Noether’s theorem, the stress-energy tensor and Hamiltonian constraints

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Abstract

Noether’s theorem is reviewed with a particular focus on an intermediate step between global and local
gauge and coordinate transformations, namely linear transformations. We rederive the well known result
that global symmetry leads to charge conservation (Noether’s first theorem), and show that linear symmetry
allows for the current to be expressed as a four divergence. Local symmetry leads to identical conservation
of the current and allows for the expression of the charge as two dimensional surface integral (Noether’s
second theorem). In the context of coordinate transformations, an additional step (Poincaré symmetry) is
of physical interest and leads to the definition of the symmetric Belinfante stress-energy tensor, which is
then shown to be identically zero in generally covariant first order theories. The intermediate step of linear
symmetry turns out to be important in general relativity when the customary first order Lagrangian is used,
which is covariant only under affine transformations. In addition, we derive explicitly the canonical stress-
energy tensor in second order theories in its identically conserved form. Finally, we analyze the relations
between the generators of local transformations, the corresponding currents and the Hamiltonian constraints.

1 Introduction

Our motivation originates in the following question: Why is it that in gravity theory, the stress-energy (pseudo)-
tensors usually presented in literature are always in a form such that they are identically divergence free, while
this is not the case for special relativistic theories? One could try to attribute this to the special role played by
the gravitational field, which is supposed to define the geometry of spacetime itself. But on the other hand, one
can also take a less geometric point of view and forget about metrical structures and geometry in general. We
are then left with a set of points $x^i$ (the spacetime manifold) and fields defined on this set, one of which be-
ing the gravitational field $g_{ik}$. From this viewpoint, there is no particular reason why the stress-energy tensor for
the gravitational field should have any essentially different properties from that of any other field.

The answer to the above question is given by Noether’s second theorem. While global symmetry of a
Lagrangian theory leads to a conservation law (first Noether theorem), invariance of the theory under the same,
but localized symmetry leads to identical conservation of the same current, in the sense that the charge can be
written in the form of a surface integral. The exact formulation of the theorem can be found in the correspond-
ing literature, in particular in Noether’s original article [1]. In this paper, we will review what it means concretely
in the case of gauge and in particular of coordinate transformations. The answer to our initial question is
provided by the fact that gravitational theories are covariant under general coordinate transformations, while
special relativistic theories are only Poincaré invariant.

Since the results presented in this paper were already known to Noether herself, there is no need to provide
a large reference list. Our aim is, on one hand, to present the results in a mathematically simple form (as
opposed to fiber bundle descriptions) for the cases of physical interest, and, on the other hand, to highlight the relevance of the different degrees of *locality* of the symmetry.

Our main focus lies on the stress-energy tensor and the conservation of energy and momentum, but in order to illustrate the procedure, and in particular the role of the *intermediate* step between global and local transformations, we start with a simple example of an internal gauge symmetry. In the first five sections of this paper, we assume that the Lagrangian $\mathcal{L}$ does not depend on second and higher derivatives of the fields. We refer to such theories as *first order* theories. Second order theories will be considered in section 6.

The paper is structured as follows. In the remaining part of the Introduction, we derive Noether’s theorem for first order non-abelian gauge theory. Then, in sections 2 to 4, we turn to coordinate transformations and the stress-energy tensor. We start with global translations and Poincaré symmetry in section 2, go over to affine symmetry in section 3, and finally to general covariance in section 4, where, in each step, the consequences of the symmetry on the conservation law for the canonical as well as for the Belinfante stress-energy tensor are discussed. In section 5, we apply those results to concrete theories, in particular to general relativity and to Einstein-Cartan theory. In section 6, we generalize the formalism to include theories based on Lagrangians containing second derivatives of the fields. Further, in section 7, we briefly analyze the relations between the Belinfante and the Hilbert (or metric) stress-energy of the matter fields. Finally, in section 8, we analyze the relations between the generators of gauge and coordinate transformations, the corresponding currents and the first class Hamiltonian constraints of the theory.

We begin by considering a Lagrangian $\mathcal{L}$ depending on a matter field $\psi$ and a gauge field $A^\alpha_i$, and assume that $\mathcal{L}$ is invariant under the following transformation

$$
\delta A^\alpha_i = \varepsilon^\alpha_i + c^\alpha_i A^\gamma_k \varepsilon^\beta_k, \quad \delta \psi = -i \varepsilon^\alpha \sigma_\alpha \psi.
$$

The variation of $\mathcal{L}$ then reads

$$
\delta \mathcal{L} = 0 = \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial \psi_i} \delta \psi_i + \frac{\partial \mathcal{L}}{\partial A^\alpha_k} \delta A^\alpha_k + \frac{\partial \mathcal{L}}{\partial A^\alpha_{k,i}} \delta A^\alpha_{k,i}.
$$

Taking into account the field equations $\partial \mathcal{L}/\partial \psi = (\partial \mathcal{L}/\partial \psi_i)_i$ and $\partial \mathcal{L}/\partial A^\alpha_k = (\partial \mathcal{L}/\partial A^\alpha_{k,i})_i$, we find

$$
0 = \left[ -i \frac{\partial \mathcal{L}}{\partial \psi_i} \sigma_\alpha \psi + \frac{\partial \mathcal{L}}{\partial A^\gamma_k} \varepsilon^\beta_k \right] \varepsilon^\alpha_i + \left[ -i \frac{\partial \mathcal{L}}{\partial \psi_i} \sigma_\alpha \psi + \frac{\partial \mathcal{L}}{\partial A^\gamma_k} c^\alpha_i \varepsilon^\beta_k \right] + \left( \frac{\partial \mathcal{L}}{\partial A^\alpha_{k,i}} \right)_i \varepsilon^\alpha_i + \frac{\partial \mathcal{L}}{\partial A^\alpha_{k,i}} \varepsilon^\alpha_{i,k}.
$$

Let us define the current by $J^\alpha_i = -i \frac{\partial \mathcal{L}}{\partial \psi_i} \sigma_\alpha \psi + \frac{\partial \mathcal{L}}{\partial A^\gamma_k} c^\alpha_i \varepsilon^\beta_k$. In a first step, we assume invariance of $\mathcal{L}$ under global transformations, i.e., $\varepsilon^\alpha_i = 0$. We then find the well known conservation law

$$
J^\alpha_{i,i} = 0.
$$

Next, we assume that the Lagrangian is invariant (in addition) under linear transformations, i.e., $\varepsilon^\alpha = \varepsilon^\alpha(x)$, with $\varepsilon^\alpha_{i,k} = 0$. This leads to the relation

$$
J^\alpha_i = -\left( \frac{\partial \mathcal{L}}{\partial A^\alpha_{i,k}} \right)_k,
$$

which expresses the current in the form of a divergence. In a last step, we assume local gauge invariance (i.e., general $\varepsilon^\alpha(x)$), which gives us a third relation

$$
\frac{\partial \mathcal{L}}{\partial A^\alpha_{i,k}} = 0.
$$

In order to shorten our expressions, we omit the terms in $\bar{\psi}$, which are similar to those in $\psi$.  

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To summarize, global invariance leads to the conservation law (2) for the current (and thus to charge conservation). Linear invariance allows us to write the current in the form (3), and finally, local symmetry leads to (4), which tells us that the expression (3) is the divergence of an antisymmetric quantity, which as such has an identically vanishing divergence. Thus, in a locally invariant theory, the current can be written in a form such that it is identically conserved. In particular, this means that the charge (integration over a spacelike hypersurface)

$$Q_\alpha = \int J_\alpha^i d\sigma_i = \int J_\alpha^{0} d^3x$$

(5)
can be written in the form (greek indices from the middle of the alphabet, $\mu, \nu \ldots$ refer to the spatial components and run from 1 to 3)

$$Q_\alpha = -\oint \frac{\partial L}{\partial A_{\alpha,k}^{i}} d\sigma_{ik} = -\oint \frac{\partial L}{\partial A_{\alpha,\mu}^{0}} d^2\sigma_{\mu},$$

(6)
i.e., in the form of a two dimensional surface integral. Note that this would not be possible without the relation (4). In the abelian case, equation (6) is nothing but the well known expression of the charge in form of an integral of $\vec{E}$ over a closed surface.

Similar results will be derived in the next sections for coordinate transformations.

2 Translations and Poincaré transformations

We consider now a generic Lagrangian $L = L(\varphi, \varphi_{,i})$, where $\varphi$ denotes collectively all the fields (including the gravitational field, whenever present). Under a global coordinate transformation $x^i \rightarrow x^i - \xi^i$, with constant $\xi^i$, all fields (scalar, vector, tensor, spinor) transform as

$$\delta \varphi = \varphi_{,i} \xi^i,$$

(7)

where $\delta \varphi$ denotes the change of $\varphi$ at the point with the same coordinates, i.e., $\delta \varphi(x) = \varphi'(x) - \varphi(x)$. If we assume invariance of the action under global coordinate translations, the Lagrangian must transform as scalar and we find

$$\delta L = L_{,i} \xi^i = \frac{\partial L}{\partial \varphi} \delta \varphi + \frac{\partial L}{\partial \varphi_{,i}} \delta \varphi_{,i}$$

which leads to the conservation law

$$\tau_{k,i} = 0,$$

(8)

with the canonical stress-energy tensor

$$\tau_{k}^{i} \equiv \frac{\partial L}{\partial \varphi_{,i}} \varphi_{,k} - \delta_{k}^{i} L.$$ 

(9)

Note that, although $L$ will, in generally covariant theories, be a scalar density rather than a scalar, we will nevertheless stick to the definition (9) (thus including the factor $\sqrt{-g}$ into $\tau_{k}^{i}$).

Thus, our first result is that global translational symmetry leads to a conserved energy-stress tensor. It is not hard to see that every Lagrangian that does not depend explicitly on $x^i$ possesses that symmetry.

Before we turn to linear transformations, we consider Poincaré transformations, which are of particular physical interest. The fields now transform as

$$\delta \varphi = \xi^{i} \varphi_{,i} + \frac{1}{2} \varepsilon^{ik} (S \varphi)_{ki},$$

(10)
with $\xi^i = a^i + \varepsilon^i_k x^k$ (constant $a^i$ and $\varepsilon^i_k$) and $\varepsilon^{ik} = \varepsilon^i_j \eta^{jk} = -\varepsilon^{ki}$. By $(S\varphi)_{ki}$, we denote the action of the Lorentz group on the field in question. Explicitely, we have

\[(S\varphi)_{ki} = 0 \quad \text{scalar} \quad (11)\]
\[(SA_l)_{ki} = \eta_{kl} A_l - \eta_{il} A_k \quad \text{vector} \quad (12)\]
\[(S h_{lm})_{ki} = \eta_{kl} h_{im} - \eta_{il} h_{km} + \eta_{km} h_{il} - \eta_{im} h_{kl} \quad \text{tensor} \quad (13)\]

and similar for contravariant or mixed tensors $A^i, h^{ik}, h^i_k$. Note that, since we are ultimately interested in generally covariant theories, we have to assume that spinors (spin 1/2) are described by fields transforming as scalars under Lorentz transformations² (since else, the generalization to general linear and diffeomorphism transformations would not be possible with finite dimensional representations). Such fields differ from true scalar fields (spin 0) by their behavior under local Lorentz gauge transformation, unrelated to the coordinate transformations we consider here.

Assuming Poincaré invariance of the action, and thus $\delta \mathcal{L} = \mathcal{L},_{\xi} \xi$, we find after some simple manipulations, apart from (8), the additional relation

\[(\tau^{ik} + \frac{1}{2} S^{mik},_{m}) \varepsilon_{ik} = 0, \quad (14)\]

Thus, since $\varepsilon^{ik}$ is arbitrary (but antisymmetric), we get

\[\tau^{[ik]} + \frac{1}{2} S^{mik},_{m} = 0. \quad (15)\]

This result can be compared with (3), but it is less strong since it determines only the form of the antisymmetric part or $\tau^{ik}$. However, (15) can be used for a different purpose, namely the symmetrization of $\tau^{ik}$. Indeed, we can define the so-called Belinfante tensor (see [2] and [3])

\[T^{ik} \equiv \tau^{ik} + \frac{1}{2} [S^{k|m} - S^{m|k} - S^{kmi}],_{m}, \quad (16)\]

which obviously satisfies $T^{ik},_{i} = 0$, since the expression in brackets is antisymmetric in $im$. In other words, $T^{ik}$ differs from $\tau^{ik}$ only by a so-called relocalization term of the form $C^{imk},_{m}$, with $C^{imk} = -C^{mik}$, i.e., a term whose divergence $C^{imk},_{m,i}$ vanishes identically and which leads only to two dimensional surface terms in the field momentum. Also, using (15), it follows immediately that we have

\[T^{ik} = T^{ki}. \quad (17)\]

Therefore, we can also define a spin current density in the form $\sigma^{lki} = T^{il} x^{k} - T^{ik} x^{l}$, satisfying $\sigma^{lki},_{i} = 0$, which is not possible with the asymmetric tensor $\tau^{ik}$.

We will further discuss the Belinfante tensor later on.

²Spin 3/2 fields $\psi_l$ carry an additional vector index and transform according to (12).
3 Linear transformations

We now replace the Lorentz group by general linear transformations, considering Lagrangians with an affine symmetry. The fields now transform as

\[ \delta \varphi = \xi^i \varphi, + \frac{1}{2} \varepsilon^i_k (\sigma \varphi)^k, \quad (18) \]

with \( \xi^i = a^i + \varepsilon^i_k x^k \), with general (constant) \( \varepsilon^i_k \), and where the action of the general linear group on the fields is given by

\[
\begin{align*}
(\sigma \varphi)^k_i &= 0 \quad \text{scalar} \\ (\sigma A_l)^k_i &= 2 \delta^k_l A_i \quad \text{vector} \\ (\sigma h_{lm})^k_i &= 2 (\delta^k_l h_{im} + \delta^k_m h_{il}) \quad \text{tensor}.
\end{align*}
\]

(19) (20) (21)

Let us also define

\[ L_{mk}^i = \frac{\partial \mathcal{L}}{\partial (\sigma \varphi)^k_m} (\sigma \varphi)^i_k. \quad (22) \]

Assuming invariance of the action means that \( \mathcal{L} \) transforms as scalar density, and thus, \( \delta \mathcal{L} = (\mathcal{L} \xi^i)_i = \mathcal{L}, \xi^i + \mathcal{L} \xi^i. \) For the rest, the argument goes just as in the case of the Lorentz symmetry, with the only difference that we end up with 16 (instead of 6) independent equations. The result is

\[ \tau^i_k = - \frac{1}{2} L_{mk}^i (23) \]

This is now in full analogy to equation (3), in the sense that invariance under linear transformations (together with the global translations) dictates the form of the conserved current \( \tau^i_k \), which is again in the form of a divergence.

In contrast to the Lorentz case, where the Minkowski metric \( \eta_{ik} \) had been introduced by hand, no metric is needed for the above arguments. However, in order to make contact with the results obtained for the Poincaré group, let us define

\[ L_{mi}^k = \eta^{kl} L_{mi}^l. \quad (24) \]

Then, we have obviously \( S_{mk}^i = L_{mek}^i \), and from the expressions (16) and (22), we can derive the following expression for the symmetric Belinfante tensor

\[ T^{ik} = - \frac{1}{4} [L^{mik} - L^{ikm} + L^{mk} + L^{mi} - L^{kim}],m \quad (25) \]

A few remarks are in order at this point. First, if \( \tau^i_k \) has the form of a relocalization term, i.e., if \( L_{mk}^i \) in (23) is antisymmetric in \( im \), then it follows from (23) that \( T^{ik} = 0 \) identically. Second, it should be noted that (23) and (24) are very strong relations. Indeed, for a scalar field, e.g., they lead (in view of (19)) immediately to \( \tau^i_k = 0 \). The reason, however, is also obvious: It is not possible to construct a theory that is invariant under general linear transformations only with scalar fields. For gauge fields (i.e., vector fields), we find

\[ \tau^i_k = - \left( \frac{\partial \mathcal{L}}{\partial (\partial A^i_m)} A^k_m, m \right). \quad (26) \]

We see that \( \tau^i_k = 0 \) is identically satisfied if \( \partial \mathcal{L}/\partial A^i_{m,m} \) is antisymmetric in \( im \). This is indeed the case in the physically relevant situations, where the derivatives of the gauge fields enter only via the Yang-Mills tensor \( F^\alpha_{ik} \). However, there could be exceptions, in particular concerning derivative couplings, e.g., of the tetrad field \( e^i_k \) in gravitational theories without independent connection. This turns out not to be the case though, as we will now show in general.
4 General covariance

From the explicit form (23), we see that only vector and tensor fields give explicit contributions to \( \tau^i_k \). Thus, the question is what kind of restriction do we get on the form of those contributions from general covariance of the theory? Well, this is not difficult to find out. Under general coordinate transformations \( x^i \rightarrow x^i - \xi^i \), the vector and tensor fields transform as

\[
\delta A_l = \xi^i A_{l,i} + \xi_{i,l} A_l, \quad (27)
\]

\[
\delta g_{lm} = \xi^i g_{lm,i} + \xi_{i,m} g_{li} + \xi_{i,l} g_{mi}, \quad (28)
\]

while for the Lagrangian, we must have

\[
\delta L = (\xi^i L)_{,i}. \quad \text{(29)}
\]

The same result is found for the case of tensor fields, but this is not really of interest, since there is no generally covariant action for a tensor field containing only first derivatives. In any case, for the physically important case, namely the vector fields, we find

\[
\frac{\partial L}{\partial A^i_{(t,m)}} A_k = 0,
\]

or, in other words,

\[
L^m_{ik} = -L^m_{ki}. \quad (29)
\]

Moreover, as mentioned before, a consequence of (29) is that the \( \text{Belinfante tensor} \) vanishes identically (a result that has been found in [4]).

As a result, we have shown that in generally covariant theories, the canonical stress-energy tensor can be written in the form of a relocalization term, and thus, the momentum vector in form of a two dimensional surface integral (see also [5]).

\[
P_k = \int \tau^i_k d\sigma_i = \int \tau^0_k d^3x
\]

can be written as a two dimensional surface integral

\[
P_k = \frac{1}{2} \oint L^m_{ik} d\sigma_{mi} = -\frac{1}{2} \oint L^\mu_{ik} d^2\sigma_{\mu}. \quad (30)
\]

Moreover, as mentioned before, a consequence of (29) is that the Belinfante tensor vanishes identically (a result that has been found in [4]).

As a result, we have shown that in generally covariant theories, the canonical stress-energy tensor can be written in the form of a relocalization term, and thus, the momentum vector in form of a two dimensional surface integral (see also [5]), while the Belinfante tensor is identically zero. This holds for theories based on Lagrangians containing only first derivatives of the fields.

5 Applications

5.1 Special relativity

The physically relevant theories that are generally covariant contain necessarily the gravitational field. Nevertheless, in order to illustrate our formalism, let us consider the following Lagrangian which is independent of any metrical background

\[
\mathcal{L} = \varepsilon^{iklm} A_{i,k} A_{l,m}. \quad \text{(30)}
\]

Those relations are simply the Lie derivatives of the corresponding fields.
This Lagrangian is a total divergence, and therefore, the field equations are identically satisfied for any $A_i$. The stress-energy tensor, according to (9) is of the form

$$\tau^i_k = 2\epsilon^{ijpq} A_{q,p} A_{i,k} - \delta^i_k \epsilon^{rsqp} A_{r,s} A_{p,q},$$

while from (20), we find

$$\tau^i_k = 2\epsilon^{ijpq} A_{q,p} A_{k,i},$$

which is obviously identically conserved. Also, the difference of both expressions, $\epsilon^{imnp} F_{km} F_{pq} - \frac{1}{2} \delta^i_k \epsilon^{mnp} F_{lm} F_{pq}$ is easily shown to be zero. A similar calculation shows that the Belinfante tensor $T^{ik}$ is zero.

### 5.2 Einstein-Cartan theory

Einstein-Cartan theory is based on a tetrad field $e^a_i$ and a Lorentz connection $\Gamma^{ab}_{,c}$ (Latin indices from the beginning of the alphabet $a, b, c \ldots$ refer to tangent space, with Minkowski metric $\eta_{ab} = \text{diag}(1, -1, -1, -1)$.) For completeness, we consider the Einstein-Cartan-Dirac-Maxwell Lagrangian

$$\mathcal{L} = \frac{e}{2} e^a_i e^b_k R_{ik}^{ab} - \frac{e}{4} F_{ik} F_{ik} + e^i_j \left( \bar{\psi} \gamma^m D_m \psi - D_m \bar{\psi} \gamma^m \psi \right) - m \bar{\psi} \psi,$$

with $e = \text{det} e^a_i$, $R_{ik}^{ab} = \Gamma_{ik}^{ab} - \Gamma^{ab}_{,i,k} + \Gamma^a_{ci} \Gamma^{cb}_{,k} - \Gamma^a_{ck} \Gamma^{cb}_{,i}$, $F_{ik} = A_{k,i} - A_{i,k}$ and $D_i \psi = \partial_i \psi - i \bar{\psi} \sigma_{ab} \psi A^a$. From (20) and (21), we find

$$\tau^i_k = -(e e^a_i e^b_k \Gamma_{ik}^{ab})_{,m} - (e F^{im} A_k)_{,m}.$$

Obviously $\tau^i_{k,i} = 0$ identically, and $T^{ik} = 0$, according to our general theorem. The expression is easily generalized to non-abelian gauge fields $A^a_i$.

In order to evaluate the momentum $P_k$ in (30), it is useful to make a few general considerations. First, it is clear that, in the presence of radiation, we will, in general, not find a finite expression when the surface of integration is extended to spatial infinity. On the other hand, if we assume that the leading order contributions have the behavior $A_i = O(\frac{1}{r})$, and thus $F_{ik} = O(\frac{1}{r^2})$, then it is clear that the second term in (32) does not contribute to $P_k$.

The same holds for non-abelian gauge fields, and in particular for $\Gamma^{ab}$ itself, if we include in $\mathcal{L}$ terms quadratic in the curvature $R_{ik}^{ab}$. The only exception to this is the tetrad field, which can be assumed to behave like $e^a_i = \delta^a_i + O(\frac{1}{r})$. Thus, quite generally, the only contributions that enter explicitly the momentum $P_k$ are those stemming from the Einstein-Cartan-Lagrangian (first term in (22)), and eventually from additional terms in the Lagrangian, quadratic in the torsion tensor $T_{ik}^{a} = e^a_{k,i} + \Gamma^a_{bi} e^b_k - e^a_{i,k} - \Gamma^a_{bk} e^b_k$. Such terms, however, will in general also modify the Newtonian limit of the gravitational theory.

Note that for the same reasons, relocalization terms are usually considered not to modify the momentum $P_k$, and are therefore used as a tool to modify the stress-energy tensor according to our will, without changing the physically important quantity, which is $P_k$. If this were generally true, however, then in generally covariant theories, the momentum would always be zero, since we can always write the stress-energy tensor in the form of a relocalization term. It is therefore very important that, as we have argued, the linear Einstein-Cartan term $eR$ provides an exception to this. On the other hand, for the same reason, it is quite a questionable procedure, in the framework of those theories, to modify the stress-energy tensor by relocalization terms, and, e.g., define the Belinfante tensor. Those procedures were invented in special relativistic theory exactly because they are supposed not to modify the momentum, and not simply to overcome our dissatisfaction with an asymmetric tensor. After all, as we have shown in the introduction, the electric current density too can be written in the form of a relocalization term, i.e., $j^i = C^i_{,k}$, with antisymmetric $C^{ik}$. However, would anyone ever come up...
with the idea to add an additional relocalization term because he does not like the specific form of \( j^i \)? And even if he did, he would certainly take care that at least the charge itself is not modified by this procedure.

Let us return to the expression (32). Recall that the connection can be split into a torsionless part and the contortion tensor \( \Gamma^i_{ab} = \hat{\Gamma}^i_{ab} + K^i_{ab} \), where, in Einstein-Cartan theory, the contortion tensor is directly expressed in terms of the spin of the spinor field. In the absence of spinor fields, we have \( K^i_{ab} = 0 \), and we can evaluate \( \hat{\Gamma}^i_{ab} \) using

\[
e_a^i, k + \hat{\Gamma}^a_{bb} e_i^b = e_i^a \hat{\Gamma}^i_{ab},
\]

where \( \Gamma^i_{ab} \) are the Christoffel symbols. For the Schwarzschild metric

\[
ds^2 = (1 - \frac{M}{r})dt^2 - (1 - \frac{M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

and using a diagonal tetrad, we find

\[
P^k = 2\pi M \delta^0_k,
\]

i.e., the energy is just one half of the mass, \( P^0 = m/2 \) \((M = 2km, \text{ with } 8\pi k = 1 \text{ in our units})\). The same result was found in the preprint version of [3]. Note that the same result emerges from the Reisner-Nordstroem solution (since only the leading order term of \( g^0_0 = -1/g_{rr} = 1 - \frac{M}{r} + \frac{q^2}{r^2} \) contributes to the surface integrals at spatial infinity).

Since \( \tau^i_k \) is asymmetric, and \( T^{ik} = 0 \) identically, we cannot formulate a conservation law for the angular momentum based on the stress-energy tensor. However, we can exploit the Lorentz gauge symmetry of the theory, i.e., the invariance of the Lagrangian under

\[
\delta e^a_i = \varepsilon^a_b e^b_i, \quad \delta \Gamma^i_{ab} = -D_i \varepsilon^a_b, \quad \delta \psi = -\frac{i}{4} \varepsilon^a_{bc} \sigma^{ab} \psi,
\]

where \( \varepsilon^{ab} = -\varepsilon^{ba} \) and \( D_i \varepsilon^{ab} = \varepsilon^{ab}_i + \Gamma^a_{ci} \varepsilon^{cb} + \Gamma^b_{ci} \varepsilon^{ac} \). Let us define the following current

\[
J^k_{ab} = \frac{1}{2} \frac{\partial L}{\partial e^a_i, k} e^b_i - \frac{1}{2} \frac{\partial L}{\partial e^b_i, k} e^a_i - \frac{i}{4} \frac{\partial L}{\partial \psi, k} \sigma_{ab} \psi - \frac{\partial L}{\partial \Gamma^c_{bi, k}} \Gamma^c_{ai} - \frac{\partial L}{\partial \Gamma^c_{ai, k}} \Gamma^c_{bi},
\]

which is antisymmetric in \( ab \) and can be interpreted as angular momentum density current. Performing the same steps as in the introduction, we find from global symmetry (\( \varepsilon^{ab} \) constant)

\[
J^k_{ab, k} = 0.
\]

and from linear symmetry (\( \varepsilon^{ab, i, k} = 0 \)) the explicit form

\[
J^k_{ab} = \left( \frac{\partial L}{\partial \Gamma^a_{bi, k}} \right)_{i},
\]

and finally, from local symmetry that the expression in parentheses is antisymmetric in \( ki \), such that the current is identically conserved and the corresponding charge can be put into the form of a two dimensional surface integral. Note that for the particular case of Einstein-Cartan theory, the first two terms in (34) are automatically absent.

Although we focused, in this section, mainly on Einstein-Cartan theory, it is obvious that the results are also valid for any candidate of Poincaré gauge theory, i.e., for any theory with an independent Lorentz connection and with a Lagrangian at most quadratic in curvature and torsion. We refer to [6], where the Hamiltonian analysis of those theories has been carried out. The analysis is easily extended to metric affine theories (with an independent general linear connection, see [7]), provided one finds a way to couple the spinor fields to the connection, see, e.g., [8].

Finally, we should also mention that in some sense, the Lorentz gauge group is more closely connected to the coordinate transformation group than conventional internal gauge groups. In order to perform Wigner's
classification of elementary particles, you need to analyze the behavior of the fields under both the diffeomorphism and the Lorentz group. The link is provided by the fact that in the flat limit, \( e^a_i = \delta^a_i \), and the residual transformation freedom is then, apart from global coordinate translations, a global Lorentz gauge rotation and a simultaneous global Lorentz coordinate transformation (with the same parameters), such that \( e^a_i = \delta^a_i \) remains unchanged. In other words, the Poincaré transformation group of special relativity (on which the particle classification is based) emerges in the flat limit as a combination of both the Lorentz gauge group and the diffeomorphism group. As a result, it is not unnatural to consider combinations of both groups right from the start.

That is, instead of the conventional Lie derivatives \( \delta \varphi = \partial \varphi - i \xi \), \( \frac{1}{4} \xi_{[i} \varphi_{k]} \) related to pure diffeomorphism invariance, one can consider modified Lie derivatives under which the theory is still invariant, as a result of the additional Lorentz gauge invariance. In this way, one can derive in quite a natural way alternative expressions for the conserved energy-momentum currents. The details of this procedure have been recently worked out in [9].

5.3 Tetrad gravity

A more conservative way to incorporate spinor fields into general relativity is by simply replacing the metric tensor with the tetrad field, without introducing an independent connection. The gravitational field then couples to the spinor field via \( \hat{\nabla}^i \psi = \partial^i \psi - \frac{i}{4} \hat{\Gamma}^{ab}_{ik} \psi \), where \( \hat{\Gamma}^{ab}_{ik} \) is a function of the tetrad field and its derivatives that can be evaluated from \( e^a_{ik} + \Gamma^a_{bk} e^b_k = e^a_i \). The free gravitational Lagrangian is taken in the form

\[
\mathcal{L} = -\frac{1}{2} e^{ijkl} \tau_{ijkl} - \frac{1}{2} \tau^{ijkl} \Gamma_{[ijk]} + \frac{1}{2} \tau^{im} \tau^{ijl},
\]

where \( \tau^{ijkl} = e^a_{ik} - e^a_{ij} \). This Lagrangian coincides up to a surface term with the Hilbert-Einstein Lagrangian, but it has the additional feature that it does not contain second derivatives of the tetrad field, and it is generally covariant under coordinate transformations. Note also that \( \mathcal{L} \) is invariant under (local) Lorentz rotations \( \delta e^a_i = \varepsilon^a_b e^b_i \). For this Lagrangian, we find

\[
L^{mi} = e^{\tau_{mi}} - \tau^{im} - \delta^i_k \tau^{m}{}_{k} - \delta^m_{k} \tau^{i}{}_{l},
\]

which is antisymmetric in \( im \), and thus leads an identically conserved \( \tau^i_k \) and to \( T^{ik} = 0 \) for the Belinfante tensor.

Note that, as a result of the derivative couplings, in the presence of spinor fields (with the same Dirac Lagrangian as in (31), where \( \nabla^i \psi \) is replaced by \( \hat{\nabla}^i \psi \)) we find an additional term in the form

\[
L^{mi(D)} = 4[\sigma^{mi} - \sigma^{im}] + \sigma^{im},
\]

where \( \sigma_{ab}^{im} = \frac{\partial L}{\partial (\tau^{im})_m} \) is the so-called spin density of the spinor field. (This is not a conserved quantity, though.) Again, \( L^{mi(D)} \) leads to a surface term. (There can also be additional terms, from gauge vector fields, in the form of the second term in (32) which, as we have argued before, do not contribute explicitly to the momentum.)

The theory has a local Lorentz symmetry

\[
\delta e^a_i = \varepsilon^a_b e^b_i, \quad \delta \psi = -\frac{i}{4} \varepsilon^{ab} \sigma_{ab} \psi,
\]

which could eventually be used to define an angular momentum density for the system. However, from the above symmetry, we find relations formally identical to (33), (35) and (36), without the terms involving \( \Gamma^{ab} \).
In particular therefore, from (36), we find that the current vanishes identically. Thus, in this theory, it is not possible to formulate a conservation law for angular momentum, neither from the stress-energy tensor, nor using the local Lorentz symmetry. Note however that you can use the vanishing of the expression (34) in order to simplify the stress-energy tensor obtained from (35) and (39), by observing that the second term in (34) is again the spin density $\sigma_{ab}^k$. We then find $L^{mk} = -2\sigma^{km}$ (for the total expression, (35) plus (36)), and therefore, from (15), $\tau^{[ik]} = \sigma^{km} \delta_{ik}$, showing that in the absence of spinor fields, $\tau^{ik}$ is automatically symmetric.

5.4 General relativity

Finally, we consider classical general relativity. Since we have restricted our formalism to Lagrangians that do not contain second derivatives of the fields, we start with the Lagrangian

$$L = -\frac{1}{2} \sqrt{-\eta} \left[ g^{ik} (\Gamma^m_{im} \Gamma^m_{kl} - \Gamma^m_{ik} \Gamma^m_{lm}) \right],$$

which is equivalent to the Hilbert-Einstein Lagrangian $-\frac{1}{2} \sqrt{-\eta} R$. Apart from $g_{ik}$, there might be gauge vectors $A^i_j$ and scalar fields $\varphi$, which do not contribute explicitly to $P_k$ (and do not appear at all in the Belinfante tensor $T^{ik}$). We assume that there are no derivative couplings of $g_{ik}$ to any other fields, as is indeed the case with gauge fields and minimally coupled scalars. (Spinors cannot be described within a purely metric formalism.) After some algebra, we derive from (41) the following expression

$$L^{mk} = \sqrt{-\eta} \left[ g^{ik} \delta^l_k - g^{kl} \delta^m_i + g^{m}q_{iq}g^{ik} - g^{ik}g_{iq}g^{lm} \right] + \left[ \sqrt{-\eta} \left( -2g^{im}\delta^k_i + g^{ik}\delta^m_i + g^{km}\delta^l_i \right) \right] \delta_{ik}.$$  

The expression in the first row is antisymmetric in $mk$ and is thus not only identically conserved, but moreover, it does not contribute to the Belinfante tensor $T^{ik}$, as we have established previously. The expression in the second row, however, is not antisymmetric in $mk$. This is because $L$ is not covariant under general coordinate transformations. It is not hard to see that nevertheless, the divergence of this expression vanishes identically.

The expression is in the form of what we will call in the next section a second order relocalization term. Also, since $L$ is covariant under linear transformations, the relation $T^{ik} = -\frac{1}{2} L^{mk}_{i,k,m}$ is still valid. Note, however, that in order to evaluate the total canonical stress-energy tensor, you have to add the contributions from the vector fields to $L^{mk}_{i,k,m}$. (Although they will in general not contribute to $P_k$ anyway.)

Finally, since $L^{mk}$ is not antisymmetric in $mk$, the Belinfante tensor $T^{ik}$ does not identically vanish, and indeed, we find from (25)

$$T^{ik} = \frac{1}{2} \left[ \sqrt{-\eta} \left( \eta^{ik} g^{lm} - g^{im} \eta^{kl} - g^{km} \eta^{li} + g^{ik} \eta^{lm} \right) \right] \delta_{ik},$$

This is exactly the form derived in (3) by the direct application of the Belinfante relation (10) to the canonical tensor (and a subsequent use of the Einstein field equations) and the tensor $T^{ik}$ is known as Papapetrou tensor. Note that matter fields (vector and scalar) do not contribute to $T^{ik}$, since they lead to antisymmetric (and zero) contributions in $L^{mk}_{i,k,m}$, and therefore $T^{ik}$ represents the total stress-energy tensor of the system.

It is interesting to remark that $T^{ik}$, although apparently not written in the form of a relocalization term (see, however, section 6), is identically conserved and it is also not hard to show that the momentum can again be written as a two dimensional surface integral, although two steps have to be performed to achieve this. (In the integral over $T^{ik}$ from (34), split first one of the indices $l$ or $m$ into time and space components, and then the other one. The term with both $l = m = 0$ does not contribute, while the remaining terms lead to surface
integrals.) It turns out (see [3]) that the same Belinfante tensor emerges from the full (covariant) Hilbert-Einstein Lagrangian. (Not the same canonical tensor $\tau^i_k$, though.) This shows that the statement given at the end of section 4 (i.e., that in generally covariant theories, the Belinfante tensor vanishes identically) does not generalize in its full extend to generally covariant theories with second derivatives in the Lagrangian. (However, there is a generalization of that statement, in the sense that the stress-energy tensor $\tau^i_k$ is still identically conserved and the momentum can be expressed in terms of two dimensional surface integrals, similar as is the case with $T^{ik}$ here. We will show this in section 6.)

We have evaluated the momentum emerging from $\tau^i_k$ and $T^{ik}$ and find $P_k = 4\pi M \delta^0_k = m \delta^0_k$ for the Schwarzschild metric in both cases, which is the double value of that found in Einstein-Cartan theory. This is in accordance with the results found in [3], where in addition, the tensor $T^{ik}$ has been shown to be equivalent, as far as the leading order contributions are concerned, to the corresponding Landau-Lifshitz tensor and the Weinberg tensor. (Note that only the leading order contributions contribute to the surface integrals at infinity anyway.)

The Belinfante tensor being symmetric, we can formulate a conservation law for the angular momentum density $T^{ik}x^m - T^{im}x^k$, which would not be possible from $\tau^i_k$ or from any other line of argumentation (as was the case in Einstein-Cartan theory).

As a final remark, it is important to have in mind that the fact that the stress-energy tensor (both $\tau^i_k$ and $T^{ik}$) is identically conserved can not be derived merely from the symmetry of $L$ under affine transformations, but it emerges here rather as an additional, unexpected result. Otherwise stated, not every Lagrangian covariant only under affine transformations leads to an identically conserved current. The specific Lagrangian considered here, however, has the additional property that it is equal, up to a four divergence, to a generally covariant Lagrangian, albeit one of second order. As a result, under general coordinate transformations $\delta x^i = \xi^i$, the Lagrangian transforms as $\delta L = (\xi^i L)_{,i} + \Phi^{,i}$, i.e., it picks up an additional four divergence. Quite obviously, such a transformation behavior is sufficient for the field equations to be generally covariant.

5.5 Discussion

Summarizing our results, we found that in Einstein-Cartan theory, the canonical tensor is identically conserved, while the Belinfante tensor vanishes. Nevertheless, it is possible to establish a conservation law for angular momentum, based on the local Lorentz gauge invariance of the theory. Thus, in a sense, there is no need for a symmetrized stress-energy tensor. The separation of the angular momentum from the stress-energy tensor (which, after all, is the current corresponding to translational invariance) seems to be satisfying also from a logical point of view, if we recall that the spin structure of the spinor field was completely separated from the coordinate transformations (i.e., $\psi$ is treated as a scalar field), and was transferred to a tangent space with an inherent Lorentz symmetry. Thus, it should also be that same Lorentz symmetry that is responsible for the angular momentum conservation. (The argument is not absolute, though, since after all, the angular momentum is not entirely given in terms of intrinsic spin.)

On the other hand, in general relativity, we have a non-vanishing symmetric Belinfante tensor, and we can therefore formulate an angular momentum conservation law based on this tensor. Nevertheless, we see two problems with this procedure. First, in the context of a generally covariant theory (or at least, covariant under the general linear group, in the first order theory), we arbitrarily pick out a specific subgroup, namely the Lorentz group, and modify the canonical stress-energy tensor, which was already in the form of a relocalization term, by a relocalization term (of the same order of magnitude) such that the resulting tensor allows for the formulation of the conservation laws corresponding to that subgroup, namely angular momentum conservation. This, in our opinion, is not quite in the spirit of general relativity, which is based on the full diffeomorphism invariance, and has no inherent preferred subgroups. (Again, the argument is not absolute, since nevertheless,
the Poincaré group (or eventually the de Sitter group, in presence of a cosmological constant) emerges as
ground state symmetry of the theory, if the correct signature of the metric is assumed a priori.) Second, the
metric framework of general relativity does not allow for the presence of spinor fields. Therefore, as soon as
we deal with spinor fields, we have to go over to the tetrad formulation, which has therefore to be considered
as more fundamental. However, as we have established, the tetrad formulation leads to a vanishing Belinfante
tensor, and there is no obvious way to define a conserved angular momentum current. Therefore, the apparent
success of the Belinfante tensor in general relativity seems to be only a coincidence that does not generalize to
the more fundamental formulation of the theory. And most importantly, the absence of a conserved angular
momentum in the tetrad formulation of general relativity provides an argument in favor of Einstein-Cartan
theory with independent Lorentz connection.

Our opinion is therefore that the use of the Belinfante symmetrization procedure should be confined to
the purpose it was initially designed for, namely to special relativistic theories, where the Poincaré symmetry
is an inherent ingredient right from the start, and where the integrated momentum is not influenced by the
relocalization terms as a result of the asymptotic behavior of the fields in conventional theories. Both of those
requirements are violated in general relativity.

Nevertheless, in the framework of general relativity, the tensor (43) can certainly have its usefulness, if
interpreted correctly and most importantly, if used consistently. It represents, after all, a conserved current
and is, in this aspect, no different from the Weinberg or the Landau-Lifshitz tensor (see 3). But it is also
clear that any expression of the form (43), with \( g^{lm} \) replaced with any other symmetric tensor, \( \eta^{lm} \) replaced
with any other symmetric tensor, and \( \sqrt{-g} \) replaced with any other function, is identically conserved too, is
symmetric too, and is also related to the canonical tensor by a relocalization term (of second order, see next
section). Thus, modifying the canonical tensor by relocalization terms and requiring Poincaré invariance does
in no way fix the form of the stress-energy tensor.

6 Second order field theory

For the sake of completeness, we now extend our analysis to theories based on Lagrangians containing second
derivatives of the fields (we refer to such theories as second order). The second Noether theorem tells us that
again, invariance under a local symmetry leads to an identically conserved current. We will establish the explicit
form of the current corresponding to general coordinate covariance.

As can be expected, the analysis for second order theories contains one more step, namely global, linear,
quadratic and finally general gauge or coordinate transformations. We start again with gauge theory as a
warmup exercise, but since we do not know of any physically relevant second order gauge theory anyway, we
confine ourselves this time to the abelian theory.

Thus, we consider a theory invariant under

\[
\delta A_i = \varepsilon, \quad \delta \psi = -i\varepsilon \psi.
\]

Having in mind that the field equations (for a generic field \( \varphi \)) in second order theories are of the form

\[
\frac{\partial L}{\partial \varphi} - \left(\frac{\partial L}{\partial \varphi_m}\right)_m + \left(\frac{\partial L}{\partial \varphi_m,\varphi_l}\right)_{m,l} = 0,
\]

and requiring \( \delta L = 0 \) under (44), first with constant \( \varepsilon \), then with linear \( \varepsilon \), then with second order \( \varepsilon \) (i.e.,
\[ \varepsilon = \varepsilon_{ik} x^i x^i \] with constant \( \varepsilon_{ik} \)) and finally with general \( \varepsilon(x) \), we arrive at the following four equations

\[
J^{i,\varepsilon} = 0
\]
\[
J^i = \left( \frac{\partial L}{\partial \psi_{i,k}},k \right) - \left( \frac{\partial L}{\partial A_{i,k}},k \right) - \left( \frac{\partial L}{\partial A_{i,k,l}},k,l \right) \tag{47}
\]

\[
0 = \frac{\partial L}{\partial \psi_{i,k}} - \frac{\partial L}{\partial A_{i,k}} \tag{48}
\]

\[
0 = \frac{\partial L}{\partial A_{i,k,l}} \tag{49}
\]

where the current has been defined as

\[
J^i = -\frac{\partial L}{\partial \psi_{i,i}} \psi_{i,k} - \frac{\partial L}{\partial A_{i,k,i}} \psi_{i,k} + \left( \frac{\partial L}{\partial A_{i,k,i}} \right)_{k,l} \psi_{i,l} \tag{50}
\]

Note that in (49), symmetrization over the three indices is understood. The situation is essentially the same as in first order theories (see equations (2), (3), (4)). Global symmetry (40) leads to current conservation, linear symmetry (47) enables us to write the current in the form of a divergence, and finally, local symmetry (in this case, quadratic and cubic) shows that the current is identically conserved. More specifically, (48) shows that the two first terms in (47) are identically divergence free, while (49) shows the same for the last term in (47).

Note that, if we take into account (48) and (49), then the first two terms in (47) are in the form of the previously encountered relocalization terms, \(C_{ik}^k\) with antisymmetric \(C_{ik}\). It is interesting that the last term, although identically conserved, is not of the same form, but rather in the form \(C_{ikl}^{kl}\) such that the totally symmetric part is zero, \(C_{i(kl)} = 0\). It is not hard to show that such a term in the current \(J^i\) leads again to a two dimensional surface term for the charge \(\int J^i d^3x\), namely \((\kappa, \lambda = 1, 2, 3)\)

\[
\int C^{0kl}_{i,k} d^3x = \int C^{0kl}_{i,k} d^2\sigma_{kl} + \int C^{000}_{i,k} d^2\sigma_{k} = \int C^{000}_{i,k} d^3x = \int C^{000}_{i,k} d^2\sigma_{k}, \tag{52}
\]

where we use the fact that \(C^{000} = 0\). Therefore, we will refer to such terms as second order relocalization terms\(^6\).

Note that the stress-energy tensor \(\tau_{i,k}\) consists of such a term.

We now turn to coordinate transformations in second order theories (see [10]). Thus, we consider again transformations of the form

\[
\delta \phi = \xi^i \phi_{,i} + \frac{1}{2} \xi^i_{,k} (\sigma \phi)^k, \tag{51}
\]

where the expressions for \((\sigma \phi)^k\) are given in (19), (20) and (21). Then, we require invariance of the action, i.e., \(\delta L = (L \xi^i)_{,i}\) and derive the corresponding conservation laws. The manipulations are simple and we give only the results here.

From invariance under global translations \((\xi^i = a^i = const)\), we find the conservation law for the canonical stress-energy tensor

\[
\tau_{i,k} = 0 \tag{52}
\]

with

\[
\tau_{i} = \frac{\partial L}{\partial \phi_{,i}} - \frac{\partial L}{\partial \phi_{,k}} \phi_{,i} - \frac{\partial L}{\partial \phi_{,k,i}} \phi_{,i} - \frac{\partial L}{\partial \phi_{,k,l}} \phi_{,i,l}. \tag{53}
\]

Invariance under general linear transformations \((\xi^i = \epsilon^i_{k} x^k)\), with constant \(\epsilon^i_{k}\), determines again the form of the stress-energy tensor

\[
\tau_{i} = -\left[ \frac{\partial L}{\partial \phi_{,k,l}} \phi_{,i} \right]_{,l} - \frac{1}{2} \left[ \frac{\partial L}{\partial \phi_{,k,l}} (\sigma \phi)^k \right]_{,l} - \frac{\partial L}{\partial \phi_{,m,l}} [(\sigma \phi)^k_{,i}]_{,l} + \frac{1}{2} \left[ \frac{\partial L}{\partial \phi_{,m,l}} (\sigma \phi)^k \right]_{,m,l}. \tag{54}
\]

\(^6\)The quantities \(C_{ik}^k\) and \(C_{ikl}^{kl}\) are also referred to as superpotentials.
Invariance under second order transformations \((\xi^i = \varepsilon_{ik} x^i x^k)\) leads to
\[
\left[ \frac{\partial L}{\partial \varphi_{,k,m}} \varphi_{,i} + \frac{1}{2} \frac{\partial L}{\partial \varphi_{,m}} (\sigma \varphi)^k_{,i} + \frac{\partial L}{\partial \varphi_{,j,m}} [(\sigma \varphi)^k_{,i}]_l \right]_{(km)} = 0 \tag{55}
\]
where the subscript \((km)\) means that the expression in brackets has to be symmetrized in \(km\) (no differentiation).

Finally, the requirement of general covariance \((\xi^i(x)\) arbitrary) leads to
\[
\left[ \frac{\partial L}{\partial \varphi_{,m,l}} (\sigma \varphi)^k_{,i} \right]_{(kml)} = 0, \tag{56}
\]
where the expression is totally symmetrized in \(kml\) (no differentiation). Quite obviously, (55) shows that the three first terms in (54) are in the form of a relocalization term \(C_{kl}^{ij}\), with \(C_{kl}^{ij} = -C_{lk}^{ij}\), while (56) shows that the last term has the form of a second order relocalization term \(C_{klm}^{ij}\), with \(C_{klm}^{ij} = 0\). Thus, \(\tau^k_i\) is identically conserved and \(P_k\) can be written as a two dimensional surface integral, in accordance with the second Noether theorem.

From (54), it is also clear why the Belinfante tensor cannot vanish in second order theories. That is because the Belinfante symmetrization procedure is based on the definition (16). More precisely, the Belinfante procedure for second order theories (see [3]) consists in writing (54) in the form
\[
\tau^k_i = -\frac{1}{2} \tilde{L}_{ik}^{m,k}, \tag{57}
\]
with
\[
\tilde{L}_{ik}^{m,k} = 2 \frac{\partial L}{\partial \varphi_{,k,m}} \varphi_{,i} + \frac{\partial L}{\partial \varphi_{,m}} (\sigma \varphi)^k_{,i} + 2 \frac{\partial L}{\partial \varphi_{,j,m}} [(\sigma \varphi)^k_{,i}]_j - \left[ \frac{\partial L}{\partial \varphi_{,j,m}} (\sigma \varphi)^k_{,i} \right]_{ij}, \tag{58}
\]
and then defining (compare (16)) the Belinfante tensor
\[
T^{ik} = \tau^{ik} + \frac{1}{2} [\tilde{S}^{ikm} - \tilde{S}^{mki} - \tilde{S}^{kmi}]_{,m}, \tag{59}
\]
where \(\tilde{S}^{mki} = \tilde{L}^{m[kli]}\) and \(\tilde{L}^{mki} = \eta^{il} \tilde{L}^{mkl}_i\). This tensor is easily shown to be symmetric if the theory is Poincaré invariant, i.e., if at least the antisymmetric part of (54) is satisfied. Moreover, if the theory is invariant under the affine group, i.e., if (54) is satisfied, then \(T^{ik}\) can be written in the form of equation (25), i.e.,
\[
T^{ik} = -\frac{1}{4} [\tilde{L}^{mki} - \tilde{L}^{ikm} + \tilde{L}^{mki} + \tilde{L}^{kmi} - \tilde{L}^{kim}]_{,m} = -\frac{1}{2} [\tilde{L}^{(mki)} - \tilde{L}^{(ik)m} + \tilde{L}^{(mk)i}]_{,m}. \tag{60}
\]

If the theory is generally covariant, then we find from (55) for the symmetric part of \(\tilde{L}^{mki}\) the following relation
\[
\tilde{L}^{(mk)i} = -\frac{1}{2} \left[ \frac{\partial L}{\partial \varphi_{,j,m}} (\sigma \varphi)^k_{,i} \right]_{,j} - \frac{1}{2} \left[ \frac{\partial L}{\partial \varphi_{,j,k}} (\sigma \varphi)^m_{,i} \right]_{,j}, \tag{61}
\]
from which you can explicitly evaluate \(T^{ik}\). It does not identically vanish, but it is nevertheless identically conserved, as a result of (56). In the case of general relativity, with the covariant, second order Lagrangian
\( \mathcal{L} = \frac{1}{2} \sqrt{-g} R \), it leads again to the expression (43), as is easily verified, in accordance with the results of [3].

It is interesting to remark that the evaluation of \( T^{ik} \) involves a lot less computation than that of \( \tau^i_k \).

We wish to point out, however, that you can always write the last term in (54) in the form of a first order relocalization term too. The has been shown in [10] for the general case. Specifically, in general relativity, this term is equal to

\[
-\frac{1}{2} \left[ \sqrt{-g} \left( -2g^{lm} \delta^k_i + g^{lk} \delta^m_i + g^{km} \delta^l_i \right) \right]_{m,l},
\]

which corresponds to the second line in (42) (hence the identical Belinfante tensors in first and second order theory). An equivalent form of this expression is

\[
\left[ \sqrt{-g} \left( g^{lm} \delta^k_i - g^{km} \delta^l_i \right) \right]_{l,m},
\]

which is now of the form \(-\frac{1}{2} \hat{L}^{lk}_{i,l}\), with \( \hat{L}^{lk}_{i,l} \) antisymmetric in \( kl \). Together with the three first terms in (54), we can now write the stress-energy in the form \( \tau^k_i = -\frac{1}{2} \hat{L}^{lk}_{i,l} \), with antisymmetric \( \hat{L}^{lk}_{i,l} \), i.e., in the form of a first order relocalization term (although still equal to (57)). There is no reason to prefer one or the other form, meaning that the generalization of the Belinfante procedure to second order theory is not really as unambiguous as it has been presented in [3]. Indeed, defining the Belinfante tensor using \( \hat{L}^{lki}_{i} \) in (60) leads to a vanishing tensor.

Finally, we note that (61) reduces in the case of first order theories to (29), which completes the proof of the results of section 4 without explicit reference to the vector or tensor nature of the fields.

The above results do not mean that there is any fundamental difference between first and second order theories. In both kind of theories, the stress-energy tensor is given in the form of relocalization terms and can thus obviously be annihilated by adding appropriate relocalization terms, of first or second order. Also, in both cases, relocalization terms modify the momentum vector in the case of the linear Hilbert-Einstein or Einstein-Cartan type theories. The only difference comes from the following: In first order theories, if you eliminate the antisymmetric part of \( \tau^i_k \), making it thus suitable for the formulation of conservation laws for the full Poincaré group, then you automatically annihilate the complete tensor, while in second order theories, you can eliminate the antisymmetric part and still have a non-vanishing tensor. However, as we have argued before, in generally covariant theories, it is rather arbitrary to pick out the Poincaré group and modify the stress-energy tensor accordingly. There seem to be only two reasonable (albeit extreme) points of view: Either, we allow for modifications of \( \tau^i_k \) by relocalization terms or we don’t. If we do, then we should use them in order to construct the tensor that has the largest possible symmetry. This is quite obviously the tensor that vanishes identically, since it is trivially fully covariant. If we do not, then we should simply stick to the canonical tensor in its initial form. Everything else is, in the framework of generally covariant theories, an ad hoc procedure and can only be of limited usefulness. (See, however, the remarks at the end of section 5.)

\(^{7}\)It is understood that the matter part of the Lagrangian is first order, and thus does not explicitly contribute to \( T^{ik} \). If this is not the case, you will have, apart from \( L \), additional terms in \( T^{ik} \), which, however, do not contribute to \( P_k \) if the usual asymptotical behavior is assumed. A similar assumption has been made in [4], where the Belinfante procedure was applied to the canonical tensor of the gravitational field, and then brought into the form (36) by the use of the gravitational field equations. The result is then the Belinfante tensor of gravity plus the Hilbert tensor of matter. However, the Hilbert tensor is not equal to the Belinfante tensor of the matter field if the matter Lagrangian contains second order derivatives (see next section). The only generally valid form for the total Belinfante tensor \( T^{ik} \) is therefore (60) with (61).
7 Matter stress-energy and the Hilbert tensor

In this section, we split the Lagrangian into a gravitational and a matter part, \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m \), and briefly examine the relations between the canonical stress-energy for the matter fields, the corresponding Belinfante tensor and the so-called Hilbert tensor. Since the relation between canonical and Hilbert tensor has been discussed in detail in our previous article [11], and since the generalization of the discussion to the Belinfante tensor is rather simple, in view of the relations we have already obtained in the previous sections, we will confine ourselves to give a very short exposition of the issue. The subject has also been covered, e.g., in [12] and [13].

We consider purely metric theories (e.g., general relativity), and for (some kind of) completeness, we allow, for the moment, derivative couplings (first order) of the metric to the matter fields. Nevertheless, non-minimal couplings of the form \( \sqrt{-g} \phi^2 R \), are still excluded, since they contain second derivatives of the metric coupling to \( \phi \). Moreover, we confine ourselves to first order theories, i.e., we assume that no second derivatives of the matter fields occur in the Lagrangian.

The variation of the matter Lagrangian then is of the form
\[
\delta \mathcal{L}_m = \frac{\partial \mathcal{L}_m}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}_m}{\partial g_{lm}} \delta g_{lm} + \frac{\partial \mathcal{L}_m}{\partial g_{lm,i}} \delta g_{lm,i},
\]
where \( \phi \) denotes collectively the matter fields (i.e., all the dynamical fields, except for \( g_{lm} \)). Note that the matter fields are supposed to satisfy the field equations, while for \( g_{lm} \), no field equation can be used, since the gravitational part \( \mathcal{L}_0 \) is not included in our Lagrangian. In other words, \( g_{lm} \) is treated as background field.

Under a coordinate transformation, the fields transform according to (51), and in particular, for the metric we have
\[
\delta g_{lm} = g_{lm,k} \xi^k + \xi^k g_{lm} + \xi^m g_{lk}.
\]
The matter Lagrangian is assumed to be a scalar density, i.e.,
\[
\delta \mathcal{L}_m = (\mathcal{L}_m \xi^k)_k.
\]

The Hilbert stress-energy tensor is defined by
\[
T^{lm} = -2 \delta \mathcal{L}_m / \delta g_{lm} = -2 \frac{\partial \mathcal{L}_m}{\partial g_{lm}} + 2 \left( \frac{\partial \mathcal{L}_m}{\partial g_{lm,i}} \right)_{,i},
\]
where, in accordance with our previous conventions, we have defined \( T^{lm} \) as tensor density. In view of the invariance of \( \mathcal{L}_m \) under coordinate transformations, \( T^{lm} \) satisfies the covariant conservation law \( T^{lm} = T^{lm} + \Gamma^{lm}_{k}T^{lk} = 0 \). The derivation of this law is found in any textbook on general relativity (or, see [11]). Note however that it holds independently of the specific form of the gravitational Lagrangian.

With this definition, and using \( T^{lm} = 0 \), we derive for \( \xi^i = a^i \) the following conservation law
\[
\tau^i_k - T^i_k = 0,
\]
where \( \tau^i_k \) is defined as in (53) (but with \( \mathcal{L}_m \)), and
\[
T^i_k = T^{im} g_{mk}.
\]
Next, for linear transformations, we get
\[
\tau^i_k - T^i_k + \frac{1}{2} L^{mi} g_{km} + 2 \left[ \frac{\partial \mathcal{L}_m}{\partial g_{li,m}} g_{kl} \right]_{,m} + \frac{\partial \mathcal{L}_m}{\partial g_{lm,i}} g_{lm,k} = 0,
\]
where \( L^{mi} \) is defined as in (24). Requiring general covariance, we find the additional relation
\[
\frac{1}{2} L^{(mi)} g_{kl} + 2 \left[ \frac{\partial \mathcal{L}_m}{\partial g_{li,m}} g_{kl} \right]_{,m} = 0,
\]
showing that $\tau^i_k - T^i_k + \frac{\partial L}{\partial g_{ik}} g_{lm,k}$ is identically conserved. It is not hard to see from the definition (16) that the Belinfante tensor is not equal to $T^{ik}$, nor does it reduce to $T^{ik}$ in the flat limit $g_{ik} = \eta_{ik}$. However, in the case where derivative couplings are absent, (16) simplifies to $\tau^i_k - T^i_k = -\frac{1}{2} L^{mi}_{k,m}$, and, since (66) reduces to $L^{(mi)}_{k} = 0$, the Belinfante tensor $T^{ik}$ is now easily shown to be directly related to $T^{ik}$, namely, we have

$$T^{ik} = T^{i}_{m} \eta^{mk},$$

(67)

where we recall that $T^{i}_{m} = g_{mk} T^{ik}$, and thus, since the second index of $T^{ik}$ is lowered with the Minkowski metric, we can equivalently write

$$T^{i}_{k} = T^{i}_{k}.$$  

(68)

In summary, in first order theories without derivative couplings, the Belinfante tensor is indeed equal to the Hilbert tensor (when written in mixed form). Note that the Belinfante (matter) tensor $T^{ik}$ is, strictly speaking, not symmetric (since it is not equal to $T^{ik}$ in contravariant form). A symmetric tensor is obtained by raising the second index of $T^{i}_{k}$ with $g^{ik}$ instead of $\eta^{ik}$. This, however, is not really of importance, since $T^{i}_{k}$ is not conserved anyway in a curved background (neither is $T^{ik}$), and in general, neither will the angular momentum be conserved.

The equality of the Belinfante and the Hilbert tensors in first order theories in the framework of metrical theories of gravity can give rise to two different interpretations. The first one is that it provides a strong argument to the standpoint that the Belinfante procedure can indeed be used to derive (or rather define) the correct stress-energy tensor not only in special relativity, but also in metric theories of gravity, and in particular for the gravitational Lagrangian itself. A second, quite different point of view consists in interpreting the Hilbert tensor as a covariant generalization of the Belinfante tensor, a point of view equally strongly supported by the above equality. According to the second interpretation, the stress-energy tensor of the gravitational field itself is then given by the variation of $L$ with respect to the metric, and quite obviously, the total stress-energy tensor would vanish in view of the gravitational field equations. This interpretation is supported by the fact that in generally covariant first order theories, the total Belinfante tensor is indeed identically zero, while in second order theories (like general relativity), one could modify the Belinfante symmetrization procedure, including second order relocalization terms, in a way that again, the total tensor would vanish.

We have carried out a similar analysis of the relation between the canonical and the corresponding (generalized Hilbert) tensor $T^{i}_{m} = -\frac{\partial L}{\partial g^{ik}}$ in the framework of Einstein-Cartan theory and also in tetrad gravity in [11]. It was shown in particular that both tensors are not in general related by a relocalization term (because of the additional field $\Gamma^{ab}_{i}$ in the first case, and because of the derivative couplings to spinor fields in the second case), and it is not hard to extend the analysis to include the Belinfante tensor. The result is that there is no equality (not even up to a relocalization term) in those theories between the Hilbert and the Belinfante tensors. For instance, in Einstein-Cartan theory, one can derive the relation $T^{i}_{k} = T^{i}_{a} e^{a}_{k} - \sigma^{ab}_{i} \Gamma^{a}_{k}$, with $\sigma^{ab}_{i} = \partial L_{m}/\partial \Gamma^{ab}_{i}$. Equality holds in the absence of spinor fields $\sigma^{ab}_{i} = 0$ or in the limit of vanishing gravity, $e^{a}_{i} = \delta^{a}_{i}, \Gamma^{ab}_{k} = 0$. Similarly, in tetrad gravity, we have equality only in those limits.

Moreover, the above result for metric theories does not generalize to matter Lagrangians containing second derivatives. This is actually quite obvious, since if it would, then the same would be true for the gravitational part of the Lagrangian, and then, the total Belinfante tensor (adopting the definition of [3], i.e., using (16) and (68)) would be equal to the variation of the total Lagrangian with respect to $g_{ik}$, and thus zero, which is, however, not the case in second order theories, as we have shown in the previous section. The equality between Hilbert and Belinfante tensor can be achieved by modifying the Belinfante relation for second order theories, as we have done in the previous section, rewriting the second order relocalization term in the form of a first order term, such that the total Belinfante tensor vanishes again. However, in view of the physical irrelevance of matter Lagrangians with second derivatives, we do not pursue this idea further.
8 Generators and Hamiltonian constraints

In this section, we briefly review the relation between the conserved charges, the generators of the corresponding symmetries and the (first class) Hamiltonian constraints. The issue is of fundamental importance for the quantization of field theories with local symmetries and has been pioneered by Bergmann and Dirac. We confine ourselves to a brief analysis of the important cases (internal gauge symmetry and general covariance) and refer the reader to the initial articles, in particular [14], [15], [16], [17] and [18].

8.1 Gauge theory

In order to present our equations in a covariant form, we use the covariant formalism of [19]. In particular, we will use integrals over spacelike hypersurfaces $\sigma$ defined by $\Phi(x) = 0$, with (timelike) normal vector $n_i = \Phi, i$. The normal vector is understood to be normalized, $g^{ik} n_i n_k = 1$. Note that this normalization is only a convention used for greater convenience in intermediate steps. Our initial and final relations should not depend on this normalization, since we do not want to assert any special meaning to the metric $g_{ik}$, which is considered to be a field like any other field.

Further, we define the canonical momenta by

$$\pi^i = \frac{\partial L}{\partial \dot\varphi^i}, \quad (69)$$

where we put the index into parentheses to remind of the fact that only the part normal to the hypersurface, $\pi = \pi^i n_i$, corresponds to the physical momentum known from conventional Hamiltonian theory.

We start again with non-abelian gauge theory. The conserved current density has been derived in the form

$$J^i_\alpha = -i \frac{\partial L}{\partial \psi^i_\alpha} + \frac{\partial L}{\partial A^\alpha_{k,i}} c^\beta_{\alpha\gamma} A^\gamma_k = -i \pi^i_\psi \psi^\sigma_\alpha \psi + \pi^k_\beta c^\beta_{\alpha\gamma} A^\gamma_k \quad (70)$$

Next, in view of equation (3) we define

$$\tilde{J}^i_\alpha = -i \frac{\partial L}{\partial A^\alpha_{k,i}} = -\pi^k_\alpha \quad (71)$$

and in view of equation (4),

$$\hat{J}^{ki}_\alpha = \frac{\partial L}{\partial A^{\alpha}_{i,k}} = \pi^i_\alpha \quad (72)$$

as well as the corresponding charges

$$Q = \int \varepsilon_\alpha J^i_\alpha d\sigma, \quad \tilde{Q} = \int \varepsilon_\alpha \tilde{J}^i_\alpha d\sigma, \quad \hat{Q} = \int [\varepsilon_\alpha \hat{J}^{ki}_\alpha]_k d\sigma, \quad (73)$$

where $\varepsilon_\alpha(x)$ is an arbitrary parameter.

As a result of equation (3), we have the identically satisfied relation $J^i_\alpha + \tilde{J}^i_\alpha = 0$, and as a result of (4), $\hat{J}^{ik} + \hat{J}^{ki} = 0$. It is not hard to recognize that the corresponding equation $Q + \tilde{Q} = 0$ is equivalent to the secondary (first class) constraints of the theory. Consider, e.g., flat space and choose $n_i = \delta^0_i$ (that is, the hypersurface $t = \text{const}$). Then, the integrand of $Q + \tilde{Q}$ reduces to $\varepsilon_\alpha [-i \pi^i_\psi \psi^\sigma_\alpha \psi + \pi^k_\beta c^\beta_{\alpha\gamma} A^\gamma_k - \pi^\mu_{\alpha, \mu}]$, where

\[Note that this is the opposite of the expression (3), if (4) is taken into account.\]
In order not to overload the notation, we have omitted the arguments of the fields. It is understood that all integrals without specifying a hypersurface, but they do not appear in the combinations of the generators we are interested in. If we choose the hypersurface $t = \text{const}$ for $\mu = 1, 2, 3$, we find the more conventional form\(^9\)

$$[\hat{Q}, A^\mu_0] = -\varepsilon^\mu_\mu, \quad [\hat{Q}, A^0_0] = -\varepsilon^3_0.$$  

\(^9\)Note that the same results can be directly derived by the use of the more familiar, but not covariant generators

$$Q = \int \varepsilon^\alpha J^{(0)}_\alpha \, d^3x, \quad \hat{Q} = \int \varepsilon^\alpha j^{(0)}_\alpha \, d^3x, \quad \hat{Q} = \int \varepsilon^\alpha j^{(0)}_\alpha \, d^3x,$$

instead of $[\hat{Q}, A^\mu_0] = -\varepsilon^\mu_\mu, \quad [\hat{Q}, A^0_0] = -\varepsilon^3_0$, where we identify directly the physical momenta with $\pi = \pi^{(0)}$, and take the commutation relations to be (at equal times) $[\varphi(x), \pi(y)] = \delta^{(3)}(x - y)$. 

\(\mu = 1, 2, 3\). In the last term, we have exploited the antisymmetry of $\pi^i_\alpha$ resulting from (41). This is exactly the secondary constraint arising in Yang-Mills-Dirac theory. In the context of quantization in the Coulomb gauge, it is usually referred to as Gauss’ law. More generally, the relation $\hat{J}^i_\alpha + \hat{J}^i_\beta = 0$ is equivalent to the field equations for the Yang-Mills field, but the fact that it can be derived directly from the symmetry of the theory indicates the presence of constraints. The field equations for $\psi$ for instance cannot be derived in this way. 

In a similar way, the relation $\hat{j}^{(k)} = 0$ is related to the primary constraints. Consider again the case $n_i = \delta^i_0$. Then, we have $\pi^i_\alpha = \pi^{(0)}_\alpha$, and therefore $\pi^0_\alpha = 0$, which is the well-know primary constraint of Yang-Mills theory. Alternatively, with the help of equation (111) of Appendix A, we can express the primary constraints simply as $P = 0$.

Next, we analyze the relations of $Q$, $\hat{Q}$ and $\hat{Q}$ to the generators of the gauge transformation. For this, we assume the following commutation relations between fields and canonical momenta 

$$[\varphi(x), \varphi(y)] = 0, \quad [\pi^{(i)}(x), \pi^{(m)}(y)] = 0, \quad [\varphi(x), \pi^{(i)}(y)] = \delta^i_s(x - y),$$

where $x, y$ are assumed to be separated by a spacelike distance. For details on our formalism, we refer to Appendix A. In the case of spinor fields, the above relations are replaced by anticommutation relations, e.g., 

$$\{\psi^M(x), \pi^{(i)}_N(y)\} = \delta^M_N \delta^i_s(x - y),$$

where $M, N$ denote the spinor components. It is not of our concern here where the (anti)commutation relations come from, be it from a classical Poisson bracket (see 19) or be it from postulating them in a quantum theory. They are simply assumed to be valid.

We are now ready to evaluate the commutation relations between the fields and the charges defined in (73). For the manipulations involved, a few helpful rules are provided in Appendix A. Here, we give only the results. Note that in order to derive those results, two dimensional surface terms have been omitted during the calculations. This is permitted if we assume that the parameter $\varepsilon^\alpha$ has an appropriate asymptotical behavior, or if we simply assume that it vanishes outside of a certain region.

We find the following relations

$$[Q, A^\beta_1] = -\varepsilon^\alpha c^0_\alpha \gamma A^\gamma_1, \quad [Q, \psi] = i\varepsilon^\alpha \sigma_\alpha \psi,$$

$$[\hat{Q}, A^\beta_1] = -\varepsilon^\beta_1 + \varepsilon^\beta_m n^m n_1 + \int \varepsilon^\beta [\delta^i_s(x - y)]_i d\sigma_l, \quad [\hat{Q}, \psi] = 0,$$

$$[\hat{Q}, A^\beta_0] = -\varepsilon^\beta_m n^m n_1 - \int \varepsilon^\beta [\delta^i_s(x - y)]_i d\sigma_l, \quad [\hat{Q}, \psi] = 0.$$
as well as \([\hat{Q}, A^\beta_0] = [\hat{Q}, A^\beta_\mu] = 0\).

Those results are very interesting. First, we see that the charge \(Q\) does not (as is occasionally stated) generate gauge transformations on the fields. Rather, it generates homogeneous transformations on both \(A^\beta_1\) and \(\psi\), in the adjoint (or vector) representation on \(A^\beta_1\) (a rotation in isospin space) and in the fundamental representation on \(\psi\).

The inhomogeneous part is generated by \(\hat{Q} + \hat{Q}\), namely we have
\[
[\hat{Q} + \hat{Q}, A^\beta_1] = -\varepsilon^\beta_i.
\]

In order to find the generator of the full gauge transformation, we need all three parts, i.e.,
\[
[Q + \hat{Q}, A^\beta_1] = -\varepsilon_i^\beta - \varepsilon^\beta_\gamma A^\gamma_1, \quad [Q + \hat{Q}, \psi] = i\varepsilon^\alpha\sigma_\alpha\psi,
\]
which is exactly the transformation \(11\) we started from. Both \(78\) and \(79\) are independent of the hypersurface. Another combination of interest is the generator \(Q + \hat{Q}\), because it is identically zero (related to the secondary constraint). If we choose the hypersurface \(t = \text{const.}\), we find for the spatial components \([Q + \hat{Q}, A^\beta_1] = -\varepsilon_i^\beta - \varepsilon^\beta_\gamma A^\gamma_1\). The fact that an operator that vanishes weakly (i.e., as a result of the constraints) generates gauge transformations on the propagating fields (consider, e.g., the Coulomb gauge), is interpreted in quantum theory as the expression for the fact that physical states have to be singlets under the gauge group. That is, free bosons \(A^\beta_1\) cannot be part of the physical particle spectrum (see, e.g., [24]). This should be somehow alerting, considering the fact that we will now go over to spacetime symmetries.

### 8.2 General covariance

The analysis is quite similar to the previous case. In view of the relations \(11\), \(23\) and \(29\), we define\(^{10}\)
\[
\tau^i_k = \pi^{(i)}\varphi,_{k} - \delta^i_k L,
\]
\[
\bar{\tau}^i_k = -\frac{1}{2} L^{i m}_{k, m} = -\frac{1}{2} [m^{(i)}(\sigma \varphi)^m]_{k, m},
\]
and
\[
\bar{\tau}^{m i}_{k} = \frac{1}{2} L^{m i}_{k, k} = \frac{1}{2} \pi^{m} (\sigma \varphi)^i_k.
\]

Next, we introduce the corresponding charges\(^{11}\)
\[
P = \int \varepsilon^k \tau^i_k d\sigma_i, \quad \bar{P} = \int \varepsilon^k \bar{\tau}^i_k d\sigma_i, \quad \bar{P} = \int [\varepsilon^k \bar{\tau}^{m i}_{k, k}]_{m, m} d\sigma_i.
\]

The primary constraints \(\bar{P} = 0\) arise again from the fact that \(\bar{\tau}^{m i}_{k}\) is antisymmetric in \(mi\), while the secondary constraints are expressed by \(P + \bar{P} = 0\). The explicit form will depend on the nature of the field, i.e., on the form of \((\sigma \varphi)^i_k\). Note also that the relation \(\tau^i_k + \bar{\tau}^i_k = 0\) is equivalent to a field equation. In general relativity, this would be the Einstein equation. In other theories, it could be a combination of field equations.

In order to evaluate the commutation relations with \(\varphi\), it is important to recall that in \(30\), the Lagrangian is expressed in terms of the field and its derivatives. It is not understood that field derivatives are to be replaced

\(^{10}\) Note that again, the second expression \(31\) corresponds to the opposite of \(29\).

\(^{11}\) Again, there is the more familiar choice \(P = \int \varepsilon^k \tau^i_k d^3x\), \(\bar{P} = \int \varepsilon^k \bar{\tau}^i_k d^3x\) and \(\bar{P} = \int [\varepsilon^k \bar{\tau}^{m i}_{k, k}]_{m} d^3x\), involving only the components \(\pi^{(i)}\) of the momenta, which satisfy \([\varphi(x), \pi(y)] = 3(x - y)\)
by momenta. This cannot be done in a unique way anyway, since \[10\] is not a Legendre transformation. In other words, \(\tau^i_k\) has nothing to do with the field Hamiltonian, which is indeed a Legendre transformation (see \[10\] for more details and on the definition of the Hamiltonian in the manifestly covariant formalism).

We find the following commutation relations

\[
\begin{align*}
[P, \varphi] &= -\varphi_{,k} \varepsilon^k, \\
[\tilde{P}, \varphi] &= -\frac{1}{2} \varepsilon^k_{,m} (\varphi \varphi)^m_k + \frac{1}{2} [(\varphi \varphi)^m_k \varepsilon^k_i], i, n_m n^i + \frac{1}{2} \int [\delta^i_j (x - y)]_i, i (\varphi \varphi)^m_k \varepsilon^k d\sigma_m, \\
[\hat{P}, \varphi] &= -\frac{1}{2} (\varphi \varphi)^k_m \varepsilon^m_{,i}, i, n_k n^i - \frac{1}{2} \int [\delta^i_j (x - y)]_i, i (\varphi \varphi)^m_k \varepsilon^k d\sigma_m.
\end{align*}
\]

In particular, for the sum \(\tilde{P} + \hat{P}\), we have

\[
[\tilde{P} + \hat{P}, \varphi] = -\frac{1}{2} \varepsilon^k_{,m} (\varphi \varphi)^m_k.
\]

The situation is in complete analogy to the previous case. The charge \(P\) (i.e., the canonical field momentum in the strict sense) generates what is usually referred to as spacetime translations, in the sense that it tells us the evolution of \(\varphi\) from one point to another, \(\varphi(x) - \varphi(x') = -\varphi_{,k} \varepsilon^k\). On the other hand, the operator \(\tilde{P} + \hat{P}\) generates passive coordinate transformations as they are usually considered in general relativity, i.e., \(\varphi(x) - \varphi'(x') = -\frac{1}{2} \varepsilon^k_{,m} (\varphi \varphi)^m_k\). And finally, the operator \(P + \tilde{P} + \hat{P}\) generates the active coordinate transformations we started from, i.e., the Lie derivatives of the field

\[
[P + \tilde{P} + \hat{P}, \varphi] = -\varphi_{,k} \varepsilon^k - \frac{1}{2} \varepsilon^k_{,m} (\varphi \varphi)^m_k = \varphi'(x) - \varphi(x),
\]

with \(x'^i = x^i + \varepsilon^i\).

A few remarks are in order at this point. The relation \[84\] is in the form one expects for a momentum operator. It is the operator one wants to have in a quantum theory. And it is based on the canonical stress-energy tensor. Now, since in the framework of general relativity, \(\tau^i_k + \tau^i_k = 0\) is equivalent to Einstein’s equation, i.e., to \(\delta \mathcal{L}/\delta g_{ik} = 0\), the tensor corresponding to the gauge generator \(P + \tilde{P}\) is obviously given by \(-\sqrt{-g} G^{ik} + T^{ik}\), where \(G^{ik}\) is the Einstein tensor and \(T^{ik}\) the Hilbert tensor (density) for the matter fields. In particular, concentrating on the matter part, we see that \(T^{ik}\), which is also equal to the Belinfante (matter) tensor \(T^{ik}\), is related to a gauge transformation of the form (for the hypersurface \(t = \text{const}\)) \(\delta \varphi = -\varepsilon^k \varphi_{,k} - \frac{1}{2} \varepsilon^k_{,m} (\varphi \varphi)^m_k + \frac{1}{2} [(\varphi \varphi)_{k}^{0} \varepsilon^k]_0\). This makes it rather hard to interpret the quantity \(\int T^{ik} \varepsilon^k x\) as field momentum. Quite obviously, this interpretation should be reserved to the corresponding expression with the canonical tensor.

In short, the Belinfante is not directly related to the generators of coordinate transformations and as such, its integral over space should not be directly interpreted as energy and momentum.

A second remark concerns the same generator, \(P + \tilde{P}\), which is zero (secondary constraint), and generates again gauge transformations on the propagating fields, quite similar as in the case of internal gauge theories. For instance, for the spatial components of a tensor (e.g., the metric) we find (for the hypersurface \(t = \text{const}\)) \([P + \tilde{P}, g_{\mu \nu}] = -g_{\mu \nu, k} \varepsilon^k - \varepsilon^k \partial_\mu g_{\nu k} - \varepsilon^k g_{\mu k}\) for transformations restricted to the hypersurface, we have \(\varepsilon^k = (0, \varepsilon^i)\), and thus, we find that \([P + \tilde{P}, g_{\mu \nu}]\) is equal to the Lie derivative of \(g_{\mu \nu}\) in the three dimensional subspace. One is thus tempted to conclude that physical states have to be singlets under coordinate transformations, i.e., scalars. This would not only exclude gravitons, but also photons and vector fields in general. While for the latter, we can argue that they are considered usually on a given background (with a reduced symmetry), in the case of the gravitational theory, the solution to this problem is more profound and can be found in the fact that
general relativity (or similar theories, e.g., Einstein-Cartan) is actually a theory with a spontaneous symmetry breaking, in the sense that the vacuum is given by $g_{ik} = \delta_{ik}$ (or eventually a de Sitter metric) and thus allows only for Poincaré (or de Sitter) transformations. This invalidates the previous arguments, and gravitons can now be interpreted as fields propagating on this background. (Note that similarly, massive gauge bosons in the Salam-Weinberg model are also part of the physical spectrum.)

Finally, we caution that we have assumed in our analysis that the theory is free of second class constraints. Second class constraints arise in particular in the Dirac theory, but also in Einstein-Cartan theory (see [31]) and are not related to gauge symmetries. Their presence leads to modifications of the canonical (anti)commutation relations and to modifications of the above results, in particular of the relation (84). We refer to [19] for details on the Dirac theory. As to Einstein-Cartan theory, one has to take into account both the diffeomorphism relations and to modifications of the above results, in particular of the relation (84). We refer to [19] for details.

8.3 General relativity

The previous analysis is valid for generally covariant first order theories, and as such, is directly applicable to, e.g., Einstein-Cartan theory or Poincaré gauge theories in general, once we have dealt consistently with the theory is free of second class constraints. On the other hand, in order to evaluate the stress-energy tensor, we are not really interested in $L^{mk}_{i}$, but rather in $L^{mk}_{i,m}$. Therefore, as we have already indicated in section 6, we can modify the second line in (42) and write instead $[\sqrt{-g}(-2g^{lm}\delta_{i}^{k} + 2g^{lk}\delta_{i}^{m})]_{i}$. We denote by $L^{mk}_{i}$ the expression (42) with the second line replaced in that way. This is now antisymmetric in $km$. Then, we have

$$
\tilde{L}^{mk}_{i} - L^{mk}_{i} = [\sqrt{-g}(g^{lk}\delta_{i}^{m} - g^{km}\delta_{i}^{l})]_{i}. 
$$

(90)

Obviously, $\tilde{L}^{mk}_{i,m} = L^{mk}_{i,m}$. We can thus write $\tau^{k} = -\frac{1}{2} \tilde{L}^{mk}_{i,m}$, with $\tilde{L}^{mk}_{i}$ antisymmetric in $mk$. In view of (35), we introduce modified momenta $\tilde{\pi}^{kl(m)}$ by requiring

$$
\tilde{L}^{mk}_{i} = 4\tilde{\pi}^{kl(m)}g_{li},
$$

(91)

which leads to

$$
\tilde{\pi}^{kl(m)} = \pi^{kl(m)} + (\sqrt{-g}g^{kl})_{i}g^{mi} - (\sqrt{-g}g^{km})_{i}g^{li}.
$$

(92)
Note that $\tilde{\pi}_{ki(m)}$ is no longer symmetric in $ki$. Next, we postulate canonical commutation relations between fields and modified momenta in the form (for spacelike separations)

$$[g_{ik}(x), \tilde{\pi}^{ln(m)}(y)] = \frac{1}{2} (\delta_i^l \delta_k^n + \delta_k^l \delta_i^n) \delta^{mn}(x - y).$$  

(93)

Only the symmetric part of $\tilde{\pi}^{ln(m)}$ does not commute with $g_{ik}$ and contains thus the physical momenta.

The stress-energy tensor $\tau^i_k = (\partial L/\partial g_{lm,i})g_{lm,k} - \delta^i_k L$ can now be written in the form

$$\tau^i_k = \tilde{\pi}^{ln(i)} g_{lm,k} - \delta^i_k L = \tilde{\pi}^{ln(i)} g_{lm,k} - ([\sqrt{-g} g^{pl} p g^{im} - (\sqrt{-g} g^{lj} p g^{pm})]g_{lm,k} - \delta^i_k L. $$

(94)

We can now proceed as in the previous cases, i.e., we define the generators by

$$P = \int \varepsilon^k \tau^i_k d\sigma_i, $$  

(95)

as well as

$$\tilde{P} = \int \varepsilon^k \tilde{\tau}^i_k d\sigma_i $$  

(96)

with $\tau^i_k$ as above and with

$$\tilde{\tau}^i_k = -\frac{1}{2} \tilde{L}^{im}_{k,m} = -2[\tilde{\pi}^{mi(l)} g_{kl}]_m. $$

(97)

Finally, for the third generator (see (83)), we have

$$\hat{P} = \int [\varepsilon \tilde{\tau}^{mi}_k]_m d\sigma_i = 2 \int [\varepsilon \tilde{\pi}^{mp(m)}]_m d\sigma_i, $$

(98)

where $\tilde{\tau}^{mi}_k = \frac{1}{2} \tilde{L}^{mi}_k$. The primary constraints are now expressed by the fact that $\tilde{\tau}^{mi}_k$ is antisymmetric, and thus $\hat{P} = 0$, while the secondary constraints are given by $P + P = 0$. Thus, in a sense, with the modification of the canonical momenta, we have restored the primary constraints which have been lost because we started with a non-covariant Lagrangian.

The commutation relations of those generators with the metric lead consistently to the expressions (84), (85) and (86). Explicitly, if we choose again the hypersurface $t = const$, we have

$$[P, g_{ik}] = -g_{ik,m} \varepsilon^m, $$

(99)

$$[\hat{P}, g_{ik}] = -\varepsilon^m, g_{mk} - \varepsilon^m, g_{mk} + (\varepsilon^m, g_{mk})_0 \delta^0_0 + (\varepsilon^m, g_{mi})_0 \delta^0_k, $$

(100)

$$[\tilde{P}, g_{ik}] = -(g_{im} \varepsilon^m),_0 \delta^0_0 - (g_{km} \varepsilon^m),_0 \delta^0_k. $$

(101)

From this, we can see explicitly that action of the constraint $P + \hat{P} = 0$ on the spatial components, namely

$$[P + \hat{P}, g_{\mu\nu}] = -g_{\mu\nu,\lambda} \varepsilon^\lambda - \varepsilon^\lambda, g_{\kappa\nu} - \varepsilon^\lambda, g_{\kappa\mu}, $$

(102)

where we have assumed that $\varepsilon^0 = 0$, i.e., the transformation takes place on the hypersurface in question. As pointed out earlier, this is again the Lie derivative of the three dimensional metric. In Appendix B, we analyze the question whether $g_{\mu\nu}$ can indeed be assumed to represent the propagating part of $g_{ik}$ in an appropriate gauge.

In order to check whether the introduction of the new momenta $\tilde{\pi}^{ln(i)}$ is fully consistent, one has to introduce a Hamiltonian $H = \int \tau^i_k n^k d\sigma_i$ (see [19]), express it in terms of the momentum $\tilde{\pi}^{im} = \tilde{\pi}^{im(i)} n_i$, i.e., eliminate
the velocities \( g_{m,i} \), and analyze the commutation relations of \( H \) with the momenta. This is a non-trivial task, in particular one will have to deal with ordering problems, see, e.g., [21]. It is however not unusual in the context of first order general relativity to carry out modifications by hand, in order to achieve consistency between the constraints of the theory. For instance, Dirac [22] chose to modify the Lagrangian by a surface term in a way that the momenta conjugated to \( g_{0\mu} \) vanish weakly, which then represents a primary constraint.

An alternative way, and probably a more elegant one, is to start from the covariant, second order Lagrangian. Let us briefly sketch how this could work. One defines the canonical momenta

\[
\pi^i = \frac{\partial L}{\partial \varphi^i} - (\frac{\partial L}{\partial \varphi_{k,i}})_{,k}, \quad p^{m(i)} = \frac{\partial L}{\partial \varphi_{m,i}}
\]

and writes the stress-energy tensor (53) in the form

\[
\tau^k_i = \pi^{(k)} \varphi_{,i} + p^{m(k)} \psi_{m,i} - \delta^k_i \mathcal{L},
\]

where \( \psi_m = \varphi_{,m} \) plays the role of the canonical variable conjugate to \( p^{m(i)} \). The identically conserved form of the stress-energy tensor from (54) takes the simple form

\[
\tau^k_i = -\frac{1}{2} \left[ \pi^{(m)} (\sigma \varphi)^k_i + p^{i(m)} (\sigma \psi_j)^k_i \right]_{,m},
\]

where \( (\sigma \psi_j)^k_i = 2 \delta^k_j \varphi_{,i} + [(\sigma \varphi)^k_i]_{,j} \), i.e., it acts correctly on \( \psi_j \) in accordance with its total tensor structure, taking account of the additional vector index. Similarly, we express equations (55) and (56) in terms of the momenta and finally construct four charge operators, the first two expressing the constraints stemming from the equality between (104) and (105), and the other two expressing the vanishing of the expressions (55) and (56). For instance, in a metric theory, it can easily be seen from (106) that one set of (primary) constraints is related to the fact that no second time derivatives of the the components \( g_{0\mu} \) are contained in the Lagrangian. We thus see that this apparent coincidence in general relativity is actually a necessary feature of generally covariant second order theories.

The only non-vanishing commutators in second order field theory are given by

\[
[\varphi(x), \pi^{(i)}(y)] = \delta^i_s (x - y), \quad [\psi_m(x), p^{k(i)}(y)] = \delta^k_m \delta^i_s (x - y),
\]

where \( x \) and \( y \) are assumed to be separated by a spacelike separation.

9 Conclusions

We have analyzed Noether’s theorem localizing step by step the transformation group from global to local. The intermediate steps of linear (and eventually quadratic) transformations turned out to be of importance, not only in general relativity, where one usually prefers the use of a first order Lagrangian which is covariant only under affine transformations, but also in theories based on a fully covariant Lagrangian. Indeed, the restrictions on the conserved current derived from each step of locality are directly related to the first class Hamiltonian constraints of the theory. The primary, secondary, etc., constraints related to the invariance of the theory under a certain symmetry group, as well as the corresponding generators, can be directly read off from the form of the Noether current as it arises in each step.

Concerning the stress-energy tensor, we have shown that it is identically conserved whenever the theory is generally covariant and the momentum can thus be written in terms of two dimensional surface integrals. In first
order theories, the symmetric Belinfante tensor was shown to be identically zero. In second order theories, this is not necessarily the case, but the generalization of the Belinfante formula to those theories does not seem to be unique. Special attention has been paid to general relativity, which, in the first order approach, shares many features with theories based on a fully covariant Lagrangian, but at some points, modifications are necessary in order to restore the properties that have been lost as a result of the use of a non-covariant Lagrangian.

Appendix A: Covariant Hamiltonian formalism

We recall the main features of the manifestly covariant formalism for field theory used in [19].

Let \( x^i = (x^0, x^1, x^2, x^3) \) be spacetime coordinates such that a hypersurface element can be written as

\[
d\sigma_i(x) = \begin{pmatrix}
    dx^1(\sigma)dx^2(\sigma)dx^3(\sigma) \\
    dx^0(\sigma)dx^2(\sigma)dx^3(\sigma) \\
    dx^0(\sigma)dx^1(\sigma)dx^3(\sigma) \\
    dx^0(\sigma)dx^1(\sigma)dx^2(\sigma)
\end{pmatrix},
\]

(107)

where \( dx^i(\sigma) \) means that \( dx^i \) is restricted to some hypersurface \( \sigma \) defined by \( \Phi(x) = 0 \). (E.g., for the hypersurface \( x^0 = \text{const} \), we have \( dx^0 = 0 \) and \( d\sigma_i(x) = \delta_i^0 d^3x \).

In the same coordinate system, we define

\[
\delta^i(x-y) = \begin{pmatrix}
    \delta_{i0} \delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3) \\
    \delta_{i1} \delta(x^0-y^0)\delta(x^2-y^2)\delta(x^1-y^1) \\
    \delta_{i2} \delta(x^0-y^0)\delta(x^1-y^1)\delta(x^3-y^3) \\
    \delta_{i3} \delta(x^0-y^0)\delta(x^1-y^1)\delta(x^2-y^2)
\end{pmatrix}.
\]

(108)

The transformation behavior for \( \delta^i(x-y) \) under a coordinate change is found from the known transformation behavior of \( d\sigma_i(x) \) (\( \sqrt{-g} d\sigma_i \) is a vector) by requiring \( \delta^i(x-y)d\sigma_i \) to transform as scalar under general coordinate transformations. Thus, \( \delta^i(x-y) \) transforms as vector density.

Next, consider a spacelike hypersurface \( \sigma \) defined by \( \Phi(x) = 0 \), with the (timelike) normal vector \( n_i = \Phi_{,i} \). For convenience, \( \Phi(x) \) can be chosen such that \( n^i \equiv n_0 n_k g^{ik} = 1 \). Then, we have

\[
\int_\sigma f(x)\delta^i(x-y)d\sigma_i(x) = f(y)
\]

(109)

where the integration is carried out over the hypersurface \( \sigma \) containing the point \( y \). For the specific hypersurface \( t = t_0 = \text{const} \), we find\(^{12}\), e.g.,

\[
\int_\sigma f(x)\delta^i(x-y)d\sigma_i(x) = \int_\sigma f(x^0, \bar{x})\delta^{(3)}(\bar{x} - \bar{y}) \delta_{i0} \delta(x^0 \delta_{i0}, \bar{x} x = f(x^0, \bar{y}) \delta_{i0,0},
\]

(110)

where \( x^0 \) is to be taken on the hypersurface in question, i.e., \( x^0 = t_0 \). Thus, if \( y^0 = t_0 \) (i.e., if \( y \) lies on the hypersurface \( t = t_0 \), the result is simply \( f(y) \), while else, we find zero. Thus, \( \delta^i(x-y) \) can be seen as covariant generalization of the three dimensional delta function.

Two useful relations are the following (see [19])

\[
\int_\sigma f_i d\sigma_k = \int_\sigma f_k d\sigma_i.
\]

(111)

and

\[
\int_\sigma f(x)n_i d\sigma_k(x) = \int_\sigma f(x)n_k d\sigma_i(x).
\]

(112)

\(^{12}\)We use both \( x^0 \) (and \( y^0 \) etc.) as well as \( t \) for the time coordinate, while \( t_0 \) always refers to a constant.
Note that in general, \( n_t = n_t(x) \), but we will omit the argument whenever there is no danger of confusion. In order for the above relations to hold, an appropriate asymptotical behavior of \( f \) has to be assumed, such that two dimensional integrals over the boundary of \( \sigma \) can be omitted.

In particular, we have

\[
\int_\sigma f(x)\delta^i(x-y)d\sigma_i = \int_\sigma f(x)\delta^i(x-y)n^k_{\cdot k}d\sigma_i
\]

which is equal to \( f(y) \) if \( y \) lies on the hypersurface. (We take the convention that all quantities whose arguments are not written explicitly are to be taken at the point \( x \).) Let us introduce the following definitions

\[
d\sigma = n^i d\sigma_i \quad \delta_{\sigma}(x-y) = \delta^i(x-y)n_i \quad \delta_{\sigma}'(x-y) = \delta_{\sigma}(x-y)n^i = \delta^m(x-y)n_m n^i.
\]

We can thus write

\[
\int_\sigma f(x)\delta^i(x-y)d\sigma_i(x) = \int_\sigma f(x)\delta_{\sigma}'(x-y)d\sigma_i(x) = \int_\sigma f(x)\delta_{\sigma}(x-y)d\sigma = f(y) \quad (117)
\]

where for the last relation, it is assumed that \( y \) lies on the hypersurface. Moreover, we have \( \delta^i(x-y)n_i = \delta_{\sigma}'(x-y)n_i \).

Nevertheless, one should not confuse \( \delta^i(x-y) \) which is given explicitly by (111), with \( \delta_{\sigma}'(x-y) \), which is defined with respect to a specific hypersurface. In particular, for \( t = t_0 \), we have \( \delta_{\sigma}'(x-y) = \delta_{\sigma,0}g^0(y, x) \), \( \delta^i(x-y) \) leads to \( n_0 = 1/\sqrt{g^{00}} \) and \( n^i = g^{0i}/g^{00} \). In particular, in flat spacetime, we see that \( \delta_{\sigma}'(x-y) \) has only one non-vanishing component, in contrast to \( \delta_{\sigma}(x-y) \).

In order to derive the commutation relations with the charges we encounter in section 8, the following relation is useful

\[
\int_\sigma f(x)[\delta_{\sigma}'(x-y)]_i d\sigma_i(x) = -f_{,i}(y) + f_{,i}(y)n^k_{\cdot k}(y)n_i + \int_\sigma f(x)[\delta_{\sigma}'(x-y)]_i d\sigma_i(x), \quad (118)
\]

which is easily derived with the help of (111) and (112). The last term in (118) cannot be simplified without specifying a hypersurface. (Note that \( [\delta^i(x-y)]_i = 0 \), but \( [\delta_{\sigma}'(x-y)]_i \neq 0 \).) For the hypersurface \( t = const \), we find by partial integration \( -f_{,\mu}(g^{0\mu}/g^{00})\delta_{\sigma}^0 \), and thus

\[
\int_\sigma f(x)[\delta_{\sigma}'(x-y)]_i d\sigma_i(x) = -f_{,i} + f_{,0}\delta_{\sigma}^0. \quad (119)
\]

With the help of those relations, it is an easy task to evaluate the commutators \((66)-(68)\) and \((69)-(71)\).

**Appendix B: Propagating fields in linearized general relativity**

We consider a perturbation \( h_{ik} \) on a flat background \( g_{ik} = \eta_{ik} + h_{ik} \). The field equations of general relativity, \( \sqrt{-g}G_{ik} = \nabla_i h_{jk} \) to first order in \( h_{ik} \) have the form

\[
\frac{1}{2} \left[ -\Box \psi_{ik} + \psi_{i,k,i} + \psi_{i,k,j} - \eta_{ik}\psi^{lm}_{,j,m} \right] = \nabla_i h_{jk}, \quad (120)
\]

where \( \psi_{ik} = h_{ik} - \frac{1}{2} h \), and \( h = \eta^{ik} h_{ik} \). In particular, to this order, we have the conservation law \( \nabla^i h_{ik} = 0 \). Further, we have the gauge freedom of the linearized theory, \( \delta h_{ik} = \xi_{i,k} + \xi_{k,i} \), where \( \xi^i \) is assumed to be of the same order as \( h_{ik} \).

All indices are raised and lowered with \( \eta_{ik} \).
Our intention is to look for a Coulomb type gauge choice, where the non-propagating fields can be eliminated by solving the field equations (Gauss’ law in electromagnetism) and the remaining fields satisfy a conventional wave equation. Note that, with the conventional gauge choice \( \psi^{ik}, \xi = 0 \), the field equations take the form \( T_{ik} = -\frac{1}{2} \Box \psi_{ik} \), which is easy to solve, but it does not tell us anything on the number of propagating fields (there are still 6 independent fields).

A more convenient gauge choice can be read off the field equations after a 3+1 split,

\[
\begin{align*}
T_{00} &= \frac{1}{2} (\Delta \psi_{00} - \psi_{\mu \nu, \mu \nu}) \\
T_{0\mu} &= \frac{1}{2} (\Delta \psi_{0\mu} + \psi_{i,0,\mu} + \psi_{0,\mu,0} + \psi_{0,0,\mu}) \\
T_{\mu \nu} &= \frac{1}{2} (\Box \psi_{\mu \nu} + \psi_{\mu,0,\nu} + \psi_{0,\mu,\nu} + \psi_{i,\nu,\lambda,\mu} + \psi_{0,\nu,0,\mu} + \psi_{0,0,\nu,\mu}) - \frac{1}{2} \eta_{\mu \nu} (\psi^{00,0,0} + \psi^{\lambda,\lambda,\lambda} + 2 \psi^{0\lambda,0\lambda,\lambda}).
\end{align*}
\]

where \( \Delta = -\partial^\mu \partial_\mu \) and \( \Box = \partial^2 - \Delta \). We choose the following gauge conditions

\[
\psi_{\mu \nu, \mu} = 0, \quad \psi_{0, m, n} = 0.
\]

The first condition on the three four-vectors \( \psi^{\mu m} \) is analogous to the Coulomb gauge in Maxwell theory, while the second one can be seen as a Lorentz gauge on the four-vector \( \psi^{\mu m} \). To show that it is indeed possible to impose such conditions on \( \psi^{ik} \), consider the variations under gauge transformations of the quantities in question. We find

\[
\psi^{0 m, n} \rightarrow \psi^{0 m, n} + \Box \xi^\mu, \quad \psi_{\mu \nu, \mu} \rightarrow \psi_{\mu \nu, \mu} + \xi^{0, \mu, \mu} - \xi^{\mu, \nu}.
\]

Therefore, we perform first a transformation with \( \xi^0 \) satisfying \( \Box \xi^0 = -\psi^{0 m, n} \) (and with \( \xi^\mu = 0 \)), and then a transformation with \( \xi^\mu \) satisfying \( \Delta \xi^\mu = -\psi^{\mu \nu, \nu} \) (and with \( \xi^0 = 0 \)). Both transformations exists generally, and since the second transformation does not modify the result of the first, the argument is complete. Note that it is not possible to require \( \psi_{\mu \nu, \mu} = 0 \) together with \( \psi^{0 m, n} = 0 \) instead of \( \Box \xi^0 = \xi^{0, \mu, \mu} + \Delta \xi^\mu = 0 \).

The field equations take the simplified form

\[
\begin{align*}
T_{00} &= \frac{1}{2} \Delta \psi_{00} \\
T_{0\mu} &= \frac{1}{2} \Delta \psi_{0\mu} \\
T_{\mu \nu} &= -\frac{1}{2} \Box \psi_{\mu \nu} + \frac{1}{2} (\psi_{\mu,0,\nu} + \psi_{0,\mu,\nu} - \eta_{\mu \nu} \psi^{0 \lambda,0 \lambda,\lambda}).
\end{align*}
\]

The residual gauge freedom is now restricted to

\[
\Box \xi^0 = 0, \quad \xi^{0,\mu,\mu} + \Delta \xi^\mu = 0.
\]

Similar as in electrodynamics, we can eliminate the non-propagating modes by solving \( T_{00} \) and \( T_{0\mu} \). We have

\[
\begin{align*}
\psi_{00}(t, x) &= -\frac{1}{2 \pi} \int \frac{T_{00}(t, x')}{|x - x'|} d^3x', \\
\psi_{0\mu}(t, x) &= -\frac{1}{2 \pi} \int \frac{T_{0\mu}(t, x')}{|x - x'|} d^3x'.
\end{align*}
\]

It is an easy task to verify that this solution satisfies indeed the constraint \( \psi^{0 m, n} = 0 \). They are Coulomb type solutions, and for slowly moving matter distributions, the important contribution is given by \( T_{00} \), corresponding to the Newtonian potential.

To some extent, we have obtained a system quite similar to electrodynamics in the Coulomb gauge. As was to be expected, to the charge density correspond the four momentum density components \( T_{00} \), and the corresponding fields \( \psi_{00} \) can be eliminated by a Gauss type law. Also, just like \( A^\mu = 0 \), we have a remaining, symmetric transverse tensor satisfying \( \psi_{\mu \nu, \mu} = 0 \), containing the propagating modes. Further, the non-propagating modes appear as an additional
source to the tensor $\psi_{\mu\nu}$ in (124). This too is in straight analogy to the Maxwell equation $\Box A^\alpha = -j^\alpha + A^{\alpha\nu}g_{\nu0}$ in the Coulomb gauge.

To simplify equation (124), we observe that at large distances from the matter distribution (far zone), we have approximately $|x - x'| \approx |x|$, such that the four equations (130) and (131) can be approximated by

$$\psi_{0i}(t, x) = -\frac{1}{2\pi} \frac{1}{|x|} \int T_{0i}(t, x')d^3x'.$$

(132)

On the other hand, we have in the linear theory the relation $T_{ik} = 0$ which leads upon integration to the conservation law $(d/dt) \int T_{0i}d^3x = 0$. In particular, from (132), we thus have $\psi_{0i,0} = 0$, and therefore equation (128) reduces to

$$\Box \psi_{\alpha\beta} = -2T_{\alpha\beta}.$$  

(133)

We also retain that from $\psi_{0i,0} = 0$, together with our gauge conditions (124), we find

$$\psi^{d}_{,t} = 0 \text{ and } \psi^{d\alpha}_{,\beta} = 0 \text{ for } |x| >> \ell,$$

(134)

where $\ell$ characterizes the dimensions of the matter distribution. Therefore, the conventional Lorentz type gauge condition holds again in this limit. Together with (133), we see that our field equations, as far as $\psi_{\mu\nu}$ is concerned, are completely equivalent to those of the conventional approach based on the gauge $\psi_{ik}$. Nevertheless, we are one step ahead, because we did not simply forget the components $\psi_{a0}, \psi_{00}$, but we have eliminated them properly via a Gauss type law.

The above approach is probably the closest one can get to the Coulomb analogy of Maxwell’s theory. There is only one disturbing point: The field $\psi^{\mu\nu}$ satisfying the transversality condition $\psi^{\mu\nu}_{,\nu} = 0$ contains still three independent degrees of freedom, which is one more than those needed for the description of the massless spin 2 field. It is not hard to see that for plane wave solutions, the residual gauge freedom (124) can be used to make $\psi^{\mu\nu}$ traceless, thus excluding the existence of an additional scalar field. It should therefore be possible to eliminate a fifth component of $\psi^{ik}$ either by an additional Gauss type law, e.g., the trace of $\psi^{ik}$ or of $\psi^{\mu
u}$ and relate it to a fifth charge (e.g., the trace of $T^{ik}$ or of $T^{\mu\nu}$), or simply to impose an additional condition of $\psi^{\mu\nu}$ to reduce the degrees of freedom to two. However, with the gauge freedom (124), this does not seem to be possible. Neither would the appearance of a fifth charge be physically satisfying. We therefore suggest that there exists a better gauge choice, such that upon eliminating four components by a Gauss type law and imposing four conditions on the remaining components, the gauge is completely fixed, and the degrees of freedom are reduced to two. We do not know whether such a gauge has been proposed in literature.

In any case, we see that the propagating modes (in a perturbative approach) can be assumed to be contained in the spatial components of $\psi_{ik}$, which is (up to a constant) the linear approximation of $1/\sqrt{-g}g_{ik}$, the tensor with inverse determinant of $g_{ik}$. It is now an easy matter to evaluate the action of the generator $P + \hat{P}$ on this field, using (130) and (131). We find

$$[P + \hat{P}, \frac{1}{\sqrt{-g}}g_{\mu\nu}] = -\frac{1}{\sqrt{-g}}g_{\mu\lambda}\epsilon^{\lambda}_{,\nu} - \frac{1}{\sqrt{-g}}g_{\nu\lambda}\epsilon^{\lambda}_{,\mu} - \frac{1}{\sqrt{-g}}g_{\nu\lambda}\epsilon^{\lambda}_{,\mu} + \frac{1}{\sqrt{-g}}g_{\mu\nu}\epsilon^{\lambda}_{,\lambda},$$

(135)

where we have assumed that $\epsilon^0 = 0$. This is the Lie derivative of the pseudotensor $1/\sqrt{-g}g_{\mu\nu}$, confirming our assumption that the secondary constraint $P + \hat{P} = 0$ generates coordinate transformations (on the hypersurface $t = const$) on the propagating components of the field, in complete analogy to the case of non-abelian gauge theory.

Finally, we note that the gauge choice (124) is the linearized version of $(\sqrt{-g}g^{00})_{,t} = 0$ and $(\sqrt{-g}g^{\mu\nu})_{,\mu} = 0$.

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