Phantom Energy with Variable $G$ and $\Lambda$

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We have investigated a cosmological model of a phantom energy with a variable cosmological constant ($\Lambda$) depending on the energy density ($\rho$) as $\Lambda \propto \rho^{-\alpha}$, $\alpha = \text{const.}$ and a variable gravitational constant ($G$). The model requires $\alpha < 0$ and a negative gravitational constant. A negative gravitational constant may forbid black holes to form a particle horizon in a background of phantom energy. This implies that black holes are naked, and consequently the Cosmic Censorship theorem is violated. The cosmological constant evolves with time as, $\Lambda \propto t^{-2}$. For $\omega > -1$ and $\alpha < -1$ the cosmological constant, $\Lambda < 0$, $G > 0$ and $\rho$ decrease with cosmic expansion. For ordinary matter (or dark matter), i.e., $\omega > -1$ we have $-1 < \alpha < 0$ and $\beta > 0$ so that $G > 0$ increases with time and $\rho$ decreases with time. Cosmic acceleration with dust particles is granted provided $-\frac{2}{3} < \alpha < 0$ and $\Lambda > 0$.

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I. INTRODUCTION

Cosmologists have wondered whether our present universe will eventually re-collapse and end with a Big Crunch, or expand indefinitely and eventually becomes cold and empty. However, recent evidence from supernovae type I ushers into a flat universe, possibly with a cosmological constant or some other sort of negative-pressure dark energy, has suggested that our fate is accelerating (Perlmutter et al., Riess et al.). However, the data may actually be pointing toward an astonishingly different cosmic end game. Caldwell et al. explored the consequences that follow if the dark energy is a phantom energy, i.e., the sum of the pressure and energy density is negative. The positive phantom-energy density becomes infinite in a finite time, overcoming all other forms of matter, that will rapidly brings the epoch of cosmic structure to a halt (Dabrowski at al.). The phantom energy rips apart every bound matter before the Universe ends into a Big-Rip.

However, the phantom energy scenario does violate the the strong energy condition (SEC), a principle that keeps energies positive and imposes energy conservation on a global scale. It is the strong energy condition that helps to rule out wormholes, warp drives, and time machines. Dark energy requires an equation of state $p + 3\rho < 0$. The violation of the null energy condition (NEC) $p + \rho < 0$ results in energy flows faster than the speed of light. A phantom behavior is predicted by several scenarios, e.g., kinetically driven models (Chiba et al.) and some versions of braneworld cosmologies ( Sahni and Shtanov).

Another possibility for dark energy is an energy of a scalar field known as quintessence having an equation of state such that $-1 < \omega < -\frac{1}{3}$ (Caldwell et al., Ratra & Peebles, Wetterich, Turner & White, Caldwell). Assuming the quintessence field coupled minimally to gravity, one writes it lagrangian as,

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

with energy density and pressure given by

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi).$$

where a dot is a derivative w.r.t time. In the so-called tracker models the scalar field density (and its equation of state) remains close to that of the dominant background matter during most of cosmological evolution. The equation of state is given by

$$\frac{p}{\rho} = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}.$$
A comprehensive study of quintessence is investigated by Ratra and Peebles. However, A tracker potential of the form $V(\phi) \propto \phi^{-n}$, $n = \text{const.}$ is considered recently by Sahni. A minimally coupled scalar field to gravity has the general lagrangian of the form

$$L = P \equiv P(X, \phi),$$

where $X = \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi$, having an energy momentum tensor

$$T_{\mu\nu} = -\mathcal{L}g_{\mu\nu} + P' \partial_\mu \phi \partial_\nu \phi,$$

or

$$T_{\mu\nu} = P' \partial_\mu \phi \partial_\nu \phi - Pg_{\mu\nu}.$$

If $X$ is time-like vector then $T_{\mu\nu}$ is equivalent to that of a perfect fluid, $T_{\mu\nu} = (p + \rho) u_\mu u_\nu - pg_{\mu\nu}$ with energy density $\rho = 2XP' - P$, pressure $P$ and 4-velocity $u_\mu = \frac{\partial_\mu \phi}{\sqrt{2X}}$. Its equation of state amounts to

$$\omega = \frac{P}{\rho} = -1 + \frac{2X P'}{\rho} = -1 + \frac{P' \dot{\phi}^2}{\rho},$$

where $P' = \frac{\partial P}{\partial X}$. Since for dark matter $T_{00} = \rho > 0$, one has the condition $P' < 0$ for $\omega < -1$. The current observational data amounts to $\omega < -1$, however this posed a problem.

Moreover, one may include the possibility of an equation of state $p = -\rho$. This is attributed to existence of vacuum energy or the cosmological constant. At the present time it is difficult to tell which form of energy our universe consists of.

In this paper, we investigate the evolution of dark energy and phantom energy arising from the introduction of a cosmological constant that evolves as $\Lambda = \frac{33}{\rho}$, where $\alpha, \beta = \text{const.}$. With this assumption, a phantom energy arises whenever $p + \rho < 0$ and $\alpha > 0$. However, the gravitational constant becomes negative. In the present model, the dark energy models do no necessarily require the condition $p > -\frac{1}{3} \rho$. Cosmic acceleration is generated for $\alpha > 0$, $-1 < \alpha < 0$ and $\alpha < -1$. Phantom energy with variable $G$ has been recently considered by Stefancic. We have shown that during the evolution of the domain-walls the cosmological constant flips it sign. We have seen that the whole evolution of the universe is characterized by the the two constants $\alpha$ and $\beta$.

II. THE MODEL

Consider the Einstein-Hilbert action with a cosmological constant term ($\Lambda$)

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{g} (R + 2\Lambda) + S_{\text{matter}}$$

(4)

The variation of the metric with respect to $g_{\mu\nu}$ with $f(R) = R - 2\Lambda$, gives (Amarzguioui et al.)

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

(5)

where $T_{\mu\nu}$ is the energy momentum tensor of the cosmic fluid.

For an ideal fluid one has

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu},$$

(6)

where $u_\mu, \rho, p$ are the velocity, density and pressure of the cosmic fluid. Contracting Eq.(5), using Eq.(6) and taking the 00 components give, the equation

$$R f'(R) - 2 f(R) + 8\pi G T_{00} = 0,$$

(7)

and

$$f'(R) R_{00} + \frac{1}{2} f(R) + 8\pi G T_{00} = 0,$$

(8)
with $T_{00} = \rho$, $T = \rho - 3p$ and $T_{ij} = -p$ for $i, j = 1, 2, 3$. For a flat Friedmann-Lematre-Robertson-Walker metric,
\[
ds^2 = dt^2 - a^2(t) \left( dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right),
\]
one has $R_{00} = -\frac{\dot{a}}{a}$ and $R = -6\left[(\frac{\dot{a}}{a})^2 + \frac{\ddot{a}}{a}\right]$, so that Eqs.(7) and (8) yield
\[
3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi \, G\rho + \Lambda, \quad (9)
\]
and the energy conservation equation reads,
\[
\dot{\rho} + 3 \left( \frac{\dot{a}}{a} \right) (\rho + p) = 0. \quad (11)
\]
The pressure $p$ and energy density $\rho$ of an ideal fluid are related by the equation of state,
\[
p = \omega \, \rho, \quad \omega = \text{const.} \quad (12)
\]
The Einstein field equation, with time-dependent $G$ and $\Lambda$, then yields two independent equations (Eqs.(9) and (10)) having the same form as in the standard model. Hence, we now allow $\Lambda$ and $G$ to vary with time, i.e., $\Lambda = \Lambda(t)$ and $G = G(t)$.
The Bianchi identity
\[
(R^\mu{}^\nu - \frac{1}{2} R g^\mu{}^\nu); \, \mu = -(8\pi G T^\mu{}^\nu + \Lambda g^\mu{}^\nu); \, \mu = 0, \quad (13)
\]
and Eqs.(6) and (12) imply that
\[
G \dot{\rho} + 3(1 + \omega) \rho G \frac{\dot{a}}{a} + \rho \ddot{G} + \frac{\dot{\Lambda}}{8\pi} = 0,
\]
so that the energy conservation, Eq.(11), entitles that (Beesham, Abdel Rahman, Arbab)
\[
8\pi \dot{G}\rho + \dot{\Lambda} = 0. \quad (14)
\]
We consider here the ansatz
\[
\Lambda = \frac{3\beta}{\rho^\alpha}, \quad \beta, \, \alpha = \text{const.} \quad (15)
\]
Integrating Eq.(11), using Eq.(12), we obtain
\[
\rho = A a^{-3(1+\omega)}, \quad A = \text{const.} \quad (16)
\]
Substituting this in Eq.(13) using Eq.(14), one gets
\[
G = \left( \frac{-3\alpha \beta}{8\pi (1 + \alpha) A^{1+\alpha}} \right) a^{3(1+\omega)(1+\alpha)}, \quad \alpha \neq -1. \quad (17)
\]
When $\alpha = 0$, Eq.(13) implies that $\Lambda = \text{const.}$ and $G = \text{const.}$. Substituting Eqs.(15) and (16) into Eq.(9) we obtain,
\[
H = \frac{\dot{a}}{a},
\]

\[
H^2 = \left( \frac{\beta}{1 + \alpha} \right) A^{-\alpha} a^{3(1+\omega)\alpha}, \quad \alpha \neq -1. \quad (18)
\]
or
\[
H^2 = \left( \frac{\beta}{1 + \alpha} \right) \rho^{-\alpha}, \quad \alpha \neq -1. \quad (19)
\]
Using Eq.(18), Eq.(16) can be written as

$$8\pi G = \left(\frac{-3\alpha\beta}{1+\alpha}\right) \rho^{-(\alpha+1)}, \quad \alpha \neq -1. \tag{20}$$

Integrating Eq.(18), one obtains,

$$a(t) = \left(\frac{F}{A}\right)^{-1/3(1+\omega)} t^{-2/3\alpha(1+\omega)}, \quad \alpha \neq -1, \tag{21}$$

where $F = \left[\frac{-3}{2}\alpha(1+\omega)((1+\omega)/\alpha)^{1/2}\right]^{2/\alpha}$ Using Eq.(21), Eq.(17) becomes

$$G = \frac{-3\alpha\beta}{8\pi (1+\alpha)} F^{-(1+\alpha)} t^{-2(1+\alpha)/\alpha}, \quad \alpha \neq -1. \tag{22}$$

Using Eq.(21), Eq.(16) reads

$$\rho = F t^{2/\alpha}, \quad \alpha \neq 0, -1, \tag{23}$$

which shows that the energy density of the phantom increases with expansion, since $\alpha > 0$. Using Eq.(23), Eq.(15) yields

$$\Lambda = \frac{3\beta}{F^\alpha} t^{-2}. \tag{24}$$

This form is found to emerge from many of the vacuum decaying models.

### III. PHANTOM ENERGY

For an expanding universe, we require that $-\alpha(1+\omega) > 0$. Since, $\alpha > 0$, one must have the relation $\omega + 1 < 0$. This is the familiar condition for the existence of phantom energy. This relation implies that $p < -\rho$. In this case, the energy density grows with time, as it is evident from Eq.(23). This is the condition for phantom energy. It is very interesting to see that such a cosmological constant variation leads to phantom energy solution. We, however, notice that phantom energy existence requires $G < 0$, so that $G > 0$ and $\Lambda < 0$. We, therefore, see that the phantom energy has negative gravity. One may attribute this to the anti-gravitating nature of phantom energy. Such a bizarre behavior could be the reason why phantom energy has a negative pressure, unlike the ordinary matter.

In the scalar field theory, the phantom energy is model by a field with a negative kinetic energy, but negative pressure and positive energy. In our present scenario, $G < 0$ and $p < 0$, but $\rho > 0$. Hence, the two pictures of phantom energy evolution could be equivalent. However, in our present scenario both the gravitational and the cosmological constants decrease with time. Unlike the standard phantom energy model, we see that as $t \to 0, a \to 0, \rho \to 0$; and as $t \to \infty, a \to \infty, \rho \to \infty$. Consider now the case $\omega = -\frac{3}{2}$ and $\alpha = \frac{2}{3}$. In this case, $\alpha \propto t^2$, and hence, $\rho \propto t^3$ and $G \propto t^{-5}$. An equivalent case corresponds to $\alpha = 1$ and $\omega = -\frac{4}{3}$. However, in the latter case $\rho \propto t^2$ and $G \propto t^{-4}$. The present observational data favor negative values for $\omega$ rather than positive ones. In particular, models with $-1.62 < \omega < -0.74$ are favored observationally (Carroll at al.).

### IV. NEGATIVE GRAVITATIONAL CONSTANT

For phantom energy to develop, we require the gravitational constant to be negative during its dominance. Phantom energy implies that $\rho + p < 0$ and this also implies $\rho + 3p < 0$. The gravitational potential ($V$) for matter having energy density($\rho$) and pressure ($p$), satisfies the Poisson equation

$$\nabla^2 V = 4\pi G(\rho + 3p).$$

For phantom energy one has

$$\rho + 3p < 0 \quad \text{and} \quad G < 0$$
and so that the above equation does not change sign. This would mean that this equation governs phantom too. A black hole metric (Schwarzschild) with phantom energy does not lead to a particle horizon, since the Schwarzschild metric

$$ds^2 = (1 - \frac{2GM}{rc^2}) dt^2 - \frac{dr^2}{(1 - \frac{2GM}{rc^2})} - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

will be defined everywhere (except at $r = 0$). Thus, a spherical phantom star of mass $M$ would not be bounded by an event horizon. Hence, a phantom energy prohibits a black hole to be covered by a horizon. Hence, black holes become only dark energy (as a background energy with $p < 0$ implies $\beta < 0$). For the two cases, both $G > 0$, Eq.(20) will be defined everywhere (except at $r = 0$). Thus, cosmic strings in this case evolves like radiation. For the two case the energy density evolves as, $\rho \propto t^{-1}$ and $\Lambda \propto t^{-1}$. Now consider the case $-1 < \alpha < 0$, $\beta > 0$ and $\omega > -1$. This is the case for ordinary matter (or dark energy). In this case, $G > 0$, $\Lambda > 0$. Eqs.(22) and (23) imply that $G$ increases with time and $\rho$ decreases with time. For $\alpha = -\frac{1}{2}$ and $\omega = 1$, the scale factor varies as, $a \propto t^{2/3}$. Thus, stiff matter mimics ordinary matter. In this case, $G \propto t^2$ and $\rho \propto t^{-4}$. We notice that in the present scenario one has the relation $G\rho \propto H^2$. Such a relation is known to satisfy the Machian cosmology (Arbab, 1997).

V. COSMIC ACCELERATION

The present observational data usher toward an accelerated expansion of the universe (Reiss, et al., Perlmutter et al.). The deceleration parameter is given by $q = -\frac{\ddot{a}}{a}H^2$. Using Eq.(21), this yields

$$q = -\frac{2 + 3\alpha(1 + \omega)}{2}.$$  \hspace{1cm} (25)

For an accelerating universe, one must have $q < 0$, i.e., $\alpha > \frac{2}{3(1+\omega)}$. For non-phantom energy one has $1 + \omega > -1$ so that $\alpha < 0$. We treat the case $\alpha = -1$ separately.

For $\omega > -1$ and $\alpha < -1$, the scale factor grows with time. For a positive gravitational constant, $G > 0$, Eq.(20) implies $\beta < 0$, so that the cosmological constant becomes negative, $\Lambda < 0$. Eqs.(22) and (23) imply that $G$ and $\rho$ decrease with cosmic expansion. For $\alpha = -2$ and $\omega = -\frac{1}{2}$ one has $a \propto t^{2/3}$. This dark energy mimics the evolution of dust particles. Similarly for $\alpha = -2$ and $\omega = -\frac{1}{4}$, the scale factor evolves as, $a \propto t^{1/2}$. Hence, cosmic strings in this case evolves like radiation. For the two case the energy density evolves as, $\rho \propto t^{-1}$ and $\Lambda \propto t^{-1}$.

A. Acceleration with ordinary matter

In this model, it is possible to have a cosmic acceleration in the present epoch with ordinary matter (dust or radiation). To my knowledge this solution has not been considered before. This arises due to the presence of the cosmological constant of the form suggested in Eq.(15). We consider here $\omega > 0$ and $\beta > 0$. In this case we have:

- dust: $\omega = 0 \Rightarrow \alpha > -\frac{2}{3}$,
- radiation: $\omega = 1 \Rightarrow \alpha > -\frac{1}{2}$.

For the two cases, both $G > 0$ and $\Lambda > 0$. Moreover, $G$ increases while $\Lambda$ decreases with cosmic time.
B. Acceleration with dark energy

In this case, we consider $-1 < \omega < 0$. Cosmic acceleration for strings-like and domain-walls like fluid respectively imply

- cosmic strings: $\omega = -\frac{1}{3} \Rightarrow \alpha > -1$.
- domain-walls: $\omega = -\frac{2}{3} \Rightarrow \alpha > -2$.

We observe that domain-walls proceeds in two different ways:

(i) when $\alpha < -1, \beta < 0$ for $G > 0$ so that $\Lambda < 0$.
(ii) when $\alpha > -1, \beta > 0$ for $G > 0$ so that $\Lambda > 0$. Hence, during the domain-walls evolution the cosmological constant changes its sign.

VI. NEW SOLUTION

We study in this section the case $\alpha = -1$. Eqs.(14) and (15) imply that

$$G = \frac{3\beta}{8\pi}(1 + \omega) \ln(Ca), \quad C = \text{const.} \quad (26)$$

This equation implies that when $a \to 0$, $G \to -\infty$ unless $\beta$ or $1 + \omega$ becomes negative. Physically this case is meaningless. Hence, one may say that in the case $\Lambda = 3\beta\rho$, the initial singularity is avoided provided the gravitational constant is allowed to vary with time. For an expanding universe the gravitational constant must increase with cosmic time. We notice that $G$ increase with time for $\beta > 0$ ($\Lambda > 0$) and $\omega > -1$ (ordinary/dark energy), or $\beta < 0$ ($\Lambda < 0$) and $\omega < -1$ (phantom energy). Upon using this equation, Eq.(9) becomes

$$\dot{a}^2 = \beta[1 + 3(1 + \omega) \ln(Ca)] Aa^{-(1+3\omega)}. \quad (27)$$

The solution of the above equation gives the time dependence of the gravitational constant and energy density.

VII. INFLATIONARY SOLUTION

We consider here the case $\omega = -1$. In this case Eq.(16) yields $\rho = \text{const.}$ and then Eq.(15) implies $\Lambda = \text{const.}$ Now Eq.(14) implies that $G = \text{const.}$ and hence Eq.(9) gives $a \propto \exp(H_0 t), \quad H_0 = \text{const.}$. This is the standard de-Sitter expansion. We notice from Eq.(26) that during inflation the gravitational constant vanishes. This may have assisted the universe to inflate with a constant expansion rate, $H = \sqrt{\frac{3}{8\pi}}$.

VIII. CONCLUDING REMARKS

We have considered in this paper a cosmological model with a cosmological constant varies as $\Lambda = \frac{3\beta}{\rho_0^\alpha}$. We have found that cosmic acceleration is guaranteed in radiation ($-\frac{1}{2} < \alpha < 0$) and matter ($-\frac{2}{3} < \alpha < 0$) dominated epochs with $\Lambda > 0$ and $G > 0$. For dark energies (cosmic-strings/domain-walls) cosmic acceleration occurs when $\alpha < -1$. Phantom energy with $\omega < -1$ is allowed provided $\alpha > 0$ and $G < 0$. The phantom energy density varies as, $\rho \propto t^{2/\alpha}$. For instance, for $\alpha = 1$ and $\omega = -\frac{4}{3}$ the scale factor increases as, $a \propto t^2$. The negative gravitational constant is tantamount to negative kinetic energy for phantom field. The case $\omega = -1$ gives the familiar de-Sitter inflationary solution.

IX. ACKNOWLEDGMENTS

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