Optimal Control Problems of Forward-Backward Stochastic Volterra Integral Equations

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(Dedicated to Professor Xunjing Li on His 80th Birth Anniversary)

Abstract

Optimal control problems of forward-backward stochastic Volterra integral equations (FBSVIEs in short) are formulated and studied. A general duality principle is established for linear backward stochastic integral equation and linear stochastic Fredholm-Volterra integral equation with mean-field. With the help of such a duality principle, together with some other new delicate and subtle skills, Pontryagin type maximum principles are proved for two optimal control problems of FBSVIEs.

Keywords: Forward-backward stochastic Volterra integral equations, adapted M-solution, stochastic maximum principle, duality principle, stochastic Fredholm-Volterra integral equations

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1 Introduction

Let $(Ω, F, P)$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $\{W(t), t \geq 0\}$ is defined with $\mathcal{F} = \{\mathcal{F}_t \,|\, t \geq 0\}$ being its natural filtration augmented by all the $P$-null sets. We point out that assuming $W(\cdot)$ to be one-dimensional is just for the simplicity of presentation; Our results remain for the case of multi-dimensional Brownian motions.

Let us start with a classical stochastic optimal control problem. To this end, we consider the following controlled stochastic differential equation (SDE, for short):

$$\begin{cases}
  dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), & s \in [0, T], \\
  X(0) = x,
\end{cases}$$

with cost functional

$$J^0(x; u(\cdot)) = \mathbb{E}\left[h(X(T)) + \int_0^T g(s, X(s), u(s))ds\right],$$

where $b, \sigma, h, g$ are some suitable maps, the control $u(\cdot)$ is taken from some suitable set $U$, and the state process $X(\cdot)$ is valued in $\mathbb{R}^n$. In the above, components of $X(\cdot)$ could be wealth, owned commodities/assets,
inventory of products, and some economic factors (interest rates, unemployment rate), and so on. Also, \( u(\cdot) \) can be regarded as some kind of investment, production effort, labor force, transaction of assets, etc. Then our classical optimal control can be stated as follows.

**Problem (C)**. For given \( x \in \mathbb{R}^n \), find a \( \bar{u}(\cdot) \in \mathcal{U} \), called an optimal control, such that

\[
J^0(x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}} J(x; u(\cdot)).
\]

Standard results for the above Problem (C)\(^0\) can be found, say, in [19].

Now let us make some further analysis on the equation (1.1) which can be written as follows:

\[
X(t) = x + \int_0^t b(s, X(s), u(s))ds + \int_0^t \sigma(s, X(s), u(s))dW(s), \quad t \in [0, T].
\]

Although the state equation has the above integral form, it is still memoryless, in the sense that the increment \( X(t + \Delta t) - X(t) \) of the state on \([t, t + \Delta t]\) only depends on the local “driving force” \( \{(b(s, X(s), u(s)), \sigma(s, X(s), u(s)))\}, \ s \in [t, t + \Delta t]\) :

\[
X(t + \Delta t) - X(t) = \int_t^{t+\Delta t} b(s, X(s), u(s))ds + \int_t^{t+\Delta t} \sigma(s, X(s), u(s))dW(s).
\]

Whereas, in reality, memory or long-term dependence often exists. In another word, the increment \( X(t + \Delta t) - X(t) \) of the state on \([t, t + \Delta t]\) might depend on the “driving force” of non-infinitesimal time duration, say, \([t - \tau, t]\), for some \( \tau > 0 \). For example, the current production level usually depends on some renovation of production equipments some time ago, the profit of investment usually depends on the transactions some time before, the air pollution is caused by some bad production strategies some years ago, etc. To model various possible situations with memory, instead of (1.4), we may consider the following controlled (forward) stochastic Volterra integral equation (FSVIE, for short):

\[
X(t) = \varphi(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dW(s), \quad t \in [0, T].
\]

Unlike (1.4), due to the dependence of \( (b(t, s, x, u), \sigma(t, s, x, u)) \) on \( t \), even for the case \( \varphi(t) \equiv x \), we have

\[
X(t + \Delta t) - X(t) = \int_t^{t+\Delta t} b(t + \Delta t, s, X(s), u(s))ds + \int_t^{t+\Delta t} \sigma(t + \Delta t, s, X(s), u(s))dW(s)
\]

\[
+ \int_0^t \left[ b(t + \Delta t, s, X(s), u(s)) - b(t, s, X(s), u(s)) \right] ds
\]

\[
+ \int_0^t \left[ \sigma(t + \Delta t, s, X(s), u(s)) - \sigma(t, s, X(s), u(s)) \right] dW(s),
\]

which depends not only on the values of the “driving force” in \([t, t + \Delta t]\), but also on those in the whole interval \([0, t]\) up to the current time \( t \). Therefore, with suitable choices of \( b \) and \( \sigma \), it is possible to model certain memory effects through (1.5). Based on these arguments, one can use deterministic or stochastic Volterra integral equations to describe some economic models, see [3, 8, 4], for examples.

On the other hand, let us turn to the cost functional (1.2). As we know, stochastic differential utility (SDU, for short) introduced by Duffie–Epstein ([2]) can be represented by backward stochastic differential equations (BSDEs, for short). More precisely, if \( C(\cdot) \) is a consumption process and \( \xi \) is a payoff at the terminal time \( T \), then an SDU process \( Y(\cdot) \) for the pair \( (C(\cdot), \xi) \) can be modeled by the following:

\[
Y(t) = \mathbb{E} \left[ \xi + \int_t^T g(s, Y(s), C(s))ds \bigg| \mathcal{F}_t \right], \quad t \in [0, T],
\]
for some suitable map \( g \). This can also be regarded as a dynamic risk measure process associated with the pair \((\xi, C(\cdot))\). It turns out that (1.6) admits the following equivalent form,

\[
Y(t) = \xi + \int_t^T g(s, Y(s), C(s))ds - \int_t^T Z(s)dW(s), \quad s \in [0, T],
\]

which is a BSDE whose solution is a pair \((Y(\cdot), Z(\cdot))\) of \( \mathbb{F} \)-adapted processes ([8, 7, 19]). Note that if we let

\[
\xi = h(X(T)), \quad u(\cdot) = C(\cdot),
\]

then the cost functional (1.2) admits the following representation:

\[
J^0(x; u(\cdot)) = Y(0).
\]

Thanks to the further development of BSDEs, one could extend the SDU theory via more general BSDEs, namely, one may define an SDU process \((Y(\cdot), Z(\cdot))\) as the adapted solution to the following general BSDE:

\[
\begin{aligned}
    dY(t) &= -g(t, Y(t), Z(t), C(t))dt + Z(t)dW(t), \quad t \in [0, T], \\
    Y(T) &= \xi,
\end{aligned}
\]

whose equivalent integral form reads

\[
Y(t) = \xi + \int_t^T g(s, Y(s), Z(s), C(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T].
\]

Although it is very general, the above is still challenged by the following two aspects: (i) The terminal payoff/cost \( \xi \) is time-independent; (ii) the “running utility/cost” is of memoryless feature. These two lead to the time-consistency of the utility process \( Y(\cdot) \), which is a little too ideal. In reality, substantial evidence (see [1], for example) shows that people in the real life are more concerned (or impatient) about the choices (or decisions) for the immediate future, but are more rational (or patient) when facing long-term alternatives. Such a phenomenon is just one particular case of time inconsistency. Therefore inspired by the theory of backward stochastic Volterra integral equations (BSVIEs, for short) ([6, 13, 14, 15, 16, 17, 18]), we could modify the above into the following form, taking into account of our controlled FSVIE:

\[
Y(t) = \psi(t, X(t), X(T)) + \int_t^T g(t, s, X(t), X(s), Y(s), Z(t, s), Z(s, t), u(s))ds \\
- \int_t^T Z(t, s)dW(s), \quad t \in [0, T],
\]

where \((Y(\cdot), Z(\cdot, \cdot))\) is a so-called adapted \( M \)-solution of the above (see [18]). In the above, we see that both \( X(t) \) and \( X(T) \) appear in the free-term \( \psi(t, X(t), X(T)) \). A motivation of that is the following: Suppose \( X(\cdot) \) represent the production level process of certain product. One expects that the terminal level should be within a certain range determined by the current level, due to the limitation of resource, manpower, machine capacity, market demand/price, etc. Some similar explanations can be made for the appearance of both \( X(t) \) and \( X(s) \) in the integrand.

Motivated by the above arguments, in this paper we study the following controlled forward-backward stochastic Volterra integral equations (FBSVIEs, for short):

\[
\begin{aligned}
    X(t) &= \varphi(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dW(s), \\
    Y(t) &= \psi(t, X(t), X(T)) + \int_t^T g(t, s, X(t), X(s), Y(s), Z(t, s), Z(s, t), u(s))ds \\
    &\quad - \int_t^T Z(t, s)dW(s), \quad t \in [0, T],
\end{aligned}
\]
We call \((X(\cdot), Y(\cdot), Z(\cdot, \cdot))\) the state and \(u(\cdot)\) the control. In such a system, \(X(\cdot)\) and \(Y(\cdot)\) can be regarded as the portfolio process and the dynamic risk process, respectively. To introduce the cost functional, we need to separate two cases.

First of all, if the generator \(g(\cdot)\) of the BSVIE in (1.10) is independent of \(Z(s, t)\), then the state equation reads:

\[
\begin{align*}
X(t) &= \varphi(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dW(s), \\
Y(t) &= \psi(t, X(t), X(T)) + \int_t^T g(t, s, X(t), X(s), Y(s), Z(t, s), u(s))ds \\
&\quad - \int_t^T Z(t, s)dw(s), \quad t \in [0, T].
\end{align*}
\]

(1.11)

In this case, under some mild conditions, for any control \(u(\cdot)\), there exists a unique triplet \((X(\cdot), Y(\cdot), Z(\cdot, \cdot))\), called the adapted solution to (1.11), such that

\[
\begin{align*}
&\quad\quad s \mapsto (X(s), Y(s)) \text{ is } \mathbb{F}\text{-adapted on } [0, T], \\
&\quad\quad s \mapsto Z(t, s) \text{ is } \mathbb{F}\text{-adapted on } [t, T], \quad \forall t \in [0, T),
\end{align*}
\]

and (1.11) is satisfied in the usual Itô’s sense. Moreover, \(Y(\cdot) \in C_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}^n))\). Therefore, \(Y(0)\) is well-defined and for such a case, we may introduce the following cost functional:

\[
J_1(u(\cdot)) = \mathbb{E}\left[ h(X(T), Y(0)) + \int_0^T \int_t^T f(t, s, X(s), Y(s), Z(t, s), u(s))dsdt \right].
\]

(1.13)

Note that in the above case, the process \(Z(t, s)\) is only defined for \(0 \leq t \leq s \leq T\).

On the other hand, if the generator \(g(\cdot)\) depends on \(Z(s, t)\), then, by [18], under some suitable conditions, there exists a unique triplet \((X(\cdot), Y(\cdot), Z(\cdot, \cdot))\), called the adapted \(M\)-solution to (1.10), such that

\[
\begin{align*}
&\quad\quad s \mapsto (X(s), Y(s)) \text{ is } \mathbb{F}\text{-adapted on } [0, T], \\
&\quad\quad s \mapsto Z(t, s) \text{ is } \mathbb{F}\text{-adapted on } [0, T], \quad \text{a.e. } t \in [0, T],
\end{align*}
\]

and in addition to (1.10) being satisfied in the usual Itô’s sense, one also has

\[
Y(t) = \mathbb{E}Y(t) + \int_0^t Z(t, s)dw(s), \quad \text{a.s., a.e. } t \in [0, T].
\]

(1.15)

Different from the first case, in this second case, the process \(Z(t, s)\) is defined on \([0, T]^2\), and the additional relation (1.15) holds. In this second case, due to the fact that \(t \mapsto Z(t, s)\) (for \(t \leq s\)) is not necessarily continuous, we could not expect the continuity of \(t \mapsto Y(t)\). Therefore, \(Y(0)\) might not be well-defined in general. Consequently, the corresponding cost functional should not contain the term like \(h(X(T), Y(0))\) as in \(J_1(u(\cdot))\). Fortunately, a comparison theorem found in [14] suggests that in the current case, it might be more proper to use \(\mathbb{E}\int_0^T Y(s)ds\) as an alternative for \(Y(0)\). Hence, we propose the following cost functional:

\[
J_2(u(\cdot)) = \mathbb{E}\left[ h\left(X(T), \mathbb{E}\int_0^T Y(s)ds \right) + \int_0^T \int_0^T f(t, s, X(s), Y(s), Z(t, s), u(s))dsdt \right].
\]

(1.16)

From the above, we see that one can formulate two different optimal control problems. In this paper, we will establish Pontryagin type maximum principles for the optimal control problems corresponding to
the above two settings. It is known that in deriving maximum principle, besides the suitable variation of the state equation and cost functional, the key is to have a duality principle. The major contribution of this paper is the discovery of a duality principle for general linear BSVIEs, which is a significant extension of that presented in [18]. It turns out that our new duality principle involves a special type of stochastic Fredholm-Volterra integral equations, and we are able to obtain its solvability under natural conditions. It is worthy of pointing out that in contract with SDE case ([9, 10, 19]), we need to carry out all the calculations without differentiation due to the lack of Itô’s formula for stochastic integral equations.

The rest of this paper is organized as follows. In Section 2, some basic results concerning BSVIEs are recalled. In Section 3, we state two maximum principles for controlled FBSVIEs. A general dual principle for linear BSVIEs is established in Section 4. Then the stated maximum principles are proved in Section 5. Section 6 concludes the paper.

2 Results for BSVIEs Revisited

In this section, we are going to recall some relevant results for BSVIEs. To this end, let us first introduce some spaces. For \( H = \mathbb{R}^n, \mathbb{R}^{n \times m} \), etc., we denote its norm by \(| \cdot |\). For \( 0 \leq s < t \leq T \), define

\[
L^2_T(\Omega; H) = \left\{ \xi : \Omega \to H \mid \xi \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{E}[|\xi|^2 < \infty] \right\},
\]

\[
L^2_T(s, t; H) = \left\{ X : [s, t] \times \Omega \to H \mid X(\cdot) \text{ is } \mathcal{F}_T\text{-measurable}, \mathbb{E} \int_s^t |X(r)|^2 dr < \infty \right\},
\]

\[
L^2_T(\Omega; C(s, t; H)) = \left\{ X : [s, t] \times \Omega \to H \mid X(\cdot) \text{ is } \mathcal{F}_T\text{-measurable, has continuous paths,} \mathbb{E} \left( \sup_{r \in [s, t]} |X(r)|^2 \right) < \infty \right\},
\]

\[
C_{\mathcal{F}_T}(s, t; L^2(\Omega; H)) = \left\{ X : [s, t] \to L^2_T(\Omega; H) \mid X(\cdot) \text{ is continuous, } \sup_{r \in [s, t]} \mathbb{E}[|X(r)|^2 < \infty] \right\}.
\]

Also, we define

\[
L^2_T(s, t; H) = \left\{ X(\cdot) \in L^2_T(s, t; H) \mid X(\cdot) \text{ is } \mathbb{F}\text{-adapted} \right\}.
\]

The spaces \( L^2_T(\Omega; C(s, t; H)) \) and \( C_{\mathbb{F}}(s, t; L^2(\Omega; H)) \) can be defined in the same way. It should be pointed out that

\[
L^2_T(\Omega; C(s, t; H)) \subseteq C_{\mathcal{F}_T}(s, t; L^2(\Omega; H)),
\]

\[
L^2_T(\Omega; C(s, t; H)) \subseteq C_{\mathbb{F}}(s, t; L^2(\Omega; H)),
\]

and the equalities do not hold in general. Further, we denote

\[
\Delta = \left\{ (t, s) \in [0, T]^2 \mid t \leq s \right\}, \quad \Delta^* = \left\{ (t, s) \in [0, T]^2 \mid t \geq s \right\} \equiv \overline{\Delta},
\]

and let

\[
L^2_{\mathbb{F}}(\Delta; H) = \left\{ Z : \Delta \times \Omega \to H \mid s \mapsto Z(t, s) \text{ is } \mathbb{F}\text{-adapted on } [t, T], \text{ a.e. } t \in [0, T], \mathbb{E} \int_0^T \int_t^T |Z(t, s)|^2 dsdt < \infty \right\},
\]

\[
L^2(0, T; L^2_{\mathbb{F}}(0, T; H)) = \left\{ Z : [0, T]^2 \times \Omega \to H \mid s \mapsto Z(t, s) \text{ is } \mathbb{F}\text{-adapted on } [0, T], \text{ a.e. } t \in [0, T], \mathbb{E} \int_0^T \int_0^T |Z(t, s)|^2 dsdt < \infty \right\}.
\]
We denote
\[
\begin{align*}
\mathcal{H}_\Delta^2[0, T] &= L_2^0(0, T; \mathbb{R}^m) \times L_2^0(\Delta; \mathbb{R}^m), \\
\mathcal{H}^2[0, T] &= L_2^0(0, T; \mathbb{R}^m) \times L^2(0, T; L_2^0(0, T; \mathbb{R}^m)).
\end{align*}
\]

Further, we let \( \mathcal{M}^2[0, T] \) be the set of all \((y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}^2[0, T] \) such that
\[
y(t) = \mathbb{E}y(t) + \int_0^t z(t, s)dW(s), \quad \text{a.s., a.e. } t \in [0, T].
\]

Clearly, \( \mathcal{M}^2[0, T] \) is a closed subspace of \( \mathcal{H}^2[0, T] \). Also, for any \((y(\cdot), z(\cdot, \cdot)) \in \mathcal{M}^2[0, T], \) we have
\[
\mathbb{E}|y(t)|^2 = |\mathbb{E}y(t)|^2 + \mathbb{E} \int_0^t |z(t, s)|^2 ds \geq \mathbb{E} \int_0^t |z(t, s)|^2 ds, \quad \text{a.e. } t \in [0, T].
\]

The above implies that for any \( \beta \geq 0 \), there exists a constant \( K > 0 \) depending on \( \beta \) such that
\[
\| (y(\cdot), z(\cdot, \cdot)) \|^2_{\mathcal{M}_\beta^2[0, T]} \equiv \mathbb{E} \left[ \int_0^T |y(t)|^2 dt + \int_0^T \int_0^T |z(t, s)|^2 ds dt \right] \\
\leq 2\mathbb{E} \left[ \int_0^T |y(t)|^2 dt + \int_0^T \int_0^T |z(t, s)|^2 ds dt \right] \equiv 2\mathbb{E} \| (y(\cdot), z(\cdot, \cdot)) \|^2_{\mathcal{H}^2[0, T]} \\
\leq K\mathbb{E} \left[ \int_0^T e^{\beta t} |y(t)|^2 dt + \int_0^T e^{\beta t} \int_0^T |z(t, s)|^2 ds dt \right] \equiv K\mathbb{E} \| (y(\cdot), z(\cdot, \cdot)) \|^2_{\mathcal{M}_\beta^2[0, T]} \\
\leq K\| (y(\cdot), z(\cdot, \cdot)) \|^2_{\mathcal{H}^2[0, T]},
\]

which means that \( \cdot \| \cdot \|_{\mathcal{H}^2[0, T]} \) is an equivalent norm of \( \cdot \| \cdot \|_{\mathcal{M}^2[0, T]} \) on \( \mathcal{M}^2[0, T] \). Next, we let \( \mathcal{M}^2[0, T] \) be the set of all \((y(\cdot), z(\cdot, \cdot)) \in \mathcal{M}^2[0, T] \) such that \( t \mapsto z(t, s) \) is continuous on \([0, s] \). Following the ideas in, for example, Lemma 2.4 (p.133) and Proposition 2.6 (p.134) of [5], one can see that any measurable process in \( \mathcal{M}^2[0, T] \) can be approximated by the one in \( \mathcal{M}^2[0, T] \). Therefore \( \mathcal{M}^2[0, T] \) is dense in \( \mathcal{M}^2[0, T] \).

We now consider the following two types of BSVIE:
\[
Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s), \quad \text{a.s., a.e. } t \in [0, T].
\]

(2.4)
\[
Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s), \quad \text{a.s., a.e. } t \in [0, T],
\]

Note that (2.4) is a special case of (2.3) with the generator \( g(\cdot) \) independent of \( Z(s, t) \). We recall the following definition.

**Definition 2.2.** (i) A pair \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]\) is called an adapted solution to BSVIE (2.4) if for almost every \( t \in [0, T] \), \( Y(t) \) is \( \mathcal{F}_t \)-measurable, \( s \mapsto Z(t, s) \) is \( \mathbb{F} \)-adapted on \([t, T]\), and (2.4) is satisfied in the usual Itô’s sense.

(ii) A pair \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{M}^2[0, T]\) is called an adapted M-solution to BSVIE (2.3) if for almost every \( t \in [0, T] \), \( Y(t) \) is \( \mathcal{F}_t \)-measurable, \( s \mapsto Z(t, s) \) is \( \mathbb{F} \)-adapted on \([0, T]\), and (2.3) is satisfied in the usual Itô’s sense.

We point out that for BSVIE (2.4) the values \( Z(t, s) \) of \( Z(\cdot, \cdot) \) with \( t \geq s \) are irrelevant. Hence, adapted solutions \((y(\cdot), z(\cdot, \cdot)) \) of (2.4) need only belong to \( \mathcal{H}_\Delta^2[0, T]\).

The following is a hypothesis for the coefficients of BSVIE (2.3).
(H1) Let $g : [0, T]^2 \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ be measurable such that $(y, z, z') \mapsto g(t, s, y, z, z')$ is uniformly Lipschitz, and

\begin{equation}
E \int_0^T \left( \int_0^T |g(t, s, 0, 0)| ds \right)^2 dt \leq \infty, \quad y, z, z' \in \mathbb{R}^m.
\end{equation}

For BSVIE (2.4), we introduce the following hypothesis:

(H2) Let $g : [0, T]^2 \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n$ be measurable such that $(y, z) \mapsto g(t, s, y, z)$ is uniformly Lipschitz,

\begin{equation}
E \int_0^T \left( \int_0^T |g(t, s, 0, 0)| ds \right)^2 dt \leq \infty,
\end{equation}

and there exists a modulus of continuity $\rho : [0, \infty) \to [0, \infty)$ (a continuous and monotone increasing function with $\rho(0) = 0$) such that

\begin{equation}
|g(t, s, y, z) - g(t', s, y, z)| \leq \rho(|t - t'|)(|y| + |z|), \quad \forall t, t' \in [0, T], \ y, z \in \mathbb{R}^m.
\end{equation}

We have the following result concerning BSVIEs (2.3)–(2.4). One can essentially find proofs of such a result in [16, 18, 11, 12]. For convenience, we provide a direct proof.

**Theorem 2.3.** (i) Let (H1) hold. Then for any $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m)$, BSVIE (2.3) admits a unique adapted $M$-solution $(Y(\cdot), Z(\cdot, \cdot)) \in H^2[0, T]$, and the following holds:

\begin{equation}
E \int_0^T |Y(t)|^2 ds + E \int_0^T \int_0^T |Z(t, s)|^2 ds dt 
\leq K \left[ E \int_0^T |\psi(t)|^2 dt + \int_0^T \left( \int_0^T |g(t, s, 0, 0)| ds \right)^2 dt \right].
\end{equation}

If $\psi_i(\cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m)$, $g_i(\cdot)$ satisfies (H1) and $(Y_i(\cdot), Z_i(\cdot, \cdot))$ are corresponding adapted $M$-solutions, then

\begin{equation}
E \int_0^T |Y_1(t) - Y_2(t)|^2 dt + E \int_0^T \int_0^T |Z_1(t, s) - Z_2(t, s)|^2 ds dt 
\leq K \left[ \int_0^T |\psi_1(t) - \psi_2(t)|^2 dt 
+ \int_0^T \left( \int_0^T |g_1(t, s, Y_1(s), Z_1(t, s), Z_1(s, t)) - g_2(t, s, Y_1(s), Z_1(t, s), Z_1(s, t))| ds \right)^2 dt \right].
\end{equation}

(ii) Let (H2) hold. Then for any $\psi(\cdot) \in C_{\mathcal{F}_T}(0, T; L^2(\Omega; \mathbb{R}^m))$, BSVIE (2.4) admits a unique adapted solution $(Y(\cdot), Z(\cdot, \cdot)) \in C_{\mathcal{F}_T}(0, T; L^2(\Omega; \mathbb{R}^m)) \times L^2(\Delta; \mathbb{R}^m)$ such that

\begin{equation}
\sup_{t \in [0, T]} E|Y(t)|^2 + \sup_{t \in [0, T]} E \int_0^T |Z(t, s)|^2 ds 
\leq K \left\{ \sup_{t \in [0, T]} E|\psi(t)|^2 + \sup_{t \in [0, T]} E \left( \int_0^T |g(t, s, 0, 0)| ds \right)^2 \right\}.
\end{equation}
If \( g_i(\cdot) \) satisfies (H2), \( \psi_i(\cdot) \in C_{Fr}(0,T; L^2(\Omega; \mathbb{R}^m)) \), and \( (Y_i(\cdot), Z_i(\cdot, \cdot)) \in C_T(0,T; L^2(\Omega; \mathbb{R}^m)) \times L^2(\Delta; \mathbb{R}^m) \) is the unique adapted solution of BSVIE (2.4) corresponding to \( (g_i(\cdot), \psi_i(\cdot)) \), then

\[
\sup_{t \in [0,T]} \mathbb{E}[|Y_1(t) - Y_2(t)|^2 + \int_t^T |Z_1(t,s) - Z_2(t,s)|^2 ds] 
\leq K \left\{ \sup_{t \in [0,T]} \mathbb{E} \left[ |\psi_1(t) - \psi_2(t)|^2 \right] + \sup_{t \in [0,T]} \mathbb{E} \left( \int_t^T |g_1(t,s,Y_1(s),Z_1(t,s)) - g_2(t,s,Y_1(s),Z_1(t,s))| ds \right)^2 \right\}.
\]

(2.11)

Proof. First, we let \( \psi(\cdot) \in C_{Fr}([0,T]; L^2(\Omega; \mathbb{R}^m)) \), and assume that \( t \mapsto g(t,s,y,z,z') \) is continuous. For any \( (y(\cdot), z(\cdot, \cdot)) \in \mathcal{M}^2[0,T] \), consider the following BSDE parameterized by \( t \in [0,T] \):

\[
\eta(t,r) = \psi(t) + \int_r^T g(t,s,y(s),z(s,t)) ds - \int_r^T \zeta(t,s) dW(s), \quad r \in [0,T].
\]

(2.12)

Under our conditions, the above BSDE admits a unique adapted solution

\[
(\eta(t, \cdot), \zeta(t, \cdot)) \in L^2_T(\Omega; C([0,T]; \mathbb{R}^m)) \times L^2(\Delta; \mathbb{R}^m),
\]

and the following holds:

\[
\mathbb{E} \left[ \sup_{r \in [t,T]} |\eta(t,r)|^2 + \int_t^T |\zeta(t,s)|^2 ds \right] \leq K \mathbb{E} \left[ |\psi(t)|^2 + \left( \int_t^T |g(t,s,y(s),0,0,0)| ds \right)^2 \right]
\]

\[
\leq K \mathbb{E} \left\{ |\psi(t)|^2 + \int_t^T |y(s)|^2 ds + \left( \int_t^T |y(s)|^2 ds \right)^2 \right\}.
\]

(2.13)

We now let

\[
Y(t) = \eta(t,t), \quad t \in [0,T], \quad Z(t,s) = \zeta(t,s), \quad (t,s) \in \Delta,
\]

(2.14)

and define \( Z(t,s) \) with \( t \geq s \) through the following:

\[
Y(t) = \mathbb{E} Y(t) + \int_0^t Z(t,s) dW(s), \quad t \in [0,T].
\]

(2.15)

Then \( (Y(\cdot), Z(\cdot, \cdot)) \) is an adapted M-solution to the following BSVIE:

\[
Y(t) = \psi(t) + \int_t^T g(t,s,y(s),Z(t,s),z(s,t)) ds - \int_t^T Z(t,s) dW(s), \quad t \in [0,T],
\]

and from (2.13), one has

\[
\mathbb{E} \left[ |Y(t)|^2 + \int_t^T |Z(t,s)|^2 ds \right] \leq K \mathbb{E} \left\{ |\psi(t)|^2 + \left( \int_t^T |g_0(t,s)| ds \right)^2 + \int_t^T |y(s)|^2 ds + \left( \int_t^T |z(s,t)| ds \right)^2 \right\},
\]

(2.16)

where

\[
g_0(t,s) = g(t,s,0,0,0).
\]
Consequently, noting (2.1),

\[
\|(Y(\cdot), Z(\cdot, \cdot))\|_{\mathcal{H}^2[0,T]}^2 = \mathbb{E}\left\{ \int_0^T |Y(t)|^2 \, dt + \int_0^T \int_0^T |Z(t,s)|^2 \, ds \, dt \right\}
\]

\[
\leq \mathbb{E}\left\{ 2 \int_0^T |Y(t)|^2 \, dt + \int_0^T \int_0^T |Z(t,s)|^2 \, ds \, dt \right\}
\]

\[
\leq K \mathbb{E}\left\{ \left( \int_0^T |g_0(t,s)| \, ds \right)^2 + \int_0^T |\psi(t)|^2 \, dt + \int_0^T |y(t)|^2 \, dt + \int_0^T \left( \int_0^T |z(s,t)| \, ds \right)^2 \, dt \right\}
\]

\[
\leq K \mathbb{E}\left\{ \left( \int_0^T |g_0(t,s)| \, ds \right)^2 + \int_0^T |\psi(t)|^2 \, dt + \int_0^T |y(t)|^2 \, dt \right\}
\]

\[
\leq K \left\{ \left( \int_0^T |g_0(t,s)| \, ds \right)^2 + \|\psi(\cdot)\|_{L^2_x(0,T;\mathbb{R}^m)}^2 + \|(y(\cdot), z(\cdot, \cdot))\|_{\mathcal{M}^2[0,T]} \right\}.
\]

Hence, if we define \(\Theta(g(\cdot), z(\cdot, \cdot)) = (Y(\cdot), Z(\cdot, \cdot))\), then \(\Theta\) maps from \(\mathcal{M}^2[0,T]\) to \(\mathcal{M}^2[0,T]\). Note that such a map also depends on the choice of the free term \(\psi(\cdot)\) and the generator \(g(\cdot)\). We now show that the mapping \(\Theta\) can be extended to \(\mathcal{M}^2[0,T]\) and the extension, still denoted by \(\Theta\), is contractive and also has a stability property with respect to \((\psi(\cdot), g(\cdot))\). To this end, we take any \(\psi_i(\cdot) \in C_{\mathcal{F}_T}(0,T;L^2(\Omega;\mathbb{R}^m))\), \(g_i(\cdot)\) satisfies (H1) with \(t \mapsto g_i(t, s, y, z, z')\) being continuous, and \((y_i(\cdot), z_i(\cdot, \cdot)) \in \mathcal{M}^2[0,T]\) \((i = 1, 2)\). Let \((\eta_i(\cdot), \zeta_i(\cdot, \cdot))\) be the adapted solution of BSDE (2.12) with \((\psi_i(\cdot), g_i(\cdot))\) and \((y(\cdot), z(\cdot, \cdot))\) replaced by \((\psi_i(\cdot), g_i(\cdot))\) and \((y_i(\cdot), z_i(\cdot, \cdot))\), respectively. By the stability of adapted solutions to BSDEs, one has

\[
\mathbb{E}\left[ \sup_{r \in [t,T]} |\eta_i(t,r) - \eta_2(t,r)|^2 + \int_t^T |\zeta_i(t,s) - \zeta_2(t,s)|^2 \, ds \right]
\]

\[
\leq K \mathbb{E}\left\{ \left| \psi_i(t) - \psi_2(t) \right|^2 + \left( \int_t^T |g_i(t,s,y_i(s),\zeta_i(t,s),z_1(s,t)) - g_2(t,s,y_2(s),\zeta_1(t,s),z_2(s,t))| \, ds \right)^2 \right\}
\]

\[
\leq K \mathbb{E}\left\{ \left| \psi_i(t) - \psi_2(t) \right|^2 + \left( \int_t^T |\delta g(t,s)| \, ds \right)^2 \right\}
\]

\[
+ \int_t^T |y_i(s) - y_2(s)|^2 \, ds + \left( \int_t^T |z_i(s,t) - z_2(s,t)| \, ds \right)^2 \right\},
\]

with

\[
\delta g(t,s) = g_i(t,s,y_i(s),\zeta_i(t,s),z_1(s,t)) - g_2(t,s,y_1(s),\zeta_1(t,s),z_2(s,t)).
\]

Then it by defining \((Y_i(\cdot), Z_i(\cdot, \cdot))\) similar to (2.14)–(2.15), one has

\[
\mathbb{E}\left[ |Y_1(t) - Y_2(t)|^2 + \int_t^T |Z_1(t,s) - Z_2(t,s)|^2 \, ds \right]
\]

\[
\leq K \mathbb{E}\left[ \left| \psi_1(t) - \psi_2(t) \right|^2 + \left( \int_t^T |\delta g(t,s)| \, ds \right)^2 \right]
\]

\[
+ \int_t^T |y_1(s) - y_2(s)|^2 \, ds + \left( \int_t^T |z_1(s,t) - z_2(s,t)| \, ds \right)^2 \right].
\]

(2.17)
Making use of a relation similar to (2.1) for \((y_1(\cdot), z_1(\cdot, \cdot), z_2(\cdot, \cdot))\), we have
\begin{equation}
\mathbb{E} \int_0^T e^{\beta t} \left[ \int_t^T |y_1(s) - y_2(s)|^2 ds + \left( \int_t^T |z_1(s, t) - z_2(s, t)| ds \right)^2 \right] dt \\
= \mathbb{E} \left[ \int_0^T \left( \int_0^T e^{\beta s} ds \right) |y_1(s) - y_2(s)|^2 ds + \left( \int_0^T e^{\beta s} \int_0^s |z_1(s, t) - z_2(s, t)| ds \right)^2 dt \right] \\
\leq \mathbb{E} \left[ \frac{1}{\beta} \int_0^T e^{\beta t} |y_1(t) - y_2(t)|^2 dt + \frac{1}{\beta} \int_0^T e^{\beta t} \left( \int_t^T e^{-\beta s} ds \right) \left( \int_t^T e^{\beta s} |z_1(s, t) - z_2(s, t)|^2 ds \right) dt \right] \\
\leq \frac{1}{\beta} \mathbb{E} \int_0^T e^{\beta t} |y_1(t) - y_2(t)|^2 dt + \frac{1}{\beta} \mathbb{E} \int_0^T e^{\beta t} \int_0^s |z_1(s, t) - z_2(s, t)|^2 ds dt ds \\
\leq \frac{2}{\beta} \mathbb{E} \int_0^T e^{\beta t} |y_1(t) - y_2(t)|^2 dt \leq \frac{2}{\beta} \|(y_1(\cdot), z_1(\cdot, \cdot)) - (y_2(\cdot), z_2(\cdot, \cdot))\|^2_{\mathcal{M}_a^2[0,T]}.
\end{equation}
(2.18)

Consequently,
\begin{equation}
\|(Y(\cdot), Z(\cdot, \cdot)) - (Y(\cdot), Z(\cdot, \cdot))\|^2_{\mathcal{M}_a^2[0,T]} \\
\equiv \int_0^T e^{\beta t} \mathbb{E} \left[ |Y_1(t) - Y_2(t)|^2 + \int_t^T |Z_1(t, s) - Z_2(t, s)|^2 ds \right] dt \\
\leq K \int_0^T e^{\beta t} \mathbb{E} \left[ \|\psi_1(t) - \psi_2(t)\|^2 + \left( \int_t^T |\delta g(t, s)| ds \right)^2 \right] dt \\
+ \int_t^T |y_1(s) - y_2(s)|^2 ds + \left( \int_t^T |z_1(s, t) - z_2(s, t)|^2 ds \right)^2 dt \\
\leq K \mathbb{E} \left\{ \int_0^T e^{\beta t} \left[ \|\psi_1(t) - \psi_2(t)\|^2 + \left( \int_t^T |\delta g(t, s)| ds \right)^2 \right] dt \\
+ \frac{1}{\beta} \|(y_1(\cdot), z_1(\cdot, \cdot)) - (y_2(\cdot), z_2(\cdot, \cdot))\|^2_{\mathcal{M}_a^2[0,T]} \right\}.
\end{equation}
(2.19)

Hence, by letting \(\psi_1(\cdot) = \psi(\cdot)\) and \(g_1(\cdot) = g(\cdot)\), we have
\[\|\Theta(y_1(\cdot), z_1(\cdot, \cdot)) - \Theta(y_2(\cdot), z_2(\cdot, \cdot))\|^2_{\mathcal{M}_a^2[0,T]} \leq \frac{K}{\beta} \|(y_1(\cdot), z_1(\cdot, \cdot)) - (y_2(\cdot), z_2(\cdot, \cdot))\|^2_{\mathcal{M}_a^2[0,T]}\].

This implies that \(\Theta\) can be naturally extended to \(\mathcal{M}^2[0, T]\) since \(\tilde{\mathcal{M}}^2[0, T]\) is dense in \(\mathcal{M}^2[0, T]\). Moreover, since the constant \(K > 0\) appears in the right hand side of the above is independent of \(\beta\), by choosing \(\beta > 0\) large enough, we obtain that the extension of \(\Theta\), still denoted by itself, is a contraction. Hence, \(\Theta\) admits a unique fixed point \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{M}^2[0, T]\), which is the unique adapted M-solution of (2.3).

Note that (2.19) implies that for general \(\psi(\cdot) \in L^2_{F_0}(0, T; \mathbb{R}^m)\) and general generator \(g(\cdot)\) satisfying (H1), by an approximating argument, one can obtain that the corresponding BSVIE admits a unique adapted M-solution \((Y(\cdot), Z(\cdot, \cdot))\). Also, the stability estimate (2.9) follows.

On the other hand, similar to (2.19), we have
\begin{equation}
\|\|(Y(\cdot), Z(\cdot, \cdot))\|^2_{\mathcal{M}_a^2[0,T]} = \int_0^T e^{\beta t} \mathbb{E} \left[ |Y(t)|^2 + \int_t^T |Z(t, s)|^2 ds \right] dt \\
\leq K \int_0^T e^{\beta t} \mathbb{E} \left[ |\psi(t)|^2 + \left( \int_t^T |g_0(t, s)| ds \right)^2 + \int_t^T |Y(t)|^2 ds + \left( \int_t^T |Z(t, s)| ds \right)^2 \right] dt \\
\leq K \mathbb{E} \left\{ \int_0^T e^{\beta t} \left[ |\psi(t)|^2 + \left( \int_t^T |g_0(t, s)| ds \right)^2 \right] dt + \frac{1}{\beta} \|(Y(\cdot), Z(\cdot, \cdot))\|^2_{\mathcal{M}_a^2[0,T]} \right\}.
\end{equation}
which leads to the estimate (2.8).

(ii) Let \( (Y(\cdot), Z(\cdot, \cdot)) \) be the adapted solution of BSVIE (2.4). For any fixed \( t \in [0, T] \), we let \( (\eta(t, \cdot), \zeta(t, \cdot)) \) be the adapted solution to the following BSDE:

\[
\eta(t, r) = \psi(t) + \int_r^T g(t, s, Y(s), \zeta(t, s))ds - \int_r^T \zeta(t, s)dW(s), \quad r \in [0, T].
\]

Then we know that
\[ Y(t) = \eta(t, t), \quad Z(t, s) = \zeta(t, s), \quad 0 \leq t \leq s \leq T. \]

By (2.20), we have

\[
E \left[ \sup_{r \in [t, T]} |\eta(t, r)|^2 + \int_t^T |\zeta(t, s)|^2ds \right] \leq KE \left[ |\psi(t)|^2 + \left( \int_t^T |g(t, s, Y(s), 0)|ds \right)^2 \right] 
\]

\[
\leq KE \left[ |\psi(t)|^2 + \left( \int_0^T |g_0(t, s)|ds \right)^2 + \int_t^T |Y(s)|^2ds \right].
\]

Thus,
\[
E \left[ |Y(t)|^2 + \int_t^T |Z(t, s)|^2ds \right] \leq KE \left[ |\psi(t)|^2 + \left( \int_0^T |g_0(t, s)|ds \right)^2 + \int_t^T |Y(s)|^2ds \right].
\]

By Gronwall’s inequality, we obtain estimate (2.10). This also leads to

\[
E \left[ \sup_{r \in [t, T]} |\eta(t, r)|^2 + \int_t^T |\zeta(t, s)|^2ds \right] \leq KE \left[ |\psi(t)|^2 + \left( \int_0^T |g_0(t, s)|ds \right)^2 \right].
\]

Similar to (2.17), in the current case, we have

\[
E \left[ |Y_1(t) - Y_2(t)|^2 + \int_t^T |Z_1(s) - Z_2(s)|^2ds \right] \leq KE \left[ |\psi_1(t) - \psi_2(t)|^2 + \left( \int_t^T |g(t, s, Y(s), 0)|ds \right)^2 + \int_t^T |Y_1(s) - Y_2(s)|^2ds \right].
\]

Then applying Gronwall’s inequality, we obtain stability estimate (2.11). To prove the continuity of \( t \mapsto Y(t) \), we let \( t, t' \in [0, T] \) and consider the following:

\[
\eta(t, r) - \eta(t', r) = \psi(t) - \psi(t') + \int_r^T \left( g(t, s, Y(s), \zeta(t, s)) - g(t', s, Y(s), \zeta(t', s)) \right)ds
\]

\[- \int_r^T \left( \zeta(t, s) - \zeta(t', s) \right)dW(s), \quad r \in [0, T].
\]

Then the stability of adapted solutions to BSDEs implies that

\[
E \left[ \sup_{r \in [0, T]} |\eta(t, r) - \eta(t', r)|^2 + \int_0^T |\zeta(t, s) - \zeta(t', s)|^2ds \right] \leq KE \left[ |\psi(t) - \psi(t')|^2 + \left( \int_0^T |g(t, s, Y(s), 0) - g(t', s, Y(s), 0)|ds \right)^2 \right]
\]

\[
\leq KE \left[ |\psi(t) - \psi(t')|^2 + \rho(|t - t'|)^2 \left( \int_0^T |Y(s)|^2ds \right) \right]
\]

\[
\leq KE \left\{ |\psi(t) - \psi(t')|^2 + \rho(|t - t'|)^2 \left[ \int_0^T |\psi(t)|^2dt + \int_0^T \left( \int_0^T |g(t, s, 0, 0)|ds \right)^2dt \right] \right\}.
\]
Therefore, one has
\[
\begin{align*}
\lim_{t-t' \to 0} \mathbb{E}|\eta(t, r) - \eta(t', r)|^2 &= 0, \quad \text{uniformly in } r \in [0, T], \\
\lim_{t-r \to r} \mathbb{E}|\eta(t, r) - \eta(t, r')|^2 &= 0, \quad \forall (t, r) \in [0, T]^2.
\end{align*}
\]
Hence, \((t, r) \mapsto \eta(t, r)\) is continuous, i.e.,
\[
\lim_{(t', r') \to (t, r)} \mathbb{E}|\eta(t', r') - \eta(t, r)|^2 = 0, \quad \forall (t, r) \in [0, T]^2.
\]
Consequently, \(t \mapsto \eta(t, t) = Y(t)\) is continuous from \([0, T]\) to \(L^2_{F}(\Omega; \mathbb{R}^m)\), which leads to \(Y(\cdot) \in C^0(0, T; L^2(\Omega; \mathbb{R}^m))\). \(\square\)

### 3 Optimal Control Problems and Maximum Principles

Now, we consider the following controlled FBSVIE:
\[
\begin{align*}
X(t) &= \varphi(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dW(s), \\
Y(t) &= \psi(t, X(t), X(T)) + \int_t^T g(t, s, X(t), X(s), Z(t, s), Z(s, t), u(s))ds \\
& \quad - \int_t^T Z(t, s)dW(s), \quad t \in [0, T],
\end{align*}
\]
where admissible control \(u(\cdot)\) belongs to \(\mathcal{U}[0, T]\) defined by
\[
\mathcal{U}[0, T] = \left\{ u(\cdot) \in L^2_{F}(0, T; \mathbb{R}^d) \mid u(t) \in U, \ a.s., \ a.e. \ t \in [0, T] \right\},
\]
with \(U\) being a nonempty convex subset of \(\mathbb{R}^d\). For convenience, we let \(0 \in U\). Also, we will consider the following FBSVIE which is a special case of (3.1):
\[
\begin{align*}
X(t) &= \varphi(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dW(s), \\
Y(t) &= \psi(t, X(t), X(T)) + \int_t^T g(t, s, X(t), X(s), Y(t), Z(t, s), Z(s, t), u(s))ds \\
& \quad - \int_t^T Z(t, s)dW(s), \quad t \in [0, T],
\end{align*}
\]
A triple
\[
(X(\cdot), Y(\cdot), Z(\cdot, \cdot)) \in C^0(0, T; L^2(\Omega; \mathbb{R}^n)) \times L^2_{F}(0, T; \mathbb{R}^m) \times L^2(0, T; L^2_{F}(\Delta; \mathbb{R}^m))
\]
is called an adapted M-solution of (3.1) if \(X(\cdot)\) satisfies the forward stochastic Volterra integral equation (FSVIE, for short) in (3.1) and \((Y(\cdot), Z(\cdot, \cdot))\) is the adapted M-solution of the BSVIE in (3.1). Also, a triple
\[
(X(\cdot), Y(\cdot), Z(\cdot)) \in C^0(0, T; L^2(\Omega; \mathbb{R}^n)) \times C^0([0, T]; L^2(\Omega; \mathbb{R}^m)) \times L^2_{F}(\Delta; \mathbb{R}^m)
\]
is called an adapted solution of (3.2) if \(X(\cdot)\) satisfies the FSVIE in (3.2) and \((Y(\cdot), Z(\cdot, \cdot))\) is an adapted solution of BSVIE in (3.2).

The following collects some basic assumptions on the coefficients of FBSVIE (3.1).
(H3) Let \( \varphi(\cdot) \in C_T(0, T; L^2(\Omega; \mathbb{R}^n)) \), \( \psi(\cdot, 0, 0) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m) \), and let
\[
\begin{align*}
\begin{cases}
b, \sigma : [0, T]^2 \times \mathbb{R}^n \times U \times \Omega &\to \mathbb{R}^n, \\
g : [0, T]^2 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \times \Omega &\to \mathbb{R}^m
\end{cases}
\end{align*}
\]
be measurable such that
\[
s \mapsto (b(t, s, x, u), \sigma(t, s, x, u), g(t, s, x', x, y, z, z', u))
\]
is \( \mathcal{F} \)-progressively measurable on \([0, T] \),
\[
(x', x, y, z, z', u) \mapsto (b(t, s, x, u), \sigma(t, s, x, u), g(t, s, x', x, y, z, z', u), \psi(t, x', x))
\]
is continuously differentiable with uniformly bounded derivatives, and with the notation
\[
b_0(t, s) \equiv b(t, s, 0, 0), \quad \sigma_0(t, s) \equiv \sigma(t, x, 0, 0), \quad g_0(t, s) \equiv g(t, s, 0, 0, 0, 0, 0, 0),
\]
one has
\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \left( \int_0^T |b_0(t, s)|ds \right)^2 + \int_0^T |\sigma_0(t, s)|ds \right] + \mathbb{E} \int_0^T \left( \int_0^T |g_0(t, s)|ds \right)^2 dt < \infty.
\]
Further, there exists a modulus of continuity \( \rho : [0, \infty) \to [0, \infty) \) (i.e., \( \rho(\cdot) \) is continuous and strictly increasing with \( \rho(0) = 0 \)) such that
\[
|b(t, s, x, u) - b(t', s, x, u)| + |\sigma(t, s, x, u) - \sigma(t', s, x, u)| \leq \rho(|t - t'|)(1 + |x| + |u|),
\]
\[
t, t', s \in [0, T], \ (x, u) \in \mathbb{R}^n \times U.
\]

For FBSVIE (3.2), we introduce the following stronger hypothesis.

(H4) Let \( \varphi(\cdot) \in C_T(0, T; L^2(\Omega; \mathbb{R}^n)) \), \( \psi(\cdot, 0, 0) \in C_T(0, T; L^2(\Omega; \mathbb{R}^m)) \), and let
\[
\begin{align*}
\begin{cases}
b, \sigma : [0, T]^2 \times \mathbb{R}^n \times U \times \Omega &\to \mathbb{R}^n, \\
g : [0, T]^2 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \times \Omega &\to \mathbb{R}^m
\end{cases}
\end{align*}
\]
be measurable such that
\[
s \mapsto (b(t, s, x, u), \sigma(t, s, x, u), g(t, s, x', x, y, z, u))
\]
is \( \mathcal{F} \)-progressively measurable on \([0, T] \),
\[
(x', x, y, z, u) \mapsto (b(t, s, x, u), \sigma(t, s, x, u), g(t, s, x', x, y, z, u), \psi(t, x', x))
\]
is continuously differentiable with uniformly bounded derivatives, and with notation
\[
b_0(t, s) \equiv b(t, s, 0, 0), \quad \sigma_0(t, s) \equiv \sigma(t, x, 0, 0), \quad g_0(t, s) \equiv g(t, s, 0, 0, 0, 0, 0, 0),
\]
one has
\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \left( \int_0^T |b_0(t, s)|ds \right)^2 + \int_0^T |\sigma_0(t, s)|ds + \int_0^T |g_0(t, s)|ds \right]^2 < \infty.
\]
Further, there exists a modulus of continuity \( \rho : [0, \infty) \to [0, \infty) \) such that
\[
|b(t, s, x, u) - b(t', s, x, u)| + |\sigma(t, s, x, u) - \sigma(t', s, x, u)| + |\psi(t, x', x) - \psi(t', x', x)|
\]
\[
+ |g(t, s, x', x, y, z, u) - g(t', s, x', x, y, z, u)| \leq \rho(|t - t'|)(1 + |x| + |x'| + |y| + |z| + |u|),
\]
\[
t, t', s \in [0, T], \ x, x' \in \mathbb{R}^n, \ y, z \in \mathbb{R}^m, \ u \in U.
\]
The following result follows from some standard theory of stochastic Volterra integral equations and those presented in the previous section.

**Proposition 3.1.** (i) Let (H3) hold. Then for any \( u(\cdot) \in \mathcal{U}[0,T] \), (3.1) admits a unique adapted \( M \)-solution \((X(\cdot), Y(\cdot), Z(\cdot, \cdot))\). Moreover, the following holds:

\[
\sup_{t \in [0,T]} \mathbb{E}|X(t)|^2 + \mathbb{E} \int_0^T |Y(t)|^2 dt + \mathbb{E} \int_0^T \int_0^T |Z(t,s)|^2 ds dt \\
\leq K \left\{ \sup_{t \in [0,T]} \mathbb{E}|\varphi(t)|^2 + \sup_{t \in [0,T]} \mathbb{E} \left( \int_0^T |b_0(t,s)| ds \right)^2 + \sup_{t \in [0,T]} \mathbb{E} \int_0^T |\sigma_0(t,s)|^2 ds \\
+ \mathbb{E} \int_0^T |\psi(t,0,0)|^2 dt + \mathbb{E} \int_0^T \int_0^T |g_0(t,s)| ds \right\}.
\]

(3.5)

In addition, if \((\varphi_i(\cdot), b_i(\cdot), \sigma_i(\cdot), \psi_i(\cdot), g_i(\cdot))\) also satisfy (H3), \( u_i(\cdot) \in \mathcal{U}[0,T], i = 1, 2 \), and \((X_i(\cdot), Y_i(\cdot), Z_i(\cdot, \cdot))\) is the adapted \( M \)-solution of the corresponding FBSVIEs, then

\[
\sup_{t \in [0,T]} \mathbb{E}|X_1(t) - X_2(t)|^2 dt + \mathbb{E} \int_0^T |Y_1(t) - Y_2(t)|^2 dt + \mathbb{E} \int_0^T \int_0^T |Z_1(t,s) - Z_2(t,s)|^2 ds dt \\
\leq K \left\{ \sup_{t \in [0,T]} \mathbb{E}|\varphi_1(t) - \varphi_2(t)|^2 + \sup_{t \in [0,T]} \mathbb{E} \left( \int_0^T |b_1(t,s, X_1(s), u_1(s)) - b_2(t,s, X_1(s), u_2(s))| ds \right)^2 \\
+ \mathbb{E} \int_0^T \left( \int_0^T |g_1(t,s, X_1(t), X_1(s), Y_1(s), Z_1(t,s), Z_1(t,s), u_1(s)) - g_2(t,s, X_1(t), X_1(s), Y_1(s), Z_1(t,s), Z_1(t,s), u_2(s))| ds \right)^2 dt \right\}.
\]

(3.6)

(ii) Let (H4) hold. Then for any \( u(\cdot) \in \mathcal{U}[0,T] \), (3.2) admits a unique adapted solution \((X(\cdot), Y(\cdot), Z(\cdot, \cdot))\). Moreover, the following holds:

\[
\sup_{t \in [0,T]} \mathbb{E}|X(t)|^2 + \sup_{t \in [0,T]} \mathbb{E}|Y(t)|^2 + \sup_{t \in [0,T]} \mathbb{E} \int_t^T |Z(t,s)|^2 ds \\
\leq K \left\{ \sup_{t \in [0,T]} \mathbb{E}|\varphi(t)|^2 + \sup_{t \in [0,T]} \mathbb{E} \left( \int_0^T |b_0(t,s)| ds \right)^2 + \sup_{t \in [0,T]} \mathbb{E} \int_0^T |\sigma_0(t,s)|^2 ds \\
+ \sup_{t \in [0,T]} \mathbb{E} |\psi(t,0,0)|^2 + \sup_{t \in [0,T]} \mathbb{E} \left( \int_0^T |g_0(t,s)| ds \right)^2 \right\}.
\]

(3.7)

In addition, if \((\varphi_i(\cdot), b_i(\cdot), \sigma_i(\cdot), \psi_i(\cdot), g_i(\cdot))\) satisfy (H4), \( u_i(\cdot) \in \mathcal{U}[0,T], i = 1, 2 \), and \((X_i(\cdot), Y_i(\cdot), Z_i(\cdot, \cdot))\) is the adapted \( M \)-solution of the corresponding FBSVIEs, then

\[
\sup_{t \in [0,T]} \mathbb{E}|X_1(t) - X_2(t)|^2 dt + \sup_{t \in [0,T]} \mathbb{E}|Y_1(t) - Y_2(t)|^2 dt + \sup_{t \in [0,T]} \mathbb{E} \int_0^T |Z_1(t,s) - Z_2(t,s)|^2 ds dt \\
\leq K \left\{ \sup_{t \in [0,T]} \mathbb{E}|\varphi_1(t) - \varphi_2(t)|^2 + \sup_{t \in [0,T]} \mathbb{E} \left( \int_0^T |b_1(t,s, X_1(s), u_1(s)) - b_2(t,s, X_1(s), u_2(s))| ds \right)^2 \\
+ \mathbb{E} \int_0^T \left( \int_0^T |g_1(t,s, X_1(t), X_1(s), Y_1(s), Z_1(t,s), Z_1(t,s), u_1(s)) - g_2(t,s, X_1(t), X_1(s), Y_1(s), Z_1(t,s), Z_1(t,s), u_2(s))| ds \right)^2 dt \right\}.
\]

(3.8)
We see that whether the generator \( g(t, s, y, z, z') \) depends on \( z' \) will have different regularity of \( Y(\cdot) \) in general. Therefore, we will introduce two different optimal control problems.

First, we consider state equation (3.2). Since for such a case, \( Y(0) \) is well defined, we may introduce the cost functional as follows:

\[
J_1(u(\cdot)) = E\left[h(X(T), Y(0)) + \int_0^T \int_t^T f(t, s, X(s), Y(s), Z(t, s), u(s)) ds dt\right].
\]

For the involved functions \( h \) and \( f \) in (3.9), we impose the following hypothesis.

**H5** Let \( h : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R} \) be measurable such that \( x, y \mapsto h(x, y) \) are continuously differentiable with the derivatives of \( h \) being bounded by \( K(1 + |x| + |y|) \) and \( K(1 + |x| + |y| + |z| + |u|) \), respectively.

Now, we state our first optimal control problem.

**Problem (C1).** With the state equation (3.2), find \( \bar{u}(\cdot) \) such that

\[
J_1(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U[0, T]} J_1(u(\cdot)).
\]

Any \( \bar{u}(\cdot) \in U[0, T] \) satisfying (3.10) is called an *optimal control* of Problem (C1). The corresponding state process, denoted by \( (\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot)) \), is called an *optimal state process*, and \( (\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot)) \) is called an *optimal 4-tuple of Problem (C1)*. To make the statement of the maximum principle for Problem (C1) simpler, let us introduce the following notations which will also be used in Section 5. For any given optimal 4-tuple \( (\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot)) \) of Problem (C1), we denote

\[
b_x(t, s) = b_x(t, s, \bar{X}(s), \bar{u}(s)), \quad b_u(t, s) = b(t, s, \bar{X}(s), \bar{u}(s)).
\]

The notations \( \sigma_x(t, s), \sigma_u(t, s), \psi_x(t), \psi_{x'}(t), g_x(t, s), g_{x'}(t, s), g_y(t, s), g_{y'}(t, s), g_z(t, s), g_{z'}(t, s), g_u(t, s), h_x, h_y, f_x(t, s), f_y(t, s), \) and \( f_u(t, s) \) are similar. Also, for any scalar valued function, say \( x \mapsto f(t, s, x, y, z, u) \), \( f_x(t, s, x, y, z, u) \) is regarded as row vector, i.e., \( \mathbb{R}^{1 \times n} \)-valued. Such a convention will be consistent with vector valued functions, say, \( x \mapsto \psi(t, x, x') \) for which \( \psi_x(t, x, x') \) takes values in \( \mathbb{R}^{m \times n} \). We now state the following maximum principle.

**Theorem 3.2.** Let (H4) and (H5) hold. Let \( (\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot)) \) be an optimal 4-tuple of Problem (C1). Then

\[
\langle \int_s^T [b_u(t, s)^T p(t) + \sigma_u(t, s)^T q(t, s)] dt + b_u(T, s)^T \mu(T) + \sigma_u(T, s)^T \nu(s) \rangle + g_u(0, s)^T \lambda(s) + \int_0^s f_u(t, s)^T dt + \int_0^s g_u(t, s)^T \xi(t) dt, v - \bar{u}(s) \rangle \geq 0,
\]

\[\forall v \in U, \text{ a.e. } t \in [0, T], \text{ a.s. }.
\]
where 

\[
\begin{align*}
\lambda(t) &= \mathbb{E}(h_y^T) + \int_0^t g_x(0,s)^T \lambda(s)dW(s), \\
\xi(t) &= g_y(0,t)^T \lambda(t) + \int_0^t f_y(s,t)^T ds + \int_t^T f_x(t,s)^T dW(s) \\
&\quad + \int_0^t g_y(s,t)^T \mathbb{E}_t[\xi(s)] ds + \int_t^T g_x(t,s)^T \mathbb{E}_t[\xi(t)] dW(s), \\
\mu(t) &= h_x^T + \psi_x(0)^T \lambda(T) + \int_0^T \psi_x(s)^T \xi(s) ds - \int_t^T \nu(s) dW(s), \\
p(t) &= b_x(T,t)^T \mu(T) + \sigma_x(T,t)^T \nu(t) + g_x(0,t)^T \lambda(t) + \int_t^T f_x(s,t)^T ds \\
&\quad + \left(\psi_x(t)^T + \int_t^T g_x(s,t)^T ds\right) \xi(t) + \int_0^T g_x(s,t)^T \xi(s) ds \\
&\quad + \int_t^T \left(b_u(s,t)^T p(s) + \sigma_u(s,t)^T q(s,t)\right) ds - \int_t^T q(t,s) dW(s).
\end{align*}
\]

(3.12)

Note that in (3.12), \(\lambda(\cdot)\) solves an FSDE, \(\xi(\cdot)\) solves a special type of stochastic Fredholm integral equation with mean-field. We will show in the next section that such an equation admits a unique solution \(\xi(\cdot)\) which is not required to be \(\mathbb{F}\)-adapted. The equation for \((\mu(\cdot), \nu(\cdot))\) is a BSDE, and that for \((p(\cdot), q(\cdot, \cdot))\) is a BSVIE. We see that the system (3.12) is a decoupled system.

Next let us consider the state equation (3.1). For this case, we introduce the following cost functional:

\[
J_2(u(\cdot)) = \mathbb{E} \left[ h \left( X(T), \mathbb{E} \int_0^T Y(s) ds \right) + \int_0^T \int_0^T f(t, s, X(s), Y(s), Z(t, s), u(s)) ds dt \right].
\]

(3.13)

For the involved functions \(h\) and \(f\) in (3.13), we impose the following hypothesis.

**(H6)** Let

\[
h : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}, \quad f : [0, T]^2 \times \mathbb{R}^n \times \mathbb{R}^m \times U \times \Omega \to \mathbb{R}
\]

be measurable such that

\[
(x, y) \mapsto h(x, y), \quad (x, y, z, u) \mapsto f(t, s, x, y, z, u)
\]

are continuously differentiable with the derivatives of \(h\) and \(f\) being bounded by \(K(1 + |x| + |y|)\) and \(K(1 + |x| + |y| + |z| + |u|)\), respectively.

We may pose the following problem.

**Problem (C2).** With the state equation (3.1), find \(\bar{u}(\cdot)\) such that

\[
J_2(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J_2(u(\cdot)).
\]

(3.14)

Any \(\bar{u}(\cdot) \in \mathcal{U}[0, T]\) satisfying (3.14) is called an optimal control of Problem (C2). The corresponding state process, denoted by \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot))\), is called an optimal state process, and \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot))\) is called an optimal 4-tuple of Problem (C2). We have the following maximum principle.
Theorem 3.3. Let (H3) and (H6) hold. Let \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot))\) be the corresponding M-solution of FBSVIE (3.1). Then

\[
\langle b_u(T, t)^T \mu(T) + \sigma_u(T, t)^T \nu(t) + \int_0^T f_u(s, t)^T ds + \int_0^T g_u(s, t)^T \xi(s) ds + \int_t^T b_u(s, t)^T p(s) ds + \int_t^T \sigma_u(s, t)^T q(s, t) ds, v - \bar{u}(t) \rangle \geq 0, \\
\forall v \in U, \text{ a.s., a.e. } t \in [0, T],
\]

where \((\xi(\cdot), \mu(\cdot), \nu(\cdot), p(\cdot), g(\cdot, \cdot))\) solves the following adjoint equation:

\[
\begin{align*}
\xi(t) &= h_T^T + \int_0^T f_y(s, t)^T ds + \int_0^T f_z(s, t)^T dW(s) \\
&+ \int_0^T g_y(s, t)^T E_x[\xi(s)] ds + \int_0^T E_x[g_z(s, t)^T \xi(s)] dW(s) \\
&+ \int_0^T g_z(s, t)^T E_x[\xi(t)] dW(s), \\
\mu(t) &= h_T^T + \int_0^T \psi_x(s)^T \xi(s) ds - \int_t^T \nu(s) dW(s), \\
p(t) &= b_x(T, s)^T \mu(T) + \sigma_x(T, s)^T \nu(s) + \int_0^T f_x(s, s)^T ds \\
&+ \left( \psi_x(s)^T + \int_t^T g_x(s, t)^T ds \right) \xi(t) + \int_0^s g_x(s, t)^T \xi(s) ds \\
&+ \int_t^T \left[ b_x(s, t)^T p(s) + \sigma_x(s, t)^T q(s, t) \right] ds - \int_t^T q(t, s) dW(s).
\end{align*}
\]

We see that different from Theorem 3.2, in the above the adjoint equation only consists of three equations, the equation for \(\lambda(\cdot)\) is not necessary here. Actually, we will see that the equation for \(\lambda(\cdot)\) is used to take care of the term involving \(Y(0)\) in the cost functional. Again, (3.15) is also decoupled.

4 Duality Principles

The aim of this section is to establish a duality principle between the following linear BSVIE:

\[
Y(t) = \psi(t) + \int_t^T \left[ A(t, s)Y(s) + B(t, s)Z(t, s) + C(t, s)Z(t, s) \right] ds \\
- \int_t^T Z(t, s) dW(s), \quad t \in [0, T],
\]

and its adjoint equation, where \((Y(\cdot), Z(\cdot, \cdot))\) is the unique M-solution associated with \(\psi(\cdot) \in L_{F^T}^2(0, T; \mathbb{R}^m)\).

We introduce the following hypothesis for the coefficients of the above equation.

\textbf{(H7)} The maps \(A, B, C : [0, T]^2 \times \Omega \to \mathbb{R}^{m \times m}\) are measurable and bounded such \(s \mapsto (A(t, s), B(t, s), C(t, s))\) is \(F\)-adapted on \([t, T]\) for every \(t \in [0, T]\).

By Theorem 2.3, we know that under \textbf{(H7)}, for any \(\psi(\cdot) \in L_{F^T}^2(0, T; \mathbb{R}^m)\), linear BSVIE (4.1) admits a unique adapted M-solution \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{M}^2[0, T]\). Note that in [18] (see Theorem 5.1 there), a duality principle was established for the case that \(Z(t, s)\) does not appear (or \(B(\cdot, \cdot) = 0\)). The significance here in the current paper is that we have discovered the adjoint equation of (4.1) with all the interested terms
appearing and we have the well-posedness of such an equation. We now introduce the adjoint equation for (4.1). For any \((\alpha(\cdot), \beta(\cdot, \cdot)) \in L^2_\mathbb{F}(0, T; \mathbb{R}^m) \times L^2(0, T; L^2_\mathbb{F}(0, T; \mathbb{R}^m))\), consider the following stochastic integral equation:

\[
\xi(t) = \alpha(t) + \int_0^t A(s, t)^T \mathbb{E}_t[\xi(s)] ds + \int_0^t \mathbb{E}_s[C(s, t)^T \xi(s)] dW(s)
\]

\[
+ \int_0^t \beta(t, s) dW(s) + \int_t^T B(t, s)^T \mathbb{E}_s[\xi(t)] dW(s), \quad \text{a.s., a.e. } t \in [0, T],
\]

where \(\mathbb{E}_t[\xi] \triangleq \mathbb{E}[\xi | \mathcal{F}_t]\) for any integrable random variable \(\xi\) and \(r \in [0, T]\). We call (4.2) the **adjoint equation** of linear BSVIE (4.1). It is seen that (4.2) is a mean-field stochastic Fredholm-Volterra type integral equation with some special structure, whose unknown is an \(\mathcal{F}_t\)-measurable process \(\xi(\cdot)\). Unlike usual BSDEs or BSVIEs, in the above, we do not require \(\xi(\cdot)\) to be \(\mathbb{F}\)-adapted. We now state the duality principle.

**Theorem 4.1.** Let (H7) hold. Then for any \((\alpha(\cdot), \beta(\cdot, \cdot)) \in L^2_\mathbb{F}(0, T; \mathbb{R}^n) \times L^2(0, T; L^2_\mathbb{F}(0, T; \mathbb{R}^n))\), equation (4.2) admits a unique solution \(\xi(\cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m)\). Further, if \((Y(\cdot), Z(\cdot, \cdot))\) is the unique \(M\)-solution for (4.1) with \(\psi(\cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^n)\), then

\[
\mathbb{E} \int_0^T (\psi(t), \xi(t)) dt = \mathbb{E} \left\{ \int_0^T (Y(t, \alpha(t))) dt + \int_0^T \int_0^T (Z(t, s), \beta(t, s)) ds dt \right\}.
\]

**Proof.** We first prove the well-posedness of the adjoint equation (4.2). Let \((\alpha(\cdot), \beta(\cdot, \cdot)) \in L^2_\mathbb{F}(0, T; \mathbb{R}^n) \times L^2(0, T; L^2_\mathbb{F}(0, T; \mathbb{R}^n))\) be given. Suppose \(\xi(\cdot)\) is a solution to (4.2). For almost all \(t \in [0, T]\) and any \(r \in [t, T]\), applying \(\mathbb{E}_t\) on the both sides of (4.2), one gets

\[
\mathbb{E}_t[\xi(t)] = \alpha(t) + \int_0^t \beta(t, s) dW(s) + \int_t^T B(t, s)^T \mathbb{E}_s[\xi(t)] dW(s)
\]

\[
+ \int_0^t A(s, t)^T \mathbb{E}_t[\xi(s)] ds + \int_0^t \mathbb{E}_s[C(s, t)^T \xi(s)] dW(s)
\]

\[
= \alpha(t) + \int_0^t \beta(t, s) dW(s) + \int_t^T \left( B(t, s)^T \mathbb{E}_s[\xi(t)] + \beta(t, s) \right) dW(s)
\]

\[
+ \int_0^t A(s, t)^T \mathbb{E}_t[\xi(s)] ds + \int_0^t \mathbb{E}_s[C(s, t)^T \xi(s)] dW(s), \quad \forall r \in [t, T], \ t \in [0, T].
\]

On the other hand, if \(\xi(\cdot)\) satisfies (4.4) for any \(r \in [t, T]\), then taking \(r = T\), we recover (4.2). Thus, to solve (4.2), it suffices to solve (4.4).

For any \(\tilde{\xi}(\cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m)\), and almost every \(t \in [0, T]\), consider the following family of stochastic differential equation with parameter \(t\):

\[
\lambda(t, r) = \alpha(t) + \int_0^t \beta(t, s) dW(s) + \int_t^T \left( B(t, s)^T \lambda(t, s) + \beta(t, s) \right) dW(s)
\]

\[
+ \int_0^t A(s, t)^T \mathbb{E}_t[\xi(s)] ds + \int_0^t \mathbb{E}_s[C(s, t)^T \tilde{\xi}(s)] dW(s), \quad r \in [t, T].
\]

It admits a unique solution \(\lambda(t, \cdot)\) which is \(\mathbb{F}\)-adapted on \([t, T]\), and

\[
\mathbb{E}|\lambda(t, r)|^2 \leq K \mathbb{E}\left\{ |\alpha(t)|^2 + \int_0^r |\beta(t, s)|^2 ds + \int_t^T |\lambda(t, s)|^2 ds + \int_t^T |\tilde{\xi}(s)|^2 ds \right\}, \quad r \in [t, T].
\]

Hence, it follows from Gronwall’s inequality that

\[
\mathbb{E}|\lambda(t, r)|^2 \leq K \mathbb{E}\left\{ |\alpha(t)|^2 + \int_0^r |\beta(t, s)|^2 ds + \int_t^T |\tilde{\xi}(s)|^2 ds \right\}, \quad r \in [t, T].
\]
We define
\[ \xi(t) = \lambda(t, T), \quad \text{a.e. } t \in [0, T]. \]
Then for any \( r \in [t, T] \), from (4.5), one has
\[ E_r[\xi(t)] = E_r[\lambda(t, T)] = \lambda(t, r), \tag{4.7} \]
and it follows from (4.6) that
\[ E[\xi(t)]^2 \leq K E\left\{ |\alpha(t)|^2 + \int_0^T |\beta(t, s)|^2 ds + \int_0^t |\tilde{\xi}(s)|^2 ds \right\}, \quad \text{a.e. } t \in [0, T]. \]
Thus, \( \xi(\cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m) \), and we have defined a map \( \tilde{\xi}(\cdot) \mapsto \xi(\cdot) \) from \( L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m) \) to itself. We now show that this map is a contraction. To this end, we take any \( \bar{\mu} > 0 \), it follows that
\[ \lambda_1(t, r) - \lambda_2(t, r) = \int_t^r B(t, s)^T \left( \lambda_1(t, s) - \lambda_2(t, s) \right) dW(s) \]
\[ + \int_0^t A(s, t)^T E_s[\tilde{\xi}_1(s) - \tilde{\xi}_2(s)] ds + \int_0^t E_s [C(s, t)^T (\tilde{\xi}_1(s) - \tilde{\xi}_2(s))] dW(s). \]
Hence,
\[ E|\lambda_1(t, r) - \lambda_2(t, r)|^2 \leq K E\left( \int_t^r |\lambda_1(t, s) - \lambda_2(t, s)|^2 ds + \int_0^t |\tilde{\xi}_1(s) - \tilde{\xi}_2(s)|^2 ds \right). \]
By Grownall’s inequality, we obtain
\[ E|\lambda_1(t, r) - \lambda_2(t, r)|^2 \leq K E \int_0^t |\tilde{\xi}_1(s) - \tilde{\xi}_2(s)|^2 ds, \quad \forall r \in [t, T]. \]
With the definition
\[ \xi_i(t) = \lambda_i(t, T), \quad t \in [0, T], \text{a.e.} \]
one obtains
\[ E|\xi_1(t) - \xi_2(t)|^2 \leq K E \int_0^t |\tilde{\xi}_1(s) - \tilde{\xi}_2(s)|^2 ds, \quad \text{a.e. } t \in [0, T]. \]
Now, for any \( \mu > 0 \), it follows that
\[ E \int_0^T e^{-\mu t}|\xi_1(t) - \xi_2(t)|^2 dt \leq K E \int_0^T e^{-\mu t} \int_0^t |\tilde{\xi}_1(s) - \tilde{\xi}_2(s)|^2 ds dt \]
\[ = K E \int_0^T \left( \int_s^T e^{-\mu t} dt \right) |\tilde{\xi}_1(s) - \tilde{\xi}_2(s)|^2 ds \leq K \mu \int_0^T e^{-\mu t} |\tilde{\xi}_1(s) - \tilde{\xi}_2(s)|^2 ds. \]
Since \( K \) in the above is an absolute constant, by choosing \( \mu > 0 \) large, we get that the map \( \tilde{\xi}(\cdot) \mapsto \xi(\cdot) \) is a contraction in \( L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^m) \) with a weighted norm. Hence, it admits a unique fixed point which is the unique solution of (4.4).

Next, we prove the duality relation (4.3).

For any given \( \psi(\cdot) \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R}^n) \), let \( (Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{M}^2[0, T] \) be the unique adapted M-solution of (4.1). Then we have
\[ Y(t) = E Y(t) + \int_0^t Z(t, s) dW(s), \quad \text{a.e. } t \in [0, T]. \]
Hence, for any \((\alpha(\cdot), \beta(\cdot, \cdot)) \in L^2_\mathcal{F}(0, T; \mathbb{R}^n) \times L^2(0, T; L^2_\mathcal{F}(0, T; \mathbb{R}^n))\), we have,

\[
\mathbb{E}\left\{ \int_0^T \langle Y(t), \alpha(t) \rangle + \int_0^T \beta(t, s)dW(s) \rangle dt + \int_0^T \int_t^T \langle Z(t, s), \beta(t, s) \rangle dsdt \right\} \\
= \mathbb{E}\left\{ \int_0^T \langle Y(t), \alpha(t) \rangle dt + \int_0^T \int_t^T \langle Z(t, s), \beta(t, s) \rangle dsdt \right\}. 
\]  

(4.9)

On the other hand, let \(\xi(\cdot) \in L^2_\mathcal{F}(0, T; \mathbb{R}^m)\) be the solution to the adjoint equation (4.2) corresponding to \((\alpha(\cdot), \beta(\cdot, \cdot))\). Then

\[
\xi(t) = \alpha(t) + \int_0^t \beta(t, s)dW(s) + \int_0^t B(t, s)^T \mathbb{E}_t[\xi(t)]dW(s) \\
+ \int_0^t A(s, t)^T \mathbb{E}_s[\xi(s)]ds + \int_0^t \mathbb{E}_s[C(s, t)^T \xi(s)]dW(s) \\
= \alpha(t) + \int_0^t A(s, t)^T \mathbb{E}_s[\xi(s)]ds + \int_0^t \left( \mathbb{E}_s[C(s, t)^T \xi(s)] + \beta(t, s) \right)dW(s) \\
+ \int_0^T \left( B(t, s)^T \mathbb{E}_s[\xi(t)] + \beta(t, s) \right)dW(s) \\
= \mathbb{E}_t[\xi(t)] + \int_t^T \xi(t, s)dW(s),
\]

with

\[
\zeta(t, s) = B(t, s)^T \mathbb{E}_s[\xi(t)] + \beta(t, s), \quad s \in [t, T], \text{ a.e. } t \in [0, T]. 
\]  

(4.10)

(4.11)

Consequently,

\[
\mathbb{E} \int_0^T \langle \psi(t), \xi(t) \rangle dt \\
= \mathbb{E}\left\{ \int_0^T \langle Y(t) - \int_t^T \left[ A(t, s)Y(s) + B(t, s)Z(t, s) + C(t, s)Z(s, t) \right]ds, \xi(t) \rangle dt \\
+ \int_0^T \int_t^T \langle Z(t, s), \zeta(t, s) \rangle dsdt \right\} \\
= \mathbb{E}\left\{ \int_0^T \langle Y(t), \xi(t) \rangle - \int_0^t A(s, t)^T \xi(s)ds \right\} dt \\
+ \int_0^T \int_t^T \langle Z(t, s), \zeta(t, s) \rangle - B(t, s)^T \xi(t) \rangle dsdt - \int_0^T \int_0^T \langle Z(t, s), C(s, t)^T \xi(s) \rangle dsdt \right\} \\
= \mathbb{E}\left\{ \int_0^T \langle Y(t), \xi(t) \rangle - \int_0^t A(s, t)^T \xi(s)ds - \int_0^t \mathbb{E}_s[C(s, t)^T \xi(s)]dW(s) \rangle dt \\
+ \int_0^T \int_t^T \langle Z(t, s), \zeta(t, s) - B(t, s)^T \xi(t) \rangle dsdt \right\} \\
= \mathbb{E}\left\{ \int_0^T \langle Y(t), \int_t^t \beta(t, s)dW(s) + \alpha(t) \rangle dt + \int_0^T \int_t^T \langle Z(t, s), \beta(t, s) \rangle dsdt \right\}.
\]

where the last equality in the above follows from (4.10) and (4.11). Thanks to (4.9), our conclusion follows.

Let us recall the following duality principle found in [18], which is a corollary of Theorem 4.1.

**Corollary 4.2.** Let \(X(\cdot)\) be the solution to the following FSVIE:

\[
X(t) = \varphi(t) + \int_0^t A_0(t, s)X(s)ds + \int_0^t C_0(t, s)X(s)dW(s), \quad t \in [0, T], 
\]

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for uniformly bounded measurable \(A_0(\cdot, \cdot)\) and \(C_0(\cdot, \cdot)\), with \(s \mapsto (A_0(t, s), C_0(t, s))\) is \(\mathbb{F}\)-progressively measurable. Let \((p(\cdot), q(\cdot, \cdot))\) be the adapted \(\mathcal{M}\)-solution to the following BSVIE:

\[
p(t) = \psi(t) + \int_t^T \left[ A_0(s, t)^T p(s) + C_0(s, t)^T q(s, t) \right] ds - \int_t^T q(t, s) dW(s), \quad t \in [0, T].
\]

Then

\[
\mathbb{E} \int_0^T \langle \psi(t), X(t) \rangle dt = \mathbb{E} \int_0^T \langle \varphi(t), p(t) \rangle dt.
\]

This is a special case of Theorem 4.1 in which

\[
\alpha(t) = \varphi(t), \quad \beta(t, s) = 0,
\]

and

\[
A(t, s) = A_0(s, t)^T, \quad B(t, s) = 0, \quad C(t, s) = C_0(s, t)^T.
\]

In the current case, \(\xi(\cdot) = X(\cdot)\) is adapted. Therefore,

\[
\mathbb{E}_s[C(s, t)^T \xi(s)] = \mathbb{E}_s[C_0(t, s) X(s)] = C_0(t, s) X(s).
\]

### 5 Proofs of Theorem 3.2 and Theorem 3.3

This section is devoted to the proofs of Theorems 3.2 and 3.3.

As a standard step, to obtain the maximum principle we need to obtain the variation of the state and then use duality principle(s).

For Theorem 3.1, we have the following result.

**Theorem 5.1.** Let (H3) and (H5) hold and \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot))\) be an optimal 4-tuple of Problem (C1). Then for any \(v(\cdot) \in \mathcal{U}[0, T]\),

\[
0 \leq \mathbb{E} \left\{ h_x X_1(T) + h_y Y_1(0) + \int_0^T \left( f_x(t, s) X_1(s) + f_y(t, s) Y_1(s) 
+ f_z(t, s) Z_1(t, s) + f_u(t, s) [v(s) - \bar{u}(s)] \right) ds dt \right\},
\]

where

\[
\begin{align*}
h_x &= h_x(\bar{X}(T), \bar{Y}(0)), & h_y &= h_y(\bar{X}(T), \bar{Y}(0)), \\
f_x(t, s) &= f_x(t, s, \bar{X}(s), \bar{Y}(s), \bar{Z}(t, s), \bar{u}(s)), & f_y(t, s) &= f_y(t, s, \bar{X}(s), \bar{Y}(s), \bar{Z}(t, s), \bar{u}(s)), \\
f_z(t, s) &= f_z(t, s, \bar{X}(s), \bar{Y}(s), \bar{Z}(t, s), \bar{u}(s)), & f_u(t, s) &= f_u(t, s, \bar{X}(s), \bar{Y}(s), \bar{Z}(t, s), \bar{u}(s)),
\end{align*}
\]

and \((X_1(\cdot), Y_1(\cdot), Z_1(\cdot, \cdot))\) is the adapted solution to the following linear FBSVIE:

\[
\begin{align*}
X_1(t) &= \int_0^t \left\{ b_x(t, s) X_1(s) + b_u(t, s) [v(s) - \bar{u}(s)] \right\} ds \\
&\quad + \int_0^t \left\{ \sigma_x(t, s) X_1(s) + \sigma_u(t, s) [v(s) - \bar{u}(s)] \right\} dW(s), \\
Y_1(t) &= \psi_x(t) X_1(t) + \psi_u(t) X_1(T) + \int_t^T \left\{ g_x(t, s) X_1(s) + g_u(t, s) Y_1(s) 
+ g_z(t, s) Z_1(t, s) + g_u(t, s) [v(s) - \bar{u}(s)] \right\} ds - \int_t^T Z_1(t, s) dW(s).
\end{align*}
\]
Proof. Let $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot, \cdot), \tilde{u}(\cdot))$ be an optimal 4-tuple of Problem (C1). Fix any $v(\cdot) \in \mathcal{U}[0, T]$ and any $\varepsilon \in (0, 1)$, let $(X^\varepsilon(\cdot), Y^\varepsilon(\cdot), Z^\varepsilon(\cdot, \cdot))$ be the adapted M-solution of (3.1) corresponding to

$$u^\varepsilon(\cdot) \equiv \tilde{u}(\cdot) + \varepsilon[v(\cdot) - \tilde{u}(\cdot)] = (1 - \varepsilon)\tilde{u}(\cdot) + \varepsilon v(\cdot) \in \mathcal{U}[0, T].$$

Under (H3), one has

$$\sup_{t \in [0, T]} \mathbb{E}[X(t)]^2 + \sup_{t \in [0, T]} \mathbb{E}[X^\varepsilon(t)]^2 < \infty, \quad \forall \varepsilon \in (0, 1].$$

Note that

$$X^\varepsilon(t) = \varphi(t) + \int_0^t b(t, s, X^\varepsilon(s), u^\varepsilon(s))ds + \int_0^t \sigma(t, s, X^\varepsilon(s), u^\varepsilon(s))dW(s), \quad \text{a.e. } t \in [0, T].$$

Thus, by Proposition 3.1, we have

$$\begin{align*}
\sup_{t \in [0, T]} \mathbb{E}[X^\varepsilon(t) - \tilde{X}(t)]^2 &+ \sup_{t \in [0, T]} \mathbb{E}[Y^\varepsilon(t) - \tilde{Y}(t)]^2 + \sup_{t \in [0, T]} \mathbb{E} \int_t^T |Z^\varepsilon(t, s) - \tilde{Z}(t, s)|^2 ds \\
&\leq K \left\{ \sup_{t \in [0, T]} \mathbb{E} \left( \int_0^T |b(t, s, \tilde{X}(s), u^\varepsilon(s)) - b(t, s, \tilde{X}(s), \tilde{u}(s))|^2 ds \right) \right. \\
&\quad + \sup_{t \in [0, T]} \mathbb{E} \left( \int_0^T |\sigma(t, s, \tilde{X}(s), u^\varepsilon(s)) - \sigma(t, s, \tilde{X}(s), \tilde{u}(s))|^2 ds \right) \\
&\left. \quad + \sup_{t \in [0, T]} \mathbb{E} \left( \int_0^T |g(t, s, \tilde{X}(t), \tilde{X}(s), \tilde{Y}(s), \tilde{Z}(t, s), u^\varepsilon(s)) - g(t, s, \tilde{X}(t), \tilde{X}(s), \tilde{Y}(s), \tilde{Z}(t, s), \tilde{u}(s))|^2 ds \right) \right\} \\
&\leq K \varepsilon^2 \mathbb{E} \int_0^T |v(s) - \tilde{u}(s)|^2 ds.
\end{align*}$$

For (5.2), we firstly have

$$\sup_{t \in [0, T]} \mathbb{E}[X_1(t)]^2 \leq K \mathbb{E} \int_0^T |v(t) - \tilde{u}(t)|^2 dt.$$

Then, by Theorem 2.3, one has

$$\begin{align*}
\sup_{t \in [0, T]} \mathbb{E}[Y_1(t)]^2 &+ \sup_{t \in [0, T]} \mathbb{E} \int_t^T |Z_1(t, s)|^2 ds \leq K \mathbb{E} \int_0^T |v(t) - \tilde{u}(t)|^2 dt.
\end{align*}$$

Now, for $t, s \in [0, T]$, let

$$X^\varepsilon_1(t) = \frac{X^\varepsilon(t) - \tilde{X}(t)}{\varepsilon}, \quad Y^\varepsilon_1(t) = \frac{Y^\varepsilon(t) - \tilde{Y}(t)}{\varepsilon}, \quad Z^\varepsilon_1(t, s) = \frac{Z^\varepsilon(t, s) - \tilde{Z}(t, s)}{\varepsilon}.$$
Then we look at the following:

\[
X_1^\varepsilon(t) - X_1(t) = \int_0^t \left\{ \frac{b(t, s, X^\varepsilon(s), u^\varepsilon(s)) - b(t, s, \bar{X}(s), \bar{u}(s))}{\varepsilon} 
- b_x(t, s)X_1(s) - b_u(t, s)[v(s) - \bar{u}(s)] \right\} ds \\
+ \int_0^t \left\{ \frac{\sigma(t, s, X^\varepsilon(s), u^\varepsilon(s)) - \sigma(t, s, \bar{X}(s), \bar{u}(s))}{\varepsilon} 
- \sigma_x(t, s)X_1(s) - \sigma_u(t, s)[v(s) - \bar{u}(s)] \right\} dW(s)
\]

where

\[
b_x(t, s) = \int_0^1 b_x(t, s, \bar{X}(s) + \theta[X^\varepsilon(s) - \bar{X}(s)], \bar{u}(s) + \theta\varepsilon[v(s) - \bar{u}(s)]) d\theta,
\]

and \(b_u(t, s), \sigma_x(t, s), \sigma_u(t, s)\) are defined similarly. Then we have

\[
\mathbb{E}[X_1^\varepsilon(t) - X_1(t)]^2 \leq K \left\{ \mathbb{E}\left[ \int_0^t \left( |b_x(t, s) - b_x(t, s)| |X_1(s)| + |b_u(t, s) - b_u(t, s)| |v(s) - \bar{u}(s)| \right) ds \right]^2 \\
+ \mathbb{E}\left[ \int_0^t \left( |\sigma_x(t, s) - \sigma_x(t, s)|^2 |X_1(s)|^2 + |\sigma_u(t, s) - \sigma_u(t, s)|^2 |v(s) - \bar{u}(s)|^2 \right) ds \right]\right\}.
\]

By the dominated convergence theorem, we have

\[
\lim_{\varepsilon \to 0} \mathbb{E}[X_1^\varepsilon(t) - X_1(t)]^2 = 0, \quad \forall t \in [0, T],
\]

\[
\lim_{\varepsilon \to 0} \mathbb{E}\int_0^T |X_1^\varepsilon(t) - X_1(t)|^2 dt = 0.
\]

Next,

\[
Y_1^\varepsilon(t) - Y_1(t) = \frac{\psi(t, X^\varepsilon(T), Z^\varepsilon(T)) - \psi(t, \bar{X}(t), \bar{Z}(t))}{\varepsilon} 
- \psi_x(t)X_1(t) - \psi_x(t)X_1(T)
\]

\[
+ \int_t^T \left\{ \frac{g(t, s, X^\varepsilon(s), Y^\varepsilon(s)) - g(t, s, \bar{X}(s), \bar{Y}(s))}{\varepsilon} 
- g_x(t, s)X_1(t) - g_x(t, t)X_1(s) \\
- g_y(t, s)Y_1(t) - g_y(t, s)Y_1(s) - g_z(t, s)Z_1(t) - g_z(t, s)Z_1(s) \right\} ds
\]

\[
- \int_t^T \left( Z_1^\varepsilon(t, s) - Z_1(t, s) \right) dW(s)
\]

where

\[
\psi_x(t) = \int_0^1 \psi_x(t, \bar{X}(t) + \theta[X^\varepsilon(t) - \bar{X}(t)], \bar{Z}(t) + \theta[X^\varepsilon(T) - \bar{X}(T)]) d\theta,
\]

\[
\psi_y(t) = \int_0^1 \psi_y(t, \bar{X}(t) + \theta[X^\varepsilon(t) - \bar{X}(t)], \bar{Z}(t) + \theta[X^\varepsilon(T) - \bar{X}(T)]) d\theta,
\]

\[
\psi_z(t) = \int_0^1 \psi_z(t, \bar{X}(t) + \theta[X^\varepsilon(t) - \bar{X}(t)], \bar{Z}(t) + \theta[X^\varepsilon(T) - \bar{X}(T)]) d\theta.
\]
and $\psi_x(t), g_x(t, s), g_x^e(t, s), g_y(t), g_y^e(t, s)$ are defined similarly. Then

\[
\mathbb{E}|Y_1^r(t) - Y_1(t)|^2 + \mathbb{E} \int_t^T |Z_1^r(t, s) - Z_1(t, s)|^2 ds \\
\leq K\mathbb{E}\left|\psi_x(t)X_x(t) - \psi_x(t)X_x(t) + \psi_x^e(t)X_x^e(T) - \psi_x(t)X_x(T)\right|
\]

\[
+ \int_t^T \left\{ g_x^e(t, s)X_x^e(t) - g_x^e(t, s)X_x(t) + g_x^e(t, s)X_x^e(s) - g_x(t, s)X_x(s)\right\} ds \\
+ \left( g_y^e(t, s) - g_y(t, s) \right) Y_1(s) + \left( g_y^e(t, s) - g_y(t, s) \right) Z_1(t, s) \\
+ \left[ g_y^e(t, s) - g_y(t, s) \right][v(s) - \bar{u}(s)] ds
\]

Then by the dominated convergence theorem, we have

\[
\lim_{\varepsilon \to 0} \left( \mathbb{E}|Y_1^r(t) - Y_1(t)|^2 + \mathbb{E} \int_t^T |Z_1^r(t, s) - Z_1(t, s)|^2 ds \right) = 0, \quad \forall t \in [0, T],
\]

\[
\lim_{\varepsilon \to 0} \left( \mathbb{E} \int_t^T |Y_1^r(t)|^2 dt + \mathbb{E} \int_t^T \int_t^T |Z_1^r(t, s) - Z_1(t, s)|^2 ds dt \right) = 0
\]

Now, by the optimality of $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot))$, we have

\[
0 \leq J_1(u^e(\cdot)) - J_1(\bar{u}(\cdot)) = \mathbb{E}\left[h(X^e(T), Y^e(0)) - h(\bar{X}(T), \bar{Y}(0))\right]
\]

\[
+ \int_0^T \int_t^T \frac{f(t, s, X^e(s), Y^e(s), Z^e(t, s), u^e(s)) - f(t, s, \bar{X}(s), \bar{Y}(s), \bar{Z}(t, s), \bar{u}(s))}{\varepsilon} ds dt
\]

\[
= \mathbb{E}\left\{ h_x^e X_1^e(T) + h_y^e Y_1^e(0) + \int_0^T \int_t^T \left( f_x^e(t, s)X_x^e(s) + f_y^e(t, s)Y_x^e(s)\right) ds dt \\
+ f_x^e(t, s)Z_1^e(t, s) + f_y^e(t, s)[v(s) - \bar{u}(s)] \right\},
\]

where

\[
h_x^e = \int_0^1 h_x(\bar{X}(T) + \theta[X^e(T) - \bar{X}(T)], \bar{Y}(0) + \theta[Y^e(0) - Y(0)]) d\theta,
\]

and $h_y^e, f_x^e(t, s), f_y^e(t, s), f_y^e(t, s)$, and $f_u^e(t, s)$ are defined similarly. Passing to the limit in (5.8), we obtain (5.1).

We now present a lemma which will play an interesting role below.

**Lemma 5.2.** Suppose

\[\eta_0 = \xi + \int_0^T g^0(s, \zeta_0(s)) ds - \int_0^T \zeta_0(s) dW(s),\]

where $\xi \in L_2^2(\Omega; \mathbb{R}^m), \eta_0 \in \mathbb{R}^m$ is deterministic, $\zeta_0(\cdot) \in L_2^2(0, T; \mathbb{R}^m), g^0 : [0, T] \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m$ such that $s \mapsto g^0(s, \zeta)$ is progressively measurable for all $\zeta \in \mathbb{R}^m$, almost surely, and $\zeta \mapsto g^0(s, \zeta)$ is uniformly...
Lipschitz continuous. Let \((\eta(\cdot), \zeta(\cdot))\) be the unique adapted solution to the following BSDE:

\[
\eta(t) = \xi + \int_t^T g^0(s, \zeta(s))ds - \int_t^T \zeta(s)dW(s), \quad t \in [0, T].
\]

Then

\[
\eta_0 = \eta(0), \quad \zeta_0(s) = \zeta(s), \quad s \in [0, T], \text{ a.s.}
\]

**Proof.** Under our condition, BSDE (5.10) admits a unique adapted solution \((\eta(\cdot), \zeta(\cdot))\). Let

\[
\tilde{\eta}(t) = \eta_0 - \int_0^t g^0(s, \zeta_0(s))ds + \int_0^t \zeta_0(s)dW(s) = \xi + \int_t^T g^0(s, \zeta_0(s))ds - \int_t^T \zeta_0(s)dW(s), \quad t \in [0, T].
\]

Therefore, \((\tilde{\eta}(\cdot), \zeta_0(\cdot))\) is also an adapted solution of BSDE (5.10). Hence, by uniqueness, one must have

\[
(\tilde{\eta}(\cdot), \zeta_0(\cdot)) = (\eta(\cdot), \zeta(\cdot)).
\]

Consequently, we have (5.11). \qed

We now carry out a proof of Theorem 3.2.

**Proof of Theorem 3.2.** We begin with the variational inequality (5.1) which is rewrite here for convenience:

\[
0 \leq E\left\{ h_x X_1(T) + h_y Y_1(0) + \int_0^T \int_t^T \left( f_x(t, s) X_1(s) + f_y(t, s) Y_1(s) + f_z(t, s) Z_1(t, s) + f_u(t, s)[v(s) - \bar{u}(s)] \right) ds dt \right\},
\]

where \((X_1(\cdot), Y_1(\cdot), Z_1(\cdot, \cdot))\) is the adapted solution to the following linear FBSVIE:

1. \[
X_1(t) = \int_0^t \left\{ b_x(t, s) X_1(s) + b_y(t, s) [v(s) - \bar{u}(s)] \right\} ds
\]
2. \[
+ \int_0^t \left\{ \sigma_x(t, s) X_1(s) + \sigma_y(t, s) [v(s) - \bar{u}(s)] \right\} dW(s),
\]
3. \[
Y_1(t) = \psi_x(t) X_1(t) + \psi_y(t) Y_1(T) + \int_t^T \left\{ g_x(t, s) X_1(s) + g_y(t, s) Y_1(s) + g_z(t, s) Z_1(t, s) + g_u(t, s) [v(s) - \bar{u}(s)] \right\} ds - \int_t^T Z_1(t, s)dW(s).
\]

To obtain the maximum principle, we will use duality principles to eliminate \(Y_1(0), (Y_1(\cdot), Z_1(\cdot, \cdot)), X_1(T), \) and \(X_1(\cdot)\) one by one. Therefore, there are four steps.

**Step 1.** Eliminate \(Y_1(0)\). We note that

\[
Y_1(0) = \psi_x(0) X_1(T) + \int_0^T \left\{ g_x(0, s) X_1(s) + g_y(0, s) Y_1(s) + g_z(0, s) Z_1(0, s) + g_u(0, s) [v(s) - \bar{u}(s)] \right\} ds - \int_0^T Z_1(0, s)dW(s).
\]

By Lemma 5.2, we let \((\eta_1(\cdot), \zeta_1(\cdot))\) be the adapted solution to the following BSDE

\[
\eta_1(t) = \psi_x(0) X_1(T) + \int_t^T \left\{ g_x(0, s) X_1(s) + g_y(0, s) Y_1(s) + g_z(0, s) Z_1(s) + g_u(0, s) [v(s) - \bar{u}(s)] \right\} ds - \int_t^T \zeta_1(s)dW(s).
\]
Then one has
\[ Y_1(0) = \eta_1(0), \quad Z_1(0, s) = \zeta_1(s), \quad s \in [0, T]. \]

Now, let
\[
\begin{align*}
(5.12) \quad \left\{ 
\begin{array}{l}
d\lambda(t) = g_z(0, t)^T \lambda(t) dW(t), \quad t \in [0, T], \\
\lambda(0) = \mathbb{E}(h_y)^T.
\end{array}
\right.
\end{align*}
\]

Then
\[
\mathbb{E}[h_y Y_1(0)] = \mathbb{E} \langle \lambda(0), \eta_1(0) \rangle
= \mathbb{E} \left\{ \langle \lambda(T), \eta_1(T) \rangle + \int_0^T \left( \langle \lambda(s), g_x(0, s) X_1(s) + g_y(0, s) Y_1(s) \\
+ g_z(0, s) \zeta_1(s) + g_u(0, s) [v(s) - \bar{u}(s)] \rangle - \langle g_z(0, s)^T \lambda(s), \zeta_1(s) \rangle \right) ds \right\}
= \mathbb{E} \left\{ \langle \lambda(T), \psi_x(0) X_1(T) \rangle + \int_0^T \langle \lambda(s), g_z(0, s) X_1(s) + g_y(0, s) Y_1(s) + g_u(0, s) [v(s) - \bar{u}(s)] \rangle ds \right\}.
\]

Hence,
\[
J_1(\bar{u} \cdot; v \cdot) = \mathbb{E} \left\{ h_x^T X_1(T) + h_y^T Y_1(0) + \int_0^T \int_t^T \left( f_x(t, s)^T X_1(s) + f_y(t, s)^T Y_1(s) \\
+ f_z(t, s)^T + f_u(t, s)^T [v(s) - \bar{u}(s)] \right) ds dt \right\}
= \mathbb{E} \left\{ \langle h_x^T + \psi_x(0)^T \lambda(T), X_1(T) \rangle + \int_0^T \left( \langle g_x(0, s)^T \lambda(s) + \int_0^s f_x(t, s)^T dt, X_1(s) \rangle \\
+ \langle g_u(0, s)^T \lambda(s) + \int_0^s f_u(t, s)^T dt, v(s) - \bar{u}(s) \rangle \right) ds \right. \\
\left. + \int_0^T \langle g_y(0, s)^T \lambda(s) + \int_0^s f_y(t, s)^T dt, Y_1(s) \rangle ds + \int_0^T \int_t^T \langle f_z(t, s)^T, Z_1(t, s) \rangle ds dt \right\}.
\]

So, \( Y_1(0) \) is eliminated.

**Step 2.** Eliminate \((Y_1 \cdot) , Z_1 \cdot , \cdot \)\). We apply Theorem 4.1 with the following:

\[ A(t, s) = g_y(t, s), \quad B(t, s) = g_z(t, s), \quad C(t, s) = 0, \]

and
\[
\psi(t) = \psi_x(t) X_1(t) + \psi_x(t) X_1(T) + \int_t^T \left\{ \langle g_x(t, s) X_1(t) + g_z(t, s) X_1(s) \\
+ g_u(t, s) [v(s) - \bar{u}(s)] \rangle ds, \right\}
\]

\[
\alpha(t) = g_y(0, t)^T \lambda(t) + \int_0^t f_y(r, t)^T dr, \quad \beta(t, s) = f_z(t, s)^T.
\]

Therefore, let \( \xi(\cdot) \) solve the following:
\[
(5.13) \quad \xi(t) = g_y(0, t)^T \lambda(t) + \int_0^t f_y(r, t)^T dr + \int_t^T f_z(t, r)^T dW(r) + \int_t^T g_z(t, s)^T \mathbb{E}_t[\xi(t)] dW(s) \\
+ \int_0^t g_y(s, t)^T \mathbb{E}_t[\xi(s)] ds, \quad \text{a.e. } t \in [0, T].
\]
Hence, by Theorem 4.1,

\[
\begin{align*}
&\mathbb{E}\left\{ \int_0^T (g_y(0, t))^T \lambda(t) + \int_0^T f_y(r, t)^T dr, Y_1(t) \right\} dt + \int_0^T \int_t^T (f_z(t, s))^T, Z_1(t, s) \right\} dsdt \\
&= \mathbb{E}\left\{ \left( \psi_x(t) + \int_t^T g_x(t, s)ds \right) X_1(t) + \psi_x(t)X_1(T) + \int_t^T g_x(t, s)X_1(s)ds \\
&\quad + \int_t^T g_u(t, s)[v(s) - \bar{u}(s)]ds, \xi(t) \right\} dt.
\end{align*}
\]

(5.14)

Hence,

\[
\begin{align*}
J_1(\bar{u}(.); v(.)) &= \mathbb{E}\left\{ (h^T_x + \psi_x(0))^T \lambda(T), X_1(T) \right\} + \int_0^T \left\{ (g_x(0, s))^T \lambda(s) + \int_0^s f_x(t, s)^T dt, X_1(s) \right\} \\
&\quad + (g_u(0, s))^T \lambda(s) + \int_0^s f_u(t, s)^T dt, v(s) - \bar{u}(s) \right\} ds \\
&\quad + \int_0^T (g_y(0, s))^T \lambda(s) + \int_0^s f_y(t, s)^T dt, Y_1(s) \right\} ds + \int_0^T \int_t^T (f_z(t, s))^T, Z_1(t, s) \right\} dsdt \\
&= \mathbb{E}\left\{ (h^T_x + \psi_x(0))^T \lambda(T), X_1(T) \right\} + \int_0^T \left\{ (g_x(0, s))^T \lambda(s) + \int_0^s f_x(t, s)^T dt, X_1(s) \right\} \\
&\quad + (g_u(0, s))^T \lambda(s) + \int_0^s f_u(t, s)^T dt, v(s) - \bar{u}(s) \right\} ds \\
&\quad + \int_0^T (\psi_x(t) + \int_t^T g_x(t, s)ds) X_1(t) + \psi_x(t)X_1(T) + \int_t^T g_x(t, s)X_1(s)ds \\
&\quad + \int_t^T g_u(t, s)[v(s) - \bar{u}(s)]ds, \xi(t) \right\} dt \\
&= \mathbb{E}\left\{ (h^T_x + \psi_x(0))^T \lambda(T) + \int_0^T \psi_x(t)^T \xi(t)dt, X_1(T) \right\} + \int_0^T \left\{ (g_x(0, s))^T \lambda(s) + \int_0^s f_x(t, s)^T dt \\
&\quad + (\psi_x(s)^T + \int_s^T g_x(s, t)^T dt) \xi(s) + \int_0^s g_x(t, s)^T \xi(t)dt, X_1(s) \right\} \\
&\quad + (g_u(0, s))^T \lambda(s) + \int_0^s f_u(t, s)^T dt + \int_0^s g_u(t, s)^T \xi(t)dt, v(s) - \bar{u}(s) \right\} ds \right\}.
\end{align*}
\]

Thus, \((Y_1(\cdot), Z_1(\cdot, \cdot))\) is eliminated.

Step 3. Eliminate \(X_1(T)\). Let \((\mu(\cdot), \nu(\cdot))\) be the adapted solution to the following BSDE:

\[
\mu(t) = h^T_x + \psi_x(0)^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r)dr - \int_t^T \nu(s)dW(s), \quad t \in [0, T].
\]

Note that

\[
\begin{align*}
X_1(T) &= \int_0^T \left( b_x(T, s)X_1(s) + b_u(T, s)[v(s) - \bar{u}(s)] \right) ds \\
&\quad + \int_0^T \left( \sigma_x(T, s)X_1(s) + \sigma_u(T, s)[v(s) - \bar{u}(s)] \right) dW(s).
\end{align*}
\]
Thus,

\[
\mathbb{E}\left\{ (h_x^T + \psi_x(0))^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r) dr, X_1(T) \right\} \\
= \mathbb{E}\left\{ \int_0^T (h_x^T + \psi_x(0))^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r) dr, b_x(T, s)X_1(s) + b_u(T, s)[v(s) - \bar{u}(s)] \right\} ds \\
+ \int_0^T (\sigma_x(T, s)X_1(s) + \sigma_u(T, s)[v(s) - \bar{u}(s)]) ds \right\}
\]

Consequently,

\[ J_1(\bar{u}(\cdot); v(\cdot)) = \mathbb{E}\int_0^T \left\{ (b_x(T, s))^T (h_x^T + \psi_x(0))^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r) dr \\
+ \sigma_x(T, s)^T \nu(s) + g_x(0, s)^T \lambda(s) + \int_0^s f_x(r, s)^T dr \\
+ \psi_x'(s)^T + \int_0^T g_x'(s, r)^T \xi(r) dr \right\} ds \\
+ (b_u(T, s))^T (h_x^T + \psi_x(0))^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r) dr + \sigma_u(T, s)^T \nu(s) \\
+ g_u(0, s)^T \lambda(s) + \int_0^s f_u(r, s)^T dr + \int_0^s g_u(r, s)^T \xi(r) dr, v(s) - \bar{u}(s) \right\} ds. \]

Thus, \( X_1(T) \) is eliminated.

**Step 4.** Eliminate \( X_1(\cdot) \). We apply Corollary 4.2 with

\[ A_0(t, s) = b_x(t, s), \quad C_0(t, s) = \sigma_x(t, s), \]

and

\[ \psi(t) = b_x(T, t)^T (h_x^T + \psi_x(0))^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r) dr \\
+ \sigma_x(T, t)^T \nu(t) + g_x(0, t)^T \lambda(t) + \int_0^t f_x(r, t)^T dr \\
+ (\psi_x'(t) + \int_t^T g_x'(t, r)^T \xi(r) dr) \xi(t) + \int_0^t g_x(r, t)^T \xi(r) dr. \]
Therefore, we let \((p(\cdot), q(\cdot, \cdot))\) be the adapted M-solution to the following BSVIE:

\[
p(t) = b_x(T, t)^T \left( h_x^T + \psi_x(0)^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r) dr \right) \\
+ \sigma_x(T, t)^T \nu(t) + g_x(0, t)^T \lambda(t) + \int_0^t f_x(r, t)^T dr \\
+ \left( \psi_x(t)^T + \int_t^T g_x(t, r)^T dr \right) \xi(t) + \int_0^t g_x(r, t)^T \xi(r) dr \\
+ \int_0^T \left( b_u(s, t)^T p(s) + \sigma_u(s, t)^T q(s, t) \right) ds - \int_t^T q(t, s) dW(s).
\]

Then, by Corollary 4.2, one obtains

\[
E \int_0^T \langle b_x(T, t)^T \left( h_x^T + \psi_x(0)^T \lambda(T) + \int_0^T \psi_x(r)^T \xi(r) dr \right) \\
+ \sigma_x(T, t)^T \nu(t) + g_x(0, t)^T \lambda(t) + \int_0^t f_x(r, t)^T dr \\
+ \left( \psi_x(t)^T + \int_t^T g_x(t, r)^T dr \right) \xi(t) + \int_0^t g_x(r, t)^T \xi(r) dr, X_1(t) \rangle \ dt \\
= E \int_0^T \langle p(t), \int_0^t b_u(t, s)[v(s) - \bar{u}(s)] ds + \int_0^t \sigma_u(t, s)[v(s) - \bar{u}(s)] dW(s) \rangle \\
= E \int_0^T \int_s^t \langle b_u(t, s)^T p(t) + \sigma_u(t, s)^T q(t, s), v(s) - \bar{u}(s) \rangle \ dtds.
\]

Consequently,

\[
J_1(\bar{u}(\cdot); v(\cdot)) = E \int_0^T \left( \int_s^T [b_u(t, s)^T p(t) + \sigma_u(t, s)^T q(t, s)] \ dt \\
+ b_u(T, s)^T \mu(T) + \sigma_u(T, s)^T \nu(s) + g_u(0, s)^T \lambda(s) \\
+ \int_0^s f_u(t, s)^T dt + \int_0^s g_u(t, s)^T \xi(t) dt, v(s) - \bar{u}(s) \right) ds.
\]

Then the maximum principle follows. \(\square\)

Now, we look at Problem (C2). For that, we have the following result.

**Theorem 5.3.** Let (H4) and (H6) hold. Let \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot))\) be an optimal 4-tuple of Problem (C2). Then for any \(v(\cdot) \in \mathcal{U}[0, T],\)

\[
0 \leq E \left\{ h_x X_1(T) + \int_0^T \left[ \left( \int_0^t f_x(t, s)^T dt \right) X_1(s) + \left( h_y^T + \int_0^t f_y(t, s)^T dt \right) Y_1(s) \\
+ \left( \int_0^T f_y(t, s)^T dt \right) [v(s) - \bar{u}(s)] \right] ds + \int_0^T \int_0^s f_z(t, s)^T Z_1(t, s) ddsdt \right\},
\]

with

\[
h_x = h_x \left( \bar{X}(T), \int_0^T \bar{Y}(t) dt \right), \quad h_y = h_y \left( \bar{X}(T), \int_0^T \bar{Y}(t) dt \right),
\]

and
etc, and \((X_1(\cdot), Y_1(\cdot), Z_1(\cdot, \cdot))\) is the adapted solution to the following linear FBSVIE:

\[
\begin{aligned}
X_1(t) &= \int_0^t \left\{ b_x(t, s)X_1(s) + b_u(t, s) [v(s) - \bar{u}(s)] \right\} ds \\
&\quad + \int_t^T \left\{ \sigma_x(t, s)X_1(s) + \sigma_u(t, s) [v(s) - \bar{u}(s)] \right\} dW(s), \\
Y_1(t) &= \psi_x(t)X_1(t) + \psi_x(t)X_1(T) + \int_t^T \left\{ g_x(t, s)X_1(t) + g_x(t, s)X_1(s) + g_u(t, s)Y_1(s) \right. \\
&\quad \left. + g_x(t, s)Z_1(t, s) + g_x(t, s)Z_1(s, t) + g_u(t, s) [v(s) - \bar{u}(s)] \right\} ds - \int_t^T Z_1(t, s)dW(s),
\end{aligned}
\]

with for example,

\[
\begin{aligned}
b_x(t, s) &= b_x(t, s, \bar{X}(s), \bar{u}(s)), \quad \sigma_x(t, s) = \sigma_x(t, s, \bar{X}(s), \bar{u}(s)), \\
\psi_x(t) &= \psi_x(t, \bar{X}(t)), \quad g_x(t, s) = g_x(t, s, \bar{X}(t), \bar{Y}(s), \bar{Z}(t, s), \bar{Z}(s, t), \bar{u}(s)).
\end{aligned}
\]

and so on.

**Proof.** For optimal 4-tuple \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot))\) of Problem (C2), fix any \(v(\cdot) \in U[0, T]\), for any \(\varepsilon \in (0, 1)\), we define \((X^\varepsilon(\cdot), Y^\varepsilon(\cdot), Z^\varepsilon(\cdot, \cdot), \bar{u}(\cdot))\) similar to the proof of Theorem 5.1. Then

\[
\sup_{t \in [0, T]} E|X^\varepsilon(t) - \bar{X}(t)|^2 + E \int_0^T |Y^\varepsilon(t) - \bar{Y}(t)|^2 dt + E \int_0^T \int_0^T |Z^\varepsilon(t, s) - \bar{Z}(t, s)|^2 ds dt \\
\leq K \left\{ \sup_{t \in [0, T]} E \left( \int_0^T |b(t, s, \bar{X}(s), \bar{u}(s)) - b(t, s, \bar{X}(s), \bar{u}(s))| ds \right)^2 \\
+ E \int_0^T \left( \int_0^T |\sigma(t, s, \bar{X}(s), \bar{u}(s)) - \sigma(t, s, \bar{X}(s), \bar{u}(s))| ds \right)^2 ds \\
+ E \int_0^T \left( \int_0^T |g(t, s, \bar{X}(t), \bar{X}(s), \bar{Y}(s), \bar{Z}(t, s), \bar{Z}(s, t), \bar{u}(s)) - g(t, s, \bar{X}(t), \bar{X}(s), \bar{Y}(s), \bar{Z}(t, s), \bar{Z}(s, t), \bar{u}(s))| ds \right)^2 \right\} \\
\leq K \varepsilon^2 E \int_0^T |v(s) - \bar{u}(s)|^2 ds.
\]

For \(t, s \in [0, T]\), we still let

\[
X^\varepsilon_1(t) = \frac{X^\varepsilon(t) - \bar{X}(t)}{\varepsilon}, \quad Y^\varepsilon_1(t) = \frac{Y^\varepsilon(t) - \bar{Y}(t)}{\varepsilon}, \quad Z^\varepsilon_1(t, s) = \frac{Z^\varepsilon(t, s) - \bar{Z}(t, s)}{\varepsilon}.
\]

For the variational system (5.16), we have

\[
\sup_{t \in [0, T]} E|X_1(t)|^2 \leq K E \int_0^T |v(t) - \bar{u}(t)|^2 dt.
\]

Consequently, by Theorem 2.3,

\[
E \int_0^T |Y_1(t)|^2 dt + E \int_0^T \int_0^T |Z_1(t, s)|^2 ds dt \leq K E \int_0^T |v(t) - \bar{u}(t)|^2 dt.
\]

We also have

\[
\lim_{\varepsilon \to 0} E|X^\varepsilon_1(t) - X_1(t)|^2 = 0, \quad \forall t \in [0, T],
\]

\[
\lim_{\varepsilon \to 0} E \int_0^T |X^\varepsilon_1(t) - X_1(t)|^2 dt = 0.
\]
Next,
\[
Y_{1}^{\varepsilon}(t) - \bar{Y}_{1}(t) = \frac{\psi(t, X^{\varepsilon}(t), X^{\varepsilon}(T)) - \psi(t, \bar{X}(t), \bar{X}(T))}{\varepsilon} - \psi_{x}(t)X_{1}(t) - \psi_{x}(t)X_{1}(T)
\]
\[
+ \int_{t}^{T} \left\{ \frac{1}{\varepsilon} g(t, s, X^{\varepsilon}(t), X^{\varepsilon}(s), Y^{\varepsilon}(s), Z^{\varepsilon}(s, t), u^{\varepsilon}(s))
\right.
\]
\[
- g(t, s, \bar{X}(t), \bar{X}(s), \bar{Y}(s), \bar{Z}(s, t), \bar{u}(s)) \right\} ds
\]
\[
- \int_{t}^{T} \left( Z_{1}^{\varepsilon}(t, s) - Z_{1}(t, s) \right) dW(s)
\]
\[
= \psi_{x}(t)X^{\varepsilon}(t) - \psi_{x}(t)X_{1}(t) + \psi_{x}(t)X_{1}(T) - \psi_{x}(t)X_{1}(T)
\]
\[
+ \int_{t}^{T} \left\{ g_{x}(t, s)X_{1}^{\varepsilon}(t) - g_{x}(t, s)X_{1}(t) + g_{x}(t, s)X_{1}(s) - g_{x}(t, s)X_{1}(s)
\right.
\]
\[
+ g_{0}(t, s)Y_{1}^{\varepsilon}(s) - g_{0}(t, s)Y_{1}(s) + g_{0}(t, s)Z_{1}^{\varepsilon}(t, s) - g_{0}(t, s)Z_{1}(t, s)
\]
\[
+ g_{x}(t, s)Z_{1}^{\varepsilon}(s, t) - g_{x}(t, s)Z_{1}(s, t) + [g_{0}(t, s) - g_{0}(t, s)] [v(s) - \bar{u}(s)] \right\} ds
\]
\[
- \int_{t}^{T} \left( Z_{1}^{\varepsilon}(t, s) - Z_{1}(t, s) \right) dW(s),
\]
where
\[
\psi_{x}(t) = \int_{0}^{1} \psi_{x}(t, \bar{X}(t) + \theta[X^{\varepsilon}(t) - \bar{X}(t)], \bar{X}(T) + \theta[X^{\varepsilon}(T) - \bar{X}(T)]) d\theta,
\]
and \( \psi_{x}(g_{x}(t, s), g_{0}(t, s), g_{0}(t, s), g_{0}(t, s), g_{0}(t, s) \) and \( g_{0}(t, s) \) are defined similarly. Then
\[
\mathbb{E} \int_{0}^{T} |Y_{1}^{\varepsilon}(t) - Y_{1}(t)|^{2} dt + \mathbb{E} \int_{0}^{T} \int_{0}^{T} |Z_{1}^{\varepsilon}(t, s) - Z_{1}(t, s)|^{2} ds dt
\]
\[
\leq K \mathbb{E} \int_{0}^{T} \left[ \psi_{x}(t)X^{\varepsilon}(t) - \psi_{x}(t)X_{1}(t) + \psi_{x}(t)X_{1}(T) - \psi_{x}(t)X_{1}(T)
\right.
\]
\[
+ \int_{t}^{T} \left\{ g_{x}(t, s)X_{1}^{\varepsilon}(t) - g_{x}(t, s)X_{1}(t) + g_{x}(t, s)X_{1}(s) - g_{x}(t, s)X_{1}(s)
\right.
\]
\[
+ g_{0}(t, s)Y_{1}^{\varepsilon}(s) - g_{0}(t, s)Y_{1}(s) + g_{0}(t, s)Z_{1}^{\varepsilon}(t, s) - g_{0}(t, s)Z_{1}(t, s)
\]
\[
+ g_{x}(t, s)Z_{1}^{\varepsilon}(s, t) - g_{x}(t, s)Z_{1}(s, t) + [g_{0}(t, s) - g_{0}(t, s)] [v(s) - \bar{u}(s)] \right\} ds
\]
\[
- \int_{t}^{T} \left( Z_{1}^{\varepsilon}(t, s) - Z_{1}(t, s) \right) dW(s),
\]
Then by the dominated convergence theorem, we have
\[
\lim_{\varepsilon \to 0} \left( \mathbb{E} \int_{0}^{T} |Y_{1}^{\varepsilon}(t) - Y_{1}(t)|^{2} dt + \mathbb{E} \int_{0}^{T} \int_{0}^{T} |Z_{1}^{\varepsilon}(t, s) - Z_{1}(t, s)|^{2} ds dt \right) = 0.
\]
Now, by the optimality of \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot), \bar{u}(\cdot))\), we have

\[
0 \leq \frac{J_2(u^\varepsilon(\cdot)) - J_2(\bar{u}(\cdot))}{\varepsilon} = \frac{1}{\varepsilon} \mathbb{E}\left[h\left(X^\varepsilon(T), \int_0^T Y^\varepsilon(t)dt\right) - h\left(\bar{X}(T), \int_0^T \bar{Y}(t)dt\right)
+ \int_0^T \int_0^T \left(f(t, s, X^\varepsilon(s), Y^\varepsilon(s), Z^\varepsilon(t, s), u^\varepsilon(s))ight)dsdt\right]
\]

(5.22)

where

\[
h^\varepsilon_t = \frac{1}{\varepsilon} \mathbb{E}\left[h_x(\bar{X}(T) + \theta[X^\varepsilon(T) - \bar{X}(T)]), \int_0^T \{\bar{Y}(t) + \theta[Y^\varepsilon(t) - \bar{Y}(t)]\}dt\right]d\theta,
\]

and \(h^\varepsilon_y, f^\varepsilon_x(t, s), f^\varepsilon_y(t, s), f^\varepsilon_z(t, s),\) and \(f^\varepsilon_x(t, s)\) are defined similarly. Passing to the limit in (5.22), by the dominated convergence theorem, we have (5.15).

Now let us present a proof of Theorem 3.3.

**Proof of Theorem 3.3.** Similar to the proof of Theorem 3.2, we begin with (5.15) which is written here

\[
0 \leq \mathbb{E}\left\{h_xX_1(t) + \int_0^T \left[\int_0^T f_x(t, s)ds\right]X_1(s) + \left(h_y + \int_0^T f_y(t, s)dt\right)Y_1(s)
+ \int_0^T f_u(t, s)ds\right\}ds + \int_0^T f_z(t, s)Z_1(t, s)dsdt \equiv J_2(\bar{u}(\cdot); v(\cdot)),
\]

with \((X_1(\cdot), Y_1(\cdot), Z_1(\cdot, \cdot))\) being the adapted M-solution to the following linear FBSVIE:

\[
\begin{align*}
X_1(t) & = \int_0^t \left\{ b_x(t, s)X_1(s) + b_u(t, s)\left[v(s) - \bar{u}(s)\right]\right\}ds + \int_0^t \left\{ \sigma_x(t, s)X_1(s) + \sigma_u(t, s)\left[v(s) - \bar{u}(s)\right]\right\}dW(s), \\
Y_1(t) & = \psi_x(t)X_1(t) + \psi_x(t)X_1(T) + \int_t^T \left\{ g_x(t, s)X_1(t) + g_u(t, s)X_1(s) + g_y(t, s)Y_1(s)
+ g_z(t, s)Z_1(t, s) + g_z(t, s)Z_1(s, t) + g_u(t, s)\left[v(s) - \bar{u}(s)\right]\right\}ds - \int_t^T Z_1(t, s)dW(s).
\end{align*}
\]

(5.23)

In the current case, we are going to eliminate \((Y_1(\cdot), Z_1(\cdot, \cdot)), X_1(T)\) and \(X_1(\cdot)\) consecutively. Therefore, we will have three steps.

**Step 1.** Eliminate \((Y_1(\cdot), Z_1(\cdot, \cdot))\). To this end, we apply Theorem 4.1 with

\[
A(t, s) = g_y(t, s), \quad B(t, s) = g_z(t, s), \quad C(t, s) = g_z(t, s),
\]

and

\[
\psi(t) = \psi_x(t)X_1(t) + \psi_x(t)X_1(T) + \int_t^T \left\{ g_x(t, s)X_1(t) + g_x(t, s)X_1(s)
+ g_u(t, s)\left[v(s) - \bar{u}(s)\right]\right\}ds,
\]

\[
\alpha(t) = h^\varepsilon_y + \int_0^T f_y(t, s)ds, \quad \beta(t, s) = f_z(t, s).
\]

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Therefore, we let $\xi(\cdot)$ solve the following:

$$\xi(t) = h^T_y + \int_0^T f_y(r, t)^T dr + \int_0^T f_z(t, s)^T dW(s) + \int_t^T g_z(t, s)^T \mathbb{E}_s[\xi(t)] dW(s)$$

$$+ \int_0^t g_y(s, t)^T \mathbb{E}_t[\xi(s)] ds + \int_0^T \mathbb{E}_s[g_z(s, t)^T \xi(s)] dW(s), \quad \text{a.e. } t \in [0, T].$$

(5.24)

Then by Theorem 4.1, one has

$$\mathbb{E}\left\{ \int_0^T \left[ (h_y + \int_0^T f_y(t, s) ds) Y_1(s) ds + \int_0^T f_z(t, s) Z_1(t, s) ds dt \right] \right\}$$

$$= \mathbb{E}\left\{ \int_0^T \langle \alpha(s), Y_1(s) \rangle ds + \int_0^T \int_0^T \langle \beta(t, s), Z_1(t, s) \rangle ds dt = \mathbb{E}\int_0^T \langle \psi(t), \xi(t) \rangle dt \right\}$$

$$= \mathbb{E}\left\{ \int_0^T \langle \psi_x(t) X_1(t) + \psi_s(t) X_1(T) + \int_0^T g_z(t, s) X_1(t) + g_x(t, s) X_1(s) \right.$$ 

$$+ g_u(s, t)[v(s) - \bar{u}(s)] ds, \xi(t) \rangle dt$$

$$= \mathbb{E}\left\{ \int_0^T \langle \psi_x(t) \xi(t) dt, X_1(T) \rangle + \int_0^T \langle \psi_x(t) + \int_t^T g_z(t, s) ds \rangle \xi(t) + \int_t^T g_z(t, s)^T \xi(s) ds, X_1(t) \rangle dt \right.$$ 

$$+ \int_t^T \left\{ \int_0^T g_u(s, t)^T \xi(s) ds, v(t) - \bar{u}(t) \right\} dt \right\}$$

which leads to

$$J_2(\bar{u}(\cdot); v(\cdot)) = \mathbb{E}\left\{ \int_0^T \left[ \left( \int_0^T f_z(t, s) dt \right) X_1(s) + \left( h_y + \int_0^T f_y(t, s) dt \right) Y_1(s) \right.$$ 

$$+ \left( \int_0^T f_u(t, s) ds \right) [v(s) - \bar{u}(s)] ds + \int_0^T f_z(t, s) Z_1(t, s) ds dt \left\} \right.$$

$$= \mathbb{E}\left\{ \int_0^T \left[ \left( \int_0^T f_z(t, s) dt \right) X_1(s) + \left( \int_0^T f_u(t, s) ds \right) [v(s) - \bar{u}(s)] \right\] ds$$

$$+ \int_0^T \langle \psi_x(t) \xi(t) dt, X_1(T) \rangle + \int_0^T \left( \psi_x(t) + \int_t^T g_z(t, s) ds \right) \xi(t) + \int_t^T g_z(t, s)^T \xi(s) ds, X_1(t) \rangle dt \right.$$ 

$$+ \int_0^T \left\{ \int_0^T g_u(s, t)^T \xi(s) ds, v(t) - \bar{u}(t) \right\} dt \right\}$$

$$= \mathbb{E}\left\{ \int_0^T \left[ \left( \int_0^T f_z(t, s) dt \right) \right.$$

$$\int_0^T \langle \psi_x(t) \xi(t) dt, X_1(T) \rangle$$

$$+ \int_0^T \left( \psi_x(t) + \int_t^T g_z(t, s) ds \right) \xi(t) + \int_t^T g_z(t, s)^T \xi(s) ds, X_1(t) \rangle dt \right.$$ 

$$+ \int_0^T \left\{ \int_0^T g_u(s, t)^T \xi(s) ds + \int_0^T f_u(r, t)^T dr, v(t) - \bar{u}(t) \right\} dt \right\}.$$ 

This finishes the elimination of $(Y_1(\cdot), Z_1(\cdot, \cdot))$.

**Step 2.** Elimination of $X_1(T)$. We let $(\mu(\cdot), \nu(\cdot))$ be the adapted solution to the following BSDE:

$$\mu(t) = h^T_x + \int_0^T \psi_x(r) \xi(r) dr - \int_t^T \nu(s) dW(s), \quad t \in [0, T].$$

Note that

$$X_1(T) = \int_0^T \left( b_x(T, s) X_1(s) + b_u(T, s) [v(s) - \bar{u}(s)] \right) ds$$

$$+ \int_0^T \left( \sigma_x(T, s) X_1(s) + \sigma_u(T, s) [v(s) - \bar{u}(s)] \right) dW(s).$$
Thus,

\[ \mathbb{E} \left\{ \int_0^T \psi_x(r)^T \xi(r) dr, X_1(T) \right\} = \mathbb{E} \left\{ \mu(T), X_1(T) \right\} \]

\[ \mathbb{E} \left\{ \int_0^T \langle \mu(T), b_x(T, s)X_1(s) + b_u(T, s)[v(s) - \bar{u}(s)] \rangle ds \right. \]

\[ + \int_0^T \langle \nu(s), \sigma_x(T, s)X_1(s) + \sigma_u(T, s)[v(s) - \bar{u}(s)] \rangle ds \right\} \]

\[ = \mathbb{E} \left\{ \int_0^T \langle b_x(T, s)^T \mu(T) + \sigma_x(T, s)^T \nu(s), X_1(s) \rangle \right. \]

\[ + \langle b_u(T, s)^T \mu(T) + \sigma_u(T, s)^T \nu(s), v(s) - \bar{u}(s) \rangle ds \right\} \]

Consequently,

\[ J_2(\bar{u}(\cdot); v(\cdot)) = \mathbb{E} \int_0^T \left\{ \langle b_x(T, s)^T \mu(T) + \sigma_x(T, s)^T \nu(s), + \int_0^T f_x(r, s)^T dr \right. \]

\[ + \left( \psi_x(s)^T + \int_s^T g_x(r, s)^T dr \right) \xi(s) + \int_0^s g_x(r, s)^T \xi(r) dr, X_1(s) \right\} \]

\[ + \langle b_u(T, s)^T \mu(T) + \sigma_u(T, s)^T \nu(s), + \int_0^T f_u(r, s)^T dr + \int_0^s g_u(r, s)^T \xi(r) dr, v(s) - \bar{u}(s) \rangle ds \right\} \]

Thus, \( X_1(T) \) is eliminated.

**Step 3. Eliminate \( X_1(\cdot) \).** We now apply Corollary 4.2 with

\[ A_0(t, s) = b_x(t, s), \quad C_0(t, s) = \sigma_x(t, s), \]

and

\[ \varphi(t) = \int_0^t b_u(t, s)[v(s) - \bar{u}(s)] ds + \int_0^t \sigma_u(t, s)[v(s) - \bar{u}(s)] dW(s), \]

\[ \psi(t) = b_x(T, s)^T \mu(T) + \sigma_x(T, s)^T \nu(s) + \int_0^T f_x(r, s)^T dr \]

\[ + \left( \psi_x(s)^T + \int_s^T g_x(r, s)^T dr \right) \xi(s) + \int_0^s g_x(r, s)^T \xi(r) dr \]

Then we introduce the following BSVIE:

\[ p(t) = b_x(T, s)^T \mu(T) + \sigma_x(T, s)^T \nu(s) + \int_0^T f_x(r, s)^T dr \]

\[ + \left( \psi_x(s)^T + \int_s^T g_x(r, s)^T dr \right) \xi(s) + \int_0^s g_x(r, s)^T \xi(r) dr \]

\[ + \int_t^T \left[ b_x(s, t)^T p(s) + \sigma_x(s, t)^T q(s, t) \right] ds - \int_t^T q(t, s) dW(s), \quad t \in [0, T]. \]

Using Corollary 4.2, we have

\[ \mathbb{E} \int_0^T \langle b_x(T, s)^T \mu(T) + \sigma_x(T, s)^T \nu(s), + \int_0^T f_x(r, s)^T dr \right. \]

\[ + \left( \psi_x(s)^T + \int_s^T g_x(r, s)^T dr \right) \xi(s) + \int_0^s g_x(r, s)^T \xi(r) dr, \right. \]

\[ \left. X_1(t) \right) dt \]

\[ = \mathbb{E} \int_0^T \langle \varphi(t), X_1(t) \rangle dt = \mathbb{E} \int_0^T \langle \varphi(t), p(t) \rangle dt \]

\[ = \mathbb{E} \int_0^T \left[ \int_0^t \langle b_u(t, s)[v(s) - \bar{u}(s)], p(t) \rangle ds + \int_0^t \langle \sigma_u(t, s)[v(s) - \bar{u}(s)], q(t, s) \rangle ds \right] dt. \]
Hence,
\[
J_2(\bar{u}(-); v(\cdot)) = \mathbb{E} \int_0^T \left\{ \langle b_x(T, s)T \mu(T) + \sigma_x(T, s)T \nu(s) + \int_0^T f_x(r, s)T dr \right.
\]
\[+ (\psi_{2x}(s)T + \int_s^T g_x(s, r)T dr)\xi(s) + \int_0^s g_x(r, s)T \xi(r) dr, X_1(s) \rangle
\]
\[+ \langle b_u(T, s)T \mu(T) + \sigma_u(T, s)T \nu(s) + \int_0^T f_u(r, s)T dr + \int_0^s g_u(r, s)T \xi(r) dr, v(s) - \bar{u}(s) \rangle \right\} ds
\]
\[= \mathbb{E} \int_0^T \left\{ \langle b_u(T, s)T \mu(T) + \sigma_u(T, s)T \nu(s) + \int_0^T f_u(r, s)T dr \right.
\]
\[+ \int_s^T g_u(r, s)T \xi(r) dr + \int_0^T b_u(r, s)T p(r) dr + \int_s^T \sigma_u(r, s)T q(r, s) dr, v(s) - \bar{u}(s) \rangle \right\} ds.
\]
Then the maximum principle follows.

6 Concluding Remarks

In this paper, we have formulated two optimal control problems for forward-backward stochastic Volterra integral equations. Corresponding Pontryagin type maximum principles are established. The key contribution of this paper is the discovery of a general duality principle between a linear BSVIE and a linear Fredholm-Volterra stochastic integral equation with mean-field. This result substantially extends the corresponding result found in [16, 18]. Such a duality principle enables us to prove the maximum principles. It is known that when studying stochastic integral equations, the Itô’s formula, which is very powerful in studying SDEs, is absent. Therefore, one has to carry out all relevant calculations without differentiation. We expect that the techniques/ideas developed in this paper might have some significant impacts on other type problems when Itô’s formula is not directly applicable.

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