DIOPHANTINE PROPERTIES OF ELEMENTS OF SO(3).

V. KALOSHIN, I. RODNIANSKI

1. Introduction

The classical result of metric number theory on Diophantine properties of numbers says the following: for any $\epsilon > 0$ and a.e. $\alpha \in \mathbb{R}$ the map $n\alpha \pmod{1}$ has a constant $C = C(\alpha) > 0$ such that $n\alpha \pmod{1} > C|n|^{-1-\epsilon}$ for every integer $n$ [Kh].

Diophantine properties of numbers arise in various problems in metric number theory [Kh], smooth dynamical systems, holomorphic dynamics [HK], KAM theory [La], and others.

Generalizations of the metric number theory led to the development of the theory of simultaneous Diophantine approximations and even Diophantine approximations on manifolds. In the latter case consider manifold $M \subset \mathbb{R}^n$ defined by $n$ analytic functions $f_1, \ldots, f_n : U \subset \mathbb{R}^d \to \mathbb{R}$, $M = \{f(x) : x \in U\}$. Assume that functions $1, f_1, \ldots, f_n$ are linearly independent over $\mathbb{R}$. One of the central questions of the theory is the following conjecture made by Sprindţuk in 1980 and recently proved by D. Kleinbock and G. Margulis [KM]:

Any manifold $M \subset \mathbb{R}^n$ of the above type is extremal, i.e., for almost all $y \in M$ and any $\epsilon > 0$ there exists a positive constant $D(y)$ such that for all $q \in \mathbb{Z}^n$ and $p \in \mathbb{Z}$

$$|q \cdot y + p| \geq \frac{D(y)}{\|q\|^{n(1+\epsilon)}}. \tag{1}$$

Here $q \cdot y = \sum_{i=1}^n q_i y_i$ and $\|q\| = \max_{1 \leq i \leq n} |q_i|$.

In fact, Kleinbock-Margulis prove even a stronger statement that $M$ is strongly extremal (approximation in the sense of (1) is replaced by the notion of multiplicative approximation). The proof is based on the correspondence between approximation properties of number $y \in \mathbb{R}^n$ and behavior of certain orbits in the space of unimodular lattices in $\mathbb{R}^{n+1}$.

The analogue of the Diophantine property can be also formulated in the non-commutative setting. As far as we know very little is known in this case. However, some intuition has already been developed for the group $SU(2)(SO(3))$. We

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say that $g_1,\ldots,g_k \in SU(2)$ are Diophantine if there exists a positive constant $d(g_1,\ldots,g_k)$ such that for $n \geq 1$ and $W_n$ a word in $g_1,\ldots,g_k$ of length $n$

$$||W_n - Id|| \geq d^{-n}.$$ 

Our interest to the problem of Diophantine approximations on the group $SO(3)$ stems mainly from the question formulated in the list of open problems in the paper of A. Gamburd, D. Jakobson, and P. Sarnak (Problem 4): *The Haar generic elements $(g_1, g_2, \ldots, g_k) \in SU(2)^k$ in the sense of measure are Diophantine* $^4$. The paper $^4$ provides an elementary solution of Ruziewicz problem asserting that the Haar measure on $S^2$ is the unique finitely additive $SO(3)$ invariant measure defined on Lebesgue sets.

In what follows it is more convenient for us to pass to the group $SO(3)$ and restrict our attention to the case of two generators. Consider a subgroup $F$ generated by two elements $A, B \in SO(3)$. The group $SO(3)$ would have a Diophantine property if for almost all rotations $A, B \in SO(3)$ in the sense of measure and all reduced words $W_n \in F$ of length $n$ in $A, B, A^{-1}, B^{-1}$,

$$(2) \quad ||W_n - Id|| \geq D(A, B)^{-n}$$

for some positive constant $D(A, B)$. The presence of the words of the form $ABA^{-1}B^{-1}$ and like indicates that $F$ has to be a free subgroup. It is a classical fact that the set of elements $A, B \in SO(3)$ which do not generate a free subgroup is a countable union of analytic sets of codimension one (see also Lemma 2 for an independent demonstration). To see this it is sufficient to establish the existence of just one free subgroup of rank two. The first explicit construction of such a subgroup was given by Hausdorff in 1914 in his work on Hausdorff-Banach-Tarski paradox. A free subgroup $F$ of rank two in $SO(3)$ allows one to construct four disjoint subsets of the sphere $S^2$ such that after rotating these subsets by elements of $F$ one obtains two copies of $S^2$ minus a countable set. Modulo the issue of the countable set it follows that there is no finitely additive measure defined on all sets of $S^2$. It also follows that any finitely additive $SO(3)$ invariant measure defined on Lebesgue sets is absolutely continuous with respect to the Lebesgue measure. The Ruziewicz problem is to show that any such measure necessarily coincides with the Lebesgue measure. In the general setting, the problem is formulated for the finitely additive $SO(n + 1)$ invariant measure on $S^n$. It is interesting to note that in dimension one Banach provided a negative solution to the Ruziewicz problem. G. Margulis $^5$ and D. Sullivan $^6$ used Kazhdan property (T) to give the positive answer in dimensions $n \geq 4$. For dimensions $n = 2, 3$ the affirmative solution had been given by V. Drinfeld $^7$.

The solution of Ruziewicz problem in dimensions $n \geq 2$ can be reduced to the problem of finding a free subgroup $F \in SO(n + 1)$ with a *spectral gap* property $^8$. Namely, consider the subspace $L_0^2(S^n) = \{f \in L^2(S^n) : \int f d\mu = 0\}$. Then
$F$ is said to have a spectral gap property if there exists a positive constant $c$ such that for any $f \in L^2(S^n)$ there exists an element $g \in F$ such that $\|f \circ g - f\| \geq c\|f\|$. After passing from $SO(3)$ to its double cover $SU(2)$ the above can also be reformulated in terms of the spectra of the irreducible representations of $SU(2)$ restricted to the element $z = g_1 + g_1^{-1} + \ldots + g_k + g_k^{-1}$. Namely, let $\pi_N$ denote the irreducible representation of $SU(2)$ realized as a linear action on the space of homogeneous polynomials in two variables of degree $N$. Define $\hat{z}(\pi_N) = \pi_N(g_1) + \pi_N(g_2^{-1}) + \ldots + \pi_N(g_k) + \pi_N(g_k^{-1})$ to be an $(N+1) \times (N+1)$ matrix. Then we say that a subgroup $F$ generated by $g_1, \ldots, g_k$ has a gap if

$$\limsup_{N \to \infty} \|\hat{z}(\pi_N)\| < \|z\|.$$ 

A. Lubotzky, R. Phillips, and P. Sarnak construct explicit examples of elements $g_1, \ldots, g_k \in SU(2)$ with $k \geq 3$ which generate a subgroup with a gap. For those generators $\|\hat{z}(\pi_N)\| \leq 2\sqrt{2k-1} < 2k$ [LPS1].

Lubotzky-Phillips-Sarnak also show that the sequence of measures $\mu_N(z)$ associated with the eigenvalue distributions of $\hat{z}(\pi_N)$ has two accumulation points as $N \to \infty$. Namely, they prove that there exist two measures $\nu^{\text{even}}(z)$ and $\nu^{\text{odd}}(z)$ such that $\mu_{2N}(z) \to \nu^{\text{even}}(z)$ and $\mu_{2N+1}(z) \to \nu^{\text{odd}}(z)$. Moreover, the rate of the convergence depends on the Diophantine properties of the generators $g_1, \ldots, g_k$ of $F$. In addition they show that a free subgroup generated by the elements $g_1, \ldots, g_k \in SU(2)$ with algebraic entries is Diophantine.

In this paper we take a first step in an attempt to understand the Diophantine properties of the group $SO(3)$. We establish that almost all pairs of rotations $(A, B) \in SO(3)$ generate subgroups that satisfy a weak Diophantine condition when the conjectured exponent $n$ in (2) is replaced by $n^2$. Although, the results below are stated for the rank two subgroups of $SO(3)$ they can be easily generalized to include $SU(2)$ and higher number of generators.

It follows from the pigeonhole principle and compactness of $SO(3)$ that an exponential estimate (not super-exponential) (2) is the optimal one since the number of words of length $n$ grows exponentially with $n$. It is an easy exercise to show that for a Baire generic (residual) set of pairs $A, B \in SO(3)$ Diophantine condition is not satisfied. Therefore, the problem about Diophantine properties of elements of $SO(3)$ is another example of a property which fails on a Baire generic set, but holds on a set of full measure. Numerous examples of this phenomena appear in dynamical systems and topology (see [O], [HSY], and [Ka]).

As we mentioned above, in this paper we obtain the first result on Diophantine properties of elements of $SO(3)$. Consider $SO(3)$ with the Haar measure $\mu$ on it. We show that for an a.e. pair $(A, B) \in SO(3) \times SO(3)$ there is a constant $D > 0$ such that for any $n$ and any word $W_n(A, B)$ of length $n$ in $A$ and $B$ we have

$$\|W_n(A, B) \pm Id\| \geq D^{-n^2}.$$ 

(3)
Let us describe the approach we use to prove the result and discuss the difficulties which arise in the attempt to get an exponential lower bound as in (2). Let $A, B \in SO(3)$ be two distinct elements, $k \in \mathbb{Z}_+$, and $W_n(A, B)$ be a word of length $n$ in $A$ and $B$. Denote by $\alpha$ and $\beta$ the angles of rotations of $A$ and $B$ respectively and by $\gamma$ the angle between the axes of $A$ and $B$. Without loss of generality we can assume that the axis of rotation of $A$, denote $v_A$, is the OX-axis in the ambient $\mathbb{R}^3$ and the axis of rotation of $B$, denote $v_B$, belongs to the $(x,y)$-plane forming angle $\gamma$ with $v_A$ in the clockwise direction. Notice any word $W_n(A, B)$ is uniquely defined by a triple $(\alpha, \beta, \gamma) \in T^3$. Denote $W_n(A, B) = W_n(\alpha, \beta, \gamma)$. Now consider the 3-dimensional torus $T^3$ as a parameter space with Lebesgue measure $m$. It is clear that a set of full product Haar measure $\mu \times \mu$ on $SO(3) \times SO(3)$ corresponds to a set of full Lebesgue measure $m$ on $T^3$.

The proof presented below is based on a standard Borel-Cantelli arguments. The rough sketch is as follows. Fix a word $W_n(\alpha, \beta, \gamma)$ of length $n$ in $A$ and $B$. The goal is to estimate the measure of the set of parameters $(\alpha, \beta, \gamma) \in T^3$ for which $W_n(\alpha, \beta, \gamma)$ is at most $D^{-n^2}$ away from Id. Let $m_n(D)$ be an upper bound for the measure of the union of these sets over all words of length $n$. By Borel-Cantelli if $\sum_n m_n(D) < \infty$, then for a.e. $(\alpha, \beta, \gamma) \in T^3$ (3) holds for all except finitely many words. Increasing $D$ we satisfy those finitely many conditions and complete the proof.

It turns out that a distance of $W_n(A, B)$ to $Id$ can be represented as a trigonometric polynomial $P_n(\alpha, \beta, \gamma)$ of degree $n$ in $\alpha, \beta,$ and $\gamma$ with integer coefficients. Fix $\beta = \beta^*$ and $\gamma = \gamma^*$ and consider measure of $\alpha$’s for which $P_n(\alpha, \beta^*, \gamma^*)$ is $D^{-n^2}$-small. If a nontrivial $P_n(\alpha, \beta^*, \gamma^*)$ with integer coefficients has a zero of order $n$ in $\alpha$ then measure $\{|\alpha|: |P_n(\alpha, \beta^*, \gamma^*)| < D^{-n^2}\}$ can be as big as $D^{-n}$. Suppose we can prove that $D^{-n}$ is an upper bound. Since, there are at most $4^n$ words $W_n(A, B)$ of length $n$ we obtain that the total “bad” measure of words of length $n$ is at most $(4/D)^n$ and is exponentially small for $D > 4$.

One can think that the polynomial $P_n(\alpha, \beta^*, \gamma^*)$ with a zero in $\alpha$ of high order corresponds to the fact that the word $W_n(A, B)$ ”sticks” in a neighborhood of $Id$ and leaves this neighborhood slowly as parameters $\alpha, \beta, \gamma$ vary. This shows that a possible presence of high order degeneracies for the polynomial representing the distance from a word $W_n(A, B)$ to $Id$ raises difficulties for estimates of measure of a set where $W_n(A, B)$ is close to $Id$. In particular, possible high degeneracies stand in the way of proving the desired optimal result (2).

In the last section we present a collection of words $W_n(A, B)$ of length $n$ for which polynomial $P_n(\alpha)$ does have a zero of order $\sqrt{n}$. This shows that it is indeed possible for a word $W_n(A, B)$ to ”stick” in a neighborhood of $Id$. This degenerate collection is constructed using commutators $[A, B] = ABA^{-1}B^{-1}$. Degeneracies of high orders for trigonometric polynomials $P_n(\alpha, \beta, \gamma)$ arising as a distance from a words $W_n(\alpha, \beta, \gamma)$ to $Id$ do occur.
2. Statement of the result

Let \( A, B \in SO(3) \) be two distinct elements and \( k \in \mathbb{Z}^+ \). Denote \( \mathcal{I}_m = (s_1, r_1, \ldots, s_m, r_m) \) a set of \( 2m \) nonzero integers, \( |\mathcal{I}_m| = \sum_p (|s_p| + |r_p|) \), and \( W_{\mathcal{I}_m}(A, B) = A^{s_1}B^{r_1} \cdots A^{s_m}B^{r_m} \). So, \( W_{\mathcal{I}_m}(A, B) \) corresponds to the word defined by the multi-index \( \mathcal{I}_m \).

**Theorem 1.** For any element \( C \in SO(3) \) and \( \mu \times \mu \)-a.e. pair \( (A, B) \in SO(3) \times SO(3) \) there is a constant \( D = D(A, B) > 0 \) such that

\[
\min_{\{\mathcal{I}_m: |\mathcal{I}_m| = n\}} \|W_{\mathcal{I}_m}(A, B) - C\| \geq D^{-n^2} \quad \text{for all } n \in \mathbb{Z}^+.
\]

In other words, for \( \mu \)-generic choice of a pair \( A \) and \( B \), all possible words of length \( n \) cannot approximate ahead given element \( C \) better than \( D^{-n^2} \). The most interesting case when \( C \) is the identity.

Reformulate (4) in a different form.

**Theorem 2.** For any element \( C \in SO(3) \) and Lebesgue a.e. \( (\alpha, \beta, \gamma) \in T^3 \) there is a constant \( D = D(\alpha, \beta, \gamma) > 0 \) such that

\[
\min_{\{\mathcal{I}_m: |\mathcal{I}_m| = n\}} \|W_{\mathcal{I}_m}(\alpha, \beta, \gamma) - C\| \geq D^{-n^2} \quad \text{for all } n \in \mathbb{Z}^+.
\]

Fix a word \( W_{\mathcal{I}_m}(\alpha, \beta, \gamma) \). The idea of the proof is to show that outside of some small measure set in \( T^3 \) size of the derivative

\[
\|W'_{\mathcal{I}_m}(\alpha, \beta, \gamma)\|_2^2 = D_{\mathcal{I}_m}(\alpha, \beta, \gamma)
\]

is not too small. When the derivative with respect to \( \alpha \) is not too small the word \( W_{\mathcal{I}_m}(\alpha, \beta, \gamma) \) varies sufficiently fast with \( \alpha \) and passes the “dangerous” \( D^{-n^2} \)-neighborhood of the rotation \( C \) sufficiently quickly. This implies smallness of the “prohibited” set in the parameter space \( (\alpha, \beta, \gamma) \).

Fix \( n \in \mathbb{Z}^+ \) and denote \( \mathcal{R}_n = \{\mathcal{I}_m: |\mathcal{I}_m| = n\} \). Define

\[
\Phi_{\mathcal{I}_m}(D, C) = \{ (\alpha, \beta, \gamma) \in T^3 : \|W_{\mathcal{I}_m}(\alpha, \beta, \gamma) - C\| \leq D^{-n^2} \}
\]

\[
\Phi_n(D, C) = \bigcup_{\mathcal{I}_m \in \mathcal{R}_n} \Phi_{\mathcal{I}_m}(D, C).
\]

If for some \( D^* > 0 \) we prove that

\[
\sum_{n=1}^{\infty} m\{\Phi_n(D^*, C)\} < \infty,
\]

then for \( m \)-a.e. \( (\alpha, \beta, \gamma) \in T^3 \) (resp. \( \mu \times \mu \)-a.e. \( (A, B) \in SO(3) \times SO(3) \)) there is \( D = D(\alpha, \beta, \gamma) \geq D^* \) (resp. \( D = D(A, B) \)) such that (5) is satisfied.

To estimate measure of \( \Phi_n(D, C) \) we need to estimate measure of \( \Phi_{\mathcal{I}_m}(D, C) \) for each word \( \mathcal{I}_m \) of length \( n \), i.e. \( |\mathcal{I}_m| = n \). Define the set of parameters, where the
derivative with respect to $\alpha$ is small

$$\Phi^\alpha_{I_m} = \{(\alpha, \beta, \gamma) \in \mathbb{T}^3 : D_{I_m}(\alpha, \beta, \gamma) \leq D^{-n^2/3}\}. \quad (9)$$

Denote $H(\mathbb{R})$ the ring of quaternions $q = x_0 + ix_1 + jx_2 + kx_3, x_p \in \mathbb{R}$. Let $\bar{q} = x_0 -(ix_1+jx_2+kx_3)$ and $N(q) = q\bar{q}$. Denote $SH(\mathbb{R}) = \{q \in H(\mathbb{R}) : N(q) = 1\}$. It is well-known that there is a representation of $SO(3)$ as $SH(\mathbb{R})$ in the following form:

$$q = \cos \alpha + \sin \alpha(iv_1+jv_2+kv_3), \quad (10)$$

where $\alpha$ is the angle of rotation and a unit vector $(v_1,v_2,v_3) \in \mathbb{R}^3$ corresponds to an axis of rotation in the ambient $\mathbb{R}^3$ of an element from $SO(3)$.

**Lemma 1.** With the above notations

$$\|W_{I_m}(\alpha, \beta, \gamma)^{''\alpha}\|^2 \leq |I_m|^4. \quad (11)$$

**Proof** This follows from the quaternion representation (10). Indeed, our choice of the ambient coordinate system gives

$$W_{I_m}(\alpha, \beta, \gamma) = (\cos s_1\alpha + i\sin s_1\alpha)(\cos r_1\beta + \sin r_1\beta(i\cos \gamma + j\sin \gamma))$$

$$\ldots (\cos s_m\alpha + i\sin s_m\alpha)(\cos r_m\beta + \sin r_m\beta(i\cos \gamma + j\sin \gamma)). \quad (12)$$

Differentiating this expression twice with respect to $\alpha$ gives

$$\|W_{I_m}(\alpha, \beta, \gamma)^{''\alpha}\|^2 \leq \left(\sum_{s=1}^{k} |s_p|\right)^4 \leq |I_m|^4. \quad (13)$$

**Lemma 2.** The map $W_{I_m} : SO(3) \times SO(3) \to SO(3)$ for a nontrivial word $I_m$ is open. This, in particular, implies that a pair of random elements of $SO(3)$ form a free group.

**Remark 1.** The conclusion of Lemma 2 is a well-known fact. In particular, the statement that almost all subgroups in $SO(3)$ are free can be reduced to simply showing that there exists a free subgroup in $SO(3)$. The latter is a classical question which was solved positively first by F. Hausdorff in 1914 [Ha]. We present here a very explicit (constructive) independent proof of Lemma 2.

**Proof** Consider representation (12). To show that a trigonometric function is nontrivial with respect to, say $\alpha$, it is sufficient to establish that the highest frequency in $\alpha$ has a nonzero functional coefficient. We shall compute this functional coefficient, namely, the coefficient in front the monomial $\exp(i \text{sign}(s_m) \sum_{p=1}^{m} |s_p| \alpha)$. Notice that

if $s > 0$ $e^{i\alpha}(\cos r\beta + i \sin r\beta \cos \gamma) = (\cos r\beta + i \sin r\beta \cos \gamma)e^{i\alpha}$

if $s < 0$ $e^{i\alpha}j \sin r\beta \sin \gamma = j \sin r\beta \sin \gamma e^{-i\alpha}$
Now we describe the procedure of permuting terms with $\alpha$ to the right and particular terms with $\beta$ and $\gamma$ to the left so that after such permutations the only term which has $\alpha$ is on the right end of the word and equals $\exp\left(i \operatorname{sign}(s_m) \sum_{p=1}^{m} |s_p| \alpha \right)$.

The first step of permutation: Consider the signs of $s_1$ and $s_2$. If they are different, then we change the sign of the $s_1$-term by choosing permutation (14), otherwise, we choose (14) in both cases with $s = s_1$ and $r = r_1$. After the permutation the first term with $\alpha$ from the left is $\exp\left(i \operatorname{sign}(s_2) \sum_{p=1}^{2} |s_p| \alpha \right)$.

The second step of permutation: Consider the signs of $s_2$ and $s_3$. Use the recipe of the first step. The permutation gives the third term $\exp\left(i \operatorname{sign}(s_3) \sum_{p=1}^{3} |s_p| \alpha \right)$ and so on. Therefore, the only term which has $\exp\left(i \operatorname{sign}(s_m) \sum_{p=1}^{m} |s_p| \alpha \right)$ equals

$$\prod_{\{p: s_p s_{p-1} > 0\}} (\cos r_p \beta + i \sin r_p \beta \cos \gamma) \times$$

$$\prod_{\{p: s_p s_{p-1} < 0\}} j \sin r_p \beta \sin \gamma \exp\left(i \operatorname{sign}(s_m) \sum_{p=1}^{m} |s_p| \alpha \right).$$

This completes the proof.

Lemma 3. Let $|I_m| = n$. Then

(14) $$m\{\Phi_{I_m}(D, C)\} \leq m\{\Phi^\alpha_{I_m}(D, C)\} + 4D^{-n^2/3}n^4.$$  

Proof In the complement to the set $\Phi^\alpha_{I_m}(D, C)$ we have estimates

(15) $$\|W_{I_m}(\alpha, \beta, \gamma)\|_\alpha' \geq D^{-n^2/3} \quad \|W_{I_m}(\alpha, \beta, \gamma)\|_{\alpha\alpha} \leq n^4.$$  

For each pair $(\beta, \gamma) \in \mathbb{T}_{2, \gamma}^3$ split the circle $\mathbb{T}_{\alpha}^1$ into $\frac{Dn^2/3}{2n}$ intervals of equal length. Choose one interval and denote it by $I$. If there is a point in $(\alpha^*, \beta, \gamma) \in I$ which belongs to the complement of $\Phi^\beta_{I_m}(D, C)$, then by the Taylor formula along with (13) for each point in $I$ we have

(16) $$\|W_{I_m}(\alpha, \beta, \gamma)\|_\alpha' \geq \frac{D^{-n^2/3}}{2}.$$  

Therefore, the Taylor formula implies that measure of $\alpha \in I$ such that

(17) $$\|W_{I_m}(\alpha, \beta, \gamma) - C\| \leq D^{-n^2}$$  

is at most $2D^{-2n^2/3}$. Collecting all segments and applying Fubini’s theorem we complete the proof. Denote

(18) $$\Phi_n^\alpha(D) = \bigcup_{I_m \in \mathcal{R}_n} \Phi_{I_m}(D).$$
Lemma 3 reduces a proof of (8) to a proof of
\[
\sum_{n=1}^{\infty} m\{\Phi_n^\alpha(D^*)\} < \infty.
\]

We prove the convergence next.

**Lemma 4.** For any word \(\mathcal{I}_m\) of length \(n\) (\(\|\mathcal{I}_m\| = n\)) there is a polynomial \(P_{\mathcal{I}_m}(x_\alpha, y_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma)\) of degree \(2n + 2m\) with integer coefficients such that
\[
\|W_{\mathcal{I}_m}(\alpha, \beta, \gamma)'\|_2 = P_{\mathcal{I}_m}(\cos \alpha, \sin \alpha, \cos \beta, \sin \beta, \cos \gamma, \sin \gamma).
\]

**Proof** Consider the quaternion representation (12) differentiate it and take the sum of squares of components. Then express \(\cos s_p \alpha\) and \(\sin s_p \alpha\) (resp. \(\cos r_p \beta\) and \(\sin r_p \beta\)) as polynomials in \(\cos \alpha\) and \(\sin \alpha\) (resp. \(\cos \beta\) and \(\sin \beta\)). This gives a polynomial \(P_{\mathcal{I}_m}(\cos \alpha, \sin \alpha, \cos \beta, \sin \beta, \cos \gamma, \sin \gamma)\) with integer coefficients since all operations are with integer-coefficient trigonometric expressions.

The main idea is that a polynomial with integer coefficients can not be small on a set of large measure. In our notations for \(|\mathcal{I}_m| = n\)
\[
\Phi_{\mathcal{I}_m}^\alpha(D) = \{(\alpha, \beta, \gamma) \in \mathbb{T}^3 : P_{\mathcal{I}_m}(\cos \alpha, \sin \alpha, \cos \beta, \sin \beta, \cos \gamma, \sin \gamma) \leq D^{-n^2/3}\}.
\]

The following result for polynomials in one variable proved in the paper of S. Dani and G. Margulis [DM]. For more general statements in this direction see also Kleinbock-Margulis [KM].

**Lemma 5.** [DM, KM] Let \(F(x)\) be a polynomial of degree \(\leq n\). Denote \(\|F\|_B := \max_{x \in B} |F(x)|\). Then for any open interval \(B\)
\[
m_1\{x \in B : |F(x)| \leq \epsilon\} \leq 2n(n + 1)^{1/2} \left(\frac{\epsilon}{\|F\|_B}\right)^{1/2} m_1\{B\}.
\]

3. **Elimination of Variables and Reduction to the 1-dimensional Case**

There are several technical difficulties that complicate matters in our setup. We need to show that a certain polynomial in several variables does not spend too much time in the neighborhood of zero. In addition, we have a trigonometric polynomial which means that some of the variables are dependent. To resolve the latter we apply the procedure known as elimination of variables described in Lemma 5 of next section. The former problem is treated with the multiple application of Lemma [DM] each time reducing the number of variables.

The polynomial in question is \(P_{\mathcal{I}_m}(\cos \alpha, \sin \alpha, \cos \beta, \sin \beta, \cos \gamma, \sin \gamma)\). We need an estimate on the size of the set \(\Phi_{\mathcal{I}_m}^\alpha(D)\), defined above. The above set has
essentially the same measure as the set
\[ \mathbb{K} := \{ (x_\alpha, x_\beta, x_\gamma) \in [-1, 1]^3 : \quad P_{I_m}(x_\alpha, y_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) - \epsilon = 0, \]
\[ y_\alpha^2 + x_\beta^2 - 1 = 0, \quad y_\beta^2 + x_\gamma^2 - 1 = 0, \quad y_\gamma^2 + x_\alpha^2 - 1 = 0, \quad \text{for some } \epsilon \leq D^{-n^2/3} \} \]

We will apply elimination of variables and Lemma [DM] three times in a row. First list properties of the polynomial \( P_{I_m}(x_\alpha, y_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) \).
• \( \deg_{x_\alpha, y_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma} P \leq 2n. \)
• \( P_{I_m}(x_\alpha, y_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) = \sum_{l=0}^{n} p_l(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma)y_\alpha^l, \)
\[ |p_l| \leq H := (2^n n)^2, \quad \forall l = 0, \ldots, n. \]
Apply Lemma [3] for the polynomials \( \sum_{l=0}^{n} p_l(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma)y_\alpha^l \) and \( y_\alpha^2 + x_\beta^2 - 1 \) with \( s = r = 2n \) and \( H = (2^n n)^2 \). From the properties of the resultant \( R_e(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) \) defined in Lemma [3] it follows that
\[ \mathbb{K} \subset \{ (x_\alpha, x_\beta, x_\gamma) \in [-1, 1]^3 : \quad R_e(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) = 0, \]
\[ y_\beta^2 + x_\beta^2 - 1 = 0, \quad y_\gamma^2 + x_\gamma^2 - 1 = 0, \quad \text{for some } \epsilon \leq D^{-n^2/3} \} \]

Using estimates (27) we conclude that
\[ \mathbb{K} \subset \{ (x_\alpha, x_\beta, x_\gamma) \in [-1, 1]^3 : \quad R(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) \leq \delta, \]
\[ y_\alpha^2 + x_\beta^2 - 1 = 0, \quad y_\gamma^2 + x_\gamma^2 - 1 = 0 \}
\[ \delta := D^{-n^2/3}(2^{2n} n H + 2^{2n} n H^2) D^{-n^2/3}. \]

Observe that \( \delta \) is of the size \( D^{-n^2} \). Fix \( (x_\beta, y_\beta, x_\gamma, y_\gamma) \) satisfying \( y_\beta^2 + x_\beta^2 - 1 = 0, y_\gamma^2 + x_\gamma^2 - 1 = 0 \) and apply Lemma [DM] to the polynomial \( R(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) \) with respect to \( x_\alpha \). Let
\[ \mathbb{K}_{\beta, \gamma} := \{ x_\alpha \in [-1, 1] : \quad (x_\alpha, x_\beta, x_\gamma) \in \mathbb{K} \}. \]
It follows that
\[ m_1(\mathbb{K}_{\beta, \gamma}) \leq 16n(8n + 1)^{\frac{1}{2n}} \left( \frac{\delta}{||R(\cdot, x_\beta, y_\beta, x_\gamma, y_\gamma)||} \right)^{\frac{1}{2n}}. \]

Note that \( m_1 \) and \( m_2 \) denote one and two-dimensional Lebesgue measures correspondingly.
Define
\[ \mathbb{K}^1 := \{ (x_\beta, x_\gamma) \in [-1, 1]^2 : \quad ||R(\cdot, x_\beta, y_\beta, x_\gamma, y_\gamma)|| \leq \delta^2, \]
\[ y_\beta^2 + x_\beta^2 - 1 = 0, \quad y_\gamma^2 + x_\gamma^2 - 1 = 0 \} \]

Define
The Fubini Theorem implies that
\[ m\{\mathbb{K}\} \leq 2m_2\{\mathbb{K}^1\} + m\left\{ \bigcup_{(x_\beta, x_\gamma) \notin \mathbb{K}^1} \mathbb{K}_{x_\beta, x_\gamma} \right\}. \]

Observe also that by the Fubini Theorem and (21) the set \( \bigcup_{(x_\beta, x_\gamma) \notin \mathbb{K}^1} \mathbb{K}_{x_\beta, x_\gamma} \) obeys the following estimate on its size:
\[ m\left\{ \bigcup_{(x_\beta, x_\gamma) \notin \mathbb{K}^1} \mathbb{K}_{x_\beta, x_\gamma} \right\} \leq 64n(8n + 1)\frac{n!}{n^n} \delta^{16n}. \]

To estimate the size of the set \( \mathbb{K}^1 \) we employ the conclusions of the second part of Lemma 3. Define \( P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma) \) from the resultant \( R(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma) \) as in (33):
\[ P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma) := (16n)! \int_{-1}^{1} |R(x_\alpha, x_\beta, y_\beta, x_\gamma, y_\gamma)|^2 dx_\alpha. \]

The constant in front of the integral is introduced so that the resulting polynomial is still a polynomial with integer coefficients. Clearly,
\[ \mathbb{K}^1 \subset \{(x_\beta, x_\gamma) \in [-1, 1]^2 : |P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma)| \leq 2(16n)!\delta, \]
\[ y_\beta^2 + x_\beta^2 - 1 = 0, \quad y_\gamma^2 + x_\gamma^2 - 1 = 0\}.

Combining estimates (27), (28), and (30) we conclude that there exist positive constants \( C_1, \rho \) such that
\[ m\{\Phi^\alpha_{I_m}(D)\} \leq 2 m_2\{(x_\beta, x_\gamma) \in [-1, 1]^2 : |P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma)| \leq D^{-\rho n^2}, \]
\[ y_\beta^2 + x_\beta^2 + 1 = 0, \quad y_\gamma^2 + x_\gamma^2 - 1 = 0\} + C_1^{-n}.

The problem is now reduced to a similar two-dimensional question. We are in position to apply another round of Lemma 6 and Lemma [DM]. Reiterate the arguments above for the polynomial \( P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma) \) with properties as described in Lemma 3:

- \( \deg_{x_\beta, y_\beta, x_\gamma, y_\gamma} P_{I_m}^1 \leq 16n. \)
- \( P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma) = \sum_{l=0}^{16n} p_{1l}(x_\beta, x_\gamma, y_\gamma) y_\beta^l, \) and
- \( \max_{x_\beta, x_\gamma, y_\beta \in [-1, 1]^3} |p_{1l}(x_\beta, x_\gamma, y_\beta)| \leq H_1 := ((16n)!34^{2n+1}(2n)^4. \)

Note that by a crude estimate for any positive \( \epsilon \) and all sufficiently large \( n, H \leq 2^{n^{1+\epsilon}}. \)

Define the resultant \( R^1(x_\beta, x_\gamma, y_\gamma) \) of the polynomials \( P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma) \) and \( y_\beta^2 + x_\beta^2 - 1. \)
We obtain
\[ m_2 \{(x_\beta, x_\gamma) \in [-1,1]^2 : |P_{I_m}^1(x_\beta, y_\beta, x_\gamma, y_\gamma)| \leq D^{-\rho_1 n^2}, \]
\[ y_\beta^2 + x_\beta^2 - 1 = 0, y_\gamma^2 + x_\gamma^2 - 1 = 0 \} \leq 2 m_1 \{|K^2| + m_2 \left\{ \bigcup_{x_\gamma \in \mathbb{K}^2} \mathbb{K}_{x_\gamma}^2 \right\}, \]
(32)
\[ \mathbb{K}^2 := \{x_\gamma \in [-1,1] : \|R^1(\cdot, x_\gamma, y_\gamma)\| \leq \delta_1^{\frac{1}{2}}, y_\gamma^2 + x_\gamma^2 - 1 = 0\}, \]
\[ \delta_1 := D^{-\rho_1 n^2} (\delta_{16n}^2 (16n H_1 + 2^{32n} (16n H_1)^2 D^{-\rho_1 n^2}), \]
\[ m_2 \left\{ \bigcup_{x_\gamma \in \mathbb{K}^2} \mathbb{K}_{x_\gamma}^2 \right\} \leq 4(16n)(4(8n)+1)\frac{\delta_1^{\frac{1}{64}}}{\delta_1^{\frac{1}{64}}}. \]

Observe that \( \delta_1 \) is still of the size \( D^{-n^2} \). Therefore, there exist positive constants \( C_2, \rho_2 \) such that
\[ m\{\Phi_{I_m}^0(D)\} \leq 2 m_1 \{|x_\gamma \in [-1,1] : |P_{I_m}^2(x_\gamma, y_\gamma)| \leq D^{-\rho_2 n^2}, \]
\[ y_\gamma^2 + x_\gamma^2 - 1 = 0\} + C_2^{-n} + C_1^{-n}, \]
where the polynomial \( P_{I_m}^2(x_\gamma, y_\gamma) \) is formed from the resultant \( R^1(x_\beta, x_\gamma, y_\gamma) \) as in (33). Finally, eliminating \( y_\gamma \) and applying Lemma [DM] we can find a positive constant \( \delta_2 \) of the size \( D^{-n^2} \) such that
\[ m\{\Phi_{I_m}^0(D)\} \leq \left( \frac{\delta_2}{\|R^2(\cdot)\|} \right)^{\frac{1}{64}} + C_2^{-n} + C_1^{-n}. \]

The resultant \( R^2(x_\gamma) \) is a polynomial with integer coefficients of degree at most 128n. Therefore, \( (256n)! \int_1^{-1} |R_2(x_\gamma)|^2 \, dx_\gamma \) is a non-negative integer. If it is positive, the desired estimate immediately follows from (34). So we need to make sure that \( R_2(x_\gamma) \) is not identically zero.

The polynomial \( R_2(x_\gamma) \) was obtained via combination of elimination of variables (forming the resultant) and integration as in (33). Certainly, integration can not produce the identically zero polynomial from a nonzero one. Therefore, we need to justify the “non-degeneracy” of elimination. The basic property of the resultant \( R[P_1, P_2](x) \) of two polynomials \( P_1(x, y), P_2(x, y) \), defined below in (32), is that \( R[P_1, P_2](x_0) \) equals 0 if and only if for some \( y \in \mathbb{C} \) we have \( P_1(x_0, y) = P_2(x_0, y) = 0 \) ([IM], p.34). In our case one of polynomials, say \( P_2 \), is \( x^2 + y^2 - 1 \). If \( R[P_1, P_2](x) \equiv 0 \), then \( x = \cos \alpha, y = \sin \alpha \), and \( P_1(\cos \alpha, \sin \alpha) \) vanishes on the open set \( \alpha \in U \subset \mathbb{R} \), which implies that it is identically zero. This is in contradiction with non-degeneracy of \( W_{I_m}(\alpha, \beta, \gamma) \) (see Lemma [2]).
4. An Auxiliary Lemma

Let \( P_1(x, y) = \sum_{l=0}^{r} p_l(x) y^l \) and \( P_2(x, y) = \sum_{l=0}^{s} p_{2l}(x) y^l \) be two polynomials in \( y \) of degree \( r \) and \( s \) correspondingly. Define the resultant

\[
R[P_1, P_2](x) := \det A
\]

of \( P_1 \) and \( P_2 \) as the determinant of the following \((r + s) \times (r + s)\) matrix

\[
\begin{pmatrix}
p_{1r}(x) & \ldots & p_{10}(x) & 0 & \ldots & \ldots & 0 \\
0 & p_{1r}(x) & \ldots & p_{10}(x) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{sr}(x) & \ldots & \ldots & 0 & \ldots & p_{s0}(x) & 0 \\
0 & p_{sr}(x) & \ldots & \ldots & p_{s0}(x) & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & p_{s0}(x) & \ldots & p_{s0}(x)
\end{pmatrix}
\]

We formulate an auxiliary lemma

**Lemma 6.** Let \( P(x, y, u, v) = \sum_{l=0}^{r} p_l(x, u, v) y^l \). Assume that the coefficients \( p_l(x, u, v) \) are polynomials of \((x, u, v)\) of degree \( \leq s \) with respect to each variable: \( \deg_{x,u,v} p_l \leq s \).

Assume also that for some constant \( H \geq 1 \) there holds the following estimates

\[
\max_{x,u,v \in [-1,1]} |p_l(x, u, v)| \leq H, \quad \forall l = 0, \ldots, r.
\]

Form a resultant \( R_\epsilon(x, u, v) \) of the polynomials \( P(x, y, u, v) - \epsilon \) and \( y^2 + x^2 - 1 \). Then

- If for some \( y \) the polynomials \( P(x, y, u, v) - \epsilon = y^2 + x^2 - 1 = 0 \), the resultant \( R_\epsilon(x, u, v) = 0 \).
- \( R_\epsilon = R + \epsilon R_1 + \epsilon^2 R_2 \), where \( R(x, u, f) \) is the resultant of \( P(x, y, u, v) \) and \( y^2 + x^2 - 1 \), and

\[
\max_{x,u,v \in [-1,1]} |R_i(x, u, v)| \leq 2^r (rH)^{2-i}, \quad i = 0, \ldots, 2.
\]

Define the following polynomial of \((u, v)\):

\[
P_1(u, v) := (4(s + r))! \int_{-1}^{1} (R(x, u, v))^2 \, dx.
\]

Then

- \( \deg_{u,v} P_1 \leq 4(s + r) \).
The polynomial $P_1(u, v)$ can be written as

\[ P_1(u, v) = \sum_{i=0}^{4(s+r)} p_{1i}(u) v^i, \]

and

\[
\max_{u \in [-1,1]} |p_{1i}(u)| \leq \frac{((4(s + r) - l)!^3}{l!} 4^{r+1}(rH)^4, \quad \forall l = 0, \ldots, 4(s + r).
\]

**Proof.** The $(r + 2) \times (r + 2)$ matrix corresponding to the resultant of the polynomials $P(x, y, u, v) - \epsilon$ and $y^2 + x^2 - 1$ has the form

\[
A := \begin{pmatrix}
1 & 0 & x^2 - 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
0 & \ldots & 0 & 1 & 0 & x^2 - 1 \\
p_r(\cdot) & \ldots & \ldots & \ldots & p_0(\cdot) - \epsilon & 0 \\
0 & p_r(\cdot) & \ldots & \ldots & \ldots & p_0(\cdot) - \epsilon
\end{pmatrix}
\]

Any solution $y$ of the system $P(x, y, u, v) - \epsilon = y^2 + x^2 - 1 = 0$ produces a nontrivial kernel containing the vector $(y^{r+1}, y^r, \ldots, 1)$ of the matrix $A$. Therefore, if for fixed $(x, u, v)$ such a $y$ exists, the resultant $R_0(x, u, v)$ vanishes.

The estimate (39) is the only nontrivial remaining statement of this lemma. Its proof is based on the application of the Markov inequality:

\[
\max_{x \in [-1,1]} |F'(x)| \leq n^2 \max_{x \in [-1,1]} |F(x)|
\]

for any polynomial $F$ of degree $n$. It easily follows from (37) and (38) that

\[
\max_{u, v \in [-1,1]} |P_1(u, v)| \leq (4(s + r))! 4^{r+1}(rH)^4.
\]

The coefficient $p_{1i}(u)$ can be found from the identity

\[
p_{1i}(u) = \frac{1}{l!} \frac{d^l}{dv^l} P_1(u, 0)
\]

Using Markov’s inequality for the polynomial $P_1(u, v)$ of degree $4(s + r)$ $l$ times we conclude that

\[
|p_{1i}(u)| \leq \frac{((4(s + r) - l)!^3}{l!} 4^{r+1}(rH)^4, \quad \forall l = 0, \ldots, 4(s + r).
\]

5. **Degenerate Words**

In this section for $n = 4^m \in \mathbb{Z}_+$ we construct $\sqrt{n}$ words $W_n(\alpha, \beta, \gamma)$ such that if $P_n(\alpha, \beta, \gamma)$ is the polynomial of distance of $W_n(\alpha, \beta, \gamma)$ to $Id$, defined above, then it has a zero of order $\sqrt{n}$ with respect to $\alpha$ at any point of the form $(0, \beta, \gamma)$.
Recall that $W_n(\alpha, \beta, \gamma) = W_n(A, B)$ is a word in $A$ and $B$, defined by the angle of rotation $\alpha$ of $A$, the angle of rotation $\beta$ of $B$ and the angle $\gamma$ between the axis of rotations of $A$ and $B$ (see the introduction). Denote by $[A, B] = ABA^{-1}B^{-1}$ the commutator formed by $A$ and $B$. The idea of the construction is the following remark: For a sufficiently small $\alpha$ the angle of rotation of the commutator $[A, B]$ is of order at most $\alpha^2$. At most $\alpha^2$ because, if axis of $A$ and $B$ are $\alpha$-close, then $[A, B]$ has an angle of rotation of order at most $\alpha^3$. This follows directly from the quaternion representation (10).

Consider two rotations $A, B \in SO(3)$. Define a map

$$\phi : \begin{pmatrix} A \\ B \end{pmatrix} \mapsto \begin{pmatrix} ABA^{-1}B^{-1} \\ BAB^{-1}A^{-1} \end{pmatrix},$$

which maps a pair of rotations into a pair of commutator rotations. Define

$$\phi : \begin{pmatrix} A_{k+1} \\ B_{k+1} \end{pmatrix} \mapsto \begin{pmatrix} A_kB_kA_k^{-1}B_k^{-1} \\ B_kA_kB_k^{-1}A_k^{-1} \end{pmatrix},$$

where $A_0 = A^{\pm 1}$ and $B_0 = B^{\pm 1}$. Notice that $A_1$ and $B_1$ are rotations by an angle of order at most $\alpha^2$ provided that $\alpha$ is sufficiently small. $A_2$ and $B_2$ are rotations by an angle of order at most $\alpha^4$, and $A_k$ and $B_k$ are rotations by an angle of order at most $\alpha^{2k}$. Since there is freedom in choosing powers of $A_k^{\pm}$ and $B_k^{\pm}$ in the definition of $\phi$ it is easy to see that this construction gives at least $2^k$ words of kind $A_k^{\pm}$ and $B_k^{\pm}$. Note that $A_k^{\pm}$ and $B_k^{\pm}$ are words of length $4^k$.

Let $\rho$ be the golden mean. It is not too difficult to see that after choosing $A^{\pm 1}$ and $B^{\pm 1}$ in an appropriate way inside of the commutators one can construct a word $W_n(A, B)$ with a zero of order $n^{(\rho+1)/2} = 2^{(\rho+1)k}$ at the point $(\alpha, \beta, \gamma) = (0, \beta, \gamma)$.

All degenerations described here occur in a neighborhood of zero. It is an interesting question whether there are zeroes of high order far away from the identity element in $SO(3)$.

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