Quantum diffusion on almost commutative spectral triples and spinor bundles

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Abstract

Based on the observation that Ćaćić’s characterization of almost commutative spectral triples as Clifford module bundles can be pushed to endomorphism algebras of Dirac bundles, with the geometric Dirac operator related to the Dirac operator of the spectral triple by a perturbation, the question of complete positivity of the heat semigroups generated by connection laplacian and Dirac and Kostant’s cubic Dirac laplacians is approached using spin geometry and $C^*$-Dirichlet forms. The geometric heat semigroups for on endomorphism algebras of spinor bundles are shown to be quantum dynamical semigroups and the existence of covariant quantum stochastic flows associated to the heat semigroups on spinor bundles over reductive homogeneous spaces is established using the construction of Sinha and Goswami.

1 Introduction

Almost commutative spectral triples are a class of noncommutative geometries defined to develop the noncommutative geometric approaches to the standard model of particle physics (see, for instance, [35]). Ćaćić characterizes such spectral triples as endomorphism algebras of spinor bundles. In this short article the heat semigroup associated to almost commutative spectral triples and spinor bundles is studied. The semigroup is shown to be a quantum dynamical semigroup. When the underlying manifold is a reductive homogeneous space, using the quantum stochastic calculus of Sinha and Goswami, the existence of a quantum stochastic dilation of Evans-Hudson type is established. The construction from [34] extends to semigroups with unbounded generators, and while [8] is an alternative, the covariant construction is more natural on homogeneous spaces. This question of existence of such dilation – which can be viewed as diffusion – on the spectral triple has relevance in light of recent results of relating entropy for second quantization on the fermionic Fock space and the spectral action for the spectral triple. The appearance of Brownian bridge integrals expansion of the spectral action in [18] is also suggestive of a deeper connection between noncommutative geometry and probability as Wiener space and boson Fock spaces are isomorphic. Also, of relevance is the stochastic quantization considered by where related ideas on Grassman algebras are explored.

1.1 Organization

Section 2 introduces the quantum dynamical smigroups, dilations and relevant background; section 3 introduces Ćaćić’s results and gives the characterization in terms of spinor bundles. Section 4 considers complete positivity on product almost commutative spectral triples and considers more general almost commutative spectral triples (§ 4.2). Section 5 introduces framework from Sinha and Goswami and establishes the existence of Evans-Hudson dilation on reductive homogeneous spaces. The results are obtained by first showing that the $D^2$ defines a $C^*$-Dirichlet form and generates a completely Markov semigroup, and then over reductive homogeneous spaces the covariance and regularity conditions required by [34] are satisfied.

Some notational conventions: for Hilbert space, $H$, $B(H)$ will denote bounded operators on $H$. i will denote $\sqrt{-1}$. For $m,n \in \mathbb{N}$, $[n]$ will denote $\{1, 2 \ldots n\}$ and $\{m, m+1 \ldots n\}$ respectively. $\text{Lie}(G)$ will denote the Lie algebra of Lie group $G$. $(M, g)$ will denote a smooth manifold $M$ with a Riemannian metric $g$ on $M$.

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metric. After fixing a local orthonormal frame, \( e_j \), the connection \( \nabla_j \) will be used interchangeably for \( \nabla_{e_j} \).

Following [25], by a Riemannian connection we mean a metric connection not necessarily torsion free; the canonical Riemannian connection is taken as the torsion free Riemannian connection. Throughout we will restrict to compact spin manifolds since that is the object in the reconstruction theorems from noncommutative geometry. Additionally, we will work with even dimensional manifolds with empty boundary\(^1\). \( \text{Cl}(E, g) \) will denote the Clifford algebra (bundle) over the vector space (vector bundle) \( E \) with quadratic form \( q \). For Riemannian manifold \( (M, g) \), \( \text{Cl}(M) = \sqcup_{m \in \mathbb{M}} \text{Cl}(T^*_m M), -g \). \( \mathcal{D}^2 \) is used for geometric Dirac laplacian, i.e. associated to the given connection, and \( D^2 \) when \( D \) may not be the geometric Dirac operator.

## 2 Quantum stochastic flows

### 2.1 Quantum dynamical semigroups

Suppose \( \mathcal{A} \) is a \( * \)-algebra, then \( \text{Mat}_n(\mathcal{A}) \cong \mathcal{A} \otimes \text{Mat}_n(\mathbb{C}) \). Recall the notion of complete positivity –

**Definition 1.** For unital \( * \)-algebras, \( \mathcal{A}_1, \mathcal{A}_2 \)

1. A linear map \( T : \mathcal{A}_1 \to \mathcal{A}_2 \) is positive if \( T((\mathcal{A}_1)_+) \subseteq (\mathcal{A}_2)_+ \).

2. \( T \) is completely positive if for all \( n \in \mathbb{N} \), \( T_n := T \otimes 1_n : \text{Mat}_n(\mathcal{A}_1) \cong \mathcal{A}_1 \otimes \text{Mat}_n(\mathbb{C}) \to \mathcal{A}_2 \otimes \text{Mat}_n(\mathbb{C}) \cong \text{Mat}_n(\mathcal{A}_2) \), \( T_n([a_{ij}]) = [T(a_{ij})] \), is positive. This is equivalent to \( \sum_{i,j \in [n]} b_j^* \phi(a_{ij}) b_j \geq 0 \) with \( n \in \mathbb{N} \), \( a_i \in \mathcal{A}_1, b_i \in \mathcal{A}_2, i \in [n] \).

**Definition 2 (Quantum Dynamical Semigroups).** A quantum dynamical semigroup (Q.D.S.) on a \( C^* \)-algebra \( \mathcal{A} \) is a strongly continuous, contractive semigroup \( T_t : \mathcal{A} \to \mathcal{A} \) is completely positive map. A semigroup is conservative if for all \( t, T_t(1) = 1 \) (equivalently \( \mathcal{L}(1) = 0 \) for the generator \( \mathcal{L} \) of \( T_1 \)). The semigroup is of class \( C_0 \) if \( \lim_{t \to t_0} T_t x = T_{t_0} x \) for all \( x, t_0 \).

**Definition 3.** (Covariant quantum dynamical semigroups) Let \( G \) be a locally compact group acting on \( C^* \)-algebra by \( \alpha : G \to \text{Aut}(\mathcal{A}) \) with \( \alpha_g \) denoting \( \alpha(g) \). A quantum dynamical semigroup \( (T_t) \) is covariant with respect to \( G \) if for all \( t \geq 0, g \in G \), \( T_t \circ \alpha_g = \alpha_g \circ T_t \), equivalently \( \mathcal{L} \circ \alpha_g = \alpha_g \circ \mathcal{L} \) where \( \mathcal{L} \) generates \( (T_1) \).

**Definition 4.** A conditional expectation is a linear map, \( E : \mathcal{N} \to \mathcal{M} \), between \( * \)-algebras \( \mathcal{M}, \mathcal{N} \), satisfying \( \mathcal{M} \subseteq \mathcal{N}, E[1] = 1 \) and for any \( M_1 \in \mathcal{M}, N \in \mathcal{N} \), \( E[M_1 N M_2] = M_1 E[N] M_2 \).

**Definition 5 (Stochastic dilation).** For quantum dynamical semigroup \( (T_t, t \geq 0) \) on a \( C^* \) (or von Neumann algebra) \( \mathcal{M} \), a quantum stochastic dilation is a family of \( * \)-homomorphisms, \( j_t : \mathcal{M} \to \mathcal{N} \), where \( \mathcal{N} \) is a \( * \)-algebra with conditional expectation \( E_0 : \mathcal{N} \to \mathcal{M} \) satisfying \( T_t E_0 [j_t] = E_0 [j_t] \).

A stochastic dilation on the Fock space will be considered as a quantum stochastic dilation.

**Definition 6.** For a \( C^* \) or von Neumann algebra, \( \mathcal{A} \), \( \mathcal{A}'' \) will denote the bicommutant. \( \mathcal{A}'' \) is a von Neumann algebra. For a Hilbert space \( H \), the free Fock space, \( \Gamma_f(H) \) is the sum of the boson (symmetric) and fermion (antisymmetric) Fock spaces, \( \Gamma^s(H), \Gamma^a(H), \Gamma_f(H) = \Gamma^s(H) \oplus \Gamma^a(H) \). The symmetrization operator defines the map from free to boson Fock space, \( \Gamma_f(H) \to \Gamma^s(H) \) by \( \text{Symm}(\otimes_{i \in [n]} g_i) = 1/(n! - 1)! \sum_{\sigma \in S_n} \otimes_{i \in [n]} g_{\sigma(i)} \). For a subspace \( V \subseteq H \), \( \mathcal{E}(V) \subseteq \Gamma^s(H) \) denotes the \( \mathbb{C} \)-linear span of exponential vectors

\[
\mathcal{E}(v) = \oplus_{k \in \mathbb{N}} v^k / \sqrt{k!}, v \in V
\]

There’s an inner product on the \( \Gamma^s(H) \) induced by the inner product on \( H \), \( \langle \mathcal{E}(u), \mathcal{E}(v) \rangle = \exp \langle u, v \rangle \). The exponential vectors are linearly independent and total in the boson Fock space. We will denote boson Fock space \( \Gamma^s(H) \) by \( \Gamma(H) \).

\(^1\)The empty boundary requirement is needed as the Dirac operator may not be symmetric otherwise (see, for instance, [25, eq II.5.2]). The even dimensionality is used in two places: to use the characterization of Clifford module bundles as twisted spinor bundles, and the triviality of the center of even dimensional Clifford algebras over fields of characteristic not equal to 2([26, § 2.2]). Since the odd dimensional Clifford algebras decompose as direct sums of two copies of even dimensional Clifford algebras, the difficulty is not fundamental.
2.2 Brownian motion as a stochastic dilation

Viewing Brownian motion on \((M, g)\) as a diffusion generated by the Laplace-Beltrami operator, it's noted that the Feynman-Kac formula for a Riemannian manifold[30, Thm 3.2], \((M, g)\), for the operator \(H := \frac{1}{2} \Delta_{(M)} + V, u \in C^4(M), V \in C(M)\), with Laplace-Beltrami operator, \(\Delta_{(M)}\), acting on the \(C^2(M)\) gives \(e^{-tH}u(x) = \int_{W(M)} \exp(\int_0^t V(\omega(s)) - 1/2\kappa_M(\omega(s))ds)u(x) \, dW_M^x(d\omega)/N(u, \kappa_M, dW_M^x(d\omega))\) where \(dW_M^x(d\omega)\) denotes the Wiener measure on \((C(M), \kappa_M)\) and \(\kappa_M\) is the scalar curvature of \(M\) and \(N(u, \kappa_M, dW_M^x(d\omega))\) a normalization depending on \(\kappa_M, u\) and \(dW_M^x(d\omega)\). This can be thought of as a stochastic dilation of heat semigroup on \(C^0(M)\) to the Wiener space, \(W(M)\), on \(M\), the integral with respect to the Wiener measure playing the role of the conditional expectation.

The Ito-Wiener-Segal isomorphism [29] provides the bridge to quantum probability: for a separable Hilbert space \(H\), a stochastic process \(W = \{W(h), h \in H\}\) defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with each \(W(x) \in W\) a centered Gaussian satisfying \(\mathbb{E}(W(h), W(g)) = \langle h, g \rangle_M\), is called an isonormal Gaussian process, and for the \(\sigma\)-field \(\mathcal{G}\) generated by \(w \in W\) for an appropriate isonormal Gaussian process \(W\) (see, for instance, [29, § 1.1]), \(L^2(\Omega, \mathcal{G}, \mathbb{P})\) is isomorphic to the boson Fock space \(\Gamma(H)\). Additionally, when \(H\) is the space \(L^2(T, B, \mu)\) where \(\mu\) is \(\sigma\)-finite without atoms over a measure space \((T, B), W(h)\) can be regarded as stochastic representations, with polynomials in \(W(h)\) denoted by \(\mathcal{W}(\cdot)\). The canonical example is for \(H := L^2(\mathbb{R}^2)\) where \(\Gamma(L^2(\mathbb{R}^2)) \cong L^2(C(\mathbb{R}^2), \mathcal{P}_{\text{Wiener}})\). Through the Ito-Wiener-Segal isomorphism between Wiener space \(W(M)\) and the associated Fock space, the heat semigroup has a stochastic dilation on the Fock space. This dilation corresponds to a flow for an Evans-Hudson type quantum stochastic differential equation (qsde) introduced next. A process satisfying a qsde of this type is considered as a quantum diffusion process.

2.3 Quantum stochastic dilation of Evans-Hudson type

On a smooth manifold, \(M\), a (homogeneous) flow is a smooth map \(\phi : \mathbb{R}_+ \times M \rightarrow M, \phi_t(m) := \phi(t, m)\), satisfying \(\phi_t(s + m) = \phi(s, \phi_t(m))\), \(\phi(0, m) = m\). The flow induces a 1-parameter semigroup, \((\phi_t)_{t \geq 0} : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M), \phi_t(f) := f \circ \phi_t^{-1}\) with the infinitesimal generator \(\mathcal{L}\) following the differential equation[23],

\[
\frac{d}{dt} \phi_t(f) = \mathcal{L}(f), \quad \text{with } \phi_0(f) = f, \quad \mathcal{L}(f) = \frac{d}{dt} \bigg|_{t=0} \phi_t(f), \quad f \in \mathcal{C}^\infty(M)
\]

The classical stochastic flow can be viewed as a stochastic process \(\psi_t\) taking values in isomorphism group of \(M\) which satisfies the flow property almost surely (see, for instance, [24, Ch 3]). Now solutions to stochastic differential equations (sde) on manifolds generate stochastic flows, the stochastic version of flow equation is obtained by introducing Wiener process terms into eq 1 yields

\[
\frac{d}{dt} \hat{h}(f) = \mathcal{L}(f) + \sum_{j \in [n]} \mathcal{L}(b_j(f)) dB_j
\]

for linear maps \(b_k\), and components \(B_j\) of \(n\)-dimensional Brownian motion \(B\) on \(M\) with sample space \(\Omega\). Algebraically, \(j_t\), are now \(\ast\)-algebra homomorphisms, \(j_t : \mathcal{B}(M \times \Omega) \supset \mathcal{C}^\infty(M) \rightarrow \mathcal{C}(M \times \Omega)\) for the space of bounded measurable functions, \(\mathcal{B}(M \times \Omega)\), on \(M \times \Omega\). Note that \(\mathcal{C}^\infty(M)\) is embedded in \(\mathcal{B}(M \times \Omega)\) by trivially extending to \(M \times \Omega\). In the integral form, the quantum analog of this sde can be defined on the Fock space.

For a finite dimensional Hilbert space \(V\), set \(H = L^2(\mathbb{R}_+, V) := L^2(\mathbb{R}_+^\ast \otimes V), H\) decomposes as \(H = H_t \otimes H^\ast\) where \(H_t = L^2([0, t]) \otimes V, H^\ast = L^2([t, \infty)) \otimes V\). On the Fock space, \(\Gamma(H) = \Gamma(H_0) \otimes_{\text{alg}} \Gamma(H^\ast)\). Given an “initial” Hilbert space \(H_0\), set

\[
\hat{H}_t = H_0 \otimes \Gamma(H_t), \hat{H}^\ast = H_0 \otimes \Gamma(H^\ast), \hat{H} = H_0 \otimes \Gamma(H)
\]

then for a class of \(\mathbb{R}_+\)-indexed operator families on \(\hat{H}, \Lambda^i_j, i, j \in [0 : \dim V_0]\), called the fundamental processes (or quantum noises, which corresponds to the annihilation, creation and conservation processes on the Fock space), the quantum stochastic integral \(\int_0^t \sum_{i,j} E^i_j(d\Lambda^i_j)\) can be defined for processes \((E_t^i)_{t \in \mathbb{R}_+}\), that are regular (i.e. the map \(t \rightarrow E_t^i(u) \otimes \mathcal{E}(u)\)) is continuous with a growth condition on \(\|E_t^i\|(u_0 \otimes \mathcal{E}(u_0))\) and each \((E_t^i)\) is adapted where a process \(X_t : \hat{H} \rightarrow \hat{H}\) is adapted if there exists \(Y_t : H_0 \otimes \mathcal{E}(H_t) \rightarrow H_0 \otimes \Gamma(H_t)\), so that \(X_t = Y_t \otimes 1_{\Gamma(H^\ast)}\), i.e. \(X_t\) does not look into the future – same as

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the classical notion of adaptedness. For brevity, we do not detail the construction, but refer to standard references [32, 34, 5].

The stochastic calculus can be developed on operator algebras in similar manner to Hilbert spaces[34, Ch 5] and the stochastic flow can be defined by extending the classical picture: for a dense \(*\)-algebra \(A_0 \subset A\) with \(A \subset B(H)\) unital, the quantum stochastic flow \((j_t)_{t \geq 0}\) is family of injective \(*\)-homomorphism, \(j_t : A_0 \to B(H)\), such that for all \(a \in A\), each \(j_t(a)\) is an adapted process and there exists \(\{\lambda_j^i : i, j \in [0 : \dim V]\}\), called the structure maps, with

\[
j_t(a) = a \otimes 1 + \int_0^t \sum_{i,j} j_t(\lambda^i_j(a)) d\lambda^i_j(t)
\]

Equivalently, in differential form, \(dj_t(a) = \sum_{i,j} j_t(\lambda^i_j(a)) d\lambda^i_j(t)\) with \(j_0 = 1\). Flows of this form, with \(j_t\) satisfying some additional constraints, are called Evans-Hudson flows[32, § 27, 28]. In particular, Brownian motion on \(\mathbb{R}\) can be realized as Evans-Hudson dilation on the Fock space \(\Gamma(L^2(\mathbb{R} \geq 0))\) defined below by specializing to \(A = L^\infty(\mathbb{R})\) viewed as operators on \(H = L^2(\mathbb{R})\), \(V\) fixed as trivial and using the Fock space-Wiener space dictionary provided by the Wiener-Ito-Segal isomorphism.

**Definition 7.** (Evans-Hudson dilation[34, Def 6.0.2]) For a quantum dynamical semigroup \((T_t)_{t \geq 0}\) with generator \(L\) on \(C^*\)-algebra \(A \subset B(H)\), a family of \(*\)-homomorphisms, \((j_t)_{t \geq 0} : A \to A'' \otimes B(\Gamma(L^2(\mathbb{R} \geq 0) \otimes V))\) satisfying

- There exist maps \(J_t : A \otimes_{alg} E(L^2(\mathbb{R} \geq 0) \otimes V) \to A'' \otimes B(\Gamma(L^2(\mathbb{R} \geq 0) \otimes V))\), \(J_t(a \otimes e(f))u := j_t(a)(ue(f))\) such that for an ultra-weakly dense subalgebra \(A_0 \subset A\), \(\text{Dom}(L) \subset A_0\), on \(A_0 \otimes L^2(\mathbb{R} \geq 0) \otimes V\) the Evans-Hudson flow sde

\[
dJ_t = J_t(a \delta(dt)) + a^\delta_t(dt) + \Lambda_\sigma(dt) + 1_L(dt), \quad J_0 = 1
\]

holds, where \(a_\delta, a^\delta_t, \Lambda_\sigma, 1_L\) are structure maps as defined [34, § 5.4]; \(J_t\) as a quantum stochastic process is regular and adapted.

- \(j_t\) is a dilation of \(T_t\) in the following sense: for all \(u, v \in H, a \in A\), \(\langle uE(0), j_t(a)vE(0) \rangle = \langle v, T_t(a)u \rangle\)

## 3 Almost commutative spectral triples as spinor bundles

A spectral triple is three basic pieces of data, \((A, H, D)\), where \(D\) is symmetric operator on the Hilbert space \(H\), and a \(*\)-algebra of bounded operators on \(H, A \subset B(H)\). The operator \(D\) is allowed to be self-adjoint and unbounded but with \([D, a]\) bounded for all \(a \in A\). A compact Riemannian spin manifold \((M, g)\) can be characterized by the canonical spectral triple, \(\mathfrak{A}_M := (C^\infty(M), L^2(M, S), D_M; J_M, \gamma_M)\) where \(S\) is the spinor bundle, \(C^\infty(M)\) is the \(*\)-algebra of smooth functions interpret as operators acting on \(L^2(S)\) by multiplication and \(D_M\) is the Dirac operator associated with the Levi-Civita connection on the spinor bundle, and the data of a spectral triple has been supplemented with a \(Z_2\) grading operator \(\gamma_M\) on \(H\) and an anti-unitary operator \(J : H \to H\), called the real structure, which makes \(H\) an \(A - A\) bimodule from a left \(A\)-module. Such spectral triples can be characterized abstractly; Connes reconstruction theorem recovers the Riemannian spin structure from the abstract spectral triples[21, Thm 11.2].

A finite noncommutative space is the finite spectral triple, \(\mathfrak{A}_F := (A_F, H_F, D_F)\), with \(\dim H_F\) finite. This is supplemented with a real structure and a grading, \((J_F, \gamma_F)\). A product almost commutative spectral triple is the globally trivial bundle,

\[
M \times F := (C^\infty(M) \otimes A_F, L^2(M, S \otimes H_F), D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F)
\]

Čapić[10] expands the definition of product almost commutative spectral triples include non-trivial algebra bundles over the base space. This is formalized without appeal to the explicit product structure as an abstract almost commutative spectral triple:

**Definition 8.** ([10, Def 2.16]) A spectral triple \((A, H, D)\), \(B \subset A\) a central, unital \(*\)-subalgebra is an abstract almost-commutative spectral triple over the base \(B\) if \((B, H, D)\) is a commutative spectral triple of Dirac type[10], and for all \(a \in A, [D, a] \in A\), additionally

\[\text{References: [31]}\] note, any Markov chain on countable state space can be realized as Evans-Hudson flow.
1. For all \(a \in A, b \in B\), \([[D, b], a] = 0\).

2. \(A\) is an even finitely generated projective \(B\)-module and a \(*\)-subalgebra of the algebra \(\text{End}_{S^+B}(H_\infty)\) where \(H_\infty = \cap_{k \in \mathbb{N}} \text{Dom} D^k\)

The concrete realization of the abstract almost-commutative spectral triple is constructed by appeal to Connes's reconstruction theorem[21, Ch 11], and the following global analytic equivalent formulation is obtained, and this is formulation that we work with.

**Definition 9.** [10, Def 2.3] An almost-commutative spectral triple is a spectral triple of the form

\[(C^\infty(X, A), L^2(X, H), D_0)\]

for a compact oriented Riemannian manifold \(X, H\) a self-adjoint Clifford module bundle, \(A\) a real unital \(*\)-algebra subbundle of \(\text{End}^+_\text{Cl}(X)(H)\), and \(D_0\) is a symmetric Dirac-type operator on \(H\), where \(\text{End}^+_\text{Cl}(X)(H)\) are the even endomorphisms of \(H\) that supercommute with the Clifford action \(c : T^*X \to \text{End}(H)\) defined by \(D\).

**Remark 1.** Recalling that a \(\mathbb{Z}_2\) graded \(\mathbb{k}\)-algebra, \(A = A^0 \oplus A^1\), with \(A^i \cdot A^j \subset A^{i+j}\), the supercommutator \([\cdot , \cdot]_s\) is the map \([a^i, b^j]_s = a^ib^j - (-1)^{ij}b^ia^j\) for \(a^i, b^j \in A^i, b^j \in A^j\). As the Clifford action, \(c : T^*X \to \text{End}(H)\), and as \(\text{End}^+_\text{Cl}(X)(H)\) consists of even endomorphisms, \(\phi \in \text{End}^+_\text{Cl}(X)(H)\) commutes with \(c, \phi \circ c = c \circ \phi\).

### 3.1 Structure of Dirac bundles

Recall the Clifford algebra \(\text{Cl}(V, Q)\) is the algebra generated over the vector space \(V\) by the relation \(v^2 = Q(v)1\) where \(Q\) is a quadratic form on \(V\). It satisfies the following universal property: any linear map \(f : V \to A, V\) a vector space, \(A\) a unital associative \(\mathbb{k}\)-algebra, with \(f(v) \cdot f(v) = Q(v)1\) extends uniquely to a \(\mathbb{k}\)-algebra homomorphism \(\tilde{f} : \text{Cl}(V, Q) \to A\). \(\text{Cl}(V, Q)\) comes with a \(\mathbb{Z}_2\) grading, \(\chi(v_1 \ldots v_k) = (-1)^k\), that yields the decomposition, \(\text{Cl}(V, Q) = \text{Cl}(V, Q)^0 \oplus \text{Cl}(V, Q)^1\). Specializing to \(\mathbb{R}^n\), fix \(Q_n = \sum_{i=1}^n x_i^2\), define \(\text{Cl}^+_n = \text{Cl}(\mathbb{R}^n, Q_n)\), \(\text{Cl}^-_n = \text{Cl}(\mathbb{R}^n, -Q_n)\) and \(\text{Cl}^\omega = \text{Cl}(\mathbb{C}^n, Q_n)\) which is \(\text{Cl}^+_n \otimes \mathbb{R} \mathbb{C}, \text{Cl}^-_n \otimes \mathbb{R} \mathbb{C}\). The grading comes from the chirality operator \(\gamma_{n+1}\) on \(\text{Cl}_n\) is given by \((-1)^m e_1 \cdots e_n\) where \(e_i\)'s generate \(\text{Cl}_n\) and \(n = 2m\) if even and \(n = 2m+1\) for odd. This can be carried over to a vector bundle –

**Definition 10.** A Clifford structure on a vector bundle \(E\), is a bundle morphism \(c : T^*M \to \text{End}(E)\), \(\{c(u), c(v)\} = -2g(u, v)1\). \(c(v) \in \text{End}(E)\) denotes the “Clifford multiplication by \(v\), and the pair \((E, c)\) is the Clifford bundle. The Clifford bundle \(E \to M = \mathbb{Z}_2\) graded if there’s a decomposition \(E = E^+ \oplus E^-\) such that \(c(\alpha)\) for each \(\alpha \in T^*M\) is an odd endomorphism: \(c(\alpha)(\Gamma(E^\pm)) = \Gamma(E^\mp)\). A vector bundle with a Clifford structure is a Clifford module bundle.

**Remark 2.** Note that \(c : \Omega^1(M) \to \Gamma(\text{End}(E))\). Actually, \(c\) is the full Clifford algebra, \(c : \text{Cl}(M) \to \Gamma(\text{End}(E))\), but because \(c : T^*M \to \Gamma(\text{End}(E))\) satisfies \(\{c(u), c(v)\} = -2g(u, v)\), the full action follows by using the universal property of Clifford algebras; \((c, E)\) is a representation of \(\text{Cl}(M)\).

On any Riemannian manifold \((M, g)\), there exists a canonical Clifford bundle, \(\text{Cl}(T^*M, -g) = \text{Cl}(M)\). A Clifford module bundle is any bundle that carries an action of the Clifford bundle.

A Dirac bundle \(S\) over a \((M, g)\) is a Clifford module bundle with a connection \(\nabla^S\) that is compatible with the Clifford multiplication –

- for all \(\sigma_i \in S_x, e \in T_x M, \|e\| = 1\), \(e\) acting on \(\sigma_i\) by clifford multiplication, \(\langle e \cdot \sigma_1, e \cdot \sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle\) (as \(e^2 = -1\), this yields the skew-hermiticity, \(\langle e \cdot \sigma_1, \sigma_2 \rangle = -\langle \sigma_1, e \cdot \sigma_2 \rangle\))

- \(\nabla^S(\phi \cdot \sigma) = (\nabla^\text{Cl}(M)\phi) \cdot \sigma + \phi \cdot \nabla^S \sigma\)

For clarity it is useful to separate out the algebraic Clifford structure from the geometric piece.

**Definition 11.** A Dirac bundle, \((E, c, h, \nabla, M, g)\), is a Clifford bundle \((E, c)\) over \((M, g)\) with a hermitian metric \(h\) on \(E\) and Clifford connection, \(\nabla\), compatible with \(h\) such that for all \(\alpha \in \Omega^1(M)\) the following holds:

- \(c(\alpha) \in \text{End}(E)\) is skew-Hermitian
• For $X \in \Gamma(TM)$, $u \in \Gamma^\infty(E)$, $\nabla^M$ the Levi-Civita connection on $M$, $\nabla_X (c(\alpha)(u)) = c(\nabla^M_X \alpha)u + c(\alpha)(\nabla_X u)$.

The Dirac structure is the tuple $(\nabla, h)$ associated to $(E, c)$.

**Definition 12** (Geometric Dirac operator). A geometric Dirac operator is a Dirac operator, $\mathcal{D}$, that is associated to a $(E, c, h, \nabla)$ Dirac structure over $(M, g)$ by

$$\mathcal{D} := c \circ \nabla : \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\nabla} \Gamma(E)$$

In local coordinates, after fixing a basis $(e^i)$ of $T^*M$ and the corresponding dual basis $(e_i)$, $\nabla \in \Gamma(S \otimes T^*M)$ can be expanded in this basis as $\sum_i e^i \otimes \nabla_{e_i}$. Composed with the Clifford action this gives that the geometric Dirac operator acts by $\Gamma(S) \ni \sigma \rightarrow \sum_i e^i \cdot \nabla_{e_i} \sigma \in \Gamma(S)$. More generally Dirac operator can be defined as a first order partial differential operator on the sections of any left $\text{Cl}(M)$ module bundle.

**Definition 13.** (Generalized laplacian and Dirac operator)

- A generalized laplacian $\Delta$ is a second order differential operator on a vector bundle $E$ with $\sigma_2(L)(x, \xi) = |\xi|^2$.
- A Dirac-type operator on a Clifford module bundle $E$ with Clifford action $c$ over $(M, g)$ is a first order differential operator $D$ such that $[D, f] = c(df)$ for all $f \in C^\infty(M)$.

Every Dirac operator $D$ on the vector bundle $E$ over $M$, induces a Clifford action of $T^*M$ on $E$ by $c(df) := [D, f]$ for $f \in C^\infty(M)$, and conversely, associated to any Clifford action $c$, the operator satisfying $[D, f] = c(df)$ is a Dirac operator (see, for instance, [9, Prop 3.38]).

**Definition 14.** (Spinor bundle) For any oriented vector space $V$, $\dim V = 2k$, the spinor module is the unique $\mathbb{Z}_2$ graded Clifford module $S = S^+ \oplus S^-$ with $\text{Cl}(V) \otimes \mathbb{C} = \text{End}(S)$. The spinor bundle $S$ over a $2k$-dimensional manifold $M$ with a spin structure is the associated bundle $\text{Spin}(M) \times_{\text{Spin}(n)} S$.

Every $\mathbb{Z}_2$ graded complex $\text{Cl}(V)$-module $E$ is isomorphic to $W \otimes S$, and given $E, W$ can be recovered by $W = \text{Hom}_{\text{Cl}(V)}(S, E)$ with trivial $\text{Cl}(V)$ action, that is, the Clifford action on $E$ is the Clifford action on the $S$ component, $e \cdot (w \otimes s) := w \otimes (e \cdot s)$, and $\text{End}(W) \cong \text{End}_{\text{Cl}(V)}(E)$. The fibers, $S_x$, are isomorphic to $S$ and, and, therefore, over local a trivialization, $(U, \phi_U)$, the statements about $S$ carry over to $S^*_U$, and to the bundle $S$: every Clifford module $H$ over $M$ is a twisted spinor bundle, $H = W \otimes S$ with $W \cong \text{End}_{\text{Cl}(X)}(S, H)$, $\text{End}(W) \cong \text{End}_{\text{Cl}(X)}(H)$ (see, for instance, [9, Prop 3.35]).

This yields that on spin manifolds, associated to Clifford structures, Dirac structures exist—locally $H = W \otimes S$, the tensor product connection of the Levi-Civita connection on $S$ and any connection on $W$ is compatible with the Clifford action, and then the global version follows by a partition of unity argument.

Suppose $(C^\infty(X, A), L^2(X, H), D_0)$ is an almost commutative spectral triple with generalized Dirac operator $D_0$, $H$ a spinor bundle over compact spin manifold $M$. [10, Thm 2.17] gives a metric $Q$ on $H$ which corresponds to the Clifford action associated to $D_0$ on $H$. By above, there exists a Dirac structure on $H$ arising from the Clifford action induced by $D_0$. $H$ being a Clifford module bundle is a twisted spinor bundle $W \otimes S$ with twisting space $W$; the connection on $S$ is the spinor connection and on $W$ it can be chosen as the canonical Riemannian connection for the metric induced from $Q$. If, however, the bundle $H$ comes with a Dirac structure, then choice to use canonical Riemannian connections on the twisting space and the spinor bundle is not necessary and the given Dirac structure can be used. The canonical Riemannian connections are chosen the connection on $H$ is torsion free. However, often connections with non-vanishing torsions are of interest and arise naturally: for example, the canonical connection on a homogeneous space and connection associated to Kostant’s cubic Dirac operator may not be torsion free[1]; connections with totally anti-symmetric non-zero, torsion also have relevance to models of gravity.

**Observation 1.** Suppose $D$ is the geometric Dirac operator for the Dirac structure. Then $D_0$ and $D$ give the same Clifford action, and, therefore, $D - D_0 = A$ for some odd endomorphism, $A \in \Gamma(\text{End}^-(H))$. When $W$ is $\mathbb{Z}_2$-graded, and the $D_0, D$ are odd with respect to the grading and $\gamma D_0 = -D_0 \gamma, \gamma D = -D \gamma$, then $A$ is anticommutes with $\gamma$. Using this, it follows that $D - D_0 = A$ is in fact associated with a

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4see, for instance, [28, Prop 11.1.65, 9, Cor 3.41]
connection potential: $H = W \otimes S$ implies $A \in \Gamma(\mathrm{Cl}(X) \otimes \mathrm{End}(W))$, but as vector spaces $\mathrm{Cl}(X)$ and $\Lambda^* T X$ are isomorphic, so $A$ specifies a form $\mathrm{End}(W)$-valued form $\Omega_A$ defined in local geodesic coordinates $(e_i)$ by $\Omega_A(e_i) = \gamma A$ for each $i$. Because $\gamma A = -A \gamma$, $\Omega_A$ is anti-symmetric and, therefore, defines a metric compatible connection locally by $d + \Omega_A$. More formally, this is the statement that the Dirac operators for a Clifford action on a twisted spinor bundle are in one-one correspondence with (super)connection on the twisting space (see, for instance, [9, Ch 3]).

Remark 1 leads to the following result on structure of almost commutative type endomorphism algebra $A \subset \mathrm{End}^\times_{\mathrm{Cl}(X)}(H)$, where $H$ is Clifford module bundle over a Riemannian manifold, $(X, g)$, therefore, a twisting of the complex spinor bundle $S$, $H = W \otimes S$.

**Theorem 1.** If $\alpha \in A \subset \mathrm{End}^\times_{\mathrm{Cl}(X)}(H)$ then $\alpha = w_a \otimes 1$ for $w_a \in \mathrm{End}(W)$ up to multiplication by $f \in C(X)$. That is, as a module over $C(X)$, $C(X, A)$ is generated by endomorphisms of form $w_a \otimes 1$.

**Proof.** The proof is basically the observation that locally $\mathrm{End}(W) \cong \mathrm{End}_{\mathrm{Cl}(X)}(H)$ (see, for instance, [9, Prop 3.27]) (i.e. $A \cong W_A \subset \mathrm{End}(W)$). Now $\mathrm{End}(H)$ is the topological closure of $\mathrm{End}(W) \otimes \mathrm{End}(S)$, where $S$ is the complex spinor bundle, $\mathrm{End}(S) \cong \mathrm{Cl}(X) \otimes \mathbb{C}$.

Suppose $\alpha = \sum_i \alpha_{w,i} \otimes \alpha_{s,i} \in A \subset \mathrm{End}(W) \otimes \mathrm{Cl}(X) \otimes \mathbb{C}$ where $\alpha_{s,i} \in \mathrm{Cl}(X) \otimes \mathbb{C}$. Consider the Clifford action, $c : \mathrm{Cl}(X) \to \mathrm{End}(H)$, $v \mapsto c(v) := \sum_i w_i \otimes s_i \in \mathrm{End}(H)$ with $w_i \in \mathrm{End}(W), s_i \in \mathrm{Cl}(X) \otimes \mathbb{C}$. By construction of the twisted spinor bundle, the Clifford action on the $W$ piece is trivial so $w_i = 1$ for all $i$, therefore, $c(v) = 1 \otimes w_v$ with $w_v \in \mathrm{Cl}(X) \otimes \mathbb{C}$.

Since $\alpha$ commutes with the Clifford action, $c(v)$,

$$
\sum_i \alpha_{w,i} \otimes v_s \alpha_{s,i} = (1 \otimes v_s) \circ \sum_i \alpha_{w,i} \otimes \alpha_{s,i} = \sum_i \alpha_{w,i} \otimes \alpha_{s,i} \circ (1 \otimes v_s) = \sum_i \alpha_{w,i} \otimes \alpha_{s,i} v_s
$$

In even dimensions, the canonical complex bundle $S$ in the twisted spinor decomposition, $W \otimes S$, is irreducible Clifford module and $\mathrm{Cl}(X)$ is a central simple algebra, therefore, $v_s$ runs over all elements in $\mathrm{Cl}(X) \otimes \mathbb{C}$ As $v_s$ is arbitrary, $\alpha_{s,i}$ lie in the center of $\mathrm{Cl}(X)$. This can be seen locally – choose a basis $(e_i)$ for $T X$, then the basis for $\mathrm{Cl}(T M)$ is $(e_I)_{I \subset \{1, \ldots, \dim T X\}}$. Expressing $\alpha$ in $e_I$’s gives $\sum_i \alpha_{w,i} \otimes \alpha_{s,i} = \sum_I k_I \alpha_I \otimes e_I$ for $k_I$. Note that $e_I \cdot e_J = \pm e_J \cdot e_I$ for any $i$. Suppose $|I| > 0$. If $|J|$ is odd, then there exists $j \notin I$, and $e_I \cdot e_J = -e_J \cdot e_I$ as it commutes past each $e_i$ for $i \in I$. If $|I| = 2k$ with $e_I = e_{i_1} \cdot \ldots e_{i_{2k}}$, then $e_I \cdot e_{i_2k} = -e_{i_1} \cdot \ldots e_{i_{2k-1}}$, while $e_{i_2k} e_I = e_{i_1} \cdot \ldots e_{i_{2k-1}}$ because there are $2k - 1$ sign changes on moving across and then a final sign change from $e_{i_{2k}}^2 = -1$. Therefore, $|I| = 0$ for $e_I$ to commute with each $e_i$ but then $e_I \in \mathcal{Z}(\mathrm{Cl}(T^* X))$. Note there’s no ambiguity in the crossnorm with respect to which $\mathrm{End}(W) \otimes \mathrm{Cl}(X) \otimes \mathbb{C}$ is completed: the norm is the operator norm on $H$. The conclusion holds on the algebraic tensor product, and also the topological completion.

This is consistent with the case for commutative spectral triples where the algebra $C^\infty(M)$ acts by multiplication on the spinor bundle $L^2(S)$ and commutes with the Clifford action. Since the spinor endomorphism part is restricted to be trivial, if the twisting space is chosen as a trivial matrix bundle with Hilbert-Schmidt inner product, then a rough analogy between almost commutative and fuzzy spectral triples introduced in [7] becomes clear.

**3.2 Complete Markovity on product almost commutative spectral triples**

For real even spectral triples, $(\mathcal{A}_i, H_i, D_i; J_i, \gamma_i), i \in \{1, 2\}$, that is, the spectral triples comes with a real structure $J_i$ and a grading $\gamma_i$ such that for all $a \in \mathcal{A}_i, \gamma_i a = a \gamma_i, D_i = -D_i \gamma_i$, the product is defined by $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2, H = H_1 \otimes H_2, D := D_1 \otimes 1 + 1 \otimes D_2, \gamma = \gamma_1 \otimes \gamma_2, J = J_1 \otimes J_2$. If the second triple is not even then resulting structure does not have a grading and the adjective even is dropped. Since the first triple is even and $D_1, \gamma_1$ anti-commute, $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$. Note that $C^\infty(M, A_F) = C^\infty(M) \otimes A_F$ and the tensor products are $\mathbb{Z}_2$-graded.

**Remark 3.** While noncommutative geometry does not require that $\mathcal{A}_i$ be closed, for example, the requirement $[D_i, a]$ is bounded is only needed for $a \in \mathcal{A}_i$; however, the questions about quantum dynamical semigroups often presume norm closure. The algebras $\mathcal{A}_1, \mathcal{A}_2$ are only pre-$C^*$-algebras, but can be completed in the respective $C^*$-norm; for the canonical spectral triple, $C^\infty(M)$, will have $C(M)$ as the closure. Note that in the $\mathbb{Z}_2$-graded tensor product category, as with ungraded tensor products, the commutative $C^*$-algebras are characterized nuclear[14], so there’s no ambiguity in the norm to use.
The product almost commutative spectral triple is the product of the canonical spectral triple of a Riemannian spin manifold, \( \mathfrak{A}_M := (C^\infty(M), L^2(S), D_M; J_M, \gamma_M) \), and a finite noncommutative space, \( \mathfrak{A}_F := (A_F, H_F, D_F; J_F, \gamma_F) \),

\[
M \times F := (C^\infty(M) \otimes A_F, L^2(M, S \otimes H_F), D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F)
\]

The following technical lemma will be useful: topological questions in tensor products of \( C^* \)-algebras can be delicate; we will work with identifications as described in the statement, the point of this lemma is that such identifications behave well.

**Lemma 1.** Suppose \( A_1, A_2 \) are unital \( C^* \)-algebras. Suppose at least one of \( A_1, A_2 \) is nuclear, so there’s a unique crossnorm on \( A_1 \otimes A_2 \), then the map, \( \phi : A_1 \to A_1 \otimes A_2, a \to 1 \otimes a \), is a completely positive, homeomorphism onto its image.

**Proof.** The kernel of \( \phi \) is trivial, and \( \phi \) is positive as \( a \) positive in \( A_1 \) means \( 1 \otimes a \) is positive in \( A_1 \otimes A_2 \). Trivially \( \phi \) is unital. From the \( \mathbb{R} \)-linearity of the tensor product, it follows the map \( \phi \) preserves norms. It also follows that \( \phi \otimes 1_n \) also preserves norms, so \( \phi \) is a unital, completely contractive map, hence is completely positive. Being contractive also implies continuity. The inverse map on the image, \( \phi^{-1} \), \( 1 \otimes a \to a \), is again unital and completely contractive: the same holds for \( \phi^{-1} \) as well.

We note the following about the complete positivity of the Dirac heat semigroup for product almost commutative spectral triples:

**Theorem 2.** The complete positivity of the semigroup \( e^{-tD^2} \) is well defined.

**Proof.** By remark 3, if both spectral triples are \( \mathbb{Z}_2 \)-graded with \( \mathbb{Z}_2 \)-graded tensor product, \( C(M) \otimes A_F \), then as using the commutative algebra \( C(M) \) is nuclear in \( \mathbb{Z}_2 \)-graded tensor product category, the cross norm is unique, and there’s no ambiguity to the norm with which to complete the tensor product.

The only other ambiguity to resolve is the order of the product: \( \mathfrak{A}_F \times \mathfrak{A}_M \) versus \( \mathfrak{A}_M \times \mathfrak{A}_F \). However, the complete positivity of the semigroup generated by \( -D^2_{F \otimes M} := -D^2_F \otimes 1 - 1 \otimes D^2_M \) means the map \( e^{-tD^2_{F \otimes M}} \otimes 1_n : \text{MAT}_n[A_F \otimes C^\infty(M)] \to \text{MAT}_n[A_F \otimes C^\infty(M)] \) is completely positive for each \( t \). On the algebraic tensor product, the positivity of \( a \otimes b \) and \( b \otimes a \) is equivalent, so the oppositely ordered semigroup generated \( e^{-tD_{M \otimes F}} \otimes 1_n : \text{MAT}_n[C^\infty(M) \otimes A_F] \to \text{MAT}_n[C^\infty(M) \otimes A_F] \) is also completely positive; the result also holds in the topological completion as the norm is independent of the order of the product.

**Theorem 3.** If \( e^{-tD^2_{M}} \) and \( e^{-tD^2_{F}} \) are both completely positive then for \( D^2 = 1 \otimes D^2_M + D^2_F \otimes 1, e^{-tD^2} \) is as well. The converse holds when \( e^{-tD^2} \) is conservative.

**Proof.** Because \( 1 \otimes D^2_M \) and \( D^2_F \otimes 1 \) commute, therefore, \( e^{-tD^2} = e^{-t(1 \otimes D^2_M)} e^{-t(D^2_F \otimes 1)} = e^{-t(D^2_F \otimes 1)} e^{-t(1 \otimes D^2_M)} \). Now suppose \( e^{-tD^2_{M}} \) and \( e^{-tD^2_{F}} \) are completely positive. The tensor product of completely positive maps extends to a completely positive map with respect to the \( \| \cdot \|_{min} \) (see, for instance, [33, Thm 12.3]: the standard result is for ungraded tensor product, but it applies since commutative \( C^* \)-algebras are nuclear regardless of the grading and there’s only one cross norm across both settings). Since \( A_M \) is nuclear, \( 1 \otimes e^{-tD^2_{M}}, e^{-tD^2_{F}} \otimes 1 \) are completely positive on \( A_F \otimes A_M = A_F \otimes_{min} A_M \). Furthermore, \( 1 \otimes D^2_M, D^2_F \otimes 1 \) commute, and \( e^{-tD^2} \) is composition of completely positive maps and also completely positive.

When \( H_t := e^{-tD^2_{M}} \) is unitary, \( H_t(1 \otimes A_M) = 1 \otimes e^{-tD^2_{M}}(A_M) \). Since \( 1 \otimes A_M \) generates the \( C^* \)-algebra, \( K \otimes K A_M \cong A_M \), with \( K = \mathbb{C}, \mathbb{R} \) depending on the underlying Hilbert space, the result follows for \( e^{-tD^2_{M}} \), with symmetric argument for \( e^{-tD^2_{F}} \).

If \( e^{-tD^2_{M}} \) and \( e^{-tD^2_{F}} \) are contractive, then the composition \( e^{-tD^2} \) is contractive as well. However, the converse does not hold. The same applies to conservativeness: if \( e^{-tD^2_{M}} \) and \( e^{-tD^2_{F}} \) are multiplication by \( a \neq 0, 1 \) and \( 1/a \) respectively then the composition is identity while neither map is conservative and only one is contractive.

Notice that since \( D^2_F \) is a bounded operator, it generates a completely positive semigroup if and only if it’s completely conditionally positive; in particular, the general form of for general form for generators of completely positive uniformly continuous semigroups, i.e., bounded generators, is known (see, for instance [4]).
Example 1. On the product spectral triple, $M \times F = (\mathcal{C}^\infty(M) \otimes A_F, L^2(M, S \otimes H_F), D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F)$, it’s assumed that $D_F$ does not know about $M$. This can be generalized slightly following [10] to the picture where $M \times H_F \to M$ is a trivial bundle with a trivial connection $\nabla^F$ and $H = S \otimes (M \times H_F)$ is a twisted spinor bundle, i.e., the Clifford action takes place on $S$. Noting observation 1, the geometric Dirac operator on $H$ is given by

$$\slashed{D}_H := \slashed{D}_S \otimes 1 + c \otimes \nabla^F$$

where $c$ is the Clifford action. Ćaćić [10] defines the operator $D = \slashed{D}_H + 1 \otimes D_F$. $D$ is Dirac type operator on the spectral triple $(\mathcal{C}(M, A), L^2(M, H), D)$ where $A := L \otimes (X \times A_F)$ for $L$ a real, unital, trivial sub-bundle of $\text{End}(S)$ given by $L_x := \mathbb{R}1_{S_x}$. This follows easily as $[D, f] = [\slashed{D}_H, f] + [1 \otimes D_F, f]$ and $[1 \otimes D_F, f]$ vanishes as $f \in \mathcal{C}^\infty(M)$ on each fiber is scalar multiplication that commutes with the $D_F$ which on the fibers is a matrix of scalars.

Remark 4. Note that the fibers being $\mathbb{R}1_{S_x}$ is just a specialization of the theorem 1.

In the bundle $H$, the symmetric operator $D_{HF}$ on the fibers $(H_F)_{m \in M}$ can now vary with $m \in M$. Note the mixed term $(\slashed{D} + c \circ \slashed{D}) \otimes (\nabla^F + \Omega_F)$ that now appears even when $D_F$ is constant. Complete positivity on such bundles is addressed with same geometric methods as general almost commutative $\mathcal{L}$.

For the spinor bundle $S \to M$, we want to consider the complete positivity\(^5\) of semigroup generated by the heat operator $e^{-\theta \slashed{D}}$ on an appropriate algebra $\mathcal{A} \subset \mathcal{B}(L^2(M; \text{End}(S)))$. The algebra $\mathcal{A}$ will contain the Hilbert-Schmidt operators on $L^2(M, S)$, with Hilbert-Schmidt inner product $(f, g)_{HS}$,

$$(f, g)_{HS} := \text{Tr}_{HS}(fg^*) = \sum \langle e_i, fg^*e_i \rangle_{L^2(M, E)} = \sum \int \langle e_i, f(x)(g(x)^*e_i) \rangle_{E, x} \, dx \, dvol(M)$$

where $(e_i)$ is an orthonormal system for $L^2(M, E)$. Such systems are provided by self-adjoint elliptic operators on the bundle: if $P$ is self-adjoint elliptic operator $P : \Gamma(E) \to \Gamma(E)$, then eigenspaces of $P$, $E_\lambda := \ker(P - \lambda I)$, are finite dimensional, consist of smooth sections, and give a complete orthonormal system for $L^2(E)$, $L^2(E) = \oplus \lambda E_\lambda$. Additionally, for an elliptic operator $P : \Gamma(E) \to \Gamma(E)$ of order $m$ on vector bundle $E$ over compact $X$, on any open set $U \subset X$, $u \in L^2_s(E)$ where $L^2_s(E), s \in \mathbb{R}$ is the Sobolev space, $Pu \in C^\infty$ implies $u \in C^\infty$. Now the connection laplacian $\nabla^* \nabla$ is an elliptic operator. By [6, Thm 3.7], the closure of the connection laplacian of $E$, $\Delta^E$ is self-adjoint. Since $\Delta^E$ restricts to $\Delta^E$ over $\Gamma^\infty(E)$, and the eigenspaces consist of smooth sections, we have a basis for $L^2(E)$ in terms of smooth eigensections of $\Delta^E$ (see [25, Thm III.5.2, III.5.8, Def III.2.3]).

$\text{Tr}_{HS}$, being lower semicontinuous and faithful is permissible in the sense of Albeverio and Høegh-Krohn [2]; this means we can use to noncommutative Dirichlet form theory to consider the question of generating completely Markov semigroups; we introduce this next.

3.3 Noncommutative Dirichlet forms

Recall from [2], for a $C^*$-algebra $\mathcal{A}$ with a lower semicontinuous faithful trace $\tau$, $L^2(\mathcal{A}, \tau)$ is the completion on the pre-Hilbert space $\{x : \tau(x^*x) < \infty\}$ with inner product $\langle x, y \rangle := \tau(y^*x)$. Set $L^2_k(\mathcal{A}, \tau) := \{x \in L^2(\mathcal{A}, \tau) : x = x^*\}$.

Definition 15. A strongly continuous contraction semigroup on $L^2(\mathcal{A}, \tau)$ is symmetric if for all $x, y$, $\langle \Phi_t(x), y \rangle = \langle x, \Phi_t(y) \rangle$. Further, if $0 \leq \Phi_t \leq 1$ whenever $0 \leq t \leq 1$ then the semigroup is Markov.

Definition 16. Suppose $\mathcal{E}(x, x)$ is a closed, quadratic form on $L^2_k(\mathcal{A}, \tau)$, with dense domain $\text{Dom}(\mathcal{E})$ with $f(\text{Dom}(\mathcal{E})) = \text{Dom}(\mathcal{E})$ for $f \in \text{Lip}(\mathbb{R}, 0)$, the Banach space of Lipschitz continuous functions that fix zero, $\|f\|_{\text{lip}} := \inf\{m : \|f(x) - f(y)\| \leq m|x - y| \text{ for all } x, y \in \mathbb{R}\}$. Then $\mathcal{E}$ is a Dirichlet form if $\mathcal{E}(f(x), f(x)) \leq \|f\|^2_{\text{lip}} \mathcal{E}(x, x)$. The form $\mathcal{E}$ is completely Dirichlet if $\mathcal{E} \in \mathcal{I}_n$ is Dirichlet for each $n \in \mathbb{N}$.

---

\(^5\)Davies and Rothaus [15] considers a $C^*$-bundle structure on $\mathcal{C}(\text{M, End}(E))$ over a Riemannian manifold $(M, g)$ by defining the involution and $\mathcal{C}^*$-algebras pointwise then integrating to get the global structure. Such bundles are isomorphic to trivial bundles and do not capture the noncommutative geometry perspective. The complete positivity of heat-semigroups on the Clifford $C^*$-bundles is considered in [13, 15] and similar results on twisted spinor bundles can be derived using techniques considered here.
For a symmetric Markov semigroup $(\Phi_t)_{t \geq 0}$ on $L^2(A, \tau)$, with a positive self-adjoint generator $L$ on $L^2(A, \tau)$, $\Phi_1 = e^{-tL}$, the associated quadratic form [2, thm 2.7] is given by

$$E^L(x) := E^L(x, x) = \langle L^{1/2}x, L^{1/2}x \rangle = \| L^{1/2}x \|^2_{L^2(A, \tau)}$$

**Theorem 4.** ([2, Thm 2.7, 3.2]) Dirichlet forms are in one-one correspondence with symmetric Markov semigroups: the positive quadratic form $E^L$ associated to the generator $L$ for a symmetric Markov semigroup $\Phi_1$ is a Dirichlet form. And conversely, if $E^L(x, x)$ is a Dirichlet form on $L^2(A, \tau)$ then $H$ generates a Markov semigroup on $L^2(A, \tau)$. This extends to complete Markovity: $E^L$ is completely Dirichlet if and only if $L$ generates a completely Markov semigroup.

## 4 Complete Markovity on spinor bundles

### 4.1 Twisted spinor bundles

To examine complete positivity on almost commutative spectral triples we will need the Bochner, Weitzenböck, and Lichnerowicz–Schrödinger identities from spin geometry. Suppose $H$ is a twisted spinor bundle, with Dirac laplacian, $D^H_2$, while the associated connection laplacian is $\triangle^H$. The Riemann curvature tensor, $Rm \in \Gamma(\otimes^4 TM)$, is defined by $Rm(v_1, v_2, v_3, v_4) = [R(v_1, v_2)v_3, v_4]$, $v_i \in \text{Vect}(X)$, $Rm(v_1, v_2) \in \text{End}(T^*_m(X))$, $m \in X$ for each $v_1, v_2 \in T^*_mM$ defined by $Rm(v_1, v_2)v = (\nabla_{v_1}\nabla_{v_2}v - \nabla_{v_2}\nabla_{v_1}v - \nabla_{[v_1,v_2]}v)(m)$ for $v_i \in \text{Vect}(M)$. The symmetries of the curvature imply that $R$ can be viewed as an operator, $R : \Lambda^2 TM \otimes \Lambda^2 TM \rightarrow \mathbb{R}$. For any vector bundle $E$ with connection $\nabla^E$, $R^E_{\nabla^E} w$ will denote the curvature transformation of the bundle, $R^E_{\nabla^E} : \Gamma(E) \rightarrow \Gamma(E)$, $e \rightarrow (\nabla_{v_1}\nabla_{v_2}e - \nabla_{v_2}\nabla_{v_1}e - \nabla_{[v_1,v_2]}e)e \in \Gamma(E)$.

**Definition 17.** For the connection laplacian $\triangle = \nabla^*\nabla$ and the Dirac operator $D$ for any Dirac bundle $S$ over $X$, $n = \dim X$, with $R^S_{\nabla^S}$ the curvature transformation of $S$, $(e_i)$ the orthonormal tangent frame, the **general Bochner identity** states

$$D^2 = \triangle + 2 \mathcal{R}(\phi) := \frac{1}{2} \sum_{j,k \in [n]} e_j \cdot e_k \cdot R^S_{e_i,e_j}(\phi)$$

(4)

This specializes to the **Lichnerowicz–Schrödinger formula** when $X$ is a spin manifold, $S$ the bundle of spinors over $X$ with the canonical Riemannian connection, $D$, the Atiyah-Singer operator, where $\kappa : X \rightarrow \mathbb{R}$ is the scalar curvature, $\kappa = -\sum_{j,k \in [n]} (R^S_{e_i,e_j}(e_i), e_j)$,

$$D^2 = \triangle + \frac{1}{4} \kappa$$

(5)

On any twisted spinor bundle $S \otimes W$, with Dirac operator $D^2_{S \otimes W}$, the connection laplacian $\triangle^{S \otimes W}$, the general Bochner identity becomes **Bochner-Weitzenböck identity**

$$D^2_{S \otimes W} = \triangle^{S \otimes W} + \frac{1}{2} \kappa + \mathcal{R}^{S \otimes W} := \frac{1}{2} \sum_{j,k \in [n]} (R^S_{e_i,e_j}(e_i), e_j) \otimes R^W_{e_i,e_j} w$$

(6)

Suppose $H = S \otimes E$, that is, $H$ is the spinor bundle $S$ twisted by $E$. Using that the connection laplacian $\triangle^{S \otimes E} = -\text{Tr}((V, V') \rightarrow \nabla^2_{V,V'})$ where $\nabla^2_{V,V'} = \nabla_V \nabla_{V'} - \nabla_{\nabla_V V'}$ which in the geodesic frame is becomes

$$\triangle^{S \otimes E} = -\sum_i \nabla_{e_i} \nabla_{e_i}$$

(7)

Explicitly the tensor connection laplacian is given by:

$$\triangle^{S \otimes E} \sigma = -\sum_i (\nabla^S_i \otimes 1 + 1 \otimes \nabla^E_i) (\nabla^S_i \otimes 1 + 1 \otimes \nabla^E_i) \sigma$$

$$= -\sum_i (\nabla^S_i \nabla^S_i \otimes 1 + 2 \nabla^S_i \nabla^E_i + 1 \otimes \nabla^E_i \nabla^E_i) \sigma = \left(\triangle^S \otimes 1 - 2 \sum_i \nabla^S_i \nabla^S_i + 1 \otimes \triangle^E \right) \sigma$$

(8)

In general, to compute $e^{-t \triangle^{S \otimes E}}$ Baker-Campbell-Hausdorff formula is needed as the the terms in the expansion of $-t \triangle^{S \otimes E}$ don’t commute. A simple calculation that verifies that the terms commute when the curvatures of the bundles $E$ and $S$ vanish identically –
Lemma 5. For any vector bundle, the connection laplacian $[-\Delta^V, \sum_j \nabla_j] = \sum_{i,j} R(i,j) \nabla_i + \sum_{i,j} \nabla_i R(i,j)$.

Proof. In geodesic frame, $(e_i : i \in \text{dim } V)$, using $R(i,j) = \nabla_i \nabla_j - \nabla_j \nabla_i$, with shorthand $R(i,j) := R(e_i, e_j)$

$$-\Delta^V \sum_j \nabla_j = \sum_i \nabla_i \nabla_i \sum_j \nabla_j$$

$$= \sum_{i,j} \nabla_i \nabla_i \nabla_j = \sum_{i,j} (\nabla_i \nabla_j \nabla_i + \nabla_i R(i,j)) = \sum_{i,j} \nabla_i \nabla_i \nabla_i + \sum_{i,j} R(i,j) \nabla_i + \sum_{i,j} \nabla_i R(i,j)$$

That is, $[-\Delta^V, \sum_j \nabla_j] = \sum_{i,j} R(i,j) \nabla_i + \sum_{i,j} \nabla_i R(i,j)$. Using $R(i,j) = -R(j,i)$ and $R(i,i) = 0$,

$$\sum_{i,j} R(i,j) \nabla_i = \sum_{i,j} (R(i,j) \nabla_i + R(j,i) \nabla_j) + \sum_{i,j} R(i,j) \nabla_i = \sum_{i,j} (R(i,j) \nabla_i + R(j,i) \nabla_j) = \sum_{i,j} R(i,j) (\nabla_i - \nabla_j)$$

By the second Bianchi identity, $(\nabla_u R)(v,w) + (\nabla_v R)(w,u) + (\nabla_w R)(u,v) = 0$, when $v = u = w$, $(\nabla_u R)(u,u) = 0$, therefore

$$\sum_{i,j} \nabla_i R(i,j) = \sum_{i,j} (\nabla_i R(i,j) + \nabla_j R(j,i)) = \sum_{i,j} (\nabla_i R(i,j) + \nabla_j R(i,j)) = \sum_{i,j} (\nabla_i - \nabla_j) R(i,j)$$

This yields $[-\Delta^V, \sum_j \nabla_j] = \sum_{i,j} R(i,j) \nabla_i + \sum_{i,j} \nabla_i R(i,j)$ \hfill \Box

Corollary 6. If vector bundles $S \to X, E \to X$ are such that the curvatures $R^E, R^S$ satisfy $\sum_{i,j} R^E(i,j) \nabla_i^E + \sum_{i,j} \nabla_i^E R^E(i,j) = 0$ and $\sum_{i,j} R^S(i,j) \nabla_i^S + \sum_{i,j} \nabla_i^S R^S(i,j) = 0$ then

$$e^{-t \Delta_{S \otimes E}} = e^{-t \Delta_{S \otimes E}^1} e^{t \sum_i \nabla_i^S \otimes \nabla_i^E} e^{-t \Delta_{S \otimes E}^{1 \otimes E}}$$

Proof. Since $R^E, R^S$ satisfy the condition in lemma 5, $\sum_i \nabla_i^S \otimes \nabla_i^E$ commutes with $1 \otimes \Delta^E$ and $\Delta^S \otimes 1$. As $\Delta^S \otimes 1$ also commutes with $1 \otimes \Delta^E$, so each term in the tensor laplacian commutes with the rest and one does not need Baker-Campbell-Hausdorff formula to compute the exponential. It follows that $e^{-t \Delta_{S \otimes E}^{1 \otimes E}} = e^{-t \Delta_{S \otimes E}^{1 \otimes E}}$.

The commutativity condition (equivalently the curvature condition in lemma 5) is satisfied more generally than for vanishing curvature. For instance, in section 5.3, it is verified that for canonical connection homogeneous space $K/H$ the connection acts through the Lie algebra and the connection laplacian for bundles associated to irreducible representations of $H$, is the Casimir laplacian with an additive scalar, so it commutes with the connection.

This corollary forces the semigroup generated by $\sum_i \nabla_i^S \otimes \nabla_i^E$ to be completely positive if the curvature conditions of lemma 5 are satisfied and $\Delta_{S \otimes E}$ (and therefore, also $\Delta_{S \otimes 1}, \Delta_{1 \otimes E}$) generates a completely positive semigroup$^6$.

4.2 Complete positivity of heat semigroups

The quadratic forms associated to the laplacians are now considered using $C^*$-Dirichlet form theory. As a warm up, the following result is in the spirit of [2, Cor 4.4], however, $\Delta$ is essentially self-adjoint, not self-adjoint, so the argument is made directly from the definition, without appeal to formulation in terms of normal contractions on $C^*$-algebras. Denoting by $\mathcal{H}(L^2(E))$ the Hilbert-Schmidt operators acting on $L^2(E)$ with $\text{Tr}_{HS}$ inner product for any vector bundle, $E$, we have

Proposition 1. If the quadratic form, $\mathcal{E}_\Delta$, associated with the connection laplacian $\Delta_E$ on a vector bundle $E \to X$ with metric compatible connection, on $\mathcal{H}(L^2(E))$, $\Delta_E$ is closed, then the form is Dirichlet and completely Dirichlet. The result also holds for $\Delta_E$ replaced by $\mathcal{B}^2_E$ for the spinor bundle $E$, and any positive operator $T = S^* S$.

$^6$This can be viewed as the conditional complete positivity of $\sum_i \nabla_i^S \otimes \nabla_i^E$ on such bundles
Proof. By definition 16, with $E_{\Delta} (x, x) = \text{Tr}_{HS}(\Delta x^2) = \text{Tr}_{HS}(x \Delta x)$ where $x = x^* \in \text{Tr}(x^2) < \infty$, we need to check that for $f \in \text{Lip}(\mathbb{R}, 0)$, $f(\text{Dom}(E)) = \text{Dom}(E)$ and $E_{\Delta} (f(x), f(x)) \leq \|f\|_{\text{Lip}}^2 E_{\Delta} (x, x)$.

The condition $E_{\Delta} (f(x), f(x)) \leq \|f\|_{\text{Lip}}^2 E_{\Delta} (x, x)$ follows by noting that $x$ and $x^2$ are compact and self-adjoint and, therefore, $x^2 = \sum \alpha_i P_i$ where $x = \sum \alpha_i P_i, \alpha_i \in \mathbb{R}$ is the representation from the spectral theorem for compact self-adjoint operators. Now for $r \in \mathbb{R}$, $f(r)/r \leq \|f\|_{\text{Lip}}$ implying

$$E_{\Delta} (f(x), f(x)) = \text{Tr}_{HS}(f(x) \Delta f(x)) = \sum (e_i f(x), \nabla^* \nabla f(x)) e_i = \sum \|\nabla f(x)\|_2^2$$

Since $y \in \mathcal{H}(L^2(E))$ can be written as $(y + y^*)/2 + (y - y^*)/2$, so to show invariance of the domain, it suffices to show $f(y) \in \text{Dom}(E)$ for $y \in \text{Dom}(\Delta)$ self-adjoint. As $y \in \text{Dom}(\Delta)$ means $\text{Tr}_{HS}(y \Delta y) = \|\nabla y\|_{HS} < \infty$, $y \in \text{Dom}(E)$ follows by same estimate. Therefore, if $E_{\Delta}$ is closed, it’s Dirichlet.

Set $\Delta_n = \Delta \otimes 1_n$ for $1_n$ the identity map on $\text{Mat}_n$. Since $\Delta_n = (\Delta^* \otimes 1_n)(\Delta \otimes 1_n)$, and any $y \in \text{Mat}_n$ is diagonalizable, this analysis as before can be used. As $1_n$ is closed, $E_{\Delta_n}$ is closed if and only if $E_{\Delta}$ is closed. This establishes the claim. The same argument applies to the Dirac laplacian $\mathcal{B}$ for the spinor bundle $E$ acting on $L^2(X, E)$ and for any $T$ of the specified form.

It remains to show that the form $E_{\mathcal{B}}^2$ is closed on Hilbert-Schmidt operators on $L^2(E)$. We identify $\mathcal{B}$ with the Dirac operator extended to the $L^2$ sections, i.e. acting distributionally, which is self-adjoint. $E_{\mathcal{B}}^2$ is also identified with the extended version. Then by the Bochner identity and the fact that the curvature operator on compact manifold is self-adjoint and bounded, therefore closed, it also follows for the connection laplacian.

**Theorem 7.** Suppose $(E, h)$ is a spinor bundle over the compact Riemannian manifold $(X, g)$ with $D$ denoting the self-adjoint extension of the Dirac operator to $L^2(E)$. Let $\mathcal{H}$ be the Hilbert space of Hilbert-Schmidt operators on $L^2(E)$, $(x, y)_{HS} = \text{Tr}(x^* y)$. Then the quadratic form $E_{\mathcal{B}}^2(x, y) = q(x, y) := \text{Tr}(x^* y D^2)$ on $\mathcal{H} \times \mathcal{H}$ is closed, and therefore, $E_{\mathcal{B}}^2$ is also completely Dirichlet.

Proof. It’s enough to show for the claim for $x, y$ self-adjoint, so we work with $q(x, x) = \text{Tr}(x^* D^2)$. Let $(e_i)$ be a basis of $L^2(E)$ consisting of smooth eigensections of the laplacian $\Delta^E = \nabla^E \nabla$ associated to the connection for $D$. Note that $q$ is semibounded, since $\text{Tr}(x^2 D^2) = \sum_i (D x e_i, D x e_i) = \|D x\|_{HS} \geq 0$ where we used that $x$ is self-adjoint and the trace is cyclic, so $\text{Tr}(x^2 D^2) = \text{Tr}(x D^2 x)$.

Now suppose $(x_n)$ is a Cauchy sequence in norm $\|a\|_+ := \sqrt{\|a\|_{HS} + \|a(a)\|_{HS}}$. So $(x_n)$ is Cauchy sequence in the Hilbert space $(\mathcal{H}, \|\|_{HS})$ implying $(x_n)_{HS} \rightarrow x \in \mathcal{H}$. And $(x_n)$ being Cauchy in $\|\|_+$ also gives that $q(x_n - x_n, x_n - x_m) \rightarrow 0$. Because $q(a, x) = \|D a\|_{HS}^2$, so $(D x_n)$ is also Cauchy in $(\mathcal{H}, \|\|_{HS})$ and therefore convergent with $\lim_{n \rightarrow \infty} D x_n = g \in \mathcal{H}$.

Now if for all $i, n, x e_i, x_n e_i \in \text{Dom}(D)$, i.e., $x e_i, x_n e_i$ are weakly differentiable then we can show $g = D x$, which gives

$$\lim_{n \rightarrow \infty} q(x_n - x, x_n - x) = \lim_{n \rightarrow \infty} \|D(x_n - x)\|_{HS} = \lim_{n \rightarrow \infty} \|D x_n - g\|_{HS} = 0$$

meaning the form $q$ is closed.

To see $g = D x$ assuming $x e_i, x_n e_i$ are weakly $(L^2)$ differentiable, note that we have $x_n \xrightarrow{HS} x$, so for all $i, x_n e_i \rightarrow x e_i$, then using that $D$ is self-adjoint, hence closed on $L^2(E)$, yields

$$g e_i = \lim_{n \rightarrow \infty} (D x_n) e_i = \lim_{n \rightarrow \infty} D(x_n e_i) = D(x e_i) = (D x) e_i$$

Since $D x$ and $g$ agree on the basis $(e_i), D x = g$.

Finally, the weak differentiability of $x_n e_i$ hold since $e_i$ is smooth and $\|x_n e_i\|_{LS}^2 \leq \text{Tr}(x_n^2) = \|x_n\|_{HS}^2 < \infty$, $\|D x_n e_i\|_{LS}^2 \leq \text{Tr}(x_n D^2) = \|D x_n\|_{LS}^2 < \infty$, so $x_n e_i, D x_n e_i \in L^2(E)$. And similarly $\|x e_i\|_{HS} \leq \|x\|_{HS}$, $\lim_{n \rightarrow \infty} \|D x_n e_i\|_{HS} \leq \|g\|_{HS}$, so $x e_i \in \text{Dom}(D)$.

Alternatively, the weak differentiability is also clear from noting that $x_n e_i$ is in the Sobolev space $W^{1, 2}(E)$ for all $i, n$ where we used that $D \phi = \sum k^t \cdot \nabla x_i \phi, \langle D \phi, D \phi \rangle = \sum_1 e_i \langle k^t \cdot \nabla x_i \phi, t^* \cdot \nabla x_i \phi \rangle = \sum_1 e_i$.
\[ \sum_{kl} \delta_{kl} \langle \nabla_{k} \phi, \nabla_{l} \phi \rangle. \]

While convergence of \((x_n)\) in \(\|\cdot\|_+\) implies convergence of \(x_n e_i\) in the norm \(\|u\|_{W^{1,2}} := \|u\|_{L^2(E)} + \|\nabla u\|_{L^2(E)}\) for the \(W^{1,2}(E)\). Hence \(\lim_{n \to \infty} x_n e_i = x e_i \in W^{1,2}\) (with \(W^{1,2} = \text{Dom}(\nabla) = \text{Dom}(D)\)), and therefore is weakly differentiable and \(x e_i \in \text{Dom}(D) \subset L^2(E)\). \(\square\)

**Corollary 8.** The form associated to laplacian, \(\mathcal{E}_\Delta\), on the vector bundle \(H\) is completely Dirichlet form on \(\mathcal{H}(H)\).

Instead of using Bochner identity, one can also get at the result for the connection laplacian, \(\Delta_E\), by adjusting the same reasoning to from \(D\) to the closure \(\overline{\nabla}\) of \(\nabla\) using the results from [6] after accounting for domain and codomain of \(\nabla\) not being the same Hilbert space as for \(D\). This suggests that compactness of the underlying manifold is not essential.

Kostant’s cubic Dirac operator, \(D^{1/3}\), is the Dirac operator associated to a linear combination of the canonical and Levi-Civita connection of the reductive space \(K/H\). The laplacian, \((D^{1/3})^2\), can be expressed as the quadratic Casimir operator (i.e. the Casimir laplacian) with an additive scalar (see, for instance, [1, Thm 3.3]). By same argument it follows that it follows that it generates a quantum dynamical semigroup.

**Corollary 9.** Kostant’s cubic Dirac operator generates a quantum dynamical semigroup.

Similarly, quadratic form for a positive curvature operator, \(\mathfrak{R}^H\) can be shown to be completely Dirichlet.

**Proposition 2.** If \(\mathfrak{R} \geq 0\) then the associated form, \(\mathcal{E}_{\mathfrak{R}}\), is completely Dirichlet on \(L^2(A, \tau)\).

**Proof.** First note that the \(\mathfrak{R}^H\) at each fibre is bounded symmetric operator. To see the symmetry, note it can immediately be checked that for any Riemannian connection, the curvature transformation is skew symmetric in the sense \(\langle R_{\nabla} W_s, s' \rangle = -\langle s, R_{\nabla} W s' \rangle\). Consider each term in \(\mathfrak{R}^H\), \(\langle s, e_l \cdot e_k \cdot R_{e_l e_k} s' \rangle\) for \(s, s' \in \Gamma(H)\), and an geodesic frame \((e_i)\), since \(l \neq k\) must hold,

\[ \langle s, e_l \cdot e_k \cdot R_{e_l e_k} s' \rangle = -\langle R_{e_l e_k} e_l \cdot e_k \cdot s, s' \rangle = \langle R_{e_l e_k} e_l \cdot e_k \cdot s, s' \rangle = \langle e_l \cdot e_k \cdot R_{e_l e_k} s, s' \rangle \]

where to commute \(e_l, e_k\) past \(\nabla_{e_l}, \nabla_{e_k}\) inside \(R_{e_l e_k}\), the product rule was used with the fact that the coordinates are geodesic, so covariantly constant. As \(\mathfrak{R}^H\) varies smoothly, and the manifold is assumed to be compact, it’s bounded globally. Since everywhere defined, symmetric operators are self-adjoint, and bounded operators are closed, \(\mathfrak{R}^H\) is self-adjoint and closed. If \(\mathfrak{R}\) is non-negative, \(\mathfrak{R}^{1/2}\) exists and being bounded is closed, therefore, it follows as before that \(\mathcal{E}_{\mathfrak{R}}\) is completely Dirichlet. \(\square\)

Note that in the \(L^2(A, \|\cdot\|_{HS})\) setting the complete Markovity of the Dirac heat semigroup does not depend on the curvature unlike for \(C^*\)-bundles where for Clifford bundles it does[13].

So far we have considered \(\mathcal{H}(L^2(E))\), equivalently, \(L^2(A, \tau) = \text{Tr}_{HS}, \mathcal{A} = \mathcal{B}(L^2(E))\). But \(\mathcal{H}(L^2(E))\) is not unital which is needed to get at the dilations. So consider extension of a completely Markov semigroup \(e^{-t\mathcal{L}}\) from \(L^2(A, \tau)\) to \(\mathcal{A}\).

**Theorem 10.** The completely Markov semigroup \(e^{-t\mathcal{L}}\) extends from \(L^2(A, \tau)\) to \(\mathcal{B}(L^2(H))\) if and only if the \(e^{-t\mathcal{L}}\) is completely Markov for each \(t\) on the operator system[33], i.e. a \(*\)-closed vector space, \(\mathcal{O}(L^2(A, \tau), 1)\), generated by \(\mathcal{O}(L^2(A, \tau), 1)\) and \(\mathbb{1}\).

**Proof.** If \(e^{-t\mathcal{L}}\) is not a completely Markov family of maps on \(\mathcal{O}(L^2(A, \tau), 1)\) then obviously \(e^{-t\mathcal{L}}\) does not extend to \(\mathcal{B}(L^2(H))\). If it’s a completely Markov family, then as \(\mathcal{O}(L^2(A, \tau), 1)\) is an operator system, so as completely positive maps, \(e^{-t\mathcal{L}}\), extends to \(\mathcal{B}(L^2(H))\) by Arveson’s extension theorem[33, Thm 7.5]. Complete Markovity follows since even though Hilbert-Schmidt operators are not norm dense, they are strongly dense in \(\mathcal{B}(L^2(H))\). \(\square\)

**Corollary 11.** Suppose \(\mathcal{L}(1) = 0\) then \(e^{-t\mathcal{L}}\) is completely Markov on \(\mathcal{O}(L^2(A, \tau), 1)\) and, therefore, on \(\mathcal{B}(L^2(H))\).

**Proof.** If \(a \in \mathcal{O}(L^2(A, \tau), 1)\), then \(a = \beta 1 + a\) with \(a \in L^2(A, \tau), \beta \in \mathbb{C}\), and \(\beta 1, a\) commute. \(e^{-t\mathcal{L}(\beta 1 + a)} = e^{-t\mathcal{L}(1)} e^{-t a} = e^{-t a}\) which is completely Markov. The conclusion follows from the theorem 10. \(\square\)
5 Evans-Hudson dilation on reductive homogeneous spaces

5.1 Complete smoothness

With the results on complete Markovity, the existence of the technical handle to handle covariance and the unboundedness of the generator remains to be checked. Recalling the setup from [34, Ch 8], we start with a $C^*$-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$, $G$ is a second countable, compact Lie group with finite dimensional Lie algebra, acting by a strongly continuous representation $G \ni g \mapsto \alpha_g \in \text{Aut}(\mathcal{A})$ on $\mathcal{A}$.

**Definition 18.** Suppose $\chi_i : i \in [n]$ is the basis for the Lie algebra $\mathfrak{Lie}(G)$, and $d\mu$ the left Haar measure on $G$. The smooth algebra is defined by $\mathcal{A}_\infty = \{a : g \mapsto \alpha_g(a) \text{ is smooth for all } g \in G \text{ in norm topology}\}$.

Note that $\mathcal{A}_\infty = \cap_{k \in [n]} \text{Dom}(\partial_i)$ where $\partial_i$ is closed $*$-derivation on $\mathcal{A}$ given by the generator of the automorphism group $(\alpha_{\chi_i})_{t \in \mathbb{R}}$. $\mathcal{A}_\infty$ can be equipped with Sobolev-type norms,

$$\|a\|_n = \sum_{i_1, i_2, \ldots, i_k \leq n} ||\partial_{i_1} \cdots \partial_{i_k} (a)||$$

with $||a||_0 = \|a\|$. $\mathcal{A}_\infty$ is a Frechet algebra\(^8\).

Let $T_t$ be a conservative semigroup with generator $\mathcal{L}$ that is covariant with respect to $G$, that is, $\alpha_g$ commutes with $T_t$ for all $t$ which is equivalent to $\mathcal{L}(\alpha_g)(a) = \alpha_g(\mathcal{L}(a))$ for all $a \in \text{Dom}(\mathcal{L})$. $\mathcal{L}$ is possibly unbounded, but with $\text{Dom}(\mathcal{L}) \subset \mathcal{A}_\infty \subset \text{dom}(\mathcal{L})$. Additionally, suppose $\mathcal{L}$ is completely smooth, that is, a smooth map $\mathcal{L}$ between Frechet algebras, $\mathcal{M}_\infty, \mathcal{N}_\infty$ with respect to actions $\mu_\gamma, \eta_\gamma$ of compact Lie group $G$ on $C^*$-algebras $\mathcal{M}, \mathcal{N}$ is completely smooth if there exists a constant $C$ and $p \in \mathbb{Z}^\geq 0$ satisfying for all $n, N \geq 0$ and $\xi \in M_\infty \otimes Mat_n$,

$$\|\mathcal{L} \otimes 1_{\text{Mat}_N}(\xi)\|_n \leq C \|\xi\|_{n+p}$$

Note that bounded operators are completely smooth since from equation (9), $||\cdot||_l \leq ||\cdot||_q$ for all $l \geq q$. Complete smoothness is a regularity condition that guarantees the convergence of the iterative scheme to construct the Evans-Hudson flow.

**Lemma 12.** Suppose $W_i$ is $w_i$-completely smooth for $i \in [N]$, then any polynomial in $W_i$'s is completely smooth of some order.

**Proof.** First, since $W_i$ is $w_i$-completely smooth for $i \in [N]$, let $\|W_i \otimes 1_{\text{Mat}_N}(\xi)\|_n \leq C_i \|\xi\|_{n+w_i}$. By eq 9, so we can assume $W_i$ are $w = \max(w_i)$-completely smooth, meaning $C = \max_{[N]} C_i$, $\|W_i \otimes 1_{\text{Mat}_N}(\xi)\|_n \leq C \|\xi\|_{n+w}$ for all $i$. This gives

$$\left|\sum_{i \in [N]} W_i \otimes 1_{\text{Mat}_N}(\xi)\right|_n \leq \sum_{i \in [N]} \|W_i \otimes 1_{\text{Mat}_N}(\xi)\|_n \leq NC \|\xi\|_{n+w}$$

For $W_i W_j := W_i \otimes W_j$, $\|W_i W_j \otimes 1\|_n \|W_i \otimes 1\|_{n+w_i} \leq C_i \|W_j \otimes 1\|_{n+w_i} \leq C_j \|\xi\|_{n+w_i}$, and the conclusion follows.  \(\square\)

We note the following version of [34, Thm 8.1.28] –

**Proposition 3.** Suppose $\mathfrak{Lie}(G)$ has basis $X_i : i \in [m]$, i.e. $X_i$'s generate one-parameter subgroups, then the $\Phi[X_i : i \in [m]]$ be a polynomial degree $p$ in $X_i$'s with coefficients in $B(\mathcal{H})$, which by the Lie algebra action on $\mathcal{A}_\infty$ defines a map $\Phi : \mathcal{A}_\infty \rightarrow \mathcal{A}_\infty$, then $\Phi$ is $p$-completely smooth.

**Proof.** Set $\alpha$ as the norm of the largest coefficient of $\Phi$, wlog assume $\alpha \geq 1$. For any monomial $\Phi_i \in \Phi$, with $\xi = \sum_{|\eta|} x_\eta \otimes m_\eta$, $\Phi = \Phi[X_i : i \in [m]]$,

$$\|\Phi_i \otimes 1\|_n = \sum_{i_1 \cdots i_k \leq n} \left( \prod_{j \in [k]} X_{ij} \otimes 1 \right) \sum_{|\eta|} \Phi_i(x_\eta) \otimes m_\eta \leq \alpha \sum_{i_1 \cdots i_k \leq n+p} \left( \prod_{j \in [k]} X_{ij} \otimes 1 \right) \sum_{|\eta|} x_\eta \otimes m_\eta = \alpha \|\Phi_i \otimes 1\|_{n+p}$$

This yields $\|\Phi \otimes 1\|_n \leq N \alpha \|\xi\|_{n+p}$ where $\Phi$ has $N$ monomials.  \(\square\)

\(^8\)The algebra $\mathcal{A}_\infty$ is also used in [17, Pg 5]; however, the norms $||\cdot||_n$ are symmetrized explicitly.
Theorem 13. (Existence of Evans-Hudson dilation[34, Thm 8.1.38]) If $(T_t)$ is a conservative quantum dynamical semigroup on a unital $C^*$-algebra $\mathcal{A}$, covariant with respect to action of a second countable compact Lie group $G$, with possibly unbounded generator $L$ that is $p$-completely smooth for some $p$ and $L(\mathcal{A}_\infty) \subset \mathcal{A}_\infty \subset \text{Dom}(L)$, then the Evans-Hudson dilation exists.

By theorem 13, the existence of Evans-Hudson dilation requires that the semigroup be conservative. As remarked before, this does not hold for the semigroups $e^{-tL}$. While the commutation of the generator with the Lie group action and complete smoothness are tied to the Lie algebra structure. For homogeneous spaces, and to some extent, more generally, we work towards verifying – and working around, the hypothesis needed.

5.2 The endomorphism connection

In requiring the semigroups to be conservative, one quickly sees that the laplacian $\Delta^H$ cannot be generated conservative if it acts by composition on $\text{End}(L^2(X, H))$: fix a basis $(e_i)$ of eigensections of $\Delta^H$ for $L^2(H)$ and let $\lambda_i$ be eigenvalue for $\Delta^H$ on $e_i$, then $(e_i \otimes e_i^*)_i$ is a basis for $\text{End}(H)$. If $\Delta^H$ acts by composition on $\text{End}(H)$ then it maps $e_i \otimes e_i^*$ to $\lambda_i e_i \otimes e_i^*$, but then $\Delta^H(1) = H(\sum e_i \otimes e_i^*) = 0$ cannot hold. However, we have the following –

Observation 2. On the endomorphism bundle, the canonical connection $\nabla_{\text{End}(H)} = \nabla^H \otimes 1 + 1 \otimes \nabla^{H^*}$ is easily seen to be conservative semigroup and is uniquely determined by $\nabla^H$: if over $(U, \phi_U)$ the connection acts locally by $\nabla((\sum \sigma^j \mu_j) = \sum_j (d\sigma^j) \mu_j + \sum_j \sigma^j A\mu_j$ for a matrix of $T^*M$-valued 1-forms $\Lambda$, then the dual connection acts with matrix $\Lambda := -A^t$, and $\nabla_{\text{End}(H)}((\sum \sigma^j \mu_j \otimes \mu^j)$ is given by

$$\nabla_{\text{End}(H)}((\sum \sigma^j \mu_j \otimes \mu^j) = \sum_j (d\sigma^j) \mu_j \otimes \mu^j + \sum_j [\sigma A - A\sigma] \mu_j \otimes \mu^j$$

Additionally, from previous computations, $\Delta^\text{End}(H) = \Delta^H \otimes 1 + 2 \sum \nabla_i \otimes \nabla^{H^*} + 1 \otimes \Delta^{H^*}$, and therefore if $u^{H^*} \in \ker(\Delta^{H^*})$ is covariantly constant and non-zero, then $\Delta^\text{End}(H)(\phi \otimes u^{H^*}) = (\Delta^H \phi) \otimes u^{H^*}$, meaning $e^{-t \Delta^H}(\phi)$ on $H$ can be identified with $e^{-t \Delta^\text{End}(H)}(\phi \otimes u^{H^*})$ on $\text{End}(H)$.

In terms of the acting on $\phi \otimes \psi$ this gives that $e^{-t \Delta^H}$ acting by composition on $\phi \otimes \psi$ i.e., $e^{-t \Delta^H}$ maps $\phi \otimes \psi$ to $e^{-t \Delta^H} \phi \otimes \psi$, can be identified with $U_\phi \circ e^{-t \Delta^\text{End}(H)}(\phi \otimes u^{H^*})$, where $U_\phi$ is a change of basis for $H^*$ sending $u^{H^*}$ to $\psi$. Note all computations are in this basis and the identification between bases for $H^2$ and $H$ is no longer canonical. The existence of covariantly constant sections relates to holonomy. For homogeneous spaces with canonical connection such sections can be induced by using that the torsion and curvature tensors are covariant constant.

If $E$ is a hermitian (or euclidean) vector bundle with connection $\nabla^E$ and $H$ a Dirac bundle with connection $\nabla^H$ over $X$, then the $\phi \cdot (h \otimes e) \rightarrow (\phi \cdot h) \otimes e$ for $\phi \in \text{Cl}(X)$ defines a Clifford action on $H \otimes E$. The skew hermiticity of the action is obvious and as needed the tensor product connection $\nabla^{H \otimes E}$ satisfies

$$\nabla^{H \otimes E}(\phi \cdot (\sigma \otimes e)) = (\nabla^{\text{Cl}(X)} \phi) \cdot (\sigma \otimes e) + \phi \cdot \nabla^H \otimes E \sigma \otimes e$$

Now the Dirac and Clifford structures are local as the Clifford multiplication acts on fibres and the connections can be computed in a chart. The local structures can then be glued to get the global structure. As a special case of tensor product bundles, consider $\text{End}(H)$ for a Dirac bundle $H$. Suppose local sections $\mu_i : i \in [\dim H]$ form an orthonormal basis of $H$ in chart $(U, \phi_U)$, and the corresponding dual basis $\mu^i$ for $H^*$. Over the $U$, $\text{End}\big|_U(H)$ is just the bundle $H|_U \otimes H^*|_U$ with the fibres given by $\text{Span}(e_i \otimes e^j : i, j \in [\dim H])$. This yields: if $H$ is a Dirac bundle, then $\text{End}(H), H \otimes E$ are Dirac bundles as well. Relevantly, there’s the following observation –

Proposition 4. Semigroups generated by laplacians $\mathcal{D}^\text{End}(H), \Delta$ for the endomorphism connection are conservative.

Proof. As $1 = \sum_i \mu_i \otimes \mu^i$, $\nabla(1)$ vanishes identically over $U$, and therefore, $\Delta(1) = 0$. This implies $\mathcal{D}^\Delta(1) = 0$ as well. □

Example 2. The Clifford bundle can be viewed as the endomorphism bundle of the spinor bundle. The Clifford connection is naturally an endomorphism connection. It is a derivation on sections of the bundle, and, therefore, is zero on the identity element of the Clifford bundle.
5.3 The canonical connection laplacian

Suppose the homogeneous space $M = K/H$ for compact, connected, Lie group $K$, closed Lie subgroup $H \subset K$ is reductive with $\text{Lie}(K) = \text{Lie}(H) \oplus \mathfrak{m}$ as a vectorspace for an $\text{Ad}(H)$ invariant subspace $\mathfrak{m}$. $\mathfrak{m}$ is identified with $T_oM$ where $o = eH$ in the coset manifold $K/H$. The homogeneous space $K/H$ is principal $H$-bundle, $\pi: K \to K/H$ and carries a $K$ action. Note that if the $K$ acts effectively on reductive homogeneous space $K/H$ then $H$ is isomorphic to a subgroup of $\text{GL}(\text{dim} M, \mathbb{R})$, and the fiber bundle $\pi: K \to K/H$ is isomorphic to a sub-bundle of the principal frame bundle $F(M, \text{GL}(\text{dim} M, \mathbb{R}))$. The $K$ action is assumed to be effective. We will consider homogeneous vector bundles over $K/H$, that is, a vector bundle $E \to K/H$ is such that $K$ acts on $E$, with $gE_x = E_{gx}$, and the action $g: E_x \to E_{gx}$ is an isomorphism for all $g \in K, x \in K/H$. $H$ induces automorphism at each fiber, meaning the fibers carry a representation of $E$.

Suppose additionally that $K$ is semisimple, so the Killing form $B_K$ defines a positive definite Riemannian metric on $K$ and an inner product on $\text{Lie}(K)$ by $-B_K$ such that the reductive decomposition for $K/H$ satisfies $\mathfrak{m} = \text{Lie}(H)^\perp$ with respect to $-B_K$. By left invariance of the Killing form, the inner product on $\text{Lie}(G)$ extends to a Riemannian metric on $M = K/H$. Since the Lie group $K$ is compact and connected, the Lie algebra exponential agrees with the Riemannian exponential and is surjective. This means that Casimir laplacian commutes with action of both Lie group and the Lie algebra.

The connections of interest are invariant connections, where by the invariance of a connection under a diffeomorphism, $g: K/H \to K/H$ means $\nabla_{gX}(gY) = g_*(\nabla_X Y)$. There’s a unique $K$-invariant connection in $K$ such that if $f_t = \exp(tX)$ be the 1-parameter subgroup of $K$ corresponding to $X \in \mathfrak{m}$ with a natural lift of $o$ to $u_o$ in the principal bundle, then the orbit of $f(u_o)$ is horizontal. More intuitively, connection 1-form for the canonical connection is projection onto the $\text{Lie}(H)$; the horizontal distribution is obtained at $o$ by translating $\mathfrak{m}$ by the left $K$-action.

The canonical connection is a metric connection, but is not necessarily torsion free, instead the 11, we note the Lie product on $K/H$ Kmannian metric on carry a representation of $\Gamma(\mathfrak{m})$. This means that Casimir laplacian commutes with action of both Lie group and the Lie algebra.

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**Lemma 14.** The canonical connection laplacian for homogeneous vector bundle $E$ over $K/H$ is expressible in terms of Lie algebra action and is completely smooth.

**Proof.** Take an orthonormal basis $(Y_i)$ for $\text{Lie}(K)$ at $e \in K$ which contains an orthonormal basis $(X_i)$ for $\mathfrak{m}$. As $K$ is compact, the Riemannian exponential agrees with Lie exponential, so the basis defines geodesic local coordinates at $K$. On $K/H$, $(X_i)$’s define geodesic local coordinates on $K/H$. Therefore, $\Delta^E = \sum_i \nabla_{X_i} \nabla_{X_i}$. In any local frame $(e_j)$ for $E$ over $K/H$ about $o$, since the connection 1-form, being projection onto the $\text{Lie}(H)$, vanishes on $X_i$’s,

$$\nabla_{X_i} \left( \sum_j \phi_j e_j \right) = \sum_j X_i(\phi_j) e_j$$

implying $\Delta^{K/H} = \sum X_i^2$ locally. By $K$ invariance of the connection, this holds everywhere. Complete smoothness follows from proposition 3.

Noting the representations of $K, H$ at play here gives the connection laplacian in terms of the Casimir operators of the Lie groups –

**Proposition 5.** For a homogeneous vector bundle $E$ over $K/H$ with the canonical connection,

$$\Delta^E = -C_2(K, \Gamma(E)) + C_2(H, E)$$

where $C_2(K, \Gamma(E))$ and $C_2(H, E)$ are Casimir operators for $K$ and $H$, the representation for $K$ being the induced representation on $\Gamma(E)$ and the representation of $H$, the representation on the fibers defining $E$. The action of $H$ on the sections is pointwise.

**Proof.** Proof is contained in lemma 14, since $\Delta^{K/H} = -\sum X_i^2$ and $X_i$’s for a basis for $\mathfrak{m}$ while the remaining $(Y_i)$’s give the Casimir operator for $H$.\[\square\]
If the representation of $H$ is irreducible, then $C_2(H, E)$ is multiplicity by a constant. If not, then on each fibre it’s a bounded operator, and therefore, on $\Gamma(E)$ it’s a bounded as $E$ is finite dimensional, and $K/H$ compact. This construction can be pushed to the endomorphism bundle.

**Observation 3.** If $\phi_g$ is the identification between fibers $E_x, E_{gx}$ then coupling the duality between the fibers $E^*_x, E_x$ with $\phi_g$ gives the identification between $E^*_{gx}, E^*_x$. Therefore, the dual bundle, $E^*$, and similarly the endomorphism bundle $E \otimes E^*$, are also homogeneous. $E^*, E \otimes E^*$ are defined by $\rho^*_E, \rho_E \otimes \rho^*_E$, where $\rho_E : H \to \text{End}(E)$ is representation defining $E$ and $\rho^*_E$, the dual representation. $G$ has an induced representation on $\Gamma(E \otimes E^*)$, so the canonical connection laplacian on $E \otimes E^*$ can be expressed similarly

$$\triangle^{E \otimes E^*} = -C_2(K, \Gamma(E \otimes E^*)) + C_2(H, E \otimes E^*)$$

Since $C_2(H, E \otimes E^*)$ is a bounded operator, and $\triangle^{E \otimes E^*}$ is a polynomial in Lie algebra action and, therefore, completely smooth.

### 5.4 Quantum stochastic dilation on homogeneous spinor bundles

Throughout this section, $E, S$ are homogeneous vector bundle with the canonical connection on reductive homogeneous space $M := K/H$, $K$ compact, $H$ closed. The metric is the bi-invariant metric from the Killing form. $S$ is a homogeneous Clifford module bundle, i.e. a homogeneous twisted spinor bundle. Note the compatibility of the Clifford action with the homogeneous structure of the bundle, from which the $K$-invariance of the curvature operator will follow –

**Lemma 15.** The Clifford action $c : T^*M \to \text{End}(S)$ commutes with the $K$-action.

**Proof.** We first check that Clifford multiplication $\text{Cl}(T^*M)$ commutes with $K$-action so the Clifford action is well defined. Let $\phi_g$ be the vector space isomorphism given by the left action of $g \in K, \phi_g(s) := g \cdot s$ on $E$ between $E_x, E_{gx}$. Invariance of the metric an orthonormal basis of $(e_i)$ at $T^*_x M$ is mapped to an orthonormal basis under $\phi_g$ and this gives an identification $\text{Cl}_x$ between $\text{Cl}_{gx}$. Denoting Clifford multiplication at $x, gx$, by $\text{Cl}_x, \text{Cl}_{gx}$, since $(\phi_g(u), \phi_g(v))_{gx} = (u, v)_x$, $\phi_g : T^*_x M \to \text{Cl}(T^*_x M)$ satisfies the universal property for Clifford algebras $\phi_g(u) \cdot \text{Cl}_{ux} \phi_g(u) = (u, u)_1 \text{Cl}(T^*_x M)$, and therefore, $\phi_g$ extends to an algebra isomorphism $\phi_g : \text{Cl}(T^*_x M) \to \text{Cl}(T^*_x M)$. This means $\phi_g(u) \cdot \text{Cl}_x \phi_g(u) \cdot \phi_g(v)$ holds, and the Clifford multiplication commutes with the $K$-action.

Now consider the Clifford action $c : T^*M \to \text{End}(S)$. Since Clifford action on a twisted spinor $S := S \otimes \mathcal{W}$ bundle only acts on the spinor bundle piece $S$, it suffices to verify that for any $v \in S$, $\phi_g(\Psi(v)) = (\phi_g(\Psi))(\phi_g(v))$ where $(\phi_g(\Psi))$ is the action of $g$ on the $\Psi \in \text{End}(S)$. By linearity of $K$-action, we can assume $\Psi = f_1, f_2^*$ for a basis $(f_j)$ of $S$, then $(\phi_g(\Psi)) = (\phi_g(f_1)) \otimes (\phi_g(f_2^*))$. Since $\phi_g(f_1) = f_1 \circ \phi_g$, (equivalently, these can be explicitly written in terms of the defining representation and its dual), $\phi_g(\Psi(v)) = (\phi_g(\Psi))(\phi_g(v))$ follows. 

**Lemma 16.** The curvature operator $\mathfrak{R}^S$ from the general Bochner identity (eq 4) is completely smooth and commutes with the Lie group action of $K$, i.e., it’s $K$-covariant.

**Proof.** As $K$ is compact, $H$ closed, $K/H$ is compact, and therefore, the curvature operator is bounded and completely smooth. If $S$ is the spinor bundle, and the canonical connection is the Levi-Civita connection, i.e. $K/H$ is a symmetric space, then by equation (5), the $\mathfrak{R}^S$ is multiplication by scalar curvature $\kappa$, and its $K$-invariance is clear. More generally, the canonical connection is $K$-invariant, so by definition the curvature transformation $R^S$ commutes with the action of $K$. And as $\mathfrak{R}^S(\phi) := \frac{1}{2} \sum_{j,k \in [n]} e_{j\cdot} \cdot \text{R}^S_{e_k \cdot}(\phi)$, using that the Clifford multiplication commutes with action of $K$, the invariance holds.

**Example 3.** ([19, Ch 3]) An example where complete smoothness and the covariance are particularly transparent is when $K/H$ is a symmetric space. The Dirac laplacian $\mathcal{B}_K/H$ for the canonical connection is given by $\mathcal{B}^2 = \Omega_K + \kappa/8$ where $\kappa$ is the scalar curvature and $\Omega_K$ the Casimir operator for $K$.

**Theorem 17.** Evans-Hudson dilation exists for the semigroup $(T_t)_{t \geq 0}$ generated by the canonical connection laplacian, $\triangle^{\text{End} E}$, on $\text{End} E$.

**Proof.** As established $\triangle^{\text{End} E}$ is completely smooth and commutes with action of $K$, and $(T_t)_{t \geq 0}$ is a conservative quantum dynamical semigroup, the conclusion follows from theorem 13.

**Corollary 18.** Evans-Hudson dilation exists for the semigroup $(T_t)_{t \geq 0}$ generated by the Dirac laplacian $\mathcal{B}^{2}_{\text{End} S}$. 

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Proof. Since $\nabla_{\text{End}(S)}(1) = 0$, $\mathcal{D}_{\text{End}(S)}^2(1) = 0$. The result for $\mathcal{D}_{\text{End}(S)}^2$ is immediate from $K$-covariance and complete smoothness of $\mathcal{R}^S$ using the general Bochner identity.

Remark 5. The only obstruction to pushing the arguments for Kostant’s cubic Dirac laplacian is the conservativeness of the heat semigroup. Though, the same identification with the endomorphism connection can again be made, and the “endomorphism cubic Dirac operator” can be expanded in terms of Kostant’s cubic Dirac laplacian and the connection laplacian for the dual bundle as in section 4.1.

On almost commutative spectral triples, where the Dirac operator is a perturbation of the geometric Dirac laplacian, the endomorphism trick, along with observation 2 finds use again. The following example considers the existence of Evans-Hudson dilation with respect to the perturbed geometric Dirac laplacian on the endomorphism bundle.

Example 4. Consider the almost commutative spectral triple, $\mathfrak{A}_{K/H} := (C^\infty(K/H, A), L^2(K/H, H), D_0)$. $H$ being a bundle of Clifford modules is a twisted spinor bundle $H := W \otimes S$. Let $\mathcal{D}$ be the geometric Dirac operator on $W \otimes S$ for the canonical connection on $K/H$, with $D_0 = \mathcal{D} + B$ for some endomorphism $B$, where the Dirac operators are acting distributionally on $L^2(K/H, H)$. If the perturbation $B$ is self-adjoint and bounded, $D_0^2 = (\mathcal{D} + B)^2$ is easily seen to be completely Dirichlet: it’s just the Dirac laplacian for a connection with a different potential on the twisting space. Completely smoothness is also immediately clear by proposition 3. To handle conservativeness, we pass to the endomorphism bundle $\text{End}(H) = \text{End}(W) \otimes \text{End}(S)$ (where topological closure is implicit). Since $D_0, \text{End}(W)$ acts locally by commutator associated to potential $\Omega_B$ for the endomorphism $B$, the semigroup is conservative, and as before, the existence of Evans-Hudson dilation follows if the potential is covariant with respect to the action of $H$. An ensemble of Dirac operators, as for fuzzy spectral triples[22], can be realized by randomizing the connection potential, but now it also has a geometric interpretation through the Bochner identity.

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