Generalized squeezing operators, bipartite Wigner functions and entanglement via Wehrl’s entropy functionals

Marcelo A Marchiolli and Diógenes Galetti

Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 São Paulo, SP, Brazil
E-mail: mamarchi@ift.unesp.br and galetti@ift.unesp.br

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Abstract

We introduce a new class of unitary transformations based on the su(1, 1) Lie algebra that generalizes, for certain particular representations of its generators, well-known squeezing transformations in quantum optics. To illustrate our results, we focus on the two-mode bosonic representation and show how the parametric amplifier model can be modified in order to generate such a generalized squeezing operator. Furthermore, we obtain a general expression for the bipartite Wigner function which allows us to identify two distinct sources of entanglement, here labelled dynamical and kinematical entanglement. We also establish a quantitative estimate of entanglement for bipartite systems through some basic definitions of entropy functionals in continuous phase-space representations.

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1. Introduction

In the last few years, physics has experienced the appearance of two relatively young branches with strong appeal in both theoretical and experimental aspects. Labelled quantum information and quantum computation, these branches apparently entangled have attracted, since then, a lot of attention from researchers working in well-established areas in physics (such as, for instance, solid-state physics, nuclear physics, high-energy physics, general relativity and cosmology [1]). The interdisciplinarity provided by quantum information and quantum computation is basically focused upon fundamental physical concepts that constitute the cornerstones of quantum mechanics. Hence, concepts related to nonclassical states, superposition principle, entanglement, phase-space representations, quantum teleportation, quantum key distribution, among others, represent nowadays common words in many scientific papers covering different areas of knowledge in physics.

In particular, let us restrict our attention to quantum information theory and its description by quantum continuous variables [2], where entanglement effects can be efficiently produced in the laboratory through the adequate manipulation of continuous quadrature amplitudes of the quantized electromagnetic field. In this promising scenario, the squeezed light [3, 4] has a prominent role in the experimental implementation of continuous-variable entanglement, since the degree of imperfection in entanglement-based quantum protocols depends on ‘the amount of squeezing of the laser light involved’. From a conceptual point of view, some questions related to the entanglement measures (or separability criteria) still remain open in the specialized literature [5]. For instance, entanglement of formation [6] and concurrence [7] are now widely accepted as entanglement measures for the two-qubit case. However, it is worth noticing that different approaches to this problem exist, which are outlined by means of information-theoretic arguments [8].

The main goal of this paper is to present some contributions to certain specific topics in quantum information theory that allow us to go further in our comprehension of the entanglement process in ideal bipartite systems.
(the interaction with any dissipative environment [9] is discarded in the first moment). For this purpose, we first construct a general family of unitary transformations associated with the su(1,1) Lie algebra generators that generalizes—if one considers the one- and two-mode bosonic representations of its generators—two well-known expressions of squeezing operators [3]. In the following, we study a modified version of the parametric amplifier [10] with emphasis on obtaining the solutions of the Heisenberg equations and its respective time-evolution operator. In particular, we show the efficacy of this model in generating the generalized two-mode squeezing operator through its connection with the time-evolution operator.

The next step then consists of performing a preliminary study via the Wigner function on the qualitative aspects of entanglement for a general class of bipartite systems whose dynamics is governed by arbitrary quadratic Hamiltonians. As a by-product of this investigation, we obtain a general integral representation for the bipartite Wigner function that leads us to identify two distinct sources of entanglement (here labelled dynamical and kinematical entanglement). The last contribution refers to a direct application of the results obtained by Piattek and Leoński [11] for the intermediate correlations in continuous phase-space representations, which are based on Wehrl’s approach [12] regarding some definitions of entropy functionals and their inherent properties. In fact, these preliminary considerations on the su(1,1) Lie algebra lead us to demonstrate an important composition formula involving the product of two abstract operators, each one being defined as in equation (1) and characterized by a particular set of arbitrary c-number parameters. In fact, this section will deal with some applications of the composition formula with emphasis on unitary transformations which resemble, for certain representations of the generators $K_{\pm}$ and $K_{0}$, the squeezing transformations in quantum optics.

The Casimir operator is defined within this context through the mathematical identity

$$K^{2} := K_{0}^{2} - (1/2)\{K_{-}, K_{+}\}$$

with $[K^{2}, K_{\pm}] = [K^{2}, K_{0}] = 0$, where $\{K_{-}, K_{+}\}$ represents the anticommutation relation between the operators $K_{-}$ and $K_{+}$. Furthermore, let us also consider the abstract operator

$$T(\Omega_{\pm}, \Omega_{0}) := \epsilon^{\Omega_{K}, \Omega_{0}K_{\omega}, \Omega, K_{-}} \Omega_{+}$$

written in terms of the arbitrary c-number parameters $\Omega_{\pm}$ and $\Omega_{0}$, whose generalized normal- and antinormal-order decomposition formulae have already been established in the literature [13, 14]. These preliminary considerations on the su(1,1) Lie algebra lead us to demonstrate an important composition formula involving the product of two abstract operators, each one being defined as in equation (1) and characterized by a particular set of arbitrary c-number parameters. In fact, this section will deal with some applications of the composition formula with emphasis on unitary transformations which resemble, for certain representations of the generators $K_{\pm}$ and $K_{0}$, the squeezing transformations in quantum optics.

**Lemma 2.1** Let $T(\Omega_{\pm}, \Omega_{0})$ and $T(\Lambda_{\pm}, \Lambda_{0})$ be two abstract operators whose functional forms obey equation (1). For a given set of arbitrary c-number parameters $\{\Omega_{\pm}, \Omega_{0}, \Lambda_{\pm}, \Lambda_{0}\}$ it is always possible to verify the general composition formula

$$T(\Omega_{\pm}, \Omega_{0})T(\Lambda_{\pm}, \Lambda_{0}) = T(\Sigma_{\pm}, \Sigma_{0})$$

where $\Sigma_{\pm}$ and $\Sigma_{0}$ are solutions of the coupled set of nonlinear equations

$$A_{+} + B_{0}B_{-} \frac{A_{+} - A_{-}}{B_{0}} = \Sigma_{+}, \Sigma_{-},$$

$$\sqrt{B_{0}/A_{0}} (A_{+} + B_{-}) = (\Sigma_{+}/\beta) \sinh(\beta),$$

with $\beta = [(\Sigma_{0}/2)^{2} - \Sigma_{\pm}]^{1/2}$. The c-number functions $A_{\pm}, A_{0}, B_{\pm}$ and $B_{0}$, present in the lhs of equations (3a) and (3b), are connected with $[\Omega_{\pm}, \Omega_{0}, \Lambda_{\pm}, \Lambda_{0}]$ through the identities [14]:

$$A_{\pm} = \frac{(\Omega_{\pm}/\phi) \sinh(\phi)}{\cosh(\phi) - (\Omega_{0}/2\phi) \sinh(\phi)},$$

$$A_{0} = [\cosh(\phi) - (\Omega_{0}/2\phi) \sinh(\phi)]^{-2},$$

$$B_{\pm} = \frac{(\Lambda_{\pm}/\theta) \sinh(\theta)}{\cosh(\theta) + (\Lambda_{0}/2\theta) \sinh(\theta)},$$

$$B_{0} = [\cosh(\theta) + (\Lambda_{0}/2\theta) \sinh(\theta)]^{-2},$$

for

$$\phi = [(\Omega_{0}/2)^{2} - \Omega_{-}]^{1/2},$$

$$\theta = [(\Lambda_{0}/2)^{2} - \Lambda_{-}]^{1/2}.$$
**Proof 2.1.** Firstly, we apply the generalized normal- and antinormal-order decomposition formulae established by Ban [14] for exponential functions of the generators of $\mathfrak{su}(1, 1)$ Lie algebra to the abstract operators $T(\Omega_{\pm}, \Omega_0)$ and $T(\Lambda_{\pm}, \Lambda_0)$, that is,

$$ T(\Omega_{\pm}, \Omega_0) = e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Omega})} e^{A_{\pm} C_{\pm}}, $$

and

$$ T(\Lambda_{\pm}, \Lambda_0) = e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Lambda})} e^{A_{\pm} C_{\pm}}, $$

where the c-number functions $A_{\pm}, B_{\pm}, A_0$ and $B_0$ were defined by means of the identities (4a)–(4d). In this way, the product $T(\Omega_{\pm}, \Omega_0) T(\Lambda_{\pm}, \Lambda_0)$ can be expressed as

$$ T(\Omega_{\pm}, \Omega_0) T(\Lambda_{\pm}, \Lambda_0) = e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Omega})} e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Lambda})} e^{A_{\pm} C_{\pm}} = e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Omega})} e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Lambda})} e^{A_{\pm} C_{\pm}}. $$

(7)

with $C_{\pm}$ and $C_0$ given by [14]:

$$ C_{\pm} = \frac{B_0 B_{\pm}}{1 - B_0 B_{\pm} (A_{\pm} + B_{\pm})}, $$

$$ C_{\pm} = \frac{B_0 (A_{\pm} + B_{\pm})}{1 - B_0 B_{\pm} (A_{\pm} + B_{\pm})}, $$

$$ C_0 = \frac{B_0}{[1 - B_0 B_{\pm} (A_{\pm} + B_{\pm})]^2}. $$

The second step consists of using the relationship

$$ e^{\text{ln}(b_{\Omega})} e^{\text{ln}(b_{\Lambda})} = e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Omega})} e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Lambda})} e^{A_{\pm} C_{\pm}}. $$

for the second and third exponentials in the second equality on the rhs of equation (7), with the aim of establishing the intermediate result

$$ T(\Omega_{\pm}, \Omega_0) T(\Lambda_{\pm}, \Lambda_0) = e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Omega})} e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Lambda})} e^{A_{\pm} C_{\pm}}. $$

The rhs of this equation represents the normal-order decomposition of an abstract operator defined as

$$ T(\Sigma_{\pm}, \Sigma_0) := e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Omega})} e^{A_{\pm} C_{\pm}} e^{\text{ln}(b_{\Lambda})} e^{A_{\pm} C_{\pm}}, $$

where the c-number parameters $\Sigma_{\pm}$ and $\Sigma_0$ satisfy the following mathematical relations:

$$ A_{\pm} + A_0 C_{\pm} = \frac{(\Sigma_{\pm} / \beta) \sinh(\beta)}{\cosh(\beta) - (\Sigma_0 / 2\beta) \sinh(\beta)}, $$

$$ C_{\pm} = \frac{(\Sigma_{\pm} / \beta) \sinh(\beta)}{\cosh(\beta) - (\Sigma_0 / 2\beta) \sinh(\beta)}, $$

$$ A_0 C_0 = [\cosh(\beta) - (\Sigma_0 / 2\beta) \sinh(\beta)]^{-2} $$

with $\beta = [(\Sigma_0 / 2)^2 - \Sigma_{\pm}]^{1/2}$. Consequently, substituting the definitions of $C_{\pm}$ and $C_0$ into these relations, we obtain a coupled set of nonlinear equations that permits us not only to establish a link between $(\Sigma_{\pm}, \Sigma_0)$ and $(\Omega_{\pm}, \Omega_0, \Lambda_{\pm}, \Lambda_0)$, but also to verify the general composition formula (2).

An interesting consequence from this mathematical statement is associated with the construction process of a general family of unitary transformations where the abstract operator (1) has a central role. To carry out this task let us first establish a corollary which is directly related to lemma 2.1.

**Corollary 2.1.** For $\Omega_{\ast} = \hat{\xi}$, $\Omega_{\ast} = -\hat{\xi}^*$ and $\Omega_0 = i\omega$, with $\hat{\xi} \in \mathbb{C}$ and $\omega \in \mathbb{R}$, the abstract operator (1) represents a generator of unitary transformations associated with the $\mathfrak{su}(1, 1)$ Lie algebra.

**Proof 2.2.** Basically, the idea is obtaining a specific subset of arbitrary c-number parameters such that

$$ T(\Omega_{\pm}, \Omega_0) [T(\Omega_{\pm}, \Omega_0)]^\dagger = [T(\Omega_{\pm}, \Omega_0)] T(\Omega_{\pm}, \Omega_0) = 1. $$

Hence, let us initially investigate under what circumstances the mathematical relation

$$ T(\Omega_{\pm}, \Omega_0) [T(\Omega_{\pm}, \Omega_0)]^\dagger = 1 $$

is verified. In fact, this condition can be promptly derived from the general composition formula (2) for $A_{\pm} = \Omega_{\ast}$, $A_0 = \Omega_0^*$ and $\Sigma_{\pm} = \Sigma_0 = 0$, with the additional restrictions $\Lambda_{\pm} = -\Lambda_0$ and $\Lambda_0 = -\Omega_0$. Consequently, the equalities $\Omega_{\pm} = -\Omega_0^*$ and $\Omega_0 = -\Omega_0^*$ are satisfied only for $\Omega_{\ast} = \hat{\xi}$, $\Omega_{\ast} = -\hat{\xi}^*$ and $\Omega_0 = i\omega$, with $\hat{\xi} \in \mathbb{C}$ and $\omega \in \mathbb{R}$. This same particular subset of arbitrary c-number parameters can also be obtained from the analysis of

$$ [T(\Omega_{\pm}, \Omega_0)]^\dagger T(\Omega_{\pm}, \Omega_0) = 1, $$

which implies that $T(\hat{\xi}, \omega)$ gives a family of unitary transformations for a general class of representations associated with the $\mathfrak{su}(1, 1)$ Lie algebra.

To illustrate our results, let us consider as a first example the single-mode bosonic representation of the Heisenberg–Weyl algebra where the generators $K_a$ and $K_0$ are expressed as $K_a = (1/2)a^2$, $K_0 = (1/2)a^2$ and $K_0 = (1/2)(a^\dagger a + 1/2)$, with $a$ and $a^\dagger$ being, respectively, the boson annihilation and creation operators satisfying the well-known commutation relation $[a, a^\dagger] = 1$. In this case, the unitary operator $T(\xi, \omega)$ assumes the form

$$ T_1(\xi, \omega) = \exp\{i[a^\dagger \omega + i\omega(a^\dagger + 1/2) - \xi a^\dagger a]\}, $$

(8)

which coincides with the squeezing operator

$$ S(\xi) := e^{i/2}(a^\dagger \omega - \xi a^\dagger a), \quad \text{when} \ \ \omega = 0. $$

It is worth mentioning that, in particular, equation (8) can be promptly used to derive the quantum analogues of the Fresnel transform in classical optics [15].

Another interesting example encompasses the two-mode bosonic representation in which the generators are specifically given by $K_a = a^\dagger b^\dagger$, $K_a = ab$ and $K_0 = (1/2)(a^\dagger a + b^\dagger b + 1)$, where $[a, a^\dagger] = [b, b^\dagger] = 1$. In this context, the unitary operator

$$ T_2(\xi, \omega) = e^{i[a^\dagger b^\dagger + i(\omega/2)(a^\dagger a + b^\dagger b + 1) - \xi a b]} $$

(9)
recovery the two-mode squeezing operator

\[ S_2(\xi) = \exp(\xi a^\dagger b^\dagger - \xi^* a b) \quad \text{for} \quad \omega = 0. \]

Moreover, the action of \( T_2(\xi, \omega) \) in the annihilation operators for each mode of the electromagnetic field implies the following results:

\[
T_2(\xi, \omega) a T_2(\xi, \omega) = [\cosh(\phi) + i(\omega/2\phi) \sinh(\phi)] a + (\xi/\phi) \sinh(\phi) b^\dagger, \tag{10}
\]

\[
T_2(\xi, \omega) b T_2(\xi, \omega) = [\cosh(\phi) + i(\omega/2\phi) \sinh(\phi)] b + (\xi/\phi) \sinh(\phi) a^\dagger, \tag{11}
\]

where \( \phi = ([|\xi|^2 - (\omega/2)^2])^{1/2} \). This set of unitary transformations for \( \omega = 0 \) has its counterpart in the quantum description of physical processes involving parametric amplification [10, 16]. Recently, Pielawa et al [17] have proposed a new method for generating two-mode squeezing in high-\( Q \) resonators using a beam of atoms (which acts as a reservoir for the field) with random arrival times. In particular, the authors have used the unitary two-mode squeezing operator \( S_2(\xi) \) to bring an effective Hamiltonian—which describes resonant single-photon processes—to the well-known Jaynes–Cummings form, where the new bosonic operators are connected to the old ones by two-mode squeezing transformations. Another interesting application is based on the analogy between phonons in an axially time-dependent ion trap and quantum fields in an expanding/contracting universe, where the multimode squeezing operator represents the basic mechanism for cosmological particle creation [18].

It is worth noticing that there are some textbooks on quantum optics [3] that discuss particular cases (if one compares them with those exposed here) of one- and two-mode squeezing operators in different contexts in physics and their connections with nonclassical states of the electromagnetic field. However, the dynamical origin of the real parameter \( \omega \) present in equation (9), for example, has not been investigated so far in the literature, and this fact will be our object of study in the next section. For this purpose, we obtain the exact solutions of the Heisenberg equations for a specific nonresonant system like the parametric amplifier, and show the link between time-evolution operator and \( T_2(\xi, \omega) \).

3. Parametric amplifier

The parametric amplifier model proposed by Louisell et al [10] consists basically of two coupled modes of the electromagnetic field, which play a symmetrical role in the amplification process [16]. Such dynamical elements are usually described by the Hamiltonian \( \hbar = 1 \)

\[
H(t) = \omega_a a^\dagger a + \omega_b b^\dagger b + \kappa a b e^{i\omega t} + \kappa^* a^\dagger b^\dagger e^{-i\omega t}, \tag{12}
\]

where \( a (b) \) is the annihilation operator for the signal (idler) mode, \( \eta \) provides the frequency of the pump field (which has been assumed strong enough to be expressed in classical terms), and \( \kappa \) represents a complex constant being proportional to the second-order susceptibility of the nonlinear medium and to the amplitude of the pump. Moreover, let us introduce a small deviation \( \delta \) in the usual definition of \( \eta \) such that \( \eta = \omega_a + \omega_b + \delta \) with \( \delta/(\omega_a + \omega_b) \ll 1 \) (by hypothesis we are assuming that \( \delta \) comes from a nonperfect match between the frequencies \( \eta \) and \( \omega_a + \omega_b \)). In this case, the solutions of the Heisenberg equations are given by

\[
a(t) = e^{-i(\omega_a t + \delta/2)t} \left[ [\cosh(\phi t) + i(\delta/2) \sinh(\phi t)] a(0) - i(\kappa^*/\phi) \sinh(\phi t) b^\dagger(0) \right],
\]

\[
b(t) = e^{-i(\omega_b t + \delta/2)t} \left[ [\cosh(\phi t) + i(\delta/2) \sinh(\phi t)] b(0) - i(\kappa^*/\phi) \sinh(\phi t) a^\dagger(0) \right],
\]

plus their Hermitian conjugates for \( \varphi = [|\kappa|^2 - (\delta/2)^2]^{1/2} \) fixed. These solutions lead us to verify that \( [a(t), a^\dagger(t)] = [b(t), b^\dagger(t)] = 1 \), which implies in the unitarity of the time-evolution operator \( U(t) \). This fact leads us to show that \( n_a(t) - n_b(t) = n_a(0) - n_b(0) \) (conservation law) for \( n_a(t) := e^t c(t) c^\dagger(t) \) is easily seen to hold; therefore, the intensity correlation function \( I(n_a(t), n_b(t)) \) can be written as \( \langle n_a(t), n_b(t) \rangle = \langle n_a(0), n_b(0) \rangle > n_a(0) - n_b(0) \rangle \). For instance, when the initial state coincides with the number states \( \{n_a, n_b\}_{n_a, n_b \in \mathbb{N}} \), the second term in the rhs of this equation is reduced to \( (n_b - n_a)(n_a, n_b)_{n_a, n_b} = n_a(n_b) \) for \( n_a \neq n_b \). So, if one considers \( n_a = n_b \), there is no contribution from this term and, consequently, \( n_a(t), n_b(t) \) corresponds to the maximum violation of the Cauchy–Schwarz inequality \( \langle a^\dagger a b^\dagger b \rangle = \langle a^\dagger a^\dagger a b \rangle^2 \) for the parametric amplifier [3].

Next, we show how the time-evolution operator \( U(t) \) can be connected with the unitary operator \( T_2(\xi, \omega) \). To carry out this task, let us initially mention that the time-dependent global phase factors present in the solutions of the Heisenberg equations are obtained through the action of the rotation operator

\[
R(\omega_a, \omega_b, \delta; t) = e^{-i[\omega_a t + \delta t/2]a^\dagger a + (\omega_a + \delta t/2)b^\dagger b]}
\]

(13)

on the annihilation operators \( a(0) \) and \( b(0) \). The next step then consists of noticing that for \( \phi = \varphi t, \omega = \delta t \) and \( \xi = -i\kappa^* t \), the operator

\[
T_2(\kappa, \delta; t) = e^{-i[\kappa^* a^\dagger b^\dagger - i(\delta/2)(a^\dagger a + b^\dagger b + 1) + \kappa a b]} \tag{14}
\]

is responsible for generating the terms between braces in the solutions \( a(t) \) and \( b(t) \)—see equations (10) and (11). After these considerations, it is easy to show that

\[
U(t) := R(\omega_a, \omega_b, \delta; t) T_2(\kappa, \delta; t)
\]

(15)

provides the solutions obtained above by means of the mathematical operation sketched in the identities \( a(t) = U^t(0) a(0) U(t) \) and \( b(t) = U^t(0) b(0) U(t) \). Note that the particular nonresonant system here studied represents a first dynamical application of the results obtained in section 2 for the two-mode bosonic representation, where the connection between the time-evolution operator (15) and the unitary operator (9) is promptly established. It is worth emphasizing
that different physical systems can also be used for explaining the dynamical origin of the real parameter \( \omega \) present in
\( T_2(\xi, \omega) \) (e.g. see [17]), but this fact still deserves to be carefully investigated.

4. An initial study on entanglement via the Wigner function

Let us initially consider a specific subset of bipartite physical systems described by continuous variables such that the
elementary physical system is an element of a quantum system
that constitutes a particular element. Furthermore, let us also assume that
\( \rho(t) = \mathbf{U}(t) \rho(0) \mathbf{U}^\dagger(t) \) describes the dynamics of any bipartite system belonging to this subset
whose initial time-evolution operator \( \mathbf{U}(t) \) is related to the
Hamiltonian operator \( \mathbf{H} \); by hypothesis, the initial
density operator \( \rho(0) \) represents the system prepared at the
initial instant \( t = 0 \) in any disentangled (entangled) state. The
symmetric characteristic function for this class of bipartite
physical systems is given by
\( \mathcal{N}(\xi; t) := \text{Tr} \{ \mathbf{D}(\xi) \rho(t) \} \), where
\( \mathbf{D}(\xi) := \exp(-\mathbf{G}^\dagger \mathbf{E} \mathbf{O}) \) defines a
operator written in terms of the matrices \( \mathbf{G}^\dagger = (\xi_a^x \xi_b^x \xi_a^y \xi_b^y) \). \( \mathbf{E} = \mathbb{I} \otimes \mathbb{S} \)
with \( \mathbb{I} = \text{diag}(1, 1) \) and \( \mathbb{S} = \text{diag}(1, -1) \) being the respective
2 \times 2 identity and symplectic matrices, and \( \mathbb{G}^\dagger = (a^\dagger \hspace{0.5cm} b^\dagger \hspace{0.5cm} b \hspace{0.5cm} a) \).
Note that due to the cyclic property of the trace operation,
the symmetric characteristic function can also be evaluated through
the mathematical statement \( \text{Tr} \{ \mathbf{U}(t) \mathbf{D}(\xi) \mathbf{U}(t) \rho(0) \} \), where
the time evolution of the displacement operator plays an
important role.

Next, let us suppose that the action of the unitary
time-evolution operator on the matrix \( \mathbb{G} \) transforms the
annihilation (creation) operators \( a(a^\dagger) \) and \( b(b^\dagger) \) following the general rule
\( \mathbf{G}(t) = \mathbf{U}(t) \mathbb{G} \mathbf{U}(t)^\dagger \), where
\[
\mathbf{T}(t) = \begin{bmatrix}
\mu_a(t) & \nu_a(t) & \chi_a(t) & \eta_a(t) \\

\nu_a^*(t) & \mu_a^*(t) & \eta_a^*(t) & \chi_a^*(t) \\

\mu_b(t) & \nu_b(t) & \chi_b(t) & \eta_b(t) \\

\nu_b^*(t) & \mu_b^*(t) & \eta_b^*(t) & \chi_b^*(t)
\end{bmatrix}
\]
represents a \( 4 \times 4 \) matrix whose elements are c-number
functions determined from the Heisenberg equations for
\( \mathbf{G}(t) \) with specific initial conditions that preserve the
quantum mechanics (the subscript \( H \) indicates that
operators are in the Heisenberg picture). For instance, when
\( t = 0 \) the c-number functions should necessarily imply
(4) the mathematical identity \( \mathbf{T}(0) = \text{diag}(1, 1, 1, 1) \); while for \( t \geq 0 \)
the commutation relations
\[
[\mathbf{a}_H(t), \mathbf{a}_H^\dagger(t)] = 1, \quad [\mathbf{b}_H(t), \mathbf{b}_H^\dagger(t)] = 1,
\]
\[
[\mathbf{a}_H(t), \mathbf{b}_H(t)] = 0, \quad [\mathbf{a}_H(t), \mathbf{b}_H^\dagger(t)] = 0
\]
provide extra relations for the c-number functions which
permit us to solve completely the Heisenberg equations,
\[
|\mu_a(t)|^2 - |\nu_a(t)|^2 + |\chi_a(t)|^2 - |\eta_a(t)|^2 = 1,
\]
\[
|\mu_b(t)|^2 - |\nu_b(t)|^2 + |\chi_b(t)|^2 - |\eta_b(t)|^2 = 1,
\]
\[
\mu_a(t)\nu_a(t) - \mu_a^*(t)\nu_a^*(t) + \mu_a^*(t)\eta_a(t) - \mu_a(t)\eta_a^*(t) = 0,
\]
\[
\mu_a(t)\nu_b(t) - \mu_b^*(t)\nu_b^*(t) + \mu_b^*(t)\eta_b(t) - \mu_b(t)\eta_b^*(t) = 0,
\]
\[
\beta_a = \mu_a(t)\xi_a - \nu_a(t)\xi_a^* + \mu_a^*(t)\xi_b - \nu_b(t)\xi_b^*.
\]

The first immediate consequence of these results is that
\( \mathbf{U}(t) \mathbf{D}(\xi) \mathbf{U}(t) \) will produce a new displacement operator
\( \mathbf{D}(\eta) = \exp(-\mathbf{Y}^\dagger \mathbf{E} \mathbf{O}) \) with
\( \mathbf{Y}^\dagger = (\beta_a^\dagger \beta_b^\dagger \beta_a \beta_b) \), where
the new elements are connected with the old ones through the
equalities
\[
\beta_a = \mu_a(t)\xi_a - \nu_a(t)\xi_a^* + \mu_a^*(t)\xi_b - \nu_b(t)\xi_b^*.
\]

Note that the variables of
(5) the symmetric characteristic function \( \mathcal{N}(\xi; t) \) is thus specified in terms of the
form it takes at \( t = 0 \), which corroborates the results obtained
by Mollow and Glauber [16] for the parametric amplifier model,
and this fact will bring some insights into the study of
entanglement for bipartite systems via the Wigner function.

The Wigner function for this particular subset of bipartite
physical systems may then be defined as the four-dimensional
Fourier transform of the symmetric characteristic function
\( \mathcal{N}(\xi; t) \), that is,
\[
\mathcal{W}(\xi; t) = \int \frac{d^2\xi_a d^2\xi_b}{\pi^2} \exp(\mathbf{G}^\dagger \mathbf{E} \mathbf{O}) \mathcal{N}(\xi; t)
\]
(16)
where \( \mathbf{Y}^\dagger := (\alpha_a^\dagger \alpha_a \alpha_b^\dagger \alpha_b) \) defines a new matrix with
elements given by
\[
\gamma_a = \mu_a(t)\alpha_a - \nu_a(t)\alpha_a^* + \mu_a^*(t)\alpha_b - \nu_b(t)\alpha_b^*,
\]
\[
\gamma_b = \chi_a(t)\alpha_a - \eta_a(t)\alpha_a^* + \chi_b^*(t)\alpha_b - \eta_b(t)\alpha_b^*,
\]
and their respective complex conjugates. Thus, equation (17)
asserts that \( \mathcal{W}(\xi; t) \) can also be expressed in terms of the
form it takes at the initial time \( t = 0 \), since the new variables
\( \gamma_a(\alpha_a; \alpha_b; t) \) and \( \gamma_b(\alpha_a; \alpha_b; t) \) carry the information on
the dynamical entanglement between the variables \( \alpha_a \) and
\( \alpha_b \). Furthermore, this result permits us to show that
for a given bipartite system initially prepared in any
\[
\mathbf{Z}^\dagger := (\gamma_a^\dagger \gamma_b^\dagger \gamma_a \gamma_b)
\]
(17)
entangled state, it will remain entangled for all \( t \geq 0 \); otherwise, if the density operator is described at \( t = 0 \) as \( \mathbf{\rho}(0) = \rho_a(0) \otimes \rho_b(0) \) (disentangled state), the appearance of entanglement in the Wigner function will depend exclusively on the dynamics provided by the Hamiltonian operator \( \mathbf{H}(t) \)—here associated with the time-dependent matrix \( \mathbf{Z} \). In the following, let us apply the results obtained until now to the parametric amplifier model, where different initial states of the electromagnetic field will be considered.

Now we focus our efforts on evaluating \( \mathcal{W}(\mathbf{Z}; 0) \) for a well-known family of two-mode electromagnetic fields where, in particular, both modes are initially prepared in the coherent states, number states and thermal states, respectively. For instance, the symmetric characteristic functions in these situations are given by

\[
\begin{align*}
\mathcal{W}_{\text{coh}}(\mathbf{Y}; 0) &= e^{-\frac{1}{2}(\beta_1^2 + \beta_2^2) + 2\text{Im}(\beta_2 \beta_1^* + \beta_1 \beta_2^*)}, \\
\mathcal{W}_{\text{a}}(\mathbf{Y}; 0) &= e^{-\frac{1}{2}(\beta_1^2 + \beta_2^2)} L_{n_a}(\beta_1^2) L_{n_b}(\beta_2^2), \\
\mathcal{W}_{\text{b}}(\mathbf{Y}; 0) &= e^{-\frac{1}{2}(1 + 2 \eta_a) \beta_1^2 + (1 + 2 \eta_b) \beta_2^2},
\end{align*}
\]

while their respective Wigner functions can be expressed as

\[
\begin{align*}
\mathcal{W}_{\text{coh}}(\mathbf{Z}; 0) &= 4 e^{-2(\eta_a - \xi_1^2 + \eta_b - \xi_2^2)}, \\
\mathcal{W}_{\text{a}}(\mathbf{Z}; 0) &= 4(-1)^{n_a + n_b} e^{-2(\eta_a^2 + \eta_b^2)} L_{n_a}(4|\eta_a|^2) L_{n_b}(4|\eta_b|^2), \\
\mathcal{W}_{\text{b}}(\mathbf{Z}; 0) &= 4 \left[ (1 + 2 \eta_a)(1 + 2 \eta_b) \right]^{-1} \times e^{-2(1 + 2 \eta_a) \xi_1^2 + (1 + 2 \eta_b) \xi_2^2},
\end{align*}
\]

with \( L_n(z) \) denoting a Laguerre polynomial and \( \tilde{n}_{a|b} \) being the mean number of photons for a chaotic light field. Note that if we consider the solutions obtained from the Heisenberg equations for the parametric amplifier model (see the previous section), it is easy to show that \( \mu_{a|b}(t), \tilde{n}_{a|b}(t), v_b(t) \) and \( \chi_b(t) \) (once the further time-dependent coefficients do not exist) determine completely the c-numbers \( \beta_{a|b}(\xi_a, \xi_b; t) \) and \( \gamma_{a|b}(\alpha_a, \alpha_b; t) \). Indeed, this last step leads us to characterize precisely \( \mathcal{W}(\mathbf{Y}; 0) \) and \( \mathcal{W}(\mathbf{Z}; 0) \). Moreover, when \( \delta = 0 \), \( \delta \in \mathbb{R} \), and \( (\omega_a + \omega_b) t = \pi/2 \), equation (18) coincides with \( \mathcal{W}_{\text{EPR}}(Y_a, Y_b) \) for \( \xi_{a|b} = 0 \) (two-mode vacuum state), this function being that used by Braunstein and Kimble [22] in the theoretical description of teleportation involving continuous quantum variables.

Finally, let us say a few words about the results obtained in this section. It is important to emphasize that equation (17) permits us to describe qualitatively the entanglement of a specific subset of bipartite systems, where now we can promptly identify two distinct origins of this quantum effect: the first associated with the entanglement in the initial conditions (here labelled kinematical entanglement), while the second is responsible for the dynamical entanglement via Hamiltonian operator. In addition, the normally ordered moments

\[
\langle a^{+p}(t) a^{q}(t) b^{+r}(t) b^{s}(t) \rangle = \Gamma^{(p,q,r,s)}_{\xi_a, \xi_b, \eta_a, \eta_b} e^{i(2 |\xi_a|^2 + |\xi_b|^2)} \times \mathcal{U}(\xi_a, \xi_b^*, \eta_a, \eta_b^*; t) \bigg|_{\xi_a, \xi_b, \eta_a, \eta_b} = 0,
\]

with

\[
\Gamma^{(p,q,r,s)}_{\xi_a, \xi_b, \eta_a, \eta_b} := (-1)^{q+r-s} \frac{\partial^{p+q+r+s}}{\partial \xi_a^p \partial \xi_b^q \partial \eta_a^r \partial \eta_b^s} |_{\xi_a, \xi_b, \eta_a, \eta_b}.
\]

and \( p, q, r, s \in \mathbb{N} \), can also be used to investigate some recent proposals of inseparability criteria for continuous bipartite quantum states [21, 23]. In the next section, we will establish a reasonable measure of entanglement which is based on the results obtained by Piątek and Leoniński [11] for the intermode correlations in phase space.

5. Entropy functionals for continuous phase-space representations

In order to establish a quantitative estimate of entanglement for bipartite systems, let us introduce some basic definitions of entropy functionals in continuous phase-space representations. The first definition is based on the joint entropy [11]

\[
E[\mathcal{H}; t] := - \int \frac{d^2 \alpha_a d^2 \alpha_b}{\pi^2} \mathcal{H}(\alpha_a, \alpha_b; t) \ln \left[ \mathcal{H}(\alpha_a, \alpha_b; t) \right].
\]

where \( \mathcal{H}(\alpha_a, \alpha_b; t) := \langle \mathcal{H}(\mathbf{p}(t) a(\alpha_a, \alpha_b) \rangle \) denotes the Husimi function in the continuous coherent-state representations for a bipartite system described by the density operator \( \mathbf{p}(t) \). Note that \( E[\mathcal{H}; t] \) presents certain properties inherent to its definition which deserve to be mentioned: (i) the probability distribution function \( \mathcal{H}(\alpha_a, \alpha_b; t) \) is strictly positive and limited to the interval \([0, 1]\); consequently, the joint entropy (22) characterizes a well-defined function whose behaviour does not present any mathematical inconsistencies. Moreover, (ii) this definition essentially measures the functional correlation between the continuous variables used to describe each part of the joint system. Hence, \( E[\mathcal{H}; t] \)

2 According to Mollow and Glauber [16]: 'The fact that the Wigner function has this property is a consequence of the form taken by the Hamiltonian (12). It may be shown that whenever the Hamiltonian of a system of oscillators is given by a quadratic form in the creation and annihilation operators, the Wigner function is constant along classical trajectories. This property does not extend to systems with arbitrary Hamiltonians, as it does in the case of the classical phase-space distribution.' Beyond these fundamental features, it is worth mentioning that recent studies on the characterization and quantification of entanglement for symmetric and asymmetric bipartite Gaussian states have contributed considerably to our comprehension of this important nonclassical effect in quantum optics and quantum information theory [20, 21]. In this sense, we believe that the formalism here presented for the Wigner function can help to extend such results in order to include some important dynamical effects that allow us to better understand certain peculiarities on the entanglement process for a specific subset of bipartite physical systems described by continuous variables [22].

3 It is worth noting that the cases here studied represent a specific set of disentangled bipartite states whose characteristic and Wigner functions are written as a product of two functions associated with each mode separately, i.e., since \( \mathbf{\rho}(0) = \rho_a(0) \otimes \rho_b(0) \) we promptly obtain the relations \( \mathcal{W}(\mathbf{Y}; 0) = \mathcal{W}_{\text{a}}(\mathbf{Y}; 0) \mathcal{W}_{\text{b}}(\mathbf{Y}; 0) \) and \( \mathcal{W}(\mathbf{Z}; 0) = \mathcal{W}_{\text{a}}(\mathbf{Z}; 0) \mathcal{W}_{\text{b}}(\mathbf{Z}; 0) \). In this situation, the dynamical entanglement that originated from the Hamiltonian \( \mathbf{H}(t) \) allows us to correlate the complex variables \( \alpha_a \) and \( \alpha_b \) (or \( \xi_a \) and \( \xi_b \)) present in \( \mathcal{W}_{\text{a|b}}(\mathbf{B}_{a|b}) \).
can be considered as a natural extension of that definition employed by Wehrl [12] for information entropy.\(^4\)

The second definition consists of functionals related to the partial entropies

\[
E[\mathcal{H}^{(A)}; t] := -\int \frac{d^2 \alpha_a}{\pi} \mathcal{H}^{(A)}(\alpha_a; t) \ln \mathcal{H}^{(A)}(\alpha_a; t)
\]

(23)

and

\[
E[\mathcal{H}^{(B)}; t] := -\int \frac{d^2 \alpha_b}{\pi} \mathcal{H}^{(B)}(\alpha_b; t) \ln \mathcal{H}^{(B)}(\alpha_b; t).
\]

(24)

which depend basically on the marginal Husimi functions

\[
\mathcal{H}^{(A)}(\alpha_a; t) = \int \frac{d^2 \alpha_b}{\pi} \mathcal{H}(\alpha_a, \alpha_b; t),
\]

\[
\mathcal{H}^{(B)}(\alpha_b; t) = \int \frac{d^2 \alpha_a}{\pi} \mathcal{H}(\alpha_a, \alpha_b; t).
\]

Such marginal Husimi functions carry information on the entanglement between the subsystems ‘A’ and ‘B’, since the partial trace over the continuous variables \(\alpha_a(b)\) of a particular subsystem ‘A’ (‘B’) allows the introduction, via time-evolution operator \(U(t)\) and/or initial density operator \(\rho(0)\), of important correlations in the bipartite states.

Let us derive now some mathematical relations among these entropy functionals from the Araki–Lieb inequality [25], i.e.

\[
|E[\mathcal{H}^{(A)}; t] - E[\mathcal{H}^{(B)}; t]| \leq E[\mathcal{H}; t] \leq E[\mathcal{H}^{(A)}; t] + E[\mathcal{H}^{(B)}; t].
\]

(25)

For instance, the rhs of this inequality corresponds to the subadditivity property for Wehrl’s entropy functionals, while the equal sign reflects the complete disentanglement between the subsystems ‘A’ and ‘B’ of the joint system. In addition, the conditional entropies [11, 12]

\[
E[\mathcal{H}/\mathcal{H}^{(A)}; t] = E[\mathcal{H}; t] - E[\mathcal{H}^{(A)}; t].
\]

(26)

and

\[
E[\mathcal{H}/\mathcal{H}^{(B)}; t] = E[\mathcal{H}; t] - E[\mathcal{H}^{(B)}; t]
\]

(27)

lead us not only to establish the balance equation

\[
E[\mathcal{H}/\mathcal{H}^{(A)}; t] + E[\mathcal{H}^{(A)}; t] = E[\mathcal{H}/\mathcal{H}^{(B)}; t] + E[\mathcal{H}^{(B)}; t].
\]

(28)

but also to determine the further inequalities

\[
E[\mathcal{H}/\mathcal{H}^{(A)}; t] \leq E[\mathcal{H}^{(B)}; t]
\]

(29)

which can be used to measure the functional correlation (that is, the ‘degree of entanglement’) between the parts ‘A’ and ‘B’ of the joint system, having as reference a factorizable bipartite Husimi function \(\mathcal{H}(\alpha_a, \alpha_b; t)\). In this definition, the inferior limit of the closed interval \([0, 1]\) corresponds to the situation where the bipartite Husimi function factorizes, while its superior limit reflects the opposite situation (i.e. when the bipartite states are maximally correlated); besides, the global factor ‘2’ is associated with the number of possible combinations between the constituent parts of the system under investigation. As a first practical application of these entropy functionals we will consider two different examples of initially uncoupled bipartite states for the parametric amplifier model.

5.1. Coherent states

In this first example, we consider the bipartite pure states \(\rho(0) = |\zeta_a, \zeta_b\rangle \langle \zeta_a, \zeta_b|\) as being the initial state at \(t = 0\) of the model governed by the Hamiltonian (12). Here, \(|\zeta_a, \zeta_b\rangle = |\zeta_a\rangle \otimes |\zeta_b\rangle\), where \(|\zeta_a(b)\rangle\) characterizes the coherent states related to the signal (idler) mode of the electromagnetic field. Thus, after some nontrivial algebra, the bipartite Husimi function assumes the closed form

\[
\mathcal{H}_{\text{coh}}(\alpha_a, \alpha_b; t) = |A_0|e^{-\left(\gamma_a^2 + \gamma_b^2 + \gamma_{ab}^2\right)}/2\Re \left(\gamma_{ab}e^{i\gamma_{ab}t}\right)
\]

(30)

such that

\[
Y_{a(b)} = e^{i\alpha_{a(b)} + i/2} \left(\cosh(\varphi t) - i(\delta/2\varphi) \sinh(\varphi t)\right) \alpha_{a(b)}
\]

\[
+ i(\kappa/\varphi) e^{-i\alpha_{a(b)} + i/2}\sinh(\varphi t) \alpha_{b(a)}^*.
\]

(31)

\[
A_+ = \frac{i(\kappa/\varphi) \sinh(\varphi t)}{\cosh(\varphi t) + i(\delta/2\varphi) \sinh(\varphi t)}.
\]

\[
A_0 = [\cosh(\varphi t) + i(\delta/2\varphi) \sinh(\varphi t)]^{-1/2}.
\]

Furthermore, the marginal Husimi functions

\[
\mathcal{H}_{\text{coh}}^{(A)}(\alpha_a; t) = |A_0|e^{-|A_0|^2|\alpha_a - \gamma_a|^2}
\]

(31)

and

\[
\mathcal{H}_{\text{coh}}^{(B)}(\alpha_b; t) = |A_0|e^{-|A_0|^2|\alpha_a - \gamma_a|^2}
\]

(32)
with
\[ \epsilon_{a(b)} = \left[ \cosh(\varphi t) + i(b/2\varphi) \sinh(\varphi t) \right] \zeta_{a(b)} - i(k^*/\varphi) \sinh(\varphi t) \zeta_{bh(a)}^* e^{-i(\omega_{a(b)} + b/2)t}, \]

permit us to verify how the dynamical entanglement introduces correlations between the continuous variables \( \zeta_{a(b)} \) and \( \zeta_{bh(a)}^* \) through the label \( \epsilon_{a(b)} \). Indeed, for \( t = 0 \) it is possible to demonstrate that \( \mathcal{H}_{\text{coh}}(\alpha_a, \alpha_b; 0) \) factorizes in the product \( \mathcal{H}_{\text{coh}}^{(A)}(\alpha_a; 0) \mathcal{H}_{\text{coh}}^{(B)}(\alpha_b; 0) \). However, if one considers \( t > 0 \), the dynamics governed by the process of parametric amplification introduces significant correlations between the signal and idler modes. In the following, we will quantify these correlations with the help of equation (29).

The Gaussian structures present in the bipartite and marginal Husimi functions allow us to obtain closed form expressions for the respective joint and partial entropies, namely,

\[ E[\mathcal{H}_{\text{coh}}; t] = 2 + \ln \left[ 1 + (|\kappa|/\varphi)^2 \sinh^2(\varphi t) \right] \]

and

\[ E[\mathcal{H}_{\text{coh}}^{(A,B)}; t] = 1 + \ln \left[ 1 + (|\kappa|/\varphi)^2 \sinh^2(\varphi t) \right]. \]

Consequently, the correlation functional can be promptly evaluated as follows:

\[ C[\mathcal{H}_{\text{coh}}; t] = \frac{\ln \left[ 1 + (|\kappa|/\varphi)^2 \sinh^2(\varphi t) \right]}{1 + \ln \left[ 1 + (|\kappa|/\varphi)^2 \sinh^2(\varphi t) \right]}. \]

Besides, if one considers the limit \( \varphi t \gg 1 \) for sufficiently long time, the correlation functional goes asymptotically to 1, this value being associated with the specific dynamics given by the Hamiltonian (12). On the other hand, for \( \varphi t = 0 \) we obtain \( C[\mathcal{H}_{\text{coh}}; 0] = 0 \), which corroborates the factorization process of the bipartite Husimi function \( \mathcal{H}_{\text{coh}}(\alpha_a, \alpha_b; 0) \).

### 5.2. Thermal states

Now, let us suppose that both signal and idler modes were initially prepared in the thermal states. In this case, the initial density operator admits the expansion

\[ \rho_{\text{th}}(0) = \frac{1}{\bar{n}_{a(b)}} \int \frac{d^2 \zeta_a d^2 \zeta_b}{\pi^2} e^{-\bar{n}_{a(b)} \zeta_a^* \zeta_a - \bar{n}_{a(b)} \zeta_b^* \zeta_b} \mathbf{P}(\zeta_a, \zeta_b), \]

where \( \bar{n}_{a(b)} \) represents the average occupation number of each mode, and \( \mathbf{P}(\zeta_a, \zeta_b) = |\zeta_a, \zeta_b \rangle \langle \zeta_a, \zeta_b| \) is the projector of the coherent states. This typical example of bipartite mixed states constitutes an important starting point in the investigation process of the dynamical entanglement due to the parametric amplifier model. So, after some calculations, the bipartite Husimi function can be written as

\[ \mathcal{H}_{\text{th}}(\alpha_a, \alpha_b; t) = \left[ \mathcal{G}_{a,b}(1, 1; t) \right]^{-1} e^{-\left( \epsilon_{a(\alpha)}^2 + \epsilon_{b(\alpha)}^2 \right)} e^{2\Re\epsilon_{a(\alpha)} \epsilon_{b(\alpha)}} \]

with

\[ \epsilon_a = \frac{\mathcal{G}_{a,b}(1, 1; t)}{\mathcal{G}_{a,b}(0, 1; t)}, \quad \epsilon_b = \frac{\mathcal{G}_{a,b}(0, 1; t)}{\mathcal{G}_{a,b}(1, 1; t)}, \]

\[ \epsilon_{ab} = i(k/\varphi) \sinh(\varphi t) \left[ \cosh(\varphi t) - i(\delta/2\varphi) \sinh(\varphi t) \right] \]

\[ \times (\bar{n}_{a} + \bar{n}_{b} + 1) \left[ \mathcal{G}_{a,b}(1, 1; t) \right]^{-1} e^{(\omega_{a(b)} + b/2)t}, \]

and

\[ \mathcal{G}_{a,b}(x, y; t) = (\bar{n}_{a}x + 1)(\bar{n}_{b}y + 1) + (\bar{n}_{a} + \bar{n}_{b} + 1)(|\kappa|/\varphi)^2 \sinh^2(\varphi t), \]

while its joint entropy assumes the closed form

\[ E[\mathcal{H}_{\text{th}}; t] = 2 + \ln \left[ \mathcal{G}_{a,b}(1, 1; t) \right]. \]

In addition, the marginal Husimi functions

\[ \mathcal{H}_{\text{th}}^{(A)}(\alpha_a; t) = \left[ \mathcal{G}_{a,b}(1, 0; t) \right]^{-1} e^{-\left[ |\alpha_a| \right]^2} \]

and

\[ \mathcal{H}_{\text{th}}^{(B)}(\alpha_b; t) = \left[ \mathcal{G}_{a,b}(0, 1; t) \right]^{-1} e^{-\left[ |\alpha_b| \right]^2} \]

permit us also to determine explicitly the partial entropies

\[ E[\mathcal{H}_{\text{th}}^{(A,B)}; t] = 1 + \ln \left[ \mathcal{G}_{a,b}(1, 0; t) \right], \]

\[ E[\mathcal{H}_{\text{th}}^{(A,B)}; t] = 1 + \ln \left[ \mathcal{G}_{a,b}(0, 1; t) \right]. \]

At this moment, it becomes important to notice that the subadditivity property is not violated for any \( t \geq 0 \), the equality \( E[\mathcal{H}_{\text{th}}; 0] = E[\mathcal{H}_{\text{th}}^{(A)}; 0] + E[\mathcal{H}_{\text{th}}^{(B)}; 0] \) being consistent with the factorization of \( \mathcal{H}_{\text{th}}(\alpha_a, \alpha_b; 0) \) in the product \( \mathcal{H}_{\text{th}}^{(A)}(\alpha_a); 0) \mathcal{H}_{\text{th}}^{(B)}(\alpha_b; 0) \).

In following, let us investigate the correlation functional \( C[\mathcal{H}_{\text{th}}; t] \) for \( t > 0 \), since \( C[\mathcal{H}_{\text{th}}; 0] = 0 \) reflects the disentanglement between the bipartite mixed states. In fact, for \( t \neq 0 \) this functional assumes any values into the open interval \( (0, 1) \), the maximum value 1 being reached only for sufficiently long time. Hence, we can conclude that: (i) any initially uncoupled states become entangled in the process described by the Hamiltonian (12), which confirms the results obtained by Dodonov et al [21] through the evaluation of the inverse negativity coefficient (this measure also leads us to estimate quantitatively the ‘degree of entanglement’ of a particular class of bipartite states such as those studied in this section); furthermore, (ii) the dynamics here characterized by the physical system under investigation produces a maximum estimate of entanglement for any initially disentangled states, this value (namely, 1) being considered as a specific quantum signature of the parametric amplifier model.

### 6. Conclusions

We have established a set of interesting formal results within the scope of quantum optics and quantum information theory that allows us, among other things,

- to define a new class of unitary squeezing transformations related to \( su(1, 1) \) Lie algebra which generalizes, for certain particular representations of its generators, the one- and two-mode squeezing operators [3];
- to discuss, from the physical point of view, how the generalized two-mode squeezing operator can be generated through a slightly modified version of the parametric amplifier model [10];
● to obtain a general integral representation for the bipartite Wigner function whose integrand is expressed as a product of two terms which are responsible for the dynamical and kinematical entanglement; and finally,
● to estimate quantitatively the ‘degree of entanglement’ related to an ideal bipartite system (we are discarding the unavoidable coupling with the environment in this context) by means of a theoretical framework [11] which is based on Wehrl’s approach [12] for the entropy functionals.

In fact, equation (29) and its respective properties mentioned properly in the body of the text permit us to estimate, through an entropic approach, the entanglement effects for a wide class of electromagnetic field states, including Gaussian and non-Gaussian states. As a concluding remark, it is worth mentioning that these results also have potential applications in modern research on quantum teleportation [22], quantum tomography [4, 26] and quantum computation [27] (once continuous-variable entanglement can also be efficiently produced using squeezed light and linear optics [28]), as well as on the foundations of quantum mechanics through its extensions to the su(2) Lie algebra [29].

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References

[1] Galindo A and Martin-Delgado M A 2002 Rev. Mod. Phys. 74 347
Peres A and Terno D R 2004 Rev. Mod. Phys. 76 93
[2] Braunstein S L and van Loock P 2005 Rev. Mod. Phys. 77 513
[3] Walls D F and Milburn G J 1995 Quantum Optics (New York: Springer)
Gerry C and Knight P L 2006 Introductory Quantum Optics (Cambridge: Cambridge University Press)
Fox M 2006 Quantum Optics: An Introduction (New York: Oxford University Press)
[4] Leonhardt U 1997 Measuring the Quantum State of Light: Cambridge Studies in Modern Optics (Cambridge: Cambridge University Press)
Vahlbruch H, Chelkowski S, Danzmann K and Schnabel R 2007 New J. Phys. 9 371
[5] Alber G, Beth T, Horodecki M, Horodecki P, Horodecki R, Rötteler M, Weinifter H, Werner R and Zeilinger A 2001 Quantum Information: An Introduction to Theoretical Concepts and Experiments (Berlin: Springer)
[6] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[7] Wootters W K 1998 Phys. Rev. Lett. 80 2215
[8] Jozsa R, Robb D and Wootters W K 1994 Phys. Rev. A 49 668
Horodecki R and Horodecki M 1994 Phys. Rev. A 54 1938
Cerf N J and Adami C 1997 Phys. Rev. Lett. 79 5194
Cerf N J and Adami C 1998 Physica D 120 62
Alcaraz F C and Tsallis C 2002 Phys. Lett. A 301 105
Vedral V 2002 Rev. Mod. Phys. 74 197
Feng H 2003 J. Phys. A: Math. Gen. 36 12201
Berry D W and Sanders B C 2003 J. Phys. A: Math. Gen. 36 12255

[9] Breuer H P and Petruccione F 2002 The Theory of Open Quantum Systems (New York: Oxford University Press)
Benatti F and Floreanini R 2003 Irreversible Quantum Dynamics: (Lecture Notes in Physics) (New York: Springer)
[10] Louisell W H, Yariv A and Siegman A E 1961 Phys. Rev. 124 1646
[11] Mizrahi S S and Marchiolli M A 1993 Physica A 199 96
Pietek K and Leoński W 2001 J. Phys. A: Math. Gen. 34 4951
[12] Wehrl A 1978 Rev. Mod. Phys. 50 221
Wehrl A 1979 Rep. Math. Phys. 16 353
[13] Wódkiewicz K and Eberly J H 1985 J. Opt. Soc. Am. B 2 458
[14] Ban M 1993 J. Opt. Soc. Am. B 10 1347
[15] Shan-Jun M, Li-Yun H and Hong-Yi F 2008 Commun. Theor. Phys. 49 1295
[16] Mollow B R and Glauber R J 1967 Phys. Rev. 160 1076
Mollow B R and Glauber R J 1967 Phys. Rev. 160 1097
Dell’Anno F, De Siena S and Illuminati F 2006 Phys. Rep. 428 53
[17] Pielawa S, Morigi G, Vitali D and Davidovich L 2007 Phys. Rev. Lett. 98 240401
[18] Schützhold R, Uhlmann M, Petersen L, Schmitz H, Friedenauer A and Schütz T 2007 Phys. Rev. Lett. 99 201301
[19] Pechuk J and Wódkiewicz K 2003 Int. J. Quantum Inf. 1 153
[20] Enßlin B-G and Wódkiewicz K 2003 Phys. Rev. A 77 015809
[21] Dodonov A V, Dodonov V V and Mizrahi S S 2005 J. Phys. A: Math. Gen. 38 683
Adesso G and Illuminati F 2007 J. Phys. A: Math. Theor. 40 7821
Poon P S Y and Law K C 2007 Phys. Rev. A 76 054305
[22] Braunstein S L and Kimble H J 1998 Phys. Rev. Lett. 80 869
Milburn G J and Braunstein S L 1999 Phys. Rev. A 60 937
[23] Duan L-M, Giedke G, Cirac J I and Zoller P 2000 Phys. Rev. Lett. 84 2722
[24] Simon R 2000 Phys. Rev. Lett. 84 2726
Werner R F and Wolf M M 2001 Phys. Rev. Lett. 86 3658
Giedke G, Kraus B, Lewenstein M and Cirac J I 2001 Phys. Rev. Lett. 87 167904
Marchiolli M A, Missori R J and Roversi J A 2003 J. Phys. A: Math. Gen. 36 12275
Schuchkin E and Vogel W 2005 Phys. Rev. Lett. 95 230502
Marchiolli M A 2006 J. Mod. Opt. 53 2733
[25] Míntez F and Zyczkowski K 2004 Phys. Rev. A 69 022317
[26] Araki H and Lieb E H 1970 Commun. Math. Phys. 18 160
Lieb E H and Rudskai M B 1973 J. Math. Phys. 14 1938
[27] Marchiolli M A, Mizrahi S S and Dodonov V V 1997 Phys. Rev. A 56 4278
Marchiolli M A, Mizrahi S S and Dodonov V V 1998 Phys. Rev. A 57 3885
[28] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[29] Yazdi Z S, Turner P S Y and Law C K 2007 Phys. Rev. A 76 054309
[30] Dowling J P, Agarwal G S and Schleich W P 1994 Phys. Rev. A 49 4101
[31] Klimov A B and Romero J L 2008 J. Phys. A: Math. Theor. 41 055303