Improved weighted NND scheme for shock-capturing

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Abstract. Aimed at decreasing the numerical dissipation of weighted non-oscillatory and non-free-parameter dissipation (WNND) scheme, we present an improved counterpart for shock-capturing. The new algorithm is based on the framework of Z-type weighting procedure with new local and global smoothness indicators. The performances of the proposed scheme are evaluated on several numerical tests governed by one-dimensional Euler equations. Numerical results indicate that the improved WNND scheme has advantages over the original WNND and third-order WENO-JS and WENO-Z schemes.

1. Introduction

Non-oscillatory and non-free-parameter dissipation (NND) difference scheme [1] was proposed after analyzing the dissipation and dispersion terms in the modified differential equation [2]. Due to its property of robust, efficiency and total variation diminishing (TVD), NND scheme and its modified versions are widely used in aerodynamic applications. To obtain higher accuracy and reduce heavy usage of logical statements, Wang and Shen [3] presented the weighted NND (WNND) scheme, where nonlinear weights are introduced to replace the minmod limiter in original NND scheme.

However, the original weighted schemes, such as WNND and weighted essentially non-oscillatory (WENO) [4], behaves relative large dissipation for scale-resolving simulations. Therefore, many developments are presented continuously [5]. Among them, the modification of Z-type nonlinear weights proposed by Borges et al [6] is an alternative framework to improve the nonlinear weighted schemes. For example, Li et al [7] improved the third-order WENO scheme in this framework by devising new global smoothness indicators.

In this paper, we develop the WNND scheme after the inspiration of Ref. [6] and [7]. Numerical tests are performed to show the good performance of the improved scheme, and comparisons are also given between the WNND and the third-order WENO schemes.

2. NND and WNND schemes

In this section, a brief review is given on NND and WNND schemes. For simplicity, the one-dimensional hyperbolic conservation law is used for illustration:

$$u_t + f(u)_x = 0.$$ (1)

With the domain discretized into uniform intervals of $\Delta x$, the semi-discretized conservative scheme can be written as:

$$\left( u_i \right)_t = -\left( h_{i+1/2} - h_{i-1/2} \right)/\Delta x = -\left( \hat{f}_{i+1/2} - \hat{f}_{i-1/2} \right)/\Delta x,$$ (2)

where $\hat{f}_{i+1/2}$ approximates $h_{i+1/2} = h(x_{i+1/2})$ to a high order with $h(u)$ implicitly defined by
When applied in numerical computation, \( f(u) \) is split into two parts:
\[
  f(u) = f^+(u) + f^-(u).
\]
In this paper, Steger-Warming splitting method is used. Accordingly, \( \hat{f}_{i+1/2} \) can be expressed as:
\[
  \hat{f}_{i+1/2} = \hat{f}_{i+1}^+ + \hat{f}_{i+1}^-.
\]

2.1. NND scheme

The NND scheme proposed by Zhang [1] is given as:
\[
\begin{align*}
\hat{f}_{i+1/2}^+ &= f_i^+ + \frac{1}{2} \min \text{mod} \left( \Delta f_{i+1/2}^+, \Delta f_{i-1/2}^+ \right) \\
\hat{f}_{i+1/2}^- &= f_i^- - \frac{1}{2} \min \text{mod} \left( \Delta f_{i+1/2}^-, \Delta f_{i-1/2}^- \right)
\end{align*}
\]
(6)
where \( \Delta f_{i+1/2}^+ = f_{i+1}^+ - f_i^+ \), and the minmod limiter is given by
\[
\min \text{mod}(a,b) = \begin{cases} 
0, & ab \leq 0 \\
\text{sgn}(a) \cdot \min(|a|,|b|), & a > 0 \\
\text{sgn}(a) = 0, & a = 0 \\
-1, & a < 0
\end{cases}
\]
(7)

2.2. WNND scheme

The original WNND scheme given by Wang and Shen [3] used weighted functions to combine the two parts of the limiter in Eq. (6). The optimal order of the WNND scheme is third-order other than second-order of WNND scheme. The formula of WNND are given as
\[
\begin{align*}
\hat{f}_{i+1/2}^+ &= f_i^+ + \frac{1}{2} \left( a_i^+ \Delta f_{i+1/2}^+ + a_i^- \Delta f_{i-1/2}^+ \right) \\
\hat{f}_{i+1/2}^- &= f_i^- - \frac{1}{2} \left( a_i^+ \Delta f_{i+1/2}^- + a_i^- \Delta f_{i-1/2}^- \right)
\end{align*}
\]
(8)
where
\[
\begin{align*}
a_i^+ &= \frac{\alpha_i^+}{\alpha_i^+ + \alpha_i^-}, \quad a_i^- = \frac{\alpha_i^-}{\alpha_i^+ + \alpha_i^-} \\
\beta_i^+ &= |\Delta f_{i+1/2}^+|, \quad \beta_i^- = |\Delta f_{i-1/2}^-| \\
\beta_i^+ &= |\Delta f_{i+1/2}^-|, \quad \beta_i^+ = |\Delta f_{i-1/2}^+|
\end{align*}
\]
(9)
The linear weights has the value of \( d_i^+ = 1/6, d_i^- = 1/3 \). It is denoted as WNND-WS in the following.

3. Improvement on WNND scheme

Inspired by the improvements of WENO schemes in Ref. [6] and [7], the new nonlinear weighting method is given to replace the Eq. (9)
\[
\begin{align*}
\alpha_i^+ &= \frac{\alpha_i^+}{\alpha_i^+ + \alpha_i^-}, \quad a_i^+ = d_i^+ \left( 1 + \frac{\tau_i^+}{\varepsilon + \beta_i^+} \right), \\
\beta_i^+ &= |\Delta f_{i+1/2}^+|, \quad \beta_i^- = |\Delta f_{i-1/2}^-|, \\
\beta_i^+ &= |\Delta f_{i+1/2}^-|, \quad \beta_i^+ = |\Delta f_{i-1/2}^+|
\end{align*}
\]
(10)
where \( \tau^+ \) are the global smoothness indicators, \( d_i^+ \) and \( \beta_i^+ \) have the same formula of Eq. (9) . The improved scheme is named as WNND-L.

For a nonlinear scheme, the modified wave number can be obtained by the approximate dispersion relation (ADR) method. In general, the WNND schemes have better spectral properties compared with WENO3-JS and WENO3-Z schemes as shown in Fig. 1. Besides, the improved WNND-L scheme has better dispersion and dissipation properties than original WNND-WS.
4. Numerical tests

In this section, three problems with shock waves are performed. The governing equation is the one-dimensional Euler equation in conservation form:

$$Q_t + F_x = 0,$$

where

$$Q = (\rho, \rho u, E)^T, \quad F = (\rho u, \rho u^2 + p, (E + p)u)^T,$$

the equation of state is given by

$$p = (\gamma - 1)(E - \frac{1}{2} \rho u^2),$$

where $\rho$, $u$, $p$, and $E$ refer to the density, velocity, pressure and total energy, respectively. $\gamma$ is ratio of specific heats with $\gamma = 1.4$.

The third-order TVD Runge-Kutta method is employed for temporal discretization.

4.1. Shock tube problem of Sod

For the Sod problem, the initial conditions are:

$$\begin{cases} \rho, u, p = \begin{bmatrix} 0.125, & 0, & 0.1 \end{bmatrix} & -5 \leq x < 0, \\ \begin{bmatrix} 1.000, & 0, & 1.0 \end{bmatrix} & 0 \leq x \leq 5, \end{cases}$$

and the computation is advanced till $t = 2.0$.

The simulated density of the Sod problem is shown in Fig. 2. We can see that all the schemes are essentially non-oscillatory near the discontinuities. The hierarchy of the resolution performance from good to poor is WNND-L, WNND-WS, WENO3-Z and WENO3-JS.

4.2. Shock tube problem of Lax

For the Lax problem, the initial conditions are:

$$\begin{cases} \rho, u, p = \begin{bmatrix} 0.445, & 0.698, & 0.3528 \end{bmatrix} & -5 \leq x < 0, \\ \begin{bmatrix} 0.500, & 0.000, & 0.5710 \end{bmatrix} & 0 \leq x \leq 5, \end{cases}$$
and the final time is $t = 1.3$.

Fig. 3 presents the simulated density of the Lax problem. It can be observed that the results of WNND-L schemes show enhancement solutions compared with WNND-WS. Also the WNND schemes behave better than WENO3 schemes.

![Figure 2. Density of the Shock tube problem of Sod computed by WNND-WS, WNND-L, WENO3-JS and WENO3-Z schemes with 200 points at $t = 2.0$.](image2)

![Figure 3. Density of the Shock tube problem of Lax computed by WNND-WS, WNND-L, WENO3-JS and WENO3-Z schemes with 200 points at $t = 1.3$.](image3)

4.3. **Shock-density wave interaction**

This test problem was first proposed by Shu and Osher, and the initial conditions on the computational domain $-4.5 \leq x \leq 4.5$ are as follows:

$$
(\rho, u, p) = \begin{cases}
(3.857143, 2.629369, 10.33333) & x < 4, \\
(1.0 + 0.2 \sin(5x), 0, 1.0) & x \geq 4.
\end{cases}
$$

The numerical solutions are obtained on 400 grid points at $t = 1.8$. Density fields for WNND-WS, WNND-L, WENO3-JS and WENO3-Z schemes are shown in Fig. 4. The “exact” solution is obtained by WENO5-JS scheme with 1600 points. In Fig 4, it can be observed that WENO3-JS scheme yields
excessively damped solutions, and WENO-Z scheme show basically equivalent performance to WNND-WS. Moreover, the improved WNND-L give the best numerical solutions among the schemes.

![Figure 4. Density distribution of the shock–density wave interaction computed by WNND-WS, WNND-L, WENO3-JS and WENO3-Z schemes with 400 points at t = 1.8.](image)

5. Conclusion
In this paper, an improved WNND scheme, named as WNND-L, for compressible flows are presented. The new weighting method is based on the Z-type weighting method with new local and global smoothness indicators. Numerical tests of several one-dimensional Euler cases indicate that the WNND-L scheme behaves less dissipation and higher resolution than the original WNND-WS and third-order WENO-JS and WENO-Z schemes.

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