System of equations describing charges of multiple conductors immersed in electrostatic fields

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Abstract: Problems of electrostatic fields involving multiple conductors arise in various scientific fields. It is known that the relation between the electric potentials and the charges of multiple conductors can be expressed by a system of linear equations. However, the well-known system of equations holds true only when no other charge exists external to the conductors. In this paper, we derive an extended system of equations that is valid even if the conductors are immersed in electrostatic fields due to external charges.

Keywords: electrostatic fields, multiple conductors
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1 Introduction

Problems of electrostatic fields involving multiple conductors arise in various scientific fields, such as intrabody communications [1], capacitive wireless power
transfer systems [2], crosstalk analysis between transmission lines [3], and so on. It is known that the relation between the electric potentials and the electric charges of \( N \) conductors can be expressed as follows [4]:

\[
\sum_{j=1}^{N} c_{ij} V_j = Q_i \quad (i = 1, \ldots, N),
\]

(1)

where \( c_{ij} \) is the capacitance coefficient between conductors \( \#i \) and \( \#j \), \( V_j \) is the potential of conductor \( \#j \), \( Q_i \) is the net charge of conductor \( \#i \), and we have equality of \( c_{ij} = c_{ji} \). The capacitance coefficients can be obtained by means of electrostatic analysis, i.e., by obtaining the net charge of each conductor under the condition

\[
V_j = \begin{cases} 
1 & (j = k) \\
0 & (j \neq k)
\end{cases},
\]

(2)

the capacitance coefficients are found as \( c_{ik} = Q_i \) \( (i = 1, \ldots, N) \). Because Eq. (1) is the expression for \( Q_1, \ldots, Q_N \), they can be obtained for arbitrarily specified \( V_1, \ldots, V_N \). Alternatively, \( Q_1, \ldots, Q_N \) may be specified and Eq. (1) can be solved for \( V_1, \ldots, V_N \). In general, Eq. (1) can be solved if either \( V_i \) or \( Q_i \) is given for each \( i \).

However, Eq. (1) holds true only when no other charge exists external to the conductors. In other words, Eq. (1) cannot be applied to problems in which the conductors are immersed in electrostatic fields. Of course, if the potentials of the conductors are given, i.e., fixed to certain values, the net charges of the conductors can be obtained by solving Poisson’s equation or the corresponding integral equation. However, in most practical problems, we have to specify the net charges of the conductors instead of their potentials. In this regard, Harrington explained in his classic book that in this case the potentials of the conductors are treated as unknown constants to be obtained after the charge distributions on the conductors are obtained [5]. However, he did not show any specific procedure.

To achieve this purpose, we have to extend Eq. (1) so that it holds true even if external charges exist, and solve it with the given charges. In this paper, the extended version of Eq. (1) is derived.

2 Theory

Consider that there are \( N \) conductors in space and electric charge \( \rho^{\text{ext}} \) is external to the conductors (external charge), as shown in Fig. 1. We assume that the charge \( \rho^{\text{ext}} \) generates electric field \( E^{\text{ext}} \) in the absence of the conductors (external field) and the distributions of \( \rho^{\text{ext}} \) are not changed by the presence of the conductors.

In this situation, electric potential \( \phi \) can be separated into two components as follows:

\[
\phi = \phi^{\text{ind}} + \phi^{\text{ext}},
\]

(3)

where \( \phi^{\text{ind}} \) is the component due to charges induced on the conductors, and \( \phi^{\text{ext}} \) is the component due to the external charge \( \rho^{\text{ext}} \) and satisfies the relationship of \( E^{\text{ext}} = -\nabla \phi^{\text{ext}} \). (In this paper, potentials as functions of position are denoted by \( \phi \) whereas those as constant values on conductors are denoted by \( V \).) By using the integral expression for \( \phi^{\text{ind}} \), Eq. (3) can be rewritten as
where $\mathbf{r}$ and $\mathbf{r}'$ are the observation and the source points, respectively, $\varepsilon_0$ is the permittivity of vacuum, $S$ is the surface area of conductors, $G(\mathbf{r}, \mathbf{r}')$ is Green’s function of Laplace’s equation, $\sigma(\mathbf{r}')$ is the surface charge density on the conductors, and prime (’) on $dS'$ denotes the integral with respect to the source point $\mathbf{r}'$.

Because integrals are linear operations, Eq. (4) can be rewritten as

$$\phi(\mathbf{r}) - \phi^{\text{ext}}(\mathbf{r}) = L[\sigma(\mathbf{r})],$$

where $L$ is a linear operator representing the integral in Eq. (4). The charge distributions $\sigma(\mathbf{r})$ are uniquely determined for all conditions of $\phi(\mathbf{r}) - \phi^{\text{ext}}(\mathbf{r})$ on conductors (uniqueness theorem). Therefore, the inverse linear operator $L^{-1}$ exists such that

$$L^{-1}[\phi(\mathbf{r}) - \phi^{\text{ext}}(\mathbf{r})] = \sigma(\mathbf{r}).$$

As we all know, the method of moments (MoM) is the procedure discretizing Eq. (5) and obtaining $L^{-1}$ and the solution $\sigma(\mathbf{r})$ of Eq. (5). Now, we express $\phi(\mathbf{r})$ on conductors as

$$\phi(\mathbf{r}) = \sum_{j=1}^{N} V_j f_j(\mathbf{r}) \quad (\mathbf{r} \in S),$$

where $V_j$ is the potential of conductor #j, and $f_j(\mathbf{r})$ is the function such that

$$f_j(\mathbf{r}) = \begin{cases} 1 & (\mathbf{r} \in S_j) \\ 0 & (\mathbf{r} \in S \setminus S_j) \end{cases},$$

where $S_j$ is the surface region of conductor #j. Substituting Eq. (7) into Eq. (6) and using the linearity of $L^{-1}$, we get the expression

$$\sum_{j=1}^{N} V_j L^{-1}[f_j(\mathbf{r})] + L^{-1}[-\phi^{\text{ext}}(\mathbf{r})] = \sigma(\mathbf{r}).$$

Integrating Eq. (9) over $S_i$, we get

$$\sum_{j=1}^{N} V_j \int_{S_i} L^{-1}[f_j(\mathbf{r})]dS + \int_{S_i} L^{-1}[-\phi^{\text{ext}}(\mathbf{r})]dS = \int_{S_i} \sigma(\mathbf{r})dS.$$
Now, let us think about the meaning of Eq. (10). Because \( f_j(r) \) is 1 on \( S_j \) and 0 elsewhere, as defined in Eq. (8), \( \mathcal{L}^{-1}[f_j(r)] \) is the charge distribution when \( V_j = 1 \) and others are zero and \( \phi^{\text{ext}} = 0 \). Therefore, the integral in the first term on the left-hand side of Eq. (10) is nothing less than the capacitance coefficient between conductors \( #i \) and \( #j \), i.e.,

\[
c_{ij} \equiv \iint_{S_i} \mathcal{L}^{-1}[f_j(r)]dS. \tag{11}
\]

On the other hand, the charge distribution characterized by \( \mathcal{L}^{-1}[-\phi^{\text{ext}}(r)] \) generates the potential \( \phi^{\text{ind}} \) that equals \( -\phi^{\text{ext}} \) on the conductors, i.e.,

\[
\phi^{\text{ind}}(r) + \phi^{\text{ext}}(r) = 0 \quad (r \in S).
\tag{12}
\]

In other words, the potentials of the conductors are maintained to zero. Therefore, the surface integral of \( \mathcal{L}^{-1}[-\phi^{\text{ext}}(r)] \) on \( S_i \) is the net charge of conductor \( #i \) induced by the external field when all the conductors are grounded, which is denoted by \( q_i \) in this paper, i.e.,

\[
q_i \equiv \iint_{S_i} \mathcal{L}^{-1}[-\phi^{\text{ext}}(r)]dS. \tag{13}
\]

Of course, the integral on the right-hand side of Eq. (10) is the net charge of conductor \( #i \), i.e.,

\[
Q_i \equiv \iint_{S_i} \sigma(r)dS. \tag{14}
\]

Substituting Eqs. (11), (13), and (14) into Eq. (10), we finally get an extended version of Eq. (1):

\[
\sum_{j=1}^{N} c_{ij} V_j + q_i = Q_i \quad (i = 1, \ldots, N). \tag{15}
\]

Specifically, the capacitance coefficients can be obtained as \( c_{ik} = Q_i \) \((i = 1, \ldots, N)\) under the condition

\[
V_j = \begin{cases} 
1 & (j = k) \\
0 & (j \neq k)
\end{cases} \quad \text{and} \quad \phi^{\text{ext}} = 0. \tag{16}
\]

Similarly, \( q_i \) can be obtained as \( q_i = Q_i \) \((i = 1, \ldots, N)\) under the condition

\[
V_1 = \ldots = V_N = 0 \quad \text{and} \quad \phi^{\text{ext}} \neq 0. \tag{17}
\]

Once we obtain \( c_{ij} \) and \( q_i \) via electrostatic analysis, we can calculate the potentials of the conductors for arbitrary charge conditions by solving Eq. (15) for \( V_j \). This idea is most likely what Harrington intended. Incidentally, to our knowledge, there is no literature that shows an expression identical to Eq. (15). However, Van Bladel has showed an equivalent expression for the special case of \( N = 1 \) [4].

### 3 Numerical example

Here, we describe a simple numerical example. Fig. 2 shows the calculation model. Three cylindrical wires \( (N = 3) \) that are parallel to the z-axis are on the \( xz \)-plane of \( y = 0 \). The radii and the lengths of the wires are 1 mm and 200 mm, respectively.
The central points of wires #1–#3 are \((x, y, z) = (0, 0, 340 \text{ mm})\), \((x, y, z) = (0, 0, 120 \text{ mm})\), and \((x, y, z) = (40 \text{ mm}, 0, 230 \text{ mm})\), respectively. An infinite ground plane is on the \(xy\)-plane of \(z = 0\), and an external electric field is assumed to be vertical to the ground plane, i.e., the electric potential and field are as follows:

\[
\varphi^\text{ext} = E_0 z, \quad \mathbf{E}^\text{ext} = -E_0 \hat{z},
\]

where \(E_0 = 1 \text{ V/m}\) is the magnitude of the electric field and \(\hat{z}\) is the unit vector codirectional with the \(z\)-axis.

\[
\begin{align*}
V_i - \varphi^\text{ext}(\mathbf{r}) &= \frac{1}{\varepsilon_0} \sum_{j=1}^{N} \iint_{S_j} G(\mathbf{r}, \mathbf{r'}) \sigma(\mathbf{r'}) dS' \quad (\mathbf{r} \in S_i; \ i = 1, \ldots, N). \\
\end{align*}
\]

(20)

In the present problem, the radii of the wires are sufficiently smaller than the lengths. Therefore, Eq. (20) may be reduced to

\[
V_i - E_0 z_i = \frac{1}{\varepsilon_0} \sum_{j=1}^{3} \int_{z_j^e}^{z_j^s} G_{ij}(z_i, z_j) \lambda_j(z_j) dz_j \quad (i = 1, 2, 3),
\]

(21)

where \(z_j^e\) and \(z_j^s\) are the endpoints of wire \#j \((z_j^e < z_j^s)\), \(\lambda_j(z_j)\) is the linear charge distribution of wire \#j, and \(G_{ij}(z_i, z_j)\) is expressed as follows:

\[
G_{ij}(z_i, z_j) = \frac{1}{4\pi \sqrt{d_{ij}^2 + (z_i - z_j)^2}} - \frac{1}{4\pi \sqrt{d_{ij}^2 + (z_i + z_j)^2}}.
\]

(22)

In Eq. (22), \(d_{ij}\) is determined according to the following rule:

\[
d_{ij} = \begin{cases} 
    a & (x_i = x_j) \\
    |x_i - x_j| & (x_i \neq x_j)
\end{cases}
\]

(23)
where $a$ denotes the radii of the wires, and $x_i$ is the $x$-coordinate of wire #i. This is the so-called thin-wire approximation. Each wire is divided into 20 segments of the same length, and piecewise constant basis function is assumed on each segment.

By applying Galerkin’s MoM to Eq. (21) under the condition of Eq. (16) for $k = 1, 2, 3$, the capacitance coefficients $c_{11} \sim c_{33}$ are obtained as follows:

$$
c_{11} = 2.445 \text{ [pF]}, \quad c_{12} = -0.1379 \text{ [pF]}, \quad c_{13} = -0.4366 \text{ [pF]},
$$

$$
c_{21} = -0.1379 \text{ [pF]}, \quad c_{22} = 2.610 \text{ [pF]}, \quad c_{23} = -0.3931 \text{ [pF]},
$$

$$
c_{31} = -0.4366 \text{ [pF]}, \quad c_{32} = -0.3931 \text{ [pF]}, \quad c_{33} = 2.548 \text{ [pF]}.
$$

By conducting similar procedure for the condition of Eq. (17), $q_1 \sim q_3$ are obtained as follows:

$$
q_1 = -0.6871 \text{ [pC]}, \quad q_2 = -0.1934 \text{ [pC]}, \quad q_3 = -0.3881 \text{ [pC]}.
$$

Now, we can obtain the potentials $V_1 \sim V_3$ under arbitrary charges $Q_1 \sim Q_3$. Here, we consider that the conductors are floating and uncharged, i.e.,

$$
Q_1 = Q_2 = Q_3 = 0.
$$

Substituting Eqs. (24), (25) and (26) into Eq. (15), and solving it for $V_1 \sim V_3$, we get the following solution:

$$
V_1 = 0.3289 \text{ [V]}, \quad V_2 = 0.1258 \text{ [V]}, \quad V_3 = 0.2280 \text{ [V]}.
$$

This is a reasonable result because $V_1 \sim V_3$ approximately agree with the values of $\phi^\text{ext}$ at the central points of wires #1–#3, respectively.

### 4 Conclusion

In this paper, a system of equations describing the charges of multiple conductors immersed in electrostatic fields, which is an extended version of Eq. (1), was derived. In the derived system of equations (Eq. (15)), the charges of the conductors are the superposition of those in the absence of external electric fields and those when the conductors are grounded. If the charges of the conductors are given, the derived system of equations can be solved for the potentials of the conductors. In addition, a simple numerical example involving three parallel wires was solved by means of the MoM, and a reasonable result was obtained.

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