Accuracy of the relativistic Cowling approximation in slowly rotating stars

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ABSTRACT
We have calculated the non-radial oscillation in slowly rotating relativistic stars with the Cowling approximation. The frequencies are compared with those based on the complete linearized equations of general relativity. It is found that the results with the approximation differ by less than about 20\% for typical relativistic stellar models. The approximation is more accurate for higher-order modes as in the Newtonian case.

Key words: oscillation, neutron star

1 INTRODUCTION
In recent years, our understanding of pulsations of non-rotating relativistic stars has been much improved. In particular, the role of gravitational wave in the stars becomes much clear. There exists the oscillation mode named as w-mode (gravitational wave mode), associated with the gravitational wave. The gravitational wave is inherent in the general relativity, so that the mode becomes evident only for relativistic system. Except for this new mode, the relativity little affects the modes known in the Newtonian pulsation theory. The general relativity slightly changes the oscillation frequency and gives rise to a very slow damping of the mode. The emission of gravitational radiation implies that the oscillation frequency should be complex with a relatively tiny imaginary part. (See Andersson, Kojima, and Kokkotas (1996) and references therein for the present status of the oscillation spectra of non-rotating relativistic stars.)

The perturbation of the gravity is not so important, if the w-mode and the decay of the pulsations due to the gravitational radiation do not matter. This fact hints at the further simplification of the pulsation problem, that is, neglecting the perturbation of the gravitational field. This approximation is known as the Cowling approximation (Cowling (1941)) in the Newtonian stellar pulsation theory, and gives the same qualitative results and reasonable accuracy of the oscillation frequencies. (See, e.g., Cox (1980).) Two different prescriptions, so far, had been proposed for the relativistic Cowling approximation in non-radial pulsations of the non-rotating stars. One method is that all metric perturbations are neglected, i.e., $\delta g_{\mu\nu} = 0$ (McDermott, Van Horn & Scholl 1983). The other is that the $\delta g_{\nu t}$ component of the metric perturbations is retained in the pulsation equations (Finn 1988). Lindblom and Splinter (1990) examined the accuracy of these two versions of the approximation for the dipole p-modes, and concluded that the McDermott, Van Horn, and Scholl version is more accurate than Finn’s version. However, the Finn’s version is superior in g-mode calculations (Finn 1988).

In contrast to the oscillations in the non-rotating stars, the calculation of oscillation frequencies in rapidly rotating relativistic stars seems to be very difficult task. The equations become significantly complicated due to the relativity and rotation. See Priou (1992) for a set of the explicitly written, lengthy equations. Therefore, the calculation of normal frequencies was limited to the non-rotating and slowly rotating stars at best (Kojima 1993). The possibility to use the Cowling approximation in rotating relativistic system should be studied. Ipser and Lindblom (1992) formulated the Cowling approximation of the relativistic system, setting the metric perturbation to zero. If the approximation is applied to the stellar pulsation, the equation becomes single second order partial differential equation. The equation is manageable and will be hopefully solved in the future. The crucial point is how good the Cowling approximation is to estimate the oscillation frequency in the relativistic regime.

In this paper, we will estimate the accuracy of the Cowling approximation in the frequencies of the non-rotating star and its rotational corrections. That is, we will compare the frequencies of the Cowling approximation with those of the relativistic perturbation theory. In section 2, we present the formalism to calculate normal frequencies and the rotational shifts by the the Cowling approximation. In section 3, the comparison between both numerical results is given. Finally, section 4 is devoted to the discussion. Throughout this paper we will use units of $G = c = 1$. 

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\section{Method}

To assess the accuracy of the Cowling approximation, two distinct calculations are performed. In one calculation, the relativistic perturbation equations are solved to find normal frequencies and the rotational corrections for the pulsation of stellar models. The equations and techniques are described in Kojima (1992, 1993). In the other calculation, we adopt the relativistic Cowling approximation. We restrict ourselves to the slowly rotating case, and write down the equations of the pulsation and rotational corrections below. Normal frequencies are real numbers, because gravitational perturbations are neglected in the Cowling approximation. On the other hand, the normal frequencies are complex numbers with small imaginary part representing the damping of the gravitational radiation, when solving the exact pulsation equations. We make a comparison in the oscillation frequency only.

\subsection{Equilibrium state}

We assume that the star is slowly rotating with a uniform angular velocity \( \Omega \). In this paper we consider only the first order effect with \( \epsilon = \Omega / \sqrt{M/R^3} \), where \( R \) and \( M \) are the radius and the mass of the star, respectively. The geometry of a slowly rotating star in general relativistic description is described by

\[ ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - 2\omega r^2 \sin^2 \theta dt d\phi , \]

(1)

where \( \nu, \lambda \) and \( \omega \) are functions of \( r \) only. Given an equation of state \( p = p(\rho) \), the construction of the slowly rotating equilibrium model is well-known. (For details see, e.g., Hartle (1967), Chandrasekhar and Miller (1974).) We will assume a polytropic equation of state,

\[ p = k \rho^{(1+1/n)} , \]

(2)

where \( n \) and \( k \) are constants. The function \( \omega \) is of order \( \epsilon \), and the others, \( \nu, \lambda, p, \) and \( \rho \) are the same as in the non-rotating case.

\subsection{The Cowling approximation}

In this subsection, we will explicitly write down the pulsation equation for the slowly rotating stars up to the first order of \( \epsilon \), using the equations derived by Ipser and Lindblom (1992). The basic equation to be solved becomes second order ordinary differential equation for \( \delta V \). (See equation (38) of Ipser and Lindblom (1992).) We have

\[ \frac{d}{dr} A \frac{d}{dr} + B + \sigma^2 C \delta V = m (2\sigma \Omega C - S) \delta V , \]

(11)

where

\[ A = (\rho + p)r^2 e^{(\nu-\lambda)/2} , \]

(12)

\[ B = e^{-\nu/2} \frac{d}{dr} r^2 \frac{dp}{dr} e^{(2\nu-\lambda)/2} \left[ l(l+1) (\rho+p) e^{(\nu+\lambda)/2} \right] , \]

(13)

\[ C = \frac{(\rho + p)^2}{p \Gamma} r^2 e^{(-\nu+\lambda)/2} , \]

(14)

\[ S = (\rho + p)^2 r^2 e^{(\nu+\lambda)/2} \left[ 2\sigma \varpi e^{-\nu} + \frac{2\varpi}{\sigma r^2} \right. \]

\[ \left. - \frac{e^{-3\nu/2}}{\sigma(\rho + p)} \frac{d}{dr} \left[ (\rho + p) e^{(5\nu-\lambda)/2} \frac{d}{dr} \left( \varpi e^{-\nu} r^2 \right) \right] \right] , \]

(15)

where \( S \) and \( \Omega \) are of the order \( \epsilon \).

We will solve the eigen-value problem, equation (11) by standard perturbation method, that is, expand the potential \( \delta V \) and the frequency \( \sigma \) with respect to the small parameter \( \epsilon \) as,

\[ \delta V = \delta V_0 + \delta V_1 \epsilon + \cdots , \]

(16)

\[ \sigma = \sigma_0 + \sigma_1 \epsilon + \cdots . \]

(17)

The equation of order \( \epsilon^0 \) is for the non-rotating case,
Therefore, it is sufficient to calculate \( \delta V \) where

\[
\left[ \frac{d}{dr} A \frac{d}{dr} + B + \sigma_0^2 C \right] \delta V_0 = 0 ,
\]

and the equation of order \( \epsilon \) is

\[
\left[ \frac{d}{dr} A \frac{d}{dr} + B + \sigma_0^2 C \right] \delta V_1 + 2 \sigma_0 \sigma_1 C \delta V_0 = m \sqrt{\frac{M}{\Omega^2 R^3}} \left( 2 \sigma_0 \Omega C - S_0 \right) \delta V_0 ,
\]

where \( S_0 \) is defined by equation (13) with \( \sigma = \sigma_0 \). Multiplying \( \delta V_0^n \) to equation (19), and integrating over the star, we have the first order rotational correction of the normal frequency \( \sigma_1 \) as

\[
\sigma_1 = m \sqrt{\frac{M}{\Omega^2 R^3}} \left[ \Omega - \frac{\int_0^R S_0 |\delta V_0|^2 \, dr}{2 \sigma_0 \int_0^R C |\delta V_0|^2 \, dr} \right] .
\]

Similar expression can be found in the Newtonian pulsation equation, but is written by the Lagrangian displacement, not by the Eulerian change of the potential. Note also that the correction (20) can be determined from the eigen-function and eigen-value in the non-rotating case, i.e., \( \delta V_0 \) and \( \sigma_0 \). Therefore, it is sufficient to calculate \( \delta V_0 \) and \( \sigma_0 \).

The frequency in the Cowling approximation should be compared with the real part of eigen-frequency calculated by the exact pulsation equation. The frequency \( \sigma \) of the slowly rotating star can be parameterized as

\[
\sigma = \sigma_R + \sigma'_R \epsilon ,
\]

where

\[
\sigma_R = \sigma_0 , \quad \sigma'_R = \sigma_1 / m .
\]

The subscript \( R \) means the real part of the exact normal frequency. We will compare results of two calculations in \( \sigma_R \) and \( \sigma'_R \).

2.3 Numerical method

As mentioned in the last subsection, we need eigen-frequency and eigen-function of the pulsations in the non-rotating stars. The method to solve equation (18) is well-known and straightforward. We will briefly summarize the boundary conditions and the numerical method.

Since the basic equation (18) is a second order differential equation, two boundary conditions should be given for \( \delta V_0 \) at the center and the surface of the star. The boundary condition at the center is that the function \( \delta V_0 \) should be regular near \( r = 0 \). This condition can be written by power series as

\[
\delta V_0 = r^l (v_0 + v_2 r^2 + \cdots) .
\]

At the surface, the Lagrangian perturbation of pressure should be vanished, \( \Delta p = 0 \). This condition can be written explicitly as

\[
\frac{d \delta V_0}{dr} = - \left( \sigma_0^2 \epsilon^{\lambda - \nu} (\rho + p) \frac{dr}{dp} + \frac{1}{\rho + p \, dp} \right) \delta V_0 .
\]

We can numerically integrate the equation (18) from the center with the condition (22), and from the surface with (23) to the appropriate interior point. The eigen-value \( \sigma_0 \) can be obtained as the consistent frequency so as to match functions smoothly at the point. We also numerically calculate the rotational correction of normal frequency \( \sigma_1 \), by \( \sigma_0 \) and \( \delta V_0 \) through the equation (20).

In order to check our numerical code for the relativistic Cowling approximation, we calculated the fundamental normal mode frequencies having harmonic index \( l \) in the range \( 2 \leq l \leq 5 \) for the non-rotating stellar model used by Ipser and Lindblom (1992). Our results were in good agreement with their normal frequencies and the difference was of the order less than 0.05%.

3 NUMERICAL RESULTS

We have calculated normal frequencies for a range of the stellar models with a polytropic equation of state (1). We adapt two different polytropic indices \( n = 1.0, 1.5 \). The compactness of the star ranges from \( M/R = 0.05 \) to \( M/R = 0.20 \). The oscillation frequencies and the rotational corrections are calculated for the f-, p1-, and p2-modes having harmonics index \( l \) in the range \( 2 \leq l \leq 5 \). The results for typical stellar models are shown in Tables 1–4. Normal frequencies of the non-rotating stars and the rotational corrections are compared. Since there is not so much difference in the polytropic index, we hereafter concentrate in the model with \( n = 1 \).

We show the normal frequencies and rotational corrections of the f-modes as a function of \( M/R \) in Fig.1. The eigen-frequencies of the non-rotating star, \( \sigma_R \) are given in Fig.1(a), and the rotational corrections \( \sigma'_R \) are in Fig.1(b). The results with the exact relativistic calculation are represented by the dashed line, and those with the Cowling approximation are by the solid lines. Figure 1(a) shows that two results of the f-mode approach with the increase of \( M/R \), that is, the error monotonically decreases. For example, the relative error for \( l = 2 \) f-mode in \( M/R = 0.05 \) model is about 30% in the magnitude, but it decreases to 15% in \( M/R = 0.2 \) model. The relativistic Cowling approximation therefore gives better result in the f-mode calculation, as equilibrium model becomes more relativistic. Figure 1(b) shows that both results are always in good agreement on the rotational corrections. The relative error is always less than 3%.

Next, we show results of the p1- and p2-modes in Figs.2 and 3, respectively. Compared with the exact calculation, the dependence of \( M/R \) is similar in the oscillation frequencies and the rotational corrections. The relative error is almost constant in \( \sigma_R \), but increases in \( \sigma'_R \) with \( M/R \). The approximation for these p-modes becomes worse for the relativistic stars. However, the discrepancy is always within 10% error.

From Figs. 1–3, it is quite clear that the accuracy of the Cowling approximation increases with the spherical harmonic index, \( l \). This feature is well-known in the Newtonian Cowling approximation. (See, e.g. Cox (1980).) The neglect of the gravitational perturbation is justified due to the averaging effect of the local fluctuations for high-order modes. The Newtonian Cowling approximation also becomes better with the increase of the radial node number. (See, e.g., Robe (1964) for numerical results of the Newtonian Cowling approximation.) In the relativistic case, the accuracy also increases, but the behavior is a little complicated due to the additional parameter \( M/R \).
4 DISCUSSION

In this paper, we have compared the relativistic Cowling approximation with the perturbation calculation of general relativity. The accuracy of the Cowling approximation is considerably good, concerning the normal frequencies in the non-rotating star and the first order rotational corrections. Remarkable fact is that the relative error of the f-mode decreases with the relativistic factor, $M/R$. The approximation is also better for larger radial node number and harmonic index $l$, and greater central condensation, as in the Newtonian pulsation theory.

One of the important applications of the normal modes is to determine the critical angular velocity of the rotating stars. The maximum angular velocity is limited by the existence of the axisymmetric equilibrium state. The equilibrium state however suffers secular instability due to viscosity and/or gravitational radiation reaction (GRR). The instability sets in through $\sigma = m\Omega$ (viscous instability) or $\sigma = 0$ (GRR instability), respectively (e.g., Friedman and Schutz 1978). Form the Newtonian calculation, most significant contributions to the GRR instability are the f-modes with $l = -m \sim 4$. (See, e.g., Yoshida and Eriguchi 1995.) The present study shows that the relativistic Cowling approximation is able to give a good prediction of the oscillation frequency within 6% error for the f-modes with $l = -m = 4, 5$ in the typical neutron stars, $M/R \sim 0.2$. The critical point is however not within the regime of the linear extrapolation from the non-rotating case, as suggested in the Newtonian calculation. That is, the present results, $\sigma_R$ and $\sigma'_R$ are not sufficient, and the higher order rotational corrections are necessary. Therefore, solving normal frequency in rapidly rotating stars with the Cowling approximation would be a good method to determine the critical angular velocity.

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Table 3. The same as Table 1, but for \( n = 1.5 \) and \( M/R = 0.1 \).

| \( l \) mode | \( \sigma_{ex} \) | \( \sigma_{Cow} \) | \( \sigma_{ex}' \) | \( \sigma_{Cow}' \) |
|-------------|----------------|----------------|----------------|----------------|
| 2 \( f \)    | 1.43           | 1.68           | 0.591          | 0.609          |
| \( p_1 \)    | 2.87           | 3.18           | 0.851          | 0.855          |
| \( p_2 \)    | 4.27           | 4.56           | 0.911          | 0.911          |
| 3 \( f \)    | 1.83           | 1.97           | 0.729          | 0.734          |
| \( p_1 \)    | 3.35           | 3.55           | 0.886          | 0.889          |
| \( p_2 \)    | 4.76           | 4.98           | 0.926          | 0.926          |
| 4 \( f \)    | 2.11           | 2.21           | 0.796          | 0.798          |
| \( p_1 \)    | 3.73           | 3.88           | 0.908          | 0.910          |
| \( p_2 \)    | 5.19           | 5.36           | 0.937          | 0.938          |
| 5 \( f \)    | 2.34           | 2.41           | 0.835          | 0.837          |
| \( p_1 \)    | 4.05           | 4.16           | 0.923          | 0.925          |
| \( p_2 \)    | 5.56           | 5.69           | 0.945          | 0.946          |

Table 4. The same as Table 1, but for \( n = 1.5 \) and \( M/R = 0.2 \).

| \( l \) mode | \( \sigma_{ex} \) | \( \sigma_{Cow} \) | \( \sigma_{ex}' \) | \( \sigma_{Cow}' \) |
|-------------|----------------|----------------|----------------|----------------|
| 2 \( f \)    | 1.36           | 1.47           | 0.698          | 0.706          |
| \( p_1 \)    | 2.46           | 2.71           | 0.855          | 0.867          |
| \( p_2 \)    | 3.54           | 3.82           | 0.895          | 0.900          |
| 3 \( f \)    | 1.68           | 1.75           | 0.795          | 0.797          |
| \( p_1 \)    | 2.93           | 3.08           | 0.889          | 0.894          |
| \( p_2 \)    | 4.05           | 4.24           | 0.909          | 0.912          |
| 4 \( f \)    | 1.93           | 1.98           | 0.843          | 0.844          |
| \( p_1 \)    | 3.29           | 3.40           | 0.910          | 0.912          |
| \( p_2 \)    | 4.47           | 4.61           | 0.921          | 0.923          |
| 5 \( f \)    | 2.13           | 2.17           | 0.872          | 0.873          |
| \( p_1 \)    | 3.60           | 3.68           | 0.924          | 0.925          |
| \( p_2 \)    | 4.84           | 4.95           | 0.930          | 0.932          |

Figure caption

Fig.1(a). Normal frequencies, \( \sigma_R \) vs. relativistic factor \( M/R \) for \( f \)-mode and \( n = 1.0 \). Dashed lines and solid lines correspond to results with the relativistic perturbation calculation and the Cowling approximation, respectively. Normal frequencies are normalized by \( (M/R^3)^{1/2} \). Attached labels \( l = 2, 3, 4 \) and 5 mean the harmonic indices.

Fig.1(b). Rotational corrections, \( \sigma'_R \) vs. relativistic factor \( M/R \) for \( f \)-mode and \( n = 1.0 \). Dashed lines and solid lines correspond to results with the full calculation and the Cowling approximation, respectively. Rotational corrections are normalized by \( (M/R^3)^{1/2} \). Attached labels \( l = 2, 3, 4 \) and 5 mean the harmonic indices.

Fig.2(a). The same as Fig. 1(a), but for \( p_1 \)-mode.

Fig.2(b). The same as Fig. 1(b), but for \( p_1 \)-mode.

Fig.3(a). The same as Fig. 2(a), but for \( p_2 \)-mode.

Fig.3(b). The same as Fig. 2(b), but for \( p_2 \)-mode.
