Controlled Reflected SDEs and Neumann Problem for Backward SPDEs

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Abstract

We solve the optimal control problem of a one-dimensional reflected stochastic differential equation, whose coefficients can be path dependent. The value function of this problem is characterized by a backward stochastic partial differential equation (BSPDE) with Neumann boundary conditions. We prove the existence and uniqueness of sufficiently regular solution for this BSPDE, which is then used to construct the optimal feedback control. In fact we prove a more general result: The existence and uniqueness of strong solution for the Neumann problem for general nonlinear BSPDEs, which might be of interest even out of the current context.

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1 Introduction

Let $T \in (0, \infty)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ which satisfies the usual conditions. The filtration $\mathcal{F}$ is generated by two independent $m$-dimensional Brownian motions $W$ and $B$. We denote by $\{\mathcal{F}_t\}_{t \geq 0}$ the filtration generated by $W$, together with all $\mathbb{P}$ null sets. The predictable $\sigma$-algebra on $\Omega \times [0, +\infty)$ corresponding to $\{\mathcal{F}_t\}_{t \geq 0}$ and $\mathcal{F}$ is denoted $\mathcal{P}$, respectively.

In this paper, we consider the following stochastic optimal control problem:

$$\min_{\theta} \mathbb{E} \left[ \int_0^T f_t(X_t, \theta_t) \, dt + \int_0^T g_t(X_t) \, dL_t + \int_0^T g_t(X_t) \, dU_t + G(X_T) \right]$$

subject to

$$\begin{cases}
\frac{dX_t}{dt} = \beta_t(X_t, \theta_t) \frac{dt}{dt} + \sigma_t(X_t) \frac{dW_t}{dt} + \bar{\sigma}_t(X_t) \frac{dB_t}{dt} + dL_t - dU_t, \quad t \in [0, T]; \\
X_0 = x; \quad L_0 = U_0 = 0; \\
0 \leq X_t \leq b, \quad \text{a.s.;} \\
\int_0^T X_t \, dL_t = \int_0^T (b - X_s) \, dU_s = 0, \quad \text{a.s.,} \\
\end{cases}$$

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where $L$ and $U$ are two non-decreasing processes. The real-valued process $(X_t)_{t \in [0,T]}$ is the state process. Its drift is governed by the control $\theta$. We sometimes write $X^{s,x}_{t}$ for $0 \leq s \leq t \leq T$ to indicate the dependence of the state process on the control $\theta$, the initial time $s$ and initial state $x \in \mathbb{R}$. The set of admissible controls consists of all $\bar{\mathcal{F}}_{t}$-adapted processes $\theta$ such that the reflected stochastic differential equation (SDE) (1.2) admits a unique solution and $\theta_t \in \Theta$ a.s. for each $t \in [0,T]$ with set $\Theta \subset \mathbb{R}^n$.

Classical stochastic control problems, see e.g. [10] [11] [17], have been generalized more recently to handle the path dependent case [7] [24] [26]. We will in addition consider the problem of controlling reflected path dependent SDEs. The analysis of such control problems is motivated by the drift rate controlled queueing problem in [11], where the control problem is of ergodic/stationary type and is concerned with minimizing the long-run average cost under the Markovian framework. In contrast to that set-up, the coefficients in (1.1) and (1.2) are allowed to be random and thus can be non-Markovian; more precisely, we assume:

(A0) The coefficients $\beta, \sigma, \bar{\sigma}, f, g$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^n)$-measurable and $G$ is $\mathcal{F}_T \times \mathcal{B}(\mathbb{R})$-measurable.

We would also note that, as stated in [11], because the reflecting barriers are not discretionary and only the drift rate is controlled, the control problem does not fall in the spectrum of “singular” stochastic control. To the best of our knowledge, this is the first study on the controlled reflected SDEs with random coefficients.

The dynamic cost functional is assumed to be of the form:

$$J_t(X_t; \theta) = E^{\mathbb{P}_t} \left[ \int_t^T f_s(X_s, \theta_s) \, ds + \int_t^T g_s(X_s) \, dL_s + \int_t^T g_s(X_s) \, dU_s + G(X_T) \right].$$

The value function is given by:

$$u_t(x) \triangleq \text{ess inf}_{\theta} J_t(X_t; \theta) \big|_{X_t=x}.$$ (1.4)

In view of Peng’s seminal work [25] on non-Markovian stochastic optimal control, the dynamic programming principle suggests that the value function $u$ is the first component of the pair $(u, \psi)$ satisfying formally the following Neumann problem for backward stochastic partial differential equation (BSPDE):

$$
\begin{cases}
-du_t(x) = \left[ \frac{1}{2} (|\sigma|^2 + |\bar{\sigma}|^2) D^2 u_t(x) + \sigma D\psi_t(x) + \Phi_t(x, Du_t(x)) \right] dt - \psi_t(x) dW_t, \\
\quad (t, x) \in [0,T] \times [0,b]; \\
Du_t(0) = g_t(0), \quad Du_t(b) = g_t(b); \\
u_T(x) = G(x), \quad x \in [0,b],
\end{cases}$$

with

$$
\Phi_t(x, Du_t(x)) \triangleq \text{ess inf}_{\theta \in \Theta} \{ \beta_t(x, \theta) Du_t(x) + f_t(x, \theta) \}, \quad (t, x) \in [0,T] \times [0,b].$$

(1.5)

First, the self-contained proofs for the existence and uniqueness of strong solution are given for the Neumann problem of general nonlinear BSPDEs. Then the existence and uniqueness of strong solution to (1.5) follows immediately. However, to verify that the obtained solution is the value function and to derive the optimal feedback control for problem (1.1) - (1.2), we need to make sense of the composition of the solution of (1.5) and the controlled state process $X$, and this requires improved regularity of $u$. Inspired by the smoothing properties of the leading operators of BSPDEs (see [24]), we assume that $\bar{\sigma}$ satisfies the super-parabolicity condition:

(A1) There exists constant $\kappa$, s.t. $|\bar{\sigma}_t(x)|^2 \geq \kappa > 0$ a.s., $\forall (t, x) \in [0,T] \times \mathbb{R}$.\footnote{It is worth noting that, unlike Dirichlet problems for BSPDEs (see [29]) or Neumann problems for deterministic PDEs (see [18]), Itô formula for the square norm is not well-defined for the weak solutions of the Neumann problems for BSPDEs with a nontrivial coefficient $\sigma$ (Remark 6.1).}
Then, we take spatial derivatives on both sides of (1.5). The resulting Dirichlet problem admits a unique strong solution (see [6]), which yields additional regularity of $Du$. Finally, the generalized Itô-Kunita-Wentzell formula allows us to finish the verification.

The nonlinear BSPDE like (1.5) is called stochastic Hamilton-Jacobi-Bellman (HJB) equation, which was first introduced by Peng [25] for controlled SDEs without reflection. For the utility maximization with habit formation, a specific fully nonlinear stochastic HJB equation was formulated by Englezos and Karatzas [8] and the value function was verified to be its classical solution. The study of linear BSPDEs, on the other hand, dates back to about thirty years ago (see Benoussan [2] and Pardoux [23]). They arise in many applications of probability theory and stochastic processes, for instance in the nonlinear filtering and stochastic control theory for processes with incomplete information, as an adjoint equation of the Duncan-Mortensen-Zakai filtration equation (for instance, see [2, 13, 14, 35]). The representation relationship between forward-backward stochastic differential equations and BSPDEs yields the stochastic Feynman-Kac formula (see [13, 21, 30]). In addition, as the obstacle problems of BSPDEs, the reflected BSPDE arises as the HJB equation for the optimal stopping problems (see [3, 22, 31]).

The linear and semilinear BSPDEs have been extensively studied, we refer to [6, 13, 20, 21, 33] among many others. For the weak solutions and associated local behavior analysis for general quasi-linear BSPDEs, see [28, 29], and we refer to [12] for BSPDEs with singular terminal conditions. However, the existing literature is mainly about the BSPDEs in the whole space and Dirichlet problem, and not on the Neumann problem, though some partial results could be concluded from the semigroup method of BSPDEs [14, 34] for the cases when $\sigma \equiv 0$.

The remainder of this paper is organized as follows. In section 2, we summarize the main assumptions and results. The existence and uniqueness of strong solution for the Neumann problem of general nonlinear BSPDEs is established in Section 3, where we first give the a priori estimates of strong solutions for linear equations and then use the continuity method to prove the well-posedness for the general nonlinear cases. In Section 4, we complete the proof of the main theorem. Finally, the appendix recalls an Itô formula for the square norms of solutions of SPDEs and provides the proof for a generalized Itô-Kunita-Wentzell formula.

### 2 Preliminaries and Main Result

#### 2.1 Notation

In this paper, we use the following notation. $D$ and $D^2$ denote the first order and second order spatial partial derivative operators, respectively; the other partial derivatives are denoted by $\partial$. For a Banach space $V$ we denote by $S^p_{\mathcal{F}}([0,T];V)$, $p \in [1,\infty)$, the set of all the $V$-valued and $\mathcal{F}$-measurable càdlàg processes $(X_t)_{t\in[0,T]}$ such that

$$\|X\|_{S^p_{\mathcal{F}}([0,T];V)} = E \sup_{t\in[0,T]} \|X_t\|_V < \infty.$$  

By $L^p_{\mathcal{F}}(0,T;V)$ we denote the class of $V$-valued $\mathcal{F}$-measurable processes $(u_t)_{t\in[0,T]}$ such that

$$\|u\|_{L^p_{\mathcal{F}}(0,T;V)} = E \int_0^T \|u_t\|^p_V \, dt < \infty, \quad [1,\infty);$$

$$\|u\|_{L^\infty_{\mathcal{F}}(0,T;V)} = \text{ess sup}_{(\omega,t)\in\Omega \times [0,T]} \|u_t\|_V < \infty, \quad p = \infty.$$  

In a similarly way, we define $S^p_{\mathcal{F}}([0,T];V)$ and $L^p_{\mathcal{F}}(0,T;V)$. For the two spaces $S^p_{\mathcal{F}}([0,T];V)$ and $L^p_{\mathcal{F}}(0,T;V)$, we omit the subscript for simplicity, especially when there is no confusion on the filtration and adaptedness.
For $k \in \mathbb{N}^+$ and $p \in [1, \infty)$, $H^{k,p}([0,b])$ is the Sobolev space of all real-valued functions $\phi$ whose up-to $k$th order derivatives belong to $L^p([0,b])$, equipped with the usual Sobolev norm $\|\phi\|_{H^{k,p}([0,b])}$. By $H^{k,p}_0([0,b])$, we denote the space of all the trace-zero functions in $H^{k,p}([0,b])$. For $k=0$, $H^0([0,b]) \triangleq L^p([0,b])$. For simplicity, by $u = (u_1, \ldots, u_l) \in H^{k,p}([0,b])$, we mean $u_1, \ldots, u_l \in H^{k,p}([0,b])$ and $\|u\|_{H^{k,p}([0,b])} = \sum_{j=1}^l \|u_j\|_{H^{k,p}([0,b])}$. We use $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ to denote the norm and the inner product in the usual Hilbert space $L^2([0,b])$, and if there is no confusion, we shall also use $\langle \cdot, \cdot \rangle$ to denote the duality between Hilbert spaces $H^{k,2}([0,b])$ and their dual spaces.

Throughout this paper, we set for $k=1, 2$

\[
\mathcal{H} = \mathcal{S}_g^2(0,T; L^2([0,b])) \cap \mathcal{L}_g^2(0,T; H^{1,2}([0,b])) \times \mathcal{L}_g^2(0,T; L^2([0,b])), \\
\mathcal{H}^k = \mathcal{S}_g^2(0,T; H^{k,2}([0,b])) \cap \mathcal{L}_g^2(0,T; H^{k+1,2}([0,b])) \times \mathcal{L}_g^2(0,T; H^{k,2}([0,b])),
\]

and they are complete spaces equipped respectively with the norms

\[
\| (u, \psi) \|_{\mathcal{H}}^2 = \|u\|^2_{\mathcal{S}_g^2(0,T; L^2([0,b]))} + \|u\|^2_{\mathcal{S}_g^2(0,T; H^{1,2}([0,b]))} + \|\psi\|^2_{\mathcal{L}_g^2(0,T; L^2([0,b]))}, \quad \text{for} \ (u, \psi) \in \mathcal{H},
\]

\[
\| (u, \psi) \|_{\mathcal{H}^k}^2 = \|u\|^2_{\mathcal{S}_g^2(0,T; H^{k,2}([0,b]))} + \|u\|^2_{\mathcal{S}_g^2(0,T; H^{k+1,2}([0,b]))} + \|\psi\|^2_{\mathcal{L}_g^2(0,T; H^{k,2}([0,b]))}, \quad \text{for} \ (u, \psi) \in \mathcal{H}^k.
\]

### 2.2 Assumptions and main result

We now introduce the notion of solutions to a BSPDE.

**Definition 2.1.** Let $G \in L^2(\Omega, \mathcal{F}_T; L^2([0,b]))$. A pair of processes $(u, \psi)$ is a weak solution to the BSPDE

\[
\begin{cases}
- \partial_t u(t) = F(t, y, u, Du, D^2 u, \psi, D\psi) dt - \psi_t(y) dW_t, \quad (t, y) \in [0, T] \times [0,b]; \\
\partial_t u(0) = g_U(0), \\
Du(0) = g_t(0), \\
\partial_T u(y) = G(y), \quad y \in [0,b],
\end{cases}
\]

if $(u, \psi) \in \mathcal{H}$ and $(u, \psi)$ satisfies BSPDE (2.1) in the weak sense, i.e., for any $\varphi \in C^\infty_c((0,b))$,

\[
\langle \varphi, F(\cdot, \cdot, u, Du, D^2 u, \psi, D\psi) \rangle \in L^2(0,T; \mathbb{R})
\]

and

\[
\langle \varphi, u_t \rangle = \langle \varphi, G \rangle + \int_t^T \langle \varphi, F(s, y, u, Du, D^2 u, \psi, D\psi) \rangle ds - \int_t^T \langle \varphi, \psi_s dW_s \rangle \text{ a.s., } \forall 0 \leq t \leq T.
\]

The above $(u, \psi)$ is called a strong solution if we have improved regularity $(u, \psi) \in \mathcal{H}^1$.

For the well-posedness of BSPDE (1.5), we need further the following regularity conditions on the random coefficients.

(A2) The functions $\sigma$, $\bar{\sigma}$ and their spatial partial derivatives $D\sigma$, $D\bar{\sigma}$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable and essentially bounded by a positive constant $K > 0$. And the functions $\beta$, $f$ and the spatial partial derivative $D\beta$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^n)$-measurable with $\beta(\cdot, 0) \in L^2(0,T; \mathbb{R})$ and $|D\beta_t(x, \theta)| \leq \Lambda$ a.s. for any $(t, x, \theta) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$.

(A*) (i).

\[
G \in L^2(\Omega, \mathcal{F}_T; H^{2,2}([0,b])), \quad DG - g_T \in L^2(\Omega, \mathcal{F}_T; H^{1,2}_0([0,b])),
\]

and together with another function $\mathcal{G}$, $(g, \mathcal{G}) \in \mathcal{H}$ satisfies BSPDE $-dg_t = \mathcal{G}_t dt - \mathcal{G}_t dW_t$ in the weak sense (see Definition 2.1) with $\mathcal{G} \in L^2(0,T; L^2([0,b]))$. 

4
For any $u \in H^{2,2}([0, b])$, $\Phi(\cdot, Du), (D\Phi)(\cdot, Du) \in L^2(0, T; L^2([0, b]))$, and there exists a nonnegative constant $N$ such that for any $v_1, v_2 \in \mathbb{R}$, there holds almost surely

$$|\Phi_t(x, v_1) - \Phi_t(x, v_2)| \leq N|v_1 - v_2|, \quad \text{for all } (t, x) \in [0, T] \times [0, b].$$

(iii). There exists a $\mathcal{F} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable $\Theta$-valued function $\Pi$ such that $\Phi_t(x, y) = \beta_t(x, \Pi_t(x, y)) + f_t(x, \Pi_t(x, y))$ and for any $v \in S^2(0, T; H^{1,2}([0, b])) \cap L^2(0, T; H^{2,2}([0, b]))$, the reflected SDE (1.2) associated with drift coefficient $\beta_t(X_t, \Pi_t(X_t, v_t))$ has a unique solution.

Now, we state the main theorem, whose proof requires some preparations and will be carried out subsequently.

**Theorem 2.1.** Let assumptions (A0)-(A2) and (A*) hold. BSPDE (1.5) admits a unique strong solution $(u, \psi)$. For this strong solution, we further $(u, \psi) \in H^2$. Moreover, $u$ turns out to be the value function of the stochastic control problem (1.1), and the optimal control $\theta^*$ and state process $X^*$ are given by $\theta^* = \Pi_t(X^*_t, Du_t(X^*_t))$ and

$$
\begin{align*}
&dX^*_t = \beta_t(X^*_t, \Pi_t(X^*_t, Du_t(X^*_t))) dt + \sigma_t(X^*_t) dW_t + \bar{\sigma}_t(X^*_t) dB_t + dL_t - dU_t, \quad t \in [0, T]; \\
&X^*_0 = x; \quad L_0 = U_0 = 0; \\
&0 \leq X^*_t \leq b, \quad a.s.; \\
&\int_0^T X^*_t dL_t = \int_0^T (b - X^*_t) dU_s = 0, \quad a.s.
\end{align*}
$$

(2.3)

Assumptions (A0), (A1) and (A2) are standard to guarantee the adaptedness and super-parabolicity of BSPDE (1.5) and the well-posedness of the controlled reflected SDEs (see [19] Theorem 3.1 and Remark 3.3).

In assumption (i) of (A*), to have $(u, \psi) \in H^2$, the requirements on $G$ is standard (see $L^p$-theory of BSPDE of [3]); in view of the Skorohod conditions of RSDE (1.2), one has

$$
\int_0^T g_s(X_s) dL_s = \int_0^T g_s(0) dL_s, \quad \text{and} \quad \int_0^T g_s(X_s) dU_s = \int_0^T g_s(b) dU_s,
$$

so only the traces $g_s(0)$ and $g_s(b)$ of $g$ are involved in the control problem. In fact, assumption (i) of (A*) allows $g_s(0)$ and $g_s(b)$ to be any processes that, together with another two processes $(\zeta^0, \zeta^b)$, satisfy BSDEs of the following form:

$$
\begin{align*}
g_t(0) &= DG(0) + \int_t^T g_s^0 ds - \int_t^T \zeta^0_s dW_s; \\
g_t(b) &= DG(b) + \int_t^T g_s^b ds - \int_t^T \zeta^b_s dW_s,
\end{align*}
$$

with $g^0, g^b \in L^2(0, T; \mathbb{R})$, and we can construct (not uniquely) the time-space random function $g_t(x)$ in different ways. For instance, starting with $(g_t(0), g_t(b))$, one can construct linearly

$$
g_t(x) = g_t(0) + \frac{(g_t(b) - g_t(0)) x}{b}, \quad (t, x) \in [0, T] \times [0, b],
$$

which then satisfies assumption (i) of (A*) with

$$
\mathcal{G}_t(x) = \tilde{g}^0_t + \frac{\tilde{g}^b_t - \tilde{g}^0_t}{b} x \quad \text{and} \quad \mathcal{G}_t(x) = \zeta^0_t + \frac{\zeta^b_t - \zeta^0_t}{b} x, \quad (t, x) \in [0, T] \times [0, b].
$$

In this paper, we adopt assumption (i) of (A*) for the convenience of discussions.
By (ii) of \( (A^*) \), we assume the Lipchitz continuity of Hamiltonian function \( \Phi_t(x, v) \) with respect to \( v \), which implies \( \partial_v \Phi_t(\cdot, v) \in L^\infty(\Omega \times [0, T] \times [0, b]) \) for any \( v \in \mathbb{R}^d \). This excludes the control problems of linear-quadratic type. The quadratic case definitely needs more efforts, for which we need to deal with not only the quadratic growth but also the improved regularity in Theorem 2.1 so we would postpone the discussions on quadratic cases to a future work.

In (iii) of \( (A^*) \), \( \Pi \) is the minimizer function of \( \Phi_t(x, v) \) (see (1.6)) and for each \( u \in S^2(0, T; H^{2,2}([0, b])) \cap L^2(0, T; H^{3,2}([0, b])) \), the composite function \( \beta_t(\pi_t(x, Du_t(x))) \) is not necessarily Lipchitz continuous with respect to \( x \). Let us consider the following specific example.

**Example 2.2.** Let \( d = n = 1 \), \( \Theta = [-1, 0] \) and \( \beta_t(x, \theta) = \theta \), while \( f_t(x, \theta) = \mu|\theta| + h_t(x) \) with \( \mu \in \mathbb{R}^+ \) and \( h \in L^2(H^1([0, b])) \). \( \sigma_t(x) \) and \( \bar{\sigma}_t(x) \) are general random functions so that assumptions \( (A0) - (A2) \) are satisfied. Assume \( g_t(x) \equiv \frac{\mu^2}{\sigma^2} \) as in \( \Pi \). Let \( G(x) = \frac{\mu^2}{\sigma^2} \). Then

\[
\Phi_t(x, Du_t) = \text{ess inf}_{-1 \leq \theta \leq 0} \{ \theta Du_t(x) + |\theta| + h_t(x) \} = -(Du_t(x) - \mu)^+ + h_t(x)
\]

and

\[
\Pi_t(x, Du_t(x)) = -1_{\{Du_t(x) > \mu\}}.
\]

It is easy to check that \( (A0) - (A2) \) and (i) and (ii) of \( (A^*) \) hold.Obviously, the drift \( \beta = \Pi \), as a step function, is not necessarily Lipchitz continuous with respect to \( x \) for each \( u \in S^2(0, T; H^{2,2}([0, b])) \cap L^2(0, T; H^{3,2}([0, b])) \). In our case, given \( u \in S^2(0, T; H^{2,2}([0, b])) \cap L^2(0, T; H^{3,2}([0, b])) \), the resulting reflected SDE

\[
\begin{aligned}
dX_t &= -1_{\{Du_t(X_t) > \mu\}} dt + \sigma_t(X_t) dW_t + \bar{\sigma}_t(X_t) dB_t + dL_t - dU_t, \quad t \in [0, T]; \\
X_0 &= x; \quad L_0 = U_0 = 0; \\
0 &\leq X_t \leq b, \quad a.s.; \\
\int_0^T X_t dL_t &= \int_0^T (b - X_t) dU_s = 0, \quad a.s.,
\end{aligned}
\tag{2.4}
\]

admits a unique solution. Indeed, let \( X^0 \) be the unique solution of reflected SDE

\[
\begin{aligned}
dX^0_t &= \sigma_t(X^0_t) dW_t + \bar{\sigma}_t(X^0_t) dB_t + dL_t - dU_t, \quad t \in [0, T]; \\
X^0_0 &= x; \quad L_0 = U_0 = 0; \\
0 &\leq X^0_t \leq b, \quad a.s.; \\
\int_0^T X^0_t dL_t &= \int_0^T (b - X^0_t) dU_s = 0, \quad a.s.,
\end{aligned}
\tag{2.5}
\]

under the equivalent probability measure \( \mathbb{Q} \) with

\[
\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp\left( \int_0^T 1_{\{Du_t(X^0_t) > \mu\}} |\bar{\sigma}_t(X^0_t)|^{-1} d\mathbb{B}_s - \frac{1}{2} \int_0^T 1_{\{Du_t(X^0_t) > \mu\}} |\bar{\sigma}_t(X^0_t)|^{-2} ds \right),
\]

and Girsanov’s theorem indicates that \( X^0 \) is coinciding with the unique solution \( X \) of reflected SDE (under the original probability measure \( \mathbb{P} \)). Hence, the assumption (iii) of \( (A^*) \) is satisfied and the main theorem 2.1 applies.

### 3 Existence and uniqueness of the strong solution

In this section, we shall establish the existence and uniqueness of strong solution for the Neumann problem for general nonlinear BSPDEs, which might be of interest even out of the current context. For simplicity, we consider the 1-dimensional case, though there would be no essential difficulty for multi-dimensional extensions.
3.1 The a priori estimates

Consider the following Neumann problem:

\[
\begin{cases}
-d_{t}u_{t}(x) = \left[ \frac{1}{2} \left( |\sigma_{t}|^{2} + |\bar{\sigma}_{t}|^{2} \right) D^{2}u_{t}(x) + \sigma_{t}D\psi_{t}(x) + h_{t}(x) \right] dt \\
-\psi_{t}(x) dW_{t}, \quad (t, x) \in [0, T] \times [0, b]; \\
D_{b}u_{t}(0) = 0, \quad D_{b}u_{t}(b) = 0; \\
u_{T}(x) = G(x), \quad x \in [0, b].
\end{cases}
\]  

(3.1)

Proposition 3.1. Let (A2) hold and \( h \in L^{2}(0, T; L^{2}([0, b])) \) and \( G \in L^{2}(\Omega, \mathcal{F}_{T}; H^{1,2}([0, b])) \). Suppose \( (u, \psi) \) is a strong solution of Neumann problem (3.1). Then the strong solution is unique and it satisfies

\[
\|u\|_{H^{1}}^{2} \leq C \left\{ \|G\|_{L^{2}(\Omega, \mathcal{F}_{T}; L^{2}([0, b]))}^{2} + E \left[ \int_{0}^{T} \left( \langle h_{s}, u_{s} \rangle \|u_{s}\|^{2} + \|D\psi_{s}\|^{2} + \varepsilon \|\psi_{s}\|^{2} \right) \right] ds \right\}, \quad \forall \varepsilon > 0,
\]

and

\[
\|u\|_{H^{1}}^{2} \leq C \left\{ \|G\|_{L^{2}(\Omega, \mathcal{F}_{T}; H^{1,2}([0, b]))}^{2} + E \left[ \int_{0}^{T} \left( \langle h_{s}, u_{s} \rangle \|u_{s}\|^{2} + \|D\psi_{s}\|^{2} + \varepsilon \|\psi_{s}\|^{2} \right) \right] ds \right\}
\]

with the constants \( C \)'s depending on \( \kappa \), \( K \) and \( T \).

Proof. Step 1. Applying Itô formula (see Lemma A.1) to the square norm yields

\[
\|u_{t}\|^{2} + \int_{t}^{T} \|\psi_{s}\|^{2} ds - \|G\|^{2}
\]

\[
= \int_{t}^{T} \langle u_{s}, (|\sigma_{s}|^{2} + |\bar{\sigma}_{s}|^{2})D^{2}u_{s} + 2\sigma_{s}D\psi_{s} + 2h_{s} \rangle ds - 2 \int_{t}^{T} \langle u_{s}, \psi_{s} dW_{s} \rangle, \quad \text{a.s.} \forall t \in [0, T].
\]

In view of the Neumann boundary condition, we have

\[
\int_{t}^{T} \langle u_{s}, (|\sigma_{s}|^{2} + |\bar{\sigma}_{s}|^{2})D^{2}u_{s} \rangle ds = -\int_{t}^{T} \langle D_{b}u_{s}, (|\sigma_{s}|^{2} + |\bar{\sigma}_{s}|^{2})Du_{s} \rangle ds - \int_{t}^{T} \langle u_{s}D(|\sigma_{s}|^{2} + |\bar{\sigma}_{s}|^{2)}, Du_{s} \rangle ds
\]

\[
\leq -\kappa \int_{t}^{T} \|Du_{s}\|^{2} ds + \frac{C}{\varepsilon_{1}} \int_{t}^{T} \|u_{s}\|^{2} ds + \varepsilon_{1} \int_{t}^{T} \|D_{b}u_{s}\|^{2} ds
\]

\[
\leq -\kappa \int_{t}^{T} \|Du_{s}\|^{2} ds + \frac{C}{\varepsilon_{1}} \int_{t}^{T} \|u_{s}\|^{2} ds + \varepsilon_{1} \int_{t}^{T} \|Du_{s}\|^{2} ds, \quad \varepsilon_{1} > 0.
\]

Using Schwartz inequality, we further have

\[
\int_{t}^{T} \langle u_{s}, 2\sigma_{s}D\psi_{s} \rangle ds \leq \frac{K}{\varepsilon_{2}} \int_{t}^{T} \|u_{s}\|^{2} ds + \varepsilon_{2} \int_{t}^{T} \|D\psi_{s}\|^{2} ds, \quad \varepsilon_{2} > 0.
\]  

(3.2)

In addition, we have

\[
2E \left[ \sup_{\tau \in [t, T]} \left| \int_{\tau}^{T} \langle u_{s}, \psi_{s} dW_{s} \rangle \right| \right] \leq 4E \left[ \sup_{\tau \in [t, T]} \left| \int_{\tau}^{T} \langle u_{s}, \psi_{s} dW_{s} \rangle \right| \right]
\]

(by BDG inequality) \( \leq CE \left[ \left( \int_{t}^{T} \|u_{s}\|^{2} \|\psi_{s}\|^{2} ds \right)^{1/2} \right] \).
Applying again Itô formula (Lemma A.1) to the square norm yields
\[ \lambda E \left[ \sup_{s \in [t,T]} \|u_s\|^2 \right] + (1 - \lambda) E \left[ \|u_t\|^2 \right] + E \int_t^T \|\psi_s\|^2 \, ds \]
\[ \leq C E \left[ \|G\|^2 + \int_t^T \left( 1 + \frac{1}{\varepsilon_2} \right) \|u_s\|^2 + \|\langle h_s, u_s \rangle\| + \varepsilon_2 \|D\psi_s\|^2 \, ds + \lambda \left( \int_t^T \|u_s\|^2 \|\psi_s\|^2 \, ds \right)^{1/2} \right] \]
\[ \leq C E \left[ \|G\|^2 + \int_t^T \left( 1 + \frac{1}{\varepsilon_2} \right) \|u_s\|^2 + \|\langle h_s, u_s \rangle\| + \varepsilon_2 \|D\psi_s\|^2 \, ds + \lambda \int_t^T \|\psi_s\|^2 \, ds \right] \]
\[ + \frac{\lambda}{2} E \left[ \sup_{s \in [t,T]} \|u_s\|^2 \right] \]

with \( \lambda \in \{0, 1\} \). Applying Gronwall inequality successively for the cases \( \lambda = 0 \) and \( \lambda = 1 \), we obtain
\[ E \left[ \sup_{s \in [t,T]} \|u_s\|^2 \right] + E \int_t^T \|\psi_s\|^2 \, ds \]
\[ \leq C E \left[ \|G\|^2 + \int_t^T \|\langle h_s, u_s \rangle\| \, ds + \frac{1}{\varepsilon_2} \int_t^T \|u_s\|^2 \, ds + \varepsilon_2 \int_t^T \|D\psi_s\|^2 \, ds \right]. \quad (3.3) \]

Step 2. Taking the spatial derivatives on both sides of BSPDE (3.1), one can easily check that \((v, \Psi) \equiv (Du, D\psi)\) is a weak solution of the following Dirichlet problem:
\[ \begin{cases} 
-\partial_t v_t(x) &= \left[ \frac{1}{2} \left( |\sigma_t|^2 + |\bar{\sigma}_t|^2 \right) D^2 v_t(x) + \frac{1}{2} D \left( |\sigma_t|^2 + |\bar{\sigma}_t|^2 \right) D\psi_t(x) + \sigma_t D\Psi_t(x) + D\sigma_t \Psi_t(x) \right] dt - \Psi_t(x) \, dW_t, \\
&+ Dh_t(x), \quad (t, x) \in [0, T] \times [0, b]; \\
v_t(0) &= 0, \quad v_t(b) = 0; \\
v_T(x) &= DG(x), \quad x \in [0, b].
\end{cases} \quad (3.4) \]

Applying again Itô formula (Lemma A.1) to the square norm yields
\[ \|v_t\|^2 + \int_t^T \|\Psi_s\|^2 \, ds - \|DG\|^2 \]
\[ = \int_t^T \langle v_s, (|\sigma_s|^2 + |\bar{\sigma}_s|^2) D^2 v_s + 2\sigma_s D\Psi_s + 2Dh_s + D (|\sigma_s|^2 + |\bar{\sigma}_s|^2) Dv_s + 2D\sigma_s \Psi_s \rangle \, ds \]
\[ - 2 \int_t^T \langle v_s, \Psi_s \, dW_s \rangle, \quad \text{a.s.} \forall t \in [0, T]. \]

In view of the zero-Dirichlet condition, one has
\[ \int_t^T \langle v_s, (|\sigma_s|^2 + |\bar{\sigma}_s|^2) D^2 v_s + 2\sigma_s D\Psi_s + D (|\sigma_s|^2 + |\bar{\sigma}_s|^2) Dv_s + 2D\sigma_s \Psi_s \rangle \, ds \]
\[ = - \int_t^T \langle Dv_s, (|\sigma_s|^2 + |\bar{\sigma}_s|^2) Dv_s + 2\sigma_s \Psi_s \rangle \, ds \]
\[ \leq - \int_t^T \left( |\bar{\sigma}_s Dv_s|^2 - \varepsilon_5 ||\sigma_s Dv_s||^2 \right) \, ds + \frac{1}{1 + \varepsilon_3} \int_t^T \|\Psi_s\|^2 \, ds \]
\[ \leq - (\kappa - \varepsilon_5 K) \int_t^T \|\sigma_s Dv_s\|^2 \, ds + \frac{1}{1 + \varepsilon_3} \int_t^T \|\Psi_s\|^2 \, ds, \quad \varepsilon_5 > 0, \]
and
\[ \int_t^T \langle v_s, Dh_s \rangle \, ds = - \int_t^T \langle Dv_s, h_s \rangle \, ds. \]

Taking \( \varepsilon_3 = \frac{\kappa}{2K} \) and in a similar way to Step 1, we get
\[ E \left[ \sup_{s \in [t,T]} \| u_s \|_{L^2(\Omega)}^2 \right] + E \int_t^T \left( \| Dv_s \|_2^2 + \| \Psi_s \|_2^2 \right) \, ds \leq C E \left[ \| Dv \|_2^2 + \int_t^T \langle \langle h_s, Dv_s \rangle \rangle \right], \]
i.e.,
\[ E \left[ \sup_{s \in [t,T]} \| Dv_s \|_2^2 \right] + E \int_t^T \left( \| D^2 u_s \|_2^2 + \| D\psi_s \|_2^2 \right) \, ds \leq C E \left[ \| Dv \|_2^2 + \int_t^T \langle \langle h_s, D^2 u_s \rangle \rangle \right], \]
which, together with (3.3), implies
\[ E \left[ \sup_{s \in [t,T]} \| u_s \|_{H^1_2([0,b])}^2 \right] + E \int_t^T \left( \| u_s \|_{H^1_2([0,b])}^2 + \| \psi_s \|_{H^1_2([0,b])}^2 \right) \, ds \leq C \left[ \| G \|_{H^1_2([0,b])}^2 + \int_t^T \langle \langle h_s, u_s \rangle \rangle + \langle \langle h_s, Dv_s \rangle \rangle \right] + \frac{1}{\varepsilon_2} \int_t^T \| u_s \|_2^2 \, ds + \varepsilon_2 \int_t^T \| D\psi_s \|_2^2 \, ds, \]
with \( C \) depending only on \( \kappa, K \) and \( T \).

Noticing that
\[ \int_t^T \langle \langle h_s, u_s \rangle \rangle + \langle \langle h_s, D^2 u_s \rangle \rangle \rangle \, ds \leq \int_t^T \left( \frac{2}{\varepsilon_2} \| h_s \|_2^2 + 2\varepsilon_2 \| u_s \|_{H^1_2([0,b])}^2 \right) \, ds \]
and letting \( \varepsilon_2 \) be small enough, one obtains for any \( t \in [0,T] \),
\[ E \left[ \sup_{s \in [t,T]} \| u_s \|_{H^1_2([0,b])}^2 \right] + E \int_t^T \left( \| u_s \|_{H^1_2([0,b])}^2 + \| \psi_s \|_{H^1_2([0,b])}^2 \right) \, ds \leq C \left[ \| G \|_{H^1_2([0,b])}^2 + \int_t^T \langle \langle h_s, u_s \rangle \rangle + \langle \langle h_s, D^2 u_s \rangle \rangle \rangle \, ds \right] \]
\[ \leq C \left[ \| G \|_{H^1_2([0,b])}^2 + \int_t^T \| h_s \|_2^2 \, ds \right] \]  
(3.5)
with the constants \( C \)s depending on \( \kappa, K \) and \( T \). The uniqueness follows as an immediate consequence of the estimates. The proof is complete.

When \( \sigma \equiv 0 \), in view of estimates (3.2) and (3.3), we have

**Corollary 3.2.** Let (A2) hold with \( \sigma \equiv 0 \), and \( h \in L^2(0,T;L^2([0,b])) \), \( G \in L^2(\Omega, \mathcal{F}_T;L^2([0,b])) \). Suppose \( (u, \psi) \) is a weak solution of Neumann problem (3.1). Then the weak solution is unique and it holds that
\[ \| u \|_{L^2(0,T;L^2([0,b]))}^2 \leq C \left\{ \| G \|_{L^2(\Omega, \mathcal{F}_T;L^2([0,b]))}^2 + E \left[ \int_0^T \langle \langle h_t, u_t \rangle \rangle \right] \right\} \]
\[ \leq C \left\{ \| G \|_{L^2(\Omega, \mathcal{F}_T;L^2([0,b]))}^2 + \| h \|_{L^2(0,T;L^2([0,b]))}^2 \right\}, \]
with the constants \( C \)s depending on \( \kappa, K \) and \( T \).

**Remark 3.1.** When \( \sigma \) is not vanishing, for a weak solution \( (u, \psi) \), the estimate (3.2) makes no sense. In fact, the term \( \int_t^T \langle u_s, 2\sigma, D\psi_s \rangle \, ds \) is not well-defined. Even when we apply the integration-by-parts formula, the function \( \psi \) has no intrinsic meaning on the boundary, nor does the term \( u\sigma \psi \), because they are just restrictions to the boundary of \( L^2([0,b]) \) functions. Thus, for the Neumann problems like (1.1) and (3.1), Ito formula for the square norm is not applicable to the weak solutions when \( \sigma \) is not vanishing.
3.2 Existence and uniqueness of the strong solution

First, we consider the following Neumann problem with Laplacian operator:

\[
\begin{aligned}
    &-d_u(x) = [D^2u_t(x) + h_t(x)] dt - \psi_t(x) dW_t, \quad (t, x) \in [0, T] \times [0, b]; \\
    &D_u(0) = 0, \quad D_u(b) = 0; \\
    &u_T(x) = G(x), \quad x \in [0, b]. \\
\end{aligned}
\]  (3.6)

Proposition 3.3. Let \( h \in L^2(0, T; L^2) \) and \( G \in L^2(\Omega, \mathcal{F}_T; H^{1,2}([0, b])) \). BSPDE (3.6) admits a unique strong solution \((u, \psi)\).

Proof. The uniqueness of strong solution follows directly from Proposition 3.1. We need only to prove the existence. \textbf{Step 1.} Suppose further \( h \in L^2(0, T; H^{1,2}([0, b])) \) and \( DG \in L^2(\Omega, \mathcal{F}_T; H^{1,2}_0([0, b])) \). By the theory on the Neumann problem of deterministic parabolic PDEs (see [18] Theorem 7.20), there exists a unique strong solution \( \hat{u} \) to PDE:

\[
\begin{aligned}
    &-\partial_t \hat{u}_t(x) = D^2 \hat{u}_t(x) + h_t(x), \quad (t, x) \in [0, T] \times [0, b]; \\
    &\hat{u}_t(0) = 0, \quad \hat{u}_t(b) = 0; \\
    &\hat{u}_T(x) = G(x), \quad x \in [0, b], \\
\end{aligned}
\]  (3.7)

such that \( \hat{u}, D\hat{u}, D^2 \hat{u}, \partial_t \hat{u} \in L^2(\Omega, \mathcal{F}_T; L^2([0, T] \times [0, b])) \). Taking conditional expectations in Hilbert spaces (see [11]), set

\[
u_t = E[\hat{u}_t | \mathcal{F}_t], \quad \text{a.s., for each } t \in [0, T],
\]

which admits a version in \( S^2(0, T; L^2([0, T])) \cap L^2(0, T; H^{2,2}([0, b])) \) that together with \( \psi \in L^2(L^2()) \) satisfies \( L^2([0, b]) \)-valued BSDE:

\[
\begin{aligned}
    &-d_u(x) = [D^2u_t(x) + h_t(x)] dt - \psi_t(x) dW_t; \\
    &u_T(x) = G(x), \quad x \in [0, b]. \\
\end{aligned}
\]  (3.8)

In view of the definition of \( u \), \( u \) satisfies the zero-Neumann boundary condition. By Definition 2.4 and Corollary 3.2, it is easy to check that \((u, \psi)\) is the weak solution to BSPDE (3.6).

\textbf{Step 2.} We now prove that the constructed weak solution \((u, \psi)\) is in fact the unique strong solution of BSPDE (3.6). In a similar way to \textbf{Step 1}, it is easy to check that \( D\hat{u} \) would be the strong solution of Dirichlet problem:

\[
\begin{aligned}
    &-\partial_t \hat{v}_t(x) = D^2 \hat{v}_t(x) + Dh_t(x), \quad (t, x) \in [0, T] \times [0, b]; \\
    &\hat{v}_t(0) = 0, \quad \hat{v}_t(b) = 0; \\
    &\hat{v}_T(x) = DG(x), \quad x \in [0, b], \\
\end{aligned}
\]  (3.9)

and \((Du, D\psi)\) satisfies \( L^2([0, b]) \)-valued BSDE (3.8) associated to the coefficients \((Dh, DG)\). In particular, we have

\[
(Du, D\psi) \in S^2(0, T; L^2([0, b])) \times L^2(0, T; L^2([0, b]))
\]

and thus \((u, \psi)\) is the strong solution to BSPDE (3.6). For general \( h \in L^2(0, T; L^2([0, b])) \) and \( G \in L^2(\Omega, \mathcal{F}_T; H^{1,2}([0, b])) \), we use the the standard method of approximations, which together with the estimates in Proposition 3.1 yields the existence of strong solution. We complete the proof. \[\Box\]

We are now ready to study the general nonlinear cases. Consider the following Neumann problem:

\[
\begin{aligned}
    &-d_u(x) = \left[ \frac{1}{2} (|\sigma|^2 + |\bar{\sigma}|^2) D^2 u_t(x) + \sigma_t D\psi_t(x) + \Gamma_t(x, u, Du, D^2 u, \psi, D\psi) \right] dt \\
    &- \psi_t(x) dW_t, \quad (t, x) \in [0, T] \times [0, b]; \\
    &D_u(0) = 0, \quad D_u(b) = 0; \\
    &u_T(x) = G(x), \quad x \in [0, b]. \\
\end{aligned}
\]  (3.10)
For any \((u, \psi) \in H^{2,2}([0, b]) \times H^{1,2}([0, b]), \) \(\Gamma(t, u, Du, D^2u, \psi, D\psi) \in L^2(0, T; L^2([0, b])),\) and there exist **nonnegative** constants \(\mu\) and \(L\) such that for any \((u_i, \psi_i) \in H^{2,2}([0, b]) \times H^{1,2}([0, b]), i = 1, 2,\) there holds
\[
\|\Gamma(t, u_1, Du_1, D^2u_1, \psi_1, D\psi_1) - \Gamma(t, u_2, Du_2, D^2u_2, \psi_2, D\psi_2)\| \\
\leq \mu \left(\|D^2(u_1 - u_2)\| + \|D(\psi_1 - \psi_2)\|\right) + L \left(\|u_1 - u_2\|_{H^{1,2}([0, b])} + \|\psi_1 - \psi_2\|_{L^2([0, b])}\right),
\]
for any \(t \in [0, T].\)

**Remark 3.2.** Assumption (A3) holds for the following semi-linear functional:
\[
\Gamma(t, x, u, Du, D^2u, \psi, D\psi) = \alpha_t Du_t(x) + c_t u_t(x) + \gamma_t \psi + F_t(x, u, Du, \psi)
\]
with bounded coefficients \(\alpha, c, \gamma\) and a certain Lipschitz continuous (w.r.t. \((u, Du, \psi)\)) function \(F.\) More examples can be constructed in a similar way to [5, Remark 5.1]. It is worth noting that Assumption (A3) allows \(\Gamma\) to be fully nonlinear with a certainly small dependence on \(D^2u\) and \(D\psi.\)

**Theorem 3.4.** Let \(G \in L^2(\Omega, \mathcal{F}_T; H^{1,2}([0, b]))\) and assumptions (A0)–(A3) hold. There exists a positive constant \(\mu_0\) depending on \(\kappa, L, K\) and \(T,\) such that when \(0 \leq \mu < \mu_0,\) BSPDE (3.10) admits a unique strong solution \((u, \psi)\) satisfying
\[
\|\Gamma(\psi, u, Du, D^2u)\|_{H^{1,2}([0, b])} + \|\Gamma_0\|_{L^2(0, T; L^2([0, b]))},
\]
where \(\Gamma_0 \equiv \Gamma(\cdot, 0, 0, 0, 0, 0)\) and the constant \(C\) depends on \(\mu, L, \kappa, K\) and \(T.\)

**Proof.** We use the continuity method.

**Step 1.** For each \(\lambda \in [0, 1],\) consider the following BSPDE
\[
\begin{aligned}
-du_t(x) &= \left\{ \lambda \left[ \frac{1}{2} \left( |\sigma_t|^2 + |\bar{\sigma}_t|^2 \right) D^2u_t(x) + \sigma_t D\psi_t(x) + \Gamma_t(x, u, Du, D^2u, \psi, D\psi) \right] \\
&+ (1 - \lambda) D^2u_t(x) \right\} dt - \psi_t(x) dW_t, \quad (t, x) \in [0, T] \times [0, b]; \\
D u_t(0) &= 0, \quad D u_t(b) = 0; \\
u_T(x) &= G(x), \quad x \in [0, b].
\end{aligned}
\]

We would generalize the a priori estimates from linear cases of Proposition 3.1 to nonlinear equation (3.12). Suppose \((u, \psi)\) is a strong solution of BSPDE (3.12). Applying Proposition 3.1 to each \(t \in [0, T]\) (see also estimates (3.3) and (3.5)), we have
\[
E \left[ \sup_{s \in [t, T]} \|u_s\|^2 \right] + E \int_t^T \|\psi_s\|^2 + \|Du_s\|^2 \, ds \\
\leq C E \left[ \|G\|^2 + \int_t^T \left| \left\langle \Lambda \Gamma_s \left( \cdot, u, Du, D^2u, \psi, D\psi \right), u_s \right\rangle \right| \, ds + \frac{1}{\varepsilon} \int_t^T \|u_s\|^2 \, ds + \varepsilon \int_t^T \|D\psi_s\|^2 \, ds \right] \\
\leq C E \left[ \|G\|^2 + \|G\| \int_t^T \|\Gamma_s \left( \cdot, u, Du, D^2u, \psi, D\psi \right) \|^2 \, ds + \frac{2}{\varepsilon} \int_t^T \|u_s\|^2 \, ds + \varepsilon \int_t^T \|D\psi_s\|^2 \, ds \right] \\
\leq C_1 E \left[ \|G\|^2 + \varepsilon \int_t^T \left( \|\Gamma^*_s\|^2 + \|u_s\|^2 + \|Du_s\|^2 + \|\psi_s\|^2 \right) \, ds + \frac{2}{\varepsilon} \int_t^T \|u_s\|^2 \, ds \\
&+ \int_t^T \varepsilon (1 + \mu^2) \|D\psi_s\|^2 + \varepsilon \mu^2 \|D^2u_s\|^2 \, ds \right]
\]
\[(3.13)\]
and

\[
E \left[ \sup_{s \in [t,T]} \|u_s\|_{H^1,2([0,b])}^2 \right] + E \int_t^T \left( \|u_s\|_{H^2,2([0,b])}^2 + \|\psi_s\|_{H^1,2([0,b])}^2 \right) ds \\
\leq C E \left[ \|G\|_{H^1,2([0,b])}^2 + \int_t^T \left| \left\langle \lambda \Gamma_s(x, u, Du, D^2 u, \psi, D\psi) \right\rangle \right| ds \\
+ \int_t^T \left| \left\langle \lambda \Gamma_s(x, u, Du, D^2 u, \psi, D^2 u_s) \right\rangle \right| ds \right]
\]

\leq C E \left[ \|G\|_{H^1,2([0,b])}^2 + \left( 1 + \frac{1}{\varepsilon} \right) \int_t^T \left( \|\Gamma^0_s\|^2 + \|u_s\|^2 + \|Du_s\|^2 + \|\psi_s\|^2 \right) ds \\
+ \int_t^T E \left[ (\mu^2 + \varepsilon) \|D^2 u_s\|^2 + (\mu^2 + \varepsilon) \|D\psi_s\|^2 \right] ds, \quad \text{(3.14)}
\]

with Cs depending on \(\kappa, K, L\) and \(T\). Letting \(\varepsilon < \frac{1}{2C_1 + T}\), we have by (3.13),

\[
E \left[ \sup_{s \in [t,T]} \|u_s\|^2 \right] + E \int_t^T (\|\psi_s\|^2 + \|Du_s\|^2) ds \\
\leq C_2 E \left[ \|G\|^2 + \int_t^T \|\Gamma^0_s\|^2 ds + \frac{2}{\varepsilon} \int_t^T \|u_s\|^2 ds + \int_t^T \varepsilon(1 + \mu^2)\|Du_s\|^2 + \varepsilon\mu^2\|D^2 u_s\|^2 ds \right] \quad \text{(3.15)}
\]

with \(C_2\) independent of \((\varepsilon, \mu)\). From (3.13) and (3.14), it follows that, there exists \(\mu_0\) depending on \(\kappa, K, L\) and \(T\) such that when \(\mu < \mu_0\), letting \(\varepsilon\) be small enough, one has

\[
\|(u, \psi)\|_{H^1} \leq C \left( \|G\|_{L^2(0,T;H^1,2([0,b]))} + \|\Gamma^0\|_{L^2(0,T;L^2([0,b]))} \right),
\]  \quad \text{(3.16)}

with the constant \(C\) depending on \(\mu, \kappa, K\) and \(T\).

**Step 2.** Suppose \((u_1, \psi_1)\) and \((u_2, \psi_2)\) are two strong solutions of BSPDE (3.12). Put \((\delta u, \delta \psi) = (u_1 - u_2, \psi_1 - \psi_2)\). In a similar way to **Step 1**, applying Itô formula (Lemma A.1) to square norms of \((\delta u, \delta \psi)\), one gets estimates (3.13) and (3.14), and further (3.10) but with \((G, \Gamma^0)\) being replaced by zero values. This indicates the uniqueness of the strong solution to BSPDE (3.12).

**Step 3.** We first note that the a priori estimate (3.10) holds with the constant \(C\) being independent of \(\lambda\). Assume that BSPDE (3.12) admits a unique strong solution \((u, \psi)\) for \(\lambda = \lambda_0\). By Proposition 3.3 we have shown that this assumption is true for \(\lambda_0 = 0\). For each \((\hat{u}, \hat{\psi}) \in H^1\), the following BSPDE

\[
\begin{aligned}
- \frac{d}{dt}u(t) &= \left\{ \lambda_0 \left[ \frac{1}{2} \left( |\sigma|^2 + |\sigma_0|^2 \right) D^2 u(t) + \sigma_1 \psi(t) \right] + \Gamma(x, u, Du, D^2 u, \psi, D\psi) \right\} + (1 - \lambda_0)D^2 u(t) \\
&\quad + \left( \lambda - \lambda_0 \right) \left[ \frac{1}{2} \left( |\sigma|^2 + |\sigma_0|^2 \right) D^2 \hat{u}(t) + \sigma_1 \hat{\psi}(t) \right] + \Gamma(x, \hat{u}, D\hat{u}, D^2 \hat{u}, \hat{\psi}, D\hat{\psi}) \right\} \quad dt \\
- \psi(t) dW_t, \\
Du(t) &= 0, \quad (t, x) \in [0,T] \times [0,b]; \\
u(t, x) &= G(x), \quad x \in [0,b],
\end{aligned}
\]

has a unique strong solution \((u, \psi)\), and we can define the solution map as follows

\[
\mathcal{A}_{\lambda_0} : H^1 \rightarrow H^1, \quad (\hat{u}, \hat{\psi}) \mapsto (u, \psi).
\]

Then for any \((u_i, \psi_i) \in H^1, i = 1, 2\), in a similar way to **Step 2**, we have

\[
\|(u_1 - u_2, \psi_1 - \psi_2)\|_{H^1} \leq C \lambda \lambda_0 \left\| \frac{1}{2} \left( |\sigma|^2 + |\sigma_0|^2 \right) D^2 (u_1 - u_2) - D^2 (u_1 - u_2) + \sigma D(\psi_1 - \psi_2) \right\|.
\]
Then, for each \( x \) with 
\[
\left(0, b\right) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \) and \( A \) being an \( \mathcal{F}_t \)-adapted continuous bounded variation process satisfying \( A_0 = 0 \). Suppose \( 0 \leq X_t \leq b \) a.s. and 
\[
u_t(x) = u_0(x) + \int_0^t q_r(x) \, dr + \int_0^t \psi_r(x) \, dW_r, \quad \text{for} \ (t, x) \in [0, T] \times [0, b],
\]
holds in the weak sense with
\[(u, q, \psi) \in (L^2(0, T; [0, b])) \cap (L^2(0, T; R^d)) \times (L^2(0, T; R^d)) \times (L^2(0, T; R^d)).\]
Then, for each \( x \in [0, b] \), it holds almost surely that, for all \( t \in [0, T] \),
\[
u_t(X_t^0, x) = u_0(x) + \int_0^t \left( q_r(X_t^0, x) + \xi_r D_{\nu_t}(X_t^0, x) + \frac{1}{2} \left( |\rho|^2 + |\bar{\rho}|^2 \right) D^2 u_r(X_t^0, x) + \rho D_{\nu_t}(X_t^0, x) \right) \, dr \\
+ \int_0^t D_{\nu_t}(X_t^0, x) \, dA_r + \int_0^t \left( \psi_r(X_t^0, x) + D_{\nu_t}(X_t^0, x) \rho_r \right) \, dW_r + \int_0^t D_{\nu_t}(X_t^0, x) \rho_r \, dB_r. \quad (4.2)
\]
By Sobolev’s embedding theorem, \( H^{m,2}(\mathbb{R}) \) is continuously embedded into continuous function space \( C^{m-1} \). Thus, the equation \( (4.2) \) makes sense for each \( x \in [0, b] \) and in this way, Lemma \( 4.1 \) is similar to the first formula of Kunita [15] Pages 118-119] if we replace the bounded domain \( [0, b] \) by the whole real line \( \mathbb{R} \). To eliminate the affects of the boundary of the bounded domain, we extend the Sobolev spaces to the whole line, and the sketch of the proof is provided in the appendix. We would note that in Lemma \( 4.1 \), we consider the one-dimensional case for simplicity and that there is no essential difficulty in extending it to multi-dimensional cases.

We introduce a result on the Dirichlet problem of BSPDEs by Du and Tang [6] Theorem 3.1].

**Lemma 4.2.** Consider the following Dirichlet problem of BSPDE:
\[
\begin{cases}
-dv_t(x) = \left[ \frac{1}{2} \left( |\sigma_t|^2 + |\bar{\sigma}_t|^2 \right) D^2 v_t(x) + \sigma_t D\psi_t(x) + h_t(x) \right] \, dt \\
- \psi_t(x) \, dW_t, \\
v_t(0) = 0, \quad v_t(b) = 0; \\
v_T(x) = G(x), \quad x \in [0, b],
\end{cases} \quad (4.3)
\]
with $G \in L^2(\Omega, \mathcal{F}_T; H^{1,2}_0([0, b]))$ and $h \in L^2(0, T; L^2([0, b]))$. Under assumptions (A0) – (A2), BSPDE \ref{eq:4.4} admits a unique weak solution $(v, \psi)$ which is also the unique strong solution with

$$
\|(v, \psi)\|_{\mathcal{H}^1} \leq C \left( \|G\|_{L^2(\Omega, \mathcal{F}_T; H^{1,2}([0, b]))} + \|h\|_{L^2(0, T; L^2([0, b]))} \right)
$$

with the constant $Cs$ depending on $\kappa$, $K$ and $T$.

**Proof of Theorem 4.4** We first reduce the Neumann problem \ref{eq:1.5} to the case with zero Neumann boundary condition. In view of assumption (i) of $(A^*)$ and Definition 4.1

we have $(\hat{g}, \hat{G})$ is the strong solution of the following BSPDE

\begin{equation}
\begin{aligned}
-\frac{1}{2} \left( |\sigma|^2 + |\bar{\sigma}|^2 \right) D^2 \hat{g}_t(x) + \sigma D\hat{\psi}_t(x) + \hat{f}_t(x) &= \hat{f}_t(x) dt - \hat{\psi}_t(x) dW_t, \\
&\quad (t, x) \in [0, T] \times [0, b]; \\
D\hat{g}_t(0) &= g_t(0), \quad D\hat{g}_t(b) = g_t(b); \\
\hat{g}_T(x) &= \int_0^x g_T(y) dy, \quad x \in [0, b],
\end{aligned}
\end{equation}

with $\hat{f} \in L^2(0, T; H^{1,2}([0, b]))$ being defined

$$
\hat{f}_t(x) = - \frac{1}{2} \left( |\sigma|^2 + |\bar{\sigma}|^2 \right) D^2 \hat{g}_t(x) - \sigma D\hat{\psi}_t(x) + \bar{\hat{G}}_t(x).
$$

Thus, the existence and uniqueness of strong solution $(u, \psi)$ to BSPDE \ref{eq:1.5} is equivalent to that of the strong solution $(\hat{u}, \hat{\psi}) = (u - \hat{g}, \psi - \bar{\hat{G}})$ to the following BSPDE:

\begin{equation}
\begin{aligned}
-\frac{1}{2} \left( |\sigma|^2 + |\bar{\sigma}|^2 \right) D^2 \hat{u}_t(x) + \sigma D\hat{\psi}_t(x) + \Phi_t(x, \hat{u}_t(x) + D\hat{g}_t(x)) - \hat{f}_t(x) &= \hat{f}_t(x) dt \\
&\quad - \hat{\psi}_t(x) dW_t, \quad (t, x) \in [0, T] \times [0, b]; \\
D\hat{u}_t(0) &= 0, \quad D\hat{u}_t(b) = 0; \\
\hat{u}_T(x) &= G(x) - \hat{g}_T(x), \quad x \in [0, b].
\end{aligned}
\end{equation}

By Theorem 4.4, BSPDE \ref{eq:4.6} has a unique strong solution $(\hat{u}, \hat{\psi})$. Taking derivatives, one can easily check that

$$(v, \zeta) \triangleq (D\hat{u}, D\hat{\psi}) = (Du - D\hat{g}, D\psi - D\bar{\hat{G}}) = (Du - g, D\psi - \bar{G})$$

is a weak solution of the following Dirichlet problem:

\begin{equation}
\begin{aligned}
-\frac{1}{2} \left( |\sigma|^2 + |\bar{\sigma}|^2 \right) D^2 v_t(x) + \zeta(x) &= F_t(x) dt \\
&\quad - \zeta_t(x) dW_t, \quad (t, x) \in [0, T] \times [0, b]; \\
v_t(0) &= 0, \quad v_t(b) = 0; \\
v_T(x) &= DG(x) - g_T(x), \quad x \in [0, b],
\end{aligned}
\end{equation}

with

$$
F_t(x) = - D\hat{f}_t(x) + (D^2 \sigma' + D\bar{\sigma} \bar{\sigma}') D^2 \hat{u}_t(x) + D\sigma D\hat{\psi}_t(x) + (D\Phi)_t(x, D\hat{u}_t(x) + g_t(x)) \\
+ (\partial_t \Phi)_t(x, D\hat{u}_t(x) + g_t(x)) \left( D^2 \hat{u}_t(x) + D\hat{g}_t(x) \right).
$$

By assumption (ii) of $(A^*)$, one has $F \in L^2(0, T; L^2([0, b]))$. Then, one can conclude from Lemma 4.2 that $(v, \zeta)$ turns out to be a strong solution. Thus, $(D\hat{u}, D\hat{\psi}) = (v, \zeta) \in \mathcal{H}^1$ and moreover, $(Du, D\psi) = (D\hat{u} + g, D\hat{\psi} + \bar{G}) \in \mathcal{H}^1.$
Hence, \((u, \psi) \in H^2\). This regularity and assumption (iii) of \((A^*)\) indicate the admissibility of the control \(\theta_T^* = \Pi_t(X_t^*, u_t(X_t^*))\). For each admissible control \(\theta\), applying the generalized Itô-Kunita-Wentzell formula to \(u_t(X_t^{0,x;\theta})\) indicates that for each \(x \in [0, b]\), there holds almost surely

\[
u_t(X_t^{0,x;\theta})
\begin{align*}
&= E \left[ \int_t^T \left( \text{ess inf}_{\theta \in \Theta} \left\{ \beta_r(X_r^{0,x;\theta}, \tilde{\theta})Du_r(X_r^{0,x;\theta}) + f_r(X_r^{0,x;\theta}, \tilde{\theta}) \right\} - \beta_r(X_r^{0,x;\theta}, \theta_r)Du_r(X_r^{0,x;\theta}) \right) \, dr \bigg| \mathcal{F}_t \right] \\
&\quad + E \left[ G(X_T^{0,x;\theta}) \mid \mathcal{F}_t \right] \\
&\leq E \left[ G(X_T^{0,x;\theta}) + \int_t^T f_r(X_r^{0,x;\theta}, \theta_r) \, dr \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T].
\end{align*}
\]

Thus, for any admissible control \(\theta\), it holds almost surely,

\[
u_t(X_t^{0,x;\theta}) \leq J_t(X_t^{0,x;\theta}; \theta), \quad \forall t \in [0, T].
\]

On the other hand, in a similar way to (4.8), for each \(x \in [0, b]\),

\[
u_t(X_t^*) = E \left[ G(X_T^*) + \int_t^T f_r(X_r^*, \theta_r^*) \, dr \bigg| \mathcal{F}_t \right] = J_t(X_t^*; \theta^*)
\]

holds for all \(t \in [0, T]\) with probability 1. Hence, in view of relations (4.9) and (4.10), \(u_t(x)\) coincides with the value function and the optimal control is given by \(\theta_T^* = \Pi_t(X_t^*, Du_t(X_t^*))\) with the optimal state process \(X_t^*\) satisfying RSDE (2.3). We complete the proof. \(\square\)

\section{An Itô formula for the square norm of solutions of SPDEs}

Let \((V, \| \cdot \|_V)\) be a real reflexive and separable Banach space, and \(H\) a real separable Hilbert space. With a little notational confusion, the inner product and norm in \(H\) is denoted by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) respectively. Assume that \(V\) is densely and continuously imbedded in \(H\). Thus, the dual space \(H'\) is also continuously imbedded in \(V'\) which is the dual space of \(V\). Simply, we denote the above framework by

\[
V \hookrightarrow H \cong H' \hookrightarrow V'.
\]

We denote \(\| \cdot \|_*\) the norm in \(V'\). The dual product between \(V\) and \(V'\) is denoted by \(\langle \cdot, \cdot \rangle_{V', V}\). \((V, H, V')\) is called a Gelfand triple.

The Itô formula plays a crucial role in the theory of SPDEs (see [15, 32] for instance). In the following, we introduce a backward version. See [28, Theorem 3.2] for the proof for general cases.

\begin{lemma}
Let \(\xi \in L^2(\Omega, \mathcal{F}_T, H)\), \(F \in L^2(0,T;V')\) and \((u, \psi) \in L^2(0,T;V) \times L^2(0,T;L(\mathbb{R}^d,H))\) with \((L(\mathbb{R}^d,H), \| \cdot \|_1, \langle \cdot, \cdot \rangle_1)\) being the space of Hilbert-Schmidt operators from \(\mathbb{R}^d\) to \(H\). Assume that the following backward SDE

\[
u_t = \xi + \int_t^T F_s \, ds - \int_t^T \psi_s \, dW_s, \quad t \in [0, T],
\]

holds in the weak sense, i.e., for any \(\phi \in V\), there holds almost surely

\[
\langle \nu_t, \phi \rangle = \langle \xi, \phi \rangle + \int_t^T \langle F_s, \phi \rangle_{V', V} \, ds - \int_t^T \langle \phi, \psi_s \, dW_s \rangle, \quad t \in [0, T].
\]

Then we assert that \(u \in S^2(0,T; H)\) and the following Itô formula holds almost surely

\[
\| \nu_t \|^2 = \| \xi \|^2 + \int_t^T \left( 2 \langle F_s, u_s \rangle_{V', V} - \| \psi_s \|^2 \right) \, ds - \int_t^T 2 \langle u_s, \psi_s \, dW_s \rangle, \quad t \in [0, T].
\]

\end{lemma}
Remark A.1. In Lemma [A.1] suppose additionally $\xi \in L^2(\Omega, \mathcal{F}_T, H)$, $\hat{F} \in L^2(0, T; V')$, $(\hat{u}, \hat{\psi}) \in L^2(0, T; L(\mathbb{R}^d, H))$ and

$$\hat{u}_t = \xi + \int_t^T \hat{F}_s \, ds - \int_t^T \hat{\psi}_s \, dW_s, \quad t \in [0, T],$$

(A.3)

holds in the weak sense. Then $\hat{u} \in S^2(0, T; H)$, and applying the parallelogram rule yields that there holds almost surely

$$\langle u_t, \hat{u}_t \rangle = \langle \xi, \hat{\xi} \rangle + \int_t^T \left( \langle F_s, \hat{u}_s \rangle_{V', V} + \langle \hat{F}_s, u_s \rangle_{V', V} - \langle \hat{\psi}_s, \hat{\psi}_s \rangle_{V', V} \right) \, ds$$

$$- \int_t^T \langle \hat{u}_s, \hat{\psi}_s \rangle_{W_s} - \int_t^T \langle u_s, \hat{\psi}_s \rangle_{W_s}, \quad \forall t \in [0, T].$$

(B.4)

B Proof of Lemma 4.1

Sketch of the proof of Lemma 4.1. The Sobolev space theory allows us to extend $H^{k,2}([0, b])$ to $H^{k,2}(\mathbb{R})$ for integers $k \geq 1$. In particular, when $k = 1, 2$, the bounded linear extension operator can be constructed in the same way as follows (see [9], Pages 254-257): for each $\zeta \in H^{1,2}([0, b])$ or $\zeta \in H^{2,2}([0, b])$,

$$E(\zeta(x)) = \begin{cases} 
\zeta(x), & \text{if } x \in [0, b]; \\
\gamma(x) \left[ -3\zeta(-x) + 4\zeta(-x/2) \right], & \text{if } x \in [-b, 0]; \\
\gamma(x) \left[ -3\zeta(2b - x) + 4\zeta((2b - x)/2) \right], & \text{if } x \in [b, 2b]; \\
0, & \text{if } x \in (-\infty, -b) \cup (2b, \infty),
\end{cases}$$

where $\gamma \in C^\infty(\mathbb{R})$ satisfying $\gamma(x) = 1$ for $x \in [0, b]$ while $\gamma(x) = 0$ for $x \in (-\infty, -b/2] \cup [3b/2, \infty)$. $E\zeta$ is called an extension of $\zeta$ to $\mathbb{R}$. Then it is easy to check that

$$E u_t(x) = E u_0(x) + \int_0^t E q_r(x) \, dr + \int_0^t E \psi_r(x) \, dW_r, \quad \text{for } (t, x) \in [0, T] \times \mathbb{R},$$

(B.1)

holds in the weak sense.

Define

$$\phi(x) = \begin{cases} 
ed^x - 1 & \text{if } |x| \leq 1; \\
0 & \text{otherwise},
\end{cases} \quad \text{with } \hat{e} := \left( \int_{-1}^1 e^{-x} \, dx \right)^{-1},$$

(B.2)

and for $l \in \mathbb{N}$, set $\phi_l(x) = l \phi(lx), x \in \mathbb{R}$. Itô’s formula yields that, for each $y \in \mathbb{R}$,

$$\phi_l(X_{t^0} - y) = \phi_l(x - y) + \int_0^t \frac{D \phi_l(X_{s^0} - y)}{D \phi_l(x - y)} \, dW_r + \int_0^t \left[ \frac{D^2 \phi_l(x_{s^0} - y)}{D^2 \phi_l(x - y)} \right] \, dr$$

+ $\int_0^t \frac{D \phi_l(x - y)}{D \phi_l(x_{s^0} - y)} \, dA_r, \quad t \in [s, T].$

In view of (B.1), we have by Itô formula of Remark A.1

$$\int_\mathbb{R} \phi_l(X_{t^0} - y) E u_t(y) \, dy - \int_\mathbb{R} \phi_l(x - y) E u_0(y) \, dy - \int_0^t \int_\mathbb{R} D E u_r(y) \phi_l(X_{t^0} - y) \, dy \, dA_r$$

$$= \int_0^t \int_\mathbb{R} \left[ D E u_r(y) \xi_r + E q_r(y) + \frac{1}{2} \left( |\rho_r|^2 + |\rho_r|^2 \right) D^2 E u_r(y) + \rho_r D E u_r(y) \right] \phi_l(X_{t^0} - y) \, dy \, dr$$

$$+ \int_0^t \int_\mathbb{R} \phi_l(X_{t^0} - y) \left( \rho_r D E u_r(y) + \psi_r(y) \right) \, dy \, dW_r + \int_0^t \int_\mathbb{R} \phi_l(X_{t^0} - y) \rho_r D E u_r(y) \, dy \, dB_r, \quad \text{a.s.},$$

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for all $t \in [0,T]$. We note that as $X^{s,x} \in S^2_{\mathcal{P}}(0,T;[0,b])$, all the above integrals on $\mathbb{R}$ are taken on a compact set for almost every $\omega \in \Omega$ and thus make sense. Since the sequence of convolutions indexed by $l$ approximates to the identity and $0 \leq X_t \leq b$ a.s., letting $l \to \infty$ and recalling that $E_u_t(x) = u_t(x)$ for $(t,x) \in [0,T] \times [0,b]$, we obtain that for each $x \in [0,b]$, it holds almost surely that, for all $t \in [0,T]$,

$$ u_t(X^{0,x}_0) = u_0(x) + \int_0^t \left[ q_r(X^{0,x}_r) + \xi_r Du_r(X^{0,x}_r) + \frac{1}{2} \left( |\rho_r|^2 + |\bar{\rho}_r|^2 \right) D^2 u_r(X^{0,x}_r) + \rho_r D\psi_r(X^{0,x}_r) \right] dr $$

$$ + \int_0^t Du_r(X^{0,x}_r) dA_r + \int_0^t \left( \psi_r(X^{0,x}_r) + Du_r(X^{0,x}_r)\rho_r \right) dW_r + \int_0^t Du_r(X^{0,x}_r)\bar{\rho}_r dB_r. $$

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