Asymptotic quantum estimation theory for the displaced thermal states family

Masahito Hayashi
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
e-mail address: masahito@kusm.kyoto-u.ac.jp

Abstract
Concerning state estimation, we will compare two cases. In one case we cannot use the quantum correlations between samples. In the other case, we can use them. In addition, under the later case, we will propose a method which simultaneously measures the complex amplitude and the expected photon number for the displaced thermal states.

1 Introduction
Quantum estimation is essentially different from classical estimation regarding the following two points. The first point is that we cannot simultaneously construct the optimal estimators corresponding to respective parameters because of non-commutativity between them. It has been a serious problem since the beginning of the quantum estimation [1, 2, 3]. The second point is that we can reduce the estimation error under the assumption that we can prepare independent and identical samples of the unknown quantum state. It was pointed by Nagaoka [4, 5] concerning the large deviation theory in one-parameter estimation. The purpose of this paper is to clear the second point concerning the mean square error (MSE).

Our situation is divided into the following two cases. In the first case, we estimate the unknown state by independently measuring every sample. In this case, we may decide the \(n\)-th POVM from \(n - 1\) data which have been already given. In the second case, we estimate the unknown state by regarding \(n\)-sample system as a single composite system. In this case, we may use POVMs which are indivisible into every sample system. In order to construct these POVMs, we need to use quantum correlations between every sample. The former is called the non-quantum correlation case and the later the quantum correlation case. When the unknown state is a pure state, the errors of both are asymptotically equivalent in the first order [6]. Concerning the spin 1/2 system, see Hayashi [7].

In this paper, we formulate a general theory for the asymptotic quantum estimation. It is applied to the simultaneous estimation of the expected photon number and the complex amplitude for the quantum displaced thermal state.

2 Asymptotic Estimation Theory
In this paper, we use a quantum state family \(\mathcal{S}\) parameterized by finite parameters \(\theta^1, \ldots, \theta^d\):

\[
\mathcal{S} := \{\rho_\theta \in \mathcal{S}(\mathcal{H}) | \theta = (\theta^1, \ldots, \theta^d) \in \Theta \subset \mathbb{R}^d\},
\]

where the set \(\mathcal{S}(\mathcal{H})\) denotes the set of densities on \(\mathcal{H}\). For simplicity, we assume that \(\rho_\theta\) is nondegenerate.
2.1 Non-quantum correlation case

The non-quantum correlation case is formulated as follows. A pair $\mathcal{E}_n = \{M_k\}_{k=1}^{n}, \hat{\theta}_n$ is called a recursive estimator where $\hat{\theta}_n$ is a function estimating the unknown parameter from $n$ data, and $\{M_k\}_{k=1}^{n}$ is a sequence of POVMs $M_1, M_2(\omega_1), \ldots, M_n(\omega_1, \ldots, \omega_{n-1})$ as follows: the $n$-th POVM $M_k(\omega_1, \ldots, \omega_{k-1})$ is determined by $k-1$ data which have been already given.

A sequence $\{\mathcal{E}_n\}_{n=1}^{\infty}$ of recursive estimators is called a recursive MSE consistent estimator if

$$\lim_{n \to \infty} \int \cdots \int_n \left\| \theta_n(\omega_1, \omega_2, \ldots, \omega_n) - \theta \right\|^2 P^\mathcal{E}_n (d\omega_1, d\omega_2, \ldots, d\omega_n) = 0, \forall \theta \in \Theta,$$

where

$$P^\mathcal{E}_n (d\omega_1, d\omega_2, \ldots, d\omega_n) = \text{tr} \rho M_1(\omega_1)\text{tr} \rho M_2(\omega_1)(d\omega_2) \ldots \text{tr} \rho M_n(\omega_1, \ldots, \omega_{n-1})(d\omega_n).$$

We define the non-quantum-correlational Cramér-Rao type bound $C^\text{NQC}_\theta (G)$ for a weighted matrix $G(G$ is a $d \times d$ real positive symmetric matrix.) as:

$$C^\text{NQC}_\theta (G) := \inf \left\{ \liminf_{n \to \infty} \text{Tr} nG_{\theta}(\mathcal{E}_n) \right\} \text{ where } \mathcal{E}_n \text{ is a recursive MSE consistent estimator},$$

where the MSE matrix $V_{\theta}(\mathcal{E}_n)$ is given by:

$$V_{\theta}^{i,j}(\mathcal{E}_n) = \int \cdots \int_n \left( \hat{\theta}_n^i(\omega_1, \ldots, \omega_n) - \theta^i \right) \left( \hat{\theta}_n^j(\omega_1, \ldots, \omega_n) - \theta^j \right) P^\mathcal{E}_n (d\omega_1, \ldots, d\omega_n).$$

We have the following equation:

$$C^\text{NQC}_\theta (G) = \inf \left\{ \text{Tr} G \left( J^M_\theta \right)^{-1} \right\} \text{ where } J^M_\theta \text{ denotes the Fisher information matrix at } \theta \text{ of } \{ \text{tr} \rho M(\omega) | \theta \in \Theta \}. \text{ It is derived by Jensen’s inequality } \mathfrak{J}. \text{ Under some regular condition, we show that there exists a recursive MSE consistent estimator } \mathcal{E}_n \text{ such that } \mathfrak{J}:$$

$$n \text{ Tr} G V_{\theta}(\mathcal{E}_n) \to C^\text{NQC}_\theta (G) \text{ as } n \to \infty, \forall \theta \in \Theta.$$

According Holevo$\mathfrak{J}$, we have $J^M_\theta \leq \tilde{J}_\theta$, where $\tilde{J}_\theta$ is the RLD Fisher information matrix defined as: $\tilde{J}_{\theta,i,j} := \text{tr}(\tilde{L}_{\theta,i})^* \rho \tilde{L}_{\theta,j}$, $\tilde{L}_{\theta,i} := (\rho \tilde{L}_{\theta,i})^{-1} \frac{\partial \rho}{\partial \theta^i}$. Therefore we have the following inequality

$$C^\text{NQC}_\theta (G) \geq C^R_\theta (G),$$

where

$$C^R_\theta (G) := \inf \left\{ \text{tr} GV | V \text{ is a } d \times d \text{ real symmetric matrix } V \geq \tilde{J}_\theta^{-1} \right\}$$

$$= \text{Tr} G \text{ Re} \tilde{J}_\theta^{-1} + \text{Tr} \left| \sqrt{G} \text{ Im} \tilde{J}_\theta^{-1} \sqrt{G} \right|.$$
2.2 Quantum correlation case

Next, we formulate the quantum correlation case. For this purpose, we consider a quantum counterpart of independent and identically distributed condition. If $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are $n$ Hilbert spaces which correspond to the physical systems, then their composite system is represented by the tensor Hilbert space.

$$\mathcal{H}^{(n)} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n.$$ 

Thus, a state on the composite system is denoted by a density operator $\rho^{(n)}$ on $\mathcal{H}^{(n)}$. In particular if each system $(\mathcal{H}_1, \ldots, \mathcal{H}_n)$ is independent of each other, there exists a density $\rho_k$ on $\mathcal{H}_k$ such that

$$\rho^{(n)} = \rho_1 \otimes \cdots \otimes \rho_n, \text{ on } \mathcal{H}^{(n)}.$$

The condition:

$$\mathcal{H}_1 = \cdots = \mathcal{H}_n = \mathcal{H}, \rho_1 = \cdots = \rho_n = \rho$$

corresponds to the independent and identically distributed condition in the classical case. Therefore, we consider the parameter estimation problem for the family $\{\rho^{(n)}_\theta, \mathcal{H}| \theta \in \Theta\}$ which is called the $n$-i.i.d. extended family.

In this case, we use a sequence $\{M^n\}_{n=1}^\infty$ of POVMs where $M^n$ is a POVM on $\mathcal{H}$ whose measurable set is $\mathbb{R}^d$ as an estimator. A sequence $\{M^n\}_{n=1}^\infty$ is called an MSE consistent estimator if $\{M^n\}_{n=1}^\infty$ satisfies

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} ||\hat{\theta} - \theta||^2 \text{tr} \rho^{(n)}_\theta M^n(d\hat{\theta}) = 0, \forall \theta \in \Theta.$$ 

A recursive MSE consistent estimator can be regarded as an MSE consistent estimator because a recursive estimator $E_n = (\{M_k^n\}_{k=1}^n, \hat{\theta}_n)$ is regarded as a POVM $M(E_n)$ as follows:

$$M(E_n)(B) := \int_{\hat{\theta}_{n-1}(B)} \bigotimes_{k=1}^n M_k(\omega_1, \ldots, \omega_{k-1})(d\omega_k), \forall B \subset \mathbb{R}^d \text{ on } \mathcal{H}^{(n)}.$$ 

We define the quantum-correlational Cramèr-Rao type bound $C^{QC}_\theta(G)$ for a weighted matrix $G$ as:

$$C^{QC}_\theta(G) := \inf \left\{ \liminf_{n \to \infty} n \text{Tr} G \rho^{(n)}_\theta M^n(d\hat{\theta}) \right\}$$ 

where $V^{i,j}_\theta(M^n)$ is given by:

$$V^{i,j}_\theta(M^n) = \int_{\mathbb{R}^d} (\hat{\theta}^i - \theta^i)(\hat{\theta}^j - \theta^j) \text{tr} \rho^{(n)}_\theta M^n(d\hat{\theta}).$$ 

We have the following equation

$$C^{QC}_\theta(G) = \liminf_{n \to \infty} nC^{QC}_\theta(G),$$ 

where $C^{QC}_\theta(G)$ denotes the non-quantum-correlational Cramèr-Rao type bound for the $n$-i.i.d. extended family. From the definition of the $n$-i.i.d. extended family, we have $C^{NQC}_\theta(G) \geq nC^n_\theta(G)$. Therefore, we have the first inequality of (1).

$$C^{NQC}_\theta(G) \geq C^{QC}_\theta(G) \geq C^{R}_\theta(G).$$ (1)
It shows the second inequality of (1) that $J^M_{\theta} \leq n\tilde{J}_{\theta}$ for any POVM $M^n$ on $\mathcal{H}^{(n)}$. Therefore the difference between $C^\text{NQC}_\theta(G)$ and $C^\text{QC}_\theta(G)$ means the difference of the quantum correlation case from the non-quantum correlation case. Under some regular condition, we can show that there exists an MSE consistent estimator \(\{M^n\}_{n=1}^\infty\) such that:

$$n \operatorname{Tr} G V_{\theta}(M^n) \rightarrow C^\text{QC}_\theta(G) \text{ as } n \rightarrow \infty, \forall \theta \in \Theta.$$ 

3 Quantum displaced thermal states family

Now we consider the estimation for the the complex amplitude $\zeta$ and expected photon number $N$ for the quantum displaced thermal states family defined as:

$$S := \left\{ \rho_{\zeta,N} := \frac{1}{\pi N} \int_{\mathbb{C}} \exp \left( -\frac{|\zeta - \alpha|^2}{N} \right) |\alpha\rangle \langle \alpha| d^2\alpha \middle| \zeta \in \mathbb{C}, N > 0 \right\}.$$

3.1 Estimation of complex amplitude $\zeta$

In the case of that photon number $N$ is known, we estimate the tow unknown parameters $\zeta = (\theta^1 + i\theta^2)/\sqrt{2}$. This estimation problem is investigated by Yuen, Lax and Holevo [2, 3]. In this case they calculated the inverse $\tilde{J}^{-1}_{\theta}$ of the RLD Fisher information matrix as:

$$\tilde{J}^{-1}_{\theta} = \begin{pmatrix} N + \frac{1}{2} & i \frac{1}{2} \\ - \frac{i}{2} & N + \frac{1}{2} \end{pmatrix}.$$

They calculated the non-quantum-correlational Cramér-Rao type bound $C^\text{NQC}_\theta(G)$ as follows:

$$C^\text{NQC}_\theta(G) = C^R_\theta(G) = 2 \left( N + \frac{1}{2} \right) g_1 + \sqrt{g_1^2 - g_2^2 - g_3^2},$$

(2)

where the weighted matrix $G$ is parameterized as:

$$G = \begin{pmatrix} g_1 + g_2 & g_3 \\ g_3 & g_1 - g_2 \end{pmatrix}.$$ 

From (1) and (3), we have the following equations.

$$C^\text{NQC}_\theta(G) = C^\text{QC}_\theta(G) = C^R_\theta(G).$$

(3)

In this case, the optimal estimator is the squeezed heterodyne.

3.2 Simultaneous estimation of complex amplitude $\zeta$ and expected photon number $N$

Next we consider the case of that both of the expected photon number $N$ and the complex amplitude $\zeta$ are unknown. In this case, we estimate three unknown parameters $\theta^1, \theta^2$ and $\theta^3 = N$. The first equation of (3) isn’t held. Therefore, the squeezed heterodyne isn’t optimal. The inverse $\tilde{J}^{-1}_{\theta}$ of the RLD Fisher information matrix is calculates as:

$$\tilde{J}^{-1}_{\theta} = \begin{pmatrix} N + \frac{1}{2} & i \frac{1}{2} & 0 \\ -i \frac{1}{2} & N + \frac{1}{2} & 0 \\ 0 & 0 & N(N + 1) \end{pmatrix}.$$ 

4
Therefore we can calculated $C^R_{\theta}(G)$ as:

$$C^R_{\theta}(G) = g_0 N(N + 1) + 2 \left( N + \frac{1}{2} \right) g_1 + \sqrt{g_1^2 - g_2^2 - g_3^2},$$

if the weighted matrix $G$ can be parameterized as:

$$G = \begin{pmatrix} g_1 + g_2 & g_3 & 0 \\ g_3 & g_1 - g_2 & 0 \\ 0 & 0 & g_0 \end{pmatrix}. \quad (4)$$

If the weighted matrix $G$ can be parameterized as (4), we obtain the following equations:

$$C^{\text{NQC}}_{\theta}(G) > C^{\text{QC}}_{\theta}(G) = C^R_{\theta}(G). \quad (5)$$

A proof for $C^{\text{NQC}}_{\theta}(G) > C^{\text{QC}}_{\theta}(G)$ is omitted. The inequality $C^{\text{NQC}}_{\theta}(G) > C^{\text{QC}}_{\theta}(G)$ means that we cannot the simultaneous measurement of the photon number counting and heterodyne for a single sample.

### 3.3 Construction of an MSE consistent estimator $\{M^n\}_{n=1}^{\infty}$ attaining $C^R_{\theta}(I)$

Now, for a weighted matrix $I$, we construct an MSE consistent estimator $\{M^n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} n \text{ tr } V_\theta(M^n) = C^R_{\theta}(I), \quad \forall \theta \in \Theta.$$

It is sufficient for $C^{\text{QC}}_{\theta}(I) = C^R_{\theta}(I)$ to construct such an MSE consistent estimator.

Every POVM $M^n$ is constructed in the following step:

1. Evolve the unknown state $\rho_{\zeta,N} \otimes \cdots \otimes \rho_{\zeta,N}$ as:

$$\rho_{\zeta,N} \otimes \cdots \otimes \rho_{\zeta,N} \rightarrow U_n \rho_{\zeta,N} \otimes \cdots \otimes \rho_{\zeta,N} U^*_n = \rho_{\sqrt{\pi} \zeta,N} \otimes \rho_{0,N} \otimes \cdots \otimes \rho_{0,N} \text{ on } \mathcal{H}^{(n)},$$

where

$$U_n = \exp \phi_{n-1}(a_n^* a_1 - a_1^* a_n) \cdots \exp \phi_2(a_3^* a_1 - a_1^* a_3) \exp \phi_1(a_2^* a_1 - a_1^* a_2) \text{ on } \mathcal{H}^{(n)}$$

$$\phi_i = \arctan \frac{1}{\sqrt{i}}, \quad i = 1, 2, \ldots, n-1.$$

$a_i$ denotes the annihilation operator on $\mathcal{H}_i$.

2. Measure the first sample $\rho_{\sqrt{\pi} \zeta,N}$ by the heterodyne, then we get the estimate $\hat{\zeta}$ of the complex amplitude.

3. Measure the others by the photon counting, then we obtain $n-1$ data which obey the probability distribution $P^N(k)$:

$$P^N(k) = \frac{1}{N+1} \left( \frac{N}{N+1} \right)^k, \quad k = 0, 1, \ldots.$$

4. We obtain the estimate $\hat{N}$ of the expected photon number $N$ by the maximum likelihood estimator of the probability distribution $P^N(k_1), \ldots, P^N(k_{n-1})$. 


4 Conclusion

We formulate an asymptotic quantum estimation theory. This theory is applied to the simultaneous measurement of the photon number counting and the heterodyne for displaced thermal states. It is a future study to realize the MSE consistent estimator proposed in this paper in an actual physical system.

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References

[1] C. W. Helstrom. “Quantum Detection and Estimation Theory,” Academic Press, New York (1976).

[2] H. P. Yuen and M. Lax, Multiple-parameter quantum estimation and measurement of nonselfadjoint observables, IEEE trans. Inform. Theory, IT 19, 740 (1973).

[3] A. S. Holevo. “Probabilistic and Statistical Aspects of Quantum Theory,” North-Holland, Amsterdam (1982).

[4] H. Nagaoka, On the relation Kullback divergence and Fisher information -from classical systems to quantum systems-, in: “Proc. Society Information Theory and its Applications in Japan” (1992) (in Japanese).

[5] H. Nagaoka, Two quantum analogues of the large deviation Cramér-Rao inequality, in: “Proc. 1994 IEEE Int. Symp. on Information Theory” p.118 (1994).

[6] M. Hayashi, Asymptotic estimation theory for a finite-dimensional pure state model, J. Phys. A: Math. Gen. 31 4633 (1998).

[7] M. Hayashi, Asymptotic Quantum Parameter Estimation in Spin 1/2 System, LANL e-print quant-ph/9710040 (1997).

[8] M. Hayashi and K. Matsumoto, Quantum mechanics as a statistical model which allows a free choice of measurements in: “Large Deviation and Statistical Inference” RIMS koukyuu-roku No. 1055 RIMS, Kyoto (1998) (in Japanese).