Spin projection operators and higher-spin Cotton tensors in three dimensions

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Abstract

We elaborate on the spin projection operators in three dimensions and use them to derive a new representation for the linearised higher-spin Cotton tensors.
1 Introduction

In four dimensions (4D), there exists a remarkably simple expression for the linearised higher-spin Weyl tensors in terms of gauge prepotentials $h_{\alpha_1...\alpha_s\dot{\alpha}_1...\dot{\alpha}_s} = h_{(\alpha_1...\alpha_s)(\dot{\alpha}_1...\dot{\alpha}_s)}$, see e.g. [1, 2] and section 6.9 of [3]. In the case of an integer spin $s \geq 1$, it reads

$$C_{\alpha_1...\alpha_{2s}} = \partial_{(\alpha_1} \beta_1 ... \partial_{\alpha_s} \beta_s h_{\alpha_{s+1}...\alpha_{2s}} \dot{\alpha}_1...\dot{\alpha}_s \right).$$

(1.1)

For a half-integer spin $s + \frac{1}{2} \geq \frac{3}{2}$, with $s = 1, 2, \ldots$, we have

$$C_{\alpha_1...\alpha_{2s+1}} = \partial_{(\alpha_1} \beta_1 ... \partial_{\alpha_s} \beta_s \psi_{\alpha_{s+1}...\alpha_{2s+1}} \dot{\alpha}_1...\dot{\alpha}_s \right).$$

(1.2)

It follows from (1.1) and (1.2) that the higher-spin Weyl tensors are invariant under gauge transformations of the form

$$\delta_\zeta h_{\alpha_1...\alpha_s\dot{\alpha}_1...\dot{\alpha}_s} = \partial_{(\alpha_1} (\zeta_{\alpha_2...\alpha_s}) \dot{\alpha}_2...\dot{\alpha}_s \right),$$

(1.3)

$$\delta_\zeta \psi_{\alpha_1...\alpha_{s+1}\dot{\alpha}_1...\dot{\alpha}_s} = \partial_{(\alpha_1} (\zeta_{\alpha_2...\alpha_{s+1}}) \dot{\alpha}_2...\dot{\alpha}_s \right).$$

(1.4)
It should be remarked that there are two ways for the bosonic gauge field \( h_{\alpha(s)} \) to occur in higher-spin gauge theories. Firstly, \( h_{\alpha(s)} \) is one of the two gauge prepotentials \{\( h_{\alpha(s)} \), \( h_{\alpha(s-2)} \)\} in the Fronsdal massless integer-spin models \([4, 5]\) (see section 6.9 of \([3]\) for a review). Secondly, it is the gauge field in the Fradkin-Tseytlin conformal integer-spin theories \([6]\). In the former theories, \( C_{\alpha(2s)} \) and its conjugate are the only gauge invariant field strengths which survive on the mass shell. In the latter theories, the gauge-invariant action may be formulated in terms of \( C_{\alpha(2s)} \) and its conjugate \([1, 2]\).

In three dimensions, the Weyl tensor vanishes identically and all information about the conformal geometry of spacetime is encoded in the Cotton tensor. Spacetime is conformally flat if and only if the Cotton tensor vanishes \([7]\) (see \([8]\) for a modern proof). Linearised higher-spin extensions of the Cotton tensor in Minkowski space were constructed in \([9]\) and \([10]\) in the bosonic and fermionic cases, respectively. In terms of a gauge prepotential \( h_{\alpha_1...\alpha_n} = h_{(\alpha_1...\alpha_n)} \), with \( n > 1 \), the linearised Cotton tensor is given by the expression \([10]\)

\[
C_{\alpha(n)}(h) = \frac{1}{2n-1} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j+1} \Box^j \partial_{\alpha_1} \beta_1 \cdots \partial_{\alpha_{n-2j-1}} \beta_{n-2j-1-1} h_{\alpha_{n-2j-1}...\alpha_n} \beta_1...\beta_{n-2j-1} ,
\]

(1.5)

where \( \lfloor x \rfloor \) denotes the floor function and returns the integer part of a real number \( x \geq 0 \).

The fundamental properties of \( C_{\alpha(n)} \) are the following:

(i) \( C_{\alpha(n)} \) is invariant under gauge transformations of the form

\[
\delta_\zeta h_{\alpha(n)} = \partial_{(\alpha_1\alpha_2}\zeta_{\alpha_3...\alpha_n)} \implies \delta_\zeta C_{\alpha(n)} = 0 ;
\]

(1.6)

(ii) \( C_{\alpha(n)} \) is divergenceless

\[
\partial^{\beta\gamma} C_{\beta\gamma\alpha_1...\alpha_{n-2}} = 0 .
\]

(1.7)

Unlike the 4D relations \([11]\) and \([12]\), the expression for \( C_{\alpha(n)} \) given by \((1.5)\) is not illuminating. It is not obvious from \((1.5)\) that \( C_{\alpha(n)} \) possesses the properties \((1.6)\) and \((1.7)\). Recently it has been shown, first in the bosonic (even \( n \)) \([11]\) and later in the fermionic (odd \( n \)) \([12]\) case, that \((1.5)\) is the most general solution of the conservation

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\(^1\)The story with the fermionic gauge field \( \psi_{\alpha(s+1)} \) is analogous.

\(^2\)The compensator \( h_{\alpha(s-2)} \) transforms under \((1.3)\) by the rule \( \delta_\zeta h_{\alpha(s-2)} \) \( \propto \partial^{\beta\delta} \zeta_{\beta\alpha(s-2)}\delta\alpha(s-2) \).
equation (1.7), and the proofs are non-trivial. There exists a simple proof of this statement based on the use of $\mathcal{N} = 1$ supersymmetry [10]. However, it makes use of an embedding of the higher-spin gauge prepotentials in superfields. A simple non-supersymmetric proof of this statement is still missing.

In this letter we derive a new representation for the higher-spin Cotton tensor $C_{\alpha(n)}$ which is analogous to the 4D relations (1.1) and (1.2) and which makes obvious the properties (1.6) and (1.7). Our approach is based on the use of 3D analogues of the Behrends-Fronsdal projection operators [13, 14] (see [15, 16, 17] for modern descriptions using the two-component spinor formalism). These projection operators were generalised beyond four dimensions by Segal [18] (for recent discussions, see also [17, 19]) for integer spin values, while the half-integer-spin case was described in [17]. As will be shown below, the 3D case is somewhat special.

This paper is organised as follows. In section 2 we discuss various aspects of massive higher-spin fields. Section 3 is devoted to spin projection operators. In section 4 we derive a new representation for the higher-spin Cotton tensor $C_{\alpha(n)}$. Concluding comments are given in section 5. Our spinor conventions are summarised in the appendix.

2 On-shell massive fields in three dimensions

We start with discussing tensor fields realising irreducible massive (half-)integer spin representations of the Poincaré group in three dimensions. We restrict our attention to the case of integer and half-integer spin values; for a discussion of the anyon representations see, e.g., [20]. The 3D spin group \( \text{SL}(2, \mathbb{R}) \) is \( \text{SL}(2, \mathbb{R}) \), so that the fields of interest are real symmetric rank-\( n \) spinors, \( \Phi_{\alpha_1...\alpha_n} = \Phi(\alpha_1...\alpha_n) \equiv \Phi(\alpha(n)). \)

For \( n > 1 \), an on-shell field \( \Phi(\alpha(n))(x) \) of mass \( m \) satisfies the following differential equations [22, 23] (see also [24]):

\[
\begin{align*}
\partial^\gamma \Phi_{\beta \gamma \alpha(n-2)} &= 0, \tag{2.1a} \\
\partial^\delta (\alpha_1 \Phi_{\alpha_2...\alpha_n}) &= m \sigma \Phi(\alpha(n)), \quad \sigma = \pm 1. \tag{2.1b}
\end{align*}
\]

In the spinor case, \( n = 1 \), eq. (2.1a) is absent, and it is the Dirac equation (2.1b) which defines a massive field. The constraints (2.1a) and (2.1b) imply the mass-shell equation

\[
(\Box - m^2)\Phi(\alpha(n)) = 0. \tag{2.2}
\]
Equations (2.1a) and (2.2) prove to be equivalent to the 3D Fierz-Pauli field equations [25]. It is worth pointing out that the equations (2.1) naturally originate upon quantisation of the particle models studied in [22, 26].

Let $P_a$ and $J_{ab} = -J_{ba}$ be the generators of the 3D Poincaré group. The Pauli-Lubanski pseudo-scalar

$$ W := \frac{1}{2} \epsilon^{abc} P_a J_{bc} = -\frac{1}{2} P^{\alpha\beta} J_{\alpha\beta} \quad (2.3) $$

commutes with the generators $P_a$ and $J_{ab}$. Irreducible unitary representations of the Poincaré group are labelled by two parameters, mass $m$ and helicity $\lambda$, which are associated with the Casimir operators,

$$ P^a P_a = -m^2 \mathbb{1} \; , \quad W = m \lambda \mathbb{1} \quad (2.4) $$

The parameter $|\lambda|$ is identified with spin.

In the case of field representations, we have

$$ W = \frac{1}{2} \sigma^{\alpha\beta} M_{\alpha\beta} \; , \quad (2.5) $$

where the action of $M_{\alpha\beta} = M_{\beta\alpha}$ on a field $\Phi_{\gamma(n)}$ is defined by

$$ M_{\alpha\beta} \Phi_{\gamma_1...\gamma_n} = \sum_{i=1}^n \epsilon_{\gamma_i(\alpha} \Phi_{\beta)\gamma_1...\hat{\gamma_i}...\gamma_n} \quad (2.6) $$

where the hatted index of $\Phi_{\beta\gamma_1...\hat{\gamma_i}...\gamma_n}$ is omitted. It follows from (2.1b) and the second relation in (2.4) that the helicity of the on-shell massive field $\Phi_{\alpha(n)}$ is

$$ \lambda = \frac{n}{2} \sigma \quad (2.7) $$

In order to make contact with Wigner’s classification of unitary representations of the Poincaré group [27] and its 3D extension [28], it is more convenient to work in momentum space in which the equations (2.1) take the form

$$ p^{\beta\gamma} \Phi_{\beta\gamma\alpha(n-2)}(p) = 0 \; , \quad (2.8a) $$

$$ p^{\beta}(\alpha_1 \Phi_{\alpha_2...\alpha_n})_{\beta}(p) = -i \sigma m \Phi_{\alpha(n)}(p) \; , \quad \sigma = \pm 1 \; , \quad (2.8b) $$

where $\Phi_{\alpha(n)}(p)$ denotes the positive-energy part of the Fourier transform of $\Phi_{\alpha(n)}(x)$. We now develop some group-theoretical aspects before discussing the equations (2.8) in more detail.
Let \( q^a = (m, 0, 0) \) be the momentum of a massive particle at rest. Then an arbitrary three-momentum \( p^a \) of the particle is obtained by applying a proper orthochronous Lorentz transformation to \( p^a \), that is
\[
(p \cdot \gamma)_{\alpha\beta} := p^a (\gamma_a)_{\alpha\beta} \equiv p_{\alpha\beta} = (L(q \cdot \gamma) L^T)_{\alpha\beta},
\]  
for some matrix \( L \in \text{SL}(2, \mathbb{R}) \). It is convenient to parametrise \( L \) in terms of two linearly independent real commuting spinors
\[
L = \frac{1}{(\nu, \mu)^{1/2}} \begin{pmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{pmatrix} = \frac{1}{(\nu, \mu)^{1/2}} (\mu_\alpha, \nu_\alpha), \quad (\nu, \mu) := \nu^\alpha \mu_\alpha = -(\mu, \nu).
\]  
Here the spinors \( \mu_\alpha \) and \( \nu_\alpha \) are arbitrary modulo the condition \( (\nu, \mu) > 0 \). Note that \( \det L = 1 \). It should be remarked that (2.10) is invariant under the rescalings \( \mu_\alpha \to \rho \mu_\alpha, \nu_\alpha \to \rho \nu_\alpha \). In principle, we can use this symmetry to normalise \( (\nu, \mu) = 1 \), but we prefer to keep all expressions in the most general form.

Making use of the relations (2.9) and (2.10) gives
\[
p_{\alpha\beta} = \frac{m}{(\nu, \mu)} (\mu_\alpha \mu_\beta + \nu_\alpha \nu_\beta).
\]  
The identities \( (\mu, \mu) = (\nu, \nu) = 0 \) imply that
\[
\frac{1}{2} p^{\alpha\beta} p_{\alpha\beta} = -p^ap_a = m^2.
\]  
Since \( (q \cdot \gamma)_{\alpha\beta} = m(\gamma_0)_{\alpha\beta} = m \mathbb{1} \), it follows that \( p_{\alpha\beta} \) given by (2.9) is invariant under the transformation \( L \to L \cdot h \), where \( h \in \text{SO}(2) \). The latter group is the 3D little group in the massive case.

Since the little group \( \text{SO}(2) \) is abelian, Wigner’s wave function \( \phi^{(\lambda)}(p) \), which describes the irreducible massive representation of helicity \( \lambda \), must be one-component. It follows from (2.8) that \( \Phi_{\alpha(n)}(p) \) describes one degree of freedom (it suffices to consider the \( p^a = q^a \) case). However, even for the simplest choice \( p^a = q^a \) all components of \( \Phi_{\alpha(n)} \) are non-vanishing. It would be convenient to have an approach that provides a simple rule to read off a one-component Wigner wave function for every (half-)integer helicity. For this we will use the isomorphism between \( \text{SL}(2, \mathbb{R}) \) and \( \text{SU}(1, 1) \) described in detail in [29]. Associated with a group element \( L = (L_{\alpha\beta}) \in \text{SL}(2, \mathbb{R}) \) is the matrix \( \tilde{L} = (\tilde{L}_{\alpha\beta}) \in \text{SU}(1, 1) \) given by
\[
\tilde{L} = T^{-1}LT,
\]  

\(^4\)Strictly speaking, different types of indices have to be used for the elements of \( \text{SL}(2, \mathbb{R}) \) and \( \text{SU}(1, 1) \). To avoid a cluttered notation, we will not make such a distinction. We simply denote all operators and tensors of \( \text{SU}(1, 1) \) with a tilde.
where $T$ denotes the following unitary, unimodular matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SU(2). \quad (2.14)$$

If $\psi_\alpha$ is a spinor of $SL(2, \mathbb{R})$, the corresponding spinor $\tilde{\psi}_{\alpha}$ of $SU(1, 1)$ is given by $\tilde{\psi} = T^{-1} \psi$. More generally, associated with an arbitrary symmetric rank-$n$ $SL(2, \mathbb{R})$ spinor $\Phi_{\alpha(n)}$ is the $SU(1, 1)$ tensor $\tilde{\Phi}_{\alpha(n)}$ defined by

$$\tilde{\Phi}_{\alpha_1 ... \alpha_n} = (T^{-1})_{\alpha_1}^{\beta_1} \cdots (T^{-1})_{\alpha_n}^{\beta_n} \Phi_{\beta_1 ... \beta_n}. \quad (2.15)$$

Hence, in the $SU(1, 1)$ picture the dynamical equations (2.8a) and (2.8b) look the same except that $p_{\alpha\beta}$ and $\Phi_{\alpha(n)}$ are replaced with $\tilde{p}_{\alpha\beta}$ and $\tilde{\Phi}_{\alpha(n)}$, respectively.

We will parametrise the group elements $\tilde{L} \in SU(1, 1)$ in terms of two complex spinors $\tilde{\mu}_\alpha$ and $\tilde{\nu}_\alpha$ that are related to each other by Dirac conjugation. More specifically, every element of $SU(1, 1)$ can be represented as

$$\tilde{L} = \begin{pmatrix} \tilde{\mu}_1 & \tilde{\nu}_1 \\ \tilde{\mu}_2 & \tilde{\nu}_2 \end{pmatrix} = (\tilde{\mu}_\alpha, \tilde{\nu}_\alpha), \quad \tilde{\nu}^\alpha = \overline{\tilde{\mu}}^\alpha, \quad \tilde{\nu}^\alpha \tilde{\mu}_\alpha = 1. \quad (2.16)$$

where the Dirac conjugate $\overline{\psi} = (\overline{\psi}^\alpha)$ of a spinor $\tilde{\psi} = (\tilde{\psi}_\alpha)$ is defined by

$$\overline{\psi} = \tilde{\psi}^\dagger \sigma_3. \quad (2.17)$$

We can rewrite $\tilde{L}$ in the form

$$\tilde{L} = (\tilde{\mu}_+^\alpha, \tilde{\mu}_-^\alpha) \in SU(1, 1), \quad (2.18)$$

where ‘plus’ and ‘minus’ refer to charges with respect to the $U(1)$ action

$$\tilde{L} \rightarrow \tilde{L} \exp (i \varphi \sigma_3), \quad \varphi \in \mathbb{R}. \quad (2.19)$$

With this notation the $SU(1, 1)$ formalism is analogous to the $SU(2)$ one used within the harmonic superspace approach in four dimensions [30].

In the $SU(1, 1)$ picture, the momentum $\tilde{p}_{\alpha\beta} = (p \cdot \tilde{\gamma})_{\alpha\beta}$ is obtained from $p_{\alpha\beta} = (p \cdot \gamma)_{\alpha\beta}$ by the rule

$$\tilde{p}_{\alpha\beta} = (T^{-1})_{\alpha}^{\gamma}(T^{-1})_{\beta}^{\delta} p_{\gamma\delta} = (T^{-1} p T^{-1})_{\alpha\beta} = (\tilde{L} q \tilde{L}^\dagger)_{\alpha\beta}, \quad (2.20)$$

where the momentum of a particle at rest, $\tilde{q}_{\alpha\beta} = (q \cdot \tilde{\gamma})_{\alpha\beta}$, becomes

$$\tilde{q} = T^{-1} (q \cdot \gamma) T^{-1} = m(T^{-1})^2 = -im \sigma_1. \quad (2.21)$$
Making use of eqs. (2.16) and (2.21) gives

$$\tilde{p}_{\alpha\beta} = -im(\tilde{\mu}_\alpha \tilde{\nu}_\beta + \tilde{\mu}_\beta \tilde{\nu}_\alpha) . \quad (2.22)$$

The stability group of $\tilde{q}$ consists of all group elements $\tilde{h} \in SU(1,1)$ with the property

$$\tilde{q} = \tilde{h} q \tilde{h}^T \iff \sigma_1 = \tilde{h} \sigma_1 \tilde{h}^T . \quad (2.23)$$

Hence, the little group in the $SU(1,1)$ picture consists of the matrices

$$\tilde{h} = e^{i \varphi_3} \in SU(1,1) , \quad \varphi \in \mathbb{R} , \quad (2.24)$$

and is isomorphic to $U(1)$.

The group element $\tilde{L} \in SU(1,1)$ in (2.20) is defined modulo arbitrary right shifts

$$\tilde{L} \rightarrow \tilde{L} e^{i \varphi_3} , \quad \varphi \in \mathbb{R} . \quad (2.25)$$

This freedom may be fixed by choosing the global coset representative

$$\tilde{L}(p) = \frac{1}{\sqrt{2m(p^0 + m)}} \begin{pmatrix} p^0 + m & p^1 + ip^2 \\ p^1 - ip^2 & p^0 + m \end{pmatrix} \in SU(1,1) , \quad (2.26)$$

which parametrises the homogeneous space $SU(1,1)/U(1)$ that is diffeomorphic to the hyperbolic plane $\mathbb{H}^2$.

Now we are prepared to construct massive fields $\tilde{\Phi}^{(\pm)}_{\alpha(n)}$ of helicity $\pm n/2$. They are:

$$\tilde{\Phi}^{(+)}_{\alpha_1 \alpha_2 ... \alpha_n}(p) = \tilde{\mu}_{\alpha_1} \tilde{\mu}_{\alpha_2} ... \tilde{\mu}_{\alpha_n} \tilde{\phi}^{(+n)}(\tilde{\mu}, \tilde{\nu}) , \quad (2.27a)$$

$$\tilde{\Phi}^{(-)}_{\alpha_1 \alpha_2 ... \alpha_n}(p) = \tilde{\nu}_{\alpha_1} \tilde{\nu}_{\alpha_2} ... \tilde{\nu}_{\alpha_n} \tilde{\phi}^{(-n)}(\tilde{\mu}, \tilde{\nu}) . \quad (2.27b)$$

Indeed, from eqs. (2.22) and (2.27) it follows that

$$\tilde{p}^{\beta \gamma} \tilde{\Phi}^{(\pm)}_{\beta \gamma \alpha(n-2)} = 0 \quad (2.28)$$

and eq. (2.8a) is satisfied. Furthermore, using

$$\tilde{p}^{\alpha \beta} \tilde{\mu}_\beta = -im\tilde{\mu}_\alpha , \quad \tilde{p}^{\alpha \beta} \tilde{\nu}_\beta = im\tilde{\nu}_\alpha , \quad (2.29)$$

we obtain

$$\tilde{p}^{\beta}_{\alpha_1 \alpha_2 ... \alpha_n} \tilde{\phi}^{(\pm)}_{\beta \gamma \alpha(n-2)} = \mp im \tilde{\phi}^{(\pm)}_{\alpha(n)} , \quad (2.30)$$

and so eq. (2.8b) is also satisfied. Therefore, $\tilde{\Phi}^{(\pm)}_{\alpha(n)}(p)$ describe the irreducible massive representations of the Poincaré group with helicity $\pm n/2$. Since $\tilde{\Phi}^{(\pm)}_{\alpha(n)}(p)$ is invariant under the transformation (2.25), the wave function $\tilde{\phi}^{(\pm)}(\tilde{\mu}, \tilde{\nu})$ must possess the following homogeneity property

$$\tilde{\phi}^{(\pm n)}(e^{i\varphi} \tilde{\mu}, e^{-i\varphi} \tilde{\nu}) = e^{\mp in\varphi} \tilde{\phi}^{(\pm n)}(\tilde{\mu}, \tilde{\nu}) . \quad (2.31)$$
3 Projection operators

Having described the irreducible tensor fields carrying definite helicity we can now construct the projection operators onto these states.

3.1 On-shell projectors

We will start with the simplest case of spin 1/2. We have two spinors carrying definite helicity
\[ \tilde{\Phi}^+(\alpha) = \tilde{\mu}_\alpha \tilde{\phi}^+, \quad \tilde{\Phi}^-(\alpha) = \tilde{\nu}_\alpha \tilde{\phi}^- . \] (3.1)
This means that \( \tilde{e}^+_\alpha = \tilde{\mu}_\alpha \) and \( \tilde{e}^-_\alpha = \tilde{\nu}_\alpha \) are the polarisation spinors. Now we define the following projection operators
\[ \tilde{\Pi}^+_\alpha^\beta = \tilde{\mu}_\alpha \tilde{\nu}^\beta , \quad \tilde{\Pi}^-_\alpha^\beta = -\tilde{\nu}_\alpha \tilde{\mu}^\beta . \] (3.2)
They satisfy the following properties
\[ \tilde{\Pi}^+_\alpha^\beta \tilde{\Pi}^+_\beta^\gamma = \tilde{\Pi}^+_\alpha^\gamma , \quad \tilde{\Pi}^-_\alpha^\beta \tilde{\Pi}^-_\beta^\gamma = \tilde{\Pi}^-_\alpha^\gamma , \]
\[ \tilde{\Pi}^+_\alpha^\beta \tilde{\Pi}^-_\beta^\gamma = \tilde{\Pi}^-_\alpha^\gamma \tilde{\Pi}^+_\beta^\gamma = 0 . \] (3.3a, 3.3b)

Consider an arbitrary on-shell spinor field \( \tilde{\Phi}_\alpha(p) \). Then we obtain
\[ \tilde{\Pi}^+_\alpha^\beta \tilde{\Phi}_\beta = \tilde{\mu}_\alpha \tilde{\nu}^\beta \tilde{\Phi}_\beta , \quad \tilde{\Pi}^-_\alpha^\beta \tilde{\Phi}_\beta = -\tilde{\nu}_\alpha \tilde{\mu}^\beta \tilde{\Phi}_\beta . \] (3.4)
Comparing with eq. (3.1) we conclude that \( \tilde{\Pi}^\pm \) are the projection operators onto the states with positive and negative helicity. Using the identities
\[ \tilde{\mu}_\alpha \tilde{\nu}^\beta - \tilde{\mu}^\beta \tilde{\nu}_\alpha = \delta_\alpha^\beta , \quad \tilde{\mu}_\alpha \tilde{\nu}^\beta + \tilde{\mu}^\beta \tilde{\nu}_\alpha = \frac{i}{m} \tilde{p}_\alpha^\beta , \] (3.5)
we can also write the projection operators in the form
\[ \tilde{\Pi}^\pm_\alpha^\beta = \frac{1}{2} \left( \delta_\alpha^\beta \pm \frac{i}{m} \tilde{p}_\alpha^\beta \right) . \] (3.6)
At this stage we will remove the tilde assuming that we have performed the transformation to the \( SL(2, \mathbb{R}) \) picture \( \tilde{\Pi}^\pm \to \Pi^\pm, \tilde{p} \to p \).

Now it is clear how to construct the projectors for an arbitrary integer or half-integer spin:
\[ \Pi^{(+n)}_{\alpha(n)}^\beta(n) = \Pi^{(+)}_{\alpha_1}^\beta_1 \ldots \Pi^{(+)}_{\alpha_n}^\beta_n , \] (3.7a)
\[ \Pi^{(-n)}_{\alpha(n)} \beta(n) = \Pi^{(-)}_{\alpha_1} \beta_1 \ldots \Pi^{(-)}_{\alpha_n} \beta_n . \]  

(3.7b)

Given an arbitrary on-shell field \( \Phi_{\alpha(n)}(p) \), we define

\[ \Phi^{(\pm)}_{\alpha(n)} = \Pi^{(\pm(n)}_{\alpha(n)} \beta(n) \Phi_{\beta(n)} . \]  

(3.8)

Then it follows that \( \Phi^{(\pm)}_{\alpha(n)} \) satisfies eqs. (2.8a) and (2.8b) and, hence, it is an irreducible field. This can be checked explicitly using the identities

\[ p_\alpha^\beta \Pi^{(\pm)}_{\beta} = \mp \text{Im} \Pi^{(\pm)}_{\alpha}, \quad \varepsilon^{\alpha\beta} \Pi^{(\pm)}_{\alpha} \Pi^{(\pm)}_{\beta} = 0 . \]  

(3.9)

### 3.2 Off-shell projectors

Let us take a step further and view the projection operators (3.6) and (3.7) as acting not just on the space of on-shell fields, but on the space of arbitrary fields, whose momentum does not necessarily satisfy \( p^2 = -m^2 \). In this case we have to replace \( m \) with \( \sqrt{-p^2} \), or in the coordinate representation with \( \sqrt{\Box} \).

We introduce off-shell projection operators

\[ \Pi^{(\pm)}_{\alpha} = \frac{1}{2} \left( \delta_\alpha^\beta \pm \frac{1}{\sqrt{\Box}} \partial_\alpha^\beta \right) . \]  

(3.10)

and their higher-rank extensions (compare with (3.7) in the momentum representation)

\[ \Pi^{(+n)}_{\alpha(n)} \beta(n) = \Pi^{(+)}_{\alpha_1} \beta_1 \ldots \Pi^{(+)}_{\alpha_n} \beta_n , \quad \Pi^{(-n)}_{\alpha(n)} \beta(n) = \Pi^{(-)}_{\alpha_1} \beta_1 \ldots \Pi^{(-)}_{\alpha_n} \beta_n . \]  

(3.11a, b)

Given an off-shell field \( h_{\alpha(n)} \), the action of \( \Pi^{(\pm n)}_{\alpha(n)} \) on \( h_{\alpha(n)} \) is defined by

\[ \Pi^{(+n)}_{\alpha(n)} h_{\alpha(n)} := \Pi^{(+)}_{\alpha_1} \beta_1 \ldots \Pi^{(+)}_{\alpha_n} \beta_n h_{\beta_1 \ldots \beta_n} \equiv h^{(+)}_{\alpha(n)} , \]

(3.12a)

\[ \Pi^{(-n)}_{\alpha(n)} h_{\alpha(n)} := \Pi^{(-)}_{\alpha_1} \beta_1 \ldots \Pi^{(-)}_{\alpha_n} \beta_n h_{\beta_1 \ldots \beta_n} \equiv h^{(-)}_{\alpha(n)} . \]  

(3.12b)

The operators \( \Pi^{(+n)} \) and \( \Pi^{(-n)} \) are orthogonal projectors, since

\[ \Pi^{(+n)} \Pi^{(+n)} = \Pi^{(+n)} , \quad \Pi^{(-n)} \Pi^{(-n)} = \Pi^{(-n)} , \quad \Pi^{(+n)} \Pi^{(-n)} = 0 . \]  

(3.13)

One may also check that the following relations

\[ \partial^{\alpha_1 \alpha_2} \Pi^{(\pm)}_{\alpha_1} \Pi^{(\pm)}_{\alpha_2} = 0 , \quad \Pi^{(\pm)}_{\alpha_1} \Pi^{(\pm)}_{\alpha_2} \partial_{\beta_1 \beta_2} = 0 \]  

(3.14)
hold. The first identity in (3.14) implies that the field \( h^{(\pm)}_{\alpha(n)} \) is transverse,

\[
\partial^{\beta\gamma} h^{(\pm)}_{\beta\gamma\alpha(n-2)} = 0.
\]  (3.15)

The second identity in (3.14) implies that \( h^{(\pm)}_{\alpha(n)} \) is invariant under the gauge transformations

\[
\delta_{\zeta} h^{(\pm)}_{\alpha(n)} = \partial (\alpha_1 \alpha_2 \zeta_{\alpha_3...\alpha_n}) .
\]  (3.16)

In addition to these, one may show that \( h^{(\pm)}_{\alpha(n)} \) satisfies the identity

\[
\partial^{\beta} (h^{(\pm)}_{\alpha_1\alpha_2...\alpha_n})_{\beta} = \pm \sqrt{\Box} h^{(\pm)}_{\alpha(n)} .
\]  (3.17)

The operators \( \Pi^{(\pm)}_{\alpha(n)} \) contain terms involving the operator \( \Box^{-1/2} \) which requires a special definition. However, the sum

\[
\Pi^{[n]}_{\alpha(n)} \beta(n) := \Pi^{(+)}_{\alpha(n)} \beta(n) + \Pi^{(-)}_{\alpha(n)} \beta(n)
\]  (3.18)

is well defined since it contains only inverse powers of \( \Box \) and all terms involving odd powers of \( \Box^{-1/2} \) cancel out. An important observation is that the map \( h_{\alpha(n)} \rightarrow \Pi^{[n]}_{\alpha(n)} h_{\alpha(n)} \) projects the space of symmetric fields \( h_{\alpha(n)} \) onto the space of divergence-free fields, in accordance with (3.15). Thus our projectors (3.18) are the 3D analogues of the Behrends-Fronsdal projection operators [13, 14].

Furthermore, given an arbitrary field \( h_{\alpha(n)} \), it may be shown that

\[
\left( 1 - \Pi^{[n]} \right) h_{\alpha(n)} = \partial (\alpha_1 \alpha_2 \lambda_{\alpha_3...\alpha_n}) ,
\]  (3.19)

for some \( \lambda_{\alpha(n-2)} \).

Let \( \Phi_{\alpha(n)} \) be a field satisfying the Klein-Gordon equation (2.2), with \( n > 1 \). As a consequence of the above analysis, the following results hold:

(i) \( \Pi^{[n]} \Phi_{\alpha(n)} \) is a solution of the 3D Fierz-Pauli field equations (2.1a) and (2.2); and

(ii) \( \Pi^{(\pm)} \Phi_{\alpha(n)} \) is a solution of the equations (2.1a) and (2.1b).

We now give several examples of the spin projectors (3.18):

\[
\Pi^{[2]} h_{\alpha(2)} = \frac{1}{2} \Box \left( \partial_{\alpha_1} \beta_1 \partial_{\alpha_2} \beta_2 h_{\beta(2)} + \Box h_{\alpha(2)} \right) ,
\]  (3.20a)

\[
\Pi^{[3]} h_{\alpha(3)} = \frac{1}{2^2} \Box \left( 3 \partial_{\alpha_1} \beta_1 \partial_{\alpha_2} \beta_2 h_{\alpha_3(2)} + \Box h_{\alpha(3)} \right) ,
\]  (3.20b)
\[ \Pi^{[4]} h_{\alpha(4)} = \frac{1}{2^3 \Box^2} \left( \partial_{\alpha_1} \beta_1 \cdots \partial_{\alpha_4} \beta_4 h_{\beta(4)} + 6 \Box \partial_{(\alpha_1} \beta_1 \partial_{\alpha_2} \beta_2 h_{\alpha_3\alpha_4)\beta(2)} + \Box^2 h_{\alpha(4)} \right), \] (3.20c)

\[ \Pi^{[5]} h_{\alpha(5)} = \frac{1}{2^4 \Box^2} \left( 5 \partial_{(\alpha_1} \beta_1 \cdots \partial_{\alpha_4} \beta_4 h_{\alpha_5)\beta(4)} + 10 \Box \partial_{(\alpha_1} \beta_1 \partial_{\alpha_2} \beta_2 h_{\alpha_3\alpha_4\alpha_5)\beta(2)} + \Box^2 h_{\alpha(5)} \right), \] (3.20d)

\[ \Pi^{[6]} h_{\alpha(6)} = \frac{1}{2^5 \Box^3} \left( \partial_{\alpha_1} \beta_1 \cdots \partial_{\alpha_6} \beta_6 h_{\beta(6)} + 15 \Box \partial_{(\alpha_1} \beta_1 \cdots \partial_{\alpha_4} \beta_4 h_{\alpha_5\alpha_6)\beta(4)} + 15 \Box^2 \partial_{(\alpha_1} \beta_1 \partial_{\alpha_2} \beta_2 h_{\alpha_3\alpha_4\alpha_5\alpha_6)\beta(2)} + \Box^3 h_{\alpha(6)} \right). \] (3.20e)

All projectors may be rewritten in vector notation via the standard procedure. Namely, given a bosonic symmetric rank-(2s) spinor field \( h_{\alpha(2s)} \), with integer \( s > 0 \), we associate with it the symmetric rank-s tensor \( h_{\alpha_1 \cdots \alpha_s} \)

\[ h_{\alpha_1 \cdots \alpha_s} := \left( \frac{1}{2} \right)^s (\gamma_{\alpha_1})^{\alpha_1} \cdots (\gamma_{\alpha_s})^{\alpha_s} h_{\alpha_1 \alpha_1 \cdots \alpha_s \alpha_s}, \] (3.21a)

which is automatically traceless,

\[ \eta^{bc} h_{bc\alpha_1 \cdots \alpha_{s-2}} = 0. \] (3.21b)

Given a fermionic symmetric rank-(2s + 1) spinor field \( h_{\alpha(2s+1)} \), with integer \( s > 0 \), we associate with it the symmetric rank-s tensor-spinor \( h_{\alpha_1 \cdots \alpha_s \gamma} \) defined by

\[ h_{\alpha_1 \cdots \alpha_s \gamma} := \left( \frac{1}{2} \right)^s (\gamma_{\alpha_1})^{\alpha_1} \cdots (\gamma_{\alpha_s})^{\alpha_s} h_{\alpha_1 \alpha_1 \cdots \alpha_s \alpha_s \gamma}. \] (3.22a)

It is automatically traceless and \( \gamma \)-traceless,

\[ \eta^{bc} h_{bc\alpha_1 \cdots \alpha_{s-2} \gamma} = 0, \quad (\gamma^{b})^{\beta\gamma} h_{b\alpha_1 \cdots \alpha_{s-1} \gamma} = 0. \] (3.22b)

In vector notation, the examples [3.20] are equivalent to

\[ \Pi^{[2]} h_a = \frac{1}{\Box} \left( \Box h_a - \partial_a \partial^b h_b \right), \] (3.23a)

\[ \Pi^{[3]} h_{a\gamma} = \frac{1}{\Box} \left( \Box h_{a\gamma} - \partial_a \partial^b h_{b\gamma} - \frac{1}{2} \varepsilon_{abc} (\gamma^b)^{\delta} \partial^c \partial^d h_{db} \right), \] (3.23b)

\[ \Pi^{[4]} h_{ab} = \frac{1}{\Box^2} \left( \Box^2 h_{ab} - 2 \partial^c \partial^d h_{(a} h_{b)c} + \frac{1}{2} \eta_{ab} \partial^c \partial^d h_{cd} + \frac{1}{2} \partial_a \partial_b \partial^c \partial^d h_{cd} \right), \] (3.23c)

\[ \Pi^{[5]} h_{ab\gamma} = \frac{1}{\Box^2} \left( \Box^2 h_{ab\gamma} - 2 \partial^c \partial^d h_{(a} h_{b)c\gamma} + \frac{1}{4} \eta_{ab} \partial^c \partial^d h_{cd\gamma} + \frac{3}{4} \partial_a \partial_b \partial^c \partial^d h_{cd\gamma} \right. \]

\[ - \frac{1}{2} (\gamma^c)^{\delta} \varepsilon_{cd(a} \left[ \partial^d \partial^f h_{b)\delta} - \partial_{b)} \partial^d \partial^f \partial^g h_{fg\delta} \right], \] (3.23d)

\[ \Pi^{[6]} h_{abc} = \frac{1}{\Box^3} \left( \Box^3 h_{abc} - 3 \partial^d h_{(a} h_{bc)d} + \frac{3}{4} \Box^2 \partial^d \partial^f h_{(a} h_{bc)df} + \frac{9}{4} \partial^d \partial^f \partial^g h_{(a} h_{bc)df} \right. \]

\[ - \frac{3}{4} \Box \eta_{(a} \partial_{b)} \partial^d \partial^f \partial^g h_{dfg} - \frac{1}{4} \partial_a \partial_b \partial_c \partial^d \partial^f \partial^g h_{dfg} \right). \] (3.23e)

One may check that the conditions (3.22b) hold.
4 Linearised higher-spin Cotton tensors

Associated with a conformal gauge field $h_{\alpha(n)}$, with $n > 1$, is the linearised Cotton tensor $C_{\alpha(n)}(h)$ given by the expression (1.5). Its fundamental properties are described by the relations (1.6) and (1.7). In this section we derive a new representation for the higher-spin Cotton tensor $C_{\alpha(n)}$ which makes obvious the properties (1.6) and (1.7).

Making use of the spin projection operators, eqs. (3.11) and (3.12), it is possible to show that the following relation holds

$$h^{(\pm)}_{\alpha(n)} = \Pi^{(\pm n)} h_{\alpha(n)} = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} \times \frac{(\pm 1)^j}{\Box^{j/2}} \partial_{(\alpha_1 \ \beta_1 \ldots} \partial_{\alpha_j \beta_j} h_{\alpha_{j+1} \ldots \alpha_n)\beta_1 \ldots \beta_j} . \quad (4.1)$$

To construct the higher-spin Cotton tensor using the projectors, it is necessary to consider separately the cases of integer and half-integer spin.

We will begin with the fermionic case and set $n = 2s + 1$ for integer $s > 0$. If we take the sum of the positive and negative helicity parts of $h_{\alpha(2s+1)}$, then all terms with odd $j$ in (4.1) will vanish,

$$h^{(+)}_{\alpha(2s+1)} + h^{(-)}_{\alpha(2s+1)} = \frac{1}{2^{2s}} \sum_{j=0}^{s} \binom{2s + 1}{2j + 1} \times \frac{1}{\Box^{s-j}} \partial_{(\alpha_1 \ \beta_1 \ldots} \partial_{\alpha_{2s-2j} \beta_{2s-2j}} h_{\alpha_{2s-2j+1} \ldots \alpha_{2s+1})\beta_1 \ldots \beta_{2s-2j}} . \quad (4.2)$$

From here it follows that the fermionic Cotton tensor may be written as

$$C_{\alpha(2s+1)}(h) = \Box^s \left( \Pi^{(2s+1)} - \Pi^{(-2s-1)} \right) h_{\alpha(2s+1)} . \quad (4.3)$$

In the $n = 2s$ case, we instead take the difference of the positive and negative helicity modes, whereupon all even terms in (4.1) cancel and we obtain

$$h^{(+)}_{\alpha(2s)} - h^{(-)}_{\alpha(2s)} = \frac{1}{2^{2s-1}} \sum_{j=0}^{s-1} \binom{2s}{2j + 1} \times \frac{1}{\Box^{(2s-2j-1)/2}} \partial_{(\alpha_1 \ \beta_1 \ldots} \partial_{\alpha_{2s-2j-1} \beta_{2s-2j-1}} h_{\alpha_{2s-2j+1} \ldots \alpha_{2s+1})\beta_1 \ldots \beta_{2s-2j-1}} . \quad (4.4)$$

Therefore, we may express the bosonic Cotton tensor as

$$C_{\alpha(2s)}(h) = \Box^{s-\frac{1}{2}} \left( \Pi^{(2s)} - \Pi^{(-2s)} \right) h_{\alpha(2s)} . \quad (4.5)$$
By virtue of the identities (3.15) and (3.16), the properties (1.6) and (1.7) are made manifest when $C_{\alpha(n)}$ is represented in the form (4.3) and (4.5).

Using the identity (3.17), it is possible to show that the following relations between the derivative of the Cotton tensors and the projectors hold,

$$
\partial^\beta (\alpha_1 C_{\alpha_2...\alpha_{2s}}) = \Box^s \left( \Pi^{(+2s)} + \Pi^{(-2s)} \right) h_{\alpha(2s)}, \quad (4.6a)
$$

$$
\partial^\beta (\alpha_1 C_{\alpha_2...\alpha_{2s+1}}) = \Box^{s+\frac{1}{2}} \left( \Pi^{(+2s+1)} - \Pi^{(-2s-1)} \right) h_{\alpha(2s+1)}. \quad (4.6b)
$$

Finally, making use of the relations (4.3), (4.5) and (4.6), in conjunction with the identities

$$
\partial_{\alpha} \gamma \Pi_{\gamma}^{(\pm)} = \pm \sqrt{\Box} \Pi_{\alpha}^{(\pm)}, \quad \Pi_{\alpha}^{(\pm)} \partial_{\gamma} \beta = \pm \sqrt{\Box} \Pi_{\alpha}^{(\pm)}, \quad (4.7)
$$

we arrive at the following property

$$
C_{\alpha(n)} (\partial h) = \partial (\alpha_1 \beta C_{\alpha_2...\alpha_{n}}) (h), \quad (\partial h)_{\alpha(n)} := \partial (\alpha_1 \beta h_{\alpha_2...\alpha_{n}}). \quad (4.8)
$$

5 Concluding comments

In four dimensions, the linearised conformal higher-spin actions [6] were originally formulated in terms of the Behrends-Fronsdal projection operators [13, 14], and several years later in terms of the linearised higher-spin Weyl tensors [1, 2]. In three dimensions, making use of the relations (4.3) and (4.5) allows us to rewrite the linearised conformal higher-spin actions [9, 10]

$$
S^{(n)}[h] \propto i^n \int d^3 x h_{\alpha(n)} C_{\alpha(n)} (h) \quad (5.1)
$$

in terms of the spin projection operators. Moreover, making use of (4.6) also allows us to rewrite the massive higher-spin gauge models of [36, 37]

$$
S^{(n)}_{\text{massive}}[h] \propto i^n \int d^3 x C^{(n)}(h) \left\{ \partial^\beta (\alpha_1 - m\sigma \delta^\beta_{\alpha_1}) \right\} h_{\alpha_2...\alpha_{n} \beta}, \quad \sigma = \pm 1 \quad (5.2)
$$

in terms of the spin projection operators. The Bianchi identity (1.7) and the equation of motion derived from (5.2) are equivalent to the massive equations (2.1).

5 The choices $n = 2$ and $n = 4$ in (5.1) correspond to a U(1) Chern-Simons term [32, 33, 34, 35] and a Lorentz Chern-Simons term [34, 35], respectively.

6 The bosonic case, $n = 2, 4, \ldots$, was first described in [36].
In the $n = 2$ case, the action (5.2) proves to be proportional to that for topologically massive electrodynamics \[32, 33, 34, 35\]

$$S = -\frac{1}{4} \int d^3x \left\{ F^{ab}F_{ab} + m \sigma \varepsilon^{abc} h_a F_{bc} \right\}, \quad F_{ab} = \partial_a h_b - \partial_b h_a. \quad (5.3)$$

One may check that for $n = 4$, the action (5.2) yields linearised new topologically massive gravity \[38, 39\].

It should be pointed out that various aspects of the bosonic higher-spin Cotton tensors $C_{\alpha(n)}$, with $n$ even, were studied in \[40, 41\].

The results of this work admit supersymmetric extensions. They will be discussed elsewhere.

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A Spinor conventions

Here we summarise our notation and spinor conventions which follow \[31\]. We use the metric $\eta_{mn} = \text{diag}(-1, 1, 1)$ and normalise the Levi-Civita symbol as $\varepsilon^{012} = -\varepsilon_{012} = 1$.

In the $\text{SL}(2, \mathbb{R})$ picture, the $\gamma$-matrices with lower indices are chosen as

$$(\gamma_m)^{\alpha\beta} = (\gamma_m)^{\beta\alpha} = (1, \sigma_1, \sigma_3). \quad (A.1)$$

The spinor indices are raised and lowered,

$$\psi^{\alpha} = \varepsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta, \quad (A.2)$$

using the antisymmetric tensors $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ and $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ normalised as $\varepsilon_{12} = -1$ and $\varepsilon^{12} = 1$. The Dirac $\gamma$-matrices are

$$(\gamma_m)^{\alpha\beta} := \varepsilon^{\beta\gamma} (\gamma_m)_{\alpha\gamma} = (-i\sigma_2, \sigma_3, \sigma_1). \quad (A.3)$$

The $\gamma$-matrices have the following properties:

$$(\gamma_m)^{\alpha\rho}(\gamma_n)^{\beta\rho} = \eta_{mn} \delta^\beta_\alpha + \varepsilon_{mpq}(\gamma^p)^{\alpha\beta}, \quad (A.4a)$$
\[(\gamma^m)_{\alpha\beta}(\gamma_m)^{\rho\sigma} = -(\delta^\rho_\alpha\delta^\sigma_\beta + \delta^\rho_\beta\delta^\sigma_\alpha), \quad (A.4b)\]
\[\varepsilon_{amn}(\gamma^m)_{\alpha\beta}(\gamma^n)^{\gamma\delta} = \varepsilon_{(\alpha}(\gamma_a)_{\beta)}\delta + \varepsilon\delta(\alpha(\gamma_a)_{\beta)\gamma}. \quad (A.4c)\]

Given a three-vector \(\Phi_a\), it can equivalently be realised as a symmetric rank-2 spinor \(\Phi_{\alpha\beta} = \Phi_{\beta\alpha}\). The relationship between \(\Phi_a\) and \(\Phi_{\alpha\beta}\) is as follows:
\[
\Phi_{\alpha\beta} := (\gamma^a)_{\alpha\beta}\Phi_a, \quad \Phi_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta}\Phi_{\alpha\beta}. \quad (A.5)\]

In the \(SU(1,1)\) picture, the \(\gamma\)-matrices with lower indices are
\[
(\tilde{\gamma}_m)_{\alpha\beta} = (T^{-1})^\alpha_\alpha(T^{-1})^\beta_\beta(\gamma_m)^{\gamma\delta}, \quad (A.6)\]
where \(T\) is given by (2.14). The explicit expressions for these matrices are
\[
(\tilde{\gamma}_m)_{\alpha\beta} = (-i\sigma_1, -i1, \sigma_3). \quad (A.7)\]

For the Dirac \(\gamma\)-matrices we obtain
\[
(\gamma_m)_{\alpha\beta} = \varepsilon^{\beta\gamma}(\gamma_m)_{\alpha\gamma} = (-i\sigma_3, -\sigma_2, -\sigma_1). \quad (A.8)\]

**References**

[1] E. S. Fradkin and V. Y. Linetsky, “Cubic interaction in conformal theory of integer higher-spin fields in four dimensional space-time,” Phys. Lett. B 231, 97 (1989).

[2] E. S. Fradkin and V. Y. Linetsky, “Superconformal higher spin theory in the cubic approximation,” Nucl. Phys. B 350, 274 (1991).

[3] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, IOP, Bristol, 1995 (Revised Edition: 1998).

[4] C. Fronsdal, “Massless fields with integer spin,” Phys. Rev. D 18, 3624 (1978).

[5] C. Fronsdal, “Singletons and massless, integral-spin fields on de Sitter space,” Phys. Rev. D 20, 848 (1979).

[6] E. S. Fradkin and A. A. Tseytlin, “Conformal supergravity,” Phys. Rept. 119, 233 (1985).

[7] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton, 1926.

[8] D. Butter, S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, “Conformal supergravity in three dimensions: New off-shell formulation,” JHEP 1309, 072 (2013) [arXiv:1305.3132 [hep-th]].

[9] C. N. Pope and P. K. Townsend, “Conformal higher spin in (2+1) dimensions,” Phys. Lett. B 225, 245 (1989).
[10] S. M. Kuzenko, “Higher spin super-Cotton tensors and generalisations of the linear-chiral duality in three dimensions,” Phys. Lett. B 763, 308 (2016) [arXiv:1606.08624 [hep-th]].

[11] M. Henneaux, S. Hörtner and A. Leonard, “Higher spin conformal geometry in three dimensions and prepotentials for higher spin gauge fields,” JHEP 1601, 073 (2016) [arXiv:1511.07389 [hep-th]].

[12] M. Henneaux, V. Lekeu, A. Leonard, J. Matulich and S. Prohazka, “Three-dimensional conformal geometry and prepotentials for four-dimensional fermionic higher-spin fields,” JHEP 1811, 156 (2018) [arXiv:1810.04457 [hep-th]].

[13] R. E. Behrends and C. Fronsdal, “Fermi decay of higher spin particles,” Phys. Rev. 106, no. 2, 345 (1957).

[14] C. Fronsdal, “On the theory of higher spin fields,” Nuovo Cim. 9, 416 (1958).

[15] W. Siegel and S. J. Gates, Jr., “Superprojectors,” Nucl. Phys. B 189, 295 (1981).

[16] S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, Superspace, or One Thousand and One Lessons in Supersymmetry, Benjamin/Cummings (Reading, MA), 1983, hep-th/0108200.

[17] A. P. Isaev and M. A. Podolitsyn, “Two-spinor description of massive particles and relativistic spin projection operators,” Nucl. Phys. B 929 (2018) 452 [arXiv:1712.00833 [hep-th]].

[18] A. Y. Segal, “Conformal higher spin theory,” Nucl. Phys. B 664, 59 (2003) [hep-th/0207212].

[19] R. Bonezzi, “Induced action for conformal higher spins from worldline path integrals,” Universe 3, no. 3, 64 (2017) [arXiv:1709.00850 [hep-th]].

[20] R. Jackiw and V. P. Nair, “Relativistic wave equations for anyons,” Phys. Rev. D 43, 1933 (1991).

[21] S. P. Novikov and I. A. Taimanov, Modern Geometric Structures and Fields, American Mathematical Society, Providence, 2006.

[22] I. V. Gorbunov, S. M. Kuzenko and S. L. Lyakhovich, “On the minimal model of anyons,” Int. J. Mod. Phys. A 12, 4199 (1997) [hep-th/9607114].

[23] I. V. Tyutin and M. A. Vasiliev, “Lagrangian formulation of irreducible massive fields of arbitrary spin in (2+1) dimensions,” Teor. Mat. Fiz. 113N1, 45 (1997) [Theor. Math. Phys. 113, 1244 (1997)] [hep-th/9704132].

[24] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “On higher derivatives in 3D gravity and higher spin gauge theories,” Annals Phys. 325, 1118 (2010) [arXiv:0911.3061 [hep-th]].

[25] M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field,” Proc. Roy. Soc. Lond. A 173, 211 (1939).

[26] J. A. de Azcárraga, S. Fedoruk, J. M. Izquierdo and J. Lukierski, “Two-twistor particle models and free massive higher spin fields,” JHEP 1504, 010 (2015) [arXiv:1409.7169 [hep-th]].

[27] E. P. Wigner, “On unitary representations of the inhomogeneous Lorentz group,” Annals Math. 40, 149 (1939) [Nucl. Phys. Proc. Suppl. 6, 9 (1989)].

[28] B. Binegar, “Relativistic field theories in three dimensions,” J. Math. Phys. 23, 1511 (1982).
[29] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, American Mathematical Society, Providence, 1968.

[30] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, “Unconstrained N = 2 matter, Yang-Mills and supergravity theories in harmonic superspace,” Class. Quant. Grav. 1, 469 (1984).

[31] S. M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, “Off-shell supergravity-matter couplings in three dimensions,” JHEP 1103, 120 (2011) [arXiv:1101.4013 [hep-th]].

[32] W. Siegel, “Unextended superfields in extended supersymmetry,” Nucl. Phys. B 156, 135 (1979).

[33] J. F. Schonfeld, “A mass term for three-dimensional gauge fields,” Nucl. Phys. B 185, 157 (1981).

[34] S. Deser, R. Jackiw and S. Templeton, “Three-dimensional massive gauge theories,” Phys. Rev. Lett. 48, 975 (1982).

[35] S. Deser, R. Jackiw and S. Templeton, “Topologically massive gauge theories,” Annals Phys. 140, 372 (1982) [Erratum-ibid. 185, 406 (1988)].

[36] E. A. Bergshoeff, M. Kovacevic, J. Rosseel, P. K. Townsend and Y. Yin, “A spin-4 analog of 3D massive gravity,” Class. Quant. Grav. 28, 245007 (2011) [arXiv:1109.0382 [hep-th]].

[37] S. M. Kuzenko and M. Ponds, “Topologically massive higher spin gauge theories,” JHEP 1810, 160 (2018) [arXiv:1806.06643 [hep-th]].

[38] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin and P. K. Townsend, “Massive 3D supergravity,” Class. Quant. Grav. 27, 025010 (2010) [arXiv:0907.4658 [hep-th]].

[39] D. Dalmazi and E. L. Mendonca, “A new spin-2 self-dual model in D=2+1,” JHEP 0909, 011 (2009) [arXiv:0907.5009 [hep-th]].

[40] H. Linander and B. E. W. Nilsson, “The non-linear coupled spin 2 - spin 3 Cotton equation in three dimensions,” JHEP 1607, 024 (2016) [arXiv:1602.01682 [hep-th]].

[41] T. Basile, R. Bonezzi and N. Boulanger, “The Schouten tensor as a connection in the unfolding of 3D conformal higher-spin fields,” JHEP 1704, 054 (2017) [arXiv:1701.08615 [hep-th]].