Deconfinement of Spinons on Critical Points:
Multi-Flavor CP¹ + U(1) Lattice Gauge Theory in Three Dimensions

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In this paper, we study the three-dimensional (3D) \(N_f\)-flavor CP¹ model (a set of \(N_f\) CP¹ variables) coupled with a dynamical compact U(1) gauge field by means of Monte-Carlo simulations. This model is relevant to 2D \(s = 1/2\) quantum spin models, and has a phase transition line which separates an ordered phase of global spin symmetry from a disordered one. From a gauge theoretical point of view, the ordered phase is a Higgs phase whereas the disordered phase is a confinement phase. We are interested in the gauge dynamics just on the critical line, in particular, whether a Coulomb-like deconfinement phase is realized there. This problem is quite important to clarify low-energy excitations in certain class of quantum spin models. If the gauge dynamics is in the deconfinement phase there, spinons, which transform in the fundamental representation of the SU(\(N_f\)) symmetry, appear as low-energy excitations. By Monte-Carlo simulations, we found that the “phase structure” on the criticality strongly depends on the value of \(N_f\). For small \(N_f\), the confinement phase is realized, whereas the deconfinement phase appears for sufficient large \(N_f \geq 14\). This result strongly suggests that compact QED₃ is in a deconfinement phase for sufficiently large number of flavors of massless fermions.

I. INTRODUCTION

Some recent experiments of strongly-correlated electron systems indicate that the usual Fermi liquid theory breaks down in certain cases, and the low-energy quasi-excitations carrying fractional/exotic quantum number appear there. The fractional quantum Hall effect (FQHE)¹ is a typical example, in which composite fermions appear as relevant excitations. Another example may be quantum spin models in low spatial dimensions, which have been studied quite intensively. For certain class of \(s = \frac{1}{2}\) anti-ferromagnetic (AF) spin models in two dimensions, it is argued that low-energy excitations at a quantum phase transition point are spinons².³

For studying the above “deconfined critical point” and the quantum phase transition itself, gauge theory is quite useful. Concept of confinement and deconfinement in the gauge theory is suitable for understanding the change of particle picture happening at deconfined critical points.

In the previous paper³, we showed that the phase transition from the Néel state to the dimer state in the AF magnet corresponds to a Higgs (confinement) to confinement phase transition in the simple CP¹ model. There we were also interested in the gauge dynamics at the critical point. If the three-dimensional (3D) Coulomb-like phase is realized there as a simple loop expansion predicts, quasi-excitations are massless spinons. As the low-energy excitations are magnons (spin waves) in the Néel state and spin-triplet excitations in the dimer state, the existence of spinons at the criticality indicates breakdown of the traditional Ginzburg-Landau (GL) theory of phase transition. This is because the GL theory uses an (a set of) order parameter to describe both a phase transition by its expectation value and low-energy excitations by its fluctuations in space and time.

To study the gauge dynamics of a class of spin models in a more general framework, we introduced the 3D CP¹+U(1) lattice gauge theory in Ref.⁴. The model contains two parameters, the spin stiffness \(c_1\) and the gauge coupling \(c_2\), and describes the O(3)(CP¹) and O(4) spin models in the specific limits, \((c_2 = 0\) and \(c_2 = \infty\), respectively). However, from the calculation of both instanton density and gauge-boson mass, we concluded there that the confinement phase is realized on its critical line⁵.

In this paper, we continue to study the gauge dynamics of these spin systems defined in two spatial dimensions at zero temperature by generalizing the above 3D CP¹+U(1) model to the 3D multi-flavor CP¹+U(1) model. In particular, we explore the possibility of change of particle picture on the criticality by controlling the flavor number \(N_f\) as an adjustable parameter.

The rest of the paper is organized as follows. In Sect.2, we explain the model and its relation to the AF Heisenberg model. In Sect.3, results of Monte-Carlo simulations are shown. We calculated the specific heat, the gauge-boson mass, and the instanton density for various values of \(N_f\), and found that the deconfinement phase is realized on the critical line for sufficiently large \(N_f\). Section 4 is devoted for conclusion.
II. MULTI-FLAVOR CP\(^1\) + U(1) MODEL ON THE 3D LATTICE

Let us first define the model on the cubic lattice, and explain its relation to quantum spin models. Hereafter we use \(x\) as the site index and \(\mu = 1, 2, 3\) as the direction index. On each site \(x\), we put \(N_t\)-flavor CP\(^1\) variables \(z^\alpha_x\), where \(\alpha\) is the flavor index and takes \(\alpha = 1, \cdots, N_t\). \(z^\alpha_x\) is a two-component complex field,

\[
z^\alpha_x = \begin{pmatrix} z^\alpha_{x,1} \\ z^\alpha_{x,2} \end{pmatrix}, \quad z^\alpha_{x,1}, z^\alpha_{x,2} \in \mathbb{C}, \tag{2.1}\]

satisfying the so-called CP\(^1\) constraint,

\[
z^\alpha_{x,1} z^\alpha_{x,2} = \sum_{\alpha = 1,2} |z^\alpha_{x,\alpha}|^2 = 1 \quad \text{for each } x \text{ and } \alpha. \tag{2.2}\]

On each link \((x, x + \mu)\) we put a U(1) gauge variable, \(U_{x\mu} = \exp(\theta_{x\mu} )\) \(\theta_{x\mu} \in (-\pi, +\pi)\). The action of the model \(S\) is given as

\[
S = -\frac{c_1}{2} \sum_{x,\mu,\alpha,\sigma} (z^\alpha_{x+\mu,\alpha} U_{x\mu} z^{\alpha\sigma}_{x,\alpha} + \text{H.c.}) \\
-\frac{c_2}{2} \sum_{x,\mu<\nu} \left( \bar{U}_{x\nu} U_{x+\nu,\mu} U_{x+\mu,\nu} U_{x\mu} + \text{H.c.} \right), \tag{2.3}\]

where \(c_1\) and \(c_2\) are real parameters of the model. It is obvious that the action (2.3) has a local U(1) gauge symmetry as well as SU(2) and SU\((N_t)\) global symmetries;

\[
z_{x\alpha} \rightarrow z'_{x\alpha} = \exp(i \Lambda_x) z_{x\alpha}, \\
U_{x\mu} \rightarrow U'_{x\mu} = \exp(i \Lambda_{x+\mu}) U_{x\mu} \exp(-i \Lambda_x), \tag{2.4}\]

\[
z^\alpha_{x\alpha} \rightarrow (z^\alpha_{x\alpha})' = \sum_{b=1}^2 V_{ab} z^\alpha_{xb} \quad V \in \text{SU}(2), \tag{2.5}\]

\[
z^\alpha_{x\alpha} \rightarrow (z^\alpha_{x\alpha})' = \sum_{\beta=1}^{N_t} W_{\alpha\beta} z^\beta_{x\alpha} \quad W \in \text{SU}(N_t). \tag{2.6}\]

Hereafter we call the above SU\((2)\)(SU\((N_t)\)) symmetry the spin symmetry (flavor symmetry). The partition function \(Z\) is given by

\[
Z = \int [dU]_{\text{U}(1)} [dz]_{\text{CP}(1)} \exp(-S). \tag{2.7}\]

There are many gauge-invariant quantities composed of \(z^\alpha_x\) and \(U_{x\mu}\). Among them, typical \textit{local} combinations are a set of \(N_t\) O(3) (real three-component) spins \(n^\alpha_x\),

\[
n^\alpha_x = z^\alpha_x \sigma z^\alpha_x, \quad n^\alpha_x \cdot n^\beta_x = 1, \tag{2.8}\]

where \(\sigma = (\sigma_1, \sigma_2, \sigma_3)^t\) are the Pauli matrices. For \(N_t = 1\), the model reduces in the limit of \(c_2 = 0\) to the O(3) spin model described by the field \(n_x\) with nearest-neighbor interactions\(^4\). For \(N_t > 1\), there are other local gauge invariant objects \(\bar{M}^{\alpha\beta}_x\), which are \textit{four-component} O(4) vectors,

\[
\bar{M}^{\alpha\beta}_x = \frac{1}{\sqrt{2}} (z^\alpha_x \sigma_1 z^\beta_x, z^\alpha_x \sigma_2 z^\beta_x, z^\alpha_x \sigma_3 z^\beta_x, z^\alpha_x \sigma_4 z^\beta_x)^t, \\
\bar{M}^{\alpha\beta}_x \cdot \bar{M}^{\gamma\delta}_x = 1, \quad \bar{M}^{\beta\gamma}_x = \bar{M}^{\alpha\alpha}_x. \tag{2.9}\]

\(\bar{M}^{\alpha\beta}_x\) are complex for \(\alpha \neq \beta\), while \(\bar{M}^{\alpha\alpha}_x\) are real. In the limit of \(c_2 = 0\), one can integrate over \(U_{x\mu}\) link by link to obtain

\[
Z_{c_2=0} = \int [dz]_{\text{CP}(1)} \exp \left[ \sum_{x,\mu} \log I_0(\gamma_{x\mu}) \right], \\
\gamma_{x\mu}^2 = \frac{c^2}{4} \sum_{\alpha,\beta=1}^{N_t} \bar{M}^{\alpha\beta}_x \cdot \bar{M}^{\beta\alpha}_x + \text{H.c.} \tag{2.10}\]

where \(I_0(\gamma_{x\mu})\) is the modified Bessel function. We note that \(\bar{M}^{\alpha\beta}_x\) are \textit{not} all independent, so one needs to include extra interactions associated with the change of variables from \(z^\alpha_x\) to \(\bar{M}^{\alpha\beta}_x\) to treat them as independent O(4) complex spin vectors. For finite \(c_2\), the model involves nonlocal and/or nonpolynomial interactions among \(\bar{M}^{\alpha\beta}_x\).

On the other hand, in the limit of \(c_2 \rightarrow \infty\), the gauge configuration is restricted to \(U_{x\mu} = 1\) up to gauge transformations (2.4). Then the model reduces to an ensemble of \textit{independent} \(N_t\) O(4) nonlinear sigma models,

\[
Z_{c_2=\infty} = \int [dR]_{\text{O}(4)} \exp(c_1 \sum_{x,\mu,\alpha} \bar{R}^\alpha_{x+\mu} \cdot \bar{R}^\alpha_x), \\
\int [dR]_{\text{O}(4)} = \prod_{x,\alpha=1}^4 \int \prod_{k=1}^4 dR^\alpha_{xk} \delta(\bar{R}^\alpha_x \cdot \bar{R}^\alpha_x - 1), \\
z^\alpha_{x1} = R^\alpha_{x1} + i R^\alpha_{x2}, \quad z^\alpha_{x2} = R^\alpha_{x3} + i R^\alpha_{x4}, \tag{2.11}\]

where \(\bar{R}^\alpha_x\) is a four-component real O(4) vector, \(\bar{R}^\alpha_x = (R^\alpha_{x1}, \cdots, R^\alpha_{x4})^t\).
the disordered phase which are separated by the second-order transition line \( c_1 = c_{1c} (c_2) \). (See Fig.1.) These two phases correspond to the Higgs and the confinement phases in the U(1) gauge dynamics, respectively. In the ordered phase \( c_1 > c_{1c} \), there is a nonvanishing “spin magnetization” \( \langle z_x \sigma_z \rangle \neq 0 \), and as a result the low-energy excitations are the massless components of \( z_{xa} \), which corresponds to the spin waves in the AF magnets (see later discussion). On the other hand, in the disordered phase \( c_1 < c_{1c} \), the confinement phase is realized, and the low-energy excitations are the “spin-triplet” vector field which is nothing but the composite field, \( n_x = z_x \sigma_z \). Just on the critical line \( c_1 = c_{1c} \), there is no spontaneous symmetry breaking of the internal spin symmetry and \( z_{xa} (a = 1, 2) \) behave as gauge-interacting massless bosons. Thus one may naturally expect that a 3D Coulomb-like phase with a potential \( 1/r \) may be realized there because of the screening effect by the massless bosons \( z_{xa} \). In such a phase on the critical line, the low-energy excitations are to be “weakly interacting massless spinons” \( z_{xa} \). With this possibility in mind, we studied the gauge dynamics on the critical line, and found that the confinement phase is realized there. This result means that the CP\(^1\) model coupled with the dynamical gauge field \((2.3)\) belongs to the same universality class as the \( O(3) \) nonlinear \( \sigma \) model.

The \( N_f = 1 \) case of the CP\(^1\) model \((2.3)\) is known to be a low-energy effective field theory of the nonuniform \( s = \frac{1}{2} \) AF Heisenberg model on a square lattice\(^3\,6\,7\),

\[
H_{\text{AF}} = \sum_{xj} J_{xj} \hat{S}_x \cdot \hat{S}_{x+j} + \cdots ,
\]

where \( j \) is the spatial direction index \((j = 1, 2)\), \( \hat{S}_x \) is the quantum spin operator at site \( x \), and \( J_{xj} \) is the nonuniform exchange coupling. (See Fig.2.) The ellipses in Eq.(2.12) represent other multi-spin interactions. By varying the couplings \( J_{xj} \), the ground state of the Hamil-

\[
\hat{S}_x = \frac{1}{2} z_x^\dagger \sigma_z z_x ,
\]

and the CP\(^1\) constraint, \( \sum_a z_{xa}^\dagger z_{xa} \mid \text{phys} \rangle = \mid \text{phys} \rangle \), restricts the magnitude of the spin to \( \frac{1}{2} \). It was shown that the above Néel-dimer phase transition is nothing but the transition of the CP\(^1\) model discussed above\(^3\). Therefore our investigation on the critical behavior of the CP\(^1\) model\(^4\) indicates that the quantum phase transition in the system \((2.12)\) belongs to the same universality class with the classical phase transition in the 3D \( O(3) \) nonlinear \( \sigma \) model\(^9\).

![FIG. 2. 2D square lattice; crosses are odd sites, and filled circles are even sites. Solid bonds indicate that their exchange couplings are stronger than those on the dotted bonds.](image)

The above result supports the traditional idea for quantum phase transition that a quantum system in \( d \) spatial dimensions belongs to the same universality class as a certain classical system in \( d + 2 \) dimensions, where 2 is called dynamical critical exponent\(^10\). In recent years, however, it has been recognized that the above idea of the dynamical exponent breaks down in some cases; nontrivial physics appears at the criticality of quantum phase transitions. The spinons, as it was explained above, are a typical example of such interesting possibility.

In order to see the above interesting phenomenon of quantum phase transition, we shall extend the model. The CP\(^N\) model in the 3D continuum space-time is certainly such a model, which can be studied by the 1/N expansion\(^11\). In the leading order of the 1/N, a nontrivial infrared fixed point appears. On this fixed point, i.e., on the critical point, a nonlocal term for the gauge field \( A_\mu (x) \) like,

\[
N \int d^3 x \int d^3 y \sum_{\mu, \nu} F_{\mu \nu} (x) \frac{1}{|x-y|^2} F_{\mu \nu} (y) ,
\]

appears in the effective action due to the vacuum polarization of the massless \( z_x \). At long distances, the above
term dominates the usual Maxwell term which may exist in the original action. From (2.14), it is straightforward to calculate the potential energy $V(r)$ between the two charges separated by a distance $r$ as $V(r) \propto 1/r$. Then it is quite interesting to study the CP$^N$ models on the lattice for various values of $N$, in particular, to investigate the change in their critical behaviors.

Below we shall study the multi-flavor CP$^1$ model (2.3) numerically instead of the CP$^N$ model. The reason to choose the multi-flavor CP$^1$ model is simply a matter of simplicity and shorter computing time in Monte-Carlo simulations. In the large-$N$ limit, it is expected that the both models exhibit similar behavior.

III. NUMERICAL RESULTS

In this section, we present the results of our numerical study of the model on the 3D cubic lattice of the system size $N = L^3$, $L = 8, 12, 16$ with the periodic boundary condition for the flavor number $N_f = 1, 2, 3, 4, 5, 10, 14,$ and 18. We measured the internal energy, the specific heat, the mass of the gauge boson, and the instanton density. We observed no hysteresis in the internal energy $\langle S \rangle /N$.

A. Specific heat

We first show the results of the specific heat $C = \langle (S - \langle S \rangle)^2 \rangle /N$ measured in order to determine the phase structure.

![Fig. 4](image1.png)

FIG. 4. Specific heat $C$ vs. $c_1$ at $c_2 = 2.0$ for the $N_f = 2$ case. The system size is $8^3, 12^3$ and $16^3$. $C$ shows typical behavior of the second-order phase transition.

In Fig.4 and Fig.5 we present $C$ at the gauge coupling $c_2 = 2.0$ for $N_f = 2$ and 14, respectively. These results show a typical behavior of the second-order phase transition. To confirm the transition is of second-order, we fit these data by the finite-size scaling hypothesis (FSSH)$^{13}$. To this end, we introduce a parameter $\epsilon \equiv (c_1 - c_{1\infty}) / c_{1\infty}$ where $c_{1\infty}$ is the critical coupling in the infinite system ($L \to \infty$). Then we assume that the correlation length at $L \to \infty$ scales as $\xi \propto \epsilon^{-\nu}$ with a critical exponent $\nu$. We also assume that the maximum of $C$ at $L \to \infty$, $C_{\infty}$ diverges as $C_{\infty} \propto \epsilon^{-\sigma}$ with another exponent $\sigma$. Then FSSH predicts that the specific heat $C_L(\epsilon)$ for the system size $L$ scales as

$$C_L(\epsilon) = L^{\sigma/\nu} \phi(L^{1/\nu} \epsilon),$$

(3.1)

where $\phi(x)$ is the scaling function$^{13}$. The scaling function obtained from the data in Fig.5 is shown in Fig.6. The parameters are estimated as $\nu = 1.0, c_{1\infty} = 1.01$ and $\sigma = 0.20$. The function $\phi(x)$ is well determined, and it is obvious that the FSSH is satisfied quite well. We investigated the phase structure of the CP$^N$ ($N = 2, 3, 4$) models and also $N_f = 2, 3$ cases of the multi-flavor CP$^1$ models by calculating the specific heat. We conclude that the phase structures of both the multi-flavor CP$^1$+U(1) model and the CP$^N$+U(1) model are similar to that of the CP$^3$+U(1) model shown in Fig.1.

![Fig. 6](image2.png)

FIG. 6. Finite-size scaling function $\phi(x)$ of Eq.(3.1) determined by using $C$ of Fig.5. All the data of $L = 8, 12$ and 16 are fitted well by the single function $\phi(x)$.

B. Mass of gauge boson

Now let us turn to the gauge-boson mass. We calculate the gauge-invariant gauge-boson mass $M_G$ as follows$^{1,14}$. To define $M_G$ we first introduce a gauge-invariant operator $O(x)$,
We define the gauge correlation function,

\[ D(x_3) = \sum_{x_1, x_2} \langle O(x) \rangle e^{ip_1 x_1 + ip_2 x_2}. \]  

(3.3)

We determine the Fourier transformed field \( \tilde{O}(x_3) \) as follows,

\[ \tilde{O}(x_3) = \sum_{x_1, x_2} O(x)e^{ip_1 x_1 + ip_2 x_2}. \]  

(3.3)

We define the gauge correlation function,

\[ D_G(t) = \frac{1}{L^3} \sum_{x_3} \langle \tilde{O}(x_3) \tilde{O}(x_3 + t) \rangle. \]  

(3.4)

In the continuum, \( D_G(t) \) is expected to behave as

\[ D_G(t) = \int dp_1 \frac{e^{ip_1 t}}{p^2 + M_G^2} \propto e^{-\sqrt{p_1^2 + p_2^2 + M_G^2} t}. \]  

(3.5)

Typical behavior of the correlator \( D_G(t) \) is shown in Fig.7. We determine \( M_G \) by fitting the data \( D_G(t) \) by the exponential form (3.5). For practical calculations, we set \( p_1 = p_2 = 2\pi/L \).

\[ O(x) = \sum_{\mu, \nu=1,2} \epsilon_{\mu\nu} \text{Im} (\bar{U}_{x\mu} U_{x+\mu,\nu} U_{x+\mu,\nu} U_{x\mu}) \]

\[ = \sum_{\mu, \nu} \epsilon_{\mu\nu} \sin(-\theta_{x\mu} - \theta_{x+\mu,\nu} + \theta_{x+\mu,\nu} + \theta_{x\mu}), \]  

(3.2)

where \( \epsilon_{\mu\nu} \) is the antisymmetric tensor. Then we introduce the Fourier transformed field \( \tilde{O}(x_3) \) as follows,

\[ \tilde{O}(x_3) = \sum_{x_1, x_2} O(x)e^{ip_1 x_1 + ip_2 x_2}. \]  

(3.3)

In Fig.8 and Fig.9, we plot \( M_G \) for \( N_f = 1 \sim 5 \) and \( N_f = 10, 14, 18 \), respectively. From the results in Fig.8, it is obvious that the gauge-boson mass has the minimum in the region close to the phase transition point. The minimum of the value of \( M_G \) decreases as \( N_f \) increases as expected, but it is still nonvanishing. In the previous paper\(^4\), we observed similar behavior of \( M_G \) in the CP\(^N U(1) \) model for \( N = 1, 2, 3, 4 \).

On the other hand, \( M_G \) for \( N_f = 10, 14, 18 \) in Fig.9 shows that \( M_G \) vanishes at the criticality for \( N_f \geq 14 \). This indicates that a deconfinement phase is realized on the critical line for large \( N_f \). Appearance of the deconfinement phase stems from the shielding effect by the massless bosons \( \varphi^a \). On the critical line, low-energy excitations are massless \( \varphi^a \) and massless gauge boson \( \theta_{x\mu} \). Furthermore, we expect that topological nontrivial excitations, i.e., instantons, become irrelevant on the critical line due to a large number of the massless \( \varphi^a \). (See later discussion.)

From the data of Fig.8 and Fig.9, one can locate the minimum value of \( M_G \) along the line \( c_2 = 2.0 \) for each \( N_f \). These minima are presented in Fig.10. The minimum value of \( M_G \) seems to decrease continuously as \( N_f \) increases. By making the linear extrapolation of the data for \( N_f = 1 \sim 5 \) and 10, we estimate that \( M_G \) starts to vanish at \( N_f \approx 13.5 \). This value is regarded as the critical flavor number at which the phase on the criticality changes continuously from the confinement phase to the deconfinement Coulomb-like phase.
C. Instantons

Instantons play an important role in compact U(1) gauge theories\(^\text{15}\). Their proliferation (condensation) enhances fluctuations of U(1) gauge field and induces the confinement phase of the gauge dynamics. In the present 3D case, the instantons are just the magnetic monopoles and their condensation puts the system into the “dual” superconducting phase. The dual Meissner effect squeezes electric fluxes one-dimensionally, and as a result a pair of oppositely charged point particles separated by a distance \(r\) have the energy proportional to \(r\), i.e., they are confined.

In order to measure the instanton density, let us define instanton charge as in Ref.\(^\text{4}\) i.e., they are confined. The instanton charge is given by

\[
\Theta_{x,\mu\nu} \equiv \theta_{x\mu} + \theta_{x+\mu,\nu} - \theta_{x+\nu,\mu} - \theta_{x\nu},
\]

\[-4\pi < \Theta_{x,\mu\nu} < 4\pi. \tag{3.6}\]

We decompose \(\Theta_{x,\mu\nu}\) into its integer part \(2\pi n_{x,\mu\nu}\) \((n_{x,\mu\nu} \text{ is an integer})\) and the remaining part \(\tilde{\Theta}_{x,\mu\nu} \equiv \Theta_{x,\mu\nu} \mod (2\pi),\)

\[
\Theta_{x,\mu\nu} = 2\pi n_{x,\mu\nu} + \tilde{\Theta}_{x,\mu\nu}, \quad (-\pi < \tilde{\Theta}_{x,\mu\nu} < \pi). \tag{3.7}\]

Physically, \(n_{x,\mu\nu}\) describes the Dirac string. The instanton charge \(Q_x\) at the cube around the dual site \(\tilde{x} = x + (1 + 2 + 3)/2\) is defined as

\[
Q_x = -\frac{1}{2} \sum_{\mu,\nu,\rho} \epsilon_{\mu\nu\rho} (n_{x+\mu,\nu,\rho} - n_{x,\nu,\rho})
\]

\[
= \frac{1}{4\pi} \sum_{\mu,\nu,\rho} \epsilon_{\mu\nu\rho} (\tilde{\Theta}_{x+\mu,\nu,\rho} - \tilde{\Theta}_{x,\nu,\rho}), \tag{3.8}\]

where \(\epsilon_{\mu\nu\rho}\) is the complete antisymmetric tensor. Then it is obvious that \(Q_x\) measures the total flux emanating from the monopole(instanton) sitting at \(\tilde{x}\). The instanton density \(\rho\) is defined as

\[
\rho = \sum_x |Q_x|/N. \tag{3.9}\]

In Fig.11, we show the instanton density \(\rho\) at \(c_2 = 2.0\) as a function \(c_1\) for the \(N_f = 1\) and \(N_f = 18\) cases. As snapshots of instanton configurations in Ref.\(^\text{4}\) show, some of instantons form pairs with anti-instantons located at NN sites, i.e., instanton-anti-instanton dipoles. These dipoles are not effective for disordering the gauge-field dynamics and do not contribute to confinement. In fact, the confinement phase of the gauge dynamics is nothing but the plasma phase of the instantons as first shown by Polyakov\(^\text{17}\). On the other hand, the insulating phase of the instantons, in which almost all instantons form dipoles, is the deconfinement phase of the gauge dynamics. Then the density of isolated (single) instantons is a physical quantity which monitors whether the system is in the (de)confinement phase. Therefore in Fig.11, we also show the density of isolated instantons \(\rho_{\text{is}}\),

\[
\rho_{\text{is}} \equiv \rho - 2\rho_{\text{dp}}, \tag{3.10}\]

where \(\rho_{\text{dp}}\) is the density of NN instanton-anti-instanton dipoles defined similarly as in Eq.(3.9) (The factor 2 in front of \(\rho_{\text{dp}}\) in Eq.(3.10) comes from the fact that a dipole is composed of an instanton and an anti-instanton).

In Fig.11, both \(\rho\) and \(\rho_{\text{is}}\) almost vanish for \(c_1 > c_{1c}\) in the \(N_f = 18\) case in which \(M_G\) vanishes at the critical point. However, in the \(N_f = 1\) case, there remains a finite instanton density at the critical point. This result and the calculation of \(M_G\) indicate that the Coulomb phase is realized on the critical line for \(N_f \geq 14\).
In Eq.(3.12), $S_z(U)$ dominates over the single-plaquette term for sufficiently large $N_f$ and it determines the constant $A$ in the fitting $\rho$ as actually observed in Fig.13.

IV. CONCLUSION

In this paper we studied the multi-flavor CP$^1$ model in three dimensions by Monte-Carlo simulations. In particular, we are interested in the gauge dynamics on the critical line which separates the Higgs (Néel) and confinement (dimer) phases. On the critical line, “spinons” $z_\alpha^a$ are massless. Their fluctuations shield the confining gauge force at least partly. If the number of these spinons is sufficiently large, the confining forces may be completely shielded by them and the deconfining (Coulomb-like) force may appear instead. By calculating the gauge-boson mass and the instanton density, we found that the Coulomb-like deconfinement phase is actually realized for $N_f \geq 14$. The low-energy excitations on the critical line are the massless “spinons” $z_\alpha^a$ and massless gauge boson. Similar deconfinement phase is expected to appear on the critical points of the large-$N$ solution of the CP$^N$ model.

As far as the shielding phenomenon is concerned, massless fermions give a similar effect as massless bosons. Thus the present result indicates that the parity-preserving QED$_3$ with massless four-component-spinor fermions should have a deconfinement phase for sufficiently large flavor number of fermions, as long as the chiral symmetry is not spontaneously broken to avoid the generation of the dynamical mass. For example, in perturbation theory, gauge-interacting fermions generate the nonlocal terms like Eq.(2.14). Recently, 3D U(1) gauge theories coupled with gapless matter fields have been studied quite intensively, in particular, to answer the question whether a confinement-deconfinement phase transition takes place. The results of the present paper are in agreement with those obtained in these works.

One may wonder how the results in this paper are applied to the dynamics of realistic quantum spin models. The corresponding quantum model for the $N_f$-flavor CP$^1$ model is the SU(2)$\times$ SU($N_f$) “spin” AF magnets, whereas for the CP$^N$ model that is SU($N+1$) “spin” AF magnets. Unfortunately, as far as we know, there are no materials which have the above internal quantum degrees of freedom. However, as we explained in the introduction and also in the above, study of the strongly-correlated electron systems often reduces to the study of certain gauge models of gapless matter fields. The results in the present paper should give an important insight into the phase structures of these gauge systems and the related strongly-correlated electron systems.
The discrepancy between the two alternative results for the Néel-dimer critical point of the 2D AF Heisenberg spin model, the deconfinement phase\(^3\) or the confinement phase\(^4\), implies that the inverse effective gauge-coupling constant \(g_1\) was overestimated in Ref.\(^3\). \(g_1\) should be below the critical coupling \(g_c\) to put the system in the confinement phase to agree with Ref.\(^4\).

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