Uncertainty Principle for the Two-Sided Quaternion Windowed Linear Canonical Transform

Wen-Biao Gao · Bing-Zhao Li

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Abstract
In this paper, we investigate the (two-sided) quaternion windowed linear canonical transform (QWLCT) and study the uncertainty principles associated with the QWLCT. First, several important properties of the QWLCT such as bounded, shift, modulation and orthogonality relations are presented based on the spectral representation of the quaternionic linear canonical transform (QLCT). Second, Pitt’s inequality and the Lieb inequality for the QWLCT are explored. Moreover, we study different kinds of uncertainty principles for the QWLCT, such as the logarithmic uncertainty principle, the entropic uncertainty principle, the Lieb uncertainty principle and Donoho–Stark’s uncertainty principle. Finally, we provide a numerical example and a potential application to signal recovery by using Donoho–Stark’s uncertainty principle associated with the QWLCT.

Keywords Quaternion Fourier transform · Quaternion linear canonical transform · Quaternion windowed linear canonical transform · Uncertainty principle · Signal recovery

1 Introduction
The linear canonical transform (LCT) plays an important role in chirp signal analysis in parameter estimation, applied mathematics, signal processing, radar system analysis, filter design, phase retrieval, and pattern recognition and optics [15,37,38,41,43].

Bing-Zhao Li
li_bingzhao@bit.edu.cn
Wen-Biao Gao
wenbiaogao@163.com

1 School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 102488, P.R. China
2 Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 102488, P.R. China
However, the LCT cannot show the local LCT frequency contents as a result of its global kernel. To achieve this goal, the LCT was successfully applied to research on generalized windowed functions in [29,30]. The windowed linear canonical transform (WLCT) is a method devised to study signals whose spectral content changes with time. In [4,16,22,29,30,46], some important properties of the WLCT, such as the analogue of the Poisson summation formula, sampling formulas, the Paley–Wiener theorem and the dual window solution, were discussed. The discrete WLCT was discussed in [44]. It represents the time and LCT frequency information and was originally a local LCT distribution.

In the last few years, some authors have generalized the LCT to quaternion-valued signals, and this is known as the quaternionic linear canonical transform (QLCT). It transforms a quaternionic 2D signal into a quaternion-valued frequency-domain signal which is an effective processing tool for colour image analysis. The QLCT was first studied in [3] including prolate spheroidal wave signals and uncertainty principles. Some useful properties of the QLCT, such as linearity, the reconstruction formula, continuity, boundedness, the positivity inversion formula and the uncertainty principle, were established in [20,25–28,45]. An application of the QLCT to the study of generalized swept-frequency filters was introduced in [2]. According to the QLCT, Xiong and Fu [39] proposed the two-sided quaternion linear canonical transform associated with the real window function, called the quaternion window linear canonical transform (QWLCT), and they studied several properties and Heisenberg’s uncertainty principle for the QWLCT [39]. Unlike [39], based on the quaternion window function, we [17] investigated some properties of the two-sided QWLCT and provided an example to illustrate that the QWLCT is highly effective.

On the other hand, the uncertainty principle plays a vital role in time–frequency signal analysis and quantum mechanics [10,18,19,33–36]. It was first proposed by the German physicist W. Heisenberg in 1927 [19]. In quantum mechanics, the uncertainty principle asserts that one cannot be certain of the position and velocity of an electron (or any particle) at the same time. That is, increasing the knowledge of the position decreases the knowledge of the velocity or momentum of an electron. In signal processing, the uncertainty principle states that the product of the variances of a signal in the time and frequency domains has a lower bound. There are many different kinds of uncertainty principle, such as the Heisenberg uncertainty principle [9,17,19,23,39,40], the logarithmic uncertainty principle [8], Hardy’s uncertainty principle [9,21], Beurling’s uncertainty principle [47], the Lieb uncertainty principle [7,29] and Donoho–Stark’s uncertainty principle [1,11,31]. The Lieb uncertainty principle for the WLCT was discussed in [29], which takes the LCT version as one of its special cases. In [40], Xu et al. studied the uncertainty relation for the WLCT in two WLCT domains. Huang et al. [23] discussed the uncertainty principle and the orthogonal condition for the WLCT in one WLCT domain. In [39], based on the real window function, Heisenberg’s uncertainty principle for the QWLCT was obtained.

However, to the best of our knowledge, other kinds of uncertainty principle for the QWLCT have not been found in the literature. The aim of this paper is to obtain several uncertainty principles for the QWLCT. First, some properties of the QWLCT are reviewed. Second, based on the relationship between the QWLCT and the quaternion Fourier transform (QFT) [5,6,12,13,24,42], we establish Pitt’s inequality and the
Lieb inequality associated with the QWLCT. Third, the uncertainty principles for the QWLCT, such as the logarithmic uncertainty principle, the entropic uncertainty principle, the Lieb uncertainty principle and Donoho–Stark’s uncertainty principle, are obtained. Finally, a numerical example and a potential application in signal recovery associated with Donoho–Stark’s uncertainty principle are given.

The paper is organized as follows: Sect. 2 gives a brief introduction to some general definitions and basic properties of quaternion algebra, and QLCTs of 2D quaternion-valued signals. We present the definition and the properties of the QWLCT in Sect. 3. Section 4 provides the uncertainty principles associated with the QWLCT. We give a numerical example and a potential application in Sect. 5. In Sect. 6, some conclusions are drawn.

2 Preliminaries

In this section, we mainly review some basic facts of quaternion algebra and the QLCT, which will be needed throughout the paper.

2.1 Quaternion Algebra

Quaternion algebra is an extension of the complex number to 4D algebra. It was first invented by W. R. Hamilton in 1843 and is classically denoted as \( \mathbb{H} \) in his honour. Every element of \( \mathbb{H} \) has a Cartesian form given by

\[
\mathbb{H} = \{ q | q := [q]_0 + i[q]_1 + j[q]_2 + k[q]_3, [q]_i \in \mathbb{R}, i = 0, 1, 2, 3 \} \tag{1}
\]

where \( i, j, k \) are imaginary units obeying Hamilton’s multiplication rules (see [12])

\[
i^2 = j^2 = k^2 = -1, \\
i j = -j i = k, j k = -k j = i, k i = -i k = j.
\tag{2}
\]

Let \( [q]_0 \) and \( q = i[q]_1 + j[q]_2 + k[q]_3 \) denote the real scalar part and the vector part of quaternion number \( q = [q]_0 + i[q]_1 + j[q]_2 + k[q]_3 \), respectively. Then, from [12], the real scalar part has cyclic multiplication symmetry

\[
[pql]_0 = [qlp]_0 = [lqp]_0, \quad \forall q, p, l \in \mathbb{H}, \tag{3}
\]

the conjugate of a quaternion \( q \) is defined by \( \overline{q} = [q]_0 - i[q]_1 - j[q]_2 - k[q]_3 \) and the norm of \( q \in \mathbb{H} \) defined as

\[
|q| = \sqrt{q \overline{q}} = \sqrt{[q]_0^2 + [q]_1^2 + [q]_2^2 + [q]_3^2}. \tag{4}
\]

It is easy to verify that

\[
\overline{pq} = \overline{q} \overline{p}, |qp| = |q||p|, \quad \forall q, p \in \mathbb{H}. \tag{5}
\]
If \( 1 \leq s < \infty \), the quaternion modules \( L^s(\mathbb{R}, \mathbb{H}) \) are defined as

\[
L^s(\mathbb{R}, \mathbb{H}) := \left\{ f : \mathbb{R} \to \mathbb{H}, \| f \|_{L^s(\mathbb{R}, \mathbb{H})} = \left( \int_{\mathbb{R}} |f(x)|^s \, dx \right)^{\frac{1}{s}} < \infty \right\}. \tag{6}
\]

For \( s = \infty \), \( L^\infty(\mathbb{R}, \mathbb{H}) \) is a collection of essentially bounded measurable functions with the norm

\[
\| f \|_{L^\infty(\mathbb{R}, \mathbb{H})} = \text{ess sup} |f(x)|,
\]

if \( f \in L^\infty(\mathbb{R}, \mathbb{H}) \) is continuous then

\[
\| f \|_{L^\infty(\mathbb{R}, \mathbb{H})} = \sup |f(x)|.
\]

Now, we introduce an inner product of quaternion functions \( f \) and \( g \) defined on \( L^2(\mathbb{R}, \mathbb{H}) \), as given by

\[
(f, g)_{L^2(\mathbb{R}, \mathbb{H})} = \int_{\mathbb{R}} f(x)\overline{g(x)} \, dx, \tag{7}
\]

with symmetric real scalar part

\[
\langle f, g \rangle = \frac{1}{2} \left( (f, g) + (g, f) \right) = \int_{\mathbb{R}} [f(x)\overline{g(x)}]_0 \, dx, \tag{8}
\]

where \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( dx = dx_1 \, dx_2 \).

The associated scalar norm of \( f(x) \in L^2(\mathbb{R}, \mathbb{H}) \) is defined by both (7) and (8):

\[
\| f \|_{L^2(\mathbb{R}, \mathbb{H})}^2 = \langle f, f \rangle_{L^2(\mathbb{R}, \mathbb{H})} = \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty. \tag{9}
\]

The convolution of \( f \in L^2(\mathbb{R}, \mathbb{H}) \) and \( g \in L^2(\mathbb{R}, \mathbb{H}) \), denoted by \( f \ast g \), is defined by

\[
(f \ast g)(y) = \int_{\mathbb{R}} f(x)g(y-x) \, dx,
\]

where \( y = (y_1, y_2) \in \mathbb{R}^2 \).

**Lemma 1** If \( f, g \in L^2(\mathbb{R}, \mathbb{H}) \), then the Cauchy–Schwarz inequality holds \([35]\)

\[
|\langle f, g \rangle_{L^2(\mathbb{R}, \mathbb{H})}|^2 \leq \| f \|^2_{L^2(\mathbb{R}, \mathbb{H})} \| g \|^2_{L^2(\mathbb{R}, \mathbb{H})}. \tag{10}
\]

If and only if \( f = \lambda g \) for some quaternionic parameter \( \lambda \in \mathbb{H} \), the equality holds.
2.2 The Quaternion Linear Canonical Transform

The QLCT was first defined by Kou et al., and it is a generalization of the LCT in the framework of quaternion algebra [12,24]. Due to the noncommutativity of quaternion multiplication, there are three different types of QLCT: the left-sided QLCT, the right-sided QLCT and the two-sided QLCT. In this paper, we mainly focus on the two-sided QLCT.

**Definition 1** Let \( A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) be a matrix parameter satisfying \( \det(A_i) = 1 \), for \( i = 1, 2 \). The two-sided QLCT of signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) is defined by

\[
\mathcal{L}_{A_1, A_2} \{ f \}(w) = \int_{\mathbb{R}^2} K^i_{A_1}(x_1, \omega_1) f(x) K^j_{A_2}(x_2, \omega_2) dx,
\]

where \( w = (\omega_1, \omega_2) \in \mathbb{R}^2 \) is regarded as the QLCT domain, and the kernel signals \( K^i_{A_1}(x_1, \omega_1) \), \( K^j_{A_2}(x_2, \omega_2) \) are, respectively, given by

\[
K^i_{A_1}(x_1, \omega_1) := \begin{cases} 
\frac{1}{\sqrt{2\pi b_1}} e^{j \left( \frac{a_1}{b_1} x_1 \omega_1 - \frac{c_1}{b_1} \omega_1^2 - \frac{\pi}{4} \right)}, & b_1 \neq 0 \\
\sqrt{d_1} e^{j \frac{c_1 d_1}{2} \omega_1^2} \delta(x_1 - d_1 w_1), & b_1 = 0
\end{cases}
\]

and

\[
K^j_{A_2}(x_2, \omega_2) := \begin{cases} 
\frac{1}{\sqrt{2\pi b_2}} e^{j \left( \frac{a_2}{b_2} x_2 \omega_2 - \frac{c_2}{b_2} \omega_2^2 - \frac{\pi}{4} \right)}, & b_2 \neq 0 \\
\sqrt{d_2} e^{j \frac{c_2 d_2}{2} \omega_2^2} \delta(x_2 - d_2 w_2), & b_2 = 0
\end{cases}
\]

where \( \delta(x) \) representing the Dirac function.

From the above definition, it is noted that for \( b_i = 0, i = 1, 2 \), the QLCT of a signal is a kind of scaling and chirp multiplication operation, and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set \( b_i \neq 0 \) in the next section unless stated otherwise. Under some suitable conditions, the above QLCT is invertible and the inversion is given in the next section.

**Lemma 2** Suppose \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), then the inversion of the QLCT of \( f \) is given by

\[
f(x) = \mathcal{L}_{A_1, A_2}^{-1} \{ \mathcal{L}_{A_1, A_2} \{ f \} \}(x)
= \int_{\mathbb{R}^2} K^{-i}_{A_1}(x_1, \omega_1) \mathcal{L}_{A_1, A_2} \{ f \}(w) K^{-j}_{A_2}(x_2, \omega_2) dw.
\]

3 Quaternionic Windowed Linear Canonical Transform (QWLCT)

In this section, the generalization of the window function associated with the QLCT is discussed, which is denoted as the QWLCT. Moreover, several of their basic properties of them are presented.
3.1 The Definition of 2D QWLCT

This subsection leads to a 2D quaternion window function associated with the QLCT. Due to the noncommutative property of the multiplication of quaternions, there are three different types of QWLCTs: the two-sided QWLCT, left-sided QWLCT and right-sided QWLCT. Alternatively, we use the two-sided QWLCT to define the QWLCT.

Definition 2 (QWLCT) \[17,39\] Let \( A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) be a matrix parameter satisfying \( \det(A_i) = 1 \), for \( i = 1, 2 \). Let \( \phi \in L^2(\mathbb{R}^2, \mathbb{H})\setminus\{0\} \) be a quaternion window function. The two-sided QWLCT of a signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) with respect to \( \phi \) is defined by

\[
G^{A_1,A_2}_\phi \{f\}(w, u) = \int_{\mathbb{R}^2} K^i_{A_1}(x_1, \omega_1) f(x) \bar{\phi}(x - u) K^j_{A_2}(x_2, \omega_2) dx,
\]

where \( u = (u_1, u_2) \in \mathbb{R}^2 \), \( K^i_{A_1}(x_1, \omega_1) \) and \( K^j_{A_2}(x_2, \omega_2) \) are given by (12) and (13), respectively.

For a fixed \( u \), we have

\[
G^{A_1,A_2}_\phi \{f\}(w, u) = \mathcal{L}_{A_1,A_2} \{f(x)\bar{\phi}(x - u)\}(w).
\]

Applying the inverse QLCT to (16), we have

\[
f_u(x) = f(x)\bar{\phi}(x - u) = \mathcal{L}_{A_1,A_2}^{-1}\{G^{A_1,A_2}_\phi \{f\}(w, u)\}
\]

\[
= \int_{\mathbb{R}^2} K^{-i(x_1,\omega_1)}_{A_1} G^{A_1,A_2}_\phi \{f\}(w, u) K^{-j(x_2,\omega_2)}_{A_2} dw,
\]

where \( f_u(x) \) is called the modified signal.

Moreover, we can also obtain the relationship between the QWLCT and the QFT:

\[
G^{A_1,A_2}_\phi \{f\}(w, u) = \frac{1}{\sqrt{2\pi}b_1} e^{i \left( \frac{d_1}{2b_1^2} \omega_1^2 - \frac{\pi}{4} \right)} F_Q(h) \left( \frac{w}{b}, u \right) \frac{1}{\sqrt{2\pi}b_2} e^{i \left( \frac{d_2}{2b_2^2} \omega_2^2 - \frac{\pi}{4} \right)},
\]

where \( b = (b_1, b_2) \in \mathbb{R}^2 \), \( F_Q(f)(w) = \int_{\mathbb{R}^2} e^{-i \omega_1 x_1} f(x) e^{-i \omega_2 x_2} dx \) is the QFT of the signal \( f(x) \) and

\[
h(x, u) = e^{i \frac{d_1}{2b_1^2} x_1^2} f(x) \bar{\phi}(x - u) e^{i \frac{d_2}{2b_2^2} x_2^2}
\]

\[
= e^{i \frac{d_1}{2b_1^2} x_1^2} f_u(x) e^{i \frac{d_2}{2b_2^2} x_2^2}.
\]

3.2 Some Properties of QWLCT

In this subsection, we present several basic properties of the QWLCT. These properties were proved in \[17,39\].
Property 1 (Boundedness) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then

$$|G_{\phi}^{A_1,A_2} \{ f \} (w, u)| \leq \frac{1}{2\pi \sqrt{|b_1b_2|}} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| \phi \|_{L^2(\mathbb{R}^2, \mathbb{H})}.$$  \hfill (20)

Furthermore, we have

$$\| G_{\phi}^{A_1,A_2} \{ f \} (w, u) \|_{L^\infty(\mathbb{R}^2, \mathbb{H})} \leq \frac{1}{2\pi \sqrt{|b_1b_2|}} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| \phi \|_{L^2(\mathbb{R}^2, \mathbb{H})}.$$  \hfill (21)

Property 2 (Linearity) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function and $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$. The QWLCT is a linear operator, namely

$$[G_{\phi}^{A_1,A_2} \{ \lambda f + \mu g \}](w, u) = \lambda G_{\phi}^{A_1,A_2} \{ f \}(w, u) + \mu G_{\phi}^{A_1,A_2} \{ g \}(w, u).$$  \hfill (22)

for arbitrary real constants $\lambda$ and $\mu$.

Property 3 (Parity) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we have

$$G_{\phi}^{A_1,A_2} \{ Pf \} (w, u) = G_{\phi}^{A_1,A_2} \{ f \} (-w, -u),$$  \hfill (23)

where $P \phi(x) = \phi(-x)$ for every window function $\phi \in L^2(\mathbb{R}^2, \mathbb{H})$.

Property 4 (Shift) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we have

$$G_{\phi}^{A_1,A_2} \{ T_r f \} (w, u) = e^{i\pi \omega_1 \pi_1} e^{-i \frac{\pi \pi_1^2}{2} \pi_1} G_{\phi}^{A_1,A_2} \{ f \}(m, n) e^{i\pi \omega_2 \pi_2} e^{-i \frac{\pi \pi_2^2}{2} \pi_2},$$  \hfill (24)

where $T_r f(x) = f(x - r), r = (r_1, r_2), m = (m_1, m_2), n = (n_1, n_2) \in \mathbb{R}^2, m_i = w_i - a_i r_i, n_i = u_i - r_i, i = 1, 2$.

Property 5 (Modulation) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Let $f$ be modulation operator defined by $M_s f(x) = e^{i\omega_1 \pi_1} f(x) e^{i\omega_2 \pi_2}$ with $s = (s_1, s_2) \in \mathbb{R}^2$. Then we have

$$G_{\phi}^{A_1,A_2} \{ M_s f \} (w, u) = e^{i\pi \omega_1 \pi_1} e^{-i \frac{\pi \pi_1^2}{2} \pi_1} G_{\phi}^{A_1,A_2} \{ f \}(v, u) e^{i\pi \omega_2 \pi_2} e^{-i \frac{\pi \pi_2^2}{2} \pi_2},$$  \hfill (25)

where $v = (v_1, v_2) \in \mathbb{R}^2, v_i = w_i - s_i b_i, i = 1, 2$.

Property 6 (Inversion formula) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function, $0 < \| \phi \|^2 < \infty$ and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we have the inversion formula of the QWLCT,

$$f(x) = \frac{1}{\| \phi \|^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^{-i(x_1, \omega_1)} G_{\phi}^{A_1,A_2} \{ f \}(w, u) K_{A_2}^{-i(x_2, \omega_2)} \phi(x - u) dw du.$$  \hfill (26)

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Property 7 (Parseval’s theorem) Let $\phi, \psi \in L^2(\mathbb{R}^2, \mathbb{H})\setminus\{0\}$ be window functions and $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$. Then
\[
\langle G_{\phi}^{A_1, A_2} \{f\} (w, u), G_{\psi}^{A_1, A_2} \{g\} (w, u) \rangle = [(f, g)(\phi, \psi)]_0.
\] (27)

Based on the above theorem, we may infer the following important consequences.

(i) If $\phi = \psi$, then
\[
\langle G_{\phi}^{A_1, A_2} \{f\} (w, u), G_{\phi}^{A_1, A_2} \{g\} (w, u) \rangle = \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \langle f, g \rangle.
\] (28)

(ii) If $f = g$, then
\[
\langle G_{\phi}^{A_1, A_2} \{f\} (w, u), G_{\psi}^{A_1, A_2} \{f\} (w, u) \rangle = \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \langle \phi, \psi \rangle.
\] (29)

(iii) If $f = g$ and $\phi = \psi$, then
\[
\langle G_{\phi}^{A_1, A_2} \{f\} (w, u), G_{\phi}^{A_1, A_2} \{f\} (w, u) \rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1, A_2} \{f\} (w, u)|^2 \, dw \, du
\]
\[
= \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
\] (30)

3.3 Pitt’s Inequality

Lemma 3 (Pitt’s inequality of the QFT [7]) For $f \in S(\mathbb{R}^t, \mathbb{H})$,
\[
\int_{\mathbb{R}^t} |w|^{-\alpha} |F_Q(f)(w)|^2 \, dw \leq M_\alpha \int_{\mathbb{R}^t} |x|^{\alpha} |f(x)|^2 \, dx,
\] (31)

where $M_\alpha = \pi^\alpha \left[ \Gamma \left( \frac{2t-\alpha}{4} \right) \Gamma \left( \frac{2t+\alpha}{4} \right) \right]$, $0 \leq \alpha \leq t$ and $S(\mathbb{R}^t, \mathbb{H})$ denotes the Schwartz class and $\Gamma$ is the gamma function.

According to the above lemma, we obtain Pitt’s inequality of the QWLCT.

Theorem 1 (Pitt’s inequality of the QWLCT) For $f \in S(\mathbb{R}^t, \mathbb{H})$,
\[
\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G_{\phi}^{A_1, A_2} \{f\} (w, u)|^2 \, du \, dw
\]
\[
\leq \frac{M_\alpha \|\phi\|_{L^2(\mathbb{R}^t, \mathbb{H})}^2}{4\pi^2 |b|^{\alpha}} \int_{\mathbb{R}^t} |x|^{\alpha} |f(x)|^2 \, dx,
\] (32)

where $M_\alpha = \pi^\alpha \left[ \Gamma \left( \frac{2t-\alpha}{4} \right) \Gamma \left( \frac{2t+\alpha}{4} \right) \right]$, $0 \leq \alpha \leq t$ and $\Gamma$ is the gamma function.
Proof  From (18), we have

\[
\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G_{A1,A2}^* \{ f \}(w, u)|^2 \text{d}u \text{d}w
\]

\[
= \frac{1}{4\pi^2|b|} \int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |F_Q(h) \left( \frac{w}{b}, u \right)|^2 \text{d}u \text{d}w.
\]

Let \( \frac{w}{b} = \varphi \), then

\[
\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G_{A1,A2}^* \{ f \}(w, u)|^2 \text{d}u \text{d}w
\]

\[
= \frac{1}{4\pi^2|\varphi|} \int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |\varphi|^{-\alpha} |F_Q(h) (\varphi, u)|^2 \text{d}u \text{d}w. \]

By Lemma 3, we obtain

\[
\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G_{A1,A2}^* \{ f \}(w, u)|^2 \text{d}u \text{d}w
\]

\[
\leq \frac{1}{4\pi^2|\varphi|^\alpha} M_\alpha \int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |\varphi|^\alpha |h(x)|^2 \text{d}u \text{d}x.
\]

Using (19), the above formula becomes that

\[
\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G_{A1,A2}^* \{ f \}(w, u)|^2 \text{d}u \text{d}w
\]

\[
\leq \frac{1}{4\pi^2|\varphi|^\alpha} M_\alpha \int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |\varphi|^\alpha |f(x)|^2 \left( \int_{\mathbb{R}^t} |\varphi| |h(x)|^2 \text{d}u \right) \text{d}x.
\]

According to (6), then

\[
\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G_{A1,A2}^* \{ f \}(w, u)|^2 \text{d}u \text{d}w
\]

\[
\leq \frac{1}{4\pi^2|\varphi|^\alpha} M_\alpha \| \varphi \|_{L^2(\mathbb{R}^t, \mathbb{R}^t)}^2 \int_{\mathbb{R}^t} |\varphi|^\alpha |f(x)|^2 \text{d}x.
\]

\[\square\]

Corollary 1 When \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), Pitt’s inequality for the QWLCT leads to Pitt’s inequality for the two-sided quaternion windowed Fourier transform (QWFT) [5,7], i.e.

\[
\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G_{\phi} f(w, u)|^2 \text{d}u \text{d}w \leq M_\alpha \| \phi \|_{L^2(\mathbb{R}^t, \mathbb{R}^t)}^2 \int_{\mathbb{R}^t} |\varphi|^\alpha |f(x)|^2 \text{d}x,
\]

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where $G_{\phi}(f)(w, u)$ is the QWFT of the signal $f$.

**Corollary 2** When $\phi = 1$ and $A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, Pitt’s inequality for the QWLCT leads to Pitt’s inequality for the QFT.

**Remark 1** When the window function is normalized, namely $\|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})} = 1$, then (32) implies

$$\int_{\mathbb{R}^t} \int_{\mathbb{R}^t} |w|^{-\alpha} |G^{A_1, A_2}_\phi(f)(w, u)|^2 dudw \leq \frac{1}{4\pi^2 |b|^\alpha} M_\alpha \int_{\mathbb{R}^t} |x|^{\alpha} |f(x)|^2 dx.$$

### 3.4 Lieb Inequality

**Theorem 2** Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be window functions. For every $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $s \geq 2$. Then

$$\|G^{A_1, A_2}_\phi(f)\|_{L^s(\mathbb{R}^4, \mathbb{H})} \leq \frac{|b|^{\frac{1}{s} - \frac{1}{s}'} D_s, s' \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}, \quad (33)$$

where $D_s, s' = \left(\frac{4}{s}\right)^\frac{1}{s} \left(\frac{4}{s'}\right)^{\frac{1}{s'}}$ and $\frac{1}{s} + \frac{1}{s'} = 1$.

**Proof** According to (18) and Hausdorff–Young theorem, we have

$$\left(\int_{\mathbb{R}^2} |G^{A_1, A_2}_\phi(f)(w, u)|^s dw \right)^{\frac{1}{s}} = \frac{1}{2\pi |b|^{\frac{1}{2}}} \left(\int_{\mathbb{R}^2} |F_Q(h) \left(\frac{w}{b}, u\right)|^s dw \right)^{\frac{1}{s}} \leq \frac{1}{2\pi |b|^{\frac{1}{2}}} \left(\int_{\mathbb{R}^2} |h(x, u)|^s' dx \right)^{\frac{1}{s'}} = \frac{|b|^{\frac{1}{s} - \frac{1}{s}'} D_s, s' \|f\|_{L^s(\mathbb{R}^2, \mathbb{H})}^{\frac{1}{s'}} \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}}{2\pi},$$

where $\frac{1}{s} + \frac{1}{s'} = 1$, $s \geq 2$ and $\tilde{\phi}(x) = \phi(-x)$.

$$\|G^{A_1, A_2}_\phi(f)\|_{L^s(\mathbb{R}^4, \mathbb{H})} \leq \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |G^{A_1, A_2}_\phi(f)(w, u)|^s dw \right) du \right)^{\frac{1}{s'}} \leq \frac{|b|^{\frac{1}{s} - \frac{1}{s}'} D_s, s' \|f\|_{L^s(\mathbb{R}^2, \mathbb{H})}^{\frac{1}{s'}} \|\tilde{\phi}\|_{L^{s'}(\mathbb{R}^2, \mathbb{H})}^{\frac{1}{s'}}}{2\pi}.$$
If \( k = \frac{s}{s'} \), \( l = \frac{s'}{s} \), and \( \frac{1}{k} + \frac{1}{s'} = 1 = \frac{1}{l} + \frac{1}{s} \), then \( s' = \frac{k}{1 + k} = \frac{l}{1 + l} \), and hence, as \( |f|^{s'} \), \( |\tilde{\phi}|^{s'} \in L^k(\mathbb{R}^2, \mathbb{H}) \) and by Young inequality, we have

\[
\| |f|^{s'} \ast |\tilde{\phi}|^{s'} \|_{L^k(\mathbb{R}^2, \mathbb{H})} \leq B_k^4 B_{s'}^2 \| |f|^{s'} \|_{L^k(\mathbb{R}^2, \mathbb{H})} \| |\tilde{\phi}|^{s'} \|_{L^k(\mathbb{R}^2, \mathbb{H})},
\]

where \( B_s = \left( \frac{s}{s'} \right)^{\frac{1}{2}} \), \( \frac{1}{s} + \frac{1}{s'} = 1 \).

On the other hand,

\[
\| |f|^{s'} \|_{L^k(\mathbb{R}^2, \mathbb{H})} = \left( \int_{\mathbb{R}^2} |f(x)|^{s'} \frac{2}{s'} \, dx \right)^{\frac{1}{s'}} = \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})},
\]

and

\[
\| |\tilde{\phi}|^{s'} \|_{L^k(\mathbb{R}^2, \mathbb{H})} = \left( \int_{\mathbb{R}^2} |\tilde{\phi}(x-u)|^{s'} \frac{2}{s'} \, dx \right)^{\frac{1}{s'}} = \| \phi \|_{L^2(\mathbb{R}^2, \mathbb{H})}.
\]

Hence, we obtain

\[
\| G_{A_1,A_2}^1 \{ f \} \|_{L^*(\mathbb{R}^4, \mathbb{H})} \leq \frac{|b|^{\frac{1}{2} - \frac{1}{2}}}{2\pi} \| |f|^{s'} \ast |\tilde{\phi}|^{s'} \|_{L^k(\mathbb{R}^2, \mathbb{H})}^{\frac{1}{s'}}
\]

\[
\leq \frac{|b|^{\frac{1}{2} - \frac{1}{2}}}{2\pi} \left( B_k^4 B_{s'}^2 \| |f|^{s'} \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| |\tilde{\phi}|^{s'} \|_{L^2(\mathbb{R}^2, \mathbb{H})} \right)^{\frac{1}{s'}}
\]

\[
= \frac{|b|^{1 - \frac{1}{2}}}{2\pi} B_k^4 B_{s'}^2 \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| \phi \|_{L^2(\mathbb{R}^2, \mathbb{H})}
\]

\[
= \frac{|b|^{\frac{1}{2} - \frac{1}{2}}}{2\pi} D_{s,s'} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| \phi \|_{L^2(\mathbb{R}^2, \mathbb{H})}.
\]

\[\square\]

**Corollary 3** When \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the Lieb inequality for the QWLCT becomes the Lieb inequality for the QWFT [6].

**Remark 2** When \( \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} = \| \phi \|_{L^2(\mathbb{R}^2, \mathbb{H})} = 1 \), then (33) implies

\[
\| G_{A_1,A_2}^1 \{ f \} \|_{L^*(\mathbb{R}^4, \mathbb{H})} \leq \frac{|b|^{\frac{1}{2} - \frac{1}{2}}}{2\pi} D_{s,s'}.
\]
4 Uncertainty Principles for the QWLCT

4.1 Logarithmic Uncertainty Principle

Based on Pitt’s inequality, the logarithmic uncertainty principle for the QFT was proved in [8]. In this subsection, applying the logarithmic uncertainty principle for the QFT, we study the logarithmic uncertainty principle for the QWLCT.

Lemma 4 (Logarithmic uncertainty principle for the QFT) [8] For \( f \in S(\mathbb{R}^2, H) \),

\[
\int_{\mathbb{R}^2} \ln |x||f(x)|^2 \, dx + \int_{\mathbb{R}^2} \ln |w||F_Q(f)(w)|^2 \, dw \geq \Delta \int_{\mathbb{R}^2} |f(x)|^2 \, dx,
\]

where \( \Delta = \phi \left( \frac{1}{2} \right) - \ln(\pi), \quad \phi(t) = \frac{\Gamma'(t)}{\Gamma(t)} \).

Now we arrive at the following result.

Theorem 3 (Logarithmic uncertainty principle for the QWLCT) For \( f, \phi \in S(\mathbb{R}^2, H) \),

\[
\frac{\|\phi\|^2_{L^2(\mathbb{R}^2, H)}}{4\pi^2} \int_{\mathbb{R}^2} \ln |x||f(x)|^2 \, dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w||G_{\phi}^{A_1, A_2}(f)(w, u)|^2 \, dw \, du \geq \frac{(\Delta + \ln |b|)}{4\pi^2} \|f\|^2_{L^2(\mathbb{R}^2, H)} \|\phi\|^2_{L^2(\mathbb{R}^2, H)},
\]

where \( \Delta = \phi \left( \frac{1}{2} \right) - \ln(\pi), \quad \phi(t) = \frac{\Gamma'(t)}{\Gamma(t)} \).

Proof According to (18), we have

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w||G_{\phi}^{A_1, A_2}(f)(w, u)|^2 \, dw \, du
= \frac{1}{4\pi^2} |b| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w||F_Q(h) \left( \frac{w}{b} \cdot u \right)|^2 \, dw \, du
= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |by||F_Q(h)(y, u)|^2 \, dy \, du
= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |b||F_Q(h)(y, u)|^2 \, dy \, du
+ \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |y||F_Q(h)(y, u)|^2 \, dy \, du,
\]
from Parseval’s formula for QFT, (36) becomes

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| |G^A_\phi (f)(w, u)|^2 dw \, du \\
= \frac{1}{4\pi^2} \ln |b| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |h(w, u)|^2 dw du + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |y| |F_Q(h)(y, u)|^2 dy du \\
= \frac{\ln |b|}{4\pi^2} \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |y| |F_Q(h)(y, u)|^2 dy du.
\]

(37)

On the other hand, because \( f, \phi \in S(\mathbb{R}^2, \mathbb{H}) \), which implies \( h(x, u) \in S(\mathbb{R}^2, \mathbb{H}) \). Therefore, according to Lemma 4, we have

\[
\int_{\mathbb{R}^2} \ln |x| |h(x)|^2 dx + \int_{\mathbb{R}^2} \ln |w| |F_Q(h)(w)|^2 dw \geq \Delta \int_{\mathbb{R}^2} |h(x)|^2 dx,
\]

(38)

let \( h(x, u) = e^{i\frac{d_1}{x_1}} f(x, \phi(x - u)) e^{i\frac{d_2}{x_2}} \), multiply both sides of (38) by \( \frac{1}{4\pi^2} \) and integrating both sides of (38) with respect to \( du \), we have

\[
\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| |h(x)|^2 dx du + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| |F_Q(h)(w)|^2 dw du \\
\geq \frac{1}{4\pi^2} \Delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |h(x)|^2 dx du,
\]

(39)

that is

\[
\frac{\|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2}{4\pi^2} \int_{\mathbb{R}^2} \ln |x| |f(x)|^2 dx + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| |F_Q(h)(w)|^2 dw du \\
\geq \frac{\Delta}{4\pi^2} \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
\]

(40)

Inserting (37) into (40), we obtain

\[
\frac{\|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2}{4\pi^2} \int_{\mathbb{R}^2} \ln |x| |f(x)|^2 dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| |G^A_\phi f(w, u)|^2 dw du \\
\geq \frac{\Delta + \ln |b|}{4\pi^2} \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
\]

\[\square\]

**Corollary 4** When \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the logarithmic uncertainty principle for the QWLCT becomes the logarithmic uncertainty principle for the QWFT [7].

**Corollary 5** Suppose that \( \phi = 1 \) and \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the logarithmic uncertainty principle of the QWLCT becomes Lemma 4.
4.2 Entropic Uncertainty Principle

In this subsection, we present the entropic uncertainty principle for the QWLCT. The entropic uncertainty principle is very important in signal processing and harmonic analysis. Its locality is measured by Shannon entropy. This entropy represents an advantageous way to measure the decay of a function.

**Definition 3** The Shannon entropy of $p$ is denoted as

$$E(p) = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(p(w, u)) p(w, u) \, dw \, du,$$

where $p$ is a probability density function on $\mathbb{R}^2 \times \mathbb{R}^2$.

**Theorem 4** Let $\phi$ be a quaternion window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$ such that $f, \phi \neq 0$ and $|b_1 b_2| \geq \frac{1}{4\pi^2}$. Then

$$E(|G_{\phi}^{A_1, A_2} \{f\}|^2) \geq \frac{1}{2\pi^2} (\ln(2) - \ln(2\pi) - \ln(|b_1 b_2|)) \|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H})} \|\phi\|^2_{L^2(\mathbb{R}^2, \mathbb{H})}$$

$$- \|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H})} \|\phi\|^2_{L^2(\mathbb{R}^2, \mathbb{H})} \ln(\|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H})} \|\phi\|^2_{L^2(\mathbb{R}^2, \mathbb{H})}).$$

**Proof** Step 1: Suppose that $\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} = \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})} = 1$, according to (21), we have

$$0 \leq |G_{\phi}^{A_1, A_2} \{f\}(w, u)| \leq \|G_{\phi}^{A_1, A_2} \{f\}\|_{L^\infty(\mathbb{R}^4, \mathbb{H})}$$

$$\leq \frac{1}{2\pi \sqrt{|b_1 b_2|}} \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq 1,$$

then $\ln(|G_{\phi}^{A_1, A_2} \{f\}|) \leq 0$ and $E(|G_{\phi}^{A_1, A_2} \{f\}|) \geq 0$.

i. If the entropy $E(|G_{\phi}^{A_1, A_2} \{f\}|) = +\infty$, then (42) hold.

ii. Now assume that the entropy $E(|G_{\phi}^{A_1, A_2} \{f\}|) < +\infty$. Let $\Theta_y(z) = \frac{y^2 - z^2}{2}$ be the function defined on $(2, 3)$ and $0 < y \leq 1$, then for every $2 < z \leq 3$, we have

$$\frac{d\Theta_y}{dz}(z) = \frac{(z - 2)y^2 \ln(y) - y^2 + y^2}{(z - 2)^2}.$$ (44)

Let $h_y(z) = (z - 2)y^2 \ln(y) - y^2 + y^2$, then for every $0 < y < 1$, the function $h_y$ is differentiable on $\mathbb{R}$; hence,

$$\frac{dh_y}{dz}(z) = (z - 2)(\ln(y))^2 y^2.$$ (45)

We have that for all $0 < y < 1$, $\frac{dh_y}{dz}(z)$ is positive on $(2, 3)$, so $h_y$ is increasing on $(2, 3)$.
In addition, for all $0 < y < 1$, \( \lim_{z \to 2^+} \frac{y^z - y^2}{z - 2} = \frac{h_y(2)}{2} = 0 \), then \( h_y \) is positive which implies that \( \frac{\text{d}\Theta_y}{\text{d}z} (z) \) is positive on \((2, 3]\); hence, \( \Theta_y \) is increasing on \((2, 3] \). Therefore, \( \forall \ z \in (2, 3] \), we have

\[
y^2 \ln(y) = \lim_{z \to 2^+} \frac{y^z - y^2}{z - 2} \leq h_y(z), \tag{46}
\]

that is to say \( \forall \ z \in (2, 3] \),

\[
0 \leq \frac{y^2 - y^z}{z - 2} \leq -y^2 \ln(y). \tag{47}
\]

When \( y = 0 \) and \( y = 1 \), the inequality (47) holds. So for every \( 0 \leq y \leq 1 \), we have the inequality (47).

Step 2: According to (43), we know that for every \((w, u) \in \mathbb{R}^2 \times \mathbb{R}^2\), \( 0 \leq |G_{\phi}^{A_1, A_2} \{ f \} (w, u)| \leq 1 \), then for every \( 2 < z \leq 3 \), we obtain

\[
0 \leq \frac{|G_{\phi}^{A_1, A_2} \{ f \} (w, u)|^z - |G_{\phi}^{A_1, A_2} \{ f \} (w, u)|^2}{z - 2} \leq -|G_{\phi}^{A_1, A_2} \{ f \} (w, u)|^2 \ln(|G_{\phi}^{A_1, A_2} \{ f \} (w, u)|). \tag{48}
\]

Let \( \Phi \) be defined on \([2, +\infty)\) by

\[
\Phi(s) = \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1, A_2} \{ f \} (w, u)^s \text{d}w \text{d}u \right) - \left( \frac{|b|^\frac{1}{2} - \frac{1}{2}}{2\pi} D_{s, s'} \right)^s. \tag{49}
\]

From the Lieb inequality (33), we know that \( \Phi(s) \leq 0 \) for every \( 2 \leq s < +\infty \) and using (30), we obtain

\[
\Phi(2) = 1 - \frac{1}{\pi^2} > 0. \tag{50}
\]

Hence, \( \left( \frac{\text{d}\Phi}{\text{d}x} \right)_{x=2^+} < 0 \).

If for every \( 2 < z \leq 3 \) and \((w, u) \in \mathbb{R}^2 \times \mathbb{R}^2\), then

\[
|G_{\phi}^{A_1, A_2} \{ f \} (w, u)|^z - |G_{\phi}^{A_1, A_2} \{ f \} (w, u)|^2 \leq -|G_{\phi}^{A_1, A_2} \{ f \} (w, u)|^2 \ln(|G_{\phi}^{A_1, A_2} \{ f \} (w, u)|). \]
Then

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^z - |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2}{z - 2} \, dw \, du \\
\leq - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2 \ln(|G_{\phi}^{A_1, A_2}\{f\}(w, u)|) \, dw \, du \\
= \frac{1}{2} E(|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2) < +\infty.
\]

If for every \(3 \leq z < +\infty\) and \((w, u) \in \mathbb{R}^2 \times \mathbb{R}^2\), then

\[
\frac{|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^z - |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2}{z - 2} \\
\leq 2|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2,
\]

and

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^z - |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2 | \, dw \, du \\
\leq 2\|G_{\phi}^{A_1, A_2}\{f\}\|_{L^2(\mathbb{R}^4, \mathbb{H})} < +\infty.
\]

Therefore, for every \(2 \leq z < +\infty\) and \((w, u) \in \mathbb{R}^2 \times \mathbb{R}^2\), we have

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^z - |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2}{z - 2} \, dw \, du < +\infty. \tag{51}
\]

According to (48), we obtain

\[
\left( \frac{d}{dz} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^z \, dw \, du \right)_{z=2^+} \\
= \lim_{z \to 2^+} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^z - |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2}{z - 2} \, dw \, du \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \lim_{z \to 2^+} \frac{|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^z - |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2}{z - 2} \, dw \, du \\
= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2 \ln(|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2) \, dw \, du \\
= - \frac{1}{2} E(|G_{\phi}^{A_1, A_2}\{f\}(w, u)|^2).
\]
Moreover, by (49), we have

\[
\left( \frac{d\Phi}{dz} \right)_{z=2^+} = -\frac{1}{2} E(|G_{\phi}^{A_1,A_2}\{f\}(w,u)|^2) - \left( \frac{d}{dz} \left( \left| b \right|^{\frac{1}{2} - \frac{1}{2}} D_{z,z'} \right) \right)_{z=2^+},
\]

where \( D_{z,z'} = \left( \frac{1}{z} \right) \left( \frac{1}{z'} \right) \) and \( \frac{1}{z} + \frac{1}{z'} = 1 \).

On the other hand,

\[
\left( \frac{d}{dz} \left( \left| b \right|^{\frac{1}{2} - \frac{1}{2}} D_{z,z'} \right) \right)_{z=2^+} = \frac{1}{4\pi^2} (\ln(2\pi) + \ln(|b_1b_2|) - \ln(2)).
\]

Hence,

\[
\left( \frac{d\Phi}{dz} \right)_{z=2^+} = -\frac{1}{2} E(|G_{\phi}^{A_1,A_2}\{f\}(w,u)|^2) + \frac{1}{4\pi^2} (\ln(2) - \ln(2\pi) - \ln(|b_1b_2|)) \leq 0,
\]

we obtain

\[
E(|G_{\phi}^{A_1,A_2}\{f\}(w,u)|^2) \geq \frac{1}{2\pi^2} (\ln(2) - \ln(2\pi) - \ln(|b_1b_2|)).
\]

Step 3: For generic \( f, \phi \neq 0 \), Let \( \xi = \frac{f}{||f||_{L^2(\mathbb{R}^2,\mathbb{H})}} \) and \( \theta = \frac{\phi}{||\phi||_{L^2(\mathbb{R}^2,\mathbb{H})}} \), such that \( ||\xi||_{L^2(\mathbb{R}^2,\mathbb{H})} = ||\theta||_{L^2(\mathbb{R}^2,\mathbb{H})} = 1 \) and \( E(|G_{\phi}^{A_1,A_2}\{\xi\}(w,u)|^2) \geq \frac{1}{2\pi^2} (\ln(2) - \ln(2\pi) - \ln(|b_1b_2|)) \).

According to the definition of the QWLCT, we know that

\[
G_{\phi}^{A_1,A_2}\{\xi\}(w,u) = \frac{G_{\phi}^{A_1,A_2}\{f\}(w,u)}{||f||_{L^2(\mathbb{R}^2,\mathbb{H})} ||\phi||_{L^2(\mathbb{R}^2,\mathbb{H})}}.
\]

Hence,

\[
E(|G_{\phi}^{A_1,A_2}\{\xi\}|^2) = -\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \ln(|G_{\phi}^{A_1,A_2}\{f\}(w,u)|^2) \right. \\
- \ln(\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2) \left( |G_{\phi}^{A_1,A_2}\{f\}(w,u)|^2 \right) \\
= \frac{E(|G_{\phi}^{A_1,A_2}\{f\}|^2)}{||f||_{L^2(\mathbb{R}^2,\mathbb{H})}^2 ||\phi||_{L^2(\mathbb{R}^2,\mathbb{H})}^2} + \ln(\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2 ||\phi||_{L^2(\mathbb{R}^2,\mathbb{H})}^2) \geq \frac{1}{2\pi^2} (\ln(2) - \ln(2\pi) - \ln(|b_1b_2|)).
\]
consequently

\[
E(|G_{\phi}^{A_1,A_2}\{f\}|^2) \\
\geq \frac{1}{2\pi^2} \left( \ln(2) - \ln(2\pi) - \ln(|b_1b_2|) \right) \|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2 \\
- \|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2 \ln(\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2 \|\phi\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2).
\]

\[\square\]

**Corollary 6** When \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the entropic uncertainty principle for the QWLCT becomes the entropic uncertainty principle for the QWFT \[6\].

### 4.3 Lieb Uncertainty Principle

In this subsection, we prove the Lieb uncertainty principle for the QWLCT based on the below definition and the Lieb inequality

**Definition 4** A function \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) is \( \epsilon \)-concentrated on a measurable set \( \Omega \subseteq \mathbb{R}^2 \), where \( \Omega^c = \mathbb{R}^2 \setminus \Omega \), if

\[
\|\chi_{\Omega^c}f\|_{L^2(\mathbb{R}^2, \mathbb{H})} = \left( \int_{\Omega^c} |f(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \epsilon \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}, \tag{52}
\]

where \( \chi_{\Omega} \) denotes the characteristic function of \( \Omega \). If \( 0 \leq \epsilon \leq \frac{1}{2} \), then the most of energy is concentrated on \( \Omega \) and \( \Omega \) can be called the essential support of \( f \). If \( \epsilon = 0 \), then \( \Omega \) contains the support of \( f \).

**Theorem 5** (Lieb uncertainty principle for the QWLCT) Let \( \phi \) be a quaternion window function and \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) such that \( f \neq 0 \). Let \( \Omega \) be a measurable set of \( \mathbb{R}^2 \times \mathbb{R}^2 \) and \( \epsilon \geq 0 \). If \( G_{\phi}^{A_1,A_2}\{f\} \) is \( \epsilon \)-concentrated on \( \Omega \), then for every \( s > 2 \), we have

\[
|b|(1 - \epsilon^2)^{\frac{s}{s-2}} \left( \frac{D_{s,s'}}{2\pi} \right)^{\frac{2s}{s-2}} \leq |\Omega|,
\]

where \( |\Omega| \) is the normalized Lebesgue measure of the set \( \Omega \), \( D_{s,s'} = \left( \frac{4}{s'} \right)^{\frac{1}{s'}} \left( \frac{4}{s} \right)^{\frac{1}{s}} \) and \( \frac{1}{s} + \frac{1}{s'} = 1 \).

**Proof** Because \( G_{\phi}^{A_1,A_2}\{f\} \) is \( \epsilon \)-concentrated on \( \Omega \), according to Definition 4 and (30), it implies that

\[
\|\chi_{\Omega^c}G_{\phi}^{A_1,A_2}\{f\}\|_{L^2(\mathbb{R}^4, \mathbb{H})} \leq \epsilon \|G_{\phi}^{A_1,A_2}\{f\}\|_{L^2(\mathbb{R}^2, \mathbb{H})} = \epsilon \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|\phi\|_{L^2(\mathbb{R}^2, \mathbb{H})}; \tag{54}
\]
hence,
\[
\| \chi_{\Omega} G_{\phi}^{A_1, A_2} \{f\} \|^2_{L^2(\mathbb{R}^4, \mathbb{H})} = \| G_{\phi}^{A_1, A_2} \{f\} \|^2_{L^2(\mathbb{R}^4, \mathbb{H})} - \| \chi_{\Omega^c} G_{\phi}^{A_1, A_2} \{f\} \|^2_{L^2(\mathbb{R}^4, \mathbb{H})} \\
\geq (1 - \varepsilon^2) \| f \|^2_{L^2(\mathbb{R}^4, \mathbb{H})} \| \phi \|^2_{L^2(\mathbb{R}^4, \mathbb{H})},
\]
(55)

and applying the Hölder inequality, then
\[
\| \chi_{\Omega} G_{\phi}^{A_1, A_2} \{f\} \|^2_{L^2(\mathbb{R}^4, \mathbb{H})} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi_{\Omega}(w, u) |G_{\phi}^{A_1, A_2} \{f\}(w, u)|^2 dw \, du \\
\leq \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\chi_{\Omega}(w, u))^\frac{s}{s-2} \, dw \, du \right)^{\frac{s-2}{s}} \\
\times \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (|G_{\phi}^{A_1, A_2} \{f\}(w, u)|^2)^\frac{s}{2} \, dw \, du \right)^{\frac{2}{s}} \\
= |\Omega|^{\frac{s-2}{s}} \times \| G_{\phi}^{A_1, A_2} \{f\} \|^2_{L^s(\mathbb{R}^2, \mathbb{H})},
\]
(56)

In addition, by the Lieb inequality, the above inequality deduces that
\[
\| \chi_{\Omega} G_{\phi}^{A_1, A_2} \{f\} \|^2_{L^2(\mathbb{R}^4, \mathbb{H})} \leq |\Omega|^{\frac{s-2}{s}} \left( |b|^\frac{1}{s} - \frac{1}{2} \frac{D_{s, s'}}{2\pi} \right)^2 \| f \|^2_{L^2(\mathbb{R}^4, \mathbb{H})} \| \phi \|^2_{L^2(\mathbb{R}^2, \mathbb{H})}. \]
(57)

According to (55) and (57), we obtain
\[
1 - \varepsilon^2 \leq |\Omega|^{\frac{s-2}{s}} \left( |b|^\frac{1}{s} - \frac{1}{2} \frac{D_{s, s'}}{2\pi} \right)^2.
\]
(58)

So
\[
|b|(1 - \varepsilon^2)^{\frac{s}{s-2}} \left( \frac{D_{s, s'}}{2\pi} \right)^{\frac{2s}{s}} \leq |\Omega|.
\]
(59)

**Corollary 7** When \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the Lieb uncertainty principle for the QWLCT results in the Lieb uncertainty principle for the QWFT [6].

### 4.4 Donoho–Stark’s Uncertainty Principle

In this subsection, according to the relationship between the QWLCT and the QFT, we obtain Donoho–Stark’s uncertainty principle for QWLCT.
Lemma 5 (Donoho–Stark’s uncertainty principle for QFT) [7] Let \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) with \( f \neq 0 \) be \( \epsilon_A \)-concentrated on \( A \subseteq \mathbb{R}^2 \) and \( F_Q(f) \) be \( \epsilon_S \)-concentrated on \( S \subseteq \mathbb{R}^2 \). Then

\[
|A||S| \geq 2\pi(1 - \epsilon_A - \epsilon_S)^2. \tag{60}
\]

Inspired by Donoho–Stark’s uncertainty principle for the QFT, we can obtain Donoho–Stark’s uncertainty principle for the QWLCT.

Theorem 6 (Donoho–Stark’s uncertainty principle for QWLCT) Suppose that the nonzero signal quaternion function \( L^{-1}_{A_1, A_2}[G^{A_1, A_2}_\phi(f)](x, u) \in L^2(\mathbb{R}^2, \mathbb{H}) \) is \( \epsilon_A \)-concentrated on \( A \subseteq \mathbb{R}^2 \) and \( G^{A_1, A_2}_\phi(f)(w, u) \) is \( \epsilon_S \)-concentrated on \( S \subseteq \mathbb{R}^2 \). Then

\[
|A||bS| \geq 2\pi(1 - \epsilon_A - \epsilon_S)^2. \tag{61}
\]

Proof By applying (17) and (19), we have

\[
h(x, u) = e^{i\frac{a_1}{2b_1}x_1^2}L^{-1}_{A_1, A_2}[G^{A_1, A_2}_\phi(f)](x, u)e^{i\frac{a_2}{2b_2}x_2^2}. \tag{62}
\]

Since \( L^{-1}_{A_1, A_2}[G^{A_1, A_2}_\phi(f)](x, u) \in L^2(\mathbb{R}^2, \mathbb{H}) \) is \( \epsilon_A \)-concentrated on \( A \subseteq \mathbb{R}^2 \), then \( h(x, u) \in L^2(\mathbb{R}^2, \mathbb{H}) \) is \( \epsilon_A \)-concentrated on \( A \subseteq \mathbb{R}^2 \).

On the other hand, from \( G^{A_1, A_2}_\phi(f)(w, u) \) is \( \epsilon_S \)-concentrated on \( S \subseteq \mathbb{R}^2 \), based on the relationship (18), we know that \( F_Q(h)(\frac{w}{b}, u) \) is \( \epsilon_S \)-concentrated on \( S \subseteq \mathbb{R}^2 \), that is to say, \( F_Q(h)(w, u) \) is \( \epsilon_S \)-concentrated on \( bS \subseteq \mathbb{R}^2 \).

Hence, according to Lemma 5, we obtain

\[
|A||bS| \geq 2\pi(1 - \epsilon_A - \epsilon_S)^2.
\]

\( \square \)

Corollary 8 When \( L^{-1}_{A_1, A_2}[G^{A_1, A_2}_\phi(f)](x, u) \in L^2(\mathbb{R}^2, \mathbb{H}) \),

\[
\text{supp} L^{-1}_{A_1, A_2}[G^{A_1, A_2}_\phi(f)](x, u) \subseteq A \text{ and supp} G^{A_1, A_2}_\phi(f)(w, u) \subseteq S,
\]

then

\[
|A||bS| \geq 2\pi.
\]

5 Numerical Example and Potential Application

The uncertainty principles for the QWLCT have been studied in this paper. In this section, to show the correctness and usefulness of the theorems, a numerical example is given to verify the result, and potential applications are also presented to show the importance of the theorems.
Example: Consider a signal

\[ f(x) = \frac{1}{(\pi \beta)^{\frac{1}{2}}} e^{ix_1u_0} e^{-\frac{|x|^2}{2\beta}} e^{ix_2v_0}, \]

and the window function is

\[ \phi(x) = 2\sqrt{\frac{\pi}{\beta}} e^{ix_1u_0} e^{-\frac{|x|^2}{2\beta}} e^{ix_2v_0}, \]

where \( \beta > 0, u_0 \) and \( v_0 \) are real constants.

It follows that

\[ \| f \|_{L^2(\mathbb{R}^2, H)}^2 = \int_{\mathbb{R}^2} |f(x)|^2 dx = \frac{1}{\pi \beta} \int_{\mathbb{R}^2} e^{-\frac{|x|^2}{\beta}} dx = 1, \]

\[ \| \phi \|_{L^2(\mathbb{R}^2, H)}^2 = \int_{\mathbb{R}^2} |\phi(x)|^2 dx = \frac{4\pi^2}{\pi \beta} \int_{\mathbb{R}^2} e^{-\frac{|x|^2}{\beta}} dx = 4\pi^2. \]

Therefore, the logarithmic uncertainty principle for the QWLCT becomes

\[ \int_{\mathbb{R}^2} \ln |x| |f(x)|^2 dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| |G_{A_1,A_2}^f(w,u)|^2 dwdudu \geq \Delta + \ln |b|. \]  \hspace{1cm} (63)

By Jensen’s inequality [8], the logarithmic uncertainty principle for the QWLCT implies Heisenberg–Weyl’s uncertainty principle for the QWLCT

\[ \left( \int_{\mathbb{R}^2} |x|^2 |f(x)|^2 dx \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |w|^2 |G_{\phi}^{A_1,A_2} f(w,u)|^2 dwdudu \right)^{\frac{1}{2}} \geq \frac{|b|}{4\pi}. \]  \hspace{1cm} (64)

Let \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, u_0 = v_0 = 0 \) and \( \beta = \frac{1}{16} \), then

\[ h(x,u) = e^{i\frac{u_0}{16}x_2^2} f(x)\phi(x-u)e^{i\frac{v_0}{16}x_2^2} = \frac{2}{\beta} e^{-\frac{|x|^2}{2\beta}} e^{-\frac{|x-u|^2}{2\beta}}. \]
so

\[
F_Q(h) \left( \frac{\mathbf{w}}{\mathbf{b}}, \mathbf{u} \right) = \int_{\mathbb{R}^2} e^{-ix_1w_1/\beta} e^{-ix_2w_2/\beta} e^{-ix_1u_1/\beta} \, dx
\]

\[
= 2\pi e^{-u_1^2/4\beta - \beta w_1^2/2b_1 - u_2^2/4\beta - \beta w_2^2/2b_2}.
\]

According to (18), we have

\[
G^{A_1, A_2}_\phi \{ f \}(\mathbf{w}, \mathbf{u})
= \frac{1}{\sqrt{b_1 b_2}} e^{i\left( d_1^2 + \omega_1^2 - \pi - u_1 w_1 \right)} e^{-u_1^2/4\beta - \beta w_1^2/2b_1} e^{-u_2^2/4\beta - \beta w_2^2/2b_2} e^{i\left(d_2^2 + \omega_2^2 - \pi - u_2 w_2 \right)}.
\]

Moreover

\[
\int_{\mathbb{R}^2} |x|^2 |f(x)|^2 \, dx = \frac{1}{\pi\beta} \int_{\mathbb{R}^2} (x_1^2 + x_1^2) e^{-|x|^2/\beta} \, dx = \beta = \frac{1}{16},
\]

and

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathbf{w}|^2 |G^{A_1, A_2}_\phi \{ f \}(\mathbf{w}, \mathbf{u})|^2 \, d\mathbf{w} d\mathbf{u}
= \frac{1}{b_1 b_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (w_1^2 + w_1^2) e^{-u_1^2/2\beta - \beta w_1^2/2b_1} e^{-u_2^2/2\beta - \beta w_2^2/2b_2} \, d\mathbf{w} d\mathbf{u}
= \frac{4\pi^2}{b_1 b_2} (b_1^3 \sqrt{b_2} + b_2^3 \sqrt{b_1})
= \frac{\pi^2}{16^2}.
\]

Hence,

\[
\left( \int_{\mathbb{R}^2} |x|^2 |f(x)|^2 \, dx \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathbf{w}|^2 |G^{A_1, A_2}_\phi \{ f \}(\mathbf{w}, \mathbf{u})|^2 \, d\mathbf{w} d\mathbf{u} \right)^{\frac{1}{2}} = \frac{\pi}{4}.
\]

On the other hand,

\[
\frac{|\mathbf{b}|}{4\pi} = \frac{\sqrt{2}}{16\pi}.
\]

Then, expressions (67) and (68) verify (64). This means that Theorem 3 is verified.

The uncertainty principle has some applications in signal recovery. In [11,47], the authors studied the problem of signal recovery by uncertainty principles. [31] evaluated signal recovery problems in which there is an interplay of missing and time-limited data. The correlative result was applied to the window Fourier transform (WFT) as
well in [32]. Recently, the authors of [1] have extended signal recovery by using a local uncertainty principle for the QLCT. In this section, we will give a potential application for signal recovery by using Donoho–Stark’s uncertainty principle for the QWLCT. Let the modified signal \( f_u(x) \in L^2(\mathbb{R}^2, \mathbb{H}) \) be bandlimited to a set \( Q \) of finite measures for the QLCT. Assuming that the receiver is unable to observe all of \( f_u(x) \), a certain subset \( T \) of \( x \)-values is unobserved. Furthermore, the modified signal \( f_u(x) \) is transmitted and the received signal is corrupted by observational noise \( n(x) \). Thus, the received signal \( r(x) \) can be taken to have the form

\[
r(x) = \begin{cases} f_u(x) + n(x), & x \notin T \\ 0, & x \in T \end{cases}
\]

The receiver’s aim is to reconstruct the transmitted signal \( f_u(x) \) as nearly as possible, making use of the bandlimited hypothesis and the received data \( r(x) \). The uncertainty principle can make stable signal recovery possible as long as \( 0 < \|Q\|/\|T\| < \frac{2\pi}{|b|} \). That is, if there exists a linear operator \( P \) and a constant \( \Upsilon \) such that

\[
\|f_u - Pr\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq \Upsilon \|n\|_{L^2(\mathbb{R}^2, \mathbb{H})},
\]

where \( \Upsilon \leq \left( 1 - \sqrt{\frac{|b|}{2\pi}} \sqrt{\|Q\|/\|T\|} \right)^{-1} \). Then \( f_u \) can be stably reconstructed from \( r \). The application of the uncertainty principles for the QWLCT in signal recovery is our main research direction in the future.

Moreover, the results derived in this paper can be extended to other transforms, such as the Wigner distribution (WD) and the ambiguity function (AF) [6,14].

6 Conclusions

The QWLCT is not only a linear transform but also has similar properties to the WLCT. In this paper, Pitt’s inequality and the Lieb inequality for the QWLCT are investigated and different forms of uncertainty principles associated with the QWLCT are proposed. First, we present some important properties of the QWLCT. Second, based on the relationship between the QWLCT and the QFT, Pitt’s inequality and the Lieb inequality associated with the QWLCT are demonstrated. Third, the uncertainty principles for the QWLCT, such as the logarithmic uncertainty principle, the entropic uncertainty principle, the Lieb uncertainty principle and Donoho–Stark’s uncertainty principle are obtained. Finally, we provide a numerical example and a potential application to signal recovery by using the uncertainty principle. The results can be viewed as a generalized form of other uncertainty relations of transforms, such as the QLCT, QWFT and QFT. Further investigations on this topic are now underway, such as directional uncertainty principle for the QWLCT. Moreover, we will discuss the physical significance and engineering background of this paper in the future. They will be reported in a forthcoming paper.

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References

1. A. Achak, A. Abouelaz, R. Daher, N. Safouane, Uncertainty principles for the quaternion linear canonical transform. Adv. Appl. Clifford Algebr. 29(5), 1–19 (2019)
2. M. Bahri, Quaternion linear canonical transform application. Global J. Pure Appl. Math. 11(1), 19–24 (2015)
3. M. Bahri, R. Ashino, Simplified proof of uncertainty principle for quaternion linear canonical transform, in Abstract and Applied Analysis. (Hindawi, London, 2016), pp. 1–11
4. M. Bahri, R. Ashino, Some properties of windowed linear canonical transform and its logarithmic uncertainty principle. Int. J. Wavelets Multiresolut Inf. Process. 14(3), 1–21 (2016)
5. M. Bahri, E.S.M. Hitzer, R. Ashino, R. Vaillancourt, Windowed Fourier transform of two-dimensional quaternionic signals. Appl. Math. Comput. 216(8), 2366–2379 (2010)
6. K. Brahimi, T. Emma, Uncertainty principle for the two sided quaternion windowed Fourier transform. J. Pseudo-Differ. Oper. Oper. 11(1), 159–185 (2020)
7. K. Brahimi, T. Emma, Uncertainty principle for the two-sided quaternion windowed Fourier transform. Integral Transform. Spec. Funct. 30(9), 362–382 (2019)
8. L.P. Chen, K.I. Kou, M.S. Liu, Pitt’s inequality and the uncertainty principle associated with the quaternion Fourier transform. J. Math. Anal. Appl. 423(1), 681–700 (2015)
9. D. Cheng, K.I. Kou, Plancherel theorem and quaternion Fourier transform for square integrable functions. Complex Var. Elliptic Equ. 64(2), 223–242 (2019)
10. P. Dang, G.T. Deng, T. Qian, A sharper uncertainty principle. J. Funct. Anal. 265, 2239–2266 (2013)
11. D.L. Donoho, P.B. Stark, Uncertainty principles and signal recovery. SIAM J. Appl. Math. 49, 906–931 (1989)
12. H. Eckhard, J.S. Stephen, Quaternion and Clifford Fourier Transforms and Wavelets (Birkhäuser, Basel, 2013)
13. T. A. Ell, N. L. Bihan, S. J. Sangwine, Quaternion Fourier Transforms for Signal and Image Processing (John Wiley Sons, Ltd, 2014)
14. X.L. Fan, K.I. Kou, M.S. Liu, Quaternion Wigner-Ville distribution associated with the linear canonical transforms. Signal Process. 130, 129–141 (2017)
15. Q. Feng, B.Z. Li, Convolution and correlation theorems for the two-dimensional linear canonical transform and its applications. IET Signal Process. 10(2), 125–132 (2016)
16. W.B. Gao, B.Z. Li, Uncertainty principles for the short-time linear canonical transform of complex signals. Signal Process. 111, 1–10 (2021)
17. W.B. Gao, B.Z. Li, Quaternion windowed linear canonical transform of two-dimensional signals. Adv. Appl. Clifford Algebr. 30(1), 1–18 (2020)
18. Y.E. Haoui, S. Fahlaoui, The uncertainty principle for the two-sided quaternion Fourier transform. Mediterr. J. Math. 14(6), 1–8 (2017)
19. W. Heisenberg, Über den anschaulichen inhalt der quanten theoretischen kinematik und mechanik. Zeitschrift fur Physik. 43, 172–198 (1927)
20. B. Hu, Y. Zhou, L.D. Lie, J.Y. Zhang, Polar linear canonical transform in quaternion domain. J. Inf. Hiding Multimed. Signal Process. 6(6), 1185–1193 (2015)
21. X.X. Hu, K.I. Kou, Quaternion Fourier and linear canonical inversion theorems. Math. Methods Appl. Sci. 40(7), 2421–2440 (2017)
22. L. Huang, K. Zhang, Y. Chai, S.Q. Xu, Computation of the short-time linear canonical transform with dual window. Math. Probl. Eng. 2017, 1–8 (2017)
23. L. Huang, K. Zhang, Y. Chai, S.Q. Xu, Uncertainty principle and orthogonal condition for the short-time linear canonical transform. Signal Image Video Process. 10, 1177–1181 (2016)
24. E.M.S. Hitzer, Quaternion Fourier transform on quaternion fields and generalizations. Adv. Appl. Clifford Algebr. 17(3), 497–517 (2007)
25. K.I. Kou, J. Morais, Asymptotic behaviour of the quaternion linear canonical transform and the Bochner-Minlos theorem. Appl. Math. Comput. 247(15), 675–688 (2014)
26. K.I. Kou, J. Morais, Y. Zhang, Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis. Math. Methods Appl. Sci. 36(9), 1028–1041 (2013)

27. K.I. Kou, M. Liu, S. Tao, Herglotz’s theorem and quaternion series of positive term. Math. Methods Appl. Sci. 39(18), 5607–5618 (2016)

28. K.I. Kou, J.Y. Ou, J. Morais, Uncertainty principles associated with quaternionic linear canonical transforms. Math. Meth. Appl. Sci. 39(10), 2722–2736 (2016)

29. K.I. Kou, R.H. Xu, Y.H. Zhang, Paley-Wiener theorems and uncertainty principles for the windowed linear canonical transform. Math. Methods Appl. Sci. 35(17), 2122–2132 (2012)

30. K.I. Kou, R.H. Xu, Windowed linear canonical transform and its applications. Signal Process. 92(1), 179–188 (2012)

31. K.I. Kou, Y. Yang, C. Zou, Uncertainty principle for measurable sets and signal recovery in quaternion domains. Math. Methods Appl. Sci. 40(11), 3892–3900 (2017)

32. M.S. Liu, K.I. Kou, J. Morais, P. Dang, Sharper uncertainty principles for the windowed Fourier transform. J. Mod. Opt. 62(1), 46–55 (2015)

33. P.J. Loughlin, L. Cohen, The uncertainty principle: global, local, or both? IEEE Trans. Signal Process. 52(5), 1218–1227 (2004)

34. D. Mustard, Uncertainty principle invariant under fractional Fourier transform. J. Austral. Math. Soc. Ser. B 33, 180–191 (1991)

35. H.M. Ozaktas, O. Aytur, Fractional Fourier domains. Signal Process. 46, 119–124 (1995)

36. S. Shinde, M.G. Vikram, An uncertainty principle for real signals in the fractional Fourier transform domain. IEEE Trans. Signal Process. 49(11), 2545–2548 (2001)

37. R. Tao, B.Z. Li, Y. Wang et al., On sampling of band-limited signals associated with the linear canonical transform. IEEE Trans. Signal Process. 56(11), 5454–5464 (2008)

38. K.B. Wolf, Canonical Transforms, in Integral Transforms in Science and Engineering (Plenum Press, New York, NY, USA, 1979)

39. Z. Xiong, Y. X. Fu, A kind of windowed quaternionic linear canonical transform. Master Thesis, Hubei University, Wuhan, China, (2015)

40. G.L. Xu, X.T. Wang, X.G. Xu, Uncertainty inequalities for linear canonical transform. IET Signal Process. 3(5), 392–402 (2009)

41. T.Z. Xu, B.Z. Li, Linear Canonical Transform and Its Application (Science Press, Beijing, 2013)

42. M.H. Yel, Relationships among various 2-D quaternion Fourier transforms. IEEE Signal Process. Lett. 15, 669–672 (2008)

43. Z. Zalevsky, D. Mendlovic, M. AlperKutay et al., Improved acoustic signals discrimination using fractional Fourier transform based phase-space representations. Opt. Commun. 190(1–6), 95–101 (2001)

44. Q.Y. Zhang, Discrete Windowed Linear Canonical Transform (ICSPPC. Hong Kong, China, 2016)

45. Y.N. Zhang, B.Z. Li, Novel uncertainty principles for two-sided quaternion linear canonical transform. Adv. Appl. Clifford Algebr. 28(1), 1–15 (2018)

46. Z.C. Zhang, Sampling theorem for the short-time linear canonical transform and its applications. Signal Process. 113, 138–146 (2015)

47. X.Y. Zhu, S.Z. Zheng, Uncertainty principles for the two-sided quaternion linear canonical transform. Circuits Syst. Signal Process. 39(9), 4436–4458 (2020)

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