Analytical Method for Nonlinear Behavior of Soil under Vertically Loaded Circular Plate

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Abstract
In this paper, nonlinear springs and linear under-soil are supposed to model the nonlinear behavior of soil lying under the vertically loaded circular plate. The analysis procedure using this model is introduced. Dozens of loading test data are used to get estimation of elastic soil modulus for the analysis. The characteristics of nonlinear springs for each kind of soil are also discussed on the basis of relations between load and settlement from the field test data. Settlements of circular plate calculated by proposed analysis model are compared with the test results. The settlements calculated are in good agreement with the measured, except that a few calculations show big difference between calculation and test at higher load levels. This difference may be caused by neglect of local shear failure in analysis of this paper. This defect should be improved in future study for the proposed analysis model.

Keywords: soil; nonlinear behavior; analysis method; loading test

Introduction
Soil’s behavior has a very important role in modern building structure designs. Although soil characters have been found to be very complicated, simple vertical loading tests of circular plate are always performed to estimate the bearing capacity of building foundation. Results of these tests generally indicate nonlinear relationship between load and settlement of the plate. It becomes a common acknowledge of structure designers and researchers that the analysis of soil’s nonlinear performance cannot be neglected or avoided for foundation designs.

Although the analytical methods to predict the behavior of vertically loaded plate have been presented by some investigators, most of them deal with ultimate resistance only [for example, K. Terzaghi & R. Peck (1967)⁵], or elastic behavior only [for example, W. Steinbrenner (1936)⁶ and H. Tanahashi (2000)⁷]. Only a few of the studies investigated the vertical behavior of the plate to the ultimate state [H. Yamaguchi (1977)⁸, M. Georgiadis et al. (1988)⁹, Architectural Institute of Japan (2001)¹⁰ and so on].

Our study on the loading test is to simulate the whole process of the circular plate’s settlement from the linearly elastic state to ultimate state, to obtain the characteristics of soil for its whole process from start to fail. This paper presents a method for calculating the non-linear behavior of vertically loaded circular plate. In this method an analytical model is used, in which non-linear springs lying on the homogeneous elastic half space of soil are considered. The nonlinear relation between load and settlement is all considered in the springs and the soil is considered to be linearly elastic. Calculations using this model are also compared with tests performed in sites where is deposited by loam, sand and gravel, respectively.

Analysis Model
The circular steel plates used for the loading tests are considered to be rigid relative to soil. The analytical model consists of a circular plate, non-linear springs and homogeneous elastic soil as shown in Fig. 1.
1) The soil is linearly elastic, modeled by Boussinesq solution. For the settlement of soil on the contact surface, the Boussinesq solution can be expressed by Eq. (1).

\[ \delta = \frac{P(1 - \nu^2)}{\pi \varepsilon E_s} \quad \cdots (1) \]

where \( \delta \) represents the deformation of soil at the evaluation point, \( E_s \) is soil’s Young’s modulus, \( \nu \) is the Poisson’s ratio of soil, \( P \) is external vertical load, and \( \zeta \) is the distance from external load’s location to the evaluation point.

2) The non-linear characters of the soil are concentrated in discrete springs with no length, which are bounded by the plate and the soil. Tests on a steel pipe pile under horizontal loads indicated that the horizontal soil reaction forces are proportional to square root of horizontal displacement. In order to derive the proposed analytical method, the Boussinesq solution can be expressed by Eq. (1).

For the settlement of soil on the contact surface, the Boussinesq solution can be expressed into the elastic soil under the circular rigid plate is divided following assumptions are applied:

- Equal area for each section. For each section, the sections (see Fig. 2). The rule of dividing is to have equal area for each section. For each section, the following equation can be derived:

\[ N = k x_{sp}^{1/2} \quad \cdots (2) \]

where \( N \) is the internal force occurred in spring, \( k \) is spring’s coefficient, which is constant everywhere in the soil, and \( x_{sp} \) is the deformation of spring from its natural state. To make out the proper nonlinear relation between load and settlement of soil is to find out the proper \( k \) in Eq. (2), which will be discussed in the subsequent section.

In order to derive the proposed analytical method, the elastic soil under the circular rigid plate is divided into \( n \) sections in radial direction and circularly into \( m \) sections (see Fig. 2). The rule of dividing is to have equal area for each section. For each section, the following assumptions are applied:

1) The reactions of soil and spring are evenly distributed on the section;
2) The sum reacting force is located circularly at the middle and radially at 2/3 of length;
3) The evaluation point of the soil deformation is located circularly and radially at the middle of each section, and the discrete spring is also located at the same point.

Since the circular plate in this paper is axially symmetric, the reaction along the circumference is same and the difference is along the radial direction only.

If the displacement of rigid plate under external force, \( F \) is \( S \), the equations for spring system and soil will be respectively as follow:

For springs:

\[
\begin{bmatrix}
N_1 \\
N_2 \\
\vdots \\
N_n
\end{bmatrix}
= k
\begin{bmatrix}
x_{sp1}^{1/2} \\
x_{sp2}^{1/2} \\
\vdots \\
x_{spn}^{1/2}
\end{bmatrix}
\quad \cdots (3)
\]

For soil:

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
N_1 \\
N_2 \\
\vdots \\
N_n
\end{bmatrix}
= \begin{bmatrix}
x_{sol} \\
x_{so2} \\
\vdots \\
x_{soln}
\end{bmatrix}
\quad \cdots (4)
\]

The external load \( F \) must be balanced with the sum of forces occurred in springs:

\[ N_1 + N_2 + \cdots + N_n = F / m \quad \cdots (5) \]

And the relationship between spring’s deformation and soil’s deformation is:

\[
\begin{bmatrix}
x_{sp1} \\
x_{sp2} \\
\vdots \\
x_{spn}
\end{bmatrix}
+ \begin{bmatrix}
x_{sol} \\
x_{so2} \\
\vdots \\
x_{soln}
\end{bmatrix}
= \begin{bmatrix}
S \\
S \\
\vdots \\
S
\end{bmatrix}
\quad \cdots (6)
\]

In Eq. (6) \( n \) is the number of division in radial direction, \( x_{spi} \) is the deformation of spring \( i \), and \( x_{sol} \) is the deformation of soil section \( i \).

Let \( y_i = x_{spi}^{1/2} \), \( i = 1, \cdots, n \), and from Eqs. (3) to (6) the following equation can be derived.
The following partial matrix of Eq. (7) is the linear characteristics matrix of soil. When \( dA_j \) is defined as the area of each division, the element of the partial matrix can be expressed by the Boussinesq equation followed.

\[
[A] = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

\[
a_y = \sum_{j=1}^{n} \frac{1 - \nu^2}{\pi E \zeta_{ij}(k_j)} dA_j \quad \cdots \quad (8)
\]

\[(a_y; i = 1, \cdots, n; j = 1, \cdots, n)\]

**Analysis Procedure**

To use this method for estimation of foundation behavior in structural designs or researches, the first aim is to obtain the nonlinear characteristics of a certain kind of soil through field tests.

A computer program has been built up to decide the corresponding coefficient \( k \) of a given loading test data. The analytical procedure for each loading step is shown in Fig. 3 and detailed as follow:

1) An initial value for displacement of spring \( \{x_{sp}\}_0 \) is assumed.

2) Since the internal forces of springs are given by Eq. (3) on the basis of above assumption, the value of nonlinear coefficient \( k \) may be calculated by Eq. (5).

3) According to the action-reaction theory, the soil displacement \( \{x_{so}\} \) caused by reaction forces of soil equal to springs’ internal forces may be calculated by using Eq. (4).

4) The displacement of springs \( \{x_{sp}\}_1 \) is then given by Eq. (6).

5) If the difference between \( \{x_{sp}\}_1 \) and \( \{x_{sp}\}_0 \) is not within a specified tolerance, an iterative value to initial displacement of springs \( \{x_{sp}\}_0 \) is given and steps 1) to 4) are repeated until the difference between \( \{x_{sp}\}_1 \) and \( \{x_{sp}\}_0 \) is small enough.

Using the above procedure, \( k \) values according to load steps of the performed test can be obtained. However, the calculated \( k \) value slightly varies at every load steps. In this paper, therefore, the average of all \( k \) values is taken as the corresponding nonlinear spring coefficient of the field test data.

![Fig.3. Main Iterative Process](attachment:fig3.png)

After getting the soil’s nonlinear coefficient of \( k \) as described above, estimation of the behavior of vertically loaded circular plate can be made. The analytical procedure is as follow:

1) An initial settlement of plate \( S_0 \) is assumed.

2) Repeat the whole procedure to obtain nonlinear coefficient \( k \), except that in the second step, only the internal forces of springs are calculated and...
that in Eq. (6), the settlement \( S \) is now replaced by \( S_0 \).

3) The sum of springs’ internal forces \( F' \) can be made. If the difference between \( F' \) and \( F \) is not within a specified tolerance, an iterative value is given to settlement of plate \( S_0 \), and the steps 1) to 3) are repeated until the difference between \( F' \) and \( F \) is small enough.

**Analysis of Test Results**

In this study, dozens of data of loading tests, from Obayashigumi, Ltd. and Tokyo Soil Research, Ltd., performed in sites deposited by clayey soil including loam and sandy soil including gravel are analyzed. In this session, the estimation of elastic modulus \( E_s \) of these soils applied for the analysis is presented first. Nonlinear spring coefficient \( k \) obtained from the analysis is presented secondly. And finally, in order to prove the validity of the proposed method, comparisons between calculated and measured load-displacement curves of six tests are made.

All the tests are performed on circular steel plate with a diameter of 0.30m. In this analysis, however, the Poisson’s ratio is 0.50 for clayey soil and 0.30 for sandy soil.

1. **Estimation of elastic soil modulus \( E_s \) for analysis**

It’s assumed that the load-settlement relation from a loading test may be simulated by the following hyperbolic equation.

\[
F = \frac{S}{aS + b} \quad \cdots \quad (9)
\]

On the basis of this assumption, the elastic behavior of vertically loaded plate at lower load level is related to the constant of \( 1/b \) from Eq. (9). While the logarithmic \( F-S \) relation of vertically loaded plate has usually bifurcated straight lines. Therefore, the elastic soil modulus \( E_s \) is backfigured by using the constant \( 1/b \) of the hyperbolic function, which is decided by load-settlement relations in lower part of the bifurcated lines, based on the least square method. Fig. 4 is the case of test No.104 for an example.

While unconfined compressive strength \( q_u \) and \( N \)-value of SPT (standard penetration test) are useful and basic values for foundation design. And to extend the use of the proposed analysis method to general cases beyond the test data, from the elastic modulus for all test data obtained by the hyperbolic equation, the following relationships between the \( E_s \)-value and unconfined compressive strength \( q_u \) for clayey soil and between the \( E_s \)-value and \( N \)-value of SPT for sandy soil are derived:
For clayey soil:

\[ E_s = 138.34 \times q_u \quad \cdots (10) \]

For sandy soil:

\[ E_s = 1174.5 \times N \quad \cdots (11) \]

These results are shown in Figs. 5 and 6, in which the straight line based on the approximate function decided by the least square method is indicated. (The numbers indicated in the figures are of those used for analysis.) These two empirical equations are used for following analysis.

2. Nonlinear coefficient \( k \)

Three test data on sandy soil and three on clayey soil are applied to make analysis using the proposed method. The test results are summarized in Table 1. The elastic soil modulus for each test based on the Eqs. (10) and (11) are shown in Table 1.

The calculated \( k \) value slightly varies at every load steps. Take No.104 for an example. There are 8 steps of load-adding and so 8 steps of \( k \)-value calculating. The results in sequence are 2420.914, 2897.651, 2600.971, 2212.278, 1842.037, 1575.281, 1452.082 and 1322.757 \( (N/m^{1/2}) \). And then the average of them is taken as the \( k \)-value for this case of soil, which is 2040.496 \( (N/m^{1/2}) \). And the average values of nonlinear spring coefficients for all the 6 cases are also shown in Table 1. (The tolerance for judgment is 0.1mm for each cycle of calculation.)

3. Comparisons between measured and analyzed load-settlement behavior

Comparisons between the predicted overall load-settlement curves and the observed curves are shown in Figs.7 and 8 for three clayey soils and three sandy soils, respectively. From these figures, it is found out that the tendencies of the calculated behavior are almost similar to the observed. However, we can find out the following differences on detailed examination.

Figure 7 shows that for two tests of Tests 48 and 71, which were performed under loads as low as 7 kN and 22 kN, comparisons between calculated and measured settlements reveal reasonable overall agreement. But it is shown that for Test 104, which was made under large loads over 31 kN, the calculated settlement is less than the measured at high load levels, and that at the final load step, the settlement ratio of the measured to the calculated is as much as about double.

On the other hand, comparisons between calculated and measured settlements for three tests on the sandy soil, shown in Fig.8, indicate that theoretical results are less than those by the tests at high loads over 35 kN to 100 kN.

Under an applied vertical load, soil may react non-linearly first. Then, shear failure may occur in soil along the plate edge at load level beyond a limit. And the failure area may propagate gradually toward the center of the plate with increasing load. It was found out by Takano\textsuperscript{10)}, from vertical loading test data, that shear failure occurred progressively in the soil under the pile point. And in his same works, he established a method to deal with shear failure, which had good agreement with the test results. The proposed analytical method of our study takes account of the nonlinear behavior of the under-soil, but without consideration of shear failure. The above differences of the calculated and measured load-settlement relationships at high load level may be caused by neglect of the shear failure in this method. We will refer to the method established by Takano\textsuperscript{10)} in further study and it is reasonable to expect that the problem can be solved and the method can be improved in this way.

Conclusion

This paper presents the analytical method to simulate nonlinear behavior of vertically loaded rigid circular plate. For six tests performed on clayey soils and sandy soils, comparisons between the measured load versus settlement and the calculated are also given. The proposed analytical method basically consists of a circular rigid plate, homogeneous elastic soil and non-linear springs with no length, which is bounded by the plate and soil.

Elastic soil modulus back-figured from a number of

| Test No. | Soil Type | SPT N-Value * | \( q_u ^* \) (KN/m²) | \( E_s \) (KN/m²) by Eq. (11) or (12) | At final load step | Nonlinear Coefficient \( k \) (N/m²) |
|----------|-----------|---------------|-----------------------|--------------------------------|-------------------|-----------------------------|
| 48       | Loam      | -             | 94.08                 | 13015.027                     | 22.638           | 9.600                       | 1459.080                   |
| 71       | Loam      | -             | 143.08                | 19793.687                     | 7.350            | 49.915                      | 157.242                    |
| 104      | Loam      | -             | 169.54                | 23454.164                     | 31.360           | 14.857                      | 2040.496                   |
| T-24     | Sand      | 13            | -                     | 15267.616                     | 281.691          | 28.744                      | 2304.586                   |
| 180      | Gravel    | -             | 41                    | 48151.712                     | 109.760          | 63.592                      | 3799.748                   |
| 281      | Sand      | 56            | -                     | 65768.192                     | 205.898          | 48.145                      | 9278.601                   |

Note: * Average of values from ground surface to depth of 1.0m.
Fig. 7. Comparisons for Clayey Soil

Fig. 8. Comparisons for Sandy Soil
test data is related to $q_{u}$-value for clayey soil or SPT $N$-value for sandy soil as input-data to calculate test data by using the method.

Dealing the coefficient of nonlinear spring $k$ in computing process is presented.

This study is concluded as follows:

From comparisons between the measured and the calculated load-settlement relation, it is found out that the tendencies of the calculated behavior are almost similar to the observed, calculations are in good agreement with test data at low load levels, but the calculated settlement is less than the measured at high load levels.

The above defect of discrepancy, which may be the result of not considering the local shear failure of soil in the method, will be improved in the future study.

Acknowledgments
Thanks are given to Obayashigumi, Ltd. and Tokyo Soil Research, Ltd. for supplying test data beyond their duty.

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