Bounds on the Number of Longest Common Subsequences

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Abstract

This paper performs the analysis necessary to bound the running time of known, efficient algorithms for generating all longest common subsequences. That is, we bound the running time as a function of input size for algorithms with time essentially proportional to the output size. This paper considers both the case of computing all distinct LCSs and the case of computing all LCS embeddings. Also included is an analysis of how much better the efficient algorithms are than the standard method of generating LCS embeddings. A full analysis is carried out with running times measured as a function of the total number of input characters, and much of the analysis is also provided for cases in which the two input sequences are of the same specified length or of two independently specified lengths.

Keywords: longest common subsequences, edit distance, shortest common supersequences

1 Background and Terminologies

Let $A = a_1a_2\ldots a_m$ and $B = b_1b_2\ldots b_n$ ($m \leq n$) be two sequences over an alphabet $\Sigma$. A sequence that can be obtained by deleting some symbols of another sequence is referred to as a subsequence of the original sequence. A common subsequence of $A$ and $B$ is a subsequence of both $A$ and $B$. A longest common subsequence (LCS) is a common subsequence of greatest possible length. A pair of sequences may have many different LCSs. In addition, a single LCS may have many different embeddings, i.e., positions in the two strings to which the characters of the LCS correspond.
Most investigations of the LCS problem have focused on efficiently finding one LCS. A widely familiar $O(mn)$ dynamic programming approach goes back at least as far as the early 1970s [19, 22, 24], and many later studies have focused on improving the time and/or space required for the computation (e.g. [13, 11, 10, 18, 23, 4, 7, 5, 15, 25, 6, 4, 7, 21, 8]).

The familiar dynamic programming approach provides a basis for generating all LCSs, but the naive approach (e.g. [1]) may generate the same LCS or even the same LCS embedding many times. Other methods have been developed to generate a listing of all distinct LCSs or all LCS embeddings in time proportional to the output size (plus a preprocessing time of $O(mn)$ or less), i.e., without generating duplicates [2, 9, 20, 10]. But prior works give no indication of how long the running time may be as a function of input size. They also do not indicate how the asymptotic time of the efficient methods compares to the naive approach, so it is unclear how worthwhile it is to implement the more complex algorithms.

Section 2 of this paper obtains bounds on the amount of time that may be required to find all distinct LCSs of two input sequences of fixed total length when using an algorithm with time proportional to the output size. Technically, the time is governed by the LCS length times the number of LCSs, but we focus on bounding just the maximum possible number of distinct LCSs. (There will be little difference in the LCS lengths that maximize these two measures.)

Section 3 similarly bounds the amount of time that may be required to find all LCS embeddings. Here an exact computation of the maximum possible number of LCS embeddings is provided, and the analysis is carried out both for a fixed total number of input characters and for two input sequences of the same fixed length. In addition, a partial analysis is provided for two input sequences with independently specified lengths. Since the maximum number of LCS embeddings is achievable when there is just one distinct LCS, the results in this section also give a measure of how much more efficient it is to generate all distinct LCSs in time proportional to the output size, as compared to a method that efficiently generates all embeddings and removes duplicate LCSs.

Section 4 indicates how much more efficient a fast algorithm for generating all LCS embeddings (or all distinct LCSs) may be in comparison to the standard method that may even report the same embedding more than once. It turns out that the naive algorithm may even generate the same embedding of a single LCS exponentially many times, and we precisely quantify the asymptotic worst-case overhead.

## 2 Bounding the Number of Distinct LCSs

In this section we determine how much time an efficient algorithm for listing all distinct LCSs may require. While the actual time is the LCS length $l$ times the number of distinct LCSs (plus preprocessing time), we focus here on bounding just the maximum number of distinct LCSs. The values of $l$ that maximize these two measures will be increasingly close
as the sizes of the input sequences (and consequently \( l \)) grow. Throughout this section, we will let \( D(t) \) denote the maximum possible number of distinct LCSs for two input sequences of total length \( t \) (assuming an unbounded alphabet).

Letting \( m = \lfloor t/2 \rfloor \) and \( z = (- \lfloor t/2 \rfloor) \mod 3 \), a lower bound on \( D(t) \) follows from considering two input sequences of length \( m \) of the form \( Xefghijklm... \) and \( Ygfejihmlk... \), where \( X \) and \( Y \) are empty if \( z = 0 \), \( ab \) and \( ba \) if \( z = 1 \), or \( abcd \) and \( badc \) if \( z = 2 \):

**Theorem 1** For \( t \geq 4 \), \( D(t) \geq \frac{3(\lfloor t/2 \rfloor - 2((- \lfloor t/2 \rfloor) \mod 3))/32^{(- \lfloor t/2 \rfloor) \mod 3}}{2} \) (which implies that for \( t \) divisible by 6, \( D(t) \geq \frac{3t}{6} > 1.2t \)).

To obtain an upper bound, we begin with the following lemma. For this purpose, we define an embedding of a character of an LCS in the two input sequences \( A \) and \( B \) as an ordered pair of a position in \( A \) and a position in \( B \) from which the character may be selected when forming the LCS. We say that two character embeddings \((p, q)\) and \((p', q')\) cross if \( p < p' \wedge q' < q \) or \( p' < p \wedge q < q' \).

**Lemma 2** Consider two LCSs starting with different characters and any embedding of these two LCSs. The embeddings of the initial characters of these two LCSs must cross.

**Proof.** Suppose there are embeddings of two LCSs \( C = c_1c_2...c_l \) and \( C' = c'_1c'_2...c'_l \) such that \( c_1 \neq c'_1 \) and the embeddings of \( c_1 \) and \( c'_1 \) do not cross. Then \( cC' \) or \( c'C \) is a common subsequence, contradicting the assumption that \( C \) and \( C' \) are LCSs.

**Theorem 3** \( D(t) \leq 4t/5 < 1.32t \).

**Proof.** The proof is by induction on \( t \); the base case is easy to check. For the induction step, let \( k \) be the number of choices for the first character when constructing an LCS from the two given input strings. Since the embeddings of all these possible initial characters must cross, the sum of the two string positions corresponding to each such embedding must be at least \( k + 1 \). Furthermore, once such an embedding is chosen for the first character of the LCS, \( k + 1 \) characters of the input strings are removed from consideration for construction of the rest of the LCS, since no other character of the LCS can have an embedding that crosses the first one. Thus, \( D(t) \leq kD(t - (k + 1)) \). By the induction hypothesis, \( D(t) \leq k4^{(t - (k + 1))/5} \). Then the result follows from the observation that \( k / 4^{(k + 1)/5} \) is decreasing for \( k > 5 / \ln 4 \approx 3.6 \) and is at most 1 for integral \( k \leq 4 \).

Neither Theorem 1 nor Theorem 3 is tight; e.g., with \( t = 10 \), there are 7 distinct LCSs for the input strings \( abcdab \) and \( cbabc \). If neither input string contains repeated characters, however, we can combine Theorem 1 with the following theorem to obtain tight upper and lower bounds:
Theorem 4  \( D(t) \leq 3^{(\lfloor t/2 \rfloor - 2(\lfloor -t/2 \rfloor \text{ mod } 3))}/3^{2(\lfloor -t/2 \rfloor \text{ mod } 3)} \) if there are no repeated characters in either input sequence.

Proof. We proceed by induction as in Theorem 3, but now when we make one of \( k \) choices for the first character of the LCS, we eliminate \( 2k \) characters from possible use in the rest of the LCS. Thus, \( D(t) \leq kD(t - 2k) \). Using the induction hypothesis and considering the different cases for the values of \( \lfloor t/2 \rfloor \) and \( k \) modulo 3, the result follows as long as we can show that \( k/3^{k/3} \) is decreasing for \( k \geq 3/\ln 3 \approx 2.7 \).

Note that the results in this section also apply in the case that we require \( m = n \), by setting \( t = 2n \).

3 The Maximum number of LCS Embeddings

In this section, we determine how much time an efficient algorithm for listing all LCS embeddings may require. Utilizing the same justification as in the previous section, we neglect LCS length as a component of the running time. (It would actually be easy to incorporate LCS length into the presentation in this section, and this may be done in the full paper.) Thus we focus on computing the maximum possible number of LCS embeddings. In the full paper, we will argue that the maximum number of LCS embeddings can be achieved when there is just one distinct LCS. Therefore, we turn our attention to computing the maximum possible number of embeddings of a single LCS. This result will also indicate how much more efficient it is to generate all distinct LCS embeddings in time proportional to output size rather than to generate all embeddings (efficiently) and remove duplicates.

We begin by determining the maximum number of embeddings of an LCS of length \( l \) in two input strings of length \( m \) and \( n \). Then we perform the maximization over \( l \) in the cases of (1) \( m = n \) and (2) \( m \) and \( n \) variable but with \( m + n \) fixed at \( t \).

Lemma 5 The maximum possible number of embeddings \( E(n, m, l) \) of a single LCS of length \( l \) in two input sequences of lengths \( m \) and \( n \) is

\[
E(n, m, l) = \max_{y \leq l} \binom{m - y}{l - y} \binom{n + y - l}{y}.
\]

Proof. First, \( E(n, m, l) \geq \max_{y \leq l} \binom{m - y}{l - y} \binom{n + y - l}{y} \), because, for any \( y \leq l \), we can find \( \binom{m - y}{l - y} \binom{n + y - l}{y} \) embeddings of the string \( x^i y \) in the two strings \( a^m y b^n \) and \( a^l y b^{n + y - l} \) (where \( x^i \) represents \( n \) repetitions of the character \( x \)).

Now we prove \( E(n, m, l) \leq \max_{y \leq l} \binom{m - y}{l - y} \binom{n + y - l}{y} \) as follows. Each character of any LCS must have a fixed embedding in at least one of the two input strings \( A = a_1 a_2 \ldots a_m \) and
$B = b_1 b_2 \ldots b_n$. (Suppose, to the contrary, that $c_k$ of the LCS $C = c_1 c_2 \ldots c_l$ could be embedded into $a_i$ or $a_j$ ($i < j$) and into $b_p$ or $b_q$ ($p < q$). Then $c_1 c_2 \ldots c_{k-1}$ could be embedded in $a_1 a_2 \ldots a_{i-1}$ and in $b_1 b_2 \ldots b_{p-1}$, while $c_{k+1} c_k c_{k+2} \ldots c_l$ could be embedded in $a_{j+1} a_{j+2} \ldots a_m$ and in $b_{q+1} b_{q+2} \ldots b_n$. This contradicts the supposition that $C$ is an LCS, because we now know that $c_1 c_2 \ldots c_{k-1} c_k c_{k+1} c_{k+2} \ldots c_l$ is a common subsequence of $A$ and $B$.) Let $y$ be the number of characters of the LCS under consideration that have a fixed embedding in $A$. Then at least $l - y$ characters have a fixed embedding in $B$. Now the number of ways to embed those $l - y$ characters in $A$ is at most $\binom{m-y}{l-y}$, and the number of ways to embed into $B$ the $y$ characters fixed in $A$ is at most $\binom{n-(l-y)}{y}$.

\[ E(n, m, l) = \binom{m-y}{l-y} \binom{n+y-l}{y}, \]

where

\[ y^* = \left\lfloor \frac{l(n-l) + l - m}{2(m+n-2l)} \right\rfloor. \]

**Proof.** The result follows from Lemma 5 as long as we can show that $y^*$ (which satisfies $y^* \leq l$) is the (nonnegative integral) value of $y$ that maximizes $P(y) = m-y \binom{n-y-l}{l-y}$. To do this, we show that $P(y+1) \leq P(y)$ if and only if $y \geq \frac{l(n-l) + l - m}{m+n-2l}$ as follows:

\[
\begin{align*}
P(y+1) &\leq P(y) \iff (m-y-1)! (n+y+1-l)! \leq (m-y)! (n+y-l)! \\leq (l-y)! (m-l)! (y+1)! (n-l)! \\leq (m-y)(y+1) \\leq -y^2 + y(l-n+l-1) + ln - l^2 + l \leq -y^2 + y(m-1) + m \\leq l(n-l) + l - m \leq y(m+n-2l)
\end{align*}
\]

(Note that $m + n - 2l \geq 0$, and where it equals 0 ($n = m = l$), the result remains correct as long as we use the convention of interpreting $\binom{0}{0}$ as 1.)

Next we specialize to $m = n$.

**Lemma 7** The maximum possible number of embeddings $E(n, n, l)$ of a single LCS of length $l$ in two input sequences of length $n$ is

\[ E(n, n, l) = \binom{n-\lfloor l/2 \rfloor}{\lfloor l/2 \rfloor} \binom{n-\lfloor l/2 \rfloor}{\lfloor l/2 \rfloor} = \binom{n-\lfloor l/2 \rfloor}{n-l} \binom{n-\lfloor l/2 \rfloor}{n-l}. \]

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Proof. Substituting $n$ for $m$ in Lemma 6 gives $y^* = \lceil (l - 1)/2 \rceil = \lfloor l/2 \rfloor$. Then substituting for $m$ and $y^*$ in the expression for $E(n, m, l)$ there gives the desired result after using basic facts about floors and ceilings and the relationship \( \binom{r}{k} = \frac{r!}{k!(r-k)!} \).

Lemma 8 Let $\sigma = (5n - 1 - \sqrt{5(n+1)^2 - 4})/5$ and $\tau = (5n - \sqrt{5(n+1)^2})/5$. Then, the maximum possible number of embeddings of a single LCS in two input sequences of length $n$ is achieved with an LCS length $l^*$ that satisfies the following conditions.

1. $l^*$ can be chosen as either $\sigma$ or $\sigma + 1$ if $\sigma$ is integral.
2. Otherwise, $l^* = \lceil \sigma \rceil$ if $\lceil \sigma \rceil$ is even.
3. Otherwise, $l^* = \lfloor \tau \rfloor$ (which most often equals $\lceil \sigma \rceil$).

Proof. From Lemma 6, we see that the condition $E(n, n, l + 1) \leq E(n, n, l)$ is equivalent to:

\[
\frac{(n - \lceil \frac{l+1}{2} \rceil)!}{\lceil \frac{l+1}{2} \rceil! (n-l-1)!} \leq \frac{(n - \lceil \frac{l}{2} \rceil)!}{\lceil \frac{l}{2} \rceil! (n-l)!} \frac{(n - \lceil \frac{l}{2} \rceil)!}{\lceil \frac{l}{2} \rceil! (n-l)!}
\]

\[
\text{i.e.,}
\]

\[
(n-l)^2 \leq \left\lfloor \frac{l+1}{2} \right\rfloor (n - \lfloor l/2 \rfloor)
\]

since $\lceil \frac{l+1}{2} \rceil = \lceil \frac{l}{2} \rceil$, and $\lceil \frac{l+1}{2} \rceil = \lceil \frac{l}{2} \rceil + 1$.

Now we see that the condition $E(n, n, l + 1) \leq E(n, n, l)$ is equivalent to:

\[
5l^2 - 2(5n - 1)l + 4n(n-1) \leq 0 \quad \text{for } l \text{ even} \tag{1}
\]

\[
5l^2 - 10nl + 4n^2 - 2n - 1 \leq 0 \quad \text{for } l \text{ odd} \tag{2}
\]

For each of these two quadratic expressions in $l$, the roots $r_1$ and $r_2$ satisfy $-1 < r_1 < n$ and $n < r_2$. Since $l \leq n$, in each of the cases of $l$ even and $l$ odd, an (integral) value of $l$ maximizing $E(n, n, l)$ is $\lceil r_1 \rceil$ as long as that value is of the appropriate parity. The values of $r_1$ are $\sigma$ and $\tau$, in (1) and (2), respectively, and it may be noted that $\tau$ is never integral, since $\sqrt{5}$ is irrational.

Now, if $\sigma$ is an even integer, we see that according to (1), $l^*$ is selectable as $\sigma$ or $\sigma + 1$. Furthermore, this is the final result, since the odd value $\lceil \tau \rceil$ that maximizes $E(n, n, l)$ according to (2) is equal to $\sigma + 1$.

It is easy to check that any time $\sigma$ is integral, it is even, so it only remains to consider $\sigma$ nonintegral. We see that if $\lceil \sigma \rceil$ is even, then according to (1), $l^* = \sigma$. Furthermore, (2) will not lead to a better value of $E(n, n, l)$, since $\sigma < \tau < \sigma + \frac{1}{2}$. Finally, if $\lceil \sigma \rceil$ is odd, then either $\lceil \tau \rceil$ has the same odd value, or $\lceil \tau \rceil$ has the even value $\lceil \sigma \rceil + 1$; either way, we see from (1) and (2) that $l^* = \lceil \tau \rceil$. \qed
Theorem 9  The maximum possible number of embeddings of a single LCS in two input sequences of length \( n \) is
\[
\left( \frac{\left\lfloor \frac{1}{2} \left( 1 + \frac{1}{\sqrt{5}} \right) (n + 1) \right\rfloor}{\left\lceil \frac{5n + 1 + \sqrt{5(n + 1)^2 - 4}}{10} \right\rceil} \right) \left( \frac{\left\lfloor \frac{1}{2} \left( 1 - \frac{1}{\sqrt{5}} \right) (n + 1) \right\rfloor}{\left\lceil \frac{\phi n/\sqrt{5}}{(\phi - 1)n/\sqrt{5}} \right\rceil} \right)^{2}.
\]

Proof. The result follows from Lemma 8 as follows. Since \( \sigma < \tau < \sigma + 1 \), we know \( \lceil \tau \rceil /2 = \lceil \sigma \rceil /2 \) if \( \sigma \) is even. Thus, by Lemma 8, \( l^*/2 = \lceil \sigma \rceil /2 \). Similarly, there is an acceptable \( l^* \) with \( \lceil l^*/2 \rceil = \lceil \sigma /2 \rceil \). Now we have the following four relationships:
1. \( \lceil l^*/2 \rceil = \lfloor \lceil \sigma /2 \rceil \rfloor = \lceil \sigma /2 \rceil \)
2. \( n - \lceil l^*/2 \rceil = n - \lceil \sigma /2 \rceil = n - \sigma /2 \)
3. \( \lfloor l^*/2 \rfloor = \lfloor \lceil \tau /2 \rceil \rfloor = \lfloor (\tau + 1)/2 \rfloor = \lceil (\tau + 1)/2 \rceil \)
4. \( n - \lfloor l^*/2 \rfloor = n - \lceil (\tau + 1)/2 \rceil = n - (\tau + 1)/2 \)

Substituting these four relationships into \( E(n, n, l) = \frac{\left( n - \lfloor l/2 \rfloor \right) \left( n - \lceil l/2 \rceil \right)}{\left( n - \frac{\phi n/\sqrt{5}}{(\phi - 1)n/\sqrt{5}} \right)^{2}} \) from Lemma 7 yields the desired result.

Corollary 10  The limit as \( n \) goes to infinity of the maximum possible number of embeddings of a single LCS in two input sequences of length \( n \) is
\[
\frac{\phi^2 \sqrt{5}}{2\pi} \left( \frac{\phi^2}{\sqrt{5}} \right)^n \approx 0.932(2.62)^n / n,
\]
where \( \phi = (1 + \sqrt{5})/2 \) (the golden ratio).

Proof. We will use Stirling's approximation to the factorial:
\[
n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n (1 + O(1/n)) \quad [\text{[4, p. 111]}. \ (3)
\]
Then, the limit as \( n \) goes to infinity of the expression in Theorem 9 is:
\[
\lim_{n \to \infty} \left( \frac{\phi^2 \sqrt{5}}{2\pi} \left( \frac{\phi^2}{\sqrt{5}} \right)^n \right) \left( \frac{\phi n/\sqrt{5}}{(\phi - 1)n/\sqrt{5}} \right)^{2} = \lim_{n \to \infty} \left( \frac{\phi n/\sqrt{5}}{(\phi - 1)n/\sqrt{5}} \right)^{2}
\]

by Eqn. 3
Next, we consider the case in which the total number of characters in the two input strings is fixed, but the lengths of the individual strings are not. The following Lemma follows immediately from Lemma \( \text{[3]} \), using the fact \( \binom{r}{k} \binom{r'}{k'} \leq \binom{r+r'}{k+k'} \).

**Lemma 11** The maximum possible number of embeddings of a single LCS of length \( l \) in two input sequences of total length \( t \) is \( \binom{t-l}{l} \).

Using Lemma \( \text{[11]} \) and reasoning similar to the proof of Lemma \( \text{[8]} \), yields the following theorem.

**Theorem 12** The maximum possible number of embeddings of a single LCS in two input sequences of total length \( t \) is

\[
\left( \binom{5t + 3 + \sqrt{5(t+1)^2 + 4}}{10} / 10 \right).
\]

Finally, we can use the same type of reasoning as in Corollary \( \text{[10]} \) to reach the following conclusion:

**Corollary 13** The limit as \( t \) goes to infinity of the maximum possible number of embeddings of a single LCS in two input sequences of total length \( t \) is \( \phi \sqrt{\frac{5}{2\pi}} \phi^{t} / \sqrt{t} \approx 0.965(1.62)^{t} / \sqrt{t} \).

4 The Degree of Inefficiency of Naively Generating all LCSs

The standard “naive” method of computing the length of an LCS is a “bottom-up” dynamic programming approach based on the following recurrence for the length \( L[i,j] \) of an LCS of \( a_{1}a_{2}\ldots a_{i} \) and \( b_{1}b_{2}\ldots b_{j} \):

\[
L[i,j] = \begin{cases} 
0 & \text{if } i = 0 \text{ or } j = 0 \\
L[i-1,j-1] + 1 & \text{if } i,j > 0 \text{ and } a_{i} = b_{j} \\
\max\{L[i-1,j], L[i,j-1]\} & \text{otherwise}
\end{cases}
\]

We refer to the \( L[i,j] \) value as the rank of \([i,j]\), and we call \([i,j]\) a match if \( a_{i} = b_{j} \). In \( O(mn) \) time, one may fill an array with all the values of \( L[i,j] \) for \( 0 \leq i \leq m \land 0 \leq j \leq n \), and the length \( l \) of an LCS is read off from \( L[m,n] \). The same time bound also suffices to produce a single LCS by a “backtracing” approach starting from position \([m,n]\) of the array.
At each stage we just step from position \([i, j]\) to a position \([i - 1, j - 1]\), \([i - 1, j]\), or \([i, j - 1]\) that is responsible for the setting of \(L[i, j]\) as per (4); each match encountered generates a character of the LCS (in reverse order).

The naive approach to generate all LCS embeddings \([1]\) would be to extend the backtracing method as follows. (To generate all distinct LCSs, one could generate all embeddings and remove duplicate LCSs.) At each step, we would consider three possibilities (and continue recursively); from position \([i, j]\), we could add a character to the LCS and move to \([i - 1, j - 1]\) if \([i, j]\) is a match, and we could move to \([i - 1, j]\) or \([i, j - 1]\) if the rank there equals \(L[i, j]\) (without adding a character to the LCS and regardless of whether \([i, j]\) is a match). Whenever we reach \([0, 0]\), we can print out an LCS embedding. We could make some simple improvements such as stopping each backtrace path at any position of rank 0, but this will not change the basic degree of inefficiency as expressed in the theorem below. (Note that output size is always at least 1 rather than 0, because the empty string \(\varepsilon\) is a common subsequence of any pair of input sequences.)

\[\textbf{Theorem 14} \] The naive method of generating all LCS embeddings (or all LCSs) may require time exceeding the output size by a factor of \(\Theta\left(\binom{n+m}{m}\right)\) in the worst case.

\[\textbf{Proof.}\] For the upper bound, consider the “normalized” time \(N[i, j]\), representing the time to complete the naive backtrace procedure from position \([i, j]\), divided by \(\max\{1, L[i, j]\}\). An induction argument shows that there are positive constants \(c\) and \(d\) such that \(N[m, n] \leq c \binom{n+m}{m} - d\). It is easy to choose constants and obtain \(N[i, j] \leq c \binom{i+j}{i} - d\) for any \([i, j]\) with \(i \leq 1\) or \(j \leq 1\). Included in this result is that \(N[i, j] \leq c \binom{i+j}{i} - d\) for \(i + j < n + m\). For this final step, we perform the following case analysis, with \(l\) denoting the rank of \([m, n]\).

Case I: \([m, n]\) is not a match. Then \(N[m, n] \leq N[m - 1, n] + N[m, n - 1] + O(1)\). (It is easy to see that this relationship holds if \(N\) represents ordinary time, since the traceback from \([m, n]\) does not need to add anything to the outputs generated in the tracebacks from \([m - 1, n]\) and \([m, n - 1]\). The relationship then holds for normalized time, since the ranks of \([m - 1, n]\) and \([m, n - 1]\) can be no higher than \(l\).) The induction hypothesis can be completed by invoking the induction hypothesis and using the familiar identity

\[
\binom{r - 1}{k} + \binom{r - 1}{k} = \binom{r}{k}. \tag{5}
\]

Case II: \([m, n]\) is a match. The following three subcases cover all possibilities (albeit with overlap between cases IIA and IIB).

Case IIA: \([m - 1, n]\) is not of rank \(l\). Then, we have \(N[m, n] \leq N[m - 1, n - 1] + N[m, n - 1] + O(1)\). (Here, this relationship would not be valid with \(N\) representing ordinary time,
because every output produced in the traceback from \([m-1, n-1]\) must be augmented with an additional character corresponding to the match at position \([m, n]\). But with normalized time, the relationship can be justified as follows. \([m-1, n-1]\) is of rank \(l-1\), so \((l-1)N[m-1, n-1]\) is an upper bound on the amount of time spent in the backtrace from \([m-1, n-1]\). Furthermore, \(N[m-1, n-1]\) is an upper bound on the number of outputs produced in the backtrace from \([m-1, n-1]\) and therefore on the amount of extra time appending a single extra character to each output. Since \([m, n-1]\) is also of rank at most \(l\) and we need not add anything to the outputs of the backtrace from \([m, n-1]\), the desired relationship holds.) We can now complete the induction step in a similar fashion to Case I.

Case IIB: \([m, n-1]\) is not of rank \(l\). This case is completely analogous to Case IIA.

Case IIC: \([m-1, n]\) and \([m, n-1]\) are both of rank \(l\). Then

\[
N[m, n] \leq N[m-1, n] + N[m-1, n-1] + N[m, n-1] + O(1). \tag{6}
\]

Furthermore, since \([m, n]\) is a match, \([m-1, n-1]\) is at rank \(l-1\), which is lower than the ranks of \([m-1, n]\) and \([m, n-1]\). Thus,

\[
N[m-1, n] \leq N[m-2, n] + N[m-2, n-1] + O(1) \tag{7}
\]

\[
N[m, n-1] \leq N[m, n-2] + N[m-1, n-2] + O(1) \tag{8}
\]

Combining, Equations 6, 7, and 8, we have

\[
N[m, n] \leq N[m-2, n] + N[m-2, n-1] + N[m-1, n-1] + N[m-1, n-2] + N[m, n-2] + O(1) \tag{9}
\]

Now we are again able to complete the induction step as in Case I, using Equation 3 several times.

From the case analysis, we have concluded that the normalized time \(N[m, n]\) is \(O\left(\binom{n+m}{m}\right)\). Since the true output size of listing even just distinct LCSs is at least \(l = L[m, n]\), the overhead of the naive algorithm is \(O\left(\binom{n+m}{m}\right)\).

For the lower bound, note that an overhead of \(\Theta\left(\binom{n+m}{m}\right)\) is achievable by simply choosing sequences with no matches. Furthermore, even if we make the backtracing procedure less naive by printing outputs whenever we hit a node of rank 0, we still would spend \(\Omega\left(\binom{n+m-2}{m-1}\right) = \Omega\left(\binom{n+m}{m}\right)\) time for a pair of input strings in which the only match is at \([1, 1]\), while the true output size would be 1 even to list all embeddings of all LCSs.

We can now give a simple expression for the worst-case overhead of the naive algorithm on two input strings of equal length. This result follows from expressing \(\binom{2n}{n}\) from Theorem 14 as \(\frac{(2n)!}{(n!)^2}\) and using Equation 3.

**Corollary 15** For two input strings of length \(n\), naively generating all LCS embeddings (or all LCSs) may require time exceeding the output size by a factor of \(\Theta\left(\frac{4^n}{\sqrt{n}}\right)\) in the worst case.
Finally, we can recast the result for the case in which the total number of characters in the two input strings is fixed, but the lengths of the individual strings are not.

**Corollary 16** For two input strings of total length $t$, naively generating all LCS embeddings (or all LCSs) may require time exceeding the output size by a factor of $\Theta(2^t/\sqrt{t})$ in the worst case.

**Proof.** From Theorem 14, the worst-case overhead is based on the maximum value of $\binom{t}{m}$, which is $\binom{t}{\lceil t/2 \rceil}$. Then we proceed as for Corollary 15.

5 Conclusion

We have seen that the maximum number of distinct longest common subsequences for fixed input length is much less than the maximum number of LCS embeddings, which is much less than the maximum number of embeddings (including duplicates) obtained by generating embeddings by the standard method. Thus, it is much more efficient to generate all distinct LCSs or all LCS embeddings in time proportional to the output size than to use the standard method of generating LCS embeddings.

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