1. Introduction

Let \((W, S)\) be a finite Coxeter system with length function \(\ell\) and identity \(e\). Endow \(W\) with the Bruhat order \(\leq\), that is, \(w \leq g\) in \(W\) if and only if an expression for \(w\) can be obtained by deleting simple reflections in a reduced expression for \(g\). (If \(w \leq g\) then we necessarily have \(\ell(w) \leq \ell(g)\).) We refer the reader to \([2, 3]\) as general references for Coxeter groups and the Bruhat order.

Denote by \(W_I\) the standard parabolic subgroup of \(W\) generated by \(I \subset S\). For \(I, J \subset S\), each double coset in \(W_I/W/W_J\) has a unique minimal element. Let \(X_{IJ} = \{w \in W | w < rw, w < ws, \forall r \in I, \forall s \in J\}\) be the set of all minimal representatives of double cosets in \(W_I/W/W_J\).

Curtis \([3, \text{Theorem 1.2}]\) shows that for any \(I, J \subset S\) and \(b \in X_{IJ}\), there is a unique maximal element \(b_{\text{max}}\) in \(W_I b W_J\). This fact plays an important role in his study of Lusztig’s isomorphism theorem.

The aim of this note is to prove the following result, which seems to have escaped observation:

**Theorem 1.** Let \(I, J \subset S\) and \(u, v \in X_{IJ}\). Then \(u \leq v\) if and only if \(u_{\text{max}} \leq v_{\text{max}}\).

Double parabolic cosets arise in a variety of settings. In particular Theorem 1 is used in \([7]\) in the study of the dual canonical basis of \(O(SL_n \mathbb{C})\).

After proving our result in \(\S 2\) we give a combinatorial criterion in \(\S 3\) for the comparison of \(u\) and \(v\) (or \(u_{\text{max}}\) and \(v_{\text{max}}\)).

2. Proof

For \(I \subset S\), it is well-known that the set

\[W^I = \{u \in W | u < us, \forall s \in I\}\]

is a set of minimal length coset representatives of \(W/W_I\). Each element \(w \in W\) has therefore a unique decomposition \(w = w^I w_I\) where \(w^I \in W^I\) and \(w_I \in W_I\). Moreover \(\ell(w) = \ell(w^I) + \ell(w_I)\). The pair \((w^I, w_I)\) is generally referred to as the parabolic components of \(w\) along \(I\) (see \([2, \text{Proposition 2.4.4}]\), or \([4, 5.12]\)). It is clear that \(W^K = W^K \cap W_I\) is a set of minimal length coset representatives of \(W_I/W_K\). Moreover \(X_{IJ} = (W^I)^{-1} \cap W^J\), where \((W^I)^{-1} = \{w^{-1} | w \in W^I\}\).

Let \(w_{0,I}\) denote the unique maximal element in in \(W_I\), and let \(w_0 = w_{0,S}\) denote the longest element of \(W\). Then the parabolic components of \(w_0\) are \((w_{0,I}^I, w_{0,I})\), where \(w_{0,I}^I\) is the unique maximal element in \(W^I\) (see \([2, \S 2.5]\)). It follows that for \(K \subset I \subset S\), the unique maximal element in \(W^I_K\) is \(w_{0,I}^I = w_{0,I} w_{0,K}\).

We recall the following well-known facts:

\[\]
(i) For any $I, J \subset S$, define $I \cap bJb^{-1} = I \cap \{bsb^{-1} \mid s \in J\}$. Then we have

$$W^J = \prod_{b \in X_{IJ}} W_{I,bJb^{-1}}.$$ 

Therefore, each element $w \in W$ has a unique decomposition $(a, b, w_J)$ where $b \in X_{IJ}$, $a \in W_{I,bJb^{-1}}$ and $ab = w_J$. Moreover $\ell(w) = \ell(a) + \ell(b) + \ell(w_J)$.

(See for instance [11 §2].)

(ii) Let $w, g, x \in W$ satisfy $\ell(wx) = \ell(w) + \ell(x)$ and $\ell(gx) = \ell(g) + \ell(x)$. Then $w \leq g$ if and only if $wx \leq gx$.

(iii) From (ii) we have: if $w \leq g$, then $w^J \leq g^J$ for any $w, g \in W$ and $I \subset S$.

(iv) Deodhar’s Lemma [4]: Let $K \subset S$, $x \in W^I$ and $s \in S$. If $sx < x$ then $sx \in W^K$. If $x < sx$ then either $sx \in W^K$ or $sx = xg$.

(v) Lifting property: Let $w, g \in W$ and $s \in S$ satisfy $w < sw$ and $sg < g$. Then $w \leq g \iff w \leq sg \iff sw \leq g$ (see [12 2]).

Curtis [3, Theorem 1.2] shows that for any $I, J \subset S$ and $b \in X_{IJ}$, $h_{\text{max}} = w_{0,I,J}^{-1}b_{w_{0,J}}$ is the unique maximal element in $W_I b W_J$. Here we give a short proof of this fact. Let $w \in W_I b W_J$, then by (i) we have $w = abw_J$ with $a \in W_{I,bJb^{-1}}$. Hence $a \leq w_{0,I,J}^{-1}$ and $w \leq w_{0,J}$, and by (i) and (ii) we have $w \leq h_{\text{max}}$.

**Lemma 2.** Let $I, J \subset S$ and suppose that $u, v \in X_{IJ}$ satisfy $u \leq v$. Then for any $a \in W_{I,uJ^{-1}}$ we have $au \leq w_{0,I,J}^{-1}v$.

**Proof.** Writing $a = w_{0,I,J}^{-1}w$, we will use induction on $\ell(a)$ to show that $au \leq ov$. If $\ell(a) = 0$, then $a = e$ and $au = u \leq ov$ and $\ell(ov) = \ell(a) + \ell(v)$.

Assume therefore that $\ell(a) > 0$. Then some $s \in I$ satisfies $sa < a$ and we have $sa \in W_{I,J}^{-1}$ by Deodhar’s Lemma. Since $\ell(sa) < \ell(a)$, we have $sa u \leq ov$ by our induction hypothesis. We also have $sa u < au$ by (ii) since $a, sa \in W_I$ and $u^{-1} \in W^I$. In order to compare $ov$ and $au$ we consider two cases.

If $sa ov$ then we obtain $au \leq ov$ from (v) using $w = sau$ and $g = ov$. If $ov < sa ov$, then $au = s(sau) \leq sa ov$ by definition. Observe that $\alpha < sa$ by (ii). As $\alpha$ is the maximal element in $W_{I,J}^{-1}$, we have $\alpha \notin W_{I,J}^{-1}$ by (iv). So some $r \in I \cap J$ satisfies $\alpha r = \alpha$.

Set $t = v^{-1}rv \in J$ so that $\alpha v = ov t$. As $ov \in W^J$ (by (i)) we deduce that $(ov, t)$ are the parabolic components of $s a v$ along $J$. As $au \in W^I$ and $au \leq ov$, we obtain by (iii) that $au = (au)^J \leq (\alpha v)^J = ov$. (proved)

**Proof of Theorem 4.** Assume that $u_{\text{max}} \leq v_{\text{max}}$. First observe that from (1) and (i) we have $((h_{\text{max}})^{-1})^J = b^{-1}$ for any $b \in X_{IJ}$. Now use (iii) and the automorphism $w \mapsto w^{-1}$ of the Bruhat order to show that $u_{\text{max}} \leq v_{\text{max}}$ implies $u \leq v$.

Assume now that $u \leq v$. Using (1), (i) and (ii) we just have to show that $w_{0,I,J}^{-1}u = (u_{\text{max}})^J \leq (v_{\text{max}})^J = w_{0,I,J}^{-1}v$. But this is the special case $a = w_{0,I,J}^{-1}$ of Lemma 2 (proved).

### 3. The special case of the symmetric group

Given a permutation $w \in S_n$, we define the matrix $M(w) = (m_{i,j}(w))$ by setting $m_{i,j}(w) = \delta_{i,j}w_i$, where $w_1 \cdots w_n$ is the one-line notation of $w$. We define the related matrix $D(w) = (d_{i,j}(w))$ by $d_{i,j}(w) = \sum_{k=1}^{i-1} \sum_{l=1}^{j-1} m_{i,j}(w)$. It is well known that $u \leq v$ if and only if we have the componentwise inequality of matrices $D(u) \geq D(v)$, and we shall state a similar fact for double parabolic analogs of $M$ and $D$. 


A subset \( H \) of \( S = \{s_1, \ldots, s_{n-1}\} \) induces an equivalence relation \( \sim_H \) on \([n] = \{1, \ldots, n\}\) which is the transitive closure of the relation \( i R (i+1) \) for all \( s_i \in H \). Let \( B_1, \ldots, B_{q} \) and \( C_1, \ldots, C_{r} \) be the equivalence classes of \( \sim_I \) and \( \sim_J \), respectively, and define the matrices \( M^{I,J}(w) = (m_{i,j}^{I,J}(w)) \) and \( D^{I,J}(w) = (d_{i,j}^{I,J}(w)) \) by

\[
m_{i,j}^{I,J}(w) = \# \{ k \in B_i \mid w_k \in C_j \}, \quad d_{i,j}^{I,J}(w) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} m_{k,\ell}^{I,J}(w).
\]

It is well known (see, e.g., [6]) that \( u \) and \( v \) belong to the same double coset in \( W_I \setminus W/W_J \) if and only if \( M^{I,J}(u) = M^{I,J}(v) \). Furthermore we have the following.

**Proposition 3.** Given \( u, v \) in \( X_{I,J} \), then \( u \leq v \) (or \( u^{\max} \leq v^{\max} \)) if and only if we have the componentwise inequality of matrices \( D^{I,J}(u) \geq D^{I,J}(v) \).

**Proof.** Define \( \overline{I} = [n] \setminus \{i \mid s_i \in I\} \), \( \overline{J} = [n] \setminus \{j \mid s_j \in J\} \). Then for each \( w \in S_n \), the matrix \( D^{I,J}(w) \) is equal to the \((\overline{I}, \overline{J})\) submatrix of \( D(w) \). The “only if” direction follows immediately.

Suppose the \( u \not\in [v] \) and let \((i,j)\) be a componentwise minimal pair satisfying \( d_{i,j}(u) < d_{i,j}(v) \). If \( i > 1 \), then the fact that the matrices \( D(u) \) and \( D(v) \) weakly increase down columns and across rows, with adjacent entries differing by no more than 1, implies that \( d_{i-1,j}(u) \leq d_{i,j}(u) < d_{i,j}(v) \leq d_{i-1,j}(v) + 1 \). By the minimality of \( i \) and \( j \), this last expression is less than or equal to \( d_{i-1,j}(u) + 1 \), and for some nonnegative integer \( c \) we have \( d_{i-1,j}(u) = d_{i,j}(u) = d_{i-1,j}(v) = c \) and \( d_{i,j}(v) = c+1 \). Similarly, if \( j > 1 \) then we have \( d_{i,j-1}(u) = d_{i,j}(u) = d_{i,j-1}(v) = c \). It follows that for any values of \((i,j)\) we must have \( u_i > j \) and \( u_j^{-1} > i \).

Now let \((k, \ell)\) be the componentwise minimal pair in \( \overline{I} \times \overline{J} \) satisfying \( i \leq k, j \leq \ell \). Since \( u \in X_{I,J} \), we must also have

\[
\ell < u_i < \cdots < u_k, \quad k < u_j^{-1} < \cdots < u_{\ell-1}.
\]

Thus \( d_{k,\ell}(u) = c \). Since \( d_{k,\ell}(v) \geq c + 1 \), we conclude that \( D^{I,J}(u) \not\geq D^{I,J}(v) \). The equivalence of \( D^{I,J}(u) \geq D^{I,J}(v) \) and \( u^{\max} \leq v^{\max} \) follows from a similar argument. \( \square \)

We illustrate Proposition 3 by considering \( W = S_7 \), subsets \( I = \{s_1, s_2, s_4, s_6\} \), \( J = \{s_1, s_3, s_4, s_5\} \) of generators, and corresponding equivalence classes \( 123|4567, 12|3456|7 \). To compare minimal representatives \( u = 1342567, v = 3471526 \) of two double cosets in \( W_I \setminus W/W_I \), we use the matrices

\[
M^{I,J}(u) = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad M^{I,J}(v) = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},
\]

and compute

\[
D^{I,J}(u) = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 5 & 5 \\ 2 & 6 & 7 \end{bmatrix}, \quad D^{I,J}(v) = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 6 & 7 \end{bmatrix},
\]

and conclude that \( u \leq v \) and \( u^{\max} \leq v^{\max} \).
References

[1] F. Bergeron, N. Bergeron, R. B. Howlett and D. E. Taylor, A decomposition of the descent algebra of a finite Coxeter group, J. Algebraic Combin. 1 (1992), 23–44.

[2] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, Graduate Texts in Math. Springer (2005).

[3] C. W. Curtis, On Lusztig’s isomorphism theorem for Hecke algebras, J. Algebra 92 (1985), 348–365.

[4] V. V. Deodhar, Some characterization of Bruhat ordering on a Coxeter group and determination of the relative Möbius function, Inv. Math 39 (1977), 187–198.

[5] J. E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, UK (1990).

[6] G. James and A. Kerber, The representation theory of the symmetric group, vol. 16 of Encyclopedia of Mathematics and its Applications. Addison-Wesley Publishing Co., Reading, Mass. (1981).

[7] M. Skandera. Nonnegativity properties of the dual canonical basis, (2005). In preparation.

(Christophe Hohlweg) The Fields Institute, 222 College Street, Toronto, Ontario, M5T 3J1, CANADA
E-mail address: chohlweg@fields.utoronto.ca
URL: http://www.fields.utoronto.ca/~chohlweg

(Mark Skandera) Mathematics Department, Haverford College, 370 Lancaster Ave, Haverford, PA 19041, USA
E-mail address: mskander@haverford.edu
URL: http://www.haverford.edu/math/skandera.html