BGG CORRESPONDENCE AND RÖMER’S THEOREM
ON AN EXTERIOR ALGEBRA

KOHJI YANAGAWA

To the memory of Professor Tetsushi Ogoma

ABSTRACT. Let $E = K\langle y_1, \ldots, y_n \rangle$ be the exterior algebra. The (cohomological) distinguished pairs of a graded $E$-module $N$ describe the growth of a minimal graded injective resolution of $N$. Römer gave a duality theorem between the distinguished pairs of $N$ and those of its dual $N^*$. In this paper, we show that under Bernstein-Gel’fand-Gel’fand correspondence, his theorem is translated into a natural corollary of local duality for (complexes of) graded $S = K[x_1, \ldots, x_n]$-modules. Using this idea, we also give a $\mathbb{Z}^n$-graded version of Römer’s theorem.

INTRODUCTION

In this section, to introduce a background of the present paper, we summarize results of Aramova-Herzog [2] and Römer [11].

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$, and $M$ a finitely generated graded $S$-module. The $ij$th Betti number $\beta_{i,j}(M) = \dim_K \mathrm{Tor}_i^{S}(K, M)_j$ of $M$ is an important invariant. Following Bayer-Charalambous-Popescu [3], we say a Betti number $\beta_{k,m}(M) \neq 0$ is extremal, if $\beta_{i,j}(M) = 0$ for all $(i, j) \neq (k, m)$ with $i \geq k$ and $j - i \geq m - k$. This notion has two remarkable properties. First, a homogeneous ideal $I \subset S$ has the same extremal Betti numbers as its generic initial ideal $\mathrm{Gin}(I)$ with respect to the reverse lexicographic term order on $S$. Another important property is the following.

Theorem A (Bayer-Charalambous-Popescu, [3 Theorem 2.8]) Let $\Delta \subset 2^{\{1, \ldots, n\}}$ be a simplicial complex, and $K[\Delta] = S/I_{\Delta}$ the Stanley-Reisner ring. And let $\Delta^\vee$ be the Alexander dual complex of $\Delta$. Then $\beta_{i,i+j}(K[\Delta])$ is extremal if and only if so is $\beta_{j,i+j}(I_{\Delta^\vee})$. Moreover, if this is the case, then $\beta_{i,i+j}(K[\Delta]) = \beta_{j,i+j}(I_{\Delta^\vee})$.

We have $\beta_{i,n}(K[\Delta]) = \dim_K \tilde{H}_{n-i-1}(|\Delta|; K) =: \tilde{h}_{n-i-1}(|\Delta|)$ by Hochster’s formula. If $\beta_{i,n}(K[\Delta]) \neq 0$ then it is always an extremal Betti number. The equality $\tilde{h}_{n-i-1}(|\Delta|) = \beta_{i,n}(K[\Delta]) = \beta_{n-i,n}(I_{\Delta^\vee}) = \beta_{n-i+1,n}(K[\Delta^\vee]) = \tilde{h}_{i-2}(|\Delta^\vee|)$ induced by Theorem A corresponds to the usual Alexander duality. More generally, Theorem A gives an Alexander duality for (some) iterated Betti numbers (c.f. [9 4]).

Let $E = K\langle y_1, \ldots, y_n \rangle$ be the exterior algebra. To understand Theorem A, Aramova-Herzog [2] introduced distinguished pairs for a graded $E$-module $N$. See Definition 1.6 below. (We use a different convention to describe these pairs. See Remark 1.7.) The distinguished pairs of $N$ roughly describe the growth of the minimal graded (infinite) injective resolution of $N$. Let $K\{\Delta\} = E/J_{\Delta}$ be the
There is an equivalence $D$ and $\text{gr} S$. Definition 1.1. We say $(\text{Ext}^\bullet \omega)$, the category of graded $S$ generated graded $S$ modules is equivalent to the similar category $D^b(\text{gr} E)$ for $E$. So we will freely identify these categories. For $M$, the Krull dimension of the $0$ module is $-\infty$. So we also hold for $Z^*$, but the arguments in [2, 11] are hard to work in this context. But, since BGG correspondence also holds for $Z^*$-graded modules, our method is powerful in this context too. See §2. This part of the present paper is a continuation of the author’s previous paper [13].

1. $Z$-Graded Case

Let $W$ be an $n$-dimensional vector space over a field $K$, and $S = \bigoplus_{i \geq 0} \text{Sym}_i W$ the polynomial ring. We regard $S$ as a graded ring with $S_i = \text{Sym}_i W$. Let $\text{Gr} S$ be the category of graded $S$-modules and their degree preserving $S$-homomorphisms, and $\text{gr} S$ the full subcategory of $\text{Gr} S$ consisting of finitely generated modules. Then there is an equivalence $D^b(\text{gr} S) \cong D^b_{\text{gr} S}(\text{Gr} S)$. (For derived categories, consult [8].) So we will freely identify these categories. For $M = \bigoplus_{i \in Z} M_i$ in $\text{Gr} S$ and an integer $j$, $M(j)$ denotes the shifted module with $M(j)_i = M_{i+j}$. For $M^* \in D^b(\text{Gr} S)$, $M^*[j]$ denotes the $j$th translation of $M^*$, that is, $M^*[j]$ is the complex with $M^*[j]^i = M^{i+j}$.

So, if $M \in \text{Gr} S$, $M[j]$ is the cochain complex $\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$, where $M$ sits in the $(-j)$th position. If $M, N \in \text{Gr} S$, then $\text{Hom}_S(M, N)$ has the structure of a graded $S$-module with $\text{Hom}_S(M, N)_i = \text{Hom}_{\text{Gr} S}(M, N(i))$.

Let $\omega^* \in D^b(\text{gr} S)$ be a minimal graded injective resolution of $S(-n)[n]$. That is, $\omega^*$ is a graded normalized dualizing complex of $S$. Then $D_S(-) := \text{Hom}^\bullet_S(-, \omega^*)$ gives a duality functor from $D^b(\text{gr} S)$ to itself. The $ith$ cohomology of $D_S(M^*)$ is $\text{Ext}^i_S(M^*, \omega^*$). For $M^* \in D^b(\text{gr} S)$ and $i \in Z$, set $d_i(M^*) := \dim_S H^i(M^*)$. Here the Krull dimension of the $0$ module is $-\infty$.

Definition 1.1. We say $(d, i) \in N \times Z$ is a distinguished pair for a complex $M^* \in D^b(\text{gr} S)$, if $d = d_i(M^*)$ and $d_j(M^*) < d + i - j$ for all $j$ with $j < i$. 


Let $M^\bullet \in D^b(\text{gr} S)$ and $d = d_0(M^\bullet) \geq 0$. If $d = \max\{d_j(M^\bullet) \mid j \in \mathbb{Z}\}$, then $(d, i)$ is distinguished for $M^\bullet$. On the other hand, if $i = \min\{j \mid H^j(M^\bullet) \neq 0\}$, then $(d, i)$ is also distinguished. Thus $M^\bullet$ has several distinguished pairs in general.

In this paper, $\deg_S(M)$ denotes the multiplicity of a module $M \in \text{gr} S$ (i.e., $e(M)$ of [\text{5}] Definition 4.1.5).

**Theorem 1.2.** For $M^\bullet \in D^b(\text{gr} S)$, we have the following.

1. A pair $(d, i)$ is distinguished for $M^\bullet$ if and only if $(d, -d - i)$ is distinguished for $D_S(M^\bullet)$.

2. If $(d, i)$ is a distinguished pair for $M^\bullet$, then

$$\deg_S H^i(M^\bullet) = \deg_S \text{Ext}_S^{-d-i}(M^\bullet, \omega^\bullet).$$

**Proof.** (1) Since the statement is “symmetric”, it suffices to prove the direction $\Rightarrow$.

From the double complex $\text{Hom}^\bullet_S(M^\bullet, \omega^\bullet)$, we have a spectral sequence $E^{p,q}_2 = \text{Ext}^p_S(H^{-q}(M^\bullet), \omega^\bullet) \Rightarrow \text{Ext}^{p+q}_S(M^\bullet, \omega^\bullet)$. For simplicity, set $e^{p,q} = \text{dim}_S E^{p,q}_2$. Since $\text{Ext}^p_S(M, \omega^\bullet) \cong \text{Ext}^{p+1}_S(M, S(-n))$ for $M \in \text{gr} S$, the following inequality follows from argument analogous to [\text{5}] §8.1, Theorem 8.1.1.

$$(1.1) \quad e^{p,q}_2 = \dim_S \text{Ext}^p_S(H^{-q}(M^\bullet), \omega^\bullet) = \begin{cases} -p & \text{if } p = -d-q(M^\bullet), \\ \leq -p & \text{if } -d-q(M^\bullet) < p \leq 0, \\ -\infty & \text{otherwise}. \end{cases}$$

(I) By (1.1), we have $e^{d,-i}_2 = d$. On the other hand, we have $e^{p,q}_2 < d$ for all $(p, q) \neq (-d, -i)$ with $p + q = -d - i$. In fact, the assertion follows from (1.1) if $p > -d$. So we may assume that $p < -d$ and $q = -d - i - p > -i$. Since $(d, i)$ is distinguished, we have $d_{-q}(M^\bullet) < d + i + q = -p$. Thus $E^{p,q}_2 = 0$ in this case. Anyway, we have $e^{p,q}_2 < d$ for all $(p, q) \neq (-d, -i)$ with $p + q = -d - i$.

(II) Since $d_{i+j+1}(M^\bullet) < d+j-1 < d+j$ for all $j \geq 2$, we have that $E^{d-j,-i+j-1}_2 = 0$. So we have $E^{d-j,-i+j-1}_r = 0$ for all $r \geq 2$. Next we will show that $d = e^{d,-i}_2 = e^{d,-i}_3 = \cdots = e^{d,-i}_r$ by induction on $r$. Recall that $E^{-d,-i}_r$ is the cohomology of

$$E^{-d,-i}_{r+1} \rightarrow E^{-d,-i}_r \rightarrow E^{d+r,-i+r+1}_r.$$

But we have seen that $E^{d-r,-i+r+1}_r = 0$. Moreover, $e^{d+r,-i+r+1}_r \leq e^{d-r,-i+r+1}_r \leq d - r < d$ by (1.1), and $e^{d,-i}_r = d$ by the induction hypothesis. Thus $e^{d,-i}_r = d$. Hence $e^{d,-i}_r = d$. From this fact and (I), we have that $\dim_S \text{Ext}^{-d-i}_S(M^\bullet, \omega^\bullet) = d$.

(III) Finally, we will show that $\dim_S \text{Ext}^{-d-i-j}_S(M^\bullet, \omega^\bullet) < d+j$ for all $j > 0$. To see this, it suffices to show that $e^{d,j}_2 < d+j$ for all $j > 0$ and all $(p, q)$ with $p + q = -d - i - j$. If $p > -d - j$, the assertion is clear. If $p = -d - j$, then $q = -i$ and $d_{-q}(M^\bullet) = d < -p$. So $E^{p,q}_2 = 0$ in this case. Hence we may assume that $p < -d - j$ and $q = d + i + j + p < i$. Since $(d, i)$ is distinguished, $d_{-q}(M^\bullet) < d + (i + q) = -j - p < -p$. So we have $E^{p,q}_2 = 0$ in this case too.

(2) Since $\deg_S E^{-d,-i}_r = \deg_S E^{-d,-i}_{r+1}$ for all $r \geq 2$ by the argument in (II) of the proof of (1), we have $\deg_S E^{-d,-i}_2 = \deg_S E^{-d,-i}_\infty$. So we have

$$\deg_S \text{Ext}^{-d-i}_S(M^\bullet, \omega^\bullet) = \deg_S E^{-d,-i}_\infty = \deg_S E^{-d,-i}_2 = \deg_S \text{Ext}^{-d-i}_S(H^i(M^\bullet), \omega^\bullet),$$

where $i = \min\{j \mid H^j(M^\bullet) \neq 0\}$.
where the first equality follows from (I) and (II).

For a module $M \in \text{gr} S$ of dimension $d$, we have $\text{deg}_S M = \text{deg}_S \text{Ext}^{-d}_S(M, \omega^*)$. In fact, for a prime ideal $p \subset S$ with $\dim S/p = d$, let $x$ be a maximal $S_p$-sequence contained in $\text{Ann}_{S_p}(M_p)$, and $R := S_p/xS_p$ an artinian Gorenstein local ring. Then we have

$$\text{Ext}^{-d}_S(M, \omega^*) \otimes_S S_p \cong \text{Ext}^{n-d}_S(M_p, S_p) \cong \text{Hom}_R(M_p, R).$$

Therefore,

$$l_{S_p} (\text{Ext}^{-d}_S(M, \omega^*) \otimes_S S_p) = l_R (\text{Hom}_R(M_p, R)) = l_R (M_p) = l_{S_p} (M_p).$$

Since $\dim_S (H^i(M^*)) = d$, we have $\text{deg}_S \text{Ext}^{-d}_S(H^i(M), \omega^*) = \text{deg}_S H^i(M^*)$. \qed

Remark 1.3. Theorem 1.2 (1) also holds for a noetherian local ring $A$ admitting a dualizing complex. The part (2) also holds for $A$, if we replace $\text{deg}_S(-)$ by $l_{A_p}(- \otimes_A A_p)$ for a prime ideal $p \subset A$ with $\dim A/p = d$.

Let $V$ be the dual vector space of $W$, and $E = \bigwedge V$ the exterior algebra. We regard $E$ as a negatively graded ring with $E_{-i} = \bigwedge^i V$ (this is the opposite convention from [2, 11]). Let $\text{gr} E$ be the category of finitely generated graded $E$-modules and their degree preserving $E$-homomorphisms. Here “$E$-module” means a left and right module $N$ with $ea = (-1)^{(\text{deg} e)(\text{deg} a)}ae$ for all homogeneous elements $e \in E$ and $a \in N$.

Let $\{x_1, \ldots, x_n\}$ be a basis of $W$, and $\{y_1, \ldots, y_n\}$ its dual basis of $V$. For a complex $N^\bullet$ in $\text{gr} E$, set $L(N^\bullet) = \bigoplus_{i \in \mathbb{Z}} S \otimes_K N_i$ and $L(N^\bullet)^m = \bigoplus_{i-j=m} S \otimes_K N_i$. The differential defined by

$$L(N^\bullet)^m \supset S \otimes_K N_j^i \ni z \mapsto \sum_{1 \leq i \leq n} x_i \otimes y_i z + (-1)^m(1 \otimes \delta^i(z)) \in L(N^\bullet)^{m+1}$$

makes $L(N^\bullet)$ a cochain complex of free $S$-modules. Here $\delta^i$ is the $i$th differential map of $N^\bullet$. Moreover, $L$ gives a functor from $D^b(\text{gr} E)$ to $D^b(\text{gr} S)$.

For $M \in \text{gr} S$ and $i \in \mathbb{Z}$, we can define a graded $E$-module structure on $\text{Hom}_K(E, M_i)$ by $(af)(e) = f(ae)$. Then $\text{Hom}_K(E, M_i) \cong E_{-i} \otimes_K M_i$. Set $R(M) = \text{Hom}_K(E, M)$ and $R^i(M) = \text{Hom}_K(E, M_i)$. The differential defined by

$$R^i(M) = \text{Hom}_K(E, M_i) \ni f \mapsto [e \mapsto \sum_{1 \leq j \leq n} x_j f(y_j e)] \in \text{Hom}_K(E, M_{i+1}) = R^{i+1}(M)$$

makes $R(M)$ a cochain complex of free $E$-modules. We can also construct $R(M^\bullet)$ from a complex $M^\bullet$ in natural way. Then $R$ gives a functor from $D^b(\text{gr} S)$ to $D^b(\text{gr} E)$. See [3] for details. The following is a crucial result.

Theorem 1.4 (BGG correspondence, c.f. [3]). The functors $L$ and $R$ give a category equivalence $D^b(\text{gr} S) \cong D^b(\text{gr} E)$.

For $N \in \text{gr} E$, then $N^* := \text{Hom}_E(N, E) \cong \text{Hom}_K(N, K)(n)$ is a graded $E$-module again. $(-)^*$ gives an exact duality functor on $\text{gr} E$, and it can be extended to the duality functor $D_E$ on $D^b(\text{gr} E)$.
Proposition 1.5. For $N^\bullet \in D^b(\text{gr} E)$, we have
$$D_S \circ L(N^\bullet) \cong L \circ D_E(N^\bullet)(-2n)[2n].$$

Proof. Since $L(N^\bullet)$ consists of free $S$-modules, we have
$$D_S \circ L(N^\bullet) \cong \text{Hom}^*_S(L(N^\bullet), S(-n)[n]).$$

It is easy to see that
$$\text{Hom}^m_S(L(N^\bullet), S(-n)[n]) \cong \bigoplus_{j-i=m+n} S(-n) \otimes_K (N^i_j)^\vee,$$
where $(-)^\vee$ means the graded $K$-dual. On the other hand,
$$L \circ D_E(N^\bullet)^m = \bigoplus_{i-j=m} S \otimes_K D_E(N^\bullet)_j^i = \bigoplus_{i-j=m} S(n) \otimes_K (N^{-i}_{n-j})^\vee = \bigoplus_{j-i=m-n} S(n) \otimes_K (N^i_j)^\vee.$$

So we can easily construct a quasi-isomorphism $D_S \circ L(N^\bullet) \to L \circ D_E(N^\bullet)(-2n)[2n]$. □

For $N^\bullet \in D^b(\text{gr} E)$, we have $H^i(L(N^\bullet))_j \cong \text{Ext}^{i+j}_E(K, N^\bullet)_j$ by [6, Theorem 3.7].

So the Laurent series $P_i(t) = \sum_{j \in \mathbb{Z}} (\dim_K \text{Ext}^{i+j}_E(K, N^\bullet)_j) \cdot t^j$ is the Hilbert series of the finitely generated graded $S$-module $H^i(L(N^\bullet))$. If $H^i(L(N^\bullet)) \neq 0$, there exists a Laurent polynomial $Q_i(t) \in \mathbb{Z}[t, t^{-1}]$ such that
$$P_i(t) = \frac{Q_i(t)}{(1-t)^d},$$
where $d = d_i(L(N^\bullet)) = \dim_S H^i(L(N^\bullet))$. Set $e_i(N^\bullet) := Q_i(1) = \deg_S H^i(L(N^\bullet))$. So $d_i(L(N^\bullet))$ and $e_i(N^\bullet)$ measure the growth of the “$(-i)$-linear strand” of a minimal injective resolution of $N^\bullet$.

[6, 11] treated $d_i(L(N))$ and $e_i(N)$ for a module $N \in \text{gr} E$ more or less indirectly. But their approach is very different from ours. They use Cartan (co)homology of $N$. See [2, 11] for the definition of this (co)homology. Let $v = v_1, \ldots, v_n$ be a basis of $V$ which is generic with respect to $N$ in the sense of [2, Definition 4.7].

As [2, 11], we set $H_i(k) := H_i(v_1, \ldots, v_k; N)$ and $H^i(k) := H^i(v_1, \ldots, v_k; N)$ to be Cartan (co)homologies. Note that $H_i(n) = \text{Tot}_i^E(K, N)$, $H^i(n) = \text{Ext}_E^i(K, N)$ and $H^i(v_1, \ldots, v_k; N) \cong H_i(v_1, \ldots, v_k; N)^*$. It follows from the argument in §6 of [2] that the function $j \mapsto \dim_K H^{i+j}(k)_j$ is a polynomial function for $j \gg 0$. Moreover, $d_i(L(N)) \leq 0$ if and only if $H^{i+j}(k)_j = 0$ for $j \gg 0$ if and only if $H^{i+j}(k)_j = 0$ for all $k \leq n$ and $j \gg 0$. [2, Proposition 9.4] can be restated as follows: If $d_i(L(N)) > 0$, we have
$$d_i(L(N)) = n + 1 - \min\{ k \mid H^{i+j}(k)_j \neq 0 \text{ for all } j \gg 0 \}.$$
Definition 1.6. Let $N^\bullet \in D^b(\text{gr} E)$. We say $(d, i) \in \mathbb{N} \times \mathbb{Z}$ is a distinguished pair for $N^\bullet$ if and only if it is distinguished for $L(N^\bullet)$ (in the sense of Definition 1.4).

Remark 1.7. By [11], we see that $(d, i)$ is a distinguished pair for a module $N \in \text{gr} E$ in the above sense if and only if $(n+1-d, i)$ is a “cohomological distinguished pair” for $N$ in the sense of [11]. (Recall that $E$ is a positively graded ring in [2] [11].) [2] also use the term “distinguished pair”. But this is “homological distinguished pair” of [11], and $(d, i)$ is a distinguished pair for $N$ in our sense if and only if $(n+1-d, n-i)$ is a distinguished pair for $N^\bullet$ in the sense of [2].

Corollary 1.8 (c.f. [11] Theorem 3.8). Let $N^\bullet \in D^b(\text{gr} E)$. A pair $(d, i)$ is distinguished for $N^\bullet$ if and only if $(d, 2n-d-i)$ is distinguished for $D_E(N^\bullet)$. If this is the case, we have $e_i(N^\bullet) = e_{2n-d-i}(D_E(N^\bullet))$.

Proof. For the first statement, it suffices to prove the direction $\Rightarrow$. By Theorem 1.2, $(d, -d-i)$ is a distinguished pair for $D_S \circ L(N^\bullet) \cong L \circ D_E(N^\bullet)(-2n)[2n]$. For a complex $M^\bullet \in D^b(\text{gr} S)$, we have $H^j(M^\bullet(-2n)[2n]) = H^{2n+j}(M^\bullet)(-2n)$ and $d_j(M^\bullet(-2n)[2n]) = d_{2n+j}(M^\bullet)$. Thus $(d, 2n-d-i)$ is distinguished for $L \circ D_E(N^\bullet)$. The last equality follows from Theorem 1.2 (2).

For a module $N \in \text{gr} E$, $d_i(L(N))$ can be 0 quite often. But we have the following.

Proposition 1.9. Assume that a module $N \in \text{gr} E$ does not have a free summand. If $(d, i)$ is a distinguished pair for $N$, then we have $d > 0$.

Proof. Let $0 \rightarrow N \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ be a minimal injective resolution of $N$. For $j \geq 0$, set $\Omega_j(N) := (\ker(I^j \rightarrow I^{j+1})[-j]$. Obviously, $0 \rightarrow \Omega_j(N) \rightarrow I^j \rightarrow I^{j+1} \rightarrow \cdots$ is a minimal injective resolution. On the other hand, since $N$ does not have a free summand, $0 \rightarrow I^{-1} \rightarrow I^0 \rightarrow \cdots \rightarrow I^{j-1} \rightarrow \Omega_j(N) \rightarrow 0$ is a minimal projective resolution. If $d_i(L(N)) > 0$, then $d_i(L(\Omega_j(N))) = d_i(L(N))$ for all $j \geq 0$. If $d_i(L(N)) = 0$, then $d_i(L(\Omega_j(N))) = -\infty$ for $j \gg 0$. On the other hand, since a minimal injective resolution of $N^\bullet$ is the dual of a minimal projective resolution of $N$, we have $d_i(L(N^\bullet)) = d_i(L(\Omega_j(N^\bullet)))$ for all $i$ and all $j \geq 0$. So $N^\bullet$ and $\Omega_j(N^\bullet)$ have the same distinguished pairs. For a contradiction, we assume that $(0, i)$ is a distinguished pair for $N$. Then $(0, 2n-i)$ is a distinguished pair for $N^\bullet$ and $\Omega_j(N^\bullet)$. So $(0, i)$ is a distinguished pair for $\Omega_j(N)$ for all $j \geq 0$. This contradicts the above observation.

We say a distinguished pair $(d, i)$ is positive, if $d > 0$. Since [2] [11] study a distinguished pair for a module, they only treat a positive one.

Remark 1.10. When $N^\bullet$ is a module, Corollary was proved in [11] Theorem 3.8]. On the other hand, for positive distinguished pairs, we can prove the corollary from [11] Theorem 3.8] directly: Let $I^\bullet$ be an injective resolution of $N^\bullet$ and $P^\bullet$ a projective resolution of $I^\bullet$. From the quasi-isomorphism $f : P^\bullet \rightarrow I^\bullet$, we have the exact complex $(T^\bullet, \partial^\bullet) := \text{cone}(f)$. Then $N := \ker \partial_0$ (resp. $N^\bullet$) has the same positive distinguished pairs as $N^\bullet$ (resp. $D_E(N^\bullet)$).
A variant of BGG correspondence gives an equivalence $\operatorname{gr} E \cong D^b(\operatorname{Coh}(\mathbb{P}^{n-1}))$ of triangulated categories, where $\operatorname{gr} E$ is the stable category, and $\operatorname{Coh}(\mathbb{P}^{n-1})$ is the category of coherent sheaves on $\mathbb{P}^{n-1} = \operatorname{Proj} S$. More precisely, the composition of the functor $L : \operatorname{gr} E \to D^b(\operatorname{gr} S)$ and the natural functor $D^b(\operatorname{gr} S) \to D^b(\operatorname{Coh}(\mathbb{P}^{n-1}))$ induces this equivalence. Note that the functor $\operatorname{gr} S \ni M \mapsto \hat{M}$ in $\operatorname{Coh}(\mathbb{P}^{n-1})$ ignores modules of finite length. Hence if $d_i(M^*) = 0$ then $H^i(M^*) = 0$. In this sense, the duality in $[1]$ corresponds to a duality on $D^b(\operatorname{Coh}(\mathbb{P}^{n-1}))$.

In the rest of this section, we assume that $K$ is algebraically closed. Let $N \in \operatorname{gr} E$. Following $[1]$, we say $v \in E_{-1} = V$ is $N$-regular if $\operatorname{Ann}_N(v) = vN$. It is easy to see that $v$ is $N$-regular if and only if it is $N^*$-regular. We say $V_E(N) = \{ v \in V \mid v \text{ is not } N \text{-regular} \}$ is the rank variety of $N$ (see $[1]$). $[1]$, Theorem 3.1 states that $V_E(N)$ is an algebraic subset of $\operatorname{Spec} S$, and $\dim V_E(N) = \max\{ d_i(L(N)) \mid i \in \mathbb{Z} \}$. But $\operatorname{Ext}^*_{\mathbb{Z}}(K, N)$ has the $S$-module structure. By the same argument as $[1]$, Theorem 3.9 (see also the proof of $[7]$ Corollary 3.2 (b)), we have that

$$V_E(N) = \{ v \in V \mid \xi(v) = 0 \text{ for all } \xi \in \operatorname{Ann}_S(\operatorname{Ext}^*_E(K, N)) \}.$$

But $[\operatorname{Ext}^{*+i}_E(K, N)]_* := \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}^{j+i}_E(K, N)_j$ is an $S$-module which is isomorphic to $H^i(L(N))$ (see the proof of $[6]$ Proposition 2.3), and we have $\operatorname{Ext}^*_E(K, N) \cong \bigoplus_{j \in \mathbb{Z}} [\operatorname{Ext}^{*+i}_E(K, N)]_j$. Set

$$V_E^i(N) = \{ v \in V \mid \xi(v) = 0 \text{ for all } \xi \in \operatorname{Ann}_S([\operatorname{Ext}^{*+i}_E(K, N)]_*) \}.$$

We have $V_E^i(N) = \bigcup_i V_E^i(N)$ and $d_i(L(N)) = \dim V_E^i(N)$. For an algebraic set $X \subset \operatorname{Spec} S$ of dimension $d$, set $\operatorname{Top}(X)$ to be the union of the all irreducible components of $X$ of dimensions $d$.

**Proposition 1.11.** If $(d, i)$ is a distinguished pair for $N \in \operatorname{gr} E$, then we have $\operatorname{Top}(V_E^i(N)) = \operatorname{Top}(V_E^{2n-d-i}(N^*))$.

**Proof.** By the proof of Theorem $[12]$, $\operatorname{Ann}_S(H^i(L(N)))$ has the same top dimensional components as $\operatorname{Ann}_S(H^{n-i}(D_S \circ L(N)))$. $\square$

In the above situation, we have $V_E^i(N) \neq V_E^{2n-d-i}(N^*)$ in general.

2. Squarefree case

In this section, we regard $S = K[x_1, \ldots, x_n]$ as an $\mathbb{N}^n$-graded ring with $\deg x_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where $1$ is in the $i$th position. Similarly, $E = K(y_1, \ldots, y_n)$ is a $-\mathbb{N}^n$-graded ring with $\deg y_i = -\deg x_i$. Let $\ast \operatorname{gr} S$ (resp. $\ast \operatorname{gr} E$) be the category of finitely generated $\mathbb{Z}^n$-graded $S$-modules (resp. $E$-modules). The functors $L$ and $R$ defining the BGG correspondence $D^b(\ast \operatorname{gr} S) \cong D^b(\ast \operatorname{gr} E)$ also work in the $\mathbb{Z}^n$-graded context. That is, the functors $L : D^b(\ast \operatorname{gr} E) \to D^b(\ast \operatorname{gr} S)$ and $R : D^b(\ast \operatorname{gr} S) \to D^b(\ast \operatorname{gr} E)$ are defined by the same way as the $\mathbb{Z}$-graded case, and they give an equivalence $D^b(\ast \operatorname{gr} S) \cong D^b(\ast \operatorname{gr} E)$, see $[13]$ Theorem 4.1. Note that the dualizing
complex $\omega^\bullet$ of $S$ is $\mathbb{Z}^n$-graded, and $D_S(-) = \text{Hom}_S(-, \omega^\bullet)$ is also a duality functor on $D^b(\text{gr} S)$. Similarly, $D_E(-) = \text{Hom}_E(-, E)$ is a duality functor on $D^b(\text{gr} E)$.

As Proposition 1.5, for $N^\bullet \in D^b(\text{gr} E)$, we have $D_{S\circ L}(N^\bullet) \cong L\circ D_E(N^\bullet)(-2)[2n]$ in $D^b(\text{gr} S)$. Here we set $j := (j, j, \ldots, j) \in \mathbb{N}^n$ for $j \in \mathbb{Z}$.

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, set $\text{supp}(a) := \{ i \mid a_i > 0 \} \subset [n] := \{1, \ldots, n\}$ and $|a| = \sum_{i=1}^n a_i$. We say $a \in \mathbb{Z}^n$ is squarefree if $a_i = 0, 1$ for all $i \in [n]$. When $a \in \mathbb{Z}^n$ is squarefree, we sometimes identify $a$ with $\text{supp}(a)$.

**Definition 2.1** ([12]). We say a $\mathbb{Z}^n$-graded $S$-module $M$ is squarefree, if the following conditions are satisfied.

(a) $M$ is $\mathbb{N}^n$-graded (i.e., $M_a = 0$ if $a \notin \mathbb{N}^n$) and finitely generated.

(b) The multiplication map $M_a \ni y \mapsto (\prod x_i^{b_i}) \cdot y \in M_{a+b}$ is bijective for all $a, b \in \mathbb{N}^n$ with $\text{supp}(a+b) = \text{supp}(a)$.

For a simplicial complex $\Delta \subset 2^{[n]}$, the Stanley-Reisner ideal $I_\Delta := (\prod_{i \in F} x_i \mid F \notin \Delta)$ and the Stanley-Reisner ring $K[\Delta] := S/I_\Delta$ are squarefree modules. Note that if $M$ is squarefree then $M_a \cong M_F$ as $K$-vector spaces for all $a \in \mathbb{N}^n$ with $\text{supp}(a) = F$.

Let $S_{qs}$ be the full subcategory of $\text{gr} S$ consisting of squarefree modules. In $\text{gr} S$, $S_{qs}$ is closed under kernels, cokernels and extensions ([12, Lemma 2.3]), and we have that $D^b(S_{qs}) \cong D^b_{S_{qs}}(\text{gr} S)$. If $M^\bullet \in D^b(S_{qs})$, then $D_{S_{qs}}(M^\bullet) \in D^b_{S_{qs}}(\text{gr} S)$ (see [13]). So $D_S$ gives a duality functor on $D^b(S_{qs})$.

**Definition 2.2** (Römer [11]). A $\mathbb{Z}^n$-graded $E$-module $N = \bigoplus_{a \in \mathbb{Z}^n} N_a$ is squarefree if $N$ is finitely generated and $N = \bigoplus_{F \subset [n]} N_{-F}$.

For example, any monomial ideal of $E$ is a squarefree module. Any monomial ideal of $E$ is of the form $J_\Delta = (\prod_{i \in F} y_i \mid F \notin \Delta)$ for some simplicial complex $\Delta \subset 2^{[n]}$. We say $K\{\Delta\} := E/J_\Delta$ is the exterior face ring of $\Delta$.

Let $S_{qs}$ be the full subcategory of $\text{gr} E$ consisting of squarefree $E$-modules. Then there exist functors $S : S_{qs} \to S_{qs}$ and $E : S_{qs} \to S_{qs}$ giving an equivalence $S_{qs} \cong S_{qs}$. Here $S(N)_F = N_{-F}$ for $N \in S_{qs}$, and the multiplication map $S(N)_F \ni z \mapsto x_i z \in S(N)_{(F\cup\{i\})}$ for $i \notin F$ is given by $S(N)_F = N_{-F} \ni z \mapsto (-1)^{\alpha(i,F)} y_i z \in N_{-(F\cup\{i\})} = S(N)_{\{i\}}$, where $\alpha(i,F) = \#\{ j \in F \mid j < i \}$. For example, $S(K\{\Delta\}) = K[\Delta]$. See [11] for further information. Of course, $S$ and $E$ can be extended to the functors between $D^b(S_{qs})$ and $D^b(S_{qs})$.

If $N \in S_{qs}$, then $N^* = \text{Hom}_E(N,E)$ is squarefree again. So $(-)^*$ gives the duality functor $D_S$ on $D^b(S_{qs})$. For example, $K(\Delta)^* = J_{\Delta^\vee}$, where $\Delta^\vee = \{ F \subset [n] \mid [n] \setminus F \notin \Delta \}$ is the Alexander dual complex of $\Delta$. We have the Alexander duality functor $A := S \circ D_E \circ E$ on $S_{qs}$ (or $D^b(S_{qs})$). Of course, $A(K[\Delta]) = I_{\Delta^\vee}$.

In general, we have $A(H^i(M^\bullet)) = (H^{-i}(M^\bullet)_{[n]\setminus F})^\vee$.

An associated prime ideal of $M \in \text{gr} S$ is of the form $P_F = (x_i \mid i \notin F)$ for some $F \subset [n]$. Let $M \in S_{qs}$ be a squarefree module. A monomial prime ideal $P_F$ is a minimal prime of $M$ if and only if $F$ is a maximal element of the set $\{G \subset [n] \mid M_G \neq 0\}$. The following is a squarefree version of Definition [11].
Theorem 2.4. Let $M^\bullet \in \mathcal{D}^b(S_{13})$. A pair $(F, i)$ is distinguished for $M^\bullet$ if and only if $(F, |F| - i)$ is distinguished for $D_S(M^\bullet)$. If this is the case, $\dim_{\mathbb{K}} H^i(M^\bullet)_F = \dim_{\mathbb{K}} H^{-|F| - i}(D_S(M^\bullet))_F$.

Proof. Like the proof of Theorem 1.2, we consider the spectral sequence $E_2^{p,q} = \text{Ext}^p_S(H^{-q}(M^\bullet), \omega^\bullet) \Rightarrow \text{Ext}^{p+q}_S(M^\bullet, \omega^\bullet)$. Then $E_2^{p,q}$ is squarefree for all $p,q$ and $r \geq 2$. When we consider a distinguished pair $(F, i)$, we set

$$
\dim_{\mathbb{K}} M := \begin{cases} 
-\infty & \text{if } M_G = 0 \text{ for all } G \supset F \\
\max\{|G| \mid G \supset F, M_G \neq 0\} & \text{otherwise}
\end{cases}
$$

for $M \in S_{13}$. Set $d_i(M^\bullet) := \dim_{\mathbb{K}} H^i(M^\bullet)$ and $e_2^{p,q} := \dim_{\mathbb{K}} \text{Ext}^p_S(H^{-q}(M^\bullet), \omega^\bullet)$ for $M^\bullet \in \mathcal{D}^b(S_{13})$. We also remark that $\dim_{\mathbb{K}} M_F = l_{S_{13}}(M \otimes_{S} S_{13})$ for $M \in S_{13}$. The equation (1.1) holds in this context, and the proof of Theorem 1.2 works verbatim.

If $N^\bullet \in \mathcal{D}^b(S_{13})$, then it is easy to see that $L(N^\bullet)(-1) \in \mathcal{D}^b(S_{13})$. So $L(-) := L(-)(-1)$ gives a functor from $\mathcal{D}^b(S_{13})$ to $\mathcal{D}^b(S_{13})$. Moreover, we have $L \cong A \circ D_S \circ S$ by [13] Proposition 4.3.

Definition 2.5. Let $N^\bullet \in \mathcal{D}^b(S_{13})$. We say $(F, i)$ is a distinguished pair for $N^\bullet$ if it is a distinguished pair for $L(N^\bullet) \in \mathcal{D}^b(S_{13})$ in the sense of Definition 2.4.

The next result can be proved by the same way as Corollary 2.8 using Theorem 2.4.

Proposition 2.6. Let $N^\bullet \in \mathcal{D}^b(S_{13})$. A pair $(F, i)$ is distinguished for $N^\bullet$ if and only if $(F, 2n - |F| - i)$ is distinguished for $D_E(N^\bullet)$. If this is the case, we have

$$
\dim_{\mathbb{K}} H^i(L(N^\bullet))_F = \dim_{\mathbb{K}} H^{2n-|F|-i}(L \circ D_E(N^\bullet))_F.
$$

If $M^\bullet \in \mathcal{D}^b(\star \mathfrak{g} S)$, then $\text{Tor}^S_{p}(K, M^\bullet) := H^{-i}(\mathfrak{g} \otimes_{\mathbb{K}} P^\bullet)$ is a $\mathbb{Z}^{n}$-graded module, where $P^\bullet$ is a graded free resolution of $M^\bullet$. Set $\beta_{i,a}(M^\bullet) := \dim_{\mathbb{K}} \text{Tor}^S_{p}(K, M^\bullet)_a$ for $a \in \mathbb{Z}^n$. We say $\beta_{i,a}(M^\bullet)$ is the $(i, a)$th Betti number of $M^\bullet$. If $M^\bullet \in \mathcal{D}^b(S_{13})$ and $\beta_{i,a}(M^\bullet) \neq 0$, then $a$ is squarefree (see [13]).

Definition 2.7 (c.f. [3]). A Betti number $\beta_{i,F}(M^\bullet) \neq 0$ is extremal if $\beta_{j,G}(M^\bullet) = 0$ for all $(j, G) \neq (i, F)$ with $j \geq i$, $G \supset F$, and $|G| - j > |F| - i$.

Some of known results and backgrounds of extremal Betti numbers are found in the introduction of the present paper.

Proposition 2.8 (c.f. [2]). Let $M^\bullet \in \mathcal{D}^b(S_{13})$ and $N^\bullet := \mathcal{E}(M^\bullet) \in \mathcal{D}^b(S_{13})$. A pair $(F, i)$ is distinguished for $D_E(N^\bullet)$ if and only if $\beta_{i+|F|-n,F}(M^\bullet)$ is an extremal Betti number. If this is the case, then $\beta_{i+|F|-n,F}(M^\bullet) = \dim_{\mathbb{K}} H^i(L \circ D_E(N^\bullet))_F$. 
Proof. For $j \in \mathbb{Z}$ and $G \subset [n]$, we have the following.

$$\beta_{j,G}(M^*) = \dim_K [H^{[G]-j-n}(D_S \circ A(M^*))]|_{[n],G} \quad \text{(by \cite{13} Corollary 3.6)}$$

$$= \dim_K [H^{n+j-[G]}(A \circ D_S \circ A(M^*))]|_{G}$$

$$= \dim_K [H^{n+j-[G]}(\mathcal{L} \circ \mathcal{E} \circ A(M^*))]|_{G}$$

$$= \dim_K [H^{n+j-[G]}(\mathcal{L} \circ D_E (N^*))]|_{G}.$$ 

The assertion easily follows from this equality. \hfill \Box

Corollary 2.9. Let $M^* \in D^b(S\mathcal{Q}_S)$. A Betti number $\beta_{i,F}(M^*)$ is extremal if and only if so is $\beta_{|F|-i,F}(A(M^*))$. If this is the case, $\beta_{i,F}(M^*) = \beta_{|F|-i,F}(A(M^*))$.

Proof. If $\beta_{i,F}(M^*)$ is extremal, then $(F, n+i-|F|)$ is a distinguished pair for $D_E \circ \mathcal{E}(M^*)$ by Proposition 2.8. By Proposition 2.6, $(F, n-i)$ is a distinguished pair for $\mathcal{E}(M^*) \cong D_E \circ \mathcal{E} \circ A(M^*)$. So $\beta_{|F|-i,F}(A(M^*))$ is extremal. The converse implication can be proved by the same way. The last equality follows from Proposition 2.6. \hfill \Box

This corollary generalizes results of Bayer-Charalambous-Popescu \cite{3}, Römer \cite{11} and Miller \cite{10}. Roughly speaking, the above proof is a “complex version” of \cite{11}. But, his argument itself does not work in the $\mathbb{Z}^n$-graded context, since he uses a generic base change of $V = E_{-1}$.

For $M^* \in D^b(S\mathcal{Q}_S)$. Set proj. dim($M^*$) = $\max \{ i \mid \beta_{i,F}(M^*) \neq 0 \text{ for some } F \}$ and reg($M^*$) = $\max \{ |F|-i \mid \beta_{i,F}(M^*) \neq 0 \}$. Since Betti numbers $\beta_{i,F}(M^*)$ which give proj. dim($M^*$) or reg($M^*$) are extremal, the next result follows from Corollary 2.9.

Corollary 2.10 (c.f. \cite{10}11). If $M^* \in D^b(S\mathcal{Q}_S)$, then proj. dim($M^*$) = reg$(A(M^*))$.

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