Multimode Two-Dimensional $\mathcal{PT}$-Symmetric Waveguides

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Abstract. In this article, we apply a time-dependent Darboux transformation for the construction of $\mathcal{PT}$-symmetric multimode optical waveguides where the non-separable complex fluctuations of the refractive index confine guided modes. We focus on a family of settings based on the hyperbolic Pöschl-Teller potential well. We show that the transformed systems have a “missing” state, an extra guided mode whose analog does not exist in the original system.

1. Introduction

Classical optics and quantum mechanics can have more in common that one would expect. Indeed, there are situations where the dynamics of light beams is dictated by the same equations as the propagation of matter waves in quantum systems [1]. The intersection between the two seemingly distant areas of physics has proven to be exceedingly fruitful. In particular, it has provided optics with concepts and results known in the $\mathcal{PT}$-symmetric quantum mechanics [2, 3, 4].

In $\mathcal{PT}$-symmetric quantum mechanics, the traditional requirement on the Hermiticity of the operators is replaced by requirement of $\mathcal{PT}$-symmetry of the Hamiltonian (here $\mathcal{P}$ is space inversion and $\mathcal{T}$ is time reversal) [5, 6]. This way, potentials in terms of complex functions are permitted while the spectrum of the system is kept real. Besides its many appeals, there are some drawbacks to the theory. The scalar product needs to be redefined to get a consistent probabilistic interpretation [7, 8, 9]. It can be prohibitively difficult in many cases to find its explicit form [10]. However, as the probabilistic interpretation is not relevant in classical optics, the tools and results of non-Hermitian quantum mechanics can be applied directly to optical systems. Let us mention supersymmetry transformations that have been used effectively in the construction of optical settings [11]-[20].

The way to the common ground of optics and quantum mechanics is paved by the paraxial approximation. Let us review it briefly here. If we consider monochromatic light with a wavelength in vacuum $\lambda$, then the electric field reads as

$$\vec{E}(x, y, z) = \exp(ikn_0z) \left( \psi_T(x, y, z) + \vec{a}_z \psi_z(x, y, x) \right).$$

(1)

Here $k = 2\pi/\lambda$ and $n_0$ is a reference value of the refractive index, $\vec{a}_z$ is a unitary vector in the $z$ direction, the sub-indexes $T$ and $z$ stands for the transverse and propagation direction,
respectively. In the paraxial approximation (see e.g. [1] for more details, see also [21, 22, 23]), the Maxwell equations can be simplified to the single equation for $\vec{\psi}_T(x, y, z)$:

$$i \partial_z \vec{\psi}_T + (\partial_x^2 + \partial_y^2)\vec{\psi}_T + 2k^2x_0^2n_0\Delta n\vec{\psi}_T = 0 (2)$$

where $\Delta n(x, y, z) = n - n_0$ are the variations in refractive index with respect the reference value $n_0$. For problems where $\Delta n = \Delta n(x, z)$ we can use separation of variables in each component of $\vec{\psi}_T = e^{ik_y y}\vec{\psi}(x, z)$. Then, each vector component of $\vec{\psi}$ satisfies a time dependent Schrödinger equation $i\partial_z \psi + \partial_x^2 \psi - V \psi = 0$, where the propagation coordinate $z$ plays the role of time and the potential is proportional to the changes in index of refraction of the material, $V \propto -\Delta n$.

Now, the refractive index can be both real and complex, its imaginary part stays for gain and loss of signal in the system. $\mathcal{P}\mathcal{T}$-symmetry of the effective Schrödinger equation then reflects a balanced gain and loss that prevents the light from uncontrolled dimming or brightening.

In the following text, we will employ tools of quantum mechanics, in particular, a time-dependent Darboux transformation. We will focus on the models of optical waveguides described effectively by Schrödinger equation with $\mathcal{P}\mathcal{T}$-symmetric complex potential. The guided modes will be of our main interest. We will show that two-dimensional fluctuations of refractive index can induce a new guided mode in the multimode setting. Let us notice in this context that the time-dependent Darboux transformations were used recently in construction of non-Hermitian system in [24] and in the analysis of optical settings in [25, 26].

2. $\mathcal{P}\mathcal{T}$-Symmetry via time dependent SUSY

In this section, we briefly summarize the main aspects of the time-dependent Darboux transformation. We also discuss what are the restrictions imposed on the new systems by the requirement of $\mathcal{P}\mathcal{T}$-symmetry.

2.1. Time-dependent Darboux transformation and $\mathcal{P}\mathcal{T}$-symmetry

Let us take the following Schrödinger equation

$$S_0 \psi = i\partial_z \psi + \partial_x^2 \psi - V_0(x, z)\psi = 0, \quad x \in \mathbb{R}, \quad z \in \mathbb{R}. \quad (3)$$

We suppose that it is exactly solvable and its solutions are known. The potential term $V_0(x, z)$ has no singularities in $\mathbb{R}^2$ and it is sufficiently smooth. We can use the time-dependent Darboux transformation [27, 28, 29, 30, 31] to change the potential term in (3) while preserving its exact solvability.

The construction is based on the intertwining relation between two Schrödinger operators

$$S_1 \mathcal{L} = \mathcal{L} S_0. \quad (4)$$

It guarantees that the solutions of the equation $S_1 \phi = 0$ can be defined as $\phi = \mathcal{L} \psi$ provided that $S_0 \psi = 0$. The ansatz for the intertwining operator $\mathcal{L}$ is in the form a first order differential operator, $S_1$ is the Schrödinger operator with an altered potential term,

$$\mathcal{L} = L_1(z) \left[ \partial_x - \frac{\partial_x u(x, z)}{u(x, z)} \right], \quad S_1 = i\partial_z + \partial_x^2 - V_1(x, z). \quad (5)$$

Substituting (3) and (5) into (4), the intertwining relation can be satisfied provided that

$$V_1(x, z) = V_0(x, z) + i\partial_z \ln L_1(z) - 2\partial_x^2 \ln u(x, z) \quad (6)$$
and

\[ S_0 u(x, z) = c(z) u(x, z), \]  

see [27]. The function \( c(z) \) can alter only the phase of the solution but it has no influence on the form of the potential \( V_1 \), hence, we set it to zero, \( c(z) = 0 \). In the following, we denote by \( u \) the solution of \( S_0 u = 0 \) that is used for definition of the new potential (6) and of the intertwining operator (5). We will call it a transformation function.

In addition to the relations (6) and (7), the functions \( u(x, z) \) and \( L_1(z) \) are required to be nodeless. Otherwise, the transformation \( \mathcal{L} \) would be singular and it could not map the states from the domain of \( S_0 \) into the domain of \( S_1 \). When both \( \mathcal{L} \) and \( V_1 \) are regular, the formula (4) guarantees that the solutions of \( S_1 \phi(x, z) = 0 \) can be constructed from the solutions \( \psi(x, z) \) of (3) by \( \mathcal{L} \),

\[ \phi(x, z) = \mathcal{L} \psi(x, z). \]  

Now, we can ask the new potential \( V_1 \) to be \( \mathcal{P} \mathcal{T} \)-symmetric. We define \( \mathcal{P} \) as the reflection with respect to an axis [25],

\[ \mathcal{P} f(x, z) = f(-x, -z). \]  

The antilinear operator \( \mathcal{T} \) is defined as

\[ \mathcal{T} f(x, z) = \overline{f(x, z)}. \]  

We take as granted that \( V_0 \) is \( \mathcal{P} \mathcal{T} \)-symmetric. The requirement of \( \mathcal{P} \mathcal{T} \)-symmetry of \( V_1 \),

\[ \mathcal{P} \mathcal{T} V_1(x, z) \mathcal{P} \mathcal{T} = V_1(x, z), \]  

will impose some restricts on the choice of the transformation function \( u(x, z) \) in dependence on the actual definition of \( \mathcal{P} \). The requirement (11) reduces to

\[ 2 \partial_z^2 \ln \frac{u(x, z)}{u(-x, -z)} = i \partial_z \ln \frac{L_1(z)}{L_1(-z)}. \]  

Having reviewed all the necessary tools, let us step now to the construction of the \( \mathcal{P} \mathcal{T} \)-symmetric multimode waveguide.

3. Pöschl-Teller potential well

The Pöschl-Teller potential well is given by the expression [32],

\[ V_0 = -\frac{\hbar^2 \alpha^2}{2m} \frac{\nu(\nu - 1)}{\cosh^2(\alpha x)}, \quad \nu > 1, \]  

the corresponding time independent Schrödinger equation is then

\[ -\frac{d^2}{dx^2} \psi(x) + \left( \kappa^2 - \alpha^2 \frac{\nu(\nu - 1)}{\cosh^2(\alpha x)} \right) \psi(x) = 0, \]  

where \( \kappa^2 = -2mE/\hbar^2 \). For this work we are interested in solutions with negative energy, so we will consider \( \kappa \) as a real number. The general solution can be written as a superposition of two linearly independent solutions, one even and the other odd:

\[ \psi_e(x) = \cosh^\nu(\alpha x) \, _2F_1 \left( a, b; \frac{1}{2}; -\sinh^2(\alpha x) \right), \]  

\[ \psi_o(x) = \cosh^\nu(\alpha x) \sinh(\alpha x) \, _2F_1 \left( a + \frac{1}{2}, b + \frac{1}{2}; \frac{3}{2}; -\sinh^2(\alpha x) \right), \]
where the function \(2F_1(\cdot; \cdot; \cdot; \cdot)\) is the hypergeometric function (see [33]) and the parameters \(a\) and \(b\) are

\[
a = \frac{1}{2} \left( \nu - \frac{\kappa}{\alpha} \right), \quad b = \frac{1}{2} \left( \nu + \frac{\kappa}{\alpha} \right).
\]  

(17)

The discrete energy spectrum associated with square integrable wave functions is

\[
E_n = -\frac{\hbar^2 \alpha^2}{2m} (\nu - 1 - n)^2,
\]

(18)

where \(n\) is a whole number and \(n \leq \nu - 1\). Unnormalized eigenfunctions are then given by (15) (or by (16)) with the change \(\kappa \rightarrow \alpha(\nu - 1 - n)\) if \(n\) is even (odd), respectively.

There exists the following formula for the hypergeometric function

\[
2F_1(a, b; c; z) = (1 - z)^{-a} 2F_1\left(a, c - b; c; \frac{z}{z-1}\right) = (1 - z)^{-b} 2F_1\left(b, c - a; c; \frac{z}{z-1}\right).
\]

(19)

It works on the overlap of the domains of convergence of the series (the argument \(z\) of the hypergeometric function has to satisfy \(|z| < 1\)). The solutions can then be rewritten using this formula into the following form

\[
\psi_e(x) = \cosh^{\nu-2a}(\alpha x) \ 2F_1\left(a, \frac{1}{2} - b; 1/2; \tanh^2(\alpha x)\right)
\]

\[
= \cosh^{\nu-2b}(\alpha x) \ 2F_1\left(b, \frac{1}{2} - a; 1/2; \tanh^2(\alpha x)\right),
\]

(20)

\[
\psi_o(x) = \sinh(\alpha x) \cosh^{\nu-2a-1}(\alpha x) \ 2F_1\left(a + \frac{1}{2}, 1 - b; \frac{3}{2}; \tanh^2(\alpha x)\right)
\]

\[
= \sinh(\alpha x) \cosh^{\nu-2b-1}(\alpha x) \ 2F_1\left(b + \frac{1}{2}, 1 - a; \frac{3}{2}; \tanh^2(\alpha x)\right).
\]

(21)

These forms are more elegant because the argument stays within the radius of convergence of the hypergeometric function.

Moreover, the asymptotic behavior of the functions \(\psi_e\) and \(\psi_o\) when \(|x| \rightarrow \infty\) are well known [32]:

\[
\psi_e(x) \rightarrow 2^{-\nu} \Gamma\left(\frac{1}{2}\right) \left\{ \frac{\Gamma(b - a)}{\Gamma(b) \Gamma(1/2 - a)} 2^{2a} e^{\kappa |x|} + \frac{\Gamma(a - b)}{\Gamma(a) \Gamma(1/2 - b)} 2^{2b+1} e^{-\kappa |x|} \right\},
\]

(22)

\[
\psi_o(x) \rightarrow \pm 2^{-(\nu+1)} \Gamma\left(\frac{3}{2}\right) \left\{ \frac{\Gamma(b - a)}{\Gamma(b + 1/2) \Gamma(1 - a)} 2^{2a+1} e^{\kappa |x|} \right. \right.
\]

\[
+ \frac{\Gamma(a - b)}{\Gamma(a + 1/2) \Gamma(1 - b)} 2^{2b+1} e^{-\kappa |x|} \left\}. \right\}
\]

(23)

For the examples it is useful to know that the Wronskian \(W(\psi_e, \psi_o) = 1\), when the same value of \(\kappa\) is involved in both solutions.

4. Multimode \(\mathcal{PT}\)-symmetric waveguides

In this section we will construct a family of \(\mathcal{PT}\)-symmetric multimode waveguides based on the Pöschl-Teller potential well, moreover, the expression of the guided modes will be given. Besides
the finite number of bounded stationary states of Pöschl-Teller system that transform into the
guided modes of the two-dimensional $\mathcal{PT}$-symmetric optical setting, we will focus on the non
standard “missing” state whose analog is missing in the original system.

To perform a Darboux transformation, we need to choose a transformation function $u$ fulfilling
(7), (12) and $u(x, z) \not= 0$. To satisfy the first requirement let us start out from a superposition
of $N + M$ stationary solutions $\psi_{e}$ and $\psi_{o}$:

$$u(x, z) = \sum_{j=1}^{N} A_{j} \psi_{e_{j}} e^{i \kappa_{j} z} + i \sum_{\ell=1}^{M} B_{\ell} \psi_{o_{\ell}} e^{i \kappa_{\ell} z},$$

(24)

where $A_{j}$, and $B_{\ell}$ are $N + M$ real constants. Since solutions $\psi_{e}$ and $\psi_{o}$ depend implicitly on $\kappa$, we added the subindex $j$ and $\ell$ respectively. Here, we are only considering negative energy solutions. The imaginary unit is placed to guarantee the $\mathcal{PT}$-symmetry of the transformation solution, but we still need to define $L_{1}(z)$ to satisfy (12). To show that the selected $u$ does not vanish at any $(x, z)$ point is not trivial and it will be shown in explicit examples.

4.1. Example 1

Let us consider the following superposition

$$u(x, z) = A_{1} \psi_{e_{1}} e^{i \kappa_{1} z} + A_{2} \psi_{e_{2}} e^{i \kappa_{2} z},$$

(25)

We will take $L_{1}(z) = 1$, then (12) is satisfied. We still need to choose $A_{1}$ and $A_{2}$ such that $u$
has no zeros. In the appendix we show that $u$ is nodeless when

$$\kappa_{2} > \kappa_{1} \geq \nu, \quad |A_{2}| > |A_{1}|.$$

(26)

The potential $V_{1}$ can be calculated using (6) and (25). To obtain the guided modes we apply
the operator (5) onto every bound state of the Pöschl-Teller system. Their explicit expressions
are lengthy to present them here but their plots are shown in Fig 1.

A missing state has to be found in a special way [25]. Let us propose a preimage $v$ with the following expression:

$$v(x, z) = \left( \tilde{A}_{1} \psi_{e_{1}} + i \tilde{B}_{1} \psi_{o_{1}} \right) e^{i \kappa_{1} z} + \left( \tilde{A}_{2} \psi_{e_{2}} + i \tilde{B}_{2} \psi_{o_{2}} \right) e^{i \kappa_{2} z},$$

(27)

where $\tilde{A}_{1}$, $\tilde{B}_{1}$, $\tilde{A}_{2}$, $\tilde{B}_{2}$ are constants to be found. Notice that the constants $\kappa_{1}$ and $\kappa_{2}$ are the same as for the transformation function $u$. The function $v(x, z)$ is solution of the equation

$$i \partial_{t} v + \partial_{x}^{2} v - V_{0} v = 0.$$

The condition to fix the constants in $v$ is that the function $\mathcal{L} v = -W(u, v)/u$
vanishes when $|x| \to \infty$. Direct application of $\mathcal{L}$ onto $v$ leads to

$$\mathcal{L} v = -\frac{1}{u} \left( i A_{1} \tilde{B}_{1} e^{2 i \kappa_{1} z} + i A_{2} \tilde{B}_{2} e^{2 i \kappa_{2} z} \right) + \frac{1}{u} \left( A_{1} \tilde{A}_{2} - \tilde{A}_{1} A_{2} \right) W(\psi_{e_{1}}, \psi_{e_{2}})$$

$$+ i A_{1} \tilde{B}_{2} W(\psi_{e_{1}}, \psi_{o_{2}}) - i A_{2} \tilde{B}_{1} W(\psi_{o_{1}}, \psi_{e_{2}}) e^{(\kappa_{1}^{2} + \kappa_{2}^{2}) z}.$$  

(28)

The first term will vanish at $|x| \to \infty$ because $u \to e^{\kappa_{2} x}$. The three Wronskians behave as
$e^{(\kappa_{1}^{2} + \kappa_{2}^{2}) x}$ for large $x$. By fixing either $\tilde{A}_{1} = A_{2} = 0$ or $\tilde{A}_{1}/A_{2} = A_{1}/A_{2}$ the coefficient of $W(\psi_{e_{1}}, \psi_{e_{2}})$ will be zero, avoiding part of the problem. To deal with divergences of the second
and third Wronskians, coefficients $\tilde{B}_{1}$ and $\tilde{B}_{2}$ have to be carefully selected. Let us first define the following constants:

$$C_{e} = 2^{-\nu} \Gamma \left( \frac{1}{2} \right) \frac{\Gamma(b - a)}{\Gamma(b) \Gamma(\frac{1}{2} - a)} 2^{2a}, \quad C_{o} = 2^{-(\nu + 1)} \Gamma \left( \frac{3}{2} \right) \frac{\Gamma(b - a)}{\Gamma(b + \frac{1}{2}) \Gamma(1 - a)} 2^{2a + 1}.$$  

(29)
Then, using (22) and (23), we can obtain corresponding expressions for $W(\psi_{e,1}, \psi_{o,2})$ and $W(\psi_{o,1}, \psi_{e,2})$ at large values of $x$:

$$W(\psi_{e,1}, \psi_{o,2}) \to (\kappa_2 - \kappa_1)C_{e,1}C_{o,2}e^{(\kappa_1 + \kappa_2)x},$$  

(30)

$$W(\psi_{o,1}, \psi_{e,2}) \to (\kappa_2 - \kappa_1)C_{e,2}C_{o,1}e^{(\kappa_1 + \kappa_2)x}.$$  

(31)

To cancel the divergent behavior of the last two Wronskians in (28), $\tilde{B}_1$ and $\tilde{B}_2$ must fulfill

$$\frac{\tilde{B}_2}{\tilde{B}_1} = \frac{C_{e,2}C_{o,1}A_2}{C_{e,1}C_{o,2}A_1}.$$  

(32)

In short, the function

$$v(x, z) = i\tilde{B}_1 \left[ \psi_{o,1}e^{i\kappa_1^2z} + \frac{C_{e,2}C_{o,1}A_2}{C_{e,1}C_{o,2}A_1} \psi_{o,2}e^{i\kappa_2^2z} \right]$$  

(33)

is the preimage of the so called missing state, and the missing state, $Lv$, is an extra guided mode.

To illustrate the example, let us fix $\alpha = 1$, $\nu = 5/2$, then the Pöschl-Teller potential has two eigenfunctions, an even ground state with $\kappa_e = 3/2$ and an odd excited state with $\kappa_o = 1/2$. The plot of this potential and its eigenfunctions is shown in Fig. 1 (top left). Then, by fixing the parameters $\kappa_1 = 5/2$, $\kappa_2 = 2.8$, $A_1 = 1$, $A_2 = 2$ and using (25) and (6) the potential $V_1$ can be calculated, we present plots of its real (top center) and imaginary (top right) parts. The power densities of the guided modes are presented in the bottom row of Fig. 1. The power density of the missing state $Lv$ is shown in the bottom left, this plot shows explicitly a common phenomenon seen in $\mathcal{PT}$-symmetric system known as ‘power oscillations’. The power as a function of the propagation distance defined as $P_\phi(z) = \int_{-\infty}^{\infty} |\phi(x, z)|^2 dx$ is not a conserved quantity as opposed to what happens in Hermitian systems, in some regions the beam extracts energy from the medium and in others it loses energy (see [2, 3, 4, 25]). Here, the power oscillates from a maximum to a $\sim 13\%$ of this value. The power densities of the other two guided modes are shown in the bottom center and the bottom right, they correspond to the mapped eigenfunctions $L\psi_e e^{i\kappa_2^2z}$ and $L\psi_o e^{i\kappa_2^2z}$, respectively.

4.2. Example 2

As a second example let us take a different choice of the transformation function,

$$u(x, z) = iB_1\psi_{o,1}e^{i\kappa_1^2z} + A_2\psi_{e,2}e^{i\kappa_2^2z},$$  

(34)

and again, let us fix $L(z) = 1$ and $\kappa_1 < \kappa_2$, then the transformation functions fulfills (7) and (12). It is necessary to find parameters $\kappa_1$, $\kappa_2$, $B_1$, $A_2$ such that $u(x, z) \neq 0$, see the Appendix where discussion on the admissible choice of constants is presented. Then the potential $V_1$ can be again calculated using (6) and (34). To obtain the guided modes we apply the operator (5) onto every bound state of the Pöschl-Teller system. Explicit expressions are too long to be presented here but plots can be found in Fig 2.

A missing state can be also be found for this system. The technique is similar to previous example, first we provide a general expression for the preimage $v$, and then we fix the constants in such a way $Lv$ vanishes when $|x| \to \infty$. Let us start out from

$$v(x, z) = (\hat{A}_1\psi_{e,1} + i\hat{B}_1\psi_{o,1}) e^{i\kappa_1^2z} + (\hat{A}_2\psi_{e,2} + i\hat{B}_2\psi_{o,2}) e^{i\kappa_2^2z}.$$  

(35)
Figure 1. A confining multimode $\mathcal{PT}$-symmetric wave guide. A plot of the Pöschl-Teller potential where $\alpha = 1$, $\nu = 5/2$ along with its corresponding two eigenfunctions is presented in the top left frame. In top center and top right we show 3D plots of the real and imaginary parts of the potential $V$ where $u$ is taken as (25) with $\kappa_1 = 5/2$, $\kappa_2 = 2.8$, $A_1 = 1$, $A_2 = 2$. The intensity of the missing state $|L\psi|^2$ or extra guided mode is shown (bottom left). The intensities of the guided modes of the form $L\psi_e e^{i\kappa_2 z}$ and $L\psi_o e^{i\kappa_1 z}$ are also plotted (bottom center and bottom right, respectively), where $\kappa_e = 3/2$, $\kappa_o = 1/2$.

Notice again that $\kappa_1$ and $\kappa_2$ are the same as for $u$. Then, $Lv$ reads

$$Lv = \frac{i}{u} \left( B_1 \tilde{A}_1 e^{2i\kappa_1^2 z} - A_2 \tilde{B}_2 e^{2i\kappa_2^2 z} \right) + \frac{1}{u} \left[ i \left( A_2 \tilde{B}_1 - \tilde{A}_2 B_1 \right) W(\psi_{o,1}, \psi_{e,2}) + A_2 \tilde{A}_1 W(\psi_{e,1}, \psi_{e,2}) + B_1 B_2 W(\psi_{o,1}, \psi_{o,2}) \right] e^{i(\kappa_1^2 + \kappa_2^2)z}. \tag{36}$$

The first term vanishes at large $x$. Since all three Wronskians behaves as $e^{(\kappa_1 + \kappa_2)x}$, we must fix the constants $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$ such that they cancel each other. The coefficient of $W(\psi_{o,1}, \psi_{e,2})$ can be set equal to zero by either setting $\tilde{B}_1 = \tilde{A}_2 = 0$ or $\tilde{B}_1/\tilde{A}_2 = B_1/A_2$, both choices lead to the same answer. Now, using (22), (23) and abbreviations (29), the asymptotic behavior of the second and third Wronskinas in (36) at large $x$ can be expressed as:

$$W(\psi_{e,1}, \psi_{e,2}) \to (\kappa_2 - \kappa_1) C_{e,1} C_{e,2} e^{(\kappa_1 + \kappa_2)x}, \tag{37}$$

$$W(\psi_{o,1}, \psi_{o,2}) \to (\kappa_2 - \kappa_1) C_{o,1} C_{o,2} e^{(\kappa_1 + \kappa_2)x}. \tag{38}$$

then, by fixing

$$\frac{\tilde{A}_1}{B_2} = \frac{C_{o,1} C_{o,2} B_1}{C_{e,1} C_{e,2} A_2} \tag{39}$$

the function $Lv$ vanishes at $x \to \infty$. A similar analysis when $x \to -\infty$ gives the same conditions for the constants $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_1$. Thus, the function

$$v(x, x) = \tilde{B}_2 \left( \frac{C_{o,1} C_{o,2} B_1}{C_{e,1} C_{e,2} A_2} \psi_{e,1} e^{i\kappa_1^2 z} + i \psi_{o,2} e^{i\kappa_2^2 z} \right) \tag{40}$$
Figure 2. A confining multimode $\mathcal{PT}$-symmetric wave guide. Top left and top center plots are the real and imaginary parts, respectively, of the potential $V_1$. The parameter used for the Pöschl-Teller potential are $\alpha = 1$, $\nu = 7/2$, and for $u$ the parameters are $B_1 = 1$, $\kappa_1 = 3.6$, $A_2 = 1.01$, $\kappa_2 = 3.7$. The top right is the power density of the missing state, $|\mathcal{L}\psi|^2$. In the bottom row we plotted the intensities of the guided modes of the form $\mathcal{L}\psi$, where $\psi$ are the stationary states of the Pöschl-Teller potential.

is the preimage of the extra guided mode $\mathcal{L}\psi$.

This example is illustrated in Fig. 2. To produce the plots we started from a Pöschl-Teller potential with three bound states, and with parameters $\alpha = 1$, $\nu = 7/2$. Then, the potential $V_1$ was calculated using (6) and (34), with parameters $B_1 = 1$, $\kappa_1 = 3.6$, $A_2 = 1.01$, $\kappa_2 = 3.7$. Top left and top center subfigures are plots of the real and imaginary parts of $V_1$. This system has a missing state $\mathcal{L}\psi$, its power density is plotted (top right). In the bottom row we can see three guided modes obtained by applying $\mathcal{L}$ onto the ground state (bottom left), first excited state (bottom center) and second excited state (bottom right) of the initial system.

5. Conclusions

In this article, we discussed the construction of $\mathcal{PT}$-symmetric multi-mode optical waveguides based on the Pöschl-Teller system where the guided modes are localized due to the two-dimensional complex fluctuations of the refractive index. The procedure was based on the time-dependent Darboux transformation. We discussed the existence and construction of the exceptional “missing” state, (33) or (40), which has no analogue in the initial system and it is obtained in a nonstandard manner. Furthermore, stationary states of the Pöschl-Teller system were transformed into guided modes in the optical waveguides.

In [25], the free-particle model was used as the initial “seed” system for the Darboux transformation. Consequently, the resulting $\mathcal{PT}$-symmetric potentials were manifestly reflectionless as the incident waves were not back-scattered by the inhomogeneities of the refraction index. In this article, the seed system described by (14) ceases to be reflectionless for generic values of $\nu$. Hence, its “Darboux image”, the multi-mode optical waveguide, ceases to be transparent as well for the light beams bouncing on the wave guide. Let us mention that scattering properties of multimode $\mathcal{PT}$-symmetric systems were also discussed in [34], yet in a different settings where the waves propagated in a strip of finite width.
Appendix

We will show that the transformation function $u$ used in the examples 4.1 and 4.2 does not vanish.

**Proof that $u \neq 0$ in example 4.1** The goal is to prove there is a set of parameters $\kappa_1$, $\kappa_2$, $A_1$, $A_2$ such that $u$ as in (25) is never zero. To this end we will show first that $\psi_{e,1} \leq c \psi_{e,2}$ for a positive real $c$ when $\nu \leq \kappa_1 < \kappa_2$.

Notice that $\kappa \geq \nu$ is associated with an energy below the ground state energy. According to Sturm’s oscillation theorem (see Theorem 3.2 in [35]), solutions of the time independent Schrödinger equation with such energies will have at most one zero. Since $\psi_e$ are even solutions and $\psi_e(0) = 1$, we can say that $\psi_e$ are nodeless and positive functions. We can also say that

\begin{align*}
\psi_e'(0) &= 0, \\
\psi_e''(x) &= \left(\kappa^2 - \frac{\nu(\nu-1)}{\cosh^2(\alpha x)}\right) \psi_e(x), \quad x \in \mathbb{R}.
\end{align*}

The first property is true because the derivative of an even function is an odd function. The second property comes from the Schrödinger equation taking into account that $\kappa^2 \geq \nu^2 > \frac{\nu(\nu-1)}{\cosh^2(\alpha x)}$. As a consequence, $\psi_e'(x) > 0$ for all $x > 0$. To proceed, let us quote here explicitly the following

**Theorem 3.4 in [35]:**

Let $v_1$ and $v_2$ satisfy

$$ v_1(x) \geq v_2(x) \geq 0, \quad x \geq a. \quad (41) $$

Consider the two equations:

$$ -y''_1 + v_1 y_1 = 0, \quad -y''_2 + v_2 y_2 = 0. \quad (42) $$

If $y_1$, $y_2$ are positive increasing solutions of (42), then there exists $c > 0$ such that

$$ y_2(x) \leq c y_1(x), \quad x \geq a. \quad (43) $$

In our case, we have $v_j = \kappa_j^2 - \nu(\nu-1)\text{sech}^2(\alpha x)$, $j = 1, 2$, $v_2 \geq v_1 \geq 0$. We showed above that $\psi_{e,\kappa_0}(x)$ are positive increasing functions for $x \geq 0$. Then the theorem implies that

$$ \psi_{e,\kappa_1} \leq c \psi_{e,\kappa_2} \quad (44) $$

for a positive real constant $c$. We know that $\psi_{e,\kappa_1}(0) = \psi_{e,\kappa_2}(0) = 1$, which means that $c$ has to satisfy $c \geq 1$.

Now, the condition $u \neq 0$ holds whenever $|A_1\psi_{e,1}| \neq |A_2\psi_{e,2}|$. We can estimate the left-hand side as $|A_1|\psi_{e,1}(x) \leq c|A_1|\psi_{e,2}(x)$, $x > 0$. Hence, the inequality holds (having in mind parity properties of the wave functions) whenever we fix the constants such that

$$ c|A_1| < |A_2|. \quad (45) $$

**Proof that $u \neq 0$ in example 4.2** The proof is similar to the one of the previous example. From the oscillation theorem, we can conclude that the odd function $\psi_o(x)$ is vanishing in a single point $x_0 = 0$, $\psi_o(0) = 0$. We can assume without loss of generality that $\psi_o(x) > 0$ for $x > 0$. We also have $\psi'_o(0) \geq 0$. It follows from the fact that $\psi_o(0) = 0$ and $\psi_o(x) > 0$ for $x > 0$. We can also conclude that $\psi''_o(x) > 0$ for $x > 0$. Indeed, we have

$$ \psi''_o(x) = \left(\kappa^2 - \frac{\nu(\nu-1)}{\cosh^2(\alpha x)}\right) \psi_o(x) > 0 \quad (46) $$
(we still assume that $\kappa \geq \nu > 1$). As the second derivative is positive for $x > 0$, the first derivative is positive and increasing on this interval and so is the function itself. We can use the Theorem 3.4 from [35] that gives us

$$
\psi_{o,\kappa_1}(x) \leq c \psi_{e,\kappa_2}(x), \quad x > 0.
$$

(47)

Then we can satisfy $|A_1 \psi_{o,1}| \neq |A_2 \psi_{e,2}|$ whenever

$$
c |A_1| < |A_2|.
$$

(48)

References

[1] Lax M, Louisell W H and McKnight W B 1975 Phys. Rev. A 11 1365
[2] Makris K G, El-Ganainy R, Christodoulides D N and Musslimani Z H 2008 Phys. Rev. Lett. 100 103904
[3] Regensburger A et.al. 2012 Nature 488 167
[4] Rüter Ch E et. al. 2010 Nat. Phys. 6 192
[5] Bender C M 2007 Rep. Prog. Phys. 70 947
[6] Mostafazadeh A 2010 Int. J. Geom. Meth. Mod. Phys. 7 1191
[7] Mostafazadeh A 2003 J. Phys. A 36 7081
[8] Bender C M, Brod J, Refig A and Reuter M 2004 J. Phys. A 37 10139
[9] Znojil M 2004 Rendic. Circ. Mater. Palermo, Ser. II, Suppl. 72 211
[10] Krejčířík D, Bila H and Znojil M 2006 J. Phys. A 39 10143
[11] Míri M-A, Heinrich M, El-Ganainy R and Christodoulides D N 2013 Phys. Rev. Lett. 110 233902
[12] Míri M-A, Heinrich M and Christodoulides D N 2013 Phys. Rev. A 87 043819
[13] Heinrich M et al. 2014 Opt. Lett. 39 6130
[14] Heinrich M et al. 2014 Nat. Commun. 5 3698
[15] Laba H P and Tkachuk V M 2014 Phys. Rev. A 89 033826
[16] Midya B 2014 Phys. Rev. A 89 032116
[17] Correa F, Jakubský V and Plyushchay M 2015 Phys. Rev. A 92 023830
[18] Longhi S 2015 Opt. Lett. 40 463
[19] Longhi S 2015 J. Opt. 17 045803
[20] Macha A, Llorente R and Garcia-Meca C 2018 Phys. Rev. App. 9 014024
[21] Cruz y Cruz S and Razo R 2015 J. Phys. Conf. Ser. 624 012018
[22] Cruz y Cruz S and Rosas-Ortiz O 2015 Adv. Math. Phys. 2015 281472
[23] Cruz y Cruz S and Gress Z 2017 Ann. Phys. 383 257
[24] Cen J, Fring A and Frith T 2019 J. Phys. A: Math. Theor. 52 115302
[25] Contreras-Astorga A and Jakubský V 2019 Phys. Rev. A 99 053812
[26] Cruz y Cruz S and Razo R 2019 J. Phys. Conf. Ser. 1194 012091
[27] Bagrov V G and Samsonov B F 1996 Phys. Lett. A 210 60
[28] Cannata F, Ioffe M, Junker G and Nishnianidze D 1999 J. Phys. A: Math. Gen. 32 3583
[29] Suzko AA and Schulze-Halberg A 2009 J. Phys. A: Math. Theor. 42 295203
[30] Zelaya K and Rosas-Ortiz O 2017 J. Phys. Conf. Ser. 839 012018
[31] Contreras-Astorga A 2017 J. Phys. Conf. Ser. 839 012019
[32] Flügge S 1999 Practical quantum mechanics (Berlin: Springer-Verlag)
[33] Stegun I A and Abramowitz M 1964 Handbook of mathematical functions with formulas, graphs, and mathematical tables (New York: Dover books on intermediate advanced mathematics. US. Nat. Bureau Stand.)
[34] Ge L, Makris K G, Christodoulides D N and Feng L 2015 Phys. Rev. A 92 062135
[35] Berezin F A and Shubin M A 1991 The Schrödinger Equation (Dordrecht: Kluwer Academic Publishers)