Vafa-Witten theory and iterated integrals of modular forms

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Abstract

Vafa-Witten (VW) theory is a topologically twisted version of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. $S$-duality suggests that the partition function of VW theory with gauge group $SU(N)$ transforms as a modular form under duality transformations. Interestingly, Vafa and Witten demonstrated the presence of a modular anomaly, when the theory has gauge group $SU(2)$ and is considered on the complex projective plane $\mathbb{P}^2$. This modular anomaly could be expressed as an integral of a modular form, and also be traded for a holomorphic anomaly. We demonstrate that the modular anomaly for gauge group $SU(3)$ involves an iterated integral of modular forms. Moreover, the modular anomaly for $SU(3)$ can be traded for a holomorphic anomaly, which is shown to factor into a product of the partition functions for lower rank gauge groups. The $SU(3)$ partition function is mathematically an example of a mock modular form of depth two.
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1 Introduction

Topological field theory has been important for establishing symmetries and dualities in field theory and string theory beyond the semi-classical level \[\square \square \square \]. We consider in this article the Vafa-Witten twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills (YM) theory, or Vafa-Witten (VW)
theory for short. A generalization of electric-magnetic duality to Yang-Mills theory, S-duality, acts naturally on VW theory \[3\]. This duality, proposed by Montonen and Olive \[4\], states that YM theory with gauge group $G$ and complexified coupling constant

$$
\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \quad (1.1)
$$

has a dual description as YM theory, whose gauge group is the Langlands dual group of $G$, $L^G$, and with inverse coupling constant $-1/\tau$. Together with the periodicity of the $\theta$-angle, this generates the $SL(2,\mathbb{Z})$ S-duality group. Since the unitary groups $U(N)$ are self-dual under Langlands duality, S-duality suggests that the partition function of VW theory transforms as a modular form,

$$
Z_N\left(\frac{a\tau + b}{c\tau + d}\right) \sim Z_N(\tau), \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2,\mathbb{Z}). \quad (1.2)
$$

where $\sim$ indicates a possible pre-factor, which may depend polynomially on $\tau$. We will discuss in some more detail later the richer structure of $SU(N)$ theories under the action of $SL(2,\mathbb{Z})$.

The suggested transformation \[(1.2)\] is hard to verify for a general four-manifold and gauge group. For a few four-manifolds, such as $K3$ \[3\] and the rational elliptic surface $\frac{1}{2}K3$ \[5, 6\], modularity of the partition functions could be verified using string dualities for $U(N)$ with $N$ arbitrary. The partition functions could in these cases be expressed in terms of classical modular forms and theta series, and exhibited a rather rich structure. Depending on the four-manifold, connections to Hecke operators and quasi-modular forms were made \[5\]. For some other four-manifolds such as the complex projective plane $\mathbb{P}^2$, verification was limited to gauge group $SU(2)$ and $SO(3)$ \[3, 7, 8, 9\], which motivated the present work on gauge groups $SU(3)$ and $U(3)$.

Interestingly, for the complex projective plane $\mathbb{P}^2$ and other rational surfaces with $b_2^+ = 1$, the partition function for gauge group $SU(2)$ is expressed in terms of functions whose modular properties are more subtle than those of the classical modular forms. In particular, Vafa and Witten \[3\] expressed the VW partition function for $\mathbb{P}^2$ in terms of Zagier’s generating function of class numbers \[10\], whose modular transformations deviate from the form in Equation \[(1.2)\]. More precisely, its transformations include a shift by an integral of a modular form, such as

$$
\int_{\frac{\tau}{2}}^{i\infty} \frac{\Theta_0(u)}{(-i(\tau + u))^2} du, \quad (1.3)
$$

where $\Theta_0$ is a weight $\frac{1}{2}$ theta series defined in Equation \[(2.8)\]. The shift by such a period integral is one of the defining properties of functions known as mock modular forms \[11, 12\]. The modular anomaly of such functions can be traded for a holomorphic anomaly, by including a non-holomorphic period integral to the partition function. Such an integral is as in Equation
but with $\frac{3}{2}$ replaced by $-\bar{\tau}$. Besides their appearance in VW partition functions, mock modular forms have become an important element of the study of modular partition functions, such as in conformal field theory [13, 14, 15], Donaldson-Witten theory [16, 17, 18], AdS$_3$ gravity [19], black holes [20, 21, 22] and the moonshine phenomenon [23].

We will show in this paper that for VW theory with gauge group $SU(3)$, the modular transformations of the partition functions include a shift by iterated (double) integrals of theta series. One instance of these integrals is:

$$\int_{-\infty}^{\infty} \int_{u_2}^{\infty} \frac{\Theta_0(u_1) \Theta_0(3u_2)}{\sqrt{-(u_1 + \tau)^3(u_2 + \tau)^3}} du_1 du_2. \quad (1.4)$$

Section 6 will discuss that this modular anomaly for $SU(3)$ VW theory may be traded for a holomorphic anomaly, similarly to the discussion for gauge group $SU(2)$. The detailed structure is such that the $SU(3)$ partition function is an example of what is mathematically known as a mock modular form of depth two. Integrals of the form (1.4) (but involving weight $\frac{3}{2}$ modular forms) have recently also appeared in the study of vertex operator algebras and quantum modular forms [24]. Similar integrals for integer weight modular forms have been studied in [25], which have recently found applications in the context of Feynman amplitudes [26].

The derivation of the new results for VW partition functions is based on the holomorphic partition functions, which were derived in earlier work [27, 28]. Reference [28] expressed the partition function in terms of Appell functions of signature (2, 2), which in turn be related to indefinite theta series whose associated lattice has signature (2, 2). To determine the behavior of these functions under modular transformations, we first trade the modular anomaly of the theta series for a holomorphic anomaly by adding specific subleading, non-holomorphic terms to the kernel of the theta series. These terms were determined in Reference [29], with the aid of twistorial techniques for D-brane instanton corrections in IIB string theory [30, 22]. We will relate these non-holomorphic terms to a period integral of a modular form [29, 24]. Modular transformations of these integrals can be determined quite straightforwardly, from which we in turn can deduce the modular properties of the holomorphic partition function.

The modular transformations of the $SU(3)$ partition function have a number of interesting consequences. First of all, it confirms the proposed electric-magnetic duality of the VW theory. Given the technical difficulties involved in the partition function such as wall-crossing, and blow-up formula, this is quite remarkable. We will demonstrate moreover in Section 2.2 that for gauge group $U(3)$ the details work out neatly, such that its “holomorphic anomaly” $D_N Z_N$ factors into the product $\sim Z_1 Z_2$. More precisely, for $N = 1, 2$ and 3, it is now verified that the
holomorphic anomaly $D_NZ_N$ satisfies:\(^1\)

\[
D_NZ_N = -\frac{3i}{16\sqrt{2\pi y^2}} \sum_{k=1}^{N-1} k(N-k) Z_k Z_{N-k},
\]

(1.5)

with $y = \text{Im}(\tau)$. This gives further evidence for such a factorization to hold for generic $N$ \[^{[5,8]}\], in which case the $Z_N$ would involve $(N - 1)$-dimensional iterated integrals of modular forms, and consequently mock modular forms of depth $N - 1$. These functions will equally play a role for other four-manifolds with $b_2^+ = 1$ \[^{[8,9]}\].

Mock modular forms of higher depth have appeared in a few other instances, i.e. the open Gromov-Witten theory of elliptic orbifolds \[^{[31]}\] and quantum invariants of torus knots \[^{[32]}\]. It would be interesting to see whether the modular anomaly also has a physical or mathematical interpretation in these examples. Beyond these examples, higher depth mock modular forms may play a more general role in conformal field theory, gauge theory and string theory. In fact, realizing $U(N)$ VW theory as a bound state of $N$ M5-branes in M-theory does naturally provide links with these subjects. The $N$ M5-branes wrapped on $\mathbb{P}^2$ lead to the VW theory we consider, but they can equally be wrapped on other four-manifolds or on divisors in Calabi-Yau threefolds \[^{[33]}\]. The corresponding partition functions \[^{[34,35,36]}\] can be quite intricate \[^{[22,20,37]}\] for non-Ricci flat divisors. We hope that the results for $\mathbb{P}^2$ will help to understand the partition functions for these more complicated geometries. Finally, reducing the M5-brane degrees of freedom to two-dimensions leads to a $(0,4)$ conformal field theory whose partition function is expected to coincide with the partition function of the M5-branes \[^{[33]}\]. Partition functions of M5-branes have been connected recently to the moonshine phenomenon \[^{[38]}\], which suggests that one may also hope to find such a connection for the $SU(N)$ VW partition functions with $N \geq 3$.

### Structure of the paper

The structure of the paper is as follows. Section 2 reviews the $SU(N)$ and $U(N)$ Vafa-Witten partition functions for $N = 1, 2$, and discusses the new results for $N = 3$. The following sections derive the new results for $SU(3)$ VW theory. For a self-contained exposition, we have included Section 3, which is an introductory section on modular forms and mock modular forms. Section 4 reviews Jacobi forms and aspects of indefinite theta series for signatures $(n - 1, 1)$ and $(n - 2, 2)$, and their completions following \[^{[11,22]}\]. Section 5 applies this to building blocks of the VW partition functions, so-called (generalized) Appell-Lerch sums. We combine all the ingredients in Section 6 and determine the modularity of the partition functions.

\(^1\)We refer to Section 2.2 for the definition of $D_N$ and other details.


2 Modularity of Vafa-Witten theory

This section discusses the modularity of VW theory. We concentrate on explaining the novel results and postpone the detailed derivation to later sections. Subsection 2.1 deals with the partition functions for gauge group $U(N)$ with a fixed magnetic 't Hooft flux, such as the partition functions for $SU(2)$ and $SU(3)$. Subsection 2.2 discusses the $U(N)$ partition functions by including a sum over $U(1)$ fluxes.

2.1 Partition functions for gauge groups $SU(2)$ and $SU(3)$

The VW twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ contains a commuting BRST-like operator, $Q$. For a suitable $W$, the topologically twisted action $S_{\text{twisted}}$ of VW theory can be expressed as a $Q$-exact term $\{Q, W\}$, plus a term multiplying the complexified coupling constant $\tau$ (1.1):

$$S_{\text{twisted}} = \{Q, W\} - 2\pi i\tau(n - \Delta),$$

where $n$ denotes the instanton number,

$$n = \frac{1}{8\pi^2} \int_M \text{Tr} F \wedge F,$$

and $\Delta$ is a rational number arising due to $R^2$ couplings in the YM action on a curved four-manifold $M$ [39, 40]. To be precise, $\Delta$ equals $N\chi(M)/24$ with $\chi(M)$ the Euler characteristic of $M$.

The path integral of VW theory localizes on a set of equations whose solutions are graded by the instanton number $n$ [3]. Since $Q$-exact terms decouple from the partition function (to first approximation), the path integral includes a holomorphic $q$-series,

$$h_{N,0}(\tau) = \sum_{n \geq 0} c_N(n) q^{n - \Delta},$$

where $q = e^{2\pi i\tau}$. The coefficient $c_N(n)$ is a topological invariant of the space $\mathcal{M}_{N,n}$ of solutions to the VW equations with instanton number $n$. In fact, $c_N(n)$ is the Euler characteristic of $\mathcal{M}_{N,n}$, $\chi(\mathcal{M}_{N,n})$, for relative prime $(N, n)$. We will specialize the four-manifold $M$ in the following to the complex projective plane $\mathbb{P}^2$. A vanishing theorem holds for this four-manifold, which has the consequence that the VW equations reduce to the self-duality equation, $F = - \ast F$, or more generally the Hermitian-Yang-Mills equations [3]. The $c_N(n)$ is therefore a (weighted) Euler characteristic of the moduli space of instantons on $\mathbb{P}^2$ with instanton number $n$ [3]. In mathematical terminology, the $c_N(n)$ is a Donaldson-Thomas type invariant of the moduli space of semi-stable coherent sheaves on $\mathbb{P}^2$. 


Before considering the partition function of the \( SU(N) \) theories, it is useful to make a few comments on the more general class of theories with gauge group \( U(N) \) and fixed magnetic \(^{\prime}\)t Hooft flux \( \frac{i}{2\pi} \text{Tr} F = \mu \in H^2(\mathbb{P}^2, \mathbb{Z}_N) \cong \mathbb{Z}_N \ [3, 41, 42] \). The instanton number \( n \) for a generic flux \( \mu \) is not an integer for \( \mathbb{P}^2 \), but takes values in \( \mathbb{Z} + \frac{\mu}{2} \). S-duality maps the partition function \( h_{N,\mu} \) of this theory to linear combinations of the \( h_{N,\nu} \), \( \nu = 1, \ldots, N \). Their expected modular transformations under the generators \( S \) and \( T \) of \( SL(2, \mathbb{Z}) \) are \[ 3 \]:

\[
S : \quad h_{N,\mu} \left( -\frac{1}{\tau} \right) = \frac{1}{\sqrt{N}} (-i\tau)^{-\frac{3}{2}} (-1)^{N-1} \sum_{\nu \mod N} e^{-2\pi i \frac{\mu \nu}{N}} h_{N,\nu}(\tau),
\]

\[
T : \quad h_{N,\mu}(\tau + 1) = e^{2\pi i (-\frac{N}{4} + \frac{1}{N}(\mu + N/2)^2)} h_{N,\mu}(\tau).
\]

These transformations follow most easily from the self-duality of \( U(N) \), which will be discussed in some more detail in Subsection 2.2.

The partition functions \( h_{N,\mu} \) are the building blocks of a family of \( SU(N) \) theories, namely the set of theories with gauge groups \((SU(N)/\mathbb{Z}_k)_{n \mod k}\) where \( k \) divides \( N \). The subscript \( n \) labels different choices of allowed line operators in the theory \[43, 44\]. The partition function \( h_{N,0} \) equals the one for \( SU(N) \), while those for the other groups are linear combinations of the \( h_{N,\mu}, \mu = 0, \ldots, N - 1 \). We deduce from Equation (2.4) that the \( SU(N) \) theory is mapped to itself by \( T \) and \( ST^N \) for \( N \) odd, and \( T \) and \( ST^{2N} \) for \( N \) even. These are respectively the generators of the congruence subgroups \( \Gamma_0(N) \) and \( \Gamma_0(2N) \).²

The \( h_{N,\mu} \) can in principle be evaluated for arbitrary \((N, \mu)\) \[9, 27, 28, 45, 46\]. They take the form

\[
h_{N,\mu} = \frac{f_{N,\mu}}{\eta^{3N}},
\]

where \( \eta \) is the Dedekind eta function defined in Equation (3.5), which is a modular form of weight \( \frac{1}{2} \). The numerator, \( f_{N,\mu} \), can be thought of as the contribution to the partition function from solution spaces of smooth instantons, while \( \eta^{-3N} \) is due to boundary components where instantons become point-like.

Before describing the novel function appearing for \( SU(3) \), let us start by reviewing the functions for \( U(1) \) and \( SU(2) \). They are given by Equation (2.5) with \( f_{1,0} = 1 \) for \( U(1) \ [3, 47] \), while for \( SU(2) \), \( f_{2,0} = 3G_0 \ [3, 46, 48] \) where \( G_0 \) is the generating function of Hurwitz class numbers \[10\],

\[
G_0(\tau) = \sum_{n \geq 0} H(4n) q^n
= -\frac{1}{12} + 1 + \frac{1}{2} q + q^2 + \frac{4}{3} q^3 + \frac{3}{2} q^4 + O(q^5).
\]

²The difference between the congruence subgroups for even and odd \( N \) is a consequence of the fact that \( \mathbb{P}^2 \) is not a spin manifold. The duality group for \( SU(N) \) is \( \Gamma_0(N) \) on a spin manifold for all \( N \).
More details of this function are given in Section 3.4. Curiously, it does not transform as a modular form under $\Gamma_0(4)$. While $G_0$ is invariant under $T$, it transforms under the second generator $ST^4S$ of $\Gamma_0(4)$ as

$$G_0 \left( \frac{\omega - \tau}{4\tau - 1} \right) = i(4\tau - 1)^{\frac{3}{2}} \left( G_0(\tau) + \frac{i}{4\sqrt{2\pi}} \int_{-\frac{1}{\tau}}^{i\infty} \frac{\Theta_0(u)}{(-i(\tau + u))^{\frac{3}{2}}} du \right), \quad (2.7)$$

where $\Theta_\alpha$ is the theta series

$$\Theta_\alpha = \sum_{k \in \mathbb{Z} + \alpha} q^{k^2}. \quad (2.8)$$

We notice that the transformation (2.7) differs from the usual transformation of a modular form, by a shift of a period integral over $\Theta_0$. Such integrals were introduced and studied by Eichler [49] and Shimura [50].

The transformation (2.7) is a non-trivial confirmation of $S$-duality, since the transformation reproduces $G_0$. However the anomalous shift by a period integral requires explanation. A proposed resolution is that the partition function includes a non-holomorphic part besides the holomorphic $q$-series [3, 5]. If we define

$$\hat{f}_{2,0}(\tau, \bar{\tau}) = f_{2,0}(\tau) - \frac{3i}{4\sqrt{2\pi}} \int_{-\tau}^{i\infty} \frac{\Theta_0(u)}{(-i(\tau + u))^{\frac{3}{2}}} du, \quad (2.9)$$

then it transforms as a modular form of weight $\frac{3}{2}$ under $\Gamma_0(4)$. One may question the legitimacy of the addition of the period integral, since the form of the action (2.1) suggests that the partition function is holomorphic in $\tau$. However in the related context of topological string theory, $Q$-exact terms may contribute to the partition function as a consequence of the boundary of the moduli spaces. The partition function could acquire in this way a non-holomorphic dependence [2, 51]. It is conceivable that the non-holomorphic contribution may be derived along these lines in VW theory as being due to reducible $SU(2)$ connections $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a$ and $b$ are $U(1)$ connections with opposite 't Hooft fluxes [3, 5]. Indeed, the non-holomorphic term of $\hat{f}_{2,0}$ is multiplied in $\hat{h}_{2,0} = \hat{f}_{2,0}/\eta^6$ by $\eta^{-6} = h_1^2$. When we discuss the holomorphic anomaly for the $SU(3)$ partition function below, and for $U(2)$ and $U(3)$ in the next subsection, we will find more evidence for this interpretation.

While not in VW theory, the non-holomorphic contribution is understood in a physical context, namely IIB string theory. The $\hat{h}_{N,\mu}$ appear in D3-brane instanton corrections to the hypermultiplet geometry [30] in a suitably chosen compactification of IIB string theory. The non-holomorphic period integral appears in the form of a twistor integral [30]. The form (2.9) is characteristic for functions known as mock modular forms in mathematics [11, 12]. The function $\Theta_0$ is known as the “shadow” of $f_{2,0}$ in this context. Section 3.3 will discuss mock modular forms in more detail.
The main result of the present paper is the derivation of the modular properties of the $f_{3,\mu}$, $\mu = 0, 1$, which is again of the form given in Equation (2.5). The first terms of $f_{3,0}$ are [28, 52]

$$f_{3,0}(\tau) = \frac{1}{9} - q + 3q^2 + 17q^3 + 41q^4 + 78q^5 + 120q^6 + O(q^7).$$  \hfill (2.10)

Appendix [A] gives an explicit expression for its $q$-series and lists the first 30 coefficients, which suggest that their growth is polynomial. The function is clearly invariant under the generator $T \in SL(2,\mathbb{Z})$. We will show in Section 6 that $f_{3,0}$ transforms under the other generator of $\Gamma_0(3)$, $ST^3 S$, as:

$$f_{3,0}(-\tau^3 \tau - 1) = -(3\tau - 1)^3 \left( f_{3,0}(\tau) - \frac{i}{\pi} \left( \frac{3}{2} \right)^{\frac{3}{2}} \int_{-\frac{1}{3}}^{1} \frac{\sum_{\mu=0,1} f_{2,\mu}(\tau, -u) \Theta_\mu(3u)}{(-i(\tau + u))^{\frac{3}{2}}} du \right).$$  \hfill (2.11)

Note that the completions $\hat{f}_{2,\mu}$ appear in the integrand of the period integral. Thus even the holomorphic function $f_{3,0}$ “knows” about the non-holomorphic completion $\hat{f}_{2,\mu}$. It is quite remarkable that VW theory and the mathematical theory of invariants of moduli spaces leads to $q$-series with such rich properties. Moreover, since $\hat{f}_{2,0}$ (2.9) involves an integral over a theta series, we realize that the rhs of Equation (2.11) includes an iterated integral over theta series.

As will be explained in more detail in Section 3.3, $f_{3,0}$ is an example of a mock modular form (of depth one).

Similarly to the case of $SU(2)$, we can trade the modular anomaly by a holomorphic anomaly by adding specific non-holomorphic terms to $f_{3,0}$. Section 6.3 will demonstrate that the completion $\hat{f}_{3,0}$ is given by:

$$\hat{f}_{3,0}(\tau, \bar{\tau}) = f_{3,0}(\tau) - \frac{i}{\pi} \left( \frac{3}{2} \right)^{\frac{3}{2}} \int_{-\frac{1}{3}}^{1} \frac{\sum_{\mu=0,1} \hat{f}_{2,\mu}(\tau, -u) \Theta_\mu(3u)}{(-i(\tau + u))^{\frac{3}{2}}} du.$$  \hfill (2.12)

The non-holomorphic term has again the structure reminiscent of reducible connections. Moreover, substituting $\hat{f}_{2,0}$ in this equation shows that $\hat{f}_{3,0}$ contains a (non-holomorphic) iterated period integral over theta series.

Before moving on to gauge group $U(N)$, let us discuss an experimental observation. Note that we are free to add a vector-valued modular form of weight three to $f_{3,\mu}$ with the same multiplier system, without changing the completion. There is in fact one such a modular form given in Equation (3.13) [53]. In this way, we could for example cancel the constant term of $f_{3,0}$ by subtracting $\frac{1}{9} \Theta_{E_6}$. The resulting function is in fact a mock modular cusp form of depth two. Interestingly, the coefficients of the resulting function appear to be divisible by 9. More explicitly we have,

$$F = \frac{1}{81} \Theta_{E_6} - \frac{1}{9} f_{3,0}$$

$$= q + 3q^2 + 7q^3 + 7q^4 + 18q^5 + 14q^6 + 23q^7 + 30q^8 + O(q^9).$$  \hfill (2.13)
It would be interesting to explore why the coefficients of $f_{3,0}$ satisfy this congruence, and whether these coefficients have an independent arithmetic interpretation.

2.2 Gauge group $U(N)$ and the holomorphic anomaly

We have seen in the previous section, that the VW partition functions for $SU(2)$ and $SU(3)$ transform as a modular form, once non-holomorphic terms are added to the holomorphic generating series. We recall in this subsection that this holomorphic anomaly fits a quite elegant holomorphic anomaly equation for the $U(2)$ partition function \[5\], and derive the holomorphic anomaly equation for $U(3)$.

The partition function for gauge group $U(N)$ includes a sum over the 't Hooft fluxes (or first Chern classes) $c_1 = \frac{1}{2\pi} \text{Tr} F \in H^2(\mathbb{P}^2, \mathbb{Z})$. Fortunately, a symmetry allows us to include this sum relatively easily. Namely, addition to the field strength $F$ of

$$-2\pi i \omega k \mathbf{1}_N,$$

with $\omega$ the Kähler form, $k \in \mathbb{Z}$ and $\mathbf{1}_N$ the $N$-dimensional identity matrix, induces an isomorphism between the instanton moduli spaces $\mathcal{M}_\gamma$ and $\mathcal{M}_{\gamma'}$. The transformed vector $\gamma' = (N, c'_1, n')$ is given by

$$c'_1 = c_1 + kN \omega,$$

$$n' = n + \frac{1}{2} k^2 N + \frac{k}{2\pi} \text{Tr} F,$$

and as a result $h_{N,\mu} = h_{N,\mu+Nk}$. Due to this symmetry, the $U(N)$ partition function allows a theta-function decomposition

$$Z_N = \alpha_N \sum_{\mu \in \mathbb{Z}_N} \widehat{h}_{N,\mu} \vartheta_{N,\mu},$$

(2.15)

where the $\widehat{h}_{N,\mu}$ are appropriate completions of the $h_{N,\mu}$ (2.3), and we include a dimensionless constant $\alpha_N$, which will prove useful later in this section\(^3\). Besides the equality $h_{N,\mu} = h_{N,\mu+Nk}$, the $h_{N,\mu}$ satisfy the relation $h_{N,\mu} = h_{N,-\mu}$. There are thus only $\lfloor \frac{N}{2} \rfloor + 1$ independent $h_{N,\mu}$. As a result, the apparently $N$-dimensional representation of $SL(2,\mathbb{Z})$ in Equation (2.4), is in fact only $\lfloor \frac{N}{2} \rfloor + 1$ dimensional.

The theta series $\vartheta_{N,\mu}$ in Equation (2.15) captures the sum over $U(1)$ fluxes, and is given by

$$\vartheta_{N,\mu}(\tau, \rho) = \sum_{k \in \mathbb{Z} + \frac{N}{2} + N\mathbb{Z}} (-1)^k q^{\frac{k^2}{8}} e^{2\pi i \rho k}.$$  

(2.16)

Note we added a fugacity $\rho$ for the 't Hooft flux. We discuss these theta series in a bit more detail in Section 4.1. A few comments are in order. To properly include fermions in the theory,

\(^3\)Such constants are also familiar from Donaldson-Witten theory \[10\].
the flux \( k \) in (2.16) is shifted by \( N/2 \) for \( \mathbb{P}^2 \) such that \( F \) is a spin, connection \[54\]. Explicitly, we have

\[
k = \frac{i}{2\pi} \int_H \text{Tr} F \in \frac{N}{2} \int_H w_2 + Z = \frac{N}{2} + Z, \tag{2.17}
\]

with \( w_2 \) the second Stiefel-Whitney class of \( \mathbb{P}^2 \) and \( H \) its hyperplane. The phase \((-1)^k\) is a consequence of integrating out massive modes of the fermions \[55\].

The partition function \( Z_N \) is conjectured to transform as a modular form of mixed weight \((-\frac{3}{2}, \frac{1}{2})\) under \( SL(2, \mathbb{Z}) \). One can either arrive at this result from the topologically twisted theory \[3\], or by reducing the degrees of freedom to two dimensions using M5-branes \[5, 34, 35, 36\] and using the modularity of two-dimensional conformal field theory. More precisely, the expected transformation properties for \( \mathbb{P}^2 \) are

\[
Z_N \left( -\frac{1}{\tau}, \rho \right) = i^{-N} \tau^{-\frac{3}{2}} \bar{\tau}^{\frac{1}{2}} e^{\pi i N \rho} Z_N(\tau, \rho),
\]

\[
Z_N(\tau + 1, \rho) = i^{-N} Z_N(\tau, \rho). \tag{2.18}
\]

Combining these transformations with the transformations of \( \vartheta_{N,\mu} \) \[4, 5\], implies the transformations of the \( h_{N,\mu} \) (2.4).

Let us now turn to the holomorphic anomaly of the functions \( \tilde{h}_{N,\mu} \), which was introduced to mitigate the modular anomaly of \( h_{N,\mu} \). To demonstrate the effect of the holomorphic anomaly, we act with the wave operator

\[
D_N = \partial_\tau - \frac{i}{4\pi N} \partial_\rho^2, \tag{2.19}
\]

on \( Z_N \). The operator \( D_N \) annihilates the \( \vartheta_{N,\mu} \), and receives therefore only contributions from the non-holomorphic terms added to \( h_{N,\mu} \). Working out the details for \( N \leq 3 \), we find an interesting structure.

Clearly, \( D_1 Z_1 \) vanishes. Continuing with \( D_2 Z_2 \), we find using Equations (2.9) and (6.16)

\[
D_2 Z_2 = -\frac{3i\alpha_2}{16\pi y^2} \frac{1}{\eta^6} \left( \Theta_0 \vartheta_{2,0} + \Theta_{\frac{1}{2}} \vartheta_{2,1} \right). \tag{2.20}
\]

To write this more elegantly, we use the identity:

\[
\Theta_0 \vartheta_{2,0} + \Theta_{\frac{1}{2}} \vartheta_{2,1} = \vartheta_{1,0}^2, \tag{2.21}
\]

to arrive at \[5, 8\]

\[
D_2 Z_2 = -\frac{3i}{16\pi y^2} \frac{\alpha_2}{\alpha_1^2} Z_1^2. \tag{2.22}
\]

As mentioned before, the factorization of the anomaly into \( Z_1^2 \) is suggestive of an explanation in terms of reducible connections, where the \( U(2) \) connection is rather a \( U(1) \times U(1) \) connection.
Using the new results for $U(3)$ with fixed ’t Hooft flux, we can also determine $D_3 Z_3$. Using Equations (2.12) and (6.28), this becomes

$$D_3 Z_3 = -\frac{i \alpha_3}{\pi} \frac{3^3}{8 \pi y^2 \eta^9} \frac{1}{\eta^9} \times \sum_{\mu=0,1} \tilde{f}_{2,\mu} \left[ \Theta_{\frac{3}{2}}(3\tau) \bar{\vartheta}_{3,0}(\tau, \rho) + \Theta_{\frac{2+3\mu}{6}}(3\tau) \left( \bar{\vartheta}_{3,1}(\tau, \rho) + \bar{\vartheta}_{3,-1}(\tau, \rho) \right) \right].$$  \hspace{1cm} (2.23)

Similarly to $U(2)$, we use an identity for the products of theta series:

$$\Theta_{\frac{3}{2}}(3\tau) \bar{\vartheta}_{3,0}(\tau, \rho) + \Theta_{\frac{2+3\mu}{6}}(3\tau) \left( \bar{\vartheta}_{3,1}(\tau, \rho) + \bar{\vartheta}_{3,-1}(\tau, \rho) \right) = \vartheta_{1,0}(\tau, \rho) \vartheta_{2,\mu}(\tau, \rho).$$  \hspace{1cm} (2.24)

After substitution of this expression, we can express the rhs in terms of $Z_1$ and $Z_2$:

$$D_3 Z_3 = -\frac{i 3 \sqrt{3}}{8 \pi y^2} \frac{\alpha_3}{\alpha_1 \alpha_2} Z_1 Z_2. \hspace{1cm} (2.25)$$

Again we recognize the factorization of the holomorphic anomaly, which is in this case suggestive of reducible connections $U(1) \times U(2) \subset U(3)$.

Besides for the four-manifold $\mathbb{P}^2$, such a structure was also found for the rational elliptic surface, $\frac{1}{2}K3$ [5, Equation (3.18)], where it is related to the holomorphic anomaly of topological strings by $T$-duality. The general proposed form for the anomaly $D_N Z_N$ for four-manifolds $M$ with $b_2^+ = 1$ is [5, 8]

$$D_N Z_N = C_M(y) \sum_{k=1}^{N-1} k(N-k) Z_k Z_{N-k}. \hspace{1cm} (2.26)$$

Note that the factor $k(N-k)$ equals the number of matrix elements of a $U(N)$ connection, which vanish if the connections is reducible to $U(k) \times U(N-k) \in U(N)$. Equation (2.26) is indeed confirmed by Equations (2.22) and (2.25), if we take $\alpha_N \sim \frac{1}{\sqrt{N}}$ for $\mathbb{P}^2$. It would be interesting to derive this factor within VW theory. It is promising that the structure (2.15) with $\alpha_N \sim 1/\sqrt{N}$ does occur naturally in the hypermultiplet geometry of IIB string theory. See Equation (4.14) in Reference 30.\footnote{Note after specializing the generic case discussed in [30] to $U(N)$ VW theory on $\mathbb{P}^2$, the prefactor $1/\sqrt{p \cdot t}$ in [30, Equation (4.14)] becomes proportional to $1/\sqrt{N}$.}

A natural question is the generalization of Equation (2.26) to other four-manifolds with $b_2^+ = 1$. If in addition $b_2 > 1$, the situation is more complicated due to the effect of wall-crossing. For Hirzebruch surfaces, which are examples of such four-manifolds, it was found in Reference 9 for gauge group $U(2)$ that additional terms are present on the rhs of Equation (2.26) for a generic choice of metric, while the additional terms may vanish for special metrics. This happens most notably if the period point of the metric is proportional to the anti-canonical class $-K_M$.  

\hspace{1cm} 4
3 Modular forms and mock modular forms

We begin in this section the derivation of the results described in Section 2. We recall a number of aspects of modular forms, which will be useful in the following sections. The discussion is largely based on examples, which will return in the following sections. For more details on modular forms, the reader may consult one of the many textbooks on the subject, such as [56, 57].

3.1 Modular groups

The modular group $SL(2, \mathbb{Z})$ is defined by:

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}; \ ad - bc = 1 \right\}. \quad (3.1)$$

This group is generated by two elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Two congruence subgroups of $SL(2, \mathbb{Z})$ are relevant for us. The congruence subgroup $\Gamma_0(n) \in SL(2, \mathbb{Z})$ is defined as:

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \middle| c = 0 \mod n \right\}. \quad (3.2)$$

Its generators are $ST^nS$ and $T$. Finally, the congruence subgroup $\Gamma(n) \in SL(2, \mathbb{Z})$ is defined as

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}. \quad (3.3)$$

3.2 Modular forms

A modular form $f : \mathbb{H} \to \mathbb{C}$ of weight $k$ for $SL(2, \mathbb{Z})$ is a function which satisfies

$$f\left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau), \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (3.4)$$

We denote the space of holomorphic modular forms of weight $k$ with subexponential growth for $\tau \to i\infty$ by $M_k(SL(2, \mathbb{Z}))$. One of the powerful features of modular forms is that the space $M_k$ is finite dimensional for fixed $k$. We can similarly define modular forms for congruence subgroups such as $\Gamma_0(n)$ and $\Gamma(n)$ introduced above. We can further generalize Equation (3.4) to include modular forms with a multiplier system. These are functions which transform as in Equation (3.4), except that the right-hand side is multiplied by a phase $\varepsilon(\gamma)$ for each $\gamma \in SL(2, \mathbb{Z})$.

If the weight $k$ is half-integral, such a multiplier system is in fact required to have any non-trivial holomorphic functions satisfying Equation (3.4). For example, the Dedekind eta
function $\eta$, defined by
\[ \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \]
transforms under the generators $S$ and $T$, with the phases $\varepsilon(S) = e^{-\frac{2\pi i}{8}}$ and $\varepsilon(T) = e^{\frac{2\pi i}{24}}$:
\[ S: \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau), \]
\[ T: \quad \eta(\tau + 1) = e^{\frac{2\pi i}{24}} \eta(\tau). \]

One of the key techniques to construct modular forms is as a theta series, which are functions which involve a sum over a lattice.\(^5\) Application of Poisson resummation allows to determine the modular properties of such series relatively easily. The simplest example has associated lattice $\mathbb{Z}$, and is defined as $\Theta_0 = \sum_{k \in \mathbb{Z}} q^{k^2}$. This is a modular form for the group $\Gamma_0(4)$ with a multiplier system. Its transformation for an arbitrary element of this group is:
\[ \Theta_0\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon_d \Theta_0(\tau), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \]
where $(\zeta_3) \in \pm 1$ is the Jacobi symbol and
\[ \varepsilon_d = \begin{cases} 1, & d = 1 \mod 4, \\ i, & d = 3 \mod 4. \end{cases} \]

We end this section with a brief discussion on theta series for higher-dimensional lattices. We consider first the series $b_{3,j}$, whose corresponding lattice is the $A_2$ root lattice:\(^6\)
\[ b_{3,j}(\tau) := \sum_{k_1, k_2 \in \mathbb{Z} + \frac{i}{3}} q^{k_1^2 + k_2^2 + k_1 k_2}, \]
These functions transform under $\Gamma(3)$ for $j = 0, 1$, and form in fact a vector-valued representation of $SL(2, \mathbb{Z})$. The three functions $b_{3,j}$ with $j \in \{-1, 0, 1\}$ form a three-dimensional representation of $SL(2, \mathbb{Z})$. For the generators $S$ and $T$, they transform as
\[ S: \quad b_{3,j}\left(-\frac{1}{\tau}\right) = -\frac{i\tau}{\sqrt{3}} \sum_{\ell \mod 3} e^{-2\pi ij\ell/3} b_{3,\ell}(\tau), \]
\[ T: \quad b_{3,j}(\tau + 1) = e^{2\pi ij^2/3} b_{3,j}(\tau). \]
In particular, $b_{3,0}$ is a modular form of weight 1 for the congruence subgroup $\Gamma_0(3)$ with multiplier $(\frac{d}{3}) \[59\].

---

\(^5\)Another widely used technique is the sum over modular images, as for example in Eisenstein and Poincaré series.
\(^6\)To notation $b_{3,j}$ is chosen to match with previous work. See for example References [28, 58].
As a final example, we introduce the theta series $\Theta_{E_6}$, whose associated lattice is the six-dimensional root lattice $\Lambda_{E_6}$ of $E_6$. It is explicitly given as

$$\Theta_{E_6}(\tau) = \sum_{k \in \Lambda_{E_6}} q^{Q(k)/2}$$

$$= 1 + 72q + 270q^2 + 720q^3 + 936q^4 + \ldots,$$

where $Q(k)$ is here the quadratic form of the $E_6$ root lattice. The function $\Theta_{E_6}$ transforms as a modular form of $\Gamma_0(3)$ and its weight is 3. In terms of the $b_{3,j}$ introduced above, we have the identity

$$\Theta_{E_6} = b_{3,0}^3 + 2b_{3,1}^3.$$  \hspace{1cm} (3.13)

This function forms together with $3b_{3,0}b_{3,1}^2$ a two-dimensional representation of $SL(2,\mathbb{Z})$. After dividing these functions by $\eta^9$, they transform precisely as the $\hat{h}_{3,\mu}$ in Equation (2.4). It can be shown that $\Theta_{E_6}$ is the only holomorphic modular form with this property \cite{53}.

### 3.3 Mock modular forms

The previous subsection provided examples of holomorphic modular forms. The spaces of such functions are finite dimensional once the weight and multiplier system are fixed. However, as discussed in the previous section, these spaces are not rich enough to capture the VW partition functions for four-manifolds with $b^+_2 = 1$. We have seen for gauge group $SU(2)$, that we should consider functions, which can be expressed as a holomorphic $q$-series plus a non-holomorphic period integral. Such functions are known as mock modular forms, which have received much interest in recent years following \cite{11, 12}. We will discuss the main aspects of these functions in this subsection. See Reference \cite{60} for a recent comprehensive text book on the subject.

We restrict the discussion for simplicity to the full modular group $SL(2,\mathbb{Z})$. The generalization to congruent subgroups, such as $\Gamma_0(3)$ and $\Gamma_0(4)$ is straightforward. To introduce the notion of a mock modular form, let us first introduce the so-called “shadow map” \cite{12}. The argument of this map is a function $g : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ with mixed weight $(\ell, 2 - k + \ell)$, i.e. $g$ transforms under $SL(2,\mathbb{Z})$ as

$$g \left( \frac{a\tau + b}{c\tau + d}, \frac{a\sigma + b}{c\sigma + d} \right) = (c\tau + d)^\ell (c\sigma + d)^{2-k+\ell} g(\tau, \sigma).$$

The shadow map sends $g$ to the non-holomorphic period integral $g^*$, defined as

$$g^*(\tau, \bar{\tau}) = -i(1/2)^{k-1} \int_{-\tau}^{i\infty} \frac{g(\tau, -v)}{(-i(v + \tau))^{k-\ell}} dv.$$  \hspace{1cm} (3.15)

This function transforms under an element of $SL(2,\mathbb{Z})$ almost as a modular form of weight $k$, but the transformation includes a shift by a holomorphic period integral:

$$g^* \left( \frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = (c\tau + d)^k \left( g^*(\tau, \bar{\tau}) + i(1/2)^{k-1} \int_{-\tau}^{i\infty} \frac{g(\tau, -v)}{(-i(v + \tau))^{k-\ell}} dv \right).$$  \hspace{1cm} (3.16)
Using the shadow map, we can introduce the notions of completion, shadow and of mock modular form. Let $h$ be a holomorphic $q$-series. Then we call the sum,

$$\hat{h} = h + g^*,$$  \hspace{1cm} (3.17)

the completion of $h$, if $\hat{h}$ transforms as a modular form for some weight $k$. In this case, we call the function $g$ the “shadow” of $h$. Note that the shadow of $h$ can be obtained from $\hat{h}$ by acting with $y^{k-\ell} \partial_{\tau}$, where $y = \text{Im}(\tau)$. Of course, most $q$-series do not have a completion and shadow, since modular transformations of an arbitrary $q$-series do not transform complementary to the transformation of $g^*$ in Equation (3.16).

A mock modular form $h$ is a holomorphic $q$-series, whose shadow $g$ is required to factor as $g = f_1 f_2$, where $f_1$ is a holomorphic modular form of weight $\ell$ and $f_2$ is a holomorphic modular form of weight $2 - k + \ell$. In other words, $g$ is an element of the tensor space $M_{\ell} \otimes \overline{M}_{2-k+\ell}$. When $\ell = 0$ and $f_1$ a constant, we say that the mock modular form $h$ is “pure” [21], while when $f_1$ is a non-constant modular form, $h$ is called a “mixed” mock modular form. The class number generating function $G_0$ (2.6) is an example of a pure mock modular form for the congruence subgroup $\Gamma_0(4)$. This can be seen by comparing Equations (2.7) and (3.16), using that $\Theta_0(-\overline{\tau})$ equals the complex conjugate $\overline{\Theta_0(\tau)}$. The class of mock modular forms can be expanded further, by allowing $g$ to be a sum over products, $\sum_j f_{1,j} f_{2,j}$, with weights $\ell_j$ and $2 - k + \ell_j$ [21].

The modular transformations of the mock modular forms discussed above only involve one-dimensional period integrals. To include also functions involving higher dimensional iterated integrals in the theory of mock modular forms, the notion of “depth” of a mock modular form is introduced [61, 62]. The depth is a positive number, which gives information on the shadow $g$ of $h$. Mock modular forms of depth zero coincide with the familiar holomorphic modular forms, whose shadow vanishes, while the mock modular forms of the previous paragraph are said be of depth one. The depth is defined iteratively for $r > 1$. To this end, let us denote the space of mock modular forms of depth $r$ by $\hat{M}_{k}^r$, and the space of their completions by $\hat{M}_{k}^r$. We say that a mock modular form has depth $r$ if its associated shadow $g$ is an element of the tensor space $\hat{M}_{k}^{r-1} \otimes \overline{M}_{2-k+\ell}$ for some $\ell$.\footnote{For simplicity, we assume here that $g$ factors as a product $f_1 f_2$. The generalization to a shadow $g = \sum_j f_{1,j} f_{2,j}$ as mentioned above is straightforward.} We note that a mock modular form of depth $r$, involves $r$-dimensional iterated integrals.

We can also extend iteratively the definition of “pure” to depth $> 1$. To this end, let $g = \hat{f}_1 \hat{f}_2$, with $\hat{f}_1 \in \hat{M}_{k}^{r-1}$ and $f_2 \in M_{2-k+\ell}$. Then for $r > 1$, we say that $h$ is pure, if $\hat{f}_1$ is pure. We deduce from Equation (2.12) that the VW partition function $f_{3,0}$ is a (sum of two) pure mock modular form(s) of depth two with weight $k = 3$ and $\ell = \frac{3}{2}$.
3.4 Generating function of Hurwitz class numbers

As mentioned in the previous subsection, the class number generating function is an example of a pure mock modular form of weight $3/2$. We will discuss this function in some more detail in this section. As the name suggests, the generating function $G$ of Hurwitz class numbers $H(n)$ is defined as [10, 63]:

$$G(\tau) = \sum_{n \geq 0} H(n) q^n$$

$$= -\frac{1}{12} + \frac{1}{3} q^3 + \frac{1}{2} q^4 + q^7 + \ldots,$$

where the $H(n)$ are the Hurwitz class numbers.\(^8\) Note that the $H(n)$ vanish for $d \equiv 1, 2 \mod 4$. Using results going back to Kronecker [64], the function $G$ can be explicitly written as the following $q$-series:

$$G(\tau) = -\frac{1}{2\Theta_0(\tau + \frac{1}{2})} \sum_{n \in \mathbb{Z}} \frac{n(-1)^n q^{n^2}}{1 + q^{2n}} - \frac{1}{12} \Theta_0(\tau)^3.$$  \hspace{1cm} (3.19)

One way to view this function is as an Eisenstein series for $\Gamma_0(4)$ of weight $3/2$, from which one may understand the modular anomaly of $G$. Since the weight is $\leq 2$, the classical definition of the Eisenstein series is divergent, and the sum requires a regularization. Due to the regularization, the holomorphic $q$–series does not transform as modular form, but involves a shift by a period integral. This can be mitigated by the addition of a non-holomorphic period integral. The completion $\hat{G}$, defined by [10]

$$\hat{G}(\tau, \bar{\tau}) = G(\tau) - \frac{i}{8\sqrt{2\pi}} \int_{-\tau}^{i\infty} \frac{\Theta_0(u)}{(-i(\tau + u))^{3/2}} du,$$  \hspace{1cm} (3.20)

transforms as a modular form of weight $3/2$ under $\Gamma_0(4)$.

We conclude this subsection by noting that a vector-valued non-holomorphic modular form $\hat{G}_\mu, \mu = 0, 1$, for $SL(2, \mathbb{Z})$ can be obtained from $\hat{G}$. These functions appear in the $SU(2)$ and $SO(3)$ VW partition functions, given in Equations (2.6) and (6.14). The holomorphic parts $G_\mu$ are given by

$$G_\mu(\tau) = \sum_{n \geq 0} H(4n - \mu) q^{n-\frac{\mu}{2}}, \quad \mu = 0, 1,$$  \hspace{1cm} (3.21)

while the modular completions read

$$\hat{G}_\mu(\tau, \bar{\tau}) = G_\mu(\tau) - \frac{i}{4\sqrt{2\pi}} \int_{-\tau}^{i\infty} \frac{\Theta_\frac{1}{2}(u)}{(-i(\tau + u))^{\frac{3}{2}}} du.$$  \hspace{1cm} (3.22)

\(^8\)The Hurwitz class number $H(-D)$ is defined for $D < 0$ as the number of binary integral quadratic forms, $Q(x, y) = Ax^2 + Bxy + Cy^2$, $A, B, C \in \mathbb{Z}$, with negative discriminant $0 > D = B^2 - 4AC$, modulo the action of $SL(2, \mathbb{Z})$, and weighted by the inverse of the order of the automorphism group in $PSL(2, \mathbb{Z})$. We set furthermore $H(0) = -\frac{1}{12}$. 

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The completed functions $\hat{G}_\mu$ form a two-dimensional Weil representation of $SL(2,\mathbb{Z})$:

\[
S : \quad \hat{G}_\mu \left( -\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) = -\frac{1}{\sqrt{2}} (-i\tau)^{\frac{3}{2}} \sum_{\nu=0,1} (-1)^{\mu\nu} \hat{G}_\nu(\tau, \bar{\tau}),
\]

\[
T : \quad \hat{G}_\mu(\tau + 1, \bar{\tau} + 1) = (-i)^{\mu} \hat{G}_\mu(\tau, \bar{\tau}).
\]

(3.23)

The modular properties of the non-holomorphic period integral in Equation (3.22) are easily established. One finds for the $\Gamma_0(4)$ generator $ST^4S$:

\[
\int_{\frac{i}{4\tau-1}}^{i\infty} \frac{\Theta_0(u)}{(-i(\frac{u}{4\tau-1} + u))^{\frac{3}{2}}} du = i(4\tau - 1)^{\frac{1}{2}} \int_{-\frac{i}{\tau}}^{-\frac{i}{\tau}} \frac{\Theta_0(w)}{(-i(\tau + w))^{\frac{3}{2}}} dw,
\]

after the change of variables $u = \frac{w}{4\tau+1}$ and using Equation (3.7) for the transformation of $\Theta_0$. Then using $\int_{-\frac{i}{\tau}}^{-\frac{i}{\tau}} = \int_{i\infty}^{i\infty} - \int_{-\frac{i}{\tau}}^{i\infty}$, we confirm the transformation of the holomorphic part $G_0$ as in Equation (2.7). Note that the shift is in particular holomorphic, even though the lhs of Equation (3.24) is non-holomorphic. This transformation of the period integral then implies the transformation of the holomorphic $q$-series as in Equation (2.7).

4 Jacobi forms and indefinite theta series

The previous section introduced modular forms and mock modular forms. A well-known generalization of the single variable modular forms are Jacobi forms, which include a second elliptic variable. The elliptic variable is naturally included in theta series with a positive definite lattice, while the larger class of Jacobi forms has diverse applications in mathematics and physics. Similarly to the mock modular forms in the previous section, a “mock” variation on theta series exists. These are theta series whose associated lattice is indefinite. The techniques developed for indefinite theta series with signature $(n-1,1)$ and $(n-2,2)$, are the main tool to determine the properties of the $U(3)$ VW partition function. In fact, the completion of the class number generating function may also be derived via this route.

The outline of this section is as follows. We start with a brief review of Jacobi forms. Subsection discusses indefinite theta series for signatures $(n-1,1)$ and $(n-2,2)$, and the generalized error functions appearing in their completions. Subsequent subsections discuss decompositions of the generalized error functions, and how these can be related to (iterated) period integrals.

4.1 Jacobi forms

This subsection provides a brief introduction to Jacobi forms, and provides a few useful examples. See for more properties of these functions the textbooks [60, 65]. Jacobi forms are functions of two variables, the modular variable $\tau \in \mathbb{H}$ and the elliptic variable $z \in \mathbb{C}$. A
Jacobi form $\phi: \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ is characterized by its weight $k$ and index $m$. A Jacobi form of $SL(2, \mathbb{Z})$ satisfies for modular transformations:

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{2\pi i mc\tau + d} \phi(\tau, z),$$

(4.1)

and is quasi-periodic under shifts of $z$:

$$\phi(\tau, z + k\tau + \ell) = q^{-mk^2} w^{-2mk} \phi(\tau, z),$$

(4.2)

where $w = e^{2\pi iz}$. Similarly to modular forms, these definitions are easily modified to include congruence subgroups. One may also consider multiple elliptic variables, in which case the index $m$ becomes a matrix.

Jacobi forms with half-integer weight and index exist, if we include additional phases on the right-hand-side of Equation (4.1). Famous examples with weight $\frac{1}{2}$ and index $\frac{1}{2}$ are the Jacobi theta functions:

$$\theta_1(\tau, z) = i \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^r e^{\frac{1}{2} q^{r^2/2} e^{2\pi i rz}},$$

$$\theta_2(\tau, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} q^{r^2/2} e^{2\pi i rz},$$

$$\theta_3(\tau, z) = \sum_{n \in \mathbb{Z}} q^{n^2/2} e^{2\pi inz},$$

$$\theta_4(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2} e^{2\pi inz}.$$

(4.3)

The function $\theta_1$ is a Jacobi form for $SL(2, \mathbb{Z})$, whereas the other three transform under $\Gamma_0(4)$. We will sometimes suppress the $\tau$-dependence of the Jacobi theta functions and other Jacobi forms, thus $\theta_j(\tau, z) = \theta_j(z)$.

We define the more general family of binary theta series $\vartheta_{N,\mu}(\tau, z)$, which we encountered earlier in Equation (2.16) in the $U(N)$ VW partition function,

$$\vartheta_{N,\mu}(\tau, z) = \sum_{k \in \mathbb{Z} + \frac{N}{2} + N\mathbb{Z}} (-1)^k q^{k^2/2} e^{2\pi i k z}.$$

(4.4)

These functions form an $N$-dimensional Weil representation of $SL(2, \mathbb{Z})$, with weight $\frac{1}{2}$ and index $\frac{N}{2}$. The transformations under the generators $S$ and $T$ are

$$S: \quad \vartheta_{N,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{1}{\sqrt{N}} (-i\tau)^{\frac{1}{2}} e^{-2\pi i \frac{N^2}{2} + i\pi N^2} \sum_{\nu \mod N} e^{-2\pi i \mu \nu} \vartheta_{N,\mu}(\tau, z),$$

$$T: \quad \vartheta_{N,\mu}(\tau + 1, z) = e^{(\frac{1}{2N}(\mu + \frac{N}{2})^2)} \vartheta_{N,\mu}(\tau, z).$$

(4.5)

Theta series associated to higher dimensional, positive definite lattices provide similarly examples of Jacobi forms. For example, we can easily include an elliptic variable $z$ in $b_{3,j}$ (3.9).
encountered in the last subsection, by defining:

\[
b_{3,j}(\tau, z) := \sum_{k_1, k_2 \in \mathbb{Z}+\frac{l}{3}} e^{2\pi i z (k_1+2k_2)} q^{k_1^2+k_2^2+k_1k_2}.
\]

Under the generators \(S\) and \(T\), these \(b_{3,j}\) transform as

\[
S: \quad b_{3,j}(\frac{-1}{\tau}, z) = -\frac{i\tau}{\sqrt{3}} e^{2\pi i z^2/\tau} \sum_{\ell \mod 3} e^{-2\pi i j\ell/3} b_{3,\ell}(\tau, z),
\]

\[
T: \quad b_{3,j}(\tau+1, z) = e^{2\pi ij^2/3} b_{3,j}(\tau, z),
\]

and they satisfy the quasi-periodicity relation:

\[
b_{3,j}(\tau, z + \lambda \tau + \mu) = q^{-\lambda^2} w^{-2\lambda} b_{3,j}(\tau, z).
\]

We deduce from these relations that \(b_{3,0}\) is a Jacobi form of weight one and index one for the congruence subgroup \(\Gamma_0(3)\) with multiplier \(\left(\frac{4}{3}\right)\) [59]. For later reference, we express \(b_{3,0}\) in terms of the Jacobi theta series,

\[
b_{3,0}(\tau, z) = \theta_2(6\tau, 3z) \theta_2(2\tau, z) + \theta_3(6\tau, 3z) \theta_3(2\tau, z).
\]

### 4.2 Indefinite theta series

In the previous section, we met a few examples of theta series, whose associated lattice is definite, and which are examples of Jacobi forms. This section will consider theta series, whose associated lattice is indefinite. The relation between such theta series and the theta series for a definite lattice is similar to the relation between mock modular forms and classical modular forms, which we discussed in Section 3.3. In fact, specialization of elliptic variables of an indefinite theta series leads to mock modular forms.

As mentioned above, indefinite theta series involve a sum over an indefinite lattice \(\Lambda\). Summing over all lattice points as for example in Equation (3.13) gives a divergent series due to the indefiniteness of the lattice. To ensure convergence, we introduce a non-trivial kernel multiplying \(q^{Q(k)/2}\). Convergence can easily be ensured in this way, for example by summing only over positive definite lattice vectors. However, such a regularization does in general not lead to modular transformations. As we will discuss in more detail, modular transformation properties can be obtained for specific non-holomorphic kernels [11, 29, 66]. We will restrict in this section to signatures \((n-1, 1)\) and \((n-2, 2)\), which are sufficient to determine the modular properties of the \(U(3)\) VW theory. More general signatures are relevant for the analysis of higher rank gauge groups. References [67, 68, 69] may be useful for such an analysis.

We assume in the following that the lattice \(\Lambda\) is non-degenerate, with signature \((r, s)\) and dimension \(n = r + s\) and \(s = 1\) or \(2\). The bilinear form corresponding to \(\Lambda\) is denoted by
B, and the quadratic form by \( Q \). Let furthermore \( p \) be a characteristic vector of \( \Lambda \). We then define the theta series \( \Theta : \mathbb{H} \times \mathbb{C}^n \to \mathbb{C} \) associated to \( \Lambda \) by

\[
\Theta[\Phi]_\mu(\tau, z) = \sum_{k \in \Lambda + \mu + \frac{1}{2}p} (-1)^{B(k, p)} \Phi(k + b) q^{Q(k)/2} e^{2\pi i B(z, k)},
\]

where \( b = \Im(z)/y \). If the kernel is trivial, \( \Phi = 1 \), the series is clearly divergent for \( s > 0 \). Let us now specialize \( \Lambda \) to a Lorentzian lattice, or equivalently \( s = 1 \). See for a more comprehensive treatment [11]. The holomorphic kernel \( \Phi_1 \), which is most relevant for us has its support on a positive definite subset of \( \Lambda \). To define \( \Phi_1 \), let \( \text{sgn}(x) \) be defined by

\[
\text{sgn}(x) = \begin{cases} 
-1, & x < 0, \\
0, & x = 0, \\
-1, & x > 0.
\end{cases}
\]

(4.10)

and let \( C \) and \( C' \in \Lambda \otimes \mathbb{R} \) be two negative definite vectors, \( Q(C), Q(C') < 0 \), and which satisfy in addition \( B(C, C') < 0 \). The kernel \( \Phi_1 : \Lambda \otimes \mathbb{R} \to \{0, \pm \frac{1}{2}, \pm 1\} \) is then defined as

\[
\Phi_1(x) = \frac{1}{2} (\text{sgn}(B(C, x)) - \text{sgn}(B(C', x))).
\]

(4.11)

The conditions for \( C \) and \( C' \) ensure that the series is convergent. However, \( \Theta[\Phi_1]_\mu \) does not transform as a modular or Jacobi form, which can be understood from the fact that the support of \( \Phi_1 \) is not a lattice. Alternatively, one can check that \( \Phi_1 \) does not satisfy the conditions derived by Vignéras [66] for \( \Theta[\Phi_1]_\mu \) to transform as a modular form.

However, Zwegers [11] has demonstrated that a non-holomorphic modular form \( \hat{\Theta}_\mu = \Theta[\hat{\Phi}_1]_\mu \) can be obtained from \( \Theta[\Phi_1]_\mu \). To this end, one replaces \( \text{sgn}(x) \) in \( \Phi_1 \) by the non-holomorphic smooth function \( E_1(\sqrt{2y}x) \) with \( y = \Im(\tau) \), where \( E_1 \in C^\infty \) equals the Gaussian integral

\[
E_1(u) = 2 \int_0^u e^{-\pi t^2} dt = \int_\mathbb{R} e^{-\pi(t-u)^2} \text{sgn}(t) dt.
\]

(4.12)

One may express \( E_1 \) as a reparametrization of the error function, \( E_1(u) = \text{Erf}(\sqrt{\pi}u) \). The completed indefinite theta series \( \hat{\Theta}_\mu \) then reads:

\[
\hat{\Theta}_\mu(\tau, z) = \sum_{k \in \Lambda + \mu + \frac{1}{2}p} (-1)^{B(k, p)} \hat{\Phi}_1(k + b) q^{Q(k)/2} e^{2\pi i B(z, k)},
\]

where the kernel \( \hat{\Phi}_1 : \Lambda \otimes \mathbb{R} \to \mathbb{R} \) is given by

\[
\hat{\Phi}_1(\sqrt{2y}x) = \frac{1}{2} E_1(\sqrt{2y} B(C, x)) - \frac{1}{2} E_1(\sqrt{2y} B(C', x)).
\]

(4.14)
Here $C$ denotes the normalization of $C$, $C/\sqrt{-Q(C)}$. Note that in the limit, $y \to \infty$, $\hat{\Phi}_1 \to \Phi_1$. Similarly, if we let $C$ approach a null-vector $D$ ($Q(D) = 0$), then $E_1(\sqrt{2yB(C,x)}) \to \text{sgn}(B(D,x))$. The function $\hat{\Theta}_\mu$ transforms with weight $n/2$, and multiplier system specified by $\Lambda$. The explicit transformations are:

$$S : \quad \hat{\Theta}_\mu\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{i}{\sqrt{|\Lambda^*|/\Lambda}} \tau^{\frac{n}{2}} e^{\pi i Q(\nu)} e^{-\pi i \frac{Q(\mu)}{2}} \sum_{\nu \in \Lambda^*/\Lambda} e^{-2\pi i B(\mu, \nu)} \hat{\Theta}_\nu(\tau, z),$$

(4.15)

$$T : \quad \hat{\Theta}_\mu(\tau + 1, z) = e^{\pi i Q(\mu)} \hat{\Theta}_\mu(\tau, z),$$

with $r - s = n - 2$ in this case.

Next we consider a lattice $\Lambda$ with signature $(n-2, 2)$. A convergent theta series can also be constructed in this case, by restricting the lattice sum to positive definite vectors. To this end, we introduce the kernel $\Phi_2 : \Lambda \otimes \mathbb{R} \to \{0, \pm \frac{1}{4}, \pm \frac{1}{2}, \pm 1\}$, which involves two pairs of negative definite vectors $(C_i, C'_i)$ for $i = 1, 2$. See for additional sufficient requirements on the $C_i$ and $C'_i$ [29] Theorem 4.2 [68] [70]. Then we set

$$\Phi_2(x) = \frac{1}{4} \prod_{i=1}^{2} (\text{sgn}(B(C_i, x)) - \text{sgn}(B(C'_i, x))).$$

(4.16)

With the kernel $\Phi_2$, we can again construct a convergent theta series $\Theta_\mu$ as in Equation (4.9) for Lorentzian signature. For the same reason as for signature $(n-1, 1)$, $\Theta_\mu$ does not transform as a modular form in general, but we can derive a non-holomorphic completion of $\Theta_\mu$ similarly to the discussion for $s = 1$. To this end, each product of sgn’s in $\Phi_2$ is replaced by a generalized error function $E_2$ [29]

$$E_2(\alpha; u_1, u_2) = \int_{\mathbb{R}^2} du'_1 du'_2 e^{-\pi (u_1-u'_1)^2-\pi (u_2-u'_2)^2} \text{sgn}(u'_2) \text{sgn}(u'_1 + \alpha u'_2).$$

(4.17)

For the product $\text{sgn}(B(C_1, x)) \text{sgn}(B(C_2, x))$, the arguments $\alpha$, $u_1$ and $u_2$ of $E_2$ are given by

$$\alpha = -\frac{B(C_1, C_2)}{\sqrt{\Delta(C_1, C_2)}}, \quad \Delta(C_1, C_2) = Q(C_1) Q(C_2) - B(C_1, C_2)^2,$$

$$u_1 = -\frac{B(C_{1\perp2}, x)}{\sqrt{-Q(C_{1\perp2})}}, \quad u_2 = -\frac{B(C_2, x)}{\sqrt{-Q(C_2)}},$$

(4.18)

where $C_{1\perp2}$ is the component of $C_1$ orthogonal to $C_2$. The modular properties of $\hat{\Theta}_\mu$ are now as in Equation (4.15), with $r - s = n - 4$.

4.3 Generalized error functions and period integrals

To analyze the difference between the two kernels $\Phi_\ell$ and $\hat{\Phi}_\ell$ (for $\ell = 1, 2$), we discuss in some more detail the error function $E_1 : \mathbb{R} \to [-1, 1]$, and the generalized error function $E_2 : \mathbb{R}^2 \to [-1, 1]$. We will in particular relate these to iterated period integrals.
We start with \( E_1(u) \in C^\infty \), which is anti-symmetric and interpolates monotonically from \(-1\) for \( u \to -\infty \) to \(+1\) for \( u \to +\infty \). Let us express \( E_1(u) \) as \( \text{sgn}(u) \) plus a remainder term \( M_1(u) \):

\[
E_1(u) = \text{sgn}(u) + M_1(u). \tag{4.19}
\]

Clearly, the function \( M_1 \) is discontinuous, since \( \text{sgn} \) is discontinuous and \( E_1 \) is smooth. It has various integral representations. Using the first expression in Equation (4.12) for \( E_1 \), one may write \( M_1 \) as

\[
M_1(u) = \begin{cases} 
-\text{sgn}(u) \beta_\frac{1}{\pi}(u^2), & u \neq 0, \\
0, & u = 0.
\end{cases} \tag{4.20}
\]

with

\[
\beta_\nu(x) = \int_x^\infty u^{-\nu} e^{-\pi u} du. \tag{4.21}
\]

Using this expression, we may easily write \( M_1(u) \) as a period integral:

\[
M_1(u) = \frac{i u}{\sqrt{2\pi \alpha^2 y}} \frac{\pi}{\sqrt{-i(w + \tau)}} \int_{-\tau}^{\infty} e^{\pi i u^2 w} dw, \quad u \neq 0. \tag{4.22}
\]

For later reference, we partially integrate the integral, such that \( uM_1 \) can be expressed as the form

\[
u M_1(u) = -\frac{1}{\pi} e^{-\pi u^2} - \frac{i \sqrt{2\pi}}{2\sqrt{-i(w + \tau)}} \int_{-\tau}^{\infty} e^{\pi i u^2 w} dw. \tag{4.23}
\]

This expression holds for all \( u \in \mathbb{R} \).

We continue along the same lines with the generalized error function \( E_2 \). We can express this function as a product of \( \text{sgn} \)'s plus a remainder term \[29]\:

\[
E_2(\alpha; u_1, u_2) = \text{sign}(u_2) \text{sign}(u_1 + \alpha u_2) + \text{sign}(u_1) M_1(u_2)
+ \text{sign}(u_2 - \alpha u_1) M_1\left(\frac{u_1 + \alpha u_2}{\sqrt{1 + \alpha^2}}\right) + M_2(\alpha; u_1, u_2), \tag{4.24}
\]

where \( M_1(u) \) is given by Equation (4.20) as before. The function \( M_2 \) is discontinuous across the loci where \( u_1 = 0 \) and \( u_2 - \alpha u_1 = 0 \). To give the definition of \( M_2 \), we first introduce the iterated integral \( m_2 \),

\[
m_2(u_1, u_2) = 2u_2 \int_1^\infty dt e^{-\pi t^2 u_2^2} M_1(tu_1). \tag{4.25}
\]

In terms of this function, \( M_2 \) is defined by

\[
M_2(\alpha; u_1, u_2) = \begin{cases} 
-m_2(u_1, u_2) - m_2\left(\frac{u_1 - \alpha u_1}{\sqrt{1 + \alpha^2}}, \frac{u_1 + \alpha u_1}{\sqrt{1 + \alpha^2}}\right), & u_1 \neq 0, u_2 - \alpha u_1 \neq 0, \\
-m_2\left(\frac{u_2}{\sqrt{1 + \alpha^2}}, \frac{\alpha u_2}{\sqrt{1 + \alpha^2}}\right), & u_1 = 0, u_2 \neq 0, \\
-m_2(u_1, u_2), & u_1 \neq 0, u_2 - \alpha u_1 = 0, \\
\frac{2}{\pi} \arctan(\alpha), & u_1 = u_2 = 0.
\end{cases} \tag{4.26}
\]
Using the iterated integral $m_2$, we can in turn write $M_2$ as an iterated period integral \[29\] \[24\]. One finds for the various domains of $u_1$ and $u_2$:

- for $u_1 \neq 0$ and $u_2 - \alpha u_1 \neq 0$:

\[
\begin{align*}
- \frac{u_1 u_2}{2y} & \frac{u_1^2 + u_2^2}{q^{4y} + \frac{1}{q^{4y}}} \int_{-\pi}^{i\infty} dw_2 \int_{w_2}^{i\infty} dw_1 \frac{e^{\pi i u_1 w_1 + \pi i u_2 w_2}}{q^{2y}(w_2 + \tau)} \\
& \frac{u_1 + \alpha u_2 - u_2 - \alpha u_1}{2y(1 + \alpha^2)} \frac{u_1^2 + u_2^2}{q^{4y} + \frac{1}{q^{4y}}} \int_{-\pi}^{i\infty} dw_2 \int_{w_2}^{i\infty} dw_1 \frac{e^{\pi i (u_2 - \alpha u_1)^2 w_1 + \pi i (u_1 + u_2)^2 w_2}}{q^{2(1 + \alpha^2)y}(w_2 + \tau)},
\end{align*}
\]

- for $u_1 = 0$, $u_2 \neq 0$:

\[
\begin{align*}
- \frac{\alpha u_2^2}{2y(1 + \alpha^2)} & \frac{u_2^2}{q^{4y} + \frac{1}{q^{4y}}} \int_{-\pi}^{i\infty} dw_2 \int_{w_2}^{i\infty} dw_1 \frac{e^{\pi i u_2^2 w_1 + \pi i u_2^2 w_2}}{q^{2(1 + \alpha^2)y}(w_2 + \tau)},
\end{align*}
\]

- for $u_1 \neq 0$, $u_1 - \alpha u_2 = 0$:

\[
\begin{align*}
- \frac{u_1 u_2}{2y} & \frac{u_1^2 + u_2^2}{q^{4y} + \frac{1}{q^{4y}}} \int_{-\pi}^{i\infty} dw_2 \int_{w_2}^{i\infty} dw_1 \frac{e^{\pi i u_1^2 w_1 + \pi i u_2^2 w_2}}{q^{2y}(w_2 + \tau)},
\end{align*}
\]

- for $u_1 = u_2 = 0$:

\[
\begin{align*}
\frac{2}{\pi} \arctan \alpha.
\end{align*}
\]

### 5 Appell-Lerch sums and their completions

In this section, we will discuss a class of functions which are closely related to the indefinite theta series of the previous sections. The classical Appell-Lerch sums were introduced by Appell in 1886 \[71\] and also studied independently by Lerch \[72\], in their study of doubly periodic functions. Over the years, various generalizations have appeared in the mathematical \[73\] \[74\] and physical literature. For the latter, in particular as partition functions in conformal field theories \[14\] and topological Yang-Mills theory \[28\]. Similarly to the mock modular forms and indefinite theta series, these functions do not transform as a modular form, but this can be mitigated by the addition of a non-holomorphic, subleading term \[11\] \[29\].

Before discussing various examples, let us briefly present the general form of the functions. As in Reference \[28\], we introduce a general Appell function in terms of an $m$-dimensional positive definite lattice $\Lambda$, with associated bilinear form $B$ and quadratic form $Q$. Let furthermore $\{m_j\}_{j=1,...,n}$ be a set of $n$ vectors in the dual lattice $\Lambda^*$. The general Appell function $A_{Q,m_i}$ is then defined as:

\[
A_{Q,m_i}(\tau, u, v) = e^{2\pi i (u)} \sum_{k \in \Lambda} q^{\frac{1}{2}Q(k) + R} e^{-2\pi i B(v,k)} \prod_{j=1}^{n} \left(1 - q^{B(m_j,k)} e^{2\pi i u_j}\right),
\]

\[5.1\]
where \( \mathbf{u} \in \mathbb{C}^n \) and \( \mathbf{v} \in \mathbb{C} \times \Lambda \), and \( \ell \) is a linear function in \( \mathbf{u} \). We will say that \( A_{Q,m_i} \) has signature \((m,n)\). When we discuss the relation of these functions with indefinite theta series later in this section, the signature of the Appell function will coincide with the signature of the associated indefinite theta series.

We furthermore introduce the general Appell-Lerch sum \( \mu_{Q,m_i} \), which is almost identical to \( A_{Q,m_i} \), except that it is divided by a theta series \( \Theta_Q \) with associated quadratic form \( Q \),

\[
\mu_{Q,m_i}(\tau, \mathbf{u}, \mathbf{v}) = \frac{A_{Q,m_i}(\tau, \mathbf{u}, \mathbf{v})}{\Theta_Q(\tau, \mathbf{v})}. \tag{5.2}
\]

We will focus in the following on examples, rather than discussing the general functions introduced above. We will first recall the classical Appell-Lerch sum and its completion, followed by a discussion on the functions which appear in the partition functions of \( U(3) \) Yang-Mills theory.

5.1 The classical Appell-Lerch sum

The classical Appell-Lerch sum \( \mu(\tau, \mathbf{u}, \mathbf{v}) := \mu(\mathbf{u}, \mathbf{v}) \) is of signature \((1,1)\). Following Zwegers [11], we define it as the ratio

\[
\mu(\mathbf{u}, \mathbf{v}) = \frac{A(\mathbf{u}, \mathbf{v})}{\theta_1(v)}, \tag{5.3}
\]

with \( \theta_1 \) as in Equation (5.3). The Appell function \( A(\tau, \mathbf{u}, \mathbf{v}) := A(\mathbf{u}, \mathbf{v}) \) is defined as

\[
A(\mathbf{u}, \mathbf{v}) = e^{\pi i \mathbf{u} \cdot \mathbf{v}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n+1/2} e^{2\pi i n v}}{1 - e^{2\pi i q^n}}. \tag{5.4}
\]

Completion of \( \mu \)

As mentioned above, \( \mu \) does not transform as a (multi-variable) Jacobi form. However, a completion \( \hat{\mu}(\tau, \mathbf{u}, \mathbf{v}) = \hat{\mu}(\mathbf{u}, \mathbf{v}) \) can be defined, which does transform as Jacobi form. This function is defined as [11]:

\[
\hat{\mu}(\mathbf{u}, \mathbf{v}) := \mu(\mathbf{u}, \mathbf{v}) + \frac{i}{2} R(\mathbf{u} - \mathbf{v}), \tag{5.5}
\]

with

\[
R(\tau, \mathbf{u}) := R(\mathbf{u}) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left( \text{sgn}(n) - E_1 \left( (n+a) \sqrt{2y} \right) \right) (-1)^{n+\frac{1}{2}} e^{-2\pi i n u} q^{-n^2/2}, \tag{5.6}
\]

where \( y = \text{Im} (\tau) \), \( a = \text{Im} (u)/y \) and \( E_1 \) is the function as defined before in Equation (4.12). Then \( \hat{\mu}(\mathbf{u}, \mathbf{v}) \) exhibits various elegant modular and quasi-periodicity properties [11]. In particular, \( \hat{\mu} \) transforms under \( SL(2,\mathbb{Z}) \) transformations as a Jacobi form of weight one, and matrix-valued index \( \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \):

\[
\hat{\mu} \left( \frac{a \tau + b}{c \tau + d}, \frac{u}{c \tau + d} ; \frac{v}{c \tau + d} \right) = \varepsilon(\gamma)^{-3} (c \tau + d)^{1/2} e^{-\pi i c (u-v)^2/(c \tau + d)} \hat{\mu}(\tau, u, v),
\]

\[
24
\]
where \( \varepsilon(\gamma) \) corresponds to the multiplier system of the \( \eta \) function (3.5).

### Proof

The proof of Equations (5.5) and (5.6) is originally given in Zwegers’ thesis [11]. To aid the reader with the proof in Section 5.3, we give here a proof based on the techniques of indefinite theta functions discussed in Section 4. This technique will also be used for the Appell-Lerch sums of signature \((2, 2)\). We first expand the denominator as a geometric series, then

\[
A(u, v) = \frac{1}{2} e^{\pi i u} \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}} (\text{sgn}(\ell + \varepsilon) + \text{sgn}(n + \text{Im}(u)/y)) (-1)^n q^{n(n+1)/2 + \ell n} e^{2\pi i \ell u + 2\pi i n v},
\]

where \( 0 < \varepsilon \ll 1 \). We see that the quadratic form \( Q \) is given by

\[
Q(k) = n^2 + 2n \ell
\]

with \( k = (n, \ell) \), and the bilinear form by \( B(k, (u, v - u)) = \ell u + n v \). Comparing with Equation (4.9) and (4.11), we furthermore derive that the vectors \( C \) and \( C' \) equal \( C = (1, -1) \) and \( C' = (0, -1) \). Note that in Equation (4.9), \( b \) in the argument of \( \Phi_1 \) equals \( \text{Im}(z)/y \). However in Equation (5.7), \( \varepsilon \) is not related to \( u \) and \( v \). To express \( A \) in terms of an indefinite theta series (4.9), we write Equation (5.7) as

\[
A(u, v) = \frac{1}{2} \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z} + \frac{1}{2}} (\text{sgn}(\ell + \text{Im}(v - u)/y) + \text{sgn}(n + \text{Im}(u)/y)) (-1)^n q^{n^2/2 + n \ell} e^{2\pi i \ell u + 2\pi i n v} + \frac{1}{2} \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z} + \frac{1}{2}} (\text{sgn}(\ell - \ell_1) - \text{sgn}(\ell + \text{Im}(v - u)/y)) (-1)^n q^{n^2/2 + \ell n} e^{2\pi i \ell u + 2\pi i n v},
\]

where we shifted \( \ell \mapsto \ell - \ell_1 \). Choosing \( \varepsilon = \frac{1}{2} \), and shifting \( n \mapsto n - \ell \), brings the second line to the form

\[
-\frac{i}{2} \theta_1(v) \sum_{\ell \in \mathbb{Z} + \frac{1}{2}} (\text{sgn}(\ell) - \text{sgn}(\ell + \text{Im}(v - u)/y)) (-1)^{\ell - \frac{1}{2}} q^{\ell^2/2} e^{2\pi i \ell(u-v)}.
\]

Completing \( A(u, v) \) now amounts to replacing the \( \text{sgn}(...) \)'s on the first line by \( E_1(...) \)'s, and by subtracting the second line in Equation (5.8), because this line is not modular. Since \( Q(C', C') = 0 \), this reproduces Equations (5.5) and (5.6) as claimed. \( \square \)

We will meet later three variations of the classical Appell function \( A \) (5.4) in the refined \( U(3) \) VW partition function. See Equations (6.18) and (6.22). In preparation for Section 6.

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we will determine here their completions. The three functions \( A_j(\tau, z) = A_j(z) \) are:

\[
A_0(z) = -\frac{1}{2} \theta_3(6\tau, 6z) + \sum_{k \in \mathbb{Z}} \frac{w^{-6k} q^{3k^2}}{1 - w^6 q^{3k}}, \\
A_1(z) = \sum_{k \in \mathbb{Z}} \frac{w^{-6k+6} q^{3k^2 + \frac{1}{2}}}{1 - w^6 q^{3k+1}}, \\
A_2(z) = \sum_{k \in \mathbb{Z}} \frac{w^{-6k+6} q^{3k^2+3k+\frac{2}{3}}}{1 - w^6 q^{3k+1}},
\]

(5.10)

with \( w = e^{2\pi iz} \).

**Completion of \( A_j \)**

We will list the completions first and then give the derivation based on the completion \( \hat{\mu} \) (5.5).

We express the completions \( \hat{A}_j \) of \( A_j \) as\(^{10}\)

\[
\hat{A}_j(z) = A_j(z) - \frac{1}{2} R_{A_j}(z).
\]

The non-holomorphic terms \( R_{A_j}(z) \) equal

\[
R_{A_0}(z) = \theta_3(6\tau, 6z) R_{1,0}(6z) + \theta_2(6\tau, 6z) R_{1,\frac{1}{4}}(6z), \\
R_{A_1}(z) = \theta_3(6\tau, 6z) R_{1,\frac{1}{4}}(6z) + \theta_2(6\tau, 6z) R_{1,\frac{5}{6}}(6z), \\
R_{A_2}(z) = \theta_3(6\tau, 6z) R_{1,\frac{2}{3}}(6z) + \theta_2(6\tau, 6z) R_{1,\frac{1}{6}}(6z),
\]

(5.11)

where the \( R_{1,\alpha} \) are defined as:

\[
R_{1,\alpha}(z) := \sum_{\ell \in \mathbb{Z} + \alpha} [\text{sign}(\ell) - E_1(\sqrt{3}\ell z - \alpha)] e^{6\pi i\ell z} q^{-3\ell^2}.
\]

(5.12)

**Proof**

Let us demonstrate this completion for \( A_0 \). We first rewrite \( A_0 \) as

\[
A_0(z) = -\frac{1}{2} \theta_3(6z, 6\tau) + \sum_{k \in \mathbb{Z}} \left( \frac{w^{-6k} q^{3k^2}}{1 - w^{12} q^{6k}} + \frac{w^{-6k+6} q^{3k^2+3k}}{1 - w^{12} q^{6k}} \right),
\]

by multiplying the numerator and denominator in the sum by \( 1 + w^6 q^{3k} \). Now we can express \( A_0 \) in terms of the original \( A \) (5.4):

\[
A_0(z) = -\frac{1}{2} \theta_3(6z, 6\tau) + w^{-6} A(6\tau, 12z, -\frac{1}{2} - 6z - 3\tau) + A(6\tau, 12z, -\frac{1}{2} - 6z)
\]

(5.13)

Application of Equations (5.3) and (5.5) gives then the first line of Equation (5.11). \(\square\)

\(^{10}\)We will in the following suppress the variable \( \tau \) from the arguments of the non-holomorphic terms such as \( R_{A_\alpha}(z) \).
5.2 Example of signature (2, 1): the function $\Phi$

We will consider in this subsection the function $\Phi$, which is of signature (2, 1) and will occur in the refined VW partition functions (6.13). We define the function $\Phi(\tau, u, v) := \Phi(u, v)$ by\(^{11}\)

$$\Phi(u, v) := \frac{1}{2} + \frac{e^{2\pi i u}}{b_{3,0}(\tau, v)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{e^{2\pi i (k_1 + 2k_2)} q^{k_1^2 + k_2^2 + k_1 k_2 + 2k_1 + k_2}}{1 - e^{2\pi i u} q^{2k_1 + k_2}} \tag{5.14}$$

where $b_{3,0}$ is defined in Equation (4.6). Note that $\Phi$ involves the same quadratic form in the numerator as is associated to $b_{3,0}$, namely the quadratic form of the $A_2$ root lattice.

We will now show that this function can be expressed in terms of the classical Appell-Lerch sum $\mu$ (5.3), such that its completion and other properties can be readily determined. To this end, we make the change of variables $k = k_2$, $\ell = 2k_1 + k_2$ in the sum in Equation (5.14), such that $\Phi$ takes the form

$$\Phi(u, v) = \frac{1}{2} + \frac{e^{2\pi i u}}{b_{3,0}(\tau, v)} \left( \sum_{\ell, k \in 2\mathbb{Z}} + \sum_{\ell, k \in 2\mathbb{Z}+1} \right) \frac{e^{\pi i (\ell + 3k)} q^{\frac{1}{2} \ell^2 + \ell + \frac{3}{4} k^2}}{1 - e^{2\pi i u} q^{\ell}}$$

$$= \frac{1}{2} + \frac{e^{2\pi i u}}{b_{3,0}(\tau, v)} \left( \theta_3(6\tau, 3v) \sum_{\ell \in \mathbb{Z}} \frac{e^{\pi i (\ell^2 + 2\ell)} q^{\ell^2 + 2\ell}}{1 - e^{2\pi i u} q^{2\ell}} + \theta_2(6\tau, 3v) \sum_{\ell \in \mathbb{Z}} \frac{e^{\pi i (\ell^2 - \frac{3}{2})} q^{\ell^2 - \frac{3}{2}}}{1 - e^{2\pi i u} q^{2\ell - 1}} \right). \tag{5.15}$$

The sums over $\ell$ on the second line can now be replaced by specializations of $\mu$. We arrive in this way at

$$\Phi(u, v) = \frac{1}{2} + \frac{e^{\pi i u}}{b_{3,0}(\tau, v)} \left\{ \theta_3(6\tau, 3v) \theta_1(2\tau, v + \frac{1}{2} + \tau) \mu(2\tau, u, v + \frac{1}{2} + \tau) \right.$$  
$$+ e^{-\pi i u} q^{-\frac{1}{4}} \theta_2(6\tau, 3v) \theta_1(2\tau, v + \frac{1}{2}) \mu(2\tau, u - \tau, v + \frac{1}{2}) \} \right\}. \tag{5.16}$$

We will simplify Equation (5.16) further for later reference. Using the following identity for $\mu$ [11],

$$\mu(u + z, v + z) - \mu(u, v) = -\frac{i \eta^3 \theta_1(u + v + z) \theta_1(z)}{\theta_1(u) \theta_1(v) \theta_1(u + z) \theta_1(v + z)}, \tag{5.17}$$

we can express $\mu(2\tau, u - \tau, v + \frac{1}{2})$ in terms of $\mu(2\tau, u, v + \frac{1}{2} + \tau)$. Upon using also the relation [4.8] for $b_{3,0}$, $\Phi$ can be expressed as

$$\Phi(u, v) = \frac{1}{2} - e^{\pi i (u-v)} q^{-\frac{1}{4}} \mu(2\tau, u - \tau, v + \frac{1}{2})$$
$$- i \eta^3 \frac{\theta_4(2\tau, 0) \theta_3(6\tau, 3v) \theta_2(2\tau, u + v)}{\theta_1(2\tau, u) \theta_4(2\tau, u) \theta_2(2\tau, v)} b_{3,0}(\tau, v) \tag{5.18}.$$

For the refined VW partition functions, we will later be interested in the specializations $\Phi(4z, -2z)$ and $\Phi(4z, -2z - \tau)$. Using Equation (5.18), these can be expressed as:

$$\Phi(4z, -2z) = \frac{1}{2} - w^3 q^{-\frac{1}{4}} \mu(2\tau, 4z - \tau, -2z + \frac{1}{2}) - i \eta^3 \frac{\theta_3(6\tau, 6z)}{\theta_1(2\tau, 4z) b_{3,0}(\tau, 2z)}, \tag{5.19}$$

\(^{11}\)We hope there will be no confusion between this $\Phi$ and the kernel $\Phi_j$ used in the previous section.
We consider next two Appell-Lerch sums of signature \((2, 2)\) with \(2\) and \(5.3\) Examples of signature \((2, 2)\): the functions \(\Psi_0\) and \(\Psi_1\)

We consider next two Appell-Lerch sums of signature \((2, 2)\). The positive definite lattice corresponds again to the \(A_2\) root lattice as before. The two functions \(\Psi_j : \mathbb{H} \times \mathbb{C}^2 \rightarrow \mathbb{C}\) are defined by

\[
\Psi_0(\tau, u, v) := \frac{1}{4} + \frac{e^{2\pi i u}}{b_{3,0}(\tau, v)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{e^{2\pi i v(k_1 + 2k_2)} q^{k_1^2 + 3k_2 + k_1 k_2 + 2k_1 + k_2}}{(1 - e^{2\pi i u} q^{2k_1 + 2k_2})(1 - e^{2\pi i u} q^{k_2 - k_1})};
\]

\[
\Psi_1(\tau, u, v) := \frac{e^{2\pi i u}}{b_{3,0}(\tau, v)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{e^{2\pi i v(k_1 + 2k_2)} q^{k_1^2 + 3k_2 + k_1 k_2 + 2k_1 + k_2}}{(1 - e^{2\pi i u} q^{2k_1 + 2k_2})(1 - e^{2\pi i u} q^{k_2 - k_1})}.
\]

The completions of \(\Phi\) follow immediately by replacing \(\mu\) by \(\hat{\mu}\).

Completion of \(\hat{\Psi}_j\)

To state the modular completions \(\hat{\Psi}_j\) of \(\Psi_j\), we define \(R_{2,\mu}(z)\) by

\[
R_{2,\mu}(z) := \sum_{k_3, k_4 \in \mathbb{Z}^2 + \mu} \left[ \text{sgn}(k_3) \text{sgn}(k_4) - \text{sgn}(k_4) - E_1(\sqrt{3}y(k_4 - a)) \text{sgn}(2k_4 - k_3) 
- E_2 \left( \frac{1}{\sqrt{3}}; \sqrt{3}y(k_3 - k_4 - a), \sqrt{3}y(k_4 - a) \right) \right] \times e^{2\pi i (k_3 + k_4)z} q^{-k_3^2 - k_4^2 + k_3 k_4}
\]

with \(\mu \in \mathbb{R}^2\) and \(a = \text{Im}(z)/y\). The completions \(\hat{\Psi}_j\), \(j = 0, 1\), are then defined by

\[
\hat{\Psi}_j := \Psi_j - \frac{1}{4} R_{\Psi_j},
\]

with

\[
R_{\Psi_0}(u, v) := R_{2,0}(u - v) + 4\Phi(u, v) R_{1,0}(u - v) + 4 e^{\pi i (v - u)} q^{-\frac{1}{4}} (\Phi(u + \tau, v) - \frac{1}{2}) R_{1,\frac{1}{2}}(u - v),
\]

and

\[
R_{\Psi_1}(u, v) \equiv R_{2,\frac{1}{2}}(-1,1)(u - v)
+ 2 \Phi(u, v) \left( R_{1,\frac{1}{2}}(u - v) + R_{1,\frac{3}{2}}(u - v) \right) + 2 e^{\pi i (v - u)} q^{-\frac{1}{4}} (\Phi(u + \tau, v) - \frac{1}{2}) \left( R_{1,\frac{1}{2}}(u - v) + R_{1,\frac{3}{2}}(u - v) \right).
\]
Then the $\hat{\Psi}_j$ transform as a two-variable Jacobi form of weight 1 under the modular group $\Gamma(3)$ and index $m = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The completion of $\Psi_0$ was earlier proved in [29, Theorem 5.3]. In the following, we give a proof for the completion of $\Psi_1$.

**Proof**

The proof follows the strategy of the proof of the completion of $\mu$ in Section 5.1 and is very similar to the proof for the completion of $\Psi_0$ in Reference [29]. We first write the denominators in the summand of $\Psi_1$ as a geometric sum. In this way, we can relate $\Psi_1$ to an indefinite theta series whose associated lattice has signature $(2,2)$. However, the argument of $Q$ and associated quadratic form $\langle C \rangle$ is almost of the form of an indefinite theta series of signature $(2,2)$. Using the techniques of References [29, 66], we then determine the modular completion of the indefinite theta series, and consequently of $\Psi_1$.

Let us start by considering the Appell function $C(\tau, u, v) := C(u, v)$ obtained from $\Psi_1$ by multiplying with $b_{3,0}$,

$$C(u, v) = e^{2\pi i u} \sum_{k_1, k_2 \in \mathbb{Z} - \frac{1}{4}} \frac{e^{2\pi i u(k_1 + 2k_2)} q^{k_1^2 + k_2^2 + k_1k_2 + k_1 + k_2}}{(1 - e^{2\pi i u q^{2k_1 + k_2}})(1 - e^{2\pi i u q^{k_2 - k_1}})} \quad (5.26)$$

We expand the denominator of $C$ using a double geometric series,

$$C(u, v) = \frac{1}{4} \sum_{k_1, k_2 \in \mathbb{Z} - \frac{1}{4}} \frac{[\text{sgn}(k_3 + \epsilon) + \text{sgn}(2k_1 + k_2 + a)][\text{sgn}(k_4 + \epsilon) + \text{sgn}(k_2 - k_1 + a)]}{k_3, k_4 \in \mathbb{Z}}$$

$$\times e^{2\pi i u(k_1 + 2k_2) + u(k_3 + k_4 + 1)} q^{k_1^2 + k_2^2 + k_1k_2 + (2k_1 + k_2)(k_3 + \frac{2}{3}) + (k_2 - k_1)(k_4 - \frac{1}{3})}, \quad (5.27)$$

where $a = \text{Im}(u)/y$ and $0 < \epsilon < 1$. After shifting $k_3 \mapsto k_3 - \frac{2}{3}$ and $k_4 \mapsto k_4 - \frac{1}{3}$, $C$ takes the form

$$C(u, v) = \frac{1}{4} \sum_{k \in \mathbb{Z}^4 + \nu} [\text{sgn}(k_3 - \frac{2}{3} + \epsilon) + \text{sgn}(2k_1 + k_2 + a)][\text{sgn}(k_4 - \frac{1}{3} + \epsilon) + \text{sgn}(k_2 - k_1 + a)]$$

$$\times e^{2\pi i u(k_1 + 2k_2) + 2\pi i u(k_3 + k_4)} q^{k_1^2 + k_2^2 + k_1k_2 + (2k_1 + k_2)(k_3 + \frac{2}{3}) + (k_2 - k_1)(k_4 + \frac{1}{3})}, \quad (5.28)$$

with $\nu = \frac{1}{3}(-1, -1, -1, 1)$. We can express the second line more compactly by introducing the bilinear form $B(x, y)$ with matrix representation

$$B = \begin{pmatrix} 2 & 1 & 2 & -1 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \quad (5.29)$$

and associated quadratic form $Q(k) = B(k; k)$. The second line can then be expressed as $e^{2\pi i B(k, z) Q(k)/2}$ with $z = (0, u, v - u, v - u) \in \mathbb{C}^4$. The right hand side of Equation (5.28) is almost of the form of an indefinite theta series of signature $(2,2)$. However, the argument of
the sgn(\ldots)'s in Equation (5.28) should then match with the shift \text{Im}(z)/y as in Equation (4.9). This is not the case for the sgn(\ldots)'s with \(k_3\) and \(k_4\). Nevertheless, we can write \(C(u, v)\) as an indefinite theta function plus a correction term:

\[
C(u, v) = \frac{1}{4} \sum_{k \in \mathbb{Z}^4 + \mu} [\text{sgn}(k_3 + b - a) + \text{sgn}(2k_1 + k_2 + a)][\text{sgn}(k_4 + b - a) + \text{sgn}(k_2 - k_1 + a)]
\times e^{2\pi i B(k, z)} q^{Q(k)/2} + s(k, z, \epsilon) e^{2\pi i B(k, z)} q^{Q(k)/2},
\] (5.30)

where \(b = \text{Im}(v)\) and \(s(k, z, \epsilon)\) is given by

\[
s(k, z, \epsilon) = [\text{sgn}(k_3 - \frac{2}{3} + \epsilon) - \text{sgn}(k_3 + b - a)] \text{sgn}(k_2 - k_1 + a)
+ [\text{sgn}(k_4 - \frac{1}{3} + \epsilon) - \text{sgn}(k_4 + b - a)] \text{sgn}(2k_1 + k_2 + a)
+ \text{sgn}(k_3 - \frac{2}{3} + \epsilon) \text{sgn}(k_4 - \frac{1}{3} + \epsilon) - \text{sgn}(k_3 + b - a) \text{sgn}(k_4 + b - a).
\] (5.31)

Note that in \(s(k, z, \epsilon)\), we may replace \(-\frac{2}{3} + \epsilon\) (respectively \(-\frac{1}{3} + \epsilon\)) by 0, since this does not change the value of \(\text{sgn}(k_3 - \frac{2}{3} + \epsilon)\) for any \(k_3 \in \mathbb{Z} - \frac{1}{3}\) (respectively the value of \(\text{sgn}(k_4 - \frac{1}{3} + \epsilon)\) for any \(k_4 \in \mathbb{Z} + \frac{1}{3}\)).

We write the modular completion \(\widehat{C}\) of \(C\) as

\[
\widehat{C} = C - \frac{1}{4} R_C,
\] (5.32)

where \(R_C\) is a subleading non-holomorphic function. To determine \(R_C\), we complete the first line of Equation (5.30) using the techniques of indefinite theta series [29], and moreover subtract the non-modular second line of (5.30). In this way, one derives for \(R_C\):

\[
R_C(u, v) = \sum_{k \in \mathbb{Z}^4 + \mu} \left\{ \left( \text{sgn}(k_3) - E_1(\sqrt{3}y(k_3 + b - a)) \right) \text{sgn}(k_2 - k_1 + a)
\right.
+ \left( \text{sgn}(k_4) - E_1(\sqrt{3}y(k_4 + b - a)) \right) \text{sgn}(2k_1 + k_2 + a)
\]
\[
+ \text{sgn}(k_3) \text{sgn}(k_4) - E_2\left( \frac{1}{\sqrt{3}}; \sqrt{3}y(2k_3 - k_4 + b - a), \sqrt{3}y(k_4 + b - a) \right)
\}
\times e^{2\pi i B(k, z)} q^{Q(k)/2}.
\] (5.33)

Next we want to carry out the geometric sums in this equation. To this end, we combine the first and second line of (5.33), using the transformation

\[
k_1 \mapsto -k_1, \quad k_2 \mapsto k_2 + k_1, \quad k_3 \mapsto k_4, \quad k_4 \mapsto k_3,
\] (5.34)

which leaves both \(B(k, z)\) and \(Q(k)\) invariant. Note that this transformation does flip the conjugacy class \(\mu\) to \(-\mu\). Thus we can write the first two lines of \(R_C\) as

\[
\sum_{k \in \mathbb{Z}^4 + \mu} \left( \text{sgn}(k_4) - E_1(\sqrt{3}y(k_4 + b - a)) \right) \text{sgn}(2k_1 + k_2 + a) e^{2\pi i B(k, z)} q^{Q(k)/2}.
\] (5.35)
Our next aim is to express this sum in terms of $R_{1,\alpha}$ and to carry out the sum over $k_3$ as a geometric series. The sum over $k_3$ will in fact return the specializations of $\Phi(u, v)$ we met earlier in Equation (5.14). We start with replacing in Equation (5.35) $\text{sgn}(2k_1 + k_2 + a)$ by $\text{sgn}(2k_1 + k_2 + a) + \text{sgn}(2k_3 - k_4) - \text{sgn}(2k_3 - k_4)$. We can then carry out the sum over $k_3$ in the terms multiplying $\text{sgn}(2k_1 + k_2 + a) + \text{sgn}(2k_3 - k_4)$,

$$\sum_{k_1, k_2, k_3 \in \mathbb{Z} + \frac{1}{4}} (\text{sgn}(2k_1 + k_2 + a) + \text{sgn}(2k_3 - k_4)) e^{2\pi i B(k, z)} q^{Q(k)/2}. \tag{5.36}$$

To this end, we shift the summation variables as follows:

$$k_1 \mapsto k_1 + \frac{1}{2} k_4, \quad k_2 \mapsto k_2 - k_4, \quad k_3 \mapsto k_3, \quad k_4 \mapsto k_4,$$  \tag{5.37}

such that Equation (5.36) becomes:

$$\sum_{k_2 \in \mathbb{Z}, k_3 \in \mathbb{Z} + \frac{1}{4}} (\text{sgn}(2k_1 + k_2 + a) + \text{sgn}(2k_3 - k_4))$$

$$\times e^{2\pi i (k_1 + 2k_2 - \frac{3}{4} k_4) + 2\pi i u (k_3 + k_4)} q^{k_1^2 + k_2^2 + k_3^2 + k_4^2 + (2k_1 + k_2)(k_3 - \frac{1}{2}k_4) - \frac{3}{4} k_4^2}. \tag{5.38}$$

There are two possibilities for $k_4$, $k_4 \in 2\mathbb{Z} \pm \frac{1}{3}$ and $k_4 \in 2\mathbb{Z} \pm \frac{1}{2}$. Let us first assume that $k_4 \in 2\mathbb{Z} \pm \frac{1}{3}$. Then, since $k_3 \in \mathbb{Z} + \frac{1}{4}$, $\text{sgn}(2k_3 - k_4)$ is positive if $k_3 \geq \frac{k_4}{2} + \frac{1}{2}$ and negative for $k_3 < \frac{k_4}{2} + \frac{1}{2}$. After resumming $k_3$, we arrive therefore at:

$$2 \sum_{k_1 \in \mathbb{Z} + \frac{1}{2}, k_2 \in \mathbb{Z}} \frac{e^{2\pi i (k_1 + 2k_2 - \frac{3}{4} k_4) + 2\pi i u (\frac{3k_4}{2} + \frac{1}{2})} q^{k_1^2 + k_2^2 + k_1 k_2 - \frac{3}{4} k_4^2}}{1 - e^{2\pi i u} q^{2k_1 + k_2}} \tag{5.39}$$

$$= 2 e^{\pi i (v - u) + \pi i \frac{3}{2} k_4 (u - v)} q^{\frac{1}{2} - \frac{3}{4} k_4^2} b_{3,0}(v) (\Phi(u + \tau, v) - \frac{1}{2}),$$

where we substituted $\Phi$ (5.14). The second possibility is $k_4 \in 2\mathbb{Z} \pm \frac{1}{2}$. Carrying out the sum over $k_3 \geq \frac{k_4}{2}$ and $k_3 < \frac{k_4}{2}$ gives in this case:

$$2 \sum_{k_1, k_3 \in \mathbb{Z}} \frac{e^{2\pi i (k_1 + 2k_2) + 2\pi i \frac{3}{2} k_4 (u - v)} q^{k_1^2 + k_2^2 + k_1 k_2 - \frac{3}{4} k_4^2}}{1 - e^{2\pi i u} q^{2k_1 + k_2}} \tag{5.40}$$

$$- \sum_{k_1, k_2 \in \mathbb{Z}} e^{2\pi i (k_1 + 2k_2) + 2\pi i \frac{3}{2} k_4 (u - v)} q^{k_1^2 + k_2^2 + k_1 k_2 - \frac{3}{4} k_4^2},$$

which equals

$$2 b_{3,0}(v) \Phi(u, v) e^{2\pi i (u - v) \frac{3k_4}{2}} q^{-\frac{3}{4} k_4^2}. \tag{5.41}$$

As a result, we can express the sum over $k_4$ in the contributions of these terms to Equation (5.35) in terms of $R_{1,\alpha}$. Combining all terms we find that the first two lines of Equation (5.33)
equals
\[ 2b_{3,0}(v) (R_{1,\frac{3}{2}}(u-v) + R_{1,\frac{1}{2}}(u-v)) \Phi(u,v) \]
\[ + 2e^{\pi i(u-v)} q^{-\frac{1}{2}} b_{3,0}(v) (R_{1,\frac{1}{2}}(u-v) + R_{1,\frac{3}{2}}(u-v)) (\Phi(u+\tau,v) - \frac{1}{2}) \]
\[ - \sum_{k \in \mathbb{Z}^2 \pm \mu} \left( \text{sgn}(k_4) - E_1(\sqrt{3} y (k_4 + b - a)) \right) \text{sgn}(2k_3 - k_4) e^{2\pi i B(k,z)} q^{Q(k)/2}. \]  
(5.42)

Using the transformations:
\[ k_1 \mapsto k_1 - k_3 + k_4, \quad k_2 \mapsto k_2, \quad k_3 \mapsto k_3, \quad k_4 \mapsto k_4, \]
(5.43)
the sum over \( k_{1,2} \) decouples from the sum over \( k_{3,4} \), and can be factored out as \( b_{3,0} \). The third line of Equation (5.33) and the third line of Equation (5.42) combine to \( b_{3,0} R_{2,\frac{1}{2}}(-1,1) \) (5.22). Adding up all contributions and dividing by \( b_{3,0} \) gives the desired result.

\[ \square \]

### 5.4 Taylor expansions and iterated period integrals

A building block for the refined VW partition function \( f_{3,\mu} \) is the specialization \( \Psi_{\mu}(4z,-2z) \). The unrefined partition function involves the second Taylor coefficient in \( z \) of these functions. To derive the completion, we determine in this subsection the second derivative of the completion \( R_{\Psi_j}(5.22) \) at \( z = 0 \). To this end, we need to determine the first derivative of \( R_{1,\alpha} \) and the second derivative of \( R_{2,\alpha} \). We will express these derivatives as (iterated) integrals of theta series.

**Taylor expansion of \( R_{1,\alpha} \)**

We start with \( R_{1,\alpha} (5.12) \) and write its Taylor expansion as
\[ R_{1,\alpha}(z) = \sum_{\ell \geq 0} R_{1,\alpha}^{(\ell)} z^{\ell}. \]  
(5.44)

The first two coefficients are given by:
\[ R_{1,\alpha}^{(0)} = - \sum_{\ell \in \mathbb{Z}+\alpha} M_1(2\sqrt{3} y \ell) q^{-3\ell^2}, \]
\[ R_{1,\alpha}^{(1)} = -\sqrt{3} \frac{\pi i}{2} \int_{-\tau}^{\tau} \sum_{\ell \in \mathbb{Z}+\alpha} e^{6\pi i \ell^2 w} (-i(w+\tau))^{\frac{3}{2}} dw. \]  
(5.45)

**Proof**

The expression for \( R_{1,\alpha}^{(0)} \) follows immediately from the definition (5.12) and Equation (4.19). Note that for \( \alpha = 0 \) and \( \alpha = \frac{1}{2} \), \( R_{1,\alpha} \) vanishes.

To determine the first derivative, \( R_{1,\alpha}^{(1)} \), we first set \( F(z) = E_1(\sqrt{3} y (2\ell - a)) e^{6\pi i \ell z} \). For its first derivative, \( F^{(1)}(0) \), we find:
\[ F^{(1)}(0) = i \sqrt{3} \frac{\pi y \ell^2}{y} + 6\pi i \ell E_1(2\sqrt{3} y \ell). \]  
(5.46)
Substitution in $R^{(1)}_\alpha$ gives:

$$R^{(1)}_\alpha = - \sum_{\ell \in \mathbb{Z} + \alpha} \left[ \sqrt{\frac{2}{y}} e^{-12\pi \ell^2 y} + 6 \pi i \ell M_1(2\sqrt{3y\ell}) \right] q^{-3\ell^2}.$$  \hspace{1cm} (5.47)

Finally, substitution of Equation (4.23) gives the expression of Equation (5.45). □

**Taylor expansion of $R_{2,\beta}$**

We consider next the Taylor expansion of $R_{2,\beta}(z)$ around $z = 0$:

$$R_{2,\beta}(z) = \sum_{\ell \geq 0} R^{(\ell)}_{2,\beta} z^{\ell}. \hspace{1cm} (5.48)$$

Our interest is in two choices of $\beta$: $\beta = 0$ and $\frac{1}{3}(-1, 1)$. Since $R_{2,\beta}(z)$ is a symmetric function in $z$ for both choices, only even derivatives of $R_{2,\beta}$ are non-vanishing at $z = 0$. We have

$$R^{(0)}_{2,\beta} = \begin{cases} -\frac{1}{3}, & \beta = 0, \\ 0, & \beta = \frac{1}{3}(-1, 1), \end{cases}$$

$$R^{(1)}_{2,\beta} = 0,$$

$$R^{(2)}_{2,\beta} = -2\sqrt{3} \int_{-\pi}^{\pi} \int_{w_2}^{i\infty} \sum_{k_3, k_4 \in \mathbb{Z}^2 + \beta} \frac{e^{\frac{i\pi}{3}(2k_3 - k_4)^2 w_1 + \frac{2\pi i}{3} k_3^2 w_2}}{\sqrt{-(w_1 + \tau)^3(w_2 + \tau)^3}} dw_1 dw_2. \hspace{1cm} (5.49)$$

**Proof**

It follows from Equations (4.17) and (4.19), that the constant term equals

$$R^{(0)}_{2,\beta} = - \sum_{k_3, k_4 \in \mathbb{Z}^2 + \beta} M_2 \left( \frac{1}{\sqrt{3}}; \sqrt{y}(2k_3 - k_4), \sqrt{3y k_4} \right) q^{-k_3^2 - k_4^2 + k_3 k_4}. \hspace{1cm} (5.50)$$

This can be further simplified. To this end, we express the summand as an iterated period integral. For $2k_3 - k_4 \neq 0$ and $2k_4 - k_3 \neq 0$, this gives using Equation (4.27):

$$M_2 \left( \frac{1}{\sqrt{3}}; \sqrt{y}(2k_3 - k_4), \sqrt{3y k_4} \right) q^{-k_3^2 - k_4^2 + k_3 k_4} = \frac{\sqrt{3}}{2} (2k_3 - k_4) k_4 \int_{-\pi}^{\pi} \int_{w_2}^{i\infty} \frac{e^{\frac{i\pi}{3}(2k_3 - k_4)^2 w_1 + \frac{2\pi i}{3} k_3^2 w_2}}{\sqrt{-(w_1 + \tau)(w_2 + \tau)}} dw_1 dw_2 \hspace{1cm} (5.51)$$

Since we sum over all $k_3, k_4 \in \mathbb{Z}^2 + \beta$, we can change in the third line $k_3 \to -k_4$ and $k_4 \to k_3 - k_4$. Then the third line is cancelled by the second line. One proves similarly that the only non-vanishing contribution is due to $k_3 = k_4 = 0$. From Equation (4.30), we deduce the desired result.
To determine the second derivative, consider first \( K(z) = E_1(\sqrt{3y}(k_4 - a))e^{2\pi i(k_3+k_4)z} \). We find for the first derivative:

\[
K^{(1)}(z) = i\sqrt{\frac{3}{y}}e^{-3\pi yk_4^2 - 3\pi ya^2 + 2\pi iz(k_3 + \frac{1}{2}k_4) + 3\pi izk_4} + 2\pi i(k_3 + k_4)E_1(\sqrt{3y}(k_4 - a))e^{2\pi iz(k_3+k_4)}. \tag{5.51}
\]

Taking the second derivative at \( z = 0 \), we arrive at

\[
K^{(2)} = -2\pi \sqrt{\frac{2}{y}}(2k_3 + \frac{1}{2}k_4)e^{-3\pi yk_4^2} - 4\pi^2(k_3 + k_4)^2E_1(\sqrt{3y}k_4). \tag{5.52}
\]

Next we define:

\[
L(z) = E_2\left(\frac{1}{\sqrt{3}}; \sqrt{y}(2k_3 - k_4 - a), \sqrt{3y}(k_4 - a)\right)e^{2\pi i(k_3+k_4)z}. \tag{5.53}
\]

We find for its first derivative:

\[
L^{(1)}(z) = 2\pi i(k_3 + k_4)L(z) + i\sqrt{\frac{3}{y}}E_1(\sqrt{y}(2k_4 - k_3 - a))e^{-3\pi yk_4^2 - 3\pi ya^2 + 2\pi i(k_3 - \frac{1}{2}k_4)z + 3\pi ik_4}\tag{5.54}
\]

After differentiating one more time and setting \( z = 0 \), we find for \( L^{(2)} \):

\[
L^{(2)} = -4\pi^2(k_3 + k_4)^2E_2\left(\frac{1}{\sqrt{3}}; \sqrt{y}(2k_3 - k_4), \sqrt{3y}k_4\right)
- 2\pi \sqrt{\frac{2}{y}}(2k_4 + \frac{1}{2}k_3)E_1(\sqrt{y}(2k_4 - k_3))e^{-3\pi yk_4^2} - 2\pi \sqrt{\frac{2}{y}}(2k_3 + \frac{1}{2}k_4)E_1(\sqrt{y}(2k_3 - k_4))e^{-3\pi yk_4^2} - \frac{2\sqrt{3}}{y}e^{-4\pi y(k_3^2 - k_3k_4 + k_4^2)}. \tag{5.55}
\]

After substitution of \( K^{(2)} \) and \( L^{(2)} \) in \( R_{2\mu}^{(2)} \), we arrive at:

\[
R_{2\beta}^{(2)} = 4\pi^2 \sum_{k_3,k_4}(k_3 + k_4)^2M_2\left(\frac{1}{\sqrt{3}}; \sqrt{y}(2k_3 - k_4), \sqrt{3y}k_4\right)q^{-k_3^2 - k_4^2 + k_3k_4}
- 4\pi \sqrt{\frac{3}{y}}\sum_{k_3,k_4}(2k_3 + \frac{1}{2}k_4)\text{sgn}(2k_3 - k_4)q^{-\frac{1}{4}(2k_3 - k_4)^2}q^{\frac{3}{4}k_4^2}
+ 4\pi \sqrt{\frac{3}{y}}\sum_{k_3,k_4}(2k_3 + \frac{1}{2}k_4)E_1(\sqrt{y}(2k_3 - k_4))q^{-\frac{1}{4}(2k_3 - k_4)^2}q^{\frac{3}{4}k_4^2}
+ \frac{2\sqrt{3}}{y}b_{3,j\beta}, \tag{5.56}
\]

where \( j_\beta = 0 \) (respectively \( j_\beta = 1 \)) for \( \beta = 0 \) (respectively \( \beta = \frac{1}{2}(1,-1) \)). The first line in Equation (5.56) corresponds to combining the second term of \( K^{(2)}(0) \) and the first line of
The second line of Equation (5.56) is due to the first term of $K^{(2)}(0)$. Replacing the sgn$(\cdots)$ on the second line and $E_1$ on the third by $M_1$, we arrive at:

$$R_{2,\beta}^{(2)} = 4\pi^2 \sum_{k_3,k_4} (k_3 + k_4)^2 M_2 \left( \frac{1}{\sqrt{3}}: \sqrt{y(2k_3 - k_4)}, \sqrt{3} y k_4 \right) q^{-k_3^2 - k_4^2 + k_3 k_4}$$

$$+ 4\pi \sqrt{\frac{3}{y}} \sum_{k_3,k_4} (2k_3 + \frac{1}{2} k_4) M_1(\sqrt{y(2k_3 - k_4)}) q^{-\frac{1}{2}(2k_3 - k_4)^2} \frac{4}{q^{k_4^2}}$$

$$+ 2\sqrt{\frac{3}{y}} \tilde{b}_{3,j\beta},$$

In the following, we will write $R_{2,\beta}^{(2)}$ more concisely as a single iterated period integral. To this end, we substitute the expression for \(M_2\) as a period integral (4.27). Then the first line of Equation (5.59) becomes

$$- 4\sqrt{3}\pi^2 \sum_{k_3,k_4} (k_3 + k_4)^2 (2k_3 - k_4) k_4 \int_{-\pi}^{\infty} \int_{w_2}^{\infty} e^{\frac{\pi}{4}(2k_3 - k_4)^2 w_1 + \frac{\pi}{2} k_4^2 w_2} \frac{d w_1 d w_2}{\sqrt{(w_1 + \tau)(w_2 + \tau)}},$$

for $2k_3 - k_4 \neq 0$ and 0 otherwise. To bring $R_{2,\beta}^{(2)}$ in a simpler form, we write $\sum_{k_3,k_4}$ in Equation (5.58) as $\frac{1}{2} \sum_{k_3,k_4} + \frac{1}{2} \sum_{k_3,k_4}$ and make the transformation $k_3 \to k_4 - k_3$ and $k_4 \to k_4$ in the second sum. This shows that the first line of Equation (5.59) equals

$$- 6\sqrt{3}\pi^2 \sum_{k_3,k_4} (2k_3 - k_4)^2 k_4^2 \int_{-\pi}^{\infty} \int_{w_2}^{\infty} e^{\frac{\pi}{4}(2k_3 - k_4)^2 w_1 + \frac{\pi}{2} k_4^2 w_2} \frac{d w_1 d w_2}{\sqrt{(w_1 + \tau)(w_2 + \tau)}},$$

if $2k_3 - k_4 \neq 0$ and vanishes otherwise. To combine this line with the other lines, let us partially integrate with respect to $w_2$. This expresses Equation (5.59) as the sum of three integrals:

$$i4\sqrt{3}\pi \sum_{k_3,k_4} (2k_3 - k_4)^2$$

$$\left\{ - \frac{1}{\sqrt{2y}} q^{\frac{3}{4}k_4^2} \int_{-\pi}^{\infty} e^{\frac{\pi}{4}(2k_3 - k_4)^2 w_1} \frac{d w_1}{\sqrt{-i(w_1 + \tau)}} \right.$$\n
$$+ \int_{-\pi}^{\infty} e^{\frac{\pi}{4}(2k_3 - k_4)^2 w_1 + \frac{3\pi}{2} k_4^2 w_2} \frac{d w_2}{(-i(w_2 + \tau))}$$\n
$$\left. - \frac{i}{2} \int_{-\pi}^{\infty} \int_{w_2}^{\infty} e^{\frac{\pi}{4}(2k_3 - k_4)^2 w_1 + \frac{3\pi}{2} k_4^2 w_2} \frac{d w_1 d w_2}{\sqrt{(w_1 + \tau)(w_2 + \tau)^3}} \right\}. \quad (5.60)$$

To partially integrate the second integral to $w_2$, note that, due the symmetry $k_3 \leftrightarrow k_4$, the factor $(2k_3 - k_4)^2$ can be replaced by $2(k_3^2 + k_4^2 - k_3 k_4)$ for this term. After this substitution, the second integral in Equation (5.60) can easily be partially integrated to obtain:

$$i8\sqrt{3}\pi \sum_{k_3,k_4} (k_3^2 + k_4^2 - k_3 k_4) \int_{-\pi}^{\infty} e^{2\pi i (k_3^2 + k_4^2 - k_3 k_4) w_2} \frac{d w_2}{(-i(w_2 + \tau))}$$

$$= - \frac{2\sqrt{3}}{y} \tilde{b}_{3,j\mu} - 4i\sqrt{3} \sum_{k_3,k_4} \int_{-\pi}^{\infty} e^{2\pi i (k_3^2 + k_4^2 - k_3 k_4) w_2} \frac{d w_2}{(-i(w_2 + \tau))^2}. \quad (5.61)$$
Next, we partially integrate the third integral of Equation (5.60) to \( w \), which gives:

\[
4\sqrt{3} \int_{-\tau}^{\tau} e^{2\pi i (k_3^2 + k_4^2 - k_3 k_4) w} (-\tau(w_2 + \tau))^{3/2} \, dw_2 \\
- 2\sqrt{3} \int_{-\tau}^{\tau} \int_{w_2}^{\infty} \frac{e^{\frac{\pi i}{2}(2k_3 - k_4)^2 w_1 + \frac{3\pi i}{2} k_4^2 w_2}}{\sqrt{-(w_1 + \tau)^3(w_2 + \tau)^3}} \, dw_1 \, dw_2.
\]

(5.62)

As a result, we find that the first line of Equation (5.57) can be expressed as:

\[
- 4\pi \sqrt{3} y \sum_{k_3, k_4} (2k_3 - k_4) M_1(\sqrt{2}(2k_3 - k_4)) q^{-\frac{(2k_3 - k_4)^2}{4}} q^{\frac{3k_4^2}{4}} \\
- 2\sqrt{3} b_{3,j} - 2\sqrt{3} \sum_{k_3, k_4} \int_{-\tau}^{\tau} \int_{w_2}^{\infty} \frac{e^{\frac{\pi i}{2}(2k_3 - k_4)^2 w_1 + \frac{3\pi i}{2} k_4^2 w_2}}{\sqrt{-(w_1 + \tau)^3(w_2 + \tau)^3}} \, dw_1 \, dw_2,
\]

(5.63)

where we substituted Equation (4.22) for the first integral of Equation (5.60).

After addition of the remaining two lines of Equation (5.57), we see that the two terms with \( \tilde{b}_{3,j} \) cancel. Moreover, using the transformation \( k_3 \leftrightarrow k_4 - k_3, k_4 \leftrightarrow k_4 \), one may show that also the terms with \( M_1(\ldots) \) cancel. We thus finally arrive at

\[
R_{2,\beta}^{(2)} = -2\sqrt{3} \int_{-\tau}^{\tau} \int_{w_2}^{\infty} \sum_{k_3, k_4 \in \mathbb{Z}^2 + \beta} w \frac{e^{\frac{\pi i}{2}(2k_3 - k_4)^2 w_1 + \frac{3\pi i}{2} k_4^2 w_2}}{\sqrt{-(w_1 + \tau)^3(w_2 + \tau)^3}} \, dw_1 \, dw_2.
\]

(5.64)

where we brought the sum inside the integrand.

6 Derivation of the completed VW function

We return in this section to the VW partition functions \( h_{N,\mu} \). After discussing the general structure of the partition function and its refinement, we give explicit expressions for gauge groups \( U(2) \) and \( U(3) \) with fixed 't Hooft fluxes, and derive the completions using the previous sections.

6.1 Refined and unrefined partition functions

The holomorphic generating functions \( h_{N,\mu}(\tau) \) can be determined using algebraic-geometric techniques [27, 28, 45, 46, 47, 52]. In fact, one arrives using motivic techniques naturally at a refinement \( h_{N,\mu}(\tau, z) \) of \( h_{N,\mu}(\tau) \) (2.3), where the coefficients of \( h_{N,\mu}(\tau) \) are replaced by rational functions of an additional parameter \( w = e^{2\pi i z} \). This extra parameter arises in the partition function by including an extra fugacity \( z \) for the R-symmetry quantum number. The refinement \( h_{N,\mu}(\tau, z) \) is mathematically a generating function of (weighted) Poincaré polynomials rather than the Euler numbers of the instanton moduli spaces. We will concentrate on the modular properties of \( h_{N,\mu}(\tau) \), since they demonstrate the the action of the \( SL(2, \mathbb{Z}) \) S-duality group most distinctly.
The function $h_{N,\mu}(\tau)$ is actually only a generating function of Euler numbers, when the pair $(N, \mu)$ is relatively prime since the corresponding moduli space of HYM connections (or semi-stable sheaves) is then compact and smooth. In these cases, there is no ambiguity in the mathematical interpretation of the VW partition function as a generating function of Euler numbers or Poincaré polynomials of moduli spaces. However, when the pair $(N, \mu)$ is not coprime, the moduli spaces contain singularities, due to strictly semi-stable bundles. The topological content of the partition function is therefore more elusive. It is conceivable that evaluation of the VW path integral will lead to one of the various “types” of Euler numbers available for such spaces. While this is hard to carry out directly, we can alternatively avail of $S$-duality which relates the $h_{N,\mu}$ with different $\mu$. The $S$-duality relation (2.4) does indeed pick out a specific rational invariant of the moduli space. Namely, the multiple cover invariant $\bar{\chi}(\gamma)$. This rational invariant is explicitly given as:

$$\bar{\chi}(\gamma) = (-1)^{\dim \mathcal{M}_\gamma} \sum_{m \geq 1, m | \gamma} (-1)^{\dim \mathcal{M}_{\gamma/m}} \frac{\chi(\gamma/m)}{m^2},$$

(6.1)

where the sum runs over positive integers $m \in \mathbb{Z}_{>0}$ which divide the vector $\gamma$. Note that $(-1)^{\dim \mathcal{M}_\gamma}$ is independent of the instanton number $n$ and evaluates to $(-1)^{(N-1)(\mu^2-1)}$ for $\mathbb{P}^2$.

For a generic 't Hooft flux $\mu$, the exponentiated classical action evaluates to $q^{n + \frac{\mu^2}{2N} - \frac{N}{2}}$. The VW partition function for gauge group $U(N)$ and with fixed 't Hooft flux $\mu$, then reads

$$h_{N,\mu}(\tau) = \sum_{n \in \mathbb{Z} + \frac{\mu}{2}} \bar{\chi}(\gamma) q^{n + \frac{\mu^2}{2N} - \frac{N}{8}}.$$  

(6.2)

The VW partition function $h_{N,\mu}(\tau)$ can be derived from the refined partition function $h_{N,\mu}(\tau, z)$. The refined partition function is a generating function of the following rational functions of $w$:

$$\Omega(\gamma, w) = \sum_{\ell = -d}^{d} b_{\ell} w^\ell, \quad w \neq \pm 1,$$

(6.3)

where the positive integer $d$ equals $\dim \mathcal{M}_\gamma$. The numerator of $\Omega(\gamma, w)$ is a palindromic Laurent polynomial. More precisely, it is the intersection Poincaré polynomial of the (possibly singular) moduli space of semi-stable sheaves $\mathcal{M}_\gamma$ multiplied by $w^{-d}$ [75]. The corresponding rational invariant $\bar{\Omega}(\gamma, w)$ is defined as:

$$\bar{\Omega}(\gamma, w) = \sum_{m | \gamma} \frac{\Omega(\gamma/m, -(-w)^m)}{m}.$$  

(6.4)

These are the coefficients of the refined partition function $h_{N,\mu}(\tau, z)$:

$$h_{N,\mu}(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{\mu}{2}} \bar{\Omega}(\gamma, w) q^{n + \frac{\mu^2}{2N} - \frac{N}{8}}.$$  

(6.5)

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One can show for the projective space $\mathbb{P}^2$, that $\bar{\Omega}(\gamma, -w) = \pm \bar{\Omega}(\gamma, w)$, since all cohomology is even. Rather then taking the limit $w \to -1$, the numerical invariants $\chi$ and $\bar{\chi}$ can therefore be obtained from $\Omega$ and $\bar{\Omega}$ by taking the simpler limit $w \to 1$,

\[
\chi(\gamma) = \lim_{w \to 1} (w - w^{-1}) \Omega(\gamma, w), \\
\bar{\chi}(\gamma) = \lim_{w \to 1} (w - w^{-1}) \bar{\Omega}(\gamma, w).
\] (6.6)

Let us now turn to explicit expressions of the VW partition functions. For gauge group $U(1)$, the refined partition function $h_{1,0}(\tau, z)$ is simply the inverse of a Jacobi theta series (4.3)

\[
h_{1,0}(\tau, z) = \frac{i}{\theta_1(\tau, 2z)},
\] (6.7)

In the mathematical literature, $h_{1,0}$ is known as Göttsche’s formula [47] for the cohomology of the Hilbert schemes of points on $\mathbb{P}^2$. The VW partition function follows straightforwardly,

\[
h_{1,0}(\tau) = \lim_{z \to 0} 4\pi iz h_{1,0}(\tau, z) = \frac{1}{\eta(\tau)^3}.
\] (6.8)

Before presenting the explicit expression for $N = 2$ and 3, we explain the structure for arbitrary $N$. The refined (respectively numerical) VW partition functions $h_{N,\mu}$ factorize in terms of the $N$'th power of the $U(1)$ partition function times another function, which we denote by $g_{N,\mu}(\tau, z)$ (respectively $f_{N,\mu}(\tau)$). We have more explicitly,

\[
h_{N,\mu}(\tau, z) = g_{N,\mu}(\tau, z) h_{1}(\tau, z)^N, \\
h_{N,\mu}(\tau) = f_{N,\mu}(\tau) \eta(\tau)^{3N}.
\] (6.9)

One may think of the $g_{N,\mu}$ and $f_{N,\mu}$ in these expressions as being due to smooth instantons, and $h_{1,0}$ as due the cusps of the moduli space where instantons become point-like. Note that since $\lim_{z \to 0} z h_{N,\mu}(\tau, z)$ is finite following Equation (6.6), and $h_{1,0}(\tau, z)$ has a simple pole at $z = 0$, $g_{N,\mu}(\tau, z)$ has a zero of multiplicity $N - 1$ at $z = 0$. As a result, we can write $f_{N,\mu}$ as the $(N - 1)$’th derivative of the refined partition function:

\[
f_{N,\mu}(\tau) = \frac{1}{(N - 1)!} \left(\frac{1}{2\pi i} \partial_z\right)^{N-1} g_{N,\mu}(\tau, z) |_{z=0}.
\] (6.10)

The transformation properties of $\eta$ are given in Equation (3.6), and we are therefore left with determining the modular properties of $f_{N,\mu}$ to verify the modularity of the VW partition function $h_{N,\mu}$. We derive easily from Equation (2.4), that the expected transformation properties for the $f_{N,\mu}$ are:

\[
f_{N,\mu} \left( -\frac{1}{\tau} \right) = \frac{1}{\sqrt{N}} (-i\tau)^{\frac{3}{2}(N-1)} (-1)^{N-1} \sum_{\nu \mod N} e^{-\frac{2\pi i \nu}{N}} f_{N,\mu}(\tau),
\] (6.11)

\[
f_{N,\mu}(\tau + 1) = (-1)^\mu e^{2\pi i \frac{\mu^2}{2N}} f_{N,\mu}(\tau)
\]
Of course, it was established for \( N = 2 \) in Reference \[3\], that one needs to replace \( f_{2,\mu} \) by a suitable completion \( \hat{f}_{2,\mu} \) to arrive at functions which transform as a modular form. We rederive the completion in the next subsection. A completion is similarly required for \( N = 3 \), which will be derived in Subsection \( 6.3 \).

### 6.2 Gauge group \( U(2) \)

We review in this subsection the VW partition function for \( N = 2 \) and its modular completion. The refined partition functions are determined by Yoshioka \[45, 46\] and equal

\[
\begin{align*}
g^{2,0}(\tau, z) &= \frac{1}{2} + \frac{q^{-\frac{3}{2}}w^5}{\theta_2(2\tau, 2z)} \sum_{n \in \mathbb{Z}} \frac{q^{n^2+n}w^{-2n}}{1-w^4q^{2n}-1}, \\
g^{2,1}(\tau, z) &= \frac{q^{-\frac{1}{2}}w^3}{\theta_3(2\tau, 2z)} \sum_{n \in \mathbb{Z}} \frac{q^{n^2}w^{-2n}}{1-w^4q^{2n}-1}.
\end{align*}
\]  

(6.12)

For later reference, we note that these relations can also be expressed in terms of \( \mu \) and \( \Phi \) as

\[
\begin{align*}
g^{2,0}(\tau, z) &= \frac{1}{2} - q^{-\frac{1}{2}}w^3\mu(2\tau, 4z - \tau, -2z + \frac{1}{2}) \\
&= \Phi(\tau, 4z, -2z) + \frac{i \eta(\tau)^3\theta_3(6\tau, 6z)}{\theta_1(\tau, 4z) b_{3,0}(\tau, 2z)}, \\
g^{2,1}(\tau, z) &= -\mu(2\tau, 4z - \tau, -2z - \tau + \frac{1}{2}) \\
&= w^{-3}q^{-\frac{1}{4}}\left(\Phi(\tau, 4z, -2z - \tau) - \frac{1}{2}\right) + \frac{i \eta(\tau)^3\theta_2(6\tau, 6z)}{\theta_1(\tau, 4z) b_{3,0}(\tau, 2z)}.
\end{align*}
\]  

(6.13)

Using Equation (6.10), one may show that the VW partition functions in this case are generating functions of the Hurwitz class numbers \( H(n) \) \[3, 76\]:

\[
f^{2,\mu}(\tau) = \sum_{n \geq 0} H(4n-\mu) q^{n-\frac{3}{2}}, \quad \mu = 0, 1.
\]  

(6.14)

The modular completions of the refined partition functions \( g^{2,\mu}(\tau, z) \) follow from the modular completion of the \( \mu \)-function in Equations (5.3) and (5.6). They are given by\[12\]

\[
\begin{align*}
\tilde{g}^{2,0}(\tau, z) &= g^{2,0}(\tau, z) + \frac{1}{2} \sum_{\ell \in \mathbb{Z}} \left( \text{sgn}(\ell) - E_1(\sqrt{\gamma(2\ell - 3v)})\right) q^{-\ell^2}w^{3\ell}, \\
\tilde{g}^{2,1}(\tau, z) &= g^{2,1}(\tau, z) + \frac{1}{2} \sum_{\ell \in \mathbb{Z} + \frac{1}{2}} \left( \text{sgn}(\ell) - E_1(\sqrt{\gamma(2\ell - 3v)})\right) q^{-\ell^2}w^{3\ell}.
\end{align*}
\]  

(6.15)

Note that the \( \frac{1}{2} \) in the expression for \( g^{2,0}(\tau, z) \) (6.12) is in fact a non-subleading, holomorphic part of the completion of \( \mu(2\tau, 4z - \tau, -2z + \frac{1}{2}) \), and should therefore not be considered as part

\[\text{\footnote{We have chosen to omit } \bar{\tau} \text{ and } \bar{z} \text{ from the arguments of } \tilde{g}^{2,\mu}(\tau, z)\}.\]
of the subleading, non-holomorphic terms. Taking the derivative to \( z = 0 \), we arrive at the modular completions of \( f_{2,\mu}(\tau) \)

\[
\hat{f}_{2,\mu}(\tau, \bar{\tau}) = f_{2,\mu}(\tau) + \frac{3(1 + i)}{8\pi} \int_{-\pi}^{i\infty} \frac{\Theta_{\frac{3}{2}}(v)}{(\tau + v)^{\frac{3}{2}}} dv,
\]

\[
= f_{2,\mu}(\tau) - \frac{3i}{4\sqrt{2\pi}} \int_{-\pi}^{i\infty} \frac{\Theta_{\frac{3}{2}}(v)}{(-i(v + \tau))^\frac{3}{2}} dv,
\]

(6.16)

The \( \hat{f}_{2,\mu} \) have precisely the expected transformation properties \((6.11)\). As discussed in Section \(2.2\) the non-holomorphic period integral implies an elegant equation for the holomorphic anomaly \((2.22)\).

### 6.3 Gauge group \(U(3)\)

Next we move on to \( N = 3\). Also for this gauge group, there are only two independent ‘t Hooft fluxes and therefore only two independent partition functions, \( f_{3,\mu} \) with \( \mu = 0, 1 \). The explicit expressions for the refined partition functions are \([28]\):

\[
g_{3,0}(\tau, z) = \frac{1}{b_{3,0}(\tau, 2z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-2k_1 - 4k_2} q^{k_1^2 + k_2^2 + k_1 k_2}}{(1 - w^4 q^{2k_1 + k_2})(1 - w^4 q^{k_2 - k_1})} + \frac{2i\eta(\tau)^3}{\theta_1(\tau, 4z) b_{3,0}(\tau, 2z)} \sum_{k \in \mathbb{Z}} \frac{w^{-6k} q^{-3k^2}}{1 - w^6 q^{3k}} - \frac{\eta(\tau)^6 \theta_1(\tau, 2z)}{\theta_1(\tau, 4z)^2 \theta_1(\tau, 6z) b_{3,0}(\tau, 2z)} - g_{2,0}(\tau, z) - \frac{1}{6},
\]

(6.17)

and

\[
g_{3,1}(\tau, z) = \frac{1}{b_{3,0}(\tau, 2z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-2k_1 - 4k_2 + 6} q^{k_1^2 + k_2^2 + k_1 k_2 - \frac{3}{4}}}{(1 - w^4 q^{2k_1 + k_2 - 1})(1 - w^4 q^{k_2 - k_1})} + \frac{i\eta(\tau)^3}{\theta_1(\tau, 4z) b_{3,0}(\tau, 2z)} \left( \sum_{k \in \mathbb{Z}} \frac{w^{-6k + 6} q^{3k^2 - \frac{3}{4}}}{1 - w^6 q^{3k - 1}} + \sum_{k \in \mathbb{Z}} \frac{w^{-6k} q^{3k^2 + 3k + \frac{3}{2}}}{1 - w^6 q^{3k + 1}} \right).
\]

(6.18)

Note that a few different expressions are available for the \( f_{3,\mu} \), which are related by the blow-up formula \([28, 58]\).

The VW partition functions are obtained from Equations \((6.17)\) and \((6.18)\) using Equation \((6.10)\).\(^{13}\) The first few coefficients are:

\[
f_{3,0}(\tau) = \frac{1}{9} - q + 3 q^2 + 17 q^3 + 41 q^4 + 78 q^5 + 120 q^6 + O(q^7),
\]

\[
f_{3,1}(\tau) = 3 q^\frac{3}{4} (1 + 5 q + 12 q^2 + 23 q^3 + 38 q^4 + 55 q^5 + O(q^6)).
\]

\(^{13}\)Expressions for \( f_{3,1}(\tau) \) were also determined using localization with respect to the toric symmetry of \( \mathbb{P}^2 \) by Weist \([77]\) and Kool \([78]\).
See Appendix A for explicit expressions for the \( q \)-series of these functions, and their first 30 coefficients. These coefficients indicate that their growth is polynomial. Also note that all coefficients of \( f_{3,0} \) are integers, except the constant term. This is consistent with Equation (6.11), since the coefficients of \( \eta(\tau)^9/(9 \eta(3\tau)^3) \) are integers, except for the constant term. In the following we will determine the modular completions of \( f_{3,\mu} \) in a similar way as was discussed for \( U(2) \). The modular properties of the holomorphic \( q \)-series \( f_{3,\mu} \) (2.11) follow easily from the modular properties of the non-holomorphic terms.

**Completion of \( f_{3,0} \)**

The completion of the refined partition function \( g_{3,0}(\tau, z) \) is given by:

\[
\hat{g}_{3,0}(\tau, z) = g_{3,0}(\tau, z) - \frac{1}{12} - \frac{1}{4} R_{2,0}(\tau, 6z) - \sum_{\mu=0,1} q_{2,\mu}(\tau, z) R_{1,\mu}(\tau, 6z).
\]  

(6.20)

while the completion of the VW partition function \( f_{3,0}(\tau) \) is given:

\[
\hat{f}_{3,0}(\tau, \bar{\tau}) = f_{3,0}(\tau) - \frac{i}{\pi} \left( \frac{3}{2} \right)^{3/2} \sum_{\mu=0,1} \int_{-\tau}^{\tau} \frac{\hat{R}_{2,\mu}(\tau, -v) \Theta_2(3v)}{(-i(v + \tau))^{3/2}} dv.
\]  

(6.21)

**Proof**

We start by bringing \( g_{3,0}(\tau, z) \) in a more convenient form by substituting Equation (6.13) for \( g_{2,0}(\tau, z) \) in Equation (6.17). After combining the first term on the rhs of \( g_{2,0}(\tau, z) \) in Equation (6.13) with the first term on the rhs of Equation (6.17), one can express \( g_{3,0} \) as

\[
g_{3,0}(\tau, z) = \frac{1}{4} + \frac{w^4}{b_{3,0}(\tau, 2z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-2k_1 - 4k_2} q^{k_1^2 + k_2^2 + k_1 + 2k_2 + 2k_1 + k_2}}{(1 - w^4 q^{2k_1 + k_2})(1 - w^4 q^{2k_2 - k_1})}
\]

\[
+ \frac{2i \eta(\tau)^3}{\theta_1(\tau, 4z) b_{3,0}(\tau, 2z)} \left( -\frac{1}{2} \theta_3(6\tau, 6z) + \sum_{k \in \mathbb{Z}} w^{-6k} q^{3k^2} \frac{1}{1 - w^6 q^{3k}} \right)
\]

\[
- \frac{\eta(\tau)^6}{\theta_1(\tau, 4z)^2 \theta_1(\tau, 6z) b_{3,0}(\tau, 2z)} + \frac{1}{12}.
\]  

(6.22)

We can now determine the completion \( \hat{f}_{3,0} \) by determining the completions using the results of Section 5. Working line by line, we arrive at the following completions:

1. The first line of the rhs of Equation (6.22) equals \( \Psi_0(4z, -2z) \), a specialization of \( \Psi_0(u, v) \) which is defined in (5.21). The completion of \( \Psi_0(u, v) \) is given by Equation (5.24). Specialization of the latter, provides the completion of the first line:

\[
- \frac{1}{4} R_{2,0}(6z) - \Phi(4z, -2z) R_{1,0}(6z) - w^{-3} q^{3/2} \left( \Phi(4z + \tau, -2z) - \frac{1}{2} \right) R_{1,1}(6z).
\]  

(6.23)
2. The completion of the second line follows from Equation (5.6), and equals
\[ -\frac{i\eta(\tau)^3}{\theta_1(\tau, 4z) \theta_3(\tau, 2z)} (\theta_3(6\tau, 6z) R_{1,0}(6z) + \theta_2(6\tau, 6z) R_{1,1}(6z)) \] (6.24)

3. The first term of the third line transforms as a Jacobi form of weight 1 and index \(-36\) and does therefore not require a completion. However, the constant term \(\frac{1}{12}\) does not transform appropriately. Therefore we subtract it, such that the completion of the third line is
\[ -\frac{1}{12}. \] (6.25)

Adding the three contributions above and substitution of the \(N = 2\) partition functions \(g_{2,\mu}\) using Equation (6.13), we find the claimed expression in Equation (6.20).

Our next aim is to determine the modular completion of the VW partition function. To this end, we make a Taylor expansion of \(\hat{g}_{3,0}(\tau, z)\) around \(z = 0\). We already discussed that the constant and linear term of \(g_{3,0}(\tau, z)\) vanishes. This is in fact also the case for \(\hat{g}_{3,0}\). To see this, note that the constant \(-\frac{1}{12}\) in Equation (6.20) cancels against the constant term of \(R_{2,\mu}\) (5.38). Furthermore, the \(g_{2,\mu}\) and \(R_{1,\alpha}\) in Equation (6.20) all start with a linear term in \(z\).

Let us now determine the completion \(\hat{f}_{3,0}(\tau)\) by determining the quadratic term in the Taylor expansion of \(\hat{g}_{3,0}(\tau, z)\) using the results of Section 5.4. Substitution of Equations (5.12) and (5.48) for \(R_{1,0}^{(1)}\) and \(R_{2,0}^{(2)}\) in the \(\frac{1}{2}3\theta_3^2\hat{g}_{3,0}(\tau, z)|_{z=0}\), gives
\[ \hat{f}_{3,0}(\tau, \overline{\tau}) = f_{3,0}(\tau) + \frac{6}{4\pi i} \sum_{\mu=0,1} f_{2,\mu}(\tau) \sqrt{\frac{3}{2}} \sum_{\ell \in \mathbb{Z} + \frac{\mu}{2}} \int_{-\tau}^{i\infty} (-i(w + \tau))^\tau e^{\pi i \ell^2 w} dw \]
\[ + \frac{36}{(4\pi i)^2 2} \int_{-\tau}^{i\infty} \int_{w_2}^{i\infty} \sum_{k_3, k_4 \in \mathbb{Z}} e^{\pi i \frac{2k_3 - k_4}{2} w_1 + \frac{3\pi i k_4^2}{2} w_2} \sqrt{-((w_1 + \tau)^3(w_2 + \tau)^3)} dw_1 dw_2. \] (6.26)

We split the sum over \(k_3\) and \(k_4\) on the second line into a sum with \(k_4\) even and one with \(k_4\) odd. Then the sum splits into a sum of products of theta series:
\[ \sum_{k_3, k_4 \in \mathbb{Z}} e^{\pi i \frac{2k_3 - k_4}{2} w_1 + \frac{3\pi i k_4^2}{2} w_2} = \sum_{\mu=0,1} \Theta_\frac{\mu}{2}(w_1) \Theta_\frac{\mu}{2}(3w_2), \] (6.27)
with \(\Theta_\alpha\) as in Equation (2.8). Then using Equation (6.10), we can further simplify \(\hat{f}_{3,0}\) to the expression in Equation (6.21).

**Completion of \(f_{3,1}\)**

The completion of the refined partition function \(g_{3,1}(\tau, z)\) is given by:
\[ \hat{g}_{3,1}(\tau, z) = g_{3,1}(\tau, z) - \frac{1}{4} R_{2,\frac{1}{4}}(-1,1)(6z) \]
\[ - \frac{1}{2} g_{2,0}(\tau, z) (R_{1,\frac{1}{4}}(6z) - R_{1,\frac{1}{4}}(-6z)) \] (6.28)
\[ - \frac{1}{2} g_{2,1}(\tau, z) (R_{1,\frac{1}{4}}(6z) - R_{1,\frac{1}{4}}(-6z)). \]
From a Taylor expansion of $\hat{g}_{3,1}$, one can derive that the completion $\hat{f}_{3,1}$ of the VW partition function $f_{3,1}$ is given by:

$$\hat{f}_{3,1}(\tau, \bar{\tau}) = f_{3,1}(\tau) - \frac{i}{\pi} \left(\frac{3}{2}\right)^{\frac{3}{2}} \sum_{\mu=0,1} \int_{-\pi}^{i\infty} \frac{\hat{f}_{2,\mu}(\tau, -v) \Theta_{\frac{1}{3} + \frac{\mu}{2}}(3v)}{(-i(v + \tau))^\frac{3}{2}} dv. \quad (6.29)$$

**Proof**

The derivation of the completion of $f_{3,1}(\tau)$ is similar to the one of $f_{3,0}(z)$ discussed above. The explicit expression for the refined generating function $f_{3,1}(\tau, z)$ was given in Equation (6.18).

Section 5 provides the completion of each line as before:

1. The first line of Equation (6.18) equals the Appell-Lerch $\Psi_1(4z, -2z)$ with signature $(2, 2)$ defined in Equation (5.21). The additional non-holomorphic terms follow by specializing $R_{\Psi_1}$ (5.25):

$$-\frac{1}{4} R_{1,\frac{1}{3}(-1,1)}(6z) - \frac{1}{2} \Phi(4z, -2z)(R_{1,\frac{1}{3}}(6z) - R_{1,\frac{1}{3}}(-6z))$$

$$-\frac{1}{2} w^{-3} q^{-\frac{1}{2}} \left(\Phi(4z + \tau, -2z) - \frac{1}{2} \left(R_{1,\frac{1}{3}}(6z) - R_{1,\frac{1}{3}}(-6z)\right)\right). \quad (6.30)$$

2. The second line of Equation (6.18) can be expressed in terms of $A_1$ and $A_2$ defined in Equation (5.10). The additional terms for the completion can be read off from Equation (5.11) and read:

$$-\frac{1}{2} \theta_3(6\tau, 6z) \left(\Theta_{\frac{1}{3}}(6z) - R_{1,\frac{1}{3}}(-6z)\right) + \frac{i\eta^3}{2} \frac{\theta_1(\tau, 4z) b_{3,0}(\tau, 2z)}{\theta_3(6\tau, 6z) \left(\Theta_{\frac{1}{3}}(6z) - R_{1,\frac{1}{3}}(-6z)\right) + \theta_3(6\tau, 6z) \left(\Theta_{\frac{1}{3}}(6z) - R_{1,\frac{1}{3}}(-6z)\right)} \right). \quad (6.31)$$

Adding these terms to the holomorphic part $g_{3,1}$, and substitution of Equation (6.13) gives for $\hat{g}_{3,1}(\tau, z)$ the expression in Equation (6.28). The completion of the VW partition function $f_{3,1}(\tau)$ is obtained as for $f_{3,0}(\tau)$ above, by determining $\frac{1}{2} \partial_z^2 \hat{f}_{3,1}(\tau, z)|_{z=0}$. One arrives in this way at the expression in Equation (6.29). \qed

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A Explicit $q$-series for $f_{3,\mu}$

A.1 $q$-series

The $q$-series $f_{3,\mu}$ is defined in terms of the $g_{3,\mu}(\tau, z)$ by Equation (6.10). Based on the explicit expressions for $g_{3,\mu}(\tau, z)$, (6.18) and (6.22), we can derive explicit $q$-series for $f_{3,\mu}(\tau)$. To this end, recall the classical Eisenstein series $E_k(\tau)$ of weight $k \in 2\mathbb{N}$, which have the $q$-expansion

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1-q^n},$$

where $q = e^{2\pi i \tau}$, and $B_k$ are the Bernoulli numbers, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, etc. We define furthermore the following summands

$$S_1,\mu(k; q) = \frac{(-1 + E_2(\tau))(k - \mu + 1)}{2(1 - q^{3k-\mu})} + \frac{9(k - \mu)^2 + 33(k - \mu) + 31 - E_2(\tau)}{2(1 - q^{3k-\mu})^2}$$

$$- \frac{15(k - \mu) + 34}{(1 - q^{3k-\mu})^3} + \frac{19}{(1 - q^{3k-\mu})^4},$$

$$S_2(A, B; q) = \frac{4q^B}{(1 - q^A)(1 - q^B)} + \frac{4q^A}{(1 - q^A)(1 - q^B)} + \frac{4}{(1 - q^A)^2(1 - q^B)^2} - \frac{2(A + B + 1)q^B}{(1 - q^A)(1 - q^B)^2} - \frac{2(A + B + 1)q^A}{(1 - q^A)^2(1 - q^B)} + \frac{(A + B - 2)^2 - 8}{2(1 - q^A)(1 - q^B)}.$$ 

We then have:

- $b_{3,0} f_{3,0}$ can be expressed as:

$$b_{3,0}(\tau) f_{3,0}(\tau) = \frac{13}{1} \frac{1}{240} + \frac{1}{24} E_2(\tau) + \frac{1}{12} E_2(\tau)^2 + \frac{1}{720} E_4(\tau) - \frac{9}{2} \sum_{k \in \mathbb{Z}} k^2 q^{3k^2} + \frac{1}{6} \sum_{k_1, k_2 \in \mathbb{Z}} (k_1 + 2k_2)^2 q^{k_1^2 + k_2^2 + k_1 k_2}$$

$$+ \sum_{k \in \mathbb{Z}} S_1,0(k; q) q^{3k^2} + \sum_{k \in \mathbb{Z}} S_1,0(k; q) q^{3k^2}$$

- $b_{3,0} f_{3,1}$ can be expressed as:

$$b_{3,0}(\tau) f_{3,1}(\tau) = \sum_{k \in \mathbb{Z}} S_1,1(k; q) q^{3k^2-\frac{1}{3}}$$

$$+ \sum_{k \in \mathbb{Z}} S_2(2k_1 + k_2 - 1, k_2; q) q^{k_1^2 + k_2^2 + k_1 k_2 - \frac{1}{3}}.$$
Proof

We prove Equation (A.4) by determining the coefficient of $z^2$ in the Taylor expansion of $b_{3,0}(\tau, z) g_{3,1}(\tau, z)$. We rewrite Equation (6.18) as

\[
\begin{align*}
& b_{3,0}(\tau, z) g_{3,1}(\tau, z) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-2k_1 - 4k_2 + 6 q^{k_1^2 + k_2^2 + k_1 k_2 - \frac{4}{3}}}}{(1 - w^4 q^{2k_1 + k_2 - 1})(1 - w^4 q^{k_1 - k_2})} \\
& \quad + \frac{1}{1 - w^4} \left( \frac{w^{-6k + 6 q^{3k^2 - \frac{1}{3}}}}{1 - w^4 q^{3k - 1}} + \frac{w^{-6k + 2 q^{3k^2 + 3k + \frac{2}{3}}}}{1 - w^4 q^{3k + 1}} \right) \\
& \quad + \frac{i \eta(\tau)^3}{\theta_1(\tau, z)} \left( \sum_{k \in \mathbb{Z}} \frac{w^{-6k + 6 q^{3k^2 - \frac{1}{6}}}}{1 - w^6 q^{3k - 1}} + \sum_{k \in \mathbb{Z}} \frac{w^{-6k q^{3k^2 + 3k + \frac{2}{3}}}}{1 - w^6 q^{3k + 1}} \right), \quad (A.5)
\end{align*}
\]

where the first line is finite in the limit $z \to 0$. The second and third line have a first order pole, which cancel each other. The Taylor coefficient of $z^2$ of $e^{-\frac{(2A + 2B - 4)z}{(1 - e^{4iz})(1 - e^{-4iz})}}$ with $A, B \neq 0$ is given by $4 S_2(A, B; q)$ (A.2), which reproduces the second line of Equation (A.4). Using the Laurent expansion

\[
\frac{i \eta(\tau)^3}{\theta_1(\tau, z)} = \frac{1}{2 \pi i z} - \frac{1}{24} E_2(\tau) (2\pi i z) + \frac{1}{5760} (5 E_2(\tau)^2 + 2 E_4(\tau))(2\pi i z)^3 + \ldots, \quad (A.6)
\]

one can verify that the coefficient of $(2\pi i z)^2$ in the Taylor expansion of the sum of the second and third line of (A.5) is $4 S_{1,1}(k; q)$. We thus arrive at the claimed result for $f_{3,1}$.

To determine $f_{3,0}$, one groups terms of $b_{3,0} g_{3,0}$ which are finite for $z \to 0$, or have poles of order one and two. The claimed expression for $f_{3,0}$ (A.3) follows then in a similar way for $f_{3,1}$. □

A.2 Coefficients

Let $d_\mu(n)$ be the coefficients of $f_{3,\mu}$, $\mu = 0, 1$, defined as

\[
f_{3,\mu}(\tau) = \sum_{n \geq 0} d_\mu(n) q^{n - \frac{\mu}{3}}. \quad (A.7)
\]

Using Equations (A.3) and (A.4) it is easy to determine the first $d_\mu(n)$. Table 1 lists the first 30 coefficients $d_\mu(n)$ for $\mu = 0$ and 1.
\begin{tabular}{|c|c|c|}
\hline
\(n\) & \(d_0(n)\) & \(d_1(n)\) \\
\hline
0 & \(\frac{1}{5}\) & 0 \\
1 & -1 & 0 \\
2 & 3 & 3 \\
3 & 17 & 15 \\
4 & 41 & 36 \\
5 & 78 & 69 \\
6 & 120 & 114 \\
7 & 193 & 165 \\
8 & 240 & 246 \\
9 & 359 & 303 \\
10 & 414 & 432 \\
11 & 579 & 492 \\
12 & 626 & 669 \\
13 & 856 & 726 \\
14 & 906 & 975 \\
15 & 1194 & 999 \\
16 & 1172 & 1332 \\
17 & 1638 & 1338 \\
18 & 1569 & 1743 \\
19 & 1987 & 1716 \\
20 & 2040 & 2226 \\
21 & 2578 & 2130 \\
22 & 2340 & 2775 \\
23 & 3255 & 2625 \\
24 & 2940 & 3354 \\
25 & 3665 & 3129 \\
26 & 3642 & 4041 \\
27 & 4490 & 3735 \\
28 & 3940 & 4752 \\
29 & 5484 & 4317 \\
30 & 4734 & 5532 \\
\hline
\end{tabular}

Table 1: First 30 coefficients of the functions \(f_{3,\mu}, \mu = 0, 1\), discussed in the main text.

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