Spectral problems from quantum field theory

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Abstract. We describe how spectral functions of differential operators appear in the quantum field theory context. We formulate consistency conditions which should be satisfied by the operators and by the boundary conditions. We review some modern developments in quantum field theory and strings and show which new spectral and boundary value problems arise.

1. Introduction

There is no sharp boundary between physics and mathematics. The list of topics which are considered as being parts of these two disciplines varies in space and time. Moreover, there are topics which belong to both. Spectral geometry is just one of the fields where the interaction between physicists and mathematicians has been especially fruitful. On the other hand, stylistic and linguistic differences between traditional physical and mathematical literature are considerable, so that some extremists could even suggest that there is no boundary since there is a gap.

The most immediate aim of this paper is to show how the notions of quantum field theory (QFT) can be translated to the language of spectral theory. Also, I would like to give the reader an idea of which structures can appear in the QFT context, which structures are less likely, and which are forbidden on general grounds.

The way spectral functions appear in quantum theory may be illustrated by the following simple example. It is well known that the zero-point energy (the lowest energy level) of the harmonic oscillator with the frequency \( \omega \) is \( E_{\omega} = \frac{\hbar}{2} \omega \) (\( \hbar \) is the Planck constant). Therefore, the ground state energy (lowest energy) of a system of non-interacting harmonic oscillators with eigenfrequencies \( \omega_j \) is

\[
E_0 = \frac{\hbar}{2} \sum_j \omega_j .
\]

QFT is characterised by the presence of an infinite number of degrees of freedom, i.e. it corresponds to an infinite system of harmonic oscillators with eigenfrequencies defined by eigenvalues of a differential operator. The sum (1.1) is typically divergent, but may be regularised by relating it to the zeta function of the operator in question.

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The first spectral function which appeared in QFT was not, however, the zeta function but the heat kernel which was used by Fock \cite{32} in 1937 to represent the Green functions. Later, in 1951, this representation was used by Schwinger \cite{73} in his famous work on quantum electrodynamics. DeWitt \cite{23} made the heat kernel a standard tool to study QFT in curved space-time. In mid 1970s Dowker and Critchley \cite{26} and Hawking \cite{49} introduced the zeta function regularization thus giving a precise meaning to the idea sketched in the previous paragraph.

During the same period important developments appeared in mathematics as well. The 1949 papers by Minakshisundaram \cite{59, 60} had much influence on the theoretical physics research. The Atiyah-Singer Index Theorem \cite{4} found many applications in gauge theories. The works of Seeley \cite{74} who analysed asymptotics of the spectral functions, and of Gilkey who suggested \cite{36} the most effective way for actual calculation of these asymptotics were essential for QFT in curved space-time.

There are many books and review papers which treat the problems of common interest for spectral geometry and quantum field theory\footnote{I cannot mention all publications which may seem relevant here or in the text below. I ask the authors whose works are omitted for understanding.}. In particular, the monographs by Gilkey \cite{37} and Grubb \cite{44} contain a very detailed description of relevant mathematics. The book by Kirsten \cite{52} deals also with some physical applications as the Casimir energy and the Bose-Einstein condensation. The reviews by Barvinsky and Vilkovisky \cite{10} and by Vassilevich \cite{80} (see also a shorter version \cite{79}), as well as the book by Elizalde \cite{29} are oriented to the physicists. In the present paper I introduce some new structures which appeared recently in QFT and may be of interest for the experts in spectral geometry.

This paper is organised as follows. In the next section I briefly introduce the path integral quantisation, the effective action, and the semiclassical expansion. The leading order of this expansion is defined by spectral functions of some differential operator which may be derived from the classical action. I also discuss which properties some general properties of the operator and of the boundary conditions. This section describes objects instead of rigorously defining them. However, one can give precise mathematical sense to most of the constructions presented here. The interested reader can consult the introduction to QFT specially tailored for the mathematicians \cite{22}. Sec. 3 discusses main spectral functions appearing in the context of QFT. In particular, the divergences (“infinities”) of the effective action are defined by the heat trace asymptotics, and the “finite” part is the zeta-determinant. Quantum anomalies are related to localised zeta functions. In this section we also discuss which properties of the heat trace asymptotics are essential to have a meaningful quantum field theory. Throughout this paper we discuss bosonic fields theories and Laplace type operators. Fermionic theories give rise to operators of Dirac type. An introduction to the theory of Dirac operators with applications to QFT can be found in the monograph \cite{30} by Esposito. Sec. 4 contains examples of several new problems which appeared in physics over the recent years. In particular, sec. 4.1 is devoted to string theory. Here we discuss which boundary conditions correspond to open strings, and which to the Dirichlet-branes. We also comment on string dualities and non-commutativity (as it comes from strings). In sec. 4.2 we consider domain walls and the so-called brane-world scenario. These configurations can be described as two smooth manifolds glued together along a
common boundary. Instead of boundary conditions one has matching conditions on the interface surface in this case. Last sections contain remarks on supersymmetric theories and on non-commutative field theories respectively.

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2. Quantum Field Theory and the Path Integral

Classical Field Theory consists of (i) a Riemannian manifold $M$ called the space-time, (ii) a Hermitian vector bundle $V$ with sufficiently smooth sections $\phi$ which are called fields, and (iii) a classical action $S$ defined on $\phi$ with values in $\mathbb{R}$.

Some comments are in order. Strictly speaking, the space-time should be a pseudo-Riemannian manifold. Transitions between the Riemannian (Euclidean) and pseudo-Riemannian (Minkowski) signatures of the metric are performed by the so-called Wick rotation which introduces an imaginary time coordinate (relations between spectral theory and the Wick rotation are discussed by Fulling [33]). In this manner most (but not all) properties of an Euclidean theory may be translated to the Minkowski context. Besides, Euclidean field theories contain much of important physics and interesting mathematics of their own. In this paper we restrict ourselves to Euclidean theories.

In many cases, classical fields do not form a vector bundle since the fibres may have a more complicated geometry. However, in this work we shall restrict ourselves to the perturbative analysis, i.e. we shall work with small fluctuations about a given background field. Such small fluctuations always form a vector space for any model. We also assume given a Hermitian structure though it is not always uniquely defined. It is natural to assume that the classical action $S$ is bounded from the below\(^2\). Local minima of the classical action are called classical solutions or classical trajectories of the theory.

Fundamental theories of physics are local, i.e. the action has the form

\[(2.1)\quad S = \int_M L_{\text{int}} + \int_{\partial M} L_{\text{bou}},\]

where $L_{\text{int}}$ and $L_{\text{bou}}$ depend on fields at a given point and on finite number of their derivatives.

Probably the most popular example of a field theory model is the so called $\phi^4$ theory in $n = 4$ dimensions. For simplicity, we suppose that $V$ is a real line bundle. Then the classical action reads:

\[(2.2)\quad S_{\phi^4} = \int_M [(\nabla \phi)^2 + m^2 \phi^2 + g\phi^4]\]

Here $m^2$ and $g$ are (positive) constants.

To quantise a given classical theory one has to replace the classical fields $\phi$ by operator valued distributions in a suitably defined Hilbert space. The final aim is to be able to calculate the so-called vacuum expectation values of arbitrary polynomials of the field operators. For these objects there is a “path integral”

\(^2\)Euclidean gravity is a famous exception \[35\].
representation:\(^{3}\):

\[
\langle \phi(x_1) \ldots \phi(x_k) \rangle_0 = \frac{1}{N} \int (D\phi) \phi(x_1) \ldots \phi(x_k)e^{-\frac{i}{\hbar} S}.
\]

Here the bracket \(\langle \ldots \rangle_0\) denotes the vacuum expectation value, \(h\) is the Planck constant, \((D\phi)\) is the integration measure, \(1/N\) is a normalisation constant chosen in such a way that \(\langle 1 \rangle_0 = 1\).

Let me stress that the vacuum expectation values of field polynomials contain practically all information which is needed in QFT. The representation (2.3) has to be derived from basic principles of quantum mechanics, but it can be easily understood on its’ own. The right hand side of (2.3) is nothing else than a statistical average with the weight \(e^{-\frac{1}{\hbar} S}\). In the formal limit \(\hbar \to 0\) the path integral is dominated by the minima of the classical action \(S\) thus recovering the classical theory.

The integral (2.3) is infinite-dimensional and, as it stays, is ill-defined. Besides, the measure \((D\phi)\) may have a very complicated structure. We shall ignore these difficulties in this section and just deal with the path integral as with an ordinary integral. This will be enough to achieve a qualitative understanding of what is going on in quantum field theory.

The vacuum expectation values can be generated by taking repeated functional derivatives of the functional

\[
Z(J) = \frac{1}{N} \int (D\phi) e^{-\frac{1}{\hbar} (S + f_M J\phi)}
\]

with respect to the “external source” \(J\).

There is another functional, \(W(\bar{\phi})\), which is given by the Legendre transform of \(\ln Z(J)\). It is called the effective action and is defined by the equation

\[
e^{-\frac{1}{\hbar} W(\bar{\phi})} = \frac{1}{N} \int (D\phi) e^{-\frac{1}{\hbar} (S(\bar{\phi} + \phi) + f_M J\phi)} ,
\]

where \(J\) is not an independent variable any more. It should be expressed in terms of the background field \(\phi\) by means of

\[
\frac{\delta W(\bar{\phi})}{\delta \bar{\phi}} = -J.
\]

The effective action \(W\) contains the same information as \(Z(J)\) but is somewhat easier to analyse. Let us consider a semiclassical expansion of \(W\). This means \(\hbar \to 0\) asymptotics of the equations (2.5) and (2.6). To this end one has to use the saddle point method to evaluate the integral in (2.5). Let us expand the classical action \(S(\bar{\phi} + \phi)\) about \(\bar{\phi}\),

\[
S(\bar{\phi} + \phi) = S(\bar{\phi}) + \int \frac{\delta S}{\delta \bar{\phi}(x)} \phi(x) + \frac{1}{2} \int \int \frac{\delta^2 S}{\delta \bar{\phi}(x) \delta \bar{\phi}(y)} \phi(x)\phi(y) + O(\phi^3).
\]

the zeroth order approximation to the effective action \(W\) is just the classical action:

\[
W_0(\bar{\phi}) = S(\bar{\phi}).
\]

\(^{3}\)A simple and clean derivation of this representation can be found in the excellent textbook [31] by Faddeev and Slavnov.
Consequently, to this order

\begin{equation}
J_0 = -\frac{\delta S(\phi)}{\delta \phi},
\end{equation}

so that the linear terms in $\phi$ in the exponential \ref{2.9} are cancelled. The next approximation to the effective action is obtained by keeping the quadratic term in \ref{2.7} and performing the Gaussian integration. Usually there exists a (pseudo)differential operator $D$ such that

\begin{equation}
S_2(\bar{\phi}, \phi) := \int_x \int_y \frac{\delta^2 S(\phi)}{\delta \phi(x) \delta \phi(y)} \phi(x) \phi(y) = \int \phi D[\bar{\phi}] \phi.
\end{equation}

Therefore,

\begin{equation}
W_1(\bar{\phi}) = \frac{\hbar}{2} \ln \det(D)
\end{equation}

We stress again that we are working with bosonic theories only. For fermions the rules of functional integration are considerably different, so that \ref{2.11} is no longer true.

This semiclassical expansion is also called the loop expansion since in the language of Feynman diagrams $W_q$ is described by graphs containing $q$ loops.

One can also define the correlation functions of $\phi$ in the presence of the background field $\bar{\phi}$. In this case one has to keep the sources unrestricted to be able to vary with respect to them. Then the semiclassical expansion looks a little bit more complicated. In the leading order in $\hbar$ the result reads:

\begin{equation}
\langle \phi(x) \phi(y) \rangle \sim G(x, y),
\end{equation}

where $G(x, y)$ is the Green function, $DG = Id$.

The operator $D$ should satisfy some natural restrictions:

1. The operator $D$ should be symmetric. Otherwise, the Gaussian integral would not produce $\det(D)$.
2. Since the fundamental actions are local (an exception will be discussed in sec. 4.4 below), the operator $D$ is a partial differential operator rather than a pseudodifferential one.
3. The operator $D$ should have finite number of negative and zero modes. Path integration in the directions corresponding to negative and zero modes cannot be performed by the saddle point method. These directions should be treated separately.

Note, that infinite number of zero modes is a characteristic feature of gauge theories (cf. a mathematical introduction \cite{MaratheMartucci} by Marathe and Martucci). During the quantisation these zero modes are removed by a gauge fixing procedure \cite{31}, so that the resulting quantum theory satisfies the restriction given above.

If $M$ has a boundary, r.h.s. of \ref{2.10} is supplemented by a boundary term,

\begin{equation}
S_2(\bar{\phi}, \phi) = \int_M \phi D[\bar{\phi}] \phi + \int_{\partial M} L^2_{bou}(\bar{\phi}, \phi),
\end{equation}

where the boundary density $L^2_{bou}(\bar{\phi}, \phi)$ is quadratic in $\phi$. One has to impose some boundary conditions on $\phi$. They should be such that (i) the properties (1) and (3) listed above hold, and (ii) the boundary term in \ref{2.13} vanishes, so that the Gaussian integration produces eigenvalues of $D$ at least formally.
Let us now consider the example (2.2). Obviously, the quadratic part of the action reads:

\[ S_2(\bar{\phi}, \phi) = \int_M \phi (-\nabla^2 + m^2 + 6g\bar{\phi}^2) \phi - \int_{\partial M} \phi \nabla_n \phi , \]

where \( \nabla_n \) is a derivative with respect to inward pointing unit vector on the boundary. Clearly, in this case

\[ D = -\nabla^2 + m^2 + 6g\bar{\phi}^2 \]

is an elliptic partial differential operator. Natural boundary conditions which ensure vanishing of the boundary term in (2.14) are either Dirichlet

\[ \phi |_{\partial M} = 0 , \]

or Neumann

\[ \nabla_n \phi |_{\partial M} = 0 , \]

ones. Both guarantee strong ellipticity of the boundary value problem and satisfy all consistency conditions listed above. A somewhat less trivial fact is that these conditions are also satisfied by rather complicated mixtures of Dirichlet and Neumann conditions (cf. sec. 4).

3. Spectral Functions and QFT

As we have seen in the previous section, first non-trivial quantum correction to classical action is defined by \( \det(D) \). This quantity has to be regularised. Let us define the heat trace \( K(t, D) \) by the equation

\[ K(t, D) = \text{Tr}_{L^2} (e^{-tD}) . \]

There exist a useful representation for the determinant

\[ \ln \det(D) = -\int_0^\infty \frac{dt}{t} K(t, D) , \]

which is still divergent, but is nevertheless useful to discuss regularizations. One possible way to make sense of (3.2) is to shift the power of \( t \) (and introduce a constant \( \tilde{\mu} \) of the dimensions of mass to keep proper dimension of the effective action) \[ 26, 49 \]

\[ \ln \det(D)_s = -\tilde{\mu}^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(t, D) , \]

so that the integral converges for “typical” \( D \) for sufficiently large \( \text{Re}s \). The zeta function is defined as

\[ \zeta(s, D) := \text{Tr}_{L^2}(D^{-s}) \]

We assume that \( D \) is positive (negative and zero modes should be excluded, cf. previous section). The heat trace is then expressed as

\[ K(t, D) = \frac{1}{2\pi i} \int_{\text{Res}=c} t^{-s} \Gamma(s) \zeta(s, D) . \]

\( c \) should be sufficiently large.
We shall also need a “localised” (or “smeared”) version of the heat trace. Let $f$ be a smooth function on $M$ (or an endomorphism of the vector bundle $V$, depending on the context). Then

$$K(f; t, D) = \text{Tr}_{L^2}(fe^{-tD}).$$

Obviously, $K(t, D) = K(1; t, D)$. One can also define a localised version of the zeta function, so that the relations between the heat trace and the zeta function will hold also in the local sense.

We assume that the following asymptotic exists for $t \to +0$:

$$K(f; t, D) \simeq t^{-n/2} \left( \sum_{k=0}^{N} t^{k/2} a_k(f, D) + \sum_{j=N+1}^{\infty} t^{j/2} (a'_j(f, D) \ln t + a''_j(f, D)) \right),$$

so that we have mixed power and power-logarithm asymptotic expansion. The coefficients $a_k$ are locally computable, i.e. they can be represented as integrals of local invariants constructed from the symbol of $D$. This expansion indeed exists if $D$ is an elliptic second order operator and if the boundary conditions satisfy some additional requirements formulated by Grubb and Seeley [47, 45]. In the particular case when $D$ is of Laplace type and the boundary conditions are local, logarithms are absent ($N = \infty$). If, moreover, $M$ has no boundary, even numbered coefficients vanish.

For any operator of Laplace type there exist a unique connection $\nabla$ and a unique endomorphism $E$ such that

$$D = -(g^{\mu\nu} \nabla_\mu \nabla_\nu + E)$$

($g^{\mu\nu}$ is Riemannian metric on $M$). Let $\partial M = \emptyset$. Let $R_{\mu\nu\rho\sigma}$ be the Riemann curvature tensor, let $R_{\mu\nu} := R^\rho_{\mu\nu\rho}$ be the Ricci tensor, and let $R := R^\mu_{\mu}$ be the scalar curvature. We define the field strength of $\nabla_\mu = \partial_\mu + \omega_\mu$ by the equation $\Omega_{\mu\nu} := \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\mu \omega_\nu - \omega_\nu \omega_\mu$. Then

$$a_0(f, D) = (4\pi)^{-n/2} \int_M \text{tr}\{f\}.$$

$$a_2(f, D) = (4\pi)^{-n/2} 6^{-1} \int_M \text{tr}\{f(6E + R)\}.$$

$$a_4(f, D) = (4\pi)^{-n/2} 360^{-1} \int_M \text{tr}\{f(60\nabla^2 E + 60RE + 180E^2 + 12\nabla^2 R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + 30\Omega^{\mu\nu}\Omega_{\mu\nu})\}.$$

These expressions appeared in both mathematical [55] and physical [23] literature. The coefficient $a_6$ was first calculated by Gilkey [30], $a_8$ was obtained by Amsterdam et al. [2] and by Avramidi [5], and $a_{10}$ was calculated by van de Ven [76]. In the presence of boundaries the calculations are much more involved. For local mixed boundary conditions $a_4$ was calculated by Branson and Gilkey [17] (with minor corrections by Vassilevich [77]), and $a_5$ was done by Branson et al. [18].

Equation (3.6) implies that there is a relation between the poles of $\Gamma(s)\zeta(s, D)$ and the asymptotic expansion for the heat trace. In particular, if $a''_n(D) := a''(1, D) = 0$ the zeta function is regular at $s = 0$. This is achieved in the case $N > n$ when also the following important relation holds

$$\zeta(0, D) = a_n(D),$$
which tells that $\zeta(0, D)$ is locally computable.

For sufficiently large $\text{Re} s$ the integral (3.11) yields

$$\ln \det(D)_s = -\tilde{\mu}^2 s \Gamma(s) \zeta(s, D).$$

Now the right hand side has to be analytically continued to the physical value $s = 0$. If $\zeta(s, D)$ is regular near $s = 0$ (i.e. if $a'_n(D) = 0$) the only singularity of (3.11) at this point comes from a pole of the gamma function. One can expand near $s = 0$:

$$\ln \det(D)_s \simeq -\left(\frac{1}{s} - \gamma_E + \ln \tilde{\mu}^2\right) \zeta(0, D) - \zeta'(0, D),$$

where $\gamma_E$ is the Euler constant. The second term on the right hand side of (3.12) is nothing else than the Ray-Singer [68] definition of the determinant.

One can see that the r.h.s. of (3.12) is divergent as $s \to 0$, and therefore the one-loop effective action (2.11) is divergent as well. However, one can ensure that the sum of the zero-loop part (2.8) and of the one-loop part is convergent in the limit $s \to 0$. Let the classical action $S$ depend on fields $\phi$ and “charges” $e_j$. One can replace in $S$: $\phi \to Z_\phi \phi, e_j \to Z_e e_j$ (no summation over $j$) with $Z_{j, \phi} = 1 + h z_{j, \phi} + O(h^2)$. The constants $z$ can depend on charges and on the regularization parameter $s$ but cannot depend on $\phi$. Obviously, the classical limit $h = 0$ remains unchanged. One can try to find such values of $z$ that the sum $S(\hat{\phi}) + W_1(\hat{\phi})$ becomes regular at $s = 0$. If this is possible to do by introducing a finite number of renormalization constants $Z_j$ the theory is called multiplicatively renormalizable (at one loop). One can give a precise mathematical meaning to the renormalization procedure. Here we shall need just one simple observation. In a renormalizable theory the divergent part of the one-loop effective action should repeat the structure of the classical action. In particular, the divergent part must be local.

In order to see how this procedure works let us consider the $\phi^4$ theory (2.2) on a compact flat 4-dimensional manifold without boundary. Relevant operator is given by (2.15), it is of Laplace type, $E = -m^2 - 6g^2 \bar{\phi}^2$. Then, by using (3.9) and (3.10), one obtains

$$\zeta(0, D) = a_4 = \frac{1}{32\pi^2} \int_M \left[m^4 + 12gm^2\bar{\phi}^2 + 36g^2\bar{\phi}^4\right].$$

By (2.11) this expression defines the one-loop divergences. We see, that the pole part of the effective action indeed repeats the form of the classical action (2.2) up to a field independent term $m^4$ which can be neglected (unless we treat metric as a dynamical field, which is a much more complicated case). The remaining divergences can be removed by choosing

$$zm^2 = \frac{3g}{16\pi^2 s}, \quad z_g = \frac{9g}{16\pi^2 s},$$

so that the $\phi^4$ theory is indeed multiplicatively renormalizable (at least at the one-loop level).

We have seen that the properties of the heat trace expansion (3.11) for $f = 1$ are important for the renormalization theory. The localised heat trace asymptotics

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4This is, of course, an oversimplified example.
(f \neq 1) are also very useful. Probably the most spectacular application is quantum anomalies. Consider a family of the operators

\[ D_{[\alpha \rho]} = e^{\alpha \rho} D e^{\alpha \rho}, \]

where \( \rho \) is an endomorphism of \( V \), and \( \alpha \) is a number. Then

\[ \frac{d}{d\alpha} \zeta(s, D_{[\alpha \rho]}) = -2s \text{Tr}(\rho D^{-s} \rho) = -2s \zeta(\rho, s, D_{[\alpha \rho]}). \]

By comparing this equation to (3.12) one sees that if the localised zeta function \( \zeta(\rho, s, D_{[\alpha \rho]}) \) is regular and locally computable at \( s = 0 \), the derivative of \( \log \det(D_{[\alpha \rho]}) \) with respect to \( \alpha \) is finite for \( s \to 0 \) and local. In other words, one can take the \( s \to 0 \) limit in (3.12) to write:

\[ \frac{d}{d\alpha} \ln \det(D_{[\alpha \rho]}) = 2 \zeta(\rho, 0, D_{[\alpha \rho]}). \]

For a hermitian \( \rho \) the transformation \( D \to D_{[\alpha \rho]} \) can be made a symmetry of the quadratic part of the classical action (2.10) if accompanied by the transformation \( \phi \to e^{-\alpha \rho} \phi \) of fluctuations. If it can be promoted to a symmetry of full classical action, then r.h.s. of (3.17) is called quantum anomaly since it describes non-invariance of the quantum action with respect to the same transformations.

Given explicit form of the zeta function in (3.17) this equation can be integrated to give \( \ln \det(D_{[\rho]}) - \ln \det(D_{[0]}) \). This is especially important if \( D_{[0]} \) is trivial in some sense. Then one can obtain the effective action (determinant) itself. Probably the most famous example of such construction is the Polyakov effective action [65] which is obtained by integration of the conformal anomaly on a two dimensional manifold (i.e. when \( \rho \) is a real function). A more recent example is duality symmetries of the p-form theories considered by Gilkey et al [40].

In general, the analysis of a quantum theory at the one-loop approximation (by spectral theory methods) consists of the following steps. One starts with defining the operator \( D \) and with fixing an appropriate set of boundary conditions (cf. sec. 2). Then one has to make sure that an expansion of the type (3.7) exists. Then, if the log-term \( a_n' \) is absent and if \( a_n = a_n'' \) is local, one can analyse renormalization in the usual way and calculate the anomalies. Note, that strictly speaking it is not excluded on general grounds that the theory can be renormalised even if \( a_n' \neq 0 \). However, no example of such theory is known. As soon as the renormalization is done, one can try to calculate finite part of the effective action which is essentially defined by \( \zeta'(0, D) \) (cf. eq. (3.12)). There exist very few examples of the theories where this can be done in a closed analytical form. Therefore, one is usually restricted to various expansions of the effective action (cf. [80] for an overview).

4. Examples

During its’ earlier history Quantum Field Theory dealt with Laplace type operators either on manifolds without boundaries or with simplest (Dirichlet or Neumann) boundary conditions. Recent new developments introduced new geometries and new spectral problems to physics. Several examples will be considered in this section.
4.1. Strings. String is a one-dimensional object moving in a d-dimensional manifold called the target space. During this movement a two-dimensional submanifold \( M \) called the world surface is formed. We denote by \( X^j \) a local coordinate system in the target space, and by \( x^\mu \) a coordinate system on the world surface. Dynamics of string is defined by the embedding functions \( X^j(x) \). We assume that the target space is equipped with a Riemannian metric \( G_{ij}(X) \) and with some other fields \( B_{ij}(X), A_j(X) \), etc. Then propagation of the string is described by the world-surface action

\[
S[\sigma] = \int_M d^2x \left( \sqrt{|g|} G_{ij}(X) g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j + \epsilon^{\mu\nu} B_{ij}(X) \partial_\mu X^i \partial_\nu X^j \right) + \int_{\partial M} A_j dX^j.
\]

(4.1)

Here \( g^{\mu\nu} \) and \( \epsilon^{\mu\nu} \) are the metric tensor and the Levi-Civita tensor on the world surface respectively. We have absorbed the string tension \( \alpha' \) (which usually appears in the action) in a field redefinition. \( A^j \) plays a role of the electromagnetic potential. We see, that electric charges of the string are concentrated at the end points.

We are dealing with a two-dimensional field theory where the coordinates of the string \( X^j(x) \) play the role of fields (the same as \( \phi \) in the sections above). The quantities \( G, B \) and \( A \) (which are fields on the target space) play the role of charges (or couplings) in this two-dimensional theory. Since \( G, B \) and \( A \) are almost arbitrary functions of \( X \) we have a system with infinitely many charges (each charge corresponds to a power of \( X \) in Taylor series expansions of \( G, B \) and \( A \)).

To apply the background field formalism one has to fix a trajectory of the string \( \bar{X}(x) \) and consider small deviations \( \xi \) from this trajectory, \( X = \bar{X} + \xi \) (\( \xi \) then belongs to the tangent bundle of the target space manifold restricted to the string world surface). Then one has to expand the action in a power series of \( \xi \) up to the quadratic order. With increasing level of generality this was done e.g. by Braaten, Curtright and Zachos, Osborn, Kummer and Vassilevich.

Here we are interested in the boundary term only:

\[
S_{\text{bou}}^{\partial} = -\frac{1}{2} \int_{\partial M} d\tau \xi \left( \nabla_n + \frac{1}{2} (\nabla_\tau \Gamma + \Gamma \nabla_\tau) + \mathcal{S} \right),
\]

(4.2)

where \( \nabla \) is a connection, \( \Gamma \) and \( \mathcal{S} \) are some endomorphisms which depend on \( A(\bar{X}), G(\bar{X}), B(\bar{X}) \).

There exists a natural choice of the boundary conditions

\[
\left( \nabla_n + \frac{1}{2} (\nabla_\tau \Gamma + \Gamma \nabla_\tau) + \mathcal{S} \right) \xi |_{\partial M} = 0,
\]

(4.3)

for which the boundary action (4.2) vanishes and the operator \( D \) (not presented here explicitly) is symmetric. These boundary conditions describe “free” propagation of an open string in the target space manifold. This is probably the most important case in physics when the boundary conditions contain both normal and tangential derivatives\(^5\). Such boundary conditions (called oblique) appeared also in the mathematical papers by Grubb and by Gilkey and Smith. The study of the heat trace asymptotics was initiated by McAvity and Osborn and

\(^5\)Other examples of such boundary conditions include gravity and some solid state systems. 


then continued by Dowker and Kirsten \[27, 28\]. Avramidi and Esposito \[7, 8\] lifted some commutativity assumptions and proved a simple criterion of strong ellipticity.

In contrast to Dirichlet and Neumann boundary value problems oblique boundary conditions are strongly elliptic only if $|\Gamma|$ is sufficiently small. For large $|\Gamma|$ the operator $D$ has infinitely many negative modes, so that the heat kernel is not of trace class any more. Physically, lack of strong ellipticity means that the string endpoints are forced to move faster than light, so that the system develops instabilities. If strong ellipticity is preserved, there exists the asymptotic expansion (3.7) without log-terms, so that all coefficients are locally computable. One can then easily check that the model is renormalizable and calculate the counterterms. The condition that these counterterms vanish is very important in string theory since it is equivalent to the equations of motion of a (super-) gravity theory for the fields $G, A, B$ on the target space manifold. This is probably the most straightforward way to derive the low-energy limit of the string theory.

The condition (4.3) is not the only possible choice. Let us assume given two local complementary projectors $\Pi_+ \text{ and } \Pi_-$. Then we may impose the conditions (4.3) on $\Pi_+ \xi$ and Dirichlet boundary conditions on $\Pi_- \xi$. Such configurations, introduced in the string theory context by Dai, Leigh and Polchinski \[21\], are called Dirichlet branes (or D-branes). Physically these boundary conditions mean that the string endpoints are confined in a submanifold of the target space. Spectral properties (strong ellipticity, absence of logarithms, locality of the counterterm) of D-branes are very similar to that of the open strings. Therefore, we shall not consider them here in detail.

A very interesting idea of string theory which has not been fully explored yet from the spectral theory point of view is the string dualities. It has been observed \[21, 50, 42\] that by suitably transforming the target space fields $A, B, G$ and by interchanging oblique (open string) and Dirichlet (D-brane) boundary conditions (i.e. by exchanging the role of $\Pi_-$ and $\Pi_+$) one arrives at a quantum theory which is equivalent to the initial one. Such transformation is called target space duality (or T-duality). An important property of the duality transformations is that they map ”strong coupling” regimes to ”weak coupling” regimes \[6\]. In some simple cases, as explained by Schwarz and Tseytlin \[72\] and by Vassilevich and Zelnikov \[83\], the duality symmetry leads to equivalence of determinants of some (non-isospectral) operators. Could it be that in this way one may obtain more interesting (and yet overlooked) relations between determinants?

Another important feature of string theory is that it leads to a non-commutative geometry on the target space. To illustrate this point let us consider a rather simple particular case of the string dynamics when the world surface of the string is $M = \mathbb{R} \times \mathbb{R}_+$, the target space is $\mathbb{R}^d$ with standard flat metric $G_{ij} = \delta_{ij}$, with zero electromagnetic field $A$ and with a constant field $B$. In this case $D = -\partial_1^2 - \partial_2^2$, and the boundary condition (4.3) becomes:

\[(4.4) \quad (\partial_2 \delta_{jk} + B_{jk} \partial_1) \xi^k |_{\partial M} = 0 ,\]

where $\partial_2$ is a normal derivative, and $\partial_1$ is a tangential one. This problem can be explicitly solved. In particular, one can find the Green function $G(x, y)$. When

\[\text{One can keep in mind the example of ordinary electrodynamics. The duality transformation, which is the Hodge duality of the electromagnetic field strength, interchanges electric and magnetic fields, and also interchanges electrically charged particles with the charge $e$ with magnetically charge particles (monopoles) with the magnetic charge $\sim 1/e$.} \]
both point $x$ and $y$ are on the boundary it reads \cite{71,70} ($\tau := x^1, \ \tau' := y^1$):

\begin{equation}
G(\tau, \tau') = -C \log(\tau - \tau')^2 + \frac{1}{2} i \theta \text{sign}(\tau - \tau')
\end{equation}

where

\begin{equation}
C = (1 + B^2)^{-1}, \quad \theta = i B (1 + B^2)^{-1}
\end{equation}

and all multiplications are $d \times d$ matrix multiplications.

Next we need further input from Quantum Field Theory, which we use here without going into the derivation. One has to interpret $\tau$ as a time coordinate\footnote{This means that one has to consider the theory in a pseudoeuclidean space. In such a case the field $B$ has to be replaced by $iB$. This property may be considered as just another miracle of quantum theory, but it is important if one compares the formulae below to other results in the literature.}. The commutators of the $\xi$ is then related to time-ordered correlators:

\begin{equation}
[\xi^j(\tau), \xi^k(\tau)] = \lim_{\tau' \to \tau} \langle \xi^j(\tau_>) \xi^k(\tau_<) - \xi^k(\tau_>) \xi^j(\tau_<) \rangle,
\end{equation}

where $\tau_>$ (resp. $\tau_<$) is larger (smaller) of the two arguments $\tau$ and $\tau'$. Now we can relate the correlators to the Green function by means of \cite{21,12}:

\begin{equation}
[\xi^j(\tau), \xi^k(\tau)] = i \delta^{jk}.
\end{equation}

Since $\xi$ is a coordinate of the string endpoint, the equation (4.8) implies that the coordinates on the target space do not commute. Although the arguments given above are rather incomplete (cf. papers by Schomerus \cite{71} and by Seiberg and Witten \cite{70} where I have borrowed these arguments for more details) there are two important lessons one can learn from them. First, the non-commutativity structure if fully defined by the Green function of the operator $D$, so that it can be analysed by the spectral theory methods. Second, this effect appears anytime when the Green function has a part which is antisymmetric in the coordinates, so that this feature should be rather common.

Another question which arises in the context of strings is could we define a more general boundary conditions than the ones already considered above? Clearly, if we demand that the boundary conditions are local, we do not have much additional opportunities. There is no necessity, however, to demand locality. Physically, boundary conditions describe interactions with some states leaving exclusively on the boundary (called edge or boundary states, such states appear also in condensed matter problems). Especially in the view of the duality symmetry it is clear that such states may be non-local (cf. the example above involving electric charges and monopoles: in normal electrodynamics electric charges are local point-like objects, while magnetic monopoles are soliton-like extended objects). These arguments motivated Vassilevich to introduce spectral branes \cite{78}, i.e. to replace the local projectors $\Pi_{\pm}$ by spectral projectors of the Atiyah-Patodi-Singer \cite{23} type. There is an important difference for the APS scheme: for the bosonic string spectral boundary conditions have to be used for a second order operator of Laplace type, while the original APS proposal refers to a Dirac operator. Fortunately for the spectral branes it has been demonstrated very recently by Grubb \cite{46} that for the case in question there is an asymptotic series \cite{57} with vanishing leading logarithm, $a'_n = 0$. Therefore, one can indeed construct a well posed quantum field theory (at least in the one-loop approximation) and define such important quantities as
quantum anomalies. Actual calculations of the divergences and anomalies is still an open subject.

4.2. Domain walls and the brane-world scenario. Decomposition of manifolds was one of the main topics of this workshop. Over the recent years many exciting results on spectral invariants were obtained in this framework (see other contributions to this volume and lectures by Park and Wojciechowski [63]). Somewhat similar constructions appeared recently in theoretical physics in the context of domain walls and of the brane world scenario.

Sharp boundaries are not always good models for real physical systems. A narrow potential barrier is a much better approximation in many cases. Consider a manifold $M$ and a submanifold $\Sigma$ of the dimension $n-1$. Let

$$D[v] = D + v\delta_\Sigma.$$ (4.9)

$D$ is an operator of Laplace type. Let $h$ be the determinant of the induced metric on $\Sigma$. Then $\delta_\Sigma$ is a delta function defined such that

$$\int_M dx \sqrt{g} \delta_\Sigma f(x) = \int_\Sigma dx \sqrt{h} f(x).$$ (4.10)

The spectral problem for $D[v]$ on $M$ as it stands is ill-defined owing to the discontinuities (or singularities) on $\Sigma$. It should be replaced by a pair of spectral problems on the two sides $M^\pm$ of $\Sigma$ together with suitable matching conditions on $\Sigma$. Let $e_n$ be a unit normal to $\Sigma$ and let $x^n = 0$ on $\Sigma$. Then the matching conditions read:

$$\phi|_{x^n=+0} = \phi|_{x^n=-0}$$

$$-\nabla_n \phi|_{x^n=+0} + \nabla_n \phi|_{x^n=-0} + v\phi = 0.$$ (4.11)

The first line (continuity of $\phi$) is required to make sense of the multiplication of $\phi$ by a delta-function. The second line can be “derived” by considering the eigenvalue equation for $D[v]$ in a vicinity of $\Sigma$.

Further generalisations of this constructions are suggested by the so-called brane-world scenario of Randall and Sundrum [66, 67] which became a hot topic in theoretical physics a few years ago. According to this scenario our world is a four dimensional membrane in a five dimensional space (a similar proposal was made earlier by Rubakov and Shaposhnikov [69]). Typical form of the metric near $\Sigma$ is

$$(ds)^2 = (dx^n)^2 + e^{-\alpha|x^n|}(ds_{n-1})^2,$$ (4.12)

where $\alpha$ is a constant and where $(ds_{n-1})$ is a line element on the $(n-1)$-dimensional hypersurface $\Sigma$. Due to the presence of the absolute value of the $n$-th coordinate in (4.12), the normal derivative of the metric jumps on $\Sigma$. One can think of two smooth manifolds $M^+$ and $M^-$ glued together along their common boundary $\Sigma$. Neither Riemann tensor, nor matrix potential $E$ must be continuous on $\Sigma$. Also, the extrinsic curvatures $L^+_ab$ and $L^-_ab$ of $\Sigma$ considered as a submanifold in $M^+$ and in $M^-$ respectively are, in general, different. Together with the conditions (4.11) this defines a well-posed spectral problem for an operator of Laplace type. In this case there is a power-law asymptotic expansion for the heat trace (no log-terms) with locally computable coefficients. First several coefficients have been calculated by Bordag and Vassilevich [14], Moss [61], Gilkey, Kirsten and Vassilevich [38].

It is very well known [1] that the conditions (4.11) are not the most general matching conditions which can be defined on a surface. In general, boundary values...
of a function and of its normal derivatives are related by a $2 \times 2$ transfer matrix:

\begin{equation}
0 = \left( \begin{array}{cc}
\nabla_{\nu}^+ + S^{++}, & S^+- \\
S^{-+}, & \nabla_{\nu}^- + S^{--}
\end{array} \right) \begin{pmatrix} \phi^+ \\ \phi^-
\end{pmatrix} \bigg|_\Sigma.
\end{equation}

Note, that the transfer conditions (4.13) do not assume identification of $\phi^+$ and $\phi^-$ on $\Sigma$. In other words, there is no ad hoc relation between the restrictions of the vector bundles $V^+|_\Sigma$ and $V^-|_\Sigma$. We can even consider the situation when we have $\dim V^+ \neq \dim V^-$, i.e. the fields on $M^-$ and $M^-$ can have different structures with respect to space-time and internal symmetries. $S^{\pm \pm}$ are some matrix valued functions on $\Sigma$ (one can even consider the case when they are differential operators).

The heat trace asymptotics for the transfer problem have been analysed by Gilkey, Kirsten and Vassilevich [39].

Note, that although the conditions (4.11) can be obtained from (4.13) as a limiting case, taking asymptotic expansion of the heat trace does not commute with this limit.

A rather interesting generalisation of the constructions considered in this section consists in lifting the assumption that the metric (or the leading symbol of the operator) is continuous across $\Sigma$. Physically this means that the speed of light jumps on the interface surface $\Sigma$, like in the case of a dielectric body immersed in the vacuum. This modification, of course, complicates the problem considerably. Therefore, except for a couple of particular case calculations by Bordag et al [12, 13] very little is known about behaviour of the spectral functions in this case.

In general, relations between dynamics in the volume and on the boundary or on an interface is of much interest. Examples include the boundary state dynamics in solid state physics and in strings, the "near horizon" dynamics in physics of black holes. Celebrated AdS/CFT correspondence principle [56, 48, 84] which states that certain correlation functions in quantum conformal field theory (CFT) on the boundary can be calculated by from classical supergravity theory in the "volume" of the Anti-de Sitter (AdS) manifold is also an example of such relations. In this respect people sometimes refer to the so-called holographic principle which is usually attributed to 't Hooft. Since no strict formulation of this principle exists, it is usually understood as any possibility to make statements about physics in the volume by looking at the boundary. Here we have to remember again the results on relations between spectral invariants of various various differential operators acting on a manifold and on its boundary reported on this workshop.

4.3. Supersymmetry. Symmetry principles are the guiding rules of contemporary theoretical physics. Therefore, for a long time physicists tried to find a symmetry which would involve both bosonic and fermionic fields. The crucial difference between bosons and fermion is that the former obey a commutator algebra, while the latter an anti-commutator one. This property introduces a natural grading both in the space of the fields and in the symmetry group. An object which has to replace the space time in the case of supersymmetric theories is the supermanifold (see [23] for details). Locally a supermanifold looks as a usual manifold with additional Grassmann coordinates $\vartheta^\alpha$ which satisfy the relation $\vartheta^\alpha \vartheta^\beta = -\vartheta^\beta \vartheta^\alpha$ yielding $(\vartheta^\alpha)^2 = 0$. Functions on the supermanifold ("superfields") are defined locally through a Taylor expansion in $\vartheta$:

\begin{equation}
\Phi(x, \vartheta) = \phi(x) + \phi_\alpha(x) \vartheta^\alpha + \phi_{\alpha\beta}(x) \vartheta^\alpha \vartheta^\beta + \ldots
\end{equation}
Since \((\partial^{\alpha})^{2} = 0\) the expansion contains a finite number of terms. One can also introduce an integration and a differential structure on supermanifolds. Consequently, one can define classical actions, classical field theories, and develop a quantisation of these theories. This procedure gives rise to some natural operators. As before, quantum effective action is defined by spectral functions of these operators. Of course, these problems can be addressed locally by using the expansions in the components (of the type (4.14)). However, a much nicer approach would consists in working directly on a supermanifold without referring to a particular (super)coordinate system. A modern survey on superanalysis can be found in the monograph by Khrennikov [51], which also contains a long list of open problems.

I conclude this short section by noting that the enormous interest to supersymmetry in theoretical physics is caused mostly by aesthetic reasons, and also because supersymmetric theories are mathematically better behaved and, sometimes, even perturbatively finite. So far there are no experimental evidences in favour of supersymmetry.

4.4. Non-commutative field theories. Recent years much attention was attracted to non-commutative field theories (see reviews by Douglas and Nekrasov [25] and by Szabo [75]). Initially non-commutativity appeared in the framework of the deformation quantisation approach (which means deformations of symplectic structures in the mathematical language) of Bayen et al [11] (for a historical overview see [85]). Modern interest to the topic has been boosted by applications to the solid state physics (quantum Hall effect) and to string theory (cf. sec. 4.1).

The most rigorous way to introduce a noncommutative manifold uses the \(C^{*}\)-algebras [20]. For our purposes a somewhat simplified approach will be enough. In this approach one replaces ordinary multiplication of the functions by the Groenewold–Moyal product:

\[
(4.15) \quad (f \star g)(x) = \exp\left(\frac{i}{2} \theta^{\mu\nu} \partial_{\mu} \partial_{\nu}\right) f(y) g(x) |_{y = x}.
\]

This product as it stays is defined for smooth functions only so that it should be understood through Fourier series. The formula (4.15) is defined with respect to a coordinate system. We shall consider a torus where a suitable global coordinate system exists. It is easy to see that

\[
(4.16) \quad x^\mu \star x^\nu - x^\nu \star x^\mu = i \theta^{\mu\nu}
\]

so that the associative product (4.15) reproduces the string commutator (4.8).

It seems natural (and it is also useful for non-commutative QFT) to study spectral properties of a generalisation of the Laplace type operator:

\[
D\phi = -(\delta^{\mu\nu} \partial_{\mu} \partial_{\nu} + a^{\mu} \partial_{\mu} + b) \star \phi.
\]

Asymptotics the heat trace for (4.17) were considered on a non-commutative torus by Vassilevich [81]. These results were extended to non-commutative \(\mathbb{R}^n\) by Gayral and Iochum [34]. The main result is amazingly simple. There exists a full power-law asymptotic expansion for the heat trace, and the coefficients are integrals of universal free polynomials of \(a, b\) and their derivatives evaluated with the non-commutative product. This also shows that the heat trace coefficients have the form a “typical” classical action for non-commutative field theories. In this respect, one remembers the so called spectral action principle of Chamseddine and Connes [19] which suggests to use the heat trace asymptotics on non-commutative
manifolds to construct classical actions (this procedure is in a sense “reverse” of the renormalization).

Here a warning is in order. One cannot expect that generalisations to the non-commutative case will always be straightforward. Some operators also appearing in the context of non-commutative QFT have quite unusual spectral properties found by Vassilevich and Yurov [82].

It does not make much sense to give a list of open problems since almost all problems at the interface of non-commutative geometry and spectral geometry are “open”. This seems to be a wonderful field of research.

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