General properties of nonlinear mean field Fokker-Planck equations

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Abstract. Recently, several authors have tried to extend the usual concepts of thermodynamics and kinetic theory in order to deal with distributions that can be non-Boltzmannian. For dissipative systems described by the canonical ensemble, this leads to the notion of nonlinear Fokker-Planck equation (T.D. Frank, Non Linear Fokker-Planck Equations, Springer, Berlin, 2005). In this paper, we review general properties of nonlinear mean field Fokker-Planck equations, consider the passage from the generalized Kramers to the generalized Smoluchowski equation in the strong friction limit, and provide explicit examples for Boltzmann, Tsallis and Fermi-Dirac entropies.

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1. THE GENERALIZED KRAMERS EQUATION

1.1. Generalized stochastic processes

Nonlinear Fokker-Planck (NFP) equations have been the subject of recent activity [1, 2, 3, 4, 5]. Here, we consider a generalized Kramers equation of the form [6]:

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left( Dh(f) \frac{\partial f}{\partial v} + \xi g(f) v \right),$$

where $h(f)$ and $g(f)$ are positive functions. For $h(f) = 1$ and $g(f) = f$, Eq. (1) reduces to the familiar Kramers equation where $D$ is the diffusion coefficient and $\xi$ the friction coefficient. Usually, $\Phi(r)$ is an external potential but we can also consider the case where the potential is produced by the density $\rho(r, t) = \int f(r, v, t) dv$ according to the relation

$$\Phi(r, t) = \int u(|r - r'|) \rho(r', t) dr',$$

where $u(|r - r'|)$ is a binary potential of interaction. The nonlinear mean field Fokker-Planck equation [2] is associated to the Ito-Langevin stochastic process

$$\frac{dr}{dt} = v, \quad \frac{dv}{dt} = -\xi(f)v - \nabla\Phi + \sqrt{2D(f)} R(t),$$

$$\xi(f) = \frac{\xi g(f)}{f}, \quad D(f) = \int_0^f h(x) dx,$$

where $R(t)$ is a white noise satisfying $\langle R(t) \rangle = 0$ and $\langle R_i(t) R_j(t') \rangle = \delta_{ij} \delta(t - t')$ where $i = 1, \ldots, d$ label the coordinates of space.
1.2. The H-theorem

We introduce the energy

\[ E = \frac{1}{2} \int f v^2 d\mathbf{r} d\mathbf{v} + \frac{1}{2} \int \rho \Phi d\mathbf{r} = K + W, \]  

(5)

where \( K \) is the kinetic energy and \( W \) is the potential energy. For an external potential, we have \( W = \int \rho \Phi d\mathbf{r} \). We define the temperature by

\[ T = \frac{D}{\xi}. \]  

(6)

Therefore, the Einstein relation is preserved in this generalized thermodynamical framework. We also set \( \beta = 1/T \). We introduce the generalized entropic functional

\[ S = -\int C(f) d\mathbf{r} d\mathbf{v}, \]  

(7)

where \( C(f) \) is a convex function (\( C'' > 0 \)) satisfying [3]:

\[ C''(f) = \frac{h(f)}{g(f)}. \]  

(8)

Since the temperature is fixed (canonical description), the relevant thermodynamical potential is the generalized free energy

\[ F = E - TS. \]  

(9)

The definition of the free energy (Legendre transform) is preserved in this generalized thermodynamical framework. A straightforward calculation then shows that [3]:

\[ \dot{F} = -\int \frac{1}{\xi g(f)} \left( Dh(f) \frac{\partial f}{\partial \mathbf{v}} + \xi g(f) \mathbf{v} \right)^2 d\mathbf{r} d\mathbf{v}. \]  

(10)

Therefore, \( \dot{F} \leq 0 \) (provided that \( \xi > 0 \)). This forms an \( H \)-theorem in the canonical ensemble. The free energy \( F \) plays the role of a Lyapunov functional. Note that the NFP equation (1) can be written

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ \xi g(f) \frac{\partial}{\partial \mathbf{v}} \left( \frac{\delta F}{\delta f} \right) \right]. \]  

(11)

1.3. Stationary solutions

The steady states of Eq. (1) must satisfy \( \dot{F} = 0 \) leading to a vanishing current

\[ J \equiv Dh(f) \frac{\partial f}{\partial \mathbf{v}} + \xi g(f) \mathbf{v} = 0. \]  

(12)
Using Eqs. (6) and (8), we get

\[ C''(f) \frac{\partial f}{\partial v} + \beta v = 0, \quad (13) \]

which can be integrated into

\[ C'(f) = -\beta \left[ \frac{v^2}{2} + \lambda(r) \right], \quad (14) \]

where \( \lambda(r) \) is a function of the position. Since \( \frac{\partial f}{\partial t} = 0 \) and \( J = 0 \), the advective (Vlasov) term in Eq. (1) must also vanish leading to the condition

\[ v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = 0. \quad (15) \]

Using

\[ C''(f) \frac{\partial f}{\partial r} = -\beta \nabla \lambda, \quad C''(f) \frac{\partial f}{\partial v} = -\beta v, \quad (16) \]

we obtain \( (\nabla \lambda - \nabla \Phi) \cdot v = 0 \) which must be true for all \( v \). This yields \( \nabla \lambda - \nabla \Phi = 0 \), so that

\[ \lambda(r) = \Phi(r) + \alpha/\beta, \quad (17) \]

where \( \alpha \) is a constant. Therefore, the stationary solutions of Eq. (1) are determined by the relation [5]:

\[ C'(f) = -\beta \varepsilon - \alpha, \quad (18) \]

where \( \varepsilon = \frac{v^2}{2} + \Phi(r) \) is the individual energy. Since \( C \) is convex, this equation can be reversed to give

\[ f = F(\beta \varepsilon + \alpha), \quad (19) \]

where \( F(x) = (C')^{-1}(-x) \) is a decreasing function. Thus \( f = f(\varepsilon) \) is a decreasing function of the energy. We have \( f'(\varepsilon) = -\beta / C''(f) \leq 0 \).

\[ 1.4. \ \text{Minimum of free energy} \]

The critical points of free energy at fixed mass are determined by the variational problem

\[ \delta F + T \alpha \delta M = 0, \quad (20) \]

where \( \alpha \) is a Lagrange multiplier. We can easily establish that

\[ \delta E = \int \left( \frac{v^2}{2} + \Phi \right) \delta f \, dr \, dv, \quad \delta S = - \int C'(f) \delta f \, dr \, dv. \quad (21) \]
Therefore, the variational principle (20) gives [5]:

\[ C'(f) = -\beta \varepsilon - \alpha, \]

equivalent to Eq. (18). Therefore, a stationary solution of the GK equation (1) is a critical point of free energy \( F[f] \) at fixed mass \( M \). Furthermore, it is shown in Ref. [1, 5] that it is linearly dynamically stable if and only if it is a minimum (at least local) of \( F \) at fixed mass. Note that when \( \Phi \) is an external potential, we have

\[ \delta^2 F = -T \delta^2 S = \frac{1}{2} T \int C''(f)(\delta f)^2 dr dv \geq 0 \]

so that a critical point of \( F \) is always a minimum.

### 2. THE GENERALIZED SMOLUCHOWSKI EQUATION

We restrict ourselves to the case of a constant friction so that \( g(f) = f \) and \( h(f) = fC''(f) \). The generalized Kramers equation (1) then becomes

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ \xi \left( T fC''(f) \frac{\partial f}{\partial v} + f v \right) \right].
\]

In that case, \( \xi(f) = \xi \) and \( D(f) = Df[C(f)/f]' \). Let us derive the hydrodynamic moments of this equation [5]. Defining the density and the local velocity by

\[ \rho = \int f dv, \quad \rho u = \int f v dv, \]

and integrating Eq. (23) on velocity, we get the continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \]

Next, multiplying Eq. (23) by \( v \), integrating on the velocity and using the continuity equation (25), we obtain the momentum equation

\[
\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial P_{ij}}{\partial x_j} - \rho \frac{\partial \Phi}{\partial x_i} - \xi \rho u_i,
\]

where we have defined the pressure tensor

\[ P_{ij} = \int f w_i w_j dv, \]

where \( w = v - u(r,t) \) is the relative velocity. We now derive the generalized Smoluchowski (GS) equation from the generalized Kramers (GK) equation in the strong friction limit (see [7], Sec. 9). For \( \xi \to +\infty \) with fixed \( T \), the term in parenthesis in Eq. (23) must vanish at leading order

\[
T fC''(f) \frac{\partial f}{\partial v} + f v \simeq 0.
\]
Then, we find that the out-of-equilibrium distribution function \( f_0(r, v, t) \) is determined by

\[
C'(f_0) = -\beta \left[ \frac{v^2}{2} + \lambda(r, t) \right] + O(\xi^{-1}),
\]

(29)

where \( \lambda(r, t) \) is a constant of integration that is determined by the density according to

\[
\rho(r, t) = \int f_0 dv = \rho[\lambda(r, t)].
\]

(30)

Note that the distribution function \( f_0 \) is isotropic so that the velocity \( u(r, t) = O(\xi) \) and the pressure tensor \( P_{ij} = p\delta_{ij} + O(\xi^{-1}) \) where \( p \) is given by

\[
p(r, t) = \int \rho v^2 dv = p[\lambda(r, t)].
\]

(31)

Eliminating \( \lambda(r, t) \) between Eqs. (30) and (31), we find that the fluid is barotropic in the sense that \( p = p(\rho) \), where the equation of state is entirely determined by the generalized entropy \( C(f) \). Now, considering the momentum equation (26) in the limit \( \xi \to +\infty \), we find that

\[
\rho u = -\frac{1}{\xi} (\nabla p + \rho \nabla \Phi) + O(\xi^{-2}).
\]

(32)

Inserting this relation in the continuity equation (25), we obtain the generalized Smoluchowski equation [5]:

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (\nabla p + \rho \nabla \Phi) \right].
\]

(33)

This equation can also be obtained from a Chapman-Enskog expansion in powers of \( 1/\xi \) [6]. It monotonically decreases the free energy [5]:

\[
F[\rho] = \int \rho \frac{p(\rho')}{\rho'^2} d\rho' d\mathbf{r} + \int \rho \Phi d\mathbf{r},
\]

(34)

which can be deduced from the free energy (9) by using Eq. (29) to express \( F[f] \) as a functional \( F[\rho] \equiv F[f_0] \) of the density [6, 7]. A direct calculation leads to the \( H \)-theorem

\[
\dot{F} = -\int \frac{\xi}{\rho} (\nabla p + \rho \nabla \Phi)^2 d\mathbf{r} \leq 0.
\]

(35)

Moreover the stationary solutions of the generalized Smoluchowski equation (33) are critical points of the free energy \( F[\rho] \) at fixed mass, satisfying \( \delta F - \alpha \delta M = 0 \) where \( \alpha \) is a Lagrange multiplier. This yields \( \int \rho' p'(\rho')/\rho'^2 d\rho' = -\Phi \) leading to the condition of hydrostatic balance

\[
\nabla p + \rho \nabla \Phi = 0.
\]

(36)

After integration, we get \( \rho = \rho(\Phi) \) with \( \rho'(\Phi) \leq 0 \). This result can also be obtained by integrating \( f = f(\epsilon) \) on the velocity. Finally, a steady state of the GS equation (33) is linearly dynamically stable iff it is a (local) minimum of \( F[\rho] \) at fixed mass [1, 5].
3. EXPLICIT EXAMPLES

3.1. Isothermal systems: Boltzmann entropy

If we consider the Boltzmann entropy
\[ S_B[f] = - \int f \ln f \, dr \, dv, \]  
(37)
we get the ordinary Kramers equation
\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ \xi \left( T \frac{\partial f}{\partial v} + f v \right) \right]. \]  
(38)
The stationary solution is the Boltzmann distribution
\[ f = A e^{-\beta \varepsilon}, \]  
(39)
where \( A \) is determined by the conservation of mass. The equation of state is the isothermal one
\[ p = \rho T. \]  
(40)
In the strong friction limit, we recover the ordinary Smoluchowski equation
\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (T \nabla \rho + \rho \nabla \Phi) \right]. \]  
(41)
The free energy is the Boltzmann free energy in physical space
\[ F[\rho] = T \int \rho \ln \rho \, dr + \frac{1}{2} \int \rho \Phi \, dr. \]  
(42)
The stationary solution is the Boltzmann distribution in physical space
\[ \rho = A' e^{-\beta \Phi}, \]  
(43)
where \( A' = (2\pi/\beta)^{d/2} A. \)

3.2. Polytropes: Tsallis entropy

If we consider the Tsallis \( q \)-entropy
\[ S_q[f] = - \frac{1}{q-1} \int (f^q - f) \, dr \, dv, \]  
(44)
we obtain the polytropic Kramers equation
\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ \xi \left( T \frac{\partial f^q}{\partial v} + f^q v \right) \right]. \]  
(45)
The stationary solution is the Tsallis (or polytropic) distribution

\[ f = \left[ \mu - \frac{(q-1)\beta}{q} \varepsilon \right]^{\frac{1}{q-1}}_+, \quad (46) \]

where \( \mu \) is determined by the conservation of mass. The index \( n \) of the polytrope is

\[ n = \frac{d}{2} + \frac{1}{q-1}. \quad (47) \]

Isothermal distribution functions are recovered in the limit \( q \to 1 \) (i.e. \( n \to +\infty \)). We shall consider \( q > 0 \) so that \( C \) is convex. For \( q > 1 \), i.e. \( n > d/2 \), the distribution has a compact support (case 1) since \( f \) is defined only for \( \varepsilon \leq \varepsilon_m = q\mu / [q(q-1)] \). For \( \varepsilon \geq \varepsilon_m \), we set \( f = 0 \). For \( n = d/2 \), \( f \) is the Heaviside function. For \( q > 1 \), the distribution is defined for all energies (case 2). For large velocities, it behaves like \( f \sim v^{2n-d} \). Therefore, the density and the pressure are finite only for \( n < -1 \), i.e. \( d/(d+2) < q < 1 \). This fixes the range of allowed parameters. The equation of state is that of a polytrope \[8, 9\]

\[ p = K \rho^n, \quad \gamma = 1 + \frac{1}{n}. \quad (48) \]

For \( n > d/2 \) (case 1) the polytropic constant is

\[ K = \frac{1}{n+1} \left[ A S_d 2^{d-1} \frac{\Gamma(d/2) \Gamma(1-d/2+n)}{\Gamma(1+n)} \right]^{-1/n}, \quad (49) \]

and for \( n < -1 \) (case 2), we have

\[ K = -\frac{1}{n+1} \left[ A S_d 2^{d-1} \frac{\Gamma(d/2) \Gamma(-n)}{\Gamma(d/2-n)} \right]^{-1/n}, \quad (50) \]

where \( A = [\beta |q-1| / q]^{1/(q-1)} \). In the strong friction limit, we get the polytropic Smoluchowski equation

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (K \nabla \rho^n + \rho \nabla \Phi) \right]. \quad (51) \]

The free energy is the Tsallis free energy in physical space

\[ F[\rho] = \frac{K}{\gamma-1} \int (\rho^n - \rho) d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}. \quad (52) \]

The stationary solution is the Tsallis distribution in physical space

\[ \rho = \left[ \lambda - \frac{\gamma-1}{K\gamma} \Phi \right]^{\frac{1}{\gamma-1}}_+. \quad (53) \]

We note that a polytropic distribution with index \( q \) in phase space yields a polytropic distribution with index \( \gamma = 1 + 2(q-1)/(2+d(q-1)) \) in physical space. In this sense, Tsallis distributions are stable laws. By comparing Eq. (46) with Eq. (53) or Eqs. (9) and (44) with Eq. (52), we note that \( K \) plays the same role in physical space as the temperature \( T = 1/\beta \) in phase space. It is sometimes called a “polytropic temperature” \[9\].
3.3. Fermions: Fermi-Dirac entropy

If we consider the Fermi-Dirac entropy

\[
S_{FD}[f] = -\eta_0 \int \left\{ \frac{f}{\eta_0} \ln \frac{f}{\eta_0} + \left( 1 - \frac{f}{\eta_0} \right) \ln \left( 1 - \frac{f}{\eta_0} \right) \right\} drdv, \tag{54}
\]

we obtain the fermionic Kramers equation

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{1}{\partial_v} \left[ \xi \left( -T \eta_0 \frac{\partial}{\partial v} \ln \left( 1 - \frac{f}{\eta_0} \right) + f v \right) \right]. \tag{55}
\]

The stationary solution is the Fermi-Dirac distribution function

\[
f = \frac{\eta_0}{1 + \lambda e^{\beta \epsilon}}, \tag{56}
\]

where \(\lambda > 0\) is determined by the conservation of mass. The Fermi-Dirac distribution function \((56)\) satisfies the constraint \(f \leq \eta_0\) which is related to the Pauli exclusion principle in quantum mechanics. The isothermal distribution function \((39)\) is recovered in the non-degenerate limit \(f \ll \eta_0\) (valid at high temperatures). On the other hand in the completely degenerate limit (valid at low temperatures) the distribution is a step function corresponding to a polytrope of index \(n = d/2\). The distribution in physical space associated with the Fermi-Dirac statistics is

\[
\rho = \frac{\eta_0 S_d 2^{d-1}}{\beta^{d/2} I_{d/2-1}} (\lambda e^{\beta \Phi}), \tag{57}
\]

where \(I_n\) is the Fermi integral

\[
I_n(t) = \int_0^{+\infty} \frac{x^n}{1 + te^x} dx. \tag{58}
\]

The quantum equation of state for fermions is given in parametric form by

\[
\rho = \frac{\eta_0 S_d 2^{d-1}}{\beta^{d/2}} I_{d/2-1}(t), \quad p = \frac{\eta_0 S_d 2^d}{d \beta^{d+1}} I_d(t). \tag{59}
\]

At high temperatures we recover the classical isothermal law \(p = \rho T\) and at low temperatures we get a polytropic equation of state \(p = K \rho^\gamma\) with \(\gamma = \frac{d+2}{2}\) and \(K = \frac{1}{d+2} \left( \frac{d}{\eta_0 S_d} \right)^{2/d}\).

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