Distant Perturbations of the Laplacian in a Multi-Dimensional Space

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Abstract. We consider the Laplacian in $\mathbb{R}^n$ perturbed by a finite number of distant perturbations that are abstract localized operators. We study the asymptotic behaviour of the discrete spectrum as the distances between perturbations tend to infinity. The main results are a convergence theorem and the asymptotic expansions for the eigenvalues. Some examples of the possible distant perturbations are given; they may be a multiplication operator, second order differential operator, magnetic Schrödinger operator, integral operator, and $\delta$-potential.

1. Introduction

Spectra of self-adjoint operators with distant perturbations exhibit various interesting features and such operators were studied quite intensively. Much attention was paid to a multiple well Shrödinger operator in the case the wells were separated by a large distance (see, for instance, [6, 8, 9, 11], [4, Sect. 8.6]). The similar problem for the Dirac operator was treated in [7]. The main result of the cited works was the description of the asymptotic behaviour of the isolated eigenvalues as the distances between wells tend to infinity. Recently new problems with more complicated distant perturbations have been considered. S. Kondej and I. Veselić studied a $\delta$-potential supported by a curve which consisted of a several components [12]. In the case these components are separated by a large distance their results imply an asymptotic estimate for the lowest spectral gap. The problems with distant perturbations were considered also for the waveguides. In [2] the Dirichlet Laplacian in a planar strip was studied, and the distant perturbations were two segments of the same length on the boundary on which the boundary condition switched to the Neumann one. The asymptotic expansions for the isolated eigenvalues were constructed as the distance between Neumann segments tends to infinity. These results were generalized in [3] where we studied the Dirichlet Laplacian in a domain formed by two adjacent strips of arbitrary width coupled by two windows.
These windows were segments cut out from the common boundary of the strips. The technique employed in [3] followed the general ideas of the paper [1] where we considered the Dirichlet Laplacian in an infinite multi-dimensional tube perturbed by two distant perturbations. The perturbations were two abstract localized operators. The asymptotic expansions for the eigenvalues and the associated eigenfunctions were constructed.

In the present paper we consider the Laplacian in \( \mathbb{R}^n \), \( n \geq 1 \), perturbed by several distant perturbations. The number of the perturbations is finite but arbitrary, and each perturbation is an abstract localized operator. The restrictions for these operators are quite weak, and the results of this paper are applicable to a wide class of distant perturbations of various nature (see Section 8).

In the paper we construct the asymptotic expansions for the isolated eigenvalues and the associated eigenfunctions of the problem considered. The technique we develop is a generalization of the approach employed in [1]. Such generalization is needed since the tube considered in [1] was infinite in one dimension only. This is not the case for a multi-dimensional space considered here. The main additional ingredient we use is the technique borrowed from [13, Ch. XIV, Sect. 4]. Our approach allows us actually to reduce the original perturbed operator to a small regular perturbation of the direct sum of the limiting operators each of them is the Laplacian with one of the original perturbations. Due to this fact we believe that this approach can be employed not only for the asymptotical purposes, but also in studying other properties of the problems with distant perturbations.

The structure of the paper is as follows. In the next section we formulate the problem and present the main results. The third section is devoted to the study of the essential spectrum; we also prove the finiteness of the discrete spectrum. In the fourth section we employ the technique from [13, Ch. XIV, Sect. 4] and transform the equation for the resolvent of the both limiting and perturbed operators to a certain operator equation. We employ it in the fifth section to obtain an equation for the eigenelements of the perturbed operator. We solve this equation explicitly using the slight modification of the Birman–Schwinger approach suggested in [5]. This allows us to prove the main results in the sixth section. In the seventh section we apply the general results to certain important particular cases. The eighth section is devoted to certain examples of the distant perturbation to which the general results of this paper can be applied.

2. Problem and main results

Let \( x = (x_1, \ldots, x_n) \) be the Cartesian coordinates in \( \mathbb{R}^n \), \( n \geq 1 \). Given a bounded domain \( Q \subset \mathbb{R}^n \), by \( L_2(\mathbb{R}^n, Q) \) we denote the subset of the functions from \( L_2(\mathbb{R}^n) \) whose support lies inside \( \overline{Q} \).

Let \( \Omega_i \subset \mathbb{R}^n \), \( i = 1, \ldots, m \), be bounded non-empty domains with infinitely differentiable boundary. By \( \mathcal{L}_i : W_2^2(\Omega_i) \to L_2(\mathbb{R}^n, \Omega_i) \), \( i = 1, \ldots, m \), we indicate
linear bounded operators satisfying the relations
\[
(L_i u_1, u_2)_{L^2(\Omega_i)} = (u_1, L_i u_2)_{L^2(\Omega_i)},
\]
for all \( u_1, u_2 \in W^2_2(\Omega_i) \), where \( c_0, c_1 \) are constants independent of \( u, u_1, u_2 \), and \( c_0 < 1 \).

Since each \( u \in W^2_2(\mathbb{R}^n) \) belongs to \( W^2_2(\Omega_i) \), we can regard \( W^2_2(\mathbb{R}^n) \) as a subset of \( W^2_2(\Omega_i) \). Due to such embedding we can define the operators \( L_i \) on the space \( W^2_2(\mathbb{R}^n) \), and consider them as unbounded operators in \( L^2(\mathbb{R}^n) \).

We introduce the shift operator in \( L^2(\mathbb{R}^n) \) as \( S(a)u := u(\cdot + a) \), where \( a \in \mathbb{R}^n \).

Let \( X_i, i = 1, \ldots, m \), be points in \( \mathbb{R}^n \), and denote \( X := (X_1, \ldots, X_m) \), \( l_{i,j} := |X_i - X_j| \). We set
\[
L_X := \sum_{i=1}^m S(-X_i)L_i S(X_i).
\]
This operator is defined on \( W^2_2(\Omega_X) \), \( \Omega_X := \bigcup_{i=1}^m (\Omega_i + \{X_i\}) \), \( \Omega_i + \{X_i\} := \{x : x - X_i \in \Omega_i\} \), and maps this space into \( L^2_2(\mathbb{R}^n, \Omega_X) \). In what follows we assume that the distances between \( X_i \) increase unboundedly, i.e., \( l_{i,j} \to +\infty \), \( i \neq j \). Hence the distances between the domains \( \Omega_i + \{X_i\} \) tend to infinity, and the operator \( L_X \) can be naturally treated as the distant perturbation formed by the operators \( L_i \), \( i = 1, \ldots, m \). We can also consider \( L_X \) as an unbounded one in \( L^2_2(\mathbb{R}^n) \) having \( W^2_2(\mathbb{R}^n) \) as the domain.

The main object of our study is the operator \( \mathcal{H}_X := -\Delta + L_X \) in \( L^2(\mathbb{R}^n) \) with the domain \( W^2_2(\mathbb{R}^n) \). Here \( -\Delta \) denotes the Laplacian in \( L^2(\mathbb{R}^n) \) with the domain \( W^2_2(\mathbb{R}^n) \). Our main aim is to study the behaviour of the spectrum of \( \mathcal{H}_X \) as \( l_{i,j} \to +\infty \).

Let \( \mathcal{H}_i := -\Delta + L_i \) be the operators in \( L^2(\mathbb{R}^n) \) having \( W^2_2(\mathbb{R}^n) \) as the domain. Throughout the paper we assume that \( \mathcal{H}_i \) and \( \mathcal{H}_X \) are self-adjoint.

**Remark 2.1.** We stress that the conditions (2.1)–(2.3) do not imply that \( \mathcal{H}_i \) and \( \mathcal{H}_X \) are self-adjoint, and we can not apply neither Kato–Rellich theorem nor KLMN theorem. At the same time, if we replace the condition (2.2) by a stricter assumption that all \( L_i \) are \( -\Delta \)-bounded with the bound less than one, it will yield the self-adjointness of \( \mathcal{H}_i \) and \( \mathcal{H}_X \) by Kato–Rellich theorem.

By \( \sigma(\cdot), \sigma_{\text{ess}}(\cdot), \sigma_{\text{disc}}(\cdot) \) we denote the spectrum, the essential and the discrete spectrum of an operator. Our first result is

**Theorem 2.1.** The essential spectra of \( \mathcal{H}_i, \mathcal{H}_X \) coincide with the semi-axis \([0, +\infty)\).
The discrete spectra of these operator consist of finitely many negative eigenvalues.
The total multiplicity of the isolated eigenvalues of \( \mathcal{H}_X \) is bounded uniformly in \( l_{i,j} \) provided these distances are large enough.

We denote \( \sigma_* := \bigcup_{i=1}^m \sigma_{\text{disc}}(\mathcal{H}_i) \). We say that \( \lambda_* \in \sigma_* \) is \((p_1 + \cdots + p_m)\)-multiple, if it is a \( p_i \)-multiple eigenvalue of \( \mathcal{H}_i, i = 1, \ldots, m \). The relation \( p_i = 0 \) corresponds to the case \( \lambda_* \) is not in the spectrum of \( \mathcal{H}_i \). Let \( l_X := \min_{i \neq j} l_{i,j} \).
Theorem 2.2. Each isolated eigenvalue of $H_X$ converges to zero or to $\lambda_* \in \sigma_*$ as $l_X \to +\infty$. If $\lambda_* \in \sigma_*$ is $(p_1 + \cdots + p_m)$-multiple, the total multiplicity of the eigenvalues of $H_X$ converging to $\lambda_*$ equals $p_1 + \cdots + p_m$.

Theorem 2.3. Let $\lambda_* \in \sigma_*$ be $(p_1 + \cdots + p_m)$-multiple, and let $\lambda_i = \lambda_i(X) \xrightarrow{l_X \to +\infty} \lambda_*$, $i = 1, \ldots, p, p := p_1 + \cdots + p_m$, be the eigenvalues of $H_X$ taken counting multiplicity and ordered as follows:

$$0 \leq |\lambda_1(X) - \lambda_*| \leq |\lambda_2(X) - \lambda_*| \leq \cdots \leq |\lambda_p(X) - \lambda_*|.$$ 

These eigenvalues solve the equation

$$\det \left( (\lambda - \lambda_*) E - A(\lambda, X) \right) = 0,$$

where $E$ is the identity matrix, and satisfy the asymptotic formulas:

$$\lambda_i(X) = \lambda_* + \tau_i(X) \left( 1 + \mathcal{O} \left( \frac{n-3}{2} e^{-l_X \sqrt{-\lambda_*}} \right) \right), \quad l_X \to +\infty.$$

The matrix $A$ is defined by (5.19), (5.16), (5.13) in terms of auxiliary operators and functions introduced in (4.3), (4.5), (4.13), (5.6), (5.8), (5.10). The quantities $\tau_i = \tau_i(X) = \mathcal{O} \left( \frac{n-1}{2} e^{-l_X \sqrt{-\lambda_*}} \right), \quad l_X \to +\infty,$

are the zeroes of the polynomial $\det(\tau E - A(\lambda_*, X))$ taken counting multiplicity and ordered as follows:

$$0 \leq |\tau_1(X)| \leq |\tau_2(X)| \leq \cdots \leq |\tau_p(X)|.$$ 

The eigenfunctions associated with $\lambda_i$ obey the asymptotic representation

$$\psi_i = \sum_{j=1}^{m} S(-X_j) \sum_{q=1}^{p_j} \kappa_{\alpha_j+q}^{(i)} \psi_{j,q} + \mathcal{O} \left( \frac{n-3}{2} e^{-l_X \sqrt{-\lambda_*}} \right), \quad l_X \to +\infty,$$

$$\alpha_1 := 0, \quad \alpha_j := p_1 + \cdots + p_{j-1},$$

in $W^2_2(\mathbb{R}^n)$-norm. Here $\psi_{i,j}, j = 1, \ldots, p_i$, are the eigenfunctions of $H_i$ associated with $\lambda_*$ and orthonormalized in $L_2(\mathbb{R}^n)$. The numbers $\kappa_j^{(i)}$ are the components of the vectors

$$\kappa_i = \kappa_i(X) = \begin{pmatrix} \kappa_1^{(i)}(X) \\ \vdots \\ \kappa_p^{(i)}(X) \end{pmatrix},$$

which are the solutions to the system (5.18) for $\lambda = \lambda_i(X)$ and satisfy the condition

$$(\kappa_i, \kappa_j)_C = \begin{cases} 1, & i = j, \\ \mathcal{O} \left( \frac{n-3}{2} e^{-l_X \sqrt{-\lambda_*}} \right), & i \neq j. \end{cases}$$
As it is stated in the theorem, the leading terms of the asymptotic expansions for the eigenvalues $\lambda_i$ are determined by the matrix $A(\lambda_*, X)$. At the same time it could be a difficult problem to calculate this matrix and its eigenvalues explicitly. In the following theorems we show how to calculate the asymptotic expansions for $\lambda_i$ in a more explicit form.

We will say that a square matrix $A(X)$ satisfies the condition $(A)$, if it is diagonalizable and the determinant of the matrix formed by the normalized eigenvectors of $A(X)$ is separated from zero uniformly in $l_{i,j}$ large enough.

**Theorem 2.4.** Let the hypothesis of Theorem 2.3 hold true, and suppose that the matrix $A(\lambda_*, X)$ can be represented as

$$A(\lambda_*, X) = A_0(X) + A_1(X),$$  \hspace{1cm} (2.7)

where $A_0$ satisfies the condition $(A)$ and $\|A_1(X)\| \to 0$ as $l_X \to +\infty$. Then the eigenvalues $\lambda_i$ of $\mathcal{H}_X$ obey the asymptotic formulas

$$\lambda_i = \lambda_* + \tau^{(0)}_i(X) + \mathcal{O}(\|A_1(l)\|),$$  \hspace{1cm} (2.8)

as $l_X \to +\infty$. Here $\tau^{(0)}_i(X)$ are the roots of the polynomial $\det(\tau E - A_0(X))$ taken counting multiplicity and ordered as follows:

$$0 \leq |\tau^{(0)}_1(X)| \leq |\tau^{(0)}_2(X)| \leq \cdots \leq |\tau^{(0)}_p(X)|.$$  \hspace{1cm} (2.9)

Each of these roots satisfies the estimate

$$\tau^{(0)}_i(X) = \mathcal{O}(\|A_0(X)\|), \quad l_X \to +\infty.$$

According to this theorem, the leading terms of the asymptotics for $\lambda_i$ are determined by the first order term of the asymptotics for $A(\lambda_*, X)$. In other words, one does not need to know the matrix $A$ explicitly, but only its "principal" part. We also note that the estimate for the error term in (2.8) can be worse than one in (2.5).

Some of the eigenvalues of the matrix $A_0$ in Theorem 2.5 can be identically zero for large $l_{i,j}$. In this case the leading terms in (2.10) vanish. If it occurs, one should employ next-to-leading terms of the asymptotic expansion for $A(\lambda_*, X)$ and treat them as a part of $A_0$ in (2.7). Such an expansion for $A(\lambda_*, X)$ can be obtained by the technique employed in the seventh section (see the proof of Theorem 2.6). We do not provide such results in the paper in order not to overload the text by quite technical and bulky calculations. Instead of this, in the following theorems we show how one can employ effectively even first term in the asymptotics for $A(\lambda_*, X)$.

We denote $X_{i,j} := X_i - X_j$.

**Theorem 2.5.** Let the hypothesis of Theorem 2.3 holds true. Then the eigenvalues $\lambda_i$ satisfy the asymptotic formulas

$$\lambda_i(X) = \lambda_* + \tau^{(0)}_i(X) + \mathcal{O}(l^{-n+2}_X e^{-2l_X \sqrt{-\lambda_*}}), \quad l_X \to +\infty.$$  \hspace{1cm} (2.10)
Here $\tau_i^{(0)}$ are the roots of the polynomial $\det(\tau E - \Lambda_0)$ taken counting multiplicity and ordered in accordance with (2.9), and the hermitian matrix $\Lambda_0$ reads as follows:

$$A_i^{(0)}(X) := (L_k S(X_{k,r}) \psi_{r,s}, \psi_{k,q})_{L^2(\Omega_k)} , \quad \text{if} \quad k \neq r ,$$

$$A_i^{(0)}(X) := 0 , \quad \text{if} \quad k = r ,$$

where $k = 1, \ldots, m$, $q = 1, \ldots, p_k$, $i = \alpha_k + q$, $r = 1, \ldots, m$, $s = 1, \ldots, p_r$, $i = \alpha_r + s$. The estimates

$$\tau_i^{(0)} = O \left( l^{-n - \frac{1}{2}} e^{-l \sqrt{-\lambda^*}} \right) , \quad l_X \to +\infty ,$$

are valid.

This theorem is applicable in the general case, and it says that the leading terms of the eigenvalues asymptotics can be determined by the matrix $\Lambda_0$ introduced in (2.11). This matrix has the block structure. Each block is defined explicitly and describes the interaction between the eigenfunctions of the operators $H_r$ and $H_k$ via the perturbation $L_k$.

Given arbitrary $L_i$ and an eigenvalue of $H_i$, this eigenvalue is generally speaking simple. Once we deal with several operators $H_i$, their eigenvalues do not necessarily coincide. In other words, the usual situation is that a number $\lambda^* \in \sigma^*$ is a simple eigenvalue of one of the operators $H_i$ only. This particular but important case is addressed in the next theorem.

**Theorem 2.6.** Let $\lambda^* \in \sigma^*$ be $(1 + 0 + \cdots + 0)$-multiple, and $\psi_1$ be the associated eigenfunction of $H_1$ normalized in $L^2(\mathbb{R}^n)$. Then the asymptotic expansion for the eigenvalue $\lambda(X) \xrightarrow{l_X \to +\infty} \lambda^*$ of $H_X$ is

$$\lambda(X) = l_* - \sum_{j=2}^{m} (L_j S(X_{j,1})(H_j - \lambda^*)^{-1} L_j S(X_{j,1})\psi_1, \psi_1)_{L^2(\Omega_1)}$$

$$+ O \left( l^{-n-\frac{5}{2}} e^{-3l \sqrt{-\lambda^*}} \right)$$

as $l_X \to +\infty$. The associated eigenfunction $\psi$ satisfy the asymptotic representation

$$\psi(x, X) = \psi_1(x - X_1) + O \left( l^{-\frac{3}{2}} e^{-l \sqrt{-\lambda^*}} \right) , \quad l_X \to +\infty ,$$

in $W^2_2(\mathbb{R}^n)$-norm.

**Remark 2.2.** In this theorem the operators $(H_j - \lambda^*)$, $j = 2, \ldots, m$, are boundedly invertible since $\lambda^* \not\in \sigma(H_j)$.

In view of said before Theorem 2.6, the number $\lambda^* \in \sigma^*$ being an eigenvalue of more than one operators $H_i$ is a quite exceptional situation unless some kind of symmetry is assumed. The most possible among these exceptional cases is $\lambda^*$ being an eigenvalue of two of the operators $H_i$. This case is a subject of
Theorem 2.7. Let $\lambda_* \in \sigma_*$ be $(1 + 1 + 0 + \cdots + 0)$-multiple, and $\psi_i$, $i = 1, 2$ be the associated eigenfunctions of $H_i$ normalized in $L_2(\mathbb{R}^n)$. Then the asymptotic expansions for the eigenvalues $\lambda_i$, $i = 1, 2$, are

$$\lambda_1 = \lambda_* - \Big( L_1 S(X_{1,2}) \psi_1, \psi_1 \Big)_{L_2(\Omega_1)} + O \left( l_*^{-n+2} e^{-2l_* \sqrt{-\lambda_*}} \right),$$

$$\lambda_2 = \lambda_* + \Big( L_1 S(X_{1,2}) \psi_2, \psi_1 \Big)_{L_2(\Omega_1)} + O \left( l_*^{-n+2} e^{-2l_* \sqrt{-\lambda_*}} \right),$$

as $l_\chi \to +\infty$.

The asymptotic expansions stated in the last theorem are very similar to ones for a double-well Schrödinger operator with symmetric wells (see, for instance, [8, Th. 2.8]). At the same time, in our case the number of distant perturbations is arbitrary and no symmetry is assumed. It allows us to conclude that the symmetry in the asymptotics for $\lambda_1$ and $\lambda_2$ is not due to the symmetry of the perturbation, but is the general situation. We also stress that it holds true even for distant perturbations of different nature.

In the article we consider only the eigenvalues of $H_X$ converging to the isolated eigenvalues of $H_i$. At the same time, in accordance with Theorem 2.2, there can be also the eigenvalues converging to the threshold of the essential spectrum. Description of the behavior of such eigenvalues is an interesting but complicated problem, and we leave it to another paper. In some particular case this problem was solved successfully, see [2,9,11].

3. Proof of Theorem 2.1

Let $\Omega \subset \mathbb{R}^n$ be a bounded non-empty domain, and $\mathcal{L} : W^2_2(\Omega) \to L_2(\mathbb{R}^n, \Omega)$ be an operator satisfying the relations

$$\left( \mathcal{L} u_1, u_2 \right)_{L_2(\Omega)} = \left( u_1, \mathcal{L} u_2 \right)_{L_2(\Omega)},$$

$$\left( \mathcal{L} u, u \right) \geq -c_0 \| \nabla u \|_{L_2(\Omega)}^2 - c_1 \| u \|_{L_2(\Omega)}^2$$

for all $u, u_1, u_2 \in W^2_2(\Omega)$, where $c_0, c_1$ are constants, and $c_0$ obeys (2.3). We introduce the operator $\mathcal{H}_\mathcal{L} := -\Delta_{\mathbb{R}^n} + \mathcal{L}$ in $L_2(\mathbb{R})$ with the domain $W^2_2(\mathbb{R})$, and assume that it is self-adjoint.

Lemma 3.1. $\sigma_{\text{ess}}(\mathcal{H}_\mathcal{L}) = [0, +\infty)$.

Proof. We will employ Weyl criterion to prove the lemma. Let $\lambda \in [0, +\infty)$. By $\chi = \chi(t)$ we denote an infinitely differentiable function cut-off function being one as $r < 0$ and vanishing as $r > 1$. We introduce the sequence of the functions

$$u_p(x) := c_p |x|^{-n/2+1} J_{n/2-1}(\sqrt{\lambda} |x|) \chi(|x| - p) \in W^2_2(\mathbb{R}^n),$$

where $J_{n/2-1}$ is the Bessel function of the first kind of order $n/2-1$. Inserting $u_p$ into the left-hand side of (3.1) and taking into account that $\mathcal{L}$ is a lower bound, we get

$$\left( \mathcal{L} u_p, u_p \right)_{L_2(\Omega)} \geq -c_0 \| \nabla u_p \|_{L_2(\Omega)}^2 - c_1 \| u_p \|_{L_2(\Omega)}^2 \geq -c_0 \| u_p \|_{L_2(\Omega)}^2,$$
where \( J_{n/2-1} \) is the Bessel function of \( (n/2 - 1) \)-th order. The coefficients \( c_p \) are specified by the normalization condition \( \| u_p \|_{L^2(\mathbb{R}^n)} = 1 \). Since

\[
|x|^{-n+2} J_{n/2-1}^2 (\sqrt{\lambda} |x|) = \frac{2|x|^{-n+1}}{\pi \sqrt{\lambda}} \left( \cos^2 \left( \frac{\sqrt{\lambda} |x| - (n-3)\pi}{4} \right) + O(|x|^{-1}) \right),
\]
as \( |x| \to +\infty \), it follows that \( c_p \to 0 \). Using this fact it is easy to check that

\[
\| Lu_p \|_{L^2(\mathbb{R}^n)} \to 0, \quad \| H_L u_p \|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad p \to +\infty.
\]

Therefore, \( u_p \) is a singular sequence for \( H_L \) at \( \lambda \) and \( [0, +\infty) \subseteq \sigma_{\text{ess}}(H_L) \). The opposite inclusion can be shown completely by analogy with how the same was established in the proof of Lemma 2.1 in [1].

**Lemma 3.2.** The discrete spectrum of \( H_L \) consists of finitely many negative eigenvalues.

The proof of this lemma is the same as that of Lemma 2.2 in [1].

We apply now Lemmas 3.1, 3.2 with \( L = L_i, \Omega = \Omega_i, i = 1, \ldots, m \) and arrive at the statement of the theorem on \( H_i \). It also follows from Lemmas 3.1, 3.2 with \( L = L_X, \Omega := \Omega_X \), that the essential spectrum of \( H_X \) coincides with \( [0, +\infty) \) and the discrete spectrum consists of finitely many eigenvalues. It remains to check that the total multiplicity of these eigenvalues is independent of \( l_{i,j} \) provided these distances are large enough. Completely how the same was established in the proof of Lemma 2.2 in [1], one can check that

\[
H_X \geq H_X^{(0)} \oplus H_X^{(1)},
\]

where \( H_X^{(1)} \) is the negative Neumann Laplacian on \( \mathbb{R}^n \setminus \Omega_X \), while \( H_X^{(0)} \) denotes the operator

\[
- \text{div} \left( 1 - c_0 \sum_{i=1}^m \chi( |x-X_i| - \varepsilon ) \right) \nabla - c_1 \sum_{i=1}^m \chi( |x-X_i| - \varepsilon )
\]
in \( \Omega_X \) subject to Neumann boundary condition. Here \( \varepsilon \) is such that \( \Omega_i \subseteq \{ x : |x| < \varepsilon \} \), and the lengths \( l_{i,j} \) are supposed to be large enough so that supports of \( \chi( |x-X_i| - \varepsilon ) \) do not intersect for different \( i \). It is clear that \( H_X^{(0)} \) is unitary equivalent to the sum \( \bigoplus_{i=1}^m H_{X_i}^{(0)} \), where \( H_{X_i}^{(0)} \) is the operator

\[
- \text{div} \left( 1 - c_0 \chi( |x-X_i| - \varepsilon ) \right) \nabla - c_1 \chi( |x-X_i| - \varepsilon )
\]
in \( \{ x : |x| < \varepsilon \} \) subject to Neumann boundary condition. This sum is independent of \( l_{i,j} \) and has a finite number of negative isolated eigenvalues. By the minimax principle and (3.2) these eigenvalues give the lower bounds for the negative eigenvalues of \( H_X \). It implies that total multiplicity of the negative eigenvalues of \( H_X \) is bounded uniformly in \( l_{i,j} \) provided these quantities are large enough.
4. Reduction to an operator equation

In this section we collect some preliminaries which will be employed in the proof of Theorems 2.2–2.4.

Let \( \mathcal{L} \) and \( \mathcal{H}_\mathcal{L} \) be the operators introduced in the previous section. For any \( \varepsilon > 0 \) by \( \mathbb{S}_\varepsilon \) we indicate the set of complex numbers separated from the half-line \([0, +\infty)\) by a distance greater than \( \varepsilon \). We also assume that \( \varepsilon \) is chosen so that \( \sigma_{\text{disc}}(\mathcal{H}) \subset \mathbb{S}_\varepsilon \).

Consider the equation

\[
(\mathcal{H}_\mathcal{L} - \lambda)u = f, \tag{4.1}
\]

where \( f \in L_2(\mathbb{R}^n, \Omega^\beta) \), \( \Omega^\beta := \{ x \in \mathbb{R}^n : \text{dist}(\Omega, x) < \beta \} \), \( \beta > 0 \), \( \lambda \in \mathbb{S}_\varepsilon \). We remind that \( L_2(\mathbb{R}^n, \Omega^\beta) \) is the subset of the functions in \( L_2(\mathbb{R}^n) \) having supports in \( \overline{\Omega}^\beta \).

We are going to reduce the last equation to an operator equation in \( L_2(\Omega^\beta) \). In order to do it, we will employ the general scheme borrowed from [13, Ch. XIV, Sect. 4].

Let \( g \in L_2(\mathbb{R}^n, \Omega^\beta) \) be a function. We introduce \( v := (-\Delta_{\mathbb{R}^n} - \lambda)^{-1}g \). The function \( v \) can be represented as

\[
v(x, \lambda) := \int_{\Omega^\beta} G_n(||x - y||, \lambda)g(y) \, dy, \tag{4.2}
\]

where \( G_n(t, \lambda) := -\frac{1}{2\pi^{n+1}} t^{-\frac{n}{2}+1} H^{(1)}_{\frac{n}{2}-1}(it\sqrt{-\lambda}) \),

where \( H^{(1)}_{\frac{n}{2}-1} \) is the Hankel function of the first kind and \( (n/2 - 1) \)-th order. The branches of the roots are specified by the requirements \( \Re \sqrt{-\lambda} > 0 \), \( \Re \sqrt{-\lambda} > 0 \), \( \Im \sqrt{-\lambda} > 0 \) as \( \lambda \in \mathbb{S}_\varepsilon \).

We denote by \( \mathcal{H}_\Omega \) the operator \(-\Delta + \mathcal{L}\) in \( L_2(\Omega^\beta) \) with domain \( W^2_0(\Omega^\beta) \). Here \( W^2_0(\Omega^\beta) \) consists of the functions in \( W^2(\Omega^\beta) \) vanishing on \( \partial \Omega^\beta \). The operator \( \mathcal{H}_\Omega \) is symmetric (see (3.1)), and the operator \((\mathcal{H}_\Omega - i)^{-1}\) is therefore well-defined and is bounded as an operator in \( L_2(\Omega^\beta) \). Moreover, \( \mathcal{H}_\Omega \) is bounded as an operator from \( W^2_0(\Omega^\beta) \) into \( L_2(\Omega^\beta) \). By Banach theorem on inverse operator two last facts imply that the operator \((\mathcal{H}_\Omega - i)^{-1} : L_2(\Omega^\beta) \rightarrow W^2_0(\Omega^\beta)\) is bounded. Using this operator, we define one more function \( w := -(\mathcal{H}_\Omega - i)^{-1}Lv \).

By \( \chi_{\Omega} = \chi_{\Omega}(x) \) we indicate infinitely differentiable cut-off function being one in \( \Omega^{\beta/2} \) and vanishing outside \( \Omega^\beta \). We construct the solution to (4.1) as

\[
u(x, \lambda) = T_1(\lambda)g := v(x, \lambda) + \chi_{\Omega}(x)w(x, \lambda). \tag{4.3}
\]

This function is obviously an element of \( W^2_0(\mathbb{R}^n) \). Now we apply the operator \((\mathcal{H}_\mathcal{L} - \lambda)\) to this function:

\[
(\mathcal{H}_\mathcal{L} - \lambda)u = g + \mathcal{L}v + (-\Delta - \lambda + \mathcal{L})\chi_{\Omega}w = g + T_2(\lambda)g, \tag{4.4}
\]

\[
T_2(\lambda)g := -2\nabla \chi_{\Omega} \cdot \nabla w - w(\Delta + \lambda - i)\chi_{\Omega}. \tag{4.5}
\]
Here we have also used the identities $L \chi_\Omega w = L w = \chi_\Omega L w$. Thus, (4.1) holds true, if

$$g + T_2(\lambda)g = f.$$  \hspace{1cm} (4.6)

**Lemma 4.1.** The operators $T_1(\lambda) : L^2(\Omega^3) \to W^2_2(\mathbb{R}^n)$ and $T_2(\lambda) : L^2(\Omega^3) \to L^2(\Omega^3)$ are bounded and holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. For each solution of (4.6) the function $u$ defined by (4.3) solves (4.1). And vice versa, for each solution $u$ of (4.1) there exists unique solution $g$ of (4.6) satisfying the relation $u = T_1(\lambda)g$. The equivalence holds true for all $\lambda \in \mathbb{S}_\varepsilon$.

**Proof.** The operator $(-\Delta_{\Omega^3} - \lambda)^{-1} : L^2(\mathbb{R}^n, \Omega^3) \to W^2_2(\mathbb{R}^n)$ is bounded and holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$ that can be established by analogy with the proof of Lemma 3.1 in [1]. Since $(H_{\Omega} - i)^{-1}L : W^2_2(\Omega^3) \to W^2_{2,0}(\Omega^3)$ is a bounded operator, we conclude that the mapping $g \mapsto w$ is a bounded operator from $L^2(\mathbb{R}^n, \Omega^3)$ into $W^2_{2,0}(\Omega^3)$ being holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. Thus, the operator $T_1(\lambda) : L^2(\Omega^3) \to W^2_{2,0}(\Omega^3)$ is bounded and holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. This fact and the definition of $T_2$ imply that this operator is bounded and holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$ as an operator in $L^2(\Omega^3)$.

Let $g$ solve (4.6); as it was shown above in this case the function $u$ defined by (4.3) is a solution to (4.1). Suppose now that $u$ solves (4.1). By direct calculations one can check that the corresponding $v$, $w$ and $g$ are given by the formulas

$$w := (\Delta_{\Omega^3} + i)^{-1}L u, \quad v := u - \chi_\Omega w, \quad g = T_1^{-1}(\lambda)u := (-\Delta - \lambda)v,$$  \hspace{1cm} (4.7)

where $-\Delta_{\Omega^3}$ denotes the Dirichlet Laplacian in $\Omega^3$.

**Lemma 4.2.** The operator $(I + T_2)^{-1}$ is bounded and meromorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. The poles of this operator are simple and coincide with the isolated eigenvalues of $H_{\mathcal{L}}$. For $\lambda$ close to a $p$-multiple eigenvalue $\lambda_*$ of $H_{\mathcal{L}}$ the representation

$$(I + T_2(\lambda))^{-1} = -\sum_{i=1}^{p} \frac{\phi_i(\cdot, \psi_i)_{L^2(\Omega^3)}}{\lambda - \lambda_*} + T_3(\lambda)$$  \hspace{1cm} (4.8)

holds true. Here $\psi_i$ are the eigenfunctions associated with $\lambda_*$ and orthonormalized in $L^2(\mathbb{R}^n)$, $\phi_i := T_1^{-1}(\lambda_*)\psi_i$, and the operator $T_3 : L^2(\Omega^3) \to L^2(\Omega^3)$ is bounded and holomorphic w.r.t. $\lambda$ close to $\lambda_*$ as an operator in $L^2(\Omega^3)$. The equation (4.6) with $\lambda = \lambda_*$ is solvable, if and only if

$$(f, \psi_i)_{L^2(\Omega^3)} = 0, \quad i = 1, \ldots, p,$$  \hspace{1cm} (4.9)

and the solution reads as follows

$$g = T_3(\lambda_*)f + \sum_{j=1}^{m} c_j \phi_j,$$  \hspace{1cm} (4.10)

where $c_i$ are arbitrary constants.
Proof. It follows from (4.3), (4.4) that \((\mathcal{H}_\lambda - \lambda) T_1(\lambda) = I + T_2(\lambda)\). Therefore,

\[(\mathcal{H}_\lambda - \lambda)^{-1} = T_1(\lambda)(I + T_2(\lambda))^{-1}, \quad (I + T_2(\lambda))^{-1} = T_1^{-1}(\lambda)(\mathcal{H}_\lambda - \lambda)^{-1}, \quad (4.11)\]

where the operator \(T_1^{-1}(\lambda)\) is defined by the formulas (4.7). By analogy with the proof of Lemma 3.1 in [1] one can show that the operator \((\mathcal{H}_\lambda - \lambda)^{-1} : L_2(\mathbb{R}^n, \Omega^3) \to W_2^2(\mathbb{R}^n)\) is meromorphic w.r.t. \(\lambda \in \mathbb{R}\), its poles coincide with the isolated eigenvalues of \(\mathcal{H}_\lambda\), and for \(\lambda\) close to \(\lambda_*\) the representation

\[(\mathcal{H}_\lambda - \lambda)^{-1} = -\sum_{i=1}^p \frac{\psi_i(\lambda, \lambda_t^\prime)_{\lambda_t}}{\lambda - \lambda_*} + T_4(\lambda) \quad (4.12)\]

holds true, where the operator \(T_4(\lambda) : L_2(\mathbb{R}^n) \to W_2^2(\mathbb{R}^n)\) is bounded and holomorphic w.r.t. \(\lambda\) close to \(\lambda_*\). Hence, in view of (4.11), (4.12), and (4.7), the operator \((I + T_2)^{-1}\) is meromorphic w.r.t. \(\lambda \in \mathbb{S}\), its poles of this operator are simple and coincide with the isolated eigenvalues of \(\mathcal{H}_\lambda\), and the representation (4.8) holds true. As it also follows from (4.12), (4.1) with \(\lambda = \lambda_*\) is solvable, if and only if the relations (4.9) are valid, and the solution of (4.1) with \(\lambda = \lambda_*\) is given by the formula \(u = T_4(\lambda_*) f + \sum_{j=1}^m c_j \psi_j\), where \(c_j\) are arbitrary constants. Employing now Lemma 4.1, we conclude that the relations (4.9) are the solvability conditions for (4.7) with \(\lambda = \lambda_*\). Thus, the solution of this equation is defined uniquely up to a linear combination of the functions \(\phi_i\), \(i = 1, \ldots, m\). The formula (4.10) is valid, since for the functions \(f\) satisfying (4.9) the identity \((I + T_2(\lambda_*) T_5(\lambda_*) f = f)\) holds true due to (4.8).

Let \(\tilde{\Omega} \subset \mathbb{R}^n\) be a bounded domain with infinitely differentiable boundary, and \(\tilde{X} \in \mathbb{R}^n\) be a point. Suppose that \(l := |\tilde{X}|\) is a large parameter. We define the operator \(T_5(\lambda, \tilde{X}) : L_2(\mathbb{R}^n, \Omega^3) \to W_2^2(\tilde{\Omega})\) as

\[T_5(\lambda, \tilde{X}) := S(\tilde{X})(-\Delta_{\mathbb{R}^n} - \lambda)^{-1}. \quad (4.13)\]

**Lemma 4.3.** The operator \(T_5\) is bounded and holomorphic w.r.t. \(\lambda \in \mathbb{S}\). For any compact set \(K \subset \mathbb{S}\) the estimates

\[
\left\| \frac{\partial^l T_5}{\partial \lambda_i^l} \right\| \leq C l^{-n-2i+1} e^{-t\sqrt{\lambda}}, \quad i = 0, 1, \quad (4.14)
\]

hold true, where the constant \(C\) is independent of \(\tilde{X}\) and \(\lambda \in K\).

**Proof.** As it was said in the proof of Lemma 4.2, the operator \((-\Delta_{\mathbb{R}^n} - \lambda)^{-1} : L_2(\mathbb{R}^n, \Omega^3) \to W_2^2(\mathbb{R}^n)\) is bounded and holomorphic w.r.t. \(\lambda \in \mathbb{S}\). Therefore, the same is true for the operator \(T_5\). The estimates (4.14) follow from the asymptotics

\[G_n(t, \lambda) = -\frac{(\sqrt{\lambda} - n)^n}{2(n+1)2(\pi(n-1)/2)^{n-1}} t^{-(n-1)/2} e^{-t\sqrt{\lambda}} \left(1 + O(|\lambda|^{-1/2} t^{-1})\right), \quad (4.15)\]

as \(t \to +\infty, \lambda \in \mathbb{S}\); this formula can be differentiated w.r.t. \(\lambda\). \(\square\)
5. Equation for the eigenvalues of $\mathcal{H}_X$

In this section we will obtain the equation for the eigenvalues and the eigenfunctions of the operator $\mathcal{H}_X$ and will solve this equation explicitly.

By $\mathcal{T}^{(i)}_j$, $\mathcal{T}^{(X)}_j$, we denote the operators $\mathcal{T}_j$ from the previous section corresponding to $L = L_i$, $\Omega = \Omega_i$, and $L = L_X$, $\Omega = \Omega_X$. Let us study the structure of the operator $\mathcal{T}^{(X)}_j$ in more details.

Given $g \in L^2(\Omega^2_X)$, due to (4.2) we have

$$v_X(x, \lambda) = \int_{\Omega^2_X} G_n(|x-t|, \lambda) g(t) d\tau = \sum_{i=1}^{m} \int_{\Omega^2_i+\{x_i\}} G_n(|x-t|, \lambda) g(t) d\tau$$

$$= \sum_{i=1}^{m} \int_{\Omega^2_i} G_n(|x-X_i-t|, \lambda) g_i(t) d\tau = \sum_{i=1}^{m} (\mathcal{S}(X_i)v_i)(x, \lambda),$$

$$g_i(t) := g(X_i + t), \quad v_i(x) := \int_{\Omega^2_i} G_n(|x-t|, \lambda) g_i(t) d\tau.$$

Now we apply the operator $\mathcal{L}_X$ to $v_X$ and obtain:

$$\mathcal{L}_X v_X = \sum_{i=1}^{m} \mathcal{S}(-X_i) \mathcal{L}_i \left( v_i + \sum_{j=1}^{m} \mathcal{S}(X_{i,j}) v_j \right)$$

$$= \sum_{i=1}^{m} \mathcal{S}(-X_i) \mathcal{L}_i (v_i + \tilde{v}_i),$$

$$\tilde{v}_i := \sum_{j=1}^{m} \mathcal{S}(X_{i,j}) v_j = \sum_{j=1}^{m} \mathcal{T}_5(\lambda, X_{i,j}) g_j.$$

We introduce the functions

$$w_i := -(\mathcal{H}_{\Omega_i} - i)^{-1} \mathcal{L}_i v_i, \quad \tilde{w}_i := -(\mathcal{H}_{\Omega_i} - i)^{-1} \mathcal{L}_i \tilde{v}_i,$$

$$w_X := W_X + \overline{W}_X, \quad W_X := \sum_{i=1}^{m} \mathcal{S}(-X_i) w_i, \quad \overline{W}_X := \sum_{i=1}^{m} \mathcal{S}(-X_i) \tilde{w}_i.$$

It is obvious that $w_X, W_X, \overline{W}_X \in W^2_{2,0}(\Omega^2_X)$. Since

$$\mathcal{L}_X w_X = \sum_{i=1}^{m} \mathcal{S}(-X_i) \mathcal{L}_i (w_i + \tilde{w}_i),$$
we obtain
\[
(H_{\Omega X} - i)w_X = \sum_{i=1}^{m} \left( - (\Delta + i)S(-X_i) + S(-X_i)\mathcal{L}_i \right)(w_i + \tilde{w}_i)
\]
\[
= - \sum_{i=1}^{m} S(-X_i)\mathcal{L}_i (v_i + \tilde{v}_i) = - \mathcal{L}_X v_X ,
\]
\[
w_X = -(H_{\Omega X} - i)^{-1} \mathcal{L}_X v_X .
\]

We define the cut-off function \( \chi_{\Omega X} := \sum_{i=1}^{m} S(-X_i) \chi_{\Omega_i} \), where the function \( \chi_{\Omega_i} \) corresponds to the operator \( T_1^{(i)} \). In this case the operator \( T_1^{(X)} \) reads as follows:
\[
T_1^{(X)} g = \sum_{i=1}^{m} S(-X_i) v_i + \sum_{i=1}^{m} S(-X_i) \chi_{\Omega_i}(w_i + \tilde{w}_i)
\]
\[
= \sum_{i=1}^{m} S(-X_i)(v_i + \chi_{\Omega_i} w_i + \chi_{\Omega_i} \tilde{w}_i)
\]
\[
= \sum_{i=1}^{m} S(-X_i) (T_1^{(i)} g_i + \chi_{\Omega_i} \tilde{w}_i) .
\]

Therefore,
\[
T_2^{(X)}(\lambda, X) g = \sum_{i=1}^{m} S(-X_i) T_2^{(i)} (\lambda) g_i
\]
\[
+ \sum_{i=1}^{m} S(-X_i) \sum_{j=1}^{m} T_6^{(i,j)}(\lambda) g_j ,
\]
\[
T_6^{(i,j)}(\lambda) := \left( 2\nabla \chi_{\Omega_i} \cdot \nabla + (\Delta \chi_{\Omega_i} + (\lambda - i) \chi_{\Omega_i}) \right) \left( H_{\Omega_i} - i \right)^{-1} \mathcal{L}_i T_5(\lambda, X_{i,j}) .
\]

**Lemma 5.1.** The operators \( T_6^{(i,j)} : L_2(\mathbb{R}^n, \Omega_j^2) \rightarrow L_2(\Omega_i^2) \) are bounded and holomorphic w.r.t. \( \lambda \in \mathbb{S}_\epsilon \). The relation
\[
T_6^{(i,j)}(\lambda) = \mathcal{L}_i T_5(\lambda, X_{i,j}) + (\Delta - \mathcal{L}_i + \lambda) \chi_{\Omega_i} (H_{\Omega_i} - i)^{-1} \mathcal{L}_i T_5(\lambda, X_{i,j})
\]
is valid. For each compact set \( K \subset \mathbb{S}_\epsilon \) the estimates
\[
\left\| \frac{\partial^k T_6^{(i,j)}}{\partial \lambda^k} \right\| \leq C l_{i,j}^{-2k-1} e^{-l_{i,j} \sqrt{-\lambda}} , \quad k = 0, 1 ,
\]
hold true, where the constant \( C \) is independent of \( l_{i,j} \) and \( \lambda \in K \).

The statement of the lemma follows from the definition of \( T_6^{(i,j)} \) and Lemma 4.3.

According to Lemma 4.1, the eigenvalues of \( H_X \) are numbers for which (4.6) with \( T_2 = T_2^{(X)} \) and \( f = 0 \) has a nontrivial solution. Let \( g_X \) be a solution to this
equation. Since \( g_X = \sum_j i = 1^m S(-X_i)g_i \), due to (5.3) we conclude that (4.6) for \( g_X \) can be rewritten as

\[
\sum_{i=1}^m S(-X_i) \left( g_i + \mathcal{T}_2^{(i)}(\lambda)g_i + \sum_{j=1, j \neq i}^m \mathcal{T}_6^{(i,j)}(\lambda)g_j \right) = 0.
\]

Each term in this equation has a compact support and these supports do not intersect, if \( l_{i,j} \) are large enough. Thus, the equation obtained is equivalent to

\[
g_i + \mathcal{T}_2^{(i)}(\lambda)g_i + \sum_{j=1, j \neq i}^m \mathcal{T}_6^{(i,j)}(\lambda)g_j = 0, \quad i = 1, \ldots, m. \tag{5.5}
\]

We introduce two operators in the space \( L := \bigoplus_{i=1}^m L_2(\mathbb{R}^n, \Omega_i^\beta) \),

\[
\mathcal{T}_7(\lambda)g := (\mathcal{T}_2^{(1)}(\lambda)g_1, \ldots, \mathcal{T}_2^{(m)}(\lambda)g_m),
\]

\[
\mathcal{T}_8(\lambda, X)g := \left( \sum_{j=1, j \neq i}^m \mathcal{T}_6^{(i,j)}(\lambda)g_j, \ldots, \sum_{j=1, j \neq m}^m \mathcal{T}_6^{(m,j)}(\lambda)g_j \right), \tag{5.6}
\]

where \( g := (g_1, \ldots, g_m) \in L \). Employing these operators, we can rewrite (5.5) as

\[
g + \mathcal{T}_7(\lambda)g + \mathcal{T}_8(\lambda, X)g = 0. \tag{5.7}
\]

Let \( \lambda_s \in \sigma_* \) be \((p_1 + \cdots + p_m)\)-multiple, and \( \psi_{i,j}, i = 1, \ldots, m, j = 1, \ldots, p_i \), be the associated eigenfunctions of \( H_i \) orthonormalized in \( L_2(\mathbb{R}^n) \). We denote

\[
p := p_1 + \cdots + p_m,
\]

\[
\phi_{\alpha_1+j} := (\phi_{1,j}, 0, \ldots, 0) \in L, \quad j = 1, \ldots, p_1,
\]

\[
\phi_{\alpha_1+j} := (g_1, \phi_{1,j})_{L_2(\Omega_i^\beta)}, \quad j = 1, \ldots, p_1,
\]

\[
\phi_{\alpha_2+j} := (0, \phi_{2,j}, 0, \ldots, 0) \in L, \quad j = 1, \ldots, p_2,
\]

\[
\phi_{\alpha_2+j} := (g_2, \phi_{2,j})_{L_2(\Omega_i^\beta)}, \quad j = 1, \ldots, p_2,
\]

\[
\cdots
\]

\[
\phi_{\alpha_m+j} := (0, \ldots, 0, \phi_{m,j}) \in L, \quad j = 1, \ldots, p_m,
\]

\[
\phi_{\alpha_m+j} := (g_m, \phi_{m,j})_{L_2(\Omega_i^\beta)}, \quad j = 1, \ldots, p_m.
\]

Here \( \phi_{i,j} := (T_i(\lambda_s))^{-1} \psi_{i,j} \). Lemmas 4.2, 5.1 yield

**Lemma 5.2.** The operator \( \mathcal{T}_8 \) is bounded and holomorphic w.r.t. \( \lambda \in \mathbb{S}_x \). For each compact set \( K \subset \mathbb{S}_x \) the uniform in \( \lambda \in K \) and large \( l_{i,j} \) estimates

\[
\left\| \frac{\partial^i T_8}{\partial \lambda^i} \right\| \leq C l_x \frac{-\alpha_{i-1}}{2} e^{-l_x \sqrt{\alpha}} , \quad i = 0, 1, \tag{5.9}
\]

are valid. The operator \( \mathcal{T}_7 \) is bounded and meromorphic w.r.t. \( \lambda \in \mathbb{S}_x \). The set of its poles coincide with \( \sigma_* \). For any \( \lambda \) close to \((p_1 + \cdots + p_m)\)-multiple \( \lambda_s \in \sigma_* \) the
representation

\[(I + T_7(\lambda))^{-1} = -\sum_{i=1}^{p} \frac{\phi_i T_9^{(i)}}{\lambda - \lambda_*} + T_{10}(\lambda), \quad (5.10)\]

holds true, where the j-th component of the vector \(T_{10}(\lambda)g\) is \(T_9^{(j)}g_j\), if \(p_j \neq 0\) and \((I + T_9^{(j)}(\lambda))^{-1}g_j\), if \(p_j = 0\). The operator \(T_{10} : L \rightarrow L\) is bounded and holomorphic w.r.t. \(\lambda\) close to \(\lambda_*\). The equation \((I + T_7(\lambda_*))g = f\) is solvable, if and only if \(T_9^{(i)}f = 0\), \(i = 1, \ldots, m\). The solution of this equation is given by \(g = T_{10}(\lambda_*)f + \sum_{i=1}^{p} c_i \phi_i\), where \(c_i\) are constants.

**Lemma 5.3.** Each isolated eigenvalue of \(\mathcal{H}_X\) converges to zero or to \(\lambda_* \in \sigma_*\) as \(l_X \rightarrow +\infty\).

**Proof.** Using (2.2), (2.3), for each \(u \in W^2_1(\mathbb{R}^n)\) we obtain

\[(\mathcal{H}_X u, u)_{L^2(\mathbb{R}^n)} \geq \|\nabla u\|^2_{L^2(\mathbb{R}^n)} - c_0 \|\nabla u\|^2_{L^2(\Omega_X)} - c_1 \|u\|^2_{L^2(\Omega_X)} \geq -c_1 \|u\|^2_{L^2(\mathbb{R}^n)},\]

which implies that \(\sigma_{\text{disc}}(\mathcal{H}_X) \subset [-c_1, 0]\). We define \(\mathbb{K}_\varepsilon := [-c_1, -\varepsilon) \setminus \bigcup_{\lambda \in \sigma_*} (\lambda - \varepsilon, \lambda + \varepsilon)\). This set obeys the hypothesis of Lemma 4.2, and due to (5.9) the norm of \(T_8\) is exponentially small as \(\lambda \in \mathbb{K}_\varepsilon\) and \(l_X \rightarrow +\infty\). In accordance with Lemma 5.2, the operator \(I + T_7(\lambda)\) is boundedly invertible as \(\lambda \in \mathbb{K}_\varepsilon\). Therefore, the operator \(I + T_7(\lambda) + T_8(\lambda, X)\) is boundedly invertible as \(\lambda \in \mathbb{K}_\varepsilon\), if \(l_X\) is large enough. Thus, (5.7) has no nontrivial solution as \(\lambda \in \mathbb{K}_\varepsilon\), if \(l_X\) is large enough, and by Lemma 4.1 we conclude that the set \(\mathbb{K}_\varepsilon\) contains no eigenvalues of \(\mathcal{H}_X\), if \(l_X\) is large enough. The number \(\varepsilon\) being arbitrary completes the proof.

Let \(\lambda_* \in \sigma_*\) be \((p_1 + \cdots + p_m)\)-multiple; we are going to find non-trivial solutions of (5.7) for \(\lambda\) close to \(\lambda_*\).

Assume first that \(\lambda \neq \lambda_*\). We apply the operator \((I + T_7)^{-1}\) to this equation and substitute the representation (5.10) into the relation obtained. This procedure yields

\[g - \sum_{i=1}^{p} \frac{\phi_i T_9^{(i)} T_8(\lambda, X)g}{\lambda - \lambda_*} + T_{10}(\lambda) T_8(\lambda, X)g = 0. \quad (5.11)\]

In view of (5.9) the operator \(T_{10}(\lambda) T_8(\lambda, X)\) is small, if \(l_X\) is large enough. Thus, the operator \((I + T_7(\lambda) T_8(\lambda, X))^{-1}\) is well-defined and bounded. We apply now this operator to (5.11) and arrive at

\[g - \sum_{i=1}^{p} \frac{T_9^{(i)} T_8(\lambda, X)g}{\lambda - \lambda_*} \Phi_i = 0, \quad (5.12)\]

\[\Phi_i(\cdot, \lambda, X) := (I + T_7(\lambda) T_8(\lambda, X))^{-1} \phi_i. \quad (5.13)\]

Hence,

\[g = \sum_{i=1}^{p} \kappa_i \Phi_i, \quad (5.14)\]
where $\kappa_i$ are numbers to be found. We substitute now this identity into (5.12) and obtain

$$
\sum_{i=1}^{p} \Phi_i \left( \kappa_i - \sum_{j=1}^{p} A_{ij} \kappa_j \right) = 0,
$$

(5.15)

where

$$
A_{ij} = T_9^{(i)}(\lambda, X) \Phi_j(\cdot, \lambda, X).
$$

(5.16)

The estimates (5.9) imply that

$$
\Phi_i = \phi_i + O\left(\frac{1}{l_X} e^{-l_X \sqrt{-\lambda}}\right),
$$

(5.17)

Since the vectors $\phi_i$ are linear independent, due to the last relations the same is true for $\Phi_i$. Thus, (5.15) is equivalent to the system of linear equations

$$
(\lambda - \lambda_*) E - A(\lambda, X) \kappa = 0,
$$

(5.18)

$$
\kappa := \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_p \end{pmatrix},
\Lambda(\lambda, X) := \begin{pmatrix} A_{11}(\lambda, X) & \cdots & A_{1p}(\lambda, X) \\ \vdots & \ddots & \vdots \\ A_{p1}(\lambda, X) & \cdots & A_{pp}(\lambda, X) \end{pmatrix}.
$$

(5.19)

The corresponding solution of (5.7) is given by (5.14). Since the vectors $\Phi_i$ are linear independent, this solution is non-zero, if and only if $\kappa \neq 0$. The criterion of the existence of nontrivial solution to (5.18) is (2.4). Therefore, the number $\lambda \neq \lambda_*$ converging to $\lambda_*$ as $l_X \to +\infty$ is an eigenvalue the operator $\mathcal{H}_X$, if and only if it is a root of (2.4). The multiplicity of this eigenvalue equals to the number of linear independent solutions of the corresponding system (5.18). Let us prove that the same is true, if $\lambda = \lambda_*$. Consider (5.7) with $\lambda = \lambda_*$. If we treat $T_9(\lambda_*, X) g$ as a right-hand side, according to Lemma 5.2 this equation is solvable, if and only if $\kappa \neq 0$. The criterion of the existence of nontrivial solution to (5.18) is (2.4). Therefore, the number $\lambda \neq \lambda_*$ converging to $\lambda_*$ as $l_X \to +\infty$ is an eigenvalue the operator $\mathcal{H}_X$, if and only if it is a root of (2.4). The multiplicity of this eigenvalue equals to the number of linear independent solutions of the corresponding system (5.18). Let us prove that the same is true, if $\lambda = \lambda_*$. Consider (5.7) with $\lambda = \lambda_*$. If we treat $T_9(\lambda_*, X) g$ as a right-hand side, according to Lemma 5.2 this equation is solvable, if and only if $\kappa \neq 0$. The criterion of the existence of nontrivial solution to (5.18) is (2.4). Therefore, the number $\lambda \neq \lambda_*$ converging to $\lambda_*$ as $l_X \to +\infty$ is an eigenvalue the operator $\mathcal{H}_X$, if and only if it is a root of (2.4). The multiplicity of this eigenvalue equals to the number of linear independent solutions of the corresponding system (5.18).
6. Proof of Theorems 2.2–2.4

In view of Lemmas 5.3, 5.4 we will complete the proof of Theorem 2.2, if we prove that total number of non-trivial solutions to (5.18) associated with the roots of (2.4) equals $p$.

Throughout this section we assume that $\lambda_* \in \sigma_*$ is $(p_1 + \cdots + p_m)$-multiple and $\lambda$ belongs to a small neighbourhood of $\lambda_*$. We denote $B(\lambda, X) := (\lambda - \lambda_*) E - A(\lambda, X), \ F(\lambda, X) := \det B(\lambda, X)$.

Lemma 6.1. In the vicinity of $\lambda_*$ the function $F(\lambda, X)$ has exactly $p$ zeroes counting their orders. These zeroes converge to $\lambda_*$ exponentially fast as $l_X \to +\infty$.

Proof. The definition of the functions $A_{i,j}$ and Lemma 5.2 imply that these functions are holomorphic w.r.t. $\lambda$ and satisfy the estimates

$$|\frac{\partial^k A_{i,j}}{\partial \lambda^k}(\lambda, t)| \leq C l_X^{-\frac{n-k}{2}} e^{-l_X \sqrt{-\lambda}}, \quad k = 0, 1.$$  \hspace{1cm} (6.1)

It is clear that

$$F(\lambda, X) = (\lambda - \lambda_*)^p + \sum_{i=0}^{p-1} P_i(\lambda, X)(\lambda - \lambda_*)^i, $$

where the functions $P_i$ are holomorphic w.r.t. $\lambda$ and obey the uniform in $\lambda$ and $X$ estimate

$$|P_i(\lambda, X)| \leq C l_X^{-\frac{p-i}{2} - \frac{n-1}{2}} e^{-l_X \sqrt{-\lambda}}.$$  \hspace{1cm} (6.2)

For a sufficiently small fixed $\varepsilon > 0$ this estimate yields

$$\sum_{i=0}^{p-1} P_i(\lambda, X)(\lambda - \lambda_*)^i < |\lambda - \lambda_*|^p \quad \text{as} \quad |\lambda - \lambda_*| = \varepsilon,$$

if $l_X$ is large enough. Hence, by Rouche theorem the function $F(\lambda, X)$ has the same number of zeroes (counting orders) inside the disk $\{\lambda : |\lambda - \lambda_*| < \varepsilon\}$ as the function $\lambda \mapsto (\lambda - \lambda_*)^p$ does. The number $\varepsilon$ being arbitrary proves that all the zeroes converge to $\lambda_*$. The estimates for $P_i$ imply that $(\lambda - \lambda_*)^p = O(l_X^{-\frac{n-1}{2}} e^{-l_X \sqrt{-\lambda}})$ and therefore the zeroes converge to $\lambda_*$ exponentially fast. \hfill $\square$

Lemma 6.2. Suppose that $\lambda_1(X)$ and $\lambda_2(X)$ are different roots of (2.4), and $\kappa_1(X)$ and $\kappa_2(X)$ are the associated non-trivial solutions to (5.18) normalized by the condition

$$\|\kappa_i\|_{C^p} = 1.$$  \hspace{1cm} (6.3)

Then

$$(\kappa_1, \kappa_2)_{C^p} = O(l_X^{-\frac{n+1}{2}} e^{-l_X \sqrt{-\lambda_*}}), \quad l_X \to +\infty.$$
Proof. We indicate by \( g_j \) the solutions of (5.7) associated with \( \lambda_j \); these solutions are given by (5.14). Due to Lemma 4.3 the functions \( \tilde{v}_i \) and \( \tilde{w}_i \) corresponding to each of the vectors \( g_j \) satisfy the estimates

\[
\|L_i \tilde{v}_i\|_{L^2(\Omega^s)} = O\left(\left(\frac{n+1}{2}\right)e^{-tx\sqrt{-\lambda_i}}\right),
\]

\[
\|\tilde{w}_i\|_{W^2_2(\Omega^s)} = O\left(\left(\frac{n+1}{2}\right)e^{-tx\sqrt{-\lambda_i}}\right),
\]

as \( l_X \to +\infty \). Moreover, it follows from (5.17) that

\[
g_i = \sum_{j=1}^{p} \kappa^{(i)}_{j} \phi_{j} + O\left(\left(\frac{n+1}{2}\right)e^{-tx\sqrt{-\lambda_i}}\right),
\]

where \( \kappa^{(i)}_{j} \) are the components of the vectors \( \kappa_{i} \). In view of the relations obtained and (5.1), (5.2), (5.18), (6.1) we infer that the eigenfunctions \( \psi_{1}(x, X) \) associated with \( \lambda_{i} \) satisfy the asymptotic formulas:

\[
\psi_{i} = \sum_{j=1}^{m} S(-X_{j}) \sum_{q=1}^{p_j} \kappa^{(i)}_{j+q} \psi_{j,q} + O\left(\left(\frac{n+1}{2}\right)e^{-tx\sqrt{-\lambda_i}}\right),
\]

where, we remind, \( \psi_{i,j}, j = 1, \ldots, p_{i} \), are the eigenfunctions of \( \mathcal{H}_{i} \) associated with \( \lambda_{i} \) and orthonormalized in \( L^2(\mathbb{R}^{n}) \). Since the operator \( \mathcal{H}_{X} \) is self-adjoint, the eigenfunctions \( \psi_{i} \) are orthogonal in \( L^2(\mathbb{R}^{n}) \). Together with the established asymptotic representations for \( \psi_{i} \) it implies

\[
0 = (\psi_{1}, \psi_{2})_{L^2(\mathbb{R}^{n})} = \sum_{j=1}^{m} \sum_{q=1}^{p_j} \sum_{r=1}^{p_{i}} \kappa^{(1)}_{j+q} \kappa^{(2)}_{i+r} \left(S(-X_{j})\psi_{j,q}, S(-X_{i})\psi_{i,r}\right)_{L^2(\mathbb{R}^{n})}
\]

\[
+ O\left(\left(\frac{n+1}{2}\right)e^{-tx\sqrt{-\max\{\lambda_{i}, \lambda_{2}\}}\right).}
\]

It is clear that

\[
\left(S(-X_{j})\psi_{j,q}, S(-X_{j})\psi_{j,r}\right)_{L^2(\mathbb{R}^{n})} = \left(\psi_{j,q}, \psi_{j,r}\right)_{L^2(\mathbb{R}^{n})} = \begin{cases} 1, & q = r, \\ 0, & q \neq r, \end{cases}
\]

and for \( i \neq j \)

\[
\left(S(-X_{j})\psi_{j,q}, S(-X_{i})\psi_{i,r}\right)_{L^2(\mathbb{R}^{n})} = \left(S(X_{i,j})\psi_{j,q}, \psi_{i,r}\right)_{L^2(\mathbb{R}^{n})}
\]

\[
= \left(S(X_{i,j})\psi_{j,q}, \psi_{i,r}\right)_{L^2(\Omega^{i,j})}
\]

\[
+ \left(S(X_{i,j})\psi_{j,q}, \psi_{i,r}\right)_{L^2(\mathbb{R}^{n}\setminus\Omega^{i,j})}
\]

\[
= O\left(\left(\frac{n+1}{2}\right)e^{-l_{i,j}\sqrt{-\lambda_i}}\right),
\]
Here we have used that due to (4.15)
\[ \psi_{i,j} = C_{i,j} |x|^{-(n-1)/2} e^{-|x|\sqrt{-\lambda}} \left( 1 + \mathcal{O}(|x|^{-1}) \right), \quad |x| \to +\infty, \]
where \( C_{i,j} \) are constants. Substituting the obtained relations into (6.2), and taking into account Lemma 6.1 we arrive at the statement of the lemma. \( \square \)

Let \( \lambda(X) \xrightarrow{t_X \to +\infty} \lambda_* \) be a root of (2.4). Without loss of generality we assume that the corresponding solutions of (5.18) are orthonormalized in \( \mathbb{C}^p \). Consider the set of all such solutions to (5.18) associated with all roots of (2.4) converging to \( \lambda_* \) as \( t_X \to +\infty \), and denote these vectors as \( \kappa_i = \kappa_i(X), i = 1, \ldots, q \). In view of Lemma 6.2 the vectors \( \kappa_i \) satisfy (2.6).

**Lemma 6.3.** Let \( \lambda(X) \xrightarrow{t_X \to +\infty} \lambda_* \) be a root of (2.4) and \( \kappa_i, i = N, \ldots, N + q, q \geq 0 \), be the associated solutions to (5.18). Then the representation
\[ B^{-1}(\lambda, X) = \sum_{i=N}^{N+q} \frac{T^{(i)}_{11}(X)}{\lambda - \lambda(X)} \kappa_i(X) + B_0(\lambda, X) \]
is valid for all \( \lambda \) close to \( \lambda(X) \). Here \( T^{(i)}_{11} : \mathbb{C}^p \to \mathbb{C} \) are functionals, while the matrix \( B_0(\lambda, X) \) is holomorphic w.r.t. \( \lambda \) in a neighbourhood of \( \lambda(X) \).

**Proof.** The matrix \( B \) is meromorphic and its inverse thus has a pole at \( \lambda(X) \). Proceeding as in the proof of Lemma 5.2 in [1] one can show that the residue at this pole is \( \sum_{i=N}^{N+q} \kappa_i(X) T^{(i)}_{11}(X) \), where \( T^{(i)}_{11} : \mathbb{C}^p \to \mathbb{C} \) are functionals. We are going to prove that this pole is simple; clearly, it will complete the proof of the lemma.

Consider \( \lambda \) close to \( \lambda(X) \) and not coinciding with \( \lambda_* \) and \( \lambda(X) \). Let \( f_i \in L_2(\mathbb{R}^n, \Omega_i) \) be arbitrary functions, \( f := (f_1, \ldots, f_m) \in \mathcal{L}, \tilde{f} := \sum_{i=1}^m S(-X_i)f_i. \) Completely by analogy with (5.1)–(5.7) one can check easily that (4.6) with \( T_2 = T^{(X)}_2 \) is equivalent to
\[ g + T_7(\lambda)g + T_8(\lambda, X)g = f. \]
Proceeding as in (5.11), (5.12), one can reduce this equation to an equivalent one,
\[ g - \sum_{i=1}^p \frac{T_0^{(i)}T_8(\lambda, X)g}{\lambda - \lambda_*}\Phi_i = - \sum_{i=1}^p \frac{T_0^{(i)}f}{\lambda - \lambda_*}\Phi_i \]
\[ + (I + T_0(\lambda)T_8(\lambda, X))^{-1}T_0(\lambda)f. \]

We denote
\[ \kappa_i := \frac{T_0^{(i)}T_8(\lambda, X)g}{\lambda - \lambda_*}. \]
and apply the functionals $T_9^{(j)}T_8(\lambda, X)$ to (6.3). This procedure leads us to the equation for $\kappa$:

$$B(\lambda, X)\kappa = -\frac{1}{\lambda - \lambda_*} \Lambda(\lambda, X)h_1 + h_2,$$

where $\kappa$ is defined as in (5.19). Hence,

$$\kappa = \frac{1}{\lambda - \lambda_*} h_1 + \tilde{\kappa}, \quad \tilde{\kappa} := B^{-1} \tilde{h}, \quad \tilde{h} := h_2 - h_1,$$

where $\tilde{\kappa}_i$ are components of the vector $\tilde{\kappa}$. By Lemma 4.2 the solution to (4.6) with $T_2 = T_2^{(X)}$ has at most simple pole at $\lambda(X)$. Hence, the same is true for the vector $g$ just determined. It follows that the vector $B^{-1} \tilde{h}$ can have at most simple pole at $\lambda(X)$. The estimates (5.9) imply that

$$\tilde{h} = -h_1 + O \left( I_X \sum_{\lambda} e^{-2l_X \sqrt{-\lambda}} \right).$$

In view of this identity and the definition of $h_1$ we conclude that for any $\tilde{\kappa} \in \mathbb{C}^p$ there exists $f \in L$ such that $\tilde{h} = h_2 - h_1$, where $h_i$ are given by (6.4). Therefore, the matrix $B^{-1}$ has the simple pole at $\lambda(X)$.

Reproducing word for word the proof of Lemma 5.3 in [1] we obtain

**Lemma 6.4.** A zero $\lambda(X) \xrightarrow{I_X \rightarrow +\infty} \lambda_*$ of the function $F(\lambda, X)$ has order $q$ if and only if it is a $q$-multiple eigenvalue of $\mathcal{H}_X$.

The statement of Theorem 2.2 follows from Lemmas 5.3, 5.4, 6.1, 6.4. The proof of Theorems 2.3, 2.4 repeats verbatim et literatim the proof of Theorems 1.4, 1.5 in [1].

**7. Proof of Theorems 2.5–2.7**

**Proof of Theorem 2.5.** Let us prove first that the representation (2.7) is valid, where the matrix $A_0$ is defined in the statement of the theorem and

$$\|A_1\| = O \left( I_X^{\frac{n+1}{2}} e^{-2l_X \sqrt{-\lambda_*}} \right), \quad l_X \rightarrow +\infty.$$
Due to (5.9), (5.17) we have
\[ A_{i,j}(\lambda_s, X) = T_0^{(i)}(\lambda_s, X)\phi_j + \mathcal{O}\left(\frac{1}{X^{n+1}}e^{-2\sqrt{X}\sqrt{-\lambda_s}}\right), \quad X \to +\infty. \]

We are going to show that \( A^{(0)}_{i,j}(X) = T_0^{(i)}(\lambda_s, X)\phi_j \) and the matrix \( A_0 \) satisfies the condition (A); this will obviously imply the needed representation.

We choose \( i \) and \( j \), and let \( k, r \in \{1, \ldots, m\} \), \( q \in \{1, \ldots, p_k\} \), \( s \in \{1, \ldots, p_r\} \) be such that \( i = \alpha_k + q \), \( j = \alpha_r + s \). Then
\[
T_0^{(i)}T_0(\lambda_s, X)\phi_j = \begin{cases} 
T_0^{(k,r)}(\lambda_s)\phi_{r,s}, \psi_{k,q} & r \neq k, \\
0 & r = k.
\end{cases}
\]

Consider the case \( r \neq k \). We employ (5.4) and (2.1) and integrate by parts to obtain:
\[
(T_0^{(k,r)}(\lambda_s)\phi_{r,s}, \psi_{k,q})_{L_2(\Omega_k)} = (L_k T_0(\lambda_s, X_{k,r})\phi_{r,s}, \psi_{k,q})_{L_2(\Omega_k)} + (\Delta - \lambda_s + L_k)(\mathcal{H}_{\Omega_k} - i)^{-1}L_k T_0(\lambda_s, X_{k,r})\phi_{r,s}, \psi_{k,q})_{L_2(\Omega_k)} \quad (7.1)
\]

It follows from Lemma 4.1 and the definition of \( T_5 \) that \( T_5(\lambda_s, X_{k,r})\phi_{k,s} = S(X_{k,r})\psi_{r,s} \). Hence,
\[
T_0^{(i)}(\lambda_s, X)\phi_j = (L_k S(X_{k,r})\psi_{r,s}, \psi_{k,q})_{L_2(\Omega_k)} = A_{i,j}^{(0)}(X).
\]

Using this identity, the condition (2.1) and the equation for \( \psi_{r,s} \) and \( \psi_{k,q} \), we check that
\[
A_{i,j}^{(0)}(X) = (L_k S(X_{k,r})\psi_{r,s}, \psi_{k,q})_{L_2(\Omega_k)} = (S(X_{k,r})\psi_{r,s}, L_k \psi_{k,q})_{L_2(\Omega_k)} = (\psi_{r,s}, S(X_{k,r})L_k \psi_{k,q})_{L_2(\Omega_k)} = (\psi_{r,s}, (\Delta + \lambda_s)S(X_{k,r})\psi_{k,q})_{L_2(\Omega_k)} = (\psi_{r,s}, S(X_{k,r})\psi_{k,q})_{L_2(\Omega_k)} = (\Delta + \lambda_s)\psi_{r,s}, S(X_{k,r})\psi_{k,q})_{L_2(\Omega_k)} = (L_r \psi_{r,s}, S(X_{r,k})\psi_{k,q})_{L_2(\Omega_r)} = (L_r S(X_{r,k})\psi_{k,q}, \psi_{r,s})_{L_2(\Omega_r)} = T_{j,i}^{(0)}(X).
\]

Hence, the matrix \( A_0 \) is hermitian. The eigenvectors of \( A_0 \) are orthonormal in \( C^p \), and the determinant of the matrix formed by these vectors thus equals one. Therefore, the matrix \( A_0 \) satisfies the condition (A). Now it is sufficient to apply Theorem 2.4 to complete the proof. \( \square \)
Proof of Theorem 2.7. In the case considered the matrix $A_0$ reads as follows:

$$A_0 = \begin{pmatrix} 0 & (\mathcal{L}_1 S(X_{1,2})\psi_2, \psi_1)_{L_2(\Omega_1)} \\ (\mathcal{L}_1 S(X_{1,2})\psi_2, \psi_1)_{L_2(\Omega_1)} & 0 \end{pmatrix},$$

where we have taken in account the hermiticity of this matrix. The eigenvalues of $A_0$ are

$$\tau_1^{(0)} = -\left| (\mathcal{L}_1 S(X_{1,2})\psi_2, \psi_1)_{L_2(\Omega_1)} \right|, \quad \tau_2^{(0)} = \left| (\mathcal{L}_1 S(X_{1,2})\psi_2, \psi_1)_{L_2(\Omega_1)} \right|.$$

Applying now Theorem 2.5, we complete the proof. \qed

Proof of Theorem 2.6. Theorem 2.3 implies that the eigenvalue $\lambda(X)$ has the asymptotic expansion (2.5), where $\tau(X) = A_{11}(\lambda, X)$. It follows from the definition of $\Phi_1$ and the estimates (5.9) that

$$\Phi_1(\cdot, \lambda, X) = \phi_1 - T_{10}(\lambda_*) T_8(\lambda_*, X) \phi_1 + \mathcal{O}\left(\frac{1}{X^2} e^{-2X \sqrt{\lambda_*}}\right),$$

as $l_X \to +\infty$. Since $T_9^{(1)} T_8(\lambda_*, X) \phi_1 = 0$, we infer that

$$A_{11}(\lambda, X) = -T_9^{(1)} T_8(\lambda_*, X) T_{10}(\lambda_*) T_8(\lambda_*, X) \phi_1 + \mathcal{O}\left(\frac{1}{X^{m-2}} e^{-3X \sqrt{\lambda_*}}\right), \quad (7.3)$$

as $l_X \to +\infty$. By direct calculations we check

$$T_{10}(\lambda_*) T_8(\lambda_*, X) \phi_1 = \left(0, (1 + T_2^{(2)}(\lambda_*))^{-1} T_6^{(2,1)} \phi_1, \ldots, (1 + T_2^{(m)}(\lambda_*))^{-1} T_6^{(m,1)} \phi_1 \right),$$

where $\phi_1 := (T_6^{(1)}(\lambda_*))^{-1} \psi_1$. Using this relation and proceeding in the same way as in (7.1), we obtain

$$T_9^{(1)} T_8(\lambda_*, X) T_{10}(\lambda_*) T_8(\lambda_*, X) \phi_1 = \sum_{j=1}^m \left(\mathcal{L}_1 S(X_{1,2}) T_1^{(j)}(\lambda_*) (1 + T_2^{(j)}(\lambda_*))^{-1} T_6^{(j,1)} \phi_1, \psi_1 \right)_{L_2(\Omega_1)} \quad (7.4).$$

In view of Lemma 4.1 the function $T_1^{(j)}(\lambda_*) (1 + T_2^{(j)}(\lambda_*))^{-1} T_6^{(j,1)} \phi_1$, $j = 2, \ldots, m$, is a solution of (4.1) with $\mathcal{H}_L = \mathcal{H}_j$, $\lambda = \lambda_*$, $f = T_6^{(j,1)} \phi_1$. Since

$$T_6^{(j,1)} \phi_1 = \mathcal{L}_j T_5(\lambda_*, X_{j,1}) \phi_1 - (\mathcal{H}_j - \lambda_*) \chi_{\Omega_j}(\mathcal{H}_j - i)^{-1} \mathcal{L}_j T_5(\lambda_*, X_{j,1}) \phi_1,$$

due to (5.4), and $\mathcal{L}_j T_5(\lambda_*, X_{j,1}) \phi_1 = \mathcal{L}_j S(X_{j,1}) \psi_1$ by Lemma 4.1, we infer that

$$T_1^{(j)}(\lambda_*) (1 + T_2^{(j)}(\lambda_*))^{-1} T_6^{(j,1)} \phi_1 = (\mathcal{H}_j - \lambda_*)^{-1} \mathcal{L}_j S(X_{j,1}) \psi_1 - \chi_{\Omega_j}(\mathcal{H}_j - i)^{-1} \mathcal{L}_j S(X_{j,1}) \psi_1.$$
The support of the second term in the right-hand side of this identity lies inside $\Omega^\beta_j$. Bearing this fact in mind, from (7.4) we deduce

$$
T^{(1)}_9 T_8 (\lambda_*, X) T_{10} (\lambda_*) T_8 (\lambda_*, X) \phi_1 = \sum_{j=2}^{m} \left( L_{11} S(X_{1,j}) (H_j - \lambda_*)^{-1} L_{12} S(X_{j,1}) \psi_1, \psi_1 \right)_{L_2(\Omega_1)}.
$$

We substitute this identity and (7.3) into (2.5) and take into account that by (5.9)

$$
T^{(1)}_9 T_8 (\lambda_*, X) T_{10} (\lambda_*) T_8 (\lambda_*, X) \phi_1 = O(l^{-n+1} e^{-2l \sqrt{-\lambda_*}}), \quad l_X \to +\infty.
$$

This leads us to the claimed asymptotics for $\lambda(X)$. Since $p=1$, the system (5.18) reduces to an equation $(\lambda - \lambda_*^\beta_1 (\lambda, X)) k_1 = 0$, which has the non-trivial solution $\kappa_1 = 1$. This identity and Theorem 2.3 imply the asymptotics for $\psi(x, X)$. □

8. Examples

In this section we will give some possible examples of the operators $L_i$. Throughout this section we suppose that $\Omega_i \subset \mathbb{R}^n$ are given bounded domains with infinitely differentiable boundary. We will often omit the index "$i$" in the notations corresponding to $i$-th operator $L_i$, writing simply $L, \Omega, \mathcal{H}$, etc.

1. Potential. The simplest example of the operator $L$ is the multiplication by the compactly supported real-valued potential. This is the classical example. The case of two symmetric wells was considered in [8], and the asymptotics for the eigenvalues were obtained. These results are reproduced by our Theorem 2.7. A convergence result for two non-symmetric wells was obtained in [4, Ch. 8, Sect. 8.6]. To our knowledge, in the multiple-well case $m \geq 3$ the asymptotic expansions for the eigenvalues were not known. In view of this, the results of Theorems 2.5–2.7 applied to this example are seemed to be new, at least if $m \geq 3$.

2. Second order differential operator. A more general example is a differential operator

$$
L = \sum_{i,j=1}^{n} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0,
$$

where the coefficients $b_{ij}$ are piecewise continuously differentiable and the coefficients $b_i$ are piecewise continuous. The functions $b_{ij}$ and $b_i$ are also assumed to be complex-valued and compactly supported. We also suppose that the conditions (2.1), (2.2) hold true; the self-adjointness of the operator $\mathcal{H}$ and $\mathcal{H}_X$ follows from these conditions due to the specific definition of $L$.

The particular case of (8.1) is

$$
L = \text{div} \ G \nabla + i \sum_{i=1}^{n} \left( b_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} b_i \right) + b_0,
$$

(8.2)
where $G = G(x)$ is an $n \times n$ hermitian matrix having piecewise continuously differentiable elements, the functions $b_i = b_i(x)$ are real-valued and piecewise continuously differentiable, the potential $b_0 = b_0(x)$ is real-valued and piecewise continuous. We also suppose that the matrix $G$ and the functions $b_i$ are compactly supported and

$$\left( G(x)y, y \right)_{C^n} \geq -c_0 \|y\|_{C^n}^2, \quad x \in \overline{\Pi}, \quad y \in C^n, \quad c_0 < 1,$$

where the constant $c_0$ is independent of $x, y$. The inequality $c_0 < 1$ in fact means that the operator $L$ does not break the uniform ellipticity of the Laplacian. The matrix $G$ can be zero; in this case $L$ is a first order differential operator.

### 3. Magnetic Schrödinger operator.

Let $\mathbf{b} = (b_1, \ldots, b_n) \in C^1_0(\mathbb{R}^n)$ be a magnetic vector-potential, and $b_0 := \|\mathbf{b}\|_{L^2} + V$, where $V = V(x) \in C_0(\mathbb{R}^n)$ is an electric potential. We define the operator $L$ by the formula (8.2) with $G = 0$. Such operator describes the magnetic field with compactly supported vector-potential.

### 4. Integral operator.

The operator $L$ need not to be a differential one. For instance, it can be an integral operator

$$(Lu)(x) := \int_{\Omega} L(x, y)u(y) \, dy,$$

where the kernel $L$ is an element of $L^2(\Omega \times \Omega)$. We also assume that the function $L(\cdot, y)$ is compactly supported and the relation $L(x, y) = \overline{L(y, x)}$ holds true. Such operator satisfies the conditions (2.1), (2.2). It is also $\Delta_{\mathbb{R}^n}$-compact and therefore the operator $H$ is self-adjoint.

### 5. $\delta$-potential.

The results of the general scheme developed in the present article can be applied to the perturbing operators not even satisfying the conditions we assume for $L$. It is possible, if such operators can be reduced by a transformation to an operator satisfying needed conditions. One of such examples is $\delta$-potential supported by a manifold. Namely, let $\Gamma$ be a bounded closed $C^3$-manifold in $\mathbb{R}^n$ of codimension one and oriented by a normal vector-field $\nu = \nu(\xi)$, where $\xi = (\xi_1, \ldots, \xi_n)$ are local coordinates on $\Gamma$. Let $\varrho$ be the distance from a point to $\Gamma$ measured in the direction of $\nu$. We suppose that $\Gamma$ is so that the coordinates $(\varrho, \xi)$ are well-defined in a neighbourhood of $\Gamma$, and in this neighbourhood the mapping $(\varrho, \xi) \mapsto x$ is $C^3$-diffeomorphism. We introduce the operator

$$(H_\Gamma v)(x) := -\Delta_{\mathbb{R}^n} + b\delta(x - \Gamma)$$

as

$$H_\Gamma v = -\Delta v, \quad x \not\in \Gamma,$$

on the functions $v \in W^2_2(\mathbb{R}^n \setminus \Gamma) \cap W^1_2(\mathbb{R}^n)$ satisfying the condition

$$\left. \frac{\partial v}{\partial \varrho} \right|_{\varrho = +0} - \left. \frac{\partial v}{\partial \varrho} \right|_{\varrho = -0} = b v \big|_{\varrho = 0},$$

where $b = b(\xi) \in C^3(\Gamma)$. We reproduce now word for word the arguments of Example 5 in [1, Sect. 7] to establish

**Lemma 8.1.** There exists $C^1$-diffeomorphism $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_n)$, such that
1. The second derivatives of $\mathcal{P}$ and $\mathcal{P}^{-1}$ exist and are piecewise continuous.

2. The function $p := \det \mathcal{P}$ and the matrix

$$\mathcal{P} := \begin{pmatrix} \frac{\partial \mathcal{P}_1}{\partial x_1} & \cdots & \frac{\partial \mathcal{P}_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{P}_n}{\partial x_1} & \cdots & \frac{\partial \mathcal{P}_n}{\partial x_n} \end{pmatrix}$$

satisfy the identities

$$p^{1/2} \big|_{\rho=0} - p^{1/2} \big|_{\rho=-0} = 0, \quad \frac{\partial}{\partial \rho} p^{1/2} \big|_{\rho=0} - \frac{\partial}{\partial \rho} p^{1/2} \big|_{\rho=-0} = b,$$

$$p \equiv 1 \text{ as } |\rho| \geq \varepsilon,$$

where $\varepsilon$ is a small fixed number.

3. The mapping $(Uv)(x) := p^{-1/2}v(\mathcal{P}^{-1}(x))$ is a linear unitary operator in $L^2(\mathbb{R}^n)$ which maps the domain of the operator $\mathcal{H}_\Gamma$ onto $W^2_2(\mathbb{R}^n)$. The identity

$$\mathcal{H}_\varepsilon := U\mathcal{H}_\Gamma U^{-1} = -\Delta_{\mathbb{R}^n} + \mathcal{L}$$

holds true, where the operator $\mathcal{L}$ is given by (8.1) and the supports of $b_{i,j}$, $b_i$ lie inside $\{x : \rho \leq \varepsilon\}$.

The item 3 of this lemma implies that the original $\delta$-potential can be reduced to a differential operator (8.1) with the same spectrum. Thus, after such transformation we can apply the results of this paper to such perturbation as well.

The operator $\mathcal{L}$ in (8.4) depends on the auxiliary transformation $\mathcal{P}$. We are going to show that the leading terms of the asymptotic expansions established in Theorems 2.5–2.7 do not depend on $\mathcal{P}$.

We begin with Theorem 2.5. Let $\mathcal{L}_k = \mathcal{L}$ for some $k$, where $\mathcal{L}$ is from (8.4), and $\tilde{\psi}$ be an eigenfunction of $\mathcal{H}_\varepsilon$ associated with $\lambda_*$. The corresponding elements of the matrix $A_0$ introduced in Theorem 2.5 are

$$A^{(0)}_{i,j} = (\mathcal{L}u, \tilde{\psi})_{L^2(\Omega_{2\varepsilon})},$$

where $u = S(X_{k,r})\psi_{r,s}$, and $\Omega_{2\varepsilon} := \{x : \rho < 2\varepsilon\}$. The function $u$ satisfies the equation

$$(\Delta + \lambda_*)u = 0, \quad x \in \Omega_{2\varepsilon}. \quad (8.5)$$

The function $\psi := U^{-1}\tilde{\psi} = p^{1/2}\tilde{\psi}(\mathcal{P}(\cdot))$ is an eigenfunction of $\mathcal{H}_\Gamma$ associated with $\lambda_*$, and is thus independent of $\mathcal{P}$. The identities (8.3) imply that $\tilde{\psi} \equiv \psi$ as $\varepsilon < \rho \leq 2\varepsilon$. Employing this fact, (2.1), (8.5) and integrating by parts, we obtain

$$(\mathcal{L}u, \tilde{\psi})_{L^2(\Omega_{2\varepsilon})} = (u, \mathcal{L}\tilde{\psi})_{L^2(\Omega_{2\varepsilon})} = \left( u, (\Delta + \lambda_*)\tilde{\psi} \right)_{L^2(\Omega_{2\varepsilon})}$$

$$= \int_{\partial\Omega_{2\varepsilon}} \left( \frac{\partial \tilde{\psi}}{\partial \nu_{\varepsilon}} - \frac{\partial u}{\partial \nu_{\varepsilon}} \right) \psi \, ds,$$
where $\nu_\varepsilon$ is the outward normal to $\partial \Omega_{2\varepsilon}$. The last integral is independent of $\varepsilon$ since for any $\tilde{\varepsilon} \in (0, \varepsilon)$

$$(\Delta + \lambda_\ast) \psi = 0, \quad x \in \Omega_{2\varepsilon} \setminus \Omega_{2\tilde{\varepsilon}},$$

$$0 = (u, (\Delta + \lambda_\ast) \tilde{\psi})_{L^2(\Omega_{2\varepsilon}) \setminus \Omega_{2\tilde{\varepsilon}}}$$

$$= \int_{\partial \Omega_{2\varepsilon}} \left( u \frac{\partial \tilde{\psi}}{\partial \nu_\varepsilon} - \tilde{\psi} \frac{\partial u}{\partial \nu_\varepsilon} \right) \, ds - \int_{\partial \Omega_{2\tilde{\varepsilon}}} \left( u \frac{\partial \tilde{\psi}}{\partial \nu_\varepsilon} - \tilde{\psi} \frac{\partial u}{\partial \nu_\varepsilon} \right) \, ds.$$

Using now the boundary conditions for $\psi$ on $\Gamma$, we pass to the limit $\varepsilon \to +0$ and obtain

$$\lim_{\varepsilon \to 0} \int_{\partial \Omega_{2\varepsilon}} \left( u \frac{\partial \tilde{\psi}}{\partial \nu_\varepsilon} - \tilde{\psi} \frac{\partial u}{\partial \nu_\varepsilon} \right) \, ds = \int_{\Gamma} u \left( \frac{\partial \tilde{\psi}}{\partial \varrho} \bigg|_{\varrho=+0} - \frac{\partial \psi}{\partial \varrho} \bigg|_{\varrho=-0} \right) \, ds$$

$$= (u, b\psi)_{L^2(\Gamma)}.$$

Thus, if an operator $L_k$ describes the $\delta$-potential, the corresponding elements of the matrix $A_0$ in Theorem 2.5 are

$$A_{k,j}^{(0)} := (S(X_{k,r}) \psi_{r,s}, \psi_{k,q})_{L^2(\Gamma)},$$

where $\psi_{k,q}$ are the eigenfunctions of the operator $H_\Gamma$. In particular, if in Theorem 2.7 the operator $H_1$ is $H_\Gamma$, the asymptotic expansions for $\lambda_i$ cast into

$$\lambda_1 = \lambda_\ast - \frac{|(L_1 S(X_{1,2}) \psi_2, \psi_1)_{L^2(\Gamma)}| + O \left( l_x^{-n+\sqrt{\lambda_\ast}} \right)}{\lambda_\ast + 1},$$

$$\lambda_2 = \lambda_\ast + \frac{|(L_1 S(X_{1,2}) \psi_2, \psi_1)_{L^2(\Gamma)}| + O \left( l_x^{-n+\sqrt{\lambda_\ast}} \right)}{\lambda_\ast + 1}.$$  (8.6)

as $l_x \to +\infty$. If under the hypothesis of Theorem 2.6 the operator $L_1$ describes $\delta$-potential, the arguments similar to that given above show that the asymptotics for $\lambda(X)$ reads as follows

$$\lambda(X) = \lambda_\ast - \frac{m}{\lambda_\ast + 1} \frac{|(S(X_{1,j}) (H_j - \lambda_\ast)^{-1} L_j S(X_{j,1}) \psi_1, \psi_1)_{L^2(\Gamma)}| + O \left( l_x^{-\sqrt{\lambda_\ast}} \right)}{\lambda_\ast + 1},$$

where $\psi_1$ is the eigenfunction of $H_\Gamma$. The asymptotics for the associated eigenfunction remains the same, if by $\psi_1$ we mean the eigenfunction of $H_\Gamma$. In view of the discontinuity of $\psi_1$ on $\Gamma$, this asymptotic is valid in $W^{1}_2(\mathbb{R}^n)$-norm and in the norm of $W^{1}_2(Q)$ for each domain $Q$ separated from $\Gamma$ by a positive distance.

Suppose now that under the hypothesis of Theorem 2.6 one of the operators $L_j$, $j \geq 2$, describes the $\delta$-potential. We denote $u := (H_j - \lambda_\ast)^{-1} L_j S(X_{j,1}) \psi_1$. 

Proceeding in the same way as in (7.2), we obtain
\[
(L_1 S(X_{1,j})u, \psi_1)_{L^2(\Omega_1)} = \int_{\partial \Omega_2} \left( S(X_{1,j})\overline{\psi_1} \frac{\partial u}{\partial \nu_x} - u \frac{\partial}{\partial \nu_x} (S(X_{1,j})\overline{\psi_1}) \right) \, ds. \tag{8.7}
\]
Since \((\Delta + \lambda_*) S(X_{1,j})\psi_1 = 0\) in \(\Omega_{2\epsilon}\), it follows that
\[
L_j S(X_{1,j})\psi_1 = (H_j - \lambda_*) S(X_{1,j})\psi_1 + (\Delta + \lambda_*) S(X_{1,j})\psi_1
= \mathcal{U} \left( (H_j - \lambda_*)^{1/2} + (\Delta + \lambda_*) \right) S(X_{1,j})\psi_1.
\]
Using this relation, (8.3), and the identity \((H_j - \lambda_*)^{-1} = \mathcal{U}(H_j - \lambda_*)^{-1}\mathcal{U}^{-1}\), we obtain \(u = \mathcal{U}u\), where
\[
U = (H_j - \lambda_*)^{-1} \left( (H_j - \lambda_*)^{1/2} + (\Delta + \lambda_*) \right) S(X_{1,j})\psi_1 = \tilde{U} + U_j,
\]
\[
\tilde{U} = \lambda_0^{1/2} \psi_1 \left( \mathcal{P}\mathcal{P}(\psi) + X_{1,j} \right) - S(X_{1,j})\psi_1,
\]
and \(U_j \in W^2_0(\mathbb{R}^n \setminus \Gamma) \cap W^1_0(\mathbb{R}^n)\) is the unique solution to the problem
\[
(\Delta + \lambda_*) U_j = 0, \quad x \in \mathbb{R}^n \setminus \Gamma,
\]
\[
\frac{\partial U_j}{\partial \nu} \bigg|_{x = +0} - \frac{\partial U_j}{\partial \nu} \bigg|_{x = -0} = bU_j \bigg|_{x = 0} - bS(X_{1,j})\psi_1 \bigg|_{x = 0}.
\]
It follows from (8.3) that \(\tilde{U} = 0\), \(u = U_j\) as \(\epsilon \leq \rho \leq 2\epsilon\). Bearing these relations in mind, we substitute the obtained representation for \(u\) into (8.7) and continue our calculations:
\[
(L_1 S(X_{1,j})u, \psi_1)_{L^2(\Omega_1)} = \int_{\partial \Omega_2} \left( S(X_{1,j})\overline{\psi_1} \frac{\partial U_j}{\partial \nu_x} - U_j \frac{\partial}{\partial \nu_x} (S(X_{1,j})\overline{\psi_1}) \right) \, ds.
\]
The right hand side of this identity is independent of small \(\epsilon\) that allows us to pass to the limit \(\epsilon \to +0\) and obtain
\[
(L_1 S(X_{1,j})u, \psi_1)_{L^2(\Omega_1)} = \int_{\Gamma} S(X_{1,j})\overline{\psi_1} \left( \frac{\partial U_j}{\partial \nu} \bigg|_{x = +0} - \frac{\partial U_j}{\partial \nu} \bigg|_{x = -0} \right) \, ds
\]
\[
= (bU_j - bS(X_{1,j})\psi_1, S(X_{1,j})\psi_1)_{L^2(\Gamma)}.
\]
Finally, it leads us to the formula
\[
\lambda(X) = \lambda_* - (bU_j - bS(X_{1,j})\psi_1, \psi_1)_{L^2(\Gamma)} - \sum_{k \neq j} (L_1 S(X_{1,k})(H_k - \lambda^*)^{-1} L_k S(X_{1,k})\psi_1, \psi_1)_{L^2(\Omega_1)}
\]
\[
+ O \left( \begin{array}{c} \frac{\sqrt{m}}{\sqrt{\lambda_*}} \\ e^{-3t_X \sqrt{\lambda_*}} \end{array} \right),
\]
being valid as \(t_X \to +\infty\) if the operator \(H_j\) describes the \(\delta\)-potential.
It follows from the results of [12] that if the distant perturbation is the δ-potential supported by several curves separated by large distances, the gap between two smallest eigenvalues is estimated from below by the function exponentially small as $l_X \to +\infty$. It is in a good accordance with the last asymptotics and (8.6). Moreover, these asymptotics allow one to make the results of [12] more precise in the large-distance regime.

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