Projective Compactifications
and Einstein Metrics

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PROJECTIVE COMPACTIFICATIONS
AND EINSTEIN METRICS

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Abstract. For complete affine manifolds we introduce a definition of compactification based on the projective differential geometry (i.e., geodesic path data) of the given connection. The definition of projective compactness involves a real parameter $\alpha$ called the order of projective compactness. For volume preserving connections, this order is captured by a notion of volume asymptotics that we define. These ideas apply to complete pseudo-Riemannian spaces, via the Levi-Civita connection, and thus provide a notion of compactification alternative to conformal compactification. For each order $\alpha$, we provide an asymptotic form of a metric which is sufficient for projective compactness of the given order, thus also providing many local examples.

Distinguished classes of projectively compactified geometries of orders one and two are associated with Ricci-flat connections and non–Ricci–flat Einstein metrics, respectively. Conversely, these geometric conditions are shown to force the indicated order of projective compactness. These special compactifications are shown to correspond to normal solutions of classes of natural linear PDE (so-called first BGG equations), or equivalently holonomy reductions of projective Cartan/tractor connections. This enables the application of tools already available to reveal considerable information about the geometry of the boundary at infinity. Finally, we show that metrics admitting such special compactifications always have an asymptotic form as mentioned above.

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1. Introduction

Let $\overline{M}$ be an $(n + 1)$-dimensional manifold with boundary $\partial M$. A defining function $r$ for the boundary is a smooth function on $\overline{M}$, with zero locus $\partial M$, and such that $dr$ is nowhere vanishing on $\partial M$. Recall that a pseudo-Riemannian metric $g^o$ on the interior $M$ of $\overline{M}$ is said to be conformally compact if $g := r^2 g^o$ extends to a pseudo–Riemannian metric on $\overline{M}$, where $r$ is a defining function for the boundary; this extension meaning that $g$ is smooth and non-degenerate up to the boundary. At points where the boundary conormal is not null, the restriction of $g$ determines a conformal structure on $\partial M$ (that is independent of the choice of $r$) and the conformal compactification provides a geometric framework for relating conformal geometry, and associated field theories, to the asymptotic phenomena of
the interior (pseudo-)Riemannian geometry of one higher dimension. This notion had its origins in the work of Newman and Penrose, for treating four dimensional spacetime physics, and has remained extremely important for general relativity and related questions [6, 16, 17, 25, 31]. Conformally compact geometries have also proved to be a powerful tool for conformal invariant theory [14, 15], the geometric scattering program and related analysis [20, 21, 26, 27, 32], and the AdS/CFT program from physics [1, 12, 24, 33].

Considering other geometric compactifications of complete metrics should be very useful for many of these directions. Here we develop an effective version of this idea based on projective geometry, thereby also extending the concept to manifolds endowed with a complete affine connection. Recall that torsion free connections are projectively equivalent if they share the same geodesics up to reparametrisation; the resulting emphasis on the role of geodesics seems particularly natural for general relativity and related geometric analysis. The resulting compactifications exhibit substantial differences from conformal compactifications. This can for example be seen from the description of the natural projective compactification of Minkowski space in Section 3.5, see in particular Remark 17. More information on relations and differences between projective and conformal compactifications can be found in Remark 7.

In special settings projective compactifications have arisen in the literature. Indeed in Chapter 4 of Fefferman and Graham’s book [15] a (negative Einstein) “projectively compact metric” is given by an explicit formula linked to the formula for their “ambient metric”. Einstein and Ricci flat projectively compact structures are seen to arise naturally via suitable holonomy reductions of the canonical Cartan/tractor connection [8, 10]; each such reduction is equivalent to a special, so-called normal, solution of a certain overdetermined PDE, a point we shall take up below. These results follow a similar story for Einstein conformally compact metrics [18], and are part of a fascinating very general picture [9]. Extending a classical theme, recently there has been increased interest in the interaction between projective and pseudo-Riemannian geometries including in the context of general relativity [3, 5, 13, 22, 23, 29]. We also see the current work as adding to this developing picture.

Let us summarize the program to be followed and the results obtained. An affine connection on $M$ is projectively compact if its projective class extends to the boundary in a suitable way. This is made precise in Definition 1, and in this a parameter $\alpha \in \mathbb{R}_+$ is involved. For connections preserving a volume density the number $\alpha$ controls the volume growth asymptotics: smaller values of $\alpha$ are associated with larger volume growth near the boundary, see Proposition 3. The notion of connection volume asymptotics is defined in a general context in Definition 2, since this enables rather general comparison; for example between the volume growth of a projective compactification and the usual conformal compactification.

In Section 2.3 we treat completeness. It shown that for $\alpha \leq 2$, projectively compact connections are automatically complete, and essentially the converse holds. This is the content of Proposition 4.
A Levi-Civita connection can be projectively compact and hence we come to the notion of projective compactness of a pseudo-Riemannian metric in Definition 5. A main result of the article is Theorem 6 which describes an asymptotic form of a metric depending on $\alpha \in (0, 2]$ which is sufficient for projective compactness of order $\alpha$. This Theorem provides a class of projectively compact metrics which can be used and treated in a manner similar to conformally compact metrics. In particular, it provides a large number of local examples of such metrics.

There is an alternative interpretation of volume growth, based on the concept of defining densities, which is crucial for the second part of the article. The line bundle $E(\alpha)$ of densities of projective weight $\alpha$ is defined in Section 2.2. A "defining density" $\sigma$ for the boundary is a section of such a density bundle having $\partial M$ as its zero locus, and with the derivative of $\sigma$ nowhere zero along $\partial M$. If an affine connection admitting a parallel volume density is projectively compact of order $\alpha$ then one obtains a defining density $\sigma \in \Gamma(E(\alpha))$ (unique up to constant multiples) for the boundary.

It is thus natural to single out particularly nice projective compactifications by imposing projectively invariant differential equations on the natural defining density. In particular there are canonical overdetermined linear equations (called first BGG equations) available in the case that $\alpha = 1$ and $\alpha = 2$ [8, 11]. These equations are rather well studied, in particular it is known that there is a subclass of so-called normal solutions which correspond to reductions of projective holonomy, i.e. the holonomy of the canonical Cartan/tractor connection associated to the projective structure. Using the available results we obtain the following picture:

- On densities of weight 1 the BGG equation is second order and the natural defining density $\sigma \in \Gamma(E(1))$ of a projectively compact space $(M, \nabla)$ satisfies this equation, if and only if $\nabla$ is Ricci flat. Furthermore the boundary $\partial M$ is totally geodesic and inherits a natural projective structure. The solution $\sigma$ in this case is necessarily "normal" thus giving rise to a reduction of projective holonomy on $\overline{M}$, see Theorem 9.
- On densities of weight 2 the BGG equation is third order and for $\tau \in \Gamma(E(2))$ a nontrivial solution means that, for the corresponding connection on $M$, the symmetrized covariant derivative of the Ricci curvature vanishes. If the solution is normal and satisfies a suitable non-degeneracy condition, then we get a non–Ricci–flat Einstein metric on $M$, with Levi–Civita connection in the projective class and a conformal structure on the boundary, see Theorem 11. In Proposition 13, we show that in the latter case the metric has asymptotics of the form discussed in Theorem 6 for $\alpha = 2$.

The relation between the geometric conditions of Ricci–flatness, respectively being non–Ricci–flat Einstein, and the order of projective compactness exhibited in the above results is of a deep and fundamental nature; it is not due to any choice being made concerning the equations on the defining functions. This is shown by the following strong converses to the above results. For these results, we assume that for $\overline{M} = M \cup \partial M$ we have an affine connection $\nabla$ on $M$ with the following properties: it preserves a volume density, it does not extend to any neighborhood
of a boundary point, but its projective class does extend to all of $\overline{M}$. Under these assumptions, we prove in Theorems 10 and 12:

- If $\nabla$ is Ricci flat then it is projectively compact of order $\alpha = 1$. Moreover, the unique (up to constant factors) non–zero section of $\mathcal{E}(1)$ which is parallel for $\nabla$ is a defining density for $\partial M$ which solves the relevant first BGG equation. Thus one automatically is in the setting of Theorem 9.

- If $\nabla$ is the Levi–Civita connection of a non–Ricci–flat Einstein metric (or, more generally, an affine connection with parallel, non–degenerate Ricci curvature), then it is projectively compact of order $\alpha = 2$. Moreover, the unique (up to constant factors) non–zero section of $\mathcal{E}(2)$ which is parallel for $\nabla$ is a defining density for $\partial M$ which solves the first BGG equation on $\mathcal{E}(2)$. Thus one automatically is in the setting of Theorem 11.

Our last main result concerns projectively compact Ricci flat metrics of any signature $(p, q)$, the model example of which is discussed in Section 3.4. From the earlier results already mentioned, we know that the order of projective compactness has to be one, there is a natural defining density in $\Gamma(\mathcal{E}(1))$ which solves the first BGG equation, and that the boundary $\partial M$ inherits a projective structure. As an additional ingredient we use the fact that the metric determines a normal solution to the projective metricity equation (which again is a first BGG equation). Using this, we prove:

- The natural projective structure on the boundary inherits a reduction of projective holonomy to the orthogonal group $SO(p, q)$ (see Theorem 15). This means that the boundary is projective almost Einstein which by [8, 9] implies a stratification of $\partial M$, called a “curved orbit decomposition”, which is explicitly described in Theorem 18. The open curved orbits are Einstein (never Ricci-flat) of signature $(p-1, q)$, respectively $(p, q-1)$, depending on whether the boundary points are the limits of space–like or time–like geodesics. The closed curved orbits consist of limit points of interior null geodesics and inherit a conformal structure of signature $(p-1, q-1)$.

- In Proposition 20 we show that, locally around the points of $\partial M$ which lie in open curved orbits, the interior metric has asymptotics of the form discussed in Theorem 6 with $\alpha = 1$.

2. Projective compactifications

2.1. Projectively compact affine connections. Throughout this article we will refer to linear connections on the tangent bundle on a manifold as affine connections and all such connections will be assumed to be torsion free. Recall that two such connections are called projectively equivalent, if and only if they have the same geodesics up to paramerization. Equivalently, their contorsion tensor can be expresses in terms of a one–form $\Upsilon$ in the form

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi) \eta + \Upsilon(\eta) \xi,$$

where $\xi$, and $\eta$ are tangent vector fields. We will formally write this relation as $\tilde{\nabla} = \nabla + \Upsilon$ from now on.
In the setting of a manifold $\overline{M}$ of dimension $n + 1$ with boundary $\partial M$ and with interior $M$, the basic question we study in this article is the following. Suppose we have a connection $\nabla$ on $M$ which does not extend to $\overline{M}$, for example because it is complete. Can we projectively modify it to a connection $\hat{\nabla}$, which extends to $\overline{M}$?

Motivated by the concept of conformal compactifications of Riemannian metrics in a similar setting, we formulate this in terms of local defining functions $\rho$ for the boundary. So we assume that $\rho$ is a smooth, real valued function defined on some open subset $U \subset M$, with non–negative values, such that $\rho^{-1}(\{0\}) = \partial M \cap U$, and such that $d\rho$ is nowhere vanishing on $\rho^{-1}(\{0\})$. We will be mainly interested in the following condition in the cases $\alpha = 2$ and $\alpha = 1$, the other cases are included for completeness.

**Definition 1.** Let $\alpha$ be a positive real number. An affine connection $\nabla$ on $M$ is called *projectively compact* of order $\alpha \in \mathbb{R}^+$ if for any $x \in \partial M$, there is a neighborhood $U$ of $x$ in $M$ and a defining function $\rho : U \to \mathbb{R}$ for the boundary such that the connection

$$\hat{\nabla} = \nabla + \frac{d\rho}{\alpha \rho}$$

on $U \cap M$ extends to all of $U$.

Observe first that in case $\hat{\nabla}$ extends, the extension is uniquely determined by $\nabla$ and $\rho$ since $U \cap M$ is dense in $U$. Moreover, given one defining function $\rho$, any other defining function for the boundary can be locally written as $\tilde{\rho} = e^f \rho$ for some smooth function $f$ on $U$. One immediately computes that

$$d\tilde{\rho} = \frac{d\rho}{\rho} + \frac{1}{\alpha} df.$$

Thus the question whether $\hat{\nabla}$ extends to the boundary is actually independent of the defining function $\rho$. Note however that the parameter $\alpha$ cannot be eliminated in a straightforward way, since this would amount to replacing $\rho$ by some power of $\rho$, which then cannot be a defining function.

### 2.2. Volume asymptotics.

If a linear connection $\nabla$ on $M$ is projectively compact of any order $\alpha$ as in Definition 1, then the projective structure on $M$ defined by $\nabla$ extends to $\overline{M}$, and this extension is evidently unique. We next want to show that, for connections admitting a parallel volume density, apart from the extension of the projective structure, projective compactness of order $\alpha$ amounts to a fixed growth rate of the volume towards the boundary.

Recall that on a manifold $N$ endowed with a projective structure, there is a standard notion of *projective densities*. For any real number $w \in \mathbb{R}$, one has the bundle $E(w)$ of densities of projective weight $w$. These bundles are always trivial, but there is no preferred trivialization. Since they can be defined as associated bundles to the linear frame bundle, any connection $\nabla$ in the projective class induces linear connections on all density bundles which will be denoted by the same symbol. If $\hat{\nabla} = \nabla + \Upsilon$ for $\Upsilon \in \Omega^1(M)$ in the sense introduced in Section 2.1 then the induced connections on $E(w)$ are related by

$$\hat{\nabla}_\xi \sigma = \nabla_\xi \sigma + w \Upsilon(\xi) \sigma \quad \text{for } \sigma \in \Gamma(E(w)), \xi \in \mathfrak{X}(M).$$

This easily implies (see e.g. [10]) that given an arbitrary nowhere vanishing section of $E(w)$ with $w \neq 0$, there is a unique connection in the projective class for which...
this section is parallel. In this situation, we will call the nowhere vanishing section a \textit{scale} and refer to the resulting connection as the connection determined by that scale.

In order to allow comparison to the case of conformal structures, for which there also is an established convention, we put this into a more general context. Recall that on a general (possibly non–oriented) smooth manifold $N$ of dimension $n+1$ there is a natural line bundle whose sections can be canonically integrated. This can be defined as an associated bundle to the linear frame bundle of $N$ and if $N$ is orientable, a choice of orientation identifies this bundle with the bundle $Λ^\langle n+1 \rangle T^\ast N$ of $(n+1)$–forms, see \cite[Section 10]{28}. We will call this the bundle $\text{vol}(N)$ of \textit{volume densities} (avoiding the common name “1–densities” which might lead to confusion with the conventions mentioned above). As above, any linear connection $\nabla$ on $TN$ induces a linear connection on the line bundle $\text{vol}(N)$, which we will denote by the same symbol. We shall call the connection $\nabla$ \textit{special} if there is a non–zero section of $\text{vol}(N)$ which is parallel for the induced connection. Such a section then is clearly uniquely determined up to a constant factor. For this (or its roots as appropriate) we may use the term “the canonical density” determined by a special affine connection. This slight abuse of language should cause no confusion.

From the construction of the bundle of volume densities it follows that $\text{vol}(N)$ is always a trivial bundle, but there is no canonical trivialization. Thus one can form powers of this bundle with any real number as an exponent; this is easily established via the language of associated bundles. The relation to projective densities then can be simply expressed as $E(w) = \text{vol}(N)^{-w/(n+2)}$ if $\text{dim}(N) = n+1$.

Returning to our standard setting, this allows us to define the notion of volume asymptotics for special linear connections on the interior.

\textbf{Definition 2.} Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$, and let $\nabla$ be a special affine connection on $M$. Then $\nabla$ is said to have \textit{volume asymptotics of order} $\beta$ if and only if for each point $x_0 \in \partial M$, there is an open neighborhood $U$ of $x_0$ in $\overline{M}$, a local defining function $\rho$ for $\partial M$, and a nowhere vanishing section $\nu$ of $\text{vol}(\overline{M})$ over $U$ such that the section $\rho^{\beta}\nu|_{U \cap M}$ of $\text{vol}(M)$ is parallel for $\nabla$.

Note that the number $\beta$ is independent of the choice of defining function $\rho$, and corresponding section $\nu$. There is an alternative interpretation of volume asymptotics which will be very useful for our purposes. This is based on the notion of defining densities. Given a section $\nu$ of $\text{vol}(\overline{M})$, one can naturally view $\nu^r$ as a section of $\text{vol}(\overline{M})^r$ for any $r \in \mathbb{R}$. In particular, a choice of non–vanishing section of $\text{vol}(\overline{M})$ gives rise to non–vanishing sections of all powers and for a special affine connection on $M$ there are parallel sections of all density bundles. For any fixed bundle these are unique up to constant multiples.

Now there is a well defined notion of a defining density for $\partial M$. Saying that $\sigma \in \Gamma(\text{vol}(\overline{M})^r)$ is a defining density means that the zero locus of $\sigma$ coincides with $\partial M$ and with respect to some (or equivalently any) local trivialization of $\text{vol}(\overline{M})^r$ around points in $\partial M$, $\sigma$ is represented by a defining function. Otherwise put, for any locally non–vanishing section $\tilde{\sigma}$ of $\text{vol}(\overline{M})^r$, the uniquely defined function $\rho$
such that $\sigma = \rho \hat{\sigma}$ must be a defining function for $\partial M$. The crucial point to notice here is that this pins down the weight. If $\sigma$ is a defining density, then no power $\sigma^t$ for $t \neq 1$ can be a defining density for its zero locus.

We are now ready to clarify the relation between the order of projective compactness and volume asymptotics.

**Proposition 3.** Let $\overline{M}$ be a smooth $n + 1$–dimensional manifold with boundary $\partial M$ and with interior $M$.

(i) If $\nabla$ is a special linear connection on $M$, which is projectively compact of order $\alpha > 0$, then it has volume asymptotics of order $\frac{(n+2)}{\alpha}$. Any non–zero section of $E(\alpha)$, which is parallel for $\nabla$ extends by $0$ to a defining density for the boundary $\partial M$.

(ii) Suppose that $M$ is endowed with a projective structure, and that $\sigma \in \Gamma(E(\alpha))$ is a defining density for $\partial M$. Then one can view $\sigma$ as a scale for the restriction of the projective structure to $M$ and the affine connection $\nabla$ on $M$ determined by this scale is projectively compact of order $\alpha$.

**Proof.** (i) Fix a local defining function $\rho : U \to \mathbb{R}_{\geq 0}$ for $\partial M$ and let $\sigma$ be a nonzero section of $E(\alpha) \to M$ which is parallel for $\nabla$. Then we consider the section $\hat{\sigma} := \sigma/\rho$ of $E(\alpha)$ which is defined and nowhere vanishing over $U \cap M$. By assumption, the connection $\hat{\nabla} = \nabla + \frac{d\rho}{\rho}$ extends to the boundary. Then from the definition of $\hat{\sigma}$, formula (2) and $\nabla \sigma = 0$ we get

$$\nabla \hat{\sigma} = \nabla \frac{\sigma}{\rho} = -\frac{d\rho}{\rho^2} \sigma + \frac{1}{\rho} \nabla \sigma = -\frac{d\rho}{\rho} \hat{\sigma}.$$ 

Hence the non–zero section $\hat{\sigma}$ is parallel for $\hat{\nabla}$ over $U \cap M$, so it extends to a parallel section for $\hat{\nabla}$ on all of $U$, which is nowhere vanishing. But then $\sigma = \hat{\sigma} \rho$ shows that $\sigma$ extends by zero to a defining density. The statement on volume asymptotics then follows immediately by forming powers of order $-(n+2)/\alpha$.

(ii) As a defining density for $\partial M$, $\sigma$ is nowhere vanishing on $M$ and thus determines a connection $\nabla$ in the projective class there. For a point $x_0 \in \partial M$ choose an open neighborhood $U$ and a nowhere vanishing section $\hat{\sigma}$ of $E(\alpha)$ defined over $U$. Let $\hat{\nabla}$ be the unique connection in the projective class on $U$ such that $\hat{\nabla} \hat{\sigma} = 0$. Since $\sigma$ is a defining density for $\partial M$, there is a defining function $\rho : U \to \mathbb{R}_{\geq 0}$ for $\partial M$ such that $\sigma = \hat{\sigma} \rho$. Since $\hat{\sigma}$ is parallel for $\hat{\nabla}$, we get $\hat{\nabla} \sigma = \hat{\sigma} d\rho = \sigma \frac{d\rho}{\rho^2}$ over $U \cap M$. But then (2) shows that $\sigma$ is parallel on $U \cap M$ for the connection $\nabla + \frac{d\rho}{\alpha \rho}$, which thus has to coincide with $\nabla$. $\square$

2.3. **Completeness.** Next we derive a result related to (geodesic) completeness of an affine connection $\nabla$ which is projectively compact of some order $\alpha$. By the definition of this property, the projective class of such a connection $\nabla$ extends to all of $\overline{M}$. In particular, we have distinguished paths on all of $\overline{M}$ which are the geodesic paths of the connections in the class. We will show that, provided $\alpha \leq 2$, paths approaching the boundary $\partial M$ transversally do not reach the boundary in finite time when parameterized as geodesics for $\nabla$. Motivated by this result, we will restrict to the case $\alpha \leq 2$ from now on.
Proposition 4. Let $\nabla$ be an affine connection on $M$ which is projectively compact of some order $\alpha \leq 2$. Suppose that one has a geodesic path which reaches $\partial M$ in a point $x_0$ with tangent transversal to $\partial M$. Then a part of this path can be parameterized as a geodesic for $\nabla$ in the form $c : [0, \infty) \to \overline{M}$ in such a way that $c([0, \infty)) \subset M$ and $\lim_{t \to \infty} c(t) = x_0$.

Proof. Fix a defining function $\rho$ for $\partial M$ on a neighborhood of $x_0$ and consider the connection $\tilde{\nabla} := \nabla + \frac{d\rho}{\rho}$ which is projectively related to $\nabla$ and extends to $\overline{M}$. Then there is a unique vector $\xi$ tangent to the path at $x_0$ such that $d\rho(\xi) = 1$. Now we can consider the (parameterized) geodesic $\hat{c}$ for $\tilde{\nabla}$ emanating from $x_0$ in the direction $\xi$. For sufficiently small times, we will have $d\rho(\hat{c}'(t)) > 1/2$ and we restrict to an interval on which this is true. Then run along this curve backwards, and call the result again $\hat{c}$. So we may assume that $\hat{c} : [0, t_0] \to \overline{M}$ is a geodesic for $\tilde{\nabla}$ such that $\hat{c}([0, t_0]) \subset M$, $\hat{c}(t_0) = x_0 \in \partial M$ and $d\rho(\hat{c}'(t)) < -1/2$ for all $t \in [0, t_0]$. In particular, this implies that $f := \frac{1}{\alpha} \rho \circ \hat{c}$, where $\alpha = \rho(\hat{c}(0))$, will be an orientation reversing diffeomorphism from $[0, t_0]$ onto $[0, 1]$.

Now we know that this curve can be reparametrized as a geodesic for $\nabla$ in the form $c := \hat{c} \circ \varphi$ where $\varphi$ is a strictly increasing map defined on $[0, b)$ for an (initially unspecified) number $b \in \mathbb{R}$ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$. The relation between the connections $\nabla$ and $\hat{\nabla}$ together with the fact that $\hat{c}$ is a geodesic for $\hat{\nabla}$ shows that $\varphi$ has to satisfy a differential equation. First we get

$$0 = \nabla_{\varphi'} \hat{c}'(t) = \hat{\nabla}_{\varphi'} c'(t) - 2 \frac{d\rho(\hat{c}'(t))}{\rho(\hat{c}(t))} \hat{c}'(t).$$

Now we insert $c'(t) = \hat{c}'(\varphi(t)) \varphi'(t)$ and use that $\varphi'' \cdot \varphi' = \varphi'' / \varphi'$, and that $\hat{\nabla}_\varphi \hat{c}' = 0$. Using the definition of $f$, we conclude that

$$0 = \varphi''(t) - 2 \frac{f'(\varphi(t)) \varphi'(t)^2}{\alpha f(\varphi(t))}.$$

Dividing by $\varphi'(t)$ (which is strictly positive), we get an equality of logarithmic derivatives, which implies that $\varphi'(t) = C (f \circ \varphi)(t)^{2/\alpha}$ for some non-zero constant $C$. Since we require $\varphi'(0) = 1$ we actually get $C = 1$.

Now $f \circ \varphi$ will be an orientation reversing diffeomorphism $[0, b) \to (d, 1]$ where $d = f(\varphi(b))$, and we will use the equation on $\varphi$ we have just derived to obtain a differential equation on $\psi := (f \circ \varphi)^{-1}$. We get $\psi'(t) = 1/(f \circ \varphi)'((f \circ \varphi)^{-1}(t))$, and inserting the differential equation on $\varphi$ we conclude that

$$\psi'(t) = \frac{1}{t^{2/\alpha} f'(f^{-1}(t))}.$$  

By our assumptions $f'(f^{-1}(t))$ is strictly negative and bounded away from zero, which shows that there are positive numbers $A < B$ such that $B t^{1-2/\alpha} < \psi'(t) < A t^{1-2/\alpha}$. Integrating, and using $\psi(1) = 0$, we conclude that for $\alpha < 2$, we obtain

$$A(-1 + 2/\alpha)(-1 + t^{1-2/\alpha}) \leq \psi(t) \leq B(-1 + 2/\alpha)(-1 + t^{1-2/\alpha}),$$

while for $\alpha = 2$, we get

$$A(- \log(t)) \leq \psi(t) \leq B(- \log(t)).$$
In any case, this implies that \( \psi \) will be defined on \((0,1]\) with \( \lim_{t \to 0} \psi(t) = \infty \), which implies our claims.

2.4. **Projectively compact pseudo–Riemannian metrics.** The notion of being projectively compact of some order for affine connections introduced in 2.1 gives rise to an evident notion for pseudo–Riemannian metrics.

**Definition 5.** Let \( \overline{M} = M \cup \partial M \) be as in 2.1. For a positive real number \( \alpha \), a pseudo–Riemannian metric \( g \) on \( M \) is called **projectively compact** of order \( \alpha \) if and only if its Levi–Civita connection \( \nabla \) is projectively compact of order \( \alpha \) in the sense of Definition 1.

Observe the volume density \( \text{vol}(g) \) of a pseudo–Riemannian metric \( g \) is parallel for the Levi–Civita connection. Hence we are always in the setting of special affine connections and have a canonical parallel density of each projective weight (not just up to a constant factor). In particular, Proposition 3 always applies and shows that \( \text{vol}(g)^{-\alpha/(n+2)} \) extends to a defining density for the boundary \( \partial M \).

We next prove that a certain asymptotic form of a metric implies projective compactness of order \( \alpha \) for \( \alpha \leq 2 \). We formulate this asymptotic behavior in a form which does not depend on a choice of coordinates. Indeed, consider an open subset \( U \subset \overline{M} \) and a local defining function \( \rho : U \to \mathbb{R}_{\geq 0} \) for \( \partial M \). Then for a nowhere vanishing smooth function \( C : U \to \mathbb{R} \) consider the \((0,2)\)-tensor field \( h = h_C \) on \( U \cap M \) defined by

\[
(3) \quad h := \rho^{2/\alpha} g - C \frac{d\rho \otimes d\rho}{\rho^{2/\alpha}}.
\]

We will assume that, locally around each point in the boundary we can find a defining function \( \rho \) and a function \( C \), which satisfies certain growth conditions towards the boundary, such that the tensor field \( h \) defined by (3) extends smoothly to the boundary and that its restriction to the boundary is non–degenerate on \( T(U \cap \partial M) \subset T(U)|_{\partial M} \).

The condition just given means that, for an appropriate choice of a function \( C \) and a defining function \( \rho \), we can write the metric as

\[
(4) \quad g = \frac{h}{\rho^{2/\alpha}} + C \frac{d\rho \otimes d\rho}{\rho^{4/\alpha}}
\]

with \( h \) going to the boundary and restricting to a pseudo–Riemannian metric there. Specialized to this setting, our completeness result in Proposition 4 is nicely compatible with the result on completeness of such metrics in the case that \( C \) is constant, see [27].

It should be noted at this point that the dependence of such a form on the defining function \( \rho \) is different for different values of \( \alpha \). For \( \alpha \neq 2 \), one can absorb a constant factor in \( C \) into a constant rescaling of \( \rho \). Moreover, if \( \alpha < 2 \) the question whether \( h \) defined by (3) extends to the boundary heavily depends on the defining function. Indeed if this works for a defining function \( \rho \), then this defining function is uniquely determined up to addition of terms of the order of \( \rho^2 \).
In contrast, in the case $\alpha = 2$, the condition is independent of the choice of defining function. To see this, consider $\hat{\rho} = e^f \rho$ for a smooth function $f : U \to \mathbb{R}$. Then $d\hat{\rho} = e^f d\rho + \hat{\rho} df$ and thus $\frac{\partial}{\partial \rho} \hat{\rho} = \frac{\partial}{\partial \rho} \rho + df$. Forming the symmetric product with $d\hat{\rho}$, one immediately concludes that the tensor field $\hat{h}$ constructed from $\hat{\rho}$ according to (3) (with $\alpha = 2$) is given by

$$\hat{h} = e^f h - 2C e^f d\rho \odot df - C \hat{\rho} df \odot df.$$ 

Evidently the last two summands are smooth up to the boundary. Thus, smoothness of $h$ up to the boundary implies smoothness of $\hat{h}$ up to the boundary. Moreover,

$$\hat{h}|_{\partial M} = e^f h|_{\partial M} - 2C e^f d\rho \odot df|_{\partial M}.$$ 

Since the last term involves $d\rho$, it vanishes on $T(U \cap \partial M)$, so there the two bilinear forms are conformal. In particular, we see that one obtains a well defined conformal structure on $\partial M$ from a metric of the form (4) with $\alpha = 2$.

**Theorem 6.** Fix $\alpha \in (0, 2]$. Suppose that $g$ is a pseudo–Riemannian metric on $M$ such that for each point $x_0 \in \partial M$, we can find an open neighborhood $U$ of $x_0$ in $M$, a defining function $\rho : U \to \mathbb{R}_{\geq 0}$ for the boundary, and a nowhere vanishing smooth function $C : U \to \mathbb{R}$ such that:

- For any vector field $\zeta \in \mathfrak{X}(U)$ with $d\rho(\zeta) = 0$, the function $\rho^{-2/\alpha}\zeta \cdot C$ is smooth up to the boundary.
- The tensor field $h$ defined in (3) extends smoothly to the boundary, with the restriction to the boundary being non–degenerate as a bilinear form on the boundary tangent bundle.

Then $g$ is projectively compact of order $\alpha$.

**Proof.** Consider the Levi–Civita connection $\nabla$ of $g$ and the projectively related connection $\tilde{\nabla}$ defined as in (4). To prove that $\tilde{\nabla}$ extends to the boundary, it suffices to show that for arbitrary vector fields $\xi$ and $\eta$ defined on all of $U$, also $\tilde{\nabla}_{\xi \eta}$ extends smoothly to all of $U$. To do this, we first show that $d\rho(\tilde{\nabla}_{\xi \eta})$ is smooth up to the boundary. Further we prove that for any smooth vector field $\zeta$ on $U$ such that $d\rho(\zeta) = 0$, also $h(\tilde{\nabla}_{\xi \eta}, \zeta)$ is smooth up to the boundary. In view of our assumptions, this evidently suffices to complete the proof.

We first have to construct a vector field $\tilde{\zeta}_0$ on $U$, which plays the role of a Reeb field. Shrinking $U$ we may assume that $d\rho$ is nowhere vanishing on $U$, so its kernel $\ker(d\rho)$ defines a hyperplane distribution. By assumption the restriction of $h$ to the boundary is non–degenerate on $\ker(d\rho)$. Possibly shrinking $U$ further, we may thus assume that $h$ restricts to a non–degenerate bilinear form on $\ker(d\rho)$ on all of $U$. This implies that we can use $h$ to orthonormalize a local frame for $\ker(d\rho)$ and, again shrinking, we obtain a frame $\{\xi_1, \ldots, \xi_n\}$ for $\ker(d\rho)$ such that $h(\xi_i, \xi_j) = \delta_{ij}$ with $\epsilon_i = \pm 1$ for all $i$. Next, choose a vector field $\hat{\zeta}$ on $U$ such that $d\rho(\hat{\zeta}) = 1$ and define $\zeta_0 := \hat{\zeta} - \sum_i \epsilon_i h(\hat{\zeta}, \xi_i) \xi_i$. Then it is clear that $d\rho(\zeta_0) = d\rho(\hat{\zeta}) = 1$ and that for any vector field $\xi$ on $U$ such that $d\rho(\xi) = 0$, we have $h(\zeta_0, \xi) = 0$.

Now a general vector field $\xi \in \mathfrak{X}(U)$ can be decomposed as

$$\xi = d\rho(\xi)\zeta_0 + (\xi - d\rho(\xi)\zeta_0).$$
Again using (4), we obtain
\[ ζ \]
Differentiating in direction \( ρ \) to the right hand side.

Differentiating this by \( ξ, η \) as follows:
\[ U \]
Two vector fields defined on all of \( X(U) \) we can use (4) to compute on \( U \cap M \) as follows:
\[ g(∇ξη, ζ_0) = \frac{1}{ρ^2/α}h(∇ξη, ζ_0) + \frac{C}{ρ^2/α}dρ(∇ξη) = dρ(∇ξη)(\frac{C}{ρ^2/α} + \frac{h(ζ_0, ζ_0)}{ρ^2/α}). \]

Hence we see that smoothness of \( dρ(∇ξη) \) up to the boundary will follow from smoothness of \( dρ(∇ξη) \) up to the boundary.

On the other hand, consider any vector field \( ζ \in X(U) \) such that \( dρ(ζ) = 0 \). Again using (4), we obtain
\[ g(∇ξη, ζ) = \frac{1}{ρ^2/α}h(∇ξη, ζ), \]
so we can prove smoothness of \( h(∇ξη, ζ) \) up to the boundary by showing that \( ρ^{2/α}g(∇ξη, ζ) \) is smooth up to the boundary.

Next the Koszul formula for the Levi–Civita connection reads as
\[ 2g(∇ξη, ζ) = ξ \cdot g(η, ζ) - ζ \cdot g(ξ, ζ) + η \cdot g(ξ, ζ) + g([ξ, η], ζ) - g([ξ, ζ], η) - g(η, [ξ, ζ]). \]

To compute \( 2g(∇ξη, ζ) \), we have to add
\[ (5) \]
\[ \frac{2dρ(ξ)}{αρ}g(η, ζ) + \frac{2dρ(η)}{αρ}g(ξ, ζ) \]
to the right hand side.

Now let us first look at the case \( ζ = ζ_0 \), so we want to prove that \( ρ^{2/α}g(∇ξη, ζ_0) \) is smooth up to the boundary. Now as above, we compute on \( U \cap M \):
\[ g(η, ζ_0) = dρ(η)(\frac{1}{ρ^2/α}h(ζ_0, ζ_0) + \frac{C}{ρ^2/α}). \]

Differentiating this by \( ξ \), and ignoring terms which are smooth up to the boundary after multiplication by \( ρ^{2/α} \), we are left with \( \frac{-4Cρdρ(η)dρ(ξ)}{αρ^{2+α}g(ξ, ζ_0)}. \) The same contribution comes from \( η \cdot g(ξ, ζ_0) \). Next,
\[ -g(ξ, η) = \frac{-1}{ρ^2/α}h(ξ, η) + \frac{C}{ρ^2/α}dρ(ξ)dρ(η). \]

Differentiating in direction \( ζ_0 \) and ignoring terms which are smooth up to the boundary after multiplication by \( ρ^{2/α} \), we see that this term contributes \( \frac{4Cρdρ(ζ)dρ(ξ)}{αρ^{2+α}g(ξ, ζ_0)}. \)

Smoothness of \( h \) and \( C \) up to the boundary immediately implies that inserting two vector fields defined on all of \( U \) into \( g \), and multiplying by \( ρ^{2/α} \), the result is always smooth up to the boundary, so there comes no further contribution from the Koszul formula. But then
\[ \frac{2dρ(ξ)}{αρ}g(η, ζ_0) = \frac{2Cρdρ(ξ)dρ(η)}{αρ^{2+α}g(ξ, ζ_0)} \]
up to terms which are smooth up to the boundary after multiplication by \( ρ^{2/α} \), and the other term from (5) gives the same contribution. Thus we have verified that \( dρ(∇ξη) \) is smooth up to the boundary.

So let us turn to the case that \( dρ(ζ) = 0 \), and we have to show that \( ρ^{2/α}g(∇ξη, ζ) \) is smooth up to the boundary. Notice first that \( dρ = 0 \) and \( dρ(ζ) = 0 \) imply that
\[ \zeta \cdot d\rho(\xi) = d\rho([\zeta, \xi]) \] and likewise for \( \eta \). Together with \( \zeta \cdot \rho = 0 \) and the fact that \( \frac{1}{\rho^{2/\alpha}} \zeta \cdot C \) is smooth up to the boundary, this easily implies that

\[
\zeta \cdot g(\xi, \eta) + g([\xi, \zeta], \eta) + g([\eta, \zeta], \xi)
\]
is smooth up to the boundary after multiplication by \( \rho^{2/\alpha} \). On the other hand, \( g([\xi, \eta], \zeta) \) evidently is smooth up to the boundary after multiplication by \( \rho^{2/\alpha} \). For the remaining two summands in the Koszul formula, one easily computes that, up to terms which are smooth up to the boundary after multiplication by \( \rho^{2/\alpha} \), one gets

\[
-2d\rho(\xi) h(\eta, \zeta) \frac{1}{\alpha \rho^{(2+\alpha)/\alpha}} - 2d\rho(\eta) h(\xi, \zeta) \frac{1}{\alpha \rho^{(2+\alpha)/\alpha}},
\]
and this exactly cancels with the contribution from the two terms in (5). □

**Remark 7.** (i) Notice that for \( \alpha = 1 \) and \( C = 1 \) the asymptotic form in (4) is called a “scattering metric” in [27, Chapter 6], where it is used to develop generalizations of Euclidean scattering theory. For the more general asymptotic form \( g = \frac{d\rho^2}{\rho^2} + \frac{h}{\rho^{\alpha}} \) considered in [27, Chapter 8], our proof suggests that projective compactness forces, at least in a certain range, a relation between \( a \) and \( b \). Together with the appropriate volume asymptotics this then pins down both exponents.

(ii) The standard form of a conformally compact metric is \( g = \frac{h + dr^2}{r^2} \) for a defining function \( r \). Looking at the volume density, we see that this has volume asymptotics of order \( n + 1 = \dim(M) \) as compared to \( \frac{2n}{\alpha} \) for projectively compact metrics of order \( \alpha \). This indicates a significant difference between the two types of compactifications. In the domain of a local chart around the boundary, there is a way to formally relate compactifications of the two types. Looking at a metric of the form (4) with \( \alpha = 2 \) and \( C \) constant, one can formally put \( \rho = r^2 \) (which makes it impossible for \( r \) to be a defining function without changing the smooth structure), to obtain

\[
g = \frac{h}{r} + C \frac{d\rho^2}{\rho^2} = \frac{h + 4Chr^2}{r^2}.
\]

While this can be used to reduce some local analytical questions on metrics which are projectively compact of order two to the conformally compact case, it is not clear to what extent this relation is independent of coordinates or has some geometric meaning. Moreover, it seems not to be possible to apply a similar idea to metrics which are projectively compact of orders different from 2.

### 3. First BGG–equations and reductions of projective holonomy

In this section, we further explore the projective geometry of special affine connections and in particular of pseudo–Riemannian metrics, which are projectively compact of order one or two. By Proposition 3, in these cases, we have a canonical defining density, which is a section of \( E(1) \) respectively of \( E(2) \). Sections of each of these two bundles form the domain of a canonical projectively invariant overdetermined system of PDEs. These systems are coming from the machinery of BGG sequences, which also leads to a special class of solutions, called normal solutions. Hence we can single out particularly nice subclasses of projectively compact affine
connections and metrics by requiring that the canonical defining densities are solutions respectively normal solutions of the first BGG equation. We will analyze the meaning of these conditions, noting that for normal solutions one obtains a reduction of projective holonomy as discussed in [8] and [9].

3.1. Background on projective tractor bundles and first BGG operators.
We start with a brief review on some elements of the geometry of projective structures and a related class of projectively invariant differential operators. We will restrict our attention to the cases we need in this article and hence only discuss the projective standard cotractor bundle and its symmetric square. More information can be found in [8]. The standard cotractor bundle (or rather its dual) was introduced by T. Thomas in the 1930’s as an alternative to the canonical Cartan connection associated to a projective structure, a modern presentation can be found in [4].

Given a manifold $N$ with a projective structure, one can define the cotractor bundle $\mathcal{T}^*$ simply as the first jet–prolongation $J^1(\mathcal{E}(1))$ of the density bundle $\mathcal{E}(1)$ introduced in Section 2.2. In particular, one has the jet exact sequence

\[ 0 \to T^*N \otimes \mathcal{E}(1) \to \mathcal{T}^* \xrightarrow{\pi} \mathcal{E}(1) \to 0. \]

From now on, we will sometimes use abstract index notion, so we write $\mathcal{E}^a$ for the tangent bundle and $\mathcal{E}_a$ for the cotangent bundle, and we will indicate a tensor product with the line bundle $\mathcal{E}(w)$ by adding “($w$)” to the name of a bundle. In this notation, the jet exact sequence reads as $0 \to \mathcal{E}_a(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0$.

Choosing a connection $\nabla$ from the projective class, one obtains an induced connection on the density bundle $\mathcal{E}(1)$ and thus a splitting of the jet exact sequence, i.e. an isomorphism $\mathcal{T}^* \cong \mathcal{E}_a(1) \oplus \mathcal{E}(1)$. In this picture, we write sections of $\mathcal{T}^*$ as vectors $\left(\begin{array}{c} \sigma \\ \mu_a \end{array} \right)$ with $\sigma \in \Gamma(\mathcal{E}(1))$ and $\mu_a \in \Gamma(\mathcal{E}_a(1))$. Changing from the connection $\nabla$ to the connection $\nabla + \Upsilon_a$ as defined in Section 2.1 this identification changes as

\[ \left(\begin{array}{c} \sigma \\ \mu_a \end{array} \right) \mapsto \left(\begin{array}{c} \sigma \\ \mu_a + \Upsilon_a \sigma \end{array} \right), \]

which also shows that the projection onto the top slot and the inclusion of the bottom slot are natural bundle maps.

There is a natural connection $\nabla^{\mathcal{T}^*}$ on $\mathcal{T}^*$, which, in the identification $\mathcal{T}^* \cong \mathcal{E}_a(1) \oplus \mathcal{E}(1)$ defined by $\nabla$, is given by

\[ \nabla^{\mathcal{T}^*}_a \left(\begin{array}{c} \sigma \\ \mu_a \end{array} \right) = \left(\begin{array}{c} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab} \sigma \end{array} \right). \]

Here $P_{ab}$ denotes the (projective) Schouten–tensor. For our purposes it suffices to know that for a special affine connection in the projective class, we have $P_{ab} = \frac{1}{\dim(N)-1} \text{Ric}_{ab}$, where $\text{Ric}_{ab}$ is the usual Ricci tensor, see [4], and this is symmetric.

The description of the cotractor bundle is particularly simple for the homogeneous model of projective geometry. We consider here the model for orientable projective structures, which is the sphere $S^{n+1}$ viewed as the ray projectivization of $\mathbb{R}^{n+2} \setminus \{0\}$. In this case, one can actually identify $\mathbb{R}^{n+2} \setminus \{0\}$ with the frame
bundle of $\mathcal{E}(1)$, so densities of weight $w$ can be identified with smooth functions on $\mathbb{R}^{n+2} \setminus \{0\}$ which are homogeneous of degree $w$. Moreover, a local nowhere-vanishing section of $\mathcal{E}(1)$ can be viewed as a local section of the ray projectivization, and the connection on $S^{n+1}$ leaving that scale parallel is just the pullback of the standard flat connection on $\mathbb{R}^{n+2} \setminus \{0\}$ along the section.

Sections of the standard cotractor bundle $T^*$ can be identified with one–forms on $\mathbb{R}^{n+2} \setminus \{0\}$ which are homogeneous of degree 1, and the tractor connection is again induced from the flat connection. In particular, a parallel section of $T^*$ is equivalent to a fixed element of $\mathbb{R}$, which is viewed as a differential form on $\mathbb{R}^{n+2} \setminus \{0\}$. It works similarly for more general tractor bundles.

Apart from the tractor connection, we will need a second ingredient, the Kostant codifferential. There is an obvious natural bundle map $\partial^* : \mathcal{E}_a \otimes T^* \to T^*$ defined by $\varphi_a \otimes (\sigma^a) \mapsto (\sigma^a \rho_a)$. This can be interpreted as defining an action of the bundle $\mathcal{E}_a$ of abelian Lie algebras on the bundle $T^*$. Thus it extends to a sequence of bundle maps $\partial^* : \Lambda^k T^* N \otimes T^* \to \Lambda^{k-1} T^* N \otimes T^*$ such that $\partial^* \circ \partial^* = 0$. Hence we have bundle maps on the bundles of $T^*$–valued differential forms such that $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^* N \otimes T^*$. What we really need is the explicit description of these two subspaces in the case $k = 1$. Using the obvious extension of the vector notation from above, the end of the $\partial^*$–sequence has the form

$$\begin{pmatrix}
\mathcal{E}(1) \\
\mathcal{E}_a(1)
\end{pmatrix} \mapsto \begin{pmatrix}
\mathcal{E}_a(1) \\
\mathcal{E}_{ab}(1)
\end{pmatrix} \mapsto \begin{pmatrix}
\mathcal{E}_{ab}(1) \\
\mathcal{E}_{abc}(1)
\end{pmatrix} \mapsto \cdots$$

From the definition above it is evident that $\partial^*$ always maps a row in some column to one row below in the next column. Moreover, one may use general tools to show that the cohomology of the sequence is given by $\mathcal{E}(1)$ in degree 0 and by $\mathcal{E}_{(ab)}$ in degree one. This implies that $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset T^* N \otimes T^*$ has the form

$$\begin{pmatrix}
0 \\
\mathcal{E}_{ab}(1)
\end{pmatrix} \subset \begin{pmatrix}
0 \\
\mathcal{E}_{ab}(1)
\end{pmatrix} \subset \begin{pmatrix}
\mathcal{E}_a(1) \\
\mathcal{E}_{ab}(1)
\end{pmatrix}. \tag{9}$$

Next, we need the analogous information for the symmetric square $S^2 T^*$. Choosing a connection from the projective class, we evidently get an isomorphism $S^2 T^* \cong \mathcal{E}_{(ab)}(2) \oplus \mathcal{E}_a(2) \oplus \mathcal{E}(2)$, and we will use a vector notation with three components, similar to the case of $T^*$. Passing from $\nabla$ to $\hat{\nabla} = \nabla + \Upsilon$, this identification changes as

$$\begin{pmatrix}
\tau \\
\nu_a \\
\rho_{ab}
\end{pmatrix} \mapsto \begin{pmatrix}
\tau \\
\nu_a + \Upsilon_a \tau \\
\rho_{ab} + 2 \Upsilon_{(ab)} + \Upsilon_a \Upsilon_b 
\end{pmatrix}. \tag{10}$$

The connection on $S^2 T^*$ induced by $\nabla^{T^*}$ can be easily computed directly. It is given by

$$\nabla^{S^2 T^*}_{\tau} = \begin{pmatrix}
\nabla_a \tau - 2 \nu_a \\
\nabla_a \nu_b + P_{ab} \tau - \rho_{ab} \\
\nabla_a \rho_{bc} + 2 P_{a(b} \nu_{c)}
\end{pmatrix}. \tag{11}$$
The interpretation of $\partial^*$ as an action of the bundle $E_a$ of abelian Lie algebras on $T^*$ readily provides a similar action $\partial^* : E_a \otimes S^2 T^* \to S^2 T^*$. This then extends to a sequence of differentials defined on the bundles of differential forms with values in $S^2 T^*$. To understand $\text{im}(\partial^*) \subset \ker(\partial^*) \subset T^* N \otimes S^2 T^*$, we again write out the end of the sequence:

$$
\begin{pmatrix}
E(2) \\
E_a(2) \\
E_{(ab)}(2)
\end{pmatrix}
\overset{\partial^*}{\rightarrow}
\begin{pmatrix}
E_a(2) \\
E_{ab}(2) \\
E_{a(bc)}(2)
\end{pmatrix}
\overset{\partial^*}{\rightarrow}
\begin{pmatrix}
E_{[ab]}(2) \\
E_{[abc]}(2) \\
E_{[abc](cd)}(2)
\end{pmatrix}.
$$

As before, application of $\partial^*$ moves down one row, and the cohomologies of the sequence are known to be $E(2)$ in degree zero and $E_{(ab)}(2) \subset E_{a(bc)}(2)$ in degree one. Hence the map $\partial^*$ defined on $T^* N \otimes S^2 T^*$ must map $E_a(2)$ isomorphically onto the copy of the same bundle contained in $S^2 T^*$ and $E_{ab}(2)$ onto the copy of $E_{(ab)}(2)$ contained in that bundle. Likewise, the next map $\partial^*$ has to map $E_{[ab]}(2)$ isomorphically onto the copy of this bundle contained in the middle slot of $E_a \otimes S^2 T^*$ and it must map $E_{[abc]}(2)$ onto the kernel of the complete symmetrization in the bottom slot of this bundle. Thus we see that $\text{im}(\partial^*) \subset \ker(\partial^*) \subset T^* N \otimes S^2 T^*$ is given by

$$
E_{[ab]}(2) \subset E_{[abc]}(2) \subset E_{[abc](cd)}(2),
$$

where $F \subset E_{a(bc)}$ is the kernel of the complete symmetrization.

Now we are ready to describe the relation of the structures developed so far to the first BGG operators determined by the two bundles. The construction of BGG sequences, see e.g. the sketch in [7], shows that given any density $\psi$ of the appropriate weight, there is a unique section $L(\psi)$ of the tractor bundle with $\psi$ in the top slot such that applying the tractor connection one obtains a section of the subbundle $\ker(\partial)^*$. Then the first BGG operator is obtained by projecting this section to the quotient bundle $\ker(\partial^*) / \text{im}(\partial^*)$. In particular, $\psi$ is a solution of the first BGG--operator if and only if the covariant derivative of $L(\psi)$ actually is a section of $\text{im}(\partial^*)$. There is an obvious subclass of solutions, namely those $\psi$, for which $L(\psi)$ actually is a parallel section of the tractor bundle in question. These are the normal solutions which are the main object of study in [8] and [9].

3.2. Projective compactness of order one. We want to now treat geometric structures that are related to projective compactness of order one. As before, we are working on a smooth manifold $M$ of dimension $n + 1$ with boundary; we write $M$ for the interior of $M$ and $\partial M$ for the boundary.

Let us first assume that $\nabla$ is a special affine connection on $M$, which is projectively compact of order one. Thus by Proposition 3 there is a natural defining density for $\partial M$ which is a section of $\mathcal{E}(1)$. Next, we use the machinery developed in Section 3.1 to understand the splitting operator and the first BGG operated defined on sections of this bundle.

Given any section $\sigma \in \Gamma(\mathcal{E}(1))$, we first have to find a section $s \in \Gamma(T^*)$ with $\sigma$ in the top slot and the additional property that $\nabla^{T^*} s$ has zero in the top slot,
so then \( s = L(\sigma) \). From the definition \([8]\) of the standard tractor connection it is clear that

\[
L(\sigma) = \begin{pmatrix} \sigma \\ \nabla_a \sigma \end{pmatrix}
\]

in the splitting determined by an arbitrary connection \( \nabla \) in the projective class. Applying the tractor connection to this, we get \((\nabla_a \nabla_b + p_{ab})\sigma\), so the first BGG operator is given by

\[
\sigma \mapsto \nabla_a \nabla_b \sigma + \tilde{p}_{ab}\sigma.
\]

The existence of the splitting operator readily leads to information on special affine connections which are projectively compact of order 1:

**Proposition 8.** Let \( \nabla \) be a special affine connection on \( M \) which is projectively compact of order \( \alpha = 1 \), and let \( \sigma \in \Gamma(E(1)) \) be the canonical defining density for \( \partial M \) determined by \( \nabla \). Then we have:

(i) The section \( L(\sigma) \in \Gamma(T^*M) \) is nowhere vanishing on \( M \).

(ii) The smooth section \( P_{ab}\sigma \) of \( E_{(ab)}(1) \) over \( M \) extends smoothly to \( \overline{M} \). The restriction of this section to \( \partial M \) can be naturally viewed as a second fundamental form for the boundary. In particular, \( \partial M \subset \overline{M} \) is totally geodesic if and only if this second fundamental form vanishes identically on \( T\partial M \times T\partial M \).

**Proof.** Since \( \sigma \) is parallel for \( \nabla \), formula \([13]\) for the splitting operator shows that \( L(\sigma) = (0) \) on \( M \) in the splitting corresponding to \( \nabla \), so in particular this is nowhere vanishing on \( M \). Now choose a local defining function \( \rho \) for the boundary and consider the connection \( \tilde{\nabla} = \nabla + \frac{\partial \rho}{\rho} \) which locally extends to all of \( \overline{M} \). Then from formula \([7]\) we see that \( L(\sigma) = \binom{\sigma}{\sigma \rho d\rho} \) in the splitting corresponding to \( \tilde{\nabla} \). But from the proof of Proposition 3 we know that the density \( \frac{\sigma}{\rho} \) is parallel for \( \tilde{\nabla} \) and extends to \( \partial M \), so it is nowhere vanishing. Since \( d\rho \) is non-vanishing along \( \partial M \), (i) follows.

(ii) We have also seen already that, in the splitting defined by \( \nabla \) on \( M \), \( \nabla^{\tau*}L(\sigma) \) has zero in the top slot and \( P_{ab}\sigma \) in the bottom slot. Since this is concentrated in the bottom slot, it is independent of the choice of splitting. Now of course \( \nabla^{\tau*}L(\sigma) \) is defined on all of \( \overline{M} \) and the values along \( \partial M \) must also lie in the subbundle \( E_{(ab)}(1) \), which gives the required extension of \( P_{ab}\sigma \).

Now take the local defining function \( \rho \) and the corresponding connection \( \tilde{\nabla} \) as in the first part of the proof. From there we know that \( L(\sigma) = \binom{\sigma}{\sigma \rho d\rho} \) in the splitting determined by \( \tilde{\nabla} \). Applying the defining formula \([8]\) for the tractor connection and using that \( \frac{\sigma}{\rho} \) is extends to a density \( \tilde{\sigma} \) which is parallel for \( \tilde{\nabla} \), we see that \( \nabla^{\tau*}L(\sigma) \) has zero in the top slot and \( \tilde{\sigma} \nabla_a d\rho + \tilde{P}_{ab}\sigma \) in the bottom slot. Since \( \sigma \) vanishes along \( \partial M \), we see that the extension of \( P_{ab}\sigma \) is given by \( \tilde{\sigma} \nabla_a d\rho \) along \( \partial M \). Since \( \tilde{\sigma} \) is nowhere vanishing, we see that inserting vector fields \( \xi, \eta \) tangent to the boundary, this gives a non-zero multiple of \( d\rho(\nabla\xi\eta) \) which justifies the interpretation as a (projectively weighted) second fundamental form. \( \Box \)
Notice that in the case that the second fundamental form from part (ii) is non-degenerate, it defines a canonical conformal structure on the boundary.

Next, we can analyze the meaning of the canonical defining density being a solution, respectively normal solution, of the first BGG operator (14).

**Theorem 9.** Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Let $\nabla$ be a special affine connection on $M$ which is projectively compact of order $\alpha = 1$, and let $\sigma \in \Gamma(\mathcal{E}(1))$ be the canonical defining density for $\partial M$ determined by $\nabla$.

Then $\sigma$ is a solution of the first BGG operator defined on $\mathcal{E}(1)$ if and only if the connection $\nabla$ is Ricci flat. In this case, the boundary $\partial M$ is totally geodesic and hence inherits a projective structure. Moreover, $\sigma$ automatically is a normal solution of the first BGG operator (14) so one is in the situation of a reduction of projective holonomy as described in Theorem 3.1 of [8].

**Proof.** We have observed already that $L(\sigma) = (\sigma_0)$ in the splitting determined by $\nabla$, and using $\nabla$ to write the first BGG operator, we see that $\sigma$ is a solution if and only if $\text{Ric}_{ab}\sigma = 0$. (Recall that $P_{ab}$ is symmetric and a non-zero multiple of $\text{Ric}_{ab}$ since $\nabla$ is a special affine connection.) Hence $\sigma$ is a solution if and only if $\nabla$ is Ricci flat. It then follows from part (ii) of Proposition [3] that $\partial M$ is totally geodesic, which in turn implies that one obtains an induced projective structure. Moreover, if $\nabla$ is Ricci flat, then the formulae above immediately imply that $L(\sigma)$ is parallel, so $\sigma$ is a normal solution. \( \square \)

Conversely, given a projective structure on $\overline{M}$, a Ricci flat special connection on $M$ which lies in the projective class and does not extend to any part of the boundary must be projectively compact of order one:

**Theorem 10.** Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Suppose that $\overline{M}$ is endowed with a projective structure and that $\nabla$ is a connection over $M$ contained in (the restriction of) the projective class such that

* $\nabla$ is special, i.e. preserves a non-zero section $\kappa$ of $\text{vol}(M)$;
* $\text{Ric}\nabla = 0$;
* $\nabla$ does not extend smoothly to any neighborhood of a boundary point.

Then $\nabla$ is projectively compact of order $\alpha = 1$, $\sigma := \kappa^{-1/(n+2)}$ smoothly extends by zero to a defining density for $\partial M$, and this extension satisfies the equation of the first BGG operator (14) on $\overline{M}$. So we are in the situation of Theorem [3].

**Proof.** By construction, $\sigma \in \Gamma(\mathcal{E}(1))$ on $M$ and is nowhere vanishing there. Since $\text{Ric}\nabla = 0$, and $\nabla\sigma = 0$ it follows that $I := L(\sigma) = (\sigma_0)$ is parallel on $M$. But then $I$ extends by parallel transport to a parallel tractor on all of $\overline{M}$ and projecting this to the quotient bundle $\mathcal{E}(1)$, we obtain a smooth extension of $\sigma$ to $\overline{M}$. By continuity we see that on all of $\overline{M}$, we have $I = L(\sigma)$ and that $\sigma$ satisfies the first BGG equation (14) on $\overline{M}$.

Next we claim that (the extension of) $\sigma$ is identically zero on $\partial M$. If $q \in \partial M$ would be a point such that $\sigma(q) \neq 0$, then take an open neighborhood $U$ of $q$ on which $\sigma$ is non-vanishing. Then there is a unique connection in (the restriction of)
the projective class on $U$ for which $\sigma$ is parallel. By construction, this agrees with $\nabla$ on $U \cap \partial M$, thus providing an extension contradicting our assumptions. Hence we see that the zero locus of $\sigma$ coincides with $\partial M$. Finally, $I$ is parallel and hence nowhere zero, thus from (13) we see that for any connection $\nabla$ in the projective class, which extends to the boundary, $\nabla \sigma$ is nowhere zero along $\partial M$. Thus $\sigma$ is a defining density for $\partial M$. $\square$

3.3. Projective compactness of order two. We now want to consider structures which turn out to be related to projective compactness of order two.

First let us assume that we have an affine connection $\nabla$ on $M$ which is projectively compact of order two, and let us denote by $\tau \in \Gamma(E(2))$ a corresponding defining density (which is unique up to a constant factor). To apply the machinery from Section 3.1, we first have to describe $\nabla_a \nabla^a \tau$, $\rho_{ab} = \frac{1}{2} \nabla^a \nabla_b \tau + \tilde{P}_{ab} \tau$, and this describes $L(\tau)$. In particular, in the splitting determined by $\nabla$, we get

$$L(\tau) = \begin{pmatrix} \tau \\ 0 \\ P_{ab} \tau \end{pmatrix} \quad \nabla_a^{S^2 T^*} L(\tau) = \begin{pmatrix} 0 \\ 0 \\ \tau \nabla_a P_{bc} \end{pmatrix},$$

where as before we use that $P_{ab}$ is symmetric. Using this we get the first part of the following result.

**Theorem 11.** Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Let $\nabla$ be a special affine connection on $M$ which is projectively compact of order two, let $\text{Ric}_{ab}$ be its Ricci curvature, and let $\tau \in E(2)$ be the corresponding defining density. Then:

(i) $\tau$ is a solution of the first BGG operator if and only if $\nabla_a \text{Ric}_{bc} = 0$.

(ii) $\tau$ is a normal solution if and only if $\nabla_a \text{Ric}_{bc} = 0$. If $\text{Ric}_{ab}$ is non–degenerate, then it defines a pseudo–Riemannian Einstein–metric on $M$ with Levi–Civita connection $\nabla$. In this case, $L(\tau)$ defines a non–degenerate bundle metric (necessarily of indefinite signature) on the standard tractor bundle over $\overline{M}$. This gives rise to a reduction of projective holonomy to an orthogonal group as studied in Section 3.3 of [8] and in Section 3.1 of [9], with the closed curved orbit given by the boundary $\partial M$ and the open curved orbit given by the interior $M$. In particular, in this case, the boundary $\partial M$ inherits a canonical conformal structure.

**Proof.** From the formula for $\nabla^{S^2 T^*} L(\tau)$ above, equation (12), and the fact that $P_{ab} = \frac{1}{n} \text{Ric}_{ab}$, we immediately get (i) and the first statement in (ii). If in the setting of (ii), $\text{Ric}_{ab}$ is non–degenerate, then it can be used as a pseudo–Riemannian metric, and since $\nabla_a \text{Ric}_{bc} = 0$, $\nabla$ must be its Levi–Civita connection. But by construction, the Ricci curvature of this metric is a multiple of the metric, so it is Einstein with non–zero scalar curvature (cf. [2]).

As noted in Section 3.1 associated to the choice of the connection $\nabla$ in the projective class, there is an identification $T^* \cong E_a(1) \oplus E(1)$. Correspondingly, the standard tractor bundle decomposes as $T \cong E^a(-1) \oplus E(-1)$. Viewing sections of $S^2 T^*$ as bilinear forms on $T$, the decomposition into triples we have used has the
restrictions to the two summands in the top and bottom slots and the cross-term in the middle slot. As we have noted above, in the splitting determined by $\nabla$, we have

$$L(\tau) = \begin{pmatrix} \tau \\ 0 \\ P_{ab}\tau \end{pmatrix}. $$

Vanishing of the middle slot shows that over $M$, the decomposition $\mathcal{T} = \mathcal{E}(1) \oplus \mathcal{E}_a(1)$ determined by $\nabla$ is orthogonal for the bilinear form $L(\tau)$. Since $P_{ab}$ is a non-zero multiple of $\text{Ric}_{ab}$, non-degeneracy of $\text{Ric}_{ab}$ implies that the restriction of $L(\tau)$ to both summands is non-degenerate. This shows that $L(\tau)$ is a non-degenerate bilinear form on $\mathcal{T}$ over $M$. Since $L(\tau)$ is parallel, this is true over all of $\overline{M}$ and we get a holonomy reduction as claimed. In the references mentioned in the theorem, it is shown that the curved orbit decomposition is determined by the sign of the density $\tau$, which implies the last part. □

Again, we can also prove a nice converse to this result:

**Theorem 12.** Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Suppose that $\overline{M}$ is endowed with a projective structure and that $\nabla$ is a connection in (the restriction of) the projective class on $M$ which is the Levi–Civita connection of a non–Ricci–flat Einstein metric or, more generally, satisfies:

- $\nabla$ is special, i.e. it preserves a non–zero section $\kappa$ of $\text{vol}(M)$;
- $\nabla \text{Ric} = 0$, and $\text{Ric}$ is non-degenerate.

Suppose further that $\nabla$ does not smoothly extend to any neighborhood of a boundary point.

Then $\nabla$ is projectively compact of order $\alpha = 2$, $\tau := \kappa^{-2/(n+2)}$ smoothly extends by zero to a defining density for $\partial M$, and this extension satisfies the first BGG–equation on $\mathcal{E}(2)$ on $\overline{M}$. So we are again in the situation of a holonomy reduction as in part (ii) of Theorem 11.

**Proof.** Note that $\tau \in \Gamma(\mathcal{E}(2))$ on $M$. Since $\nabla \text{Ric} = 0$, and $\nabla \tau = 0$ it follows that $H := L(\tau)$ (as in (15)) is parallel on $M$. Now the argument follows the proof of Theorem 10 mutatis mutandis, up to that point that the zero locus of $\tau$ coincides with $\partial M$. To see that $\tau$ is indeed a defining density, observe that the projection of $H$ to the quotient bundle $\mathcal{E}(2)$ coincides with (the extension of) $\tau$, so it vanishes along $\partial M$. In the proof of Theorem 11 we have noted that this describes the restriction of $H$ to the natural line subbundle $\mathcal{E}(-1) \subset \mathcal{T}$. Non–degeneracy of $H$ then implies that the middle slot of $H$ (which describes the cross-term of the bilinear form) is nowhere vanishing along $\partial M$. Now by (15) and (10) this middle slot is a non–zero multiple of $\frac{1}{2}\tilde{\nabla} \tau$, where $\tilde{\nabla}$ is any connection in the projective class that extends to the boundary. □

Now we can analyze the section $L(\tau)$ in a similar way as for projective compactness of order one studied in Section 3.2. This only works in the setting of part (ii) of Theorem 11 so we deal with an Einstein metric with non–zero scalar curvature. In this case, we get a converse to Theorem 6.
Proposition 13. Suppose that we are in the situation of part (ii) of Theorem 11, and let $g$ be the resulting Einstein metric on $M$ and $R$ its (non–zero) scalar curvature.

Then for any local defining function $\rho$ for $\partial M$, the symmetric $\binom{n}{2}$–tensor field $\rho \frac{4R}{n(n+1)} g + \frac{d\rho^2}{\rho}$ on $TM$ extends to the boundary and there restricts to a non–degenerate symmetric bilinear form on $T\partial M$.

Proof. From the proof of Theorem 11 we see that, in the splitting determined by $\nabla$, we have

$$L(\tau) = \begin{pmatrix} \tau \\ 0 \\ \frac{1}{n(n+1)} Rg_{ab} \tau \end{pmatrix},$$

where we have used that $P_{ab} = \frac{1}{n} \text{Ric}_{ab} = \frac{1}{n(n+1)} Rg_{ab}$ for an Einstein metric. Now we compute the expression for $L(\tau)$ in the splitting determined by the connection $\hat{\nabla} = \nabla + \frac{d\rho}{2\rho}$, which extends to the boundary. By formula (10), this is given by

$$\begin{pmatrix} \tau \\ \frac{d\rho}{2\rho} \\ \left( \frac{1}{n(n+1)} Rg_{ab} + \frac{d\rho^2}{4\rho^2} \right) \tau \end{pmatrix}.$$  

Of course the top slot vanishes along $\partial M$. As we have noted in Section 2.2, $\hat{\tau} =: \tau$ is a section of $\mathcal{E}(2)$, which is parallel for $\hat{\nabla}$, and thus is nowhere vanishing on the domain of definition of $\rho$. Consequently, the middle slot of this expression approaches a non–zero multiple of $d\rho$ in each point of the boundary. The bottom slot is given by $\hat{\tau} h_{ab}$, where

$$h_{ab} = \frac{1}{n(n+1)} \rho Rg_{ab} + \frac{d\rho^2}{4\rho}$$

so $h_{ab}$ has to extend to the boundary. Then the fact that $L(\tau)$ remains non–degenerate along the boundary is equivalent to the fact that the restriction of $h_{ab}$ to the kernel of $d\rho$ is non–degenerate along the boundary. Since $\rho$ is a defining function, this kernel is $T\partial M \subset T\overline{M}|_{\partial M}$. □

3.4. The case of Ricci–flat metrics. Suppose that we have given a Ricci–flat pseudo–Riemannian metric $g$ on $M \subset \overline{M}$, let $\nabla$ be its Levi–Civita connection and consider $\sigma := \text{vol}(g)^{-1/(n+2)} \in \Gamma(\mathcal{E}(1))$. Then by Theorem 9 $\sigma$ satisfies the first BGG equation defined by (14). Likewise, a slight variant of Theorem 11 shows that $\tau := \sigma^2 \in \Gamma(\mathcal{E}(2))$ has to be a normal solution of the first BGG equation defined on the bundle $\mathcal{E}(2)$. This is easy to explain: From the proof of Theorem 9 we see that $\sigma$ is automatically a normal solution, so $s = L(\sigma)$ is a parallel section of $\mathcal{T}^*$. But then $s \circ s$ is a parallel section of the tractor bundle $S^2\mathcal{T}^*$, and thus determines a normal solution of the corresponding first BGG equation, which is clearly given by $\sigma^2$.

Remark 14. Of course $\tau = \sigma^2$ cannot be a boundary defining density. Nevertheless the other observations suggest that perhaps there could be two natural notions of a projective compactification for a Ricci flat metric. Namely these corresponding
to projective compactness of, respectively, order one and order two. However by Theorem 10, a projectively compact Ricci flat metric is necessarily of order $\alpha = 1$. That a Ricci–flat metric $g$ on $M$ cannot be projectively compact of order two can also be seen directly as follows.

Assuming that $g$ is projectively compact of order two, consider the natural defining density $\tau := \text{vol}(g)^{-2/(n+2)} \in \Gamma(E(2))$ for $\partial M$. Then from Section 3.3 we know that the section $L(\tau)$ of $S^2(T^*)$ is parallel and in the splitting defined by $\nabla$ it has $\tau$ in the top slot while the other two slots are identically zero. Thus, as a bilinear form of $T$, $L(\tau)$ has rank one over $M$, and since it is parallel, this holds over all of $M$. As in the proof of Proposition 13 we can next compute $L(\tau)$ in the splitting corresponding to the connection $\hat{\nabla} = \nabla + d\rho^2$ which by assumption extends to the boundary. This is given by

$$
\begin{pmatrix}
\tau \\
p \tau \\
p^2 \tau
\end{pmatrix}.
$$

But as before, $\hat{\tau} = \frac{\tau}{\rho}$ is parallel for $\hat{\nabla}$ and thus extends to the boundary with non–zero boundary value. The same holds for $d\rho$ and hence $d\rho^2 = d\rho \otimes d\rho$ extends to the boundary with non–zero boundary value. As before, the middle slot is just $\frac{\tau}{\rho} d\rho$, so this is fine, but the bottom slot is $\frac{\tau}{\rho^2} d\rho^2$, which cannot extend, so we obtain a contradiction.

There are nice cases of Ricci flat metrics which are projectively compact (of order one). The simplest example of this situation is provided by the homogeneous model of projective geometry, see Section 3.1. Consider the sphere $S^{n+1}$ as the ray projectivization of $\mathbb{R}^{n+2} \setminus \{0\}$. Recall from Section 3.1 that a local scale for this projective structure is determined by a local smooth section of the ray projectivization, and the corresponding connection is the pullback of the flat connection. In particular, the embedding of $S^{n+1}$ as the unit sphere of $\mathbb{R}^{n+2}$ is a global section and the corresponding pullback connection is just the Levi–Civita connection of the round metric on $S^{n+1}$. On the other hand, we can define a local section over an open hemisphere by mapping the round hemisphere to an affine hyperplane in $\mathbb{R}^{n+2}$ via central projection. The resulting connection is then the pullback of the flat connection on that affine hyperplane. This connection is the Levi Civita connection for the flat metric on $\mathbb{R}^{n+1}$ of any chosen signature. Moreover, the corresponding scale is just given by the restriction of a fixed linear functional on $\mathbb{R}^{n+2}$, which, as we have seen in Section 3.1, corresponds to a parallel standard cotractor on $S^{n+1}$. In particular, we can pass to the closed hemisphere and then the section of $E(1)$ underlying this parallel cotractor is a defining density for the boundary sphere $S^n$. In particular, this shows that the flat metric on a hemisphere obtained via central projection is projectively compact of order one.

3.5. Projectively compact Ricci flat metrics. To proceed with the analysis of this case we have to involve a new ingredient, namely the so–called projective metricity equation. This is the first BGG equation associated to the bundle $S^2 T$,
the dual of the tractor bundle giving rise to the first BGG equation on $E(2)$ as studied in Section 3.3. The relation between the first BGG equations determined by the two bundles is much more complicated than mere duality, however. The metricity equation is discussed in [10] in a way closely analogous to the discussion in Section 3.1, and we take some information from there. The natural quotient bundle of $S^2T$, on which the first BGG equation is defined is the bundle $E^{(ab)}(-2)$, a weighted version of the bundle of symmetric bilinear forms on the cotangent bundle.

The main information we need at this place concerns a manifold $N$ of dimension $n+1$ endowed with a projective structure containing the Levi-Civita connection of a pseudo–Riemannian metric $g$. Then putting $\sigma := \text{vol}(g)^{-1/(n+2)} \in \Gamma(E(1))$, and denoting by $g^{-1} \in \Gamma(S^2TN)$ the inverse of $g$, the section $\sigma^{-2}g^{-1} \in \Gamma(S^2TN(-2))$ is a solution of this first BGG operator. (This is an easy consequence of the fact that it is parallel for the connection $\nabla$ from the projective class and the BGG operator in this case is of order one.) In [10] it is shown that this solution is normal if and only if $g$ is Einstein, so in that case $L(\sigma^{-2}g^{-1})$ is a parallel section of $S^2T$, so it can be interpreted as a parallel (degenerate) bundle metric on the standard cotractor bundle $T^*$ (cf. [19] Theorem 3.1).

Similarly as in Section 3.3 it is easy to describe $L(\sigma^{-2}g^{-1})$ in the splitting determined by $\nabla$. If $g$ is Ricci flat (indeed, scalar flat is sufficient for this), then it has $\sigma^{-2}g^{-1}$ in the projecting slot and 0 in both other slots. This immediately implies that, as a bilinear form on $T^*$, the section $L(\sigma^{-2}g^{-1})$ has (constant) rank $n+1$ (i.e. corank one). Moreover, from Section 3.2 we see that the parallel section $L(\sigma) \in \Gamma(T^*)$ corresponding to $\sigma$ is concentrated in the projecting slot, which immediately implies that it spans the null–space of the degenerate bilinear tractor form $L(\sigma^{-2}g^{-1})$.

Returning to our usual setting, these observations suffice to describe the structure on the boundary induced by a projectively compact Ricci flat metric in the interior.

**Theorem 15.** Let $\overline{M}$ be a smooth manifold of dimension $n+1$ with boundary $\partial M$ and interior $M$, and suppose that $g$ is a projectively compact Ricci flat pseudo–Riemannian metric of signature $(p,q)$ on $M$. Then the order of projective compactness is one and the induced projective structure on $\partial M$, from Theorem 4, canonically inherits a holonomy reduction to the group $SO(p,q) \subset SL(n+1,\mathbb{R})$.

**Proof.** By Theorem 11 the metric is projectively compact of order one and $\sigma := \text{vol}(g)^{-1/(n+2)} \in \Gamma(E(1))$ is a defining density for $\partial M$. Consider the solution $\sigma^{-2}g^{-1}$ of the metricity equation on $M$. As discussed above, the corresponding section $L(\sigma^{-2}g^{-1})$ of $S^2T$ is parallel over $M$, so since the projective structure extends to the boundary, it extends to a parallel section over all of $\overline{M}$. As a bilinear form on $T^*$, $L(\sigma^{-2}g^{-1})$ has rank $n+1$ over $M$, so this also holds on the boundary. Moreover, the parallel section $L(\sigma) \in \Gamma(T^*)$ spans the null space of $L(\sigma^{-2}g^{-1})$ over $M$, and again this continues to hold over $\overline{M}$. Finally, we know from the proof of Proposition 8 that $L(\sigma) = \left(\frac{0}{\sigma \partial \rho}\right)$ along the boundary.
Using this, we can now nicely describe the induced projective structure on $\partial M$ as the kernel of $L(\sigma)$. Indeed, since $L(\sigma)$ is nowhere vanishing, its kernel defines a smooth corank one subbundle $\tilde{T} \subset T|_{\partial M}$. Moreover, along the boundary, $L(\sigma)$ defines a section of $T^*M \otimes \mathcal{E}(1) \cong T^*M \otimes \text{vol}(\mathcal{M})^{-1/(n+2)}$ whose pointwise kernel is $T(\partial M) \otimes \text{vol}(\mathcal{M})^{-1/(n+2)}$. Denoting by $\mathcal{N}$ the conormal bundle of the boundary, we have obtained a section of $\mathcal{N} \otimes \text{vol}(\mathcal{M})^{-1/(n+2)}$. But of course, $\text{vol}(\mathcal{M})|_{\partial M} \cong \mathcal{N} \otimes \text{vol}(\partial M)$, so $\mathcal{N} \cong \text{vol}(\partial M)^{-1} \otimes \text{vol}(\mathcal{M})|_{\partial M}$, and we can interpret $L(\sigma)$ as a non–vanishing section of $\text{vol}(\partial M)^{-1} \otimes \text{vol}(\mathcal{M})^{(n+1)/(n+2)}|_{\partial M}$. This section identifies $\text{vol}(\partial M)^{-1} \otimes \text{vol}(\mathcal{M})^{(n+1)/(n+2)}|_{\partial M}$. Taking the power of this of order $-1/(n+1)$ we obtain an isomorphism of $\mathcal{E}(1)|_{\partial M}$ with the space of densities of projective weight one on $\partial M$.

Thus we conclude that $\tilde{T} \to \partial M$ is a bundle of rank $n + 1$ which inherits the appropriate composition series for a projective standard tractor bundle. Since $L(\sigma)$ is parallel, the standard tractor connection on $\mathcal{M}$ restricts to a connection on the vector bundle $\tilde{T}$, and in [8, Theorem 3.1] it is shown that this restriction is normal. Hence we can view $\tilde{T}$ with the standard tractor bundle of the induced projective structure on $\partial M$.

By duality, the standard cotractor bundle $\tilde{T}^*$ for this structure can be identified with the quotient of $T^*|_{\partial M}$ by the line spanned by $L(\sigma)$. But then we know that $L(\sigma^{-2}g^{-1})$ descends to a non–degenerate bundle metric on this quotient bundle, which has the same signature as $g$ and by construction is parallel for the induced connection. Hence the inverse defines a non–degenerate parallel metric of signature $(p, q)$ on the standard tractor bundle, thus giving rise to the claimed holonomy reduction.

Remark 16. Note that the Theorem statement above could be strengthened without adjusting the proof. Rather than requiring the Ricci-flat Levi-Civita connection to be projectively compact it would be sufficient to assume that its projective class extends to the boundary, while the connection itself does not (along the lines of Theorem [10]).

Projective holonomy reductions to orthogonal groups have been studied in detail in Section 3.2 of [3] and in Section 3.1 of [9] and we use the results obtained there. If we start with a Riemannian metric $g$, then the reduction will be to the orthogonal group $SO(n + 1) \subset SL(n + 1, \mathbb{R})$ and this amounts to a positive Einstein Riemannian metric in the projective class. If the initial metric is pseudo–Riemannian of signature $(p, q)$ with $p, q > 0$, then the holonomy reduction induces the so–called curved orbit decomposition $\partial M = \partial M_+ \cup \partial M_0 \cup \partial M_-$ with $\partial M_\pm$ open in $\partial M$, while $\partial M_0$ (if non–empty) is an embedded hypersurface, which separates $\partial M_+$ and $\partial M_-$. On $\partial M_\pm$ the holonomy reduction determines Einstein metrics in the projective class of signature $(p - 1, q)$ and $(p, q - 1)$, respectively. On $\partial M_0$, one obtains a well defined conformal structure of signature $(p - 1, q - 1)$ whose normal conformal standard tractor bundle with its canonical connection coincides with the restriction of $\tilde{T}$. We shall see below how to describe this decomposition explicitly.
Let us analyze the orbit decomposition in the case of the homogeneous model. As in Section 3.4 we consider the flat connection on an open hemisphere in $S^{n+1}$ obtained via central projection to an affine hyperplane, and this is projectively compact (of order one) on the closed hemisphere. The corresponding parallel standard cotractor $I = L(\sigma)$ is described by the functional whose kernel projectivizes to the boundary sphere $S^n$. Now the flat connection on an affine hyperplane is the Levi–Civita connection of the flat metric of any signature, and to be definite, we consider a metric of Lorentzian signature $(n, 1)$. This metric is encoded as a parallel bilinear form on the standard cotractor bundle whose null space is spanned by $I$.

In the case of the homogeneous model, this corresponds to a fixed element $H$ of $S^2\mathbb{R}^{n+2}$, which has rank $n + 1$ with null space spanned by $I$. Then $H$ descends to a non–degenerate bilinear form on $\mathbb{R}^{(n+2)^*}/(\mathbb{R} \cdot I)$. This is the dual space of $\ker(I)$ so we can view the inverse $H^{-1}$ as a non–degenerate bilinear form on $\ker(I)$. Now of course, the boundary sphere $S^n$ can be viewed as the ray projectivization of $\ker(I) \setminus \{0\}$ and $H^{-1}$ describes a parallel section of the symmetric square of the cotractor bundle for the resulting flat projective structure. Now the orbit decomposition is described in [8] and [9]. It exactly corresponds to the restriction of $H^{-1}$ to a ray being positive definite, negative definite, and zero, respectively. So these are just the points at infinity reached by space–like, respectively time–like, respectively null lines in the original Lorentzian vector space. The open curved orbits are the spaces of positive respectively negative rays and thus the standard models for hyperbolic spaces of the appropriate signature. The closed curved orbit consists of two copies of the sphere $S^{n-1}$ viewed as the ray projectivized light cone of a Lorentzian metric, so each of the two copies is a homogeneous model of conformal geometry in Riemannian signature.

**Remark 17.** For emphasis, we point out here that the projective compactification of Minkowski space is very different from the usual conformal compactification of Minkowski space, which conformally embeds Minkowski space into a subspace of the Einstein cone. (This compactification is due to Penrose, see e.g. [16, 31]). Whereas here in the projective compactification the set of points at infinity reached by space–like geodesic rays is an open set, by contrast in the conformal compactification all such rays end in a point of the conformal infinity. It is similar for future time–like rays, and past time–like rays. They end in open caps of the boundary sphere $S^n$ of the projective compactification, while they end respectively in the two points known as “future and past time-like infinity” in the conformal compactification. In the projective compactification the end points of future directed null rays form an $S^{n-1}$ whereas in the conformal compactification the “future null infinity” is open in the boundary (and so of dimension $n$). Again it is similar for past null rays.

### 3.6. Explicit form of curved orbit decomposition.
To obtain an explicit description of the curved orbits in general, we have to analyze the parallel section $L(\sigma^{-2}g^{-1})$ determined by a Ricci flat metric which is projectively compact of order one in more detail. Dual to the description of $S^2T^*$ in Section 3.1 a choice
of connection in the projective class identifies the bundle $S^2\mathcal{T}$ with the direct sum $\mathcal{E}(-2) \oplus \mathcal{E}^a(-2) \oplus \mathcal{E}^{ab}(-2)$. Dualizing (11), we see how this identification changes when passing from $\nabla_a$ to $\tilde{\nabla}_a = \nabla_a + \Upsilon_a$:

$$
\begin{pmatrix}
\tau^{ab} \\
\lambda^a \\
\nu
\end{pmatrix} 
\mapsto 
\begin{pmatrix}
\tau^{ab} \\
\lambda^a - \tau^{ab} \Upsilon_b \\
\nu - 2\lambda^a \Upsilon_a + \tau^{ab} \Upsilon_a \Upsilon_b
\end{pmatrix}
$$

(16)

**Theorem 18.** Let $\overline{M}$ be a smooth manifold of dimension $n + 1$ with boundary $\partial M$ and interior $M$, and let $g$ be a projectively compact Ricci-flat pseudo-Riemannian metric on $M$. Let $\rho : U \to \mathbb{R}_{\geq 0}$ be a local defining function for $\partial M$, and let us write $\rho_a$ for the one-form $d\rho$.

(i) The section $\rho^{-2}g^{ab}$ of $S^2TM$ extends smoothly to the boundary and the boundary value $\tau^{ab}$ satisfies $\tau^{ab}\rho_b = 0$. Moreover, the curved orbits $\partial M_\pm$ consists of those points of $M$ in which the bilinear form $\tau^{ab}$ has rank $n$, while in points of $\partial M_0$ it has rank $n - 1$.

(ii) The section $\rho^{-3}g^{ab}\rho_b$ of $TM$ extends smoothly to the boundary, and the boundary value $\lambda^a$ satisfies $\lambda^a\rho_a = 0$.

(iii) The function $\rho^{-4}g^{ab}\rho_a\rho_b$ on $M$ extends smoothly to the boundary.

**Proof.** As before, we write $\sigma := \text{vol}(g)^{-1/(n+2)}$. Then from above we know that in the splitting determined by $\nabla$, the parallel section $L(\sigma^{-2}g^{-1})$ has $\sigma^{-2}g^{ab}$ in the top slot and zero in the other two slots. Now take a local defining function $\rho : U \to \mathbb{R}_{\geq 0}$ for the boundary, and pass to the splitting defined by $\tilde{\nabla} = \nabla + \frac{d\rho}{\rho}$ which extends to the boundary and using (16), we get

$$
L(\sigma^{-2}g^{-1}) = \begin{pmatrix}
\sigma^{-2}g^{ab} \\
-\sigma^{-2} \rho^{-1} g^{ab} \rho_b \\
\sigma^{-2} \rho^{-2} g^{ab} \rho_a \rho_b
\end{pmatrix}.
$$

Since $\sigma/\rho$ has a finite non-zero limit to the boundary, the same holds for $\sigma^{-2}\rho^2$. Pulling this out, we see that the three slots of are (up to sign) exactly the three objects claimed to smoothly extend to the boundary, so these claims follow. Next, we know that $L(\sigma^{-2}g^{-1})$ has rank $n + 1$ with null space spanned by $L(\sigma)$. This implies that $\rho_a$ lies in the null space of $\tau^{ab}$ and in the kernel of $\lambda^a$ (which also follows from the existence of the limits towards the boundary shown above). Since $\tau^{ab}$ describes the restriction of $L(\sigma^{-2}g^{-1})$ to a subspace of codimension one and is degenerate, the only possible ranks for $\tau^{ab}$ are $n$ and $n - 1$. So it remains to prove the relation to the curved orbit decomposition.

The holonomy reduction giving rise to the curved orbit decomposition comes from the inverse of the metric induced by $L(\sigma^{-2}g^{-1})$ on the quotient of $\mathcal{T}^*$ by the line spanned by $L(\sigma)$. Now this inverse is a section of the bundle $S^2\mathcal{T}^*$ of metrics on the standard cotractor bundle $\mathcal{T}$ for the induced projective structure on the boundary. The irreducible quotient of this is the bundle of densities of projective weight two, with the quotient projection coming from the restriction of the metric to the natural line subbundle in $\mathcal{T}$. By theorem 3.1 of [9], the curved orbit $\partial M_0$ coincides with the zero set of the induced section of the quotient bundle. Otherwise
put, a point $x_0$ lies in $\partial M_0$ if and only if the distinguished line in $\tilde{T}$ is isotropic for the metric defining the holonomy reduction. But this is equivalent to the fact that the dual metric is degenerate on the annihilator of this line, which is exactly the natural subbundle in $\tilde{T}^\ast$. Of course, this is equivalent to the null–space of $\tau^{ab}$ being strictly bigger than the line spanned by $\rho_a$. □

3.7. Asymptotic form. Our last task is to show that a Ricci flat metric which is projectively compact of order one admits an asymptotic form as discussed in Section 2.4, at least around points in the open curved orbits in the boundary. (In particular, this will always be true if the initial metric is Riemannian.) As we have noted there, the asymptotic form will only be available for specific defining functions, so we have to specialize the defining function appropriately.

Lemma 19. In the setting of Theorem 18 assume that $U \cap \partial M \subset \partial M_+ \cup \partial M_-$. Then, possibly shrinking $U$, one can modify the defining function $\rho$ to $\tilde{\rho}$ in such a way that the boundary value of $\tilde{\rho}^{-3}g^{ij}\tilde{\rho}_j$ vanishes identically and the function $\tilde{\rho}^{-4}g^{ij}\tilde{\rho}_i\tilde{\rho}_j$ is of the form $\nu_0 + \tilde{\rho}^2\nu_2$ for a non–zero constant $\nu_0$ and a smooth function $\nu_2$ on $U$.

Proof. By assumption, the boundary value $\tau^{ij}$ of $\rho^{-2}g^{ij}$ satisfies $\tau^{ij}\rho_j = 0$ and it has rank $n$. This implies that viewing $\tau^{ij}$ as a map from the cotangent space to the tangent space, its image will be the full annihilator of $\rho_j$. Since the boundary value $\lambda^i$ of $\rho^{-3}g^{ij}\rho_j$ satisfies $\lambda^i\rho_i = 0$, we see that there is a one–form $\varphi_j$ such that $\tau^{ij}\varphi_j = -\lambda^i$ and $\varphi_j$ is actually unique up to adding a some function times $\rho_j$.

Now denoting by $\tilde{\nabla}$ the connection in the projective class corresponding to the defining function $\rho$, we can determine what the fact that $L(\sigma^{-2}g^{-1})$ is parallel means in the splitting determined by $\tilde{\nabla}$. Using the formula for the tractor connection on $S^2T$ formula (7)] and the fact that $\sigma^{-2}g^2$ is parallel for $\tilde{\nabla}$, we get

$$\tilde{\nabla}_a(\rho^{-2}g^{ij}) = \rho^{-3}\delta^i_a g^{jk}\rho_k + \rho^{-1}\delta^i_a g^{jk}\rho_k.$$  

(17)

$$\tilde{\nabla}_a(\rho^{-3}g^{ik}\rho_k) = +\rho^{-4}\delta^i_a g^{jk}\rho_j\rho_k - \rho^{-2}\tilde{\nabla}_a g^{ij}.$$  

(18)

$$\tilde{\nabla}_a\rho^{-4}g^{jk}\rho_j\rho_k = -2\rho^{-3}\tilde{\nabla}_a g^{jk}\rho_k.$$  

(19)

Now we use this to compute $\tau^{ai}\tau^{bj}\tilde{\nabla}_i \varphi_j = \tau^{ai}\tilde{\nabla}_i \tau^{bj}\varphi_j - \tau^{ai}\varphi_j \tilde{\nabla}_i \tau^{bj}$. In the first summand, we just get a linear combination of $\tau^{ab}$ and $\tau^{ai}\tilde{\nabla}_i \tau^{bj}$. Since $\tilde{\nabla}$ is a special affine connection, $\tilde{\nabla}_i \tau^{bj}$ is symmetric, so the terms are symmetric in $a$ and $b$. From the second summand, we get $-\tau^{ai}\varphi_j \delta^b_i \lambda^j - \tau^{ai}\varphi_j \delta^b_i \lambda^b$ so this again adds a multiple of $\tau^{ab}$ plus $\lambda^a\lambda^b$. Thus we conclude that $\tau^{ai}\tau^{bj}\tilde{\nabla}_i \varphi_j$ is symmetric in $a$ and $b$, or otherwise put, the alternation of $\tilde{\nabla}_i \varphi_j$, which equals $d\varphi$ as $\tilde{\nabla}$ is torsion free, contracts trivially into $\tau^{ai}\tau^{bj}$. Since $\rho_j$ spans the null space of $\tau^{ab}$, this implies that $d\varphi = \psi \wedge df$ for some one–form $\psi$. In particular, this shows that the restriction of $\varphi$ to a one–form on $\partial M$ is closed. Possibly shrinking $U$, we may assume that there is a smooth function $f: U \cap \partial M \to \mathbb{R}$ such that $\varphi|_{\partial M} = df$. Extending $f$ arbitrarily to $U$, we conclude that the one–form $f_i := df$ has the property that $\tau^{ij}f_j = -\lambda^i$. 


Now we define $\tilde{\rho} := e^f \rho$, which implies that $\tilde{\rho} \rho_j = \tilde{\rho} f_j + e^f \rho_j$. Thus we get
\[
\tilde{\rho}^{-3} g^{i\bar{j}} \tilde{\rho}_{\bar{j}} = \tilde{\rho}^{-2} g^{i\bar{j}} f_j + e^f \tilde{\rho}^{-3} g^{i\bar{j}} \rho_j = e^{-2f}(\rho^{-2} g^{i\bar{j}} f_j + \rho^{-3} g^{i\bar{j}} \rho_j),
\]
and the term in the bracket goes to zero at the boundary by construction. Next, consider $\tilde{\rho}^{-4} g^{i\bar{j}} \tilde{\rho} \rho_j =: \nu$. The analog of (19) for $\tilde{\rho}$ implies that $\tilde{\nabla}_a \nu$ vanishes identically along the boundary. In particular, $\nu$ equals some constant $\nu_0$ along the boundary, and this constant must be non–zero, since we know that $L(\sigma^{-2}g^{-1})$ has rank $n+1$ everywhere. Hence $\nu - \nu_0$ vanishes along the boundary and thus is of the form $\tilde{\rho} \nu_1$ for some function $\nu_1$ which is smooth up to the boundary. Differentiating, we get $d\nu = \tilde{\rho} d\nu_1 + \nu_1 d\tilde{\rho}$. But we know that $d\nu$ vanishes identically along the boundary, so inserting a vector field $\xi$ such that $d\tilde{\rho}(\xi) = 1$, we see that $\nu_1$ vanishes along the boundary and thus can be written as $\tilde{\rho} \nu_2$ for a function $\nu_2$ which is smooth up to the boundary. \[\square\]

**Proposition 20.** In the setting of Theorem 18 assume that the defining function $\rho$ has the additional properties derived for $\tilde{\rho}$ in Lemma 19 above. Then putting $\nu = \rho^{-4} g^{i\bar{j}} \rho_a \rho_b$, the tensor field $h = \rho^2 g + \frac{1}{\nu} \frac{d\rho}{d\tilde{\rho}}$ satisfies the hypothesis of Theorem 18 (for $\alpha = 1$).

Proof. By assumption, $\rho^{-3} g^{i\bar{j}} \rho_j$ goes to zero on the boundary, so it is of the form $\rho t^i$ for some vector field $t^i$ which is smooth up to the boundary. Moreover, by construction $t^i \rho_i = \nu$. Now we define a tensor field $h^{i\bar{j}}$ on $U$ by
\[(20) h^{i\bar{j}} := \frac{1}{\rho} g^{i\bar{j}} - \frac{e^f}{\nu} t^i t^{\bar{j}}.\]

By Theorem 18 this is smooth up to the boundary and the boundary value coincides with the one of $\rho^{-2} g^{i\bar{j}}$ and thus has rank $n$ with its null–space spanned by $\rho_j$. By definition, $t^i \rho_j = \nu$, which shows that $h^{i\bar{j}} \rho_j = \frac{1}{\rho} g^{i\bar{j}} \rho_j - \rho^2 t^i = 0$. On the other hand, on the kernel of $t^i$, $h^{i\bar{j}}$ evidently coincides with $\frac{1}{\rho} g^{i\bar{j}}$ so it is non–degenerate there.

Now from (20), we see that $g^{i\bar{j}} = \rho^2 h^{i\bar{j}} - \frac{e^f}{\nu} t^i t^{\bar{j}}$. This represents an orthogonal decomposition of $T^* M$ with respect to $g^{i\bar{j}}$ into the line spanned by $d\rho$ and the kernel of $t^i$, so the decomposition of the space extends to the boundary. Dually, one obtains an orthogonal decomposition for $g_{i\bar{j}}$ into $\ker(d\rho)$ and the line spanned by $t^i$. With respect to this decomposition, the metric $g_{i\bar{j}}$ then clearly is the sum of $\rho^{-2} h_{i\bar{j}}$ (where $h_{i\bar{j}}$ is the inverse of $h^{i\bar{j}}$ on $\ker(d\rho)$, extended by zero on the line spanned by $t^i$) and some multiple of $\rho_i \rho_j$. This multiple can be computed by observing that $g_{i\bar{j}} t^i t^{\bar{j}} = \rho^{-4} \nu$, which shows that
\[g_{i\bar{j}} = \rho^{-2} h_{i\bar{j}} + \rho^{-4} \nu^{-1} \rho_i \rho_j.\]

To complete the proof, it thus suffices to show that the function $\nu^{-1}$ satisfies the assumptions of Theorem 18. But $d\nu^{-1} = -\nu^{-2} d\nu$, so this vanishes identically along the boundary. As in the proof of Lemma 19 we thus conclude that $\nu^{-1} = (1/\nu_0) + \rho^2 \tilde{\nu}$ for some function $\tilde{\nu}$ which is smooth up to the boundary. Thus $d\nu^{-1} = 2 \tilde{\rho} \nu \tilde{\nu} d\rho + \rho^2 d\tilde{\nu}$. For a vector field $\zeta$ on $U$ such that $d\rho(\zeta) = 0$, we thus have $\zeta \cdot \nu^{-1} = \rho^2 (\zeta \cdot \tilde{\nu})$, which completes the proof. \[\square\]
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