AN EASY WAY TO A THEOREM OF KIRA ADARICHEVA AND MADINA BOLAT ON CONVEXITY AND CIRCLES

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Dedicated to the eighty-fifth birthday of Béla Csákány

Abstract. Kira Adaricheva and Madina Bolat have recently proved that if \( U_0 \) and \( U_1 \) are circles in a triangle with vertices \( A_0, A_1, A_2 \), then there exist \( j \in \{0, 1, 2\} \) and \( k \in \{0, 1\} \) such that \( U_1 - k \) is included in the convex hull of \( U_0 \cup (\{A_0, A_1, A_2\} \setminus \{A_j\}) \). We give a short new proof for this result, and we point out that a straightforward generalization for spheres fails.

1. Aim and introduction

Our goal. The real \( n \)-dimensional space and the usual convex hull operator on it will be denoted by \( \mathbb{R}^n \) and \( \text{Conv}_{\mathbb{R}^n} \). That is, for a set \( X \subseteq \mathbb{R}^n \) of points, \( \text{Conv}_{\mathbb{R}^n}(X) \) is the smallest convex subset of \( \mathbb{R}^n \) that includes \( X \). In this paper, the Euclidean distance \( \sum_{i=1}^n (X_i - Y_i)^2 \) of \( X,Y \in \mathbb{R}^n \) is denoted by \( \text{dist}(X,Y) \). For \( P \in \mathbb{R}^2 \) and \( 0 \leq r \in \mathbb{R} \), the circle of center \( P \) and radius \( r \) will be denoted by

\[
\text{Circ}(P,r) := \{X \in \mathbb{R}^2 : \text{dist}(P,X) = r\}.
\]

Notably enough, Adaricheva and Bolat [2, Theorem 5.1] states even more than [2, Theorem 3.1]; we formulate their more general result as follows.

Corollary 1.2 (Adaricheva and Bolat [2, Theorem 5.1]). If \( C_0, C_1, C_2, U_0, \) and \( U_1 \) are circles in the plane such that \( U_i \subseteq \text{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\}) \) for \( i \in \{0, 1\} \), then there exist subscripts \( j \in \{0, 1, 2\} \) and \( k \in \{0, 1\} \) such that

\[
(1.1) \quad U_{1-k} \subseteq \text{Conv}_{\mathbb{R}^2}(U_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})).
\]

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Note that Adaricheva and Bolat [2] call the property stated in this corollary for circles the “Weak Carousel property”. Note also that [2] gives a new justification to Czéli and Kincses [10], because Theorem 5.2 and Section 6 in [2] yield that the almost-circles in [10] cannot be replaced by circles. Also, [2] motivates Czéli [9] and Kincses [13]. This paper is self-contained. For more about the background of this topic, the reader may want, but need not, to see, for example, Adaricheva and Nation [5] and [6], Czéli [8], Edelman and Jamison [11], Kashiwabara, Nakamura, and Okamoto [12], Monjardet [14], and Richter and Rogers [16].

The results of Adaricheva and Bolat [2], that is, Theorem 1.1 and Corollary 1.2 above, and our easy approach raise the question whether the most straightforward generalizations hold for 3-dimensional spheres. In Section 4, which is a by-product of our method in some implicit sense, we give a negative answer.

2. Homotheties and round-edged angles

2.1. A single circle. For \(0 < r \in \mathbb{R}\) and \(F, P \in \mathbb{R}^2\) with \(\text{dist}(F, P) > r\), let

\[ \text{Ang}(F, \text{Circ}(P, r)) \]

be the grey-filled area in Figure 1; it is called the round-edged angle determined by its focus \(F\) and spanning circle \(\text{Circ}(P, r)\). Note that \(\text{Ang}(F, \text{Circ}(P, r))\) is not bounded from the right and \(F\) is outside both \(\text{Circ}(P, r)\) and \(\text{Ang}(F, \text{Disk}(P, r))\). Note that \(\text{Ang}(F, \text{Disk}(P, r))\) includes its boundary, which consists of a circular arc called the front arc and two half-lines.

![Figure 1. Round-edged angle](image)

2.2. Externally perspective circles. First, recall or define some easy concepts and notations. For topologically closed convex sets \(W_1, W_2 \subseteq \mathbb{R}^2\), we will say that

\[ W_1 \text{ is loosely included in } W_2, \text{ in notation, } W_1 \text{ loose } \subset W_2, \]

if every point of \(W_1\) is an internal point of \(W_2\). Given \(P \in \mathbb{R}^2\) and \(0 \neq \lambda \in \mathbb{R}\), the homothety with (homothetic) center \(P\) and ratio \(\lambda\) is defined by

\[ \chi_{P,\lambda} : \mathbb{R}^2 \to \mathbb{R}^2 \text{ by } X \mapsto PX\lambda := (1 - \lambda)P + \lambda X. \]

We will not need negative ratios \(\lambda\) and we use the Polish notation for the barycentric operation \(\lambda\). Homotheties are similarity transformations. In particular, they map the center of a circle to the center of its image. If \(C_1\) and \(C_2\) are circles and \(C_2 = \chi_{P,\lambda}(C_1)\) such that \(P\) is (strictly) outside both \(C_1\) and \(C_2\) (equivalently, if \(P\) is outside \(C_1\) or \(C_2\)) and \(0 < \lambda \in \mathbb{R}\), then \(C_1\) and \(C_2\) will be called externally perspective circles. Clearly, if \(C_1\) and \(C_2\) are of different radii and none of them is inside the other, then \(C_1\) and \(C_2\) are externally perspective, \(P\) is the intersection point of their external tangent lines, and \(\lambda\) is the ratio of their radii.
Lemma 2.1. Let $\text{Circ}(P_1, r_1)$ and $\text{Circ}(P_2, r_2)$ be externally perspective circles in the plane with center $F$ of perspectivity such that $0 < r_2 < r_1$; see Figure 2. If $G$ is a point on the line segment $[F, P_2]$ such that $r_2 < \text{dist}(G, P_2) < \text{dist}(F, P_2)$, then $\text{Ang}(F, \text{Circ}(P_1, r_1)) \subset \text{Ang}(G, \text{Circ}(P_2, r_2))$; see (2.2) and Figure 2.

**Figure 2. Illustration for Lemma 2.1**

*Proof.* Clearly, $\text{Circ}(P_1, r_1) = \chi_{F, \lambda}(\text{Circ}(P_2, r_2))$ with $\lambda = r_1/r_2 > 1$. The external tangent lines of our circles intersect at $F$. Since $\chi_{F, \lambda}$ preserves tangency, it maps the circular arc $I_2$ of $\text{Circ}(P_2, r_2)$ between the tangent points onto the circular arc $I_1$ of $\text{Circ}(P_1, r_1)$ between the images of these tangent points; see the thick arcs in Figure 2. Hence, $I_2$ is strictly on the left of $I_1$ in the figure, implying the lemma. □

Lemma 2.2. If $\lambda, \mu \in \mathbb{R} \setminus \{0\}$, $F, Q \in \mathbb{R}^2$, and $R = \chi_{F, \lambda}(Q)$, then, composing maps from right to left, $\chi_{R, \mu} \circ \chi_{F, \lambda} = \chi_{F, \lambda} \circ \chi_{Q, \mu}$.

*Proof.* $\chi_{F, \lambda} \circ \chi_{Q, \mu} \circ \chi_{R, \mu}$ is clearly a homothety of ratio $\mu$ that fixes $R$. So this homothety is $\chi_{R, \mu}$, which implies the lemma. □

Lemma 2.3. If $\lambda > 1$ and $C_0$ and $C_1$ are internally tangent circles with center points $C_0^*$ and $C_1^*$, respectively, then either one of $\chi_{\lambda, C_0^*}(C_0)$ and $\chi_{\lambda, C_1^*}(C_1)$ is in the interior of the other, or $C_0 = C_1$.

**Figure 3. Illustration for Lemma 2.3**

*Proof.* We can assume that the radii $r_0$ and $r_1$ are distinct, say, $r_0 < r_1$; see Figure 3. The distance $d := \text{dist}(C_0^*, C_1^*)$ is $r_1 - r_0$. Since $\lambda r_1 = \lambda(r_0 + d) > \lambda r_0 + d$, $\chi_{\lambda, C_0^*}(C_0)$ is in the interior of $\chi_{\lambda, C_1^*}(C_1)$, as required. □
The following lemma resembles the 2-Carousel Rule in Adaricheva [1].

**Lemma 2.4.** Let $A_0$, $A_1$, and $A_2$ be non-collinear points in the plane. If $B_0$ and $B_1$ are distinct internal points of $\text{Conv}_{R^2}([A_0, A_1, A_2])$, then there exist $j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$ such that
\[
\{B_{1-k}\} \subset \text{Conv}_{R^2}(\{B_k \cup \{A_0, A_1, A_2 \setminus \{A_j\}\})).
\]

**Proof.** Since the triangle $\text{Conv}_{R^2}(\{A_0, A_1, A_2\})$ is clearly of the form (2.4)
\[
\text{Conv}_{R^2}(\{B_0, A_1, A_2\}) \cup \text{Conv}_{R^2}(\{A_0, B_0, A_2\}) \cup \text{Conv}_{R^2}(\{A_0, A_1, B_0\}).
\]
$B_1$ belongs to at least one of the triangles in (2.4). If one of these three triangles, say, $\text{Conv}_{R^2}(\{B_0, A_1, A_2\})$, contains $B_1$ as an internal point, then we let $k = 0$ and $j = 0$. Otherwise, there is a $j' \in \{0, 1, 2\}$ such that the line segment $[B_0, A_{j'}]$ contains $B_1$ in its interior, and we can clearly let $k = 1$ and $j = j'$. □

### 3. Proving Theorem 1.1 with Analytic Tools

**Proof of Theorem 1.1.** If $A_0$, $A_1$, and $A_2$ are collinear points, then the circles are of radii 0 and (1.1) holds trivially (even without $U_k$ on the right). Hence, in the rest of the proof, we assume that $A_0$, $A_1$, and $A_2$ are non-collinear points. We let
\[
T := \text{Conv}_{R^2}(\{A_0, A_1, A_2\}).
\]
Let $P_i$ and $r_i$ denote the center and the radius of $U_i$ from the theorem. Note that (3.1)
\[
r_1 = 0 \text{ implies (1.1), by (2.4) applied for } B_0 \in U_0 \text{ and } B_1 = P_1;
\]
and similarly for $r_0 = 0$. Therefore, we will assume that none of $r_0$ and $r_1$ is zero. From now on, we prove the theorem by way of contradiction. That is, we assume that $U_0$ and $U_1$ are circles satisfying the assumptions of Theorem 1.1, $r_0 r_1 > 0$, but (1.1) fails. For $0 \leq \xi \leq 1$ and $k \in \{0, 1\}$, we denote $\text{Circ}(P_k, \xi \cdot r_k)$ by $U_k(\xi)$. Let
\[
H := \{\eta \in [0, 1] : (\forall \xi \in [0, \eta]) (\exists k \in \{0, 1\}) (\exists j \in \{0, 1, 2\}) \text{ such that } U_{1-k}(\xi) \subseteq \text{Conv}_{R^2}(U_k(\xi) \cup \{A_0, A_1, A_2 \setminus \{A_j\}\})\}
\]
In other words, $H$ consists of those $\eta$ for which $U_0(\xi), U_1(\xi), A_0, A_1,$ and $A_2$ satisfy the theorem for all $\xi$ in the closed interval $[0, \eta] \subseteq [0, 1] \subseteq R$. For brevity, we let
\[
W(j, k, \xi) := \text{Conv}_{R^2}(U_k(\xi) \cup \{A_0, A_1, A_2 \setminus \{A_j\}\}); \text{ then } H := \{\eta \in [0, 1] : (\forall \xi \in [0, \eta]) (\exists k) (\exists j) (U_{1-k}(\xi) \subseteq W(j, k, \xi))\}
\]
By (3.1), $0 \in H$. Since $U_k(1) = U_k$, for $k \in \{0, 1\}$, our indirect assumption gives that $1 \not\in H$. Clearly, if $0 \leq \eta_1 \leq \eta_2 \leq 1$ and $\eta_2$ belongs to $H$, then so does $\eta_1$; in other words, $H$ is an order ideal of the poset $\langle [0, 1], \leq \rangle$. From now on,
\[
\text{(3.4) let } \xi \text{ denote the supremum of } H.
\]
We are going to show that
\[
\text{(3.5) } \xi \in H, \text{ whereby } \xi \text{ is actually the maximum of } H, \text{ and } \xi > 0.
\]
Since $r_0, r_1 > 0$ and $P_0$ and $P_1$ are internal points of the triangle $T$, it follows from Lemma 2.4 that $\xi > 0$. In order to prove the rest of (3.3) by way of contradiction, suppose that $\xi \not\in H$. However, for each $i$ such that $[1/\xi] < i \in \mathbb{N}$, in short, for each sufficiently large $i$, $\xi - 1/i \in H$. Hence, for each sufficiently large $i$, we can pick a $k_i \in \{0, 1\}$ and a $j_i \in \{0, 1, 2\}$ such that $U_{1-k_i}(\xi - 1/i) \subseteq W(j_i, k_i, \xi - 1/i)$; see (3.3). Since $\{0, 1\} \times \{0, 1, 2\}$ is finite, one of its pairs, $(k, j)$, occurs infinitely many
times in the sequence of pairs \((k_i, j_i)\). Thus, there exist a \(k \in \{0, 1\}\), a \(j \in \{0, 1, 2\}\), and an infinite set \(I \subseteq \mathbb{N}\) of sufficiently large integers \(i\) such that

\[
\text{(3.6) for all } i \in I, \text{ we have that } U_{1-k}(\xi - 1/i) \subseteq W(j, k, \xi - 1/i).
\]

Since, for all \(\eta\) and \(\zeta\), \(0 \leq \eta \leq \zeta\) implies \(W(j, k, \eta) \subseteq W(j, k, \zeta)\), (3.6) yields that

\[
\text{(3.7) for all } i \in I, \text{ we have that } U_{1-k}(\xi - 1/i) \subseteq W(j, k, \xi).
\]

Next, let \(X\) be an arbitrary point of the circle \(U_{1-k}(\xi)\). Denote by \(X_i\) the point \(X_{P_{1-k}(\xi - 1/i)/\xi}(X)\); it belongs to \(U_{1-k}(\xi - 1/i)\). Less formally, we obtain \(X_i\) as the intersection of \(U_{1-k}(\xi - 1/i)\) with the line segment connecting \(X\) and \(P_{1-k}(\xi - 1/i)/\xi\). As \(i \to \infty\), \(X_i \to X\). Combining this with \(X_i \in U_{1-k}(\xi - 1/i)\) and (3.7), we obtain that \(X\) is a limit point (AKA accumulation point or cluster point) of \(W(j, k, \xi)\). The convex hull of a compact subset of \(\mathbb{R}^n\) is compact; see, for example, Proposition 5.2.5 in Papadopoulos [15]. Hence, \(W(j, k, \xi)\) from (3.3) is a compact set; whereby it contains its limit point, \(X\). Thus, since \(X\) was an arbitrary point of \(U_{1-k}(\xi)\), we conclude that \(U_{1-k}(\xi) \subseteq W(j, k, \xi)\). By (3.3), this proves (3.5).

**Figure 4.** Illustration for (3.8)

Since \(\xi \in H\), we can assume that the indices are chosen so that \(U_1(\xi)\) is included in the grey-filled “round-backed trapezoid”

\[
\text{(3.8) } D(\xi) := \text{Conv}_{\mathbb{R}^2}(\{A_0, A_1\} \cup U_0(\xi)); \text{ see Figure 4}
\]

If \(U_1(\xi)\) was included in the interior of \(D(\xi)\), then there would be a (small) positive \(\varepsilon\) such that \(U_1(\xi + \delta) \subseteq D(\xi) \subseteq D(\xi + \delta)\) for all \(\delta \in (0, \varepsilon]\) and \(\xi + \varepsilon\) would belong to \(H\), contradicting (3.4). Therefore, \(U_1(\xi)\) is tangent to the boundary of \(D(\xi)\). Since \(\xi < 1\) and \(U_1(1) = U_1\) is still included in the triangle \(T\), \(U_1(\xi)\) cannot be tangent to the side \([A_0, A_1]\) of \(T\). If \(U_1(\xi)\) was tangent to the back arc of the “round-backed trapezoid” \(D(\xi)\) and so to \(U_0(\xi)\), then one of \(U_0 = U_0(1)\) and \(U_1 = U_1(1)\) would be...
in the interior of the other by Lemma 2.3 and this would contradict the indirect assumption that \([1,1] \) fails. Hence \( U_1(\xi) \) is tangent to one of the “legs” of \( D(\xi) \); this leg is an external tangent line \( e \) of the circles \( U_1(\xi) \) and \( U_0(\xi) \) through, say, \( A_0 \); see Figure 4. The corresponding touching points will be denoted by the figure. Let \( \lambda := \text{dist}(A_0, E_1)/\text{dist}(A_0, E_0) \); note that \( 0 < \lambda < 1 \). By well-known properties of homotheties, the auxiliary circle

\[
C := \chi_{A_0, \lambda}(U_0(\xi)), \quad \text{with center } P := \chi_{A_0, \lambda}(P_0),
\]

touches \( e \) and, thus, \( U_1(\xi) \) at \( E_1 \). Let \( f \) denote the other tangent of \( U_0(\xi) \) through \( A_0 \). Let \( A_1^* \) and \( A_2^* \) be the intersection points of \( f \) and \( e \) with the line through \( A_1 \) and \( A_2 \), respectively. Since \( U_0(1) = U_0 \) is also included in \( T \) and \( U_0(\xi) \) is a smaller circle concentric to \( U_0 \), both \( A_1^* \) and \( A_2^* \) are in the interior of the line segment \([A_0, P]\) such that \( G \) is outside \( C \) and \( G \) is so close to \( A_0 \) that the tangent lines \( e' \) and \( f' \) of \( C \) through \( G \) intersect the line segments \([A_2^*, A_2]\) and \([A_1, A_1^*]\) at some of their internal points, which we denote by \( A'_2 \) and \( A'_1 \), respectively. Since the “round-backed trapezoid” \( \text{Conv}_{\mathbb{R}^2}(\{A'_1, A'_2\} \cup C) \) is clearly the intersection of the round-edged angle \( \text{Ang}(G, C) \) and one of the half-planes determined by the line through \( A'_1 \) and \( A'_2 \), we obtain from Lemma 2.1 that \( U_0(\xi) \subseteq \text{Conv}_{\mathbb{R}^2}(\{A'_1, A'_2\} \cup C) \).

Combining this with the obvious \( \text{Conv}_{\mathbb{R}^2}(\{A'_1, A'_2\} \cup C) \subseteq \text{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup C) \), we obtain that \( U_0(\xi) \subseteq \text{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup C) \) for all \( \delta \in (0, \varepsilon) \).

Let \( r \) be the radius of \( C \). Depending on \( r \), there are two cases. First, if \( r_1 > r \), then \( C \) is inside \( U_1(\xi) \) and, consequently, also in \( U_1(\xi + \delta) \), whereby (3.10) leads to

\[
U_0(\xi + \delta) \subseteq \text{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup C) \subseteq \text{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup U_1(\xi + \delta))
\]

for all \( \delta \in (0, \varepsilon) \). This gives that \( \xi + \varepsilon \in H \), contradicting (3.4).

Second, let \( r_1 \leq r \). Now \( U_1(\xi) \) coincides with or is inside \( C \). By Lemma 2.3

\[
\chi_{P, \mu}(U_1(\xi)) \subseteq \text{Conv}_{\mathbb{R}^2}(\{A_0 \cup U_0(\xi)\})
\]

Clearly, \( C \subseteq T \), since so is \( U_0(\xi) \). Hence, we can choose a (small) positive \( \delta \) such that \( \chi_{P, \mu}(U_0(\xi)) = U_0(\xi) \) and \( \chi_{P, \mu}(C) \) are loosely included in \( T \) for every \( \mu \in [1, 1 + \delta] \). Furthermore, for every \( \mu \in [1, 1 + \delta] \),

\[
\chi_{P, \mu}(C) \subseteq \chi_{P, \mu}(\chi_{A_0, \lambda}(U_0(\xi))) \quad \text{by Lemma 2.2}
\]

\[
\chi_{A_0, \lambda}(\chi_{P, \mu}(U_0(\xi))) = \chi_{A_0, \lambda}(U_0(\xi)).
\]

Since \( 0 < \lambda < 1 \), it follows that

\[
\chi_{P, \mu}(C) \subseteq \chi_{A_0, \lambda}(U_0(\xi)) \in \text{Conv}_{\mathbb{R}^2}(\{A_0 \cup U_0(\xi)\}),
\]

whence

\[
U_1(\xi) = \chi_{P, \mu}(U_1(\xi)) \subseteq \text{Conv}_{\mathbb{R}^2}(\chi_{P, \mu}(C)) \subseteq \text{Conv}_{\mathbb{R}^2}(\{A_0 \cup U_0(\xi)\}).
\]

Since this holds for all \( \mu \in [1, 1 + \delta] \), we conclude that \( \xi(1 + \delta) \in H \). This contradicts (3.4), completing the proof of Theorem 1.1. \( \square \)
4. Examples

This section explains why we have been unable to generalize Theorem 1.1 for spheres so far. In our first example, one can change \(-1, 0\) and \(-1 - k\) to \(0, 1\) and \(1 - k\), respectively; we have chosen \(-1, 0\) and \(-1 - k\) for a technical reason.

Example 4.1. Let \(A_0, \ldots, A_3\) be the vertices of a regular tetrahedron as well as some vertices of a cube; see Figure 5. Let \(B\) and \(C\) be the middle points of the line segments \([A_0, A_1]\) and \([A_2, A_3]\), respectively, and let \(P_{-1}\) and \(P_0\) divide \([B, C]\) into three equal parts as the figure shows. Finally, let \(S_{-1}\) and \(S_0\) be spheres in the interior of the tetrahedron \(\text{Conv}_{\mathbb{R}^3}(\{A_0, \ldots, A_3\})\) with centers \(P_{-1}\) and \(P_0\) and of the same positive radius. Then, for all \(j \in \{0, 1, 2, 3\}\) and \(k \in \{-1, 0\}\),

\[
S_{-1 - k} \not\subseteq \text{Conv}_{\mathbb{R}^3}(S_k \cup \bigcup(\{A_0, A_1, A_2, A_3\} \setminus \{A_j\})).
\]

**Proof.** By symmetry, it suffices to show (4.1) only for \(j = 3\). First, let \(k = 0\). We denote by \(\pi\) the orthogonal projection of \(\mathbb{R}^3\) to the plane containing \(A_2, A_3\) and \(B\). Suppose for a contradiction that \(S_{-1} \subseteq \text{Conv}_{\mathbb{R}^3}(S_0 \cup A_0 \cup A_1 \cup A_2)\); this inclusion is preserved by \(\pi\). Since \(\pi\) commutes with the formation of convex hulls and the disk \(\pi(S_{-1})\) is not included in \(\text{Conv}_{\mathbb{R}^2}(\pi(S_0) \cup \pi(\{A_0, A_1, A_2\}))\), the grey-filled area in Figure 6 which is a contradiction. Second, if \(k = -1\), then the argument is essentially the same but the grey-filled area in Figure 6 has to be changed. □
Example 4.2. For \( t \in \{3, 4, 5, \ldots \} \), add \( t - 2 \) additional spheres to the previous example in the following way. Let \( P_1, \ldots, P_{t-2} \) divide the line segment \([P_0, P_{-1}]\) equidistantly; see Figure 6 for \( t = 4 \). This figure contains also a circular dotted arc with a sufficiently large radius; its center is far above the triangle. Besides the boundary circles of the disks \( \pi(S_0) \) and \( \pi(S_{-1}) \) from the previous example, let \( C_1, \ldots, C_{t-2} \) be additional circles with centers \( P_1, \ldots, P_{t-2} \) such that all the (little) circles are tangent to the dotted arc; this idea is taken from Czédli [8, Figure 5]. For \( i \in \{1, \ldots, t-2\} \), let \( S_i \) be the sphere obtained from \( C_i \) by rotating it around the line through \( B \) and \( C \). Note that \( \pi(S_i) \approx C_i \) in Figure 6 means that the circle \( C_i \) is the boundary of the disk \( \pi(S_i) \). Now, for all \( j \in \{0, 1, 2, 3\} \) and \( k \in \{-1, 0, \ldots, t-2\} \), \( S_k \) is not a subset of
\[
\text{Conv}_{\mathbb{R}^2}(\bigcup \{S_{-1}, S_0, \ldots, S_{t-2}\} \setminus \{S_k\}) \cup \bigcup \{\{A_0, A_1, A_2, A_3\} \setminus \{A_j\}\}
\]
while all the \( S_k \) are still included in the tetrahedron \( \text{Conv}_{\mathbb{R}^3}(\{A_0, \ldots, A_3\}) \).

Proof. Combine the previous proof and Czédli [8 Example 4.3].

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