Velocity-Field Theory, Boltzmann’s Transport Equation, Geometry and Emergent Time

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Abstract

Boltzmann equation describes the time development of the velocity distribution in the continuum fluid matter. We formulate the equation using the field theory where the velocity-field plays the central role. The properties of the matter particles, which constitute the fluid, appear as the density and the viscosity. Fluctuation is examined, and is clearly discriminated from the quantum effect. The time variable is emergently introduced through the computational process step. The collision term, for the (velocity)**4 potential (4-body interaction), is explicitly obtained and the (statistical) fluctuation is closely explained. The present field theory model does not conserve energy and is an open-system model. (One dimensional) Navier-Stokes equation, Burger’s equation, appears. In the latter part, we present a way to directly define the distribution function by use of the geometry, appearing in the energy expression, and Feynman’s path-integral.

Key Words : Boltzmann equation; velocity field theory; statistical fluctuation; computational step number; open system; geometry.

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1 Introduction

Boltzmann equation was introduced to explain the second law of the thermodynamics in the dynamical way, in 1872, by Boltzmann. We consider the (visco-elastic) fluid matter and examine the dynamical behavior using the velocity-field theory. The scale size we consider is far bigger than the atomic scale ($\sim 10^{-10}$m) and is smaller than or nearly equal to the optical microscope scale ($\sim 10^{-6}$m). The equation describes the temporal development of the distribution function $f(t, x, v)$ which shows the probability of a fluid-molecule (particle) having the velocity $v$ at the space $x$ and time $t$.

We reformulate the Boltzmann equation using the field theory of the velocity field $u(x, \ 't')$. Basically it is based on the minimal energy principle. We do not introduce time $t$. Instead of $t$, we use the computational step number $n$. The system we consider consists of the huge number of fluid-particles (molecules) and the physical quantities, such as energy and entropy, are the statistically-averaged ones. It is not obtained by the deterministic way like the classical (Newton) mechanics. We introduce the statistical ensemble by using the well-established field-theory method, the background-field method. Renormalization phenomenon occurs not from the quantum effect but from the statistical fluctuation due to the inevitable uncertainty caused by 1) the step-wise (discrete-time) formulation and 2) the continuum formulation for the space (of the real world).

The dissipative system we consider is characterized by the dissipation of energy. Even for the particle classical (Newton) mechanics, the notion of energy is somewhat obscure when the dissipation occurs. We consider the movement of a particle under the influence of the friction force. The emergent heat (energy) during the period $[t_1, t_2]$ can not be written as the following popularly-known form.

$$\int_{x_1}^{x_2} F_{\text{friction}} \, dx = [E(x(t), \dot{x}(t))]^t_{t_1} = E|_{t_2} - E|_{t_1} , \ x_1 = x(t_1) , \ x_2 = x(t_2) , \ (1)$$

where $x(t)$ is the orbit (path) of the particle. It depends on the path (or orbit) itself. It cannot be written as the form of difference between some quantity at time $t_1$ and $t_2$. In this situation, we realize the time itself should be re-considered when the dissipation occurs. Owing to Einstein’s idea of “space-time democracy”, we have stuck to the standpoint that space and time should be treated on the equal footing. We present here the step-wise approach to the time-development.

We do not use time variable. Instead we use the computational-process step number $n$. Hence the increasing of the number $n$ is identified as the time development. The connection between step $n$ and step $n - 1$ is determined...
by the *minimal energy principle*. In this sense, time is ”emergent” from the minimal energy principle. The direction of flow (arrow of time) is built in from the beginning. \[1\]

In the latter part of this paper, an approach to the statistically-averaging procedure, based on the geometry of the mechanical dynamics, is presented.

The content is described as follows. The step-wise dynamical equation is presented in Sec.2. We start with the n-th step energy functional. By regarding n steps as the time $t_n$, we derive (1 dim) Navier-Stokes equation. In Sec.3, the orbit (path) of the fluid particle is explained in this step-wise formalism. The total energy and the energy rate are also explained. The statistical fluctuation is closely explained in Sec.4. Especially the difference from the quantum effect is stressed. Using the path-integral, we take into account the fluctuation effect. Owing to the present velocity-field formalism, we can obtain Boltzmann’s equation, as described in Sec.5, up to the collision term. This step-wise approach is applied to the mechanical system in Sec.6. We take a simple dissipative model: the harmonic oscillator with friction. The trajectory is solved in the step-wise way. We find the total energy changes as the step proceeds. From the n-step energy expression we can extract the geometrical structure (the metric) of the trajectory. The metric is used, in Sec.7, to define the statistical ensemble of the system of N viscous particles. We propose some models using the geometrically-basic quantities: the length and the area. Conclusion is given in Sec.8. A few appendices are provided to supplement the text. App.A treats (1+3) dimensional field theory in this step-wise formalism. App.B is the calculation of the statistical fluctuation effect and it supplements Sec.4. An additional mechanical model (Spring-Block model) is described with the calculation result in App.C.

## 2 Emergent Time and Diffusion (Heat) Equation

We consider 1 dimensional viscous fluid, and the velocity field $\{u(x); -\infty < x < \infty\}$ describes the velocity distribution in the 1 dim space. Let us take the following energy functional\[2, 3\] of the velocity-field $u(x)$,

\[
I_n[u(x); u_{n-1}(x), \sigma_{n-1}(x), \tilde{\rho}_{n-1}(x)] = \int dx \left\{ \frac{\sigma_{n-1}}{2\tilde{\rho}_{n-1}} \left( \frac{du}{dx} \right)^2 + V(u) + u \frac{dV'(x)}{dx} + \frac{1}{2h} (u - u_{n-1})^2 \right\} + \int_0^{t_n} V(u) = \frac{m^2}{2} u^2 + \frac{\lambda}{4!} u^4,
\]

\[1\] For a recent review on the nature of time, see ref.\[1\].
Figure 1: The energy functional $I_n[u(x)]$, of the velocity-field $u(x)$. 

\[ I_n[u(x)] = \int dx \left\{ \frac{\sigma_{n-1}}{2\rho_{n-1}} \left( \frac{du}{dx} \right)^2 + V(u) + u \frac{dV^1(x)}{dx} + W_n(u) \right\} + I_0^n, \]

where $I_0^n$ is a 'constant' term which is independent of $u(x)$. Later we will fix it. $m^2$ is a parameter with the dimension of the mass density: $(\text{mass of the fluid-particle})/2l$. The quantity $V^1(x)$ is the total energy of the fluid. The velocity potential $V(u)$ has the mass term and the 4-body interaction term.\[ V^1(x) \]

\[ n = 1, 2, \cdots, \quad u = u(x), \quad u_{n-1} = u_{n-1}(x), \quad \sigma_{n-1} = \sigma_{n-1}(x), \quad \rho_{n-1} = \rho_{n-1}(x), \]

where $u_{n-1}, \rho_{n-1}$ and $\sigma_{n-1}$ are the step (n-1) distributions of the velocity, the mass-density and the viscosity respectively. $I_0^n$ is a 'constant' term which is independent of $u(x)$. Later we will fix it. $m^2$ is a parameter with the dimension of the mass density: $(\text{mass of the fluid-particle})/2l$. The quantity $V^1(x)$ is the total energy of the fluid. The velocity potential $V(u)$ has the mass term and the 4-body interaction term. \[ V^1(x) \]

$\sqrt{\lambda \sigma} = (M/L)^{3/2}T$, \[ \sqrt{h^3 \lambda \sigma} = [m^{-3} \sqrt{\lambda \sigma}] = [h \sqrt{\lambda \sigma}/m] = T. \]

More generally, the 2-body and 4-body couplings depend on $n$: $(m^2)_{n-1}$ and $\lambda_{n-1}$. Here we consider the simple case, the couplings do not depend on $n$ ('time'-independent).
position-dependent potential. \( \frac{dV^1(x)}{dx} \) is the external source (force) in this velocity-field theory. \( h \) is some constant which will be identified as the time-separation for one step. \( u_{n-1}(x), \sigma_{n-1}(x) \) and \( \bar{\rho}_{n-1}(x) \) are given distributions at the n-th step evaluation. The n-th step velocity field \( u_n(x) \) is given by the minimal principle of the n-th energy functional \( I_n(u) \). This approach is called ”discrete Morse flows method”\([2, 3, 4]\).

For simplicity we take the periodic boundary condition for the space.

\[
 u(x) = u(x + 2l) \quad ,
\]

where \( 2l \) is the periodic length. We may restrict the space region as \( -l \leq x \leq l \). The variation equation \( \delta I_n(u) = 0(u(x) \rightarrow u(x) + \delta u(x)) \) gives

\[
 \frac{1}{h}(u_n(x) - u_{n-1}(x)) = \frac{\sigma_{n-1}}{\rho_{n-1}} \frac{d^2 u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u_n} - \frac{dV^1(x)}{dx} \quad ,
\]

where we have replaced the minimal solution by \( u_n \). From the construction, we have the relation:

\[
 I_n[u_n] \leq I_n[u_{n-1}] \quad .
\]

We, however, cannot say \( I_n[u_n] \leq I_{n-1}[u_{n-1}] \). The above equation describes the n-th step velocity field \( u_n(x) \) in terms of \( u_{n-1}(x) \) and vice versa. Hence it can be used for the computer simulation.\([4]\)

We here introduce the discrete time variable \( t_n \) as the step number \( n \) of \( u_n \).

\[
 t_n = nh = n\tau_0 \times \left( \frac{h}{\tau_0} \right) \quad , \quad \tau_0 \equiv h\sqrt{\lambda\sigma_0/|m|} \quad , \quad n = 1, 2, \cdots \quad ,
\]

where \( \sigma_0 \) is the representative value (constant) of the system viscosity and \( \tau_0 \) is the time unit.\([\text{footnote} 5]\) The eq.\([5]\) is, in terms of the ‘renewed’ field \( u(x, t) \),

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\( ^4 \) The lattice Boltzmann method\([5]\) is the most suitable one.

\( ^5 \) Note \( [\tau_0]=T \). See the footnote of eq.\([2]\). Generally the time \( t_n \) can be introduced by \( t_n = f(n)h \) where \( f(n) \) is a function of \( n \). The form of \( f(n) \) defines the time coordinate. The change of the form is the transformation of the time coordinate. The simple one is \( f(n) = an + b \). In the text \( f(n) = n \) is taken. If we take \( f(n) = -n \), the time flow is introduced in the inverse way.
expressed as

\[
\frac{1}{h}(u(x, t_{n-1} + h) - u(x, t_{n-1})) = \\frac{\sigma(x, t_{n-1})}{\tilde{\rho}(x, t_{n-1})} \frac{\partial^2 u(x, t_n)}{\partial x^2} - \frac{\delta V(u(x, t_n))}{\delta u(x, t_n)} - \frac{\partial V^1(x, t_n)}{\partial x},
\]

(8)

where we use \(u(x, t_n) \equiv u_n(x)\), \(t_n = t_{n-1} + h\). As \(h \to 0\), we obtain

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\sigma(x, t)}{\tilde{\rho}(x, t)} \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\delta V(u(x, t))}{\delta u(x, t)} - \frac{\partial V^1(x, t)}{\partial x},
\]

(9)

where we have replaced both \(t_n\) and \(t_{n-1}\) by \(t\). This is, when \(\sigma = \text{const}\) and \(\tilde{\rho} = \text{const}\), 1 dim diffusion equation with the potential \(V(u)\).

We remind that the variational principle for the n-step energy functional \(I_n[u(x)]\) \(\mathcal{2}\), \(\delta I_n = I_n[u + \delta u] - I_n[u] = 0\), gives \(u_n(x)\) for the given \(u_{n-1}(x), \sigma_{n-1}(x)\) and \(\tilde{\rho}_{n-1}(x)\). We regard the increase of the step number as the time development.\(\mathcal{5}\) Taking into account the fact that, at the (n-1)th-step, the matter-particle at the point \(x\) flows at the speed of \(u_{n-1}(x)\), the energy functional \(I_n\), \(\mathcal{2}\), should be replaced by the following one\(\mathcal{2, 4}\).

\[
\tilde{I}_n[u(x)] = \int dx \{ \frac{\sigma_{n-1}}{2\tilde{\rho}_{n-1}} \left( \frac{du}{dx} \right)^2 + V(u) + u \left( \frac{dV^1(x)}{dx} \right) + \frac{1}{2h} (u(x + hu_{n-1}) - u_{n-1})^2 \} + \tilde{\rho}_0\ ,
\]

\[
V(u) = \frac{m^2}{2} u^2 + \frac{\lambda}{4!} u^4, \quad n = 1, 2, \ldots ,
\]

\(u = u(x), \ u_{n-1} = u_{n-1}(x), \ \sigma_{n-1} = \sigma_{n-1}(x), \ \tilde{\rho}_{n-1} = \tilde{\rho}_{n-1}(x)\). \(\mathcal{10}\)

Note that \(u(x) - u_{n-1}(x)\) in eq.\(\mathcal{2}\) is replaced by \(u(x + hu_{n-1}(x)) - u_{n-1}(x)\).

For the simple case of no potential, \(V = 0\) , and no external force, \(\frac{dV^1}{dx} = 0\),

\[
J_n[u(x)] = \int dx \{ \frac{\sigma_{n-1}}{2\tilde{\rho}_{n-1}} \left( \frac{du}{dx} \right)^2 + \frac{1}{2h} (u(x + hu_{n-1}(x)) - u_{n-1}(x))^2 \} +'\const' \ , \(\mathcal{11}\)

The above functional is equivalent to \(I_n[u(x)]\) with the potential.

\[
V(u) = (u(x) - u_{n-1}(x))u_{n-1}(x) \frac{du(x)}{dx} + O(h) \ .
\]

(12)

where we consider the case of sufficiently-small \(h\). Eq.\(\mathcal{5}\) gives us the following equation as the minimal equation for \(J_n[u] : \ \delta J_n[u] = 0\)

\[
\frac{1}{h}(u_n(x) - u_{n-1}(x)) = \frac{\sigma_{n-1}}{\tilde{\rho}_{n-1}} \frac{d^2 u_n}{dx^2} - u_{n-1}(x) \frac{du_n}{dx} + O(h) \ .
\]

(13)

\(\mathcal{6}\) Time is defined here by the energy-minimal principle.
Hence the step-wise recursion relation (5) is corrected as

\[
\frac{1}{h}(u_n(x) - u_{n-1}(x)) + u_{n-1}(x) \frac{du_n(x)}{dx} = \frac{\sigma_{n-1}}{\tilde{\rho}_{n-1}} \frac{d^2 u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u_n} - \frac{dV_1(x)}{dx},
\]

As done before, let us replace the step number \( n \) by the discrete time \( t_n = nh \). Taking the continuous time limit \( (h \to 0) \), we obtain

\[
\frac{\partial u(x, t)}{\partial t} + u(x, t) \frac{\partial u(x, t)}{\partial x} = \frac{\sigma(x, t)}{\tilde{\rho}(x, t)} \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\delta V(u(x, t))}{\delta u(x, t)} - \frac{\partial V_1(x, t)}{\partial x},
\]

This is, when \( \sigma = \text{const} \) and \( \tilde{\rho} = \text{const} \), Burgers’s equation (with the velocity potential \( V(u) \) and the external force \( \frac{\partial V_1}{\partial x} \)) and is considered to be 1 dimensional Navier-Stokes equation. Note that the non-linear term in the LHS of eq. (15) appears not from the potential (velocity-field interaction) but from the change \( u(x) \) in (2) to \( u(x + h u_{n-1}) \) in (10), namely, the consistency between the (space) coordinate \( x \) and the velocity field \( u(x) \) in the step-flow. The differential operator: \( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial t} \), appearing in LHS, is called Lagrange derivative. The relations between \( \sigma_{n-1} \) and \( \sigma_n \), and \( \tilde{\rho}_{n-1} \) and \( \tilde{\rho}_n \), which describe their step-flow (‘time’-development), are given in Sec. 5.

The equation (15), for the massless case \( m = 0 \), is invariant under the global Weyl transformation (the scale transformation).

\[
\begin{align*}
V^1(x, t) &\to e^{-2\varepsilon} V^1(e^{\varepsilon} x, e^{2\varepsilon} t), \\
u(x, t) &\to e^{-\varepsilon} u(e^{\varepsilon} x, e^{2\varepsilon} t), \\
\frac{\sigma(x, t)}{\tilde{\rho}(x, t)} &\to \frac{\sigma(e^{\varepsilon} x, e^{2\varepsilon} t)}{\tilde{\rho}(e^{\varepsilon} x, e^{2\varepsilon} t)}, \\
\frac{\partial}{\partial x} = \partial_x &\to e^{-\varepsilon} \partial_x, \\
\frac{\partial}{\partial t} = \partial_t &\to e^{-2\varepsilon} \partial_t, \\
&\quad t \to e^{\varepsilon} t, \\
&\quad x \to e^{\varepsilon} x,
\end{align*}
\]

where \( \varepsilon \) is the real constant parameter. When it is small: \( |\varepsilon| \ll 1 \), the variation \( \delta u \) is given by

\[
\delta u = \varepsilon \{-u + x \partial_x u + 2t \partial_t u\} + O(\varepsilon^2),
\]

For simplicity, we explain in one space-dimension (dim). The generalization to 2 dim and 3 dim is straightforward. Furthermore the ordinary field theory (not using the velocity field but the particle field) is described by this step-wise approach in App.A.

\[\text{7 If we consider the mass } m \text{ here appears in some dynamical way (, for example, through the spontaneous breakdown , } m \to e^{-\varepsilon} m \text{ makes eq. (15) global Weyl invariant.}\]
3 Space Orbit (Path) and Total Energy

The space coordinate $x$ always appears in the velocity field $u_n(x)$. We can introduce the n-th step space coordinate $x_n$ as

$$u_n(x_n) = \frac{x_{n+1} - x_n}{\tau_0} , \quad n = 0, 1, 2, \cdots$$

$$x_{n+1} = x_n + \tau_0 u_n(x_n) ,$$

(18)

where $x_0$ is a given initial position and $\tau_0 = h\sqrt{\lambda \sigma_0/|m|}$ in eq. (17). We can trace the position of the matter-point, which was at $x_0$ at the initial (0-th) step, along the step process: $x_0, x_1, x_2, \cdots$. After N steps, the matter-point reaches

$$x_N = x_{N-1} + \tau_0 u_{N-1}(x_{N-1}) = x_0 + \tau_0 \sum_{n=0}^{N-1} u_n(x_n) .$$

(19)

In terms of the continuous time,

$$x(T) = x_0 + \frac{\tau_0}{h} \int_0^T u(x(t), t) dt ,$$

(20)

where $T = N h, t_n = n h, x(t_n) = x_n, u(x, t_n) = u_n(x)$. $x(t)$ is the orbit or path of the matter-point, at $x_0$ initially, "moving" in the N steps.

At the n-th step, the total energy of the system, $E_n$, is given by

$$E_n \equiv \tilde{I}_n[u_n] , \quad \text{where} \quad \left. \frac{\delta \tilde{I}_n[u]}{\delta u(x)} \right|_{u=u_n} = 0 ,$$

$$\tilde{I}_n[u] = \int dx \left\{ \frac{\kappa_{n-1}}{2} (\frac{du}{dx})^2 + V(u) + u \frac{dV}{dx} + \frac{1}{2h} (u(x + hu_{n-1}) - u_{n-1})^2 \right\} + \tilde{I}_n^0 ,$$

$$\kappa_n(x) = \frac{\sigma_n(x)}{\rho_n(x)} ,$$

(21)

where $\kappa = \kappa(x)$ is introduced. $\tilde{I}_n^0$ is taken as

$$\tilde{I}_n^0 = -\int dx \left\{ u_n \frac{dV}{dx} + \frac{1}{2h} (u_n(x + hu_{n-1}) - u_{n-1})^2 \right\} + \int dx \left\{ \frac{\kappa_0}{2} \left( \frac{du_0}{dx} \right)^2 + V(u_0) + u_0 \frac{dV}{dx} + \frac{\rho_0}{2} u_0^2 \right\} .$$

(22)

The above one is chosen in such a way that the total energy $E_n$ keeps the initial energy (the second integral of (22)) when the dissipative terms, $\frac{\kappa_{n-1}}{2} (\frac{du}{dx})^2$ and $V(u)$, do not appear.
The system total energy $E_n$ generally changes as the step number increases.

$$W(t_n) = \frac{\hbar}{\tau_0} \frac{dE(t_n)}{dt_n} = \tilde{I}_{n+1}[u_{n+1}] - \tilde{I}_n[u_n] =$$

$$\frac{1}{\tau_0} \int dx \left\{ \frac{1}{2} \left( \kappa_n \left( \frac{du_{n+1}}{dx} \right)^2 - \kappa_{n-1} \left( \frac{du_n}{dx} \right)^2 \right) + V(u_{n+1}) - V(u_n) + (u_{n+1} - u_n) \frac{dV^1}{dx} \right. +$$

$$+ \frac{1}{2\hbar} \left\{ (u_{n+1} + hu_n \frac{du_{n+1}}{dx} - u_n)^2 - (u_n + hu_{n-1} \frac{du_n}{dx} - u_{n-1})^2 \right\} \right\} + \frac{\tilde{r}_{n+1}^0 - \tilde{r}_n^0}{\tau_0} \ (23)$$

where $E(t_n) \equiv E_n$ and $W(t_n)$ is the energy rate. From the above formula we get the expression for the energy at $t = Nh = t_n$.

$$E(t_N) = \tilde{I}_N[u_N] = \tilde{I}_{N-1}[u_{N-1}] + \tau_0 W(t_{N-1}) = \tilde{I}_1[u_1] + \tau_0 \sum_{n=1}^{N-1} W(t_n) \ . \ (24)$$

When we regard the process of the increasing step-number as the time development, the system generally does not conserve energy.\footnote{We will numerically confirm the non-conservation later in Sec.6.} $E(t_n)$ generally changes step by step. We can physically understand that the increase or decrease of the total system energy is given or subtracted by the outside (environment). The energy functional \footnote{Energy constantly comes in or goes out. App.C shows such an example.} describes the open-system dynamics. When $E(t_n)$ satisfies

$$W(t_n) = \frac{\hbar}{\tau_0} \frac{dE(t_n)}{dt_n} \rightarrow W_0(\text{constant}) \quad \text{as} \ n \rightarrow \infty \ , \ (25)$$

we say the system finally reaches the steady energy-state.\footnote{Energy does not go out and does not come in.} For the special case of $W_0 = 0$, we say the system finally reaches the constant energy-state\footnote{}.

In Sec.\footnote{We will numerically confirm the non-conservation later in Sec.6.} we treat the $W_0 = 0$ case. As the example of the more general case ($W_0 \neq 0$), another model is given in App.C.

### 4 Statistical Fluctuation Effect

We are considering the system of large number of matter-particles, hence the physical quantities, such as energy and entropy, are given by some statistical average. In the present approach, the system behavior $u_n(x) = u(x, t_n)$ is completely determined by eq.\footnote{We will numerically confirm the non-conservation later in Sec.6.} when the initial configuration $u_0(x) =$
\textit{u}(x,0) \textit{is} given. \textit{We have obtained the solution by the continuous variation} \( \delta u(x) \) to \( \tilde{I}_n[u] \). In this sense, \( u_n(x) \) \textit{is} the 'classical path'. \textit{Here} we should note that the present formalism is an \textit{effective} approach to calculate the physical properties of this \textit{statistical system}. \textit{Approximation} is made in the following points:

1) So far as \( h \neq 0 \), the \textit{finite time-increment} gives \textit{uncertainty} to the minimal solution \( u_n(x) \). \textit{This} is because we cannot specify the minimum configuration definitely, but can only do it with \textit{finite} uncertainty.

2) The real fluid matter is made of many micro particles with small but \textit{non-zero size}. The existence of the characteristic particle size gives uncertainty to the minimal solution in this (space-)continuum formalism. Furthermore the particle size is not constant but does distribute in the statistical way. The shape of each particle differs. The present continuum formalism has limitation to describe the real situation accurately.

3) The system energy generally changes step by step. The present model describes an \textit{open-system}. It means the present system \textit{energetically} interacts with the outside. Such interaction is caused by the dissipative term in \( \tilde{I}_n[u(x)] \).

We claim the fluctuation comes \textit{not} from the \textit{quantum effect} but from the \textit{statistics} due to the uncertainty which comes from the \textit{finite} time-separation and the spacial distribution of \textit{size} and \textit{shape}.

\textit{To take into account this fluctuation effect, we newly define the n-th energy functional} \( \hat{\Gamma}_n[u(x)] = \hat{\Gamma}_n[u(x); u_{n-1}(x), \sigma_{n-1}(x), \tilde{\rho}_{n-1}(x)] \) in terms of the original one \( \tilde{I}_n[u(x)], \) using the path-integral.

\[
e^{-\frac{1}{\hbar} \Gamma_n[u(x)]} = \int \mathcal{D}u(x)e^{-\frac{1}{\hbar} \tilde{I}_n[u(x)]}, \]

\[
\hat{I}_n[u(x)] = \int dx \left\{ \frac{\kappa_{n-1}}{2} \left( \frac{du}{dx} \right)^2 + V(u) + u(x) \frac{dV^1(x)}{dx} + \frac{1}{2h}(u(x + hu_{n-1}) - u_{n-1})^2 \right\} + \hat{I}_0^n, \]

\[
V(u) = \frac{m^2}{2} u^2 + \frac{\lambda}{4!} u^4, \quad (26)\]

In the above path-integral expression, \textit{all} paths \( \{u(x); -l \leq x \leq l\} \) are taken into account.

We are considering the minimal path \( u_n(x) \) as the dominant configuration and the small deviation \( q(x) \) around it.

\[
u(x) = u_n(x) + \sqrt{\alpha} q(x), \quad |\sqrt{\alpha} q| \ll |u_n|, \quad \left. \frac{\delta \hat{I}_n[u]}{\delta u} \right|_{u=u_n} = 0, \quad (27)\]
In eq.(26) and eq.(27), a new expansion parameter $\alpha$ is introduced. ($[\alpha] = [I_n] = M L^2 T^{-2}$) As the above formula shows, $\alpha$ should be small. The concrete form should be chosen depending on problem by problem. It should not include Planck constant, $\hbar$, because the fluctuation does not come from the quantum effect. It should be chosen as

1) the dimension is consistent,

2) it is proportional to the small scale parameter which characterizes the relevant physical phenomena such as the mean-free path of the fluid particle,

3) the precise value should be best-fitted with the experimental data.

The background-field method\cite{6,7} tells us to do the Taylor-expansion around $u_n$.\footnote{The background-field method was originally introduced to quantize the gravitational field theory. Here we borrow the method only to define the statistical distribution measure.}

$$I_n[u(x)] = I_n[u_n(x) + \sqrt{\alpha}q(x)] = \sum_{l=0}^{\infty} \frac{\alpha^{l/2} q(x)^l}{l!} \frac{\delta I_n[u]}{\delta u(x)^l} |_{u_n} = \sum_{l=0}^{\infty} S_l[u_n] , \quad (28)$$

Then the $n$-th energy functional, $\Gamma_n[u(x)]$, is expressed in the perturbed form up to the second order.

$$e^{-\frac{1}{\alpha} I_n[u_n(x)]]} = \int Dq(x) \exp \left( -\frac{1}{\alpha} (S_0 + S_2 + O(q^3)) \right) = e^{-\frac{1}{\alpha} I_n[u_n]} \times$$

$$\int Dq \exp \left[ \int dx \left\{ -\frac{q(x)^2}{2!} \left( \frac{\delta}{\delta u(x)} \right)^2 I_n[u] |_{u_n} + O(q^3) \right\} \right] , \quad \frac{\delta}{\delta u(x)} I_n[u] |_{u_n} = 0 ,$$

$$S_0 = I_n[u_n] , \quad S_1 = \int dx q(x) \left( \delta I_n[u] / \delta u \right) |_{u_n} = 0 ,$$

$$\frac{1}{\alpha} S_2 = \frac{q(x)^2}{2!} \left( \frac{\delta}{\delta u(x)} \right)^2 I_n[u] |_{u_n} = \frac{\kappa_{n-1}}{2} \left( \frac{dq}{dx} \right)^2 + \frac{m^2}{2} + \frac{\lambda}{2} u_n^2 q^2 + \frac{1}{2\hbar} (q + hu_{n-1}) \frac{d^2q}{dx^2}$$

$$\equiv \frac{1}{2} \frac{d}{dx} (\kappa_{n-1} q \frac{dq}{dx}) + \frac{1}{2} q Dq + O(h) , \quad D \equiv -\frac{d}{dx} \kappa_{n-1} \frac{d}{dx} + \lambda u_n^2 + m^2 + \frac{1}{h} - \frac{du_{n-1}}{dx} \quad (29)$$

where we make the Gaussian(quadartic, 1-loop) approximation.\footnote{\textit{O}(h) = (h/2)u_{n-1}^2 (dq/dx)^2 may be ignored for $h \ll 1$.}

$$e^{-\frac{1}{\alpha} I_n[u_n(x)]]} = e^{-\frac{1}{\alpha} I_n[u_n]} \times (\text{det } D)^{-1/2} ,$$

$$(\text{det } D)^{-1/2} = \exp \left\{ -\frac{1}{2} \text{Tr} \ln D \right\} = \exp \left\{ \frac{1}{2} \text{Tr} \int_0^\infty \frac{e^{-\tau D}}{\tau} d\tau + \text{const} \right\} , \quad (30)$$
where $\tau$ is called Schwinger’s proper time. ([\tau]=[D^{-1}]=L/M.)

To rigorously define the inside of the above exponent, we introduce Dirac’s abstract state vector $|x\rangle$ and $\langle x|$. 

\[
\langle x|e^{-\tau D}|y\rangle \equiv G(x, y; \tau) \quad , \\
(\frac{\partial}{\partial \tau} + D)G(x, y; \tau) = 0 \quad , \\
\lim_{\tau \to +0} G(x, y; \tau) = \delta(x - y) \quad , \\
D = -\frac{d}{dx}\kappa_{n-1}(x)\frac{d}{dx} - \bar{V}(x) \quad , \quad \bar{V}(x) = -\lambda u_n(x)^2 - m^2 - \frac{1}{h} - \frac{du_{n-1}}{dx} \quad .
\]

$(31)$

$G(x, y; \tau)$ is called heat-kernel.

In App.B, we evaluate, for the case $\kappa_{n-1} = \epsilon^{-1}$(const), 
\[
\ln(\det D)^{-1/2} = \frac{1}{2} \int_{l}^{l} d\tau \text{Tr} G(x, y)/\tau = \frac{1}{2} \int_{l}^{l} d\tau \int_{-l}^{l} dx \: G(x, x)/\tau. 
\]

Up to the first order of $\bar{V}$, the result is given by
\[
\frac{l}{\sqrt{\pi}} \sqrt{\epsilon \Lambda} = \frac{1}{2} \sqrt{\frac{\epsilon \mu}{\pi}} \int_{-l}^{l} dx \bar{V}(x) \quad ,
\]

where the infrared cut-off parameter $\mu \equiv \sqrt{\sigma_0/l}$ and the ultraviolet cut-off parameter $\Lambda \equiv h^{-1}$ are introduced. ($^{13}$ $\epsilon^{-1} \equiv \sigma_0/\tilde{\rho}_0 = 1$. We see the mass parameter $m^2$ shifts under the influence of the fluctuation. ($^{14}$)

\[
m^2 \to m^2 + \alpha \sqrt{\epsilon \mu} = m^2 + \alpha \sqrt{\frac{l \tilde{\rho}_0}{\pi \sigma_0 \sqrt{\sigma_0}}} \quad ,
\]

And the bottom of the potential shifts as
\[
V(u_{min}) \rightarrow V(u_{min}) + \alpha \left\{ \frac{1}{2} \sqrt{\frac{\epsilon \Lambda}{\pi}} - \frac{\epsilon}{2 \sqrt{\pi \epsilon \mu}} (m^2 + \frac{1}{h}) \right\} \quad ,
\]

\[
= V(u_{min}) + \alpha \left\{ \sqrt{\frac{\tilde{\rho}_0}{4 \pi \sigma_0}} \left\{ \frac{1}{\sqrt{h}} - \sqrt{\frac{l}{\sqrt{\sigma_0}}} (m^2 + \frac{1}{h}) \right\} \right\} \quad .
\]

$^{13}$ The last term in the round-brackets does not contribute due to the periodicity of $u_{n-1}(z)$.

$^{14}$ This corresponds to renormalization of "mass" $m^2$ in the field theory. When natural cut-offs (IR and UV) are there in the system model-parameters, the divergences coming from the space integral and the mode summation are effectively expressed by "large" but finite quantities.
The coupling \( \lambda \) is also shifted by the \( O(\bar{V}^2) \) correction. \[15\] The shift of these parameters corresponds to the renormalization in the field theory. \[10\] In this effective approach, we have physical cut-offs \( \mu \) and \( \Lambda \) which are expressed by the (finite) parameters appearing in the starting energy-functional. When the functional \[10\] (effectively) works well, all effects of the statistical fluctuation reduces to the simple shift of the original parameters. This corresponds to the renormalizability condition in the field theory. We consider the renormalizable case in the following.

## 5 Boltzmann’s Transport Equation

We use, for simplicity, the original names for the shifted parameters. The step-wise development equation \[14\] with \( \delta V/\delta u = m^2 u + \frac{\lambda}{3!} u^3 + u_{n-1} \frac{du_n}{dx} \), \( V_n^1 = 0 \), and the correction term \[12\] is written as

\[
\frac{1}{h} (u_n(x) - u_{n-1}(x)) = \kappa_{n-1} \frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda}{3!} u_n^3 - u_{n-1} \frac{du_n}{dx} \quad \text{or} \quad u_{n-1}(x) = \frac{u_n(x) - h \left\{ \kappa_{n-1} \frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda}{3!} u_n^3 \right\}}{1 - h \frac{du_n}{dx}}, \tag{35}
\]

where \( \kappa_n(x) = \sigma_n(x)/\tilde{\rho}_n(x) \) (eq. \[21\]). The latter form is convenient, when \( \kappa_{n-1} \) is independent of \( n \) ('time'-independent), for the ‘backward’ recursive computation: \( u_n \to u_{n-1} \). When the system reaches the equilibrium state after sufficient recursive computation \( (n \gg 1) \), we may assume \( u_{n-1}(x) = u_n(x) \equiv u^\infty(x), \sigma_{n-1} \equiv \sigma^\infty(x) \) and \( \tilde{\rho}_{n-1} \equiv \tilde{\rho}^\infty(x) \). \( u^\infty(x) \) satisfies

\[
\kappa^\infty \frac{d^2 u^\infty}{dx^2} - m^2 u^\infty - \frac{\lambda}{3!} u^\infty^3 - u^\infty \frac{du^\infty}{dx} = 0, \quad \kappa^\infty \equiv \frac{\sigma^\infty}{\tilde{\rho}^\infty}. \tag{36}
\]

Here we introduce the particle number density, \( \rho_n(x) \). \[17\] The continuity

\[16\] The coupling (\( \lambda \)) shift can be obtained from \( O(\bar{V}) \) result \[33\] by assuming the "renormalization" consistently works. Noting \( V(u_{min}) = -6m^4/\lambda, \) \( \lambda \) should shift as

\[
\lambda \rightarrow \lambda + \frac{\alpha}{6} \left( \frac{\lambda}{m^2} \right)^2 \sqrt{\frac{\rho_0}{4\pi \sigma_0}} \frac{1}{\sqrt{h}} \left\{ 1 - \frac{l}{\sqrt{\sigma_0 h}} \right\}
\]

\[16\] When \( \kappa^\infty = 1 \), a solution is \( u^\infty = \sqrt{(-3!/\lambda)m^2} \) (constant) for \( m^2 < 0 \).

\[17\] Here we list the physical dimension of some quantities appearing this section. \( [x]=L, [u_n]=[v]=L/T, [k_BT]=ML^2/T^2, \rho_n(x)=L^{-1}, \tilde{\rho}_n(x)=L^{-1}, [f_n]=T/L^2, [P_n]=ML/T^2, [q_n]=ML^2/T^3. \)
equation is given by

\[
\frac{1}{\hbar} (\rho_n(x) - \rho_{n-1}(x)) + \frac{d}{dx} (\rho_{n-1}(x) u_{n-1}(x)) = 0 .
\] (37)

This relation defines the step-flow of \( \rho_n(x) \).

The distribution function \( f_n(x,v) \) is introduced in the following way. The probability for the matter particle in the space interval \( x \sim x + dx \) and the velocity interval \( v \sim v + dv \), at the step \( n \), is given by

\[
\frac{1}{\bar{N}_n} f_n(x,v) dx dv ,
\] (38)

where \( \bar{N}_n \) is the total particle number of the system at the step \( n \). Then the \( n \)-th distribution \( f_n(x,v) \) and the equilibrium distribution \( f^\infty(x,v) \) are introduced as

\[
u^\infty(x) = \frac{1}{\rho^\infty(x)} \int v f^\infty(x,v) dv, \quad u_n(x) = \frac{1}{\rho_n(x)} \int v f_n(x,v) dv, \quad \rho_n(x) = \int dv f_n(x,v),
\]

\[
u_n(x) \to \nu^\infty(x) \quad \text{and} \quad f_n(x,v) \to f^\infty(x,v) \quad \text{as} \quad n \to \infty ,
\] (39)

where \( \nu^\infty(x) \) is the equilibrium velocity distribution. The expression \( u_n(x) \) in eq. (39) guarantees the momentum conservation at each point, \( x \).

\[
0 = \tilde{\rho}_n(x) \int dv (v - u_n(x)) f_n(x,v) .
\] (40)

The recursion relation (35) is expressed, in terms of the distribution functions, as

\[
\frac{1}{\hbar} \left[ \hat{f}_n(x + \hbar u_{n-1}(x), v) - \hat{f}_{n-1}(x,v) \right] = \\
\kappa_{n-1}(x) \frac{d^2 \hat{f}_n(x,v)}{dx^2} - m^2 \hat{f}_n(x,v) - \frac{\lambda}{3!} \hat{f}_n(x,v) u_n(x)^2 ,
\]

where \( \hat{f}_n(x,v) = \frac{f_n(x,v)}{\rho_n(x)} \), \( u_n(x) = \frac{1}{\rho_n(x)} \int v f_n(x,v) dv \), \( \kappa_{n-1}(x) = \frac{\sigma_{n-1}(x)}{\tilde{\rho}_{n-1}(x)} \). (41)

This is the Boltzmann’s transport equation for the 2-body and 4-body velocity-interactions. We can express the step-wise expression (41) in the continuous time \( t \) form as in Sec 2. This is the integrodifferential equation for \( f_n(x,v) \). The right hand side (RHS) is called collision term.

We now introduce some physical quantities used in the non-equilibrium statistical mechanics. The entropy \( S_n \) and the total particle-number \( \bar{N}_n \) are defined by

\[
S_n \equiv -k_B \int dv \int dx f_n(x,v) \ln f_n(x,v) ,
\]

\[
\bar{N}_n = \int dx \rho_n(x) = \int dx \int dv f_n(x,v) ,
\] (42)
where \( k_B \) is Boltzmann’s constant. Another physical quantities will be presented.

Besides the particle-number density \( \rho_n(x) \), we already have introduced the mass density \( \tilde{\rho}_n(x) \) at step \( n \). We here consider the case of one kind particle.

\[
\frac{\tilde{\rho}_n(x)}{\rho_n(x)} = m_1 \text{ (constant)} \quad ,
\]

(43)

where \( m_1 \) is the particle mass. In this case the total mass \( M_n \) is given by

\[
M_n \equiv \int dx \tilde{\rho}_n(x) = m_1 \int dx \rho_n(x) = m_1 \bar{N}_n \quad .
\]

(44)

The temperature distribution \( T_n(x) \), the heat current distribution \( q_n(x) \) and the pressure \( P_n(x) \) are given by

\[
\frac{1}{2} k_B T_n(x) \equiv \frac{1}{\rho_n(x)} \int dv \frac{m_1}{2} (v - u_n(x))^2 f_n(x,v) \quad ,
\]

\[
q_n(x) \equiv \int dv \frac{m_1}{2} (v - u_n(x))^3 f_n(x,v) \quad ,
\]

\[
P_n(x) \equiv m_1 \int dv (v - u_n(x))^2 f_n(x,v) = k_B \rho_n(x) T_n(x) \quad ,
\]

(45)

where \( k_B \) is Boltzmann’s constant.

Using the transport equation \( (41) \) which \( f_n(x,v) \) satisfies, we expect the following two equations are satisfied. The first one is the rephrasing the Navier-Stokes equation \( (35) \).

\[
\frac{1}{\hbar} (u_n(x) - u_{n-1}(x)) + u_{n-1} \frac{d u_n}{d x} = - \frac{1}{\tilde{\rho}_n(x)} \frac{d P_n(x)}{d x} \quad .
\]

(46)

The second one is the energy equation.

\[
\frac{1}{2} \rho_n(x) k_B \left\{ \frac{1}{\hbar} (T_n(x) - T_{n-1}(x)) + u_{n-1} \frac{d T_n}{d x} \right\} = - \frac{d q_n(x)}{d x} - P_n(x) \frac{d u_n(x)}{d x} \quad .
\]

(47)

We explain how the above two relations, \( (46) \) and \( (47) \), are valid.

We define n-th viscosity \( \sigma_n(x) \) by

\[
P_n(x) = - \sigma_n(x) \frac{d u_n}{d x} \quad .
\]

(48)

\[18\] For \( k \) kinds particles, we introduce \( \rho_n \) and \( \tilde{\rho}_n \) fields for each one: \( \{ \rho_n^i, \tilde{\rho}_n^i | i = 1, 2, \cdots, k \} \) with the relation \( \tilde{\rho}_n^i(x)/\rho_n^i(x) = m_i \).
This gives the step-flow of $\sigma_n(x)$. Then the RHS of eq.(46) is written as
\[ -\frac{1}{\rho_n(x)} \frac{dP_n(x)}{dx} = \kappa_n(x) \frac{d^2 u_n}{dx^2} + \frac{1}{\rho(x)} \frac{d\sigma_n(x)}{dx} \frac{du_n}{dx} . \] (49)
The LHS of eq.(46) is written, using Navier-Stokes equation \((45)\), as
\[ \frac{1}{h}(u_n(x) - u_{n-1}(x)) + u_{n-1} \frac{du_n}{dx} = \kappa_{n-1} \frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda}{3!} u_n^3 . \] (50)
Generally we have some cases about the 2 quantities \((19)\) and \((50)\).

Case W1: The 2 quantities coincide.
In this case, the present theory is well-defined. There is no problem.

Case W2: The 2 quantities differ, but they can be equal by changing the 2 constants, $m^2$ and $\lambda$. Namely,
\[ \kappa_n(x) \frac{d^2 u_n}{dx^2} + \frac{1}{\rho(x)} \frac{d\sigma_n(x)}{dx} \frac{du_n}{dx} = \kappa_{n-1} \frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda'}{3!} u_n^3 . \] (51)
By identifying $m^2$ and $\lambda$ by $(m^2)_{n-1}$ and $\lambda_{n-1}$, and $m^2'$ and $\lambda'$ by $(m^2)_n$ and $\lambda_n$, the above equation defines the step-flow of the 2 constants $(m^2)_n$ and $\lambda_n$.

Case W1 is the special case of W2, namely, $n$("time")-independent one. We call the both cases well-defined theory. We call other cases, where eq.(46) is not valid, ill-defined theory. The validity check of eq.(46) selects the present model, \((10)\), well-defined or not.

Next we explain how eq.(47) is valid. Noting the heat current $q_n(x)$ is, in the expression \((45)\), the higher-moment of the distribution $f_n(x,v)$ than the pressure $P_n(x)$, we newly define the heat conductivity $\omega_n(x)$ in the same way as \((18)\).
\[ q_n(x) = \omega_n(x) \frac{dT_n}{dx} . \] (52)
\[ 20 \] Noting the relation $P_n(x) = k_B \rho_n(x) T_n(x)$, \((45)\), eq.(47) is written as
\[ \frac{1}{2} \left\{ \frac{1}{h} (P - \frac{\rho_n}{\rho_{n-1}} P_{n-1}) + u_{n-1} \frac{dP_n}{dx} - u_{n-1} \frac{d\rho_n}{dx} \frac{P_n}{\rho_n} \right\} = - \frac{dq_n(x)}{dx} - P_n(x) \frac{du_n(x)}{dx} \] (53)
\[ 19 \] When $\sigma_n(x)$, defined by eq.(45), is expressed as $\sigma_n(x) = F(du_n/dx)$, the present fluid has different names depending on the form of $F$.
1) $F = \mu(const)$, Newtonian; 2) $F \propto (du_n/dx)^{n-1}$, $n > 1$, Dilatant; 3) $F \propto (du_n/dx)^{n-1}$, $n < 1$, Quasi-viscous.
20 The heat conductivity $\omega_n(x)$ appears here for the first time. We do not have this quantity in the starting energy expression eq.(14). In this point, the heat conductivity differs from the viscosity $\sigma_n(x)$. $\sigma$ is the basic physical quantity which characterize the present system, while $\omega$ is the derived quantity from other basic ones.
Using the relations, \( P_n = -\sigma_n \times \frac{du_n}{dx} \) and \( q_n = \omega_n \times \frac{dT_n}{dx} \), we obtain the differential equation for \( \omega_n(x) \) as

\[
\frac{d}{dx} \left( \omega_n \frac{dT_n}{dx} \right) = \frac{1}{2} \left\{ \frac{1}{\rho_n} (\sigma_n \frac{du_n}{dx} - \rho_n \sigma_{n-1} \frac{du_{n-1}}{dx}) + u_{n-1} \frac{d}{dx} (\sigma_n \frac{du_n}{dx}) - u_n \frac{d}{dx} (\rho_n \frac{du_n}{dx}) \right\} + \sigma_n (\frac{du_n(x)}{dx})^2 
\]

This differential equation determines \( \omega_n(x) \) with no use of \( q_n(x) \). Hence eq. (54) fixes \( \omega_n(x) \) independently of eq. (52). The (numerical) equality check of \( \omega_n(x) \) obtained from both equations is the validity check of eq. (47). When the equality do hold or do not, we call the present model, (10), well-defined or ill-defined respectively.

In the remaining sections, we present an alternative approach to the distribution function \( f_n(x, v) \).

6 Classical and Quantum Mechanics and Its Trajectory Geometry

We can treat the classical mechanics and its quantization (the quantum mechanics, not the quantum field theory) in the same way. In this case, the model is simpler than the previous case (space-field theory) and we can see the geometrical structure clearly. Let us begin with the energy function of a system variable, \( x \), (1 degree of freedom). For example the position (in
1 dimensional space) of the harmonic oscillator with friction. We take the following n-th energy function to define the step flow.

\[ K_n(x) = V(x) + \frac{\eta}{2\hbar}(x-x_{n-1})^2 + \frac{m}{2\hbar^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0, \]  

(55)

where \( V(x) \) is the general potential and \( K_n^0 \) is a constant which does not depend on \( x \). For the harmonic oscillator \( V(x) = kx^2/2 \) where \( k \) is the spring constant. \( \eta \) is the viscosity and \( m \) is the particle mass. \(^{21}\)

We assume \( x_{n-1} \) and \( x_{n-2} \) are given values. As in Sec.\(^4\) the n-th step position \( x_n \) is given by the minimal principle of the n-th energy function \( K_n(x) : \delta K_n = 0, x \to x + \delta x. \)

\[ \frac{\delta V}{\delta x} \bigg|_{x=x_n} + \frac{\eta}{\hbar}(x_n - x_{n-1}) + \frac{m}{\hbar^2}(x_n - 2x_{n-1} + x_{n-2}) = 0, \]

(56)

With the time \( t_n \), the continuous limit \((\hbar \to 0)\) gives us

\[ \frac{dV(x)}{dx} + \frac{\eta}{\hbar}x + m\frac{d^2x}{dt^2} = 0, \]

(57)

where \( t_n = \eta h \to t, x_n = x(t_n) \to x(t), (x_n - x_{n-1})/h = dx/dt|_{t_n} \to dx/dt, (x_n - 2x_{n-1} + x_{n-2})/h = d^2x/dt^2|_{t_n} \to d^2x/dt^2. \) For the case of \( V = kx^2/2, \) this is the harmonic oscillator with the friction \( \eta. \) See Fig.\(^2\)

This is a simple dissipative system. \(^{22}\)

The recursion relation (55) gives us, for the initial data \( x_0 \) and \( x_1 \), the series \( \{x_n = x(t_n) | n = 0, 1, 2, \cdots \} \). This is the classical ’path’. The fluctuation of the path comes from the uncertainty principle of the quantum mechanics in this case. (We are treating the system of 1 degree of freedom. No statistical procedure is necessary.) As the time-interval \( \hbar \) tends to zero, the energy uncertainty grows \((\Delta t \cdot \Delta E \geq \hbar)\). Hence the path \( x_n \), obtained by the recursion relation (56), has more uncertainty as \( \hbar \) goes to 0. As in Sec.\(^4\) we can generalize the n-th energy function \( K_n(x) \), (55), to the following one \( \Gamma(x_{n-1}, x_{n-2}) \) in order to take into account the quantum effect.

\[ e^{-\frac{\hbar}{2}\Gamma(x_{n-1}, x_{n-2})} = \int_{-\infty}^{\infty} dx \ e^{-\frac{\hbar}{2}K_n(x)} \]

\[ K_n(x) = V(x) + \frac{\eta}{2\hbar}(x-x_{n-1})^2 + \frac{m}{2\hbar^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0. \]

\(^{21}\) Here we list the dimension of parameters and variables in this section. \([x] = [x_n] = L, [\eta] = [\eta_n] = L/T, [t] = [t_n] = T, [q] = T^{1/2}M^{-1/2}, [\hbar] = ML^2/T, [m] = M, [\eta] = M/T, [h] = T, [K_n] = [V] = ML^2/T^2, [k] = M/T^2, [\sqrt{\hbar}] = M^{1/2}, [\sqrt{\hbar}M^2] = M^{1/2}T. \) Some ones, such as \( t, m, h, q \) and \( V, \) appearing before this section have different dimensions in this section.

\(^{22}\) The eq. (57) is compared with the eq. (10) for the case of the no external force:

\[ \tilde{\rho}_0 \frac{\delta V(u(x, t))}{\delta u(x, t)} - \sigma \frac{\partial^2 u(x, t)}{\partial x^2} + \tilde{\rho}_0 \frac{\partial u(x, t)}{\partial t} = 0, \]

where we notice the friction term in eq. (57) corresponds to the dissipative term in eq. (10).
We can evaluate the quantum effect by the expansion around the classical value \( x_n : x = x_n + \sqrt{\hbar} q \) where \( \hbar \) is Planck constant. \(^{23}\)

\[
\frac{\delta K_t}{\delta x} \bigg|_{x=x_n} = \frac{\delta V}{\delta x} \bigg|_{x=x_n} + \frac{\eta}{\hbar} (x_n - x_{n-1}) + \frac{m}{\hbar^2} (x_n - 2x_{n-1} + x_{n-2}) = 0 ,
\]

\[
\Gamma_n \equiv \Gamma(x_n; x_{n-1}, x_{n-2}) = K_n(x_n) + \frac{\hbar}{2m} \ln(k + \frac{\eta}{\hbar} + \frac{m}{\hbar^2}) , \quad (59)
\]

where the final expression is for the oscillator model: \( V = kx^2/2 \). The quantum effect does not depend on the step number \( n \). It means the quantum effect contributes to the energy as an additional fixed constant at each step.

The energy rate is obtained as

\[
h \frac{dK_n(t_n)}{dt_n} = K_{n+1}(x_{n+1}) - K_n(x_n) = \Gamma_n + \frac{\hbar}{2m} \ln(k + \frac{\eta}{\hbar} + \frac{m}{\hbar^2}) ,
\]

\[
= V(x_{n+1}) - V(x_n) + \frac{\eta}{\hbar} (x_{n+1} - x_{n-1})^2 - (x_n - x_{n-1})^2 \]

\[
+ \frac{m}{2\hbar^2} \left((x_{n+1} - 2x_n + x_{n-1})^2 - (x_n - 2x_{n-1} + x_{n-2})^2\right) K_0^1 + K_0^0 , \quad (60)
\]

The present system is again an open system, and the energy generally changes.

In terms of the position difference \( x_n - x_{n-1} \equiv \Delta x_n \) and the velocity difference \( (x_n - 2x_{n-1} + x_{n-2})/\hbar \equiv v_n - v_{n-1} \equiv \Delta v_n \), we can rewrite the energy at \( n \)-step and read the metric as follows. \(^{24}\)

\[
K_n(x_n) = V(x_n) + \frac{\eta}{\hbar} (x_n - x_{n-1})^2 + \frac{m}{2\hbar^2} (x_n - 2x_{n-1} + x_{n-2})^2 + K_0^0
\]

\[
= \frac{1}{\hbar^2} \left\{ V(x_n)(\Delta t)^2 + \frac{\eta \hbar}{2}(\Delta x_n)^2 + \frac{m \hbar^2}{2}(\Delta v_n)^2 \right\} + K_0^0 , \quad (61)
\]

where \( \hbar \) (time increment) in the first term within the round brackets is replaced by \( \Delta t \). This shows us the metric for the n-step energy function is given by

\[
(\Delta s_n)^2 \equiv 2\hbar^2 K_n(x_n) = 2V(x_n'/\sqrt{\eta \hbar})(\Delta t)^2 + (\Delta x_n')^2 + (\Delta v_n')^2 , \quad (62)
\]

\[
x_n' \equiv \sqrt{\eta \hbar} x_n , \quad v_n' \equiv \sqrt{m \hbar^2} v_n ,
\]

where, for the oscillator model, \( V(x_n'/\sqrt{\eta \hbar}) = (k'/2)x_n'^2 \). \(^{25}\) Eq.\(^{(62)}\) shows the energy-line element \( \Delta s^2 \) in the \( (t, x_n', v_n') \) space. \(^{25}\) Note that the

\(^{23}\) Do not confuse \( \hbar \) (Planck constant/2\( \pi \)) with \( \hbar \) (time interval).

\(^{24}\) \( v_n \equiv (x_n - x_{n-1})/\hbar = \Delta x_n/\hbar \)

\(^{25}\) In eq.\(^{(62)}\), the first term shows the potential part, the second one the kinetic part and the third one a new term. In ref.\(^{11}\) and ref.\(^{12}\), the hysteresis term appears as a new one.
above metric is along the path \( x_n = x(t_n), \) \( v_n = v(t_n) = (x(t_n) - x(t_{n-1}))/h \) given by (56). The metric is used, in the next section, as the geometrical basis for fixing the statistical ensemble.

We take the freedom of the value \( K^0_n \) in the following way.

\[
K^0_n = -V(x_n) - \frac{m}{2h^2}(x_n - 2x_{n-1} + x_{n-2})^2 + V(x_0) + \frac{m}{2h^2}(x_1 - x_0)^2.
\] (63)

This is chosen in such a way that the step \( n \) energy \( K_n(x_n) \), for the no dissipation (\( \eta = 0 \)), does not depend on the step number \( n \) and the value is the total energy at the initial step (last 2 terms in (63)). The graphs of movement and energy change, for various viscosities, are shown in Fig. 9. For the no friction case, the oscillator keeps the initial energy (Fig. 1). When the viscous effect appears, the energy changes step by step, and finally reaches a constant nonzero value (Fig. 6, Fig. 7, Fig. 9). We understand the finally-remaining energy (constant) as the dissipative one. Physically (in the real matter) the energy is realized as the pressure and the temperature which characterize the particle’s ”environment”(out-side world).

For the resonate case \( (4k/m = (\eta/m)^2) \), both the movement and the energy are large. The notable advantageous evidence of the step-wise solution (56) over the analytic one of (57), is that we need not treat the 3 cases, \( 4k/m > \eta^2/m^2 \), \( 4k/m < \eta^2/m^2 \) and \( 4k/m = \eta^2/m^2 \), separately.

7 Statistical Ensemble, Geometry and Initial Condition

In this section, we consider some statistical ensemble of the classical mechanical system taken in the previous section. Namely, we take \( N \) ’copies’ of the classical model and regard them as a set of (1 dimensional) particles, where the dynamical configuration distributes in the probabilistic way. \( N \) is a large number. The set has \( N \) degrees of freedom: \( x_1, x_2, \cdots, x_N \). As the physical systems, (1 dimensional) viscous gas and viscous liquid are examples. Each particle obeys the (step-wise) Newton’s law (59) with different initial conditions. \( N \) is so large that we do not or can not observe the initial data. Usually we do not have interest in the trajectory of every particle and do not observe it. We have interest only in the macroscopic quantities such as total energy and total entropy. The \( N \) particles (fluid molecules) in

\footnote{Note that the particle does not move at the final stage.}

\footnote{For example, \( N \sim 10^{23} \).}

\footnote{We are considering \( N \)-body problem where each particle moves (fluid flows) with friction. We aproach it using the effective 1-body energy function (57).}
Figure 3: Harmonic oscillator with no friction, Movement, $h=0.1, \sqrt{k/m}=0.01, \eta/m=0.0$, total step no = 20000. The step-wise solution correctly reproduces the analytic solution: $x(t) = \sin(\omega t)$, $0 \leq t \leq 2000$, $x(0) = 0$, $\dot{x}(0) = 1$. 

![Graph showing the movement of a harmonic oscillator with no friction](image-url)
Figure 4: Harmonic oscillator with no friction, Energy change, $h=0.1, \sqrt{k/m}=0.01, \eta/m=0.0$, total step no = 20000.

\[ \text{Kn(Xn)/m  Energy at n-Step} \]

\[ n(\text{Step No}) \times h(\text{Step Intvl}) \times \omega(\text{Freq}) \]

“EnefrHO.dat”

NO FRICTION

$h=0.1, \omega=0.01, \eta/m=0.0$
Figure 5: Harmonic oscillator with friction, Movement, $h=0.1, \sqrt{k/m}=0.01$, $\eta/m=0.005$ (Elasticity dominate) and 0.03 (Viscosity dominate), total step no $=20000$. The step-wise solution \(60\) correctly reproduces the analytic solution: (1)Elast.Dom. \(x(t) = e^{-\eta' t/2} \sin(\sqrt{4\omega^2 - \eta'^2} \ t/2) , \ 0 \leq t \leq 2000 \) , \(x(0) = 0, \dot{x}(0) = \sqrt{4\omega^2 - \eta'^2}/2 = (1.94 \times 10^{-2})/2\) (2)Visco.Dom. \(x(t) = e^{-\eta' t/2} \sinh(\sqrt{\eta'^2 - 4\omega^2} \ t/2) , \ 0 \leq t \leq 2000 \) , \(x(0) = 0, \dot{x}(0) = \sqrt{\eta'^2 - 4\omega^2}/2 = (2.24 \times 10^{-2})/2\).
Figure 6: Harmonic oscillator with friction, Energy change, \( h=0.1, \sqrt{k/m}=0.01, \eta/m=0.005 \) (Elasticity dominate), total step no =20000.
Figure 7: Harmonic oscillator with friction, Energy change, $h=0.1, \sqrt{k/m}=0.01, \eta/m=0.03$ (Viscosity dominate), total step no = 20000.
Figure 8: Harmonic oscillator with friction, Movement, $h=0.1, \sqrt{k/m}=0.01,$
$\eta/m=0.02$ (Resonant), total step no =20000. The step-wise solution \[ x(t) = te^{-\omega t} \], $0 \leq t \leq 2000$, $x(0)=0$, $\dot{x}(0)=1$.
Figure 9: Harmonic oscillator with friction, Energy change, $h=0.1, \sqrt{k/m}=0.01, \eta/m=0.02$ (Resonant), total step no = 20000.
Figure 10: The path \( \{(y(t), w(t), t)|0 \leq t \leq \beta\} \) of line in 3D bulk space \((X, P, t)\).

The present system are "moderately" interacting each other in such way that each particle almost independently moves except that energy is exchanged.

As the statistical ensemble, we adopt the Feynman’s idea of "path-integral" \([13, 14, 15, 16, 17, 18, 12, 11]\). We take into account all possible paths \(\{y_n\}\). \(\{y_n\}\) need not satisfy \((56)\) nor certain initial condition. As the measure for the summation (integral) over all paths, we propose the following ones based on the geometry of \((62)\). As the first measure, we construct it in terms of the length, using the "Dirac-type" metric \([12, 11]\).

\[
(ds^2)_D \equiv 2V(X)dt^2 + dX^2 + dP^2 \quad \text{on-path}(X = y(t), P = w(t)) \to (2V(y) + \dot{y}^2 + \dot{w}^2)dt^2,
\]

\[
L_D = \int_0^\beta ds|_{\text{on-path}} = \int_0^\beta \sqrt{2V(y) + \dot{y}^2 + \dot{w}^2}dt = \hbar \sum_{n=0}^{\beta/\hbar} \sqrt{2V(y_n) + \dot{y}_n^2 + \dot{w}_n^2},
\]

\[
d\mu = e^{-\frac{1}{\alpha}L_D}dydw, \quad e^{-\beta F} = \int \prod_n dy_n dw_n e^{-\frac{1}{\alpha}L_D} \quad (64)
\]

where \(\alpha\) is a parameter with the dimension of length ([\(\alpha\]=L) and \(V(X) = (k/2)X^2\). See Fig.10 As explained in Sec.4, it is appropriately chosen problem by problem. \(\beta\) is introduced to restrict the \(t\)-region \((0 \leq t \leq \beta)\) and, in
In this context, should be regarded as a part of the choice of the ensemble. \( \beta \) plays the role of the inverse temperature. Among all possible paths \( \{y_n\} \), the minimal length \( (\delta L_D = 0) \) solution, (56), gives the dominant path \( \{x_n\} \).

The second choice is constructed using the "Standard-type" metric.

\[
(ds^2)_S = \frac{1}{dt^2} [(ds^2)_D]^2 \quad \text{on-path}(X = y(t), P = w(t)) \rightarrow (2V(y) + \dot{y}^2 + \dot{w}^2)dt^2 ,
\]

\[
L_S = \int_0^\beta ds|_{\text{on-path}} = \int_0^\beta (2V(y) + \dot{y}^2 + \dot{w}^2)dt = h \sum_{n=0}^{\beta/h} (2V(y_n) + \dot{y}_n^2 + \dot{w}_n^2) ,
\]

\[
d\mu = e^{-\frac{\beta L_S}{h}} \mathcal{D}y \mathcal{D}w , \quad e^{-\beta F} = \int \prod_n dy_n dw_n e^{-\frac{\beta L_S}{h}} \tag{65}
\]

where we should notice \( dt (= h > 0) \) is non-zero. In both cases above we take the metric of the 3 dimensional (bulk) space-time \((X, P, t)\), which is consistently chosen with the trajectory metric (62). Note that the standard case partly has the same expression as the free energy (trace of the density matrix) expression in the Feynman’s textbook[19].

Another choice of path is making use of surfaces instead of lines. Let us consider the following 2 dim surface in the 3 dim manifold \((X, P, t)\). Assuming \( Z_2 \) invariance both in \( X \) and in \( P \), the general one is

\[
\frac{X^2}{a(t)^2} + \frac{P^2}{b(t)^2} = 1 \quad , \quad 0 \leq t \leq \beta \tag{66}
\]

where \( a(t) \) and \( b(t) \) are arbitrary (non-negative) functions of \( t \). We take \( a(t) = b(t) \equiv r(t) \) for simplicity. See Fig.11 By varying the form of \( \{r(t) : 0 \leq t \leq \beta\} \), we obtain different surfaces. Regarding each of them as a path used in the Feynman’s path-integral, we obtain the following statistical ensemble. First the induced metric \( g_{ij} \) on the surface (66) is given as

\[
(ds^2)_D|_{\text{on-path}} = 2V(X)dt^2 + dX^2 + dP^2|_{\text{on-path}} = \sum_{i,j=1}^{2} g_{ij} dX^i dX^j ,
\]

\[
(g_{ij}) = \left( \begin{array}{cc}
1 + \frac{2V}{r^2} X^2 & \frac{2V}{r^2} X \\
\frac{2V}{r^2} P & 1 + \frac{2V}{r^2} P^2
\end{array} \right) \tag{67}
\]

\footnote{\( \beta/h = N \) should be an (large) integer. The increment \( h \) is the (inverse) temperature unit as well as the time unit. From the dimensional analysis \( k_B^{-1} \eta^2/\beta \) corresponds to the temperature. Here \( k_B \) is Boltzmann’s constant and \( \eta^2 \equiv \hbar' \) is the combination of the friction coefficient \( \eta \) and some length scale \( \tilde{l} \) (\( \tilde{l} = L \)) such as the mean free path of the fluid particle. Note that \( \hbar' \) has the same dimension as \( h \). \( \hbar' = \hbar = ML^2/T \).}
where \((X^1, X^2) = (X, P)\). Then the area of the surface \((66)\) is given by

\[
A = \int \sqrt{\det g_{ij}} d^2X = \int \sqrt{1 + \frac{2V}{r^2}} dXdP ,
\]

(68)

We consider all possible surfaces of \((66)\). The statistical distribution is, using the area \(A\), given by

\[
e^{-\beta F} = \int_{r(0)}^{\infty} d\rho \int r(0) = \rho \prod_t DX(t) DP(t)e^{-\frac{A}{\alpha}} ,
\]

(69)

In relation to Boltzmann’s equation (Sec.5), we have directly defined the distribution function \(f(t, x, v)\) using the geometrical quantities in the 3 dim bulk space.

In App.C, another model called “Spring-Block” model is explained. This is the simplified model of the earthquake. The same thing in this section is valid by taking the potential \(V(x)\) as

\[
V(X) = \frac{k'}{2} X^2 + (\bar{p} - \bar{V}'t)k'X , \quad k' = \frac{k}{\eta h} , \quad \bar{p} = \sqrt{\eta hl} , \quad \bar{V}' = \sqrt{\eta hV} .
\]

(70)
8 Conclusion

We have presented the field theory approach to Boltzmann’s transport equation. The collision term is explicitly obtained. Time is not used, instead the step number \( n \) plays the role. We have presented the \( n \)-th energy functional \( I_n[u(x)] \) which gives the step \( n \) configuration \( u_n(x) \) from the minimal energy principle. We regard the step flow (the increase of \( n \)) as the evolution of the system, namely, time-development. Navier-Stokes equation is obtained by identifying time \( t \) as \( nh \) \((7)\). Fluctuation effect due to the micro structure and micro (step-wise) movement is taken into account by generalizing the \( n \)-th energy functional \( I_n[u(x)] \), \((10)\), to \( \Gamma[u(x); u_{n-1}(x)] \), \((26)\), where the classical path \( u_n(x) \) is dominant but all possible paths are taken into account (path-integral). Renormalization is explicitly done. The total energy generally does not conserve. The system is an open one, namely, the energy comes in from or go out to the outside. In the latter part we have presented a direct approach to the distribution function \( f_n(x, v) \) based on the geometry emerging from the mechanical (particle-orbit) dynamics. We have examined the dissipative system using the minimal (variational) principle which is the key principle in the standard field theory\((20)\).

9 Appendix A \((3+1)\)D Scalar Field Theory

3+1 dimensional scalar field is here treated in the present step-wise formalism. We start with the following \( n \)-th step energy functional.

\[
I_n[\phi(x)] = \int d^3x \left\{ \frac{1}{2} (\nabla \phi)^2 + V(\phi) + \frac{1}{2h^2}(\phi - 2\phi_{n-1} + \phi_{n-2})^2 \right\} \quad n = 2, 3, \cdots ,
\]

\[
V(\phi) = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 , \quad \phi = \phi(x) , \quad \phi_{n-1} = \phi_{n-1}(x) , \quad \phi_{n-2} = \phi_{n-2}(x) ,
\]

where \( \phi_{n-2} \) and \( \phi_{n-1} \) are given. \((x) = (x_1, x_2, x_3) \) is the 3 dimensional spacial coordinates. The step \( n \) configuration \( \phi_n(x) \) is defined to be the energy minimal one.

\[
\frac{\delta I_n}{\delta \phi} \bigg|_{\phi = \phi_n} = -\nabla^2 \phi_n + \frac{\delta V}{\delta \phi} \bigg|_{\phi = \phi_n} + \frac{1}{h^2}(\phi_n - 2\phi_{n-1} + \phi_{n-2}) = 0 \ ,
\]

\[
\frac{\delta V}{\delta \phi} = m^2 \phi + \frac{\lambda}{3!} \phi^3 . \quad (72)
\]

Using the step-time notation: \( \phi_n(x) \equiv \phi(x, t_n) , \quad t_n = nh \), we obtain, in the continuous time limit \( h \rightarrow +0 \),

\[
(\partial_t^2 - \nabla^2 + m^2)\phi + \frac{\lambda}{3!} \phi^3 = 0 . \quad (73)
\]
This is the (3+1) dim $\phi^4$ scalar field equation.

\section{Appendix B Calculation of Fluctuation Effect}

In Sec. 4, we have developed the method of calculating the statistical fluctuation occurring in the (1-dim) viscous fluid matter. The background-field method is taken. At the 1-loop approximation, the key quantity to calculate the energy functional $\Gamma[u(x)]$ is the heat-kernel $G(x, y; \tau)$ given by (see eq. (31))

\[
|x| e^{-\tau D}|y] \equiv G(x, y; \tau),
\]

\[
(\partial_{\tau} + D)G(x, y; \tau) = 0, \quad \lim_{\tau \to +0} G(x, y; \tau) = \delta(x - y),
\]

\[
D = -\epsilon^{-1} \frac{\partial^2}{\partial x^2} - \bar{V}(x), \quad \bar{V}(x) = -\lambda u_n(x)^2 - m^2 - \frac{1}{h} - \frac{du_{n-1}}{dx}, \quad (74)
\]

where $x|$ and $< x|$ are Dirac's abstract state vectors. $\epsilon \equiv \tilde{\rho}_0/\sigma$ is explicitly written to show the dimension consistency. In the text, we take $\epsilon = 1$. For the calculation, in this appendix, we change the scale of $\tau$ and $D$ as follows.

\[
\tau \to \frac{\tau}{\epsilon} = \tilde{\tau}, \quad D \to \epsilon D = \tilde{D},
\]

\[
\tilde{D} = -\frac{\partial^2}{\partial x^2} - \epsilon \bar{V}(x), \quad [\tilde{\tau}] = L^2, \quad [\tilde{D}] = L^{-2}, \quad [\epsilon] = 1/LM. \quad (75)
\]

In the following within this appendix, for simplicity we omit the symbol of 'tilde'.

The kernel is formally solved as

\[
G(x, y; \tau) = G_0(x - y; \tau) + \int dz \int_0^\tau d\omega S(x - z; \tau - \omega) e^{V(z)} G(z, y; \omega) \quad (76)
\]

where $G_0(x - y; \tau)$ and the (heat-)propagator $S(x - y; \tau)$ are given by

\[
G_0(x - y; \tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-k^2 \tau + ik(x-y)} = \frac{1}{\sqrt{4\pi \tau}} e^{\frac{(x-y)^2}{4\tau}}, \quad \tau > 0,
\]

\[
S(x - y; \tau) = \int \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{dk^0}{2\pi} \exp\left\{-ik^0 \tau + ik(x - y)\right\} = \theta(\tau) G_0(x - y; \tau) \quad (77)
\]

They satisfy

\[
(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2}) G_0(x - y; \tau) = 0, \quad \tau > 0, \quad \lim_{\tau \to +0} G_0(x - y; \tau) = \delta(x - y),
\]

\[
(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2}) S(x - y; \tau) = \delta(\tau) \delta(x - y), \quad \lim_{\tau \to +0} S(x - y; \tau) = \delta(x - y) \quad (78)
\]
Up to the first order of $\bar{V}$,
\[
G(x, x; \tau) = \frac{1}{\sqrt{4\pi \tau}} + \int dz \int_0^\tau d\omega S(x - z; \tau - \omega) e^{\bar{V}(z)} G_0(z - x; \omega) + O(\bar{V}^2) \quad (79)
\]

The second term is evaluated as
\[
\int dz \int_0^\tau d\omega G_0(x - z; \tau - \omega) e^{\bar{V}(z)} G_0(z - x; \omega) = \int dz \int_0^\tau d\omega \frac{1}{4\pi (\tau - \omega)\omega} e^{\bar{V}(z)} \exp\left\{ -\frac{\tau}{4(\tau - \omega)} (x - z)^2 \right\} \quad (80)
\]

Finally the contribution to $\ln(\det D)^{-1/2} = \frac{1}{2} \int_0^\infty (d\tau/\tau) \text{Tr} G(x, y; \tau) = \frac{1}{2} \int_0^\infty (d\tau/\tau) \int_{-l}^l dx G(x, x; \tau)$ is evaluated as
\[
\frac{l}{\sqrt{\pi}} \sqrt{\epsilon \Lambda} - \frac{1}{2\sqrt{\pi}} \int_{-l}^l dz \left( \lambda u_n(z)^2 + m^2 + \frac{1}{\hbar} + \frac{du_{n-1}(z)}{dz} \right) \quad (81)
\]

where the infrared cut-off parameter $\mu \equiv \sqrt{\sigma/l}$ and the ultraviolet cut-off parameter $\Lambda \equiv h^{-1}$ are introduced. \[30\]

### 11 Appendix C Spring-Block Model

In Sec.6 the movement of the harmonic oscillator with friction was examined. Here we treat the movement of a block which is pulled by the spring which moves at the constant speed $\bar{V}$. The block moves on the surface with friction.

We take the following $n$-th energy function to define the step flow.
\[
K_n(x) = V(x) - \hbar n k \bar{V} x + \frac{\eta}{2\hbar}(x - x_{n-1})^2 + \frac{m}{2\hbar^2}(x - 2x_{n-1} + x_{n-2})^2 + K_0^n \quad ,
\]

\[
V(x) = \frac{kx^2}{2} + \bar{V} x \quad (82)
\]

where $\eta$ is the friction coefficient and $m$ is the block mass. The potential $V(x)$ has two terms: one is the harmonic oscillator with the spring constant $k$, and the other is the linear term of $x$ with a new parameter $\bar{l}$ (the natural length of the spring). $\bar{V}$ is the velocity (constant) with which the front-end

---

\[30\] The dimensions of these parameters are $[\mu]=[\Lambda]=M/L$. The space-integral part ($\int dx \cdots$) in (81) is evaluated as $\int_{-l}^l dx \exp\left\{ -\frac{(\tau/4(\tau - \omega)) (x - z)^2}{\omega} \right\} \sim \int_{-\infty}^\infty \exp\left\{ \eta \right\} = 2\sqrt{\pi (\tau - \omega)\omega/\tau}$ where $l$ is safely extended to infinity.
of the spring moves. $K_0^n$ is a constant which does not depend on $x$. It will be fixed later. The $n$-th step $x_n$ is determined by the energy minimum principle: 
\[ \delta K_n(x) |_{x=x_n} = 0. \]

\[
\frac{k}{m}(x_n + \bar{l} - nh\bar{V}) + \eta \frac{1}{m h} (x_n - x_{n-1}) + \frac{1}{h^2} (x_n - 2x_{n-1} + x_{n-2}) = 0 \quad \text{or} \quad
x_n = \frac{\omega^2 (-\bar{l} + nh\bar{V}) + \frac{\eta}{m} x_{n-1} + \frac{1}{h^2} (2x_{n-1} - x_{n-2})}{\omega^2 + \frac{\eta}{m} + \frac{1}{h^2}}, \quad \omega \equiv \sqrt{\frac{k}{m}}, \quad \eta' \equiv \frac{\eta}{m}. \quad (83)
\]

For the continuous limit: $h \to 0, nh = t_n \to t, (x_n - x_{n-1})/h \to \dot{x}, (x_n - 2x_{n-1} + x_{n-2})/h^2 \to \ddot{x}$, the above recursion relation reduces to the following differential equation:
\[ m\ddot{x} = k(\bar{V}t - x - \bar{l}) - \eta \dot{x}. \quad (84) \]

We keep the step-wise approach. The system energy given by $K_n(x_n)$. Taking the constant term $K_0^n$ as
\[
K_0^n = -V(x_n) - \frac{m}{2h^2} (x_n - 2x_{n-1} + x_{n-2})^2 + V(x_0) + \frac{m}{2h^2} (x_1 - x_0)^2 + hn\bar{V}x_n + h\bar{V}x_n, \quad (85)
\]
the energy is given as
\[
K_n(x_n) = hn\bar{V}x_n + \frac{\eta}{2h} (x_n - x_{n-1})^2 + V(x_0) + \frac{m}{2h^2} (x_1 - x_0)^2, \quad (86)
\]

\[ ^{31} \text{This equation is called spring-block model and is used to explain some aspect (stick-slip motion, etc) of the earthquake.} \]
We have taken the constant term $K^0_n$, (85), in such a way that the system keeps the constant energy when the energy dissipation does not occur ($\eta = 0, \bar{V} = 0$). The last two terms in (85) comes from the following relation.

$$\frac{d}{dt} \left( \frac{1}{2} k \dot{x}^2 - k \bar{V} x t + V(x) \right) = -\eta \dot{x}^2 - k \bar{V} x . \quad (87)$$

Note that the $\bar{V}$-terms appear both hand-sides, the total derivative terms and the dissipative terms. \[32\]

The graphs of movement $(x_n, \text{eq.}(83))$ and energy change $(K_n(x_n), \text{eq.}(86))$ are shown in Fig.[13] and Fig.[14] respectively. From the graph of Fig.[14] we see this system starts with the stick-slip motion and reaches the steady energy-state as $n \to \infty$, (25).

12 Acknowledgment

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References

[1] G. W. Gibbons, arXiv: 1111.0457, 2011, ”The emergent nature of time and the complex numbers in quantum cosmology”

[2] N. Kikuchi, An approach to the construction of Morse flows for variational functionals in "Nematics-Mathematical and Physical Aspects", eds. J. -M. Coron, J. -M. Ghidaglia and Hélein, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 332, Kluwer Acad. Pub., Dordrecht-Boston-London, 1991, p195-198

[3] N. Kikuchi, A method of constructing Morse flows to variational functionals. Nonlin. World 131(1994)

\[32\] The role of last 2 terms of (85) are as follows. The first one $(+hnk\bar{V}x_n)$ is for canceling the term $-hnk\bar{V}x$ in $K_n(x)$, (32). The second one $(+hk\bar{V}x_n)$ is for taking care of the second dissipative term of the RHS of (87), $-k\bar{V}x$. 

35
Figure 13: Spring-Block Model, Movement, $h=0.0001, \sqrt{k/m}=10.0$, $\eta/m=1.0$, $\bar{V}=1.0$, $\bar{l}=1.0$, total step no =20000. The step-wise solution correctly reproduces the analytic solution:

$$x(t) = e^{-\eta' \bar{V} t / \bar{V}} \left\{ \frac{(\eta' / \omega^2 - 1)(\sin \Omega t)}{\Omega} + \bar{V} (t - \eta' / \omega^2) \right\} - \bar{l} + \bar{V} (t - \eta' / \omega^2),$$

$$\Omega = (1/2) \sqrt{4\omega^2 - \eta'^2} = 9.99, \quad 0 \leq t \leq 2, \quad x(0) = -\bar{l}, \quad \dot{x}(0) = 0.$$
Figure 14: Spring-Block Model, Energy Change, $h=0.0001, \sqrt{k/m}=10.0, \eta/m=1.0, V=1.0, l=1.0$, total step no = 20000.
[4] L. Ma and I. Witt, arXiv: 1203.2225[math.DG] ”Discrete Morse flow for Ricci flow and porous media equation”

[5] S. Succi, ”The Lattice Boltzmann Equation”, Oxford University Press Inc., New York, 2001

[6] B. S. DeWitt, Phys. Rev. 162, 1195, 1239(1967)

[7] G. ’t Hooft, Nucl.Phys.B62(1973)444

[8] J. Schwinger, Phys.Rev.82(1951)664

[9] M.L. Bellac, F. Mortessagne and G.G. Batrouni, ”Equilibrium and Non-Equilibrium Statistical Thermodynamics”, Cambridge Univ. press, Cambridge, 2004

[10] S. Weinberg, ”The Quantum Theory of Fields I”, Cambridge Univ. press, Cambridge-New York-Melbourne, 1995, p499

[11] S. Ichinose, J.Phys:Conf.Ser.258(2010)012003, [arXiv:1010.5558] Proc. of Int. Conf. on Science of Friction 2010 (Ise-Shima, Mie, Japan, 2010.9.13-18).

[12] S. Ichinose, [arXiv:1004.2573] ”Geometric Approach to Quantum Statistical Mechanics and Minimal Area Principle”, 2010, 28 pages.

[13] S. Ichinose, Prog.Theor.Phys.121(2009)727, [arXiv:0801.3064v8[hep-th]].

[14] S. Ichinose, ”Casimir Energy of 5D Warped System and Sphere Lattice Regularization”, [arXiv:0812.1263][hep-th], US-08-03, 61 pages.

[15] S. Ichinose, ”Casimir Energy of AdS5 Electromagnetism and Cosmological Constant Problem”, Int.Jour.Mod.Phys.24A(2009)3620, Proc. of Int. Conf. on Particle Physics, Astrophysics and Quantum Field Theory: 75 Years since Solvay (Nov.27-29, 2008, Nanyang Executive Centre, Singapore), [arXiv:0903.4971]

[16] S. Ichinose, J. Phys. : Conf.Ser.222(2010)012048. Proceedings of First Mediterranean Conference on Classical and Quantum Gravity (09.9.14-18, Kolymbari, Crete, Greece). [arXiv:1001.0222][hep-th]

[17] S. Ichinose, ”New Regularization in Extra Dimensional Model and Renormalization Group Flow of the Cosmological Constant”, Proceedings of the Int. Workshop on ‘Strong Coupling Gauge Theories in LHC Era’(09.12.8-11, Nagoya Univ., Nagoya, Japan) p407, eddited by H. Fukaya et al, World Scientific. [arXiv:1003.5041][hep-th]
[18] S. Ichinose, J. Phys. : Conf.Ser.384(2012)012028, Proceedings of DSU2011 (2011.9.26-30, Beijin, China). [arXiv:1205.1316][hep-th]

[19] R.P. Feynman, "Statistical Mechanics", W.A.Benjamin,Inc., Massachusetts, 1972

[20] W. Yourgrau and S. Mandelstam, "Variational Principles in Dynamics and Quantum Theory", Dover Publications, Inc., New York, 1952, 1968, 1979