SIGMA THEORY AND TWISTED CONJUGACY-II: HOUGHTON GROUPS AND PURE SYMMETRIC AUTOMORPHISM GROUPS

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Abstract Let \( \phi : \Gamma \to \Gamma \) be an automorphism of a group \( \Gamma \). We say that \( x, y \in \Gamma \) are in the same \( \phi \)-twisted conjugacy class and write \( x \sim_{\phi} y \) if there exists an element \( \gamma \in \Gamma \) such that \( y = \gamma x \phi(\gamma^{-1}) \). This is an equivalence relation on \( \Gamma \) called the \( \phi \)-twisted conjugacy. Let \( R(\phi) \) denote the number of \( \phi \)-twisted conjugacy classes in \( \Gamma \). If \( R(\phi) \) is infinite for all \( \phi \in \text{Aut}(\Gamma) \), we say that \( \Gamma \) has the \( R_{\infty} \)-property.

The purpose of this note is to show that the symmetric group \( S_\infty \), the Houghton groups and the pure symmetric automorphism groups have the \( R_{\infty} \)-property. We show, also, that the Richard Thompson group \( T \) has the \( R_{\infty} \)-property. We obtain a general result establishing the \( R_{\infty} \)-property of finite direct product of finitely generated groups.

This is a sequel to an earlier work by Gonçalves and Kochloukova, in which it was shown, using the sigma-theory due to Bieri-Neumann-Strebel, that for most of the groups \( \Gamma \) considered here, \( R(\phi) = \infty \) where \( \phi \) varies in a finite index subgroup of the automorphisms of \( \Gamma \).

1. Introduction

Let \( \Gamma \) be a group and let \( \phi : \Gamma \to \Gamma \) be an endomorphism. Then \( \phi \) determines an action \( \Phi \) of \( \Gamma \) on itself where, for \( \gamma \in \Gamma \) and \( x \in \Gamma \), we have \( \Phi_{\gamma}(x) = \gamma x \phi(\gamma^{-1}) \). The orbits of this action are called the \( \phi \)-twisted conjugacy classes. We write \( x \sim_{\phi} y \) if \( x \) and \( y \) are in the same \( \phi \)-twisted conjugacy class. Note that when \( \phi \) is the identity automorphism, the orbits are the usual conjugacy classes of \( \Gamma \). We denote by \( \mathcal{R}(\phi) \) the set of all \( \phi \)-twisted conjugacy classes and by \( R(\phi) \) the cardinality \( \# \mathcal{R}(\phi) \) of \( \mathcal{R}(\phi) \). We say that \( \Gamma \) has the \( R_{\infty} \)-property if \( R(\phi) = \infty \), that is if \( \mathcal{R}(\phi) \) is infinite, for every automorphism \( \phi \) of \( \Gamma \).

The problem of determining which groups have the \( R_{\infty} \)-property—more briefly the \( R_{\infty} \)-problem—has attracted the attention of many...
researchers after it was discovered that all non-elementary Gromov-
hyperbolic groups have the $R_\infty$-property. See [18] and [12]. It is par-
ticularly interesting when the group in question is finitely generated or
countable. The notion of twisted conjugacy arises naturally in fixed
point theory, representation theory, algebraic geometry and number
theory. In recent years the $R_\infty$-problem has emerged as an active re-
search area.

Recall that Houghton introduced a family of groups $H_n$, $n \geq 2$, de-
fined as follows: Let $M_n := \{1, 2, \ldots, n\} \times \mathbb{N}$. The group $H_n$ con-
sists of all bijections $f : M_n \rightarrow M_n$ such that there exists integers
t_1, \ldots, t_n such that $f(j, s) = (j, s + t_j)$ for all $s \in \mathbb{N}$ sufficiently
large and all $j \leq n$. Note that necessarily $\sum_{1 \leq j \leq n} t_j = 0$. Let
$Z = \{(t_1, \ldots, t_n) \mid \sum_{1 \leq j \leq n} t_j = 0\} \subset \mathbb{Z}^n \cong \mathbb{Z}^{n-1}$. One has a sur-
jective homomorphism $\tau : H_n \rightarrow Z \cong \mathbb{Z}^{n-1}$ sending $f$ to its translation
part $(t_1, \ldots, t_n)$ (with notation as above). It is easily verified that $\tau$
is surjective with kernel the group of all
finitary
permutations of $M_n$. K. S. Brown [6] showed that
$H_n$ is finitely presented for $n \geq 3$ and that it is $\text{FP}_{n-1}$ but not $\text{FP}_n$. Note that the above definition of $H_n$
makes sense even for $n = 1$ and we have $H_1 \cong S_\infty$. However, we treat
the group $S_\infty$ separately and we shall always assume that $n \geq 2$ while
considering the family $H_n$.

Next we recall the group $G_n$, the group of pure symmetric automor-
phisms of the free group $F_n$ of rank $n \geq 2$. Fix a basis $x_k, 1 \leq k \leq n,$
of $F_n$. Denote by $\alpha_{ij} \in \text{Aut}(F_n), 1 \leq i \neq j \leq n$, the automorphism
defined as $x_i \mapsto x_j x_i x_j^{-1}, x_k \mapsto x_k, 1 \leq k \leq n, k \neq i$. The group $G_n$
is the subgroup of $\text{Aut}(F_n)$ generated by $\alpha_{ij}, 1 \leq i \neq j \leq n$. McCool
showed that $G_n$ is finitely presented where the generating relations are:
(i) $[\alpha_{ij}, \alpha_{kl}] = 1$, whenever $i, j, k, l$ are all different;
(ii) $[\alpha_{ik}, \alpha_{jk}] = 1$ and $[\alpha_{ij} \alpha_{kj}, \alpha_{ik}] = 1$ whenever $i, j, k$ are all different.

It was shown by Gonçalves and Kochloukova [14] that $R(\phi) = \infty$
for all $\phi$ in a finite index subgroup of the group of all automorphisms
of $\Gamma$ where $\Gamma = H_n, G_n$. Our main result is the following theorem. We
give two proofs for the case of Houghton groups, neither of which use
$\Sigma$-theory. However we still need to use the results of [14] in the case of
$G_n$.

\textbf{Theorem 1.1.} The following groups have the $R_\infty$-property:
(i) The group $S_\infty$ of finitary permutations of $\mathbb{N}$,
(ii) the Houghton groups $H_n, n \geq 2$, and,
(iii) the group $G_n, n \geq 2$, of pure symmetric automorphisms of a free
group of rank $n$. 
Recall that Richard Thompson constructed three groups finitely presented infinite groups $F \subset T \subset V$ around 1965 and showed that $T$ and $V$ are simple. The groups $F, T$, and $V$ arise as certain homeomorphism groups of the reals, the circle, and the Cantor set respectively. Since then these constructions have been generalized by G. Higman [17]. See also K. S. Brown [6], R. Bieri and R. Strebel [3], and M. Stein [23]. For an introduction to the Thompson groups $F, T, V$ see [10].

**Theorem 1.2.** The Richard Thompson group $T$ has the $R_\infty$-property.

As the group $T$ is simple, $\Sigma$-theory yields no information about the $R_\infty$-property. The above theorem was first proved by Burillo, Matucci, and Ventura [8]. Shortly thereafter, Gonçalves and Sankaran [15] also independently obtained the same result.

In section §2 we make some preliminary observations concerning the $R_\infty$-property which will be needed for our purposes. Theorem 1.1 will be established in §3. The $R_\infty$-property of the group $T$ will be proved in §4. In §5 we consider the $R_\infty$-property finite direct product of groups and obtain a strengthening of a result of Gonçalves and Kochloukova [14].

This is a sequel to the paper [14] by Gonçalves and Kochloukova. We reassure the reader that this paper can be read independently of it. Although results from [14] are used, we develop our own proof techniques to go forward.

If $f : X \to Y$ is a map of sets, we shall always write the argument to the right of $f$; thus $f(x)$ denotes the image of $x \in X$ under $f$.

2. Preliminaries

We begin by recalling some general result concerning twisted conjugacy classes of an automorphism of a group and that of its restriction to a normal subgroup. We obtain a criterion for a periodic automorphism to have infinitely many twisted conjugacy classes. We shall also briefly recall the notion of the Bieri-Neumann-Strebel invariant and give its known description in the case of Houghton groups and the pure symmetric automorphism groups.

2.1. **Addition formula.** The following lemma may be found in [16, §2]. For any element $g \in G$, we shall denote by $\iota_g$ the inner automorphism $x \mapsto gxg^{-1}$ of $G$. When $N$ is a normal subgroup of $G$, we shall abuse notation and denote by the same symbol $\iota_g$ the automorphism of $N$ got by restriction of $\iota_g$ to $N$. 
Lemma 2.1. Suppose that we have a commutative diagram of homomorphisms of groups where the vertical arrows are isomorphisms and horizontal rows are short exact sequence:

\[
\begin{array}{cccccc}
1 & \to & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \to & 1 \\
\downarrow{\theta'} & & \downarrow{\theta} & & \downarrow{\bar{\theta}} & & \\
1 & \to & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \to & 1
\end{array}
\]

Then:

(i) One has an exact sequence of (pointed) sets \( \mathcal{R}(\theta') \xrightarrow{i^*} \mathcal{R}(\theta) \xrightarrow{p_*} \mathcal{R}(\bar{\theta}) \to \{0\} \). That is, \( p_* \) is surjective and \( \text{Im}(i_*) = p_*^{-1}(\{N\}) \).

(ii) (Addition Formula): Suppose that \( \mathcal{R}(\bar{\theta}) < \infty \) and that \( \text{Fix}(\iota_{\alpha N} \circ \bar{\theta}) = \{N\} \) for all \( \alpha \in G \). Then \( \mathcal{R}(\theta) < \infty \) if and only if \( \mathcal{R}(\iota_{\alpha} \theta') < \infty \) for all \( \alpha \in G \). Moreover, the following addition formula holds if \( \mathcal{R}(\theta) < \infty \):

\[
\mathcal{R}(\theta) = \sum_{[\alpha N] \in \mathcal{R}(\bar{\theta})} \mathcal{R}(\iota_{\alpha} \theta').
\]

Proof. We prove only (ii), assertion (i) being well-known and easy.

Let \( \alpha \in G \), and \( x, y \in N \). Suppose that \( x, y \) are in the same \( \iota_{\alpha} \theta' \)-twisted conjugacy class when regarded as elements of \( G \). That is \( y = zx\iota_{\alpha} \theta(z^{-1}) \) for some \( z \in G \). Applying the projection \( p : G \to G/N \) and denoting \( xN \) by \( \bar{x} \) we obtain \( \bar{1} = \bar{y} = \bar{z} \iota_{\alpha} \bar{\theta}(\bar{z}^{-1}) \) and so \( \bar{z} \in \text{Fix}(\iota_{\alpha} \bar{\theta}) = \{\bar{1}\} \). Hence \( z \in N \) and so \( y \) and \( x \) are \( \iota_{\alpha} \theta' \)-twisted conjugates. Thus we see that distinct \( \iota_{\alpha} \theta' \)-conjugacy classes map to distinct \( \iota_{\alpha} \theta \)-conjugacy classes. This implies that \( \mathcal{R}(\theta') < \infty \) if \( \mathcal{R}(\theta) < \infty \).

Also, \( y = zx\iota_{\alpha} \theta(z^{-1}) \) implies that \( y\alpha = z(x\alpha) \theta(z^{-1}) \) and so \( y\alpha, x\alpha \in G \) are \( \theta \)-twisted conjugates. Thus translation on the right by \( \alpha \) of \( \theta' \)-twisted conjugacy classes of \( N \) are contained \( \theta \)-twisted conjugacy classes.

Suppose that \( x\alpha, x\beta \) are in the same \( \theta \)-twisted conjugacy class where \( x \in N, \alpha, \beta \in G \). Then there exists a \( u \in G \) such that \( x\beta = ux\alpha \theta(u^{-1}) \). Applying the projection we obtain that \( \bar{\beta} = \bar{u} \bar{\alpha} \bar{\theta}(\bar{u}^{-1}) \). Thus \( \bar{\alpha} \) and \( \bar{\beta} \) are \( \theta \)-twisted conjugates. This implies that the right translates of \( \theta' \)-twisted conjugacy classes by elements \( \alpha, \beta \) belong to the same \( \theta \)-twisted conjugacy class only if that \( \bar{\alpha} \) and \( \bar{\beta} \) are \( \bar{\theta} \)-twisted conjugates. Conversely, if \( \bar{\alpha} \) and \( \bar{\beta} \) are \( \bar{\theta} \) twisted conjugates, then by reversing the arguments, we see that the translates by \( \alpha \) and \( \beta \) of \( \theta' \)-twisted conjugacy classes are contained in the same \( \theta \)-twisted conjugacy class. This establishes the addition formula. \( \square \)

Remark 2.2. Note that if \( G/N \cong \mathbb{Z}^n, n < \infty \), and if \( 1 \) is not an eigenvalue of the matrix of \( \bar{\theta} \) with respect to a basis of \( G/N \), then for
any \( \alpha \in G \), \( \text{Fix}(\iota_{\alpha N} \circ \bar{\theta}) = \text{Fix}(\bar{\theta}) \) consists only of the trivial element. So the lemma implies that, if \( R(\theta') = \infty \), then \( R(\theta) = \infty \).

### 2.2. Periodic outer automorphisms

Let \( \Gamma \) be a group with infinitely many conjugacy classes. Then, for any automorphism \( \phi : \Gamma \rightarrow \Gamma \), and any \( g \in G \), \( R(\phi) = R(\iota_g \circ \phi) \) where \( \iota_g \) denotes the inner automorphism \( x \mapsto gxg^{-1} \). Indeed it is readily seen that the \( \bar{\phi} \)-twisted conjugacy classes are the same as the left translation by \( g \) of the \( \iota_g \circ \phi \)-twisted conjugacy classes. Thus \( \Gamma \) has the \( R_\infty \)-property if and only if \( R(\bar{\phi}) = \infty \) for a set of coset representatives of \( \text{Out}(\Gamma) = \text{Aut}(\Gamma)/\text{Inn}(\Gamma) \). We have the following lemma. Compare [15].

**Lemma 2.3.** Let \( \theta \in \text{Aut}(\Gamma) \) and let \( n \geq 1 \). Suppose that \( \{ x^n \mid x \in \text{Fix}(\theta) \} \) is not contained in the union of finitely many \( \theta^n \)-twisted conjugacy classes of \( \Gamma \). Then \( R(\theta) = \infty \).

**Proof.** Let \( x \sim_{\theta} y \) in \( \Gamma \) where \( x, y \in \text{Fix}(\theta) \). Thus there exists an \( z \in \Gamma \) such that \( y = z^{-1}x\theta(z) \). Applying \( \theta^i \) both sides, we obtain \( y = \theta^i(z^{-1})x\theta^{i+1}(z) \) as \( x, y \in \text{Fix}(\theta) \). Write \( \phi := \theta^n \). Multiplying these equations successively for \( 0 \leq i < n \), we obtain

\[
y^n = \prod_{0 \leq i < n} \theta^i(z^{-1})x\theta^{i+1}(z) = z^{-1}x^n\theta^n(z) = z^{-1}x^n\phi(z).
\]

That is, \( y^n \sim_{\phi} x^n \). Our hypothesis says that there are infinitely many elements \( x_k \in \text{Fix}(\theta), k \geq 1 \), such that the \( x_k^n \) are in pairwise distinct \( \phi \)-twisted conjugacy classes of \( \Gamma \). Hence we conclude that \( R(\theta) = \infty \). \( \square \)

**Remark 2.4.** When \( \theta^n = \iota_\gamma \) is an inner automorphism, we see from the above lemma that \( R(\theta) = \infty \) if \( \{ x^n\gamma \mid x \in \text{Fix}(\theta) \} \) is not contained in a finite union of conjugacy classes of \( \Gamma \). If \( \theta^n = \text{id} \), then \( R(\theta) = \infty \) if \( \text{Fix}(\theta) \) contains elements of order \( k \) for arbitrarily large values of \( k \in \mathbb{N} \).

### 2.3. \( \Sigma \)-theory of \( H_n \) and \( G_n \)

Bieri, Neumann, and Strebel [2] introduced, for any finitely generated group \( \Gamma \), an invariant \( \Sigma(\Gamma) \) which is a certain open subset—possibly empty—of the character sphere \( S(\Gamma) := \text{Hom}(\Gamma, \mathbb{R}) \setminus \{ 0 \} / \mathbb{R}_{>0} \) where the action of the multiplicative group of positive reals is via scalar multiplication. The automorphism group \( \text{Aut}(\Gamma) \) acts on \( S(\Gamma) \) where \( \phi^* : S(\Gamma) \rightarrow S(\Gamma) \) is defined as \( [\chi] \mapsto [\chi \circ \phi], [\chi] \in S(\Gamma) \), for \( \phi \in \text{Aut}(\Gamma) \). This action preserves the subspace \( \Sigma(\Gamma) \) and hence also its complement \( \Sigma^c(\Gamma) \). If the image of the homomorphism \( \eta : \text{Aut}(\Gamma) \rightarrow \text{Homeo}(\Sigma^c(\Gamma)) \) is a finite group, then \( K = \ker(\eta) \) is a finite index subgroup of \( \text{Aut}(\Gamma) \) which fixes every character class in \( \Sigma^c(\Gamma) \). This happens, for example, if \( \Sigma^c(\Gamma) \) is a non-empty finite set. If \( \Sigma^c(\Gamma) \) contains a discrete character class \( [\chi] \), that is,
a class represented by a character $\chi$ whose image $\chi(\Gamma) \subset \mathbb{R}$ is infinite cyclic, then it was observed by Gonçalves and Kochloukova [14] that the character $\chi$ itself is fixed by the action of $K$ on $\text{Hom}(\Gamma, \mathbb{R})$. That is, $\chi \circ \phi = \chi$ for all $\phi \in K \subset \text{Aut}(\Gamma)$. This easily implies that $R(\phi) = \infty$ by Lemma 2.1(i), taking $G = \Gamma, N = \ker \chi, \theta = \phi$ in the notation of that lemma, so that $\bar{\theta} = \id$.

When $\Gamma = G_n, n \geq 3$, the group of pure symmetric automorphisms of $F_n$, L. Orlandi-Korner [22] has determined $\Sigma^c(\Gamma)$. When $\Gamma = H_n$, the Houghton group, Brown [7] computed the set $\Sigma^c(\Gamma)$. Using these results, Gonçalves and Kochloukova, showed that if $\Gamma$ is any one of the Houghton group, Brown [7] computed the set $\Sigma^c(\Gamma)$. Using these results, Gonçalves and Kochloukova, showed that if $\Gamma$ is any one of the groups $H_n, n \geq 2, G_m, m \geq 3$, the image of $\eta : \text{Aut}(\Gamma) \to \text{Homeo}(\Sigma^c(\Gamma))$ is finite.

In the case of the Houghton group $H_n, n \geq 2$, it turns out that $\Sigma^c(H_n)$ is a finite set of discrete character classes $[\chi_j], 1 \leq j \leq n$. Explicitly, $\chi_j : H_n \to \mathbb{Z}$ may be taken to be $-\pi_i \circ \tau$ where $\tau : H_n \to \mathbb{Z}$ is the translation part (see §1) and $\pi_i : \mathbb{Z} \to \mathbb{Z}$ is the restriction of to $\mathbb{Z} \subset \mathbb{Z}^n$ of the $i$-th projection (see [7]). (Recall from §1 that $\mathbb{Z} = \{(t_1, \ldots, t_n) \in \mathbb{Z}^n \mid \sum_{1 \leq j \leq n} t_j = 0\}$.) Thus $\text{Homeo}(\Sigma^c(H_n)) \cong S_n$ is finite and so is the image of $\eta : \text{Aut}(H_n) \to \text{Homeo}(\Sigma^c(H_n))$. As already remarked $R(\phi) = \infty$ for all $\phi \in \ker(\eta)$. The lemma below will not be used in the sequel but included here for illustrative purposes.

**Lemma 2.5.** Suppose that $\eta(\phi) : \Sigma^c(H_n) \to \Sigma^c(H_n)$ is not an $n$-cycle. Then $R(\phi) = \infty$.

**Proof.** Since $\eta(\phi)$ is not an $n$-cycle, the orbit of $[\chi_1]$ under $\eta(\phi)$ consists of at most $n - 1$ elements. Since $\chi_1$ are all discrete, the orbit of $\chi_1 \in \text{Hom}(H_n, \mathbb{R})$ consists of at most $n - 1$ elements. (In fact the orbit of $\chi_1$ is a subset of $\{\chi_j \mid 1 \leq j \leq n\}$.) Now the orbit sum $\lambda := \sum_{1 \leq j \leq k} \chi_j \phi^j$ is a non-zero character since any $n - 1$ elements of $\chi_j, 1 \leq j \leq n$ form a basis of $\text{Hom}(H_n, \mathbb{R})$. It follows that, since $\phi^*(\lambda) = \lambda, R(\phi) = \infty$. □

If $\phi^* : \Sigma^c(H_n) \to \Sigma^c(H_n)$ is an $n$-cycle, then the orbit sum is zero and the above argument fails. In fact, it is easily seen that every possible permutation of $\Sigma^c(H_n)$ may be realized as $\eta(\phi)$ for some $\phi \in \text{Aut}(H_n)$, that is, $\eta : \text{Aut}(H_n) \to \text{Homeo}(\Sigma^c(H_n)) \cong S_n$ is surjective.

### 3. Proof of Theorem 1.1

Let $X$ be an infinite set. We will only be concerned with the case when $X$ is countably infinite. We shall denote by $S_\infty(X)$ the group of all finitary permutations of $X$, that is those permutations which fixes all but finitely many elements of $X$. The group of all permutations of $X$ will be denoted by $S(X)$. We shall denote $S(X)$ (resp. $S_\infty(X)$) simply
by $S_\omega$ (resp. $S_\infty$) when $X$ is clear from the context. If $x = (x_k)_{k\in\mathbb{Z}}$ is a doubly infinite sequence in $X$ of pairwise distinct elements, we regard it as an element of $S(X)$ where $x(x_k) = x_{k+1}$ and $x(a) = a$ if $a \neq x_k \forall k \in \mathbb{Z}$. Two such sequences $x = (x_k)$ and $y = (y_k)$ define the same permutation if and only if $y$ is a shift of $x$, that is, there exists an $n$ such that $x_k = y_{k+n}$ for all $k \in \mathbb{Z}$. Thus, the sequence $x = (x_k)_{k\in\mathbb{Z}}$ is just the infinite cycle defined as $(x_0, x_1, \ldots)$. Two such sequences $x, y$ of pairwise distinct elements, we regard them to be conjugate in $S(X)$. Any $f \in S(X)$ is uniquely expressible as a product of disjoint cycles. Such an expression of $f$ is its cycle decomposition. The cycle type of an $f \in S(X)$ is the function $c(f) : \mathbb{N} \cup \{\infty\} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ where $c(f)(\alpha)$ is the number of $\alpha$-cycles in the cycle decomposition of $f$ if that number is finite, otherwise it is $\infty$ for $\alpha \in \mathbb{N} \cup \{\infty\}$. As in the case $S_\infty(X)$, if $f$ and $g$ have the same cycle type, then they are conjugate in $S(X)$. We need a criterion for $f$ and $g$ to be conjugate by an element of $S_\infty(X)$.

**Lemma 3.1.** Let $x = (x_k)_{k\in\mathbb{Z}}, y = (y_k)_{k\in\mathbb{Z}} \in S_\omega(X)$ be two disjoint infinite cycles and let $(a, b) \in S_\infty$.

(i) If $a = x_0, b = x_k, k > 0$, then $(a, b)x = uv, a$ product of disjoint cycles $u = (u_j)_{j\in\mathbb{Z}} \in S_\omega, v \in S_\infty$, defined as

$$u_j = \begin{cases} x_j, & j < 0, \\ x_{j+k}, & j \geq 0, \end{cases}$$

and $v = (x_0, \ldots, x_{k-1}) \in S_\infty$.

(ii) If $a = x_0, b = y_0$, then $(a, b)xy = uv$, where $u = (u_j)_{j\in\mathbb{Z}}, v = (v_j)_{j\in\mathbb{Z}}$ are disjoint infinite cycles defined as

$$u_j = \begin{cases} x_j, & j < 0, \\ y_j, & j \geq 0, \end{cases} \quad \text{and} \quad v_j = \begin{cases} y_j, & j < 0, \\ x_j, & j \geq 0. \end{cases}$$

\[\square\]

If $k \in \mathbb{N}$, we denote by $\mathbb{N}_{>k}$ the set of all integers greater than $k$. Note that $S_\infty = \cup_{k>2} S_k$ where $S_k$ is the subgroup consisting of permutations of $\mathbb{N}$ which fixes all $n > k$. In particular, the group $S_\infty$ is generated by transpositions $(i, i+1), i \geq 1$. The alternating group $A_\infty$ equals the commutator subgroup $[S_\infty, S_\infty]$, has index 2 in $S_\infty$ and is simple. The conjugacy class of any element of $S_\infty$ is determined by its cycle type, as in the case of finite symmetric groups. The group $S_\infty$ is a normal subgroup of $S_\omega = S(\mathbb{N})$. In particular, any bijection $f : \mathbb{N} \to \mathbb{N}$ defines an automorphism $\iota_f \in \text{Aut}(S_\infty)$ by restricting the inner automorphism determined by $f \in S_\omega$. Moreover $\iota_f$ is the identity automorphism only
if $f$ is equals the identity map. The following lemma is perhaps well-known, although we could not find a reference for it.

**Lemma 3.2.** The homomorphism $\iota : S_\omega \to \text{Aut}(S_\infty)$ is an isomorphism of groups.

**Proof.** By the discussion above, it only remains to show that $\iota$ is surjective. Let $\theta : S_\infty \to S_\infty$ be an isomorphism. We claim that $\theta$ maps transpositions to transpositions. Indeed, let $\tau = (1,2)$ so that $\theta(\tau)$ is an element of order 2. Hence $\theta(\tau) := \xi$ is a product of disjoint transpositions $\xi = \tau_1 \ldots \tau_k$. To establish our claim, we must show that $k = 1$. (Since $\tau$ is an odd permutation, $k$ must be odd. This fact is not required here.) Without loss of generality, we assume that $\tau_i = (i, k + i), 1 \leq i \leq k$, so that $\xi = (1, k + 1)(2, k + 2)\ldots(k, 2k).

The centralizer of $\tau$ is the subgroup $Z(\tau) = \{\sigma \in S_\infty \, | \, \sigma \tau = \tau \sigma\} = S_2 \times S_\infty(\mathbb{N}_{\geq 3}) \subset S_\infty$.

The automorphism $\theta$ maps the $Z(\tau)$ isomorphically onto $Z(\theta(\tau)) = Z(\xi)$. On the other hand, if $\sigma \in S_\infty$, then $\sigma \xi \sigma^{-1} = \sigma \tau_1 \ldots \tau_k \sigma^{-1} = (\sigma(1), \sigma(k + 1)) \ldots (\sigma(k), \sigma(2k))$. Thus $\xi = \sigma \xi \sigma^{-1}$ holds if and only if $\sigma \in S_{2k} \times S(\mathbb{N}_{>2k})$ and $|\sigma(j + k) - \sigma(j)| = k$ for $1 \leq j \leq k$. That is, $Z(\xi) = Z_0 \times S(\mathbb{N}_{>2k})$ where $Z_0 \subset S_{2k}$ is the centralizer of $\xi \in S_{2k}$. Denote by $\beta : S_k \to S_{2k}$ the monomorphism $(\beta(h))(j) = h(j), \beta(h)(j + k) = h(j) + k, 1 \leq j \leq k$. Then we see that $\sigma \in Z_0$ if and only if $\sigma = \tau_{i_1} \ldots \tau_{i_r} \beta(h)$ for some $h \in S_k$ and $1 \leq i_1 < \ldots < i_r \leq k$; moreover the element $h$ and the (possibly empty) sequence $i_1, \ldots, i_r$ are determined uniquely by $\sigma$. Consider the homomorphism $Z_0 \to (\mathbb{Z}/2\mathbb{Z})^k \times S_k$ defined as $\sigma \mapsto (t, h)$ where $t_j = 1$ or 0 according as $j$ belongs to $\{i_1, \ldots, i_r\}$ or not. (Here the group $S_k$ operates on $(\mathbb{Z}/2\mathbb{Z})^k$ by permuting the coordinates.) It is easily seen that this is an isomorphism of groups. In particular $Z_0$ has order $2^k k!$.

Since $\theta$ is an automorphism, we must have $S_2 \times S(\mathbb{N}_{>2}) \cong Z(\tau) \cong Z(\theta(\tau)) = Z(\xi) \cong Z_0 \times S(\mathbb{N}_{>2k})$. Now $Z(\xi)$ has a normal subgroup of index $2^{k+1} k!$, namely the alternating group on $\mathbb{N}_{>2k}$. On the other hand, the only finite index proper normal subgroups of $Z(\tau) \cong S_2 \times S_\infty$ are $1 \times A_\infty$, which has index 4, and $S_2 \times A_\infty$ and $1 \times S_\infty$, each of which has index 2. It follows that $k = 1$. Thus $\theta((1,2)) = (1,2)$. By the same argument $\theta((1,3)) = (i, j)$. We assert that either $i \in \{1, 2\}$ or $j \in \{1, 2\}$. In fact if $\{i, j\} \cap \{1, 2\} = \emptyset$ or if $\{i, j\} = \{1, 2\}$, then $\theta((1,2))$ and $\theta((1,3))$ commute, a contradiction. Suppose that $i \in \{1, 2\}$ and $j \notin \{1, 2\}$. We set $f(1) = i$. In general, $f(k)$ is defined to be the unique number $l$ where $\theta((k, k + 1)) = (l, a), \theta((k, k + 2)) = (l, b)$. Then $f : \mathbb{N} \to \mathbb{N}$ is a monomorphism. It is a surjection since $\theta$ is. It is readily seen that $\theta = \iota_f$. □
Corollary 3.3. Suppose that $S_\infty$ is a characteristic subgroup of a group $H$ contained in $S_\omega$. Then the automorphism group of $H$ is isomorphic to the normalizer of $N(H)$ of $H$ in $S_\omega$. In particular, every automorphism of $H$ is the restriction to $H$ of a unique inner automorphism of $S_\omega$.

Proof. We shall denote by the same symbol $\iota_f$ to denote the conjugation by $f \in S_\omega$ or its restriction to any subgroup normalized by $f$.

It is evident that $\iota : N(H) \to \text{Aut}(H)$ defined as $f \mapsto \iota_f$ defines an homomorphism. (Here $\iota_f(h) = fhf^{-1}$ $\forall h \in H$.) This is a monomorphism since $\iota_f$ is non-trivial on $S_\infty \subset H$ if $f$ is not the identity.

Let $\phi : H \to H$ be any automorphism and let $f \in S_\omega$ be the element such that $\phi|_{S_\infty} = \iota_f$. We claim that $\phi = \iota_f$. Suppose that $u := \phi(h), \iota_f(h) = fhf^{-1} =: v$ for some $h \in H$. We must show that $u(i) = v(i)$ for all $i \in \mathbb{N}$. It suffices to show that $\{u(i), u(j)\} = \{v(i), v(j)\}$ for all $i,j \in \mathbb{N}, i \neq j$. Let $i,j \in \mathbb{N}$, $i \neq j$. Now consider the transposition $(a,b) \in S_\infty$ such that $\iota_f(a,b) = \phi(a,b) = (i,j)$. We have $\phi(h(a,b)h^{-1}) = \phi(h)\phi(a,b)\phi(h^{-1}) = u(i,j)u^{-1} = (u(i), u(j))$, while $\iota_f(h(a,b)h^{-1}) = \iota_f(h)\iota_f(a,b)\iota_f(h^{-1}) = v(i,j)v^{-1} = (v(i), v(j))$. Therefore $(u(i), u(j)) = (v(i), v(j)) \in S_\infty$ since $\iota_f$ and $\phi$ agree on $S_\infty$. This implies that $\{u(i), u(j)\} = \{v(i), v(j)\}$, completing the proof. \(\square\)

3.1. $S_\infty$ has the $R_\infty$-property. Let $\theta \in \text{Aut}(S_\infty)$. In view of Lemma 3.2 $\theta = \iota_f$ for some $f \in S_\omega$. Let $x, y \in S_\infty$ and suppose that $y = \zeta x \theta (z^{-1}) = \zeta x f z^{-1} f^{-1}$ for some $z \in S_\infty$. Then $y f = z (x f) z^{-1} f^{-1}$ for some $z \in S_\infty$. For any cycle (finite or infinite) $u = (u_j)$, we have that $zu^{-1}$ is the cycle $(z u_j)$. Any $z \in S_\infty$ moves only finitely many elements of \(\mathbb{N}\). Hence when $u$ is an infinite cycle $z(u_j) = u_j$ for all but finitely many $j \in \mathbb{Z}$. For an arbitrary element $u$ expressed as a product of pairwise disjoint cycles, $u(\alpha) = (u(\alpha)_{j})$, the element $zu^{-1}$ being a product of $z u(\alpha) z^{-1}$, we see that $zu(\alpha) z^{-1} = u(\alpha)$ for all but a finitely many $\alpha$, and, moreover, if $u(\alpha) = (u(\alpha)_{j})_{j \in \mathbb{Z}}$ is an infinite cycle, then $z(u(\alpha)_{j}) = u(\alpha)_{j}$ for all but finitely many $j \in \mathbb{Z}$.  \(^1\)

Lemma 3.4. Suppose that $f \in S_\omega$ has an infinite cycle $u$, then there exist infinitely many transpositions $\tau_k \in S_\infty$ such that $\tau_j f \neq z \tau_k f z^{-1}$ for any $z \in S_\infty$.

Proof. Fix an infinite cycle $u = (u_\alpha)_{\alpha \in \mathbb{Z}}$ that occurs in the cycle decomposition of $f$. Let $\tau_\alpha = (u_0, u_\alpha), \alpha \geq 1$. Then we claim that $\tau_\alpha f$ and $\tau_\beta f$ are not conjugates if $\alpha \neq \beta$. To see this, we apply Lemma

\(^1\)There is a mild abuse of notation here; $u(\alpha)$ is not to be confused with the value of $u$ at $\alpha$. We will use Greek letters as labels in such situations.
3.1 to compute $\tau_3u, \alpha \geq 1$. Note that the cycles that occur in $\tau_3u$ also occur in the cycle decomposition of $\tau_3f$. This is true in particular of the infinite cycle, denoted $v(\alpha)$, that occurs in $\tau_3u$.

Now $v(\alpha)p = v(\beta)p = u_p$ for all $p < 0$ and $\alpha, \beta \geq 1$, and, when $\alpha \neq \beta$, $u_{p+\alpha} = v(\alpha)p \neq v(\beta)p = u_{p+\beta}, p \geq 0$. This implies that the $zv(\beta)z^{-1}$ cannot occur in $\tau_3f$ for any $z \in S_\infty$ if $\alpha \neq \beta$ in its cycle decomposition by the assertion made in the paragraph above the statement of the lemma. Hence $\tau_3f \neq z_{\tau_3f}z^{-1}$ for any $z \in S_\infty$. \hfill \Box

We are now ready to prove the first assertion of Theorem 1.1.

**Theorem 3.5.** The group $S_\infty$ has the $R_\infty$-property.

**Proof.** Let $\theta = \iota_f \in \text{Aut}(S_\infty)$ where $f \in S_\omega$. We need to show that there exists pairwise distinct elements $\tau_j \in S_\infty, j \in \mathbb{N}$ such that $\tau_jf \neq z_{\tau_jf}z^{-1}$ for any $z \in S_\infty$ if $j \neq k$. Since $S_\infty$ has infinitely many conjugacy classes, the assertion holds for $f \in S_\infty$, we need only consider the case $f \notin S_\infty$. Thus, in the cycle decomposition of $f$, either (i) there exist an infinite cycle, or, (ii) all the cycles are finite and there are infinitely many of them.

Case (i). In this case the assertion has already been established in Lemma 3.4.

Case (ii). Suppose that $f = \prod_{\alpha \in \mathbb{N}} u(\alpha)$ where the $u(\alpha)$ are all finite cycles having length $\ell(\alpha)$ at least 2 for every $\alpha \in \mathbb{N}$. Let $J := \{\alpha \in \mathbb{N} | \ell(\alpha) \geq 3\}$. We break up the proof into two subcases depending on whether $J$ is infinite or not.

Subcase (a): $J$ is infinite. Let $J_k \subset J$ be the set consisting of the first $k$ elements of $J$ (with respect to the usual ordering on $J \subset \mathbb{N}$). Write $u(\alpha) = (u(\alpha)_1, \ldots, u(\alpha)_{\ell(\alpha)})$ and set $U(\alpha) := \{u(\alpha)_i | 1 \leq i \leq \ell(\alpha)\}, \alpha \in \mathbb{N}$. Consider the collection of pairwise disjoint transpositions $\lambda_\alpha = (u(\alpha)_1, u(\alpha)_2), \alpha \in J$, and let $\tau_k = \prod_{\alpha \in J_k} \lambda_\alpha$. Note that $\lambda_\alpha u(\alpha) = (u(\alpha)_1, (u(\alpha)_2, \ldots, u(\alpha)_{\ell(\alpha)}) = (u(\alpha)_2, \ldots, u(\alpha)_{\ell(\alpha)})$ fixes only $u(\alpha)_1$ in the set $U(\alpha)$ as $\ell(\alpha) \geq 3$. Then $\tau_k, \prod_{\alpha \in J_k} u(\alpha)$ fixes only $u(\alpha)_1 \in \mathbb{N}, \alpha \in J_k$, in the set $\cup_{\alpha \in J_k} U(\alpha)$. Let $F_0 = \text{Fix}(f)$. Then $\text{Fix}(\tau_kf) = F_0 \cup \{u(\alpha)_1 | \alpha \in J_k\} =: F_k$.

Suppose that $\tau_jf = z_{\tau_jf}z^{-1}$ with $z \in S_\infty$ with $j \neq k$. Then $z$ defines a bijection $\zeta : F_j \to F_k$ between the fixed sets of $\tau_jf$ and $\tau_kf$. Clearly this is a contradiction if $\text{Fix}(f) = F_0$ is finite. Assume that $F_0 \subset \mathbb{N}$ is infinite. Since $z \in S_\infty$, it fixes all but finitely many elements of $F_0$. Let $L := \{m \in F_0 | z(m) \neq m\}$. Note that $\zeta$ restricts to the identity on $F_0 \setminus L$. Therefore $\zeta$ restricts to a bijection between $L \cup \{u(\beta)_1 | \beta \in J_j\}$ and $L \cup \{u(\beta)_1 | \beta \in J_k\}$. Since $j \neq k$, $L$ is finite and $L \subset F_0$ is disjoint from $\{u(\beta)_1 | \beta \in J_n\}; n = j, k$, this is a contradiction.
Subcase (b): The set \( J \) is finite; we set \( K = \mathbb{N} \setminus J \) and define \( K_j, j \in \mathbb{N} \), to be the set of first \( \alpha \) elements of \( K \). Again we set \( \lambda_\alpha = (u(\alpha)_1), u(\alpha)_2) = u(\alpha), \alpha \in K \). Now, if \( \alpha \in K \), we have \( \lambda_\alpha u(\alpha) = id \), that is, \( \lambda_\alpha u(j) \) fixes both points of \( U(\alpha) \). We set \( \tau_j := \prod_{\alpha \in K_j} \lambda_\alpha, F_j := \text{Fix}(\tau_j f) = F_0 \cup_{\alpha \in K_j} U(\alpha) \). Arguing exactly as above we see that for any \( z \in S_\infty, \tau_j f = z\tau_k f z^{-1} \) implies \( j = k \), completing the proof. \( \square \)

3.2. Houghton groups. As in the introduction, \( H_n, n \geq 2 \), denotes the Houghton group. We first describe the group of outer automorphisms of \( H_n \). Recall from §1 that one has an exact sequence

\[ 1 \to S_\infty(M_n) \to H_n \xrightarrow{\tau} Z \to 1 \]

where \( \tau: H_n \to Z \) send \( f \in H_n \) to the translation part \( (t_1, \ldots, t_n) \in Z \) of \( f \). The group \( S_\infty(M_n) \) is the commutator subgroup of \( H_n \) if \( n \geq 3 \). When \( n = 2 \), the commutator subgroup is the alternating group \( A_\infty(M_2) \) which has index 2 in \( S_\infty(M_2) \). In any case, \( S_\infty = S_\infty(M) \) is characteristic in \( H_n \) as \( H_n/S_\infty \) is the maximal torsion-free abelian quotient of \( H_n \).

**Lemma 3.6.** Let \( \phi: H_n \to H_n, n \geq 2 \), be an automorphism. Then \( \phi \) is inner if and only if \( \phi: Z \to Z \) is the identity automorphism.

**Proof.** It is trivial to see that any inner automorphism of \( H_n \) induces the identity automorphism of \( Z \). For the converse, suppose that \( \phi: H_n \to H_n \) induces the identity automorphism of \( Z \).

Let \( f \in S(M_n) \) be such that \( \iota f(H_n) = H_n \).

Consider the element \( h_p : M_n \to M_n, 1 \leq p < n, \) in \( H_n \) defined follows: \( ^2 \)

\[ h_p(i, k) = \begin{cases} 
(p, k + 1), & \text{if } i = p, k \geq 1 \\
(n, k - 1), & \text{if } i = n, k > 1 \\
(p, 1), & \text{if } i = n, k = 1 \\
(i, k), & \text{if } i \neq p, n.
\end{cases} \]

Thus \( h_p \) permutes \( \{p, n\} \times \mathbb{N} \) in a single cycle:

\[ h_p = (\ldots, (n, 2), (n, 1), (p, 1), (p, 2), \ldots, (p, k), \ldots) \]

and so \( fh_pf^{-1} \) is the cycle

\[ fh_pf^{-1} = (\ldots, f(n, 2), f(n, 1), f(p, 1), f(p, 2), \ldots, f(p, k), \ldots) \in H_n. \]

The only infinite cycles in \( H_n \) are those whose terms, except for a finite part of the cycle, are consecutive numbers along two rays, say \( \{i_n\} \times \mathbb{N} \) and \( \{i_p\} \times \mathbb{N} \), in the negative and positive directions respectively of the cycle \( fh_pf^{-1} \). Therefore we have \( \tau(fh_pf^{-1}) = e_{i_p} - e_{i_n} \). Moreover, there

\(^2\)The element \((p, k) \in M_n\) should not be confused with the transposition in \( S(\mathbb{N}) \).
exist integers \(t_n, t_p\) such that \(f(n, k) = (i_n, k + t_n), f(p, k) = (i_p, k + t_p)\) for sufficiently large \(k\). Clearly \(i_n\) and \(t_n\) are independent of \(p\). Since \(f\) is a bijection, the association \(p \mapsto i_p\) is a permutation \(\pi_f \in S_n\), and 
\[\sum_{1 \leq q \leq n} t_q = 0.\]
Note that \(\pi_f = id\) if and only if \(f \in H_n\).

Since \(S_\infty\) is characteristic in \(H_n\), by Corollary 3.3, \(\phi = \iota_g\) for a unique \(g \in S(M_n)\). We claim that \(g \in H_n\). Since \(\tau(ghg^{-1}) = \tau(\phi(h)) = \tau(h)\ \forall h \in H_n\), we have \(\pi_g(q) = q\) for all \(q \leq n\) and so we have \(g \in H_n\).

The group \(S_n\) acts on the set \(M_n = \{1, \ldots, n\} \times \mathbb{N}\) in the obvious manner, by acting via the identity on \(\mathbb{N}\). This defines an action \(\psi\) of \(S_n\) on the group \(S(M_n)\) defined as \(f \mapsto \sigma \circ f \circ \sigma^{-1}\) which preserves the subgroup \(H_n\). Thus we obtain a homomorphism \(\psi : S_n \to \text{Aut}(H_n)\). It is readily seen that \(\tau(\psi_i(h)) = \sigma(\tau(h)) \\forall h \in H_n\), where \(\sigma\) acts on \(Z \subset \mathbb{Z}^n\) by permuting the standard basis elements \(e_1, \ldots, e_n\). In particular \(\psi\) is a monomorphism. Let \(\bar{\psi} : S_n \to \text{Out}(H_n)\) be the composition of \(\psi\) with the projection \(\text{Aut}(H_n) \to \text{Out}(H_n)\).

**Proposition 3.7.** The homomorphism \(\bar{\psi} : S_n \to \text{Out}(H_n)\) is an isomorphism and so \(\text{Aut}(H_n) = \text{Inn}(H_n) \rtimes S_n \cong H_n \rtimes S_n\).

**Proof.** Lemma 3.6 shows that \(\bar{\psi}\) is a monomorphism. We shall show that it is surjective.

Let \(\phi \in \text{Aut}(H_n)\). Write \(\phi = \iota_f\) for a (unique) \(f \in S(M_n)\). With notations as in the proof of Lemma 3.6, let \(\pi := \pi_f \in S_n\).

Consider the automorphism \(\psi_\pi^{-1}\phi =: \theta\). We have \(\tau(\theta(h_p)) = \pi^{-1}(\tau(\phi(h_p))) = \pi^{-1}(\tau(fh_pf^{-1})) = \pi^{-1}((e_{\pi(p)} - e_{\pi(n)}) = e_p - e_n = \tau(h_p), 1 \leq p < n\). Since the group \(Z\) is generated by \(\tau(h_p), 1 \leq p < n\), it follows by Lemma 3.6 that \(\theta\) is inner. Hence \(\bar{\psi}(\pi) = \phi \mod \text{Inn}(H_n)\).

Finally note that \(\text{Inn}(H_n) \cong H_n\) since the centre of \(H_n\) is trivial. 

The above description of \(\text{Aut}(H_n)\) had been obtained by Burillo, Cleary, Martino, and Röver [9] and also by Cox [11]. Our proof is essentially based on Corollary 3.3, which is applicable in a more general context.

**Theorem 3.8.** The Houghton group \(H_n\) has the \(R_\infty\)-property for any \(n \geq 2\).

We shall give two proofs. The first one uses the structure of the automorphism group of \(H_n\) and is more direct. The second one uses the result of Theorem 3.5 and the addition formula (Lemma 2.1).

**First proof.** First observe that there are infinitely many conjugacy classes in \(H_n\) since two elements in \(S_\infty = S_\infty(M_n) \subset H_n\) are conjugates in \(H_n\) only if they have the same cycle type. It follows that \(R(\phi) = \infty\)
for any inner automorphism $\phi$ of $H_n$. Therefore, to show that $R(\phi) = \infty$ for an arbitrary $\phi \in \text{Aut}(H_n)$, it suffices to show that $R(\phi) = \infty$ for all $\phi$ in a set of coset representatives of elements of $\text{Out}(H_n)$. Thus we need only show that $R(\psi_\sigma) = \infty$ for any $\sigma \in S_n$, where $\psi : S_n \to \text{Aut}(H_n)$ is as defined in the paragraph above Proposition 3.7. We shall use Lemma 2.3 and Remark 2.4 to achieve this.

For $k \geq 1$, consider the element $\xi_k$ defined as the product of $k$-cycles $((i,1), \ldots, (i,k)) \in H_n$, $1 \leq i \leq n$. Explicitly,

$$\xi_k(i,j) = \begin{cases} (i,j+1) & \text{if } 1 \leq j < k, \\ (i,1) & \text{if } j = k, \\ (i,j) & \text{if } j > k, \end{cases}$$

for all $i \leq n$. Then $\xi_k$ is fixed by $\psi_\sigma$ for every $\sigma \in S_n$. Thus, $\{\xi^n_k \mid k \geq 1\}$ contains elements of arbitrarily large orders and so by Remark 2.4 it follows that $R(\psi_\sigma) = \infty$ for all $\sigma \in S_n$, completing the proof.

Second proof. Consider the exact sequence $1 \to S_\infty(M_n) \to H_n \to Z \to 0$. As remarked already, $S_\infty(M_n)$ is characteristic in $H_n$ and we have $Z \cong \mathbb{Z}^{n-1}$. Thus any automorphism $\theta$ of $H_n$ restricts to an automorphism $\theta'$ of $S_\infty(M_n)$ and induces an automorphism $\bar{\theta}$ of $Z$. If $R(\theta') = \infty$ then, by Lemma 2.1 (i), we have $R(\theta) = R(\bar{\theta}) = \infty$. Now suppose that $R(\theta) < \infty$. Then $\text{Fix}(\bar{\theta}) = 0$. Since $Z$ is abelian and since $R(\theta') = \infty$ by Theorem 3.5, the addition formula (Lemma 2.1(ii)) yields $R(\theta) = R(\theta') = \infty$, completing the proof. \hfill \Box

3.3. The group of pure symmetric automorphisms. Recall that $G_n \subset \text{Aut}(F_n), n \geq 2$, denotes the group of pure symmetric automorphisms of the free group $F_n$ of rank $n$. A presentation for $G_n$, obtained by McCool, was recalled in §1. It is immediate from this presentation that the abelianization $G_n^{ab} = G_n/[G_n, G_n]$ is isomorphic to $\mathbb{Z}^{n^2-n}$ with basis the images $\bar{\alpha}_{ij}, 1 \leq i \neq j \leq n$. We denote by $\{\chi_{ij} \mid 1 \leq i \neq j \leq n\}$ the basis of $\text{Hom}(G_n^{ab}, \mathbb{Z})$ dual to the basis $\{\bar{\alpha}_{ij} \mid 1 \leq i \neq j \leq n\}$. We shall denote by the same symbol $\chi_{ij}$ the composition $G_n \to G_n^{ab} \xrightarrow{\chi_{ij}} \mathbb{Z} \hookrightarrow \mathbb{R}$. We will assume that $n \geq 3$, leaving out $G_2$ which is isomorphic to a free group of rank 2, which is known to have the $R_\infty$-property.

We begin by recalling the explicit description of $\Sigma^e(G_n)$ due to Orlandi-Korner [22].

Let $A_{ij} := \mathbb{R} \chi_{ij} + \mathbb{R} \chi_{ji}$, $B_{ijk} := \mathbb{R} (\chi_{ij} - \chi_{kj}) + \mathbb{R} (\chi_{jk} - \chi_{ik}) + \mathbb{R} (\chi_{ki} - \chi_{ji})$, $i,j,k$ pairwise distinct. Note that $A_{ij} = A_{ji}$ and $B_{ijk} = B_{pqr}$ if $\{i,j,k\} = \{p,q,r\}$. Let $S$ be union of vector subspaces $S = \bigcup A_{pq} \cup \bigcup B_{ijk} \subset \text{Hom}(G_n, \mathbb{R})$ where the unions are over all pairs of distinct number $p,q \leq n$ and all pairwise distinct numbers $i,j,k \leq n$. It was
shown by Orlandi-Korner [22] that \( \Sigma^c(G_n) \) is the image of \( S \setminus \{0\} \subset \text{Hom}(G_n, \mathbb{R}) \setminus \{0\} \).

Let \( S_n \) denote the semidirect product \( C^n_2 \rtimes S_n \) where \( S_n \) acts on \( C^n_2 \) by permuting the coordinates. Here \( C_2 = \{1, -1\} \). The group \( S_n \) acts effectively on \( F_n \) the free group with basis \( \{x_1, \ldots, x_n\} \) where \( \pi \in S_n \) permutes the generators: \( \pi(x_j) = x_{\pi(j)}, 1 \leq j \leq n \) and the action of the \( k \)-th factor of \( C^n_2 \) is given by the automorphism \( t_k(x) = x_k^{-1}, t_k(x_j) = x_j, j \neq k \). Thus \( S_n \) is a subgroup of \( \text{Aut}(F_n) \). It is readily verified that \( S_n \) normalizes \( G_n \): \( t_k \alpha_{i,j} t_k^{-1} = \alpha_{i,j} \) if \( k = j \) and equals \( \alpha_{i,j} \) otherwise; if \( \pi \in S_n \), then \( \pi \alpha_{i,j} \pi^{-1} = \alpha_{\pi(i),\pi(j)} \) for all \( i,j \). In particular \( \pi^*(A_{ij}) = A_{\pi(i)\pi(j)}, \pi^*(B_{ijk}) = B_{\pi(i)\pi(j)\pi(k)} \) for all \( \pi \in S_n \). Thus we have the following

**Lemma 3.9.** Let \( n \geq 3 \). The action of the group \( S_n \subset \text{Aut}(G_n) \) on \( \Sigma^c(G_n) \) is defined by \( \pi^*(\chi_{i,j}) = \chi_{\pi(i)\pi(j)}, t^*(\chi_{i,j}) = t_i t_j \chi_{i,j} \), for all \( \pi \in S_n, t = (t_1, \ldots, t_n) \in C^n_2 \). \( \square \)

The following proposition is refinement of a statement of Gonçalves and Kochloukova in proof of [14, Theorem 4.11]).

**Proposition 3.10.** There exists a homomorphism \( \eta : \text{Aut}(G_n) \to S_n \) which is surjective such that \( \phi^*(\chi_{i,j}) = \epsilon_{i,j} \chi_{\sigma(i)\sigma(j)}, 1 \leq i \neq j \leq n \), where \( \epsilon_{i,j} \in \{1,-1\} \) and \( \sigma = \eta(\phi) \in S_n \). In particular, \( \text{Aut}(G_n) \cong K \rtimes S_n \) where \( K = \ker(\eta) \).

**Proof.** Since \( \phi^* \) is a linear isomorphism of \( \text{Hom}(G_n, \mathbb{R}) \) and since \( \phi^* : \Sigma^c(G_n) \to \Sigma^c(G_n) \) is a homeomorphism, \( \phi^* \) preserves the collections of subspaces \( A := \{A_{ij} \mid 1 \leq i < j \leq n\} \) and \( B := \{B_{ijk} \mid 1 < i < j < k \leq n\} \). Note that \( B \) is non-empty since \( n \geq 3 \). In our notation \( A_{pq}, B_{pqr} \) it is not assumed that \( p < q < r \).

It is readily seen that \( (A_{pq} + A_{rs}) \cap B_{ijk} = 0 \) unless \( \{p, q, r, s\} = \{i, j, k\} \). On the other hand \( (A_{ij} + A_{ik}) \cap B_{ijk} = \mathbb{R}(\chi_{k,i} - \chi_{j,i}) \). It follows that \( \phi^* \) preserves the collection of 1-dimensional spaces \( C := \{\mathbb{R}(\chi_{k,i} - \chi_{j,i}) \mid i, j, k \text{ pairwise distinct}\} \).

Let \( \phi^*(A_{ij}) = A_{pq}, \phi^*(A_{ik}) = A_{rs} \), where \( i, j, k \) are pairwise distinct. Then \( \{p, q\} \cap \{r, s\} \) is a singleton, say \( s = p \) so that \( \phi^*(A_{ik}) = A_{pr} \) and \( \phi^*(B_{ijk}) = B_{pqr} \). For, otherwise \( (A_{ij} + A_{ik}) \cap B_{ijk} \) is one-dimensional whereas \( \phi^*((A_{ij} + A_{ik}) \cap B_{ijk}) = (A_{pq} + A_{pr}) \cap \phi^*(B_{ijk}) = 0 \).

In view of the fact that \( \phi^* \) stabilizes \( C \), we have

\[
\phi^*(\chi_{k,i} - \chi_{j,i}) = a(\chi_{r,p} - \chi_{q,p}).
\]

On the other hand \( \chi_{k,i} \in A_{ik} \) and so \( \phi^*(\chi_{k,i}) \in \phi^*(A_{ik}) = A_{pr} \) and so \( \phi^*(\chi_{k,i}) = b\chi_{r,p} + c\chi_{r,p} \) for some \( b, c \in \mathbb{R} \); similarly \( \phi^*(\chi_{j,i}) = b'\chi_{q,p} +

for some \( b', c' \in \mathbb{R} \). Therefore

\[
\phi^*(\chi_{k,i} - \chi_{j,i}) = b\chi_{p,r} + c\chi_{r,p} - b'\chi_{q,p} + c'\chi_{p,q}.
\]

Equating coefficients we see that

\[
\sum_{i,j} (\phi^*(\chi_{k,i}) - \phi^*(\chi_{j,i})) = \sum_{i,j} (b\chi_{p,r} + c\chi_{r,p} - b'\chi_{q,p} + c'\chi_{p,q}).
\]

Comparing (\( \ast \)) and (\( \ast \ast \)) we see that \( b = 0 = c' \), that is, \( \phi^*(\chi_{k,i}) = c\chi_{r,p} \) and \( \phi^*(\chi_{j,i}) = b'\chi_{q,p} \). Since \( \phi^* : \text{Hom}(G_n; \mathbb{R}) \to \text{Hom}(G_n, \mathbb{R}) \) preserves the lattice \( \text{Hom}(G_n, \mathbb{Z}) \) and since \( \chi_{k,i}, \chi_{j,i} \) are part of a \( \mathbb{Z} \)-basis of \( \text{Hom}(G_n, \mathbb{Z}) \), we see that \( c, b' = \pm 1 \).

To complete the proof, we define the permutation \( \sigma \in S_n \) associated to \( \phi \in \text{Aut}(G_n) \) as \( \sigma(i) = p \), (with notation as above). Note that \( \sigma \) is indeed a bijection since \( \phi^* \) is an isomorphism. We define \( \eta : \text{Aut}(G_n) \to S_n \) by \( \eta(\phi) = \sigma \). Then \( \eta \) is a homomorphism of groups. It is surjective since its restriction to \( S_n \subset S_n \) is the identity by Lemma 3.9. This also shows that \( \eta \) splits, completing the proof.

**Remark 3.11.** It seems plausible that there exists a surjective homomorphism \( \tau : \text{Aut}(G_n) \to S_n \) such that \( \phi^*(\chi_{i,j}) = t_it_j\chi_{\sigma(i),\sigma(j)} \) for \( 1 \leq i \neq j \leq n \), where \( \tau(\phi) = (t_1, \ldots, t_n) \in C_n^2 \). \( \sigma = \eta(\phi) \in S_n \). This would then imply that \( \text{Aut}(G_n) \cong N \times S_n \) for a suitable subgroup \( N \subset \text{Aut}(G_n) \).

The above proposition says that the matrix of \( \phi^* \) with respect to the basis \( \{\chi_{i,j} \mid 1 \leq i \neq j \} \) (ordered by, say, the lexicographic ordering of the indices \( i, j \)), is of the form \( \phi^* = DP \) where \( D \) is a diagonal matrix with eigenvalues \( \pm 1 \) and \( P \) is a permutation matrix.

**Lemma 3.12.** Let \( T = DP \) where \( D, P \in M_m(\mathbb{R}) \) are such that \( D \) is a diagonal matrix and \( P \) is a permutation matrix. If \( P = P_1, \ldots, P_k \) is a cycle decomposition then there exist eigenvectors \( v_1, \ldots, v_k \) which are linearly independent.

**Proof.** The cycle decomposition allows us to express \( \mathbb{R}^n \) as a direct sum \( V_1 \oplus \cdots \oplus V_k \) where \( V_j \) is spanned by \( \{e_i \mid P_j(i) \neq i\} \). Specifically, if \( P_j = (i_1, \ldots, i_k) \). Then \( v_{j} := e_{i_1} + d_{i_1}e_{i_2} + \cdots + d_{i_{k-1}}e_{i_k} \) which is the sum of the vectors in the \( DP \)-orbit of \( e_{i_1} \), is an eigenvector of \( T \) with eigenvalue \( d_{i_1} \cdots d_{i_k} \). Evidently \( v_1, \ldots, v_k \) are linearly independent. \( \square \)

We will use the above lemma to construct two linearly independent eigenvectors of \( \phi^* \) (with further properties that will be relevant for our purposes). Let \( \sigma = \eta(\phi) \neq id \) and \( \phi^* = DP \) with \( D \) diagonal and \( P \) a permutation transformation (with respect to the basis \( \{\chi_{i,j}\} \)). Suppose that \( \sigma \) has a \( k \)-cycle in its cycle decomposition where \( k > 2 \). Choose any \( i \) that occurs in it and let \( j := \sigma(i) \). Then \( \chi_{i,j} \) and \( \chi_{j,i} \) do not occur in the same orbit of \( DP \) and so \( v_{i,j} := \sum_{0 \leq r < k}(DP)^r(\chi_{i,j}) \) and \( v_{j,i} := \sum_{0 \leq r < k}(DP)^r(\chi_{j,i}) \) are eigenvectors of the same eigenvalue \( \epsilon \in \{1, -1\} \). Without loss of generality we assume that \( i = 1, j = 2 \) and set \( v_{1,2} := u, v_{2,1} = v \). Suppose that there is no such \( k \)-cycle in \( \sigma \).
Then $\sigma$ is a product of disjoint transpositions. Without loss of generality, suppose that the transposition $(1,3)$ occurs in the decomposition. Since $n > 2$, either $\sigma$ has a fixed point, say 2, or $n > 3$ and, say the transposition $(2,4)$ occurs in the decomposition. In the first case $u := \chi_{1,2} + d_{1,2}\chi_{3,2}$, $v := \chi_{2,1} + d_{2,1}\chi_{2,3}$ are eigenvectors of $P$ and in the latter case, $u := \chi_{1,2} + d_{1,2}\chi_{3,4}$ and $v := \chi_{2,1} + d_{2,1}\chi_{4,3}$ are eigenvectors of $P$. Thus in all case $\chi_{1,2}$ occurs in $u$ and $\chi_{2,1}$ occurs in $v$ where $u, v$ are eigenvectors of $\phi$. If 1 is an eigenvalue of $\phi^*$, then $\phi$ has a non-zero fixed element so $R(\phi) = \infty$. So assume that $\phi^*(u) = -u, \phi^*(v) = -v$.

Then there exists elements $\beta, \gamma \in G_n$ such that $\phi(\beta) = -\beta, \phi(\gamma) = -\gamma$ where $\alpha_{1,2}, \alpha_{2,1}$ occur in $\beta, \gamma$ respectively, with coefficient 1.

Denote by $\Gamma_2 := \Gamma_2(G_n)$ the commutator subgroup of $G_n$ and by $\Gamma_3 := \Gamma_3(G_n)$ the subgroup $[G_n, \Gamma_2] \subset \Gamma_2$. Thus $G_n/\Gamma_3$ is a two-step-nilpotent group and we have the following exact sequences:

$$1 \to \Gamma_3 \to G_n \to G_n/\Gamma_3 \to 1,$$

$$1 \to \Gamma_2/\Gamma_3 \to G_n/\Gamma_3 \to G_n/\Gamma_2 \to 1$$

Since $\Gamma_3$ and $\Gamma_2$ are characteristic in $G_n$, any automorphism of $G_n$ restricts to automorphisms of $\Gamma_2$ and $\Gamma_3$ and hence induces automorphisms of the quotients $G/\Gamma_3, \Gamma_2/\Gamma_3$ and $G_n/\Gamma_2 = G_n^\text{ab}$. Denote by $\theta \in \text{Aut}(G_n/\Gamma_3)$ the automorphism defined by $\phi$ and $\theta'$, the restriction of $\theta$ to $\Gamma_2/\Gamma_3$. With notations as above, $[\beta, \gamma] \Gamma_3 \in \Gamma_2/\Gamma_3$ satisfies $\theta'([\beta, \gamma] \Gamma_3) = [\beta, \gamma] \Gamma_3$. By using the addition formula (Lemma 2.1), we conclude that $R(\theta) = \infty$, provided $[\beta, \gamma]/\Gamma_3$ is of infinite order. Granting this for the moment, by the first part of the same lemma we conclude that $R(\phi) = \infty$ using the first exact sequence above. Since $\phi \in \text{Aut}(G_n)$ was arbitrary, we conclude that $G_n$ has the $R_\infty$-property. So all that remains is to show that $[\beta, \gamma] \Gamma_3$ is not a torsion element.

We use the fact that under the surjection $\psi : G_n \to G_2$ that maps $\alpha_{i,j}$ to $\alpha_{i,j}$ when $\{i, j\} = \{1, 2\}$ and the remaining $\alpha_{i,j}$ to 1, $\Gamma_k$ maps onto $\Gamma_k(G_2)$, $k=2,3$. Let $\beta_2, \gamma_2 \in G_2$ be the images of $\beta, \gamma$ respectively under $\psi$. Then $\beta_2 = \alpha_{1,2}, \gamma_2 = \alpha_{2,1} \in G_2^\text{ab}$. Therefore $[\beta_2, \gamma_2] \Gamma_3(G_2) = [\alpha_{1,2}, \alpha_{2,1}] \Gamma_3(G_2)$. Since $G_2$ is a free group with basis $\{\alpha_{1,2}, \alpha_{2,1}\}$ we see that $[\alpha_{1,2}, \alpha_{2,1}] \Gamma_3(G_2)$ generates an infinite cyclic group. Hence the same is true of $[\beta, \gamma] \Gamma_3$. This completes the proof of part (iii) of the main theorem, which is restated below:

**Theorem 3.13.** The group $G_n, n \geq 2$, has the $R_\infty$-property. □
Recall, from §1, the description of the Richard Thompson group $T$ as the group of all orientation preserving piecewise linear homeomorphisms of $S = I/{\{0,1\}}$ with slopes in the multiplicative group generated by $2 \in \mathbb{R}_{>0}$ and break points in $\mathbb{Z}[1/2]$. We regard the Thompson group $F$ as the subgroup of $T$ consisting of elements which fix the element $1 \in S^1$. In this section we prove the following result.

**Theorem 4.1.** ([8], [15]). The Richard Thompson group $T$ has the $R_\infty$-property.

The fact that $T$ has the $R_\infty$ property has been proved first by Burillo, Matucci, and Ventura [8] (see also [15]). The crucial point in the proofs of the result above is the same in both [8] and [15] and both the proofs rely on the description of the outer automorphism of $T$ (recalled in Theorem 4.2 below). However, since the approaches before getting to the main point are slightly different, we provide our proof here which may contain some features that are useful for other situations (such as in Remark 4.7 below).

It is readily seen that the reflection map $r$ defined as $r(x) = 1 - x, x \in [0,1]$, induces an automorphism $\rho : T \to T$ defined as $\rho(f) = r \circ f \circ r^{-1} = r \circ f \circ r$. We now state the following result of Brin.

**Theorem 4.2.** (Brin [4]) The group of inner automorphisms of $T$ is of index two in $\text{Aut}(T)$ and the quotient group $\text{Out}(T)$ is generated by $\rho$.

As observed in §2.2, for any group $\Gamma$ and any automorphism $\phi \in \text{Aut}(\Gamma)$, and any $g \in \Gamma$, $R(\phi) = \infty$ if and only if $R(\phi \circ \iota_g) = \infty$. Therefore, to establish the $R_\infty$-property for $\Gamma$, it is enough to show that $R(\phi) = \infty$ for a set of coset representatives of $\text{Out}(\Gamma)$. In the case $\Gamma = T$, in view of Theorem 4.2 due to Brin, we need only show that $R(\rho) = \infty$ and $R(id) = \infty$. The latter equality is established in Proposition 4.5 as an easy consequence of Lemma 4.4 below. Since $\rho^2 = id$, we may apply Remark 2.4 to show that $R(\rho) = \infty$. The main idea is to make use of homeomorphisms in $\text{Fix}(\rho)$, whose supports have arbitrarily large number of disjoint intervals in $S^1$. (This was also the idea used in the proof by Burillo, Matucci, and Ventura [8].)

**Definition 4.3.** Let $X$ be a Hausdorff topological space.

(i) The support of $f \in \text{Homeo}(X)$ is the open set $\text{supp}(f) := \{x \in X \mid f(x) \neq x\}$.

(ii) Let $\sigma : \text{Homeo}(X) \to \mathbb{N} \cup \{\infty\}$ be defined as follows: $\sigma(id) = 0$, if
f \neq id, \sigma(f) \text{ is the number of connected components of } \text{supp}(f) \text{ if it is finite, otherwise } \sigma(f) = \infty.

**Lemma 4.4.** Let \( \Gamma \subset \text{Homeo}(X) \) and let \( \sigma \) be as defined above. Suppose that \( \theta \in \text{Homeo}(X) \) normalizes \( \Gamma \). Then \( \sigma(f) = \sigma(\theta f \theta^{-1}) \).

**Proof.** It is clear that the number of connected components of an open set \( U \subset X \) remains unchanged under a homeomorphism of \( X \). The lemma follows immediately from the observation that \( \text{supp}(\theta f \theta^{-1}) = \theta(\text{supp}(f))) \). \( \square \)

**Proposition 4.5.** The groups \( F \) and \( T \) have infinitely many conjugacy classes.

**Proof.** This follows from Lemma 4.4 on observing that \( F \) has elements \( f \) with \( \sigma(f) \) any arbitrary prescribed positive integer. Since \( F \subset T \), the same is true of \( T \) as well. \( \square \)

**Lemma 4.6.** Suppose that \( h : \mathbb{R} \rightarrow \mathbb{R} \) is an orientation preserving homeomorphism. Then \( \text{supp}(h) = \text{supp}(h^k) \) for any non-zero integer \( k \).

**Proof.** Since \( \text{supp}(h) = \text{supp}(h^{-1}) \) we may assume that \( k > 0 \). Since \( h \) is orientation preserving, it is order preserving. Suppose that \( x \in \text{supp}(h) \) so that \( h(x) \neq x \). Say, \( x < h(x) \). Then applying \( h \) to the inequality we obtain \( h(x) < h^2(x) \) so that \( x < h(x) < h^2(x) \). Repeating this argument yields \( x < h(x) < \cdots < h^k(x) \) and so \( x \in \text{supp}(h^k) \). The case when \( x > h(x) \) is analogous. Thus \( \text{supp}(h) \subset \text{supp}(h^k) \). On the other hand, if \( x \notin \text{supp}(h) \), then \( h(x) = x \) and so \( h^k(x) = x \) for all \( k \). Therefore equality should hold, completing the proof. \( \square \)

We are now ready to prove our main theorem.

**Proof of Theorem 4.1:** By Theorem 4.2(ii), \( \text{Out}(T) \cong \mathbb{Z}/2\mathbb{Z} \) generated by \( \rho \). By Proposition 4.5, \( R(id) = \infty \). It only remains to verify that \( R(\rho) = \infty \). We apply Remark 2.4 with \( \theta = \rho, n = 2, \gamma = 1 \). It remains to show that \( \text{Fix}(\rho) \) has infinitely many elements \( h \) such that the \( h^2 \) are pairwise non-conjugate.

Let \( k \geq 1 \). Let \( f_k \in F \subset T \) be such that \( \text{supp}(f_k) \subset (0, 1/2) \) and has exactly \( k \) components. Thus, \( \sigma(f_k) = k \). (It is easy to construct such an element.) Then \( \text{supp}(\rho(f_k)) = \text{supp}(r f_k r^{-1}) = r(\text{supp}(f_k)) \subset (1/2, 1) \) is disjoint from \( \text{supp}(f_k) \subset (0, 1/2) \). In particular \( f_k, \rho(f_k) = \rho(f_k) . f_k =: h_k, \text{supp}(h_k) = \text{supp}(f_k) \cup r(\text{supp}(f_k)) \) and so \( \sigma(h_k) = 2k \). Moreover, since \( \rho^2 = 1 \), we see that \( h_k \in \text{Fix}(\rho) \). By Lemma 4.6, we have \( \sigma(h_k^2) = \sigma(h_k) = 2k \). It follows that \( h_k^2 \) are pairwise non-conjugate in \( T \), completing the proof. \( \square \).
Remark 4.7. In the case of the generalized Thompson groups $T_{n,r}$, suppose that $\theta \in \text{Aut}(T_{n,r})$ is a torsion element, say of order $m$, our method of proof of Theorem 4.1 can be applied to show that $R(\theta) = \infty$. In fact, applying a theorem of McCleary and Rubin [20], to the group $T_{n,r}$, we obtain that the automorphism group of $T_{n,r}$ equals its normalizer in the group of all homeomorphisms of the circle $\mathbb{S}^1 = [0, r]/\{0, r\}$. Let $\theta \in \text{Aut}(T_{n,r})$ and $f \in \mathbb{S}^1$ is such that $\theta(x) = xf^{-1}$ with $f \in \text{Homeo}(\mathbb{R}/r\mathbb{Z})$. Suppose $f^m = \gamma \in T_{n,r}$ so that $\theta$ represents a torsion element of $\text{Out}(T_{n,r})$. If $\gamma = 1$, our method of proof of Theorem 4.1 can be applied to show that $R(\theta) = \infty$. See [15] for details. However, when $\gamma \neq 1$, it is not clear to us how to find elements of $\text{Fix}(\theta)$ satisfying the hypotheses of Lemma 2.3. Our approach yields no information about automorphisms which represent non-torsion elements in the outer automorphism group. The study of the $R_\infty$ property for the groups $T_{n,r}$ is a work in progress.

5. DIRECT PRODUCT OF GROUPS

It was shown in [14, Theorem 4.8] that if $G = G_1 \times \cdots \times G_n$ where each $G_i$ is a finitely generated group with the property that $\Sigma^i(G_i)$ is a finite set of discrete character classes, not all of them empty, then there exists a finite index subgroup $H$ of $\text{Aut}(G)$ such that $R(\phi) = \infty$ for all $\phi \in H$. Further, when each $G_i$ is a generalized Richard Thompson group $F_{n,\infty}$, $n_i \geq 2$, then $G$ itself has the $R_\infty$-property.

We shall strengthen the above result here. We make use of a result of Meinert (as did Gonçalves and Kochloukova [14]) that describes the $\Sigma$-invariant of a direct product which is recalled below. (Meinert’s theorem describes the $\Sigma$-invariant in the more general setting of a graph product of groups.)

Let $G = G_1 \times \cdots \times G_n$ and let $r_j = rk(G_j^{ab})$ so that $S(G_j) \cong \mathbb{S}^{r_j-1}$. We will assume that $r_1 \geq 1$. Then $S(G) = \prod_{1 \leq j \leq n} \text{Hom}(G_j, \mathbb{R}) \setminus \{0\}/\sim \cong \mathbb{S}^{r-1}$ and so $S(G) \cong \mathbb{S}^{r-1}, r := \sum_{1 \leq j \leq n} r_j$. It is understood that $S(G_j) = \emptyset$ if $r_j = 0$. The sphere $S(G_i)$ is identified with the subspace of $S(G)$ with the set of points with $j$-th coordinate equal to zero for all $j \neq i$. Observe that $S(G_i) \cap S(G_j) = \emptyset$ if $i \neq j$. In order to emphasize this we shall write $S(S_i) \cup S(S_j)$ to denote their union, thought of as subspaces of $S(G)$.

Recall that $\Sigma^c(G)$ denotes the complement of $\Sigma^1(G) \subset S(G)$.

Theorem 5.1. (Meinert [21]) Let $G = G_1 \times \cdots \times G_n$ be finitely generated and let $r_1 = rk(G_1^{ab}) > 0$. With the above notations, $\Sigma^c(G)$ equals $\sqcup_{1 \leq j \leq n} \Sigma^c(G_j)$. $\square$
We will exploit the fact that any \( \phi \in \text{Aut}(G) \) induces a homeomorphism of the character sphere \( S(G) \) which preserves its rational structure. Recall that an element \([\chi] \in S(G)\) is called discrete (or rational) if \( \text{Im}(\chi) \subset \mathbb{R} \) is infinite cyclic, equivalently, \( \chi \) may be chosen to take values in \( \mathbb{Q} \subset \mathbb{R} \). The set of rational points in \( S(G) \) is denoted \( S_{\mathbb{Q}}(G) \). We denote by \( D_{\mathbb{Q}}(G) \) the set of isolated rational points in \( \Sigma^c(G) \). The set of all limit points of \( D_{\mathbb{Q}}(G) \) which are contained in \( S_{\mathbb{Q}}(G) \) will be denoted by \( L_{\mathbb{Q}}(G) \). Also, we denote by \( L(G) \) the set of all limit points of \( \Sigma^c(G) \). Since \( \Sigma^c(G) \) is closed, \( L_{\mathbb{Q}}(G) \) and \( L(G) \) are subsets of \( \Sigma^c(G) \). Any homeomorphism of \( \Sigma^c(G) \) induced by an automorphism of \( G \) maps \( D_{\mathbb{Q}}(G), L_{\mathbb{Q}}(G), L(G) \) respectively onto itself.

We are now ready to prove the following theorem. The proof is essentially the same in spirit as that of [14, Theorem 3.3]. See also [14, §4c].

**Theorem 5.2.** Suppose that \( G = G_1 \times \cdots \times G_n, n \geq 1 \), is finitely generated and that any one of the following holds: (i) the set \( D_{\mathbb{Q}}(G_1) \) is non-empty, finite, and is contained in an open hemisphere and \( D_{\mathbb{Q}}(G_j) \) is finite (possibly empty) for \( 2 \leq j \leq n \), (ii) the set \( L_{\mathbb{Q}}(G_1) \) is non-empty, finite, and is contained in an open hemisphere and \( L_{\mathbb{Q}}(G_j) \) is a finite set (possibly empty) for \( 2 \leq j \leq n \), (iii) the set \( L(G_1) \cap S_{\mathbb{Q}}(G_1) \) is a non-empty finite set contained in an open hemisphere and the set \( L(G_j) \cap S_{\mathbb{Q}}(G_j) \) is finite (possibly empty) for \( 2 \leq j \leq n \). Then \( G \) has the \( R_{\infty} \)-property.

**Proof.** Let \( \phi \in \text{Aut}(G) \). We shall show that there exists a discrete character \( \lambda \in \text{Hom}(G, \mathbb{R}) \) such that \( \lambda \circ \phi = \lambda \). By the discussion in §2.3, it follows that \( R(\phi) = \infty \) and it follows that \( G \) has the \( R_{\infty} \)-property.

First we suppose that \( n = 1 \). The theorem, then, is essentially due to Gonçalves and Kochloukova [14]. Let \( \phi \in \text{Aut}(G) \) and let \( \phi^* : \Sigma^c(G) \to \Sigma^c(G) \) be the induced map, defined as \( \phi^*([\chi]) = [\chi \circ \phi] \). The map \( \phi^* : \Sigma^c(G) \to \Sigma^c(G) \) being a homeomorphism, it maps isolated points to isolated points. Moreover, \( \phi^* \) preserves the set of all rational points in \( \Sigma^c(G) \). It follows that \( \phi^*(W) = W \) where \( W \) is one of the sets \( D_{\mathbb{Q}}(G), L_{\mathbb{Q}}(G) \) or \( L(G) \cap S_{\mathbb{Q}}(G) \).

In each of the cases (i)-(iii), we see that there is a non-empty finite set of rational character classes \( W(G) \subset S_{\mathbb{Q}}(G) \) which is contained in an open hemisphere and is mapped to itself by \( \phi^* \). Suppose that \( [\chi] \in W(G) \) and that the orbit of \( [\chi] \) under \( \phi^* \), namely the set \( \{(\phi^*)^j([\chi]) = [\chi \circ \phi^j] \mid j \in \mathbb{N}\} \), has \( k \) elements. Then the orbit sum \( \lambda := \sum_{0 \leq j < k} \chi \circ \phi^j \in \text{Hom}(G, \mathbb{R}) \) is a non-zero discrete character invariant under \( \phi^* \), as was to be shown.
Now let \( n = 2 \). By Meinert’s theorem (Theorem 5.1) \( D_\mathbb{Q}(G) = D_\mathbb{Q}(G_1) \sqcup D_\mathbb{Q}(G_2) \), \( L_\mathbb{Q}(G) = L_\mathbb{Q}(G_1) \sqcup L_\mathbb{Q}(G_2) \) and \( L(G) = L(G_1) \sqcup L(G_2) \).

Case (i). Let \( [\chi] \in D_\mathbb{Q}(G_1) \). Consider the \( \phi^* \)-orbit of \( [\chi] \), namely, \( \{(\phi^k)^*([\chi]) = [\chi \circ \phi^k] \mid k \in \mathbb{Z} \} \). This set is finite since it is contained in \( D_\mathbb{Q}(G) = D_\mathbb{Q}(G_1) \sqcup D_\mathbb{Q}(G_2) \), which is finite. Suppose that \( [\chi \circ \phi^j] \mid 0 \leq j < q \), are the distinct rational points in the orbit. We claim that the orbit sum \( \lambda := \sum_{0 \leq j < q} \chi \circ \phi^j \) is a non-zero character such that \( \lambda \circ \phi = \lambda \). To see that \( \lambda \in \text{Hom}(G, \mathbb{R}) \) is non-zero, note that its restriction to \( G_1 \) is the character \( \lambda_J = \sum_{j \in J} \chi \circ \phi^j \) where \( J := \{j < q \mid [\chi \circ \phi^j] \in D_\mathbb{Q}(G_1) \} \). Since \( D_\mathbb{Q}(G_1) \) is contained in an open hemisphere, the characters \( \chi \circ \phi^j \), \( j \in J \), are in an open half-space of \( \text{Hom}(G_1, \mathbb{R}) \). Therefore the same is true of their sum, \( \lambda_1 \), and we conclude that \( \lambda \neq 0 \).

It is clear that \( \lambda \circ \phi = \lambda \) since \( [\lambda \circ \phi] = [\lambda] \) and \( \lambda \) is rational. As observed in the first para of §2.3, this implies that \( R(\phi) = \infty \).

Proof in cases (ii) is almost identical, starting with a \( [\chi] \in L_\mathbb{Q}(G_1) \). We need only observe that \( \phi^*(L_\mathbb{Q}(G)) = L_\mathbb{Q}(G) \) and that as in case (ii), \( L_\mathbb{Q}(G) = L_\mathbb{Q}(G_1) \sqcup L_\mathbb{Q}(G_2) \) is finite. The orbit sum \( \lambda := \sum_{0 \leq j < q} \chi \circ \phi^j \) is again a non-zero character which is discrete and satisfies \( \lambda \circ \phi = \lambda \).

Again we conclude that \( R(\phi) = \infty \).

Case (iii). Again we start with a \( \chi \in L(G_1) \cap S_\mathbb{Q}(G_1) \) and proceed as in case (ii). We leave the details to the reader.

Finally, let \( n \geq 3 \) be arbitrary. Let \( H = G_2 \times \cdots \times G_n \). Again by Meinert’s theorem \( D_\mathbb{Q}(H) = \sqcup_{2 \leq j \leq n} D_\mathbb{Q}(G_j) \) and similar expressions hold for \( L_\mathbb{Q}(H) \) and \( L(H) \cap S_\mathbb{Q}(H) \). Our hypotheses on \( G_j \) implies that one of the sets \( D_\mathbb{Q}(H), L_\mathbb{Q}(H), L(G) \cap S_\mathbb{Q}(G) \) is finite depending on case (i), (ii), and (iii) respectively. Since \( G = G_1 \times H \), we are now reduced to the situation where \( n = 2 \), which has just been established. This completes the proof.

We conclude the paper with the following examples.

**Examples 5.3.** (i) Examples of groups with \( D_\mathbb{Q}(G) \) non-empty, finite, and contained in an open hemisphere are known. These include non-polycyclic nilpotent-by-finite groups of type \( FP_\infty \), the generalized Richard Thompson groups \( F_{n,\infty} \), the double of a knot group \( K \) with non-finitely generated commutator subgroup (thus \( G \cong K \star_{\mathbb{Z}_2} K \)). For details see [14, §4].

(ii) Examples of groups with \( D_\mathbb{Q}(G) \) and \( L_\mathbb{Q}(G) \) being finite sets are finite groups, the Houghton groups [6], the pure symmetric automorphism groups [22], finitely generated infinite groups with finite abelian-ization (which include the generalized Richard Thompson groups \( T_{n,r} \), ...
see [6, p. 64]), \( \mathbb{Z}^n, n \geq 1 \), and the free groups of rank \( n \geq 2 \). Another class of such groups is provided by [2, Theorem 8.1]. Consider a finitely generated group \( G \) which is a subgroup of the group of all orientation preserving PL-homeomorphisms of the interval \([0, 1]\). The group \( G \) is said to be irreducible if there is no \( G \) fixed point in \((0, 1)\). The logarithms of the slopes near the end points \( 0, 1 \), define characters \( \chi_0, \chi_1 : G \to \mathbb{R} \) respectively. We recall that two characters \( \lambda, \chi \) are independent if \( \lambda(\ker(\chi)) = \lambda(G) \) and \( \chi(\ker(\lambda)) = \chi(G) \). It was shown in [2, Theorem 8.1] that \( \Sigma^r(G) = \{[\chi_0], [\chi_1]\} \) if \( G \) is irreducible and \( \chi_0, \chi_1 \) are independent. (These points may not be in \( S_\mathbb{Q}(G) \); see [2, p. 470].)

(iii) Let \( G = G_1 \times G_2 \) where \( G_1 \) is a finite product of groups (with \( G_1 \) non-trivial) as in example (i) and \( G_2 \), a finite product of groups as in example (ii) above. Then \( G \) has the \( R_\infty \)-property. Since there are continuously many pairwise non-isomorphic two generated infinite simple groups, taking \( G_2 \) to be any one of them, we obtain a continuous family of groups with \( R_\infty \)-property.

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