On The Hamiltonian Formalism Of The Tetrad-Gravity With Fermions

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Abstract

We extend the analysis of the Hamiltonian formalism of the d-dimensional tetrad-connection gravity to the fermionic field by fixing the non-dynamic part of the spatial connection to zero [1]. Although the reduced phase space is equipped with complicated Dirac brackets, the first-class constraints which generate the diffeomorphisms and the Lorentz transformations satisfy a closed algebra with structural constants analogous to that of the pure gravity. We also show the existence of a canonical transformation leading to a new reduced phase space equipped with Dirac brackets having a canonical form leading to the same algebra of the first-class constraints.

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1 Introduction

It is well-known that along the development of various canonical gravity formalisms, the interest on incorporating matter to the theory has immensely
increased. In \cite{2}, the analysis of the covariant Hamiltonian of the tetrad-gravity coupled to the fermionic field was carried out in the second order formalism. Note that in the presence of fermions the torsion is not zero on-shell. This means that the first order formalism is not equivalent to the correspondent second order Lagrangian obtained by substituting the torsion-free spin connection in place of the connection, unless modifying the action by an appropriate term \cite{3}.

The first order Hamiltonian formalism of the Ashtekar complex connection \cite{4} coupled to the fermionic matter has been done in \cite{5}. To avoid the complex nature of the Ashtekar connection the action is modified by the Holst term \cite{6} depending on a new parameter known as the Barbero-Immirzi parameter \cite{7}. This permits the construction of the Hilbert space in loop quantum gravity by reducing, in the time gauge, the Lorentz’s manifest invariance of the action to the compact SO(3) subgroup \cite{8}. The Holst term does not affect the classical equation of motion but this unphysical parameter appears in the expressions of the observables at the quantum level \cite{9}. Even at the classical level the effect of this unphysical parameter must be observed via the no vanishing torsion which emerges in the effective action under the form of current-current interaction \cite{10}. In the time gauge of the $su(2)$ valued Ashtekar-Barbero connection formalism, a detailed canonical treatment of non-minimal coupling of fermion generalizing the results of \cite{12} is done in \cite{11}. All these works start from the very beginning by the A.D.M. decomposition of the tetrad components in terms of lapse and shift. This leads to an algebra of first-class constraints involving structure functions meaning that the symmetries generated by these constraints are not based on true Lie groups.

Even in the case where the A.D.M. phase space is extended to the Lagrange multipliers (the lapse and the shift) as in \cite{13}, the analysis of the covariant Hamiltonian formalism of super gravity theories leads to an algebra of constraints which closes with structure functions.

In this paper we extend the Hamiltonian formalism of $d-$dimensional tetrad-gravity minimally coupled to the fermionic field, where the non-dynamic part of the spatial connection is fixed to zero \cite{1}. This analysis is performed without the A.D.M. decomposition of the tetrad components and is free of the Barbero-Immirzi parameter which is peculiar to the 4–dimension. In this framework we show the consistency of the Hamiltonian formalism and establish the Dirac brackets of the reduced phase space from which we derive the algebra of the first-class constraints which closes with structure constants.

The paper is organized as follows: In section II we start with the action of the tetrad-gravity coupled with the fermionic field where the non-dynamic part of the spacial connection is fixed to zero. This hypothesis permits us to
get a consistent Hamiltonian formulation of the tetrad gravity coupled with
the fermionic field. In section III we establish the Dirac brackets of the re-
duced phase space elements where the second-class constraints are eliminated
as strong equalities. We show that even if the Dirac brackets of the connec-
tion with itself and with the fermionic field have complicated non polynomial
expressions, the reduced first-class constraints are polynomial and satisfy the
same closed algebra (with structural constants) as that of the pure gravity.
We show the existence of a canonical transformation leading to a new reduced
phase space which is canonical in terms of the Dirac brackets. The first-class
constraints defined on this new canonical phase space satisfy the same closed
algebra with structure constants. In the appendix A we collect the properties
of the projectors which allows us to fix the non-dynamic projected part of
the connection to zero.

2 Hamiltonian formalism

In this chapter we will extent the analysis of the Hamiltonian formalism of the
d–dimensional tetrad-gravity to the fermionic field by fixing the non-dynamic
part of the spatial connection to zero. The fixing of the non-dynamic part
of the connection to zero is necessary to avoid the constraints resulting from
the evolution equations of the non-dynamic part of the spatial connection
which are difficult to analyze [1].

As usual, to pass to the Hamiltonian formalism, we will suppose that the
manifold \( \mathcal{M} \) has topology \( R \times \Sigma \), where \( t \in R \) represents the time which
is the evolution parameter of \( d - 1 \) dimensional hypersurfaces \( \Sigma_t \) in the
d–dimensional manifold \( \mathcal{M} \). We perform the Legendre transformations from
the action \( S(e, \omega, \Psi) \)

\[
S(e, \omega_1, \Psi) = \int_{\mathcal{M}} \left( eA^{aK|I} (\partial_t \omega_{1aKL} - D_{1a} \omega_{tKL}) - eA^{aK+bL} \frac{\Omega_{1abKL}}{2} \right) d^d x \\
+ \int_{\mathcal{M}} e \left( \frac{i}{2} \left( \overline{\Psi} \gamma_I \partial_t \Psi - (\partial_t \overline{\Psi}) \gamma_I \Psi \right) \right) d^d x \\
+ \int_{\mathcal{M}} \left( e e^{aI} \frac{i}{2} \left( \overline{\Psi} \gamma_I D_a \Psi - (D_a \overline{\Psi}) \gamma_I \Psi \right) - e m \overline{\Psi} \Psi \right) d^d x \\
+ \int_{\mathcal{M}} e \left( \frac{i}{2} \left( \overline{\Psi} (\gamma_I \frac{\sigma_{KL}}{2} + \frac{\sigma_{KL}}{2} \gamma_I) \Psi \right) \right) \omega_{1KL} d^d x
\]

where the spatial connection \( \omega_{aKL} \) valued in \( so(1, d - 1) \) Lie-algebra is re-
stricted to its dynamic part $\omega_{1a KL} = P_{1KaL} P_{dQ} \omega_{dPQ}$ by fixing its non-dynamic part $\omega_{2a KL} = P_{2KaL} P_{dQ} \omega_{dPQ}$ to zero (A.10) where $P_{1KaL}$ and $P_{2KaL}$ are projectors (A.9). The covariant derivative is expressed in terms of the dynamic part of the connection, $D_{1a} \omega_{tKL} = P_{1KaL} P_{dQ} d\omega_{tPQ}$ and $\Omega_{1abKL} = \partial_a \omega_{bKL} - \partial_b \omega_{aKL} + \omega_{1aK} \gamma_{1bNL} - \omega_{1bK} \gamma_{1aNL}$ is the curvature of $\omega_{1aKL}$. $x^\mu$ are local coordinates of the $d$-dimensional manifold $\mathcal{M}$, the Greek letters $\mu, \nu \in [0, 1, \ldots, d-1]$ denote space-time indices ($t$ represents the time $t = x^t = x^0$ and $x^a$ with the space indices $a, b \in [1, \ldots, d-1]$ are local coordinates of $\Sigma_t$), $I_0, \ldots, I_{d-1} \in [0, \ldots, d-1]$ denote internal indices of the tensorial representation spaces of the Lorentz group, $e_{\mu K}$ are the components of the co-tetrad one-form valued in the vectorial representation space endowed with the flat metric $\eta_{IJ} = \text{diag}(-1, 1, \ldots, 1)$, $\eta_{IL}$ is the inverse of $e_{\mu L}$, $e_{\mu K} e_{\nu L} = \delta^K_L$, $e_{\mu K} e_{\nu K} = \delta^\nu_\mu$. The metric $\eta_{IL}$ and its inverse $\eta^{IL}$ are used to lower and to lift the Lorentz indices and to determine the metric $g_{\mu \nu} = e_{\mu I} e_{\nu J} \eta_{IJ}$ of the tangent space manifold $\mathcal{M}$ and $e_{A \mu K \nu L} = e(e_{\mu K} e_{\nu L} - e_{\nu K} e_{\mu L})$. $\Psi$ is the Dirac spinors of mass $m$, with components $\Psi^A$ where the Dirac indices take values $A, B \in \{1, 2, \ldots, 2(2^{d/2})\}$ for even dimension $d$ and $A, B \in \{1, 2, \ldots, 2(2^{d-2})\}$ for odd dimension $d$. $\bar{\Psi} = \Psi^\dagger \gamma^0$ is the Dirac conjugate and $\gamma^I$ are the Dirac Matrices satisfying

$$\gamma^K \gamma^L + \gamma^L \gamma^K = 2 \eta^{KL}.$$  

The covariant derivative acts on the spinor fields as $D_\mu \Psi = \partial_\mu \Psi + \omega_{1aKL} \frac{\sigma_{KL}}{2} \Psi$ and $D_\mu \bar{\Psi} = \partial_\mu \bar{\Psi} - \omega_{1aKL} \frac{\sigma_{KL}}{2}$ where

$$\sigma^{KL} = \frac{1}{4}(\gamma^K \gamma^L - \gamma^L \gamma^K)$$

obeying the relations

$$[\sigma^{KL}, \sigma^{PQ}] = \eta^{KQ} \sigma^{LP} + \eta^{LP} \sigma^{KQ} - \eta^{KP} \sigma^{LQ} + \eta^{LQ} \sigma^{KP}$$

and

$$[\sigma^{KL}, \gamma^I] = \delta^K_I \gamma^L - \delta^L_I \gamma^K.$$  

The Lagrangian density of (1) is invariant under the infinitesimal gauge transformations

$$\delta e_{\mu K} = \theta^K_N e_{\mu N}, \delta \omega_{tNM} = -D_t \theta_{NM}, \delta \omega_{1aKL} = -D_{1a} \theta_{KL},$$

$$\delta \Psi = \frac{\sigma^{KL}}{2} \Psi \theta_{KL}$$

and

$$\delta \bar{\Psi} = -\bar{\Psi} \frac{\sigma^{KL}}{2} \theta_{KL}.$$
subject to the conditions \[ D_2 \theta_{KL} = P_{2aKL} \partial^Q \partial_a \theta_{KL} = 0, \text{ and } \partial_2 \theta_{KL} = P_{2aKL} \partial^Q \partial_a \theta_{KL} = 0. \quad (4) \]

Note that these conditions do not restrict the gauge parameters \( \theta_{KL} \). They restrict only the gauge transformations of the dynamic part of the connection.

The conjugate momenta \( \pi^{\beta N} \), \( \mathcal{P}^{aKL}_1 \) and \( \mathcal{P}^{tKL} \) of the co-tetrad \( e_{\beta N}, \omega_{1aKL} \) and \( \omega_{tKL} \) are derived from the action (1)

\[
\pi^{\beta N}(x) = \frac{\delta S}{\delta \partial_t e_{\beta N}(x)} = 0, \quad \mathcal{P}^{aKL}_1(x) = \frac{\delta S}{\delta \partial_t \omega_{1aKL}(x)} = eA^{aKL}(x)
\]

and

\[
\mathcal{P}^{tKL}(x) = \frac{\delta S}{\delta \partial_t \omega_{tKL}(x)} = 0.
\]

The spinors \( \Psi \) and \( \Psi \) are considered to be anticommuting fields whose fermionic momenta \( \Pi \) and \( \Pi \) are obtained from the left functional derivatives

\[
\Pi = \frac{\delta S}{\delta \partial_t \Psi(x)} = \frac{i}{2} e^tN \overline{\Psi}_N \gamma_N \text{ and } \Pi = \frac{\delta S}{\delta \partial_t \overline{\Psi}(x)} = \frac{i}{2} e^tN \gamma_N \Psi.
\]

The bosonic phase space elements obey the following non-zero fundamental Poisson brackets at fixed time

\[
\{ e_{aI}(\vec{x}), \pi^{\beta N}(\vec{y}) \} = \delta_\beta^\gamma \delta_a^\gamma \delta(\vec{x} - \vec{y}),
\]

\[
\{ \omega_{IJ}(\vec{x}), \mathcal{P}^{tKL}(\vec{y}) \} = \frac{1}{2} (\delta_I^K \delta_J^L - \delta_I^L \delta_J^K) \delta(\vec{x} - \vec{y})
\]

\[
\{ \omega_{1aIJ}(\vec{x}), \mathcal{P}^{tKL}_1(\vec{y}) \} = P_{1aIJ} \mathcal{P}^{tKL}(\vec{y})
\]

where \( \vec{x} \) denotes the local coordinates \( x^a \) of \( \Sigma_t \). The fermionic ones obey the following non-zero fundamental anticommuting Poisson brackets

\[
\{ \Psi_A(\vec{x}), \Pi_B(\vec{y}) \}_+ = \delta_{AB} \delta(\vec{x} - \vec{y})
\]

and

\[
\{ \overline{\Psi}_A(\vec{x}), \overline{\Pi}_B(\vec{y}) \}_+ = \delta_{AB} \delta(\vec{x} - \vec{y}).
\]

The expressions of the conjugate momenta lead to the primary bosonic constraints

\[
\pi^{tN} \simeq 0, \quad \mathcal{P}^{tKL} \simeq 0, \quad \pi^{bN} \simeq 0, \quad C_{aKL} = \mathcal{P}^{aKL}_1 - eA^{aKL} \simeq 0 \quad (5)
\]
and to the fermionic

\[ C = \Pi - \frac{i}{2} e e^{tN} \overline{\gamma}_N, \quad \overline{C} = \Pi - \frac{i}{2} e e^{tN} \gamma_N \Psi \]  

(6)

satisfying the following non-zero Poisson brackets

\[ \{ \pi^{aN}(\vec{x}), C^{bKL}(\vec{y}) \} = -e B_a^{\alpha KNtKbL} \delta(\vec{x} - \vec{y}), \]  

(7)

\[ \{ \pi^{aN}(\vec{x}), C(\vec{y}) \} = \frac{i}{2} e A_a^{\alpha tKtM} \overline{\Psi}(\vec{y}) \gamma_M \delta(\vec{x} - \vec{y}), \]  

(8)

\[ \{ \pi^{aN}(\vec{x}), \overline{C}(\vec{y}) \} = \frac{i}{2} e A_a^{\alpha tKtM} \gamma_M \Psi(\vec{y}) \delta(\vec{x} - \vec{y}) \]  

(9)

and

\[ \{ C_A(\vec{x}), C_B(\vec{y}) \}_+ = -i e e^{tI} \gamma_{IAB} \delta(\vec{x} - \vec{y}). \]  

(10)

The total Hamiltonian is

\[ H_T = \int_\Sigma (\pi^{tN} \Lambda_{tN} + P^{tKL} A_{tKL} + \pi^{bN} \Lambda_{bN} + C_1^{aKL} \frac{A_{1aKL}}{2} + C \Lambda - \Lambda C) + H_0 \]  

(11)

where

\[ H_0 = \int_\Sigma \left( e A_a^{KbL} \frac{\Omega_{1abKL}}{2} + e A_a^{KtL} D_{1a} \omega_{tKL} \right) - e \left( e^{aK} \frac{i}{2} \left( \overline{\Psi} \gamma_K D_a \Psi - (D_a \overline{\Psi}) \gamma_K \Psi \right) - m \overline{\Psi} \Psi \right) \]  

\[ - \int_\Sigma e^{tK} \frac{i}{2} \left( \overline{\Psi} \gamma_K \sigma^{IJ}_{\frac{1}{2}} + \frac{\sigma^{IJ}_{\frac{1}{2}}}{2} \gamma_K \right) \Psi \omega_{tIJ}, \]  

\[ \Lambda_{tN}, A_{tKL}, \Lambda_{bN} \] and \( A_{1bKL} \) are the bosonic Lagrange multiplier fields and \( \Lambda \) and \( \overline{\Lambda} \) are the fermionic multiplier fields enforcing the primary constraints.

The consistency requires that these primary constraints must be preserved under the time evolution given by the total Hamiltonian (11). In order to satisfy the Jacobi identities, the calculation is given in terms of Poisson brackets projected by \( P_1^{KaL} P_{dq} \) when projected elements of the phase space \( \omega_{tKL} \) and \( P_{1KL}^{a} \) are involved [11]:

\[ \{ \pi^{tN}, H_T \} = -e B^{tNaKbL} \frac{\Omega_{1abKL}}{2} + \frac{i}{2} e A^{tNaK} \left( \overline{\Psi} \gamma_K D_a \Psi - (D_a \overline{\Psi}) \gamma_K \Psi \right) \]  

\[ -e e^{tN} m \overline{\Psi} \Psi = P^N \simeq 0, \]  

(12)
\[ \{\mathcal{P}^{KL}, \mathcal{H}_T\} = D_a e A^{aKL} + i \frac{e e^I}{2} T^I \bar{\Psi} \left( \frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \right) \Psi = M^{KL} \simeq 0, \] (13)

\[ \{\pi^{bN}, \mathcal{H}_T\} = -e B^{bNtKL} \left( \frac{A_{aKL}}{2} - D_{la} \omega_{tKL} \right) - e B^{bNakl} \frac{\Omega_{\lambda eKL}}{2} \]
\[ + \frac{i}{2} e A^{bNtKL} (\bar{\Psi} \gamma_K \Lambda - \bar{\Lambda} \gamma_K \Psi) \]
\[ + \frac{i}{2} e A^{bNaK} (\bar{\Psi} \gamma_K D_a \Psi - (D_a \bar{\Psi}) \gamma_K \Psi) - e e^bN \frac{\bar{m}}{2} \]
\[ + \frac{i}{2} e A^{bNtI} \left( \bar{\Psi} \left( \frac{\gamma_I \sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_I \right) \right) \omega_{tKL} = \mathcal{R}^{bN} = 0, \] (14)

\[ \{C^{aKL}, \mathcal{H}_T\} = e A^{bNtKL} \left( \Lambda_{bN} + \omega_{tI} M_{eN} e_{bM} \right) + D_a e A^{cKL} \]
\[ + \frac{i}{2} (e e^a I \bar{\Psi}(\frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_I)) \omega_{tKL} = \mathcal{R}^{aKL} = 0, \] (15)

\[ \{C, \mathcal{H}_T\} = C_{fer} \simeq 0 = -i e e^I N \bar{\Lambda}, \gamma_N - \frac{i}{2} e A^{bNtLM} \Lambda_{bN} \bar{\Psi} \gamma_M - \mu \bar{m} \]
\[ - D_a \left( \frac{i}{2} e e^a M \bar{\Psi} \right) \gamma_M - \frac{i}{2} e e^a M (D_a \bar{\Psi}) \gamma_M \]
\[ + \frac{i}{2} e e^t M \bar{\Psi} \left( \gamma_M \frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_M \right) \omega_{tKL} \] (16)

and

\[ \{\bar{C}, \mathcal{H}_T\} = \bar{C}_{fer} \simeq 0 = -i e e^I N \gamma_N \Lambda - \frac{i}{2} e A^{bNtLM} \Lambda_{bN} \gamma_M \Psi + \epsilon \Psi \]
\[ - D_a \left( \frac{i}{2} e e^a M \gamma_M \Psi \right) - \frac{i}{2} e e^a M (D_a \Psi) \]
\[ - \frac{i}{2} e e^t M \left( \gamma_M \frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_M \right) \omega_{tKL}. \] (17)

These consistency conditions show that the evolution of the constraints \(\pi^{tN}\) and \(\mathcal{P}^{tNM}\) lead to secondary constraints \(P^N\) and \(M^{NM}\) while the constraints \(\pi^{bN}, C^{aKL}, C\) and \(\bar{C}\) lead to equations for Lagrange multipliers.

Now we have to check the consistency of the secondary constraints. For \(M^{KL}\) we combine (14-17), the properties (A.2) of the B-matrix and the identity

\[ T^K_N e A^{bNtKL} + T^L_N e A^{bNtKL} = \frac{1}{2} \left( \epsilon^K_{bN} e B^{bNtKL} T^I_{LM} - \epsilon^L_{bN} e B^{bNtLM} T^I_{KM} \right), \] (18)
true for any antisymmetric tensor $T_{NM} = -T_{MN}$, to get

$$\{ M^{KL}, \mathcal{H}_T \}$$

$$= -\frac{1}{2} \left( e_b^K \, R^{bL} - e_b^K \, R^{bK} \right) + \frac{1}{2} \left( \bar{\nabla} \sigma^{KL} \right) \mathcal{C}_{fer} + \frac{1}{2} \left( \bar{\Psi} \sigma^{KL} \right) \mathcal{C}_{fer} - \frac{1}{2} \left( \omega^K_{LN} M^{NL} + \omega^L_{MN} M^{KN} \right)$$

which, when (14-17) are satisfied, reduces to

$$\{ M^{KL}, \mathcal{H}_T \} = \frac{1}{2} \left( e_t^K P^L - e_t^L P^K \right) - \left( \omega^K_{LN} M^{NL} + \omega^L_{MN} M^{KN} \right) \simeq 0 \quad (19)$$

ensuring the consistency of the constraint $M^{KL}$.

In what follows we consider instead of the constraint $P^N$ its temporal projection

$$\mathcal{D}_t = e_{tN} P^N = -e A^{aKbL} \frac{\Omega_{1abKL}}{2} + \frac{i}{2} e \epsilon^a \left( \bar{\Psi} \gamma_K D_a \Psi - D_a \bar{\Psi} \gamma_K \Psi \right) - e n \bar{\Psi} \Psi \quad (20)$$

and its smeared spatial projection

$$\mathcal{D}_{sp}(\vec{N}) = -\int_{\Sigma} N^a (e_a N^P + \omega_{1aKL} M^{KL}) + \int_{\Sigma} e A^{aKtL} \mathcal{L}_{\vec{N}}(\omega_{1aKL}) + \frac{i}{2} e \epsilon^a \left( \bar{\Psi} \gamma_K \mathcal{L}_{\vec{N}} \Psi - \mathcal{L}_{\vec{N}} \bar{\Psi} \gamma_K \Psi \right) \quad (21)$$

where $\mathcal{L}_{\vec{N}}(\omega_{1aKL}) = N^b \partial_b \omega_{1aKL} + \partial_a (N^b) \omega_{1bKL}$ is the Lie-derivative along the arbitrary $(d - 1)$ dimensional vector field $\vec{N}$ tangent to $\Sigma$. This Lie-derivative $\mathcal{L}_{\vec{N}}$ treats the temporal components $e_{tN}$ and $\omega_{1tKL}$ as well as the Lorentz indices as scalars, i.e., $\mathcal{L}_{\vec{N}}(e_{tN}) = N^a \partial_a e_{tN}$, $\mathcal{L}_{\vec{N}}(\omega_{tKL}) = N^a \partial_a \omega_{tKL}$, $\mathcal{L}_{\vec{N}} \Psi = N^b \partial_b \Psi$ and $\mathcal{L}_{\vec{N}} \bar{\Psi} = N^b \partial_b \bar{\Psi}$.

A straightforward computation gives for the evolution of the constraint $\mathcal{D}_t$

$$\{ \mathcal{D}_t, \mathcal{H}_T \} = (\Lambda_N + \omega_{tN}^M e_{tM}) P^N - D_c (e_c N^R \mathcal{C}^C_N) + R^{aKL} \left( \frac{A_{1aKL}}{2} - D_a \omega_{tKL} \right)$$

$$+ (\Lambda_c N - \omega_{cN}^M e_{cM}) \mathcal{C}_{rfer} + \mathcal{C}_{fer} (\Lambda + \frac{\sigma^{KL}}{2} \Psi \omega_{tKL}) - (\Lambda - \bar{\Psi} \sigma^{KL} \omega_{tKL}) \bar{\mathcal{C}}_{fer}$$

leading to

$$\{ \mathcal{D}_t, \mathcal{H}_T \} = (\Lambda_N + \omega_{tN}^M e_{tM}) P^N \simeq 0 \quad (22)$$
when (14-17) are satisfied. For
\[
\Lambda_N = -\omega_{tN} M e_{tM} \implies \pi^N \Lambda_N = -\frac{1}{2} (\pi^N e^M_{tM} - \pi^M e^N_{tM}) \omega_{tNM}
\]  
(23)

(22) vanishes strongly. The consistency of \( D_{sp} (\vec{N}) \) gives
\[
\{ D_{sp} (\vec{N}) , \mathcal{H}_T \} = - \int_{\Sigma} (P^N \mathcal{L}_N(e_{tN}) + M^{KL} \mathcal{L}_N(\omega_{tKL})) - \mathcal{R}^{aKL} \mathcal{L}_N(\omega_{1aKL})
\]
\[- \mathcal{R}^{cN} \mathcal{L}_N(e_{cN}) - \mathcal{C}_{fer} \mathcal{L}_N \Psi + \mathcal{L}_N(\overline{\Psi}) \mathcal{C}_{fer}
\]
leading to
\[
\{ D_{sp} (\vec{N}) , \mathcal{H}_T \} = - \int_{\Sigma} (P^N \mathcal{L}_N(e_{tN}) + M^{KL} \mathcal{L}_N(\omega_{tKL}))
\]
when (14-17) are satisfied.

This shows that the set of constrains is complete meaning that the total Hamiltonian \( \mathcal{H}_T \) is consistent providing that (14-17) are satisfied.

Now we have to solve equations (14-17) to determine the Lagrange multipliers and then insert their expressions into the Hamiltonian. As consequence of \( \mathcal{R}^{dPQ} = P_{1KL}^{dPQ} \mathcal{R}^{aKL} \) and the rank \( d(d-1) \) of the projector \( P_{1KL}^{dPQ} \), there are as many equations as multipliers of Lagrange \( \Lambda_{bN} \). This is an indication to uniquely solve (15) by multiplying it by \( B_{bNtKL} \) and using (A.6) to get
\[
\Lambda_{bN} = -\omega_{tN} M e_{bM} + D_b e_{tN} - e^{-1} B_{bNtKL} M^{KL}
\]
\[- \frac{i}{4} A_{bKL} \overline{\Psi} (\gamma_N \frac{\sigma_{KL}}{2} + \frac{\sigma_{KL}}{2} \gamma_N) \Psi
\]
\[\simeq -\omega_{tN} M e_{bM} + D_b e_{tN} - \frac{i}{4} A_{bKL} \overline{\Psi} (\gamma_N \frac{\sigma_{KL}}{2} + \frac{\sigma_{KL}}{2} \gamma_N) \Psi. \]  
(24)

Similarly for \( A_{1aKL} \) which has the same number of components as equations (14) whose solution is
\[
\frac{1}{2} A_{1aKL} = D_{1a} \omega_{tKL} - B_{bNtKL} B_{bNC_{PdQ}} \Omega_{1cdPQ} \frac{1}{2}
\]
\[+ \frac{i}{2} B_{bNtKal} A_{bNtLM} (\overline{\Psi} \gamma_M \Lambda - \overline{\Psi} \gamma_M \Psi) - B_{bNtKae} \overline{\Psi} m \overline{\Psi}
\]
\[+ \frac{i}{2} B_{bNtKal} A_{bNC_{M}} (\overline{\Psi} \gamma_M D_e \Psi - D_e \overline{\Psi} \gamma_M \Psi)
\]
\[+ \frac{i}{2} B_{bNtKal} A_{bNtLM} (\overline{\Psi} \gamma_M \frac{\sigma_{PQ}}{2} + \frac{\sigma_{PQ}}{2} \gamma_M) \omega_{tPQ}. \]
By multiplying on the left (16) and on the right (17) by \( \frac{i}{g_{tt}} e^{tM} \gamma_M \) and using

\[
-\frac{i}{2} e^{bNtM} \Lambda_{bN} = \frac{i}{2} \omega_N^M e^{tN} + D_b \left( \frac{i}{2} e^{bM} \right) + \frac{i}{2} M^{NM} e_{tN} \\
\approx \frac{i}{2} \omega_N^M e^{tN} + D_b \left( \frac{i}{2} e^{bM} \right)
\]

deduced from (24), we obtain the expressions of the fermionic Lagrange multipliers

\[
\Lambda = -\frac{\sigma^{KL}}{2} \omega_{tKL} - \frac{i}{g_{tt}} m e^{tM} \gamma_M \Psi - \frac{1}{g_{tt}} e^{bK} e^{tM} \gamma_K D_b \Psi \quad (26)
\]

and

\[
\Lambda = \Psi \frac{\sigma^{KL}}{2} \omega_{tKL} + \frac{i}{g_{tt}} m e^{tM} \overline{\Psi} \gamma_M - \frac{1}{g_{tt}} e^{bK} e^{tM} D_b \overline{\Psi} \gamma_K \gamma_M \quad (27)
\]

where \( g_{tt} = e^{tN} e^t_N \). The Insertion of (26-27) in (25) gives

\[
\frac{1}{2} A_{1aKL} = D_{1a} \omega_{tKL} - B_{bNtKab} \left( B^{bNcPd} \delta \frac{\Omega_{1aPQ}}{2} \right) \\
- \frac{i}{2} A^{bNcM} \left( \overline{\Psi} \gamma_M D_c \Psi - D_c \overline{\Psi} \gamma_M \Psi \right) + \frac{i}{2g_{tt}} A^{bNtM} e^{cM} e^{tM} \left( \overline{\Psi} \gamma_I \gamma_M D_c \Psi - D_c \overline{\Psi} \gamma_M \gamma_I \gamma_P \Psi \right) + \frac{\rho}{g_{tt}} e^{bP} e^{tM} \overline{\Psi} \Psi.
\]

which is inserted with (23), (24) and (26) in (11) to give the total Hamil-
tonian

\[ H'_T = \int_{\Sigma} \mathcal{P}^{KL} A_{tKL} - \int_{\Sigma} \left( \frac{1}{2} (\pi^{\mu K} e^L_{e \mu} - \pi^{\mu L} e^K_{e \mu}) + D_a (C^{aKL} + e A^{aKL}) \right) + \left( C + \frac{i}{2} e e^{I} \nabla_I \right) \frac{\sigma^{KL}}{2} \Psi + \nabla_I \left( C + \frac{i}{2} e e^{I} \nabla_I \right) \omega_{tKL} \]

+ \int_{\Sigma} \left( e A^{aKLbK} \frac{\Omega_{1abKL}}{2} - \frac{i}{2} e e^{aK} (\nabla_J D_d \Psi - D_d \nabla_J \Psi) + e \nabla_I \right) \frac{\sigma^{KL}}{2} \Psi + \nabla_I \left( C + \frac{i}{2} e e^{I} \nabla_I \right) \omega_{tKL} \]

+ \int_{\Sigma} \pi^{aKL} \left( D_a e^{tH} - \frac{i}{4} A^{aKL} \nabla_I \left( C + \frac{i}{2} e e^{I} \nabla_I \right) \omega_{tKL} \right) \]

- \int_{\Sigma} C^{aKL} B_{bKLdK} \left( B^{bKL} D_d \Psi - D_d \nabla_J \Psi + D_d \nabla_J \Psi \right) \]

+ \frac{i}{2} g^M e^{tM} \frac{\epsilon^{aKL}}{2} \left( \Psi \nabla_I \nabla_I \nabla_I \Psi \right) - \int_{\Sigma} i e^{tM} \left( C + \frac{i}{2} e e^{I} \nabla_I \right) \Psi (C + \frac{i}{2} e e^{I} \nabla_I \Psi) \]

- \int_{\Sigma} \frac{1}{2} g^M e^{tM} \frac{\epsilon^{aKL}}{2} \left( \Psi \nabla_I \nabla_I \nabla_I \Psi \right) = 0 \tag{28} \]

With the Hamiltonian \( H'_T \), the consistency of the constraint \( \mathcal{P}^{tNM} \) leads to

\[ \{ \mathcal{P}^{tKL}, H'_T \} = D_a (C^{aKL} + e A^{aKL}) + \frac{1}{2} (\pi^{\mu K} e^L_{e \mu} - \pi^{\mu L} e^K_{e \mu}) \]

+ \left( C + \frac{i}{2} e e^{I} \nabla_I \right) \frac{\sigma^{KL}}{2} \Psi + \nabla_I \left( C + \frac{i}{2} e e^{I} \nabla_I \right) \omega_{tKL} \]

= \frac{1}{2} M^{KL} \simeq 0 \tag{29} \]

where \( M^{KL} \) is the new constraints replacing \( M^{KL} \). By using the relations

\[ e^{tN} e^{tKL} (B_{bKLdK}) = B_{bKLdK}, \quad e^{tN} \left\{ \pi^{tL}, \frac{e^{tM}}{g^M} \right\} = -\frac{e^{tM}}{g^M} \quad \text{and} \quad e^{tN} \left\{ \pi^{tL}, e^{tM} \right\} = 0 \]

we can rewrite the total Hamiltonian \( (28) \) as

\[ H'_T = \int_{\Sigma} \left( \frac{1}{2} \mathcal{P}^{tKL} A_{tKL} - D'_t - M^{tKL} \omega_{tKL} \right) \]

where \( D'_t \) is the projection of the evolution of \( \pi^{tN} \), i.e.,

\[ e^{tN} \left\{ \pi^{tN}, H'_T \right\} = -e^{tN} \omega^{tM}_{tL} \pi^{tM} + D^{tN} e^{tN} \]

= -e^{tN} \omega^{tM}_{tL} \pi^{tM} + D'_t \simeq 0 \implies D'_t \simeq 0. \]

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By using \( e_{cN} \frac{\delta}{\delta e_{cN}} (e^{-1} B_{bMtKaL}) = e^{-1} B_{bMcKaL} \), \( C_{1}^{aKL} B_{bMcKaL} = 0 \) and

\[
\int_{\Sigma} C_{1}^{aKL} B_{bMtKaL} N^{c} e_{cN} C'(N^{bMcP}dQ) \Omega_{1edPQ} \Omega_{1edPQ} = - \int_{\Sigma} \left( C_{1}^{aKL} L_{N}^{c} (\omega_{1aKL}) + D_{a}(eA^{aKL}) N^{c} \omega_{1aKL} \right)
\]

we get for the projection \( e_{cN} P^N \), the following smeared combination of constraints

\[
\int_{\Sigma} \left( C_{1}^{aKL} e^{-1} B_{cMtKaL} eB^{tMcP}dQ \Omega_{1edPQ} \right)
\]

This is completed by adding the constraint \( \pi^{tN} L_{N}^{c} (e_{tN}) \) to get

\[
D_{sp}'(\vec{N}) = \int_{\Sigma} \left( \pi^{\mu M} L_{N}^{c} (e_{\mu M}) + (C_{1}^{aKL} + eA^{aKL}) L_{N}^{c} (\omega_{1aKL}) \right)
\]

A straightforward computation shows that \( D_{sp}'(\vec{N}) \) satisfies

\[
\{ D_{sp}'(\vec{N}), D_{sp}'(\vec{N}') \} = D_{sp}'(L_{N}^{c}(\vec{N'}) - L_{N}^{c}(\vec{N})) = D_{sp}'([\vec{N}, \vec{N}']),
\]

where \([\vec{N}, \vec{N}']\) is the Lie bracket. The constraint \( D_{sp}'(\vec{N}) \) generates spatial diffeomorphisms of the phase space elements

\[
\{ e_{\mu N}, D_{sp}'(\vec{N}) \} = L_{N}^{c}(e_{\mu N}), \{ \omega_{1aNM}, D_{sp}'(\vec{N}) \} = L_{N}^{c}(\omega_{1aNM}),
\]
\[
\{\Psi, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(\Psi), \quad \{\overline{\Psi}, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(\overline{\Psi}), \\
\{\pi^{\mu N}, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(\pi^{\mu N}), \quad \{C^{aNM}, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(C^{aNM}), \\
\{C, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(C) \quad \text{and} \quad \{\overline{C}, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(\overline{C})
\]

from which we deduce that the Poisson brackets of \(\mathcal{D}'_{\text{sp}}(N)\) with the primary constraints weakly vanish and the constraints \(\mathcal{D}'_t\) and \(M^{KL}\) transform like scalar densities of weight one

\[
\{\mathcal{D}'_t, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(\mathcal{D}'_t) = \partial_t(N^c\mathcal{D}'_t) \\
\implies \{\mathcal{D}'_{\text{sp}}(\hat{N}), \mathcal{D}'_t(M)\} = \mathcal{D}'_t(\mathcal{L}_{\hat{N}}(M))
\]

and

\[
\{M^{KL}, \mathcal{D}'_{\text{sp}}(\hat{N})\} = \mathcal{L}_{\hat{N}}(M^{KL}) = \partial_t(N^cM^{KL}) \\
\implies \{\mathcal{D}'_{\text{sp}}(\hat{N}), \mathcal{M}(\theta)\} = \mathcal{M}(\mathcal{L}_{\hat{N}}(\theta)).
\]

Here \(\mathcal{D}'_t(M) = \int M \mathcal{D}'_t\) and \(\mathcal{M}(\theta) = \int M^{KL}\theta_{KL} - \delta \omega_{1KL}\) where \(\theta_{KL} = \delta t \omega_{1KL}\) are dimensionless and infinitesimal arbitrary local parameters subject to the condition \([31]\). The above Poisson brackets imply that the constraint \(\mathcal{D}'_{\text{sp}}(\hat{N})\) is of first-class.

For the smeared constraint \(\mathcal{M}(\theta)\) which acts as generators of the infinitesimal Lorentz transformation group, we get

\[
\{e_{\mu N}, \mathcal{M}(\theta)\} = \theta_{N}^{L} e_{\mu L}, \quad \{\omega_{1aNM}, \mathcal{M}(\theta)\} = -\mathcal{D}_{1a}(\theta_{NM}), \quad (31) \\
\{\Psi, \mathcal{M}(\theta)\} = \frac{\sigma_{KL}}{2} \Psi_{KL}, \quad \{\overline{\Psi}, \mathcal{M}(\theta)\} = -\overline{\Psi} \frac{\sigma_{KL}}{2} \theta_{KL}. \quad (32)
\]

\[
\{\pi^{\mu N}, \mathcal{M}(\theta)\} = \theta_{N}^{L} \pi_{\mu L}, \quad \{C^{aNM}, \mathcal{M}(\theta)\} = \theta_{N}^{L} C^{aLM} + \theta_{M}^{L} C^{aNL} \quad (33) \\
\{C, \mathcal{M}(\theta)\} = -C \frac{\sigma_{KL}}{2} \theta_{KL}, \quad \{\overline{C}, \mathcal{M}(\theta)\} = -\overline{C} \frac{\sigma_{KL}}{2} \theta_{KL} \quad (34)
\]

from which we deduce that the Poisson brackets of \(\mathcal{M}(\theta)\) with the primary constraints \(\pi^{\mu N}, C^{aNM}, C\) and \(\overline{C}\) weakly vanish. Since \(\mathcal{M}(\theta)\) treats the space-time indices as scalars, the transformations \([31][34]\) make easy the computation of transformation it generates. the constraints \(M^{KL}\) transform as tensor

\[
\{M^{NM}, \mathcal{M}(\theta)\} = \theta_{N}^{L} M^{LM} + \theta_{M}^{L} M^{NL},
\]
leading to the \(so(1,d-1)\) Lie algebra

\[
\left\{ M^{NM}(\vec{x}), M^{KL}(\vec{y}) \right\}_D = (\eta^{NL}M^{MK}(-\vec{x}) + \eta^{MK}M^{NL}(-\vec{x})) \\
-\eta^{NK}M^{ML}(\vec{x}) - \eta^{ML}M^{NK}(\vec{x})) \delta(\vec{x} - \vec{y}),
\]

and \(D'_t\) is a scalar under Lorentz transformations, implying

\[
\left\{ D'_t(\vec{x}), M(\theta) \right\} = 0 \Rightarrow \left\{ D'_t(M), M(\theta) \right\} = 0
\]

showing that the constraints \(M^{KL}\) are also of first-class.

Now, what remains is to study the consistency of the constraint \(D'_t\). Since \(D'_t\) commutes weakly in terms of Poisson brackets with \(D'_t(\vec{N})\) and \(M(\theta)\), it remains to calculate its Poisson brackets with the primary constraints \(\pi^{aN}, C^{aNM}_1, C\) and \(\overline{C}\). A straightforward computation gives

\[
\left\{ \pi^{cN}(\vec{x}), D'_t(\vec{y}) \right\} \simeq +\left( eB^{cNaKbL} \frac{\Omega_{abKL}}{2} \right) \\
-\frac{i}{2} eA^{cNaK}(\overline{\Psi}_K D_a \Psi - D_a \overline{\Psi}_K \Psi) + e^{cN} \overline{\Psi} \Psi \delta(\vec{x} - \vec{y}) \\
-\frac{i}{2} eB^{cNtKd} \left( \overline{\Psi}_Q \gamma_P \gamma_1 D_d \Psi - D_d \overline{\Psi}_Q \gamma_P \gamma_1 \Psi \right) \\
+ \frac{i}{2g_{tt}} e^{dP} A^{bMP} \left( \overline{\Psi}_Q \gamma_1 D_d \Psi - D_d \overline{\Psi}_Q \gamma_1 \Psi \right) \\
+ \frac{i}{m_1 } \left( e^{cN} \overline{\Psi} \Psi \right) \delta(\vec{x} - \vec{y}) \\
-\frac{i}{2g_{tt}} e^{dP} A^{bMP} \left( \overline{\Psi}_Q \gamma_1 D_d \Psi - D_d \overline{\Psi}_Q \gamma_1 \Psi \right) \\
+ \frac{i}{m_1 } \left( e^{cN} \overline{\Psi} \Psi \right) \delta(\vec{x} - \vec{y})
\]

where \(\simeq\) means that only the terms which are not proportional to the primary constraints have been kept. \(\left\{ \pi^{cN}(\vec{x}), D'_t(\vec{y}) \right\}\) weakly vanishes as a consequence of (A.6) applied in the third line.

For the Poisson brackets of the constraint \(C^{aNM}_1\) with \(D'_t\) we get

\[
\left\{ C^{aKL}_1(\vec{x}), D'_t(\vec{y}) \right\} \\
\simeq -(D_b eA^{bKL})_1 \delta(\vec{x} - \vec{y}) \\
-\frac{i}{2} \left( e^{aN}(\overline{\Psi}_{KL} \gamma_N) + \frac{\sigma^{KL}}{2} \gamma_N \Psi \right) \delta(\vec{x} - \vec{y}) \\
-\frac{i}{2} eB^{cNtKd} \left( \overline{\Psi}_Q \gamma_P \gamma_1 \Psi \right) \\
+ \frac{i}{4} eB^{cNtKd} A^{cPQ} \left( \overline{\Psi}_Q \gamma_N \frac{\sigma^{PQ}}{2} + \frac{\sigma^{PQ}}{2} \gamma_N \Psi \right) \delta(\vec{x} - \vec{y}).
\]

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The first term of the second hand gives
\[ D_{b}eA^{bKaL} = B_{c^{M}p^{d}q^{e}}B^{c^{M}K^{a}l}D_{b}eA^{bP^{d}q^{e}} \]
\[ = B_{c^{M}p^{d}q^{e}}B^{c^{M}K^{a}l}(eB^{M}K^{d}q^{e}D_{b}e_{i} + eB^{M}P^{d}q^{e}D_{b}e_{i}) \]
\[ = -eB^{c^{M}K^{a}l}D_{b}e_{i}M - B_{c^{M}p^{d}l}B^{c^{M}K^{a}l}D_{b}eA^{bP^{d}q^{e}}. \]

The relation
\[ B_{c^{M}p^{d}q^{e}}e^{d_{N}} = ( -B_{c^{M}p^{d}q^{e}}e^{l_{N}} + \frac{1}{2} A_{c^{M}P^{l}}\delta^{N}_{P} + A_{c^{Q}M}d^{N}_{P} + A_{c^{P}q^{d}}\delta^{N}_{M}) \]
deduced from (A.3), and
\[ \frac{1}{2}(A_{c^{M}P^{l}}\delta^{N}_{P} + A_{c^{Q}M}d^{N}_{P} + A_{c^{P}q^{d}}\delta^{N}_{M})(\overline{\Psi}_{N}\frac{\sigma_{PQ}}{2} + \frac{\sigma_{PQ}}{2}\gamma_{N}\Psi) = 0 \]
imply for the second term
\[ \frac{i}{2}(e^{aN}(\overline{\Psi}_{N}\frac{\sigma_{K}^{L}}{2} + \frac{\sigma_{K}^{L}}{2}\gamma_{N}\Psi))_{1} \]
\[ = -\frac{i}{2}B_{c^{M}p^{d}q^{e}}B^{c^{M}K^{a}l}e^{d_{N}}(\overline{\Psi}_{N}\frac{\sigma_{PQ}}{2} + \frac{\sigma_{PQ}}{2}\gamma_{N}\Psi) \]
\[ = \frac{i}{2}B_{c^{M}p^{d}l}B^{c^{M}K^{a}l}e^{l_{N}}(\overline{\Psi}_{N}\frac{\sigma_{PQ}}{2} + \frac{\sigma_{PQ}}{2}\gamma_{N}\Psi) \]
\[ - \frac{i}{4}B^{c^{M}K^{a}l}A_{c^{P}q^{d}}(\overline{\Psi}_{M}\frac{\sigma_{PQ}}{2} + \frac{\sigma_{PQ}}{2}\gamma_{M}\Psi) \]
leading to
\[ \{C_{1}^{aK}(\vec{x}), D_{t}^{1}(\vec{y})\} \]
\[ \simeq B_{c^{M}p^{d}q^{e}}B^{c^{M}K^{a}l}(D_{b}eA^{bP^{d}q^{e}} + \frac{i}{2}(e^{l}N(\overline{\Psi}_{N}\frac{\sigma_{PQ}}{2} + \frac{\sigma_{PQ}}{2}\gamma_{N}\Psi))\delta(\vec{x} - \vec{y}) \]
\[ \simeq B_{c^{M}p^{d}q^{e}}B^{c^{M}K^{a}l}M^{KL}\delta(\vec{x} - \vec{y}) \simeq 0. \]

For the fermionic constraints, we get
\[ \{C(\vec{x}), D_{t}^{1}(\vec{y})\} \simeq ((D_{a}\frac{i}{2}e^{aK}\overline{\Psi} + \frac{i}{2}e^{aK}D_{a}\overline{\Psi})\gamma_{K} + e_{M}\overline{\Psi})\delta(\vec{x} - \vec{y}) \]
\[ \frac{i}{2}e^{aN}K(\overline{\Psi}_{K}(D_{b}e_{i}N) \]
\[ - \frac{i}{4}A_{c^{P}q^{d}}(\overline{\Psi}_{N}\frac{\sigma_{PQ}}{2} + \frac{\sigma_{PQ}}{2}\gamma_{N}\Psi)\delta(\vec{x} - \vec{y}) \]
\[ - \frac{im}{g^{d}}e^{l}M\overline{\Psi}(\vec{y})\gamma_{M}\{C(\vec{x}), \overline{C}(\vec{y})\} \]
\[ + \frac{1}{g^{d}}e^{l}M_{a}D_{a}\overline{\Psi}(\vec{y})\gamma_{K}\gamma_{M}\{C(\vec{x}), \overline{C}(\vec{y})\}. \]
The relations

\[ e^{A^N K} D e^{l_N} = D e^{A^N K} e^{l_N} - D e^{A^N K} e^{l_N}, \]

\[ A^{cN K} A_{cPQ} = \delta^{cN}_{cP} e^{cK} e^{l_Q} - \delta^{cN}_{cQ} e^{cK} e^{l_P} - \delta^{cK}_{cP} e^{cN} e^{l_Q} + \delta^{cK}_{cQ} e^{cN} e^{l_P}, \]

\[ (\delta^{cN}_{cP} e^{cK} e^{l_Q} - \delta^{cN}_{cQ} e^{cK} e^{l_P}) \Psi (\gamma_N \frac{\gamma_P}{2} + \frac{\gamma_P}{2} \gamma_N) = 0, \]

\[ e^{M N} e^{l_N} \gamma_M \gamma_N = g^{t_t} \] and \( (10) \) lead to

\[ \{ C(\vec{x}), D'_t(\vec{y}) \} \simeq -i \frac{1}{2} \left( D e^{A^N K} \right) \]

\[ + i e^{dP} \left( \gamma_P \frac{\gamma_P}{2} + \frac{\gamma_P}{2} \gamma_P \right) e^{l_N} \Psi \gamma_K \delta(\vec{x} - \vec{y}) \]

\[ \simeq -i \frac{1}{2} M^{N K} e^{l_N} \Psi \gamma_K \delta(\vec{x} - \vec{y}) \simeq 0. \]

The same computation shows

\[ \{ C(\vec{x}), D'_t(\vec{y}) \} \simeq -i \frac{1}{2} M^{N K} \gamma_K \Psi e^{l_N} \delta(\vec{x} - \vec{y}) \simeq 0. \]

Finally, a long and direct calculation shows that the Poisson bracket of the smeared scalar constraint with itself is strongly equal to \((M \partial_a M' - M' \partial_a M) \) times a linear combination of the primary constraints \( C^a_{1KL}, C^a_{1KL} \) as

\[ \{ D'_t(M), D'_t(M') \} = \int_{\Sigma} ((M \partial_a M' - M' \partial_a M)(F^a(C_1) + G^a(C, \overline{C})) \simeq 0 \] (35)

where

\[ F^a(C_1) = \left( \frac{1}{2(d - 2)} g^{t_t} C^a_{1KL} A_{bK1L} e^{bQ} (me^{l_Q} \overline{\Psi}) \right) \]

\[ - e^{dL} e^{t_P} i \left( \overline{\Psi} \gamma_P \gamma_I D_d \Psi - D_d \overline{\Psi} \gamma_I \gamma_P \Psi \right) \]

\[ + \left( \frac{1}{2(d - 2)} g^{t_t} C^a_{1KL} e^{dK} e^{t_P} e^{t_M} (m \overline{\Psi} (\gamma_L \gamma_P \gamma_I \gamma_M + \gamma_M \gamma_J \gamma_P \gamma_L \Psi) \right) \]

\[ - e^{dL} (\overline{\Psi} \gamma_L \gamma_P \gamma_I \gamma_M \gamma_I D_d \Psi - D_d \overline{\Psi} \gamma_I \gamma_M \gamma_J \gamma_P \gamma_L \Psi) \]

and

\[ G^a(C, \overline{C}) = \left( \frac{1}{(g^{t_t})^2} e^{l_N} e^{M} e^{K} (im (C^{\gamma_N} \gamma_K \gamma_M \Psi + \overline{\Psi} \gamma_N \gamma_K \gamma_M \overline{C})) \right) \]

\[ + e^{dL} (C^{\gamma_N} \gamma_K \gamma_M \gamma_I D_d \Psi - D_d \overline{\Psi} \gamma_I \gamma_N \gamma_K \gamma_M \overline{C}). \]
Note that as opposed in the pure gravity where the Poisson bracket of scalar constraint with itself vanishes strongly [1], in presence of the fermionic matter, this Poisson bracket vanishes only weakly. This shows that the set of constraints is complete and closed meaning that the total Hamiltonian (28) is consistent.

3 The Dirac brackets

Before starting the constraint processing, let us note that instead of directly analyzing the total Hamiltonian (11) we can take the non-dynamic part of the connection \( \omega_{2aKL} \) and its conjugate moment \( P_{aKL} \) as primary constraints satisfying

\[
\{ \omega_{2aKL}(x), P_{bPQ}^a(y) \} = P_{2K_aL}^{bPQ} \delta(x - y) \tag{36}
\]

and contribute to the total Hamiltonian by adding the term

\[
\int \Sigma \left( \frac{B_{aKL}^2}{2} + \frac{A_{aKL}^2}{2} \right)
\]

where \( B_{aKL}^2 \) and \( A_{aKL} \) are Lagrange multiplier fields.

In order to satisfy the Jacobi identities we project the brackets acting on the projected elements of the phase space \( \omega_{2aKL} \) and \( P_{aKL} \) as

\[
\{ \pi^{\mu N}, \omega_{2aKL} \} = \left\{ \pi^{\mu N}, P_{2K_aL}^{bPQ} \omega_{bPQ} \right\}_2 = \left\{ \pi^{\mu N}, P_{2K_aL}^{PbQ} \right\}_2 \omega_{bPQ} = P_{2K_aL}^{cRS} \left\{ \pi^{\mu N}, P_{2cRS}^{dNM} \right\} P_{2dNM}^{PbQ} \omega_{bPQ} = 0
\]
due to \( P(\delta P) = 0 \) for any projector \( P \). The same computation gives

\[
\{ \pi^{\mu N}, P_{2aKL} \} = 0
\]

which show that the Poisson brackets between these constraints and the constraints \( \pi^{\mu N}, C^{aK}_1, C \) and \( \overline{C} \) vanish. The consistency of the constraint \( \omega_{2aKL} \) leads to \( A_{aKL} = 0 \) and the consistency of \( P_{2aKL} \) determines the multiplier fields

\[
\{ \omega_{2aKL}, H_T \} = A_{aKL} = 0
\]

\[
\{ P_{2aKL}, H_T \} = -\frac{B_{aKL}^2}{2} + \left\{ P_{2aKL}, H_0 \right\} = 0
\]

where \( H_0 \) is expressed in term \( \omega_{aKL} = \omega_{1aKL} + \omega_{2aKL} \). The first-class constraints of the previous chapter take the same form plus terms proportional...
to the constraint $\omega_{2aKL}$ that can be ignored. As a result, the constraints $\omega_{2aKL}$ and $\mathcal{P}_{2KL}$ are of second-class and their Poisson brackets with the other constraints vanish and so can just be discarded from the theory leading to the total Hamiltonian (11).

In this section we consider the second-class constraints $\pi^{aN}$, $C_{1}^{aKL}$, $C$ and $\overline{C}$ as strong equalities by eliminating them from the theory leading to the reduced Hamiltonian

$$\mathcal{H}_{T}' = -\int_{\Sigma} \left( \mathcal{D}'_{i} + M^{tKL}_{r} \omega_{tKL} \right)$$

where

$$M^{tKL} = (\pi^{tK} e_{t}^{L} - \pi^{tL} e_{t}^{K}) + 2D_{e}(eA^{aKL}) + iee^{tI} \Psi \gamma_{I} \left( \frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_{t} \right)$$

are the reduced Lorentz constraints and

$$\mathcal{D}'_{i} = -eA^{abKL} \frac{\Omega^{abKL}}{2} + \frac{i}{2} e^{aKL} (\overline{\Psi} \gamma_{K} D_{a} \Psi - D_{a} \overline{\Psi} \gamma_{K} \Psi) - em \overline{\Psi}$$

is the reduced scalar constraint. The diffeomorphism constraint reduces to

$$\mathcal{D}_{sp}(\overline{N}) = \int_{\Sigma} \left( \pi^{tK} \mathcal{L}_{\overline{N}}(e_{tK}) + eA^{aKL} \mathcal{L}_{\overline{N}}(\omega_{1aKL}) + \frac{i}{2} e^{tK} (\overline{\Psi} \gamma_{K} \mathcal{L}_{\overline{N}} \Psi - \mathcal{L}_{\overline{N}} \overline{\Psi} \gamma_{K} \Psi) \right).$$

In this case the algebra of the first-class constraints must be computed in terms of projected Dirac brackets defined from the projected Poisson brackets as

$$\{A, B\}_{D} = \{A, B\} - \{A, C_{i}\} \{C_{i}, C_{j}\}^{-1} \{C_{j}, B\}$$

where $C_{i} = (\pi^{aN}, C_{1}^{aKL}, C, \overline{C})$ are the set of the second-class constraints. The non-zero elements of the inverse super matrix $\{C_{i}, C_{j}\}^{-1}$ are given by

$$\{\pi^{bN}(\overline{x}), C_{1}^{aKL}(\overline{y})\}^{-1} = e^{-1} B_{bNtK} \delta(\overline{x} - \overline{y}) = -\{C_{1}^{aKL}(\overline{x}), \pi^{bN}(\overline{y})\}^{-1},$$

$$\{\pi^{bN}(\overline{x}), C_{A}(\overline{y})\}^{-1} = 0 = \{\pi^{bN}(\overline{x}), \overline{C}_{A}(\overline{y})\}^{-1},$$

$$\{C_{A}(\overline{x}), \overline{C}_{B}(\overline{y})\}_+^{-1} = i(e^{tt})^{-1} e^{tI} \gamma_{IAB} \delta(\overline{x} - \overline{y}) = \{\overline{C}_{B}(\overline{y}), C_{A}(\overline{x})\}_+^{-1},$$

$$\{C_{A}(\overline{x}), C_{1}^{aKL}(\overline{y})\}^{-1} = -\frac{1}{2} (e^{tt})^{-1} e^{tI} B_{bNtK} A^{bNTM} (\gamma_{I} \gamma_{M} \Psi)_{A} \delta(\overline{x} - \overline{y})$$

$$= \{C_{1}^{aKL}(\overline{y}), C_{A}(\overline{x})\}^{-1},$$

18
\{ C_A(\vec{x}), C_1^{aKL}(\vec{y}) \}^{-1} = \frac{1}{2} (eg^\mu)^{-1} e^H B_{bNKal} A^{bNLM} (\overline{\Psi}_{\gamma M} \gamma_I) A^\delta(\vec{x} - \vec{y})

= \{ C_1^{aKL}(\vec{y}), C_A(\vec{x}) \}^{-1}

and

\{ C_1^{aKL}(\vec{x}), C_{1PQ}^{b}(\vec{y}) \}^{-1} = -\frac{i}{4} (eg^\mu)^{-1} e^H B_{dNKLal} A^{dNLM} B_{cJlpq} A^{cJR} \times 

\overline{\Psi} (\gamma_M \gamma_I \gamma_R - \gamma_R \gamma_I \gamma_M) \Psi (\vec{x} - \vec{y})

from which we deduce the following non-zero Dirac brackets of the reduced phase space \( e_{aN}, \omega_{1aKL}, e_{1K}, \pi^{IK}, \Psi \) and \( \overline{\Psi} \) as

\{ e_{1N}(\vec{x}), \pi^{LM}(\vec{y}) \} = \delta_N^M \delta(\vec{x} - \vec{y}), \quad \{ e_{aN}(\vec{x}), \omega_{1bKL}(\vec{y}) \} = e^{-1} B_{aNtKbl} \delta(\vec{x} - \vec{y}),

\{ \omega_{1aKL}(\vec{x}), \omega_{1bPQ}(\vec{y}) \} = \{ C_1^{aKL}(\vec{x}), C_{1PQ}^{b}(\vec{y}) \}^{-1}, \quad (38)

\{ \Psi_A(\vec{x}), \omega_{1aKL}(\vec{y}) \} = \{ C_A(\vec{x}), C_1^{aKL}(\vec{y}) \}^{-1} = -\{ \omega_{1aKL}(\vec{y}), \Psi_A(\vec{x}) \}, \quad (39)

\{ \overline{\Psi}_A(\vec{x}), \omega_{1aKL}(\vec{y}) \} = \{ \overline{C}_A(\vec{x}), C_1^{aKL}(\vec{y}) \}^{-1} = -\{ \omega_{1aKL}(\vec{y}), \overline{\Psi}_A(\vec{x}) \}, \quad (40)

and

\{ \Psi_A(\vec{x}), \overline{\Psi}_B(\vec{y}) \} = -\{ C_A(\vec{x}), \overline{C}_B(\vec{y}) \}^{-1} = \{ \overline{\Psi}_B(\vec{y}), \Psi_A(\vec{x}) \}, \quad (41)

A direct computation leads to the Dirac bracket between the spatial diffeomorphism constraints as

\{ D_{sp}^r(\vec{N}), D_{sp}^r(\vec{N}') \} = D_{sp}^r(\vec{N}, \vec{N}') \quad (42)

and to the spacial diffeomorphism transformations of the reduced phase space elements as

\delta e_{\mu N} = \{ e_{\mu N}, D_{sp}^r(\vec{N}) \} = \mathcal{L}_{\vec{N}} e_{\mu N}, \quad \delta \omega_{1aKL} = \{ \omega_{1aKL}, D_{sp}^r(\vec{N}) \} = \mathcal{L}_{\vec{N}} \omega_{1aKL},

\delta \Psi_A = \{ \Psi_A, D_{sp}^r(\vec{N}) \} = \mathcal{L}_{\vec{N}} \Psi_A \quad \text{and} \quad \delta \overline{\Psi}_A = \{ \overline{\Psi}_A, D_{sp}^r(\vec{N}) \} = \mathcal{L}_{\vec{N}} \overline{\Psi}_A

from which we deduce that the scalar and Lorentz constraints transform like scalar densities of weight one

\{ D_{sp}^r(\vec{N}), D_{sp}^r(\vec{M}) \} = -\mathcal{L}_{\vec{N}}(D_{sp}^r) = -\partial_a(N^a D_{sp}^r)

\implies \{ D_{sp}^r(\vec{N}), D_{sp}^r(\vec{M}) \} = D_{sp}^r(\mathcal{L}_{\vec{N}}(\vec{M})) \quad (43)
\[ \left\{ \mathcal{D}_{sp}^r(\widehat{N}), M^{rKL} \right\}_D = \mathcal{L}_{\widehat{N}} M^{rKL} = -\partial_a (N^a M^{rKL}) \]

\[ \Rightarrow \left\{ \mathcal{D}_{sp}^r(\widehat{N}), \mathcal{M}^r(\theta) \right\}_D = \mathcal{M}^r (\mathcal{L}_{\widehat{N}} (\theta)). \] (44)

The reduced phase space elements transform under \( M^r(\theta) \) as infinitesimal gauge transformations

\[ \delta e_{\mu N} = \{ e_{\mu N}, M^r(\theta) \}_D = \theta^M_N e_{\mu M}, \quad \delta \omega_{1aKL} = \{ \omega_{1aKL}, M^r(\theta) \}_D = -D_{1a} \theta_{KL}, \]

\[ \delta \Psi = \{ \Psi, M^r(\theta) \}_D = \frac{\sigma_{KL}}{2} \Psi \theta_{KL}, \quad \delta \overline{\Psi} = \{ \overline{\Psi}, M^r(\theta) \}_D = -\overline{\Psi} \frac{\sigma_{KL}}{2} \theta_{KL} \]

which imply that \( M^{rKL} \) transforms like a contravariant tensor

\[ \left\{ M^{rKL}, M^r(\theta) \right\}_D = \theta^K_P M^{rPL} + \theta^L_P M^{rKP} \]

leading to the \( so(1, d-1) \) Lie algebra. From the transformations of the reduced phase space the smeared scalar constraint \( \mathcal{D}^r_i(M) \) transforms like a scalar

\[ \left\{ M^r(\theta), \mathcal{D}^r_i(M) \right\}_D = 0. \] (45)

The Dirac bracket between the smeared scalar constraints consist of the following three parts: the first part concerns the purely gravitational part

\[ \left\{ \int M e^A a^{Kb} \Omega_{1abKL} \frac{\Omega_{1cdNM}}{2}, - \int M' e^A c^{NdM} \frac{\Omega_{1cdNM}}{2} \right\}_D \]

\[ = - \int \partial_a M \partial_c M' \frac{i}{4g^t} e^A e^{aK} e^{cN} e^{tI} \Psi (\gamma_K \gamma_I \gamma_N - \gamma_N \gamma_I \gamma_K) \Psi \]

\[ + \int (M \partial_a M' - M' \partial_a M) \]

\[ + \frac{i}{4g^t} e^{tI} e^A B_{dRNdM} A^{dRS} D_c (\gamma_K \gamma_I \gamma_S - \gamma_S \gamma_I \gamma_K) \Psi \]

which does not vanish as in the pure gravity. This is due to the non vanishing Dirac brackets of the connection with itself in the presence of the fermionic
field. The fermionic part gives

\[
\begin{aligned}
&\left\{ \int -M' \frac{i}{2} e e^{aK} (\overline{\psi} \gamma_K D_a \psi - D_a \overline{\psi} \gamma_K \psi), -M' \frac{i}{2} e e^{cN} (i \overline{\psi} \gamma_N D_c \psi - i D_c \overline{\psi} \gamma_N \psi) \right\}_D \\
+ &\left\{ \int -M' \frac{i}{2} e e^{aK} (\overline{\psi} \gamma_K D_a \psi - D_a \overline{\psi} \gamma_K \psi), \int M' e m \overline{\psi} \psi \right\}_D - (M \Leftrightarrow M') \\
+ &\left\{ \int M e m \overline{\psi} \psi, \int M' e m \overline{\psi} \psi \right\}_D
\end{aligned}
\]

\[
= \int (M \partial_a M' - M' \partial_a M) \frac{i}{4 g t} e e^{aK} e^{bK} e^{tL} (\overline{\psi} \gamma_K \gamma_I \gamma_L D_b \psi - D_b \overline{\psi} \gamma_L \gamma_I \gamma_K \psi)
\]

\[
+ \int \frac{1}{8 g t} (e^{aK} e^{bL} e^{tM} (\gamma_L \gamma_K \gamma_I \gamma_N - \gamma_N \gamma_I \gamma_K)) B_{\overline{\psi} \gamma_S \gamma_I} A^{dRtM} \overline{\psi} (\gamma_K \gamma_I \gamma_M - \gamma_M \gamma_I \gamma_K))
\]

\[
- \int \frac{i}{4 g t} \gamma_N D_c \psi - D_c \overline{\psi} \gamma_N \psi - M' e m \overline{\psi} \psi \right\}_D
\]

and the third part gives

\[
\begin{aligned}
&(M \Leftrightarrow M') = \int \frac{i}{2 g t} \partial_a M \partial_a \partial_c M' e e^{aK} e^{cN} e^{tI} (\overline{\psi} \gamma_K \gamma_I \gamma_N - \gamma_N \gamma_I \gamma_K) \psi \\
+ &\int (M \partial_a M' - M' \partial_a M) \frac{i}{4 g t} e e^{aK} e^{bL} e^{tM} (\overline{\psi} \gamma_K \gamma_I \gamma_L D_b \psi - D_b \overline{\psi} \gamma_L \gamma_I \gamma_K \psi)
\]

\[
- \int \frac{1}{8 g t} e^{aK} e^{bL} e^{tM} (\gamma_L \gamma_K \gamma_I \gamma_N - \gamma_N \gamma_I \gamma_K)) B_{\overline{\psi} \gamma_S \gamma_I} A^{dRtM} \overline{\psi} (\gamma_K \gamma_I \gamma_M - \gamma_M \gamma_I \gamma_K))
\]

\[
- \frac{i}{4 g t} e^{aK} e^{tI} (\overline{\psi} \gamma_K \gamma_I \gamma_L D_b (e e^{bL} \overline{\psi} - D_b (e e^{bL} \overline{\psi} \gamma_L \gamma_I \gamma_K \psi) + \frac{e m}{g t} e^{aM} \overline{\psi} \psi)
\]

\[
- \left\{ \int -M \partial_a M' - M' \partial_a M \right\} \times
\]

\[
(\frac{i}{4 g t} e^{aK} B_{eRNdM} A^{dRtS} D_c (e A^{cN} D_M) \overline{\psi} (\gamma_K \gamma_I \gamma_S - \gamma_S \gamma_I \gamma_K)) \psi.
\]
These three parts cancel each other leading to
\[ \{D^r_i(M), D^r_i(M')\}_D = 0 \]
which shows that the Dirac bracket of the reduced scalar constraints strongly vanishes. This shows that the reduced first-class constraints satisfy a closed algebra with structure constants.

It is easy to show that the physical degrees of freedom of the reduced phase space match with those of the \( d \)-dimensional general relativity coupled with the fermionic field.

Note that the reduced first-class constraints are polynomial but not the Dirac brackets of the reduced phase space elements. In addition, the Dirac brackets of the dynamic spacial connection with itself and with the fermionic field do not vanish.

To get a new reduced phase space which is canonical with respect to the Dirac brackets, we consider the following canonical transformation
\[ e_aN \rightarrow e_aN, \]
\[ \omega_{1aKL} \rightarrow \mathcal{P}^{cN}(e, \omega_1, \Psi, \overline{\Psi}) = eB^{cNtK}aL\omega_{1aKL} + \frac{i}{2}eA^{cNtK}aL\overline{\Psi}_I\gamma_K\gamma^I e^I, \]
\[ \Psi \rightarrow \Psi, \]
\[ \overline{\Psi} \rightarrow \Pi(e, \overline{\Psi}) = iee^I\overline{\Psi}_I \]
which results from the fact that the new variables \( e_aN, \mathcal{P}^{cN}, \Psi \) and \( \Pi \) of the reduced phase space result from the new symplectic form of the action which differs from the one of (1) by a total derivative with respect of time
\[
\begin{align*}
\int_M & \left( eA^aKL\partial_t \omega_{1aKL} + ee^I i2 \overline{\Psi}_I \partial_t e^I - i2 (\partial_t e^I) \gamma_I e^I \right) \\
& = \int_M \left( eB^{cNtK}aL\omega_{1aKL} + i2 eA^{cNtK}aL\overline{\Psi}_I\gamma_K\gamma^I e^I \partial_t e^I + iee^I\overline{\Psi}_I \partial_t e^I \right).
\end{align*}
\]

Note that the the rank of the projector \( P_{1aKL}^{a\mathcal{Q}} \) is \( d(d-1) \), implying that the number of the independent components of \( \omega_{1aKL} \) is equal to that of the co-tetrad \( e_aN \) and that of \( \mathcal{P}^{aN} \).

The primary constraints become
\[ \pi^{cN} \rightarrow C^{aN} = \mathcal{P}^{cN} - eB^{cNtK}aL\omega_{1aKL} - \frac{i}{2}eA^{cNtK}aL\overline{\Psi}_I\gamma_K\gamma^I e^I \approx 0, \]
\[ C_1^{aKL} \rightarrow C_1^{aKL} = \mathcal{P}_1^{aKL} \approx 0, \]
\[ C \rightarrow C = \Pi - iee^I\overline{\Psi}_I \gamma_I \approx 0 \text{ and} \]
\[ \overline{C} \rightarrow \overline{C} = \overline{\Pi} \approx 0 \]

(47)
but the super matrix \( \{C_i, C_j\} \) and its inverse \( \{C_i, C_j\}^{-1} \) keep the same form. This new reduced phase space is canonical in the sense that their non-zero Dirac brackets are given by

\[
\{ e_a N, P^{bM}(\vec{y}) \}_D = \delta_b^a \delta_N^M \delta(\vec{x} - \vec{y}), \\
\{ \Psi_A(\vec{x}), \Pi_B(\vec{y}) \}_+ = \delta_{AB} \delta(\vec{x} - \vec{y}).
\]

These canonical relations can be obtained either by using directly the Dirac brackets with the constraints (47) or by using the Dirac brackets between the previous reduced phase space variables.

The inverse of the canonical transformation (46) gives \( \omega_{1aKL} \) and \( \Psi \) as functions of the canonical phase space

\[
\omega_{1aKL}(e, P, \Psi, \Pi) = P_{bM} e^{-1} B_{bMtKaL} - B_{bMtKaL} i A^{bM} N \gamma_N \Psi
\]

and

\[
\Psi(e, \Pi) = -i(e g^{tt})^{-1} e^{tI} \Pi \gamma_I
\]

leading to polynomial constraints of first-class of the form

\[
D^r_{sp}(\vec{N}) = \int_\Sigma \left( \pi^{tK} L^{N}(e_{tK}) + P^{aK} L^{N}(e_{aK}) + \Pi L^{N}(\Psi) \right)
\]

and

\[
M^r(\theta) = \int_\Sigma \left( \partial_a (e A^{aKtL}) + \frac{1}{2} (P^{aK} e_{aL} - P^{aL} e_{aK}) + \frac{\Pi \sigma^{KL}}{2} \Psi \right) \theta_{KL} = \int_\Sigma M^{rKL} \frac{\theta_{KL}}{2}
\]

but the scalar constraint \( D^r(M) \) take a non polynomial form.

A calculation analogous to that performed in [1], to demonstrate Jacobi’s identities, shows that the dynamic spatial connection (49) satisfies the same projected Dirac brackets as that of the reduced phase (38-40). This leads to the same closed algebra of reduced first-class constraints expressed in terms of the new reduced phase space equipped with the canonical Dirac brackets (48).

4 Conclusion

We have showed in this paper that a modified action of the tetrad-gravity coupled with the fermionic field where the non-dynamic part of the connection is fixed to zero leads to a consistent Hamiltonian formalism in any
dimension $d \geq 3$ free of the Barbero-Immirzi parameter. Contrarily to the different Hamiltonian formalisms based on the ADM construction where the algebra of the first-class constraint is closed with structure functions, here we are in presence of an algebra of first-class constraints which closes with structure constants. Therefore, this algebra generates true Lie-group transformations expressing the local invariance of the tetrad-gravity coupled with the fermionic matter under Lorentz transformations $M^r(\theta)$ and diffeomorphism whose smeared generators are the scalar constraint $D_t(M)$ and the vector constraint $D_{sp}(\bar{N})$. The scalar function $M$ and the spatial vector field $\bar{N}$ may be interpreted as the usual lapse and shift respectively although they do not result from the A.D.M. formalism [15]. The absence of the structure functions is due to the fact that the scalar function $M$ and the spatial vector field $\bar{N}$ are introduced here only as smeared functions independently to the decomposition A.D.M of the tangent space which requires the metric which appears in the structure functions [16].

Although the Dirac brackets of the reduced phase space between the dynamic spatial connection with itself and with the fermionic fields are complicated, the first-class constraints satisfy a closed algebra with structure constant analogous to the one of the pure gravity. We have also showed that a canonical transformation leads to a new reduced phase space endowed with Dirac brackets having a canonical form leading to the same algebra with structure constants.

Now we can investigate the contribution of the torsion to the connections. For that, we solve the Hamiltonian equations in terms of the Dirac brackets. The temporal evolution of the tetrad components is given by

$$\frac{de_{aN}}{dt} = \{e_{aN}, \mathcal{H}_T\}_D = D_a e_{tN} - B a_{NtK\mu L} e^{\mu I} \frac{i e}{2} \bar{\Psi} (\gamma_I \frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_I) \Psi$$

$$+ B a_{NtKL} M^{rKL} - \omega_{tN}^M e_{aM}$$

where the constraint $M^{rKL}$ is expressed with $\pi^{tN} = 0$. Modulo the constraint $M^{rKL}$, which is the equation obtained by varying the action by connection $\omega_{tKL}$, we get

$$D_t e_{aN} - D_a e_{tN} = - B a_{NtK\mu L} e^{\mu I} \frac{i e}{2} \bar{\Psi} (\gamma_I \frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_I) \Psi. \quad (51)$$

The solutions of this equation and $M^{rKL} = 0$ give the expression of the connection in terms of the tetrad and the fermionic field. First, the equation

$$M^{rKL} = 2 D_a (eA^{aKL}) + i e e^{I\mu} \bar{\Psi} (\gamma_I \frac{\sigma^{KL}}{2} + \frac{\sigma^{KL}}{2} \gamma_I) \Psi = 0 \quad (52)$$
can be solved for the torsion by splitting the connection as \( \omega_{\mu KL} = \tilde{\omega}_{\mu KL} + C_{\mu KL} \) \([17]\) where \( \tilde{\omega}_{\mu KL} = e_{\nu K} \nabla_{\mu} e_{L} = e_{\nu K} (\partial_{\mu} e_{L}^{\nu} + \Gamma_{\mu\nu}^{\rho} e_{L}^{\rho}) \) is the free-torsion part of the connection compatible with the tetrad and \( C_{\mu KL} \) is the contorsion. Since \( P_{2K_{\mu L} N b_{M} \omega_{\nu b N M}} \) is the solution of the homogeneous equation \([52]\), the general solution is \( \omega_{a KL}^{s} = \tilde{\omega}_{a KL}^{s} + C_{a KL}^{s} \) and so the equations \([51]\) and \([52]\) are expressed with the full solution \( \omega_{a KL}^{s} \). The solution of \([52]\) is given by

\[
D_{a} e_{b N} - D_{b} e_{a N} = C_{a N}^{M} e_{b M} - C_{b N}^{M} e_{a M} = B_{a N b K L} e_{0}^{I} e_{I}^{J} (\gamma_{I} \frac{\sigma_{K L}}{2} + \frac{\sigma_{K L}}{2} \gamma_{I}) \Psi^{s} (A.1)
\]

where, as a consequence of \((A.8)\), the second term of the right hand side is the solution of the homogeneous equation \([52]\). \([51]\) and \([53]\) are exactly the solutions of the equations obtained by varying with respect of the connection the action of gravity coupled minimally with the fermions.

We end this work by noting that the main purpose of the canonical formalism of the gravity is its canonical quantization which is made off-shell with the connection not with its free-torsion part and the contortion which result from the solutions of equation of motion.

5 Appendix A

In this appendix, we collect the properties of the functions \( A, B \) \([1], [18]\) and the projectors used in the Hamiltonian analysis.

\[
e A^{\mu K \nu L} = \frac{\delta}{\delta e_{\mu K}} e^{\nu L} = \frac{1}{(d - 2)!} \epsilon_{I_{0}...I_{d-3} K L} e_{\mu I_{0}}...e_{\mu I_{d-3}} e^{\nu_{0}...\nu_{d-3} \mu \nu} = e (e^{\mu K} e^{\nu L} - e^{\nu K} e^{\mu L}) = -e A^{\nu K \mu L} = -e A^{\mu L \nu K} (A.1)
\]

\[
e B^{\beta N \mu K \nu L} = \frac{1}{(d - 3)!} \epsilon_{I_{0}...I_{d-4} N K L} e_{\mu I_{0}}...e_{\mu I_{d-4}} e^{\nu_{0}...\nu_{d-4} \beta \mu} = e (e^{\beta N} A^{\mu K \nu L} + e^{\beta K} A^{\mu L \nu N} + e^{\beta L} A^{\mu N \nu K}) = e (e^{\beta N} A^{\mu K \nu L} + e^{\rho N} A^{\nu K \beta L} + e^{\nu N} A^{\beta K \mu L}) = \frac{\delta}{\delta e_{\beta N}} e A^{\mu K} (A.2)
\]
with its inverse

\[
B_{\mu\nu K\alpha L} = \frac{1}{2} \left( \epsilon_{\mu N} \frac{A_{\nu K\alpha L}}{d-2} + \epsilon_{\nu N} \frac{A_{\alpha K\mu L}}{d-2} + \epsilon_{\alpha N} A_{\mu K\nu L} \right)
\]

in the sense that

\[
B_{\mu\nu K\alpha L} B_{\mu\nu P Q} = \delta^{P}_{\alpha} (\delta_{Q}^{P} \delta_{K}^{L} - \delta_{L}^{P} \delta_{K}^{Q})
\]

and

\[
B_{\mu\nu K\alpha L} B_{\rho M\sigma K\alpha L} = \delta^{M}_{\mu} \delta^{\sigma}_{N} \delta^{P}_{\alpha}
\]

We see from (A.3) that, unlike \(A_{\mu K\nu L}\) and \(B_{\mu\nu K\alpha L}\) which are antisymmetric in interchange of two indices of the same nature, \(B_{\mu K\nu L}\) is only antisymmetric on the indices \(\mu\) and \(\nu\) and on the indices \(K\) and \(L\). As a consequence of the antisymmetric of the indices \(\mu\), \(\nu\) and \(\alpha\) of \(B_{\mu K\nu L}\), (A.5) gives

\[
B_{cNtK\alpha L} B^{bMtK\alpha L} = \delta^{M}_{N} \delta^{b}_{c}
\]

for \(\sigma = \nu = t\) and

\[
B_{cNdK\alpha L} B^{bMtK\alpha L} = 0
\]

for \(\sigma = t\) and \(\nu = d\). (A.4) leads to

\[
B^{aNhPQ} B_{aNh\mu L} = \delta^{t}_{\mu} (\delta_{Q}^{P} \delta_{L}^{K} - \delta_{L}^{P} \delta_{K}^{Q})
\]

A straightforward computation shows that the contraction of \(B_{cNdK\alpha L}\) which one of the components \(e_{cN}, e_{dN}, e_{aL}, e_{cK}, e_{tN}, e_{tK}\) or \(e^{KL}\) vanishes. From (A.4), (A.5) and (A.6) we deduce that

\[
P_{1KaL}^{pdQ} = B_{bNhK\alpha L} B^{bNhPdQ} \quad \text{and} \quad P_{2KaL}^{pdQ} = \frac{1}{2} B_{bNhK\alpha L} B^{bNhPdQ}
\]

are projectors:

\[
P_{1KaL}^{pdQ} + P_{2KaL}^{pdQ} = \frac{1}{2} \delta_{a}^{d} (\delta_{Q}^{P} \delta_{L}^{K} - \delta_{L}^{P} \delta_{K}^{Q})
\]

\[
P_{1KaL}^{NhM} P_{1NhM}^{pdQ} = P_{1KaL}^{pdQ}, \quad P_{2KaL}^{NhM} P_{2NhM}^{pdQ} = P_{2KaL}^{pdQ}
\]

and

\[
P_{1KaL}^{NhM} P_{2NhM}^{pdQ} = 0.
\]
Their ranks are given by their trace
\[ \frac{1}{2} \delta_d^a (\delta^K_P \delta^L_Q - \delta^K_P \delta^L_P) P_{1KaL}^{PdQ} = d(d-1) \]
and
\[ \frac{1}{2} \delta_d^a (\delta^K_Q \delta^L_P - \delta^K_P \delta^L_P) P_{2KaL}^{PdQ} = \frac{1}{2} d(d-1)(d-3). \]

Inserting \( \omega_{2aKL} \) in the symplectic part of the action, we get
\[
e^{AaKtL} \partial_t \omega_{2aKL} = \partial_t (e^{AaKtL} \omega_{2aKL}) - \partial_t (e^{AaKtL}) \omega_{2aKL} = e^{B^NtKaN} \partial_t \omega_{2aKL} = 0 \quad (A.10)
\]
which shows that \( \omega_{2aKL} \) is non-dynamic part in the sense that it does not contribute to the symplectic part of the action. Similarly for
\[
D_{2a} \omega_{1KL} = P_{2KaL}^{PdQ} D_d \omega_{1PQ} = e^{AaKtL} D_{2a} \omega_{1KL} = 0.
\]

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