**Research Article**

**Products of Toeplitz Operators on the 2-Analytic Bergman Space**

Bo Zhang, Yixin Yang, and Yufeng Lu

1School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China
2School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

Correspondence should be addressed to Yixin Yang; yangyixin@dlut.edu.cn

Received 24 May 2021; Revised 6 September 2021; Accepted 15 September 2021; Published 4 October 2021

Academic Editor: Nikolai L. Vasilevski

Copyright © 2021 Bo Zhang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1Bo Zhang, Yixin Yang, and Yufeng Lu

2School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

1School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China

1Introduction

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ equipped with the normalized Lebesgue area measure $dA(z) = (1/\pi)dx dy$, and let $L^2(\mathbb{D}, dA)$ denote the Lebesgue space on $\mathbb{D}$. For $n \in \mathbb{Z}^+$, let $A^2_n$ denote the $n$-analytic Bergman space, that is, the subspaces of $L^2$ consisting of $n$-differentiable functions such that $\partial^n f / \partial z^n = 0$, where

$$\partial^n z = \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

As we know, $A^2_n$ is a Hilbert subspace with the inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(\omega) g(\omega) dA(\omega),$$

where $f, g \in A^2_n$.

The planar Beurling transform is the singular integral operator given by

$$Sf(z) = \int_{\mathbb{C}} \frac{f(\omega)}{(\omega - z)^2} dA(\omega), \quad z \in \mathbb{C}.$$  

It is well known that the Beurling transform is a unitary operator acting on $L^2(\mathbb{C}, dA)$ (see [1], p. 364). For $\mathbb{D} \subset \mathbb{C}$, the compression of the Beurling transform to $L^2$ is a bounded linear operator acting on $L^2$ defined by

$$S_D f(z) = \int_{\mathbb{D}} \frac{f(\omega)}{(\omega - z)^2} dA(\omega), \quad f(z) \in L^2.$$  

The $n$-analytic Bergman projection $P_n$ is defined to be the orthogonal projection of $L^2$ onto $A^2_n$. The singular integral operator $S_D$ is related to $P_n$, and it is known (see [2]) that

$$P_n = I - (S_D)^n, \quad n \in \mathbb{Z}^+.$$  

For a function $u \in L^{\infty}$, the Toeplitz operator $T_u$ with symbol $u$ on $A^2_n$ is defined by

$$T_u f = P_n(uf), \quad f \in A^2_n.$$  

$n$-analytic functions play an important role in mathematical, and the space $A^2_n$ has been intensively studied. More details about the structure of these spaces can be found in paper [3–5] and Balk’s book [6].

Zero-product problem is a very important question in the operator theory. For Toeplitz operators, we have the general zero-product problem. Namely, if $f$ and $g$ are bounded functions such that $T_f T_g = 0$, then must one of the functions be zero? Ahern and Cučković (see [7]) obtained an

$$...$$
affirmative answer for Toeplitz operators on $A^2$ when one of the functions is radial. Le (see [8, 9]) generalized this result to more than two functions. Ćučković and Le (see [10]) gave a positive answer when both functions are harmonic. While the general zero-product problem (even on $A^2$) is still far from being solved, it is known that Toeplitz operators with radial symbols are diagonal with respect to the standard orthonormal basis of $A^2$. However, this is not the case on $A^2$ when $n \geq 2$. Then, Ćučković and Le (see [10]) raised the following open question:

**Question 1.** Let $f$ and $g$ be bounded functions, one of which is radial. If $T_f T_g = 0$ on $A^2_n$ (or more generally, $T_f T_g$ has finite rank), must one of these functions be zero?

In this paper, we give a partial answer to this question on the 2-analytic Bergman space $A^2_2$. We show that if $g$ is a radial function satisfying a Mellin transform condition, then $T_f T_g = 0$ if and only if $f$ is a zero function.

## 2. Some Preliminary Results

We adopt the following boundary conditions for the binomial coefficients:

$$
\binom{n}{m} = 0, \text{where } n = 0, \pm 1, \pm 2, \cdots \text{ and } m = 1, 2, \cdots,
$$

$$
\binom{n}{m + n} = 0, \text{where } n = 0, 1, 2, \cdots \text{ and } m = 1, 2, \cdots.
$$

(7)

An orthogonal basis in the space $A^2_n$ is given by (see [3, 11])

$$
\phi_{jk} = \sqrt{k + j - 1} \frac{1}{(k + j - 2)!} \frac{\partial^{k+j-2}}{\partial z^{k-j} \partial \bar{z}^{j-1}} (|z|^2 - 1)^{k+j-2},
$$

(8)

where $k = 1, 2, \cdots$ and $j = 1, 2, \cdots, n$. The orthogonal basis can also be written as

$$
\phi_{jk} = \sqrt{k + j - 1} \sum_{i=0}^{j-1} \frac{(-1)^i}{i!} \binom{j-1}{i} \binom{j+k-i-2}{j-1} z^{k-j} \bar{z}^{j-1},
$$

(9)

where $k = 1, 2, \cdots$ and $j = 1, 2, \cdots, n$. For $n = 2$, we have the following lemma.

**Lemma 2.** An orthogonal basis in $A^2_2$ is given by

$$
\phi_{1k} = \sqrt{k} z^{k-1},
$$

$$
\phi_{2k} = \sqrt{k + 1} \left( \frac{k^2 - 1}{k} z - (k - 1) z^{k-2} \right),
$$

(10)

where $k = 1, 2, \cdots$.

For each $z \in \mathbb{D}$, since the point evaluation at $z$ is a bounded linear functional on $A^2_n$, there exists a unique reproducing kernel function $K(z, w) \in A^2_n$ such that

$$
g(z) = \int_{\mathbb{D}} g(w) K(z, w) dA(w), \quad z \in \mathbb{D},
$$

(11)

for every $g \in A^2_n$. On 2-analytic Bergman space $A^2_2$,

$$
K(z, w) = \sum_{k=1}^{\infty} \phi_{1k}(z) \phi_{1k}^*(w) + \sum_{k=1}^{\infty} \phi_{2k}(z) \phi_{2k}^*(w).
$$

(12)

The Mellin transform $\hat{g}$ of a function $g \in L^1([0, 1], rd\omega)$ is defined by

$$
\hat{g}(z) = \int_0^1 g(s) s^{z-1} ds.
$$

(13)

It is easy to see that $\hat{g}$ is well defined and analytic on the right half-plane $\{z : \Re z \geq 2\}$. Ćučković and Rao (see [12]) first used the Mellin transform to study Toeplitz operators on the classical Bergman space. For notational convenience, we define $\phi_{10} = \phi_{20} = 0$ and $a_0 = b_0 = c_0 = d_0 = 0$. For some Toeplitz operators on 2-analytic Bergman space $A^2_2(\mathbb{D})$, we obtain the following lemmas.

**Lemma 3.** Let $g$ be a bounded radial function. Then, for each $p = 1, 2, \cdots$, we have

$$
\int_{\mathbb{D}} g(r) r^{p-1} K(z, w) dA(w) = a_p \phi_{1p}(z) + b_p \phi_{2p+1}(z),
$$

$$
\int_{\mathbb{D}} g(r) r^{p-1} \bar{w} K(z, w) dA(w) = c_p \phi_{1p-1}(z) + d_p \phi_{2p}(z),
$$

(14)

where $a_p = 2 \sqrt{p + 1} \tilde{g}(2p)$, $b_p = 2 \sqrt{p + 1} \int (p + 1) \tilde{g}(2p + 2) - p \tilde{g}(2p)$, $c_p = 2 \sqrt{p + 1} \tilde{g}(2p)$, and $d_p = 2 \sqrt{p + 1} [p \tilde{g}(2p + 2) - (p - 1) \tilde{g}(2p + 2)]$.

**Proof.** For each $p = 1, 2, \cdots$, since $g$ is a bounded radial function, thus

$$
\int_{\mathbb{D}} g(r) r^{p-1} K(z, w) dA(w) = \int_{\mathbb{D}} g(r) r^{p-1} \left[ \sum_{k=1}^{\infty} \phi_{1k}(z) \phi_{1k}^*(w) \right] dA(w)
$$

$$
\quad + \sum_{k=1}^{\infty} \phi_{2k}(z) \phi_{2k}^*(w) \right] dA(w)
$$

$$
= \sum_{k=1}^{\infty} \phi_{1k}(z) \left[ \int_{\mathbb{D}} g(r) r^{p-1} \sqrt{k} z^{k-1} dA(w) \right.
$$

$$
\quad + \sum_{k=1}^{\infty} \phi_{2k}(z) \left. \left[ \int_{\mathbb{D}} g(r) r^{p-1} \frac{\sqrt{k+1}}{k} z^{k-1} dA(w) \right] \right. \right.
$$

$$
= 2 \sqrt{p + 1} \tilde{g}(2p) \phi_{1p}(z) + 2 \sqrt{p + 2} \cdot [(p + 1) \tilde{g}(2p + 2) - p \tilde{g}(2p)] \phi_{2p+1}(z),
$$

\]
Applying Lemma 4, we conclude that radial Toeplitz operators on $A^2_d$ are not diagonal. The following corollary is an immediate consequence of Lemma 4.

**Corollary 5.** Let $g$ be a bounded radial function. Then, for each $p, q \in \mathbb{Z}^+$,

$$
\langle T_g \phi_{1,p}, \phi_{1,q} \rangle = \begin{cases} 
\sqrt{p}q_p, & \text{if } q = p, \\
0, & \text{if } q \neq p.
\end{cases}
$$

$$
\langle T_g \phi_{1,p}, \phi_{2,q} \rangle = \begin{cases} 
\sqrt{p}b_p, & \text{if } q = p + 1, \\
0, & \text{if } q \neq p + 1.
\end{cases}
$$

$$
\langle T_g \phi_{2,p}, \phi_{1,q} \rangle = \begin{cases} 
\sqrt{p + 1} \left[ p c_p - (p - 1) a_{p-1} \right], & \text{if } q = p - 1, \\
0, & \text{if } q \neq p - 1.
\end{cases}
$$

$$
\langle T_g \phi_{2,p}, \phi_{2,q} \rangle = \begin{cases} 
\sqrt{p + 1} \left[ p d_p - (p - 1) b_{p-1} \right], & \text{if } q = p, \\
0, & \text{if } q \neq p.
\end{cases}
$$

3. Products of Two Toeplitz Operators

A bounded function $f$ is said to be quasihomogeneous of degree $k \in \mathbb{Z}$ if

$$
f \left( r e^{i \theta} \right) = e^{i k \theta} f(r),
$$

where $g(r)$ is a radial function (see [14]). For any function $f \in L^2(D, dA)$, it has the polar decomposition, i.e.,

$$
f \left( r e^{i \theta} \right) = \sum_{k \in \mathbb{Z}} f_k(r),
$$

where $f_k(r)$ are radial functions in $L^2(0, 1, rdr)$ (see [12]). A direct calculation gives the following lemma.

**Lemma 6.** Let $f$ be a bounded function. Then, for each $p, q \in \mathbb{Z}^+$,

$$
\langle f \phi_{1,p}, \phi_{1,q} \rangle = 2 \sqrt{pq} \tilde{f}_{q-p}(p + q),
$$

$$
\langle f \phi_{1,p}, \phi_{2,q} \rangle = 2 \sqrt{p} \left( q + 1 \right) \tilde{f}_{q-p-1}(p + q + 1)
$$

$$
- (q - 1) \tilde{f}_{-q-p-1}(p + q - 1),
$$

$$
\langle f \phi_{2,p}, \phi_{1,q} \rangle = 2 \sqrt{(p + 1)q} \tilde{f}_{q-p-1}(p + q + 1)
$$

$$
- (p - 1) \tilde{f}_{q-p+1}(p + q - 1),
$$

$$
\langle f \phi_{2,p}, \phi_{2,q} \rangle = 2 \sqrt{(p + 1)(q + 1)} \left[ pq \tilde{f}_{q-p}(p + q + 2) + (p + q - 2pq) \tilde{f}_{q-p}(p + q) + (p - 1)(q - 1) \tilde{f}_{q-p}(p + q - 2) \right].
$$
Proof. For all $p, q \in \mathbb{Z}^+$, it is easy to verify that
\[
\left< f \phi_p, f \phi_q \right> = \sum_{k \in \mathbb{Z}} \left< e^{ikr} f_k(r) \sqrt{p} z^{p-1}, \sqrt{q} z^{q-1} \right>
= \sum_{k \in \mathbb{Z}} \sqrt{p} q^{(k+1)/2} \left< e^{ikr} f_k(r) z^{p-1}, z^{q-1} \right>
= 2 \sqrt{pq} f_{-q-p}(p+q).
\]

Similarly, the rest of the lemma can be proved. \(\square\)

When considering the product of two Toeplitz operators, we often use the Mellin convolution. If $f, g \in L^1([0, 1], r dr)$, then their Mellin convolution is given by
\[
(f * g)(r) = \int_0^r f\left(\frac{t}{r}\right) g(t) \frac{dt}{t}, 0 \leq t < 1.
\]

The Mellin convolution theorem (see [15]) states that
\[
\widehat{f * g}(s) = \widehat{f}(s) \widehat{g}(s),
\]
and if $f$ and $g$ are bounded, then so is $f * g$.

It is well known that the Mellin transform is uniquely determined by its value on an arithmetic sequence of integers. The following results (see [15], p. 102, [16]) will be needed later.

**Theorem 7.** Suppose $f$ is a bounded analytic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2, \cdots$, where
\[
\inf \left\{ |z_1| \right\} > 0, \quad \sum_{n \geq 1} \Re \left( \frac{1}{z_n} \right) = \infty.
\]

Then, $f$ vanishes identically on $\{z : \Re z > 0\}$.

**Remark 8.** Using this theorem, we can see that if $g \in L^1([0, 1], r dr)$ and if there exists a sequence $\{n_k\}_{k \geq 0} \subset \mathbb{N}$ such that
\[
\widehat{g}(n_k) = 0, \quad \sum_{k \geq 0} \frac{1}{n_k} = \infty,
\]
then, $\widehat{g}(z) = 0$ for all $z \in \{z : \Re z > 2\}$, by the Müntz-Szasz theorem (see [17], p. 312), $g = 0$.

For $p \in \mathbb{Z}^+$, we obtain
\[
\widehat{g}(p) = \int_0^1 g(s) s^{p-1} ds.
\]

The numbers $\widehat{g}(p)$ can also be called the moment Mellin sequence of $g$. Let
\[
A(p) = \begin{pmatrix} a_{p-1} & b_{p-1} \\ pc_p - (p-1)a_{p-1} & pd_p - (p-1)b_{p-1} \end{pmatrix}.
\]

$A(p)$ is closed related to the moment Mellin sequence of $g$, and we have the following lemma.

**Lemma 9.** Let $p$ be a fixed positive integer. Then, the following statements hold:

(i) $a_p = c_p = 0$ if and only if $\widehat{g}(2p) = 0$

(ii) $b_p = 0$ if and only if $(p+1)\widehat{g}(2p + 2) - p\widehat{g}(2p) = 0$

(iii) $d_p = 0$ if and only if $p\widehat{g}(2p + 2) - (p-1)\widehat{g}(2p) = 0$

(iv) $|A(p+1)| = 0$ if and only if $(r^2 \widehat{g} * r^2 \widehat{g})(2p) - (r^4 \widehat{g} * \widehat{g})(2p) = 0$

Proof. From Lemma 3, it is easy to check that (i), (ii), and (iii) hold.

To prove (iv), in fact, for a fixed $p \in \mathbb{Z}^+$,
\[
|A(p+1)| = \left| \begin{pmatrix} a_p & b_p \\ (p+1)c_{p+1} - pa_p & (p+1)d_{p+1} - pb_p \end{pmatrix} \right|
= \left| (p+1)a_pb_{p+1} - (p+1)b_pc_{p+1} \right|
= 4(p+1)^2 \sqrt{p(p+2)} \left\{ \widehat{g}(2p + 4)\widehat{g}(2p) - |g \wedge (2p + 2)|^2 \right\}.
\]

It follows that $|A(p+1)| = 0$ if and only if
\[
|g \wedge (2p + 2)|^2 - \widehat{g}(2p + 4)\widehat{g}(2p) = 0.
\]

Using $\widehat{g}(2p + 2) = \widehat{r^2 g}(2p)$, $\widehat{g}(2p + 4) = \widehat{r^4 g}(2p)$, and Mellin convolution (24), we get the above equality is equivalent to
\[
(r^2 \widehat{g} * r^2 \widehat{g})(2p) - (r^4 \widehat{g} * \widehat{g})(2p) = 0.
\]

**Lemma 10.** Let $g$ be a bounded radial function. The function $g = 0$ if and only if there exists a sequence $\{p_k\}_{k \geq 0} \subset \mathbb{Z}^+$ such that
\[
\sum_{k \geq 0} \frac{1}{p_k} = \infty \quad \text{such that} \quad (r^2 \widehat{g} * r^2 \widehat{g})(2p_k) = (r^4 \widehat{g} * \widehat{g})(2p_k).
\]

Proof. If the function $g = 0$, then $r^2 g * r^2 g = r^4 g * g = 0$, for each $p \in \mathbb{Z}^+$,
\[
(r^2 \widehat{g} * r^2 \widehat{g})(2p) = (r^4 \widehat{g} * \widehat{g})(2p) = 0.
\]

This proves the sufficient condition.
Next, we prove the necessary condition. Suppose \( \{ p_k \}_{k=0}^{\infty} \subset \mathbb{Z}^+ \),
\[
\sum_{k=0}^{\infty} \frac{1}{p_k} = \infty,
\]
(34)
\[(r^2 \overline{g} \ast r^2 g)(2p_k) = (r^4 \overline{g} \ast g)(2p_k).
\]
Using Remark 8, we have
\[(r^2 \overline{g} \ast r^2 g)(z) = (r^4 \overline{g} \ast g)(z),
\]
for all \( z \in \{ z : \text{Re} z > 2 \} \). Therefore, for each \( p \in \mathbb{Z}^+ \),
\[
[g \wedge (2p + 2)]^2 = \overline{g}(2p + 4) \overline{g}(2p).
\]
That is, \( \{ \overline{g}(2p) \}_{p=1}^{\infty} \) is a geometric sequence. There exists a constant \( a \) such that
\[
\overline{g}(2p + 2) = a \cdot \overline{g}(2p).
\]
Then,
\[(r^2 \overline{g} - ag)(2p) = 0.
\]
(38)
Since \( \{ 2p \}_{p=1}^{\infty} \subset \mathbb{Z}^+ \) is a sequence and \( \sum_{p=1}^{\infty} (1/2p) = \infty \), by Remark 8, \( (r^2 - a)g = 0 \), which implies \( g = 0 \).

For each \( p, q \in \mathbb{Z}^+ \), let \( b_{11}(p, q) = \langle f \phi_{1p}, \phi_{1q} \rangle \), \( b_{12}(p, q) = \langle f \phi_{1p}, \phi_{2q} \rangle \), \( b_{21}(p, q) = \langle f \phi_{2p}, \phi_{1q} \rangle \), and \( b_{22}(p, q) = \langle f \phi_{2p}, \phi_{2q} \rangle \). Let
\[
B(p, q) = \begin{pmatrix}
    b_{11}(p, q) & b_{12}(p, q) \\
    b_{21}(p + 1, q) & b_{22}(p + 1, q)
\end{pmatrix}.
\]
(39)
The first main result of this paper is the following theorem.

**Theorem 11.** Let \( f \) be a bounded function and \( g \) be a bounded radial function. Then, \( T_f T_g = 0 \) on \( A_2^\infty \) if and only if for each \( p, q \in \mathbb{Z}^+ \), \( A(p)B(p - 1, q) = 0 \).

**Proof.** Using the fact that \( f \) is a bounded function, we have
\[
f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} f_k(r).
\]
(40)
If \( T_f T_g = 0 \), then for each \( p, q \in \mathbb{Z}^+ \),
\[
\left\langle T_f T_g \phi_{1p}, \phi_{1q} \right\rangle = 0,
\]
\[
\left\langle T_f T_g \phi_{1p}, \phi_{2q} \right\rangle = 0.
\]
(41)
By Lemma 4,
\[
a_p \langle f \phi_{1p}, \phi_{1q} \rangle + b_p \langle f \phi_{2p+1}, \phi_{1q} \rangle = 0,
\]
\[
a_p \langle f \phi_{1p}, \phi_{2q} \rangle + b_p \langle f \phi_{2p+1}, \phi_{2q} \rangle = 0,
\]
from which we conclude that
\[
(a_p, b_p)B(p, q) = 0.
\]
(42)
Since \( p \) is arbitrary, it follows that
\[
(a_{p-1}, b_{p-1})B(p - 1, q) = 0.
\]
(43)
Analogously, for each \( p, q \in \mathbb{Z}^+ \), it is easily verified that
\[
\left\langle T_f T_g \phi_{2p}, \phi_{1q} \right\rangle = 0,
\]
\[
\left\langle T_f T_g \phi_{2p}, \phi_{2q} \right\rangle = 0,
\]
(44)
thus, we get
\[
\left(p c_p - (p - 1)d_{p-1}, p d_p - (p - 1)b_{p-1}\right)B(p - 1, q) = 0.
\]
(46)
The above equations are equivalent to
\[
A(p)B(p - 1, q) = 0.
\]
(47)
This completes the proof of the theorem.

For \( p = 1, 2, \cdots \), firstly if \( a_p = c_p = 0 \), then \( \overline{g}(2p) = 0 \), using Remark 8, we get \( g = 0 \). Now, if \( b_p = 0 \), then
\[
(2p + 2)\overline{g}(2p + 2) - 2p\overline{g}(2p) = 0.
\]
(48)
Letting \( \zeta = 2p \), we have
\[
\zeta \overline{g}(\zeta) = (\zeta + 2)\overline{g}(\zeta + 2).
\]
(49)
It is easy to see that the function \( \zeta \overline{g}(\zeta) \) is a periodic function with a period 2. Using the same argument as the one at the end of Section 2 in [12], we conclude that \( \zeta \overline{g}(\zeta) \) must be a constant function. Hence,
\[
\overline{g}(\zeta) = \frac{C}{\zeta},
\]
(50)
where \( C \) is a constant and it is clear that \( g \) is also a constant. Finally, if \( d_p = 0 \), then
\[
2p \overline{g}(2p + 2) - (2p - 2)\overline{g}(2p) = 0;
\]
(51)
that is,
\[
2p \overline{g} \cdot r^2(2p) - (2p - 2) \overline{g} \cdot r^2(2p - 2) = 0.
\]
(52)
Similarly, we can also conclude that \( r^2 \cdot g \) is a constant. Thus, if \( g \) is a bounded radial function, it must be zero. Finally, we obtain the following lemma.

**Lemma 12.** Let \( g \) be a bounded radial function. Then, the following statements hold:

(i) \( a_p = c_p = 0 \) for all \( p \in \mathbb{Z}^+ \) if and only if \( g = 0 \)

(ii) \( b_p = 0 \) for all \( p \in \mathbb{Z}^+ \) if and only if \( g \) is a constant

(iii) \( d_p = 0 \) for all \( p \in \mathbb{Z}^+ \) if and only if \( g = 0 \)

**Remark 13.** In Lemma 12, the condition “for all \( p \in \mathbb{Z}^+ \)” can also be replaced by “a sequence \( \{p_k\}_{k \geq 0} \subset \mathbb{Z}^+ \) satisfying \( \sum_{k=0}^{\infty} (1/p_k) = \infty \).”

In Theorem 11, if \( a_p = 0 \) or \( c_p = 0 \), or \( d_p = 0 \), then \( g = 0 \), so it is clear that \( T_f T_g = 0 \). If \( b_p = 0 \), then \( g \) is a constant; it is also easy to see that if \( g \) is not zero and \( T_f T_g = 0 \), then \( f \) must be zero. If \( |A(p+1)| = 0, A(p+1) \) is not invertible. On the other hand, when \( |A(p+1)| \neq 0 \), then \( A(p+1) \) is an invertible matrix. For a bounded radial function \( g \) such that \( |A(p+1)| \neq 0 \), if \( T_f T_g = 0 \), is it necessary that \( f = 0 \)? The second main theorem of this paper answers this question by giving a sufficient and necessary condition.

**Theorem 14.** Let \( g \) and \( f \) be bounded functions and \( g \) be a bounded radial function satisfying

\[
(r^2 \cdot g \ast r^2 g)(2p) \neq (r^4 \cdot g \ast g)(2p),
\]

for each \( p \in \mathbb{Z}^+ \). Then, \( T_f T_g = 0 \) on \( A_2 \) if and only if \( f = 0 \).

**Proof.** If \( f \) is a zero function, it is obvious that \( T_f T_g = 0 \).

Now, we assume \( T_f T_g = 0 \) and we shall prove \( f = 0 \). If \( g \) is a bounded radial function and for each \( p \in \mathbb{Z}^+ \),

\[
(r^2 \cdot g \ast r^2 g)(2p) \neq (r^4 \cdot g \ast g)(2p),
\]

then, by the Mellin convolution theorem (24), it follows that

\[
[g \land (2p + 2)]^2 \neq \overline{g}(2p + 4) \overline{g}(2p).
\]

For each \( p \in \mathbb{Z}^+ \),

\[
|A(p+1)| = 4(p+1)^2 \sqrt{p(p+2)} \{ \overline{g}(2p + 4) \overline{g}(2p) - [g \land (2p + 2)]^2 \}.
\]

Applying (55), we get \( |A(p+1)| \neq 0 \), that is, \( A(p+1) \) is an invertible matrix. If \( T_f T_g = 0 \) and for each \( q \in \mathbb{Z}^+ \), we get

\[
A(p+1)B(p, q) = 0.
\]

Since \( A(p+1) \) is invertible,

\[
B(p, q) = \begin{pmatrix} b_{11}(p, q) & b_{12}(p, q) \\ b_{21}(p+1, q) & b_{22}(p+1, q) \end{pmatrix} = 0.
\]

Thus, \( b_{11}(p, q) = 0 \), by Lemma 6, we have

\[
\hat{f}_q(p + q) = 0.
\]

That is,

\[
\hat{f}_k(k + 2p) = 0,
\]

where \( k = q - p \). Since \( p \) and \( q \) are arbitrary elements in \( \mathbb{Z}^+ \), by Remark 8, we obtain \( f_k = 0 \) for all \( k \in \mathbb{Z} \). It follows that \( f = 0 \). This completes the proof of the theorem.

**Example 1.** Let \( g = r^m \), where \( m \in \mathbb{Z}^+ \). Then, for each \( p \in \mathbb{Z}^+ \),

\[
(r^2 \cdot g \ast r^2 g)(2p) = \left( \frac{1}{2p + 2 + m} \right)^2,
\]

\[
(r^4 \cdot g \ast g)(2p) = \frac{1}{(2p + m)(2p + m + 4)}.
\]

Obviously, \( (r^2 \cdot g \ast r^2 g)(2p) \neq (r^4 \cdot g \ast g)(2p) \). It is easy to see that \( T_f T_g = 0 \) on \( A_2 \) if and only if \( f = 0 \).

In the following, we discuss when condition (53) is not satisfied.

**Case 1.** If \( g \) is a bounded radial function and for each \( p \in \mathbb{Z}^+ \),

\[
(r^2 \cdot g \ast r^2 g)(2p) = (r^4 \cdot g \ast g)(2p).
\]

Then, using the Mellin convolution theorem (24), we have

\[
[g \land (2p + 2)]^2 = \overline{g}(2p + 4) \overline{g}(2p).
\]

That is, \( \{\overline{g}(2p)\}_{p=1}^{\infty} \) is a geometric sequence. Using Lemma 10, we get \( g \) must be zero. It is clear that \( T_f T_g = 0 \).

**Case 2.** If \( g \) is a bounded radial function and for some \( p \in \mathbb{Z}^+ \),

\[
(r^2 \cdot g \ast r^2 g)(2p) = (r^4 \cdot g \ast g)(2p).
\]

(1) If there exists a sequence \( \{p_k\}_{k \geq 0} \subset \mathbb{Z}^+ \) satisfying \( \sum_{k=0}^{\infty} (1/p_k) = \infty \) such that

\[
(r^2 \cdot g \ast r^2 g)(2p_k) = (r^4 \cdot g \ast g)(2p_k),
\]
then, by using Lemma 10, we get that \( g \) must be zero function.

(2) If there exists a finite sequence \( \{p_k\} \subset \mathbb{Z}^+ \), or an infinite sequence \( \{p_k\} \subset \mathbb{Z}^+ \) satisfying \( \sum_{k=0}^{\infty} \frac{1}{p_k} < \infty \), such that

\[
(r^2 g \ast r^2 g)(2p_k) = (r^4 g \ast g)(2p_k),
\]

then, the radial function \( g \) may not be zero function. For example, if \( \{p_k\} = \{p_1\} \) is finite sequence and \( p_1 = 1 \), there exist some nonzero bounded radial functions \( g \) such that

\[
(r^2 g \ast r^2 g)(2) = (r^4 g \ast g)(2).
\]

Let \( g = ar^2 + br^4 \), where \( a, b \in \mathbb{R} \). Then,

\[
\begin{align*}
\bar{g}(4) &= \frac{a}{6} + \frac{b}{8}; \\
\bar{g}(6) &= \frac{a}{8} + \frac{b}{10}; \\
\bar{g}(2) &= \frac{a}{4} + \frac{b}{6}.
\end{align*}
\]

When \( a = 360, b = -720 + 120\sqrt{6} \), a direct calculation shows that condition (67) is satisfied. In this case, we can prove that \( A(2) \) is not invertible. As

\[
\begin{align*}
a_1 &= 2\bar{g}(2); \\
b_1 &= 2\sqrt{3}[2\bar{g}(4) - \bar{g}(2)]; \\
c_1 &= 4\sqrt{3}(4); \\
d_1 &= 4\sqrt{3}[2\bar{g}(6) - \bar{g}(4)],
\end{align*}
\]

then

\[
A(2) = \begin{pmatrix} a_1 & b_1 \\ 2c_2 - a_1 & 2d_2 - b_1 \end{pmatrix} = \begin{pmatrix} 2\bar{g}(2) & 2\sqrt{3}[2\bar{g}(4) - \bar{g}(2)] \\ 4\bar{g}(4) - 2\bar{g}(2) & 2\sqrt{3}[4\bar{g}(6) - 4\bar{g}(4) + \bar{g}(2)] \end{pmatrix}.
\]

Since \( g = ar^2 + br^4 \), it follows from (68) and (70) that

\[
A(2) = \begin{pmatrix} a + b & \sqrt{3}(a + b) \\ \frac{a}{12} + \frac{b}{15} & 2\sqrt{3}(\frac{a}{12} + \frac{b}{15}) \end{pmatrix}.
\]

When \( a = 360, b = -720 + 120\sqrt{6} \), a direct calculation shows that \( |A(2)| = 0 \) and \( A(2) \) is not invertible.

**Remark 15.** For a nonzero function \( g \) whose related matrices are \( A(p), p \in \mathbb{Z}^+ \), if there exist matrices \( B(p), p \in \mathbb{Z}^+ \) such that

(i) \( B(p) \) are not all zero

(ii) For each \( p \in \mathbb{Z}^+ \), \( A(p)B(p) = 0 \)

then, we can construct a nonzero function \( f \), such that \( T_f T_g = 0 \). The following example solves (i) and (ii) for a fixed \( p \). However, it is still unknown if (i) and (ii) hold for all \( p \in \mathbb{Z}^+ \), and we will study this question in the future work.

**Example 2.** Suppose \( g = 360r^2 + (-720 + 120\sqrt{6})r^4 \). Then

\[
A(2) = \begin{pmatrix} 60 + 40\sqrt{6} & -603 + 60\sqrt{2} \\ -60 + 20\sqrt{6} & -36\sqrt{3} + 48\sqrt{2} \end{pmatrix}.
\]

As \( A(2) \) is not invertible, there exist some nonzero matrix \( B \) such that \( A(2)B = 0 \). For example,

\[
B = \begin{pmatrix} \sqrt{6} & 2\sqrt{3} \\ \sqrt{2} + 2\sqrt{3} & 2 + 2\sqrt{6} \end{pmatrix}.
\]

For each \( p = 1, 2, \cdots \), analogous to Lemma 3, we define

\[
\begin{align*}
a'_p &= 2\sqrt{p}\bar{f}(2p), \\
b'_p &= 2\sqrt{p + 2} \left[(p + 1)\bar{f}(2p + 2) - pf(2p)\right], \\
c'_p &= 2\sqrt{p - 1}\bar{f}(2p), \\
d'_p &= 2\sqrt{p + 1} \left[p\bar{f}(2p + 2) - (p - 1)\bar{f}(2p)\right],
\end{align*}
\]

and \( a'_0 = b'_0 = c'_0 = d'_0 = 0 \).

In Theorem 14, if \( f \) and \( g \) are all bounded radial function and there exists a sequence \( \{p_k\}_{k=0}^{\infty} \subset \mathbb{Z}^+ \),

\[
\sum_{k=0}^{\infty} \frac{1}{p_k} = \infty, \text{such that} \quad (r^2 g \ast r^2 g)(2p_k) \neq (r^4 g \ast g)(2p_k),
\]

the conclusion is still valid; then, we have the following corollary.

**Corollary 16.** Let \( f \) and \( g \) be bounded radial functions. Suppose there exists a sequence \( \{p_k\}_{k=0}^{\infty} \subset \mathbb{Z}^+ \),

\[
\sum_{k=0}^{\infty} \frac{1}{p_k} = \infty, \text{such that} \quad (r^2 g \ast r^2 g)(2p_k) \neq (r^4 g \ast g)(2p_k),
\]

If \( T_f T_g = 0 \) on \( A^2 \), then \( f = 0 \).
Proof. For \( p \in \mathbb{Z}^+ \), define
\[
B_p = \begin{pmatrix}
\sqrt{p}a_p' \\
\sqrt{p+2}[(p+1)c'_{p+1} - pa_p']
\end{pmatrix},
\] (77)

By the hypothesis, \( f \) is a bounded radial function, it follows from Lemma 4 and Theorem 11 that \( T_f T_g = 0 \) if and only if for each \( p \in \mathbb{Z}^+ \),
\[
A_p B_{p-1} = 0.
\] (78)

Let \( g \neq 0 \) and there exists a sequence \( \{p_k\}_{k \geq 1} \subset \mathbb{Z}^+ \),
\[
\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty, \text{such that } (r^2 g \ast r^2 g)(2p_k) \neq (r^4 g \ast g)(2p_k).
\] (79)

Then, it follows that \( A(p_k + 1) \) is an invertible matrix. Combining this with \( A_{p_k+1} B_{p_k} = 0 \), we get \( B_{p_k} = 0 \). It follows that
\[
a'_{p_k} = 2\sqrt{p_k} \hat{f}(2p_k) = 0.
\] (80)

This implies that \( \hat{f}(2p_k) = 0 \), combing with
\[
\sum_{k=1}^{\infty} \frac{1}{2p_k} = \infty,
\] (81)

and using Remark 8, we get \( f = 0 \).

For \( p \in \mathbb{Z}^+ \), if \( f \) and \( g \) are bounded radial functions, it follows from Lemma 4 that
\[
T_f T_g(\phi_{1.p}) = \lambda_{11}(p)\phi_{1.p} + \lambda_{12}(p)\phi_{2.p+1},
\]
\[
T_f T_g(\phi_{2.p}) = \lambda_{21}(p)\phi_{1.p-1} + \lambda_{22}(p)\phi_{2.p},
\] (82)

where
\[
\lambda_{11}(p) = p\alpha_p a'_{p} + \sqrt{p}b_p' \left[ (p+1)c'_{p+1} - pa_p' \right],
\]
\[
\lambda_{12}(p) = p\alpha_p b_p' + \sqrt{p}b_p' \left[ (p+1)d'_{p+1} - pb_p' \right],
\]
\[
\lambda_{21}(p) = \sqrt{p+1} \left[ pc_p - (p-1)\alpha_{p-1} \right] \sqrt{p-1}a'_{p-1}
\]
\[
+ \left[ pd_p - (p-1)b_{p-1} \right] \sqrt{p+1} \left[ pc'_{p} - (p-1)\alpha_{p-1} \right],
\]
\[
\lambda_{22}(p) = \sqrt{p+1} \left[ pc_p - (p-1)\alpha_{p-1} \right] \sqrt{p-1}b'_{p-1}
\]
\[
+ \left[ pd_p - (p-1)b_{p-1} \right] \sqrt{p+1} \left[ pd_p - (p-1)b_{p-1} \right].
\] (83)

If \( T_f T_g \) has a finite rank, there exists \( N \in \mathbb{Z}^+ \), for all \( p > N \), such that
\[
T_f T_g(\phi_{1.p}) = 0,
\]
\[
T_f T_g(\phi_{2.p}) = 0.
\] (84)

As in Corollary 16, there exists a sequence \( \{p_k\}_{k \geq 0} \) meet the conditions, where \( \{p_k\} \subset \mathbb{Z}^+ \) and \( p_k > N \); using properties of Mellin transform, we can obtain that \( T_f T_g \) has a finite rank if and only if \( f = 0 \).

Remark 17. As in Corollary 16, let \( f \) and \( g \) are bounded radial functions and there exists a sequence \( \{p_k\}_{k \geq 0} \subset \mathbb{Z}^+ \),
\[
\sum_{k=0}^{\infty} \frac{1}{p_k} = \infty, \text{such that } (r^2 g \ast r^2 g)(2p_k) \neq (r^4 g \ast g)(2p_k).
\] (85)

Then, \( T_f T_g \) has finite rank if and only if \( f = 0 \).

The following question is the general zero-product problem on \( A_n^2 \) when \( n \geq 3 \).

Question 18. Let \( f \) be a bounded function and \( g \) be a bounded radial function. Suppose that \( T_f T_g = 0 \) on \( A_n^2 \) when \( n \geq 3 \), can we obtain any similar conclusions?

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

References
[1] E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.
[2] Y. I. Karlovich and L. V. Pessoa, “Poly-Bergman projections and orthogonal decompositions of \( L^2 \)-spaces over bounded domains,” *Operator Theory: Advances and Applications*, vol. 181, pp. 263–282, 2008.
[3] A. D. Košćelj, “The kernel function of a Hilbert space of functions that are polyanalytic in the disc,” *Doklady Akademii Nauk SSSR*, vol. 232, pp. 277–279, 1977.
[4] A. K. Ramazanov, “On the structure of spaces of polyanalytic functions,” *Matematicheskie Zametki*, vol. 72, no. 5, pp. 750–764, 2002.
[5] N. L. Vasilevski, "On the structure of Bergman and poly-Bergman spaces," *Integral Equations and Operator Theory*, vol. 33, no. 4, pp. 471–488, 1999.
[6] M. B. Balk, *Polyanalytic Functions*, Akademie Verlag, Berlin, 1991.
7] P. Ahern and Ž. Čučković, “Some examples related to the Brown-Halmos theorem for the Bergman space,” ActaScientiarum Mathematicarum, vol. 70, pp. 373–378, 2004.
8] T. Le, “Finite-rank products of Toeplitz operators in several complex variables,” Integral Equations and Operator Theory, vol. 63, no. 4, pp. 547–555, 2009.
9] T. Le, “A refined Luecking’s theorem and finite rank products of Toeplitz operators,” Complex Analysis and Operator Theory, vol. 4, no. 2, pp. 391–399, 2010.
10] Ž. Čučković and T. Le, “Toeplitz operators on Bergman spaces of polyanalytic functions,” The Bulletin of the London Mathematical Society, vol. 44, no. 5, pp. 961–973, 2012.
11] L. V. Pessoa, “Planar Beurling transform and Bergman type spaces,” Complex Analysis and Operator Theory, vol. 8, no. 2, pp. 359–381, 2014.
12] Ž. Čučković and N. V. Rao, “Mellin transform, monomial symbols, and commuting Toeplitz operators,” Journal of Functional Analysis, vol. 154, no. 1, pp. 195–214, 1998.
13] R. M. Barrera-Castelán, E. A. Maximenko, and G. Ramos-Vázquez, “Radial operators on polyanalytic weighted Bergman spaces,” Boletín de la Sociedad Matemática Mexicana, vol. 27, no. 2, 2021.
14] L. Zakariasy, “The rank of Hankel operators on harmonic Bergman spaces,” Proceedings of the American Mathematical Society, vol. 131, no. 4, pp. 1177–1180, 2003.
15] R. Remmert, “Classical topics in complex function theory,” in Graduate Texts in Mathematics, vol. 172, Springer, New York, 1998.
16] X. T. Dong and Z. H. Zhou, “Products of Toeplitz operators on the harmonic Bergman space,” Proceedings of the American Mathematical Society, vol. 138, no. 5, pp. 1765–1773, 2010.
17] W. Rudin, Real and Complex Analysis, McGraw-Hill Book Co., New York, Third edition, 1987.