Two dimensional electron transport in disordered and ordered distributions of magnetic flux vortices

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Abstract

We have considered the conductivity properties of a two dimensional electron gas (2DEG) in two different kinds of inhomogeneous magnetic fields, i.e. a disordered distribution of magnetic flux vortices, and a periodic array of magnetic flux vortices. The work falls in two parts. In the first part we show how the phase shifts for an electron scattering on an isolated vortex, can be calculated analytically, and related to the transport properties through the differential cross section. In the second part we present numerical results for the Hall conductivity of the 2DEG in a periodic array of flux vortices found by exact diagonalization. We find characteristic spikes in the Hall conductance, when it is plotted against the filling fraction. It is argued that the spikes can be interpreted in terms of “topological charge” piling up across local and global gaps in the energy spectrum.

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I. INTRODUCTION

Over the last decade the two dimensional electron gas (2DEG) have been exposed to a wide range of physical experiments, in which the electrons have been perturbed by different configurations of electrostatic potentials, with or without a homogeneous perpendicular magnetic field. These experiments have shown new kinds of oscillations in the magnetoconductivity, with a periodicity not given by the geometry of the Fermi surface, as is the case with the Shubnikov-de Haas oscillations, but given by the interaction of the two length scales given respectively by the magnetic length, and by the spatial structure of the potential, e.g. the Weiss oscillations [1]. More recently, there have been increasing interest in systems where the 2DEG is exposed to an inhomogeneous perpendicular magnetic field. In such systems the inhomogeneities in the magnetic field acts as perturbations of the 2DEG, relative to the homogeneous magnetic field, where the band structure consists of the completely flat Landau bands. The inhomogeneous magnetic field appears in the Hamiltonian in the form of a non trivial vector potential. In the case of a periodic variation in the magnetic field, it is possible to construct a periodic vector potential, if and only if the flux through the unit cell of the field is equal to a rational number, when measured in units of the flux quantum \( \phi_0 = \hbar/e \). Under these special circumstances the Hamiltonian is periodic, and Bloch states can be used as a basis for the calculation of response properties of the electron gas.

In this paper we have considered a special class of spatially varying magnetic fields which consists of flux vortices, that are either distributed at random or placed in a regular lattice structure. A system consisting of a 2DEG penetrated by a random distribution of magnetic flux vortices, have been experimentally realized by Geim et al. [2,3]. They made a sandwich construction of a GaAs/GaAlAs sample with a 2DEG at the interface, and a type II superconducting lead film, electrically disconnected from the 2DEG. When the system was placed in an external magnetic field, and cooled below the transition temperature of the film, the magnetic field penetrated the film, and thereby the 2DEG, in the form of Abrikosov vortices. When the external magnetic field is weak, below 100G, the vortices will
be well separated, and the 2DEG therefore sees a very inhomogeneous magnetic field. In the experiments conducted by Geim et al. the flux pinning in the film was strong, resulting in a disordered distribution of flux vortices. This is the physical situation which we investigate in Sec. [II] below. In very clean films of type II superconducting material, the flux vortices will order in a periodic array, i.e. an Abrikosov lattice, and thereby create a periodic magnetic field at the 2DEG. This is the situation which we analyse in Sec. [III].

Several authors have investigated the transport properties of 2DEG’s in different kinds of inhomogeneous magnetic fields. Peeters and Vasilopoulos [4] have made a theoretical study of the magnetoconductivity in a 2DEG in the presence of a magnetic field, which was modulated weakly and periodically along one direction. They found large oscillations in the longitudinal resistivity as a function of the applied magnetic field strength. These oscillations are due to the interference between the two length scales given respectively by the period of the lateral variation of the magnetic field, and by the magnetic length corresponding to the average background field. The oscillations are reminiscent of the Weiss oscillations, but have a higher amplitude and a shifted phase, relative to the magneto resistance oscillations induced by the periodic electrostatic potential.

The problem of how the transport properties of the 2DEG is modified by the presence of a random distribution of flux vortices, have been treated earlier by A. V. Khaetskii [5], and also by Brey and Fertig [6]. The approach used by these authors are basically similar to the one we have presented in Sec. [II], i.e. based on the Boltzmann transport equation. The main difference being that while Khaetskii have treated the scattering in certain limiting cases, including the semiclassical, and Brey and Fertig have calculated the scattering cross section numerically, we have found an analytic expression for the scattering cross section for electrons scattering on an idealized vortex. This has enabled us to study the scattering in more detail, and to observe scattering resonances.

In Sec. [III] we will address the “paradox” of how the Hall effect can disappear in the following situation: We imagine a 2DEG in a regular 2D-lattice of flux vortices, with the magnetic field from a single vortex exponentially damped with an exponential length $\xi$,
in units of the lattice spacing. We take the total flux from a single vortex to be $\phi_0/2$, as is the case when the vortices come from a superconductor. When $\xi \gg 1$, the field is homogeneous and the Hall conductivity is $\sigma_H = p e^2/h$, where $p$ is the number of filled bands. In the other limit i.e. when $\xi \to 0$, the time reversal symmetry is restored, and the Hall conductivity vanish. The fact that the system has time reversal symmetry when the vortices are infinitely thin, can be seen by subtracting a Dirac string carrying one quantum of magnetic flux $\phi_0$, from each flux vortex. The introduction of the Dirac strings can not change any physical quantities, and the procedure therefore establishes that the system, with infinitely thin vortices with flux $+\phi_0/2$, is equivalent with the system with reversed flux $-\phi_0/2$, through each vortex. The paradoxical situation arises because it is known from general arguments \[\text{[7]}\], that the contribution to the total Hall conductivity from a single filled nondegenerate band is a topological invariant, and therefore cannot change gradually. The situation is even more clear cut if we imagine a periodic array of exponential flux vortices carrying one flux quantum $\phi_0$ each. Then the limit $\xi = \infty$ corresponds to a homogeneous magnetic field, while the opposite limit $\xi = 0$ coresponds to free particles. Incidentally this scheme can be used to establish an interpolating path between the multifractal structure known as Hofstadters butterfly \[\text{[8]}\] which is a plot of the allowed energy levels for electrons on a lattice in a homogeneous magnetic field, and the corresponding plot for lattice-electrons in no field, which is completely smooth \[\text{[9]}\].

The plan of the paper is as follows. In Sec. \[\text{II}\] we concentrate on the theory of electrons scattering on a single vortex, and the physical consequences for the resistivities. First we review the classical scattering theory in Sec. \[\text{II A}\] before we discuss thre quantum theory of scattering in Sec. \[\text{II B}\]. The longitudinal and transverse resistivities are discussed in Sec. \[\text{II C}\], and resonance scattering is demonstrated in Sec. \[\text{II D}\]. The case of a 2DEG in a periodic array of flux vortices, is the subject of Sec. \[\text{II}\]. In the first part, Sec. \[\text{II A} - \text{II D}\], of this section the general theory of electron motion in a periodic magnetic field is reviewed, and in the second part, Sec. \[\text{II E} - \text{II F}\], we present the results of the numerical calculations.
II. SINGLE VORTEX SCATTERING

In this section we will consider the consequences of the introduction of magnetic flux vortices into a 2 dimensional electron gas, in the approximation where each vortex is treated as an individual scattering center. The vortices are assumed to be distributed at random, homogeneously over the sample. The average separation between the vortices is assumed so large, that we can neglect interference from multiple scattering events. In the experimental situation the mean free path $l_f$ from impurity and phonon scattering, may be very long compared to the average separation between the vortices, due to the very clean samples and liquid Helium temperature. This means that multiple scattering and interference may have important consequences. Nevertheless we will stick to the simplifying picture of vortices as individual scatterers in this section. As we shall see, the gross features observed in experiments on this system, can be accounted for within this approximation.

A. The Classical Cross Section

We will start by calculating the differential cross section for an electron scattering on a flux vortex within the framework of classical mechanics. This will provide a reference frame, and allow us to speak unambiguously about the classical limit. In the calculations we shall use an ideal vortex, which has a circular cross section with constant magnetic field inside, and zero magnetic field outside

$$B(r) = \begin{cases} B_0 = \frac{\phi}{\pi R_v^2} & \text{for } r < R_v \\ 0 & \text{for } r > R_v. \end{cases} \quad (1)$$

Here $R_v$ is the radius, and $\phi$ is the total flux carried by the vortex. The classical orbit is found as the solution to Newton’s equation of motion with the force given by the Lorentz expression $\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$. It consists, as is well known, of straight line segments outside the vortex, and an arc of a circle inside, with radius of curvature given by the cyclotron radius $l_c = v/\omega_c$, with $v$ being the particle velocity, and $\omega_c = eB/m$ the cyclotron frequency. Inside
the vortex the orbit is an arc of a circle, and as it is impossible to draw a circle that only cut
the circumference of the vortex once, it is a simple geometrical consequence that a particle
obeying the laws of classical mechanics, and which initially is outside the vortex, can never
become trapped inside the vortex. It is clear that the classical scattering is controlled by
the single parameter $\gamma = l_c/R_v$, which is the ratio between the radius of the cyclotron orbit,
and the radius of the flux vortex. Our first objective is therefore, for a given $\gamma$, to find
the relation between the impact parameter $b$, and the scattering angle $\theta$. Fig. 1 shows the
geometry of the scattering, and the definition of the impact parameter $b$, and angles $\phi, \psi, \theta$.
Let us define the reduced impact parameter $\beta = b/R_v$, which is bounded to the interval
$-1 < \beta < 1$. By inspecting Fig. 1 it is observed that the following relations hold

$$\beta = \sin \phi$$  \hspace{1cm} (2)
$$\tan \psi = \frac{\gamma + \beta}{\sqrt{1 - \beta^2}}$$  \hspace{1cm} (3)
$$\gamma \sin \frac{\theta}{2} = \sin(\psi - \phi)\text{sign}(\gamma + \beta),$$  \hspace{1cm} (4)

where the sign of $\gamma$ is dictated by the direction of the magnetic field inside the vortex. We
take $\gamma$ to be positive. After a small amount of arithmetic $\phi$ and $\psi$ are eliminated, and we
have

$$\sin \frac{\theta}{2} = \text{sign}(\gamma + \beta)\sqrt{\frac{1 - \beta^2}{\gamma^2 + 2\gamma\beta + 1}}.$$  \hspace{1cm} (5)

This relation gives the scattering angle as a function of the impact parameter. In classical
scattering the scattering angle is always uniquely determined, once the impact parameter is
given, in contrast to the inverse, i.e. the impact parameter as a function of the deflection
angle.

In an experiment one would measure the number of particles per time $\frac{dN(\theta)}{dt}d\theta$ scattered
to an interval $d\theta$ about the angle $\theta$. This will of course depend on the incoming flux of
particles $j$, defined as the number of particles per time that cross a unit length perpendicular
to the flow. Therefore we write

$$\frac{dN(\theta)}{dt} = j \frac{d\sigma}{d\theta}.$$  \hspace{1cm} (6)
The differential cross section $\frac{d\sigma}{d\theta}$ gives the total weight of impact parameters, which give scattering into the direction $\theta$. If, for a given angle $\theta$, we label the different values of the impact parameter $b_1, b_2, \ldots b_p$, which result in scattering into $\theta$, then we have

$$\frac{d\sigma}{d\theta} = \sum_{i=1}^{p} \left| \frac{db_i}{d\theta} \right|.$$  \hfill (7)

Equation 7 has at most two solutions, which are easily found to be

$$\beta_{\pm}(\theta) = -\gamma \sin^2 \theta / 2 \pm \cos \theta / 2 \sqrt{1 - \gamma^2 \sin^2 \theta / 2}.$$  \hfill (8)

The solutions have to obey the auxiliary conditions $|\beta| \leq 1$, and $\text{sign}(\gamma + \beta) = \text{sign}(\theta)$. Furthermore we have

$$\frac{d\beta_{\pm}}{d\theta} = -\gamma \cos \theta / 2 \sin \theta / 2 \mp \sin \theta / 2 \frac{1 + \gamma^2 \cos \theta}{2\sqrt{1 - \gamma^2 \sin^2 \theta / 2}},$$  \hfill (9)

from which the differential cross-section can be calculated from Eqn. 7, still having the auxiliary conditions in mind. Examples of cross-sections and trajectories are shown in Fig. 2. The integrated cross section

$$\sigma_{\text{tot}} = \int_{-\pi}^{\pi} d\theta \left[ \frac{d\sigma}{d\theta} \right],$$  \hfill (10)

is equal to the total weight of impact parameters, which hit the vortex. It is equal to the diameter of the vortex $\sigma_{\text{tot}} = 2R_v$, as is always the case in classical scattering. Let us imagine an electron at the Fermi surface scattering off the vortex. Then we have $l_c = v_F / \omega_c = mv_F / eB = \hbar k_F / eB$. If we furthermore take the flux of the vortex to be a fraction $f$ of the flux quantum $\phi_0$, the flux density becomes $B = (f \phi_0) / \pi R_v^2 = 2f \hbar / eR_v^2$. The dimensionless cyclotron radius is then given by $\gamma = l_c / R_v = k_F R_v / 2f$, and this parameter we call $\kappa / 2f$. In the quantum regime $\kappa = 2\pi R_v / \lambda_F$, and $f$ are the natural parameters to characterize the scattering. This identification of parameters allows us to compare the predictions of classical and quantum theory below.
B. Quantum Scattering

In this section we shall consider the electron scattering off a magnetic flux vortex within the framework of quantum scattering theory. We will calculate the differential cross section, from which the longitudinal and transverse conductivities can be found from the theory of Sec. II C. The quantum nature of the electron radically alters the picture of the scattering process, when the wavelength of the electron is comparable to, or longer than the diameter of the vortex. In the limit of very small electron wavelength, the scattering can essentially by described by the laws of geometrical optics, and thereby classical mechanics.

We will again take an idealized cylindrical vortex, with constant magnetic field inside, and zero field outside, Eqn. 1. This vortex is completely symmetric under any rotation about the center axis. This symmetry can also be made a symmetry of the Hamiltonian, by choosing a proper gauge when writing down the vector potential. In cylindrical coordinates $A = e_r A_r + e_\theta A_\theta$, we have

$$B(r) = \partial_r A_\theta - \frac{1}{r} \partial_\theta A_r + \frac{A_\theta}{r}. \quad (11)$$

When $B$ is invariant under rotation, this equation has the simple solution

$$A_r = 0, \quad A_\theta(r) = \frac{1}{r} \int_0^r dr' r' B(r'), \quad (12)$$

which in our case give the vector potential $A = e_\theta A_\theta$ with

$$A_\theta(r) = \begin{cases} \frac{\phi r}{2\pi R_v^2} & \text{for } r < R_v \\ \frac{\phi}{2\pi r} & \text{for } r > R_v. \end{cases} \quad (13)$$

The Hamiltonian is given by the expression

$$H = \frac{1}{2m}(p + eA)^2 = -\frac{\hbar^2}{2m} \left\{ \partial_r^2 + \frac{1}{r} \partial_r + \left( \frac{1}{r} \partial_\theta + \frac{ie}{\hbar} A_\theta \right)^2 \right\}. \quad (14)$$

Here we have taken the charge of the electron to be $-e$. The Hamiltonian is rotationally invariant, and therefore commutes with the angular momentum about the symmetry axis, $L_z$. Consequently $L_z$ and $H$ have common eigenstates. The canonical momentum of a
charge-$q$ particle in a magnetic field is given by the expression $\mathbf{p} = m\mathbf{v} + q\mathbf{A} = \frac{\hbar}{i} \nabla$, and the operator for the angular momentum about the $z$-axis is $L_z = [\mathbf{r} \times \mathbf{p}]_z = \frac{\hbar}{i} \partial_\theta$. The eigenstates of $L_z$ are $e^{il\theta}$, and the requirement that the wave function have no cut when the vector potential is non-singular, reduces the possible values of $l$ to the set of positive and negative integers. We can now separate the variables of the common eigenstates of $L_z$ and $H$, and write

$$\phi_{kl}(r, \theta) = R_{kl}(r)e^{il\theta},$$

(15)

where $k$ is an energy label $E = \frac{\hbar^2 k^2}{2m}$. Let us introduce the flux quantum $\phi_0 = \hbar/e$ and the dimensionless fraction $f = \phi/\phi_0$. The differential equations for the radial part of the wave function, takes a particularly simple form if we write it down in dimensionless variables $\xi = r/R_v$ and $\kappa = kR_v = 2\pi R_v/\lambda$. The energy variable $\kappa$ measures the size of the vortex compared to the electron wavelength. In terms of $\kappa$ and $f$ the classical limit will be $\kappa, f \gg 1$.

With these definitions, the equation for the radial part of the wave function for $\xi < 1$ is

$$R'' + \frac{1}{\xi}R' + \left(\kappa^2 - \left(\frac{l}{\xi} + f\xi\right)^2\right)R = 0.$$  

(16)

And for $\xi > 1$ we have

$$R'' + \frac{1}{\xi}R' + \left(\kappa^2 - \frac{(l + f)^2}{\xi^2}\right)R = 0.$$  

(17)

Inside the vortex an analytical solution to the radial equation can be found by the following procedure essentially due to L. Page [10, 11]. First we make the substitutions $\rho = \sqrt{2f}\xi$ and $w = \frac{\kappa^2}{2f}$ (we assume $f > 0$), which results in the equation

$$R'' + \frac{1}{\rho}R' + \left(w - \left(\frac{l + \rho}{\rho}\right)^2\right)R = 0,$$

(18)

for $0 < \rho < \sqrt{2f}$. Next we write the radial function as

$$R_i(\rho) = \rho^m e^{-\rho^2/4}V_i(\rho),$$

(19)

with $m = |l|$, and insert this into Eqn. [18]. Hereby we get an equation for $V_i$
\[ V''_l + \left( \frac{2m+1}{\rho} - \rho \right) V'_l + (w - l - m - 1)V_l = 0. \]  

(20)

This equation can be further simplified by making the substitution \( x = \rho^2/2 \). Finally we have the equation

\[ xV''_l + (m + 1 - x)V'_l - \frac{1}{2}(m + l + 1 - w)V_l = 0. \]

(21)

This differential equation belongs to a class of equations known as Kummer’s equation. Kummer’s equation is a member of an even bigger class of equations of the form \( \sum_{p=0}^{n}(a_p + b_p x) \frac{d^p y}{dx^p} = 0 \), which can all be solved by Laplace’s method [12]. Kummer’s equation is solved by the confluent hypergeometric functions \( M \) and \( U \) (in the notation of Abramowitz and Stegun [13]). The complete solution to Eqn. 21 can be written

\[ V_l(x) = c_1 M\left(\frac{1}{2}(l + m + 1 - w), m + 1, x\right) + c_2 U\left(\frac{1}{2}(l + m + 1 - w), m + 1, x\right). \]

(22)

It turn out that in order that \( R_l(\xi) \) be regular as \( \xi \to 0 \), we must take \( c_2 = 0 \), and we can therefore write down the solution to the Schrödinger equation inside the vortex

\[ \phi_{\kappa l}(\xi, \theta) = C_1 \xi^{|l|} e^{-\frac{1}{2}f\xi^2} M\left(\frac{1}{2}(l + |l| + 1 - \frac{\kappa^2}{2f}), |l| + 1, f\xi^2\right) e^{i\theta}. \]

(23)

Here \( C_1 \) is a normalization constant which we will not need to evaluate.

Outside the vortex the radial equation is just the differential equation for ordinary Bessel functions of the first kind. We therefore immediately have for \( \xi > 1 \)

\[ \phi_{\kappa l}(\xi, \theta) = (A_l J_{l+f}(\kappa \xi) + B_l Y_{l+f}(\kappa \xi)) e^{i\theta}. \]

(24)

The two constants \( A_l, B_l \) are found from the requirement, that the wavefunction has to be continuously differentiable at the boundary of the vortex. There is no need to normalize the wave functions \( \phi_{\kappa l} \), as the normalization constant will drop out of the final expression.

In quantum scattering theory one seeks a particular eigenstate of the Hamiltonian which belongs to the continuous part of the spectrum, and which far away from the scattering center has a direction in which it represents an incoming flow of particle current. Let us consider an eigenstate corresponding to the energy \( E_k = \frac{\hbar^2 k^2}{2m} \)
We want $\psi_k$ to represent the scattering of particles which are incident along the x-axis, as indicated in Fig. 3. The asymptotic boundary condition on $\psi_k$ is therefore that it far to the left of the origin represents an uniform current of incoming particles.

The vector potential gives a contribution to the current $j$, so that the plane wave in the direction of the x-axis is altered from the field free form $e^{ikr\cos \theta}$. The particle current density is given by

$$j = \frac{\hbar}{2mi} \left\{ \Psi^\dagger \left[ (\nabla + ie\frac{\hbar}{\hbar}A)\Psi \right] - \left[ (\nabla + ie\frac{\hbar}{\hbar}A)\Psi \right]^\dagger \Psi \right\}, \quad (26)$$

and it is straightforward to check that the correct form for a state with uniform current in the direction of the x-axis is $e^{ikr\cos \theta - if\theta}$. If the flux through the vortex is not an integer number of flux quanta, i.e. if the fraction $f$ is not integer, then the factor $e^{-if\theta}$ is not single valued as it stands, and we have to introduce a cut to make it so. We only enforce the asymptotic boundary condition along the negative x-axis, so the cut can be placed anywhere outside this region. Let us for a moment introduce the principal angle $[\theta]$, defined by $[\theta] = \theta$ when $c < \theta < c + 2\pi$, and otherwise given by periodicity. Then the single valued factor $e^{if[\theta]}$ has a cut along the half line $\theta = c$. We want to express the plane wave, as a sum over partial waves, and therefore we consider the inner product with $e^{ikr\sin \theta}$, $l$ integer

$$\int_{\frac{c}{2} + \frac{\pi}{2}}^{\frac{c}{2} + 2\pi} d\theta e^{ikr\cos \theta - if[\theta]-il\theta} = \int_{c}^{c+2\pi} d\theta e^{ikr\cos \theta - i(l+f)\theta} = e^{i\frac{\pi}{2}(l+f)} \int_{c}^{c+\frac{\pi}{2}} d\theta e^{ikr\sin \theta - i(l+f)\theta} = i^{(l+f)} J_{l+f}(kr), \quad (27)$$

where the last equality sign holds if $c = -\frac{3\pi}{2}$. This means that the cut is placed along the positive y-axis, as shown in Fig. 3. The function $J_\nu(z)$ is known as Anger’s function [14], and coincide with Bessel’s $J_\nu(z)$ when $\nu$ is an integer. For $\nu$ not an integer the Anger function has the nice property that is goes asymptotically as the Bessel function for large arguments, i.e.

$$J_\nu(z) = J_\nu(z) + \frac{\sin \frac{\pi \nu}{z}}{\pi z} \left[ 1 - \frac{\nu}{z} + O(|z|^{-2}) \right], \quad (28)$$
for $z \to \infty$. Let us remark that if the cut is placed somewhere else the integral will not give Anger’s function, but asymptotically it will still go as some combination of Bessel functions.

Let us now subtract the plane wave from $\psi_k$

$$\psi_k(r, \theta) - e^{ikr \cos \theta - if\theta} = -i^{l+f} J_{l+f}(\kappa \xi) + b_l A_l J_{l+f}(\kappa \xi) + b_l B_l Y_{l+f}(\kappa \xi). \tag{29}$$

In the asymptotic region the involved functions can be expanded to give

$$\psi_k(r, \theta) - e^{ikr \cos \theta - if\theta} = \frac{1}{\sqrt{2\pi \kappa \xi}} \left\{ [b_l A_l - i^{l+f} - ib_l B_l] e^{i\kappa \xi - i(l+f)\frac{\pi}{2} - i\frac{\pi}{4}} + [b_l A_l - i^{l+f} + ib_l B_l] e^{-i\kappa \xi + i(l+f)\frac{\pi}{2} + i\frac{\pi}{4}} \right\}. \tag{30}$$

This combination of terms are the sum of an incoming and an outgoing circular wave. The coefficient multiplying the incoming wave must vanish, as all the ingoing current should be represented by the plane wave. We therefore have the condition

$$b_l A_l - i^{l+f} + ib_l B_l = 0. \tag{31}$$

The radial differential equations are real and linear, and therefore $A_l$ and $B_l$ are real numbers by construction. The phase shifts $\delta_l$ are defined by the relation

$$\delta_l = \arctan \frac{B_l}{A_l}. \tag{32}$$

We note that the phase shifts are independent of the arbitrary normalization of the wave functions. If we write $A_l = C_l \cos \delta_l$ and $B_l = C_l \sin \delta_l$, then Eqn. \[31\] can be solved to give $b_l C_l = i^{l+f} e^{-i\delta_l}$. The outgoing circular wave which represents the scattered current, is given by the expression

$$F_l(\theta) \frac{e^{ikr}}{\sqrt{r}} = \frac{e^{ikr}}{\sqrt{r/R_0}} \sum_l F_l e^{il\theta}. \tag{33}$$

Putting things together, we get the following expression

$$\mathcal{F}_l = -\sqrt{\frac{2}{\pi \kappa}} e^{i\pi - i\delta_l} \sin \delta_l. \tag{34}$$

It is seen that $\mathcal{F}_l$ is a function of $l$ only through the phase shifts $\delta_l$, we can therefore write $\mathcal{F}_l = \mathcal{F} [\delta_l]$. From the above expression we can in principle calculate $\mathcal{F}(\theta)$, and thereby the
differential cross section $\frac{d\sigma}{d\theta} = |F(\theta)|^2$. But this is only practically feasible if the sum over $l$ converges so fast that we can approximate it with a finite sum. The way things are stated above, $F_l$ does not go to zero for $l \to -\infty$, but rather goes to a constant value. The reason for this is that we have not singled out the Aharonov-Bohm contribution to the scattering amplitude $F(\theta)$, which is of a singular nature. We will now pass to the Aharonov-Bohm limit, that is the limit where $R_v \to 0$, while the flux is kept constant. We will then be able to express the effect of the finite radius of the vortex, as the difference in the scattering amplitude from the Aharonov-Bohm result

$$F(\theta) = \delta F(\theta) + F^{AB}(\theta).$$  \hfill (35)

When treating potential scattering for potentials with finite range, the scattering wave function is directly written as the sum of the incoming plane wave and the outgoing circular wave. The same thing can not be done here, as can be seen by the fact that such a sum must have a cut, while the true wave function can not have any cuts. This is a consequence of the long range of the vector potential which far away from the scattering center falls off as $1/r$. The term $F(\theta) e^{ikr}/\sqrt{r}$ must therefore be interpreted as the largest term in an asymptotic expansion of the part of the wave function carrying the outward particle current \cite{Fetter}. This is not different from the procedure used to calculate cross sections for scattering on long range scalar potentials, i.e. Coulomb.

1. The Aharonov-Bohm Limit

In the limit of vanishing radius of the flux vortex $R_v = 0$, we are left with only one dimensionfull variable $k$, and therefore it is impossible to express the scattering amplitude in dimensionless form. When $R_v \to 0$ the eigenstates of the Hamiltonian are everywhere given by the expression

$$\phi_{kl}(r, \theta) = (A_l J_{l+\frac{1}{2}}(kr) + B_l Y_{l+\frac{1}{2}}(kr)) e^{il\theta}.$$  \hfill (36)
The physical demand that we have to impose on the solutions $\phi_{kl}$ is that of boundedness at the origin. Let us put $\nu = l + f$, the order of the Bessel functions. The properties of $J_\nu(z)$ and $Y_\nu(z)$ for $z \to 0$, we infer from the relations

$$J_\nu(z) = \frac{(1/2)z^\nu}{\nu} \sum_{k=0}^{\infty} \frac{(-1/4z^2)^k}{k! \Gamma(\nu + k + 1)} \quad (37)$$

$$Y_\nu(z) = \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi}, \quad (38)$$

valid for all values of $\nu$ and $z$. For $\nu > 0$ $J_\nu$ is regular, and $Y_\nu$ is irregular, and we therefore have to take $A_l = 1$, $B_l = 0$ for $l + f > 0$. For $\nu < 0$ the condition for regularity is seen to be $B_l/A_l = -\tan \pi\nu$. In order to select one of the two solutions of $\tan \delta_{lAB} = -\tan \pi(l + f) = -\tan \pi f$, we impose the additional condition that the radial wave function shall be positive around $r = 0$. Which translates into $(-1)^l \sin \delta_{lAB} \sin \pi f < 0$. All in all this means that the phase shifts in the Aharonov-Bohm limit are

$$\delta_{lAB} = \begin{cases} 0 & \text{for } l + f > 0 \\ -\pi(l + f) & \text{for } l + f < 0 \end{cases} \quad (39)$$

modulo $2\pi$. We note that it is only the fractional part of $f$ which play a role in this limit, any integer part may be absorbed in a redefinition of $l$. We also note that the phase shifts are independent of the electron wavelength. If we take $0 < f < 1$, the calculation of the scattering amplitude look like this

$$F^{AB}(\theta) = \sqrt{\frac{2}{\pi k}} e^{i\frac{\pi}{4} + i\pi f} \sin \pi f \sum_{l=1}^{\infty} e^{-il\theta} = \frac{1}{\sqrt{2\pi k}} e^{-i\frac{\pi}{4} + i\pi f - i\frac{\theta}{2}} \sin \frac{\pi f}{2} \sin \frac{\theta}{2}. \quad (40)$$

The differential cross section for scattering on an infinitely thin vortex carrying a magnetic flux $f\phi_0$ is therefore

$$\left[ \frac{d\sigma}{d\theta} \right]_{AB} = \frac{1}{2\pi k} \frac{\sin^2 \pi f}{\sin^2 \theta/2}. \quad (41)$$

A result first derived by Y. Aharonov and D. Bohm in 1959, by a slightly different approach [15]. The Aharonov-Bohm cross section corresponding to $f = 1/2$ is plotted in Fig. 4 for reference. The AB cross section is completely symmetric under reflection $\theta \mapsto -\theta$, for
all values of $f$, and can not give rise to a net transverse force on the electrons. On the other hand in analogy with ordinary impurities, it can give rise to a finite lifetime of the electron states, and thereby give a contribution to the longitudinal resistance of the system. We note that the cross section is periodic in the flux, with period equal to the flux quantum, and that it completely vanish when the flux is equal to an integer number of flux quanta. This fact, that the scattering amplitude $F^{AB}(\theta)$ disappers when the stringlike vortex contains an integer number of flux quanta, has useful consequences.

The AB cross section is non integrable because of the singularity at $\theta = 0$, and we can therefore not calculate $\sigma_{tot}$. This is due to the long range nature of the interaction, i.e. the vector potential, which fall of only as $1/r$. This situation is not different from scattering on an ordinary scalar potential with long range, such as the Coulomb potential.

2. Phase Shifts for Scattering on Vortex with Finite Radius

In this section we will briefly describe how the phase shifts for scattering on a cylindrical vortex with finite radius are calculated. The equation from which the phase shifts are derived, is simply the equation which results from the demand that the logarithmic derivative of the radial wave function must be continuous at the boundary of the vortex

$$\frac{1}{R_i^c} \left. \frac{dR_i^c}{d\xi} \right|_{\xi=1} = \frac{1}{R_i^c} \left. \frac{dR_i^c}{d\xi} \right|_{\xi=1}. \quad (42)$$

Inside the vortex we have

$$E_i \equiv \left. \frac{1}{R_i^c} \frac{dR_i^c}{d\xi} \right|_{\xi=1} = |l| - f + 2f \frac{a_l M(a_l + 1, b_l + 1, f)}{b_l M(a_l, b_l, f)}, \quad (43)$$

where we have defined the parameters $a_l = \frac{1}{2}(|l| + |l| + 1 - \kappa^2/2l)$, and $b_l = |l| + 1$. Outside the vortex the logarithmic derivative reads

$$\left. \frac{1}{R_i^c} \frac{dR_i^c}{d\xi} \right|_{\xi=1} = \frac{\gamma_l + \gamma_l \tan \delta_l}{J_{l+f}(\kappa) + Y_{l+f}(\kappa) \tan \delta_l}, \quad (44)$$

where we have introduced the abbreviations $\gamma_l = \kappa J_{l+f}(-1) - (l+f) J_{l+f}(\kappa)$ and $\gamma_l = \kappa Y_{l+f}(-1) - (l+f) Y_{l+f}(\kappa)$. It is now simple to solve for $\delta_l$. 

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\[
\tan \delta_l = \frac{j_l - E_l J_{l+f}(\kappa)}{E_l Y_{l+f}(\kappa) - y_l}.
\]  

(45)

The \(\tan \delta_l\)'s are the bricks from which cross sections and transport coefficients can be build up. The presented curves and cross sections have all been calculated with phase shifts found by this expression with the help of Mathematica, which have implementations of all the involved special functions.

3. Asymmetric Scattering on Vortex with Finite Radius

For a vortex with finite radius, we want to express the scattering amplitude \(F(\theta)\) exclusively in terms of the dimensionless parameters \(\kappa\) and \(f\). The procedure for calculating the scattering amplitude, for scattering on a vortex with finite radius, is as follows. First the \(\delta F_l\)'s are found from

\[
\delta F_l = F[\delta_{kl}] - F[\delta_{l}^{AB}],
\]

(46)

where \(\delta_{kl}\) is the phase shift calculated numerically by the formulas given in the above section, and \(F[\cdot]\) is given by Eqn. [34]. For small values of \(\kappa\) the \(\delta F_l\)'s vanishes rapidly when \(|l|\) increases, making it possible to approximate the sum \(\sum_l \delta F_l e^{i\theta}\) well by a finite number of terms. We have found that for \(\kappa < 10\), of order 20 terms are needed at most. The scattering amplitude is then simply given by

\[
F(\theta) = \sum_{l=1}^{l_{max}} \delta F_l e^{i\theta} + F^{AB}(\theta),
\]

(47)

where the dimensionless AB cross section is given by \(F^{AB}(\theta) = \sum_l F[\delta_{l}^{AB}] e^{i\theta}\). From which we get the dimensionless cross section

\[
\frac{d\varsigma}{d\theta} = |F(\theta)|^2.
\]

(48)

Plots of differential cross sections calculated by this procedure is shown in Fig. [3]. We note that the degree of asymmetry is determined by the size of the parameter \(\kappa = kR_v\). The classical limit is approached when \(\kappa \gg 1\). The quantum cross section, unlike the classical
one, gives a finite probability of scattering to both sides of the vortex for all parameter values, but not necessarily in all directions.

C. Conductivity of 2DEG in Vortex Field

In this section we will make an estimate of the contribution to the resistance of a 2DEG, from a random distribution of flux vortices. The experimental situation we have in mind, is that of Geim et. al. [2,3], who placed a thin film of lead on top of a GaAs/GaAlAs heterostructure. When the temperature is lowered below the critical temperature, and the magnetic field is below $H_{c2}$, the magnetic field penetrates the superconductor in the form of Abrikosov vortices. At the 2DEG the field will be confined to regions of radius (or exponential length) $\lambda_s$, each threaded by a magnetic flux $\phi_0/2$. The vortices are assumed to be distributed randomly, with an average separation $d$ given by the strength of the external $B$-field according to the relationship $d^2B_{\text{ext}} = \phi_0/2$.

The Boltzmann equation, linearized in the external electric field, reads

$$-e\mathbf{v} \cdot \mathbf{E} \frac{\partial f^0}{\partial \epsilon} = \int \frac{d^2q}{(2\pi)^2} \left\{ f_{p+q}w_{p+q\rightarrow p} - f_p w_{p\rightarrow p+q} \right\} - \frac{f_p - f^0}{\tau_{\text{imp}}}. \quad (49)$$

Here the transition probabilities $w_{p\rightarrow p+q}$ are potentially asymmetrical quantities, due to the time reversal breaking magnetic field in the vortices. As argued by B. I. Sturman [16], the correct form of the collision integral, even in the absence of detailed balance, is the one given in Eqn. [19]. The electron-vortex scattering will be elastic. In order to solve the Boltzmann equation we Fourier transform, and write

$$f(k, \theta) = \sum_{n=-\infty}^{\infty} e^{-in\theta} f_n(k) \quad (50)$$

$$w(k, \theta) = \sum_{n=-\infty}^{\infty} e^{-in\theta} w_n(k), \quad (51)$$

where $\theta$ is the angle between $\mathbf{k}$ and $\mathbf{E}$. The electron-vortex collision integral is diagonal when Fourier transformed, and we get the following equation for the $n$’th component of the distribution function
\[- evE \frac{\partial f^0}{\partial e} \left\{ \delta_{n,1} + \delta_{n,-1} \right\} = - n_e \{ w_0 - w_n \} f_n - \frac{f_n - f^0 \delta_{n,0}}{\tau_{imp}}. \tag{52} \]

The current is given by

\[
\dot{j} = -2e \int \frac{d^2k}{(2\pi)^2} v f(k, \theta) = - \frac{e^2 \epsilon_F E}{\pi \hbar^2 n_e} \left\{ \begin{array}{c}
\text{Re} \left[ \frac{1}{w_1 - w_0} \right] \\
\text{Im} \left[ \frac{1}{w_1 - w_0} \right]
\end{array} \right\}, \tag{53} \]

from which the conductivities can be read off. The resistivities are found by inverting the conductivity tensor.

1. Longitudinal Resistivity

The longitudinal resistivity, obtained by the above procedure, is

\[
\rho_{xx} = \frac{m}{ne^2} \left( \frac{1}{\tau_v} + \frac{1}{\tau_{imp}} \right), \tag{54} \]

where the transport scattering time for the electrons, due to scattering on the vortices, is

\[
\frac{1}{\tau_v} = n_e v_F \int_{-\pi}^{\pi} d\theta (1 - \cos \theta) |F(\theta)|^2. \tag{55} \]

Let us introduce a dimensionless quantity \( \zeta \), by writing the contribution to the longitudinal resistivity, from the electron-vortex scattering, as

\[
\rho^v_{xx} = \frac{\hbar}{e^2} \frac{n_v}{n} \zeta. \tag{56} \]

The dimensionless parameter \( \zeta \), which is explicitly given by the expression

\[
\zeta = \kappa \int_{-\pi}^{\pi} d\theta (1 - \cos \theta) \frac{d\zeta}{d\theta}, \tag{57} \]

is characterizing the efficiency of a single vortex, to scatter the electrons from the front to the back of the Fermi circle. It is straightforward to do the integral and obtain the following sum

\[
\zeta = 4 \sum_{l=-\infty}^{\infty} \frac{t_l(t_l - t_{l+1})}{(1 + t_l^2)(1 + t_{l+1}^2)}. \tag{58} \]
where \( t_i = \tan \delta_i \) is found from Eqn. 53.

In the Aharonov-Bohm limit we have an analytical expression for the cross section. It turns out that in this limit the integrand in Eqn. 55 is constant. We have in this limit
\[
\zeta = 2 \sin^2 \pi f,
\]
or in other words
\[
\frac{1}{\tau_{AB}} = n_v v_F \frac{\sin^2 \pi f}{\pi k} = \omega_c \frac{\sin^2 \pi f}{\pi f}, \tag{59}
\]
where \( \omega_c \) is the cyclotron frequency corresponding to the flux density of the external magnetic field. We note that \( \zeta \) vanishes in the Aharonov-Bohm limit, when the flux fraction \( f \) is integer, as it should. We can use this expression to make an estimate of the relative resistance change due to the vortices \( \Delta \rho_{xx}/\rho_{xx} = 2 l_f n_v \sin^2 \pi f/k_F \). With \( f = 1/2 \), \( l_f = 5 \mu m \), \( n = 10^{11} \text{cm}^{-2} \) and \( B = 100G \), we get \( \Delta \rho_{xx}/\rho_{xx} = 0.6 \), which is a significantly higher value than is observed experimentally. Let us now include the effects of the finite radius of the vortices. Fig. 6 shows several \( \zeta(\kappa) \) curves for different values of the flux fraction \( f \). It is seen that in general broader vortices give lower \( \zeta \), i.e. less resistance. The curve \( f = 1/2 \) corresponds to the physical Abrikosov vortices, if the difference in the cross sectional shape is ignored. In the very low field limit, that is less than 100G, the observed increase in resistivity is linear in the applied field, and the relative change at \( B = 100G \) is in the range \( 10^{-2} - 10^{-3} \), much less than the present estimate gives.

The theory we have outlined here is only valid when the vortices are well separated, this amounts to assuming \( d \gg R_v \). The crossover to a different kind of behaviour observed in experiment, which is seen in both \( \rho_{xx} \) and \( \rho_{xy} \), appear about 100G, and we believe that this is the flux density where the broad Abrikosov vortices begin to interfere.

2. Transverse Resistivity

The Hall resistivity, obtained from Eqn. 53, we can write as
\[
\rho_{xy} = \alpha \frac{B}{nc}, \tag{60}
\]
where \( B \) is the externally applied homogeneous magnetic field. Here \( B/\text{ne} \) is the Hall resistivity of a 2DEG in a homogeneous magnetic field, and \( \alpha \) is a dimensionless number, which describe the effects of the field being inhomogeneous. The dimensionless quantity \( \alpha \) is given by the expression

\[
\alpha = \frac{k_F}{2\pi f} \int_{-\pi}^{\pi} d\theta \sin \theta |F(\theta)|^2 = \frac{\kappa}{2\pi f} \int_{-\pi}^{\pi} d\theta \sin \theta \frac{dc}{d\theta}.
\]  

(61)

Again, the integral can be expressed as a sum over terms involving only the parameters \( t_l = \tan \delta_l \), which are given by Eqn. 45

\[
\alpha = 2\pi f \sum_{l=-\infty}^{\infty} \frac{t_l t_{l+1} (t_{l+1} - t_l)}{(1 + t_l^2)(1 + t_{l+1}^2)}.
\]  

(62)

Curves showing \( \alpha \) as a function of \( \kappa \) for different values of the flux fraction \( f \), have been plotted in Fig. 8. The classical limit is realized when \( R_v/\lambda_F \gg 1 \) together with \( f \gg 1 \), and we have \( \kappa = 2\pi R_v/\lambda_F \) so none of the curves shown reach the classical regime. The number of terms, which must be included in the sum over \( l \), grows rapidly with increasing \( \kappa \) and \( f \), thus making it difficult to reach the true classical regime by this technique.

The \( \alpha \)-curve for \( f = 1/2 \) we can compare with the experimental Hall factor measured by Geim et al. [2]. The overall qualitative behaviour is in good agreement, when the very idealized shape of the vortices, we have used in our calculation, is taken into account. To make a quantitative test we have fitted our curve to the experimental curve by tuning the radius of the vortex \( R_v \). The best fit is obtained for a vortex radius \( R_v = 30\text{nm} \), and this is nearly an order of magnitude smaller than the exponential length estimated by Geim to be 100nm. This may indicate that most of the flux in the vortices are concentrated in a narrow core.

It is seen in Fig. 8 that the \( \alpha \) curves corresponding respectively to \( f = 1/4 \), and \( f = 3/4 \) does not seem to converge to the level \( \alpha = 1 \) as one would expect from the classical calculation. This is another manifestation of the Aharonov-Bohm phenomenon. Consider the ordinary double slit experiment with an infinitely thin solenoid hidden behind the middle obstacle. The interference pattern which can be observed behind the arrangement when it is
hit by an incident plane wave, is symmetric when the flux through the AB-solenoid is equal to zero, modulo the flux quantum. In this case the interference pattern will have a local maxima right in the middle. When the flux is equal to half a flux quantum, modulo the flux quantum, the interference pattern will again be symmetric, but this time with a node in the middle. In both the cases \( \phi = \phi_0 \) and \( \phi = \phi_0/2 \) there are no net scattering of electrons to either side. But for general fluxes \( f\phi_0 \), with \( 2f \) not equal to an integer, the interference pattern is not symmetric, and this is the reason that the \( f = 1/4 \) and \( f = 3/4 \) \( \alpha \)-curves does not converge to the “classical” level at \( \alpha = 1 \). When \( \kappa \gg 1 \) and the scattering inside the vortex has become classically behaived, the vortex still plays the role of an obstacle that gives rise to an interference pattern, and the vector potential outside the vortex deflects the pattern and thereby gives rise to the asymmetry we observe in Fig. 8. We emphasize that in the limit where the diameter of the vortex (the obstacle) goes to zero, there is no net scattering to either side.

D. Multi Flux Quantum Vortex and Resonance Scattering

When the total amount of magnetic flux inside the vortex is increased, the \( \alpha \) and \( \zeta \) spectra acquire more structure. In Fig. 9 we have shown \( \alpha \) and \( \zeta \) curves for a flux vortex carrying a total of 10 flux quanta. The structure seen in the plots is an effect of the resonant scattering which takes place when the energy of the incoming particle is close to one of the Landau quantization energies corresponding to the magnetic field strength inside the vortex. The magnetic field in the vortex vanishes outside a finite range – the radius of the vortex – and there are therefore no real Landau levels in the sense of stationary eigenstates, but only metastable states. In the dimensionless units we are working with, the Landau quantization energies \( E_p = \hbar \omega_c (p + 1/2) \) corresponds to

\[
\kappa_p = 2\sqrt{|f|(p + \frac{1}{2})}, \quad p = 0, 1, 2, \ldots
\]  

These values are in excellent agreement with the resonances seen in Fig. 9, where the first eight resonances corresponding to \( p = 0, \ldots, 7 \), are clearly distinguished. At the resonance
energies the typical time the particle spends in the scattering region, i.e. inside the vortex, is much longer than it is away from the resonance. The time the particle spends inside the vortex at a resonance, can be thought of as the lifetime of the corresponding metastable state. The inverse lifetime is proportional to the width of the resonance, that is strictly speaking the width of the peak in the partial wave cross section $\sigma_l$, corresponding to the $l$ quantum number of the metastable Landau state.

It is easy to interpret the small peaks in the $\zeta$-curve in Fig. 9, appearing at the resonance energies. Because when the electron spends longer time in the scattering region, it loses knowledge of where it came from, resulting in an enhanced probability of being scattered in the backwards direction. The $\alpha$-curve is a measure of asymmetric scattering, and we can therefore interpret the dips seen in Fig. 9 along the same line of reasoning as for the $\zeta$-peaks. The electron spends longer time in the scattering region, thereby losing knowledge of what is left and what is right. To explain why the $\alpha$-curve is asymmetric in $\kappa$ around the resonances one could look at it this way: For increasing $\kappa$ the electron scattering becomes more and more classical, giving rise to the overall increasing background in $\alpha$. But every time a new scattering channel is opened, the asymmetry of the scattering is suppressed, due to the lack of knowledge effect, thus resulting in a sawtooth like curve.

III. HALL EFFECT IN A REGULAR ARRAY OF FLUX VORTICES

A. Introduction

Recently measurements were made by Geim et al. of the Hall resistivity of low density 2DEG’s in a random distribution of flux vortices, at very low magnetic field strengths. A profound suppression of the Hall resistivity was found, for 2DEG’s with Fermi wavelengths of the same order of magnitude as the diameter of the flux vortices. This indicates that we are dealing with a phenomenon of quantum nature. These measurements were made by placing a thin lead film on top of a GaAs/GaAlAs heterostructure. When a perpendicular
magnetic field is applied, the magnetic field penetrates the superconducting lead film and also the heterostructure in the form of flux vortices each carrying half a flux quantum $\phi_0/2$ of magnetic flux. Due to the strong flux vortex pinning in the films Geim have used, the vortices were positioned in a random configuration.

In this section we will consider the hypothetical experiment where one instead of a “dirty” film, places a perfectly homogeneous type II superconducting film, on top of the 2DEG. The film do not have to be made of a material which is type II superconducting in bulk form. A film of a type I superconducting material will also display a mixed state if the thickness of the film is below the critical thickness $d_c$. Experimentally perfect Abrikosov flux vortex lattices have been observed in thin films of lead with thickness $d < d_c \approx 0.1 \mu m$, [17].

If one succeeds to make such a sandwich construction, one has an ideal system for investigating how a 2 dimensional electron gas behaves in a periodic magnetic field. When the magnetic field exceeds $H_{c1}$, which can be extremely low, the superconducting film will enter the mixed phase, and form an Abrikosov lattice of flux vortices. The Abrikosov lattice in the superconductor will give rise to a periodic magnetic field at the 2DEG, and moreover as the strength of the applied magnetic field is varied the only difference at the 2DEG, is that the lattice constant of the periodic magnetic field varies. The Abrikosov lattice is most often a triangular lattice with hexagonal symmetry, although other lattices have been observed (e.g. square) in special cases where the atomic lattice structure impose a symmetry on the flux lattice, [17]. For simplicity the model calculations which we have done, were made for a system with a square lattice of flux vortices, but we do not expect this to influence the overall features of the results. From the point of view of the 2DEG it is important that the flux vortices carry half a flux quantum $\frac{\phi_0}{2} = \frac{\hbar}{2e}$ due to the $2e$-charge of the Cooper pairs in the superconductor. The magnetic field from a single flux vortex fall of exponentially with the distance from the center of the vortex. This exponential decay is characterised by a length $\lambda_s$, which essentially is the London length of the superconductor, proportional to one over the square root of the density of Cooper pairs. The length $\lambda_s$ can be varied by changing the temperature, or the material of the superconductor.
The other characteristic lengths of the system are the Fermi wavelength \( \lambda_F = \sqrt{\frac{2\pi}{n}} \), where \( n \) is the density of the 2DEG, the lattice constant \( a \) of the periodic magnetic field, and the mean free path \( l_f = v_F \tau \). The mean free path we assume to be very large compared to \( a \) and \( \lambda_s \). The magnetic field is only varying appreciably when \( a \) is larger than \( \lambda_s \). This means that the magnetic flux density of the applied field should be appreciably less than \( \phi_0/(\pi \lambda_s^2) \), which typically is of order 1000Gauss. In the limit where \( \lambda_s \ll a, \lambda_F \), the vortices can be considered magnetic strings, and the electrons experiences a periodic array of Aharonov-Bohm scatterers. In this case the value of the flux through each vortex is crucial. If for instance the flux had been one flux quantum \( \phi_0 = \frac{h}{e} \), the electrons would not have been able to feel the vortices at all. But in the real world the vortices from the superconductor carry \( \frac{\phi_0}{2} = \frac{h}{2e} \) of flux, and therefore this limit is nontrivial. The electrons has for instance a band structure quite different from that of free electrons. In the mathematical limit of infinitely thin vortices each carrying half a flux quantum, there cannot be any Hall effect. This is most easily seen by subtracting one flux quantum from each vortex to obtain a flux equal to minus half a flux quantum through each vortex. As we have discussed earlier the introduction of the Dirac strings can not change any physics, and the procedure therefore shows that the system is equivalent to it’s time reversed counterpart, thereby eliminating the possibility of a Hall effect.

In this study we have ignored the electron spin throughout, in order to keep the model simple. From the point of view of the phenomena we are going to describe, the effect of the electron spin will be to add various small corrections.

B. Electrons in a periodic magnetic field

1. Magnetic translations

It is a general result for a charged particle in a spatially periodic magnetic field \( B(x, y) \), that the eigenstates of the system can be labeled by Bloch vectors taken from a Brillouin
zone, if and only if the flux through the unit cell of the magnetic field is a rational number $p/q$ times the flux quantum. The standard argument for this fact is made by introducing magnetic translation operators. To introduce magnetic translation operators in an inhomogeneous magnetic field we first make the following observation. The periodicity of the field can be stated $B(r + R) = B(r)$, for $R$ belonging to a Bravais lattice. But this implies that the difference between the vector potentials $A(r + R)$ and $A(r)$ must be a gauge transformation

$$\nabla \times \{A(r + R) - A(r)\} = B(r + R) - B(r) = 0.$$  \hspace{1cm} (64)

We introduce the gauge potential $\chi_R$ and write

$$A(r + R) = A(r) + \nabla \chi_R(r).$$ \hspace{1cm} (65)

The function $\chi_R$ is only defined modulo an arbitrary additive constant which have no physical effect. The Hamiltonian of the electrons is

$$H = \frac{1}{2m} (p + eA(r))^2.$$ \hspace{1cm} (66)

The ordinary translation operators $T_R = \exp[I\hbar R \cdot p]$ do not commute with the Hamiltonian, because they shift the argument of the vector potential from $r$ to $r + R$, but as we just have seen this can be undone with a gauge transformation. We therefore introduce the magnetic translation operators as the combined symmetry operation of an ordinary translation and a gauge transformation

$$M_R = \exp[-ie\hbar \chi_R(r)] \exp[i\hbar R \cdot p].$$ \hspace{1cm} (67)

The operator $M_R$ is unitary, as it is the product of two unitary operators, and therefore has eigenvalues of the form $e^{i\lambda}$. Let us denote the primitive vectors of the Bravais lattice $a$ and $b$. We can find common eigenstates of $M_a$, $M_b$ and $H$, if and only if they all commute with each other. The magnetic translations each commute with the Hamiltonian by construction, and furthermore we have
If the flux $\phi$ through the unit cell is a rational number $p/q$ ($p$ and $q$ relatively prime) times the flux quantum $\phi_0$, $M_\alpha$ and $M_\beta$ commute. In this case the cell spanned by $qa$ and $b$ is called the magnetic unit cell. Let us define $c = qa$. The possible eigenvalues of $M_c$ are phases $e^{2\pi i k_1}$, where we can restrict $|k_1| < 1/2$, and equivalently for $M_b$. We can therefore label the common eigenstates $|k, n\rangle$, where $k = k_1c^* + k_2b^*$, and $c^*, b^*$ are the primitive vectors of the reciprocal lattice. The vector $k$ is restricted to the magnetic Brillouin zone. An arbitrary magnetic translation of an eigenstate with a Bravais lattice vector $R = nc + mb$ can now be written $M_R|k, n\rangle = (M_c)^n(M_b)^m|k, n\rangle = \exp[ik \cdot R]|k, n\rangle$, showing that the eigenstate is a Bloch state. In this case we can speak of energy bands forming a band structure in the usual sense. When the flux through the elementary unit cell $(a, b)$ is an irrational number of flux quanta, the situation is different. The irrational number can be reached as the limit where $p$ and $q$ get very large, and consequently the Brillouin zone get very small and collapses in the limit.

2. The Dirac vortex viewpoint

In this section we will show, how it is possible to argue in a slightly different way from the previous section, and hereby in a simpler way obtain the vector potential of a periodic magnetic field. Let us again assume a rectangular unit cell $(a, b)$, $B(x + a, y) = B(x, y + b) = B(x, y)$ etc., to keep the notation simple. The magnetic field enter the Hamiltonian only through the vector potential. The question is therefore if one can choose a gauge such that the vector potential will be translationnally invariant relative to a unit cell $(c, d)$ $A(x + c, y) = A(x, y + d) = A(x, y)$ etc. It is clear that if such a periodic $A$-field exists, then the total flux $\Phi_{cd}$ through the unit cell $(c, d)$ will be zero, as it is given by the line integral of $A$ around the boundary of the unit cell $(c, d)$

$$\Phi_{cd} = \oint_{\partial(c,d)} A \cdot dl,$$

(69)
which is zero by the periodicity. We remark that due to the relation $B = \nabla \times A$, the cell $(c, d)$ will be bigger than or equal to $(a, b)$. If the flux through the unit cell of the magnetic field is not zero, but equal to a rational number times the flux quantum, $\Phi_{ab} = \frac{p}{q} \phi_0$ ($p$ and $q$ relatively prime), a trick can be applied to make the flux $\Phi_{cd}$ become zero. It is a basic fact, apparently first observed by Dirac \[18\], that a particle with charge $e$ cannot feel an infinitely thin solenoid carrying a flux equal to an integer multiple of the flux quantum $\phi_0 = \frac{h}{e}$. Such a stringlike object carrying one flux quantum is sometimes called a Dirac vortex. On a lattice the Dirac vortex goes through the center of a plaquette, and the electron can therefore never enter the core of it. The lattice Dirac vortex can be made to disappear by a gauge transformation.

To find the periodic vector potential take an enlarged unit cell $(c, d) = (qa, b)$, so that $\Phi_{cd} = p\phi_0$ and put by hand $p$ counter Dirac vortices through the cell, to obtain zero net flux. Then a divergence free vector potential can be build for instance by Fourier transform

$$B(Q) = \frac{1}{cd} \int_{(c,d)} \frac{d^2r}{2\pi} \exp[-i\mathbf{Q} \cdot \mathbf{r}] B(r),$$

$$A(r) = \sum_{Q \neq 0} \left( \begin{array}{c} iQ_y \\ -iQ_x \end{array} \right) \frac{\exp[i\mathbf{Q} \cdot \mathbf{r}]}{Q^2} B(Q),$$

where the sum is over $Q$ in the reciprocal lattice. Here we have used continuum notation, but it is straightforward to write down the lattice equivalents of the expressions.

C. Lattice calculation of Hall conductivity

We have calculated the Hall conductance of the 2DEG in the vortex field by a numerical lattice method. This we do because the calculations then reduces to linear algebra operations on finite size matrices, which can be implemented in a C++ program on a computer. The idea is to consider an electron moving on a discrete lattice, rather than in continuum space. We know, although we are not going to prove it here, that in the limit where the discrete lattice becomes finegrained compared to all other characteristic length of the system, the
continuum theory is recovered. Here we assume that the original Bravais lattice has square lattice symmetry, with a lattice parameter which we call $a$. The discrete micro lattice is then introduced as a finegrained square lattice inside the unit cell of the Bravais lattice. The lattice parameter of the micro lattice we then take as $a/d$, where $d$ is some large number, in order to keep the two lattices commensurable. The condition that we have to impose on the micro lattice, in order that it is a good approximation to the continuum, can then be stated

$$a/d \ll a, \lambda_F, \lambda_s, \ldots.$$  

(72)

In the numerical calculations we have made, we have taken $d = 10$.

The tight-binding calculations are made with the Hamiltonian

$$H = - \sum_{ij\tau\tau'} t_{i+\tau,j+\tau'} c_{i+\tau}^\dagger c_{j+\tau'}.$$  

(73)

Here $i, j$ are Bravais lattice vectors, and $\tau, \tau'$ are vectors indicating the sites in the basis. The matrix elements $t_{i+\tau,j+\tau'}$ are taken non-zero only between nearest neighbour sites. The matrix element between two nearest neighbour sites $\tau$ and $\tau'$ are complex variables $t_{\tau,\tau+\epsilon_\mu} = te^{iA_\mu(\tau)}$ with a phase given by the vector potential $A_\mu(\tau)$ residing on the link joining the sites. The translation invariance of the Hamiltonian can then be stated $t_{i+\tau+l,j+\tau'+l} = t_{i+\tau,j+\tau'}$, for all vectors $l$ belonging to the Bravais lattice. Let us introduce the system on which our calculations were made as an example. Fig. 10 shows the unit cell with its internal structure i.e. the basis. There are $N = d \cdot d$ sites in the basis. The length of the links we write as $a/d$, where $a$ is the side of the unit cell, with area $\Omega = a^2$. The vectors $\tau = (\tau_1, \tau_2)\frac{a}{d}$, $\tau_1, \tau_2 = 0, 1, \ldots d - 1$ are offsets into the basis, while the vectors $i = (i_1, i_2)a$, $i_1, i_2 \in \mathbb{Z}$ indicate the cells in the Bravais lattice. The operator $c_{i+\tau}^\dagger$, for a given $\tau$, is defined on the Bravais lattice, and accordingly it can be resolved as a Fourier integral over the Brillouin zone as

$$c_{j+\tau}^\dagger = \frac{\Omega}{(2\pi)^2} \int_{BZ} d^2 q e^{-i(q \cdot j + \tau)} c_{q,\tau}^\dagger.$$  

(74)

(It should be noted that the factor $e^{-i\tau \cdot q}$ is arbitrary and included here for later convenience).

Inserting this and using the translation invariance, the Hamiltonian can be rewritten as
\( H = \frac{\Omega}{(2\pi)^2} \int_{BZ} d^2kH_k, \) \tag{75}

where we have introduced

\[ H_k = - \sum_{j,\tau,\tau'} t_{\tau,j+\tau'} e^{ik\cdot(j+\tau'-\tau)} c_{k,\tau'}^\dagger c_{k,\tau} \] \tag{76}

It is seen that \( H_k \) only mixes the \( N \) states \( |k\tau\rangle \), i.e. it is an \( N \times N \) matrix. The \( N \) eigenvalues of \( H_k \) are the energies of the \( N \) tight binding Bloch states with wavevector \( k \). Let us denote the eigenstates of \( H_k \) by \( u^\alpha_k \)

\[ H_k u^\alpha_k = E^\alpha_k u^\alpha_k \] \tag{77}

where \( \alpha = 1, 2, \ldots N \) and \( E^\alpha_k \leq E^{\alpha+1}_k \). From the \( N \) dimensional vector \( u^\alpha_k \) we can construct the eigenstate \( \Psi^\alpha_k \) of the Hamiltonian \( H \)

\[ \langle j+\tau|\Psi^\alpha_k \rangle = e^{ik\cdot(j+\tau)} u^\alpha_k(\tau). \] \tag{78}

It is straightforward to verify that this is the correct Bloch eigenstate of \( H \). The bandstructure can be calculated directly by diagonalising the \( N \times N \) matrices, \( H_k \), for representative choices of \( k \) in the Brillouin zone. Before one can compare the spectrum obtained from this calculation with that of a continuum system, a scaling of the energies is required. To scale the energy to the spectrum of a particle with an effective mass \( m \), we have to take

\[ t = \hbar^2d^2/ma^2, \]  
and

\[ \epsilon^\alpha_k = E^\alpha_k + 4t. \]  

The Hall conductivity can be calculated by the same method as in the homogeneous magnetic field \cite{7,19}. We have used a single particle Kubo formula to calculate the Hall conductance

\[ \sigma_{xy} = \frac{i\hbar}{A_0} \sum_{E^\alpha < E_F < E^\beta} \frac{(J_x\alpha\beta)(J_y)_{\beta\alpha} - (J_y\alpha\beta)(J_x)_{\alpha\beta}}{(E^\alpha - E^\beta)^2}, \] \tag{79}

where \( J_x, J_y \) are the currents in the \( x,y \) directions, and the sum is over single particle states \( |\alpha, k\rangle \) with energies below and above the Fermi level \( E_F \). The area of the system is denoted \( A_0 \). All quantities are diagonal in \( k \), and therefore this index is suppressed. The summation
is composed of a discrete sum over bands, and an integral over the Brillouin zone for each band. The Brillouin zone shown in Fig. 11 is doubly connected because the states on the edges is to be identified according to the translation invariance. This gives the Brillouin zone the topology of a torus $T^2$, with two basic non contractible loops. The current operator can be written
\begin{equation}
J = \frac{\Omega}{(2\pi)^2} \int_{BZ} d^2 k J_k, \tag{80}
\end{equation}
where $J_k = \frac{e}{\hbar} \frac{\partial H}{\partial k}$. By use of some simple manipulations and completeness, it is straightforward to refrase Eqn. 79
\begin{equation}
\sigma_{xy} = \frac{ie^2}{\hbar A_0} \sum_{E < E_F} \left( \left\langle \frac{\partial \alpha}{\partial k_x} \right| \frac{\partial \alpha}{\partial k_y} \right) - \left\langle \frac{\partial \alpha}{\partial k_y} \right| \frac{\partial \alpha}{\partial k_x} \right), \tag{81}
\end{equation}
where $\left| \frac{\partial \alpha}{\partial k_{\mu}} \right|$ is shorthand for $\frac{\partial}{\partial k_{\mu}} |\beta, k\rangle$. This formula was first derived by Thouless, Kohmoto, Nightingale and den Nijs [7], for a noninteracting 2 dimensional electron gas in a periodic scalar potential, and a commensurate perpendicular magnetic field. It requires some comments to be meaningful. In order to calculate $\left| \frac{\partial \alpha}{\partial k_{\mu}} \right|$ it is necessary to consider the difference $(|\alpha, k + \delta k_\mu\rangle - |\alpha, k\rangle)/\delta k_\mu$. But this difference is not well defined as it stands, as the phase of the states is arbitrary. Rather than representing the state $u_k$ by a single vector in $C^N$, it should be represented by a class of vectors which differ only by a phase. These equivalence classes are sometimes called rays. To compare states locally, we need to project this $U(1)$ degree of freedom out. This is done by demanding the wave function to be real, when evaluated in a fixed point, i.e. $u_\alpha^\tau(\tau_i) = \langle \tau_i | \alpha, k \rangle \in R$. If the wave function happens to be zero in $\tau_i$, some other point $\tau_j$ must be used. When a band has a non-zero Hall conductivity, it is not possible to find a single $\tau$ which work for all the states in the Brillouin zone. The change from $\tau_i$ to $\tau_j$ which shifts the phase of the states, is analogous to a gauge transformation on the set of states. The special combination of terms which appear in Eqn. 79 is gauge invariant with respect to these special “gauge transformations”. If we let $|\chi^\alpha\rangle$ denote a state which is obtained from $|\alpha, k\rangle$ by fixing the phase according to the above scheme, the following formula for the contribution to the Hall conductivity from a single band $\alpha$, is well defined.
\[
\sigma_{xy}^\alpha = \frac{e^2}{h} \frac{1}{2\pi i} \int_{BZ} d^2 k \left\{ \left\langle \frac{\partial \chi^\alpha}{\partial k_y} \left| \frac{\partial \chi^\alpha}{\partial k_x} \right\rangle - \left\langle \frac{\partial \chi^\alpha}{\partial k_x} \left| \frac{\partial \chi^\alpha}{\partial k_y} \right\rangle \right\} \right\}. \quad (82)
\]

It has been shown in detail by Kohmoto [19], for the homogeneous magnetic field case, that this expression is equal to minus \( \frac{e^2}{\pi} \) times the first Chern number of a principal fibre bundle over the torus. As the first Chern number is always an integer, this has the physical consequence that whenever the Fermi energy lies in an energy gap, the Hall conductance is quantized. We will use this result to interpret certain peaks in the \( \sigma_{xy} \)-spectra we have calculated. When the Fermi level is not in an energy gap of the system, we will have to use Eqn. 81 to calculate \( \sigma_{xy} \). As we shall see, in this case there is no topological quantization of the Hall conductivity.

**D. Energy band crossing**

In this section we study the effect on the Hall conductivity of an energy band crossing. This has previously been discussed in different contexts by several authors [20–22].

When the shape of the magnetic field is varied, controlled by some outer parameter \( \xi \), it will happen for certain parameter values \( \xi_0 \), that two bands cross, see Fig. 12. This is the consequence of the Wigner-von Neumann theorem, which states that three parameters are required in the Hamiltonian in order to produce a degeneracy not related to symmetry. Here the parameters are \( k_x, k_y \) and the outer parameter \( \xi \), which in our calculation is the exponential length of the flux vortices from the superconductor. When the energy difference \( E^+ - E^- \) between the two bands considered is much smaller than the energy distance to the other bands, the Hamiltonian can be restricted to the subspace spanned by the two states \( |+, k^0\rangle \) and \( |-, k^0\rangle \). The point in the Brillouin zone where the degeneracy occur we denote \( k^0 \). The Hamiltonian \( H(k) \) is diagonal for \( k = k^0 \), and we denote the diagonal elements respectively \( E_0 + \epsilon \) and \( E_0 - \epsilon \). For small deviations of \( k \) from \( k^0 \) the lowest order corrections to the Hamiltonian is offdiagonal elements \( \Delta(k) \) linear in \( k - k^0 \). Without essential loss of generality we can assume that \( \epsilon \) is independent of \( k \). Then \( \epsilon \) plays the role of the outer parameter controlling the band crossing. The Hamiltonian is then approximated by
\[ H(k) = \begin{pmatrix} \epsilon & \Delta^* \\ \Delta & -\epsilon \end{pmatrix} + E_0. \] (83)

The off-diagonal element is expanded as
\[ \Delta(k) = \alpha(k_x - k^0_x) + \beta(k_y - k^0_y) \] (84)
with \( \alpha = \frac{\partial}{\partial k_x} \langle -, k^0 | H_k | +, k^0 \rangle \), and \( \beta = \frac{\partial}{\partial k_y} \langle -, k^0 | H_k | +, k^0 \rangle \).

We want to find the consequences of the energy band degeneracy, on the topological Hall quantum numbers of the bands. Let us define
\[ B_{\pm}(k) = \left\{ \frac{\partial}{\partial k_y} \frac{\partial}{\partial k_x} - \frac{\partial}{\partial k_x} \frac{\partial}{\partial k_y} \right\}. \] (85)
Then the interesting quantities are the integrals of \( B_+(k) \) and \( B_-(k) \), around a small neighbourhood of the degeneracy point \( k^0 \). It turns out that it is the two numbers \( \alpha \) and \( \beta \) that control what happens.

In general \( \alpha \) and \( \beta \) will be nonzero complex numbers — nonzero because we have assumed the degeneracy to be of first order. Let us first consider the degenerate case where \( \alpha \) and \( \beta \) are linearly dependent, i.e. \( \alpha/\beta \) is real, or otherwise stated \( \text{Im}(\alpha^*\beta) = 0 \). Then by a linear transformation we can write the Hamiltonian
\[ h(\kappa) = (\kappa_1 + \kappa_2)\sigma^1 + \gamma \sigma^3 = \begin{pmatrix} \gamma & \kappa_1 + \kappa_2 \\ \kappa_1 + \kappa_2 & -\gamma \end{pmatrix}, \] (86)
where now \( \gamma \) is the dimensionless parameter of the crossing, and \( \kappa_1, \kappa_2 \) are the rescaled dimensionless momentum variables. The \( \sigma^\mu \)’s refers to the Pauli matrices. This Hamiltonian is real, and we can therefore also choose the eigenstates to be real, and this will clearly lead to a vanishing \( B(k) \). In this case we therefore conclude that there is no exchange of topological charge. Here the word topological charge is used to denote Hall quanta.

Let us now treat the general case where \( \alpha \) and \( \beta \) are linearly independent, i.e. \( \text{Im}(\alpha^*\beta) \neq 0 \). In this case, we can by a linear transformation write \( \Delta(k)/E_0 = \kappa_1 + i\kappa_2 = \kappa e^{i\theta} \), which defines the scaled momentum variables \( \kappa, \theta \). (Here \( E_0 \) is some constant with dimension of energy.) This reduces the Hamiltonian to the form
\[ h(\kappa, \theta) = \kappa_1 \sigma^1 + \kappa_2 \sigma^2 + \gamma \sigma^3 = \begin{pmatrix} \gamma & \kappa e^{-i\theta} \\ \kappa e^{i\theta} & -\gamma \end{pmatrix}. \] (87)

Let us define \( \lambda = \sqrt{\gamma^2 + \kappa^2} \). Then the eigenvalues of \( h(\kappa, \theta) \) are \( \pm \lambda \) and the two corresponding eigenstates are

\[ |\pm, \kappa\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm \sqrt{1 \pm \gamma/\lambda} \\ e^{i\theta} \sqrt{1 \mp \gamma/\lambda} \end{pmatrix}. \] (88)

In order to calculate the integral of the \( B \)-function we need to express it in terms of the \( \kappa, \theta \)-variables. The Jacobian of the transformation is given by the expression

\[ dk_x dk_y = \frac{\kappa d\kappa d\theta}{|\alpha_r \beta_i - \alpha_i \beta_r|}, \] (89)

and

\[ B_{\pm}(\kappa, \theta) = \frac{\alpha_r \beta_i - \alpha_i \beta_r}{\kappa} \left\{ \left( \frac{\partial \pm}{\partial \theta} \left| \frac{\partial \pm}{\partial \kappa} \right. \right) - \left. \left( \frac{\partial \pm}{\partial \kappa} \left| \frac{\partial \pm}{\partial \theta} \right. \right) \right\} = \pm (\alpha_r \beta_i - \alpha_i \beta_r) \frac{i\gamma}{2\lambda^3}, \] (90)

where the indices \( r, i \) refer to the real and imaginary parts respectively. We can now calculate the contribution to the Hall conductivity from each of the bands, from the area around \( k^0 \) given by \( |\kappa| < \kappa_c \), where \( \kappa_c \) is some local cutoff parameter which limit the integration to the area where the approximation leading to the Hamiltonian Eqn. \[ \text{is valid} \]

\[ \Delta \sigma_{xy}^{\pm} = \frac{e^2}{\hbar} \frac{1}{2\pi i} \int dk_x \int dk_y B(k) \]

\[ = \frac{e^2}{\hbar} \frac{1}{2\pi i} \text{sign}[\text{Im}(\alpha^* \beta)] \int_{\kappa_c}^{0} d\kappa \int_{0}^{2\pi} d\theta \frac{i\kappa \gamma}{2\lambda^3} \]

\[ = \frac{e^2}{2\hbar} \text{sign}[\alpha^* \beta] \frac{\gamma}{|\gamma|} \int_{0}^{\kappa_c/|\gamma|} u \frac{du}{(1 + u^2)^{3/2}} \]

\[ = \pm \frac{e^2}{2\hbar} \text{sign}[\alpha^* \beta] \text{sign}[\gamma], \] (91)

where the last equality sign is valid when \( \kappa_c/|\gamma| \gg 1 \), i.e. close to the crossing where \( \gamma = 0 \). Here the factor \( \text{sign}[\gamma] \) signals that the two bands exchange exactly one topological conductivity quantum \( \frac{e^2}{h} \), at the crossing. This is not surprising, because we know that the total contribution from the states in a single band is always an integer times the quantum
Moreover it is readily seen that in the hypothetical situation of a $n$’th order degeneracy, i.e. one for which $\Delta = \kappa^n e^{i\theta}$, $n$ quanta are exchanged. The total topological charge in the band structure is conserved. The topological charge can flow around and rearrange itself inside a band, but only be exchanged between bands in lumps equal to an integer multiplum of the conductivity quantum $\frac{e^2}{h}$. When the radius of the flux vortices is gradually shrunked to zero, the Hall effect has to disappear. There are two mechanisms with which the Hall effect can be eliminated. The first is by moving the topological charge up through the band structure by exchanging quanta, resulting in a net upward current of topological charge, eventually moving the charge up above the Fermi surface, where it has no effect. The second mechanism is by rearranging the topological charge inside the bands, so that each band has a large negative charge in the bottom, and a large positive charge in the top, but arranged in such a clever way that charge neutrality is more or less retained for all energies. This second mechanism will also give a net displacement of topological charge up above the Fermi energi, because in general the Fermi surface cuts a great many bands, and for all these bands the large negative charge, which they have in their bottom part, will be uncompensated. To use the language of electricity theory, we can say that every band gets extremely polarized, resulting in a net upward displacement current, in analogy with the situation in a strongly polarized dielectric. Our numerical calculations indicate, that it is the second mechanism which is responsible for the elimination of the Hall effect, as the radius of the vortices shrinks to zero.

When two bands are nearly degenerate for some $\mathbf{k}^0$, each of the bands have concentrated half a quantum in a small area in k-space around $\mathbf{k}^0$, and in general the topological charge piles up across local and global gaps in the energy spectrum. This is the reason for the oscillatory and spiky behaviour of the Hall conductance as a function of electron density, that is seen on the calculated spectra below. It is also the reason why the numerical integration involved in the actual evaluation of the Hall conductivity, is more tricky than one could wish. In particular it indicates that it is not true as sometimes conjectured, e.g. [23], that
the Hall conductivity is smoothly distributed in k-space, and that it therefore should vary smoothly between the quantized values at the energy gaps, as the Fermi energy is swept through a band.

E. Numerical results

1. Transverse Conductivity

We have calculated the transverse conductivity $\sigma_{xy}$ as a function of the integrated density of states, for electrons in a square lattice of flux vortices, for a series of varying cross sectional shapes of the flux vortices. Each of the field configurations consists of a square lattice of flux vortices with a given exponential length $\lambda_s$. The parameter which vary from calculation to calculation, is the dimensionless ratio $\xi = \lambda_s/a$, where $a$ is the length of the edges of the quadratic unit cell. The unit cell is shown on Fig. 10 and contain, as we have already discussed, two vortices and a counter Dirac vortex. In order to do the tight binding calculation a micro lattice is introduced in the unit cell. In all the numerical calculations we present, the micro lattice is $10 \times 10$. This gives 100 energy bands distributed symmetrically about the center on an energy scale. Out of these only the lower part, say band 1 to 20, approximate the real energy bands well, while the rest is significantly affected by the finite size of the micro lattice. A careful examination of the vector potential reveal that the symmetry of the Hamiltonian is very high for the particular choice of unit cell shown in Fig. 10. The field from a single vortex we have taken as $B_0 e^{-(|\tau_x| + |\tau_y|)/\lambda_s}$ instead of the more realistic $B_0 e^{-|\tau|/\lambda_s}$. With this choice the vector potential can be written down analytically in closed form. This makes the calculations simpler, and does not break any symmetry that is not already broken by the introduction of the micro lattice. (We have made calculations of the band structure, with both kinds of flux vortices, and the differences are indeed very small). The energy spectrum is invariant under the changes $(k_x, k_y) \mapsto (\pm k_x, \pm k_y), (\pm k_y, \pm k_x)$. This fact is exploited to present the band structures in
an economic way. The labels $\Gamma$, $X$ and $M$ correspond to the indicated points in the Brillouin zone, Fig. [11].

In Fig. [13] we have plotted a selection of typical bandstructures which illustrates the crossover from the completely flat Landau bands in the homogeneous magnetic field $\xi = \infty$, to the bandstructure of electrons in a square lattice of Aharonov-Bohm scatterers ($f = 1/2$) at $\xi = 0$. The bandstructures have been found by direct numerical diagonalization of the Hamiltonian. (See also Fig. [10]).

In Fig. [14] some typical results of the numerical calculations of the Hall conductivity are shown. In general we have no reason to expect, that the Hall conductivity should be isotropic as a function of the angle between the current and the flux vortex lattice. The results we present is for a current running along the diagonal of the square lattice, i.e. along the $x$-axes in Fig. [10]. It is seen, that whenever there is a gap in the spectrum, the Hall conductivity gives the quantized value in agreement with the discussion in the last section. At Fermi energies not lying in a gap $\sigma_{xy}$ always tend to be lower than the value it has in the homogeneous field. And in the limit $\xi \rightarrow 0$, $\sigma_{xy}$ vanishes altogether. In this limit the electron sees a periodic array of Aharonov-Bohm scatterers, each carrying half a flux quantum, and there is no preferential scattering to either side. We observe that when the flat Landau bands starts to get dispersion, the contribution to the Hall effect is no longer distributed equally in the Brillouin zone. Instead it piles up across local and global gaps in the spectrum resulting in the spiky $\sigma_{xy}$ spectra. Finally, we note that the amplitude of the fluctuations of $\sigma_{xy}$ is of order $\frac{e^2}{h}$, consistent with this picture.

In the calculations presented in Fig. [14], the filling fraction is limited to values below 30. This limitation comes from the fact, that we are only able to handle matrices of limited size in the numerical calculations. Filling fractions below 30 correspond to very low density electron gases, and our calculations belong therefore to the “quantum” regime, i.e. to the regime where $\lambda_F \gg \lambda_s$. According to the discussion in Sec. [4], we expect a cross-over to a semiclassical regime, for electron gases with higher density, where $\lambda_F \ll \lambda_s$, with a qualitatively different behaviour.
It is an important question whether it is possible to observe these features of the Hall conductivity in experiments. The conditions, which are necessary, are that the mean free path \( l \) is long compared to all other lengths, and that \( a, \lambda_s < \lambda_F \). The last condition indicates that we are concerned with quantum magneto transport, in the sense that the vector potential is included in the proper quantum treatment of the electrons. This is in contrast to the case \( \lambda_F \ll a, \lambda_s \) where the electron transport can be treated as the semiclassical motion of localized wavepackets in a slowly varying magnetic field. Let us assume, in order to make some estimates, that the superconductor has a London length somewhat less than 1000 Å, resulting in an exponential length of the vortices \( \lambda_s \) of about 1000 Å at the 2DEG, after the broadening due to the distance between the superconductor and the 2DEG has been taken into account. In order to have a variation in the magnetic field we should have \( a > \lambda_s \), and to be in the quantum regime \( \lambda_F > a, \lambda_s \). This gives the estimate for the electron density \( n < 6 \cdot 10^{10} \text{cm}^{-2} \), which is not unrealistic. The effect of the impurities is (to first order), to give the electrons a finite lifetime. This gives a finite longitudinal conductivity \( \sigma_{xx} \) and broadens the density of states. It also introduces localized states at the band edges (Lifshitz tails). If the field is nearly homogeneous, we have the standard quantum Hall picture with mobility edges above and below every Landau band, resulting in the formation of plateaus in \( \sigma_{xy} \), which only can be observed at much higher magnetic fields, of order \( 10^5 \text{G} \), where the filling fraction is of order one. On the other hand when the amount of impurities is low, that is \( k_F l \gg 1 \), all these effects will be small, and we expect that the features of \( \sigma_{xy} \), shown in Fig. 14 will have observable consequences in

\[
\rho_{xy} = \frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}.
\]

In Fig. 16 we have plotted the bandstructure of electrons in a square lattice of vortices carrying one flux quantum each. The calculation has been made with the same basis as the bandstructures shown in Fig. 13, the only difference is that all fluxes have been multiplied by a factor of 2, as can be seen from the spacing between the Landau bands. In the limit where \( \xi \to 0 \), and the vortex lattice becomes a regular array of Aharonov-Bohm scatterers, we recover the well-known bandstructure of free electrons. This is in agreement with our
discussion of the AB-vortex in Sec. [II].

2. Exchange of topological quanta

An example of exchange of topological quanta between neighbouring bands is shown in Fig. 17. The figure is an enlargement of the band structure around the $X$ point, showing an accidental degeneracy between the 3’ed and 4’th band, which occur about $\xi = 0.035$. Also indicated is the Hall conductance of each band in units of $\frac{e^2}{h}$, found by numerical integration. At the degeneracy it is not possible to define the Hall conductance for the individual bands. On the Brillouin zone torus there are two $X$ points $X_1, X_2$, and the bands have a simple (1. order) degeneracy in each. The numerical integration shows that two topological quanta are transferred from the lower to the upper band, and this is in full agreement with the discussion of Sec. [III.D]. Exchange of topological quanta between bands is a common phenomena as $\xi$ is varied, and this particular example has only been chosen as an illustration of the general phenomena.

3. Transverse conductivity in a disordered vortex phase

When the vortices come from a thin film of superconducting material, which have many impurities and crystal lattice defects acting as pinning centers for the vortices, the distribution of vortices will be disordered rather than forming a regular Abrikosov lattice. The effect of the disorder will be to wash out the distinctive features of the band structure, i.e. to average out the characteristic fluctuations in $\sigma_{xy}$, leaving a smooth curve in Fig. 14 with a characteristic dimensionless proportionality constant $s(\xi)$, in the form $\sigma_{xy} = \frac{e^2}{h} s(\xi) \nu$, where $\nu$ is the number of electrons in the magnetic unit cell $\nu = na^2 = n \frac{\Phi_0}{B}$ (the filling fraction). With this conjecture we can estimate the normalized Hall conductivity $s(\xi)$ by making a linear fit to the calculated $\sigma_{xy}(\nu)$ distribution. In the experimental situation $\lambda_s$ is constant, and this makes $s(\xi)$ a function of the applied magnetic field through the relationship $\xi = \lambda_s/a = \lambda_s \sqrt{B/\Phi_0}$. In Fig. 18 we have plotted $s(B)$ for a vortex exponential length.
$\lambda_s = 80$nm. The $s(B)$-curve shows essentially that the Hall effect of a dilute distribution of vortices is strongly suppressed compared to the Hall effect of a homogeneous magnetic field with the same average strength. This is in good qualitative agreement with what is seen in the experiments of Geim et al. When doing experiments, one is not directly measuring the conductivities, but rather the resistivities $\rho_{xx}, \rho_{xy}$. The experiments of Geim et al. cover the parameter range from $\lambda_F \ll \lambda_s$ at high 2DEG densities, down to the value $\lambda_F/\lambda_s = 0.7$ for the 2DEG with the lowest density experimentally obtainable, where the new phenomena begin to occur. Our numerical calculations belongs to the other side of this cross-over where $\lambda_F \gg \lambda_s$. The physical picture of this cross-over can be stated as follows. On the high density side the magnetic field varies slowly over the size of an electron wavepacket for electrons at the Fermi energy, with the result that the wavepacket more or less behaves as a classical particle. On the other side of the crossover $\lambda_F \gg \lambda_s$ the magnetic field varies rapidly over the lengthscale of a wavepacket, for an electron at the Fermi energy, and this introduces new phenomena of an essential quantum character.

F. Superlattice potential

The general picture we have outlined so far of energy bands having dispersion, with the dispersion giving rise to a non trivial behaviour of the Hall conductivity, is not limited to the inhomogeneous magnetic field. The dispersion could have another origin for instance a superlattice potential. To illustrate this a series of calculations have been made on a 2DEG in a homogeneous magnetic field, and a scalar potential which we have taken as a square lattice cosine potential. This system have commensurability problems because of the two “interfering” length scales, given respectively by the magnetic length $l_B = \sqrt{\frac{\hbar}{eB}}$, and the period of the superlattice potential $a$. To make things as simple as possible we have fixed the period of the cosine potential $a$, and the magnetic field strength $B$, and only varied the amplitude of the cosine potential. Furthermore the flux density of the magnetic field is tuned so that the flux through one unit cell of the cosine potential is exactly one flux
quantum. The cosine potential is

\[ U(x, y) = V_0 (\cos 2\pi x/a + \cos 2\pi y/a), \quad (92) \]

and the magnetic field \( B = \phi_0/a^2 \). The dimensionless parameter controlling the shape of the energy band structure is in this case

\[ v = \frac{V_0}{\hbar \omega_c}, \quad (93) \]

Examples of the \( \sigma_H \)-spectra are shown in Fig. [1]. It is observed that although the spectra look different from the vortex lattice spectra, they have the same spiky nature. The spikes have the same interpretation as in the vortex lattice system. Local spikes are due to local gaps in the spectra. That is when two bands are close to each other for some \( k \) vector in the Brillouin zone, the result is a pile up of topological charge across the gap, and this gives a spike in the \( \sigma_H \)-spectra when the Fermi energy is swept across the gap. Global spikes, that is spikes which go all the way up to the diagonal line indicating the Landau limit, are due to global gaps in the energy spectrum, combined with the topological quantization.

\section*{IV. CONCLUSION}

In Sec. [II] of this paper we have considered the longitudinal and the transverse resistivities of a 2DEG in a disordered distribution of flux vortices, within the theoretical framework where each scattering event is treated independently, and the electrons are non-interacting. The general features observed in experiments are in agreement with the results we have outlined, but we do not have quantitative agreement. If we use the radius \( R_v \) of the vortices as a fitting parameter, then the longitudinal resistance fits to a radius of the order \( R_v \approx 1000 \text{nm} \), while the transverse resistivity fits best at \( R_v \approx 30 \text{nm} \). The radius of the real vortices, is estimated by Geim to be \( R_v \sim 100 \text{nm} \). This point requires analysis on a more elaborate level, in order to be resolved.

In Sec. [III] we have considered a new kind of experiment where a 2DEG is placed in a periodic magnetic field varying on a length scale \( \lambda_s \), comparable to (or less than) the Fermi
wavelength $\lambda_F$ of the electrons. In this limit, where it is necessary to include explicitly the vector potential in a quantum treatment of the electron motion, we expect the 2DEG to exhibit new phenomena. We have presented numerical results for a non-interacting 2DEG without impurities showing characteristic spikes of the Hall conductivity versus filling fraction, which can be understood in terms of local and global energy gaps in the spectrum.

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\[ E_p = \hbar \omega_c (p + |l| + \frac{l + |l|}{2} + \frac{1}{2}), \quad p = 0, 1, 2, \ldots, \]

(94)

the Landau quantization energies. These polynomial solutions to Eqn. 21 was found by L. Page in the summer of 1930 [10], who also showed that the wave functions diverged for $\xi \to \infty$ when the energy was not equal to one of the above levels, thereby establishing the Landau quantization of electrons in a homogeneous extended magnetic field. This work was contemporary with, and independent of, Landau’s familiar and more elegant solution of the problem.
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FIGURES

FIG. 1. The scattering geometry for classical scattering on an idealized cylindrical vortex with constant magnetic field inside and no field outside.

FIG. 2. Differential cross section and classical trajectories for four different values of the parameter $\gamma = l_c/R_v = 0.025, 0.40, 1.00, 2.50$

FIG. 3. The geometry of the scattering situation.

FIG. 4. The dimensionless Aharonov-Bohm cross section, defined as $F^{AB}(\theta) = \sum_l F[\delta_l^{AB}]e^{il\theta}$. Here plotted for $f = 1/2$. The different curves are cross sections corresponding to different values of $\kappa$, and they have been translated relative to each other in order not to overlap too much. The horizontal lines indicate the zero level for the different curves. The lowest curve corresponds to $\kappa = 0.25$, and the other curves to respectively $1.00, 2.50, 5.00$, with $\kappa$ increasing upwards. The scale of the plot can be read off the distance between the horizontal lines, which is equal to 1.

FIG. 5. Differential cross sections plotted relative to the Aharonov-Bohm cross section, for an electron scattering on a magnetic flux vortex with finite radius. The curves have been obtained by subtracting the dimensionless AB cross section from the calculated finite radius cross sections. This have been done in order to single out the effect of the finite radius. All plots are in the same scale, and this includes the preceding figure, see caption of Fig. 4. Furthermore all plots are with the same values of $\kappa = 0.25, 1.00, 2.50, 5.00$ increasing upwards.

FIG. 6. Resistance efficiency of single vortex. Curves show $\zeta(\kappa)$ for different values of the flux fractions $f$.

FIG. 7. Resistance efficiency $\zeta(\gamma)$, and Hall efficiency factor $\alpha(\gamma)$ calculated from the classical cross section for scattering on a magnetic flux tube. The parameter $\gamma$ is given by the cyclotron radius divided by the radius of the flux tube $\gamma = l_c/R_v$. 
FIG. 8. The efficiency of a dilute distribution of vortices in producing Hall effect, compared to a homogeneous magnetic field with the same average flux density. Curves show $\alpha$ as a function of $\kappa$ for different values of the flux $f$.

FIG. 9. These plots of $\alpha$ and $\zeta$ for a vortex with $f = 10$, shows a striking structure of resonances at the values of $\kappa$ corresponding to the Landau quantization energies.

FIG. 10. (A) The unit cell with basis. The large circles indicates the position of the Abrikosov vortices, and the small circle indicate the position of the Dirac vortex with the counter flux. The micro lattice shown here is $6 \times 6$, whereas all the numerical results we have presented are obtained with a micro lattice of $10 \times 10$ sites. (B) Four concatenated unit cells, showing the square lattice of Abrikosov vortices.

FIG. 11. The Brillouin zone with the conventional symmetry labels.

FIG. 12. Schematic energy band crossing, controlled by an outer parameter $\gamma = \xi - \xi_0$.

FIG. 13. Bandstructures for 2D electrons in a square lattice of Abrikosov vortices with $f = 1/2$. The different plots show bandstructures corresponding to various values of the parameter $\xi$ equal to the ratio between the exponential length of the magnetic field from a single vortex, and the lattice parameter. The flux through a single vortex is half a flux quantum.

FIG. 14. Calculated Hall conductivity versus filling fraction, for various values of the ratio $\xi = \lambda_s / a$. These calculations are made on the same system as the bandstructures of Fig. 3.4, that is a square lattice of Abrikosov vortices with $f = 1/2$. Each of the spectra are made as follows. For 2000 equidistant Fermi energies $\epsilon_F$, the total Hall conductivity $\sigma_H(\epsilon_F)$, and the integrated density of states $\nu(\epsilon_F)$ are calculated numerically. This is done by 20 pages of C++ code, running on a workstation for 24 hours. The $x$-axes indicates the integrated density of states in units of filled bands. The $y$-axes indicates the total Hall conductivity in units of the conductivity quantum $e^2/h$. The diagonal line in the plots indicate the Hall conductivity in a homogeneous magnetic field.
FIG. 15. The density of Hall effect, or “topological charge”, plotted as function of the filling fraction. It is seen that for $\xi \to 0$ the distribution gets strongly polarized, with a negative contribution to the Hall effect at the bottom part of the bands, and a positive contribution at the uppermost part of the bands.

FIG. 16. Bandstructures for 2D electrons in a square lattice of vortices carrying one flux quantum each, $f = 1$. The different plots shows bandstructures corresponding to various values of the parameter $\xi$ equal to the ratio between the exponential length of the magnetic field from a single vortex, and the lattice parameter. The bandstructures have been calculated using the basis shown in Fig. 10, with the only difference that here the flux through each of the vortices are $\phi_0$, and the counter Dirac flux is $-2\phi_0$.

FIG. 17. Exchange of topological quanta. The figure shows an enlargement of the $f = 1/2$ band structure around the $X$ point, where a degeneracy between the 3’rd and 4’th band occur. (See Fig. 13). The parameter $\gamma$ appearing in the figure is defined as $\gamma = \xi - \xi_0$, with $\xi_0 = 0.035$. The numbers give the Hall conductance of the bands in units of $\frac{e^2}{h}$, found by numerical integration.

FIG. 18. The normalized Hall conductivity $s(B)$ for vortices of exponential length $\lambda_s = 80\text{nm}$.

FIG. 19. Calculated Hall conductivity versus filling fraction, for a 2DEG in a homogeneous magnetic field, and a square lattice cosine potential, in the special case where the magnetic flux density is exactly equal to one flux quantum per unit cell area. The dimensionless parameter $v$ indicated in the plots, is equal to the amplitude of the cosine potential divided by the Landau energy $\hbar \omega_c$. The $x$-axes indicates the integrated density of states in units of filled bands. The $y$-axes indicates the total Hall conductivity in units of the conductivity quantum $e^2/h$. The diagonal lines in the plots indicate the Hall conductivity in a homogeneous magnetic field, without any potential. For further details see the caption of Fig. 14.