ON SOME FROBENIUS TYPE DIVISIBILITY RESULTS IN A PREMODULAR CATEGORY

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Abstract

In this note some new Frobenius type divisibility results are obtained for premodular categories. In particular, we extend [Yu20, Corollary 3.4] from the settings of super-modular categories to arbitrary pseudo-unitary premodular categories.

1. Introduction

In [Yu20] the author showed that in a slightly-degenerate braided fusion category one has that $\dim(C) \leq \frac{\dim(Y)^2}{2} \in \mathbb{A}$ for any simple object $Y$ of $C$. The main goal of this note is to generalize this result in the settings of a premodular fusion category with pointed Müger center, i.e. all the objects of this center are invertible.

Our first main result is the following:

Theorem 1.1. Let $C$ be a premodular category and $D$ a fusion subcategory of $C$. If $D \cap Z_2(C) = \text{Vec}$ then

$$\frac{\dim(C)}{\dim(Y)^2} \in \mathbb{A}$$

for any simple object $Y$ in $D$.

Moreover, in the case of a pointed Müger center we prove that:

Theorem 1.3. Let $C$ be a pseudo-unitary premodular fusion category with $Z_2(C) \subseteq C_{pt}$.

1. For any simple object $C$ of $Y$ one has

$$\frac{\text{FPdim}(C)\text{FPdim}(Z_2(C))}{\text{FPdim}(Y)^2} \in \mathbb{A}.$$ 

2. If the action of $Z_2(C)$ on the simple objects of $C$ is free then

$$\frac{\text{FPdim}(C)}{\text{FPdim}(Z_2(C))\text{FPdim}(Y)^2} \in \mathbb{A}.$$
Recall that $\mathcal{C}_{pt}$ denotes the maximal pointed subcategory of $\mathcal{C}$. Note that the second item of the above Theorem generalizes [Yu20, Corollary 3.4] in the case of a pseudo-unitary premodular category.

Shortly, this note is organized as follows. In Section 2 we recall the basic properties of fusion categories needed through this paper. In Subsection 2.1 we briefly recall the adjoint algebra and the characters of simple objects of pivotal fusion categories as defined in [Sh17].

In Section 3 we construct a Hecke type algebras associated to the right cosets of a fusion subcategory of a given fusion category. In the same section we prove orthogonality relations for Hecke algebras. In Section 4 we prove the main results stated above.

2. Preliminaries

In this section we recall the main properties of premodular fusion categories that are needed through the paper. For the basic theory of fusion categories, we refer the reader to [EGNO15]. Recall that a fusion category is by definition, a semisimple finite tensor category.

Throughout this note $\mathcal{C}$ denotes a fusion category over $\mathbb{C}$ with eventual additional properties. Let $\text{Irr}(\mathcal{C}) := \{X_0, X_1, \ldots, X_m\}$ be a complete set of representatives for the isomorphism classes of simple objects of $\mathcal{C}$. By $K_0(\mathcal{C})$ we denote the Grothendieck ring of $\mathcal{C}$ and let $K(\mathcal{C}) := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$.

Recall that a pivotal structure of a rigid monoidal category $\mathcal{C}$ is an isomorphism $j : \text{id}_\mathcal{C} \to (\cdot)^\ast$ of monoidal functors. A pivotal monoidal category is a rigid monoidal category endowed with a pivotal structure. For any pivotal structure one can construct categorical dimensions $\dim(X)$ of any object $X$ of $\mathcal{C}$. In fact, the map $\dim : K(\mathcal{C}) \to \mathbb{C}, [X] \mapsto \dim(X)$ is an algebra morphism.

A pivotal structure a on a fusion category $\mathcal{C}$ is called spherical, see [EGNO15] if $\dim(V) = \dim(V^\ast)$ for any object $V$ of $\mathcal{C}$. A tensor category is spherical if it is equipped with a spherical structure. Given a fusion category there is a unique algebra morphism $\text{FPdim} : K(\mathcal{C}) \to \mathbb{C}$ such that $\text{FPdim}(X_i) > 0$ for any $X_i \in \text{Irr}(\mathcal{C})$. Then $\text{FPdim}$ is called the Frobenius-Perron morphism.

Recall that a braided fusion category is called premodular if it has a spherical structure. Equivalently, a premodular category is a braided fusion category equipped with a twist (also called a balanced structure), see [EGNO15] Section 8.10 for details.

A fusion category $\mathcal{C}$ is called pseudo-unitary if $\text{FPdim}(\mathcal{C}) = \text{dim}(\mathcal{C})$. If such is the case, then by [ENO05, Proposition 8.23], $\mathcal{C}$ admits a
unique spherical structure with respect to which the categorical dimensions of each simple coincides with its Frobenius-Perron dimension, i.e. $\text{FPdim}(X) = \dim(X)$ for any object $X$ of $\mathcal{C}$.

2.1. **Adjoint algebra and internal characters.** Let $\mathcal{C}$ be a fusion category and $Z(\mathcal{C})$ be its Drinfeld center. The forgetful functor $F : Z(\mathcal{C}) \to \mathcal{C}$ admits a right adjoint functor $R : \mathcal{C} \to Z(\mathcal{C})$ and let $Z := FR : \mathcal{C} \to \mathcal{C}$. Then $Z$ is a Hopf comonad and by [Shi17, Section 2.6] one has that

\begin{equation}
Z(V) \simeq \int_{X \in \mathcal{C}} X \otimes V \otimes X^*
\end{equation}

It is known that $A := R(1)$ has the structure of central commutative algebra in $Z(\mathcal{C})$.

We denote by $m : A \otimes A \to A$ the multiplication and by $u : 1 \to A$ the unit of the adjoint algebra $A = Z(1)$.

The vector space $CE(\mathcal{C}) := \text{Hom}_{\mathcal{C}}(1, A)$ is called the set of central elements. For $a, b \in CE(\mathcal{C})$, we set $ab := m \circ (a \otimes b)$. Then the set $CE(\mathcal{C})$ is a $\mathbb{C}$-algebra with respect to this operation. The space $CF(\mathcal{C}) := \text{Hom}_{\mathcal{C}}(A, 1)$ is called the space of class functions of $\mathcal{C}$. The space $CF(\mathcal{C})$ is a $\mathbb{C}$-algebra with the multiplication defined $f \star g := f \circ Z(g) \circ \delta_1$ for any $f, g \in CF(\mathcal{C})$. Here $\delta : Z \to Z^2$ is the comultiplication structure of the Hopf comonad $Z$, see [Shi17] for details. To any object $X$ of $\mathcal{C}$ Shimizu assigned in [Shi17, Section 3] a class function $\text{ch}(X)$ called the internal character. It is defined as a partial trace of the canonical left action $\rho_X : A \otimes X \to 1$. By [Shi17, Theorem 3.10] one has that $\text{ch}(X \otimes Y) = \text{ch}(X)\text{ch}(Y)$ for any two objects $X$ and $Y$ of $\mathcal{C}$ and $K(\mathcal{C}) \simeq CF(\mathcal{C})$ as algebras.

There is also a non-degenerate paring $\langle \ , \ \rangle : CF(\mathcal{C}) \times CE(\mathcal{C}) \to 1$, given by $\langle f, a \rangle \text{id}_1 = f \circ a$, for all $f \in CF(\mathcal{C})$ and $a \in CE(\mathcal{C})$.

For any simple object $X_i$ of $\mathcal{C}$ we denote by $\chi_i := \text{ch}(X_i)$ its associated character. The central element space $CE(\mathcal{C})$ has a basis of primitive orthogonal idempotents $E_i$ such that $\langle \chi_i, E_j \rangle = \delta_{i,j}d_i$ for all $i, j$.

Recall $R : \mathcal{C} \to Z(\mathcal{C})$ is a right adjoint to the forgetful functor $F : Z(\mathcal{C}) \to \mathcal{C}$. As explained in [Shi17 Theorem 3.8] this adjunction gives an isomorphism of algebras

\begin{equation}
CF(\mathcal{C}) \xrightarrow{\cong} \text{End}_{Z(\mathcal{C})}(R(1)), \quad \chi \mapsto Z(\chi) \circ \delta_1.
\end{equation}

Since $Z(\mathcal{C})$ is also fusion category we can write $R(1) = \bigoplus_{j=0}^m C^j$ as a direct sum of simple objects in $Z(\mathcal{C})$. Recall that $C^j$ are called **conjugacy classes** for $\mathcal{C}$. 
For the rest of this section let \( \mathcal{C} \) be a fusion category with a commutative Grothendieck ring \( K(\mathcal{C}) \). Since \( CF(\mathcal{C}) \) is also commutative semisimple \( \mathbb{C} \)-algebra has

\[
(2.3) \quad CF(\mathcal{C}) \simeq \bigoplus_{j=0}^{m} \mathbb{C}F_j
\]

where \( F_j \) are the central primitive idempotents of \( \mathcal{C} \). We define \( \mathcal{J} := \{0, \ldots, m\} \).

Since \( CF(\mathcal{C}) \) is a commutative algebra it follows from Equation (2.2) that \( R(1) \) is multiplicity free, i.e. each conjugacy class \( C_j \) appears with multiplicity 1 inside \( R(1) \). Moreover we fix the canonical bijection \( F_j \leftrightarrow C_j \) for which the image of \( F_j \) under the canonical adjunction isomorphism from Equation (2.2) is the projection of \( R(1) \) on \( C_j \).

With this bijection, the second orthogonality from [Shi17, Theorem 6.10] can be written as:

\[
(2.4) \quad \sum_{i=0}^{m} \mu_i(\chi_i)\mu_k(\chi_i^*) = \delta_{l,k} \frac{\dim(C)}{\dim(C^k)}
\]

where \( \mu_j : CF(\mathcal{C}) \to \mathbb{C} \) is the set of algebra maps of \( CF(\mathcal{C}) \). In terms of Equation (2.3) one has \( \mu_j(F_k) = \delta_{j,k} \).

Following [Bur20] we use the following definition:

**Definition 2.5.** To any fusion subcategory \( \mathcal{D} \) we associate a set of indices \( \mathcal{J}_\mathcal{D} \subseteq \mathcal{J} \) such that:

\[
(2.6) \quad \lambda_\mathcal{D} = \sum_{j \in \mathcal{J}_\mathcal{D}} F_j.
\]

In this settings, Equation (4.20) from [Bur20] gives that

\[
(2.7) \quad \sum_{j \in \mathcal{J}_\mathcal{D}} \dim(C^j) = \frac{\dim(C)}{\dim(D)}.
\]

3. Hecke algebras of cosets

For a fusion subcategory \( \mathcal{D} \) of \( \mathcal{C} \) we denote by \( \simeq^\mathcal{D} \) the equivalence relation on the set of simple objects \( \text{Irr}(\mathcal{C}) \) of \( \mathcal{C} \). It is given by \( X \simeq^\mathcal{D} Y \) if and only if there is a simple object \( S \) of \( \mathcal{D} \) such that \( X \) is a constituent of \( Y \otimes S \). An equivalence class of \( \simeq^\mathcal{D} \) is called a right coset of \( \mathcal{C} \) with respect to \( \mathcal{D} \). We denote by \( (\mathcal{C}/\mathcal{D})_r \) the set of equivalence classes (right cosets) with respect to \( \mathcal{D} \). It was shown in [BB15] that

\[
(3.1) \quad X \simeq^\mathcal{D} Y \iff \frac{[X]R_\mathcal{D}}{FPdim(X)} = \frac{[Y]R_\mathcal{D}}{FPdim(Y)}
\]
where \( R_D \) is the regular element of the fusion subcategory \( D \), \( R_D = \sum_{X \in \text{Irr}(D)} \text{FPdim}(X)[X] \in K(C) \). In this case it can be shown that

\[
[X]R_D \text{FPdim}(X) = [Y]R_D \text{FPdim}(Y) = R_m \text{FPdim}(R_m)
\]

where \( m \in (C/D)_r \) is the right coset corresponding to \( X \) and \( Y \) and \( R_m := \sum_{Z \in m} \text{FPdim}(Z)[Z] \in K(C) \) is the regular element corresponding to any coset \( m \).

Based on Equation (3.2) it is easy to verify that if \( K(C) \) is commutative then for all \( m, n, p \in (C/D)_r \)

\[
\frac{R_m}{\text{FPdim}(R_m)} \frac{R_n}{\text{FPdim}(R_n)} = \sum_{p \in (C/D)_r} H_{mn}^p \frac{R_p}{\text{FPdim}(R_p)}
\]

where \( H_{mn}^p \) are (integral) scalars given by

\[
H_{mn}^p = (\sum_{Z \in p} \text{FPdim}(Z)N^{Z}_{XY})
\]

where \( X \in m \) and \( Y \in n \) are arbitrarily chosen representatives for each coset \( m, n \in (C/D)_r \). Thus in the case \( K(C) \) is commutative, the vector subspace \( \mathcal{H}(D) := \{R_m \mid m \in (C/D)_r\} \) of \( K(C) \) generated by the regular elements \( R_m \) is in fact a subalgebra of \( K(C) \). It is called the Hecke algebra of \( C \) with respect to \( D \). Note that Equation (3.2) implies \( \mathcal{H}(D) = K(C)R_D \) inside \( K(C) \).

**Proposition 3.4.** Let \( D \) be a fusion subcategory of a pseudo-unitary fusion category \( C \) with a commutative Grothendieck ring. Then the dimension of the Hecke algebra \( \mathcal{H}(D) \) equals the cardinality of \( J_D \).

**Proof.** If \( C \) is pseudo-unitary then as above \( K(C) \simeq \text{CF}(C) \) and \( R_D \) corresponds to the idempotent cointegral \( \lambda_D \). Then from Equation (2.3) in this case one has:

\[
\mathcal{H}(D) = \text{CF}(C)\lambda_D = \bigoplus_{j \in J_D} \text{CF}(C)F_j = \bigoplus_{j \in J_D} \text{CF}_j
\]

by Equation (2.3). \( \square \)

**General Hecke orthogonality.** For any \( t \in (C/D)_r \) let \( t^* := \{X^* \mid X \in t\} \). It is easy to see that in the case of a commutative Grothendieck ring, if \( t \in (C/D)_r \) is a right coset then \( t^* \) is also a right coset. Indeed, since \( K(C) \) is commutative one has \( ([X])R_D = R_D^*[X] = R_D[X^*] \) since \( R_D^* = R_D \). For any \( t \in (C/D)_r \) choose a representative simple
object $X_t$ of $C$ belonging to this coset such that $X_t^* = X_{t^*}$. Denote also $\chi_t := \chi(X_t) \in \text{CF}(C)$.

Equation (2.6) implies that

$$\mathcal{H}(D) = \text{CF}(C)\lambda_D = \bigoplus_{j \in J_D} \text{CF}^j$$

Therefore $\hat{\mathcal{H}}(D) = \{\mu_j, \ | j \in J_D\}$ are the all linear characters of $\mathcal{H}(D)$.

**Theorem 3.5.** Let $D \subseteq C$ be a fusion subcategory of a pseudo-unitary fusion category $C$ with commutative Grothendieck ring. Then we have the following:

1. **The first orthogonality relation for $\mathcal{H}(D)$:**

   $$(3.6) \quad \sum_{t \in (C/D)} \frac{\text{FPdim}(R_t)}{\text{FPdim}(X_t^*)} \mu_k(\chi_t) \mu_l(\chi_{t^*}) = \delta_{l,k} \frac{\text{FPdim}(C)}{\text{FPdim}(C^l)}, \text{ for any } k, l \in J_D,$$

   where $R_t = \sum_{\chi_i \in D_t} d_i \chi_i$.

2. **The second orthogonality relation for $\mathcal{H}(D)$:**

   $$(3.7) \quad \sum_{k=0} \frac{\text{FPdim}(C^k)\mu_k(\chi_t)\mu_k(\chi_{t^*})}{\text{FPdim}(X_t^*)} = \delta_{s,t} \frac{\text{FPdim}(C)}{\text{FPdim}(R_t)}, \text{ for all } t, s \in (C/D)_r.$$

**Proof.** Since $C$ is a pseudo-unitary fusion category one has $\lambda_D = \frac{R_D}{\text{FPdim}(R_D)}$ for any fusion subcategory and $\text{dim}(C) = \text{FPdim}(C)$. Moreover $Z(C)$ is also pseudo-unitary and therefore $\text{dim}(C^k) = \text{FPdim}(C^k)$ for any conjugacy class of $C$.

Let $j \in J_D$. If $X_i \in t$ for some $t \in (C/D)_r$ by applying $\mu_j$ to Equation (3.1) one has

$$ (3.8) \quad \frac{\mu_j(\chi_i)}{\text{FPdim}(X_i)} = \frac{\mu_j(\chi_t)}{\text{FPdim}(X_t)} = \mu_j(\frac{R_t}{\text{FPdim}(R_t)}). $$

Note that since $j \in J_D$ Equation (2.6) implies that $\mu_j(\lambda_D) = 1$.

The second orthogonality relation for $C$ from Equation (2.4) can be written as

$$ \sum_{t \in (C/D)_r} \left( \sum_{X_i \in t} \mu_l(\chi_i) \mu_k(\chi_{i^*}) \right) = \delta_{l,k} \frac{\text{FPdim}(C)}{\text{FPdim}(C^k)}.$$
If \( l, k \in \mathcal{J}_D \), using Equation (3.8) one obtains Equation (3.6) from above. Indeed,

\[
\delta_{l,k} \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(\mathcal{C}^k)} = \sum_{t \in (\mathcal{C}/\mathcal{D})_r} \sum_{X_i \in t} \mu_l(\chi_i)\mu_k(\chi_{i'}) \\
= \sum_{t \in (\mathcal{C}/\mathcal{D})_r} \sum_{X_i \in t} \frac{\text{FPdim}(X_i)\mu_l(\chi_i)}{\text{FPdim}(X_i)} \frac{\text{FPdim}(X_i)\mu_k(\chi_{i'})}{\text{FPdim}(X_{i'})} \\
= \sum_{t \in (\mathcal{C}/\mathcal{D})_r} \frac{\mu_k(\chi_{i'})\mu_l(\chi_i)}{\text{FPdim}(X_i)^2} (\sum_{X_i \in t} \text{FPdim}(X_i)^2) = \\
= \sum_{t \in (\mathcal{C}/\mathcal{D})_r} \frac{\mu_k(\chi_{i'})\mu_l(\chi_i)}{\text{FPdim}(X_i)^2} \text{FPdim}(R_t)
\]

The second orthogonality relation follows as usually from the fact that for complex matrices \((BA = I_n \implies AB = I_n)\). Indeed, denote

\[
a_{tk} := \sqrt{\frac{\text{FPdim}(\mathcal{C}^k)\text{FPdim}(R_t)}{\text{FPdim}(\mathcal{C})}} \frac{\mu_k(\chi_t)}{\text{FPdim}(X_t)} , \quad k \in \mathcal{J}_D, t \in (\mathcal{C}/\mathcal{D})_r
\]

and

\[
b_{lt} := \sqrt{\frac{\text{FPdim}(\mathcal{C}^l)\text{FPdim}(R_t)}{\text{FPdim}(\mathcal{C})}} \frac{\mu_l(\chi_{t'})}{\text{FPdim}(X_{t'})} , \quad l \in \mathcal{J}_D, t \in (\mathcal{C}/\mathcal{D})_r
\]

With these notations it is easy to see that the equality \(BA = I_n\) (where \(n := |(\mathcal{C}/\mathcal{D})_r| = |\mathcal{J}_D|\)) is equivalent to the first orthogonality relations. Then the second orthogonality relations are equivalent to \(AB = I_n\).

Indeed,

\[
\delta_{s,t} = \sum_{k=0}^{m} a_{tk} b_{ks} \\
= \sum_{k=0}^{m} \sqrt{\frac{\text{FPdim}(\mathcal{C}^k)\text{FPdim}(R_t)}{\text{FPdim}(\mathcal{C})}} \frac{\mu_k(\chi_t)}{\text{FPdim}(X_t)} \sqrt{\frac{\text{FPdim}(\mathcal{C}^k)\text{FPdim}(R_s)}{\text{FPdim}(\mathcal{C})}} \frac{\mu_k(\chi_{s'})}{\text{FPdim}(X_{s'})} \\
= \sqrt{\frac{\text{FPdim}(R_t)\text{FPdim}(R_s)}{\text{FPdim}(X_t)\text{FPdim}(X_s)}} \sqrt{\text{FPdim}(\mathcal{C}^k)\text{FPdim}(R_t)\text{FPdim}(R_s)\sum_{k=0}^{m} \text{FPdim}(\mathcal{C}^k)\mu_k(\chi_t)\mu_k(\chi_{s'})}
\]

\[\square\]

**Corollary 3.9.** Let \(\mathcal{D}\) be a fusion subcategory of a pseudo-unitary fusion category \(\mathcal{C}\). With the above notations one has:

1. For any right coset \(t \in (\mathcal{C}/\mathcal{D})_r\) and any representative \(X_t \in t\) one has

\[
(3.10) \quad \frac{\text{FPdim}(X_t)^2\text{FPdim}(\mathcal{C})}{\text{FPdim}(R_t)} \in \mathbb{A}.
\]
(2) Suppose that $\mathcal{D} \subseteq \mathcal{C}_{pt}$ acts freely on the simple objects of $\mathcal{C}$. Then

$$\frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{C})^j} \in \mathbb{A}$$

for all $j \in \mathcal{J}_D$.

**Proof.** (1) It follows from Equation (3.7) for $s = t$ since $\text{FPdim}(\mathcal{C}^k) \in \mathbb{A}$ and $\mu_k(\chi_t) \in \mathbb{A}$.

(2) Note that in this case, since $\mathcal{D}$ acts freely, one has $\text{FPdim}(R_m) = \text{FPdim}(\mathcal{D}) \text{FPdim}(X_m^2)$. Then equation (3.6) can be written as

$$\sum_{t \in (\mathcal{C}/\mathcal{D})_r} \mu_k(\chi_t) \mu_l(\chi_t^*) = \delta_{k,l} \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{C})^k},$$

for any $k, l \in \mathcal{J}_D$.

Put $k = l$ above, and since $\mu_k(\chi_t) \in \mathbb{A}$ the divisibility result follows.

□

**Lemma 3.12.** Suppose that $\mathcal{D}, \mathcal{A}$ are fusion subcategories of a fusion category $\mathcal{C}$. If $m \in (\mathcal{C}/\mathcal{D})_r$ is a right coset of $\mathcal{C}$ with respect to $\mathcal{D}$ then the set $\mathcal{A} \cap m$, if not empty, is a right coset $\mathcal{A}$ with respect to $\mathcal{A} \cap \mathcal{D}$.

**Proof.** Let $X, Y \in \mathcal{A} \cap m$ be any two simple objects. We have to show that $X \simeq_{\mathcal{A} \cap \mathcal{D}} Y$ as objects of $\mathcal{A}$. Since $X, Y$ are in the same right coset with respect to $\mathcal{D}$, by definition, there is $Z \in \mathcal{D}$ such that $m(X, Y \otimes Z) > 0$. This implies $m(Z, Y^* \otimes X) > 0$ which in turn gives that $Z \in \mathcal{A}$ since both $X, Y \in \mathcal{A}$. Thus $Z \in \mathcal{D} \cap \mathcal{A}$. It follows that $X, Y$ are in the same coset of $\mathcal{A}$ with respect to $\mathcal{A} \cap \mathcal{D}$. □

4. **Divisibility results for premodular categories**

4.1. **Definition of the braided partition function $M$.** Let $\mathcal{C}$ be a premodular (i.e. braided and spherical) fusion category. We keep all the notations from the previous section, in particular $\text{Irr}(\mathcal{C}) = \{X_0, X_1, \ldots, X_m\}$ and $d_i := \dim(X_i)$ for all $i$. By [Shi17, Example 6.14] there is $\mathbb{C}$-algebra map $f_Q : \text{CF}(\mathcal{C}) \to \text{CE}(\mathcal{C})$ given by the following formula:

$$f_Q(\chi_i) = \sum_{i'=0}^m \frac{s_{ii'}}{d_i} E_{i'}. \hspace{1cm} (4.1)$$

where $S = (s_{ij})$ is the $S$-matrix of $\mathcal{C}$.

Given a fusion subcategory $\mathcal{D}$ of a braided fusion category $\mathcal{C}$, the notion of Müger centralizer of $\mathcal{D}$ was introduced in [DGNO10]. Two objects $X$ and $Y$ of $\mathcal{C}$ centralize each other if $c_{X,Y} c_{Y,X} = \text{id}_{X \otimes Y}$. 

The centralizer $\mathcal{D}'$ is defined as the fusion subcategory $\mathcal{D}'$ of $\mathcal{C}$ generated by all simple objects $X$ of $\mathcal{C}$ centralizing any object objects $X$ of $\mathcal{D}$. In the premodular case $X_i$ centralizes $X_j$ if and only if $s_{ij} = d_i d_j$, (see also [M"ug03]). In particular, the centralizer $\mathcal{C}'$ of $\mathcal{C}$ is also denoted by $\mathcal{Z}_2(\mathcal{C})$ and it is called the M"uger center of $\mathcal{C}$.

Since $\chi_i$ is an idempotent element of $\text{CE}(\mathcal{C})$ one may write:

$$f_Q(F_j) = \sum_{i \in A_j} E_i$$

for some subset $A_j \subseteq \{0, \ldots, m\}$. Note that the set $A_j$ might be empty precisely when $f_Q(F_j) = 0$. Denote by $A_j \subseteq J := \{0, 1, \ldots, m\}$ the set of all indices $i$ such that $A_j$ not a empty set. Since $f_Q(1) = 1$ we obtain in this way a partition for the set of all irreducible representations $\text{Irr}(\mathcal{C}) = \bigcup_{j \in J} A_j$, where, to be precise, $A_j = \{[X_i] \mid i \in A_j\}$. For any index $0 \leq i \leq m$ we denoted by $M(i)$ the unique index $j \in J$ such that $i \in A_j$. One obtains a (unique) function $M : \{0, 1, \ldots, m\} \to J$ with the property that $E_i f_Q(F_{M(i)}) \neq 0$ for all $i \in \{0, 1, \ldots, m\}$.

Since $\text{CF}(\mathcal{C})$ is a semisimple commutative algebra it follows that $F_j$ form a $\mathbb{C}$-linear basis for $\text{CF}(\mathcal{C})$. Then for any irreducible character $\chi_i = \sum_{j=0}^m \alpha_{ij} F_j$ for some $\alpha_{ij} \in \mathbb{C}$.

The following lemma generalizes [CW10, Equation (22)].

**Lemma 4.2.** With the above notations, for all $0 \leq i, i' \leq m$ one has

$$\frac{\alpha_{iM(i')}}{d_i} = \frac{s_{ii'}}{d_{i'i'}} = \frac{\alpha_{i'M(i')}}{d'_{i'}}.$$  

(4.3)

**Proof.** Since $f_Q(F_j) = 0$ if $j \notin J$ it follows that $f_Q(\chi_i) = \sum_{j=0}^m \alpha_{ij} f_Q(F_j) = \sum_{j \in J} \sum_{i' \in A_j} \alpha_{iM(i')} E_{i'} = \sum_{i' = 0}^m \alpha_{iM(i')} E_{i'}$. Thus

$$f_Q(\chi_i) = \sum_{i' = 0}^m \alpha_{iM(i')} E_{i'}.$$  

(4.4)

for all $i$. Comparing with Equation (4.1), for all indices $i$ and $i'$ one has $\alpha_{iM(i')} = \frac{s_{ii'}}{d_{i'i'}}$. Now Equation (4.3) follows since $s_{ii'} = s_{i'i'}$. \hfill $\square$

Recall from [EGNO15] Proposition 8.13.11], that for any simple object $X_i \in \text{Irr}(\mathcal{C})$ there is a character $\psi_{[X_i]} : \text{CF}(\mathcal{C}) \to \mathbb{C}$ given by $\psi_{[X_i]}(\chi_i) = \frac{s_{ii}}{d_i}$. Using Equation (4.3) it can be easily seen that

$$\psi_{[X_i]} = \mu_{M(i)}, \text{ for all } i \in I.$$  

(4.5)

Indeed, for all indices $i'$ one has:

$$\psi_{[X_i]}(\chi_i') = \frac{s_{ii'}}{d_i} = \alpha_{i'M(i')} = \mu_{M(i')}(\chi_i').$$
Next Theorem is a generalization of the corresponding result for semisimple Hopf algebras obtained in [CW10, Theorem 4.3].

**Theorem 4.6.** With the above notations one has

\[ f_Q(\chi_i) = \frac{d_i}{\dim(CM(i))} C_{M(i)} \]

**Proof.** Let \( \chi_i = \sum_{j=0}^m \alpha_{ij} F_j \) as above. Using Equation (4.3) and Equation (4.4) it follows that

\[ f_Q(\chi_i) = \sum_{i'=0}^m \alpha_{i'M(i')} E_{i'} = d_i \left( \sum_{i'=0}^m \frac{\alpha_{i'M(i')}}{d_{i'}} E_{i'} \right). \]

Note that by [Bur20, Equations (4.3) and (4.8)] one has

\[ C_j = \dim(C^j) \left( \sum_{i'=0}^m \frac{1}{d_{i'}} \alpha_{i'j} E_{i'} \right) \]

Then for \( j = M(i) \) Equation (4.8) becomes \( f_Q(\chi_i) = \frac{d_i}{\dim(CM(i))} C_{M(i)}. \)

\[ \square \]

4.2. Cosets with respect to \( \mathcal{Z}_2(C) \). Recall the notion of cosets with respect to a fusion subcategory from Section 3.

**Theorem 4.10.** Two simple objects \( X_i, C_{i'} \) of a pseudo premodular fusion category \( C \) are in the same coset with respect to \( \mathcal{Z}_2(C) \) if and only if \( M(i) = M(i') \).

**Proof.** Let \( \chi_i := \chi(X_i) \) and \( \chi_{i'} := \chi(X_{i'}). \) Suppose that the two characters \( \chi_i \) and \( \chi_{i'} \) are in the same coset with respect to \( \mathcal{Z}_2(C) \). Then as above from Equation (3.2) one has:

\[ \frac{\chi_i \lambda C'}{d_i} = \frac{\chi_{i'} \lambda C'}{d_{i'}}. \]

On the other hand \( f_Q(\lambda C') = 1 \) by [Bur20, Corollary 5.8], and applying \( f_Q \) to the above Equation one has:

\[ f_Q(\chi_i \lambda C') = \frac{1}{\dim(CM(i))} C_{M(i)} = f_Q(\chi_{i'} \lambda C') = \frac{1}{\dim(CM(i'))} C_{M(i')} \]

which proves that \( M(i) = M(i'). \)

Therefore every set \( \tilde{A}_j := \{ X_i \mid M(i) = j \} \) is a union of right cosets with respect to \( C' \). Clearly, by its definition, the number of the non-empty sets \( \tilde{A}_j \) equals the cardinality of \( \mathcal{J}_2 \).

Note that the relation \( f_Q(\lambda \mathcal{Z}_2(C)) = 1 \) implies also that \( \mathcal{J}_2 = \mathcal{J}_{\mathcal{Z}_2(C)}. \) Then from Proposition (3.3) it follows that

\[ \dim \mathcal{H}(\mathcal{Z}_2(C)) = |\mathcal{J}_2|. \]
On the other hand, Equation (4.11) implies that the number of right cosets also equals the cardinal of \( \mathcal{J}_2 \). Thus each \( \mathcal{A}_j \) consists of a single right coset with respect to \( \mathcal{C}' \) and the proof is finished. \( \square \)

For any \( j \in \mathcal{J}_2 \) we denote by \( \mathcal{R}_j := \{ X_i \mid M(i) = j \} \). The above theorem implies that \( \mathcal{R}_j \) with \( j \in \mathcal{J}_2 \) are exactly the right cosets of \( \mathcal{C} \) with respect to \( \mathbb{Z}_2(\mathcal{C}) \). Denote also \( \mathcal{R}_j := \sum_{X_i \in \mathcal{R}_j} d_i \chi_i \in \text{CF}(\mathcal{C}) \) the regular part of their characters.

### 4.3. On the dimension of the cosets

For any premodular fusion category \( \mathcal{D} \) of \( \mathcal{C} \) we denote by \( \mathcal{R}(\mathcal{D})_j := \mathcal{D} \cap \mathcal{R}_j \), the intersection of \( \mathcal{D} \) with each coset \( \mathcal{R}_j \) of \( \mathcal{C} \) with respect to \( \mathcal{C}' \). We denote also their regular parts by

\[
\mathcal{R}(\mathcal{D})_j := \sum_{X_i \in \mathcal{R}(\mathcal{D})_j} d_i \chi_i \in \text{CF}(\mathcal{C}).
\]

**Proposition 4.12.** Let \( \mathcal{C} \) be a premodular fusion category and \( \mathcal{D} \) a fusion subcategory of \( \mathcal{C} \). With the above notations it follows that

\[
\mathcal{J}_D = \{ M(i) \mid \chi_i \in \text{Irr}(\mathcal{D}) \}
\]

and

\[
\dim(\mathcal{R}(\mathcal{D})_j) = \dim(\mathcal{D} \cap \mathcal{C}') \dim(\mathcal{C})^j, \text{ for all } j \in \mathcal{J}_D.
\]

**Proof.** Since \( \mathcal{C} \) is spherical one has \( d_i = d_i^* \) for all \( i \). Recall by [Shi17, Equation (6.8)] that \( \lambda_D = \frac{1}{\dim(\mathcal{D})} \left( \sum_{\chi_i \in \text{Irr}(\mathcal{D})} d_i \chi_i \right) \) and Theorem 4.7 gives

\[
f_Q(\lambda_D) = \frac{1}{\dim(\mathcal{D})} \left( \sum_{\chi_i \in \text{Irr}(\mathcal{D})} \frac{d_i d_i^*}{\dim(C_{M(i)})} C_{M(i)} \right).
\]

Applying now the Fourier transform to the last equality, since \( C_j := \mathcal{F}^{-1}(F_j) \) it follows that

\[
\mathcal{F}(f_Q(\lambda_D)) = \frac{1}{\dim(\mathcal{D})} \left( \sum_{\chi_i \in \text{Irr}(\mathcal{D})} \frac{d_i d_i^*}{\dim(C_{M(i)})} F_{M(i)} \right)
\]

\[
= \frac{1}{\dim(\mathcal{D})} \left( \sum_{j \in \mathcal{J}_2} \left( \sum_{\{ \chi_i \in \text{Irr}(\mathcal{D}) \mid M(i) = j \}} d_i d_i^* \frac{F_j}{\dim(\mathcal{C})} \right) \right)
\]

\[
= \frac{1}{\dim(\mathcal{D})} \sum_{\{ j \in \mathcal{J}_2 \mid j = M(i), \chi_i \in \text{Irr}(\mathcal{D}) \}} \frac{\dim(\mathcal{R}(\mathcal{D})_j)}{\dim(\mathcal{C})} F_j.
\]
since, from its definition, one has $\dim(R(\mathcal{D})_j) := \left(\sum_{\{\chi_i \in \text{Irr}(\mathcal{D}) \mid M(i) = j\}} d_i^2\right)$.

On the other hand by [Bur20, Theorem 1] one can write

$$\mathcal{F}(f_Q(\lambda_D)) = \frac{\dim(D')}{\dim(C)} \lambda_{D'} = \frac{\dim(D')}{\dim(C)} \left(\sum_{j \in J_{D'}} F_j\right)$$

Note that the sphericality of $\mathcal{C}$ implies by [EGNO13, Prop 4.8.4] that the coefficients of $F_j$ in both above formulae for $\mathcal{F}(f_Q(\lambda_D))$ are non-zero scalars. Comparing the coefficients of $F_j$ in the above two formulae for $\mathcal{F}(f_Q(\lambda_D))$ it follows that $J_{D'} = \{M(i) \mid \chi_i \in \text{Irr}(\mathcal{D})\}$ and

$$\dim(R(\mathcal{D})_j) = \dim(C) \frac{\dim(D') \dim(D)}{\dim(C)}$$

which implies the result since by [DGNO10, Theorem 3.10] one has

$$\dim(D) \dim(D') = \dim(C) \dim(\mathcal{D} \cap \mathcal{C}').$$

\[\square\]

**Corollary 4.16.** With the above notations it follows that

$$\frac{\dim(C) \dim(C' \cap \mathcal{D})}{\dim(R(\mathcal{D})_j)} \in \mathbb{A}$$

for all $j \in J_{D'}$.

**Proof.** Note that equation (4.14) gives that $\frac{\dim(C)}{\dim(C')} = \frac{\dim(C) \dim(C' \cap \mathcal{D})}{\dim(R(\mathcal{D})_j)}$.

Since $C_j$ is a simple object of $\mathcal{Z}(\mathcal{C})$ the result follows. \[\square\]

**Corollary 4.18.** If $\mathcal{C}$ is an integral braided fusion category of free square dimension and $\mathcal{D} \cap \mathcal{Z}_2(\mathcal{C}) = \text{Vec}$ then $\mathcal{D}$ is pointed.

**Corollary 4.19.** With the above notations it follows that

$$\dim(R_j) = \dim(C') \dim(C' \cap \mathcal{D}), \text{ for all } j \in J_2.$$ 

**Proof.** It is Equation (4.14) for $\mathcal{D} = \mathcal{C}$. \[\square\]

### 4.4. Proof of Theorem 1.1

**Proposition 4.21.** Let $\mathcal{D}$ be a fusion subcategory of a pseudo-unitary premodular category $\mathcal{C}$. With the above notations one has:

$$\mathcal{D} = \bigoplus_{j \in J_{D'}} R(\mathcal{D})_j$$

is the decomposition of $\mathcal{D}$ in cosets with respect to $\mathcal{D} \cap \mathcal{C}'$. 

Proof. By Lemma 3.12 each non-empty set $R(D)_j$ is a coset of $D$ with respect to $D \cap Z_2(C)$. On the other hand we will show that
\[
\sum_{j \in J_{D'}} \text{FPdim}(R(D)_j) = \text{FPdim}(D).
\]
This proves that $R(D)_j$ with $j \in J_{D'}$ are all the right cosets. Indeed, adding all the Equations (4.14) for $j \in J_{D'}$ one obtains that
\[
\sum_{j \in J_{D'}} \text{FPdim}(R(D)_j) = \text{FPdim}(D \cap C') (\sum_{j \in J_{D'}} \text{FPdim}(C^j)).
\]
On the other hand, since $C$ is pseudo-unitary note that by Equation 2.7 one has
\[
(4.22) \sum_{j \in J_{D'}} \text{FPdim}(C^j) = \frac{\text{FPdim}(C)}{\text{FPdim}(D')} = \frac{\text{FPdim}(D)}{\text{FPdim}(D \cap Z_2(C))}
\]
and Equation (4.15) gives that
\[
\sum_{j \in J_{D'}} \text{FPdim}(R(D)_j) = \text{FPdim}(D). \quad \square
\]

Next one can obtain a proof of Theorem 1.1 as follows:

Proof. By Proposition 4.21 we may suppose that $Y \in R(D)_j$ for some $j \in J_{D'}$. If $D \cap Z_2(C) = \text{Vec}$ then $R(D)_j$ is a coset of $\text{Vec}$ inside $D$ and therefore it consists of a single element $Y$ of $D$. Thus, in this case $\text{FPdim}(R(D)_j) = \dim(Y)^2$ and the result follows from Equation (4.17).

4.5. Proof of Theorem 1.3. In this subsection we prove our main second result:

Proof. For the first item suppose that $Y = X_i$ and $M(i) = j$. Since $C'$ is pointed all the simple objects in $R_j$ have the same dimension, namely $\text{FPdim}(Y)$. If $r_j := |R_j|$ and $G_Y$ is the stabilizer of $Y$ under the action of $Z_2(C)$ it follows that
\[
\text{FPdim}(R_j) = r_j \text{FPdim}(Y)^2 = \frac{\text{FPdim}(C')}{|G_Y|} \text{FPdim}(Y)^2
\]
since $r_j |G_Y| = \text{FPdim}(C')$. Then by Proposition 4.12 one has:
\[
(4.23) \quad \text{FPdim}(C^j) = \frac{\text{FPdim}(R_j)}{\text{FPdim}(C')} = \frac{\text{FPdim}(Y)^2}{|G_Y|}
\]
Note that $\frac{\text{FPdim}(C)}{\text{FPdim}(C')} \in \mathbb{A}$ since $C^j$ is a simple object of $Z(C)$. Thus
\[
(4.24) \quad \frac{\text{FPdim}(C) |G_Y|}{\text{FPdim}(Y)^2} \in \mathbb{A}
\]
and the first divisibility result follows.
If the action of $C'$ is free then by the second item of Corollary 3.9

$$\frac{\text{FPdim}(C)}{\text{FPdim}(\mathbb{Z}_2(C))\text{FPdim}(C')} \in A$$

for all $j \in J_{\mathbb{Z}_2(C)}$. Since in this case $|G_Y| = 1$ it follows by Equation (4.23) that $\text{FPdim}(C^j) = \text{FPdim}(Y)^2$ and therefore

$$\frac{\text{FPdim}(C)}{\text{FPdim}(\mathbb{Z}_2(C))\text{FPdim}(Y)^2} \in A.$$

□

Remark 4.25. Note that for any simple object $X_i$ of $C$ Equation (3.10) gives that

$$\text{FPdim}(X_i)^2\text{FPdim}(C) = \text{FPdim}(C')\text{FPdim}(C^{M(i)})$$

since $\text{FPdim}(R_{M(i)}) = \text{FPdim}(\mathbb{Z}_2(C))\text{FPdim}(C^{M(i)})$

Remark 4.27. As mentioned in the introduction, the divisibility of Equation (1.5) generalizes [Yu20, Corollary 3.4.] in the pseudo-unitary settings.

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