Two-dimensional scattering of a deformed Coulomb potential

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Abstract. Scattering theory is a perturbation theory studying self-adjoint operators on a Hilbert space. For quantum theory, we have a Laplacian being perturbed by varying potentials. Scattering amplitudes are found by normalizing ingoing asymptotic states and finding the amplitude of the associated outgoing asymptotic state. The Smorodinsky-Winternitz potential is a deformation of the Coulomb potential found by looking for quantum systems with dynamical symmetries. Because it has this symmetry, the potential remains superintegrable despite the change in potential. Superintegrability has been shown to be associated with exact solvability of the bound states in terms of special functions. In this presentation, the scattering amplitude of the system will be found, thus extending exact solvability to the scattering states.

1. Introduction
Superintegrable systems are Hamiltonians that have $2d-1$ ($d$ for dimension) integrals of motion. Any Hamiltonian that is time independent has at least one integral of motion, the Hamiltonian itself. This is denoted by the Poisson bracket equaling zero: $\{H, H\} = 0$. A superintegral system has other integrals of motion $L_i$, such that $\{H, L_i\} = 0$. There is a conjecture that all bound states of a superintegrable system are exactly solvable [1]. In this paper, we consider an analogous claim for scattering states. Superintegrability also makes a Hamiltonian multiseparable i.e. separable in multiple coordinate systems as long as the integrals of motion are second order in the momentum. A feature that we can exploit to solve for the bound or scattering states of a specific Hamiltonian system.

Finding the bound or scattering states of a Hamiltonian is done by taking our time-independent Schroedinger equation: $H\psi = E\psi$ and solving for $\psi$. Usually $E$ is set to equal $k^2$. Bound states are when $E$ is negative, or $k \in \mathbb{C} \setminus \mathbb{R}$ and scattering is when $E$ is positive, or $k \in \mathbb{R}$.

In this paper we will be looking for the scattering amplitude, or the amplitude of the spherical wave scattered off a central potential when hit by a plane wave. This means we want to build a solution $\psi$ in the form of

$$\psi \approx e^{ikx} + f_c e^{ikr}/r$$  \hspace{1cm} (1)

This is an approximation of the wave function a large distance from the origin and corresponds to a plane wave incoming from the left (the $e^{ikx}$ term) and an outgoing spherical wave attenuated by the scattering coefficient $f_c$, when the time evolution operator is applied.
The Coulomb potential itself $V = -\frac{Q}{r}$, in a Hamiltonian $H = -\Delta + V$ is a superintegrable system. Therefore, we will start by showing how to find it’s scattering coefficient. Then this same process will be used to find the scattering states of the Smorodinsky-Winternitz potential:

$$V = -\frac{Q}{r} + \frac{a}{4r^2 \cos^2 \left(\frac{\theta}{2}\right)} + \frac{b}{4r^2 \sin^2 \left(\frac{\theta}{2}\right)}.$$  

This potential was found in [2] by searching for superintegrable systems in two dimensions.

2. Coulomb Scattering

To begin, we will demonstrate how to find the scattering amplitude of the regular Coulomb potential. This is demonstrated in [3] and [4], and we will be providing an overview here. However, we will not be getting a solution exactly like Equation 1 because the Coulomb potential is long-range. Thus there will be attenuation of the plane wave and the spherical wave, which will appear as we build the scattering solution.

First, we take the time-independent Schroedinger equation with the Hamiltonian in spherical coordinates:

$$-\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \psi(r, \theta) - \frac{1}{r^2} \frac{\partial}{\partial \theta} \psi(r, \theta) - \frac{Q}{r} \psi(r, \theta) = k^2 \psi(r, \theta).$$

Next, exploiting multiseparability, the equation above is converted to parabolic coordinates using the following identities:

$$u = r - x = r(1 - \cos \theta)$$
$$v = r + x = r(1 + \cos \theta).$$

After some careful work, the parabolic Laplacian can be found:

$$\Delta = \frac{4}{u + v} (u \partial_u^2 + v \partial_v^2) + \frac{2}{u + v} (\partial_u + \partial_v).$$

Since the Hamiltonian is $H = -\Delta - \frac{Q}{r}$, the Schroedinger equation is the following:

$$-\frac{4}{u + v} (u \partial_u^2 + v \partial_v^2) \psi(u, v) - \frac{2}{u + v} (\partial_u + \partial_v) \psi(u, v) - \frac{2Q}{u + v} \psi(u, v) = k^2 \psi(u, v).$$

Next, the ansatz solution $\psi(u, v) = Ne^{ikx}G(u) = Ne^{ik\left(\frac{v-u}{2}\right)}G(u)$ with $N$ as a normalization constant, will be substituted in. This is possible because the plane wave is separable in parabolic coordinates, and this anstaz will also be separable, therefore we can look for the form of $G(u)$:

$$(-2ik - 2Q)G(u) + (-2 - 4iku)G'(u) - 4uG''(u) = 0.$$  

Performing another change of coordinates, by taking $\xi = iku$, the above becomes

$$\xi G''(\xi) + \left(\frac{1}{2} - \xi\right) G'(\xi) + \frac{Q}{2ik} G(\xi) = 0.$$  

This is Kummer’s differential equation, [5, Eq. 13.2.1], which has the general form

$$\xi W''(\xi) + (c - \xi) W'(\xi) - aW(\xi) = 0.$$  

It has two solutions, which are the following:
\begin{align*}
M(a, b, z) &= \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \\
U(-m, b, z) &= (-1)^m \sum_{s=0}^{m} \binom{m}{s} (b + s)_{m-s} (-z)^s.
\end{align*}

Defining \( \eta = -\frac{Q}{2k} \), our solutions can be written as:

\[ \psi(u, v) = Ne^{ikx} \left( c_1 U(-i\eta, \frac{1}{2}; iku) + c_2 M(-i\eta, \frac{1}{2}; iku) \right). \quad (2) \]

However, the Kummer \( U \) function has a branch point at \( z = 0 \), and since we want our solution \( \psi \) to be differentiable, we can set \( c_1 = 0 \). The next step is to look at the asymptotics of this solution. As \( z \to 0 \), the Kummer \( M \) equation has the following asymptotics [5, Eq. 13.2.13]:

\[ M(a, b, z) \sim 1 + O(z). \]

For the scattering solution, we want \( \psi(u, v) \to e^{ikx} \) as \( u \to 0 \). Therefore, coefficient \( c_2 \) can be set to one. Finally, we have the following solution:

\[ \psi(u, v) = Ne^{ikx} M(-i\eta, \frac{1}{2}; iku). \]

Now we check the asymptotics of \( \psi \) as \( z \to \infty \), which will give us our scattering state. From [5, 13.7.2], we have

\[ M(a, b, z) \sim \sum_{s=0}^{\infty} \frac{(1 - a)_s}{s!} z^{-s} + \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b - a)} \sum_{s=0}^{\infty} \frac{(a)_s (a-b+1)_s}{s!} (z)^{-s}. \]

Which is valid for \(-\frac{3}{2} \pi + \delta \leq \pm\text{ph} z \leq \frac{3}{2} \pi - \delta \). For our purposes, we only need the \( s = 0 \) terms. The solutions for large \( z \) are then:

\[ \psi(u, v) \sim Ne^{ikx} \left( e^{iku (iku)^{-i\eta} - \frac{1}{2}} \frac{1}{\Gamma(-i\eta)} + e^{\pi i\eta} (iku)^{i\eta} \frac{1}{\Gamma(\frac{1}{2} + i\eta)} \right). \]

The task at hand is now to rearrange this to get a form similar to Equation 1. First some terms are factored out:

\[ \psi(u, v) \sim Ne^{ikx} e^{\pi \eta} \left( \frac{1}{\Gamma(\frac{1}{2} + i\eta)} \left( iku \right)^{i\eta} \frac{\Gamma(\frac{1}{2} + i\eta)}{\Gamma(-i\eta)} e^{iku (iku)^{-i\eta} - \frac{1}{2} e^{-\pi i\eta}} \right). \]

Using the identity \( i = e^{i\frac{\pi}{2}} \), the above becomes:

\[ \psi(u, v) \sim Ne^{ikx} e^{\pi \eta} \left( e^{-\frac{\pi \eta}{2}} (ku)^{i\eta} \frac{\Gamma(\frac{1}{2} + i\eta)}{\Gamma(-i\eta)} e^{iku (iku)^{-i\eta} (iku)^{-\frac{1}{2} e^{-\pi i\eta}}} \right). \]

Factoring out \( e^{-\frac{\pi \eta}{2}} \):

\[ \psi(u, v) \sim Ne^{ikx} e^{\frac{\pi \eta}{2}} \left( \frac{1}{\Gamma(\frac{1}{2} + i\eta)} \left( iku \right)^{i\eta} \frac{\Gamma(\frac{1}{2} + i\eta)}{\Gamma(-i\eta)} e^{iku (iku)^{-i\eta} (iku)^{-\frac{1}{2} e^{-\pi i\eta}}} \right). \]
Then recalling that \( u = r - x = r(1 - \cos \theta) = 2r \sin^2 \frac{\theta}{2} \), we have the identity:

\[
(iku)^{-\frac{1}{2}}(ku)^{-i\eta} = \left(2ikr \sin^2 \frac{\theta}{2}\right)^{-\frac{1}{2}} \left(2kr \sin^2 \frac{\theta}{2}\right)^{-i\eta} = (2ik)^{-\frac{1}{2}} \left(\sin \frac{\theta}{2}\right)^{-1-2i\eta} r^{-\frac{1}{2}} e^{-i\eta \log(2kr)}.
\]

Which we use to get our solution:

\[
\psi(u,v) \sim \frac{Ne^{-\frac{u\eta}{2}}}{\Gamma\left(\frac{1}{2} + i\eta\right)} \left(e^{ikx + i\eta \log(k(r-x))} + \frac{\Gamma\left(\frac{1}{2} + i\eta\right)}{\Gamma\left(-i\eta\right)}(2ik)^{-\frac{1}{2}} \left(\sin \frac{\theta}{2}\right)^{-1-2i\eta} e^{ikr - i\eta \log(2kr)}\right).
\]

Thus, we can see that there is some attenuation of both the plane wave and the spherical wave since the Coulomb potential is long-range. In the end, we can still take the coefficient of the attenuated spherical wave to be the scattering coefficient, \( f_c \). This turns out to be

\[
f_c = \frac{\Gamma\left(\frac{1}{2} + i\eta\right)}{\Gamma\left(-i\eta\right)}(2ik)^{-\frac{1}{2}} \left(\sin \frac{\theta}{2}\right)^{-1-2i\eta}.
\]

The same method can be used for the case when our ansatz is \( \psi(u,v) = N e^{ikx} F(v) \). The purpose of this is to look at a wave incoming to the origin from the left instead of leaving the origin to the right. This yields the solution of

\[
\psi \approx e^{ikx - i\eta \log kv} + \frac{\Gamma\left(\frac{1}{2} - i\eta\right)}{\Gamma(i\eta)}(-2ik)^{-\frac{1}{2}} \left(\cos \frac{\theta}{2}\right)^{-1+2i\eta} e^{-ikr + i\eta \log 2kr}/\sqrt{r}.
\]

3. Partially Caged Coulomb Potential

The Hamiltonian for the fully caged Coulomb potential in parabolic coordinates is as follows:

\[
H = -\frac{4}{u + v}(u\partial_u^2 + v\partial_v^2) - \frac{2}{u + v}(\partial_u + \partial_v) - \frac{2Q}{u + v} + \frac{a}{2u(u + v)} + \frac{b}{2u(u + v)}.
\]

We create the partially caged Hamiltonian by setting either \( a \) or \( b \) to zero. In this way the Hamiltonian applied to the plane wave solution \( \psi(u,v) = Ne^{ikx} F(v) \) remains separable in parabolic coordinates. For now, we will set \( a = 0 \). This creates a singularity along the positive x-axis. Therefore we wish to recreate a plane wave in our scattering solution as traveling to the origin from the negative x-axis. Now we will build a scattering solution starting with the ansatz \( \psi(u,v) = Ne^{ikx} F(v) G(u) \), along with extra conditions. Our potential is \( V = -\frac{2Q}{u + v} + \frac{b}{2u(u + v)} \), which creates a singularity along the positive x-axis. Thus our solution requires that

\[
\lim_{u \to 0} \psi(u,v) = 0 \quad \text{or} \quad \lim_{u \to 0} G(u) = 0.
\]

The solution should also be a plane wave along the negative x-axis. This leads to the requirement:

\[
\lim_{u \to 0} \psi(u,v) = Ne^{ikx}.
\]

Or identically:

\[
\lim_{v \to 0} F(v) = 1 \quad \text{and} \quad \lim_{u \to +0} G(u) = 1
\]

Lastly, the scattering solution will occur at the following asymptotics: \( v \to \infty \) and \( u \to 0 \). Using these requirements, we can build the scattering solution. Taking the ansatz \( \psi(u,v) = \)}
\[ N e^{ikr \omega - i\theta} \mathcal{L}(v) F(v) G(u) \] and plugging it in the the equation \( H \psi(u, v) = k^2 \psi(u, v) \) where \( H \) is the Hamiltonian of our partially caged potential, we produce two differential equations:

\[
(1 + 2iku)F'(v) + 2vF''(v) + (Q - A)F(v) = 0 \\
(1 - 2iku)G'(u) + 2uG''(u) + (A - b)G(u) = 0
\]

To clean up the notation, set \( \beta = \sqrt{1 + 2b} \). Thus we get the following solutions:

\[
F(v) = c_1 U \left( \frac{i}{2\pi} (A - Q) , \frac{1}{2}, -ikv \right) + c_2 M \left( \frac{i}{2\pi} (A - Q) , \frac{1}{2}, -ikv \right) \\
G(u) = u \frac{1}{2}(1 + \beta) \left( d_1 U \left( \frac{iA}{2\pi} + \frac{i}{4}(1 + \beta) , 1 + \frac{1}{2} \beta , iku \right) + d_2 M \left( \frac{iA}{2\pi} + \frac{i}{4}(1 + \beta) , 1 + \frac{1}{2} \beta , iku \right) \right)
\]

By taking the separation constant \( A \) to be zero, \( G(u) \) becomes:

\[
G(u) \approx d_2 u \frac{1}{2}(1 + \beta) \left( \frac{e^{iku}(iku) \frac{iA}{2\pi} + \frac{i}{4}(1 + \beta) - (1 + \frac{1}{2} \beta)}{G \left( \frac{iA}{2\pi} + \frac{i}{4}(1 + \beta) \right)} + \frac{(-iku) \frac{iA}{2\pi} - \frac{i}{4}(1 + \beta)}{G \left( \frac{iA}{2\pi} + \frac{i}{4}(1 + \beta) \right)} \right)
\]

Then by setting \( d_2 = \frac{1}{2} \Gamma \left( \frac{3}{2}, 1 + \frac{1}{2} \beta \right) (-ik)^{-\frac{1}{2}} \), we get that \( G(u) \approx 1 \) as \( u \to \infty \).

Combining everything together into \( \psi(u, v) = Ne^{ikr} G(u) F(v) \), we get the full wave solution:

\[
\psi(u, v) = Ne^{ikr \omega - i\theta} \Gamma \left( \frac{3}{4} + \frac{1}{2} \beta \right) (-ik)^{-\frac{1}{2} \beta} \frac{1}{4}(1 + \beta) u \frac{1}{4}(1 + \beta) M \left( \frac{1}{4}(1 + \beta) , 1 + \frac{1}{2} \beta , iku \right) M \left( -\frac{iQ}{2\pi}, \frac{1}{2}, -iku \right)
\]

Then, by taking \( u \to 0 \) and \( v \to \infty \), the scattering solution is found:

\[
\psi \approx \frac{1}{4}(1 + \beta) \left( e^{ikx - in \log kv} + \frac{1}{\Gamma(n \log kv)} (-2ik)^{-\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{-1 + 2in} e^{-ikr + in \log 2kr} \right)
\]

This means that the scattering coefficient is then:

\[
f_c = \frac{1}{\Gamma(n \log kv)} (-2ik)^{-\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{-1 + 2in}
\]

It can be noted that if \( b \to 0 \), then \( G(u) \) becomes simply the Kummer M function. Thus when \( u \to 0 \), this becomes \( 1 \) and the wave becomes \( \psi = Ne^{ikr} F(v) \), yielding the same scattering solutions seen previously. The following graphics show the scattering solution of Equation 5 with \( b = 1 \).

In Figures 1, 2 and 3, you can see that waves with higher energies diminish the effect of the Coulomb potential. Conversely, the higher the charge Q of the Coulomb potential, the more the spherical wave dominates. The extended singularity remains a visible regardless of the energy and potential.
4. Fully Caged Coulomb Potential

Now it is time to look at the full deformation:

\[
V = -\frac{Q}{r} + \frac{a}{4r^2 \cos^2 \left(\frac{\phi}{2}\right)} + \frac{b}{4r^2 \sin^2 \left(\frac{\phi}{2}\right)}.
\]

Unfortunately with this potential, we need to have a plane wave that is not incoming along the x-axis, since the entire x-axis is now a singularity. A plane wave coming in along the y-axis would solve this. However, the plane wave \(e^{iky} = e^{ik\sqrt{uv}}\) is no longer separable in parabolic coordinates. What we will look for instead is solutions in spherical coordinates. Then we can recreate the plane wave and spherical wave form of Equation 1 by summing these solutions. For example, for \(\psi\) separable,

\[
\psi(r, \phi) = R(r)S(\phi).
\]

We hope to find a scattering solution, \(\psi_s\), of the form

\[
\psi_s(r, \phi) \approx e^{iky} + f_c e^{ikr} = \sum_j \sum l R_j(r)S_l(\phi).
\]

In order to do this, take our Schroedinger equation in spherical coordinates and separated \(\psi\):

\[
- \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} - V \right) R(r)S(\phi) = k^2 R(r)S(\phi).
\]

Seperating this:

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{Q}{r} + k^2 - \frac{A^2}{r^2} \right) R(r) = 0
\]

\[
\left( -\frac{\partial^2}{\partial \phi^2} + \frac{a}{4 \cos^2 \left(\frac{\phi}{2}\right)} + \frac{b}{4 \sin^2 \left(\frac{\phi}{2}\right)} - A^2 \right) S(\phi) = 0.
\]

Solving yields:

\[
R(r) = c_1 M_{a,b}(2ikr).
\]

Where \(M_{a,b}(z)\) is the Whittaker M function.

\[
S(\phi) = c_2 \sqrt{\sin \phi} \left( \cos \frac{\phi}{2} \right)^\alpha \left( \sin \frac{\phi}{2} \right)^\beta P_n^{(\alpha,\beta)}(-\cos \phi).
\]
Here \( P_n^{(\alpha,\beta)}(z) \) are Legendre polynomials, and \( \alpha = \sqrt{a+4}, \beta = \sqrt{b+4} \), and \( n = \frac{1}{2}(1 + \alpha + \beta) - A \).

The asymptotic limit of Whittaker M function for a large argument is the following:

\[
M_{\kappa,\mu}(z) \sim \frac{\Gamma(1 + 2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} e^{\frac{1}{2}z \pm (\frac{1}{2} + \mu - \kappa)i \kappa} e^{1/2z} z^{-\kappa} + \frac{\Gamma(1 + 2\mu)}{\Gamma(\frac{1}{2} + \mu + \kappa)} e^{-1/2z} z^{\kappa}
\]

This yields a partial scattering solution;

\[
\psi_n \approx \left( \frac{\Gamma(1 - n + \frac{1}{2}(\alpha + \beta) + i\eta)}{\Gamma(1 - n + \frac{1}{2}(\alpha + \beta) - i\eta)} e^{ikr - i\eta \log 2kr} + \frac{e^{-ikr + i\eta \log 2kr}}{\sqrt{r}} \right) S_n(\phi).
\]

5. Conclusion

An exact solutions for the scattering of the partially caged Coulomb potential were found. This is an example of how exact solvability does extend to scattering states. Strong hints of the exact solution for the scattering of the full Smordinsky-Winternitz potential were observed. Future work includes finding the scattering coefficient for the fully caged potential, as well finding the scattering coefficients for the Tremblay-Turbiner-Winternitz potential [6]. This is a further modification of the Coulomb potential and is the following:

\[
V = -\frac{Q}{r} + \frac{at^2}{4r^2 \cos^2\left(\frac{\phi}{2}\right)} + \frac{bt^2}{4r^2 \sin^2\left(\frac{\phi}{2}\right)}.
\]

The continuation of this work would not only help us find if exactly solvable bound states extend to scattering states, but also if Hamiltonian systems with no bound states are exactly solvable.

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