A note on friezes of type $\Lambda_4$ and $\Lambda_6$

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Abstract

We point out a certain connection between Conway-Coxeter friezes of triangulations and $p$-angulated generalisation of frieze patterns recently introduced by Holm and Jørgensen in [8]: the friezes of type $\Lambda_p$ coincide with Conway-Coxeter friezes of certain triangulations for $p = 4$ and $p = 6$ in every odd row.

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1 Settings and Theorem

Frieze patterns where defined by Coxeter in [6].

Definition 1 Let $n > 0$ be an integer. A frieze pattern of width $n$ consists of $n + 4$ infinite horizontal rows of non-negative real numbers, with an offset between even and odd rows. The rows are numbered 0 through $n + 3$, starting from below, and they satisfy:

(i) Rows number 0 and $n + 3$ consist of zeroes. Rows number 1 and $n + 2$ consist of ones. Rows number 2 through $n + 1$ consist of positive real numbers.

(ii) Each “diamond” $\begin{array}{cc} a & b \\ c & d \end{array}$ satisfies $ad - bc = 1$. Row number 2 (the first non-trivial row from below) is called the quiddity row.

Furthermore, we call any $n$-tuple of successive entries of the quiddity row quiddity sequence.

Conway and Coxeter proved that triangulations of polygons are in bijection with integral frieze patterns in [4] and [5]. Holm and Jørgensen recently generalized in [8] this relation by studying certain frieze patterns over $O_K$, the ring of algebraic integers in the field $K = \mathbb{Q}(\lambda_{p_1}, \ldots, \lambda_{p_s})$, for $s$ and $\lambda_{p_i}$, $1 \leq i \leq s$ as defined below.

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Definition 2 Let $p \geq 3$ be an integer. A frieze pattern is of type $\Lambda_p$ if the quiddity row consists of (necessarily) positive integral multiples of

$$\lambda_p = 2 \cos \left( \frac{\pi}{p} \right).$$

One of the main results of [8] is the following theorem:

Theorem 3 There is a bijection between $p$-angulations of the $(n+3)$-gon and frieze patterns of type $\Lambda_p$ and width $n$.

Definition 4 A polygon dissection $D$ of a polygon $P$ is a set of pairwise non-crossing diagonals. It splits $P$ into subpolygons $P_1, \ldots, P_s$ where $P_j$ is a $p_j$-gon for some $p_j$. Observe that $D$ is a $p$-angulation if and only if $p_1 = \cdots = p_s = p$. We denote the set of all $p$-angular dissections $D$ of a $(p-2)s+2$-gon $P$ by $\mathcal{D}_p((p-2)s+2)$.

A dissection $D$ of $P$ into subpolygons $P_i$, where $P_i$ is a $p_i$-gon, is mapped to a frieze pattern $F$ constructed as follows: To each vertex $\alpha$ of $P$, associate the sum

$$\sum_{P_i \text{ is incident with } \alpha} \lambda_{p_i}.$$

Furthermore, for every vertex $\alpha$, we associate

$$q_\alpha := |\{P_j \mid P_j \text{ is incident with } \alpha\}|.$$

The quiddity sequence of a frieze pattern of a $p$-angulation of a $(p-2)s+2$-gon is

$$(\lambda_p q_0, \lambda_p q_1, \ldots, \lambda_p q_{(p-2)s+1}). \tag{1}$$

We will replace the term frieze pattern by frieze for the rest of the note, see [8, Sect.1] for details. Figure 1 shows a 4-angulation $D_0 \in \mathcal{D}_4(10)$ with diagonals $\{1,4\}$, $\{4,9\}$ and $\{5,8\}$. Therefore $q_0 = q_2 = q_3 = q_6 = q_7 = 1$, $q_1 = q_5 = q_8 = q_9 = 2$ and $q_4 = 3$. Figure 2 shows the associated frieze of type $\Lambda_4$.

In this note, we consider $p$-angulations for $p = 4$ and $p = 6$ and want to prove the following theorem:

Theorem 5 Let $p = 4$ or $p = 6$. For every $p$-angulation $D$ of a polygon $P$, there exists a uniquely determined triangulation $T_D$ such that the frieze of type $\Lambda_p$ associated to $D$ and the Conway-Coxeter frieze of $T_D$ coincide in every odd row.
Figure 2: The frieze $F_{D_0}$ of type $\Lambda_4$ of $D_0$ has width 7 (number of non-trivial rows).

2 Friezes of type $\Lambda_4$ and associated Conway-Coxeter friezes.

There is a bijection between quadrangulations on the $2s + 2$-gon and noncrossing trees on $s + 1$ vertices as defined in [7].

Definition 6 A noncrossing tree in the regular polygon with $s + 1$ vertices is a set of edges between the vertices of the polygon, with the following properties

- edges do not cross pairwise
- any two vertices are connected by a sequence of edges
- there is no loop made of edges.

We follow [3] for describing this bijection. Assume that the vertices of the $2s + 2$-gon have been coloured black and white alternating. Then every edge of the quadrangulation connects one black and one white vertex. Every quadrangle contains a unique black-black diagonal. The collection of these diagonal edges form a noncrossing tree. Conversely, by drawing a noncrossing tree using the black vertices of the $2s + 2$-gon, one can consider all black-white edges that do not cross the edges of the noncrossing tree. This gives back the quadrangulation. This construction is shown in Figure 3.

Figure 3: The bijection between quadrangulation and noncrossing trees.

Definition 7 To every quadrangulation $D$ of a polygon $P$ we associate the triangulation $T_D$ consisting of diagonals defined by the dissection $D$ of $P$ together with the edges of the corresponding noncrossing tree.
Figure 4: The triangulation $T_{D_0}$ of the dissection $D_0$.

\[
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 1 & 2 & 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 1 & 2 & 3 & 4 & 1 & 2 & 2 & 2 & 1 & 4 \\
8 & 2 & 2 & 5 & 11 & 3 & 1 & 1 & 3 & 7 & 3 & 2 \\
5 & 1 & 3 & 7 & 13 & 5 & 1 & 3 & 7 & 13 & 5 & 1 \\
8 & 2 & 1 & 10 & 8 & 2 & 2 & 2 & 2 & 18 & 5 & 11 \\
3 & 1 & 3 & 7 & 3 & 3 & 3 & 3 & 1 & 5 & 11 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{array}
\]

Figure 5: The Conway-Coxeter frieze $F_{T_{D_0}}$ of $T_{D_0}$.

Remark 8

Note that the association $D \mapsto T_D$ from Definition 7 agrees with the map in [1] between 4-gonal dissections and 2-color valid triangulations in many cases, but in general it is different.

Figure 4 shows the associated triangulation of $D_0$.

For the triangulation $T_D$ associated to a dissection $D$, define for every vertex $0 \leq \alpha \leq 2s+1$,

\[t_\alpha := |\{P_j \mid P_j \text{ is incident with } \alpha\}|.\]

The quiddity sequence of the Conway-Coxeter frieze is then by definition $(t_0, t_1, \ldots, t_{2s+1})$. Figure 5 shows the Conway-Coxeter frieze of $T_{D_0}$. Note that every row with an odd number (the initial row of zeros has row number 0) coincide with the frieze $F_{D_0}$ of type $\Lambda_4$ in Figure 2.

In order to prove the following Lemma, we need an equivalent combinatorial object to a dissection and its properties (see also [1]). The dual graph of a $p$-angular dissection $D$ of a polygon $P$ has a vertex for every subpolygon $P_j$ and two vertices are connected by an edge if the corresponding $p$-gons share a common edge. An ear is a $p$-gon $P_j$ which consists of $p-1$ boundary edges of $P$ and exactly one diagonal of $D$. The dual graph of any dissection $D$ of a polygon $P$ is a tree. Every tree has at least two leaves and every leave of the dual graph of $D$ of $P$ corresponds to an ear of $D$ of $P$.

Recall that for a dissection $D$ of a polygon into $s$ quadrilaterals, $q_\alpha$ denotes the number of quadrilaterals incident with vertex $\alpha$ for $0 \leq \alpha \leq 2s+1$. We can now state the following Lemma:
Lemma 9 For every 4-angulation $D \in \mathcal{D}_4(2s + 2)$ and its associated triangulation $T_D$, the following equalities hold:

$$t_\alpha = \begin{cases} q_\alpha & \text{if } \alpha \text{ is even} \\ 2q_\alpha & \text{otherwise.} \end{cases} \quad (2)$$

Proof. The proof is done by induction. For $s = 1$, the statement is true. Let $s > 1$ and let $D \in \mathcal{D}_4(2s + 2)$. Let $\{\alpha, \alpha + 3\}$ be an ear of $D$. Cutting off this ear from $D$, as shown in Figure 6, gives a 4-gonal dissection $D'$ with $2s$ vertices and hence equation (2) holds for all vertices $\beta \in [0, 2s + 1] \setminus \{\alpha, \alpha + 1, \alpha + 2, \alpha + 3\}$ by induction hypothesis. Denote

Figure 6: Cutting off an ear from $D$ gives a quadrangulation $D'$ on $2s$ vertices. Here $\alpha$ is odd.

the number of triangles and quadrilaterals incident with vertices $\alpha$ and $\alpha + 3$ in $T_{D'}$ and $D'$ by $t'_\alpha$, $q'_\alpha$ and $t'_{\alpha+3}$, $q'_{\alpha+3}$ respectively. There are 2 cases, as vertex $\alpha$ is either coloured black or white. If $\alpha$ is odd and coloured black, then $t'_\alpha = 2q'_\alpha$ and $t'_{\alpha+3} = q'_{\alpha+3}$. As the triangulation $T_D$ includes the diagonal $\{\alpha, \alpha + 2\}$, we obtain the equations

$$t_\alpha = t'_\alpha + 2 = 2(q'_\alpha + 1)$$
$$t_{\alpha+1} = 1 = q_{\alpha+1}$$
$$t_{\alpha+2} = 2 = 2q_{\alpha+2}$$
$$t_{\alpha+3} = t'_{\alpha+3} + 1 = q'_{\alpha+3} + 1.$$

As the number of incident quadrilaterals increases by 1 only for the 4 vertices $\alpha, \alpha + 1, \alpha + 2, \alpha + 3$ in $D$ compared to $D'$, the claimed equalities (2) hold for all vertices. The arguments in case of even $\alpha$ can be shown analogous. \hfill \square

Proposition 10 Let $D$ be a quadrangulation of $P$ and $F_D$ the associated frieze of type $\Lambda_4$. If $T_D$ is the associated triangulation of $D$ and $F_{T_D}$ the corresponding Conway-Coxeter frieze, then $F_D$ and $F_{T_D}$ coincide in every second row.

Proof. Set $\lambda := \lambda_4$. By Lemma 9 the quiddity sequence of $T_D$ for $D \in \mathcal{D}_4(2s + 2)$ is

$$(t_0, t_1, t_2, t_3, \ldots, t_{2s}, t_{2s+1}) = (g_0, 2g_1, q_2, 2g_3, \ldots, q_{2s}, 2g_{2s+1}).$$

The proof consists of the following three steps:

1.) Calculate the row number 3 of the friezes $F_D$ and $F_{T_D}$ by the diamond rule and show that they coincide.

2.) Analyse the entries of the 2 friezes in even rows.

3.) Use 2.) to show that odd rows of the 2 friezes coincide.

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1.) By using the quiddity sequences given by \(1\) for \(p = 4\), we calculate the third row of both friezes \(F_D\) and \(F_{T_D}\). Let’s consider the following part of the frieze \(F_D\):
\[
\begin{array}{ccccccc}
\cdots & 1 & 1 & 1 & \cdots \\
\cdots & \lambda q_{2k} & \lambda q_{2k+1} & \lambda^2 q_{2k+2} & \lambda q_{2k+3} & \cdots \\
\cdots & \lambda^2 q_{2k} q_{2k+1} - 1 & \lambda^2 q_{2k+1} q_{2k+2} - 1 & \lambda^2 q_{2k+2} q_{2k+3} - 1 & \cdots \\
\end{array}
\]

The corresponding part of the frieze \(F_{T_D}\) is by Lemma \(9\) of the form
\[
\begin{array}{ccccccc}
\cdots & 1 & 1 & 1 & \cdots \\
\cdots & q_{2k} & 2q_{2k+1} & q_{2k+2} & 2q_{2k+3} & \cdots \\
\cdots & 2q_{2k} q_{2k+1} - 1 & 2q_{2k+1} q_{2k+2} - 1 & 2q_{2k+2} q_{2k+3} - 1 & \cdots \\
\end{array}
\]

and as this repeats periodically with period \(2s + 2\) and as \(\lambda^2 = 2\), the third row of the friezes coincide. The numbers are positive integers.

2.) We now want to show the connection between the rows with even numbers of the friezes \(F_D\) and \(F_{T_D}\), although the values of the former are not integers. Let’s assume that there exists a \(j\) s.t. row number \(2j\) of \(F_D\) has the entries
\[
\cdots \lambda a_0 \lambda a_1 \lambda a_2 \lambda a_3 \cdots \lambda a_{2s} \lambda a_{2s+1} \lambda a_0 \cdots (3)
\]

and that row number \(2j\) of \(F_{T_D}\) has entries
\[
\cdots a_0 2a_1 a_2 a_3 \cdots a_{2s} 2a_{2s+1} a_0 \cdots (4)
\]

with \(a_\alpha \in \mathbb{Z}_{>0}\) for \(0 \leq \alpha \leq 2s + 1\). Furthermore assume that the two friezes coincide in row number \(2j + 1\). Note that this assumption is valid for \(j = 1\) as shown above. Then for \(i \in [0, 2s + 1]\) and \(b_i \in \mathbb{Z}_{>0}\) the row number \(2j + 2\) of \(F_D\) and \(F_{T_D}\) is (shown in the third row of each pattern)
\[
\begin{array}{ccccccc}
\cdots & \lambda a_{2k-1} & \lambda a_{2k} & \lambda a_{2k+1} & \lambda a_{2k+2} & \cdots \\
\cdots & b_{2k-2} \lambda a_{2k-1} & b_{2k-1} \lambda a_{2k} & b_{2k} \lambda a_{2k+1} & b_{2k+1} \lambda a_{2k+2} & \cdots \\
\end{array}
\]

and
\[
\begin{array}{ccccccc}
\cdots & 2a_{2k-1} & a_{2k} & 2a_{2k+1} & a_{2k+2} & \cdots \\
\cdots & b_{2k-2} 2a_{2k-1} & b_{2k-1} a_{2k} & b_{2k} \lambda a_{2k+1} & b_{2k+1} \lambda a_{2k+2} & \cdots \\
\end{array}
\]

respectively. Define \(c_k = \frac{b_k b_{k+1}}{2a_{k+1}}\) for \(0 \leq k \leq 2s + 1\) (with \(b_{2s+2} := b_0\) and \(a_{2s+2} := a_0\)). Note that this definition involves a shift of the indices to the right from row number \(2j\) to row number \(2j + 2\). In other words, the diamonds are of the form \(b_k b_{k+1} - 2a_{k+1} c_k = 1\). We obtain that the row number \(2j + 2\) of \(F_D\) and \(F_{T_D}\) is
\[
\cdots \lambda c_0 \lambda c_1 \lambda c_2 \lambda c_3 \cdots \lambda c_{2s} \lambda c_{2s+1} \lambda c_0 \cdots
\]

and
\[
\cdots c_0 2c_1 c_2 2c_3 \cdots c_{2s} 2c_{2s+1} c_0 \cdots
\]

respectively. All values \(c_j\) for \(0 \leq j \leq 2s + 1\) are positive integers. This is trivial for even indices, as all values are entries of a Conway-Coxeter frieze. For odd indices, \(c_{2k-1}\) is a positive integer too by using a well known dependency of entries of a Conway-Coxeter frieze, stated in \([6] (6.6)\). In particular, \(2c_{2k-1}\) fulfills the equation
\[
2c_{2k-1} = t_{2k-1} \cdot b_{2k} - 2a_{2k+1} = 2(q_{2k-1} \cdot b_{2k} - a_{2k+1})
\]
and as $q_{2k-1}, b_{2k}$ and $a_{2k+1}$ are integers, $c_{2k-1}$ is also an integer. It is positive as being an entry of a Conway-Coxeter frieze.

3.) Now assume that there exists a $j$ such that the two friezes coincide in rows number $2j + 1$, and that row number $2j + 2$ of $F_{TD}$ and $F_D$ are of the form ([3]) and ([4]) respectively. Note that this assumption is valid for $j = 1$. For $i \in [0, 2s + 1]$ let $b_i \in \mathbb{Z}_{>0}$. Then row number $2j + 3$ of $F_D$ and $F_{TD}$ has the following entries (shown in the third row of each pattern):

$$\begin{array}{cccccc}
\cdots & \lambda a_{2k} & b_{2k+1} & \lambda a_{2k+1} & b_{2k+2} & \lambda a_{2k+2} & b_{2k+3} & \cdots \\
\cdots & \frac{\lambda^2 q_{2k} q_{2k+1} - 1}{b_{2k+1}} & \lambda a_{2k+1} & \frac{\lambda^2 q_{2k+1} q_{2k+2} - 1}{b_{2k+2}} & \lambda a_{2k+2} & \frac{\lambda^2 q_{2k+2} q_{2k+3} - 1}{b_{2k+3}} & \lambda a_{2k+3} & \cdots \\
\cdots & \frac{2q_{2k+1} q_{2k+2} - 1}{b_{2k+2}} & 2a_{2k+1} & \frac{2q_{2k+2} q_{2k+3} - 1}{b_{2k+3}} & 2a_{2k+2} & \frac{2q_{2k+3} q_{2k+4} - 1}{b_{2k+4}} & 2a_{2k+3} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{array}$$

As $\lambda^2 = 2$, row number $2j + 3$ of the frieze $F_D$ and $F_{TD}$ coincide. Furthermore, as $F_{TD}$ is an integral frieze arising from a triangulation, all the entries of row $2j + 3$ are positive integers. □

As every triangulation defines a unique Conway-Coxeter frieze, combined with the fact that all even rows are determined (see ([3]) and ([4]) Proof of Proposition [10]) proves Theorem 5 for $p = 4$.

3 Friezes of type $\Lambda_6$ and associated Conway-Coxeter frieze.

We use similar arguments as in previous section, but we have to associate different triangulations to a 6-angular dissection $D \in \mathcal{D}_6(4s + 2)$ of a polygon $P$ with $4s + 2$ vertices. Therefore we colour even vertices white and odd vertices black again. Then every diagonal of $D$ connects a black and a white vertex. We triangulate every 6-gon $P_\alpha$ by inserting edges between all pairs of black vertices. An example is shown in Figure 7, where the bold edges are diagonals of $D$. Let $q_\alpha$ be the number of 6-gons incident with a vertex $0 \leq \alpha \leq 4s + 1$ in $D$ and $t_\alpha$ be the number of triangles incident with vertex $\alpha$ as in previous section. The quiddity sequences of $D_1$ and $T_{D_1}$ are

$$(\sqrt{3}, 2\sqrt{3}, \sqrt{3}, \sqrt{3}, 3\sqrt{3}, \sqrt{3}, 2\sqrt{3}, \sqrt{3}, 3\sqrt{3}, 2\sqrt{3}, \sqrt{3}, 2\sqrt{3}, \sqrt{3}, 3\sqrt{3})$$

and

$$(1, 6, 1, 3, 1, 3, 3, 2, 3, 1, 3, 1, 6, 1, 6, 1, 3)$$

Figure 7: The triangulation $T_{D_1}$ associated to a 6-angular dissection $D_1$ (bold diagonals).
respectively. We obtain by the same arguments as in Lemma 9, that for \( T_D \) and \( 0 \leq \alpha \leq 4s+1 \),
\[
t_{\alpha} = \begin{cases} 
q_{\alpha} & \text{if } \alpha \text{ is even} \\
3q_{\alpha} & \text{otherwise.}
\end{cases}
\] (5)

Defining \( \lambda := \lambda_6 = 2 \cos \left( \frac{\pi}{6} \right) = \sqrt{3} \) and slightly changing the form of rows with even numbers \( [4] \) of the frieze \( F_{T_D} \) to
\[
\cdots \ a_0 \ 3a_1 \ a_2 \ 3a_3 \ \cdots \ a_{4s} \ 3a_{4s+1} \ a_0 \ \cdots
\] (6)
we obtain that the friezes \( F_D \) of type \( \Lambda_6 \) and the Conway-Coxeter frieze \( F_{T_D} \) of the assigned triangulation \( T_D \) coincide in every row with odd number by the same calculations as in Proposition \( 10 \) and the fact that \( \lambda^2 = 3 \). Hence Theorem 5 is fulfilled for \( p = 6 \).

4 Remarks and Outlook

Similar results for other values of \( p > 3 \) are unlikely, as the values \( \lambda_4 \) and \( \lambda_6 \) are (the only) square roots of integers in the set of all \( \lambda_p \).

This note gives a partial answer to question (i) of \([8]\). A possible connection may arise from complete exceptional sequences over the path algebra of a Dynkin quiver of type \( A_n \), studied by Araya in \([2]\) where a key ingredient are noncrossing trees.

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