Dynamical Yang-Baxter maps and Hopf algebroids associated with s-sets

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Abstract

An s-set is an algebraic generalization of the regular s-manifold introduced by Kowalski, one of the generalized symmetric spaces in differential geometry. We prove that suitable s-sets give birth to dynamical Yang-Baxter maps, set-theoretic solutions to a version of the quantum dynamical Yang-Baxter equation. As an application, Hopf algebroids and rigid tensor categories are constructed by means of these dynamical Yang-Baxter maps.

1 Introduction

The quantum Yang-Baxter equation [2, 3, 35, 36] is closely related to algebraic structures, for example, the quantum group [7, 12], the Hopf algebra [11, 21, 33], and the triple system [22]. Analogously, the quantum dynamical Yang-Baxter equation [10, 11], a generalization of this equation, produces Hopf algebroids [4, 5, 6, 17, 18, 24, 25, 34]. In fact, Felder’s dynamical R-matrix [10], a solution to the quantum dynamical Yang-Baxter equation, yields a Hopf algebroid called the elliptic quantum group [8] through the Faddeev-Reshetikhin-Takhtajan construction [9].

In a similar way, suitable dynamical Yang-Baxter maps [26, 28], set-theoretic solutions to a version of the quantum dynamical Yang-Baxter equation...
equation (see Remark 2.5), produce Hopf algebroids [30, 31]. This Hopf algebroid implies two rigid tensor categories, one of which is a category consisting of finite-dimensional L-operators.

Our purpose is to find dynamical Yang-Baxter maps giving birth to Hopf algebroids and rigid tensor categories.

Several studies clarified how to construct the dynamical Yang-Baxter maps [19, 20, 26, 27, 29]. In [13], suitable homogeneous pre-systems produce the dynamical Yang-Baxter maps. The homogeneous pre-system is a generalization of the homogeneous system [14] in differential geometry, an algebraic feature of the reductive homogeneous space with suitable conditions.

It is natural to try to relate geometric structures to the construction of the dynamical Yang-Baxter map that can provide with Hopf algebroids and rigid tensor categories.

Kowalski [15, Definition II.2] presented a notion of the regular s-manifold, a generalization of the symmetric space in the sense of [16]. The regular s-manifold is a $C^\infty$-manifold $M$ with a differentiable multiplication $M \times M \ni (x, y) \mapsto x \cdot y \in M$ such that the maps $s_x : M \ni y \mapsto x \cdot y \in M$ satisfy the following:

1. $s_x(x) = x$ for any $x \in M$;
2. every map $s_x$ is a diffeomorphism;
3. $s_x \circ s_y = s_{s_x(y)} \circ s_x$ for any $x, y \in M$;
4. for each $x \in M$, the tangent map $(s_x)_* : T_x(M) \to T_x(M)$ has no fixed vector except the null vector.

The aim of this paper is to construct Hopf algebroids and rigid tensor categories by introducing a notion of the s-set (Definition 3.1), a generalization of the regular s-manifold from the algebraic point of view. In this construction, suitable s-sets give birth to the dynamical Yang-Baxter maps through the homogeneous pre-systems. This paper gives another way to relate the dynamical Yang-Baxter map to differential geometry.

The organization of this paper is as follows. In Section 2, we give a brief exposition of the homogeneous pre-system, the dynamical Yang-Baxter map, the Hopf algebroid, and the rigid tensor category [13, 30]. Section 3 discusses the construction of ternary operations by means of the s-sets. In Section 4, we apply the results of Section 3 to get the dynamical Yang-Baxter maps via the homogeneous pre-systems from suitable s-sets. The last section, Section 5, is devoted to the study of the Hopf algebroids associated with the above
dynamical Yang-Baxter maps. Each Hopf algebroid can produce the rigid tensor category consisting of finite-dimensional L-operators.

2 Summary of homogeneous pre-systems, dynamical Yang-Baxter maps, Hopf algebroids, and rigid tensor categories

In this section, we summarize without proofs the relevant material on homogeneous pre-systems, dynamical Yang-Baxter maps, Hopf algebroids, and rigid tensor categories [13, 30], to render this paper as self-contained as possible.

We first introduce quasigroups [23, 32].

Definition 2.1. A nonempty set $Q$ with a binary operation $Q \times Q \ni (u, v) \mapsto uv \in Q$ is called a quasigroup, iff:

1. for any $u, w \in Q$, there uniquely exists $v \in Q$ such that $uv = w$;
2. for any $v, w \in Q$, there uniquely exists $u \in Q$ such that $uv = w$.

That is to say, the left and the right translations on the quasigroup are both bijective. On account of this fact, we define the map $\backslash : Q \times Q \to Q$ by

$$v = u \backslash w \Leftrightarrow uv = w \quad (u, v, w \in Q). \quad (2.1)$$

Any group is a quasigroup; on the other hand, the quasigroup is not always associative.

Example 2.2 ([30]). Let $Q_5 := \{0, 1, 2, 3, 4\}$. We define a binary operation on this set $Q_5$ by Table 1. Here $04 = 0$. Each element of $Q_5$ appears once and only once in each row and in each column of Table 1, and this set $Q_5$ is hence a quasigroup [23 Theorem I.1.3]. The binary operation on $Q_5$ is not associative, because $(12)3 = 1 \neq 4 = 1(23)$.

Definition 2.3. A pair $(S, \eta)$ of a nonempty set $S$ and a ternary operation $\eta : S \times S \times S \to S$ is called a homogeneous pre-system [13], iff the ternary operation $\eta$ satisfies: for any $x, y, u, v, w \in S$,

$$\eta(x, y, x) = y;$$
$$\eta(x, y, \eta(u, v, w)) = \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)). \quad (2.2)$$
This homogeneous pre-system \((S, \eta)\) satisfying
\[
\eta(x, y, z) = \eta(w, \eta(x, y, w), z) \quad (\forall x, y, z, w \in S),
\]
(2.3)
together with a suitable quasigroup, can produce a dynamical Yang-Baxter map [13]. Let \(H\) and \(X\) be nonempty sets with a map \(H \times X \ni (\lambda, x) \mapsto \lambda x \in H\).

**Definition 2.4.** A map \(\sigma(\lambda) : X \times X \to X \times X \ (\lambda \in H)\) is a dynamical Yang-Baxter map, iff \(\sigma(\lambda)\) satisfies a version of the quantum dynamical Yang-Baxter equation
\[
\sigma_{12}(\lambda X^{(3)})\sigma_{23}(\lambda)\sigma_{12}(\lambda X^{(3)}) = \sigma_{23}(\lambda)\sigma_{12}(\lambda X^{(3)})\sigma_{23}(\lambda) \quad (\forall \lambda \in H). \tag{2.4}
\]
Here, the maps \(\sigma_{12}(\lambda X^{(3)}), \sigma_{23}(\lambda) : X \times X \times X \to X \times X \times X\) are defined by
\[
\sigma_{12}(\lambda X^{(3)})(x, y, z) = (\sigma(\lambda z)(x, y), z); \sigma_{23}(\lambda)(x, y, z) = (x, \sigma(\lambda)(y, z)).
\]

**Remark 2.5.** For a map \(\sigma(\lambda) : X \times X \to X \times X \ (\lambda \in H)\), we set \(R(\lambda)(x, y) := \sigma(\lambda)(y, x) \ (\lambda \in H, x, y \in X)\). Then the following conditions are equivalent:

1. the map \(\sigma(\lambda)\) satisfies (2.4);
2. the map \(R(\lambda)\) satisfies
\[
R_{23}(\lambda)R_{13}(\lambda X^{(2)})R_{12}(\lambda) = R_{12}(\lambda X^{(3)})R_{13}(\lambda)R_{23}(\lambda X^{(1)}) \tag{2.5}
\]
for any \(\lambda \in H\) [26] (2.1).

Throughout this paper, both (2.4) and (2.5) are called versions of the quantum dynamical Yang-Baxter equation (see also Remark 2.9).
Let \((S, \eta)\) be a homogeneous pre-system satisfying (2.3), and let \(Q\) be a quasigroup, isomorphic to \(S\) as sets. We denote by \(\pi : Q \to S\) the (set-theoretic) bijection that gives this isomorphism. We define the ternary operation \(\mu\) on \(S\) by

\[
\mu(a, b, c) = \eta(b, a, c) \quad (a, b, c \in S).
\]

Proposition 2.6. The ternary operation \(\mu\) satisfies:

\[
\mu(a, \mu(a, b, c), \mu(\mu(a, b, c), d)) = \mu(a, b, \mu(b, c, d)) \quad (\forall a, b, c, d \in S);
\]
\[
\mu(\mu(a, b, c), d) = \mu(\mu(a, b, \mu(b, c, d)), \mu(b, c, d)) \quad (\forall a, b, c, d \in S).
\]

For \(\lambda \in Q\), we define the map \(\sigma(\lambda) : Q \times Q \to Q \times Q\) by

\[
\sigma(\lambda)(u, v) = (h(\lambda, v, u) \backslash ((\lambda v)u), \lambda \backslash h(\lambda, v, u)).
\]
Here, \(h(\lambda, v, u) = \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda v), \pi((\lambda v)u)))\) \((\lambda, u, v \in Q)\) and see (2.1) for \(h(\lambda, v, u) \backslash ((\lambda v)a)\). Proposition 2.6 implies (2.4), and we have the following as a result.

Proposition 2.7. The map \(\sigma(\lambda)\) is a dynamical Yang-Baxter map.

Furthermore, we assume that this dynamical Yang-Baxter map \(\sigma(\lambda)\) satisfies:

(1) the set \(S\) is finite (and so is the set \(Q\));

(2) for any \(b, c, d \in S\), there uniquely exists \(a \in S\) such that

\[
\mu(a, b, c) = d;
\]

(3) for any \(a, c, d \in S\), there exists a unique solution \(b \in S\) to (2.7);

(4) for any \(a, b, d \in S\), there exists a unique solution \(c \in S\) to (2.7).

This dynamical Yang-Baxter map \(\sigma(\lambda)\) produces the Hopf algebroid \(A_{\sigma}\) [30, Sections 3 and 4]. We will briefly describe it as below.

Let \(\mathbb{K}\) be an arbitrary field, and let \(M_Q\) denote the \(\mathbb{K}\)-algebra of all \(\mathbb{K}\)-valued maps on the set \(Q\). We define a map \(T_a : M_Q \to M_Q \ (a \in Q)\) by

\[
T_a(f)(\lambda) = f(\lambda a) \quad (f \in M_Q, \lambda, a \in Q).
\]

Let \(L_{ab}, (L^{-1})_{ab} \ (a, b \in Q)\) be indeterminates. We define the set \(AQ\) by

\[
AQ := (M_Q \otimes_{\mathbb{K}} M_Q) \bigsqcup \{L_{ab} | a, b, \in Q\} \bigsqcup \{(L^{-1})_{ab} | a, b, \in Q\}.
\]

\(A_{\sigma}\) is the quotient of the free \(\mathbb{K}\)-algebra \(\mathbb{K}(AQ)\) on the set \(AQ\) by two-sided ideal \(I_{\sigma}\) whose generators are:
Here the symbol $+$ in $\xi + \xi'$ means the addition in the algebra $\mathbb{K}\langle AQ \rangle$, while the symbol $+$ in $(\xi + \xi')(\in AQ)$ is the addition in the algebra $M_Q \otimes \mathbb{K} M_Q$. The notations of the scalar products and products in the other generators are similar.

Here $\delta_{ab}$ denotes Kronecker’s delta symbol.

$(f \otimes 1_{M_Q})L_{ac} - L_{ac} f \otimes 1_{M_Q} \quad \quad (1_{M_Q} \otimes f)L_{ab} - L_{ab} f \otimes 1_{M_Q}$.

Here $1_{M_Q}$ defined by $1_{M_Q}(\lambda) = 1 (\lambda \in Q)$ is the unit of $M_Q$ (for $T_a$, see (2.8)).

$\sum_{x,y \in Q}(\sigma_{ac}^{xy} \otimes 1_{M_Q})L_{yd}L_{xb} - \sum_{x,y \in Q}(1_{M_Q} \otimes \sigma_{bd}^{xy})L_{cy}L_{ax}$

Here $\sigma_{ac}^{xy} \in M_Q$ is defined by

$$\sigma_{ac}^{xy}(\lambda) = \begin{cases} 1, & \text{if } \sigma(\lambda)(x, y) = (a, c); \\ 0, & \text{otherwise}. \end{cases}$$

Proposition 2.8. This algebra $A_\sigma$ is a Hopf algebroid.
sions and an isomorphism $L_V : V \otimes K \rightarrow K \otimes V$ of $\mathcal{V}_G$ satisfying

$$(\sigma \otimes \text{id}_V)(\text{id}_{KQ} \otimes L_V)(L_V \otimes \text{id}_{KQ}) = (\text{id}_{KQ} \otimes L_V)(L_V \otimes \text{id}_{KQ})(\text{id}_V \otimes \sigma).$$

for any $\lambda \in Q$, where $\text{Hom}_K(V, W)$ is the $K$-vector space of $K$-linear maps from $V$ to $W$. In addition, the composition $fg$ of morphisms $f$ and $g$ is defined by $(fg)(\lambda) := f(\lambda) \circ g(\lambda)$ ($\lambda \in Q$).

$\mathcal{V}_G$ is a tensor category. The tensor product $V \otimes W$ of objects $V = \bigoplus_{\beta \in G} V_\beta$ and $W = \bigoplus_{\gamma \in G} W_\gamma$ is $V \otimes W = \bigoplus_{\alpha \in G} (V \otimes W)_\alpha$, where $(V \otimes W)_\alpha := \bigoplus_{\beta, \gamma \in G, \alpha = \gamma \beta} V_\beta \otimes_K W_\gamma$. Here, $\gamma \beta$ is the multiplication of $\gamma$ and $\beta$ in the group $G$. In addition, the tensor product $f \otimes g$ of morphisms $f : U \rightarrow V$ and $g : W \rightarrow Y$ is a map $(f \otimes g)(\lambda) = \sum_{\alpha \in G} (\sum_{\beta, \gamma \in G, \alpha = \gamma \beta} (f \otimes g)(\lambda)_{\beta, \gamma}) (\lambda \in Q)$. Here, $(f \otimes g)(\lambda)_{\beta, \gamma} \in \text{Hom}_K(U_\beta \otimes_K W_\gamma, V \otimes Y)$ is defined by

$$(f \otimes g)(\lambda)_{\beta, \gamma}(u_\beta \otimes w_\gamma) := f(\gamma(\lambda))(u_\beta) \otimes g(\lambda)(w_\gamma) \quad (u_\beta \in U_\beta, w_\gamma \in W_\gamma).$$

The unit $I$ is the $K$-vector space $\mathbb{K} = \bigoplus_{\alpha \in G} I_\alpha$ with

$$I_\alpha = \begin{cases} \mathbb{K}, & \text{if } \alpha = 1_G (= \text{id}_Q); \\ \{0\}, & \text{otherwise}. \end{cases}$$

The left and right unit constraints with respect to this unit $I$ are defined by $l_V(\lambda) = \bigoplus_{\alpha \in G} l_V(\lambda)_\alpha$ and $r_V(\lambda) = \bigoplus_{\alpha \in G} r_V(\lambda)_\alpha$ ($\lambda \in Q$). Here,

$$l_V(\lambda)_\alpha : (I \otimes V)_\alpha = I_{1G} \otimes_K V_\alpha = \mathbb{K} \otimes_K V_\alpha \ni a \otimes v \mapsto av \in V_\alpha \subset V;$$

$$r_V(\lambda)_\alpha : (V \otimes I)_\alpha = V_\alpha \otimes_K I_{1G} = V_\alpha \otimes_K \mathbb{K} \ni v \otimes a \mapsto av \in V_\alpha \subset V.$$

We are now in a position to construct the rigid tensor category $\text{Rep}_{\mathcal{V}_G}(\sigma)_f$. For $a \in Q$, we define $\deg(a) \in G$ by $\deg(a)(\lambda) := \lambda a$ ($\lambda \in Q$). This definition is unambiguous, because $Q$ is a quasigroup. Let $\mathbb{K}Q$ denote the $K$-vector space with the basis $Q$. This $\mathbb{K}Q$ is an object of $\mathcal{V}_G$, since $\mathbb{K}Q = \bigoplus_{\alpha \in G} (\mathbb{K}Q)_\alpha$ and

$$(\mathbb{K}Q)_\alpha = \begin{cases} \mathbb{K}a(= \mathbb{K}), & \text{if } \exists \alpha \in G \text{ such that } \alpha = \deg(a); \\ \{0\}, & \text{otherwise}. \end{cases}$$

This is well defined on account of the definition of the quasigroup. We can regard the map $\sigma(\lambda) : Q \times Q \rightarrow Q \times Q$ ($\lambda \in Q$) as a $K$-linear map on $\mathbb{K}Q \otimes_K \mathbb{K}Q$, and this $\sigma : \mathbb{K}Q \otimes \mathbb{K}Q \rightarrow \mathbb{K}Q \otimes \mathbb{K}Q$ is a morphism of the category $\mathcal{V}_G$.

Remark 2.9. A version of the quantum dynamical Yang-Baxter equation \cite{24} for the dynamical Yang-Baxter map $\sigma(\lambda)$ is exactly the same as the braid relation in the category $\mathcal{V}_G$ \cite[Proposition 4.5]{30}.
The morphism \( f : (V, L_V) \to (W, L_W) \) is a morphism \( f : V \to W \) of \( \mathcal{V}_G \) such that \( (\text{id}_\mathcal{K}_Q \otimes f)L_V = L_W(f \otimes \text{id}_\mathcal{K}_Q) \).

The tensor product \((V, L_V) \boxtimes (W, L_W)\) is \((V \otimes W, (L_V \otimes \text{id}_W)(\text{id}_V \otimes L_W))\), and the tensor product of morphisms is exactly the same as that of \( \mathcal{V}_G \): \( f \boxtimes g := f \otimes g \). The unit is \((I, r^{-1}_\mathcal{K}_Q l_\mathcal{K}_Q)\).

From [30, Section 5], we have the following.

**Proposition 2.10.** \( \text{Rep}_{\mathcal{V}_G}(\sigma)_f \) is a rigid tensor category.

### 3 Ternary operations from s-sets

Let \( M \) be a non-empty set and \( s_x : M \to M \) a bijection for each \( x \in M \). An s-set \((M, \{s_x\}_{x \in M})\) is a generalization of the regular s-manifold [15] (see Introduction) in the generalized symmetric spaces from the algebraic point of view.

**Definition 3.1.** A pair \((M, \{s_x\}_{x \in M})\) is an s-set, iff the bijections \( s_x \) satisfy

\[
s_x \circ s_y = s_{s_x(y)} \circ s_x
\]

for any \( x, y \in M \).

For a simple example, we note that any group \( G \) makes an s-set \((G, \{s_x\})\). Here, the bijection \( s_x \) \((x \in G)\) is defined by \( s_x(y) = xyx^{-1} \) \((y \in G)\).

In this section, we will show that every s-set \((M, \{s_x\})\) can produce ternary operations \( \eta \) on the set \( M \) satisfying (2.2).

Let \( I = (i_1, i_2, \ldots, i_l) \in \mathbb{Z}^l \) \((l \geq 1)\), and we write

\[
w_I(X, Y) = \begin{cases} X^{i_1}Y^{i_2} \cdots X^{i_{l-1}}Y^{i_l}, & \text{if } l \text{ is even;} \\ X^{i_1}Y^{i_2} \cdots Y^{i_{l-1}}X^{i_l}, & \text{if } l \text{ is odd.} \end{cases}
\]

Here, \( w_I(X, Y) \) is an element of the quotient of the free algebra on the set \( \{X, X^{-1}, Y, Y^{-1}\} \) by the two-sided ideal whose generators are \( XX^{-1} - 1, X^{-1}X - 1, YY^{-1} - 1, \) and \( Y^{-1}Y - 1 \).

Let \( \eta_I \) denote the ternary operation on an s-set \((M, \{s_x\})\) defined by

\[
\eta_I(x, y, z) = w_I(s_x, s_y)(z) \quad (x, y, z \in M). \tag{3.2}
\]

**Theorem 3.2.** The ternary operation \( \eta := \eta_I \) satisfies (2.2).
Proof. The proof is by induction on the length \( l(w_I) := |i_1| + |i_2| + \cdots + |i_t| \) of the word \( w_I(X,Y) \).

If \( l(w_I) = 0 \), then \( w_I \) is an empty word, and \( \eta_I(x,y,z) = w_I(s_x, s_y)(z) = z \) as a result. An easy computation shows \((2.2)\).

If \( l(w_I) = 1 \), then \( w_I(X,Y) = X, X^{-1}, Y, Y^{-1} \). We give the proof only for the case that \( w_I(X,Y) = X^{-1} \). Because \( w_I(X,Y) = X^{-1}, \eta_I(x,y,z) = s_x^{-1}(z) \). By substituting \( s_x^{-1}(u) \) into \( y \) in \((3.1)\),

\[ s_x s_{s_x^{-1}(u)} = s_u s_x, \]

and consequently, the right-hand-side of \((2.2)\) is \( s_x^{-1}s_u^{-1}(w) \), which is exactly the left-hand-side of \((2.2)\).

If \( l(w_I) \geq 2 \), then there exists a word \( w'(X,Y) \) whose length is less than \( l(w_I) \) such that \( w_I(X,Y) = Xw'(X,Y) \), or \( w_I(X,Y) = X^{-1}w'(X,Y) \), or \( w_I(X,Y) = Yw'(X,Y) \), or \( w_I(X,Y) = Y^{-1}w'(X,Y) \). For example, if \( w_I(X,Y) = X^{-3}Y^{-2}XY^8 \), then we set \( w'(X,Y) = X^{-2}Y^{-2}XY^8 \), which satisfies \( w_I(X,Y) = X^{-1}w'(X,Y) \).

We prove only for the case that \( w_I(X,Y) = X^{-1}w'(X,Y) \). Since \( w_I(X,Y) = X^{-1}w'(X,Y) \), \( \eta_I(x,y,z) = s_x^{-1}w'(s_x, s_y)(z) \), and the right-hand-side of \((2.2)\) is

\[ s_x^{-1}s_x^{-1}w'(s_x, s_y)/(u) s_x^{-1}w'(s_x, s_y)(v)) s_x^{-1}w'(s_x, s_y)(w). \]

Substituting \( w'(s_x, s_y)(u) \) into \( u \) in \((3.3)\) gives

\[ s_x^{-1}s_x^{-1}w'(s_x, s_y)(u) s_x^{-1}w'(s_x, s_y)(v)) s_x^{-1}w'(s_x, s_y)(w). \]

and, in the same manner, we can see that \( s_x^{-1}s_x^{-1}w'(s_x, s_y)(v)) s_x^{-1}w'(s_x, s_y)(w). \)

Now \((3.4)\) becomes

\[ s_x^{-1}s_x^{-1}w'(s_x, s_y)(u) s_x^{-1}w'(s_x, s_y)(v)) s_x^{-1}w'(s_x, s_y)(w). \]

Lemma 3.3. Let \( X, Y, Z, X^{-1}, Y^{-1}, Z^{-1} \) be indeterminates satisfying

\[ XX^{-1} = X^{-1}X = YY^{-1} = Y^{-1}Y = ZZ^{-1} = Z^{-1}Z = 1, \]

and let \( w(X,Y) \) be a word of \( X, X^{-1}, Y, \) and \( Y^{-1} \). Then \( w(ZXZ^{-1}, ZYZ^{-1}) = Zw(X,Y)Z^{-1} \). Here, we regard \( w(X,Y) \) as an element of the quotient of the free algebra on the set \( \{X,X^{-1},Y,Y^{-1},Z,Z^{-1}\} \) by the two-sided ideal whose generators are \( XX^{-1} - 1, X^{-1}X - 1, YY^{-1} - 1, Y^{-1}Y - 1, ZZ^{-1} - 1, \) and \( Z^{-1}Z - 1 \).
The proof of this lemma is obvious, because \((ZX^{\pm 1}Z^{-1})^i = ZX^{\pm 1}Z^{-1}\) for any \(i \in \mathbb{Z}\).

On account of this lemma, \((3.6)\) is
\[
\begin{align*}
  s_{x}^{-1}w'(s_x, s_y(u)) & \cdot s_{x}^{-1}w'(s_{w'(s_x, s_y)(u)}, s_{w'(s_x, s_y)(v)})w'(s_x, s_y)(w). \\
\end{align*}
\]  

(3.7)

We define the ternary operation \(\eta'\) on \(M\) by \(\eta'(x, y, z) = w'(s_x, s_y)(z)\) \((x, y, z \in M)\). Because of the fact that
\[
\begin{align*}
  w'(s_{w'(s_x, s_y)(u)}, s_{w'(s_x, s_y)(v)})w'(s_x, s_y)(w) & = \eta'(\eta'(x, y, u), \eta'(x, y, v), \eta'(x, y, w)), \\
\end{align*}
\]
the induction hypothesis, and \((3.5), (3.7)\) is
\[
\begin{align*}
  & = s_{x}^{-1}s_{x}^{-1}w'(s_x, s_y(u))s_{x}^{-1}\eta'(x, y, \eta'(u, v, w)) \\
  & = s_{x}^{-1}s_{x}^{-1}w'(s_x, s_y(u))w'(s_x, s_y)w'(s_u, s_v)(w). \\
\end{align*}
\]

(3.8)

We will prove the following claim later.

**Claim 3.4.** For any \(x, y, u \in M\),
\[
s_{w'(s_x, s_y)(u)} \circ w'(s_x, s_y) = w'(s_x, s_y) \circ s_u. \\
\]  

(3.9)

By virtue of this claim, the right-hand-side of \((3.8)\) is exactly the same as the left-hand-side of \((2.2)\), which is the desired conclusion. \(\square\)

**Proof of Claim 3.4.** The proof is similar to that of the theorem; we will prove it by induction on the length \(l(w')\) of the word \(w'(X, Y)\).

The proof of the cases that \(l(w') = 0, 1\) is obvious.

If \(l(w') \geq 2\), then there exists a word \(w''(X, Y)\) whose length is less than \(l(w')\) such that \(w'(X, Y) = Xw''(X, Y)\), or \(w'(X, Y) = X^{-1}w''(X, Y)\), or \(w'(X, Y) = Yw''(X, Y)\), or \(w'(X, Y) = Y^{-1}w''(X, Y)\). We prove only for the case \(w'(X, Y) = X^{-1}w''(X, Y)\). Since \(w'(X, Y) = X^{-1}w''(X, Y)\), the left-hand-side of \((3.9)\) is
\[
\begin{align*}
  & = s_{x}^{-1}s_{w''(s_x, s_y)(u)}s_{x}^{-1}w''(s_x, s_y). \\
\end{align*}
\]

(3.10)

By substituting \(w''(s_x, s_y)(u)\) into \(u\) in \((3.8), (3.10)\) is \(s_{x}^{-1}s_{w''(s_x, s_y)(u)}w''(s_x, s_y)\).

By the induction hypothesis, this is \(s_{x}^{-1}w''(s_x, s_y)s_u = w'(s_x, s_y)s_u\), which is exactly the right-hand-side of \((3.9)\). This establishes the formula. \(\square\)
4 Dynamical Yang-Baxter maps from s-sets

This section is devoted to the construction of the dynamical Yang-Baxter maps (Definition 2.4) via the homogeneous pre-systems (Definition 2.3) by means of suitable s-sets (Definition 3.1).

Let $R$ be a ring with the unit $1(\neq 0)$, $M$ a left $R$-module, and $r$ an invertible element of the ring $R$. We define $s_x : M \to M$ ($x \in M$) by

$$s_x(y) = (1 - r)x + ry \quad (y \in M).$$

**Proposition 4.1.** $(M, \{s_x\})$ is an s-set.

In fact, the inverse of the map $s_x$ is

$$s_x^{-1}(y) = (1 - r^{-1})x + r^{-1}y,$$

and it is a simple matter to show (3.1).

Let $I = (i_1, i_2, \ldots, i_l) \in \mathbb{Z}^l$ ($l \geq 2$). We denote by $\Phi_I(X)$ the following polynomial of the variables $X$ and $X^{-1}$.

$$\Phi_I(X) = \begin{cases} 
1 + \sum_{j=1}^{l}(-1)^jX\sum_{m=1}^{j}i_m, & \text{if } l \text{ is even;} \\
1 + \sum_{j=1}^{l-1}(-1)^jX\sum_{m=1}^{j}i_m, & \text{if } l \text{ is odd.}
\end{cases} \quad (4.1)$$

**Proposition 4.2.** $\eta_I(x, y, z) = (\Phi_I(r) - r^d)x + (1 - \Phi_I(r))y + r^dz$ for any $x, y, z \in M$. Here, $\eta_I(x, y, z)$ is defined in (3.2) and $d := i_1 + i_2 + \cdots + i_l$.

**Proof.** The proof of the proposition is by induction on $l$. For the proof of the $l = 2$ case, we need

**Lemma 4.3.** For any integer $i$,

$$s_x^i(z) = (1 - r^i)x + r^iz \quad (\forall x, z \in M).$$

As a corollary of this lemma,

$$s_x^i s_y^j(z) = (1 - r^i)x + (r^i - r^{i+j})y + r^{i+j}z \quad (i, j \in \mathbb{Z}, x, y, z \in M),$$

which immediately induces the $l = 2$ case. The rest of the proof is straightforward. 

As a corollary, we find

**Corollary 4.4.** If the invertible element $r \in R$ satisfies that $\Phi_I(r) = 0$ in $R$, then $(M, \eta)$ is a homogeneous pre-system (see Definition 2.3) satisfying (2.3).
The proof is obvious, since
\[ \eta_I(x, y, z) = -r^dx + y + rdz \quad (x, y, z \in M), \] (4.2)
if \( \Phi_I(r) = 0 \).

**Example 4.5.** Let \( k \geq 2 \) be a positive integer, and let \( r \in \mathbb{C} \) be a primitive \( k \)-th root of unity. We denote by \( \Phi_k \) the cyclotomic polynomial of level \( k \). If \( k = 6 \), then \( \Phi_I(X) := \Phi_6(X) = 1 - X + X^2 \) satisfies (4.1) \((I = (1, 1))\). Because \( \Phi_I(r) = 0 \), any \( \mathbb{C} \)-vector space \( V \) produces a homogeneous pre-system satisfying (2.3) on account of Corollary 4.4.

If \( k = 10, 14, 15, 18, 20, 21, 22, 24, 26, 28, 33, 34, 35, 36, 38, 39, 40 \), then any primitive \( k \)-th root \( r \) of unity can also give birth to a homogeneous pre-system satisfying (2.3).

**Remark 4.6.** The \( s \)-set \((V, \{s_x\})\) for any finite-dimensional \( \mathbb{C} \)-vector space \( V \) in the above example is a regular \( s \)-manifold of order \( k \) (see [15, Definition II.43]). Here, we regard \( V \) as an \( \mathbb{R} \)-vector space of \( 2 \dim \mathbb{C} V \) dimensions.

**Example 4.7.** Let \( R := \mathbb{Z}/5\mathbb{Z} \). The invertible element \( r := 2 \in R \) is a root of \( \Phi_{(2,1)}(X) = 1 - X^2 + X^3 \), and hence \( r \) with any \( (\text{left}) \) \( R \)-module \( M \) gives a homogeneous pre-system satisfying (2.3) according to Corollary 4.4. In fact, \( \eta_{(2,1)}(x, y, z) = -3x + y + 3z \). In a similar fashion, the invertible element \( 3 \in R \) also provides with such a homogeneous pre-system, because \( 3 \) is a root of \( \Phi_{(2,-1)}(X) = 1 - X^2 + X \).

**Example 4.8.** Let \( R := \mathbb{Z}/5\mathbb{Z} \). The invertible element \( r := 2 \in R \) is also a root of \( \Phi_{(2,1,1)}(X) = \Phi_{(2,1)}(X) = 1 - X^2 + X^3 \), and hence \( r \) with any \( (\text{left}) \) \( R \)-module \( M \) gives a homogeneous pre-system satisfying (2.3) according to Corollary 4.4. In this case, \( \eta_{(2,1,1)}(x, y, z) = -x + y + z \), because \( r^d = r^4 = 1 \) in \( R \). On account of (2.6), this \( \eta_{(2,1,1)} \) yields the ternary operation \( \mu \) in [30 (4.13)].

Let \( Q \) be a quasigroup (Definition 2.1), isomorphic to the set \( M \) as sets. If the element \( r(\in R) \) satisfies \( \Phi_I(r) = 0 \), then it follows from Proposition 2.7 and Corollary 4.4 that this \( s \)-set \((M, \{s_x\})\) with the quasigroup \( Q \) gives birth to the dynamical Yang-Baxter map \( \sigma(\lambda) \) (Definition 2.3).

## 5 Hopf algebroids from finite \( s \)-sets and rigid tensor categories

Let \( R \) be a ring with the unit \( 1(\neq 0) \), \( M \) a left \( R \)-module, \( r(\in R) \) an invertible element satisfying \( \Phi_I(r) = 0 \) (for \( \Phi_I \), see (4.1)), and \( Q \) a quasigroup, isomorphic to the set \( M \) as sets (see Definition 2.1).
In this last section, we will construct Hopf algebroids by means of the dynamical Yang-Baxter maps $\sigma(\lambda)$ in Section 4. In order to define the Hopf algebroid, we restrict our attention to the case that $M$ is finite and that $|M| > 1$ (it follows that $|Q| > 1$). We remark that $R = \mathbb{Z}/5\mathbb{Z}$ in Examples 4.7 and 4.8 is a field and that any (left) $R$-module $M$ is an $R$-vector space. Hence, each $M(\neq \{0\})$ in Examples 4.7 and 4.8 of finite dimensions is finite and satisfies $|M| > 1$.

**Proposition 5.1.** The ternary operation $\mu$ on $M$ satisfies:

1. for any $b, c, d \in M$, there exists a unique solution $a \in M$ to (2.7);
2. for any $a, c, d \in M$, there exists a unique solution $b \in M$ to (2.7);
3. for any $a, b, d \in M$, there exists a unique solution $c \in M$ to (2.7).

The proof is clear from (4.2) and the fact that the element $r$ is invertible in the ring $R$. By virtue of Proposition 2.8 we have the following.

**Theorem 5.2.** $A_\sigma$ is a Hopf algebroid.

The homogeneous pre-system $\eta_{(2,1,1)}$ in Example 4.8 yields a Hopf algebroid $A_\sigma$ in \cite{30}.

Furthermore, we have the following from Proposition 2.10.

**Theorem 5.3.** $\text{Rep}_{\{\gamma_G\}}(\sigma)_f$ is a rigid tensor category.

In view of Proposition 5.1 (2) (see \cite{30} Proposition 4.5), $(\mathbb{K}Q, \sigma) \in \text{Rep}_{\{\gamma_G\}}(\sigma)_f$, and this object is not the unit of $\text{Rep}_{\{\gamma_G\}}(\sigma)_f$, because $|Q| > 1$. The rigid tensor category $\text{Rep}_{\{\gamma_G\}}(\sigma)_f$ is hence non-trivial (see \cite{31} Remark 3.9).

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