Tubular neighborhoods of nodal sets and diophantine approximation

Dmitry Jakobson, Dan Mangoubi

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TUBULAR NEIGHBORHOODS OF NODAL SETS
AND DIOPHANTINE APPROXIMATION

By DMITRY JAKOBSON and DAN MANGOUBI

Abstract. We give upper and lower bounds on the volume of a tubular neighborhood of the nodal set of an eigenfunction of the Laplacian on a real analytic closed Riemannian manifold $M$. As an application we consider the question of approximating points on $M$ by nodal sets, and explore analogy with approximation by rational numbers.

1. Introduction and main results. Let $(M, g)$ be a real analytic closed Riemannian manifold. In the first part of this paper we give upper and lower bounds on the volume of tubular neighborhoods of nodal sets. Consider the eigenequation

$$\Delta \phi_\mu + \mu^2 \phi_\mu = 0,$$

where $\Delta$ is the Laplace–Beltrami operator on $M$. We denote the nodal set $\{\phi_\mu = 0\}$ by $N_\mu$. Consider the tubular neighborhood of the nodal set

$$(1.1) \quad T_{\mu, \delta} = \{x \in M : \text{dist} (x, N_\mu) < \delta\}.$$

We prove:

Theorem 1.2. Let $(M, g)$ be a real analytic closed Riemannian manifold. Then there exist $C_1, C_2, C_3 > 0$ such that

$$C_1 \mu \delta \leq \text{Vol} (T_{\mu, \delta}) \leq C_2 \mu \delta,$$

whenever $\mu \delta \leq C_3$.

The problem of estimating the size of the $(n - 1)$-Hausdorff measure of the nodal set was raised by Yau ([Yau82], problem #74). Yau also conjectured optimal bounds. An optimal lower bound in dimension two was proved by Brüning [Bru78] and by Yau independently. An upper bound in dimension two was proved by Donnelly and Fefferman [DF90] and by Dong [Don92]. In the case of real

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THEOREM 1.3. [DF88, Theorem 1.2] Let $(M, g)$ be a closed real analytic Riemannian manifold. Then, there exist $C_4, C_5 > 0$ such that $C_4 \mu \leq \text{Vol}_{n-1}(N_\mu) \leq C_5 \mu$, where $\text{Vol}_{n-1}$ is the $(n - 1)$-dimensional Hausdorff measure on $M$.

From this perspective, we see that Theorem 1.2 describes a regularity property of the nodal set. For example, the upper bound implies that the nodal set does not have too many needles or very narrow branches, while the lower bound says that the nodal set doesn’t curve too much.

For the proof of Theorem 1.2 we need to study the behavior of eigenfunctions in all scales $0 < \delta \leq 1/\mu$ ($1/\mu$ is called the wavelength). Roughly, we show that for most points $x$, $\phi_\mu(x)$ is comparable to the average of $\phi_\mu$ on a ball of radius $\delta$ centered at $x$. This study is the content of Sections 2-6, and it extends the work of Donnelly and Fefferman in Section 5 of [DF88], where they consider scales comparable to the wavelength $C_1/\mu \leq \delta \leq C_2/\mu$.

Donnelly and Fefferman showed that by considering an analytic continuation of $\phi_\mu$, one can treat our problem by studying polynomials in dimension one, and then applying an induction argument. We adopt this approach here. The key proposition is Proposition 4.4. Most of its proof goes without change from the proof of Proposition 5.11 in [DF88]. We had to adjust the arguments from [DF88] in two main points. The first is the proof in dimension one, where we added the parameter $\delta$ to the proofs in [DF88], and showed that everything goes through. The second is in the proof of Proposition 3.7 where the change of variables argument is different and more subtle than the parallel argument in [DF88].

The proof of the lower bound in Theorem 1.2 is given in Section 7. It is based on the behavior of eigenfunctions in scales comparable to the wavelength and on the Brunn-Minkowski inequality.

The idea of the proof of the upper bound in Theorem 1.2 was suggested to the authors by C. Fefferman. We give the proof in Section 8. A proof of the upper bound by different methods can be found in [Yom08]. Our proof is based on the upper bound in Theorem 1.3 and our study of eigenfunctions in all scales $\delta \leq C/\mu$.

In Section 10 we consider the special case where $\dim(M) = 2$. We show that the lower bound is true for any smooth surface and the upper bound is true for any smooth surface which satisfies Yau’s conjecture.

In the second part of the paper we make an attempt to look simultaneously on the ensemble of nodal sets which belong to different eigenvalues. Consider first a simple example: Eigenfunctions on $M = [0, \pi]$ with the standard metric and (say) with Dirichlet boundary conditions. Then

$$
\mu_k = k, \quad \phi_k(x) = \sin(kx), \quad N_k = \left\{ \frac{\pi j}{k} : 0 \leq j \leq k \right\}.
$$

Accordingly, the set $N_k$ is $\pi/(2k)$-dense in $M$. Interestingly, a similar result holds on any smooth Riemannian manifold (see e.g. [Brü78]):
PROPOSITION 1.4. There exists $C > 0$ (which depends only on $M, g$) such that

$$B(x, C/\mu) \cap \mathcal{N}(\phi_{\mu}) \neq \emptyset$$

for any $x \in M$ and $\mu > 0$.

Here $B(x, r)$ denotes the ball of radius $r$ centered at $x \in M$. Thus $\mathcal{N}_\mu$ is $C/\mu$-dense in $M$.

To study the rate of approximation by $\mathcal{N}_\mu$ as $\mu \to \infty$ in more detail, consider again the case of $M = [0, \pi]$ where approximating by points in $\mathcal{N}_k$ is equivalent (after rescaling by $\pi$) to approximating by rationals with denominator $k$. It is well-known (see e.g. [Khi97]) that the distance from any $x \in [0, 1]$ to the $m$-th convergent of its continued fraction expansion $p_m/q_m$ is $O(1/q_m^2)$. However, the denominator $q_m$ of the $m$-th continued fraction grows exponentially in $m$ for $x \notin \mathbb{Q}$ ([Khi97]).

Denote by $||x||$ the distance from $x \in \mathbb{R}$ to the nearest integer. The following proposition can be found in [Khi97] and is proved by an application of the Borel-Cantelli Lemma.

PROPOSITION 1.5. If $\sum_q \psi(q)$ converges, then for Lebesgue-almost all $x$, there exist only finitely many $q$ such that $||qx|| < \psi(q)$.

Taking $\psi(q) = C/q^{1+\varepsilon}$ in Proposition 1.5 we conclude that:

COROLLARY 1.6. Given $C, \varepsilon > 0$, for Lebesgue-almost all $x \in [0, 1]$ the inequality

$$|x - p/q| < C/q^{2+\varepsilon}$$

has finitely many integer solutions $(p, q)$.

Here, almost all $x \in M = [0, \pi]$ cannot be approximated by points in $\mathcal{N}_k$ to within $C/k^{2+\varepsilon}$ infinitely often. We prove an analogous statement for any real analytic manifold $M$.

To characterize the rate of approximation by nodal sets, we make the following definition:

Definition 1.7. Given $b > 0$ (exponent), and $C > 0$ (constant), let $M(b, C)$ be the set of all $x \in M$ such that there exists an infinite sequence of eigenvalues $\mu_k \to \infty$ for which

$$B \left( x, \frac{C}{\mu_k^b} \right) \cap \mathcal{N}(\phi_{\mu_k}) \neq \emptyset.$$ 

For example, Proposition 1.4 implies that $M(1, C) = M$ for some $C > 0$. Also, Corollary 1.6 implies that for $M = [0, \pi]$, we have $\text{Vol}(M(2+\varepsilon, C)) = 0 \forall C, \varepsilon > 0$. We prove:
Theorem 1.8. Let \((M, g)\) be a closed real analytic Riemannian manifold of dimension \(n\). Then for any \(C > 0, \varepsilon > 0\),

\[
\text{Vol}(M(n + 1 + \varepsilon, C)) = 0.
\]

The proof consists of Theorem 1.2, the Borel–Cantelli Lemma and Weyl’s asymptotics of eigenvalues.

1.1. A reader’s guide. In Sections 2–4 we study eigenfunctions in small scales. The key proposition is Proposition 4.4, which roughly shows that for most points \(x, \phi_\mu(x)\) is comparable to the average of \(\phi_\mu\) on a ball of radius \(\delta\). On a first reading one may assume this proposition. In Section 5 we show how Proposition 4.4 implies geometric information on the nodal set and its neighborhood. Section 6 is a technical section which helps us treat the scales \(\delta = C_1/\mu\) with \(C_1\) large. The results of Sections 4, 5 and 6 are combined in Section 7 in order to prove the lower bound in Theorem 1.2. Section 8 gives the line of proof of the upper bound in Theorem 1.2. On a first reading one may start with this section and move to sections 4 and 5 when necessary. In Section 9 we combine the upper bound in Theorem 1.2 with Weyl’s Law and the Borel-Cantelli Lemma in order to establish Theorem 1.8. In Section 10 we discuss Theorem 1.2 for smooth surfaces. In Section 11 we discuss possible extensions of the approximation result.

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2. Holomorphic functions in small scales: dimension 1. In this section we describe the behavior of holomorphic functions of one variable in small scales. The proofs in this section follow closely the proofs in section 5 of [DF88].

We denote by $B_r \subset \mathbb{C}$ the disk $|z| \leq r$. Suppose $F$ is holomorphic on $B_3$ and satisfies the following growth assumption:

\[(2.1) \quad \sup_{B_2} |F| \leq |F(0)| e^{C_1 \mu}.
\]

Let $I \subset \mathbb{R}$ denote the interval $[-1, 1]$. Let $0 < \delta < 1/\mu$ be given. We decompose $I$ into disjoint subintervals of sizes $\delta < |I_\nu| < 2\delta$. Given $x \in I$, we denote by $I_x$ the subinterval to which $x$ belongs ($I_x$ is defined outside a set of measure 0). We denote by $A_{I_x} F$ the average of $F$ on $I_x$. The main proposition of this section is:

**Proposition 2.2.** Assume $F$ satisfies (2.1). For all $\varepsilon > 0$ there exists a subset $E_\varepsilon \subseteq I$ of measure $|E_\varepsilon| \leq C_2 \varepsilon \mu \delta$ such that

\[
\frac{1}{C_3(\varepsilon)} \leq \frac{|F(x)|}{A_{I_x}|F|} \leq C_3(\varepsilon) \quad \forall x \in I \setminus E_\varepsilon,
\]

with $C_3(\varepsilon) = e^{11/\varepsilon^2}$.

Proposition 2.2 generalizes Proposition 5.1 from [DF88]. The main new point here is the introduction of the parameter $\delta$ of the subdivision, while in [DF88] the size of the subdivision is taken to be comparable to $1/\mu$. A minor technical difference is that here we also allow subdivisions with non-fixed size of the subintervals. This will serve us in the change of variable argument in the proof of Proposition 3.7.

The first step we make is a reduction to polynomials. It is shown in section 5 of [DF88]

**Lemma 2.3.** [DF88, Lemma 5.2] $F$ has at most $C_4 \mu$ zeroes in $B_{3/2}$.

Denote the set of zeroes of $F$ in $B_r$ by $Z_r(F)$. Fix $r < 3/2$ close to $3/2$ so that $F$ does not have zeroes on $|z| = r$. Let $P(z) := \prod_{\alpha \in Z_r(F)} (z - \alpha)$. $P$ is a polynomial of degree $d \leq C_4 \mu$. Let $f(z) = \log |P(z)|$. The next lemma shows that we can assume $F(z) = P(z)$.

**Lemma 2.4.** (i)

\[
|\log |F(x)| - \log |F(y)|| - (f(x) - f(y))| \leq C_5 \forall \forall x, y \in I_\nu.
\]

(ii)

\[
C_6 \frac{|P(x)|}{A_{I_x}|P|} \leq \frac{|F(x)|}{A_{I_x}|F|} \leq C_7 \frac{|P(x)|}{A_{I_x}|P|}.
\]
Proof. Let
\[ B_r(z, \alpha) = \frac{(z - \alpha)/r}{1 - \bar{\alpha}z/r^2} \]
be the Blaschke factor. We write
\[ F(z) = e^{g(z)} \prod_{\alpha \in Z_r(F)} B_r(z, \alpha). \]

We calculate
\[ \log |F(x)| - \log |F(y)| = \Re(g(x) - g(y)) \]
\[ + (\log |P(x)| - \log |P(y)|) \]
\[ - \sum_{\alpha \in Z_r(F)} (\log |r^2 - \alpha x| - \log |r^2 - \alpha y|). \]

The first term on the right-hand side of (2.5) is handled by Lemma 5.3 (iii) of [DF88]:
\[ |\Re(g(x) - g(y))| \leq \max_{I_{\nu}} |\nabla \Re(g)||x - y| \leq C_8 \mu \delta. \]

To bound the third term in the right-hand side of (2.5) one should only check by direct computation that
\[ \sup_{|x| \leq 1} \left| (\log |r^2 - \alpha x|) \right| \leq \sup_{|x| \leq 1} \left| \left( \Re \frac{-\alpha}{r^2 - \alpha x} \right) \right| \leq \frac{1}{r - 1} = C_9. \]

The conclusion of part (i) of the Lemma follows.

Part (i) says that
\[ \forall y \in I_x, \ e^{-C_3 \frac{|P(y)|}{|P(x)|}} \leq \frac{|F(y)|}{|F(x)|} \leq e^{C_3 \frac{|P(y)|}{|P(x)|}}. \]

It only remains to integrate over \( I_x \) in order to conclude part (ii).

We now turn to bound \( |f(x) - f(y)| \). For each \( \nu \) we decompose \( f \) into a good part and a bad part. Let \( A_\nu \) be the set of all roots \( \alpha \) for which \( \text{dist}(\alpha, I_\nu) < \delta \).
\[ g_\nu := \sum_{\alpha \notin A_\nu} \log |x - \alpha|, \quad b_\nu := \sum_{\alpha \in A_\nu} \log |x - \alpha|. \]
We now define bad subsets $E_{j, \varepsilon}$:

$E_{1, \varepsilon} := \{x \in I: \left| f'(x) \right| > 1/(\varepsilon \delta) \}$,
$E_{2, \varepsilon} := \{x \in I: \exists \alpha, |x - \alpha| < \varepsilon \delta \}$,
$E_{3, \varepsilon} := \bigcup \{I_\nu: |A_\nu| > 1/\varepsilon \}$,
$E_{4, \varepsilon} := \bigcup \{I_\nu: \int_{I_\nu} |g''_\nu(x)| \, dx > 1/(\varepsilon \delta) \}$,
$E_{5, \varepsilon} := E_{1, \varepsilon} \cup E_{2, \varepsilon} \cup E_{3, \varepsilon} \cup E_{4, \varepsilon}$,
$E_{6, \varepsilon} := \bigcup \{I_\nu: |I_\nu \cap E_{5, \varepsilon}|/|I_\nu| > 1/2 \}$,
$E_\varepsilon := E_{5, \varepsilon} \cup E_{6, \varepsilon}$.

**Lemma 2.6.** Let $x \in I_\nu \setminus (E_{2, \varepsilon} \cup E_{3, \varepsilon})$. Then,

$$\forall y \in I_\nu, \quad b_\nu(x) - b_\nu(y) \geq -\frac{1}{\varepsilon} \log \frac{3}{\varepsilon},$$

**Proof.** For all $y \in I_\nu$,

$$b_\nu(y) = \sum_{\alpha \in A_\nu} \log |y - \alpha| \leq (\log \varepsilon \delta)/\varepsilon,$$

while $b_\nu(x) \geq (\log (\varepsilon \delta))/\varepsilon$. \qed

**Lemma 2.7.** Let $x \in I_\nu \setminus (E_{2, \varepsilon} \cup E_{3, \varepsilon})$. Then, $|b'_\nu(x)| \leq 1/(\varepsilon^2 \delta)$.

**Proof.** Since $x \not\in E_{2, \varepsilon} \cup E_{3, \varepsilon}$, $|b'_\nu(x)| \leq \sum_{\alpha \in A_\nu} \frac{1}{|x - \alpha|} \leq (1/\varepsilon) \cdot 1/(\varepsilon \delta)$. \qed

**Lemma 2.8.** Let $x \in I_\nu \setminus (E_{1, \varepsilon} \cup E_{2, \varepsilon} \cup E_{3, \varepsilon})$. Then,

$$|g'_\nu(x)| \leq 2/(\varepsilon^2 \delta).$$

**Proof.** Since $x \not\in E_{1, \varepsilon}$, $|f'(x)| \leq 1/(\varepsilon \delta)$. By Lemma 2.7,

$$|b'_\nu(x)| \leq 1/(\varepsilon^2 \delta).$$

It only remains to observe that $|g'_\nu| \leq |f'| + |b'_\nu|$.

**Lemma 2.9.** Suppose $I_\nu \not\subseteq E_{5, \varepsilon}$. Then $\max_{I_\nu} |g'_\nu| \leq 3/(\varepsilon^2 \delta)$.

**Proof.** By Lemma 2.8 $\exists x_\nu \in I_\nu$ such that

$$|g'_\nu(x_\nu)| \leq 2/(\varepsilon^2 \delta).$$

Also, from the definition of $E_{4, \varepsilon}$ and the fundamental theorem of calculus

$$\forall x \in I_\nu, \quad |g'_\nu(x) - g'_\nu(x_\nu)| \leq 1/(\varepsilon \delta).$$

Together we obtain $\forall x \in I_\nu, \quad |g'_\nu(x)| \leq 3/(\varepsilon^2 \delta)$. \qed

**Lemma 2.10.** Suppose $I_\nu \not\subseteq E_{5, \varepsilon}$. Then

$$\forall x, y \in I_\nu, \quad |g_\nu(x) - g_\nu(y)| \leq 6/\varepsilon^2.$$

**Proof.** The proof is an immediate corollary of Lemma 2.9. \qed
LEMMA 2.11. Let $x \in I_\nu \setminus E_{5,\varepsilon}$.

$$\forall y \in I_\nu, \ f(x) - f(y) \geq -9/\varepsilon^2.$$ 

Proof. $f(x) - f(y) = (g_\nu(x) - g_\nu(y)) + (b_\nu(x) - b_\nu(y))$. It only remains to combine Lemmas 2.6 and 2.10. □

LEMMA 2.12. Let $x \in I_\nu \setminus E_\varepsilon$. Then

$$e^{-9/\varepsilon^2}/4 \leq \frac{\mathcal{E}^{f(x)}}{A_{U_\nu} \mathcal{E}^f} \leq 4e^{9/\varepsilon^2}$$

Proof. On the one hand, Lemma 2.11 gives

$$\frac{\mathcal{E}^{f(x)}}{(\int_{I_\nu} \mathcal{E}^f \, dx)/|I_\nu|} \leq \frac{\mathcal{E}^{f(x)}}{(\int_{I_\nu \setminus E_\varepsilon} \mathcal{E}^f \, dx)/|I_\nu|} \leq \frac{e^{f(x) - \max_{I_\nu} f}}{(\int_{I_\nu \setminus E_\varepsilon} e^{f(x) - \max_{I_\nu} f} \, dx)/|I_\nu|} \leq \frac{1}{|E_\varepsilon|/|I_\nu|} e^{-9/\varepsilon^2}/2 \leq \frac{1}{e^{-9/\varepsilon^2}/4} = 4e^{9/\varepsilon^2}.$$ 

On the other hand, Lemma 2.11 also gives

$$\frac{\mathcal{E}^{f(x)}}{(\int_{I_\nu} \mathcal{E}^f \, dx)/|I_\nu|} \geq \frac{\mathcal{E}^{f(x)}}{\mathcal{E}^{\max_{I_\nu} f}} = e^{f(x) - \max_{I_\nu} f} \geq e^{-9/\varepsilon^2}.$$ □

We now turn to estimating the size of the bad subset $E_\varepsilon$.

LEMMA 2.13. $|E_{4,\varepsilon}| < C_9 \varepsilon \mu \delta$.

Proof. This follows from the properties of the Hilbert Transform. We imitate the proof of Lemma 5.4 in [DF88] with a little more detail.

We recall the definition and some basic properties of the Hilbert Transform. Let $u \in L^2(\mathbb{R})$. Let $\text{sgn}$ be the sign function on $\mathbb{R}$. Let $\mathcal{F}$ be the Fourier Transform on $L^2(\mathbb{R})$. Define the Hilbert Transform $\mathcal{H}u$ by

$$\mathcal{F}(\mathcal{H}u) = \frac{i}{2} \text{sgn} \cdot \mathcal{F}(u).$$

From this definition it is clear that $\mathcal{H}$ is a bounded operator on $L^2(\mathbb{R})$. Observe that

$$f'(x) = \sum_{\alpha} \Re \left( \frac{1}{x - \alpha} \right).$$
We may assume \( \forall \alpha, \Im \alpha \leq 0 \). Consider first the case where \( \forall \alpha, \alpha \notin \mathbb{R} \). Let \( q_{\alpha}(x) = -\Im(1/(x - \alpha)) \), and \( q = \sum_{\alpha} q_{\alpha} \). Then, \( q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and by Theorem 3 in III.2.3 of [Ste70] \( Hq = f' \). From the fact that \( \operatorname{sgn} f' = 2\delta_0 \) and by basic properties of the Fourier Transform one sees that if \( u \in L^2(\mathbb{R}) \) has a compact support and \( x \notin \text{Supp} u \), then

\[
(Hu)(x) = \int_{\mathbb{R}} \frac{u(y)}{x-y} dy
\]

(see also exc. 1.9 in [GS94] and Theorem 5 in III.3.3 of [Ste70]). We have verified that the conditions of Theorem 3 in I.5 of [Ste93] are fulfilled for the Hilbert Transform. We conclude that the Hilbert Transform is of weak type \((1,1)\) and we get

\[
|\{|f'| > 1/(\varepsilon \delta)\}| \leq C_{10}\varepsilon\delta \|q\|_1 \leq C_{11}\varepsilon\mu\delta.
\]

(2.14)

Finally, we move to the case where \( \exists \alpha \in \mathbb{R} \). Define \( g_t(x) := f'(x - it) \). A small calculation shows that \( g_t \to f' \) in measure as \( t \to 0 \). Since we can apply the considerations above to \( g_t \) we conclude that the assertion in the lemma is true with \( C_9 = 2C_{11} \).

\[\text{LEMMA 2.15.}\]

\[|E_{2,\varepsilon}| \leq C_{12}\varepsilon\mu\delta.\]

\[\text{Proof.}\] Proof is obvious. \(\square\)

\[\text{LEMMA 2.16.}\]

\[|E_{3,\varepsilon}| \leq C_{13}\varepsilon\mu\delta.\]

\[\text{Proof.}\] This is an immediate corollary of Lemma 2.3. \(\square\)

\[\text{LEMMA 2.17.}\] \(|E_{4,\varepsilon}| \leq C_{14}\varepsilon\mu\delta.\)

\[\text{Proof.}\] We observe that

\[
g''_\nu(x) = -\sum_{\alpha \notin A_\nu} \Re \left( \frac{1}{(x-\alpha)^2} \right).
\]

Hence,

\[
\sum_{\nu} \int_{I_\nu} |g''_\nu(x)| \, dx \leq \sum_{\nu} \sum_{\alpha \notin A_\nu} \int_{I_\nu} \left| \frac{1}{(x-\alpha)^2} \right| \, dx
\]

\[
= \sum_{\nu} \sum_{\alpha \notin A_\nu} \int_{I_\nu} \frac{1}{|x-\alpha|^2} \, dx
\]

\[
\leq \sum_{\alpha} \int_{|x-\alpha| > \delta} \frac{1}{|x-\alpha|^2} \, dx \leq \mu/\delta.
\]
On the other hand
\[ \sum \nu \int_{I_\nu} |g''_\nu(x)| \, dx \geq \sum_{\nu, \nu \subseteq E_\nu} \int_{I_\nu} |g''_\nu(x)| \, dx \geq \# \{ \nu : I_\nu \subseteq E_\nu \} / (\delta \varepsilon). \]

Together, we get that \# \{ \nu : I_\nu \subseteq E_\nu \} \leq \varepsilon \mu. Hence, \(|E_{\varepsilon,4}| \leq C_{14} \varepsilon \mu \).

**Lemma 2.18.** \(|E_{\varepsilon,4}| \leq C_{15} \varepsilon \mu \)

**Proof.** Let \( N \) be the number of intervals \( I_\nu \) for which \(|I_\nu \cap E_{5,\varepsilon}|/|I_\nu| > 1/2. \)

We have
\[ C_{16} \varepsilon \mu \delta \leq |E_{5,\varepsilon}| = \sum |I_\nu \cap E_{5,\varepsilon}| \geq N|I_\nu|/2 \geq N\delta/2. \]

Hence, \( N \leq 2C_{16} \varepsilon \mu \). It follows that \(|E_{6,\varepsilon}| \leq 4C_{16} \varepsilon \mu \delta \).

This completes the proof of Proposition 2.2.

### 3. Holomorphic functions in small scales: dimension \( n > 1 \).

In this section we prove an analog of Proposition 2.2 in dimension \( n > 1 \). We adjust the proof of Proposition 5.11 in [DF88].

Let \( F \) be a holomorphic function defined in the polydisk \( B^n_3 = B_3 \times \cdots \times B_3 \subseteq \mathbb{C}^n \). Let \( I = [1, 1] \subseteq \mathbb{R} \) and \( Q = I^0 \). Assume \( F \) satisfies

\[ \sup_{B_2^n} |F| \leq |F(0)| e^{C_{1.1} \mu}. \tag{3.1} \]

Given \( 0 < \delta < C_2/\mu \), decompose \( Q \) into subboxes \( Q_\nu \) in the following way:

First, we define \( n \) decompositions of \( I \) into intervals \( \{I^{(k)}_i\} \) where \( \delta < |I^{(k)}_i| < 2\delta \)

\( 1 \leq k \leq n \) and \( 1 \leq l \leq N(k) \). Given a multi-index \( \nu = (\nu_1, \ldots, \nu_n), 1 \leq \nu_k \leq N(k) \), we set \( Q_\nu = I^{(1)}_{\nu_1} \times \cdots \times I^{(n)}_{\nu_n} \). Given \( x \in Q \), we denote by \( Q_x \) the subbox which contains \( x \). We prove:

**Proposition 3.2.** Let \( F \) satisfy (3.1) and \( F \geq 0 \) on \( Q \). Assume that \( F \equiv 1 \) on each of the hyperplanes \( z_i = 0 \). For all \( \varepsilon > 0 \) there exists a subset \( E_\varepsilon \subseteq Q \) of measure \(|E_\varepsilon| \leq C_3 \varepsilon \mu \delta \) such that

\[ \frac{1}{C_4(\varepsilon)} \leq \frac{F(x)}{A \nu Q_x F} \leq C_4(\varepsilon) \quad \forall x \in Q \setminus E_\varepsilon, \tag{3.3} \]

with \( C_4(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \).

**Proof.** For \( n = 1 \), the proposition reduces to Proposition 2.2. For \( n > 1 \), let \( E_\varepsilon \) be the subset of all \( x \in Q \) for which the inequalities in (3.3) are not true with \( C_4(\varepsilon) = e^{11n/\varepsilon^2} \). Given \( \zeta = (z_1, \ldots, z_{n-1}) \in B_3^{n-1} \) we define

\[ F_\zeta(z) = F(z', z). \]
$F_{z'}$ has the following properties:

- $F_{z'}$ is defined in $B_3$.
- If $z' \in B_2^{n-1}$, then $\sup_{B_2} |F_{z'}| \leq e^{C_1 \mu}$.
- If $Q' = I^n$ and $x' \in Q'$ then $F_{x'} \geq 0$ on the interval $[-1, 1]$.
- If $x' \in Q'$ then $F_{x'}(0) = 1$.

We have checked that for $x' \in Q'$,

$F_{x'}$ satisfies the conditions in Proposition 2.2.

Let $Q'_{\nu} = I_{n-1}$. If $x' \in Q'_{\nu}$ then $G_{x'}(0) = 1$.

It is easy to check that

- $G_{t'}$ is defined in $B_2^{n-1}$.
- $\sup_{B_2} |G_{t'}| \leq e^{C_1 \mu}$.
- $G_{t'}(x') \geq 0$ for $x' \in Q'$.
- $G_{t'}(z') = 1$ whenever one of the coordinates $z_i = 0$.

Thus, by the induction hypothesis applied to $G_{t'}$ and the decomposition $Q'_\nu$, we get a corresponding bad subset $E_{\nu}(G_{t'})$. We set $E_{\nu} := E_{\nu}(G_{t'}) \times I_{n-1}$. Let $E_{\nu} := \bigcup_{1 \leq i \leq N(n)} E_{\nu}^i$.

Claim 3.4. $E_{\nu} \subseteq E_{\nu}^i \cup E_{\nu}^{(n)}$.

Proof. Let $x \in Q' \setminus (E_{\nu}^i \cup E_{\nu}^{(n)})$. Since $x' \notin E_{\nu}(G_{t'})$ we have

$$e^{-11(n-1)/\varepsilon^2} \leq \frac{G_{t'}(x')}{A_{Q'^i} G_{t'}} \leq e^{11(n-1)/\varepsilon^2}.$$  \hspace{1cm} (3.5)

Since $x_n \notin E_{\nu}(F_{x'})$ we have

$$e^{-11/\varepsilon^2} \leq \frac{F_{x'}(x_n)}{A_{Q'^i} F_{x'}} \leq e^{11/\varepsilon^2}. $$  \hspace{1cm} (3.6)

Now recall that

$$F_{x'}(x_n) = F(x), \quad A_{Q'^i} F_{x'} = G_{t'}(x'),$$

and observe that $A_{Q'^i} G_{t'} = A_{Q'} F$. To complete the proof of Claim 3.4 we multiply (3.5) by (3.6). \qed

It only remains to check that the size of $E_{\nu}$ is not too big: By Claim 3.4 we know that $E_{\nu} \setminus E_{\nu}'$ is a measurable set all of whose intersections with lines parallel...
to the $x_n$-axis are measurable sets of sizes $\leq C_5 \varepsilon \mu \delta$. By Fubini’s Theorem, we get $|E_e \setminus E'_{e}| \leq C_6 \varepsilon \mu \delta$. By Fubini’s Theorem, we get $|E_e \setminus E'_{e}| \leq \sum_{l} |E_{e}(G_l)| \delta \leq C_7 N(n) \varepsilon \delta^2 \mu$. But $N(n) \leq 2/\delta$.

This completes the proof of Proposition 3.2, since $|E_e| \leq |E_e \setminus E'_{e}| + |E'_{e}|$.

We now remove the technical assumption in proposition 3.2. The main proposition of this section is:

**Proposition 3.7.** Let $F$ satisfy (3.1) and $F \geq 0$ on $Q$. There exists a cube $R \subseteq Q$ independent of $F$ with the following property: Suppose $\mu \delta < C_8$. We decompose $R$ into boxes $R_{e,l}$ of sides $\delta < \ell_{l}^{(k)} < 2 \delta$. Then, there exists a subset $E_{e} \subseteq R$ of measure $|E_{e}| \leq C_9 \varepsilon \mu \delta$ such that

\[
\frac{1}{C_{10}(\varepsilon)} \leq \frac{F(x)}{A_{U_{R,\delta}} F} \leq C_{10}(\varepsilon) \quad \forall x \in R \setminus E_{e},
\]

with $C_{10}(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

**Proof.** We construct $R$ in the same way as in [DF88]:

**Lemma 3.9.** [DF88, Lemma 5.10] There exists a map $W: \mathbb{R}^n \to \mathbb{R}^n$ which extends to a map $\hat{W}: \mathbb{C}^n \to \mathbb{C}^n$, with the following properties:

1. $W$ is a polynomial map.
2. $\hat{W}(B_2^n) \subseteq B_3^n$.
3. $\hat{W}(B_1^n) \subseteq B_2^n$.
4. $W(Q) \subseteq Q$.
5. $W$ maps the hyperplanes $x_i = 0$ to 0.
6. $W$ is a local diffeomorphism outside the hyperplanes $x_i = 0$.

Let $U \subseteq Q$ be an open set which is mapped diffeomorphically onto $W(U)$ and has a positive distance from any hyperplane $x_i = 0$. We let $R \subseteq W(U)$ be any cube with sides parallel to the sides of $Q$.

Let us now describe the bad subset $E_{e}$. We begin with:

**Lemma 3.10.** There exists a finite number of subdivisions $D_i$ of $Q$ into boxes $Q_{\nu,i}$ of sides $\delta < \ell_{i}^{(k)} < 2 \delta$ such that every set of diameter $< \delta / 2$ is contained in a box $Q_{\nu,i}$ for some $\nu$ and $i$.

The function $\tilde{F} = F \circ \hat{W} / F(0)$ satisfies the conditions of Proposition 3.2. So, given any of the subdivisions $D_i$ of Lemma 3.10 we can find an exceptional set $\tilde{E}_{e,l} \subseteq Q$ corresponding to $\tilde{F}$. Let $\tilde{E}_{e} = \cup l \tilde{E}_{e,l}$. Set $E^0_{e} = W(\tilde{E}_{e} \cap U) \cap R$.

Call $R_{\alpha} \varepsilon$-bad if $|E^0_{e} \cap R_{\alpha}| / |R_{\alpha}| > 1/2$. Let $B_{\varepsilon}$ be the union of all $\varepsilon$-bad $R_{\alpha}$’s. Finally, set $E_{e} = E^0_{e} \cup B_{\varepsilon}$. We now estimate the size of $E_{e}$.

**Lemma 3.11.** $|E_{e}| \leq C_{11} \varepsilon \mu \delta$. 


Proof of Lemma. Since the Jacobian of the map $W$ is bounded on $U$, and $|\tilde{E}_\varepsilon| \leq C_{12}\varepsilon\mu\delta$ we conclude that $|E_\varepsilon^0| \leq C_{13}\varepsilon\mu\delta$. We estimate $|B_\varepsilon|$

$$C_{13}\varepsilon\mu\delta \geq |E_\varepsilon^0| \geq \sum_{\text{bad } R_\alpha, s} |E_\varepsilon^0 \cap R_\alpha| \geq (1/2)\#(\text{bad } R_\alpha, s)|R_\alpha| \geq C_{14}|B_\varepsilon|.$$ 

We got $|E_\varepsilon| \leq |E_\varepsilon^0| + |B_\varepsilon| \leq C_{11}\varepsilon\mu\delta$.

The last step is to check that (3.8) is true: Let $R_\alpha$ be a subbox of $R$ with sides $C_{15}\delta \leq l_\alpha^{(k)} \leq 2C_{15}\delta$, where $C_{15}$ is small enough. Look at $\tilde{R}_\alpha = W^{-1}(R_\alpha)$. Since $W^{-1}$ has a bounded Jacobian on $R$, $\tilde{R}_\alpha$ is a set of diameter $< \delta/2$. Let $D$ be one of the subdivisions of $Q$ from Lemma 3.10 whose one of its boxes $Q_\nu$ contains $\tilde{R}_\alpha$.

It follows from Proposition 3.2 that $\tilde{F}(y_1)/\tilde{F}(y_2) \leq C_4(\varepsilon)^2 \forall y_1, y_2 \in Q_\nu \setminus \tilde{E}_\varepsilon$. Hence, if we let $x_0 \in R_\alpha \setminus E_\varepsilon$ and $y_0 = W^{-1}(x_0)$, then $y_0 \in \tilde{R}_\alpha \setminus \tilde{E}_\varepsilon$ and we obtain

$$\text{Av}_{R_\alpha} F = \frac{1}{|R_\alpha|} \int_{R_\alpha} F(x) \, dx = \frac{1}{|R_\alpha|} \int_{R_\alpha \setminus E_\varepsilon} F(x) \, dx$$

$$= \frac{1}{|R_\alpha|} \int_{R_\alpha \setminus E_\varepsilon} \tilde{F}(y)|J_W| \, dy \geq \frac{1}{C_4(\varepsilon)^2|R_\alpha|} \int_{R_\alpha \setminus E_\varepsilon} \tilde{F}(y_0)|J_W| \, dy$$

$$= \frac{|R_\alpha \setminus E_\varepsilon|}{C_4(\varepsilon)^2|R_\alpha|} F(x_0) \geq F(x_0)/(2C_4(\varepsilon)^2).$$

On the other hand,

$$\text{Av}_{R_\alpha} F = \frac{1}{|R_\alpha|} \int_{R_\alpha} F(x) \, dx = \frac{1}{|R_\alpha|} \int_{R_\alpha} \tilde{F}(y)|J_W| \, dy$$

$$\leq \frac{1}{|R_\alpha|} \int_{Q_\nu} \tilde{F}(y)|J_W| \, dy$$

$$\leq \frac{C_{16}|Q_\nu|}{|R_\alpha|} \frac{1}{|Q_\nu|} \int_{Q_\nu} \tilde{F}(y) \, dy \leq C_{17} \text{Av}_{Q_\nu} \tilde{F}$$

$$\leq C_{17}C_4(\varepsilon)\tilde{F}(y_0) = C_{18}(\varepsilon)F(x_0).$$

Inequalities (3.12) and (3.13) complete the proof of Proposition 3.7.

4. Eigenfunctions in small scales on real analytic manifolds. Let $\phi_\mu$ be an eigenfunction. Let $V$ be a small open set in which the metric $g$ can be developed in power series. We identify $V$ with a ball $B(0, \rho_0) \subseteq \mathbb{R}^n$. We prove:

PROPOSITION 4.1. There exists a cube $R \subseteq V$ with the following property: Suppose $\mu\delta < C_1$. We subdivide $R$ into boxes $R_\varepsilon$ of sides $\delta < l_\nu^{(k)} < 2\delta$. Then for all
\( \varepsilon > 0 \) there exists a subset \( E_\varepsilon \subseteq R \) of measure \( |E_\varepsilon| \leq C_2 \varepsilon \mu \delta \) such that

\[
\frac{1}{C_3(\varepsilon)} \leq \frac{\phi_\mu(x)^2}{A_{\mu R_\varepsilon} \phi_\mu^2} \leq C_3(\varepsilon) \quad \forall x \in R \setminus E_\varepsilon,
\]

with \( C_3(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \).

**Proof.** We consider an analytic continuation of \( \phi_\mu \). In order to avoid confusion we denote by \( B_R(0, \rho_0) \), \( B_C(0, \rho_1) \) balls in \( \mathbb{R}^n \), \( \mathbb{C}^n \) respectively.

**Lemma 4.2.** [DF88, Lemma 7.3] \( \phi_\mu |_{B_R^c(0, \rho_0)} \) has an analytic continuation \( F \) defined on \( B_C^c(0, \rho_1) \) for some \( \rho_1 < \rho_0 \). Moreover, the function \( F \) satisfies

\[
\sup_{B_C^c(0, \rho_1)} |F| \leq e^{C_4} \sup_{B_R^c(0, \rho_0)} |\phi_\mu|.
\]

The crucial point is that the domain to which the function \( \phi_\mu \) can be continued is independent of \( \mu \).

Let \( \rho_2 = \rho_1 / C_5 \) with \( C_5 \) large so that the polydisk \( B_{\rho_2}^n \subseteq B_C^c(0, \rho_1) \). We now recall the Donnelly-Fefferman Growth Bound:

**Theorem 4.3.**

\[
\sup_{B_{\rho_2}^n} |\phi_\mu| \leq e^{C_6(\rho_0/\rho_2)} \sup_{B(0, \rho_2)} \phi_\mu.
\]

Lemma 4.2 and Theorem 4.3 give

\[
\sup_{B_{\rho_2}^n} |F| \leq e^{C_7} \sup_{B(0, \rho_2)} \phi_\mu.
\]

Now, shift the coordinate system to be centered on the point \( x \in B(0, \rho_2) \) for which \( \phi_\mu(x) = \sup_{B(0, \rho_2)} \phi_\mu \). We get that

\[
\sup_{B_{\rho_2}^n} |F| \leq e^{C_7} |F(0)|.
\]

Hence, we can conclude the proof by applying Proposition 3.7 to \( F^2 \).

We need a slightly different version of this proposition. We say that \( R_\alpha \) touches \( R_\beta \) if they have at least one vertex in common. Each box \( R_\alpha \) touches at most \( 3^n \) boxes. Let us denote by \( R_\alpha^* \) the union of the box \( R_\alpha \) with all boxes which touch \( R_\alpha \). There exist \( 3^n \) subdivisions \( D_i \) of \( R \) such that each box of \( D_i \) is equal to \( R_\alpha^* \) for some \( \alpha \). Let \( E_\varepsilon = \bigcup_i E_{\varepsilon, i} \) where \( E_{\varepsilon, i} \) is the bad subset corresponding to the
subdivision $D_i$ according to Proposition 4.1. $|E_\varepsilon| \leq C_8 \varepsilon \mu \delta$. These considerations prove the following version of Proposition 4.1.

**Proposition 4.4.** There exists a cube $R \subseteq V$ with the following property: Suppose $\mu \delta < C_9$. We subdivide $R$ into boxes $R_\nu$ of sides $\delta < l^{(k)}_\nu < 2\delta$. There exists a subset $E_\varepsilon \subseteq R$ of measure $|E_\varepsilon| \leq C_8 \varepsilon \mu \delta$ such that

$$\frac{1}{C_{10}(\varepsilon)} \leq \frac{\phi(x)^2}{A_{R_\nu} \phi_{R_\nu}^2} \leq C_{10}(\varepsilon) \quad \forall x \in R \setminus E,$$

with $C_{10}(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

5. **Good boxes — bad boxes.** Let $F$ be a nonnegative function defined on a cube $R$. Let $D$ be a subdivision of $R$. We divide the boxes $R_\nu$ into good and bad. We show that in a vicinity of a good box we have a bounded $L^2$-growth, and that the geometry is under control. We show that the proportion of bad boxes is small.

We always assume that the sides of all boxes of a subdivision are of comparable sizes. Moreover, we assume that any two boxes $R_{\nu_1}, R_{\nu_2}$ satisfy

$$\frac{\max \text{ side } (R_{\nu_1})}{\min \text{ side } (R_{\nu_2})} \leq 5.$$

We recall that $R_{\nu}^*$ denotes the union of $R_{\nu}$ with its $3^n - 1$ neighbors.

**Definition 5.1.** Let $E \subseteq R$ be such that

$$\frac{1}{A} \leq \frac{F(x)}{A_{R_{\nu}} F} \leq A \quad \forall \nu \forall x \in R_{\nu} \setminus E.$$

We say that $(F, D, E, A)$ is true.

**Definition 5.2.** Given $E \subseteq R$, we say that $R_{\nu}$ is $E$-good if $|E \cap R_{\nu}|/|R_{\nu}| < \omega_n 10^{-2n}$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. Otherwise $R_{\nu}$ is called $E$-bad.

The next lemma shows that in the vicinity of any good box we have bounded growth. $2Q$ denotes a box concentric with $Q$, whose sides are parallel to the sides of $Q$ and twice as large.

**Lemma 5.3.** Suppose that $(F, D, E, A)$ is true. Let $R_{\nu}$ be $E$-good. Let $B \subseteq R_{\nu}$ be a ball such that $2B \subseteq R_{\nu}^*$ and whose radius $r \geq \text{side}(R_{\nu})/20$. Then,

$$\int_B F \, dx \geq C_1 A^{-1} \int_{2B} F \, dx.$$
Proof.

\[ \int_B F(x) \, dx \geq \int_{B \setminus E} F(x) \, dx \geq A^{-1} \int_{B \setminus E} AVR^* \, F \, dx \]
\[ = A^{-1} \frac{|B \setminus E|}{|R^*_\nu|} \int_{R^*_\nu} F \, dx \geq A^{-1} \frac{|B| - |E \cap R^*_\nu|}{|R^*_\nu|} \int_{2B} F \, dx \]
\[ \geq \omega_\mu (20^{-n} - 10^{-2n}) A^{-1} \frac{|R^*_\nu|}{|R^*_\nu|} \int_{2B} F \, dx \]
\[ \geq C_1 A^{-1} \int_{2B} F \, dx. \]

The next proposition shows that the geometry in good cubes is controlled.

**Proposition 5.4.** Suppose \((\phi^2_\mu, D, E, A)\) is true. Let \(R^*_\nu\) be \(E\)-good. Let \(B = B(o, 2r) \subseteq R^*_\nu\) be a ball of radius \(2r\), with \(r > \text{side}(R^*_\nu)/20\). Let \(B^* = B \cap \{\phi_\mu > 0\}\). Similarly, define \(B^-\). Suppose \(\phi_\mu(o) = 0\). Then

\[ \frac{1}{C_2 (\log A)^{n-1}} \leq \frac{\text{Vol}(B^*)}{\text{Vol}(B^-)} \leq C_2 (\log A)^{n-1}. \]

**Proof.** After rescaling the ball \(B\) to the unit ball, \(\phi_\mu\) becomes a solution \(\varphi\) of an elliptic equation

\[ - \partial_i (a^{ij} \partial_j \varphi) - (2r \mu)^2 q \varphi = 0. \]

It’s important to observe that the coefficients are bounded independently of \(\mu\), and the zero order coefficient is small. Thus, by Lemma 5.3 and by elliptic regularity

\[ \sup_{B_{1/4}} |\varphi| \leq C_3 \|\varphi\|_{L^2(B_1)} \leq C_4 A^{1/2} \|\varphi\|_{L^2(B_{1/2})} \leq C_5 A^{1/2} \sup_{B_{1/2}} |\varphi|. \]

Recall now:

**Theorem 5.7.** [Man08, Theorem 4.7] Let \(\varphi\) satisfy equation (5.5) in the unit ball \(B_1\). Suppose \(\varphi(0) = 0\) and satisfies (5.6). Then

\[ \frac{\text{Vol}(B_1 \cap \{\varphi > 0\})}{\text{Vol}(B_1)} \geq \frac{C_6}{(\log A)^{n-1}}. \]

By symmetry, we have a lower bound also on \(\text{Vol}(B_1 \cap \{\varphi < 0\}) / \text{Vol}(B_1)\). Thus, we get upper and lower bounds on the ratio between the volumes of the positivity and the negativity sets of \(\varphi\). \qed
The last lemma in this section shows that the proportion of bad cubes is small.

**Lemma 5.9.**

\[
\frac{\#(E\text{-bad boxes})}{\#(all\ boxes)} \leq C_7 |E|.
\]

**Proof.**

\[
|E| \geq \sum_{\text{bad } R_\nu\text{'s}} |E \cap R_\nu| \geq \omega n 10^{-n} \#(\text{bad } R_\nu\text{'s}) |R_\nu| \geq C_8 \frac{\#(\text{bad } R_\nu\text{'s})}{\#(all\ boxes)}.
\]

6. **Refinements of subdivisions.** In this section we analyze what happens when we pass from a fine subdivision \(D_1\) to a subdivision \(D\) of whose \(D_1\) is a refinement. We show, roughly, that if a box of \(D\) is composed of smaller good boxes then it is also good. The results in this section are applied in the proof of the lower bound in Theorem 1.2.

Let \(D_1\) be a subdivision of a cube \(R\) obtained by a refinement of \(D\). If the sides of every box in \(D\) are partitioned into \(\leq M\) intervals, we write \([D : D_1] \leq M\). Let \(D^2\) be the subdivision of \(R\) which is formed by the centers of the boxes in \(D_1\).

Throughout this section \(F\) is a nonnegative function defined on \(R\). In the next proposition we use the terminology from Definitions 5.1 and 5.2.

**Proposition 6.1.** Suppose that \((F, D^1, E^1, A)\) and \((F, D^2, E^2, A)\) are true. Let \(B\) be the union of all boxes \(R_\nu\) of \(D\) for which \(R_\nu^*\) contains an \(E^1\)-bad box of \(D_1\) or an \(E^2\)-bad box of \(D^2\). Assume \([D : D^1] \leq M\). Then, \((F, D, E^1 \cup E^2 \cup B, C_1 A^{8M+1})\) is true.

**Proof.** Let us denote the boxes of \(D\) by \(R_\nu\), and the boxes of \(D^1\) by \(R_\alpha\). Let \(E = E^1 \cup E^2 \cup B\).

**Lemma 6.2.** For all \(x \in R_\nu \setminus E\), \(y \in R_\nu^* \setminus E\) there exists a sequence of \(4M\) points \(x = x_1, \ldots, x_{4M} = y\) in \(R_\nu^* \setminus E\) such that any consecutive pair \(x_k, x_{k+1}\) is contained in a box of \(D^1\) or is contained in a box of \(D^2\).

**Proof of Lemma.** The only point to observe is that if \(R_\nu \subseteq E\) then all boxes \(R_\alpha\) contained in \(R_\nu^*\) are \(E^1\)-good. So \(|E^1 \cap R_\alpha| / |R_\alpha| < \omega n 10^{-2n} < 2^{-n}/2\). Similarly for \(E^2\).

Let \(x \in R_\nu \setminus E\). For any \(y \in R_\nu^* \setminus E\), let \(x_1, \ldots, x_{4M}\) be a sequence of points as in Lemma 6.2. Since \((F, D^1, E^1, A)\) and \((F, D^2, E^2, A)\) are true we have \(F(x_k)/F(x_{k+1}) \leq A^2\), and we get \(F(x)/F(y) \leq A^{8M}\).
Now,
\[
\frac{1}{|R_\nu^*|} \int_{R_\nu^*} F(y) \, dy \geq \frac{1}{|R_\nu^*|} \int_{R_\nu^* \setminus E} F(y) \, dy \\
\geq \frac{|R_\nu^* \setminus E|}{|R_\nu^*|} \frac{F(x)}{A^M} \geq 3^{-n} (1 - 2\omega_\nu 10^{-2n}) \frac{F(x)}{A^M}.
\]

The last inequality is true since \( R_\nu \) contains no \( E_1 \)-bad boxes neither \( E_2 \)-bad boxes.

Conversely, let \( x \in R_\nu \) and let \( J \) be the set of \( \alpha \)'s for which \( R_\alpha \subseteq R_\nu^* \). For all \( \alpha \in J \), let \( x_\alpha \in R_\alpha \setminus (E_1 \cup E_2) \). Such points exist, since \( R_\nu \not\subseteq B \).

Then,
\[
\frac{1}{|R_\nu^*|} \int_{R_\nu^*} F(y) \, dy = \frac{1}{|R_\nu^*|} \sum_{\alpha \in J} \int_{R_\alpha} F(y) \, dy \\
\leq \frac{1}{|R_\nu^*|} \sum_{\alpha \in J} AF(x_\alpha) |R_\alpha^*| \leq 3^n A^{8M+1} F(x).
\]

7. Proof of the lower bound in Theorem 1.2. First, we prove the following proposition which is announced in the introduction of [DF88].

PROPOSITION 7.1. There exists a finite collection of balls \( B_i = B(x_i, r) \) centered at \( x_i \) of radius \( r = C_1/\mu \) which satisfy:

(i) \( \phi_\mu(x_i) = 0 \), and

(ii) their doubles \( 2B_i = B(x_i, 2r) \) are pairwise disjoint.

(iii) Denote by \( B_i^+ \) the set \( \{ \phi_\mu > 0 \} \cap B_i \). Similarly, we define \( B_i^- \). Then

\[
C_2 < \frac{\text{Vol}(B_i^+)}{\text{Vol}(B_i^-)} < C_3,
\]

(iv) \( \sum_i \text{Vol}(B_i) > C_4 \text{Vol}(M) \).

Proof. It is enough to prove the proposition in a coordinate neighborhood \( V \). It is well known that there exists a constant \( C_5 \) such that every cube of side \( C_5/\mu \) contains a zero of \( \phi_\mu \) (see [Br"u78]). We can decompose \( V \) into small cubes \( R_\nu \) whose side is of size \( \delta = 3C_5/\mu \). We call this subdivision \( \mathcal{D} \). Each cube \( R_\nu \) contains a zero \( x_\nu \) of \( \phi_\mu \) in its middle third. We now take a refinement \( \mathcal{D}^1 \) of \( \mathcal{D} \). We partition each side of a cube \( R_\nu \) into \( M \) intervals of equal sizes, where \( \mu \delta / M \) is small enough in order to apply Proposition 4.4. We deduce that \( (\phi_\mu^2, \mathcal{D}^1, E_1^\epsilon, \epsilon) \) is true (cf. Definitions 5.1 & 5.2). If \( \mathcal{D}^2 \) is the subdivision obtained by taking the centers of cubes belonging to \( \mathcal{D}^1 \), then the same proposition gives that \( (\phi_\mu^2, \mathcal{D}^2, E_2^\epsilon, \epsilon) \) is true. Let \( B \) be as in Proposition 6.1, and let \( E = E_1^\epsilon \cup E_2^\epsilon \cup B \). Then, \( (\phi_\mu^2, \mathcal{D}, E, \epsilon) \) is true.
For each \( E \)-good cube \( R_\nu \) we pick a ball \( B_\nu \subset R_\nu \) whose center is \( x_\nu \) and whose radius = \( \delta/6 \). By Proposition 5.4
\[
\frac{1}{C_{8}(\varepsilon)} \leq \frac{\text{Vol}(B_\nu^+)}{\text{Vol}(B_\nu)} \leq C_{8}(\varepsilon).
\]
The crucial point is to estimate the number of \( E \)-good cubes. By Lemma 5.9, the proportion of \( E \)-good cubes is \( \geq (1 - C_9|E|) \) (which can be negative). It only remains to estimate \(|E|: |E_1^1| \leq C_{10}\varepsilon\mu\delta, |E_2^1| \leq C_{11}\varepsilon\mu\delta \) and
\[
|B| \leq \delta_n^3(\#(E_3^1)\text{-bad cubes} + \#(E_2^2)\text{-bad cubes}) \leq C_{12}(|E_1^1| + |E_2^2|) \leq C_{13}\varepsilon\mu\delta.
\]
So \(|E| \leq C_{14}\varepsilon\mu\delta\). To conclude, we take \( \varepsilon \) small enough in order that the proportion of good cubes is \( \geq 70\% \).

\( \square \)

**Proof of Theorem 1.2: Lower Bound.** The next proposition gives a lower bound in a good ball.

**PROPOSITION 7.2.** Let \( B(x, r) \) be one of the balls described above. Then we have \( \text{Vol}(T_{\mu,\delta} \cap 2B) \geq C_{15}r^{n-1}\delta \), whenever \( \mu\delta < C_{16} \).

**Proof:** Let \( (B^+)_{\delta} \) be a \( \delta \)-neighborhood of \( B^+ \), and similarly for \( (B^-)_{\delta} \). Since \( T_{\mu,\delta} \cap 2B \supseteq (B^+)_{\delta} \cap (B^-)_{\delta} \), it is clear that
\[
\text{Vol}(T_{\mu,\delta} \cap 2B) \geq \text{Vol}(B^+)_{\delta} + \text{Vol}(B^-)_{\delta} - \text{Vol}(B(x, r + \delta)).
\]
Assume first that the metric \( g \) is flat on \( 2B \). By the Brunn-Minkowski Inequality [Fed69, §3.2.41] we know
\[
\text{Vol}(B^+)_{\delta} \geq \text{Vol}(B^+) + n\omega_n^{1/n}\delta \text{Vol}(B^+)^{1-1/n},
\]
where \( \omega_n \) is the volume of the \( n \)-dimensional unit ball. We have the same inequality for \( (B^-)_{\delta} \). Set \( \text{Vol}(B^+) = \alpha \text{Vol}(B) \), and \( \text{Vol}(B^-) = (1 - \alpha) \text{Vol}(B) \). We have
\[
\text{Vol}(T_{\mu,\delta} \cap 2B) \geq \text{Vol}(B) - \text{Vol}(B(x, r + \delta))
\]
\[
+ n\omega_n^{1/n}\delta \text{Vol}(B)^{1-1/n} \left(\alpha^{1-1/n} + (1 - \alpha)^{1-1/n}\right)
\]
\[
\geq \omega_n(r^n - (r + \delta)^n) + n\omega_n r^{n-1}\delta \left(\alpha^{1-1/n} + (1 - \alpha)^{1-1/n}\right).
\]
At this point one observes that when \( \alpha \) is bounded away from 0 and 1 we have \( \alpha^{1-1/n} + (1 - \alpha)^{1-1/n} > 1 + C_{17} \). So, if we take \( \delta/r = C_{18}\mu\delta \) small enough then the last expression in (7.3) is positive and we obtain
\[
\text{Vol}(T_{\mu,\delta} \cap 2B) \geq C_{19}n\omega_n r^{n-1}\delta.
\]
Finally, since the metric $g$ is comparable to a flat metric on a small ball, we have a similar inequality also for $g$. 

To conclude the proof of the lower bound in Theorem 1.2 we observe that due to Proposition 7.1 (iv) the number of balls in Proposition 7.1 is $> C_{20} \mu^n$. So, by Proposition 7.2 $\text{Vol}(T_{\mu, \delta}) > C_{21} \delta / \mu^{n-1} \cdot \mu^n = C_{22} \mu \delta$. 

8. Proof of the upper bound in Theorem 1.2. In this section we estimate from above the volume of a tubular neighborhood of the nodal set. The proof is based on the study in Section 4 of eigenfunctions in small scales.

Let $V = \{V_k\}$ be a covering of $M$ by small open sets. Let $R_k \subseteq V_k$ be a cube preferred by Proposition 4.4. The next lemma shows that it is enough to estimate the volume of $T_{\mu, \delta}$ in preferred cubes.

**Lemma 8.1.** There exists a covering $V = \{V_k\}$ on $M$ with the following properties:

(a) $V$ is a finite covering.

(b) the metric $g$ can be developed in power series in each chart $V_k$.

(c) $M = \bigcup_k R_k$ for some choice of cubes $R_k \subseteq V_k$ preferred by Proposition 4.4.

We defer the proof of this Lemma to Section 8.1.

Now, let $R \subseteq V$ be a preferred cube. We decompose it into boxes $R_\nu$, where the sides of $R_\nu$ are of sizes $\delta < l_\nu < 2\delta$. We will denote this subdivision by $D$.

**Definition 8.2.** We call $R_\nu$ a nodal box if $\mathcal{N}_\mu \cap R_\nu \neq \emptyset$.

Let us denote the set of nodal boxes $R_\nu$ by $\text{Nod}$ and recall that $R_\nu^*$ denotes the union of $R_\nu$ with its $3^n - 1$ neighbors.

**Lemma 8.3.** $T_{\mu, \delta} \subseteq \bigcup_{R_\nu \in \text{Nod}} R_\nu^*$.

It remains to estimate the number of nodal boxes. Fix $\varepsilon = 1$. Proposition 4.4 tells us that $(\phi_\mu^2, D, E, C_1)$ is true (cf. Def. 5.1 & 5.2).

**Lemma 8.4.** The number of $E$-good nodal cubes is $\leq C_2 \text{Vol}_{n-1}(\mathcal{N}_\mu) / \delta^{n-1}$.

**Proof.** We begin with:

**Claim 8.5.** Let $R_\nu$ be an $E$-good nodal cube. Then

$$\text{Vol}_{n-1}(\mathcal{N}_\mu \cap R_\nu^*) \geq C_3 \delta^{n-1}.$$  \hfill (8.6)

**Proof of Claim.** First we see from the Brunn-Minkowski Inequality as in Proposition 7.2 that

$$\liminf_{t \to 0} \frac{\text{Vol}(T_{\mu, t})}{t} \geq C_4 \delta^{n-1}.$$  \hfill (8.7)
Since \( N_\mu \) is an analytic set, it is rectifiable ([Fed69, Theorem 3.4.8 (13)]) and thus ([Fed69, Theorem 3.2.39]), the limit in (8.7) exists and equals \( \text{Vol}_{n-1}(N_\mu \cap R^*_\nu) \).

Summing up (8.6) over all good nodal cubes we arrive at

\[
3^n \text{Vol}(N_\mu) \geq \sum_{\nu} \text{Vol}(N_\mu \cap R^*_\nu) \\
\geq \sum_{\text{good nodal } R_\nu} \text{Vol}(N_\mu \cap R^*_\nu) \\
\geq C_4 \#(\text{good nodal } R_\nu) \delta^{n-1}.
\]

**Proof of Theorem 1.2.** By Lemma 5.9 we know that the number of \( E \)-bad nodal cubes is \( \leq C_5 \mu / \delta^{n-1} \). By Lemma 8.4 and Theorem 1.3 the number of \( E \)-good nodal cubes is \( \leq C_6 \mu / \delta^{n-1} \). Together, we get that the number of nodal cubes is \( \leq C_7 \mu / \delta^{n-1} \). By Lemma 8.3 \( \text{Vol}(T_\mu, \delta) \leq C_8 \#(\text{Nod}) \delta^n \leq C_9 \mu \delta \).

**8.1. Proof of Lemma 8.1.** The following lemma is clear by compactness of \( M \).

**Lemma 8.9.** There exists \( \rho_0 > 0 \) such that for all \( p \), the metric \( g \) can be developed in power series in \( B(p, \rho_0) \).

Let \( \rho_1 = \rho_0 / C \) with \( C \) large enough.

**Lemma 8.10.** Every ball \( B(p, \rho_1) \) contains a preferred cube \( R \) which contains \( p \).

**Proof.** We identify \( B(p, \rho) \) with the Euclidean ball \( B(0, \rho) \) by working in geodesic coordinates. Suppose that the point \( x_0 \in R \subseteq B(0, \rho_0) \). Let \( x_1 \in B(0, \rho_0) \) with \( |x_1| = |x_0| =: r \). From Proposition 4.4 we know that \( R \) is independent of \( \mu \). By symmetry considerations, or just by examining the proof of Proposition 3.7 we see that any orthogonal transformation in \( B(0, \rho_0) \) takes \( R \) to another preferred cube.

Now, given \( p \), let \( q \) be any point on \( M \) such that \( \text{dist}(p, q) = r \). The geodesic ball \( B(q, \rho_0) \) contains a preferred cube \( R_1 \) which contains \( p \). Take a cube \( R \) in \( R_1 \cap B(p, \rho_1) \) which contains \( p \).

**Proof of Lemma 8.1.** By lemma 8.10 we can cover \( M \) by preferred cubes. Then by compactness of \( M \) we can extract a finite covering by preferred cubes.

**9. Approximation by nodal sets.**

**Proof of Theorem 1.8.** The proof proceeds similarly to the proof of Corollary 1.6. Fix \( C, \varepsilon > 0 \). Let \( T_{k, \delta} \) be the tubular neighborhood of \( N(\phi_k) \) of radius
\[ \delta_k = C/\mu_k^{n+1+\varepsilon}. \]  

By Theorem 1.2, \( \text{Vol}(T_k, \delta_k) \leq C/\mu_k^{n+1+\varepsilon} \). We conclude that

(9.1) \[ \sum_k \text{Vol}(T_k, \delta_k) \leq C \sum_k \mu_k^{-n-\varepsilon}. \]

By Weyl’s Law \([\text{Wey12, Hör68}]\) we know that

\[ \mu_k \asymp Ck^{1/n}. \]

Hence

\[ \sum_k \text{Vol}(T_k, \delta_k) \leq C \sum_k k^{1-\varepsilon/n} \]

is finite. So, by the Borel-Cantelli Lemma (see e.g. \([\text{Fel68}]\)) we obtain:

\[ \text{Vol}(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} T_k, \delta_k) = 0. \]

10. Dimension two.

**THEOREM 10.1.** Let \((\Sigma, g)\) be a smooth (i.e. \(C^\infty\)) closed Riemannian surface. Then there exist \(C_1, C_2 > 0\) such that

\[ C_1 \mu \delta \leq \text{Vol}(T_{\mu, \delta}) \leq C_2 \text{length}(N_\mu) \delta. \]

In particular, Theorem 1.2 is true for surfaces which satisfy Yau’s conjecture.

We recall from \([\text{DF90}]\) that for any smooth surface length \((N_\mu) \leq C_3 \mu^{3/2}.\)

Hence, if we modify the proof of Theorem 1.8 according to Theorem 10.1 we obtain

**PROPOSITION 10.2.** Let \((\Sigma, g)\) be a closed compact surface with a smooth metric \(g\). Then we have \(\text{Vol}(M(7/2 + \varepsilon, C)) = 0\) for all \(C, \varepsilon > 0\).

10.1. Lower bound in Theorem 10.1. This is basically Brüning’s argument. We can cover a fixed portion of \(\Sigma\) with pairwise disjoint balls \(B_i = B(x_i, r)\) of radius \(r = c/\mu\) and such that \(\phi_\mu(x_i) = 0\). The set \(N_\mu \cap B(x_i, r)\) is of length \(\geq r\).

Moreover, in local coordinates it has a projection of length \(\geq cr\) on one of the axes. This implies that \(T_{\mu, \delta} \cap B(x_i, r)\) has area \(\geq cr\delta\). Summing up over all the balls \(B_i\) we obtain:

\[ \text{Vol}(T_{\mu, \delta}) \geq c_1 c_2^2 \cdot c_2 \delta/\mu = c_3 \mu \delta. \]

10.2. Upper bound in Theorem 10.1: first proof. Let an eigenfunction \(\phi_\mu\) have nodal domains \(\Omega_1, \ldots, \Omega_{N(\mu)}\). Given \(\partial \Omega_j \subset N_\mu\), let \(L_\delta(t)\) denote the interior
parallel of $\partial \Omega_k$ at the distance $t$ inside $\Omega_j$. It is clear that

$$\text{area}(A_\mu) = \sum_{j=1}^{N} \int_{t=0}^{\delta} \text{length}(L_j(t)) \, dt. \quad (10.3)$$

The following inequality can be found in [Sav01, Proposition A.1.iv]:

$$\text{length}(L_j(t)) \leq \text{length}(\partial \Omega_j) + R(\Omega_j) \max \left\{ \int_{\Omega_j} K^+ - 2\pi \chi(\Omega_j), 0 \right\}. \quad (10.4)$$

Here $K^+$ denotes the positive part of the Gauss curvature, $\chi(\Omega_j)$ is proportional to the number $m_j = m_j(\mu)$ of connected components of $\partial \Omega_j$, and $R(\Omega_j)$ denotes the inner radius of $\Omega_j$. We substitute (10.4) into (10.3) and sum over $1 \leq j \leq N$. By Proposition 1.4 we know that $R(\Omega_j) \leq C/\mu$. We get the estimate

$$\frac{\text{area}(A_\mu)}{\delta} \leq 2 \cdot \text{length}(N_\mu) + \frac{C \int_{M} K^+}{\mu} + \frac{4\pi C N(\mu)}{\mu} \sum_{j=1}^{N} m_j(\mu). \quad (10.5)$$

As $\mu_j = \mu \to \infty$, the second term goes to zero. It remains to estimate the third term. One can construct a connected graph on $M$ whose edges will include all arcs of $N_\mu$, and show using Euler’s formula that

$$\sum_{j=1}^{N} m_j \leq 2(N + g - 1),$$

where $g$ denotes the genus of the surface $M$. Also, by Courant’s nodal domain theorem

$$N = N(\mu_k) \leq k + 1.$$

We recall that by [Wey12, Hör68] in dimension two $\mu_k \asymp C \sqrt{k}$, hence $N(\mu_k) \leq C \mu_k^2$. It follows that the third term in the right-hand side of (10.5) is less than $C \mu$. Substituting everything back into (10.5) and recalling that $\text{length}(N_\mu) \geq C \mu$ (see [Brü78]) we get the desired estimate. \hfill $\square$

10.3. Upper bound in Theorem 10.1: second proof. This proof was communicated by M. Sodin.

It suffices to give a proof for the neighborhood of $N_\mu$ of size $\delta/3$. We cover $M$ with cubes of side $C\delta$ (large cubes), as well as by cubes of side $C\delta/3$ (small cubes). One can easily arrange that each cube intersects a bounded number of
other cubes. For every small cube, there exists a unique concentric large cube whose side is three times larger. To estimate the area of \( T_{\mu, \delta} \), it suffices to estimate the volume of the union \( B_j \) of all small cubes which intersect the nodal set \( N_{\mu} \).

Indeed, if \( x \in T_{\mu, \delta} \), then \( N_{\mu} \) intersects either the small cube containing \( x \), or one of the 8 neighboring small cubes, so the volume of \( T_{\mu, \delta} \) is at most \( 9 \cdot \text{vol}(B_j) \).

We distinguish several cases:

(i) \( N_{\mu} \) intersects a small cube \( Q \), but any connected component of \( N_{\mu} \cap Q \) doesn't intersect the boundary of the big concentric cube \( Q' \).

(ii) \( N_{\mu} \) intersects a small cube \( Q \), and there exists a connected component of \( N_{\mu} \cap Q \) that intersects the boundary of the big concentric cube \( Q' \).

In case (i) there is at least one nodal domain contained in \( Q' \), so by the Faber-Krahn Inequality (see [EK96, Ch. 7, Th. 1]) we get that the area of this nodal domain is \( > C/\mu^2 \). By the Isoperimetric Inequality, the length of \( N_{\mu} \cap Q' \) is at least \( C/\mu \geq C\delta \).

In case (ii), the length of \( N_j \cap Q' \) is at least \( \delta/3 \).

Hence, we conclude that the number of \( Q' \) for which \( Q \) satisfies case (i) or case (ii) is \( \ll \text{length}(N_{\mu})/\delta \). Accordingly, the sum of the areas of those cubes is

\[
(10.6) \quad \ll \text{length}(N_{\mu})/\delta \cdot \delta^2 \leq C \text{length}(N_{\mu})\delta.
\]

11. Discussion. For a given \( M \) it seems interesting to find

\[
E(M) := \sup\{ b : \text{vol}(M(b, C)) > 0 \text{ for some } C > 0 \}.
\]

Theorem 1.8 implies that on real-analytic \( n \)-dimensional manifolds, \( E(M) \leq n+1 \). In dimension one, it follows from the theory of continued fractions that \( E(M) = 2 \) for \( M = [0, \pi] \). In fact, \( M(2, \pi) = M \) while \( \text{Vol}(M(2 + \varepsilon, C)) = 0 \forall \varepsilon > 0 \).

The same result likely holds for separable systems (examples include surfaces of revolution, Liouville tori and quantum completely integrable systems [TZ02]). In such systems one can separate variables and choose a basis of eigenfunctions that (in appropriate coordinates) have the form \( \phi(x_1, \ldots, x_n) = \prod \psi_j(x_j) \), where \( \psi_j \) are solutions of 2nd order differential equations. Accordingly, \( \mathcal{N}(\phi) \) forms a “grid” of hypersurfaces determined by zeros of \( \psi_j \)'s, and approximation by \( \mathcal{N}(\phi) \) reduces to a series of one-dimensional problems.

As a model example we consider an \( n \)-dimensional cube

\[
M(n) = \prod_{j=1}^{n} [0, \pi/\alpha_j],
\]

with Dirichlet boundary conditions, where for simplicity we assume \( \{\alpha_j^2\}_{j=1}^r \) are linearly independent over \( \mathbb{Q} \). Then the eigenvalues have the form \( \sum_{j=1}^{n} \alpha_j^2 m_j^2 \) (where \( m_j \in \mathbb{N} \)) and are simple, while the corresponding eigenfunctions have the
\[ \phi(m_1, \ldots, m_n; x_1, \ldots, x_n) = \prod_{j: m_j \neq 0} \sin (m_j \alpha_j x_j). \]

**Proposition 11.1.** \( E(M(n)) = 2 \) for all \( n \).

**Proof of Proposition 11.1.** We first make a change of variables \( y_j = \pi \alpha_j x_j \). This change of variables will only affect constants in the rate of approximation by nodal sets; it won’t affect the exponent. In the rescaled coordinates, nodal sets have the form

\[ \mathcal{N}(\phi(m_1, \ldots, m_n)) = \bigcup_{j: m_j \neq 0} \mathcal{A}_j, \]

where \( \mathcal{A}_j := \{ (y_1, \ldots, y_n): y_j = k_j/m_j, \quad 0 \leq k_j \leq m_j \} \). We first show that:

**Claim 11.3.** \( E(M(n)) \geq 2 \).

**Proof.** Let \( (y_1, \ldots, y_n) \in M \) be an arbitrary point on \( M \); we have \( 0 \leq y_j \leq 1 \). We can assume without loss of generality that \( y_j \in \mathbb{Q}, \forall 1 \leq j \leq n \), since the set of such points has the full measure. Consider next the continued fraction expansion of its first (say) coordinate,

\[ y_1 = [0; a_1, a_2, \ldots], \]

where we use the notation of [Khi97]. Let \( p_k/q_k, k = 1, 2, \ldots \) be the corresponding continued fractions. Then the points \((p_k/q_k, y_2, \ldots, y_n) \in \mathcal{N}(\phi(q_k, 0, \ldots, 0))\), and the Claim follows from the well-known inequality [Khi97]

\[ |y_1 - p_k/q_k| < 1/q_k^2. \]

We next show that:

**Claim 11.4.** \( E(M(n)) \leq 2 \).

**Proof.** It suffices to show that \( \text{Vol} (M(2 + \varepsilon, C)) = 0 \) for all \( C, \varepsilon > 0 \). Let \( y = (y_1, \ldots, y_n) \in M(2 + \varepsilon, C) \). As before, we may assume that \( y_j \notin \mathbb{Q} \). We know that there exists a sequence of eigenvalues \( \mu_k \to \infty \) such that

\[ d(y, \mathcal{N}(\phi_{\mu_k})) < C/\mu_k^{2+\varepsilon}. \]

Since all distances on \([0, 1]^n\) are equivalent, we may define \( d(x, y) = \max_{1 \leq j \leq n} |x_j - y_j| \).

In view of (11.2), it follows that for some \( 1 \leq j \leq n \) (say, for \( j = 1 \)), there exists a sequence of integers \( q_k, k = 1, 2, \ldots \), such that \( q_k \to \infty \) and \( |y_1 - p_k/q_k| < C/q_k^{2+\varepsilon} \) for some \( 0 \leq p_k \leq q_k \). The Claim now follows from Corollary 1.6. This also finishes the proof of Proposition 11.1.

For manifolds with ergodic geodesic flows (e.g., in negative curvature), eigenfunction behavior has been studied using *random wave model* [Ber77]. In addition,
percolation model [BS02] has been used to study the statistics of nodal domains in chaotic systems. We refer the reader to [FGS04] and references therein for a nice discussion about applicability of those models for studying various questions about eigenfunctions of chaotic systems.

In the opinion of the authors, it would be difficult to use these models directly to predict the “best possible” rate of approximation by nodal sets. The reason is that these models describe a single eigenfunction on a scale of $C/\mu$ (several wavelengths). However (as shown by the example of $M = [0, \pi]$) for a given $x \in M$ the values of $\mu$ giving the best approximation of $x$ by $\mathcal{N}(\phi_\mu)$ can grow exponentially. It thus seems difficult to take into account simultaneous behavior of all eigenfunctions in such a large energy range. However, one can probably expect that $E(M) > 2$ for such manifolds (in contrast to the integrable case), due to irregularity of nodal lines for such systems.

It also seems interesting to study “level sets” $M(b)$ for the approximation exponent $b$, e.g., defined by

$$M(b) := \cup_{C} M(b, C) \setminus (\cup_{a < b} \cup_{C} M(a, C)).$$

Remark 11.5. It should follow from the results of [JL99] that the conclusion of Theorem 1.8 should also hold for level sets of eigenfunctions (since the level set of an eigenfunction is a nodal set of a linear combination of that eigenfunction with a constant eigenfunction). It seems interesting to determine which level sets are $C/\mu$-dense (like nodal sets).

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