Correlation lengths and $E_8$ mass spectrum of the dilute $A_3$ lattice model

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Abstract

The exact perturbation approach is used to derive the elementary correlation lengths $\xi_i$ and related mass gaps $m_i$ of the two-dimensional dilute $A_L$ lattice model in regimes 1 and 2 for $L$ odd from the Bethe Ansatz solution. In regime 2 the $A_3$ model is the $E_8$ lattice realisation of the two-dimensional Ising model in a magnetic field at $T = T_c$. The calculations for the $A_3$ model in regime 2 start from the eight thermodynamically significant string types found in previous numerical studies. These string types are seen to be consistent in the ordered high field limit. The eight masses obtained reduce with the approach to criticality to the $E_8$ masses predicted by Zamolodchikov, thus providing a further direct lattice determination of the $E_8$ mass spectrum.

1 Introduction

There is an intimate relationship between conformal field theory, integrable field theory and integrable lattice models in statistical mechanics [1]. In particular, massive integrable field theory can be considered as conformal field theory perturbed by some scalar relevant operator. The perturbed Hamiltonian is

$$H = H_c + \lambda \int \Phi(r)d^2(r). \quad (1.1)$$

The canonical example is the Ising model. The $\Phi_{(1,3)}$ perturbation is thermal and introduces a single correlation length into the system. The off-critical

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The system is described by scalar combinations of massive Majorana fermions of mass proportional to the inverse of the correlation length. Here the integrable field theory can be considered as \( c = \frac{1}{2} \) conformal field theory perturbed by the energy density \( \Phi_{(1,3)} \) having the conformal dimensions \((\frac{1}{2}, \frac{1}{2})\).

In a remarkable advance, Zamolodchikov [2,3] considered the \( \Phi_{(1,2)} \) magnetic perturbation and showed that there are a number of nontrivial local integrals of motion and thus an integrable field theory. In this case the \( c = \frac{1}{2} \) conformal field theory is perturbed by the spin operator of dimension \((\frac{1}{16}, \frac{1}{16})\). Zamolodchikov then conjectured the S-matrix and mass spectrum of this field theory. The masses coincide with the components of the Perron-Frobenius vector of the Cartan matrix of the Lie algebra \( E_8 \). They are

\[
\begin{align*}
m_2/m_1 &= 2 \cos \frac{\pi}{5} = 1.618033 \ldots \\
m_3/m_1 &= 2 \cos \frac{\pi}{30} = 1.989043 \ldots \\
m_4/m_1 &= 4 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} = 2.404867 \ldots \\
m_5/m_1 &= 4 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} = 2.956295 \ldots \\
m_6/m_1 &= 4 \cos \frac{\pi}{5} \cos \frac{\pi}{30} = 3.218340 \ldots \\
m_7/m_1 &= 8 \cos^2 \frac{\pi}{5} \cos \frac{7\pi}{30} = 3.891156 \ldots \\
m_8/m_1 &= 8 \cos^2 \frac{\pi}{5} \cos \frac{2\pi}{15} = 4.783386 \ldots 
\end{align*}
\]

Very soon after, numerical tests were performed to check that these masses were indeed present in the lattice model, namely the two-dimensional Ising model in a magnetic field at \( T = T_c \). The first few masses were convincingly observed in the related quantum Ising chain in a magnetic field via finite-size diagonalisation [4-6] and the truncated fermionic state space method [7].

Zamolodchikov also pointed out the possibility of an integrable off-critical lattice model corresponding to the integrable magnetic perturbation of the field theory. In a further development, an integrable lattice realisation of the \( E_8 \) Ising model was provided by the dilute \( A_3 \) model [8]. In this model the elliptic nome plays the role of magnetic field. The calculation of the bulk free energy of the dilute \( A_3 \) model in the appropriate regime gives the magnetic Ising exponent \( \delta = 15 \) [8], which also follows from the calculation of the local height probability [9]. The expected Ising magnetic surface exponent \( \delta_s = -\frac{15}{7} \) follows from the excess surface free energy [10]. The study of the thermodynamics of the dilute \( A_3 \) model revealed the entire \( E_8 \) mass spectrum in the scaling limit [11]. In particular, the resulting integral equations in the thermodynamic Bethe Ansatz calculations are those discussed earlier based on the Lie algebra \( E_8 \) [12]. The \( E_8 \) structure in the dilute \( A_3 \) model in regime 2 has also been established by expressing the one-dimensional configuration sums appearing.
in the local height probability [9] in terms of fermionic sums which explicitly involve the \( E_8 \) root system [13]. This “fermionic sum = bosonic sum” expression yields the \( E_8 \) Rogers-Ramanujan identity for the \( \chi_{1,1}^{(3,4)} \) Virasoro character [14]. The Fourier transform results for the single particle excitations [11] have been inverted to obtain analytic expressions for the excitation energies of the eight quasiparticles in the Hamiltonian version of the model [15].

Including the groundstate, the thermodynamic Bethe Ansatz relied on the input of nine sets of thermodynamically significant string solutions of the Bethe equations. These string types were checked numerically as far as possible at criticality [11]. The stability of these string types away from criticality has been examined in detail [16,17]. One of the string types, associated with the mass \( m_4 \), was seen to differ from that proposed in [11]. String distributions were observed up to mass \( m_5 \), with a number of elementary excitations up to mass \( m_1 + m_4 \). However, root distributions associated with the masses \( m_6 \) to \( m_8 \) were not observed, presumably because of the inherent numerical difficulties.

In this paper we explicitly derive the inverse correlation lengths, and thus the mass gaps, of the dilute \( A_3 \) lattice model. We use the string solutions given in [11,17] and apply Baxter’s exact perturbative approach [19], as used for example in the calculation of correlation lengths in the cyclic solid-on-solid (CSOS) model [20]. In particular, we obtain the mass gaps given in (1.2) as criticality is approached.

The outline of the paper is as follows. The dilute \( A_L \) lattice model is defined along with the corresponding Bethe equations in Section 2. The bulk free energy is derived via the exact perturbation approach in Section 3. The eigenvalue expression for the leading excitations in regime 1 is derived for \( L \) odd in Section 4. The eigenvalue expressions in regime 2 for \( L = 3 \) associated with the eight \( E_8 \) masses are derived in Section 5. These results are collected together in one formula in Section 6, where the corresponding correlation lengths and masses are given. The paper concludes with a discussion of the results in Section 7. Some intermediary results are given in the Appendices.

2 The dilute \( A_L \) lattice model

The dilute \( A_L \) model is an exactly solvable, restricted solid-on-solid model defined on the square lattice. At criticality, the model can be constructed [8,21] from the dilute \( O(n) \) model [22,23]. Each site of the lattice can take one of \( L \) possible (height) values, subject to the restriction that neighbouring

\footnote{The result for the first correlation length has been given in [18].}
sites of the lattice either have the same height, or differ by ±1. The off-critical Boltzmann weights of the allowed height configurations of an elementary face of the lattice are [8]

\[
W\left( \begin{array}{cc} a & a \\ a & a \end{array} \right) = \frac{\vartheta_1(6\lambda - u)\vartheta_1(3\lambda + u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} - \frac{S(a + 1)\vartheta_4(2a\lambda - 5\lambda)}{S(a)\vartheta_4(2a\lambda + 5\lambda)} \vartheta_1(u)\vartheta_1(3\lambda - u),
\]

\[
W\left( \begin{array}{cc} a \pm 1 & a \\ a & a \end{array} \right) = W\left( \begin{array}{cc} a & a \\ a & a \pm 1 \end{array} \right) = \frac{\vartheta_1(3\lambda - u)\vartheta_4(\pm 2a\lambda + \lambda - u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)},
\]

\[
W\left( \begin{array}{cc} a & a \pm 1 \\ a & a \pm 1 \end{array} \right) = W\left( \begin{array}{cc} a + 1 & a + 1 \\ a & a \end{array} \right) = \frac{\vartheta_1(\pm 2a\lambda + 3\lambda)\vartheta_4(\pm 2a\lambda - \lambda)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \left[ S(a) \frac{(a + 1)\vartheta_4(2a\lambda - 5\lambda)}{\vartheta_4(2a\lambda + 5\lambda)} - \frac{S(a)\vartheta_1(2\lambda)\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda) - \vartheta_4(\pm 2a\lambda + 5\lambda)} \right],
\]

\[
W\left( \begin{array}{cc} a \pm 1 & a \\ a & a \mp 1 \end{array} \right) = \frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)},
\]

\[
W\left( \begin{array}{cc} a \pm 1 & a \\ a & a \mp 1 \end{array} \right) = - \frac{S(a - 1)S(a + 1)}{S^2(a)} \left[ \frac{\vartheta_1(u)\vartheta_1(\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \right].
\]
The crossing factors $S(a)$ are defined by

$$S(a) = (-1)^a \frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)}$$

(2.2)

and $\vartheta_1(u)$, $\vartheta_4(u)$ are standard elliptic theta functions of nome $p$,

$$\vartheta_1(u) = \vartheta_1(u, p) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} \left(1 - 2p^{2n} \cos 2u + p^{4n}\right) \left(1 - p^{2n}\right),$$

(2.3)

$$\vartheta_4(u) = \vartheta_4(u, p) = \prod_{n=1}^{\infty} \left(1 - 2p^{2n-1} \cos 2u + p^{4n-2}\right) \left(1 - p^{2n}\right).$$

(2.4)

In the above weights the range of the spectral parameter $u$ and the variable $\lambda$ are given by $0 < u < 3\lambda$ with

$$\lambda = \frac{s}{r} \pi,$$

(2.5)

where $r = 4(L + 1)$, with $s = L$ in regime 1 and $s = L + 2$ in regime 2. We do not consider the other regimes here. The thermal Ising point occurs in regime 1 with $L = 2$ and the magnetic Ising point occurs in regime 2 with $L = 3$.

The row transfer matrix of the dilute A models is defined on a periodic strip of width $N$ as

$$T_{\{a\}}^{\{b\}} = \prod_{j=1}^{N} W \begin{pmatrix} b_j & b_{j+1} \\ a_j & a_{j+1} \end{pmatrix},$$

(2.6)

where $\{a\}$ is an admissible path of heights and $a_{N+1} = a_1$, $b_{N+1} = b_1$. For convenience we take $N$ even.

The eigenvalues of the transfer matrix are [11,24,25]

$$\Lambda(u) = \omega \left[ \frac{\vartheta_1(2\lambda - u) \vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^{N} \frac{\vartheta_1(u - u_j + \lambda)}{\vartheta_1(u - u_j - \lambda)}$$

$$+ \left[ \frac{\vartheta_1(u) \vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^{N} \frac{\vartheta_1(u - u_j) \vartheta_1(u - u_j - 3\lambda)}{\vartheta_1(u - u_j - \lambda) \vartheta_1(u - u_j - 2\lambda)}$$
\[ + \omega^{-1} \left[ \frac{\vartheta_1(u)}{\vartheta_1(2\lambda)} \vartheta_1(\lambda - u) \right]^N \prod_{j=1}^N \frac{\vartheta_1(u - u_j - 4\lambda)}{\vartheta_1(u - u_j - 2\lambda)}, \]  

(2.7)

where the \( N \) roots \( u_j \) are given by the Bethe equations

\[ \omega \left[ \frac{\vartheta_1(\lambda - u_j)}{\vartheta_1(\lambda + u_j)} \right]^N = -\prod_{k=1}^N \frac{\vartheta_1(u_j - u_k - 2\lambda)}{\vartheta_1(u_j - u_k + 2\lambda)} \frac{\vartheta_1(u_j - u_k + \lambda)}{\vartheta_1(u_j - u_k - \lambda)} \]  

(2.8)

and \( \omega = \exp(i \pi \ell/(L + 1)) \) for \( \ell = 1, \ldots, L \).

There are several methods at hand to calculate the correlation length. Here we apply the perturbative approach initiated by Baxter [19,20]. For \( L \) odd this involves perturbing away from the high magnetic field limit at \( p = 1 \). We thus introduce the conjugate variables

\[ w = e^{-2\pi u/\epsilon} \quad \text{and} \quad x = e^{-\pi^2/\epsilon_0}, \]  

(2.9)

where nome \( p = e^{-\epsilon} \). The relevant conjugate modulus transformations are

\[ \vartheta_1(u, p) = \left( \frac{\pi}{\epsilon} \right)^{1/2} e^{-(u-\pi/2)^2/\epsilon} E(w, q^2), \]  

(2.10)

\[ \vartheta_4(u, p) = \left( \frac{\pi}{\epsilon} \right)^{1/2} e^{-(u-\pi/2)^2/\epsilon} E(-w, q^2), \]  

(2.11)

where \( q = e^{-\pi^2/\epsilon} \) and

\[ E(z, p) = \prod_{n=1}^{\infty} \frac{1 - p^{n-1}z}{1 - p^n z^{-1}} \frac{1 - p^n}{1 - p^n}. \]  

(2.12)

In the ordered limit \( (p \to 1 \text{ with } u/\epsilon \text{ fixed}) \) the Boltzmann weights for \( L \) odd reduce to

\[ W \begin{pmatrix} d & c \\ a & b \end{pmatrix} \sim w^{H(d,a,b)} \delta_{a,c}. \]  

(2.13)

The function \( H(d, a, b) \) is given explicitly in [9], being required for the calculation of the local height probabilities. In this limit the row transfer matrix eigenspectra breaks up into a number of distinct bands labelled by integer powers of \( w \). In regime 1 there are \( \frac{1}{2}(L + 1) \) ground states and in regime 2 there are \( \frac{1}{2}(L - 1) \) ground states, each with eigenvalue \( \Lambda_0 = 1 \). The bands of excitations are relevant to the calculation of the correlation lengths.
The number of states in the $w$ band is $\frac{1}{2}(L-1)N$ in regime 1 and $\frac{1}{2}(L-3)N$ in regime 2. These correspond to introducing in all but one of the ground state paths $\{a\}$ a single non-ground state height, in any position. In particular, note that there are no excitations in the $w$ band for $L = 3$ in regime 2. Thus for the magnetic Ising model the leading excitations are in the $w^2$ band. These are harder to count, arising from a variety of both single and multiple deviations from ground state paths. However, we observe numerically that (apart from when $N = 2$) there are $4N$ states in this $w^2$ band.

In our numerical investigation of the transfer matrix eigenspectrum we associate a given value of $\ell$ with each eigenvalue by comparing the eigenspectrum at criticality ($p = 0$) with the eigenspectrum of the corresponding $O(n)$ loop model for finite $N$. Each eigenvalue can then be tracked to the ordered limit. In this way the band of largest eigenvalues is seen to have the values $\ell = 1, \ldots, \frac{1}{2}(L + 1)$ in regime 1 and $\ell = 1, \ldots, \frac{1}{2}(L - 1)$ in regime 2, i.e. one value of $\ell$ for each ground state.

Setting $w_j = e^{-2\pi u_j/\ell}$, the eigenvalues (2.7) can be written

\[
\Lambda(w) = \omega \left[ \frac{E(x^{4s}/w, x^{2r}) E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r}) E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^{N} w_j^{1-2s/r} \frac{E(x^{2s}w/w_j, x^{2r})}{E(x^{2s}w/w, x^{2r})} \\
+ \frac{x^{2s}}{w} \left[ \frac{E(w, x^{2r}) E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r}) E(x^{6s}, x^{2r})} \right]^N \times \prod_{j=1}^{N} w_j \frac{E(w/w_j, x^{2r}) E(x^{6s}w_j/w, x^{2r})}{E(x^{2s}w_j/w, x^{2r}) E(x^{4s}w_j/w, x^{2r})} \\
+ \omega^{-1} \left[ \frac{x^{2s}}{x^{4s}} \frac{E(w, x^{2r}) E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r}) E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^{N} w_j^{2s/r} \frac{E(x^{8s}w_j/w, x^{2r})}{E(x^{4s}w_j/w, x^{2r})}.
\]

(2.14)

The Bethe equations (2.8) are now

\[
\omega \left[ w_j \frac{E(x^{2s}w_j, x^{2r})}{E(x^{2s}w_j, x^{2r})} \right]^N = -\prod_{k=1}^{N} w_k^{2s/r} \frac{E(x^{2s}w_j/w_k, x^{2r}) E(x^{4s}w_k/w_j, x^{2r})}{E(x^{2s}w_k/w_j, x^{2r}) E(x^{4s}w_j/w_k, x^{2r})}.
\]

(2.15)

We are now ready to investigate the ordered limit.

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2 Strictly speaking we compare with the eigenspectrum of the corresponding vertex model with seam $\omega$. 

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7
3 Free energy

The calculation of the largest eigenvalue in the thermodynamic limit proceeds from the $x \rightarrow 0$ limit with $w$ fixed in a similar manner to that for the eight-vertex \[19\] and CSOS \[20\] models. We make repeated use of the properties

$$E(z,p) = E(p/z,p) = -zE(z^{-1},p).$$

(3.1)

Assuming that the roots $w_j$ are on the unit circle, as observed in our numerical calculations, only the first term in the eigenvalue expression (2.14) survives in this limit, with

$$\Lambda_0 \sim \omega(w_1 \ldots w_N)^{(r-2s)/r}.$$  

(3.2)

The Bethe equation (2.15) gives

$$w^N + \omega^{-1}(w_1 \ldots w_N)^{2s/r} = 0,$$  

(3.3)

where we write $w = w_j$. We consider this equation for all complex $w$ and equate the left hand side to $\prod_{j=1}^{N}(w - w_j)$ so that

$$(w_1 \ldots w_N) = \omega^{-1}(w_1 \ldots w_N)^{2s/r} \Rightarrow (w_1 \ldots w_N)^{(r-2s)/r} = \omega^{-1}.$$  

(3.4)

Recall that we have taken $N$ even. Thus from (3.2) the largest eigenvalue is $\Lambda_0 = 1$ in the limit $x \rightarrow 0$. Further, we expect the product of the roots to obey (3.4) away from $x = 0$. Each of the degenerate ground states has a different root distribution $\{w_j\}$ on the unit circle, depending on $\ell$.

To perturb about $x = 0$, we define the auxiliary functions

$$A(z) = \prod_{k=0}^{\infty} (1 - x^{2rk}z)^N,$$  

(3.5)

$$F_0(w) = \prod_{j=1}^{N} \prod_{k=0}^{\infty} (1 - x^{2rk}w/w_j),$$  

(3.6)

$$G_0(1/w) = \prod_{j=1}^{N} \prod_{k=1}^{\infty} (1 - x^{2rk}w_j/w),$$  

(3.7)

where $A(z)$ is known, and $F_0(w)$ and $G_0(1/w)$, which depend upon the $w_j$, are the unknowns to be found. Then the Bethe equation (2.15) may be written as
\[ w^N \frac{A(x^{2s}/w)G_0(1/x^{4s}w)G_0(1/x^{2r-2s}w)}{A(x^{2r-2s}/w)G_0(1/x^{2s}w)G_0(1/x^{2r-4s}w)} + \]

\[ (w_1 \ldots w_N) \frac{A(x^{2s}w)F_0(x^{2s}w)F_0(x^{2r-4s}w)}{A(x^{2r-2s}w)F_0(x^{4s}w)F_0(x^{2r-2s}w)} = 0, \tag{3.8} \]

which is again an \( N \)th order equation with roots \( w_1, \ldots w_N \), so that the left hand side may be equated to

\[ \prod_{j=1}^{N} (w - w_j) = \begin{cases} w^N \prod_{j=1}^{N} (1 - w_j/w), & \text{w large}, \\ (w_1 \ldots w_N) \prod_{j=1}^{N} (1 - w/w_j), & \text{w small}. \end{cases} \tag{3.9} \]

These products can be expressed in terms of the auxiliary functions,

\[ \prod_{j=1}^{N} (1 - w/w_j) = \frac{F_0(w)}{F_0(x^{2r}w)}, \quad \prod_{j=1}^{N} (1 - w_j/w) = \frac{G_0(1/x^{2r}w)}{G_0(1/w)}. \tag{3.10} \]

If we then further define

\[ F_0(w) = \frac{F_0(w)}{F_0(x^{2r-4s}w)}, \quad G_0(1/w) = \frac{G_0(1/w)}{G_0(1/x^{2r-4s}w)}, \tag{3.11} \]

we obtain two equations,

\[ F_0(w) = \frac{A(x^{2s}w)F_0(x^{2s}w)}{A(x^{2r-2s}w)F_0(x^{4s}w)}, \]

\[ G_0(1/w) = \frac{A(x^{2r+2s}/w)G_0(x^{2s}/w)}{A(x^{6s}/w)G_0(x^{4s}/w)}. \tag{3.12} \]

by equating the dominant terms for \(|w| > 1\) and for \(|w| < 1\). These equations can be solved by iteration to give

\[ F_0(w) = \prod_{m=0}^{\infty} \frac{A(x^{(12m+2)s}w)A(x^{(12m+4)s}w)}{A(x^{(12m+8)s}w)A(x^{(12m+10)s}w)} \times \frac{A(x^{(12m+4)s+2r}w)A(x^{(12m+6)s+2r}w)}{A(x^{(12m-2)s+2r}w)A(x^{12sm+2r}w)}, \tag{3.13} \]

\[ G_0(1/w) = \prod_{m=0}^{\infty} \frac{A(x^{(12m+12)s}/w)A(x^{(12m+14)s}/w)}{A(x^{(12m+6)s}/w)A(x^{(12m+8)s}/w)} \times \frac{A(x^{(12m+2)s+2r}/w)A(x^{(12m+4)s+2r}/w)}{A(x^{(12m+8)s+2r}/w)A(x^{(12m+10)s+2r}/w)}. \tag{3.14} \]
The eigenvalue (2.14) can be expressed in terms of the auxiliary functions as

$$
\Lambda_0 = \frac{A(x^{4s}/w)A(x^{6s}/w)A(x^{2r-4s}w)A(x^{2r-6s}w)}{A(x^{4s})A(x^{6s})A(x^{2r-4s})A(x^{2r-6s})}F_0(x^{2s}w)G_0(1/x^{2s}w)
$$

$$
+ \frac{A(w)A(x^{6s}/w)A(x^{2r}/w)A(x^{2r-6s}w)}{A(x^{4s})A(x^{6s})A(x^{2r-4s})A(x^{2r-6s})}F_0(w/x^{2s})G_0(x^{2s}/w)
$$

$$
+ \frac{A(w)A(w/x^{2s})A(x^{2r}/w)A(x^{2r+2s}/w)}{A(x^{4s})A(x^{6s})A(x^{2r-4s})A(x^{2r-6s})}F_0(w/x^{4s})G_0(x^{4s}/w)
$$

(3.15)

where we have made use of (3.4). When the above solutions for $F_0(w)$ and $G_0(1/w)$ are used, we can show that all three terms are identical, so that we finally obtain

$$
\Lambda_0/3 = \frac{A(x^{4s}/w)A(x^{6s}/w)A(x^{2r-4s}w)A(x^{2r-6s}w)}{A(x^{4s})A(x^{6s})A(x^{2r-4s})A(x^{2r-6s})}
$$

$$
\prod_{m=0}^{\infty} \frac{A(x^{(12m+4)s}w)A(x^{(12m+6)s}w)}{A(x^{(12m+10)s}w)A(x^{(12m+12)s}w)}
$$

$$
\times \frac{A(x^{(12m+6)s+2r}w)A(x^{(12m+8)s+2r}w)}{A(x^{12ms+2r}w)A(x^{12ms+2r+2r}w)}
$$

$$
\times \frac{A(x^{12ms+2r}/w)A(x^{12ms+2r}w)}{A(x^{(12m+6)s+2r}/w)A(x^{(12m+8)s+2r}/w)}
$$

$$
\times \frac{A(x^{(12m+10)s}/w)A(x^{(12m+12)s}/w)}{A(x^{(12m+4)s}/w)A(x^{(12m+6)s}/w)}.
$$

(3.16)

The factor of 3 will not survive the thermodynamic limit. Using the identity

$$
\frac{1}{N} \log A(w) = -\sum_{k=1}^{\infty} \frac{w^k}{k(1-x^{2rk})}
$$

(3.17)

and defining the free energy per site as $f = -N^{-1} \log \Lambda_0$ we are able to write

$$
f = -\sum_{k=1}^{\infty} \frac{(1-w^k)(1-x^{6sk}w^{-k})(x^{4sk} + x^{(2r-6s)k})(1+x^{2sk})}{k(1-x^{2rk})(1+x^{6sk})}.
$$

(3.18)

Our final result is thus

$$
f = 4 \sum_{k=1}^{\infty} \frac{\text{ch}[(5\lambda - \pi)\pi k/\epsilon] \text{ch}(\pi \lambda k/\epsilon) \text{sh}(\pi uk/\epsilon) \text{sh}[(3\lambda - u)\pi k/\epsilon]}{k \text{sh}(\pi^2 k/\epsilon) \text{ch}(3\pi \lambda k/\epsilon)}.
$$

(3.19)
This is in agreement with the previous calculations via the inversion relation method \cite{8,9}. The singular behaviour follows as

\[ f \sim \begin{cases} 
  p^{1+\delta} & \text{if } L \text{ odd}, \\
  p^{2-\alpha} & \text{if } L \text{ even},
\end{cases} \quad \text{as } p \to 0, \tag{3.20} \]

where the exponents are given by \cite{8,9}

\[ \delta = \frac{3L}{L+4}, \quad \alpha = \frac{2(L-2)}{3L} \quad \text{regime 1}, \]
\[ \delta = \frac{3(L+2)}{L-2}, \quad \alpha = \frac{2(L+4)}{3(L+2)} \quad \text{regime 2}. \tag{3.21} \]

These include the Ising values \( \alpha = 0 \) for \( L = 2 \) in regime 1 and \( \delta = 15 \) for \( L = 3 \) in regime 2.

4 Excitations in regime 1

In regime 1, we observe numerically that the leading eigenvalue in the \( w \) band has \( \ell = \frac{1}{2}(L+1)+1 \). The corresponding root distribution has \( N-1 \) roots on the unit circle and a 1-string excitation located exactly at \( w_N = -x^r \). In general we assume that the 1-string excitations are located at \( w_N = bx^r \) with \( |b| \sim 1 \) and the remainder on the unit circle, i.e. \( w_j = a_j \) for \( j = 1, \ldots, N-1 \). The Bethe equations split into two sets, the first of which is

\[ \omega \left[ a_k \frac{E(x^{2s}/a_k)}{E(x^{2s}a_k)} \right]^N = -(A_{N-1}b)^{2s/r} \frac{E(x^{r-2s}b/a_k)E(x^{r-4s}b/a_k)}{E(x^{r-2s}a_k/b)E(x^{r-4s}a_k/b)} \]
\[ \times \prod_{j=1}^{N-1} \frac{E(x^{2s}a_j/a_k)E(x^{4s}a_j/a_k)}{E(x^{2s}a_j/a_k)E(x^{4s}a_j/a_k)}, \tag{4.1} \]

for \( k = 1, \ldots, N-1 \), where we have defined \( \prod_{j=1}^{m} a_j = A_m \). For \( k = N \) the other equation is

\[ \omega \left[ \frac{E(x^{r-2s}b)}{E(x^{r-2s}/b)} \right]^N = -(A_{N-1}b)^{2s/r} \prod_{j=1}^{N-1} \frac{E(x^{r-2s}a_j/b)E(x^{r-4s}b/a_j)}{E(x^{r-2s}b/a_j)E(x^{r-4s}a_j/b)}. \tag{4.2} \]

Taking the limit \( x \to 0 \), we obtain

\[ a^N + \omega^{-1}(A_{N-1}b)^{2s/r} = 0, \tag{4.3} \]
\[ -\omega^{-1}(A_{N-1}b)^{2s/r} = 1. \tag{4.4} \]
which taken together give $a^N = 1$, where $a = a_k$. This equation is of order $N$, but recall there are only $N - 1$ of the $a_k$’s. The extra degree of this equation defines a hole at $a_N$. Further, from (4.3) and (4.4) we also have
\[ \omega^{-1}(A_{N-1}b)^{2s/r} = A_N = A_{N-1}a_N = -1. \] (4.5)

To perform the perturbation, we define in addition to $A(w)$,
\[
X(w) = \prod_{k=0}^{\infty} (1 - x^{2rk}w/b),
\] (4.6)
\[ Y(1/w) = \prod_{k=1}^{\infty} (1 - x^{2rk}b/w), \]
\[ R(w) = \prod_{k=0}^{\infty} (1 - x^{2rk}w/a_N), \]
\[ S(1/w) = \prod_{k=1}^{\infty} (1 - x^{2rk}a_N/w), \]
(4.8)
and the functions we will need to find,
\[ F(w) = \prod_{j=1}^{N} \prod_{k=0}^{\infty} (1 - x^{2rk}w/a_j), \]
\[ G(1/w) = \prod_{j=1}^{N} \prod_{k=1}^{\infty} (1 - x^{2rk}a_j/w). \] (4.10)

The Bethe equation for $k = 1, \ldots, N - 1$ can now be written
\[
a^N \frac{A(x^{2s}/a)}{A(x^{2r-2s}/a)} \frac{Y(1/x^{r+4s}a)Y(1/x^{r-2s}a)}{Y(1/x^{r+2s}a)Y(1/x^{r-4s}a)}
\times \frac{S(1/x^{4s}a)S(1/x^{2r-4s}a)G(1/x^{4s}a)G(1/x^{2r-2s}a)}{S(1/x^{4s}a)S(1/x^{2r-2s}a)G(1/x^{2s}a)G(1/x^{2r-4s}a)}
\times A(x^{2s}a)X(x^{r-4s}a)X(x^{r+2s}a)
\times R(x^{4s}a)R(x^{2r-2s}a)F(x^{2s}a)F(x^{2r-4s}a)
\times R(x^{2s}a)R(x^{2r-4s}a)F(x^{4s}a)F(x^{2r-2s}a) = 0. \] (4.11)

As before, we equate the l.h.s. of this to
\[
\prod_{j=1}^{N} (a - a_j) = \begin{cases} 
a^N G(1/x^{2r}a)/G(1/a), & a \text{ large}, \\
A_N F(a)/F(x^{2r}a), & a \text{ small}. \end{cases} \] (4.12)
We are thus led to define

\[ \mathcal{F}(a) = \frac{F(a)}{F(x^{2r-4s}a)}, \quad \mathcal{G}(1/a) = \frac{G(1/a)}{G(1/x^{2r-4s}a)}, \] (4.13)

which satisfy the recurrences

\[ \mathcal{F}(a) = \frac{A(x^{2s}a)X(x^{r-4s}a)X(x^{r+2s}a)}{A(x^{2r-2s}a)X(x^{r-2s}a)X(x^{r+4s}a)} \times \frac{R(x^{4s}a)R(x^{2r-2s}a)\mathcal{F}(x^{2s}a)}{R(x^{2s}a)R(x^{2r-4s}a)\mathcal{F}(x^{4s}a)}, \]

\[ \mathcal{G}(1/a) = \frac{A(x^{2r+2s}/a)Y(1/x^{r-2s}a)Y(1/x^{r-8s}a)}{A(x^{6s}/a)Y(1/x^{r}a)Y(1/x^{r-6s}a)} \times \frac{S(1/a)S(1/x^{2r-6s}a)\mathcal{G}(x^{2s}/a)}{S(x^{2s}/a)S(1/x^{2r-8s}a)\mathcal{G}(x^{4s}/a)}. \] (4.14)

Their solutions are

\[ \mathcal{F}(a) = \mathcal{F}_0(a) \frac{X(x^{r-4s}a)R(x^{2r}a)}{X(x^{r}a)R(x^{2r-4s}a)} \times \prod_{m=0}^{N-1} \frac{(1 - x^{(12m+6)s}a/a_N)(1 - x^{(12m+8)s}a/a_N)}{(1 - x^{(12m+2)s}a/a_N)(1 - x^{(12m+12)s}a/a_N)}, \] (4.15)

\[ \mathcal{G}(1/a) = \mathcal{G}_0(1/a) \frac{Y(1/x^{r-4s}a)S(1/a)}{Y(1/x^{r}a)S(x^{4s}/a)} \times \prod_{m=0}^{N-1} \frac{(1 - x^{(12m+6)s}a_N/a)(1 - x^{(12m+14)s}a_N/a)}{(1 - x^{(12m+2)s}a_N/a)(1 - x^{(12m+12)s}a_N/a)}, \] (4.16)

where \( \mathcal{F}_0 \) and \( \mathcal{G}_0 \) are the functions used in the previous Section.

We can now construct the eigenvalue, which in terms of the auxiliary functions reads

\[ \Lambda = A_{N-1}w \frac{A(x^{2r-4sw})A(x^{2r-6sw})A(x^{4s}/w)A(x^{6s}/w)}{A(x^{4s})A(x^{6s})A(x^{2r-4s})A(x^{2r-6s})} \times \frac{R(x^{2r-2sw})S(1/x^{2r-2sw})}{R(x^{2sw})S(1/x^{2sw})} \]

\[ \times \frac{X(x^{r+2sw})Y(1/x^{r+2sw})}{X(x^{r-2sw})Y(1/x^{r-2sw})} \frac{\mathcal{F}(x^{2sw})\mathcal{G}(1/x^{2sw})}{\mathcal{F}(x^{2sw})\mathcal{G}(1/x^{2sw})} + \frac{A(w)A(x^{2r-6sw})A(x^{2r}/w)A(x^{6s}/w)}{A(x^{4s})A(x^{6s})A(x^{2r-4s})A(x^{2r-6s})}. \]
\begin{align}
& \times \frac{R(x^{2r-4s}w)R(w/x^{2s})}{R(w) R(x^{2r-6s}w)} \frac{S(1/x^{2r-4s}w)S(x^{2s}/w)}{S(1/w) S(1/x^{2r-6s}w)} \\
& \times \frac{X(x^{r-6s}w)X(w/x^r)}{X(x^{r-4s}w)X(w/x^{r+2s})} \frac{Y(1/x^{r-6s}w)Y(x^r/w)}{Y(1/x^{r-4s}w)Y(x^{r+2s}/w)} \\
& \times \frac{\mathcal{F}(w)\mathcal{G}(1/w)}{\mathcal{F}(w/x^{2s})\mathcal{G}(x^{2s}/w)} \frac{a_{N-1}b}{A(x^{4s})A(x^{6s})A(x^{2r-4s})A(x^{2r-6s})} \\
& + \times \frac{R(x^{2r-8s}w)S(1/x^{2r-8s}w)}{R(x^{r-8s}w)Y(1/x^{r-8s}w)} \frac{1}{X(x^{r-8s}w)Y(x^{r+4s}/w)} \frac{\mathcal{F}(w/x^{4s})\mathcal{G}(x^{4s}/w)}{F(w/x^{4s})G(x^{4s}/w)}.
\end{align}

Making use of the solutions (4.15) and (4.16), the identities in Appendix A, and (4.5) on the prefactors, we can show that this simplifies to

\begin{align}
\Lambda/\Lambda_0 &= wA_{N-1} \frac{E(x^{8s}w/a_N, x^{12s})E(x^{10s}w/a_N, x^{12s})}{E(x^{4s}w/a_N, x^{12s})E(x^{2s}w/a_N, x^{12s})}.
\end{align}

If both the hole \( a_N \), and \( b \) are equal to \(-1\), then we have

\begin{align}
\frac{\Lambda}{\Lambda_0} &= w \frac{E(-x^{2s}/w, x^{12s})}{E(-x^{4s}/w, x^{12s})} \frac{E(-x^{6s}/w, x^{12s})}{E(-x^{2s}/w, x^{12s})}.
\end{align}

At the isotropic point \( w = x^{3s} (u = 3\lambda/2) \) this reduces to

\begin{align}
\frac{\Lambda}{\Lambda_0} &= x^s \frac{E^2(-x^{8s}, x^{12s})}{E^2(-x^{5s}, x^{12s})} = \left[ \frac{\partial_4(\pi/12, \pi/6\lambda)}{\partial_4(3\pi/12, \pi/6\lambda)} \right]^2.
\end{align}

We can also express the second Bethe equation (4.2) in terms of the auxiliary functions. This yields an equation for the parameters \( b \) and \( a_N \) of the form

\begin{align}
\frac{A(x^{r-2s}b)A(x^{r+2s}/b)}{A(x^{r-2s}/b)A(x^{r+2s}b)} &= R(x^{r-2s}b)R(x^{r+4s}b) S(1/x^{r-2s}b)S(1/x^{r+4s}b) \\
& + R(x^{r-4s}b)R(x^{r+2s}b) S(1/x^{r-4s}b)S(1/x^{r+2s}b) \\
& \times F(x^{r-2s}b)F(x^{r+4s}b) G(1/x^{r-2s}b)G(1/x^{r+4s}b) \\
& \times F(x^{r-2s}b)F(x^{r+4s}b) G(1/x^{r-2s}b)G(1/x^{r+4s}b).
\end{align}
Since this cannot be expressed in terms of $F$ and $G$, we have to re-construct

\[ F(a) = \prod_{\ell=0}^{\infty} F(x^{(2r-4s)\ell}a), \quad G(1/a) = \prod_{\ell=0}^{\infty} G(1/x^{(2r-4s)\ell}a), \quad (4.22) \]

from the definitions (4.13). Making use of identities listed in Appendix A we finally obtain a result,

\[ \frac{E(x^r a_N/b, x^{2r-4s})}{E(x^r b/a_N, x^{2r-4s})} = 1, \quad (4.23) \]

in which neither the original nome $x^{2r}$ nor $x^{12s}$ appears. This is obviously satisfied by $a_N = b$.

5 Excitations in regime 2 for $L = 3$

For $L = 3$ in regime 2 extensive numerical investigations of the Bethe equations (2.8) have led to a conjecture for the thermodynamically significant strings [11,16,17]. For example, the leading excitation $\Lambda_1$, corresponding to mass $m_1$, is a 2-string with $\ell = 2$. However, this state is originally a 1-string for small $p$ and small $N$. Such behaviour has been discussed in [17]. By tracking this state numerically with increasing $p$ we see that this 2-string is exactly located at $-x^{\pm 11}$ in the limit $p = 1$. Moreover, it lies in the $w_2^2$ band of excitations. The string structure for the other eigenvalues ($\Lambda_j$) or masses ($m_j$) used in the thermodynamic Bethe Ansatz approach were checked numerically as far as possible at criticality [11]. The stability of these strings types away from criticality has been examined in detail [16,17]. The conjectured string positions are listed in Table 1. With the exception of mass $m_4$, these are the string positions given in [11]. Rather than the 5-string, $\pm 8, \pm 12, 16$, proposed in [11] we consider the 7-string found in [17]. In the latter study, string distributions for elementary excitations up to and including mass $m_5$ were located, including a number of composite excitations up to mass $m_1 + m_4$. However, the root distributions associated with the masses $m_6$ to $m_8$ were not observed, presumably because of the inherent numerical difficulties.

Here we show that each of the tabulated sets of strings provides a consistent solution in the ordered and large $N$ limits. However, this is also true for the proposed 5-string for mass $m_4$. According to our calculations, both the 5-string and the 7-string distributions yield the same mass $m_4$. We shall return to this point in the discussion in Section 7.

The calculations in this Section are necessarily complicated. The reader not
Table 1
String positions and corresponding eigenvalue bands for the eight elementary masses $m_j$ of the dilute $A_3$ model in regime 2. The strings are in units of $\pi i/32$.

| $j$ | string positions | band |
|-----|------------------|------|
| 1   | ±11              | 2    |
| 2   | ±5, ±15          | 2    |
| 3   | ±10, ±12         | 3    |
| 4   | ±1, ±7, ±13, 16  | 3    |
| 5   | ±9, ±11, ±13     | 4    |
| 6   | ±6, ±10, ±14, 16 | 4    |
| 7   | ±8, ±10, ±12, ±14| 5    |
| 8   | ±7, ±9, ±11, ±13, ±15 | 6 |

interested in the specific details may wish to skip to Sections 6 and 7, where the results obtained are summarised and discussed.

5.1 Mass $m_1$

We begin the perturbation argument with the structure $w_j = a_j$ for $j = 1, \ldots, N - 2$ with $w_{N-1} = b_1 x^{-11}$ and $w_N = b_2 x^{11}$. However, from the Bethe equations for $k = N - 1$ and $k = N$ (see Appendix B) we can show that $b_1 = b_2 = b$. The Bethe equation for the other roots $a_k = a$ is then

$$-\omega \left[ a \frac{E(x^{10}/a)}{E(x^{10} a)} \right]^N = (A_{N-2} b^2)^{5/8} \frac{a^2}{b^2} \frac{E(x^9 b/a) E(x^{11} b/a)}{E(x^9 a/b) E(x^{11} a/b)} \prod_{j=1}^{N-2} \frac{E(x^{10} a/a_j) E(x^{20} a_j/a)}{E(x^{10} a_j/a) E(x^{20} a_j/a)},$$

(5.1)

where as before, $\prod_{j=1}^m a_j = A_m$. In the $x \to 0$ limit this gives the equation

$$a^{N-2} + \frac{1}{\omega} (A_{N-2} b^2)^{5/8} / b^2 = 0,$$

(5.2)

which is an $(N - 2)$th order equation for $N - 2$ zeros, so that, unlike in regime 1, there are no holes in this case. Equating this as usual with $\prod_{j=1}^{N-2} (a - a_j)$,
we obtain
\[
\frac{1}{\omega} (A_{N-2} b^2)^{5/8} = A_{N-2} b^2, \tag{5.3}
\]
which we substitute into the other Bethe equations (B.1) and (B.2) in this limit, to give
\[
\left[ \frac{1}{\omega} (A_{N-2} b^2)^{5/8} \right]^2 = \frac{(A_{N-2} b^2)^2}{b^{2N}} \Rightarrow b^{2N} = 1. \tag{5.4}
\]
Note that the value \( b = -1 \), determined numerically for the leading 2-string excitation, satisfies this equation. In principle these equations can be solved consistently to determine the total number of 2-string excitations.

To proceed with the calculation we use the auxiliary functions \( A(w), X(w), Y(1/w) \) defined in (3.5), (4.6) and (4.7) with the appropriate choices \( s = 5, r = 16 \), and define in analogy the “unknown functions”,
\[
\begin{align*}
F_1(w) &= \prod_{j=1}^{N-2} \prod_{k=0}^{\infty} (1 - x^{32k} w/a_j), \\
G_1(1/w) &= \prod_{j=1}^{N-2} \prod_{k=1}^{\infty} (1 - x^{32k} a_j/w), \tag{5.5}
\end{align*}
\]
along with the functions we will actually solve for,
\[
\begin{align*}
\mathcal{F}_1(w) &= F_1(w)/F_1(x^{12} w), \\
\mathcal{G}_1(1/w) &= G_1(1/w)/G_1(1/x^{12} w). \tag{5.6}
\end{align*}
\]
In fact, although we must define \( F_i(w) \) and \( G_i(w) \) for each \( m_i \), we can define
\[
\begin{align*}
\mathcal{F}_i(w) &= F_i(w)/F_i(x^{12} w), \\
\mathcal{G}_i(1/w) &= G_i(1/w)/G_i(1/x^{12} w), \tag{5.7}
\end{align*}
\]
for \( i \neq 4, 6 \). Rearranging the Bethe equation (5.1) and equating the dominant terms, as in regime 1, gives
\[
\begin{align*}
\mathcal{F}_1(a) &= \frac{A(x^{10} a) X(x^{21} a) X(x^{23} a) F_1(x^{10} a)}{A(x^{22} a) X(x^{9} a) X(x^{11} a) F_1(x^{20} a)}, \\
\mathcal{G}_1(1/a) &= \frac{A(x^{42} /a) Y(1/x^3 a) Y(1/x a) G_1(x^{10} /a)}{A(x^{30} /a) Y(x^9 /a) Y(x^{11} /a) G_1(x^{20} /a)}. \tag{5.8}
\end{align*}
\]
We solve these recursively, using identities for $X(w)$ and $Y(1/w)$ given in Appendix A, so obtaining

\[
\mathcal{F}_1(a) = \mathcal{F}_0(a) \frac{X(x^{23}a)X(x^{33}a)}{X(x^{11}a)X(x^{55}a)} \times \prod_{m=0}^{\infty} \left\{ \frac{(1 - x^{60m+31}a/b)(1 - x^{60m+39}a/b)}{(1 - x^{60m+9}a/b)(1 - x^{60m+19}a/b)} \right. \\
\left. \times \frac{(1 - x^{60m+49}a/b)(1 - x^{60m+81}a/b)}{(1 - x^{60m+51}a/b)(1 - x^{60m+61}a/b)} \right\}, \tag{5.9}
\]

and similarly

\[
\mathcal{G}_1(1/a) = \mathcal{G}_0(1/a) \frac{Y(1/xa)Y(x^{41}/a)}{Y(x^{11}/a)Y(x^{21}/a)} \times \prod_{m=1}^{\infty} \left\{ \frac{(1 - x^{60m-31}b/a)(1 - x^{60m-21}b/a)}{(1 - x^{60m-9}b/a)(1 - x^{60m-11}b/a)} \right. \\
\left. \times \frac{(1 - x^{60m+11}b/a)(1 - x^{60m+21}b/a)}{(1 - x^{60m+9}b/a)(1 - x^{60m+41}b/a)} \right\}. \tag{5.10}
\]

We are now in a position to evaluate the eigenvalue (2.15). Expressed in terms of the auxiliary functions and using (5.3) it reads

\[
\Lambda_1 = \frac{w^2 A(x^2w)A(x^{12}w)A(x^{20}/w)A(x^{30}/w)}{b^2 A(x^2)A(x^{12})A(x^{20})A(x^{30})} \times \frac{X(x^{21}w)X(x^{31}w)Y(1/x^{31}w)Y(1/x^{21}w)}{X(xw)X(x^{11}w)Y(1/x^{11}w)Y(1/xw)} \mathcal{F}_1(x^{10}w)\mathcal{G}_1(1/x^{10}w) \\
+ \frac{A(w)A(x^2w)A(x^{30}/w)A(x^{32}/w)X(w/x^{11})X(w/x^9)X(x^{11}w)X(x^{13}w)}{A(x^2)A(x^{12})A(x^{20})A(x^{30})} \times \frac{X(w/x^{21})X(xw)X(x^{23}w)}{Y(1/x^{13}w)Y(1/x^{11}w)Y(x^9/w)Y(x^{11}/w)} \mathcal{F}_1(w)\mathcal{G}_1(1/w) \\
+ \frac{A(w/x^{10})A(w)A(x^{32}/w)A(x^{42}/w)X(w/x^{19})X(x^{3}w)}{A(x^2)A(x^{12})A(x^{20})A(x^{30})} \times \frac{X(w/x^{31})X(w/x^9)}{Y(1/x^{3}w)Y(x^9/w)} \mathcal{F}_1(w/x^{20})\mathcal{G}_1(x^{20}/w). \tag{5.11}
\]

Using (5.9) and (5.10) we find that this result may be written in terms of elliptic functions as\footnote{We point out here that all three terms in the eigenvalue expression are identical, once $\mathcal{F}_1$ and $\mathcal{G}_1$ are substituted. This feature is common to all masses $m_i$ ($i = 1, \ldots, 8$) when the appropriate functions $\mathcal{F}_i$ and $\mathcal{G}_i$ have been found and used. So}
\[ \frac{\Lambda_1}{\Lambda_0} = \frac{w^2}{b^2} \frac{E(xb/w, x^{60}) E(x^{11}b/w, x^{60})}{E(xw/b, x^{60}) E(x^{11}w/b, x^{60})} \times \frac{E(x^{31}w/b, x^{60}) E(x^{41}w/b, x^{60})}{E(x^{31}b/w, x^{60}) E(x^{41}b/w, x^{60})}. \] (5.12)

Setting \( b = -1 \), the leading excitation for \( L = 3 \) in regime 2, which clearly lies in the \( w^2 \) band, is thus

\[ \frac{\Lambda_1}{\Lambda_0} = \frac{w^2}{E(-x/w, x^{60}) E(-x^{11}/w, x^{60})} \times \frac{E(-x^{31}/w, x^{60}) E(-x^{41}/w, x^{60})}{E(-x^{31}/w, x^{60}) E(-x^{41}/w, x^{60})}. \] (5.13)

At the isotropic point \( w = x^{15} \) this reduces to

\[ \frac{\Lambda_1}{\Lambda_0} = x^{28} \frac{E^2(-x^4, x^{60}) E^2(-x^{14}, x^{60})}{E^2(-x^{16}, x^{60}) E^2(-x^{26}, x^{60})} = \left[ \frac{\vartheta_4(\frac{\pi}{15}, p^{8/15}) \vartheta_4(\frac{7\pi}{30}, p^{8/15})}{\vartheta_4(\frac{4\pi}{15}, p^{8/15}) \vartheta_4(\frac{13\pi}{30}, p^{8/15})} \right]^2. \] (5.14)

where we have written the excitation in terms of the original nome.

5.2 Mass \( m_2 \)

We begin the perturbation argument with the structure \( w_j = a_j \) for \( j = 1, \ldots, N-4 \) with \( w_{N-3} = b_1 x^{-5}, w_{N-2} = b_2 x^5, w_{N-1} = b_3 x^{-15} \) and \( w_N = b_4 x^{15} \). From the Bethe equations for \( k = N-3, \ldots, N \) (see Appendix B.2) we can show that \( b_1 = b_2 = b_3 = b_4 = b \). The Bethe equation for the other roots \( a_k = a \) is then

\[ -\omega \left[ a \frac{E(x^{10}/a)}{E(x^{10}a)} \right]^N = (A_{N-4} b^4)^{5/8} a^4 E(x^5b/a) E(x^5b/a) \prod_{j=1}^{N-4} \frac{E(x^{10}a_j/a) E(x^{20}a_j/a)}{E(x^{10}a_j/a) E(x^{20}a_j/a)} \] (5.15)

In the \( x \to 0 \) limit this gives the equation

\[ a^{N-4} + \frac{1}{\omega} (A_{N-4} b^4)^{5/8} b^4 = 0. \] (5.16)

in discussion of the subsequent masses, we suppress the second and third terms of the eigenvalue expression.
Equating this as usual with $\prod_{j=1}^{N-4}(a - a_j)$, we obtain

$$\frac{1}{\omega}(A_{N-4}b^4)^{5/8} = A_{N-4}b^4, \quad (5.17)$$

(which we later apply to prefactors in $\Lambda_2$). From the other Bethe equations (B.6)-(B.9) in this limit,

$$\left[\frac{1}{\omega}(A_{N-4}b^4)^{5/8}\right]^4 = \frac{(A_{N-4}b^4)^4}{b^{2N}} \Rightarrow b^{2N} = 1. \quad (5.18)$$

We define functions of the roots we wish to find as

$$F_2(w) = \prod_{j=1}^{N-4} \prod_{k=0}^{\infty} (1 - x^{32k}w/a_j),$$

$$G_2(1/w) = \prod_{j=1}^{N-4} \prod_{k=1}^{\infty} (1 - x^{32k}a_j/w). \quad (5.19)$$

Treating the Bethe equation (5.15) as before gives, in terms of the functions defined in (5.7), the recurrences

$$F_2(a) = \frac{A(x^{10}a)X(x^{27}a)X(x^{29}a)F_2(x^{10}a)}{A(x^{22}a)X(x^{3}a)X(x^{5}a)F_2(x^{20}a)},$$

$$G_2(1/a) = \frac{A(x^{42}/a)Y(1/x^{9}a)Y(1/x^{7}a)G_2(x^{10}/a)}{A(x^{30}/a)Y(x^{15}/a)Y(x^{17}/a)G_2(x^{20}/a)}. \quad (5.20)$$

Solving these we obtain

$$F_2(a) = F_0(a) \frac{X(x^{29}a)X(x^{39}a)}{X(x^{5}a)X(x^{15}a)} \times \prod_{m=0}^{\infty} \left\{ \frac{(1 - x^{60m+27}a/b)(1 - x^{60m+33}a/b)}{(1 - x^{60m+3}a/b)(1 - x^{60m+13}a/b)} \times \frac{(1 - x^{60m+37}a/b)(1 - x^{60m+43}a/b)}{(1 - x^{60m+57}a/b)(1 - x^{60m+67}a/b)} \right\}, \quad (5.21)$$

and similarly

$$G_2(1/a) = G_0(1/a) \frac{Y(1/x^{7}a)Y(x^{3}/a)}{Y(x^{17}/a)Y(x^{27}/a)}.$$
We now substitute these into the eigenvalue expression, the first term of which gives

\[
\Lambda_2 = \frac{w^2}{b^2} \frac{A(x^2 w)A(x^{12} w)A(x^{20} w)A(x^{30} w) X(x^{15} w)X(x^{25} w)}{A(x^2)A(x^{12})A(x^{20})A(x^{30})X(x^7 w)X(x^{17} w)} \\
\times \frac{Y(1/x^{25} w)Y(1/x^{15} w)F_2(x^{10} w)G_2(1/x^{10} w)}{Y(1/x^{17} w)Y(1/x^7 w)}. \tag{5.23}
\]

This in turn gives an expression in elliptic functions analogous to (5.12), but we immediately set \( b = -1 \), yielding the leading 4-string excitation in the \( w^2 \) band for \( L = 3 \) to be

\[
\frac{\Lambda_2}{\Lambda_0} = \frac{w^2}{b^2} \frac{E(-x^7/w, x^{60}) E(-x^{17}/w, x^{60})}{E(-x^7 w, x^{60}) E(-x^{17} w, x^{60})} \\
\times \frac{E(-x^{37} w, x^{60}) E(-x^{47} w, x^{60})}{E(-x^{37} w, x^{60}) E(-x^{47} w, x^{60})}. \tag{5.24}
\]

At the isotropic point \( w = x^{15} \) this reduces to

\[
\frac{\Lambda_2}{\Lambda_0} = x^{-10} \frac{E^2(-x^2, x^{60}) E^2(-x^8, x^{60})}{E^2(-x^{22}, x^{60}) E^2(-x^{28}, x^{60})} \\
= \left[ \frac{\vartheta_4(\frac{2\pi}{30}, p^{8/15}) \vartheta_4(\frac{2\pi}{15}, p^{8/15})}{\vartheta_4(\frac{2\pi}{15}, p^{8/15}) \vartheta_4(\frac{11\pi}{30}, p^{8/15})} \right]^2. \tag{5.25}
\]

### 5.3 Mass \( m_3 \)

We begin the perturbation argument with the structure \( w_j = a_j \) for \( j = 1, \ldots, N - 4 \) with \( w_{N-3} = b_1 x^{-10} \), \( w_{N-2} = b_2 x^{10} \), \( w_{N-1} = b_3 x^{-20} \) and \( w_N = b_4 x^{20} \). From the Bethe equations for \( k = N - 3, \ldots, N \) (see Appendix B.3) we can show that \( b_1 = b_2 = b_3 = b_4 = b \). The Bethe equation for the other roots is

\[
-\omega \left[ \frac{E(x^{10}/a)}{E(x^{10}/a)} \right]^N = (A_{N-4} b^4)^{5/8} a^4 \frac{b^4}{b^4}
\]
\[ \frac{E(x^{8b/a})E^2(x^{10b/a})E(x^{12b/a})}{E(x^{8a/b})E^2(x^{10a/b})E(x^{12a/b})} \prod_{j=1}^{N-4} \frac{E(x^{10a/a_j})E(x^{20a/a_j})}{E(x^{10a_j/a})E(x^{20a_j/a})}. \]  

(5.26)

In the \( x \rightarrow 0 \) limit this gives the equation

\[ a^{N-4} + \frac{1}{\omega} (A_{N-4}b^4)^{5/8}/b^4 = 0, \]

(5.27)

which we equate the lhs as usual with \( \prod_{j=1}^{N-4} (a - a_j) \) to obtain

\[ \frac{1}{\omega} (A_{N-4}b^4)^{5/8} = A_{N-4}b^4. \]

(5.28)

Using this with the other Bethe equations (B.14)-(B.17) in the \( x \rightarrow 0 \) limit we obtain

\[ \left[ \frac{1}{\omega} (A_{N-4}b^4)^{5/8} \right]^4 = \left( \frac{A_{N-4}b^4}{b^3N} \right)^4 \Rightarrow b^{3N} = 1. \]

(5.29)

Note that since \( N \) is even, \( b = -1 \) satisfies this equation.

We define the appropriate functions for the unknown roots,

\[ F_3(w) = \prod_{j=1}^{N-4} \prod_{k=0}^{\infty} (1 - x^{32k}w/a_j), \]

\[ G_3(1/w) = \prod_{j=1}^{N-4} \prod_{k=1}^{\infty} (1 - x^{32k}a_j/w), \]

(5.30)

then from (5.26) come the recurrences,

\[ F_3(a) = \frac{A(x^{10a}) X(x^{20a}) X^2(x^{22a}) X(x^{24a})}{A(x^{22a}) X^2(x^{20a}) X(x^{12a}) X(x^{24a})} F_3(x^{10a}), \]

\[ G_3(1/a) = \frac{A(x^{42/a}) Y(1/x^{2a}) Y^2(1/x^{2a}) Y(1/a)}{A(x^{30/a}) Y(x^{5/a}) Y^2(x^{10/a}) Y(x^{12/a})} G_3(x^{10/a}). \]

(5.31)

The solutions are

\[ F_3(a) = F_0(a) \frac{X(x^{24a}) X(x^{32a}) X(x^{34a})}{X(x^{10a}) X(x^{12a}) X(x^{20a})} \]

\[ \times \prod_{m=0}^{\infty} \left\{ \frac{(1 - x^{60m+22a/b})(1 - x^{60m+30a/b})(1 - x^{60m+32a/b})}{(1 - x^{60m+8a/b})(1 - x^{60m+10a/b})(1 - x^{60m+18a/b})} \right\}, \]

22
\[
\left\{ \frac{1 - x^{60m+38}a/b}{1 - x^{60m+52}a/b} \right\} \left( \frac{1 - x^{60m+40}a/b}{1 - x^{60m+60}a/b} \right) \left( \frac{1 - x^{60m+48}a/b}{1 - x^{60m+62}a/b} \right), \quad (5.32)
\]

\[
\mathcal{G}_3(1/a) = \mathcal{G}_0(1/a) \frac{Y(1/a)Y(1/x^2a)Y(x^8/a)}{Y(x^{12}/a)Y(x^{20}/a)Y(x^{22}/a)} \times \prod_{m=1}^{\infty} \left\{ \frac{(1 - x^{60m-32}b/a)(1 - x^{60m-30}b/a)(1 - x^{60m-22}b/a)}{(1 - x^{60m-18}b/a)(1 - x^{60m-10}b/a)(1 - x^{60m-8}b/a)} \times \frac{(1 - x^{60m+12}b/a)(1 - x^{60m+20}b/a)(1 - x^{60m+22}b/a)}{(1 - x^{60m-2}b/a)(1 - x^{60m}b/a)(1 - x^{60m+8}b/a)} \right\}. \quad (5.33)
\]

In terms of these functions the eigenvalue may be represented as

\[
\Lambda_3 = -\frac{u^3}{b^3} A(x^2w)A(x^{12}w)A(x^{20}/w)A(x^{30}/w) X(x^{20}/w)X(x^{22}/w)X(x^{30}/w) \times \frac{Y(1/x^{30}w)Y(1/x^{22}w)Y(1/x^{20}w)}{Y(1/x^{12}w)Y(1/x^{10}w)Y(1/x^2w)} F_3(x^{10}w)G_3(1/x^{10}w). \quad (5.34)
\]

Thus, application of the perturbation argument has yielded the leading excitation in the \(w^3\) band to be

\[
\frac{\Lambda_3}{\Lambda_0} = u^3 \frac{E(-x^2/w, x^{60}) E(-x^{10}/w, x^{60}) E(-x^{12}/w, x^{60})}{E(-x^2/w, x^{60}) E(-x^{10}/w, x^{60}) E(-x^{12}/w, x^{60})} \times \frac{E(-x^{32}/w, x^{60}) E(-x^{40}/w, x^{60}) E(-x^{42}/w, x^{60})}{E(-x^{32}/w, x^{60}) E(-x^{40}/w, x^{60}) E(-x^{42}/w, x^{60})}, \quad (5.35)
\]

where we have put \(b = -1\). At the isotropic point \(w = x^{15}\) this reduces to

\[
\frac{\Lambda_3}{\Lambda_0} = \left[ \frac{\varphi_4(13\pi/60, p^{8/15}) \varphi_4(\pi/12, p^{8/15}) \varphi_4(\pi/20, p^{8/15})}{\varphi_4(17\pi/60, p^{8/15}) \varphi_4(5\pi/12, p^{8/15}) \varphi_4(4\pi/20, p^{8/15})} \right]^2. \quad (5.36)
\]

### 5.4 Mass \(m_4\)

We begin the perturbation argument with the 7-string \(w_j = a_j\) for \(j = 1, \ldots, N - 7\) with \(w_{N-6} = b_1 x^{-1}\), \(w_{N-5} = b_2 x\), \(w_{N-4} = b_3 x^{-7}\), \(w_{N-3} = b_4 x^7\), \(w_{N-2} = b_5 x^{-13}\), \(w_{N-1} = b_6 x^{13}\) and \(w_N = b_7 x^{16}\). From the Bethe equations for \(k = N - 6, \ldots, N\), we show in Appendix B.4 that \(b_1 = b_4 = b_5 = \alpha\) and \(b_2 = b_3 = b_6 = \beta\). We further let \(b_7 = b\). This feature is different from that seen for the previous masses (or will see for \(m_5, m_7, m_8\)). However, in the final analysis, our answer does not depend upon \(\alpha\) or \(\beta\).
The other Bethe roots satisfy

\[
-\omega \left[ \frac{E(x^{10}/a)}{E(x^{10}a)} \right]^{N} = (A_{N-7}a^{3}b^{3})^{5/8} \frac{a^{6}}{a^{2}b^{2}}
\]

\[
\times \frac{E(x^{7}/a)E(x^{21}/a)E(x^{27}/a)E(x^{15}/a)E(x^{15}/a)E(x^{21}/a)}{E(x^{21}/a)E(x^{21}/a)E(x^{21}/a)E(x^{21}/a)}
\times \frac{E(x^{7}/a)E(x^{21}/a)E(x^{21}/a)E(x^{21}/a)}{E(x^{21}/a)E(x^{21}/a)E(x^{21}/a)E(x^{21}/a)}
\times \frac{E(x^{4}/a)E(x^{6}/a)\prod_{j=1}^{N-7}E(x^{10}/a)E(x^{10}/a)}{E(x^{6}/a)E(x^{6}/a)}.
\]

(5.37)

In the \(x \to 0\) limit this gives the equation

\[
a^{N-6} + \frac{1}{\omega} (A_{N-7}a^{3}b^{3})^{5/8} / a^{2}b^{2} = 0,
\]

(5.38)

which is an \((N-6)\)th order equation for \(N-7\) zeros, so that there will be a hole, \(a_{N-6}\), in this case, as there was in regime 1. Equating the lhs of (5.38) with \(\prod_{j=1}^{N-6} (a-a_{j})\) we obtain

\[
\frac{1}{\omega}(A_{N-7}a^{3}b^{3})^{5/8} = A_{N-6}a^{2}b^{2} = A_{N-7}a_{N-6}a^{2}b^{2}.
\]

(5.39)

From the other Bethe equations (B.22)-(B.28) in the \(x \to 0\) limit,

\[
\left( -\frac{1}{\omega}(A_{N-7}a^{3}b^{3})^{5/8} \right)^{7} = \frac{(A_{N-7})^{6}a^{2}b^{2}}{b^{2N}}.
\]

(5.40)

Now (5.38) must be satisfied by each of the \(a_{j}, j=1, \ldots, N-7\),

\[
a_{j}^{N-6} = -A_{N-7}a_{N-6},
\]

(5.41)

so that forming the product of these \((N-7)\) equations,

\[
(A_{N-7})^{N-6} = -(A_{N-7}a_{N-6})^{N-7} \Rightarrow (a_{N-6})^{N} = -A_{N-7}(a_{N-6})^{7}.
\]

(5.42)

Taking (5.39), (5.40) and (5.42) together gives

\[
(b^{2}a_{N-6})^{N} = 1,
\]

(5.43)

which is clearly satisfied by \(b = -1\) and \(a_{N-6} = -1\). We need to define, in analogy with the previous auxiliary functions,
and the functions

\[ X_\alpha(w) = \prod_{k=0}^{\infty} (1 - x^{32k}w/\alpha), \quad Y_\alpha(1/w) = \prod_{k=1}^{\infty} (1 - x^{32k}\alpha/w), \quad (5.44) \]

\[ X_\beta(w) = \prod_{k=0}^{\infty} (1 - x^{32k}w/\beta), \quad Y_\beta(1/w) = \prod_{k=1}^{\infty} (1 - x^{32k}\beta/w), \quad (5.45) \]

\[ R_4(w) = \prod_{k=0}^{\infty} (1 - x^{32k}w/a_{N-6}), \quad (5.46) \]

\[ S_4(1/w) = \prod_{k=1}^{\infty} (1 - x^{32k}a_{N-6}/w), \quad (5.47) \]

for the as-yet-unknown roots. In place of the usual definition, we have

\[
\mathcal{F}_4(w) = \frac{F_4(w)X_\alpha(xw)X_\beta(x^7w)}{F_4(x^{12}w)X_\alpha(x^5w)X_\beta(x^{11}w)},
\]

\[
\mathcal{G}_4(w) = \frac{G_4(1/w)Y_\alpha(1/xw)Y_\beta(1/x^7w)}{G_4(1/x^{12}w)Y_\alpha(1/x^5w)Y_\beta(1/x^{11}w)}. \quad (5.49)
\]

Rearranging the Bethe equations (5.37) gives recurrences as usual, with

\[
\mathcal{F}_4(a) = \frac{A(x^{10}a)X(x^{26}a)X(x^{28}a)R_4(x^{20}a)R_4(x^{22}a)\mathcal{F}_4(x^{10}a)}{A(x^{22}a)X(x^4a)X(x^6a)R_4(x^{10}a)R_4(x^{12}a)\mathcal{F}_4(x^{20}a)},
\]

\[
\mathcal{G}_4(1/a) = \frac{A(x^{42}/a)Y(1/x^6a)Y(1/x^8a)S_4(1/x^2a)S_4(1/a)\mathcal{G}_4(x^{10}/a)}{A(x^{30}/a)Y(x^{14}/a)Y(x^{16}/a)S_4(x^8/a)S_4(x^{10}/a)\mathcal{G}_4(x^{20}/a)}. \quad (5.50)
\]

which we solve using identities for the various auxiliary functions listed in Appendix A to obtain

\[
\mathcal{F}_4(a) = \mathcal{F}_0(a)\frac{X(x^{28}a)X(x^{38}a)R_4(x^{32}a)}{X(x^6a)X(x^{16}a)R_4(x^{12}a)} \times \prod_{m=0}^{\infty} \left\{ \frac{(1 - x^{60m+26}a^{2}/6)(1 - x^{60m+34}a^{2}/6)(1 - x^{60m+36}a^{2}/6)}{(1 - x^{60m+26}a^{2}/6)(1 - x^{60m+14}a^{2}/6)(1 - x^{60m+4}a^{2}/6)} \times \frac{(1 - x^{60m+14}a^{2}/6)(1 - x^{60m+30}a^{2}/a_{N-6})(1 - x^{60m+40}a^{2}/a_{N-6})}{(1 - x^{60m+6}a^{2}/6)(1 - x^{60m+10}a^{2}/a_{N-6})(1 - x^{60m+6}a^{2}/a_{N-6})} \right\}, \quad (5.51)
\]
and similarly

\[ G_4(1/a) = G_0(1/a) \frac{Y(1/x^6 a)Y(x^4/a) S_4(1/a)}{Y(x^{16}/a)Y(x^{20}/a) S_4(x^{20}/a)} \times \prod_{m=1}^{\infty} \left\{ \frac{(1 - x^{60m-36 b/a})(1 - x^{60m-26 b/a})(1 - x^{60m+16 b/a})}{(1 - x^{60m-14 b/a})(1 - x^{60m-6 b/a})(1 - x^{60m-4 b/a})} \right\} \times \frac{(1 - x^{60m+26 b/a})(1 - x^{60m-30 a N_{-6}}/a)(1 - x^{60m+20 a N_{-6}}/a)}{(1 - x^{60m+4 b/a})(1 - x^{60m-10 a N_{-6}}/a)(1 - x^{60m+2 a N_{-6}}/a)} \]  \hspace{1cm} (5.52)

The eigenvalue expression is

\[ \Lambda_4 = -\frac{w^3}{a_{N-6} b^2} \frac{E(x^{6 b}/w, x^{60}) E(x^{10 a_{N-6}}/w, x^{60}) E(x^{14 b}/w, x^{60})}{E(x^{60} w/b, x^{60}) E(x^{100} w/a_{N-6}, x^{60}) E(x^{140} w/b, x^{60})} \times \frac{E(x^{36} w/b, x^{60}) E(x^{40} w/a_{N-6}, x^{60}) E(x^{44} w/b, x^{60})}{E(x^{36} b/w, x^{60}) E(x^{40} a_{N-6} w, x^{60}) E(x^{44} b/w, x^{60})} \]  \hspace{1cm} (5.53)

which, in elliptic functions is

\[ \Lambda_4 = \frac{w^3}{a_{N-6} b^2} \frac{E(-x^{6}/w, x^{60}) E(-x^{10}/w, x^{60}) E(-x^{14}/w, x^{60})}{E(-x^{6} w, x^{60}) E(-x^{10} w, x^{60}) E(-x^{14} w, x^{60})} \times \frac{E(-x^{36} w, x^{60}) E(-x^{40} w, x^{60}) E(-x^{44} w, x^{60})}{E(-x^{36} w, x^{60}) E(-x^{40} w, x^{60}) E(-x^{44} w, x^{60})} \]  \hspace{1cm} (5.54)

Finally, with \( b = a_{N-6} = -1 \),

\[ \Lambda_4 = w^3 \frac{E(-x^{6}/w, x^{60}) E(-x^{10}/w, x^{60}) E(-x^{14}/w, x^{60})}{E(-x^{6} w, x^{60}) E(-x^{10} w, x^{60}) E(-x^{14} w, x^{60})} \times \frac{E(-x^{36} w, x^{60}) E(-x^{40} w, x^{60}) E(-x^{44} w, x^{60})}{E(-x^{36} w, x^{60}) E(-x^{40} w, x^{60}) E(-x^{44} w, x^{60})} \]  \hspace{1cm} (5.55)

At the isotropic point \( w = x^{15} \) this becomes

\[ \frac{\Lambda_4}{\Lambda_0} = \left[ \vartheta_4(\frac{3\pi}{20}, p^{8/15}) \vartheta_4(\frac{\pi}{12}, p^{8/15}) \vartheta_4(\frac{5\pi}{60}, p^{8/15}) \right]^2 \]  \hspace{1cm} (5.56)

5.5 Mass \( m_5 \)

We begin the perturbation argument with \( w_j = a_j \) for \( j = 1, \ldots, N - 6 \) and \( w_{N-5} = b_1 x^{-13}, w_{N-4} = b_2 x^{13}, w_{N-3} = b_3 x^{-11}, w_{N-2} = b_4 x^{11}, w_{N-1} = b_5 x^{-9}, \)
$w_N = b_6 x^9$. In Appendix B.5 we show that the $b_i$ are equal, and we call them $b$. The Bethe equation for the other roots is

$$-\omega \left[ \frac{E(x^{10}/a)}{E(x^{10}a)} \right]^N = (A_{N-6} b^6)^{5/8} \frac{a^6}{b^6} \frac{E(x^7b/a)E^2(x^9b/a)}{E^2(x^{13}b/a)E^2(x^9a/b)}$$

$$\times \frac{E^2(x^{11}b/a)E(x^{13}b/a)}{E^2(x^{11}a/b)E(x^{13}a/b)} \prod_{j=1}^{N-6} \frac{E(x^{10}a/a_j)E(x^{20}a_j/a)}{E(x^{10}a_j/a)E(x^{20}a_j/a)}. \quad (5.57)$$

In the $x \to 0$ limit this gives the equation

$$a^{N-6} + \frac{1}{\omega} (A_{N-6} b^6)^{5/8} / b^6 = 0, \quad (5.58)$$

which is an $(N-6)$th order equation for $N-6$ zeros, so that no hole needs to be considered. Equating the lhs with $\prod_{j=1}^{N-6} (a - a_j)$ gives

$$\frac{1}{\omega} (A_{N-6} b^6)^{5/8} = A_{N-6} b^6. \quad (5.59)$$

From the other Bethe equations (B.31),

$$\left[ \frac{1}{\omega} (A_{N-6} b^6)^{5/8} \right]^6 = \frac{(A_{N-6} b^6)^6}{b^{4N}} \Rightarrow b^{4N} = 1. \quad (5.60)$$

For the unknown roots we define the functions

$$F_5(w) = \prod_{j=1}^{N-6} \prod_{k=0}^{\infty} (1 - x^{32k} w/a_j),$$

$$G_5(1/w) = \prod_{j=1}^{N-6} \prod_{k=1}^{\infty} (1 - x^{32k} a_j/w), \quad (5.61)$$

which, after rearranging (5.57), obey the recurrences

$$\mathcal{F}_5(a) = \frac{A(x^{10}a) X(x^{19}a)X^2(x^{21}a)X^2(x^{23}a)X(x^{25}a) \mathcal{F}_5(x^{10}a)}{A(x^{22}a) X(x^7a)X^2(x^9a)X^2(x^{11}a)X(x^{13}a) \mathcal{F}_5(x^{20}a)^6},$$

$$\mathcal{G}_5(1/a) = \frac{A(x^{42}/a) Y(1/x^5a)Y^2(1/x^3a)Y^2(1/xa)Y(x/a) \mathcal{G}_5(x^{10}/a)}{A(x^{30}/a) Y(x^7/a)Y^2(x^9/a)Y^2(x^{11}/a)Y(x^{13}/a) \mathcal{G}_5(x^{20}/a)}. \quad (5.62)$$

The solutions are
\[ F_5(a) = F_0(a) \frac{X(x^{25}a)X(x^{31}a)X(x^{33}a)X(x^{35}a)}{X(x^9a)X(x^{11}a)X(x^{13}a)X(x^{19}a)} \times \prod_{m=0}^{\infty} \left\{ \begin{array}{l} (1 - x^{6m+23}a/b)(1 - x^{6m+9}a/b)(1 - x^{6m+31}a/b) \\ (1 - x^{6m+7}a/b)(1 - x^{6m+9}a/b)(1 - x^{6m+11}a/b) \\ (1 - x^{6m+33}a/b)(1 - x^{6m+37}a/b)(1 - x^{6m+39}a/b) \\ (1 - x^{6m+17}a/b)(1 - x^{6m+53}a/b)(1 - x^{6m+59}a/b) \\ (1 - x^{6m+41}a/b)(1 - x^{6m+47}a/b) \\ (1 - x^{6m+61}a/b)(1 - x^{6m+63}a/b) \end{array} \right\}, \]

\[ G_5(1/a) = G_0(1/a) \frac{Y(1/x^3a)Y(1/x^3a)Y(x/a)Y(x^7/a)}{Y(x^{13}/a)Y(x^{19}/a)Y(x^{21}/a)Y(x^{23}/a)} \times \prod_{m=1}^{\infty} \left\{ \begin{array}{l} (1 - x^{6m-33}b/a)(1 - x^{6m-31}b/a)(1 - x^{6m-29}b/a) \\ (1 - x^{6m-17}b/a)(1 - x^{6m-11}b/a)(1 - x^{6m-9}b/a) \\ (1 - x^{6m-23}b/a)(1 - x^{6m+13}b/a)(1 - x^{6m+19}b/a) \\ (1 - x^{6m-7}b/a)(1 - x^{6m-3}b/a)(1 - x^{6m-1}b/a) \\ (1 - x^{6m+21}b/a)(1 - x^{6m+23}b/a) \\ (1 - x^{6m+1}b/a)(1 - x^{6m+7}b/a) \end{array} \right\}, \]

which we next substitute into the eigenvalue expression

\[ \Lambda_5 = \frac{w^4 A(x^2w)A(x^{12}w)A(x^{20}/w)A(x^{30}/w)}{b^4 A(x^2)A(x^{12})A(x^{20})A(x^{30})} \times \frac{X(x^{19}w)X(x^{21}w)X(x^{23}w)X(x^{29}w)}{X(x^{3}w)X(x^{9}w)X(x^{11}w)X(x^{13}w)} \times \frac{Y(1/x^{29}w)Y(1/x^{23}w)Y(1/x^{21}w)Y(1/x^{19}w)}{Y(1/x^{3}w)Y(1/x^{1}w)Y(1/x^{11}w)Y(1/x^{13}w)} \times \mathcal{F}_5(x^{10}w)G_5(1/x^{10}w), \]

to obtain (with \( b = -1 \))

\[ \frac{\Lambda_5}{\Lambda_0} = \frac{w^4 E(-x^3/w)E(-x^9/w)E(-x^{11}/w)E(-x^{13}/w)}{E(-x^3)E(-x^9)E(-x^{11})E(-x^{13})} \times \frac{E(-x^{33})E(-x^{39})E(-x^{41})E(-x^{43})}{E(-x^{33}/w)E(-x^{39}/w)E(-x^{41}/w)E(-x^{43}/w)}. \]

The elliptic functions are of nome \( x^{60} \). At the isotropic point \( w = x^{15} \) the leading excitation reduces to

\[ \frac{\Lambda_5}{\Lambda_0} = \left[ \frac{\vartheta_4(\pi/30, p^{8/15}) \vartheta_4(\pi/10, p^{8/15}) \vartheta_4(\pi/15, p^{8/15}) \vartheta_4(\pi/30, p^{8/15})}{\vartheta_4(\pi/10, p^{8/15}) \vartheta_4(\pi/30, p^{8/15}) \vartheta_4(d\pi/30, p^{8/15}) \vartheta_4(\pi/15, p^{8/15})} \right]^2. \]

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5.6 Mass $m_6$

We begin with $w_j = a_j$ for $j = 1, \ldots, N - 7$ and $w_{N-6} = b_1 x^{-14}$, $w_{N-5} = b_2 x^{14}$, $w_{N-4} = b_3 x^{-10}$, $w_{N-3} = b_4 x^{10}$, $w_{N-2} = b_5 x^{-6}$, $w_{N-1} = b_6 x^6$, $w_N = b_7 x^{16}$. In Appendix B.6 we show that $b_1 = b_2 = b_5 = b_6 = b_7 = b$ and that $b_3 = b_4 = \alpha$. This feature is similar to that for $m_4$; again, in the final analysis, our answer does not depend upon $\alpha$. The Bethe equation for the other roots is

$$
-\omega \left[ \frac{E(x^{10}/a)}{E(x^{10}a)} \right]^N = \left( A_{N-7} b^5 \alpha^2 \right)^{5/8} \frac{a^7}{b^6 \alpha} \\
\times \frac{E(x^2 b/a) E(x^4 b/a) E(x^6 b/a) E(x^{10} b/a) E(x^{14} b/a)}{E(x^2 a/b) E(x^4 a/b) E(x^6 a/b) E(x^8 a/b) E(x^{14} a/b)} \\
\times \frac{E(x^{10} a/\alpha) E(x^{12} a/\alpha) E(x^{30} a/\alpha)}{E(x^{10} a/\alpha) E(x^{12} a/\alpha) E(x^{30} a/\alpha)} \prod_{j=1}^{N-7} \frac{E(x^{10} a_j/a_j) E(x^{20} a_j/a_j)}{E(x^{10} a_j/a) E(x^{20} a_j/a)}.
$$

(5.68)

In the $x \to 0$ limit this gives

$$
a^{N-7} + \frac{1}{\omega} \left( A_{N-7} b^5 \alpha^2 \right)^{5/8} / b^6 \alpha = 0
$$

(5.69)

as the equation for the $(N - 7)$ roots. Equating this with $\prod_{j=1}^{N-7} (a - a_j)$ gives

$$
\frac{1}{\omega} \left( A_{N-7} b^5 \alpha^2 \right)^{5/8} = A_{N-7} b^6 \alpha.
$$

(5.70)

From the other Bethe equations (B.36) we obtain

$$
\left[ \frac{1}{\omega} \left( A_{N-7} b^5 \alpha^2 \right)^{5/8} \right]^7 = \frac{(A_{N-7} b^6 \alpha)^7}{b^{4N}} \Rightarrow b^{4N} = 1.
$$

(5.71)

We define

$$
F_6(w) = \prod_{j=1}^{N-7} \prod_{k=0}^{\infty} (1 - x^{32k} w/a_j),
$$

$$
G_6(1/w) = \prod_{j=1}^{N-7} \prod_{k=1}^{\infty} (1 - x^{32k} a_j/w),
$$

(5.72)

and in place of the usual definition,

$$
\mathcal{F}_6(w) = \frac{F_6(w)}{F_6(x^{12}w)} X_\alpha(x^{10}w) / X_\alpha(x^2w),
$$

29
where $X_\alpha (w)$ and $Y_\alpha (1/w)$ have the same definition (5.44) as they did for $m_4$. We solve the recurrences

$$G_6 (w) = \frac{G_6 (1/w)}{G_6 (1/w^2)} \frac{Y_\alpha (1/x^{10} w)}{Y_\alpha (1/x^2 w)}, \tag{5.73}$$

which are obtained from (5.68), giving

$$F_6 (a) = \frac{A(x^{10} a) X(x^{15} a) X(x^{22} a) X(x^{26} a) X(x^{30} a)}{A(x^{22} a) X(x^{2} a) X(x^{4} a) X(x^{8} a) X(x^{14} a)} \frac{F_6 (x^{10} a)}{F_6 (x^{20}/a)},$$

$$G_6 (1/a) = \frac{A(x^{20} a) X(x^{2} a) Y(1/x^{10} a) Y(1/x^{6} a) Y(1/x^{4} a) Y(x^2/a)}{A(x^{30} a) Y(x^{6} a) Y(x^{12} /a) Y(x^{14} /a) Y(x^{16} /a) Y(x^{18} /a)} \times \frac{G_6 (x^{10} a)}{G_6 (x^{20} /a)}, \tag{5.74}$$

$$F_6 (a) = F_6 (a) \frac{X(x^{28} a) X(x^{30} a)}{X(x^{14} a) X(x^{16} a)} \times \prod_{m=0}^{\infty} \left\{ \frac{(1 - x^{60 m - 2 a} \frac{b}{a}) (1 - x^{60 m + 28 \frac{a}{b}}) (1 - x^{60 m + 32 \frac{a}{b}})}{(1 - x^{60 m + 2 a} \frac{b}{a}) (1 - x^{60 m + 4 \frac{a}{b}}) (1 - x^{60 m + 6 \frac{a}{b}})} \right\}$$

$$\times \frac{(1 - x^{60 m + 34 \frac{a}{b}}) (1 - x^{60 m + 36 \frac{a}{b}}) (1 - x^{60 m + 38 \frac{a}{b}})}{(1 - x^{60 m + 8 \frac{a}{b}}) (1 - x^{60 m + 12 \frac{a}{b}}) (1 - x^{60 m + 14 \frac{a}{b}})}$$

$$\times \frac{(1 - x^{60 m + 42 \frac{a}{b}}) (1 - x^{60 m + 44 \frac{a}{b}})}{(1 - x^{60 m + 58 \frac{a}{b}}) (1 - x^{60 m + 58 \frac{a}{b}})} \tag{5.75}$$

$$G_6 (1/a) = G_6 (1/a) \frac{Y(x^2/a) Y(x^4/a)}{Y(x^{16}/a) Y(x^{18}/a)} \times \prod_{m=1}^{\infty} \left\{ \frac{(1 - x^{50 m - 38 \frac{b}{a}}) (1 - x^{60 m - 36 \frac{b}{a}}) (1 - x^{60 m - 34 \frac{b}{a}})}{(1 - x^{60 m - 14 \frac{b}{a}}) (1 - x^{60 m - 12 \frac{b}{a}}) (1 - x^{60 m - 8 \frac{b}{a}})} \right\}$$

$$\times \frac{(1 - x^{60 m - 32 \frac{b}{a}}) (1 - x^{60 m - 28 \frac{b}{a}}) (1 - x^{60 m - 26 \frac{b}{a}})}{(1 - x^{60 m - 6 \frac{b}{a}}) (1 - x^{60 m - 4 \frac{b}{a}}) (1 - x^{60 m - 2 \frac{b}{a}})}$$

$$\times \frac{(1 - x^{60 m + 16 \frac{b}{a}}) (1 - x^{60 m + 18 \frac{b}{a}})}{(1 - x^{60 m + 2 \frac{b}{a}}) (1 - x^{60 m + 4 \frac{b}{a}})} \tag{5.76}$$

Substituted into the eigenvalue expression

$$\Lambda_6 = \frac{w^4 A(x^{2} w) A(x^{12} w) A(x^{20}/w) A(x^{30}/w)}{b^4 A(x^{2}) A(x^{12}) A(x^{20}) A(x^{30})} \times \frac{X(x^{24} w) X(x^{26} w) Y(1/x^{26} w) Y(1/x^{24} w)}{X(x^{6} w) X(x^{8} w) Y(1/x^{8} w) Y(1/x^{6} w)}$$

30
\[ \times \mathcal{F}_6(x^{10}w) G_6(1/x^{10}w), \quad (5.77) \]

this gives (with \( b = -1 \) and with elliptic nome \( x^{60} \)),

\[ \frac{\Lambda_6}{\Lambda_0} = w^4 \frac{E(-x^6/w) E(-x^8/w) E(-x^{12}/w) E(-x^{14}/w)}{E(-x^6w), E(-x^8w), E(-x^{12}w) E(-x^{14}w)} \times \frac{E(-x^{36}/w) E(-x^{38}/w) E(-x^{42}/w) E(-x^{44}/w)}{E(-x^{36}/w) E(-x^{38}/w) E(-x^{42}/w) E(-x^{44}/w)}. \quad (5.78) \]

At the isotropic point this reduces to

\[ \frac{\Lambda_6}{\Lambda_0} = \left[ \partial_4 \left( \frac{3\pi}{30}, p^{8/15} \right) \partial_4 \left( \frac{7\pi}{60}, p^{8/15} \right) \partial_4 \left( \frac{\pi}{36}, p^{8/15} \right) \partial_4 \left( \frac{\pi}{60}, p^{8/15} \right) \right]^2. \quad (5.79) \]

\subsection*{5.7 Mass \( m_7 \)}

We begin with \( w_j = a_j \) for \( j = 1, \ldots, N - 8 \) and \( w_{N-7} = b_1x^{-14}, w_{N-6} = b_2x^{14}, w_{N-5} = b_3x^{-12}, w_{N-4} = b_4x^{12}, w_{N-3} = b_5x^{-10}, w_{N-2} = b_6x^{10}, w_{N-1} = b_7x^{-8}, w_N = b_8x^8 \). We show in Appendix B.7 that the \( b_i \) are all equal. The Bethe equation for the other roots is

\[ -\omega \left[ \frac{E(x^{10}/a)}{E(x^{10}/a)} \right]^N = \left( A_{N-8}b^8 \right)^{5/8} \frac{E(x^{6}b/a) E^2(x^{6}b/a) E^2(x^{10}b/a)}{b^8 E(x^{6}a/b) E^2(x^{6}a/b) E^2(x^{10}a/b)} \times \frac{E^2(x^{12}b/a) E(x^{14}b/a) \prod_{j=1}^{N-8} E(x^{10}a/a_j) E(x^{20}a_j/a)}{E^2(x^{12}a/b) E(x^{14}a/b) E(x^{20}a/a_j)}. \quad (5.80) \]

In the \( x \to 0 \) limit this gives

\[ a^{N-8} + \frac{1}{\omega} (A_{N-8}b^8)^{5/8} b^8 = 0, \quad (5.81) \]

which is an \((N - 8)\)th order equation for \( N - 8 \) zeros, so that there is again no hole. Equating this with \( \prod_{j=1}^{N-8} (a - a_j) \) we obtain

\[ \frac{1}{\omega} (A_{N-8}b^8)^{5/8} = A_{N-8}b^8, \quad (5.82) \]

and from Bethe equations (B.40),

\[ \left[ \frac{1}{\omega} (A_{N-8}b^8)^{5/8} \right]^8 = \frac{(A_{N-8}b^8)^8}{b^{5N}} \Rightarrow b^{5N} = 1. \quad (5.83) \]
Defining

\[ F_7(w) = \prod_{j=1}^{N-8} \prod_{k=0}^{\infty} (1 - x^{32k} w/a_j), \]  
(5.84)

\[ G_7(1/w) = \prod_{j=1}^{N-8} \prod_{k=1}^{\infty} (1 - x^{32k} a_j/w), \]  
(5.85)

and rearranging the Bethe equation (5.80) as before, gives the recurrences

\[
F_7(a) = \frac{A(x^{10}a) X(x^{18}a) X^2(x^{20}a) X^2(x^{22}a) X^2(x^{24}a) X^2(x^{26}a)}{A(x^{22}a) X(x^{6}a) X^2(x^{8}a) X^2(x^{10}a) X^2(x^{12}a) X(x^{14}a)} 
\times \frac{F_7(x^{10}a)}{F_7(x^{20}a)},
\]

\[
G_7(1/a) = \frac{A(x^{42}/a) Y(1/x^{4}a) Y^2(1/x^{2}a) Y^2(1/a) Y(1/a)}{A(x^{30}/a) Y(x^{6}/a) Y^2(x^{8}/a) Y^2(x^{10}/a) Y^2(x^{12}/a) Y(x^{14}/a)} 
\times \frac{G_7(x^{10}/a)}{G_7(x^{20}/a)},
\]

(5.86)

The solutions are

\[
F_7(a) = F_0(a) \frac{X(x^{26}a) X(x^{30}a) X(x^{32}a) X(x^{34}a) X(x^{36}a)}{X(x^{8}a) X(x^{10}a) X(x^{12}a) X(x^{14}a) X(x^{18}a)} 
\times \prod_{m=0}^{\infty} \left\{ \frac{(1 - x^{60m+24 \frac{a}{b}})(1 - x^{60m+28 \frac{a}{b}})(1 - x^{60m+30 \frac{a}{b}})}{(1 - x^{60m+6 \frac{a}{b}})(1 - x^{60m+8 \frac{a}{b}})(1 - x^{60m+10 \frac{a}{b}})} \right\}.
\]

(5.87)

\[
G_7(1/a) = G_0(1/a) \frac{Y(1/x^{4}a) Y(1/x^{2}a) Y(1/a) Y(1/a)}{Y(x^{14}/a) Y(x^{18}/a) Y(x^{20}/a) Y(x^{22}/a) Y(x^{24}/a)} 
\times \prod_{m=1}^{\infty} \left\{ \frac{(1 - x^{60m-34 \frac{b}{a}})(1 - x^{60m-32 \frac{b}{a}})(1 - x^{60m-30 \frac{b}{a}})}{(1 - x^{60m-16 \frac{b}{a}})(1 - x^{60m-12 \frac{b}{a}})(1 - x^{60m-10 \frac{b}{a}})} \right\}.
\]

(5.88)
Substitution into

\[
\Lambda_7 = -w^{5} b^{3} \frac{A(x^2 w) A(x^{12} w) A(x^{20} / w) A(x^{30} / w)}{A(x^2) A(x^{12}) A(x^{20}) A(x^{30})} \times \frac{X(x^{18} w) X(x^{20} w) X(x^{22} w) X(x^{24} w) X(x^{28} w)}{X(x^4 w) X(x^{8} w) X(x^{10} w) X(x^{12} w) X(x^{14} w)} \times \frac{Y(1/x^{28} w) Y(1/x^{24} w) Y(1/x^{22} w) Y(1/x^{20} w) Y(1/x^{18} w)}{Y(1/x^{14} w) Y(1/x^{12} w) Y(1/x^{10} w) Y(1/x^{8} w) Y(1/x^{4} w)} \times \mathcal{F}_7(x^{10} w) \mathcal{G}_7(1/x^{10} w),
\]

(5.89)

yields the result (with \(b = -1\) and elliptic nome \(x^{60}\))

\[
\frac{\Lambda_7}{\Lambda_0} = w^{5} \frac{E(-x^4 / w) E(-x^8 / w) E(-x^{10} / w) E(-x^{12} / w) E(-x^{14} / w)}{E(-x^{14} / w) E(-x^{38} / w) E(-x^{40} / w) E(-x^{42} / w) E(-x^{44} / w)} \frac{E(-x^{34} w) E(-x^{38} w) E(-x^{40} w) E(-x^{42} w) E(-x^{44} w)}{E(-x^{34} / w) E(-x^{38} / w) E(-x^{40} / w) E(-x^{42} / w) E(-x^{44} / w)}. \]

(5.90)

This reduces to

\[
\frac{\Lambda_7}{\Lambda_0} = \left[ \frac{\varphi_4(\frac{11\pi}{60}) \varphi_4(\frac{7\pi}{60}) \varphi_4(\frac{\pi}{12}) \varphi_4(\frac{\pi}{20}) \varphi_4(\frac{\pi}{60})}{\varphi_4(\frac{19\pi}{60}) \varphi_4(\frac{13\pi}{60}) \varphi_4(\frac{5\pi}{12}) \varphi_4(\frac{7\pi}{20}) \varphi_4(\frac{3\pi}{60})} \right]^2,
\]

(5.91)

at the isotropic point, with the elliptic functions of nome \(p^{8/15}\).

5.8 Mass \(m_8\)

We begin with the distribution \(w_j = a_j\) for \(j = 1, \ldots, N - 10\) and \(w_{N-9} = b_1 x^{-15}, w_{N-8} = b_2 x^{15}, w_{N-7} = b_3 x^{-13}, w_{N-6} = b_4 x^{13}, w_{N-5} = b_5 x^{-11}, w_{N-4} = b_6 x^{11}, w_{N-3} = b_7 x^{-9}, w_{N-2} = b_8 x^{9}, w_{N-1} = b_9 x^{-7}, w_N = b_{10} x^{7}\). We show in Appendix B.8 that the \(b_i\) are equal. The Bethe equation for the other roots is

\[
-\omega \left[ \frac{E(x^{10} / a)}{E(x^{10} a)} \right]^N = (A_{N-10} b^{10})^{5/8} a^{10} \frac{E(x^{5} b / a) E^2(x^7 b / a) E^2(x^9 b / a)}{E^2(x^7 a / b) E^2(x^9 a / b)} \times \frac{E^2(x^{11} b / a)}{E^2(x^{11} a / b)} \frac{E^2(x^{13} b / a)}{E^2(x^{13} a / b)} \frac{E(x^{15} b / a)}{E(x^{15} a / b)} \prod_{j=1}^{N-10} \frac{E(x^{10} a / a_j) E(x^{20} a_j / a)}{E(x^{10} a_j / a) E(x^{20} a / a_j)}. \]

(5.92)

In the \(x \to 0\) limit this gives

\[
a^{N-10} + \frac{1}{\omega} (A_{N-10} b^{10})^{5/8} b^{10} = 0,
\]

(5.93)
for the $N - 10$ zeros. We equate this with $\prod_{j=1}^{N-10}(a - a_j)$ to obtain

$$\frac{1}{\omega}(A_{N-10}b^{10})^{5/8} = A_{N-10}b^{10}, \tag{5.94}$$

and from Bethe equations (B.43),

$$\left[\frac{1}{\omega}(A_{N-10}b^{10})^{5/8}\right]^{10} = \frac{(A_{N-10}b^{10})^{10}}{b^{6N}} \Rightarrow b^{6N} = 1. \tag{5.95}$$

The functions

$$F_8(w) = \prod_{j=1}^{N-10} \prod_{k=0}^{\infty} (1 - x^{32k} w/a_j), \tag{5.96}$$

$$G_8(1/w) = \prod_{j=1}^{N-10} \prod_{k=1}^{\infty} (1 - x^{32k} a_j/w), \tag{5.97}$$

must as a consequences of (5.92) obey the recurrences

$$F_8(a) = \frac{A(x^{10}a) X(x^{17}a)X^2(x^{19}a)X^2(x^{21}a)X^2(x^{23}a)X^2(x^{25}a)}{A(x^{22}a) X(x^{3}a)X^2(x^{5}a)X^2(x^{9}a)X^2(x^{11}a)X^2(x^{13}a)}$$

$$\times \frac{X(x^{27}a) F_8(x^{10}a)}{X(x^{15}a) F_8(x^{20}a)} \tag{5.98}$$

$$G_8(1/a) = \frac{A(x^{42}/a) Y(1/x^{7}a)Y^2(1/x^{5}a)Y^2(1/x^{3}a)}{A(x^{30}/a) Y(x^{5}/a)Y^2(x^{7}/a)Y^2(x^{9}/a)}$$

$$\times \frac{Y^2(1/x a)Y^2(x/a)Y(x/a) G_8(x^{10}/a)}{Y^2(x^{11}/a)Y^2(x^{13}/a)Y(x^{15}/a) G_8(x^{20}/a)} \tag{5.99}$$

Solving for them, we obtain

$$F_8(a) = F_0(a) \frac{X(x^{27}a) X(x^{29}a)X(x^{31}a)X(x^{33}a)X(x^{35}a)X(x^{37}a)}{X(x^{17}a)X(x^{19}a)X(x^{11}a)X(x^{13}a)X(x^{15}a)X(x^{17}a)}$$

$$\times \prod_{m=0}^{\infty} \left\{ \begin{array}{c} (1 - x^{60m+25 a/b})(1 - x^{60m+27 a/b})(1 - x^{60m+29 a/b}) \\ (1 - x^{60m+5 a/b})(1 - x^{60m+7 a/b})(1 - x^{60m+9 a/b}) \end{array} \right\} \tag{5.100}$$

$$\left\{ \begin{array}{c} (1 - x^{60m+11 a/b})(1 - x^{60m+13 a/b}) \\ (1 - x^{60m+3 a/b})(1 - x^{60m+33 a/b}) \end{array} \right\} \left\{ \begin{array}{c} (1 - x^{60m+15 a/b}) \\ (1 - x^{60m+35 a/b}) \end{array} \right\} \left\{ \begin{array}{c} (1 - x^{60m+39 a/b}) \\ (1 - x^{60m+55 a/b}) \end{array} \right\} \left\{ \begin{array}{c} (1 - x^{60m+57 a/b}) \\ (1 - x^{60m+59 a/b}) \end{array} \right\} \left\{ \begin{array}{c} (1 - x^{60m+41 a/b}) \\ (1 - x^{60m+43 a/b}) \end{array} \right\} \left\{ \begin{array}{c} (1 - x^{60m+45 a/b}) \\ (1 - x^{60m+61 a/b}) \end{array} \right\} \left\{ \begin{array}{c} (1 - x^{60m+63 a/b}) \\ (1 - x^{60m+65 a/b}) \end{array} \right\} \left\{ \begin{array}{c} (1 - x^{60m+67 a/b}) \\ (1 - x^{60m+69 a/b}) \end{array} \right\}.$$
At the isotropic point this reduces to

\[ \mathcal{G}_8(1/a) = \mathcal{G}_0(1/a) \frac{Y(1/x^5 a)Y(1/x^3 a)Y(1/x a)Y(x/a)Y(x^3/a)Y(x^5/a)}{Y(x^{15}/a)Y(x^{17}/a)Y(x^{19}/a)Y(x^{21}/a)Y(x^{23}/a)Y(x^{25}/a)} \times \prod_{m=1}^{\infty} \left\{ \left( 1 - x^{60m - 35 \frac{b}{a}} \right) \left( 1 - x^{60m - 33 \frac{b}{a}} \right) \left( 1 - x^{60m - 31 \frac{b}{a}} \right) \right\} \right] . \] (5.101)

The eigenvalue expression is

\[ \Lambda_8 = \frac{w^6}{b^6} \frac{A(x^2 w)A(x^{12} w)A(x^{20}/w)A(x^{30}/w)}{A(x^2)A(x^{12})A(x^{20})A(x^{30})} \times \frac{X(x^{17} w)X(x^{19} w)X(x^{21} w)X(x^{23} w)X(x^{25} w)X(x^{27} w)}{X(x^5 w)X(x^7 w)X(x^9 w)X(x^{11} w)X(x^{13} w)X(x^{15} w)} \times \frac{Y(1/x^{27} w)Y(1/x^{25} w)Y(1/x^{23} w)Y(1/x^{21} w)Y(1/x^{19} w)}{Y(1/x^{17} w)Y(1/x^{15} w)Y(1/x^{13} w)Y(1/x^{11} w)Y(1/x^{9} w)Y(1/x^{7} w)} \times \frac{Y(1/x^{17} w)}{Y(1/x^{5} w)} \mathcal{F}_8(x^{10} w) \mathcal{G}_8(1/x^{10} w). \] (5.102)

Thus the leading excitation in the \( w^6 \) band, with \( b = -1 \) and elliptic nome \( x^{60} \), is

\[ \frac{\Lambda_8}{\Lambda_0} = \frac{w^6 E(-x^5/w)E(-x^7/w)E(-x^9/w)E(-x^{11}/w)}{E(-x^5/w)E(-x^7/w)E(-x^9/w)E(-x^{11}/w)} \times \frac{E(-x^{13}/w)E(-x^{15}/w)E(-x^{35}/w)E(-x^{37}/w)}{E(-x^{13}/w)E(-x^{15}/w)E(-x^{35}/w)E(-x^{37}/w)} \times \frac{E(-x^{39}/w)E(-x^{41}/w)E(-x^{43}/w)E(-x^{45}/w)}{E(-x^{39}/w)E(-x^{41}/w)E(-x^{43}/w)E(-x^{45}/w)}. \] (5.103)

At the isotropic point this reduces to

\[ \frac{\Lambda_8}{\Lambda_0} \left[ \begin{array}{c} \vartheta_4(\frac{\pi}{3}) \vartheta_4(\frac{\pi}{15}) \vartheta_4(\frac{\pi}{30}) \vartheta_4(\frac{\pi}{15}) \vartheta_4(\frac{\pi}{30}) \vartheta_4(0) \\ \vartheta_4(\frac{\pi}{3}) \vartheta_4(\frac{11\pi}{30}) \vartheta_4(\frac{2\pi}{5}) \vartheta_4(\frac{11\pi}{30}) \vartheta_4(\frac{2\pi}{5}) \vartheta_4(\frac{\pi}{2}) \end{array} \right] \right]^2, \] (5.104)

where the elliptic functions are of nome \( p^{8/15} \).
6 General formula and correlation lengths

Having obtained the relevant eigenvalues in the thermodynamic limit we are now in a position to calculate the correlation lengths and related mass gaps.

6.1 Correlation lengths

Recalling that our transfer matrix acts in the vertical direction, we consider the pair correlation function between two sites in the same column of the lattice separated by distance $l$. The correlation length defining the decay of the correlation function for large $l$ can be obtained either by integrating over the relevant bands of eigenvalues and applying the method of steepest descent, or equivalently via the leading eigenvalue at the isotropic point (see, e.g., [19,20]). Here we simply follow the latter approach.

Define the quantity

$$r_j(u) = \lim_{N \to \infty} \frac{\Lambda_j(u)}{\Lambda_0(u)}.$$  \hspace{1cm} (6.1)

The various correlation lengths follow as

$$\xi_j^{-1} = -\log r_j(u),$$  \hspace{1cm} (6.2)

where we are to understand that we take the relevant leading eigenvalue at the isotropic point $u = 3\lambda/2$.

6.2 Regime 1

For $L$ odd in regime 1, we derived the general result (4.18). For the leading eigenvalue in the band, our numerical checks confirm that the hole is located at $a_N = -1$, with the excitation parameter $b = -1$, as expected. Thus using (4.20) gives the result

$$\xi^{-1} = 2 \log \left[ \frac{\vartheta_4(\frac{5\pi}{12}, p^{\pi/6\lambda})}{\vartheta_4(\frac{7\pi}{12}, p^{\pi/6\lambda})} \right].$$  \hspace{1cm} (6.3)

The correlation length diverges at criticality, with

$$\xi \sim \frac{1}{4\sqrt{3}} p^{-\nu_h} \quad \text{as} \quad p \to 0,$$  \hspace{1cm} (6.4)
where the correlation length exponent $\nu_h$ is given by

$$\nu_h = \frac{r}{6s} = \frac{2(L + 1)}{3L}. \quad (6.5)$$

### 6.3 $L = 3$ regime 2

The magnetic Ising value $\lambda = \frac{\pi}{16}$ for $L = 3$ in regime 2 has been of particular interest. Our results for the eight eigenvalue expressions obtained in Section 5 can be summarised in the compact form

$$r_j(w) = w^{n(a)} \prod_a \frac{E(-x^a/w)E(-x^{30+a}w)}{E(-x^a w)E(-x^{30+a}/w)}, \quad (6.6)$$

where the numbers $a$ and $n(a)$ are given in Table 2. The $E_8$ numbers $a$ have already appeared in [15] for the related Hamiltonian. The number $n(a)$ denotes the relevant band of eigenvalues.

### Table 2

Parameters appearing in the eigenvalue expression (6.6).

| $j$ | $n(a)$ | $a$          |
|-----|--------|--------------|
| 1   | 2      | 1, 11        |
| 2   | 2      | 7, 13        |
| 3   | 3      | 2, 10, 12    |
| 4   | 3      | 6, 10, 14    |
| 5   | 4      | 3, 9, 11, 13 |
| 6   | 4      | 6, 8, 10, 12 |
| 7   | 5      | 4, 8, 10, 12, 14 |
| 8   | 6      | 5, 7, 9, 11, 13, 15 |

In the original notation (6.6) reads

$$r_j(u) = \prod_a \vartheta_4\left(\frac{a\pi}{60} - \frac{8u}{15}, p^{8/15}\right) \vartheta_3\left(\frac{a\pi}{60} + \frac{8u}{15}, p^{8/15}\right) \vartheta_4\left(\frac{a\pi}{60} + \frac{8u}{15}, p^{8/15}\right), \quad (6.7)$$

where $\vartheta_3(u) = \vartheta_4(u + \frac{\pi}{2})$. The zero-momentum excitation energies of the related Hamiltonian, obtained by inverting the Fourier transforms in the ther-
odynamic Bethe Ansatz approach [11,15], are recovered on taking the logarithmic derivative of this expression evaluated at \( u = 0 \) and applying a Landen transformation.

The first correlation length is given by

\[
\xi_1^{-1} = 2 \log \left[ \frac{\vartheta_4\left(\frac{13\pi}{30}, p^{8/15}\right)}{\vartheta_4\left(\frac{2\pi}{15}, p^{8/15}\right) \vartheta_4\left(\frac{7\pi}{30}, p^{8/15}\right)} \right]. \tag{6.8}
\]

All eight fundamental correlation lengths can be written

\[
m_j = \xi_j^{-1} = 2 \sum_a \log \frac{\vartheta_4\left(\frac{2\pi a}{30} + \frac{\pi}{4}, p^{8/15}\right)}{\vartheta_4\left(\frac{2\pi a}{30} - \frac{\pi}{4}, p^{8/15}\right)}. \tag{6.9}
\]

In particular,

\[
m_j \sim 8 p^{8/15} \sum_a \sin \frac{a\pi}{30} \quad \text{as} \quad p \to 0. \tag{6.10}
\]

This is the formula obtained by McCoy and Orrick [15], from which the \( E_8 \) masses in (1.2) are recovered by virtue of trig identities.

We see that there is surprisingly little variation in the masses (6.9) as a function of the magnetic field-like variable \( p \). In the limit \( p \to 1 \) the mass ratios are given exactly by \( 1, 2, 2, 3, 3, 4, 5 \), which are to be compared with the \( E_8 \) values in (1.2).

### 6.4 Universal magnetic Ising amplitude

Making use of the Poisson summation formula in the free energy (3.19) at the magnetic Ising value \( \lambda = \frac{5\pi}{16} \) we find

\[
f \sim 4 \sqrt{3} \frac{\pi}{\cos \frac{\pi}{30}} p^{16/15} \quad \text{as} \quad p \to 0. \tag{6.11}
\]

On the other hand, from (6.8) we have

\[
\xi_1 \sim \frac{1}{8 \sqrt{3} \sin \frac{\pi}{5}} p^{-8/15} \quad \text{as} \quad p \to 0. \tag{6.12}
\]
Combining these results gives the universal magnetic Ising amplitude

\[
f \xi_1^2 = \frac{1}{16\sqrt{3}\sin \frac{\pi}{6} \cos \frac{\pi}{30}} = 0.061 \, 728 \, 589 \ldots \text{ as } p \to 0. \tag{6.13}
\]

This result has been predicted earlier by other means. Namely by thermodynamic Bethe Ansatz calculations based on the \( E_8 \) scattering theory\[26,5,7,27\] (see also Ref. \[28\] in the context of the form-factor bootstrap approach). Here it is obtained explicitly from the lattice model.

## 7 Conclusion

We have applied the exact perturbation approach to the Bethe Ansatz solution of the dilute \( A_L \) lattice model to derive the free energy per site and the excitation energies in regimes 1 and 2 for \( L \) odd. In the dilute \( A_L \) model the elliptic nome \( p \) is magnetic field-like for \( L \) odd and is temperature-like for \( L \) even. The particular point in regime 2 for \( L = 3 \) has attracted considerable recent attention, being in the same universality class as the Ising model in a magnetic field. We have specifically considered the case \( L \) odd for which the method perturbs from the ordered high field limit. Our result for the free energy (3.19) is in agreement with that obtained via the inversion relation method for general \( L \) \[8,9\].

In regime 1 the leading excitations in the \( w \) band are 1-strings, characterised by the string excitation parameter \( b \) and a single hole. Our final result for the related correlation length is given in (6.3). The excitations are considerably more complicated in regime 2, where we have concentrated on the case \( L = 3 \). Here our results for the leading eigenvalue associated with each mass are summarised in (6.6) and (6.7). The corresponding inverse correlation lengths or masses are given in (6.9), all of which give the magnetic Ising correlation length exponent \( \nu_h = \frac{8}{15} \). In particular, the \( E_8 \) masses (1.2) predicted by Zamolodchikov appear with the approach to criticality.

Of course these masses have been obtained earlier in the scaling limit via the thermodynamic Bethe Ansatz approach \[11\]. Although the calculations are somewhat complicated our approach nevertheless provides in principle a means of classifying and counting all excitations in the eigenvalue bands. We have not pursued this classification in detail, rather contenting ourselves with the location of the leading eigenvalue in each band, of relevance to the correlation lengths. As for the thermodynamic Bethe Ansatz, our calculations rely upon the key input of the string configurations. We have taken as our starting point the eight thermodynamically significant string excitation types revealed in the previous numerical studies \[11,16,17\]. The one exception is that
we have considered the 7-string configuration [17] for mass $m_4$ (see Table 1) rather than the 5-string [11]. However, although we have not included here our calculations based on the 5-string, we find that the 5-string configuration is also consistent in the ordered limit. Moreover, we find that both the 5-string and the 7-string lead to the same eigenvalue expression and thus mass. The origin of this behaviour is seen in the 7-string calculation where the string parameters $\alpha$ and $\beta$ (see eqn (B.30) do not appear in the final result (5.54). In both cases, it is the unpaired string at $bx^{16}$ which ultimately drives the calculation.

The derivation of the correlation length for $L \neq 3$ in regime 2 is complicated. In this regime the leading excitation in the $w$ band has $\ell = \frac{1}{2}(L - 1) + 1$ and, like the leading 2-string in the $w^2$ band for $L = 3$, it begins life for small $N$ and $p \simeq 0$ as a 1-string. We have not pursued this further. Nevertheless we have numerically observed that the final result (4.19) for regime 1 also applies to the leading $w$ band excitation in regime 2. We thus believe that the correlation length (6.8) and the corresponding exponents

$$\nu_h = \frac{r}{6s} = \frac{2(L + 1)}{3(L + 2)}$$  \hspace{1cm} (7.1)

also hold in regime 2 for $L \neq 3$. This result includes $\nu_h = \frac{8}{15}$ for $L = 3$.

The correlation length exponents are seen to satisfy the general scaling relation $2\nu_h = 1 + 1/\delta$, which follows from the general relation $f \xi^2 \sim \text{constant}$, where $f \sim p^{1+1/\delta}$ is the singular part of the bulk free energy and the exponents $\delta$ are those following from the singular behaviour of (3.19) [8,9]. The same correlation length exponents should hold for $L$ even, for which the integrable perturbation is thermal-like. The scaling relation is now $2\nu_t = 2 - \alpha$, where $\nu_t$ is as given in (6.5), (7.1) and $\alpha$ in [8,9]. In particular, (6.5) gives the Ising value $\nu_t = 1$ for $L = 2$ in regime 1, as expected.

Another approach to the results given here is to use the inversion relation method, as was originally used to obtain the bulk [8,9] and excess surface [10] free energies. This approach can also be applied to the excitations.\footnote{Yet another would involve integral equations for the root densities, as done for the eight-vertex model [29].} This has been done, for example, for the eight-vertex [30] and the Andrews-Baxter-Forrester models [31]. Although in principle completely avoiding the string hypothesis the inversion relation method still requires strong assumptions on the analyticity properties of the eigenvalues. Our explicit results for $r_j(u)$ give these analyticity properties \textit{a posteriori}. It remains to be seen if these properties can be established \textit{a priori}, thus allowing an alternative derivation.
of the mass gaps. We note that the inversion relation

$$r_j(u) r_j(u + 3\lambda) = 1$$

(7.2)

is indeed satisfied by our results. Alternatively, $r_j(w) r_j(x^{30}w) = 1$ for $L = 3$ in regime 2, which is seen to hold trivially in view of (6.6). There is a further relation

$$r_j(u) r_j(u + 2\lambda) = r_j(u + \lambda),$$

(7.3)

which is also easily seen to be satisfied by our results.

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5 We thank A. Klümper for pointing this relation out to us.
Appendices

A Auxiliary functions

The simplest identities for the auxiliary functions (4.6)-(4.8) are

\[ \frac{X(x^n w)}{X(x^{n+2r} w)} = (1 - x^n w/b), \quad (A.1) \]
\[ \frac{Y(x^{n-2r} / w)}{Y(x^n / w)} = (1 - x^n b/w), \quad (A.2) \]
\[ \frac{R(x^n w)}{R(x^{n+2r} w)} = (1 - x^n w/a_N), \quad (A.3) \]
\[ \frac{S(x^{n-2r} / w)}{S(x^n / w)} = (1 - x^n a_N/w). \quad (A.4) \]

When solving the recurrence relations (in either regime), their generalisations,

\[ \prod_{m=0}^{\infty} \frac{X(x^{12ms+n} w)}{X(x^{12ms+n+2r} w)} = \prod_{m=0}^{\infty} (1 - x^{12ms+n} w/b), \quad (A.5) \]
\[ \prod_{m=0}^{\infty} \frac{Y(x^{12ms+n-2r} / w)}{Y(x^{12ms+n} / w)} = \prod_{m=0}^{\infty} (1 - x^{12ms+n} b/w), \quad (A.6) \]

are used. In regime 1 we also require

\[ \prod_{m=0}^{\infty} \frac{R(x^{12ms+n} w)}{R(x^{12ms+n+2r} w)} = \prod_{m=0}^{\infty} (1 - x^{12ms+n} w/a_N), \quad (A.7) \]
\[ \prod_{m=0}^{\infty} \frac{S(x^{12ms+n-2r} / w)}{S(x^{12ms+n} / w)} = \prod_{l=0}^{\infty} (1 - x^{12ms+n} a_N/w). \quad (A.8) \]

In regime 2 we treat only \( L = 3 \), so that \( s = 5 \) and \( r = 16 \). For mass \( m_4 \) we need the analogous identities,

\[ \prod_{m=0}^{\infty} \frac{R_4(x^{60m+n} w)}{R_4(x^{60m+n+32} w)} = \prod_{m=0}^{\infty} (1 - x^{60m+n} w/a_{N-6}), \quad (A.9) \]
\[ \prod_{m=0}^{\infty} \frac{S_4(x^{60m+n-32} / w)}{S_4(x^{60m+n} / w)} = \prod_{l=0}^{\infty} (1 - x^{12ms+n} a_{N-6}/w). \quad (A.10) \]

Of course, the regime 2 auxiliary functions \( X_\alpha(w) \) and \( X_\beta(w) \) obey the same identities as \( X(w) \), with \( b \) therein replaced by \( \alpha \) and \( \beta \), respectively. Simi-
larly $Y_\alpha(1/w)$ and $Y_\beta(1/w)$ obey the same identities as $Y(1/w)$, with again $b$ replaced by $\alpha$ and $\beta$ as appropriate.

When substituting $F$ and $G$ into the $k = N$ Bethe equation in regime 1 we use

\[
\prod_{i=0}^{\infty} \frac{X(x(2i-4)s+n+w)}{X(x(2i-4)s+n+2w)} = \prod_{i=0}^{\infty} \left(1 - x(2i-4)s+n+w/b\right), \quad (A.11)
\]
\[
\prod_{i=0}^{\infty} \frac{Y(x(2i-4)s+n-2r+w)}{Y(x(2i-4)s+n+w)} = \prod_{i=0}^{\infty} \left(1 - x(2i-4)s+n+b/w\right), \quad (A.12)
\]
\[
\prod_{i=0}^{\infty} \frac{R(x(2i-4)s+n+w)}{R(x(2i-4)s+n+2w)} = \prod_{i=0}^{\infty} \left(1 - x(2i-4)s+n+w/a_N\right), \quad (A.13)
\]
\[
\prod_{i=0}^{\infty} \frac{S(x(2i-4)s+n-2r+w)}{S(x(2i-4)s+n+w)} = \prod_{i=0}^{\infty} \left(1 - x(2i-4)s+n+a_N/w\right). \quad (A.14)
\]

To combine the Bethe equations from the strings in Appendix B, once the functions $F_i$ and $G_i$ have been found, we use

\[
\prod_{m=0}^{\infty} \frac{A(x^{60m+n}/b)}{A(x^{60m+n+32}/b)} = \prod_{m=0}^{\infty} \left(1 - x^{60m+n}/b\right)^N, \quad (A.15)
\]
\[
\prod_{m=0}^{\infty} \frac{A(x^{60m+n}/b)}{A(x^{60m+n+32}/b)} = \prod_{m=0}^{\infty} \left(1 - x^{60m+n}/b\right)^N, \quad (A.16)
\]
\[
\prod_{m=0}^{\infty} \frac{A(x^{60m+60+n}/b)}{A(x^{60m+60+n+32}/b)} = X(x^n)^{-N} \prod_{m=0}^{\infty} \left(1 - x^{60m+60+n}/b\right)^N, \quad (A.17)
\]
\[
\prod_{m=0}^{\infty} \frac{A(x^{60m+60+n}/b)}{A(x^{60m+60+n+32}/b)} = Y(x^n)^{-N} \prod_{m=0}^{\infty} \left(1 - x^{60m+60+n}/b\right)^N. \quad (A.18)
\]

These last four relations are just special cases of more general identities (which we do not require) which would have $12s$ in place of $60$ and $2r$ in place of $32$.

### B Bethe equations for the masses

In this Appendix we consider the Bethe equations for the eight string types corresponding to the $E_8$ masses. Their chief usefulness is to establish analytically that the coefficient of each member of a string is the same. (In the case of $m_4$ and $m_6$, these equations give equality only between subsets of the coefficients, and this feature was explicitly included in the calculations in the body of the paper.) Secondly, once the auxiliary functions (which give the remainder of the Bethe roots, the $a_k$) are substituted into the string equations,
it is possible to establish a condition on the string coefficient $b$ in which the same patterns in the exponents of $x$ arise as do in the eigenvalues (or masses) themselves. Such higher level Bethe equations for the string parameters have been discussed, for example, for the CSOS model [20]. In each case these equations ensure that the corresponding eigenvalue expression reduces to an $N$th root of unity at $w = 1$. This must be so, as the row transfer matrix reduces to a shift operator at this point.

Because some of the strings are quite long, it will be convenient to introduce the notation $\prod_{i=1}^{m} b_i = B_m$.

### B.1 Mass $m_1$

With the roots $w_j = a_j$ for $j = 1, \ldots, N - 2$ and $w_{N-1} = b_1 x^{-11}$, $w_N = b_2 x^{11}$ the Bethe equations for $k = N - 1$ and $k = N$ are

$$\begin{align*}
-\omega \left[ x^{-10} \frac{E(x^{21}/b_1)}{E(x/b_1)} \right]^N &= (A_{N-2}B_2)^{-3/8} b_1^{N} x^{-N-18} \\
\times &\frac{E(x^{10} b_2/b_1) E(x^{12} b_2/b_1)}{E(b_1/b_2) E(x^{2} b_2/b_1)} \prod_{j=1}^{N-2} \frac{E(x a_j/b_1) E(x^{31} a_j/b_1)}{E(x^{21} a_j/b_1) E(x^{33} a_j/b_1)},
\end{align*}$$

(B.1)

$$\begin{align*}
-\omega \left[ x^{10} \frac{E(x b_2)}{E(x^{21} b_2)} \right]^N &= (A_{N-2}B_2)^{-3/8} b_2^{N} x^{N+18} \\
\times &\frac{E(b_1/b_2) E(x^{2} b_2/b_1)}{E(x^{10} b_2/b_1) E(x^{12} b_2/b_1)} \prod_{j=1}^{N-2} \frac{E(x^{21} b_2/a_j) E(x^{9} a_j/b_2)}{E(x a_j/b_2) E(x^{31} a_j/b_2)}.
\end{align*}$$

(B.2)

Taken individually as $x \to 0$, these equations imply that

$$1 - \frac{b_1}{b_2} = O(x^{9N}) \Rightarrow \frac{b_1}{b_2} = 1 + O(x^{9N}).$$

(B.3)

Thus in the thermodynamic ($N \to \infty$) limit $b_1 = b_2 = b$.

Forming the product of (B.1) and (B.2), we see that many factors cancel and those remaining can be written in terms of the auxiliary functions (5.5) and (5.6) for $m_1$, namely

$$\left[ \frac{E(x b) E(x^{21}/b)}{E(x/b) E(x^{21} b)} \right]^N = \frac{b^{2N}}{\mathcal{F}_1(x^{9} b) \mathcal{F}_1(x^{11} b) \mathcal{G}_1(1/x^{9} b) \mathcal{G}_1(1/x^{11} b)}.$$
Using the functions (5.9) and (5.10) and identities from Appendix A, we obtain

\[
\left[ \frac{E(xb, x^{60})E(x^{11}b, x^{60})E(x^{31}b, x^{60})E(x^{41}b, x^{60})}{E(x/b, x^{60})E(x^{11}/b, x^{60})E(x^{31}b, x^{60})E(x^{41}b, x^{60})} \right]^N = b^{2N}.
\]

(B.5)

This defining relationship is the higher level Bethe equation to be satisfied by the string parameter \( b \). Note that it is trivially satisfied by \( b = -1 \). Compare also the pattern of exponents in this equation with the expression (5.12) for \( \Lambda_1 \). Taken together they ensure that the eigenvalue is an \( N \)th root of unity at \( w = 1 \).

**B.2 Mass \( m_2 \)**

With the roots \( w_j = a_j \) for \( j = 1, \ldots, N - 4 \) and \( w_{N-3} = b_1 x^{-5} \), \( w_{N-2} = b_2 x^5 \), \( w_{N-1} = b_3 x^{-15} \), \( w_N = b_4 x^{15} \) the Bethe equations for \( k = N-3, N-2, N-1, N \), respectively, are

\[
-\omega \left[ b_1 x^{-5} \frac{E(x^{15}/b_1)}{E(x^{5}b_1)} \right]^N = (A_{N-4} B_4)^{5/8} b_1^2 b_4^{-2} x^{-18}
\]

\[
\times \frac{E(b_1/b_2)E(x^{30}b_2/b_1)E(x^{10}b_3/b_1)E(x^{20}b_1/b_3)E(x^{8}b_4/b_1)E(x^{10}b_4/b_1)}{E(x^{10}b_1/b_2)E(x^{20}b_2/b_1)E(b_3/b_1)E(x^{30}b_1/b_3)E(b_1/b_4)E(x^{30}b_4/b_1)}
\]

\[
\times \prod_{j=1}^{N-4} \frac{E(x^{5}b_1/a_j)E(x^{25}a_j/b_1)}{E(x^{15}a_j/b_1)E(x^{15}b_1/a_j)},
\]

(B.6)

\[
-\omega \left[ b_2 x^5 \frac{E(x^{5}/b_2)}{E(x^{5}b_2)} \right]^N = (A_{N-4} B_4)^{5/8} b_2^2 b_3^{-2} x^{18}
\]

\[
\times \frac{E(x^{10}b_1/b_2)E(x^{20}b_2/b_1)E(b_3/b_2)E(x^{30}b_2/b_3)E(b_2/b_4)E(x^{30}b_4/b_2)}{E(b_1/b_2)E(x^{30}b_2/b_1)E(x^{8}b_3/b_2)E(x^{10}b_2/b_3)E(x^{10}b_4/b_2)}
\]

\[
\times \prod_{j=1}^{N-4} \frac{E(x^{15}b_2/a_j)E(x^{15}a_j/b_2)}{E(x^{5}a_j/b_2)E(x^{25}b_2/a_j)},
\]

(B.7)

\[
-\omega \left[ x^{-10} \frac{E(x^{25}/b_3)}{E(x^{5}/b_3)} \right]^N = (A_{N-4} B_4)^{5/8} A_{N-4}^{-2} b_3^{2N-6} b_2^{-2} x^{-8N-6}
\]

\[
\times \frac{E(b_3/b_1)E(x^{30}b_1/b_3)E(x^{8}b_2/b_3)E(x^{10}b_2/b_3)E(x^{20}b_4/b_3)E(x^{18}b_4/b_3)}{E(x^{10}b_3/b_1)E(x^{20}b_1/b_3)E(b_3/b_2)E(x^{30}b_2/b_3)E(x^{8}b_4/b_3)E(x^{10}b_4/b_3)}
\]

\[
\times \prod_{j=1}^{N-4} \frac{E(x^3a_j/b_3)E(x^{5}a_j/b_3)}{E(x^{25}a_j/b_3)E(x^{5}b_3/a_j)},
\]

(B.8)
\(-\omega \left[ x^{10} \frac{E(x^5 b_4)}{E(x^{25} b_4)} \right]^N = (A_{N-4} B_4)^{5/8} A_{N-4}^{-2} b_{N-4}^{2N-6} b_1^{-2} x^{8N+6} \)

\[
\times \frac{E(b_1/b_4) E(x^{30} b_4/b_1) E(x^{10} b_2/b_4) E(x^{20} b_1/b_3) E(x^{10} b_4/b_3) E(x^{30} b_4/b_1) E(x^{10} b_3/b_2) E(x^{20} b_4/b_3) E(x^{18} b_4/b_3) \times \prod_{j=1}^{N-4} \frac{E(x^3 a_j/b_4) E(x^5 a_j/b_4)}{E(x^{25} a_j/b_4) E(x^5 a_j/b_4)}.
\]

(B.9)

Taken individually as \( x \to 0 \), these equations imply that

\[
\begin{align*}
\frac{(1 - \frac{b_1}{b_4}) (1 - \frac{b_2}{b_4})}{(1 - \frac{b_1}{b_2})} &= O(x^{5N}), & \frac{(1 - \frac{b_1}{b_3})}{(1 - \frac{b_1}{b_4})} &= O(x^{2N}), \\
\frac{(1 - \frac{b_2}{b_5}) (1 - \frac{b_3}{b_5})}{(1 - \frac{b_2}{b_3})} &= O(x^{5N}), & \frac{(1 - \frac{b_3}{b_4})}{(1 - \frac{b_3}{b_5})} &= O(x^{2N}).
\end{align*}
\]

(B.10)

We cannot read off the relationship between the \( b_i \) in this case, as we could for \( m_1 \), but we see that these relations are satisfied if

\[
b_1 = b_4 + O(x^{5N}), \quad b_2 = b_2 + O(x^{3N}), \quad b_3 = b_4 + O(x^{3N}).
\]

(B.11)

Thus we conclude \( b_1 = b_2 = b_3 = b_4 = b \) and \( B_4 = b^4 \).

The product of the four Bethe equations can be written in terms of the auxiliary functions (5.19) for \( m_2 \), with

\[
\left[ \frac{E(x^{15}/b) E(x^{25}/b)}{E(x^{15} b) E(x^{25} b)} \right]^N = b^{2N} \left[ \mathcal{F}_2(x^3 b) \mathcal{F}_2(x^5 b) \mathcal{F}_2(x^{15} b) \mathcal{F}_2(x^{17} b) \right]^{-1} \times \left[ \mathcal{G}_2(1/x^3 b) \mathcal{G}_2(1/x^5 b) \mathcal{G}_2(1/x^{15} b) \mathcal{G}_2(1/x^{17} b) \right]^{-1}.
\]

(B.12)

Using the functions (5.21) and (5.22) and identities from Appendix A, we obtain

\[
\left[ \frac{E(x^7 b, x^{60}) E(x^{13} b, x^{60}) E(x^{37} b, x^{60}) E(x^{43} b, x^{60})}{E(x^7/b, x^{60}) E(x^{13}/b, x^{60}) E(x^{37}/b, x^{60}) E(x^{43}/b, x^{60})} \right]^N = b^{2N}.
\]

(B.13)

### B.3 Mass \( m_3 \)

With the roots \( w_j = a_j \) for \( j = 1, \ldots, N-4 \) and \( w_{N-3} = b_1 x^{-10}, w_{N-2} = b_2 x^{10}, w_{N-1} = b_3 x^{-20}, w_N = b_4 x^{20} \) the Bethe equations for \( k = N-3, N-2, N-1, N \)
\[-\omega \left[ b_1 x^{-10} \frac{E(x^{20}/b_1)}{E(b_1)} \right]^N = (A_{N-4} B_4)^{5/8} b_2^2 b_1^2 x^{-38} \]
\[-\omega \left[ x^{10} \frac{E(b_2)}{E(x^{20}b_2)} \right]^N = (A_{N-4} B_4)^{5/8} b_2^2 b_1^2 x^{-38} \]
\[-\omega \left[ x^{-10} \frac{E(x^{30}/b_3)}{E(x^{10}b_3)} \right]^N = (A_{N-4} B_4)^{5/8} A_{N-4}^2 b_3^2 A_{N-4}^{2N-8} x^{-18N + 32} \]
\[-\omega \left[ x^{10} \frac{E(x^{10}b_4)}{E(x^{20}b_4)} \right]^N = (A_{N-4} B_4)^{5/8} A_{N-4}^2 b_4^2 A_{N-4}^{2N-8} x^{18N - 32} \]

Taken individually as \( x \to 0 \), these equations imply that

\[
(1 - \frac{b_1}{b_2})(1 - \frac{b_3}{b_1}) = O(x^{10N}), \quad (1 - \frac{b_3}{b_1}) = O(x^{8N}), \\
(1 - \frac{b_1}{b_2})(1 - \frac{b_2}{b_4}) = O(x^{10N}), \quad (1 - \frac{b_2}{b_4}) = O(x^{8N}).
\]
from which we can read off

\[ b_1 = b_3 + O(x^{8N}), \quad b_2 = b_4 + O(x^{8N}), \quad b_1 = b_2 + O(x^{2N}). \] (B.19)

Thus we conclude \( b_1 = b_2 = b_3 = b_4 = b \) and \( B_4 = b^4 \).

Expressing the product of the four Bethe equations in terms of the \( m_3 \) auxiliary functions (5.30) gives

\[
\left[ \frac{E(x^{20}/b)E(x^{22}/b)E(x^{30}/b)}{E(x^{20}/b)E(x^{22}/b)E(x^{30}/b)} \right]^N = b^{3N} \left[ F_3(x^8/b)F_3(x^{10}/b)G_3(1/x^8/b)G_3^2(1/x^{10}/b)G_3(1/x^{12}/b) \right]^{-1}. \] (B.20)

Using the functions (5.32) and (5.33) and identities from Appendix A we obtain, in terms of elliptic functions (of nome \( x^{60} \), not \( x^{32} \)),

\[
\left[ \frac{E(x^{10}/b)E(x^{12}/b)E(x^{32}/b)E(x^{40}/b)E(x^{42}/b)}{E(x^{2}/b)E(x^{10}/b)E(x^{12}/b)E(x^{32}/b)E(x^{40}/b)E(x^{42}/b)} \right]^N = b^{3N}. \] (B.21)

B.4 Mass \( m_4 \)

With the roots \( w_j = a_j \) for \( j = 1, \ldots, N - 7 \) and \( w_{N-6} = b_1 x^{-1} \), \( w_{N-5} = b_2 x \), \( w_{N-4} = b_3 x^{-7} \), \( w_{N-3} = b_4 x^7 \), \( w_{N-2} = b_5 x^{-13} \), \( w_{N-1} = b_6 x^{13} \), \( w_N = b_7 x^{16} \) the Bethe equations for \( k = N - 6, \ldots, N \) are

\[
-\omega \left[ b_1 x^{-1} \frac{E(x^{11}/b_1)}{E(x^{9}/b_1)} \right]^N = (A_{N-7} B_7)^{5/8} b_1 a_7^{2} (b_5 b_6 b_7)^{-2} x^{-6} \\
\times \frac{E(x^{8}/b_2)E(x^{10}/b_1/b_2)E(x^{14}/b_3/b_1)E(x^{16}/b_1/b_3)E(x^{2}/b_4/b_1)E(x^{28}/b_4/b_1)}{E(x^{12}/b_2/b_1)E(x^{18}/b_1/b_2)E(x^{4}/b_3/b_1)E(x^{26}/b_1/b_3)E(x^{18}/b_4/b_1)E(x^{20}/b_4/b_1)} \\
\times \frac{E(x^{8}/b_5/b_1)E(x^{10}/b_5/b_1)E(x^{4}/b_6/b_1)E(x^{2}/b_7/b_1)E(x^{5}/b_7/b_1)}{E(x^{12}/b_6/b_1)E(x^{18}/b_1/b_6)E(x^{4}/b_7/b_1)E(x^{2}/b_1/b_6)E(x^{5}/b_1/b_7)} \\
\times \prod_{j=1}^{N-7} \frac{E(x^{9}/b_1/a_j)E(x^{21}/a_j/b_1)}{E(x^{11}/a_j/b_1)E(x^{19}/b_1/a_j)}.
\] (B.22)

\[
-\omega \left[ b_2 \frac{E(x^{9}/b_2)}{E(x^{11}/b_2)} \right]^N = (A_{N-7} B_7)^{5/8} b_2^{2} (b_5 b_6 b_7)^{-2} x^{-6} \\
\times \frac{E(x^{12}/b_2/b_1)E(x^{18}/b_1/b_2)E(x^{18}/b_3/b_3)E(x^{12}/b_3/b_2)E(x^{26}/b_2/b_4)E(x^{20}/b_2/b_4)}{E(x^{8}/b_1/b_2)E(x^{10}/b_1/b_2)E(x^{2}/b_3/b_2)E(x^{28}/b_2/b_3)E(x^{16}/b_4/b_2)E(x^{18}/b_4/b_2)}
\]
\[
\times E(x^8b_2/b_5) E(x^6b_5/b_2) E(x^2b_6/b_2) E(b_2/b_6) E(x^5b_7/b_2) E(x^3b_7/b_2) \\
E(x^4b_2/b_5) E(x^2b_2/b_5) E(x^8b_2/b_6) E(x^{10}b_2/b_6) E(x^7b_2/b_7) E(x^5b_2/b_7)
\]
\[
\times \prod_{j=1}^{N-7} \frac{E(x^{11}b_2/a_j) E(x^{19}a_j/b_2)}{E(x^9a_j/b_2) E(x^{21}b_2/a_j)}, \tag{B.23}
\]

\[
-\omega \left[ b_3 x^{-7} \frac{E(x^{17}/b_3)}{E(x^3b_3)} \right]^N = (A_{N-7}B_7)^{5/8} b_3^{4} b_4 b_6 - 2 x^{-34}
\]
\[
\times E(x^4b_3/b_1) E(x^{26}b_1/b_3) E(x^2b_2/b_2) E(x^{28}b_2/b_3) E(x^4b_4/b_3) E(x^2b_4/b_3)
\]
\[
E(x^{14}b_3/b_1) E(x^{16}b_1/b_3) E(x^{18}b_3/b_3) E(x^{12}b_3/b_2) E(x^6b_3/b_4) E(x^8b_3/b_4)
\]
\[
\times \frac{E(x^4b_5/b_3) E(x^{16}b_5/b_3) E(x^{10}b_6/b_3) E(x^8b_6/b_3) E(x^{13}b_7/b_3) E(x^{11}b_7/b_3)}{E(x^3b_5/b_3) E(x^6b_5/b_4) E(x^2b_3/b_4) E(x^6b_7/b_3) E(x^2b_7/b_3) E(x^3b_7/b_3)}
\]
\[
\times \prod_{j=1}^{N-7} \frac{E(x^{3}b_3/a_j) E(x^{27}a_j/b_3)}{E(x^{17}a_j/b_3) E(x^{13}b_3/a_j)}, \tag{B.24}
\]

\[
-\omega \left[ b_4 x^{7} \frac{E(x^3/b_4)}{E(x^4b_4)} \right]^N = (A_{N-7}B_7)^{5/8} b_4^{4} (b_5 b_3) - 2 x^{-34}
\]
\[
\times E(x^{18}b_4/b_1) E(x^{20}b_4/b_1) E(x^{16}b_4/b_2) E(x^{18}b_4/b_2) E(x^6b_3/b_4) E(x^8b_3/b_4)
\]
\[
E(x^2b_4/b_1) E(x^{28}b_4/b_1) E(x^2b_4/b_4) E(x^{20}b_4/b_2) E(x^4b_4/b_3) E(x^2b_4/b_3)
\]
\[
\times \frac{E(x^2b_5/b_4) E(x^6b_4/b_4) E(x^4b_4/b_4) E(x^6b_4/b_6) E(x^4b_4/b_7) E(x^3b_4/b_7)}{E(x^{10}b_4/b_5) E(x^8b_4/b_5) E(x^{16}b_4/b_6) E(x^{14}b_4/b_6) E(x^{11}b_4/b_7) E(x^{13}b_4/b_7)}
\]
\[
\times \prod_{j=1}^{N-7} \frac{E(x^{17}b_4/a_j) E(x^{13}a_j/b_4)}{E(x^3a_j/b_4) E(x^{27}b_4/a_j)}, \tag{B.25}
\]

\[
-\omega \left[ x^{-10} \frac{E(x^{23}/b_5)}{E(x^3b_5)} \right]^N = (A_{N-7}B_7)^{5/8} A_{N-7}^{-2} b_5^{2N-8} (b_1 b_2 b_4) - 2 x^{-28-4N}
\]
\[
\times E(b_1/b_5) E(x^2b_1/b_5) E(x^4b_2/b_5) E(x^2b_2/b_5) E(x^4b_3/b_5) E(x^6b_5/b_3)
\]
\[
E(x^8b_5/b_1) E(x^{10}b_5/b_1) E(x^8b_2/b_5) E(x^6b_5/b_2) E(x^{14}b_5/b_3) E(x^{16}b_5/b_3)
\]
\[
\times \frac{E(x^4b_5/b_4) E(x^8b_4/b_5) E(x^{16}b_6/b_5) E(x^{14}b_6/b_5) E(x^{10}b_7/b_5) E(x^{17}b_7/b_5)}{E(x^2b_5/b_4) E(x^6b_5/b_4) E(x^6b_6/b_5) E(x^7b_7/b_5) E(x^{9}b_7/b_5)}
\]
\[
\times \prod_{j=1}^{N-7} \frac{E(x^3a_j/b_5) E(xa_j/b_5)}{E(x^{23}a_j/b_5) E(x^7a_j/b_5)}, \tag{B.26}
\]

\[
-\omega \left[ x^{10} \frac{E(x^3b_6)}{E(x^3b_6)} \right]^N = (A_{N-7}B_7)^{5/8} A_{N-7}^{-2} b_6^{2N-8} (b_1 b_2 b_3) - 2 x^{28+4N}
\]
\[
\times E(x^6b_1/b_6) E(x^8b_1/b_6) E(x^{10}b_2/b_6) E(x^{10}b_2/b_6) E(x^2b_3/b_6) E(b_3/b_6)
\]
\[
E(x^4b_6/b_1) E(x^4b_6/b_1) E(x^2b_6/b_2) E(x^2b_6/b_2) E(b_2/b_6) E(x^{10}b_6/b_3) E(x^8b_6/b_3)
\]
$$\times E(x^{16} b_4/b_6) E(x^{14} b_4/b_6) E(x^4 b_4/b_5) E(x^5 b_6/b_5) E(x^7 b_6/b_7) E(x^{23} b_7/b_6)$$
$$\times \frac{N-7}{N} E(x^{23} b_6/a_j) E(x^7 a_j/b_6)$$
$$\times \frac{E(x^{26} b_6/a_j) E(x^4 a_j/b_7)}{E(x^{26} b_7/a_j) E(x^4 b_7/a_j)}.$$  \hfill (B.27)

$$-\omega \left[ \frac{E(x^6 b_7)}{E(x^{26} b_7)} \right]^N = \left( A_{N-7} B_7 \right)^{5/8} A_{N-7}^2 b_7^{N-10} (b_1 b_2)^{-2}$$
$$\times \frac{E(x^3 b_1/b_7) E(x^5 b_1/b_7) E(x^7 b_2/b_7) E(x^5 b_2/b_7) E(x^7 b_7/b_3) E(x^3 b_7/b_3)}{E(x^7 b_7/b_1) E(x^5 b_7/b_1) E(x^5 b_7/b_2) E(x^3 b_7/b_2) E(x^{13} b_7/b_3) E(x^{11} b_7/b_3)}$$
$$\times \frac{E(x^{11} b_4/b_7) E(x^{13} b_4/b_7) E(x^7 b_7/b_5) E(x^9 b_7/b_5) E(x^{13} b_7/b_6) E(x^{17} b_7/b_6)}{E(x^7 b_4/b_7) E(x^7 b_7/b_5) E(x^{17} b_7/b_5) E(x^7 b_7/b_6) E(x^{23} b_7/b_6)}$$
$$\times \frac{N-7}{N} E(x^{26} b_6/a_j) E(x^4 a_j/b_7)$$
$$\times \frac{E(x^6 b_7/a_j) E(x^4 b_7/a_j)}{E(x^{26} b_7/a_j) E(x^4 b_7/a_j)}. \quad \hfill (B.28)$$

Taken individually as \(x \to 0\) these equations imply

\[
(1 - \frac{b_1}{b_5}) = O(x^N), \quad (1 - \frac{b_2}{b_6}) = O(x^N),
\]
\[
(1 - \frac{b_3}{b_4}) = O(x^7N), \quad (1 - \frac{b_4}{b_6}) = O(x^7N),
\]
\[
(1 - \frac{b_5}{b_4}) = O(x^6N), \quad (1 - \frac{b_5}{b_6}) = O(x^6N), \quad \text{B.29}
\]

which fall into two unconnected groups. Notice also that the seventh equation \((B.28)\) does not provide any link between \(b_7\) and the other \(b_i\). We are only able to conclude that

\[b_1 = b_4 = b_5 = \alpha, \quad b_2 = b_3 = b_6 = \beta, \quad \text{and} \quad b_7 = b, \quad \text{B.30}\]

with \(B_7 = \alpha^3 \beta^3 b\).

**B.5 Mass \(m_5\)**

We consider the roots \(w_j = a_j\) for \(j = 1, \ldots, N - 6\) and \(w_{N-5} = b_1 x^{-13}\), \(w_{N-4} = b_2 x^{13}\), \(w_{N-3} = b_3 x^{-11}\), \(w_{N-2} = b_4 x^{11}\), \(w_{N-1} = b_5 x^{-9}\), \(w_N = b_6 x^9\). In the \(x \to 0\) limit, the Bethe equations for the latter six roots are

\[x^{-10N} = \frac{-1}{\omega} (A_{N-6} B_6)^{5/8} A_{N-6}^{-2} \beta_1^{-2N-12} x^{-4N-36} E(b_6/b_1)^{-1},\]

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The parameter $b$ where the elliptic functions are of nome $x$ is thus found to obey the equation

$$x^{10N} = -\frac{1}{\omega} (A_{N-6}B_6)^{5/8} A_{N-6}^{-2} b_2^{2N-12} x^{4N+36} E(b_2/b_5),$$

$$x^{-10N} = -\frac{1}{\omega} (A_{N-6}B_6)^{5/8} A_{N-6}^{-1} b_3^{N-4} b_2^{-2} x^{-N-52} [E(b_3/b_6) E(b_4/b_3)]^{-1},$$

$$x^{10N} = -\frac{1}{\omega} (A_{N-6}B_6)^{5/8} A_{N-6}^{-1} b_4^{N-4} b_2^{-2} x^{N+52} E(b_4/b_3) E(b_5/b_4),$$

$$[b_5 x^{-9}]^N = -\frac{1}{\omega} (A_{N-6}B_6)^{5/8} b_5^2 (b_4 b_6)^{-2} x^{-52} [E(b_5/b_4) E(b_2/b_5)]^{-1},$$

$$[b_6 x^9]^N = -\frac{1}{\omega} (A_{N-6}B_6)^{5/8} b_6^4 (b_3 b_5)^{-2} x^{52} E(b_6/b_1) E(b_3/b_6). \quad (B.31)$$

We conclude from the first, third and sixth of these that

$$b_1 = b_6 + O(x^{6N}), \quad b_3 = b_6 + O(x^{3N}), \quad b_3 = b_4 + O(x^{6N}). \quad (B.32)$$

From the second and fifth we have

$$b_2 = b_5 + O(x^{6N}), \quad b_4 = b_5 + O(x^{3N}). \quad (B.33)$$

The fourth confirms $b_3 = b_4 + O(x^{6N})$, so that we may conclude that the $b_i$ are equal with $B_6 = b^6$.

In terms of the $m_5$ auxiliary functions (5.63) and (5.64) for the product of the six Bethe equations give

$$\left[ \frac{E(x^{19}/b) E(x^{21}/b) E(x^{23}/b) E(x^{29}/b)}{E(x^{19}b) E(x^{21}b) E(x^{23}b) E(x^{29}b)} \right]^N = b^{4N} \left[ \mathcal{F}_5(x^7b)^2 \mathcal{F}_5^2(x^9b) \mathcal{F}_5^2(x^{11}b) \mathcal{F}_5(x^{13}b) \right]^{-1} \times \left[ \mathcal{G}_5(1/x^7b)^2 \mathcal{G}_5^2(1/x^9b) \mathcal{G}_5^2(1/x^{11}b) \mathcal{G}_5(1/x^{13}b) \right]^{-1}. \quad (B.34)$$

The parameter $b$ is thus found to obey the equation

$$\left[ \frac{E(x^3b) E(x^9b) E(x^{11}b) E(x^{13}b)}{E(x^3b) E(x^9b) E(x^{11}b) E(x^{13}b)} \right]^N \times \left[ \frac{E(x^{31}/b) E(x^{39}/b) E(x^{41}/b) E(x^{43}/b)}{E(x^{31}b) E(x^{39}b) E(x^{41}b) E(x^{43}b)} \right]^N = b^{4N}, \quad (B.35)$$

where the elliptic functions are of nome $x^{60}$. 

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We consider the roots $w_j = a_j$ for $j = 1, \ldots, N - 7$ and $w_{N-6} = b_1x^{-14}$, $w_{N-5} = b_2x^{14}$, $w_{N-4} = b_3x^{-10}$, $w_{N-3} = b_4x^{10}$, $w_{N-2} = b_5x^{-6}$, $w_{N-1} = b_6x^6$, $w_N = b_7x^{16}$. In the $x \to 0$ limit, the Bethe equations for the latter seven roots are

\[
x^{-10N} = -\frac{1}{\omega}(A_{N-7}B_7)^{5/8}A_{N-7}^{-2}b_{1N}^{-1}b_0^{-2}x^{-6N-26}E(b_1/b_0)^{-1},
\]

\[
x^{10N} = -\frac{1}{\omega}(A_{N-7}B_7)^{5/8}A_{N-7}^{-2}b_2^{-12}b_5^{-12}x^{6N+26}E(b_5/b_2),
\]

\[
[b_3x^{-10}]^N = -\frac{1}{\omega}(A_{N-7}B_7)^{5/8}b_3^4(b_4b_0)^{-2}x^{-58}E(b_3/b_4)^{-1},
\]

\[
[b_4x^{10}]^N = -\frac{1}{\omega}(A_{N-7}B_7)^{5/8}b_4^4(b_3b_5)^{-2}x^{58}E(b_3/b_4),
\]

\[
[b_5x^{-6}]^N = -\frac{1}{\omega}(A_{N-7}B_7)^{5/8}b_5^6(b_2b_4b_6)^{-2}x^{-40}E(b_6/b_5)
\]

\[
\times \frac{E(b_1/b_0)E(b_6/b_5)}{E(b_7/b_5)E(b_5/b_2)},
\]

\[
[b_6x^{-6}]^N = -\frac{1}{\omega}(A_{N-7}B_7)^{5/8}b_6^6(b_1b_3b_5)^{-2}x^{-40}E(b_1/b_0)
\]

\[
\times \frac{E(b_6/b_5)}{E(b_7/b_5)E(b_5/b_2)},
\]

\[
x^{10N} = -\frac{1}{\omega}(A_{N-7}B_7)^{5/8}E(b_7/b_5)
\]

\[
\times \frac{E(b_1/b_0)}{E(b_6/b_7)}.
\]

We notice immediately that the third and fourth equations decouple from the others to give $b_3 = b_4 + O(x^{10N})$, so we set $b_3 = b_4 = \alpha$. From the remaining five equations we may conclude that (to $O(x^{2N})$ or $O(x^{4N})$) the remaining $b_i$ are equal, with $B_7 = b^5\alpha^2$.

The product of the seven Bethe equations may be written in terms of the auxiliary functions (5.72) to give

\[
\left[\frac{E(x^{24}/b)E(x^{26}/b)E(x^{20}/\alpha)}{E(x^{24}/b)E(x^{26}/b)E(x^{20}/\alpha)}\right]^{N}
\]

\[
= b^{4N}\left[F_6(x^{18})F_6(x^{24})F_6(x^{26})F_6(x^{28})F_6(x^{30})F_6(x^{14})\right]
\]

\[
\times \left[\frac{G_6(1/x^{20})G_6(1/x^{28})G_6(1/x^{26})G_6(1/x^{24})G_6(1/x^{18})}{G_6(1/x^{14})G_6(1/x^{8})G_6(1/x^{6})G_6(1/x^{4})G_6(1/x^{2})}\right]
\]

\[
\times \left[\frac{F_6(x^{2})F_6(x^{20})F_6(x^{22})G_6(1/x^{2})G_6(1/x^{20})G_6(1/x^{22})}{F_6(x^{10})F_6(x^{12})F_6(x^{30})G_6(1/x^{10})G_6(1/x^{12})G_6(1/x^{30})}\right].
\]

(B.37)

Now, while some of these factors may be grouped to give the functions $F_6$ and $G_6$ in (5.73), for the others we must use
\[ F_6(w) = \prod_{l=0}^{\infty} F_6(x^{12l}w) \frac{X_\alpha(x^{12l+2}w)}{X_\alpha(x^{12l+10}w)}, \]
\[ G_6(1/w) = \prod_{l=0}^{\infty} G_6(x^{12l+1}/w)^{-1} \frac{Y_\alpha(x^{12l+2}/w)}{Y_\alpha(x^{12l+10}/w)}, \] (B.38)

which is similar to what we did in regime 1. By applying identities from Appendix A, we find that all factors involving \( \alpha \) cancel. The final result, in terms of elliptic functions of nome \( x^{60} \), is

\[
\left[ \frac{E(x^6b)E(x^8b)E(x^{12}b)E(x^{14}b)}{E(x^6/b)E(x^8/b)E(x^{12}/b)E(x^{14}/b)} \right]^N
\times \left[ \frac{E(x^{36}/b)E(x^{38}/b)E(x^{42}/b)E(x^{44}/b)}{E(x^{36}b)E(x^{38}b)E(x^{42}b)E(x^{44}b)} \right]^N = b^{4N}. \] (B.39)

B.7 Mass \( m_7 \)

For the roots \( w_j = a_j \) for \( j = 1, \ldots, N-8 \) and \( w_{N-7} = b_1 x^{-14}, w_{N-6} = b_2 x^{14}, w_{N-5} = b_3 x^{-12}, w_{N-4} = b_4 x^{12}, w_{N-3} = b_5 x^{-10}, w_{N-2} = b_6 x^{10}, w_{N-1} = b_7 x^{-8}, \) \( w_N = b_8 x^{8} \) the last eight Bethe equations in the \( x \to 0 \) limit give

\[
x^{-10N} = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} A_{N-8}^2 b_1^{2N-16} x^{-6N-32} E(b_8/b_1)^{-1},
\]
\[
x^{10N} = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} A_{N-8}^{-2} b_2^{2N-16} x^{6N+32} E(b_2/b_7),
\]
\[
x^{-10N} = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} A_{N-8}^{-2} b_3^{2N-16} b_8^{-2} x^{-2N-62} [E(b_5/b_8)E(b_6/b_3)]^{-1},
\]
\[
x^{10N} = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} A_{N-8}^{-2} b_4^{2N-16} b_7^{-2} x^{2N+62} E(b_4/b_5)E(b_7/b_4),
\]
\[
\left[ b_5 x^{-10} \right]^N = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} b_5^{4} (b_5 b_6) x^{-72} [E(b_5/b_6)E(b_4/b_5)]^{-1},
\]
\[
\left[ b_6 x^{10} \right]^N = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} b_6^{4} (b_5 b_7) x^{-72} E(b_6/b_3)E(b_5/b_6),
\]
\[
\left[ b_7 x^{-8} \right]^N = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} b_7^{6} (b_4 b_5 b_8) x^{-62} [E(b_7/b_4)E(b_2/b_7)]^{-1},
\]
\[
\left[ b_8 x^{8} \right]^N = -\frac{1}{\omega} (A_{N-8} B_8)^{5/8} b_8^{6} (b_3 b_5 b_7) x^{62} E(b_8/b_1)E(b_3/b_8). \] (B.40)

In this case we can conclude that the \( b_i \) are equal, to \( O(x^{4N}) \) or \( O(x^{6N}) \).

The product of the eight Bethe equations can be written in terms of the \( m_7 \) auxiliary functions (5.87) and (5.88) as

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\[
\left[ \frac{E(x^{18}/b) E(x^{20}/b) E(x^{22}/b) E(x^{24}/b) E(x^{26}/b)}{E(x^{18}b) E(x^{20}b) E(x^{22}b) E(x^{24}b) E(x^{26}b)} \right]^N = b^{5N} \left[ \mathcal{F}_7(x^6b) \mathcal{F}_7^2(x^{10}b) \mathcal{F}_7^2(x^{12}b) \mathcal{F}_7(x^{14}b) \right]^{-1} \times \left[ \mathcal{G}_7(1/x^6b) \mathcal{G}_7^2(1/x^{10}b) \mathcal{G}_7^2(1/x^{12}b) \mathcal{G}_7(1/x^{14}b) \right]^{-1}. \quad (B.41)
\]

Or finally, again with elliptic nome \( x^{60} \),

\[
\left[ \frac{E(x^{4b}/b) E(x^{8b}/b) E(x^{10b}/b) E(x^{12b}/b) E(x^{14b}/b)}{E(x^{4b}/b) E(x^{8b}/b) E(x^{10b}/b) E(x^{12b}/b) E(x^{14b}/b)} \right]^N \times \left[ \frac{E(x^{34b}/b) E(x^{38b}/b) E(x^{40b}/b) E(x^{42b}/b) E(x^{44b}/b)}{E(x^{34b}/b) E(x^{38b}/b) E(x^{40b}/b) E(x^{42b}/b) E(x^{44b}/b)} \right]^N = b^{5N}. \quad (B.42)
\]

### B.8 Mass \( m_8 \)

For the roots \( w_j = a_j \) for \( j = 1, \ldots, N-10 \) and \( w_{N-9} = b_1 x^{-15} \), \( w_{N-8} = b_2 x^{15} \), \( w_{N-7} = b_3 x^{-13} \), \( w_{N-6} = b_4 x^{13} \), \( w_{N-5} = b_5 x^{-11} \), \( w_{N-4} = b_6 x^{11} \), \( w_{N-3} = b_7 x^{-9} \), \( w_{N-2} = b_8 x^9 \), \( w_{N-1} = b_9 x^{-7} \), \( w_N = b_{10} x^7 \) the last ten Bethe equations in the \( x \to 0 \) limit give

\[
x^{-2N} = -\frac{1}{\omega} (A_{N-10} B_{10})^{5/8} A_{N-10}^{-2} b_{10}^{2N-20} x^{-20} E(b_1/b_0)^{-1},
\]

\[
x^{2N} = \frac{1}{\omega} (A_{N-10} B_{10})^{5/8} A_{N-10}^{-2} b_{10}^{2N-20} x^{20} E(b_2/b_0),
\]

\[
x^{-6N} = -\frac{1}{\omega} (A_{N-10} B_{10})^{5/8} A_{N-10}^{-2} b_{10}^{2N-18} b_{10}^{-2} x^{-58} [E(b_3/b_{10}) E(b_8/b_3)]^{-1},
\]

\[
x^{6N} = \frac{1}{\omega} (A_{N-10} B_{10})^{5/8} A_{N-10}^{-2} b_{10}^{2N-18} b_{10}^{-2} x^{58} E(b_4/b_7) E(b_9/b_4),
\]

\[
x^{-9N} = -\frac{1}{\omega} (A_{N-10} B_{10})^{5/8} A_{N-10}^{-1} b_5^{N-6} (b_9 b_{10})^{-2} x^{-82} [E(b_5/b_8) E(b_6/b_5)]^{-1},
\]

\[
x^{9N} = \frac{1}{\omega} (A_{N-10} B_{10})^{5/8} A_{N-10}^{-1} b_5^{N-6} (b_7 b_0)^{-2} x^{82} E(b_6/b_5) E(b_7/b_0),
\]

\[
[b_7 x^{-9}]^N = -\frac{1}{\omega} (A_{N-10} B_{10})^{5/8} b_7^6 (b_0 b_7 b_{10})^{-2} x^{-82} [E(b_7/b_0) E(b_4/b_7)]^{-1},
\]

\[
[b_8 x^9]^N = \frac{1}{\omega} (A_{N-10} B_{10})^{5/8} b_8^6 (b_5 b_7 b_9)^{-2} x^{82} E(b_8/b_3) E(b_5/b_8),
\]

\[
[b_9 x^7]^N = -\frac{1}{\omega} (A_{N-10} B_{10})^{5/8} b_{10}^8 (b_4 b_9 b_{10})^{-2} x^{-68} [E(b_9/b_4) E(b_2/b_9)]^{-1},
\]

\[
[b_{10} x^7]^N = -\frac{1}{\omega} (A_{N-10} B_{10})^{5/8} b_{10}^8 (b_3 b_5 b_{10})^{-2} x^{68} E(b_{10}/b_1) E(b_3/b_{10}).
\]

(B.43)
We are able to conclude (to $O(x^N)$, $O(x^{2N})$, $O(x^{5N})$ or $O(x^{8N})$) that the $b_i$ are equal, so that $B_{10} = b_{10}^6$. In terms of the functions (5.100) and (5.101), the product of the ten Bethe equations gives

\[
\frac{\left[ E(x^{17}/b)E(x^{19}/b)E(x^{21}/b)E(x^{23}/b)E(x^{25}/b)E(x^{27}/b) \right]^N}{E(x^{17}b)E(x^{19}b)E(x^{21}b)E(x^{23}b)E(x^{25}b)E(x^{27}b)} = b^{6N} \left[ \mathcal{F}_8(x^{5}b)\mathcal{F}_8^2(x^{7}b)\mathcal{F}_8^2(x^{9}b)\mathcal{F}_8^2(x^{11}b)\mathcal{F}_8^2(x^{13}b)\mathcal{F}_8(x^{15}b) \right]^{-1} \times \left[ G_8(1/x^{5}b)G_8^2(1/x^{7}b)G_8^2(1/x^{9}b)G_8^2(1/x^{11}b)G_8^2(1/x^{13}b)G_8(1/x^{15}b) \right]^{-1},
\]

which can be written as

\[
\frac{\left[ E(x^{5}b)E(x^{7}b)E(x^{9}b)E(x^{11}b)E(x^{13}b)E(x^{15}b) \right]^N}{E(x^{5}b)E(x^{7}b)E(x^{9}b)E(x^{11}b)E(x^{13}b)E(x^{15}b)} \times \left[ \frac{E(x^{35}/b)E(x^{37}/b)E(x^{39}/b)E(x^{41}/b)E(x^{43}/b)E(x^{45}/b)}{E(x^{35}b)E(x^{37}b)E(x^{39}b)E(x^{41}b)E(x^{43}b)E(x^{45}b)} \right]^N = b^{6N}.
\]

again with elliptic nome $x^{60}$. 

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