A Factorization of a Quadratic Pencils of Accretive Operators and Applications

Fairouz Bouchelaghem and Mohammed Benharrat

Abstract. A canonical factorization is given for a quadratic pencil of accretive operators in a Hilbert space. We establish a criterion in order that the linear factors, into which the pencil splits, generates a holomorphic semi-group of contraction operators. As an application, we study a result of existence, uniqueness, and maximal regularity of the strict solution for complete abstract second-order differential equation in the non-homogeneous case. An illustrative example is also given.

Mathematics Subject Classification. Primary 47A10, 47A56.

Keywords. Accretive operators, Quadratic operator pencil, Spectral theory, Semigroup of contractions.

1. Introduction

Many problems in mathematical physics and mechanics can be described by the following second order linear differential equation

\[ u''(t) - 2Bu'(t) - Cu(t) = 0, \]  

where \( u(t) \) is a vector-valued function in an appropriate (finite or infinite dimensional) Hilbert space \( \mathcal{H} \), \( B \) and \( C \) are linear (bounded or unbounded) operators on \( \mathcal{H} \). Properties of the differential equation (1.1) are closely connected with spectral properties of a quadratic pencil

\[ Q(\lambda) = \lambda^2I - 2\lambda B - C, \quad (\lambda \in \mathbb{C}); \]  

which is obtained by substituting exponential functions \( u(t) = \exp(\lambda t)x, \) \( x \in \mathcal{H} \) into (1.1). In many applications \( B \) and \( C \) are self-adjoint positive definite operators. An important and subtle problem in the theory of such operator pencils is to factoring them and studying the spectral properties of the factors. Under some assumptions, Krein and Langer [13] proved that a

This work was supported by the Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO) and the Algerian research Project: PRFU, No. C00L03ES310120180002 (DGRSDT).
self-adjoint polynomial of the form (1.2) can always be written as a product of two linear factors as follows

\[ \lambda^2 I - 2\lambda B - C = (\lambda I - Z_1)(\lambda I - Z_2), \]  

(1.3)

with \( Z_1 \) and \( Z_2 \) are a roots of the quadratic operator equation

\[ Q(Z) = Z^2 I - 2BZ - C = 0. \]  

(1.4)

Of particular interest is the separation of spectral values of \( Q \) between the spectra of the roots. Such separation may be complicated, even in the case of eigenvalues, see [23] and references therein. The factorization theorems have been studied extensively also for the self-adjoint quadratic operator pencils under the extra condition of strong and weak damping. For the exhaustive survey on these topics, please see the two seminal books [17] and [18] and the references therein.

But some models of continuous mechanics are reduced to differential equation (1.1) with sectorial operators, see [1,3,8,12] and references therein. In this cases methods, developed for self-adjoint operators, cannot be applied.

The main objective of the manuscript is to find sufficient conditions on, in general, unbounded linear accretive operators \( B \) and \( C \) on the Hilbert space, under which a factorization (1.3) is possible. The approach is based on the perturbation theory of accretive operators. We also obtain a criterion in order that the linear factors, into which the pencil splits, generates an holomorphic semi-group of contraction operators. We apply this result to establish a theorem of existence, uniqueness, and maximal regularity of the strict solution of an abstract second order evolutionary equations generated by such pencils in the non-homogeneous case.

2. Accretive Operators Framework

In this section, we introduce the notation and the operator theoretic framework used in the rest of our work. Throughout this paper \( \mathcal{H} \) is a complex Hilbert space with an inner product \( < \cdot,\cdot> \) and norm \( \|\cdot\| \). Let \( B(\mathcal{H}) \) denote the Banach space of all bounded linear operators on \( \mathcal{H} \). Given a linear operator \( T \) on \( \mathcal{H} \) we denote by \( \mathcal{D}(T) \), \( \mathcal{N}(T) \), and \( \mathcal{R}(T) \) the domain, the null space and the range of \( T \), respectively. For a closable densely defined linear operator \( T \) in some Hilbert space \( \mathcal{H} \) we denote by \( \sigma(T) = \sigma_p(T) \) the spectrum, and by \( \sigma_p(T) \) the point spectrum of \( T \). For \( \lambda \in \rho(T) \), the inverse \( (\lambda I - T)^{-1} \) is, by the closed graph theorem, a bounded operator on \( \mathcal{H} \) and will be called the resolvent of \( T \) at the point \( \lambda \).

Recall that a linear operator \( T \) with domain \( \mathcal{D}(T) \) in a complex Hilbert space \( \mathcal{H} \) is said to be accretive if

\[ \text{Re} < Tx, x > \geq 0 \quad \text{for all } x \in \mathcal{D}(T) \]

or, equivalently if

\[ \|(\lambda + T)x\| \geq \lambda\|x\| \quad \text{for all } x \in \mathcal{D}(T) \text{ and } \lambda > 0. \]

An accretive operator \( T \) is called maximal accretive, or \( m \)-accretive for short, if one of the following equivalent conditions is satisfied:
1. $T$ has no proper accretive extensions in $\mathcal{H}$;
2. $T$ is densely defined and $\mathcal{R}(\lambda + T) = \mathcal{H}$ for some (and hence for every) $\lambda > 0$;
3. $T$ is densely defined and closed, and $T^*$ is accretive.

In particular, every m-accretive operator is accretive and closed densely defined, its adjoint is also m-accretive (cf. [14], p. 279). Furthermore,
\[
(\lambda + T)^{-1} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \| (\lambda + T)^{-1} \| \leq \frac{1}{\lambda} \quad \text{for} \quad \lambda > 0.
\]

In particular, a bounded accretive operator is m-accretive. Óta showed in [22, Theorem 2.1] that, if $T$ is closed and an accretive such that there is a positive integer $n$ with $\mathcal{D}(T^n)$ is dense in $\mathcal{H}$ and $\mathcal{R}(T^n) \subset \mathcal{D}(T)$, then $T$ is bounded. In particular, for a closed and accretive operator $T$, if $\mathcal{R}(T)$ is contained in $\mathcal{D}(T)$, or in $\mathcal{D}(T^*)$, then $T$ is automatically bounded, see also [22, Theorem 3.3]. Also, if $T$ is maximal accretive, then
\[
\mathcal{N}(T) = \mathcal{N}(T^*) \quad \text{and} \quad \mathcal{N}(T) \subseteq \mathcal{D}(T) \cap \mathcal{D}(T^*). \tag{2.1}
\]

The numerical range is very useful set by what we can we define the accretive operators. For a linear operator $T : \mathcal{D}(T) \to \mathcal{H}$ it is defined by
\[
W(T) := \{ < Tx, x > : x \in \mathcal{D}(T), \quad \text{with} \quad \| x \| = 1 \}, \tag{2.2}
\]
It is well-known that $W(T)$ is a convex set of the complex plane (the Toeplitz–Hausdorff theorem), and in general is neither open nor closed, even for a closed operator $T$. Clearly, an operator $T$ is accretive when $W(T)$ is contained in the closed right half-plane
\[
W(T) \subset \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Re}(z) \geq 0 \}.
\]

Further, if $T$ is m-accretive operator then $W(T)$ has the so-called spectral inclusion property
\[
\sigma(T) \subset \overline{W(T)}. \tag{2.3}
\]

A linear operator $T$ in a Hilbert space $\mathcal{H}$ is called sectorial with vertex $z = 0$ and semi-angle $\omega \in [0, \pi/2)$, or $\omega$-accretive for short, if its numerical range is contained in a closed sector with semi-angle $\omega$,
\[
W(T) \subset \mathcal{S}(\omega) := \{ z \in \mathbb{C} : | \arg z | \leq \omega \} \tag{2.4}
\]
or, equivalently,
\[
| \text{Im} \ < Tx, x > \ | \leq \tan \omega \ \text{Re} \ < Tx, x > \quad \text{for all} \ x \in \mathcal{D}(T).
\]

An $\omega$-accretive operator $T$ is called m-$\omega$-accretive, if it is m-accretive. We have $T$ is m-$\omega$-accretive if and only if the operators $e^{\pm i\theta}T$ is m-accretive for $\theta = \frac{\pi}{2} - \omega$, $0 < \omega \leq \pi/2$. The resolvent set of an m-$\omega$-accretive operator $T$ contains the set $\mathbb{C} \setminus \mathcal{S}(\omega)$ and
\[
\| (T - \lambda I)^{-1} \| \leq \frac{1}{\text{dist} (\lambda, \mathcal{S}(\omega))}, \quad \lambda \in \mathbb{C} \setminus \mathcal{S}(\omega).
\]

In particular, m-$\pi/2$-accretivity means m-accretivity. A 0-accretive operator is symmetric. An operator is positive if and only if it is m-0-accretive.
It is known that the \( C_0 \)-semigroup \( T(t) = \exp(-tT), \ t \geq 0 \), has contractive and holomorphic continuation into the sector \( S(\pi/2\omega) \) if and only if the generator \( T \) is \( m-\omega \)-accretive, see [14, Theorem V-3.35].

Recently, the authors of [2] obtained a precise localization of the numerical range of one-parameter semigroup \( T(t) = \exp(-tT), \ t \geq 0 \), generated by an \( m-\omega \)-accretive operator, \( \omega \in [0, \pi/2) \). More precisely, by [2, Theorem 3.4], we have

\[
W(\exp(-tT)) \subseteq \Omega(\omega) = \{ z \in \mathbb{C} : |\text{Im} \sqrt{z}| \leq \frac{1}{2}(1 - |z|) \tan(\omega) \}, \quad t \geq 0,
\]

with limiting cases: \( \Omega(0) = [0, 1] \) and \( \Omega(\pi/2) = \mathbb{D} \). In particular, the family \( \exp(-tT), t \geq 0 \), is a quasi-sectorial contractions semigroup in the terminology of [2].

We mention that if \( T \) is \( m \)-accretive, then for each \( \alpha \in (0, 1) \) the fractional powers \( T^\alpha, 0 < \alpha < 1 \), are defined by the following Balakrishnan formula, see [4],

\[
T^\alpha x = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-1} T(\lambda + T)^{-1} x dt,
\]

for all \( x \in D(T) \). The operators \( T^\alpha \) are \( m-(\alpha\pi)/2 \)-accretive and, if \( \alpha \in (0, 1/2) \), then \( D(T^\alpha) = D(T^{*\alpha}) \). It was proved in [15, Theorem 5.1] that, if \( T \) is \( m \)-accretive, then \( D(T^{1/2}) \cap D(T^{*1/2}) \) is a core of both \( T^{1/2} \) and \( T^{*1/2} \) and the real part \( \text{Re} T^{1/2} := (T^{1/2} + T^{*1/2})/2 \) defined on \( D(T^{1/2}) \cap D(T^{*1/2}) \) is a selfadjoint operator. Further, by [15, Corollary 2],

\[
D(T) = D(T^*) \implies D(T^{1/2}) = D(T^{*1/2}) = D(T^{1/2}_R) = D[\phi],
\]

where \( \phi \) is the closed form associated with the sectorial operator \( T \) via the first representation theorem [14, Sect. VI.2.1] and \( T^R \) is the non-negative selfadjoint operator associated with the real part of \( \phi \) given by \( \text{Re} \phi := (\phi + \phi^*)/2 \).

3. A Canonical Factorization of Monic Quadratic Operator Pencils

In this section, we will investigate a canonical factorization of quadratic operator pencils \( Q \) of the form

\[
Q(\lambda) = \lambda^2 I - 2\lambda B - C,
\]

on a Hilbert space with domain \( D(Q) = D(B) \cap D(C) \), where \( \lambda \in \mathbb{C} \) is the spectral parameter and the two operators \( B \) and \( C \) with domain \( D(C) \) and \( D(B) \), respectively, satisfy one of the following conditions,

(C.1) there exists \( \alpha \geq 0, 0 \leq \beta < 1 \) and \( \delta \geq 0 \) such that

\[
\text{Re} \langle B^2 x, C x \rangle \geq -\alpha \|x\|^2 - \beta \|B^2 x\|^2 - \delta \|B^2 x\| \|x\|,
\]

for all \( x \in D(B^2) \subset D(C) \).
(C.2) $C$ is $B^2$-bounded with lower bound $< 1$, i.e. there exists $a \geq 0$ and $0 \leq b < 1$ such that
\[ \|Cx\|^2 \leq a\|x\|^2 + b\|B^2x\|^2, \quad \text{for all } x \in \mathcal{D}(B^2) \subset \mathcal{D}(C). \]

(C.3) $I + C(B^2 + t_0)^{-1}$ is boundedly invertible, for some $t_0 > 0$.

(C.4) $B$ is accretive and $\mathcal{D}(B) \subset \mathcal{D}(C)$.

(C.5) $B$ is accretive and $C$ is bounded.

**Proposition 3.1.** Let $B^2$ be $m$-accretive and $C$ is accretive. If the operator $B$ and $C$ verifies one of the conditions above, then the operator $\Lambda = B^2 + C$ with domain $\mathcal{D}(B^2)$ is $m$-accretive.

**Proof.** Assume (C.1), then by [21, Theorem 3.10] we can prove that $B^2 + C$ is $m$-accretive. For the convince of the reader we give a detailed proof of this fact and adapted to the Hilbert case. First, we have $B^2 + C$ is accretive and densely defined. We show that $B^2 + C$ is closed. In fact, it follows from (C.1) that
\[
\|B^2x\|^2 = \text{Re} < B^2x, B^2x > \\
\leq \text{Re} < (B^2 + C)x, B^2x > + \alpha \|x\|^2 + \beta \|B^2x\|^2 + \delta \|B^2x\| \|x\|, \\
\text{for all } x \in \mathcal{D}(B^2). \]

So, we have
\[
(1 - \beta) \|B^2x\|^2 \leq [\delta \|x\| + \|B^2 + C\|\|B^2x\| + \alpha \|x\|^2, \\
\text{for all } x \in \mathcal{D}(B^2). \]

Solving this inequality, we obtain
\[
\|B^2x\| \leq \frac{1}{1 - \beta} \|B^2 + C\|\|x\| + \kappa \|x\| \tag{3.2}
\]

for all $x \in \mathcal{D}(B^2)$, with $\kappa = \frac{\delta + \sqrt{\alpha(1 - \beta)}}{1 - \beta}$. On the other hand, since $\mathcal{D}(B^2) \subset \mathcal{D}(C)$, with $\mathcal{D}(B^2)$ dense in $\mathcal{H}$, there exists a constant $\vartheta > 0$, such that
\[
\|Cx\| \leq \vartheta(\|x\| + \|B^2x\|), \tag{3.3}
\]

for all $x \in \mathcal{D}(B^2)$. Now, let a sequence $(x_n)_n \subset \mathcal{D}(B^2)$ such that $x_n \rightarrow x$ and $(B^2 + C)x_n \rightarrow y$. Applying the inequality (3.2) to $x$ replaced by $x_n - x_m$, we see that the sequence $(B^2x_n)_n$ converge. Since $B^2$ is closed we conclude that $B^2x_n \rightarrow B^2x$ and $x \in \mathcal{D}(B^2)$. By (3.3), we have
\[
\|(B^2 + C)(x_n - x)\| \leq \vartheta \|x_n - x\| + (1 + \vartheta) \|B^2(x_n - x)\|. \\
\]

Hence $(B^2 + C)x_n \rightarrow (B^2 + C)x$ and $y = (B^2 + C)x$, which shows $B^2 + C$ is closed. On the other hand, we have
\[
\text{Re} < B^2x, tCx > \geq -t\alpha \|x\|^2 - t\beta \|B^2x\|^2 - t\delta \|B^2x\| \|x\|, \\
\text{for all } 0 \leq t \leq 1. \quad \text{Since } t\beta < 1, \text{ by the same argument, we assert that } \\
(B^2 + tC) \text{ is closed for all } 0 \leq t \leq 1. \quad \text{Hence, } (B^2 + tC) \text{ is closed and accretive for all } 0 \leq t \leq 1. \]

By [21, Lemma 3.1], the dimension of $\mathcal{R}(B^2 + C + \lambda)^\perp$ and $\mathcal{R}(B^2 + \lambda)^\perp$ are the same for all $\lambda > 0$. Since $B^2$ is $m$-accretive, we conclude that $B^2 + C$ is $m$-accretive.
If (C.2) holds, the result follows by [9, Theorem 2.]. Also (C.2) implies (C.1) in the case of $\gamma = 0$, see [20, Remark 4.4]. In fact, setting $\alpha = a/2$ and $\beta = (b+1)/2$, we have that for $x \in \mathcal{D}(B^2)$,

$$\|Cx\|^2 \leq 2\alpha \|x\|^2 + (2\beta - 1)\|B^2 x\|^2 \leq \|(B^2 + C)x\|^2 - \|B^2 x\|^2 + 2\alpha \|x\|^2 + 2\beta \|B^2 x\|^2$$

$$= 2(\text{Re} < B^2 x, Cx > + \alpha \|x\|^2 + \beta \|B^2 x\|^2) + \|Cx\|^2.$$

Hence $\text{Re} < B^2 x, Cx > + \alpha \|x\|^2 + \beta \|B^2 x\|^2 \geq 0$.

Assume that (C.3) is satisfied. Since $B^2 + C$ is densely defined and accretive, it suffices to show that $\mathcal{R}(B^2 + C + t_0) = \mathcal{H}$. But this follows immediately from

$$B^2 + C + t_0 = (I + C(B^2 + t_0)^{-1})(B^2 + t_0),$$

and clearly $B^2 + C + t_0$ is invertible.

Now, we consider (C.4). Since $B$ is an accretive operator, by [11, Theorem 1.2], we have for an arbitrary $\nu > 0$,

$$\|Bx\|^2 \leq \nu \|x\|^2 + \frac{1}{\nu} \|B^2 x\|^2, \quad (3.4)$$

for all $x \in \mathcal{D}(B^2)$. Since $\mathcal{D}(B) \subset \mathcal{D}(C)$, with $\mathcal{D}(B)$ dense in $\mathcal{H}$, there exists a constant $\eta > 0$, such that

$$\|Cx\|^2 \leq \eta \|Bx\|^2,$$

for all $x \in \mathcal{D}(B)$. It follows that

$$\|Cx\|^2 \leq \eta(\nu \|x\|^2 + \frac{1}{\nu} \|B^2 x\|^2),$$

for all $x \in \mathcal{D}(B^2)$. Choosing $\nu > 0$ so large that $\frac{\eta}{\nu} < 1$, we get $C$ is $B^2$-bounded with lower bound $< 1$.

Finally, clearly (C.5) is a particular case of (C.4). \qed

Remark 3.2. 1. In (C.3), if we assume further $\mathcal{D}(B^2) \subset \mathcal{D}(C)$, then by [25, Proposition 2.12], the lower bound $b$ in (C.2) is equal to

$$\sup_{t>0} \|C(B^2 + t)^{-1}\|.$$

Hence, if we assume further, $\|C(B^2 + t_0)^{-1}\| < 1$ for some $t_0 > 0$, so $I + C(B^2 + t_0)^{-1}$ is boundedly invertible, for some $t_0 > 0$. In this case (C.3) implies (C.2).

2. If the condition (C.4) is satisfied, clearly $\mathcal{D}(Q) = \mathcal{D}(B)$.

3. In (C.4) and (C.5), $B$ is m-accretive. Indeed, choosing $\nu > 0$ so large that $\frac{1}{\nu} < 1$ in (3.4), we obtain $B$ is $B^2$-bounded with lower bound $< 1$.

Then $B^2 + B$ with domain $\mathcal{D}(B^2)$ is m-accretive. Now, let us remark that

$$\left(\frac{1}{4}I + B^2 + B\right)x = \left(\frac{1}{2}I + B\right)^2 x.$$
for all \( x \in D(B^2) \). Since the operator on the left-hand side is invertible, then \((\frac{1}{2}I + B)^2\) is invertible, so \(\frac{1}{2}I + B\) is also invertible. It follows that \( B \) is m-accretive.

In the sequel, we assume that \( B^2 \) be m-accretive and \( C \) is accretive verify the condition (C.1), unless otherwise specified.

Now, we state some properties of the operator \( \Lambda = B^2 + C \). The first important one is the existence and uniqueness of its square root. This is an immediate consequence of [14, Theorem 3.35, p. 281].

**Corollary 3.3.** The operator \( \Lambda \) admits unique square root \( \Lambda^{\frac{1}{2}} \) \( m-(\pi/4) \)-accretive operator with \( D(B^2) \) is a core of \( \Lambda^{\frac{1}{2}} \) (that is, the closure of the restriction of \( \Lambda^{\frac{1}{2}} \) to \( D(B^2) \) is again \( \Lambda^{\frac{1}{2}} \)).

**Proposition 3.4.** If \( C \) is \( \theta \)-accretive, with \( 0 \leq \theta < \pi/2 \), then
\[
N(\Lambda) \subset N(B^2) \cap N(C^*).
\]

**Proof.** (1) Let \( x \in D(B^2), x \neq 0 \), such that \( \Lambda x = 0 \), as before, we have
\[
\text{Re} \langle \Lambda x, x \rangle = \text{Re} \langle B^2x, x \rangle + \text{Re} \langle Cx, x \rangle,
\]

then
\[
\text{Re} \langle B^2x, x \rangle \leq \text{Re} \langle \Lambda x, x \rangle \quad \text{and} \quad \text{Re} \langle Cx, x \rangle \leq \text{Re} \langle \Lambda x, x \rangle.
\]

Therefore, \( \text{Re} \langle \Lambda x, x \rangle = 0 \) implies that \( \text{Re} \langle B^2x, x \rangle = 0 \) and \( \text{Re} \langle Cx, x \rangle = 0 \). On the other hand, since \( C \) is \( \theta \)-accretive, with \( 0 \leq \theta < \pi/2 \), then
\[
|\text{Im} \langle Cx, x \rangle| \leq \tan(\theta)\text{Re} \langle Cx, x \rangle.
\]

Thus,
\[
\text{Im} \langle Cx, x \rangle = 0 \quad \text{and} \quad \text{Im} \langle B^2x, x \rangle = -\text{Im} \langle Cx, x \rangle = 0,
\]

hence
\[
\langle B^2x, x \rangle = 0 \quad \text{and} \quad \langle Cx, x \rangle = 0.
\]

Since \( B^2 \) is m-accretive operator, we conclude that \( x \in N(B^2) \) and \( x \in N(C^*) \). \( \Box \)

Now, we define the linear factors \( Z_1 \) and \( Z_2 \) by
\[
Z_1 = B + \Lambda^{\frac{1}{2}}
\]
and
\[
Z_2 = B - \Lambda^{\frac{1}{2}}
\]
with domain \( D(B) \cap D(\Lambda^{\frac{1}{2}}) \), into which the quadratic pencil (3.1) can be decomposed.

**Proposition 3.5.** Assume that \( B(D(B^2)) \subset D(B^2) \) and \( \Lambda^{\frac{1}{2}}(D(B^2)) \subset D(B^2) \). \( Q \) takes the following form,
\[
Q(\lambda)x = \frac{1}{2}(\lambda I - Z_1)(\lambda I - Z_2)x + \frac{1}{2}(\lambda I - Z_2)(\lambda I - Z_1)x,
\]
(3.5)
for all \( x \in \mathcal{D}(B^2) \).

In particular, if \( B\Lambda^{\frac{1}{2}} = \Lambda^{\frac{1}{2}}B \) on \( \mathcal{D}(B^2) \), then \( Q \) admits the following canonical factorization,

\[
Q(\lambda)x = (\lambda I - Z_1)(\lambda I - Z_2)x = (\lambda I - Z_2)(\lambda I - Z_1)x, \quad (3.6)
\]

for all \( x \in \mathcal{D}(B^2) \).

Proof. We have \( \mathcal{D}(B^2) \subset \mathcal{D}(Z_1) = \mathcal{D}(Z_2) = \mathcal{D}(\Lambda^{\frac{1}{2}}) \cap \mathcal{D}(B) \) and \( \mathcal{D}(B^2) \subset \mathcal{D}(C) \).

The fact that \( B(\mathcal{D}(B^2)) \subset \mathcal{D}(B^2) \) and \( \Lambda^{\frac{1}{2}}(\mathcal{D}(B^2)) \subset \mathcal{D}(B^2) \), we have

\[
\mathcal{D}(B^2) \subset \mathcal{D}(B\Lambda^{\frac{1}{2}}), \quad \mathcal{D}(B^2) \subset \mathcal{D}(\Lambda^{\frac{1}{2}}B) \quad \text{and} \quad \mathcal{D}(B^2) \subset \mathcal{D}(Z_1^2). \]

Now, we can verify that

\[
Z_1^2x - BZ_1x - Z_1Bx - Cx = 0,
\]

for all \( x \in \mathcal{D}(B^2) \), hence on \( \mathcal{D}(B^2) \), we have

\[
Q(\lambda) = Q(\lambda) - (Z_1^2 - BZ_1 - Z_1B - C)
= \lambda^2I - 2\lambda B - C - Z_1^2 + BZ_1 + Z_1B + C
= \lambda^2I - Z_1^2 - B(\lambda - Z_1) - (\lambda - Z_1)B
= \frac{1}{2}(\lambda - Z_1)(\lambda + Z_1 - 2B) + \frac{1}{2}(\lambda + Z_1 - 2B)(\lambda - Z_1)
= \frac{1}{2}(\lambda I - Z_1)(\lambda I - Z_2) + \frac{1}{2}(\lambda I - Z_2)(\lambda I - Z_1).
\]

Now, if \( B\Lambda^{\frac{1}{2}} = \Lambda^{\frac{1}{2}}B \) on \( \mathcal{D}(B^2) \), we obtain (3.6). \( \square \)

In the sequel, we investigate some properties of the operators \( Z_1 \) and \( Z_2 \).

Proposition 3.6. Assume that \( \mathcal{D}(\Lambda^{\frac{1}{2}}) \subset \mathcal{D}(B) \), then for any \( \varepsilon > 0 \), there exist \( r_1, r_2 > 0 \), such that \( Z_1 + r_1 \) and \( -Z_2 + r_2 \) are \( m\)-\( \psi \)-accretive operators with \( \psi = \pi/4 + \varepsilon \).

In particular, \(-Z_1 - r_1 \) and \(-Z_2 - r_2 \) generates holomorphic \( C_0 \)-semigroup of contraction operators \( T_1(z) \) and \( T_2(z) \) of angle \( \pi/2 - \psi \).

Proof. Now assume that \( \mathcal{D}(\Lambda^{\frac{1}{2}}) \subset \mathcal{D}(B) \). It follows that

\[
\|Bx\| \leq a\|x\| + b\|\Lambda^{\frac{1}{2}}x\| \quad (3.7)
\]

for all \( x \in \mathcal{D}(\Lambda^{\frac{1}{2}}) \) and for some nonnegative constants \( a \) and \( b \). On the other hand, since \( \Lambda^{\frac{1}{2}} \) is \( m\)-\((\pi/4)\)-accretive and both \( B \) and \( -B \) satisfy (3.7), by [14, Theorem IX-2.4], we obtain the desired results. \( \square \)

Remark 3.7. If \( a = 0 \) in (3.7), we have \( r_1 = r_2 = 0 \) (cf. [14, Theorem IX-2.4]).

Proposition 3.8. If \( \mathcal{D}(\Lambda^{\frac{1}{2}}) \subset \mathcal{D}(B) \) and \( B \) is accretive, then \( Z_1 \) is \( m\)-\( \pi/4 \)-accretive operators. In particular, \(-Z_1 \) generates holomorphic \( C_0 \)-semigroup \( T_1(z) \) of angle \( \pi/4 \).

Further, if \( B \) is \( \theta \)-accretive with \( 0 \leq \theta < \pi/2 \), then \( \mathcal{N}(Z_1) = \mathcal{N}(B) \cap \mathcal{N}(\Lambda^{\frac{1}{2}}) \).
Proof. By [19, Theorem 6.10], we have for an arbitrary \( \rho > 0 \),
\[
\| \Lambda^\frac{1}{2} x \|^2 \leq \frac{1}{\pi^2} \left( \rho \|x\|^2 + \frac{1}{\rho} \|\Lambda x\|^2 \right),
\]
for all \( x \in D(B^2) \).

Thus by (3.8) and (3.7), we obtain
\[
\| Bx \|^2 \leq 2a \left( 1 + \frac{\rho}{\pi^2} \right) \|x\|^2 + \frac{2b}{\pi^2 \rho^2} \|\Lambda x\|^2,
\]
for all \( x \in D(B^2) \) and an arbitrary \( \rho > 0 \). Thus
\[
\| B(t + \Lambda^\frac{1}{2})^{-1} x \|^2 \leq 2a \left( 1 + \frac{\rho}{\pi^2} \right) \| (t + \Lambda^\frac{1}{2})^{-1} x \|^2 + \frac{2b}{\pi^2 \rho} \| \Lambda (t + \Lambda^\frac{1}{2})^{-1} x \|^2,
\]
for all \( x \in \mathcal{H} \).

Hence
\[
\left\| B(t + \Lambda^\frac{1}{2})^{-1} \right\|^2 \leq \frac{2a}{t^2} \left( 1 + \frac{\rho}{\pi^2} \right) + \frac{2b}{\pi^2 \rho} \left\| \Lambda (t + \Lambda^\frac{1}{2})^{-1} \right\|^2.
\]

Letting \( t \) to \(+\infty\), we assert that
\[
M = \sup_{t > 0} \left\| B(t + \Lambda^\frac{1}{2})^{-1} \right\| < \frac{2b}{\pi^2 \rho^2}.
\]
(cf. [25, Proposition 2.12]). Since \( \rho \) is arbitrary, we can choose it such that
\[
\frac{2b}{\pi^2 \rho^2} < 1.
\]
Thus \( Z_1 \) is m-accretive. Since \( B \) is accretive and \( \Lambda^\frac{1}{2} \) is m-(\( \pi/4 \))-accretive, then \( Z_1 \) is m-(\( \pi/4 \))-accretive. By [14, Theorem IX-1.24], the factor \(-Z_1\) generates holomorphic \( C_0 \)-semigroup \( T_1(z) \) of angle \( \pi/4 \).

The inclusion \( \mathcal{N}(B) \cap \mathcal{N}(\Lambda^\frac{1}{2}) \subset \mathcal{N}(Z_1) \) is obvious. Conversely, let \( x \in \mathcal{D}(Z_1), \, x \neq 0 \), such that \( Z_1 x = 0 \), we have
\[
\Re < Z_1 x, x > = \Re < B x, x > + \Re < \Lambda^\frac{1}{2} x, x >,
\]
then
\[
\Re < B x, x > \leq \Re < Z_1 x, x > \quad \text{and} \quad \Re < \Lambda^\frac{1}{2} x, x > \leq \Re < Z_1 x, x >.
\]
Therefore, \( \Re < Z_1 x, x > = 0 \) implies that \( \Re < B x, x > = 0 \) and \( \Re < \Lambda^\frac{1}{2} x, x > = 0 \). On the other hand, we have
\[
|\Im < B x, x >| \leq \tan(\theta) \Re < B x, x >
\]
and
\[
|\Im < \Lambda^\frac{1}{2} x, x >| \leq \Re < \Lambda^\frac{1}{2} x, x >.
\]
Thus,
\[
\Im < B x, x > = 0 \quad \text{and} \quad \Im < \Lambda^\frac{1}{2} x, x > = 0,
\]
hence
\[
< B x, x > = 0 \quad \text{and} \quad < \Lambda^\frac{1}{2} x, x > = 0.
\]
Since $B$ is m-$\theta$-accretive (see Remark 3.2) and $\Lambda^{\frac{1}{2}}$ is m-(\pi/4)-accretive, we conclude that 

$$Bx = 0 \quad \text{and} \quad \Lambda^{\frac{1}{2}}x = 0.$$ 

Consequently, $\mathcal{N}(Z_1) \subset \mathcal{N}(B) \cap \mathcal{N}(\Lambda^{\frac{1}{2}})$. \hfill \Box

**Remark 3.9.** In Proposition 3.1, if we assume only $B$ is m-accretive, $C + B^2$ need not be m-accretive, because $B^2$ fails to be accretive (with the same vertex as $B$) even in the case of an accretive matrix $B$ with numerical range contained in a sector of angle less than $\pi/4$, as the following example shows.

**Example.** Let $\mathcal{H} = \mathbb{C}^2$ and 

$$B = \begin{bmatrix} 4 - i & 4i \\ 4i & 16 + 4i \end{bmatrix}.$$ 

For $x = (x_1, x_2) \in \mathbb{C}^2$, we have 

$$\text{Re} \ <Bx, x> = 4|x_1|^2 + 16|x_2|^2$$

and 

$$\text{Im} \ <Bx, x> = -|x_1|^2 + 8\text{Re}(x_1 \overline{x_2}) + 4|x_2|^2 \leq 3|x_1|^2 + 8|x_2|^2 < \text{Re} \ <Bx, x>.$$ 

Thus 

$$W(B) \subset S_{\pi/4}.$$ 

However, for $x = (1, 0)$, we have 

$$<B^2x, x> = -1 - 8i,$$

it follows that $W(B^2)$ is not a subset of the right half complex plane.

**Remark 3.10.** The operator pencil $Q$ is not necessarily an accretive, because we can find an eigenvalues not located in the closed right half-plane. Indeed, let $\lambda$ be an eigenvalue of $Q$ and $v \in \mathcal{D}(Q)$ its corresponding eigenvector with $\|v\| = 1$. Let us remark that if $\lambda = 0$, then $0 = <Cv, v>$ and hence $0 \in W(C)$. In the sequel we assume that $\lambda \neq 0$ with $\lambda = \alpha + i\beta$. Then 

$$<Q(\lambda)v, v> = 0,$$

and consequently, taking real and imaginary parts, 

$$(\alpha^2 - \beta^2) - 2\alpha\text{Re} <Bv, v> + 2\beta\text{Im} <Bv, v> - \text{Re} <Cv, v> = 0,$$

and 

$$2\alpha\beta - 2\beta\text{Re} <Bv, v> - 2\alpha\text{Im} <Bv, v> - \text{Im} <Cv, v> = 0.$$ 

It follows that 

$$\text{Re} <Cv, v> = (\alpha^2 - \beta^2) - 2\alpha\text{Re} <Bv, v> + 2\beta\text{Im} <Bv, v>,$$

and 

$$\text{Im} <Cv, v> = 2\alpha\beta - 2\beta\text{Re} <Bv, v> - 2\alpha\text{Im} <Bv, v>.$$
Since \( \text{Re} \langle Cv, v \rangle \geq 0 \), we obtain from the first relation,
\[
2\alpha \text{Re} \langle Bv, v \rangle \leq \alpha^2 - \beta^2 + 2\beta \text{Im} \langle Bv, v \rangle.
\]
The fact that \( |\text{Im} \langle Bv, v \rangle| \leq \text{Re} \langle Bv, v \rangle \), we get
\[
2\alpha \text{Re} \langle Bv, v \rangle \leq \alpha^2 - \beta^2 + 2|\beta| \text{Re} \langle Bv, v \rangle.
\]
Thus
\[
2(\alpha - |\beta|) \text{Re} \langle Bv, v \rangle \leq \alpha^2 - \beta^2.
\]
Now, if assume \( |\alpha| \leq |\beta| \), it follows that
\[
(\alpha - |\beta|) \text{Re} \langle Bv, v \rangle \leq 0.
\]
Consequently, \( \alpha \leq |\beta| \).

Remark 3.11. Similar results can be obtained by interchanging the role of \( B^2 \) and \( C \), be careful with domains. In this case we have \( \mathcal{D}(C) \subset \mathcal{D}(B^2) \subset \mathcal{D}(B) \).

4. An Application to an Abstract Second-Order Differential Equation

Let us consider, in the complex Hilbert space \( \mathcal{H} \), the following abstract second order differential equation
\[
u''(x) - 2Bu'(x) - Cu(x) = f(x), \quad x \in (0, 1), \tag{4.1}
\]
under the boundary conditions
\[
u(0) = u_0, \quad \nu(1) = u_1, \tag{4.2}
\]
where \( B \) and \( C \) are two closed operators in a Hilbert space with domains \( \mathcal{D}(B) \) and \( \mathcal{D}(C) \), respectively, \( f \in L^p(0, 1; \mathcal{H}) \), \( 1 < p < \infty \) and \( u_0, u_1 \) are given elements in \( \mathcal{H} \). We seek for a strict solution \( \nu \) to (4.1)–(4.2), i.e. a function \( \nu \) such that
\[
\begin{aligned}
i) & \; \nu \in W^{2,p}(0, 1; \mathcal{H}) \cap L^p(0, 1; \mathcal{D}(C)), \; \nu' \in L^p(0, 1; \mathcal{D}(B)), \\
iin) & \; \nu \text{ satisfies } (4.1) - (4.2).
\end{aligned}
\tag{4.3}
\]

Theorem 4.1. Let \( B \) and \( C \) two operators in a Hilbert space \( \mathcal{H} \) such that
\begin{enumerate}
\item \( B^2 \) is \( m \)-accretive and \( C \) is accretive satisfy one of conditions of Proposition 3.1.
\item \( \mathcal{D}((B^2 + C)^{1/2}) \subset \mathcal{D}(B). \)
\item \( B(\mathcal{D}(B^2)) \subset \mathcal{D}(B^2) \) and \( (B^2 + C)^{1/2}(\mathcal{D}(B^2)) \subset \mathcal{D}(B^2). \)
\item \( (B^2 + C)^{-1/2} \) exist and bounded.
\item \( B(B^2 + C)^{1/2} = (B^2 + C)^{1/2}B \) on \( \mathcal{D}(B^2). \)
\item \( f \in L^p(0, 1; \mathcal{H}) \) with \( 1 < p < \infty \).
\end{enumerate}

Then the problem (4.1)–(4.2) has a classical solution \( \nu \) if and only if
\[
Z_1^2 e^{-Z_1} u_0, \quad Z_1^2 e^{-Z_1} u_1 \in L^p(0, 1; \mathcal{H}).
\]
In this case, \( u \) is uniquely determined by

\[
\begin{align*}
  u(x) &= (I - e^{Z_2-Z_1})^{-1}e^{xZ_2}u_0 + (I - e^{Z_2-Z_1})^{-1}e^{-(1-x)Z_1}u_1 \\
  &\quad - (I - e^{Z_2-Z_1})^{-1}e^{xZ_2}e^{-Z_1}\left(u_1 - (Z_2 - Z_1)^{-1}\int_0^1 e^{(1-s)Z_2}f(s)ds\right) \\
  &\quad - (I - e^{Z_2-Z_1})^{-1}e^{-(1-x)Z_1}e^{Z_2}\left(u_0 + (Z_2 - Z_1)^{-1}\int_0^1 e^{-sZ_1}f(s)ds\right) \\
  &\quad + (I - e^{Z_2-Z_1})^{-1}(Z_2 - Z_1)^{-1}e^{xZ_2}\int_0^1 e^{-sZ_1}f(s)ds \\
  &\quad - (I - e^{Z_2-Z_1})^{-1}(Z_2 - Z_1)^{-1}e^{-(1-x)Z_1}\int_0^1 e^{-(1-s)Z_2}f(s)ds \\
  &\quad + (Z_2 - Z_1)^{-1}\int_0^x e^{(x-s)Z_2}f(s)ds - (Z_2 - Z_1)^{-1}\int_x^1 e^{(x-s)Z_1}f(s)ds.
\end{align*}
\]

**Proof.** Under the assumptions, by Proposition 3.6 and Remark 3.7, the factors \(-Z_1\) and \(Z_2\) generates bounded holomorphic \(C_0\)-semigroup \((e^{-tZ_1})_{t \geq 0}\) and \((e^{tZ_2})_{t \geq 0}\), respectively. Also, \(D(Z_1) = D(Z_2) = D(\Lambda^{1/2})\) and

\[
D(Z_1Z_2) = \{x \in D(Z_2); Z_2x \in D(Z_1)\} = \{x \in D(Z_2); Z_2x \in D(Z_2)\} = D(Z_2^2),
\]

\[
D(Z_2Z_1) = \{x \in D(Z_1); Z_1x \in D(Z_2)\} = \{x \in D(Z_1); Z_1x \in D(Z_1)\} = D(Z_1^2).
\]

But

\[
D(Z_1^2) = \left\{x \in D(\Lambda^{1/2}); Z_1x \in D(\Lambda^{1/2})\right\}
\]

and

\[
D(Z_2^2) = \left\{x \in D(\Lambda^{1/2}); Z_2x \in D(\Lambda^{1/2})\right\}.
\]

The fact that, \(B(D(B^2)) \subset D(B^2)\) and \(\Lambda^{1/2}(D(B^2)) \subset D(B^2)\), we obtain \(D(B^2) \subset D(Z_1^2)\) and \(D(B^2) \subset D(Z_2^2)\), with \(D(B^2)\) densely defined on \(H\). Furthermore, \(e^{-tZ_1}u_0 \in D(Z_1^n)\) and \(e^{tZ_2}u_1 \in D(Z_2^n)\) for all \(u_0, u_1 \in H, t > 0\) and \(n \in \mathbb{N}\). Hence \(u(x) \in D(C)\) for all \(x \in (0, 1)\). Since the two \(C_0\)-semigroups are holomorphic, \(u(.)\) can be differentiated any numbers of times. Now, by taking \(-B\) instead \(B\), \(A = -C\), \(L = -Z_1\) and \(M = Z_2\) in [8, Theorem 5.], all assumptions of this theorem are fulfilled. Hence we obtain the desired result. \(\square\)

5. An Example of a Second-order Partial Differential Equation

The aim of this section is to use the obtained results to discuss the existence, uniqueness, and maximal regularity of the strict solution for the following
non-homogeneous second order differential equation,

\[
\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x, y) - 2i p_0(y) \frac{\partial^2 u}{\partial y \partial x}(x, y) - 2i p_1(y) \frac{\partial u}{\partial x}(x, y) - \alpha p_0(y) \frac{\partial u}{\partial y}(x, y) \\
- (\alpha p_1(y) + \beta) u(x, y) + \gamma u(x, y) = f(x, y), \quad x \in (0, 1), \quad y \in (0, 1)
\end{aligned}
\]

where,

\begin{itemize}
  \item \( f \in L^p(0, 1; L^2(0, 1; \mathbb{C})) \), \( 1 < p < \infty \),
  \item \( \alpha \in \mathbb{R}, \beta \in \mathbb{C}, p_0, p_1 \in C^1(0, 1) \) and \( p_0(x) \neq 0 \) for all \( x \in [0, 1] \).
  \item \( \gamma = -\left( \frac{r + 1}{4\varepsilon} M_1 + M_2 \right) \), with \( r > 0 \) and \( \varepsilon \) are arbitrary and chosen such that \( m_0 - \varepsilon(1 + r)M_1 > 0 \), for some nonegative constants \( m_0, M_1 \) and \( M_2 \) are described below.
\end{itemize}

The second order differential equation \((E)\) is equivalent to

\[
\frac{\partial^2 u}{\partial x^2}(x, y) - 2i B \frac{\partial u}{\partial x}(x, y) - C u(x, y) + \gamma u(x, y) = f(x, y), \quad x \in (0, 1), \quad y \in (0, 1).
\]

with the boundary conditions

\[
u(0, y) = u_0(y), \quad u(1, y) = u_1(y), \quad y \in (0, 1),
\]

where,

\[
\begin{aligned}
B &= p_0 \frac{\partial}{\partial y} + p_1, \quad D(B) = \{ \psi \in H^1(0, 1) : \psi(0) = \psi(1) = 0 \}
\end{aligned}
\]

and

\[
\begin{aligned}
C &= \alpha p_0 \frac{\partial}{\partial y} + (\alpha p_1 + \beta), \quad D(C) = \{ \phi \in H^1(0, 1) : \phi(0) = \phi(1) = 0 \}.
\end{aligned}
\]

with \( \phi(y) = u(x, y) \) and \( \psi(y) = \frac{\partial u}{\partial x}(x, y) \), \( x \in (0, 1), \quad y \in (0, 1) \). We seek for a strict solution \( u(., y) \) to \((5.1)-(5.2)\), i.e. a function \( u(., y) \in L^2(0, 1; \mathbb{C}) \) such that \( 4.3 \) holds. This will be done by the following preparatory results.

**Claim 1.** The operator \(-B^2 - \gamma I\) is \(m, \omega\)-accretive, with \( \omega = \arctan \left( \frac{1}{r} \right) \).

**Proof.** For \( \psi \in D(B^2) \subset \{ \psi \in H^2(0, 1) : \psi(0) = \psi(1) = 0 \} \subset D(B) \), we have

\[
-B^2 \psi = \varphi_0 \psi'' + \varphi_1 \psi' + \varphi_2 \psi,
\]

with \( \varphi_0 = -p_0^2, \varphi_1 = -p_0(p_0' + 2p_1) \) and \( \varphi_2 = -(p_1^2 + p_0 p_1') \). Under the assumptions there exists a nonegative constants \( m_0, M_0 \) and \( M_1 \) such that

\[
- \varphi_0 > m_0 > 0, \quad |\varphi_1 - \varphi_0'| \leq M_1, \quad \text{and} \quad |\varphi_2| \leq M_2.(5.3)
\]

By \([14, \text{Example V-3.34}]\), \(-B^2\) is \(m, \omega\)-accretive operator with vertex \( \gamma \), where

\[
\gamma = -\left( \frac{r + 1}{4\varepsilon} M_1 + M_2 \right), \quad \omega = \arctan \left( \frac{1}{r} \right), \quad r > 0 \text{ and } \varepsilon \text{ is chosen such that}
\]
\( m_0 - \varepsilon(1 + r)M_1 > 0 \). Hence the operator \(-B^2 - \gamma I\) is m-\(\omega\)-accretive, with 
\[ \omega = \arctan \left( \frac{1}{r} \right). \]

Claim 2. If \(\alpha p_1 + \text{Re}(\beta) - \frac{\alpha}{2} p'_0 \geq 0\) then \(C\) is an accretive operator.

Proof. Let \(\psi \in D(C)\), we have
\[
<C\psi, \psi> = \alpha \int_0^1 p_0(y)\psi'(y)\bar{\psi}(y)dy + \int_0^1 (\alpha p_1(y) + \beta) |\psi(y)|^2 dy.
\]
By integration by parts,
\[
<C\psi, \psi> = -\alpha \int_0^1 p_0(y)\psi(y)\bar{\psi}'(y)dy + \int_0^1 (\alpha p_1(y) + \beta - \alpha p'_0(y)) |\psi(y)|^2 dy.
\]
Also
\[
<C\psi, \psi> = \alpha \int_0^1 p_0(y)\psi(y)\bar{\psi}'(y)dy + \int_0^1 (\alpha p_1(y) + \bar{\beta}) |\psi(y)|^2 dy.
\]
Thus
\[
\text{Re} <C\psi, \psi> = \int_0^1 (\alpha p_1 + \text{Re}(\beta) - \frac{\alpha}{2} p'_0) |\psi(y)|^2 dy.
\]
Hence the desired result.

If we take \(\alpha = 1\) and \(\beta = 0\) in Claim 2, we obtain

Claim 3. If \(p_1 - \frac{1}{2} p'_0 \geq 0\) then \(B\) is an accretive operator. In particular, by Remark 3.2, \(B\) is m-accretive.

Claim 4. If \(p_1 - \frac{1}{2} p'_0 \geq 0\) and \(\alpha p_1 + \text{Re}(\beta) - \frac{\alpha}{2} p'_0 \geq 0\), then \(-\Lambda = -B^2 + C - \gamma I\) with domain \(D(B^2)\) is m-accretive. Also, \(-\Lambda\) admits an unique square root \((-\Lambda)^{1/2}\) m-(\(-\pi/4\))-accretive.

Proof. By Claim 1. \(-B^2 - \gamma I\) with domain \(D(B^2)\) is m-accretive, by Claim 2. \(C\) is an accretive and by Claim 3. \(B\) is an accretive operator. Also, \(D(B) = D(C)\). Now the desired result holds from the \((C.4)\) and Proposition 3.1.

Claim 5. If \(p''_0\) is continuous on \([0, 1]\), then \((-B^2 + C - \gamma I)^{-1}\) exist and bounded.

Proof. As before; for \(\psi \in D(B^2) \subset \{\psi \in H^2(0, 1) : \psi(0) = \psi(1) = 0\} \subset D(B)\), we have
\[
[-B^2 + C - \gamma I]\psi = \varphi_0 \psi'' + (\varphi_1 + \alpha p_1) \psi' + (\varphi_2 + \alpha p_1 + \beta - \gamma) \psi,
\]
with \(\varphi_0 = -p''_0\), \(\varphi_1 = -p_0(p'_0 + 2p_1)\) and \(\varphi_2 = -(p''_1 + p_0p'_1)\). Since \(p''_0\) and \(p'_1\) are continuous on \([0, 1]\), it follows that \(\varphi_0', \varphi_1' + \alpha p_1'\) and \(\varphi_2 + \alpha p_1 + \beta - \gamma\) are continuous on \([0, 1]\). By a similar way as in [14, Sect. 3-III. p. 146-149], we prove that \((-B^2 + C - \gamma I)^{-1}\) exist and bounded.

Combining Claim 4. Corollary 3.3, Propositions 3.6 and 3.8, we obtain,
Claim 6. The operators $Z_1 = iB + (-\Lambda)^{1/2}$ and $Z_2 = iB - (-\Lambda)^{1/2}$ with domain $\mathcal{D}(\Lambda^{1/2}) \subset \mathcal{D}(B)$ are $B^2$-bounded and closed operators. Furthermore, the closure of the restriction of $Z_i$ to $\mathcal{D}(B^2)$ is again $Z_i$, $i = 1, 2$, $-Z_1$ generates holomorphic $C_0$-semigroup $T_1(z)$ of angle $\pi/4$ and $Z_2$ generates holomorphic $C_0$-semigroup $T_2(z)$ of angle $\pi/4$.

We are now ready to state the following existence and uniqueness result.

Theorem 5.1. Let the equation (E) on $\mathcal{H} = L^2(0, 1; \mathbb{C})$. Assume that
1. $f \in L^p(0, 1; \mathcal{H})$, $1 < p < \infty$,
2. $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$, $p_0 \in C^2(0, 1)$, $p_1 \in C^1(0, 1)$ and $p_0(x) \neq 0$ for all $x \in [0, 1]$,
3. $p_1 - \frac{1}{2} p_0' \geq 0$ and $\alpha(p_1 - \frac{1}{2} p_0') + \text{Re}(\beta) \geq 0$,
4. $\gamma = -\left(\frac{r + 1}{4\varepsilon} M_1 + M_2\right)$, with $r > 0$ and $\varepsilon$ are arbitrary and chosen such that $m_0 - \varepsilon(1 + r)M_1 > 0$, for some nonnegative constants $m_0$, $M_1$ and $M_2$ are given by (5.3),
5. $B(-\Lambda)^{1/2} = (-\Lambda)^{1/2}B$ on $\mathcal{D}(B^2)$.

Then the problem (5.1)–(5.2) has a classical solution $u$ if and only if

$$Z_1^2 e^{-Z_1} u_0, \quad Z_2^2 e^{-Z_1} u_1 \in L^p(0, 1; \mathcal{H}).$$

In this case, $u$ is uniquely determined as in Theorem 4.1.

Proof. Thus the restriction of $Z_1$ and $-Z_2$ to $\mathcal{D}(B^2)$ are $m-(\pi/4)$-accretive operators. Also; by Claim 5., the inverse of $(-\Lambda)^{1/2}$ exist and bounded. Thus, all assumptions of Theorem 4.1 are fulfilled. Consequently, we get the desired result. \hfill $\square$

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Aglazin, S.D., Kiiko, I.A.: Numerical-analytic investigation of the flutter of a panel of arbitrary shape in a design. J. Appl. Math. Mech. 61, 171–174 (1997)
[2] Arlinskii, Y.M., Zagrebnov, V.: Numerical range and quasi-sectorial contractions. J. Math. Anal. Appl. 366, 33–43 (2010)
[3] Artamonov, N.: Estimates of solutions of certain classes of second-order differential equations in a Hilbert space. Sbornik Math. 194(8), 1113–1123 (2003)
[4] Balakrishnan, A.V.: Fractional powers of closed operators and the semi-groups generated by them. Pacific J. Math. 10, 419–437 (1960)
[5] Ben-Israel, A., Greville, T.N.E.: Generalized Inverses: Theory and Applications, 2nd edn. Springer, New York (2003)
[6] Duffin, R.J.: A minimax theory for overdamped networks. J. Rational. Mech. Anal. 4, 221–233 (1955)
[7] Eschwé, D., Langer, M.: Variational principles for eigenvalues of selfadjoint operator functions. Integral Equ. Oper. Theory 49, 287–321 (2004)
[8] Favini, A., Labbas, R., Maingot, S., Tanabe, H., Yagi, A.: A simplified approach in the study of elliptic differential equations in UMD spaces and new applications. Funkcialaj Ekvacioj 51, 165–187 (2008)
[9] Gustafson, K.: A perturbation lemma. Bull. Am. Math. Soc. 72, 334–338 (1966)
[10] Gustafson, K., Rao, D.: Numerical Range, the Field of Values of Linear Operators and Matrices. Springer, New York (1997)
[11] Hayashi, M., Ozawa, T.: On Landau–Kolmogorov inequalities for dissipative operators. Proc. Am. Math. Soc. 145, 847–852 (2017)
[12] Ilyushin, A.A., Kiiko, I.A.: Vibrations of a rectangle plate in a supersonic aerodynamics and the problem of panel flutter. Moscow Univ. Mech. Bull. 49, 40–44 (1994)
[13] Krein, M.G., Langer, H.: On the theory of quadratic pencils of self-adjoint operators, Dokl. Akad. Nauk SSSR 154 (1964), 1258–1261 (Russian); English transl., Soviet Math. Dokl. 5, 266–269 (1964)
[14] Kato, T.: Perturbation Theory for Linear Operators. Springer, New York (1995)
[15] Kato, T.: Fractional powers of dissipative operators. Proc. Jpn. Acad. 13(3), 246–274 (1961)
[16] Kato, T.: On an inequality of Hardy, Littlewood, and Polya. Adv. Math. 7, 217–218 (1971)
[17] Markus, A.S.: Introduction to the spectral theory of polynomial operator pencils. Translations of Mathematical Monographs, vol. 71. American Mathematical Society, Providence, RI (1988)
[18] Möller, M., Pivovarchik, V.: Spectral Theory of Operator Pencils, Hermite–Biehler Functions, and their Applications, Birkhäuser (2015)
[19] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, Berlin-Heidelberg-New York (1983)
[20] Okazawa, N.: Perturbations of linear m-accretive operators. Proc. Am. Math. Soc. 37(1), 169–174 (1973)
[21] Okazawa, N.: On the perturbation of linear operators in Banach and Hilbert spaces. J. Math. Soc. Jpn. 34, 677–701 (1982)
[22] Ōta, S.: Closed linear operators with domain containing their range. Proc. Edinburgh Math. Soc. 27, 229–233 (1984)
[23] Shkalikov, A.A.: Strongly damped pencils of operators and solvability of the corresponding operator-differential equations. Math. USSR Sb. 63, 97–119 (1989)
[24] Stampfli, J.G.: Minimal range theorems for operators with thin spectra. Pac. J. Math. 23, 601–612 (1967)
[25] Yoshikawa, A.: On Perturbation of closed operators in a Banach space. J. Fac. Sci. Hokkaido Univ. 22, 50–61 (1972)
[26] Yosida, K.: A perturbation theorem for semigroups of linear operators. Proc. Jpn. Acad. 41, 645–64 (1965)
Fairouz Bouchelaghem
Département de Mathématiques
Université Oran 1-Ahmed Ben Bella
BP 1524, Oran-El M’naouar
31000 Oran
Algeria
e-mail: fairouzbouchelaghem@yahoo.fr

Mohammed Benharrat
Département de Mathématiques et Informatique
Ecole Nationale Polytechnique d’Oran-Maurice Audin (Ex. ENSET d’Oran)
BP 1523 Oran-El M’naouar
31000 Oran
Algeria
e-mail: mohammed.benharrat@enp-oran.dz;
mohammed.benharrat@gmail.com

Received: September 3, 2020.
Accepted: October 3, 2021.