Black hole mergers, gravitational waves and scaling relations

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Observations of gravitational waves provide new opportunities to study our Universe. In particular, mergers of stellar black holes are the main targets of the current gravitational wave experiments. In order to make accurate predictions, it is however necessary to simulate the mergers in numerical general relativity, which requires high performance computing. Yet numerical simulation codes are optimized for specific mass scales, and may not be adapted to the study of other mass scales. In particular, primordial black holes can have masses ranging from the Planck mass to millions of solar masses, and simulations of primordial black hole mergers over the whole mass range are currently beyond the capabilities of the numerical codes. In this letter, we derive scaling relations, which can be used to rescale simulations of stellar black hole mergers and gravitational waves to any mass scale, hence allowing to perform precise simulations at any mass scale. In addition we study the domain of validity of the rescaling.

I. INTRODUCTION

After the discovery of gravitational waves (GWs) by LIGO [1], studies of mergers of stellar mass black holes (BHs) and of the resulting emission of gravitational waves have multiplied, either with semi-analytical descriptions or with numerical general relativity simulations. In parallel in absence of discovery of new particles at the LHC and in dark matter detection experiments, the nature of dark matter is still actively searched for, and the fact that primordial black holes (PBHs) can constitute dark matter is now considered as a viable possibility [2, 3]. Contrary to stellar black holes, the mass of primordial black holes spans from values as low as the Planck mass to millions of solar masses, and the merger of such PBHs can generate GWs with frequencies and amplitudes very different from those accessible to LIGO and VIRGO. Yet such PBH-generated GWs may be accessible to future GW experiments such as eLISA [4], and it is important to correctly model them. However because the masses of the involved PBHs can be very different from the stellar BHs upon which numerical simulation codes have been built, it may be very difficult to simulate numerically their mergers, and in particular the final states, which require numerical simulations for a correct description. Using the global scale invariance of general relativity in vacuum, we derive in the following scaling description. By definition of the metric and its relation with the local flat coordinates, the metric is scale invariant under transformation (1). Under this transformation, we obtain for the proper time \( \tau \), Christoffel symbol \( \Gamma^\alpha_{\mu\nu} \) and Riemann tensor \( R^\sigma_{\mu\nu\kappa} \):

\[
d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \longrightarrow \lambda^2 d\tau^2
\]

\[
\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left( \frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \longrightarrow \lambda^{-1} \Gamma^\sigma_{\mu\nu}
\]

\[
R^\sigma_{\mu\nu\kappa} = \frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\sigma_{\mu\kappa}}{\partial x^\nu} + \Gamma^\sigma_{\lambda\nu} \Gamma^\lambda_{\mu\kappa} - \Gamma^\sigma_{\lambda\kappa} \Gamma^\lambda_{\mu\nu} \longrightarrow \lambda^{-2} R^\sigma_{\mu\nu\kappa}
\]

In addition the four velocity vector \( u^\mu = dx^\mu / d\tau \) is scale invariant, and the Ricci tensor \( R_{\mu\nu} \), the curvature scalar
$R$ and the Weyl conformal tensor $C_{\mu\nu\rho\sigma}$ scale similarly to the Riemann tensor $R^\rho_{\mu\nu\sigma}$. The geodesic equation

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \quad (3)$$

is also scale invariant, meaning that its solutions are unchanged and only the coordinates are dilated.

Concerning Einstein’s field equations (EFE) with a cosmological constant $\Lambda$ and a stress-energy tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (4)$$

it is clear that the EFEs are not scale invariant. Indeed the cosmological constant is not scale invariant, nor is $T_{\mu\nu}$ in the general case. The scale invariance is retrieved in absence of cosmological constant and in the vacuum.

The description of the merger of black holes and generation of gravitational waves fulfills this condition: the GWs are perturbations of the metric in the vacuum, and the BHs are described as metrics with horizons. In absence of matter outside the BHs, the EFEs are written in the vacuum, and the cosmological constant is too small to have any effect during a merger. Let us consider the Kerr metric $\mathbf{g}$ in the BoyerLindquist coordinates $\mathbf{g}$, which describes a rotating black hole of mass $M$ and angular momentum $J$:

$$d\tau^2 = (dt - a \sin^2 \theta d\phi)^2 \frac{\Delta}{\Sigma} - \left(\frac{dr^2}{\Delta} + d\theta^2\right) \Sigma - \left((r^2 + a^2) d\phi - adt\right)^2 \frac{\sin^2 \theta}{\Sigma}, \quad (5)$$

where $(r, \theta, \phi)$ are spherical coordinates, $a = J/M$, $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - R_\Sigma r + a^2$ and $R_\Sigma = 2GM$ the Schwarzschild radius. The scale invariance of the metric implies that:

$$R_\Sigma \rightarrow \lambda R_\Sigma \iff M \rightarrow \lambda M \quad (6)$$

$$a \rightarrow \lambda a \iff J \rightarrow \lambda^2 J \quad (7)$$

under transformation $\mathbf{g}$. The transformation of $J$ is compatible with the standard definition of angular momentum $J = m \vec{x} \times \vec{v}$ when applying the coordinate and mass transformations $\mathbf{g}$ and $\mathbf{g}$. Also the Kerr dimensionless spin parameter $a^* = a/M$, which is equal to 0 for Schwarzschild BHs and 1 for extremal Kerr BHs, is scale invariant. Regarding the mass transformation, a remark is in order: For a BH the mass is not a real physical mass but a measure of the horizon radius.

We can now derive the following scaling rules for the merger of $n$ BHs of masses $M_i$, spins $J_i$ and momentum $\vec{P}_i$ with positions $x_i$ (i = 1...n) at time $t$:

$$M_i \rightarrow \lambda M_i \quad t \rightarrow \lambda t \quad (8)$$

$$\vec{P}_i \rightarrow \lambda \vec{P}_i \quad \vec{x}_i \rightarrow \lambda \vec{x}_i \quad \vec{v}_i \rightarrow \lambda^2 \vec{v}_i \quad a^*_i \rightarrow \lambda a^*_i$$

Consequently the local densities scale as $\rho \rightarrow \lambda^{-2} \rho$, the accelerations as $\vec{a} \rightarrow \lambda^2 \vec{a}$ and the velocities scale invariant.

Similarly, since gravitational waves can be considered as perturbations of the metric, their frequency $f$, wavelength $\lambda$, energy $E$, amplitude (or metric perturbation) $h$, speed $v$ and stress-energy tensor scale as

$$f \rightarrow \lambda^{-1} f \quad \Lambda \rightarrow \lambda \Lambda \quad (9)$$

$$E \rightarrow \lambda E \quad h \rightarrow h \quad v \rightarrow \lambda^{-2} T_{\mu\nu} \rightarrow \lambda^{-2} T_{\mu\nu}.$$

Using these relations, it is possible to rescale the results of numerical simulations.

### III. SIMULATIONS OF BH MERGERS

In this section, we simulate mergers of binary black holes and emission of gravitational waves. Such simulations are computationally intensive, and each of our simulations necessitated about one week on a 64-processor server.

The evolution of binary black holes is composed of different phases. The first one, which is the longest, is the inspiral phase described using post-Newtonian techniques. The second one is the plunge and merger phase, which can be only described by numerical relativity. The last one is the ringdown phase described by perturbation methods.

To solve the Einstein equations, we use the Einstein Toolkit code $\mathbf{g}$ $\mathbf{g}$. The initial data are generated via the TwoPunctures routine $\mathbf{g}$ for a merger with a near circular orbit. We consider only this case since for a binary orbit with a given eccentricity $e$, the emission of gravitational waves leads to a decrease in eccentricity. The evolution is performed using the BSSN formulation $\mathbf{g}$ via the McLachlan routine $\mathbf{g}$.

To compute the properties of the emitted gravitational waves, we use the Newman-Penrose formalism $\mathbf{g}$ where the Weyl scalar $\psi_4$ and the GW polarization amplitudes $h_{+\times}$ are related by:

$$h_{+} - i h_{\times} = \psi_4 = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \psi_4^m(t, r) Y_{lm}(\theta, \phi). \quad (10)$$

This corresponds to the decomposition of $\psi_4$ into $s = -2$ spin-weighted spherical harmonics. The dominant modes for the gravitational wave strain $h$ are the $l = m = 2$ modes: $h_{+\times}^{22}$. Considering the scaling relations, since $h_{+\times}$ are scale invariant, we have $\psi_4 \rightarrow \lambda^{-2} \psi_4$.

To demonstrate the advantages of the scaling presented in the previous section, we first perform two simulations of Schwarzschild BH mergers. The first one is the merger of two BHs of masses equal to 0.5 solar mass, which can be taken as the reference simulation. The second one is a similar simulation with a scale factor $\lambda = 0.01$. All the initial quantities are rescaled using the relations given in the previous section (e.g. the mass of the BHs is 0.005 solar mass, ...). In Figure $\mathbf{g}$, we show the real part of the Weyl tensor and the GW strain for $l = m = 2$, as...
a function of time. To fulfill the scale invariance, the time and Weyl tensor have been adimensioned using the total initial mass, as given in the axis labels. As expected the two simulations give similar results in these scale invariant parameter planes. The main difference is a high-frequency numerical noise, which comes from interpolation procedures between the different path grids. In order to reduce this noise, it is possible to increase and adjust the strength of the Kreiss-Oliger artificial dissipation term \cite{19} or change the finite differential order, but this requires re-running simulations with adjusted parameters until the correct precision is reached. On the contrary, the reference simulation does not show any instability, because Einstein Toolkit has been optimized for stellar BHs and the addition of a dissipation term has only limited consequences. Therefore, using the scaling avoids the numerical instabilities by using optimal mass scales instead of running into numerical problems for very small or very large masses.

Figure 2 shows $\psi_{22}^2$ and $h_{22}^2$ for a similar merger, for two Kerr BHs with reduced spin parameters $a^* = \pm 0.5$, as a function of time. Similarly to the previous figure, we made two simulations of BH mergers with total masses of $1 M_\odot$ and $0.01 M_\odot$. We see that the numerical noise has in the case of Kerr BHs, a larger impact which leads to a shift at later times. This can only be reduced using a very finely adjusted Kreiss-Oliger dissipation strength and/or a change of finite differential method, after which it is mandatory to run many simulations to reach a correct precision.

To summarize, it can be very difficult to reach a correct precision in simulation of mergers of very light or very massive BHs because of the numerical noise or instabilities. In particular we have not been able to perform simulations using Einstein Toolkit for BHs of masses below $10^{-5} M_\odot$. On the other hand, using the scaling relations derived in the previous section, it is possible to avoid this difficulty by performing simulations in the optimal mass ranges for the numerical relativity codes, and to scale them to any total mass and spin. This is especially important when the spins are non-negligible and lead to larger uncertainties. Such scaling relations can also be used to test numerical codes outside of their optimal mass scales.

IV. DOMAIN OF VALIDITY OF THE SCALING

A. Charged spinning black holes

We first discuss the case of charged and rotating black holes, which are described by the Kerr-Newman metric \cite{20}, which is similar to Eq. 3, with $\Sigma = r^2 - R_\odot r + a^2 + \ldots$
$R^2_Q$, where $R_Q$ is related to the black hole charge $Q$ by
\begin{equation}
R_Q = \sqrt{\frac{G}{4\pi\epsilon_0}} Q, \tag{11}
\end{equation}
and $\epsilon_0$ is the void permittivity. The scale invariance can be restored if the charge transforms as
\begin{equation}
Q \longrightarrow \lambda Q. \tag{12}
\end{equation}
Such a rotating charge induces an electromagnetic potential such as
\begin{equation}
A_{\mu} = \left( \frac{rR_Q}{\Sigma}, 0, 0, -\frac{a^*R_Qr\sin^2 \theta}{\Sigma G} \right), \tag{13}
\end{equation}
which is invariant under our set of transformations. The electromagnetic field therefore transforms as
\begin{equation}
F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \longrightarrow \lambda^{-1}F_{\mu\nu} \tag{14}
\end{equation}
and the associated stress-energy tensor as
\begin{equation}
T^{\mu\nu} = \epsilon_0 \left( F^{\alpha\beta} g_{\alpha\beta} F^{\nu\gamma} - \frac{1}{4} g^{\mu\nu} F^{\delta\gamma} F_{\delta\gamma} \right) \longrightarrow \lambda^{-2}T^{\mu\nu}, \tag{15}
\end{equation}
which is the behaviour required to let the EFEs invariant in presence of a source term. Concerning Maxwell’s equations
\begin{equation}
\frac{\partial}{\partial x^\nu} \left( \sqrt{-\det(g_{\mu\nu})} g^{\alpha\beta} F_{\alpha\beta} g^3 \right) = J^\mu, \tag{16}
\end{equation}
they are invariant when the current four-vector $J^\mu$ is zero or scales as $\lambda^{-2}$. Consequently the electromagnetic waves emitted by a merger of charged black holes in empty space respects the same scale properties as for gravitational waves, given in Eq. [3]. There is a strong parallel between Einstein’s field equations and Maxwell’s equations: in presence of source terms the scaling properties are generally broken.

B. Einstein’s equation with source terms

We can now consider the presence of source terms in the EFEs. For a perfect fluid in thermodynamic equilibrium, the stress-energy tensor reads:
\begin{equation}
T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P g^{\mu\nu}, \tag{17}
\end{equation}
where $\rho$ and $P$ are the density and pressure of the fluid. In order to leave the EFEs scale invariant the density and pressure need to transform under the scale transformation as:
\begin{equation}
(\rho, P) \longrightarrow \lambda^{-2}(\rho, P). \tag{18}
\end{equation}
Such a transformation does not hold in the general case, but it is valid for collisionless particles with negligible pressure or when $P \propto \rho$. In addition, when processes dominated by the weak or strong interactions occur the scaling is broken, because weak and strong interactions have limited ranges which act at scales different from gravity and electromagnetism. Similarly quantum effects, which involve the Planck constant, are not scale invariant either.

C. Black holes and expansion rate of the Universe

We now consider the case of light PBHs more carefully. As they originate from the early Universe, they can be affected by the expansion. Since expansion is time-dependent but affects space, it breaks the scaling of time and space. A simple rule would be to consider that a rescaling is possible as long as the duration of the merger $T_{\text{merger}}$ is much larger than the Hubble time $t_{\text{Hubble}}(t)$ at cosmological time $t$, such as
\begin{equation}
T_{\text{merger}} \ll t_{\text{Hubble}}(t) \equiv \frac{a(t)}{\dot{a}(t)}, \tag{19}
\end{equation}
where $a(t)$ is the cosmological scale factor. To calculate the merger time, we approximated the trajectory of the BHs to an equilibrium circular orbit with an orbital decay rate $dD(t)/dt$. At lowest order, this expression can be calculated by the quadrupole expression [17] and an integration of the orbital decay gives, for the binary separation distance $D(t)$ of the coalescence of two Schwarzschild BHs of Schwarzschild radius $R_s$, the following scale invariant expression:
\begin{equation}
D(t) = 4 \left( \frac{R_s^3}{20} (T_{\text{merger}} - t) \right)^{1/4}, \tag{20}
\end{equation}
which is valid at $t < T_{\text{merger}}$. Assuming a $\Lambda$CDM model in a flat Universe, using Planck 2018 cosmological parameters [21], considering 60 e-folds for inflation, we show in Figure 3 the individual BH mass of binary mergers for which $T_{\text{merger}} = t_{\text{Hubble}}(t)$, as a function of the age of the Universe $t$. BHs with masses above the lines merge faster than the expansion. The line corresponding to the maximal PBH mass has been obtained by assuming that the biggest BH has a Schwarzschild radius equal to the Hubble radius. In particular, at our present epoch, mergers of stellar black holes are affected by the expansion only if their initial distance is larger than about $10^6$ km.

D. Black holes and Hawking evaporation

Light PBHs are expected to vanish via emission of Hawking radiation [22]. Since the lifetime of a BH is typically proportional to its mass cubed [23], the scaling of the mass proportionally to the spacetime scaling is not possible anymore. Therefore, the scaling can be applied only if the duration of the merger is
FIG. 3. BH mass corresponding to a merger time equal to the Hubble time, as a function of the age of the Universe, for different initial merger BH distances \( D \) given as numbers of the Schwarzschild radius. The dashed line corresponds to the maximum mass of BHs.

FIG. 4. In the time vs. BH mass plane, the solid lines correspond to the evaporation time for a BHs of mass \( M \) and spin \( a^* = 0 \) (red) and \( a^* = 0.99 \) (blue). The dashed lines correspond to the merger time for two identical Schwarzschild BHs of mass \( M \), for different initial distances \( D \) given as numbers of the Schwarzschild radius.

much smaller than the lifetime of the BHs. In Figure 4 we show the evaporation time of BHs as a function of their mass, for Schwarzschild BHs and for nearly extremal Kerr BHs with \( a^* = 0.99 \). The evaporation time has been computed with the public program BlackHawk [24]. For comparison, typical durations of mergers of two black holes obtained from Eq. (20) are also plotted as a function of the mass of the system, assuming that both BHs have identical masses, for initial distances \( D = (10 R_s, 10^3 R_s, 10^6 R_s, 10^9 R_s) \). Hence, the rescaling is typically correct down to masses of about \( 10^{-4}, 10^3, 10^{12} \) grams, respectively, below which evaporation has to be taken into account and breaks the rescaling. Since PBHs having not completely evaporated today have masses above \( 10^{14} \) g [25, 26], the mergers of the surviving PBHs are not affected by Hawking evaporation.

V. CONCLUSIONS AND PERSPECTIVES

In this letter we have proposed a transformation to rescale the results from simulations of stellar black hole mergers, from primordial to supermassive black holes. We have shown that these rescaling relations are valid for Schwarzschild and Kerr spinning black holes, as well as for Kerr-Newman charged spinning black holes, and that they can be extended to the gravitational and electromagnetic waves emitted by such mergers. Indeed, while simulations of stellar BH mergers are accurately performed using numerical codes, numerical problems can limit precision for very small or very large masses, and the scaling relations can be used to rescale the numerical simulations performed at optimal scales. We have shown that there are a few limitations to the rescaling: first, in the early Universe, the expansion of the Universe can prevent the merger because it increases the distances faster than the merger decreases them. Then, for light primordial black holes, Hawking evaporation can be faster than the merger. Finally, in presence of energy with a non-negligible pressure or a pressure not proportional to the density, the Einstein’s field equations are not scale invariant any more, and the scaling relations cannot apply. In this case however, if the energy is much smaller than the black hole masses, it can be possible to use a post-processing procedure: first, the merger of black holes is simulated in absence of additional energy and the metric computed at each step of the merger. Second, the additional energy is injected a posteriori in the obtained geometry and its behaviour simulated at each step time in the metric provided by the simulation of the merger in empty space.

In this study we have derived scaling relations for numerical simulations of black hole mergers, but such rescaling methods can be derived and used for simulations of other phenomena, in particular in electromagnetism.

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