Abelian Toda solitons revisited

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Abstract

We present a systematic and detailed review of the application of the method of Hirota and the rational dressing method to abelian Toda systems associated with the untwisted loop groups of complex general linear groups. Emphasizing the rational dressing method, we compare the soliton solutions constructed within these two approaches, and show that the solutions obtained by the Hirota’s method are a subset of those obtained by the rational dressing method.

1 Introduction

Two-dimensional Toda equations associated with loop groups\(^1\) are very interesting examples of completely integrable systems, see, for example, the monographs [1, 2]. They possess soliton solutions having a nice physical interpretation as interacting extended objects. Actually there is no a clear definition of a soliton solution. In the present paper we call a solution of equations an \(n\)-soliton solution, if it depends on \(n\) linear combinations of independent variables.

Soliton solutions for Toda equations can be constructed with the help of various methods. As far as we know, first explicit solutions of Toda equations associated with loop groups were found by Mikhailov [3]. He used the rational dressing method being a version of the inverse scattering method [4]. Note that in general the solutions obtained by Mikhailov are not soliton solutions. Besides, they are described by a redundant set of parameters.

Another method used here is the Hirota’s one. Its essence [5] is a change of the dependent variables which introduces the so called \(\tau\)-functions. Here the final goal is to come to some special bilinear partial differential equations which are solved then perturbatively. The soliton solutions arise when the perturbation series truncates at some finite order. This method was applied to affine Toda systems, for example, in the papers [6, 7, 8, 9, 10]. The main disadvantage of the Hirota’s method is that there is no a regular method to find the desired transformation from the initial dependent variables.

\(^1\)Sometimes one deals with Toda equations associated with affine groups being central extensions of loop groups. Usually it is possible to construct solutions of the equations associated with affine groups starting from solutions of the equations associated with loop groups.
variables to $\tau$-functions. Therefore, sometimes it is used in combination with other methods that helps to obtain a desired ansatz, see, for example, the papers [11, 12].

There are also two additional approaches to the problem, being a development of the Leznov–Saveliev method [13, 14, 15], and of the Bäcklund–Darboux transformation [16, 17, 18, 19, 20]. These methods give the same soliton solutions as the Hirota’s one and are not in the scope of the present paper.

Basic purpose of our review is to reproduce, in a possibly systematic and detailed way, the application of the Hirota’s and rational dressing methods to Toda systems associated with the untwisted loop groups of complex general linear groups, making an emphasis on the rational dressing method, and compare the soliton solutions constructed along these approaches. We show that all soliton solutions obtained by the Hirota’s method are contained among the solutions obtained by the rational dressing method.

## 2 Equations

### 2.1 Zero-curvature representation of Toda equations

It is well known that Toda equations can be formulated as the zero-curvature condition for a connection of a special form on the trivial fiber bundle $\mathbb{R}^2 \times \mathcal{G} \rightarrow \mathbb{R}^2$, where $\mathcal{G}$ is a Lie group with the Lie algebra $\mathfrak{g}$. The connection under consideration can be identified with a $\mathfrak{g}$-valued 1-form $\mathcal{O}$ on $\mathbb{R}^2$. One can decompose such a connection over basis 1-forms as

$$\mathcal{O} = \mathcal{O}_- dz^- + \mathcal{O}_+ dz^+, $$

where $z^-, z^+$ are the standard coordinates on the base manifold $\mathbb{R}^2$, and the components $\mathcal{O}_-, \mathcal{O}_+$ are $\mathfrak{g}$-valued functions on it. Let us assume that the connection $\mathcal{O}$ is flat, that means that its curvature is zero. This condition in terms of the components has the form

$$\partial_- \mathcal{O}_+ - \partial_+ \mathcal{O}_- + [\mathcal{O}_-, \mathcal{O}_+] = 0. \quad (2.1)$$

One can consider this relation as a system of partial differential equations. In a sense, this system is trivial, and its general solution is well known. It is given by the relations

$$\mathcal{O}_- = \Phi^{-1} \partial_- \Phi, \quad \mathcal{O}_+ = \Phi^{-1} \partial_+ \Phi, $$

where $\Phi$ is an arbitrary mapping of $\mathbb{R}^2$ to $\mathcal{G}$. Actually the triviality of the zero-curvature condition is due to its gauge invariance. That means that if a connection $\mathcal{O}$ satisfies (2.1) then for an arbitrary mapping $\Psi$ of $\mathbb{R}^2$ to $\mathcal{G}$ the gauge transformed connection

$$\mathcal{O}^\Psi = \Psi^{-1} \mathcal{O} \Psi + \Psi^{-1} d\Psi, \quad (2.2)$$

satisfies (2.1) as well.

To obtain a nontrivial integrable system out of the zero-curvature condition one imposes on the connection $\mathcal{O}$ some restriction destroying the gauge invariance. For the case of Toda equations they are the grading and gauge fixing conditions which are introduced as follows.

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2 We use the usual notation $\partial_- = \partial / \partial z^-$ and $\partial_+ = \partial / \partial z^+$. 

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Suppose that the Lie algebra $\mathfrak{g}$ is endowed with a $\mathbb{Z}$-gradation,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, \quad [\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l},$$

and that a positive integer $L$ is such that the grading subspaces $\mathfrak{g}_k$, where $0 < |k| < L$, are trivial. The grading condition states that the components of $\mathcal{O}$ have the form

$$\mathcal{O}_- = \mathcal{O}_{-0} + \mathcal{O}_{-L}, \quad \mathcal{O}_+ = \mathcal{O}_{+0} + \mathcal{O}_{+L},$$

(2.3)

where $\mathcal{O}_{-0}$ and $\mathcal{O}_{+0}$ take values in $\mathfrak{g}_0$, while $\mathcal{O}_{-L}$ and $\mathcal{O}_{+L}$ take values in $\mathfrak{g}_{-L}$ and $\mathfrak{g}_{+L}$ respectively. There is a residual gauge invariance. Indeed, the gauge transformation (2.2) with $\Psi$ taking values in the Lie subgroup $\mathfrak{g}_0$ corresponding to the subalgebra $\mathfrak{g}_0$ does not violate the validity of the grading condition (2.3). Therefore, one imposes an additional condition, called the gauge fixing condition, of the form

$$\mathcal{O}_{+0} = 0.$$ 

After that one can show that the components of the connection $\mathcal{O}$ can be represented as

$$\mathcal{O}_- = \mathcal{E}^{-1}\partial_+ \mathcal{E} + \mathcal{F}_-, \quad \mathcal{O}_+ = \mathcal{E}^{-1}\mathcal{F}_+ \mathcal{E},$$

(2.4)

where $\mathcal{E}$ is a mapping of $\mathbb{R}^2$ to $\mathfrak{g}_0$, $\mathcal{F}_-$ and $\mathcal{F}_+$ are some mappings of $\mathbb{R}^2$ to $\mathfrak{g}_{-L}$ and $\mathfrak{g}_{+L}$. One can easily get convinced that the zero-curvature condition is equivalent to the equality

$$\partial_+ (\mathcal{E}^{-1}\partial_+ \mathcal{E}) = [\mathcal{F}_-, \mathcal{E}^{-1}\mathcal{F}_+ \mathcal{E}]$$

(2.5)

and the relations

$$\partial_+ \mathcal{F}_- = 0, \quad \partial_- \mathcal{F}_+ = 0.$$ 

(2.6)

One supposes that the mappings $\mathcal{F}_-$ and $\mathcal{F}_+$ are fixed and considers (2.5) as an equation for $\mathcal{E}$ called the Toda equation. When the group $\mathfrak{g}_0$ is abelian the corresponding Toda equations are called abelian.

Thus, a Toda equation associated with a Lie group $\mathcal{G}$ is specified by a choice of a $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ and mappings $\mathcal{F}_-$, $\mathcal{F}_+$ satisfying the conditions (2.6). To classify the Toda equations associated with a Lie group $\mathcal{G}$ one should classify $\mathbb{Z}$-gradations of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$.

Two remarks are in order. First, let $\Sigma$ be an isomorphism from a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ to a Lie algebra $\mathfrak{h}$. One can consider $\mathfrak{h}$ as a $\mathbb{Z}$-graded Lie algebra with grading subspaces $\mathfrak{h}_k = \Sigma(\mathfrak{g}_k)$. In such situation, one says that $\mathbb{Z}$-gradations of $\mathfrak{g}$ and $\mathfrak{h}$ are conjugated by $\Sigma$. Now let the Lie algebra $\mathfrak{g}$ of the Lie group $\mathcal{G}$ be supplied with a $\mathbb{Z}$-gradation, and $\sigma$ be an isomorphism from $\mathcal{G}$ to a Lie group $\mathcal{H}$. Denote by $\Sigma$ the isomorphism from the Lie algebra $\mathfrak{g}$ to the Lie algebra $\mathfrak{h}$ of the Lie group $\mathcal{H}$ induced by the isomorphism $\sigma$. It is clear that if $\mathcal{E}$ is a solution of the Toda equation (2.5), then the mapping

$$\mathcal{E}' = \sigma \circ \mathcal{E}$$

(2.7)

\textsuperscript{3}It can be shown that rejecting this restriction we do not come to new Toda equations, see the paper [21].

\textsuperscript{4}We assume for simplicity that $\mathcal{G}$ is a subgroup of the group formed by invertible elements of some unital associative algebra $\mathcal{A}$. In this case $\mathfrak{g}$ can be considered as a subalgebra of the Lie algebra associated with $\mathcal{A}$. Actually one can generalize our consideration to the case of an arbitrary Lie group $\mathcal{G}$. 

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satisfies the Toda equation (2.5) with the mappings $F_-, F_+$ replaced by the mappings

$$F'_- = \Sigma \circ F_-, \quad F'_+ = \Sigma \circ F_+.$$  \hfill (2.8)

In other words, the solutions to two Toda equations under consideration are connected via the isomorphism $\sigma$, and in this sense these equations are equivalent. Thus, to describe really different Toda equations it suffices to consider Lie groups and $\mathbb{Z}$-gradations of their Lie algebras up to isomorphisms.

Secondly, let $\Theta_-$ and $\Theta_+$ be some mappings of $\mathbb{R}^2$ to $G_0$ which satisfy the conditions

$$\partial_+ \Theta_- = 0, \quad \partial_- \Theta_+ = 0.$$ 

If a mapping $\Xi$ satisfies the Toda equation (2.5), then the mapping

$$\Xi' = \Theta_+^{-1} \Xi \Theta_-,$$  \hfill (2.9)

satisfies the Toda equation (2.5) where the mappings $F_-, F_+$ are replaced by the mappings

$$F'_- = \Theta_+^{-1} F_- \Theta_-, \quad F'_+ = \Theta_+^{-1} F_+ \Theta_+.$$  \hfill (2.10)

Again, the Toda equations for $\Xi$ and $\Xi'$ are not actually different, and it is natural to use the transformations (2.10) to make the mappings $F_-, F_+$ as simple as possible.

If the mappings $\Theta_-$ and $\Theta_+$ are such that

$$\Theta_+^{-1} F_- \Theta_- = F_-, \quad \Theta_+^{-1} F_+ \Theta_+ = F_+.$$ 

then the mapping $\Xi'$ satisfies the same Toda equation as the mapping $\Xi$. Hence, in this case the transformation described by relations (2.9) is a symmetry transformation for the Toda equation under consideration.

### 2.2 Toda equations associated with loop groups of complex simple Lie groups

Let $\mathfrak{g}$ be a finite dimensional real or complex Lie algebra. The loop Lie algebra of $\mathfrak{g}$, denoted $\mathcal{L}(\mathfrak{g})$, is defined alternatively either as the linear space $C^\infty(S^1, \mathfrak{g})$ of smooth mappings of the circle $S^1$ to $\mathfrak{g}$, or as the linear space $C^\infty_{2\pi}(\mathbb{R}, \mathfrak{g})$ of smooth $2\pi$-periodic mappings of the real line $\mathbb{R}$ to $\mathfrak{g}$ with the Lie algebra operation defined in both cases pointwise, see, for example, [22, 23, 24]. In this paper we adopt the second definition and think of the circle $S^1$ as consisting of complex numbers of modulus one. There is a convenient way to supply $\mathcal{L}(\mathfrak{g})$ with the structure of a Fréchet space, so that the Lie algebra operation becomes a continuous mapping, see, for example, [25, 23, 24].

Now, let $G$ be a finite dimensional Lie group with the Lie algebra $\mathfrak{g}$. We define the loop group of $G$, denoted $\mathcal{L}(G)$, as the set $C^\infty(S^1, G)$ of smooth mappings of $S^1$ to $G$ with the group law defined pointwise. We assume that $\mathcal{L}(G)$ is supplied with the structure of a Fréchet manifold modeled on $\mathcal{L}(\mathfrak{g})$ in such a way that it becomes a Lie group, see, for example, [25, 23, 24]. The Lie algebra of the Lie group $\mathcal{L}(G)$ is naturally identified with the loop Lie algebra $\mathcal{L}(\mathfrak{g})$.

Let $A$ be an automorphism of a finite dimensional Lie algebra $\mathfrak{g}$ satisfying the relation $A^M = \text{id}_\mathfrak{g}$ for some positive integer $M$. The twisted loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ is a

\footnote{Note that we do not assume that $M$ is the order of the automorphism $A$. It can be an arbitrary multiple of the order.}
subalgebra of the loop Lie algebra $L(g)$ formed by elements $\xi$ that satisfy the equality

$$\zeta(\epsilon_M \bar{p}) = A(\zeta(\bar{p})), \tag{2.10}$$

where $\epsilon_M = e^{2\pi i / M}$ is the $M$th principal root of unity. Similarly, given an automorphism $a$ of a Lie group $G$ that satisfies the relation $a^M = \text{id}_G$, we define the twisted loop group $L_{a,M}(G)$ as the subgroup of the loop group $L(G)$ formed by the elements $\chi$ satisfying the equality

$$\chi(\epsilon_M \bar{p}) = a(\chi(\bar{p})). \tag{2.11}$$

The Lie algebra of a twisted loop group $L_{a,M}(G)$ is naturally identified with the twisted loop Lie algebra $L_{A,M}(g)$, where we denote by $A$ the automorphism of the Lie algebra $g$ corresponding to the automorphism $a$ of the Lie group $G$.

It is clear that loop groups and loop Lie algebras are partial cases of twisted loop groups and twisted loop Lie algebras respectively. Therefore, below by a loop group we mean either a usual loop group or a twisted loop group, and by a loop Lie algebra we mean either a usual loop Lie algebra or a twisted loop Lie algebra.

Now let us discuss the form of the Toda equations associated with a loop group $L_{a,M}(G)$. First of all note that the group $L_{a,M}(G)$ and its Lie algebra $L_{A,M}(g)$ are infinite dimensional manifolds. It appears that it is convenient to reformulate the zero curvature representation of the Toda equations associated with $L_{a,M}(G)$ in terms of finite dimensional manifolds. To this end we use the so-called exponential law, see, for example, [26, 27].

Let $\mathcal{M}, \mathcal{N}, \mathcal{P}$ be three finite dimensional manifolds, and $\mathcal{N}$ be compact. Consider a smooth mapping $\mathcal{F}$ of $\mathcal{M}$ to $\mathcal{C}^\infty(\mathcal{N}, \mathcal{P})$. This mapping induces a mapping $f$ of $\mathcal{M} \times \mathcal{N}$ to $\mathcal{P}$ defined by the equality

$$f(\bar{m}, \bar{n}) = (\mathcal{F}(\bar{m}))(\bar{n}). \tag{2.12}$$

It can be proved that the mapping $f$ is smooth. Conversely, if one has a smooth mapping of $\mathcal{M} \times \mathcal{N}$ to $\mathcal{P}$, reversing the above equality one defines a mapping of $\mathcal{M}$ to $\mathcal{C}^\infty(\mathcal{N}, \mathcal{P})$, and this mapping is also smooth. Thus, we have the following canonical identification

$$\mathcal{C}^\infty(\mathcal{M}, \mathcal{C}^\infty(\mathcal{N}, \mathcal{P})) = \mathcal{C}^\infty(\mathcal{M} \times \mathcal{N}, \mathcal{P}).$$

It is this equality that is called the exponential law.

In the case under consideration the connection components $\mathcal{O}_-$ and $\mathcal{O}_+$ entering the equality (2.1) are mappings of $\mathbb{R}^2$ to the loop Lie algebra $L_{A,M}(g)$. We will denote the corresponding mappings of $\mathbb{R}^2 \times S^1$ to $g$ by $\omega_-$ and $\omega_+$, and call them also the connection components. The mapping $\Phi$ generating the connection is a mapping of $\mathbb{R}^2$ to $L_{a,M}(G)$. Denoting the corresponding mapping of $\mathbb{R}^2 \times S^1$ by $\varphi$ we write

$$\varphi^{-1} \partial_- \varphi = \omega_-, \quad \varphi^{-1} \partial_+ \varphi = \omega_+. \tag{2.11}$$

Having in mind that the mapping $\varphi$ uniquely determines the mapping $\Phi$, we say that the mapping $\varphi$ also generates the connection under consideration.

The relations (2.4) are equivalent to the equalities

$$\omega_- = \gamma^{-1} \partial_- \gamma + f_-, \quad \omega_+ = \gamma^{-1} f_+ \gamma,$$
where $\gamma$ is a smooth mapping of $\mathbb{R}^2 \times S^1$ to $G$ corresponding to the mapping $\Xi$, $f_-$ and $f_+$ are smooth mappings of $\mathbb{R}^2 \times S^1$ to the Lie algebra $\mathfrak{g}$ of $G$ corresponding to the mappings $\mathcal{F}_-$ and $\mathcal{F}_+$. The mappings $f_-$ and $f_+$ satisfy the conditions

$$\partial_+ f_- = 0, \quad \partial_- f_+ = 0,$$

which follow from the conditions (2.6). The Toda equation (2.5) in the case under consideration is equivalent to the equation

$$\partial_+ (\gamma^{-1} \partial_- \gamma) = [f_-, \gamma^{-1} f_+ \gamma].$$

To classify Toda equations associated with loop groups one has to classify $\mathbb{Z}$-gradations of the corresponding loop Lie algebras. This problem was partially solved in the paper [24], see also [21]. In these papers the case of loop Lie algebras of complex simple Lie algebras was considered and for this case a wide class of the so-called integrable $\mathbb{Z}$-gradations [24] with finite dimensional grading subspaces was described. Actually it was shown that when $\mathfrak{g}$ is a complex simple Lie algebra any integrable $\mathbb{Z}$-gradation of a loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ with finite dimensional grading subspaces is conjugated by an isomorphism to the standard gradation of another loop Lie algebra $\mathcal{L}_{A',M'}(\mathfrak{g})$, where the automorphisms $A$ and $A'$ differ by an inner automorphism of $\mathfrak{g}$. In particular, if $A$ is an inner (outer) automorphism of $\mathfrak{g}$, then $A'$ is also an inner (outer) automorphism of $\mathfrak{g}$.

Assume now that $G$ is a finite dimensional complex simple Lie group, then its Lie algebra $\mathfrak{g}$ is a complex simple Lie algebra. Consider a Toda equation associated with a loop group $\mathcal{L}_{a,M}(G)$. The corresponding $\mathbb{Z}$-gradation of $\mathcal{L}_{A,M}(\mathfrak{g})$ is conjugated by an isomorphism to the standard $\mathbb{Z}$-gradation of an appropriate loop Lie algebra $\mathcal{L}_{A',M'}(\mathfrak{g})$. Since the automorphisms $A$ and $A'$ differ by an inner automorphism of $\mathfrak{g}$, the automorphism $A'$ can be lifted to an automorphism $a'$ of $G$, and the isomorphism from $\mathcal{L}_{A,M}(\mathfrak{g})$ to $\mathcal{L}_{A',M'}(\mathfrak{g})$ under consideration can be lifted to an isomorphism from $\mathcal{L}_{a,M}(G)$ to $\mathcal{L}_{a',M'}(G)$. Actually this means that the initial Toda equation associated with $\mathcal{L}_{a,M}(G)$ is equivalent to a Toda equation associated with $\mathcal{L}_{a',M'}(G)$ arising when we supply $\mathcal{L}_{A',M'}(\mathfrak{g})$ with the standard $\mathbb{Z}$-gradation.

The grading subspaces for the standard $\mathbb{Z}$-gradation of a loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ are

$$\mathcal{L}_{A,M}(\mathfrak{g})_k = \{ \xi \in \mathcal{L}_{A,M}(\mathfrak{g}) \mid \xi = \lambda^k x, \ A(x) = \epsilon^k_M x \},$$

where by $\lambda$ we denote the restriction of the standard coordinate on $\mathbb{C}$ to $S^1$. It is very useful to realize that every automorphism $A$ of the Lie algebra $\mathfrak{g}$ satisfying the relation $A^M = \text{id}_\mathfrak{g}$ induces a $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ with the grading subspaces\(^{6}\)

$$\mathfrak{g}[k]_M = \{ x \in \mathfrak{g} \mid A(x) = \epsilon^k_M x \}, \quad k = 0, \ldots, M - 1.$$

Vice versa, any $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ defines in an evident way an automorphism $A$ of $\mathfrak{g}$ satisfying the relation $A^M = \text{id}_\mathfrak{g}$. A $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ is called an inner or outer type gradation, if the associated automorphism $A$ of $\mathfrak{g}$ is of inner or outer type respectively.

In terms of the corresponding $\mathbb{Z}_M$-gradation the grading subspaces for the standard $\mathbb{Z}$-gradation of a loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ are

$$\mathcal{L}_{A,M}(\mathfrak{g})_k = \{ \xi \in \mathcal{L}_{A,M}(\mathfrak{g}) \mid \xi = \lambda^k x, \ x \in \mathfrak{g}[k]_M \}.$$

\(^{6}\)We denote by $[k]_M$ the element of the ring $\mathbb{Z}_M$ corresponding to the integer $k$.  

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**Note:** The text contains mathematical expressions and notations that are typical in advanced algebraic and geometric contexts, specifically focusing on the classification of Toda equations and their associated loop Lie algebras. The text employs concepts such as automorphisms, gradations, and loops in the context of Lie algebras and groups, aiming to classify and understand the equations associated with loop groups.
It is evident that for the standard $\mathbb{Z}_M$-gradation the subalgebra $L_{a,M}(g)_0$ is isomorphic to the subalgebra $g_{[0]}_{[M]}$ of $g$, and the Lie group $L_{a,M}(G)_0$ is isomorphic to the connected Lie subgroup $G_0$ of $G$ corresponding to the Lie algebra $g_{[0]}_{[M]}$. Hence, the mapping $\gamma$ is actually a mapping of $R^2$ to $G_0$. The mappings $f_-$ and $f_+$ are given by the relation

$$f_-(\bar{m}, \bar{p}) = \bar{p}^{-L}c_-(\bar{m}), \quad f_+(\bar{m}, \bar{p}) = \bar{p}^Lc_+(\bar{m}), \quad \bar{m} \in R^2, \quad \bar{p} \in S^1,$$

where $c_-$ and $c_+$ are mappings of $R^2$ to $g_{-}[L]_M$ and $g_{+}[L]_M$ respectively. For the connection components $\omega_-$ and $\omega_+$ we have

$$\omega_- = \gamma^{-1}\partial_-\gamma + \lambda^{-L}c_-, \quad \omega_+ = \lambda^L\gamma^{-1}c_+\gamma,$$

(2.14)

and the Toda equation (2.13) can be written as

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma].$$

(2.15)

The conditions (2.12) imply that

$$\partial_+c_-=0, \quad \partial_-c_+=0. \quad (2.16)$$

It is natural to call an equation of the form (2.15) also a Toda equation.

Let $b$ be an automorphism of the Lie group $G$ and $B$ be the corresponding automorphism of the Lie algebra $g$. The mapping $\sigma$ defined by the equality

$$\sigma(\chi) = b \circ \chi, \quad \chi \in L_{a,M}(G),$$

is an isomorphism from $L_{a,M}(G)$ to $L_{a',M}(G)$, where $a'$ is an automorphism of $G$ defined as

$$a' = bab^{-1}.$$

It is clear that the mapping $\Sigma$ defined by the equality

$$\Sigma(\zeta) = B \circ \zeta, \quad \zeta \in L_{A,M}(g),$$

is an isomorphism from $L_{A,M}(g)$ to $L_{A',M}(g)$, where $A'$ is an automorphism of $g$ corresponding to the automorphism $a'$ of $G$. With such isomorphism in the case under consideration the transformations (2.7) and (2.8) take the form

$$\gamma' = b \circ \gamma,$$

$$c'_- = B \circ c_-, \quad c'_+ = B \circ c_+.$$

If a mapping $\gamma$ satisfies the Toda equations (2.15), then the mapping $\gamma'$ satisfies the Toda equation (2.15) where the mappings $c_-, c_+$ are replaced by the mappings $c'_-$ and $c'_+$. Actually this means that Toda equations of the form (2.15) defined by means of conjugated $\mathbb{Z}_M$-gradations are equivalent.

The transformations (2.9) and (2.10) take now the forms

$$\gamma' = \eta^{-1}_+\gamma\eta_-, \quad (2.17)$$

$$c'_- = \eta^{-1}_-c_-\eta_-, \quad c'_+ = \eta^{-1}_+c_+\eta_+, \quad (2.18)$$

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where \( \eta_- \) and \( \eta_+ \) are some mappings of \( \mathbb{R}^2 \times S^1 \) to \( G_0 \) that satisfy the conditions
\[
\partial_+ \eta_- = 0, \quad \partial_- \eta_+ = 0.
\] (2.19)

Again, if a mapping \( \gamma \) satisfies the Toda equations (2.15), then the mapping \( \gamma' \) satisfies the Toda equation (2.15) where the mappings \( c_- \), \( c_+ \) are replaced by the mappings \( c'_- \) and \( c'_+ \). If the mappings \( \eta_- \) and \( \eta_+ \) are such that
\[
\eta_-^{-1} c_- \eta_- = c_- \quad \text{and} \quad \eta_+^{-1} c_+ \eta_+ = c_+.
\] (2.20)

then the transformation (2.17) is a symmetry transformation for the Toda equation under consideration.

Thus, if \( G \) is a finite dimensional complex simple Lie group, then the Toda equation associated with a loop group \( \mathcal{L}_{a,M}(G) \) and defined with the help of an integrable \( \mathbb{Z} \)-gradation of \( \mathcal{L}_{A,M}(g) \) with finite dimensional grading subspaces is equivalent to an equation of the form (2.15). To describe all nonequivalent Toda equations of this type one has to classify finite order automorphisms of the Lie algebra \( g \) or, equivalently, its \( \mathbb{Z}_M \)-gradations up to conjugations.\(^7\) This problem was solved quite a long time ago, see, for example, [28, 29]. However, it appeared that the classification described in [28, 29] is not convenient for classification of Toda equations. Restricting to the case of loop Lie algebras of complex classical Lie algebras one can use another classification based on a convenient block matrix representation of the grading subspaces [30, 21]. Let us describe the main points of the resulting classification of Toda equations.

Each element \( x \) of the complex classical Lie algebra \( g \) under consideration is considered as a \( p \times p \) block matrix \( (x_{\alpha \beta}) \), where \( x_{\alpha \beta} \) is an \( n_{\alpha} \times n_{\beta} \) matrix. Certainly, the sum of the positive integers \( n_{\alpha} \) is the size \( n \) of the matrices representing the elements of \( g \).

For the case of Toda systems associated with the loop groups \( \mathcal{L}_{a,M}(\text{GL}_n(\mathbb{C})) \), where \( a \) is an inner automorphism of \( \text{GL}_n(\mathbb{C}) \), the integers \( n_{\alpha} \) are arbitrary. For the other cases they should satisfy some restrictions dictated by the structure of the Lie algebra \( g \).

The mapping \( \gamma \) has the block diagonal form
\[
\gamma = \begin{pmatrix}
\Gamma_1 \\
\Gamma_2 \\
\vdots \\
\Gamma_p
\end{pmatrix}.
\]

For each \( \alpha = 1, \ldots, p \) the mapping \( \Gamma_\alpha \) is a mapping of \( \mathbb{R}^2 \) to the Lie group \( \text{GL}_{n_{\alpha}}(\mathbb{C}) \). For the case of Toda systems associated with the loop groups \( \mathcal{L}_{a,M}(\text{GL}_n(\mathbb{C})) \), where \( a \) is an inner automorphism of \( \text{GL}_n(\mathbb{C}) \), the mappings \( \Gamma_\alpha \) are arbitrary. For the other cases they satisfy some additional restrictions.

The mapping \( c_+ \) has the following block matrix structure:
\[
c_+ = \begin{pmatrix}
0 & C_{+1} & & \\
0 & 0 & \ddots & \\
& \ddots & \ddots & \ddots \\
C_{+0} & 0 & \cdots & C_{+(p-1)}
\end{pmatrix},
\]

\(^7\)Strictly speaking, we have to consider only conjugations by automorphisms of \( g \) which can be lifted to automorphisms of \( G \).
where for each \( \alpha = 1, \ldots, p - 1 \) the mapping \( C_{+\alpha} \) is a mapping of \( \mathbb{R}^2 \) to the space of \( n_{\alpha} \times n_{\alpha+1} \) complex matrices, and \( C_{-0} \) is a mapping of \( \mathbb{R}^2 \) to the space of \( n_p \times n_1 \) complex matrices. The mapping \( c_- \) has a similar block matrix structure:

\[
c_- = \begin{pmatrix}
0 & C_{-0} \\
C_{-1} & 0 \\
& \ddots & \ddots \\
& & C_{-(p-1)} & 0
\end{pmatrix},
\]

where for each \( \alpha = 1, \ldots, p - 1 \) the mapping \( C_{-\alpha} \) is a mapping of \( \mathcal{M} \) to the space of \( n_{\alpha+1} \times n_\alpha \) complex matrices, and \( C_{-0} \) is a mapping of \( \mathcal{M} \) to the space of \( n_1 \times n_p \) complex matrices. The conditions (2.16) imply

\[
\partial_+ C_{-\alpha} = 0, \quad \partial_- C_{+\alpha} = 0, \quad \alpha = 0, 1, \ldots, p - 1.
\]

For the case of Toda systems associated with the loop groups \( \mathcal{L}_{a,M}(\text{GL}_n(\mathbb{C})) \), where \( a \) is an inner automorphism of \( \text{GL}_n(\mathbb{C}) \), the mappings \( C_{\pm\alpha} \) are arbitrary. For the other cases they should satisfy some additional restrictions.

It is not difficult to show that the Toda equation (2.15) is equivalent to the following system of equations for the mappings \( \Gamma_{\alpha} \):

\[
\begin{align*}
\partial_+ (\Gamma_1^{-1} \partial_- \Gamma_1) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + C_{-0} \Gamma_p^{-1} C_{+0} \Gamma_1, \\
\partial_+ (\Gamma_2^{-1} \partial_- \Gamma_2) &= -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2, \\
&\vdots \\
\partial_+ (\Gamma_{p-1}^{-1} \partial_- \Gamma_{p-1}) &= -\Gamma_{p-1}^{-1} C_{+(p-1)} \Gamma_p C_{-(p-1)} + C_{-(p-2)} \Gamma_{p-2}^{-1} C_{+(p-2)} \Gamma_{p-1}, \\
\partial_+ (\Gamma_p^{-1} \partial_- \Gamma_p) &= -\Gamma_p^{-1} C_{+0} \Gamma_1 C_{-0} + C_{-(p-1)} \Gamma_{p-1}^{-1} C_{+(p-1)} \Gamma_p.
\end{align*}
\]

(2.21)

It appears that in the case under consideration without any loss of generality one can assume that the positive integer \( L \), entering the construction of Toda equations, is equal to 1. Note also that if any of the mappings \( C_{+\alpha} \) or \( C_{-\alpha} \) is a zero mapping, then the equations (2.21) are actually equivalent to a Toda equation associated with a finite dimensional group or to a set of two such equations.

### 2.3 Abelian Toda equations associated with loop groups of complex general linear Lie groups

There are three types of abelian Toda equations associated with \( \mathcal{L}_{a,M}(\text{GL}_n(\mathbb{C})) \).

#### 2.3.1 First type

The abelian Toda equations of the first type arise when the automorphism \( A \) is defined by the equality

\[
A(x) = h x h^{-1}, \quad x \in \mathfrak{gl}(\mathbb{C}),
\]
where $h$ is a diagonal matrix with the diagonal matrix elements
\[ h_{kk} = e^{n-k+1}, \quad k = 1, \ldots, n. \] (2.22)

The corresponding automorphism $a$ of $\text{GL}_n(\mathbb{C})$ is defined by the equality
\[ a(g) = hgh^{-1}, \quad g \in \text{GL}_n(\mathbb{C}), \] (2.23)

where again $h$ is a diagonal matrix determined by the relation (2.22). Here the integer $M$ is equal to $n$, and $A$ is an inner automorphism which generates a $\mathbb{Z}_n^n$-gradation of $\text{gl}_n(\mathbb{C})$. The block matrix structure related to this gradation is the matrix structure itself. In other words, all blocks are of size one by one. The mappings $\Gamma_\alpha$ are just complex functions on $\mathbb{R}^2$. The Toda equations under consideration have the form (2.21) with $p = n$.

Let us describe the action of the transformations (2.17) and (2.18) on the equations (2.21) in the case under consideration. The mappings $\eta_-$ and $\eta_+$ have a diagonal form and we denote
\[ (\eta_-)_{\alpha\alpha} = H_{-\alpha}, \quad (\eta_+)_{\alpha\alpha} = H_{+\alpha}, \quad \alpha = 1, \ldots, n. \]

The functions $H_{-\alpha}$ and $H_{+\alpha}$ satisfy the relations
\[ \partial_+ H_{-\alpha} = 0, \quad \partial_- H_{+\alpha} = 0, \]

which follow from the relations (2.19). In terms of the functions $C_{-\alpha}$ and $C_{+\alpha}$ the transformations (2.17) and (2.18) look as
\[ \Gamma'_\alpha = H_{+(\alpha+1)}^{-1} \Gamma_\alpha H_{-\alpha}, \quad C'_{-\alpha} = H_{-(\alpha+1)}^{-1} C_{-\alpha} H_{-\alpha}, \quad C'_{+\alpha} = H_{+(\alpha+1)}^{-1} C_{+\alpha} H_{+(\alpha+1)}. \] (2.24) (2.25)

Assume that the functions $C_{-\alpha}$ and $C_{+\alpha}$ have no zeros. Let us show that in this case the functions $H_{-\alpha}$ and $H_{+\alpha}$ can be chosen in such a way that $C'_{-\alpha} = C_-$ and $C'_{+\alpha} = C_+$ for some functions $C_-$ and $C_+$ which have no zeros and are subject to the conditions
\[ \partial_+ C_- = 0, \quad \partial_- C_+ = 0. \] (2.26)

Indeed, let $C_-$ and $C_+$ be some functions which satisfy the equalities
\[ C_-^n = \prod_{\alpha=1}^{n} C_{-\alpha}, \quad C_+^n = \prod_{\alpha=1}^{n} C_{+\alpha}. \]

One can verify that the transformations (2.25) with
\[ H_{-\alpha} = \prod_{\beta=\alpha}^{n} \frac{C_-}{C_-^\beta}, \quad H_{+\alpha} = \prod_{\beta=\alpha}^{n} \frac{C_+^\beta}{C_+}, \]

give the desired result, $C'_{-\alpha} = C_-$ and $C'_{+\alpha} = C_+$. The methods to find soliton solutions described below work for arbitrary functions $C_-$ and $C_+$. However, to simplify
formulas we will only consider the case when \( C_- = m \), and \( C_+ = m \), where \( m \) is a nonzero constant. In other words, we will assume that
\[
c_- = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_+ = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.
\] (2.27)

The equations under consideration take in this case the form
\[
\partial_+ \left( \Gamma_1^{-1} \partial_- \Gamma_1 \right) = -m^2 (\Gamma_1^{-1} \Gamma_2 - \Gamma_p^{-1} \Gamma_1),
\]
\[
\partial_+ \left( \Gamma_2^{-1} \partial_- \Gamma_2 \right) = -m^2 (\Gamma_2^{-1} \Gamma_3 - \Gamma_1^{-1} \Gamma_2),
\]
\[
\vdots
\]
\[
\partial_+ \left( \Gamma_{n-1}^{-1} \partial_- \Gamma_{n-1} \right) = -m^2 (\Gamma_{n-1}^{-1} \Gamma_n - \Gamma_{n-2}^{-1} \Gamma_{n-1}),
\]
\[
\partial_+ \left( \Gamma_n^{-1} \partial_- \Gamma_n \right) = -m^2 (\Gamma_n^{-1} \Gamma_1 - \Gamma_{n-1}^{-1} \Gamma_n).
\] (2.28)

It is worth to note that when the functions \( C_- \) and \( C_+ \) are real one can come to the Toda equations with \( C_- = m \) and \( C_+ = m \) by an appropriate change of the coordinates \( z^- \) and \( z^+ \).

The symmetry transformations (2.20) for the system under consideration are described by the relations (2.24) where \( H_{-\alpha} = H_- \) and \( H_{+\alpha} = H_+ \) for some functions \( H_- \) and \( H_+ \) satisfying the conditions
\[
\partial_+ H_- = 0, \quad \partial_- H_+ = 0.
\]
In particular, multiplication of all \( \Gamma_\alpha \) by the same constant is a symmetry transformation.

Defining
\[
\Gamma = \Gamma_1 \Gamma_2 \ldots \Gamma_n,
\]
one can easily see that
\[
\partial_+ (\Gamma^{-1} \partial_- \Gamma) = \sum_{\alpha=1}^{n} \partial_+ (\Gamma_\alpha^{-1} \partial_- \Gamma_\alpha).
\]

Equations (2.28) give
\[
\partial_+ (\Gamma^{-1} \partial_- \Gamma) = 0,
\]
therefore,
\[
\Gamma = \Gamma_+ \Gamma_-^{-1},
\]
for some functions \( \Gamma_- \) and \( \Gamma_+ \) which satisfy the relations
\[
\partial_+ \Gamma_- = 0, \quad \partial_- \Gamma_+ = 0.
\]
Thus, if we perform the symmetry transformation (2.24) with \( H_{-\alpha} \) and \( H_{+\alpha} \) given by
\[
H_{-\alpha} = \Gamma_-^{1/n}, \quad H_{+\alpha} = \Gamma_+^{1/n},
\]

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we will obtain functions $\Gamma'_i$ which satisfy the Toda equations (2.28) and obey the equality
\[
\Gamma' = \Gamma'_1\Gamma'_2\ldots\Gamma'_n = 1.
\]
Actually this means that via appropriate symmetry transformations we can reduce solutions of the abelian Toda equations associated with the loop group of $GL_n(\mathbb{C})$ under consideration to solutions of the corresponding Toda equations associated with the loop group of $SL_n(\mathbb{C})$.

Suppose now that we have a solution of the equations (2.28) with $\Gamma = 1$. The mappings $\Gamma_\alpha$ for $\alpha = 1,\ldots,n-1$ are independent. Introduce a new set of $n-1$ independent mappings $\Phi_\alpha, \alpha = 1,\ldots,n-1$, defined as
\[
\Phi_\alpha = \prod_{\beta=1}^{\alpha} \Gamma_\beta.
\]
It is easy to show that the inverse transition to the mappings $\Gamma_\alpha$ is described by the equalities
\[
\Gamma_1 = \Phi_1, \quad \Gamma_2 = \Phi_1^{-1}\Phi_2, \quad \ldots \quad \Gamma_{n-1} = \Phi_{n-2}^{-1}\Phi_{n-1}, \quad \Gamma_n = \Phi_{n-1}^{-1},
\]
and that the mappings $\Phi_\alpha$ satisfy the equations
\[
\partial_+(\Phi_1^{-1}\partial_-\Phi_1) = -m^2(\Phi_1^{-2}\Phi_2 - \Phi_{n-1}\Phi_1),
\partial_+(\Phi_2^{-1}\partial_-\Phi_2) = -m^2(\Phi_1\Phi_2^{-2}\Phi_3 - \Phi_{n-1}\Phi_1),
\vdots
\partial_+(\Phi_{n-2}^{-1}\partial_-\Phi_{n-2}) = -m^2(\Phi_{n-3}\Phi_{n-2}^{-2}\Phi_{n-1} - \Phi_{n-1}\Phi_1),
\partial_+(\Phi_{n-1}^{-1}\partial_-\Phi_{n-1}) = -m^2(\Phi_{n-2}\Phi_{n-1}^{-2} - \Phi_{n-1}\Phi_1).
\]
This system can be written in a more symmetric form. To this end one introduces an additional mapping $\Delta_0$, which satisfies the equation
\[
\partial_+(\Delta_0^{-1}\partial_-\Delta_0) = -m^2\Phi_{n-1}\Phi_1,
\]
and denotes
\[
\Delta_\alpha = \Delta_0\Phi_\alpha, \quad \alpha = 1,\ldots,n-1.
\]
It is easy to see that the mappings $\Delta_\alpha, \alpha = 0,1,\ldots,n-1$, satisfy the equations
\[
\partial_+(\Delta_0^{-1}\partial_-\Delta_0) = -m^2\Delta_{n-1}\Delta_0^{-2}\Delta_1,
\partial_+(\Delta_1^{-1}\partial_-\Delta_1) = -m^2\Delta_0\Delta_1^{-2}\Delta_2,
\vdots
\partial_+(\Delta_{n-2}^{-1}\partial_-\Delta_{n-2}) = -m^2\Delta_{n-3}\Delta_{n-2}^{-2}\Delta_{n-1},
\partial_+(\Delta_{n-1}^{-1}\partial_-\Delta_{n-1}) = -m^2\Delta_{n-2}\Delta_{n-1}^{-2}\Delta_0,
\]
which can be written as
\[
\partial_+(\Delta_\alpha^{-1}\partial_-\Delta_\alpha) = -m^2\prod_{\beta=0}^{\alpha-1} \Delta_\beta^{-\alpha_\beta},
\]
where \( a_{\alpha \beta} \) are the matrix elements of the Cartan matrix of an affine Lie algebra of type \( A^{(1)}_{n-1} \):

\[
(a_{\alpha \beta}) = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
-1 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}.
\]

(2.31)

The equations (2.30) are of the standard form of the Toda equations associated with the \( A^{(1)}_{n-1} \) affine Lie group.

### 2.3.2 Second type

The abelian Toda equations of the second and third types arise when we use outer automorphisms of \( \mathfrak{gl}_n(\mathbb{C}) \). For the equations of the second type \( n \) is odd, and for the equations of the third type \( n \) is even.

Consider first the case of an odd \( n = 2s - 1 \), \( s \geq 2 \). In this case an abelian Toda equation arises when the automorphism \( A \) is defined by the equality

\[
A(x) = -h(B^{-1}x B)h^{-1},
\]

(2.32)

where \( ^tx \) means the transpose of \( x \), \( h \) is a diagonal matrix with the diagonal matrix elements

\[
h_{kk} = e^{n-k+1}_{2s-2} = e^{2s-k}_{4s-2},
\]

and \( B \) is an \( n \times n \) matrix of the form

\[
B = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}.
\]

The order of the automorphism \( A \) is \( 2n = 4s - 2 \) and the integer \( p \) is \( 2s - 1 \). The mapping \( \gamma \) is a diagonal matrix, and the mappings \( \Gamma_\alpha \) are mappings of \( \mathbb{R}^2 \) to \( \mathbb{C}^\times \) subject to the constraints

\[
\Gamma_1 = 1, \quad \Gamma_{2s-\alpha+1} = \Gamma_\alpha^{-1}, \quad \alpha = 2, \ldots, s.
\]

The mappings \( C_{\pm \alpha} \) are complex functions satisfying the equality

\[
C_{\pm 0} = C_{\pm 1},
\]

(2.33)

and for \( s > 2 \) the equalities

\[
C_{\pm (2s-\alpha)} = -C_{\pm \alpha}, \quad \alpha = 2, \ldots, s - 1.
\]

(2.34)

Let us choose the mappings \( \Gamma_\alpha, \alpha = 2, \ldots, s \), as a complete set of mappings parameterizing the mapping \( \gamma \). Taking into account the equalities (2.33) and (2.34) we come
to the following set of independent equations equivalent to the Toda equation under consideration

\[
\begin{align*}
\partial_+(\Gamma_2^{-1}\partial_-\Gamma_2) &= -C_{+2}C_{-2}\Gamma_2^{-1}\Gamma_3 + C_{+1}C_{-1}\Gamma_2,
\partial_+(\Gamma_3^{-1}\partial_-\Gamma_3) &= -C_{+3}C_{-3}\Gamma_3^{-1}\Gamma_4 + C_{+2}C_{-2}\Gamma_2^{-1}\Gamma_3,
\vdots \\
\partial_+(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}) &= -C_{+s-1}C_{-s-1}\Gamma_{s-1}^{-1}\Gamma_s + C_{+s-2}C_{-s-2}\Gamma_{s-2}^{-1}\Gamma_{s-1},
\partial_+(\Gamma_s^{-1}\partial_-\Gamma_s) &= -C_{+s}C_{-s}\Gamma_s^{-1}\Gamma_{s+1} + C_{+s-1}C_{-s-1}\Gamma_{s-1}^{-1}\Gamma_s.
\end{align*}
\]  

(2.35)

As above, in the case when the functions \(C_{-\alpha}\) and \(C_{+\alpha}\) have no zeros, using the transformation (2.17) and (2.18), one can show that the equations (2.35) are equivalent to the same equations, where \(C_{-\alpha} = C_-\) and \(C_{+\alpha} = C_+\) for some functions \(C_-\) and \(C_+\) which have no zeros and are subject to the conditions (2.26). If these functions are real, then with the help of an appropriate change of the coordinates \(z_-\) and \(z_+\) we can come to the equations

\[
\begin{align*}
\partial_+(\Gamma_2^{-1}\partial_-\Gamma_2) &= -m^2(\Gamma_2^{-1}\Gamma_3 - \Gamma_2), \\
\partial_+(\Gamma_3^{-1}\partial_-\Gamma_3) &= -m^2(\Gamma_3^{-1}\Gamma_4 - \Gamma_2^{-1}\Gamma_3), \\
\vdots \\
\partial_+(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}) &= -m^2(\Gamma_{s-1}^{-1}\Gamma_s - \Gamma_{s-2}^{-1}\Gamma_{s-1}), \\
\partial_+(\Gamma_s^{-1}\partial_-\Gamma_s) &= -m^2(\Gamma_s^{-2} - \Gamma_{s-1}^{-1}\Gamma_s),
\end{align*}
\]

where \(m\) is a nonzero constant, see also the papers [3, 31].

For \(s = 2\) denoting \(\Gamma_2\) by \(\Gamma\) we have the equation

\[
\partial_+(\Gamma^{-1}\partial_-\Gamma) = -m^2(\Gamma^{-2} - \Gamma).
\]

Putting \(\Gamma = \exp(F)\) we obtain

\[
\partial_+\partial_-F = -m^2[\exp(-2F) - \exp(F)].
\]

This is the Tzitzéica equation [32] which is now usually called the Dodd–Bullough–Mikhailov equation [33, 3].

Let us show how the above equations are related to the Toda equations associated with the \(A_{2s-2}^{(2)}\) affine Lie group. Assume that \(s > 2\). Introduce an additional mapping \(\Delta_0\) which satisfies the equation

\[
\partial_+(\Delta_0^{-1}\partial_-\Delta_0) = -\frac{m^2}{2}\Gamma_2
\]  

(2.36)

and denote

\[
\Delta_\alpha = 2^{-\alpha}\Delta_0^{\frac{\alpha+1}{2}}\prod_{\beta=2}^{s} \Gamma_\beta, \quad \alpha = 1, \ldots, s - 2, \quad \Delta_{s-1} = 2^{-s+2}\Delta_0^s\prod_{\beta=2}^{s} \Gamma_\beta.
\]

Now one can get convinced that the mappings \(\Delta_\alpha, \alpha = 0, 1, \ldots, s - 1\), satisfy the equations of the form (2.30) where \(n = s\) and \(a_{\alpha\beta}\) are the matrix elements of the Cartan
matrix of an affine Lie algebra of type \( A_{2s-2}^{(2)} \):

\[
(a_{\alpha \beta}) = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -2 & 2
\end{pmatrix}.
\]

In the case of \( s = 2 \) we again define the mapping \( \Delta_0 \) with the help of the relation (2.36) and denote

\[
\Delta_1 = 2^{-1/3} \Delta_0^2 \Gamma_2.
\]

After an appropriate rescaling of the coordinates \( z^- \) and \( z^+ \) we come to the equations (2.30) where \( n = 2 \) and \( a_{\alpha \beta} \) are the matrix elements of the Cartan matrix of an affine Lie algebra of type \( A_{2}^{(2)} \):

\[
(a_{\alpha \beta}) = \begin{pmatrix}
2 & -1 \\
-4 & 2
\end{pmatrix}.
\]

### 2.3.3 Third type

In the case of an even \( n = 2s \), \( s \geq 2 \), to come to an abelian Toda system one should use again the automorphism \( A \) defined by the relation (2.32) where now

\[
B = \begin{pmatrix}
1 & 1 \\
1 & \ddots & 1 \\
\vdots & \ddots & \ddots & 1 \\
-1 & \cdots & -1 & 1
\end{pmatrix}
\]

and \( h \) is a diagonal matrix with the diagonal matrix elements

\[
h_{11} = e_{2n-2}^{n-1} = e_{4s-2}^{2s-1} = -1, \quad h_{ii} = e_{2n-2}^{n-i+1} = e_{4s-2}^{2s-i+1}, \quad i = 2, \ldots, n.
\]

The number \( p \) characterizing the block structure is equal to \( n - 1 = 2s - 1 \), \( n_1 = 2 \), and \( n_\alpha = 1 \) for \( \alpha = 2, \ldots, 2s - 1 \).

The mapping \( \Gamma_1 \) is a mapping of \( \mathbb{R}^2 \) to the Lie group \( \text{SO}_2(\mathbb{C}) \) which is isomorphic to \( \mathbb{C}^\times \). Actually \( \Gamma_1 \) is a \( 2 \times 2 \) complex matrix valued function satisfying the relation

\[
J_2^{-1} \Gamma_1 J_2 = \Gamma_1^{-1},
\]

where

\[
J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

It is easy to show that \( \Gamma_1 \) has the form

\[
\Gamma_1 = \begin{pmatrix}
(G_1)_{11} & 0 \\
0 & (G_1)_{11}^{-1}
\end{pmatrix},
\]

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where \((\Gamma_1)_{11}\) is a mapping of \(\mathbb{R}^2\) to \(\mathbb{C}^\times\). The mappings \(\Gamma_\alpha, \alpha = 2, \ldots, 2s - 1\), are mappings of \(\mathbb{R}^2\) to \(\mathbb{C}^\times\) satisfying the relations

\[
\Gamma_{2s-\alpha+1} = \Gamma_\alpha^{-1}.
\]

The mappings \(C_{-1}, C_{+0}\) are complex \(1 \times 2\) matrix valued functions, the mappings \(C_{-0}, C_{+1}\) are complex \(2 \times 1\) matrix valued functions. Here one has

\[
C_{-0} = J_2^{-1}C_{-1}, \quad C_{+0} = tC_{+1}J_2. \tag{2.37}
\]

The mappings \(C_{\pm \alpha}, \alpha = 2, \ldots, p - 1 = 2s - 2\), are just complex functions, satisfying for \(s > 2\) the equalities

\[
C_{\pm (2s-\alpha)} = -C_{\pm \alpha}, \quad \alpha = 2, \ldots, s - 1. \tag{2.38}
\]

The mappings \((\Gamma_1)_{11}\) and \(\Gamma_\alpha, \alpha = 2, \ldots, s\), form a complete set of mappings parameterizing the mapping \(\gamma\). Taking into account the equalities (2.37) and (2.38) we come to the following set of independent equations equivalent to the Toda equation under consideration:

\[
\partial_+((\Gamma_1)_{11}^{-1})_1 = -((C_{+1})_{11}(C_{-1})_{11}(\Gamma_1)_{11}^{-1})_1 + ((C_{+1})_{12}(C_{-1})_{12}\Gamma_2(\Gamma_1)_{11},
\partial_+(\Gamma_2^{-1} \partial_+ \Gamma_2) = -C_{+2}C_{-2}(\Gamma_2^{-1})_3
\]
\[
+ ((C_{+1})_{11}(C_{-1})_{11}(\Gamma_1)_{11}^{-1})_1 + ((C_{+1})_{12}(C_{-1})_{12}\Gamma_2(\Gamma_1)_{11},
\partial_+(\Gamma_3^{-1} \partial_+ \Gamma_3) = -C_{+3}C_{-3}(\Gamma_3^{-1})_4 + C_{+2}C_{-2}(\Gamma_2^{-1})_3,
\]
\[
\vdots
\]
\[
\partial_+(\Gamma_{s-1}^{-1} \partial_+ \Gamma_{s-1}) = -C_{+(s-1)}C_{-(s-1)}(\Gamma_{s-1}^{-1})_1 + C_{+1}C_{-1}(\Gamma_{s-2}^{-1})_2,
\partial_+(\Gamma_{s}^{-1} \partial_+ \Gamma_{s}) = -C_{+s}C_{-s}(\Gamma_{s}^{-1})_2 + C_{+1}C_{-1}(\Gamma_{s-1}^{-1})_1.
\]

As well as for the first two types, under appropriate conditions these equations can be reduced to the equations with \(C_{-\alpha} = m, C_{+\alpha} = m\) for \(\alpha = 2, \ldots, s\), \((C_{-1})_{11} = (C_{-1})_{12} = m/\sqrt{2}\) and \((C_{+1})_{11} = (C_{+1})_{21} = m/\sqrt{2}\), where \(m\) is a nonzero constant.\(^8\)

Thus, we come to the equations

\[
\partial_+(\Gamma_1^{-1})_1 = -\frac{m^2}{2}(\Gamma_1^{-1} - \Gamma_1)_2,
\]
\[
\partial_+(\Gamma_2^{-1})_2 = -m^2(\Gamma_2^{-1})_3 + \frac{m^2}{2}(\Gamma_1^{-1} + \Gamma_1)_2,
\partial_+(\Gamma_3^{-1})_3 = -m^2(\Gamma_3^{-1})_2 + \frac{m^2}{2}(\Gamma_2^{-1})_3,
\]
\[
\vdots
\]
\[
\partial_+(\Gamma_{s-1}^{-1} \partial_+ \Gamma_{s-1}) = -m^2(\Gamma_{s-1}^{-1})_1 + \frac{m^2}{2}(\Gamma_{s-2}^{-1})_2,
\partial_+(\Gamma_{s}^{-1} \partial_+ \Gamma_{s}) = -m^2(\Gamma_{s}^{-2})_1 + \frac{m^2}{2}(\Gamma_{s-1}^{-1})_2,
\]

where slightly abusing notation we denote \((\Gamma_1)_{11}\) by \(\Gamma_1\).

\(^8\)This choice is convenient for applications of the rational dressing method.
Introduce now an additional mapping $\Delta_0$ which satisfies the equation
\[
\partial_+ (\Delta_0^{-1} \partial_- \Delta_0) = -\frac{m^2}{2} \Gamma_1 \Gamma_2
\]
and denote
\[
\Delta_1 = \Delta_0 \Gamma_1, \quad \Delta_\alpha = 2^{\alpha(\alpha-1)/(2s-1)-\alpha+1} \Delta_0^2 \prod_{\beta=1}^\alpha \Gamma_\beta, \quad \alpha = 2, \ldots, s.
\]
The mappings $\Delta_\alpha$, $\alpha = 0, 1, \ldots, s$, satisfy the equations which, after an appropriate rescaling of the coordinates $z^-$ and $z^+$, take the form (2.30), where now $n = s + 1$ and $a_{\alpha\beta}$ are the matrix elements of the Cartan matrix of an affine Lie algebra of type $A_{2s-1}^{(2)}$:
\[
(a_{\alpha\beta}) = \begin{pmatrix}
2 & 0 & -1 & \cdots & 0 & 0 & 0 \\
0 & 2 & -1 & \cdots & 0 & 0 & 0 \\
-1 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -2 & 2
\end{pmatrix}.
\]

3 Soliton solutions

In this section we compare two methods used to construct soliton solutions of the abelian Toda systems associated with the loop groups of the complex general linear groups. We restrict ourselves to the abelian Toda equations of the first type which have the form (2.28).

3.1 Hirota’s method

It is convenient to treat the system (2.28) as an infinite system
\[
\partial_+ (\Gamma_\alpha^{-1} \partial_- \Gamma_\alpha) = -m^2 (\Gamma_\alpha^{-1} \Gamma_{\alpha+1} - \Gamma_{\alpha-1}^{-1} \Gamma_\alpha),
\]
where the functions $\Gamma_\alpha$ are defined for arbitrary integer values of the index $\alpha$ in the periodic way,
\[
\Gamma_{\alpha+n} = \Gamma_\alpha.
\]

following the Hirota’s approach [6, 7, 8, 9, 10], one introduces $\tau$-functions connected with $\Gamma_\alpha$ by the relation
\[
\Gamma_\alpha = \tau_\alpha / \tau_{\alpha-1},
\]
where we assume that the $\tau$-functions are defined for all integer values of the index $\alpha$. This change of variables is the essence of the Hirota’s method. The periodicity condition (3.2) in terms of the $\tau$-functions takes the form
\[
\tau_{\alpha+n} / \tau_\alpha = \tau_{\alpha-1+n} / \tau_{\alpha-1}.
\]
It means that the ratio $\tau_{\alpha+n} / \tau_\alpha$ does not depend on $\alpha$. Noting that
\[
\Gamma = \Gamma_1 \Gamma_2 \ldots \Gamma_n = \tau_n / \tau_0,
\]
we write
\[ \tau_{\alpha+n} = \Gamma \tau_{\alpha} . \]

As was explained in the previous section, with the appropriate symmetry transformation one can make \( \Gamma = 1 \). We will assume that the corresponding symmetry transformation was performed, and, therefore,
\[ \tau_{\alpha+n} = \tau_{\alpha} . \quad (3.4) \]

The equations (3.1) in terms of the \( \tau \)-functions look as
\[ \partial_+ (\tau^{-1}_a \partial_- \tau_a) - \partial_+ (\tau^{-1}_{a-1} \partial_- \tau_{a-1}) = -m^2 (\tau_{a-1} \tau_a^2 \tau_{a+1} - \tau_{a-2} \tau_a^2 \tau_a) . \]

Consider the following decoupling of the above equations
\[ \partial_+ (\tau^{-1}_a \partial_- \tau_a) = m^2 (1 - \tau_{a-1} \tau^{-2}_a \tau_{a+1}) . \quad (3.5) \]

It is evident that if the \( \tau \)-functions satisfy these equations, then the functions \( \Gamma_a \) defined by (3.3) satisfy the system (2.28). Moreover, it is easy to show that in this case the functions
\[ \Delta_a = \exp(-m^2 z^+ z^-) \tau_a \]
satisfy the system (2.29).

It is convenient to rewrite the equations (3.5) in the form
\[ \tau_a \partial_+ \partial_- \tau_a - \partial_+ \tau_a \partial_- \tau_a = m^2 (\tau_a^2 - \tau_{a-1} \tau_{a+1}) . \quad (3.6) \]

These equations are of the Hirota bilinear type. Their solutions, leading to multi-soliton solutions of the system (2.28), can be found perturbatively in the following way.

Consider a series expansion of the functions \( \tau_a \) in some parameter \( \varepsilon \) which will be set to one at the final step of the construction. So we represent the functions \( \tau_a \) in the form
\[ \tau_a = \tau^{(0)}_a + \varepsilon \tau^{(1)}_a + \varepsilon^2 \tau^{(2)}_a + \ldots , \quad (3.7) \]
and assume that \( \tau_a^{(0)} \) are constants. The periodicity condition (3.4) gives
\[ \tau^{(k)}_{\alpha+n} = \tau^{(k)}_a , \quad k = 0, 1, \ldots . \]

Let us try to solve equations (3.6) order by order in \( \varepsilon \). Actually our goal is to find solutions for which the series (3.7) truncates at some finite order in \( \varepsilon \). In such a case we have an exact solution.

Using the expansion (3.7), one obtains
\[ \tau_a \partial_+ \partial_- \tau_a = \sum_{k=0}^{\infty} \varepsilon^k \sum_{\ell=0}^{k} \tau^{(k-\ell)}_a \partial_+ \partial_- \tau^{(\ell)}_a . \]

Similarly one has
\[ \partial_+ \tau_a \partial_- \tau_a = \sum_{k=0}^{\infty} \varepsilon^k \sum_{\ell=0}^{k} \partial_+ \tau^{(k-\ell)}_a \partial_- \tau^{(\ell)}_a . \]
Now, using the equality
\[ \tau_\alpha^2 - \tau_\alpha \tau_{\alpha+1} = \sum_{k=0}^{\infty} \varepsilon_k \sum_{\ell=0}^{k} \left( \tau_\alpha^{(\ell)} \tau_{\alpha+1}^{(k-\ell)} - \tau_{\alpha-1}^{(\ell)} \tau_\alpha^{(k-\ell)} \right), \]
we see that the equations (3.6) are equivalent to the equations
\[ \sum_{\ell=0}^{k} \left( \tau_\alpha^{(k-\ell)} \partial_+ \tau_\alpha^{(\ell)} - \partial_+ \tau_\alpha^{(k-\ell)} \partial_+ \tau_\alpha^{(\ell)} \right) = m^2 \sum_{\ell=0}^{k} \left( \tau_\alpha^{(\ell)} \tau_\alpha^{(k-\ell)} - \tau_{\alpha-1}^{(\ell)} \tau_{\alpha+1}^{(k-\ell)} \right), \] (3.8)
which can be solved step by step starting from \( k = 0 \).

For \( k = 0 \) one has
\[ \tau_{\alpha+1}^{(0)} \tau_\alpha^{(0)} - \tau_\alpha^{(0)} \tau_{\alpha-1}^{(0)} = 0, \]
that can be rewritten as
\[ \tau_\alpha^{(0)} / \tau_{\alpha-1}^{(0)} = \tau_\alpha^{(0)} / \tau_{\alpha-1}^{(0)}. \]

It is clear that the general solution to this relation is
\[ \tau_\alpha^{(0)} = \tau_0^{(0)} d^\alpha, \] (3.9)
where \( d \) is an arbitrary constant. Recall that the Toda equations (3.1) are invariant with respect to the multiplication of all \( \Gamma_\alpha \) by the same constant. From the point of view of the \( \tau \)-functions this is equivalent to the multiplication of the function \( \tau_\alpha \) by the \( \alpha \)th power of the constant. Hence, different values of the constant \( d \) in the relation (3.9) correspond to the functions \( \Gamma_\alpha \) connected by a rescaling. Moreover, dividing all \( \tau \)-functions by the same constant we do not change the functions \( \Gamma_\alpha \). Therefore, actually without any loss of generality, one can put
\[ \tau_\alpha^{(0)} = 1. \] (3.10)

Using this equality, we rewrite (3.8) as
\[ \partial_+ \partial_- \tau_\alpha^{(k)} - m^2 \sum_{\beta=0}^{n-1} a_{\alpha\beta} \tau_\beta^{(k)} = - \sum_{\ell=1}^{k-1} \left( \tau_\alpha^{(k-\ell)} \partial_+ \partial_- \tau_\alpha^{(\ell)} - \partial_+ \tau_\alpha^{(k-\ell)} \partial_- \tau_\alpha^{(\ell)} \right) + m^2 \sum_{\ell=1}^{k-1} \left( \tau_\alpha^{(\ell)} \tau_\alpha^{(k-\ell)} - \tau_{\alpha-1}^{(\ell)} \tau_{\alpha+1}^{(k-\ell)} \right), \] (3.11)
where \( a_{\alpha\beta} \) are the matrix elements of the Cartan matrix (2.31) of an affine Lie algebra of type \( A^{(1)}_{n-1} \). Thus, we see that at each step we should solve a system of linear differential equations.

In particular, for \( k = 1 \) one has to solve the system of equations
\[ \partial_+ \partial_- \tau_\alpha^{(1)} - m^2 \sum_{\beta=0}^{n-1} a_{\alpha\beta} \tau_\beta^{(1)} = 0. \] (3.12)

It is easy to find solutions of these equations using the eigenvectors \( \theta_\rho \) of the Cartan matrix \( (a_{\alpha\beta}) \) which are given by
\[ (\theta_\rho)_\alpha = \varepsilon_n^{(\alpha+1)\rho}, \quad \rho = 0, 1, \ldots, n - 1. \] (3.13)
Here the corresponding eigenvalues are
\[ \kappa_\rho = 2 - \epsilon_n^0 - \epsilon_n^{-\rho} = 4 \sin^2(\pi \rho / n). \]

Let us assume that the functions \( \tau^{(1)}_\alpha \) are of the form
\[
\tau^{(1)}_\alpha = \sum_{i=1}^r E_{\alpha i},
\]
where
\[
E_{\alpha i} = \epsilon_n^{(a+1)\rho_i} \exp\left[m \kappa_\rho \left(\zeta_i^{-1}z^- + \zeta_i z^+\right) + \delta_i\right].
\]
Here \( \rho_i \) is an integer from the interval from 1 to \( n - 1 \), \( \zeta_i \) and \( \delta_i \) are arbitrary complex numbers. Note that the choice \( \rho_i = 0 \) is excluded because it gives a constant contribution to the \( \tau \)-functions which can be included into \( \tau^{(0)}_\alpha \). Then after a corresponding rescaling one can satisfy the normalization (3.10). For definiteness we assume that
\[ \kappa_\rho = -i(\epsilon_n^{\rho/2} - \epsilon_n^{-\rho/2}) = 2 \sin(\pi \rho / n). \]

Certainly, the ansatz (3.14) does not give a general solution to the equations (3.12) but it ensures truncation of the expansion (3.7).

For \( k = 2 \) the equations (3.11) have the form
\[
\partial_+ \partial_- \tau^{(2)}_\alpha - m^2 \sum_{\beta=0}^{n-1} a_\alpha \delta \tau^{(2)}_\beta \nonumber = -\tau^{(1)}_\alpha \partial_+ \partial_- \tau^{(1)}_\alpha + \partial_+ \tau^{(1)}_\alpha \partial_- \tau^{(1)}_\alpha + m^2 (\tau^{(1)}_\alpha \tau^{(1)}_\alpha - \tau^{(1)}_{\alpha-1} \tau^{(1)}_{\alpha+1}).
\]
Using the equalities
\[
\partial_- E_{\alpha i} = m \kappa_\rho \zeta_i^{-1} E_{\alpha i}, \quad \partial_+ E_{\alpha i} = m \kappa_\rho \zeta_i E_{\alpha i},
\]
one obtains
\[
-\tau^{(1)}_\alpha \partial_+ \partial_- \tau^{(1)}_\alpha + \partial_+ \tau^{(1)}_\alpha \partial_- \tau^{(1)}_\alpha + m^2 (\tau^{(1)}_\alpha \tau^{(1)}_\alpha - \tau^{(1)}_{\alpha-1} \tau^{(1)}_{\alpha+1})
\]
\[
= \frac{m^2}{2} \sum_{i_1, i_2=1}^r \left[ \kappa_\rho \kappa_\rho \left(\zeta_{i_1}^{-1}z^- + \zeta_{i_2}^{-1}z^+\right) - \kappa_{\rho_1}^2 - \kappa_{\rho_2}^2 + \kappa_{\rho_{i_1} \rho_{i_2}}^2 \right] E_{\alpha i_1} E_{\alpha i_2}. \tag{3.17}
\]

Note that in the case of \( r = 1 \) we have
\[
-\tau^{(1)}_\alpha \partial_+ \partial_- \tau^{(1)}_\alpha + \partial_+ \tau^{(1)}_\alpha \partial_- \tau^{(1)}_\alpha + m^2 (\tau^{(1)}_\alpha \tau^{(1)}_\alpha - \tau^{(1)}_{\alpha-1} \tau^{(1)}_{\alpha+1}) = 0
\]
and we can put \( \tau^{(2)}_\alpha = 0 \). This gives the one-soliton solutions
\[
\Gamma_\alpha = \frac{1 + \epsilon_n^{\rho(a+1)} \exp\left[m \kappa_\rho (\zeta^{-1}z^- + \zeta z^+) + \delta\right]}{1 + \epsilon_n^{\rho_\alpha} \exp\left[m \kappa_\rho (\zeta^{-1}z^- + \zeta z^+) + \delta\right]}. \tag{3.18}
\]

In the case when \( r > 1 \) the equality (3.17) suggests to look for \( \tau^{(2)}_\alpha \) of the form
\[
\tau^{(2)}_\alpha = \frac{1}{2} \sum_{i_1, i_2=1}^r \eta_{i_1 i_2} E_{\alpha i_1} E_{\alpha i_2}.
\]
Here one can easily find that

\[ \partial_+ \partial_- \tau_\alpha^{(2)} - m^2 \sum_{\beta=0}^{n-1} a_{\alpha \beta} \tau_\beta^{(2)} = \frac{m^2}{2} \sum_{i_1, i_2=1}^{\ell} \left[ \kappa_{\rho_{i_1}} \kappa_{\rho_{i_2}} (\zeta_{i_1} \zeta_{i_2}^{-1} + \zeta_{i_1}^{-1} \zeta_{i_2}) + \kappa_{\rho_{i_1}}^2 + \kappa_{\rho_{i_2}}^2 - \kappa_{\rho_{i_1}} \kappa_{\rho_{i_2}} - \kappa_{\rho_{i_1}}^2 + \kappa_{\rho_{i_2}}^2 \right] \eta_{i_1 i_2} E_{ai_1} E_{ai_2} \]

and, therefore,

\[ \eta_{i_1 i_2} = \frac{\kappa_{\rho_{i_1}} \kappa_{\rho_{i_2}} (\zeta_{i_1} \zeta_{i_2}^{-1} + \zeta_{i_1}^{-1} \zeta_{i_2}) - \kappa_{\rho_{i_1}}^2 - \kappa_{\rho_{i_2}}^2 + \kappa_{\rho_{i_1}} \kappa_{\rho_{i_2}} - \kappa_{\rho_{i_1}}^2 + \kappa_{\rho_{i_2}}^2}{\kappa_{\rho_{i_1}} \kappa_{\rho_{i_2}} (\zeta_{i_1} \zeta_{i_2}^{-1} + \zeta_{i_1}^{-1} \zeta_{i_2}) + \kappa_{\rho_{i_1}}^2 + \kappa_{\rho_{i_2}}^2 - \kappa_{\rho_{i_1}} \kappa_{\rho_{i_2}} - \kappa_{\rho_{i_1}}^2 + \kappa_{\rho_{i_2}}^2} \]

that can be written as

\[ \eta_{i_1 i_2} = \frac{(\zeta_{i_1} \zeta_{i_2}^{-1} + \zeta_{i_1}^{-1} \zeta_{i_2}) - 2 \cos[\pi(\rho_{i_1} - \rho_{i_2})/n]}{(\zeta_{i_1} \zeta_{i_2}^{-1} + \zeta_{i_1}^{-1} \zeta_{i_2}) + 2 \cos[\pi(\rho_{i_1} + \rho_{i_2})/n]} \]  (3.19)

The quantities \( \eta_{i_1 i_2} \) are symmetric with respect to the indices \( i_1, i_2 \) and they turn to zero when \( i_1 = i_2 \). Hence one can write

\[ \tau_\alpha^{(2)} = \sum_{1 \leq i_1 < i_2 \leq r} \eta_{i_1 i_2} E_{ai_1} E_{ai_2} \]

It can be shown that when \( r = 2 \) one can choose \( \tau_\alpha^{(3)} = 0 \). In general, it can be shown that for \( \ell \leq r \) one can choose

\[ \tau_\alpha^{(\ell)} = \sum_{1 \leq i_1 < i_2 < \ldots < i_\ell \leq r} \left( \prod_{1 \leq j < k \leq \ell} \eta_{i_j i_k} \right) E_{ai_1} E_{ai_2} \ldots E_{ai_\ell} \]

and \( \tau_\alpha^{(\ell)} = 0 \) for \( \ell > r \). In other words, the equations (3.6) have the following solutions

\[ \tau_\alpha = 1 + \sum_{i=1}^{r} E_{ai} + \sum_{\ell=2}^{r} \left[ \sum_{1 \leq i_1 < i_2 < \ldots < i_\ell \leq r} \left( \prod_{1 \leq j < k \leq \ell} \eta_{i_j i_k} \right) E_{ai_1} E_{ai_2} \ldots E_{ai_\ell} \right] \]  (3.20)

3.2 Rational dressing

Since for any \( \tilde{m} \in \mathbb{R}^2 \) the matrices \( c_- (\tilde{m}) \) and \( c_+ (\tilde{m}) \) commute,\(^9\) it is obvious that

\[ \gamma = I_n, \]  (3.21)

where \( I_n \) is the \( n \times n \) unit matrix, is a solution to the Toda equation (2.15). Denote a mapping of \( \mathbb{R}^2 \times S^1 \) to \( \text{GL}_n(\mathbb{C}) \) which generates the corresponding connection by \( \varphi \). Using the equalities (2.11) and (2.14) and remembering that in our case \( L = 1 \), we write

\[ \varphi^{-1} \partial_- \varphi = \lambda^{-1} c_-, \quad \varphi^{-1} \partial_+ \varphi = \lambda c_+ , \]

\(^9\)Actually in the case under consideration \( c_- \) and \( c_+ \) are constant mappings.
where the matrices $c_-$ and $c_+$ are defined by the relation (2.27).

To construct more interesting solutions to the Toda equations we will look for a mapping $\psi$, such that the mapping

\[ \varphi' = \varphi \psi \]  

(3.22)

would generate a connection satisfying the grading condition and the gauge-fixing

\[ \omega_{+,0} = 0. \]

For any $\bar{m} \in \mathbb{R}^2$ the mapping $\tilde{\psi}_m$ defined by the equality $\tilde{\psi}_m(\bar{p}) = \psi(\bar{m}, \bar{p})$, $\bar{p} \in S^1$, is a smooth mapping of $S^1$ to $\text{GL}_n(\mathbb{C})$. Recall that we treat $S^1$ as a subset of the complex plane which, in turn, will be treated as a subset of the Riemann sphere. Assume that it is possible to extend analytically each mapping $\tilde{\psi}_m$ to all of the Riemann sphere. As the result we get a mapping of the direct product of $\mathbb{R}^2$ and the Riemann sphere to $\text{GL}_n(\mathbb{C})$ which we also denote by $\psi$. Suppose that for any $\bar{m} \in \mathbb{R}^2$ the analytic extension of $\tilde{\psi}_m$ results in a rational mapping regular at the points 0 and $\infty$, hence the name rational dressing.

Below, for each point $\bar{p}$ of the Riemann sphere we denote by $\psi_p$ the mapping

\[ \psi_p(\bar{m}) = \psi_p(\bar{m}, \bar{p}), \]

which we also denote by $\psi_p$. Suppose that for any $\bar{m} \in \mathbb{R}^2$ and $\bar{p} \in S^1$ we should have

\[ \psi_p(\bar{m}, \epsilon_n \bar{p}) = h \psi_p(\bar{m}, \bar{p}) h^{-1}, \]  

(3.23)

where $h$ is a diagonal matrix described by the relation (2.22). The equality (3.23) means that two rational mappings coincide on $S^1$, therefore, they must coincide on the entire Riemann sphere.

A mapping, satisfying the equality (3.23), can be constructed by the following procedure. Let $\chi$ be an arbitrary mapping of the direct product of $\mathbb{R}^2$ and the Riemann sphere to $\text{GL}_n(\mathbb{C})$. Let $\hat{a}$ be a linear operator acting on $\chi$ as

\[ \hat{a} \chi(\bar{m}, \bar{p}) = h \chi(\bar{m}, \epsilon_n^{-1} \bar{p}) h^{-1}. \]

It is easy to get convinced that the mapping

\[ \psi = \sum_{k=1}^{n} \hat{a}^k \chi \]

satisfies the relation $\hat{a} \psi = \psi$ which is equivalent to the equality (3.23). Note that $\hat{a}^n \chi = \chi$.

To construct a rational mapping satisfying (3.23) we start with a rational mapping regular at the points 0 and $\infty$ and having poles at $r$ different nonzero points $\mu_i$, $i = 1, \ldots, r$. Concretely speaking, we consider a mapping $\chi$ of the form

\[ \chi = \left( I_n + n \sum_{i=1}^{r} \frac{\lambda}{\lambda - \mu_i} P_i \right) \chi_0, \]

where $P_i$ are some smooth mappings of $\mathbb{R}^2$ to the algebra Mat$_n(\mathbb{C})$ of $n \times n$ complex matrices and $\chi_0$ is a mapping of $\mathbb{R}^2$ to the Lie subgroup of $\text{GL}_n(\mathbb{C})$ formed by the elements $g \in \text{GL}_n(\mathbb{C})$ satisfying the equality

\[ hgh^{-1} = g. \]

(3.24)
Actually this subgroup coincides with the subgroup $G_0$. The averaging procedure leads to the mapping

$$\psi = \left( I_n + \sum_{i=1}^{r} \sum_{k=1}^{n} \frac{\lambda}{\lambda - \epsilon_n^k \mu_i} h^k P_i h^{-k} \right) \psi_0,$$

where $\psi_0 = n\chi_0$. It is convenient to assume that $\mu_i^n \neq \mu_j^n$ for all $i \neq j$.

Denote by $\psi^{-1}$ the mapping of $\mathbb{R}^2 \times S^1$ to $\text{GL}_n(\mathbb{C})$ defined by the relation

$$\psi^{-1}(\bar{m}, \bar{p}) = (\psi(\bar{m}, \bar{p}))^{-1}.$$

Suppose that for any fixed $\bar{m} \in \mathbb{R}^2$ the mapping $\tilde{\psi}_m^{-1}$ of $S^1$ to $\text{GL}_n(\mathbb{C})$ can be extended analytically to a mapping of the Riemann sphere to $\text{GL}_n(\mathbb{C})$ and as the result we obtain a rational mapping of the same structure as the mapping $\psi$,

$$\psi^{-1} = \psi_0^{-1} \left( I_n + \sum_{i=1}^{r} \sum_{k=1}^{n} \frac{\lambda}{\lambda - \epsilon_n^k \mu_i} h^k Q_i h^{-k} \right),$$

with the pole positions satisfying the conditions $\nu_i \neq 0$, $\nu_i^n \neq \nu_j^n$ for all $i \neq j$, and additionally $\nu_i^n \neq \nu_j^n$ for any $i$ and $j$. We will denote the mapping of the direct product of $\mathbb{R}^2$ and the Riemann sphere to $\text{GL}_n(\mathbb{C})$ again by $\psi^{-1}$.

By definition, the equality

$$\psi^{-1} \psi = I_n$$

is valid at all points of the direct product of $\mathbb{R}^2$ and $S^1$. Since $\psi^{-1} \psi$ is a rational mapping, the above equality is valid at all points of the direct product of $\mathbb{R}^2$ and the Riemann sphere. Hence, the residues of $\psi^{-1} \psi$ at the points $\nu_i$ and $\mu_i$ should be equal to zero. Explicitly we have

$$Q_i \left( I_n + \sum_{j=1}^{r} \sum_{k=1}^{n} \frac{\nu_j}{\nu_j - \epsilon_n^k \mu_j} h^k P_j h^{-k} \right) = 0,$$

$$\left( I_n + \sum_{j=1}^{r} \sum_{k=1}^{n} \frac{\mu_i}{\mu_i - \epsilon_n^k \nu_j} h^k Q_j h^{-k} \right) P_i = 0.$$

In this case due to the relation (3.23) the residues of $\psi^{-1} \psi$ at the points $\epsilon_n^k \mu_i$ and $\epsilon_n^k \nu_i$ vanish for $k = 1, \ldots, n$. We will discuss later how to satisfy these relations, and now let us consider what connection is generated by the mapping $\varphi'$ defined by (3.22) with the mapping $\psi$ possessing the prescribed properties.

Using the representation (3.22), we obtain for the components of the connection generated by $\varphi'$ the expressions

$$\omega_- = \psi^{-1} \partial_- \psi + \lambda^{-1} \psi^{-1} c_- \psi,$$

$$\omega_+ = \psi^{-1} \partial_+ \psi + \lambda \psi^{-1} c_+ \psi.$$

We see that the component $\omega_-$ is a rational mapping which has simple poles at the points $\mu_i, \nu_i$ and zero.\footnote{Here and below discussing the holomorphic properties of mappings and functions we assume that the point of the space $\mathbb{R}^2$ is arbitrary but fixed.} Similarly, the component $\omega_+$ is a rational mapping which has
simple poles at the points \( \mu_i, v_i \) and infinity. We are looking for a connection which satisfies the grading and gauge-fixing conditions. The grading condition in our case is the requirement that for each point of \( \mathbb{R}^2 \) the component \( \omega_- \) is rational and has the only simple pole at zero, while the component \( \omega_+ \) is rational and has the only simple pole at infinity. Hence, we demand that the residues of \( \omega_- \) and \( \omega_+ \) at the points \( \mu_i \) and \( v_i \) should vanish. In this case, as above, due to the relation (3.23) the residues of \( \omega_- \) and \( \omega_+ \) at the points \( \epsilon_n^k \mu_i \) and \( \epsilon_n^k v_i \) vanish for \( k = 1, \ldots, n \).

The residues of \( \omega_- \) and \( \omega_+ \) at the points \( v_i \) are equal to zero if and only if

\[
(\partial_- Q_i - v_i^{-1} Q_i c_-) \left( I_n + \sum_{j=1}^r \sum_{k=1}^n \frac{v_i}{v_i - \epsilon_n^k \mu_j} h^k \bar{P}_j h^{-k} \right) = 0, \tag{3.31}
\]

\[
(\partial_+ Q_i - v_i Q_i c_+) \left( I_n + \sum_{j=1}^r \sum_{k=1}^n \frac{v_i}{v_i - \epsilon_n^k \mu_j} h^k \bar{P}_j h^{-k} \right) = 0, \tag{3.32}
\]

respectively. Similarly, the requirement of vanishing of the residues at the points \( \mu_i \) gives the relations

\[
\left( I_n + \sum_{j=1}^r \sum_{k=1}^n \frac{\mu_i}{\mu_i - \epsilon_n^k v_j} h^k Q_j h^{-k} \right) (\partial_- P_i + \mu_i^{-1} c_- P_i) = 0, \tag{3.33}
\]

\[
\left( I_n + \sum_{j=1}^r \sum_{k=1}^n \frac{\mu_i}{\mu_i - \epsilon_n^k v_j} h^k Q_j h^{-k} \right) (\partial_+ P_i + \mu_i c_+ P_i) = 0. \tag{3.34}
\]

To obtain the relations (3.31)–(3.34) we made use of the equalities (3.27), (3.28).

Suppose that we have succeeded in satisfying the relations (3.27), (3.28) and (3.31)–(3.34). In such a case from the equalities (3.29) and (3.30) it follows that the connection under consideration satisfies the grading condition.

It is easy to see from (3.30) that

\[
\omega_+ (\bar{m}, 0) = \psi_0^{-1}(\bar{m}) \partial_+ \psi_0(\bar{m}).
\]

Taking into account that \( \omega_+(\bar{m}) = \omega_+(\bar{m}, 0) \), we conclude that the gauge-fixing constraint \( \omega_+(0) = 0 \) is equivalent to the relation

\[
\partial_+ \psi_0 = 0. \tag{3.35}
\]

Assuming that this relation is satisfied, we come to a connection satisfying both the grading condition and the gauge-fixing condition.

Recall that if a flat connection \( \omega \) satisfies the grading and gauge-fixing conditions, then there exist a mapping \( \gamma \) from \( \mathbb{R}^2 \) to \( G \) and mappings \( c_- \) and \( c_+ \) of \( \mathbb{R}^2 \) to \( g_{-1} \) and \( g_{+1} \), respectively, such that the representation (2.14) for the components \( \omega_- \) and \( \omega_+ \) is valid. In general, the mappings \( c_- \) and \( c_+ \) parameterizing the connection components may be different from the mappings \( c_- \) and \( c_+ \) which determine the mapping \( \varphi \). Let us denote the mappings corresponding to the connection under consideration by \( \gamma' \), \( c'_- \) and \( c'_+ \). Thus, we have

\[
\psi^{-1} \partial_- \psi + \lambda^{-1} \psi^{-1} c_- \psi = \gamma'^{-1} \partial_- \gamma' + \lambda^{-1} c'_-, \tag{3.36}
\]

\[
\psi^{-1} \partial_+ \psi + \lambda \psi^{-1} c_+ \psi = \lambda \gamma'^{-1} c'_+ \gamma'. \tag{3.37}
\]
Note that $\psi_\infty$ is a mapping of $\mathbb{R}^2$ to the Lie subgroup of $GL_n(\mathbb{C})$ defined by the relation (3.24). Recall that this subgroup coincides with $G_0$, and denote $\psi_\infty$ by $\gamma$. From the relation (3.36) we obtain the equality

$$\gamma'^{-1}\partial_-\gamma' = \gamma^{-1}\partial_-\gamma.$$

The same relation (3.36) gives

$$\psi_0^{-1}c_-c_0 = c'.$$

Impose the condition $\psi_0 = I_n$, which is consistent with (3.35). Here we have

$$c' = c_-. $$

Finally, from (3.37) we obtain

$$\gamma'^{-1}c_+\gamma' = \gamma^{-1}c_+. $$

We see that if we impose the condition $\psi_0 = I_n$, then the components of the connection under consideration have the form given by (2.14) where $\gamma = \psi_\infty$.

Thus, to find solutions to Toda equations under consideration, we can use the following procedure. Fix $2r$ complex numbers $\mu_i$ and $\nu_i$. Find matrix-valued functions $P_i$ and $Q_i$ satisfying the relations (3.27), (3.28) and (3.31)–(3.34). With the help of (3.25), (3.26), assuming that

$$\psi_0 = I_n,$$

construct the mappings $\psi$ and $\psi^{-1}$. Then, the mapping

$$\gamma = \psi_\infty$$

satisfies the Toda equation (2.15).

Let us return to the relations (3.27), (3.28). It can be shown that, if we suppose that the matrices $P_i$ and $Q_i$ are of maximum rank, then we get the trivial solution of the Toda equation given by (3.21). Hence, we will assume that $P_i$ and $Q_i$ are not of maximum rank. The simplest case here is given by matrices of rank one which can be represented as

$$P_i = u_i^t w_i, \quad Q_i = x_i^t y_i,$$

where $u, w, x$ and $y$ are $n$-dimensional column vectors. This representation allows one to write the relations (3.27) and (3.28) as

$$t y_i + \sum_{j=1}^{r} \sum_{k=1}^{n} \frac{v_i}{v_i - \epsilon_n^k \mu_j} (t y_i h^k u_j)^j w_j h^{-k} = 0,$$  

$$u_i + \sum_{j=1}^{r} \sum_{k=1}^{n} \frac{\mu_i}{\mu_i - \epsilon_n^k v_j} h^k x_j (t y_j h^{-k} u_i) = 0.$$  

Using the identity

$$\sum_{k=0}^{n-1} \frac{z \epsilon_n^{-jk}}{z - \epsilon_n^k} = n \frac{z^n - |j|_n}{z^n - 1},$$

where $|j|_n$ is the residue of division of $j$ by $n$, we can rewrite (3.39) in components terms,

$$y_{ik} + n \sum_{j=1}^{r} (R_k)_{ij} w_{jk} = 0.$$
Here the \( r \times r \) matrices \( R_k \) are defined as
\[
(R_k)_{ij} = \frac{1}{\nu_i^n - \mu_j^n} \sum_{\ell=1}^n \nu_i^{n-|\ell-k|} \mu_j^{n-|\ell-k|} u_{i\ell} y_{j\ell}.
\]
The same identity (3.41) allows one to write component form of (3.40) as
\[
u ik + n \sum_{j=1}^r x_{jk} (S_k)_{ji} = 0, \tag{3.43}
\]
where
\[
(S_k)_{ji} = -\frac{1}{\nu_j^n - \mu_i^n} \sum_{\ell=1}^n \nu_j^{k-\ell} \nu_i^{n-k-\ell} y_{j\ell} u_{i\ell}.
\]
With the help of the equality
\[
n - 1 - |i - 1|_n = |i|_n
\]
it is straightforward to demonstrate that
\[
(S_k)_{ji} = -\frac{\mu_i}{\nu_j} (R_{k+1})_{ji}.
\]
Consequently, we can write the equation (3.43) as
\[
u ik - n \mu_i \sum_{j=1}^r x_{jk} \frac{1}{\nu_j} (R_{k+1})_{ji} = 0. \tag{3.44}
\]
We use the equations (3.42) and (3.44) to express the vectors \( w_i \) and \( x_i \) via the vectors \( u_i \) and \( y_i \),
\[
w_{ik} = -\frac{1}{n} \sum_{j=1}^r (R_{k-1}^{-1})_{ij} y_{jk}, \quad x_{ik} = \frac{1}{n} \sum_{j=1}^r u_{jk} \frac{1}{\nu_j} (R_{k+1}^{-1})_{ji} u_i.
\]
As the result, we come to the following solution of the relations (3.27) and (3.28):
\[
(P_i)_{\ell\ell} = -\frac{1}{n} u_{ik} \sum_{j=1}^r (R_{\ell}^{-1})_{ij} y_{jk}, \quad (Q_i)_{\ell\ell} = \frac{1}{n} \sum_{j=1}^r u_{jk} \frac{1}{\nu_j} (R_{k+1}^{-1})_{ji} v_i y_{i\ell}.
\]
Further, it follows from (3.39) and (3.40) that, to fulfill also (3.31)–(3.34), it is sufficient to satisfy the equations
\[
\partial_- y_i = v_i^{-1} t_- c_- y_i, \quad \partial_+ y_i = v_i t_+ c_+ y_i, \tag{3.45}
\]
\[
\partial_- u_i = -\mu_i^{-1} c_- u_i, \quad \partial_+ u_i = -\mu_i c_+ u_i. \tag{3.46}
\]
The \( n \)-dimensional column vectors \( \theta_{\rho} \), defined by the relation (3.13), are eigenvectors of the matrices \( t_- c_- \), \( t_+ c_+ \), \( c_- \) and \( c_+ \),
\[
t_- c_- \theta_{\rho} = m e_n^\rho \theta_{\rho}, \quad t_+ c_+ \theta_{\rho} = m e_n^{-\rho} \theta_{\rho}, \quad c_- \theta_{\rho} = m e_n^\rho \theta_{\rho}, \quad c_+ \theta_{\rho} = m e_n^{-\rho} \theta_{\rho},
\]
and form a basis in the space \( \mathbb{C}^n \). Hence, the general solution of the equations (3.45) and (3.46) can be written in the form
\[
u ik = \sum_{\rho=1}^n c_{i\rho} e_n^{k\rho} e^{-Z_\rho(\mu_i)}, \quad y_{ik} = \sum_{\rho=1}^n d_{i\rho} e_n^{k\rho} e^{Z_\rho(\nu_i)},
\]

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where \(c_{ip}, d_{ip}\) are arbitrary constants and

\[
Z_\rho(\mu) = m(\epsilon^{-\rho}_n \mu^{-1}z^{-} + \epsilon^\rho_n \mu z^{+}).
\]

Thus, we see that it is possible to satisfy (3.27), (3.28) and (3.31)–(3.34). This gives us solutions of the Toda equations (2.28). Let us show that they can be written in a simple determinant form.

Using (3.38) and (3.25), one gets

\[
\gamma = \psi_\infty = I_n + \sum_{i=1}^{r} \sum_{k=1}^{n} h^k P_i h^{-k}.
\]

For the matrix elements of \(\gamma\) this gives the expression

\[
\gamma_{k\ell} = \delta_{k\ell} \left( 1 + n \sum_{i=1}^{r} (P_i)_{kk} \right) = \delta_{k\ell} \left( 1 - \sum_{i,j=1}^{r} u_{ij} (R^{-1}_\alpha)_{ij} y_{\alpha j} \right).
\]

Hence, we have

\[
\Gamma_\alpha = 1 - \sum_{i,j=1}^{r} u_{i\alpha} (R^{-1}_\alpha)_{ij} y_{\alpha j}.
\]

To this expression can also be given the form

\[
\Gamma_\alpha = 1 - t_{\alpha} u_{\alpha} R^{-1}_\alpha y_{\alpha},
\]

where \(u_{\alpha}\) and \(y_{\alpha}\) are \(r\)-dimensional column vectors with the components \(u_{i\alpha}\) and \(y_{i\alpha}\), respectively.

We assume for convenience that the functions \(u_{i\alpha}\) and \(y_{i\alpha}\) are defined for arbitrary integral values of \(\alpha\) and

\[
u_{i,\alpha+n} = u_{i\alpha}, \quad y_{i,\alpha+n} = y_{i\alpha}.
\]

The periodicity of \(R_\alpha\) in the index \(\alpha\) follows from the definition. It appears that it is more appropriate to use quasi-periodic quantities \(\tilde{u}_{\alpha}, \tilde{y}_{\alpha}\) and \(\tilde{R}_{\alpha}\) defined as

\[
\tilde{u}_{\alpha} = M^\alpha u_{\alpha}, \quad \tilde{y}_{\alpha} = N^{-\alpha} y_{\alpha},
\]

\[
\tilde{R}_{\alpha} = N^{-\alpha} R_\alpha M^\alpha,
\]

where \(N\) and \(M\) are diagonal \(r \times r\) matrices given by

\[
N_{ij} = \nu_i \delta_{ij}, \quad M_{ij} = \mu_i \delta_{ij}.
\]

For these quantities one has quasi-periodicity conditions

\[
\tilde{u}_{\alpha+n} = M^n \tilde{u}_{\alpha}, \quad \tilde{y}_{\alpha+n} = N^{-n} \tilde{y}_{\alpha},
\]

\[
\tilde{R}_{\alpha+n} = N^{-n} \tilde{R}_{\alpha} M^n.
\]

The expression of the matrix elements of the matrices \(\tilde{R}_{\alpha}\) through the functions \(\tilde{y}_{i\alpha}\) and \(\tilde{u}_{i\alpha}\) has a remarkably simple form [3]

\[
(\tilde{R}_{\alpha})_{ij} = \frac{1}{\nu_i^n - \mu_j^n} \left( \mu_j^n \sum_{\beta=1}^{\alpha-1} \tilde{y}_{i\beta} \tilde{u}_{j\beta} + \nu_i^n \sum_{\beta=\alpha}^{n} \tilde{y}_{i\beta} \tilde{u}_{j\beta} \right).
\]

(3.47)
In terms of the quasi-periodic quantities, for the functions \( \Gamma_\alpha \) we have
\[
\Gamma_\alpha = 1 - \tilde{u}_\alpha \tilde{R}_\alpha^{-1} \tilde{y}_\alpha,
\]
and it can be shown that
\[
\Gamma_\alpha = \frac{\det(\tilde{R}_\alpha - \tilde{y}_\alpha \tilde{u}_\alpha)}{\det \tilde{R}_\alpha}.
\]
Using the explicit form of \( \tilde{R}_\alpha \), one comes to the equality
\[
\tilde{R}_{\alpha+1} = \tilde{R}_\alpha - \tilde{y}_\alpha \tilde{u}_\alpha.
\]
Therefore, one can write [3]
\[
\Gamma_\alpha = \frac{\det \tilde{R}_{\alpha+1}}{\det \tilde{R}_\alpha}.
\] (3.48)

### 3.3 Solitons through the rational dressing

To obtain a one-soliton solution one puts \( r = 1 \). In this case \( \tilde{R}_\alpha \) are ordinary functions for which one has the expression
\[
\tilde{R}_\alpha = \frac{1}{v^n - \mu^n} \sum_{\rho,\sigma = 1}^{n} c_{\rho} d_{\sigma} e^{-Z_{\rho}(\mu) + Z_{-\sigma}(v)} \left[ \mu^n \sum_{\beta = 1}^{\alpha-1} \mu^{\beta} v - \beta \epsilon_n^{(\rho+\sigma)\beta} + v^n \sum_{\beta = \alpha}^{n} \mu^{\beta} v - \beta \epsilon_n^{(\rho+\sigma)\beta} \right].
\]
It is not difficult to verify that
\[
\mu^n \sum_{\beta = 1}^{\alpha-1} \mu^{\beta} v - \beta \epsilon_n^{(\rho+\sigma)\beta} + v^n \sum_{\beta = \alpha}^{n} \mu^{\beta} v - \beta \epsilon_n^{(\rho+\sigma)\beta} = (v^n - \mu^n) \mu^{\alpha} v^{-\alpha} \epsilon_n^{(\rho+\sigma)\alpha} \frac{\epsilon_n^{(\rho+\sigma)\alpha}}{1 - \mu v^{-1} \epsilon_n^{\rho+\sigma}}.
\]
Thus one obtains the following expression for \( \tilde{R}_\alpha \):
\[
\tilde{R}_\alpha = \mu^{\alpha} v^{-\alpha} \sum_{\rho,\sigma = 1}^{n} c_{\rho} d_{\sigma} e^{-Z_{\rho}(\mu) + Z_{-\sigma}(v)} \frac{\epsilon_n^{(\rho+\sigma)\alpha}}{1 - \mu v^{-1} \epsilon_n^{\rho+\sigma}}.
\]

To obtain a solution which depends on only one combination of \( z^- \) and \( z^+ \) we suppose that \( c_{\rho} \) is different from zero for only one value of \( \rho \) which we denote by \( I \), and that \( d_{\sigma} \) is different from zero for only two values of \( \sigma \) which we denote by \( J \) and \( K \). In this case we arrive at a simplified version of \( \tilde{R}_\alpha \), that is
\[
\tilde{R}_\alpha = \mu^{\alpha} v^{-\alpha} c_I e^{-Z_I(\mu)} \left[ d_J e^{Z_J(v)} \frac{\epsilon_n^{(I+J)\alpha}}{1 - \mu v^{-1} \epsilon_n^{I+J}} + d_K e^{Z_K(v)} \frac{\epsilon_n^{(I+K)\alpha}}{1 - \mu v^{-1} \epsilon_n^{I+K}} \right],
\]
and the corresponding solution can be written as
\[
\Gamma_\alpha = \mu v^{-1} \epsilon_n^{I+J} \frac{1 + d_n^{(K-J)(\alpha+1)} e^{Z_K(v) - Z_J(v)} e^{Z_J(v) - Z_J(v)}}{1 + d_n^{(K-J)\alpha} e^{Z_K(v) - Z_J(v)}},
\]
where
\[
d = \frac{d_K (1 - \mu v^{-1} \epsilon_n^{I+J})}{d_J (1 - \mu v^{-1} \epsilon_n^{I+K})}.
\]
Making use of the freedom in multiplying a solution by a constant, we can write the obtained solution as (3.18), where \( \rho = K - J \), \( \kappa_\rho \) is defined by (3.16), \( \zeta = -ie_n^{-\nu/(K+J)}/v \), and \( \delta \) is a constant introduced by the relation \( \exp \delta = d \). Thus we arrive at the one-soliton solution obtained before by the Hirota’s method.

In the case of \( r > 1 \) (multi-soliton construction) we suppose that for any \( i \) the coefficients \( c_{i\rho} \) are different from zero for only one value of \( \rho \) which we denote by \( I_i \), and that the coefficients \( d_{i\sigma} \) are different from zero for only two values of \( \sigma \) which we denote by \( J_i \) and \( K_i \). This leads to the following expression for the matrix elements of the matrices \( \tilde{R}_\alpha \):

\[
(\tilde{R}_\alpha)_{ij} = v_i^{-\alpha} e_n^{I_i} d_{ij} e^{Z-J_i(v_i)} \\
	imes \left[ \frac{1}{1 - \mu_j v_i^{-1} e_n^{I_i}} + \left( \frac{\mu_j v_i^{-1} e_n^{I_i}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \right] \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right).
\]

Immediately we see from (3.48) that the solution in question has the form

\[
\Gamma_\alpha = \frac{\prod_{i=1}^r \mu_i v_i^{-1} e_n^{I_i} + 1}{\det \tilde{R}_\alpha^{r+1}} \left( \frac{\det \tilde{R}_\alpha^r}{\det \tilde{R}_\alpha} \right),
\]

where the matrices \( \tilde{R}_\alpha^r \) are defined by

\[
(\tilde{R}_\alpha^r)_{ij} = \frac{1}{1 - \mu_j v_i^{-1} e_n^{I_i}} + \left( \frac{\mu_j v_i^{-1} e_n^{I_i}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right).
\]

Using the matrices \( D \) defined in appendix A.1, we rewrite the expression for \( \tilde{R}_\alpha^r \) in the form

\[
(\tilde{R}_\alpha^r)_{ij} = D_{ij}(v e_n^{-J}, \mu e_n^J) + \left( \frac{\mu_j v_i^{-1} e_n^{I_i}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right).
\]

It is clear that instead of \( \tilde{R}_\alpha^r \) one can use in the relation (3.49) the matrices \( \tilde{R}_\alpha^{r'} \) defined as

\[
(\tilde{R}_\alpha^{r'})_{ij} = \delta_{ij} + \sum_{k=1}^r D_{ik}(v e_n^{-K}, \mu e_n^K) D_{kj}^{-1}(v e_n^{-J}, \mu e_n^J).
\]

Using the equality (A.3), one comes to the expression

\[
(\tilde{R}_\alpha^{r''})_{ij} = \frac{v_i \prod_{\ell=1}^r (v_\ell e_n^{-J} - v_{\ell'} e_n^{-J})}{\epsilon_n^J \prod_{\ell=1}^r (v_\ell e_n^{-J} - v_{\ell'} e_n^{-J})} (T_\alpha)_{ij} \frac{v_j \prod_{\ell=1}^r (v_\ell e_n^{-J} - v_{\ell'} e_n^{-J})}{\epsilon_n^J \prod_{\ell=1}^r (v_\ell e_n^{-J} - v_{\ell'} e_n^{-J})},
\]

where

\[
(T_\alpha)_{ij} = \delta_{ij} + \left( \frac{\mu_j v_i^{-1} e_n^{I_i}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right) \left( \frac{\epsilon_n^{(K_i - J_i)\alpha}}{1 - \mu_j v_i^{-1} e_n^{I_i}} \right).
\]
and, with a slight abuse of notation,
\[
d_i = d_K^J_i \prod_{\ell=1}^r (v_i e^{-K_i} - v_\ell e^{-J_\ell}) \prod_{\ell=1}^r (v_i e^{-J_i} - \mu_\ell e^{-L_\ell})
\]
\[
d_i = d_J^K_i \prod_{\ell=1}^r (v_i e^{-J_i} - v_\ell e^{-J_\ell}) \prod_{\ell=1}^r (v_i e^{-K_i} - \mu_\ell e^{-L_\ell}).
\]

Utilizing the expression (3.50) and having in mind the freedom in multiplying a solution by a constant, we write the solution under consideration as follows:

\[
\Gamma_\alpha = \frac{\det T_{\alpha+1}}{\det T_\alpha}.
\]

Defining \(\rho_i = K_i - J_i, \zeta_i = -ie^{-K_i}v_i\) and introducing constants \(\delta_i\) satisfying the relations \(\exp \delta_i = d_i\), one can write

\[
(T_\alpha)_{ij} = \delta_{ij} + \epsilon_n^{\rho_i} \exp[m\kappa_i (\zeta_i^- z^- + \zeta_i^+ z^+)] + \delta_i \frac{\epsilon_n^{-\rho_i/2} \zeta_i^- - \epsilon_n^{-\rho_i/2} \zeta_i^+}{\epsilon_n^{-\rho_i/2} \zeta_i^- - \epsilon_n^{-\rho_j/2} \zeta_j^+}. \tag{3.51}
\]

It is proved in appendix A.2 that

\[
\det T_{\alpha+1} = 1 + \sum_{i=1}^r E_{ai} + \sum_{\ell=2}^r \left[ \sum_{1\leq i_1<\ldots<i_\ell\leq r} \left( \prod_{1\leq j<k\leq \ell} \eta_{ij_k} \right) E_{ai_1} E_{ai_2} \cdots E_{ai_\ell} \right], \tag{3.52}
\]

where the functions \(E_{ai}\) and \(\eta_{ij_k}\) are defined by the relations (3.15) and (3.19) respectively. Thus, we come to the multi-soliton solutions which coincide with those obtained by the Hirota’s method. The Hirota’s \(\tau\)-functions (3.20) are given by the equality

\[
\tau_\alpha = \det T_{\alpha+1}.
\]

It is clear that the quantities \(\eta_{ij_k}\) here make the same sense as do the normal ordering coefficients effectively describing the interaction between solitons in the vertex operators approach of Olive, Turok and Underwood [14, 15]. We refer the reader to the papers [10, 34, 35] for some more specific properties of such coefficients.

### 4 Conclusion

In this paper we have considered abelian Toda systems associated with the loop groups of the complex general linear groups. We have reviewed two different approaches to construct soliton solutions to these equations in the untwisted case, namely, the Hirota’s and rational dressing methods. Subsequently, basic ingredients representing soliton solutions within the frameworks of these methods have been explicitly related. As we have seen in section 3.2, the rational dressing method allows one to construct solutions to the loop Toda equations, presenting them as the ratio of the determinants of specific matrices (3.47), (3.48), and they actually represent a class of solutions being wider than that formed by the soliton solutions of the Hirota’s method in section
3.1: By setting the initial-data coefficients arising in the rational dressing method to some specific values we have shown in section 3.3 that the Hirota’s soliton solutions are contained among the solutions constructed by the rational dressing approach.  

Note also that the reduction to the systems based on the loop groups of the complex special linear groups can easily be performed.  

Our consideration can be generalized to Toda systems based on other loop groups, such as twisted loop groups of the complex general linear groups, twisted and untwisted loop groups of the complex orthogonal and symplectic groups. However, one should take into account that the change of field variables in the Hirota’s method is more tricky there, and besides, when applying the rational dressing to obtain soliton solutions, one faces that the pole positions of the dressing meromorphic mappings and their inverse ones are to be related, just due to the group conditions. These circumstances make part of the formulae more intricate than in the general linear case considered in the present paper. We will address to this problem and present our results in some future publications.  

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Appendix

A.1 Some properties of the matrices $D$

In this appendix we investigate $r \times r$ matrices $D(f, g)$ with matrix elements given by the equality

$$D_{ij}(f, g) = \frac{1}{1 - \frac{1}{f_i} g_j} = \frac{f_i}{f_i - g_j}.$$  

Let us show that for the matrix elements of the inverse matrix $D^{-1}(f, g)$ one has the representation

$$D_{ij}^{-1}(f, g) = \frac{\prod_{\ell=1}^{r} (f_{i} - g_{\ell}) \prod_{\ell=1}^{r} (f_{j} - g_{\ell})}{f_{i} \prod_{\ell=1}^{r} (g_{\ell} - g_{i}) \prod_{\ell=1}^{r} (f_{j} - f_{\ell})}.$$  

(A.1)

To prove the above equality one has to demonstrate that

$$\sum_{k=1}^{r} \frac{f_{i} \prod_{\ell=1}^{r} (f_{\ell} - g_{k}) \prod_{\ell=1}^{r} (f_{j} - g_{\ell})}{f_{j} (f_{i} - g_{k}) \prod_{\ell=1}^{r} (g_{\ell} - g_{k}) \prod_{\ell=1}^{r} (f_{j} - f_{\ell})} = \delta_{ij}.$$  

(A.2)
Consider the set of meromorphic functions of $z$ defined as

$$F_{ij}(f, g, z) = \prod_{\ell=1, \ell \neq j}^{r} (f_{\ell} - z) \prod_{\ell=1}^{r} (g_{\ell} - z).$$

The residue of $F_{ij}(f, g, z)$ at infinity is equal to zero, therefore, the sum of the residues at the point $f_i$ and at the points $g_{\ell}, \ell = 1, \ldots, r,$ is also zero. Hence we have the following equality

$$\sum_{k=1}^{r} \frac{\prod_{\ell=1, \ell \neq j}^{r} (f_{\ell} - g_{k})}{(f_{i} - g_{k}) \prod_{\ell=1, \ell \neq k}^{r} (g_{\ell} - g_{k})} = -\frac{\prod_{\ell=1, \ell \neq j}^{r} (f_{\ell} - f_{i})}{\prod_{\ell=1}^{r} (g_{\ell} - f_{i})},$$

and, therefore,

$$\sum_{k=1}^{r} \frac{f_{i} \prod_{\ell=1, \ell \neq j}^{r} (f_{\ell} - g_{k}) \prod_{\ell=1}^{r} (f_{j} - g_{\ell})}{f_{j} (f_{i} - g_{k}) \prod_{\ell=1, \ell \neq k}^{r} (g_{\ell} - g_{k}) \prod_{\ell=1, \ell \neq j}^{r} (f_{j} - f_{\ell})} = \frac{f_{i} \prod_{\ell=1, \ell \neq j}^{r} (f_{i} - f_{\ell}) \prod_{\ell=1}^{r} (f_{j} - g_{\ell})}{f_{j} \prod_{\ell=1}^{r} (f_{i} - g_{\ell}) \prod_{\ell=1, \ell \neq j}^{r} (f_{j} - f_{\ell})}.$$

Now, taking into account the identity

$$\prod_{\ell=1, \ell \neq j}^{r} \frac{(f_{i} - f_{\ell})}{(f_{j} - f_{\ell})} = \delta_{ij},$$

we see that the relation (A.2) is true. Thus the equivalent relation (A.1) is also true.

In a similar way one can prove the validity of the equality

$$\sum_{k=1}^{r} D_{ik}(\tilde{f}, g) D_{kj}^{-1}(f, g) = \frac{\tilde{f}_{i} \prod_{\ell \neq j}^{r} (\tilde{f}_{i} - f_{\ell}) \prod_{\ell=1}^{r} (f_{j} - g_{\ell})}{f_{j} \prod_{\ell=1}^{r} (f_{i} - g_{\ell}) \prod_{\ell \neq j}^{r} (f_{j} - f_{\ell})}.$$  \hspace{1cm} (A.3)

### A.2 Proof of relation (3.52)

Proceeding from the relation (3.51), one obtains

$$(T_{\alpha+1})_{ij} = \delta_{ij} + E_{ai} \frac{\tilde{f}_{i} - f_{i}}{\tilde{f}_{i} - f_{j}}.$$
where
\[
\tilde{f}_i = e_n \rho_i/2 \zeta_i, \quad f_i = e_n \rho_i/2 \zeta_i. \tag{A.4}
\]
and the functions \(E_{ai}\) are defined by the relation (3.15). Then it is not difficult to get convinced that
\[
det T_{\alpha+1} = 1 + \sum_{i=1}^{r} E_{ai} + \sum_{\ell=2}^{r} \left[ \sum_{1 \leq i_1 < i_2 < \ldots < i_{\ell} \leq r} \eta_{i_1 i_2 \ldots i_{\ell}} E_{a_i_1} E_{a_i_2} \cdots E_{a_i_{\ell}} \right], \tag{A.5}
\]
where
\[
\eta_{i_1 \ldots i_\ell} = \sum_{\pi \in S_\ell} \text{sgn}(\pi) \left( \prod_{j=1}^{\ell} \frac{\tilde{f}_{i_j} - f_{i_j}}{f_{i_j} - f_{\pi(i_j)}} \right).
\]
As is customary, we denote by \(S_\ell\) the symmetric group on the set \{1, 2, \ldots, \ell\} and by \(\text{sgn}(\pi)\) the signature of the permutation \(\pi\).

For \(\ell = 2\) one has
\[
\eta_{i_1 i_2} = 1 - \frac{(\tilde{f}_{i_1} - f_{i_1})(\tilde{f}_{i_2} - f_{i_2})}{(\tilde{f}_{i_1} - f_{i_2})(\tilde{f}_{i_2} - f_{i_1})} = \frac{(f_{i_1} - f_{i_2})(\tilde{f}_{i_2} - f_{i_1})}{(f_{i_1} - f_{i_2})(\tilde{f}_{i_2} - f_{i_1})}.
\]
Using the definition (A.4) of \(f_i\) and \(\tilde{f}_i\), we see that the quantities \(\eta_{i_1 i_2}\) coincide with the coefficients \(\eta_{i_1 i_2}\) defined by the relation (3.19).

Let us prove by induction that
\[
\eta_{i_1 i_2 \ldots i_{\ell}} = \prod_{1 \leq j < k \leq \ell} \eta_{i_j i_k}. \tag{A.6}
\]
Certainly, for \(\ell = 2\) the equality (A.6) is valid. Suppose that it is valid up to some fixed value of \(\ell\) and show that it is valid for its value incremented by one.

The group \(S_\ell\) can be identified with a subgroup of \(S_{\ell+1}\) formed by the permutations \(\pi \in S_{\ell+1}\) satisfying the condition \(\pi(\ell + 1) = \ell + 1\). Denote by \(\pi_m, m = 1, \ldots, \ell,\) the transposition exchanging \(m\) and \(\ell + 1\) and represent the group \(S_{\ell+1}\) as the union of the right cosets \(S_\ell \pi_m\). This allows us to write
\[
\eta_{i_1 \ldots i_{\ell+1}} = \eta_{i_1 \ldots i_{\ell}} - \sum_{m=1}^{\ell} \sum_{\pi \in S_\ell} \text{sgn}(\pi) \left( \prod_{j=1}^{\ell+1} \frac{\tilde{f}_{i_j} - f_{i_j}}{f_{i_j} - f_{\pi(i_j)}} \right). \tag{A.7}
\]
It is not difficult to realize that
\[
\sum_{\pi \in S_\ell} \text{sgn}(\pi) \left( \prod_{j=1}^{\ell+1} \frac{\tilde{f}_{i_j} - f_{i_j}}{f_{i_j} - f_{\pi(i_j)}} \right) = \frac{(\tilde{f}_{i_m} - f_{i_m})(\tilde{f}_{i_{\ell+1}} - f_{i_{\ell+1}})}{(\tilde{f}_{i_m} - f_{i_{\ell+1}})(\tilde{f}_{i_{\ell+1}} - f_{i_m})} \eta_{i_1 \ldots i_{\ell}} | \tilde{f}_{i_m} = \tilde{f}_{i_{\ell+1}}.
\]
and that
\[
\eta_{i_1 \ldots i_{\ell}} | \tilde{f}_{i_m} = \tilde{f}_{i_{\ell+1}} = \prod_{j=1}^{\ell} \frac{(\tilde{f}_{i_{\ell+1}} - \tilde{f}_{i_j})(\tilde{f}_{i_m} - f_{i_j})}{(\tilde{f}_{i_{\ell+1}} - f_{i_j})(\tilde{f}_{i_m} - f_{i_j})} \eta_{i_1 \ldots i_{\ell}}.
\]
Using these equalities in (A.7), we obtain
\[ \eta_{i_1\ldots i_{\ell+1}} = \eta_{i_1\ldots i_{\ell}} \]

\[ + \frac{\prod_{j=1}^{\ell}(\tilde{f}_{i_{\ell+1}} - f_{i_j})}{\prod_{j=1}^{\ell}(\tilde{f}_{i_{\ell+1}} - f_{i_j})} \sum_{m=1}^{\ell} \frac{\prod_{j=1}^{\ell+1}(\tilde{f}_{i_m} - f_{i_j})}{\prod_{j=1}^{\ell+1}(\tilde{f}_{i_m} - f_{i_j})}. \]  
(A.8)

Now consider a meromorphic function of \( z \) defined as
\[ F(f, \tilde{f}, z) = \prod_{j=1}^{\ell+1}(z - f_{i_j}) \frac{(z - f_{i_{\ell+1}}) \prod_{j=1}^{\ell+1}(z - \tilde{f}_{i_j})}{(z - f_{i_{\ell+1}}) \prod_{j=1}^{\ell+1}(z - f_{i_j})}. \]

The equality of the sum of the residues of \( F(f, \tilde{f}, z) \) to zero gives the relation
\[ \sum_{m=1}^{\ell} \frac{\prod_{j=1}^{\ell}(\tilde{f}_{i_m} - f_{i_j})}{\prod_{j=1}^{\ell}(\tilde{f}_{i_m} - f_{i_j})} = - \frac{\prod_{j=1}^{\ell+1}(f_{i_{\ell+1}} - f_{i_j})}{\prod_{j=1}^{\ell+1}(f_{i_{\ell+1}} - f_{i_j})} - \frac{\prod_{j=1}^{\ell}(\tilde{f}_{i_{\ell+1}} - f_{i_j})}{\prod_{j=1}^{\ell}(\tilde{f}_{i_{\ell+1}} - f_{i_j})}. \]

Using it in (A.8), we come to the equality
\[ \eta_{i_1\ldots i_{\ell+1}} = \eta_{i_1\ldots i_{\ell}} \prod_{j=1}^{\ell} \frac{(f_{i_j} - f_{i_{\ell+1}})(\tilde{f}_{i_{\ell+1}} - \tilde{f}_{i_j})}{(f_{i_j} - f_{i_{\ell+1}})(\tilde{f}_{i_{\ell+1}} - f_{i_j})} \]
\[ = \eta_{i_1\ldots i_{\ell}} \prod_{j=1}^{\ell} \eta_{i_i i_{\ell+1}}, \]
that gives (A.6). It is clear that (A.5) and (A.6) prove the validity of (3.52).

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