WIRED CYCLE-BREAKING DYNAMICS FOR UNIFORM SPANNING FORESTS

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We prove that every component of the wired uniform spanning forest (WUSF) is one-ended almost surely in every transient reversible random graph, removing the bounded degree hypothesis required by earlier results. We deduce that every component of the WUSF is one-ended almost surely in every supercritical Galton–Watson tree, answering a question of Benjamini, Lyons, Peres and Schramm [Ann. Probab. 29 (2001) 1–65].

Our proof introduces and exploits a family of Markov chains under which the oriented WUSF is stationary, which we call the wired cycle-breaking dynamics.

1. Introduction. The uniform spanning forests (USFs) of an infinite, locally finite, connected graph $G$ are defined as infinite-volume limits of uniformly chosen random spanning trees of large finite subgraphs of $G$. These limits can be taken with respect to two extremal boundary conditions, free and wired, giving the free uniform spanning forest (FUSF) and wired uniform spanning forest (WUSF), respectively (see Section 2 for detailed definitions). The study of uniform spanning forests was initiated by Pemantle [12], who, in addition to showing that both limits exist, proved that the wired and free forests coincide in $\mathbb{Z}^d$ for all $d$ and that they are almost surely a single tree if and only if $d \leq 4$. The question of connectivity of the WUSF was later given a complete answer by Benjamini, Lyons, Peres and Schramm (henceforth referred to as BLPS) in their seminal work [3], in which they proved that the WUSF of a graph is connected if and only if two independent random walks on the graph intersect almost surely [3], Theorem 9.2.

After connectivity, the most basic topological property of a forest is the number of ends its components have. An infinite connected graph $G$ is said to be $k$-ended if, over all finite sets of vertices $W$, the graph $G \setminus W$ formed by deleting $W$ from $G$ has a maximum of $k$ distinct infinite connected components. In particular, an infinite tree is one-ended if and only if it does not contain any simple bi-infinite paths and is two-ended if and only if it contains a unique simple bi-infinite path.

Components of the WUSF are known to be one-ended for several large classes of graphs. Again, this problem was first studied by Pemantle [12], who proved that the USF on $\mathbb{Z}^d$ has one end for $2 \leq d \leq 4$ and that every component has at most

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two ends for \( d \geq 5 \). (For \( d = 1 \), the forest is all of \( \mathbb{Z} \) and is therefore two-ended.)

A decade later, BLPS [3], Theorem 10.1, completed and extended Pemantle’s result, proving in particular that every component of the WUSF of a Cayley graph is one-ended almost surely if and only if the graph is not itself two-ended. Their proof was then adapted to random graphs by Aldous and Lyons [1], Theorem 7.2, who showed that all WUSF components are one-ended almost surely in every transient reversible random rooted graph with bounded vertex degrees. Taking a different approach, Lyons, Morris and Schramm [9] gave an isoperimetric condition for one-endedness, from which they deduced that all WUSF components are one-ended almost surely in every transient transitive graph and every non-amenable graph.

In this paper, we remove the bounded degree assumption from the result of Aldous and Lyons [1]. We state our result in the natural generality of reversible random rooted networks. Recall that a network is a locally finite, connected (multi)graph \( G = (V, E) \) together with a function \( c : E \to (0, \infty) \) assigning a positive conductance \( c(e) \) to each unoriented edge \( e \) of \( G \). For each vertex \( v \), the conductance \( c(v) \) of \( v \) is defined to be the sum of the conductances of the edges adjacent to \( v \), where self-loops are counted twice. Locally finite, connected graphs without specified conductances are considered to be networks by setting \( c \equiv 1 \). The WUSF of a network is defined in Section 2 and reversible random rooted networks are defined in Section 5.

**Theorem 1.1.** Let \( (G, \rho) \) be a transient reversible random rooted network and suppose that \( \mathbb{E}[c(\rho)^{-1}] < \infty \). Then every component of the wired uniform spanning forest of \( G \) is one-ended almost surely.

The condition that the expected inverse conductance of the root is finite is always satisfied by graphs, for which \( c(\rho) = \deg(\rho) \geq 1 \). In Example 5.1, we show that the theorem can fail in the absence of this condition.

Theorem 1.1 applies (indirectly) to supercritical Galton–Watson trees conditioned to survive, answering positively Question 15.4 of BLPS [3].

**Corollary 1.2.** Let \( T \) be a supercritical Galton–Watson tree conditioned to survive. Then every component of the wired uniform spanning forest of \( T \) is one-ended almost surely.

Previously, this was known only for supercritical Galton–Watson trees with offspring distribution either bounded, in which case the result follows as a corollary to the theorem of Aldous and Lyons [1], or supported on a subset of \([2, \infty)\), in which case the tree is non-amenable and we may apply the theorem of Lyons, Morris and Schramm [9].

Our proof introduces a new and simple method, outlined as follows. For every transient network, we define a procedure to “update an oriented forest at an edge”,

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in which the edge is added to the forest while another edge is deleted. Updating oriented forests at randomly chosen edges defines a family of Markov chains on oriented spanning forests, which we call the **wired cycle-breaking dynamics**, for which the oriented wired uniform spanning forest measure is stationary (Proposition 3.2). This stationarity allows us to prove the following theorem, from which we show Theorem 1.1 to follow by known methods.

**Theorem 1.3.** Let $G$ be any network. If the wired uniform spanning forest of $G$ contains more than one two-ended component with positive probability, then it contains a component with three or more ends with positive probability.

The case of recurrent reversible random rooted graphs remains open, even under the assumption of bounded degree. In this case, it should be that the single tree of the WUSF has the same number of ends as the graph (this prediction appears in [1]). BLPS proved this for transitive recurrent graphs [3], Theorem 10.6.

1.1. **Consequences.** The one-endedness of WUSF components has consequences of fundamental importance for the **Abelian sandpile model**. Járai and Werning [7] proved that the infinite-volume limit of the sandpile measures exists on every graph for which every component of the WUSF is one-ended almost surely. Furthermore, Járai and Redig [6] proved that, for any graph which is both transient and has one-ended WUSF components, the sandpile configuration obtained by adding a single grain of sand to the infinite-volume random sandpile can be stabilized by finitely many topplings (their proof is given for $\mathbb{Z}^d$ but extends to this setting, see [5]). Thus, a consequence of Theorem 1.1 is that these properties hold for the Abelian sandpile model on transient reversible random graphs of unbounded degree.

Theorem 1.1 also has several interesting consequences for random plane graphs, which we address in upcoming work with Angel, Nachmias and Ray. In particular, we deduce from Theorem 1.1 that every Benjamini–Schramm limit of finite planar graphs is almost surely Liouville, that is, does not admit non-constant bounded harmonic functions.

2. **The wired uniform spanning forest.** In this section, we briefly define the wired uniform spanning forest and introduce the properties that we will need. For a comprehensive treatment of uniform spanning trees and forests, as well as a detailed history of the subject, we refer the reader to Chapters 4 and 10 of [11].

**Notation and orientation.** Throughout this paper, the graphs on which the USFs and USTs are defined will be connected and locally finite unless stated otherwise. We do not distinguish notationally between oriented and unoriented trees, forests or edges. Whether or not a tree, forest or edge is oriented will be clear from context. Edges $e$ are oriented from their tail $e^-$ to their head $e^+$, and have reversal $-e$. An oriented tree or forest is a tree or forest together with an orientation of its edges.
Given an oriented tree or forest in a graph, we define the past of each vertex \( v \) to be the set of vertices \( u \) for which there is a directed path from \( u \) to \( v \) in the oriented tree or forest.

For a finite connected graph \( G \), we write \( \text{UST}_G \) for the uniform measure on the set of spanning trees (i.e., connected cycle-free subgraphs containing every vertex) of \( G \), considered for measure-theoretic purposes to be functions from \( E \) to \( \{0, 1\} \). More generally, if \( G \) is a finite network, we define \( \text{UST}_G \) to be the probability measure on spanning trees of \( G \) for which the measure of a tree \( t \) is proportional to the product of the conductances of its edges.

There are two extremal (with respect to stochastic ordering) ways to define infinite volume limits of the uniform spanning tree measures. Let \( G \) be an infinite network and let \( V_n \) be an increasing sequence of finite connected subsets of \( V \) such that \( \bigcup V_n = V \), which we call an exhaustion of \( G \). For each \( n \), let the network \( G_n \) be the subgraph of \( G \) induced by \( V_n \) together with the conductances inherited from \( G \). The weak limit of the measures \( \text{UST}_{G_n} \) is known as the free uniform spanning forest: for each finite subset \( S \subset E \),

\[
\text{FUSF}_G(S \subseteq F) := \lim_{n \to \infty} \text{UST}_{G_n}(S \subseteq T).
\]

Alternatively, at each step of the exhaustion we define a network \( G^*_n \) by identifying ("wiring") \( V \setminus V_n \) into a single vertex \( \partial_n \) and deleting all the self-loops that are created, and define the wired uniform spanning forest to be the weak limit

\[
\text{WUSF}_G(S \subseteq F) := \lim_{n \to \infty} \text{UST}_{G^*_n}(S \subseteq T).
\]

Both limits were shown (implicitly) to exist for every network and every choice of exhaustion by Pemantle [12], although the WUSF was not defined explicitly until the work of Häggström [4]. As a consequence, the limits do not depend on the choice of exhaustion. Both measures are supported on spanning forests (i.e., cycle-free subgraphs containing every vertex) of \( G \) for which every connected component is infinite. The WUSF is usually much more tractable, thanks in part to Wilson’s algorithm rooted at infinity, which both connects the WUSF to loop-erased random walk and allows us to sample the WUSF of an infinite network directly rather than by passing to an exhaustion.

Wilson’s algorithm [13] is a remarkable method of generating the UST on a finite or recurrent network by joining together loop-erased random walks. It was extended to generate the WUSF of transient networks by BLPS [3]. Let \( G \) be a network, and let \( \gamma \) be a path in \( G \) that is either finite or transient, that is, visits each vertex of \( G \) at most finitely many times. The loop-erasure \( \text{LE}(\gamma) \) is formed by erasing cycles from \( \gamma \) chronologically as they are created. Formally, \( \text{LE}(\gamma)_i = \gamma_{t_i} \) where the times \( t_i \) are defined recursively by \( t_0 = 0 \) and \( t_i = 1 + \max\{t \geq t_{i-1} : \gamma_t = \gamma_{t_{i-1}}\} \). (In the presence of multiple edges, a path is not determined by its vertex-trajectory. However, the definition of the loop-erasure extends to this setting in the obvious way. Similarly, when performing Wilson’s algorithm in the presence
of multiple edges, we consider the random walks and their loop-erasures to be random paths in the graph.) Let \( \{ v_j : j \in \mathbb{N} \} \) be an enumeration of the vertices of \( G \) and define a sequence of forests in \( G \) as follows:

1. If \( G \) is finite or recurrent, choose a root vertex \( v_0 \) and let \( F_0 \) include \( v_0 \) and no edges (in which case we call the algorithm Wilson’s algorithm rooted at \( v_0 \)). If \( G \) is transient, let \( F_0 = \emptyset \) (in which case we call the algorithm Wilson’s algorithm rooted at infinity).
2. Given \( F_i \), start an independent random walk from \( v_{i+1} \) stopped if and when it hits the set of vertices already included in \( F_i \).
3. Form the loop-erasure of this random walk path and let \( F_{i+1} \) be the union of \( F_i \) with this loop-erased path.
4. Let \( F = \bigcup F_i \).

This is Wilson’s algorithm: the resulting forest \( F \) has law \( \text{UST}_G \) in the finite case \( \cite{13} \) and \( \text{WUSF}_G \) in the infinite case \( \cite{3} \), and is independent of the choice of enumeration.

We also consider oriented spanning trees and forests. Let \( \text{OUST}_{G_n^*} \) denote the law of the uniform spanning tree of \( G_n^* \) oriented towards the boundary vertex \( \partial_n \), so that every vertex of \( G_n^* \) other than \( \partial_n \) has exactly one oriented edge emanating from it in the tree, while \( \partial_n \) does not have any oriented edges emanating from it. Wilson’s algorithm on \( G_n^* \) rooted at \( \partial_n \) may be modified to produce an oriented tree with law \( \text{OUST}_{G_n^*} \) by considering the loop-erasered paths in step (2) to be oriented chronologically. If \( G \) is transient, making the same modification to Wilson’s algorithm rooted at infinity yields a random oriented forest, known as the oriented wired uniform spanning forest \( \cite{3} \) of \( G \) and denoted \( \text{OWUSF}_G \). The proof of the correctness of Wilson’s algorithm rooted at infinity \( \cite{3} \), Theorem 5.1, also shows that, when \( G_n \) is an exhaustion of a transient network \( G \), the measures \( \text{OUST}_{G_n^*} \) converge weakly to \( \text{OWUSF}_G \).

3. Wired cycle-breaking dynamics. Let \( G \) be an infinite transient network and let \( \mathcal{F}(G) \) denote the set of oriented spanning forests \( f \) of \( G \) such that every vertex has exactly one oriented edge emanating from it in \( f \). For each \( f \in \mathcal{F}(G) \) and oriented edge \( e \) of \( G \), the update \( U(f, e) \in \mathcal{F}(G) \) of \( f \) is defined by the following procedure:

**DEFINITION 3.1 (Updating \( f \) at \( e \)).** If \( e \) or its reversal \( -e \) is already included in \( f \), or is a self-loop, let \( U(f, e) = f \). Otherwise:

- If \( e^+ \) is in the past of \( e^- \) in \( f \), so that there is a directed path \( \langle e_1, \ldots, e_k, d \rangle \) from \( e^+ \) to \( e^- \) in \( f \), let
  \[
  U(f, e) = f \cup \{-e, -e_1, \ldots, -e_k\} \setminus \{d, e_k, \ldots, e_1\}.
  \]
- Otherwise, if \( e^+ \) is not in the past of \( e^- \) in \( f \), let \( d \) be the unique oriented edge of \( f \) with \( d^- = e^- \) and let \( U(f, e) = f \cup \{e\} \setminus \{d\} \).
FIG. 1. Updating an oriented spanning forest (left, solid black) of $\mathbb{Z}^2$ (dashed black) at an oriented edge $e$ (left, blue) to obtain a new oriented spanning forest (right, solid black). Arrow heads represent orientations of edges.

See Figure 1 for examples. Note that in either case, as unoriented forests, we have simply that $U(f, e) = f \cup \{e\} \setminus \{d\}$; the change in orientation in the first case ensures that every vertex has exactly one oriented edge emanating from it in $U(f, e)$, so that $U(f, e) \in \mathcal{F}(G)$.

Let $v$ be a vertex of $G$. We define the wired cycle-breaking dynamics rooted at $v$ to be the Markov chain on $\mathcal{F}(G)$ with transition probabilities

$$p^v(f_0, f_1) = \frac{1}{c(v)} c\{e : e^- = v \text{ and } U(f_0, e) = f_1\}.$$ 

That is, we perform a step of the dynamics by choosing an oriented edge randomly from the set $\{e : e^- = v\}$ with probability proportional to its conductance, and then updating at this edge. Dynamics of this form for the UST on finite graphs are well known; see [11], Section 4.4.

To explain our choice of name for these dynamics, as well as our choice to consider oriented forests, let us give a second, equivalent, description of the update
rule. If $e$ or its reversal $-e$ is already included in $f$, or is a self-loop, let $U(f, e) = f$. Otherwise:

- If $e^+$ and $e^-$ are in the same component of $f$, then $f \cup e$ contains a (not necessarily oriented) cycle. Break this cycle by deleting the unique edge $d$ of $f$ that is both contained in this cycle and adjacent to $e^-$, letting $\tilde{U}(f, e) = f \cup \{e\} \setminus \{d\}$.
  - If $e^+$ was not in the past of $e^-$ in $f$, let $U(f, e) = \tilde{U}(f, e)$.
  - Otherwise, if $e^+$ was in the past of $e^-$ in $f$, then there exists an oriented path from $e^-$ to $d^+$ in $\tilde{U}(f, e)$. Let $U(f, e)$ be the oriented forest obtained by reversing each edge in this path.

- If $e^+$ and $e^-$ are not in the same component of $f$, we consider $e$ together with the two infinite directed paths in $f$ beginning at $e^-$ and $e^+$ to constitute a wired cycle, or “cycle through infinity”. Break this wired cycle by deleting the unique edge $d$ in $f$ such that $d^- = e^-$, letting $U(f, e) = f \cup \{e\} \setminus \{d\}$.

The benefit of taking our forests to be oriented is that it allows us to define these wired cycles unambiguously. If every component of the WUSF of $G$ is one-ended almost surely, then there is a unique infinite simple path from each of $e^-$ and $e^+$ to infinity, so that wired cycles are already defined unambiguously and the update rule may be defined without reference to an orientation.

**Proposition 3.2.** Let $G$ be an infinite transient network. Then for each vertex $v$ of $G$, $\text{OWUSF}_G$ is a stationary measure for the wired cycle-breaking dynamics rooted at $v$, that is, for $p^v(\cdot, \cdot)$.

**Proof.** Let $\langle V_n \rangle_{n \geq 1}$ be an exhaustion of $G$. We may assume that $V_n$ contains $v$ and all of its neighbours for all $n \geq 1$.

Let $\mathcal{T}(G_n^*)$ denote the set of spanning trees of $G_n^*$ oriented towards the boundary vertex $\partial_n$. For each $t \in \mathcal{T}(G_n^*)$ and oriented edge $e$ with $e^- = v$, we define the update $U(t, e)$ of $t$ at $e$ by the same procedure (Definition 3.1) as for $f \in \mathcal{F}(G)$.

**Proposition 3.3.** $U(T_n, E) \overset{d}{=} T_n$ for every $n \geq 1$.

Proposition 3.3 is a slight variation on the classical Markov Chain-Tree theorem [2, 8, 11]: Define a Markov chain on $\mathcal{T}(G_n^*)$, as we did on $\mathcal{F}(G)$, by

$$p^v(t_0, t_1) = \frac{1}{c(v)} c(\{e : e^- = v \text{ and } U(t_0, e) = t_1\}).$$

The claimed equality in distribution is equivalent to $\text{OUST}_{G_n^*}$ being a stationary measure for $p^v(\cdot, \cdot)$, and so it suffices to verify that $\text{OUST}_{G_n^*}$ satisfies the detailed balance equations for $p^v(\cdot, \cdot)$. This verification, which is both straightforward and similar to that of the classical Markov Chain-Tree theorem, is omitted.

To complete the proof, we show that $U(T_n, E)$ converges to $U(F, E)$ in distribution. It might at first seem that this convergence holds trivially, but in fact some
work is required: Updating $F$ or $T_n$ at $E$ requires knowledge of whether or not $E^+$ is in the past of $E^-$, which cannot necessarily be obtained by observing the tree or forest only within a finite set. A priori, it is therefore possible that $E^+\in T_n$ is in the past of $E^-$, due to the existence of a very long oriented path from $E^+$ to $E^-$ in $T_n$ that disappears in the limit, obstructing the claimed convergence in distribution. This behaviour will be ruled out by Lemma 3.4.

By the Skorokhod representation theorem, there exist random variables $\langle T_n \rangle_{n\geq 1}$ and $F$, defined on some common probability space, such that $T_n$ has law $\text{OUST}_{G^*_n}$ for each $n$, $F$ has law $\text{OWUSF}_{G}$, and $T_n$ converges to $F$ almost surely as $n$ tends to infinity. Let $E$ be an oriented edge chosen randomly from the set $\{e : e^- = v\}$ with probability proportional to its conductance, independently of $\langle T_n \rangle_{n\geq 1}$ and $F$. We write $\mathbb{P}$ for the probability measure under which $\langle T_n \rangle_{n\geq 1}$, $F$ and $E$ are sampled as indicated. It suffices to prove that $U(T_n, E)$ converges to $U(F, E)$ in probability with respect to $\mathbb{P}$.

Given $F$, let $R$ be the length of the longest finite simple path in $F$ connecting $v$ to one of its neighbours in $G$ that is in the same component as $v$ in $F$. Since $T_n$ converges to $F$ almost surely, there exists a random $N$ such that $T_n$ and $F$ coincide on the ball $B_R(v)$ of radius $R$ about $v$ in $G$ for all $n \geq N$.

We claim that, with probability tending to one, $F$ and $T_n$ agree about whether or not $E^+$ is in the past of $v$.

**Lemma 3.4.** Consider the events

$$\mathcal{P} = \{ E^+ \text{ is in the past of } v \text{ in } F \} \quad \text{and} \quad \mathcal{P}_n = \{ E^+ \text{ is in the past of } v \text{ in } T_n \}.$$  

The probability of the symmetric difference $\mathcal{P} \triangle \mathcal{P}_n$ converges to zero as $n \to \infty$.

**Proof.** Given $E$, the probability that $E^+$ is in the past of $v$ in $T_n$ is, by Wilson’s algorithm, the probability that $v$ is contained in the loop-erasure of a random walk from $E^+$ to $\partial_n$ in $G^*_n$. Since $G$ is transient, this probability converges to the probability that $v$ is contained in the loop-erased random walk from $E^+$ in $G$. This probability is exactly the probability that $E^+$ is in the past of $v$ in $F$, and so

$$\mathbb{P}(\mathcal{P}_n) \longrightarrow \mathbb{P}(\mathcal{P}).$$

If $\mathbb{P}(\mathcal{P}) \in \{0, 1\}$, we are done. Otherwise, on the event $\mathcal{P}$, there is by definition a finite directed path from $E^+$ to $v$ in $F$. This directed path is also contained in $T_n$ for all $n \geq N$ and so

$$\mathbb{P}(\mathcal{P}_n|\mathcal{P}) \longrightarrow 1.$$  

Combining these two above limits gives

$$\mathbb{P}(\mathcal{P}_n|\neg \mathcal{P}) = \frac{\mathbb{P}(\mathcal{P}_n) - \mathbb{P}(\mathcal{P}_n|\mathcal{P})\mathbb{P}(\mathcal{P})}{\mathbb{P}(\neg \mathcal{P})} \longrightarrow 0$$

as $n \to \infty$. 

and hence
\[ P(P \triangle P_n) = P(P) - P(P \cap P_n) + P(P_n \cap \neg P) \]
\[ = P(P) - P(P_n | P)P(P) + P(P_n | \neg P)P(\neg P) \]
\[ \xrightarrow{n \to \infty} P(P) - P(P) + 0 = 0. \]
\[ \square \]

Let \( r \geq 1 \). Observe that on the event \( \{ T_n \text{ and } F \text{ coincide on the ball of radius } \max\{R, r\} \text{ about } v \} \setminus (P \triangle P_n) \), \( U(F, E) \) and \( U(T_n, E) \) coincide on the ball of radius \( r \) about \( v \). By Lemma 3.4 and the definition of \( P \), the probability of this event converges to 1 as \( n \to \infty \), and consequently \( U(T_n, E) \) converges to \( U(F, E) \) in probability with respect to \( P \). \( \square \)

3.1. Update-tolerance. Let \( G \) be a transient network and let \( F \) be a sample of OWUSF\(_G\). An immediate consequence of Proposition 3.2 is that for each oriented edge \( e \) of \( G \), the law of \( U(F, e) \) is absolutely continuous with respect to the law of \( F \).

**COROLLARY 3.5.** Let \( G \) be a transient network and let \( e \) be an oriented edge of \( G \). Then for every event \( A \subset \mathcal{F}(G) \),
\[ \text{OWUSF}_G(F \in A) \geq \frac{c(e)}{c(e^-)} \text{OWUSF}_G(U(F, e) \in A). \]

**PROOF.** By Proposition 3.2,
\[ \text{OWUSF}_G(F \in A) = \sum_{\hat{e}^- = e^-} \frac{c(\hat{e})}{c(e^-)} \text{OWUSF}_G(U(F, \hat{e}) \in A) \]
\[ \geq \frac{c(e)}{c(e^-)} \text{OWUSF}_G(U(F, e) \in A). \]
\[ \square \]

We refer to this property as *update-tolerance* by analogy to the well-established theories of insertion- and deletion-tolerant invariant percolation processes [11], Chapters 7 and 8.

4. **Proof of Theorem 1.3.**

**PROOF.** Let \( G \) be a network such that the WUSF of \( G \) contains at least two two-ended connected components with positive probability. Since \( G \)'s WUSF is therefore disconnected with positive probability, Wilson’s algorithm implies that \( G \) is necessarily transient. The *trunk* of a two-ended tree is defined to be the unique bi-infinite simple path contained in the tree, or equivalently the set of vertices and
edges in the tree whose removal disconnects the tree into two infinite connected components.

Let $F_0$ be a sample of $\text{OWUSF}_G$. By assumption, there exists a (non-random) path $({\gamma}_i)_{i=0}^n$ in $G$ such that, with positive probability, $\gamma_0$ and $\gamma_n$ are in distinct two-ended components of $F_0$, $\gamma_n$ is in the trunk of its component, and $\gamma_i$ is not in the trunk of $\gamma_n$’s component for $i < n$. Write $\mathcal{A}_γ$ for this event.

For each $1 \leq i \leq n$, let $e_i$ be an edge with $e_i^- = \gamma_i$ and $e_i^+ = \gamma_{i-1}$, and let $F_i \in \mathcal{F}(G)$ be defined recursively by

$$F_i = U(F_{i-1}, e_i) \quad \text{for } 1 \leq i \leq n.$$  

We claim that on the event $\mathcal{A}_γ$, the component containing $\gamma_n$ in the updated forest $F_n$ has at least three ends. Applying update-tolerance (Corollary 3.5) iteratively will then imply that the probability of the $\text{WUSF}$ containing a component with three or more ends is at least

$$\text{OWUSF}_G(\mathcal{A}_γ) \prod_{i=1}^n \frac{c(e_i)}{c(\gamma_i)},$$

which is positive as claimed.

First, notice that $\gamma_i$’s component in $F_i$ has at least two ends for each $0 \leq i \leq n$. This may be seen by induction on $i$. The component of $\gamma_0$ in $F_0$ is two-ended by assumption, while for each $0 \leq i < n$:

- If $\gamma_{i+1}$ is in the same component as $\gamma_i$ in $F_i$, then the component containing $\gamma_{i+1}$ in the updated forest $F_{i+1}$ has the same number of ends and the same vertex set as the component of $\gamma_i$ in $F_i$.

- If $\gamma_{i+1}$ is in a different component to $\gamma_i$ in $F_i$, then the component containing $\gamma_{i+1}$ in $F_{i+1}$ is equal to the union of the component of $\gamma_i$ in $F_i$, the edge $e_i$, and the past of $\gamma_{i+1}$ in $F_i$. Thus, the component of $\gamma_{i+1}$ in $F_{i+1}$ has at least as many ends as the component of $\gamma_i$ in $F_i$.

This induction also shows that for every $0 \leq i \leq n$, the component of $F_i$ containing $\gamma_i$ has vertex set equal to the union of the vertices in the component of $F_0$ containing $\gamma_0$, and the pasts of the vertices $\gamma_j$ in $F_j$ for $0 \leq j < i$. By definition of the event $\mathcal{A}_γ$, the vertex $\gamma_i$ is not in the trunk of $\gamma_n$’s component in $F_0$ for any $i < n$, and so in particular $\gamma_n$ is not in the past of $\gamma_i$ in $F_{i-1}$ for any $i < n$, so that $\gamma_n$ and $\gamma_n$ are in different components of $F_{n-1}$. Furthermore, since neither endpoint of $e_i$ is contained in the trunk of $\gamma_n$’s component in $F_0$ for any $0 \leq i \leq n - 1$, the trunk of $\gamma_n$’s component in $F_0$ is still contained in $F_{n-1}$. From this, we see that $\gamma_n$’s component in $F_n$ has at least three ends as claimed. See Figure 2 for an illustration. □
When we update along a path (blue arcs) connecting a two-ended component to the trunk of another two-ended component (with each edge oriented backwards), a three-ended component is created. Edges whose removal disconnects their component into two infinite connected components are bold.

5. Reversible random networks and the proof of Theorem 1.1. A rooted network \((G, \rho)\) is a network \(G\) together with a distinguished vertex \(\rho\), the root. An isomorphism of graphs is an isomorphism of rooted networks if it preserves the conductances and the root. A random rooted network \((G, \rho)\) is a random variable taking values in the space of isomorphism classes of random rooted networks (see [1] for precise definitions, including that of the topology on this space). Similarly, we define doubly-rooted networks to be networks together with an ordered pair of distinguished vertices. Let \((G, \rho)\) be a random rooted network and let \((X_n)_{n \geq 0}\) be simple random walk on \(G\) started at \(\rho\). We say that \((G, \rho)\) is reversible if the random doubly-rooted networks \((G, \rho, X_n)\) and \((G, X_n, \rho)\) have the same distribution

\[
(G, \rho, X_n) \overset{d}{=} (G, X_n, \rho)
\]

for every \(n\), or equivalently for \(n = 1\). Be careful to note that this is not the same as the reversibility of the random walk on \(G\), which holds for any network. Reversibility is essentially equivalent to the related property of unimodularity. We refer the reader to [1] for a systematic development and overview of the beautiful theory of reversible and unimodular random rooted graphs and networks, as well as many examples.

We now deduce Theorem 1.1 from Theorem 1.3. Our proof that the WUSF cannot have a unique two-ended component is adapted closely from Theorem 10.3 of [3].

**Proof of Theorem 1.1.** Let \((G, \rho)\) be a reversible random rooted network such that \(\mathbb{E}[c(\rho)^{-1}] < \infty\). Biasing the law of \((G, \rho)\) by the inverse conductance \(c(\rho)^{-1}\) (that is, reweighting the law of \((G, \rho)\) by the Radon–Nikodym derivative \(c(\rho)^{-1}/\mathbb{E}[c(\rho)^{-1}]\)) gives an equivalent unimodular random rooted network, as can be seen by checking involution invariance of the biased measure [1], Proposition 2.2. This allows us to apply Theorem 6.2 and Proposition 7.1 of [1] to deduce that every component of the WUSF of \(G\) has at most two ends almost surely.
Theorem 1.3 then implies that the WUSF of $G$ contains at most one two-ended component almost surely.

Suppose for contradiction that the WUSF contains a single two-ended component with positive probability. Recall that the trunk of this component is defined to be the unique bi-infinite path in the component, which consists exactly of those edges and vertices whose removal disconnects the component into two infinite connected components.

Let $\langle X_n \rangle_{n \geq 0}$ be a random walk on $G$ started at $\rho$, and let $F$ be an independent random spanning forest of $G$ with law $WUSF_G$, so that (since $WUSF_G$ does not depend on the choice of exhaustion of $G$) the sequence $\langle (G, X_n, F) \rangle_{n \geq 0}$ is stationary. If the trunk of $F$ is at some distance $r$ from $\rho$, then $X_r$ is in the trunk with positive probability, and it follows by stationarity that $\rho$ is in the trunk of $F$ with positive probability. We will show for contradiction that in fact the probability that the root is in the trunk must be zero.

Recall that, for each $n$, the forest $F$ may be sampled by running Wilson’s algorithm rooted at infinity, starting with the vertices $\rho$ and $X_n$. If we sample $F$ in this way and find that both $\rho$ and $X_n$ are contained in $F$’s unique trunk, we must have had either that the random walk started from $\rho$ hit $X_n$, or that the random walk started from $X_n$ hit $\rho$. Taking a union bound,

$$P(\rho \text{ and } X_n \text{ in trunk}) \leq P(\text{random walk started at } X_n \text{ hits } \rho) + P(\text{random walk started at } \rho \text{ hits } X_n)$$

By reversibility, the two terms on the right-hand side are equal and hence

$$P(\rho \text{ and } X_n \text{ in trunk}) \leq 2P(\text{random walk started at } X_n \text{ hits } \rho).$$

The probability on the right-hand side is now exactly the probability that simple random walk started at $\rho$ returns to $\rho$ at time $n$ or greater, and by transience this converges to zero. Thus,

$$P(\rho \text{ and } X_n \text{ in trunk}) = E[1(\rho \text{ in trunk})1(X_n \text{ in trunk})] \xrightarrow{n \to \infty} 0$$

and so

$$(\star) \quad E \left[ 1(\rho \text{ in trunk}) \frac{1}{n} \sum_{i=1}^{n} 1(X_i \text{ in trunk}) \right] \xrightarrow{n \to \infty} 0.$$ 

Let $\mathcal{I}$ be the invariant $\sigma$-algebra of the stationary sequence $\langle (G, X_n, F) \rangle_{n \geq 0}$. The Ergodic theorem implies that

$$\frac{1}{n} \sum_{i=1}^{n} 1(X_i \text{ in trunk}) \xrightarrow{a.s., n \to \infty} P(\rho \text{ in trunk}|\mathcal{I}).$$

Finally, combining this with (\star) and the Dominated Convergence theorem gives

$$E[1(\rho \text{ in trunk}) \cdot P(\rho \text{ in trunk}|\mathcal{I})] = E[P(\rho \text{ in trunk}|\mathcal{I})^2] = 0.$$
It follows that $\mathbb{P}(\rho \text{ in trunk}) = 0$, contradicting our assumption that $F$ had a unique two-ended component with positive probability. □

**Proof of Corollary 1.2.** Given a probability distribution $\langle p_k; k \geq 0 \rangle$ on $\mathbb{N}$, the augmented Galton–Watson tree $T$ with offspring distribution $\langle p_k \rangle$ is defined by taking two independent Galton–Watson trees $T_1$ and $T_2$, both with offspring distribution $\langle p_k \rangle$, and then joining them by a single edge between their roots. Lyons, Pemantle and Peres [10] proved that $T$ is reversible when rooted at the root of the first tree $T_1$; See also [1], Example 1.1.

If the distribution $\langle p_k \rangle$ is supercritical (i.e., has expectation greater than 1), then the associated Galton–Watson tree is infinite with positive probability and on this event is almost surely transient [11], Chapter 16. Thus, Theorem 1.1 implies that every component of $T'$s WUSF is one-ended almost surely on the event that either $T_1$ or $T_2$ is infinite.

Recall that for every connected graph $G$ and every edge $e$ of $G$ which has a positive probability of not being included in $G$’s WUSF, the law of $G$’s WUSF conditioned not to contain $e$ is equal to $\text{WUSF}_{G \setminus \{e\}}$ [3], Proposition 4.2, where, if $G \setminus \{e\}$ is disconnected, $\text{WUSF}_{G \setminus \{e\}}$ is defined to be the union of independent samples of WUSFs of the two connected components of $G \setminus \{e\}$. Let $e$ be the edge between the roots of $T_1$ and $T_2$ that was added to form the augmented tree $T$. On the positive probability event that $T_1$ and $T_2$ are both infinite, running Wilson’s algorithm on $T$ started from the roots of $T_1$ and $T_2$ shows, by transience of $T_1$ and $T_2$, that $e$ has positive probability not to be included in $T$’s WUSF. On this event, $T$’s WUSF is distributed as the union of independent samples of $\text{WUSF}_{T_1}$ and $\text{WUSF}_{T_2}$. It follows that every component of $T_1$’s WUSF is one-ended almost surely on the event that $T_1$ is infinite. □

**Example 5.1 (\(\mathbb{E}[c(\rho)^{-1}] < \infty\) is necessary).** Let $(T, o)$ be a 3-regular tree with unit conductances rooted at an arbitrary vertex $o$. Form a network $G$ by adjoining to each vertex $v$ of $T$ an infinite path, and setting the conductance of the $n$th edge in each of these paths to be $2^{-n-1}$. Let $o_n$ be the $n$th vertex in the added path at $o$. Define a random vertex $\rho$ of $G$ which is equal to $o$ with probability 4/7 and equal to the $n$th vertex in the path at $o$ with probability $3/(7 \cdot 2^n)$ for each $n \geq 1$. The only possible isomorphism classes of $(G, \rho, X_1)$ are of the form $(G, o_n, o_{n+1})$, $(G, o_{n+1}, o_n)$, $(G, o, o_1)$, $(G, o_1, o)$, or $(G, o, o')$, where $o'$ is a neighbour of $o$ in $T$. This allows us to easily verify that $(G, \rho)$ is a reversible random rooted network:

$$\mathbb{P}((G, \rho, X_1) = (G, o_n, o_{n+1})) = \mathbb{P}((G, \rho, X_1) = (G, o_{n+1}, o_n)) = \frac{1}{7 \cdot 2^n}$$

for all $n \geq 1$ and

$$\mathbb{P}((G, \rho, X_1) = (G, o, o_1)) = \mathbb{P}((G, X_1, \rho) = (G, o, o_1)) = \frac{4}{7}.$$
When we run Wilson’s algorithm on $G$ started from a vertex of $T$, every excursion of the random walk into one of the added paths is erased almost surely. It follows that the WUSF of $G$ is simply the union of the WUSF of $T$ with each of the added paths, and hence every component has infinitely many ends almost surely.

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