A holonomy invariant anisotropic surface energy in a Riemannian manifold

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Abstract

In this paper, we investigate a holonomy invariant elliptic anisotropic surface energy for hypersurfaces in a complete Riemannian manifold, where “holonomy invariant” means that the elliptic parametric Lagrangian (i.e., a Finsler metric) of the Riemannian manifold used to define the anisotropic surface energy is constant along each holonomy subbundle of the tangent bundle of the Riemannian manifold. First we obtain the first variational formula for this anisotropic surface energy. Next we shall introduce the notions of an anisotropic convex hypersurface, an anisotropic equifocal hypersurface and an anisotropic isoparametric hypersurface for this anisotropic surface energy. Also, we shall introduce the notion of an anisotropic tube for this anisotropic surface energy. We prove that anisotropic tubes over a one-point set in a symmetric space are anisotropic convex hypersurfaces and that anisotropic tubes over a certain kind of reflective submanifold in a symmetric space are anisotropic isoparametric and anisotropic equifocal hypersurfaces.

Introduction

H. Federer ([F1,2,3]) studied the elliptic parametric functional given by a parametric Lagrangian of general degree in the Euclidean space from the point of wide view of geometric measure theory, where we note that his study can be apply to the study of the elliptic parametric functional for (smooth) submanifolds of general codimension in the Euclidean space because the parametric Lagrangian is of general degree. B. White ([W]) studied the elliptic parametric functional for (smooth) submanifolds (of general codimension) in the Euclidean space in detail. In particular, U. Clarenz ([Cl]) studied the elliptic parametric functional for (smooth) hypersurfaces in the Euclidean space in more detail. On the other hand, Koiso and Palmer ([KP1-3, Palm]) studied a special elliptic parametric functional (which they called an anisotropic surface energy) for (smooth) hypersurfaces in the Euclidean space.

In the case where the ambient space is a general complete oriented Riemannian manifold, Lira and Melo ([LM]) have recently introduced an elliptic parametric functional for (smooth) hypersurfaces as follows. Let $\tilde{M}$ be an $(n+1)$-dimensional complete oriented Riemannian manifold and $M$ be an $n$-dimensional compact oriented manifold, where $M$ may have the boundary. Let $\tilde{F}$ be a positive $C^\infty$-function over the tangent bundle $T\tilde{M} \setminus \{0\}$ of $\tilde{M}$ satisfying the homogeneity condition

\[(H) \quad \tilde{F}(tX) = t\tilde{F}(X) \quad (X \in T\tilde{M} \setminus \{0\}, \ t > 0).\]

Then $\tilde{F}$ is called a parametric Lagrangian (of $\tilde{M}$). Furthermore, if it satifies the elliptic

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condition
\[(E) \quad (\tilde{\nabla}d\tilde{F})_X(Y,Y) > 0 \quad (X \in T\tilde{M} \setminus \{0\}, \ Y(\neq 0) \in T_X(T\tilde{M}) \text{ s.t. } \tilde{g}(X,Y) = 0),\]
then it is said to be elliptic, where $\tilde{g}$ is the Sasaki metric of $T\tilde{M}$ and $\tilde{\nabla}$ is the Riemannian connection of $\tilde{g}$. Note that an elliptic parametric Lagrangian means a Finsler metric. Let $f$ be an immersion of $M$ into $\tilde{M}$. They studied the following type of functional:
\[\mathcal{F}(f) := \int_{x \in M} \bar{F}(\xi_x) dV,\]
where $\xi$ is the unit normal vector field of $f$ (compatible with the orientations of $M$ and $\tilde{M}$) and $dV$ is the volume element of the induced metric on $M$ by $f$. They called $\mathcal{F}$ an elliptic parametric functional. In this paper, we shall call $\mathcal{F}$ an anisotropic surface energy following to the terminology of [KP1-3]. In particular, if $\bar{F}$ is constant along each holonomy subbundle of $T\tilde{M}$ (i.e., horizontally constant in the sense of [LM]), then we shall say that $\mathcal{F}$ and $\bar{F}$ are holonomy invariant. Note that the anisotropic surface energy treated by Koiso and Palmer is holonomy invariant. Lira and Melo ([LM]) stated that the horizontally constancy (=holonomy invariance) of the elliptic anisotropic surface energy need to be imposed in order that the anisotropic mean curvature (which is a variational notion) of $f$ is given as the trace of the anisotropic analogue (which is a hypersurface theoretical notion) of the shape operator of $f$.

Motivation 1. As in the above statement by Lira and Melo, we consider that the elliptic anisotropic surface energy need to be assumed to be holonomy invariant in order to study the variational problem for the above elliptic anisotropic surface energy from the view of point of the hypersurface theory.

Motivation 2. In the case where the ambient space is a standard space, it is important to give examples of critical points (i.e., hypersurfaces with constant anisotropic mean curvature) of the holonomy invariant elliptic anisotropic surface energy over the class of all volume-preserving variations.

If the holonomy group of the ambient space $\tilde{M}$ is big, then the holonomy invariance of the elliptic anisotropic surface energy is a strong constraint condition. In particular, if $\tilde{M}$ is rotationally symmetric (for example, a rank one symmetric space), then the only holonomy invariant elliptic anisotropic surface energy is a constant-multiple of the volume functional. Hence we are interested in the case where the holonomy group of the ambient space is small.

J. Ge and H. Ma [GM] studied the anisotropic surface energy treated by Koiso-Palmer. For this anisotropic surface energy, they introduced the notions of an anisotropic principal curvature and an anisotropic parallel translation of a hypersurface in the Euclidean space. They proved that a hypersurface is with constant anisotropic principal curvatures if and only if the anisotropic parallel hypersurfaces of the hypersurface are of constant anisotropic mean curvature (i.e., the hypersurface is anisotropic isoparametric in the sense of [GM]). Furthermore, they obtained a Cartan identity for a hypersurface with constant anisotropic principal curvatures and classified complete hypersurfaces with constant anisotropic principal curvatures in terms of this identity.
Motivation 3. In a (general) complete Riemannian manifold, we should introduce the notions similar to the above notions introduced in [GM] and study the relations between the notions. Also, we should give examples of the notions in a standard complete Riemannian manifold.

Under the above motivations, in this paper, we shall first give the first variational formula for a holonomy invariant elliptic anisotropic surface energy satisfying some convexity condition in a complete Riemannian manifold, which is a special one of the first variational formula for a (not necessarily holonomy invariant) elliptic anisotropic surface energy given by Lira and Melo ([LM]) but cannot be derived directly from their formula (see Section 1). Furthermore, by using this formula, we investigate a critical point of this anisotropic surface energy over the class of all variations (or over that of all volume-preserving variations) (see Section 1). Next, for this anisotropic surface energy, we introduce the notions of an anisotropic convex hypersurface, an anisotropic equifocal hypersurface and an anisotropic isoparametric hypersurface (see Section 2). Next, for this anisotropic surface energy, we introduce the notion of an anisotropic tube over a submanifold (Section 3) and prove that anisotropic tubes over an one-point set (which are called anisotropic geodesic spheres and are the notion corresponding to the Wulff shape in the Euclidean case) in a symmetric space of compact type or non-compact type are anisotropic convex hypersurfaces (see Section 4) and that anisotropic tubes over a certain kind of reflective submanifold in a symmetric space of compact type or non-compact type are anisotropic isoparametric and anisotropic equifocal hypersurfaces (see Section 5). Finally, we prove that the anisotropic equifocality is equivalent to the anisotropic isoparametricity in the case where the ambient space is a symmetric space of non-negative curvature (see Section 6).

1 Holonomy invariant anisotropic surface energy

In this section, we shall define a holonomy invariant anisotropic surface energy in a complete Riemannian manifold, which is treated in this paper, and obtain the first variational formula for this anisotropic surface energy.

Let $\tilde{M}$ be an $(n+1)$-dimensional complete (oriented) Riemannian manifold. Denote by $\langle \cdot , \cdot \rangle$ and $\nabla$ the (Riemannian) metric and the Riemannian connection of $\tilde{M}$, respectively. Also, denote by $\tau_c$ the parallel translation along a curve $c$ in $\tilde{M}$, $\Phi_p$ the holonomy group of $\tilde{M}$ at $p(\in \tilde{M})$ and $P^\mathrm{hol}_v$ the holonomy subbundle of $T\tilde{M}$ through $v(\in T\tilde{M})$. Also, denote by $S^n(1)_p$ the unit sphere centered at the origin in $T_p\tilde{M}$. Fix $p_0 \in \tilde{M}$. For simplicity, set $\Phi := \Phi_{p_0}$ and $S^n(1) := S^n(1)_{p_0}$. It is clear that $\Phi$ acts on $S^n(1)$. Denote by $\pi_\Phi$ the orbit map of this action. For each $p \in \tilde{M}$, take a shortest geodesic $\gamma_p : [0, 1] \to \tilde{M}$ connecting $p_0$ to $p$ (i.e., $\gamma_p(0) = p_0$, $\gamma_p(1) = p$). Note that the choice of $\gamma_p$ is not unique for each $p$ belonging to the cut locus (which is denoted by $C$) of $p_0$. In the sequel, we fix the choices of $\gamma_p$’s ($p \in C$). For simplicity, set $\tau_p := \tau_{\gamma_p}$. Let $M$ be a $n$-dimensional compact (oriented) manifold, which may have boundary, and $f$ an immersion of $M$ into $\tilde{M}$. Denote by $\xi$ the unit normal vector field of $f$ (compatible with the orientations of $M$ and $\tilde{M}$). Define a map $\nu : M \to S^n(1)$ by

$$\nu(x) = \tau_{f(x)}^{-1}(\xi_x) \quad (x \in M).$$
Set $\tilde{\nu} := \pi_\Phi \circ \nu$. Note that $\nu$ depends on the choices of $\gamma_p$’s ($p \in C$) but $\tilde{\nu}$ is independent of their choices. Under fixed choices of $\gamma_p$’s ($p \in C$), we call $\nu$ the Gauss map of $f$. Denote by $C^\infty(S^n(1))_\Phi$ the ring of all $\Phi$-invariant $C^\infty$-functions over $S^n(1)$. Take an elliptic parametric Lagrangian $\tilde{F}$ of $\tilde{M}$. Assume that $\tilde{F}$ is holonomy invariant, that is, $\tilde{F}|_{\tilde{F}_p}^{\text{hol}}$ is constant for any $\nu \in T\tilde{M} \setminus \{0\}$. Denote by $F_p$ the restriction of $\tilde{F}$ to $S^n(1)_p$. In particular, we denote $F_{p_0}$ by $F$ for simplicity. Then, since $\tilde{F}$ is holonomy invariant, $F_p = F \circ \tau_p^{-1}$ holds for any $p \in \tilde{M}$ and $F$ is $\Phi$-invariant. Thus $\tilde{F}$ is determined by $F$, that is, any holonomy invariant elliptic parametric Lagrangian (of $\tilde{M}$) is constructed from a $\Phi$-invariant $C^\infty$-function over $S^n(1)$. Furthermore, assume that $F$ satisfies the following convexity condition:

\[(C)\quad \nabla^S \text{grad } F + F \text{id} > 0,\]

where $\nabla^S$ is the Riemannian connection of $S^n(1)$, $\text{id}$ is the identity transformation of $TS^n(1)$. Note that the orbit space $S^n(1)/\Phi$ is an $(r - 1)$-dimensional orbifold in the case where $\tilde{M}$ is a symmetric space of rank $r$. In particular, if $\tilde{M}$ is a symmetric space of rank one, then $S^n(1)/\Phi$ is of dimension zero and hence $F$ must be constant and hence the above anisotropic surface energy $F$ is equal to the (usual) volume functional up to a constant-multiple. Thus, if $\tilde{M}$ is a symmetric space, then we are interesting in the case where the rank of the symmetric space is higher. Denote by $\text{Imm}(M, \tilde{M})$ the set of all $(C^\infty)$-immersions of $M$ into $\tilde{M}$. In this paper, we shall investigate the holonomy invariant elliptic anisotropic surface energy $F : \text{Imm}(M, \tilde{M}) \to \mathbb{R}$ by

\[F(f) := \int_{x \in M} \tilde{F}(\xi_x) dV,\]

where $dV$ is the volume element of the induced metric on $M$ by $f$. Note that $\tilde{F}(\xi_x) = (F \circ \nu)(x)$ ($x \in M$) and hence $F \circ \nu$ is independent of the choices of $\gamma_p$’s ($p \in C$). Denote by $H$ the mean curvature of $f$ (with respect to $\xi$). Define a function $H_F$ over $M$ by

\[(1.1) \quad (H_F)_x := (F \circ \nu)(x) H_x - (\text{div}(f_\ast^{-1}(\tau_{f(y)}((\text{grad} F)_{\nu(y)}))))_x \quad (x \in M),\]

where $\text{div}(\cdot)$ is the divergence of $(\cdot)$ with respect to the induced metric on $M$ by $f$. Here we note that $\tau_{f(y)}((\text{grad} F)_{\nu(y)}) \in f_\ast(T_y M)$ for any $y \in M$ (see Figure 1). We call $H_F$ the anisotropic mean curvature of $f$. If $H_F = 0$, then we call $f : M \hookrightarrow \tilde{M}$ an anisotropic minimal hypersurface. Also, if $H_F$ is constant, then we call $f : M \hookrightarrow \tilde{M}$ a hypersurface with constant anisotropic mean curvature.

Figure 1.
Take \( v \in T_p\tilde{M} \) and a curve \( c : [0, \varepsilon) \to \tilde{M} \) with \( c'(0) = v \), we define a linear map \( \tau_v^{\text{hol}} : T_{p_0}\tilde{M} \to T_{p_0}\tilde{M} \) by

\[
\tau_v^{\text{hol}} := \frac{d}{ds}
\left|_{s=0} \left( \tau_{c(0) \cdot c|[0,s]}^{-1} c(\gamma_{c(0)} c|[0,s]) \right) \right.,
\]

where \( \tau_{c(0) \cdot c|[0,s]}^{-1} \) is the parallel translation along the product \( \gamma_{c(0)} \cdot c|[0,s] \cdot \gamma_{c(s)}^{-1} \) of the curves \( \gamma_{c(0)} \cdot c|[0,s] \) and \( \gamma_{c(s)}^{-1} \). This linear map \( \tau_v^{\text{hol}} \) is described explicitly in the case where \( \tilde{M} \) is a symmetric space (see Lemma 2.1).

Let \( f_t (\varepsilon < t < \varepsilon) \) be a \((C^\infty)\)-variation of \( f \) in \( \text{Imm}(M, \tilde{M}) \) and define \( f : M \times (-\varepsilon, \varepsilon) \to \tilde{M} \) by \( f(x, t) := f_t(x) ((x, t) \in M \times (-\varepsilon, \varepsilon)) \). Denote by \( g_t \) the metric on \( M \) induced from \( \langle , \rangle \) by \( f_t \) and \( \nabla^f \) the Riemannian connection of \( g_t \). Set \( \nabla := \nabla^0 \). In the sequel, denote by \( \langle , \rangle \) all metrics unless necessary. Let \( \pi_M \) be the natural projection of \( M \times (-\varepsilon, \varepsilon) \) onto \( M \). Denote by \( \nabla_f \) and \( \nabla^f \) the covariant derivative along \( f \) and \( \tilde{f} \) for \( \nabla \), respectively. For a vector bundle \( E \), denote by \( \Gamma(E) \) the space of all \((C^\infty)\)-sections of \( E \). Also, denote by \( TM \) the tangent bundle of \( M \) and \( \pi_M^*(TM) \) the bundle induced from \( TM \) by \( \pi_M \). For \( X \in \Gamma(TM) \), we define \( X \in \pi_M^*(TM) \) by \( X(x, t) := X_x ((x, t) \in M \times (-\varepsilon, \varepsilon)) \). Let \( \nabla' \) be the connection of \( \pi_M^*(TM) \) satisfying

\[
(\nabla'_X Y)(x, t) := (\nabla_X Y)_x \quad ((x, t) \in M \times (-\varepsilon, \varepsilon)) \quad \text{and} \quad \nabla'_{\partial t} \nabla = 0
\]

for any \( X, Y \in \Gamma(TM) \). Set

\[
\text{Imm}_{b,f}(f) := \{ \hat{f} \in \text{Imm}(M, \tilde{M}) \mid \exists \text{a boundary fixing variation } f_t (0 \leq t \leq 1) \quad \text{s.t.} \quad f_0 = f \text{ and } f_1 = \hat{f} \}
\]

and

\[
\text{Imm}_{b,f;v,p}(f) := \{ \hat{f} \in \text{Imm}(M, \tilde{M}) \mid \exists \text{a boundary fixing and volume preserving variation } f_t (0 \leq t \leq 1) \text{ s.t.} \quad f_0 = f \text{ and } f_1 = \hat{f} \}.
\]

We obtain the following first variational formula for \( F \).

**Theorem 1.1.** Let \( f_t (\varepsilon < t < \varepsilon) \) be a boundary-fixing variation of \( f \) and \( V = f_t(V_T) + \psi \xi \) the variational vector field of \( f_t \), where \( V_T \in \Gamma(TM) \) and \( \psi \in C^\infty(M) \). Then we have

\[
\frac{d}{dt}
\bigg|_{t=0} \mathcal{F}(f_t) = -\int_M \psi H_F dV,
\]

where \( H_F \) is the anisotropic mean curvature of \( f \) and \( dV \) is the volume element of the induced metric on \( M \) by \( f \).

**Proof.** For simplicity, denote by \( \nu_t \) the Gauss map of \( f_t \). Denote by \( g_t, A^t, h_t, H_t \) and \( \xi_t \) the induced metric, the shape operator, the second fundamental form, the mean curvature and a unit normal vector field of \( f_t \), respectively. Also, denote by \( dV_t \) the volume element of \( g_t \). For simplicity, set \( A := A^0, h := h_0, H := H_0 \) and \( \xi := \xi_0 \). Then we have

\[
\frac{d}{dt}
\bigg|_{t=0} \mathcal{F}(f_t) = \int_{x \in M} \frac{d}{dt}
\bigg|_{t=0} \left( F(\nu_t(x)) \right) (dV_t)_x + \int_{x \in M} \frac{d}{dt}
\bigg|_{t=0} \left( F(\nu_t(x)) \right) dV_x + \int_{x \in M} \frac{d}{dt}
\bigg|_{t=0} \left( F(\nu(x)) \right) dV_t.
\]

(1.2)
Also, we have

\[
\frac{d}{dt} \bigg|_{t=0} (dV_t)_x = (-H_x \psi(x) + (\text{div } V_T)_x)dV_x
\]

and

\[
\frac{d}{dt} \bigg|_{t=0} F(\nu_t(x)) = \left\langle (\text{grad } F)_{\nu(x)}, \frac{d}{dt} \bigg|_{t=0} \nu_t(x) \right\rangle.
\]

Set \( \beta(t) := f_t(x) \). By a simple calculation, we can show

\[
\tilde{\nabla}^f_{\partial t} |_{t=0} \xi_t = -f_x(A(V_T)_x + (\text{grad } \psi)_x).
\]

On the other hand, we have

\[
\begin{align*}
\frac{d}{dt} \bigg|_{t=0} \nu_t(x) &= \frac{d}{dt} \bigg|_{t=0} \tau^{-1}_{f_t(x)}(\xi_t) \\
&= \frac{d}{dt} \bigg|_{t=0} \tau^{-1}_{\gamma_{f(x)}(\cdot)[0,t]}(\tau^{-1}_{\gamma_{f(x)}(\cdot)[0,t]}(\xi_x)) \\
&= \left( \frac{d}{dt} \bigg|_{t=0} \tau^{-1}_{\gamma_{f(x)}(\cdot)[0,t]} \right) (\tau^{-1}_{f(x)}(\xi_x)) + \tau^{-1}_{f(x)}(\tilde{\nabla}^f_{\partial t} |_{t=0} \xi_t) \quad (1.6)
\end{align*}
\]

Take a curve \( c : [0, \varepsilon) \rightarrow M \) with \( c'(0) = (V_T)_x \). Then we have

\[
\begin{align*}
\tau^{-1}_{f(x)}(f_x A(V_T)_x) &= -\tau^{-1}_{f(x)}(\nabla^f_{(V_T)_x}) \xi \\
&= -\frac{d}{ds} \bigg|_{s=0} \tau^{-1}_{\gamma_{f(s)}(\cdot)[0,s]}(\xi_{f(s)}) \\
&= \tau^{-1}_{f_x((V_T)_x)}(\tau^{-1}_{f(x)}(\xi_x)) - \nu_x((V_T)_x). \quad (1.7)
\end{align*}
\]

From (1.4) – (1.7), we have

\[
\begin{align*}
\frac{d}{dt} \bigg|_{t=0} F(\nu_t(x)) &= -(\text{div}(\psi(f_x^{-1}(\tau_{f(x)}(\text{grad } F)_{\nu(x)})))_x + \psi(x)(\text{div}(f_x^{-1}(\tau_{f(x)}(\text{grad } F)_{\nu(x)})))_x \\
&\quad + \langle (\text{grad } F)_{\nu(x)}, \tau^{-1}_{V_x} \rangle \tau^{-1}_{\gamma_{f(x)}(\cdot)[0,s]}(\xi_{f(s)}) - \langle (\text{grad } F)_{\nu(x)}, \tau^{-1}_{V_x}\rangle \tau^{-1}_{\gamma_{f(x)}(\cdot)[0,s]}(\xi_{f(s)}) \\
&\quad + (\text{div}((F \circ \nu) V_T))_x - (F \circ \nu)(x)(\text{div}(V_T))_x,
\end{align*}
\]

where we use \( \text{div}(\phi Y) = \phi \text{div } Y + \langle \text{grad } \phi, Y \rangle \quad (\forall \phi \in C^\infty(M), \forall Y \in \Gamma(TM)) \). Since \( F \) is \( \Phi \)-invariant, we have

\[
\langle (\text{grad } F)_{\nu(x)}, \tau^{-1}_{V_x} \rangle = \langle (\text{grad } F)_{\nu(x)}, \tau^{-1}_{\gamma_{f(x)}(\cdot)[0,s]}(\xi_{f(s)}) \rangle = 0,
\]

where we note that \( \tau^{-1}_{V_x} \) and \( \tau^{-1}_{\gamma_{f(x)}(\cdot)[0,s]}(\xi_{f(s)}) \) are elements of the holonomy algebra of \( \tilde{M} \) at \( p_0 \). Also, since \( V \) vanishes on \( \partial M \), we have

\[
\int_M \text{div}(\psi(f_x^{-1}(\tau_{f(x)}(\text{grad } F)_{\nu(x)})))dV = \int_M \text{div}((F \circ \nu) V_T)dV = 0.
\]
From (1.2), (1.3), (1.8), (1.9) and (1.10), we obtain the desired variational formula. q.e.d.

Remark 1.1. This first variational formula cannot be derived directly from the first variational formula by Lira and Melo (see the proof of [LM, Theorem 1]) for a (not necessarily holonomy invariant) elliptic anisotropic surface energy.

From this first variational formula, we obtain the following result.

Theorem 1.2. (i) $f$ is a critical point of $\mathcal{F}|_{\text{Imm}_{b,t}}(f)$ if and only if $H_F$ vanishes.
(ii) $f$ is a critical point of $\mathcal{F}|_{\text{Imm}_{b,t}\times\mathbb{R}}(f)$ if and only if $H_F$ is constant.

Proof. The statement (i) follows from the first variational formula in Theorem 1.1 directly. When the variation $f_t (-\varepsilon < t < \varepsilon)$ is volume-preserving, we have $\int_M \psi dV = 0$, where $\psi$ is as in Theorem 1.1. Hence the statement (ii) also follows from the first variational formula in Theorem 1.1 directly. q.e.d.

2 Anisotropic equifocal hypersurfaces and anisotropic isoparametric hypersurfaces

We use the notations in the previous section. In this section, we shall introduce the notions of an anisotropic convex hypersurface, an anisotropic equifocal hypersurface and an anisotropic isoparametric hypersurface for the holonomy invariant elliptic anisotropic surface energy $\mathcal{F}$. Assume that $\partial M = \emptyset$. Define a transversal vector field $\xi_F$ of $f$ by

\begin{equation}
(\xi_F)_x := (F \circ \nu)(x)\xi_x + \tau_f(x)(\text{grad } F)_{\nu(x)} \quad (x \in M).
\end{equation}

We call $\xi_F$ a anisotropic transversal vector field of $f$. Take $X \in T_x M$. Let $\beta : [0, \varepsilon) \to M$ be a curve with $\beta'(0) = X$. Then, since $F$ is $\Phi$-invariant, we have

\begin{equation}
\tilde{\nabla}_X^{\mathcal{F}} \xi_F = X(F \circ \nu)\xi_x - (F \circ \nu)(x)f_*(A_x X) + \tau_f(x)((\tau^{\text{hol}}_X)^{-1}(\text{grad } F)_{\nu(x)})
+ \tau_f(x)\left(\tilde{\nabla}_{\nu_*(X)} \text{grad } F - \langle \nu_*(X), \text{grad } F \rangle \nu(x)\right)
= -(F \circ \nu)(x)f_*(A_x X) + f_*(\tilde{\nabla}_X (\tau^{-1}_f((\text{grad } F)_{\nu(x)}))).
\end{equation}

So, define a $(1,1)$-tensor field $A^F$ on $M$ by

\begin{equation}
A^F_x X = (F \circ \nu)(x)A_x X - \nabla_X (\tau^{-1}_f((\text{grad } F)_{\nu(x)}))
\end{equation}

for any $x \in M$ and $X \in T_x M$. We call $A^F$ a anisotropic shape operator of $f$ and the eigenvalues of $A^F$ anisotropic principal curvatures of $f$ at $x$. It is easy to show that $\text{Tr } A^F$ coincides with $H_F$. If all anisotropic principal curvatures of $f$ are positive (or negative) at each point of $M$, then we say that $f : M \hookrightarrow \tilde{M}$ is anisotropic convex. Note that, in the case where $\tilde{M}$ is a $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, we have

$$f_*(\tilde{\nabla}_X (\tau^{-1}_f((\text{grad } F)_{\nu(x)}))) = \tilde{\nabla}^S_{\nu_*(X)} \text{grad } F = -\tilde{\nabla}^S_{f_*(A_x X)} \text{grad } F$$

and hence

$$f_*(A^F_x X) = (F \circ \nu)(x)f_*(A_x X) + \tilde{\nabla}^S_{f_*(A_x X)} \text{grad } F,$$
where we identify $f_*(T_xM)$ with $T_{f(x)}S^n(1)$ under the identification of $T_{f(x)}\mathbb{R}^{n+1}$ and $T_{f_0}\mathbb{R}^{n+1}$. Denote by $R$ the curvature tensor of $\tilde{M}$ and $R(\xi_F)$ the anisotropic normal Jacobi operator $R(\cdot, \xi_F)\xi_F$. Let $\gamma^F_\tau$ be the geodesic in $\tilde{M}$ whose initial velocity vector is equal to $(\xi_F)_x$. Take $X \in T_xM$. Let $Y_X$ be the Jacobi field along $\gamma^F_\tau$ with $Y_X(0) = f_*(X)$ and $Y_X'(0) = -f_*(A_xF X)$. We call $Y_X$ an anisotropic $M$-Jacobi field. If $Y_X(0) = 0$ for some $X \neq 0 \in T_xM$, then we call $s_0$ (resp. $\gamma^F_x(s_0)$) an anisotropic focal radius (resp. an anisotropic focal point) of $f$ at $x$. Also, for an anisotropic focal radius $s_0$ of $f$ at $x$, we call \{ $X \in T_xM \mid Y_X(s_0) = 0$ \} the nullity space of $s_0$ and its dimension the multiplicity of $s_0$. If the set of all anisotropic focal radii of $f$ at $x$ is independent of the choice of $x \in M$, then we call $f : M \hookrightarrow \tilde{M}$ an anisotropic equifocal hypersurface. Also, if the set of all anisotropic principal curvatures of $f$ at $x$ is independent of the choice of $x \in M$, then we call $f : M \hookrightarrow \tilde{M}$ a hypersurface with constant anisotropic principal curvatures. Next we shall introduce the notion of an anisotropic isoparametric hypersurface. Define a map $f_t : M \to \tilde{M}$ by

$$f_t(x) := \exp f(x)(t(\xi_F)_x) \quad (x \in M),$$

where $\exp f(x)$ is the exponential map of $\tilde{M}$ at $f(x)$. It is easy to show that $f_t$ is an immersion for each $t$ sufficiently close to zero. If $f_t$ is an immersion, then we call $f_t : M \hookrightarrow \tilde{M}$ an anisotropic parallel hypersurface of $f : M \to \tilde{M}$ of distance $t$. Also, if $f_t$ is not an immersion but the differential of $f_t$ at each point of $\tilde{M}$ is of constant rank, then we call $f_t(M)$ an anisotropic focal submanifold of $f : M \hookrightarrow \tilde{M}$ (corresponding to anisotropic focal radius $t$). If, for each $t$ sufficiently close to zero, $f_t : M \hookrightarrow \tilde{M}$ is of constant anisotropic mean curvature, we call $f : M \hookrightarrow \tilde{M}$ an anisotropic isoparametric hypersurface.

We consider the case where $\tilde{M}$ is a symmetric space. Then a Jacobi field $Y$ along a geodesic $\gamma$ in $\tilde{M}$ is described as

$$(2.4) \quad Y(s) = \tau_{\gamma|_{[0,s]}} \left( D^{co}_{s\gamma'}(Y(0)) + sD^{si}_{s\gamma'}(Y'(0)) \right),$$

where $D^{co}_{s\gamma'}(Y)$ (resp. $D^{si}_{s\gamma'}(Y)$) is given by

$$D^{co}_{s\gamma'} := \cos(s\sqrt{R(\gamma'(0))}) \quad \text{(resp. } D^{si}_{s\gamma'} := \frac{\sin(s\sqrt{R(\gamma'(0))})}{s\sqrt{R(\gamma'(0))}} \text{)}.$$ 

In particular, the anisotropic $M$-Jacobi field $Y_X$ of $f$ is described as

$$(2.5) \quad Y_X(s) = \tau_{\gamma^F_x|_{[0,s]}} \left( D^{co}_{s(\xi_F)_x}(f_*(X)) - sD^{si}_{s(\xi_F)_x}(f_*(A_xF X)) \right),$$

where $D^{co}_{s(\xi_F)_x}$ (resp. $D^{si}_{s(\xi_F)_x}$) is defined in a similar way to $D^{co}_{s\gamma'}(Y)$ (resp. $D^{si}_{s\gamma'}(Y)$). According to (2.5), the anisotropic focal radii of $f$ at $x$ coincide with zero points of the function

$$\rho(s) := \det \left( D^{co}_{s(\xi_F)_x} \circ f_* - s(D^{si}_{s(\xi_F)_x} \circ f_* \circ A_xF) \right).$$

In particular, in the case where $\tilde{M}$ is a Euclidean space, $D^{co}_{s(\xi_F)_x} = D^{si}_{s(\xi_F)_x} = \text{id}$ and hence the anisotropic focal radius of $f$ at $x$ are equal to the inverse numbers of anisotropic principal curvatures of $f$ at $x$. 

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At the end of this section, we give an explicit description of the linear map $\tau_{v}^{\text{hol}}$ defined in the previous section in the case where $\tilde{M}$ is a symmetric space.

**Lemma 2.1.** For $w \in T_{p_{0}}\tilde{M}$, we have

$$\tau_{v}^{\text{hol}}(w) = R_{p_{0}} \left( \gamma'_{p}(0), \frac{\text{id} - D^{co}_{\gamma'_{p}(0)}}{\text{ad}(\gamma'_{p}(0))^2} \left( (\exp_{p_{0}} - 1)_{*} \gamma'_{p}(0) \right) \right) w.$$ 

*Proof.* Set $\varpi := (\exp_{p_{0}})_{*}^{-1} \gamma'_{p}(0)(v)$. Define a 2-parameter map $\delta : [0, 1]^{2} \to \tilde{M}$ by

$$\delta(s, t) := \exp_{p_{0}}(s(\gamma'_{p}(0) + t\varpi)), \quad ((s, t) \in [0, 1]^{2}).$$

Let $Y$ be the vector field along $\gamma_{p}$ defined by $Y := \frac{\partial \delta}{\partial t} \bigg|_{t=0}$. Since $Y$ is the Jacobi field along $\gamma_{p}$ with $Y(0) = 0$ and $Y'(0) = \varpi$, it is described as

$$Y(s) = \tau_{\gamma_{p}[0, s]} \left( sD^{si}_{\gamma'_{p}(0)\varpi} \right).$$

Let $\tilde{w}$ be the vector field along $\delta$ with $\tilde{w}(0, 0) = w$ such that $s \mapsto \tilde{w}(s, t)$ is parallel along $s \mapsto \delta(s, t)$ for each $t \in [0, 1]$. Define a 3-parameter map $\tilde{\delta} : [0, 1]^{3} \to \tilde{M}$ by

$$\tilde{\delta}(s, t, u) := \exp_{\tilde{\delta}(s, t)}(u\tilde{w}(s, t)), \quad ((s, t, u) \in [0, 1]^{3}).$$

Clearly we have

$$\tau_{v}^{\text{hol}}(w) = -\tau_{p}^{-1} \left( \tilde{\nabla}_{\varpi} \frac{\partial \tilde{\delta}}{\partial u} \bigg|_{s=1, t=0, u=0} \right).$$

Define a vector field $Z$ along $\delta$ by

$$Z := \tilde{\nabla}_{\varpi} \frac{\partial \tilde{\delta}}{\partial u} \bigg|_{u=0}.$$ 

Then, by using (2.6), we can show

$$\left( \tilde{\nabla}_{\varpi} Z \right) \bigg|_{t=0} = \left( \tilde{\nabla}_{\varpi} \tilde{\nabla}_{\varpi} \tilde{w} \right) \bigg|_{t=0} + R_{p_{0}} \left( \frac{\partial \tilde{\delta}}{\partial s} \bigg|_{t=u=0}, \frac{\partial \tilde{\delta}}{\partial t} \bigg|_{t=u=0} \right) \tilde{w} \bigg|_{t=0} = R_{p_{0}} \left( \gamma'_{p}(s), Y(s) \right) \tilde{w} \bigg|_{t=0} = \tau_{\gamma_{p}[0, s]} \left( R_{p_{0}} \left( \gamma'_{p}(0), sD^{si}_{\gamma'_{p}(0)\varpi} \right) w \right).$$

Also, we have $Z|_{s=t=0} = 0$. Hence we obtain

$$Z(s, t) = \tau_{\gamma_{p}[0, s]} \left( \int_{0}^{s} R_{p_{0}} \left( \gamma'_{p}(0), sD^{si}_{\gamma'_{p}(0)\varpi} \right) w \, ds \right).$$
In particular, we obtain

\[
Z_{(1,0)} = \tau_p \left( \int_0^1 R_{p_0} (\gamma'_p(0), sD_{\gamma'_p(0)}^s) w \, ds \right)
\]

\[
= \tau_p \left( R_{p_0} (\gamma'_p(0), \left( \int_0^1 sD_{\gamma'_p(0)}^s \, ds \right) (\nabla) w \right) .
\]

This relation together with (2.7) implies the desired relation. q.e.d.

3 Anisotropic tubes

We use the notations in Sections 1 and 2. In this section, we introduce the notion of anisotropic tube over a submanifold. Let \( B \) be an embedded submanifold in \( \tilde{M} \) and \( \pi_B : T^{1,1}B \to B \) the unit normal bundle of \( B \). For a positive number \( r \), we define \( f^F_{F,r} : T^{1,1}B \to \tilde{M} \) by

\[
f^F_{F,r}(v) := \exp_{\pi_B(v)} \left( r \left( \tilde{F}(v)v + (\grad(F_{\pi_B(v)}))v \right) \right) \quad (v \in T^{1,1}B).
\]

Set \( t^F_r(B) := f^F_{F,r}(T^{1,1}B) \). If \( f^F_{F,r} \) is an immersion, then we call \( t^F_r(B) \) the anisotropic tube over \( B \) of radius \( r \). For an anisotropic tube, we can show the following fact.

Theorem 3.1. Let \( B \) be a complete embedded submanifold in \( \tilde{M} \) and \( f : M \hookrightarrow \tilde{M} \) a complete hypersurface in \( \tilde{M} \). If \( B \) is an anisotropic focal submanifold of \( f(M) \) corresponding to anisotropic focal radius \( r \), then \( f(M) = t^F_r(B) \) holds.

Proof. By the assumption, we have \( B = f_r(M) \). Set \( \varepsilon := \frac{r}{|\gamma'|} \). Let \( \xi(s) \) and \( \xi_F(s) \) \( (0 \leq s < \varepsilon r) \) be the unit normal vector field and the anisotropic normal vector field of the parallel hypersurface \( f_s(M)(= t^F_{-r}(B)) \) of \( f(M) \), respectively. Take \( x \in M \) and set \( p := f_r(x) \). Since \( f_s(x) = \gamma(\xi_F(0))_x(s) \), we have \( (\xi_F(s))_x = \gamma(\xi_F(0))_x(s) \). Also, since \( (\xi_F(s))_x = \tilde{F}(\xi(s))_x + (\grad(F_{\xi(s)}))_x \), we obtain

\[
\gamma(\xi_F(0))_x(r) = \tilde{F}(\lim_{s \to r} \xi(s))_x \lim_{s \to r} \xi(s)_x + (\grad(F_{\xi(s)}))_x \lim_{s \to r} \xi(s)_x .
\]

Set \( v := \lim_{s \to r} \xi(s)_x \). Then we have

\[
f(x) = \gamma(\xi_F(0))_x(0) = \gamma(\xi_F(0))_x(r) = \exp_p(-r \gamma(\xi_F(0))_x(r))
\]

\[
= \exp_p \left( -r \left( \tilde{F}(v)v + (\grad(F_{\xi})v) \right) \right) = f^F_{F,-r}(v) \in t^F_{-r}(B).
\]

Thus it follows from the arbitrariness of \( x \) that \( f(M) \subset t^F_{-r}(B) \). Furthermore, it follows from the completeness of \( f(M) \) that \( f(M) = t^F_{-r}(B) \). q.e.d.
4 The anisotropic geodesic spheres

We use the notations in Sections 1-3. For simplicity, set $\text{Exp} := \exp_{p_0}$. In this section, we consider the anisotropic tube $t^F_r(p_0)$ over $\{p_0\}$. We call this tube the anisotropic geodesic sphere of $r$ centered at $p_0$. For example, in the setting of [KP1-3, Palm], the Wulff shape is the anisotropic geodesic sphere of radius 1 centered at the origin. Define $\hat{f}^F_{p_0,r} : S^n(1) \to T_{p_0} \tilde{M}$ by

$$\hat{f}^F_{p_0,r}(v) := r(F(v)v + (\text{grad} F)_v) \quad (v \in S^n(1)).$$

Clearly we have $f^F_{p_0,r} = \text{Exp} \circ \hat{f}^F_{p_0,r}$.

**Assumption.** Let $r$ be such a sufficiently small positive constant as $t^F_r(p_0)$ does not intersect with the cut locus of $p_0$.

Assume that $\tilde{M}$ is a symmetric space of compact type or non-compact type. Let $G$ be the identity component of the isometry group of $\tilde{M}$ and $G_{p_0}$ the isotropy group of $G$ at $p_0$. For simplicity, set $K := G_{p_0}$. Also, let $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition. The space $\mathfrak{p}$ is identified with $T_{p_0} \tilde{M}$. Fix $v \in \mathfrak{p}$. Take a maximal abelian subspace $\mathfrak{a}_v$ of $\mathfrak{p}$ containing $v$, where “abelian” means that $R(w_1, w_2) = 0$ holds for any elements $w_1$ and $w_2$ of $\mathfrak{a}_v$. Then it is shown that $R(w) := R(\cdot, w)w's$ ($w \in \mathfrak{a}_v$) are simultaneously diagonalizable. Hence they have common eigenspace decomposition. Let $\mathfrak{p} = \mathfrak{a}_v \oplus (\oplus_{i=1}^k \mathfrak{p}_i^v)$ be their common eigenspace decomposition. It is clear that there exist an element $\alpha^v_i$ of the dual space $\mathfrak{a}_v^*$ of $\mathfrak{a}_v$ such that, for each $w \in \mathfrak{a}_v$, $R(w) = \varepsilon \alpha^v_i(w)^2 \text{id}$ holds on $\mathfrak{p}_i^v$, where $\text{id}$ is the identity transformation of $\mathfrak{p}$ and $\varepsilon = 1$ (resp. $\varepsilon = -1$) in the case where $G/K$ is of compact type (resp. of non-compact type). Note that $\alpha^v_i$ is unique up to the ($\pm$)-multiple. Set $\Delta^v := \{\pm \alpha^v_1, \cdots, \pm \alpha^v_k\}$. For convenience, we denote $\mathfrak{p}^v_i$ (resp. $\mathfrak{a}^v_i$) by $\mathfrak{p}_i^v$ (resp. $\mathfrak{p}_i^v$). The sysytem $\Delta^v$ gives a root system and it is isomorphic to the (restricted) root system of the symmetric pair $(G, K)$. Hence, if $\alpha, \beta \in \Delta^v$ and if $\beta = a\alpha$ for some constant $a$, then $a = \pm 1$ or $\pm 2$.

Denote by $\xi$ and $\nu$ the outward unit normal vector field and the Gauss map of $f^F_{p_0,r}$, respectively. For $\nu$, we have the following fact.
Lemma 4.1. Assume that \( \widetilde{M} \) is a symmetric space of compact type or non-compact type. For any \( v \in S^n(1) \), \( \nu(v) = v \) holds.

Proof. For simplicity, set \( f := f^F_{p_0, r} \) and \( \tilde{f} := \tilde{f}^F_{p_0, r} \). Take \( X \in T_v S^n(1) \). Let \( c(t) (-\varepsilon < t < \varepsilon) \) be a curve in \( S^n(1) \) with \( c'(0) = X \). Then we have

\[
\begin{align*}
&f_* X = \left. \frac{d}{dt} \right|_{t=0} f(c(t)) \\
&= r \exp_{*f(v)} \left( \frac{d}{dt} \bigg|_{t=0} (F(c(t))c(t) + (\nabla F)_{c(t)}) \right) \\
&= r \exp_{*f(v)} ((XF)v + F(v)X + \nabla^0_X F \nabla F),
\end{align*}
\]

where \( \nabla^0 \) is the Euclidean connection of \( T_{p_0} M \), \( \iota \) is the inclusion map of \( S^n(1) \) into \( T_{p_0} \widetilde{M} \) and \( (\nabla^0)^\iota \) is the covariant derivative along \( \iota \) induced from \( \nabla^0 \). On the other hand, since \( F \) is \( \Phi \)-invariant (hence invariant with respect to the linear isotropy action \( K \mapsto p \)), we may assume that \( (\nabla F)_{c(t)} \in a_v \) by retaking \( a_v \) if necessary. First we consider the case of \( X \in a_v \). Then we have \( \nabla^0_X \nabla F \in a_v \). Hence, since \( (XF)v + F(v)X + \nabla^0_X \nabla F \) belongs to \( a_v \), we can derive

\[
\exp_{*f(v)} ((XF)v + F(v)X + \nabla^0_X \nabla F) = \tau_{f(v)}((XF)v + F(v)X + \nabla^0_X \nabla F).
\]

From (4.1) and this relation, we have

\[
(f_* X, \tau_{f(v)}(v)) = r((XF)v + F(v)X + \nabla^0_X \nabla F, v) = r(XF - \langle \nabla F, X \rangle) = 0.
\]

Next we consider the case of \( X \in a_v^\perp \). Take a curve \( \hat{c} : (-\varepsilon, \varepsilon) \rightarrow T_{p_0} \widetilde{M} \) with \( \hat{c}(0) = \hat{f}(v) \) and \( \hat{c}'(0) = (XF)v + F(v)X + \nabla^0_X \nabla F \), where \( \varepsilon \) is a small positive number. Define a map \( \delta : (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow \widetilde{M} \) by

\[
\delta(t, s) := \exp(s\hat{c}(t)) ((t, s) \in (-\varepsilon, \varepsilon) \times [0, 1]).
\]

Set \( Y := \delta_* \left( \left. \frac{\partial}{\partial t} \right|_{t=0} \right). \) This vector field \( Y \) is the Jacobi field along \( \gamma_{f(v)} \) with \( Y(0) = 0 \) and \( Y'(0) = (XF)v + F(v)X + \nabla^0_X \nabla F \). Hence it follows from (2.4) that

\[
\exp_{*\hat{f}(v)} ((XF)v + F(v)X + \nabla^0_X \nabla F) = Y(1) = \tau_{f(v)}(D_{\hat{f}(v)}(\triangledown^0_{\hat{f}(v)}((XF)v + F(v)X + \nabla^0_X \nabla F))).
\]

On the other hand, from \( X \in a_v^\perp \), we have \( XF = 0 \) and \( \nabla^0_X \nabla F \in a_v^\perp \). From these facts, we can show that

\[
\exp_{*\hat{f}(v)} ((XF)v + F(v)X + \nabla^0_X \nabla F) \in \tau_{v}(a_v^\perp). \]

Hence the relation (4.2) follows from (4.1). Thus, in both cases, we can derive (4.2). Therefore, from the arbitrariness of \( X \), we obtain \( \tau_{f(v)}(v) = \xi_v \), that is, \( \nu(v) = v \). q.e.d.
Denote by \( r_{\widetilde{M}} \) the first conjugate radius of \( \widetilde{M} \). For the anisotropic geodesic sphere, we have the following fact.

**Theorem 4.2.** Assume that \( \widetilde{M} \) is an irreducible symmetric space of compact type or non-compact type and that it is of rank greater than one. Then the following statements hold:

(i) \( r < \frac{1}{2 \max_{v \in S_n(1)} \| F(v) v + (\text{grad } F) c(v) \|} \), then \( t^F_\tau(p_0) \) is anisotropic convex.

(ii) \( t^F_\tau(p_0) \) does not have constant anisotropic principal curvatures.

(iii) \( t^F_\tau(p_0) \) is anisotropic isoparametric.

(iv) If \( \widetilde{M} \) is of compact type, then \( t^F_\tau(p_0) \) is not anisotropic equifocal.

(v) If \( \widetilde{M} \) is of non-compact type, then \( t^F_\tau(p_0) \) is anisotropic equifocal.

**Proof.** For simplicity, set \( f := f^F_{p_0,r} \) and \( \tilde{f} := \tilde{f}^F_{p_0,r} \). Denote by \( A \) the shape operator of \( f \) (for \( \xi \)). Also, denote by \( \xi_F \) the anisotropic transversal vector field of \( f \) and \( A_F \) the anisotropic shape operator of \( f \). Take \( v \in S^n(1) \). According to Lemma 4.1, we have \( \nu(v) = v \). From this fact, it is easy to show that \( \gamma'_{f(v)}(1) = r(\xi_F)_v \). For simplicity, denote \( \gamma_{f(v)} \) by \( \gamma \). Take \( X \in T_v(S^n(1)) \) and let \( c : (-\varepsilon, \varepsilon) \to S^n(1) \) be a curve with \( c'(0) = X \), where \( \varepsilon \) is a small positive number. Define a map \( \delta : (-\varepsilon, \varepsilon) \times [0, 1] \to \widetilde{M} \) by

\[
\delta(t, s) := \text{Exp}(s\tilde{f}(c(t))) \quad ((t, s) \in (-\varepsilon, \varepsilon) \times [0, 1]).
\]

Set \( Y := \delta_s \left( \partial \frac{\partial}{\partial t} \big|_{t=0} \right) \), which is a Jacobi field along \( \gamma \). We have \( Y(0) = 0 \) and

\[
Y'(0) = \frac{d}{dt} \bigg|_{t=0} \left( r \left( F(c(t)) c(t) + (\text{grad } F) c(t) \right) \right) = r \left( (XF)v + F(v)X + (\nabla^0)c_X \text{grad } F \right)
\]

So, according to (2.4), we have

\[
(4.3) \quad Y(s) = \tau_{\gamma|_{[0,s]}} \left( \lambda_{\lambda,\gamma}^D \left( F(v)X + \nabla_X^s \text{grad } F \right) \right).
\]

Hence we have

\[
(4.4) \quad f_*X = Y(1) = r \tau_{\gamma} \left( \lambda_{\lambda,\gamma}^D \left( F(v)X + \nabla_X^s \text{grad } F \right) \right).
\]

On the other hand, we have

\[
(4.5) \quad f_*(A^F_v(X)) = -\nabla^f_X \xi_F = -\frac{1}{r} \left( \nabla^\delta_X \delta_s \left( \frac{\partial}{\partial s} \right) \right) \bigg|_{t=0,s=1} = \frac{1}{r} Y'(1) = -\tau_{\gamma} \left( \lambda_{\lambda,\gamma}^D \left( F(v)X + \nabla_X^s \text{grad } F \right) \right).
\]

Let \( a_v, \Delta^v \) and \( p^v_\gamma \) be as above. Since \( F \) is \( \Phi \)-invariant (hence invariant with respect to the linear isotropy action \( K \curvearrowright p \)), we may assume that \( (\text{grad } F)_v \in a_v \) retaking \( a_v \) if necessary.
First we consider the case of $X \in \mathfrak{a}_v \supset \text{Span}\{v\}$. Then, since $(\text{grad } F)_w \in \mathfrak{a}_v$ for any $w \in S^n(1) \cap \mathfrak{a}_v$, we have $\nabla^S_X \text{grad } F \in \mathfrak{a}_v$. Hence it follows from (4.4) that

\begin{align}
(4.6) \quad f_* X &= r\tau_\gamma (F(v)X + \nabla^S_X \text{grad } F).
\end{align}

Also it follows from (4.5) that

\begin{align}
(4.7) \quad f_*(A^F_\nu X) &= -\tau_\gamma (F(v)X + \nabla^S_X \text{grad } F).
\end{align}

Therefore, we obtain

\begin{align}
(4.8) \quad A^F_\nu X &= -\frac{1}{r} X.
\end{align}

Also, we have

\begin{align}
(4.9) \quad \left(D^c_{s(\xi_F)_v} \circ f_* - s(D^i_{s(\xi_F)_v} \circ f_* \circ A^F_\nu)\right)(X) &= \left(1 + \frac{s}{r}\right)f_* X.
\end{align}

Next we consider the case of $X \in p^\nu_\alpha (\alpha \in \Delta^v)$. Then there exist a one-parameter subgroup $\{k_t\}_{t \in \mathbb{R}}$ in $K$ such that $\frac{d}{dt}\bigg|_{t=0} k_t \cdot v = X$. Since $k_t|_{T,S^n(1)}$ coincides with the parallel translation along $t \mapsto k_t \cdot v$ in $S^n(1)$ and since grad $F$ is $K$-invariant, we obtain $\nabla^S_X \text{grad } F = 0$. Hence it follows from (4.4) that

\begin{align}
(4.10) \quad f_* X &= \frac{F(v) \sin(r\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v)))}{\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v))} \tau_\gamma(X).
\end{align}

Also it follows from (4.5) that

\begin{align}
(4.11) \quad f_*(A^F_\nu X) &= -F(v) \cos(r\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v)))\tau_\gamma(X).
\end{align}

Therefore, we obtain

\begin{align}
(4.12) \quad A^F_\nu X &= -\frac{\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v))}{\tan(r\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v)))} X,
\end{align}

where $\frac{\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v))}{\tan(r\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v)))}$ means 0 if $\tilde{M}$ is of compact type and if $r\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v)) = \pm \frac{\pi}{2}$. Here, when $\tilde{M}$ is of compact type, we note that $r\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v))$ is smaller than $\pi$ because $W(r)$ does not intersect with $C$ (i.e., $||\tilde{f}(v)|| = r||((\xi_F)_v)||$ is smaller than $r\tilde{M}$) and $r\tilde{M} < \frac{\pi ||((\xi_F)_v)||}{|\alpha(\tau^{-1}_F((\xi_F)_v)))|}$. Also, we have

\begin{align}
(4.13) \quad \left(D^c_{s(\xi_F)_v} \circ f_* - s(D^i_{s(\xi_F)_v} \circ f_* \circ A^F_\nu)\right)(X) &= \left(\cos(s\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v))) + \frac{\sin(s\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v)))}{\tan(r\sqrt{\epsilon}\alpha(\tau^{-1}_F((\xi_F)_v)))}\right) f_* X.
\end{align}
According to (4.8) and (4.12), the spectrum Spec $A^F_v$ of $A^F_v$ is given by

\begin{equation}
\text{Spec } A^F_v = \left\{ -\frac{\sqrt{2}r\alpha(\tau^{-1}_{f(v)}((\xi F)_v))}{\tan(r\sqrt{2}r\alpha(\tau^{-1}_{f(v)}((\xi F)_v)))} \middle| \alpha \in \Delta^v_+ \right\} \cup \left\{ -\frac{1}{r} \right\},
\end{equation}

where $\Delta^v_+$ is the positive root system of $\Delta^v$ under some lexicographic ordering of $a^*_v$.

Since $\hat{M}$ is an irreducible symmetric space of rank greater than one, $\alpha((\xi F)_v)$ depends on the choice of $v \in S^n(1)$. Therefore $t^F_r(p_0)$ does not have constant anisotropic principal curvatures and it is not anisotropic isoparametric. Thus the statements (ii) and (iii) follow. In particular, if $r < \frac{r_{\tilde{M}}}{2\max_{v \in S^n(1)} \| F(v) v + (\text{grad } F)_v \|}$, then $r\sqrt{2}r\alpha(\tau^{-1}_{f(v)}((\xi F)_v))$ is smaller than $\frac{\pi}{2}$ in the case where $\tilde{M}$ is of compact type. Hence, in this case, it follows from (4.14) that $W_F(r)$ is anisotropic convex. Thus the statement (i) follows. From (4.9) and (4.13), the set $\mathcal{AFR}_v$ of all anisotropic focal radii of $t^F_r(p_0)$ at $v$ is given by

$$\mathcal{AFR}_v = \left\{ \begin{cases} -r + \frac{j\pi}{\alpha(\tau^{-1}_{f(v)}((\xi F)_v))} & \alpha \in \Delta^v_+, \ j \in \mathbb{Z} \\ \{-r\} & \text{($\tilde{M}$ : compact type)} \end{cases} \right\} \quad \text{($\tilde{M}$ : non - compact type)}.$$

Hence we obtain the statement (v). Also, since $\alpha((\xi F)_v)$ depends on the choice of $v \in S^n(1)$, we obtain the statement (iv) \quad \text{q.e.d.}

Remark 4.1. If $\tilde{M}$ is irreducible and of rank greater than one, then geodesic spheres in $\tilde{M}$ are not isoparametric and they have not constant principal curvatures. On the basis of this fact, we can conjecture the statements (ii) and (iii) of this theorem in advance.

5 Anisotropic tubes over certain kind of reflective submanifolds in a symmetric space

We use the notations in Sections 1-4. Let $\tilde{M}$ be a symmetric space of compact type or non-compact type. In this section, we shall show that anisotropic tubes over a reflective singular orbit of a Hermann action of cohomogeneity one on $\tilde{M}$ are anisotropic equivocal and anisotropic isoparametric hypersurface (for $F$). Fix $p_0 \in \hat{M}$. Let $G$ and $K$ be as in the previous section. Let $\theta$ be the involution of $G$ with $(\text{Fix } \theta)_0 \subset K \subset \text{Fix } \theta$, where $\text{Fix } \theta$ is the fixed point group of $\theta$ and $(\text{Fix } \theta)_0$ is the identity component of $\text{Fix } \theta$. Here, in the case where $\hat{M}$ is of compact type (resp. non-compact type), we give $\tilde{M}$ the $G$-invariant metric induced from the $-\langle \ , \rangle_K$ (resp. $\langle \ , \rangle_K$), where $\langle \ , \rangle_K$ is the Killing form of the Lie algebra of $G$. Let $H$ be a symmetric subgroup of $G$ (i.e., $(\text{Fix } \tau)_0 \subset H \subset \text{Fix } \tau$ for some involution $\tau$ of $G$). The natural action of $H$ on $\tilde{M}$ is called a Hermann action (see [HPTT], [Kol], [Koi2]). In the sequel, we assume that the $H$-action is of cohomogeneity one and commutative, where "commutative" means that $\theta \circ \tau = \tau \circ \theta$. Let $\mathfrak{g}, \mathfrak{f}$ and $\mathfrak{h}$ be the Lie algebras of $G, K$ and $H$, respectively. We denote the involutions of $\mathfrak{g}$ induced from $\theta$ and $\tau$ by the same symbols $\theta$ and $\tau$, respectively. Set $\mathfrak{p} := \text{Ker}(\theta + \text{id})$ and $\mathfrak{q} := \text{Ker}(\tau + \text{id})$. For simplicity, set $B := H_{p_0}$, which is reflective because $\theta \circ \tau = \tau \circ \theta$. Note that $\mathfrak{p}, \mathfrak{p} \cap \mathfrak{h}$ and $\mathfrak{p} \cap \mathfrak{q}$ are identified with $T_{p_0} \hat{M}, T_{p_0} B$ and $T_{p_0}^\perp B$, respectively. Let $\exp(\cdot : T\hat{M} \rightarrow \hat{M})$ be
the exponential map of \( \tilde{M} \) and \( \exp^{\tilde{G}} \) be the exponential map of \( \tilde{G} \). From \( \theta \circ \tau = \tau \circ \theta \), it follows that \( p = p \cap \mathfrak{h} \oplus p \cap \mathfrak{q} \). We define a map \( \tilde{f}^E_{B,r} : T^{1,1}B \to TM \) by

\[
\tilde{f}^E_{B,r}(v) := r \left( \tilde{F}(v) + (\text{grad} \tilde{F}|_{S^n(1)_{p_B(v)}}) v \right) \quad (v \in T^{1,1}B).
\]

Then we have \( f^E_{B,r} = \exp \circ \tilde{f}^E_{B,r} \). Denote \( \nabla^B \) the normal connection of \( B \) and \( P^\text{hol}_v \) the holonomy subbundle of \( TM \) through \( v \). Then we can show the following fact.

**Lemma 5.1.** The holonomy invariant elliptic parametric Lagrangian \( \tilde{F} \) is constant over \( T^{1,1}B \).

**Proof.** Take any curve \( c \) in \( B \). Since \( B \) is totally geodesic, the parallel translations along \( c \) with respect to \( \nabla^B \) and \( \tilde{\nabla} \) coincide with each other. Hence \( P^\text{hol}_v \) is included by \( T^{1,1}B \) for each \( v \in T^{1,1}B \). On the other hand, since the \( H \)-action on \( \tilde{M} \) is of cohomogeneity one, we can derive that the fibre \( \pi_B^{-1}(p_0) \) is an orbit of the subaction by \( H \cap K \) of the linear isotropy group action \( K \rhd T_{p_0}M \) (which is the holonomy group action of \( \tilde{M} \) at \( p_0 \)). Hence we obtain \( T^{1,1}B = P^\text{hol}_v \) for any \( v \in \pi_B^{-1}(p_0) \). Therefore, \( \tilde{F} \) is constant over \( T^{1,1}B \) because \( \tilde{F} \) is holonomy invariant.

Denote by \( X^L_v \) the horizontal lift of \( X(\in T_p \tilde{M}) \) to \( v(\in T_p \tilde{M}) \), where \( v \) is regarded as a point of the fibre of \( TM \) over \( p \). Also, denote by \( c^L_v \) the horizontal lift of a curve \( c : [0,\varepsilon) \to \tilde{M} \) to \( v(\in T_{c(0)} \tilde{M}) \), where \( v \) is regarded as a point of the fibre of \( TM \) over \( c(0) \). Then we can show the following fact.

**Lemma 5.2.** Let \( X \) be a tangent vector of \( B \) at \( p \), \( \gamma_X \) the geodesic in \( B \) with \( \gamma'_X(0) = X \) and \( v \) be an element of \( T^{1,1}_pB \). Then we have

\[
\frac{\partial}{\partial t} \tilde{F}^E_{B,r}(v)(t) = 0.
\]

**Proof.** From the holonomy invariantness of \( \tilde{F} \) and \( (\gamma_X)^L_v(t) = \tau_{\gamma_X(t)}(v) \), we can derive

\[
\left. \frac{\partial}{\partial t} \right|_{t=0} \left( (\text{grad} \tilde{F}|_{S^n(1)_{\gamma_X(t)}})(\gamma_X)^L_v(t) \right) = 0.
\]

From this relation and Lemma 5.1, the desired relation follows directly. q.e.d.

Denote by \( H \) (resp. \( V \)) be the horizontal (resp. vertical) distribution on \( T^{1,1}B \), where “horizontality” means that so is with respect to \( \nabla^B \). Then we can show the following fact.

**Theorem 5.3.** Assume that \( r < \frac{\text{max}_{v \in S^n(1)}|F(v) + (\text{grad} F)_v|}{2} \). Then the following statements (i)-(iii) hold:
(i) \( f_{B,p}^* \) is an embedding, and \( t_{B}^*(B) \) is an anisotropic equifocal and anisotropic isoparametric hypersurface.

(ii) \( t_{B}^*(B) \) has constant anisotropic principal curvatures.

Proof. For simplicity, set \( \tilde{f} := f_{B,p}^*, \ f := f_{B,p}^* \) and \( M := t_{B}^*(B) \). Denote by \( \xi, \xi_F \) and \( A^F \) the unit normal vector field, the anisotropic normal vector field and the anisotropic shape operator of \( f \), respectively. For any \( p \in B \), denote by \( G_p \) the isotropy group of \( G \) at \( p \). For convenience, set \( K^p := G_p \). Let \( \mathfrak{g}^p \) be the Lie algebra \( K^p \) and \( \mathfrak{g} = \mathfrak{g}^p + \mathfrak{p}^p \) be the canonical decomposition associated with the symmetric pair \((G, K^p)\). Let \( \gamma^0_\mathfrak{p}(t) = \gamma^0_\mathfrak{g}(t) \) be the common eigenspace decomposition of \( f \) and \( \sigma^0_\gamma \) be the parallel translation along \( \gamma^0_\mathfrak{g}(t) \). Take any \( v \in \mathfrak{p}^p \). Let \( \mathfrak{a}_v \) be a maximal abelian subspace of \( \mathfrak{p}^p \) containing \( v \). Let \( \mathfrak{p}^p = \mathfrak{a}_v \oplus (\oplus_{i=1}^k \mathfrak{p}_i^p) \) be the dual space \( \mathfrak{a}_v^* \) of \( \mathfrak{a}_v \) such that, for each \( \mathfrak{v} \in \mathfrak{a}_v \), \( R_i(\mathfrak{v}) = \varepsilon \alpha^0_i(\mathfrak{v})^2 \) holds on \( \mathfrak{p}_i^p \), where \( \varepsilon = 1 \) (resp. \( \varepsilon = -1 \)) in the case where \( M \) is of compact type (resp. of non-compact type). Set \( \Delta^\mathfrak{v} := \{ \pm \alpha^0_1, \cdots, \pm \alpha^0_k \} \). For convenience, we denote \( \mathfrak{p}_i^p \) (resp. \( \mathfrak{a}_v \)) by \( \mathfrak{p}_i^v \) (resp. \( \mathfrak{a}_v^* \)).

Take any \( p \in B \) and any \( v \in \pi^{-1}_B(p) \). Also, take any \( w \in \mathfrak{V}_v \). Let \( c : (-a, a) \rightarrow \pi^{-1}_B(p) \) be a curve with \( c(0) = w \), where \( a \) is a positive number. Define a map \( \delta : (-a, a) \times [0, 1] \rightarrow \tilde{M} \) by \( \delta(t, s) = \exp_{\pi_B(p)}(s\tilde{f}(c(t))) \), \( (t, s) \in (-a, a) \times [0, 1] \). Set \( Y := \frac{\partial \delta}{\partial t} \bigg|_{t=0} \), which is a Jacobi field along \( \gamma_{f(v)} \) with \( Y(0) = 0 \). Also we have

\[
Y'(0) = \frac{d}{dt} \bigg|_{t=0} r \left( \tilde{F}(c(t))c(t) + (\grad(\tilde{F}|_{S^n(1)p}))(c(t)) \right) = r \left( w\tilde{F} + \tilde{F}(v)w + (\nabla^0)w\grad(\tilde{F}|_{S^n(1)p}) \right) = r \left( \tilde{F}(v)w + \nabla^S_w\grad(\tilde{F}|_{S^n(1)p}) \right),
\]

where \( \nabla^0 \) is the Euclidean connection of \( T_p\tilde{M} \), \( \iota \) is the inclusion map of \( S^n(1)p \) into \( T_p\tilde{M} \), \( (\nabla^0)^\iota \) is the covariant derivative along \( \iota \) induced from \( \nabla^0 \) and \( \nabla^S \) is the Riemannian connection of \( S^n(1)p \). Since \( \tilde{F} \) is holonomy invariant and \( w \) belongs to \( \mathfrak{a}_v^* \), we can derive \( \nabla^S_w\grad(\tilde{F}|_{S^n(1)p}) = 0 \) (see the proof of Theorem 4.2). Hence we have \( Y'(0) = r\tilde{F}(v)w \).

Therefore \( Y \) is described as

\[
Y(s) = \tau_{\gamma_{f(v)}(0, s)}^p \left( sr\tilde{F}(v)D^{s\gamma^0_i}_{f(v)}w \right).
\]

Therefore, we obtain

\[
f_*w = Y(1) = \tau_{f(v)}^p \left( r\tilde{F}(v)D^{s\gamma^0_i}_{f(v)}w \right).
\]

Take \( v \in \pi^{-1}_B(p) \) and \( X \in T_pB \). Let \( \gamma_X : (-a, a) \rightarrow B \) be the geodesic in \( B \) with \( \gamma_X(0) = X \) (i.e., \( \gamma_X(t) = \exp_p tX \)) and \( \tilde{v} \) the parallel unit normal vector field of \( B \) along \( \gamma_X \) with \( \tilde{v}(0) = v \). Define a map \( \delta : (-a, a) \times [0, 1] \rightarrow \tilde{M} \) by \( \delta(t, s) = \exp_{\gamma_X(t)}(s\tilde{f}(\tilde{v}(t))) \), \( (t, s) \in (-a, a) \times [0, 1] \). Set \( \tilde{Y} := \frac{\partial \delta}{\partial t} \bigg|_{t=0} \), which is a Jacobi field along \( \gamma_{\tilde{f}(v)} \) with \( \tilde{Y}(0) = X \). Since \( B \) is totally geodesic, we have

\[
\nabla^X_{\tilde{v}} \bigg|_{t=0} = 0.
\]
which implies that \( \tilde{v}(t) = (\gamma_X)_v^L(t) \) that is, \( \tilde{v}'(0) = X_v^L \). On the other hand, since \( \tilde{F} \) is holonomy invariant, we have \((X_v^L)\tilde{F} = 0 \). From these facts and Lemma 5.2, we can derive

\[
\hat{Y}'(0) = \frac{\tilde{\gamma}_w^X}{\tilde{r}} \bigg|_{t=0} \tilde{f}(\tilde{v}) = 0.
\]

Therefore \( \hat{Y} \) is described as

\[
(5.3) \quad \hat{Y}(s) = \tau_{\tilde{f}(v)[0,s]}\left(D_i\tilde{f}(v)X_1\right).
\]

Therefore, we obtain

\[
(5.4) \quad f_s(X_v^L) = \hat{Y}(1) = \tau_{\tilde{f}(v)}(D_i\tilde{f}(v)X).
\]

Since \( r < \frac{r}{2\max_{u \in S^n(1)} ||F(u)|| + (\text{grad } F)||u||} \) by the assumption and \( r_M < \min_{\alpha \in \Delta_v^u} \frac{\pi}{\min_{\alpha \in \Delta_v^u} \alpha(u)} \), we have \( |\alpha(\tilde{f}(v))| < \frac{\pi}{2} \) for any \( \alpha \in \Delta_v^u \). Therefore, it follows from (5.2) and (5.4) that \( f \) is a immersion. Furthermore, it is easy to show that \( f \) is an embedding. From (5.2) and (5.4), we have

\[
\langle f_s w, \tau_{\tilde{f}(v)}(v) \rangle = \langle f_s(X_v^L), \tau_{\tilde{f}(v)}(v) \rangle = 0.
\]

From the arbitrariness of \( w \) and \( X \), these relations imply that \( \tau_{\tilde{f}(v)}(v) = \xi_v \). Furthermore, from this fact, we can derive

\[
(5.5) \quad (\tau_{\tilde{f}(v)})'(1) = r(\xi_F)_v.
\]

This fact implies that \( f_\varepsilon(v) = p \). Hence it follows from the arbitrariness of \( v \) that \( f_\varepsilon(\pi_{B^{-1}}(p)) = \{p\} \). Hence we obtain \( f_\varepsilon(T^{1-1}B) = B \). Thus \( B \) is an anisotropic focal submanifold of \( f(M) \). From (5.5), we have

\[
(5.6) \quad f_s(A^F_v w) = -\tilde{\nabla}_w^f \xi_F = -\frac{1}{r} \left( \tilde{\nabla}_w^f \delta_v^s \left( \frac{\partial}{\partial s} \right) \right) \bigg|_{t=0,s=1} = -\frac{1}{r} \tilde{Y}'(1)
\]

and

\[
(5.7) \quad f_s(A^F_v (X_v^L)) = -\tilde{\nabla}_w^f \xi_F = -\frac{1}{r} \left( \tilde{\nabla}_w^f \delta_v^s \left( \frac{\partial}{\partial t} \right) \right) \bigg|_{t=0,s=1} = -\frac{1}{r} \tilde{Y}'(1)
\]

Assume that \( w \in p^*_v \). Then it follows from (5.2) and (5.6) that

\[
(5.8) \quad A^F_v w = -\frac{\sqrt{\varepsilon} \alpha((\tau_{\tilde{f}(v)})^{-1}((\xi_F)_v))}{\tan(r \sqrt{\varepsilon} \alpha((\tau_{\tilde{f}(v)})^{-1}((\xi_F)_v)))} w,
\]

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where \( \frac{\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}}{\tan(r\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}))} \) means 0 if \( \widetilde{M} \) is of compact type and if \( r\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v) = \pm \frac{\pi}{2} \). Also, we have

\[
\begin{align}
(D^o_{s(\xi_{F})} \circ f_s - s(D^i_{s(\xi_{F})} \circ f_s \circ A_v^F)) \tag{5.10} \\
&= \left( \cos(s\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v)}) + \frac{\sin(s\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v})}{\tan(r\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}))} \right) f_s w.
\end{align}
\]

Assume that \( X \in p^u_a \). Then it follows from (5.4) and (5.7) that

\[
A_v^F(X_v^L) = \sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}) \tan(r\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}))X_v^L.
\]

Also, we have

\[
\begin{align}
(D^o_{s(\xi_{F})} \circ f_s - s(D^i_{s(\xi_{F})} \circ f_s \circ A_v^F)) \tag{5.11} \\
&= \left( \cos(s\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v)}) \\
&\quad - \sin(s\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}) \right) \tan(r\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v})) f_s(X_v^L).
\end{align}
\]

According to (5.8) and (5.10), we obtain

\[
\text{Spec } A_v^F = \left\{ \begin{array}{l}
- \frac{\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v})}{\tan(r\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}))} \bigg| \alpha \in \Delta^u_+ \text{ s.t. } p^u_a \cap q \neq \{0\} \\
\cup \left\{ \sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v}) \tan(r\sqrt{\varepsilon}(\tau_{f}^p)^{-1}(\xi_{F})_{v})) \bigg| \alpha \in \Delta^u_+ \cup \{0\} \text{ s.t. } p^u_a \cap h \neq \{0\} \right\}
\end{array} \right.,
\]

where \( p^u_a \) means \( a_u \). Also, according to (5.9) and (5.11), we obtain

\[
\begin{align}
\text{Spec } A_v^F & = \left\{ \begin{array}{l}
- \frac{j\pi}{\alpha(\tau_{f}^p)^{-1}(\xi_{F})_{v})} \bigg| \alpha \in \Delta^u_+ \text{ s.t. } p^u_a \cap q \neq \{0\} \\
\cup \left\{ \frac{r+2j+1}{2\alpha(\tau_{f}^p)^{-1}(\xi_{F})_{v})} \bigg| \alpha \in \Delta^u_+ \text{ s.t. } p^u_a \cap h \neq \{0\} \right\}
\end{array} \right.
\]

(5.13) \( \text{AFR}_v = \) (when \( \widetilde{M} \) : compact type)

\( \) (when \( \widetilde{M} \) : non−compact type).

Take a geodesic \( \tilde{\gamma} \) in \( B \) with \( \tilde{\gamma}(0) = p \). Set \( \tilde{X} := \tilde{\gamma}'(0) \). We have \( \tilde{\gamma}(t) = \text{Exp}_t \tilde{X} \) and \( \tau_{\gamma}|_{[0,t]} = (\exp^G t \tilde{X})_{*} p \). Set \( g_t := \exp^G t \tilde{X} \). According to Lemma 5.2, we have

\[
\begin{align}
(\tau_{f(\gamma^L_{v}(t))}^p)^{-1}(\xi_{F})_{v} & = \frac{1}{r} \tilde{f}(\gamma^L_{v}(t)) \\
& = \frac{1}{r} (g_t)_{*} p (\tau_{f(\gamma^L_{v}(t))}^p)^{-1}(\xi_{F})_{v}) \\
& = (g_t)_{*} p (\tau_{f(\gamma^L_{v}(t))}^p)^{-1}(\xi_{F})_{v}),
\end{align}
\]

where \( \tau_{f(\gamma^L_{v}(t))}^p \) is defined in similar to \( \tau_{f}^p \). From this fact, we can derive \( \text{Spec } A_{\tilde{\gamma}^L_{v}(t)}^F = \text{Spec } A_v^F \) and \( \text{AFR}_{\tilde{\gamma}^L_{v}(t)} = \text{AFR}_v \). Therefore, it follows from the arbitrariness of \( \gamma \) and
\( P^\text{hol}_v = T^\perp 1B \) that Spec \( A^F \) and \( AFR \) are independent of the choice of \( \bullet \in T^\perp 1B \). Therefore, \( M \) is anisotropic equifocal and has constant anisotropic principal curvatures. For each \( t \) close to 0, the anisotropic parallel hypersurface \( f_t : T^\perp 1B \to \hat{M} \) is given by

\[
f_t(v) := \exp_{\pi_B(v)} \left( \left( \frac{t}{r} + 1 \right) \hat{f}(v) \right) \quad (v \in T^\perp 1B).
\]

Set \( M_t := f_t(T^\perp 1B) \). As above, it is shown that \( M_t \)'s have constant anisotropic principal curvatures and hence they have constant anisotropic mean curvature. That is, \( M \) is anisotropic isoparametric. This completes the proof. 

\( \text{q.e.d.} \)

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![Diagram 3](image3.png)

**Figure 3.**

![Diagram 4](image4.png)

**Figure 4.**
At the end of this section, we give the list of all (commutative) Hermann actions $H \ltimeseq G/K$ of chomogeneity one on irreducible symmetric spaces $G/K$ of non-compact type and rank greater than one and the reflective singular orbit $B = H(eK)$ (see [BT,Koi3]).

### Table 1.

| $H \ltimeseq G/K$ | $B = H(eK)$ | Remark |
|-------------------|--------------|---------|
| $SO_0(k-1,n-k)$ $\ltimeseq$ | $SO_0(k-1,n-k)/SO(k-1) \times SO(n-k)$ | $2 \leq k < n/2$ |
| $SO_0(k,n-k)/SO(k) \times SO(n-k)$ | | |
| $SO_0(k,n-k-1)$ $\ltimeseq$ | $SO_0(k,n-k-1)/SO(k) \times SO(n-k-1)$ | $2 \leq k < n/2$ |
| $SO_0(k,n-k)/SO(k) \times SO(n-k)$ | | |
| $SO_0(k-1,k)$ $\ltimeseq$ | $SO_0(k-1,k)/SO(k-1) \times SO(k)$ | $k \geq 3$ |
| $SO_0(k)/SO(k) \times SO(k)$ | | |
| $SL(3,\mathbb{R}) \cdot U(1)$ $\ltimeseq$ | $(SL(3,\mathbb{R})/SO(3)) \times \mathbb{R}$ | |
| $SO_0(3,3)/SO(3) \times SO(3)$ | | |
| $SU(1,n-1) \cdot U(1)$ $\ltimeseq$ | $SU(1,n-1)/SO(1) \times U(n-1)$ | $n \geq 3$ |
| $SU(1,n-1) \cdot U(1)$ $\ltimeseq$ | | |
| $SO_0(2,2n-2)/SO(2) \times SO(2n-2)$ | | |
| $SU(k,n-k)$ $\ltimeseq$ | $SU(k-1,n-k)/SO(1) \times U(n-1)$ | $2 \leq k < n/2$ |
| $SU(k,n-k)/SU(k) \times U(n-k)$ | | |
| $SU(k,n-k-1)$ $\ltimeseq$ | $SU(k,n-k-1)/SU(k) \times U(n-k-1)$ | $2 \leq k < n/2$ |
| $SU(k,n-k)/SU(k) \times U(n-k)$ | | |
| $SU(k-1,k)$ $\ltimeseq$ | $SU(k-1,k)/SU(k-1) \times U(k)$ | $k \geq 3$ |
| $SU(k)/SU(k) \times U(k)$ | | |
| $Sp(1,n-1)$ $\ltimeseq$ | $Sp(1,n-1)/Sp(1) \times Sp(n-1)$ | $n \geq 3$ |
| $Sp(1,n-1) \cdot U(1)$ $\ltimeseq$ | | |
| $SO(2,2n-2)/SO(2) \times U(2n-2)$ | | |
| $H \curvearrowright G/K$ | $B = H(eK)$ | Remark |
|---------------------------|----------------|---------|
| $Sp(k - 1, n - k) \curvearrowright$ | $Sp(k - 1, n - k)/Sp(k - 1) \times Sp(n - k)$ | $2 \leq k < n/2$ |
| $Sp(k, n - k)/Sp(k) \times Sp(n - k)$ | | |
| $Sp(k, n - k - 1) \curvearrowright$ | $Sp(k, n - k - 1)/Sp(k) \times Sp(n - k - 1)$ | $2 \leq k < n/2$ |
| $Sp(k, n - k - 1)/Sp(k) \times Sp(n - k)$ | | |
| $Sp(k - 1, k) \curvearrowright$ | $Sp(k - 1, k)/Sp(k - 1) \times Sp(k)$ | $k \geq 2$ |
| $Sp(k - 1, k)/Sp(k) \times Sp(k)$ | | |
| $Sp(2, C) \curvearrowright$ | $Sp(2, C)/Sp(2)$ | |
| $Sp(2, 2)/Sp(2) \times Sp(2)$ | | |
| $SL(n - 1, \mathbb{R}) : \mathbb{R} \curvearrowright SL(n, \mathbb{R})/SO(n)$ | $(SL(n - 1, \mathbb{R})/SO(n - 1)) \times \mathbb{R}$ | $n \geq 3$ |
| $SU^*(2n - 2) : \mathbb{R} \curvearrowright SU^*(2n)/Sp(n)$ | $(SU^*(2n - 2)/Sp(n - 1)) \times \mathbb{R}$ | $n \geq 3$ |
| $SO^*(2n - 2) \curvearrowright SO^*(2n)/U(n)$ | $SO^*(2n - 2)/U(n - 1)$ | $n \geq 4$ |
| $SU(1, 3) : U(1) \curvearrowright SU^*(8)/U(4)$ | $SU(1, 3)/SU(4)$ | |
| $SU^*(4) : U(1) \curvearrowright SU^*(8)/U(4)$ | $SU^*(4)/Sp(2)$ | |
| $Sp(1, \mathbb{R}) \times Sp(n - 1, \mathbb{R}) \curvearrowright$ | $(Sp(n, \mathbb{R})/U(n))$ | |
| $SO(n - 1, \mathbb{C}) \curvearrowright SO(n, \mathbb{C})/SO(n)$ | $SO(n, \mathbb{C})/SO(n)$ | $n \geq 5$ |
| $SL(3, \mathbb{C}) : SO(2, C) \curvearrowright SO(6, \mathbb{C})/SO(6)$ | $(SL(3, \mathbb{C})/SU(3)) \times (SO(2, C)/SO(2))$ | |
| $SL(n - 1, \mathbb{C}) \times \mathbb{C} \curvearrowright$ | $(SL(n - 1, \mathbb{C})/SU(n - 1)) \times \mathbb{R}$ | $n \geq 3$ |
| $SL(n - 1, \mathbb{C})/SU(n)$ | |
| $SL(3, \mathbb{R}) \curvearrowright SL(3, \mathbb{C})/SU(3)$ | $SL(3, \mathbb{C})/SU(3)$ | |
| $Sp(2, C) \curvearrowright SL(4, \mathbb{C})/SU(4)$ | $Sp(2, C)/Sp(2)$ | |
| $SU(1, 2) \curvearrowright SL(3, \mathbb{C})/SU(3)$ | $SU(1, 2)/SU(1) \times SU(2)$ | |
| $Sp(1, \mathbb{C}) \times Sp(n - 1, \mathbb{C}) \curvearrowright$ | $(Sp(n, \mathbb{C})/Sp(1)) \times (Sp(n - 1, \mathbb{C})/Sp(n - 1))$ | $n \geq 3$ |
| $F_4^\times \curvearrowright E_6^\times/Sp(6) \times SU(2)$ | $F_4^\times/Sp(3) \times Sp(1)$ | |
| $F_4^\times \curvearrowright E_6^\times/Spin(14) \times U(1)$ | $F_4^\times/Spin(9)$ | |
| $SU^*(6) \times SU(2) \curvearrowright E_6^\times/Spin(26)/F_4$ | $SU^*(6) \times SU(2)/Sp(3) \times Sp(1)$ | |
| $SO_0(1, 9) : U(1) \curvearrowright E_6^\times/Spin(26)/F_4$ | $SO_0(1, 9) : U(1)/SO(1) \times SO(9)$ | |
| $SO_0(4, 5) \times F_4^\times/Sp(3) \times Sp(1)$ | $SO_0(4, 5)/SO(4) \times SO(5)$ | |
| $SO(9, C) \times F_4^\times/F_4$ | $SO(9, C)/SO(9)$ | |

Table 1 (continued).

The dual action $H^* \curvearrowright G^*/K$ of a (commutative) Hermann action $H \curvearrowright G/K$ is defined in a natural manner, where $G^*$ is the compact dual of $G$ with respect to $K$ and $H^*$ is the compact dual of $H$ with respect to $H \cap K$. The dual actions of Hermann actions in Table 1 are all of (commutative) Hermann actions of cohomogeneity one on irreducible symmetric space of compact type and rank greater than one.
6 The equivalenceness of the anisotropic equifocality and the anisotropic isoparametricness

We use the notations in Sections 1-5. Assume that $\partial M = \emptyset$. In the case where $\widetilde{M}$ is a Euclidean space, J. Ge and H. Ma ([GM]) showed that $f$ is of constant anisotropic principal curvatures (i.e., anisotropic equifocal) if and only if $f$ is anisotropic isoparametric. We obtain the following similar result in the case where $\widetilde{M}$ is a symmetric space of non-negative curvature.

Theorem 6.1. In the case where $\widetilde{M}$ is a symmetric space of non-negative curvature, the anisotropic equifocality is equivalent to the anisotropic isoparametricness.

Assume that $\widetilde{M}$ is a symmetric space $G/K$ of non-negative curvature, where $G$ and $K$ are as in the previous section. We shall prove this theorem by reducing to the investigation of the lift of $f(M)$ by a Riemannian submersion of a Hilbert space onto $G/K$. We suffice to show the statement of this theorem in the case where $G/K$ is simply connected. In the sequel, we assume that $G/K$ is simply connected. In this case, $G/K$ is decomposed irreducibly as $G/K = \left( \prod_{i=1}^{l} G_i/K_i \right) \times \mathbb{R}^r$ ($G_i/K_i$ : a simply connected irreducible symmetric space of compact type, $r$ : a non-negative integer). Let $H^0([0,1], \mathfrak{g}_i)$ be the (separable) Hilbert space of all $L^2$-integrable paths in the Lie algebra $\mathfrak{g}_i$ of $G_i$ (having $[0,1]$ as the domain) and $H^1([0,1], G_i)$ the Hilbert Lie group of all $H^1$-paths in $G_i$ (having $[0,1]$ as the domain), where we give $G_i$ the bi-invariant metric inducing the metric of $G_i/K_i$ and $\mathfrak{g}_i$ the Ad$(G_i)$-invariant inner product compatible with the bi-invariant metric. Let $\phi_i : H^0([0,1], \mathfrak{g}_i) \to G_i$ be the parallel transport map for $G_i$, that is, $\phi(u) := g_u(1)$ ($u \in H^0([0,1], \mathfrak{g}_i)$), where $g_u$ is the element of $H^1([0,1], G_i)$ with $g_u(0) = e_i$ and $(R_{g_u(t)})_{s}^{-1}(g_u'(t)) = u(t) (0 \leq t \leq 1)$, where $e_i$ is the identity element of $G_i$. See [TT], [Ch], [HLO] and [Koi1,3] about the investigation of submanifold geometry in a symmetric space of compact type by using the parallel transport map. The group $H^1([0,1], G_i)$ acts on $H^0([0,1], \mathfrak{g}_i)$ isometrically as the action of the Gauge transformations to connections as follows:

$$ (g * u)(t) := \text{Ad}(g(t))(u(t)) - (R_{g(t)})_{s}^{-1}(g'(t)) $$

$$(g \in H^1([0,1], G_i), u \in H^0([0,1], \mathfrak{g}_i), 0 \leq t \leq 1).$$

Also, let $\pi_i : G_i \to G_i/K_i$ be the natural projection. Set $\hat{\phi}_i := \pi_i \circ \phi_i$, which is a Riemannian submersion of $H^0([0,1], \mathfrak{g}_i)$ onto $G_i/K_i$. Set $P(G_i, e_i \times K_i) := \{ g \in H^1([0,1], G_i) \mid (g(0), g(1)) \in \{ e_i \} \times K_i \}$. This subgroup $P(G_i, e_i \times K_i)$ acts on $H^0([0,1], \mathfrak{g}_i)$ freely and $\hat{\phi}_i : H^0([0,1], \mathfrak{g}_i) \to G_i/K_i$ is regarded as a $P(G_i, e_i \times K_i)$-bundle. For simplicity, set $V := \left( \prod_{i=1}^{l} H^0([0,1], \mathfrak{g}_i) \right) \times \mathbb{R}^r$, $\tilde{G} := \prod_{i=1}^{l} P(G_i, e_i \times K_i)$ and $\tilde{\phi} := (\prod_{i=1}^{l} \hat{\phi}_i) \times \text{id}_{\mathbb{R}^r}$.

Note that $\tilde{\phi} : V \to G/K$ is regarded as a $\tilde{G}$-bundle. We consider $\bar{F} := \tilde{F} \circ \tilde{\phi} : TV \to \mathbb{R}$ as a parametric Lagrangian of $V$. Note that $\bar{F}$ is not holonomy invariant. Denote by $\widetilde{M}$ the induced bundle $f^\ast V := \{ (x, u) \in M \times V \mid f(x) = \tilde{\phi}(u) \}$ of this bundle $\tilde{\phi} : V \to G/K$ by $f$ and define an immersion $\tilde{f} : \widetilde{M} \hookrightarrow H^0([0,1], \mathfrak{g})$ by $\tilde{f}(x, u) := u ((x, u) \in \widetilde{M})$. Note that $\tilde{f}(\widetilde{M}) = \tilde{\phi}^{-1}(f(M))$. In 1989, Terng ([T]) introduced the notion of a proper Fredholm submanifold in the Hilbert space and, in 2006, Heintze-Liu-Olmos ([HLO]) introduced the
notion of a regularizable submanifold in the Hilbert space. According to Lemma 6.2 of [HLO], the hypersurface \( \tilde{f} : \tilde{M} \hookrightarrow V \) is a regularizable (proper Fredholm) hypersurface. The horizontal lift \( \xi^L \) of \( \xi \) is a unit normal vector field of \( \tilde{f} \). Denote by \( \hat{A} \) the shape operator of \( \tilde{f} \) for \( \xi^L \). Since \( \hat{A} : M \hookrightarrow V \) is a regularizable hypersurface, for any \( (x, u) \in \tilde{M} \), \( \hat{A}(x, u) \) is a compact self-adjoint operator, and the regularized trace \( \text{Tr}_r \hat{A}(x, u) \) of \( \hat{A}(x, u) \) and the trace \( \text{Tr} \hat{A}^2(x, u) \) exist. See [HLO] (or [Koi4]) about the definition of the regularized trace. The regularized mean curvature \( \hat{H} \) of \( \tilde{f} \) is defined by \( \hat{H}(x, u) := \text{Tr}_r \hat{A}(x, u) \) \((x, u) \in \tilde{M}\). Denote by \( \hat{\phi}_M \) the natural projection of \( \tilde{M} \) onto \( M \) (i.e., \( \hat{\phi}_M(x, u) = x \) \((x, u) \in \tilde{M}\)). Define a transversal vector field \( \xi_F \) of \( \tilde{f} \) by

\[
(6.1) \quad (\xi_F)(x, u) := (F \circ \nu \circ \hat{\phi}_M)(x, u)\xi^L_u + \tilde{f}_* (\text{grad} (F \circ \nu \circ \hat{\phi}_M)) \quad ((x, u) \in \tilde{M}).
\]

We call \( \xi_F \) an anisotropic transversal vector field of \( \tilde{f} \). Note that \( \tilde{f}^*(\xi^L_u) = (F \circ \nu \circ \hat{\phi}_M)(x, u) \).

Denote by \( \hat{\nabla} \) the Riemannian connection of \( V \) and \( \nabla \) the Riemannian connection of the metric of \( M \) induced by \( \tilde{f} \). For any \( X \in T_{(x, u)}\tilde{M} \), we can show

\[
\hat{\nabla}_X \xi_F = -(F \circ \nu \circ \hat{\phi}_M)(x, u)\tilde{f}_* (\hat{\Delta}(x, u) X) + \tilde{f}_* (\nabla_X \text{grad} (F \circ \nu \circ \hat{\phi}_M)) \quad (\text{mod } T^\perp \tilde{M}),
\]

where \( \hat{\nabla} \tilde{f} \) is the covariant derivative along \( \tilde{f} \) for \( \hat{\nabla} \). So, define a \((1,1)\)-tensor field \( \hat{A}^F \) on \( \tilde{M} \) by

\[
(6.2) \quad \hat{A}^F_{(x, u)} X = (F \circ \nu \circ \hat{\phi}_M)(x, u)\hat{A}(x, u) X - \hat{\nabla}_X \text{grad} (F \circ \nu \circ \hat{\phi}_M)
\]

for any \( (x, u) \in \tilde{M} \) and \( X \in T_{(x, u)}\tilde{M} \). We call \( \hat{A}^F \) an anisotropic shape operator of \( \tilde{f} \) and the eigenvalues of \( \hat{A}^F \) anisotropic principal curvatures of \( \tilde{f} \). Since \( F \circ \nu \circ \hat{\phi}_M \) is \( G \)-invariant, \( \hat{\nabla} \text{grad} (F \circ \nu \circ \hat{\phi}_M) \) is a compact self-adjoint regularizable operator. On the other hand, so is also \( \hat{A}(x, u) \). Hence so is also \( \hat{A}^F \). Define a function \( \hat{H}_F \) over \( \tilde{M} \) by \( \hat{H}_F := \text{Tr}_r \hat{A}^F \), where \( \text{Tr}_r \hat{A}^F \) is the regularized trace of \( \hat{A}^F \). We call \( \hat{H}_F \) an anisotropic regularized mean curvature of \( \tilde{f} \). Define a map \( \hat{f}_t : \tilde{M} \to V \) by

\[
\hat{f}_t(x, u) := \hat{f}(x, u) + t(\xi_F)(x, u) \quad ((x, u) \in \tilde{M}).
\]

Since \( \hat{f}(\tilde{M}) \) is \( G \)-invariant and \( \hat{\phi}(\hat{f}(\tilde{M})) \) is compact, it is shown that \( \hat{f}_t \) is an immersion for each \( t \) sufficiently close to zero. If \( \hat{f}_t \) is an immersion, then we call \( \hat{f}_t : \tilde{M} \hookrightarrow V \) an anisotropic parallel hypersurface of \( \hat{f} : \tilde{M} \hookrightarrow V \). If, for each \( t \) sufficiently close to zero, \( \hat{f}_t : \tilde{M} \hookrightarrow V \) is of constant anisotropic regularized mean curvature, we call \( \hat{f} : \tilde{M} \hookrightarrow V \) an anisotropic isoparametric hypersurface. Also, if the spectrum of \( \hat{A}^F_{(x, u)} \) is independent of the choice of \( (x, u) \in \tilde{M} \), then we call \( \hat{f} : \tilde{M} \hookrightarrow V \) a hypersurface with constant anisotropic principal curvatures. The following facts follow directly.

**Lemma 6.2.** (i) \( \xi_F \) is the horizontal lift of \( \xi \).

(ii) \( \hat{H}_F = \hat{H}_F \circ \hat{\phi}_M \) holds.

**Proof.** The statement (i) follows from (2.1) and (6.1) directly. Also, the statement (ii) follows from (2.3) and (6.2) directly. \( \text{q.e.d.} \)
By using this lemma, we shall prove Theorem 6.1.

**Proof of Theorem 6.1.** First we shall show that \( f \) is anisotropic equifocal if and only if \( \tilde{f} \) is of constant anisotropic principal curvatures. Denote by \( \gamma^F_{(x,u)} \) the geodesic in \( V \) whose initial velocity vector is equal to \( (\tilde{\xi}_F)_{(x,u)} \). Take \( X \in T_{(x,u)}M \). The anisotropic \( \tilde{M} \)-Jacobi field \( Y_X \) along \( \gamma^F_{(x,u)} \) with \( Y_X(0) = \tilde{f}_s \) and \( Y'_X(0) = -\tilde{f}_s(\tilde{A}^F_{(x,u)}X) \) is described as

\[
Y_X(s) = \tilde{f}_s X - s\tilde{f}_s(\tilde{A}^F_{(x,u)}(X)).
\]

From this description, anisotropic focal radii of \( \tilde{f} \) at \( (x, u) \) are equal to the inverse numbers of anisotropic principal curvatures. Also, since \( \tilde{\phi} \) is a Riemannian submersion and \( \tilde{\xi}_F \) is the horizontal lift of \( \xi_F \), the anisotropic focal radii of \( f \) at \( x \) are equal to those of \( \tilde{f} \) with considering their multiplicities. From these facts, it follows that \( f \) is anisotropic equifocal if and only if \( \tilde{f} \) has constant anisotropic principal curvatures.

Next we shall show that \( \tilde{f} \) has constant anisotropic principal curvatures if and only if \( \tilde{f} \) is anisotropic isoparametric. Take a positive number \( \epsilon \) such that \( \tilde{f}_t (\epsilon < t < \epsilon) \) are immersions. Denote by \( \tilde{\xi}_F \) and \( \tilde{A}^F \) the anisotropic transversal vector field and the anisotropic shape operator of the parallel hypersurface \( \tilde{f}_t : \tilde{M} \hookrightarrow V (\epsilon < t < \epsilon) \), respectively. Also, denote by \( \tilde{A}^F \) the anisotropic regularized mean curvature of \( \tilde{f}_t \). Easily we can show that \( \tilde{\xi}_F = \tilde{\xi}_F \) in \( V \). Assume that \( \tilde{A}^F_{(x,u)}(X) = \lambda X \). Let \( Y_X \) be the anisotropic \( \tilde{M} \)-Jacobi field along \( \gamma^F_{(x,u)} \) with \( Y_X(0) = \tilde{f}_s \) and \( Y'_X(0) = -\tilde{f}_s(\tilde{A}^F_{(x,u)}X) \). Since \( Y_X(t) = \tilde{f}_s X \) and \( Y'_X(t) = -\tilde{f}_s(\tilde{A}^F_{(x,u)}(X)) \), and since \( Y_X \) is described as in (6.3), we have

\[
(\tilde{f}_s)_s X = (1 - t\lambda)\tilde{f}_s X,
\]

and

\[
(\tilde{f}_s)_s((\tilde{A}^F_{(x,u)}(X)) = \lambda\tilde{f}_s X.
\]

From these relation, we obtain

\[
(\tilde{A}^F_{(x,u)}(X) = \frac{\lambda}{1 - t\lambda} X.
\]

Denote by \( \text{Spec}(\cdot) \) the spectrum of \( (\cdot) \). Set \( m_\lambda := \dim \ker (\tilde{A}^F_{(x,u)} - \lambda \text{id}) \) for each \( \lambda \in \text{Spec} \tilde{A}^F_{(x,u)} \). From (6.4), we have

\[
\text{Spec}(\tilde{A}^F_{(x,u)} = \left\{ \frac{\lambda}{1 - t\lambda} \middle| \lambda \in \text{Spec} \tilde{A}^F_{(x,u)} \right\}
\]

and \( \dim \ker (\tilde{A}^F_{(x,u)} - \frac{\lambda}{1 - t\lambda} \text{id}) = m_\lambda \). Hence we have

\[
(\tilde{H}^F_{(x,u)} = \sum_{\lambda \in \text{Spec} \tilde{A}^F_{(x,u)}} \frac{m_\lambda}{1 - t\lambda} ,
\]

where the right-hand side means the regularized series. Hence, by using Lemma 4.1 in [HLO], we can show that \( \tilde{f} \) has constant anisotropic principal curvatures if and only if \( \tilde{f} \) is anisotropic isoparametric.
Finally we shall show that $f$ is anisotropic isoparametric if and only if so is $\hat{f}$. Denote by $H_{F}^{t}$ the anisotropic mean curvature of $f_{t}$. According to (i) of Lemma 6.2, we have $\hat{\phi} \circ \hat{f}_{t} = f_{t} \circ \hat{\phi}_{\hat{M}'}$. Hence we have $\hat{H}_{F}^{t} = H_{F}^{t} \circ \hat{\phi}_{\hat{M}'}$ in similar to (ii) of Lemma 6.2. This implies that $f$ is anisotropic isoparametric if and only if so is $\hat{f}$. Therefore the statement of Theorem 6.1 follows.

References

[BCO] J. Berndt, S. Console and C. Olmos, Submanifolds and holonomy, Research Notes in Mathematics 434, CHAPMAN & HALL/CRC Press, Boca Raton, London, New York Washington, 2003.

[BT] J. Berndt and H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces with a totally geodesic singular orbit, Tohoku Math. J. 56 (2004) 163-177.

[Ch] U. Christ, Homogeneity of equifocal submanifolds, J. Differential Geom. 62 (2002) 1-15.

[Cl] U. Clarenz, Enclosure theorems for extremals of elliptic parametric functionals, Calc. Var. 15 (2002) 313-324.

[CW] S. Carter and A. West, Partial tubes about immersed manifolds, Geom. Dedicata 54 (1995) 145-169.

[F1] H. Federer, Geometric Measure Theory, Grundlehren math. Wiss. 153, Berlin, Heidelberg, New York:Springer, 1969.

[F2] H. Federer, Real flat chains, cochains and variational problems, Indiana Univ. Math. J. 24 (1974) 351-407.

[F3] H. Federer, Colloquium Lectures on Geometric Measure Theory, Bulletin of the Amer. Math. Soc. 84(3), May 1978.

[GM] J. Ge and H. Ma, Anisotropic isoparametric hypersurfaces in Euclidean spaces, Ann. Glob. Anal. Geom. 41 (2012) 347-355.

[HLO] E. Heintze, X. Liu and C. Olmos, Isoparametric submanifolds and a Chevalley type restriction theorem, Integrable systems, geometry, and topology, 151-190, AMS/IP Stud. Adv. Math. 36, Amer. Math. Soc., Providence, RI, 2006.

[H] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, 1978.

[Koi1] N. Koike, On proper Fredholm submanifolds in a Hilbert space arising from submanifolds in a symmetric space, Japan. J. Math. 28 (2002) 61–80.

[Koi2] N. Koike, Actions of Hermann type and proper complex equifocal submanifolds, Osaka J. Math. 42 (2005) 599-611.

[Koi3] N. Koike, Complex hyperpolar actions with a totally geodesic orbit, Osaka J. Math. 44 (2007) 491-503.

[Koi4] N. Koike, Collapse of the mean curvature flow for equifocal submanifolds, Asian J. Math. 15 (2011) 101-128.

[Kol] A. Kollross, A classification of hyperpolar and cohomogeneity one actions, Trans. Amer. Math. Soc. 354 (2002) 571-612.

[KP1] M. Koiso and B. Palmer, Geometry and stability of surfaces with constant anisotropic mean curvature, Indiana Univ. Math. J. 54 (2005) 1817-1852.

[KP2] M. Koiso and B. Palmer, Anisotropic capillary surfaces with wetting energy, Calc. Var. 29 (2007) 295-345.

[KP3] M. Koiso and B. Palmer, Anisotropic umbilic points and Hopf’s Theorem for surfaces with constant anisotropic mean curvature, Indiana Univ. Math. J. 59 (2010) 79-90.
[LM] J.H.S. de Lira and M. Melo, Hypersurfaces with constant anisotropic mean curvature in Riemannian manifolds, Calc. Var. 50 (2014) 335-364.

[Pala] R. S. Palais, Morse theory on Hilbert manifolds, Topology 2 (1963) 299–340.

[Palm] B. Palmer, Stability of the Wulff shape, Proc. Amer. Math. Soc. 126 (1998) 3661-3667.

[PT] R.S. Palais and C.L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. 1353, Springer, Berlin, 1988.

[T] C.L. Terng, Proper Fredholm submanifolds of Hilbert space, J. Differential Geometry 29 (1989) 9-47.

[TT] C.L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, J. Differential Geometry 42 (1995) 665-718.

[W] B. White, The space of m-dimensional surfaces that are stationary for a parametric elliptic integrand, Indiana Univ. Math. J. 36 (1987) 567-602.

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