A Deontic Logic Analysis of Autonomous Systems’ Safety

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ABSTRACT

We consider the pressing question of how to model, verify, and ensure that autonomous systems meet certain obligations (like the obligation to respect traffic laws), and refrain from impermissible behavior (like recklessly changing lanes). Temporal logics are heavily used in autonomous system design; however, as we illustrate here, temporal (alethic) logics alone are inappropriate for reasoning about obligations of autonomous systems. This paper proposes the use of Dominance Act Utilitarianism (DAU), a deontic logic of agency, to encode and reason about obligations of autonomous systems. We use DAU to analyze Intel’s Responsibility-Sensitive Safety (RSS) proposal as a real-world case study. We demonstrate that DAU can express well-posed RSS rules, formally derive undesirable consequences of these rules, illustrate how DAU could help design systems that have specific obligations, and how to model-check DAU obligations.

CCS CONCEPTS
- Computer systems organization → Robotic autonomy;
- Computing methodologies → Modeling methodologies; Model verification and validation; Knowledge representation and reasoning.

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1 OBLIGATIONS, PERMISSIONS AND NORMS FOR AUTONOMOUS VEHICLES

There is now a realistic prospect that Autonomous ground Vehicles (AVs) will be deployed on public roads in the next few years, with Waymo already charging customers for self-driving taxi in Arizona [10]. While companies produce ‘event reports’ to regulators, there is a worrying sparsity of rigorous verification methods, and of external independent assessment, of the vehicles’ performance. The most pressing issue is that of verifying safety. So far, the vast majority of the work in formal verification of AVs used the tools of alethic temporal logic (like Linear [17] or Metric Temporal Logic [13]) to express behavioral specifications of system models. Alethic logic is the logic of necessity and possibility: for example, if \( p \) is a predicate, \( \Box p \) says that \( p \) is true in every accessible world - that is, \( p \) is necessary. Possibility is then formalized as \( \Diamond p := \neg \Box \neg p \): saying that \( p \) is possible is the same as saying that it is not the case that \( \neg p \) is necessary. And so on. The best known instantiation of this in Verification is LTL [15], in which an accessible world is a moment in the (linear) future. Thus \( \Box p \) formalizes ’\( p \) is true in every future moment’, and \( \Diamond p \) formalizes ’\( p \) is true in some future moment’.

It is, however, equally important to think in terms of obligations and permissions of the autonomous system: for instance, we may wish to say that ‘It is obligatory for the AV to not rear-end a car’, or ‘It is permissible to drive on the shoulder if the car ahead brakes suddenly’. Obligations, permissions and prohibitions are also pervasive when discussing ethical questions: what should the AV do when faced with two equally unsavory but inevitable alternatives? Obligations and permissions are collectively called norms and statements about them are called normative statements. A prominent example of a proposed normative system for Autonomous Vehicles (AVs) is Intel’s Responsibility-Sensitive Safety (RSS) [21], which states what the AV should and should not do to avoid accidents. It is essential to logically formalize proposed norms for autonomous systems to enable automatic reasoning about their logical consistency, consequences, and automate system design. While all current work in AV verification and testing uses temporal logics [24], which are types of alethic logic, it has been understood for over 70 years that the logic of norms is different from that of necessity [16]: applying alethic logic rules to normative statements leads to conclusions that are intuitively paradoxical or undesirable. Consider the following statements:

A. The car will eventually change lanes: this is a statement about possibility. It says nothing about whether the car plays an active role in the lane change (e.g., perhaps it will hit a slippery road patch).

B. The car sees to it that it changes lanes: this is a statement about agency. It tells us that the car is an active agent in the lane change, or is choosing to change lanes.

C. The car can change lanes: this is a statement about ability. The car might be able to do something, but have no ‘choice’ or agency in the matter.

D. The car ought to change lanes: this is a statement about obligation, a concept not captured in the first three statements.

These are qualitatively different statements and there is no a priori equivalence between any two of them. The logic we adopt should reflect this: its operators and inference rules should model these aspects. Alethic logics like LTL cannot do so.

We now give a simple but fundamental example, drawn from [16], illustrating this point. (In Section 2 we give an AV-specific example.)
One might be tempted to formalize obligation using the necessity operator $\Box$ that is, formalize ‘The AV should stay in its lane’ by $\Box$stay-in-lane. However, in alethic logic, $\Box p \implies p$: if $p$ is necessarily true then it is true. If we interpret $\Box$ as obligation this reads as $\text{Obligatory } p \implies p$: this is clearly non-sensical because agents sometimes violate their obligations so some obligatory things are not true. This leads us to a major question in studying obligations: the automatic derivation of what an agent should do when some primary obligations are violated. I.e. we wish to study statements of the form $\text{Obligatory } p \land \neg p \implies \ldots$. This is simply impossible in pure alethic logic, since $\Box p \land \neg p \implies q$ is trivially true for any $p$ and $q$. Thus alethic logics (including common temporal logics like LTL, MTL or CTL [6]) are not appropriate, on their own, for automatic reasoning about norms.

Deontic logic [7] has been developed specifically to reason about normative statements, starting with von Wright [23]. It is widely used in contract law, including software contracts. There are many flavors of deontic logic [11]. In this paper, we adopt Dominance Act Utilitarianism (DAU) developed by Horty [12] because it explicitly models all four aspects above: necessity, agency, ability and obligation. It includes a temporal logic as a component so we can describe temporal behaviors essential to system design, and it uses branching time, essential for modeling uncontrollable environments.

To assess whether DAU is appropriate for reasoning about the norms of autonomous systems, we formalize a subset of Intel’s Responsibility-Sensitive Safety, or RSS, in DAU. RSS proposes a set of norms or rules that, if followed by all cars in traffic, would lead to zero accidents [21]. The RSS proposal is expressed in the language of norms or rules that, if followed by all cars in traffic, would lead to zero accidents [21]. The RSS proposal is expressed in the language of norms or rules that, if followed by all cars in traffic, would lead to zero accidents [21]. The RSS proposal is expressed in the language of norms or rules that, if followed by all cars in traffic, would lead to zero accidents [21]. The RSS proposal is expressed in the language of norms or rules that, if followed by all cars in traffic, would lead to zero accidents [21]. The RSS proposal is expressed in the language of norms or rules that, if followed by all cars in traffic, would lead to zero accidents [21].

The development of DAU in [12] uses a restricted temporal logic, but that is immaterial here.

2. DOMINANCE ACT UTILITARIANISM

2.1 A deontic logic over branching time

This section summarizes the main aspects of DAU developed in [12], starting with classical branching time models. Let $Tree$ be a set of moments with an irreflexive, transitive ordering relation $<, <$ such that for any three moments $m_1, m_2, m_3$ in $Tree$, if $m_1 < m_3$ and $m_2 < m_3$ then either $m_1 < m_3$ or $m_2 < m_3$. There is a unique root moment of the tree satisfying $root < m$ for all $m' = root$. A history is a maximal linearly ordered set of moments from $Tree$. Intuitively, it is a branch of the tree that extends infinitely. Given a moment $m \in Tree$, the set of histories that go through $m$ is $H_m := \{h \mid m \in h\}$. See Fig. 1. We will frequently refer to moment/history pairs $m/h$, where $m \in Tree$ and $h \in H_m$.

Definition 1. [12, Def. 2.2] With $AP$ a set of atomic propositions, a branching time model is a tuple $\mathcal{M} = (Tree, <, v)$ where $Tree$ is a tree of moments with ordering $<$ and $v$ is a function that maps $m/h$ pairs in $\mathcal{M}$ to sets of atomic propositions from $2^{AP}$.

A branching time model can be seen as the result of executing a non-deterministic automaton that models all agents in the system. While we will frequently speak of one agent’s obligations for simplicity, the reader should keep in mind that a model $\mathcal{M}$ can represent the possible evolutions of several agents.

We will use CTL* as the tense logic on branching time models - see [6] for details.1 $\text{CTL}^*$ includes computational tree logic (CTL) and linear temporal logic (LTL), and has become widely used in model checking. $\text{CTL}^*$ can produce sentences like $\phi := \exists X(p) \land \ldots$

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1The development of DAU in [12] uses a restricted temporal logic, but that is immaterial here.
∀ ♦□(ϕ) which can be interpreted as ‘there exists a path where ϕ holds at the next state, and all paths will eventually always satisfy ϕ’. CTL* allows us to formalize the temporal evolution of events along a given history h (e.g., ♦ϕ), and quantify over histories passing through a moment m (e.g., ∀ϕ meaning ‘for all histories, φ holds’). In this paper, to retain a uniform satisfaction relation like [12], we will speak of formulas holding or not at an m/h pair: for a pair m/h in a model M, we write M, m/h ⊨ ϕ, where it is always the case that h ∈ H_m. There should be no confusion as a CTL* path formula is evaluated along h and a state formula is evaluated at m.

A formula ϕ is identified at moment m with the set of histories where it holds

$$\phi_m^A := \{h \in H_m \mid M, m/h \models \phi\}$$  (1)

Where there’s no risk of ambiguity, we drop M from the notation, writing $|\phi|_m$, etc.

The rest of this section is dedicated to the exposition of the properly deontic aspects of DAU.

**Choice.** Let Agent be a set of agents, which represent, for example, the cars in traffic. Consider an agent α ∈ Agent and a given model M. Then at every moment m, α is faced with a choice of actions which we denote by Choice^α_m. Intuitively, an action causes some histories from H_m to no longer be realizable, while others still are. Thus we can identify each action K ∈ Choice^α_m with the set of histories that are still realizable after taking the action, and we may write K ⊆ H_m. See moments and actions in Fig. 1. Choice^α_m must obey certain constraints which we relegate to Appendix A.

**Agency.** Agency is defined via the Chellas ‘sees to it’ operator cstit, named in honor of Brian Chellas who introduced an analogous operator in [5]. (Saying ‘John sees to it that the window is open’ means that John ensures the window is open.) Intuitively, an agent sees to it that A by taking action K at m/h iff, whatever other history h’ could’ve resulted from the action, A is true at m/h’ as well. Thus, the non-determinism does not prevent α from achieving A. Let Choice^α_m (h) be the unique action that contains h. In Fig. 1 Choice^α_m (h_1) = K_1 = {h_1, h_2, h_3, h_4} and h_5 does not satisfy A. Optimal^α_m = {K_2, K_3} and so α has no obligations at m’ since there is no formula ϕ s.t. $|\phi|_m' \not\subseteq K_4 \cup K_5$ (See Def. 6). Finally, m/h_5 ∪ α cstit : A because K_2 \subseteq |A|_m and H_m = |A|_m = {h_1, h_2, h_3, h_5, h_6}.

**Definition 2 (Chellas sttit).** [12, Def. 2.7] With agent α and formula ϕ

$$M, m/h = [\alpha cstit : \phi] \iff \text{Choice}^α_m (h) \subseteq |\phi|_m^A$$

See Fig. 1. We also define a deliberative sttit operator, which captures the notion that an agent can only truly be said to do something if it also has the choice of not doing it.

**Definition 3 (Deliberative sttit).** [12, Def. 2.8] With α and ϕ as before,

$$M, m/h = [\alpha \text{ distit} : \phi] \iff \text{Choice}^α_m (h) \subseteq |\phi|_m^A \text{ and } |\phi|_m^A \not\subseteq H_m$$

Thus [α distit : A] iff some histories don’t satisfy A but α’s choice ensures A. See Fig. 1. The operators cstit and distit are not interchangeable and they fulfill complementary roles. This paper focuses on obligation statements of the following form.

**Definition 4 (Obligations).** Let α be an agent. An obligation A is either a CTL* formula, or a statement of the form [α cstit : ϕ] or ¬[α distit : ϕ] where ϕ is a CTL* formula.

Like Eq. (1) for CTL* formula, we identify an obligation A at moment m with the set of histories where it holds

$$|A|^A_m := \{h \in H_m \mid M, m/h = A\}$$  (2)

Obligations can be used in sttit formulations, similarly to formulas in definitions 2 and 3:

$$M, m/h = [\alpha cstit : A] \iff \text{Choice}^α_m (h) \subseteq |A|^A_m$$
Optimal actions. To speak of an agent’s obligations, we will need to speak of ‘optimal actions’, those actions that bring about an ideal state of affairs. We make the simplifying assumption that all agents in the system collaborate to achieve a common goal. This is consistent with the RSS assumption that all agents are following the same rules to avoid collisions anywhere in traffic. Let Value : Hroot → ℝ be a value function that maps histories of M to utility values from the real line ℜ. This value represents the utility associated by all the agents to this common history.

Definition 5. A utilitarian stit frame is a tuple \((\text{Tree}, <, \text{Agent, Choice, Value})\) where Tree and < are as in branching time frames, Agent is a set of agents, Choice is a choice mapping (which is specialized as Choice\(_a^m\) for each agent and moment), and Value is a value function. A utilitarian stit model is a model based on a utilitarian stit frame. If Choice\(_a^m\) is finite for every \(a \in \text{Agent} \) and \(m\), the model is said to be finite-choice.

All models in what follows are finite-choice utilitarian stit models. Given two sets of histories \(X, Y\) we order them as

\[
X \leq Y \iff \text{Value}(h) \leq \text{Value}(h') \quad \forall h \in X, h' \in Y
\]

(3)

Let State\(_a^m\) := Choice\(_a^m\) Agent::(\(a\)) be the set of background states against which \(a\)'s decisions are to be evaluated. These are other agents’ independent actions. Given two actions \(K, K'\) in Choice\(_a^m\), \(K \leq K'\) iff \(K \cap S \leq K' \cap S\) for all \(S \in\) State\(_a^m\). That is, \(K'\) dominates \(K\) iff it is preferable to it regardless of what the other agents do (known as sure-thing reasoning). Strict inequalities are naturally defined. Optimal actions are given by [12]

\[
\text{Optimal}_a^m := \{ K \in \text{Choice}_a^m \mid \exists K' \in \text{Choice}_a^m. K \prec K' \}
\]

(4)

Optimal\(_a^m\) is non-empty in finite-choice utilitarian stit models [12, Thm. 4.10].

Dominance Ought. Intuitively we will want to say that at moment \(m\), agent \(a\) ought to see it to that \(A\) iff \(A\) is a necessary condition of all the histories considered ideal at moment \(m\). This is formalized in the following dominance Ought operator, which is pronounced “\(a\) ought to see it to that \(A\) holds”.

Definition 6 (Dominance ought). With \(a\) an agent and \(A\) an obligation in a model \(M\),

\[
M, m/h = \ominus [\text{a cstit : A}] \iff K \subseteq |\text{Choice}_a^m| \text{ for all } K \in \text{Optimal}_a^m
\]

See Fig. 1 for examples. If \(K \subseteq |\text{Choice}_a^m|\) we say that \(K\) guarantees \(A\). Note that the dominance Ought is only defined with the cstit operator and not dstit; this is because it leads to a simpler logic. The dominance ought satisfies a number of pleasing logical properties; we refer the reader to [12, Ch. 4].

Conditional obligation. It is often necessary to say that an obligation is imposed only under certain conditions. Where \(A\) and \(B\) are obligations, the statement

\[
M, m/h = \ominus ([\text{a cstit : A}]/B)
\]

(5)

expresses that \(a\) ought to see to it that \(A\) under the condition that \(B\) holds.

Definition 7 (Conditional ought). With \(a\) an agent and \(A, B\) as obligations in a model \(M\),

\[
M, m/h = \ominus ([\text{a cstit : A}]/B) \iff K \subseteq |\text{Choice}_a^m| \text{ for all } K \in \text{Optimal}_a^m
\]

where Optimal\(_a^m\) is the set of actions available to \(a\) that are optimal if we ignore \(B\)-violating histories [12]. We note that conditional obligation is not the same as \(B \implies \ominus [\text{a cstit : A}]\).

Conditional obligation only considers \(B\)-guaranteeing dominating histories, while this latter formula still considers all optimal actions, not only those that guarantee the truth of \(B\).

Syntax. We now summarize the syntax of DAU statements. Obligations are generated as follows.

\[
A := \phi \mid [\text{dstit : A}] \mid \neg A
\]

where \(\phi \in\) CTL*, and the semantics of \([\text{dstit : A}]\) were given in Def. 3. Ought statements are in one of two forms:

\[
\ominus [\text{a cstit : A}] \text{ or } \ominus [\text{a cstit : A}]/B
\]

where \(a\) is an agent and \(A\) and \(B\) are obligations. The semantics were given in Def. 6.

2.2 Alethic Logic vs DAU for Analyzing AV Behavior

We now offer an AV-specific example of the advantage that a DAU formalization offers over pure temporal logic. Specifically, DAU allows deriving obligations over time by construction and in a uniform manner; attempts to do so using pure temporal logic are unsatisfactory. Consider the stit model in Fig. 2, which models the situation on the left: agent \(a\) could either stay in its lane behind the slower \(b\) (\(K_1\)), or pass \(b\) by going into the opposite lane \(K_2\) and risk a head-on collision. Every history in \(K_1\) is deemed preferable to every history in \(K_2\) because \(K_1\) eliminates the risk of collision, so we assign history values accordingly, as shown. If the agent does \(K_2\), then it needs to get back into its lane. Thus at \(m'\), every history in \(K_2\) is preferable to every history in \(K_1\), and this is reflected in the values. Naturally, the histories in \(K_1\) at \(m\) satisfy \(\psi := (\Diamond \neg \psi)\) (\(a\) does not pass, i.e., does not change lanes), those in \(K_2\) at \(m'\) satisfy \(\psi := \Diamond [\Diamond t] \psi, t = 0, 1, 2\), which says that \(a\) changes lanes in at most \(t\) time steps \(\Diamond [\Diamond t] \psi\). Moreover, suppose \(K_1\) histories at \(m\) satisfy some arbitrary formula \(\chi\). The following obligations are then automatically derived from the stit model:

\[
\begin{align*}
\text{At } m, \quad & \ominus [\text{a cstit : } \psi \wedge \chi] \\
\text{At } m', \quad & \ominus [\text{a cstit : } \varphi_2] \quad (\text{since } \varphi_2 \text{ is true if } \varphi_0 \text{ or } \varphi_1 \text{ are})
\end{align*}
\]

(6)

(7)

Thus it emerges that at \(m\), \(a\) ought to not change lanes. Also at \(m\), \(a\) ought to see to it that \(\chi\) - which may have nothing to do with how the values were assigned to the histories. E.g., \(\chi\) might constrain

5This is not a well-formed DAU expression, but we can extend the logic to give this expression its natural definition as \(\neg B \vee [a \text{ cstit : A}]\).

6In DAU, \(\ominus [\text{a cstit : } \psi] \wedge \ominus [\text{a cstit : } \varphi]\) is equivalent to \(\ominus [\text{a cstit : } \psi \wedge \varphi]\)
Within the next two states). However, at α's obligations then. Another method may be to specify behavior through reactive implications, e.g. "oncoming-traffic ⇒ change lanes", but this sort of explicit rule must be built in by a human designer. The conclusion is that there is a need to use a logic that captures preferences and derives obligations from them, as well as what agents are able and unable to do; a logic of agency and obligation.

3 FORMALIZING RSS IN DAU
Responsibility-Sensitive Safety, or RSS, is a proposal put forth by Intel’s Mobileye division [21]. It proposes rules or requirements that, if followed by all cars in traffic, would lead to zero accidents.

Our objective here is to formalize some of the RSS rules in the language of Dominance Act Utilitarianism (DAU), and study their logical consequences. Three important points must be made:

(A) The formalization does not depend on the dynamical equations that govern the cars because we wish our conclusions to be independent of these lower-level concerns. This is consistent with the standard AV control architecture where a logical planner decides what to do next (‘change lanes’ or ‘turn right’) and a lower-level motion planner executes these decisions. Our logical analysis concerns the logical planner.

(B) We are not trying to formalize general traffic laws or driving scenarios, which is outside the scope of this paper. We are only formalizing the RSS rules.

(C) Every formalization, in any logic, can always be refined. We are not aiming for the most detailed formalization; we aim for a useful formalization.

We have three objectives in doing so: demonstrating the usefulness of DAU in a real use case; highlighting the ambiguities implicit in such proposals, which would go unnoticed without formalization; and automating the checking of logical consistency and deriving of conclusions. We first present the RSS rules in natural language (Section 3.1), then their formalization (Section 3.2), and finally we analyze the rules’ logical consequences.

3.1 The RSS rules
The rules for Responsibility-Sensitive Safety are [21]:

RSS1. Do not hit someone from behind.
RSS2. Do not cut-in (to a neighboring lane) recklessly.
RSS3. Right-of-way is given, not taken.
RSS4. Be careful of areas with limited visibility.
RSS5. If you can avoid an accident without causing another one, you must do it.
RSS6. To change lanes, you should not wait forever for a perfect gap: i.e., you should not wait for a gap large enough to get into even when the other car, already in the lane, maintains its current motion.

RSS6 is derived directly from the following in [21, Section 3]: "the interpretation of the duty-of-care law should lead to [...] an agile driving policy rather than an overly-defensive driving which inevitably would confuse other human drivers and will block traffic [...] . As an example of a valid, but not useful, interpretation is to assume that in order to be "careful" our actions should not affect other road users. Meaning, if we want to change lane we should find a gap large enough such that if other road users continue their own motion uninterrupted we could still squeeze-in without a collision. Clearly, for most societies this interpretation is over-cautious and will lead the AV to block traffic and be non-useful.”

Note that, consistently with points (A)-(C) above, this is stated without any reference to dynamics or specific scenarios. The RSS authors are concerned that overlay cautious driving might lead to unnatural...
traffic, so RSS aims to allow cars to move a bit assertively, and defines correct reactions to that. We will not study RSS4 and 5 as they are currently too vague for formalization.

### 3.2 Formalization of RSS Rules

**Formalizing RSS1.** Let \( \phi \) be a formula denoting 'Hit someone from behind'. A plausible formalization of RSS1 is then

\[
RSS1. \quad \Box[\alpha \text{ cstit}: \neg \phi]
\]

That is, \( \alpha \) ought to see to it that it does not hit anyone from behind. However, suppose that \( \alpha \) finds itself, through no fault of its own, in a situation where a collision is unavoidable at time \( m \), that is, \( \mathcal{H}_m = \{\phi\}^M_\alpha \). Then we can show that RSS1 cannot be met. This is something we know at design time. There isn’t much value in specifying obligations that remain in force even when they become impossible to meet, since we can’t design controllers for them. A better formalization of RSS1 would automatically, as a matter of logic, remove the obligation when a collision becomes unavoidable. This can be done using \( \text{dstit} \) of Def. 3 as follows:

\[
RSS1r. \quad \Box[\alpha \text{ cstit}: \neg[\alpha \text{ dstit}: \phi]]
\]

This says that \( \alpha \) should see to it that it does not deliberately ensure an accident \( \phi \). This form of obligation is called\( \text{refrains from hitting anyone from behind.} \) RSS1 and RSS1r are not logically equivalent. If \( \{\phi\}^M_\alpha = \mathcal{H}_m \), then \( \Box[\alpha \text{ dstit}: \phi] \) is necessarily false, and RSS1r is trivially satisfied since \( \Box[\alpha \text{ cstit}: \top] \) is a theorem of DAU. Thus RSSr does not impose unrealistic obligations on the agent. Of course, a test engineer should then examine why the inevitable situation arose in the first place - but that is a separate debugging effort. The control engineer can now focus on designing a controller that meets the more realistic RSSr.

**Formalizing RSS2.** Define two CTL* formulas, \( \psi : \) a non-reckless cut-in, and \( \psi_r : \) a reckless cut-in. Then RSS2 is formalizable as

\[
RSS2. \quad \Box[\alpha \text{ cstit}: \forall \Box[\psi \lor \psi_r \implies \neg \psi_r]].
\]

That is, \( \alpha \) should see to it that always, if a cut-in happens, then it is a non-reckless cut-in.

**Formalizing RSS3.** Formalizing this rule requires some care. First, note that RSS3 should probably be amended to say that 'Right-of-way is given, not taken, and some car is given the right-of-way - otherwise, traffic comes to a standstill.' We will first focus on formalizing the prohibition (nobody should take the r-o-w), then we will formalize the positive obligation (somebody must be given it).

Let \( \text{Agent} = \{a, b, c, \ldots\} \) be a finite set of agents. Define the atomic propositions \( \text{GROW}_a : \beta \) gives right-of-way to \( a \) and \( p_a : \) \( a \) proceeds/drives through the conflict region. Then \( \text{TROW}_a := p_a \land \neg(\text{GROW}_a \land \text{GROW}_b \land \ldots) \) formalizes taking the r-o-w: \( a \) proceeds without being given the right of way by everybody. We could now express the prohibition in RSS3: every \( \alpha \) ought to see to it that it does not take the r-o-w:

\[
RSS3\text{prohib0.} \quad \bigwedge_{a \in \text{Agent}} \Box[\alpha \text{ cstit}: \neg \text{TROW}_a] \quad (8)
\]

The difficulty with this formulation is that it could lead to \( \alpha \) being obliged to force everybody else to give it the r-o-w - something over which, a priori, it has no control. To see this, we need the following, whose proof is omitted due to lack of space.

**Theorem 1.** Given obligations \( A \) and \( B \), \( \Box[\alpha \text{ cstit}: A \lor B] \land (\forall \neg \phi) \rightleftharpoons \Box[\alpha \text{ cstit}: B] \)

In other words, if \( \alpha \) has an obligation to fulfill \( A \) or \( B \) at \( m \), but every available history violates \( A \lor (\neg \phi,A) \), then its obligation is effectively to fulfill \( B \). Applied to Eq. (8) with \( A = \neg p_a \) and \( B = \neg \neg p_a \text{GROW}_a \), Thm. 1 says that if \( \alpha \) is in a situation where it has no choice but to proceed (e.g. as a result of slippage on a wet road, say), then its obligation is to see to it that everybody else gives it the right-of-way, which is unreasonable.

To remedy this, we first formalize the positive obligation: somebody must be given the right-of-way. This seems to be a group obligation: the group must give r-o-w to one of its members. Group obligations are formally defined in [12, Ch. 6]. Therefore, we define an atomic proposition \( g_a : \) r-o-w is Granted to \( \alpha \). Then we formalize

\[
RSS3\text{pos.} \quad \Box[\text{Agent cstit}: \exists \forall_a \text{Agent } g_a] \quad (9)
\]

This says the group \( \text{Agent} \) has an obligation to give r-o-w to someone, and the only choice is in \( \alpha \) who gets it. We now come back to formalizing the prohibition:

\[
RSS3\text{prohib.} \quad \bigwedge_{a \in \text{Agent}} \Box[\alpha \text{ cstit}: \neg (g_a \implies \neg p_a)] \quad (10)
\]

Finally, we formalize RSS3 as the conjunction RSS3prohib \( \land RSS3\text{pos} \).

**Formalizing RSS6.** This rule says that if the car wants to change lanes, it shouldn’t wait for the perfect gap (otherwise, traffic is stalled). First, let’s formalize ‘waiting for the perfect gap’, that is, waiting until the other car, already in the lane, gives the AV the right-of-way (e.g., by slowing down). Let the atomic proposition \( w_a \) mean ‘\( a \) wants to change lanes’ and recall that \( p_a \) means ‘\( a \) proceeds through the conflict region’ while \( g_a \) means ‘\( a \) is Granted the right-of-way’. For conciseness, let’s introduce the bounded Release operator \( \text{R}_N \), which informally says that over the next \( N \) steps, either \( \psi \) does not hold at all, or it does and \( \phi \) holds continuously until \( \psi \) holds.

\[
\psi \text{R}_N \phi = \psi \lor (\phi \land X\psi) \lor (\phi \land X\phi \land X^2\psi) \lor \ldots \\
\ldots \lor (\phi \land X\phi \land \ldots \land X^{N-1} \phi \land X^N\psi) \lor (\phi \land X\phi \land \ldots \land X^N \phi)
\]

Then \( \neg p_a \text{R}_N g_a \) says that \( \alpha \) waits for the perfect gap up to \( N \) steps (but we don’t know what happens after this). \( [\alpha \text{ dstit}: \neg p_a \text{R}_N g_a] \) formalizes the agent deliberately seeing to it that it waits to be given the right-of-way, when it doesn’t have to. Finally,

\[
RSS6. \quad \Box[\alpha \text{ cstit}: \neg [\alpha \text{ dstit}: \neg p_a \text{R}_N g_a]]/w_a \quad (11)
\]

formalizes that \( \alpha \) ought to refrain from seeing to it that it waits for the right-of-way given that it wants to change lanes. This obligation does not delay the lane change - in particular, it does not require the car to wait for the perfect gap. It also does not rush \( a \): it can wait if it wishes to. We emphasize that RSS assertive driving requires that an AV sometimes force its way, as expressed in (11).
In DAU, obligations are automatically derived from the stit model via Def. 2. Given an obligation that we want the system to have, how should we structure the stit model so that it has that obligation? This is similar to synthesis-from specifications, an active research area in programming and in Cyber-Physical Systems. This section gives an example where it is possible to manually partially infer the stit model structure from the RSS obligations.

Consider again the RSS3prohib and RSS6 statements (Eqs. (10) and (11)).

**Proposition 1.** A stit model has both obligations RSS3prohib and RSS6 at m if for every optimal action $K \in \text{Optimal}^m$, it holds that $|K| \geq 2$, and there exist a history $\bar{h} \in K$ and a moment $m^\prime > m$ in $\bar{h}$ such that $\text{Choice}_{\alpha}(m^\prime) \geq 2$, $2 \bar{h} \in \text{CTL}^\alpha \{\neg \alpha \Rightarrow \neg \alpha\}$ and $\bar{h}$ is not in any optimal action at $m^\prime$.

The proof is omitted due to lack of space. The conclusion of the Proposition, illustrated in Fig. 3, is counter-intuitive: it necessitates the existence of a history $\bar{h}$ along which one of the formulas, $\Box (\neg \alpha \Rightarrow \neg \alpha)$, is violated. But since the inferred structure places $\bar{h}$ in a non-optimal action (via $\text{Value}$), this doesn’t lead to an obligation violation.

### 3.4 Application: undesirable consequence of RSS star-calculations

One of the main tenets of RSS is that an AV is only responsible for avoiding potential accidents between itself and other cars (so-called ‘star calculations’); interactions between 2 other cars are not its concern [21, Remarks 1 and 8]. Yet everyday driving experience makes clear that our actions can be faulted for at least facilitating an accident; e.g., by repeated braking, I may cause the car behind me to do the same, leading the car behind it to rear-end it. Or I might make a sudden lane change over two lanes, causing the car in the lane next to me to over-react when I speed past it, and collide with someone else. We now show how this intuition is automatically captured by the DAU logic, and that RSS star-calculations lead to undesirable behavior of the AV.

Let $\phi \in \text{CTL}^\alpha$ denote a formula expressing “Accident between two other cars”, and assume the accident is such that $a$ can facilitate it as in the above 2 examples. Then $[\alpha \text{ dstit } : \phi]a$ says that $a$ (deliberately) sees to it that the accident happens even though it could avoid doing so; given what we assumed about this accident, this means $a$ facilitates the accident. Then $[\alpha \text{ dstit } : \neg [\alpha \text{ dstit } : \phi]]$ expresses that $a$ sees to it that it does not facilitate the accident; this is a form of refraining. Finally, $[\alpha \text{ dstit } : \neg [\alpha \text{ dstit } : \neg [\alpha \text{ dstit } : \phi]]]$ says that $a$ refrains from refraining, that is, $a$ does not refrain from facilitating the accident (even though it could). The RSS position is that it is OK for $a$ to refrain from refraining [21, Remarks 1 and 8], as formalized here.

However, refraining from refraining is the same as doing. Formally [12, 2.3.3.]

$$[\alpha \text{ dstit } : \neg [\alpha \text{ dstit } : \neg [\alpha \text{ dstit } : \phi]]] \equiv [\alpha \text{ dstit } : \phi]$$

And we argue that this matches our intuition: to not refrain from facilitating an accident even though one could be the same as facilitating it. In other words, under this formalization, the RSS position is tantamount to allowing AVs to facilitate accidents between others - clearly, an undesirable conclusion. This aspect of RSS, therefore, needs refinement to take into account longer-range interactions between traffic participants.

### 4 SYSTEM DESIGN AND MODEL CHECKING DAU OBLIGATIONS

The system designer’s job is to design a system that has the right obligations; it is then the control engineer’s job to design a controller that makes the system meet these obligations. In DAU, obligations are automatically derived from stit models/trees, but designers usually model an agent as an automaton or a similar structure. The question then naturally poses itself: given an agent model, how do we verify whether it has a given obligation? Answering this question is a crucial design step: there is no point designing a controller that meet the wrong obligations. This can be cast as a model-checking question, which this section tackles. All proofs are in the appendices.

#### 4.1 Modeling an agent

**Definition 8 (STT Automaton).** Let $AP$ be a finite set of atomic propositions. A stit automaton $T$ is a tuple $T = (Q, q_0, K, F, \Delta, L, w, \lambda)$, where $Q$ is a finite set of states, $q_0$ is the initial state, $K$ is a finite set of actions ($K \subset 2^{H_{root}}$), $F \subset Q$ is a set of final states, $\Delta \subset Q \times K \times Q$ is a finite transition relation such that if $(q, K', q')$ and $(q, K', q')$ are in $\Delta$ then $K = K'$, $L : Q \to 2^{AP}$ is a labeling function, $w : \Delta \to R$ is a weight function, and $\lambda : R^+ \to R$ is an accumulation function.

Denote by $\Delta(q) \subset \Delta$ the set of outgoing transitions from $q$ ($\Delta(q) = \{ (q, K', q') | \in \Delta \}$), by $Post(q, K)$, $\{ q' \in (q, K', q') \in \Delta(q) \}$ the successors of $q$ under $K$, and by $Post(q) = \cup K(q, K', q') \in \Delta(q)$ the set of all successors of $q$. Finally, denote by $T[q_0]$ the initial state of $T$ when there’s a need to clarify the automaton. Note that $T$ is a type of non-deterministic weighted automaton. Its
The cstit model-checking problem is: Given a stit automaton $T$ that models an agent $a$ and an obligation $A$, determine whether $\mathcal{M}_T, \text{root}\mid h \models [\text{a cstit: } A]$ for some $h \in H_{\text{root}}$. The case of conditional oaths $\otimes([\text{a cstit: } A]/B)$ is similarly handled and we omit the details.

4.2 Model checking algorithm

The cstit model-checking problem is: Given a stit automaton $T$ that models an agent $a$ and an obligation $A$, determine whether $\mathcal{M}_T, \text{root}\mid h \models [\text{a cstit: } A]$ for some $h \in H_{\text{root}}$. The case of conditional oaths $\otimes([\text{a cstit: } A]/B)$ is similarly handled and we omit the details.

Figure 4: Left: a stit model generated by executing the stit automaton $T$ (transition weights not shown). Center and right: Automata $T_n$ and $T_n'$ used in Algorithm 1. $T_1$ only has $K_1$ as first action, and $T_1'$ is obtained by renaming states of $T_1$ and 'addingâZN a copy of $T_1$ to it. Executions of $T_1'$ are simply the execution of $T$ that start with $K_1$.

unweighted counterpart $T^u$ is a classical transition system; thus for a CTL* formula $\phi$, we could model-check whether $T^u \models \phi$. A set of agents is modeled by the product of all individual stit automata, which is itself a stit automaton. (When taking the product, we must define how weights are combined and how to construct the product’s accumulation function, which are application-specific considerations.) Therefore the rest of this section applies to stit automata, whether they model one or multiple agents. We will continue to refer to one agent $a$ for simplicity.

From automata to stit models. Let $S^a$ denote the set of infinite sequences $(a_i)_{i \in \mathbb{N}}$ with $a_i \in S$. An execution of a stit automaton $T$ is a sequence $\pi \in \Delta^\omega$ of transitions of the form $\pi = (q_0, K_0, q_1)(q_1, K_1, q_2) \ldots$. The corresponding sequence of actions $K_0, K_1, \ldots \in K^\omega$ is called a strategy. Because of non-determinism, a strategy can produce multiple executions. An execution of the automaton generates a stit model in the natural way: starting in state $q_0$ the automaton takes an infinite sequence of actions from $K$, thus non-deterministically traversing an infinite number of transitions $e$ from $A$. These sequences of transitions form the histories in the corresponding stit model, with every transition $e$ adding a moment to the histories. The value(s) of those histories are obtained by accumulating $w(e)$ along the traversed transitions using function $\lambda$. See Fig. 4 for an example. The formal construction and proof are in Appendix B.

**Theorem 2.** The structure $\mathcal{M}_T$ obtained by executing a stit automaton $T$ is a utilitarian stit model with finite Choice$_a^m$ for every agent $a$ and moment $m$.

Given the structure of an obligation given in Def. 4, the model-checking problem can be broken down into two parts: what is the set of optimal actions at root, Optimal$_{\text{root}}^a$? And out of these optimal actions, which ones guarantee the truth of $A$? (Recall Eqs. (3)-(4): action optimality is determined solely by the Value function, and not by which obligations its histories satisfy.) If all optimal actions guarantee $A$, then by Def. 2, $\mathcal{M}_T$ has obligation $A$ at root/h. The algorithm is presented in Algorithm 1 page 9. In it, $\models_{\text{CTL}^*}$ denotes the classical CTL* satisfaction relation.

**Theorem 3.** Algorithm 1 returns True iff $\mathcal{M}_T, \text{root}\mid h \models [\text{a cstit: } A]$. It has complexity $O(2m(|T| + c_1 + |T| \cdot 2^{\omega_A}))$, where $c_1$ is the cost of computing the minimum and maximum values of a strategy executed on automaton $T$ and $|T|$ is the number of states and transitions in $T$.

The proof is in Appendix C. This algorithm can be amended to accept a conditional obligation $\otimes([\text{a cstit: } A]/B)$ by accepting only those actions $K$ in Optimal$_{\text{root}}^a$ that guarantee $A$ and $B$. The computation of the minimum and maximum values of a strategy’s execution line 8 clearly depends on the function $\lambda$ used for accumulating weights along the execution: e.g., if $\lambda$ is addition and all the weights are positive, then all executions have infinite value, and every future is ideal, which is a comforting thought but of little interest in modeling the real world. This question is related to but distinct from temporal logic accumulation [2] and quantitative languages [4]. We give now one example of a $\lambda$ that can model real-world phenomena, and lead to finite values of $u_h$. Take $\lambda = \min$. For instance, if $w(q, K, q')$ is the time-to-collision resulting from action $K$ then Value$(h)$ is the shortest time-to-collision encountered along the history, and an optimal strategy is one with the highest minimum time-to-collision. It’s a simple matter to prove that $u_h$ is the maximum weight of any reachable transition from $q_0$, which can be computed in a finite number of steps. (Unfortunately, different $\lambda$s will, in general, require different customized analyzes.)
Data: A stit automaton $T = (Q, q_0, K, F, \Delta, L, w, \lambda)$, an obligation $A$
Result: $M_T$ with $h = \ominus[\alpha \STIT \vdash A]$

1. Set $root = 0$
2. Set $Choice_{root} = \{K \in K \mid (q_0, K, q') \in \Delta$ for some $q'\} = \{K_1, \ldots, K_m\}$
3. // First step: find optimal actions at root
4. for $1 \leq n \leq m$
5.     /* Construct automaton $T_n$ s.t. every execution of $T_n$ is an execution of $T$ starting with $q_0 \in K_n$. See Fig. 4. */
6.     Create automaton $T_n$ by deleting all transitions $(q_0, K, q')$ with $K \neq K_n$
7.     Create a copy $T^\tau_n$ of $T_n$
8.     Create the automaton $T_n^\tau$ as a union of $T^\tau_n$ and $T$, with every transition $(q, K, q', \tau)$ in $T^\tau_n$ replaced by a transition $(q, K, q', \tau)$
9.     Compute the max value, $u_n$, and min value, $\ell_n$, of any $T_n^\tau$ strategy starting at $q_0$
10. end
11. // An interval $[\ell_n, u_n]$ is un-dominated if there is no other interval $[\ell'_n, u'_n]$, computed in the above for-loop, s.t. $\ell'_n > u_n$
12. Find all un-dominated intervals $[\ell_n, u_n]$
13. Set $Optimal_{root} = \{K_n \in Choice_{root} \mid [\ell_n, u_n]$ is un-dominated$\}$
14. /* Second step: decide whether all actions $K$ in $Optimal_{root}$ guarantee $A$, i.e., $K \subseteq \{\alpha \STIT \vdash A\}$. */
15. for $K_n \in Optimal_{root}$
16.     if $A$ is a CTL* formula then
17.         /* Does every execution of $T$ starting with $K_n$ satisfy $A$? */
18.         Use CTL* model-checking to check whether $T_n \Vdash_{CTL*} \forall A$
19.         if $T_n \Vdash_{CTL*} \forall A$ // Optimal action $K_n$ does not guarantee $A$
20.             then
21.                 return False
22.         end
23.     end
24.     else if $A = [\alpha \STIT \vdash \phi]$ with $\phi \in CTL*$ then
25.         // This is true iff $h_{root} = \phi|_{root}$
26.         Model-check whether $T \Vdash_{CTL*} \forall \phi$
27.         /* This is true iff $K_n$ guarantees $\phi$, is not equiv. to line 26 */
28.         Model-check whether $T_n \Vdash_{CTL*} \forall \phi$
29.         if $T \Vdash_{CTL*} \forall \phi$ or $T_n \Vdash_{CTL*} \forall \phi$
30.             then
31.                 return False
32.         end
33.     end
34. end
35. Return True

Algorithm 1: Model checking DAU.

5 CONCLUSIONS
We have demonstrated the use of Dominance Act Utilitarianism in formalizing safety norms for autonomous vehicles. Our objective was to assess the feasibility and utility of doing so: we expressed safety norms from RSS in DAU; found undesirable consequences in these norms; and showed that system designers can automatically derive a formalized system’s obligations and objectives.

It is desirable next to enrich the interaction between deontic and temporal modalities, e.g., to express things like ‘in the next planning cycle the AV must see to it that it changes lanes’. This then allows reasoning about obligation propagation through time [3]. It will be equally important to study obligation inheritance between groups and individuals: e.g., if it is the group’s obligation to give the right-of-way, what does that imply for individual obligations? Given that deontic logics were developed for ethical analysis, this work also opens the way to formally considering ethical implications of system design. In our experience even framing technical specifications as obligations can make explicit an implicit norm. Addressing ethical considerations is necessary to build trust in autonomous systems, and this work suggests it may be possible to formalize a system’s ethical constraints, and analyze the moral implications of its design. These and other considerations will ultimately determine the suitability of DAU for AV design and verification.

A MORE ELEMENTS OF DOMINANCE ACT UTILITARIANISM
Agent choice. The choice mapping $Choice_{root}^m$ in a general deontic stit model obeys

- The actions in $Choice_{root}^m$ partition the set $H_m = K \cap K' = \emptyset$ for every $K, K'$ and $\forall K \in Choice_{root}^m H_m$. There is no loss of generality in this constraint, it is a formality that allows us to maintain the useful tree structure.
- Independence of agents: given any group of agents $\Gamma \subseteq Agent$, $\forall \alpha \in \Gamma Choice_{alpha}^m H_m = \emptyset$. That is, the actions of one agent do not prevent the choice of action available to any other agent at the same moment $m$.
- No choice between undivided histories: If two histories are still undivided at $m$ (that is, they share a moment $m' > m$) then they belong to the same action $K$ in $Choice_{alpha}^m$.

B CONSTRUCTION OF $M_T$ AND PROOF OF THM. 2
We give the formal construction of stit model $M_T$ from stit automaton $T$, then prove Thm. 2. The construction is as follows (see Fig. 4).

- Initialization: set iteration $i = 1$, $q = q_0$, $root = 0$, $S = \{q_0, root\}$, $Tree = \{\text{root}\}$.
- Expansion: Set $S' = \emptyset$. For every couple $(q, m) \in S$,
Exp1) set $Choice_{alpha}^m = \{K : (q, K, q') \in \Delta(q)$ for some $q'\}$: the agent has a choice of actions at $m$ from the actions that label the transitions out of $q$.
Exp2) For every $K \in Choice_{alpha}^m$ and every $q' \in Post(q, K)$, add a new moment $m_K(q', i)$ to $Tree$ with $m_K(q', i) > m$, and such that the history ending with the moments $(m, m_K(q', i))$ belongs to action $K$. Also, add the couple $(q', m_K(q', i))$ to $S'$.
Exp3) Set the label map \( v(m/h) = L(q) \) for every history \( h \) passing through \( m \).

- **Update:** Set \( S = S' \). For the next iteration, set \( i = i + 1 \). Goto **Expansion**.
- **Valuation:** For every history \( h \) constructed in the Expansion loop, its value is computed as \( Value(h) = \lambda (v(e_j)) \) where \( e_j \)'s are the transitions taken while constructing \( h \). (\( \lambda \) must be such that infinite accumulation yields a finite value).

Theorem 2. We first verify that \( M_T \) is a branching time model (Def. 1). The ordering between moments is irreflexive and transitive by construction.

Take 3 moments \( m_1, m_2 \) and \( m_3 \) s.t. \( m_1 < m_2 \) and \( m_2 < m_3 \). Moments are only added in Exp2 so \( m_3 = m_K(q', i) \) for some \( q', i \), and by construction there is a unique moment \( m_K'(q, i - 1) \) at level \( i - 1 \) s.t. \( m_K'(q, i - 1) > m_K(q, i - 1) \). By a simple inductive argument, there is a unique moment \( m_K'(q, i - j) \) at level \( j \) s.t. \( m_K'(q, i - j) > m_K(q, i - j) \) for every \( j > i \). Thus the sequence of moments that are smaller than \( m(q, i) \) forms a chain (a linear order) to which must belong both \( m_2 \) and \( m_3 \), so either \( m_2 < m_3 \) or \( m_3 < m_2 \).

The tree is rooted at 0 as can be easily established by induction on \( i \).

The function \( v \) in Exp3 plays the role of the stit model’s label map.

We now show that \( Choice_m^m \) satisfies the constraints of Appendix A on choices:

- The actions in \( Choice_m^m \) partition \( H_m \): indeed, take a history starting at \( m = m_K(q, i - 1) \). It is expanded in Exp2 only, by \( m_K'(q', i) \) say, and the expanded history \( (m, m_K'(q', i)) \) is assigned to only one action. Thus the histories \( (m, m_K'(q', i)), q' \in \text{Post}(q, K'), K' \in Choice_m^m \) are partitioned among the actions at \( m \). By definition of the automaton transition relation, two different actions must lead to two different states \( q', q'' \) so the newly created moments \( m_K'(q', i + 1) \) and \( m_K''(q'', i + 1) \) at the next iteration \( i + 1 \), and which expand these histories, are different. Therefore, two histories that were in different actions at \( m \) will never share a moment after \( m \). Thus the actions at \( m \) partition \( H_m \).

- Independence of agents: this is automatically guaranteed by using an automaton that models the product of all stit automata.

- No choice between undivided histories: as established in the first bullet of the proof, histories that in different actions at \( m \) will never share a moment after \( m \). Therefore, two histories that share a moment at \( m' > m \) must be in the same action at \( m \).

Finally, \( Choice_m^m \) is finite for each moment since, as can be seen in Exp1, \( Choice_m^m \) is (isomorphic to) a subset of \( \Delta \) and the latter is finite.

QED □

C PROOF OF THM. 3

Recall that by executing a stit automaton, a stit model is created (Appendix B).

**Lemma 1.** The histories generated by \( T_n^m \) are exactly the histories of \( T \) whose first action is \( K_n \), modulo a re-naming of the states.

**Proof.** Recall that \( T_n^m \) has two components, namely a copy \( T_n^m \) of \( T_n \) and a copy of \( T \). See Fig. 4. \( T_n \) is obtained by removing transitions from \( T \), thus every history generated by \( T_n \) is a valid \( T \)-history.

Every history generated by \( T_n \) starts with \( K_n \) by construction. So every history \( h \) of \( T_n^m \) starts with \( K_n \), because it starts in \( T_n^m \).

Case 1: \( h \) never leaves \( T_n^m \). Then \( T_n^m \) is nothing but a renaming of \( T_n \) and we’ve already established that a history of \( T_n \) is a history of \( T \), so this case is done.

Case 2: \( h \) leaves \( T_n^m \). That is, a transition takes the execution into the \( T \)-copy. Up to the transition, \( h \) is a history of \( T \) as established in Case 1. The transition itself, say \( (q, K, T, q_0) \), is a valid transition of \( T \) (modulo re-naming) since it was created by replacing a \( T \) transition of the form \( (T, q, K, T, q_0) \). Once in the \( T \)-copy, the history of course continues to be a valid history of \( T \). QED. □

**Lemma 2.** The set computed at line 13 is indeed \( Optima^m \).

**Proof.** Every history of \( T_n^m \) starts with \( K_n \) so \( \ell_n = \min \{ Value(h) \mid h \in K_n \} \) and \( u_n = \max \{ Value(h) \mid h \in K_n \} \). By definition of action dominance, \( K_n \subseteq K_n^* \) in \( T \) iff \( u_n \leq \ell_n \). So \( \{ \ell_n, u_n \} \) is un-dominated iff its action \( K_n \) is un-dominated and must be optimal. QED. □

**Lemma 3.** If line 21 is executed, then \( K_n \not\subseteq \{A| root \}\).

**Proof.** If \( T_n^m \not\subseteq \{A| root \} \) this means some execution \( h \) of \( T_n^m \) violates \( A \). By Lemma 1 \( h \) is also a history of \( T \) starting with the optimal action \( K_n \), so that \( K_n \not\subseteq \{A| root \} \). QED.

**Lemma 4.** If line 30 is executed, then \( M, root/ h \not\subseteq \{a cstit : \phi \}\).

**Proof.** \( T \models_{\text{CTL}^*} \forall \phi \) iff every history of \( T \) satisfies \( \phi \) and so \( H_{root} = \{ \phi | root \} \); in this case, by definition of \( dstit \), root/ \( h \not\subseteq \{a cstit : \phi \} \). Again this is also a history of \( T \) which belongs to the optimal \( K_n \) so that \( K_n \not\subseteq \{\phi | root \} \). QED. □

**Thm. 3.** We need to establish that the algorithm returns True iff \( K \models \{A| root \} \) for every optimal \( K \). The set of optimal actions is computed at line 13 by Lemma 2. The for-loop at line 15 visits each optimal action in turn. Line 37 is executed iff none of the ‘return False’ statements preceding it are executed; namely, iff \( K \models \{A| root \} \) by Lemma 3 in Case A is CTL*, or iff \( H \models \{\phi | root \} \) and \( K \models \{\phi | root \} \) in the case of line 24 by Lemma 4 (and of the case line 33 is similarly treated). These are the definition of root/ \( h \models \{a cstit : A \} \).

For the complexity, the first for-loop takes \( 2 |T| \) operations per iteration to create the automata copies and \( 2 |c_2| \) to compute \( \ell_n \) and \( u_n \). Finding the un-dominated intervals takes \( m - 1 \) comparisons to find the largest \( \ell_n \) and \( m \) to compare each \( u_n \) to max \( \ell_n \). The second for-loop does at the most two \( \text{CTL}^* \) model-checking runs per optimal action; each run has complexity \( O(|T| \cdot 2^{|\phi|}) \) and there are at most \( m \) optimal actions. The total is then \( O(2m(|T| + c_2)) + 2m - 1 + 2m(|T|2^{|\phi|}) \).

QED.

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