Identical particles and entanglement are both fundamental components of quantum mechanics. However, when identical particles are condensed in a single spatial mode, the standard notions of entanglement, based on clearly identifiable subsystems, break down. This has led many to conclude that such systems have limited value for quantum information tasks, compared to distinguishable particle systems. To the contrary, we show that any entanglement formally appearing amongst the identical particles, including entanglement due purely to symmetrization, can be extracted into an entangled state of independent modes, which can then be applied to any task. In fact, the entanglement of the mode system is in one-to-one correspondence with the entanglement between the inaccessible identical particles. This settles the long-standing debate about the resource capabilities of such states, in particular spin-squeezed states of Bose-Einstein condensates, while also revealing a new perspective on how and when entanglement is generated in passive optical networks. Our results thus reveal new fundamental connections between entanglement, squeezing, and indistinguishability.

In this Letter, we re-examine systems of indistinguishable particles, resurrecting legitimate meaning for their entanglement structure. Using intuition similar to [33], entanglement should be given meaning only when it can be extracted onto distinguishable registers via operations which themselves do not contribute any entanglement. Such entanglement can then be applied to standard quantum information tasks. Remarkably, we show that this extractable entanglement exactly corresponds with the entanglement one would find within a naive multiparticle description. Specifically, identical particle entanglement can be transferred, with unit probability, onto independent modes using elementary operations. Thus, symmetrization entanglement is a fundamental, ubiquitous, and readily-extractable resource for standard quantum information tasks. Our results demonstrate the usefulness of single-mode BECs for many tasks beyond metrology, and reveal new insight on how and when entanglement is generated in passive optical networks.
In the decomposition $\mathcal{H}_N = \mathcal{H}^\otimes N$, each particle is its own subsystem. On the other extreme, we can consider a bipartition, grouping the first $N_X$ particles into one subsystem and the remaining $N_Y$ into another, giving $\mathcal{H}_N = \mathcal{H}^\otimes N_X \otimes \mathcal{H}^\otimes N_Y =: \mathcal{H}_X \otimes \mathcal{H}_Y$. Here we will focus mainly on bipartite entanglement; extensions to multipartite scenarios are analogous (e.g., see Appendix). Alternatively, we can use a ‘second quantization’ basis that more accurately describes the accessible degrees of freedom. For $N$ identical particles in mode $A$, the symmetric states $\{|n, N-n\rangle\}$ enumerate composite states that have $n$ spin-down and $N-n$ spin-up particles. These can be obtained by symmetrizing single-particle states:

$$|n, N-n\rangle_A = \frac{1}{\sqrt{\binom{N}{n}}} S \left[ |0\rangle_1 \ldots |0\rangle_n |1\rangle_{n+1} \ldots |1\rangle_N \right],$$

(1)

where $S$ generates a sum over all unique permutations with $n$ spin-down particles out of $N$ and $\binom{N}{n}$ is the normalization. These states form an orthonormal basis for the symmetric subspace on which all physical states live. The symmetric subspace can also be generated using creation operators: $\hat{a}_k^{\dagger} |k, l\rangle_A = \sqrt{k+1} |k+1, l\rangle_A$ and $\hat{a}_l^{\dagger} |k, l\rangle_A = \sqrt{l+1} |k, l+1\rangle_A$.

**Mode-splitting.** In the typical setting, bipartite entanglement is defined relative to two parties with independent, accessible subsystems. In contrast, in the full $N$-particle state space, the subsystems which appear to be entangled are inherently inaccessible. Intuitively, we might imagine getting at this entanglement by somehow splitting the particles up into physically distinguishable modes. For instance, we could let a BEC cloud spread until it separates into distinct clusters, or we could use a more tunable operation such as tunneling between neighboring modes. The occupied output modes then provide some physically accessible degrees of freedom, and we can safely speak of entanglement between these modes.

But there are a few obvious concerns. First, we will still not have any access to the particle pseudo-labels that characterize the original state’s entanglement. If we find that there is a single particle in output mode $C$, we can use this information to distinguish this particle in future experiments. But relative to the original pseudo-label basis (for any non-trivial bipartition), we still cannot have any access to the particle pseudo-labels. How might this affect the entanglement we can extract?

To explore these issues, we consider the example of a beamsplitter transformation from optics. For BECs, this is equivalent to a tunnelling operation where particles can leak from mode $A$ into a neighbouring mode $B$ via a Hamiltonian of the form $H \sim \sum_{k=0,1} (\hat{b}_k^{\dagger} \hat{a}_k + \hat{a}_k^{\dagger} \hat{b}_k)$. We denote the modes post-tunnelling by $C$ and $D$. Suppose we initially have the 3-particle state $|\phi_m\rangle_A = |2, 1\rangle_A$, a symmetric state with 2 spin down and 1 spin up particles. Because of symmetrization, this state is entangled in the pseudo-label basis (for any non-trivial bipartition). We then apply a splitting transformation $\hat{a}_k^{\dagger} \rightarrow r \hat{c}_k^{\dagger} + t \hat{d}_k^{\dagger}$ ($k = 0, 1$); this operation, which is insensitive to the internal degrees of freedom, transfers single particles from mode $A$ into $C$ ($D$) with amplitude $r \ (t)$. The other mode $B$ initially has no particles. The final state is

$$|\phi_{out}\rangle_{CD} = r^3 |2, 1\rangle_C |0, 0\rangle_D$$

$$+ \sqrt{3} r^2 t \frac{1}{\sqrt{3}} \left[ |2, 0\rangle_C |0, 1\rangle_D + \sqrt{2} |1, 1\rangle_C |1, 0\rangle_D \right]$$

$$+ \sqrt{3} r^2 t \frac{1}{\sqrt{3}} \left[ |0, 1\rangle_C |2, 0\rangle_D + \sqrt{2} |1, 0\rangle_C |1, 1\rangle_D \right]$$

$$+ r^3 |0, 0\rangle_C |2, 1\rangle_D .$$

(2)

We have ordered the output state with respect to different possibilities for local particle numbers. In the first/last case (all particles in one mode), the output state is the same as the input state, with no mode entanglement. For the other cases, there is clearly entanglement between the output modes. Even if we project onto fixed local particle numbers to respect superselection rules [43], we have, on average, non-zero entanglement in the output state. This entanglement is now a valid resource in the LOCC paradigm. Evidently, correlated single-mode states have some many-body coherence properties [37] that may lead to mode entanglement after splitting. In fact, we recognize a conceptual connection with the widely-known notion from continuous-variable optics ([44–47]) that beamsplitters transform non-classical states (e.g., squeezed states) into mode entangled states.

But how does this output mode entanglement relate to the input state’s apparent pseudo-label entanglement? For concreteness, suppose we group particles 1 and 2 into subsystem $X$ and particle 3 into subsystem $Y$. To classify the entanglement, we put $|\phi_m\rangle_A$ into Schmidt form:

$$|2, 1\rangle_A = \frac{1}{\sqrt{3}} \left( |0\rangle_1 |0\rangle_2 |1\rangle_3 + |0\rangle_1 |1\rangle_2 |0\rangle_3 + |1\rangle_1 |0\rangle_2 |0\rangle_3 \right)$$

$$= \frac{1}{\sqrt{3}} \left( |0\rangle_1 |0\rangle_2 \right) |1\rangle_3 + \frac{1}{\sqrt{3}} \sqrt{2} S |0\rangle_1 |1\rangle_2$$

$$= \frac{1}{\sqrt{3}} \left( |0\rangle_X |0, 1\rangle_Y + \sqrt{2} |1, 1\rangle_X |1, 0\rangle_Y \right).$$

(3)

In the last line we have rewritten the states within the fictitious subsystems $X$ and $Y$ in second-quantized form.
We now make the crucial observation that Eq. (3) is algebraically equivalent to the mode-split state in Eq. (2) for the case where \((N_C, N_D) = (N_X, N_Y) = (2, 1)\), as considered in this example. In fact, we can establish a general equivalence.

**Schmidt equivalence of particle and mode states.** Take any single-mode basis state \(|n, N-n\rangle_A\), and fix a bipartition into \((N_X, N_Y)\) particles. Consider the same state after it has been split by any transformation \(\hat{a}_k \rightarrow r_{k}^1 + t_{k}^1\), with \(k = 0, 1\), and \(|r|^2 + |t|^2 = 1\), followed by projection onto local particle numbers \((N_C, N_D)\). If \((N_C, N_D) = (N_X, N_Y)\) or \((N_Y, N_X)\), then the Schmidt form of the input state (in the given particle bipartition) is equivalent to the Schmidt form of the output state (in the mode bipartition).

**Proof:** The Schmidt form of the final state, which we denote by \(|n, N-n\rangle_{(N_C,N_D)}\), can be straightforwardly but laboriously obtained by writing the input state as \(|n, N-n\rangle_A = \sum_{n_0} \omega^{n}_{n_0} a^d_0 |n, 0\rangle_A\), transforming \(\hat{a}_k \rightarrow r_{k}^1 + t_{k}^1\), then projecting onto terms with fixed local particle numbers \((N_C, N_D)\), i.e., those with prefactor \(\sim v^{N_C} N_D\); see Appendix. Once normalized, this automatically yields the Schmidt form: \(|n, N-n\rangle_{(N_C,N_D)} = \sum_{n_k} \lambda_{n_k} |n_k, N_k-N_k\rangle_{K}\), where the local states of mode \(K = C, D\) are second quantization basis states: \(|n_k\rangle = |n_k, N_k-n_k\rangle\), and the sum is over all valid \((n_C, n_D)\) such that \(n_C + n_D = n\). The Schmidt coefficients are calculated to be \(\lambda_{n_k} = \sqrt{(n_k)! (N_k-n_k)! / n!}\).

In first-quantization, we begin with Eq. (1). We subdivide this state into parts \(X, Y\), where \(X\) contains the pseudo-labels \(1, \ldots, N_X\) and \(Y\) the rest (in fact, by symmetrization, the specific order will not matter). After symmetrizing, we collect terms that have the same number \(n_X\) of spin-down states within \(X\); \(Y\) will contain the remaining \(n_Y = n - n_X\). For every pair \((n_X, n_Y)\), both parts have a symmetrized form:

\[|n, N-n\rangle_A = \frac{1}{\sqrt{\binom{N}{n_X} \binom{N}{n_Y}}} \sum_{n_X+n_Y=n} [S \mid v_{n_X} \rangle \mid v_{n_Y} \rangle], \quad (4)\]

where \(|v_{n_X}\rangle = |0\rangle_1 \ldots |0\rangle_{n_X} |1\rangle_{n_X+1} \ldots |1\rangle_{N_X}\) and analogously for \(Y\). Comparing to Eq. (1), we see that \(S \mid v_{n_Z} \rangle = \sqrt{n_Z! n_{Z-n_Z}!} \mid Z_X \rangle \mid Z_Y \rangle\) for \(Z = X, Y\). Since the states \(|n_Z, N_Z-n_Z\rangle_Z\) are orthonormal, this is the Schmidt form, with coefficients \(\lambda_{n_X,n_Y} = \sqrt{\binom{N_X}{n_X} \binom{N_Y}{n_Y} / n!}\). Thus, if \((N_C, N_D) = (N_X, N_Y)\) or \((N_Y, N_X)\), then the particle Schmidt form and the mode Schmidt form are in one-to-one correspondence.

This equivalence has strong consequences. The single-mode state \(|n, N-n\rangle_A\) and its two-mode equivalents \(|n, N-n\rangle_{(N_C,N_D)}\) not only have the same exact entanglement structure, but the former states can be easily mapped to the latter. This holds as well for arbitrary superpositions \(|\phi\rangle_A = \sum_n \theta_n|n, N-n\rangle_A\), since the entanglement properties within any bipartition are completely determined by the coefficients \(|\theta_n|\) and the Schmidt structure of the basis vectors. By linearity, the algebraic correspondence also holds for mixed states. Thus, by enacting the isomorphism \(|n, N-n\rangle_A \rightarrow |n, N-n\rangle_{(N_C,N_D)} \forall n\), (with \((N_C, N_D) = (N_X, N_Y)\)), we can map any single-mode state into its two-mode version, where the entanglement structure is not only preserved, but is readily accessible. To emphasize, although we cannot individually access the identical particles, their overall state is, in fact, accessible, since it can be mapped faithfully onto distinguishable mode subsystems. We will call any protocol that achieves the isomorphism \(|n, N-n\rangle_A \rightarrow |n, N-n\rangle_{(N_C,N_D)} \forall n\), ideal mode splitting.

Mode splitting does not fit in the framework of LOCC, and the process appears to ‘create’ entanglement (this is a well-known property of beamsplitters). By the above isomorphism, the structure and amount of mode entanglement, for fixed local particle numbers, is completely determined from the input state. Thus, mode splitting is a mechanism for faithfully transferring correlations from inaccessible identical particles onto accessible modes. If the splitting is sufficiently passive (we give formal conditions below), all mode entanglement comes from the initial state, and no more entanglement can be generated than what appears from the \(N\)-particle decomposition. Finally, it is not a practical problem if a non-ideal mode splitting creates extra entanglement (e.g., by having a non-vacuum state in input mode \(B\)); such entanglement is nevertheless a useful resource. However, we cannot interpret such entanglement as coming solely from the input state.

**Probabilistic mode splitting and mode mixing.** Along with tunnelling/beam-splitting, what other operations achieve the desired isomorphism? Besides the basic ability to coherently map one mode into two, there are three other important components. First, to put particle and mode entanglement on the same footing (in terms of subsystem size), and to exclude superpositions of different local particle numbers from consideration, we must project onto fixed particle numbers for each output mode. Second, the operation should not introduce any extra particles. Finally, the process should not (de-)excite the particles, so that the total number of excitations will be preserved. The reasoning for the latter two requirements is similar: such operations could lead to entanglement in the output mode when none is apparent in the input partitioning. Consider the initial state \(|N,0\rangle_A\), which has no pseudo-label entanglement. If the output system contains a spin-up particle (either externally added or internally excited), then output states of the form \(|M,1\rangle_{(N_C,N_D)}\) could appear, which are mode entangled for all \(N_C, N_D \neq 0\). Obviously this entanglement is not representative of the initial state’s entanglement structure.
These basic requirements are necessary and sufficient to give the desired isomorphism — at least probabilistically — ensuring that the splitting itself does not contribute any entanglement [48]. To show this, we consider the slightly more general situation of mode mixing, from the input space \( \mathcal{H}_{\text{in}} = \text{span}\{\varphi_n, N_{\mathbf{A}, N_{\mathbf{B}}}\} \) to the output space \( \mathcal{H}_{\text{out}} = \text{span}\{\varphi_n, N_{\mathbf{C}, N_{\mathbf{D}}}\} \), where the basis states encompass all combinations with fixed global particle numbers \( N_{\mathbf{A}} + N_{\mathbf{B}} = N_{\mathbf{A}} \mathbf{B} \). This extra generality will be useful for our deterministic mode-splitting protocol later. For particle excitation, we use the operators \( \hat{J}_{\mathbf{AB}}^+ := \hat{a}_0 \hat{a}_1^\dagger + \hat{b}_0 \hat{b}_1^\dagger \) and \( \hat{J}_{\mathbf{CD}}^+ := \hat{c}_0 \hat{c}_1^\dagger + \hat{d}_0 \hat{d}_1^\dagger \). The conjugate operators \( \hat{J}_{\mathbf{KL}}^+ := (\hat{J}_{\mathbf{KL}}^+)^\dagger \) model de-excitation. We denote the parameters \( (N_{\mathbf{A}}, N_{\mathbf{B}}; N_{\mathbf{C}}, N_{\mathbf{D}}) \) by the shorthand \( \{N\} \).

Theorem 1: Necessary and sufficient conditions for mode-mixing. Let \( M : \mathcal{H}_{\text{in}} \rightarrow \mathcal{H}_{\text{out}} \) be a linear particle-preserving map, with \( N_{\mathbf{AB}} = N_{\mathbf{CD}} = N \). Then \( M \) has the effect \( M [n, N - n]_{(N_{\mathbf{A}, N_{\mathbf{B}}})} = \sum C \{N\} [n, N - n]_{(N_{\mathbf{C}, N_{\mathbf{D}}})} \) for every choice, this protocol faithfully extracts the corresponding entangled state. We emphasize that during the extraction protocol, intermediate states could have quantitatively more entanglement than the final state. One should not interpret this as meaning that the protocol does not extract all available entanglement. Rather, we remember that the entanglement content is relative to the choice of bipartition of the initial state; for every choice, this protocol faithfully extracts the corresponding entangled state.

Relation to Spin-squeezing. These results reveal new operational meaning for single-mode spin-squeezed states. Squeezing information can be used to bound the expected mode entanglement, even without performing the splitting. Using only a few simple collective spin measurements [12], we can obtain the two-particle reduced state \( \rho_{pq} \), which is the same for every pseudo-label pair. We can then bound the output state’s entanglement using any monogamous measure \( \mathcal{E} \):

\[
\mathcal{E}(\rho_{\mathbf{C}, \mathbf{D}}) \geq \mathcal{E}(\rho_{\mathbf{C}_1, \mathbf{D}_1}) \geq \sum_j \mathcal{E}(\rho_{\mathbf{C}_j, \mathbf{D}_j}) = N_{\mathbf{D}} \mathcal{E}(\rho_{\mathbf{C}_1, \mathbf{D}_1}),
\]

where the first relation follows from tracing out all qubits in \( \mathbf{C} \) but \( \mathbf{C}_1 \), the second represents monogamy, and the third is from symmetry; a similar inequality holds for \( N_{\mathbf{D}} \leftrightarrow N_{\mathbf{C}} \). For a broad class of spin-squeezed states created by standard methods, there is a quantita-
tive relationship between the spin-squeezing parameter \( \xi^2 < 1 \) and the concurrence [49] for any \( \rho_{pq} \) [50–53]. We can leverage this to bound the tangle [54, 55] (generalized concurrence), a measure which quantifies the usefulness of a state for bipartite channel discrimination [56]:

\[
\tau(\rho_{C:D}) \geq \max\{N_C, N_D\} \left[ 1 - \frac{\xi^2}{N-1} \right]^2.
\]

Thus, spin-squeezed states, and the squeezing parameter \( \xi^2 \), acquire new operational meaning thanks to our results.

**Conclusion.** We have shown that identical-particle entanglement can be easily and faithfully extracted and used as a resource for standard quantum information tasks. Practically, such entanglement is naturally occurring and quite robust [57–59]. In optics, the idea to use non-classical states and beamsplitters to create entanglement has appeared many times. However, because the second quantization formalism is dominant, and because the particle superselection rules are not relevant for photons, the connection between entanglement in a discrete identical particle basis and beam-splitter generated entanglement was not previously uncovered. For massive particles, it is perhaps more natural to begin with the \( N \)-particle state space, but the notion of splitting and mixing modes is not as prevalent. Our results illuminate new connections between entanglement, squeezing, and indistinguishability in both scenarios.

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Appendix

Appendix 1: Algebraic form of multimode and multiparticle states

In this section, we carry out the straightforward but algebraically laborious calculation of the (generalized) Schmidt form of the state $|n, N-n\rangle_A$ after a (multi-)mode splitting transformation. For simplicity, we change the notation slightly from the main text: particles with internal state $i$ in mode $K$ are associated with the creation operator $\hat{a}^\dagger_{iK}$. We also use here the same labels $A, B, C, \ldots$ for both input and output modes, and fix a total number of modes. Symbolically, we represent the last mode in the list by $Z$, but this does not imply any specific number of modes. The initial state is thus given by

$$
|n, N-n\rangle_A = \frac{\hat{a}^\dagger_{0K} \hat{a}^\dagger_{0L}}{\sqrt{n!}(N-n)!} |\text{vac}\rangle. 
$$

(8)

An arbitrary linear transformation amongst the creation operators has the form $\hat{a}^\dagger_{iK} \rightarrow \sum_j \alpha_{iKL} \hat{a}^\dagger_{jL} + \beta_{iKL} \hat{a}_{jL}$, with $\alpha_{iL}, \beta_{iKL} \in \mathbb{C}$ for $i, j \in \{0, 1\}$ and $K, L \in \{A, B, \ldots, Z\}$. When the total number of particles is preserved and the transformation is independent of the internal state, we have the simplifications $\beta_{iKL} = 0$ and $\alpha_{iKL} = \alpha_{KL} \delta_{ij}$. Applying such a transformation to the state (8) gives

$$
|\phi_{\text{out}}\rangle = \left(\sum_K \alpha_{AK} \hat{a}^\dagger_{0K}\right)^n \left(\sum_L \alpha_{AL} \hat{a}^\dagger_{1L}\right)^{N-n} \frac{\sqrt{n!(N-n)!}}{\prod_K \alpha_{AK}^{n_K}}, 
$$

(9)

where the sums are over all output modes $K/L = A, B, C, \ldots, Z$. From this expression, we carry out multinomial expansions

$$
\left(\sum_K \alpha_{AK} \hat{a}^\dagger_{0K}\right)^n = \sum_{n_A+\cdots+n_Z=n} \frac{n!}{n_A! \cdots n_Z!} \prod_K (\alpha_{AK} \hat{a}^\dagger_{0K})^{n_K},
$$

(10)

$$
\left(\sum_L \alpha_{AL} \hat{a}^\dagger_{1L}\right)^{N-n} = \sum_{m_A+\cdots+m_Z=N-n} \frac{(N-n)!}{m_A! \cdots m_Z!} \prod_L (\alpha_{AL} \hat{a}^\dagger_{1L})^{m_K}. 
$$

(11)

Since $(\hat{a}^\dagger_{0K})^{n_K} (\hat{a}^\dagger_{1K})^{m_K} |\text{vac}\rangle = \sqrt{n_K! m_K!} |n_K, m_K\rangle_K$ for mode $K$, the output state $|\phi_{\text{out}}\rangle$ becomes

$$
\sum_{n_A+\cdots+n_Z=n} \frac{n!(N-n)!}{n_A! \cdots n_Z! m_A! \cdots m_Z!} \otimes_{K} \alpha_{AK}^{n_K+m_K} |n_K, m_K\rangle_K. 
$$

(12)

We now group terms where each mode $K$ has a fixed number of particles $N_K$, i.e., $|\phi_{\text{out}}\rangle = \sum_{N_A=0}^{N} \ldots \sum_{N_Z} w_{N_A, \ldots, N_Z} |\phi_{\text{out}}\rangle_{N_A, \ldots, N_Z}$. These terms can be identified by the condition $n_K + m_K = N_K$. We simplify the coefficients by multiplying with the unit term $\prod_K \alpha_{AK}^{N_K}$, which yields the normalized states

$$
|\phi_{\text{out}}\rangle_{N_A, \ldots, N_Z} = \sum_{n_A+\cdots+n_Z=n} \frac{(N_A)_{n_A}}{(n_A)!} \cdots \frac{(N_Z)_{n_Z}}{(n_Z)!} \otimes_{K} |n_K, N_K - n_K\rangle_K 
$$

(13)

with weights $w_{N_A, \ldots, N_Z} = \sqrt{\frac{N_A! \cdots N_Z!}{N!}} \prod_K \alpha_{AK}^{N_K}$. Since the states $\{\otimes_K |n_K, N_K - n_K\rangle_K\}$ are orthonormal, Eq. (13) has the form of a generalized Schmidt decomposition. For two output modes, we recover the standard Schmidt decomposition, which appeared in the main text.

To get the multipartition form in the $N$-particle basis, we use the trick of continually splitting a single-partition into two parts. Suppose we want to partition the $N$ particles in the initial state $|n, N-n\rangle_A$ into groups containing $N_a, N_b, N_c, \ldots, N_\varsigma$. Without loss of generality, we form a bipartition of the first $N_a$ particles and the remaining $N_\Sigma = N - N_a$. By the bipartite decomposition, Eq. (4), we have

$$
|n, N-n\rangle_A = \frac{1}{\sqrt{N_a!}} \sum_{n_{a'+\cdots+n_{a''}=n}} \left[\mathcal{S} |v_{a'}\rangle \right] \left[|S |v_{a''}\rangle\right], 
$$

(14)

where $|v_{a'}\rangle = |0\rangle \cdots |0\rangle |1\rangle_{n_{a'+\cdots+n_{a''}} \ldots} |1\rangle_{N_\Sigma}$ and $|v_{a''}\rangle$ has an analogous form. Rewriting the state $\mathcal{S} |v_{a'}\rangle$ in second quantized form, it becomes $\sqrt{\binom{N_a}{n_a}} |n_a, N_a-n_a\rangle_a$.

The second partition can now be further subdivided into two parts, containing $N_b$ and $N_\Sigma = N - N_a - N_b$ particles. Continuing in this way, and rewriting each partition in second quantization, we end up with

$$
|\phi_{\text{m}}\rangle_A = \sum_{n_{a'+\cdots+n_{a''}=n}} \frac{(N_a')_{n_{a'}}}{(n_{a'})!} \cdots \frac{(N_\varsigma')_{n_{\varsigma'}}}{(n_{\varsigma'})!} \prod_{\kappa} |n_{\kappa}, N_\kappa - n_{\kappa}\rangle_\kappa, 
$$

(15)

which is in one-to-one correspondence to the multimode split state in Eq. (13) when $(N_A, \ldots, N_Z)$ is some permutation of $(N_a, \ldots, N_\varsigma)$. Thus, an arbitrary multimode transformation of the form $\hat{a}^\dagger_{iK} \rightarrow \sum_L \alpha_{KL} \hat{a}^\dagger_{iL}$,
with respect to the tell us that $M$ must have a local block-diagonal structure where the excitation, we define the local operators $\hat{\sigma}_p$ by

$$\langle i | \hat{\sigma}_p | j \rangle = \delta_{i_p, j_p} \langle i_p^+ | j_p \rangle,$$

where $i_p \in \{0, 1\}$ is the $p$th entry of $i$, labeling the internal state of particle $p$. Similarly, $K_p$ is the $p$th entry of $K$, labeling the external mode which particle $p$ occupies. For pre-mixing states, $K_p \in \{A, B\}$; for post-mixing states, $K_p \in \{C, D\}$.

Intuitively, we begin with the same operational requirements as before, namely that the operation preserves particle numbers and does not excite the system. Particle preservation is captured simply by requiring that the operator $M$ maps between the $N$-particle state spaces $\mathcal{H}_{\text{in}} = \text{span}\{i | K \}$ and $\mathcal{H}_{\text{out}} = \text{span}\{i | K \}$ and let $\mathcal{H}_{\text{mix}} = \text{span}\{i | K \}$.

For (de-)excitation, we define the local operators $\hat{\sigma}_p^\pm$ on particle $p$ by

$$\hat{\sigma}_p^+ | i | K \rangle = \delta_{i_p, 0} | i_p^+ | K \rangle,$$

$$\hat{\sigma}_p^- | i | K \rangle = \delta_{i_p, 1} | i_p^- | K \rangle,$$

where the $p$th element of $i_p^\pm$ is $i_p \pm 1$ and all others are the same as in $i$. If we demand that $M$ commutes with both $\hat{\sigma}_p^+$ and $\hat{\sigma}_p^-$ for every $p$, we can straightforwardly derive the following set of conditions, which hold for all $i, j, K, L$:

$$\langle i | K | M | j | L \rangle = \begin{cases} 
| i_p^+ | K | M | j_p | L \rangle & \text{if } i_p = 0, j_p = 0 \\
0 & \text{if } i_p = 0, j_p = 1 \\
0 & \text{if } i_p = 1, j_p = 0 
\end{cases}$$

These are actually quite stringent conditions. They tell us that $M$ must have a local block-diagonal structure with respect to the $N$-particle state space:

$$M = \bigotimes_{p=1}^N M^{(p)},$$

$$M^{(p)} = \sum_{i=0}^1 \sum_{K=C,D} \sum_{L=A,B} C_{KL}^{(p)} | i | K \langle i | L \rangle.$$

Thus, $M$ simply maps particle $p$ from mode $L = A, B$ to mode $K = C, D$ with amplitude $C_{KL}^{(p)}$, without considering nor changing its internal state. Even more, if the particles are truly identical, there can be no dependence on the pseudo-label $p$, so in fact, we must have $C_{KL}^{(p)} = C_{KL}$ and $M = [M^{(1)}]^{\otimes N}$.

**Theorem 1a.** Let $M : \mathcal{H}_{\text{in}} \rightarrow \mathcal{H}_{\text{out}}$ and let $N_A + N_B = N_C + N_D = N$. If $M \in \text{span}\{i | K \}$, then $M$ has the effect $M | n, N - n \rangle_{(N_A, N_B)} = M | n, N - n \rangle_{(N_C, N_D)} \forall n$, where each $C\{N\} \in \mathbb{C}$.

**Proof:** First fix the number combination $\{N\}$ and consider the collective excitation operator $\hat{J}^+ := \sum_{p=1}^N \hat{\sigma}_p^+$. On two-mode symmetric states, for any local particle numbers $(N_X, N_Y)$, $\hat{J}^+$ has the following effect:

$$\hat{J}^+ | n, N_{XY} - n \rangle_{(N_X, N_Y)} = 
\sqrt{n(N_{XY} - n + 1)} | n - 1, N_{XY} - n + 1 \rangle_{(N_X, N_Y)}.$$  

Since $M$ commutes with each $\hat{\sigma}_p^+$, it will also commute with $\hat{J}^+$. Similar to Theorem 1, we can directly work out a recurrence relation on the matrix elements $M_{mn}^{(N)}$ (cf. Eq. (5)):

$$M_{m+1,n+1}^{(N)} = M_{mn}^{(N)} \sqrt{(n+1)(N-n)} / \sqrt{(m+1)(N-m)}.$$  

Furthermore, $M$ will also commute with the operator $\hat{n}_0 = \sum_{p=1}^N \hat{\sigma}_p^- \hat{\sigma}_p^+$, which has the effect

$$\hat{n}_0 | n, N_{XY} - n \rangle_{(N_X, N_Y)} = n | n, N_{XY} - n \rangle_{(N_X, N_Y)}.$$  

This implies that $M_{mn}^{(N)} (m-n) = 0,$

hence, $M^{(N)}$ is diagonal. Considering the recurrence relation (23), we conclude that $M^{(N)}$ must be a multiple of the identity:

$$M_{mn}^{(N)} = C\{N\} \delta_{mn}.$$  

For non-fixed bipartition sizes, we will instead have the general form $M = \sum_{(N)} M^{(N)}$ with $M^{(N)}$ as above. Thus, $M$ carries out the desired transformation. \(\blacksquare\)

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