Unadjusted Langevin algorithm for non-convex weakly smooth potentials

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Abstract: Discretization of continuous-time diffusion processes is a widely recognized method for sampling. However, the canonical Euler Maruyama discretization of the Langevin diffusion process, referred as Unadjusted Langevin Algorithm (ULA), studied mostly in the context of smooth (gradient Lipschitz) and strongly log-concave densities, is a considerable hindrance for its deployment in many sciences, including statistics and machine learning. In this paper, we establish several theoretical contributions to the literature on such sampling methods for non-convex distributions. Particularly, we introduce a new mixture weakly smooth condition, under which we prove that ULA will converge with additional log-Sobolev inequality. We also show that ULA for smoothing potential will converge in $L_2$-Wasserstein distance. Moreover, using convexification of nonconvex domain \cite{24} in combination with regularization, we establish the convergence in Kullback-Leibler (KL) divergence with the number of iterations to reach $\epsilon$-neighborhood of a target distribution in only polynomial dependence on the dimension. We relax the conditions of \cite{31} and prove convergence guarantees under isoperimetry, and non-strongly convex at infinity.

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1. Introduction

Over the last decade, Bayesian inference has become one of the most prevalent inferring instruments for a variety of disciplines, including the computational statistics and statistical learning [30]. In general, Bayesian inference seeks to generate samples of the posterior distribution of the form:

\[ \rho(x) = \frac{e^{-U(x)}}{\int_{\mathbb{R}^d} e^{-U(y)} dy}, \]

where the function \( U(x) \), also known as the potential function, is often convex. The most conventional approaches, random walks Metropolis Hastings [30], often struggle to select a proper proposal distribution for sampling. As a result, it has been proposed to consider continuous dynamics which inherently leaves the objective distribution \( \rho \) invariant. Probably one of the most well-known example of these continuous dynamic applications is the over-damped Langevin diffusion [10] associated with \( U \),

\[ dX_t = -\nabla U(X_t) dt + \sqrt{2} dB_t, \]

where \( B_t \) is a \( d \)-dimensional Brownian motion and its Euler-Maruyama discretization hinges on the following update rule:

\[ x_{k+1} = x_k - \eta_k \nabla U(x_k) + \sqrt{2\eta_k} \xi_k, \]
where \((\eta_k)_{k \geq 1}\) is a sequence of step sizes, which can be kept constant or decreasing to 0, and \(\xi_k \sim N(0, I_d)\) \((I_d\) denotes identity matrix dimension \(d\), are independent Gaussian random vectors. It can be seen that Euler-Maruyama discretization, also known as Langevin Monte Carlo (LMC) algorithm, does not involve knowledge of \(U\), but gradient of \(U\) instead, which makes it ideally applicable where we typically only know \(U\) up to a normalizing constant. Owing to its simplicity, efficiency, and well understood properties, there are various applications using LMC \([33, 9, 28, 34, 23]\). Much of the theory of convergence of sampling used to focus on asymptotic convergence, failing to provide a detailed study of dimension dependence. Recently, there is a surge of interests in non-asymptotic rates of convergence, which include dimension dependence, especially polynomial dependence on the dimension of target distribution; see, e.g., \([8, 10, 12, 15, 3, 14, 7, 22, 25, 26]\). However, there is a critical gap in the theory of discretization of an underlying stochastic differential equation (SDE) to the broad spectrum of applications in statistical inference. In particular, the application of techniques from SDEs traditionally requires \(U(x)\) to have Lipschitz-continuous gradients. This requirement prohibits many typical utilizations \([13]\). \([6]\) has recently established an original approach to deal with weakly smooth (possibly non-smooth) potential problems through smoothing. Their technique rests on results obtained from the optimization community, perturbing a gradient evaluating point by a Gaussian. However, \([6]\) analyzes over-damped Langevin diffusion in the contexts of convex potential functions while many applications involve sampling in high dimensional spaces have non-convex settings. In another research, \([16]\) proposes a very beautiful result using tail growth for weakly smooth and weakly dissipative potentials. By using degenerated convex and modified log Sobolev inequality, they prove that LMC gets \(\epsilon\)-neighborhood of a target distribution in KL divergence with the convergence rate of \(\tilde{O}(d^{1/\alpha} + \frac{1}{\alpha(\frac{1}{\alpha} - 1(\beta/1))} \epsilon^{1/\alpha})\) where \(\alpha\) and \(\beta\) are degrees of weakly smooth and dissipative defined in the next section.

Unfortunately, (weakly) smooth conditions typically can not cover a mixture of distributions with different tail growth behaviors, which prohibit a large range of applications. Therefore, we first introduce an \(\alpha\)-mixture weakly smooth condition to overcome the limitation of the current weakly smooth condition. Under our novel condition and log-Sobolev inequality, we will recover the ULA convergence results \([31]\). In addition, we also show that ULA for smoothing potential will converge in \(L_2\)-Wasserstein distance. Since log-Sobolev inequality is preserved under bounded perturbation, we will extend the results based on convexification of a non-convex domain \([24]\). Our contributions can be outlined as follows.

First, for a potential function \(U\), which satisfies an \(\alpha\)-mixture weakly smooth, and \(\gamma\)-log Sobolev inequality, we prove that ULA achieves the convergence rate of

\[
\tilde{O}\left(d^{\frac{1}{\alpha} - \frac{1}{\gamma}} \epsilon^{\frac{1}{\gamma}}\right)
\]

in KL-divergence.

Second, our convergence results also cover sampling from non-convex po-
tentials, satisfying $\alpha$-mixture weakly smooth, 2-dissipative and $\gamma$-Poincaré or $\gamma$-Talagrand with convergence rate in KL divergence of
\[
\tilde{O}
\left(d^{\frac{\alpha}{2}+1} \gamma \left(\frac{\alpha}{2}\right) \epsilon^{-\frac{1}{\alpha}}\right).
\] (1.5)

Third, we further investigate the case of $\alpha$-mixture weakly smooth, 2-dissipative and non strongly convex outside the ball of radius $R$ and obtain the convergence rate of
\[
\tilde{O}
\left(d^{\frac{1}{2}} \gamma \left(\frac{\alpha}{2}\right) \epsilon^{5} \left(2 \sum_i L_i R_i^{1+\alpha} + 4 L N R^2 + 4 \max \{L_i\} R^{1+\alpha} \right) \left(\frac{1}{\alpha} + 1\right) \epsilon^{-\frac{1}{\alpha}}\right).
\] (1.6)

Finally, our convergence results remain valid under finite perturbations, indicating that it is applicable to an even larger class of potentials. Last but not least, convergence in KL divergence implies convergence in total variation and in $L_2$-Wasserstein metrics based on Pinsker inequalities, which in turn gives convergence rates of $O(\epsilon^{-\frac{1}{2}})$ and $O(\epsilon^{-\frac{1}{2}} d^{\frac{\alpha}{2}})$ in place of $O(\epsilon^{-\frac{1}{\alpha}})$ in each case above, respectively for total variation and $L_2$-Wasserstein metrics.

The rest of the paper is organized as follows. Section 2 sets out the notation, definition and smoothing properties necessary to give our main results in section 3. Section 4 extends the convexification of non convex domain of \cite{24} for strongly convex outside the ball to non-strongly convex outside the ball, and employs this outcome in combination with regularization to obtain convergence in KL divergence for potentials which satisfy log Sobolev, Talagrand, Poincaré inequalities, or non-strongly convex at infinity while Section 5 presents our conclusions.

2. Preliminaries

This section provides the notational conventions, assumptions, and some auxiliary results used in this paper. We let $|s|$, for a real number $s \in \mathbb{R}$, denote its absolute value and use $\langle \ , \ \rangle$ to specify inner products. We use $\|x\|_p$ to denote the $p$-norm of a vector $x \in \mathbb{R}^d$ and throughout the paper, we drop the subscript and just write $\|x\|_2 := \|x\|_2$ whenever $p = 2$. For a function $U : \mathbb{R}^d \to \mathbb{R}$, which is twice differentiable, we use $\nabla U(x)$ and $\nabla^2 U(x)$ to denote correspondingly the gradient and the Hessian of $U$ with respect to $x$. We use $A \succeq B$ if $A - B$ is a positive semi-definite matrix. We use big-oh notation $O$ in the following sense that if $f(x) = O(g(x))$ implies $\lim_{x \to \infty} \sup \frac{f(x)}{g(x)} < \infty$ and $\tilde{O}$ suppresses the logarithmic factors.

2.1. Assumptions on the potential $U$

The objective of this paper is to sample from a distribution $\pi \propto \exp(-U(x))$, where $x \in \mathbb{R}^d$. While sampling from the exact distribution $\pi(x)$ is generally computationally demanding, it is largely adequate to sample from an approximated distribution $\tilde{\pi}(x)$ which is in the vicinity of $\pi(x)$ by some distances. In this paper, we suppose some of the following conditions hold:
Assumption 2.1. \((\alpha\text{-mixture weakly smooth})\) There exist \(0 < \alpha = \alpha_1 < \ldots < \alpha_N \leq 1, i = 1, \ldots, N\) so that for all probability distributions \(p\) and \(\pi\) the following inequality holds for every \(x, y \in \mathbb{R}^d\), we obtain
\[
\|\nabla U(x) - \nabla U(y)\| \leq \sum_{i=1}^N L_i \|x-y\|^{\alpha_i}
\]
where \(\nabla U(x)\) represents an arbitrary sub-gradient of \(U\) at \(x\).

Assumption 2.2. \((\alpha, \ell)\text{-weakly smooth})\) There exist \(0 \leq \ell, 0 < L < \infty\) and \(\alpha \in [0, 1]\) so that for all \(\|x\| \geq R\), we obtain
\[
\|\nabla U(x) - \nabla U(y)\| \leq L \left(1 + \|x-y\|^\ell\right) \|x-y\|^\alpha,
\]
where \(\nabla U(x)\) represents an arbitrary sub-gradient of \(U\) at \(x\).

Assumption 2.3. \((\mu, \theta)\text{-degenerated convex outside the ball})\) There exist some \(\mu > 0, 1 \geq \theta \geq 0\) so that for every \(\|x\| \geq R\), the potential function \(U(x)\) satisfies
\[
\nabla^2 U(x) \geq m (\|x\|) I_d \quad \text{where} \quad m(r) = \mu \left(1 + r^\theta\right)^{-\frac{\theta}{2}}.
\]

Assumption 2.4. \((\beta\text{-dissipativity})\) There exist some \(\beta \geq 1, a, b > 0\) such that for every \(\|x\| \geq R\), \(\langle \nabla U(x), x \rangle \geq a \|x\|^\beta - b\).

Assumption 2.5. \((\text{LSI} (\gamma))\) There exist some \(\gamma > 0\), so that for all probability distributions \(p(x)\) absolutely continuous w.r.t. \(\pi(x)\), we obtain
\[
\|\nabla U(x)\| \leq \frac{1}{2\gamma} I(p|\pi).
\]

Assumption 2.6. \((\text{PI} (\gamma))\) There exist some \(\gamma > 0\), so that for all smooth functions \(g: \mathbb{R}^d \to \mathbb{R}\), we obtain
\[
\var_{\pi} (g) \leq \frac{1}{\gamma} E_\pi \left[ \|\nabla g\|^2 \right] \quad \text{where} \quad \var_{\pi} (g) = E_\pi [g^2] - E_\pi [g]^2.
\]

Assumption 2.7. \((\text{non-strongly convex outside the ball})\) For every \(\|x\| \geq R\), the potential function \(U(x)\) is positive semi-definite, that is for every \(y \in \mathbb{R}^d\), \(\langle y, \nabla^2 U(x) y \rangle \geq 0\).

Assumption 2.8. The function \(U(x)\) has stationary point at zero \(\nabla U(0) = 0\).

Remark 2.1. Assumption 2.8 is imposed without loss of generality. Condition 2.1 often holds for a mixture of distribution with different tail growth behaviors. It is straightforward to generalize condition 2.1 from the mixture of two distributions with the same constant \(L\), so we will consider condition 2.2 to simplify the proof while optimize the convergence rate. Condition 2.2 is an extension of \(\alpha\text{-weakly smooth} or \alpha\text{-Hölder} continuity of the (sub)gradients of \(U\) (that is when \(\ell = 0\), we recover normal \(\alpha\text{-weakly smooth}).

2.2. Smoothing using p-generalized Gaussian smoothing

A feature that follows straightforwardly from Assumption 2.1 is that for \(\forall x, y \in \mathbb{R}^d\):

Lemma 2.1. If potential \(U: \mathbb{R}^d \to \mathbb{R}\) satisfies an \(\alpha\text{-mixture weakly smooth} for some \(0 < \alpha = \alpha_1 < \ldots < \alpha_N \leq 1, i = 1, \ldots, N\) so that \(0 < L_i < \infty\), then:
In particular, if the potential $U : \mathbb{R}^d \to \mathbb{R}$ satisfies $(\alpha, \ell)$-weakly smooth for some $\alpha + \ell \leq 1$ and $\alpha \in (0, 1]$, then:

$$U(y) \leq U(x) + \langle \nabla U(x), y - x \rangle + \sum_i \frac{L_i}{1 + \alpha_i} \|y - x\|^{1 + \alpha_i}.$$  \hfill (2.1)

Proof. See Appendix B.1 \hfill \Box

Here, to deal with the heavy tail behavior of some distributions in the mixture, we use $p$-generalized Gaussian smoothing. Particularly, for $\mu \geq 0$, $p$-generalized Gaussian smoothing $U_\mu$ of $U$ is defined as:

$$U_\mu(y) := E_{\xi}[U(y + \mu \xi)] = \frac{1}{\kappa} \int_{\mathbb{R}^d} U(y + \mu \xi) e^{-\|\xi\|_p^p / p} d\xi,$$

where $\kappa \overset{def}{=} \int_{\mathbb{R}^d} e^{-\|\xi\|_p^p / p} d\xi = 2^{\frac{pd(1 - \alpha)}{p}} \pi^{\frac{d}{2}}$ and $\xi \sim N_p(0, I_{d \times d})$ (the $p$-generalized Gaussian distribution). The rationale for taking into account the $p$-generalized Gaussian smoothing $U_\mu$ rather than $U$ is that it typically benefits from superior smoothness properties. In particular, $U_\mu$ is smooth albeit $U$ is not. In addition, $p$-generalized Gaussian smoothing is more generalized than Gaussian smoothing in the sense that it contains all normal distribution when $p = 2$ and all Laplace distribution when $p = 1$. This family of distributions allows for tails that are either heavier or lighter than normal and in the limit as well as containing all the continuous uniform distribution. More importantly, we prove that a smoothing potential $U_\mu(x)$ is actually smooth (gradient Lipschitz). This property is novel and potentially useful in the optimization or sampling process, especially when the potential exhibits some sort of weakly smooth behaviors. Here to simplify the proof, we consider $p \in \mathbb{R}^+$, $2 \geq p \geq 1$ and some primary features of $U_\mu$ based on adapting those results of [27].

Lemma 2.2. If potential $U : \mathbb{R}^d \to \mathbb{R}$ satisfies an $\alpha$-mixture weakly smooth for some $0 < \alpha = \alpha_1 = \ldots = \alpha_N \leq 1$, $i = 1, \ldots, N$ $0 < L_i < \infty$, then:

(i) $\forall x \in \mathbb{R}^d : |U_\mu(x) - U(x)| \leq \sum_i L_i \mu^{1 + \alpha_i} d^{\frac{1 + \alpha_i}{p}}$,

(ii) $\forall x \in \mathbb{R}^d : \|\nabla U_\mu(x) - \nabla U(x)\| \leq \sum_i L_i \mu^{\alpha_i} d^{\frac{\alpha_i}{p}}$,

(iii) $\forall x, y \in \mathbb{R}^d : \|\nabla U_\mu(y) - \nabla U_\mu(x)\| \leq \sum_i \frac{L_i}{\mu^{1 - \alpha_i}} d^{\frac{\alpha_i}{p}} \|y - x\|.$

In particular, if the potential $U : \mathbb{R}^d \to \mathbb{R}$ satisfies $(\alpha, \ell)$-weakly smooth for some $\alpha + \ell \leq 1$ and $\alpha \in [0, 1]$, then:

(i) $\forall x \in \mathbb{R}^d : |U_\mu(x) - U(x)| \leq 2L \mu^{1 + \ell + \alpha} d^{\frac{1 + \ell + \alpha}{p}},$
(ii) $\forall x \in \mathbb{R}^d: \|\nabla U_\mu(x) - \nabla U(x)\| \leq L_\mu^{\alpha} d^{1+\frac{4}{p}}$,  
(iii) $\forall x, y \in \mathbb{R}^d: \|\nabla U_\mu(y) - \nabla U_\mu(x)\| \leq \frac{L_\mu}{\mu^{1-\alpha}} d^{\frac{2}{p}} \|y - x\|$. 

Proof. See Appendix B.2

3. Convergence in KL divergence along ULA under LSI

In this section we review the definition of KL divergence and the convergence of KL divergence along the Langevin dynamics in continuous time under log-Sobolev inequality. We then derive our main result for ULA algorithm under LSI.

3.1. Recall KL divergence along Langevin dynamics

Let $p, \pi$ be probability density functions with respect to the Lebesgue measure on $\mathbb{R}^d$. KL divergence of $p$ with respect to $\pi$ is defined as

$$H(p|\pi) \triangleq \int_{\mathbb{R}^d} \log \frac{p(x)}{\pi(x)} \pi(x) dx. \quad (3.1)$$

By definition, KL divergence can be considered as a measure of asymmetric “distance” of a probability distribution $p$ from a base distribution $\pi$. $H(p|\pi)$ is always nonnegative and equals zero only when $p$ equals $\pi$ in distribution. KL divergence is a rather strong measure of distance, which upper bounds a variety of distance measures. We provide the definition of other measures in Appendix A. In general, convergence in KL divergence implies convergence in total variation by Csiszar-Kullback-Pinsker inequality. In addition, under log-Sobolev inequality with constant $\gamma$, KL divergence also controls the quadratic Wasserstein $W_2$ distance by $W_2(p, \pi)^2 \leq \frac{2}{\gamma} H(p|\pi)$.

The Langevin dynamics for target distribution $\pi = e^{-U}$ is a continuous-time stochastic process $(X_t)_{t \geq 0}$ in $\mathbb{R}^d$ that progresses following the stochastic differential equation:

$$dX_t = -\nabla U(X_t) \, dt + \sqrt{2} \, dW_t \quad (3.2)$$

where $(W_t)_{t \geq 0}$ is the standard Brownian motion in $\mathbb{R}^d$.

If $(X_t)_{t \geq 0}$ updates following the Langevin dynamics (3.2), then their probability density function $(p_t)_{t \geq 0}$ will satisfy the following the Fokker-Planck equation:

$$\frac{\partial p_t}{\partial t} = \nabla \cdot (p_t \nabla U) + \Delta p_t = \nabla \cdot \left(p_t \nabla \log \frac{p_t}{\pi}\right). \quad (3.3)$$

Here $\nabla \cdot$ is the divergence and $\Delta$ is the Laplacian operator. In general, by evolving along the Langevin dynamics, a distribution will get closer to its target distribution $\pi$. From [31] Lemma 1, we have
\[
\frac{d}{dt}(H(p_t | \pi)) = -\mathbb{E}_\pi \left\| \nabla \log p_t \right\|^2.
\] (3.4)

Since \(\mathbb{E}_\pi \left\| \nabla \log p_t \right\|^2 \geq 0\), the identity (3.4) exhibits that KL divergence with respect to \(\pi\) is diminishing along the Langevin dynamics, thus the distribution \(p_t\) actually converges to \(\pi\). When \(\pi\) fulfills log-Sobolev inequality (LSI), \([31]\) Lemma 2 shows that

\[
H(p_t | \pi) \leq e^{-2\gamma t} H(p_0 | \pi).
\] (3.5)

Hence, KL divergence converges exponentially fast along the Langevin dynamics. Log-Sobolev inequality can be thought as a relaxation of log-concavity in continuous time. LSI was originally initiated by \([17]\) for the scenario of Gaussian measure, characterizes concentration of measure and sub-Gaussian tail property, to name a few. \([2]\) broadened it to strongly log-concave measure, where \(\pi\) enjoys LSI with constant \(\gamma\) whenever \(- \log \pi\) is \(\gamma\)-strongly convex. However, LSI is more general than strongly log-concave condition since it is preserved under bounded perturbation \([18]\), Lipschitz mapping, tensorization, among others. Therefore, we will study the KL convergence under log-Sobolev inequality.

### 3.2. Main result: KL divergence along Unadjusted Langevin Algorithm

In general, a practical algorithm often needs to be discretized \([20]\) but the discretization algorithms are often more complicated and require more assumptions. In this section, we investigate the behavior of KL divergence along the Unadjusted Langevin Algorithm (ULA) in discrete time. In order to sample from a target distribution \(\pi = e^{-U} \) in \(\mathbb{R}^d\), the updating rule for the discretized ULA algorithm with step size \(\eta > 0\) is defined as

\[
x_{k+1} = x_k - \eta \nabla U(x_k) + \sqrt{2\eta} z_k
\] (3.6)

where \(z_k \sim N(0, I)\) is an independent standard Gaussian random variable in \(\mathbb{R}^d\). As \(x_k\) is updated following ULA, let \(p_k\) represent its probability distribution. It is known that ULA converges to a biased limiting distribution \(\pi_\eta \neq \pi\) for any fixed \(\eta > 0\). Thus, \(H(p_k | \pi)\) does not converge to 0 along ULA, as it has an asymptotic bias \(H(\pi_\eta | \pi) > 0\). When the true target distribution \(\pi\) complies with an \(\alpha\)-mixture weakly smoothness and LSI, we can prove convergence in KL divergence along ULA. A key observation is that ULA algorithm will converge uniformly in time if the discretization error between the ULA output in one iteration and the Langevin dynamics is bounded. This technique has been used in many papers, \([31, 8]\). Our proof structure is similar to that of \([31]\), whose analysis needs stronger assumptions.

Let \(x_{k+1} \sim p_{k+1}\) be the output of one step of ULA (3.6) from \(x_k \sim p_k\), we have
Lemma 3.1. Suppose \( \pi \) is \( \gamma \)-log-Sobolev, \( \alpha \)-mixture weakly smooth, \( \max \{ L_i \} = L \geq 1 \). If \( 0 < \eta < \left( \frac{\gamma}{9N^2L^3} \right)^{\frac{1}{3}} \) and then along each step of ULA (3.6),

\[
H(p_{k+1}|\pi) \leq e^{-\gamma\eta}H(p_k|\pi) + 2\eta^{\alpha+1}D_3,
\]

where

\[
D_3 = \sum_{i} 10N^3L^6 + 16NL^4 + 8N^2L^4d_\pi^\frac{1}{2} + 4NL^2d.
\]

In particular, if \( \pi \) is \( \gamma \)-log-Sobolev, \( (\alpha, \ell) \)-weakly smooth with \( 0 < \alpha + \ell \leq 1 \). If \( 0 < \eta < \left( \frac{\gamma}{9N^2L^3} \right)^{\frac{1}{3}} \), then along each step of ULA (3.6),

\[
H(p_{k+1}|\pi) \leq e^{-\gamma\eta}H(p_k|\pi) + 2\eta^{\alpha+1}D'_3,
\]

where

\[
D'_3 = 16L^{2+2\alpha+2\ell} + 4L^{2+2\alpha}d_\pi^\frac{1}{2} + 4L^{2}\alpha+\ell.
\]

Proof. See Appendix C.3. \( \square \)

By using this component, we obtain the following theorem.

Theorem 3.1. Suppose \( \pi \) is \( \gamma \)-log-Sobolev, \( \alpha \)-mixture weakly smooth, \( \max \{ L_i \} = L \geq 1 \), and for any \( x_0 \sim p_0 \) with \( H(p_0|\pi) = C_0 < \infty \), the iterates \( x_k \sim p_k \) of LMC with step size \( \eta \leq 1 \wedge \frac{1}{\ell} \wedge \left( \frac{\gamma}{9N^2L^3} \right)^{\frac{1}{3}} \) satisfies

\[
H(p_k|\pi) \leq e^{-\gamma\eta k}H(p_0|\pi) + \frac{8\eta^{\alpha}D_3}{3\gamma},
\]

Then, for any \( \epsilon > 0 \), to achieve \( H(p_k|\pi) < \epsilon \), it suffices to run ULA with step size \( \eta \leq 1 \wedge \frac{1}{\ell} \wedge \left( \frac{\gamma}{9N^2L^3} \right)^{\frac{1}{3}} \wedge \left( \frac{3\gamma}{16D_3} \right)^{\frac{1}{3}} \) for \( k \geq \frac{1}{\gamma\eta} \log \frac{2H(p_0|\pi)}{\epsilon} \) iterations.

Proof. See Appendix C.4. \( \square \)

If we initialize with a Gaussian distribution \( p_0 = N(0, \frac{1}{L}I) \), we have the following lemma.

Lemma 3.2. Suppose \( \pi = e^{-U} \) is \( \alpha \)-mixture weakly smooth. Let \( p_0 = N(0, \frac{1}{L}I) \).

Then \( H(p_0|\pi) \leq U(0) - \frac{d}{2} \log \frac{2\pi}{L} + \sum_{i} \frac{L}{1+\alpha_i} \left( \frac{\gamma}{\alpha_i} \right)^{\frac{1}{2}} = O(d) \).

Proof. See Appendix C.1. \( \square \)

Therefore, Theorem 3.1 states that to achieve \( H(p_k|\pi) \leq \epsilon \), ULA has iteration complexity \( \tilde{O} \left( \frac{d \gamma^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}} \gamma \sqrt{\alpha}} \right) \). By Pinsker’s inequality, we have \( TV(p_k|\pi) \leq \sqrt{\frac{H(p_k|\pi)}{2}} \) which implies that to get \( TV(p_k|\pi) \leq \epsilon \), it is enough to obtain \( H(p_k|\pi) \leq 2\epsilon^2 \). This bound indicates that the number of iteration to reach \( \epsilon \) accuracy for total variation is \( \tilde{O} \left( \frac{d \gamma^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}} \gamma \sqrt{\alpha}} \right) \). On the other hand, from
Talagrand inequality, which comes from log-Sobolev inequality, we know that $W_2^2(p_k, \pi) \leq H(p_k|\pi)$, by replacing this in the bound above, we obtain the number of iteration for $L_2$-Wasserstein distance is $\tilde{O}\left(\frac{1}{\gamma^2} \cdot \frac{1}{\alpha^2} \cdot \frac{1}{\epsilon^2}\right)$.

### 3.3. Sampling via smoothing potential

Inspired by the approach of [6], we study the convergence of the discrete-time process for the smoothing potential that have the following form:

$$U_\mu(x) := \mathbb{E}_\xi[U(y + \mu \xi)].$$

Observe from Lemma 2.2 that $U(\cdot)$ is $\alpha$-mixture weakly smooth but $U_\mu(x)$ is smooth. Recall that ULA in terms of the smoothing potential $U_\mu$ can be specified as:

$$x_{k+1} = x_k - \eta \nabla U_\mu(x_k) + \sqrt{2\eta} \zeta_k,$$

where $\zeta_k \sim N(0, I_{d \times d})$ are independent Gaussian random vectors. In general, we do not have access to an oracle of $\nabla U_\mu(x)$, so rather than working with $\nabla U_\mu(x)$ as specified by Eq. 3.12, we need to use an estimate of the gradient:

$$g_\mu(x) = \nabla U(x + \mu \xi)$$

where $\xi \sim N_p(0, I_d)$. Based on the above estimate of the gradient, we obtain the following result.

**Lemma 3.3.** For any $x_k \in \mathbb{R}^d$, $g_\mu(x_k, \zeta_k) = \nabla U_\mu(x_k) + \zeta_k$ is an unbiased estimator of $\nabla U_\mu$ such that

$$\text{Var}[g_\mu(x_k, \zeta_k)] \leq 4N^2L_\mu^2\mu^2d^2\tilde{\mu}.$$ 

**Proof.** See Appendix D.1.

Let the distribution of the $k$th iterate $x_k$ be represented by $\pi_{\mu,k}$, and let $\pi_\mu \propto \exp(-U_\mu)$ be the distribution with $U_\mu$ as the potential. First, we prove that the $p$-generalized Gaussian smoothing does not alter the objective distribution substantially in term of the Wasserstein distance, by bounding $W_2(\pi, \pi_\mu)$.

**Lemma 3.4.** Assume that $\pi \propto \exp(-\pi)$ and $\pi_\mu \propto \exp(-U_\mu)$ and $\pi$ has a bounded second moment, that is $\int \|x\|^2 \pi(x)dx = E_2 < \infty$. We deduce the following bounds

$$W_2^2(\pi, \pi_\mu) \leq 8.24NL\mu^{1+\alpha}d^2\tilde{\mu} E_2.$$

for any $\mu \leq 0.05$.

**Proof.** See Appendix D.3.

We then derive a result on mixing times of Langevin diffusion with stochastic estimated gradients under log-Sobolev inequality condition, which enables us to bound $W_2(\pi_{\mu,k}, \pi_\mu)$. Our main outcome is stated in the subsequent theorem.
Theorem 3.2. Suppose πμ is γ1-log-Sobolev, α-mixture weakly smooth, with max {L_i} = L ≥ 1 and \( \int ||x||^2 \pi(x)dx = E_2 < \infty \) and for any \( x_0 \sim p_0 \) with \( H(p_0|\pi) = C_0 < \infty \), the iterates \( x_k \sim p_k \) of ULA with step size \( \eta \leq \min \left\{ 1, \frac{1}{4\gamma_i} \left( \frac{\gamma_1}{13NL^3}\right)^\frac{1}{2} \right\} \)

\( (3.14) \)
satisfies

\[
W_2(\pi_{\mu,K}, \pi) \leq e^{-\frac{2\alpha}{\gamma_1} \eta k} \sqrt{H(p_0|\pi_{\mu})} + \sqrt{\frac{8\eta^2 D_4}{3\gamma_1}} + 3\sqrt{N}L_2 d^2 \eta^2,
\]

where \( D_4 = \sum \{ 10N^3L^6 + 16NL^4 + 8N^2L^4d^2 + 4NL^2d + 8N^2L^2d^2 \} \).

Then, for any \( \epsilon > 0 \), to achieve \( W_2(\pi_{\mu,K}, \pi) < \epsilon \), it suffices to run ULA with step size \( \eta \leq \frac{1}{4\gamma_i} \left( \frac{\gamma_1}{13NL^3}\right)^\frac{1}{2} \left( \frac{C_0}{6\sqrt{D_4}} \right)^\frac{1}{2} \left( \frac{\epsilon}{9\sqrt{N}L_2 d^2} \right)^\frac{1}{2} \) for \( k \geq \frac{2\gamma_1 \eta \log \frac{3\sqrt{H(p_0|\pi_{\mu})}}{\gamma_1}}{\epsilon} \) iterations.

\( \Box \)

4. Extended result

Since log-Sobolev inequalities are preserved under bounded perturbations by \cite{18}'s theorem, we provide our extended results through convexification of non-convex domain \cite{24, 35}. Convexification of non-convex domain is an original approach proposed by \cite{24, 35}, developed and apply to strongly convex outside a compact set by \cite{35}. We would like to emphasize that it is non trivial to apply their results in our case since the requirement of strong convexity. Before starting our extension, we need an additional lemma, taken from \cite{24, 35}, for our proof.

Lemma 4.1. \cite{24} Lemma 2. Let us define \( \Omega = \mathbb{R}^d \setminus \mathbb{B}(0,R) \) where \( \mathbb{B}(0,R) \) is the open ball of radius \( R \) centered at 0, and define \( V(x) = \inf \{ \sum \lambda_i U(x_i) \} \) where the infimum is running over all possible convex combination of points \( x_i \) (that is \( \lambda_i \geq 0, \sum \lambda_i = 1 \) and \( \sum \lambda_i x_i = x \)). Then for \( \forall x \in \mathbb{B}(0,R), V(x) \) can be represented as a convex combination of \( U(x_i) \) such that \( ||x_i|| = R \), that is \( V(x) = \inf \left\{ \sum \lambda_j U(x_j) \right\} \) where \( \lambda_j \geq 0, \sum \lambda_j = 1 \) and \( \sum \lambda_j x_j = x \) and \( ||x|| = R \). Then, \( \inf_{||x||=R} U(x) \leq V(x) \leq \sup_{||x||=R} U(x) \).

\( \Box \)

Adapted techniques from \cite{24} for non-strongly convex and α-mixture weakly smooth potentials, we derive a tighter bound for the difference between constructed convex potential and the original one in the following lemma.
Lemma 4.2. For $U$ satisfying $\alpha$-mixture weakly smooth and $(\mu, \theta)$-degenerated convex outside the ball radius $R$, there exists $\hat{U} \in C^1(\mathbb{R}^d)$ with a Hessian that exists everywhere on $\mathbb{R}^d$, and $\hat{U}$ is $((1 - \theta) \frac{\mu}{2}, \theta)$-degenerated convex on $\mathbb{R}^d$ (that is $\nabla^2 \hat{U}(x) \succeq (1 - \theta) \frac{\mu}{2} (1 + \|x\|^2)^{\frac{\mu}{2}} I_d$), such that
\[
\sup \left( \hat{U}(x) - U(x) \right) - \inf \left( \hat{U}(x) - U(x) \right) \leq \sum_i L_i R^{1+\alpha_i} + \frac{4\mu}{(2-\theta)} R^{2-\theta}.
\] (4.1)

Proof. See Appendix E.2.

Remark 4.1. This result can be applied to potential with degenerated convex outside the ball. Setting $\mu = 0$ implies a result for potential with non-strongly convex outside the ball, while setting $\theta = 0$ implies a result for potential with strongly convex outside the ball. The constant could be improved by a factor of 2 if we take $\epsilon$ defined in the proof to be arbitrarily small.

4.1. ULA convergence under $\gamma$–Poincaré inequality, $\alpha$-mixture weakly smooth and $2$–dissipativity

In general, $PI$ is weaker than $LSI$. In order to apply the previous results of log Sobolev inequalities, we will also need $2$–dissipativity assumption. First, using convexification of non-convex domain result above, we have the following lemma for bounded perturbation.

Lemma 4.3. For $U$ satisfying $\gamma$–Poincaré, $\alpha$-mixture weakly smooth, there exists $\hat{U} \in C^1(\mathbb{R}^d)$ with a Hessian that exists everywhere on $\mathbb{R}^d$, and $\hat{U}$ is log-Sobolev on $\mathbb{R}^d$ such that
\[
\sup \left( \hat{U}(x) - U(x) \right) - \inf \left( \hat{U}(x) - U(x) \right) \leq 2 \sum_i L_i R^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha}.
\] (4.2)

Proof. See Appendix E.3.

Using bounded perturbation theorem, this result implies $\pi$ satisfies a log-Sobolev inequality, which in turn give us the following result.

Theorem 4.1. Suppose $\pi$ is $\gamma$–Poincaré, $\alpha$-mixture weakly smooth with $\alpha_N = 1$ and $2$–dissipativity (i.e. $\langle \nabla U(x), x \rangle \geq a \|x\|^2 - b$) for some $a, b > 0$, and for any $x_0 \sim p_0$ with $H(p_0|\pi) = C_0 < \infty$, the iterates $x_k \sim p_k$ of ULA with step size $\eta \leq 1 \wedge \frac{1}{4\gamma_3} \wedge \left(\frac{\gamma_3}{16L^{\gamma_3}}\right)^\frac{1}{2}$ satisfies
\[
H(p_k|\pi) \leq e^{-\gamma_3 \eta k} H(p_0|\pi) + \frac{8\eta^a D_3}{3\gamma_3},
\] (4.3)
where $D_3$ is defined as in equation (3.8) and

$$M_2 = \int \|x\|^2 e^{-\tilde{U}(x)} dx = O(d) \quad (4.4)$$

$$\zeta = \sqrt{2 \left( \frac{2 \left( b + (L + \frac{\alpha}{a}) R^2 + aR^2 + d \right)}{a} + M_2 \right) e^{4 \left( 2 \sum L_i R_i^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha} \right) / \gamma}} \quad (4.5)$$

$$A = \left( 1 - \frac{L}{2} \right) \frac{8}{a^2} + \zeta, \quad (4.6)$$

$$B = 2 \left[ \frac{2 \left( (b + 4 \left( L + \frac{\alpha}{a} \right) R^2 + aR^2 + d \right)}{a} + M_2 \right] \left( 1 - \frac{L}{2} + \frac{1}{\zeta} \right), \quad (4.7)$$

$$\gamma_3 = \frac{2 \gamma e^{-2(\sum L_i R_i^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha})}}{[A \gamma + (B + 2) e^{4(2 \sum L_i R_i^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha})}]}.$$  

Then, for any $\epsilon > 0$, to achieve $H(p_k|\pi) < \epsilon$, it suffices to run ULA with step size $\eta \leq 1 \wedge \frac{1}{\gamma_3} \wedge \left( \frac{36\gamma_3}{16D^2} \right)^{\frac{3}{2}} \wedge \left( \frac{3\gamma_3}{16D^2} \right)^{\frac{5}{2}}$ for $k \geq \frac{1}{\gamma_3} \log \frac{2H(p_k|\pi)}{\epsilon}$ iterations.

Proof. See Appendix E.5. \hfill \Box

From Theorem 4.1, LMC can achieve $H(p_k|\pi) \leq \epsilon$, with iteration complexity of $\tilde{O} \left( \frac{d^2}{\epsilon^{\frac{1}{2}} \gamma_3^{\frac{1}{2} + 1}} \right)$ where

$$\gamma_3 = O \left( \frac{1}{d \gamma e^{5(2 \sum L_i R_i^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha})}} \right)$$

so the number of iteration needed is

$$\tilde{O} \left( \frac{d^2 \gamma_3^2 + 1 e^{5(2 \sum L_i R_i^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha})} (\gamma_3^{\frac{1}{2} + 1})}{\gamma_3^{\frac{1}{2} + 1} e^{\frac{1}{\epsilon^{\frac{1}{2}}}}} \right).$$

Similar as before, from Pinsker’s inequality, the number of iteration to reach $\epsilon$ accuracy for total variation is

$$\tilde{O} \left( \frac{d^2 \gamma_3^2 + 1 e^{5(2 \sum L_i R_i^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha})} (\gamma_3^{\frac{1}{2} + 1})}{\gamma_3^{\frac{1}{2} + 1} e^{\frac{1}{\epsilon^{\frac{1}{2}}}}} \right). \quad (4.8)$$

To have $W_\alpha(p_k, \pi) \leq \epsilon$, it is sufficient to choose $H(p_k|\pi) = \tilde{O} \left( \epsilon^4 d^{-2} \right)$, which in turn implies the number of iteration for $W_\alpha(p_k, \pi) \leq \epsilon$ is

$$\tilde{O} \left( \frac{d^2 \gamma_3^2 + 1 e^{5(2 \sum L_i R_i^{1+\alpha_i} + 4L_N R^2 + 4LR^{1+\alpha})} (\gamma_3^{\frac{1}{2} + 1})}{\gamma_3^{\frac{1}{2} + 1} e^{\frac{1}{\epsilon^{\frac{1}{2}}}}} \right). \quad (4.9)$$
4.2. ULA convergence under non-strongly convex outside the ball, α-mixture weakly smooth and 2-dissipativity

Using convexification of non-convex domain result above, we obtain the following lemma.

Lemma 4.4. Suppose π is non-strongly convex outside the ball of radius R, α-mixture weakly smooth with αN = 1 and 2-dissipativity (i.e. ⟨∇U(x), x⟩ ≥ a ||x||^2 - b) for some a, b > 0, there exists \( U \in C^1(\mathbb{R}^d) \) with a Hessian that exists everywhere on \( \mathbb{R}^d \), and \( \hat{U} \) is convex on \( \mathbb{R}^d \) such that

\[
\sup \left( \hat{U}(x) - U(x) \right) - \inf \left( \hat{U}(x) - U(x) \right) \leq 2 \sum_i L_i R^{1+\alpha_i}. \tag{4.10}
\]

Proof. It comes directly from Lemma 4.2.

Based on result in previous section, we get the following result.

Theorem 4.2. Suppose π is non-strongly convex outside the ball \( \mathbb{B}(0, R) \), α-mixture weakly smooth with αN = 1 and 2-dissipativity (i.e. ⟨∇U(x), x⟩ ≥ a ||x||^2 - b) for some a, b > 0, and for any \( x_0 \sim p_0 \) with \( H(p_0|\pi) = C_0 < \infty \), the iterates \( x_k \sim p_k \) of LMC with step size \( \eta \) satisfies

\[
H(p_k|\pi) \leq e^{-\gamma_3^n k} H(p_0|\pi) + \frac{8\eta^n D_3}{3\gamma_3}, \tag{4.11}
\]

where \( D_3 \) is defined as in equation (3.8) and for some universal constant \( K \),

\[
M_2 = \int ||x||^2 e^{-\hat{U}(x)} dx = O(d) \tag{4.12}
\]

\[
\zeta = K \sqrt{\frac{64d \left[ \frac{2 \left( b + (L + \frac{\lambda_p}{2}) R^2 + aR^2 + d \right)}{a} + M_2 \right]}{a + b + 2aR^2 + 3 \left( \frac{ae^{-4(4LN R^2 + 4R^2)^{\alpha}}}{4L^2} \right)}} \tag{4.13}
\]

\[
A = \left( 1 - \frac{L}{2} \right) \frac{8}{a^2} + \zeta, \tag{4.14}
\]

\[
B = 2 \left[ \frac{2 \left( \left( b + 4(L + \frac{\lambda_p}{2}) R^2 + aR^2 \right) + d \right)}{a} + M_2 \right] \left( 1 - \frac{L}{2} + \frac{1}{\zeta} \right), \tag{4.15}
\]

\[
\gamma_3 = \frac{2e^{-\left( 2\sum_i L_i R_i^{1+\alpha_i} + 4L^2 R^2 + 4LR_1^{1+\alpha} \right)}}{A + (B + 2)32K^2d \left( \frac{a + b + 2aR^2 + 3}{a} \right) e^{4(4L^2R^2 + 4L^2R^2)^{1+\alpha}}} = \frac{1}{O(d)}. \tag{4.16}
\]

Then, for any \( \epsilon > 0 \), to achieve \( H(p_k|\pi) < \epsilon \), it suffices to run ULA with step size \( \eta \) satisfies

\[
\eta \leq 1 \wedge \left( \frac{\sqrt{\gamma_3}}{10\gamma_3} \right) \wedge \left( \frac{\gamma_3}{10\gamma_3} \right)^{\frac{1}{2}} \wedge \left( \frac{\gamma_3}{10\gamma_3} \right)^{\frac{1}{4}} \text{ for } k \geq \frac{1}{\gamma_3 \eta} \log \frac{2H(p_0|\pi)}{\epsilon} \text{ iterations.}
\]

Proof. See Appendix E.5.
5. Conclusion

In this article, we derive polynomial-dimension theoretical assurances of unadjusted LMC algorithm for a family of potentials that are $\alpha$-mixture weakly smooth and isoperimetric (i.e. log Sobolev, Poincaré, and Talagrand). In addition, we also investigate the family of potential which is non-strongly convex outside the ball and 2-dissipative. The analysis we proposed is an extension of the recently published paper [31] in combination with the convexification of non-convex domain [24]. There are a number of valuable potential directions which one can explore, among them we speculate some here. It is potential to broaden our results to apply underdamped LMC or higher order LMC to these class of potential while the computational complexity remains polynomial dependence on $d$. Another fascinating question is whether it is feasible to sampling from distributions with non-smooth and totally non-convex structure and integrate into derivative-free LMC algorithm.

Appendix A: Measure definitions and isoperimetry

Let $p, \pi$ be probability distributions on $\mathbb{R}^d$ with full support and smooth densities, define the Kullback-Leibler (KL) divergence of $p$ with respect to $\pi$ as

$$H(p|\pi) \triangleq \int_{\mathbb{R}^d} p(x) \log \frac{p(x)}{\pi(x)} \, dx.$$  \hspace{1cm} (A.1)

Likewise, we denote the entropy of $p$ with

$$H(p) \triangleq -\int p(x) \log p(x) \, dx$$  \hspace{1cm} (A.2)

and for $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel $\sigma$-field of $\mathbb{R}^d$, define the relative Fisher information and total variation metrics correspondingly as

$$I(p|\pi) \triangleq \int_{\mathbb{R}^d} p(x) \| \nabla \log \frac{p(x)}{\pi(x)} \|^2 \, dx,$$  \hspace{1cm} (A.3)

$$TV(p, \pi) \triangleq \sup_{A \in \mathcal{B}(\mathbb{R}^d)} \left| \int_A p(x) \, dx - \int_A \pi(x) \, dx \right|.$$  \hspace{1cm} (A.4)

Furthermore, we define a transference plan $\zeta$, a distribution on $(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d))$ (where $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ is the Borel $\sigma$-field of $(\mathbb{R}^d \times \mathbb{R}^d)$) so that $\zeta(A \times \mathbb{R}^d) = p(A)$ and $\zeta(\mathbb{R}^d \times A) = \pi(A)$ for any $A \in \mathcal{B}(\mathbb{R}^d)$. Let $\Gamma(P, Q)$ designate the set of all such transference plans. Then for $\beta > 0$, the $L_\beta$-Wasserstein distance is formulated as:

$$W_\beta(p, \pi) \triangleq \left( \inf_{\zeta \in \Gamma(P,Q)} \int_{x,y \in \mathbb{R}^d} \|x - y\|^\beta \, d\zeta(x, y) \right)^{1/\beta}.$$  \hspace{1cm} (A.5)

Note that although KL divergence is an asymmetric measure of distance between probability distributions, it is the preferred measure of distance here since it also
implies total variation distance via Pinsker’s inequality. In addition, KL divergence also governs the quadratic Wasserstein $W_2$ distance under log-Sobolev, Talagrand, and Poincaré inequalities defined below.

**Definition A.1.** The probability distribution $p$ satisfies a logarithmic Sobolev inequality with constant $\gamma > 0$ (in short: $\text{LSI}(\gamma)$) if for all probability distribution $p$ absolutely continuous w.r.t. $\pi$,

$$H(p|\pi) \leq \frac{1}{2\gamma} I(p|\pi). \quad (A.6)$$

**Definition A.2.** The probability distribution $p$ satisfies a Talagrand inequality with constant $\gamma > 0$ (in short: $\text{T}(\gamma)$) if for all probability distribution $p$, absolutely continuous w.r.t. $\pi$, with finite moments of order 2,

$$W_2(p, \pi) \leq \sqrt{\frac{2H(p|\pi)}{\gamma}}. \quad (A.7)$$

**Definition A.3.** The probability distribution $p$ satisfies a Poincaré inequality with constant $\gamma > 0$ (in short: $\text{PI}(\gamma)$) if for all smooth function $g: \mathbb{R}^d \to \mathbb{R}$,

$$\text{Var}_p(g) \leq \frac{1}{\gamma} E_p[\|\nabla g\|^2], \quad (A.8)$$

where $\text{Var}_p(g) = E_p[g^2] - E_p[g]^2$ is the variance of $g$ under $p$.

**Appendix B: Proofs of $p$-generalized Gaussian smoothing**

**B.1. Proof of $\alpha$-mixture weakly smooth property**

**Lemma B.1.** If potential $U: \mathbb{R}^d \to \mathbb{R}$ satisfies $\alpha$-mixture weakly smooth then:

$$U(y) \leq U(x) + \langle \nabla U(x), y - x \rangle + \sum_i \frac{L_i}{1 + \alpha_i}\|y - x\|^{1+\alpha_i}.$$ 

In particular, if potential $U: \mathbb{R}^d \to \mathbb{R}$ satisfies $(\alpha, \ell)$-weakly smooth for some $\alpha + \ell \leq 1$ and $\alpha \in [0,1]$, then:

$$U(y) \leq U(x) + \langle \nabla U(x), y - x \rangle + \frac{L}{1 + \alpha}\|y - x\|^{1+\alpha} + \frac{L}{1 + \ell + \alpha}\|y - x\|^{1+\ell+\alpha}.$$
Proof. We have

\[
|U(x) - U(y) - \langle \nabla U(y), x - y \rangle | \\
= \left| \int_0^1 \langle \nabla U(y + t(x - y)), x - y \rangle dt - \langle \nabla U(y), x - y \rangle \right| \\
\leq \int_0^1 \| \nabla U(y + t(x - y)) - \nabla U(y) \| \| x - y \| dt \\
\leq \int_0^1 \sum_i L_i t^{\alpha_i} \| x - y \|^{\alpha_i} \| x - y \| dt \\
= \sum_i \frac{L_i}{1 + \alpha_i} \| x - y \|^{1 + \alpha_i},
\]

where the first line comes from Taylor expansion, the third line follows from Cauchy-Schwarz inequality and the fourth line is due to Assumption 2.1. This gives us the desired result. By replacing Assumption 2.1 with Assumption 2.2, we immediately get

\[
U(y) \leq U(x) + \langle \nabla U(x), y - x \rangle + \frac{L}{1 + \alpha} \| y - x \|^{1 + \alpha} + \frac{L}{1 + \ell + \alpha} \| y - x \|^{1 + \ell + \alpha}.
\]

\[\square\]

B.2. Proof of p-generalized Gaussian smoothing properties

Lemma B.2. If potential \( U : \mathbb{R}^d \to \mathbb{R} \) satisfies \( \alpha \)-mixture weakly smooth then:

(i) \( \forall x \in \mathbb{R}^d : |U_\mu(x) - U(x)| \leq \sum_i L_i \mu^{1 + \alpha_i} d^{\frac{1 + \alpha_i}{p}} \),

(ii) \( \forall x \in \mathbb{R}^d : \| \nabla U_\mu(x) - \nabla U(x) \| \leq \sum_i L_i \mu^{\alpha_i} d^{\frac{\alpha_i}{p}} \),

(iii) \( \forall x, y \in \mathbb{R}^d : \| \nabla U_\mu(y) - \nabla U_\mu(x) \| \leq \sum_i \frac{L_i}{\mu^{1 - \alpha_i}} d^{\frac{\alpha_i}{p}} \| y - x \| \).

In particular, if the potential \( U : \mathbb{R}^d \to \mathbb{R} \) satisfies \((\alpha, \ell)\)-weakly smooth for some \( \alpha + \ell \leq 1 \) and \( \alpha \in [0, 1] \), then:

(i) \( \forall x \in \mathbb{R}^d : |U_\mu(x) - U(x)| \leq 2L \mu^{1 + \ell + \alpha} d^{\frac{1 + \ell + \alpha}{p}} \),

(ii) \( \forall x \in \mathbb{R}^d : \| \nabla U_\mu(x) - \nabla U(x) \| \leq 2L \mu^{\alpha} d^{\frac{\alpha}{p}} \),

(iii) \( \forall x, y \in \mathbb{R}^d : \| \nabla U_\mu(y) - \nabla U_\mu(x) \| \leq \frac{L}{\mu^{1 - \alpha}} d^{\frac{\alpha}{p}} \| y - x \| \).

Proof. (i) Since \( U_\mu(x) = \mathbb{E}_\xi[U(x + \mu \xi)] \), \( U(x) = \mathbb{E}_\xi[U(x)] \) and \( \mathbb{E}_\xi \mu \langle \nabla U(x), \xi \rangle = 0 \), we have

\[
U_\mu(x) - U(x) = \mathbb{E}_\xi [U(x + \mu \xi) - U(x) - \mu \langle \nabla U(x), \xi \rangle].
\]
By the definition of the density of \( p \)-generalized Gaussian distribution \(^1\), we also have:

\[
U_\mu(x) - U(x) = \frac{1}{\kappa} \int_{\mathbb{R}^d} [U(x + \mu\xi) - U(x) - \mu \langle \nabla U(x), \xi \rangle] e^{-\|\xi\|_p^p / p} d\xi.
\]

Applying Eq. 2.2 and previous inequality:

\[
|U_\mu(x) - U(x)| = \frac{1}{\kappa} \int_{\mathbb{R}^d} [U(x + \mu\xi) - U(x) - \mu \langle \nabla U(x), \xi \rangle] e^{-\|\xi\|_p^p / p} d\xi
\]

\[
\leq \sum_i \frac{L_i \mu^{1+\alpha_i}}{\kappa(1+\alpha_i)} \int_{\mathbb{R}^d} \|\xi\|^{(1+\alpha_i)} e^{-\|\xi\|_p^p / p} d\xi
\]

\[
= \sum_i \frac{L_i \mu^{1+\alpha_i}}{(1+\alpha_i)} E[\|\xi\|^{(1+\alpha_i)}].
\]

If \( p \leq 2 \) then \( \|\xi\| \leq \|\xi\|_p \) and we get

\[
|U_\mu(x) - U(x)| \leq \sum_i \frac{L_i \mu^{1+\alpha_i}}{(1+\alpha_i)} E[\|\xi\|^{(1+\alpha_i)}]
\]

\[
\leq \sum_i \frac{L_i \mu^{1+\alpha_i}}{(1+\alpha_i)} E[\|\xi\|^{2}]^{\frac{1+\alpha_i}{2}}
\]

\[
\leq \sum_i \frac{L_i \mu^{1+\alpha_i}}{(1+\alpha_i)} (d+1)\frac{2\alpha_i}{2}\frac{2\alpha_i}{2}
\]

\[
\leq \sum_i \frac{L_i \mu^{1+\alpha_i}}{(1+\alpha_i)} d^{\frac{1+\alpha_i}{2}}
\]

\[
\leq \sum_i \frac{L_i \mu^{1+\alpha_i}}{(1+\alpha_i)} d^{\frac{\alpha_i}{2}}
\]

where step 1 follows from Jensen inequality and \( 0 \leq \alpha \leq 1 \), step 2 is from Lemma F.16 below in which if \( \xi \sim N_p(0, I_d) \) then \( d^{|\frac{\alpha}{2}} \leq E[\|\xi\|_p^n] \leq [d + \frac{d}{2}]^\frac{\alpha}{2} \) where \( |x| \) denotes the largest integer less than or equal to \( x \), and the last step is by simplification when \( d \) is large enough and \( \mu \) is small enough. By replacing Assumption 2.1 with Assumption 2.2, for \( \mu \) is small enough, we immediately get

\[
|U_\mu(x) - U(x)| \leq 2L \mu^{1+\ell+\alpha} d^{\frac{1+\ell+\alpha}{2}}.
\]

(ii). We adapt the technique of [27] to \( p \)-generalized Gaussian smoothing. Let \( y = x + \mu\xi \), then \( U_\mu(x) \) is rewritten in another form as

\[
U_\mu(x) = E_\mu[U(x + \mu\xi)]
\]

\[
= \frac{1}{\kappa \mu} \int_{\mathbb{R}^d} U(y) e^{-\frac{1}{\mu \kappa} \|y-x\|_p^p} dy.
\]
Now taking the gradient with respect to $x$ of $U_\mu(x)$ gives
\[
\nabla_x U_\mu(x) = \frac{1}{\kappa \mu} \nabla_x \int_{\mathbb{R}^d} U(y) e^{-\frac{1}{\mu \rho_p} \|y-x\|_p^p} dy.
\]

By Fubini Theorem with some regularity (i.e. $\mathbb{E}[|U(y)|] < \infty$), we can exchange the gradient and integral and get
\[
\nabla_x U_\mu(x) = \frac{1}{\kappa \mu} \int_{\mathbb{R}^d} \nabla_x \left( U(y) e^{-\frac{1}{\mu \rho_p} \|y-x\|_p^p} \right) dy
\]
\[
= \frac{1}{\kappa \mu} \int_{\mathbb{R}^d} U(y) \nabla_x \left( e^{-\frac{1}{\mu \rho_p} \|y-x\|_p^p} \right) dy
\]
\[
= \frac{1}{\kappa \mu} \int_{\mathbb{R}^d} U(y) e^{-\frac{1}{\mu \rho_p} \|y-x\|_p^p} \left( \frac{1}{\mu^p} \|y-x\|_p^p \nabla_x (\|y-x\|_p) \right) dy
\]
\[
= \frac{1}{\kappa \mu} \int_{\mathbb{R}^d} U(y) e^{-\frac{1}{\mu \rho_p} \|y-x\|_p^p} \left( \frac{1}{\mu^p} \|y-x\|_p \circ |y-x|^{p-2} \right) dy.
\]

where $\circ$ stands for the Hadamard product and $|\cdot|$ is used for absolute value of each component of the vector $y-x$. Therefore, by changing variable back to $\xi$, we deduce
\[
\nabla_x U_\mu(x) = \frac{1}{\kappa} \int_{\mathbb{R}^d} U(x + \mu \xi) e^{-\frac{1}{\mu} \|\xi\|_p^p} \frac{1}{\mu} \xi \circ |\xi|^{p-2} d\xi
\]
\[
= \mathbb{E}_\xi \left[ \frac{U(x + \mu \xi) \xi \circ |\xi|^{p-2}}{\mu} \right].
\]

In addition, if $\xi \sim N_p(0, I_d)$, $\mathbb{E}(\xi) = \frac{1}{\kappa} \int \xi e^{-\frac{1}{\mu} \|\xi\|_p^p} d\xi = 0$ and then $\nabla_\xi \mathbb{E}(\xi) = 0$. Since $\xi e^{-\frac{1}{\mu} \|\xi\|_p^p}$ is bounded, we can exchange the gradient and the integral and get
\[
\nabla_\xi \frac{1}{\kappa} \int \xi e^{-\frac{1}{\mu} \|\xi\|_p^p} d\xi = \frac{1}{\kappa} \int \nabla_\xi \left( \xi e^{-\frac{1}{\mu} \|\xi\|_p^p} \right) d\xi
\]
\[
= \frac{1}{\kappa} \int e^{-\frac{1}{\mu} \|\xi\|_p^p} d\xi + \frac{1}{\kappa} \int \xi \nabla_\xi \left( e^{-\frac{1}{\mu} \|\xi\|_p^p} \right) d\xi
\]
\[
= 0 - \frac{1}{\kappa} \int \xi e^{-\frac{1}{\mu} \|\xi\|_p^p} \|\xi\|_p^{p-1} \nabla_\xi \left( \|\xi\|_p \right) d\xi
\]
\[
= 0 - \frac{1}{\kappa} \int \xi \cdot \xi \circ |\xi|^{p-2} e^{-\frac{1}{\mu} \|\xi\|_p^p} d\xi,
\]

which implies
\[
\frac{1}{\kappa} \int \xi \cdot \xi \circ |\xi|^{p-2} e^{-\frac{1}{\mu} \|\xi\|_p^p} d\xi = 1. \tag{B.1}
\]

On the other hand, we also have $\frac{1}{\kappa} \int e^{-\frac{1}{\mu} \|\xi\|_p^p} d\xi = 1$ so $\nabla_\xi \int e^{-\frac{1}{\mu} \|\xi\|_p^p} d\xi = 0$. By exchange the gradient and the integral and we also get
\[ 0 = \nabla \xi \int e^{-\frac{1}{p}\|\xi\|^p} d\xi = \int \nabla \xi e^{-\frac{1}{p}\|\xi\|^p} d\xi = \int \nabla \xi (e^{-\frac{1}{p}\|\xi\|^p}) d\xi = -\int e^{-\frac{1}{p}\|\xi\|^p} \xi \circ |\xi|^{p-2} d\xi \]

which implies that
\[ E\xi [\xi \circ |\xi|^{p-2}] = 0. \quad (B.2) \]

From B.1 and B.2, we obtain
\[
\|\nabla U_{\mu}(x) - \nabla U(x)\| = \left\| \frac{1}{\kappa} \int_{\mathbb{R}^d} \left[ \frac{U(x + \mu \xi) - U(x)}{\mu} - \langle \nabla U(x), \xi \rangle \right] \xi \circ |\xi|^{p-2} e^{-\frac{1}{p}\|\xi\|^p} d\xi \right\|
\leq \frac{1}{\kappa \mu} \int_{\mathbb{R}^d} \|U(x + \mu \xi) - U(x) - \mu \langle \nabla U(x), \xi \rangle \| e^{-\frac{1}{p}\|\xi\|^p} \|\xi \circ |\xi|^{p-2}\| d\xi
\leq \frac{2}{\kappa} \sum_{i} \frac{L_i \mu^{\alpha_i}}{1 + \alpha_i} \int_{\mathbb{R}^d} \|\xi\|^{\alpha_i + 1} e^{-\frac{1}{p}\|\xi\|^p} \|\xi \circ |\xi|^{p-2}\| d\xi
= \frac{1}{\kappa} \sum_{i} \frac{L_i \mu^{\alpha_i}}{1 + \alpha_i} \int_{\mathbb{R}^d} \|\xi\|^{\alpha_i + 1} e^{-\frac{1}{p}\|\xi\|^p} \|\xi \circ |\xi|^{p-1}\| d\xi,
\]

where step 1 follows from Jensen inequality, step 2 is due to 2.2 and the last step follows from component-wise operation of norm. If \( p \leq 2 \), by using generalized Holder inequality, \( \|\xi^{p-1}\| \) can be bounded as follow:

\[
\|\xi^{p-1}\| \leq \|\xi^{p-1}\|_p
= \|\xi^{p-1} \cdot 1_d\|_p
\leq \|\xi\|_{p-1} \|1_d\|_{2-p}^{2-p}
= \|\xi\|_{p-1} d^{\frac{2-p}{2-p}}. \quad (B.3)
\]
As a result, if $1 \leq p \leq 2$ we have

$$
\|\nabla U_\mu(x) - \nabla U(x)\| \leq \sum_i \frac{L_i \mu^{\alpha_i}}{\kappa (1 + \alpha_i)} \int_{\mathbb{R}^d} \|\xi\|^{\alpha_i+1} \|\xi\|_p^{p-1} e^{-\frac{1}{p} \|\xi\|_p^p} d\xi
$$

$$
\leq 1 \sum_i \frac{L_i \mu^{\alpha_i}}{(1 + \alpha_i)} d^{2-\frac{2}{p}} E \left[ \|\xi\|_p^{p+\alpha_i} \right]
$$

$$
\leq 2 \sum_i \frac{L_i \mu^{\alpha_i}}{(1 + \alpha_i)} d^{2-\frac{2}{p}} E \left[ \|\xi\|_p^{2p} \right]^{\frac{p+\alpha}{2p}}
$$

$$
\leq 3 \sum_i \frac{L_i \mu^{\alpha_i}}{(1 + \alpha_i)} d^{2-\frac{2}{p}} (d + p)^{\frac{p+\alpha}{2p}}
$$

$$
\leq \sum_i L_i \mu^{\alpha_i} d^{\frac{2}{p}}
$$

where step 1 is from $\|\xi\| \leq \|\xi\|_p$, step 2 follows from Jensen inequality and $\alpha \leq p$, step 3 is due to 2.2 and in the last two steps we have used simplification for large enough $d$ and small enough $\mu$. By replacing Assumption 2.1 with Assumption 2.2, for $\mu$ is small enough, we immediately get

$$
\|\nabla U_\mu(x) - \nabla U(x)\| \leq 2L \mu^\alpha d^{\frac{2}{p}}.
$$

iii) In this case, using Eqs. 2.2 and B.2, we get:

$$
\nabla U_\mu(x) = \frac{1}{\kappa} \int_{\mathbb{R}^d} \left[ \frac{U(x + \mu \xi) - U(x)}{\mu} \right] \xi \circ |\xi|^{p-2} e^{-\frac{1}{p} \|\xi\|_p^p} d\xi.
$$

Let $V(x) = U(x + \mu \xi) - U(x)$, from above equation, we obtain

$$
\|\nabla U_\mu(y) - \nabla U_\mu(x)\|
$$

$$
= \left\| \frac{1}{\mu \kappa} \int_{\mathbb{R}^d} (V(y) - V(x)) e^{-\frac{1}{p} \|\xi\|_p^p} \xi \circ |\xi|^{p-2} d\xi \right\|
$$

$$
= \left\| \frac{1}{\mu \kappa} \int_{\mathbb{R}^d} \int_0^1 \langle \nabla V (ty + (1-t)x), y-x \rangle dt e^{-\frac{1}{p} \|\xi\|_p^p} \xi \circ |\xi|^{p-2} d\xi \right\|
$$

$$
= \left\| \frac{1}{\mu \kappa} \int_{\mathbb{R}^d} \int_0^1 \langle \nabla U (ty + (1-t)x + \mu \xi) - \nabla U (ty + (1-t)x), y-x \rangle dt e^{-\frac{1}{p} \|\xi\|_p^p} \xi \circ |\xi|^{p-2} d\xi \right\|
$$

$$
\leq \frac{1}{\mu \kappa} \int_{\mathbb{R}^d} \int_0^1 \|\nabla U (ty + (1-t)x + \mu \xi) - \nabla U (ty + (1-t)x)\| |y-x| dt e^{-\frac{1}{p} \|\xi\|_p^p} \left\| \xi \circ |\xi|^{p-2} \right\| d\xi
$$

$$
\leq \sum_i \frac{L_i}{\mu^{1-\alpha_i} \kappa} \int_{\mathbb{R}^d} \|\xi\|^{\alpha_i} |y-x| e^{-\frac{1}{p} \|\xi\|_p^p} \left\| \xi^{p-1} \right\| d\xi.
$$
Since $p \leq 2$ we have
\[
\|\nabla U_{\mu}(y) - \nabla U_{\mu}(x)\| \\
\leq \sum_{i} \frac{L_i}{\mu^{1-\alpha_i}} d^{\frac{2-p}{2-p}} \mathbb{E} (\|\xi\|_{p-1}^{p-1+\alpha}) \|y - x\| \\
\leq \sum_{i} \frac{L_i}{\mu^{1-\alpha_i}} d^{\frac{2-p}{2-p}} \mathbb{E} (\|\xi\|_{p}^{p-1+\alpha}) \|y - x\| \\
\leq \sum_{i} \frac{L_i}{\mu^{1-\alpha_i}} d^{\frac{2-p}{2-p}} (d + \frac{p}{2}) \mathbb{E} (\|\xi\|_{p}^{p-1+\alpha}) \|y - x\| \\
\leq \sum_{i} \frac{L_i}{\mu^{1-\alpha_i}} d^{\frac{2}{2-p}} \|y - x\|,
\]
where step 1 follows from Jensen inequality and $\alpha_i \leq 1$, step 2 is due to 2.2 and in the last two step is because of simplification for large enough $d$ and small enough $\mu$. By replacing Assumption 2.1 with Assumption 2.2, for $\mu$ is small enough, we immediately get
\[
\|\nabla U_{\mu}(y) - \nabla U_{\mu}(x)\| \leq \frac{L}{\mu^{1-\alpha}} d^{\frac{2}{2-p}} \|y - x\|.
\]

\[\square\]

Appendix C: Proofs under LSI

C.1. Proof of Lemma 3.2

Lemma C.1. Suppose $\pi = e^{-U}$ satisfies $\alpha$-mixture weakly smooth. Let $p_0 = N(0, \frac{1}{L} I)$. Then $H(p_0 | \pi) \leq U(0) - \frac{d}{2} \log \frac{2n}{\sqrt{L}} + \sum_{i} L_i \frac{d}{1+\alpha_i} \left( \frac{d}{L} \right)^{\frac{1+\alpha_i}{2}} = O(d)$. 

Proof. Since $U$ is mixture weakly smooth, for all $x \in \mathbb{R}^d$ we have
\[
U(x) \leq U(0) + \langle \nabla U(0), x \rangle + \sum_{i} \frac{L_i}{1+\alpha_i} \|x\|^{1+\alpha_i} \\
= U(0) + \sum_{i} \frac{L_i}{1+\alpha_i} \|x\|^{1+\alpha_i}.
\]
Let $X \sim \rho = N(0, \frac{1}{L} I)$. Then
\[
\mathbb{E}_\rho[U(X)] \leq U(0) + \sum_{i} \frac{L_i}{1+\alpha_i} \mathbb{E}_\rho (\|x\|^{1+\alpha_i}) \\
\leq U(0) + \sum_{i} \frac{L_i}{1+\alpha_i} \mathbb{E}_\rho (\|x\|^2)^{\frac{1+\alpha_i}{2}} \\
\leq U(0) + \sum_{i} \frac{L_i}{1+\alpha_i} \left( \frac{d}{L} \right)^{\frac{1+\alpha_i}{2}}.
\]
Recall the entropy of $\rho$ is $H(\rho) = -\mathbb{E}_\rho[\log \rho(X)] = \frac{d}{2} \log \frac{2\Pi e}{L}$. Therefore, the KL divergence is

$$
\mathbb{E}(\rho|\pi) = \int \rho (\log \rho + U) \, dx
= -H(\rho) + \mathbb{E}_\rho[U]
\leq U(0) - \frac{d}{2} \log \frac{2\Pi e}{L} + \sum_i \frac{L_i}{1 + \alpha_i} \left( \frac{d}{L} \right)^{1 + \alpha_i}
= O(d).
$$

This is the desired result.

\[ \square \]

C.2. Proof of Lemma 3.2

Lemma C.2. Assume $\pi = e^{-U(x)}$ is $\alpha$-mixture weakly smooth. Then

$$
\mathbb{E}_\pi \left[ \left\| \nabla U(x) \right\|^2 \right] \leq 2 \left( \sum_i L_i \right)^2 \mu^\frac{d}{2},
$$

In particular, if $\pi = e^{-U(x)}$ is $(\alpha, \ell)$-weakly smooth. Then

$$
\mathbb{E}_\pi \left[ \left\| \nabla U(x) \right\|^{2\alpha} \right] \leq L^{2\alpha} \frac{\mu^\frac{d}{2} \mu^{\frac{d}{2} - \alpha}}{d^{\frac{d}{2} - \alpha}},
$$

$$
\mathbb{E}_\pi \left[ \left\| \nabla U(x) \right\|^{2\ell + 2\alpha} \right] \leq L^{2(\ell + \alpha)} \frac{\mu^\frac{d}{2} \mu^{\frac{d}{2} - \alpha}}{d^{\frac{d}{2} - \alpha}},
$$

for $d$ sufficiently large.

Proof. Since $\pi$ is stationary distribution, we have

$$
\frac{d}{dt} \mathbb{E}_\pi [U_\mu (x)] = \int \left( \langle \triangle U_\mu (x), \nabla U_\mu (x) \rangle - \langle \nabla U (x), \nabla U_\mu (x) \rangle \right) \pi (x) \, dx = 0.
$$

So

$$
\mathbb{E}_\pi \langle \nabla U (x), \nabla U_\mu (x) \rangle = \mathbb{E}_\pi (\triangle U_\mu (x))
\leq \sum_i \frac{L_i}{\mu^{1 - \alpha_i}} d^\frac{d}{2},
$$

where the last step comes from Lemma 2.2 that $\nabla U_\mu (x)$ is $\sum_i \frac{L_i}{\mu^{1 - \alpha_i}} d^\frac{d}{2}$-Lipschitz, $\nabla^2 U_\mu (x) \leq \left( \sum_i \frac{L_i}{\mu^{1 - \alpha_i}} d^\frac{d}{2} \right) I$. In addition,
\[ E_\pi \langle \nabla U (x), \nabla U_\mu (x) \rangle = E_\pi \left[ \| \nabla U (x) \|^2 \right] + E_\pi \langle \nabla U (x), \nabla U_\mu (x) - \nabla U (x) \rangle \]
\[ \geq E_\pi \left[ \| \nabla U (x) \|^2 \right] - E_\pi \| \nabla U (x) \| \| \nabla U_\mu (x) - \nabla U (x) \| \]
\[ \geq E_\pi \left[ \| \nabla U (x) \|^2 \right] - \sqrt{E_\pi \left[ \| \nabla U (x) \|^2 \right]} \sum_i L_i \mu^{\alpha_i} d^{\frac{3}{p}}, \]

where step 1 follows from Young inequality and the last step comes from Cauchy inequality and Lemma 2.2. From quadratic inequality
\[ E_\pi \left[ \| \nabla U (x) \|^2 \right] - \sqrt{E_\pi \left[ \| \nabla U (x) \|^2 \right]} \sum_i L_i \mu^{\alpha_i} d^{\frac{3}{p}} \leq \sum_i L_i \mu^{1-\alpha_i} d^{\frac{2}{p}} \]
and since \[ \sqrt{E_\pi \left[ \| \nabla U (x) \|^2 \right]} \geq 0 \] we obtain
\[ \sqrt{E_\pi \left[ \| \nabla U (x) \|^2 \right]} \leq \frac{1}{2} \left[ \left( \sum_i L_i \mu^{\alpha_i} \right)^{\frac{2}{q}} + 4 \sum_i \frac{L_i}{\mu^{1-\alpha_i}} d^{\frac{2}{p}} + \sum_i L_i \mu^{\alpha_i} d^{\frac{3}{p}} \right]. \]

Simply choose \( \mu = 1 \), we get
\[ E_\pi \left[ \| \nabla U (x) \|^2 \right] \leq \frac{1}{4} \left[ \left( \sum_i L_i \right)^{\frac{2}{q}} + 4 \left( \sum_i L_i \right) \frac{d^{\frac{2}{p}}}{\mu^{1-\alpha}} + \sum_i L_i \mu^{\alpha} d^{\frac{3}{p}} \right]^2 \]
\[ \leq 2 \left( \sum_i L_i \right)^{\frac{2}{q}} d^{\frac{2}{p}}, \]
for large enough \( d \). If we replace Assumption 2.1 by Assumption 2.2, we can choose \( p = 2 \) and \( \mu = \frac{1}{d^{1+\alpha}} \), we deduce
\[ E_\pi \left[ \| \nabla U (x) \|^2 \right] \leq \frac{1}{4} \left[ \sqrt{L^2 \mu^{2\alpha} d^{\frac{2}{p}} + 4 \frac{L d^{\frac{2}{p}}}{\mu^{1-\alpha}} + L \mu^{\alpha} d^{\frac{3}{p}}} \right]^2 \]
\[ \leq L^2 d^{\frac{2}{1+\alpha}}, \]
for \( d \) large enough as desired. Since \( \alpha \leq 1 \), \( x \to x^\alpha \) is concave function. By Jensen inequality
\[ E_\pi \left[ \| \nabla U (x) \|^{2\alpha} \right] \leq \left( E_\pi \left[ \| \nabla U (x) \|^2 \right] \right)^\alpha \]
\[ \leq L^{2\alpha} d^{\frac{2\alpha}{1+\alpha}}, \]
Similarly, $\ell + \alpha \leq 1$, by Jensen inequality we also have

$$
\mathbb{E}_\pi \left[ \| \nabla U(x) \|^2 \right]^{\ell + \alpha} \leq \left( \mathbb{E}_\pi \left[ \| \nabla U(x) \|^2 \right] \right)^{\ell + \alpha} \leq L^{2(\ell + \alpha)} d_3^{\frac{\gamma}{2(\ell + \alpha)}},
$$

as desired.

\[\square\]

### C.3. Proof of Lemma 3.1

**Lemma C.3.** Suppose $\pi$ is $\gamma$–log-Sobolev, $\alpha$-mixture weakly smooth with $\max \{ L_i \} = L \geq 1$. If $0 < \eta \leq \left( \frac{\gamma}{9NL^3} \right)^\frac{1}{\alpha}$, then along each step of ULA (3.6),

$$
H(p_{k+1} | \pi) \leq e^{-\gamma \eta} H(p_k | \pi) + 2\eta^{\alpha + 1} D_3,
$$

where $D_3 = \sum_i 10N^3 L_i^6 + 16NL^4 + 8N^2 L d_3^\gamma + 4NL^2 d$.

In particular, if $\pi$ is $\gamma$–log-Sobolev, $(\alpha, \ell)$–weakly smooth with $0 < \alpha + \ell \leq 1$. If $0 < \eta \leq \left( \frac{\gamma L}{2L^2 + \alpha} \right)^\frac{1}{\alpha}$, then along each step of ULA (3.6),

$$
H(p_{k+1} | \pi) \leq e^{-\gamma \eta} H(p_k | \pi) + 2\eta^{\alpha + 1} D'_3,
$$

where $D'_3 = 16L^{2+2\alpha+2\ell} + 4L^{2+2\alpha}d_3^{\frac{\gamma}{2(\alpha + \ell)}} + 4L^{2d\alpha + \ell}$.

**Proof.** We adapt the proof of [31]. First, recall that the discretization of the LMC is

$$
x_{k,t} = x_k - t\nabla U(x_k) + \sqrt{2t} z_k,
$$

where $z_k \sim N(0, I)$ is independent of $x_k$. Let $x_k \sim p_k$ and $x^* \sim \pi$ with an optimal coupling $(x_k, x^*)$ so that $\mathbb{E}[[x_k - x^*]^2] = W_2(p_k, \pi)^2$. Let $D_i = 8NL_i^{2+2\alpha} \left( \sum_j L_j \right)^2 + 16L_i^{2+2\alpha} + 8L_i^2 \left( \sum_i L_i \right)^2 d_3^{\gamma} + 4L_i^2 d_3^\gamma$, we deduce
\begin{align*}
L_i^2 E_{p_k} \left[ \left\| \nabla U(x_k) + \frac{\sqrt{2t}}{L} z_k \right\|^{2\alpha_i} \right] \\
\leq 2L_i^4 t^{2\alpha_i} E_{p_k} \left[ \left\| \nabla U(x_k) \right\|^{2\alpha_i} \right] + 4L_i^2 t^{2\alpha_i} E_{p_k} \left[ \left\| z_k \right\|^{2\alpha_i} \right] \\
\leq 2L_i^4 t^{2\alpha_i} E_{p_k} \left[ \left\| \nabla U(x_k) \right\|^{2\alpha_i} \right] + 4L_i^2 t^{2\alpha_i} E_{p_k} \left[ \left\| z_k \right\|^2 \right]^{\alpha_i} \\
\leq 4L_i^4 t^{2\alpha_i} E \left[ \left\| \nabla U(x_k) - \nabla U(x^*) \right\|^{2\alpha_i} + \left\| \nabla U(x^*) \right\|^{2\alpha_i} \right] + 4L_i^2 t^{2\alpha} d^{\alpha_i} \\
\leq 4L_i^4 t^{2\alpha_i} E \left( \sum_i L_i \left\| x_k - x^* \right\|^{2\alpha_i} \right)^{2\alpha_i} + 4L_i^2 t^{2\alpha} E \left\| \nabla U(x^*) \right\|^2 + 4L_i^2 t^{2\alpha} d^{\alpha_i} \\
\leq 8L_i^2 t^{2\alpha_i} N \sum_j L_i^{2\alpha_i} E \left[ \left\| x_k - x^* \right\|^{2\alpha_i} \right] + 4L_i^2 t^{2\alpha} E \left\| \nabla U(x^*) \right\|^2 \\
+ 4L_i^2 t^{2\alpha} + 4L_i^2 t^{2\alpha} d^{\alpha_i} \\
\leq 8NL_i^2 t^{2\alpha_i} \eta^{2\alpha} \left( \sum_j L_i \right)^2 + 1 + 1 \right) E \left[ \left\| x_k - x^* \right\|^{2\alpha_i} \right] + 4L_i^2 t^{2\alpha} E \left\| \nabla U(x^*) \right\|^2 \\
+ \left( 8NL_i^2 t^{2\alpha_i} \left( \sum_j L_i \right)^2 + 1 + 16L_i^2 t^{2\alpha_i} + 8L_i^2 \left( \sum_i L_i \right)^2 d^{\frac{\alpha_i}{2}} + 4L_i^2 d^{\alpha_i} \right) \eta^{\alpha_i} \\
\leq \frac{16N}{\gamma} \left( \sum_j L_i \right)^2 + 1 \right) L_i^{2+2\alpha_i} \eta^{2\alpha_i} H(p_k|\pi) + D_1 \eta^{\alpha_i}, \quad (C.3)
\end{align*}

where step 1 follows from Lemma F.13 in Appendix F, step 2 is from $\alpha \leq 1$ and Jensen's inequality, step 3 comes from normal distribution, and step 4 follows our Assumption 2.2, and in step 5 we have used $\alpha_i \leq 1$ and the last step is due to Talagrand inequality which comes from log-Sobolev inequality and Lemma F.16 in Appendix F below. Similarly, we get
\[ E_{p_k} \| \nabla U(x_k) - \nabla U(x_{k,t}) \|_2^2 \leq \sum_i L_i^2 E_{p_k} \| \tilde{x}_{k,t} - x_k \|^{2\alpha_i} \]
\[ = \sum_i L_i^2 E_{p_k} \| -t \nabla U(x_k) + \sqrt{2t} z_k \|^{2\alpha_i} \leq \sum_i \frac{16N}{\gamma} \left( \left( \sum_j L_j \right)^2 + 1 \right) L_i^{2+2\alpha_i} \eta^{2\alpha_i} H(p_k|\pi) + \left( \sum_i D_1 \eta^{\alpha_i} \right) \]
\[ \leq \frac{20N^3}{\gamma} L^6 \eta^{2\alpha} H(p_k|\pi) + D_3 \eta^\alpha \] \hspace{1cm} (C.4)

where step 1 follows from Assumption 2.2, step 2 comes from similar reasoning as equation (C.3), and the last step comes from \( \eta \leq \frac{1}{L} \) and definition of \( D_3 \). Therefore, from [31] Lemma 3, the time derivative of KL divergence along LMC is bounded by

\[ \frac{d}{dt} H(p_{k,t}|\pi) \leq -\frac{3}{4} I(p_{k,t}|\pi) + \mathbb{E}_{p_k} \left[ \| \nabla U(x_{k,t}) - \nabla U(x_k) \|_2^2 \right] \]
\[ \leq -\frac{3}{4} I(p_k|\pi) + \frac{20N^3}{\gamma} L^6 \eta^{2\alpha} H(p_k|\pi) + D_3 \eta^\alpha \]
\[ \leq -\frac{3\gamma}{2} H(p_{k,t}|\pi) + \frac{20N^3}{\gamma} L^6 \eta^{2\alpha} H(p_k|\pi) + D_3 \eta^\alpha, \]

where in the last inequality we have used the definition A.1 of LSI inequality. Multiplying both sides by \( e^{\frac{3\gamma}{2}t} \), and integrating both sides from \( t = 0 \) to \( t = \eta \) we obtain

\[ e^{\frac{3\gamma}{2} \eta} H(p_{k+1}|\pi) - H(p_k|\pi) \leq 2 \left( \frac{\frac{3\gamma}{4} - 1}{3\gamma} \right) \left( \frac{20N^3}{\gamma} L^6 \eta^{2\alpha} H(p_k|\pi) + D_3 \eta^\alpha \right) \] \hspace{1cm} (C.5)
\[ \leq 2\eta \left( \frac{20N^3}{\gamma} L^6 \eta^{2\alpha} H(p_k|\pi) + D_3 \eta^\alpha \right) \] \hspace{1cm} (C.6)

where the last line holds by \( c^c \leq 1 + 2c \) for \( 0 < c = \frac{3\gamma}{4} \eta < 1 \). Rearranging the term of the above inequality and using the facts that \( 1 + \eta^{1+2\alpha} \frac{40N^3}{\gamma} L^6 \leq 1 + \frac{2\eta}{2} \) when \( \eta \leq \left( \frac{\gamma}{9N^2 \gamma L^3} \right)^{\frac{1}{2}} \) and \( e^{-\frac{3\gamma}{2} \eta} \leq 1 \) leads to

\[ H(p_{k+1}|\pi) \leq e^{-\frac{3\gamma}{2} \eta} \left( 1 + \eta^{1+2\alpha} \frac{40N^3}{\gamma} L^6 \right) H(p_k|\pi) + 2\eta^\alpha D_3 \]
\[ \leq e^{-\gamma \eta} H(p_k|\pi) + 2\eta^\alpha D_3. \] \hspace{1cm} (C.7)

as desired.
C.4. Proof of Theorem 3.1

Theorem C.1. Suppose $\pi$ is $\gamma$-log-Sobolev, $\alpha$-mixture weakly smooth with $\max \{ L_i \} = L \geq 1$, and for any $x_0 \sim p_0$ with $H(p_0|\pi) = C_0 < \infty$, the iterates $x_k \sim p_k$ of ULA with step size

$$\eta \leq \min \left\{ 1, \frac{1}{4\gamma}, \left( \frac{\gamma}{9N^2L^3} \right)^{\frac{1}{\alpha}} \right\}$$

(C.8)

satisfies

$$H(p_k|\pi) \leq e^{-\frac{2\gamma}{L} \eta k} H(p_0|\pi) + 2\eta^{\alpha+1} D_3,$$

(C.9)

where $D_3 = \sum_i 10N^3L^6 + 16NL^4 + 8N^2L^4d^3 + 4NL^2d$. Then, for any $\epsilon > 0$, to achieve $H(p_k|\pi) < \epsilon$, it suffices to run LMC with step size

$$\eta \leq \min \left\{ 1, \frac{1}{4\gamma}, \left( \frac{\gamma}{9N^2L^3} \right)^{\frac{1}{\alpha}}, \left( \frac{3c\gamma}{16D_3} \right)^{\frac{1}{\alpha}} \right\}$$

(C.10)

for $k \geq \frac{1}{\gamma \eta} \log \frac{2H(p_0|\pi)}{\epsilon}$ iterations.

Proof. Applying inequality C.9 recursively, and using the inequality $1-e^{-c} \geq \frac{3}{4}c$ for $0 < c = \gamma \eta \leq \frac{1}{4}$ we obtain

$$H(p_k|\pi) \leq e^{-\gamma \eta k} H(p_0|\pi) + 2\eta^{\alpha+1} D_3 \left( 1 - e^{-\gamma \eta} \right)$$

$$\leq e^{-\gamma \eta k} H(p_0|\pi) + \frac{2\eta^{\alpha+1} D_3}{\frac{3}{4} \gamma \eta}$$

$$\leq e^{-\gamma \eta k} H(p_0|\pi) + \frac{8\eta^{\alpha} D_3}{3\gamma}.$$  

(C.11)

Note that last inequality holds if we choose $\eta$ such that it satisfies

$$\eta \leq \min \left\{ 1, \frac{1}{4\gamma}, \left( \frac{\gamma}{9N^2L^3} \right)^{\frac{1}{\alpha}} \right\}.$$ 

Given $\epsilon > 0$, if we further assume $\eta \leq \left( \frac{3c\gamma}{16D_3} \right)^{\frac{1}{\alpha}}$, then the above implies $H(p_k|\pi) \leq e^{-\gamma \eta k} H(p_0|\pi) + \frac{\eta}{2}$. This means for $k \geq \frac{1}{\gamma \eta} \log \frac{2H(p_0|\pi)}{\epsilon}$, we have $H(p_k|\pi) \leq \frac{\eta}{2} + \frac{\eta}{2} = \epsilon$, as desired. \qed

Appendix D: Proof of sampling via smoothing potential

D.1. Proof of Lemma 3.3

Lemma D.1. For any $x_k \in \mathbb{R}^d$, then $g_{\mu}(x_k, \zeta_k) = \nabla U_{\mu}(x_k) + \zeta_k$ is an unbiased estimator of $\nabla U_{\mu}$ such that

$$\text{Var} [g_{\mu}(x_k, \zeta_k)] \leq 4N^2L^2\mu^{2\alpha}d^{\frac{2\alpha}{\alpha}}.$$
Proof. Recall that by definition of $U_\mu$, we have $\nabla U_\mu(x) = \mathbb{E}_\zeta[U(x + \mu \zeta)]$, where $\zeta \sim N_p(0, I_{d \times d})$, and is independent of $\zeta_1$. Clearly, $\mathbb{E}_{\zeta_1}[g(x, \zeta_1)] = \nabla U_\mu(x)$. We now proceed to bound the variance of $g(x, \zeta_1)$. We have:

\[
\mathbb{E}_{\zeta_1} [\nabla U_\mu(x) - g(x, \zeta_1)]^2 \\
\leq \mathbb{E}_{\zeta_1} [\mathbb{E}_\zeta[U(x + \mu \zeta)] - \nabla U(x + \mu \zeta_1)]^2 \\
\leq \mathbb{E}_{\zeta_1, \zeta} [\nabla U(x + \mu \zeta) - \nabla U(x + \mu \zeta_1)]^2, \\
\leq N \sum_i L_i^2 \mathbb{E}_{\zeta_1, \zeta}[\|\mu(\zeta - \zeta_1)\|^{2\alpha_i}] \\
\leq N \sum_i L_i^2 \mu^{2\alpha_i} \mathbb{E}_{\zeta_1, \zeta}[\|\zeta - \zeta_1\|^{2\alpha_i}] \\
\leq 2N \sum_i L_i^2 \mu^{2\alpha_i} \left( \mathbb{E} [\|\zeta\|^{2\alpha_i}] + \mathbb{E} [\|\zeta_1\|^{2\alpha_i}] \right) \\
\leq 2N \sum_i L_i^2 \mu^{2\alpha_i} \left( (\mathbb{E} [\|\zeta\|^2])^{\alpha_i} + (\mathbb{E} [\|\zeta_1\|^2])^{\alpha_i} \right) \\
\leq 4N \sum_i L_i^2 \mu^{2\alpha_i} d^{2\alpha_i} \\
\leq 4N^2 L^2 \mu^{2\alpha_i} d^{2\alpha_i},
\]
as claimed. \hfill \Box

D.2. Proof of Lemma 3.2

Before proving Theorem 3.2, we need an additional lemma.

Lemma D.2. [[31] modified Lemma 3] Suppose $x_{k,t}$ is the interpolation of the discretized process (1.2). Let $p_{k,t}$, $p_k$ and $p_{k|\zeta}$ denote its distribution, the joint distribution of $x_{k,t}$ and $x_k$ and the joint distribution of $x_{k,t}$ and $\zeta$ respectively. Here $g(x_k, \zeta)$ is an estimate of $\nabla U(x_k)$ with noise $\zeta$ such that $\mathbb{E}_{\zeta}g(x_k, \zeta) = \nabla U(x_k)$. Then

\[
\frac{d}{dt} H(p_{k,t}|\pi_\mu) \leq -\frac{3}{4} I(p_{k,t}|\pi_\mu) + \mathbb{E}_{p_{k|\zeta}} [\|\nabla U(x_{k,t}) - g(x_k, \zeta)\|^2]. \hspace{1cm} (D.1)
\]

Proof. The steps follow exactly as in Lemma 3 and we provide the proof here for completeness. For each $t > 0$, let $p_{k|\zeta}(x_k, \zeta)$ denote the distributions of $x_k$ and $\zeta$ conditioned on $x_{k,t}$, and $p_{k|t}(x_k, \zeta)$ denote the distributions of $x_{k,t}$ conditioned on $x_k$ and $\zeta$. Following Fokker-Planck equation, we have

\[
\frac{\partial p_{k|\zeta}(x_{k,t})}{\partial t} = \nabla \cdot (p_{k|\zeta}(x_{k,t}) g(x_k, \zeta)) + \triangle p_{k|\zeta}(x_{k,t}), \hspace{1cm} (D.2)
\]

which integrating with respect to $x_k$ and $\zeta$ achieves
\begin{align}
\frac{\partial p_{k,t}(x)}{\partial t} &= \int \int \frac{\partial p_{k|\zeta}(x)}{\partial t} p_{k|\zeta}(x_k, \zeta) dx_k d\zeta \\
&= \int \int (\nabla \cdot (p_{k|\zeta}(x_k, \zeta)g(x_k, \zeta)) + \Delta p_{k|\zeta}(x_k)) \, dx_k d\zeta \\
&= \int \int (\nabla \cdot (p_{k|\zeta}(x_k, \zeta)g(x_k, \zeta))) + \Delta p_{k,t}(x) \\
&= \nabla \cdot (p_{k,t}(x) \int \int p_{k|\zeta}(x_k)g(x_k, \zeta) dx_k d\zeta) + \Delta p_{k,t}(x) \tag{D.3} \\
&= \nabla \cdot (p_{k,t}(x)\mathbb{E}_{p_{k|\zeta}}[g(x_k, \zeta)|x_{k,t} = x]) + \Delta p_{k,t}(x). \tag{D.4}
\end{align}

Combining with \( \int p_t \frac{\partial}{\partial t} \log \frac{p_t}{\pi_\mu} \, dx = \int \frac{\partial p_t}{\partial t} \, dx = \frac{4}{D} \int p_t \, dx = 0 \), we get the following inequality for time derivative of KL-divergence.

\begin{align}
\frac{d}{dt}H(p_{k,t}||\pi_\mu) &= \frac{d}{dt} \int p_{k,t}(x) \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \, dx \\
&= \int \frac{\partial p_{k,t}}{\partial t} (x) \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \, dx \\
&= \int \left[ \nabla \cdot \left( p_{k,t}(x)\mathbb{E}_{p_{k|\zeta}}[g(x_k, \zeta)|x_{k,t} = x] \right) \right] \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \, dx \\
&+ \int \left[ \Delta p_{k,t}(x) \right] \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \, dx \\
&\overset{(i)}{=} \int \left[ \nabla \cdot \left( p_{k,t}(x)\mathbb{E}_{p_{k|\zeta}}[g(x_k, \zeta)|x_{k,t} = x] \right) \right] \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \, dx \\
&+ \int \left[ \nabla \cdot \left( \nabla \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) - \nabla U(x) \right) \right] \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \, dx \\
&\overset{(ii)}{=} - \int p_{k,t}(x) \left[ \mathbb{E}_{p_{k|\zeta}}[g(x_k, \zeta)|x_{k,t} = x], \nabla \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \right] \, dx \\
&- \int p_{k,t}(x) \left[ \nabla \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) - \nabla U(x), \nabla \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \right] \, dx \\
&= -I(p_{k,t}||\pi_\mu) \\
&+ \int p_{k,t}(x) \left[ \nabla U(x) - \mathbb{E}_{p_{k|\zeta}}[g(x_k, \zeta)|x_{k,t} = x], \nabla \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \right] \, dx \\
&= -I(p_{k,t}||\pi_\mu) + \mathbb{E}_{p_{k|\zeta}} \left[ \nabla U(x_{k,t}) - g(x_k, \zeta), \nabla \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \right] \\
&\overset{(iii)}{\leq} -I(p_{k,t}||\pi_\mu) \\
&+ \mathbb{E}_{p_{k|\zeta}} \| \nabla U(x_{k,t}) - g(x_k, \zeta) \|^2 + \frac{1}{4} \mathbb{E}_{p_{k|\zeta}} \left\| \nabla \log \left( \frac{p_{k,t}(x)}{\pi_\mu(x)} \right) \right\|^2 \\
&= -3 \frac{1}{4} I(p_{k,t}||\pi_\mu) + \mathbb{E}_{p_{k|\zeta}} \| \nabla U(x_{k,t}) - g(x_k, \zeta) \|^2 \tag{D.5}
\end{align}
in which equality (i) is follows from $\nabla p_{k,t} = \nabla (\nabla p_{k,t})$, equality (ii) follows from the divergence theorem, inequality (iii) follows from $\langle u, v \rangle \leq \|u\|^2 + \frac{1}{4}\|v\|^2$, and in the last step, the expectation is taken with respect to both $x_k, x_{k,t}$ and $\zeta$.

We now ready to state and prove Theorem 3.2.

**Theorem D.1.** Suppose $\pi$ is $\gamma_1$-log-Sobolev, $\alpha$-mixture weakly smooth, $L = 1 \lor \max \{L_i\}$, and for any $x_0 \sim p_0$ with $H(p_0|\pi) = C_0 < \infty$, the iterates $x_k \sim p_k$ of ULA with step size $\eta \leq \min \{1, \frac{1}{4\gamma_1}, (\frac{\gamma_1}{13N^2 L^3})^{\frac{1}{2}}, \frac{3c_\gamma_1}{16D_4}\}$ (D.6) satisfies

$$H(p_k|\pi) \leq e^{-\frac{3\eta}{2\gamma_1} k} H(p_0|\pi) + 2\eta^{\alpha+1} D_4,$$

(D.7)

where $D_4 = \sum_i 10N^3 L^6 + 16NL^4 + 8N^2 L^3 \eta^2 + 4NL^2 d + 8N^2 L^2 d^{\frac{2\alpha}{\alpha-1}}$. Then, for any $\epsilon > 0$, to achieve $H(p_k|\pi) < \epsilon$, it suffices to run LMC with step size $\eta \leq \min \{1, \frac{1}{4\gamma_1}, (\frac{\gamma_1}{13N^2 L^3})^{\frac{1}{2}}, \frac{3c_\gamma_1}{16D_4}\}$ (D.8) for $k \geq \frac{2}{\gamma_1} \log \frac{3H(p_0|\pi)}{\epsilon}$ iterations.

**Proof.** We adapt the proof of [31]. First, recall that the discretization of the ULA is

$$x_{k,t} = x_k - \eta g(x_k, \zeta) + \sqrt{2\eta} z_k,$$

where $z_k \sim N(0, I)$ is independent of $x_k$. Let $x_k \sim p_k$ and $x^* \sim \pi$ with an optimal coupling $(x_k, x^*)$ so that $E[\|x_k - x^*\|^2] = W_2(p_{x_k}, \pi_{x_k})^2$. Choosing $\mu = \sqrt{\eta}$, we have

$$E_{p_k|x_k} \|\nabla U(x_{k,t}) - g(x_k, \zeta)\|^2$$

$$\leq 2 E_{p_{k|x_k}} \|\nabla U(x_{k,t}) - \nabla U(x_k)\|^2 + \|\nabla U(x_k) - g(x_k, \zeta)\|^2$$

$$\leq \frac{40N^3}{\gamma_1} L^6 \eta^{2\alpha} H(p_k|\pi) + D_3 \eta^\alpha + 8N^2 L^2 \mu^{2\alpha} d^{\frac{2\alpha}{\alpha-1}}$$

$$\leq \frac{40N^3}{\gamma_1} L^6 \eta^{2\alpha} H(p_k|\pi) + D_4 \eta^\alpha,$$

where step 1 follows from Young inequality and Assumption 2, step 2 comes from equation (C.4), and the last step comes from $\eta \leq \frac{1}{L}$ and $\eta \leq 1$ and
the definition of $D_4$. Therefore, from Lemma 3.2, the time derivative of KL divergence along LMC is bounded by

\[
\frac{d}{dt} H(p_k|\pi_\mu) \leq -\frac{3}{4} I(p_k|\pi_\mu) + \frac{40N^3}{\gamma_1} L^6 \eta^{2\alpha} H(p_k|\pi_\mu) + D_4 \eta^\alpha
\]

where in the last inequality we have used the definition A.1 of LSI inequality. Multiplying both sides by $e^{\frac{3\gamma_1}{2} t}$, and integrating both sides from $t = 0$ to $t = \eta$ we obtain

\[
e^{\frac{3\gamma_1}{2} \eta} H(p_{k+1}|\pi_\mu) - H(p_k|\pi_\mu) \leq 2 \left( \frac{e^{\frac{3\gamma_1}{2} \eta} - 1}{\frac{3\gamma_1}{2}} \right) \left( \frac{40N^3}{\gamma_1} L^6 \eta^{2\alpha} H(p_k|\pi_\mu) + D_4 \eta^\alpha \right)
\]

\[
\leq 2\eta \left( \frac{40N^3}{\gamma_1} L^6 \eta^{2\alpha} H(p_k|\pi) + D_4 \eta^\alpha \right)
\]

(D.10)

where the last line holds by $e^c \leq 1 + 2c$ for $0 < c = \frac{3\gamma_1}{2} \eta < 1$. Rearranging the term of the above inequality and using the facts that $1 + \eta^{1+2\alpha} \frac{80N^3}{\gamma_1} L^6 \leq 1 + \frac{\eta^{\frac{2\alpha}{1+2\alpha}}}{3\gamma_1}$ when $\eta \leq \left( \frac{\gamma_1}{13N^2 L^3} \right)^{\frac{1}{\alpha}}$ and $e^{\frac{3\gamma_1}{2} \eta} \leq 1$ leads to

\[
H(p_{k+1}|\pi_\mu) \leq e^{-\frac{3\gamma_1}{2} \eta} \left( 1 + \eta^{1+2\alpha} \frac{80N^3}{\gamma_1} L^6 \right) H(p_k|\pi_\mu) + 2\eta^{\alpha+1} D_4
\]

\[
\leq e^{-\gamma_1 \eta} H(p_k|\pi_\mu) + 2\eta^{\alpha+1} D_4.
\]

(D.11)

Applying this inequality recursively, and using the inequality $1 - e^{-c} \geq \frac{c}{4}$ for $0 < c = \gamma_1 \eta \leq \frac{1}{4}$ we obtain

\[
H(p_k|\pi_\mu) \leq e^{-\gamma_1 \eta k} H(p_0|\pi_\mu) + \frac{2\eta^{\alpha+1} D_4}{1 - e^{-\gamma_1 \eta}}
\]

\[
\leq e^{-\gamma_1 \eta k} H(p_0|\pi_\mu) + \frac{2\eta^{\alpha+1} D_4}{\frac{4}{\gamma_1} \eta}
\]

\[
\leq e^{-\gamma_1 \eta k} H(p_0|\pi_\mu) + \frac{8\eta^\alpha D_4}{3\gamma_1}.
\]

(D.12)

Note that last inequality holds if we choose $\eta$ such that it satisfies

\[
\eta \leq \min \left\{ 1, \frac{1}{4\gamma_1}, \left( \frac{\gamma_1}{13N^2 L^3} \right)^{\frac{1}{\alpha}} \right\}.
\]

From Lemma 3.4, by choosing $\mu = \sqrt{\eta}$ small enough so that $W_2(\pi, \pi_\mu) \leq 3\sqrt{N} L \eta \frac{2}{d^*}$. Since $\pi$ satisfies log-Sobolev inequality, by triangle inequality we also get
Given $\epsilon > 0$, if we further assume $\eta \leq \left( \frac{e^{\gamma}}{9 \sqrt{NLE \delta^2}} \right)^{\frac{2}{\epsilon}}$, then the above inequality implies $H(p_k | \pi_\mu) \leq \frac{1}{\sqrt{\eta}} e^{- \frac{2\epsilon}{\eta} \gamma} H(p_0 | \pi_\mu) + \frac{2}{\epsilon} \eta^{\frac{1}{4}} \sqrt{D_4 + 3 \sqrt{NLE\eta \delta^2}}$. This means for $k \geq \frac{2}{\gamma \eta} \log \frac{3 \sqrt{H(p_0 | \pi_\mu) \gamma \epsilon}}{\epsilon}$, we have $H(p_k | \pi) \leq \frac{\epsilon}{3} + \frac{2}{3} = \epsilon$, as desired. \hfill \Box

### D.3. Proof of Lemma 3.4

**Lemma D.3.** Assume that $\pi \propto \exp(-\pi)$ and $\pi_\mu \propto \exp(-U_\mu)$ and $\pi$ has a bounded second moment, that is $\int \|x\|^2 \pi(x)dx = E_2 < \infty$. We deduce the following bounds

$$W_2^2(\pi, \pi_\mu) \leq 8.24NL\mu^{1+\alpha}d^\delta E_2,$$

for any $\mu \leq 0.05$.

**Proof.** This proof adapts the technique of the proof of [11]'s Proposition 1. Without loss of generality we may assume that $\int \exp(-U(x))dx = 1$. We first give upper and lower bounds to the normalizing constant of $\pi_\mu$, that is

$$c_\mu \triangleq \int_{\mathbb{R}^d} \pi(x) \exp(-U_\mu(x) - U(x))dx.
= \mathbb{E}_\pi \left( \exp(-U_\mu(x) - U(x)) \right).$$

The constant $c_\mu$ is an expectation of $\exp(-U_\mu(x) - U(x))$ with respect to the density $\pi$ so it can be trivially upper bounded by $e^M$ and lower bounded by $e^{-M}$ where $|U_\mu(x) - U(x)| \leq \sum_{i} L_i \mu^{1+\alpha} d^\delta = M$. Now we control the distance between densities $\pi$ and $\pi_\mu$ at any fixed $x \in \mathbb{R}^d$:

$$|\pi(x) - \pi_\mu(x)| = \pi(x) \left| 1 - \frac{\exp(-U_\mu(x) - U(x))}{c_\mu} \right|
\leq \pi(x) \left\{ \left( 1 - \frac{\exp(-U_\mu(x) - U(x))}{e^M} \right) + \exp(-U_\mu(x) - U(x)) \left( \frac{1}{c_\mu} - \frac{1}{e^M} \right) \right\}
\leq \pi(x) \left( 1 - e^{-2M} + e^{2M} - 1 \right)
\leq \pi(x) \left( 2M + e^{2M} - 1 \right).$$
The first inequality is from triangle inequality of absolute value, second inequality is trivial while the last inequality follows from $1 - e^{-x} \leq x$ for any $x \geq 0$. To bound $W_2$, we use an inequality from [32] (Theorem 6.15, page 115):

$$W_2^2(\pi, \pi_\mu) \leq 2 \int_{\mathbb{R}^d} ||x||^2_2 |\pi(x) - \pi_\mu(x)| \, dx.$$  

Combining this with the bound on $|\pi(x) - \pi_\mu(x)|$ shown above, we have

$$W_2^2(\pi, \pi_\mu) \leq 2 \int_{\mathbb{R}^d} ||x||^2_2 \pi(x) \left(2M + e^{2M} - 1\right) \, dx$$

$$\leq 2 \left(2M + e^{2M} - 1\right) E_x \left[||x||^2\right]$$

$$\leq 2 \left(2M + e^{2M} - 1\right) E_2$$

$$\leq 8.24 \sum_i L_i \mu^{1+\alpha} d_i^2 E_2$$

$$\leq 8.24 NL \mu^{1+\alpha} d^2 E_2,$$

where in the last inequality $M < 0.05$ ensures that $e^{2M} - 1 \leq 2.12M$. This gives the desired result.

\[\square\]

**Appendix E: Convexification of non-convex domain**

**E.1. Proof of Lemma 4.1**

**Lemma E.1.** For function $V$ defined as

$$V(\; x ) = \inf \{ \{ x_i \} \subset \Omega, \begin{cases} \sum \lambda_i U(\; x_i ) \end{cases} \quad \text{s.t.} \sum \lambda_i = 1 \}$$

\forall \; x \in \mathbb{B}(0, R), \inf_{||x||=R} U(x) \leq V(\; x ) \leq \sup_{||x||=R} U(x) .

**Proof.** First, by definition of $V$ inside $\mathbb{B}(0, R)$, we show that for any linear combination of the form $\sum_i \lambda_i U(\; x_i )$ where $\sum \lambda_i = 1$, we can find another representation $\sum_j \lambda_j U(\; x_j )$ where $\sum \lambda_j = 1$ and $\|x_j\| = R$ such that $\sum_j \lambda_j U(\; x_j ) \leq \sum_i \lambda_i U(\; x_i )$. This follows straightforwardly as follows.

For any $x_j \in \{ \; x_i \}$, such that $\|x_j\| > R$, there exists a new convex combination $\{ \; x_i \} \cup \{ \bar{x}_j \}$ with $\|x_j\| = R$, such that $\sum_i \lambda_i U(\; x_i ) \geq \bar{\lambda}_j U(\; x_j ) + \sum_{i\neq j} \bar{\lambda}_i U(\; x_i )$. In this case, we choose $\bar{x}_j$ where $\|x_j\| = R$, such that:

$$\bar{x}_j = \frac{1 - \bar{\lambda}_j}{1 - \lambda_j} x + \frac{\lambda_j - \bar{\lambda}_j}{1 - \lambda_j} x_j, \; \lambda_j < \bar{\lambda}_j < 1,$$

$$= \bar{\lambda}_j x_j + \left(\frac{1 - \bar{\lambda}_j}{1 - \lambda_j}\right) \left(\sum_{i\neq j} \lambda_i x_i \right).$$  \hspace{1cm} (E.2)
Since $U$ is convex on $\Omega$,

$$U(\bar{x}_j) \leq \bar{\lambda}_j U( x_j) + \left( \frac{1- \bar{\lambda}_j}{1- \lambda_j} \right) \left( \sum_{i \neq j} \lambda_i U( x_i) \right). \quad (E.3)$$

On the other hand, $x$ can be represented as a convex combination of $\{ x_i \} \cup \{ \bar{x}_j \} \setminus \{ x_j \}$:

$$x = \frac{\lambda_j}{\bar{\lambda}_j} \bar{x}_j + \left( 1 - \frac{\lambda_j}{\bar{\lambda}_j} \right) \left( \sum_{i \neq j} \lambda_i x_i \right) = \bar{\lambda}_j \bar{x}_j + \sum_{i \neq j} \tilde{\lambda}_i x_i, \quad (E.4)$$

and that

$$\sum_i \lambda_i U( x_i) \geq \frac{\lambda_j}{\bar{\lambda}_j} U(\bar{x}_j) + \left( 1 - \frac{\lambda_j}{\bar{\lambda}_j} \right) \left( \sum_{i \neq j} \lambda_i U( x_i) \right)$$

$$= \bar{\lambda}_j U(\bar{x}_j) + \sum_{i \neq j} \tilde{\lambda}_i U( x_i). \quad (E.5)$$

As a result, $V(x)$ can be represented as

$$V( x) = \inf_{\{ x_j \} \subset \Omega, \ s.t., \sum_j \lambda_j = 1} \left\{ \sum_j \lambda_j U( x_j) \right\}. \quad (E.6)$$

By the representation of $V$ inside $B(0, R)$, we obtain $\inf_{\|x\| = R} U(\bar{x}) \leq V( x) \leq \sup_{\|x\| = R} U(\bar{x})$. \hfill \Box

**E.2. Proof of Lemma 4.2**

**Lemma E.2.** For $U$ satisfying $\alpha$-mixture weakly smooth and $(\mu, \theta)$-degenerated convex outside the ball radius $R$, there exists $\hat{U} \in C^1(\mathbb{R}^d)$ with a Hessian that exists everywhere on $\mathbb{R}^d$, and $\hat{U}$ is $((1 - \theta) \frac{\mu}{2}, \theta)$-degenerated convex on $\mathbb{R}^d$ (that is $\nabla^2 \hat{U}(x) \succeq (1 - \theta) \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\theta} I_d$), such that

$$\sup \left( \hat{U}( x) - U( x) \right) - \inf \left( \hat{U}( x) - U( x) \right) \leq \sum_i L_i R^{1+\alpha_i} + \frac{4\mu}{(2 - \theta)} R^{2-\theta}. \quad (E.7)$$

**Proof.** Following closely to [24]’s approach, let $g( x) = \frac{\mu}{2(2 - \theta)} \left( 1 + \|x\|^2 \right)^{-\theta} x$ for $0 \leq \theta < 1$. The gradient of $g( x)$ is $\nabla g( x) = \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\theta} x$ and the
Hessian of \( g(x) \) is
\[
\nabla^2 g(x) = \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}} \begin{bmatrix} I_d \end{bmatrix} - \theta \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}-1} x x^T
\]
\[
\geq \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}} \begin{bmatrix} I_d \end{bmatrix}.
\]
(E.8)

On the other hand, we also have
\[
\nabla^2 g(x) = \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}} \begin{bmatrix} I_d \end{bmatrix} - \theta \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}-1} x x^T
\]
\[
= \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}-1} \begin{bmatrix} I_d + I_d \|x\|^2 \theta \|x\|^2 x x^T \\
\|x\|^2 \end{bmatrix}
\]
\[
= \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}-1} \begin{bmatrix} I_d \|x\|^2 (1 - \theta) \|x\|^2 + \theta \|x\|^2 \left( I_d - \frac{x x^T}{\|x\|^2} \right) \end{bmatrix}
\]
\[
\geq \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}-1} \left( 1 - \theta \right) \|x\|^2 + 1 \begin{bmatrix} I_d \end{bmatrix}
\]
\[
\geq \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}-1} \left( 1 - \theta \right) \left( \|x\|^2 + 1 \right) \begin{bmatrix} I_d \end{bmatrix}
\]
\[
\geq (1 - \theta) \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}} \begin{bmatrix} I_d \end{bmatrix}.
\]
(E.9)

We adapt \cite{35} by denoting \( \hat{U}(x) = U(x) - g(x) \). Since \( U(x) \) is \((\mu, \theta)\)-degenerated convex outside the ball, we deduce for every \( \|x\| \geq R \),
\[
\nabla^2 \hat{U}(x) = \nabla^2 U(x) - \nabla^2 g(x)
\]
\[
\geq \mu \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}} \begin{bmatrix} I_d \end{bmatrix} - \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}} \begin{bmatrix} I_d \end{bmatrix}
\]
\[
\geq \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\theta}{2}} \begin{bmatrix} I_d \end{bmatrix},
\]
(E.10)

which implies that \( \hat{U}(x) \) is \((\frac{\mu}{2}, \theta)\)-degenerated convex outside the ball. Now, we construct \( \hat{U}(x) \) so that it is twice differentiable, degenerated convex on all \( \mathbb{R}^d \) and differs from \( U(x) \) less than \( 4LR^{1+\alpha} + 4LR^{1+\ell+\alpha} + \frac{4}{\sqrt{2-\theta}} R^{2-\theta} \).

Based on the same construction of \cite{24}, we first define the function \( V \) as the convex extension \cite{35} of \( \hat{U} \) from domain \( \Omega = \mathbb{R}^d \setminus B(0, R) \) to its convex hull \( \Omega^\circ \). \( V(x) = \inf \{ \sum \lambda_i \hat{U}(x_i) \} \) for every \( x \in \mathbb{R}^d \). Since \( \hat{U}(x) \) is convex in \( \Omega \), \( V(x) = \hat{U}(x) \) for \( x \in \Omega \). By Lemma 4.1, \( V(x) \) is convex on the entire domain \( \mathbb{R}^d \) and \( V(x) \) can be represented as
\[
V(x) = \inf_{\{ x_i \} \subset \Omega} \left\{ \sum \lambda_j \hat{U}(x_j) \right\}.
\]
(E.11)

\[
\text{s.t.} \sum \lambda_j = 1 \}
\]
Therefore, \( \forall x \in \mathbb{B}(0,R), \inf_{\|z\|=R} \hat{U}(\bar{x}) \leq V(\bar{x}) \leq \sup_{\|z\|=R} \hat{U}(\bar{x}) \). Next we construct \( \hat{V}(\bar{x}) \) to be a smoothing of \( V \) on \( \mathbb{B}(0,R+\epsilon) \). Consider the function \( \varphi(x) \) of a variable \( x \) in \( \mathbb{R}^d \) defined by

\[
\varphi(x) = \begin{cases} 
C e^{-1/(1-\|x\|^2)} & \text{if } \|x\| < 1 \\
0 & \text{if } \|x\| \geq 1 
\end{cases}
\]  

(E.12)

where the numerical constant \( C \) ensures normalization. Let \( \varphi_\delta(x) = \delta^{-d} \varphi(\delta^{-1}x) \) be a smooth function supported on the ball \( \mathbb{B}(0,\delta) \). Define

\[
\hat{V}(\bar{x}) = \int V(y) \varphi_\delta(x - y) dy = \int V(x - y) \varphi_\delta(y) dy = E_y{[V(x - y)]}.
\]  

(E.13)

The third equality implies that for any \( x \) and \( z \in \mathbb{R}^d \),

\[
\left\langle \nabla \hat{V}(\bar{x}), \nabla \hat{V}(\bar{z}), x - z \right\rangle = \left\langle \nabla E_y{[V(x - y)]} - \nabla E_y{[V(z - y)]}, x - z \right\rangle
\]

\[
\leq \left\langle E_y{[\nabla V(x - y)]} - E_y{[\nabla V(z - y)]}, x - z \right\rangle
\]

\[
= \left\langle E_y{[\nabla V(x - y) - \nabla V(z - y)]}, x - z \right\rangle
\]

\[
= E_y{[\nabla V(x - y) - \nabla V(z - y)]}, x - z \geq 0, \quad \text{by (E.14)}
\]

where step 1 follows from exchangeability of gradient and integral and the last line is because of convexity of \( V \), which indicates \( \hat{V} \) is a smooth and convex function on \( \mathbb{R}^d \). Also, note that the definition of \( \hat{V} \) implies that \( \forall \|x\| < R + \epsilon \),

\[
\inf_{\|\bar{x}\| < R + \epsilon + \delta} \hat{V}(\bar{x}) \leq \hat{V}(\bar{x}) \leq \sup_{\|\bar{x}\| < R + \epsilon + \delta} \hat{V}(\bar{x}).
\]  

(E.15)

And by Lemma 4.1, for \( \forall \|\bar{x}\| < R + \epsilon \)

\[
\inf_{\bar{z} \in \mathbb{B}(0, R + \epsilon + \delta) \setminus \mathbb{B}(0, R)} \hat{U}(\bar{z}) \leq \hat{V}(\bar{x}) \leq \sup_{\bar{z} \in \mathbb{B}(0, R + \epsilon + \delta) \setminus \mathbb{B}(0, R)} \hat{U}(\bar{z}).
\]  

(E.16)

Finally, we construct the auxiliary function:

\[
\hat{U}(\bar{x}) - g(\bar{z}) = \begin{cases} 
\hat{U}(\bar{x}), \|x\| \geq R + 2\epsilon \\
\alpha(\bar{x})\hat{U}(\bar{x}) + (1 - \alpha(\bar{x}))\hat{V}(\bar{x}), \quad R + \epsilon < \|x\| < R + 2\epsilon \\
\hat{V}(\bar{x}), \|x\| \leq R + \epsilon
\end{cases}
\]  

(E.17)

where \( \alpha(x) = \frac{1}{2} \cos \left( \pi \left( \frac{\|x\|^2}{\epsilon(2R + 3\epsilon)^2} - \frac{(R+\epsilon)^2}{\epsilon(2R + 3\epsilon)^2} \pi \right) \right) + \frac{1}{2} \). Here we know that \( \hat{U}(\bar{x}) \) is convex and smooth in \( \mathbb{R}^d \setminus \mathbb{B}(0, R) \); \( \hat{V}(\bar{x}) \) is also convex and smooth.
in $\mathbb{R}^d \setminus B(0, R + \epsilon)$. Hence for $R + \epsilon < \|x\| < R + 2\epsilon$,

$$
\nabla^2 \left( \hat{U}(x) - g(x) \right) = \nabla^2 \hat{U}(x) + \nabla^2 \left( (1 - \alpha(x))(\hat{V}(x) - \hat{U}(x)) \right) \\
= \alpha(x) \nabla^2 \hat{U}(x) + (1 - \alpha(x)) \nabla^2 \hat{V}(x) \\
- \nabla^2 \alpha(x) \left( \hat{V}(x) - \hat{U}(x) \right) - 2 \nabla \alpha(x) \left( \nabla \hat{V}(x) - \nabla \hat{U}(x) \right)^T \\
\geq -\nabla^2 \alpha(x) \left( \hat{V}(x) - \hat{U}(x) \right) - 2 \nabla \alpha(x) \left( \nabla \hat{V}(x) - \nabla \hat{U}(x) \right)^T.
$$

(E.18)

Note that for $R + \epsilon < \|x\| < R + 2\epsilon$, we have

$$
\|\nabla g(x) - \nabla g(x - y)\| \\
= \left\| \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} \left( x - \frac{\mu}{2} \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} (x - y) \right) \right\| \\
\leq \left\| \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} x - \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} (x - y) \right\| \\
+ \left\| \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} (x - y) - \frac{\mu}{2} \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} (x - y) \right\| \\
\leq \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} \|y\| + \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} - \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} \|x - y\| \\
\leq \frac{\mu}{2} \left( 1 + (R + \epsilon)^2 \right)^{-\frac{\nu}{2}} \delta + \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} \|x - y\| \\
\leq \frac{\mu}{2} \left( 1 + (R + \epsilon)^2 \right)^{-\frac{\nu}{2}} \delta + \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} \|x - y\| \\
\leq \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} \|x - y\| \\
\leq \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} \|x - y\| \\
\leq \frac{\mu}{2} \left( 1 + (R + \epsilon)^2 \right)^{-\frac{\nu}{2}} \delta + \frac{\mu}{2} \left( 1 + \|x\|^2 \right)^{-\frac{\nu}{2}} \left( 1 + \|x - y\|^2 \right)^{-\frac{\nu}{2}} \|x - y\|. 
$$

(E.20)

where 1 follows from Lemma F.15, while 2 is due to triangle inequality. As a
\[ \left\| \nabla \tilde{V}(x) - \nabla \tilde{U}(x) \right\| = \int \left\| \nabla \tilde{U}(x - y) - \nabla \tilde{U}(x) \right\| \varphi_\delta(y) dy \]
\[ \leq \sum_i L_i \delta^{\alpha_i} + \left\| \nabla g(x) - \nabla g(x - y) \right\| \]
\[ \leq NL\delta^\alpha + \frac{\mu}{2} \left(1 + (R + \epsilon)^2\right)^{-\frac{\beta}{2}} \delta \tag{E.21} \]
\[ + \frac{\mu}{2} \left(1 + (R + \epsilon)^2\right)^{-\frac{\beta}{2}} \left(1 + (R + \epsilon - \delta)^2\right)^{-\frac{\beta}{2}} \tag{E.22} \]

On the other hand, we also acquire
\[ |\tilde{U}(x) - \tilde{U}(x - y)| \]
\[ \leq \max \{ \langle \nabla U(x - y), y \rangle, \langle \nabla U(x), -y \rangle \} + \sum_i L_i \frac{L}{1 + \alpha_i} |y|^{\alpha_i + 1} + |g(x) - g(x - y)| \]
\[ \leq \max \{ \langle \nabla U(x - y), y \rangle, \langle \nabla U(x), -y \rangle \} + \sum_i L_i \frac{L}{1 + \alpha_i} |y|^{\alpha_i + 1} \]
\[ + \left| \frac{\mu}{2 (2 - \theta)} \left(1 + \|x\|^2\right)^{1 - \frac{\beta}{2}} - \frac{\mu}{2 (2 - \theta)} \left(1 + \|x - y\|^2\right)^{1 - \frac{\beta}{2}} \right| \]
\[ \leq \max \left\{ \sum_i L_i \|x - y\|^{\alpha_i} \|y\|, \sum_i L_i \|x\|^{\alpha_i} \|y\| \right\} \]
\[ + \sum_i L_i \frac{L}{1 + \alpha_i} \|y\|^{\alpha_i + 1} \left( \frac{\mu}{2 (2 - \theta)} \left(1 + \|x\|^2\right) - \left(1 + \|x - y\|^2\right) \right) \tag{E.23} \]
\[ \leq L \|y\| \max \left\{ \sum_i L_i \|x - y\|^{\alpha_i}, \sum_i L_i \|x\|^{\alpha_i} \right\} \]
\[ + \sum_i L_i \frac{L}{1 + \alpha_i} \|y\|^{\alpha_i + 1} \left( \frac{\mu}{2 (2 - \theta)} \left(\|x\| - \|x - y\|\right) \left(\|x\| + \|x - y\|\right) \right) \tag{E.24} \]
\[ \leq L \|y\| \max \left\{ \sum_i L_i \|x - y\|^{\alpha_i}, \sum_i L_i \|x\|^{\alpha_i} \right\} \]
\[ + \sum_i L_i \frac{L}{1 + \alpha_i} \|y\|^{\alpha_i + 1} \left( \frac{\mu}{2 (2 - \theta)} \left(\|x\| + \|x - y\|\right) \|y\| \right) \tag{E.25} \]
\[ \leq L \|y\| \max \left\{ \sum_i L_i \|x - y\|^{\alpha_i}, \sum_i L_i \|x\|^{\alpha_i} \right\} + \sum_i L_i \frac{L}{1 + \alpha_i} \|y\|^{\alpha_i + 1} \left( \frac{\mu}{2 (2 - \theta)} \left(\|x\| + \|x - y\|\right) \|y\| \right), \tag{E.26} \]

where 1 follows again from Lemma F.15 and the last inequality is because of
triangle inequality. Hence for $R + \epsilon < \|x\| < R + 2\epsilon$, $\|y\| \leq \delta$,

$$
\hat{V}(x) - \hat{U}(x) = \int \left( \hat{U}(x - y) - \hat{U}(x) \right) \varphi(\|y\|) d y 
\leq L \|y\| \max \left\{ \sum_i L_i \|x - y\|^\alpha_i, \sum_i L_i \|x\|^\alpha_i \right\} + 
+ \sum_i \frac{L}{1 + \alpha_i} \|y\|^\alpha_i + \frac{\mu}{2(2 - \theta)} (\|x\| + \|x - y\|) \|y\|
\leq L \delta \left[ \sum_i L_i (R + 2\epsilon + \delta)^\alpha_i \right] + 
+ \sum_i \frac{L}{1 + \alpha_i} \delta^{\alpha_i + 1} + \frac{\mu}{(2 - \theta)} (R + 2\epsilon + \delta) \delta
$$

Therefore, when $R + \epsilon < \|x\| < R + 2\epsilon$,

$$
\nabla^2 \left( \hat{U}(x) - g(x) \right) \succeq \frac{(R + \epsilon)^2 \pi}{\epsilon (2R + 3\epsilon)} \left( L \delta \left[ \sum_i L_i (R + 2\epsilon + \delta)^\alpha_i \right] \right) I_d 
- \frac{(R + \epsilon)^2 \pi}{\epsilon (2R + 3\epsilon)} \left( \sum_i \frac{L}{1 + \alpha_i} \delta^{\alpha_i + 1} \frac{\epsilon}{m \|y\|^\alpha_i} (R + 2\epsilon + \delta) \delta \right) I_d 
- \frac{(R + \epsilon)^4 \pi^2}{\epsilon^2 (2R + 3\epsilon)} \left( NL \delta^\alpha + \frac{\mu}{2} \left( 1 + (R + \epsilon)^2 \right)^{-\frac{\alpha}{2}} \delta \right) I_d 
- \frac{(R + \epsilon)^4 \pi^2}{\epsilon^2 (2R + 3\epsilon)} \left( \frac{2(R + \epsilon - \delta)^2 \delta}{(1 + (R + \epsilon)^2)^2} \frac{\epsilon^2}{2 + (R + \epsilon - \delta)^2} \right) I_d. \tag{E.27}
$$

Taking the limit when $\delta \to 0^+$, we obtain that for $R + \epsilon < \|x\| < R + 2\epsilon$, $\nabla^2 \left( \hat{U}(x) - g(x) \right)$ is positive semi-definite; hence it is positive semi-definite on the entire $R^d$, or $\hat{U}(x) - g(x)$ is convex on $R^d$. From (E.16), we know that for $R + \epsilon < \|x\| < R + 2\epsilon$,

$$
\inf_{\bar{x} \in B(0,R+2\epsilon) \setminus \{0\}} \hat{U}(\bar{x}) \leq \hat{U}(x) - g(x) \leq \sup_{\bar{x} \in B(0,R+2\epsilon) \setminus \{0\}} \hat{U}(\bar{x}). \tag{E.28}
$$
Therefore,
\[
\sup (\hat{U}(x) - U(x)) - \inf (\hat{U}(x) - U(x)) \\
= \sup (\hat{U}(x) - g(x) - \hat{U}(x)) - \inf (\hat{U}(x) - g(x) - \hat{U}(x))
\] (E.29)
\[
\leq 2 \left( \sup_{\bar{x} \in B(0,R+2\epsilon) \setminus B(0,R)} \tilde{U}(\bar{x}) - \inf_{\bar{x} \in B(0,R+2\epsilon) \setminus B(0,R)} \tilde{U}(\bar{x}) \right)
\leq 2 \left( \sup_{\bar{x} \in B(0,R+2\epsilon)} \tilde{U}(\bar{x}) - \inf_{\bar{x} \in B(0,R+2\epsilon)} \tilde{U}(\bar{x}) \right).
\] (E.30)

Since \( U \) is \( (\alpha, \ell) \)-weakly smooth and \( \nabla U(0) = 0 \), we deduce
\[
|U(x) - U(0)| = |U(x) - U(0) - \langle x, \nabla U(0) \rangle| \\
\leq \sum_{i} L_i \frac{\|x\|^1 + \alpha_i}{1 + \alpha_i} \\
\leq \sum_{i} \frac{L_i}{1 + \alpha_i} (R + 2\epsilon)^{1 + \alpha_i} \\
\leq \sum_{i} L_i R^{1 + \alpha_i},
\] (E.31)

and
\[
|g(x)| = \left| \frac{\mu}{2(2 - \theta)} \left( 1 + \|x\|^2 \right)^{1 - \frac{\theta}{2}} \right| \\
\leq \frac{\mu}{2(2 - \theta)} \left( 1 + (R + 2\epsilon)^2 \right)^{1 - \frac{\theta}{2}} \\
\leq \frac{\mu}{(2 - \theta)} R^{2 - \theta}.\] (E.32)

So for \( \forall \|x\| \leq (R + 2\epsilon) \), \( \epsilon \) is sufficiently small,
\[
\sup_{\bar{x} \in B(R+2\epsilon)} \hat{U}(\bar{x}) - \inf_{\bar{x} \in B(R+2\epsilon)} \hat{U}(\bar{x}) \leq \sum_{i} L_i R^{1 + \alpha_i} + \frac{2\mu}{(2 - \theta)} R^{2 - \theta}.
\]

As a result, we get
\[
\sup (\hat{U}(x) - U(x)) - \inf (\hat{U}(x) - U(x)) \leq 2 \sum_{i} L_i R^{1 + \alpha_i} + \frac{4\mu}{(2 - \theta)} R^{2 - \theta}.
\]

\[\square\]

Remark E.1. When \( \theta = 0 \), the \( (\mu, \theta) \)-degenerated convex outside the ball is equivalent to the \( \mu \)-strongly convex outside the ball, we achieve a result for strongly convex outside the ball similar to [24] but for \( (\alpha, \ell) \)-weakly smooth instead of smooth. The constant could be improved by a factor of 2 if we take \( \epsilon \) to be arbitrarily small.
E.3. Proof of lemma 4.3

Lemma E.3. For $U$ satisfying $\gamma$–Poincaré, $\alpha$-mixture weakly smooth with $\alpha_N = 1$ and 2–dissipative, there exists $\tilde{U} \in C^1(\mathbb{R}^d)$ with a Hessian that exists everywhere on $\mathbb{R}^d$, and $\tilde{U}$ is log-Sobolev on $\mathbb{R}^d$ such that

$$\sup \left( \tilde{U}(x) - U(x) \right) - \inf \left( \tilde{U}(x) - U(x) \right) \leq 2 \sum_i L_i R^{1+\alpha_i} + 4L N R^2 + 4L R^{1+\alpha}. \quad (E.33)$$

Proof. First, given $R > 0$, let $\overline{U}(x) := U(x) + \frac{L_N + \lambda_0}{2} \|x\|^2$ for $\lambda_0 = \frac{2L}{R^{2+\alpha}}$, we obtain the following property

$$\langle \nabla \overline{U}(x) - \nabla \overline{U}(y), x - y \rangle$$

$$= \left\langle \nabla \left( U(x) + \frac{L_N + \lambda_0}{2} \|x\|^2 \right) - \nabla \left( U(y) + \frac{L_N + \lambda_0}{2} \|y\|^2 \right), x - y \right\rangle$$

$$= \langle \nabla U(x) - \nabla U(y) + (L_N + \lambda_0) (x - y), x - y \rangle$$

$$\geq - \sum_{i < N} L_i \|x - y\|^{1+\alpha} + \lambda_0 \|x - y\|^2$$

$$\geq \frac{\lambda_0}{2} \|x - y\|^2 \text{ for } \|x - y\| \geq \left( \frac{NL}{\lambda_0} \right)^{\frac{1}{1+\alpha}} = R, \quad (E.35)$$

where (i) follows from Assumption 2.2. This implies that $\overline{U}(x)$ is $\lambda_0$–strongly convex outside the ball $B_R = \{ x : \|x\| \leq R \}$. Though $\overline{U}(x)$ behaves differently than Lemma 4.2 assumptions, with some additional verifications, we still can apply Lemma 4.2 to derive the result. We sketch the proof as follows. There exists $\tilde{U} \in C^1(\mathbb{R}^d)$ with a Hessian that exists everywhere on $\mathbb{R}^d$,

$$\tilde{U}(x) - \frac{\lambda_0}{4} \|x\|^2 = \begin{cases} \tilde{U}(x), \|x\| \geq R + 2\epsilon \\ \alpha(x) \tilde{U}(x) + (1 - \alpha(x)) \tilde{V}(x), R + \epsilon < \|x\| < R + 2\epsilon \\ \tilde{V}(x), \|x\| \leq R + \epsilon \end{cases} \quad (E.36)$$

where $\alpha(x)$ is defined as before. Both $\tilde{U}(x)$ and $\tilde{V}(x)$ are convex and smooth in $\mathbb{R}^d \setminus B(0, R)$ and for $R + \epsilon < \|x\| < R + 2\epsilon$, $\|y\| \leq \delta$,

$$\nabla^2 \left( \tilde{U}(x) - \frac{\lambda_0}{4} \|x\|^2 \right) \geq - \nabla^2 \alpha(x) \left( \tilde{V}(x) - \tilde{U}(x) \right) - 2 \nabla \alpha(x) \left( \nabla \tilde{V}(x) - \nabla \tilde{U}(x) \right)^T. \quad (E.37)$$
In this case, we have

\[
\left\| \nabla \mathcal{V}(x) - \nabla \mathcal{U}(x) \right\| = \left\| \nabla \int \left( \mathcal{U}(x - y) - \mathcal{U}(x) \right) \varphi_{\delta}(y) dy \right\|
\]

\[
\leq \left\| \nabla \int \left( \mathcal{U}(x - y) - \mathcal{U}(x) \right) \varphi_{\delta}(y) dy \right\|
\]

\[
+ \lambda_0 \int \|y\| \varphi_{\delta}(y) dy
\]

\[
\leq \left\| \int \left( \nabla \mathcal{U}(x - y) - \nabla \mathcal{U}(x) \right) \varphi_{\delta}(y) dy \right\| + \lambda_0 \delta
\]

\[
\leq \sum_i L_i \delta^{\alpha_i + 1} + \lambda_0 \delta,
\]  

(E.38)

where 1 holds by triangle inequality and the last line is because of \((\alpha, \ell)\)–weakly smooth assumption, while

\[
\left| \mathcal{U}(x) - \mathcal{U}(x - y) \right|
\]

\[
\leq \left| \mathcal{U}(x) - \mathcal{U}(x - y) \right| + \left| \frac{L + \lambda_0}{2} \|x\|^2 - \frac{L + \lambda_0}{2} \|x - y\|^2 \right|
\]

\[
\leq \{ \langle \nabla \mathcal{U}(x - y), y \rangle \vee \langle \nabla \mathcal{U}(x), -y \rangle \} + \sum_i \frac{L_i}{1 + \alpha_i} \|y\|^{\alpha_i + 1}
\]

\[
+ \frac{L_N + \lambda_0}{2} \left( \|x\|^2 - \|x - y\|^2 \right)
\]

\[
\leq \{ \langle \nabla \mathcal{U}(x - y), y \rangle \vee \langle \nabla \mathcal{U}(x), -y \rangle \} + \sum_i \frac{L_i}{1 + \alpha_i} \|y\|^{\alpha_i + 1}
\]

\[
+ \frac{L_N + \lambda_0}{2} (\|x\| - \|x - y\|) (\|x\| + \|x - y\|)
\]  

(E.39)

\[
\leq \left\{ \left( \sum_i L_i \|x - y\|^{\alpha_i} \right) \|y\| \vee \left( \sum_i L_i \|x\|^{\alpha_i} \right) \|y\| \right\} + \sum_i \frac{L_i}{1 + \alpha_i} \|y\|^{\alpha_i + 1} + \frac{L_N + \lambda_0}{2} \|y\| \max \{ \|x - y\|, \|x\| \}
\]  

(E.40)

\[
\leq \sum_i L_i (R + 2\epsilon + \delta)^{\alpha_i \delta}
\]

(E.41)

\[
+ \sum_i \frac{L_i}{1 + \alpha_i} \delta^{\alpha_i + 1} + \frac{L_N + \lambda_0}{2} (R + 2\epsilon + \delta) \delta,
\]  

(E.42)

where 1 is due to triangle inequality, 2 follows from Assumption 1, and the last line holds by plugging in all the limits. Taking the limit when \(\delta \to 0^+\), and for sufficiently small \(\epsilon\), we obtain \(\hat{\mathcal{U}}(x) - \frac{\lambda_0}{2} \|x\|^2\) is convex on all \(\mathbb{R}^d\) or \(\hat{\mathcal{U}}(x)\) is
\(\frac{\lambda}{2}\)-strongly convex. By definition of \(\overline{U}\), for \(R + \epsilon < \|x\| < R + 2\epsilon\) we obtain

\[
|\overline{U}(x) - \overline{U}(0)| \leq |U(x) - U(0) - \langle x, \nabla U(0) \rangle| + \frac{L_N + \lambda_0}{2} \|x\|^2
\]

\[
\leq + \sum_i \frac{L_i}{1 + \alpha_i} ||x||^{\alpha_i+1} + \frac{L_N + \lambda_0}{2} \|x\|^2
\]

\[
\leq + \sum_i \frac{L_i}{1 + \alpha_i} (R + 2\epsilon + \delta)^{\alpha_i+1} + \frac{L_N + \lambda_0}{2} (R + 2\epsilon + \delta)^2
\]

\[
\leq \sum_i L_i R^{1+\alpha_i} + (L_N + \lambda_0) R^2.
\] (E.44)

As a result, from Lemma 4.2 we deduce

\[
\sup \left( \hat{U}(x) - \overline{U}(x) \right) - \inf \left( \hat{U}(x) - \overline{U}(x) \right) \leq \sum_i L_i R^{1+\alpha_i} + 2 (L_N + \lambda_0) R^2.
\] (E.45)

Let \(\hat{U}(x) = \hat{U}(x) - \left( \frac{L_N}{2} + \frac{\lambda_0}{2} \right) \|x\|^2\) then for \(\|x\| > R + 2\epsilon + \delta, \hat{U}(x) = \overline{U}(x)\) so \(\hat{U}(x) = U(x)\). For \(\|x\| \leq R + 2\epsilon + \delta\), we have

\[
\sup \left( \hat{U}(x) - U(x) \right) - \inf \left( \hat{U}(x) - U(x) \right)
\]

\[
\leq \sup \left( \hat{U}(x) + \frac{L_N + \lambda_0}{2} \|x\|^2 - \overline{U}(x) \right) - \inf \left( \hat{U}(x) + \frac{L_N + \lambda_0}{2} \|x\|^2 - \overline{U}(x) \right)
\]

\[
\leq \sup \left( \hat{U}(x) - \overline{U}(x) \right) - \inf \left( \hat{U}(x) - \overline{U}(x) \right) + (L_N + \lambda_0) (R + 2\epsilon + \delta)^2
\]

\[
\leq 2 \sum_i L_i R^{1+\alpha_i} + 2 (L_N + \lambda_0) R^2 + 2 (L_N + \lambda_0) R^2
\]

\[
\leq 2 \sum_i L_i R^{1+\alpha_i} + 4LNR^2 + 4LR^{1+\alpha}.
\] (E.46)

So for every \(x \in \mathbb{R}^d\),

\[
\sup \left( \hat{U}(x) - U(x) \right) - \inf \left( \hat{U}(x) - U(x) \right) \leq 2 \sum_i L_i R^{1+\alpha_i} + 4LNR^2 + 4LR^{1+\alpha}.
\]

Since \(U(x)\) is \(PI(\gamma)\), and using [21]'s Lemma 1.2 we have, \(\hat{U}(x)\) is Poincaré with constant

\[
\gamma_1 = \gamma e^{-4(2 \sum_i L_i R^{1+\alpha_i} + 4LNR^2 + 4LR^{1+\alpha})}.
\]

On the other hand, we know that \(\nabla^2\hat{U}(x) = \nabla^2\hat{U}(x) - (L_N + \frac{\lambda_0}{2}) I \succeq -LI\) for since \(\hat{U}(x)\) is \(\frac{\lambda}{2}\)-strongly convex, which implies that \(\nabla^2\hat{U}(x)\) is lower bounded by \(-LI\). In addition, for \(\|x\| > R + 2\epsilon + \delta\) from \(2\)-dissipative assumption, we
have for some \( a, b > 0 \), 
\[
\left\langle \nabla \tilde{U}(x), x \right\rangle \geq a \|x\|^2 - b,
\]
while for \( \|x\| \leq R + 2\epsilon + \delta \)
\[
\left\langle \nabla \tilde{U}(x), x \right\rangle \geq \langle -\nabla \left((L_N^2 + \lambda_0^2)\|x\|^2\right), x \rangle \\
\quad \geq -\left(L_N + \frac{\lambda_0}{2}\right) R^2.
\]
so for every \( x \in \mathbb{R}^d \),
\[
\left\langle \nabla \tilde{U}(x), x \right\rangle \geq a \|x\|^2 - \left(L_N + \frac{\lambda_0}{2}\right) R^2 - aR^2.
\]

By [5]'s Theorem 1.9, \( \tilde{U}(x) \) satisfies a defective log Sobolev. In addition, by Rothaus’ lemma, a defective log-Sobolev inequality together with the PI\((\gamma_1)\) implies the log-Sobolev inequality with the log Sobolev constant is
\[
\gamma_2 = \frac{2}{2 + a_1 + \frac{\lambda_0}{2}} \left(1 - L^2 + \frac{1}{\zeta}\right),
\]
where
\[
A = \left(1 - \frac{L}{2}\right) \frac{8}{a^2} + \frac{\zeta}{2},
\]
\[
B = 2 \left[ \frac{1}{2} \left(\frac{(b + 4 (L_N + \frac{\lambda_0}{2}) R^2 + aR^2) + d)}{a}\right) + M_2 \right] \left(1 - \frac{L}{2} + \frac{1}{\zeta}\right).
\]

where \( M_2 = \int \|x\|^2 e^{-\tilde{U}(x)} dx \). But it is well known from Lemma 10 that \( M_2 = O(d) \), so the log-Sobolev constant is just \( O(d) \). This concludes the proof. \( \square \)

**E.4. Proof of lemma 4.1**

**Theorem E.1.** Suppose \( \pi \) is a \( \gamma \)-Poincaré, \( \alpha \)-mixture weakly smooth with \( \alpha_N = 1 \) and 2-dissipativity (i.e. \( \langle \nabla U(x), x \rangle \geq a \|x\|^2 - b \)) for some \( a, b > 0 \), and for
any \(x_0 \sim \pi_0\) with \(H(p_0|\pi) = C_0 < \infty\), the iterates \(x_k \sim p_k\) of LMC with step size \(\eta \leq 1 \wedge \frac{1}{4\gamma_3} \wedge \left(\frac{\gamma_3}{16L^+}\right)^\frac{\alpha}{2}\) satisfies
\[
H(p_k|\pi) \leq e^{-\gamma_3 k} H(p_0|\pi) + \frac{8\eta^2 D_3}{3\gamma_3}, \tag{E.50}
\]
where \(D_3\) is defined as in equation (3.8) and
\[
\begin{align*}
M_2 &= \int \|x\|^2 e^{-U(x)} \, dx = O(d) \tag{E.51} \\
\zeta &= \sqrt{2 \left[ \frac{2 \left( b + \left( L + \frac{\gamma_3}{a} \right) \right) R^2 + a R^2 + d }{ a } + M_2 \right] e^{\left( 2 \sum_i L_i R^{i+\alpha_i} + 4L_N R^2 + 4LR^{i+\alpha_i} \right)}}, \tag{E.52} \\
A &= (1 - \frac{L}{2}) \frac{8}{a^2} + \zeta, \tag{E.53} \\
B &= 2 \left[ \frac{2 \left( b + 4 \left( L + \frac{\gamma_3}{a} \right) R^2 + a R^2 + d \right) }{ a } + M_2 \right] (1 - \frac{L}{2} + \frac{1}{\zeta}), \tag{E.54} \\
\gamma_3 &= 2 \frac{e^{-\left( 2 \sum_i L_i R^{i+\alpha_i} + 4L_N R^2 + 4LR^{i+\alpha_i} \right)}}{A\gamma + (B + 2) e^{\left( 2 \sum_i L_i R^{i+\alpha_i} + 4L_N R^2 + 4LR^{i+\alpha_i} \right)}}. \tag{E.55}
\end{align*}
\]
Then, for any \(\epsilon > 0\), to achieve \(H(p_k|\pi) < \epsilon\), it suffices to run ULA with step size \(\eta \leq 1 \wedge \frac{1}{12\gamma_3} \wedge \left(\frac{\gamma_3}{16L^+}\right)^\frac{\alpha}{2}\) for \(k \geq \frac{1}{\gamma_3 \eta} \log \frac{2H(p_0|\pi)}{\epsilon}\) iterations.

Proof. From Lemma E.3, we can optimize over \(\zeta\) and get
\[
\zeta = \sqrt{2 \left[ \frac{2 \left( b + \left( L + \frac{\gamma_3}{a} \right) \right) R^2 + a R^2 + d }{ a } + M_2 \right] e^{\left( 2 \sum_i L_i R^{i+\alpha_i} + 4L_N R^2 + 4LR^{i+\alpha_i} \right)}}. \tag{E.52}
\]
By using Holley Stroock perturbation theorem [18], we have \(U(x)\) is log-Sobolev on \(\mathbb{R}^d\) with constant
\[
\gamma_3 = 2 \frac{e^{-\left( 2 \sum_i L_i R^{i+\alpha_i} + 4L_N R^2 + 4LR^{i+\alpha_i} \right)}}{A\gamma + (B + 2) e^{\left( 2 \sum_i L_i R^{i+\alpha_i} + 4L_N R^2 + 4LR^{i+\alpha_i} \right)}}. \tag{E.54}
\]
Applying theorem 3.1, we get the desired result. \(\square\)

E.5. Proof of lemma 4.1

Lemma E.4. If \(U\) satisfies Assumption 2.4, then
\[
U(x) \geq \frac{a}{2\beta} \|x\|^\beta + U(0) - \sum_i \frac{L_i}{\alpha_i + 1} R^{\alpha_i+1} - b. \tag{E.55}
\]
Proof. Using the technique of [16], let \( R = \left( \frac{2a}{\alpha} \right)^{\frac{1}{\beta}} \), we lower bound \( U(x) \) when \( \|x\| \leq R \),

\[
U(x) = U(0) + \int_0^1 \langle \nabla U(tx), x \rangle \, dt \\
\geq U(0) - \int_0^1 \|\nabla U(tx)\| \|x\| \, dt \\
\geq U(0) - \sum_i L_i \|x\|^{\alpha_i + 1} \int_0^1 t^{\alpha_i} \, dt \\
\geq U(0) - \sum_i \frac{L_i}{\alpha_i + 1} \|x\|^{\alpha_i + 1} \\
\geq U(0) - \sum_i \frac{L_i}{\alpha_i + 1} R^{\alpha_i + 1} \quad (E.56)
\]

For \( \|x\| > R \), we can lower bound \( U \) as follows.

\[
U(x) = U(0) + \int_0^\|x\| \langle \nabla U(tx), x \rangle \, dt + \int_1^1 \langle \nabla U(tx), tx \rangle \, dt \\
\geq U(0) - \|x\| \int_0^\|x\| \sum_i L_i \|tx\|^{\alpha_i} \, dt + \int_1^1 \frac{1}{t} \langle \nabla U(tx), tx \rangle \, dt \\
\geq U(0) - \|x\| \int_0^\|x\| \sum_i L_i \|tx\|^{\alpha_i} \, dt + \int_1^1 \frac{1}{t} \left( a \|tx\|^\beta - b \right) \, dt \\
\geq U(0) - \sum_i L_i \|x\|^{\alpha_i + 1} \int_0^\|x\| t^{\alpha_i} \, dt + \int_1^1 \frac{1}{t} \left( a \|tx\|^\beta - b \right) \, dt \\
\geq \frac{1}{2} U(0) - \sum_i \frac{L_i}{\alpha_i + 1} \|x\|^{\alpha_i + 1} \left( R^{\alpha_i + 1} \|x\|^{\alpha_i + 1} \right) + \frac{a}{2} \|x\|^\beta \int_1^1 t^{\beta - 1} \, dt \\
\geq \frac{1}{2} U(0) - \sum_i \frac{L_i}{\alpha_i + 1} \left( R^{\alpha_i + 1} \|x\|^{\alpha_i + 1} \right) + \frac{a}{2} \|x\|^\beta \left( 1 - \frac{R^\beta}{\|x\|^\beta} \right) \\
\geq \frac{a}{2\beta} \|x\|^\beta + U(0) - \sum_i \frac{L_i}{\alpha_i + 1} R^{\alpha_i + 1} - b, \quad (E.57)
\]

where 1 follows from Assumption 2.4 and 2 uses the fact that if \( t \geq \frac{R}{\|x\|} \) then \( a \|tx\|^\beta - b \geq \frac{a}{2} \|tx\|^\beta \). Now, since for \( \|x\| \leq R \), \( \frac{a}{2\beta} \|x\|^\beta \leq b \), we combine the inequality for \( \|x\| \leq R \) and get

\[
U(x) \geq \frac{a}{2\beta} \|x\|^\beta + U(0) - \sum_i \frac{L_i}{\alpha_i + 1} R^{\alpha_i + 1} - b. \quad (E.58)
\]

\( \square \)
E.6. Proof of Lemma 5

**Lemma E.5.** Assume that $U$ satisfies Assumption 2.4, then for $\pi = e^{-U}$ and any distribution $\rho$, we have

$$\frac{4\beta}{a} \left[ H(\rho|\pi) + \tilde{d} + \tilde{\mu} \right] \geq E_{\rho} \left[ \|x\|^{\beta} \right], \quad (E.59)$$

where

$$\tilde{\mu} = \frac{1}{2} \log \left( \frac{2}{\beta} \right) + \sum_{i} \frac{L_i}{\alpha_i + 1} \left( \frac{2b}{a} \right)^{\frac{\alpha_i + 1}{\beta}} + b + |U(0)|, \quad (E.60)$$

$$\tilde{d} = \frac{d}{\beta} \left[ \frac{\beta}{2} \log(\pi) + \log \left( \frac{4\beta}{a} \right) + (1 - \frac{\beta}{2}) \log(\frac{d}{2e}) \right]. \quad (E.61)$$

**Proof.** Let $q(x) = e^{\frac{a}{4\beta} \|x\|^{\beta} - U(x)}$ and $C_q = \int e^{\frac{a}{4\beta} \|x\|^{\beta} - U(x)} dx$. First, we need to bound $\log C_q$. Using Lemma E.4, we have

$$U(x) \geq \frac{a}{2\beta} \|x\|^{\beta} + U(0) - \sum_{i} \frac{L_i}{\alpha_i + 1} \left( \frac{2b}{a} \right)^{\frac{\alpha_i + 1}{\beta}} - b. \quad (E.62)$$

Regrouping the terms and integrating both sides gives

$$\int e^{\frac{a}{4\beta} \|x\|^{\beta} - U(x)} dx \leq e^{-U(0) + \sum L_i \frac{\alpha_i + 1}{1 - \frac{\beta}{2}} \left( \frac{2b}{a} \right)^{\frac{\alpha_i + 1}{\beta}}} \int e^{\frac{a}{4\beta} \|x\|^{\beta}} dx$$

$$= \frac{2\pi^{d/2}}{\beta} \left( \frac{a}{\beta} \right)^{\frac{d}{2}} e^{-U(0) + \sum L_i \frac{\alpha_i + 1}{1 - \frac{\beta}{2}} \left( \frac{2b}{a} \right)^{\frac{\alpha_i + 1}{\beta}}} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - 1 \right)}$$

$$\leq \frac{2\pi^{d/2}}{\beta} \left( \frac{4\beta}{a} \right)^{\frac{d}{2}} \frac{d}{\beta} \frac{d}{\beta - 2} e^{\frac{d}{2} - \frac{1}{2}} e^{-U(0) + \sum L_i \frac{\alpha_i + 1}{1 - \frac{\beta}{2}} \left( \frac{2b}{a} \right)^{\frac{\alpha_i + 1}{\beta}}} + b, \quad (E.63)$$

where the equality on the second line comes from using polar coordinates and the third line follows from an inequality for the ratio of Gamma functions [19].
Plugging this back into the previous inequality and taking logs, we deduce

\[
\log(C_q) = \log \left( \int e^{\frac{\beta}{2} \|x\|^2 - U(x)} dx \right)
\]

\[
\leq \frac{d}{2} \log(\pi) + \frac{d}{\beta} \log \left( \frac{4\beta}{a} \right) + \frac{d}{\beta} - \frac{d}{2} \log \left( \frac{d}{2e} \right)
\]

\[+ \left( \frac{d}{\beta} + \frac{1}{2} \right) \log \left( \frac{2}{\beta} \right) + \sum_i \frac{L_i}{\alpha_i + 1} \left( \frac{2b}{a} \right)^{\frac{\alpha_i + 1}{2}} + b + |U(0)|
\]

\[\leq \frac{d}{\beta} \left[ \frac{\beta}{2} \log(\pi) + \log \left( \frac{4\beta}{a} \right) + (1 - \frac{\beta}{2}) \log \left( \frac{d}{2e} \right) \right]
\]

\[+ \frac{1}{2} \log \left( \frac{2}{\beta} \right) + \sum_i \frac{L_i}{\alpha_i + 1} \left( \frac{2b}{a} \right)^{\frac{\alpha_i + 1}{2}} + b + |U(0)|
\]

\[\leq \tilde{d} + \tilde{\mu},
\]

as definitions of \( \tilde{d} \) and \( \tilde{\mu} \). Using this bound on \( \log C_q \) we get

\[
H(\rho|\pi) = \int \rho \log \frac{\rho}{\pi} + \int \rho \log \frac{q/C_q}{\pi}
\]

\[= H(\rho|q/C_q) + E_\rho \left[ \log \frac{q/C_q}{e^{-U}} \right]
\]

\[\geq \left( \frac{a}{4\beta} \right) E_\rho \left[ \|x\|^2 \right] - \log(C_q)
\]

\[\geq \frac{a}{4\beta} E_\rho \left[ \|x\|^2 \right] - \tilde{d} - \tilde{\mu},
\]

where (1) follows from definition of \( C_q \) and the fact that relative information is always non-negative. Rearranging the terms completes the proof. \( \square \)

**Theorem E.2.** Suppose \( \pi \) is non-strongly convex outside the ball \( B(0, R) \), \( \alpha \)-mixture weakly smooth with \( \alpha_N = 1 \) and 2-dissipativity (i.e. \( \langle \nabla U(x), x \rangle \geq a \|x\|^2 - b \)) for some \( a, b > 0 \), and for any \( x_0 \sim p_0 \) with \( H(p_0|\pi) = C_0 < \infty \), the iterates \( x_k \sim p_k \) of LMC with step size \( \eta \leq 1 \wedge \frac{1}{4\gamma_3} \wedge \left( \frac{2\gamma_3}{10\gamma_3 + \gamma_3} \right)^\frac{1}{2} \) satisfies

\[
H(p_k|\pi) \leq e^{-\gamma_3 k} H(p_0|\pi) + \frac{8\eta^\alpha D_\alpha}{3\gamma_3},
\]

(E.67)
where $D_3$ is defined as in equation (3.8) and for some universal constant $K$,

$$M_2 = \int \|x\|^2 e^{-\tilde{U}(x)} dx = O(d)$$  \hspace{1cm} (E.68)

$$\zeta = K \sqrt{\frac{64d}{(b+L+\frac{\lambda x}{2})R^2 + aR^2 + d} + M_2 \left( \frac{a+b+2aR^2 + 3}{ae^{-4(4LN^2R^2+4LR^{1+\alpha})}} \right)}$$  \hspace{1cm} (E.69)

$$A = (1 - \frac{L}{2}) \frac{8}{a^2} + \zeta,$$  \hspace{1cm} (E.70)

$$B = 2 \left[ \frac{2((b+4(L+\frac{\lambda x}{2})R^2 + aR^2) + d)}{a} + M_2 \right] \left( 1 - \frac{L}{2} + \frac{1}{\zeta} \right),$$  \hspace{1cm} (E.71)

$$\gamma_3 = \frac{2e^{-2 \sum_i L_i R_i^{1+\alpha_i} + 4LN^2R^2 + 4LR^{1+\alpha}}}{A + (B+2)32K^2d \left( \frac{a+b+2aR^2 + 3}{a} \right)^e} = \frac{1}{O(d)}.$$  \hspace{1cm} (E.72)

Then, for any $\epsilon > 0$, to achieve $H(p_0|x) < \epsilon$, it suffices to run ULA with step size $\eta \leq 1 \wedge \frac{1}{4\gamma_3} \wedge \left( \frac{3\gamma_3}{16d^2} \right) \wedge \left( \frac{3\gamma_3}{16d^2} \right)$ for $k \geq \frac{1}{\gamma_3} \log \frac{2H(p_0|x)}{\epsilon}$ iterations.

Proof. Using Lemma 2, there exists $\tilde{U}(x) \in C^1(R^d)$ with its Hessian exists everywhere on $R^d$, and $\tilde{U}$ is convex on $R^d$ such that

$$\sup \left( \tilde{U}(x) - U(x) \right) - \inf \left( \tilde{U}(x) - U(x) \right) \leq 2 \sum_i L_i R_i^{1+\alpha_i}. \hspace{1cm} (E.72)$$

We can prove by two different approaches.

First approach: Since $\tilde{U}$ is convex, by Theorem 1.2 of [4], $\tilde{U}$ satisfies Poincaré inequality with constant

$$\gamma \geq 4K^2 \int \|x - E_\pi(x)\|^2 \pi(x) dx$$

$$\geq \frac{1}{8K^2 \left( E_\pi \left( \|x\|^2 \right) + \|E_\pi(x)\|^2 \right)}$$

$$\geq \frac{1}{16K^2 E_\pi \left( \|x\|^2 \right)},$$

where $K$ is a universal constant, step 1 follows from Young inequality and the last line is due to Jensen inequality. In addition, for $\|x\| > R + 2\epsilon + \delta$ from 2–dissipative assumption, we have for some $a$, $b > 0$, $\langle \nabla \tilde{U}(x), x \rangle = \langle \nabla U(x), x \rangle \geq a \|x\|^2 - b$, while for $\|x\| \leq R + 2\epsilon + \delta$ by convexity of $\tilde{U}$

$$\langle \nabla \tilde{U}(x), x \rangle \geq 0$$

$$\geq a \|x\|^2 - a (R + 2\epsilon + \delta)^2$$

$$\geq a \|x\|^2 - 2aR^2.$$
so for every $x \in \mathbb{R}^d$,
\[
\langle \nabla \hat{U}(x), x \rangle \geq a \|x\|^2 - (b + 2aR^2).
\]

Therefore, $\hat{U}(x)$ also satisfies $2-$dissipative, which implies
\[
E_\pi \left( \|x\|^2 \right) \leq 2d \left( \frac{a + b + 2aR^2 + 3}{a} \right),
\]
so the Poincaré constant satisfies
\[
\gamma \geq \frac{1}{32K^2d \left( \frac{a + b + 2aR^2 + 3}{a} \right)}.
\]

From [21]'s Lemma 1.2, we have $U$ satisfies Poincaré inequality with constant
\[
\gamma \geq \frac{1}{32K^2d \left( \frac{a + b + 2aR^2 + 3}{a} \right)} e^{-4 \sum_i L_i R^{1+\alpha_i}}.
\]

Now, applying previous section result, we derive the desired result.

Second approach. By employing Lemma F.16, combined with $2-$dissipative assumption, we get:
\[
\int e^{\frac{a}{8} \|x\|^2 - U(x)} dx \leq e^{(\hat{d} + \hat{\mu})},
\]
which in turn implies
\[
\int e^{\frac{a}{8} \|x\|^2 - \hat{U}(x)} dx \leq e^{(\hat{d} + \hat{\mu}) + 2 \sum_i L_i R^{1+\alpha_i}}.
\]

Let $\mu_1 = \frac{\int e^{\frac{a}{8p} \|x\|^2 - \hat{U}(x)} dx}{\int e^{\frac{a}{8p} \|x\|^2 - \hat{U}(x)} d\mu_1}$ and assume that $\mu_2 = \frac{\int e^{\frac{a}{8p} \|x\|^2 - \hat{U}(x)} dx}{\int e^{\frac{a}{8p} \|x\|^2 - \hat{U}(x)} d\mu_2}$. We have $\mu_1$ is $\frac{a}{8p}$ strongly convex or log Sobolev with constant $\frac{a}{8p}$ and by Cauchy Schwarz inequality, we have
\[
\left\| \frac{d\mu_2}{d\mu_1} \right\|^p_{L^p(\mu_1)} = \left( \int e^{\frac{a}{8p} \|x\|^2} d\mu_1 \right)^{\frac{p}{2}} \left( \int e^{\frac{a}{8p} \|x\|^2} d\mu_1 \right)^{\frac{p}{2}} \leq \left( \int e^{\frac{a}{8p} \|x\|^2} d\mu_1 \right)^{\frac{p}{2}} \left( \int e^{\frac{a}{8p} \|x\|^2 - \hat{U}(x)} dx \right)^{\frac{p}{2}}\left( \int e^{\frac{a}{8p} \|x\|^2 - \hat{U}(x)} dx \right)^{\frac{p}{2}}.
\]

Since
\[
|U(x) - U(0)| = |U(x) - U(0) - \langle x, \nabla U(0) \rangle| \leq \sum_{i<N} \frac{L_i}{1 + \alpha_i} \|x\|^{1+\alpha_i} + \frac{L_N}{2} \|x\|^2
\]
this implies $U(x) \leq |U(0)| + \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} \|x\|^{1 + \alpha_i} + \frac{L_N}{2} \|x\|^2$ which in turn indicates
\[
\int e^{\frac{a}{|x|^p}} \|x\|^2 - \tilde{U}(x) \, dx \geq \int e^{\frac{a}{|x|^p}} \|x\|^2 - |U(0)| - \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} \|x\|^{1 + \alpha_i} - \frac{L_N}{2} \|x\|^2 - 2 \sum_i L_i R^{1 + \alpha_i} \, dx
\]
\[
\geq e^{-|U(0)| - 2 \sum_i L_i R^{1 + \alpha_i}} \int e^{\frac{a}{|x|^p}} \|x\|^2 - \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} \|x\|^{1 + \alpha_i} - \frac{L_N}{2} \|x\|^2 \, dx
\]
\[
\geq e^{-|U(0)| - 2 \sum_i L_i R^{1 + \alpha_i} - \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} + \frac{L_N}{2}} \int e^{\frac{a}{|x|^p}} \|x\|^2 \, dx
\]
\[
\geq \frac{\pi^{\frac{d}{2}}}{\left( \frac{a}{16p} + \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} + \frac{L_N}{2} \right)^{\frac{d}{2}}} e^{-|U(0)| - 2 \sum_i L_i R^{1 + \alpha_i}}.
\] (E.77)

On the other hand,
\[
\int e^{\frac{a}{|x|^p}} \|x\|^2 - \tilde{U}(x) \, dx \leq \int e^{\frac{a}{|x|^p}} \|x\|^2 - \tilde{U}(x) \, dx
\]
\[
\leq \int e^{\frac{a}{|x|^p}} \|x\|^2 - \tilde{U}(x) \, dx
\]
\[
\leq e^{(\tilde{d} + \tilde{\mu}) + 2 \sum_i L_i R^{1 + \alpha_i}}.
\] (E.78)

Combining this with previous inequality, we obtain
\[
\frac{\|d\mu_2\|^p}{\|d\mu_1\|_{L^p(\mu_1)}} \leq \left( \frac{\pi^{\frac{d}{2}}}{\left( \frac{a}{16p} + \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} + \frac{L_N}{2} \right)^{\frac{d}{2}}} e^{-|U(0)| - 2 \sum_i L_i R^{1 + \alpha_i}} \right)^{p + \frac{1}{2}} e^{(\tilde{d} + \tilde{\mu}) + 2 \sum_i L_i R^{1 + \alpha_i}}.
\] (E.79)

Taking logarithm of $\Lambda$ we get
\[
\log \Lambda = \frac{(p + \frac{1}{2})}{p} \log \left( \frac{\pi^{\frac{d}{2}}}{\left( \frac{a}{16p} + \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} + \frac{L_N}{2} \right)^{\frac{d}{2}}} e^{-|U(0)| - 2 \sum_i L_i R^{1 + \alpha_i}} \right)^{p + \frac{1}{2}} e^{(\tilde{d} + \tilde{\mu}) + 2 \sum_i L_i R^{1 + \alpha_i}}
\]
\[
= \frac{(p + \frac{1}{2})}{p} \left( \tilde{d} + \frac{d}{2} \log \left( \frac{a}{8p} + \sum_{i < N} \frac{L_i}{\Gamma + \alpha_i} + \frac{L_N}{2} \right) - \frac{d}{2} \log (\pi) \right)
\]
\[
+ \frac{(p + \frac{1}{2})}{p} \left( \tilde{\mu} + 2 \sum_i L_i R^{1 + \alpha_i} + |U(0)| + \sum_{i < N} \frac{L_i}{1 + \alpha_i} \right)
\]
\[
= \tilde{O}(d).
\] (E.80)
Since $\mu_2$ is log concave, from Lemma 9, we have for some universal constant $C$ (not depending on $d$), it is log Sobolev with constant

$$C(\Lambda, p) = \frac{1}{\Lambda} \frac{p - 1}{\log \Lambda} \frac{1}{C^{8p} p^{1 + O(d)}}$$

From this, by using Holley-Stroock perturbation theorem, we obtain $\mathcal{U}(x)$ is log Sobolev on $\mathbb{R}^d$ with constant $\frac{1}{\Lambda} e^{-2 \sum_i L_i R_i^{1+\alpha}}$. Now, applying theorem 3.1, we derive the desired result.

Appendix F: Proof of additional lemmas

Lemma F.13. For any $0 \leq \varpi \leq k \in \mathbb{N}^+$, we have

$$\|x + y\|^\varpi \leq 2^{k-1} (\|x\|^\varpi + \|y\|^\varpi).$$

Proof. Let’s consider functions $f_k(u) = 2^{k-1} (u^{\varpi} + 1) - (1 + u)^{\varpi}$. We prove $f_k(u) \geq 0$ for every $u \geq 0$ by induction. For $k = 1$, since $0 \leq \varpi \leq 1$, we have $f_1'(u) = \varpi u^{\varpi - 1} - \varpi (1 + u)^{\varpi - 1} \geq 0$. This implies $f_1(u)$ increases on $[0, \infty]$ and since $f(0) = 0$, which in turn indicates $f(u) \geq 0$. Therefore, the statement is true for $k = 1$.

Assume that it is true for $k = n$, we will show that it is also true for $k = n + 1$. If we differentiate $f_{n+1}(u)$ we get

$$f_{n+1}'(u) = 2^n \varpi u^{\varpi - 1} - \varpi (1 + u)^{\varpi - 1} = \varpi (2^n u^{\varpi - 1} - (1 + u)^{\varpi - 1}) \geq 0,$$

for $1 \leq \varpi \leq n + 1$ by induction assumption while for $0 \leq \varpi \leq 1$, $u^{\varpi - 1} - (1 + u)^{\varpi - 1} \geq u^{\varpi - 1} - (1 + u)^{\varpi - 1} \geq 0$. Hence, $f$ increases on $[0, \infty]$ and since $f(0) = 2^{k-1} - 1 \geq 0$, this implies $f \geq 0$.

Applying to our case for $0 \leq \varpi \leq k$,

$$2^{k-1} (\|x\|^\varpi + \|y\|^\varpi) = \|x\|^\varpi 2^{k-1} \left( 1 + \left( \frac{\|y\|}{\|x\|} \right)^\varpi \right) \geq \|x\|^\varpi \left( 1 + \left( \frac{\|y\|}{\|x\|} \right)^\varpi \right)^\varpi = (\|x\| + \|y\|)^\varpi \geq (\|x + y\|)^\varpi,$$

which conclude our proof.
Lemma F.14. For $\theta > 0$, $f(r) = m(r) r^2 = \mu (1 + r^2)^{-\theta} r^2 \geq \frac{\mu}{2} r^{2-\theta} - \frac{\mu_2}{2} \frac{r^2}{r^2}$, and for $\theta = 0$, $f(r) = \mu r^2$.

Proof. For $\theta = 0$, it is straightforward. For $\theta > 0$, from Lemma 2 above, for $r \geq 2^{\frac{1}{r}}$

$$f(r) = \mu (1 + r^2)^{-\theta} r^2 \\
\geq \mu (1 + r^2)^{-1} r^2 \\
= \mu (r^{2\theta} - 1)^{-1} r^2 (r^\theta - 1) \\
\geq \mu \mu^{2-2\theta} (r^\theta - 1) \\
\geq \frac{\mu}{2} r^{2-\theta}.$$ (F.4)

For $r < 2^{\frac{1}{r}}$, $f(r) \geq 0 \geq \mu \mu^{2-\theta} - \frac{\mu_2}{2} \frac{r^2}{r^2}$ which concludes statement. \(\square\)

Lemma F.15. $f(\theta) = \left| \left(1 + \|x\|^2 \right)^{\frac{\theta}{2}} - \left(1 + \|x - y\|^2 \right)^{\frac{\theta}{2}} \right|$ is increasing function.

Proof. If $\|x\| \geq \|x - y\|$, we have $f(\theta) = \left(1 + \|x\|^2 \right)^{\frac{\theta}{2}} - \left(1 + \|x - y\|^2 \right)^{\frac{\theta}{2}}$.

Differentiate $f$ with respect to $\theta$ gives

$$f'(\theta) = \frac{1}{2} \ln \left(1 + \|x\|^2 \right) \left(1 + \|x\|^2 \right)^{\frac{\theta}{2}} \\
- \frac{1}{2} \ln \left(1 + \|x - y\|^2 \right) \left(1 + \|x - y\|^2 \right)^{\frac{\theta}{2}} \\
\geq 0.$$ (F.5)

Similarly, if $\|x\| \leq \|x - y\|$ we also obtain $f'(\theta) \geq 0$, which implies that $f$ increases as desired. \(\square\)

Lemma F.16. If $\xi \sim N_p(0, I_d)$ then $d\|\xi\|_p \leq E(\|\xi\|_p^d) \leq [d + \frac{n}{2}] \frac{\theta}{2}$ where $[x]$ denotes the largest integer less than or equal to $x$. If $n = kp$, then $E(\|\xi\|_p^d) = d...(d + k - 1)$.

Proof. From [29], we have $E(\|\xi\|_p^d) = p\frac{\Gamma \left( \frac{d+n}{p} \right)}{\Gamma \left( \frac{n}{p} \right)}$.

Since $\Gamma$ is an increasing function,

$$\frac{\Gamma \left( \frac{d+n}{p} \right)}{\Gamma \left( \frac{n}{p} \right)} \geq \frac{\Gamma \left( \frac{n}{p} \right)}{\Gamma \left( \frac{n}{p} \right)} = \frac{\Gamma \left( \frac{d+n}{p} \right)}{\Gamma \left( \frac{n}{p} \right)} \geq d\|\xi\|_p.$$

If $n = kp$ for $k \in N$ then $E(\|\xi\|_p^d) = p\frac{\Gamma \left( \frac{n}{p} \right)}{\Gamma \left( \frac{n}{p} \right)} \geq d\|\xi\|_p$. If $n \neq kp$, let
\[ \left\lfloor \frac{n}{p} \right\rfloor = k. \] Since \( \Gamma \) is log-convex, by Jensen’s inequality for any \( p \geq 1 \), we acquire \[
abla \log \Gamma \left( \frac{d}{p} + \left\lfloor \frac{n}{p} \right\rfloor + 1 \right) \geq \log \Gamma \left( \frac{d+n}{p} \right) > 0.
\]
Raising \( e \) to the power of both sides, we get
\[
\Gamma \left( \frac{d}{p} \right)^{1 - \frac{n}{p \left\lfloor \frac{n}{p} \right\rfloor + p}} \Gamma \left( \frac{d}{p} + \left\lfloor \frac{n}{p} \right\rfloor + 1 \right)^{\frac{n}{p \left\lfloor \frac{n}{p} \right\rfloor + p}} \geq \Gamma \left( \frac{d+n}{p} \right),
\]
which implies that
\[
\left[ \frac{\Gamma \left( \frac{d+n}{p} \right)}{\Gamma \left( \frac{d}{p} \right)} \right]^{\frac{n}{p \left\lfloor \frac{n}{p} \right\rfloor + p}} \geq \frac{\Gamma \left( \frac{d+n}{p} \right)}{\Gamma \left( \frac{d}{p} \right)}.
\]
Combining with \( E(||\xi||_p^n) = p \frac{\Gamma \left( \frac{d+n}{p} \right)}{\Gamma \left( \frac{d}{p} \right)} \) gives the conclusion. \( \square \)

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