Review Article

Boundary Value Problem for a Second-Order Difference Equation with Resonance

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In this paper, we study the existence and multiplicity of nontrivial solutions of a second-order discrete boundary value problem with resonance and sublinear or superlinear nonlinearity. The main methods are based on the Morse theory and the minimax methods. In addition, some examples are given to illustrate our results.

1. Introduction

Let \( Z \) and \( \mathbb{R} \) be the sets of integers and real numbers, respectively. For \( a, b \in Z \), \( Z(a, b) \) denotes the discrete interval \( \{a, a + 1, \ldots, b\} \) if \( a \leq b \).

In this paper, we consider the existence and multiplicity of nontrivial solutions for the following discrete Dirichlet boundary value problem:

\[
\begin{align*}
\Delta [p(k)\Delta u(k - 1)] + q(k)u(k) + f(k, u(k)) & = 0, \quad k \in Z(1, T), \\
u(0) = u(T + 1) & = 0,
\end{align*}
\]

(1)

where the forward difference operator is defined by \( \Delta u(k) = u(k + 1) - u(k) \); \( p(k) \) and \( q(k) \) are real valued on \( Z \), and \( p(k) \) is nonzero; and \( f(k, \cdot) \in C^1 (\mathbb{R}, \mathbb{R}) \) satisfies \( f(k, 0) = 0 \) for each \( k \in Z(1, T) \). Clearly, problem (1) has the trivial solution \( u = 0 \).

In the recent years, the existence of solutions for nonlinear difference equations has been widely studied by many authors. We note that these results were usually obtained by means of critical point theory, for example, the existence of ground state solutions [1], homoclinic orbits [2–5], the boundary value problem, and periodic solutions [6–8].

The boundary value problem (1) may be regarded as a discrete analogue of the following boundary value problem:

\[
\begin{align*}
(p(t)u'(t))' + q(t)u(t) + f(t, u(t)) & = 0, \quad 0 < t < 1, \\
u(0) = u(1) & = 0,
\end{align*}
\]

(2)

which has been successfully applied to the modeling of astrophysics, gas dynamics, and chemically reacting system, see [9–11].

By various methods and techniques, many authors studied the similar second-order difference equation under various boundary value conditions. For example, Agarwal in [10] considered the existence of solutions for the boundary value problem (1) by the contraction mapping principle and Brouwer fixed point theorems in Euclidean space. Yu and Guo in [12] first used the critical point theory to study the following discrete boundary value problem:

\[
\begin{align*}
\Delta [p(k)\Delta u(k - 1)] + q(k)u(k) & = f(k, u(k)), \quad k \in Z(1, T), \\
u(0) & = A, \\
u(T + 1) & = B,
\end{align*}
\]

(3)

where \( A \) and \( B \) are the constants, and they proved the existence of solutions of problem (3) when the nonlinear term \( f \) is sublinear or superlinear. We refer the readers to [13–16] and the reference therein for more information.

It is well known that the resonance exists in many real-world applications, and the equations with resonance have
been extensively studied in various fields [17–21]. In fact, it is more difficult to study the boundary value problem under the case of resonance because the resonance case can change the local geometric properties of the critical points [17]. In this paper, we study the existence and multiplicity of nontrivial solutions for problem (1) at resonance by means of the interaction between the nonlinearity and the spectrum of the symmetric matrix $P + Q$, where $P + Q$ denotes a matrix whose elements are given by $p(k)$ and $q(k)$ for $k \in Z(1, T)$.

Note that in [12], the authors obtained the existence of one solution for (3) via the variational methods or the saddle point theorem. In the case where $A = B$, the solution could be trivial, which was not considered in [12]. In this paper, we will show that the boundary value problem (1) has at least one nontrivial solution. In fact, our Theorem 1 and Corollary 1 extend and complement the existence results given in [12] and establish the existence of multiple solutions.

Our results are obtained by combining the Morse theory [22–24], critical group computation, and the minimax methods including the local linking [17, 25]. We prove the existence of nontrivial solutions for problem (1) at resonance by the relationship between these nonlinear techniques and methods. In order to apply these ideas to identify the unknown critical points, we compute the corresponding critical groups of functional $J$. Obviously, when we consider the existence of nontrivial solutions of problem (1) at resonance, we need to overcome much more difficulties than those in the literature [12].

We consider the $T$-dimensional Banach space:

$$S = \{u : Z(0, T + 1) \longrightarrow R \text{ such that } u(0) = u(T + 1) = 0\}.$$  \hspace{1cm} (4)

The space $S$ is a Hilbert space with the inner product

$$\langle u, v \rangle = \sum_{k=1}^{T} u(k)v(k), \ \forall u, v \in S,$$  \hspace{1cm} (5)

dowered with the norm

$$\|u\| = \sqrt{\langle u, u \rangle} = \left(\sum_{k=1}^{T} |u(k)|^2\right)^{1/2}.$$  \hspace{1cm} (6)

Now, we define the $C^1$-functional $J$ on $S$ as follows:

$$J(u) = \sum_{k=1}^{T-1} \left( \frac{1}{2} (p(k) (\Delta u(k-1))^{2} - q(k) u^2(k)) - \sum_{k=1}^{T} F(k, u(k)) \right),$$  \hspace{1cm} (7)

for every $u \in S$, where $F(k, t) = \int_{0}^{1} f(k, s)ds$, $(k, t) \in Z(1, T) \times R$. We can compute the Fréchet derivative as

$$\langle J'(u), v \rangle = \sum_{k=1}^{T} \left( p(k) \Delta u(k-1) \Delta v(k-1) - q(k) u(k)v(k) - f(k, u(k))v(k) \right)$$

$$= - \sum_{k=1}^{T} \left( \Delta (p(k) \Delta u(k-1)) + q(k) u(k) - f(k, u(k)) \right)v(k),$$  \hspace{1cm} (8)

for all $u, v \in S$. It is clear that the critical points of $J$ are the solutions of problem (1).

For convenience, we identify $u \in S$ with $u = (u(1), u(2), \ldots, u(T)) \in R^T$. Thus, we rewrite $J(u)$ and $\langle J'(u), v \rangle$ as

$$J(u) = \frac{1}{2} u^T (P + Q)u - \sum_{k=1}^{T} F(k, u(k)),$$  \hspace{1cm} (9)

$$\langle J'(u), v \rangle = u^T (P + Q)v - \sum_{k=1}^{T} f(k, u(k))v(k),$$

further,

$$\langle J''(u)v, w \rangle = u^T (P + Q)v - \sum_{k=1}^{T} f(k, u(k))v(k),$$  \hspace{1cm} (10)

where $u^T$ denotes the transpose of $u$ and $P$ and $Q$ are the $T \times T$ symmetric matrices given by

$$P = \begin{pmatrix}
    p(1) + p(2) & -p(2) & 0 & \cdots & 0 & 0 \\
    -p(2) & p(2) + p(3) & -p(3) & \cdots & 0 & 0 \\
    0 & -p(3) & p(3) + p(4) & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & p(T - 1) + p(T) & -p(T) \\
    0 & 0 & 0 & \cdots & p(T) + p(T + 1) & 0 \\
\end{pmatrix},$$  \hspace{1cm} (11)

$$Q = \begin{pmatrix}
    -q(1) & 0 & 0 & \cdots & 0 & 0 \\
    0 & -q(2) & 0 & \cdots & 0 & 0 \\
    0 & 0 & -q(3) & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & -q(T - 1) & 0 \\
    0 & 0 & 0 & \cdots & 0 & -q(T) \\
\end{pmatrix}.$$
Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$. For $c \in \mathbb{R}$, we write $J' = \{u \in E: J(u) \leq c\}$ and $K = \{u \in E: J'(u) = 0\}$.

In order to obtain the critical points of the functional $J$ on $E$, we recall some basic concepts and results of the Morse theory.

**Definition 1** (see [22]). $J$ is said to satisfy the Palais–Smale condition (PS condition for short), if any sequence $\{u_n\} \subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in $E$.

In the Morse theory, functional $J$ is always required to satisfy the deformation condition (D), which was introduced by Bartsch and Li [26]. Bartolo et al. [27] proved that $J$ satisfies the deformation condition (D) if $J$ satisfies the PS condition.

Assume that $J \in C^1(E, \mathbb{R})$, and let $u_0 \in K$ be an isolated critical point of $J$ with $J(u_0) = c \in \mathbb{R}$, and $U$ is a neighborhood of $u_0$, containing the unique critical point, then we call

$$C_q(J, u_0) = \{J' \cap U, J' \cap U\backslash u_0\}, \quad q \in \mathbb{Z},$$

the $q$-th critical group of $J$ at $u_0$, where $H_q(\cdot, \cdot)$ stands for the $q$-th singular relative homology group with integer coefficients [22, 23].

We say that $u_0$ is a homological nontrivial critical point of $J$, if at least one of its critical groups is nontrivial.

Let $J \in C^1(E, \mathbb{R})$ satisfy the PS condition and all critical values of $J$ be greater than some $a \in \mathbb{R}$, then the group

$$C_q(J, \infty) = H_q(E, J'^*), \quad q \in \mathbb{Z},$$

is called the $q$-th critical group of $J$ at infinity [26].

Assume $\#K < \infty$ and $J$ satisfies the PS condition. The Morse-type numbers of the pair $(E, J'^*)$ are defined by

$$M_q = M_q(E, J'^*) = \sum_{u \in X} \dim C_q(J, u),$$

and the Betti numbers of the pair $(E, J'^*)$ are $\beta_q = \dim C_q(J, \infty)$. By the Morse theory [22, 23], the following relations hold:

$$\sum_{j=0}^{q} (-1)^{q-j} M_j \geq \sum_{j=0}^{q} (-1)^{q-j} \beta_j, \quad \forall q \in \mathbb{Z},$$

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q.$$

By (14), it follows that $M_q \geq \beta_q$ for all $q \in \mathbb{Z}$. Hence, when $\beta_q \neq 0$ for some $q \in \mathbb{Z}$ and $J$ satisfies the PS condition, then $J$ must have a critical point $u$ satisfying $C_q(J, u_0) \neq 0$. If $J$ has only one critical point $u_0$, then $C_q(J, \infty) \equiv C_q(J, u_0), \quad \forall q \in \mathbb{Z}$. If $C_q(J, \infty) \neq C_q(J, u_0)$ for some $q$, then $J$ must have another critical point $u_1 \neq u_0$. Further, if $u$ and $v$ are two critical points of $J$ and $C_q(J, u) \neq C_q(J, v)$ for some $q$, then $u \neq v$.

We need the following lemmas to prove the main results.

**Lemma 1** (see [17]). Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the (D) condition and be bounded from below. If $J$ has a critical point that is homological nontrivial and is not the minimizer of $J$, then $J$ has at least three critical points.

**Lemma 2** (see [17, 25]). Assume that $J$ has a critical point $u = 0$ with $J(0) = 0$. If $J$ has a local linking at 0 with respect to $E = V \oplus W$ and $h = \dim V < \infty$, i.e., there exists $\gamma > 0$ such that

$$\begin{cases}
J(u) \leq 0, & u \in V, \|u\| \leq \gamma, \\
J(u) > 0, & u \in W, 0 < \|u\| \leq \gamma.
\end{cases}$$

Then, $C_q(J, 0) \equiv 0$. Moreover, if $h = \mu(0)$ or $h = \mu(0) + \nu(0)$, then $C_q(J, 0) \equiv \delta_{q\mu} \mathbb{Z}$, where $\mu(0)$ and $\nu(0)$ denote the Morse index and the nullity of $J$ at 0, respectively.

From Lemma 2, if $u = 0$ is a trivial critical point and $J$ has a local linking structure at 0, then 0 is a homological nontrivial critical point of $J$.

**Lemma 3** (see [26]). Let $E$ be a Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the PS condition. Suppose $E$ splits as $E^+ \oplus E^-$ such that $J$ is bounded from below on $E^+$ and $J(u) \to -\infty$ for $u \in E^-$ as $\|u\| \to \infty$. Then, $C_q(J, \infty) \neq 0$ for $h = \dim E^-$.

**Lemma 4** (see [28]). Let $Y \subset B \subset A \subset X$ be topological spaces and $q \in \mathbb{Z}$. If

$$H_q(A, B) \neq 0,$$

$$H_q(X, Y) \neq 0,$$

then

$$H_{q+1}(X, A) \neq 0 \lor H_{q+1}(B, Y) \neq 0.$$
\[ \lambda_i |t|^2 \leq 2F(k, t) \leq \bar{\lambda}|t|^2, |t| \leq \delta, \forall k \in \mathbb{Z}(1, T). \] (22)

**Theorem 1.** Assume that the matrix \( P + Q \) is positive definite and \( f(k, t) \) satisfies (\( G_1 \)) and (\( G_2 \)). Then, problem (1) has at least two nontrivial solutions in \( S \).

To prove Theorem 1, we need a series of lemmas.

---

**Proof.** If the matrix \( P + Q \) is positive definite and (\( G_1 \)) holds. Then, \( J \) satisfies the PS condition.

Let \( S(\lambda) \) denote the eigenspace of \( \lambda_i \) for some \( l \in \mathbb{Z}(1, -1) \), \( V \) denote the subspace of \( S \) spanned by the eigenfunctions corresponding to the eigenvalues \( \lambda_{1, \ldots, \lambda_{l-1}}, \) and \( W \) denote the subspace of \( S \) spanned by the eigenfunctions corresponding to the eigenvalues \( \lambda_{l+1, \ldots, \lambda_{T}}, \) then we are given an orthogonal decomposition

\[ S = V \oplus S(\lambda) \oplus W. \] (29)

---

**Lemma 5.** If the matrix \( P + Q \) is positive definite and (\( G_1 \)) holds. Then, \( J \) satisfies the PS condition.

Proof. For any sequence \( \{u_n\} \subset S \), with \( J'(u_n) \rightarrow 0 \) as \( n \rightarrow +\infty \), there exists a positive constant \( C \in \mathbb{R} \) such that \( |J(u_n)| \leq C \). By (\( G_1 \)), we have

\[ J(u) \geq \frac{1}{2} \lambda_1 |u|^2 - a_1 T^{(1 - \theta/2)} |u|^{\theta} - a_2 T \rightarrow +\infty, \quad \text{as } |u| \rightarrow +\infty. \] (28)

---

**Lemma 6.** If the matrix \( P + Q \) is positive definite and (\( G_1 \)) holds. Then, \( J \) is coercive on \( S \), that is, \( J(u) \rightarrow +\infty \) as \( \|u\| \rightarrow +\infty \).

Proof. If the matrix \( P + Q \) is positive definite, then \( \lambda_i > 0 \), \( \forall i \in \mathbb{Z}(1, T). \) Since (\( G_1 \)) is satisfied, it implies that there exist constants \( a_1 > 0 \) and \( a_2 > 0 \) such that

\[ F(k, t) \leq a_1 |t|^\theta + a_2, \quad \text{for all } k \in \mathbb{Z}(1, T), t \in \mathbb{R}, \] (25)

then

\[ J(u) = \frac{1}{2} u^T (P + Q)u - \sum_{k=1}^{T} F(k, u(k)) \geq \frac{1}{2} \lambda_1 \|u\|^2 \]

\[ - a_1 \sum_{k=1}^{T} |u(k)|^{\theta} - a_2 T. \] (26)

Since

\[ \sum_{k=1}^{T} |u(k)|^{\theta} \leq T^{(1 - \theta/2)} |u|^{\theta}, \] (27)

we have

\[ -C\theta - \|u_n\| \leq \theta J'(u_n) (u_n) = \left( \frac{\theta}{2} - 1 \right) u_n^T (P + Q)u_n - \sum_{k=1}^{T} (\theta F(k, u_n(k)) - f(k, u_n(k))u_n(k)) \]

\[ = \left( \frac{\theta}{2} - 1 \right) u_n^T (P + Q)u_n - \sum_{|u_n(k)| \leq M} (\theta F(k, u_n(k)) - f(k, u_n(k))u_n(k)) \]

\[ - \sum_{|u_n(k)| > M} (\theta F(k, u_n(k)) - f(k, u_n(k))u_n(k)) \leq \left( \frac{\theta}{2} - 1 \right) \lambda_i \|u_n\|^2 - L, \] (24)

where \( L = \min_{(k, t) \in \mathbb{Z}(1, T) \times [-M, M]} \left( \theta F(k, t) - f(k, t)t \right). \) Then, for any \( n \in \mathbb{Z}, \)

\[ 1 - \frac{\theta}{2} \lambda_i \|u_n\|^2 - \|u_n\| \leq C\theta - L. \] (23)

Since \( 1 < \theta < 2 \), thus \( \{u_n\} \) is bounded in \( S \) and the Bolzano–Weierstrass theorem implies that \( \{u_n\} \) has a convergent subsequence. □

---

**Lemma 7.** Assume that (\( G_2 \)) holds. Then, \( J \) has a local linking at the origin with respect to \( S = (V \oplus S(\lambda)) \oplus W \) and \( h = \dim(V \oplus S(\lambda)) < T. \)

Proof. Let \( \gamma = \delta \), for each \( u \in V \oplus S(\lambda) \) and \( \|u\| \leq \delta \) implies \( |u(k)| \leq \delta, \forall k \in \mathbb{Z}(1, T). \) By (\( G_2 \)), we have

\[ J(u) = \frac{1}{2} u^T (P + Q)u - \sum_{k=1}^{T} F(k, u(k)) \]

\[ \leq \lambda_1 \|u\|^2 - \sum_{k=1}^{T} F(k, u(k)) \]

\[ \leq \sum_{k=1}^{T} \left( \frac{\lambda_1}{2} |u(k)|^{\theta} - F(k, u(k)) \right) \leq 0. \] (30)

For each \( u \in W, 0 < \|u\| \leq \delta \) implies \( |u(k)| \leq \delta, \forall k \in \mathbb{Z}(1, T). \) We have the following estimates:
This completes the proof. □

Proof. Proof of Theorem 1. By Lemmas 5 and 6, \( J \) satisfies the PS condition and is coercive, hence \( J \) is bounded from below. Combining Lemma 7 with Lemma 2, the trivial solution \( u = 0 \) is homological nontrivial and is not a minimizer. We apply Lemma 1 to conclude that problem (1) has at least two nontrivial solutions in \( S \).

Remark 1

(i) The conclusion of Theorem 1 holds, if the condition \((G_2)\) is replaced by the following condition:

\[ (G'_2) \text{ There exist } \delta > 0, \lambda_i \neq \lambda_{i+1} \text{ for some } i < T, \text{ and } \lambda \in (\lambda_i, \lambda_{i+1}) \text{ such that } \lambda_1|t|^2 \leq tf(k,t) \leq \lambda|t|^2, \text{ for } |t| \leq \delta, \forall k \in \mathbb{Z}(1, T). \]

(ii) The condition \((G_2)\) means that problem (1) is resonant near 0 at the eigenvalue \( \lambda_i \) from the right side, and \((G_2)\) contains the situation \( \limsup_{|t| \to 0} (2F(k,t)|t|^2) = \lambda \in [\lambda_i, \lambda_{i+1}] \).

(iii) If \( f(k,0) = 0 \) for each \( k \in \mathbb{Z}(1,T) \), then problem (1) admits the trivial solution \( u = 0 \). In Theorem 1, we find two nontrivial solutions for (1). From \((G_1)\) and \((G_2)\), we see that the existence of nontrivial solutions for problem (1) depends on the behaviors of the term \( f(k,t) \) or \( F(k,t) \) at 0 and at infinity \( \forall k \in \mathbb{Z}(1, T) \).

Example 1. Let \( T = 2 \), and we consider the boundary value problem (1) with

\[
F(k,t) = \begin{cases} 
    t, & \text{if } |t| < 1, \\
    \frac{t}{|t|}, & \text{if } |t| \geq 1,
\end{cases}
\]

for all \( k \in \mathbb{Z}(1,2) \). Then,

\[
F(k,t) = \begin{cases} 
    \frac{t^2}{2}, & \text{if } |t| < 1, \\
    |t| - \frac{1}{2}, & \text{if } |t| \geq 1.
\end{cases}
\]

Fix \( \theta = 3/2 \), \( p(k) = 1 \), and \( q(k) = 0 \) \( \forall k \in \mathbb{Z}(1,2) \), then the matrix \( P + Q \),

\[
P + Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},
\]

is positive definite and admits two distinct eigenvalues given by \( \lambda_1 = 1 \) and \( \lambda_2 = 3 \).

For each \( k \in \mathbb{Z}(1,2) \), we have

\[
\frac{3}{2} F(k,t) - tf(k,t) = \frac{1}{2} |t| - \frac{3}{4} \to +\infty, \text{ as } |t| \to +\infty,
\]

thus the condition \((G_1)\) holds.

Let \( \delta = 1 \), we see that

\[
|t|^2 = 2F(k,t) < 3|t|^2, \quad |t| \leq 1, \forall k \in \mathbb{Z}(1, 2),
\]

then, the condition \((G_2)\) holds. Clearly, the conditions of Theorem 1 are satisfied, and problem (1) admits at least two nontrivial solutions in \( S \).

In fact, if \( |u(k)| \geq 1 \) for \( k \in \mathbb{Z}(1,2) \), then problem (1) is

\[
\begin{aligned}
    & -2u(1) + u(2) + \frac{u(1)}{|u(1)|} = 0, \\
    & u(1) - 2u(2) + \frac{u(2)}{|u(2)|} = 0, \\
    & u(0) = u(3) = 0.
\end{aligned}
\]

We can show that \( \{u(0) = 0, u(1) = 1, u(2) = 1, u(3) = 0\} \) and \( \{u(0) = 0, u(1) = -1, u(2) = -1, u(3) = 0\} \) are the only two nontrivial solutions of problem (38) in \( S \).

Theorem 2. Assume that the matrix \( P + Q \) is negative definite, and the conditions \((G_1)\) and \((G_2)\) hold. Then, problem (1) possesses at least one nontrivial solution.

Proof. We now show that we can find at least one nontrivial critical point of functional \( J \).

Since the matrix \( P + Q \) is negative definite, in this case, \( \lambda_i < 0 \) for every \( i \in \mathbb{Z}(1,T) \) and the inequality (20) holds. If \( u \in S \) satisfies \( ||u|| \geq M \) by \((G_1)\), we have

\[
J(u) \leq \frac{\lambda_T}{2} ||u||^2 \to -\infty, \text{ as } ||u|| \to \infty.
\]

We notice that if \( -J \) is continuity and coercive, then \( J \) has at least a local maximum \( u_1 \) in \( S \) by the above inequality. This means that \( C_q(J, u_1) \equiv \delta_{q,T}Z \). If \( u_1 = 0 \), then \( C_q(J, 0) \equiv \delta_{q,T}Z \), noticing Lemma 7, this contradicts Lemma 2. Hence, \( 0 \) is different from \( u_1 \).

Theorem 3. Assume that matrix \( P + Q \) is negative semi-definite, and

\[
(G_3) tf(k,t) > 0, \quad \text{for } t \neq 0, k \in \mathbb{Z}(1, T).
\]
Then, problem (1) has no nontrivial solution.

Proof. If we assume that (1) has a nontrivial solution, then $J(u)$ has a nonzero critical point $\Pi$. Since $J'(\Pi) = 0$ and the matrix $P + Q$ is negative semidefinite, one has

$$\Pi^T(P + Q)\Pi = \sum_{k=1}^{T} f(k, \Pi(k))\Pi(k) \leq 0. \quad (41)$$

This contradicts with $(G_3)$. The proof is complete.

If the matrix $P + Q$ is just nonsingular, we may suppose that its first positive eigenvalue is $\lambda_1 > 0$. We denote its eigenvalues as follows:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{T-1} < \lambda_1 \leq \lambda_{T+1} \leq \cdots \leq \lambda_T. \quad (42)$$

Then, we have

$$\lambda_1\|u\|_1^2 \leq u^T(P + Q)u \leq \lambda_T||u||_T^2, \quad \forall u \in V \oplus S(\lambda_1),$$

$$\lambda_{T+1}\|u\|_1^2 \leq u^T(P + Q)u \leq \lambda_T||u||_T^2, \quad \forall u \in W. \quad (43)$$

$\boxdot$

Theorem 4. If the matrix $P + Q$ is just nonsingular and $2 \leq l < T$, we assume that $(G_1)$ and the following conditions hold:

(i) $(G_1)$ $\lim_{t \to 0} (f(k, t)/t) = \lambda_l$, $\forall k \in \mathbb{Z} \setminus (1, T)$

(ii) $(G_2)$ There exists $\delta > 0$ such that $2F(k, t) \geq \lambda_l|t|^2$ for $0 < |t| \leq \delta$, $\forall k \in \mathbb{Z} \setminus (1, T)$

Then, problem (1) has at least one nontrivial solution.

Proof. We show that $J$ has a local linking at $0$. By $(G_1)$, let $\varepsilon \in (0, \lambda_1 - \lambda_l)$, then there exists $\gamma \in (0, \delta)$ such that

$$|2F(k, t) - \lambda_l|t|^2| < \varepsilon|t|^2, |t| \leq \gamma, \forall k \in \mathbb{Z} \setminus (1, T). \quad (44)$$

Since $(G_2)$ holds, then we obtain

$$0 \leq 2F(k, t) - \lambda_l|t|^2 < \varepsilon|t|^2, |t| \leq \gamma, \forall k \in \mathbb{Z} \setminus (1, T). \quad (45)$$

For every $u \in W$ and $0 < \|u\| \leq \gamma$, one has $|u(k)| \leq \gamma$ $\forall k \in \mathbb{Z} \setminus (1, T)$. Then,

$$J(u) = \frac{1}{2}u^T(P + Q)u - \sum_{k=1}^{T} F(k, u(k)) \geq \frac{\lambda_{T+1}}{2}\|u\|_1^2 - \frac{\lambda_1}{2}\sum_{k=1}^{T}|u(k)|^2 - \varepsilon \sum_{k=1}^{T}|u(k)|^2 = \frac{\lambda_{T+1} - \lambda_1 - \varepsilon}{2}\|u\|_1^2 > 0. \quad (46)$$

For every $u \in V \oplus S(\lambda_1)$, $\|u\| \leq \gamma$ implies $|u(k)| \leq \gamma$ $\forall k \in \mathbb{Z} \setminus (1, T)$. Thus, we get

$$J(u) = \frac{1}{2}u^T(P + Q)u - \sum_{k=1}^{T} F(k, u(k)) \geq \frac{\lambda_{T+1}}{2}\|u\|_1^2 - \frac{\lambda_1}{2}\sum_{k=1}^{T}|u(k)|^2 = 0. \quad (47)$$

We can apply Lemma 2 to conclude that $J$ has a local linking at 0, then $C_{\lambda_1}(J, 0) \neq \emptyset$, $h = \dim(V \oplus S(\lambda_1))$, and 0 is not local maximum in $S$.

Let $\zeta$ be the element of $S(\lambda_1)$, and owing to (G_2), we have

$$\langle J''(0)\zeta, \zeta \rangle = \sum_{k=1}^{T} (\lambda_l - f'(k, 0))|\zeta(k)|^2 = 0, \quad (48)$$

therefore, $\nu(0) = \text{dim}S(\lambda_1) > 1$ and $C_{\lambda_1}(J, 0) \cong \mathbb{Z}^{T-1+\nu(0)}$. We see that $J$ satisfies the PS condition from $(G_1)$, and we have

$$J(u) \leq \frac{\lambda_{T+1}}{2}\|u\|_1^2 \longrightarrow - \infty, \quad \text{as} \quad \|u\| \longrightarrow \infty, \quad (49)$$

for every $u \in V$.

On the other hand, we have

$$J(u) \geq \frac{1}{2}\|u\|_1^2 - a_1T^{1-\theta/2}|u|^\theta - a_2T \longrightarrow + \infty, \quad \text{as} \quad \|u\| \longrightarrow \infty, \quad (50)$$

for every $u \in S(\lambda_1) \oplus W$. Hence, $J$ is bounded from below on $S(\lambda_1) \oplus W$.

It can be easily seen that the functional $J$ satisfies all the assumptions of Lemma 3; hence, we have

$$C_{\lambda_1}(J, \infty) \neq \emptyset. \quad (51)$$

We note that $l - 1 \neq l - 1 + \nu(0)$, and then $J$ has a critical point $u_1 \neq 0$ such that $C_{\lambda_1}(J, u_1) \neq \emptyset$. Thus, $J$ has at least one nontrivial solution $u_1$. $\boxdot$

3. Superlinear Case

In this section, we will study the case where $f(k, t)$ is superlinear in $t$ at infinity for each $k \in \mathbb{Z} \setminus (1, T)$. We give the following assumption:

(i) $(G_b)$ There exist $\beta \in (2, \infty)\text{and} M_1 > 0$ such that

$$tf(k, t) \geq \beta F(k, t) > 0, \quad \text{for}\quad |t| \geq M_1, \forall k \in \mathbb{Z} \setminus (1, T). \quad (52)$$

We apply a similar method in [18] to look for nontrivial critical points of $f$ in $S$.

In order to prove the main results of this section, we need to make use of the following lemmas.

Lemma 8. If $(G_b)$ holds, then $J$ satisfies the PS condition.

Proof. Suppose that the eigenvalues of the matrix $P + Q$ are $\lambda_1, \lambda_2, \ldots, \lambda_T$, let $\lambda_{\max} = \max|\lambda_1, |\lambda_2, \ldots, |\lambda_T| > 0$. Using $(G_b)$, there exist constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$F(k, t) \geq \alpha_1|t|^{\beta} - \alpha_2, \quad \text{for all}\quad k \in \mathbb{Z} \setminus (1, T), t \in \mathbb{R}. \quad (53)$$

For any sequence $\{u_n\} \subset E, |J(u_n)| \leq C$ for some $C \in \mathbb{R}$ and $J'(u_n) \longrightarrow 0$ as $n \longrightarrow + \infty$, then we have

$$-C \leq J(u_n) = \frac{1}{2}\|u_n\|_1^2 - \sum_{k=1}^{T} F(k, u_n(k)) \leq \frac{1}{2}\|u_n\|_1^2 - a_3\sum_{k=1}^{T}|u_n(k)|^\beta + a_4T. \quad (54)$$

Since
Lemma 9. Assume that \((G_6)\) holds. Then, there exists a constant \(D > 0\) such that

\[
\sum_{k=1}^{T} |u_n(k)|^2 \leq T^{(1-2\beta)} \left( \sum_{k=1}^{T} |u_n(k)|^\beta \right)^{(2/\beta)},
\]
then,

\[
\sum_{k=1}^{T} |u_n(k)|^\beta \geq T^{(1-2\beta)} \|u_n\|^\beta.
\]

We have

\[-C \leq f(u_n) \leq \frac{1}{2} \lambda_{\max} |u_n|^2 - a_4 T^{(1-2\beta)} |u_n|^\beta + a_4 T.
\]

For any \(n \in \mathbb{Z}\),

\[a_4 T^{(1-2\beta)} |u_n|^{\beta} \leq \frac{1}{2} \lambda_{\max} |u_n|^2 \leq a_4 T + C.
\]

Note that \(\beta > 2\), thus \(\{u_n\}\) is bounded in \(S\), and the Bolzano–Weierstrass theorem implies that \(\{u_n\}\) has a convergent subsequence. \(\square\)

Let \(\beta < (-3TM_1/2)D\), \(s > 0\), and

\[J(su) = \frac{s^2}{2} \| (P + Q)u - \sum_{k=1}^{T} F(k, su(k)) \| \leq s, \quad \text{for } u \in S^{T-1}.
\]

We can compute the derivative of \(J(su)\) with respect to \(s\) and obtain

\[
\frac{dJ(su)}{ds} = s(u^T (P + Q)u - \sum_{k=1}^{T} u(k) f(k, su(k))\right).
\]

\[
\leq \frac{2}{s} \left( \sum_{k=1}^{T} F(k, su(k)) - \frac{1}{2} \sum_{k=1}^{T} su(k) f(k, su(k)) + \bar{\alpha} \right)
\]

\[
\leq \frac{2}{s} \left( \left( \frac{1}{\beta} - \frac{1}{2} \right) \sum_{|u(k)| > M_1} |u(k) f(k, u(k)) + \frac{3TM_1}{2} D + \bar{\alpha} \right)
\]

\[
\leq \frac{2}{s} \left( \left( \frac{1}{\beta} - \frac{1}{2} \right) \sum_{|u(k)| > M_1} su(k) f(k, su(k)) < 0.
\]
By the implicit function theorem, there exists a unique \( \varphi \in C(S^{T-1}, \mathbb{R}) \) such that \( s = \varphi(u) \) and \( f(\varphi(u)u) = \tilde{a} \) for \( u \in S^T \). If \( u \neq 0 \), let \( \tilde{\varphi}(u) = ((1/\|u\|)\varphi(\|u\|)) \), then \( \tilde{\varphi} \in C(S(0\|u\|)) \) and \( f(\tilde{\varphi}(u)tu) = \tilde{a} \), for \( u \in S\{0\} \). Further, if \( f(u) = \tilde{a} \), then \( \tilde{\varphi}(u) = 1 \).

Now, we define a function

\[
\Phi(u) = \begin{cases} 
\tilde{\varphi}(u), & \text{if } f(u) \geq \tilde{a}, \\
1, & \text{if } f(u) \leq \tilde{a}.
\end{cases}
\]  

(65)

Obviously, \( \Phi : S\{0\} \to \mathbb{R} \) is a continuous function.

We set a map \( \mu : [0, 1] \times (S\{0\}) \to S\{0\} \) as \( \mu(s, u) = (1-s)u + s\tilde{\varphi}(u)u \) since \( f^\mu \) is a strong deformation retract of \( S\{0\} \), that is, \( f^\mu \equiv S\{0\} \equiv S^{T-1} \).

**Theorem 5.** Assume that the conditions \( (G_2) \) and \( (G_3) \) hold. Then, problem (1) has at least one nontrivial solution.

**Proof.** Since the conditions \( (G_2) \) and \( (G_3) \) hold, it follows from Lemma 2, Lemma 7, and ([23], Theorem 4.2) that there exists a \( \varepsilon > 0 \) such that

\[
H_h(J^\varepsilon, J^\varepsilon) = C_h(J, 0) \neq 0,
\]  

(66)

where \( h = \dim(V \oplus S(\lambda_i)) \).

By Lemma 9, we have

\[
H_h(S, J^\varepsilon) \equiv H_h(S, S^{T-1}) = 0.
\]  

(67)

Using Lemma 4,

\[
H_{h+1}(S, J^\varepsilon) \neq 0 \text{ or } H_{h-1}(J^-\varepsilon, J^\varepsilon) \neq 0.
\]  

(68)

Hence, \( J \) has a critical point \( u \neq 0 \) for which

\[
|J(u)| > e\tilde{a} \leq |J(u)| < -\varepsilon.
\]  

(69)

The proof is complete.

Note that the conclusion of Theorem 5 holds if the symmetric matrix \( P + Q \) is positive definite or negative definite or just nonsingular.

**Example 2.** Let \( T = 3 \), and we study the boundary value problem (1) with

\[
f(k, t) = f(t) = \begin{cases} 
2t, & \text{if } |t| < \frac{1}{4}, \\
\frac{t}{\sqrt{|t|}}, & \text{if } \frac{1}{4} \leq |t| \leq 1, \\
|t|^4t, & \text{if } |t| > 1,
\end{cases}
\]  

(70)

for \( k \in \mathbb{Z}(1, 3) \).

Then,

\[
F(k, t) = F(t) = \begin{cases} 
2t^2, & \text{if } |t| < \frac{1}{4}, \\
\frac{2}{3}|t|^{3/2} - \frac{1}{48}, & \text{if } \frac{1}{4} \leq |t| \leq 1, \\
\frac{|t|^4}{4} + \frac{19}{48}, & \text{if } |t| > 1.
\end{cases}
\]  

(71)

Let \( \beta = 3, \ p(k) = 1, \) and \( q(k) = 0 \) for each \( k \in \mathbb{Z}(1, 3) \). The matrix \( P + Q \),

\[
P + Q = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix},
\]  

(72)

admits three distinct eigenvalues given by

\[
\lambda_1 = 2 - \sqrt{2}, \lambda_2 = 2, \lambda_3 = 2 + \sqrt{2}.
\]  

(73)

Clearly, we have

\[
2|t|^2 = 2F(k, t)(2 + \sqrt{2})|t|^2, \quad |t| \leq \frac{1}{4}, \forall k \in \mathbb{Z}(1, 3),
\]  

(74)

and

\[
t^2F(k, t) - 3F(k, t) = \frac{|t|^4}{4} - \frac{19}{16} \to +\infty, \quad |t| \to +\infty, \forall k \in \mathbb{Z}(1, 3).
\]  

(75)

This means that the conditions \( (G_2) \) and \( (G_3) \) hold. All conditions of Theorem 5 are verified, and then problem (1) admits one nontrivial solution in \( S \).

**Corollary 1.** Let \( (G_2) \) and \( (G_3) \) hold and \( \dim(V \oplus S(\lambda_i)) \neq 1 \neq T \). Then, problem (1) possesses at least two nontrivial solutions.

**Proof.** We notice that condition \( (G_3) \) implies (53). Combining with (56), we have

\[
J(u) = \frac{1}{2}u^T(P + Q)u - \sum_{k=1}^{T} F(k, u(k))
\]  

\[
\leq \frac{\lambda_1}{2}\|u\|^2 - a_3 \sum_{k=1}^{T} \|u(k)\|^\beta + a_4T
\]

\[
\leq \frac{\lambda_1T}{2}\|u\|^2 - a_3T(1-\beta/2)\|u\|^\beta
\]  

\[
+ a_4T \to -\infty \text{ as } \|u\| \to \infty.
\]

Hence, \( -J \) is coercive, and \( J \) has a local maximum at some \( u_1 \in S \). This means \( C_{eq}(J, u_1) \equiv \delta_{u_1} \mathbb{Z} \). Note that the condition \( (G_3) \) holds, so \( u_i \neq 0 \) by Lemma 7. Now, \( \dim(V \oplus S(\lambda_i)) \neq 1 \neq T \), combining with (68) and (69), there exists another nontrivial critical point \( u_2 \neq u_1 \) of \( J \).

**Example 3.** Let \( T = 3 \), and we consider the following boundary value problem (1) with

\[
f(k, t) = f(t) = \begin{cases} 
-2t, & \text{if } |t| < \frac{1}{4}, \\
\frac{-t}{\sqrt{|t|}}, & \text{if } \frac{1}{4} \leq |t| \leq 1, \\
-|t|^4t, & \text{if } |t| > 1,
\end{cases}
\]  

(77)

for \( k \in \mathbb{Z}(1, 3) \).
Complexity

Then,

\[
F(k, t) = F(t) = \begin{cases} 
-t^2, & \text{if } |t| < \frac{1}{4}, \\
\frac{2}{3}|t|^{3/2} + \frac{1}{48} & \text{if } \frac{1}{4} \leq |t| \leq 1, \\
-\frac{|t|^4}{4} - \frac{19}{48} & \text{if } |t| > 1. 
\end{cases}
\]  

(78)

Let \( p(k) = 1 \) and \( q(k) = 7/2 \) for each \( k \in \mathbb{Z}(1, 3) \). Then, the matrix

\[
P + Q = \begin{pmatrix} \frac{3}{2} & -1 & 0 \\ -1 & \frac{3}{2} & -1 \\ 0 & -1 & \frac{3}{2} \end{pmatrix},
\]  

(79)

is negative definite, and the eigenvalues are given by

\[
\lambda_1 = \frac{3 + 2\sqrt{2}}{2}, \quad \lambda_2 = \frac{3}{2}, \quad \lambda_3 = \frac{2\sqrt{2} - 3}{2}.
\]  

(80)

Let \( \delta = 1/4 \) and \( \beta = 8 \), then we have

\[
-\frac{3 + 2\sqrt{2}}{2}|t|^{3/2} < 2F(k, t) < -\frac{3}{2}|t|^3, \quad |t| \leq \frac{1}{4}, \quad \forall k \in \mathbb{Z}(1, 3),
\]  

(81)

and

\[
tf(k, t) - 8F(k, t) = |t|^3 + \frac{19}{6} \to +\infty, \quad \text{as } |t| \to +\infty, \quad \forall k \in \mathbb{Z}(1, 3).
\]  

(82)

Then, the conditions \( (G_2) \) and \( (G_3) \) hold. Moreover, \( \dim S(\lambda_1) = 1 \), and obviously, \( \dim S(\lambda_1) \pm 1 \neq 3 \). Then, problem (1) admits at least two nontrivial solutions in S by Corollary 1.

4. Conclusion

In this paper, we have studied the existence of one or multiple nontrivial solutions of the boundary value problem (1) for two cases: (i) \( f \) is sublinear and (ii) \( f \) is superlinear. Examples are provided to illustrate the applicability of our results.

We point out that our work is different from the previous works and the results of this paper improve, extend, and complement some related results in the literature [12]. Since our solutions are obtained by the variational approach combining with the Morse theory, functional \( J \) is required to satisfy the PS conditions, and some computations on the critical group of a local linking-type critical point are used to deal with the resonance at zero.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Both the authors contributed equally to the writing of this paper and they read and approved the final manuscript.

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References

[1] G. H. Lin, Z. Zhou, and J. S. Yu, “Ground state solutions of discrete asymptotically linear Schrödinger equations with bounded and non-periodic potentials,” Journal of Dynamics and Differential Equations, vol. 32, no. 2, pp. 527–555, 2020.

[2] Z. Zhou and J. S. Yu, “Homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity,” Acta Mathematica Sinica, English Series, vol. 29, no. 9, pp. 1809–1822, 2013.

[3] Z. Zhou and D. F. Ma, “Multiplicity results of breathers for the discrete nonlinear Schrödinger equations with unbounded potentials,” Science China Mathematics, vol. 58, no. 4, pp. 781–790, 2015.

[4] G. H. Lin and Z. Zhou, “Homoclinic solutions of discrete \( \phi \)-Laplacian equations with mixed nonlinearities,” Communications on Pure & Applied Analysis, vol. 17, pp. 1723–1747, 2018.

[5] Q. Q. Zhang, “Homoclinic orbits for discrete Hamiltonian systems with local super-quadratic conditions,” Communications on Pure & Applied Analysis, vol. 18, pp. 425–434, 2019.

[6] P. Mei and Z. Zhou, “Periodic and subharmonic solutions for a 2nth-order \( p \)-Laplacian difference equation containing both advances and retardations,” Open Mathematics, vol. 16, no. 1, pp. 1435–1444, 2018.

[7] Y. H. Long and C. L. Chen, “Existence of multiple solutions to second-order discrete Neumann boundary value problems,” Applied Mathematics Letters, vol. 83, pp. 7–14, 2018.

[8] Z. Zhou and J. X. Ling, “Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with \( \phi \)-Laplacian,” Applied Mathematics Letters, vol. 91, pp. 28–34, 2019.

[9] J. S. W. Wong, “On the generalized Emden–Fowler equation,” SIAM Review, vol. 17, no. 2, pp. 339–360, 1975.

[10] R. P. Agarwal, Equations and Inequalities Theory, Methods, and Applications, Marcel Dekker, New York, NY, USA, 2000.

[11] J. S. Yu and B. Zheng, “Modeling Wolbachia infection in mosquito population via discrete dynamical model,” Journal of Difference Equations and Applications, vol. 25, pp. 1549–1567, 2019.

[12] J. S. Yu and Z. M. Guo, “On boundary value problems for a discrete generalized Emden–Fowler equation,” Journal of Differential Equations, vol. 231, pp. 18–31, 2006.

[13] F. M. Atici and A. Cabada, “Existence and uniqueness results for discrete second-order periodic boundary value problems,”

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References

[1] G. H. Lin, Z. Zhou, and J. S. Yu, “Ground state solutions of discrete asymptotically linear Schrödinger equations with bounded and non-periodic potentials,” Journal of Dynamics and Differential Equations, vol. 32, no. 2, pp. 527–555, 2020.

[2] Z. Zhou and J. S. Yu, “Homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity,” Acta Mathematica Sinica, English Series, vol. 29, no. 9, pp. 1809–1822, 2013.

[3] Z. Zhou and D. F. Ma, “Multiplicity results of breathers for the discrete nonlinear Schrödinger equations with unbounded potentials,” Science China Mathematics, vol. 58, no. 4, pp. 781–790, 2015.

[4] G. H. Lin and Z. Zhou, “Homoclinic solutions of discrete \( \phi \)-Laplacian equations with mixed nonlinearities,” Communications on Pure & Applied Analysis, vol. 17, pp. 1723–1747, 2018.

[5] Q. Q. Zhang, “Homoclinic orbits for discrete Hamiltonian systems with local super-quadratic conditions,” Communications on Pure & Applied Analysis, vol. 18, pp. 425–434, 2019.

[6] P. Mei and Z. Zhou, “Periodic and subharmonic solutions for a 2nth-order \( p \)-Laplacian difference equation containing both advances and retardations,” Open Mathematics, vol. 16, no. 1, pp. 1435–1444, 2018.

[7] Y. H. Long and C. L. Chen, “Existence of multiple solutions to second-order discrete Neumann boundary value problems,” Applied Mathematics Letters, vol. 83, pp. 7–14, 2018.

[8] Z. Zhou and J. X. Ling, “Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with \( \phi \)-Laplacian,” Applied Mathematics Letters, vol. 91, pp. 28–34, 2019.

[9] J. S. W. Wong, “On the generalized Emden–Fowler equation,” SIAM Review, vol. 17, no. 2, pp. 339–360, 1975.

[10] R. P. Agarwal, Equations and Inequalities Theory, Methods, and Applications, Marcel Dekker, New York, NY, USA, 2000.

[11] J. S. Yu and B. Zheng, “Modeling Wolbachia infection in mosquito population via discrete dynamical model,” Journal of Difference Equations and Applications, vol. 25, pp. 1549–1567, 2019.

[12] J. S. Yu and Z. M. Guo, “On boundary value problems for a discrete generalized Emden–Fowler equation,” Journal of Differential Equations, vol. 231, pp. 18–31, 2006.

[13] F. M. Atici and A. Cabada, “Existence and uniqueness results for discrete second-order periodic boundary value problems,”
J. S. Yu, Z. M. Guo, and X. F. Zou, "Positive periodic solutions of second order self-adjoint difference equations," *Journal of the London Mathematical Society*, vol. 71, pp. 146–160, 2005.

H. H. Liang and P. X. Weng, "Existence and multiple solutions for a second-order difference boundary value problem via critical point theory," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 511–520, 2007.

X. M. He and X. Wu, "Existence and multiplicity of solutions for nonlinear second order difference boundary value problems," *Computers & Mathematics with Applications*, vol. 57, no. 1, pp. 1–8, 2009.

J. Q. Liu and J. B. Su, "Remarks on multiple nontrivial solutions for quasi-linear resonant problems," *Journal of Difference Equations and Applications*, vol. 258, no. 1, pp. 209–222, 2001.

S. B. Liu, "Existence of solutions to a superlinear p-Laplacian equation," *Electronic Journal of Differential Equations*, vol. 2001, no. 66, pp. 1–6, 2001.

J. S. Liu, S. L. Wang, and J. M. Zhang, "Multiple solutions for boundary value problems of second-order difference equations with resonance," *Journal of Difference Equations and Applications*, vol. 374, no. 1, pp. 187–196, 2011.

F. H. Tan and Z. M. Guo, "Periodic solutions for second-order difference equations with resonance at infinity," *Journal of Difference Equations and Applications*, vol. 18, no. 1, pp. 149–161, 2012.

S. L. Wang, J. S. Liu, J. M. Zhang, and F. Zhang, "Existence of non-trivial solutions for resonant difference equations," *Journal of Difference Equations and Applications*, vol. 19, no. 2, pp. 209–222, 2013.

J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, NY, USA, 1989.

K. C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser-Boston, Cambridge, MA, USA, 1993.

Y. H. Long, H. P. Shi, and X. Q. Deng, "Nontrivial periodic solutions to delay difference equations via Morse theory," *Open Mathematics*, vol. 16, no. 1, pp. 885–896, 2018.

J. B. Su, "Multiplicity results for asymptotically linear elliptic problems at resonance," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 397–408, 2003.

T. Bartsch and S. J. Li, "Critical point theory for asymptotically quadratic functionals and applications to problems with resonance," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 3, pp. 419–441, 1997.

P. Bartolo, V. Benci, and D. Fortunato, "Abstract critical point theorems and applications to nonlinear problems with "strong" resonance at infinity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 9, pp. 981–1012, 1983.

K. Perera, "Critical groups of critical points produced by local linking with applications," *Abstract and Applied Analysis*, vol. 3, no. 3-4, pp. 437–446, 1998.