Lovelock gravity from entropic force

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Abstract In this paper, we first generalize the formulation of entropic gravity to $(n + 1)$-dimensional spacetime and derive Newton’s law of gravity and Friedmann equation in arbitrary dimensions. Then, we extend the discussion to higher order gravity theories and propose an entropic origin for Gauss–Bonnet gravity and more general Lovelock gravity in arbitrary dimensions. As a result, we are able to derive Newton’s law of gravitation as well as the corresponding Friedmann equations in these gravity theories. This procedure naturally leads to a derivation of the higher dimensional gravitational coupling constant of Friedmann/Einstein equation which is in complete agreement with the results obtained by comparing the weak field limit of Einstein equation with Poisson equation in higher dimensions. Our strategy is to start from first principles and assuming the entropy associated with the apparent horizon given by the expression previously known via black hole thermodynamics, but replacing the horizon radius $r_+$ with the apparent horizon radius $R$. Our study shows that the approach presented here is powerful enough to derive the gravitational field equations in any gravity theory and further supports the viability of Verlinde’s proposal.
Keywords  Entropic · Gravity · Thermodynamics

1 Introduction

Nowadays, it is a general belief that there should be some deep connection between gravity and thermodynamics. Indeed, this connection has a long history since the discovery of black holes thermodynamics in 1970s by Bekenstein and Hawking [1–3]. The studies on the profound connection between gravity and thermodynamics have been continued [4–7] until in 1995 Jacobson [8] disclosed that the Einstein field equation is just an equation of state for spacetime and in particular it can be derived from the the first law of thermodynamics together with the relation between the horizon area and entropy. Inspired by Jacobson’s arguments, an overwhelming flood of papers has appeared which attempt to show that there is indeed a deeper connection between gravitational field equations and horizon thermodynamics. It has been shown that the gravitational field equations in a wide variety of theories, when evaluated on a horizon, reduce to the first law of thermodynamics and vice versa. This result, first pointed out in [9], has now been demonstrated in various theories including $f(R)$ gravity [10], cosmological setups [11–18], and in braneworld scenarios [19–23]. For a recent review on the thermodynamical aspects of gravity and complete list of references see [24]. The deep connection between horizon thermodynamics and gravitational field equations, help to understand why the field equations should encode information about horizon thermodynamics. These results prompt people to take a statistical physics point of view on gravity.

A remarkable new perspective was recently suggested by Verlinde [25] who claimed that the laws of gravitation are no longer fundamental, but rather emerge naturally from the second law of thermodynamics as an “entropic force”. Similar discoveries are also made by Padmanabhan [26] who observed that the equipartition law for horizon degrees of freedom combined with the Smarr formula leads to the Newton’s law of gravity. This may imply that the entropy links general relativity with the statistical description of unknown spacetime microscopic structure when a horizon is present. The investigations on the entropic gravity has attracted a lot of interest recently [27–56].

On the other hand, the effect of string theory on classical gravitational physics is usually investigated by means of a low energy effective action which describes gravity at the classical level. This effective action consists of the Einstein–Hilbert action plus curvature-squared (Gauss–Bonnet) term and also higher order derivatives curvature terms. Lovelock gravity [57,58] which is a natural generalization of Einstein gravity in higher dimensional spacetimes contains higher order derivatives curvature terms, however there are no terms with more than second order derivatives of metric in equations of motion just as in Gauss–Bonnet gravity. Since the Lovelock tensor contains metric derivatives no higher than second order, the quantization of the linearized Lovelock theory is ghost-free [59–61].

Since the entropic gravity is fundamentally based on the holographic principle, one expects that entropic gravity can be generalized to any arbitrary dimension [25]. The motivation for studying higher dimensional gravity originates from string theory, which is a promising approach to quantum gravity. String theory predicts that space-
time has more than four dimensions. Another striking motivation for studying higher dimensional gravity comes from AdS/CFT correspondence conjecture \[62,63\], which associates an \(n\)-dimensional conformal field theory with a gravitational theory in \((n+1)\) dimensions. The generalization of this duality is embodied by the holographic principle \[64,65\], which posits that the entropy content of any region of space is defined by the bounding area of the region. These considerations have provided us enough motivation to study the formulation of the entropic gravity in \((n+1)\)-dimensional spacetime.

The main purpose of the present paper is to apply the idea of entropic force to higher gravity theories (string inspired gravity theories) such as Gauss–Bonnet and more general Lovelock gravity. The entropic origin for these gravity theories, which we shall propose in this paper, are novel and further supports the arguments in \[25\]. In \[25\] Verlinde proposed only the entropic origin for Einstein gravity in four dimensional spacetime and then it was applied to the cosmological setup for deriving the Friedmann equations only in Einstein gravity \[27,28\]. Since the field equations of Gauss–Bonnet and Lovelock theory are non-trivial only in higher dimensions \((n > 3)\), thus in this paper we first generalize the formalism of entropic gravity to all higher dimensions. This allows us to derive the Newton’s law of gravity as well as the \((n+1)\)-dimensional Friedmann equation in Einstein, Gauss–Bonnet and more general Lovelock gravity.

This paper is organized as follow. In the next section we generalize the formalism of entropic gravity to arbitrary dimensions and derive successfully Newton’s law of gravitation as well as Friedmann equation in all higher dimensions. In Sect. 3, we derive Newton’s law of gravity and the \((n+1)\)-dimensional Friedmann equation in Gauss–Bonnet theory from the entropic gravity perspective. In Sect. 4, we generalize our study to the more general Lovelock gravity. The last section is devoted to conclusions and discussions on the strengths and weaknesses of our assumptions and the obtained results.

\section*{2 Entropic gravity in \((n+1)\)-dimensions}

According to Verlinde, when a particle is on one side of screen and the screen carries a temperature, it will experience an entropic force equal to

\[ F = -T \frac{\Delta S}{\Delta x}. \] (1)

By definition, \(F\) is a force resulting from the tendency of a system to increase its entropy. Note that \(\Delta S > 0\) and hence the sign of the force is determined by how one chooses the definition of \(\Delta x\) as it relates to the proposed system. Here \(\Delta x\) is the displacement of the particle from the holographic screen, while \(T\) and \(\Delta S\) are the temperature and the entropy change on the screen, respectively. Suppose we have a mass distribution \(M\) which is distributed uniformly inside an screen \(\Sigma\). We have also a test mass \(m\) which is located outside the screen. The surface \(\Sigma\) surrounds the mass distribution \(M\) has a spherically symmetric property, while the test mass \(m\) is assumed to be very close to \(\Sigma\) comparing to its reduced Compton wavelength \(\lambda_m = \frac{\hbar}{mc}\). Now, consider an \((n+1)\)-dimensional spacetime with \(n\) spacial dimensions. The mass \(M\) induces a holographic screen \(\Sigma_n\) at distance \(R\) that has encoded on it gravitational information. The volume and area of this \(n\)-sphere are
\[ V_n = \Omega_n R^n, \quad \Sigma_n = n \Omega_n R^{n-1}, \]  

(2) 

where 

\[ \Omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad \Gamma\left(\frac{n}{2} + 1\right) = \frac{n}{2} \left(\frac{n}{2} - 1\right)! . \]  

(3) 

According to the holographic principle, the screen encodes all physical information contained within its volume in bits on the screen. The maximal storage space, or total number of bits, is proportional to the area \( \Sigma_n \). Let us denote the number of used bits by \( N \). It is natural to assume that this number is proportional to the area \( \Sigma_n \), namely

\[ \Sigma_n = N Q, \]  

(4) 

where \( Q \) is a constant which should be specified later. Since \( N \) denotes the number of bits, thus for one unit change we find \( \Delta N = 1 \). Therefore, from relation (4) one gets \( \Delta \Sigma_n = Q \). Motivated by Bekenstein’s area law of black hole entropy, we assume the entropy of the \((n - 1)\)-dimensional holographic screen obeys the area law, namely

\[ S = \frac{k_B c^3 \Sigma_n}{4\hbar G_{n+1}}, \]  

(5) 

where

\[ G_{n+1} = 2\pi^{1-n/2} \Gamma\left(\frac{n}{2}\right) \frac{c^3 \nu^{n-1}}{\hbar}, \]  

(6) 

is the \((n + 1)\)-dimensional gravitational constant [56]. We also assume the entropy change

\[ \Delta S = \frac{k_B c^3 \Delta \Sigma_n}{4\hbar G_{n+1}} = \frac{k_B c^3 Q}{4\hbar G_{n+1}}. \]  

(7) 

is one fundamental unit of entropy when \( \Delta \chi = \frac{\hbar}{mc} \), and the entropy gradient points radially from the outside of the surface to inside. Assuming that the total energy of the system,

\[ E = Mc^2, \]  

(8) 

is evenly distributed over the bits. Then according to the equipartition law of energy [66], the total energy on the screen is

\[ E = \frac{1}{2} N k_B T. \]  

(9)
Combining Eqs. (4), (8) and (9), we find

\[ T = \frac{2Mc^2Q}{\Sigma_n k_B}. \] (10)

Finally, inserting Eqs. (7) and (10) as well as relation \( \Delta x = \frac{\hbar}{mc} \) in Eq. (1), after using relation \( \Sigma_n = n\Omega_n R^{n-1} \), it is straightforward to show that the entropic force yields the \( (n+1) \)-dimensional Newton’s law of gravitation

\[ F = -\frac{Mm}{R^{n-1}} \left[ \frac{Q^2c^6}{2n\Omega_n h^2 G_{n+1}} \right]. \] (11)

This is nothing but the Newton’s law of gravitation in arbitrary dimensions provided we define

\[ Q^2 = \frac{2h^2}{c^6} n\Omega_n G_{n+1}^2. \] (12)

For \( n = 3 \) we have \( G_4 = G = \ell_P^2 c^3 / h \) and the above expression reduces to \( Q^2 = 8\pi \ell_P^4 \) [35]. It is important to note that the relations \( N = \frac{\Delta c^3}{Gh} \) and \( \Delta S = 2\pi k_B \) were postulated by Verlinde [25]. Our assumption here differs a bit from Verlinde postulates. For example, we have taken \( \Delta S = k_B c^3 Q / 4\pi G \) instead of \( \Delta S = 2\pi k_B \). Indeed, \( \Delta S \) in our paper is \( \frac{1}{\sqrt{8\pi}} \times \Delta S \) Verlinde. So we do not expect to have exactly \( Q = \frac{G\hbar}{c^4} \).

Although our main assumptions \( \Sigma_n \propto N \) and \( \Delta S \propto k_B \) are similar but the constants of proportionality are just assumption for later convenience [25]. Combining Eq. (12) with (11) we reach

\[ F = -G_{n+1} \frac{Mm}{R^{n-1}}. \] (13)

As the next step, we generalize the study to the cosmological setup. Assuming a homogeneous and isotropic Friedmann–Robertson–Walker (FRW) spacetime which is described by the line element

\[ ds^2 = h_{\mu\nu} dx^\mu dx^\nu + R^2 d\Omega_{n-1}^2. \] (14)

Here \( R = a(t) r \), \( x^0 = t \), \( x^1 = r \), and \( h_{\mu\nu} = \text{diag} (-1, a^2/(1 - kr^2)) \) is the two dimensional metric, while \( d\Omega_{n-1}^2 \) is the metric of \( (n-1) \)-dimensional unit sphere. The dynamical apparent horizon can be determined using relation \( h^{\mu\nu} \partial_\mu R \partial_\nu R = 0 \). It is a straightforward calculation to show that the radius of the apparent horizon for the FRW universe becomes

\[ R = ar = \frac{1}{\sqrt{H^2 + k/a^2}}. \] (15)
We also assume the matter source in the FRW universe is a perfect fluid with stress-energy tensor
\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \]  
(16)

Conservation of energy-momentum in \((n + 1)\)-dimensions leads to the following continuity equation
\[ \dot{\rho} + nH(\rho + p) = 0, \]  
(17)
where \(H = \dot{a}/a\) is the Hubble parameter. First of all, we derive the dynamical equation for Newtonian cosmology. Consider a compact spatial region \(V_n\) with a compact boundary \(\Sigma_n\), which is a sphere with physical radius \(R = a(t)r\). If we combine the gravitational force (13) with the second law of Newton for the test particle \(m\) near the screen \(\Sigma_n\), then we obtain
\[ F = m\ddot{R} = m\ddot{ar} = -G\frac{Mm}{R^{n-1}}. \]  
(18)

The total physical mass \(M\) in the spatial region \(V_n\) is defined as \([27,28]\)
\[ M = \int dV (T_{\mu\nu}u^\mu u^\nu) = \Omega_n R^n \rho, \]  
(19)
where \(\rho = M/V_n\) is the energy density of the matter inside the the volume \(V_n = \Omega_n R^n\). Combining Eqs. (18) and (19) we reach
\[ \frac{\ddot{a}}{a} = -G\frac{M}{R^{n-1}} \rho. \]  
(20)
This is the dynamical equation for \((n + 1)\)-dimensional Newtonian cosmology. In four dimensional spacetime where \(n = 3\), we recover the well-known formula,
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho. \]  
(21)

In order to derive the \((n + 1)\)-dimensional Friedmann equations of the FRW universe, let us notice that the quantity which produces the acceleration in a dynamical background is the active gravitational mass \(M [67]\) rather than the total mass \(M\). To determine the active gravitational mass, we should express \(M\) in terms of energy-momentum tensor \(T_{\mu\nu}\). The key point here is to connect the energy momentum \(T_{\mu\nu}\) with the spacetime curvature with the use of the Tolman–Komar’s definition of active gravitational mass. The active gravitational mass in \((n + 1)\)-dimension is defined as \([27,28]\)
\[ M = \frac{n - 1}{n - 2} \int dV \left( T_{\mu \nu} - \frac{1}{n - 1} T g_{\mu \nu} \right) u^\mu u^\nu. \]  
(22)
It is a matter of calculation to show that
\[ M = \frac{\Omega_n R^n}{n-2} [(n-2)\rho + np] = \frac{2\pi^{n/2} R^n}{n(n-2)\left(\frac{n}{2} - 1\right)!} [(n-2)\rho + np]. \tag{23} \]

Now, we can combine Eq. (23) with (18) provided we replace \( M \) in Eq. (18) with induced active gravitational mass \( \mathcal{M} \). This can be done because according to the weak equivalence principle of general relativity, the active gravitational mass of a system (here the universe) in general relativity is equal to its total mass in Newtonian gravity. We find
\[ \ddot{a} \frac{a}{a} = -G_{n+1} \Omega_n [(n-2)\rho + np] = -\frac{2G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} [(n-2)\rho + np]. \tag{24} \]

This is the acceleration equation for the dynamical evolution of the FRW universe in \((n+1)\)-dimensional spacetime. Multiplying \( \dot{a} \dot{a} \) on both sides of Eq. (24), and using the continuity equation (17), after integrating we find
\[ H^2 + \frac{k}{a^2} = \frac{2G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} \rho, \tag{25} \]
where \( k \) is an integration constant. When \( n = 3 \), we have \( \Omega_3 = \frac{4\pi}{3} \) and one recovers the standard Friedmann equation
\[ H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho. \tag{26} \]

Is it worth noting that in the literature the \((n+1)\)-dimensional Friedmann equation, in Einstein gravity, usually is written as
\[ H^2 + \frac{k}{a^2} = \frac{2\kappa_n}{n(n-1)} \rho, \tag{27} \]
with \( \kappa_n = 8\pi G_{n+1} \) (sometimes it is also written \( \kappa_n = 8\pi G \)). Let us note that the coupling in the r.h.s of this equation differs from that we derived in Eq. (25) for \( n \geq 4 \). A question then arises, which one is the correct Einstein gravitational constant? We believe that the coupling constant we derived from the entropic force approach is the correct one. To show this, let us note that the root of the factor \( 8\pi \) in Eq. (26) and also (27) is the relation
\[ R_{00} = \nabla^2 \phi, \tag{28} \]
where \( \phi \) is the Newtonian gravitational potential and \( R_{00} \) is the (00) component of the Ricci tensor. Now, the coefficient in the Poisson equation, i.e. \( 4\pi \) has been obtained using the Gauss law for 3-dimensional space. Thus we should first derive the correct coefficient for \( n \)-dimensional space. Applying Guass’s law for an \( n \)-dimensional volume, one finds the Poisson equation for arbitrary fixed dimension [68]
\[ \nabla^2 \phi = \frac{2G_{n+1} \pi^{n/2}}{(\frac{n}{2} - 1)!} \rho, \]  
(29)

On the other hand for \( n \geq 3 \), one finds [68]

\[ R_{00} = \left( \frac{n-2}{n-1} \right) \kappa_n \rho, \]  
(30)

Comparing Eqs. (28), (29) and (30) gives us the following modified Einstein gravitational constant for arbitrary \( n \geq 3 \) dimensions

\[ \kappa_n = \frac{2(n-1)\pi^{n/2}G_{n+1}}{(n-2) \left( \frac{n}{2} - 1 \right)!}. \]  
(31)

Substituting relation (31) in (27), immediately shows that the correct form of the Friedmann equation in \( n \geq 3 \) dimension is the expression we derived in Eq. (25). This is a remarkable result and shows that the approach presented here is powerful enough to derive the correct form of the gravitational field equations.

### 3 Gauss–Bonnet entropic gravity

Next we study the entropic force idea in Gauss–Bonnet gravity. This theory contains a special combination of curvature-squared term, added to the Einstein–Hilbert action. The key point which should be noticed here is that in Gauss–Bonnet gravity the entropy of the holographic screen does not obey the area law. The lagrangian of the Gauss–Bonnet correction term is given by

\[ L_{GB} = R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}. \]  
(32)

The low energy effective action of heterotic string theory naturally produces the Gauss–Bonnet correction term. The Gauss–Bonnet term does not have any dynamical effect in four dimensions since it is just a topological term in four dimensions. Static black hole solutions of Gauss–Bonnet gravity have been found and their thermodynamics have been investigated in ample details [69–75]. The entropy of the static spherically symmetric black hole in Gauss–Bonnet theory has the following expression [73–75]

\[ S = \frac{k_B c^3}{4\hbar G_{n+1}} \left[ 1 + \frac{n-1}{n-3} \tilde{\alpha} \right], \]  
(33)

where \( \Sigma_n \) is the horizon area and \( r_+ \) is the horizon radius. In the above expression \( \tilde{\alpha} = (n-2)(n-3)\alpha \), where \( \alpha \) is the Gauss–Bonnet coefficient which is positive [69–72], namely \( \alpha > 0 \). We assume the entropy expression (33) also holds for the apparent horizon of the FRW universe in Gauss–Bonnet gravity [11]. The only change
we need to apply is the replacement of the horizon radius \( r_+ \) with the apparent horizon radius \( R \), namely

\[
S = \frac{k_B c^3 \Sigma_n}{4 h G_{n+1}} \left[ 1 + \frac{n - 1}{n - 3} \frac{2\tilde{\alpha}}{R^2} \right].
\]

(34)

For \( n = 3 \) we have \( \tilde{\alpha} = 0 \), thus the Gauss–Bonnet correction term contributes only for \( n \geq 4 \) as we mentioned. In this case, the change in the entropy becomes

\[
\Delta S = \frac{k_B c^3 \Delta \Sigma_n}{4 h G_{n+1}} + \frac{n - 1}{n - 3} \frac{k_B c^3 \tilde{\alpha}}{2 h G_{n+1}} \Delta \left( \frac{\Sigma_n}{R^2} \right).
\]

(35)

Using the relation \( \Sigma_n = n \Omega_n R^{n-1} \) we have \( \Delta \Sigma_n R = (n - 1) \Sigma_n \Delta R \). Combining this expression with Eq. (35) after using relation \( \Delta \Sigma_n = Q \), we obtain

\[
\Delta S = \frac{Q k_B c^3}{4 G_{n+1} h} \left[ 1 + \frac{2\tilde{\alpha}}{R^2} \right].
\]

(36)

Inserting Eqs. (10), (12) and (36) in Eq. (1) we find

\[
F = -G_{n+1} \frac{M m}{R^{n-1}} \left[ 1 + \frac{2\tilde{\alpha}}{R^2} \right].
\]

(37)

This is the Newton’s law of gravitation in Gauss–Bonnet gravity resulting from the entropic force approach. In the absence of Gauss–Bonnet term (\( \tilde{\alpha} = 0 \)) one recovers Eq. (13). It is worth mentioning that the correction term in Eq. (37) can be comparable to the first term only when \( R \) is very small, namely for strong gravity. This implies that the correction makes sense only at the very small distances. When \( R \) becomes large, i.e. for weak gravity, the modified Newton’s law reduces to the usual Newton’s law of gravitation.

Finally, we derive the \((n + 1)\)-dimensional Friedmann equation of FRW universe in Gauss–Bonnet gravity using the approach we developed in the previous section. In the presence of Gauss–Bonnet term Eq. (20) is modified as

\[
\frac{\ddot{a}}{a} = -\frac{2 G_{n+1} \pi^{n/2}}{n \left( \frac{n}{2} - 1 \right)!} \rho \left[ 1 + \frac{2\tilde{\alpha}}{R^2} \right].
\]

(38)

Note that \( R = a(t) r \) is a function of time. Eq. (38) is the dynamical equation for \((n + 1)\)-dimensional Newtonian cosmology in Gauss–Bonnet gravity. The main difference between this equation and Eq. (20) is that the correction term depends explicitly on the radius \( R \). In order to remove this confusion, we suppose that for Newtonian cosmology the spacetime is Minkowskian with \( k = 0 \). In this case we have \( R = 1/H \), and thus we can rewrite Eq. (38) as

\[
\frac{\ddot{a}}{a} = -\frac{2 G_{n+1} \pi^{n/2}}{n \left( \frac{n}{2} - 1 \right)!} \rho \left[ 1 + 2\tilde{\alpha} \left( \frac{\dot{a}}{a} \right)^2 \right].
\]

(39)
Combining Eq. (38) with (23), after replacing $M$ by $\mathcal{M}$, we get

$$\frac{\ddot{a}}{a} = -\frac{2G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} [(n-2)\rho + np] \left[ 1 + \frac{2\tilde{\alpha}}{R^2} \right].$$

(40)

Thus we have derived the acceleration equation for the dynamical evolution of the FRW universe in Gauss–Bonnet theory. Multiplying $\dot{a}a$ on both sides of Eq. (40), and using the continuity equation (17), we get

$$d\left(\dot{a}^2\right) = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} \left[ d(\rho a^2) + \frac{2\tilde{\alpha}}{r^2} \frac{d(\rho a^2)}{a^2} \right].$$

(41)

Integrating yields

$$H^2 + \frac{k}{a^2} = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} \rho \left[ 1 + \frac{2\tilde{\alpha}}{\rho R^2} \int \frac{d(\rho a^2)}{a^2} \right].$$

(42)

Now, in order to calculate the correction term we need to find $\rho = \rho(a)$. Suppose we assume a constant equation of state parameter, $w = p/\rho$, integrating the continuity equation (17) immediately yields

$$\rho = \rho_0 a^{-n(1+w)},$$

(43)

where $\rho_0$, an integration constant, is the present value of the energy density. Inserting relation (43) in Eq. (42), after integration, we obtain

$$H^2 + \frac{k}{a^2} = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} \rho \left[ 1 + \frac{2\tilde{\alpha}}{\rho R^2} \left( \frac{n(1+w) - 2}{n(1+w)} \right) \right].$$

(44)

Using Eq. (15) we can further rewrite the above equation as

$$\left( H^2 + \frac{k}{a^2} \right) \left[ 1 + 2\tilde{\alpha} \left( H^2 + \frac{k}{a^2} \right) \frac{n(1+w) - 2}{n(1+w)} \right]^{-1} = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} \rho.$$  

(45)

Next, we expand the above equation up to the linear order of $\tilde{\alpha}$. We find

$$\left( H^2 + \frac{k}{a^2} \right) + \alpha' \left( H^2 + \frac{k}{a^2} \right)^2 = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!} \rho,$$

(46)

where we have defined

$$\alpha' \equiv \frac{2\tilde{\alpha}[2 - n(1+w)]}{n(1+w)}.$$  

(47)
and we have neglected $O(\tilde{\alpha}^2)$ terms and higher powers of $\tilde{\alpha}$. This is due to the fact that at the present time $R \gg 1$ and hence $H^2 + k/a^2 \ll 1$. Indeed for the present time where the apparent horizon radius becomes large, the correction term is relatively small and the usual Friedman equation is recovered. Thus, the correction make sense only at the early stage of the universe where $a \to 0$. When $a \to 0$, even the higher powers of $\tilde{\alpha}$ should be considered. With expansion of the universe, the modified Friedman equation reduces to the usual Friedman equation. From Eq. (46) we see that the correction term explicitly depends on the matter content through the equation of state parameter, $w$, where we have assumed to be a constant.

Equation (46) is the $(n + 1)$-dimensional Friedmann equation in Gauss–Bonnet Gravity. The Friedmann equation obtained here from entropic force approach is in good agreement with that obtained from the gravitational field equation in Gauss–Bonnet gravity [76]. This fact further supports the viability of Verlinde formalism.

4 Lovelock entropic gravity

Finally we generalize our discussion to a more general case, the so-called Lovelock gravity, which is a generalization of the Gauss–Bonnet gravity. The most general lagrangian which keeps the field equations of motion for the metric of second order, as the pure Einstein–Hilbert action, is Lovelock lagrangian [57]. This lagrangian is constructed from the dimensionally extended Euler densities and can be written as

$$ L = \sum_{i=0}^{m} c_i L_i, \quad (48) $$

where $c_i$ and $L_i$ are arbitrary constant and Euler density, respectively. $L_0$ set to be one, so $c_0$ plays the role of the cosmological constant, $L_1$ and $L_2$ are, respectively, the usual curvature scalar and Gauss–Bonnet term. In an $(n + 1)$-dimensional spacetime $m = [n/2]$. The entropy of the spherically symmetric black hole solutions in Lovelock theory can be expressed as [77]

$$ S = \frac{k_B c^3}{4hG_{n+1}} \sum_{i=1}^{m} \frac{i(n-1)}{n-2i+1} \hat{c}_i r^{2-2i}. \quad (49) $$

where $\Sigma_n = n\Omega_n r_+^{n-1}$ is the horizon area. In the above expression the coefficients $\hat{c}_i$ are given by

$$ \hat{c}_0 = \frac{c_0}{n(n-1)}, \quad \hat{c}_1 = 1, \quad \hat{c}_i = c_i \prod_{j=3}^{2m} (n+1-j) \quad i > 1. \quad (50) $$

Note that in expression (49) for entropy, the cosmological constant term $\hat{c}_0$ doesn’t appear. This is a reasonable result, and due to the fact that the black hole entropy depends only on its horizon geometry. We further assume the entropy expression (49)
are valid for a FRW universe bounded by the apparent horizon in the Lovelock gravity provided we replace the horizon radius \( r_+ \) with the apparent horizon radius \( R \), namely

\[
S = \frac{k_B c^3}{4\hbar G_{n+1}} \sum_{i=1}^{m} \frac{i(n-1)}{n-2i+1} \hat{c}_i R^{2-2i}.
\]  

(51)

It is easy to show that, the first term in the above expression leads to the well-known area law. The second term yields the apparent horizon entropy in Gauss–Bonnet gravity. The change in the general entropy expression of Lovelock gravity is obtained as

\[
\Delta S = \frac{k_B c^3 Q}{4\hbar G_{n+1}} \sum_{i=1}^{m} i \hat{c}_i R^{2-2i}.
\]  

(52)

where we have used Eq. (4). Inserting Eqs. (10), (12) and (52) in Eq. (1) one finds

\[
F = -G_{n+1} \frac{M m}{R^{n-1}} \sum_{i=1}^{m} i \hat{c}_i R^{2-2i}.
\]  

(53)

Thus we have derived the Newton’s law of gravitation in Lovelock gravity resulting from the entropic force. It is obvious that the first term of the above expression yields the famous Newton’s law of gravity, and the others terms will be important only for strong gravity or small distances. In this manner, the dynamical equation for \((n+1)\)-dimensional Newtonian cosmology takes the following form

\[
\ddot{a} = -\frac{2G_{n+1}\pi^{n/2}}{n(n-2)(n/2-1)!} \rho \left( \hat{c}_i \right)^{2i-2}.
\]  

(54)

The acceleration equation for the dynamical evolution of the FRW universe in \((n+1)\)-dimensional Lovelock gravity is obtained following the method developed in the previous section. The result is

\[
\ddot{a} = -\frac{2G_{n+1}\pi^{n/2}}{n(n-2)(n/2-1)!} \left( \rho + np \right) \sum_{i=1}^{m} i \hat{c}_i R^{2-2i}.
\]  

(55)

Multiplying \(\dot{a}a\) on both sides of Eq. (55), and using the continuity equation (17), after integrating, we get

\[
H^2 + \frac{k}{a^2} = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)(n/2-1)!} \rho \left[ 1 + \sum_{i=2}^{m} \frac{i \hat{c}_i}{\rho a^2 R^{2i-2}} \int \frac{d(\rho a^2)}{a^{2i-2}} \right].
\]  

(56)
Using Eq. (43), we can perform the integration. We obtain

\[ H^2 + \frac{k}{a^2} = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!\rho} \left[ 1 + \sum_{i=2}^{m} \frac{[2 - n(1 + w)]i\hat{c}_i}{[2(2 - i) - n(1 + w)]} \times \frac{1}{R^{2(i-1)}} \right]. \]  

(57)

Equation (57) can be rewritten in the following form

\[ \left( H^2 + \frac{k}{a^2} \right) \left[ 1 + \sum_{i=2}^{m} \frac{(2 - n(1 + w))i\hat{c}_i}{[2(2 - i) - n(1 + w)]} \right]^{-1} = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!\rho}. \]  

(58)

At the present time where \( R \gg 1 \), we can expand the l.h.s of the above equation. Using Eq. (15), we reach

\[ \left( H^2 + \frac{k}{a^2} \right) \left[ 1 - \sum_{i=2}^{m} \frac{(2 - n(1 + w))i\hat{c}_i}{[2(2 - i) - n(1 + w)]} \left( H^2 + \frac{k}{a^2} \right)^{i-1} \right] = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!\rho}. \]  

(59)

If we define

\[ \beta_i \equiv \frac{i[2 - n(1 + w)]\hat{c}_i}{n(1 + w) - 2(2 - i)}, \]  

(60)

then we can write Eq. (59) in the following form

\[ \left( H^2 + \frac{k}{a^2} \right) + \sum_{i=2}^{m} \beta_i \left( H^2 + \frac{k}{a^2} \right)^{i} = \frac{4G_{n+1}\pi^{n/2}}{n(n-2)\left(\frac{n}{2} - 1\right)!\rho}. \]  

(61)

In this way we derive the \((n + 1)\)-dimensional Friedmann equations in Lovelock gravity from the entropic force approach which is consistent with the result obtained from different methods [13,58]. When \( \beta_i = 0 \) \((i \geq 2)\), one recovers the standard Friedmann equation in Einstein gravity. The first term in summation of the above equation is the Gauss–Bonnet leading correction term derived in the previous section provided we define \( \hat{c}_2 = \tilde{a} \). In this case \( \beta_2 \) is exactly the \( \alpha' \) of the previous section.

5 Conclusions and discussions

According to Verlinde’s argument, the total number of bits on the holographic screen is proportional to the area, \( A \), and can be specified as \( N = \frac{Ac^3}{2\hbar} \). Indeed, the derivation of Newton’s law of gravity as well as Friedmann equations, in Verlide formalism,
depend on the entropy-area relationship $S = \frac{A c^3}{4G\hbar}$, where $A = 4\pi R^2$ represents the area of the horizon [25]. However, it is well known that the area formula of black hole entropy no longer holds in higher derivative gravities. So it would be interesting to see whether one can derive Newton’s law of gravity as well as the corresponding Friedmann equations in these gravities in the framework of entropic force perspective developed by Verlinde [25].

In this paper, starting from first principles and assuming the entropy associated with the holographic screen/apparent horizon given by the expression previously known via black hole thermodynamics, we were able to derive the Newton’s law of gravity as well as the cosmological equations (Friedmann equations) governing the evolution of the universe in any gravity theory including Einstein, Gauss–Bonnet and more general Lovelock gravity in arbitrary dimensions. We derived the Newton’s law of gravitation from entropic force directly. Then, we derived the Friedmann equation by equating the mass in Newtonian gravity with active gravitational mass. In our derivation the assumption that the entropy of the apparent horizon of FRW universe in Gauss–Bonnet and Lovelock gravity have the same form as the spherically symmetric black hole entropy in these gravities, but replacing the black hole horizon radius by the apparent horizon radius, plays a crucial role. Interestingly enough, we found that the higher dimensional gravitational coupling constant of Friedmann/Einstein equation can be derived naturally from this approach which coincides with the result obtained by comparing the weak field limit of Einstein equation with Poisson equation in higher dimension. Our study shows that the approach here is powerful enough to derive the gravitational field equations in any gravity theory. The results obtained here in the framework of Gauss–Bonnet gravity and more general Lovelock gravity further support the viability of Verlinde’s formalism.

Finally, let us discuss the strengths and weaknesses of our assumptions and justify the results. First of all, we started from the entropy expression, and then following Verlinde’s approach [25] we derived the corresponding Friedmann equations. This is indeed a well-known approach for deriving Friedmann equations from thermodynamics [8, 10, 13]. However, in obtaining the field equations from the thermodynamic identity, a logic problem occurs. Wald [78, 79] has already shown that one can construct an $S$ for which $TdS = dE$ will hold for on-shell solutions of field equations in any diffeomorphism invariant theory and the existence of such a function is inevitable. We also know that, $S$ can be taken to be the Noether charge associated with the diffeomorphism invariance of the theory. But, in the absence of field equations the Noether current can be interpreted as entropy current if and only if field equations are assumed to hold; it is in this on shell context that Wald showed that it is entropy [78, 79]. So we have no independent justification for demanding the existence of the matter entropy flux. Until we come up with such a justification- without using field equations- we can prove that “field equations imply local entropy balance at local horizons” but not “local entropy balance at local horizons imply field equations”. The issue at stake is not mathematics but logic.

Nevertheless, this is not a critical problem for deriving Friedmann equations. The reason is that, we have neither used the action of the theory nor its field equations to derive Friedmann equations. We just assumed that previously, local static black
hole solutions of the theory and their thermodynamics were known (like [8,10,13]). This does not necessarily imply that the dynamical equation describing the evolution of the universe in that theory is also known and can be derived from thermodynamics argument. Basically, in order to derive the Friedmann equations one should vary the action with respect to dynamical FRW metric. Our approach is quite different and shows that without varying the action one can still arrive at Friedmann equations from thermodynamics.

Besides, starting from entropy expression and deriving the Friedmann equations from thermodynamic arguments is neither a trivial task nor a general feature of any gravity theory. Indeed, it was argued that for Horava–Lifshitz cosmology it is not possible to arrive at Friedmann equation by using the entropy expression of black hole in that theory, and applying the first law of thermodynamics on the apparent horizon [80]. In other words, in Horava–Lifshitz gravity, the Friedmann equations derived from varying the action differ from the one derived from thermodynamics. The reason of failure seems to be due to the fact that Horava–Lifshitz gravity is not diffeomorphism invariant [81]. This indicates that deriving Friedmann equations from thermodynamic approach does not work for all gravity theories, and it should be checked for each theory separately.

Furthermore, the key assumption for deriving the Friedmann equations is equating the physical mass $M$ with active gravitational mass $\mathcal{M}$. The reason for equating these masses comes from the fact that according to the weak equivalence principle, the active gravitational mass of a system (here the universe) in general relativity is equal to its total inertial mass in Newtonian gravity. Indeed, we have only one physical mass (energy) content of the universe surrounded by the apparent horizon with radius $R$. This is not strange, because according to the correspondence principle, Newtonian gravity is the weak field limit of Einstein gravity. Note that accepting the equivalence principle does not necessarily imply that we have assumed (and employed) general relativity in our formulation. Besides, equating the mass in Newtonian gravity and general relativity is not new and strange. For example, in Schwarzschild black holes (with $G = c = 1$) which is a solution of general relativity, we also equate the geometrical mass $m$ resulting from general relativity, with physical mass $M$ in Newtonian gravity, i.e., $m = M$ by taking the weak gravitational field limit of general relativity.

In addition, replacing the horizon radius $r_+$ of black hole with the apparent horizon $R$ can be justified as follows. The entropy expression depends on the gravity theory. In cosmological setup, if one derive the Friedmann equation from varying the action and then study the thermodynamics of the apparent horizon, one finds that the temperature and entropy associated with the apparent horizon have the same expressions as for black hole horizon. This is not true for the cosmological event horizon. That is to say, the temperature and entropy of cosmological event horizon differ from that of black black hole. This is due to the fact that the Friedmann equations describe local properties of spacetimes, while the cosmological event horizon is determined by global properties of spacetimes. That is, one cannot determine local properties at each point in a spacetime using global properties of the spacetime, of course, although the inverse is certainly true. It should be pointed out that the apparent horizon is determined locally. The above argument implies that the apparent horizon is a physical boundary from the thermodynamical point of view for which all laws of thermodynamics hold on it.
This fact is exactly similar to black hole horizon and justify why the entropy expression for black hole horizon and apparent horizon are usually taken the same in each gravity theory.

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