NON JORDAN GROUPS OF DIFFEOMORPHISMS AND ACTIONS OF COMPACT LIE GROUPS ON MANIFOLDS

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Abstract. A recent preprint of Csikós, Pyber and Szabó [3] proves that the diffeomorphism group of $T^2 \times S^2$ is not Jordan. The purpose of this paper is to generalize the arguments of Csikós, Pyber and Szabó in order to obtain many other examples of compact manifolds whose diffeomorphism group fails to be Jordan. In particular we prove that for any $\epsilon > 0$ there exist manifolds admitting effective actions of arbitrarily large $p$-groups $\Gamma$ all of whose abelian subgroups have at most $|\Gamma|^{\epsilon}$ elements. Finally, we also recover some results on nonexistence of effective actions of compact connected semisimple Lie group on manifolds.

1. Introduction

A group $G$ is said to be Jordan if there is some constant $C$ such that any finite subgroup $\Gamma$ of $G$ contains an abelian subgroup whose index in $\Gamma$ is at most $C$, see [16]. In their preprint [3], Csikós, Pyber and Szabó prove that the diffeomorphism group of the product of the torus $T^2 = S^1 \times S^1$ with the two dimensional sphere $S^2$ is not Jordan. This is the first known example of a compact manifold whose diffeomorphism group is not Jordan. Previously, Popov [17] had given an example of a connected open 4-manifold with non Jordan diffeomorphism group. In contrast, there are many examples of manifolds whose diffeomorphism group is known to be Jordan: these include all compact manifolds of dimension at most 3, all compact manifolds with nonzero Euler characteristic, homology spheres, the connected sum of a torus and an arbitrary compact connected manifold, and open contractible manifolds, see [8, 9, 10, 11, 20].

Denote by $M_d$ the projectivisation of the complex vector bundle $L_d \oplus \mathbb{C} \rightarrow T^2$, were $L_d \rightarrow T^2$ is a degree $d$ line bundle and $\mathbb{C} \rightarrow T^2$ is the trivial line bundle. Csikós, Pyber and Szabó base their proof in two facts. First, $M_d$ is diffeomorphic to $T^2 \times S^2$ for any even $d$. Second, for positive $d$ there is a finite group $\Gamma_d$ of order $d^3$ acting effectively on $M_d$ such that any abelian subgroup $A$ of $\Gamma_d$ has at most $d^2$ elements. The group $\Gamma_d$ is a Heisenberg group, and its action on $M_d$ is induced by an effective linear action on $L_d$. Picking algebraic structures on $T^2$ and the vector bundle $L_d \oplus \mathbb{C}$ the action of $\Gamma_d$ can be taken to be algebraic. This is the key ingredient in a paper of Zarhin [19] that gives the first example of algebraic manifold whose group of birational transformations is not Jordan.

A slightly different way to present the arguments in [3] is the following. For any (non necessarily even) integer $d$ the complex vector bundle $V := L_d \oplus L_d^{-1} \rightarrow T^2$ has degree 0. By the classification of complex vector bundles over compact connected surfaces this implies that $V$ can be trivialized. Furthermore one can pick a $\Gamma_d$-invariant trivialization.
of \( \det V \), because the action of \( \Gamma_d \) naturally induced on \( \det V \) factors through a free action of \( \mathbb{Z}_d^2 \). Using a \( \Gamma_d \)-invariant Hermitian metric on \( V \) with respect to which the section of \( \det V \) defining the trivialization has constant norm equal to 1, the bundle \( E \) of unitary frames compatible with the trivialization of \( \det V \) turns out to be a (trivial) SU(2) principal bundle, and the action of \( \Gamma_d \) on \( L_d \) gives an effective action on \( E \).

Now we can identify \( T^2 \times S^2 \) with \( E \times_{\text{SU}(2)} S^2 \), where SU(2) acts on \( S^2 \) via the quotient map \( \text{SU}(2) \to \text{SO}(3, \mathbb{R}) \) and the identification of \( \text{SO}(3, \mathbb{R}) \) with the orientation preserving isometries of \( S^2 \). If \( d \) is odd, the action of \( \Gamma_d \) on \( E \) induces an effective action on \( E \times_{\text{SU}(2)} S^2 = T^2 \times S^2 \). The parity restriction on \( d \) is a consequence of the fact that there is a subgroup \( \{ \pm 1 \} \subset \text{SU}(2) \) acting trivially on \( S^2 \). Hence this point of view gives a slightly less general construction than [3], since the latter allows to construct effective actions of \( \Gamma_d \) on \( T^2 \times S^2 \) for even \( d \); this discrepancy comes from the fact that \( E \times_{\text{SU}(2)} S^2 \) can be identified in a \( \Gamma_d \)-equivariant way with \( \mathbb{P}(L_d^2 \oplus C) \). On the other hand, this point of view immediately suggests that \( S^2 \) can be replaced by any manifold admitting an effective action of \( \text{SU}(2) \) or \( \text{SO}(3, \mathbb{R}) \). For example, since any sphere of dimension at least 2 supports such actions, we deduce that the diffeomorphism group of the product \( T^a \times S^b \) of a torus and a sphere of dimensions \( a \geq 2 \) and \( b \geq 2 \) is not Jordan.

Before stating our main theorem we introduce some terminology and conventions. We define the kernel of the action of a group \( G \) on a space \( X \) to be the subgroup of \( G \) consisting of those elements that act trivially on \( X \). We say that the action of \( G \) on \( X \) is almost effective if its kernel is finite. All manifolds and actions of groups on manifolds which we consider are implicitly assumed to be smooth. In this paper by a natural number we mean a strictly positive integer. The set of natural numbers is denoted as usually by \( \mathbb{N} \). For any \( k \in \mathbb{N} \) we denote by \( T^k \) the \( k \)-dimensional torus \( (S^1)^k \).

For any pair \((\tau, r) \in \mathbb{N}^2\) we denote by \( \text{SU}(\tau)^r \) the direct product of \( r \) copies of the special unitary group \( \text{SU}(\tau) \).

We are now ready to state the main result of this paper.

**Theorem 1.1.** There exist functions \( \tau, M : \mathbb{N} \to \mathbb{N} \) such that for any \( n, r \in \mathbb{N} \) the following property is satisfied. Suppose that a manifold \( X \) supports an almost effective action of \( \text{SU}(\tau(n))^r \) with kernel \( H \). For any prime \( p \) not dividing \( |H| \) and satisfying \( p > M(n) \) and \( p \equiv 1 \mod n + 1 \), there exists a finite \( p \)-group \( \Gamma \) acting effectively on \( T^{2nr} \times X \) such that \( |\Gamma| = p^{2n+r} \) and:

1. if \( r = 1 \) then any abelian subgroup of \( \Gamma \) has at most \( p^{n+1} \) elements;
2. if \( r > 1 \) then any abelian subgroup of \( \Gamma \) has at most \( p^{2+r+4nr} \) elements.

The manifold \( X \) need not be compact.

By Dirichlet theorem (see e.g. [13, §8.4]), for any \( n, h \in \mathbb{N} \) there are infinitely many primes \( p \) that do not divide \( h \) and satisfy \( p > M(n) \) and \( p \equiv 1 \mod n + 1 \). Hence, once we fix \( X \) and an almost effective action of \( \text{SU}(\tau(n))^r \) on \( X \) the previous theorem applies to infinitely many primes.

Our motivation to state the result referring to \( p \)-groups comes from the main theorem in [12], according to which to test whether the diffeomorphism group of a manifold is Jordan it suffices to consider finite subgroups of \( G \) whose cardinal is divisible by at most two different primes (actually the result in [12] applies more generally to any group \( G \) admitting a constant \( R \) such that any elementary \( p \)-group contained in \( G \) has rank at
most $R$, for any prime $p$; diffeomorphism groups have always this property, by a theorem of Mann and Su \cite{7}. A priori the diffeomorphism group of a manifold might fail to be Jordan but still satisfy Jordan's property restricted to $p$-groups, and Theorem \ref{mainthm} makes it clear that if $X$ satisfies the hypothesis of the theorem then the diffeomorphism group of $T^{2n} \times X$ does not even have this property.

The $p$-groups obtained in the proof of Theorem \ref{mainthm} are all 2-step nilpotent. It seems an interesting question to explore whether there are compact smooth manifolds admitting actions of $k$-step nilpotent $p$-groups for some $k \geq 3$ and arbitrarily large primes $p$.

Given a nontrivial finite group $\Gamma$ define

$$
\lambda(\Gamma) := \max \left\{ \frac{\log |A|}{\log |\Gamma|} \mid A \text{ abelian subgroup of } \Gamma \right\},
$$

and for any group $G$ containing arbitrarily big finite subgroups consider the quantity

$$
\Lambda(G) := \inf \{ \lambda \mid \exists \{\Gamma_i\}_{i \in \mathbb{N}}, \text{ each } \Gamma_i \text{ is a finite subgroup of } G, |\Gamma_i| \to \infty, \lambda(\Gamma_i) \to \lambda \};
$$

if the size of the finite subgroups of $G$ is uniformly bounded, then define $\Lambda(G) = 1$. In particular, if $\Lambda(G) < \epsilon$ for some $\epsilon > 0$, then $G$ contains arbitrarily large finite subgroups $\Gamma$ all of whose abelian subgroups have size at most $|\Gamma|^{\epsilon}$.

Obviously, $\Lambda(G) \in [0,1]$ for any $G$. If $\Lambda(G) < 1$ then $G$ is not Jordan, and the difference $1 - \Lambda(G)$ gives some measure of how far $G$ is from being Jordan. The main result in \cite{3} implies that $\Lambda(\text{Diff}(T^2 \times S^2)) \leq 2/3$, while our theorem implies that if $X$ supports an almost effective action of $\text{SU}(\tau(n))$ then $\Lambda(\text{Diff}(T^{2n} \times X)) \leq (n+1)/(2n+1)$ for any $n$. Moreover, if $X$ supports an almost effective action of the product of $\text{SU}(\tau(n))$, then $\Lambda(\text{Diff}(T^{2nr} \times M)) \leq (2 + r + 4n/r)/(2n + r)$ for any $n$. In particular, for any $\epsilon > 0$ there exist manifolds $Y$ such that $\Lambda(\text{Diff}(Y)) < \epsilon$.

We next describe the main building block in the proof of Theorem \ref{mainthm}. For any natural number $n$ and any prime $p$, define $\Gamma_{n,p}$ to be the group generated by elements $a_1, \ldots, a_n, b_1, \ldots, b_n, f$ with the relations $a_i^p = b_i^p = [a_i, a_j] = [b_i, b_j] = [a_i, f] = [b_i, f] = 1$ for every $i,j$, $[a_i, b_j] = 1$ for very $i \neq j$, and $[a_i, b_i] = f$ for every $i$. The group $\Gamma_{n,p}$ has $p^{2n+1}$ elements and no abelian subgroup of $\Gamma_{n,p}$ has more than $p^{n+1}$ elements (see Lemma \ref{21} below). We have:

**Theorem 1.2.** Given $n \in \mathbb{N}$ there exists some $\tau(n), M(n) \in \mathbb{N}$ such that for any prime $p$ satisfying $p > M(n)$ and $p \equiv 1 \mod n + 1$ the group $\Gamma_{n,p}$ acts effectively on the trivial vector bundle $T^{2n} \times \mathbb{C}^{\tau(n)}$ by vector bundle automorphisms and leaving invariant a nowhere vanishing section of the determinant bundle $T^{2n} \times \Lambda^{\tau(n)} \mathbb{C}^{\tau(n)}$.

To deduce Theorem \ref{mainthm} from Theorem \ref{12} we use some standard constructions of fiber bundles and a group theoretical result of Olshanskii \cite{13}.

As explained above, when $n = 1$ Theorem \ref{12} follows from the fact that for any degree $d$ the vector bundle $L_d \oplus L_d^{-1} \to T^2$ is trivial as a smooth vector bundle. Hence, we may take $M(1) = 1$ and $\tau(1) = 2$. An immediate consequence of Theorem \ref{mainthm} is that if a manifold $X$ has the property that $T^2 \times X$ is Jordan then $X$ does not support any almost effective action of $\text{SU}(2)$. Since any compact connected semisimple Lie group contains a subgroup isomorphic either to $\text{SU}(2)$ or to $\text{SO}(3,\mathbb{R}) \simeq \text{SU}(2)/\{\pm \text{Id}\}$ (see for example Theorem 19.1 in \cite{2}), it follows that $X$ does not admit any effective action of a compact
connected semisimple Lie group. In view of the main theorem in [S] this implies the following.

Corollary 1.3. Suppose that $X$ is a $d$-dimensional compact manifold admitting a finite unramified covering $\tilde{X} \to X$ and that there exist classes $\alpha_1, \ldots, \alpha_d \in H^1(\tilde{X}; \mathbb{Z})$ such that $\alpha_1 \cup \cdots \cup \alpha_d \neq 0$. Then any compact connected Lie group acting effectively on $X$ is abelian.

This applies in particular to the connected sum of a torus and any other manifold. Corollary 1.3 is not a new result (see [H, Theorem 2.1], [R, Theorem A]; see also [S, Theorem 2.5], which is slightly more restrictive), but the proof we obtain is new.

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2. Proof of Theorem 1.2

Fix some odd prime $p$ and a natural number $n$.

2.1. The group $\Gamma_{n,p}$ and the bundle $Q_{n,p}$ over the torus $T^{2n}$. We begin by reviewing the construction of some standard generalizations of Heisenberg $p$-groups and their action on bundles over $T^{2n}$. Let

$$X = \mathbb{R}^n \times T^n \times S^1.$$  

Define a free action of $\mathbb{Z}^n$ on $X$ by setting, for any $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, $t \in \mathbb{R}^n$, $\theta = (\theta_1, \ldots, \theta_n) \in T^n$, and $\nu \in S^1$:

$$d \cdot (t, \theta, \nu) = \left( t + d, \theta, \theta_1^{d_1} \cdots \theta_n^{d_n} \nu \right).$$

Denote by $Q_{n,p} = X/\mathbb{Z}^n$ the orbit space of this action. The projection of $X$ to the first two factors gives $Q_{n,p}$ a structure of principal $S^1$-bundle over $T^{2n}$.

Let $\mu = \exp(2\pi i/p)$ and let $\psi_j = (1, \ldots, 1, \mu, 1, \ldots, 1) \in T^n$, where the entry $\mu$ is in the $j$-th position. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$. Define diffeomorphisms $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \phi \in \text{Diff}(X)$ by the formulas

$$\alpha_j(t, \theta, \nu) = (t + p^{-1}e_j, \theta, \theta \nu), \quad \beta_j(t, \theta, \nu) = (t, \psi_j \theta, \nu), \quad \phi(t, \theta, \nu) = (t, \theta, \mu \nu).$$

Let $\Gamma$ be the group generated by $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \phi$. One easily checks that $[\alpha_i, \alpha_j] = [\beta_i, \beta_j] = \beta_i^p = 1$ for every $i, j$, that $[\alpha_i, \beta_j] = 1$ for every $i \neq j$, and that $[\alpha_j, \beta_j] = \phi$ for every $j$. Furthermore, $\phi$ is central in $\Gamma$ and has order $p$, so $[\alpha_j^p, \beta_j] = 1$ for every $j$. Hence each $\alpha_j^p$ is central in $\Gamma$. Let $\Gamma_Z \subseteq \Gamma$ be the subgroup generated by $\alpha_1^p, \ldots, \alpha_n^p$. The action of $\Gamma_Z$ on $X$ coincides with the action of $\mathbb{Z}^n$ defined in [H]; more precisely, we can identify the diffeomorphism $\alpha_j^p$ with the action of $e_j \in \mathbb{Z}^n$. So the quotient group $\Gamma_{n,p} = \Gamma/\Gamma_Z$ acts effectively on $Q_{n,p}$. The group $\Gamma_{n,p}$ is obviously the same as the one defined before the statement of Theorem 1.2.

Denote by $\overline{\alpha}_j, \overline{\beta}_j$ and $\overline{\phi}$ the classes in $\Gamma_{n,p}$ of the elements $\alpha_j, \beta_j$ and $\phi$. The group $\Gamma_{n,p}$ sits in a short exact sequence

$$0 \to \mathbb{Z}_p \to \Gamma_{n,p} \xrightarrow{\eta} (\mathbb{Z}_p)^{2n} \to 0,$$
where \( \eta(\tau_j) \in \mathbb{Z}^{2n}_p \) (resp. \( \eta(\overline{\tau}_j) \)) is the tuple with 0's everywhere except in the position \( j \) (resp. \( r + j \)), where the entry is 1 (consequently, \( \eta(\overline{\delta}) = 0 \)). Hence \( \Gamma_{n,p} \) has \( p^{2n+1} \) elements. In terms of the standard symplectic form \( \omega : \mathbb{Z}_p^{2n} \times \mathbb{Z}_p^{2n} \rightarrow \mathbb{Z}_p \) defined by

\[
\omega((x_1, \ldots, x_n, y_1, \ldots, y_n), (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n)) = \sum (x_j y'_j - x'_j y_j)
\]

we have \([\zeta, \gamma] = \delta^{\omega(\eta(\zeta), \eta(\gamma))}\) for any \( \zeta, \gamma \in \Gamma_{n,p} \). Hence any abelian subgroup of \( \Gamma_{n,p} \) projects via \( \eta \) to an \( \omega \)-isotropic subspace of \( \mathbb{Z}_p^{2n} \). Since \( \omega \) is non degenerate, \( \omega \)-isotropic subspaces of \( \mathbb{Z}_p^{2n} \) have dimension at most \( n \), so their cardinal is at most \( p^n \). We have thus proved the following.

**Lemma 2.1.** No abelian subgroup of \( \Gamma_{n,p} \) has more than \( p^{n+1} \) elements.

The action of \( \Gamma_{n,p} \) on \( Q_{n,p} \) lifts an action of \( \Gamma_{n,p} \) on \( T^{2n} \) which is not effective, since two elements \( \gamma, \gamma' \in \Gamma_{n,p} \) act via the same diffeomorphism of \( T^{2n} \) if and only if \( \eta(\gamma) = \eta(\gamma') \). Hence the action of \( \Gamma_{n,p} \) induces an effective action of \( \mathbb{Z}_p^{2n} \) on \( T^{2n} \), which of course is nothing but the diagonal action

\[
(z_1, \ldots, z_{2n}) \cdot \theta_1, \ldots, \theta_{2n}) = (e^{2\pi i z_1/p} \theta_1, \ldots, e^{2\pi i z_{2n}/p} \theta_{2n}).
\]

The quotient space of this action of \( \mathbb{Z}_p^{2n} \) on \( T^{2n} \) can be identified with \( T^{2n} \) itself, in such a way that the projection to the quotient space is the map

\[
w : T^{2n} \rightarrow T^{2n}, \quad w(\theta_1, \ldots, \theta_{2n}) = (\theta_{1}^{p}, \ldots, \theta_{2n}^{p}).
\]

**Lemma 2.2.** Suppose that \( \Gamma_{n,p} \) acts on the trivial line bundle \( T^{2n} \times \mathbb{C} \) by vector bundle automorphisms lifting the action of \( \Gamma_{n,p} \) on \( T^{2n} \). Then there is a nowhere vanishing \( \Gamma_{n,p} \)-equivariant section \( \sigma : T^{2n} \rightarrow \mathbb{C} \).

That \( \sigma \) is an equivariant section means that \( \gamma \cdot (\theta, \sigma(\theta)) = (\gamma \cdot \theta, \sigma(\gamma \cdot \theta)) \) for every \( \gamma \in \Gamma_{n,p} \) and every \( \theta \in T^{2n} \).

**Proof.** Let \( L = T^{2n} \times \mathbb{C} \). Take an action of \( \Gamma_{n,p} \) on \( L \) by vector bundle automorphisms lifting the action of \( \Gamma_{n,p} \) on \( T^{2n} \). There is a unique smooth map \( c : \Gamma_{n,p} \times T^{2n} \rightarrow \mathbb{C}^* \) satisfying

\[
\gamma \cdot (\theta, w) = (\gamma \cdot \theta, c(\gamma, \theta) w)
\]

for every \( \gamma \in \Gamma_{n,p}, \theta \in T^{2n} \) and \( w \in \mathbb{C} \). The condition that \( c \) defines an action of \( \Gamma_{n,p} \) on \( L \) is equivalent to the following cocycle condition:

\[
c(\gamma' \gamma, \theta) = c(\gamma', \gamma \cdot \theta) c(\gamma, \theta)
\]

for every \( \gamma, \gamma' \in \Gamma_{n,p} \) and \( \theta \in T^{2n} \).

This implies in particular that \( c(1, \theta) = 1 \) for every \( \theta \), where 1 denotes the identity element. We claim that for any \( \gamma \in \Gamma_{n,p} \) there is a map

\[
\tilde{c}_\gamma : T^{2n} \rightarrow \mathbb{C}
\]

such that \( c(\gamma, \theta) = \exp(\tilde{c}_\gamma(\theta)) \) for every \( \theta \). This is obvious if \( \gamma = 1 \), because \( c(1, \theta) = 1 \).

So let us assume that \( \gamma \neq 1 \) and that, contrary to the claim, the map \( \tilde{c}_\gamma \) does not exist. Then there exists some \( \gamma \in \Gamma_{n,p} \) and a map \( h : S^1 \rightarrow T^{2n} \) so that the map \( c_{\gamma, h} : S^1 \rightarrow \mathbb{C}^* \) defined as \( c_{\gamma, h}(\chi) = c(\gamma, h(\chi)) \) has nonzero index: \( \text{Ind}(c_{\gamma, h}) \neq 0 \). For any \( \delta \in \Gamma_{n,p} \) let \( \mu_\delta : T^{2n} \rightarrow T^{2n} \) be the map \( \theta \mapsto \delta \cdot \theta \). Since \( \mu_\delta \) is homotopic to the identity, we have

\[
\text{Ind}(c_{\gamma, \mu_\delta \cdot h}) = \text{Ind}(c_{\gamma, h}).
\]
Using the cocycle condition and induction on \( k \in \mathbb{N} \) we obtain
\[
c(\gamma^k, \theta) = \prod_{j=0}^{k-1} c(\gamma, \gamma^j \cdot \theta)
\]
for every \( \gamma \) and \( k \). Taking \( k = \text{ord}(\gamma) + 1 \) (so that \( \gamma^k = \gamma \)), applying the previous formula to \( \theta = h(\chi) \) for each \( \chi \in S^1 \), and using the fact that the index of maps \( S^1 \to \mathbb{C}^* \) is additive with respect to pointwise multiplication, we deduce
\[
\text{Ind}(c_{\gamma,h}) = \sum_{j=0}^{k-1} \text{Ind}(c_{\gamma,\mu_{j,0}h}) = k \text{Ind}(c_{\gamma,h}),
\]
where the second equality follows from (6). Since \( \gamma \neq 1 \), we have \( k \geq 2 \) and hence \( \text{Ind}(c_{\gamma,h}) = 0 \), contrary to our assumption. So the claim is proved.

Now let \( a = \overline{a}_1, b = \overline{b}_1 \) and \( f = \overline{\phi} \). We next claim that the action of \( f \) on \( L \) is trivial (so in particular the action of \( \Gamma_{n,p} \) is not effective). In order to prove the claim, note that since \( f \) acts trivially on \( T^2n \), its action on \( L \) will be given by \( f \cdot (\theta, w) = (\theta, g(\theta)w) \) for some smooth map \( g : T^2n \to \mathbb{C}^* \). Since \( f \) has order \( p \), \( g \) must take values in the set of \( p \)-roots of unity. Applying the cocycle condition to both sides of the equality \( ab = ba \) we obtain
\[
c(a, b \cdot \theta)c(b, \theta) = c(ba, f \cdot \theta)c(f, \theta) = c(ba, \theta)g(\theta) = c(b, a \cdot \theta)c(a, \theta)g(\theta)
\]
for any \( \theta \). It follows that
\[
\tilde{c}_a(b \cdot \theta) + \tilde{c}_b(\theta) = \tilde{c}_b(a \cdot \theta) + \tilde{c}_a(\theta) + g(\theta),
\]
where \( \theta \) is arbitrary and \( \tilde{g} : T^2n \to \mathbb{C} \) satisfies \( \exp(\tilde{g}(\theta)) = g(\theta) \), so \( \exp(\tilde{g}(\theta)w) = g(\theta)p = 1 \) for every \( \theta \). Since each \( \tilde{c}_a \) is smooth, (7) implies that \( \tilde{g} \) is smooth, and hence the condition \( \exp(\tilde{g}(\theta)) = 1 \) implies that \( \tilde{g} \) must be equal to some constant, say \( \tilde{g}_0 \in \mathbb{C} \). Now let \( q \in T^2n \) be any point and let \( \Theta = \Gamma_{n,p} \cdot q = \mathbb{Z}^2n_p \cdot q \subset T^2n \) be its orbit. Summing both sides of (7) as \( \theta \) runs over the elements of \( \Theta \) and using the fact that \( \Theta \) is invariant under the action of both \( a \) and \( b \) we deduce that \( 0 = \sum_{\theta \in \Theta} \tilde{g}(\theta) = |\Theta|\tilde{g}_0 \). Hence \( \tilde{g}_0 = 0 \), so \( g(\theta) = 1 \) for every \( \theta \) and the claim is proved.

The previous claim implies that the action of \( \Gamma_{n,p} \) on \( L \) factors through the morphism \( \eta : \Gamma_{n,p} \to \mathbb{Z}^2n_p \). Since \( \mathbb{Z}^2n_p \) acts freely on \( T^2n \), the quotient \( L/\mathbb{Z}^2n_p \) has a natural structure of line bundle over \( T^2n/\mathbb{Z}^2n_p \). In other words, there must exist some line bundle \( \Lambda \to T^2n \) and a \( \Gamma_{n,p} \)-equivariant isomorphism \( L \simeq w^*\Lambda \), where \( w \) is the map (5). This implies that \( c_1(L) = w^*c_1(\Lambda) \). But \( w^* : H^2(T^2n; \mathbb{Z}) \to H^2(T^2n; \mathbb{Z}) \) is multiplication by \( p^2 \), and hence is injective. Since \( c_1(L) = 0 \), it follows that \( c_1(\Lambda) = 0 \). Hence \( \Lambda \) is the trivial line bundle. Taking a nowhere vanishing section \( s \) of \( \Lambda \), we obtain by pullback the desired nowhere vanishing \( \Gamma_{n,p} \)-equivariant section \( \sigma \) of \( L \).

2.2. Cohomology of \( T^{2n} \). Denote the line bundle associated to \( Q_{n,p} \) by
\[
L_{n,p} = Q_{n,p} \times_{S^1} \mathbb{C},
\]
where \( S^1 \) acts on \( \mathbb{C} \) with weight 1. Choose a generator \( \sigma \in H^1(S^1; \mathbb{Z}) \). Let \( \pi_j : T^{2n} = (S^1)^{2n} \to S^1 \) denote the projection to the \( j \)-th factor and define cohomology classes
Furthermore, \( u_i, v_i \in H^1(T^{2n}; \mathbb{Z}) \) for any \( 1 \leq i \leq n \) by \( u_i = \pi^* u_i \) and \( v_i = \pi^* v_i \). Define also the class
\[
\Omega = \sum_{i=1}^{n} u_i \cup v_i \in H^2(T^{2n}; \mathbb{Z}).
\]

**Lemma 2.3.** We have \( c_1(L_{n,p}) = p\Omega \).

**Proof.** Let \( \Pi = (\pi_1, \pi_{n+1}) : T^{2n} \to S^1 \times S^1 = T^2 \). It follows from the definition of \( Q_{n,p} \) that \( L_{n,p} = \bigotimes_{i=1}^{n} \Pi_i^* L_{1,p} \). Hence it suffices to prove the lemma in the case \( n = 1 \). Using the notation of Subsection 2.1 we identify \( Q_{1,p} \) with the quotient of the trivial principal bundle
\[
\pi : \mathcal{Q}_{1,p} = \mathbb{R} \times S^1 \times S^1 \to \mathbb{R} \times S^1,
\]
where \( \pi \) is the projection to the first two factors, under the action of \( \mathbb{Z} \) given by \( k \cdot (t, \theta, \nu) = (t + k, \theta, \theta^k \nu) \). Then \( d_A := d - ipt \, dt \) is a \( \mathbb{Z} \)-invariant connection, so it descends to a connection on \( Q_{1,p} \). Its curvature on \( \mathcal{Q}_{1,n} \) is equal to \(-ip \, dt \wedge d\theta\), so we may compute, using the orientation of \( T^{2n} \) given by \( dt \wedge d\theta \)
\[
\deg Q_{1,n} = \frac{i}{2\pi} \int_{0}^{1} \left( \int_{S^1} -ip \, d\theta \right) dt = p.
\]
Since the integral of \( \Omega \) with respect to the same orientation is equal to 1, the result follows. \( \square \)

For any \( m \in \mathbb{N} \) we define \([m] := \{1, \ldots, m\} \). For any subset \( I = \{i_1 < \ldots < i_k\} \subset [n] \) define \( u_I = u_{i_1} \cup \cdots \cup u_{i_k} \) and \( v_I = v_{i_1} \cup \cdots \cup v_{i_k} \). A simple computation shows that

\[
\Omega^k = \frac{n!}{(n-k)!} \sum_{I \subset [n], |I| = k} (-1)^k u_I \cup v_I.
\]

Given a permutation \( \sigma \in S_n \) define a new permutation \( \sigma' \in S_{2n} \) by the condition \( \sigma'(i) = \sigma(i) \) and \( \sigma'(n+i) = n + \sigma(i) \) for every \( 1 \leq i \leq n \). Let \( \nu_\sigma : T^{2n} \to T^{2n} \) be the diffeomorphism defined by
\[
\nu_\sigma(\theta_1, \ldots, \theta_{2n}) = (\theta_{\sigma'(1)}, \ldots, \theta_{\sigma'(2n)}).
\]
For any real number \( t \) denote by \( \langle t \rangle \) the integer part of \( t \).

**Lemma 2.4.** For any \( k \in [n] \) there exist rational numbers \( \{a_{k,1}, \ldots, a_{k,(n/k)}\} \), where each \( a_{k,j} \) depends on \( k, j, n \) but not on \( p \), such that
\[
\prod_{\sigma \in S_n} (1 + \nu_\sigma^*(u_{[k]} \cup v_{[k]})) = 1 + \sum_{j=1}^{(n/k)} a_{k,j} \Omega^{jk}.
\]
Furthermore, \( a_{k,1} \neq 0 \).

**Proof.** The formula in the lemma follows from (5). The nonvanishing of \( a_{k,1} \) is a consequence of the fact that if for some \( \sigma \in S_n \) we have \( \sigma([k]) = [k] \) then \( \nu_\sigma^*(u_{[k]} \cup v_{[k]}) = u_{[k]} \cup v_{[k]} \) i.e., there is no minus sign for any choice of \( \sigma \). \( \square \)
2.3. Some equivariant vector bundles over $T^{2n}$.

**Lemma 2.5.** Let $\sigma_k \in H^k(S^{2k};\mathbb{Z})$ be a generator. For any natural number $k$ and any integer $\delta$ there exists a complex vector bundle $E_k(\delta) \to S^{2k}$ of rank $k$ satisfying $c_k(E_k(\delta)) = \delta (k-1)! \sigma_k$.

**Proof.** By [6, Ch. 20, Corollary 9.8] there exists some class $\epsilon(\delta) \in \tilde{K}(S^{2k})$ with $c_k(\epsilon(\delta)) = \delta (k-1)! \sigma_k$. By [6, Ch. 9, Theorem 3.8] there exists a vector bundle $\xi \to S^{2k}$ such that $\epsilon(\delta) = [\xi] - [(\text{rk} \xi)]$. Finally, by [6, Ch. 9, Remark 3.7], the vector bundle $\xi$ is stably equivalent to a vector bundle on $S^{2k}$ of rank $k$. We take $E_k(\delta)$ to be any such vector bundle. $\square$

**Lemma 2.6.** For any $k \in [n]$ and any $\delta \in \mathbb{Z}$ there exists a vector bundle $F_k(\delta)$ of rank $k$ over $T^{2n}$ satisfying $c_k(F_k(\delta)) = \delta (k-1)! (u[k] \cup v[k])$ and $c_j(F_k(\delta)) = 0$ for any $j \neq k$.

**Proof.** Let $\pi^k : T^{2n} \to T^{2k}$ denote the projection

$$(\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{2n}) \mapsto (\theta_1, \ldots, \theta_k, \theta_{n+1}, \ldots, \theta_{n+k})$$

and let $q_k : T^{2k} \to S^{2k}$ be a degree $\pm 1$ map. We may suppose that the generator $\sigma_k \in H^k(S^{2k};\mathbb{Z})$ in Lemma 2.5 satisfies $q_k^* \sigma_k = u[k] \cup v[k]$. Then we define $F_k(\delta) = (\pi^k)^* q_k^* E_k(\delta)$, where $E_k(\delta)$ is any bundle as given by Lemma 2.5. $\square$

Fix for any $k \in [n]$ and any integer $\delta$ a vector bundle $F_k(\delta)$ over $T^{2n}$ with the properties specified in Lemma 2.6. Define, for any $k$ and $\delta$,

$$F_k(\delta) := \bigoplus_{\sigma \in S_n} \nu^\sigma F_k(\delta).$$

The vector bundle $F_k(\delta)$ has rank $kn!$. Lemma 2.4 implies that the total Chern class of $F_k(\delta)$ is

$$c(F_k(\delta)) = 1 + \sum_{j=1}^{\lfloor n/k \rfloor} \delta^j a_{k,j} \Omega^{jk},$$

with $a_{k,1} \neq 0$.

**Lemma 2.7.** There exists $M \in \mathbb{N}$, depending only on $n$, with the following property. Suppose that $b_1, \ldots, b_n \in \mathbb{Z}$ are all divisible by $M$. Then there exist $\delta_1, \ldots, \delta_n \in \mathbb{Z}$ with the property that

$$\prod_{j=1}^n c(F_j(\delta_j)) = 1 + \sum_{j=1}^n b_j \Omega^j.$$

**Proof.** Denote for any $k \in \mathbb{N}$ satisfying $k \leq n$ and any $m \in \mathbb{Z}$

$$\mathcal{H}^{\geq k}(m) = \{ \alpha \in H^*(T^{2n};\mathbb{Z}) \mid \alpha = 1 + \alpha_k \Omega^k + \cdots + \alpha_n \Omega^n, a_j \in m\mathbb{Z} \text{ for each } j \}.$$

We claim that for any $1 \leq k \leq n$ there exists some integer $m_k$ such that any $\alpha \in \mathcal{H}^{\geq k}(m_k)$ can be written as $\prod_{j=k}^n c(F_j(\delta_j))$ for a suitable choice of $\delta_k, \ldots, \delta_n \in \mathbb{Z}$. Of course the case $k = 1$ is the lemma we want to prove.
We prove the claim by descending induction on $k$. Consider first the case $k = n$. Choose $m_n$ so that any element of $m_n \mathbb{Z}$ is an integral multiple of $a_{n,1}$. Since for any integer $b_n$ we have $1 + m_nb_n\Omega^i = c(F_n(m_nb_n, a_{n,1}))$ and $m_nb_n/a_{n,1}$ is an integer, we are done in this case. Now assume that the claim has been proved for some $k = i + 1$ ($1 \leq i < n$) and an integer $m_{i+1}$. Let $m'_i \in \mathbb{Z}$ be chosen in such a way that any element of $m'_i \mathbb{Z}$ is an integral multiple of $a_{i,1}$, let $m''_i \in \mathbb{Z}$ be chosen in such a way that $m''_ia_{i,j} \in m_{i+1}\mathbb{Z}$ for every $1 < j$, and let $m_i := m'_i m''_i$. We next prove that this choice of $m_i$ has the desired property.

If $\alpha = 1 + \alpha_i\Omega^i + \cdots + \alpha_n \in \mathcal{I}^{\geq i}(m_i)$ then $\delta_i := \alpha_i/a_{i,1}$ belongs to $m''_i \mathbb{Z}$. Consequently, in the development $c(F_i(\delta_i)) = 1 + \sum_{j \geq 1}^{\langle n/i \rangle} \gamma_j \Omega^j$ we have $\gamma_j \in m_{i+1}\mathbb{Z}$ for every $j > 1$. This implies that the series $\alpha' = 1 + \sum_{j \geq 1} \alpha'_j \Omega^j$, defined by the property that $\alpha = c(F_i(\delta_i))\alpha'$, belongs to $\mathcal{I}^{\geq i+1}(m_{i+1})$. By the induction hypothesis we may write $\alpha' = \prod_{j=i+1}^{n} c(F_j(\delta_j))$ for some $\delta_{i+1}, \ldots, \delta_n \in \mathbb{Z}$. Hence $\alpha = \prod_{j=i}^{n} c(F_j(\delta_j))$, so the claim is proved. \qed

Let $w : T^{2n} \to T^{2n}$ be the map (5). Define for any $k$ and $\delta$

$$G_k(\delta) := w^*F_k(\delta).$$

Since the fibers of $w$ are the orbits of the action of $\mathbb{Z}^{2n}_p$ on $T^{2n}$, the vector bundle $F_k$ carries a natural action of $\mathbb{Z}^{2n}_p$ lifting the action on $T^{2n}$. The action of $\mathbb{Z}^{2n}_p$ on $G_k(\delta)$ can be promoted to an action of $\Gamma_{n,p}$ via the projection map $\eta : \Gamma_{n,p} \to \mathbb{Z}^{2n}_p$. Of course, this action of $\Gamma_{n,p}$ is not effective.

Applying Künneth it follows that the morphism induced in cohomology by $w$ is

$$w^i : H^i(T^{2n}; \mathbb{Z}) \to H^i(T^{2n}; \mathbb{Z}), \quad w^i(\alpha) = p^i\alpha.$$

Hence we have

$$c(G_k(\delta)) = 1 + \sum_{j=k}^{\langle n/k \rangle} \delta^j p^{2jk} a_{k,j} \Omega^{jk}.$$

The next lemma follows immediately from Lemma 2.7

**Lemma 2.8.** There exists $M \in \mathbb{N}$, depending only on $n$, with the following property. Suppose that $b_1, \ldots, b_n \in \mathbb{Z}$ are such that $b_j$ is divisible by $M p^{2j}$ for each $j$. Then there exist $\delta_1, \ldots, \delta_n \in \mathbb{Z}$ with the property that

$$\prod_{j=1}^{n} c(G_j(\delta_j)) = 1 + \sum_{j=1}^{n} b_j \Omega^j.$$

**Proposition 2.9.** Let $M$ be as in Lemma 2.8. Suppose that $p \equiv 1 \mod n + 1$. There exist integers $a_1, \ldots, a_{n+1}, \delta_1, \ldots, \delta_n \in \mathbb{Z}$ such that $p$ does not divide any of the $a_j$’s and furthermore

$$\prod_{j=1}^{n+1} (1 + a_j M p \Omega^j) \prod_{j=1}^{n} c(G_j(\delta_j)) = 1.$$

**Proof.** Since $p \equiv 1 \mod n + 1$, the group of invertible elements $(\mathbb{Z}_{p^n})^*$ in $\mathbb{Z}_{p^n}$ has order $(p - 1)p^{n-1}$ divisible by $n + 1$. Since $(\mathbb{Z}_{p^n})^*$ is cyclic (equivalently, $(\mathbb{Z}_{p^n})^*$ has primitive roots, see e.g. [13, §2.8]), this implies that $R = \{\alpha \in (\mathbb{Z}_{p^n})^* \mid \alpha^{n+1} = 1\}$ is a cyclic
Lemma 2.10. There exists some number bundle $V$ denote the trivial complex vector bundle of rank $Z$. We claim that each $\sigma_j(a_1, \ldots, a_{n+1})$ is divisible by $p^n$. Equivalently, $\sigma_j(a_1, \ldots, a_{n+1}) = 0$ in $\mathbb{Z}_{p^n}$. Let $\alpha \in R$ be a generator. Since multiplication by $\alpha$ induces a permutation of the elements of $R$, we have, for any $1 \leq j \leq n$, $\alpha^j \sigma_j(a_1, \ldots, a_{n+1}) = \sigma_j(\alpha a_1, \ldots, \alpha a_{n+1}) = \sigma_j(a_1, \ldots, a_{n+1})$, which implies

$$\sigma_j(a_1, \ldots, a_{n+1}) = 0.$$

We next prove that $\alpha^j - 1$ is an invertible element of $\mathbb{Z}_{p^n}$. Since $\alpha$ is a generator of $S$ and $j \leq n$, we have $\alpha^j \neq 1$. So if $\alpha^j - 1$ were not invertible then we could write $\alpha^j = 1 + \beta p^e$ for some $1 \leq e \leq n - 1$ and some invertible $\beta \in \mathbb{Z}_{p^n}$. But then $\alpha^{j+1} = 1 + (n + 1)\beta p^e + \beta' p^{e+1}$ for some $\beta'$. Since $p$ does not divide $n + 1$, it follows that $(n + 1)\beta \in \mathbb{Z}_{p^n}$, so $(\alpha^{n+1})^j = (\alpha^j)^{n+1} \neq 1$, a contradiction. Finally, since $\alpha^j - 1$ is invertible, (11) implies $\sigma_j(a_1, \ldots, a_{n+1}) = 0$, which is what we wanted to prove.

Let now

$$s := \sum_{j=1}^{n} s_j M^j p^j \Omega^j$$

and define integers $b_1, \ldots, b_n$ by the condition that

$$\sum_{k \geq 1} (-1)^k s^k = \sum_{j=1}^{n} b_j \Omega^j.$$

It is easy to prove, using the fact that each $s_j$ is divisible by $p^n$, that $b_j$ is divisible by $M p^{2j}$ for each $j$. By Lemma 2.8, there exist integers $\delta_1, \ldots, \delta_n$ such that

$$\prod_{j=1}^{n} c(G_j(\delta_j)) = 1 + \sum_{j=1}^{n} b_j \Omega^j.$$

Since $1 + \sum b_j \Omega^j = 1 + \sum_{k \geq 1} (-1)^k s^k$ is the inverse of $1 + s = \prod (1 + a_j M p \Omega)$, the numbers $a_i, \delta_j$ and $e_k$ satisfy (10).

2.4. Trivial equivariant vector bundles on $T^{2n}$: proof of Theorem 1.2. Let $C^r$ denote the trivial complex vector bundle of rank $r$ over the torus $T^{2n}$.

**Lemma 2.10.** There exists some number $r_0$ with the property that for any complex vector bundle $V \to T^{2n}$ with vanishing Chern classes there is an isomorphism of vector bundles $V \oplus C^{r_0} \simeq C^{r_0} V^{+r_0}$.

**Proof.** We first claim that if a complex vector bundle over $T^{2n}$ has vanishing Chern classes then it represents the trivial element in $K^0(T^{2n})$. This follows from combining two facts. First, $K^*(T^{2n})$ has no torsion: this is a consequence of the isomorphisms $K^0(S^1) \simeq K^{-1}(S^1) \simeq \mathbb{Z}$ (see Example 2.8.1) and Kunneth theorem for $K$-theory (see Proposition 3.3.15). The second fact is that if $X$ is a topological space which is homeomorphic to a finite CW-complex and $K^*(X)$ is torsion free, then the Chern
character \( \text{ch} : K^*(X) \to H^*(X; \mathbb{Q}) \) is injective (see the Corollary in \([1] \text{ §2.4}\)). Since a vector bundle with vanishing Chern classes has trivial Chern character, the claim follows. To deduce the lemma from the claim, apply \([6] \text{ Ch. 9, Theorem 1.5}\).

We are now ready to prove Theorem 1.2. Assume that \( p \equiv 1 \mod n + 1 \). Let \( M \) be as in Lemma 2.8 and let \( a_1, \ldots, a_{n+1}, \delta_1, \ldots, \delta_n \in \mathbb{Z} \) be as in Proposition 2.9. Consider the following vector bundle over \( T^{2n} \):

\[
V_{n,p} = \bigoplus_{j=1}^{n+1} L_{n,p}^{a_j M} \oplus \bigoplus_{j=1}^{n} G_j(\delta_j).
\]

By Proposition 2.9 all Chern classes of \( V_{n,p} \) are trivial.

The action of \( \Gamma_{n,p} \) on \( L_{n,p} \) induces actions on its powers \( L_{n,p}^{a_j M} \). Combining these actions with those on the bundles \( G_j(\delta_j) \), we obtain an action of \( \Gamma_{n,p} \) on \( V_{n,p} \) by vector bundle automorphisms.

**Lemma 2.11.** If \( p > M \) then the action of \( \Gamma_{n,p} \) on \( V_{n,p} \) is effective.

**Proof.** It suffices to prove that the action of \( \Gamma_{n,p} \) on any of the summands \( L_{n,p}^{a_j M} \) is effective. If for some \( j \) there were a nontrivial element \( \gamma \in \Gamma_{n,p} \) acting trivially on \( L_{n,p}^{a_j M} \) then its action on \( T^{2n} \) would be trivial, i.e., \( \gamma \in \text{Ker} \eta \), see the exact sequence (2). Any nontrivial element \( \theta \in \text{Ker} \eta \simeq \mathbb{Z}_p \) acts on the circle bundle \( Q_{n,p} \) via the action of a nontrivial \( p \)-root of unity \( \mu_a \in S^1 \). Then the action of \( \theta \) on \( L_{n,p}^{a_j M} \) is via multiplication by \( \mu^{a_j M}_a \). Since neither \( a_j \) nor \( M \) are divisible by \( p \), we have \( \mu^{a_j M}_a \neq 1 \). Hence \( \Gamma_{n,p} \) acts effectively on each of the line bundles \( L_{n,p}^{a_j M} \).

Since \( \text{rk} G_j(\delta_j) = \text{rk} F_j(\delta_j) = jn! \) we have

\[
\text{rk} V_{p,n} = n + 1 + \frac{n(n + 1)}{2}n!.
\]

Since the right hand side is independent of \( p \), we may use Lemma 2.10 to conclude the existence of some natural number \( \tau = \tau(n) \), depending on \( n \) but not on \( p \), with the property that

\[
V_{p,n} \oplus \mathbb{C}^\tau - \text{rk} V_{p,n} \simeq \mathbb{C}^\tau
\]

as vector bundles. Taking the trivial lift of the action of \( \Gamma_{n,p} \) to \( \mathbb{C}^\tau - \text{rk} V_{p,n} \), we obtain an action of \( \Gamma_{n,p} \) on \( \mathbb{C}^\tau \) which by the previous lemma is effective as soon as \( p > M \). Since \( M \) only depends on \( n \), the proof of Theorem 1.2 is complete.

**3. Proof of Theorem 1.1**

Set the functions \( \tau, M : \mathbb{N} \to \mathbb{N} \) to be those of Theorem 1.2. By Theorem 1.2 for any prime \( p \) satisfying \( p > M(n) \) and \( p \equiv 1 \mod n + 1 \) there is an effective action of \( \Gamma_{n,p} \) on \( W := T^{2n} \times \mathbb{C}^{\tau(n)} \) by vector bundle automorphisms. By Lemma 2.2 the induced action on the determinant line bundle \( T^{2n} \times \Lambda^{\tau(n)} \mathbb{C}^{\tau(n)} \) admits an equivariant nowhere vanishing section \( \sigma : T^{2n} \to \Lambda^{\tau(n)} \mathbb{C}^{\tau(n)} \). Using the standard averaging trick, we may take a \( \Gamma_{n,p} \)-invariant Hermitian structure \( h_0 \) on \( W \). Multiplying \( h_0 \) by \( |\sigma|^{-1/\tau(n)}_{h_0} \in \mathbb{R}_{>0} \) we get
an invariant Hermitian metric \( h \) with respect to which \( \sigma \) has constant norm equal to 1. Then the bundle \( E \) of \( h \)-unitary frames \((\theta, w_1), \ldots, (\theta, w_{\tau(n)})\) \( \in W \) such that
\[
w_1 \wedge \cdots \wedge w_{\tau(n)} = \sigma(\theta)
\]
is isomorphic to \( T^{2n} \times SU(\tau(n)) \) and it carries an effective action of \( \Gamma_{n,p} \) by principal bundle automorphisms. Hence there exists a cocycle
\[
c : \Gamma_{n,p} \times T^{2n} \rightarrow SU(\tau(n))
\]
such that for any \( \gamma \in \Gamma_{n,p} \) and any \( (\theta, h) \in T^{2n} \times SU(\tau(n)) \) we have
\[
\gamma \cdot (\theta, h) = (\gamma \cdot \theta, c(\gamma, \theta)h),
\]
where \( \Gamma_{n,p} \) acts on \( T^{2n} \) via the map \( \eta : \Gamma_{n,p} \rightarrow \mathbb{Z}_{p}^{2n} \) in (2) and formula (4). The cocycle condition is \( c(\gamma', \gamma, \theta) = c(\gamma', \gamma, \theta) \cdot c(\gamma, \theta) \) for any \( \theta \in T^{2n} \) and any \( \gamma, \gamma' \in \Gamma_{n,p} \). The fact that the action of \( \Gamma_{n,p} \) is effective is equivalent to the condition that for any nontrivial \( \gamma \in \text{Ker} \eta \) and any \( \theta \in T^{2n} \) the element \( c(\gamma, \theta) \) is nontrivial.

3.1. Proof of (1). Suppose that \( X \) is a manifold with an almost effective action of \( SU(\tau(n)) \). Let \( H \) be the kernel of this action. Define an action of \( \Gamma_{n,p} \) on \( T^{2n} \times X \) by setting
\[
\gamma \cdot (\theta, x) = (\gamma \cdot \theta, c(\gamma, \theta) \cdot x)
\]
for every \( \gamma \in \Gamma_{n,p} \) and any \( x \in X \). The fact that this defines an action of \( \Gamma_{n,p} \) follows from the cocycle condition satisfied by \( c \).

**Lemma 3.1.** If \( p \) does not divide \( |H| \) then the action of \( \Gamma_{n,p} \) on \( T^{2n} \times X \) is effective.

**Proof.** If an element \( \gamma \in \Gamma \) acts trivially on \( T^{2n} \times X \) then we must have \( \eta(\gamma) = 0 \), so \( c(\gamma, \theta) \) is a nontrivial element of order \( p \) for every \( \theta \) (indeed, if \( \gamma \cdot \theta = \theta \) then the cocycle condition reads \( c(\gamma^k, \theta) = c(\gamma, \theta)^k \) for every \( k \)). If \( p \) does not divide \( |H| \) then \( c(\theta, \gamma) \) does not belong to \( H \), which implies that \( \gamma \) acts nontrivially on \( T^{2n} \times X \).

Setting \( \Gamma := \Gamma_{n,p} \), Lemma 2.1 concludes the proof of statement (1) of the theorem.

3.2. Proof of (2). Now assume that \( r > 1 \). Then \( E^r \) can be identified with a trivial principal \( SU(\tau(n))^r \) bundle over \( T^{2nr} \), and it carries an effective action of \( (\Gamma_{n,p})^r \). We next prove that \( (\Gamma_{n,p})^r \) contains a subgroup \( \Gamma \) with \( p^{2n+r} \) elements which does not contain any abelian subgroup with more than \( p^{2+r+4n/r} \) elements. The following result is due to Olshanskii (see Lemma 2 in [14]):

**Lemma 3.2.** Suppose that \( k \) satisfies the condition \( 4n < r(k - 1) \). Then there exists a set of symplectic forms \( \{\omega_1, \ldots, \omega_r\} \) in \( V := \mathbb{Z}_{p}^{2n} \) with the property that no \( k \)-dimensional subspace of \( V \) is isotropic for all the forms \( \omega_1, \ldots, \omega_r \) simultaneously.

Let \( \{\omega_1, \ldots, \omega_r\} \) be as in the lemma, with \( k = 2 + 4n/r \). Let \( A_1, \ldots, A_r \) be elements of \( \text{GL}(2n, \mathbb{Z}_p) \) such that
\[
\omega_j(u, v) = \omega(A_j u, A_j v)
\]
for each \( j \) and each \( u, v \in \mathbb{Z}_{p}^{2n} \), where \( \omega \) is the standard symplectic form (3). Define
\[
\Gamma = \{ (\gamma_1, \ldots, \gamma_r) \in (\Gamma_{n,p})^r | A_1^{-1} \eta(\gamma_1) = A_2^{-1} \eta(\gamma_2) = \cdots = A_r^{-1} \eta(\gamma_r) \},
\]
where $\eta : \Gamma_{n,p} \to \mathbb{Z}_p^{2n}$ is the morphism in the exact sequence (2). Consider the projection

$$\eta' : \Gamma \to \mathbb{Z}_p^{2n}, \quad \eta'(\gamma_1, \ldots, \gamma_r) = A_1^{-1} \eta(\gamma_1).$$

Then there is an exact sequence

$$0 \to \mathbb{Z}_p^r \to \Gamma \xrightarrow{\eta'} \mathbb{Z}_p^{2n} \to 0,$$

and two elements $\gamma = (\gamma_1, \ldots, \gamma_r)$ and $\gamma' = (\gamma'_1, \ldots, \gamma'_p)$ of $\Gamma$ commute if, for each $j$,

$$0 = \omega(\eta(\gamma_j), \eta(\gamma'_j)) = \omega(A_j A_j^{-1} \eta(\gamma_j), A_j A_j^{-1} \eta(\gamma'_j)) = \omega(A_j A_1^{-1} \eta(\gamma_1), A_j A_1^{-1} \eta(\gamma'_1)) = \omega(A_j \eta'(\gamma), A_j \eta'(\gamma')) = \omega_j(\gamma, \gamma').$$

So if $A \subset \Gamma$ is an abelian subgroup then $\eta'(A)$ is a subspace of $\mathbb{Z}_p^{2n}$ which is isotropic with respect to all forms $\omega_1, \ldots, \omega_r$ simultaneously. Hence, $\eta'(A)$ has dimension at most $k$ and consequently $A$ contains at most $p^{r+k}$ elements.

Now the proof of (2) is concluded arguing exactly as in (1), replacing $\text{SU}(\tau(n))$ by $\text{SU}(\tau(n))^\tau$.

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