The Betti Numbers of the Ordered Configuration Space of the Once-Punctured Torus are Polynomial in the Number of Points

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Abstract

We prove that for \( k \) at least 3 the \( k \)-th Betti number of the ordered configuration space of the once-punctured torus is a polynomial in the number of points of degree \( 2k - 2 \). We do this by building on the work of Pagaria [Pag20] who proved the same growth relation for Betti numbers of the ordered configuration space of the torus. Since the once-punctured torus is an open manifold, the homology groups of its ordered configuration space are secondary representation stable in the sense of Miller and Wilson [MW19]. We use the growth rate of the Betti numbers of the ordered configuration space of the once-punctured torus to show that this space exhibits a secondary representation stability pattern not yet seen in other surfaces. The homology groups of the ordered configuration space of Euclidean space are well-understood; in contrast, much less is known of the homology groups of the ordered configuration spaces of positive-genus surfaces. Our computations are the first to demonstrate that secondary representation stability is a non-trivial phenomenon in positive-genus surfaces.

1 Introduction

For a topological space \( X \), let

\[ F_n(X) := \{(x_1, \ldots, x_n) | x_i \in X, x_i \neq x_j \text{ if } i \neq j\} \subseteq X^n \]

denote the ordered configuration space of \( n \) distinct points on \( X \). When \( X = \mathbb{R}^d \), the homology groups of \( F_n(\mathbb{R}^d) \) are isomorphic to the operad \( Pois^d(n) \); for more see [Sin06]. For most other manifolds explicit descriptions of the homology groups of their ordered configuration spaces are unknown. In his recent paper, Pagaria [Pag20, Corollary 2.9] proved that for \( k \geq 3 \), the \( k \)-th Betti number of the ordered configuration space of the once-punctured torus was polynomial in the number of marked points and of degree \( 2k - 2 \). We build on this to prove

**Theorem 1.1.** Let \( T^\circ \) denote the once-punctured torus. Then, for \( k \geq 3 \), the \( k \)-th Betti number of \( F_n(T^\circ) \) is a polynomial in \( n \) of degree \( 2k - 2 \). For \( k = 0, 1, 2 \), the \( k \)-th Betti number of \( F_n(T^\circ) \) is a polynomial in \( n \) of degree 0, 1, 3, respectively.

The symmetric group \( S_n \) acts on the configuration space \( F_n(X) \) by permuting the coordinates. When \( X \) is an open manifold like the once-punctured torus, the ordered configuration spaces have additional structure. If \( X \) is an open manifold of dimension \( d \), then there is an embedding

\[ e : X \sqcup \mathbb{R}^d \to X. \]

Such an embedding exists, for example, by Kupers and Miller [KM15, Lemma 2.4].

\[ T^\circ \] 

\[ \mathbb{R}^2 \]

\[ T^\circ \]

Figure 1: The embedding \( e : T^\circ \sqcup \mathbb{R}^2 \to T^\circ \)
The embedding induces an inclusion of ordered configuration spaces:

\[ \iota : F_{n-1}(X) \to F_n(X) \]

by setting

\[ \iota(x_1, \ldots, x_{n-1}) \mapsto (e(x_1), \ldots, e(x_{n-1}), e(0)), \]

where 0 denotes 0 \in \mathbb{R}^d. Thus, \( \iota \) maps a configuration of \( n-1 \) points in \( X \) to its image under \( e \) and adds a new point corresponding to the image under \( e \) of the origin in \( \mathbb{R}^d \).

![Figure 2: The inclusion \( \iota : F_4(T^\circ) \to F_5(T^\circ) \)](image)

The embedding also induces a map on the product of two configuration spaces:

\[ \iota' : F_{n-2}(X) \times F_2(\mathbb{R}^d) \to F_n(X) \]

given by

\[ \iota'((x_1, \ldots, x_{n-2}), (x'_1, x'_2)) = (e(x_1), \ldots, e(x_{n-2}), e(x'_1), e(x'_2)). \]

![Figure 3: The inclusion \( \iota' : F_4(T^\circ) \to F_6(T^\circ) \)](image)

The inclusions \( \iota \) and \( \iota' \) induce maps on homology:

\[ \iota_* : H_k(F_{n-1}(X)) \to H_k(F_n(X)) \quad \text{and} \quad \iota'_* : H_{k-1}(F_{n-2}(X)) \otimes H_1(F_2(\mathbb{R}^d)) \to H_k(F_n(X)). \]

The symmetric group action on ordered configuration space induces an action of \( \mathbb{Q}[S_n] \) on \( H_k(F_n(X); \mathbb{Q}) \) for all \( k \).

**Theorem 1.2.** (Church–Ellenberg–Farb [CEF15, Theorem 6.4.3] in the orientable case and Miller–Wilson [MW19, Theorem 3.12] in the general case) Let \( X \) be a connected, noncompact \( d \)-manifold with \( d \geq 2 \). For \( k \leq \frac{n-1}{2} \),

\[ \mathbb{Q}[S_n] \cdot \iota_* (H_k(F_{n-1}(X); \mathbb{Q})) = H_k(F_n(X); \mathbb{Q}). \]

We define another stabilization map, also denoted \( \iota'_* \), that leads to a notion of secondary representation stability:

\[ \iota'_* : H_{k-1}(F_{n-2}(X)) \to H_k(F_n(X)), \]

by pairing a class in \( H_{k-1}(F_{n-2}(X)) \) with the class in \( H_1(F_2(\mathbb{R}^d)) \) corresponding to the point \( n \) orbiting the point labeled \( (n-1) \) counterclockwise. Miller and Wilson [MW19] were able to show that for \( n \) large with respect to \( k \), the homology groups of ordered configuration space of an open manifold were secondary representation stable in that they satisfied the following theorem.
Theorem 1.3. (Miller–Wilson [MW19, Theorem 1.2]) Let $X$ be a connected noncompact finite type 2-manifold. There is a function $r : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ tending to infinity such that for $k \leq \frac{n-1}{2} + r(n),$

$$\mathbb{Q}[S_n] \cdot (\iota_*(H_k(F_{n-2}(X); \mathbb{Q})) + \iota'_*(H_{k-1}(F_{n-2}(X), \mathbb{Q}))) = H_k(F_n(X); \mathbb{Q}).$$

In their paper, Miller and Wilson [MW19, Propositions 3.33 and 3.35] calculated $\mathbb{Q}[S_n]$-span of the image of $\iota'_*$ for $\mathbb{R}^2$ and $k = \frac{n}{2}$ and for general surfaces for $k = 0$ and $k = \frac{n}{2}$. In the two first cases the homology groups were already known, and in the last case the image of $\iota'_*$ is homologically trivial, i.e., the image of $\iota'_*$ is 0. No other examples have been computed to the author’s knowledge.

Since $T^\circ$ is a connected non-compact finite type 2-manifold, the homology groups of its ordered configuration spaces are secondary representation stable. Moreover, theorem 1.1 implies

Corollary 1.4. Let $T^\circ$ denote the once-punctured torus. Then, for sufficiently large $n$,

$$\mathbb{Q}[S_n] \cdot (\iota_*(H_n(F_{2n-3}(T^\circ); \mathbb{Q})) + \iota'_*(H_{n-1}(F_{2n-4}(T^\circ), \mathbb{Q}))) = H_n(F_{2n-2}(T^\circ); \mathbb{Q});$$

moreover, this is the first example of secondary representation stability in the homology groups of a configuration space where the homology groups are unknown or the image of $\iota'_*$ is homologically non-trivial.

This paper was inspired by Jeremy Miller and Jenny Wilson’s paper on secondary representation stability for ordered configuration spaces of manifolds [MW19]. I would like to thank John Wiltshire-Gordon for sharing his computations of small-degree Betti numbers of $F_n(T^\circ)$ with Miller and Wilson; these computations were an indirect inspiration for this paper. Jenny was incredibly helpful to me in both understanding her paper and writing this one. I would also like to thank Andrew Snowden for insightful conversations on TCAs and Karen Butt for her comments on this paper.

2 First and Second Order Representation Stability

We introduce the language of $\text{FI}$-mod and $\text{FIM}^+\text{-mod}$. These category-theoretic constructions allow us to formalize the concepts of first and second order representation stability.

Definition 1. Let $\text{FB}$ be the category whose objects are all finite (possibly empty) sets and whose morphisms are bijective maps.

Every finite set is isomorphic to $[n] := \{1, \ldots, n\}$ for some $n$; this provides an equivalence between $\text{FB}$ and its full subcategory that has one set $[n]$ for each $n \in \mathbb{Z}_{\geq 0}^+$.

Definition 2. A $\text{FB}$-module over the ring $R$ is a covariant functor from $\text{FB}$ to the category of $R$-modules.

For an $\text{FB}$-module $W$ and a finite set $S$, let $W_S$ denote the corresponding $R$-module. When $S$ is the set $[n]$, we write $W_n$ for $W_{[n]}$. Each $W_n$ carries an action of $S_n$ arising from the equivalence $S_n \simeq \text{End}_{\text{FB}}([n])$, so $\{W_n\}$ is a sequence of symmetric group representations.

Definition 3. Let $\text{FI}$ be the category whose objects are all finite (possibly empty) sets and whose morphisms are injective maps.

Just as for $\text{FB}$, there is an equivalence between $\text{FI}$ and its full subcategory that has one set $[n]$ for each $n \in \mathbb{Z}_{\geq 0}^+$.

Definition 4. An $\text{FI}$-module over the ring $R$ is a covariant functor from $\text{FI}$ to the category of $R$-modules. Similarly, an $\text{FI}$-(homotopy)-space is a covariant functor from $\text{FI}$ to the (homotopy)-category of topological spaces.

Much like an $\text{FB}$-module, an $\text{FI}$-module is a sequence of symmetric group representations; however, there are relations between representations of different degrees. Let $V$ be an $\text{FI}$-module, if $\iota_{n,m}, n < m$, denotes the standard inclusion of $[n]$ into $[m]$, $(\iota_{n,m})_* : V_n \to V_m$ must be $S_n$-equivariant; moreover, $(\iota_{n,m})_*(V_n)$ must be invariant under the action of $S_{m-n}$.
We want to build up an FI-module from an FB-module. For an $S_d$-representation $W_d$, let
\[ M(W_d)_n := \bigoplus_{A \subseteq [n], |A| = d} W_A. \]

Letting $n$ vary over the nonnegative integers we see that $M(W_d)$ is an FI-module.

Since an FB-module is a sequence of symmetric group representations, we can apply $M(-)$ to every degree of an FB-module $W$; this gives a functor that induces an FI-module structure on $W$:
\[ M(W) := \bigoplus_{d \geq 0} M(W_d). \]

**Definition 5.** An FI-module $V$ is *generated* by a set $S \subseteq \bigsqcup_{n \geq 0} V_n$ if $V$ is the smallest FI-submodule containing $S$. If there is some finite set $S$ that generates $V$, then $V$ is *finitely generated*. If $V$ is generated by $\bigsqcup_{0 \leq n \leq d} V_n$, then $V$ is *generated in degree* $\leq d$.

We want to recover a generating set for an FI-module $V$, preferably a minimal one. One such generating set consists of subrepresentations of $V_n$, for all $n$, not arising from the FI-structure in smaller degrees. We use the language of FI-homology to formalize this.

**Definition 6.** The *zeroth FI-homology group* of an FI-module $V$ in degree $n$, denoted $H_0^{\text{FI}}(V)_n$, are the $S_n$ representations not arising from $V_A$, for all $A \subseteq [n], |A| = n - 1$:
\[ H_0^{\text{FI}}(V)_n := V_n \setminus \bigoplus_{A \subseteq [n], |A| = n-1} V_A. \]

We have special notation when $V$ is the homology of the ordered configuration space of an open manifold $X$:

**Definition 7.** Given $i, n \geq 0$, let $W_i^X(n)$ denote the sequence of minimal generators
\[ W_i^X(n) := H_0^{\text{FI}} \left( H_{n+i}^F(F(X); \mathbb{R}) \right)_n. \]

Note that $H_0^{\text{FI}}(V)_n$ is an $S_n$ representation, so $\{H_0^{\text{FI}}(V)_n\}_n$ is a sequence of symmetric group representations, i.e., an FB-module. Thus, we can think of $H_0(-)$ as a functor from FI-mod to FB-mod.

**Definition 8.** A *based set* $S_*$ is a set with a distinguished element $* \in S_*$, the basepoint. A map of based sets $f : S_* \to T_*$ takes $* \in S_*$ to $* \in T_*$. Then, FI# is the category whose objects are based finite sets and whose morphisms are maps of based sets that are injective away from the basepoint, i.e., if $f : S_* \to T_*$ is an FI# morphism, $|f^{-1}(t)| \leq 1$ for all $t \in T_*, t \neq *$.

**Definition 9.** An *FI#-module* over the commutative ring $R$ is a covariant functor from FI# to the category of $R$-modules. Similarly, an *FI#-homotopy-space* is a functor from FI# to the (homotopy)-category of topological spaces.

An FI#-module can be viewed as FI-module by forgetting the morphisms in FI# that aren’t injections. Similarly, an FI#-module can also be seen to be an FI$^{op}$-module by only considering surjective morphisms in FI#. Moreover, FI# is equivalent to FI$^{op}$.

**Theorem 2.1.** (Church–Ellenberg–Farb [CEF15, Theorem 4.1.5]) The category of FI#-modules is equivalent to the category of FB-modules via the equivalence of categories
\[ M(-) : \text{FB-Mod} \rightleftarrows \text{FI#-Mod} : H_0^{\text{FI}}(-). \]

Thus, every FI#-module $V$ is of the form $\bigoplus_{n=0}^{\infty} M(H_0^{\text{FI}}(V)_n)$. 

Recall the definition of the ordered configuration space of $n$ points on a open manifold $X$ and the inclusion map $\iota: F_{n-1}(X) \rightarrow F_n(X)$ defined in the introduction. The inclusion $\iota$ is well-behaved up to homotopy with respect to the symmetric group action on the indices making $F_n(X)$ an FI-homotopy-space. Taking the homology groups of these ordered configuration spaces gives us a sequence of FI-modules: for fixed $k \geq 0$, $H_k(F_n(X))$ is an FI-module. We can say even more, namely that forgetful map
\[
\pi : F_n(X) \rightarrow F_{n-1}(X),
\]
given by forgetting the last coordinate
\[
\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}),
\]
is well behaved with respect to the symmetric group action, and $F_n(X)$ is an FI#-homotopy-space. Fixing $k$, this makes $H_k(F_n(X))$ an FI#-module.

Irreducible representations of $S_m$ correspond to partitions of $m$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ is a partition of $m$, then we write $V_\lambda$ for the irreducible representation of $S_m$ corresponding to $\lambda$. Let $\lambda[n]$ denote the partition of $n$ of the form $(n-m, \lambda_1, \lambda_2, \ldots, \lambda_l)$. Every partition of $n$ can be uniquely written in this form for some $\lambda$. We let $V(\lambda)_n$ denote the $S_n$-representation
\[
V(\lambda)_n := \begin{cases} 0 & n < m + \lambda_1 \\ V_\lambda[n], & n \geq m + \lambda_1. \end{cases}
\]

**Definition 10.** Let $V_n$ be a sequence of rational $S_n$-representations with decomposition into irreducible constituents
\[
V_n = \bigoplus_{\lambda} c_\lambda^n V(\lambda)_n.
\]
Then $V_n$ is (uniformly) multiplicity stable if there exists some $N \geq 0$ such that, for all $\lambda$ and for all $n \geq N$, the multiplicities $c_\lambda^n = c_\lambda^N$ are independent of $n$.

**Theorem 2.2.** (Church–Ellenberg–Farb [CEF15, Theorem 6.4.3] in the orientable case and Miller–Wilson [MW19, Theorem 3.12] in the general case) Let $X$ be a connected, non-compact $d$-manifold with $d \geq 2$. For $n \geq 2k + 1$, $H_k(F_n(X); \mathbb{Z})$ is multiplicity stable.

For a thorough overview of FI and FI-mod see [CEF15] or [Wil18].

**Definition 11.** A *matching* of a set $A$ is a set of disjoint 2-element subsets of $A$, and a matching is perfect if the union of these subsets is $A$.

The category FI ignores the data of the complement of the image of a morphism; by insisting on a perfect matching on the complement of the image we get the category FIM.

**Definition 12.** Let $FIM$ denote the category whose objects are finite sets and whose morphisms are injective maps $f : A \rightarrow B$ along with a perfect matching on $B \setminus f(A)$.

Morphisms between two objects $A, B$ of FIM exist only when $|A|$ and $|B|$ have the same parity. If there are morphisms from $A$ to $B$, then the symmetric group $S_m$, $m = \frac{|B| - |A|}{2}$, acts on the perfect matching $B_1, \ldots, B_m$ on the complement of the image of $A$ in $B$ by permuting the ordering:
\[
\sigma \cdot (B_1, \ldots, B_m) = (B_{\sigma(1)}, \ldots, B_{\sigma(m)}).
\]

This inspires the definition of $FIM^+$, a category enriched over $R$-mod.

**Definition 13.** Let $FIM^+$ be the category whose objects are finite sets and whose module of morphisms $f$, consist of injective maps with a perfect matching on the complement quotiented by a signed symmetric group action:
\[
\frac{R\langle (f : A \rightarrow B, B_1, \ldots, B_m)|f \text{ is injective, } |B_i| = 2, B = im(f) \sqcup B_1 \cdots \sqcup B_m \rangle}{\langle (f, B_1, \ldots, B_m) = \text{sign}(\sigma)(f, B_{\sigma(1)}, \ldots, B_{\sigma(m)}) \text{ for all } \sigma \in S_m \rangle}.
\]
**Definition 14.** An $\text{FIM}^+$-module over the ring $R$ is a covariant functor from $\text{FIM}^+$ to the category of $R$-modules.

Many of the definitions for FI-modules can be adapted to $\text{FIM}^+$-modules including that of finite generation.

**Definition 15.** An $\text{FIM}^+$-module $W$ is generated by a set $S \subseteq \bigsqcup_{n \geq 0} W_n$ if $W$ is the smallest $\text{FIM}^+$-submodule containing $S$. If there is some finite set $S$ that generates $W$, then $W$ is finitely generated. If $W$ is generated by $\bigsqcup_{0 \leq n \leq d} V_n$, then $W$ is generated in degree $\leq d$.

One could rephrase the definitions and results for FI-mod and $\text{FIM}^+$-mod in the language of (skew)-twisted commutative algebras, see [SS12] and [NSS19] for example.

Miller and Wilson [MW19] showed that the sequence of minimal generators of the homology groups of the ordered configuration space of an open manifold formed an $\text{FIM}^+$-module, i.e., $W^X_i(n)$ is an $\text{FIM}^+$-module. Moreover, they proved that for an open manifold $X$, the homology groups of its ordered configuration spaces are secondary representation stable in the sense that they satisfy the following theorem.

**Theorem 2.3.** (Miller–Wilson [MW19, Theorem 1.4]) If $K$ is a field of characteristic zero and $X$ is a connected non-compact manifold of finite type and dimension at least two, then, for each $i \geq 0$, the sequence of minimal generators

$$W^X_i(n) = H_0^{\text{FI}} \left( H_{n+i}(\text{F}(X); K) \right)_n$$

is finitely generated as an $\text{FIM}^+$-module.

To prove this theorem, Miller and Wilson used the complex of injective words and a Noetherianity result of Nagpal, Sam, and Snowden [NSS19]. Furthermore, they computed some explicit examples of this secondary representation stability for the homology groups of the ordered configuration space on an open manifold in [MW19].

**Proposition 2.4.** (Miller–Wilson [MW19, Proposition 3.33])

$$W^X_0(2n) \cong \bigoplus_{\lambda \in D_{2n}} V_\lambda$$

where a partition $\lambda$ of $2n$ is in $D_{2n}$ if and only if when the associated Young diagram is cut in two along the upper staircase, then the resultant two skew subdiagrams are symmetric under reflection in the line of slope $-1$.

They made some calculations for generalized surfaces:

**Proposition 2.5.** (Miller–Wilson [MW19, Proposition 3.35]) Let $X$ be a connected non-compact surface. If $X$ is not orientable or of genus greater than zero, then

$$W^X_0(0) \cong \mathbb{Z} \quad \text{and} \quad W^X_0(2n) \equiv 0 \text{ for } n > 0.$$
These explicit examples of secondary representation stability are edge cases. When $X = \mathbb{R}^2$ the FIM$^+$-module $W_0^{(2n)}(2n)$ is a “free” FIM$^+$-module and all the homology groups of $F_*(\mathbb{R}^2)$ are known. For the second set of cases explicit calculation gives $W_0^{(2n)}(0) = H_1^{F_0}(H_0(F(X)))_0 = H_0(F_0(X)) \cong \mathbb{Z}$, a previously known result. The case of $W_0^{(2n)}(2n)$ for $n > 0$ lies on the opposite extreme of $W_0^{(2n)}(2n)$, as it is always the zero FIM$^+$-module.

We seek an example of secondary representation stability that lies between these extremes; namely, a finitely generated FIM$^+$-module arising from the homology groups of the ordered configuration space of a surface that is neither free nor zero and where the homology groups are unknown.

3 Ordered Configuration Space of the Once-Punctured Torus

In a recent paper [Pag20], Pagaria used a filtration on the Kriz model to show that Betti numbers of the ordered configuration space of the torus were polynomial in the number of points, and that for large $k$, the $k$-th Betti number $b_k$ was of degree $2k - 2$.

**Theorem 3.1.** (Pagaria [Pag20, Corollary 2.9]) For $k \geq 3$ the Betti numbers of $F_n(T)$ are of the form

$$b_k = c_k \binom{n}{2k-2} + o(n^{2k-2}),$$

where $c_k \geq \binom{2k-3}{k-3}$.

For $k \leq 5$ the Betti numbers are

- $b_0 = 1$,
- $b_1 = 2n$,
- $b_2 = 2 \binom{n}{3} + 3 \binom{n}{2} + n$,
- $b_3 = 14 \binom{n}{4} + 8 \binom{n}{3} + 2 \binom{n}{2}$,
- $b_4 = 32 \binom{n}{6} + 74 \binom{n}{5} + 33 \binom{n}{4} + 5 \binom{n}{3}$,
- $b_5 = 63 \binom{n}{8} + 427 \binom{n}{7} + 490 \binom{n}{6} + 154 \binom{n}{5} + 18 \binom{n}{4}$.

We can use Pagaria’s theorem to calculate the Betti numbers of the ordered configuration space of the once-punctured torus.

**Theorem 3.2.** Let $T^o$ denote the once-punctured torus. For $k \geq 3$, the $k$-th Betti number of the ordered configuration space of $n$ points on $T^o$, $b_k(F_n(T^o))$ is polynomial in $n$ of degree $2k - 2$. When $k = 0, 1, 2$, it is polynomial in $n$ of degree 0, 1, 3, respectively. For $k \leq 5$ the Betti numbers are given by the following formulae:

- $b_0(F_n(T^o)) = 1$
- $b_1(F_n(T^o)) = 2n$
- $b_2(F_n(T^o)) = 2 \binom{n}{3} + 5 \binom{n}{2}$
- $b_3(F_n(T^o)) = 14 \binom{n}{4} + 18 \binom{n}{3}$
- $b_4(F_n(T^o)) = 32 \binom{n}{6} + 106 \binom{n}{5} + 79 \binom{n}{4}$
- $b_5(F_n(T^o)) = 63 \binom{n}{8} + 490 \binom{n}{7} + 853 \binom{n}{6} + 432 \binom{n}{5}$.
Proof. The torus $T$ can be thought of as an additive group, namely $T \simeq \mathbb{R}^2 \setminus \mathbb{Z}^2$. This allows us to decompose the ordered configuration space of the torus as a product:

$$F_n(T) \simeq T \times F_{n-1}(T^\circ)$$

$$(x_1, \ldots, x_n) \mapsto x_1 \times (x_2 - x_1, \ldots, x_n - x_1),$$

where coordinates in $F_{n-1}(T^\circ)$ are taken modulo $\mathbb{Z}^2$. Here $x_1$ is the location of the puncture in $T^\circ$.

Since Poincare polynomials respect product decompositions, we can write

$$P(F_n(T)) = P(T) \times P(F_{n-1}(T^\circ))$$

$$= (1 + 2t + t^2)P(F_{n-1}(T^\circ)).$$

These equations can be rearranged to give the Poincare polynomial for $F_{n-1}(T^\circ)$ in terms of the Poincare polynomial for $F_n(T)$:

$$P(F_{n-1}(T^\circ)) = \frac{P(F_n(T))}{1 + 2t + t^2} = \sum_{i=0}^{\infty} (-1)^i(i + 1)t^i P(F_n(T)),$$

where the second equality arises by expanding $(1 + 2t + t^2)^{-1}$ as a Taylor series in $t$.

By explicitly expressing the Poincare polynomials in terms of Betti numbers, i.e., noting

$$P = \sum_{i=0}^{\infty} b_i t^i,$$

we have

$$\sum_{k=0}^{\infty} b_k(F_{n-1}(T^\circ)) t^k = \left( \sum_{i=0}^{\infty} (-1)^i(i + 1)t^i \right) \left( \sum_{j=0}^{\infty} b_j(F_n(T)) t^j \right) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^k m(k + 1 - m) b_m(F_n(T)) t^k.$$

This gives us a formula for the Betti numbers of $F_{n-1}(T^\circ)$ in terms of the Betti numbers for $F_n(T)$:

$$b_k(F_{n-1}(T^\circ)) = \sum_{m=0}^{k} (-1)^k m(k + 1 - m) b_m(F_n(T)).$$

By replacing $n - 1$ with $n$ we see that

$$b_k(F_n(T^\circ)) = \sum_{m=0}^{k} (-1)^k m(k + 1 - m) b_m(F_{n+1}(T)),$$

e.g.,

$$b_0(F_n(T^\circ)) = b_0(F_{n+1}(T)),$$

$$b_1(F_n(T^\circ)) = b_1(F_{n+1}(T)) - 2b_0(F_{n+1}(T)).$$

$$b_2(F_n(T^\circ)) = b_2(F_{n+1}(T)) - 2b_1(F_{n+1}(T)) + 3b_0(F_{n+1}(T)),$$

etc.

Now apply theorem 3.1, which states that $b_k(F_n(T))$ is a polynomial in $n$ of degree $2k - 2$, and for $k = 0, 1, 2$, it is a polynomial of degree 0, 1, 3. Reindexing from $n$ to $n + 1$ does not change this. For $k \geq 3$, the above calculations prove that $b_k(F_n(T^\circ))$ can be written as a linear combination of $k+1$ polynomials in $n$ of degree $\leq 2k - 2$, and that the degree $2k - 2$ term has leading coefficient $\frac{1}{(2k-2)!}$. When $k = 0, 1$, or 2, $b_k(F_n(T^\circ))$ can be expressed as linear combination of $k+1$ polynomials in $n$ of degree $\leq 0, 1$, or 3, respectively, with the highest degree term having leading coefficient 1, 1, or $\frac{1}{4}$, respectively. Thus, for $k \geq 3$, $b_k(F_n(T^\circ))$ is polynomial in $n$ of degree $2k - 2$. When $k = 0, 1, 2$, it is polynomial in $n$ of degree 0, 1, 3, respectively.

To get the formulae for $b_k(F_n(T^\circ))$ for $k \leq 5$ apply the formula

$$b_k(F_n(T^\circ)) = \sum_{m=0}^{k} (-1)^k m(k + 1 - m) b_m(F_{n+1}(T))$$

to the calculations for Betti numbers of the configuration space of the torus computed in Pagaria’s paper. \qed
Since $T^o$ is a non-compact manifold, we can apply the results of Church, Ellenberg, and Farb [CEF15] and Miller and Wilson [MW19] to study the sequence of homology group generators.

**Corollary 3.3.** The sequence of minimal generators

$$W_2^{T^o}(n) = H_0^{FI} \left( H_{2+2}(F(T^o); \mathbb{R}) \right)_n$$

is finitely generated as an $\text{FIM}^+$-module. Moreover, it is not stably zero and it arises from a set of spaces whose homology groups are unknown.

**Proof.** Since $T^o$ is an open manifold, Theorem 2.3 applies, proving that this sequence is finitely generated as an $\text{FIM}^+$-module. Since $H_k(F(T^o); \mathbb{R})$ is $\text{FI}^+$-module, by theorem 2.1 it is of the form $M(W) = \bigoplus_{d \geq 0} M(W_d)$, for some $\text{FB}$-module $W$. Recall that $M(W_d)_n = \bigoplus_{d \geq 0, A \subseteq [n], |A| = d} W_A$, so

$$M(W)_n = \bigoplus_{d \geq 0} M(W_d)_n = \bigoplus_{d \geq 0, A \subseteq [n], |A| = d} W_A.$$ 

The dimension of $M(W_d)_n$ is a polynomial in $n$ of degree $d$:

$$\dim(M(W_d)_n) = \dim(W_d) \cdot \binom{n}{d}$$

Since $H_k(F(T^o); \mathbb{R})$ is of the form $M(W) = \bigoplus_{d \geq 0} M(W_d)$ and $b_k(F_n(T^o))$ is a polynomial in $n$ of degree $2k-2$ for all $k \geq 3$, we see that

$$H_k(F(T^o); \mathbb{R}) = \bigoplus_{d \geq 0} M(W_d)$$

by theorem 3.2 and that $W_{2k-2}$ is not the zero module. Setting $k = \frac{n+2}{2}$, we see that

$$H_{2+2}(F(T^o); \mathbb{R}) = \bigoplus_{d \geq 0} M(W_n).$$

By theorem 2.1 $H_0^{FI}(M(W)) \cong W$, so $W_2^{T^o}(n) = H_0^{FI} \left( H_{2+2}(F(T^o); \mathbb{R}) \right)_n \cong W_n$ is not zero for $n > 0$. 

This is the first example of a stably-nonzero secondary representation stability sequence in positive genus, and in fact the first stably non-zero example for any manifold for which the homology groups of the ordered configuration spaces were not already explicitly known.

**Corollary 3.4.** The $\text{FIM}^+$-module $W_0^{T^o}(n)$ is stably zero for all $n > 0$, and the $\text{FIM}^+$-module $W_1^{T^o}(n) \neq 0$ for $n = 1, 2$, but is stably zero for all $n \geq 3$.

**Proof.** Our proofs of corollary 3.3 and theorem 3.2 show that for $k \geq 3$ there is no $2k$ term in

$$H_k(F(T^o); \mathbb{R}) = \bigoplus_{d \geq 0} M(W_d).$$

So $W_2^{T^o}(n)$ is stably zero for all $n \geq 3$. When $k = 1, 2$, the computations of the $k$-th Betti numbers in theorem 3.2 show that

$$H_1(F(T^o); \mathbb{R}) = \bigoplus_{d \geq 0} M(W_d)$$

and

$$H_2(F(T^o); \mathbb{R}) = \bigoplus_{d \geq 0} M(W_d);$$

moreover, the computations shows that the top terms are not zero. This proves that $W_1^{T^o}(n) \neq 0$ for $n = 1, 2$. These computations also show that $W_0^{T^o}(n)$ is stably zero for all $n > 0$, providing a new proof of 2.5 in the case $X = T^o$. 

\[\square\]
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