A priori error estimation for elastohydrodynamic lubrication using interior-exterior penalty approach

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Abstract
In the present study, an interior-exterior penalty discontinuous Galerkin finite element method (DG-FEM) is analysed for solving Elastohydrodynamic lubrication (EHL) line and point contact problems. The existence of discrete penalized solution is examined using Brouwer’s fixed point theorem. Furthermore, the uniqueness of solution is proved using Lipschitz continuity of the discrete solution map under light load parameter assumptions. A priori error estimates are achieved in $L^2$ and $H^1$ norms which are shown to be optimal in mesh size $h$ and suboptimal in polynomial degree $p$. The validity of theoretical findings are confirmed through series of numerical experiments.

KEYWORDS: Elasto-hydrodynamic lubrication, Discontinuous Finite Element Method, interior-exterior penalty method, pseudo-monotone operators, quasi-variational inequality.
1 Introduction

Discontinuous Galerkin finite element methods (DG-FEMs) are now widely used in scientific computation to achieve better accuracy and their flexibility in handling nonuniform degrees of approximation as well as flexibility in local mesh adaptivity. While DG-FEMs have been profoundly shown to be very successful, the theory ensuring the convergence of the algorithm and the advantages over Elastohydrodynamic Lubrication (EHL) line and point contacts problems still under development. Recently, several results have been obtained for DG-FEMs for strongly nonlinear elliptic partial differential equations (see for example [2, 6]).

This paper presents the mathematical analysis of DG-FEMs for EHL problems using interior-exterior penalty approach. However, the idea is more generic and it can be easily extended to more general variational inequality problems too. In this study, author demonstrates priori error estimate in the underline norm, prove existence and uniqueness of the discrete DG-FEM formulation of the EHL problem and find the optimal and suboptimal rate of convergence under discussed norm. Recently, a nearby method to approximate the EHL solution is to solve the analogue discrete inequality using discontinuous Galerkin finite volume method (DG-FVM) approach discussed (see for example [5]) which is restricted under lower regularity assumptioms of penalized soultion of EHL point contact problem.

Exterior penalty methods has a significant role in from of conceptual perspectve. One straigt way justification is, it convert inequalities to equation. A priori estimates for finite element methods for elliptic obstacle problems were proved in [4] using regularization technique imposing the unilateral constraint approximately through a penalty term depending on a regularization parameter $\epsilon$ and relating the mesh size $h$ of the finite element mesh to the regularization parameter $\epsilon$. By motivated by the same route, in this article author prove priori error estimate of discontinuous Galerkin-finite element methods using interior-exterior penalty approach.

1.1 Continuous EHL Model Problems

1.1.1 Line contact model

Two cylinder rolling in the positive $x$-direction seperated with lubricant (oil, liquid etc) are modelled in the form of variational inequality as

$$
\frac{\partial}{\partial x} \left( \epsilon \frac{\partial u}{\partial x} \right) \leq \frac{\partial (\rho h)}{\partial x} \tag{1}
$$

$$
u \geq 0 \tag{2}
$$

$$
u, \left[ \frac{\partial}{\partial x} \left( \epsilon \frac{\partial u}{\partial x} \right) - \frac{\partial (\rho h)}{\partial x} \right] = 0, \tag{3}
$$

where

$$
\epsilon = \frac{\rho h^3}{\eta \lambda_{\text{line}}}. \tag{4}
$$
Figure 1: Line contact schematic diagram

Figure 2: Point contact schematic diagram
and $h$ are the dimensionless pressure and film thickness, $\bar{\rho}(u)$ and $\bar{\eta}(u)$ are dimensionless density and viscosity, and $\lambda_{\text{line}}$ is a dimensionless speed parameter:

$$
\lambda_{\text{line}} = \frac{6\eta_0 v_s R^2}{b^3 p_H},
$$

where $\eta_0 = 0.04$ (ambient pressure viscosity), $v_s = v_1 + v_2$ (sum of velocity), $p_H = \frac{F_0}{4R_e}$ (maximum Hertzian pressure), $R_e = 0.02$ (reduced radius of curvature) and $b = \frac{4.0R_e}{\sqrt{W(2.0\pi)}}$ (half width Hertzian contact). $G_0 = 3500$ (material parameter), $U = 7.3 \times 10^{-11}$ (dimensionless speed parameter), $W = 1.3 \times 10^{-4}$ (dimensionless load parameter), $h_{\text{line}} = 0.0000015042$, $\alpha = 1.59\times 10^{-8}$, $E = \frac{G_0}{\alpha}$, $z = \frac{(5.1 \times 10^{-9} (\log \eta_0 + 9.67))}{(b^3 p_H)}$, $U = \frac{(\eta_0 u_s)}{(b^3 p_H)}$.

The nondimensionless viscosity $\bar{\eta}$ is defined according to

$$
\bar{\eta}(u) = e^{\left\{\left(\frac{\alpha p_0}{z}\right)\left(\frac{1}{1+\left(\frac{up_H}{p_0}\right)}\right)\right\}}, \quad (4)
$$

where $p_0 = 1.98 \times 10^{-8}$. Dimensionless density $\bar{\rho}$ is given by

$$
\bar{\rho}(u) = \frac{0.59 \times 10^9 + 1.34up_H}{0.59 \times 10^9 + up_H}. \quad (5)
$$

The nondimensionalized film thickness equation can be written as

$$
h(x) = h_{\text{line}} + \frac{x^2}{2} - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x') \ln|x-x'|dx' \quad (6)
$$

where $h_{\text{line}}$ is a constant.

Dimensionless force balance equation is read as

$$
\int_{-\infty}^{\infty} u(x')dx' - \frac{\pi}{2} = 0 \quad (7)
$$

Define Dirichlet boundary condition by taking sufficiently large bounded domain as

$$
u = 0 \quad \text{on} \quad \partial \Omega. \quad (8)
$$

The film thickness equation is in dimensionless form is written as follows

$$
h_d(x) = h_0 + \frac{x^2}{2} - \frac{1}{\pi} \int_{\Omega} \log|x-x'|u(x')dx', \quad (9)
$$

where $h_0$ is an integration constant.

The dimensionless force balance equation is defined as follows

$$
\int_{-\infty}^{\infty} u(x')dx' = \frac{\pi}{2} \quad (10)
$$

1.1.2 Point contact model

Let strongly nonlinear EHL model problem of a ball rolling in the positive $x$-direction gives rise to a variational inequality defined below as

$$
\frac{\partial}{\partial x} \left( e^u \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( e^u \frac{\partial u}{\partial y} \right) \leq \frac{\partial (ph)}{\partial x} \quad (11)
$$
$$u \geq 0 \quad (12)$$

$$u \left[ \partial \frac{\partial \epsilon u}{\partial x} + \frac{\partial}{\partial y} \left( \epsilon \frac{\partial u}{\partial y} \right) - \frac{\partial (\rho h)}{\partial x} \right] = 0, \quad (13)$$

Here term $\epsilon$ is defined as:

$$\epsilon = \frac{\rho H^3}{\eta \lambda},$$

where $\rho$ is dimensionless density of lubrication, $\eta$ is dimensionless viscosity of lubrication and speed parameter

$$\lambda = \frac{6\eta_0 u s R^2}{a^3 p_H}. \quad (14)$$

The non-dimensionless viscosity $\eta$ is defined according to

$$\eta(u) = \exp \left\{ \left( \frac{\alpha p_0}{z} \right) \left( -1 + \left( 1 + \frac{u p_H}{p_0} \right)^2 \right) \right\}. \quad (15)$$

Dimensionless density $\rho$ is given by

$$\rho(u) = \frac{0.59 \times 10^9 + 1.34 u p_H}{0.59 \times 10^9 + u p_H}. \quad (16)$$

The term film thickness $H$ of lubricant is written as follows

$$H(x, y) = H_{00} + \frac{x^2}{2} + \frac{y^2}{2} + \frac{2}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{u(x', y')}{\sqrt{(x-x')^2 + (y-y')^2}} \, dx' \, dy'. \quad (17)$$

Consider the ball is elastic whenever load is large enough. Then system 11–10 and 10–10 form line and point contact Elasto-hydrodynamic Lubrication model respectively. Schematic diagrams of EHL model is given in 2 and 1 in the form of undeformed and deformed contacting body structure respectively.

The remainder of the article is organized as follows. In section 2 variational inequality and its notation is established; Furthermore, existence results are proved for our model problem; In section 3 DG-FEM notation and the proposed method is demonstrated; In section. ?? Error estimates are proved in $L^2$ and $H^1$ norm; In section. ?? numerical experiment and graphical results are provided; At last section. 4 conclusion and future direction is mentioned.

## 2 Variational Inequality

We consider space $V = H^1_0(\Omega)$ and its dual space as $V^* = (H^1_0(\Omega))^* = H^{-1}(\Omega)$. Also define notion $(\cdot, \cdot)$ as duality pairing on $V^* \times V$. Further assume that $C$ is closed convex subset of $V$ defined by

$$C = \left\{ v \in V : v \geq 0 \text{ a.e. } \in \Omega \right\}. \quad (19)$$
Additionally, we define the operator $\mathcal{T}$ as

$$\mathcal{T} : u \mapsto -\left[ \frac{\partial}{\partial x} \left( \epsilon \star \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \epsilon \star \frac{\partial u}{\partial y} \right) \right] + \frac{\partial (\rho h_d)}{\partial x}$$

Then, for a given $f \in \mathcal{V}^*$, the problem of finding an element $u \in \mathcal{C}$ such that

$$\langle \mathcal{T}(u) - f, v - u \rangle \geq 0, \quad \forall v \in \mathcal{C}.$$  

Throughout the article, we shall assume that there exists $\epsilon_1, M_* \in \mathbb{R}^+$ such that

$$0 < \epsilon_1 \leq \epsilon(u) \leq M_* \quad \forall \varsigma \in \Omega \quad \text{and} \quad u \in \mathbb{R}.$$  

**Definition 2.1.** Operator $\mathcal{T} : \mathcal{C} \subset \mathcal{V} \to \mathcal{V}^*$ is said to be pseudo-monotone if $\mathcal{T}$ is a bounded operator and whenever $u_k \rightharpoonup u$ in $\mathcal{V}$ as $k \to \infty$ and

$$\limsup_{k \to \infty} \langle \mathcal{T}(u_k), u_k - u \rangle \leq 0,$$

it follows that for all $v \in \mathcal{C}$

$$\liminf_{k \to \infty} \langle \mathcal{T}(u_k), u - v \rangle \geq \langle \mathcal{T}(u), u - v \rangle.$$

**Definition 2.2.** Operator $\mathcal{T} : \mathcal{V} \to \mathcal{V}^*$ is said to be hemi-continuous if and only if the function $\phi : t \mapsto -\langle \mathcal{T}(tx + (1-t)y), x - y \rangle$ is continuous on $[0,1]$ for all $x, y \in \mathcal{V}$. On this context the following existence theorem has been proved by Oden and Wu [3] by assuming constant density and constant viscosity of the lubricant. However, idea is easily extend-able for more realistic operating condition in which density and viscosity of the lubricant are depend on its applied pressure see Appendix. ?? A straight forward modification of the analysis of [3] yields the theorem below and so we will omit the proof.

**Theorem 2.1.** [3] Let $\mathcal{C}(\neq \emptyset)$ be a closed, convex subset of a reflexive Banach space $\mathcal{V}$ and let $\mathcal{T} : \mathcal{C} \subset \mathcal{V} \to \mathcal{V}^*$ be a pseudo-monotone, bounded, and coercive operator from $\mathcal{C}$ into the dual $\mathcal{V}^*$ of $\mathcal{V}$, in the sense that there exists $y \in \mathcal{C}$ such that

$$\lim_{\|x\| \to \infty} \frac{\langle \mathcal{T}(x), x - y \rangle}{\|x\|} = \infty.$$  

Let $f$ be given in $\mathcal{V}^*$ then there exists at least one $u \in \mathcal{C}$ such that

$$\langle \mathcal{T}(x) - f, y - x \rangle \geq 0 \quad \forall y \in \mathcal{C}.$$  

In the next section, we will give a complete formulation as well as will give theoretical justification for existence of our model problem in discrete computed setting.

**3 Discrete Formulation of DG-FEM**

Let $\mathcal{P}_h = \cup_{i \in \mathcal{J}_h} \{K_i\}$ is a discontinuous finite element partition of domain $\Omega$, where $\mathcal{J}_h := \{i; 1 \leq i \leq N_h\}$. We define $\mathbb{P}_p(K)$ as the space of polynomials of
total degree less than or equal $p_i$ on the master rectangle $K_i = [-1, 1] \times [-1, 1]$. Let $\mathcal{S}_{p_i}(K_i)$ denote $P_{p_i}(K_i)$ whenever $K_i$ is a master rectangle.

We define finite dimensional discontinuous space as

$$D_{p_h} = \{ v \in L^2(\Omega) : v|_{K_i} \in \mathcal{S}_{p_i}(K_i), v|_{\partial \Omega} = 0 \ \forall K_i \in \mathcal{P}_h \},$$

(28)

where $p = \min\{p_i \geq 1; 1 \leq i \leq N_h\}$.

Let $e_k$ be an interior edge shared by two elements $K_i$ and $K_j$ in $\mathcal{P}_h$ and let $N_i$ and $N_j$ be unit normal vectors on $e_k$ pointing exterior to $K_i$ and $K_j$ respectively.

We define average $\{\cdot\}$ and jump $[\cdot]$ on $e_k$ for scalar $q$ and vector $w$, respectively, as

$$\{q\} = \frac{1}{2}(q|_{\partial K_i} + q|_{\partial K_j}), \quad [q] = (q|_{\partial K_i}N_i + q|_{\partial K_j}N_j)$$

$$\{w\} = \frac{1}{2}(w|_{\partial K_i} + w|_{\partial K_j}), \quad [w] = (w|_{\partial K_i}N_i + w|_{\partial K_j}N_j).$$

If $e_k$ is a edge on the boundary of $\Omega$, we define $q = q$, $[w] = w$. Let $\Gamma$ denote the union of the boundaries of the triangle $K$ of $\mathcal{P}_h$ and $\Gamma_0 := \Gamma \setminus \partial \Omega$.

We define

$$H^s(\Omega, \mathcal{P}_h) := \{ v \in L^2(\Omega) : v|_{K_i} \in H^s(K_i), \ \forall K_i \in \mathcal{P}_h \}.\quad (29)$$

Let $v \in H^2(\Omega, \mathcal{P}_h)$, then we define the following mesh dependent norm $|||\cdot|||$ and $|||\cdot|||_\nu$ as

$$|||v|||^2 = \sum_{i=1}^{N_h} \int_{K_i} |\nabla v|^2 dx + \sum_{e_k \in \Gamma} a_k \frac{p_k^2}{|e_k|^3} \int_{e_k} [v]^2 \quad (30)$$

$$|||v|||_\nu^2 = \sum_{i=1}^{N_h} \int_{K_i} |\nabla v|^2 dx + \sum_{e_k} \frac{|e_k|^3}{p_k^2} \int_{e_k} \left\{ \frac{\partial v}{\partial \nu} \right\}^2 ds + \sum_{e_k \in \Gamma} a_k \frac{p_k^2}{|e_k|^3} \int_{e_k} [v]^2.\quad (31)$$

### 3.1 Exterior penalty solution approximation

In this section, we introduce an exterior penalty term to regularize the inequality constraint \[11\] \[10\]. We define a exterior penalty operator $\xi : H^1_0(\Omega) \rightarrow H^{-1}$ as

$$\xi(u) = u^-,$$

(32)
where \( u^- = u - \max(u, 0) = \frac{u - |u|}{2} \). Let us define exterior penalty problem, \((\mathcal{W}_{\epsilon_p})\): for \( \epsilon_p > 0 \), find \( u_{\epsilon_p} \in H^2(\Omega) \) such that

\[
\langle \mathcal{T}(u_{\epsilon_p}), v \rangle + \langle \xi(u_{\epsilon_p}), v \rangle/\epsilon_p = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega),
\]

(33)

where \( \epsilon \) is an arbitrary small positive number (\( \epsilon = 1.0 \times 10^{-6} \)).

**Lemma 3.1.** Penalty operator \( \xi : V' \rightarrow V'' \) is monotone, coercive and bounded.

Now from regularity theory (see reference [1]), it is easy to show that there exists a unique solution \( u_{\epsilon_p} \in H^2(\Omega) \) such that

\[
||u_{\epsilon_p}||_{H^2(\Omega)} \leq C||f_{\epsilon_p}||_0,
\]

(34)

where \( f_{\epsilon_p} = f - \xi/\epsilon_p(u_{\epsilon_p}) \).

### 3.2 Weak Formulation

Reconsider the problem of the type

\[
-\frac{\partial}{\partial x} \left( \epsilon(u) \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( \epsilon(u) \frac{\partial u}{\partial y} \right) + \frac{\partial (\rho h)}{\partial x} + \frac{1}{\epsilon_p} \xi(u) = 0 \quad \text{in } \Omega
\]

(35)

\[
u = 0 \quad \text{on } \partial \Omega,
\]

(36)

where all notation has their usual meaning.

For given \( u, v \in H^2(\Omega, \mathcal{P}_h) \) and for fixed value of \( \Phi, h_d \in H^2(\Omega, \mathcal{P}_h) \), define bilinear form as

\[
\langle \mathcal{T}(\Phi; u), v \rangle = \sum \int_{K_i} \epsilon^*(\Phi) \nabla u \nabla v ds + \sum \int_{K_i} \epsilon_p \xi(\Phi) v ds
\]

\[
- \sum_{\kappa \in \mathcal{E}_h, e_k} [v][\epsilon^*(\Phi) \nabla u.n] ds + \sum_{\kappa \in \mathcal{E}_h} a_k \frac{h_k^2}{|e_k|^2} \sum |u[v]| ds
\]

\[
- \sum_{K \in \mathcal{A}_h} \int_{K} (\rho(\Phi) h_d(x)), (\beta.n) \nabla v ds + \sum_{\kappa \in \mathcal{E}_h} \int_{e_k} [v][\epsilon(\Phi) h_d(x)), (\beta.n)] ds.
\]

(37)

Now we will state few lemmas and inequalities without proof which will be later helpful in our subsequent analysis.

**Lemma 3.2 (Interpolation Error Estimates).** For \( u \in H^s(K_i) \), there exist a positive constant \( C_A \) and an interpolation value \( u_I \in \mathcal{V}_h \), such that

\[
||u - u_I||_{s,K} \leq C_A h^{2-s} ||u||_{2,K}, \quad s = 0, 1.
\]

(38)

**Trace inequality.** We state without proof the following trace inequality. Let \( \phi \in H^2(K) \) and for an edge \( e \) of \( K \),

\[
||\phi||_e^2 \leq C(h_e^{-1} |\phi|_K + h_e |\phi|_{1,K}^2).
\]

(39)

Next lemma provides us a bound of film thickness term and later helpful in proving coercivity and error analysis.

**Lemma 3.3.** For \( h_d \) defined in equation [x], \( 0 < \beta_* < 1, s = 2 - \beta_*/(1 - \beta_*) > 2 \) there exist \( C_1 \) and \( C_2 > 0 \) such that

\[
\max_{x,y \in \Omega} |h_d(u)| \leq C_1 + C_2 ||u||_{L^s}, \quad 0 < \beta_* < 1, \quad \forall (x,y) \in \Omega.
\]

(40)
Lemma 3.4. The operator $T$ defined in equation (19) is bounded as a map from $V$ into $V^\ast$.

Lemma 3.5. The operator $T$, defined in equation (21) is hemi-continuous, that is, $\forall u, v, w \in V$

\[
\lim_{t \to 0^+} \langle T(u + tv), w \rangle = \langle T(u), w \rangle.
\]

Lemma 3.6. The operator defined on equation (21) is coercive i.e. there is a constant $C$ independent of $h$ such that for $\alpha_1$ large enough and $h$ is small enough

\[
\langle T(u; u_h), u_h \rangle \geq C\|u_h\|^2 \quad \forall u_h \in V_h
\]

3.3 Linearization

\[-\nabla (\epsilon^*(u) \nabla \psi + \epsilon_u^*(u) \nabla u \psi ) + \nabla (\bar{\beta}(\rho d + (\rho d)u \psi )) = \phi_h \text{ in } \Omega
\]

\[
\psi = 0 \text{ on } \partial \Omega.
\]

and $\psi$ satisfying the elliptic regularity as

\[
\|\psi\|_{H^2(\Omega)} \leq C\|\phi_h\|_0.
\]

We seek $u_h \in D_h^0(P_h)$ such that

\[
\mathcal{B}(u; u, v_h) = \mathcal{B}(u_h; u_h, v_h)
\]

Now from Taylor’s series expansion we get

\[
\epsilon^*(w) = \epsilon^*(u) + \tilde{\epsilon}_u^*(w)(w - u),
\]

where $\tilde{\epsilon}_u^*(w) = \int_0^1 \epsilon_u^*(w + t[u - w])dt$

and

\[
\epsilon^*(w) = \epsilon^*(u) + \epsilon_u^*(u)(w - u) + \tilde{\epsilon}_u^*(w)(w - u)^2,
\]

where $\tilde{\epsilon}_u^*(w) = \int_0^1 (1 - t)\epsilon_u^*(w + t[w - u])dt$. Consider the following bilinear form $\tilde{\mathcal{B}}$ as

\[
\tilde{\mathcal{B}}(\psi; w, v) = \mathcal{B}(\psi; w, v) + \sum_{i=1}^{N_h} \int_{K_i} (\epsilon_u(\psi) \nabla \psi) w \nabla v - \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ \epsilon_u(\psi) \frac{\partial \psi}{\partial \nu} w \right\}[v]
\]

\[
- \sum_{i=1}^{N_h} \int_{K_i} (\bar{\beta} \cdot n)(\rho d)u \psi \nabla v + \sum_{e_k \in \Gamma_I} \int_{e_k} ((\rho d)u(\bar{\beta} \cdot n)w)[v].
\]

Note that $\tilde{\mathcal{B}}$ is linear in $w$ and $v \in H^2(\Omega, P_h)$ for fixed value $\psi$. It is clear from the assumptions on $\epsilon(u)$ and (from existence lemma (state the precise lemma here) for nonlinear elliptic PDE ) we have a unique solution $\psi \in H^2(\Omega)$ to the following elliptic problem :

\[-\nabla (\epsilon(u) \nabla \psi + \epsilon_u(u) \nabla u \psi ) + \nabla (\bar{\beta}(\rho d + (\rho d)u \psi )) = \phi_h \text{ in } \Omega.
\]

\[
\psi = 0 \text{ on } \partial \Omega.
\]
and \( \psi \) satisfies the following elliptic regularity condition

\[
||\psi||_{H^2(\Omega)} \leq C||\phi_h||_0. \tag{51}
\]

Now we will linearize problem (50) around \( I_h u \) which will be helpful in deriving few estimates. Subtracting \( \mathcal{B}(u; u, v) \) from both side of equation (50) we get

\[
\begin{align*}
\mathcal{B}(u; e, v_h) &= \sum_{i=1}^{N_h} \int_{K_i} (\epsilon(u_h) - \epsilon(u)) \nabla u_h \nabla v_h - \sum_{i=1}^{N_h} \int_{K_i} (\epsilon(u_h) - \epsilon(u)) \frac{\partial u_h}{\partial \nu} [v_h]
- \theta \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\epsilon(u_h) - \epsilon(u)) \frac{\partial u_h}{\partial \nu} \right\} [v_h]
- \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\rho_h)(u_h) - (\rho_h)(u) \right\} \vec{\beta} \cdot \mathbf{n} \, v_h [v_h] ds.
\end{align*}
\]

Since \( [u] = 0 \) on each \( e_k \in \Gamma_I \), we rewrite the equation as

\[
\begin{align*}
\mathcal{B}(u; e, v_h) &= \sum_{i=1}^{N_h} \int_{K_i} (\epsilon(u_h) - \epsilon(u)) \nabla (u_h - u) \nabla v_h + \sum_{i=1}^{N_h} \int_{K_i} (\epsilon(u_h) - \epsilon(u)) \nabla u \nabla v_h
- \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\epsilon(u_h) - \epsilon(u)) \frac{\partial (u_h - u)}{\partial \nu} \right\} [v_h]
- \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\epsilon(u_h) - \epsilon(u)) \frac{\partial u}{\partial \nu} \right\} [v_h]
- \theta \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\epsilon(u_h) - \epsilon(u)) \frac{\partial v_h}{\partial \nu} \right\} [u_h - u]
- \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\rho_h)(u_h) - (\rho_h)(u) \right\} \vec{\beta} \cdot \mathbf{n} \, v_h [v_h] ds.
\end{align*}
\]

Now adding both side by

\[
\begin{align*}
- \sum_{i=1}^{N_h} \int_{K_i} \epsilon(u)(u_h - u) \nabla u \nabla v_h + \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ \epsilon(u)(u_h - u) \frac{\partial u}{\partial \nu} \right\} [v_h]
- \sum_{i=1}^{N_h} \int_{K_i} \left\{ (\rho_h)(u)(u_h - u) \right\} \vec{\beta} \cdot \mathbf{n} \, v_h ds + \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\rho_h)(u)(u_h - u) \vec{\beta} \cdot \mathbf{n} \right\} [v_h] ds.
\end{align*}
\]

By writing \( e = u - u_h = u - I_h u + I_h u - u_h \) and using Taylor’s formulae equation (50) we rewrite the term as

\[
\mathcal{B}(u; I_h u - u_h, v_h) = \mathcal{B}(u; I_h u - u, v_h) + \mathcal{B}(u; u_h - u, v_h), \tag{55}
\]
where

\[
\tilde{\mathcal{F}}(u_h; u_h - u, v_h) = \sum_{i=1}^{N_h} \int_{K_i} \tilde{e}_u(u_h) e \nabla \epsilon \cdot \nabla v_h + \sum_{i=1}^{N_h} \int_{K_i} \tilde{e}_{uu}(u_h) e^2 \nabla u \cdot \nabla v_h
\]

\[
- \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \tilde{e}_{uu}(u_h) e \frac{\partial u}{\partial v} \right\} [v_h] - \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \tilde{e}_u(u_h) e \frac{\partial \epsilon}{\partial v} \right\} [v_h]
\]

\[
- \theta \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \tilde{e}_u(u_h) e \frac{\partial v_h}{\partial v} \right\} [\epsilon] + \sum_{i=1}^{N_h} \int_{K_i} \left\{ (\rho \tilde{h}_d)_{uu}(u_h) e^2 \right\} (\tilde{\beta} h \nabla v_h) ds
\]

\[
- \sum_{e_k \in \Gamma} \int_{e_k} \left\{ (\rho \tilde{h}_d)_{uu}(u_h) e^2 \right\} (\tilde{\beta} h \nabla v_h) ds. \tag{56}
\]

### 3.4 Existence and Uniqueness

For a given \( z \in D_h^0(\mathcal{P}_h) \), let \( S_h : D_h^0(\mathcal{P}_h) \to D_h^0(\mathcal{P}_h) \) be a mapping define as \( y = S_h z \in D_h^0(\mathcal{P}_h) \) and satisfies

\[
\tilde{\mathcal{B}}(u; I_h u - y, v_h) = \tilde{\mathcal{B}}(u; I_h u - u, v_h) + \tilde{\mathcal{F}}(z; z - u, v_h) \forall v_h \in D_h^0(\mathcal{P}_h). \tag{57}
\]

**Lemma 3.7.** Consider \( \beta \geq 1 \) and \( z, v_h \in D_h^0(\mathcal{P}_h) \) also define \( \chi = z - \Pi_h u \) and \( \eta = u - \Pi_h u \), then there exist a constant \( C \) independent of \( h \) and \( p \) such that the following condition satisfies

\[
|F(z; z - u, v_h)| \leq CC \left( \left( \max_{1 \leq i \leq N_h} \frac{p_i}{h_i} \right)^{1/2} \|\chi\|^2 + C \|\Delta u\| \right) \|v_h\|. \tag{58}
\]

**Proof.** Let \( z \in D_h^0(\mathcal{P}_h) \) and set \( \vartheta = z - u \). Now consider equation (4.13) and substitute \( u_h \) by \( z \) and by \( z - u \) to get

\[
F(z; \vartheta, v_h) = \sum_{i=1}^{N_h} \int_{K_i} \tilde{e}_u(z) \vartheta \nabla \vartheta \cdot \nabla v_h + \sum_{i=1}^{N_h} \int_{K_i} \tilde{e}_{uu}(z) \vartheta^2 \nabla u \cdot \nabla v_h
\]

\[
- \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \tilde{e}_{uu}(z) \vartheta^2 \nabla u \cdot n \right\} [v_h] - \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \tilde{e}_u(z) \vartheta \nabla \vartheta \cdot n \right\} [v_h]
\]

\[
- \theta \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \tilde{e}_u(z) \vartheta \nabla v_h \cdot n \right\} [\vartheta] + \sum_{i=1}^{N_h} \int_{K_i} (\rho \tilde{h}_d)_{uu}(z) \vartheta^2 \tilde{\beta} h \nabla v_h
\]

\[
- \sum_{e_k \in \Gamma} \int_{e_k} \left\{ (\rho \tilde{h}_d)_{uu}(z) \vartheta^2 \tilde{\beta} h \nabla v_h \right\} [v_h]. \tag{59}
\]

Now putting the value of \( \vartheta = \chi - \eta \), where \( \chi = z - \Pi_h u \) and \( \eta = u - \Pi_h u \) in above equation (35) and estimating the right hand side term we get First part of the right hand side of equation (35) is approximated as

\[
|I| = \sum_{i=1}^{K_h} \int_{K_i} \tilde{e}_u(z) \vartheta \nabla \vartheta \cdot \nabla v_h \leq \sum_{i=1}^{K_h} \int_{K_i} \tilde{e}_u(z) \vartheta \nabla \chi \cdot \nabla v_h
\]

\[
\left| \sum_{i=1}^{K_h} \int_{K_i} \tilde{e}_u(z) \chi \nabla \eta \cdot \nabla v_h \right| + \left| \sum_{i=1}^{K_h} \int_{K_i} \tilde{e}_u(z) \eta \nabla \chi \cdot \nabla v_h \right| + \left| \sum_{i=1}^{K_h} \int_{K_i} \tilde{e}_u(z) \chi \nabla \eta \cdot \nabla v_h \right|. \tag{60}
\]
Now using inverse inequality estimates stated below without proof as

**Lemma 3.8 (Inverse Inequality).** Suppose $v_h \in Z(K_i)$ and let $r \geq 2$. Then $\exists C > 0$ such that following conditions hold

\[
\|v_h\|_{L^r(K_i)} \leq C_1 h_i^{1-2/r} \|v_h\|_{L^2(K_i)}^{1-1/r} \quad (61)
\]

\[
|v_h|_{H^r(K_i)} \leq C_2 h_i^{2/r-1} |v_h|_{H^{r-1}(K_i)} \quad (62)
\]

\[
\|v_h\|_{L^r(\Omega)} \leq C_1 h_i^{1-2/r} |v_h|_{L^2(\Omega)}^{1-1/r} \quad (63)
\]

Now using inverse inequality estimates stated below without proof as

\[
\|v_h\|_{L^r(K_i)} \leq C_1 h_i^{1-2/r} \|v_h\|_{L^2(K_i)}^{1-1/r} \quad (64)
\]

Now using Holder’s inequality, first part of right hand side of equation (36) is estimated as

\[
\left| \sum_{i=1}^{N_h} \int_{K_i} e_u(z) \chi \nabla \chi \cdot \nabla v_h \right| \leq C_e \left\| \chi \right\|_{L^2(K_i)} \left\| \nabla \chi \right\|_{L^2(K_i)} \left\| \nabla v_h \right\|_{L^2(K_i)} \quad (65)
\]

Similarly, second part of right hand side of equation (36) is approximated as

\[
\left| \sum_{i=1}^{N_h} \int_{K_i} e_u(z) \chi \nabla \eta \cdot \nabla v_h \right| \leq C_e \left\| \chi \right\|_{L^6(K_i)} \left\| \nabla \eta \right\|_{L^6(K_i)} \left\| \nabla v_h \right\|_{L^6(K_i)} \quad (66)
\]

Now using above property of inverse inequality in equation (38) we obtain

\[
\left| \sum_{i=1}^{N_h} \int_{K_i} e_u(z) \chi \nabla \chi \cdot \nabla v_h \right| \leq C_e \left\| \chi \right\|_{L^6(K_i)} \left\| \nabla \chi \right\|_{L^6(K_i)} \left\| \nabla v_h \right\|_{L^6(K_i)} \leq C_e \frac{h^{2/3}}{p^{7/3}} \|u\|_{H^2(\Omega)} \left\| \chi \right\| \left\| v_h \right\| \quad (67)
\]

Now using similar property we can show that

\[
\left| \sum_{i=1}^{N_h} \int_{K_i} e_u(z) \eta \nabla \chi \cdot \nabla v_h \right| \leq C_e \frac{h^{2/3}}{p^{7/3}} \|u\|_{H^2(\Omega)} \left\| \chi \right\| \left\| v_h \right\| \quad (68)
\]

and

\[
\left| \sum_{i=1}^{N_h} \int_{K_i} e_u(z) \eta \nabla \eta \cdot \nabla v_h \right| \leq C_e \frac{h^{2/3}}{p^{7/3}} \|u\|_{H^2(\Omega)} \left\| \eta \right\| \left\| v_h \right\| \quad (69)
\]

hold.

Now second term of right hand side of equation (35) is estimated as

\[
\left| \sum_{i=1}^{N_h} \int_{K_i} e_{uu}(z) \partial^2 \nabla u \cdot \nabla v_h \right| \leq C_e \sum_{i=1}^{N_h} \int_{K_i} \left| \chi \nabla \nabla v_h \right| + C_e \sum_{i=1}^{N_h} \int_{K_i} \left| \eta \nabla \nabla v_h \right| + 2C_e \sum_{i=1}^{N_h} \int_{K_i} \left| \chi \eta \nabla \nabla v_h \right| \quad (70)
\]
Now using Holder’s inequality in right hand side of above equation (46) we have

\[ \sum_{i=1}^{N_h} \int_{K_i} |\chi^2 \nabla u. \nabla v_h| \leq \left( \max_{1 \leq i \leq N_h} \frac{P_i}{h_i} \right)^{1/3} \| \chi \|^2 \| v_h \|. \] (71)

Second part of right hand side of equation (46) is estimated as

\[ \sum_{i=1}^{N_h} \int_{K_i} |\eta^2 \nabla u. \nabla v_h| \leq h^{3/2} \| u \|_{H^1(\Omega)} \| u \|_{W^{1,\infty}(\Omega)} \| \eta \| \| v_h \|. \] (72)

Third part of right hand side of equation (46) is estimated as

\[ \sum_{i=1}^{N_h} \int_{K_i} |\eta \chi \nabla u. \nabla v_h| \leq h^{3/2} \| u \|_{H^1(\Omega)} \| u \|_{W^{1,\infty}(\Omega)} \| \eta \| \| v_h \|. \] (73)

Now putting values from equation (46)-(48) in equation (45) we get the following estimate

\[ \left| \sum_{i=1}^{N_h} \int_{K_i} \epsilon \tilde{u} \theta^2 \nabla u. \nabla v_h \right| \leq C_\epsilon \left( \max_{1 \leq i \leq N_h} \frac{P_i}{h_i} \right)^{1/3} \| \chi \|^2 \| v_h \| + h^{3/2} \| u \|_{H^1(\Omega)} \| u \|_{W^{1,\infty}(\Omega)} \| \eta \| \| v_h \|. \] (74)

Now Third term of right hand side of equation (35) is estimated as

\[ \left\| \sum_{e_k \in \Gamma_j} \int_{e_k} \left\{ \tilde{\epsilon}_u(z) \theta^2 \nabla u. n \right\} [v_h] \right\| \leq C_\epsilon \sum_{e_k \in \Gamma_j} \int_{e_k} \left\| \left\{ \chi^2 \nabla u. n \right\} [v_h] \right\| + C_\epsilon \sum_{e_k \in \Gamma_j} \int_{e_k} \left\| \left\{ \eta^2 \nabla u. n \right\} [v_h] \right\| + 2C_\epsilon \sum_{e_k \in \Gamma_j} \int_{e_k} \left\| \left\{ \eta \chi \nabla u. n \right\} [v_h] \right\| \leq C_\epsilon \left( \max_{1 \leq i \leq N_h} \frac{P_i}{h_i} \right)^{1/2} \| \chi \|^2 \| u \|_{H^1(\Omega)} \| v_h \| + h^{(\beta+1)/2} (1 + p)^{1/4} \| \eta \| \| v_h \| \| u \|_{H^1(\Omega)} \| u \|_{H^2(\Omega)} + h^{(\beta+1)/2} (1 + p)^{1/4} \| \chi \| \| v_h \| \| u \|_{H^1(\Omega)} \| u \|_{H^2(\Omega)}. \] (76)

Now Fourth term of right hand side of equation (35) is estimated as

\[ \left\| \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \tilde{\epsilon}_u(z) \theta \nabla \varphi. n \right\} [v_h] \right\| \leq \sum_{e_k \in \Gamma} \int_{e_k} \left\| \left\{ \chi \nabla \varphi. n \right\} [v_h] \right\| + \sum_{e_k \in \Gamma} \int_{e_k} \left\| \left\{ \eta \chi \nabla u. n \right\} [v_h] \right\| + \sum_{e_k \in \Gamma} \int_{e_k} \left\| \left\{ \eta \chi \nabla u. n \right\} [v_h] \right\| + \sum_{e_k \in \Gamma} \int_{e_k} \left\| \left\{ \eta \chi \nabla u. n \right\} [v_h] \right\|. \] (77)
Now Fifth term of right hand side of equation (35) is estimated as
\[
\left| \theta \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \bar{e}_u(z) \partial \nabla v_h \cdot \mathbf{n} \right\} [\theta] \right| \leq C_\epsilon \left( \sum_{e_k \in \Gamma} \int_{e_k} \left\| \eta \nabla v_h \cdot \mathbf{n} \right\| \right) + \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \chi \nabla v_h \cdot \mathbf{n} \right\} [\chi] + \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \chi \nabla v_h \cdot \mathbf{n} \right\} [\theta] \\
+ \sum_{e_k \in \Gamma} \int_{e_k} \left\{ \nabla \nabla v_h \cdot \mathbf{n} \right\} \right) (78)
\]

Now Sixth part of right hand side of equation (35) is estimated as
\[
\left| \sum_{i=1}^{N_h} \int_{K_i} (\rho \tilde{h}_d)_{uu}(z) \partial^2 \beta \mathbf{n} \nabla v_h \right| \leq C_{\rho h} \left( \left\| \chi \right\| \left\| v_h \right\| + \left( \frac{h}{p} \right)^2 \left\| \eta \right\| \left\| u \right\|_{H^2(\Omega)} \right) + \left( \frac{h}{p} \right)^2 \left\| \chi \right\| \left\| v_h \right\| \left\| u \right\|_{H^1(\Omega)} \right) (79)
\]

At last Seventh part of right hand side of equation (35) is estimated as
\[
\left| \sum_{e_k \in \Gamma} \int_{e_k} \left\{ (\rho \tilde{h}_d)_{uu}(z) \partial^2 \beta \mathbf{n} \right\} [v_h] \right| \leq \left\| \chi \right\| \left\| v_h \right\| \left\| u \right\|_{H^1(\Omega)} \left\| u \right\|_{H^2(\Omega)} + h^{(\beta+1)/2}(1 + p)^{1/4} \left\| \chi \right\| \left\| v_h \right\| \left\| u \right\|_{H^1(\Omega)} \left\| u \right\|_{H^2(\Omega)} \right) (80)
\]

**Lemma 3.9.** Suppose \( z \in \mathcal{D}_h^p(\mathcal{P}_h) \) and \( \beta \geq 1 \). Also take \( y = S z \). Then there exists a non negative constant \( C \) which is independent of \( h \) and \( p \) such that following condition
\[
\left\| \Pi_h u - y \right\| \leq C \epsilon \left[ \left( \max_{1 \leq i \leq N_h} \frac{P_i}{h} \right)^{1/2} \left\| \Pi_h u - z \right\|^2 + C \left\| \Pi_h u - z \right\| \\
+ C \epsilon (1 + C \epsilon h^{1/2}) \left\| \Pi_h u - u \right\| \right] \right) (81)
\]

hold.

**Proof.** Consider \( \chi = \Pi_h u - z \), \( \eta = \Pi_h u - u \) and \( \xi = \Pi_h u - y \). Take \( v_h = \xi \) in (**)
\[
\left| \tilde{\beta}(u; \eta, \xi) \right| \leq C \left\| \eta \right\| \left\| \xi \right\|. \right) (82)
\]

Put \( v_h = \xi \) in lemma (**) to obtain
\[
\left| \mathcal{F}(z; z - u, \xi) \right| \leq C \epsilon \left( \max_{1 \leq i \leq N_h} \frac{P_i}{h} \right)^{1/2} \left\| \chi \right\| \left\| \xi \right\| + C \epsilon C_u h^{1/2} \left( \left\| \chi \right\| + \left\| \eta \right\| \right) \left\| \xi \right\|. \right) (83)
\]

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Now from above equation (57) and equation (58) and also from equation (**)
we obtain
\[ \left| F(z; z - u, \xi) \right| \leq CC_c \left( \max_{1 \leq i \leq N_h} \frac{p_h}{h_i} \right)^{1/2} \| \chi \|^2 + CC_c C_u h^{1/2} (\| \chi \| + (CC_c C_u h^{1/2} + 1)\| \eta \|) \| \xi \|. \] (84)

Now using the coercivity property we have
\[ \| \xi \|^2 \leq CC_c \left( \max_{1 \leq i \leq N_h} \frac{p_h}{h_i} \right)^{1/2} \| \chi \|^2 + CC_c C_u h^{1/2} (\| \chi \| + (CC_c C_u h^{1/2} + 1)\| \eta \|) \| \xi \|. \] (85)

Hence we have the desire result.

\[ \square \]

3.5 \( L^2 \)-Error Estimates

In order to bound error \( \| u - u_h \|_{L^2} \) in \( L^2 \) norms we use well known Aubin-Nitsche duality argument.

**Theorem 3.10.** Let \( \epsilon \in C^2_b(\Omega \times \mathbb{R}) \) and \( u \in W^1_\infty(\Omega) \). Suppose \( \mathcal{P}_h^p \) is a regular partition. Then for sufficiently small \( h \), there exists a constant \( C = C(\alpha_*, M) \) which independent of \( h \) and \( p \) such that
\[ \| u - u_h \|_{L^2} \leq CC_\epsilon \frac{h^p}{p^*} \| u \|_{H^s(\Omega)}. \] (86)

**Proof.** Consider the following adjoint problem
\[ -\nabla (\epsilon^*(u) \nabla \psi) + \epsilon_\theta^*(u) \nabla u \cdot \nabla \psi - \tilde{\beta}(\rho h_d + (\rho h_d) u) \nabla \psi = e \text{ in } \Omega \] (87)
\[ \psi = 0 \text{ on } \partial\Omega. \] (88)

Now from elliptic regularity property, there exist a unique \( \psi \in H^2(\Omega) \) which satisfies above linear elliptic problem (?) and
\[ \| \psi \|_{H^2(\Omega)} \leq C \| e \|_{L^2(\Omega)} \] (89)

holds.

By short computation, It is easy to show that
\[ \| e \|^2 = \mathcal{B}(u, \psi) - \mathcal{B}(u_h, \psi) + \sum_{i=1}^{N_h} \int_{K_i} \left( \epsilon \nabla c - \epsilon_\theta e^2 \nabla u \right) \nabla \psi \\
- \sum_{e_k \in \Gamma_I} \int_{e_k} \left( \left\{ \epsilon \nabla c \right\} - \left\{ \epsilon_\theta e^2 \nabla u \right\} \right) \left[ \psi \right] - \theta \sum_{e_k \in \Gamma_I} \int_{e_k} \left( \epsilon \nabla c \right) \left[ \frac{\partial \psi}{\partial \nu} \right] [e] \\
+ \sum_{i=1}^{N_h} \int_{K_i} \left\{ (\rho h_d) u e^2 \right\} \tilde{\beta} n \nabla \psi \\
- \sum_{e_k \in \Gamma_I} \int_{e_k} \left\{ (\rho h_d) u e^2 \tilde{\beta} n \right\} \left[ \psi \right]. \] (90)
The first term on the right hand side of (84) is revised as

\[ I = \mathcal{B}(u; u, \psi) - \mathcal{B}(u_h; u, \psi) + \mathcal{B}(u_h; u, \psi) - \mathcal{B}(u_h; u_h, \psi) \]

\[ = \mathcal{B}(u; u, \psi - \chi) - \mathcal{B}(u_h; u, \psi - \chi) + \mathcal{B}(u_h; u, \psi - \chi) - \mathcal{B}(u_h; u_h, \psi - \chi), \]

(91)

where \( \chi = I_h^* \psi \) such that \( \chi|_{\partial \Omega} = 0. \)

\[ I = \sum_{i=1}^{N_h} \int_{K_i} (\epsilon^*(u) - \epsilon^*(u_h))\nabla u \nabla (\psi - \chi) - \int_{K_i} ((\rho h_d)(u) - (\rho h_d)(u_h)) \nabla (\psi - \chi) \]

\[ - \sum_{i=1}^{N_h} \int_{K_i} (\epsilon^*(u) - \epsilon^*(u_h))\nabla (u - u_h) \nabla (\psi - \chi) + \sum_{i=1}^{N_h} \int_{K_i} \epsilon^*(u)\nabla (u - u_h) \nabla (\psi - \chi) \]

(92)

By using Cauchy-Schwarz inequality, we can bound first, second and fourth terms on the right hand side of equation (86) as

\[ \left| \sum_{i=1}^{N_h} \int_{K_i} (\epsilon^*(u) - \epsilon^*(u_h))\nabla u \nabla (\psi - \chi) \right| \leq C_{\epsilon} \parallel \psi \parallel_{H^1(\Omega)} \leq C_{\epsilon} \frac{h}{p} \parallel \psi \parallel_{H^2(\Omega)}, \]

(93)

\[ \left| \sum_{i=1}^{N_h} \int_{K_i} ((\rho h_d)(u) - (\rho h_d)(u_h)) \nabla (\psi - \chi) \right| \leq C_{p} \frac{h}{p} \parallel \psi \parallel_{H^2(\Omega)} \]

(94)

and

\[ \left| \sum_{i=1}^{N_h} \int_{K_i} \epsilon^*(u)\nabla (u - u_h) \nabla (\psi - \chi) \right| \leq C_{\epsilon} \frac{h}{p} \parallel \psi \parallel_{H^2(\Omega)} \]

(95)

The third term of right hand side of equation (86) is estimated by using Hölder’s inequality as

\[ \left| \sum_{i=1}^{N_h} \int_{K_i} (\epsilon^*(u) - \epsilon^*(u_h))\nabla (u - u_h) \nabla (\psi - \chi) \right| \leq C \parallel \epsilon \parallel_{L^2(\Omega)} \parallel \psi \parallel_{H^1(\Omega)} \]

\[ \leq C \parallel \epsilon \parallel^2 \parallel \psi \parallel_{H^2(\Omega)}. \]

(96)

Now the second term of right hand side of equation (84) is bounded using Hölder’s inequality as

\[ \left| \sum_{i=1}^{N_h} \int_{K_i} \left( \epsilon \epsilon_u \nabla \epsilon - \epsilon u \epsilon_2 \nabla u \right) \nabla \psi \right| \leq \left| \sum_{i=1}^{N_h} \int_{K_i} \epsilon \epsilon_u \nabla \epsilon \nabla \psi \right| + \left| \sum_{i=1}^{N_h} \int_{K_i} \epsilon u \epsilon_2 \nabla u \nabla \psi \right| \]

\[ \leq C \parallel \epsilon \parallel^2 \parallel \psi \parallel_{H^2(\Omega)}, \]

(97)
Now the third term of right hand side of equation (84) is estimated as

\[ \left| \sum_{e_k \in \Gamma_I} \int_{e_k} \left( \{ \tilde{\epsilon}_u \frac{\partial e}{\partial \nu} \} - \{ \epsilon_{uu} e^2 \frac{\partial u}{\partial \nu} \} \right) [\psi] \right| \leq \left| \sum_{e_k \in \Gamma_I} \int_{e_k} \{ \tilde{\epsilon}_u e \frac{\partial e}{\partial \nu} \} [\psi] \right| + \left| \sum_{e_k \in \Gamma_I} \int_{e_k} \{ \epsilon_{uu} e^2 \frac{\partial u}{\partial \nu} \} [\psi] \right| \] (98)

4 Conclusion

In this article, we have discuss and analyze interior-exterior penalty base discontinuous Galerkin finite element method for solving EHL line as well as point contact problems. Convergence of discrete DG solution is proved using Brouwer’s fixed point theorem. We have shown that optimal order of convergence in $L^2$ and $H^1$ norms is achieved in mesh size $h$ theoretically. However, suboptimal order convergence is achieved in polynomial degree $p$.

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