Quantum quench from a thermal initial state

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Abstract – We consider a quantum quench in a system of free bosons, starting from a thermal initial state. As in the case where the system is initially in the ground state, any finite subsystem eventually reaches a stationary thermal state with a momentum-dependent effective temperature. We find that this can, in some cases, even be lower than the initial temperature. We also study lattice effects and discuss more general types of quenches.

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Introduction. – The problem of a quantum quench, \textit{i.e.} an instantaneous change in the parameters that determine the dynamics of a quantum system, has been recently studied in various theoretical models. These include systems of free bosons or fermions in which the quenched parameter is the energy gap \cite{1–3}, integrable models described by conformal field theory \cite{1,2,4–6} and exactly solvable spin chains \cite{4,7,8}. Several studies have also been carried out in non-integrable models using numerical methods or approximation schemes \cite{9–12}. Perhaps one of the most interesting results is that in the thermodynamic limit it is possible under quite general conditions that for large times an arbitrarily large subsystem tends to a stationary state with thermal characteristics, a process we can call thermalization \cite{1–3,13}.

To be more specific, let us imagine a system of coupled harmonic oscillators or equivalently a free field theory, described by a general dispersion relation with some energy gap or “mass” \( m_0 \) and maximum group velocity of excitations \( c \). Assume that the system initially lies on the ground state of the initial Hamiltonian \( H_0 \) and at time \( t = 0 \) the mass is suddenly changed from \( m_0 \) to a different value \( m \). For \( t > 0 \), the state of the system evolves according to quantum mechanics (\textit{unitary} evolution), \textit{i.e.} it is fully isolated from its environment. After this quench there is an extensive energy excess in comparison with the ground state of the final Hamiltonian \( H \), which is distributed over the excitation levels of \( H \). It then turns out that the two-point correlation function, which in free systems contains all the information required to determine their state, acquires for sufficiently large times the form of the correlation function of a system at thermal equilibrium with a momentum-dependent effective temperature \( \beta_{\text{eff}}(k) \) \cite{1,2,14}. Note that this momentum dependence is expected for a free theory since each momentum mode-evolution evolves independently of the others and can therefore thermalize to a different temperature. For any local observable, the quantum interference between all the momentum modes gives rise to a single effective temperature, which, however, may be dependent on the chosen observable. However, the large distance behaviour of the two-point correlation function is determined only by the zero-momentum mode \( \beta_{\text{eff}}(0) \). In the so-called deep quench limit where \( m_0 \gg m \), the effective temperature \( \beta_{\text{eff}}^{-1}(k) \) is asymptotically equal to \( m_0/4 \) and becomes momentum independent \cite{1,2}.

In most studies made so far, the initial state of the system is the ground state of \( H_0 \). Here instead, we assume that, before the quench, the system is prepared in a thermal state. Then we perform the quench exactly as before keeping the system isolated from the environment (the coupling to the environment is studied in [15]). For brevity we can call this a thermal quantum quench, in contrast to the previous case that we will call a pure quantum quench. Few other studies including the effect of a thermal initial state exist \cite{8,12,16}, but consider different kinds of quenches. The solution of this problem is in general prohibitively difficult and effective numerical methods must be used to extract useful physical quantities. However, in free field theory the solution is particularly simple and allows us to investigate the effect of the initial

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temperature $\beta_0$ on the time evolution of the system. Especially we can study the crossover from the “hot” limit, i.e. when the initial temperature dominates, to the “cold” limit when it can be ignored, in which case we recover the pure quantum quench results. It turns out that after such a thermal quench the system also thermalizes in the sense described earlier. An intuitive guess would perhaps be that the final effective temperature $\beta_{\text{eff}}$ is just the sum of the initial temperature $\beta_0^{-1}$ and that corresponding to an equal pure quench of the mass $\beta_{\text{eff}}^{-1}$. However, this is not true and $\beta_{\text{eff}}^{-1}$ can instead be lower than $\beta_0^{-1}$. In the hot deep quench limit ($\beta_0^{-1} \gg m_0 > m$), for example, the final temperature is shown to be half of the initial one.

In the present paper we provide the solution of the problem of a thermal quantum quench in a system of free bosons. We start with a simple harmonic oscillator where we calculate the analogue of the correlation function as a function of time. Then we use this to find the correlation function in a bosonic free field theory with a relativistic dispersion relation and study its large time as well as real space behaviour. We thus identify its stationary part and, by comparison to the thermal expression, derive the effective temperature, whose properties in several limits are discussed. Furthermore we investigate several other types of quenches and derive general expressions for the momentum distribution of excitations in lattice models also extending to the fermionic case.

**Harmonic oscillator.** – We first consider a single harmonic oscillator that initially lies in a thermal state with temperature $\beta_0$. Then at $t = 0$ we quench its frequency from $\omega_0$ to $\omega$. The time evolution of the position operator after the quench is given by the Heisenberg equations of motion $\dot{x}(t) + \omega^2 x(t) = 0$ with solution $x(t) = x(0) \cos \omega t + p(0) \sin \omega t / \omega$. The computation of the time ordered correlator $\mathcal{T}\{x(t_1) x(t_2)\}$ thus reduces to finding the values of $x^2$, $x p$, $p x$, and $p^2$ in the initial state. These values are thermal statistical averages of quantum expectation values given in general by $\mathcal{O}(\beta) = \text{Tr}(e^{-\beta H} \mathcal{O}) / Z(\beta) = \sum_n e^{-\beta E_n} \langle n | \mathcal{O} | n \rangle / Z(\beta)$. For a simple harmonic oscillator of frequency $\omega_0$ and at temperature $\beta_0$, since the energy levels are $E_n = (n + \frac{1}{2}) \omega_0$, the partition function is $Z(\beta_0) = 1 / (2 \sinh(\omega_0 \beta_0 / 2))$. From the equipartition theorem and the equation $E(\beta) = -\partial(\ln Z(\beta)) / \partial \beta$ for the energy of a thermal state, or by direct calculation, we find

$$\langle x^2 \rangle_{\beta_0} = \frac{1}{2 \omega_0} \coth \frac{\beta_0 \omega_0}{2}, \quad \langle p^2 \rangle_{\beta_0} = \frac{\omega_0}{2} \coth \frac{\beta_0 \omega_0}{2}.$$ 

Also $[x, p] = i$ and $\langle [x, p] \rangle_{\beta_0} = 0$. Using all of the above equations we can show that the time ordered correlator is

$$C_{\beta_0}(t_1, t_2) = \mathcal{T}\{x(t_1) x(t_2)\} = \frac{1}{2 \omega_0} e^{-i \omega |t_1 - t_2|} + \frac{\omega_0}{4} \left( \frac{1}{\omega_0} + \frac{1}{\omega_2} \right) \coth \frac{\beta_0 \omega_0}{2} \cos \omega (t_1 - t_2) + \frac{\omega_0}{4} \left( \frac{1}{\omega_0} - \frac{1}{\omega_2} \right) \coth \frac{\beta_0 \omega_0}{2} \cos \omega (t_1 + t_2).$$  

(1)

The first term is the Feynman propagator in $0 + 1$ dimensions and is the only one that survives if $\omega_0 = \omega$ and $\beta_0 \to \infty$. The breaking of time-translational invariance due to the quench is apparent in the last term that depends on $t_1 + t_2$. Also notice that for $\beta_0 \to \infty$ we recover the pure quantum quench propagator, since the initial state is then the ground state of the initial Hamiltonian

$$C(t_1, t_2) = \frac{1}{2 \omega_0} e^{-i \omega |t_1 - t_2|} + \frac{\omega_0}{4 \omega_0 \omega^2} \cos \omega (t_1 - t_2) + \frac{\omega_0^2 - \omega_0^2}{4 \omega_0 \omega^2} \cos \omega (t_1 + t_2).$$  

(2)

Obviously the harmonic oscillator does not thermalize but oscillates. However, in a free field theory, we shall argue that the interference between momentum modes needed to calculate local observables is responsible for effective thermalization.

From the above results we find that the energy of the oscillator before the quench is

$$\langle H_0 \rangle_{\beta_0} = \frac{1}{2} \omega_0 \coth(\beta_0 \omega_0 / 2)$$  

(3)

while after the quench it is

$$\langle H \rangle_{\beta_0} = \frac{\omega_0^2 + \omega_0^2}{4 \omega_0} \coth(\beta_0 \omega_0 / 2)$$  

(4)

i.e. it changes by a factor $(E - E_0) / E_0 = ((\omega / \omega_0)^2 - 1) / 2$. This means that the work done on the system at the quench is positive or negative, depending on whether the frequency increases or decreases, respectively. Notice that for $\omega \ll \omega_0$ the system loses half of its energy.

**Bosonic free field theory.** – Let us now move on to a bosonic free field theory with Hamiltonian of the general form

$$H = \int d^d k \left( \frac{1}{2} \pi_k^2 + \frac{1}{2} \omega_k^2 \phi_k^2 \right),$$  

(5)

and assume a relativistic dispersion relation $\omega_k^2 = c^2 k^2 + m^2 c^4$ in which we quench the mass from $m_0$ to $m$. We will later discuss other types of quenches with different dispersion relations or different quench parameters, but we now specialize to this case which contains most of the physical features of the general cases. For brevity we set the speed $c = 1$.

The propagator in the mixed momentum-time representation is just that of a single harmonic oscillator (1) with frequencies $\omega_k$ and $\omega_0$, defined by $m$ and $m_0$, respectively. The real space propagator is the Fourier transform of the latter and we can easily check, using the stationary-phase method, that for large times and finite separations the integration over all momenta leads to the $(t_1 + t_2)$-dependent part of the propagator decreasing with time under quite general conditions (for $m \neq 0$ or even for $m = 0$ in 3d (see footnote(1)). This property is already

1Even though this condition does not hold for $m = 0, d = 1$, we can still talk about thermalization in 1d critical systems, since what is physically meaningful is not the propagator, but the correlation function of vertex operators, i.e. imaginary exponentials of the field operator, and the latter does become stationary [2].
known for the pure quench propagator (2) [1,2,14] and we now simply observe that it also holds in the more general thermal case (1). According to the above, the correlation function tends for large times to the stationary form
\[
\tilde{C}_{\beta}(k; t_1, t_2) = \frac{1}{2\omega_k} e^{-i\omega_k |t_1 - t_2|} + \left[ \frac{\omega_k}{4} \left( \frac{1}{\omega_k} + \frac{1}{\omega_k^2} \right) \cos \omega_k (t_1 - t_2) \right].
\]

Furthermore one can compare this to the thermal or Matsubara propagator in real time which gives the correlations in a system at thermal equilibrium
\[
G_{\beta}(k; t_1, t_2) = \frac{1}{2\omega_k} e^{-i\omega_k |t_1 - t_2|} + \frac{\omega_k (t_1 - t_2)}{\omega_k (e^{\beta\omega_k} - 1)}.
\]

Again as in the pure quench case, the stationary propagator (6) is of thermal form, where the effective temperature \(\beta_{\text{eff}}\) is determined by equating the coefficients of \(\cos \omega_k (t_1 - t_2)\) in the compared expressions, giving
\[
\beta_{\text{eff}}(k) = \frac{1}{\omega_k} \ln \frac{(\omega_k - \omega_0)^2 + e^{\beta_0\omega_0} (\omega_k + \omega_0)^2}{(\omega_k + \omega_0)^2 + e^{\beta_0\omega_0} (\omega_k - \omega_0)^2}.
\]

The effective temperature obtained in this way turns out to be a function of the momentum: in a free theory the different momentum modes do not interact with each other and there is no reason why they should all acquire the same temperature. As before this property is not new; it is also true for a pure quantum quench where [2]
\[
\beta_{\text{eff}}^*(k) = \frac{2}{\omega_k} \ln \frac{\omega_k + \omega_0^2}{\omega_k - \omega_0^2}.
\]

Note that the three temperatures \(\beta_0, \beta_{\text{eff}}^*, \text{ and } \beta_{\text{eff}}\) satisfy the following symmetric relation:
\[
\tan \left( \frac{\beta_{\text{eff}}(k) \omega_k}{2} \right) = \tan \left( \frac{\beta_0 \omega_0}{2} \right) = \frac{\beta_{\text{eff}}(k) \omega_k}{2}.
\]

Let us investigate the main features of the effective temperature. We can distinguish the following limiting cases:

- Cold quench \(\omega_0^2 \gg \beta_0^{-1}\): we asymptotically reproduce the pure quantum quench result (9).

- Hot quench \(\beta_0^{-1} \gg \omega_0^2\):

\[
\beta_{\text{eff}}(k) = \frac{2\beta_0}{1 + \omega_k^2 / \omega_0^2}.
\]

For \(\omega_0^2 > \omega_k\), this leads to the interesting conclusion that the system becomes "colder" after the quench, unlike the pure quench case where the system becomes "hotter" since the temperature rises from zero to a finite value. In particular in the deep quench limit \(\omega_0^2 \gg \omega_k\) the final temperature is half of the initial one. This can be explained by the comments after (4) along with the fact that from (3) the energy of a harmonic oscillator at thermal equilibrium with temperature \(\beta^{-1} \gg \omega\) tends to the classical value \(\beta^{-1}\).

**An alternative effective temperature.** Ideally the effective temperature should be a single number describing the asymptotic state after the quench. It is then important that this number should not depend on the particular quantity we measure. The zero-momentum \(\beta_{\text{eff}}(k)\) is by definition in a free theory a good measure for the large-distance behaviour of any correlation function (we have derived it from the two-point function, but in a free theory this is enough to calculate all of them). Also in a conformal field theory the zero-momentum effective temperature is well-defined and independently observable [2]. However, one could wonder whether physical quantities involving higher momentum modes can spoil this result. To this end, we propose another estimate of the effective temperature by the comparison of the field fluctuations \(\langle \phi^2(x = 0, t \to \infty) \rangle\) in the two cases: after the quench and at thermal equilibrium

\[
\int d^d k \left[ \frac{\omega_k}{4} \left( \frac{1}{\omega_0^2} + \frac{1}{\omega_k^2} \right) \coth \frac{\beta_0 \omega_k}{2} - \frac{1}{2\omega_k} \right] =
\]

\[
\int d^d k \frac{1}{\omega_k (e^{\beta \omega_k} - 1)}.
\]
Table 1: Asymptotic behaviour of the integrals $I_1, I_2$ and $I_{th}$ in the limit $\beta^{-1}_0 \gg m_0 \gg m$.

| $d$ | 1      | 2                      | 3                      |
|-----|--------|------------------------|------------------------|
| $I_1$ | $\pi m_0/8m$ | $-\ln(m/m_0)/4$ | $m_0^3/8$ |
| $I_2$ | $\pi/4\beta_0m$ | $-\ln(\beta_0 m)/2\beta_0$ | $\pi^2/6\beta_0^2$ |
| $I_{th}$ | $\pi/2\beta_0 m$ | $-\ln(\beta m)/\beta$ | $\pi^2/6\beta^2$ |

Such an estimate involves an average over all momentum modes. For this reason we will call it “average” effective temperature and denote it by $\tilde{\beta}$. In the massless limit, both integrals are infrared divergent in $1d$ or $2d$, but the two sides compensate each other.

Equation (13) can be solved numerically or even analytically in the interesting limit $\beta^{-1}_0 \gg m_0 \gg m$ mentioned above. Then we expect $\tilde{\beta}$ to be small too and we can use the relevant asymptotic form of the thermal integral

$$I_{th} = \int_0^\infty \frac{k^{d-1}dk}{\omega_k(e^{\beta k} - 1)},$$

which is easily worked out for any number of spatial dimensions $d$ (see table 1). The integral of the left-hand side of (13) can be split into two parts: the pure quench part which can be calculated exactly

$$I_1 = \int_0^\infty k^{d-1}dk \left[ \frac{\omega_{0k}}{4} \left( \frac{1}{\omega_{0k}^2} + \frac{1}{\omega_k^2} \right) - 1 \right],$$

and the remaining $\beta_0$-dependent part which turns out to dominate in the above limit

$$I_2 = \int_0^\infty k^{d-1}dk \frac{\omega_{0k}}{4} \left( \frac{1}{\omega_{0k}^2} + \frac{1}{\omega_k^2} \right) \left[ \coth \frac{\beta_0 \omega_{0k}}{2} - 1 \right].$$

Table 1 summarizes the asymptotic behaviour of the above integrals in the hot deep quench limit. The comparison shows that in $1d$ and $2d$, $\tilde{\beta} = 2\beta_0$ as happens for $\beta_{eff}(k)$ for small momenta. In $3d$ however, $\tilde{\beta} = \beta_0$, i.e. the average temperature tends to the initial temperature instead of its half. As a conclusion we observe that, even though the effective temperature for small momenta is always half of the initial one, the averaging of momenta involved in the calculation of the field fluctuations and the fact that in $3d$ the dominant role of small momenta is reduced, result in the average effective temperature remaining the same as the initial one. Note also that in $2d$ there are logarithmic corrections to the asymptotic behaviour which could render comparison with data difficult.

Interestingly enough, in the above limit $\tilde{\beta} = \beta_{eff}(k = 0)$ in $1d$ and $2d$ (and, as we will see, in any dimension for a lattice model for a sufficiently deep quench). This suggests that $\beta_{eff}$ could have a sounding physical meaning even beyond the small-momentum features.

**Two-point correlation in real space.** – It is worth to give a quick look at the equal-time correlation in real space. The reason is twofold. Firstly we can explicitly see how the zero-momentum effective temperature $\beta_{eff}(k = 0)$ describes its large distance asymptotic. Second, we can understand how the pure quench horizon effect [1,2,4] is smoothed by finite temperature.

From (1) the real space propagator for $t_1 = t_2 = t$ is

$$\frac{d^d k}{(2\pi)^d} e^{ik \cdot r} \frac{\omega_k^2 + \omega_{0k}^2 + (m^2 - m_0^2) \cos 2\omega_k t}{4\omega_k^2 \omega_{0k}} \coth \frac{\beta_0 \omega_{0k}}{2},$$

where, compared to the pure quench [2], there is only the additional coth factor. Let us firstly consider the massless (conformal) evolution with $m = 0$ and $d = 1$. For large $r, t$ and $t > r/2$ a saddle point argument gives that the correlation function is linear with a slope given by $\beta_{eff}(0)$ as $C(r, t) \sim \beta_{eff}(0)^{-1}(t - r/2)$. Figure 2 shows how this limit is reached for several quenches. Notice that in this regime the full correlation function is described only by $\beta_{eff}(0)$, that is a single number encoding all features of the quench. The massive evolution gives a less trivial saddle point that superimposes slowly decaying large oscillations that are analogous to those found in the pure quench [2] and will not be discussed further.

From fig. 2 it is also evident how the finite temperature smooths the horizon, which is not sharp as in the pure case. The interpretation of this fact is straightforward: quasi-particles emitted at $t = 0$ from a distance smaller than the thermal correlation length $\propto \beta_0$ are entangled and generate correlations between two points at distance $r$ faster than particles emitted from the same point (or inside the true correlation length $m_0^{-1} \ll \beta_0$). One is then tempted to conclude that the effect of the temperature is similar to $m_0^{-1}$, but this is only partially true. In fact a finite $m_0$, not only smooths the horizon, but also produces a shift in the zero-time correlation (see the corresponding curve in the figure).

**Free bosons on the lattice.** – Let us now consider a system of free bosons on a lattice whose Hamiltonian, if expressed in terms of the field operators, takes the form

$$H = \sum_{k \in \text{BZ}} \left( \frac{1}{2} \omega^2 k^2 + \frac{1}{2} \omega_{0k}^2 \phi^2_{0k} \right),$$

(14)
with a lattice dispersion relation, typically of the form
\[ \omega_k^2 = m^2 c^4 + 2 c^2 \sum_j (1 - \cos k_j a), \]  
(15)
where \( m \) is again the energy gap, \( a \) is the lattice spacing and \( j \) enumerates the \( d \) space coordinates. For convenience we set \( c = 1 \) and \( a = 1 \).

The only essential difference with the previous treatment is in the dispersion relation and the fact that the momentum \( k \) now takes values in the first Brillouin zone (BZ) \([-\pi, +\pi]^d\). This obviously affects all quantities that involve a sum over all momenta, but not the small-momentum ones. First of all we have to verify that the \((t_1 + t_2)\)-dependent part of the propagator is still decaying with time. Although there are now additional stationary points in \( \omega_k \) that determine the asymptotic behaviour of the \( k \)-integral (specifically the edges of the Brillouin zone) their contributions also decay with time as before. Therefore thermalization also occurs in the lattice. Secondly the computation of the average effective temperature is easier since the natural cutoff introduced by the lattice spacing simplifies the asymptotic analysis of the momentum integrals. More explicitly, in the deep quench limit, we can assume that \( m_0 \) is much larger than all \( k \) in the Brillouin zone. Then the left-hand side of (13) can be written in any dimension as \( (m_0/4) \coth(\beta_0 m_0/2) \sum_k (1/\omega_k^2) \) and the right-hand side becomes in the same limit \( \beta^{-1} \sum_k (1/\omega_k^2) \). Thus the average effective temperature is
\[ \beta^{-1} = \frac{1}{4} m_0 \coth(\beta_0 m_0/2), \]
(16)
which is the same as (12), showing that in the lattice the two effective temperatures are equal in any dimension.

**Other quenches.** Up to now we have considered only quenches of the mass parameter of a relativistic dispersion relation. Other possibilities can also be studied in exactly the same way using (1) as the starting point, checking the thermalization condition and estimating the effective temperature in various limits. For example one could quench the speed of sound \( c \) or assume a classical dispersion relation \( \omega_k = \Delta + k^2/2 m^* \) and quench independently the energy gap \( \Delta \) or the effective mass \( m^* \). However not all of these possibilities are physically meaningful in the context of a continuous field theory where the small-scale degrees of freedom have been completely ignored as presumably not playing a crucial role. This means that the quench should not alter the large-\( k \) behaviour of the spectrum. This explains the emergence of ultraviolet divergences in the field fluctuations, in any dimension, when one considers quenches of the speed of sound \( c \). Similar divergences occur at a quench of the effective mass \( m^* \) in the classical dispersion relation above, except for 1d and \( \Delta \neq 0 \), where we find that the system thermalizes. This also happens for all dimensions in the case of a quench of the energy gap \( \Delta \) only.

The conclusion is that as long as the quench does not alter the large-\( k \) behaviour of the spectrum in such a way as to cause divergences, thermalization occurs for a broad class of gapped and gapless dispersion relations. However the two simple types of dispersion relations we considered so far are not sufficiently adjustable to allow for any quench that would not affect the large-\( k \) behaviour, other than a quench of the energy gap. Conversely, in lattice models, because of the presence of a natural momentum cutoff preventing divergences, more elaborate quenches are possible and well-defined. If for example we perform a (pure) quench of the speed of sound of the lattice dispersion relation \( \omega_k^2 = \Delta^2 + 2 c^2 \sum_j (1 - \cos k_j) \), with \( \Delta \neq 0 \) kept fixed, then the propagator thermalizes as before, but the momentum distribution of quasiparticles as given by the stationary part of the propagator exhibits a maximum at a nonzero \( k \) and it vanishes at \( k = 0 \). This means that after such a quench the initial pure state feeds higher excitation levels rather than the lowest ones which remain empty. The case \( \Delta = 0, d = 1 \) is again special: the propagator does not tend to a constant value, i.e. it does not thermalize in the usual sense, but rather increases for large times as \( \ln(t/a) \). However, as mentioned earlier, in the gapless regime the correlation functions of vertex operators are physically more meaningful\(^2\).

More complex quenches are possible in free lattice models with non-trivial spectra. In any case the procedure for the computation of the time evolution is the same and general expressions can be readily obtained. Let us consider a system of free bosons described by a general Hamiltonian of the form
\[ H = \sum_k A_k a_k^\dagger a_k + \frac{1}{2} B_k (a_{-k}^\dagger a_k^\dagger + a_{-k} a_k), \]
(17)
where \( A_k, B_k \) are even functions of \( k \) so that \( H \) is Hermitian. Suppose that the initial temperature is \( \beta_0 \) when we quench the parameters of the model from \( A_0, B_0 \) to \( A, B \) and that we wish to find the time evolution of the momentum distribution \( n_k(t) = \langle a_k^\dagger a_k \rangle \). To this end we need to diagonalise each of the two Hamiltonians by means of a Bogoliubov transformation. Then to calculate \( n_k(t) = \text{Tr} \{ e^{-\beta_0 H_0} e^{i H t} a_k^\dagger a_k e^{-i H t} \} / \text{Z(\beta_0)} \), we first have to express \( a_k^\dagger a_k \) in the basis in which \( H \) is diagonal so that the action of the time evolution operator \( e^{i H t} \) can be worked out easily. Next we have to rewrite the operators in the basis in which \( H_0 \) is diagonal so that the expectation value on the initial thermal state can also be worked out, using the fact that the momentum distribution in this basis is equal to the Bose-Einstein distribution \( 1 / (\exp(\beta_0 B_k) - 1) \). We finally obtain
\[ n_k(t) = \frac{A_k}{E_k} \coth\left( \frac{\beta_0 E_k}{2} \right) - \frac{B_k^2 \sin^2 E_k t + 1}{2}. \]
(18)
\(^2\)In fact, in the Luttinger model [9], the logarithmic behaviour of the boson propagator leads to a power law time decay of the fermionic two-point correlation function.
As usual, the momentum distribution does not equilibrate itself, but since any measurement is done on a finite region of real space, the oscillating part of $n_k(t)$ is unobservable for sufficiently large times due to the momentum averaging that comes into play. This argument however relies on the functional form of the excitation spectrum $E_k$ and especially its stationary points. If this is such that the oscillating part can be neglected, the observable distribution is the stationary part

$$
\bar{n}_k = \frac{A_{0k}A^2_k}{2E_{0k}E_k^2} \coth \frac{\beta_0E_{0k}}{2} - \frac{1}{2}.
$$

(19)

If we had a system of fermions instead of bosons then we could use the same form (17) for the Hamiltonian but with $B_k$ being an odd function of $k$ and imaginary. For clarity let us redefine it by the substitution $B_k \rightarrow iB_k$ so that it is real. The energy spectrum would then be $E_k^2 = A_k^2 + B_k^2$ and following the same procedure we would find

$$
n_k(t) = \frac{A_{0k}}{E_{0k}} \tanh \frac{\beta_0E_{0k}}{2} \left( \frac{B_k^2}{E_k^2} \sin^2 E_k t - \frac{1}{2} \right) + \frac{1}{2},
$$

(20)

and

$$
\bar{n}_k = \frac{1}{2} - \frac{A_{0k}^2}{2E_{0k}E_k^2} \tanh \frac{\beta_0E_{0k}}{2}.
$$

(21)

The above expressions can be used to determine the momentum distribution after any quench of a free lattice model. Note that these results are in agreement with earlier studies (for example [8,10]).

Conclusions. – We have explored various cases of quenches in free models that lead to stationary behaviour for large times and investigated its effective thermal properties, also suggesting a way to estimate an average effective temperature from the field fluctuations. We showed that if the initial temperature of the system is nonzero and higher than other parameters then the quench can lower it significantly. We expect that several of our findings are not only valid for systems admitting a free particle representation, but are general aspects of thermal quantum quenches.

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