Systematic relativistic quantum effects on screening of fusion rates in white dwarfs

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ABSTRACT

Relativistic electron degeneracy effects are dominant in ultra-dense plasmas (UDP), such as those found in white dwarfs. These effects can be treated systematically by obtaining an expansion of the screening length in inverse powers of $\hbar^2$. In general, our theory leads to an $\mathcal{O}(10)$ effect on the enhancement of fusion rates in white dwarfs. Further, it is shown analytically for these stellar conditions that Bose statistics of nuclei have a negligible effect on the screening length, in consonance with Monte Carlo simulations found in literature.

Subject headings: screening, fusion, plasma, quantum corrections
1. Introduction

The evolution of a white dwarf is marked by fusion reactions between carbon and oxygen nuclei. These reactions are enhanced by the screening of the Coulomb interaction between fusing nuclei by the surrounding plasma. Many papers have estimated this enhancement factor (Hamada and Salpeter 1961; Salpeter and Van Horn 1969; Janovici 1977; Slattery et al. 1982; Itoh et al. 1990; Ogata 1997; Ichimaru and Kitamura 1999; Pollock and Militzer 2004; Comparat et al. 2005; Gasques et al. 2005). It is difficult to gauge the accuracy of these calculations. Our goal here is to provide a systematic basis for the estimation of the fusion enhancement factor in white dwarfs.

It is possible that the techniques developed in this paper may be applicable to studying other elementary particle processes in dense stellar interiors (Itoh et al. 2002; Langanke and Martinez-Pinedo 2003).

Nuclei in white dwarfs are submerged primarily in a sea of electrons. This is because the probability of finding a nucleus, say, in a two-component UDP (one species of nuclei of charge \(Ze\), and electrons) is down by a factor of \(1/Z\), compared to finding an electron. Consequently, we might expect that Coulomb screening will be dominated by the sea of electrons. These electrons are not only degenerate, but also relativistic since the associated fermionic chemical potential is much larger than the thermal energy. Physically, the question of the dominant mode of screening boils down to how closely these energetic, relativistic electrons surround a given nuclei, compared to the proximity of another nucleus. Coulomb repulsion will keep charged nuclei approximately a Debye-Huckel length away from each other. We find in this paper that the energetic nature of these relativistic electrons keeps them away farther than one might expect on naive grounds, approximately a Debye length as well.

Our earlier paper (Chitanvis 2007) was directed towards understanding the screening effect of electronic quantum fluctuations on fusion reactions near the center of the sun (Salpeter 1954; Gruzinov and Bahcall 1998). That paper showed quantum effects are negligible, via a systematic expansion of the screening length in powers of \(\hbar^2\), putting to rest a controversy that has ebbed and flowed over the years (Bahcall et al. 2002). In this paper we consider a plasma where relativistic quantum effects dominate, which is quite opposite to the solar plasma. And so we contemplate the theory in our previous paper in inverse powers of \(\hbar^2\), which will apply when the effects of electron degeneracy dominate.

This paper also provides an analytical underpinning to the numerical techniques used in the past to study fusion in UDP (Ogata 1997; Pollock and Militzer 2004). These papers show that the effect of indistinguishability of nuclei on the screening of fusion rates is small. Further, our technique provides an alternative, systematic method for the calculation of the
enhancement factors of fusion rates in UDP. Our estimates of enhancement are consistent with those given by Ichimaru and Kitamura (Ichimaru and Kitamura 1999), and with recent results of Gasques et al. (Gasques et al. 2005).

2. Screening Formalism for relativistic degenerate electrons

Let us begin with the classical Poisson-Boltzmann equation for a single species of ions and electrons:

\[-\nabla^2 \phi = 4\pi \rho\]
\[\rho = \rho_+ + \rho_-\]
\[\rho_+ = e n Z \exp\left(-Ze\phi/k_B T\right)\]
\[\rho_- = -e n Z \exp\left(e\phi/k_B T\right)\] (1)

where \(e\) is the magnitude of the electronic charge, \(k_B\) is Boltzmann’s constant, \(n\) is the average number density, \(Ze\) is the ionic charge, and \(T\) is the temperature of the system. We shall work in the linear regime, by retaining only terms first order in \(\phi\), leaving to the next section a discussion of nonlinear terms resulting from the Boltzmann distribution:

\[\nabla^2 \phi \approx \left(\frac{4\pi n (Z^2 n + Zn)e^2}{k_B T}\right) \phi\]
\[\equiv \Lambda_0^{-2} \phi\]
\[\Lambda_0 = \sqrt{\frac{k_B T}{4\pi n e^2 (Z^2 + Z)}}\] (2)

where \(\Lambda_0\) is the classical screening length.

We shall now generalize this method to one where nuclei are treated classically, but electrons are treated quantum mechanically (Chitanvis 2007). The quantum-mechanical version of the linearized Poisson-Boltzmann equation for a single species of ions and electrons may be written in analogy with Eqn. [1]
\[- \nabla^2 \phi = 4\pi \rho \]
\[\rho = \rho_+ + \rho_-\]
\[\rho_+ \approx e n Z \left( 1 - \frac{Ze\phi}{k_B T} \right)\]
\[\rho_- = -e |\psi(\{\vec{r}\})|^2\]  
\[(3)\]

where \( \psi \) is the many-body quantum wave-function for electrons, and \( \{\vec{r}\} \) refers collectively to the electrons in the system, and \( \phi \) is the electrostatic potential.

We now invoke the following scaled variables, in order to ease subsequent calculations:

\[\tilde{\phi} = \frac{Ze\phi}{k_B T}\]
\[\tilde{\psi} = \Lambda^{3/2} \psi\]
\[\Lambda = \sqrt{\frac{k_B T}{4\pi Z^2 ne^2}}\]
\[\vec{r}' = \frac{\vec{r}}{\Lambda}\]
\[\Gamma' = \frac{Ze^2}{\Lambda k_B T}\]  
\[(4)\]

where \( \Gamma' \) is defined differently from the usual plasma parameter. Notice also that the scalar potential has been scaled differently than in our previous paper [Chitanvis 2007]. This is done to allow for a convenient analysis of higher order contributions in the next section. \( \Lambda \) is the average distance maintained between nuclei.

Note that the first of Eqns(4) shows that we are using \( k_B T \) as the energy scale. The electrostatic potential is then given by:

\[\nabla'^2 \tilde{\phi} = (\tilde{\phi} + 4\pi \Gamma'|\tilde{\psi}|^2 - 1)\]  
\[(5)\]

where \( \psi \) is a four-vector corresponding to the Dirac spinor. This equation may be obtained from a Lagrangian density:

\[\mathcal{L}_0 = -\frac{1}{2} |\nabla \tilde{\phi}|^2 - v(\tilde{\phi}, \tilde{\psi})\]
\[v(\tilde{\phi}, \tilde{\psi}) = \frac{1}{2} \tilde{\phi}^2 + 4\pi \tilde{\phi} \Gamma'|\tilde{\psi}|^2 - \tilde{\phi}\]  
\[(6)\]
The corresponding Hamiltonian density can be easily derived:

\[ H_0 = \frac{1}{2} |\nabla \tilde{\phi}|^2 + v(\tilde{\phi}, \tilde{\psi}) \] (7)

We will now introduce second-quantized notation to deal with the statistics of electrons:

\[ v(\tilde{\phi}, \tilde{\psi}) \rightarrow v(\tilde{\phi}, \tilde{\psi}_\pm) = \frac{1}{2} \tilde{\phi}^2 - \tilde{\phi} + 4\pi \tilde{\phi} \Gamma'(\psi^+ \psi) \] (8)

where \( \tilde{\psi} \) are Grassmann variables corresponding to Dirac electrons. The co-existence of Grassmann variables and scalars in Eqn.8 is not problematic, since we shall use this discussion solely to define a partition function for the entire system. And soon thereafter we shall integrate over the electron degrees of freedom, so that only a functional involving the scalar potential survives.

The total Hamiltonian \( \mathcal{H} \) for the system, including the relativistic, quantum-mechanical part for the electrons is:

\[ \mathcal{H} = H_0 + \mathcal{H}_Q \]

\[ \mathcal{H}_Q = \mathcal{R}_Q \tilde{\psi}^+(\tilde{i}\gamma \cdot \nabla' + a) \tilde{\psi} - \mu \tilde{\psi}^+ \tilde{\psi} \] (9)

where \( \tilde{\gamma} \) denote the Dirac gamma matrices, corresponding to the \( x, y, z \) directions. \( \mu \) is the electronic chemical potential, which shall be determined by demanding charge neutrality. The relativistic quantum correction has been encapsulated in the following dimensionless parameter:

\[ \mathcal{R}_Q = \frac{\hbar c}{\Lambda k_B T} \] (10)

where \( c \) is the speed of light.

The parameter \( a \) in Eqn. 9 is related to the rest mass of the electron:

\[ a = \frac{mc\Lambda}{\hbar} \] (11)

where \( m \) is the mass of the electron. It turns out that the chemical potential \( \mu \gg k_B T \) under conditions representative of white dwarfs. It follows that the relativistic approximation employed in Eqn.9 is valid.

The partition function may be written in scaled variables as:
\[ Z = \int \mathcal{D}\tilde{\phi} \mathcal{D}^2\tilde{\psi}_{\pm} \exp\left( -\int d^3x' (\mathcal{H}_0 + \mathcal{H}_Q) \right) \tag{12} \]

where it is understood that \( k_B T = 1 \) in the units we are using. The quadratic nature of the energy functional in Eqn \[12\] allows us to perform the functional integration over the Grassmann variables associated with the electronic degrees of freedom (Ramond 1981), allowing us to obtain:

\[
Z \sim \int \mathcal{D}\tilde{\phi} \exp\left( -\int d^3x' \left[ (1/2)|\nabla\tilde{\phi}|^2 + (1/2)\tilde{\phi}^2 - \tilde{\phi} \right] \right) \sqrt{\text{Det}(\mathcal{F})}
\]

\[
\text{Det}(\mathcal{F}) = \exp(Tr \ln(\mathcal{F}))
\]

\[
\mathcal{F} \equiv R_Q \left( -i\gamma \cdot \nabla + a \right) + 4\pi \Gamma' \tilde{\phi} - \beta \mu
\tag{13}
\]

where \( \beta = 1/(k_B T) \). Having integrated over the electronic degrees of freedom, we are left with an effective energy density in terms of the electrostatic potential alone. We could have also done things the other way, integrating over the electrostatic potential in the partition function, leaving a quartic in the fermionic variables, as is conventionally done. Our procedure can be said to have bosonized our plasma, since we now only have the scalar potential to investigate. We shall show below how our method leads to useful insights into the statistics of the plasma.

We now need to evaluate the determinant of the operator obtained in the process of performing the quadratic functional integral over fermionic variables. This is conveniently performed in Fourier space, and to quadratic order in the electrostatic field variable:

\[
Tr \ln(\mathcal{F}) \approx Tr \int \frac{d^3k}{(2\pi)^3} \left( \ln(R_Q(\gamma \cdot \vec{k} - \beta \mu)) + b \frac{\tilde{\phi}(k)}{\gamma \cdot \vec{k} + a - \beta \mu/R_Q} - b^2 \left( \frac{\tilde{\phi}(k)}{\gamma \cdot \vec{k} + a - \beta \mu/R_Q} \right)^2 \right)
\tag{14}
\]

where \( \tilde{\phi}(k) \) is the Fourier transform of \( \tilde{\phi} \) and \( b \) is defined as:

\[
b = \frac{4\pi \Gamma'}{R_Q}
\tag{15}
\]

For temperatures in the range of \( 2 \times 10^8 K \), and using a number density for our plasma \( \sim 2 \times 10^{32} cm^{-3} \) (which corresponds to a mass density of \( 3 \times 10^9 g - cm^{-3} \)), we find with
$Z \sim 10$, that $\Lambda \sim 1 \text{Fermi}(10^{-12} \text{cm})$, $\Gamma' \sim 10$, and $R_Q \sim 10^8$. This is an indication that quantum corrections are dominant in this system. Hence $4\pi\Gamma'/R_Q \sim 10^{-2} << 1$ is an excellent choice for a perturbation parameter, so that the quadratic expansion considered in Eqn. 14 will be sufficient for our purpose.

Terms devoid of field variables in Eqn. 14 will be ignored. For the particular case of the UDP considered here, the cubic term ignored in the above expansion is of $O(10^{-2})$ compared to the bilinear terms we retained. Thus, higher order terms can be safely ignored. We must also consider for consistency higher order terms in the expansion of the Boltzmann distribution corresponding to the nuclear charges. We shall do so in the next section.

We can now determine the chemical potential by demanding charge neutrality for the system. This is accomplished by considering the $k \to 0$ limit of the terms in the energy functional that are linear in the electrostatic potential $\tilde{\phi}(k)$, and setting the coefficient of the linear term to zero.

$$\frac{\mu}{k_B T} = (8\pi \Gamma' + mc^2/k_B T) >> 1$$

(16)

Then the corresponding Lagrangian density in real space may be written to order $b^2$ as follows:

$$\mathcal{L}_{\text{effective}} = -\frac{1}{2} |\tilde{\nabla} \tilde{\phi}(\vec{r})|^2$$

$$+ \frac{1}{2} (\tilde{\phi}(\vec{r}))^2 - V_{\text{effective}}(\phi(\vec{r}))$$

$$V_{\text{effective}}(\phi(\vec{r})) \approx 2b^2 \tilde{\phi}(\vec{r})((\nabla^2 - 4b^2)^{-1} - 8b^2(\nabla^2 - 4b^2)^{-2})\tilde{\phi}(\vec{r})$$

(17)

where $b = 4\pi \Gamma' / \Delta Q$. The equation of motion obtained by extremizing the above Lagrangian with respect to variations in the field $\chi$ is linear, albeit of order six:

$$((\Delta - 4b^2)^2(1 + \Delta) + (\Delta + 4b^2))\tilde{\phi} = 0$$

$$\Delta \equiv \nabla^2$$

(18)

Upon factorizing, the trinomial in Eqn. 18 may be cast as:

$$(\Delta - s_3)(\Delta - s_2)(\Delta - s_1)\tilde{\phi}(\vec{r}) = 0$$

(19)
where $s_1, s_2, s_3$ are the roots of the trinomial in Eqn.18. These can be found easily using Mathematica. Upon careful examination of these roots, only one yields the correct limit for the screening length given by $1/\sqrt{s_1}, 1/\sqrt{s_2}, 1/\sqrt{s_3}$ in the limit that $h \to \infty$. The screening length must go to infinity in this limit due to quantum fluctuations. One must remember that quantum fluctuations increase the screening length (Chitanvis 2007). We will denote this root by $s_1$. It may be written down simply as follows:

$$s_1 \approx 4.93b^2 << 1$$  \hspace{1cm} (20)

For this limiting case, the dimensionless screening length is then $\sqrt{4.93/b} >> 1$. Hence we shall choose the physically interesting solution to satisfy:

$$(\nabla^2 - s_1)\tilde{\phi}(\vec{r}) = 0$$  \hspace{1cm} (21)

and this will be sufficient to guarantee that the sixth order differential equation is automatically satisfied. One may verify the screening length argument by noting that in one dimension $\exp(-x\sqrt{4.93/b})$ solves the differential equation. We have used here a finite-temperature formalism involving not only a relativistic quantum electron gas, but also an admixture of positive nuclei to study screening in the limit that electronic quantum fluctuations dominate. In our case, the screening length turns out to be:

$$\lambda_{\text{screening}} \approx \frac{\Lambda(T)}{4\pi \sqrt{4.93Z\alpha}}$$

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$$  \hspace{1cm} (22)

This screening length will be discussed in much detail in section 4. It may be compared to the relativistic Thomas-Fermi screening length at zero temperature for a pure electron gas (Janovic 1977; Itoh et al. 2002):

$$\lambda_{\text{Thomas-Fermi}} \approx \sqrt{\frac{\pi}{4\alpha}}\ell_F$$

$$\ell_F = \frac{1}{5.2 \times 10^{10}(Z\rho_6/A)^{1/3}}$$  \hspace{1cm} (23)

where $\rho_6$ is the mass density in powers of $10^6$, and $A$ is the atomic weight.
Upon inserting values for variables that are representative of white dwarfs, it turns out that $\lambda_{\text{screening}} \sim 10^{-12}$ cm, while $\lambda_{\text{Thomas-Fermi}}/\lambda_{\text{screening}} \sim 10$. We ascribe this difference to two factors, viz., finite temperature effects, and the fact that we are considering here a mixture of positively charged nuclei and electrons. One can see then that our approach will provide a higher fusion enhancement factor.

3. Non-linear screening effects of nuclear charges

In the previous section, we discussed in detail how non-linear terms arising from electron degeneracy are safely of much smaller magnitude than those retained. The issue appears to get turned around when one considers non-linear terms arising from higher order terms in the expansion of the Boltzmann distribution corresponding to nuclear charges. This is because the coefficients of the nonlinear terms are larger than those found for degenerate electrons in the previous section. The issue may be compactly discussed by noting that retention of the quadratic term in Eqn. 8 leads to the following cubic modification of Eqn. 8:

$$v \to v - \frac{1}{6} \phi^3$$

The corresponding partition function for our classical system resembles one for a Euclidean scalar quantum field theory (Ramond 1981). Effects of the non-linear terms can be evaluated perturbatively, using Feynman diagrams. Feynman diagrams can be used to estimate the leading order contribution from this cubic term to the self-energy of the system (or, equivalently, the dielectric constant, or the screening length). It turns out that the actual value of the contribution is numerically quite small. This suggests that the linear screening approximation we retained in the previous section is an acceptable approximation.

In order to perform this calculation, we formally ignored terms of $\mathcal{O}(b) \sim 10^{-2} << (1/6)$ in our diagrammatics. We then used Mathematica to formulate symbolically the lowest order contributions from the cubic potential. The term that survives is a polarization-like diagram which comes from terms of $\mathcal{O}((1/6)^2)$. The calculation is done as usual in momentum space. At this point the momentum variable is simply set to zero, so that we get an expression for the inverse square of the screening length:

$$\hat{\Sigma}(k)_{\text{polarization-like}}(\vec{k}) = \frac{1}{12} \int \frac{d^3 p}{(2\pi)^3} \hat{G}_0(\vec{k}) \hat{G}_0(\vec{p} - \vec{k})$$

$$\hat{G}_0(\vec{p}) = \frac{1}{p^2 + 1}$$

(25)
where $\hat{\Sigma}(k)$ is the usual self-energy. It can be easily related to the dielectric constant of the UDP. We shall restrict attention to $k = 0$, when the self energy it reduces to the inverse square of the screening length, and is sufficient to allow us to gauge its magnitude relative to the degenerate contribution in the previous section.

Whence:

$$\Lambda_{\text{polarization-like}}^{-2} \equiv \hat{\Sigma}(k = 0)_{\text{polarization-like}} = \frac{1}{96\pi} \approx 3.3 \times 10^{-3} \text{ (white dwarfs)}$$

$$\Lambda_{\text{polarization-like}} \approx 1.7 \times 10^{-11}\text{ cm (white dwarfs)} \quad (26)$$

We see that the screening length from the cubic term in the energy functional (due to nuclear charges alone) is larger by an order of magnitude than that obtained in the previous section by an order of magnitude ($\Lambda_{\text{screening}} \sim 10^{-12}\text{ cm}$ for a white dwarf, from Eqn[19]). This difference in the screening due to nuclear and electronic charges means that electrons will be predominantly in closer proximity to a nucleus than another another nucleus. Since we are interested in screening at extremely short distances, we can therefore ignore the contribution to screening from Eqn. [26].

The conclusions of this section may have to be modified if higher order terms turn out to be larger. For that purpose, we retain the quartic term in the energy functional arising from the Boltzmann distribution for the nuclear charges, so that Eqn[24] is modified as follows:

$$v \to v - \frac{1}{3!} \tilde{\phi}^3 + \frac{1}{4!} \tilde{\phi}^4 \quad (27)$$

The lowest order contribution from the quartic term is the setting-sun diagram(Ramond 1981). There are no cross-terms at this order between the cubic and quartic terms.

$$\Sigma_{\text{setting-sun}}(p) = \frac{1}{6} \int \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \hat{G}_0(k_1)\hat{G}_0(k_2)\hat{G}_0(|\vec{p} - \vec{k}_1 - \vec{k}_2|) \quad (28)$$

Using $t^{-1} = \int d\lambda \exp(-\lambda t)$, converting the momentum integrals to center of mass and relative co-ordinates, and using dimensional regularization, we can compute the self-energy for $p = 0$ in the following form:
\[
\Sigma_{\text{setting-sun}}(0) = \left(\frac{1}{6}\right) \frac{L I}{8\pi^4}
\]

\[
L = \int_0^\infty \frac{\exp(-\xi x)}{x^{1-\epsilon}} \, dx
\]

\[
I = \int_0^\infty \frac{\ln(\xi) - \ln(1+y)}{(1+4y)^{1+\epsilon}} \, dy
\]

where \(\xi \to 0^+\), a small-distance cut-off, and \(\epsilon \to 0^+\) have been inserted to guarantee convergence. Using Mathematica, it can be shown that:

\[
L = (\xi)^{-\epsilon} \Gamma'(\epsilon)
\]

\[
I = \frac{\ln(\xi)}{4\epsilon} - \frac{1}{16} \left(\frac{3^{-\epsilon} 4 \csc(\pi \epsilon)}{\epsilon} + \Gamma'(-\epsilon) \ 2F_1(1,1;2-\epsilon;1/4)/\Gamma'(2-\epsilon)\right)
\]

Employing Laurent expansions within Mathematica in powers of \(\epsilon\), and upon using counter-terms that account for divergences, one obtains the following correction to the square of the screening length, correct to lowest order in the quartic term:

\[
\Sigma_{\text{setting-sun}}(0) =
\]

\[
\frac{1}{48\pi^4} \left(-\gamma - \log(\xi) + \frac{1}{48} \left(-2\pi^2 + 12\Gamma'\log\left(\frac{4}{3}\right)\right) - \frac{1}{16} \left((2\log^2(3) - \gamma_1)\right)\right)
\]

\[
\sqrt{\xi} = \frac{a}{\Lambda}
\]

where \(a << \Lambda\) is a microscopic length cut-off required to render the integrals finite, \(\Gamma' \approx 0.577\) is Euler’s constant, and \(\gamma_1 \approx 0.572\) is the value of the derivative of the Hypergeometric function which appears in Eqn.30. A logarithmic dependence of our answer on this cut-off implies a relative insensitivity to this parameter. It is clear that the theory used in this section is certainly not valid at the nuclear level, and so we will use \(\xi = a/\Lambda \sim 10^2\). A cursory examination shows that this particular diagram yields a small contribution \(\mathcal{O}(10^{-4})\) contribution to the self-energy. Then following the argument above, for the cubic term, this screening contribution can be ignored as well. Thus there appears to be a trend for the classical, nonlinear terms to be small.

Of course, further issues may arise in this perturbative argument as even higher order terms arising from the Boltzmann distribution are contemplated. We will leave these questions for future investigation. It may be possible to extend to white dwarfs the methods
utilized by Brown et al.\cite{Brown2006} for obtaining the screening length for a dilute, highly charged plasma.

4. Quantum effects of nuclei on screening

Over the years, researchers have delved into the importance of applying a quantum-statistical treatment to the nuclei surrounding the ones undergoing fusion. This is a reasonable point to investigate, given the extremely high densities available in white dwarfs. Path integral Monte Carlo (PIMC) techniques have led to the discovery that the bosonic nature of nuclei\cite{Itoh1990, Ogata1997, Pollock2004} makes a calculable, small contribution to the screening length, or equivalently, to the dielectric constant of the UDP under consideration. Here we provide an analytical underpinning to that observation. The second point that needs to be reinforced is that quantum effects of nuclei in the UDP are small in general, compared to the electronic contribution. The argument is basically that of Born and Oppenheimer, who showed that the nuclear mass is so large compared to the mass of an electron that an adiabatic approximation can be applied. That argument has to be extended to finite temperatures.

We begin by considering a UDP consisting of spin-zero nuclei, in addition to a sea of degenerate electrons. Thus the Poisson-Boltzmann equation (Eqn. 8) will be replaced by:

\[
\nabla \phi + 4\pi Z \phi \Gamma' \Psi \Psi^\dagger + 4\pi \phi \Gamma' \psi \psi^\dagger = -4\pi Z \phi \Gamma' \Psi \Psi^\dagger + 4\pi \phi \Gamma' \psi \psi^\dagger
\]

(32)

where $\Psi^\dagger, \Psi$ are the second-quantized creation and annihilation operators corresponding to bosonic nuclei, and we have continued to assume that the photons are numerous and hot that they can be treated classically.

Equation 9 must be modified to account for the free Hamiltonian of the nuclei:

\[
\mathcal{H}_Q \rightarrow \mathcal{H}_Q + \mathcal{H}_N
\]

\[
\mathcal{H}_N = \Delta_Q(M) |\vec{\nabla} \Psi|^2
\]

\[
\Delta_Q(M) = \left( \frac{\hbar^2 \Lambda^{-2}}{2 M k_B T} \right)
\]

(33)

where $\Delta_Q(M)$ is defined in terms of the mass $M$ of the nucleus under consideration. It has been assumed that the interaction between nuclei are dominated by the Coulomb potential, and we have employed a non-relativistic treatment for nuclei.
The corresponding functional integrals involving the nuclear field variables can be done, just as the fermionic degrees of freedom were accounted for. The net result for the partition function is:

\[ Z \sim \int D\tilde{\phi} \exp\left(-\int d^3x' \left(1/2\right)|\nabla \tilde{\phi}|^2\right) \sqrt{\text{Det}(\mathcal{F})} \text{Det}(\mathcal{B})^{-1} \]  

(34)

where

\[ Tr \ln(\mathcal{B}^{-1}) \equiv -\int \frac{d^3k}{(2\pi)^3} \ln(4\pi Z\Gamma'\hat{\phi}(k) + \Delta_Q(M) k^2) \]  

(35)

Using parameters representative of a white dwarf, and given that \( M \sim 2 \times 10^4m, Z \sim 10 \), it turns out that \( \Delta_Q(M) \sim 10^{-4}\Delta_Q(m) \), so that \( \Delta_Q(M)/4\pi Z\Gamma' \sim 10^{-3} \). This clearly shows that quantum effects, including Bose statistics of nuclei in the UDP are negligible in circumstances representative of a white dwarf. Thus, we may continue to use a classical treatment for nuclei, as was done in the previous section. One can regain from Eqn.35 the Boltzmann approximation using the method outlined by Chitanvis (Chitanvis 2007). This will yield systematic, extremely small quantum corrections to the Boltzmann approximation in powers of \( \hbar^2 \). Considering the nuclei as fermions does not change this conclusion.

Quantum effects of nuclei are much smaller than the quantum effects due to electrons, primarily due to the large mass difference. As such they can be ignored. Our conclusions concerning the effects of indistinguishability are in general agreement with Itoh (Itoh et al. 1990), Ogata (Ogata 1997) and Pollock and Militzer (Pollock and Militzer 2004). These authors did not consider the effects of electron degeneracy on the same footing as the nuclear quantum effects.

5. Results

We shall compare our results with those found in published literature. For succinctness, we shall quote the enhancement factors of pycnonuclear reaction rates given in the review article of Ichimaru and Kitamura (Ichimaru and Kitamura 1999) and results by Gasques et al. (Gasques et al. 2005).

Our approach provides an integrated, first-principles, systematic theory of screening effects due to electrons and associated nuclei. As such, it is possible for us to estimate the accuracy with which we can calculate our enhancement factors. In order to do that for white dwarfs, we must first generalize our formalism to a binary ionic mixture (BIM), e.g.
a mixture of $^{12}C$, $^{16}O$ nuclei and associated electrons, as representative of the species in a white dwarf. This is easily accomplished via the following substitutions in all our formulae:

\[
Z^2 \rightarrow \bar{Z}^2 = \frac{n_1Z_1^2 + n_2Z_2^2}{\bar{n}} \\
n \rightarrow \bar{n} = n_1 + n_2
\]

where $n_1$, $n_2$ are the average number densities of each species in the UDP. These substitutions arise naturally through a re-derivation of our theory for a BIM. Generalizations to more than two components is straightforward.

Ichimaru uses improvements over a standard procedure to obtain net enhancement factors. First, the screening length/dielectric constant of an electron gas in a jellium of positive ions is obtained. Then various sophisticated methods are used separately to obtain screening effects due to nuclei surrounding the moieties undergoing fusion. Physically reasonable mixture rules are utilized to obtain the overall enhancement of nuclear rates in a UDP, caused by the screening of the nuclear Coulomb repulsion by intervening charges. It is possible to gauge the accuracy of such calculations by using our systematic approach, where the magnitude of terms neglected can be estimated. In general we find our estimates for fusion rate enhancement in white dwarfs do agree with those of Ichimaru and Kitamura (Ichimaru and Kitamura 1999). One case where there is severe disagreement can be attributed to the breakdown of approximations made in this paper (see Table I). The values of the classical plasma parameter $\Gamma_{\text{classical}} = \bar{Z}^2e^2/r_0kT << 170$ ($r_0$ is the mean-free distance between particles) for the cases listed in Table I. As such there is no concern regarding the UDP being close to a crystallized state.

Gasques et al. (Gasques et al. 2005) utilize a re-parameterized version of the enhancement factor obtained by Slattery et al. (Slattery et al. 1982). The enhancement factor of Gasques et al is defined in terms of the plasma parameter of a one-component classical plasma. In our case, we have a binary mixture. It is nevertheless instructive to compare to our results the classical enhancement using a positively charged gas, with each nuclei carrying a charge $Ze$. In general, there is a reasonable agreement.

We note in passing that in order to obtain agreement between different approaches for calculating the fusion enhancement factor, it was essential to assume a relativistic description for the degenerate electrons in a white dwarf, while simultaneously considering a background of positive charges representing nuclei, at finite temperature. Assuming a non-relativistic description for electrons caused us to obtain enhancement factors that are much more conservative.
The numerical values obtained for our screening length and comparisons to Ichimaru’s enhancement factor (Ichimaru and Kitamura 1999) and that of a classical gas of nuclei (Gasques et al. 2005) have been encapsulated for white dwarfs near ignition in Table I.

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| Quantity | Current theory | (Ichimaru and Kitamura 1999) | (Gasques (2005)) |
|----------|---------------|-----------------------------|------------------|
| Model WD1 | Density | $3.0 \times 10^9 g - cm^{-3}$ | → |
|           | T (Kelvin) | $1.8 \times 10^8 K$ | → |
|           | Composition | $^{12}C, e^-$ | → |
|           | $\Gamma_{\text{enhance}}$ | 9.02 | 12.09 | 11.84 |
| Model WD2 | Density | $9.0 \times 10^9 g - cm^{-3}$ | → |
|           | T (Kelvin) | $1.1 \times 10^8 K$ | → |
|           | Composition | $^{12}C, e^-$ | → |
|           | $\Gamma_{\text{enhance}}$ | 32.7 | 23.10 | 27.96 |
| Model WD3 | Density | $9.0 \times 10^9 g - cm^{-3}$ | → |
|           | T (Kelvin) | $3.4 \times 10^7 K$ | → |
|           | Composition | $^{12}C, e^-$ | → |
|           | $\Gamma_{\text{enhance}}$ | 90.45 | 20.76 | 190.3 |
| Model WD4 | Density | $9.0 \times 10^9 g - cm^{-3}$ | → |
|           | T (Kelvin) | $1.1 \times 10^8 K$ | → |
|           | Composition | $^{12}C$ (75%), $^{16}O$ (25%)$e^-$ | → |
|           | $\Gamma_{\text{enhance}}$ | 36.60 | 23.12 | 31.62 |

Table 1: Comparison of corrected rate factors for white dwarfs near ignition, between our calculation and two previous calculations (Ichimaru and Kitamura 1999; Gasques et al. 2005). The enhancement of the fusion rate is calculated as $\exp(\Gamma_{\text{enhance}})$, $\Gamma_{\text{enhance}} = e^2/(\Lambda_{\text{screening}} k_B T)$. Our quantum-influenced screening length $\Lambda_{\text{screening}}$ is defined in dimensionless terms as $1/\sqrt{s_1}$ via Eqn[19]. The different scenarios for white dwarfs, viz., models WD1 – WD4 are described in Ichimaru’s paper (Ichimaru and Kitamura 1999). The reason for the major discrepancy for model 3 is that the temperature for this case is sufficiently low so that the approximations made in this paper do not hold.