Abstract. A characteristic sample for a language $L$ and a learning algorithm $L$ is a finite sample of words $T_L$ labeled by their membership in $L$ such that for any sample $T \supseteq T_L$ consistent with $L$, on input $T$ the learning algorithm $L$ returns a hypothesis equivalent to $L$. Which $\omega$-automata have characteristic sets of polynomial size, and can these sets be constructed in polynomial time? We address these questions here.

In brief, non-deterministic omega-automata of any of the common types, in particular Büchi, do not have polynomial-size characteristic samples. Families of DFAs (FDFAs) possess polynomial-sized characteristic samples that can be constructed in polynomial time. Since deterministic omega automata of types Büchi, co-Büchi, and parity can be polynomially translated to FDFAs it follows that they too possess characteristic samples that can be constructed in polynomial time; however, no corresponding polynomial time learning algorithms is known. For deterministic omega automata that are isomorphic to their right congruence automata, the fully informative languages, polynomial time algorithms for constructing characteristic samples and learning from them is given.

The algorithms for constructing characteristic sets in polynomial time for the different omega automata (of types Büchi, coBüchi, parity, Rabin, Street, or Muller), require deterministic polynomial time algorithms for (1) equivalence of the respective omega automata, and (2) testing membership of the language of the automaton in the informative classes, which we provide.

Contents

1. Introduction 3
2. Preliminaries 5
3. Notions of Learnability and Teachability 8
4. Which representations are efficiently teachable/learnable? 12
   4.1. Negative results for nondeterministic classes 12
   4.2. Some positive results 12
   4.3. Relation to learning with equivalence and membership queries 13
5. The informative classes are efficiently teachable/learnable 15
   5.1. Duality 15
   5.2. The default acceptor 15
| Section | Title                                                                 | Page |
|---------|-----------------------------------------------------------------------|------|
| 5.3     | Strongly connected components                                           | 16   |
| 5.4     | A decreasing forest of SCCs of an automaton                            | 18   |
| 5.5     | Proving efficient teachability of the informative classes — overview  | 18   |
| 6       | The sample $T_{Aut}$ for the automaton                                 | 19   |
| 6.1     | Existence of short distinguishing words                                | 19   |
| 6.2     | Defining the sample $T_{Aut}$ for the automaton                        | 20   |
| 6.3     | Learning the automaton from $T_{Aut}$                                  | 20   |
| 7       | The samples $T_{Acc}$ and learning algorithms for IMA and IBA          | 21   |
| 7.1     | Muller acceptors                                                       | 21   |
| 7.2     | Büchi acceptors                                                       | 22   |
| 8       | The sample $T_{Acc}$ and learning for IPA                              | 23   |
| 8.1     | Constructing the Canonical Forest and Coloring of a DPA                | 23   |
| 8.2     | Constructing $T_{IPA}^{IA}$                                            | 26   |
| 8.3     | The learning algorithm $L_{IPA}^{IA}$                                  | 26   |
| 9       | The sample $T_{Acc}$ and learning for IRA                              | 28   |
| 9.1     | Singleton normal form for a Rabin acceptor                            | 28   |
| 9.2     | An ordering on sets of states                                          | 28   |
| 9.3     | The learning algorithm $L_{IRA}^{IA}$                                  | 29   |
| 9.4     | Constructing $T_{IRA}^{IA}$                                            | 29   |
| 9.5     | Correctness of $L_{IRA}^{IA}$                                          | 31   |
| 10      | Constructing characteristic samples in polynomial time                 | 32   |
| 10.1    | Computing $T_{Aut}$                                                    | 32   |
| 10.2    | The problems of inclusion and equivalence                              | 32   |
| 10.3    | Computing $T_{Acc}$                                                    | 33   |
| 11      | Inclusion algorithms                                                  | 34   |
| 12      | Inclusion and equivalence for DPAs, DBAs, DCAs                         | 35   |
| 12.1    | Searching for $w$ with given minimum colors in two acceptors          | 35   |
| 12.2    | An inclusion algorithm for DPAs                                        | 35   |
| 13      | An inclusion algorithm for DRAs                                        | 36   |
| 14      | An inclusion algorithm for DMAs                                        | 37   |
| 14.1    | Reduction of DMA inclusion to DBA/DMA inclusion                         | 38   |
| 14.2    | A DBA/DMA inclusion algorithm                                          | 40   |
| 15      | Computing the automaton $M_{\sim L}$                                  | 42   |
| 16      | Testing membership in $I_{X_{A}}$                                     | 43   |
| 16.1    | Testing membership in $I_{BA}$                                        | 43   |
| 16.2    | Testing membership in $I_{PA}$                                        | 44   |
| 16.3    | Testing membership in $I_{RA}$                                        | 45   |
| 16.4    | Testing membership in $I_{RA}$                                        | 48   |
| 16.5    | Variants of the testing algorithms                                    | 49   |
| 16.6    | Efficient teahability of the informative classes                      | 49   |
| 17      | Discussion                                                            | 49   |
|         | References                                                             | 50   |
1. Introduction

With the growing success of machine learning in efficiently solving a wide spectrum of problems, we are witnessing an increased use of machine learning techniques in formal methods for system design. One thread in recent literature uses general purpose machine learning techniques for obtaining more efficient verification/synthesis algorithms. Another thread, following the automata theoretic approach to verification [Var95, KVW00] works on developing grammatical inference algorithms for verification and synthesis purposes. Grammatical inference (aka automata learning) refers to the problem of automatically inferring from examples a finite representation (e.g. an automaton, a grammar, or a formula) for an unknown language [dlH10]. The term model learning [Vaa17] was coined for the task of learning an automaton model for an unknown system. A large body of works has developed learning techniques for different automata types (e.g. visibly-pushdown automata [KMV06], I/O automata [AV10], register automata [HSJC12], symbolic automata [DD17], program automata [MS18], probabilistic grammars [NFZ21], lattice automata [FS22]) and has shown its usability in a diverse range of tasks.\footnote{E.g., tasks such as black-box checking [PVY99], specification mining [ABL02], assume-guarantee reasoning [CGP03], regular model checking [HV05], learning verification fixed-points [VSA05], learning interfaces [NA06], analyzing botnet protocols [CBS10] or smart card readers [CPP14], finding security bugs [CPP14], error localization [CCK+15], and code refactoring [MR04, SHV16].}

In grammatical inference, the learning algorithm does not learn a language, but rather a finite representation of it. The complexity of learning algorithms may vary greatly by switching representations. For instance, if one wishes to learn regular languages, she may consider representations using deterministic finite automata (DFAs), non-deterministic finite automata (NFAs), regular expressions, linear grammars, etc. Since the translation results between two such formalisms are not necessarily polynomial, a polynomial learnability result for one representation does not necessarily imply a polynomial learnability result for another representation. Let $C$ be a class of representations $\mathcal{C}$ with a size measure $\text{size}(\cdot)$ (e.g. for DFAs the size measure can be the number of states in the minimal DFA). We extend $\text{size}(\cdot)$ to the languages recognized by representations in $\mathcal{C}$ by defining $\text{size}(L)$ to be the minimum of $\text{size}(\mathcal{C})$ over all $\mathcal{C}$ representing $L$. In this paper we restrict attention to automata representations, namely, acceptors.

There are various learning paradigms considered in the grammatical inference literature, roughly classified into passive and active. We mention here the two central ones. In passive learning the model of learning from finite data refers to the following problem: given a finite sample $T \subseteq \Sigma^* \times \{0, 1\}$ of labeled words, a learning algorithm $L$ should return an acceptor $\mathcal{C}$ that agrees with the sample $T$. That is, for every $(w, l) \in T$ the following holds: $w \in \llbracket \mathcal{C} \rrbracket$ iff $l = 1$ (where $\llbracket \mathcal{C} \rrbracket$ is the language accepted by $\mathcal{C}$). The class $\mathcal{C}$ is identifiable in the limit using polynomial time and data (in short ILPTD) if and only if there exists a polynomial time algorithm $L$ that takes as input a labeled sample $T$ and outputs an acceptor $\mathcal{C} \in \mathcal{C}$ that is consistent with $T$, and $L$ also satisfies the following condition. If $L$ is any language recognized by an automaton from class $\mathcal{C}$, then there exists a labeled sample $T_L$ consistent with $L$ of length bounded by a polynomial in $\text{size}(L)$, and for any labeled sample $T$ consistent with $L$ such that $T_L \subseteq T$, on input $T$ the algorithm $L$ produces an acceptor $\mathcal{C}$ that recognizes $L$. In this case, $T_L$ is termed a characteristic sample for the algorithm $L$. The definition of ILPTD does not require that the characteristic set can be computed in polynomial time, which is also desired. The definition of efficiently teachable adds this
requirement. In §3 we define several notions related to efficient teachability and learnability, the stronger one is efficiently teachable/learnable. The question which representations of ω-regular languages are efficiently teachable/learnable is the focus of this paper.

In active learning the model of query learning [Ang87] assumes the learner communicates with an oracle that can answer certain types of queries about the language. The most common type of queries are membership queries (is w ∈ L where L is the unknown language) and equivalence queries (is [A] = L where A is the current hypothesis for an acceptor recognizing L). Equivalence queries are typically assumed to return a counterexample, i.e., a word in [A] \ L or in L \ [A].

With regard to ω-automata (automata on infinite words) most of the works consider query learning using membership queries and equivalence queries. The representations learned so far include: (L)S [FCC08], a non-polynomial reduction to finite words; families of DFAs (FDFA) [AF14, AF16, ABF16, LCZL17]; strongly unambiguous Büchi automata (SUBA) [AAF20]; mod-2-multiplicity automata (M2MA) [AAFG22]; and deterministic weak parity automata (DWPA) [MP95]. Among these only the latter two are known to be learnable in polynomial time using membership queries and proper equivalence queries.2 We show in §4 that the classes FDFA, M2MA and DWPA are efficiently teachable/learnable. The classes DBA, DCA, DPA and SUBA are efficiently teachable but are not known to be efficiently teachable/learnable.

One of the main obstacles in obtaining a polynomial learning algorithm for regular ω-languages is that they do not in general have a Myhill-Nerode characterization; that is, there is no theorem correlating the states of a minimal automaton of some of the common automata types (Büchi, parity, Muller, etc.) to the equivalence classes of the right congruence of the language. The right congruence relation for an ω-language L relates two finite words x and y iff there is no infinite suffix z differentiating them, that is x ∼ L y (for x, y ∈ Σ*) iff ∀z ∈ Σω, xz ∈ L ⇐⇒ yz ∈ L. The quest for finding a polynomial query learning algorithm for a subclass of the regular ω-languages, lead to studying subclasses of languages for which such a relation holds. These languages are termed fully informative [AF18]. We use IBA, ICA, IPA, IRA, ISA, IMA to denote the classes of languages that are fully informative of type Büchi, coBüchi, parity, Rabin, Streett, and Muller, respectively. A language L is said to be fully informative of type X for X ∈ {B, C, P, R, S, M} if there exists a deterministic automaton of type X that recognizes L and is isomorphic to the automaton derived from ∼L. While many properties of these classes are now known, in particular that they span the entire hierarchy of ω-regular properties [Wag75], a polynomial learning algorithm for them is not known.

We show (in §5-9) that the classes IBA, ICA, IPA, IRA, ISA, IMA can be identified in the limit using polynomial time and data. We further show (in §10) that there is a polynomial time algorithm to compute a characteristic sample given an acceptor C ∈ IXA. To show that these classes are also efficiently teachable we need polynomial time algorithms for inclusion and equivalence of automata of these types. Such an algorithm is known to exist for the classes NBA, NCA, NPA, since these classes have inclusion algorithms in NL. For the other classes, we are not aware of results in the literature showing a polynomial time algorithm. We provide such algorithms in §11-14.

The last part of this manuscript (§15-16) is devoted to the question of deciding whether a given automaton A of type X is isomorphic to its right congruence, or if this is not

2Query learning with an additional type of query, loop-index queries, was studied for deterministic Büchi automata [MO20].
the case whether there exists an automaton $\mathcal{M}'$ of the same type that recognizes the same language and is isomorphic to its right congruence, namely whether the given automaton recognizes a language in the class $\mathbb{I}KA$. Using this result we can show that a teacher can construct a characteristic sample not only given an acceptor which is isomorphic to the right congruence of the language, but also given an acceptor which is not, but is equivalent to such an acceptor. We conclude in §17 with a short discussion.

2. Preliminaries

**Automata.** An *automaton* is a tuple $\mathcal{M} = \langle \Sigma, Q, q, \delta \rangle$ consisting of a finite alphabet $\Sigma$ of symbols, a finite set $Q$ of states, an initial state $q_0 \in Q$, and a transition function $\delta : Q \times \Sigma \to 2^Q$. We extend $\delta$ to domain $Q \times \Sigma^*$ in the usual way: $\delta(q, \varepsilon) = q$ and $\delta(q, \sigma x) = \bigcup_{q' \in \delta(q, \sigma)} \delta(q', x)$ for all $q \in Q$ and $\sigma, x \in \Sigma$.

We define the size of an automaton to be $|\Sigma| \cdot |Q|$. A state $q \in Q$ is *reachable* iff there exists $x \in \Sigma^*$ such that $q = \delta(q, x)$. For $q \in Q$, $\mathcal{M}'$ is the automaton $\mathcal{M}$ with its initial state replaced by $q$. We say that $\mathcal{A}$ is *deterministic* if $|\delta(q, \sigma)| \leq 1$ and *complete* if $|\delta(q, \sigma)| \geq 1$, for every $q \in Q$ and $\sigma \in \Sigma$. For deterministic automata we abbreviate $\delta(q, \sigma) = \{q'\}$ as $\delta(q, \sigma) = q'$. Two automata $\mathcal{M}$ and $\mathcal{M}'$ with the same alphabet $\Sigma$ are isomorphic if there exists a bijection $f$ from the states $Q$ of $\mathcal{M}$ to the states $Q'$ of $\mathcal{M}'$ such that $f(q) = q'$ and for every $q \in Q$ and $\sigma \in \Sigma$, $\{f(r) \mid r \in \delta(q, \sigma)\} = \delta'(f(q), \sigma)$.

We assume a fixed total ordering on $\Sigma$, which induces the *shortlex* total ordering on $\Sigma^*$, defined as follows. For $x, y \in \Sigma^*$, $x$ precedes $y$ in the shortlex ordering if $|x| < |y|$ or $|x| = |y|$ and $x$ precedes $y$ in the lexicographic ordering induced by the ordering on $\Sigma$.

A *run* of an automaton on a finite word $v = a_1 a_2 \ldots a_n$ is a sequence of states $q_0, q_1, \ldots, q_n$ such that $q_0 = q_i$, and for each $i \geq 1$, $q_i \in \delta(q_{i-1}, a_i)$. A *run* on an infinite word is defined similarly and consists of an infinite sequence of states. For an infinite run $\rho = q_0, q_1, \ldots$, we define the set of states visited infinitely often, denoted $\text{inf}_{\mathcal{M}}(\rho)$, as the set of $q \in Q$ such that $q = q_i$ for infinitely many indices $i \in \mathbb{N}$. This is abbreviated to $\text{inf}(\rho)$ if $\mathcal{M}$ is understood.

**The Product of Two Automata.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two deterministic complete automata with the same alphabet $\Sigma$, where for $i = 1, 2$, $\mathcal{M}_i = \langle \Sigma, Q_i, (q_i)_i, \delta_i \rangle$. Their product automaton, denoted $\mathcal{M}_1 \times \mathcal{M}_2$, is the deterministic complete automaton $\mathcal{M} = \langle \Sigma, Q, q, \delta \rangle$ such that $Q = Q_1 \times Q_2$ is the set of ordered pairs of states of $\mathcal{M}_1$ and $\mathcal{M}_2$; the initial state $q_i = ((q_i)_1, (q_i)_2)$ is the pair of initial states of the two automata; and for all $(q_1, q_2) \in Q$ and $\sigma \in \Sigma$, $\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma))$. For $i = 1, 2$, let $\pi_i$ be projection onto the $i$-th coordinate, so that for a subset $S$ of $Q$, $\pi_1(S) = \{q_1 \mid \exists q_2(q_1, q_2) \in S\}$, and analogously for $\pi_2$.

**Acceptors.** By augmenting an automaton $\mathcal{M} = \langle \Sigma, Q, q, \delta \rangle$ with an acceptance condition $\alpha$, obtaining a tuple $\mathcal{A} = \langle \Sigma, Q, q, \delta, \alpha \rangle$, we get an acceptor, a machine that accepts some words and rejects others. We may also denote $\mathcal{A}$ by $(\mathcal{M}, \alpha)$. An acceptor accepts a word if at least one of the runs on that word is accepting. If the automaton is not complete, a given word $w$ may not have any run in the automaton, in which case $w$ is rejected.

For finite words the acceptance condition is a set $F \subseteq Q$ and a run on a word $v$ is accepting if it ends in an accepting state, i.e., if $\delta(v)$ contains an element of $F$. For infinite words, there are various acceptance conditions in the literature, and we consider six of
them: Büchi, coBüchi, parity, Rabin, Streett, and Muller, all based on the set of states visited infinitely often in a given run. For each model we define the related quantity of the size of the acceptor, taking into account the acceptance condition.

A Büchi or coBüchi acceptance condition is a set of states $F \subseteq Q$. A run $\rho$ of a Büchi acceptor is accepting if it visits $F$ infinitely often, that is, $\inf(\rho) \cap F \neq \emptyset$. A run $\rho$ of a coBüchi acceptor is accepting if it visits $F$ only finitely many times, that is, $\inf(\rho) \cap F = \emptyset$. The size of a Büchi or coBüchi acceptor is the size of its automaton.

A parity acceptance condition is a map $\kappa : Q \rightarrow \mathbb{N}$ assigning to each state a natural number termed a color (or priority). A run of a parity acceptor is accepting if the color visited infinitely often is odd. The size of a parity acceptor is the size of its automaton.

A Rabin or Streett acceptance condition consists of a finite set of pairs of sets of states $\alpha = \{(G_1, B_1), \ldots, (G_k, B_k)\}$ for some $k \in \mathbb{N}$ and $G_i \subseteq Q$ and $B_i \subseteq Q$ for $i \in [1..k]$. A run of a Rabin acceptor is accepting if there exists an $i \in [1..k]$ such that $G_i$ is visited infinitely often and $B_i$ is visited finitely often. A run of a Streett acceptor is accepting if for all $i \in [1..k]$, $G_i$ is visited finitely often or $B_i$ is visited infinitely often. The size of a Rabin or Streett acceptor is the sum of the size of its automaton and $k - 1$.

A Muller acceptance condition is a set of sets of states $\alpha = \{F_1, F_2, \ldots, F_k\}$ for some $k \in \mathbb{N}$ and $F_i \subseteq Q$ for $i \in [1..k]$. A run of a Muller acceptor is accepting if the set $S$ of states visited infinitely often in the run is a member of $\alpha$. The size of a Muller acceptor is the sum of the size of its automaton and $k - 1$.

The set of words accepted by an acceptor $A$ is denoted by $[A]$. Two acceptors $A$ and $B$ are equivalent if they accept the same language, that is, $[A] = [B]$. For a state $q$, the acceptor $A^q$ is the acceptor $A$ with its automaton initial state replaced by $q$. We say that the $\omega$-word $w$ is accepted from state $q$ iff $w \in [A^q]$.

We use NBA, NCA, NPA, NRA, NSA, NMA (resp., DBA, DCA, DPA, DRA, DSA, DMA) for non-deterministic (resp. deterministic) Büchi, coBüchi, parity, Rabin, Streett, and Muller acceptors. We use NBA, NCA, NPA, NMA, NRA, NSA (resp., DBA, DCA, DPA, DRA, DSA, DMA) for the corresponding class of representations. We use $[X]$ for the class of languages accepted by an acceptor in $X$. $[NCA]$ is the same as $[DCA]$, and $[DCA]$ and $[DBA]$ are distinct proper subclasses of the regular $\omega$-languages. The other classes are the full class of regular $\omega$-languages.

Some relationships between the models. We observe the following known relationships.

Claim 2.1. Let $A$ be an acceptor of one of the types NBA, NCA, NPA, NRA, NSA, or NMA with $n$ states over the alphabet $\Sigma$. There is an equivalent complete acceptor $A'$ of the same type whose size is at most $|\Sigma|$ larger. $A'$ may be taken to be deterministic if $A$ is deterministic.

Claim 2.2. (1) Let $B = \langle \Sigma, Q, q_0, \delta, F \rangle$, where $B$ is an NBA. Define the NPA $P = \langle \Sigma, Q, q_0, \delta, \kappa \rangle$ where $\kappa(q) = 1$ if $q \in F$ and $\kappa(q) = 2$ otherwise. Then $B$ and $P$ are equivalent and have the same size. $P$ is deterministic if $B$ is.

(2) Let $B = \langle \Sigma, Q, q_0, \delta, F \rangle$, where $B$ is an NBA. Define the NPA $R = \langle \Sigma, Q, q_0, \delta, \{(F, \emptyset)\} \rangle$.

Then $B$ and $R$ are equivalent and have the same size. $R$ is deterministic if $B$ is.

(3) Let $C = \langle \Sigma, Q, q_0, \delta, F \rangle$, where $C$ is a NCA. Define the NSA $S = \langle \Sigma, Q, q_0, \delta, \{(F, \emptyset)\} \rangle$.

Then $S$ and $C$ are equivalent and have the same size. $S$ is deterministic if $C$ is.

(4) Let $B = C = \langle \Sigma, Q, q_0, \delta, F \rangle$, where $B$ is a complete DBA and $C$ is a complete DCA.

Then $B$ and $C$ are the same size and the languages they recognize are complements of each other, that is, $[B] = \Sigma^* \setminus [C]$.
(5) Let $\mathcal{R} = \mathcal{S} = \langle \Sigma, Q, q_0, \delta, \{(G_1, B_1), \ldots, (G_k, B_k)\} \rangle$, where $\mathcal{R}$ is a complete DRA and $\mathcal{S}$ is a complete DSA. Then $\mathcal{R}$ and $\mathcal{S}$ have the same size and the languages they recognize are complements of each other, that is, $[\mathcal{R}] = \Sigma^\omega \setminus [\mathcal{S}]$.

**Right congruence.** An equivalence relation $\sim$ on $\Sigma^*$ is a *right congruence* if $x \sim y$ implies $x\sigma \sim y\sigma$ for all $x, y \in \Sigma^*$ and $\sigma \in \Sigma$. The index of $\sim$, denoted $|\sim|$ is the number of equivalence classes of $\sim$. For a word $x \in \Sigma^*$ the notation $[x]_\sim$ denotes the equivalence class of $\sim$ that contains $x$.

With a right congruence $\sim$ of finite index one can naturally associate a complete deterministic automaton $M_\sim = \langle \Sigma, Q, q_0, \delta \rangle$ as follows: the set of states $Q$ consists of the equivalence classes of $\sim$. The initial state $q_0$ is the equivalence class $[\varepsilon]_\sim$. The transition function $\delta$ is defined by $\delta([u]_\sim, \sigma) = [u\sigma]_\sim$ for all $\sigma \in \Sigma$. Also, given a complete deterministic automaton $M = \langle \Sigma, Q, q_0, \delta \rangle$, we can naturally associate with it a right congruence as follows: $x \sim_M y$ iff $M$ reaches the same state of $M$ when reading $x$ or $y$, that is, $\delta(q_0, x) = \delta(q_0, y)$.

Given a language $L \subseteq \Sigma^*$ its *canonical right congruence* $\sim_L$ is defined as follows: $x \sim_L y$ iff $\forall z \in \Sigma^*$ we have $xz \in L \iff yz \in L$. The Myhill-Nerode Theorem states that a language $L \subseteq \Sigma^*$ is regular iff $\sim_L$ is of finite index. Moreover, if $L$ is accepted by a complete DFA $\mathcal{A}$, then $\sim_M$ refines $\sim_L$, where $M$ is the automaton of $\mathcal{A}$. Finally, any complete DFA of minimum size that accepts $L$ has an automaton that is isomorphic to $M_{\sim_L}$.

For an $\omega$-language $L \subseteq \Sigma^\omega$, its *canonical right congruence* $\sim_L$ is defined similarly, by quantifying over $\omega$-words. That is, $x \sim_L y$ iff $\forall z \in \Sigma^\omega$ we have $xz \in L \iff yz \in L$. If $L$ is a regular $\omega$-language then $\sim_L$ is of finite index, and for any complete DBA (resp., DCA, DPA, DRA, DSA, DMA) $\mathcal{A}$ that accepts $L$, $\sim_M$ refines $\sim_L$, where $M$ is the automaton of the acceptor.

However, for regular $\omega$-languages, the relation $\sim_L$ does not suffice to obtain a “Myhill-Nerode” characterization. In particular, for a regular $\omega$-language $L$ there may be no way to define an acceptance condition for $M_{\sim_L}$ that yields a DBA (resp., DCA, DPA, DRA, DSA, DMA) that accepts $L$. As an example consider the language $L = (a + b)^*(bba)^\omega$. Then $\sim_L$ consists of just one equivalence class, because for any $x \in \Sigma^*$ and $w \in \Sigma^\omega$ we have that $xw \in L$ iff $w$ has $(bba)^\omega$ as a suffix. But a DBA (resp., DCA, DPA, DRA, DSA, DMA) that accepts $L$ clearly needs more than a single state.

**The classes IIBA, ICA, IPA, IRA, ISA and IMA.** In light of the lack of a Myhill-Nerode result for regular $\omega$-languages, we define a restricted type of deterministic Büchi (resp., coBüchi, parity, Rabin, Streett, Muller) acceptor. A DBA (resp., DCA, DPA, DRA, DSA, DMA) $\mathcal{A}$ that accepts the language $L = [\mathcal{A}]$ is an IBA (resp., ICA, IPA, IRA, ISA, IMA) if it is complete and its automaton is isomorphic to $M_{\sim_L}$. A DBA $\mathcal{B}$ is in IIBA (resp., IICA, IPA, IRA, ISA, IMA) if there exists an IBA (resp, ICA, IPA, IRA, ISA, IMA) $\mathcal{A}$ such that $[\mathcal{B}] = [\mathcal{A}]$. We note that every state of an IBA (resp., ICA, IPA, IRA, ISA, IMA) is reachable because every state of $M_{\sim_L}$ is reachable.

Despite the fact that each of these classes is a proper subset of its corresponding deterministic class (e.g., IIBA is a proper subset of DIBA), these classes are more expressive than one might first conjecture. It was shown in [AF18] that in every class of the infinite Wagner hierarchy [Wag75] there are languages in IMA and IPA. Moreover, in a small experiment
reported in [AF18], among randomly generated Muller automata, the vast majority turned out to be in \( \text{IMA} \).

3. Notions of Learnability and Teachability

In this section we define notions of (efficient) learnability and teachability, and discuss the relations between them. A summary of the presented notions is provided in Table 1.

**Examples and samples.** Because we require finite representations of examples, \( \omega \)-words in our case, we work with ultimately periodic words, that is, words of the form \( u(v)^\omega \) where \( u \in \Sigma^* \) and \( v \in \Sigma^+ \). It is known that two regular \( \omega \)-languages are equivalent iff they agree on the set of ultimately periodic words [Buc62, CNP93], so this choice is not limiting.

The example \( u(v)^\omega \) is concretely represented by the pair \((u, v)\) of finite strings, and its length is \( |u| + |v| \). A labeled example is a pair \((u(v)^\omega, l)\), where the label \( l \) is either 0 or 1. A sample is a finite set of labeled examples such that no example is assigned two different labels. The length of a sample is the sum of the lengths of the examples that appear in it. A sample \( T \) and a language \( L \) are consistent with each other if and only if for every labeled example \( (u(v)^\omega, l) \in T, l = 1 \) iff \( u(v)^\omega \in L \). A sample \( T \) and an acceptor \( A \) are consistent with each other if and only if \( T \) is consistent with \( [A] \). The following results give two useful procedures on examples that are computable in polynomial time.

**Proposition 3.1.** Let \( u_1, u_2 \in \Sigma^* \) and \( v_1, v_2 \in \Sigma^+ \). If \( u_1(v_1)^\omega \neq u_2(v_2)^\omega \) then they differ in at least one of the first \( \ell \) symbols, for \( \ell = \max(|u_1|, |u_2|) + |v_1| \cdot |v_2| \).

Let \( \text{suffixes}(u(v)^\omega) \) denote the set of all \( \omega \)-words that are suffixes of \( u(v)^\omega \).

**Proposition 3.2.** The set \( \text{suffixes}(u(v)^\omega) \) consists of at most \( |u| + |v| \) different examples: one of the form \( u'(v)^\omega \) for every nonempty suffix \( u' \) of \( u \), and one of the form \( (v_2v_1)^\omega \) for every division of \( v \) into a non-empty prefix and suffix as \( v = v_1v_2 \).

**Characteristic Samples, Learner, Teacher.** Let \( C \) be a class of representations over an infinite domain \( D \) of examples, both represented by finite strings over a finite alphabet. We assume that \( C \) satisfies the following properties.

(C1) There is an algorithm to decide membership of \( C \) in \( C \) given \( C \).
(C2) There is an algorithm to decide membership of \( w \) in \([C]\) given \( w \) and \( C \in C \).
(C3) For any sample \( T \) there exists \( C \in C \) consistent with \( T \).

A learner for \( C \) is a function \( L \) that maps a sample \( T \) to some \( C_T \in C \) such that \( C_T \) and \( T \) are consistent. A teacher for \( C \) is a function \( T \) that maps \( C \in C \) to a sample \( T_C \) such that \( T_C \) and \( C \) are consistent. Note that learners and teachers need not be computable. A sample \( T \) is a characteristic sample for \( C \) and a learner \( L \) if \( T \) is consistent with \( C \) and for every sample \( T' \supseteq T \) consistent with \( C \) we have \([L(T')]=[C]\). The intuition is that additional information consistent with \( C \) beyond \( T \) will not cause the learner to change its mind about \([C]\) being the correct hypothesis.

A class \( C \) has characteristic samples for a learner \( L \) if there exist a teacher \( T \) such that for every \( C \in C \), \( T(C) \) is a characteristic sample for \( C \) and \( L \). A class \( C \) has characteristic samples if it has characteristic samples for some learner.

**Proposition 3.3.** Under the assumptions (C1), (C2), and (C3), there is a computable learner \( L \) such that class \( C \) has characteristic samples for \( L \).
Proof. The learner $L$ can be taken to be the classic algorithm of identification by enumeration. Given a sample $T$, the elements of $C$ are enumerated in a fixed order $C_1, C_2, \ldots$, and the output is $C_n$ for the least $n$ such that $C_n$ is consistent with $T$. This is possible because membership in $C$ is decidable, whether $C$ is consistent with $T$ is decidable, and for any sample $T$ there exists some $C \in C$ consistent with $T$.

The teacher $T$ with input $C$ finds the least $n$ such that $[C_n] = [C]$, and for each $m < n$ determines an example $x_m$ that distinguishes $[C_m]$ from $[C_n]$. The output sample consists of all $x_m$ with $m < n$, labeled according to $C$. $\square$

We cannot necessarily take the teacher $T$ in this construction to be computable. Fisman et al. [FFZ23] show that there exists a class $C$ that satisfies our assumptions and has characteristic samples such that there is no computable function $T$ to construct a characteristic sample for every language in the class. $^3$

Moving from this general setting, we consider polynomial bounds on the size of characteristic samples and the running times of the teacher and learner. From this point on, we assume that $C$ satisfies the following properties.

(P1) **Polynomial time class membership.** There is a polynomial time algorithm to decide whether $C \in C$ given $C$.

(P2) **Polynomial time word membership.** There is a polynomial time algorithm to decide whether $w \in [C]$ given $w$ and $C$.

(P3) **Polynomial time default hypothesis construction.** There is a polynomial time algorithm that returns $C_T \in C$ consistent with a given sample $T$.

A class $C$ is **concisely distinguishable** if there exists a polynomial $p(n)$ such that for every pair $C_1$ and $C_2$ such that $[C_1] \neq [C_2]$, there exists a word $w$ in the symmetric difference of $[C_1]$ and $[C_2]$ such that $|w| \leq p(size(C_1) + size(C_2))$.

**Concise Characteristic Samples.** A class $C$ has concise characteristic samples if there exists a polynomial $p(n)$, a teacher $T$, and a learner $L$ such that for every $C \in C$, $T(C)$ is a characteristic sample for $C$ and $L$, and $|T(C)| \leq p(size([C]))$. Note that the polynomial bound is in terms of the size of the smallest representation of $[C]$. If $C$ has concise characteristic samples then $C$ is concisely distinguishable.

**Efficient Teachability.** We say that $C$ is **efficiently teachable** if there exist a polynomial time teacher $T$ and a learner $L$ such that $T(C)$ is a characteristic sample for $C$ and $L$ for every $C \in C$.

**Proposition 3.4.** If class $C$ is efficiently teachable, then $C$ has concise characteristic samples.

Proof. Let $T$ be a polynomial time teacher and $L$ a learner witnessing the fact that $C$ is efficiently teachable. To see that $C$ has concise characteristic samples, we define the teacher $T'$ as follows. On input $C$, let $C'$ minimize $size(C)$ subject to $[C'] = [C]$. Then $T'$ outputs $T(C')$, which is a characteristic sample for $C$ and $L$ and is of size bounded by a polynomial in $size([C])$. $\square$

$^3$The characteristic samples for the class are of cardinality 2, but have no computable bound on their length.
Efficient Learnability. We say that $C$ is efficiently learnable if there exist a teacher $T$ and a polynomial time learner $L$ such that $T(C)$ is a characteristic sample for $C$ and $L$ for every $C \in C$.

As Gold noted, this definition, assuming (P1)-(P3) hold, is superfluous because the learning algorithm of identification by enumeration used in the proof of Prop. 3.3 can be modified to run in polynomial time and the teacher adjusted appropriately, creating (potentially) a ridiculously large characteristic sample, to make the learner satisfy the requirement of running in polynomial time. The argument follows.

**Proposition 3.5.** Given assumptions (P1), (P2), and (P3), $C$ is efficiently learnable.

**Proof.** The learner $L$ on input $T$ of length $n$ simulates the algorithm of identification by enumeration for $n^2$ steps. It checks whether the last hypothesis $C$ output by the simulation is consistent with $T$ and outputs $C$ if so. Otherwise, it returns the default hypothesis consistent with $T$. Given $C \in C$, the teacher $T$ takes the characteristic sample $T_C$ from Prop. 3.3 and adds enough additional examples of $C$ that there is time for $L$’s simulation of identification by enumeration to converge to its final answer. Note that since the domain $D$ is infinite it is always possible to add more labeled words to the sample. □

**Requiring Both Teacher and Learner To Be Efficient.** We say that $C$ is efficiently teachable/learnable if there exist a polynomial time teacher $T$ and a polynomial time learner $L$ such that $T(C)$ is a characteristic sample for $C$ and $L$ for every $C \in C$. Note that because of Prop. 3.5, this is not the conjunction of efficiently teachable and efficiently learnable. In Example 3.8 we show a class $C_2$ that is efficiently learnable, efficiently teachable, but not efficiently teachable/learnable.

**Comparisons.** We relate the definitions above to the definition of identification in the limit using polynomial time and data introduced by Gold [Gol78] and refined by de la Higuera [dlH97], who also showed that it is closely related to a model of a learner and a helpful teacher introduced by Goldman and Mathias [GM96].

Using the terminology of this paper, a class $C$ is identifiable in the limit using polynomial time and data if there exists a polynomial $p(n)$, a teacher $T$, and a polynomial time learner $L$ such that for every $C \in C$, the sample $T(C)$ is a characteristic sample for $C$ and $L$, and $|T(C)| \leq p(size([C]))$. This definition can be viewed as correcting the deficiency of the concept of efficiently learnable (indicated by Prop. 3.5) by a polynomial bound on the size of the characteristic sample. In this terminology, “polynomial time” refers to the polynomial running time of $L$, and “polynomial data” refers to the polynomial bound on the size of the sample $T(C)$. The latter is not a worst-case measure; there could be arbitrarily large finite samples for which $L$ outputs an incorrect hypothesis.

Because the definition of $C$ being identifiable in the limit using polynomial time and data simply adds the requirement that the learner be computable in polynomial time to the definition of $C$ having concise characteristic samples, we immediately have the following.

**Lemma 3.6.** If $C$ is identifiable in the limit using polynomial time and data then $C$ has concise characteristic samples.

Comparing with the concept of being efficiently teachable/learnable, we note the following differences. Identifiability in the limit using polynomial time and data only requires the existence of a characteristic sample, and the bound on the length of the characteristic
sample is in terms of the size of the smallest representation of [C]. For the definition of efficiently teachable/learnable, the characteristic sample must not only exist, but be computable in polynomial time, and the bound on the length of the characteristic sample is in terms of size(C).

**Lemma 3.7.** If C is efficiently teachable/learnable then C is identifiable in the limit using polynomial time and data.

**Proof.** Let T and L be a polynomial time teacher and learner witnessing that C is efficiently teachable/learnable. Let C ∈ C be given. Let C′ ∈ C minimize size(C′) subject to [C′] = [C]. Then T(C′) is a characteristic sample for C and L of length polynomial in size([C]). Thus C is identifiable in the limit using polynomial time and data. □

**Example 3.8.** We consider an example of these concepts. For any positive integer n, let F(n) be the sequence of primes in its prime factorization in non-decreasing order. Given a finite sequence of primes ℓ = (p_1, p_2, ..., p_k), let M(ℓ) = p_1 · p_2 · ... · p_k, their product. We assume a standard binary representation of positive integers. Then M(ℓ) is computable in polynomial time, while it is widely thought that F(n) is not. Also, M(F(n)) = n.

Let N_+ denote the set of positive integers, and let F(N_+) = {F(n) | n ∈ N_+}. We define two classes as follows.

1. C_1 has domain F(N_+) and consists of all finite subsets of at least two elements of F(N_+) together with the set of all n ∈ N_+, where [n] = {F(n)}.
2. C_2 has domain N_+ and consists of all finite subsets of at least two elements of N_+ together with the set of all F(n) such that n ∈ N_+, where [F(n)] = {n}.

Because primality can be decided in polynomial time, there is a polynomial time algorithm to test whether F(n) = (m_1, m_2, ..., m_k) given n and (m_1, m_2, ..., m_k) as inputs; therefore (P1) and (P2) hold for C_1 and C_2. To see that (P3) holds for C_1, assume that T is any finite sample of elements from F(N_+) and let T_0 and T_1 be the negative and positive examples in T, respectively. Then let ℓ_1 and ℓ_2 be distinct elements of N_+ not in T_0 ∪ T_1. The hypothesis T_1 ∪ {ℓ_1, ℓ_2} is consistent with T and can be generated in polynomial time. Similarly, to see that (P3) holds for C_2, assume that T is any finite sample of elements from N_+ and let T_0 and T_1 be the negative and positive examples in T, respectively. Let n_1 and n_2 be distinct positive integers not in T_0 ∪ T_1. The hypothesis T_1 ∪ {n_1, n_2} is consistent with T and can be generated in polynomial time.

By Prop. 3.5, both classes are efficiently learnable. They both have characteristic samples of polynomial size, consisting of just the finitely many positive examples of each concept. Under the assumption that F(n) cannot be computed in polynomial time, C_1 is
identifiable in the limit using polynomial time and data but not efficiently teachable, $\mathcal{C}_2$ is efficiently teachable but not efficiently teachable/learnable.

4. Which representations are efficiently teachable/learnable?

4.1. Negative results for nondeterministic classes. The classes NBA, NPA, NMA, NCA do not have concise characteristic sets [AFS20]. The proof is by constructing a family of languages \{$L_n\}_{n \in \mathbb{N}}$ with an acceptor of size quadratic in $n$ for which at least one word of length at least exponential in $n$ must be included in any characteristic sample for $L_n$.\footnote{A negative result regarding query learning of NBA, NPA, and NMA was obtained by Angluin et al. [AAF20]. That result makes a plausible assumption of cryptographic hardness, which is not required here.}

Since an NBA (resp. NCA) is a special case of NRA (resp. NSA) the same is true for NRA and NSA.

**Theorem 4.1.** The classes NBA, NPA, NMA, NCA, NRA and NSA do not have concise characteristic sets, and therefore are neither identifiable in the limit using polynomial time and data nor efficiently teachable.

4.2. Some positive results. The founding positive result in this area is the following theorem of Gold.

**Theorem 4.2** ([Gol78], Theorem 4). DFAs are efficiently teachable/learnable.

A family of DFAs (FDFA) is a representation of $\omega$-regular languages that consists of a leading automaton and a set of DFAs (termed progress DFAs), one corresponding to each state of the leading automaton [AF14]. The method of Gold’s proof can be applied directly to the class FDFA. The learner first learns the automaton for the canonical right congruence $\sim_L$ for $L = [C]$, and then learns each progress automaton individually.

**Corollary 4.3.** The class FDFA is efficiently teachable/learnable.

A class $\mathcal{C}_1$ is efficiently embeddable into a class $\mathcal{C}_2$ if there exists a polynomial time computable embedding function $e$ mapping $\mathcal{C}_1$ to $\mathcal{C}_2$ such that for all $C_1 \in \mathcal{C}_1$ we have $[C_1] = [e(C_2)]$.

**Lemma 4.4.** If $\mathcal{C}_1$ is efficiently embeddable into $\mathcal{C}_2$ and $\mathcal{C}_2$ is efficiently teachable, then $\mathcal{C}_1$ is efficiently teachable.

**Proof.** Let $e$ be a polynomial time embedding of $\mathcal{C}_1$ into $\mathcal{C}_2$. Let $T_2$ and $L_2$ witness the polynomial teachability of $\mathcal{C}_2$. Define $T_1$ with input $C_1 \in \mathcal{C}_1$ to output $T_2(e(C_1))$. $T_1$ runs in polynomial time because $e$ and $T_2$ do. Define $L_1$ on input a sample $T$ to apply $L_2$ to $T$ to obtain $C_2 \in \mathcal{C}_2$. If there exists $C_1 \in \mathcal{C}_1$ such that $[C_1] = [C_2]$, then $L_1$ outputs one such $C_1$ of minimum size. If there is no such $C_1$, $L_1$ outputs a default choice $C_1 \in \mathcal{C}_1$ consistent with $T$.

To see that $T_2(e(C_1))$ is a characteristic sample for $C_1$ and $L_1$, assume that $T \supseteq T_2(e(C_1))$ is consistent with $C_1$. Then $e(C_1)$ is consistent with $T$, and $L_2(T)$ outputs $C_2$ such that $[C_2] = [e(C_1)] = [C_1]$. Thus $L_1$ outputs $C_1'$ of minimum size equivalent to $C_1$. \qed
Note that even if the learner $L_2$ runs in polynomial time, the learner $L_1$ in this construction may not. The classes $DBA$, $DCA$, and $DPA$ can be efficiently embedded into FDFAs [ABF18], which implies the following.

**Corollary 4.5.** The classes $DBA$, $DCA$, and $DPA$ are efficiently teachable, but are not known to be efficiently teachable/learnable.

### 4.3. Relation to learning with equivalence and membership queries.

In the paradigm of *learning with membership and equivalence queries*, a learning algorithm can access an oracle that truthfully answers two types of queries about the target concept $C$, and its goal is to halt and output a representation of $[C]$. In a *membership query*, or MQ, the learning algorithm provides a word $w$ and the answer is 1 or 0 depending on whether $w \in [C]$ or not. In an *equivalence query*, or EQ, the learning algorithm provides a representation $C' \in C$, and the answer from the oracle is either “yes”, if $[C'] = [C]$, and otherwise is an arbitrarily chosen element of $[C'] \oplus [C]$ (a *counterexample* to the conjecture that $C'$ is correct). If the learning algorithm successfully learns every $C \in C$ and at every point its running time is bounded by a polynomial in $\text{size}([C])$ and the length of the longest counterexample seen to that point, we say that $C$ is polynomially learnable using membership and equivalence queries.

Bohn and Löding [BL21] prove the following consequence of learnability with membership and equivalence queries.

**Theorem 4.6 (Bohn and Löding [BL21]).** Suppose the class $C$ satisfies properties (P2) and (P3) and is concisely distinguishable. If $C$ is polynomially learnable using membership and equivalence queries, it is also identifiable in the limit using polynomial time and data.

Can this result be strengthened to conclude that $C$ is polynomially teachable/learnable? To answer this question, we examine the proof in more detail. Let $A$ be a learning algorithm using membership and equivalence queries that learns $C$ in polynomial time. Given $C$, $T$ constructs a sample $T_C$ by simulating $A$ and answering its queries according to $C$ as follows. A membership query with $w$ is answered by determining whether $w \in [C]$ (using property (P2)). For an equivalence query with $C'$, if $[C'] \neq [C]$, then $w$ is chosen to be the shortlex least element of $[C'] \oplus [C]$ and returned as the counterexample to the simulation of $A$. The counterexample $w$ is of length polynomial in the sum of the sizes of $C'$ and $C$ by the assumption of concise distinguishability. If instead $[C'] = [C]$, then the sample $T_C$ is constructed of all the strings $w$ that appeared in membership queries or as counterexamples returned to equivalence queries during the simulation, labeled to be consistent with $C$. Because of the polynomial running time of $A$ and the choice of shortest counterexamples, the length of $T_C$ is bounded by a polynomial in $\text{size}([C])$.

The corresponding learning algorithm $L$ takes a sample $T$ as input and simulates the learning algorithm $A$, attempting to answer its queries using $T$ as follows. For a membership query with $w$, if $w$ is an example in $T$, the answer is its label in $T$. If $w$ is not an example in $T$, then $L$ outputs a default $C_T$ consistent with $T$ and halts (using property (P3)). For an equivalence query with $C$, $L$ checks whether $C$ is consistent with $T$ (using property (P2)). If it is consistent, then it outputs $C$ and halts. If it is not consistent, it finds the shortlex least $w$ that is an example in $T$ whose label is not consistent with $C$ and returns $w$ as the counterexample to $A$’s equivalence query. The running time of $L$ is polynomial in the length of $T$. It is because of the choice of the shortlex counterexample by both $T$ and $L$. 


that $T$ can anticipate exactly the queries that will be made in the simulation of $A$ by $L$, even when the sample $T$ is a superset of the characteristic sample $T_G$.

What must we assume in order that the function $T$ in this construction can be computed in polynomial time? We say that $C$ has \textit{polynomial time equivalence testing} if there exists a polynomial time algorithm that takes two representations $C_1$ and $C_2$ and determines whether $[C_1] = [C_2]$. If, in addition, such an algorithm returns the shortlex least element of $[C_1] \oplus [C_2]$ when they are unequal, we say that $C$ has \textit{polynomial time equivalence testing with shortlex least counterexamples}.

\textbf{Theorem 4.7.} Suppose the class $C$ satisfies properties (P2) and (P3) and is concisely distinguishable. Suppose also that $C$ has polynomial time equivalence testing with shortlex least counterexamples. If $C$ is polynomially learnable using membership and equivalence queries, it is also efficiently teachable/learnable.

A Büchi automaton is \textit{unambiguous} if no word has more than one run starting in an initial state and visiting an accepting state infinitely often. It is \textit{strongly unambiguous} if no word has more than one run visiting an accepting state infinitely often. Modulo-2 multiplicity automata (M2MAs) are an algebraic representation of $\omega$-automata.

\textbf{Theorem 4.8.} The class $M_2MA$ is efficiently teachable/learnable. The class $SUBA$ is efficiently teachable, but is not known to be efficiently teachable/learnable.

\textit{Proof.} M2MAs were shown to be polynomially learnable using membership and equivalence queries [BBB+00]. Membership and equivalence of M2MAs can be decided in polynomial time, and a shortlex least counterexample returned in case of inequivalence [DKV09, Sak09]. It follows from Theorem 4.7 that M2MAs are efficiently teachable/learnable.

The domains for M2MAs and SUBAs are different, but there is a polynomial time translation $t$ that converts a SUBA $A$ into an M2MA $M$ such that $[M] = \{uSv \mid (u, v) \in [A]\}$ [AAF20]. By slightly generalizing the proof of Lemma 4.4, we may conclude that SUBAs are polynomially teachable.

Maler and Pnueli [MP95] give an algorithm that learns the class $DWPA$ in polynomial time using membership and equivalence queries. Membership and equivalence are decidable in polynomial time, thus by Theorem 4.6 $DWPA$ is identifiable in the limit using polynomial time and data. But it is not known whether there is a polynomial time algorithm for equivalence with shortlex counterexamples, so we are unable to apply Theorem 4.7. However, DWPAs are a special case of DPAs and the following is a corollary of Theorem 16.8 for DPAs.

\textbf{Corollary 4.9.} The class $DWPA$ is efficiently teachable/learnable.

The learning algorithm of Maler and Pnueli for $DWPA$ exploits the fact that this class does have a one-to-one relationship between states of the minimal DWPA for a language $L$ and the equivalence classes of the right congruence $\sim_L$. We term languages for which such a correspondence exists \textit{fully informative}. The class $DWPA$ is a small sub-class of the class of the fully informative $\omega$-regular languages — there are fully informative languages along every level of the Wagner hierarchy [AF18], whereas $DWPA$ is one of the lowest level in the hierarchy. The rest of the paper is dedicated to showing that fully informative languages of any of the considered $\omega$-automata types (Büchi, coBüchi, pairty, Muller, Rabin and Streett) are (concisely and) efficiently teachable/learnable.
5. The Informative Classes are Efficiently Teachable/Learnable

The rest of the paper is devoted to showing that the informative classes are efficiently teachable/learnable. This section covers some preliminary issues and gives an overview of the milestones needed to prove that the informative classes are efficiently teachable/learnable.

5.1. Duality. There are reductions of the problem of concise/efficient teachability/learnability between IBA and ICA and between IRA and ISA, using the duality between these types of acceptors. Consequently, we focus on the classes IBA, IPA, IRA, and IMA in what follows.\(^5\)

**Proposition 5.1.** IBA (resp., IRA) is efficiently teachable/learnable if and only if ICA (resp., ISA) is.

**Proof.** Let \( \mathcal{A} \) be an ICA. Because \( \mathcal{A} \) is deterministic and complete, if we let \( \mathcal{A}' \) denote the IBA with the same components as \( \mathcal{A} \), then \( \mathcal{A}' \) accepts the complement of the language \( \mathcal{A} \), by Claim 4.

We modify the characteristic sample for \( \mathcal{A}' \) by complementing all its labels to get a characteristic sample for \( \mathcal{A} \). The algorithm to learn an ICA from a sample \( T \) is obtained by complementing all the labels in the sample \( T \) and calling the algorithm to learn an IBA from a sample. The resulting IBA, now considered to be an ICA, is returned as the answer.

Exactly the same conversion may be done with acceptors of types IRA and ISA, by Claim 5.

5.2. The Default Acceptor. One condition of the definition of being efficiently teachable/learnable is that the learning algorithm must run in polynomial time and return an acceptor of the required type that is consistent with the input sample \( T \), even if the sample \( T \) does not subsume a characteristic sample. To meet this condition, we use the strategy of Gold’s construction, that is, the learning algorithm optimistically assumes that the sample includes a characteristic sample, and if that assumption fails to produce an acceptor consistent with the sample, the algorithm instead produces a default acceptor to ensure that its hypothesis is consistent with the sample. Alternatively, one can use the the generalization of the RPNI algorithm to learning \( \omega \)-words, which avoids this simple default strategy [BL21].

**Proposition 5.2.** There is a polynomial time algorithm that takes a sample \( T \) and returns a DBA (resp., DCA, DPA, DRA, DSA, DMA) consistent with \( T \).

**Proof.** Given a sample \( T \), we find the shortest prefix of each example \( u(v)^\omega \) in \( T \) that distinguishes it from all other examples in \( T \) and place these prefixes in a deterministic prefix-tree automaton. By Prop. 3.1, this prefix-tree automaton can be constructed in time polynomial in the length of the sample \( T \). For each leaf state we define self-transitions on each symbol in \( \Sigma \), and if the automaton is incomplete, we add a new dead state with self-transitions on each \( \sigma \in \Sigma \), and define all undefined transitions to go to the dead state.

For a DBA, the acceptance condition \( F \) consists of all the leaf states that are prefixes of positive examples in \( T \). For a DMA, the acceptance condition consists of \( \{q \mid q \in F\} \). For a DCA, the acceptance condition consists of the dead state (if one was added) and all

---

\(^5\)The results regarding the classes IBA (and ICA), IPA, and IMA were obtained in [AFS20]; here we extend them to the classes IRA (and ISA). Results for identifiability in the limit using polynomial time and data (but not efficient teachability) of the classes IRA (and ISA) have also been provided in [BL21] using a different algorithm.
The corresponding prefixes are \textit{abbb}, \textit{aba}, and \textit{abba}, and the default acceptor of type DBA for \( T \) is shown in Figure 1.

![Figure 1. Default acceptor of type DBA for \( T = \{(a(b)^\omega, 1), ((ab)^\omega, 1), (ab(baa)^\omega, 0)\}\). Leaf states are marked with * and the dead state with d.](image)

the leaf states that are not prefixes of positive examples in \( T \). The DBA thus constructed may be transformed to a DPA or a DRA using Claim 1 or Claim 2, and the DCA may be transformed to a DSA using Claim 3.

As an example of this construction, let the sample be

\[ T = \{(a(b)^\omega, 1), ((ab)^\omega, 1), (ab(baa)^\omega, 0)\}. \]

The corresponding prefixes are \textit{abbb}, \textit{aba}, and \textit{abba}, and the default acceptor of type DBA for \( T \) is shown in Figure 1.

5.3. \textbf{Strongly connected components}. The acceptance conditions that we consider are all based on the set of states visited infinitely often in a run of the automaton on an input \( w \in \Sigma^\omega \). We consider only acceptors whose automata are deterministic and complete, so for any \( w \in \Sigma^\omega \) there is exactly one run, which we denote \( \rho(w) \), of the automaton on input \( w \). Thus we may define \( \inf(w) = \inf(\rho(w)) \), the set of states visited infinitely often in this unique run. In the run \( \rho(w) \), there is some point after which none of the states visited finitely often is visited. Because each state in \( \inf(w) \) is visited infinitely often, for any states \( q_1, q_2 \in \inf(w) \), there exists a non-empty word \( x \in \Sigma^* \) such that \( \delta(q_1, x) = q_2 \) and for each prefix \( x' \) of \( x \), \( \delta(q_1, x') \in \inf(w) \), that is, the path from \( q_1 \) to \( q_2 \) on \( x \) does not visit any state outside the set \( \inf(w) \).

These properties characterize the following definition. Given an automaton \( \mathcal{M} \), a \textit{strongly connected component} (SCC) of \( \mathcal{M} \) is a nonempty set of states \( C \) such that for every \( q_1, q_2 \in C \), there exists a nonempty string \( x \in \Sigma^* \) such that \( \delta(q_1, x) = q_2 \) and for any prefix \( x' \) of \( x \), \( \delta(q_1, x') \in C \).

Note that an SCC need not be maximal, and that a singleton state set \( \{q\} \) is an SCC if and only if the state \( q \) has a self-loop, that is, \( \delta(q, \sigma) = q \) for some \( \sigma \in \Sigma \). There is a close relationship between SCCs and the set of states visited infinitely often in a run.

**Proposition 5.3.** Let \( \mathcal{M} \) be a complete deterministic automaton and \( w \in \Sigma^\omega \). Then \( \inf(w) \) is an SCC of \( \mathcal{M} \). If \( w \) is the ultimately periodic word \( u(v)^\omega \), then \( \inf(w) \) may be computed in time polynomial in the size of \( \mathcal{M} \) and the length of \( u(v)^\omega \).

**Proposition 5.4.** For any deterministic automaton \( \mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle \) and any reachable SCC \( C \) of \( \mathcal{M} \), there exists an ultimately periodic word \( u(v)^\omega \) of length at most \(|Q| + |C|^2 \) such that \( C = \inf(w) \). Such a word may be found in time polynomial in \(|Q| \) and \(|\Sigma| \).

**Proof.** Because \( C \) is reachable, a word \( u \in \Sigma^* \) of minimum length such that \( \delta(q_i, u) \in C \) may be found by breadth first search. The length of \( u \) is at most \(|Q| \). If \( C = \{q\} \), then

\[ \inf(w) = \{q\} \]
there is at least one symbol $\sigma \in \Sigma$ such that $\delta(q, \sigma) = q$. Then the $\omega$-word $w = u(\sigma)^\omega$ is such that $C = \text{inf}(w)$. The length of this ultimately periodic word is at most $|Q| + 1$.

If $C$ contains at least two states, let $q_1, \ldots, q_k$ be the states in $C$ that are not $q$. Then for each $i$, there exist two nonempty finite words $x_i$ and $y_i$ each of length at most $n$ such that $\delta(q, x_i) = q_i$ and $\delta(q_i, y_i) = q$, and the path on $x_i$ from $q$ to $q_i$ and the path on $y_i$ from $q_i$ to $q$ do not visit any states outside of $C$. The words $x_i$ and $y_i$ may be found in polynomial time by breadth-first search. Then the word $w = u(x_1y_1 \cdots x_ky_k)^\omega$ is such that $\text{inf}(w) = C$. The length of this ultimately periodic word is at most $|Q| + |C|^2$. □

We let $\text{Witness}(C, M)$ denote the ultimately periodic word $u(v)^\omega$ returned by the algorithm described in the proof above for the reachable SCC $C$ of automaton $M$.

**Proposition 5.5.** If $C_1$ and $C_2$ are SCCs of automaton $M$ and $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is also an SCC of $M$.

If $M$ is an automaton and $S$ is any set of its states, define $\text{SCCs}(S)$ to be the set of all $C$ such that $C \subseteq S$ and $C$ is an SCC of $M$. Also define $\text{maxSCCs}(S)$ to be the maximal elements of $\text{SCCs}(S)$ with respect to the subset ordering. The following is a consequence of Prop. 5.5.

**Proposition 5.6.** If $M$ is an automaton and $S$ is any set of its states, then the elements of $\text{maxSCCs}(S)$ are pairwise disjoint, and every set $C \in \text{SCCs}(S)$ is a subset of exactly one element of $\text{maxSCCs}(S)$.

There are some differences in the terminology related to strong connectivity between graph theory and omega automata, which we resolve as follows. In graph theory, a path of length $k$ from $u$ to $v$ in a directed graph $(V, E)$ is a finite sequence of vertices $v_0, v_1, \ldots, v_k$ such that $u = v_0$, $v = v_k$ and for each $i$ with $i \in [1..k]$, $(v_{i-1}, v_i) \in E$. Thus, for every vertex $v$, there is a path of length 0 from $v$ to $v$. A set of vertices $S$ is strongly connected if and only if for all $u, v \in S$, there is a path of some nonnegative length from $u$ to $v$ and all the vertices in the path are elements of $S$. Thus, for every vertex $v$, the singleton set $\{v\}$ is a strongly connected set of vertices. A strongly connected component of a directed graph is a maximal strongly connected set of vertices. There is a linear time algorithm to find the set of strong components of a directed graph [Tar72].

In this paper, we use the terminology SCC and maximal SCC to refer to the definitions from the theory of omega automata, and the terminology graph theoretic strongly connected components to refer to the definitions from graph theory. We use the term trivial strong component to refer to a graph theoretic strongly connected component that is a singleton vertex $\{v\}$ such that there is no edge $(v, v)$.

If $M$ is an automaton, we may define a related directed graph $G(M)$ whose vertices are the states of $M$ and whose edges $(q_1, q_2)$ are the pairs of states such that $q_2 \in \delta(q_1, \sigma)$ for some $\sigma \in \Sigma$. Then for any set $S$ of states of $M$, the maximal SCCs in $S$, $\text{maxSCCs}(S)$, are the graph theoretic strongly connected components of the subgraph of $G(M)$ induced by $S$, with any trivial strong components removed.

**Proposition 5.7.** For automaton $M$ and any subset $S$ of its states, $\text{maxSCCs}(S)$ can be computed in time linear in the size of $M$. 

5.4. A decreasing forest of SCCs of an automaton. Let $\mathcal{M}$ be a deterministic automaton, and let $S$ be a subset of its states. A **decreasing forest of SCCs of $\mathcal{M}$ rooted in $S$** is a finite rooted forest $\mathcal{F}$ in which every node $C$ is an SCC of $\mathcal{M}$ that is contained in $S$, and the following properties are satisfied.

1. The roots of $\mathcal{F}$ are the elements of $\text{maxSCCs}(S)$.
2. Whenever $D_1, \ldots, D_k$ are the children of node $C$, we have $D_1 \cup \ldots \cup D_k \subseteq C$. Also, letting $\Delta(C) = C \setminus (D_1 \cup \ldots \cup D_k)$, the children $D_1, \ldots, D_k$ are exactly the elements of $\text{maxSCCs}(C \setminus \Delta(C))$.

**Proposition 5.8.** Let $\mathcal{M}$ be a deterministic automaton, $S$ a subset of its states, and $\mathcal{F}$ a decreasing forest of SCCs of $\mathcal{M}$ rooted in $S$. Then the following are true.

1. The roots of $\mathcal{F}$ are pairwise disjoint.
2. The children of any node are pairwise disjoint.
3. $\mathcal{F}$ has at most $|S|$ nodes.
4. For any $D \subseteq S$ that is an SCC of $\mathcal{M}$, there is a unique node $C$ in $\mathcal{F}$ such that $D \subseteq C$ and $D$ is not a subset of any of the children of $C$.

**Proof.** The roots of $\mathcal{F}$ are the elements of $\text{maxSCCs}(S)$, which are pairwise disjoint. The children of a node $C$ are the elements of $\text{maxSCCs}(C \setminus \Delta(C))$, which are pairwise disjoint. The sets $\Delta(C)$ for nodes $C$ in $\mathcal{F}$ are contained in $S$, nonempty, and pairwise disjoint, so the number of nodes is at most $|S|$. If $D \subseteq S$ is an SCC of $\mathcal{M}$, then $D$ is a subset of exactly one of the roots of $\mathcal{F}$, say $C_1$. If $D \cap \Delta(C_1) \neq \emptyset$, then $D$ is not a subset of any of the children of $C_1$. Otherwise, $D$ must be a subset of exactly one of the children of $C_1$, say $C_2$. If $D \cap \Delta(C_2) \neq \emptyset$, then $D$ is not a subset of any of the children of $C_2$. Continuing in this way, we eventually arrive at the required node $C$. 

Given a decreasing forest $\mathcal{F}$ of SCCs of automaton $\mathcal{M}$ rooted in $S$, and an SCC $D \subseteq S$, we denote by $\text{Node}(D, \mathcal{F})$ the unique node $C$ of $\mathcal{F}$ such that $D \subseteq C$ and $D$ is not a subset of any of the children of $C$. We note that if $C = \text{Node}(D, \mathcal{F})$ then $D \cap \Delta(C) \neq \emptyset$. If $D$ is a child of some $C$ in $\mathcal{F}$, we define merging $D$ into $C$ as the operation of removing $D$ from $\mathcal{F}$ and making the children of $D$ (if any) direct children of $C$.

**Proposition 5.9.** Let $\mathcal{M}$ be a deterministic automaton and $S$ a subset of its states. Let $\mathcal{F}$ be a decreasing forest of SCCs of $\mathcal{M}$ rooted in $S$. Let $D$ be a child of $C$ in $\mathcal{F}$ and let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by merging $D$ into $C$. Then $\mathcal{F}'$ is also a decreasing forest of SCCs of $\mathcal{M}$ rooted in $S$.

**Proof.** After the merge, the roots of $\mathcal{F}$ remain the elements of $\text{maxSCCs}(S)$. Let $D_1, \ldots, D_k$ be the children of $C$ in $\mathcal{F}$, where $D = D_k$, and let $E_1, \ldots, E_\ell$ be the children of $D$ in $\mathcal{F}$. Because the union of $E_1, \ldots, E_\ell$ is a proper subset of $D = D_k$ and the union of $D_1, \ldots, D_k$ is a proper subset of $C$, the union of $D_1, \ldots, D_k-1$ with the union of $E_1, \ldots, E_\ell$ is a proper subset of $C$, therefore the union of the children of $C$ in $\mathcal{F}$ is a proper subset of $C$. Also, the children of $C$ in $\mathcal{F}$ are the maximum SCCs of $(C \setminus \Delta_{\mathcal{F}}(C))$, and no other nodes are affected, so $\mathcal{F}'$ is a decreasing forest of SCCs of $\mathcal{M}$ rooted in $S$. 

5.5. Proving efficient teachability of the informative classes — overview. We can show that a class is efficiently teachable by first showing that it is identifiable in the limit using polynomial time and data, and then showing a teacher that can construct the required characteristic sets in polynomial time. To show that a class is identifiable in the limit using
polynomial time and data there are two parts: (i) finding a sample of words $T_L$ of size polynomial in the size of the given acceptor $A$ for the language $L$ at hand, called the characteristic sample, and (ii) providing a polynomial time learning algorithm $L$ that for every given sample $T$ returns an acceptor consistent with $T$, and, moreover, for any sample $T$ that subsumes $T_L$, returns an acceptor that accepts $L$.

The definition of an acceptor has two parts: (a) the definition of the automaton and (b) the definition of the acceptance condition. Correspondingly, we view the characteristic sample as a union of two parts: $T_{Aut}$ (to specify the automaton) and $T_{Acc}$ (to specify the acceptance condition). In §6 we discuss the construction of $T_{Aut}$, which is common to all the classes we consider, as they all are isomorphic to $M_{\sim L}$ for the target language $L$. We also describe a polynomial time algorithm to construct the automaton using the labeled words in $T_{Aut}$.

Because the acceptance conditions differ, $T_{Acc}$ is different for each type of acceptor we consider. In §7 we describe the construction of $T_{Acc}$ for acceptors of types IMA and IBA and learning algorithms for acceptors of these types, showing that IMA, IBA, and ICA are identifiable in the limit using polynomial time and data. In §8 we describe the construction of $T_{Acc}$ for acceptors of type IPA and a learning algorithm for acceptors of this type, showing that IPA is identifiable in the limit using polynomial time and data. In §9 we describe the construction of $T_{Acc}$ for acceptors of type IRA and a learning algorithm for acceptors of this type, showing that IRA and ISA are identifiable in the limit using polynomial time and data.

In §10 we show that the characteristic samples we have defined can be computed in polynomial time in the size of the acceptor. These results rely on polynomial time algorithms for the inclusion and equivalence problems for the acceptors. These are described in Sections 11, 12, 13, and 14.

This does not yet entail that the classes IXA for $X \in \{IBA, ICA, IPA, IMA, IRA, ISA\}$ are efficiently teachable/learnable. This is because IBA (for instance) includes also DBAs that are not IBA but have equivalent IBA. In §15 we show the right congruence automaton $M_{\sim L}$ can be computed in polynomial time, given a DBA, DCA, DPA, DRA, DSA, or DMA accepting $L$, which yields a polynomial time algorithm to test whether a DBA is an IBA, and similarly for the other acceptor types. In §16, we consider the harder problem of whether a DBA accepts a language in IBA, and give polynomial time algorithms for DBAs, DCAs, DPAs, DRAs, DSAs and DMA. With these results we can finally claim that the IXA classes are efficiently teachable/learnable.

6. The sample $T_{Aut}$ for the automaton

In this section we describe construction of the $T_{Aut}$ part of the sample. We first show that if two states of the automaton are distinguishable, they are distinguishable by words of length polynomial in the number of states of the automaton.

6.1. Existence of short distinguishing words. Let $A$ be an acceptor of one of the types DBA, DCA, DPA, DRA, DSA, or DMA over alphabet $\Sigma$. We say that states $q_1$ and $q_2$ of $M$ are distinguishable if there exists a word $w \in \Sigma^\omega$ that is accepted from one state but not the other, that is, $w \in [A^{q_1}] \setminus [A^{q_2}]$ or $w \in [A^{q_2}] \setminus [A^{q_1}]$. In this case we say that $w$ is a distinguishing word.
Proposition 6.1. If two states of a complete DBA, DCA, DPA, DRA, DSA, or DMA of \( n \) states are distinguishable, then they are distinguishable by an ultimately periodic \( \omega \)-word of length bounded by \( O(n^4) \).

Proof. We prove the result for a DMA. Because any DBA, DCA, DPA, DRA, or DSA is equivalent to a DMA with the same automaton, this result holds for these types of acceptors as well. Let \( \mathcal{A} \) be a complete DMA of \( n \) states such that the states \( q_1 \) and \( q_2 \) are distinguishable. Then there exists an \( \omega \)-word \( w \) that is accepted from exactly one of the two states, that is, \( w \) is accepted by exactly one of \( \mathcal{A}^{q_1} \) and \( \mathcal{A}^{q_2} \).

Let \( \mathcal{M}_i \) denote the automaton of \( \mathcal{A} \) with its initial state replaced by \( q_i \) for \( i = 1, 2 \). Let \( \mathcal{M} \) denote the product automaton \( \mathcal{M}_1 \times \mathcal{M}_2 \). The number of states of \( \mathcal{M} \) is \( n^2 \). By Prop. 5.3, \( \inf_M(w) \) is a reachable SCC \( C \) of \( \mathcal{M} \), and by Prop. 5.4 there exists an ultimately periodic word \( u(v)\omega \) of length bounded by \( O(n^4) \) such that \( \inf_M(u(v)\omega) = C \). Then for \( i = 1, 2 \), \( \inf_{M_i}(u(v)\omega) = \pi_i(C) = \inf_M(u(v)) \), so \( u(v)\omega \) is also accepted by exactly one of \( \mathcal{A}^{q_1} \) and \( \mathcal{A}^{q_2} \), and \( u(v)\omega \) distinguishes \( q_1 \) and \( q_2 \). \( \Box \)

6.2. Defining the sample \( T_{Aut} \) for the automaton. We now define the \( T_{Aut} \) part of the characteristic sample, given an acceptor \( \mathcal{A} = (\Sigma, Q, q_0, \delta, \alpha) \) that is an IBA, ICA, IPA, IRA, ISA, or IMA. This construction is analogous to that of the corresponding part of a characteristic sample for a DFA, with distinguishing experiments that are ultimately periodic \( \omega \)-words instead of finite strings.

Let \( \mathcal{M} \) be the automaton of \( \mathcal{A} \) and let \( n \) be the number of states of \( \mathcal{M} \). Because \( \mathcal{A} \) is an IBA, ICA, IPA, IRA, ISA, or IMA, every state is reachable and every pair of states is distinguishable. We define a distinguished set of \( n \) access strings for the states of \( \mathcal{M} \) as follows. For each state \( q \), access\((q)\) is the least string \( x \) in the shortlex ordering such that \( \delta(q, x) = q \). Given \( \mathcal{A} \), the access strings may be computed in polynomial time by breadth-first search.

Because every pair of states is distinguishable, by Prop. 6.1, there exists a set \( E \) of at most \( n \) distinguishing experiments, each of length at most \( n^2 + n^4 \), that distinguish every pair of states. The issue of computing \( E \) is addressed in §10. The sample \( T_{Aut} \) consists of all the examples in \( (S \cdot E) \cup (S \cdot \Sigma \cdot E) \), labeled to be consistent with \( \mathcal{A} \). There are at most \( (1 + |\Sigma|)n^2 \) labeled examples in \( T_{Aut} \), each of length bounded by a polynomial in \( n \). A learner using \( T_{Aut} \) is described next.

6.3. Learning the automaton from \( T_{Aut} \). We now describe a learning algorithm \( L_{Aut} \) and prove the following.

Theorem 6.2. The algorithm \( L_{Aut} \) with a sample \( T \) as input runs in polynomial time and returns a deterministic complete automaton \( \mathcal{M} \). Let \( \mathcal{A} \) be an acceptor of type IBA, ICA, IPA, IRA, ISA, or IMA. If \( T \) is consistent with \( \mathcal{A} \) and subsumes \( T_{Aut} \) then the returned automaton \( \mathcal{M} \) is isomorphic to the automaton of \( \mathcal{A} \).

Algorithm \( L_{Aut} \) on input \( T \) constructs a set \( E \) of words that serve as experiments used to distinguish candidate states. For each \( (u(v)\omega, l) \) in \( T \), all of the elements of \( suffixes(u(v)\omega) \) are placed in \( E \). Two strings \( x, y \in \Sigma^* \) are consistent with respect to \( T \) if and only if there does not exist any \( u(v)\omega \in E \) such that the examples \( xu(v)\omega \) and \( yu(v)\omega \) are oppositely labeled in \( T \).
Starting with the empty string $\varepsilon$, the algorithm builds up a prefix-closed set $S$ of finite strings as follows. Initially, $S_1 = \{\varepsilon\}$. After $S_k$ has been constructed, the algorithm considers each $s \in S_k$ in shortlex order, and each symbol $\sigma \in \Sigma$ in the ordering defined on $\Sigma$. If there exists no $s' \in S_k$ such that $s\sigma$ is consistent with $s'$ with respect to $T$, then $S_{k+1}$ is set to $S_k \cup \{s\sigma\}$ and $k$ is set to $k + 1$. If no such pair $s$ and $\sigma$ is found, then the final set $S$ is $S_k$.

In the second phase, the algorithm uses the strings in $S$ as names for states and constructs a transition function $\delta$ using $S$ and $E$. For each $s \in S$ and $\sigma \in \Sigma$, there is at least one $s' \in S$ such that $s\sigma$ and $s'$ are consistent with respect to $T$. The algorithm selects any such $s'$ and defines $\delta(s, \sigma) = s'$. Once $S$ and $\delta$ are defined, the algorithm returns the automaton $M = (\Sigma, S, \varepsilon, \delta)$.

Proof of Thm. 6.2. $E$ may be computed in time polynomial in the length of $T$, by Prop. 3.2. Because the default acceptor for $T$ has a polynomial number of states and is consistent with $T$, the number of distinguishable states, and the number of strings added to $S$, is bounded by a polynomial in the length of $T$. The returned automaton $M$ is deterministic and complete by construction.

Assume the sample $T$ is consistent with $A$ and subsumes $T_{Aut}$. For any pair of states of $A$, the set $E$ includes an experiment to distinguish them. Also, if $x$ and $y$ reach the same state of $A$, there is no experiment in $E$ that distinguishes them. Then the set $S$ is precisely the access strings of $A$. The choice of $s'$ for $\delta(s, \sigma)$ is unique in each case, and the returned automaton $M$ is isomorphic to the automaton of $A$.

Although the processes of constructing $T_{Aut}$ and learning an automaton from it are the same for acceptors of types IBA, ICA, IPA, IRA, ISA, or IMA, different types of acceptance condition require different kinds of characteristic samples and learning algorithms.

In the following sections we describe for each type of acceptor the corresponding sample $T_{Acc}$ and learning algorithm. Each learning algorithm takes as input an automaton $M$ and a sample $T$ and returns in polynomial time an acceptor of the appropriate type consistent with $T$. We show that for each type of acceptor $A$, if the input automaton $M$ is isomorphic to the automaton of $A$ and the sample $T$ is consistent with $A$ and subsumes the $T_{Acc}$ for $A$, then the learning algorithm returns an acceptor that is equivalent to $A$. This learning algorithm is then combined with $L_{Aut}$ to prove $ILPTD$ for the relevant class of languages.

7. The samples $T_{Acc}$ and learning algorithms for IMA and IBA

The straightforward cases of Muller, Büchi, and coBüchi acceptance conditions are covered in this section. Subsequent sections cover the cases of parity, Rabin, and Street acceptance conditions, which are somewhat more involved.

7.1. Muller acceptors. Let $A$ be an IMA with acceptance condition $\alpha = \{F_1, \ldots, F_k\}$. By Prop. 5.3, we may assume that each $F_i$ is a reachable SCC of $A$. The sample $T_{Acc}^{IMA}$ consists of $k$ positive examples, one for each set $F_i$. The example for $F_i$ is $(u(v)^\omega, 1)$ where $\text{inf}(u(v)^\omega) = F_i$. These examples may be found in polynomial time in the size of $A$ by Prop. 5.4.

The learning algorithm $L_{Acc}^{IMA}$ takes as input a deterministic complete automaton $M$ and a sample $T$. It constructs an acceptance condition $\alpha'$ as follows. For each positive labeled example $(u(v)^\omega, 1) \in T$, it computes the set $C = \text{inf}_M(u(v)^\omega)$ and makes $C$ a member of $\alpha'$. 
Once the set \( \alpha' \) is complete, the algorithm checks whether the DMA \((M, \alpha')\) is consistent with \(T\). If so, it returns \((M, \alpha')\); if not, it returns the default acceptor of type DMA for \(T\).

**Theorem 7.1.** Algorithm \(L_{\text{Acc}}^{\text{IMA}}\) runs in time polynomial in the sizes of the inputs \(M\) and \(T\). Let \(A\) be an IMA. If the input automaton \(M\) is isomorphic to the automaton of \(A\), and the sample \(T\) is consistent with \(A\) and subsumes \(T_{\text{Acc}}^{\text{IMA}}\), then algorithm \(L_{\text{Acc}}^{\text{IMA}}\) returns an IMA \((M, \alpha')\) equivalent to \(A\).

**Proof.** The construction of \(\alpha'\) can be done in time polynomial in the sizes of \(M\) and \(T\) by Prop. 5.3. The returned acceptor is consistent with \(T\) by construction.

Assume \(M\) is isomorphic to the automaton of \(A\) and that \(T\) is consistent with \(M\). For ease of notation, assume the isomorphism is the identity. Then for each positive example \((u(v)^\omega, 1)\) in \(T\), the set \(F = \inf(u(v)^\omega)\) must be in \(\alpha\), so \(\alpha'\) is a subset of \(\alpha\).

If \(T\) subsumes \(T_{\text{Acc}}^{\text{IMA}}\), then for every set \(F \in \alpha\) there is a positive example \((u(v)^\omega, 1)\) in \(T\) with \(F = \inf(u(v)^\omega)\). Thus the set \(F\) is added to \(\alpha'\), and \(\alpha\) is a subset of \(\alpha'\). Thus, \((M, \alpha')\) is equivalent to \(A\), and because \(T\) is consistent with \(A\), the IMA \((M, \alpha')\) is returned by \(L_{\text{Acc}}^{\text{IMA}}\).

**Theorem 7.2.** The class \(\text{IMA}\) is identifiable in the limit using polynomial time and data.

**Proof.** Let \(A\) be an IMA accepting a language \(L\). The characteristic sample \(T_L = T_{\text{Aut}} \cup T_{\text{Acc}}^{\text{IMA}}\) is of size polynomial in the size of \(A\).

The combined learner \(L_{\text{Acc}}^{\text{IMA}}\) takes a sample \(T\) as input and runs \(L_{\text{Aut}}\) on \(T\) to produce an automaton \(M\) and then runs \(L_{\text{Acc}}^{\text{IMA}}\) on \(M\) and \(T\) and returns the resulting acceptor. It runs in polynomial time in the size of \(T\) because it is the composition of two polynomial time algorithms, and the acceptor it returns is guaranteed to be consistent with \(T\).

If the sample \(T\) is consistent with \(A\) and subsumes \(T_L\), then by Thm. 6.2 the automaton \(M\) returned by \(L_{\text{Aut}}\) is isomorphic to the automaton of \(A\). Then by Thm. 7.1 the acceptor returned by \(L_{\text{Acc}}^{\text{IMA}}\) with inputs \(M\) and \(T\) is an IMA equivalent to \(A\).

### 7.2. Büchi acceptors.

The case of Büchi acceptors is nearly as straightforward as that of Muller acceptors. Let \(A\) be an IBA with \(n\) states and acceptance condition \(F\). For every state \(q\) of \(A\), if there is an \(\omega\)-word \(w\) such that \(A\) rejects \(w\) and \(q \in \inf(w)\), then there is an example \(u(v)^\omega\) of length \(O(n^2)\) such that \(A\) rejects \(u(v)^\omega\) and \(q \in \inf(u(v)^\omega)\), by Prop. 5.4. The negative labeled example \((u(v)^\omega, 0)\) is included in \(T_{\text{Acc}}^{\text{IBA}}\).

The learning algorithm \(L_{\text{Acc}}^{\text{IBA}}\) takes as input a deterministic complete automaton \(M\) and a sample \(T\). The acceptance condition \(F'\) consists of all the states \(q\) of \(M\) such that for no negative example \((u(v)^\omega, 0)\) in \(T\) do we have \(q \in \inf_M(u(v)^\omega)\). Once \(F'\) has been computed, the algorithm checks whether the DBA \((M, F')\) is consistent with the sample \(T\). If so, it returns \((M, F')\); if not, it returns the default acceptor of type DBA for \(T\).

**Theorem 7.3.** Algorithm \(L_{\text{Acc}}^{\text{IBA}}\) runs in time polynomial in the sizes of the inputs \(M\) and \(T\). Let \(A\) be an IBA. If the input automaton \(M\) is isomorphic to the automaton of \(A\), and the sample \(T\) is consistent with \(A\) and subsumes \(T_{\text{Acc}}^{\text{IBA}}\), then algorithm \(L_{\text{Acc}}^{\text{IBA}}\) returns an IBA \((M, F')\) equivalent to \(A\).

**Proof.** The construction of \(F'\) can be done in time polynomial in the sizes of \(M\) and \(T\) by Prop. 5.3. The returned acceptor is consistent with \(T\) by construction.
Assume the input $\mathcal{M}$ is isomorphic to the automaton of $\mathcal{A}$, and that $T$ is consistent with $\mathcal{A}$ and subsumes $T_{\text{IA}}^{\text{IBA}}$. For ease of notation, assume the isomorphism is the identity. We show that the DBA $(\mathcal{M}, F')$ is equivalent to $\mathcal{A}$.

If $\mathcal{A}$ rejects the word $u(v)^\omega$ then let $C = \inf_{\mathcal{M}}(u(v)^\omega)$. Because $T$ subsumes $T_{\text{Acc}}^{\text{IBA}}$, for each $q \in C$, there is a negative example $(u'(v')^\omega, 0)$ in $T$ such that $q \in \inf_{\mathcal{M}}(u'(v')^\omega)$. Thus no $q \in C$ is in $F'$ and $(\mathcal{M}, F')$ also rejects $u(v)^\omega$.

Conversely, if $\mathcal{A}$ accepts the word $u(v)^\omega$, then there is at least one state $q \in F$ such that $q \in \inf_{\mathcal{M}}(u(v)^\omega)$. Because $T$ is consistent with $\mathcal{A}$, there is no negative example $(u(v)^\omega, 0)$ in $T$ such that $q \in \inf_{\mathcal{M}}(u(v)^\omega)$, so $q \in F'$ and $(\mathcal{M}, F')$ also accepts $u(v)^\omega$. Thus $(\mathcal{M}, F')$ is equivalent to $\mathcal{A}$. Because $T$ is consistent with $\mathcal{A}$, the IBA $(\mathcal{M}, F')$ is returned by $L_{\text{Acc}}^{\text{IBA}}$.

**Theorem 7.4.** The classes $\mathbb{IBA}$ and $\mathbb{ICA}$ are identifiable in the limit using polynomial time and data.

**Proof.** The result for $\mathbb{ICA}$ follows from that for $\mathbb{IBA}$ by Prop. 5.1. Let $\mathcal{A}$ be an IBA accepting language $L$. The characteristic sample $T_L = T_{\text{Aut}} \cup T_{\text{IA}}^{\text{IBA}}$ is of size polynomial in $\text{size}(\mathcal{A})$.

The combined learning algorithm $L_{\text{IBA}}$ takes a sample $T$ as input and runs $L_{\text{Aut}}$ to get a deterministic complete automaton $\mathcal{M}$. It then runs $L_{\text{IBA}}$ on inputs $\mathcal{M}$ and $T$, and returns the resulting acceptor. $L_{\text{IBA}}$ runs in polynomial time in the length of $T$ and returns a DBA consistent with $T$.

If the sample $T$ is consistent with $\mathcal{A}$ and subsumes $T_L$ then $L_{\text{Aut}}$ returns an automaton $\mathcal{M}$ isomorphic to the automaton of $\mathcal{A}$ by Thm. 6.2. Then the acceptor returned by $L_{\text{IBA}}$ is of size polynomial in $\text{size}(\mathcal{A})$.

8. The sample $T_{\text{Acc}}$ and learning for IPA

The construction of $T_{\text{Acc}}^{\text{IPA}}$ for an IPA $\mathcal{P}$ builds on the construction of the canonical forest of SCCs for $\mathcal{P}$, whose construction and properties are described next. Roughly speaking, the purpose of the canonical forest for a given parity automaton $\mathcal{P}$ is to expose a set of words that if placed in the sample will lead a smart learner to correctly determine a coloring function for the constructed automaton. It is thus not surprising, that while developed for a different motivation, it has similarities with Carton and Maceiras’s algorithm to compute the minimal number of colors for a given parity automaton [CM99].

8.1. Constructing the Canonical Forest and Coloring of a DPA. Let $\mathcal{P} = (\Sigma, Q, q_0, \delta, \kappa)$ be a complete DPA. We extend the coloring function $\kappa$ to nonempty sets of states by $\kappa(S) = \min\{k(q) \mid q \in S\}$, the minimum color of any state in $S$. We define the $\kappa$-parity of $S$ to be 1 if $\kappa(S)$ is odd, and 0 if $\kappa(S)$ is even. A word $w \in \Sigma^\omega$ is accepted by $\mathcal{P}$ iff the $\kappa$-parity of $\inf(w)$ is 1. Note that the union of two sets of $\kappa$-parity $b$ is also of $\kappa$-parity $b$. For any nonempty $S \subseteq Q$, we define $\text{minStates}(S) = \{q \in S \mid \kappa(q) = \kappa(S)\}$, the states of $S$ that are assigned the minimum color among all states of $S$.
8.1.1. The minStates-Forest. We describe an algorithm to construct the minStates-forest of \( \mathcal{P} \). The roots of the minStates-forest are the elements of \( \text{maxSCCs}(Q) \), each marked as unprocessed. If \( C \) is unprocessed, then \( D = \text{maxSCCs}(C \setminus \text{minStates}(C)) \) is computed. If \( D \) is empty, \( C \) becomes a leaf in the forest, and is marked as processed. Otherwise, \( C \) is marked as processed and the elements of \( D \) are made the children of \( C \) and are marked as unprocessed.

**Proposition 8.1.** Let \( \mathcal{P} = \langle \Sigma, Q, q_0, \delta, \kappa \rangle \) be a complete DPA with automaton \( \mathcal{M} \). Let \( \mathcal{F} \) be the minStates-forest of \( \mathcal{P} \). Then \( \mathcal{F} \) is a decreasing forest of SCCs of \( \mathcal{M} \) rooted in \( Q \), and can be computed in polynomial time. For any SCC \( D, \kappa(D) = \kappa(\text{Node}(D, \mathcal{F})) \).

**Proof.** Referring to the construction of the minStates-forest \( \mathcal{F} \), its roots are the elements of \( \text{maxSCCs}(Q) \). When a node \( C \) is processed, the nonempty set \( \text{minStates}(C) \) is removed and the maximum SCCs (if any) of the result become the children of \( C \), so the union of the children of \( C \) is a proper subset of \( C \), and the children are the maximum SCCs of \( C \setminus \Delta(C) \).

Let \( D \) be an SCC. Then \( D \subseteq Q \) and for the node \( C = \text{Node}(D, \mathcal{F}) \), we have that \( D \subseteq C \) and \( D \) is not a subset of any child of \( C \). Thus \( D \cap \text{minStates}(C) \neq \emptyset \), because otherwise \( D \) would be a subset of some child of \( C \). This implies that \( \kappa(D) = \kappa(C) \). The minStates-forest of \( \mathcal{P} \) can be computed in polynomial time because it has at most \( |Q| \) nodes, and each set \( \text{maxSCCs}(S) \) can be computed in polynomial time by Prop. 5.7.

8.1.2. The Canonical Forest and Coloring. The canonical forest of \( \mathcal{P} \) is constructed as follows, starting with the minStates-forest of \( \mathcal{P} \). While there exist in the forest a node \( D \) and its parent \( C \) of the same \( \kappa \)-parity, one such pair \( D \) and \( C \) is selected, and the child node \( D \) is merged into the parent node \( C \). When no such pair remains, the result is the canonical forest of \( \mathcal{P} \), denoted \( \mathcal{F}^*(\mathcal{P}) \). The canonical forest \( \mathcal{F}^*(\mathcal{P}) \) can be computed from \( \mathcal{P} \) in polynomial time.

From the canonical forest \( \mathcal{F}^*(\mathcal{P}) \), we define the canonical coloring \( \kappa^* \). The states in \( (Q \setminus \bigcup \text{maxSCCs}(Q)) \) are not contained in any SCC of \( \mathcal{P} \) and do not affect the acceptance or rejection of any \( \omega \)-word. For definiteness, we assign them \( \kappa^*(q) = 0 \). For a root node \( C \) of \( \kappa \)-parity \( b \), we define \( \kappa^*(q) = b \) for all \( q \in \Delta(C) \). Let \( C \) be an arbitrary node of \( \mathcal{F}^*(\mathcal{P}) \). If the states of \( \Delta(C) \) have been assigned color \( k \) by \( \kappa^* \) and \( D \) is a child of \( C \), then the states of \( \Delta(D) \) are assigned color \( k + 1 \) by \( \kappa^* \). Clearly \( \kappa^* \) can be computed from \( \mathcal{P} \) in polynomial time.

**Example 8.2.** Figure 2(a) shows the graph \( G(\mathcal{P}) \) of a DPA \( \mathcal{P} \) with states \( a \) through \( m \), labeled by the colors assigned by \( \kappa \). Figure 2(b) shows the minStates-forest of \( \mathcal{P} \), with the nodes labeled by their \( \kappa \)-parities. Figure 2(c) shows the canonical forest \( \mathcal{F}^*(\mathcal{P}) \) of \( \mathcal{P} \), with the nodes labeled by their \( \kappa \)-parities. Figure 2(d) shows the graph \( G(\mathcal{P}) \) re-colored using the canonical coloring \( \kappa^* \).

**Theorem 8.3.** Let \( \mathcal{P} = \langle \Sigma, Q, q_0, \delta, \kappa \rangle \) be a complete DPA with automaton \( \mathcal{M} \). The canonical forest \( \mathcal{F}^*(\mathcal{P}) \) is a decreasing forest of SCCs of \( \mathcal{M} \) rooted in \( Q \) and has the following properties.

1. For any SCC \( D \), both \( D \) and \( \text{Node}(D, \mathcal{F}^*(\mathcal{P})) \) have the same \( \kappa \)-parity.
2. For every node \( C \) of \( \mathcal{F}^*(\mathcal{P}) \), the \( \kappa \)-parity of \( C \) is the same as the \( \kappa^* \)-parity of \( C \).
3. The children in \( \mathcal{F}^*(\mathcal{P}) \) of a node \( C \) of \( \kappa \)-parity \( b \) are the maximal SCCs \( D \subseteq C \) of \( \kappa \)-parity \( 1 - b \).
Proof. Because $F^*(P)$ is obtained from the $\text{minStates}$-forest of $P$ by a sequence of merges, $F^*(P)$ is a decreasing forest of SCCs of $M$ rooted in $Q$ by Prop. 5.9. Let $F_0$ denote the $\text{minStates}$-forest of $P$, and let $F_i$ denote the forest after $i$ merges have been performed in the computation to produce the canonical acceptor $F^*(P)$.

By Prop. 8.1, for any SCC $D$, $\kappa(D) = \kappa(\text{Node}(D, F_0))$, so property (1) holds for $F_0$. We show by induction that it holds for each $F_i$ and therefore for $F^*(P)$. Assume that property (1) holds of $F_i$, and $F_{i+1}$ is obtained from $F_i$ by merging child node $D$ into parent node $C$. Let $D'$ be any SCC. If $\text{Node}(D', F_i) = \text{Node}(D', F_{i+1})$, then because property (1) holds in $F_i$, we have that the $\kappa$-parity of $D'$ is the same as the $\kappa$-parity of $\text{Node}(D', F_{i+1})$. Otherwise, it must be that $D = \text{Node}(D', F_i)$ and $C = \text{Node}(D', F_{i+1})$. Because $D$ is only merged to $C$ if they are of the same $\kappa$-parity, this implies that the $\kappa$-parity of $D'$ is the same as the $\kappa$-parity of $C$. Thus, property (1) holds also in $F_{i+1}$.

For property (2), we note that the $\kappa$-parity and the $\kappa^*$-parity of each root node of $F^*(P)$ is the same. Suppose $C$ is a node of $F^*(P)$ whose $\kappa$-parity and $\kappa^*$-parity are equal to $b$, and $D$ is a child of $C$. Then by the construction of $F^*(P)$, the $\kappa$-parity of $D$ is $1 - b$. And by the definition of $\kappa^*$, the $\kappa^*$-parity of the elements of $\Delta(D)$ is the opposite of the $\kappa^*$-parity of the elements of $\Delta(C)$. Because the $\kappa^*$-parity of $C$ is $b$, the $\kappa^*$-parity of $D$ is also $1 - b$.

For property (3), let $D$ be any maximal SCC of $P$ that is contained in $C$ and has $\kappa$-parity $1 - b$. Then $\text{Node}(D, F_0)$ is contained in $C$ and has the same $\kappa$-parity as $D$. Since
\[ D \text{ is maximal, we must have } D = \text{Node}(D, \mathcal{F}_0), \text{ and any nodes on the path between } C \text{ and } D \text{ must have } \kappa\text{-parity } b \text{ and must be merged into } C \text{ to form } \mathcal{F}^*(\mathcal{P}). \text{ Thus, } D \text{ is a child of } C \text{ in } \mathcal{F}^*(\mathcal{P}). \]

Conversely, if \( D \) is a child of \( C \) in \( \mathcal{F}^*(\mathcal{P}) \) then \( D \subseteq C \) and the \( \kappa \)-parity of \( D \) is \( 1 - b \). Assume \( D' \) is an SCC such that \( D' \subseteq C \), the \( \kappa \)-parity of \( D' \) is \( 1 - b \) and \( D \subseteq D' \). Then \( D'' = \text{Node}(D', \mathcal{F}_0) \) is a descendant of \( C \) in \( \mathcal{F}_0 \) that has \( \kappa \)-parity \( 1 - b \), and \( D \subseteq D'' \), so because there is another node of parity \( 1 - b \) on the path between \( D \) and \( C \) in \( \mathcal{F}_0 \), \( D \) cannot be a child of \( C \) in \( \mathcal{F}^*(\mathcal{P}) \), a contradiction. \( \square \)

Replacing the coloring function of \( \mathcal{P} \) by the canonical coloring does not change the \( \omega \)-language accepted.

**Theorem 8.4.** Let \( \mathcal{P} = \langle \Sigma, Q, q_0, \delta, \kappa \rangle \) be a complete DPA, and \( \mathcal{P}' \) be \( \mathcal{P} \) with the canonical coloring \( \kappa^* \) for \( \mathcal{P} \) in place of \( \kappa \). Then \( \mathcal{P} \) and \( \mathcal{P}' \) recognize the same \( \omega \)-language.

**Proof.** Let \( w \) be an \( \omega \)-word and let \( D = \inf(w) \). This is an SCC of the (common) automaton of \( \mathcal{P} \) and \( \mathcal{P}' \). Let \( C = \text{Node}(D, \mathcal{F}^*(\mathcal{P})) \). Then \( D \cap \Delta(C) \neq \emptyset \), and \( \kappa^*(D) = \kappa^*(C) \), by the definition of \( \kappa^* \). The \( \kappa^* \)-parity of \( C \) is the same as the \( \kappa \)-parity of \( C \), by property (2) of Thm. 8.3. The \( \kappa \)-parity of \( C \) is the same as the \( \kappa \)-parity of \( D \), by property (1) of Thm. 8.3. Thus, the \( \kappa^* \)-parity of \( D \) is the same as the \( \kappa \)-parity of \( D \), and \( w \in \mathcal{P} \) iff \( w \in \mathcal{P}' \). \( \square \)

### 8.2. Constructing \( T_{\text{IPA}}^\text{Acc} \)

We now describe the construction of \( T_{\text{IPA}}^\text{Acc} \), the second part of the characteristic sample for an IPA \( \mathcal{P} \) with the automaton \( \mathcal{M} \) of \( n \) states. The sample \( T_{\text{IPA}}^\text{Acc} \) consists of one example \( u(v)^\omega = \text{Witness}(\mathcal{C}, \mathcal{M}) \) of length \( O(n^2) \) for each reachable SCC \( \mathcal{C} \) in the canonical forest \( \mathcal{F}^*(\mathcal{P}) \). The example \( u(v)^\omega \) is labeled 1 if it is accepted by \( \mathcal{P} \) and 0 otherwise. Thus \( T_{\text{IPA}}^\text{Acc} \) contains at most \( n \) labeled examples, each of length \( O(n^2) \).

### 8.3. The learning algorithm \( L_{\text{IPA}}^\text{Acc} \)

Given a complete deterministic automaton \( \mathcal{M} = \langle \Sigma, Q, q_0, \delta \rangle \) and a sample \( T \) as input, the learning algorithm \( L_{\text{IPA}}^\text{Acc} \) attempts to construct a coloring of the states of \( \mathcal{M} \) consistent with \( T \).

The algorithm first constructs the set \( Z \) of all \( C \subseteq Q \) such that for some labeled example \((u(v)^\omega, l) \in T \) we have \( C = \inf_M(u(v)^\omega) \). If two examples with different labels are found to yield the same set \( C \), this is evidence that the automaton \( \mathcal{M} \) is not correct, and the learning algorithm returns the default acceptor of type DPA for \( T \).

Otherwise, each set \( C \) in \( Z \) is associated with the label of the one or more examples that yield \( C \). The set \( Z \) is partially ordered by the subset relation. The learning algorithm then attempts to construct a rooted forest \( \mathcal{F}' \) with nodes that are elements of \( Z \), corresponding to the canonical forest of the target acceptor. Initially, \( \mathcal{F}' \) contains as roots all the maximal elements of \( Z \). If these are not pairwise disjoint, it returns the default acceptor of type DPA for \( T \). Otherwise, the root nodes are all marked as unprocessed.

For each unprocessed node \( C \) in \( \mathcal{F}' \), it computes the set of all \( D \in Z \) such that \( D \subseteq C \), \( D \) has the opposite label to \( C \), and \( D \) is maximal with these properties, and makes \( D \) a child of \( C \) and marks \( D \) as unprocessed. When all the children of a node \( C \) have been determined, the algorithm checks two conditions: (1) that the children of \( C \) are pairwise disjoint, and (2) there is at least one \( q \in C \) that is not in any child of \( C \). If either of these conditions fail, then it returns the default acceptor of type DPA for \( T \). If both conditions are satisfied, then the node \( C \) is marked as processed. When there are no more unprocessed nodes, the construction of \( \mathcal{F}' \) is complete. Note that \( \mathcal{F}' \) has at most \(|Q|\) nodes.
When the construction of $\mathcal{F}'$ is complete, for each node $C$ in $\mathcal{F}'$ let $\Delta(C)$ denote the elements of $C$ that do not appear in any of its children. Then the learning algorithm assigns colors to the elements of $Q$ starting from the roots of $\mathcal{F}'$, as follows. If $C$ is a root with label $l$, then $\kappa'(q) = l$ for all $q \in \Delta(C)$. If the elements of $\Delta(C)$ have been assigned color $k$ and $D$ is a child of $C$, then $\kappa'(q) = k + 1$ for all $s \in \Delta(D)$. When this process is complete, any uncolored states $q$ are assigned $\kappa'(q) = 0$.

If the resulting DPA $(\mathcal{M}, \kappa')$ is consistent with the sample $T$, the algorithm $L^\text{IPA}_\text{Acc}$ returns $(\mathcal{M}, \kappa')$. If not, it returns the default acceptor of type DPA for $T$.

**Theorem 8.5.** Algorithm $L^\text{IPA}_\text{Acc}$ runs in time polynomial in the sizes of the inputs $\mathcal{M}$ and $T$. Let $\mathcal{P}$ be an IPA. If the input automaton $\mathcal{M}$ is isomorphic to the automaton of $\mathcal{P}$, and the sample $T$ is consistent with $\mathcal{P}$ and subsumes $T^\text{IPA}_\text{Acc}$, then algorithm $L^\text{IPA}_\text{Acc}$ returns an IPA $(\mathcal{M}, \kappa')$ equivalent to $\mathcal{P}$.

**Proof.** The construction of $\kappa'$ can be done in time polynomial in the sizes of $\mathcal{M}$ and $T$. The returned acceptor is consistent with $T$ by construction.

Assume the input $\mathcal{M} = (\Sigma, Q, q_0, \delta)$ is isomorphic to the automaton of $\mathcal{P}$, and that $T$ is consistent with $\mathcal{P}$ and subsumes $T^\text{IPA}_\text{Acc}$. For ease of notation, assume the isomorphism is the identity. We show that the forest $\mathcal{F}'$ constructed by the learning algorithm is equal to the canonical forest of $\mathcal{F}^\ast(\mathcal{P})$, the coloring $\kappa'$ is equal to the canonical coloring $\kappa^\ast$, and therefore the acceptor $(\mathcal{M}, \kappa')$ is equivalent to $\mathcal{P}$.

The roots of $\mathcal{F}^\ast(\mathcal{P})$ are the maximal SCCs contained in $Q$, and for each such root $C$, $T^\text{IPA}_\text{Acc}$ contains an example $(u(v)^\omega, l)$ such that $C = \inf(u(v)^\omega)$. Thus, the set of maximal elements of $Z$ is equal to the set of roots of $\mathcal{F}^\ast(\mathcal{P})$.

Let $C$ be any node of $\mathcal{F}^\ast(\mathcal{P})$, and let $D$ be a child of $C$ in $\mathcal{F}^\ast(\mathcal{P})$. Then $D$ is an SCC, $D \subseteq C$, the parity of $D$ is opposite to the parity of $C$, and $D$ is maximal in the subset $\mathcal{F}$ with these properties, by property (3) of Thm. 8.3. In the sample $T^\text{IPA}_\text{Acc}$ there is an example $(u(v)^\omega, l)$ with $D = \inf(u(v)^\omega)$, so $D$ is an element of $Z$, and will be made a child of $C$ in $\mathcal{F}'$ because $D \subseteq C$, the label $l$ is the opposite of the label of $C$, and $D$ is maximal in $Z$ with these properties. Conversely, if $D$ is made a child of $C$ in $\mathcal{F}'$, then $D \subseteq C$, the label of $D$ is opposite to the label of $C$ (that is, they are of opposite $\kappa$-parity), and $D$ is maximal in $Z$ with these properties. This implies $D$ is a child of $C$ in $\mathcal{F}^\ast(\mathcal{P})$, by property (3) of Thm. 8.3.

By induction, $\mathcal{F}'$ is equal to $\mathcal{F}^\ast(\mathcal{P})$, and therefore $\kappa'$ is equal to the canonical coloring $\kappa^\ast$. Then the IPA $(\mathcal{M}, \kappa')$ is equivalent to $\mathcal{P}$, by Thm. 8.4. Because $T$ is consistent with $\mathcal{P}$, the IPA $(\mathcal{M}, \kappa')$ is returned by $L^\text{IPA}_\text{Acc}$. □

**Theorem 8.6.** The class $\mathcal{IP}_A$ is identifiable in the limit using polynomial time and data.

**Proof.** Let $\mathcal{P}$ be an IPA accepting the language $L$. The characteristic sample $T_L = T_{\text{Aut}} \cup T^\text{IPA}_\text{Acc}$ for $\mathcal{P}$ is of size polynomial in the size of $\mathcal{P}$.

The combined learning algorithm $L^\text{IPA}$ with a sample $T$ as input first runs $L_{\text{Aut}}$ on $T$ to get a complete deterministic automaton $\mathcal{M}$ and then runs $L^\text{IPA}_\text{Acc}$ on inputs $\mathcal{M}$ and $T$ and returns the resulting acceptor. The running time of $L^\text{IPA}$ is polynomial in the length of $T$ and the returned acceptor is consistent with $T$.

Assume the sample $T$ is consistent with $\mathcal{P}$ and subsumes $T_L$. By Thm. 6.2, the automaton $\mathcal{M}$ is isomorphic to the automaton of $\mathcal{P}$. By Thm. 8.5, the acceptor $(\mathcal{M}, \kappa')$ is equivalent to $\mathcal{P}$. Because it is consistent with $T$, the IPA $(\mathcal{M}, \kappa')$ is the acceptor returned by $L^\text{IPA}$. □
9. The sample $T_{\text{Acc}}$ and learning for IRA

In this section we introduce some terminology, establish a normal form for Rabin acceptors, define an ordering on sets of states of an automaton, and then describe the learning algorithm $L_{\text{Acc}}^{\text{IRA}}$ and sample $T_{\text{Acc}}^{\text{IRA}}$ for IRAs. We then prove that the classes $\text{IRA}_t$ and $\text{ISA}_t$ are ILPTD.

Let $\mathcal{R} = \langle \Sigma, Q, q_0, \delta, \alpha \rangle$ be a Rabin acceptor, where the acceptance condition $\alpha = \{(G_1, B_1), \ldots, (G_k, B_k)\}$ is a set of ordered pairs of states. We say that an $\omega$-word $w$ satisfies a pair of state sets $(G, B)$ iff $\inf(w) \cap G \neq \emptyset$ and $\inf(w) \cap B = \emptyset$. Also, $w$ satisfies the acceptance condition $\alpha$ iff there exists $i \in [1..k]$ such that $w$ satisfies $(G_i, B_i)$. Then an $\omega$-word $w$ is accepted by $\mathcal{R}$ iff $w$ satisfies the acceptance condition $\alpha$ of $\mathcal{R}$.

9.1. Singleton normal form for a Rabin acceptor. We say that a Rabin acceptor $\mathcal{R}$ is in singleton normal form iff for every pair $(G_i, B_i)$ in its acceptance condition we have $|G_i| = 1$, that is, every $G_i$ is a singleton set. To avoid extra braces, we abbreviate the pair $(\{q\}, B)$ by $(q, B)$.

Every Rabin acceptor may be put into singleton normal form by a polynomial time algorithm.

**Proposition 9.1.** Let $\mathcal{R} = \langle \Sigma, Q, q_0, \delta, \alpha \rangle$ be a Rabin acceptor where $\alpha = \{(G_1, B_1), \ldots, (G_k, B_k)\}$. Define the acceptance condition $\alpha'$ to contain $(q, B_i)$ for every $i \in [1..k]$ and $q \in G_i$, and let $\mathcal{R}' = \mathcal{R}$ with $\alpha$ replaced by $\alpha'$. Then $\mathcal{R}'$ is in singleton normal form and accepts the same language as $\mathcal{R}$. Also, $\mathcal{R}'$ is of size at most $|Q|$ times the size of $\mathcal{R}$.

**Proof.** If an $\omega$-word $w$ satisfies a pair $(q, B)$ of $\alpha'$, then there exists a pair $(G_i, B_i)$ of $\alpha$ with $q \in G_i$ and $B_i = B$, so $w$ also satisfies the pair $(G_i, B_i)$ in $\alpha$. Conversely, if $w$ satisfies a pair $(G_i, B_i)$ in $\alpha$, then there exists $q \in G_i$ such that $q \in \inf(w)$, so $w$ also satisfies $(q, B_i)$ in $\alpha'$. Each pair $(G_i, B_i)$ in $\alpha$ is replaced by at most $|Q|$ pairs in $\alpha'$.

9.2. An ordering on sets of states. Given an automaton, we define an ordering $\preceq$ on sets of its states that is used to coordinate between the characteristic sample and the learning algorithm for an IRA.

Let $\mathcal{M} = \langle \Sigma, Q, q_0, \delta \rangle$ be a deterministic complete automaton in which every state is reachable from the start state. Recall from §6.2 that $\text{access}(q)$ is the shortlex least string $s \in \Sigma^*$ such that $\delta(q_0, s) = q$.

For a set of states $S$, we define $\text{access}(S)$ to be the sequence of values $\text{access}(q)$ for $q \in S$, sorted into increasing shortlex order. We define the total ordering $\preceq$ on sets of states of $\mathcal{M}$ as follows. If $S_1, S_2 \subseteq Q$, then $S_1 \preceq S_2$ iff either $|S_1| < |S_2|$ or $|S_1| = |S_2|$ and the sequence $\text{access}(S_1)$ is less than or equal to the sequence $\text{access}(S_2)$ in the lexicographic ordering, using the shortlex ordering on $\Sigma^*$ to compare the component entries. For example, if $\text{access}(S_1) = \langle \epsilon, a, baa \rangle$, $\text{access}(S_2) = \langle a, ba \rangle$, and $\text{access}(S_3) = \langle \epsilon, ab, ba \rangle$ then $S_2 \preceq S_1 \preceq S_3$. 

9.3. The learning algorithm $L^{IRA}_{Acc}$. We begin with the description of the learning algorithm $L^{IRA}_{Acc}$, which is used in the definition of the sample $T^{IRA}_{Acc}$ in the next section. The inputs to $L^{IRA}_{Acc}$ are a deterministic complete automaton $M$ and a sample $T$. The algorithm attempts to construct a singleton normal form Rabin acceptance condition $\beta$ to produce an acceptor $(M, \beta)$ consistent with $T$.

The processing of an example $w$ from $T$ depends only on $\inf_M(w)$. The algorithm first computes the set $\{\inf_M(w) | (w, 1) \in T\}$, sorts its elements into decreasing order $C_1, C_2, \ldots, C_t$ using $\leq$, and for each $i \in [1..t]$ chooses a positive example $(z_i, 1) \in T$ with $C_i = \inf_M(z_i)$.

At each stage $k$ of the learning algorithm, $\beta_k$ is a singleton normal form acceptance condition. Initially, $\beta_0 = \emptyset$ (which is satisfied by no words) and $k = 0$.

The main loop processes the positive examples $z_i$ for $i = 1, 2, \ldots, t$. If $z_i$ is accepted by $(M, \beta_k)$, then the algorithm goes on to the next positive example. Otherwise, we say that the example $z_i$ causes the update of $\beta_k$. Let $G = \inf_M(z_i)$ and $B = Q \setminus G$ and let $S_k$ be the set of pairs $(q, B)$ such that $q \in G$ and there is no negative example $(w, 0)$ in $T$ such that $w$ satisfies $(q, B)$. Then $\beta_{k+1}$ is set to $\beta_k \cup S_k$ and $k$ is set to $k + 1$.

When the positive examples $z_1, z_2, \ldots, z_t$ have been processed, let $\beta = \beta_k$ for the final value of $k$. If the Rabin acceptor $(M, \beta)$ is consistent with $T$, then it is returned. If not, the learning algorithm returns the default acceptor of type DRA for $T$.

**Proposition 9.2.** Let $R$ be an IRA in singleton normal form with acceptance condition $\alpha$ and assume that the input $M$ is an automaton isomorphic to the automaton of $R$. Assume the sample $T$ is consistent with $R$. For each $k$, if $z_k$ is the example that causes the update of $\beta_k$, then $(M, \beta_k)$ accepts $z_j$ for all $j < i$. Moreover, $(M, \beta)$ is consistent with $T$.

**Proof.** No pair $(q, B)$ is added to the acceptance condition if there is a negative example in $T$ that satisfies it, so $(M, \beta)$ is consistent with all the negative examples in $T$.

Consider any positive example $(w, 1)$ from $T$. There exists $i$ with $\inf_M(w) = \inf_M(z_i)$. When $z_i$ is processed by the algorithm, if it is already accepted by the current $(M, \beta)$, then it (and the word $w$) is also accepted by every subsequent hypothesis, including $(M, \beta)$, because pairs are not removed from $\beta_k$.

Otherwise, $z_i$ causes the update to $\beta_k$, and among the pairs in $S_k$ that are added to $\beta_k$ is at least one that $z_i$ satisfies. To see this, note that $z_i$ must satisfy some pair in $\alpha$, say $(q, B_j)$. Thus, $q \in \inf(z_i)$ and $B_j \cap \inf(z_i) = \emptyset$. The pair $(q, B)$ where $B = Q \setminus \inf(z_i)$ has $B_j \subseteq B$. Thus any $\omega$-word that satisfies $(q, B)$ will also satisfy $(q, B_j)$. Because $T$ is consistent with $R$, there can be no negative example in $T$ satisfying $(q, B)$, so pair $(q, B)$ is part of $S_k$ and is added to $\beta_k$. The word $z_i$ (and the word $w$) satisfies $(q, B)$ and is therefore accepted by $(M, \beta_{k+1})$ and every subsequent hypothesis, including $(M, \beta)$. \qed

9.4. Constructing $T^{IRA}_{Acc}$. In this section we describe the construction of the sample $T^{IRA}_{Acc}$, which conveys the acceptance condition of an IRA. Let $R = (\Sigma, Q, q_0, \delta, \alpha)$ be a deterministic complete IRA of $n$ states in singleton normal form, and let $M$ be its automaton. The construction of the sample $T^{IRA}_{Acc}$ proceeds in stages, simulating the learning algorithm $L^{IRA}_{Acc}$ on the portion of the sample constructed so far to determine what examples still need to be added.

Initially, $\gamma_0 = \emptyset$ and $k = 0$. The acceptance condition $\gamma_k$ tracks the learning algorithm’s $\beta_k$. The set of words accepted by $(M, \gamma_k)$ is always a subset of the set of words accepted by
(\(M, \alpha\)). The main loop is as follows. If \((M, \gamma_k)\) is equivalent to \((M, \alpha)\) then the construction of \(T_{\text{IRA}}\) is complete.

Otherwise, let \(D_k\) be the set of \(\omega\)-words that satisfy \(\alpha\) but not \(\gamma_k\). Let \(C\) be the \(\preceq\)-largest set in \(\{\inf(w) \mid w \in D_k\}\), and let \(w_{k+1} = \text{Witness}(C, M)\), an ultimately periodic word of length \(O(n^2)\). Then \(w_{k+1}\) is added as a positive example to \(T_{\text{IRA}}\).

Let \(B = Q \setminus \inf(w_{k+1})\). Define \(P_k\) to be the set of all \((q, B)\) such that \(q \in \inf(w_{k+1})\) and there is no \(\omega\)-word \(w'\) that satisfies \((q, B)\) but not \(\alpha\). Set \(\gamma_{k+1} = \gamma_k \cup P_k\).

For each \(q \in \inf(w_{k+1})\) such that there is some \(\omega\)-word \(w'\) that satisfies \((q, B)\) but not \(\alpha\), let \(u(v)^w = \text{Witness}(\inf(w'), M)\) for some such \(w'\) and include \((u(v)^w, 0)\) as a negative example in \(T_{\text{Acc}}\). The example \(u(v)^w\) is of length \(O(n^2)\). Then set \(k\) to \(k+1\) and continue with the main loop.

We prove a polynomial bound on the number of examples added to the sample \(T_{\text{IRA}}\), thus showing that its length is bounded by a polynomial in the size of \(R\).

**Proposition 9.3.** If the acceptance condition \(\alpha\) is in singleton normal form and has \(m\) pairs, then at most \(m\) positive examples and at most \(m|Q|\) negative examples are added to \(T_{\text{Acc}}\).

**Proof.** We say an acceptance condition \(\gamma\) covers a pair \((q, B)\) iff every \(\omega\)-word \(w\) that satisfies \((q, B)\) also satisfies \(\gamma\). We will show that after each positive example \(w_{k+1}\) is added to \(T_{\text{Acc}}\), the condition \(\gamma_{k+1}\) covers at least one pair in \(\alpha\) that was not covered by \(\gamma_k\).

Suppose not, and let \(k+1\) be the least index for which \(\gamma_{k+1}\) does not cover a pair of \(\alpha\) that was not covered by \(\gamma_k\). Because \(w_{k+1}\) is an example that satisfies \(\alpha\) but not \(\gamma_k\), there must be a pair \((q, B_j)\) of \(\alpha\) that is satisfied by \(w_{k+1}\). Note that \(\gamma_k\) does not cover the pair \((q, B_j)\). Then \(q \in \inf(w_{k+1})\) and letting \(B = Q \setminus \inf(w_{k+1})\), \(B_j \subseteq B\). The pair \((q, B)\) will be added to \(\gamma_k\) in constructing \(\gamma_{k+1}\) because every word that satisfies \((q, B)\) also satisfies \((q, B_j)\).

If \(\gamma_{k+1}\) does not cover \((q, B_j)\), there must be a word \(w'\) that satisfies \((q, B_j)\) but not \((q, B)\). So \(q \in \inf(w')\) and \(B_j \cap \inf(w') = \emptyset\) but \(B \cap \inf(w') \neq \emptyset\). Let \(B' = Q \setminus \inf(w')\), so \(B_j \subseteq B'\). We have \(B \cap B' \subseteq B\). Because \(\inf(w_{k+1})\) and \(\inf(w')\) are SCCs that overlap in \(q\), their union is an SCC as well. Let \(w''\) be an \(\omega\)-word such that \(\inf(w'')\) is the union of \(\inf(w_{k+1})\) and \(\inf(w')\). Then \(Q \setminus \inf(w'') = B \cap B'\). Note that \(w''\) satisfies \((q, B_j)\) because \(B_j \subseteq B \cap B'\) and thus is a positive example of \(\alpha\).

Because \(B \cap B'\) is a proper subset of \(B\), \(\inf(w'')\) is a proper superset of \(\inf(w_{k+1})\) and the positive example \(w''\) would have been considered before \(w_{k+1}\) in the construction of \(T_{\text{Acc}}\). (We can imagine all the positive examples of \(\alpha\) being considered in order to find a maximum positive counterexample at each stage.) At that time, it was either passed over because (1) the current \(\gamma_r\) already covered it, or (2) it contributed a new pair to the current \(\gamma_r\) to yield \(\gamma_{r+1}\). In case (1), there is some pair \((q', B'')\) in \(\gamma_r\) that is satisfied by \(w''\). Then \(q'' \in \inf(w'')\) and \(B'' \cap \inf(w'') = \emptyset\). Recall \(\inf(w'')\) is the union of \(\inf(w_{k+1})\) and \(\inf(w')\). Thus, \(B'' \cap \inf(w_{k+1}) = \emptyset\) and \(B'' \cap \inf(w') = \emptyset\). Note that \(q'' \in \inf(w_{k+1})\) or \(q'' \in \inf(w')\). If \(q'' \in \inf(w_{k+1})\), \(w_{k+1}\) satisfies the pair \((q'', B'')\) in \(\gamma_{k+1}\), a contradiction, because \(w_{k+1}\) is not accepted by \(\gamma_k\) and \(r \leq k\). And if \(q'' \in \inf(w')\) then \(w''\) satisfies the pair \((q'', B'')\) in \(\gamma_{r+1}\), a contradiction, because \(w''\) is not accepted by \(\gamma_{k+1}\) and \(r \leq k\).

In case (2), the positive example \(w''\) contributes at least one term \((q'', B'')\) to \(\gamma_{r+1}\). In this case \(B'' = B \cap B'\) and \(q'' \in \inf(w'')\). Thus, \(q'' \in \inf(w_{k+1})\) or \(q'' \in \inf(w')\), so \(w_{k+1}\) or
w' satisfies the term \((q'', B'')\) of \(\gamma_{r+1}\), a contradiction because \(r + 1 \leq k\) and neither \(w_{k+1}\) nor \(w'\) is covered by \(\gamma_k\).

Thus, each positive example added to \(T_{\text{Acc}}\) covers a new pair of \(\alpha\), and at most \(m\) positive examples can be added. Each positive example added requires at most \(|Q|\) negative examples to avoid adding incorrect pairs, so at most \(m|Q|\) negative examples are added. \(\square\)

9.5. Correctness of \(L^{\text{IRA}}\). We prove the correctness of the learning algorithm \(L^{\text{IRA}}\) and show that the classes \(\text{IRA}\) and \(\text{ISA}\) are identifiable in the limit using polynomial time and data.

**Theorem 9.4.** Algorithm \(L^{\text{IRA}}\) runs in time polynomial in the sizes of the inputs \(M\) and \(T\). Let \(R\) be an IRA. If the input automaton \(M\) is isomorphic to the automaton of \(R\), and the sample \(T\) is consistent with \(R\) and subsumes \(T^{\text{IRA}}\), then algorithm \(L^{\text{IRA}}\) returns an IRA \((M, \beta)\) equivalent to \(R\).

**Proof.** By Prop. 5.3, \(L^{\text{IRA}}\) can construct the sequence \(z_1, z_2, \ldots, z_\ell\) and the successive acceptance conditions \(\beta_k\) in time polynomial in the size of \(M\) and the length of \(T\).

Assume \(R\) is an IRA, that \(M\) is isomorphic to the automaton of \(R\), and that the sample \(T\) is consistent with \(R\) and subsumes \(T^{\text{IRA}}\). For ease of notation, we assume that the isomorphism is the identity.

We show by induction that for each \(k\), the acceptance condition \(\beta_k\) in the learning algorithm \(L^{\text{IRA}}\) is the same as the acceptance condition \(\gamma_k\) in the construction of \(T^{\text{IRA}}\). This is true for \(k = 0\) because \(\beta_0 = \gamma_0 = \emptyset\).

Assume that \(\beta_k = \gamma_k\) for some \(k \geq 0\). If \((M, \gamma_k)\) is equivalent to \(R = (M, \alpha)\), then also \((M, \beta_k)\) is equivalent to \(R\), and none of the remaining positive examples cause any additions to \(\beta_k\). Thus this is the final value of \(k\), so \(\beta = \beta_k\) and \((M, \beta)\) is equivalent to \(R\).

If \((M, \gamma_k)\) accepts a proper subset of the language accepted by \((M, \alpha)\), then in the construction of sample \(T^{\text{IRA}}\), \(D_k\) is equal to the \(\omega\)-words accepted by \((M, \alpha)\) but not by \((M, \gamma_k)\). This causes the positive example \((w_{k+1}, 1)\) to be added to \(T^{\text{IRA}}\), where \(\inf_M(w_{k+1})\) is \(\preceq\)-largest in the set \(\{\inf_M(w) \mid w \in D_k\}\).

In the learning algorithm, because \((M, \beta_k)\) does not accept \(w_{k+1}\), Prop. 9.2 implies that there must be an example \(z_i\) that causes the update to \(\beta_k\), and all of the examples \(z_1, \ldots, z_{i-1}\) are accepted by \((M, \beta_k)\). Because for every positive example \((w, 1)\) in \(T\) there exists \(j\) such that \(\inf_M(w) = \inf_M(z_j)\), there must be some \(r\) such that \(\inf_M(w_{k+1}) = \inf_M(z_r)\). Moreover, \(i \leq r\).

If \(i < r\), then \(\inf_M(w_i)\) is strictly \(\preceq\)-larger than \(\inf_M(w_r)\), which contradicts the choice of \(w_{k+1}\) by the sample construction procedure, because \(z_i\) is accepted by \((M, \alpha)\) but not \((M, \gamma_k)\). Thus \(i = r\) and the example \(z_i = w_{k+1}\) is the element that causes the update to \(\beta_k\). The negative examples included in \(T^{\text{IRA}}\) for the positive example \(w_{k+1}\) ensure that the update to \(\beta_k\) is the same as the update to \(\gamma_k\), and \(\beta_{k+1} = \gamma_{k+1}\).

Because \(\beta_k = \gamma_k\) for all \(k\), for the final value of \(k\), \(\beta = \beta_k = \gamma_k\), and therefore \((M, \beta)\) is equivalent to \(R\). Because the IRA \((M, \beta)\) is consistent with \(T\), it is the acceptor returned by \(L^{\text{IRA}}\). \(\square\)

**Theorem 9.5.** The classes \(\text{IRA}\) and \(\text{ISA}\) are identifiable in the limit using polynomial time and data.
Proof. By Prop. 5.1 it suffices to prove this for IRA. Let $R$ be an IRA in singleton normal form accepting the language $L$. The characteristic sample $T_L = T_{Aut} \cup T_{IRA}^{Acc}$ is of size polynomial in the size of $R$.

The combined learning algorithm $L_{IRA}$ with a sample $T$ as input first runs $L_{Aut}$ on $T$ to get a deterministic complete automaton $M$ and then runs $L_{IRA}^{Acc}$ on inputs $M$ and $T$ and returns the resulting acceptor. $L_{IRA}$ runs in time polynomial in the length of $T$ and returns a DRA consistent with $T$.

Now assume that the sample $T$ is consistent with $R$ and subsumes $T_L$. Then by Thm. 6.2, the automaton $M$ is isomorphic to the automaton of $R$. By Thm. 9.4, the acceptor returned by $T_{IRA}$ is an IRA $(M, \beta)$ is equivalent to $R$, and this is the acceptor also returned by $L_{IRA}$.

10. Constructing characteristic samples in polynomial time

The definition of identification in the limit using polynomial time and data requires that a characteristic sample exist and be of polynomial size, but says nothing about the cost of computing it. An additional desirable property is that a characteristic sample be computable in polynomial time given an acceptor $A$ as input. Recall that when this holds, we say that the class is efficiently teachable/learnable. We now show that given an acceptor that is fully informative we can design efficient teachers, i.e., algorithms that run in polynomial time and compute the characteristic samples we have defined. This is conditioned on having polynomial time algorithms for equivalence (that are given in §11-§14). To claim the class $IA_X$ is efficiently teachable/learnable we also need to show that we can construct such sets when starting with an acceptor that is not, say an IBA, but has an equivalent IBA acceptor (and similarly for the other classes). This is done in §15-§16.

10.1. Computing $T_{Aut}$. For $T_{Aut}$, we need to be able to decide for two states $q_1$ and $q_2$ of an acceptor $A$ whether there exists an $\omega$-word that distinguishes them, and if so, to return such a word. We are thus led to consider the problems of inclusion and equivalence.

10.2. The problems of inclusion and equivalence. The inclusion problem is the following. Given as input two $\omega$-acceptors $A_1$ and $A_2$ over the same alphabet, determine whether the language accepted by $A_1$ is a subset of the language accepted by $A_2$, that is, whether $[A_1] \subseteq [A_2]$. If so, the answer should be “yes”; if not, the answer should be “no” and a witness, that is, an ultimately periodic $\omega$-word $u(v)^\omega$ accepted by $A_1$ but rejected by $A_2$.

The equivalence problem is similar: the input is two $\omega$-acceptors $A_1$ and $A_2$ over the same alphabet, and the problem is to determine whether they are equivalent, that is, whether $[A_1] = [A_2]$. If so, the answer should be “yes”; if not, the answer should be “no” and a witness, that is, an ultimately periodic $\omega$-word $u(v)^\omega$ that is accepted by exactly one of $A_1$ and $A_2$.

If we have a procedure to solve the inclusion problem, at most two calls to it will solve the equivalence problem. We describe polynomial time algorithms to solve the inclusion problem for DBAs, DCAs, and DPAs in §11 and 12, for DRAs and DSAs in §13, and for DMAs in §14. Referring to those sections, we obtain polynomial time algorithms to solve the equivalence problem for DBAs, DCAs, and DPAs from Thm. 11.1 and Thm. 12.2, for
DRAs and DSAs from Thm. 13.2, and for DMAs from Thm. 14.6. Thus, we have the following.

**Theorem 10.1.** Given an acceptor \( A \) of type IBA, ICA, IPA, IRA, ISA, or IMA, the sample \( T_{\text{Aut}} \) for the automaton portion of \( A \) may be computed in polynomial time.

*Proof.* Given an acceptor \( A \) and two states \( q_1 \) and \( q_2 \), to determine whether there is an \( \omega \)-word that distinguishes them, we call the relevant polynomial time equivalence algorithm on the acceptors \( A^{q_1} \) and \( A^{q_2} \), which returns a distinguishing word \( u(v)^\omega \) if they are not equivalent. \( \square \)

### 10.3. Computing \( T_{\text{Acc}} \)

For \( T_{\text{Acc}} \), the requirements depend on the type of acceptor.

**Proposition 10.2.** Given an IBA \( B \), the sample \( T_{\text{IBA}}^{\text{IBA}} \) can be computed in time polynomial in the size of \( B \).

*Proof.* Given an IBA \( B = (\Sigma, Q, q_0, \delta, F) \) with automaton \( M \), the sample \( T_{\text{IBA}}^{\text{IBA}} \) described in §7.2 is computed as follows. For each SCC \( C \in \text{maxSCCs}(Q \setminus F) \), let \( u(v)^\omega = \text{Witness}(C, M) \) and include the negative example \((u(v)^\omega, 0)\) in \( T_{\text{IBA}}^{\text{IBA}} \). To see that these examples are sufficient, suppose that \( q \in Q \) and \( w \in \Sigma^\omega \) are such that \( B \) rejects \( w \) and \( q \in \text{inf}(w) \). Then \( D = \text{inf}(w) \) is an SCC of \( B \) contained in \( Q \setminus F \), so it is contained in some \( C \in \text{maxSCCs}(Q \setminus F) \), and there is a negative example \((u(v)^\omega, 0)\) in \( T_{\text{IBA}}^{\text{IBA}} \) such that \( \text{inf}(u(v)^\omega) \subseteq C \). Because \( D \subseteq C \), we have \( q \in C \). \( \square \)

**Proposition 10.3.** Given an IPA \( P \), the sample \( T_{\text{IPA}}^{\text{IPA}} \) can be computed in time polynomial in the size of \( P \).

*Proof.* Given an IPA \( P \) with automaton \( M \), the computation of \( T_{\text{IPA}}^{\text{IPA}} \) proceeds as described in §8.2. That is, the canonical forest \( F^*(P) \) is computed in polynomial time, and for each node \( C \) in the forest, \( u(v)^\omega = \text{Witness}(C, M) \) is computed and \((u(v)^\omega, l)\) is added to \( T_{\text{IPA}}^{\text{IPA}} \), where \( l \) is the label of node \( C \) in the canonical forest. \( \square \)

**Proposition 10.4.** Given an IRA \( R \), the sample \( T_{\text{IRA}}^{\text{IRA}} \) can be computed in time polynomial in the size of \( R \).

*Proof.* Given an IRA \( R = (M, \alpha) \), the computation of \( T_{\text{IRA}}^{\text{IRA}} \) proceeds as described in §9.4. At each stage of the computation, it is necessary to find an \( \omega \)-word \( u(v)^\omega \) with the \( \preceq \)-largest \( \text{inf}_M(u(v)^\omega) \) that is accepted by \((M, \alpha)\) and rejected by \((M, \gamma_k)\). Thm. 13.2 gives a polynomial time algorithm that not only tests the inclusion of two DRAs, but returns a witness \( u(v)^\omega \) with the \( \preceq \)-largest \( \text{inf}_M(u(v)^\omega) \) in the case of non-inclusion, because \( M \) is isomorphic to \( M \times M \). \( \square \)

**Proposition 10.5.** Given an IMA \( A \), the sample \( T_{\text{IMA}}^{\text{IMA}} \) can be computed in time polynomial in the size of \( A \).

*Proof.* Given an IMA \( A = (M, \mathcal{F}) \), the sample \( T_{\text{IMA}}^{\text{IMA}} \) described in §7.1 is computed as follows. For each \( F \in \mathcal{F} \) determine whether \( F \) is a reachable SCC of \( M \), and if so, compute \( u(v)^\omega = \text{Witness}(F, M) \) and add \((u(v)^\omega, 1)\) to the sample \( T_{\text{IMA}}^{\text{IMA}} \). \( \square \)

**Theorem 10.6.** Let \( A \) be an IBA, IPA, IRA, or IMA accepting the \( \omega \)-language \( L \). Then the characteristic sample \( T_L \) for \( A \) may be computed in polynomial time in the size of \( A \).
Proof. By Thm. 10.1, the sample $T_{Aut}$ can be computed in polynomial time in the size of $A$, and by Prop. 10.2, 10.3, 10.4, or 10.5 the sample $T_{Acc}^{IBA}$, $T_{Acc}^{IPA}$, $T_{Acc}^{IRA}$, or $T_{Acc}^{IMA}$ can also be computed in polynomial time in the size of $A$.

Note that Thm. 10.6 does not imply that the class $IXA$ for $X \in \{B, P, R, M\}$ is efficiently teachable/learnable, since this class also has representations by non-isomorphic automata.

11. Inclusion algorithms

We show that there are polynomial time algorithms for the inclusion problem for DBAs, DCAs, DPAs, DRAs, DSAs, and DMAs. Recall that two calls to an inclusion algorithm suffice to solve the equivalence problem. By Claim 4, the inclusion and equivalence problems for DCAs are efficiently reducible to those for DBAs, and vice versa. Also, by Claim 1, the inclusion and equivalence problems for DBAs are efficiently reducible to those for DPAs. Thus it suffices to consider the inclusion problem for DPAs, DRAs, and DMAs.

Remark. In the case of DFAs, a polynomial algorithm for the inclusion problem can be obtained using polynomial algorithms for complementation, intersection and emptiness (since for any two languages $L_1 \subseteq L_2$ if and only if $L_1 \cap \overline{L_2} = \emptyset$). However, a similar approach does not work in the case of DPAs; although complementation and emptiness for DPAs can be computed in polynomial time, intersection cannot [Bok18, Theorem 9].

For the inclusion problem for DBAs, DCAs, and DPAs, Schewe [Sch10, Sch11] gives the following result.

Theorem 11.1 ([Sch10]). The inclusion problems for DBAs, DCAs, and DPAs are in NL.

Because NL (nondeterministic logarithmic space) is contained in polynomial time, this implies the existence of polynomial time inclusion and equivalence algorithms for DBAs, DCAs, and DPAs. For the sake of completeness, and to address the problem of returning a witness we include a proof sketch.

Proof sketch. For $i = 1, 2$, let $P_i = \langle \Sigma, Q_i, (q_i)_i, \delta_i, \kappa_i \rangle$ be a DPA. It suffices to guess two states $q_1 \in Q_1$ and $q_2 \in Q_2$, and two words $u \in \Sigma^*$ and $v \in \Sigma^+$, and to check that for $i = 1, 2$, $\delta_i((q_i)_i, u) = q_i$ and $\delta_i(q_i, v) = q_i$, and also, that the smallest value of $\kappa_1(q)$ in the loop in $P_1$ from $q_1$ to $q_1$ on input $v$ is odd, while the smallest value of $\kappa_2(q)$ in the loop in $P_2$ from $q_2$ to $q_2$ on input $v$ is even. If these checks succeed, then $[P_1]$ is not a subset of $[P_2]$, and the ultimately periodic word $u(v)^\omega$ is a witness.

Logarithmic space is enough to record the two guessed states $q_1$ and $q_2$ as well as the current minimum values of $\kappa_1$ and $\kappa_2$ as the loops on $v$ are traversed in the two automata. The words $u$ and $v$ need only be guessed symbol-by-symbol, using a pointer in each automaton to keep track of its current state.

This approach does not seem to work in the case of testing DRA or DMA inclusion, because the acceptance conditions would seem to require keeping track of more information than would fit in logarithmic space. To supplement the proof sketch for Schewe’s theorem, in the next section we give an explicit polynomial time algorithm for testing DPA inclusion. In the following sections, we give polynomial time algorithms for testing inclusion for DRAs and DMAs, which are novel results.
12. Inclusion and equivalence for DPAs, DBAs, DCAs

In this section we describe an explicit polynomial time algorithm for the inclusion problem for two DPAs, which yields algorithms for DBAs and DCAs. If $P = (\Sigma, Q, q_0, \delta, \kappa)$ is a complete DPA and $w \in \Sigma^\omega$, we let $P(w)$ denote the minimum color visited by $P$ infinitely often on input $w$, that is, $P(w) = \kappa(\inf(w))$.

12.1. Searching for $w$ with given minimum colors in two acceptors. We first describe an algorithm that searches for an $\omega$-word that yields specified minimum colors in two different DPAs over the same alphabet.

For $i = 1, 2$, let $P_i = (\Sigma, Q_i, (q_i)_i, \delta_i, \kappa_i)$ be a DPA, and let $M_i$ be the automaton of $P_i$. Given inputs of $P_1$ and $P_2$ and two nonnegative integers $k_1$ and $k_2$, the Colors algorithm constructs the product automaton $M = M_1 \times M_2$ and the set $Q' = \{(q_1, q_2) \in Q_1 \times Q_2 \mid \kappa_1(q_1) \geq k_1 \land \kappa_2(q_2) \geq k_2\}$.

The algorithm then computes $S = \max\text{SCCs}(Q')$ for the automaton $M$, and loops through the SCCs $C \in S$ checking whether $C$ is reachable in $M$, $\min(\kappa_1(\pi_1(C))) = k_1$, and $\min(\kappa_2(\pi_2(C))) = k_2$. If so, it returns the ultimately periodic word $u(v)^\omega = \text{Witness}(C, M)$. If none of the elements $C \in S$ satisfies this condition, then the answer “no” is returned.

**Theorem 12.1.** The algorithm Colors takes as input two DPAs $P_1$ and $P_2$ over the same alphabet and two nonnegative integers $k_1$ and $k_2$, runs in polynomial time, and determines whether there exists an $\omega$-word $w$ such that $P_1(w) = k_1$ and $P_2(w) = k_2$. If not, it returns the answer “no”. If so, it returns an ultimately periodic $\omega$-word $u(v)^\omega$ such that $P_1(u(v)^\omega) = k_1$ and $P_2(u(v)^\omega) = k_2$.

**Proof.** The polynomial running time of the algorithm follows from Props. 5.4 and 5.7. To see the correctness of the algorithm, suppose first that it returns an ultimately periodic word $u(v)^\omega$. This occurs only if it finds an SCC $C$ of $M$ such that $C$ is reachable in $M$, $\kappa_1(\pi_1(C)) = k_1$, and $\kappa_2(\pi_2(C)) = k_2$. Then for $i = 1, 2$, $\pi_i(C)$ is the set of states visited infinitely often by $M_i$ on the input $u(v)^\omega$, which has minimum color $k_i$.

To see that the algorithm does not incorrectly answer “no”, suppose $w$ is an $\omega$-word such that for $i = 1, 2$, $P_i(w) = k_i$. Let $D_i = \inf_{M_i}(w)$, an SCC of $M_i$. No state in $D_i$ has a color less than $k_i$, so if $D = \inf_M(w)$, then $D \subseteq Q'$. Also, $D$ is a reachable SCC in $M$.

Then $D$ is contained in some element $C$ of $\max\text{SCCs}(Q')$. Because there are no states $(q_1, q_2)$ in $C$ with $\kappa_1(q_1) < k_1$ or $\kappa_2(q_2) < k_2$, we must have $\kappa_i(\pi_i(C)) = k_i$ for $i = 1, 2$. Also, $C$ is reachable in $M$ because $D$ is. Thus, the algorithm will find at least one such $C$ and return $u(v)^\omega$ such that $\inf_M(u(v)^\omega) = C$.

12.2. An inclusion algorithm for DPAs. The inclusion problem for DPAs $P_1$ and $P_2$ over the same alphabet can be solved by looping over all odd $k_1$ in the range of $\kappa_1$ and all even $k_2$ in the range of $\kappa_2$, calling the Colors algorithm with inputs $P_1$, $P_2$, $k_1$, and $k_2$. If the Colors algorithm returns any witness $u(v)^\omega$, then $u(v)^\omega \in [P_1] \setminus [P_2]$, and $u(v)^\omega$ is returned as a witness of non-inclusion. Otherwise, by Thm. 12.1, there is no $\omega$-word $w$ accepted by $P_1$ and not accepted by $P_2$, and the answer “yes” is returned for the inclusion problem. Note that for $i = 1, 2$, the range of $\kappa_i$ has at most $|Q_i|$ distinct elements. Thus we have the following.
Theorem 12.2. There are polynomial time algorithms for the inclusion and equivalence problems for two DPAs over the same alphabet.

From Claims 1 and 4, we have the following.

Theorem 12.3. There are polynomial time algorithms for the inclusion and equivalence problems for two DBAs (or DCAs) over the same alphabet.

13. AN INCLUSION ALGORITHM FOR DRAs

In this section we describe a polynomial time algorithm to solve the inclusion problem for two DRAs. The algorithm returns a \( \preceq \)-largest witness in the case of non-inclusion.

**Algorithm 1 SubInc\(^{\text{DRA}} \)**

**Input:** Two DRAs \( \mathcal{R}_1 = (\mathcal{M}_1, \alpha_1) \) and \( \mathcal{R}_2 = (\mathcal{M}_2, \alpha_2) \) in singleton normal form, where \( \alpha_1 = \{ (q_1', B_1') \} \) and \( \alpha_2 = \{ (q_2', B_2') \} \), and a set \( S \) of states of \( \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \)

**Output:** \( u(\omega) \in [\mathcal{R}_1] \setminus [\mathcal{R}_2] \) with \( \text{inf}_\mathcal{M}(u(\omega)) \subseteq S \) if such exists, else “none”.

\[ M = \mathcal{M}_1 \times \mathcal{M}_2 \]

\[ W \leftarrow \emptyset \]

\[ S' \leftarrow S \setminus \{(q_1, q_2) \in S \mid q_1 \in B' \} \]

\[ C \leftarrow \text{maxSCCs}(S') \]

for each reachable \( C \in \mathcal{C} \) such that \( q_1' \in \pi_1(C) \) do

if for no \( j \in [1,k] \) is \( q_1' \in \pi_2(C) \) and \( B''_j \cap \pi_2(C) = \emptyset \) then

\[ W \leftarrow W \cup \{ \text{Witness}(C, \mathcal{M}) \} \]

▷ A new candidate witness

else

\[ J = \{ q_1'' \mid j \in [1,k], B''_j \cap \pi_2(C) = \emptyset \} \]

\[ S'' \leftarrow C \setminus \{ (q_1, q_2) \in C \mid q_2 \in J \} \]

Call SubInc\(^{\text{DRA}} \) recursively with \( \mathcal{R}_1, \mathcal{R}_2, \) and \( S'' \)

if the returned value is \( u(\omega) \) then

\[ W \leftarrow W \cup \{ u(\omega) \} \]

if \( W \) is \( \emptyset \) then

return “none”

else

Let \( u(\omega) \in W \) have the \( \preceq \)-largest value of \( \text{inf}_\mathcal{M}(u(\omega)) \)

return \( u(\omega) \)

The algorithm SubInc\(^{\text{DRA}} \) takes as input two DRAs \( \mathcal{R}_1 = (\mathcal{M}_1, \alpha_1) \) and \( \mathcal{R}_2 = (\mathcal{M}_2, \alpha_2) \) in singleton normal form, where \( \alpha_1 \) consists of a single pair \((q_1', B')\). It also takes as input a subset \( S \) of the state set of the product automaton \( \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \). The problem it solves is to determine whether there exists an \( \omega \)-word \( u(\omega) \) with \( \text{inf}_\mathcal{M}(u(\omega)) \subseteq S \) such that \( u(\omega) \in [\mathcal{R}_1] \setminus [\mathcal{R}_2] \). If there is such a word, the algorithm returns one with the \( \preceq \)-largest value of \( \text{inf}_\mathcal{M}(u(\omega)) \), and otherwise, it returns “none”.

Proposition 13.1. For \( i = 1,2 \) let \( \mathcal{R}_i = (\mathcal{M}_i, \alpha_i) \) be a DRA in singleton normal form. Assume \( \alpha_1 = \{ (q_1', B') \} \) and \( \alpha_2 = \{ (q_2', B_2') \} \). Let \( \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \), and let \( S \) be a subset of the states of \( \mathcal{M} \). Then with inputs \( \mathcal{R}_1, \mathcal{R}_2, \) and \( S \), the algorithm SubInc\(^{\text{DRA}} \) runs in polynomial time and returns \( u(\omega) \in [\mathcal{R}_1] \setminus [\mathcal{R}_2] \) with the \( \preceq \)-largest value of \( \text{inf}_\mathcal{M}(u(\omega)) \) contained in \( S \), if such exists, else it returns “none”.

Proof. When the element \( u(v)^\omega = \text{Witness}(\mathcal{M}, C) \) is added to \( W \), we have that \( C \) is a reachable SCC of \( \mathcal{M} \) contained in \( S \), \( q' \in B_1(C) \), and \( B' \cap \pi_1(C) = \emptyset \) (because \( S' \) contains no elements \( (q_1, q_2) \) with \( q_1 \in B' \)), so \( (\mathcal{M}_1, \{(q', B')\}) \) accepts \( u(v)^\omega \). Also, we have that for no \( j \in [1..k] \) do we have \( q''_j \in \pi_2(C) \) and \( B''_j \cap \pi_2(C) = \emptyset \), so \( \mathcal{R}_2 \) rejects \( u(v)^\omega \). Thus, any returned \( u(v)^\omega \) is a witness to the non-inclusion of \( [(\mathcal{M}_1, \{(q', B')\})] \) in \( [\mathcal{R}_2] \) with \( \inf_{\mathcal{M}}(u(v)^\omega) \subseteq S \).

We now show by induction on the recursive calls that if \( w \) is any \( \omega \)-word such that \( \inf_{\mathcal{M}}(w) \subseteq S \), \( (\mathcal{M}_1, \{(q', B')\}) \) accepts \( w \), and \( \mathcal{R}_2 \) rejects \( w \), then \( \text{SubInc}^{\text{DRA}} \) returns a witness \( u(v)^\omega \) such that \( \inf_{\mathcal{M}}(u(v)^\omega) \) is at least as large as \( \inf_{\mathcal{M}}(w) \) in the \( \preceq \)-ordering. Let \( D = \inf_{\mathcal{M}}(w) \). Then \( D \) is a reachable SCC of \( \mathcal{M} \) such that \( q' \in \pi_1(D) \), \( B' \cap \pi_1(D) = \emptyset \), and for no \( j \in [1..k] \) do we have \( q''_j \in \pi_2(D) \) and \( B''_j \cap \pi_2(D) = \emptyset \). Then \( D \subseteq S' \) because \( B' \cap \pi_1(D) = \emptyset \). Thus, \( D \) must be a subset of exactly one of the elements \( C \) of \( \text{maxSCCs}(S') \). Then \( C \) is reachable, \( q' \in \pi_1(C) \), and \( B' \cap \pi_1(C) = \emptyset \) (because \( C \) is a subset of \( S' \)).

If \( C \) is such that for no \( j \in [1..k] \) do we have \( q''_j \in \pi_2(C) \) and \( B''_j \cap \pi_2(C) = \emptyset \), then a witness \( u(v)^\omega = \text{Witness}(C, \mathcal{M}) \) is added to \( W \), and we have that \( C = \inf_{\mathcal{M}}(u(v)^\omega) \) is at least as large in the \( \preceq \)-ordering as \( D = \inf_{\mathcal{M}}(w) \), because \( D \subseteq C \).

Otherwise, the set \( J = \{q''_j | j \in [1..k], B''_j \cap \pi_2(C) = \emptyset\} \) is non-empty, and the algorithm removes from \( C \) all the states \( (q_1, q_2) \) such that \( q_2 \in J \) to form the set \( S'' \). Because \( D \subseteq C \), if \( B''_j \cap \pi_2(C) = \emptyset \), then also \( B''_j \cap \pi_2(D) = \emptyset \). Thus, if for any \( q''_j \in J \) we have \( q''_j \in \pi_2(D) \), this would violate the assumption that \( \mathcal{R}_2 \) rejects \( w \). Hence, \( D \subseteq S'' \), and by the inductive assumption on the recursive calls, the recursive call to \( \text{SubInc}^{\text{DRA}} \) returns a witness \( u(v)^\omega \) such that \( \inf_{\mathcal{M}}(u(v)^\omega) \) is at least as large in the \( \preceq \)-ordering as \( \inf_{\mathcal{M}}(w) \). Because the top-level algorithm returns \( u(v)^\omega \) to maximize \( \inf_{\mathcal{M}}(u(v)^\omega) \) with respect to \( \preceq \), it will be at least as large as \( \inf_{\mathcal{M}}(w) \).

For the polynomial running time, we note that all the SCCs \( C \) considered are distinct elements of a decreasing forest of SCCs for the automaton \( \mathcal{M} \), and so there can be at most as many as the number of states of \( \mathcal{M} \).

\[ \square \]

**Theorem 13.2.** There are polynomial time algorithms to solve the inclusion and equivalence problems for two DRAs (resp. DSAs) \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). In the case of non-inclusion or non-equivalence, these algorithms return a witness \( u(v)^\omega \) with the \( \preceq \)-largest value of \( \inf_{\mathcal{M}}(u(v)^\omega) \), where \( \mathcal{M} \) is the product of the automata of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \).

**Proof.** It suffices to consider just DRAs, by Claim 5. Given two DRAs \( \mathcal{R}_1 = (\mathcal{M}_1, \alpha_1) \) and \( \mathcal{R}_2 = (\mathcal{M}_2, \alpha_2) \), we may assume they are in singleton normal form. Then for each pair \( (q_1, B_1) \) in \( \alpha_1 \), we call \( \text{SubInc}^{\text{DRA}} \) with inputs \( (\mathcal{M}_1, \{(q_1, B_1)\}), \mathcal{R}_2 \), with \( S \) equal to the whole state set of \( \mathcal{M} \). If all of these calls return “none”, then \( [\mathcal{R}_1] \) is a subset of \( [\mathcal{R}_2] \), and the answer returned is “yes”. Otherwise, one or more calls return a witness, and \( u(v)^\omega \) is returned such that \( \inf_{\mathcal{M}}(u(v)^\omega) \) is \( \preceq \)-largest among the witnesses returned by the calls. The running time and correctness follow from the running time and correctness guarantees of \( \text{SubInc}^{\text{DRA}} \).  

\[ \square \]

**14. An inclusion algorithm for DMAs**

In this section we develop a polynomial time algorithm to solve the inclusion problem for two DMAs over the same alphabet. The proof proceeds in two parts: (1) a polynomial time reduction of the inclusion problem for two DMAs to the inclusion problem for a DBA and
a DMA, and (2) a polynomial time algorithm for the inclusion problem for a DBA and a DMA.

14.1. Reduction of DMA inclusion to DBA/DMA inclusion. We first reduce the problem of inclusion for two arbitrary DMAs to the inclusion problem for two DMAs where the first one has just a single final state set. For $i = 1, 2$, define the DMA $U_i = \langle Q_i, \Sigma, (q_i)_1, \delta_i, F_i \rangle$, where $F_i$ is the set of final state sets for $U_i$. Let the elements of $F_1$ be $\{F_1, \ldots, F_k\}$, and for each $j \in [1..k]$, let

$$U_{1,j} = \langle Q_1, \Sigma, (q_i)_1, \delta_1, \{F_j\} \rangle,$$

that is, $U_{1,j}$ is $U_1$ with $F_j$ as its only final state set. Then by the definition of DMA acceptance,

$$[U_1] = \bigcup_{j=1}^k [U_{1,j}],$$

which implies that to test whether $[U_1] \subseteq [U_2]$, it suffices to test for all $j \in [1..k]$ that $[U_{1,j}] \subseteq [U_2]$.

**Proposition 14.1.** Suppose $L$ is a procedure that solves the inclusion problem for two DMAs over the same alphabet, assuming that the first DMA has a single final state set. Then there is an algorithm that solves the inclusion problem for two arbitrary DMAs over the same alphabet, say $U_1$ and $U_2$, which simply makes $|F_1|$ calls to $L$, where $F_1$ is the family of final state sets of $U_1$.

Next we describe a procedure $\text{SCCtoDBA}$ that takes as inputs a deterministic automaton $M$, an SCC $F$ of $M$, and a state $q \in F$, and returns a DBA $B(M, F, q)$ that accepts exactly $L(M, F, q)$, where $L(M, F, q)$ is the set of $\omega$-words $w$ that visit only the states of $F$ when processed by $M$ starting at state $q$, and visits each of them infinitely many times.

Assume the states in $F$ are $\{q_0, q_1, \ldots, q_{m-1}\}$, where $q_0 = q$. The DBA $B(M, F, q)$ is $\langle Q', \Sigma, q_0, \delta', \{q_0\} \rangle$, where we define $Q'$ and $\delta'$ as follows. We create new states $r_{i,j}$ for $i, j \in [0..m-1]$ such that $i \neq j$, and denote the set of these by $R$. We also create a new dead state $d_0$. Then the set of states $Q'$ is $Q \cup R \cup \{d_0\}$.

For $\delta'$, the dead state $d_0$ behaves as expected: for all $\sigma \in \Sigma$, $\delta'(d_0, \sigma) = d_0$. For the other states in $Q'$, let $\sigma \in \Sigma$ and $i \in [0..m-1]$. If $\delta(q_i, \sigma)$ is not in $F$, then in order to deal with runs that would visit states outside of $F$, we define $\delta'(q_i, \sigma) = d_0$ and, for all $j \neq i$, $\delta'(r_{i,j}, \sigma) = d_0$.

Otherwise, for some $k \in [0..m-1]$ we have $q_k = \delta(q_i, \sigma)$. If $k = (i + 1) \mod m$, then we define $\delta'(q_i, \sigma) = q_k$, and otherwise we define $\delta'(q_i, \sigma) = r_{k, (i+1) \mod m}$. For all $j \in [0..m-1]$ with $j \neq i$, if $k = j$, we define $\delta'(r_{i,j}, \sigma) = q_k$, and otherwise we define $\delta'(r_{i,j}, \sigma) = r_{k,j}$.

Intuitively, for an input from $L(M, F, q)$, in $B(M, F, q)$ the states $q_i$ are visited in a repeating cyclic order: $q_0, q_1, \ldots, q_{m-1}$, and the meaning of the state $r_{i,j}$ is that at this point in the input, $M$ would be in state $q_i$, and the machine $B(M, F, q)$ is waiting for a transition that would arrive at state $q_j$ in $M$, in order to proceed to state $q_j$ in $B(M, F, q)$.

---

\footnote{This construction is reminiscent of the construction transforming a generalized Büchi into a Büchi automaton [Var08, Cho74], by considering each state in $F$ as a singleton set of a generalized Büchi, but here we need to send transitions to states outside $F$ to a sink state.}
example of the construction is shown in Fig. 3; the dead state and unreachable states are omitted for clarity.

**Lemma 14.2.** Let $M$ be a deterministic automaton with alphabet $\Sigma$ and states $Q$, and let $F$ be an SCC of $M$ and $q \in F$. With these inputs, the procedure $SCCtoDBA$ runs in polynomial time and returns the DBA $B(M, F, q)$, which accepts the language $L(M, F, q)$ and has $|F|^2 + 1$ states.

**Proof.** Suppose $w$ is in $L(M, F, q)$. Let $q = s_0, s_1, s_2, \ldots$ be the sequence of states in the run of $M$ from state $q$ on input $w$. This run visits only states in $F$ and visits each one of them infinitely many times. We next define a particular increasing sequence $i_{k, \ell}$ of indices in $s$, where $k$ is a positive integer and $\ell \in [0, m - 1]$. These indices mark particular visits to the states $q_0, q_1, \ldots, q_m$ in repeating cyclic order. The initial value is $i_{1,0} = 0$, marking the initial visit to $q_0$. If $i_{k, \ell}$ has been defined and $\ell < m - 1$, then $i_{k, \ell + 1}$ is defined as the least natural number $j$ such that $j > i_{k, \ell}$ and $s_j = q_{\ell + 1}$, marking the next visit to $q_{\ell + 1}$. If $\ell = m - 1$, then $i_{k+1,0}$ is defined as the least natural number $j$ such that $j > i_{k, \ell}$ and $s_j = q_0$, marking the next visit to $q_0$.

There is a corresponding division of $w$ into a concatenation of finite segments $w_{1,1}, w_{1,2}, \ldots, w_{1,m-1}, w_{2,0}, \ldots$ between consecutive elements in the increasing sequence of indices. An inductive argument shows that in $B(M, F, q)$, the prefix of $w$ up through $w_{k, \ell}$ arrives at the state $q_\ell$, so that $w$ visits $q_0$ infinitely often and is therefore accepted by $B(M, F, q)$.

Conversely, suppose $B(M, F, q)$ accepts the $\omega$-word $w$. Let $s_0, s_1, s_2, \ldots$ be the run of $B(M, F, q)$ on $w$, and let $t_0, t_1, t_2, \ldots$ be the run of $M$ starting from $q$ on input $w$. An inductive argument shows that if $s_n = q_i$ then $t_n = q_i$, and if $s_n = r_{i,j}$ then $t_n = q_i$. Because the only way the run $s_0, s_1, \ldots$ can visit the final state $q_0$ infinitely often is to progress through the states $q_0, q_1, \ldots, q_m$ in repeating cyclic order, the run $t_0, t_1, \ldots$ must visit only states in $F$ and visit each of them infinitely often, so $w \in L(M, F, q)$.

The DBA $B(M, F, q)$ has a dead state, and $|F|$ states for each element of $F$, for a total of $|F|^2 + 1$ states. The running time of the procedure $SCCtoDBA$ is linear in the size of $M$ and the size of the resulting DBA, which is polynomial in the size of $M$.

We now show that this construction may be used to reduce the inclusion of two DMAs to the inclusion of a DBA and a DMA. Recall that if $A$ is an acceptor and $q$ is a state of $A$, then $A^q$ denotes the acceptor $A$ with the initial state changed to $q$. 

![Figure 3. Example of the construction of $B(M, F, q)$ with $F = \{q_0, q_1, q_2\}$ and $q = q_0$.](image)
Lemma 14.3. Let $U_1$ be a DMA with automaton $M_1$ and a single final state set $F_1$. Let $U_2$ be an arbitrary DMA over the same alphabet as $U_1$, with automaton $M_2$ and family of final state sets $F_2$. Let $M$ denote the product automaton $M_1 \times M_2$ with unreachable states removed. Then $[U_1] \subseteq [U_2]$ iff for every state $(q_1, q_2)$ of $M$ with $q_1 \in F_1$ we have $[B(M_1, F_1, q_1)] \subseteq [U_2]$.

Proof. Suppose that for some state $(q_1, q_2)$ of $M$ with $q_1 \in F_1$, we have $w \in [B(M_1, F_1, q_1)] \setminus [U_2]$. Let $C_1$ be the set of states visited infinitely often in $B(M_1, F_1, q_1)$ on input $w$, and let $C_2$ be the set of states visited infinitely often in $U_2^w$ on input $w$. Then $C_1 = F_1$ and $C_2 \notin F_2$. Let $u$ be a finite word such that $M(u) = (q_1, q_2)$. Then $inf_{M_2}(uw) = C_1 = F_1$ and $inf_{M_2}(uw) = C_2$, so $uw \notin [U_1] \setminus [U_2]$.

Conversely, suppose that $w \in [U_1] \setminus [U_2]$. For $i = 1, 2$ let $C_i = inf_{M_i}(w)$. Note that $C_1 = F_1$ and $C_2 \notin F_2$. Let $w = xw'$, where $x$ is a finite prefix of $w$ that is sufficiently long that the run of $M_1$ on $w$ does not visit any state outside $C_1$ after $x$ has been processed, and for $i = 1, 2$ let $q_i = M_i(x)$. Then $(q_1, q_2)$ is a (reachable) state of $M$, $q_1 \in F_1$, and the $\omega$-word $w'$, when processed by $M_1$ starting at state $q_1$ visits only states of $C_1 = F_1$ and visits each of them infinitely many times, that is, $w' \in [B(M_1, F_1, q_1)]$. Moreover, when $w'$ is processed by $M_2$ starting at state $q_2$, the set of states visited infinitely often is $C_2$, which is not in $F_2$. Thus, $w' \in [B(M_1, F_1, q_1)] \setminus [U_2]$.

To turn this into an algorithm to test inclusion for two DMAs, $U_1$ with automaton $M_1$ and a single final state set $F_1$ that is an SCC of $M_1$ and $U_2$ with automaton $M_2$, we proceed as follows. Construct the product automaton $M = M_1 \times M_2$ with unreachable states removed, and for each state $(q_1, q_2)$ of $M$, if $q_1 \in F_1$, construct the DBA $B(M_1, F_1, q_1)$ and the DMA $U_2^{q_2}$ and test the inclusion of language accepted by the DBA in the language accepted by the DMA. If all of these tests return “yes”, then the algorithm returns “yes” for the inclusion question for $U_1$ and $U_2$. Otherwise, for the first test that returns “no” and a witness $w(v)^\omega$, the algorithm finds by breadth-first search a minimum length finite word $u'$ such that $M(u') = (q_1, q_2)$, and returns the witness $u'w(v)^\omega$.

Combining this with Prop. 14.1, we have the following.

Theorem 14.4. Let $L$ be an algorithm to test inclusion for an arbitrary DBA and an arbitrary DMA over the same alphabet. There is an algorithm to test inclusion for an arbitrary pair of DMAs $U_1$ and $U_2$ over the same alphabet whose running time is linear in the sizes of $U_1$ and $U_2$ plus the time for at most $k \cdot |Q_1| \cdot |Q_2|$ calls to the procedure $L$, where $k$ is the number of final state sets in $U_1$, and $Q_i$ is the state set of $U_i$ for $i = 1, 2$.

14.2. A DBA/DMA inclusion algorithm. In this section, we give a polynomial time algorithm $\text{DBAInDMA}$ to test inclusion for an arbitrary DBA and an arbitrary DMA over the same alphabet.

Assume the inputs are a DBA $B = (M_1, F_1)$ and a DMA $U = (M_2, F)$. The overall strategy of the algorithm is to seek an SCC $C$ of $M = M_1 \times M_2$ such that $\pi_1(C) \cap F_1 \neq \emptyset$ and $\pi_2(C) \notin F$. If such a $C$ is found, the algorithm calls $\text{Witness}(C, M)$, which returns $w(v)^\omega$ such that $\text{inf}_{M_1}(w(v)^\omega) = C$. Because $\text{inf}_{M_1}(w(v)^\omega) = \pi_1(C)$ and $\pi_1(C) \cap F_1 \neq \emptyset$, $w(v)^\omega \in [B]$, and because $\text{inf}_{M_2}(w(v)^\omega) = \pi_2(C)$ and $\pi_2(C) \notin F$, $u(v)^\omega \notin [U]$. The details are given in Algorithm 2.

Theorem 14.5. The $\text{DBAInDMA}$ algorithm runs in polynomial time and solves the inclusion problem for an arbitrary DBA $B$ and an arbitrary DMA $U$ over the same alphabet.
Algorithm 2 DBAinDMA

Input: A DBA $B = (M_1, F_1)$ and a DMA $U = (M_2, F)$, where $Q_i$ is the state set of $M_i$ for $i = 1, 2$.

Output: $u(v)^\omega \in [B] \setminus [U]$ if such exists, else “yes”.

1. $M = M_1 \times M_2$
2. $C \leftarrow \text{maxSCCs}(Q_1 \times Q_2)$
3. for each reachable $C \in C$ such that $\pi_1(C) \cap F_1 \neq \emptyset$
   if $\pi_2(C) \not\in F$
     return Witness$(C, M)$
   else
     for each $F \in F$ such that $F \subseteq \pi_2(C)$ and each $q \in F$
       $S \leftarrow \{(q_1, q_2) \in Q_1 \times Q_2 \mid q_1 \in \pi_1(C) \wedge q_2 \in F \setminus \{q\}\}$
       $D \leftarrow \text{maxSCCs}(S)$
       for each $D \in D$
         if $\pi_1(D) \cap F_1 \neq \emptyset$ and $\pi_2(D) \not\in F$
           return Witness$(D, M)$

return “yes”

Proof. Suppose the returned value is a witness $u(v)^\omega$. Then the algorithm found an SCC $E$ with $\pi_1(E) \cap F_1 \neq \emptyset$ and $\pi_2(E) \not\in F$ and returned Witness$(E, M)$. In this case, the returned value is correct.

Suppose for the sake of contradiction that the algorithm incorrectly returns the answer “yes”, that is, there exists an $\omega$-word $w$ such that $w \in [B]$ and $w \not\in [U]$. Let $C'$ denote $\inf_M(w)$. Then because $w \in [B]$, $\pi_1(C') \cap F_1 \neq \emptyset$, and because $w \not\in [U]$, $\pi_2(C') \not\in F$.

Then $C'$ is a subset of a unique SCC $C \in \text{maxSCCs}(Q_1 \times Q_2)$ and $\pi_1(C) \cap F_1 \neq \emptyset$. It must be that $\pi_2(C) \in F$, because otherwise the algorithm would have returned Witness$(C, M)$. Consider the collection

$$R = \{F \in F \mid \pi_2(C') \subseteq F \subseteq \pi_2(C)\},$$

of all the $F \in F$ contained in $\pi_2(C)$ that contain $\pi_2(C')$. The collection $R$ is nonempty because $C' \subseteq C$, and therefore $\pi_2(C') \subseteq \pi_2(C)$, and $\pi_2(C) \in F$, so at least $\pi_2(C)$ is in $R$. Let $F'$ denote a minimal element of $R$ in the subset ordering.

Then $\pi_2(C') \subseteq F'$ but because $\pi_2(C') \not\in F$, it must be that $\pi_2(C') \neq F'$. Thus, there exists some $q \in F'$ that is not in $\pi_2(C')$. When the algorithm considers this $F'$ and $q$, then because $\pi_2(C') \subseteq F' \setminus \{q\}$, $C'$ is contained in $R$ and therefore is a subset of a unique SCC $D$ in maxSCCs$(R)$.

Because $C' \subseteq D$, and $\pi_1(C') \cap F_1 \neq \emptyset$, we have $\pi_1(D) \cap F_1 \neq \emptyset$. Also, $\pi_2(C') \subseteq \pi_2(D) \subseteq F'$, but because $q \not\in \pi_2(D)$, $\pi_2(D)$ is a proper subset of $F'$. When the algorithm considers this $D$, because $\pi_1(D) \cap F_1 \neq \emptyset$, it must find that $\pi_2(D) \in F$, or else it will return Witness$(D, M)$. But then $\pi_2(D)$ is in $R$ and is a proper subset of $F'$, contradicting our choice of $F'$ as a minimal element of $R$. Thus, if the algorithm outputs “yes”, this is a correct answer.

Combining Thm. 14.4, Thm. 14.5, and the reduction of equivalence to inclusion, we have the following.

Theorem 14.6. There are polynomial time algorithms to solve the inclusion and equivalence problems for two arbitrary DMAs over the same alphabet.
15. Computing the Automaton $M_{\sim L}$

In this section we use polynomial time algorithms to construct the automaton $M_{\sim L}$ of the right congruence relation $\sim_L$ of the language $L$ accepted by an acceptor $A$ of one of the types DBA, DCA, DPA, DRA, DSA, or DMA. This gives a polynomial time algorithm to test whether a given DBA (resp., DCA, DPA, DRA, DSA, DMA) is of type IBA (resp., ICA, IPA, IRA, ISA, IMA).

Recall that $A^q$ is the acceptor $A$ with the initial state changed to $q$. If $q_1$ and $q_2$ are two states of $A$, testing the equivalence of $A^{q_1}$ to $A^{q_2}$ determines whether these two states have the same right congruence class, and, if not, returns a witness $u(v)^\omega$ that is accepted from exactly one of the two states. The following is a consequence of Theorems 12.2, 13.2, and 14.6.

**Proposition 15.1.** There is a polynomial time procedure to test whether two states of an arbitrary DBA, DCA, DPA, DRA, DSA or DMA $A$ have the same right congruence class, returning the answer “yes” if they do, and returning “no” and a witness $u(v)^\omega$ accepted from exactly one of the states if they do not.

We now describe an algorithm $\text{RightCon}$ that takes as input a DBA (or DCA, DPA, DRA, DSA, or DMA) $A$ accepting a language $L$ and returns a deterministic automaton $M$ isomorphic to the right congruence automaton of $L$, i.e., $M_{\sim L}$.

**Algorithm 3 RightCon**

**Input:** An acceptor $A = \langle \Sigma, Q, q_\iota, \delta, \alpha \rangle$ of type DBA, DCA, DPA, DRA, DSA, or DMA

**Output:** A deterministic automaton $M$ isomorphic to $M_{\sim L}$, where $L = [A]$

1. $Q' \leftarrow \{ \varepsilon \}$
2. $q'_\iota \leftarrow \varepsilon$
3. $\delta'$ is initially undefined
4. while there exists $x \in Q'$ and $\sigma \in \Sigma$ such that $\delta'(x, \sigma)$ is undefined do
   1. $q_1 \leftarrow \delta(q_\iota, x\sigma)$
   2. if there exists $y \in Q'$ such that $[A^{q_1}] = [A^{q_2}]$ for $q_2 = \delta(q_\iota, y)$ then
      1. Define $\delta'(x, \sigma) = y$
   else
      1. $Q' \leftarrow Q' \cup \{ x\sigma \}$
      2. Define $\delta'(x, \sigma) = x\sigma$
5. return $M = \langle \Sigma, Q', q'_\iota, \delta' \rangle$

Assume the input acceptor is $A = \langle \Sigma, Q, q_\iota, \delta, \alpha \rangle$. The $\text{RightCon}$ algorithm constructs a deterministic automaton $M = \langle \Sigma, Q', q'_\iota, \delta' \rangle$ in which the states are elements of $\Sigma^*$ and $q'_\iota = \varepsilon$. The set $Q'$ initially contains just $\varepsilon$, and $\delta'$ is completely undefined.

While there exists a word $x \in Q'$ and a symbol $\sigma \in \Sigma$ such that $\delta'(x, \sigma)$ has not yet been defined, loop through the words $y \in Q'$ and ask whether the states $\delta(q_\iota, x\sigma)$ and $\delta(q_\iota, y)$ have the same right congruence class in $A$. If so, then define $\delta'(x, \sigma)$ to be $y$. If no such $y$ is found, then the word $x\sigma$ is added as a new state to $Q'$, and the transition $\delta'(x, \sigma)$ is defined to be $x\sigma$.

This process must terminate because the elements of $Q'$ represent distinct right congruence classes of $L$, and $M_{\sim L}$ cannot have more than $|Q|$ states. When it terminates, the automaton $M = \langle \Sigma, Q', q'_\iota, \delta' \rangle$ is isomorphic to the right congruence automaton of $A$, $M_{\sim L}$.
Theorem 15.2. The RightCon algorithm with input an acceptor \( A \) (a DBA, DCA, DPA, DRA, DSA, or DMA) accepting \( L \), runs in polynomial time and returns \( M \), a deterministic automaton isomorphic to \( M_{\sim L} \).

To test whether a given DBA (resp., DCA, DPA, DRA, DSA, DMA) \( A \) is an IBA (resp., ICA, IPA, IRA, ISA, IMA), we run the RightCon algorithm on \( A \) and test the returned automaton \( M \) for isomorphism with the automaton of \( A \). If they isomorphic, then \( A \) is an IBA (resp. ICA, IPA, IRA, ISA, IMA), otherwise it is not. This proves the following.

Theorem 15.3. There is a polynomial time algorithm to test whether a given DBA (resp., DCA, DPA, DRA, DSA, DMA) is an IBA (resp., ICA, IPA, IRA, ISA, IMA).

16. Testing membership in \( \mathbb{IBA} \)

In the previous section we showed that there is a polynomial time algorithm to test whether a given DBA \( B \) is an IBA. However, we can also ask the following harder question. Given a DBA \( B \) that is not an IBA, is \( \llbracket B \rrbracket \in \mathbb{IBA} \), that is, does there exist an IBA \( B' \) such that \( \llbracket B' \rrbracket = \llbracket B \rrbracket \)? This section shows that there are such polynomial time algorithms for DBAs, DCAs, DPAs, DRAs, DSA, and DMAs. The algorithms first compute the right congruence automaton \( M = M_{\sim L} \), where \( L = \llbracket A \rrbracket \), and then attempt to construct an acceptance condition \( \alpha \) of the appropriate type such that \( \llbracket (M, \alpha) \rrbracket = L \).

16.1. Testing membership in \( \mathbb{IBA} \). We describe the algorithm TestInIB that takes as input a DBA \( B \) and returns an IBA accepting \( \llbracket B \rrbracket \) if \( \llbracket B \rrbracket \in \mathbb{IBA} \), and otherwise returns “no”. By Claim 4, the case of a DCA is reduced to that of a DBA.

Algorithm 4 TestInIB

Input: A DBA \( B \)
Output: If \( \llbracket B \rrbracket \in \mathbb{IBA} \) then return an IBA accepting \( \llbracket B \rrbracket \), else return “no”

\[ M \leftarrow \text{RightCon}(B) \]
\[ F \leftarrow \emptyset \]
for each state \( q \) of \( M \) do
  if \( \llbracket (M, \{q\}) \rrbracket \subseteq \llbracket B \rrbracket \) then
    \[ F \leftarrow F \cup \{q\} \]
  if \( \llbracket (M, F) \rrbracket = \llbracket B \rrbracket \) then
    return \((M, F)\)
  else
    return “no”

Theorem 16.1. The algorithm TestInIB takes a DBA \( B \) as input, runs in polynomial time, and returns an IBA accepting \( \llbracket B \rrbracket \) if \( \llbracket B \rrbracket \in \mathbb{IBA} \), and otherwise returns “no”.

Proof. The algorithm calls the RightCon algorithm, and also the inclusion and equivalence algorithms from Thm. 12.3, which run in polynomial time in the size of \( B \). If the algorithm returns an acceptor, it is an IBA accepting \( \llbracket B \rrbracket \).

To see that the algorithm does not incorrectly return the answer “no”, suppose \( B' \) is an IBA accepting \( \llbracket B \rrbracket \). Then because \( M \) is isomorphic to \( M_{\sim L} \), we may assume that \( B' = (M, F') \). For every state \( q \in F' \), the inclusion query with \((M, \{q\})\) will answer “yes”,
so $q$ will be added to $F$. Thus, $F' \subseteq F$, and $[(\mathcal{M}, F)]$ subsumes $[(\mathcal{M}, F')]$. Every state $q$ added to $F$ preserves the condition that $[(\mathcal{M}, F)]$ is a subset of $[\mathcal{B}]$, so the final equivalence check will pass, and $(\mathcal{M}, F)$ will be returned.

\section{Testing membership in \ipa\textsuperscript{A}}

We describe the algorithm \texttt{TestInIP} that takes as input a DPA $\mathcal{P}$ and returns an IPA accepting $[\mathcal{P}]$ if $[\mathcal{P}] \in \ipa\textsuperscript{A}$, and otherwise returns “no”.

\begin{algorithm}[H]
\caption{\texttt{TestInIP}}
\begin{algorithmic}
\State \textbf{Input:} A DPA $\mathcal{P}$
\State \textbf{Output:} If $[\mathcal{P}] \in \ipa\textsuperscript{A}$ then return an IPA accepting $[\mathcal{P}]$, else return “no”
\State $\mathcal{M} = (\Sigma, Q, q_0, \delta) \leftarrow \texttt{RightCon}(\mathcal{P})$
\State Define $\kappa(q) = 0$ for all states $q \in Q$
\For{$k = 1$ to $|Q|$}
\If{$[(\mathcal{M}, \kappa)] = [\mathcal{P}]$}
\State \Return $(\mathcal{M}, \kappa)$
\ElseIf{$k$ is odd}
\While{$[\mathcal{P}]$ is not a subset of $[(\mathcal{M}, \kappa)]$}
\State Let $u(v)\omega$ be the returned witness
\State Define $\kappa(q) = k$ for all $q \in \inf_{\mathcal{M}}(u(v)\omega)$
\EndWhile
\Else
\While{$[\mathcal{P}]$ is not a superset of $[(\mathcal{M}, \kappa)]$}
\State Let $u(v)\omega$ be the returned witness
\State Define $\kappa(q) = k$ for all $q \in \inf_{\mathcal{M}}(u(v)\omega)$
\EndWhile
\EndIf
\EndIf
\EndFor
\State \Return “no”
\end{algorithmic}
\end{algorithm}

\textbf{Theorem 16.2.} The algorithm \texttt{TestInIP} takes a DPA $\mathcal{P}$ as input, runs in polynomial time, and returns an IPA accepting $[\mathcal{P}]$ if $[\mathcal{P}] \in \ipa\textsuperscript{A}$, and otherwise returns “no”.

\textbf{Proof.} The algorithm calls the \texttt{RightCon} algorithm and the inclusion and equivalence algorithms for DPAs from Thm. 12.2, which run in polynomial time in the size of $\mathcal{P}$. Below we show that each while loop terminates after at most $|Q|$ iterations. If the algorithm returns an acceptor, then the acceptor is an IPA accepting $[\mathcal{P}]$.

To see that the algorithm does not incorrectly return the answer “no”, suppose $\mathcal{P}'$ is an IPA accepting $[\mathcal{P}]$. We may assume that $\mathcal{P}' = (\mathcal{M}, \kappa^*)$, where $\kappa^*$ is the canonical coloring of $\mathcal{P}'$. We prove inductively that the final coloring $\kappa$ is equal to $\kappa^*$. To do so, we consider the conditions after the for loop has been completed $\ell$ times: (1) if $\ell$ is even then $[(\mathcal{M}, \kappa)] \subseteq [\mathcal{P}]$, and if $\ell$ is odd, then $[\mathcal{P}] \subseteq [(\mathcal{M}, \kappa)]$, and (2) for all $q \in Q$, if $\kappa^*(q) \leq \ell$ then $\kappa(q) = \kappa^*(q)$, and if $\kappa^*(q) > \ell$ then $\kappa(q) = \ell$.

The initialization of $\kappa(q) = 0$ for all $q \in Q$ implies that these two conditions hold for $\ell = 0$. Suppose the conditions hold for some $\ell \geq 0$. If the equivalence check at the start of the next iteration returns “yes” then the correct IPA $(\mathcal{M}, \kappa)$ is returned. Otherwise, $k = \ell + 1$; we consider the cases of odd and even $k$.

If $k$ is odd, then by condition (1), $[(\mathcal{M}, \kappa)] \subseteq [\mathcal{P}]$ and at least one witness $u(v)\omega$ accepted by $\mathcal{P}$ and rejected by $[(\mathcal{M}, \kappa)]$ will be processed in the while loop. Consider such a witness $u(v)\omega$ and let $C = \inf_{\mathcal{M}}(u(v)\omega)$. Then because $\kappa(q) = \kappa^*(q)$ if $\kappa(q) \leq \ell$, it must be that $\kappa^*(C) > \ell$ and $\kappa(C) = \ell$. For all $q \in C$, $\kappa(q)$ is set to $k = \ell + 1$, so at least one state changes $\kappa$-color from $\ell$ to $\ell+1$. This can happen at most $|Q|$ times, so the while loop for
this $k$ must terminate after at most $|Q|$ iterations. No state $q$ with $\kappa^*(q) \leq \ell$ has its $\kappa$-value changed, so when the while loop is terminated, we have that $\kappa^*(q) \leq \ell$ implies $\kappa(q) = \kappa^*(q)$.

Consider any state $q$ with $\kappa^*(q) \geq \ell + 1$. By property (2), at the start of this iteration of the for loop, $\kappa(q) = \ell$. Referring to the canonical forest $F^*$ for $P'$, the state $q$ is in $\Delta(D)$ for some node $D$ of $F^*$. The node $D$ is a descendant (or possibly equal to) some node $C$ for which the states $q \in \Delta(C)$ all have $\kappa^*(q) = \ell + 1$. Thus, as long as the value of $\kappa(q)$ remains $\ell$, the SCC $C$ will have $\kappa(C)$ even and $\kappa^*(C)$ odd, and the while loop cannot terminate. But we have shown that it does terminate, so after termination we must have $\kappa(q) = k = \ell + 1$. Thus, after this iteration of the for loop, property (2) holds for $\ell + 1$.

The case of even $k$ is dual to the case of odd $k$. Because the range of $\kappa^*$ is $[0..\ell]$ for some $\ell \leq |Q|$, the equivalence test must return “yes” before the for loop completes, at which point the IPA $(M, \kappa)$ is returned.

\[ \square \]

16.3 Testing membership in $\mathbb{IRA}$. We describe the algorithm TestInIR that takes as input a DRA $R$ and returns an IRA accepting $[R]$ if $[R] \in \mathbb{IRA}$, and otherwise returns “no”. By Claim 5, the case of a DSA is reduced to that of a DRA.

We first show that given a DRA $R$ such that $[R] \in \mathbb{IRA}$, there is an IRA equivalent to $R$ whose size is bounded by a polynomial in the size of $R$.

**Lemma 16.3.** Let $R$ be a DRA in singleton normal form whose acceptance condition has $m$ pairs, and assume $[R] \in \mathbb{IRA}$. Let $M$ be the right congruence automaton of $[R]$ with state set $Q$ and assume $|Q| = n$. Then there exists an acceptance condition $\alpha$ in singleton normal form with at most $m n$ pairs such that $(M, \alpha)$ accepts $[R]$.

**Proof.** Let $R = (M_1, \alpha_1)$, where all the states of $M_1$ are reachable, and let the function $f$ map each state of $M_1$ to the state of its right congruence class in $M$. It suffices to show that for each $(q, B) \in \alpha_1$ there exists an acceptance condition $\alpha'$ of $M$ containing at most $n$ pairs such that $[(M_1, \{(q, B)\})] \subseteq [(M, \alpha')] \subseteq [R]$. Taking the union of these $\alpha'$ conditions for all $m$ pairs $(q, B) \in \alpha_1$ yields the desired acceptance condition $\alpha$ for $M$.

Because we assume $[R] \in \mathbb{IRA}$, there exists an IRA $(M, \alpha_2)$ in singleton normal form that accepts $[R]$. Given any $u(v)^\omega$ in $[R]$, let $C = \inf_{M_1}(u(v)^\omega)$. Then $f(C) = \inf_{M}(u(v)^\omega)$ and there exists $(q', B') \in \alpha_2$ such that $q' \in f(C)$, and $f(C) \cap B' = \emptyset$. Then also $[(M, \{(q', Q \setminus f(C))\})] \subseteq [R]$. To see this, consider any $u'(v')^\omega$ with $D = \inf_{M}(u'(v')^\omega)$ and $q' \in D$ and $D \cap (Q \setminus f(C)) = \emptyset$. Then $D$ is a subset of $f(C)$, $D \cap B' = \emptyset$, $u'(v')^\omega$ satisfies $(q', B')$, and $u'(v')^\omega \in [R]$.

Given a pair $(q, B) \in \alpha_1$ the construction of the initially empty acceptance condition $\alpha'$ proceeds as follows. Let $C_0$ be the maximum SCC of $M_1$ that contains $q$ and excludes $B$. If $C_0$ is empty, then $(M_1, \{(q, B)\})$ does not accept any words, and the empty condition $\alpha'$ suffices. If $C_0$ is nonempty, then there is an element $u(v)^\omega$ of $[(M_1, \{(q, B)\})]$ such that $C_0 = \inf_{M_1}(u(v)^\omega)$ and there is a pair $(q_0, B_0)$ in $\alpha_2$ such that $q_0 \in f(C_0)$ and $B_0 \cap f(C_0) = \emptyset$. We add the pair $(q_0, Q \setminus f(C_0))$ to $\alpha'$ and note that by the argument in the preceding paragraph, $[(M, \alpha')] \subseteq [R]$.

If $[(M_1, \{(q, B)\})] \subseteq [(M, \alpha')]$ then $\alpha'$ is the desired acceptance condition. If not, there exists a word $u(v)^\omega$ such that for $C = \inf_{M_1}(u(v)^\omega)$ we have $q \in C$ and $C \cap B = \emptyset$, but either $q_0 \notin f(C)$ or $f(C) \cap (Q \setminus f(C_0)) \neq \emptyset$. Because $C_0$ is the maximum SCC of $M_1$ containing $q$ and excluding $B$, we have $C \subseteq C_0$, so $f(C) \subseteq f(C_0)$ and therefore $q_0 \notin f(C)$. Let $C_1$ be the maximum SCC $C$ of $M_1$ such that $C \subseteq C_0$, $q \in C$, and $q_0 \notin f(C)$. This is not empty,
so there is a word \( u'(v')^\omega \) such that \( C_1 = \inf M_1(u'(v')^\omega) \), which is in \( [\mathcal{R}] \) because it satisfies \((q, B)\). Thus there exists a pair \((q_1, B_1)\) in \( \alpha_2 \) that is satisfied by \( u'(v')^\omega \), and we add the pair \((q_1, Q \setminus f(C_1))\) to the acceptance condition \( \alpha' \). As above, we have \( [\mathcal{M}, \alpha'] \subseteq [\mathcal{R}] \).

If now \( [(\mathcal{M}_1, \{(q, B)\})] \subseteq [(\mathcal{M}, \alpha')] \), then \( \alpha' \) is the desired acceptance condition. If not, we repeat this step again. In general, after \( k \) steps of this kind, \( \alpha' \) consists of \( k \) pairs of the form \((q_i, Q \setminus f(C_i))\) for \( i \in [0..k - 1] \), where all of the states \( q_i \) are distinct and \( C_{i+1} \subseteq C_i \) for \( i \in [0, k - 2] \). Because \( Q \) has \( n \) states, there can only be \( n \) repetitions of this step before \( \alpha' \) satisfies the required condition, and thus \( \alpha' \) has at most \( n \) pairs.

The algorithm **TestInIR** is based on the algorithm to learn Horn sentences by Angluin, Frazier, and Pitt [AFP92], using the analogy between singleton normal form for Rabin automata and propositional Horn clauses. **TestInIR** maintains for each state \( q \) of the right congruence automaton \( \mathcal{M} \) an ordered sequence \( S_q \) of SCCs of \( \mathcal{M} \), each of which corresponds to a positive example of \([\mathcal{R}]\). At each iteration, the algorithm uses these sequences and inclusion queries with \([\mathcal{R}]\) to construct an acceptance condition \( \alpha \) for a hypothesis \((\mathcal{M}, \alpha)\), which it tests for equivalence to \( \mathcal{R} \). In the case of non-equivalence, the witness is a positive example of \([\mathcal{R}]\) that is used to update the sequences \( S_q \).

In **TestInIR** the test of whether \( C \cup C_i \) is positive is implemented by calling **Witness**\((C \cup C_i)\) and testing the resulting word \( u(v)^\omega \) for membership in \([\mathcal{R}]\).

**Algorithm 6 TestInIR**

**Input:** A DRA \( \mathcal{R} = (\mathcal{M}_1, \alpha_1) \) in singleton normal form with \( |\alpha_1| = m \)

**Output:** If \([\mathcal{R}] \in \text{IRA} \) then return an IRA accepting \([\mathcal{R}]\), else return “no”

\[
\mathcal{M} \leftarrow \text{RightCon}(\mathcal{R})
\]

Let \( Q \) be the states of \( \mathcal{M} \) and \( n = |Q| \)

For each \( q \in Q \) initialize a sequence \( S_q \) to be empty

for \( k = 1, mn^2 \) do

  for all \( q \in Q \) do

    \( \alpha_q = \emptyset \)

    for all \( C \in S_q \) do

      for all \( q' \in C \) do

        if \( [(\mathcal{M}, \{(q', Q \setminus C)\}] \subseteq [\mathcal{R}] \) then

          \( \alpha_q = \alpha_q \cup \{(q', Q \setminus C)\} \)

    \( \alpha \leftarrow \bigcup_{q \in Q} \alpha_q \)

    if \( [(\mathcal{M}, \alpha)] = [\mathcal{R}] \) then

      return \((\mathcal{M}, \alpha)\)

  else

    Let \( u(v)^\omega \) be the witness returned

    Let \( C = \inf M(u(v)^\omega) \)

    for all \( q \in C \) do

      if there is some \( C_i \in S_q \) such that \( C \not\subseteq C_i \) and \( C \cup C_i \) is positive then

        Let \( i \) be the least such \( i \) and replace \( C_i \) by \( C_i \cup C \)

      else

        Add \( C \) to the end of the sequence \( S_q \)

  return “no”

Because pairs are only added to \( \alpha \) that preserve inclusion in \([\mathcal{R}]\), it is clear that any witness \( u(v)^\omega \) returned in response to the test of equivalence of \((\mathcal{M}, \alpha)\) and \( \mathcal{R} \) is a positive
example of $[\mathcal{R}]$. Note also that all elements of $S_q$ are SCCs of $\mathcal{M}$ that contain $q$. The proof of correctness and running time of TestInIR depends on the following two lemmas.

**Lemma 16.4.** Assume that $\mathcal{R}'$ is an IRA equivalent to the target DRA $\mathcal{R}$. Consider a positive example $u(v)^\omega$ of $[\mathcal{R}]$ returned in response to the test of equivalence of $(\mathcal{M}, \alpha)$ and $\mathcal{R}$, and let $C = \inf_{\mathcal{M}}(u(v)^\omega)$. Let $(q', B)$ be a pair of $\mathcal{R}'$ such that $q' \in C$ and $C \cap B = \emptyset$. If for some $q \in C$ and some $C_i$ in $S_q$ we have $C_i \cap B = \emptyset$ then for some $j \leq i$, the element $C_j$ of $S_q$ will be replaced by $C_j \cup C$.

**Proof.** Assume that there is no such replacement for $j < i$. When $i$ is considered, we have $q \in C_i$ and $q \in C$, so $C_i \cup C$ is the union of overlapping SCCs and therefore an SCC. Then $C_i \cup C$ is positive because $q' \in C$, so $q' \in C_i \cup C$, and $C \cap B = \emptyset$ and $C_i \cap B = \emptyset$ by hypothesis, so $(C_i \cup C) \cap B = \emptyset$ and the word Witness$(C_i \cup C)$ is a positive example of $[\mathcal{R}]$ because it satisfies $(q', B)$.

To see that $C \nsubseteq C_i$, we assume to the contrary. Then $(q', Q \setminus C_i)$ is an element of $\alpha$. To see this, we show that $[([\mathcal{M}], \{(q', Q \setminus C_i)\})] \subseteq [\mathcal{R}]$. Let $u'(v'^\omega)$ with $D = \inf_{\mathcal{M}}(u'(v'^\omega))$ satisfy $(q', Q \setminus C_i)$. Then $q' \in D$ and $D \cap (Q \setminus C_i) = \emptyset$, and therefore $D \subseteq C_i$ and $C_i \cap B = \emptyset$ by hypothesis. Thus $u'(v'^\omega)$ satisfies $(q', B)$ and is in $[\mathcal{R}]$. Because $(q', Q \setminus C_i)$ is an element of $\alpha$, $u(v)^\omega$ is accepted by $(\mathcal{M}, \alpha)$ because $q' \in C$ and $C \cap (Q \setminus C_i) = \emptyset$ (because we assume $C \nsubseteq C_i$). But this means that $u(v)^\omega$ cannot be a witness to the non-equivalence of $(\mathcal{M}, \alpha)$ and $\mathcal{R}$, a contradiction.

Thus, the conditions for $C_i$ to be replaced by $C_i \cup C$ are satisfied.

**Lemma 16.5.** Assume that $\mathcal{R}' = (\mathcal{M}, \alpha')$ is an IRA in singleton normal form equivalent to the target DRA $\mathcal{R}$. The following two conditions hold throughout the algorithm TestInIR.

1. For all $q \in Q$, elements $C_i$ of $S_q$, and $(q', B) \in \alpha'$, if $C_i$ satisfies $(q', B)$ then for no $j < i$ do we have $C_j \cap B = \emptyset$.
2. For all $q \in Q$, elements $C_i$ and $C_j$ of $S_q$ with $j < i$, and $(q', B) \in \alpha'$, if $C_i$ satisfies $(q', B)$ then $C_j$ does not satisfy $(q', B)$.

**Proof.** We first show that condition (1) implies condition (2). Let $q \in Q$, $C_i$ and $C_j$ be in $S_q$ with $j < i$ and $(q', B) \in \alpha'$. If $C_i$ satisfies $(q', B)$ then by condition (1), we have $C_j \cap B = \emptyset$, so $C_j$ does not satisfy $(q', B)$.

We now prove condition (1) by induction on the number of witnesses to non-equivalence. The condition holds of the empty sequences $S_q$. Suppose conditions (1) and (2) hold of the sequences $S_q$, and the witness to non-equivalence is $u(v)^\omega$ with $C = \inf_{\mathcal{M}}(u(v)^\omega)$. If $q \notin C$ then $S_q$ is not modified, so assume $q \in C$. We consider two cases, depending on whether $C$ is added to the end of $S_q$ or causes some $C_\ell$ to be replaced by $C_\ell \cup C$.

Assume that $C$ is added to the end of $S_q$ and property (1) fails to hold. Then it must be that for $C_i = C$, some $C_j$ in $S_i$ with $j < i$ and some $(q', B) \in \alpha'$, $C_i$ satisfies $(q', B)$ and $C_j \cap B = \emptyset$. By Lemma 16.4, because $C_j \cap B = \emptyset$, $C$ should cause $C_\ell$ for some $\ell \leq j$ to be replaced by $C \cup C_\ell$ rather than being added to the end of $S_q$, a contradiction.

Assume that $C$ causes $C_\ell$ in $S_q$ to be replaced by $C_\ell \cup C$ and property (1) fails to hold. Then it must be for some $C_i$ in $S_q$ and pair $(q', B)$ in $\alpha'$, either (i) $i > \ell$ and $C_i$ satisfies $(q', B)$ and $(C_\ell \cup C) \cap B = \emptyset$, or (ii) $i < \ell$ and $C_\ell \cup C$ satisfies $(q', B)$ and $C_i \cap B = \emptyset$. In case (i), it must be that $C_i$ satisfies $(q', B)$ and $C_\ell \cap B = \emptyset$, which contradicts the assumption that property (1) holds before $C$ is processed, because $\ell < i$. In case (ii), $q' \in C_\ell \cup C$ and $(C_\ell \cup C) \cap B = \emptyset$. Thus $C \cap B = \emptyset$ and $C_\ell \cap B = \emptyset$. If $q' \in C_\ell$, then $C_\ell$ satisfies $(q', B)$, violating the assumption that property (2) holds before $C$ is processed. If $q' \in C$ then $C$
satisfies \((q', B)\) and because \(C_i \cap B = \emptyset\), by Lemma 16.4, for some \(j \leq i\) we have \(C_j\) replaced by \(C_j \cup C\), a contradiction because \(\ell > i\). Thus, in either case property (1) holds after \(C_\ell\) is replaced by \(C_\ell \cup C\).

**Theorem 16.6.** The algorithm \texttt{TestInIR} takes as input a DRA \(R\), runs in polynomial time, and returns an IRA accepting \([R]\) if \([R]\) \(\in\) IRA, and otherwise returns “no”.

**Proof.** Assume the input is DRA \(R = (M_1, \alpha_1)\) in singleton normal form with \(|\alpha_1| = m\). The algorithm \texttt{TestInIR} computes the right congruence automaton \(M\) of \([R]\), which has \(n\) states, at most the number of states of \(R\). The main loop of the algorithm is executed at most \(mn^3\) times, and each execution makes calls to the inclusion and equivalence algorithms for DRAs, and runs in time polynomial in the size of \(R\), so the overall running time of \texttt{TestInIR} is polynomial in the size of \(R\).

Clearly, if \([R] \not\in\) IRA, then the test of equivalence between \((M, \alpha)\) and \(R\) will not succeed, and the value returned will be “no”. Assume that \([R] \in\) IRA. Then by Lemma 16.3, there is an IRA \(R' = (M, \alpha')\) in singleton normal form equivalent to \(R\) such that \(|\alpha'| \leq mn\). For every state \(q \in Q\), each member of the sequence \(S_q\) satisfies some pair \((q', B)\) in \(\alpha'\), and by Lemma 16.5, no two members of \(S_q\) can satisfy the same pair, so the length of each \(S_q\) is bounded by \(mn\). Each positive counterexample must either add another member to at least one sequence \(S_q\) or cause at least one member of some sequence \(S_q\) to increase in cardinality by 1. The maximum cardinality of any member of any \(S_q\) is \(n\), and the total number of sequences \(S_q\) is \(n\), so no more than \(mn^3\) positive counterexamples can be processed before the test of equivalence between \((M, \alpha)\) and \(R\) succeeds and \((M, \alpha)\) is returned. 

16.4. Testing membership in \(\text{IMA}\). We describe the algorithm \texttt{TestInIM} that takes as input a DMA \(U\) and returns an IMA accepting \([U]\) if \([U] \in\) IMA, and otherwise returns “no”.

**Algorithm 7 TestInIM**

**Input:** A DMA \(U = (\Sigma, Q, q_\text{i}, \delta, F)\)

**Output:** If \([U] \in\) IMA then return an IMA accepting \([U]\), else return “no”

\[
\begin{align*}
M &\leftarrow \text{RightCon}(U) \\
F' &\leftarrow \emptyset \\
\text{while } ([M, F']) \neq [U] \text{ do} \\
&\quad \text{Let } u(v)^\omega \text{ be the witness returned} \\
&\quad \text{Let } C = \inf_M(u(v)^\omega) \\
&\quad \text{if } u(v)^\omega \in [U] \text{ then} \\
&\quad \quad F' \leftarrow F' \cup \{C\} \\
&\quad \text{else} \\
&\quad \quad \text{return “no”} \\
\text{return } (M, F')
\end{align*}
\]

**Theorem 16.7.** The algorithm \texttt{TestInIM} takes as input a DMA \(U\), runs in polynomial time, and returns an IMA accepting \([U]\) if \([U] \in\) IMA, and otherwise returns “no”.

**Proof.** The algorithm calls the \texttt{RightCon} algorithm and also the DMA equivalence algorithm from Thm. 14.6, which run in polynomial time. When there is a witness \(u(v)^\omega\) accepted by \(U\), there is a set \(F \in \mathcal{F}\) whose image in \(M\) is added to \(F'\), so there can be no
more such witnesses than the number of sets in $F$. After this, there must be a successful equivalence test or a witness rejected by $U$, either of which terminates the while loop. Thus, the overall running time is polynomial in the size of $U$.

If the algorithm returns an acceptor $(M, F')$, then the acceptor is an IBA that accepts $[U]$. To see that the algorithm does not incorrectly return the answer “no”, assume that $U' = (M, F'')$, where $F''$ contains no redundant sets. Then the first witness will be a word accepted by $U$ that will add an element of $F''$ to $F'$. This continues until all the elements of $F''$ have been added to $F'$, at which point the while loop terminates with equivalence.

16.5. Variants of the testing algorithms. A variant of the task considered above is the following. Given the right congruence automaton $M$ of a language $[B]$ in $IBA$, and access to information from certain queries about $[B]$, learn an acceptance condition $\alpha$ such that $(M, \alpha)$ accepts $[B]$. In the case of $IBA$, the algorithm TestInIB could be modified to perform this task using just equivalence queries with respect to $[B]$. Similarly, equivalence queries would suffice in the case of $IMA$. For $IPA$, subset and superset queries with respect to the target language would suffice. And for $IRA$, TestInIR could be modified to use membership and equivalence queries with respect to the target language, relying on negative examples to remove incorrect pairs rather than using subset queries.

16.6. Efficient teachability of the informative classes. We can finally claim that the informative classes are efficiently teachable/learnable.

**Theorem 16.8.** The classes $IBA, ICA, IPA, IMA, IRA, ISA$ are efficiently teachable/learnable.

**Proof.** By Theorems 7.2, 7.4, 8.6, and 9.5 the classes $IMA$, $IBA$, $ICA$, $IPA$, $IRA$ and $ISA$ are identifiable in the limit using polynomial time and data. It remains to show that the characteristic samples can be computed in polynomial time for any acceptor in the class.

Let $A$ be an acceptor of type DXA for $X \in \{B, C, P, R, S, M\}$. By Theorems 16.1, 16.2, 16.6, 16.7, there are polynomial time algorithms that return an equivalent acceptor $A'$ in $IXA$ if such exists and “no” otherwise. By Thm.10.6 given an acceptor $A'$ of type IBA, ICA, IPA, IRA, ISA, or $IMA$, the characteristic sample $T_L$ for $A$ may be computed in polynomial time in the size of $A$. It follows that the classes $IMA$, $IBA$, $ICA$, $IPA$, $IRA$ and $ISA$ are efficiently teachable/learnable. □

17. Discussion

We asked which classes of representations of $\omega$-regular languages are efficiently teachable/learnable. The non-deterministic acceptors $NBA$, $NCA$, $NPA$, $NRA$, $NSA$, and $NMA$ do not have polynomial size characteristic sets, thus are neither efficiently identifiable in the limit with polynomial time and data nor efficiently teachable. We showed that the classes of representations $FDFA$ and $M^2MA$ are efficiently teachable/learnable, and that the classes $SUBA$, $DBA$, $DCA$, and $DPA$ are efficiently teachable, though they are not known to be efficiently teachable/learnable. We note that some recent results were obtained for passive learning $DBA$ and $DPA$ [BL22, BL23] but from characteristic samples that may be exponential in the size of the minimal representations. Thus, these results do not settle the question of
whether these classes are efficiently teachable/learnable or identifiable in the limit using polynomial time and data.

Focusing on their sub-classes of informative languages, IBA, ICA, IPA, IRA, ISA, and IMA we have shown that they are efficiently teachable/learnable. To obtain these results we have given new polynomial time algorithms to test inclusion and equivalence for DBAs, DCAs, DPAs, DRAs, DSAs, and DMAs. We have given a polynomial time algorithm to compute the right congruence automaton $\mathcal{M}_{\sim L}$ for a language $L$ specified by a DBA, DCA, DPA, DRA, DSA, or DMA. This yields a polynomial time algorithm to test whether an acceptor $\mathcal{A}$ of type DBA is of type IBA, and similarly for acceptors of types DCA, DPA, DRA, DSA, and DMA. Moreover, we have given a polynomial time algorithm to test whether an acceptor $\mathcal{A}$ of type DBA accepts a language in the class IBA, and similarly for acceptors of types DCA, DPA, DRA, DSA and DMA.

The questions of whether the full deterministic classes DBA, DCA, DPA, DRA, DSA, and DMA are efficiently teachable/learnable or identifiable in the limit using polynomial time and data remains open. Another intriguing open question is whether the classes IBA, ICA, IPA, IRA, ISA, and IMA can be learned by polynomial time algorithms using membership and equivalence queries. However, this question is not easier than whether the corresponding deterministic classes can be learned by polynomial time algorithms using membership and equivalence queries, by the result of Bohn and L"oding [BL21].

**References**

[AAF20] D. Angluin, T. Antonopoulos, and D. Fisman. Strongly unambiguous Büchi automata are polynomially predictable with membership queries. In 28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain, pages 8:1–8:17, 2020.

[AAFG22] Dana Angluin, Timos Antonopoulos, Dana Fisman, and Nevin George. Representing regular languages of infinite words using mod 2 multiplicity automata. In Foundations of Software Science and Computation Structures - 25th International Conference, FOSSACS, pages 1–20, 2022.

[ABF16] Dana Angluin, Udi Boker, and Dana Fisman. Families of DFAs as acceptors of omega-regular languages. In 41st International Symposium on Mathematical Foundations of Computer Science, MFCS 2016, August 22-26, 2016 - Kraków, Poland, pages 11:1–11:14, 2016.

[ABF18] Dana Angluin, Udi Boker, and Dana Fisman. Families of dfas as acceptors of $\omega$-regular languages. *Logical Methods in Computer Science*, 14(1), 2018.

[ABL02] Glenn Ammons, Rastislav Bodik, and James R. Larus. Mining specifications. In *Conference Record of POPL 2002: The 29th SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, Portland, OR, USA, January 16-18, 2002, pages 4–16, 2002.

[AF14] Dana Angluin and Dana Fisman. Learning regular omega languages. In *Algorithmic Learning Theory - 25th International Conference, ALT 2014, Bled, Slovenia, October 8-10, 2014. Proceedings*, pages 125–139, 2014.

[AF16] Dana Angluin and Dana Fisman. Learning regular omega languages. *Theor. Comput. Sci.*, 650:57–72, 2016.

[AF18] D. Angluin and D. Fisman. Regular omega-languages with an informative right congruence. In *GandALF*, volume 277 of *EPTCS*, pages 265–279, 2018.

[AFF92] D. Angluin, M. Frazier, and L. Pitt. Learning conjunctions of Horn clauses. *Machine Learning*, 9:147–164, 1992.

[AFS20] Dana Angluin, Dana Fisman, and Yaara Shoval. Polynomial identification of omega-automata. In *Proc. 26nd Int. Conf. on Tools and Algorithms for the Construction and Analysis of Systems*, pages 325–343, 2020.

[Ang87] Dana Angluin. Learning regular sets from queries and counterexamples. *Inf. Comput.*, 75(2):87–106, 1987.
[AV10] Fides Aarts and Frits Vaandrager. Learning I/O automata. In Paul Gastin and François Laroussinie, editors, *CONCUR 2010 - Concurrency Theory: 21th International Conference, CONCUR 2010, Paris, France, August 31-September 3, 2010. Proceedings*, pages 71–85, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.

[BBB⁺00] Amos Beimel, Francesco Bergadano, Nader H. Bshouty, Eyal Kushilevitz, and Stefano Varricchio. Learning functions represented as multiplicity automata. *J. ACM*, 47(3):506–530, May 2000.

[BL21] L. Bohn and C. Löding. Constructing deterministic ω-automata from examples by an extension of the RPNI algorithm. In *46th Int. Symp. on Mathematical Foundations of Computer Science*, pages 20:1–20:18, 2021.

[BL22] León Bohn and Christof Löding. Passive learning of deterministic büchi automata by combinations of dfas. In *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France*, pages 114:1–114:20, 2022.

[BL23] León Bohn and Christof Löding. Constructing deterministic parity automata from positive and negative examples. *CoRR*, abs/2302.11043, 2023.

[Bok18] Udi Boker. Why these automata types? In *LPAR-22. 22nd International Conference on Logic for Programming, Artificial Intelligence and Reasoning, Awassa, Ethiopia, 16-21 November 2018*, pages 143–163, 2018.

[Büc62] J.R. Büchi. On a decision method in restricted second order arithmetic. In *International Congress on Logic, Methodology and Philosophy*, pages 1–11. Stanford Univ. Press, 1962.

[CBSS10] C. Y. Cho, D. Babic, E. C. R. Shin, and D. Song. Inference and analysis of formal models of botnet command and control protocols. In *Proceedings of the 17th ACM Conference on Computer and Communications Security, CCS 2010, Chicago, Illinois, USA, October 4-8, 2010*, pages 426–439, 2010.

[CKK⁺15] Martin Chapman, Hana Chockler, Pascal Kesseli, Daniel Kroening, Ofer Strichman, and Michael Tautschnig. Learning the language of error. In *Automated Technology for Verification and Analysis - 13th International Symposium, ATVA 2015, Shanghai, China, October 12-15, 2015, Proceedings*, pages 114–130, 2015.

[CPPdR14] Georg Chalupar, Stefan Pecherstorfer, Erik Poll, and Joeri de Ruiter. Automated reverse engineering using Lego®. In *8th USENIX Workshop on Offensive Technologies (WOOT 14)*, San Diego, CA, August 2014. USENIX Association.

[DD17] D. Drews and L. D’Antoni. Learning symbolic automata. In *Tools and Algorithms for the Construction and Analysis of Systems - 23rd International Conference, TACAS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings, Part I*, pages 173–189, 2017.

[DKV09] Manfred Droste, Werner Kuich, and Heiko Vogler, editors. *Handbook of Weighted Automata*, chapter 4: Rational and Recognizable Series, by Jaques Sakarovitch, pages 105–174. Springer-Verlag Berlin Heidelberg, 2009.

[dlH97] Colin de la Higuera. Characteristic sets for polynomial grammatical inference. *Machine Learning*, 27(2):125–138, 1997.

[dlH10] Colin de la Higuera. *Grammatical inference: learning automata and grammars*. Cambridge University Press, 2010.

[FCC⁺08] A. Farzan, Y-F. Chen, E.M. Clarke, Y-K. Tsay, and B-Y. Wang. Extending automated compositional verification to the full class of omega-regular languages. In *TACAS*, pages 2–17, 2008.
Dana Fisman, Hadar Frenkel, and Sandra Zilles. Inferring symbolic automata. Log. Methods Comput. Sci., 19(2), 2023.

Dana Fisman and Sagi Saadon. Learning and characterizing fully-ordered lattice automata. In Automated Technology for Verification and Analysis - 20th International Symposium, ATVA 2022, Virtual Event, October 25-28, 2022, Proceedings, pages 266–282, 2022.

Sally A. Goldman and H. David Mathias. Teaching a smarter learner. J. Comput. Syst. Sci., 52(2):255–267, 1996.

E. Mark Gold. Complexity of automaton identification from given data. Information and Control, 37(3):302–320, 1978.

Falk Howar, Bernhard Steffen, Bengt Jonsson, and Sofia Cassel. Inferring canonical register automata. In Verification, Model Checking, and Abstract Interpretation - 13th International Conference, VMCAI 2012, Philadelphia, PA, USA, January 22-24, 2012. Proceedings, pages 251–266, 2012.

Peter Habermehl and Tomáš Vojnar. Regular model checking using inference of regular languages. Electr. Notes Theor. Comput. Sci., 138(3):21–36, 2005.

Viraj Kumar, P. Madhusudan, and Mahesh Viswanathan. Minimization, learning, and conformance testing of boolean programs. In CONCUR 2006 - Concurrency Theory, 17th International Conference, CONCUR 2006, Bonn, Germany, August 27-30, 2006, Proceedings, pages 203–217, 2006.

Orna Kupferman, Moshe Y. Vardi, and Pierre Wolper. An automata-theoretic approach to branching-time model checking. J. ACM, 47(2):312–360, 2000.

Y. Li, Y-F. Chen, L. Zhang, and D. Liu. A novel learning algorithm for b"uchi automata based on family of dfas and classification trees. In Tools and Algorithms for the Construction and Analysis of Systems - 23rd International Conference, TACAS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings, Part I, pages 208–226, 2017.

T. Margaria, O. Niese, H. Raffelt, and B. Steffen. Efficient test-based model generation for legacy reactive systems. In HLDVT, pages 95–100. IEEE Computer Society, 2004.

Jakub Michaliszyn and Jan Otop. Learning deterministic automata on infinite words. In 24th European Conference on Artificial Intelligence (ECAI), pages 2370–2377, 2020.

O. Maler and A. Pnueli. On the learnability of infinitary regular sets. Inf. Comput., 118(2):316–326, 1995.

Roman Manevich and Sharon Shoham. Inferring program extensions from traces. In ICGI, volume 93 of Proceedings of Machine Learning Research, pages 139–154. PMLR, 2018.

W. Nam and R. Alur. Learning-based symbolic assume-guarantee reasoning with automatic decomposition. In ATVA, volume 4218 of Lecture Notes in Computer Science, pages 170–185. Springer, 2006.

Dolav Nitay, Dana Fisman, and Michal Ziv-Ukelson. Learning of structurally unambiguous probabilistic grammars. In Thirty-Fifth AAAI Conference on Artificial Intelligence, AAAI 2021, Thirty-Third Conference on Innovative Applications of Artificial Intelligence, IAAI 2021, The Eleventh Symposium on Educational Advances in Artificial Intelligence, EAAI 2021, Virtual Event, February 2-9, 2021, pages 9170–9178, 2021.

D. Peled, M. Y. Vardi, and M. Yannakakis. Black box checking. In FORTE, pages 225–240, 1999.

Jacques Sakarovitch. Elements of Automata Theory. Cambridge University Press, USA, 2009.

Sven Schewe. Beyond hyper-minimisation—minimising dbas and dpas is np-complete. In Kamal Lodaya and Meena Mahajan, editors, IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2010, December 15-18, 2010, Chennai, India, volume 8 of LIPIcs, pages 400–411. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2010.

Sven Schewe. Minimisation of deterministic parity and buchi automata and relative minimisation of deterministic finite automata, 2011.
[SHV16] Mathijs Schuts, Jozef Hooman, and Frits W. Vaandrager. Refactoring of legacy software using model learning and equivalence checking: An industrial experience report. In Integrated Formal Methods - 12th International Conference, IFM 2016, Reykjavik, Iceland, June 1-5, 2016, Proceedings, pages 311–325, 2016.

[Tar72] Robert Endre Tarjan. Depth-first search and linear graph algorithms. SIAM J. Comput., 1(2):146–160, 1972.

[Vaa17] F. Vaandrager. Model learning. Commun. ACM, 60(2):86–95, 2017.

[Var95] Moshe Y. Vardi. An automata-theoretic approach to linear temporal logic. In Banff Higher Order Workshop, volume 1043 of Lecture Notes in Computer Science, pages 238–266. Springer, 1995.

[Var08] Moshe Y. Vardi. Automata-theoretic model checking revisited. In Hardware and Software: Verification and Testing, 4th International Haifa Verification Conference, HVC 2008, Haifa, Israel, October 27-30, 2008. Proceedings, page 2, 2008.

[VSVA05] Abhay Vardhan, Koushik Sen, Mahesh Viswanathan, and Gul Agha. Using language inference to verify omega-regular properties. In Tools and Algorithms for the Construction and Analysis of Systems, 11th International Conference, TACAS 2005, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2005, Edinburgh, UK, April 4-8, 2005, Proceedings, pages 45–60, 2005.

[Wag75] K. W. Wagner. A hierarchy of regular sequence sets. In 4th Symposium on Mathematical Foundations of Computer (MFCS), pages 445–449, 1975.