C^1 \) functional
\[
J_c(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V_c(x)|u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (|x|^{-1} \ast |u|^2)|u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} G(ex, |u|^2) \, dx
\]
defined in \( H_c \). Moreover, we denote by \( N_c \) the Nehari manifold of \( J_c \),
\[
N_c := \{ u \in H_c \setminus \{0\} : J'_c(u)[u] = 0 \},
\]
and define the number \( c_e \) by
\[
c_e = \inf_{u \in N_c} J_c(u).
\]

Let \( H^*_c \) be open subset \( H_c \) given by
\[
H^*_c = \{ u \in H_c : ||u||_{H^1} > 0 \},
\]
and \( S^*_c = S_c \cap H^*_c \), where \( S_c \) is the unit sphere of \( H_c \). Note that \( S^*_c \) is a non-complete \( C^{1,1} \)-manifold of codimension 1, modeled on \( H_c \) and contained in \( H^*_c \). Therefore, \( H_c = T_u S^*_c \oplus \mathbb{R} u \) for each \( u \in T_u S^*_c \), where \( T_u S^*_c = \{ v \in H_c : \langle u, v \rangle_c = 0 \} \).

Arguing as in [29, Lemma 3.1], we can show that the functional \( J_c \) satisfies the Mountain Pass Geometry [37].

**Lemma 3.1.** For any fixed \( e > 0 \), the functional \( J_c \) satisfies the following properties:

(i) there exist \( \beta, r > 0 \) such that \( J_c(u) \geq \beta \) if \( ||u||_c = r \);
(ii) there exists \( e \in H_c \) with \( ||e||_c > r \) such that \( J_c(e) < 0 \).
Due to $f$ is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $N_\varepsilon$ and the incompleteness of $S^*\varepsilon$.

**Lemma 3.2.** Assume that (V1)–(V2) and (f1)–(f4) are satisfied, then the following properties hold:

(A1) For any $u \in H^s_\varepsilon$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = J_\varepsilon(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in $(t_u, \infty)$;

(A2) There is $\tau > 0$ independent on $u$ such that $t_u \geq \tau$ for all $u \in S^*\varepsilon$. Moreover, for each compact $\mathcal{W} \subset S^*\varepsilon$ there is $C_{\mathcal{W}} > 0$ such that $t_u \leq C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;

(A3) The map $\tilde{m}_\varepsilon : H^1_\varepsilon \to N_\varepsilon$ given by $\tilde{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon = \tilde{m}_\varepsilon|_{S^\varepsilon}$ is a homeomorphism between $S^\varepsilon$ and $N_\varepsilon$. Moreover, $m_\varepsilon^{-1}(u) = \frac{u}{||u||^2\varepsilon}$;

(A4) If there is a sequence $\{u_n\} \subset S^\varepsilon$ such that $\text{dist}(u_n, \partial S^\varepsilon) \to 0$, then $\lim_{n \to \infty} m_\varepsilon(u_n) = \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

**Proof.** (A1) Arguing as in [29, Lemma 3.1], it follows that $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Thus, $\max_{t \geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u$ satisfying $g_u'(t_u) = 0$ and $t_u u \in N_\varepsilon$. Now, we show that $t_u$ is unique. Arguing by contradiction, suppose that there exist $t_1 > t_2 > 0$ such that $g_u'(t_1) = g_u'(t_2) = 0$. Then, for $i = 1, 2$,

$$t_i ||u||^2_\varepsilon + \frac{t_i^2}{2} \int |(x^{-1} \ast |u|^2)|u|^2 dx = \int g(ex, t_i^2 |u|^2) t_i |u|^2 dx. $$

Hence,

$$\frac{||u||^2_\varepsilon}{t_i^2} + \int |(x^{-1} \ast |u|^2)|u|^2 dx = \int \frac{g(ex, t_i^2 |u|^2)}{t_i^2} dx,$$

which implies that

$$\left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) \int \left(\frac{g(ex, t_1^2 |u|^2)}{t_1^2 |u|^2} - \frac{g(ex, t_2^2 |u|^2)}{t_2^2 |u|^2}\right) |u|^4 dx$$

$$ \geq \int \left(\frac{g(ex, t_1^2 |u|^2)}{t_1^2 |u|^2} - \frac{g(ex, t_2^2 |u|^2)}{t_2^2 |u|^2}\right) |u|^4 dx$$

$$+ \int \left(\frac{g(ex, t_1^2 |u|^2)}{t_1^2 |u|^2} - \frac{g(ex, t_2^2 |u|^2)}{t_2^2 |u|^2}\right) |u|^4 dx$$

$$= \int \left(\frac{V_0}{K} \frac{1}{t_1^2 |u|^2} - \frac{f(t_1^2 |u|^2) + t_1^4 |u|^6}{t_1^2 |u|^2}\right) |u|^4 dx$$

$$+ \frac{1}{K} \left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) \int V_0 |u|^2 dx.$$

Since $t_1 > t_2 > 0$, we have

$$\frac{1}{K} \int \frac{V_0}{K} \frac{1}{t_1^2 |u|^2} - \frac{f(t_1^2 |u|^2) + t_1^4 |u|^6}{t_1^2 |u|^2} |u|^4 dx$$

$$+ \frac{1}{K} \int V_0 |u|^2 dx$$

$$\leq \frac{1}{K} \int V_0 |u|^2 dx \leq \frac{1}{K} ||u||^2_\varepsilon,$$

which is a contradiction. Therefore, $\max_{t \geq 0} g_u(t)$ is achieved at a unique $t = t_u$ so that $g_u'(t) = 0$ and $t_u u \in N_\varepsilon$. 


(A2) For $\forall u \in S^+_c$, it follows that

$$t_u + t_u^2 \int_{\mathbb{R}^3} (|x|^{-1} \ast |u|^2)|u|^2 \, dx = \int_{\mathbb{R}^3} g(ex, t_u^2 |u|^2) t_u |u|^2 \, dx.$$ 

From (g2), the Sobolev embeddings and $4 < q < 6$, it is easy to obtain

$$t_u \leq \chi t_u^2 \int_{\mathbb{R}^3} |u|^4 \, dx + C \varepsilon t_u^{-1} \int_{\mathbb{R}^3} |u|^{8} \, dx + t_u^{5} \int_{\mathbb{R}^3} |u|^{6} \, dx \leq C_1 \varepsilon t_u^{3} + C_2 \varepsilon t_u^{7} + C_3 \varepsilon^5,$$

which implies $t_u \geq \tau$ for some $\tau > 0$. If $W \subset S^+_c$ is compact, and suppose by contradiction that there is $\{u_n\} \subset W$ with $t_n := t_{u_n} \to \infty$. Since $W$ is compact, there exists $u \in W$ such that $u_n \to u$ in $H_c$. Using the proof of [29, Lemma 3.1(ii)], it follows that $J_c(t_n u_n) \to -\infty$.

On the other hand, let $v_n := t_n u_n \in N_c$, from (g4), (g5), (g6) and $\theta > 4$, it yields that

$$J_c(v_n) - \frac{1}{\theta} f'_c(v_n)v_n \geq \frac{1}{2} \frac{1}{\theta^2} \|v_n\|_e^2 + \frac{1}{\theta} \int_{\mathbb{R}^3} (|x|^{-1} \ast |v_n|^2)|v_n|^2 \, dx$$

$$+ \frac{1}{\theta} \int_{\mathbb{R}^3} g(ex, |v_n|^2)|v_n|^2 - \frac{1}{2} G(ex, |v_n|^2) \, dx$$

$$\geq \frac{1}{2} \frac{1}{\theta} (\|v_n\|_e^2 - \frac{1}{\theta} \int_{\mathbb{R}^3} V(x)|v_n|^2 \, dx)$$

$$\geq \frac{1}{2} \frac{1}{\theta} (1 - \frac{1}{\theta}) \|v_n\|_e^2.$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_e = t_n$, we may obtain

$$0 \leq \frac{1}{\theta} \frac{1}{\theta} (1 - \frac{1}{\theta}) \leq \frac{J_c(v_n)}{t_n^2} \leq 0$$

as $n \to \infty$, which yields a contradiction. This completes the proof of (A2).

(A3) We first show that $\tilde{m}_c, m_c$ and $m_c^{-1}$ are well defined. Indeed, by (A2), for each $u \in H^+_c$, there is a unique $\tilde{m}_c(u) \in N_c$. On the other hand, if $u \in N_c$, then $u \in H^+_c$. Otherwise, we have $\|\text{supp}(u) \cap \Lambda_c\| = 0$ and by (g5) it follows that

$$\|u\|_e^2 \leq \|u\|_e^2 + \int_{\mathbb{R}^3} (|x|^{-1} \ast |u|^2)|u|^2 \, dx = \int_{\mathbb{R}^3} g(ex, |u|^2)|u|^2 \, dx$$

$$= \int_{\mathbb{R}^3} g(ex, |u|^2)|u|^2 \, dx$$

$$\leq \frac{1}{K} \int_{\mathbb{R}^3} V(x)|u|^2 \, dx$$

$$\leq \frac{1}{K} \|u\|_e^2$$

which is impossible since $K > 2$ and $u \neq 0$. Thus, $m_c^{-1}(u) = \frac{u}{\|u\|_e} \in S^+_c$ is well defined and continuous. From

$$m_c^{-1}(m_c(u)) = m_c^{-1}(t_u u) = \frac{t_u u}{t_u \|u\|_e} = u, \quad \forall u \in S^+_c,$$

we know that $m_c$ is a bijection. Now we prove $\tilde{m}_c : H^+_c \to N_c$ is continuous. Let $\{u_n\} \subset H^+_c$ and $u \in H^+_c$ such that $u_n \to u$ in $H_c$. By (A2), there is a $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in N_c$, i.e.,

$$t_n^2 \|u_n\|_e^2 + t_n^2 \int_{\mathbb{R}^3} (|x|^{-1} \ast |u_n|^2)|u_n|^2 \, dx = \int_{\mathbb{R}^3} g(ex, t_n^2 |u_n|^2) t_n^2 |u_n|^2 \, dx, \quad \forall n \in N,$$
and passing to the limit as $n \to \infty$ in the last inequality, it follows that
\[
t_0^2\|u\|_{L^2_c}^2 + t_0^2 \int \left((|x|^{-1} \ast |u|^2)|u|^2\right)dx = \int g(ex, t_0^2|u|^2)dx,
\]
which implies that $t_0u \in N_c$ and $t_u = t_0$. This proves $\widehat{m}_e(u_n) \to \widehat{m}_e(u)$ in $H^*_c$. Thus, $\widehat{m}_e$ and $m_e$ are continuous and (A3) is proved.

(A4) Let $\{u_n\} \subset S^*_c$ be a subsequence such that $\text{dist}(u_n, \partial S^*_c) \to 0$, then for each $\nu \in S^*_c$ and $n \in N$, we have $|u_n| = |u_n - \nu|$ a.e. in $\Lambda_c$. Thus, by (V1), (V2) and the Sobolev embedding, for any $t \in [2, 6]$, there exists $C_t > 0$ such that
\[
\|u_n\|_{L^t(A_c)} \leq \inf_{\nu \in \partial S^*_c} \|u_n - \nu\|_{L^t(A_c)}.
\]
for all $n \in N$. From (g2), (g3) and (g5), for each $t > 0$, it follows that
\[
\int G(ex, t^2|u_n|^2)dx \leq \int \left(F(t^2|u_n|^2) + \frac{6^6|u_n|^6}{6}\right)dx + \frac{t^2}{K} \int \Lambda_c V(ex)|u_n|^2dx
\]
\[
\leq C_1t^{6^6} \int |u_n|^6dx + C_2t^4 \int |u_n|^6dx + \frac{t^2}{K} \int |u_n|^2dx + \frac{t^2}{K} \|u_n\|_{L^2_c}^2
\]
\[
\leq C_1t^{6^6} \text{dist}(u_n, \partial S^*_c)^6 + C_2t^4 \text{dist}(u_n, \partial S^*_c)^4 + C_3t^4 \text{dist}(u_n, \partial S^*_c)^4 + \frac{t^2}{K} \|u_n\|_{L^2_c}^2.
\]
Therefore,
\[
\limsup_{n} \int G(ex, t^2|u_n|^2)dx \leq \frac{t^2}{K}, \quad \forall t > 0.
\]

On the other hand, from the definition of $m_e$ and the last inequality, for all $t > 0$, we have
\[
\liminf_n J_e(m_e(u_n)) \geq \liminf_n J_e(tu_n)
\]
\[
\geq \liminf_n \frac{t^2}{2} \|u_n\|_{L^2_c}^2 - \frac{t^2}{K}
\]
\[
= \frac{K - 2}{2K} t^2
\]
which implies that
\[
\liminf_n \left\{ \frac{1}{2} \|m_e(u_n)\|_{L^2_c}^2 + \frac{1}{4} \int \left(|x|^{-1} \ast |m_e(u_n)|^2\right)|m_e(u_n)|^2dx \right\} \geq \liminf_n J_e(m_e(u_n)) \geq \frac{K - 2}{2K} t^2, \quad \forall t > 0.
\]
Since $t > 0$ is arbitrary, we can show that $\|m_e(u_n)\|_{L^2_c} \to \infty$ and $J_e(m_e(u_n)) \to \infty$ as $n \to \infty$.

At this point we define the function
\[
\tilde{\Psi}_e : H^*_c \to \mathbb{R},
\]
by $\tilde{\Psi}_e(u) = J_e(\widehat{m}_e(u))$ and denote by $\Psi_e := (\tilde{\Psi}_e)|_{S^*_c}$.

From Lemma 3.2, arguing as in [35, Corollary 10] we may obtain the following lemma.

**Lemma 3.3.** Assume that (V1)–(V2) and (f1)–(f4) hold, then
As in [35], we have the variational characterization of the infimum of $J_e$:

$$\inf_{S_e} \Psi_e = \inf_{N_e} J_e.$$  

Proof. Assume that \( \{u_n\} \subseteq H_e \) is a bounded (PS)$_c$ sequence of \( J_e \). If \( \{u_n\} \subseteq N_e \) is a bounded (PS)$_c$ sequence of \( J_e \), then \( \{\tilde{m}_c(u_n)\} \) is a (PS)$_c$ sequence of \( \Psi_e \); \( \{u_n\} \) is a critical point of \( \Psi_e \) if and only if \( m_c(u) \) is a critical point of \( J_e \). Moreover, the corresponding critical values coincide and

$$\inf_{S_e} \Psi_e = \inf_{N_e} J_e.$$ 

As in [35], we have the variational characterization of the infimum of \( J_e \) over \( N_e \):

$$c_e = \inf_{u \in N_e} J_e(u) = \inf_{u \in H_e} \sup_{t>0} J_e(tu) = \inf_{u \in S_e} \sup_{t>0} J_e(tu). \quad (3.3)$$

Lemma 3.4. Let \( \{u_n\} \) be a (PS)$_c$ sequence for \( J_e \), where \( \epsilon > 0 \), then \( \{u_n\} \) is bounded in \( H_e \).

Proof. Assume that \( \{u_n\} \subseteq H_e \) is a (PS)$_c$ sequence for \( J_e \), that is, \( J_e(u_n) \to \epsilon > 0 \) and \( J'_e(u_n) \to 0 \). From (g4), (g5) and \( \epsilon < \theta < 6 \), it follows that

$$d + o_n(1) + o_n(1)\|u_n\| \geq J_e(u_n) - \frac{1}{\theta} J'_e(u_n)[u_n]$$

$$= \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^3} |(x|^{-1} * |u_n|^2)|u_n|^2 \, dx$$

$$+ \int_{\mathbb{R}^3} \frac{1}{\theta} g(ex, |u_n|^2)|u_n|^2 - \frac{1}{2} G(ex, |u_n|^2) \, dx$$

$$\geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^3} G(ex, |u_n|^2) \, dx$$

$$\geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^3} V(ex)|u_n|^2 \, dx$$

Since \( \theta > 2 \), from the above inequalities we know that \( \{u_n\} \) is bounded in \( H_e \).

The following lemma provides a range of levels in which the functional \( J_e \) verifies the Palais-Smale condition.

Lemma 3.6. The functional \( J_e \) satisfies the (PS)$_c$ condition at any level \( c \in (0, \frac{1}{4} S^2) \), where \( S \) is the best constant for the Sobolev inequality

$$S \left( \int_{\mathbb{R}^3} |v|^6 \, dx \right)^{1/3} \leq \int_{\mathbb{R}^3} (|\nabla v|^2 + |v|^2) \, dx, \quad \text{for} \ v \in H^1(\mathbb{R}^3, \mathbb{R}).$$

Proof. Let \( \{u_n\} \subseteq H_e \) be a (PS)$_c$ sequence for \( J_e \). By Lemma 3.4, \( \{u_n\} \) is bounded in \( H_e \). Thus, up to a subsequence, \( u_n \rightharpoonup u \) in \( H_e \) and \( u_n \to u \) in \( L^{ap}(\mathbb{R}^3, C) \) for all \( 1 \leq r < 6 \) as \( n \to +\infty \).

Step 1: We show that for any given \( \xi > 0 \), for \( R \) large enough,

$$\limsup_{n} \int_{B_R(0)} (|\nabla A_n|^2 + V(e(x)|u_n|^2) \, dx \leq \xi. \quad (3.4)$$
Let $R > 0$ such that $A_e \subset B_{R/2}(0)$ and let $\phi_R \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \quad x \in B_{R/2}(0), \quad \phi_R = 1 \quad x \in B_R(0), \quad 0 \leq \phi_R \leq 1, \quad \text{and} \quad |\nabla \phi_R| \leq C/R$$

where $C > 0$ is a constant independent of $R$. Since the sequence $(\phi_R u_n)_n$ is bounded in $H_e$, we have

$$J'_e(u_n)[\phi_R u_n] = o_n(1),$$

that is

$$\text{Re} \int_{\mathbb{R}^3} \nabla A_n u_n \overline{\nabla A_n (\phi_R u_n)} dx + \int_{\mathbb{R}^3} V_e(x)|u_n|^2 \phi_R dx + \int_{\mathbb{R}^3} (|x|^{-1} * |u_n|^2)|u_n|^2 \phi_R dx$$

$$= \int_{\mathbb{R}^3} g(ex, |u_n|^2)|u_n|^2 \phi_R dx + o_n(1).$$

Since $\nabla A_n (u_n \phi_R) = \overline{u_n} \nabla \phi_R + \phi_R \overline{\nabla A_n u_n}$, using (5), we have

$$\int_{\mathbb{R}^3} (|\nabla A_n u_n|^2 + V_e(x)|u_n|^2) \phi_R dx \leq \int_{\mathbb{R}^3} g(ex, |u_n|^2)|u_n|^2 \phi_R dx - \text{Re} \int_{\mathbb{R}^3} \overline{u_n} \nabla A_n u_n \nabla \phi_R dx + o_n(1)$$

$$\leq \frac{1}{R} \int_{\mathbb{R}^3} V_e(x)|u_n|^2 \phi_R dx - \text{Re} \int_{\mathbb{R}^3} \overline{u_n} \nabla A_n u_n \nabla \phi_R dx + o_n(1).$$

By the definition of $\phi_R$, the Hölder inequality and the boundedness of $(u_n)_n$ in $H_e$, we obtain

$$\left(1 - \frac{1}{R}\right) \int_{\mathbb{R}^3} (|\nabla A_n u_n|^2 + V_e(x)|u_n|^2) \phi_R dx \leq \frac{C}{R} \|u_n\|_2 \|\nabla A_n u_n\|_2 + o_n(1) \leq \frac{C_1}{R} + o_n(1)$$

and so (3.4) holds.

From the Sobolev embedding and (3.4), we have that for any $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that,

$$\|u_n - u\|_{L(\mathbb{R}^3)} \leq \|u_n - u\|_{L(B_R(0))} + \|u_n - u\|_{L(B_R(0))}$$

$$\leq \zeta + C\left(\|u_n\|_{H_0(B_R(0))} + \|u\|_{H_0(B_R(0))}\right)$$

$$\leq C_1 \zeta$$

where $r \in [2, 6)$ and $n$ large enough. From this, we can obtain that

$$u_n \to u \quad \text{in} \quad L'(\mathbb{R}^3, \mathbb{C}), \quad \text{for any} \quad r \in [2, 6). \quad (3.5)$$

By (2), since $\phi|_u| : L^{12/5}(\mathbb{R}^3, \mathbb{R}) \to D^{1,2}(\mathbb{R}^3, \mathbb{R})$ is continuous, from (3.5) we can get

$$\phi|_u| \to \phi|_u| \quad \text{in} \quad D^{1,2}(\mathbb{R}^3, \mathbb{R}), \quad (3.6)$$

$$\int_{\mathbb{R}^3} \phi|_u| |u_n|^2 dx \to \int_{\mathbb{R}^3} \phi|_u| |u|^2 dx. \quad (3.7)$$

Using the boundedness of sequence $(u_n)_n$ and the Sobolev embedding again, for any $\varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$, we have

$$\text{Re} \int_{\mathbb{R}^3} \left(\nabla A_n u_n \overline{\nabla A_n \varphi} dx + V_e(x)u_n \overline{\varphi} dx \to \text{Re} \int_{\mathbb{R}^3} \left(\nabla A_n u \overline{\nabla A_n \varphi} + V_e(x)u \overline{\varphi} \right) dx, \quad (3.8)$$

$$\text{Re} \int_{\mathbb{R}^3} g(ex, |u_n|^2)u_n \overline{\varphi} dx \to \text{Re} \int_{\mathbb{R}^3} g(ex, |u|^2)u \overline{\varphi} dx. \quad (3.9)$$
By (3.7)-(3.9), the Hölder inequality and the Sobolev embeddings, we have
\[
\text{Re} \int_{\mathbb{R}^3} \phi |w_n|^2 \varphi dx - \text{Re} \int_{\mathbb{R}^3} \phi |u|^2 \varphi dx = \text{Re} \int_{\mathbb{R}^3} (\phi |w_n| - \phi |u|) \varphi dx
\]
\[
= \text{Re} \int_{\mathbb{R}^3} (\phi |u_n| - \phi |u|) \varphi dx + \text{Re} \int_{\mathbb{R}^3} (\phi |u_n| - \phi |u|) \varphi dx
\]
\[
\leq C \|\nabla \phi |w_n|\|_{L^{1,2}(\mathbb{R}^3, \mathbb{R})} \|u_n - u\|_{L^{1,2}(\mathbb{R}^3, \mathbb{R})} \|\varphi\|_{L^{1,2}(\mathbb{R}^3, \mathbb{R})} + C \|\nabla (\phi |u_n| - \phi |u|)\|_{L^{1,2}(\mathbb{R}^3, \mathbb{R})} \|u\|_{L^{1,2}(\mathbb{R}^3, \mathbb{R})} \|\varphi\|_{L^{1,2}(\mathbb{R}^3, \mathbb{R})}
\]
\[
\rightarrow 0, \quad \text{as } n \to \infty. \quad (3.10)
\]

By (3.8)-(3.10) and \( f'(u_n) \to 0 \), we have \( f'(u) = 0 \) and
\[
\|u\|_{L^6}^2 + \int (|x|^{-1} \ast |u|^2)|u|^2 dx = \int g(ex, |u|^2)|u|^2 dx.
\]

Step 2:
\[
\lim_n \int_{\mathbb{R}^3} g(ex, |u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} g(ex, |u|^2)|u|^2 dx. \quad (3.11)
\]

Using \( u_n \to u \) in \( L^r_{\text{loc}}(\mathbb{R}^3, \mathbb{C}) \), for all \( 1 \leq r < 6 \) again, up to a subsequence, we have that
\[
|u_n| \to |u| \text{ a.e. in } \mathbb{R}^3 \text{ as } n \to +\infty,
\]
then
\[
g(ex, |u_n|^2)|u_n|^2 \to g(ex, |u|^2)|u|^2 \text{ a.e. in } \mathbb{R}^3 \text{ as } n \to +\infty.
\]

By (g5) and (3.4), for any \( \zeta > 0 \), there exists \( R > 0 \) large enough, we have
\[
\int_{B_R^c(0)} \left( g(ex, |u_n|^2)|u_n|^2 - g(ex, |u|^2)|u|^2 \right) dx \leq \frac{2}{R} \int_{B_R^c(0)} (|\nabla A_n|^2 + V(ex)|u_n|^2) dx < \frac{2\zeta}{R}.
\]
Thus,
\[
\lim_n \int_{B_R^c(0)} g(ex, |u_n|^2)|u_n|^2 dx = \int_{B_R^c(0)} g(ex, |u|^2)|u|^2 dx.
\]

Now, we show that
\[
\lim_n \int_{B_R(0)} g(ex, |u_n|^2)|u_n|^2 = \int_{B_R(0)} g(ex, |u|^2)|u|^2 dx.
\]
From the definition of \( g \), we have that
\[
g(ex, |u_n|^2)|u_n|^2 \leq f(|u_n|^2)|u_n|^2 + |u_n|^6 + \frac{V_0}{R}|u_n|^2, \quad \text{for any } x \in \mathbb{R}^3 \setminus \Lambda_e. \quad (3.12)
\]
Since \( B_R(0) \cap (\mathbb{R}^3 \setminus \Lambda_e) \) is bounded, from the above estimate, \((f1), (f2)\), the Sobolev embedding and the Lebesgue Dominated Convergence Theorem, we can infer
\[
\lim_n \int_{B_R(0) \cap (\mathbb{R}^3 \setminus \Lambda_e)} g(ex, |u_n|^2)|u_n|^2 = \int_{B_R(0) \cap (\mathbb{R}^3 \setminus \Lambda_e)} g(ex, |u|^2)|u|^2 dx. \quad (3.13)
\]
If we can prove that
\[
\lim_{n} \int_{B_{R}(0) \setminus \Lambda_{e}} g(\varepsilon x, |u_{n}|^{2}) u_{n}^{2} = \int_{B_{R}(0)} g(\varepsilon x, |u|^{2}) |u|^{2} \, dx,
\]
from (3.13) and (3.14), it yields (3.11). Now, in order to show that (3.14) holds, we only need to prove the following limit holds
\[
\lim_{n} \int_{\Lambda_{e}} |u_{n}|^{6} \, dx = \int_{\Lambda_{e}} |u|^{6} \, dx.
\]
Using the boundedness of \((u_{n})_{n}\) in \(H_{e}\) and the diamagnetic inequality (2.1), we may assume that
\[
|\nabla |u||^{2} \rightarrow \mu \quad \text{and} \quad |u_{n}|^{6} \rightarrow \nu
\]
in the sense of measures. Moreover, by the diamagnetic inequality (2.1) and (3.4), \((u_{n})_{n}\) is a tight sequence in \(H^{1}(\mathbb{R}^{3}, \mathbb{R})\), thus, using the concentration-compactness principle in [37], we can find an at most countable index \(I\), sequences \((x_{i}) \subset \mathbb{R}^{3}, (\mu_{i}), (v_{i}) \subset (0, \infty)\) such that
\[
\mu \geq |\nabla |u||^{2} dx + \sum_{i \in I} \mu_{i} \delta_{x_{i}},
\]
\[
v = |u|^{6} + \sum_{i \in I} v_{i} \delta_{x_{i}} \quad \text{and} \quad SUV_{1}^{1/3} \leq \mu_{i}
\]
for any \(i \in I\), where \(\delta_{x_{i}}\) is the Dirac mass at the point \(x_{i}\). Let us show that \((x_{i})_{i \in I} \cap \Lambda_{e} = \emptyset\). Assume, by contradiction, that \(x_{i} \in \Lambda_{e}\) for some \(i \in I\). For any \(\rho > 0\), we define \(\psi_{\rho}(x) = \psi(\frac{x-x_{i}}{\rho})\) where \(\psi \in C_{c}^{\infty}(\mathbb{R}^{3}, [0, 1])\) such that \(\psi = 1\) in \(B_{1}, \psi = 0\) in \(\mathbb{R}^{3} \setminus B_{2}\) and \(|\nabla \psi|_{L^{\infty}(\mathbb{R}^{3}, \mathbb{R})} \leq 2\). We suppose that \(\rho > 0\) such that \(\text{supp}(\psi_{\rho}) \subset \Lambda_{e}\).

Since \((\psi_{\rho} u_{n})\) is bounded in \(H_{e}\), we can see that \(J_{e}(\psi_{\rho} u_{n}) = o_{n}(1)\), that is
\[
\text{Re} \int_{\mathbb{R}^{3}} \nabla A_{e} u_{n} \nabla (\psi_{\rho} u_{n}) \, dx + \int_{\mathbb{R}^{3}} V_{\varepsilon}(x)|u_{n}|^{2} \psi_{\rho} \, dx + \int_{\mathbb{R}^{3}} (|x|^{-1} * |u_{n}|^{2})|u_{n}|^{2} \psi_{\rho} \, dx
\]
\[
= \int_{\mathbb{R}^{3}} g(\varepsilon x, |u_{n}|^{2}) |u_{n}|^{2} \psi_{\rho} \, dx + o_{n}(1) = \int_{\mathbb{R}^{3}} f(|u_{n}|^{2}) |u_{n}|^{2} \psi_{\rho} \, dx + \int_{\mathbb{R}^{3}} |u_{n}|^{6} \psi_{\rho} \, dx + o_{n}(1).
\]
Since \(\nabla A_{e} (u_{n} \psi_{\rho}) = i u_{n} \nabla \psi_{\rho} + \psi_{\rho} \nabla A_{e} u_{n}\), using (g5), we have
\[
\int_{\mathbb{R}^{3}} |\nabla A_{e} u_{n}|^{2} \psi_{\rho} \, dx \leq \int_{\mathbb{R}^{3}} f(|u_{n}|^{2}) |u_{n}|^{2} \psi_{\rho} \, dx + \int_{\mathbb{R}^{3}} |u_{n}|^{6} \psi_{\rho} \, dx - \text{Re} \int_{\mathbb{R}^{3}} i u_{n} \nabla A_{e} u_{n} \nabla \psi_{\rho} \, dx + o_{n}(1).
\]
Using the diamagnetic inequality (2.1) again, it follows that
\[
\int_{\mathbb{R}^{3}} |\nabla |u||^{2} \psi_{\rho} \, dx \leq \int_{\mathbb{R}^{3}} f(|u_{n}|^{2}) |u_{n}|^{2} \psi_{\rho} \, dx + \int_{\mathbb{R}^{3}} |u_{n}|^{6} \psi_{\rho} \, dx - \text{Re} \int_{\mathbb{R}^{3}} i u_{n} \nabla A_{e} u_{n} \nabla \psi_{\rho} \, dx + o_{n}(1).
\]
Due to the fact that \(f\) has the subcritical growth and \(\psi_{\rho}\) has the compact support, we have that
\[
\lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{3}} f(|u_{n}|^{2}) |u_{n}|^{2} \psi_{\rho} \, dx = \lim_{\rho \rightarrow 0} \int_{\mathbb{R}^{3}} f(|u|^{2}) |u|^{2} \psi_{\rho} \, dx = 0.
\]
It’s also easy to show that
\[
\lim_{\rho \rightarrow 0} \sup_{n} \int_{\mathbb{R}^{3}} \text{Re} \int_{\mathbb{R}^{3}} i u_{n} \nabla A_{e} u_{n} \nabla \psi_{\rho} \, dx = 0.
\]
Then, taking into account (3.16), (3.18), (3.19) and (3.20), we can conclude that \(v_{i} \geq \mu_{i}\). Together with the inequality \(S_{V_{1}}^{1/3} \leq \mu_{i}\) in (3.17), we have \(v_{i} \geq S_{V_{1}}^{3/2}\). Now, from (f3), (g4) and (g5), we have
\[
c = J_{e}(u_{n}) - \frac{1}{4} J_{e}'(u_{n})[u_{n}] + o_{n}(1)
\]
\[
\begin{align*}
\frac{1}{4} \|u_n\|^2 + \int_{\mathbb{R}^3} \left( \frac{1}{4} g(e_x, |u_n|^2) |u_n|^2 - \frac{1}{2} G(e_x, |u_n|^2) \right) dx + o_n(1) \\
\geq \frac{1}{4} \|u_n\|^2 + \int_{\mathbb{R}^3} \left( \frac{1}{4} g(e_x, |u_n|^2) |u_n|^2 - \frac{1}{2} G(e_x, |u_n|^2) \right) dx \\
+ \frac{1}{12} \int_{A_c} |u_n|^6 dx + o_n(1)
\end{align*}
\]

From the above arguments and (3.17), we have

\[
\begin{align*}
c \geq \frac{1}{4} \sum_{i \in I_{\mathbb{R}^3}} \psi_{\rho}(x_i) \mu_i + \frac{1}{12} \sum_{i \in I_{\mathbb{R}^3}} \psi_{\rho}(x_i) \nu_i \\
\geq \frac{1}{4} \mu_i + \frac{1}{12} \nu_i \\
\geq \frac{1}{3} S^{3/2}
\end{align*}
\]

which gives a contradiction. This means that (3.15) holds.

**Step 3:** From \( J'_c(u_n) |u_n| \to 0 \), \( J'_c(u) = 0 \), (3.7) and (3.15), we have

\[
\lim_{n} \|u_n\|^2 = \|u\|^2,
\]

and the proof is completed.

Since \( f \) is only assumed to be continuous, the following result is required for multiplicity result in the next section.

**Corollary 3.1.** The functional \( \Psi_c \) satisfies the \((PS)_c\) condition on \( S^*_c \) at any level \( c \in (0, \frac{1}{2} S^3) \).

**Proof.** Let \( \{u_n\} \subset S^*_c \) be a \((PS)_c\) sequence for \( \Psi_c \) where \( c \in (0, \frac{1}{2} S^3) \). Then \( \Psi_c(u_n) \to c \) and \( \|\Psi'_c(u_n)\| \to 0 \), where \( \| \cdot \| \) is the norm in the dual space \((T_u S^*_c)^*\). By Lemma 3.3(B3), we know that \( \{m_c(u_n)\} \) is a \((PS)_c\) sequence for \( J_c \) in \( H_c \). From Lemma 3.5, we know that there exists a \( u \in S^*_c \) such that, up to a subsequence, \( m_c(u_n) \to m_c(u) \) in \( H_c \). By Lemma 3.2(A3), we obtain

\[
u_n \to u \text{ in } S^*_c,
\]

and the proof is completed. \(\square\)
4 Multiple solutions for the modified problem

4.1 The autonomous problem

Now, we study the following limit problem

\[
\begin{align*}
- \Delta u + V_0 u + (|x|^{-1} * |u|^2)u &= f(u^2)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\
- \Delta \phi &= u^2, & \text{in } \mathbb{R}^3, \\
u &\in H^1(\mathbb{R}^3, \mathbb{R}), \ u(x) > 0, & \text{in } \mathbb{R}^3.
\end{align*}
\]

(4.1)

The solutions of problem (4.1) are the critical points of the \(C^1\)-functional defined by

\[
I_0(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_0 u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} F(u^2) dx - \frac{1}{6} \int_{\mathbb{R}^3} (u^6)^{\frac{1}{2}} dx.
\]

Let

\[
\mathcal{N}_0 := \{u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\} : I_0'(u)[u] = 0\}
\]

and

\[
c_{V_0} := \inf_{u \in \mathcal{N}_0} I_0(u).
\]

Let \(H_0 := H^1(\mathbb{R}^3, \mathbb{R})\) and define by \(H_0^+\) the open set of \(H_0\) given by

\[
H_0^+ = \{u \in H_0 : |\text{supp}(u^2)| > 0\},
\]

and \(S_0^0 = S_0 \cap H_0^+\), where \(S_0\) be the unit sphere of \(H_0\).

As in Section 3, \(S_0^0\) is a non-complete \(C^{1,1}\)-manifold of codimension 1, modeled on \(H_0\) and contained in \(H_0^+\). Therefore, \(H_0 = T_0 S_0^0 \oplus \mathbb{R}u\) for each \(u \in T_0 S_0^0\), where \(T_0 S_0^0 = \{v \in H_0 : \langle u, v \rangle_0 = 0\}\).

Now, arguing as in Lemma 3.2, we have the following important property.

**Lemma 4.1.** Let \(V_0\) be given in (V1) and suppose that (f1)–(f4) are satisfied, then the following properties hold:

(a1) For any \(u \in H_0^+\), let \(g_u : \mathbb{R}^+ \rightarrow \mathbb{R}\) be given by \(g_u(t) = I_0(tu)\). Then there exists a unique \(t_u > 0\) such that \(g_u'(t) > 0\) in \((0, t_u)\) and \(g_u'(t) < 0\) in \((t_u, \infty)\);

(a2) There is a \(\tau > 0\) independent on \(u\) such that \(t_u > \tau\) for all \(u \in S_0^0\). Moreover, for each compact \(\mathcal{W} \subset S_0^0\) there is \(C_{\mathcal{W}}\) such that \(t_u \leq C_{\mathcal{W}},\) for all \(u \in \mathcal{W}\);

(a3) The map \(\tilde{m} : H_0^+ \rightarrow \mathcal{N}_0\) given by \(\tilde{m}(u) = t_u u\) is continuous and \(m_0 = \tilde{m}|_{S_0^0}\) is a homeomorphism between \(S_0^0\) and \(\mathcal{N}_0\). Moreover, \(m^{-1}(u) = \frac{u}{||u||_0}\);

(a4) If there is a sequence \(\{u_n\} \subset S_0^0\) such that \(\text{dist}(u_n, \partial S_0^0) \rightarrow 0\), then \(\|m(u_n)\|_0 \rightarrow \infty\) and \(I_0(m(u_n)) \rightarrow \infty\).

We shall consider the functional defined by

\[
\Psi_0(u) = I_0(\tilde{m}(u)) \quad \text{and} \quad \Psi_0 := \Psi_0|_{S_0^0},
\]

arguing as in [35, Proposition 9 and Corollary 10], the following result holds.

**Lemma 4.2.** Let \(V_0\) be given in (V1) and suppose that (f1)–(f4) are satisfied, then

\[
\begin{align*}
\Psi_0'(u)v &= \frac{\|\tilde{m}(u)||_0}{||u||_0} I_0'(\tilde{m}(u))[v], \quad \forall u \in H_0^+ \text{ and } \forall v \in H_0;
\end{align*}
\]
(b2) \( \Psi_0 \in C^1(S_0^+, \mathbb{R}) \) and
\[
\Psi_0'(u)v = \| m(u) \|_0 I_0'(m(u))[v], \quad \forall v \in T_u S_0^+;
\]
(b3) If \( \{ u_n \} \) is a \((PS)_c\) sequence of \( \Psi_0 \), then \( \{ m(u_n) \} \) is a \((PS)_c\) sequence of \( I_0 \). If \( \{ u_n \} \subset N_0 \) is a bounded \((PS)_c\) sequence of \( I_0 \), then \( m^{-1}(u_n) \) is a \((PS)_c\) sequence of \( \Psi_0 \);
(b4) \( u \) is a critical point of \( \Psi_0 \) if and only if \( m(u) \) is a critical point of \( I_0 \). Moreover, the corresponding critical values coincide and
\[
\inf_{S_0^+} \Psi_0 = \inf_{N_0} I_0.
\]

Similar to the previous argument, we also have the following variational characterization of the infimum of \( I_0 \) over \( N_0 \):
\[
c_{V_0} = \inf_{u \in N_0} I_0(u) = \inf_{u \in H_0 \setminus \{ 0 \}} \sup_{t > 0} I_0(tu) = \inf_{u \in S_0^+} \sup_{t > 0} I_0(tu). \tag{4.2}
\]
From [26, Lemma 2.6], we have \( 0 < c_{V_0} < \frac{1}{2} S_{\frac{3}{2}}^2 \).

**Lemma 4.3.** Let \( \{ u_n \} \subset H_0 \) be a \((PS)_c\) sequence for \( I_0 \) with \( c \in (0, \frac{1}{2} S_{\frac{3}{2}}^2) \) such that \( u_n \rightharpoonup 0 \). Then, one of the following alternatives occurs:

(i) \( u_n \to 0 \) in \( H_0 \) as \( n \to +\infty \);
(ii) there is a sequence \( \{ y_n \} \subset \mathbb{R}^3 \) and constants \( R, \beta > 0 \) such that
\[
\liminf_n \int_{B_R(y_n)} |u_n|^2 \, dx \geq \beta.
\]

**Remark 4.1.** From Lemma 4.3 we see that if \( u \) is the weak limit of \((PS)_{c_{V_0}}\) sequence \( \{ u_n \} \) of the functional \( I_0 \), then we have \( u \neq 0 \). Otherwise we have that \( u_n \rightharpoonup 0 \) and if \( u_n \not\to 0 \), from Lemma 4.3 it follows that there is a sequence \( \{ y_n \} \subset \mathbb{R}^3 \) and constants \( R, \beta > 0 \) such that
\[
\liminf_n \int_{B_R(y_n)} |u_n|^2 \, dx \geq \beta > 0.
\]
Then set \( v_n(x) = u_n(x + z_n) \), it is easy to see that \( \{ v_n \} \) is also a \((PS)_{c_{V_0}}\) sequence for the functional \( I_0 \), it is bounded, and there exists \( v \in H_0 \) such that \( v_n \rightharpoonup v \) in \( H_0 \) with \( v \neq 0 \).

**Lemma 4.4.** Assume that \( V \) satisfies (V1), (V2) and \( f \) satisfies (f1)–(f4), then problem (4.1) has a positive ground state solution.

**Proof.** First of all, it is easy to show that \( c_{V_0} > 0 \). Moreover, if \( u_0 \in N_0 \) satisfies \( I_0(u_0) = c_{V_0} \), then \( m^{-1}(u_0) \in S_0 \) is a minimizer of \( \Psi_0 \), so that \( u_0 \) is a critical point of \( I_0 \) by Lemma 4.2. Now, we show that there exists a minimizer \( u \in N_0 \) of \( I_0|_{N_0} \). Since \( \inf_{S_0^+} \Psi_0 = \inf_{N_0} I_0 = c_{V_0} \) and \( S_0 \) is a \( C^1 \) manifold, by Ekeland’s variational principle, there exists a sequence \( u_n \subset S_0 \) with \( \Psi_0(u_n) \to c_{V_0} \) and \( \Psi_0'(u_n) \to c_{V_0} \) as \( n \to \infty \). Put \( m_n = m(u_n) \in N_0 \) for \( n \in N \). Then \( I_0(u_n) \to c_{V_0} \) and \( I_0'(u_n) \to 0 \) as \( n \to \infty \) by Lemma 4.2(b3). Similar to the proof of Lemma 3.4, it is easy to know that \( \{ u_n \} \) is bounded in \( H_0 \). Thus, we have \( u_n \rightharpoonup u \) in \( H_0 \), \( u_n \rightharpoonup u \) in \( L^r_0(\mathbb{R}^3) \), \( 1 \leq r < 6 \) and \( u_n \to u \) a.e. in \( \mathbb{R}^3 \), thus \( I_0'(u_n) = 0 \). From [26, Lemma 2.6], we know that \( c_{V_0} < \frac{1}{2} S_{\frac{3}{2}}^2 \). Moreover, from Remark 4.1, we have that \( u \neq 0 \). Now, by Lemma 2.1,
\[
c_{V_0} \leq I_0(u) - I_0(u) - \frac{1}{|\partial|} I_0'(u)[u]
\]
\[
= \left( \frac{1}{2} - \frac{1}{|\partial|} \right) \| u \|_0^2 + \frac{1}{4} \int_{\mathbb{R}^3} \left( |x|^{-1} \ast |u|^2 \right) |u|^2 \, dx + \int_{\mathbb{R}^3} \left( \frac{1}{|\partial|} (f'(u^2) u^2 - \frac{1}{2} F(u^2)) \right) dx + \frac{1}{12} \int_{\mathbb{R}^3} (u^6) \, dx
\]
\[
\leq \liminf_n \left( \frac{1}{2} - \frac{1}{|\partial|} \right) \| u_n \|_0^2 + \frac{1}{4} \int_{\mathbb{R}^3} \left( |x|^{-1} \ast |u_n|^2 \right) |u_n|^2 \, dx + \int_{\mathbb{R}^3} \left( \frac{1}{|\partial|} (f(u_n) u_n^2 - \frac{1}{2} F(u_n^2)) \right) dx
\]
Let \( \text{Lemma 4.5.} \)

\[
\begin{align*}
\text{Lemma 4.6.}\ & \text{The numbers } c_e, c_{V_0} \text{ satisfy the following inequality} \\
\lim_{\epsilon \to 0} c_e = c_{V_0} < \frac{1}{3} S^*_0.
\end{align*}
\]

**Proof.** Let \( \eta \in C_c^\infty(\mathbb{R}^3, [0, 1]) \) be a cut-off function such that \( \eta = 1 \) in \( B_\rho/2 \) and \( \text{supp}(\eta) = B_\rho \subset A \) for some \( \rho > 0 \). Let us define \( \omega_\epsilon(x) := \eta(x)\omega(x)e^{iA(0)x} \), where \( \eta(x) = \eta(\epsilon x) \) for \( \epsilon > 0 \), \( \omega \) is a positive and radial ground state solution of problem (4.1). We observe that \( \|\omega_\epsilon\| = \eta(\epsilon) \omega_\epsilon \in H_\sigma \) in view of Lemma 2.2. Arguing as in [16, Lemma 4.1] or [24, Lemma 4.6], we obtain

\[
\lim_{\epsilon \to 0} \|\omega_\epsilon\|^2 = \|\omega\|^2
\]  

and

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} |x|^{-1} * |\omega_\epsilon|^2 dx = \int_{\mathbb{R}^3} |x|^{-1} * |\omega|^2 dx.
\]  

It is also easy to check that

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} |\omega_\epsilon|^6 dx = \int_{\mathbb{R}^3} |\omega|^6 dx.
\]  

Let \( t_\epsilon > 0 \) be the unique number such that

\[
f_\epsilon(t_\epsilon \omega_\epsilon) = \max_{t \geq 0} f_\epsilon(t \omega_\epsilon).
\]
Then $t_\varepsilon$ satisfies

$$
t_\varepsilon^2 \|\omega_\varepsilon\|^2 + t_\varepsilon^2 \int (|x|^{-1} \cdot |\omega_\varepsilon|^2)|\omega_\varepsilon|^2 \, dx = \int g(x, t_\varepsilon^2 |\omega_\varepsilon|^2) t_\varepsilon^2 |\omega_\varepsilon|^2 \, dx
$$

$$
= \int f(t_\varepsilon^2 |\omega_\varepsilon|^2) t_\varepsilon^2 |\omega_\varepsilon|^2 \, dx + \int t_\varepsilon^6 |\omega_\varepsilon|^6 \, dx,
$$

where we use $\text{supp}(\eta) \subset A$ and the definition of $g(x, t)$. Moreover, combining the facts that $\eta = 1$ in $B_{\rho/2}$, $u$ is a positive continuous function and hypothesis $(f_4)$, we have

$$
\frac{1}{t_\varepsilon^2} \int (|x|^{-1} \cdot |\omega_\varepsilon|^2)|\omega_\varepsilon|^2 \, dx = \frac{1}{t_\varepsilon^2} \int f(t_\varepsilon^2 |\omega_\varepsilon|^2) |\omega_\varepsilon|^2 \, dx + \int t_\varepsilon^6 |\omega_\varepsilon|^6 \, dx \\
\geq \frac{1}{t_\varepsilon^2} \int f(t_\varepsilon^2 \eta^2 (|x|) |\omega|^2 (x)) \eta^2 (|x|) |\omega|^2 (x) \, dx \\
\geq \frac{1}{t_\varepsilon^2} \int f(t_\varepsilon^2 \omega^2 (z)) \omega^2 (z) \, dz \\
\geq \frac{1}{t_\varepsilon^2} \int f(t_\varepsilon^2 \omega^2 (z)) \omega^2 (z) \, dz \\
\geq \frac{f(t_\varepsilon^2 \eta^2)}{t_\varepsilon^2} \int \omega^2 (z) \, dz
$$

for all $0 < \varepsilon < 1$ and where $y = \min\{\omega(z) : |z| \leq \rho/2\}$.

If $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$, by $(f_4)$, we deduce that $\int_{\mathbb{R}^3} (|x|^{-1} \cdot |\omega_\varepsilon|^2)|\omega_\varepsilon|^2 \, dx \to +\infty$ which contradicts (4.5).

Therefore, up to a subsequence, we may assume that $t_\varepsilon \to t_0 \geq 0$ as $\varepsilon \to 0$.

If $t_\varepsilon \to 0$, using the fact that $f$ is increasing, the Lebesgue dominated convergence theorem and relation (4.5), we obtain

$$
\|\omega_\varepsilon\|^2 + t_\varepsilon^2 \int (|x|^{-1} \cdot |\omega_\varepsilon|^2)|\omega_\varepsilon|^2 \, dx = \int f(t_\varepsilon^2 |\omega_\varepsilon|^2) |\omega_\varepsilon|^2 \, dx + \int t_\varepsilon^6 |\omega_\varepsilon|^6 \, dx \to 0, \text{ as } \varepsilon \to 0
$$

which contradicts (4.3). Thus, we have $t_0 > 0$ and

$$
t_0^2 \int (|\nabla \omega|^2 + V_0 \omega^2) \, dx + t_0^2 \int (|x|^{-1} \cdot |\omega|^2)|\omega|^2 \, dx = \int f(t_0 \omega^2) t_0 \omega^2 \, dx + \int t_0^6 |\omega|^6 \, dx,
$$

so that $t_0 \omega \in N_{V_0}$. Since $\omega \in N_{V_0}$, we obtain that $t_0 = 1$ and so, using the Lebesgue dominated convergence theorem, we get

$$
\lim_{\varepsilon \to 0} \int F(t_\varepsilon |\omega_\varepsilon|^2) \, dx = \int F(\omega^2) \, dx.
$$

Hence

$$
\lim_{\varepsilon \to 0} I_\varepsilon(t_\varepsilon \omega_\varepsilon) = I_{V_0}(u) = c_{V_0}.
$$

Since $c_\varepsilon \leq \max_{t \geq 0} I_\varepsilon(t \omega_\varepsilon) = I_\varepsilon(t_\varepsilon \omega_\varepsilon)$, we can conclude that $\limsup_{\varepsilon \to 0} c_\varepsilon \leq c_{V_0}$. Moreover, by (3.3), (4.2) and $I_{V_0}(u) \leq I_\varepsilon(u)$ for any $u \in H_\varepsilon$, we have $c_{V_0} \leq c_\varepsilon$. Then $c_{V_0} \leq \liminf_{\varepsilon \to 0} c_\varepsilon$. Combining with the previous arguments, we conclude that $\lim_{\varepsilon \to 0} c_\varepsilon = c_{V_0} < \frac{1}{2} S^2$.

**Remark 4.2.** From Lemma 4.1 and Lemma 3.5, we see that for $\varepsilon > 0$ small, problem (3.2) has a ground state solution $u_\varepsilon$ such that $I_\varepsilon(u_\varepsilon) = c_\varepsilon$ and $I'_\varepsilon(u_\varepsilon) = 0$. 
4.2 The technical results

By the Ljusternik-Schnirelman category theory, in this subsection we prove a multiplicity result for the modified problem (3.2). We first provide some useful preliminary results.

Let $\delta > 0$ such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^3, \mathbb{R})$ is a positive ground state solution of the limit problem (4.1), and $\eta \in C^\infty(\mathbb{R}^3, [0, 1])$ is a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leq t \leq \delta/2$ and $\eta(t) = 0$ if $t > \delta$.

For any $y \in M$, let us introduce the function

$$\Psi_{\epsilon,y}(x) := \eta((\epsilon x - y)\omega \frac{\epsilon x - y}{\epsilon}) \exp \left( i\tau_y \left( \frac{\epsilon x - y}{\epsilon} \right) \right),$$

where

$$\tau_y(x) := \sum_i A_i(y)x_i.$$

Let $t_\epsilon > 0$ be the unique positive number such that

$$\max_{t \in [0, \epsilon]} J_t(t\Psi_{\epsilon,y}) = J_{t_\epsilon}(t_\epsilon \Psi_{\epsilon,y}).$$

Note that $t_\epsilon \Psi_{\epsilon,y} \in N_\epsilon$.

Let us define $\Phi_\epsilon : M \to N_\epsilon$ as

$$\Phi_\epsilon(y) := t_\epsilon \Psi_{\epsilon,y}.$$

By construction, $\Phi_\epsilon(y)$ has compact support for any $y \in M$.

Moreover, arguing as in Lemma 4.1, the energy of above function has the following behavior as $\epsilon \to 0^+$.

**Lemma 4.7.** The limit

$$\lim_{\epsilon \to 0^+} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}$$

holds uniformly in $y \in M$.

Now we define the barycenter map.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $Y : \mathbb{R}^3 \to \mathbb{R}^3$ defined by setting

$$Y(x) := \begin{cases} x, & \text{if } |x| < \rho, \\ \rho x/|x|, & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\epsilon : N_\epsilon \to \mathbb{R}^3$ is defined by

$$\beta_\epsilon(u) := \frac{1}{\|u\|_4^4} \int_{\mathbb{R}^3} Y(\epsilon x)|u(x)|^4 dx.$$

**Lemma 4.8.** The limit

$$\lim_{\epsilon \to 0^+} \beta_\epsilon(\Phi_\epsilon(y)) = y$$

holds uniformly in $y \in M$.

**Proof.** Assume by contradiction that there exists $\kappa > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| \geq \kappa. \tag{4.6}$$

Using the change of variable $z = (\epsilon_n x - y_n)/\epsilon_n$, we can see that

$$\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3} (Y(\epsilon_n z + y_n) - y_n) \eta^4(|\epsilon_n z|) \omega^4(z) dz}{\int_{\mathbb{R}^3} \eta^4(|\epsilon_n z|) \omega^4(z) dz}.$$
Taking into account \((y_n) \subset M \subset M_\delta \subset B_p\) and the Lebesgue Dominated Convergence Theorem, we can obtain that
\[
|\beta_{c_n}(\Phi_{c_n}(y_n)) - y_n| = o_n(1),
\]
which contradicts (4.6).

Now, we prove the following useful compactness result.

**Proposition 4.1.** Let \(c_n \to 0^+\) and \((u_n) \subset N_{c_n}\) be such that \(J_{c_n}(u_n) \to c_{V_\delta}\). Then there exists \((\tilde{y}_n) \subset \mathbb{R}^3\) such that the sequence \((|y_n|) \subset H^1(\mathbb{R}^3, \mathbb{R})\), where \(v_n(x) := u_n(x + \tilde{y}_n)\), has a convergent subsequence in \(H^1(\mathbb{R}^3, \mathbb{R})\).

Moreover, up to a subsequence, \(y_n := c_n\tilde{y}_n \to y \in M\) as \(n \to +\infty\).

**Proof.** Since \(J'_{c_n}(u_n)[u_n] = 0\) and \(J_{c_n}(u_n) \to c_{V_\delta}\), arguing as in the proof of Lemma 3.2 and recalling that 
\[
\beta_{c_n}(\Phi_{c_n}(y_n)) = \beta_{c_n}(\Phi_{c_n}(y_n)) - y_n = o_n(1),
\]
we deduce that \(\tilde{y}_n \to y\) as \(n \to +\infty\). By the diamagnetic inequality (2.1), we have
\[
\beta_{c_n}(\Phi_{c_n}(y_n)) \leq \beta_{c_n}(\Phi_{c_n}(y_n)) + o_n(1),
\]
which yields \(\beta_{c_n}(\Phi_{c_n}(y_n)) \to c_{V_\delta}\) as \(n \to +\infty\).

Since the sequences \((|y_n|)\) and \((\tilde{y}_n)\) are bounded in \(H^1(\mathbb{R}^3, \mathbb{R})\) and \(|y_n| \to 0\) in \(H^1(\mathbb{R}^3, \mathbb{R})\), then \((t_n)\) is also bounded and so, up to a subsequence, we may assume that \(t_n \to t_0 \geq 0\).

We claim that \(t_0 > 0\). Indeed, if \(t_0 = 0\), then, since \((|y_n|)\) is bounded, we have \(\tilde{y}_n \to 0\) in \(H^1(\mathbb{R}^3, \mathbb{R})\), that is \(I_0(\tilde{y}_n) \to 0\), which contradicts \(c_{V_\delta} > 0\).

Thus, up to a subsequence, we may assume that \(\tilde{y}_n \to \tilde{v} := t_0\tilde{v} \neq 0\) in \(H^1(\mathbb{R}^3, \mathbb{R})\), and, by Lemma 4.5, we can derive that \(\tilde{v}_n \to \tilde{v}\) in \(H^1(\mathbb{R}^3, \mathbb{R})\), which gives \(|v_n| \to v\) in \(H^1(\mathbb{R}^3, \mathbb{R})\).

Now we show the final part, namely that \((y_n)\) has a subsequence such that \(y_n \to y \in M\). Assume by contradiction that \((y_n)\) is not bounded and so, up to a subsequence, \(|y_n| \to +\infty\) as \(n \to +\infty\). Choose \(R > 0\) such that \(A \subset B_R(0)\). Then for \(n\) large enough, we have \(|y_n| > 2R\), and, for any \(x \in B_{R/\epsilon_n}(0)\),
\[
|\epsilon_n x + y_n| \geq |y_n| - |\epsilon_n x| > R.
\]

Since \(u_n \in N_{c_n}\), using (V1) and the diamagnetic inequality (2.1), we get that
\[
\int_{\mathbb{R}^3}(|\nabla|v_n|^2 + V_0|v_n|^2)dx \leq \int_{\mathbb{R}^3} g(|\epsilon_n x + y_n|, |v_n|^2)|v_n|^2dx
\]
\[
\leq \int_{B_{R/\epsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2dx + \int_{B_{R/\epsilon_n}(0)} f(|v_n|^2)|v_n|^2dx + \int_{B_{R/\epsilon_n}(0)} |v_n|^6dx.
\]

Since \(|v_n| \to v\) in \(H^1(\mathbb{R}^3, \mathbb{R})\) and \(\tilde{f}(t) \leq V_0/K\), we can see that (4.8) yields
\[
\min \left\{1, V_0 \left(1 - \frac{1}{R}\right)\right\} \int_{\mathbb{R}^3}(|\nabla|v_n|^2 + |v_n|^2)dx = o_n(1),
\]
that is \(|v_n| \to 0\) in \(H^1(\mathbb{R}^3, \mathbb{R})\), which contradicts to \(v \neq 0\).

Therefore, we may assume that \(y_n \to y_0 \in \mathbb{R}^3\). Assume by contradiction that \(y_0 \not\in A\). Then there exists \(r > 0\)
such that for every $n$ large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \bar{A}^c$. Then, if $x \in B_{\epsilon x}(0)$, we have that $|\epsilon_n x + y_n - y_0| < 2r$ so that $\epsilon_n x + y_n \in \bar{A}^c$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \bar{A}$.

To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using the Fatou’s lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t \in [0, T]} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n)$, we obtain

$$c_{V_0} = I_0(\tilde{v}) < \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \tilde{v}|^2 + V(\tilde{v})|\tilde{v}|^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} * |\tilde{v}|^2)|\tilde{v}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} F(|\tilde{v}|^2) dx - \frac{1}{6} \int_{\mathbb{R}^3} |\tilde{v}|^6 dx$$

$$\leq \liminf_{n} \left( \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \tilde{v}|^2 + V(\epsilon_n x + y_n)|\tilde{v}|^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} * |\tilde{v}|^2)|\tilde{v}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} F(|\tilde{v}|^2) dx \right)$$

$$\leq \liminf_{n} \left( \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla |u_n|^2|^2 + V(\epsilon_n z)|u_n|^2) dz + \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} * |u_n|^2)|u_n|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} F(|u_n|^2) dz \right)$$

$$\leq \liminf_{n} \inf J_{\epsilon_n}(tu_n) \leq \liminf_{n} \inf J_{\epsilon_n}(u_n) = c_{V_0}$$

which is impossible and the proof is complete.

Let now

$$\tilde{N}_\epsilon := \{ u \in N_\epsilon : J_c(u) \leq c_{V_0} + h(\epsilon) \},$$

where $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \uparrow$, $h(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. For any $z \in M$, since, by Lemma 4.7, $|J_c(\Phi_\epsilon(y)) - c_{V_0}| \rightarrow 0$ as $\epsilon \rightarrow 0^+$, we get that $\tilde{N}_\epsilon \neq \emptyset$ for any $\epsilon > 0$ small enough.

The relation between $\tilde{N}_\epsilon$ and the barycenter map is as follows.

**Lemma 4.9.** We have

$$\lim_{\epsilon \rightarrow 0^+} \sup_{u \in \tilde{N}_\epsilon} \text{dist}(\beta_\epsilon(u), M_\delta) = 0.$$

**Proof.** Let $\epsilon_n \rightarrow 0^+$ as $n \rightarrow +\infty$. For any $n \in \mathbb{N}$, there exists $u_n \in \tilde{N}_\epsilon$, such that

$$\sup_{u \in \tilde{N}_\epsilon} \inf_{y \in M_\delta} |\beta_\epsilon(u) - y| = \inf_{y \in M_\delta} |\beta_\epsilon(u_n) - y| + o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_{n} |\beta_\epsilon(u_n) - y_n| = 0.$$

By the diamagnetic inequality (2.1), we can see that $I_0(t|u_n|) \leq J_{\epsilon_n}(tu_n)$ for any $t > 0$. Therefore, recalling that $\{u_n\} \subset \tilde{N}_\epsilon \subset N_\epsilon$, we can deduce that

$$c_{V_0} \leq \max_{t \geq 0} I_0(t|u_n|) \leq \max_{t \geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n) \leq c_{V_0} + h(\epsilon_n) \quad (4.9)$$

which implies that $J_{\epsilon_n}(u_n) \rightarrow c_{V_0}$ as $n \rightarrow +\infty$.

Then, Proposition 4.1 implies that there exists $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $y_n = \epsilon_n \tilde{y}_n \in M_\delta$ for $n$ large enough.

Thus, making the change of variable $z = x - \tilde{y}_n$, we get

$$\beta_\epsilon(u_n) = y_n + \frac{\int_{\mathbb{R}^3} Y(\epsilon_n z + y_n)|u_n(z + \tilde{y}_n)|^4 dz}{\int_{\mathbb{R}^3} |u_n(z + \tilde{y}_n)|^4 dz}.$$

Since, up to a subsequence, $|u_n| \rightarrow \tilde{y}_n$ converges strongly in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\epsilon_n z + y_n \rightarrow y \in M$ for any $z \in \mathbb{R}^3$, we conclude. \qed
4.3 Multiplicity of solutions for problem (3.2)

Finally, we present a relation between the topology of $M$ and the number of nontrivial solutions of the modified problem (3.2).

**Theorem 4.1.** For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\epsilon}_\delta > 0$ such that, for any $\epsilon \in (0, \tilde{\epsilon}_\delta)$, problem (3.2) has at least $\text{cat}_{M_\delta}(M)$ nontrivial solutions.

**Proof.** For any $\epsilon > 0$, we define the function $\pi_\epsilon : M \to S^*_\epsilon$ by

$$\pi_\epsilon(y) = m_\epsilon^{-1}(\Phi_\epsilon(y)), \forall y \in M.$$  

By Lemma 4.7 and Lemma 3.3(B4), it follows that

$$\lim_{\epsilon \to 0} \Psi_\epsilon(\pi_\epsilon(y)) = \lim_{\epsilon \to 0} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}, \text{ uniformly in } y \in M.$$  

Therefore, there is a number $\hat{\epsilon} > 0$ such that the set $S^*_\epsilon := \{ u \in S^*_\epsilon : \Psi_\epsilon(u) \leq c_{V_0} + h(\epsilon) \}$ is nonempty, for all $\epsilon \in (0, \hat{\epsilon})$, since $\pi_\epsilon(M) \subset S^*_\epsilon$. Here $h$ is given in the definition of $\tilde{N}_\epsilon$.

Given $\delta > 0$, by Lemma 4.7, Lemma 3.2(A3), Lemma 4.8, and Lemma 4.9, we can find $\tilde{\epsilon}_\delta > 0$ such that for any $\epsilon \in (0, \tilde{\epsilon}_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\epsilon} \Phi_\epsilon(M) \xrightarrow{m_\epsilon^{-1}} \pi_\epsilon(M) \xrightarrow{m_\epsilon} \Phi_\epsilon(M) \xrightarrow{\beta_\epsilon} M_\delta$$

is well defined and continuous. From Lemma 4.8, we can choose a function $\Theta(\epsilon, z)$ with $|\Theta(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$, for all $\epsilon \in (0, \hat{\epsilon})$ such that $\beta_\epsilon(\Phi_\epsilon(z)) = z + \Theta(\epsilon, z)$ for all $z \in M$. Define $H(t, z) = z + (1-t)\Theta(\epsilon, z)$. Then $H : [0, 1] \times M \to M_\delta$ is continuous. Clearly, $H(0, z) = \beta_\epsilon(\Phi_\epsilon(z))$, $H(1, z) = z$ for all $z \in M$. That is, $H(t, z)$ is a homotopy between $\beta_\epsilon \circ \Phi_\epsilon = (\beta_\epsilon \circ m_\epsilon) \circ \pi_\epsilon$ and the embedding $i : M \to M_\delta$. This fact implies that

$$\text{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M)) \geq \text{cat}_{M_\delta}(M). \tag{4.10}$$

By Corollary 3.1 and the abstract category theorem [35], $\Psi_\epsilon$ has at least $\text{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))$ critical points on $S^*_\epsilon$. Therefore, from Lemma 3.3(B4) and (4.10), we have that $J_\epsilon$ has at least $\text{cat}_{M_\delta}(M)$ critical points in $\tilde{N}_\epsilon$, which implies that problem (3.2) has at least $\text{cat}_{M_\delta}(M)$ solutions. \hfill \qed

5 Proof of Theorem 1.1

In this section we shall show that the solutions $u_\epsilon$ obtained in Theorem 4.1 satisfy

$$|u_\epsilon(x)|^2 \leq a \text{ for } x \in \Lambda_\epsilon^c$$

for $\epsilon$ small and prove the main result of this paper.

Arguing as in [29] or [41], the following uniform result holds.

**Lemma 5.1.** Let $c_n \to 0^+$ and $u_n \in \tilde{N}_{c_n}$ be a solution of problem (3.2) for $\epsilon = c_n$. Then $J_{c_n}(u_n) \to c_{V_0}$. Moreover, there exists $\{ \tilde{y}_n \} \subset \mathbb{R}^3$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $\{ |v_n| \}$ is bounded in $L^\infty(\mathbb{R}^3, \mathbb{R})$ and

$$\lim_{|x| \to +\infty} |v_n(x)| = 0 \text{ uniformly in } n \in \mathbb{N}.$$  

Now it’s the position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We want to show that there exists $\tilde{\epsilon}_\delta > 0$ such that for any $\epsilon \in (0, \tilde{\epsilon}_\delta)$ and any $u_\epsilon \in \tilde{N}_\epsilon$ solution of problem (3.2), it holds

$$\|u_\epsilon\|_{L^\infty(\Lambda_\epsilon^c)} \leq a. \tag{5.1}$$
We argue by contradiction and assume that there is a sequence $\varepsilon_n \to 0$ such that for every $n$ there exists $u_n \in \tilde{N}_{\varepsilon_n}$ which satisfies $f'_{\varepsilon_n}(u_n) = 0$ and
\[
\|u_n\|_{L^2(\mathcal{A}_{\varepsilon_n}^c)}^2 > a.
\]
(5.2)

As in Lemma 5.1, we have that $f_{\varepsilon_n}(u_n) \to c_{V_0}$, and therefore we can use Proposition 4.1 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $y_n := \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$. Then, we can find $r > 0$, such that $B_{r}(y_n) \subset A_{\varepsilon_n}$, and so $B_{r/\varepsilon_n}(\tilde{y}_n) \subset A_{\varepsilon_n}$ for all $n$ large enough.

Using Lemma 5.1, there exists $R > 0$ such that $|v_n|^2 \leq a$ in $B_{R}^c(0)$ and $n$ large enough, where $v_n = u_n(x + \tilde{y}_n)$. Hence $|u_n|^2 \leq a$ in $B_{R}^c(\tilde{y}_n) \subset B_{R}^c(\tilde{y}_n)$ and $n$ large enough. Moreover, if $n$ is so large that $r/\varepsilon_n > R$, then $A_{\varepsilon_n}^c \subset B_{r/\varepsilon_n}(\tilde{y}_n) \subset B_{R}^c(\tilde{y}_n)$, which gives $|u_n|^2 \leq a$ for any $x \in A_{\varepsilon_n}^c$. This contradicts (5.2) and proves the claim.

Let now $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \check{\varepsilon}_\delta\}$, where $\varepsilon_\delta > 0$ is given by Theorem 4.1. Then we have $\text{cat}_{\mathcal{M}}(M)$ nontrivial solutions to problem (3.2). If $u_\varepsilon \in \tilde{N}_\varepsilon$ is one of these solutions, then, by (5.1) and the definition of $g$, we conclude that $u_\varepsilon$ is also a solution to problem (2.2).

Finally, we study the behavior of the maximum points of $|\hat{u}_\varepsilon|$, where $\hat{u}_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ is a solution to problem (1.1), as $\varepsilon \to 0^+$. Take $\varepsilon_n \to 0^+$ and the sequence $(u_n)$ where each $u_n$ is a solution of (3.2) for $\varepsilon = \varepsilon_n$. From the definition of $g$, there exists $y \in (0, a)$ such that
\[
g(e_n, t^2) t^2 \leq \frac{V_0}{K} t^2, \quad \text{for all } x \in \mathbb{R}^3, \ |t| \leq y.
\]

Arguing as above we can take $R > 0$ such that, for $n$ large enough,
\[
\|u_n\|_{L^2(B_R^c(\tilde{y}_n))} < y.
\]
(5.3)

Up to a subsequence, we may also assume that for $n$ large enough
\[
\|u_n\|_{L^2(B_R^c(\tilde{y}_n))} \geq y.
\]
(5.4)

Indeed, if (5.4) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_{\infty} \leq y$. Thus, since $f'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$, using (g5) and the diamagnetic inequality (2.1) that
\[
\int_{\mathbb{R}^3} (||\nabla|u_n| |^2 + V_0 |u_n|^2) dx \leq \int_{\mathbb{R}^3} g(e_n x, |u_n|^2) |u_n|^2 dx \leq \frac{V_0}{K} \int_{\mathbb{R}^3} |u_n|^2 dx
\]
and, being $K > 2$, $\|u_n\| = 0$, which is a contradiction.

Taking into account (5.3) and (5.4), we can infer that the global maximum points $p_n$ of $|u_{\varepsilon_n}|$ belongs to $B_R(\tilde{y}_n)$, that is $p_n = q_n + \tilde{y}_n$, for some $q_n \in B_R$. Recalling that the associated solution of problem (1.1) is $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we can see that a maximum point $\eta_{\varepsilon_n}$ of $|\hat{u}_n|$ is $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$. Since $q_n \in B_R$, $\varepsilon_n \tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of $V$ allows to conclude that
\[
\lim_{n} V(\eta_{\varepsilon_n}) = V_0.
\]

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$\Box$
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