Intermediate energy spectrum of five colour QCD  
at one loop

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Abstract  
I consider the monopole condensate of five colour QCD. The naïve  
lowest energy state is unobtainable at one-loop for five or more colours due  
to simple geometry. The consequent adjustment of the vacuum condensate  
generates a hierarchy of confinement scales in a natural Higgs-free manner.  
QCD and QED-like forces emerge naturally, acting upon matter fields that  
may be interpreted as down quarks, up quarks and electrons.

1 Introduction  
It is already known [1, 2, 3], that SU(N) QCD can lower the energy of its  
vacuum with a monopole background field along the Abelian directions,  
where the Abelian components are equal in magnitude but orthogonal in  
real space [2, 3]. This orthogonality, while of no special consequence in  
SU(3) QCD in three space dimensions, does have consequences when the  
number of Abelian directions is greater than three. As noted originally  
by Flyvbjerg [2], SU(N ≥ 5) QCD cannot realise its true minimum because  
four orthogonal vectors cannot fit in three dimensions. I shall call a  
system kept from reaching its true lowest energy state by a lack of spatial  
dimensions dimensionally frustrated.

The research in this chapter seeks to identify the monopole condensate  
of five-colour QCD, or at least a good candidate for it, and examine the  
consequences. It assumes the dual superconductor model of confinement  
[4, 5, 6, 7, 8] in which chromomagnetic monopole-antimonopole pairs play  
the dual role to Cooper pairs, restricting the electric component of the  
chromodynamic field to flux tubes. This model is by no means proven but  
the case for it is very strong. I shall handle the monopole degrees of freedom  
with the Cho-Faddeev-Niemi-Shabanov decomposition [9, 10, 11] which  
specifies the internal directions corresponding to the Abelian generators  
in a gauge-covariant way, automatically introducing the monopole field in the process. It is explained in detail in section 2.

At extremely high energies where the effects of confinement are not  
significant, the dynamics are simply those of SU(5) QCD in the far ultra- 
violet. I shall show however that the dimensionally frustrated condensate
is anisotropic and this causes some colours to be confined more tightly
than others. Even more interesting, white combinations do not neces-
sarily need to contain all five colours as one would expect. The first
three colours, labelled red, blue and green, form unconfined combinations
among themselves, as do the additional two colours, which I have called
ultraviolet and infrared. (Actually the confining effect is not quite zero
but is much smaller than other effects and is therefore ignored.) This is
but one way in which the $SU(3)$ symmetry naturally breaks off from the
rest of the symmetry group.

Examination of the gluon dynamics in section 4 finds that those cor-
responding to one particular root vector are confined more tightly than
the rest. At intermediate energies these gluons drop out of the dynamics,
causing the coupling constants of those that remain to scale differently and
again leading to the separation of $SU(3)$. The emergence of an unconfined
Abelian gauge field that can be identified with the photon is demonstrated
in section 5. While these sections are chiefly reviews of work already com-
pleted [12], section 6 contains new material concerning the matter field
representations and identiﬁes the neutral, or white, colour combinations
as well as the natural emergence of both up and down quarks and the
electron, all with the correct relative electric and colour charges.

While this is strictly speaking an examination of a one-loop effect in
$SU(5)$ QCD, the prospect of grand unification does arise. The conclu-
sion of this chapter is that its effective theory does not include weak nuclear
decay but that it could well describe a uniﬁcation of QCD with QED. The
value of such a uniﬁcation is also discussed along with the predictions of
this theory in section 7 before ending with a summary in section 8.

2 The Cho-Faddeev-Niemi-Shabanov decom-
position

My treatment of the monopole condensate rests on the Cho-Faddeev-
Niemi-Shabanov decomposition [7, 9, 10, 11]. I use the following notation:
The Lie group $SU(N)$ has $N^2 - 1$ generators $\lambda^{(j)}$, of which $N - 1$ are
Abelian generators $\Lambda^{(i)}$. For simplicity, I specify the gauge transfor-
mapped Abelian directions (Cartan generators) with

$$\hat{n}_i = U^\dagger \Lambda^{(i)} U.$$  \hspace{1cm} (1)

In the same way, I replace the standard raising and lowering operators
$E_{\pm \alpha}$ for the root vectors $\alpha$ with the gauge transformed ones

$$E_{\pm \alpha} \to U^\dagger E_{\pm \alpha} U,$$  \hspace{1cm} (2)

where $E_{\pm \alpha}$ refers to the gauge transformed operator throughout the rest
of this chapter.

Gluon fluctuations in the $\hat{n}_i$ directions are described by $c^{(i)}_\mu$. The gauge
field of the covariant derivative which leaves the $\hat{n}_i$ invariant is

$$g V_\mu \times \hat{n}_i = -\partial_\mu \hat{n}_i.$$  \hspace{1cm} (3)
In general this is
\[ V_\mu = \zeta^{(i)}_\mu \hat{n}_i + B_\mu, \quad B_\mu = g^{-1} \partial_\mu \hat{n}_i \times \hat{n}_i, \] (4)
where summation is implied over \( i \). \( B_\mu \) can be attributed to non-Abelian monopoles, as indicated by the \( \hat{n}_i \) describing the homotopy group \( \pi_2[SU(N)/U(1)^{\otimes(N-1)}] \approx \pi_1[U(1)^{\otimes(N-1)}] \). The monopole field strength
\[ H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + gB_\mu \times B_\nu, \] (5)
has only Abelian components, ie.
\[ H^{(i)}_{\mu\nu} \hat{n}_i = H_{\mu\nu}, \] (6)
where \( H^{(i)}_{\mu\nu} \) has the eigenvalue \( H^{(i)} \). Since I am only concerned with magnetic backgrounds, \( H^{(i)} \) is considered the magnitude of a background magnetic field \( H^{(i)} \). The field strength of the Abelian components \( \zeta^{(i)}_\mu \) also lies in the Abelian directions as expected and is shown by
\[ F_{\mu\nu} = F^{(i)}_{\mu\nu} \hat{n}_i, \] (7)
where
\[ F^{(i)}_{\mu\nu} = \partial_\mu \zeta^{(i)}_\nu - \partial_\nu \zeta^{(i)}_\mu. \] (8)
The Lagrangian of the Abelian and monopole components is
\[ -\frac{1}{4} (F^{(i)}_{\mu\nu} \hat{n}_i + H_{\mu\nu})^2 \] (9)
The dynamical degrees of freedom (DOF) perpendicular to \( \hat{n}_i \) are denoted by \( X_\mu \), so if \( A_\mu \) is the gluon field then
\[ A_\mu = V_\mu + X_\mu = \zeta^{(i)}_\mu \hat{n}_i + B_\mu + X_\mu, \] (10)
where
\[ X_\mu \perp \hat{n}_i, \quad X_\mu = g^{-1} \hat{n}_i \times D_\mu \hat{n}_i, \quad D_\mu = \partial_\mu + gA_\mu \times . \] (11)
Because \( X_\mu \) is orthogonal to all Abelian directions it can be expressed as a linear combination of the raising and lowering operators \( E_{\pm\alpha} \), which leads to the definition
\[ X^{(\pm\alpha)}_\mu \equiv E_{\pm\alpha} \text{Tr}[X_\mu E_{\pm\alpha}], \] (12)
so
\[ X^{(-\alpha)}_\mu = X^{(+\alpha)\dagger}_\mu. \] (13)
\( H^{(\alpha)}_{\mu\nu} \), defined by
\[ H^{(\alpha)}_{\mu\nu} = \alpha_j H^{(j)}_{\mu\nu}. \] (14)
is the monopole field strength tensor felt by \( X^{(\alpha)}_\mu \). I also define the background magnetic field
\[ H^{(\alpha)} = \alpha_j H^{(j)}, \] (15)
whose magnitude \( H^{(\alpha)} \) is \( H^{(\alpha)}_{\mu\nu} \)'s non-zero eigenvalue. Since both \( B_\mu, X_\mu \) contain off-diagonal degrees of freedom, it is worth clarifying that \( X_\mu \) contains the quantum fluctuations taking place on a generally non-trivial background whose topology is contained in the monopole field \( B_\mu \).
3 The Vacuum State of five-colour QCD

The one-loop effective energy of five-colour QCD is given by [2, 3]

\[
\mathcal{H} = \sum_{\alpha > 0} \|H^{(\alpha)}\|^2 \left[ \frac{1}{5g^2} + \frac{11}{48\pi^2} \ln \frac{H^{(\alpha)}}{\mu^2} \right]
\]

(16)

which is minimal when

\[
H^{(\alpha)} = \mu^2 \exp \left( -\frac{1}{2} \frac{4\pi^2}{55g^2} \right).
\]

(17)

This neglects an alleged imaginary component [13] which has been called into serious question recently [14, 15, 16, 17, 18, 19, 3] with more and more studies finding that it is only an artifact of the quadratic approximation. Taking this to be the case, I employ the Savvidy vacuum. This can be criticised for lacking Lorentz covariance but I argue that it is likely to match the true vacuum at least locally.

Since

\[
\|H^{(1,0,0,0)}\| = \|H^{(1)}\|,
\]

\[
\|H^{(\pm \frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0)}\|^2 = \frac{1}{4} \|H^{(1)}\|^2 + \frac{3}{4} \|H^{(2)}\|^2 \pm \frac{\sqrt{3}}{2} H^{(1)} \cdot H^{(2)},
\]

\[
\|H^{(\pm \frac{1}{2}, \frac{\sqrt{11}}{4}, \frac{\sqrt{7}}{4}, 0)}\|^2 = \frac{1}{4} \|H^{(1)}\|^2 + \frac{1}{12} \|H^{(2)}\|^2 + \frac{2}{3} \|H^{(3)}\|^2 \pm \frac{\sqrt{2}}{3} H^{(1)} \cdot H^{(2)}
\]

\[
\pm \frac{1}{2\sqrt{3}} H^{(1)} \cdot H^{(3)} + \frac{\sqrt{5}}{3} H^{(2)} \cdot H^{(3)},
\]

\[
\|H^{(0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0)}\| = \frac{1}{3} \|H^{(2)}\|^2 + \frac{2}{3} \|H^{(3)}\|^2 \pm \frac{\sqrt{2}}{3} H^{(2)} \cdot H^{(3)},
\]

\[
\|H^{(0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0)}\|^2 = \frac{3}{8} \|H^{(3)}\|^2 + \frac{5}{8} \|H^{(4)}\|^2 \pm \frac{\sqrt{15}}{4} H^{(3)} \cdot H^{(4)}.
\]

\[
\|H^{(0, \frac{\sqrt{7}}{4}, \frac{\sqrt{7}}{4}, 0)}\|^2 = \frac{3}{8} \|H^{(2)}\|^2 + \frac{1}{24} \|H^{(3)}\|^2 + \frac{\sqrt{5}}{8} \|H^{(4)}\|^2 \pm \sqrt{\frac{1}{16}} H^{(2)} \cdot H^{(3)}
\]

\[
- \sqrt{\frac{15}{4}} H^{(2)} \cdot H^{(4)} + \frac{\sqrt{5}}{\sqrt{48}} H^{(3)} \cdot H^{(4)}.
\]

\[
\|H^{(\pm \frac{1}{2}, \frac{\sqrt{11}}{4}, \frac{\sqrt{7}}{4}, 0)}\|^2 = \frac{1}{4} \|H^{(1)}\|^2 + \frac{1}{12} \|H^{(2)}\|^2 \pm \frac{\sqrt{3}}{2} H^{(1)} \cdot H^{(2)}
\]

\[
\pm \frac{1}{\sqrt{3}} H^{(1)} \cdot H^{(3)} \pm \frac{\sqrt{5}}{\sqrt{48}} H^{(2)} \cdot H^{(3)}.
\]

it follows that

\[
\|H^{(1)}\| = \|H^{(j)}\|, \quad H^{(1)} \perp H^{(j)}, \quad i \neq j,
\]

(19)

which means that the chromomagnetic field components must be equal in magnitude but mutually orthogonal in the lowest energy state. However
Table 1: Candidate parallel components for vacuum condensate. The column on the left is for parallel vectors, the column on the right is for antiparallel vectors. $\Delta H$ should be multiplied by $H^{11}_{\text{pert}}$.

| $H^{(1)}$ = +$H^{(3)}$ | $\Delta H$ | $H^{(1)}$ = −$H^{(3)}$ | $\Delta H$ |
|------------------------|------------|------------------------|------------|
| $H^{(1)}$ = +$H^{(2)}$ | 1.06381    | $H^{(1)}$ = −$H^{(2)}$ | 1.06381    |
| $H^{(1)}$ = +$H^{(3)}$ | 0.857072   | $H^{(1)}$ = −$H^{(3)}$ | 0.857072   |
| $H^{(1)}$ = +$H^{(4)}$ | 0.715651   | $H^{(1)}$ = −$H^{(4)}$ | 0.715651   |
| $H^{(2)}$ = +$H^{(3)}$ | 1.01655    | $H^{(2)}$ = −$H^{(3)}$ | 0.656584   |
| $H^{(2)}$ = +$H^{(4)}$ | 0.882589   | $H^{(2)}$ = −$H^{(4)}$ | 0.577976   |
| $H^{(3)}$ = +$H^{(4)}$ | 1.00042    | $H^{(3)}$ = −$H^{(4)}$ | 0.540983   |

three dimensional space can only accommodate three mutually orthogonal vectors. Since the number of Cartan components, i.e., components corresponding to Abelian generators, is always $N - 1$ in $SU(N)$ it follows that QCD with more than four colours cannot achieve such an arrangement.

One could substitute the Cartan basis $H^{(i)}$ but this leads to intractable equations that cannot be solved analytically. It is reasonable to expect that the lowest attainable energy state is only slightly different from (16) and that this difference is due to the failure of mutual orthogonality. I therefore propose the ansatz that all Cartan components are equal in magnitude to what they would be in the absence of dimensional frustration, and that their relative orientations in real space are chosen so as to minimise the energy. In practice this means that three of the four are mutually orthogonal and the remaining one is a linear combination of those three. This remainder will increase the effective energy through its scalar products with the mutually orthogonal vectors but not all scalar products contribute equally. This follows from the form of the root vectors in eq. (18). This means that the orientation of the remaining real space vector in relation to the mutually orthogonal ones impacts the effective energy.

A little thought reveals that the lowest energy state should have only one scalar product contribute to it. The problem of finding the lowest available energy state therefore reduces to finding the scalar product that contributes to it the least. The six candidates are

$$H^{(1)} \cdot H^{(2)}, H^{(1)} \cdot H^{(3)}, H^{(1)} \cdot H^{(4)}, H^{(2)} \cdot H^{(3)}, H^{(2)} \cdot H^{(4)}, H^{(3)} \cdot H^{(4)}.$$  \hspace{1cm} (20)

As can be seen from table 1, $H^{(3)} = -H^{(4)}$ (antiparallel) yields the lowest effective energy when all other scalar products are zero.

Substituting this result into (18) finds that all $H^{(\alpha)}$ have the same magnitude except for those that couple to $H^{(4)}$, namely $H^{(4)} \left(\sqrt{\frac{8}{5}}, \sqrt{\frac{8}{5}}, \sqrt{\frac{8}{5}}\right)$, where $?_i$ indicates that there are several possible values. The other background field strengths are

$$\|H^{(\alpha)}\|^2 = H^2,$$  \hspace{1cm} (21)
while the strongest is

$$\| H(0.0, -\sqrt{3}/\sqrt{8}, \sqrt{5}/\sqrt{8}) \|^2 = H^2 \left(1 + \sqrt{15}/4\right),$$

(22)

and the weakest are

$$\| H(?, ?, 1\sqrt{24}, \sqrt{5}/\sqrt{8}) \|^2 = H^2 \left(1 - 5/48\right).$$

(23)

Remember that the negative signs are affected by $H^{(3)}, H^{(4)}$ being antiparallel.

Assuming the dual superconductor model of confinement [4, 5, 6, 7, 8], it follows that different valence gluons and even different quarks (in the fundamental representation) will be confined with different strengths and therefore at different length scales. Those that feel the background $H(0.0, -\sqrt{3}/\sqrt{8}, \sqrt{5}/\sqrt{8})$ will be confined the most strongly, those that feel the backgrounds of the form $H(?, ?, 1\sqrt{24}, \sqrt{5}/\sqrt{8})$ will be confined least strongly. The remainder will be confined with intermediate strength.

At highest energy then, we have the full dynamics of $SU(5)$ QCD. Moving down to some intermediate energy however, finds that the dynamics associated with the root vector $\left(0, 0, -\sqrt{3}/\sqrt{8}, \sqrt{5}/\sqrt{8}\right)$ are confined out of the dynamics. The remaining gluons interact among themselves. Moving to lower energy scales I find that those dynamics are all removed in their turn except for those corresponding to the root vectors $\left(?, ?, 1\sqrt{24}, \sqrt{5}/\sqrt{8}\right)$, almost leaving an $SU(2)$ gauge field interaction. I say ‘almost’ because I shall later demonstrate that the form of the monopole condensate is sufficiently different from the $SU(2)$ condensate to alter the dynamics, producing three confined $U(1)$ gauge fields, two of which are contained within $SU(3)$, a further unconfined $U(1)$ gauge field that may be identified with the photon, and three copies of the valence gluons of $SU(2)$. At lowest energies only the unconfined gauge field remains. In this way a hierarchy of confinement scales and effective dynamics emerges naturally, without the introduction of any ad. hoc. mechanisms like the Higgs field.

4 Intermediate Energy Dynamics

In constructing the hierarchical picture above, I began with $SU(5)$ and finished with $U(1)$ but had no apparent gauge group governing the dynamics at the intermediate energy scale. The dynamics of this energy scale will prove to be quite interesting.

To facilitate the discussion I introduce a notation inspired by the Dynkin diagram of $SU(5)$. The root vectors implicitly specified in eq. (18) are all linear combinations of a few basis vectors, which according to Lie algebra representation theory can be chosen for convenience. I take the
basis vectors

\((1, 0, 0, 0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0\right), \left(0, -\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{3}, 0\right), \left(0, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right), \) \(\tag{24}\)

which I shall each represent by

\[OXXX, XOXX, XXOX, XXXO,\] \(\tag{25}\)

respectively. The remaining root vectors are sums of these basis vectors. In this notation their representation contains an 'O' if the corresponding basis vector is included and 'X' if it is not. For example the root vector

\(\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0\right) = (1, 0, 0, 0) + \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0\right), \) \(\tag{26}\)

is represented by

\[OOXX = OXXX + XOXX.\] \(\tag{27}\)

When convenient, a '?' is used to indicate that either 'O' or 'X' might be substituted.

In addition to its brevity, this notation has the nice feature of making obvious which root vectors can be combined to form other root vectors because there are no root vectors with an 'X' with 'O's on either side. There is no OXXO for example.

The confinement of \(X_{n}^{(0,0,-\frac{5}{2}, \frac{9}{2})}\), the valence gluon corresponding to XXXO, out of the dynamics directly affects only those remaining valence gluons that couple to it, those of root vectors of the form ??O?. The remaining gluons, corresponding to the root vectors OXXX, XOXX and OXOX (collectively given by ??XX), may still undergo the full set of interactions available to them at higher energies. It is easy to see that these are the root vectors that comprise the group \(SU(3)\), to which the other valence gluons couple forming two six dimensional representations. Subsequent discussion shall extend the X,O,? notation to include the valence gluons corresponding to a root vector. Whether it is the gluon or the root vector that is meant will be clear from context.

Consider the beta function, or to be less imprecise, the scaling of the various gluon couplings. I shall now demonstrate that the loss to confinement of the root vector XXXO causes unequal corrections to the running of the couplings for different gluons. Since this is only an introductory paper the following analysis is only performed to one-loop.

The gluons ??XX, corresponding to the above-mentioned \(SU(3)\), retain their original set of interactions. Performing the standard perturbative calculation \[2\] therefore yields the standard result for \(SU(5)\) QCD. The remaining gluons do not. The absence of the maximally confined XXXO restricts their three-point vertices to those of \(SU(4)\), since all root vectors are now of the form ??X. The same is not true of the four-point interactions, but the exceptions do not contribute to the scaling of the coupling constant at one-loop \[20\]. We have then that the \(SU(3)\) subgroup’s coupling scales differently from the rest of the unconfined gluons when the maximally confined valence gluons XXXO drop out.
The beta function is proportional to the number of colours in pure QCD at one-loop, so as the length scale increases, the coupling among gluons within the SU(3) subgroup initially grows faster than the couplings involving the other gluons. As noted above, the SU(3) couplings will initially scale as in the five-colour theory, while the remainder scale as though there were only four colours. This specific behaviour must soon change due to both non-perturbative contributions and because the non-SU(3) gluons have a weaker coupling. A detailed understanding requires a nonperturbative analysis well beyond the scope of this chapter. Indeed, the application of one-loop perturbation theory at anything other than the far ultraviolet is questionable in itself. The point remains that the SU(3) subgroup ??XX separates from the remaining gluons by its stronger coupling strength.

The symmetry reduction that takes place in this model is suggestive of boson mass generation but there appears to be no obvious specific mechanism. Kondo et. al. have argued for the spontaneous generation of mass through various non-trivial mechanisms [18, 21, 3]. This is consistent with the well-studied correlation between confinement and chiral symmetry breaking (see [22, 23, 24, 25] and references therein).

5 The emergence of QED

Neglecting off-diagonal gluons, the equality \( H^{(3)} = -H^{(4)} \) allows the change in variables

\[
\begin{align*}
  c^{(3)}_\mu \hat{n}_3 &\rightarrow \frac{1}{2}(c^{(3)}_\mu \hat{n}_3 + c^{(4)}_\mu \hat{n}_4) + \frac{1}{2}(c^{(3)}_\mu \hat{n}_3 - c^{(4)}_\mu \hat{n}_4) = \frac{1}{\sqrt{2}}(A_\mu + E_\mu),
  \\
  c^{(4)}_\mu \hat{n}_4 &\rightarrow \frac{1}{2}(c^{(3)}_\mu \hat{n}_3 + c^{(4)}_\mu \hat{n}_4) - \frac{1}{2}(c^{(3)}_\mu \hat{n}_3 - c^{(4)}_\mu \hat{n}_4) = \frac{1}{\sqrt{2}}(A_\mu - E_\mu).
\end{align*}
\]

Substituting eqs (28) into the Abelian dynamics finds that the antisymmetric combination \( E_\mu \) couples to the background

\[ H(\hat{n}_3 - \hat{n}_4), \]

but the symmetric combination \( A_\mu \) does not. Again by the dual superconductor model the former is confined (along with \( c^{(1)}_\mu \hat{n}_1, c^{(2)}_\mu \hat{n}_2 \)) while the latter is not. Since the electromagnetic field is long range it is natural to interpret \( A_\mu \) as the photon.

The rotation from \( c^{(3)}_\mu \hat{n}_3, c^{(4)}_\mu \hat{n}_4 \) to \( E_\mu, A_\mu \) in interactions with valence gluons is only meaningful if the gluon in question couples either to both of \( c^{(3)}_\mu \hat{n}_3 \) and \( c^{(4)}_\mu \hat{n}_4 \) or to neither of them. Otherwise the combination of \( E_\mu \) and \( A_\mu \) is ill-defined because it is not unique, ie. if the valence gluon couples to \( c^{(3)}_\mu \hat{n}_3 \) but not to \( c^{(4)}_\mu \hat{n}_4 \) then arbitrary multiples of \( c^{(3)}_\mu \hat{n}_3 \) may be added to the interaction term, yielding arbitrary mixtures of \( E_\mu \) and \( A_\mu \). The gluons for which this occurs have root vectors of the form ??OX and their electric charge is ill-defined. This is of little consequence in practice because we shall see in section that such gluons have no sources at intermediate energies.
6 Matter field representations

I have shown how the coupling of the gluons to the monopole background determines their confinement strength and subsequent phenomenology. I now consider the matter fields and focus in particular on the fundamental representation of SU(5). The confinement of the fundamental representation is determined by the maximal stability group \[26, 27\], which for SU(5) is \(U(4) \approx SU(4) \otimes U(1)\), where for any given element \((\cdots \psi \cdots)^T\), the SU(4) acts only on the remaining orthogonal elements while the U(1) causes it inconsequential phase changes. This latter U(1) describes the monopole condensate contributing to the confinement and is given by the corresponding weight of the fundamental representation. As a concrete example, consider the fundamental element \((0 0 0 0 \psi)^T\). The SU(4) of its maximal stability group are the matrices

\[
\begin{pmatrix}
T_i & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

where \(T_i\) are the standard SU(4) matrices \(T_1 \ldots T_{15}\), while the \(U(1)\) is generated by \(T_{24} = \frac{1}{\sqrt{20}} \text{diag}(1 1 1 1 - 4)\). Therefore the only component of the chromomonopole condensate that contributes to the confinement of \((0 0 0 0 \psi)^T\) is that generated by \(T_{24}\). This is what would be expected based on the weights of the fundamental representation, and indeed the main result of \[27\] is that the weight of the representation determines which components of the monopole condensate contribute to a given particle’s confinement.

The weights of the fundamental representation of SU(5) are

\[
\begin{align*}
(1 0 0 0 0)^T & : \left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}, \frac{1}{\sqrt{40}}\right) \\
(0 1 0 0 0)^T & : \left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}, \frac{1}{\sqrt{40}}\right) \\
(0 0 1 0 0)^T & : \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{24}}, \frac{1}{\sqrt{40}}\right) \\
(0 0 0 1 0)^T & : \left(0, 0, -\frac{\sqrt{3}}{8}, \frac{1}{\sqrt{40}}\right) \\
(0 0 0 0 1)^T & : \left(0, 0, 0, -\frac{\sqrt{2}}{5}\right)
\end{align*}
\]

If all Abelian components of the chromomonopole condensate were of equal magnitude and mutually orthogonal in real space so that cross-terms could be neglected then they would all be confined at equal length scales and nothing remarkable would happen. However, we already know that such is not the case.
First consider the first three lines in equation (30). They all have identical (small) dependence on $H^{(3)}, H^{(4)}$, and the same dependencies on $H^{(1)}, H^{(2)}$ as in $SU(3)$ QCD. Consider now that the last two lines show no dependence on $H^{(1)}, H^{(2)}$, and it is only natural to equate the first three elements with the quark colours of the standard model. This is supported by noting that the two additional weight entries would provide very little additional confinement because the cross terms between $H^{(3)}, H^{(4)}$ have negative sign due to their antiparallelism. In fact the total contribution squared of $H^{(3)}, H^{(4)}$ is

$$
\left( \frac{1}{\sqrt{24}} H^{(3)} + \frac{1}{\sqrt{40}} H^{(4)} \right)^2 = H^2 \frac{1}{60} (4 - \sqrt{15}).
$$

As can be seen from the bracket on the right-hand-side, the cross terms almost cancel this contribution entirely. According to the naïve interpretation of the dual superconductor model employed by this paper this corresponds to extremely weak confinement. Since it is inconsequential compared to the QCD confinement and might very well scale to zero at lower energies anyway (although I have not shown this!) I shall assume that this corresponds to an unconfined state.

This all ties in rather nicely with the result of section 4 in which the corresponding $SU(3)$ dynamics separate from the remaining dynamics through stronger scaling of the coupling constant.

The remaining weights have non-zero elements only in the third and fourth position. The final weight corresponds to a colour charge which I shall refer to as infrared ($\text{ir}$), whose confinement is slightly stronger than that of the QCD colours discussed above, while the penultimate one corresponds to the colour charge ultraviolet ($\text{uv}$), whose confinement is nearly twice as strong as that of the QCD colours. (I shall refer to both of these charges as the invisible colours.) This occurs because there are positively contributing cross terms between $H^{(3)}, H^{(4)}$ (remember their antiparallelism). It follows that ultraviolet must be combined into some neutral combination with infrared at a very small length scale. Only combined with infrared does ultraviolet form an unconfined physical state.

Note that the third and fourth entries of the sum of the ultraviolet and infrared weights are exactly negative three times those of the QCD quark weights. We have already seen that such a combination effectively feels no confining effect from the background condensate. Remembering that the third and fourth Abelian directions provide the unconfined photon $A_{\mu}$ of section 5 gives the electric charge ratio between QCD quark states and white states. In conventional QCD the white states comprise both white combinations of QCD quark colours (hadrons) and truly colourless particles (leptons), and it is simply a fortunate coincidence that both have integer multiples of the electron charge. In this model, states carrying both of the invisible colours but no QCD colours are white and electrically charged, as are white combinations of the QCD colours. From the above discussion of the weights it follows that both these cases also have the same electric charge, up to a negative sign. It will therefore be natural to interpret the state with both invisible colours but no QCD colours as the electron/positron.
The reader may recall that the third and fourth Abelian directions also provide the confined photon $E_\mu$. This may confuse some. While it may provide some additional interactions at close range, $E_\mu$ is confined, not confining, so the electric charges are still free to separate to very large distances.

Based on the proceeding discussion the fundamental representation can be shown as

$$[1] = (r b g w \bar{w})^T,$$

where the electric charge of the QCD quarks is implicit in the colour. The red, blue and green quark colours are exchanged by the corresponding $SU(3)$ but the invisible colours are not exchanged except at extremely high energies due to the extra-strong confinement of the ultraviolet charge. At highest energies the dynamics are those of $SU(5)$ in the extreme weak-coupling limit. Electric charge has no meaning at such energies.

At intermediate energies at which the gluons XXXO have been confined out of the dynamics not only is there no available gluon to exchange the invisible colours, but the ultraviolet colour itself is confined just as QCD colours are confined at larger distances. This not only removes ultraviolet from the effective dynamics but the infrared as well, because the all-white combinations involving the infrared, apart from a meson-like bound state, require the ultraviolet. This confinement of infrared leaves no source for gluons of the form $??OX$ which, as discussed in section 1, have an ill-defined electric charge. They may still occur in gluon-antigluon pairs but this is no threat to electric charge.

The intermediate dynamics consist primarily of conventional QCD and QED, but there is no obvious way to include the $SU(2)$ of weak nuclear decay or to turn a QCD quark into a lepton.

I now turn to the asymmetric representation

$$[2] = [1] \otimes_{AS} [1] = \begin{bmatrix} 0 & r/b & r/g & r/w & r/\bar{w} \\ -r/b & 0 & b/g & b/w & b/\bar{w} \\ -r/g & -b/g & 0 & g/w & g/\bar{w} \\ -r/w & -b/w & -g/w & 0 & w/\bar{w} \\ -r/\bar{w} & -b/\bar{w} & -g/\bar{w} & -w/\bar{w} & 0 \end{bmatrix}, (33)$$

where $[1]$ is the fundamental representation and $\otimes_{AS}$ indicates an anti-symmetric cross-product. The top-left-hand corner has the same interpretation as in conventional GUT theories. Red/blue is effectively antigreen etc, and the $U(1)$ (in this case electric) charge is double that of the QCD quarks in the fundamental representation. In other words the $3 \times 3$ block matrix in the top-left-hand corner can be associated with the anti-up quark when the fundamental representation contains the down quark. The remaining entries of $[2]$ contain either an ultraviolet or an infrared colour charge, which confines them out of intermediate energy level dynamics. The one exception contains both invisible colour charges and is therefore a colourless state with electric charge negative three times that of the down quark, i.e. a positron as discussed above. The effective dynamics of this representation, and its complex conjugate $[3]$, are dominated by the colour and electric interactions of the up quark and electric interactions of the electron/positron.
7 Predictions and prospects for grand unification

This approach yields no weak interaction dynamics but there is a gluon-mediated exchange between the up quarks and the electron. This is reminiscent of proton decay in conventional GUTS, in which a down quark becomes a positron and an up quark becomes an anti-up quark which forms a meson with the remaining up quark. In this case however, an up quark becomes an electron and the mediating gluon carries a charge of $+\frac{2}{3}$. This cannot be absorbed by anything within the proton so proton decay is forbidden. It could however be absorbed by an anti-up quark so that an extremely high energy collision between a proton and an anti-proton yielding an electron-positron pair and a neutral pair of pions, by which I mean either two $\pi^0$s or one $\pi^+$ and one $\pi^-$. Unfortunately such a sequence can also occur through ordinary standard model interactions so this is not much of a prediction.

It was natural to hope that dimensional frustration might yield a Higgsless GUT but it makes a good start, unifying $SU(3)_c$ with $U(1)_{EM}$, the forces acting on the right-handed matter fields, which do not feel the weak nuclear force. While I cannot yet claim to have done so, a physically realistic unification of $SU(3)_c$ with $U(1)_{EM}$ would be a unification of the forces affecting right-handed matter fields.

Assuming that nature does employ this mechanism to unify the strong nuclear and electromagnetic interactions of right-handed particles, what new phenomena could we expect to see? Obviously there would be new hadrons containing the invisible quarks. These can be divided into two types. There are mesons, we could call them invisible mesons, composed of invisible quark-antiquark pairs. Remember that such pairs can be taken not just from the fundamental representation which contains pure invisible states, but also from the $[2]$ representation which contains states carrying both a QCD colour and an invisible colour. From QCD colour symmetry such states would be mixed, so the distinguishable invisible mesons are

$$u\bar{w} - \overline{u\bar{w}}, \quad \bar{u} - \bar{w}, \quad u/[rbg] - \overline{u/[rbg]}, \quad \bar{u}/[rbg] - \bar{w}/[rbg].$$

(34)

It is, of course, possible that these states also mix, either with each other or with standard model mesons. This requires further study and is beyond the scope of this paper.

There is one more invisible meson which is listed separately because it is already known. Recall that the positron is the quark in the $[2]$ representation with the colour charge $u/w$. The invisible meson associated with the positron is obviously positronium, so

$$e^+ - e^- \iff u/w - \overline{u/w}.$$  

(35)

Another obvious combination is the quark pair made up of each invisible colour from the fundamental representation, $u - \bar{w}$, and of course its antimatter partner. Next there are particles made up of those quarks in $[2]$ that carry both a QCD colour and an invisible colour, i.e. $\{rbg\}/u$ and $\{rbg\}/\bar{w}$. An unconfined combination needs each QCD colour in equal
numbers and each of the invisible colours in equal numbers, so at least six quarks are needed.

The quark combinations in the last paragraph are neither mesonic nor baryonic and may be candidates for dark matter. There may also be fancier combinations involving more quarks or even gluons similar to the exotic states discussed in relation to conventional QCD.

Conventional GUTs predict proton decay and monopole production. Dimensional frustration predicts neither of these. It does predict an anomalous scattering however. If an electron is fired into a proton at sufficiently high energies it may turn into an up quark and emit a mediating boson that is absorbed by an up quark and turns it into an electron. The experimenter sends in an electron and sees an electron emerge so this is just a scattering experiment, but it is additional scattering to the electromagnetic interaction already observed in deep inelastic scattering experiments. The strength of the scattering for a given electron energy has not yet been calculated and requires the mediating gluon mass which is currently unknown, although it is natural to expect that it is very massive so that extremely high energy scattering experiments would be needed. It is worth noting however that while reaching ever higher energies is becoming increasingly difficult, an experiment of this kind does not, by modern standards, require sophisticated detection equipment or calculations as it is only measuring electron scattering. Again, further work is required.

8 Conclusion

I have studied the long known but generally ignored result that QCD with five or more colours has an altered vacuum state due to the limited dimensionality of space, a condition dubbed ‘dimensional frustration’. It appears to lead to a unified theory of the strong and electromagnetic interactions, which is not the conventional approach to grand unification, but these two forces are the only ones acting on the right-handed matter fields. Identification of the physical vacuum encounters an intractable set of non-analytic equations but further analysis was enabled by a well-motivated ansatz.

Assuming the dual superconductor model, a range of confinement scales emerged with one root vector (XXXO) being confined more strongly than all the rest, while some others are less tightly confined (??OO). Gluons remaining at intermediate energy scales exhibit unconventional dynamics because only some of them couple to the XXXO. An SU(3) subset have stronger interactions among themselves at increasing length scales, suggesting the separation of QCD dynamics at lower energy. In addition to a weakly confined $U(1)$ and off-diagonal SU(2) generators, there also emerged a single, unconfined $U(1)$ gauge field consistent with the photon. The theory appears to be a unification of QCD with electromagnetism. Such a unification, if consistent with experiment, is of interest to the standard model in which right-handed matter fields only couple to those two forces.

Study of the matter field representations found that the fundamental representation [1] comprises the three colours of down quark and two
more so-called invisible colours, named ultraviolet and infrared. These two
colour charges combined will neutralise each other, as do white combina-
tions of the QCD colours. According to the area law of the Wilson loop
combined with the non-Abelian Stokes theorem, the asym-
metric arrangement of the chromomonopole condensate gives the invisible
colours, especially ultraviolet, an extremely short confinement scale.

The antisymmetric combination of [1] with itself, denoted [2], contains
the anti-up quark and the positron, as well as various combinations of
the QCD and invisible colours. (The remaining matter representations
[3], [4] are the complex conjugates of [1], [2].) Again the invisible-coloured
quarks have a very short confinement scale and make little contribution
the intermediate energy dynamics so that the effective theory reduces
to QCD and electromagnetism.

Much work remains to be done. The masses of gauge bosons cou-
ppling the QCD colours to the invisible colours need to be calculated. The
breaking off of QCD symmetry, both through gluon interaction strength
and quark confinement scales, suggests a strong symmetry breaking that
should render these bosons very massive.

Dimensional frustration is a natural, almost inevitable, means of gen-
erating a hierarchy in QCD with five or more colours without resorting
to contrived symmetry breaking methods such as the Higgs field. Even a
simplistic analysis such as this finds a rich phenomenology.

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