Dynamic Stiffness of Non-Loaded and Loaded Beams

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Abstract. The problem of the bending stiffness of reinforced concrete beams is considered. The research so far has shown that static stiffness (determined on the basis of deflections) is lower than dynamic stiffness (determined on the basis of eigenfrequencies). The analyses focus on eigenfrequencies determined in different ways. The dynamic rigidities of cracked RC beams were found to differ between non-loaded and loaded beams.

1. Introduction
By measuring the parameters of a structure one can assess its condition. One of the basic parameters of reinforced concrete beams, measured for diagnostic purposes, are their deflections. Knowing the deflections, the static scheme and the load one can calculate a beam’s bending stiffness $EI_{\text{eff}}$ which provides valuable information about the condition of the beam. In the case of RC beams under bending, stiffness $EI_{\text{eff}}$ is not constant and depends on the degree of load advancement – a ratio of the moment acting on the beam to the latter’s resistance to bending. The stiffness of an element is not constant (decreases) mainly due to cracking (resulting in stiffness degradation).

There are several theories on the basis of which one can calculate beam stiffness $EI_{\text{eff}}$, taking into account cracking. Most of the theories (e.g. [1, 2]) take into account a combination of the beam’s stiffness in stage I (before cracking) and in stage II (after cracking). According to [1], the relation for stiffness is as follows:

$$EI_{\text{eff}} = \frac{E_{cm} I_{II}}{1 - \beta \left( \frac{M_{cr}}{M} \right)^2 \left( 1 - \frac{I_{II}}{I_I} \right)},$$  \hspace{1cm} (1)

where: $E_{cm}$ – the mean Young’s modulus of the concrete, $M_{cr}$ – the cracking moment, $M$ – the maximum bending moment acting on the beam, $I_I, I_{II}$ – moments of inertia in respectively stage I and II, $\beta$ – a coefficient depending on the type of load (1.0 for single short-term loading, 0.5 for sustained loads or many cycles of repeated loading).

According to [2], the stiffness of a beam can be calculated from the following relation:

$$EI_{\text{eff}} = E_{cm} \left( \frac{M_{cr}}{M} \right)^3 I_I + \left[ 1 - \left( \frac{\alpha M_{cr}}{M} \right)^3 \right] I_{II},$$ \hspace{1cm} (2)

Another parameter measured when evaluating the condition of a structure (e.g. by means of operational modal analysis [3]) is eigenfrequency. Knowing the eigenfrequency for the given loading...
diagram and the vibrating mass one can calculate dynamic stiffness $EI_d$. The latter is different from static stiffness $EI_{eff}$. Numerous papers (e.g. [4-7]) have been devoted to determining the dynamic stiffness of RC beams on the basis of eigenfrequencies. Usually it is proposed (on the basis of a series of test results) to modify relation (1) or (2) so that they can be used to calculate dynamic stiffness. However, it should be noted that because of the methodology used the analyses made by different researchers yield, in most cases, different results.

Two most often used methods of determining the eigenfrequencies of RC beams are discussed in this paper. In one of the methods the measurements are performed on elements loaded with solely their dead weight. In the other method, ballast masses are used besides the dead load.

2. Dynamic stiffness acc. to [6]

As part of this research experiments on suspended elements (free-free beams) were conducted in order to determine their dynamic stiffness. The eigenfrequencies of the beams, in which only their dead load was the vibrating mass, were recorded. The eigenfrequencies would decrease as the beams were loaded at an increasing rate in each step of the three-point bending test (figure 1a). Dynamic tests were conducted after the beams had been completely unloaded and suspended on elastic ropes (figure 1b). The advantages of testing beams in this way are presented in [8]. One of the advantages is that the influences of the decreasing stiffness and the increasing mass are separated. Thus the influence of the increasing forces of inertia was eliminated in the experiments.

![Figure 1. Test stands acc. to [6] (dimensions in mm).](image)

On the basis of the experiments the following modified form of relation (1) was proposed:

$$EI_d = \frac{\alpha_B E_{cm} I_{II}}{1 - \beta \left( \frac{M_{cr}}{M} \right)^{2\beta_B} \left( 1 - \frac{I_{II}}{I_I} \right)},$$

(3)

where: $\alpha_B$ – a coefficient of conversion from the static Young’s modulus of concrete to the dynamic Young’s modulus, $\beta_B$ – a coefficient modifying the character of the change in stiffness after the beam cracks (in stage II).

Studies (e.g. [9, 10]) have shown that coefficient $\alpha_B$ for ordinary concrete ranges from 1.10 to 1.30 (hence $E_{cm} < E_d$). This is mainly due to the fact that static Young’s modulus $E_{cm}$ is determined through the classic axial compression test on a cylindrical specimen subjected to a load of $0.1 f_c - 0.3 f_c$ (where $f_c$ is the compressive strength of the concrete). In the present experiments the dynamic Young’s modulus of the concrete was determined using specimens loaded with only their dead weight. As it is known, the Young’s modulus of concrete depends on the degree of strain and it is inversely proportional to the latter.
Parameter $\beta_B$ pertains to the rate of dynamic stiffness decrease after cracking. Studies carried out by the author showed that the decrease was the more intensive, the lower the reinforcement ratio. The economic tension reinforcement ratio of a typical RC beam is in a range of 0.7%-1.5%. As part of the present research two series of beams with tension reinforcement ratio $\rho = 0.65\%$ (9 elements) and $1.38\%$ (3 elements) were tested. One can assume that the above values come from the extremes of the recommended economic reinforcement ratio interval. If parameter $\beta_B$ is assumed to vary linearly within the interval, the following relation can be proposed:

$$\beta_B = -2.22 \rho + 0.0669 .$$  \hfill (4)

Figure 2 shows static and dynamic rigidities as a function of the bending moment, calculated from relations (1) and (3) for the two extreme reinforcement ratios. The values of breaking moment $M_R$ and cracking moment $M_{cr}$ have been marked on the diagram.

![Figure 2. Static and dynamic stiffness versus bending moment.](image)

3. Dynamic stiffness acc. to [4]
Tests of the eigenfrequencies of RC beams conducted differently than in [6] are reported in [4]. In the latter case, four-point bending tests with ballast masses were carried out, which means that the eigenfrequencies of the loaded beams were measured. The load would be increased stepwise after each recording of eigenfrequencies and deflections. The test stand and one of the three tested series of beams are schematically shown in fig. 3. The beams in the other two series were not considered since their reinforcement ratios were much higher (1.76% and 2.64%). Otherwise the comparisons presented further in this paper could not have been made due to the fact that the relations given in section 2 do not take into account so high reinforcement ratios.

![Figure 3. Test stand and beam, acc. to [4] (dimensions in mm).](image)
The tested beams were made of concrete with mean compressive strength $f_{cm} = 41.37 \text{ MPa}$ and mean static Young’s modulus $E_{cm} = 30.48 \text{ GPa}$. The above values were determined using cylindrical specimens made of the same batch of concrete mixture and cured in the same conditions as the beams. Ribbed reinforcing bars made of steel with mean yield strength $f_{ym} = 276 \text{ MPa}$ and mean Young’s modulus $E_{sm} = 200 \text{ GPa}$ were used as the reinforcement.

On the basis of the test results relation (2) was modified to the form:

$$EI_d = E_{cm} \left[ \left( \frac{\alpha M_{cr}}{M} \right) I_1 + \left( 1 - \frac{\alpha M_{cr}}{M} \right) I_B \right].$$

The tests showed that empirical coefficient $\alpha$ was in an interval of 0.6-0.8. Moreover, eigenfrequencies would decrease during the tests due to the degradation of stiffness and to the simultaneous increase in the vibrating mass (as the load on the hangers was increased). The authors of [4] took this into account by giving a relation for dynamic stiffness (calculated on the basis of experimental eigenfrequencies), covering the beam’s dead weight and the suspended ballast masses.

4. Comparative analyses

Comparative analyses concerning the relations presented in [6] were carried out using the experimental results and the relations given in [4]. Considering the fairly representative test elements, the similar dimensions of the elements and the similar properties of the concrete and the steel used in [4] and [6], such comparative analyses are justified. Figure 4 shows the dynamic stiffness of the beams, calculated on the basis of the measured eigenfrequencies of two tested elements [4]. The curves in the diagram are described by relation (5) for the extreme parameter $\alpha$ values (0.6 and 0.8). Using linear regression it was found that the experimental results are best described by relation (5) with parameter $\alpha = 0.785$, which is also shown in the diagram.

For the beam shown in figure 3 dynamic stiffness $EI_d$ values were calculated from relations (3) and (5), while static stiffness $EI_{eff}$ values were calculated from relations (1) and (2). The calculation results are compared in the diagram in figure 5.
The dynamic stiffness calculated from formula (3) has the highest value and the smallest gradient of decrease after cracking. The dynamic stiffness value calculated from relation (5) is lower. The static rigidities have the lowest values. The static stiffness values calculated from respectively relation (1) and (2) are comparable. However, this fact is not elaborated since static stiffness is not the main subject of this paper.

5. Conclusions
The following conclusions can be drawn from the analyses:

- The dynamic stiffness values determined on the basis of the eigenfrequencies of the RC beam are higher than the values of its static stiffness determined on the basis of its deflections.
- The dynamic stiffness value determined on the basis of the eigenfrequency of the non-loaded RC beam with cracks is higher than the dynamic stiffness value of the RC beam with cracks and ballast masses.
- The observed behaviour of the beams indicates that the influences of stiffness and inertia on eigenfrequencies interact, which is reflected in the dynamic stiffness values. The author does not mean here the natural decrease in eigenfrequencies as the vibrating mass is increased (observed for any type of structure), but the effect of this mass on the stiffness of the structure.
- The presented results can be exploited in, e.g., the diagnosis of structures, based on eigenfrequency measurements.

The analyses should be regarded as preliminary, nevertheless their results indicate that such studies are worthwhile. Currently experiments on a larger number of beams differing in their reinforcement ratios are being planned.

References
[1] EN 1992-1-1: Design of concrete structures - Part 1-1: General rules and rules for buildings
[2] ACI 380: Building Code Requirements for Reinforced Concrete Buildings
[3] Grosel J, Sawicki W and Pakos W 2014 Application of classical and operational modal analysis for examination of engineering structures, Procedia Engineering 91 pp 136–41
[4] Jerath S and Shibani M M 1985 Dynamic Stiffness and Vibration of Reinforced Concrete Beams, ACI Journal 82–18 pp 196–202
[5] Johns K C and Belanger M D 1981 Dynamic Stiffness of Concrete Beams, ACI Journal 78–18 pp 201–5
[6] Musial M 2012 Static and dynamic stiffness of reinforced concrete beams, Archives of Civil and Mechanical Engineering 12 pp 186–91
[7] Numayr K S, Al-Jallamdeh S A and Al-Akhras N M 2002 Dynamic stiffness of reinforced concrete beams, Structures & Buildings 156 pp 373–9
[8] Musial M 2016 Laboratory tests of reinforced concrete beams with the use of operational modal analysis, Materials Science Forum 866 pp 124–8
[9] Musial M and Grosel J 2016 Determining the Young's modulus of concrete by measuring the eigenfrequencies of concrete and reinforced concrete beams, Construction and Building Materials 121 pp 44–52
[10] Musial M 2015 Young's modulus test in two directions on the basis of eigenfrequencies in concrete beams, Technical Transactions 3-B pp 133–42