Quantum Poincaré Subgroup of q-Conformal Group and q-Minkowski Geometry

by

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Abstract

We construct quantum deformation of Poincaré group using as a starting point $SU(2, 2)$ conformal group and twistor-like definition of the Minkowski space. We obtain quantum deformation of $SU(2, 2)$ as a real form of multi-parametric $GL(4, C)_{q_{ij}, r}$. It is shown that Poincaré subgroup exists for special nonstandard one-parametric deformation only, the deformation parameter $r$ being equal to unity. This leads to commuting affine structure of the corresponding Minkowski space and simple structure of the corresponding Lie algebra, the deformation of the group being non-trivial.
1 Introduction

At present there is well developed theory of quantum semisimple matrix Lie groups and algebras [1]-[3]. But as is well known in classical case, Minkowski geometry is intrinsically connected with inhomogeneous Poincaré group. General theory of q-deformation of inhomogeneous groups is absent. Method of quantum deformations of inhomogeneous groups via projective space approach was considered in [4]. In this approach inhomogeneous transformations are represented by block-triangular matrix so that general theory of matrix quantum groups can be applied. Unfortunately, the standard quantum deformation in this case leads to the appearance of additional dilatations.

Another way to attack the problem is to start from quantum deformation of Poincaré Lie algebra [5]-[10]. As is well known we can consider q-Lie algebra from two closely related point of view [1]: as quantum universal enveloping (QUE) algebra and as dual space to q-group space. According to these forms there are two approaches to q-Poincaré algebra problem.

In one of them [8] q-Poincaré algebras are derived from the action of generators on the noncommuting coordinates of quantum space-time. One parametric deformation once more forces to include dilatations (note that the authors of [10] do not meet this problem because they do not consider coproduct and hence the full Hopf algebra structure of q-Poincaré algebra).

Another approach [9] is based on contraction of QUE anti-de Sitter algebra \( U_q(O(3,2)) \), the deforming parameter \( q \) being also contracted [11]. As a result one obtains QUE Poincaré algebra and deformed dynamics of quantum field theoretical model (deformed Klein-Gordon equation) defined on the space-time with usual commutative geometry. Important problem here is to find corresponding q-group of space-time transformations (contraction on q-group level is not well understood yet, especially in the case of q-parameter contraction). The attempts to find QUE Poincaré algebra as a Hopf subalgebra of \( U_q(O(3,2)) \) or \( U_q(SU(2,2)) \) [11], [12] meet the same problem as in the case of inhomogeneous q-group: one has to add generators of dilatations to define selfconsistent coproduct. In [11] the general consideration of the problem based on Cartan automorphisms and Bruhat decomposition of non-compact algebras was done in the context of canonical procedure for q-deformation of non-compact Lie algebras. In particular, it was shown that Poincaré containing Hopf subalgebra of the quantum conformal algebra has to include dilatations, i.e. to be the 11-generators Weyl algebra (parabolic-type subalgebra).

We would like to note that it is very natural to look for q-Poincaré group as a q-subgroup of q-conformal one because of at least two reasons: i) to start from \( SU(2,2) \) is very desirable from the physical point of view because of well known important role of the conformal group at high energy and small distances where we hope noncommutative geometry becomes essential (direct construction of the q-Poincaré group does not provide automatically that it is a q-subgroup of some quantum deformation of conformal group); ii) conformally invariant field theory has the problems because of ultra-violet divergences, but the most attractive aim of q-deformation is to remove any divergences, so to start from q-conformal symmetry is logically self-consistent approach.

In this paper we apply general idea of multiparametric deformation for obtaining the inhomogeneous groups developed in our previous work [14], to physically interesting case of the q-Poincaré group. To avoid q-group contractions, we use as a starting point the twistor approach (see, e.g. [14]) to the definition of Minkowski space and Poincaré group.
2 Quantum deformation of the Poincaré group

Let us remind some basic facts concerning $G(2, 4)$ and its transformations by conformal and Poincaré group in classical case [15]. The Grassmann manifold $G(2, 4)$ is a set of 2-dimensional complex subspaces $C^2$ of 4-dimensional space $C^4$. The homogeneous coordinates of $G(2, 4)$ form $2 \otimes 4$ matrix $Z$ with matrix elements $Z_{\alpha j}(\alpha = 1, 2; j = 1, \ldots, 4)$. Grassmann manifold is topologically non-trivial and can be covered by six patches (big cells) $U_{ij}(i < j; i, j = 1, \ldots, 4)$ defined by the conditions

$$Z \in U_{ij} \iff \det Z_{\alpha j} \neq 0.$$ 

The $2 \otimes 2$ submatrix $Z_{\alpha j}$ is formed by $i$ and $j$ rows of matrix $Z$. On any big cell $U_{ij}$ one can introduce inhomogeneous coordinates $z = Z_{(kl)}Z^{-1}_{(ij)}$ (here $k, l \neq i, j$). In fact each $U_{ij}$ is a complexified Minkowski space with coordinates $x^\mu \in C$ defined by the relation $z = x^\mu \sigma_\mu$ ($\sigma_\mu$ are the Pauli matrices). If $z = z^+$, then $x^\mu \in R$ are the coordinates of the usual Minkowski space $\mathcal{M}$. The action of the Poincaré group is deduced from the action of $SU(2, 2)$ on $G(2, 4)$:

$$Z \rightarrow Z' = TZ, \quad T \in SU(2, 2). \quad (1)$$

Let us choose the big cell $U_{34}$ for definiteness, write the matrix $T \in SU(2, 2)$ in a block form,

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (2)$$

where $A, B, C, D$ are $2 \otimes 2$ matrices and choose hermitian metric $\Phi$ of the group $SU(2, 2)$ in the form

$$\Phi = \begin{pmatrix} 0 & i1_2 \\ -i1_2 & 0 \end{pmatrix}. \quad (3)$$

Matrices $T$ obey the unitarity condition

$$T^\dagger \Phi T = \Phi. \quad (4)$$

The Poincaré subgroup of the $SU(2, 2)$ is defined by

$$C = 0, \quad \det A = 1. \quad (5)$$

Note that such simple and convenient form of conditions distinguishing Poincaré group out of conformal one is possible due to antidiagonal form of the metric $\Phi$. In this case we have from (4) and (3),

$$B^+ D = D^+ B, \quad (6)$$

$$A^+ = D^{-1}, \quad (7)$$

$$\det A = \det D = 1. \quad (8)$$

The group action (1) on the inhomogeneous coordinates $z$ takes the form

$$z \rightarrow z' = AzA^+ + BA^+. \quad (9)$$

The hermitian matrices $z \in \mathcal{M}$ are transformed by (1) to the hermitian ones because of conditions (3),(7) and the Minkowski length $ds^2 = \det(z_1 - z_2)$ is invariant. Thus (1) are completely equivalent to the usual Poincaré group transformations.
To "quantize" this construction we have to find, first of all, a real form \(SU(2,2)_q\) of \(SL(4,C)_{q_{ij},r}\) corresponding to the hermitian metric \(\Phi\). In other words, we must find a multiparametric R-matrix consistent with involution,

\[ T^* = \Phi(T^{-1})^t\Phi, \tag{10} \]

followed from unitarity condition (1). Here \((T^{-1})_{ij}\) is antipode of \(T_{ij}\) defined in a usual way for \(SL(4,C)\) group \([1], [16]\). Multiparametric R-matrix for \(GL(4,C)_q\) has a form \([16]\)

\[ R_{ij}^{kl} = \delta_i^k \delta_j^l \left( \delta_{ij} + \Theta^{ij} q_{ij}^{-1} + \Theta^{ij} q_{ji} r^{-2} \right) + \delta_i^k \delta_j^l \Theta^{ij} (1 - r^{-2}), \tag{11} \]

\(\Theta^{ij} = 1\) for \(i > j\) and 0 else. As usual, for the \(SU(m,n)\) case we put \(q_{ij}, r^2 \in \mathbb{R}\) \([1], [17]\) and thus from defining relations for \(T_{ij}\),

\[ R_{ik,rs} T_{rv} T_{sw} = T_{kb} T_{ia} R_{ab,vw}, \tag{12} \]

we obtain equations

\[ R_{ik,rs} T^*_{sv} T^*_{rv} = T^*_{ia} T^*_{kb} R_{ab,vw}, \]

which can be written due to (10) in the form

\[ \tilde{R}_{sr,ki} T_{rv} T_{sw} = T_{kb} T_{ia} \tilde{R}_{uv,ab}, \tag{13} \]

where

\[ \tilde{R}_{sr,ki} = \Phi_{sa} \Phi_{rb} R_{ab,cd} \Phi_{ck} \Phi_{di}. \tag{14} \]

Comparison of (12) and (13) yields the identity

\[ \tilde{R}_{sr,ki} = R_{ik,rs}, \]

which in turn gives for deformation parameters the conditions

\[ r^2 = 1, \; q_{ij} = q_{ji} \quad (\hat{i} = 5 - i, \hat{j} = 5 - j), \]

followed from the explicit form of the R-matrix (11). In our case of \(GL(4,C)_{q_{ij},r}\) group, this means

\[ r^2 = 1, \; q_{12} = q_{34}, \; q_{13} = q_{24} \quad (q_{ij} \in \mathbb{R}). \tag{15} \]

We would like to stress that \(r^2 = 1\) condition is the consequence of the antidiagonal form of the metric \(\Phi\), which is appropriate for picking out the Poincaré subgroup. As we shall show later, the triviality of parameter \(r\) leads to extremely simple structure of the corresponding deformed Lie algebra. This is in accordance with Drinfeld’s theorem \([4]\) on uniqueness of the Lie-algebra deformation (see also \([18]\)).

To construct q-Poincaré subgroup \(\mathcal{P}_q\) of \(SU(2,2)\) we have to require centrality of \(det_q A\) and \(det_q D = det_q (A^+)^{-1}\) according to condition (8). As is shown in \([19]\) this leads to the conditions

\[ q_{12} = q_{34} = 1, \quad q_{14} = q_{23} = q, \quad q_{13} = q_{24} = q^{-1}, \tag{16} \]

which does not contradict the real form conditions (15). Taking the deformation parameters according to (16), we can put \(det A = det D = 1\) to obtain consistent q-Poincaré subgroup \(\mathcal{P}_q\) of \(SU(2,2)_q\) with a structure of real Hopf algebra inherited from \(SU(2,2)_q\) \([19]\). 

3
We can further restrict our q-group to obtain Lorentz subgroup formed by the matrices

\[ T = \begin{pmatrix} A & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \]  

Matrix elements of \( A, A^+ \) belong to \(*\)-Hopf subalgebra of \( \mathcal{P}_q \), so matrices (17) define q-Lorentz subgroup of \( \mathcal{P}_q \). Very unusual property of our deformation is the commutativity of elements of \( A \) with each other and \( A^+ \) with each other. The only nontrivial CR are those between elements of \( A \) and \( A^+ \). This is the simplest possible deformation of Lorentz group.

Condition \( A = (A^+)^{-1} \) picks out \( SO(3)_q \) rotation subgroup. But our deformation is inconsistent with such condition [19] because elements of \( A \) between themselves and with those of \( A^+ \) commute differently. So there is no q-rotation subgroup of q-Lorentz group in our case. The absence of this subgroup is not very important from the physical point of view because, as we hope, q-Poincaré symmetry has meaning at extremely high energy, so that \( SO(3)_q \) subgroup can not play essential physical role. Of course, this fact can have strong influence on q-Poincaré representation theory since subgroup of rotations is a small group of fixed momentum for massive states in \( q = 1 \) case and so is important ingredient for induced representations of the Poincaré group.

3 Quantum Minkowski geometry

The commutation relations for homogeneous coordinates \( Z \) are the same as for the two last columns of the matrix \( T \). In this case they are saved under the group transformations. Moreover, due to the centrality of \( \det Z_{(34)} \) (in analogy with \( \det D \)) we can put \( \det Z_{(34)} = 1 \) and define inhomogeneous coordinates

\[ z = Z_{(12)}Z_{(34)}^{-1} \equiv \begin{pmatrix} z^1 & z^4 \\ z^2 & z^3 \end{pmatrix} . \]

Using once more the CR (12), we obtain

\[ z^i z^j = Q_{ij} z^j z^i , \]  

where

\[ Q_{ij} = q^2 \quad \text{if} \quad j = i + 1(mod 4), \quad Q_{ji} = Q_{ij}^{-1} , \]

\[ Q_{ij} = 1 , \quad \text{if} \quad j = i + 2(mod 4) \text{ or } i = j . \]

To obtain deformed Minkowski length \( l_q \) which is invariant under q-Poincaré transformations, we have to take into account that the matrix elements of submatrix \( A \) do not commute with those of \( A^+ \) so we can not put \( l_q = \det z \) though this is a central element of the algebra, but assume a more general expression \( l_q = \det_q z \equiv z_1 z_4 - q^n z_2 z_3 \), with some integer \( n \) (\( l_q \) is central for any \( n \)). Straightforward calculations show that if \( n = 2 \) then the determinant factorizes

\[ \det_q (AzA^+) = (\det A)(\det_q z)(\det A^+) = \det_q z . \]
and the deformed Minkowski length

\[ l_q = \det q z = g_{ij}^q z^i z^j \]

is invariant under homogeneous transformations due to the conditions \( \det A = \det A^+ = 1 \). Here \( g_{ij}^q \) is q-analog of classical \( SO(2, 2) \) metric \( g_{ij} \):

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -q^2 \\
1 & 0 & 0 & 0 \\
0 & -q^2 & 0 & 0
\end{pmatrix}.
\] (19)

Note that we consider the quantum analog of usual Minkowski space but for the sake of convenience use coordinates with other reality conditions

\[(z^1)^* = z^1, \quad (z^3)^* = z^3, \quad (z^2)^* = z^4, \quad (z^4)^* = z^2, \] (20)
in contrast to the usual pure real ones \( (x^\mu)^* = x^\mu \). One can straightforwardly check that the involution (20) is consistent with the defining relations (18). The inhomogeneous transformations have the form

\[
z \rightarrow (z')^\alpha_\beta = (AzA^+)^\alpha_\beta + (BA^+)^\alpha_\beta
\] (21)

and direct inspection gives

\[[(AzA^+)^\alpha_\beta, (BA^+)^\alpha_\beta] = 0 \quad \forall \alpha, \beta = 1, 2.\] (22)

From this relation we can deduce the affine structure of the quantum Minkowski space: identity (22) implies that we can consider a set of coordinates \( (z_t)^\alpha_\beta \), where \( t \) is discrete or continuous index to distinguish different “points” of q-space-time and put

\[z^i_t z^j_{t'} = Q_{ij} z^j_{t'} z^i_t \quad \forall \ t, \ t'.\] (23)

Now we can define the invariant of the complete Poincaré group:

\[ds_q^2 = \det q (z_t - z_{t'}).\] (24)

Coordinates of q-Minkowski space \( \mathcal{M}_q \) form the vector representation of \( \mathcal{P}_q \). Construction of complete theory of \( \mathcal{P}_q \)-representations is a complicated problem because of necessity to develop integral calculus on noncommutative space and due to the absence of rotational subgroup (the small subgroup for massive states in classical case). But spinorial representations of q-Lorentz subgroup can be derived rather easily in analogy with the usual ones (see e.g. [21]).

A spinor \( \xi_\alpha \) and its conjugate \( \bar{\xi}^\dot{\alpha} \) \((\alpha = 1, 2)\) are transformed by q-Lorentz submatrix according to

\[
\begin{pmatrix}
\xi' \\
\bar{\xi}'
\end{pmatrix} =
\begin{pmatrix}
A & 0 \\
0 & (A^+)^{-1}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\bar{\xi}
\end{pmatrix}.
\]

As we have already mentioned matrix elements of \( A \) commute with each other. Hence \( \xi_1 \xi_2 = \xi_2 \xi_1 \). The same is true for \( (A^+)^{-1} \) and components of \( \bar{\xi} \). So q-Lorentz invariants \( \xi_\alpha \xi_\beta \epsilon^{\alpha \beta} \) and \( \bar{\xi}^\dot{\alpha} \bar{\xi}^\dot{\beta} \epsilon_{\dot{\alpha} \dot{\beta}} \) can be constructed with help of usual antisymmetric tensors \( \epsilon_{\alpha \beta} \) and
where the invariant metric has the form as on the q-Minkowski space. Consider the spinors with upper indices spinors.

To commutativity of elements of matrix \( A \), the isotropic Minkowski vector one has to use two different spinors (two-columns matrix homogeneous part of coordinates transformations (21). Note, however, that to obtain non-

\( \bar{\xi} \) are defined by transformations of spinors \( \xi \). In the classical limit \( q = 1 \) one obtains usual normalization factor \( \sqrt{a(j_1, j_2)} \) of symmetrical spinors.

Now we can obtain realization of q-Poincaré transformations on functions \( \psi(z) \) of non-commuting coordinates (i.e. fields from the physical point of view). Let \( \psi_{\sigma\rho} \in V_{j_1j_2} \) and so is transformed by q-Lorentz group as multispinor (26). Then q-Poincaré transformations
can be realized on the maps $\psi : \mathcal{M}_q \rightarrow V_{j_1j_2}$ in the way quite analogous to the classical case (cf. e.g. [22]):
\[
\psi_{\alpha\dot{\rho}}(z) = \mathcal{D}(A)_{\alpha}^{\lambda} \psi_{\lambda\dot{\rho}}(z'') \mathcal{D}((A^+)^{-1})_{\dot{\rho}}^{\dot{\sigma}},
\]
where $z''$ is the coordinate matrix transformed by antipode $A^{-1}$
\[
z'' = A^{-1}z(A^+)^{-1} - A^{-1}B .
\]

4 q-Poincaré algebra

As generators of q-translations it is natural to use q-derivatives (cf. [8]). Thus first of all, we have to derive formulae for differential calculus on our Minkowski space. This can be done in analogy with [23],[8], and gives the result
\[
\partial_i \partial_j = Q_{ij} \partial_j \partial_i ,
\]
\[
dz^i dz^j = -Q_{ij} dz^j dz^i ,
\]
\[
\partial_j z^i = \delta^i_j + Q_{ij} z^i \partial_j ,
\]
(28)

To find q-Poincaré algebra, we can start from general anzatz for the action of q-Lorentz generators $L$ on Minkowski coordinates,
\[
L z^i = \alpha_i z^i L + A^i_j z^j .
\]
(29)
The first term in the r.h.s. is diagonal because of the diagonal R-matrix. The unknown matrix elements $A^i_j$ and $\alpha_i$ can be defined from conditions of metric invariance,
\[
L q = l_q L
\]
(30)
and from the defining relations invariance,
\[
L(z^i z^j - Q_{ij} z^j z^i) = (z^i z^j - Q_{ij} z^j z^i)L .
\]
(31)

One can show that these conditions leads to unique solution for q-generators (29) (i.e. unique solution for $\alpha_i$ and $A^i_j$). Obviously, it would be desirable to find a form of q-Lie algebra CR analogous to the covariant tensorial form of nondeformed Lorentz Lie algebra ($SO(2, 2)$ in our case):
\[
[M^{mn}, M^{pk}] = g^{mk} M^{np} + g^{np} M^{mk} - g^{mp} M^{nk} - g^{nk} M^{mp} ,
\]
(32)
where $g^{mn}$ is inverse to (19) at $q = 1$. This can be achieved most easily by constructing the spinless representation of q-Lie algebra by q-differential operators, i.e. by the deformation of usual Killing vectors
\[
M^{mn} = z^m \partial^n - z^n \partial^m , \quad \partial^m = g^{mn} \partial_n ,
\]
on commuting Minkowski space. For this aim we introduce the two-index q-Lorentz generators $L^{mn} = -Q_{mn} L^{nm}$ and put
\[
L^{mn} = z^m \partial^n - Q_{mn} z^n \partial^m , \quad \partial^m = g^{mn} \partial_n .
\]
(33)
One can check that (33) satisfies (30), (31) and is in one to one correspondence with the
Poincaré Lie algebra:
monoms
Coproduct ∆ for generators of q-Poincaré algebra can be read off from the action oners). For example, ∆(L
where we used the symbol
Q
comma. In particular, 
Here factors in r.h.s contain all ordered pairs of indices from two sets in l.h.s divided by
Commuting coordinates
Now it is possible to introduce commuting coordinates
z
and is q-Poincaré invariant and central element of the algebra. Here
Counity
is defined, as usual, \( \epsilon(L^{mn}) = \epsilon(\partial_k) = 0 \), and antipode S is defined from
the defining property
\( \mu(id \otimes S)\Delta(L^{mn}) = \mu(S \otimes id)\Delta(L^{mn}) = 0 \) (here \( \mu \) is the algebra multiplication).
As we noted earlier, according to the Drinfeld’s uniqueness theorem [4] our q-Lie algebra has to be simply connected with usual, nondeformed one because Lie algebra deformation parameter is trivial. From the geometrical and physical point of view the most natural way to establish such correspondence is the following.
Let us introduce q-tetrade in the q-space-time \( M_q \), i.e. four q-vectors \( e_a^m \), so that
\[ e_a^m e_b^m = Q_{mn} e_b^m e_a^m \quad (a, b = 1, \ldots, 4). \]
One can show that q-determinant defined by
\[ e \equiv e_{mnpk} e_1^m e_2^n e_3^p e_4^k = \epsilon^{abcd} e_a^1 e_b^2 e_c^3 e_d^4 \]
and is q-Poincaré invariant and central element of the algebra. Here \( \epsilon_{ijkl}^q \) is q-deformed anti-symmetric Levi-Cevita tensor: \( \epsilon_{1234}^q = 1 \) and \( \epsilon_{ijkl}^q = -Q_{ji} \epsilon_{jkl}^q = -Q_{kj} \epsilon_{ikl}^q = -Q_{ik} \epsilon_{ijl}^q \).
Now it is possible to introduce commuting coordinates \( z^a \) on \( M_q \) through the relations
\[ z^m = z^a e_a^m \]
or
\[ z^a = e_a^m z^m, \]
where \( e_a^m \) are the elements of the inverse matrix of the q-tetrade defined with help of q-determinant \( e \)
\[ e_a^m = (e^{-1})_{l...nmp...k} e_1^l \ldots e_a^r e_{a+1}^p \ldots e_4^k Q(l...r, m) \]
(generalization of the inverse matrix formula in usual case). It is easy to see that
\[ e_m^a e_n^b = Q_{mn} e_b^m e_a^a \]
and that $z^a$ $(a = 1, \ldots, 4)$ are commuting coordinates. Such q-tetrad and commuting coordinates can be introduced in the q-space with commutative affine structure only, i.e. if vectors of translation commute with homogeneous part of coordinates (see (22)). We expect that in general q-affine space CR of $z^a$ are defined by parameter $r$ and it may be useful to separate different parameters of deformation with help of analogous q-bein.

Now we are ready to show why the existence of globally defined q-tetrad leads to trivial q-algebra but non-trivial q-group. Indeed, using differential operator representation (33) one has

$$L^{mn} = e_a^m z^a e_b^n \partial^b - Q_{mn} e_b^n z^b e_a^m \partial^a = e_a^m e_b^n (z^a \partial^b - z^b \partial^a) = e_a^m e_b^n M^{ab}$$

where we used

$$\partial^m = e_a^m \partial^a .$$

Obviously one can define inverse transformation

$$M^{ab} = e_a^b e_m^a L^{mn}$$

Now all the properties of the q-algebra including commutators, Casimir operators, etc. can be derived from the well known properties of the usual Poincaré algebra of $(M^{ab}, \partial^c)$.

From the other hand the relation is not so simple in the group transformations case. For the sake of simplicity let us consider homogeneous transformations only (inclusion of the translations does not change anything). In vector notations the q-Lorentz transformations has a form

$$z^m = (\Lambda_q)_n^m \otimes z^n .$$

Substituting (38) in (43) one obtains

$$z^a = \Lambda^a_b z^b$$

with

$$\Lambda^a_b = e^b_m (\Lambda_q)_n^m \otimes e_a^n$$

and the matrix elements of $\Lambda^a_b$ commute with each other if

$$e^b_n (\Lambda_q)_k^m = Q_{mn} (\Lambda_q)_k^n e^b_n .$$

However, it is impossible to derive inverse formula, i.e. to construct $\Lambda_q$ from $\Lambda$. Indeed,

$$z^m = (\Lambda_q)_n^m \otimes z^n = e^m_a z^a = (e^m_a \Lambda^a_b e^n_b) z^n \equiv S^m_n z^n .$$

But $S^m_n$ nontrivially commute with $z^k$ and, hence, can not be the element of the quantum group transformation matrix $\Lambda_q$ by definition. This shows non-trivial character of the q-Poincaré group deformation. Another important property of the q-deformation that confirms this statement is the absence of the $O(3)_q$-subgroup.

5 Conclusion

Starting from the quantum conformal group $SU(2, 2)_q$, we have defined a self-consistent quantum Poincaré group as its q-subgroup, q-Minkowski space and Lie algebra. We present the form of higher spinor q-Lorentz representations, realization of q-Poincaré
transformations on the fields defined on q-Minkowski space. The essential points of our construction are twistor approach to the definition of Minkowski space and multiparametric deformation of the linear projective groups. We would like to stress that if one considers q-Poincaré group as a q-subgroup of quantum conformal group and use known multiparametric R-matrix [16], one should obtain the group we have constructed. It has diagonal R-matrix and commuting affine structure of corresponding q-Minkowski space. This leads to great simplification of most formulae and make them quite analogous to the classical ones. Due to commuting affine structure one can introduce globally defined q-tetrade that connects commuting and q-commuting coordinates as well as usual and q-Lie algebra. We hope that such relatively simple form of deformation will be useful for construction of induced representation theory of the deformed Poincaré group, developing the q-deformed field theory etc. As an illustration of the application of our q-group to the general analysis of the q-deformed quantum field theory let us consider very briefly the relation between q-deformed space-time symmetry and q-oscillators. Remind that usual quantum fields $\psi(z)$ have the following relation with creation and destruction operators $a^\dagger(p), a(p)$ (we consider the real field for simplicity)

$$\psi^N(z) = (2\pi)^{-3/2} \int \left[ v^N_A(p) a^A(p) e^{-i(px)} + v^N_B(p) a^B(p) e^{i(px)} \right] \frac{dp}{2\omega(p)} \quad (44)$$

(in our simple deformation case one can define integration due to relation with commuting variables). Here $N$ is (multi)index of the representation and $v^N_A$ is the corresponding wave function (see, e.g. [22]). One can see easily the analogy between (14) and the relation (38) for q-coordinates $z^m$ and commuting one $z^a$, the wave function $v^N_A$ playing the role of the q-bein in the space of the representation. So in our case creation and destruction operators obey the usual CR in analogy with commuting coordinates $z^a$. Thus in general case non-trivial q-oscillator relations can be connected with algebra deformation parameter $r$ only, and CR between components of wave functions are defined by other (existing in our case) deformation parameters.

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