Wolff potential estimates for quasilinear elliptic equations with generalized Orlicz growth

By Allami BENYAICHE and Ismail KHLIFI.

Ibn Tofail University, Department of Mathematics, B.P: 133, Kenitra-Morocco.

Abstract: In this paper, we consider a quasilinear elliptic equation involving a signed Radon measure in the Musielak-Orlicz-Sobolev space. We establish pointwise estimates for solutions of such equations in terms of a Wolff type potential.

Mathematics Subject Classification (2010). 31C45. 35J60. 31C15.

Keywords. Wolff potential, generalized Φ-function, Musielak-Orlicz-Sobolev space, Dirichlet problem, Caccioppoli inequality

1 Introduction

In the classical potential theory, the solution of the following linear elliptic equation, involving a nonnegative Radon measure μ,

\[-Δu = μ, \tag{1.1}\]

considered in a Green domain Ω ⊂ ℝ^N, can be represented as

\[u(x) = G_Ω^μ(x) + h(x).\]

Where \(G_Ω^μ\) is the Green potential of μ, and \(h\) is a harmonic function. In contrast with this linear situation, this representation is not available in the nonlinear case as

\[-\text{div}(|∇u|^{p-2}∇u) = μ. \tag{1.2}\]

Indeed, by the fundamental works of Kilpelinen and Mal in [16, 17], we have just pointwise estimates for solutions of equation (1.2) in terms of Wolff potential

\[c_1 W_{1,p}^μ(x, R) ≤ u(x) ≤ c_2 \inf_{B(x,R)} u + c_3 W_{1,p}^μ(x, 2R), \tag{1.3}\]
where \( B(x, R) \subset \Omega \) and \( W_{1,p}^\mu(x, R) \) is the Wolff potential of the measure \( \mu \) defined by
\[
W_{1,p}^\mu(x, R) = \int_0^R \left( \frac{\mu(B(x, s))}{s^{n-p}} \right)^{\frac{1}{p-1}} \, ds. \tag{1.4}
\]

Next, Trudinger and Wang \cite{31} worked on a new method using Poisson modification and Harnack inequality for such equations involving a signed Radon measure. Mikkonen has treated the weighted situation in \cite{26}. A. Björn - J. Björn \cite{4} and Hara \cite{7} have developed the proof of potential estimates in the metric measure spaces. For further informations about the Wolff potential estimates and its extensions, we refer to \cite{13, 18, 20, 24, 27}.

In this paper, we study the potential estimates of solutions of the following equation:
\[
-\text{div} A(x, \nabla u) = \mu, \tag{1.5}
\]
where \( A: \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) is a Carathéodory function satisfying the growth condition
\[
A(x, \xi) \cdot \xi \approx G(x, |\xi|),
\]
for some generalized \( \Phi \)-functions \( G(\cdot) \) (see section 2), and \( \mu \) is a signed Radon measure. The natural setting to study such equations is the generalized Orlicz-Sobolev spaces (called also Musielak-Orlicz-Sobolev spaces). We have in mind the principal classes: \( G(x, t) = t^p \), \( G(x, t) = t^{p(x)} \), \( G(x, t) = t^p + a(x)t^q \). For other classes we can see the monograph of P. Hästö, P. Harjulehto \cite{8}. The generalized Orlicz spaces are motivated by their applications in the image processing \cite{10} and fluid mechanics \cite{32}. For detailed motivation of our context and additional references we refer to introduction of \cite{9}.

For this end, we define the following Wolff potential, associated to \( G(\cdot) \) and \( \mu^+ \), by
\[
W_{G(\cdot), \mu^+}^\mu(x, R) := \int_0^R g^{-1}\left( x, \frac{\mu^+(B(x, s))}{s^{n-1}} \right) \, ds, \tag{1.6}
\]
where \( g(x, \cdot) \) is the right-hand derivative of \( G(x, \cdot) \).

In the variable exponent case, \( G(x, t) = t^{p(x)} \), Yu. Alkhutov, O. Krasheninnikova in \cite{1} and T. Lukkari, FY. Maeda, N. Marola in \cite{22} gave a proof of the two-side Wolff potential bounded. In the Orlicz case, \( G(x, t) = G(t) \), the problem has been studied by J. Mal in \cite{23} and recently by I. Chlebicka, F. Giannetti, A. Zatorska-Goldstein in \cite{5}.

For our potential estimates, the major difficulty is that the function \( G(\cdot, t) \) is just measurable, which does not allow us to choose test functions containing \( G(\cdot, t) \).
The idea that makes the proof possible is using the upper envelope function $G^+$ and lower $G^-$ and the condition $(A_{1,n})$ to link these two functions. Note that this condition plays the role of the logarithmic H"older continuity in the variable exponent case. Using the Lorentz norm, Harnack estimates and other techniques, we establish pointwise estimates for solutions of such equations in terms of the Wolff potential (1.6), which also gives us another approach different from the previous methods.

This paper’s main results are the following (see Theorem 5.1 and Theorem 5.2).

**Theorem 1.1.** Let $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$, $(A_0)$ and $(A_{1,n})$. Let $u$ be a nonnegative weak solution to (1.5) with nonnegative Radon measure $\mu$ in $\Omega$, and let $B = B(x_0, R) \subset 2B \subset \Omega$. If $u \in L^\gamma(B)$ with $\gamma > 0$ and lower semicontinuous at $x_0$, then there exists a constant $C = C(c_1, c_2, g_0, g_0^0, N, \|u\|_\infty, B) > 0$ such that

$$u(x_0) \geq CW^\mu_{G(\cdot)}(x_0, R) + \text{ess inf }_{2B} u - 2R.$$ 

**Theorem 1.2.** Let $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$, $(A_0)$ and $(A_{1,n})$. Let $u$ be a solution to equation (1.5) in $\Omega$ with a signed Radon measure $\mu$, such that $|\mu| \in \left(W_{1,1}^1(G(\cdot))(\Omega)^*\right)$. If $u \in L^{\chi,\infty}(\frac{1}{2}B)$ with $\chi' = \frac{N}{N-1} (g_0 - 1)$ and lower semicontinuous at $x_0$. Then, for any $\gamma > 0$, there exists a constant $C = C(c_1, c_2, g_0, g_0^0, \gamma, N, \|u^+\|_{\chi,\infty, B}, \|u^+\|_{\infty, B\setminus\frac{1}{2}B}) > 0$, such that

$$u^+(x_0) \leq C \left( R + \left( \int_{\frac{1}{2}B \setminus B} (u^+)^\gamma \, dx \right)^{\frac{1}{\gamma}} + W^\mu_{G(\cdot)}(x_0, 2R) \right).$$

Compared to the classical case (1.3), our estimates contain extra terms $(-2R)$ in Theorem (1.1) and $(+R)$ in Theorem (1.2). Note that, in the Orlicz case, $G(x, t) = G(t)$, these terms are not needed. This comes from the fact that condition $(A_{1,n})$ is always satisfied.

The paper is organized as follows. In Section 2, we give some properties of generalized $\Phi$-functions and Musielak-Orlicz-Sobolev spaces. In Section 3, we introduce weak solutions and the weak comparison principle. In Section 4, we use the monotone operator’s theory to prove solutions to the Dirichlet problem with Sobolev Boundary values. In Section 5, we establish lower and upper pointwise estimates for solutions in terms of the Wolff potential defined by (1.6).
2 Preliminaries

We briefly introduce our assumptions. More information about, generalized $\Phi$-functions and Musielak-Orlicz-Sobolev spaces, can be found in J. Musielak monograph [28] and P. Harjulehto, P. Hästö monograph [8]. We denote, $\Omega$ a bounded domain of $\mathbb{R}^N$ with $N \geq 2$, $L^0(\Omega)$ the set of measurable functions on $\Omega$. $C$ is a generic constant whose value may change between appearances.

**Definition 2.1.** A function $G : \Omega \times [0, \infty) \rightarrow [0, \infty]$ is called a generalized $\Phi$-function, denoted by $G(\cdot) \in \Phi(\Omega)$, if the following conditions hold

- For each $t \in [0, \infty)$, the function $G(\cdot, t)$ is measurable.
- For a.e $x \in \Omega$, the function $G(x, \cdot)$ is an $\Phi$-function, i.e.
  1. $G(x, 0) = \lim_{t \to 0^+} G(x, t) = 0$ and $\lim_{t \to \infty} G(x, t) = \infty$;
  2. $G(x, \cdot)$ is increasing and convex.

Note that, a generalized $\Phi$-function can be represented as

$$G(x, t) = \int_0^t g(x, s) \, ds,$$

where $g(x, \cdot)$ is the right-hand derivative of $G(x, \cdot)$. Furthermore for each $x \in \Omega$, the function $g(x, \cdot)$ is right-continuous and nondecreasing, so we have the following inequality

$$g(x, a) b \leq g(x, a) a + g(x, b) b \quad \text{for } x \in \Omega \text{ and } a, b \geq 0 \quad (2.1)$$

We denote $G_B^+(t) := \sup_B G(x, t)$, $G_B^-(t) := \inf_B G(x, t)$. We say that $G(\cdot)$ satisfies

($SC$) : If there exist two constants $g_0, g^0 > 1$ such that,

$$1 < g_0 \leq \frac{tg(x, t)}{G(x, t)} \leq g^0.$$

($A_0$) : If there exists a constant $c_0 > 1$ such that,

$$\frac{1}{c_0} \leq G(x, 1) \leq c_0, \text{ a.e } x \in \Omega.$$

($A_{1,n}$) : If there exists $\beta \in (0, 1)$ such that, for every $x, y \in B_R \subset \Omega$ with $|B_R| \leq 1$, we have

$$G_B(x, \beta t) \leq G_B(y, t) \quad \text{when } t \in \left[1, \frac{1}{2R}\right].$$
Note that the structure condition \((SC)\) is used by I.B. Simonenko \([30]\) and G.M. Libermann \([19]\) in the \(\Phi\)-functions situation. Also, by M. Mihăilescu, V. Radulescu for the generalized \(\Phi\)-functions in \([25]\). The condition \((A_0)\) is considered by J. Musielak in \([28]\) and P. Harjulehto - P. Hästö in \([10]\) for to restrict attention to the essentially "unweighted" case. The condition \((A_{1,n})\) is used by P. Harjulehto and P. Hästö in \([12]\) and by the authors, in \([8]\), for the study of local regularity of solutions to equation (3.1). Recently, this condition has been formulated in a general situation in \([2]\).

Under the structure condition \((SC)\) \([25]\), we have the following inequalities

\[
\sigma^{g_0} G(x, t) \leq G(x, \sigma t) \leq \sigma^{g_0} G(x, t), \quad \text{for } x \in \Omega, \ t \geq 0 \text{ and } \sigma \geq 1. \tag{2.2}
\]

\[
\sigma^{g_0} G(x, t) \leq G(x, \sigma t) \leq \sigma^{g_0} G(x, t), \quad \text{for } x \in \Omega, \ t \geq 0 \text{ and } \sigma \leq 1. \tag{2.3}
\]

We define \(G^*(\cdot)\) the conjugate \(\Phi\)-function of \(G(\cdot)\), by

\[
G^*(x, s) := \sup_{t \geq 0} (st - G(x, t)), \quad \text{for } x \in \Omega \text{ and } s \geq 0.
\]

Note that \(G^*(\cdot)\) is also a generalized \(\Phi\)-function and can be represented as

\[
G^*(x, t) = \int_0^t g^{-1}(x, s) \, ds,
\]

with \(g^{-1}(x, s) := \sup\{t \geq 0 : g(x, t) \leq s\}\).

By this definition, \(G(\cdot)\) and \(G^*(\cdot)\) satisfies the following Young inequality

\[
st \leq G(x, t) + G^*(x, s), \quad \text{for } x \in \Omega \text{ and } s, t \geq 0.
\]

Furthermore, we have the equality if \(s = g(x, t) \) or \(t = g^{-1}(x, s)\). So, if \(G(\cdot)\) satisfies \((SC)\), we have the following inequality

\[
G^*(x, g(x, t)) \leq (g^0 - 1) G(x, t), \quad \forall x \in \Omega, t \geq 0. \tag{2.4}
\]

**Remark 2.1.** Following \([29]\), if \(G(\cdot) \in \Phi(\Omega)\) satisfies \((SC)\), then \(G^*(\cdot)\) satisfies the structure condition:

\[
\frac{g^0}{g^0 - 1} \leq \frac{t g^{-1}(x, t)}{G^*(x, t)} \leq \frac{g_0}{g_0 - 1}.
\]

**Definition 2.2.** We define the generalized Orlicz space, also called Musielak-Orlicz space, by

\[
L^{G(\cdot)}(\Omega) := \{ u \in L^0(\Omega) : \lim_{\lambda \to 0} \rho_{G(\cdot)}(\lambda |u|) = 0 \},
\]

5
where $\rho_{G(\cdot)}(t) = \int_{\Omega} G(x, t) \, dx$. If $G(\cdot)$ satisfies $(SC)$, then

$$L^{G(\cdot)}(\Omega) = \{ u \in L^0(\Omega) : \rho_{G(\cdot)}(|u|) < \infty \}.$$ 

On the generalized Orlicz space, we define the following norms

- Luxembourg norm $\|u\|^{G(\cdot)} = \inf \{\lambda > 0 : \rho_{G(\cdot)}(\lambda u) \leq 1\}$.
- Orlicz norm $\|u\|^{0 \cdot G(\cdot)} = \sup \{|\int_{\Omega} u(x)v(x) \, dx| : v \in L^{G^*(\cdot)}(\Omega), \rho_{G^*(\cdot)}(v) \leq 1\}$.

These norms are equivalent. Precisely, we have

$$\|u\|^{G(\cdot)} \leq \|u\|^{0 \cdot G(\cdot)} \leq 2 \|u\|^{G(\cdot)}.$$

Furthermore, by definition of Orlicz norm and Young inequality, we have

$$\|u\|^{G(\cdot)} \leq \|u\|^{0 \cdot G(\cdot)} \leq \int_{\Omega} G(x, |u|) \, dx + 1.$$ 

(2.5)

The following proposition establishes properties of convergent sequences in generalized Orlicz spaces.

**Proposition 2.1.** Let $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$. For any sequence $(u_i) \in L^{G(\cdot)}(\Omega)$, we have the following properties

1. Fatou lemma: If $u_i \to u$ almost everywhere, then

$$\int_{\Omega} G(x, |u(x)|) \, dx \leq \liminf_{i \to \infty} \int_{\Omega} G(x, |u_i(x)|) \, dx.$$

2. $\|u_i\|^{G(\cdot)} \to 0$ (resp. $1; \infty$) $\iff$ $\int_{\Omega} G(x, |u_i(x)|) \, dx \to 0$ (resp. $1; \infty$).

3. The functions $G(\cdot)$ and $G^*(\cdot)$ satisfy the Hölder inequality

$$\left| \int_{\Omega} u(x)v(x) \, dx \right| \leq 2 \|u\|^{G(\cdot)} \|v\|^{G^*(\cdot)} , \text{ for } u \in L^{G(\cdot)}(\Omega) \text{ and } v \in L^{G^*(\cdot)}(\Omega).$$

The relation between a modular and its norm, under the structure condition $(SC)$, is given by the following proposition.

**Proposition 2.2.** Let $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$. Then the following relations hold true
1. $\|u\|_{G^{-1}}\leq \rho_{G^{-1}}(u) \leq \|u\|_{G^{-1}}^0$, $\forall u \in L^{G^{-1}}(\Omega)$ with $\|u\|_{G^{-1}} \geq 1$.

2. $\|u\|_{G^{-1}}^0 \leq \rho_{G^{-1}}(u) \leq \|u\|_{G^{-1}}^0$, $\forall u \in L^{G^{-1}}(\Omega)$ with $\|u\|_{G^{-1}} \leq 1$.

**Definition 2.3.** We define the generalized Orlicz-Sobolev space by

$$W^{1, G^{-1}}(\Omega) := \{u \in L^{G^{-1}}(\Omega) : |\nabla u| \in L^{G^{-1}}(\Omega), \text{ in the distribution sense}\},$$

equipped with the norm

$$\|u\|_{1, G^{-1}} = \|u\|_{G^{-1}} + \|\nabla u\|_{G^{-1}}.$$

**Remark 2.2.** The proposition 2.2 remains true for the norm of the generalized Sobolev-Orlicz spaces (see [25]).

**Definition 2.4.** $W^{1, G^{-1}}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1, G^{-1}}(\Omega)$.

Note that, if $G(\cdot) \in \Phi(\Omega)$ satisfies the condition $(SC)$ and $(A_0)$, then $W^{1, G^{-1}}(\Omega)$ is a Banach, separable and reflexive space.

### 3 Quasilinear elliptic equations

Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a function satisfying the following assumptions:

1. $(x, \xi) \to A(x, \xi)$ is a Carathodory function.

2. There exists a positive constant $c_1$ such that

$$A(x, \xi) : \xi \geq c_1 g(x, |\xi|)|\xi|, \text{ for } x \in \Omega \text{ and } \xi \in \mathbb{R}^N.$$ 

3. There exists a positive constant $c_2$ such that

$$|A(x, \xi)| \leq c_2 g(x, |\xi|), \text{ for } x \in \Omega \text{ and } \xi \in \mathbb{R}^N.$$ 

4. $(A(x, \xi_1) - A(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$, for $x \in \Omega$ and $\xi_1, \xi_2 \in \mathbb{R}^N$ with $\xi_1 \neq \xi_2$.

Under the previous conditions, we consider the following quasilinear elliptic equation.

$$-\text{div} A(x, \nabla u) = 0.$$  \hfill (3.1)
Definition 3.1. A function $u \in W^{1,G(\cdot)}(\Omega)$ is a solution to equation (3.1) in $\Omega$ if
\[ \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = 0 \]
whenever $\varphi \in C_0^\infty(\Omega)$.

Definition 3.2. A function $u \in W^{1,G(\cdot)}(\Omega)$ is a supersolution (resp. subsolution) to equation (3.1) in $\Omega$ if
\[ \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0 \ (\text{resp.} \leq 0), \]
whenever $\varphi \in C_0^\infty(\Omega)$ nonnegative.

By Giorgi-Nash-Moser theory for solutions to equation (3.1) (see [3, 12]), we have the following Harnack estimates.

Lemma 3.1. Let $G(\cdot) \in \Phi(\Omega)$ satisfies (SC), $(A_0)$ and $(A_{1,n})$. If $u \in W^{1,G(\cdot)}(B(R)) \cap L^\infty(B(R))$ is a subsolution to equation (3.1) in $B(R)$, then for any $q > 0$, there is a constant $C = C(q, c_1, c_2, g_0, \beta, N, \|u\|_{\infty,B}) > 0$ such that
\[ \text{ess sup}_{\frac{1}{2}B} u^+ \leq C \left( \int_{B(R)} \pi^q \, dx \right)^{\frac{1}{q}}, \]
with $\pi = u^+ + R$.

Under the previous conditions $(a_1), (a_2), (a_3)$ and $(a_4)$, we consider the following quasilinear elliptic equation with data measure.
\[ -\text{div} A(x, \nabla u) = \mu. \quad (3.2) \]

Definition 3.3. Let $\mu$ be a signed Radon measure in $(W^{1,G(\cdot)}(\Omega))^*$. A function $u \in W^{1,G(\cdot)}(\Omega)$ is a solution of the equation (3.2) in $\Omega$ if
\[ \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\mu, \]
whenever $\varphi \in C_0^\infty(\Omega)$.

From the density of $C_0^\infty(\Omega)$, the class of test functions can be extended to $W^{1,G(\cdot)}_0(\Omega)$ in (3.2).
Lemma 3.2. Let $\mu$ be a signed Radon measure in $(W^{1,G}(\Omega))^*$. If $u \in W^{1,G}(\Omega)$ is a solution of the equation (3.2) in $\Omega$, then
\[ \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\mu, \]
whenever $\varphi \in W^{1,G}_0(\Omega)$.

By the monotone condition of $A$, we have the following weak comparison principle [14].

Lemma 3.3. Let $u, v \in W^{1,G}(\Omega)$. If
\[ \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx \leq \int_{\Omega} A(x, \nabla v) \cdot \nabla \varphi \, dx, \]
for all nonnegative $\varphi \in W^{1,G}_0(\Omega)$ and $(u - v)^+ \in W^{1,G}_0(\Omega)$, then $u \leq v$, a.e in $\Omega$.

4 Existence of solution

After a preliminary list of lemmas, we use the monotone operator’s theory to prove the existence of solutions of the Dirichlet problem to equation (3.2) with Sobolev boundary values.

With a similar proof to the classical case, we have the following lemma.

Lemma 4.1. Let $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$ and $(A_0)$. Let $(u_i)$ be a sequence in $L^{G(\cdot)}(\Omega)$. If $u_i \to u$ in $L^{G(\cdot)}(\Omega)$, then there exists a subsequence $(u_{i_j})$ of $(u_i)$ which converge to $u$, a.e in $\Omega$.

Using arguments similar to those used in [15], we can prove the following lemma.

Lemma 4.2. Let $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$ and $(A_0)$. Let $(u_i)$ be a bounded sequence in $L^{G(\cdot)}(\Omega)$. If $u_i \to u$, a.e in $\Omega$, then $u_i$ converge to $u$ weakly in $L^{G(\cdot)}(\Omega)$.

Theorem 4.1. Let $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$ and $(A_0)$. Let $\mu$ be a signed Radon measure in $(W^{1,G}_0(\Omega))^*$ and $\theta \in W^{1,G}(\Omega)$. Then there exists $u \in W^{1,G}(\Omega)$ be a solution of equation (3.2) such that $u - \theta \in W^{1,G}_0(\Omega)$.

Proof: Define a mapping $T : W^{1,G}_0(\Omega) \to (W^{1,G}_0(\Omega))^*$ such that
\[ \langle Tw, \varphi \rangle := \int_{\Omega} A(x, \nabla (w + \theta)) \cdot \nabla \varphi \, dx. \]
The mapping \( T \) is well defined, indeed by the condition (a3),

\[
|\langle T w, \varphi \rangle| \leq c_2 \int_{\Omega} g(x, |\nabla (w + \theta)|)|\nabla \varphi| \, dx.
\]

Using inequality (2.1), for \( a = |\nabla (w + \theta)| \) and \( b = |\nabla \varphi| \), and the condition (SC), we have

\[
|\langle T w, \varphi \rangle| \\
\leq c_2 \left( \int_{\Omega} g(x, |\nabla (w + \theta)|)|\nabla (w + \theta)| \, dx + \int_{\Omega} g(x, |\nabla \varphi|)|\nabla \varphi| \, dx \right) \\
\leq c_2 g^0 \left( \int_{\Omega} G(x, |\nabla (w + \theta)|) \, dx + \int_{\Omega} G(x, |\nabla \varphi|) \, dx \right) \tag{4.1}
\]

We apply the general result [21] which asserts that if \( T \) is a bounded, coercive, demicontinuous map, then for all \( \mu \in (W^{1,G(\cdot)}_0 (\Omega))^* \) the equation \( T w = \mu \) has a solution \( w \in W^{1,G(\cdot)}_0 (\Omega) \).

- **i)** By the inequality (4.1), the map \( T \) is bounded.

- **ii)** Next, we show that \( T \) is coercive. For any \( w \in W^{1,G(\cdot)}_0 (\Omega) \) and \( \varphi \in W^{1,G(\cdot)}_0 (\Omega) \), by the condition (a2) and, the condition (SC), we have

\[
\langle T w, w \rangle = \int_{\Omega} (A(x, \nabla (w + \theta)) \cdot \nabla w) \, dx \\
\geq c_1 \int_{\Omega} g(x, |\nabla (w + \theta)||\nabla w| \, dx \\
\geq c_1 \int_{\Omega} g(x, |\nabla (w + \theta)||\nabla (w + \theta)| \, dx - c_1 \int_{\Omega} g(x, |\nabla (w + \theta)||\nabla \theta| \, dx \\
\geq c_1 g_0 \int_{\Omega} G(x, |\nabla (w + \theta)|) \, dx - c_1 \int_{\Omega} g(x, |\nabla (w + \theta)||\nabla \theta| \, dx.
\]

Using inequality (2.1), for \( a = |\nabla (w + \theta)| \) and \( b = g^0|\nabla \theta| \), and the condition (SC), we get

\[
\int_{\Omega} g(x, |\nabla (w + \theta)||\nabla \theta| \, dx \leq \frac{1}{g^0} \int_{\Omega} g(x, |\nabla (w + \theta)||\nabla (w + \theta)| \, dx \\
+ \frac{1}{g^0} \int_{\Omega} g(x, g^0|\nabla \theta|) g^0|\nabla \theta| \, dx \\
\leq \int_{\Omega} G(x, |\nabla (w + \theta)|) \, dx + (g^0)^9 \int_{\Omega} G(x, |\nabla \theta|) \, dx.
\]

Hence,

\[
\langle T w, w \rangle \geq c_1 (g_0 - 1) \int_{\Omega} G(x, |\nabla (w + \theta)|) \, dx - c_1 (g^0)^9 \int_{\Omega} G(x, |\nabla \theta|) \, dx.
\]
Choosing $w$ sufficiently large, we can assume that $\|\nabla \theta\|_{G(\cdot)} + \frac{1}{2} \leq \frac{1}{2} \|\nabla w\|_{G(\cdot)}$. Then

$$1 \leq \|\nabla w\|_{G(\cdot)} \leq \|\nabla (w + \theta)\|_{G(\cdot)} + \|\nabla \theta\|_{G(\cdot)} \leq \|\nabla (w + \theta)\|_{G(\cdot)} + \frac{1}{2} \|\nabla w\|_{G(\cdot)}.$$  

So,

$$1 \leq \|\nabla w\|_{G(\cdot)} \leq 2 \|\nabla (w + \theta)\|_{G(\cdot)}.$$  

Hence, by Proposition 2.2, we have

$$\langle T w, w \rangle \geq \frac{c_1 (g_0 - 1)}{2 g_0} \frac{\|\nabla w\|_{G(\cdot)}^{g_0 - 1}}{\|\nabla w\|_{G(\cdot)}} \int_{\Omega} G(x, |\nabla \theta|) \, dx.$$  

The right hand side goes to $+\infty$ as $\|\nabla w\|_{G(\cdot)} \to \infty$. Hence, $T$ is coercive.

-i-) Now we show $T$ is demicontinuous. In fact, let $w^i$ be a sequence that converges to an element $w$ in $W^{1, G(\cdot)}_0(\Omega)$. By Lemma 4.1, there exists a subsequence $w^i_j$ of $w^i$, such that $w^i_j \to w$, a.e. in $\Omega$.

Since the mapping $\xi \mapsto A(x, \xi)$ is continuous, then

$$A(x, \nabla w^i_j) \to A(x, \nabla w), \text{ a.e. in } \Omega.$$  

Or by the condition $(a_3)$, Remark 2.1 and inequalities (2.2), (2.4), we have

$$\int_{\Omega} G^*(x, |A(x, \nabla w^i_j)|) \, dx \leq \int_{\Omega} G^*(x, c_2 g(x, |\nabla w^i_j|)) \, dx \leq \max \left( c_2, (c_2) \frac{g_0}{g_0 - 1} \right) \int_{\Omega} G^*(x, g(x, |\nabla w^i_j|)) \, dx \leq (g^0 - 1) \max \left( c_2, (c_2) \frac{g_0}{g_0 - 1} \right) \int_{\Omega} G(x, |\nabla w^i_j|) \, dx.$$  

Hence, from the inequality (2.5), the $L^{G^*(\cdot)}(\Omega)$—norms of $A(x, \nabla w^i_j)$ is uniformly bounded. So, by Lemma 4.2, we have

$$A(x, \nabla w^i_j) \rightharpoonup A(x, \nabla w),$$

weakly in $L^{G^*(\cdot)}(\Omega)$. Since the weak limit is independent of the choice of the subsequence, it follows that

$$A(x, \nabla w_i) \rightharpoonup A(x, \nabla w),$$

weakly in $L^{G^*(\cdot)}(\Omega)$. Consequently, for all $\varphi \in W^{1, G(\cdot)}_0(\Omega)$,

$$\langle T w_i, \varphi \rangle \to \langle T w, \varphi \rangle.$$  

Hence, $T$ is demicontinuous on $W^{1, G(\cdot)}_0(\Omega)$.

Therefore, $u = w + \theta$ is a solution to equation (3.2) \qed
5 Wolff potential bounded

In this Section, we proof pointwise potential bounds for solutions. First, we introduce the Wolff potential in the generalized Orlicz setting.

**Definition 5.1.** Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^n$ and $R > 0$. We define the Wolff potential of $\mu$ by

$$W^\mu_{G(\cdot)}(x, R) := \int_0^R g^{-1} \left( x, \frac{\mu(B(x, s))}{s^{n-1}} \right) ds.$$ 

**Examples 5.1.** In the variable exponent case, $G(x, t) = \frac{t p(x)}{p(x)}$ (see [1, 22])

$$W^\mu_{p(\cdot)}(x, R) = \int_0^R \left( \frac{\mu(B(x, s))}{s^{n-p(x)}} \right)^{\frac{1}{p(x)-1}} ds = \int_0^R \left( \frac{\mu(B(x, s))}{s^{n-1}} \right)^{\frac{1}{p(x)-1}} ds.$$ 

In the Orlicz case, $G(x, t) = G(t)$, (see [23])

$$W^\mu_G(x, R) = \int_0^R g^{-1} \left( \frac{\mu(B(x, s))}{s^{n-1}} \right) ds.$$ 

The following lemma establishes that the functions $G$ is an $\Phi$-function [3].

**Lemma 5.1.** If $G(\cdot) \in \Phi(\Omega)$ satisfies $(SC)$ and $(A_0)$, then $G^-$ is an $\Phi$-function and $1 < g_0 \leq \frac{\tilde{t} \tilde{g}(t)}{G^-(t)} \leq g^0$, where $\tilde{g}$ is the right-hand derivative of $G^-$. 

The following lemma gives a more flexible characterization of $(A_{1,n})$ [8].

**Lemma 5.2.** Let $\Omega \subset \mathbb{R}^N$ be convex, $G(\cdot) \in \Phi(\Omega)$ and $0 < r \leq s$. Then $G(\cdot)$ satisfies $(A_{1,n})$ if, and only if, there exists $\beta > 0$ such that, for every $x, y \in B_R \subset \Omega$ with $|B_R| \leq 1$, we have

$$G_B(x, \beta t) \leq G_B(y, t) \quad \text{when} \quad t \in \left[ r, \frac{s}{R} \right].$$ 

5.1 Potential lower bounded

In the follows, let $B_R \subset \Omega$ with $|B_R| \leq 1$. The following Lemma gives the Caccioppoli type estimate of supersolution to equation (3.1) (see [3]).

**Lemma 5.3.** Let $u$ be a nonpositive supersolution of (3.1) in a ball $2B \Subset \Omega$, $\eta \in C^\infty_0(2B)$ with $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq \frac{1}{R}$. Then, there exits a constant $C$ such that

$$\int_{\frac{3}{2}B} G(x, |\nabla u|) \eta^\rho \, dx \leq C \int_{\frac{3}{2}B} G^+(\frac{-u}{R}) \, dx.$$
Theorem 5.1. Let $G(\cdot) \in \Phi(\Omega)$ satisfies (SC), $(A_0)$ and $(A_{1,n})$. Let $u$ be a nonnegative weak solution to (1.5) with nonnegative Radon measure $\mu$ in $\Omega$, and let $B = B(x_0, R) \subset 2B \subset \Omega$. If $u \in L^1(B)$ with $\gamma > 0$ and lower semicontinuous at $x_0$, then there exists a constant $C = C \left( c_1, c_2, g_0, g_0^0, N, \frac{\|u\|_{L^1(B)}}{|B|} \right) > 0$ such that

$$u(x_0) \geq CW_{G(\cdot)}(x_0, R) + \inf_{2B} u - 2R.$$ 

Proof: We set $a = \inf_{2B} u$ and, $b = \inf_{B} u$, $v = \min\{u, b\} - a + R$, $u_j = \min\{u, j\}$. Choose $\omega = \eta \eta^0$ such that $\eta \in C_0^\infty(\frac{3}{2}B)$ with $0 \leq \eta \leq 1$, and $|\nabla \eta| \leq \frac{C}{R}$, we have

$$(b - a + R)\mu(B) \leq \int_{\frac{3}{2}B} \omega \, d\mu$$

$$= \int_{\frac{3}{2}B} A(x, \nabla u) \cdot \nabla \omega \, dx$$

$$\leq \int_{\frac{3}{2}B} (A(x, \nabla u) \cdot \nabla \eta) \eta^0 \, dx + \int_{\frac{3}{2}B} (A(x, \nabla u) \cdot \nabla \eta) \eta^0 \eta^0 - 1 v \, dx.$$ 

By the conditions (a$_3$) and (SC), we have

$$I_1 := \int_{\frac{3}{2}B} (A(x, \nabla u) \cdot \nabla \eta) \eta^0 \, dx$$

$$\leq c_2 g_0^0 \int_{\frac{3}{2}B} G(x, |\nabla v|) |\nabla \eta| \eta^0 \, dx.$$ 

By the conditions (a$_3$), we have

$$I_2 := \int_{\frac{3}{2}B} (A(x, \nabla u) \cdot \nabla \eta) \eta^0 \eta^0 - 1 v \, dx$$

$$\leq c_2 \int_{\frac{3}{2}B} g(x, |\nabla v|) |\nabla \eta| \eta^0 \eta^0 - 1 v \, dx.$$ 

As $v \leq b - a + R$ and $|\nabla \eta| \leq \frac{C}{R}$, we have

$$I_2 \leq C \frac{b - a + R}{R} \int_{\frac{3}{2}B} g(x, |\nabla v|) |\nabla \eta| \eta^0 \eta^0 - 1 \, dx.$$ 

Using inequality (2.1) for $a = |\nabla v|$ and $b = \frac{b - a + R}{\eta R}$, and the condition (SC), we get

$$I_2 \leq C \left( \int_{\frac{3}{2}B} G(x, |\nabla v|) \eta^0 \, dx + \int_{\frac{3}{2}B} G \left( x, \frac{b - a + R}{R} \right) \, dx \right).$$
Collecting the previous estimations of $I_1$ and $I_2$, we obtain

$$(b - a + R)\mu(B) \leq C \left( \int_{\frac{3}{2}B} G(x, |\nabla v|)\eta^0 \, dx + \int_{\frac{3}{2}B} G \left( x, \frac{b - a + R}{R} \right) \, dx \right).$$

Or, by Lemma 5.3, we have

$$\int_{\frac{3}{2}B} G(x, |\nabla (v - (b - a + R))|)\eta^0 \, dx \leq C \int_{\frac{3}{2}B} G^+ \left( \frac{b - a + R - v}{R} \right) \, dx.$$

Hence,

$$(b - a + R)\mu(B) \leq C \int_{\frac{3}{2}B} G^+ \left( \frac{b - a + R}{R} \right) \, dx.$$

Since,

$$1 \leq \frac{b - a + R}{R} \leq \frac{b + 1}{R} \leq \frac{\|u\|_{\infty, B}}{\|B\|} + 1.$$

Then by Lemma 5.2, there exists a constant $C > 0$ depend of $\frac{\|u\|_{\infty, B}}{\|B\|}$ such that

$$G^+ \left( \frac{b - a + R}{R} \right) \leq CG \left( x_0, \frac{b - a + R}{R} \right).$$

Hence,

$$(b - a + R)\mu(B) \leq CR^nG \left( x_0, \frac{b - a + R}{R} \right).$$

So, by the condition $(SC)$, we have

$$\mu(B) \leq CR^{n-1}g \left( x_0, \frac{b - a + R}{R} \right).$$

From Remark 2.1, condition $(SC)$ and inequalities (2.2), (2.3), we have

$$CR^{g^{-1}} \left( x_0, \frac{\mu(B)}{R^{n-1}} \right) \leq \text{ess inf}_B u - \text{ess inf}_{2B} u + R. \quad (5.1)$$

Let $R_j := 2^{1-j}R$. Iterating inequality (5.1), we get

$$C \sum_{j=1}^{\infty} R_j g^{-1} \left( x_0, \frac{\mu(B_j)}{R_j^{n-1}} \right) \leq \sum_{j=1}^{\infty} (\text{ess inf}_B u - \text{ess inf}_{B_j} u + R_j)$$

$$\leq \lim_{k \to \infty} (\text{ess inf}_B u) - \text{ess inf}_{2B} u + \sum_{j=1}^{\infty} R_j.$$

14
As \( u \) is lower semicontinuous at \( x_0 \), then
\[
C \sum_{j=1}^{\infty} R_j g^{-1} \left( x_0, \frac{\mu(B_j)}{R_j^{n-1}} \right) \leq u(x_0) - \text{ess inf } u + 2R.
\]

Or, by Remark 2.1, condition \((SC)\) and inequalities \((2.2), (2.3)\), we have
\[
\int_{R_{j+1}}^{R_j} g^{-1} \left( x_0, \frac{\mu(B(x_0,s))}{s^{n-1}} \right) ds \leq CR_j g^{-1} \left( x_0, \frac{\mu(B_j)}{R_j^{n-1}} \right).
\]

Hence,
\[
W_{G(.)}^\mu(x_0, R) \leq C \sum_{j=1}^{\infty} R_j g^{-1} \left( x_0, \frac{\mu(B_j)}{R_j^{n-1}} \right).
\]

Therefore,
\[
CW_{G(.)}^\mu(x_0, R) + \text{ess inf } \frac{u - 2R}{2B} \leq u(x_0).
\]

This gives the claim. \( \square \)

### 5.2 Potential upper bounded

In order to give the potential upper bound estimate, we briefly recall some results on Lorentz spaces [6].

**Definition 5.2.** Let \( f \in L^0(\Omega) \). For \( q > 0 \), we define
\[
\|f\|_{L^q, \infty}^{q, \infty}_\Omega := \sup_{t > 0} \left\| \{ x \in \Omega : |f(x)| \geq t \} \right\|^{\frac{1}{q}},
\]
\[
\|f\|_{L^q, 1}^{q, 1}_\Omega := q \int_0^{\infty} \left\| \{ x \in \Omega : |f(x)| \geq t \} \right\|^{\frac{1}{q}} dt.
\]

By \( L^{q, \infty}(\Omega) \) (resp., \( L^{q, 1}(\Omega) \)), we denote the space of all measurable functions \( f \) on \( \Omega \) such that \( \|f\|_{L^q, \infty}^{q, \infty}_\Omega < \infty \) (resp., \( \|f\|_{L^q, 1}^{q, 1}_\Omega < \infty \)). Such space are called Lorentz space.

**Proposition 5.1.** We have the following properties

1. \( L^{q, 1}(\Omega) \subset L^q(\Omega) \subset L^{q, \infty}(\Omega) \).

2. For any nonnegative constant \( l \),
\[
\|f^+\|_{L^q, \infty}^{q, \infty}_\Omega \leq l|\Omega|^{\frac{1}{q}} + \left\| (f - l)^+ \right\|_{L^q, \infty}^{q, \infty}_\Omega.
\]
3. For \( q > 1 \) and \( f, g \in L^0(\Omega) \), we have the Hölder type inequality

\[
\left| \int_{\Omega} fg \, dx \right| \leq \| f \|_{\Omega}^{q^*,1} \| g \|_{\Omega}^{q,\infty},
\]

where \( \frac{1}{q} + \frac{1}{q^*} = 1 \).

Lemma 5.4. Let \( G(\cdot) \in \Phi(\Omega) \) satisfies (SC) and (A_0). If \( (t_i) \) a family in \( \mathbb{R}^+ \) such that \( \sup_{i > 0} t_i < \infty \), then there exists a constant \( C \) such that

\[
g(x, \sup_{i > 0} t_i) \leq C \sup_{i > 0} g(x, t_i).
\]

Proof: For any \( \epsilon < 1 \), there exists \( i_0 > 0 \) such that

\[
\sup_{i > 0} t_i < t_{i_0} + \epsilon.
\]

Then, by inequalities (2.2), (2.3) and (A_0), we have

\[
g(x, \sup_{i > 0} t_i) \leq g(x, t_{i_0} + \epsilon)
\leq \frac{g_0}{g_0} 2^{g_0 - 1} (g(x, t_{i_0}) + g(x, \epsilon))
\leq \frac{g_0}{g_0} 2^{g_0 - 1} (\sup_{i > 0} g(x, t_i)) + \frac{g_0}{g_0} 2^{g_0 - 1} g(x, 1).
\]

Taking the limit \( \epsilon \to 0 \), we get

\[
g(x, \sup_{i > 0} t_i) \leq \frac{g_0}{g_0} 2^{g_0 - 1} \sup_{i > 0} g(x, t_i),
\]

so the claim holds with \( C = \frac{g_0}{g_0} 2^{g_0 - 1} \).

Theorem 5.2. Let \( G(\cdot) \in \Phi(\Omega) \) satisfies (SC), (A_0) and (A_{1,n}). Let \( u \) be a solution to equation (1.5) in \( \Omega \) with a signed Radon measure \( \mu \), such that \( |\mu| \in \left( W^{1,G(\cdot)}(\Omega) \right)^* \). If \( u \in L^{N',\infty}(\frac{1}{2}B) \) with \( \chi' = \frac{N}{N-1}(g_0 - 1) \) and lower semicontinuous at \( x_0 \). Then, for any \( \gamma > 0 \), there exists a constant \( C = C(c_1, c_2, g_0, g^0, \gamma, N, \| u^+ \|_{N, B(\frac{1}{2}B)^*}, \| u^+ \|_{\infty, B(\frac{1}{2}B)^*}) > 0 \), such that

\[
u^+(x_0) \leq C \left( R + \left( \int_{B(\frac{1}{2}B)} \nu^+(x)^{\gamma} \, dx \right)^{\frac{1}{\gamma}} + W^{\mu^+}_{G(\cdot)}(x_0, 2R) \right).
\]

16
Proof: Let \( u \) be a solution to equation (3.2) and \( A = B \setminus \frac{1}{2}B \). Since \( |\mu| \in \left( W^{1,G} \right)^*(\Omega) \) if and only if \( \mu^+, \mu^- \in \left( W^{1,G} \right)^*(\Omega) \), then by Theorem 4.1, there exists \( v \in W^{1,G}(A) \) be the solution to equation
\[
-\text{div}A(x, \nabla v) = -\mu^- \text{ in } A,
\]
such that \( v - u \in W^{1,G}(A) \).

**First step:** Fix \( \Psi \in W^{1,G}(\frac{3}{4}B) \) such that \( 0 \leq \Psi \leq 1 \), we show the following inequality
\[
\int_B A(x, \nabla v) \cdot \nabla \Psi \, dx \leq 2\mu^+(B). \tag{5.2}
\]
Indeed, from the definition of \( u \) and \( v \), we have
\[
0 \leq \int_A \varphi \, d\mu^+ = \int_A (A(x, \nabla u) - A(x, \nabla v)) \cdot \nabla \varphi \, dx. \tag{5.3}
\]
for any nonnegative \( \varphi \in W^{1,G}(A) \). So, by Lemma 3.3 and inequality (5.3), we have \( v \leq u \), a.e in \( A \). Extend \( v \) as \( v = u \) outside of \( A \). Using the inequality (5.3) with \( \varphi = I_\epsilon(u - v)\Psi \) where \( I_\epsilon(t) = \epsilon^{-1} \min\{t, \epsilon\} \), we obtain
\[
\int_A (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla I_\epsilon(u - v) \, dx
\]
\[
\leq \int_A (A(x, \nabla u) - A(x, \nabla v)) \cdot \nabla I_\epsilon(u - v) \Psi \, dx
\]
\[
\leq \int_A (A(x, \nabla u) - A(x, \nabla v)) \cdot \nabla I_\epsilon(u - v) \, dx.
\]
Again, we use the inequality (5.3) with \( \varphi = I_\epsilon(u - v) \), we get
\[
\int_A (A(x, \nabla u) - A(x, \nabla v)) \cdot \nabla I_\epsilon(u - v) \, dx \leq \int_B I_\epsilon(u - v) \, d\mu^+ \leq \mu^+(B).
\]
Hence,
\[
\int_A ((A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \Psi) I_\epsilon(u - v) \, dx \leq \mu^+(B).
\]
Take the limit \( \epsilon \to 0 \), we get
\[
\int_{\{x \in A : u(x) > v(x)\}} (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \Psi \, dx \leq \mu^+(B).
\]
Since, $\nabla u = \nabla v$ a.e on $\{x \in A : u(x) = v(x)\}$ and, $u \geq v$ a.e in $A$, $v = u$ outside of $A$, then
\[
\int_B (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \Psi \, dx \leq \mu^+(\overline{B}). \tag{5.4}
\]

On the other hand, as $\Psi \in W^{1,G(\cdot)}_0(\overline{B})$. Then, by the definition of solution $u$, we have
\[
\int_B A(x, \nabla u) \cdot \nabla \Psi \, dx = \int_B \Psi \, d\mu \leq \mu^+(\overline{B}). \tag{5.5}
\]

Combining the two inequalities (5.4) and (5.5), we obtain the inequality (5.2)
\[
\int_B A(x, \nabla v) \cdot \nabla \Psi \, dx \leq 2 \mu^+(\overline{B}).
\]

**Second step:** We show the following inequality
\[
\frac{1}{\frac{1}{2} |B|} \left\| u^+ \right\|_{\frac{\chi}{N-1}B} \leq C \left( \left( \int_{B \setminus \frac{1}{2}B} (u^+)\gamma \, dx \right)^{\frac{1}{\gamma}} + Rg^{-1} \left( x_0, \frac{\mu^+(\overline{B})}{R^{n-1}} \right) + R \right), \tag{5.6}
\]
where $\chi' = \frac{N}{N-1}(g_0 - 1)$. In one hand, we have $v$ is a subsolution in $A$ to equation (3.1), then by Lemma 3.1
\[
\text{ess sup}_S v^+ \leq C \left( \int_A \overline{v}^\gamma \, dx \right)^{\frac{1}{\gamma}}, \tag{5.7}
\]

where $S = \cup_{x \in \partial \frac{1}{2}B} B(x, \frac{R}{4})$ and $\overline{v} = v^+ + R$. Denoting $l = \text{ess sup}_S v^+$, then, by the conditions (SC) and (a2), for any positive constant $k$, we have
\[
\begin{align*}
c_1 g_0 & \int_B G (x, |\nabla \min\{(v^+ - l)^+, k\}|) \, dx \\
& \leq c_1 \int_B g(x, |\nabla \min\{(v^+ - l)^+, k\}|) |\nabla \min\{(v^+ - l)^+, k\}| \, dx \\
& \leq \int_B A(x, \nabla \min\{(v^+ - l)^+, k\}) \cdot \nabla \min\{(v^+ - l)^+, k\} \, dx \\
& \leq \int_B A(x, \nabla v) \cdot \nabla \min\{(v^+ - l)^+, k\} \, dx.
\end{align*}
\]

Note that $(v^+ - l)^+ = 0$ a.e on $S$, so $(v^+ - l)^+ \in W^{1,G(\cdot)}_0(\frac{3}{4}B)$. Then, by the inequality (5.2) for $\Psi = k^{-1} \min\{(v^+ - l)^+, k\} \in W^{1,G(\cdot)}_0(\frac{3}{4}B)$, we get
\[
\int_B A(x, \nabla v) \cdot \nabla \min\{(v^+ - l)^+, k\} \, dx \leq 2k \mu^+(\overline{B}).
\]
Hence,
\[ \int_B G(x, |\nabla \min\{(v^+ - l)^+, k\}|) \, dx \leq C k \mu^+(B). \] (5.8)

In the other hand, by the Sobolev inequality for the function

\[ G - \min\{(v^+ - l)^+, k\} \in W^{1,1}(\frac{3}{4} B), \]

there exists a constant \( C > 0 \) such that

\[ \left( \int_{\frac{3}{4} B} G\left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right)^{\chi} \, dx \right)^{\frac{1}{\chi}} \]
\[ \leq CR \int_{\frac{3}{4} B} |\nabla G\left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right)| \, dx \]
\[ \leq CR \int_{\frac{3}{4} B} \tilde{g}\left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right) \, dx \]
\[ \leq C \int_{\frac{3}{4} B} \tilde{g}\left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right) \, dx. \]

where \( \chi := 1^* = \frac{N}{N - 1}. \)

Using inequality (2.1), for \( a = \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \) and \( b = 2 g^0 C |\nabla \min\{(v^+ - l)^+, k\}| \), we have

\[ \left( \int_{\frac{3}{4} B} G\left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right)^{\chi} \, dx \right)^{\frac{1}{\chi}} \]
\[ \leq \frac{1}{2 g^0} \int_{\frac{3}{4} B} \tilde{g}\left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right) \, dx \]
\[ + \frac{1}{2 g^0} \int_{\frac{3}{4} B} \tilde{g}\left( C |\nabla \min\{(v^+ - l)^+, k\}| \right) C |\nabla \min\{(v^+ - l)^+, k\}| \, dx \]
\[ \leq \frac{1}{2} \int_{\frac{3}{4} B} G\left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right) \, dx + \frac{C g^0}{2} \int_{\frac{3}{4} B} G^- \left( |\nabla \min\{(v^+ - l)^+, k\}| \right) \, dx. \]

As,

\[ \int_{\frac{3}{4} B} G^- \left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right) \, dx \leq \left( \int_{\frac{3}{4} B} G^- \left( \min\left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R}\right\} \right)^{\chi} \, dx \right)^{\frac{1}{\chi}}. \]
Then, there exists a constant $C > 0$ such that
\[
\left( \int_{\frac{1}{2}B} G^- \left( \min \left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R} \right\} \chi \, dx \right) \right)^{\frac{1}{\chi}} \leq C \int_{\frac{1}{2}B} G^- \left( |\nabla \min \{(v^+ - l)^+, k\}| \right) \, dx.
\] (5.9)

Combining the inequalities (5.8) and (5.9), we obtain
\[
\left( \int_{\frac{1}{2}B} G^- \left( \min \left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R} \right\} \chi \, dx \right) \right)^{\frac{1}{\chi}} \leq C k \frac{\mu^+(B)}{R^n}. \quad (5.10)
\]

Otherwise, by Lemma 5.1, we have
\[
\left( \int_{\frac{1}{2}B} G^- \left( \min \left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R} \right\} \chi \, dx \right) \right)^{\frac{1}{\chi}} \geq \left( \int_{\{x \in \frac{1}{2}B : (v^+ - l)^+ \geq k\}} G^- \left( \min \left\{ \frac{(v^+ - l)^+}{R}, \frac{k}{R} \right\} \chi \, dx \right) \right)^{\frac{1}{\chi}} \geq G^- \left( \frac{k}{R} \right) \frac{|\{x \in \frac{1}{2}B : (v^+ - l)^+ \geq k\}|^{\frac{1}{\chi}}}{|\frac{1}{2}B|^{\frac{1}{\chi}}}. \]

Then, by the inequality (5.10), we have
\[
\tilde{g} \left( \frac{k}{R} \right) \frac{|\{x \in \frac{1}{2}B : (v^+ - l)^+ \geq k\}|^{\frac{1}{\chi'}}}{|\frac{1}{2}B|^{\frac{1}{\chi'}}} \leq C \frac{\mu^+(B)}{R^{n-1}}.
\]

So,
\[
\tilde{g} \left( \frac{k}{R} \right) \left( \frac{|\{x \in \frac{1}{2}B : (v^+ - l)^+ \geq k\}|^{\frac{1}{\chi'}}}{|\frac{1}{2}B|^{\frac{1}{\chi'}}} \right)^{\gamma_0 - 1} \leq C \frac{\mu^+(B)}{R^{n-1}}.
\]

where $\chi' = \chi(\gamma_0 - 1)$. Using inequality (2.3), we get
\[
\tilde{g} \left( \frac{k}{R} \right) \left( \frac{|\{x \in \frac{1}{2}B : (v^+ - l)^+ \geq k\}|^{\frac{1}{\chi'}}}{|\frac{1}{2}B|^{\frac{1}{\chi'}}} \right) \leq C \frac{\mu^+(B)}{R^{n-1}}.
\]

20
Using Lemma 5.4 and definition of Lorentz norms, we obtain

\[ \tilde{g} \left( \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}} + 1} \right) \leq C \frac{\mu^+(B)}{R^{n-1}}. \] (5.11)

Since,

\[ 1 \leq \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} + 1 \leq \frac{\left\| v^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} + 1 \leq \frac{\left\| v^+ \right\|_{\frac{1}{2}B}^\infty}{R} + 1. \]

Then, by Lemma 5.2, there exists a constant \( C \) depend of \( \left\| v^+ \right\|_{\frac{1}{2}B}^\infty \) such that

\[ G \left( x_0, \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} + 1 \right) \leq CG^{-} \left( \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} + 1 \right). \]

So, by the condition \((SC')\) and Lemma 5.1, we have

\[ g \left( x_0, \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} + 1 \right) \leq C \tilde{g} \left( \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} + 1 \right). \]

Hence, by the condition \((A_0)\) and the inequality (5.11), we have

\[ g \left( x_0, \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} \right) \leq C \tilde{g} \left( \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} \right) + \tilde{g}(1) \leq C \left( \frac{\mu^+(B)}{R^{n-1}} + 1 \right). \]

So,

\[ \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} \leq Rg^{-1} \left( x_0, C \left( \frac{\mu^+(B)}{R^{n-1}} + 1 \right) \right). \]

From Remark 2.1, the condition \((SC')\), inequalities (2.2), (2.3) and the condition \((A_0)\), we have

\[ \frac{\left\| (v^+ - l)^+ \right\|_{\frac{1}{2}B}^\infty}{R\left| \frac{1}{2}B \right|^{\frac{1}{1}}} \leq CRg^{-1} \left( x_0, \left( \frac{\mu^+(B)}{R^{n-1}} \right) \right) + CRg^{-1}(1) \leq CRg^{-1} \left( x_0, \left( \frac{\mu^+(B)}{R^{n-1}} \right) \right) + CR. \]
Using Proposition 5.1 and inequality (5.7), we obtain
\[
\frac{1}{|\frac{1}{2} B|^\frac{1}{N}} \|v^+\|_{\frac{1}{2} B}^{',\infty} \leq l \leq \frac{1}{|\frac{1}{2} B|^\frac{1}{N}} \|(v^+ - l)^+\|_{\frac{1}{2} B}^{',\infty}
\leq C \left( \int_{A} v^\gamma \, dx \right)^{\frac{1}{\gamma}} + CRg^{-1} \left( x_0, \frac{\mu^+(B)}{R^{n-1}} \right) + CR.
\]

Since \( u = v \) in \( \frac{1}{2} B \) and \( v \leq u \) in \( B \), we get the inequality (5.6)
\[
\frac{1}{|\frac{1}{2} B|^\frac{1}{N}} \|u^+\|_{\frac{1}{2} B}^{',\infty} \leq C \left( \int_{B \setminus \frac{1}{2} B} (u^+)^\gamma \, dx \right)^{\frac{1}{\gamma}} + CRg^{-1} \left( x_0, \frac{\mu^+(B)}{R^{n-1}} \right) + CR.
\]

**Iteration step:** We show the following inequality
\[
u^+(x_0) \leq C \left( R + \left( \int_{B \setminus \frac{1}{2} B} (u^+)^\gamma \, dx \right)^{\frac{1}{\gamma}} + W^\mu(x_0, 2R) \right).
\]

Let \( B_0 = B_R \), for \( j = 0, 1, \ldots \), take \( R_j = 2^{-j} R \), \( B_j = B_{R_j} \). Also, for \( \delta \in (0, 1) \) we consider a sequence
\[
l_0 := 0, \quad l_{j+1} := l_j + \frac{1}{\delta^{\frac{1}{N}}} \frac{1}{|B_{j+1}|^{\frac{1}{N}}} \|(u - l_j)^+\|_{B_{j+1}}^{',\infty}.
\]

By the definition of \( l_j \) and the inequality (5.5) for \( (u - l_j)^+ \), we have
\[
l_{j+1} - l_j = \frac{1}{\delta^{\frac{1}{N}}} \frac{1}{|B_{j+1}|^{\frac{1}{N}}} \|(u - l_j)^+\|_{B_{j+1}}^{',\infty}
\leq C \frac{1}{\delta^{\frac{1}{N}}} \left( \int_{B_{j+1}\setminus B_j} ((u - l_j)^+)^\gamma \, dx \right)^{\frac{1}{\gamma}} + R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) + R_j
\leq C \frac{1}{\delta^{\frac{1}{N}}} \left( \int_{B_{j}} ((u - l_j)^+)^\gamma \, dx \right)^{\frac{1}{\gamma}} + R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) + R_j.
\]

(5.13)
If, we choose $\gamma \leq \chi'$, then by Proposition 5.1, for $q = \frac{\chi'}{\gamma}$, we have

$$
\left(\int_{B_j} ((u - l_j)^+)^{\gamma} \, dx \right)^{\frac{1}{\gamma}} = \left(\int_{\{x \in B_j : u > l_j\}} ((u - l_j)^+)^{\gamma} \, dx \right)^{\frac{1}{\gamma}} \\
\leq C \frac{1}{|B_j|^{\frac{1}{\gamma}}} \|\{x \in B_j : u(x) \geq l_j\}\|_{L^{\frac{\chi'}{\gamma}}(B_j)}^{\frac{1}{\gamma}} \cdot \left(\|(u - l_j)^+\|_{L^{\chi'}(B_j)}^{\chi'} \right)^{\frac{1}{\gamma}} \\
\leq C \frac{1}{|B_j|^{\frac{1}{\gamma}}} \|\{x \in B_j : u(x) \geq l_j\}\|_{L^{\frac{\chi'}{\gamma}}(B_j)}^{\frac{1}{\gamma}} \cdot \|(u - l_j)^+\|_{L^{\chi'}(B_j)}^{\chi'} \cdot \delta |B_j|^{\frac{1}{\gamma'}}.
$$

Since,

$$
|\{x \in B_j : u(x) \geq l_j\}| = |\{x \in B_j : u(x) - l_j \geq l_j - l_{j-1}\}| \\
= \left(\|\{x \in B_j : \frac{u(x) - l_{j-1}}{l_j - l_{j-1}} \geq 1\}\|_{L^{\chi'}} \right)^{\chi'} \\
\leq \delta |B_j|\left(\|(u - l_{j-1})^+\|_{L^{\chi'}(B_j)}^{\chi'} \right)^{\chi'} \\
\leq \delta |B_j|^{\frac{1}{\gamma'}}.
$$

Moreover, by the definition of $l_j$, we have

$$
\|(u - l_{j-1})^+\|_{L^{\chi'}(B_j)} = (l_j - l_{j-1})\delta \frac{1}{\gamma'} |B_j|^{\frac{1}{\gamma'}}.
$$

Then,

$$
\left(\int_{B_j} ((u - l_j)^+)^{\gamma} \, dx \right)^{\frac{1}{\gamma}} \leq C \delta^{\frac{1}{\gamma}} \|\{x \in B_j : u(x) \geq l_j\}\|_{L^{\frac{\chi'}{\gamma}}(B_j)}^{\frac{1}{\gamma}} \cdot \|(u - l_j)^+\|_{L^{\chi'}(B_j)}^{\chi'} \\
\leq C \delta^{\frac{1}{\gamma}} \|\{x \in B_j : u(x) \geq l_j\}\|_{L^{\frac{\chi'}{\gamma}}(B_j)}^{\frac{1}{\gamma}} \cdot \|(u - l_{j-1})^+\|_{L^{\chi'}(B_j)}^{\chi'} \\
\leq C \delta^{\frac{1}{\gamma}} \left(\frac{1}{\gamma'} \right)^{\chi'} (l_j - l_{j-1})^{\frac{1}{\gamma'}} \\
\leq C \delta^{\frac{1}{\gamma}} (l_j - l_{j-1}).
$$

Hence, by the inequality (5.13), we have

$$
l_{j+1} - l_j \leq C \delta^{\frac{1}{\gamma}} \cdot \frac{1}{\gamma'} (l_j - l_{j-1}) + \frac{C}{\delta^{\frac{1}{\gamma'}}} R_j g^{-1}\left(x_0, \frac{\mu^+ (B_j)}{R_j^{\alpha - 1}}\right) + CR_j.
$$

We choose $\delta > 0$ such that $C \delta^{\frac{1}{\gamma}} \cdot \frac{1}{\gamma'} \leq \frac{1}{2}$, we get

$$
l_{j+1} - l_j \leq \frac{1}{2} (l_j - l_{j-1}) + CR_j g^{-1}\left(x_0, \frac{\mu^+ (B_j)}{R_j^{\alpha - 1}}\right) + CR_j.
$$
Hence,
\[ l_{k+1} - l_1 = \sum_{j=1}^{j=k} (l_{j+1} - l_j) \leq \frac{1}{2} (l_k - l_0) + C \sum_{j=1}^{j=k} R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) + C \sum_{j=1}^{j=k} R_j. \]

By the definition of \( l_1 \) and inequality (5.6), we have
\[
l_1 = \frac{1}{\delta^x} \frac{1}{|B_1|^x} \left\| u^+ \right\|_{\chi', \infty}^{B_1} \text{ and inequality (5.6), we have}
\leq \frac{C}{\delta^x} \left( \left\{ \int_{B_0 \setminus B_1} (u^+)^{\gamma} \, dx \right\}^\frac{1}{\gamma} + R_0 g^{-1} \left( x_0, \frac{\mu^+(B_0)}{R_0^{n-1}} \right) + R_0 \right).
\]

So,
\[
l_{k+1} \leq \frac{1}{2} l_k + C \left( \int_{B_0 \setminus B_1} (u^+)^{\gamma} \, dx \right)^\frac{1}{\gamma} + C \sum_{j=0}^{j=k} R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) + C \sum_{j=0}^{j=k} R_j.
\]

Taking the limit \( k \to \infty \), then
\[
\frac{1}{2} l_\infty \leq C \left( \int_{B_0 \setminus B_1} (u^+)^{\gamma} \, dx \right)^\frac{1}{\gamma} + C \sum_{j=0}^{\infty} R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) + C \sum_{j=0}^{\infty} R_j,
\]
where \( l_\infty = \lim_{k \to \infty} l_k \). Or, by Proposition 5.1 and the definition of \( l_j \), we have
\[
\frac{1}{|B_j|^x} \left\| u^+ \right\|_{\chi', \infty}^{B_j} \leq l_\infty + \frac{1}{|B_j|^x} \left\| (u - l_\infty)^+ \right\|_{\chi', \infty}^{B_j} \leq l_\infty + \frac{1}{|B_j|^x} \left\| (u - l_{j-1})^+ \right\|_{\chi', \infty}^{B_j} \leq l_\infty + \delta^x (l_j - l_{j-1}).
\]

Taking the upper limit, we obtain
\[
\limsup_{j \to \infty} \frac{1}{|B_j|^x} \left\| u^+ \right\|_{\chi', \infty}^{B_j} \leq l_\infty \leq C \left( R + \left( \int_{B_0 \setminus B_1} (u^+)^{\gamma} \, dx \right)^\frac{1}{\gamma} + \sum_{j=0}^{\infty} R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) \right).
\]

As \( u \) is lower semicontinuous at \( x_0 \), then
\[
u(x_0) \leq C \left( R + \left( \int_{B_0 \setminus B_1} (u^+)^{\gamma} \, dx \right)^\frac{1}{\gamma} + \sum_{j=0}^{\infty} R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) \right).
\]

24
Or, by Remark 2.1, the condition $(SC)$ and inequalities (2.2), (2.3), we have

\[ R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) = g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) \int_{R_j}^{2R_j} ds \leq C \int_{R_j}^{2R_j} g^{-1} \left( x_0, \frac{\mu^+(B(x_0, s))}{s^{n-1}} \right) ds. \]

So,

\[ \sum_{j=0}^{\infty} R_j g^{-1} \left( x_0, \frac{\mu^+(B_j)}{R_j^{n-1}} \right) \leq C \int_{0}^{2R} g^{-1} \left( x_0, \frac{\mu^+(B(x_0, s))}{s^{n-1}} \right) ds. \]

Therefore,

\[ u(x_0) \leq C \left( R + \left( \int_{B_0 \setminus B_1} (u^+)^\gamma \, dx \right)^{\frac{1}{\gamma}} + W_{G(\cdot)}^{\mu^+}(x_0, 2R) \right). \]

This complete the proof. \( \Box \)

**Remark 5.1.** If \( G(\cdot) \in \Phi(\Omega) \) satisfies the conditions of the theorem 1.4 in [2] and \( u \) be a nonnegative weak solution to (3.2) with nonnegative Radon measure \( \mu \) in \( B_R \). Then the weak Harnack inequality holds, so \( u \in L^\gamma(B_R) \) for any \( \gamma < \gamma_0 \) with

\[ \gamma_0 := \begin{cases} \frac{N(g_0-1)}{N-g_0} & \text{if } g_0 < N \\ \infty & \text{if } g_0 \geq N \end{cases} \]

As \( \chi' < \gamma_0 \), then \( u \in L^{\chi'}(B_R) \).

**References**

[1] Alkhutov Yu., Krasheninnikova O.: Continuity at boundary points of solutions of quasilinear elliptic equations with nonstandard growth. Izvestiya RAN, seriya matematicheskaya. 68 (2004), pp. 3-60.

[2] Benyaiche A., Harjulehto P., Peter Hästö P., Karppinen A.: The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth. (2020), [arXiv:2006.06276](https://arxiv.org/abs/2006.06276).

[3] Benyaiche A., Khlifi I.: Harnack Inequality for Quasilinear Elliptic Equations in Generalized Orlicz-Sobolev Spaces. Potential Analysis. 53 (2020), pp. 631643.

[4] Björn A., Björn J.: Nonlinear Potential Theory on Metric Spaces. European Mathematical Society, Zürich. 2011.

25
[5] Chlebicka I., Giannetti F., Zatorska-Goldstein A.: Wolff potentials and local behaviour of solutions to measure data elliptic problems with Orlicz growth. (2020), [arXiv:2006.02172].

[6] Grafakos L.: Classical Fourier Analysis. Graduate Texts in Mathematics. Springer, New York. 2008.

[7] Hara T.: Wolff potential estimates for Cheeger p-harmonic functions. Collectanea Mathematica. 69 (2018), pp. 407-426.

[8] Harjulehto P., Hästö P.: Orlicz Spaces and Generalized Orlicz Spaces. Springer-Verlag, Cham. 2019.

[9] Harjulehto P., Hästö P.: Boundary regularity under generalized growth conditions. Zeitschrift für Analysis und ihre Anwendungen. 38 (2019), pp. 73-96.

[10] Harjulehto P., Hästö P.: Double phase image restoration. Journal of Mathematical Analysis and Applications. (2020). https://doi.org/10.1016/j.jmaa.2019.123832.

[11] Harjulehto P., Hästö P., Kinn R.: Generalized Orlicz spaces and related PDE. Nonlinear Analysis: Theory, Methods & Applications. 143 (2016), pp. 155-173.

[12] Harjulehto P., Hästö P., Toivanen O.: Hölder regularity of quasiminimizers under generalized growth conditions. Calculus of Variations and Partial Differential Equations. 56 (2017), article 22.

[13] Hedberg L.I., Wolff Th.H.: Thin Sets in Nonlinear Potential Theory. Annales de l’Institut Fourier. 33 (1983), pp. 161-187.

[14] Heinonen J., Kilpeläinen T., Martio O.: Nonlinear Potential Theory of Degenerate Elliptic Equations. Clarendon Press. 1993.

[15] Hewitt E., Stromberg K.: Real and Abstract Analysis. Springer, Berlin. 1965.

[16] Kilpeläinen T., Mal J.: Degenerate elliptic equations with measure data and nonlinear potentials. Annali della Scuola normale superiore di Pisa, Classe di scienze. 19 (1992), pp. 591-613.

[17] Kilpeläinen T., Mal J.: The Wiener test and potential estimates for quasilinear elliptic equations. Acta Mathematica. 172 (1994), pp. 137-161.
[18] Korte R., Kuusi T.: A note on the Wolff potential estimate for solutions to elliptic equations involving measures. Advances in Calculus of Variations. 3 (2010), pp. 99-113.

[19] Lieberman G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations. Communications in Partial Differential Equations. 16 (1991), pp. 311361.

[20] Lindqvist P., Martio O.: Two theorems of N. Wiener for solutions of quasilinear elliptic equations. Acta Mathematica. 155 (1985), pp. 153-171.

[21] Lions J.L.: Quelques Mthodes De Rsolution Des Problmes Aux Limites Non Linaires. Dunod, GauthierVillars, Paris. 1969.

[22] Lukkari T., Maeda F.Y., Marola N.: Wolff potential estimates for elliptic equations with nonstandard growth and applications. Forum Mathematicum. 22 (2010), pp. 1061-1087.

[23] Mal J.: Wolff potential estimates of superminimizers of Orlicz type Dirichlet integrals. Manuscripta Mathematica. 110 (2003), pp. 513-525.

[24] Maz’ya V. G.: On the continuity at a boundary point of solutions of quasilinear elliptic equations. Vestnik Leningrad Univ. 25 (1970), pp. 42-55 (Russian).

[25] Mihailescu M., Radulescu V.: Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces. Annales de l’Institut Fourier. 58 (2008), pp. 20872111.

[26] Mikkonen P.: On the Wolff potential and quasilinear elliptic equations involving measures. Annales Academiae Scientiarum Fennicae, Mathematica, Dissertationes. 104 (1996), pp. 171.

[27] Mingione G., Palatucci G.: Developments and perspectives in Nonlinear Potential Theory. Nonlinear Analysis. 194 (2020), article 111452.

[28] Musielak J.: Orlicz spaces and modular spaces. Springer, Berlin. 1983.

[29] Rao M., Ren Z.: Theory of Orlicz spaces. M. Dekker. 1991.

[30] Simonenko I.B.: Interpolation and extrapolation in Orlicz spaces (in Russian). Matematicheskii Sbornik. 63 (1964), pp. 536553.
[31] Trudinger N., Wang X.-J.: *On the weak continuity of elliptic operators and applications to potential theory*. American Journal of Mathematics. 124 (2002), pp. 369-410.

[32] Swierczewska-Gwiazda A.: *Nonlinear parabolic problems in Musielak-Orlicz spaces*. Nonlinear Analysis. 98 (2014), pp. 4865.