The electromagnetic interaction of ultrarelativistic heavy ions

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Abstract

The validity of a delta-function approximation for the electromagnetic interaction of relativistic heavy ions is investigated. The production of $e^+e^-$ pairs, with electron capture, is used as a test of the approximation.

The production of $e^+e^-$ in peripheral collisions of relativistic heavy ions has attracted a great amount of theoretical interest due to its non-perturbative character. The calculations are hard to perform and it is common to find substantial differences between the cross sections calculated within several approaches [1]. A good simplification of the problem has been found by Baltz and collaborators [2]. They have shown that if one makes a gauge transformation in the wavefunction of the form $\psi = \exp \{-i\chi(r,t)\} \psi'$, where

$$\chi(r,t) = \frac{Z\alpha}{v} \ln \left[ \gamma(z-vt) + \sqrt{b^2 + \gamma^2 (z-vt)^2} \right]$$

(1)

the interaction induced by the electromagnetic field of an ultrarelativistic particle is gauge transformed to (in our units $\hbar = c = m_e = 1$)

$$V(\rho,z,t) = \phi(\rho,z,t) (1-v\hat{\alpha}_z) - \phi(\rho = 0, z, t) (1-\hat{\alpha}_z/v) \text{ ,}$$

(2)

where $\phi(\rho,z,t)$ is the Lienard-Wiechert potential at a point $r = (\rho, z)$, generated by a relativistic particle with velocity $\mathbf{v} = v\mathbf{\hat{z}}$ and impact parameter $b$,
\[ \phi(\rho, z, t) = \gamma Z \alpha \left[ (b - \rho)^2 + \gamma^2 (z - vt)^2 \right]^{-1/2} . \]  

In these expressions \( \gamma = (1 - v^2)^{-1/2} \) is the Lorentz contraction factor, and \( \hat{\alpha}_z \) is the third of the Dirac matrices.

The second part of eq. (2) acts as a regularization term of the modified potential. It removes the divergence at \( b = 0 \). This new potential is very useful since the Lorentz contraction yields a delta-function in the longitudinal variables when \( \gamma \gg 1 \), and \( b \) is not too large. Evidently, this is a great simplification since delta-function interactions always lead to a considerable decrease of integration steps in perturbative as well as in non-perturbative calculations.

In [2] the formal derivation of the delta-function interaction was obtained by expanding the gauge transformed potential (2) into multipoles. Further manipulation of the multipole expansion and comparison with numerical calculations have shown that (2) can be expressed as

\[ V(\rho, z, t) = \delta(z - t) \, Z \alpha \, (1 - \hat{\alpha}_z) \ln \left( \frac{b - \rho}{b^2} \right) . \]  

We will show that this expression can be obtained in a simpler way. The derivation is useful to study the validity of the delta-function approximation. In particular we will test the approximation in a solvable problem, namely the production of \( e^+e^- \) pairs in which the electron is captured in an orbit around one of the nuclei (bound-free pairs).

Using the Bethe-integral [3], the potential (3) can be written in the form

\[ \phi(\rho, z, t) = Z \alpha \, \frac{1}{2\pi^2} \int d^3q \, \frac{e^{-iq \cdot u} e^{iq \cdot r}}{q^2 - v^2q_z^2} , \]  

where \( u = b + vt \) and \( q = (q_t, q_z) \). For relativistic particles we can replace \((1 - v\hat{\alpha})\) and \((1 - \hat{\alpha}/v)\) by \((1 - \hat{\alpha})\) in the interaction (2). As shown in ref. [3] this amounts to neglect a very small \((\sim \mathcal{O}(1/\gamma^2))\) piece of the longitudinal part of the interaction. However, it is important to keep the other \( v \) factors in their respective places, as they give rise to
important combinations of $\gamma$ factors in the matrix elements. Moreover, the integral in (5) diverges logarithmically as $v \to 1$.

The exact interaction is then given by

$$V (\rho, z, t) = Z\alpha (1 - \hat{\alpha}_z) \frac{1}{2\pi^2} \int d^3q \ e^{-i\mathbf{q} \cdot \mathbf{r}} \frac{[e^{i\mathbf{q} \cdot \mathbf{r}} - e^{i\mathbf{q} \cdot \mathbf{z}}]}{q^2 + q_z^2/\gamma^2}, \quad (6)$$

where the denominator of the integrand in (5) has been rewritten in terms of $\gamma$. The interaction given by eq. (4) is a limit of this integral when we set $q_z^2/\gamma^2 = 0$. It is clear from the above equation that neglecting this factor yields the delta-function in (4). However, to emphasize the restrictions on $q_t$ and $q_z$ let us define

$$\Phi (\rho, z, t) \equiv V (\rho, z, t) Z\alpha (1 - \hat{\alpha}_z) = \frac{1}{\pi} \int d^2q_t \frac{1}{q_t^2} \exp (-i\mathbf{q}_t \cdot \mathbf{b}) \left[ \exp (i\mathbf{q}_t \cdot \rho) - 1 \right] \Phi_z (q_t, z, t), \quad (7)$$

where

$$\Phi_z (q_t, z, t) = \frac{q_t^2}{2\pi} \int dq_z \frac{e^{iq_z(z-vt)}}{q_t^2 + q_z^2/\gamma^2} = \frac{\gamma q_t}{2} e^{-\gamma|z-vt|}. \quad (8)$$

Now, using $\lim_{\Lambda \to \infty} (\Lambda/2) e^{-\Lambda|x|} = \delta (x)$, we see that for $\gamma \to \infty$, $\Phi_z$ does not depend on $q_t$, and assumes the form of a delta function: $\Phi_z (z, t) = \delta (z - vt)$.

In this limit, we can write (7) as

$$\Phi (\rho, z, t) = \delta (z - t) \Phi_\rho (\rho, t), \quad (9)$$

with

$$\Phi_\rho (\rho, t) = \frac{1}{\pi} \int d^2q_t \frac{1}{q_t^2} \exp (-i\mathbf{q}_t \cdot \mathbf{b}) \left[ \exp (i\mathbf{q}_t \cdot \rho) - 1 \right] = 2 \int \frac{dq_t}{q_t} \left\{ J_0 [q_t |\rho - b|] - J_0 (q_t b) \right\}, \quad (10)$$

where $J_0$ is the cylindrical Bessel function. The integral over each Bessel function diverges, but their difference does not. To show this we regularize the integrals by using
\[
\int dx \frac{xJ_0(ax)}{x^2 + k^2} = K_0(ak),
\]

where \(K_0\) is the modified cylindrical Bessel function. Taking the limit \(k \to 0\), and using \(K(ak) \simeq \ln(ak)\), for small values of \(ak\), we get

\[
\Phi(\rho, t) = \ln \left(\frac{b - \rho}{b^2}\right).
\]

This is the solution of the Coulomb potential of a unit charge in 2-dimensions. An easy way to see this is to use Gauss law for the electric field in two dimensions. One obtains \(E \simeq 1/b\), where \(b\) is the distance to the charge. Since \(E = -\partial \Phi / \partial b\), the logarithmic form of \(\Phi\) is evident.

The above derivation illustrates the validity of the approximation in terms of the transverse momentum transfer \(q_t\). It should fail for very soft processes, i.e., those for which \(q_t \to 0\). Also, it requires that \(q_z\) is small compared to \(\gamma\). As shown in ref. 3, \(q_z\) values in the range of one up to \(\gamma\) units of the electron mass contribute appreciably to the integrals involved in the production of free, and of bound-free, \(e^+e^-\) pairs. It is thus important to check the validity of the approximation (4) in a concrete case. We will do this for the production of bound-free pairs. The full calculation uses the interaction given by equation (6). For comparison, a similar calculation with the term \(\xi = q_z/\gamma\) replaced by zero in the denominator of (6) is equivalent to the use of the interaction (4).

In ref. 4 it was shown that the Coulomb distortion of the positron wavefunction is an important effect in calculations of bound-free pair production. The right magnitude of the differential cross sections depends on this effect. This is not relevant in our case, since we are only interested in the relative change of the cross sections and probabilities by using the exact and the delta-function interaction, respectively. Therefore, for simplicity, we will use plane-waves for the positron wavefunction and first-order perturbation theory. The energy transfer from the field to the created pair is given by \(\omega = \varepsilon + 1\), where \(\varepsilon\) is the positron
energy, neglecting the atomic binding energy of the captured electron. Using the interaction in the form (6), the amplitude for bound-free pair production is given by

\[ a_{fi} = i \int_{-\infty}^{\infty} dt \ e^{i\omega t} \langle \psi_{e^+} | V | \psi_{e^-} \rangle = \frac{iZ\alpha}{\pi} \int d^2 q_t \frac{e^{-i\mathbf{q}_t \cdot \mathbf{b}}}{q_t^2 + \xi^2} F (\mathbf{q}_t) , \]  

(13)

where the integral over time yields

\[ F (\mathbf{q}_t) = \int d^3 r \ \exp \left( i \frac{\omega z}{v} \right) \left[ \exp \left( i \mathbf{q}_t \cdot \mathbf{p} \right) - 1 \right] \langle \psi_{e^+} | (1 - \hat{\alpha}_z) | \psi_{e^-} \rangle , \]  

(14)

and \( \xi \) is now given by \( \xi = \omega/\gamma v \).

Using plane waves for the positron wavefunction and a hydrogenic K-orbital function for the electron, the above matrix element is given by

\[ F (\mathbf{q}_t) = f (\mathbf{q}_t) \nabla (1 - \hat{\alpha}_z) u , \]  

(15)

where

\[ f (\mathbf{q}_t) = 8\sqrt{\pi} (Z\alpha)^{5/2} \left\{ \frac{1}{1/a_H^2 + |\mathbf{Q} - \mathbf{p}|^2} - \frac{1}{1/a_H^2 + |\mathbf{Q}_0 - \mathbf{p}|^2} \right\} . \]  

(16)

In these equations \( a_H = 1/\alpha = 5.29 \times 10^4 \) fm is the Bohr radius, \( \nabla \) (\( u \)) is the positron (electron) spinor, \( p = \sqrt{\varepsilon^2 - 1} \) is the positron momentum, \( \mathbf{Q} = (\mathbf{q}_t, \omega/v) \), and \( \mathbf{Q}_0 = (0, \omega/v) \). Integrating the square modulus of (13) over \( \mathbf{b} \) yields a delta function \( \delta (\mathbf{q}_t - \mathbf{q}_t') \). Furthermore, performing the spin averages we get for the differential cross section in terms of the positron energy

\[ \frac{d\sigma}{d\varepsilon d\Omega} = \frac{(Z\alpha)^2}{2\pi^3} (\varepsilon - 1) p \int d^2 q_t \frac{|f (\mathbf{q}_t)|^2}{(q_t^2 + \xi^2)^2} . \]  

(17)

The integral over the positron scattering angle can be done analytically, as well as the remaining integral over \( \mathbf{q}_t \).

To test the delta-function interaction we define the function

\[ \Delta (\varepsilon) = \left[ (d\sigma/d\varepsilon)_{\xi=0} - d\sigma/d\varepsilon \right] / (d\sigma/d\varepsilon)_{\xi=0} , \]  

(18)
which does not depend on $Z$.

In figure 1 we plot the function $\Delta (\varepsilon)$ for SPS, RHIC and LHC heavy ion energies. The positron energies are given in MeV units. For SPS the above formulation applies directly, assuming that the electron is captured in the target and neglecting the atomic screening effects. For RHIC and LHC a transformation of (17) back to the laboratory system was performed. We notice that the delta-function interaction works very well for positron energies of the order of MeV for SPS and RHIC and up to 100 MeV for LHC. The approximation worsens abruptly at a certain positron energy. The value of $\varepsilon$ where this occurs is a function of $\gamma$. In fact, we expect that $(d\sigma/d\varepsilon)_{\xi=0}$ starts to differ substantially from $d\sigma/d\varepsilon$ for $\xi$ of the order of one, i.e., for $\varepsilon/\gamma \simeq 1$. It is thus more appropriate to plot $\Delta$ as a function of $\varepsilon/\gamma$. This is shown in figure 2 for the same laboratory energies as before. In this figure $\varepsilon/\gamma$ is given in units of the electron mass. We see that all curves collapse into approximately a single one. The differences between the results for RHIC and for LHC are imperceptible.

These results show that the calculations with the delta-function interaction differ from the calculations with the exact potential for positron energies $\varepsilon > \sim 100 \gamma mc^2$. For SPS and RHIC this implies positron energies of the order of a few MeV, and for LHC a few hundred MeV.

In the frame of reference of the nucleus where the electron is captured, the positrons move in the very forward direction, within an angle of the order of $1/\gamma$ along the projectile incident direction. Thus, to study the impact parameter dependence of the production probabilities we can safely use $p = p_z$. The differential probability for pair production is given by

$$dP(b, \varepsilon) \frac{d\Omega}{8\pi^5} (\varepsilon - 1) p \left| \int d^2 q_t \left( \frac{-i}{q_t^2 + \xi^2} - \frac{1}{q_t^2 + \eta^2} \right) f(\xi b) \right|^2. \tag{19}$$

Inserting the term inside brackets of eq. (18) in the integral above and neglecting terms of order $1/a_H^2$, one gets the result

$$\frac{2\pi i}{\xi^2 - \eta^2} \left\{ \frac{1}{\xi^2 - \eta^2} \left[ \xi K_1 (\xi b) - \eta K_1 (\eta b) \right] - \frac{b}{2} K_0 (\xi b) \right\} = \frac{\pi i \xi}{\eta^2} K_0 (\xi b), \tag{20}$$
where $\xi = \omega/\gamma v$ and $\eta = \omega/v - p$.

We define another function $\Delta (b, \varepsilon)$ to test the impact parameter dependence of the delta-function interaction,

$$
\Delta (b, \varepsilon) = \left[ \frac{dP}{d\xi d\varepsilon d\Omega} \right]_{\xi=0} \frac{dP d\varepsilon d\Omega}{(dP/d\xi d\Omega)_{\xi=0}}.
$$

(21)

In figure 3 we plot the function $\Delta (b, \varepsilon)$ for SPS, RHIC and LHC heavy ion energies, as a function of $b/\gamma$. For comparison we use two values of the positron energy; $\varepsilon = 1$ MeV, and $\varepsilon = 10$ MeV. For $\varepsilon = 1$ MeV all curves agree and one observes that the approximation is good up to impact parameters of the order of $b \simeq 0.1 \gamma/\varepsilon$. This is confirmed by looking at the curves for $\varepsilon = 10$ MeV. Then the agreement at SPS bombarding energies is not perfect, even for the smaller impact parameters. But, for RHIC and LHC energies the results are basically equal. These results originate from the function given in eq. (20) which drops sharply to zero at $\xi \simeq 1$, i.e., at positron energies of the order of $b \simeq \gamma/\varepsilon$. This is the so-called adiabatic limit. The electromagnetic field has photon energy components up to $1/t_{int}$, where $t_{int}$ is the interaction time. This time equals $(b/\gamma)/v \simeq b/\gamma$ for relativistic collisions [3].

In conclusion, the present study has shown that the delta-function interaction yields reasonable results as long as $\omega b/\gamma \lesssim 0.1$. As seen in figure 3, this amounts to $b \lesssim 0.1 \gamma/\omega$. As observed in ref. [3], the most effective impact parameters for this process are of the order of $b \simeq 1/m$. We also see in figures 1 and 2 that the differential cross sections $d\sigma/d\omega$ are well described up to energies of the order of $0.1\gamma$.

For other situations, e.g., nuclear fragmentation due to the electromagnetic interaction in relativistic heavy ion collisions, the most effective impact parameter is given by $b \simeq R$, where $R \simeq 10$ fm. We thus expect that the delta-function interaction works well for $\varepsilon \lesssim 0.1 \gamma$ MeV. Note that $\gamma$ is the Lorentz factor in the frame of reference of one of the nuclei, i.e., $\gamma = 2\gamma_c^2 - 1$, where $\gamma_c$ is the collider Lorentz factor. Thus $\gamma$ is huge for RHIC and LHC energies, and the approximation works well for all energies of practical interest in nuclear fragmentation.
The basic idea of the delta-function interaction is that the electromagnetic field of a relativistic charge looks like a very thin pancake. Those processes which do not involve too large energy transfers, will not be sensitive to the spatial variation of the field. Then the delta-function is a good approximation. For typical energy transfers of the order of 10-100 MeV in nuclear fragmentation, the approximation works well for $b \lesssim 0.01 - 0.1\gamma$ fm. To calculate total cross sections it is always necessary to account for those large impact parameters at which the delta-function approximation fails. Similar conclusions have been drawn in a recent article on projectile-electron loss in relativistic collisions with atomic targets [5].

Acknowledgments

The author is a fellow of the John Simon Guggenheim foundation. He is grateful to Francois Gelis and Tony Baltz for useful discussions. This work has been authored under Contract N0. DE-AC02-98CH10886 with the U.S. Department of Energy. Partial support from the Brazilian funding agency MCT/FINEP/CNPQ(PRONEX), under contract No. 41.96.0886.00, is also acknowledged.

Figure Captions

1. Relative difference between the bound-free pair production spectrum calculated with the delta-function approximation and with the exact interaction, respectively (see eq. (18)). The comparison is done for SPS, RHIC and LHC heavy ion energies. The positron energies are given in MeV units in the laboratory system.

2. Same as in figure 1, but as a function of the positron energies divided by the Lorentz gamma factor. The positron energies are given in units of the electron rest mass.

3. Relative difference between the bound-free pair production probabilities calculated with the delta-function approximation and with the exact interaction, respectively (see eq. (21)). The comparison is done for SPS, RHIC and LHC heavy ion energies.
and for two different positron energies. The impact parameter divided by the Lorentz factor, $b/\gamma$, is given in units of the electron Compton wavelength, $\hbar/mc$. 
REFERENCES

[1] U. Eichmann, J. Reinhardt, and W. Greiner, Phys. Rev. A 61, 062710 (2000); A. Baltz, F. Gelis, L. McLerran and A. Peshier, nucl-th/0101024

[2] A.J. Baltz, M.J. Rhoades-Brown, and J. Weneser, Phys. Rev. A 44, 5569 (1991); Phys. Rev. A 48, 2002 (1993); Phys. Rev. A 47, 3444 (1993); Phys. Rev. A 50, 4842 (1994); A.J. Baltz, Phys. Rev. A 52, 4970 (1995)

[3] C.A. Bertulani and G. Baur, Phys. Reports 163, 299 (1988)

[4] C. A. Bertulani and D. Dolci, Nucl. Phys. A683, 635 (2001)

[5] A.B. Voitkiv, C. Müller, and N. Grün, Phys. Rev. A62, 062701 (2000)
