HEAT CONTENT, HEAT TRACE, AND ISOSPECTRALITY

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Abstract. We study the heat content function, the heat trace function, and questions of isospectrality for the Laplacian with Dirichlet boundary conditions on a compact manifold with smooth boundary in the context of finite coverings and warped products.

1. Introduction

1.1. The spectral resolution. Let $\mathcal{M} := (M, g)$ be a compact Riemannian manifold of dimension $m$ with smooth non-empty boundary $\partial M$. Let $\text{dvol}_M$ and $\text{dvol}_{\partial M}$ be the Riemannian measures on $M$ and on $\partial M$, respectively. Let $\Delta_M := \delta d$ be the scalar Laplacian with Dirichlet boundary conditions, i.e.

$$\text{Domain}(\Delta_M) = \{ \phi \in C^\infty(M) : \phi|_{\partial M} = 0 \}.$$ 

There is a complete orthonormal basis $\{\phi_n\}$ for $L^2(\mathcal{M})$ where $\phi_n \in C^\infty(M)$, where $\phi_n|_{\partial M} = 0$, and where $\Delta_M \phi_n = \lambda_n \phi_n$; these are the Dirichlet eigenfunctions. The collection

$$S(\Delta_M) := \{ \phi_n, \lambda_n \}$$

is called a spectral resolution of $\Delta_M$. If one orders the eigenfunctions so

$$0 \leq \lambda_1 \leq ... \leq \lambda_n \ldots ,$$

then one has the Weyl estimate [19] that $\lambda_n \sim n^{2/m}$ as $n \to \infty$. We set

$$\text{Spec}(\Delta_M) := \{ \lambda_1, \lambda_2, ... \}$$

where each eigenvalue is repeated according to multiplicity. Two Riemannian manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be isospectral if $\text{Spec}(\Delta_{\mathcal{M}_1}) = \text{Spec}(\Delta_{\mathcal{M}_2})$. We refer to [11] for further details concerning isospectrality.

1.2. Operators of Laplace type. It is convenient to work in slightly greater generality – this will be important in Section 3 when we discuss warped products. An operator $D$ is said to be of Laplace type if the leading symbol of $D$ is given by the metric tensor or, equivalently, if we may express in any system of local coordinates $x = (x_1, ..., x_m)$ the operator $D$ in the form:

$$D = - \{ g^{ij} \partial_{x_i} \partial_{x_j} + a^i \partial_{x_i} + b \}$$

where we adopt the Einstein convention and sum over repeated indices; here $a^i$ and $b$ are smooth functions and $g^{ij}$ is the inverse of the metric $g_{ij} := g(\partial_{x_i}, \partial_{x_j})$. Let

$$\text{dvol}_M = gdx^1 ... dx^m$$

where $g = \text{det}(g_{ij})^{1/2}$. The scalar Laplacian $\Delta_M$ is of Laplace type since

(1.a) \hspace{1cm} \Delta_M = - g^{-1} \partial_{x_i} g^{ij} \partial_{x_j} = - \left( g^{ij} \partial_{x_i} \partial_{x_j} + \{ g^{-1} \partial_{x_i} (gg^{ij}) \} \partial_{x_j} \right)$.
1.3. The heat equation. Let $\phi \in C^\infty(M)$ define the initial temperature of the manifold. Let $D$ be an operator of Laplace type on $C^\infty(M)$. The subsequent temperature distribution $u := e^{-tD}\phi$ for $t > 0$ is defined by the following relations, we refer to [8] for a further discussion of the heat process:

\begin{align}
(\partial_t + D)u &= 0 \quad \text{(evolution equation),} \\
\lim_{t \downarrow 0} u(\cdot, t) &= \phi \quad \text{in } L^2 \quad \text{(initial condition),} \\
u(\cdot, t)|_{\partial M} &= 0 \quad \text{(boundary condition).}
\end{align}

The special case that $D = \Delta_M$ is of particular interest. Let

$$
\sigma_n(\phi) := \int_M \phi(x)\bar{\phi}_n(x)\, d\text{vol}_M
$$

be the \textit{Fourier coefficients}. We may then express

$$
u(x, t) = \sum_n e^{-t\lambda_n} \sigma_n(\phi)\phi_n(x).
$$

1.4. The heat content. Let $\rho$ be the specific heat and let $\phi$ be the initial temperature of the manifold. The total heat energy content is then defined to be:

\begin{equation}
\beta(\phi, \rho, D)(t) := \int_M u(x; t)\rho(x)\, d\text{vol}_M.
\end{equation}

The heat content is expressible for the Laplacian in terms of the Fourier coefficients:

$$
\beta(\phi, \rho, \Delta_M)(t) = \sum_n e^{-t\lambda_n} \sigma_n(\phi)\sigma_n(\rho).
$$

We shall assume $\rho$ and $\phi$ are smooth henceforth. We refer to [2] for some results in the non-smooth setting where $\phi$ is allowed to blow up near the boundary and to [8] where the boundary is polygonal.

The total heat energy $\beta_M(t)$ of $M$ is defined by taking $\phi(x) = \rho(x) = 1$;

\begin{equation}
\beta_M(t) := \beta(1, 1, \Delta_M)(t) = \sum_n e^{-t\lambda_n} \sigma_n(1)^2.
\end{equation}

The total heat energy content of the manifold is a scalar function which is an isometry invariant of the manifold. For example, if $M = ([0, \pi], dx^2)$ is the interval with the standard metric, then

$$
\Delta_M = -\partial_x^2, \quad \text{Spec}(\Delta_M) = \left\{ n^2 \right\}_{n=1}^\infty, \\
S(\Delta_M) = \left\{ \sqrt{\frac{2}{\pi}} \sin(nx), n^2 \right\}_{n=1}^\infty, \quad \sigma_n(1) = \left\{ \begin{array}{ll}
2\sqrt{\frac{2}{\pi}} & \text{if } n \equiv 1 \mod 2 \\
0 & \text{if } n \equiv 0 \mod 2
\end{array} \right., \\
\beta_M(t) = \sum_{k=0}^{\infty} \frac{1}{(1 + 2k)^2} e^{-(1+2k)^2t}.
$$

1.5. The heat trace. Let $D$ be an operator of Laplace type on $C^\infty(M)$. The operator $e^{-tD}$ is an infinitely smoothing operator. If $f \in C^\infty(M)$ is an auxiliary function which is used for localization or smoothing, then $fe^{-tD}$ is of trace class and $\text{Tr}_{L^2}(fe^{-tD})$ is well defined. We shall assume that $f$ is smooth and refer to [4] for some results in the non-smooth setting where $f$ is allowed to blow up near the boundary. We also refer to [16] for results concerning Riemann surfaces with corners.

If we take $f = 1$ and let $D = \Delta_M$, then

$$
\text{Tr}_{L^2}(e^{-t\Delta_M}) = \sum_n e^{-t\lambda_n}
$$

is a spectral invariant which determines $\text{Spec}(\Delta_M)$. 
1.6. Local invariants. We can extract locally computable invariants from the heat content and from the heat trace as follows. Let \( D \) be an operator of Laplace type on \( C^\infty(M) \) and let \( f, \rho, \phi \in C^\infty(M) \). Work of Greiner [12] and of Seeley [17] can be used to show that there is a complete asymptotic expansion

\[
\text{Tr}(fe^{-tD}) \sim \sum_{n=0}^{\infty} a_n(f, D)t^{(n-m)/2} \quad \text{as} \quad t \downarrow 0.
\]

Similarly, see the discussion in [2, 4], there is a complete asymptotic expansion:

\[
\beta(\phi, \rho, D)(t) \sim \sum_{n=0}^{\infty} \beta_n(\phi, \rho, D)t^{n/2}.
\]

These invariants are locally computable and have been studied by many authors; we refer to [10] for a more complete discussion of the history of the subject. To simplify the discussion, we shall only consider the special case where \( D = \Delta_M \) and where \( f = \rho = \phi = 1 \). We define the following local isometry invariants of the manifold:

\[
a_n(M) := a_n(1, \Delta_M) \quad \text{and} \quad \beta_n(M) := \beta_n(1, 1, \Delta_M).
\]

Let indices \( i, j, k, l \) range from 1 to \( m \) and index a local orthonormal frame \( \{e_1, ..., e_m\} \) for the tangent bundle of \( M \). Let \( R_{ijkl} \) be the components of the Riemann curvature tensor; our sign convention is chosen so that \( R_{1221} = +1 \) on the sphere of radius 1 in \( \mathbb{R}^3 \). Let \( \rho_{ij} := R_{ikkj} \) be the Ricci tensor and let \( \tau := \rho_{ii} \) be the scalar curvature. Near the boundary we normalize the choice of the local frame by requiring that \( e_m \) is the inward unit geodesic normal. We let indices \( a, b, c, d \) range from 1 through \( m - 1 \) and index the restricted orthonormal frame \( \{e_1, ..., e_{m-1}\} \) for the tangent bundle of the boundary. Let \( L_{ab} := g(\nabla_e e_b, e_m) \) be the components of the second fundamental form. We can use the Levi-Civita connection on \( M \) to multiply covariantly differentiate a tensor defined on the interior; we let ‘\('\)' denote the components of such a tensor. Similarly, we can use the Levi-Civita connection of \( \partial M := (\partial M, g|_{\partial M}) \) to multiply covariantly differentiate a tensor defined on the boundary; we let ‘\('\)' denote the components of such a tensor. The difference between ‘\('\)' and ‘\('\)' is measured by the second fundamental form.

**Theorem 1.1.**

1. \( a_0(M) = (4\pi)^{-m/2} \text{Volume}(M) \).
2. \( a_1(M) = -\frac{1}{4}(4\pi)^{-m/2} \text{Volume}(\partial M) \).
3. \( a_2(M) = \frac{1}{2}(4\pi)^{-m/2} \left\{ \int_M \tau \ d\text{vol}_M + \int_{\partial M} 2L_{aa} \ d\text{vol}_{\partial M} \right\} \).
4. \( a_3(M) = -\frac{1}{360}(4\pi)^{-m/2} \int_{\partial M} \left\{ 16\tau + 8R_{amam} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab} \right\} \ d\text{vol}_{\partial M} \).
5. \( a_4(M) = \frac{1}{360}(4\pi)^{-m/2} \int_M \left( 12R_{kk} + 5\tau^2 - 2|\rho|^2 + 2|R^2| \right) \ d\text{vol}_M + \frac{1}{360}(4\pi)^{-m/2} \int_{\partial M} \left\{ -18\tau_m + 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab} + 4R_{abc'b}L_{ac} + 24L_{aabb} + \frac{1}{20}L_{aabc}L_{ac} + \frac{8}{89}L_{ab}L_{bc}L_{ac} \right\} \ d\text{vol}_{\partial M} \).

**Theorem 1.2.**

1. \( \beta_0(M) = \text{Volume}(M) \).
2. \( \beta_1(M) = -\frac{2}{\sqrt{\pi}} \text{Volume}(\partial M) \).
3. \( \beta_2(M) = \int_{\partial M} \frac{1}{2} L_{aa} \ d\text{vol}_{\partial M} \).
4. \( \beta_3(M) = -\left\{ \frac{2}{\sqrt{\pi}} \int_{\partial M} \left( \frac{1}{10}L_{aa}L_{bb} - \frac{1}{2}L_{ab}L_{ab} - \frac{1}{10}R_{amam} \right) \ d\text{vol}_{\partial M} \right\} \).
5. \( \beta_4(M) = \int_{\partial M} \left\{ -\frac{1}{10}L_{ab}L_{ab}L_{cc} + \frac{1}{8}L_{ab}L_{ac}L_{bc} - \frac{1}{10}R_{ambm}L_{ab} + \frac{1}{10}R_{abc'b}L_{ac} + \frac{1}{8}\tau_m \right\} \ d\text{vol}_{\partial M} \).
Although formulas for $a_5(M)$ and $\beta_5(M)$ are known, we have omitted them in the interests of brevity. Formulas generalizing those in Theorems 1.1 and 1.2 are available in the more general setting to study the invariants $a_n(f, D)$ and $\beta_n(f, \rho, D)$ for an arbitrary vector valued operator $D$ of Laplace type; again, we shall omit details in the interests of brevity and instead refer to the discussion in [10]. Although we have chosen to work with Dirichlet boundary conditions, similar formulas exist for Neumann, transfer, transmittal, and spectral boundary conditions. The history of this subject is a vast one and beyond the scope of this brief article to give in any depth. We refer to [13] for a more detailed discussion of elliptic boundary conditions.

1.7. Relating the heat trace and heat content. McDonald and Meyers [15] have constructed additional invariants involving exit time moments which determine both the heat trace and the heat content; we also refer to related work [14] by these authors in the context of graphs.

It is difficult in general, however, to relate the heat trace and the heat content directly. In particular, there is no obvious relation between the formulas given in Theorems 1.1 and 1.2 when $n \geq 3$. It is clear that $\text{Tr}(e^{-t\Delta_M})$ is determined by $\text{Spec}(\Delta_M)$ and it is clear that $\beta_M(t)$ is determined by the full spectral resolution $S(\Delta_M)$. It is not known, however, if the full heat content function $\beta_M(t)$ or in particular the heat content asymptotic coefficients $\beta_k(M)$ might be determined by $\text{Spec}(\Delta_M)$ alone. More specifically, one does not know if there are Dirichlet isospectral manifolds with different heat content functions. In the remainder of this brief note, we shall present some results which relate to this question. In Section 2 we discuss finite coverings and in Section 3 we discuss warped products.

2. Finite coverings

2.1. Notational conventions. We suppose that $\pi : M_1 \rightarrow M_2$ is a finite $k$-sheeted covering of compact manifolds with boundary. We assume that $M_2$ is equipped with a Riemannian metric $g_2$ and choose the induced metric $g_1 := \pi^*g_2$ on $M_1$. Thus $\pi$ is a local isometry and $\text{Volume}(M_1) = k \text{Volume}(M_2)$. Since

$$|\pi^*\phi|_{L^2(M_1)}^2 = k |\phi|_{L^2(M_2)}^2,$$

pullback $\pi^*$ is an injective closed map from $L^2(M_2)$ to $L^2(M_1)$.

2.2. Heat trace and heat content asymptotics. The invariants $a_n(M)$ and $\beta_n(M)$ are locally computable. Since integration is multiplicative under finite coverings, the following result is immediate:

**Theorem 2.1.** Let $M_1 \rightarrow M_2$ be a finite $k$-sheeted Riemannian cover. Then $a_n(M_1) = ka_n(M_2)$ and $\beta_n(M_1) = k\beta_n(M_2)$ for all $n$.

2.3. Heat trace. We begin by presenting an example to show that there are examples where $\text{Tr}_{L^2}(e^{-t\Delta_{M_1}}) \neq k\text{Tr}_{L^2}(e^{-t\Delta_{M_2}})$ despite the fact that the heat content function is multiplicative under finite coverings. Let

$$M_1 := ([0,4\pi], d\theta^2)/0 \sim 4\pi \quad \text{and} \quad M_2 := ([0,2\pi], d\theta^2)/0 \sim 2\pi;$$

$M_1$ may be identified with the circle of radius 2 in $\mathbb{R}^2$ and $M_2$ may be indentified with the circle of 1 in $\mathbb{R}^2$. The natural projection from $M_1 \rightarrow M_2$ can be regarded as the double cover of the circle by the circle induced by the map $z \rightarrow \frac{1}{2}z^2$. Then

$$S(\Delta_{M_1}) = \left\{ \frac{1}{\sqrt{4\pi}}e^{-\frac{1}{2}k^2t} \frac{1}{2}k^2 \right\}_k = \infty, \quad \text{Tr}_{L^2} \left\{ e^{-t\Delta_{M_1}} \right\} = 1 + 2 \sum_{k=1}^{\infty} e^{-\frac{1}{2}k^2t},$$

$$S(\Delta_{M_2}) = \left\{ \frac{1}{\sqrt{4\pi}}e^{-k^2t} \right\}_k = \infty, \quad \text{Tr}_{L^2} \left\{ e^{-t\Delta_{M_2}} \right\} = 1 + 2 \sum_{k=1}^{\infty} e^{-k^2t}.$$
Consequently
\[ \text{Tr}_{L^2} \{ e^{-t\Delta_{M_1}} \} \neq 2 \text{Tr}_{L^2} \{ e^{-t\Delta_{M_1}} \} . \]

Although this example is in the category of closed manifolds, we can construct other examples as follows. Let \( N = ([0, \pi], d\theta^2) \) be a manifold with boundary. Let \( \tilde{M}_i := N \times M_i \) and let \( \pi \) act only on the second factor. Since
\[ e^{-t\Delta_{N \times M}} = e^{-t\Delta_N} e^{-t\Delta_M}, \]
one has:
\[ \text{Tr}_{L^2} \{ e^{-t\Delta_{\tilde{M}_1}} \} = \sum_{\ell=1}^{\infty} e^{-\ell^2t} \left\{ 1 + 2 \sum_{k=1}^{\infty} e^{-\frac{1}{4}k^2t} \right\} \]
\[ \neq 2 \sum_{\ell=1}^{\infty} e^{-\ell^2t} \left\{ 1 + 2 \sum_{k=0}^{\infty} e^{-k^2t} \right\} = 2 \text{Tr}_{L^2} \{ e^{-t\Delta_{M_2}} \} . \]

2.4. Heat content asymptotics. It is perhaps somewhat surprising that in contrast to the situation with the heat trace asymptotics discussed in Section 2.3 that one has:

**Theorem 2.2.** Let \( M_1 \to M_2 \) be a finite \( k \)-sheeted Riemannian cover. Then \( \beta(M_1)(t) = k\beta(M_2)(t) \).

**Proof.** Let \( \{ \lambda_n, \phi_n \} \) be a spectral resolution of \( \Delta_{M_2} \). Let \( c_n = \sigma_n(1) \) be the associated Fourier coefficients. We use Equation (2.a) to see that
\[ 1 = \sum_n c_n \phi_n \text{ in } L^2(M_2) \text{ implies } 1 = \sum_n c_n \pi^* \phi_n \text{ in } L^2(M_1) . \]
Since \( \Delta_{M_1} \pi^* \phi_n = \pi^* \Delta_{M_1} \phi_n = \lambda_n \pi^* \phi_n \) and since \( \pi^* \phi_n \) satisfy Dirichlet boundary conditions, we have
\[ \{ e^{-t\Delta_{M_1}} \} 1 = \sum_n e^{-t\lambda_n} c_n \pi^* \phi_n = \pi^* \{ e^{-t\Delta_{M_2}} \} 1 . \]
Consequently
\[ \beta_{M_1}(t) = \langle e^{-t\Delta_{M_1}} \pi^*1, \pi^*1 \rangle_{L^2(M_1)} = \langle \pi^* e^{-t\Delta_{M_2}} 1, \pi^*1 \rangle_{L^2(M_1)} \]
\[ = k \langle e^{-t\Delta_{M_2}} 1, 1 \rangle_{L^2(M_2)} = k\beta(M_2)(t) . \]

2.5. Summary. Theorems 2.1 and 2.2 show that a Sunada construction involving finite coverings will not produce isospectral manifolds with different heat content functions as only the order of the cover is detected. If \( M \) is a Riemannian manifold which has constant sectional curvature +1, then \( M \) is said to be a spherical space form. If \( M \) is closed and if the fundamental group \( \pi_1(M) \) is cyclic, then \( M \) is said to be a lens space. Ikeda [6, 7] and other authors have studied questions of isospectrality for spherical space forms; we refer to [9, 10] for further details as the literature is an extensive one. These examples can easily be modified to the category of manifolds with boundary by punching out a small disk from \( M_2 \) and then lifting to get a spherical spaceform with boundary. Since there are spherical space forms with the same fundamental group which are not isospectral, neither the heat trace asymptotics nor the full heat content function determine either the spectrum of the manifold or the isometry type of the manifold.
3. Warped product metrics

3.1. Notational conventions. Let \( \mathcal{N} = (N, g_N) \) be a smooth Riemannian manifold of dimension \( n \) with smooth boundary \( \partial N \), let \( \mathcal{M} = (M, g_M) \) be a closed Riemannian manifold of dimension \( m \), and let \( f \in \mathcal{C}^\infty(N) \). We consider the warped product

\[
\mathcal{N} \times_f \mathcal{M} := (N \times M, g_N + e^{\frac{2}{m}f} g_M).
\]

The normalizing constant \( \frac{2}{m} \) is chosen so that one has the following relationship between the volume elements:

\[
(3.1) \quad \text{dvol}_{\mathcal{N} \times_f \mathcal{M}} = e^f \text{dvol}_N \cdot \text{dvol}_M.
\]

We define an auxiliary operator of Laplace type on \( \mathcal{C}^\infty(N) \) by setting:

\[
D_{\mathcal{N}, f} := e^{-f} \Delta_N e^f.
\]

Note that this operator is no longer self-adjoint if \( f \) is non-constant; this operator does, however, have the same spectrum as \( \Delta_N \) since it is conjugate to this operator. We may then use Equation (1.1) to see that

\[
(3.2) \quad \Delta_{\mathcal{N} \times_f \mathcal{M}} = D_{\mathcal{N}, f} + e^{-\frac{2}{m}f} \Delta_M.
\]

3.2. The heat content. Let \( \beta(\phi, \rho, D)(t) \) be the generalized heat content function defined in Equation (1.0).

Theorem 3.1.

\begin{enumerate}
\item \( \beta(\mathcal{N} \times_f \mathcal{M})(t) = \text{Volume}(\mathcal{M}) \cdot \beta(1, e^f, D_{\mathcal{N}, f})(t) \).
\item If \( \text{Volume}(\mathcal{M}_1) = \text{Volume}(\mathcal{M}_2) \), then \( \beta(\mathcal{N} \times_f \mathcal{M}_1)(t) = \beta(\mathcal{N} \times_f \mathcal{M}_2)(t) \).
\end{enumerate}

Proof. Let \( u := e^{-tD_{\mathcal{N}, f}} \cdot 1 \) be the solution of Equation (1.1) on \( N \) with initial condition \( \phi(\cdot) = 1 \) which is defined by the operator \( D_{\mathcal{N}, f} \). Extend \( u \) to \( N \times M \) to be independent of the second variable. We apply Equation (3.1). Since \( \Delta_M u = 0 \), \( u \) satisfies Equation (1.1) on \( N \times M \) with initial condition \( \phi(\cdot) = 1 \) using the operator \( \Delta_{\mathcal{N} \times_f \mathcal{M}} \). Thus we also have that

\[
u = e^{-t\Delta_{\mathcal{N} \times_f \mathcal{M}}} \cdot 1.
\]

One may now use Equation (3.1) to compute:

\[
\beta(\mathcal{N} \times_f \mathcal{M})(t) = \int_{N \times M} u(x_N; t)e^f \text{dvol}_N \cdot \text{dvol}_M = \text{Volume}(M) \cdot \beta(1, e^f, D_{\mathcal{N}, f})(t).
\]

This establishes Assertion (1); Assertion (2) follows from Assertion (1). □

Theorem 3.1 shows that the heat content does not even determine the dimension of the underlying manifold as only the volume of the manifold \( M \) appears in this formula. On the other hand, Equation (1.5c) shows that the dimension of the underlying manifold is determined by the heat trace. Consequently, we once again see that the heat content function does not determine the underlying spectrum.
3.3. Isospectrality. We conclude our discussion by showing that isospectrality is preserved by the warped product construction.

Theorem 3.2.

(1) Let $\text{Spec}(\Delta_M) = \{\mu_i\}_{i=1}^{\infty}$. Then

$$\text{Spec}(\Delta_N \times fM) = \bigcup_{i=1}^{\infty} \text{Spec} \left( \Delta_N + \mu_i e^{-\frac{2}{m}\ell f} \right).$$

(2) If $\text{Spec}\{\mathcal{M}_1\} = \text{Spec}\{\mathcal{M}_2\}$, then

$$\text{Spec}\{\Delta_N \times f\mathcal{M}_1\} = \text{Spec}\{\Delta_N \times f\mathcal{M}_2\}.$$ 

Proof. Let $\mathcal{M}$ be a Riemannian manifold. Let $\{\Phi_i, \mu_i\}$ be a spectral resolution of $\Delta_M$. We decompose

$$L^2(\mathcal{N} \times f\mathcal{M}) = \bigoplus_i L^2(\mathcal{N}) \cdot \Phi_i.$$ 

Let $\mu_i e^{-\frac{2}{m}\ell f}$ act by scalar multiplication. We use Equation (3.3) to see that the decomposition of Equation (3.3) induces a corresponding decomposition

$$\Delta_N \times f\mathcal{M} = \bigoplus_i \left\{ e^f \Delta_N e^{-f} + \mu_i e^{-\frac{2}{m}\ell f} \right\}.$$ 

Assertion (1) now follows since

$$\text{Spec}\left\{ e^f \Delta_N e^{-f} + \mu_i e^{-\frac{2}{m}\ell f} \right\} = \text{Spec}\left\{ \Delta_N + \mu_i e^{-\frac{2}{m}\ell f} \right\}.$$ 

Assertion (2) follows from Assertion (1). \hfill \square 

We may take $N = [0, \pi]$ and assume that $f(0) = f(\pi) = 0$. We then have that $\partial(N \times M)$ is isometric to the disjoint union of two copies of $M$. Since there are many pairs of isospectral closed manifolds which are not isometric, Theorem 3.2 provides examples of isospectral manifolds with boundary given by warped products which are not isometric.

3.4. Conclusion. Theorems 1.1 and 1.2 show that the volume of the interior, the volume of the boundary, and the dimension of $M$ are determined by the heat trace. Thus Theorem 3.1 shows that a warped product construction involving isospectral manifolds with a suitably chosen manifold with boundary will not produce isospectral manifolds with different heat content functions. Theorem 3.1 does show, however, that there exist manifolds with the same heat content function which are not isospectral. If we take $f = 1$ and apply the argument of Theorem 3.1, we see that the heat content function does not determine the dimension of the manifold.

There exist spherical space forms $\mathcal{M}_1$ and $\mathcal{M}_2$ which are isospectral but not diffeomorphic. If we take $\mathcal{N} = ([a, b], dx^2)$ with $0 < a < b$ and if we take as a warping function $f(x) = x^2$, then the resulting warped products $\mathcal{P}_i := \mathcal{N} \times f\mathcal{M}_i$ are flat isospectral manifolds whose boundaries are not diffeomorphic.

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