ENTROPY IN THE CATEGORY OF PERFECT COMPLEXES WITH COHOMOLOGY
OF FINITE LENGTH

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Abstract. Let \( R \) be a Noetherian local ring. For an endomorphism \( \varphi : R \to R \) with zero-dimensional closed fiber, the total derived inverse image functor \( \mathbb{L}\varphi^* \) induces an exact (triangulated) functor from the category of perfect complexes over \( R \) with cohomology of finite length, to itself. The triangulated entropy of this endofunctor, as defined in \([3]\), exists. We show that when \( R \) is regular, the triangulated entropy of the total derived inverse image functor \( \mathbb{L}\varphi^* \) is equal to the local entropy of \( \varphi \), as defined in \([7]\). We also compute the triangulated entropy of the total derived inverse image functor of the Frobenius endomorphism over a complete local ring of equal characteristic \( p \), and show that this entropy is equal to \( d \cdot \log(p) \), where \( d \) is the dimension of the ring.

1. Introduction

Recently in \([3]\) a notion of entropy was defined and studied for an exact (triangulated) endofunctor of a triangulated category with generator. In this note we will specialize to a specific triangulated category, namely the category of perfect complexes with cohomology of finite length over a Noetherian local ring \( R \), and will study the entropy of the total derived inverse image functor \( \mathbb{L}\varphi^* \), where \( \varphi : R \to R \) is an endomorphism with zero-dimensional closed fiber. Our goal is to compute the triangulated entropy of the functor \( \mathbb{L}\varphi^* \) in terms of the local entropy (cf. \([7]\)) of the endomorphism \( \varphi \). We find that local entropy is always less than or equal to triangulated entropy. When \( R \) is regular we are able to show that the two entropies are equal. We use this result to compute the triangulated entropy of the total derived inverse image functor of the Frobenius endomorphism over a complete local ring of equal characteristic \( p \), and show that this entropy is equal to \( d \cdot \log(p) \), where \( d \) is the dimension of the ring.

The organization of the content of this paper is as follows: In Sections 2 and 3 we recall the definitions of triangulated and local entropies. In Section 4 we review the definition of the category of perfect complexes with cohomology of finite length over a Noetherian local ring \( R \) and show that in this category every nonzero object is a generator. We also explain why the total derived inverse image functor \( \mathbb{L}\varphi^* : D(R) \to D(R) \), where \( \varphi : R \to R \) is an endomorphism with zero-dimensional closed fiber, induces an exact (triangulated) functor from the category of perfect complexes with cohomology of finite length over \( R \), to itself. The aim of Section 5 is to show that it makes sense to speak of the triangulated entropy of the functor \( \mathbb{L}\varphi^* \), viewed as an endofunctor of the category of perfect complexes with cohomology of finite length over \( R \). Finally, Section 6 contains our main results, as described above.

2. Entropy of exact (triangulated) endofunctors

Let \( D \) be a triangulated category. Recall that a subcategory of \( D \) is called thick if it is triangulated, contains every object isomorphic to any of its objects, and contains all direct summands of its objects ([11] Definition 2.1.6, p. 74]). An object \( G \) of \( D \) is called a (classical) generator if the smallest thick subcategory of \( D \) containing \( G \) is equal to \( D \) itself (cf. [2] Section 2.1]). Let \( D \) be a triangulated category with a generator \( G \). To say that \( G \) is a generator is equivalent to saying that for every object \( E \) of \( D \) there is an object \( E' \)

\[ Date: January 4, 2016. \]

\[ Key words and phrases. Entropy, Local entropy, Triangulated categories, Perfect complexes, Exact endofunctors, Derived categories. \]
and a tower of (distinguished) triangles

\[
0 = F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_{k-1} \rightarrow F_k \cong E \oplus E'
\]

with \( k \geq 0 \) and \( n_i \in \mathbb{Z} \).

**Definition 2.1** ([3, Definition 2.1]). Let \( E_1 \) and \( E_2 \) be objects in a triangulated category \( D \). The complexity of \( E_2 \) with respect to \( E_1 \) is the function \( \delta_t(E_1, E_2): \mathbb{R} \rightarrow [0, \infty] \) given by

\[
\delta_t(E_1, E_2) = \inf \left\{ \sum_{i=1}^{k} e_i n_i : \begin{array}{c}
0 \\
E_1[n_1] \\
E_1[n_2] \\
\vdots \\
E_1[n_k]
\end{array} \rightarrow \begin{array}{c}
A_1 \\
A_2 \\
\vdots \\
A_{k-1} \\
E_2 \oplus E'_2
\end{array} \right\}.
\]

Note that \( \delta_t(E_1, E_2) = +\infty \) if and only if \( E_2 \) does not lie in the thick subcategory generated by \( E_1 \). The complexity function \( \delta_t(\cdot, \cdot) \) has the following properties:

**Proposition 2.2** ([3, Proposition 2.3]). Let \( E_1, E_2 \) and \( E_3 \) be objects in a triangulated category \( D \). Then

(a) (Triangle Inequality) \( \delta_t(E_1, E_3) \leq \delta_t(E_1, E_2) \cdot \delta_t(E_2, E_3) \).

(b) (Retraction) \( \delta_t(F(E_1), F(E_2)) \leq \delta_t(E_1, E_2) \) for any exact (triangulated) functor \( F: D \rightarrow D' \).

**Definition 2.3** ([3, Definition 2.5]). Let \( F: D \rightarrow D \) be an exact (triangulated) endofunctor of a triangulated category \( D \) with a generator \( G \). The entropy of \( F \) is the function \( h(F): \mathbb{R} \rightarrow [0, \infty] \) of \( t \) given by

\[
h(F) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta_t(G, F^n(G)).
\]

It is shown in [3, Lemma 2.5] that \( h(F) \) is well-defined, i.e., the limit exists and is independent of the choice of generator \( G \).

### 3. Entropy of local endomorphisms

The notion of local entropy associated with an endomorphism of finite length of a Noetherian local ring was introduced in [7]. We recall its definition.

**Definition 3.1** ([7, Definition 1]). A local homomorphism \( f: (R, m) \rightarrow (S, n) \) of Noetherian local rings is said to be of finite length if the ideal \( f(m)S \) is \( n \)-primary, or equivalently, if the closed fiber of \( f \) is of dimension zero. These conditions are also equivalent to the statement that if \( p \) is a prime ideal of \( S \) such that \( f^{-1}(p) = m \), then \( p = n \).

**Definition 3.2** ([7, Theorem 1]). Let \( (R, m) \) be a Noetherian local ring and \( \varphi: R \rightarrow R \) an endomorphism of finite length. Let \( \text{length}_R(\cdot) \) denote the length of an \( R \)-module. The local entropy of \( \varphi \) is given by

\[
h_{\text{loc}}(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log (\text{length}_R(R/\varphi^n(m)R)).
\]

It is shown in [7, Theorem 1] that \( h_{\text{loc}}(\varphi) \) is well-defined, i.e., the limit exists. Furthermore, it is shown that local entropy is non-negative and can be calculated using any module of finite length ([7, Proposition 18]). In particular, any ideal of definition of \( R \) can be used to calculate local entropy:

**Lemma 3.3** (cf. [7, Proposition 18]). Let \( (R, m) \) be a Noetherian local ring and \( \varphi: R \rightarrow R \) an endomorphism of finite length. If \( q \) is an \( m \)-primary ideal of \( R \), then

\[
h_{\text{loc}}(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log (\text{length}_R(R/\varphi^n(q)R)).
\]
Remark 3.4. In a Cohen-Macaulay Noetherian local ring of dimension \(d\), a sequence of \(d\) elements in the maximal ideal form a system of parameters if and only if they form a (maximal) regular sequence. The reader can find a proof of this fact in [9, Theorem 17.4].

4. Perfect complexes with cohomology of finite length

In Section 4 we will work in the category of perfect complexes with finite length cohomology over a Noetherian local ring \(R\) and for an endomorphism of finite length \(\varphi : R \to R\) we will study relations between \(h_{\text{loc}}(\varphi)\) and \(h_t(\varphi^*)\), where \(\varphi^*\) is the total derived inverse image functor. Our goal in this section is to show that it makes sense to speak of \(h_t(\varphi^*)\). We have also collected a number of definitions and standard facts about the category of perfect complexes. The main references for this section are [1, 14, 15].

**Definition 4.1.** Let \(R\) be a commutative ring and let \(D(R)\) denote the derived category of the category of \(R\)-modules. A strictly perfect complex on \(R\) is a bounded complex of projective \(R\)-modules of finite type. The category of perfect complexes over \(R\), which we will denote by \(D(R)_{\text{perf}}\), is the full subcategory of \(D(R)\) consisting of all complexes that are quasi-isomorphic to a strictly perfect complex on \(R\).

The category \(D(R)_{\text{perf}}\) is a thick subcategory of \(D(R)\), by [1, Exposé I, Propositions 4.10, 4.17] or [15, Proposition 2.2.13].

**Remark 4.2.** For definitions of strictly perfect complexes and perfect complexes on schemes see [1, Exposé I, Definitions 2.1 and 4.7] or [15, Definitions 2.2.2 and 2.2.10].

Let \(f : S \to R\) be a homomorphism of commutative rings. Denote the homotopy category of complexes of \(S\)-modules (respectively \(R\)-modules) by \(\mathbf{K}(S)\) (respectively \(\mathbf{K}(R)\)). Let \(f^* : \mathbf{K}(S) \to \mathbf{K}(R)\) be the inverse image functor, that is, the functor that sends a complex of \(S\)-modules \(E^*\) to the complex of \(R\)-modules \(E^* \otimes_S R\). Let \(L_f^* : D(S) \to D(R)\) denote the total derived inverse image functor. Recall that \(L_f^*\) is an exact (triangulated) functor.

**Remark 4.3.** For a morphism \(f : Y \to X\) of schemes, the total inverse image functor \(L_f^*\) was generally only defined as a functor \(D^{-}(X) \to D^{-}(Y)\) in [14] and [15, p. 99]. Spaltenstein extended the definition of \(L_f^*\) to a functor \(D(X) \to D(Y)\) in [13, Proposition 6.7]. Over an affine scheme, however, this will not make any difference because the category of perfect complexes over an affine scheme is equivalent to the triangulated category obtained from the category of bounded chain complexes of finitely generated projective modules by inverting the quasi-isomorphisms.

**Proposition 4.4** (cf. [14, Exposé I, Corollaire 4.19.1] or [15, p. 303, 2.5.1]). Let \(f : S \to R\) be a homomorphism of commutative rings and \(L_f^* : D(S) \to D(R)\) the total derived inverse image functor. Then \(L_f^*\) induces a functor \(D(S)_{\text{perf}} \to D(R)_{\text{perf}}\).

**Proof.** For \(E^*\) a strictly perfect complex of \(S\)-modules, \(f^*E^*\) is clearly a strictly perfect complex of \(R\)-modules and this complex represents \(L_f^*E^*\). Since (over an affine scheme) any perfect complex is (globally) quasi-isomorphic to a strictly perfect complex, the result follows immediately.

**Definition 4.5** (cf. [14, Definition 3.2]). Let \(R\) be a commutative ring and \(E^*\) a complex of \(R\)-modules. The cohomological support of \(E^*\) is the subspace \(\text{Supp}(E^*) \subseteq \text{Spec } R\) of those prime ideals \(p \in \text{Spec } R\) at which the complex \(E^*_p\) of \(R_p\)-modules is not acyclic.

Thus \(\text{Supp}(E^*) = \bigcup_{n \in \mathbb{Z}} \text{Supp } H^n(E^*)\) is the union of the supports in the classic sense of the cohomology modules of \(E^*\).

Suppose now that \((R, m)\) is a Noetherian local ring and denote by \(D_m(R)_{\text{perf}}\) the full subcategory of \(D(R)_{\text{perf}}\) consisting of perfect complexes \(E^*\) with \(\text{Supp } E^* \subseteq \{m\}\). Clearly a perfect complex \(E^*\) is in \(D_m(R)_{\text{perf}}\) if and only if \(H^n(E^*)\) is an \(R\)-module of finite length for every \(n \in \mathbb{Z}\). The proof of the next proposition is straightforward.

**Proposition 4.6** (cf. [14, Example 3.9.1]). Let \((R, m)\) be a Noetherian local ring. Then \(D_m(R)_{\text{perf}}\) is a thick subcategory of \(D(R)_{\text{perf}}\).
Proposition 4.7. Let \( f: (S, \mathfrak{n}) \to (R, \mathfrak{m}) \) be a homomorphism of finite length of Noetherian local rings. Then the total derived inverse image functor \( \mathbb{L}f^*: D(S)_{\text{perf}} \to D(R)_{\text{perf}} \) (cf. Proposition 4.7) induces an exact (triangulated) functor \( D_\mathfrak{n}(S)_{\text{perf}} \to D_\mathfrak{m}(R)_{\text{perf}} \).

Proof. This statement quickly follows from the following more general fact proved in [14] Lemma 3.3: “Let \( X \) be a quasi-compact and quasi-separated scheme (e.g., an affine scheme). Let \( E^* \) be a perfect complex on \( X \). If \( Y \) is a quasi-compact and quasi-separated scheme and \( f: Y \to X \) is a morphism of schemes, then \( \text{Supph}(\mathbb{L}f^*E^*) = f^{-1}(\text{Supph}(E^*)) \).” Note that denoting by \( ^a f \) the morphism \( \text{Spec} R \to \text{Spec} S \) corresponding to the given homomorphism \( f: S \to R \), we have \((^a f)^{-1}(\mathfrak{n}) = \mathfrak{m}\), as \( f \) is assumed to be of finite length.

Finally, we show that the category \( D_\mathfrak{m}(R)_{\text{perf}} \) has generators:

Lemma 4.8. Let \( (R, \mathfrak{m}) \) be a Noetherian local ring. Then every nonzero object in \( D_\mathfrak{m}(R)_{\text{perf}} \) is a generator, in the sense defined in Section [2].

Proof. This statement quickly follows from Proposition 4.6 and the following more general fact that was first proved in [5] Proof of Theorem 11. Also see [10] Lemma 1.2 or [14] Lemma 3.14: “Let \( R \) be a commutative Noetherian ring and let \( E^*, F^* \in D(R)_{\text{perf}} \) be two perfect complexes on \( R \). Suppose that \( \text{Supph}(E^*) \subseteq \text{Supph}(F^*) \). Then \( E^* \) is in the smallest thick subcategory of \( D(R)_{\text{perf}} \) containing \( F^* \)” \( \square \)

In this section we have shown that the category \( D_\mathfrak{m}(R)_{\text{perf}} \) of perfect complexes with cohomology of finite length on a Noetherian local ring \( (R, \mathfrak{m}) \) is a triangulated category in which every nonzero object is a generator. Moreover, given an endomorphism of finite length \( \varphi: R \to R \), the exact (triangulated) functor \( \mathbb{L}\varphi^*: D(R) \to D(R) \) induces an exact (triangulated) functor \( \mathbb{L}\varphi^*: D_\mathfrak{m}(R)_{\text{perf}} \to D_\mathfrak{m}(R)_{\text{perf}} \). Therefore, it makes sense to speak of \( h_\mathfrak{m}(\mathbb{L}\varphi^*) \) as defined in Definition 2.3.

5. Connections between triangulated entropy and local entropy

Let \( D_\mathfrak{m}(R)_{\text{perf}} \) be the category of perfect complexes with cohomology of finite length over a Noetherian local ring \( (R, \mathfrak{m}) \).

Lemma 5.1. Let \( \varphi: R \to R \) be an endomorphism of finite length of a Noetherian local ring \( (R, \mathfrak{m}) \) and let \( \mathbb{L}\varphi^*: D_\mathfrak{m}(R)_{\text{perf}} \to D_\mathfrak{m}(R)_{\text{perf}} \) be the total derived inverse image functor. Let \( G \in D_\mathfrak{m}(R)_{\text{perf}} \) be a generator. Assume that \( H^i(G) = 0 \) for \( |i| > N \) and let \( B := \max\{\text{length}_R(H^k(G)) \mid -N \leq k \leq N\} \). Then for any integer \( n \geq 1 \) and any real number \( t \):

\[
\text{length}_R(H^0(\mathbb{L}\varphi^n\ast(G))) \leq Be^{N|t|} \cdot \delta_t(G, \mathbb{L}\varphi^n\ast(G)).
\]

Proof. Fix an integer \( n \geq 1 \) and consider a tower of distinguished triangles in \( D_\mathfrak{m}(R)_{\text{perf}} \) of the form:

\[
0 = E_0 \to E_1 \to E_2 \to \cdots \to E_{k-1} \to E_k \cong \mathbb{L}\varphi^n\ast(G) \oplus E' \to 0,
\]

where \( E_i = G[n_i] \) and \( G[n_0] \to G[n_1] \to \cdots \). Let \( S := \{i \mid -N \leq n_i \leq N\} \). Since \( H^0(\cdots) \) is a cohomological functor ([3] Definition 1.5.2, p. 39), it quickly follows that for any distinguished triangle \( X \to Y \to Z \to X[1] \in D_\mathfrak{m}(R)_{\text{perf}} \):

\[
\text{length}_R(H^k(\mathbb{L}\varphi^n\ast(G))) \leq \text{length}_R(H^k(Y)) + \text{length}_R(H^k(Z)).
\]

Using this inequality one can immediately check that in the above tower of distinguished triangles:

\[
\text{length}_R(H^k(\mathbb{L}\varphi^n\ast(G))) \leq \text{length}_R(H^k(G[n_k])) + \cdots + \text{length}_R(H^k(G[n_2])) + \text{length}_R(H^k(G[n_1])).
\]

We will use this inequality, below.
First assume that $t \leq 0$. Then
\[
\sum_{i=1}^{k} e_{n,t}^{i} \geq \sum_{i \in S} e_{n,t}^{i} \geq e^{Nt} |S|
\]
\[
\geq \frac{e^{Nt}}{B} \sum_{i \in S} \text{length}_R(H^{n_i}(G))
\]
\[
= \frac{e^{Nt}}{B} \sum_{i \in S} \text{length}_R(H^0(G[n_i]))
\]
\[
\geq \frac{e^{Nt}}{B} \cdot \text{length}_R(H^0(\L \varphi^*(G))).
\]

Hence, when $t \leq 0$ we see that $\text{length}(H^0(\L \varphi^*(G))) \leq Be^{-Nt} \cdot \delta_t(G, \L \varphi^*(G))$.

Next, assume that $t > 0$. Then
\[
\sum_{i=1}^{k} e_{n,t}^{i} \geq \sum_{i \in S} e_{n,t}^{i} \geq e^{-Nt} |S|
\]
\[
\geq \frac{e^{-Nt}}{B} \sum_{i \in S} \text{length}_R(H^{n_i}(G))
\]
\[
= \frac{e^{-Nt}}{B} \sum_{i \in S} \text{length}_R(H^0(G[n_i]))
\]
\[
\geq \frac{e^{-Nt}}{B} \cdot \text{length}_R(H^0(\L \varphi^*(G))).
\]

Hence, when $t > 0$ we see again that $\text{length}(H^0(\L \varphi^*(G))) \leq Be^{-Nt} \cdot \delta_t(G, \L \varphi^*(G))$. \hfill \qed

**Corollary 5.2.** Let $\varphi: R \to R$ be an endomorphism of finite length of a Noetherian local ring $(R,m)$ and let $\L \varphi^*: D_{m}(R)_{\text{perf}} \to D_{m}(R)_{\text{perf}}$ be the total derived inverse image functor. Then $h_{\text{loc}}(\varphi) \leq h_t(\L \varphi^*)$ for any real number $t$.

**Proof.** Let $\{x_1, \ldots, x_d\}$ be a system of parameters of $R$ and let $q$ be the ideal of $R$ that they generate. Let $G^*(\underline{x})$ be the Koszul complex over $R$ constructed from $x_1, \ldots, x_d$. Then $G^*(\underline{x})$ is a generator for the triangulated category $D_m(R)_{\text{perf}}$. Since $H^0(G^*(\underline{x})) = R/q$ and tensor product is a right-exact functor, it quickly follows that $H^0(\L \varphi^*(G^*(\underline{x}))) = R/\varphi^*(q)R$. The desired inequality $h_{\text{loc}}(\varphi) \leq h_t(\L \varphi^*)$ follows by taking the logarithm, dividing by $n$, and passing to the limit as $n \to \infty$ on both sides of (5.1), and using Lemma 3.3. \hfill \qed

**Lemma 5.3.** Let $(R,m)$ be a regular local ring of dimension $d$. Suppose $\varphi: R \to R$ is an endomorphism of finite length, and let $\L \varphi^*: D_{m}(R)_{\text{perf}} \to D_{m}(R)_{\text{perf}}$ be the total derived inverse image functor. Let $\{x_1, \ldots, x_d\}$ be a regular system of parameters of $R$ and let $G^*(\underline{x})$ be the Koszul complex over $R$ constructed from $x_1, \ldots, x_d$. The following inequality holds for any integer $n \geq 1$ and any real number $t$:
\[
\delta_t(G^*(\underline{x}), \L \varphi^*(G^*(\underline{x}))) \leq \text{length}_R(R/\varphi^n(m)R).
\]

**Proof.** The idea behind the proof is that since $R$ is regular, every finitely generated $R$-module has finite projective dimension and is therefore an object of the category $D(R)_{\text{perf}}$. Hence, one can work with modules instead of complexes. To be more precise, consider a composition series of $R/\varphi^n(m)R$ over $R$:
\[
0 = N_\ell \subseteq N_{\ell-1} \subseteq \ldots \subseteq N_1 \subseteq N_0 = R/\varphi^n(m)R,
\]
with successive quotients $N_{i-1}/N_i$ isomorphic to the residue field $k = R/m$, for $1 \leq i \leq \ell$. Now, beginning from the left end of the composition series (5.2) we will inductively build a tower of triangles as follows: the
Koszul complex $G^*(x)$ provides a bounded free resolution of $k \cong N_{t-1}$. There exists (cf. [12, Theorem 16, p. 82]) a bounded free resolution $L^*_{t-2}$ of $N_{t-2}$ that completes the diagram

$$
\begin{array}{c}
G^*(x) \\
\downarrow \\
0 \rightarrow N_{t-1} \rightarrow N_{t-2} \rightarrow k \rightarrow 0
\end{array}
$$

into a diagram with exact rows

$$
0 \rightarrow G^*(x) \rightarrow L^*_{t-2} \rightarrow G^*(x) \rightarrow 0
$$

(5.3)

We use the distinguished triangle $G^*(x) \rightarrow L^*_{t-2} \rightarrow G^*(x) \rightarrow G^*(x)[1]$ in $D_m(R)_{perf}$ associated to the first row in (5.3) (cf. [6, Proposition 1.7.5, p. 46]) to build the beginning of our tower of triangles, as follows:

$$
\begin{array}{c}
G^*(x) \\
\downarrow \\
0 \rightarrow G^*(x) \rightarrow L^*_{t-2} \rightarrow G^*(x)
\end{array}
$$

Next, there exists (cf. [12, Theorem 16, p. 82]) a bounded free resolution $L^*_{t-3}$ of $N_{t-3}$ completing the diagram

$$
\begin{array}{c}
L^*_{t-2} \\
\downarrow \\
0 \rightarrow N_{t-2} \rightarrow N_{t-3} \rightarrow k \rightarrow 0
\end{array}
$$

into a diagram with exact rows

$$
0 \rightarrow L^*_{t-2} \rightarrow L^*_{t-3} \rightarrow G^*(x) \rightarrow 0
$$

(5.4)

We use the distinguished triangle $L^*_{t-2} \rightarrow L^*_{t-3} \rightarrow G^*(x) \rightarrow L^*_{t-2}[1]$ in $D_m(R)_{perf}$ associated to the first row in (5.4) to build the next triangle in our tower of triangles, as follows:

$$
\begin{array}{c}
L^*_{t-3} \\
\downarrow \\
0 \rightarrow G^*(x) \rightarrow L^*_{t-2} \rightarrow L^*_{t-3}
\end{array}
$$
Continuing this process, we obtain a tower of triangles in $D_m(R)_{\text{perf}}$:

\[
0 \rightarrow G^\bullet(x) \rightarrow L^\bullet_{L-2} \rightarrow \cdots \rightarrow L^\bullet_1 \rightarrow L^\bullet_0 \cong R/\varphi^n(m)R,
\]

in which the quasi-isomorphism $L^\bullet_0 \cong R/\varphi^n(m)R$ exists because $L^\bullet_0$ provides a bounded free resolution $L^\bullet_0 \rightarrow N_0 \cong R/\varphi^n(m)R \rightarrow 0$. We conclude:

\[
\delta_t (G^\bullet(x), L \varphi^{n^*} (G^\bullet(x))) \leq \sum_{1}^{t} e^{\theta t} = \ell = \text{length}_R (R/\varphi^n(m)R).
\]

\[\square\]

**Theorem 5.4.** Let $(R, m)$ be a regular local ring of dimension d. Suppose $\varphi: R \rightarrow R$ is an endomorphism of finite length, and let $L \varphi^*: D_m(R)_{\text{perf}} \rightarrow D_m(R)_{\text{perf}}$ be the total derived inverse image functor. Then $h_t(L \varphi^*)$ is constant and equal to $h_{\text{loc}}(\varphi)$.

**Proof.** Let $\{x_1, \ldots, x_d\}$ be a regular system of parameters of $R$. Let $G^\bullet(x)$ be the Koszul complex over $R$ constructed from $x_1, \ldots, x_d$. Then $G^\bullet(x)$ is a generator for the triangulated category $D_m(R)_{\text{perf}}$. By Corollary 5.2 we have $h_{\text{loc}}(\varphi) \leq h_t(L \varphi^*)$. On the other hand by Lemma 5.3 for any integer $n \geq 1$ and any real number $\ell$,

\[
\delta_t (G^\bullet(x), L \varphi^{n^*} (G^\bullet(x))) \leq \text{length}_R (R/\varphi^n(m)R).
\]

Taking the logarithm, dividing by $n$, and passing to the limit as $n \rightarrow \infty$ in (5.4), we obtain $h_t(L \varphi^*) \leq h_{\text{loc}}(\varphi)$. This concludes the proof. $\square$

**Corollary 5.5.** Let $k$ be a field and $R = k[[x_1, \ldots, x_d]]$. Suppose $\xi_1, \ldots, \xi_d$ are positive integers and let $\varphi: R \rightarrow R$ be the endomorphism that maps $x_i \mapsto x_i^{\xi_i}$ for $1 \leq i \leq d$. Then $h_t(L \varphi^*)$ is constant and equal to $\sum_{i=1}^{d} \log(\xi_i)$.

**Proof.** By Theorem 5.4 the entropy $h_t(L \varphi^*)$ is constant and equal to $h_{\text{loc}}(\varphi)$. Thus, it suffices to show that $h_{\text{loc}}(\varphi) = \sum_{i=1}^{d} \log(\xi_i)$. This is done by induction on dim $R$. If dim $R = 1$ then $R = k[[x_1]]$, $\varphi^n(x_1) = x_1^{\xi_1}$ and $R/\varphi^n(x_1)$ has a composition series of length $\xi^n$

\[
0 \subseteq (X_1^{\xi_1^{-1}})/(X_1^{\xi_1}) \subseteq \cdots \subseteq (X_1^n)/(X_1^{n+1}) \subseteq (X_1)/(X_1^{\xi_1}) \subseteq R/(X_1^{\xi_1^n}).
\]

We see by Definition 3.2 that $h_{\text{loc}}(\varphi) = \log(\xi_1)$. Next, assume the result holds for the ring $k[[x_1, \ldots, x_{d-1}]]$. We want to prove it for the ring $k[[x_1, \ldots, x_d]]$. We define a homomorphism $\alpha: k[[Y]] \rightarrow k[[x_1, \ldots, x_d]]$ by mapping $Y$ to $x_d$. This homomorphism is flat (cf. [9] Theorem 23.1, p. 179). If we equip $k[[Y]]$ with the endomorphism $\psi: k[[Y]] \rightarrow k[[Y]]$ that maps $Y$ to $Y^{\xi_d}$, then $\alpha \circ \psi = \varphi \circ \alpha$. By [8] Theorem 1 and using the induction hypothesis we see that $h_{\text{loc}}(\varphi) = \sum_{i=1}^{d} \log(\xi_i)$. $\square$

**Proposition 5.6.** Let $(R, m)$ be an arbitrary Noetherian local ring with an endomorphism $\varphi: R \rightarrow R$ of finite length. Assume there exists a homomorphism of finite length $\xi: S \rightarrow R$, where $(S, n)$ is a regular local ring, and suppose there is an endomorphism $\psi: S \rightarrow S$ of finite length, such that $\xi \circ \psi = \varphi \circ \xi$. Let $L \varphi^*: D_m(R)_{\text{perf}} \rightarrow D_m(R)_{\text{perf}}$ and $L \psi^*: D_n(S)_{\text{perf}} \rightarrow D_n(S)_{\text{perf}}$ be the corresponding total derived inverse image functors. If $h_{\text{loc}}(\varphi) = h_{\text{loc}}(\psi)$, then $h_t(L \varphi^*)$ is constant and equal to $h_{\text{loc}}(\varphi)$.

**Proof.** Let $\{x_1, \ldots, x_d\}$ be a regular system of parameters of $S$, $d = \dim S$, and let $y_i = \xi(x_i)$ for $1 \leq i \leq d$. Let $G^\bullet_S(y)$ and $G^\bullet_N(y)$ be the Koszul complexes over $S$ and $R$, respectively, constructed from $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$. The complexes $G^\bullet_S(y)$ and $G^\bullet_N(y)$ are generators for the triangulated categories $D_n(S)_{\text{perf}}$ and $D_m(R)_{\text{perf}}$, respectively. Since $\xi$ is a homomorphism of finite length, by Proposition 4.7 the restriction of the total derived inverse image functor $L \xi^*: D(S) \rightarrow D(R)$ to $D_n(S)_{\text{perf}}$ provides an exact (triangulated)
functor $D_n(S)^{perf} \to D_m(R)^{perf}$. It is clear that $L\xi^* (G^*_S(x)) = G^*_R(y)$. Since $\xi \circ \psi = \varphi \circ \xi$, it follows that $L\xi^* \circ L\psi^* = L\varphi^* \circ L\xi^*$. Using Proposition 2.3 for any integer $n \geq 1$ and any real number $t$ we obtain:

$$
\delta_t(G^*_R(y), L\varphi^{n*}(G^*_R(y))) = \delta_t(L\xi^* (G^*_S(x)), L\varphi^{n*}(L\xi^* (G^*_S(x))))
$$

$$
= \delta_t(L\xi^* (G^*_S(x)), L\xi^* (L\psi^{n*}(G^*_S(x))))
$$

$$
\leq \delta_t(G^*_S(x), L\psi^{n*}(G^*_S(x))).
$$

By taking the logarithm, dividing by $n$, and passing to the limit as $n \to \infty$ we obtain $h_t(L\varphi^*) \leq h_t(L\psi^*)$. From Corollary 5.5 and Theorem 5.3 we get:

$$
h_{loc}(\varphi) \leq h_t(L\varphi^*) \leq h_t(L\psi^*) = h_{loc}(\psi).
$$

Hence, if $h_{loc}(\varphi) = h_{loc}(\psi)$, then $h_t(L\varphi^*)$ is constant and equal to $h_{loc}(\varphi)$.

**Corollary 5.7.** Let $(R, m)$ be an arbitrary complete Noetherian local ring of prime characteristic $p$ and of dimension $d$. Suppose $f_R: R \to R$ is the Frobenius endomorphism and let $Lf_R^*: D_m(R)^{perf} \to D_m(R)^{perf}$ be the total derived inverse image functor. Then $h_t(Lf_R^*)$ is constant and equal to $d \cdot \log(p)$.

**Proof.** Let $\{x_1, \ldots, x_d\}$ be a system of parameters of $R$. We recall that $R$ is a module-finite extension of the regular ring $S := k[X_1, \ldots, X_d]$ via the injective ring homomorphism $\xi: S \to R$ that maps $X_i$ onto $x_i$, for $1 \leq i \leq d$ (cf. [9, Theorem 29.4, p. 225]). Let $f_S$ be the Frobenius endomorphism of $S$. By [7, Theorem 1] the local entropy of the Frobenius endomorphism of a Noetherian local ring of prime characteristic $p$ and of dimension $d$ is equal to $d \cdot \log(p)$. That is, $h_{loc}(f_R) = h_{loc}(f_S) = d \cdot \log(p)$. Since $\xi \circ f_S = f_R \circ \xi$, the result follows from Proposition 5.6. 

**References**

[1] P. Berthelot, A. Grothendieck, L. Illusie: *Théorie des Intersections et Théorème de Riemann-Roch*, Lecture Notes in Mathematics 225, Springer-Verlag, 1971, doi:10.1007/BFb0066283

[2] A. Bondal, M. Van Den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Moscow Math. Journal, 3 (2003), 1-36.

[3] G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich: *Dynamical systems and categories*, Contemporary Mathematics, 621 (2014), 133-170, doi:10.1090/conm/621

[4] R. Hartshorne: *Residues and Duality*, Lecture Notes in Mathematics 20, Springer-Verlag, Berlin, 1966, doi:10.1007/BFb0090482

[5] M. J. Hopkins: *Global methods in homotopy theory*, in: Homotopy Theory (conference at Durham 1985), 79-96, London Mathematical Society Lecture Note Series 117, Cambridge University Press, 1987, doi:10.1017/cbo9781107325746

[6] M. Kashiwara, P. Schapira: *Sheaves on Manifolds*, Grundlehren der mathematischen Wissenschaften 292, Springer-Verlag, Berlin, 1990, doi:10.1007/978-3-662-02661-8

[7] M. Majidi-Zolbanin, N. Miasnikov, L. Szpiro: *Entropy and flatness in local algebraic dynamics*, Publicacions Matemàtiques 57 (2013), 509-544, doi:10.5565/pubmat57213_12

[8] M. Majidi-Zolbanin: *On additivity of local entropy under flat extensions*, (2014), arXiv:1404.1541v2 [math.AC]

[9] H. Matsumura: *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge, 1986, doi:10.1017/CBO9781139171762

[10] A. Neeman: *The chromatic tower for $D(R)$*, Topology 31, Pages 519-532, (1992), doi:10.1016/0040-9383(92)90047-L

[11] A. Neeman: *Triangulated categories*, Annals of Mathematics Studies 148, Princeton University Press, Princeton, 2001, doi:10.1515/9781400827122

[12] D. G. Northcott: *An introduction to homological algebra*, Cambridge University Press, London, 1960, doi:10.1017/cbo9780511556915

[13] N. Spaltenstein: *Resolutions of unbounded complexes*, Compositio Mathematica, 65, 121-154 (1988).

[14] R. W. Thomason, *The classification of triangulated subcategories*, Compositio Mathematica, 105, 1-27 (1997).

[15] R. W. Thomason, T. F. Trobaugh: *Higher algebraic K-theory of schemes and of derived categories* in: The Grothendieck Festschrift vol. III, Progress in Mathematics 88, 247-435, Birkhäuser, 1990, doi:10.1007/978-0-8176-4576-2

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