PARTIAL DESINGULARIZATION

ANDRÉ BELOTTO DA SILVA, EDWARD BIERSTONE, AND RAMON RONZON LAVIE

Abstract. We address the following question of partial desingularization preserving normal crossings. Given an algebraic (or analytic) variety $X$ in characteristic zero, can we find a finite sequence of blowings-up preserving the normal-crossings locus of $X$, after which the transform $X'$ of $X$ has only singularities from an explicit finite list of minimal singularities, which we define using the determinants of circulant matrices. In the case of surfaces, for example, the pinch point or Whitney umbrella is the only singularity needed in addition to normal crossings.

We develop techniques for factorization (splitting) of a monic polynomial with regular (or analytic) coefficients, satisfying a generic normal crossings hypothesis, which we use together with resolution of singularities techniques to find local circulant normal forms of singularities. These techniques in their current state are enough for a positive answer to the question above, for $\dim X \leq 4$, or in arbitrary dimension if we preserve normal crossings only of order at most three. In these cases, minimal singularities have smooth normalization.

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1. Introduction

The goal of partial desingularization as described in this article is to understand the nature of the singularities that have to be admitted after a sequence of blowings-up $\sigma : X' \to X$ whose centres are restricted to lie over the complement of the normal crossings locus $X^{nc}$ of an algebraic or analytic variety $X$.

This study is motivated by the following question. Given $X$ (defined over a field $K$ of characteristic zero), can we find a sequence of blowings-up $\sigma : X' \to X$ such

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that \( \sigma \) preserves the normal crossings locus of \( X \), and \( X' \) has only normal crossings singularities? Roughly speaking, a variety has normal crossings at a point \( a \) if it can be defined by a monomial equation
\[
(1.1) \quad x_1x_2 \cdots x_k = 0
\]
in local coordinates at \( a \). But the definition of normal crossings and the answer to the preceding question depend on the meaning of local coordinates.

**Definitions and Remarks 1.1.** We say that an algebraic variety \( X \) has simple normal crossings (snc) at \( a \) if there is an embedding of an open neighbourhood of \( a \) in a smooth variety \( Z \), and a system of regular coordinates (or a regular system of parameters) \((x_1, \ldots, x_n)\) for \( Z \) at \( a \), with respect to which \( X \) is defined by an equation \((1.1)\). (In this case, we say, more precisely, that \( X \) has simple normal crossings \( \text{snc}(k) \) of order \( k \) at \( a \).)

Simple normal crossings at \( a \) is equivalent to the condition that (the restrictions of \( X \) to) all irreducible components are smooth (or empty) and transverse at \( a \).

We say that an algebraic or analytic variety \( X \) has normal crossings (nc) at \( a \) (or, more precisely, normal crossings \( \text{nc}(k) \) of order \( k \) at \( a \)) if \( X \) is again defined locally by an equation \((1.1)\), except that here \((x_1, \ldots, x_n)\) is an \( \acute{e} \)tale (or local analytic, or formal) coordinate system at \( a \) (perhaps after a finite extension of the ground field \( \mathbb{K} \)). In particular, normal crossings is the basic notion in the analytic case.

The normal crossings locus \( X^{nc} \) of \( X \) denotes the set of points of \( X \) having only normal crossings singularities. (\( X^{nc} \) includes all smooth points of \( X \).)

**Examples 1.2.** The nodal curve \( y^2 = x^2 + x^3 \) has normal crossings but not simple normal crossings at the origin. The curve \( y^2 + x^2 = 0 \) is nc at 0, but snc if and only if \( \sqrt{-1} \in \mathbb{K} \). Whitney’s umbrella \( z^2 - wx^2 = 0 \) is nc, but not snc, at every nonzero point of the \( w \)-axis \( z = x = 0 \).

We will take the ground field \( \mathbb{K} \) to be \( \mathbb{C} \) throughout the rest of the article, though all results for algebraic varieties hold over any given algebraically closed field \( \mathbb{K} \) of characteristic zero.

The answer to the question above is yes for snc; see [2], [4, Section 12], [6, Section 3], [10], [16]. There are also many interesting variants of the question for snc; for example, [7], [8], [17]. On the other hand, the answer to the question is no for nc, in general.

**Example 1.3.** The answer is no for Whitney’s umbrella \( X : z^2 - wx^2 = 0 \), which has a non-nc singularity called a pinch point \( \text{pp} \) at 0. There is no proper birational morphism that eliminates the pinch point without also modifying nc points, according to the following argument of Kollár [10] (where the question of Theorem 1.4 below and higher-dimensional analogues also was raised). At a nonzero point of the \( w \)-axis, \( X \) has two local analytic branches. If we go around a small circle about 0 in the \( w \)-axis, these branches are interchanged. This phenomenon continues to hold after any birational morphism that is an isomorphism over the generic point of the \( w \)-axis.

On the other hand, we have the following result.

**Theorem 1.4 ([6]).** For any two-dimensional algebraic (or analytic) variety \( X \), there is a morphism \( \sigma : X' \to X \) given by a finite sequence of smooth blowings-up preserving the normal crossings locus \( X^{nc} \), such that \( X' \) has only nc and pp singularities.
Whitney’s umbrella $X$ has smooth normalization; for example, if we set $w = v^2$, then $z^2 - wx^2$ factors as $(z - vx)(z + vx)$, and the morphism to $X$ from the smooth variety defined by either of the factors is a finite birational morphism. See also Proposition 2.1. Normalization plays an important part in classical approaches to resolution of singularities. In particular, smooth normalization, when it exists, is a relatively simple one-shot method to resolve singularities.

**Conjecture 1.5.** For any algebraic (or analytic) variety $X$, there is a finite composite of admissible smooth blowings-up $\sigma : X' \to X$ (see Definition 1.6 following), preserving $X^{\text{nc}}$, such that $X'$ has only singularities from an explicit finite list (which we call minimal singularities).

In the case of an analytic variety, the morphism $\sigma$ in Conjecture 1.5 should be understood to mean a morphism over a given relatively compact open subset of $X$.

**Definition 1.6.** A smooth blowing-up (i.e., a blowing-up with smooth centre $C$) is admissible if

1. locally, there are regular coordinates with respect to which $C$ is a coordinate subspace and each component of the exceptional divisor $E$ is a coordinate hypersurface (in this case, we say that $C$ and $E$ are snc);
2. the Hilbert-Samuel function $H_{X,x}$ is locally constant (as a function of $x$) on $C$.

In the case that $X$ is a hypersurface (see §1.3 below), condition (2) is equivalent to the condition that the order $\text{ord}_x X$ is locally constant on $C$. Definition 1.6 corresponds to the properties satisfied by the blowings-up involved in resolution of singularities in characteristic zero \cite{9}, \cite{4}, \cite{5}. A reader can safely choose not to focus on (2) since we use the Hilbert-Samuel function only to reduce the general to the hypersurface case (see §1.2 and Section 6).

Minimal singularities will be defined using the determinants of circulant matrices, and include the circulant singularities of §1.1 and Section 2, which are higher-dimensional versions of the pinch point. The following theorems summarize our general results on the conjecture above.

**Theorem 1.7.** Conjecture 1.5 is true for $\dim X \leq 4$.

**Theorem 1.8.** Let $X^{\text{nc}(3)}$ denote the set of normal crossings points of $X$ of order at most three. Then the analogue of Conjecture 1.5 with $X^{\text{nc}}$ replaced by $X^{\text{nc}(3)}$ is true (in any dimension).

**Remark 1.9.** In both theorems, the partial desingularization $X'$ of $X$ has smooth normalization. This phenomenon is particular to $\dim X \leq 4$, or to normal crossings of low order as in Theorem 1.8 (see Remark 1.17).

Conjecture 1.5 for $\dim X \leq 3$, and the analogue of Theorem 1.8 for $X^{\text{nc}(2)}$, are established in \cite{3}. The main novelties of the current article are:

- the development of general techniques for factorization (or splitting) of a monic polynomial with regular coefficients which satisfies a generic normal crossings hypothesis (Section 3; see also Theorem 1.13 below);
- the use of resolution of singularities techniques together with such splitting results to obtain normal forms of products of circulant singularities (Sections 4 and 5; see also Theorem 1.16);
techniques to move away limits of singularities that occur in a neighbourhood of a circulant point, to leave only minimal singularities in certain distinguished components of the exceptional divisor (Section 3).

These techniques are enough to prove Theorems 1.7 and 1.8 as we show in Section 6 following an inductive strategy presented in §6.2 and discussed in a concluding remark in §6.5. Looking forward, we mention recent progress on the techniques above in the general case, in Remark 1.17.

We can, in fact, prove more precise versions of Theorems 1.7 and 1.8, for a pair $(X, E)$, where $E$ is an snc divisor; see Theorems 6.1, 6.2 and 6.14.

Remark 1.10. The term minimal singularities comes from [6]; although the resemblance to “minimal” in the minimal model program is not coincidental, the meaning is not the same. Minimal singularities may be compared also to the singularities of the image a generic morphism of smooth varieties $X \to Z$, where $\dim Z = \dim X + 1$ (see, for example, [14]), or to the singularities of stable mappings of differentiable manifolds $X \to Z$, $\dim Z = \dim X + 1$. The notions coincide if $\dim X \leq 2$, but not in general.

The resolution of singularities techniques used in the article involve the desingularization invariant $inv$ of [4], [5]. As an illustration of our use of these techniques, let us sketch a proof of Theorem 1.4.

Proof of Theorem 1.4. We consider a hypersurface $X$ in 3 variables. Then the triple normal crossings nc(3) points of $X$ are isolated, and the nc(2) locus has codimension two in the ambient smooth variety (codimension one in $X$). We can blow up with smooth centres in the complement of nc(3), without modifying nc(2), until the maximal value of the desingularization invariant $inv$ equals the value $inv(nc(2))$ that it takes at an nc(2) point. Then the locus of points where $inv = inv(nc(2))$ is a smooth curve $C$ in the strict transform of $X$.

A basic understanding of the desingularization invariant (which we will recall and use in the article) shows that, at any point of $C$, we can choose local coordinates in which (the strict transform of) $X$ is given by an equation

$$ z^2 - w^k x^2 = 0, \tag{1.2} $$

where $w$ is an exceptional divisor; then $X$ is nc(2) on $\{ z = x = 0, w \neq 0 \}$.

Then, by finitely many blowings-up with centre $\{ z = w = 0 \}$, we can transform $X$ to either

$$ z^2 - x^2 = 0 \quad \text{nc(2)} $$

or

$$ z^2 - wx^2 = 0 \quad \text{pp} $$

(according as $k$ is even or odd).

(Note that, in any case, $z^2 - w^k x^2$ splits as a polynomial in $w^{1/2}, x, z$.)

The exponent $k$ appearing in (1.2) is, in fact, a local invariant of $X$, and the preceding blowings-up defined in local coordinates extend to global admissible blowings-up (see Theorem 1.16 and §4.3). \hfill \Box

1.1. Circulant singularities. Our minimal singularities, in general, are products of circulant singularities, described in detail in Section 2 following (see also (1.3) below), together with their neighbours. (For example, given any singularity that
has to be admitted after blowing-up sequences preserving nc, any neighbouring singularity must also be admitted. See also Section 6.

Circulant singularities were introduced in [6], [3] (where they were called cyclic singularities); we give a description in terms of circulant matrices (suggested by Franklin Vera Pacheco) in Section 2, which is convenient for studying their branching behaviour. A circulant singularity \( cp(k) \) of order \( k \) is a singularity which must be admitted as a limit of \( nc(k) \), after a blowing-up sequence preserving normal crossings. In particular, \( pp = cp(2) \) and \( \text{smooth} = cp(1) \). The circulant singularity \( cp(k) \) is the singularity at the origin of the hypersurface in \( \geq k + 1 \) variables given by

\[
\Delta_k(x_0, w^{1/k}x_1, \ldots, w^{(k-1)/k}x_{k-1}) = 0,
\]

where

\[
\Delta_k(X_0, X_1, \ldots, X_{k-1}) = \prod_{\ell=0}^{k-1} (X_0 + \varepsilon^\ell X_1 + \cdots + \varepsilon^{(k-1)\ell} X_{k-1})
\]

with \( \varepsilon = e^{2\pi i/k} \), is the determinant of the circulant matrix in \( k \) indeterminates \( (X_0, X_1, \ldots, X_{k-1}) \); see (2.3). For example,

\[
\begin{align*}
\text{cp}(2) &: \quad \Delta_2(z, w^{1/2}x) = z^2 - wx^2, \\
\text{cp}(3) &: \quad \Delta_3(z, w^{1/3}y, w^{2/3}x) = z^3 + wy^3 + w^2x^3 - 3wxyz.
\end{align*}
\]

Examples 1.11. (1) Minimal singularities in 4 variables; i.e., \( \dim X = 3 \) [3]. The complete list of minimal singularities in 4 variables comprises \( \text{cp}(3) \) and its (singular) neighbours, together with \( nc(4) \), \( \text{cp}(2) \) and \( \text{smooth} \times \text{cp}(2) \), where the latter means product as ideals; i.e., \( y(z^2 - wx^2) = 0 \). The neighbours of \( \text{cp}(3) \) are \( nc(2) \), \( nc(3) \), and the following singularity of order 2:

\[
\Delta_3(z, w^{1/3}y, w^{2/3}x) = 0.
\]

The latter was called a degenerate pinch point in [3].

In general, the minimal singularities in \( n+1 \) variables include all those which occur in \( \leq n \) variables (understood as formulas in \( n+1 \) variables where not all variables appear), together with \( nc(n+1) \) and all singularities in small neighbourhoods of products of circulant singularities that make sense as limits of \( nc(k) \), \( k = n \) (see Theorem 1.16). But the following shows that this list is not exhaustive.

(2) Minimal singularities in 5 variables; i.e., \( \dim X = 4 \) (see Section 6). Minimal singularities in 5 variables include the following limits of 4-fold normal crossings \( nc(4) \): \( \text{cp}(4) \), \( \text{smooth} \times \text{cp}(3) \), \( \text{cp}(2) \times \text{cp}(2) \), \( \text{smooth} \times \text{smooth} \times \text{cp}(2) = nc(2) \times \text{cp}(2) \). The circulant singularity \( \text{cp}(4) \) is the vanishing locus of

\[
\Delta_4(x_0, w^{1/4}x_1, w^{2/4}x_2, w^{3/4}x_3).
\]

Following are the neighbours of \( \text{cp}(4) \):

\[
\begin{align*}
(1) & \quad \Delta_4(x_0, w^{1/4}x_1, w^{2/4}x_2, w^{3/4}), \\
(2) & \quad \Delta_4(x_0, w^{1/4}x_1, w^{2/4}, w^{3/4}x_3), \\
(2') & \quad \Delta_4(x_0, w^{1/4}x_1, w^{2/4}, w^{3/4}x_2x_3), \\
(3) & \quad \Delta_4(x_0, w^{1/4}x_1, w^{2/4}, w^{3/4}).
\end{align*}
\]
Items (1), (2) and (3) in this list are the non-nc singularities in an arbitrarily small neighbourhood of \( \text{cp}(4) \) (except for the latter itself), while \((2')\) illustrates a phenomenon that does not appear in fewer than 5 variables; \((2')\) has to be admitted as a limit of singularities of the form \((2)\). In \((2')\), \(x_2\) is an exceptional divisor. For details, see Section 6.

1.2. **Approach to the main problems.** We use the desingularization invariant \( \text{inv} \) and the resolution of singularities algorithm of [4], [5] to reduce our main problems to a study of the singularities of a hypersurface \( X \) near a point in the closure of the \( \text{nc}(k) \)-locus, for given \( k \), where \( X \) has a convenient description in suitable local étale or analytic coordinates.

Normal crossings singularities are singularities of hypersurfaces. We say that \( X \) is a hypersurface if, locally, \( X \) can be defined by a principal ideal on a smooth variety. (We say that \( X \) is an embedded hypersurface if \( X \hookrightarrow Z \), where \( Z \) is smooth and \( X \) is defined by a principal ideal on \( Z \).) Conjecture 1.5 can be reduced to the case of a hypersurface using [4, 5]. Indeed, the desingularization algorithm of these articles involves blowing up with smooth centres in the maximum strata of the Hilbert-Samuel function. The latter determines the local embedding dimension, so the algorithm first eliminates points of embedding codimension \( \geq 1 \) without modifying nc points. (Recall that if \( H \) is the Hilbert-Samuel function of the local ring of a variety at a given point \( a \), then the minimal embedding dimension at \( a \) is \( H(1) - 1 \).)

Let \( X \hookrightarrow Z \) denote an embedded hypersurface, \( \dim Z = n \). Then, for any \( k \leq n \), the \( \text{nc}(k) \)-locus of \( X \) is a smooth subspace of \( X \) of codimension \( k \) in \( Z \).

The desingularization invariant \( \text{inv} \) is upper-semicontinuous with respect to the lexicographic ordering, and the locus of points where \( \text{inv} \) takes a given value is smooth. The value \( \text{inv}(\text{nc}(k)) \) of \( \text{inv} \) at a normal crossings point of order \( k \) (in year zero; i.e., before we start blowing up) is

\[
(1.4) \quad \text{inv}(\text{nc}(k)) = (k, 0, 0, \ldots, 1, 0, \infty),
\]

where there are \( k \) pairs before \( \infty \).

We remark that the condition \( \text{inv}(a) = \text{inv}(\text{nc}(k)) \) does not, in general, imply that \( X \) is nc at \( a \). For example, if \( X \) is the affine variety \( x_1^k + \cdots + x_n^k = 0 \), then \( \text{inv}(a) = (k, 0, 1, 0, \ldots, 1, 0, \infty) \), where there are \( k \) pairs, but \( X \) is not nc at 0 if \( k > 2 \).

More details of \( \text{inv} \) and the desingularization algorithm will be recalled in §4.2. We also refer the reader to [5] and to the *Crash course on the desingularization invariant* [6] Appendix A. Note, in particular, that \( \text{inv} \) is defined recursively over a sequence of admissible blowings-up in the desingularization algorithm. In year \( j \) (i.e., after \( j \) blowings-up), in general, \( \text{inv} \) depends on the previous blowings-up; it is not simply the year zero \( \text{inv} \) computed as if year \( j \) were year zero.

Our approach to the main problems involves a general inductive or recursive strategy which is the subject of Section 6 below; it is based on the following inductive formulation of Conjecture 1.5: given \( k \), there is a finite composite of admissible smooth blowings-up \( \sigma : X' \to X \), preserving normal crossings of order up to \( k \), such that \( X' \) has only minimal singularities. Theorems 1.7 and 1.8 both follow from this assertion. We will, in fact, need a more precise version of the assertion for a pair \( (X, E) \), where \( X \hookrightarrow Z \) is an embedded hypersurface and \( E \subset Z \) is an snc divisor; see Section 6.
The inductive step of the argument can be described roughly in the following way (though it is needed actually in the context of a pair \((X, E)\); see Claim 6.4).

- Blow up following the desingularization algorithm until \(\text{inv} \leq \text{inv}(\text{nc}(k))\). Then the locus of points where \(\text{inv} = \text{inv}(\text{nc}(k))\) is a smooth closed subspace \(S_k\) of codimension \(k\) in \(Z\). We can further blow up to eliminate any component of \(S_k\) on which \(X\) is not generically \(\text{nc}(k)\).
- Modify non-nc points of \(S_k\) to get minimal singularities in \(\Sigma_k = S_k \cup D_k\), where \(D_k\) is a distinguished subset of the exceptional divisor, and only normal crossings in \(U \setminus \Sigma_k\), for some neighbourhood \(U\) of \(\Sigma_k\).
- Apply the inductive hypothesis in the complement of \(\Sigma_k\); the centres of blowing up are closed in \(X\) in the case of \(S_k\), or extend to to global admissible centres when we deal with a pair \((X, E)\), in general.

Modification of the stratum \(S = S_k\) itself involves three main steps: splitting, described in Theorem 1.13 following, cleaning to get circumantal normal form as in Theorem 1.16 and moving away limits of singularities in a neighbourhood of a (product) circulant point to leave only minimal singularities in the distinguished centres when we deal with a pair \((X, E)\), in general.

1.3. Splitting techniques and circulant normal form. The non-nc\((k)\) points of \(X\) in \(S = S_k\) form a proper closed subspace \(T\) (see Lemma 3.7). After resolving the singularities of \(T\) if necessary, we can assume that, given \(a \in S\), we can choose étale (or analytic) local coordinates

\[
(w, u, x, z) = (w_1, \ldots, w_r, u_1, \ldots, u_q, x_1, \ldots, x_{k-1}, z)
\]

for \(Z\) at \(a\), in which \(X\) is given by \(f(w, u, x, z) = 0\), where

\[
f(w, u, x, z) = z^k + a_1(w, u, x)z^{k-1} + \cdots + a_k(w, u, x),
\]

the coefficients \(a_i(w, u, x)\) are regular (or analytic) functions, \(S = \{z = x = 0\}\), the exceptional divisor is \(w_1 \cdots w_r = 0\), and the complement of \(\{z = x = 0, w_1 \cdots w_r = 0\}\) maps isomorphically onto the original set of \(\text{nc}(k)\) points of \(X\) in \(S\). (It follows that every coefficient \(a_i\) vanishes to order at least \(i\) with respect to \((x, z)\) at \(a\).)

We are interested in the splitting or factorization of \(f\) at \(a\) as

\[
f(w, u, x, z) = \prod_{j=1}^{k} (z - b_j(w, u, x)),
\]

where each \(b_j\) belongs to the ideal generated by \(x_1, \ldots, x_{k-1}\). For example, at an \(\text{nc}(k)\) point, there is a formal splitting \((1.6)\), where each \(b_j\) has order 1.

From the latter generic splitting condition, it follows (at least in the algebraic case) that there is a unique splitting of \(f\) in \(\mathbb{C}(w)[[u, x]][z]\), where each \(b_j(u, w, x) \in \mathbb{C}(w)[u, x]\). Here \(\mathbb{C}(w)\) denotes an algebraic closure of the field of fractions \(\mathbb{C}(w)\) of the polynomial ring \(\mathbb{C}[w]\) (see 3.3).

For example, if there is a single \(w\) variable, then \(f\) splits over \(\mathbb{C}(w^{1/p})[[u, x]]\), for some \(p\), by the Newton-Puiseux theorem and elementary Galois theory, and we can take \(p = k\), if \(f\) is irreducible (see Lemmas 3.4 and 3.5).

Following is a simple basic example which illustrates Theorem 1.13 following, and also shows that the conclusion in this result cannot, in general, be strengthened.

**Example 1.12.** Let

\[
f(w, x, z) = z^2 + (w^3 + x)x^2.
\]
Then \( f \) (or the subvariety \( X \) of \( \mathbb{A}^3_k \) defined by \( f(w, x, z) = 0 \)) is \( \text{nc}(2) \) at every nonzero point of the \( w \)-axis \( \{ x = z = 0 \} \). The function \( f \) does not split over \( \mathbb{C}[w, x] \), but we can write

\[
f(v^2, x, z) = z^2 + v^6 \left( 1 + \frac{x}{v^6} \right) x^2,
\]

so that \( f(w, x, z) \) splits in \( \mathbb{C}(w^{1/2})[x][z] \).

Note that \( f(v^2, x, z) \) is not normal crossings at 0 as a formal power series in \( \mathbb{C}[v, x, z] \), but it is normal crossings in \( \mathbb{C}(v)[x, z] \) (i.e., as a formal power series in \( (x, z) \) with coefficients in the field \( \mathbb{C}(v) \)).

Consider the blowing-up \( \sigma \) of the origin in \( \mathbb{A}^3_k \). The \( w \)-axis lifts to the \( w \)-chart of \( \sigma \), given by substituting \((w, wx, wz) = (x, x, x) \) for \((w, x, z) \), and the strict transform of \( X \) is given by \( f' = 0 \) in the \( w \)-chart, where

\[
f'(w, x, z) := w^{-2} f(w, wx, wz) = z^2 + w(w^2 + x)x^2.
\]

After two more blowings-up of the origin, we get

\[
f''(w, x, z) = z^2 + w^3(1 + x)x^2,
\]

so that \( f''(w, x, z) \) splits over \( \mathbb{C}[w^{1/2}, x] \) (or \( f'(v^2, x, z) \) splits in an étale neighbourhood of the origin).

After an additional cleaning blowing-up, with centre \( \{ z = w = 0 \} \), we get a pinch point.

**Theorem 1.13** (limits of \( \text{nc}(k) \) in \( n = k + 1 \) variables). Let

\[
f(w, x_1, \ldots, x_{k-1}, z) = z^k + a_1(w, x)z^{k-1} + \cdots + a_k(w, x),
\]

where the coefficients \( a_i(w, x) \) are regular (or analytic) functions. If \( f(w, x, z) \) is \( \text{nc}(k) \) on \( \{ z = x = 0, w \neq 0 \} \), then, after a finite number of blowings-up of 0, \( f \) splits over \( \mathbb{C}[w^{1/p}, x] \), for some positive integer \( p \).

Theorem 1.13 is proved in Section 3 using the splitting over \( \mathbb{C}(w)[x] \) together with a multivariate Newton-Puiseux theorem due to Soto and Vicente [15], to show that the powers of \( w \) in the denominators of the roots are bounded linearly with respect to the degree with respect to \( x \) in the numerators.

**Question 1.14.** Consider the general case,

\[
f(w_1, \ldots, w_r, u_1, \ldots, u_q, x_1, \ldots, x_{k-1}, z)
\]

\[
= z^k + a_1(w, u, x)z^{k-1} + \cdots + a_k(w, u, x),
\]

where \( f(w, u, x, z) \) is \( \text{nc}(k) \) on \( \{ z = x = 0, w_1 \cdots w_r \neq 0 \} \). Is it true that, after finitely many blowings-up with successive centres of the form \( \{ z = x = w_j = 0 \} \), for some \( j \), \( f \) splits over \( \mathbb{C}[u, w^{1/p}, x] \), for some \( p \), where \( w^{1/p} := (w_1^{1/p}, \ldots, w_r^{1/p}) \)?

We give a positive answer to this question in the case \( k \leq 3 \), under the additional hypothesis \( \text{inv}(0) = \text{inv}(\text{nc}(k)) \); see Proposition 5.3.

Theorem 1.16 following ties together the splitting theorem 1.13 with the notion of circulant singularity.

**Remark 1.15.** In Theorem 1.16 and throughout the article, it is convenient to continue to use the same notation \( X \) instead of, for example, \( X_j \) for the strict transform of \( X = X_0 \) after \( j \) blowings-up.
Theorem 1.16 (circulant normal form). Consider an embedded hypersurface \( X \hookrightarrow Z \), as above, and assume that \( n := \dim Z = k + 1 \). Let \( U \) denote an open subset of \( Z \). Assume that (after a sequence of \( \text{inv} \)-admissible blowings-up of \( U \); cf. \[1.2\] the maximum value of \( \text{inv} \) on \( U \) is \( \text{inv}(\text{nc}(k)) \) and that \( X \) is generically \( \text{nc}(k) \) on the stratum \( S := \{ \text{inv} = \text{inv}(\text{nc}(k)) \} \) in \( U \); in particular, \( S \) is a smooth curve in \( U \). Then there is a finite sequence of admissible blowings-up of \( U \) (in fact, admissible for the truncated invariant \( \text{inv} \); see \[1.2\]), preserving the \( \text{nc}(k) \)-locus, after which \( X \) is a product of circulant singularities at every point of \( S \); i.e., \( X \) can be defined locally at every point of \( S \) by an equation of the form

\[
\prod_{i=1}^{s} \Delta_{k_i} \left( y_{i0}, w^{1/k_i} y_{i1}, \ldots, w^{(k_i - 1)/k_i} y_{ik_i - 1} \right) = 0,
\]

in suitable \( \text{étale} \) (or local analytic) coordinates \((w, (y_{i\ell})_{\ell=0,\ldots,k_i-1}, i=1,\ldots,s)\), where \( k_1 + \cdots + k_s = k \).

The equation \[1.8\] defines a variety with smooth normalization.

Note that \( \text{nc}(k) \) is itself a product of circulant singularities (each of order 1). Theorem \[1.16\] is proved in Section 4. A proof of Conjecture \[1.5\] following our approach requires analogues of Theorems \[1.13\] and \[1.16\] for \( k + 1 < n \); we prove these results for \( k \leq 3 \) in Section 5. The techniques of Sections 3, 4 and 5 are put together in Section 6 for modification of the strata \( S_k \) as described in \[1.2\] above. An overview the the proofs of Theorems \[1.7\] and \[1.8\] is given in \[6.2\].

Remark 1.17. Looking forward. A positive answer to Question \[1.14\] and a general version of Theorem \[1.14\] have been recently obtained by the first two authors, and are planned for a forthcoming article. The extension of Theorem \[1.16\] requires a more general formulation of the idea of a product of circulant singularities \[1.8\]; in particular, the analogous factors may include fractional powers of more than a single exceptional variable \( w \), so the normalization need no longer be smooth.

2. Circulant singularities

Circulant singularities provide a generalization to arbitrary dimension of the pinch point singularity that occurs at the origin of Whitney’s umbrella \( z^2 - wy^2 = 0 \).

Given indeterminates \( X = (X_0, X_1, \ldots, X_{k-1}) \), we define the circulant matrix

\[
C_k(X_0, X_1, \ldots, X_{k-1}) := \begin{pmatrix}
X_0 & X_1 & \cdots & X_{k-1} \\
X_{k-1} & X_0 & \cdots & X_{k-2} \\
\vdots & \vdots & \ddots & \vdots \\
X_1 & X_2 & \cdots & X_0
\end{pmatrix}.
\]

See [12] for a nice introduction to circulant matrices.

The circulant matrix \( C_k(X_0, X_1, \ldots, X_{k-1}) \) has eigenvectors

\[
V_\ell = (1, \varepsilon^\ell, \varepsilon^{2\ell}, \ldots, \varepsilon^{(k-1)\ell}),
\]

\( \ell = 0,\ldots, k-1 \), where \( \varepsilon = e^{2\pi i/k} \). The corresponding eigenvalues are

\[
Y_\ell = X_0 + \varepsilon^\ell X_1 + \cdots + \varepsilon^{(k-1)\ell} X_{k-1}, \quad \ell = 0,\ldots, k-1.
\]
Let $\Delta_k$ denote the determinant $\det C_k$. Then
\[ \Delta_k(X_0, \ldots, X_{k-1}) = Y_0 \cdots Y_{k-1} \]
\[ = \prod_{\ell=0}^{k-1} (X_0 + \varepsilon^\ell X_1 + \cdots + \varepsilon^{(k-1)\ell} X_{k-1}). \quad (2.3) \]

Given indeterminates $(w, x_0, \ldots, x_{k-1})$, set
\[ P_k(w, x_0, \ldots, x_{k-1}) := \Delta_k(x_0, w^{1/k} x_1, \ldots, w^{(k-1)/k} x_{k-1}) \]
\[ = \prod_{\ell=0}^{k-1} (x_0 + \varepsilon^\ell w^{1/k} x_1 + \cdots + \varepsilon^{(k-1)\ell} w^{(k-1)/k} x_{k-1}) \]
\[ \quad (2.4) \]

Then $P_k(w, x_0, \ldots, x_{k-1})$ is an irreducible polynomial. We define the *circulant* or *circulant point* singularity $cp(k)$ as the singularity at the origin of the variety $X$ defined by the equation $P_k(w, x_0, \ldots, x_{k-1}) = 0$; i.e., by the equation $\Delta_k(x_0, w^{1/k} x_1, \ldots, w^{(k-1)/k} x_{k-1}) = 0$.

(In [6, 3], a circulant point is called a “cyclic point”.)

For example, $cp(2)$ is the pinch point, and $cp(3)$ is given by
\[ P_3(w, z, y, x) = z^3 + wy^3 + w^2 x^3 + 3 wxyz. \]

**Proposition 2.1.** **Circulant singularities have smooth normalization.**

**Proof.** If we set $w = \varepsilon^k$, then $P_k(w, x_0, \ldots, x_{k-1})$ factors as
\[ \prod_{\ell=0}^{k-1} (x_0 + \varepsilon^\ell w^{1/k} x_1 + \cdots + \varepsilon^{(k-1)\ell} w^{(k-1)/k} x_{k-1}), \]
and the morphism $\nu$ to $X$ of the smooth hypersurface defined by any of the factors is a finite birational morphism. Therefore, $\nu$ is the normalization of $X$ (up to isomorphism): [13, III.8, Thm. 3]. See Corollary 3.14 below for an elementary proof of the proposition. \hfill \Box

**Remark 2.2.** We rewrite (2.2),
\[ \begin{pmatrix}
Y_0 \\
Y_1 \\
Y_2 \\
\vdots \\
Y_{k-1}
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \varepsilon^1 & \varepsilon^2 & \cdots & \varepsilon^{k-1} \\
1 & \varepsilon^2 & \varepsilon^4 & \cdots & \varepsilon^{2(k-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \varepsilon^{k-1} & \varepsilon^{2(k-1)} & \cdots & \varepsilon^{(k-1)^2}
\end{pmatrix}
\begin{pmatrix}
X_0 \\
X_1 \\
X_2 \\
\vdots \\
X_{k-1}
\end{pmatrix} \quad (2.6) \]

The rows (and the columns) of the matrix in (2.6) are the eigenvectors $V_0, \ldots, V_{k-1}$.

Recall that
\[ \sum_{l=0}^{k-1} \varepsilon^{i\ell} = \begin{cases} k, & i = 0, \\ 0, & i = 1, \ldots, k - 1. \end{cases} \]

The inverse of the linear transformation (2.6) is
formal power series ring

Puiseux series in

(3.1)

Consider a monic polynomial

Analogously, we can consider splitting in

(3.2)

where, for each \( j \), \( b_j(w, y) \in \mathbb{C}[w - w_0, y - y_0] \) and \( b_j(w, y) \) vanishes when \( (w - w_0, y - y_0) = (0, 0) \).

Analogously, we can consider splitting in \( \overline{\mathbb{C}(w)}[y - y_0][z] \), etc.

**Lemma 3.1.** Consider \( f(w, y, z) \) as in (3.1). Suppose that \( f \) splits formally at a point \( (w, y, z) = (w_0, y_0, 0) \). Then \( f \) splits in \( \overline{\mathbb{C}(w)}[y - y_0][z] \).

**Proof.** We can assume that \( y_0 = 0 \). There is an isomorphism of \( \overline{\mathbb{C}(w)} \) with \( \overline{\mathbb{C}(w - w_0)} \) induced by the isomorphism \( w \mapsto w_0 + (w - w_0) \) of \( \mathbb{C}[w] \) to \( \mathbb{C}[w - w_0] \), so it is enough to show that \( f \) splits in \( \overline{\mathbb{C}(w - w_0)}[y][z] \).

The roots \( b_j(w, y) \in \mathbb{C}[w - w_0, y] \) are algebraic over \( \mathbb{C}[w - w_0, y] \). The result follows since algebraicity is preserved by partial differentiation and by evaluation (i.e., by setting \( z = 0, y = 0 \)). \( \square \)

**Remark 3.2.** In the analytic case, assume that \( f(w, y, z) \in \mathcal{O}(W \times U)[z] \), where \( W \) and \( U \) are open subsets of \( \mathbb{C}^r \) and \( \mathbb{C}^m \) (respectively). Then Lemma 3.1 still holds, with the conclusion \( f \in \text{Frac}(\mathcal{O}(W)[y - y_0][z] \), where Frac denotes the field of fractions.
Indeed, we can extend all results of this section to the analytic case by replacing the ring \( \mathbb{C}(w) \) with \( \text{Frac}(\mathcal{O}(W)) \).

**Remark 3.3.** We will be interested in Lemmas 3.1 and 3.4 following in a situation where \( y = (u, x) = (u_1, \ldots, u_q, x_1, \ldots, x_{k-1}) \), \( y_0 = (u_0, 0) \), the vanishing locus \( \{w_1 \cdots w_r = 0\} \) represents an exceptional divisor, and \( f(w, u, x, z) \) in \( \text{nc}(k) \) at every point of \( \{z = x = 0, w_1 \cdots w_r \neq 0\} \) (see 1.3).

**Lemma 3.4.** Consider \( f(w, y, z) \) as in \( 3.3 \). Suppose that \( f \) splits in \( \overline{\mathbb{C}(w)[y]}[z] \). Then there is a finite and normal extension \( L \) of \( \mathbb{C}(w) \) in \( \mathbb{C}(w) \) such that \( f \) splits in \( L[y - y_0][z] \).

**Proof.** We can assume that \( y_0 = 0 \). By the hypothesis, \( f \) splits in \( \overline{\mathbb{C}(w)[y]}[z] \) as

\[
(3.3) \quad f = \prod_{j=1}^{k'} g_j^{m_j}, \quad g_j(w, y, z) = z - b_j(w, y), \quad j = 1, \ldots, k',
\]

where \( k' \leq k \), each \( m_j \) is a positive integer, and the \( b_j(w, y) \) are distinct elements of \( \mathbb{C}(w)[y] \). (The decomposition in this form is unique.)

Consider the formal expansions

\[
b_j(w, y) = \sum_{\gamma \in \mathbb{N}^m} b_{j,\gamma} y^\gamma, \quad j = 1, \ldots, k',
\]

where the coefficients \( b_{j,\gamma} \in \overline{\mathbb{C}(w)} \). Set \( M := \overline{\mathbb{C}(w)} \) and let \( L \) denote the subfield of \( M \) generated over \( \mathbb{C}(w) \) by the \( b_{j,\gamma}, \gamma \in \mathbb{N}^m \). We will show that \( L \) is a normal extension of \( \mathbb{C}(w) \) with finite automorphism group, and therefore a finite extension.

First, consider \( \sigma \in \text{Aut}_{\mathbb{C}(w)} M \), where the latter denotes the group of field automorphisms over \( M \) over \( \mathbb{C}(w) \). We claim that \( \sigma L = L \); i.e., \( \sigma \) induces an automorphism of \( L \) over \( \mathbb{C}(w) \). Indeed, the action of \( \sigma \) on \( M \) extends to an action on \( \mathbb{C}(w)[y][z] \) which fixes \( f \) but permutes the elements \( g_j \) by uniqueness of the decomposition \( (3.3) \). Therefore, \( \sigma \) fixes the set \( \{b_{j,\gamma} : j = 1, \ldots, k', \gamma \in \mathbb{N}^m\} \) (not the elements of this set). In other words, \( \sigma L = L \).

We claim, moreover, that \( L \) is a normal extension of \( \mathbb{C}(w) \); i.e., any irreducible polynomial \( p(t) \in \mathbb{C}(w)[t] \) which has a root \( a_1 \) in \( L \), splits in \( L \). First of all, the algebraic closure \( M = \overline{\mathbb{C}(w)} \) is trivially a normal extension of \( \mathbb{C}(w) \), so that \( \mathbb{C}(w) \) is the fixed point set of \( \text{Aut}_{\mathbb{C}(w)} M \), by the fundamental theorem of Galois theory. Now, \( \text{Aut}_{\mathbb{C}(w)} M \) maps to a subgroup \( S \) of the permutation group of the roots of \( p(t) \), and \( \prod_{\tau \in S}(t - \tau(a_1)) \) is fixed by \( \text{Aut}_{\mathbb{C}(w)} M \), so it is a polynomial over \( \mathbb{C}(w) \). This polynomial cannot be a nontrivial factor of \( p \) because \( p \) is irreducible, so we get the claim.

Now consider \( \sigma \in \text{Aut}_{\mathbb{C}(w)} L \). As above, \( \sigma \) induces a permutation of the \( g_j \). Moreover, \( \sigma \) is determined by its action on \( \{b_{j,\gamma}\} \), and therefore by its action on the \( g_j \); i.e., \( \text{Aut}_{\mathbb{C}(w)} L \) embeds as a subgroup of the finite group of permutations of \( \{g_j\} \), as required.

\( \square \)

**Lemma 3.5.** Assume that \( w \) is a single variable.

1. With the hypotheses of Lemma 3.4, \( f(w, y - y_0, z) \) splits in \( \mathbb{C}(w^1/r)[y - y_0][z] \), for some \( p \).
(2) Suppose \( f(w, y, z) \) is a monic polynomial \((\ref{eq:k1})\) in \( \mathbb{C}[w, y][z] \) which splits in \( \mathbb{C}(w^{1/p})[y-y_0][z] \). Then \( f = \prod_{i=1}^l f_i \), where each \( f_i \in \mathbb{C}(w)[y-y_0][z] \) is an irreducible monic polynomial in \( z \) of degree \( k_i \), which splits in \( \mathbb{C}(w^{1/k_i})[y-y_0][z] \), and \( k_1 + \cdots + k_l = k \).

**Proof.** (1) is an immediate consequence of Lemma \( \ref{lem:k4} \). For (2), we can again assume that \( y_0 = 0 \). Write
\[
 f(v^p, y, z) = \prod_{j=1}^k (z - b_j(v, y)),
\]
where each \( b_j \in \mathbb{C}(v)[y] \). The \( p \)th roots of unity \( e^{2\pi il/p}, l = 1, \ldots, p, \) have the structure of a cyclic group \( \mathbb{Z}_p \). Let \( \varepsilon = e^{2\pi i/p} \). Then the ordered set \( \{b_j(\varepsilon v, y)\} \) is a permutation of the set of roots \( \{b_j(v, y)\} \); say, \( b_j(\varepsilon v, y) = b_{\sigma(j)}(v, y) \). Then \( b_1(\varepsilon^2 v, y) = b_{\sigma(1)}(\varepsilon v, y) = b_{\sigma(2)}(v, y), \) and \( b_1(\varepsilon^l v, y) = b_{\sigma(l)}(v, y), \) for all \( l \). So there is a homomorphism of \( \mathbb{Z}_p \) onto a cyclic subgroup \( \mathbb{Z}_m \) of the group of permutations of the roots \( b_j \), for some \( m \leq k \).

Then, after reordering, \( \prod_{j=1}^m (z - b_j(v, y)) \) is invariant under the action of \( \mathbb{Z}_p \) and, therefore, an element of \( \mathbb{C}(v^p)[y][z] \). If \( m < k \), this means that \( f(w, y, z) \) is not irreducible in \( \mathbb{C}(w)[y][z] \).

We can assume that \( f(w, y, z) \) is irreducible in \( \mathbb{C}(w)[y][z] \). Then \( m = k \). In general, the group of homomorphisms \( \mathbb{Z}_p \to \mathbb{Z}_m \) is isomorphic to \( \mathbb{Z}_d \), where \( d = \gcd(p, m) \). Any \( h \in \mathbb{Z}_d \) corresponds to the homomorphism \( \mathbb{Z}_p \to \mathbb{Z}_m \) given by \( \varepsilon \mapsto \varepsilon^{hp/d} \). If \( f \) is irreducible, then \( m = k \) and \( \mathbb{Z}_p \to \mathbb{Z}_k \) is onto, so that \( d = k \) and \( f(v^k, y, z) \) splits as required. \( \square \)

**Remark 3.6.** Likewise, if \( w \) is a single variable and \( f(w, y, z) \) is a monic polynomial \((\ref{eq:k1})\) in \( \mathbb{C}[w, y][z] \) which splits in \( \mathbb{C}[w^{1/p}, y][z] \), then \( f = \prod_{i=1}^l f_i \), where each \( f_i \in \mathbb{C}[w, y][z] \) is an irreducible monic polynomial in \( z \) of degree \( k_i \), which splits in \( \mathbb{C}[w^{1/k_i}, y][z] \), and \( k_1 + \cdots + k_l = k \).

Note that the polynomial \( f(w, x, z) = z^2 + (w^2 + x)x^2 \) is irreducible in \( \mathbb{C}[w, x][z] \) but not in \( \mathbb{C}(w)[x][z] \).

### 3.2. Generic normal crossings and the discriminant

Let \( X \hookrightarrow Z \) denote an embedded hypersurface \( (Z \) smooth). For any \( k \in \mathbb{N} \), \{ \( x \in X : X \) is nc\( (k) \) at \( x \) \} is a smooth subspace of \( X \) of codimension \( k \) in \( Z \).

**Lemma 3.7.** The set of non-normal crossings points of \( X \) is a closed algebraic (or analytic) subset. If \( Y \) is an irreducible subset of \( X \) and \( X \) is generically nc\( (k) \) on \( Y \), for some \( k \in \mathbb{N} \), then \{ \( x \in Y : X \) is not nc\( (k) \) at \( x \) \} is a proper closed algebraic (or analytic) subset of \( Y \).

**Proof.** This is a simple consequence of the following two facts. (1) The desingularization invariant \( \text{inv} = \text{inv}_X \) (in year zero) is Zariski upper-semicontinuous on \( X \). (2) \( X \) is nc\( (k) \) at a point \( a \) if and only if \( \text{inv}_X(a) = (k, 0, 1, 0, \ldots, 1, 0, \infty) \) (with \( k \) pairs) and \( X \) has \( k \) local analytic branches at \( a \) (equivalently, there are precisely \( k \) points in the fibre of the normalization of \( X \) over \( a \); see \([2 \text{ Thm. 3.4}]\)). \( \square \)

Lemmas \( \ref{lem:k8} \) and \( \ref{lem:k9} \) following deal with the question of splitting in terms of the discriminant. These results will be used in Section \( \ref{sec:k5} \). Lemma \( \ref{lem:k34} \) in the case \( k = 3 \) was proved in \([3 \text{ Lemmas 3.4, 3.5}]\), but the general proof below is much simpler.
Let \( f \) denote a regular function, written in étale local coordinates
\[
(w, u, x, z) = (w_1, \ldots, w_r, u_1, \ldots, u_q, x_1, \ldots, x_{k-1}, z)
\]
as
\[
(3.4) \quad f(w, u, x, z) = z^k + a_1(w, u, x)z^{k-1} + \cdots + a_k(w, u, x).
\]
Let \( D(w, u, x) \) denote the discriminant of \( f(w, u, x, z) \) as a polynomial in \( z \). The
discriminant \( D \) is a weighted homogeneous polynomial of degree \( k(k-1) \) in the
coefficients \( a_i \), where each \( a_i \) has weight \( i \).

**Lemma 3.8.** Assume that \( f \) is in the ideal generated by \( x_1, \ldots, x_{k-1}, z \), and that
\( f \) splits formally (into \( k \) factors of order 1) at every point where \( x = z = 0 \) and
\( w_1 \cdots w_r \neq 0 \). Then \( D \) factors in an étale neighbourhood of \( a = 0 \) as
\[
(3.5) \quad D = \Phi^2 \Psi,
\]
where \( \Phi \) is in the ideal generated by \( x_1, \ldots, x_{k-1} \), and \( \Psi \) is nonvanishing outside
\({w_1 \cdots w_r = 0}\).

**Proof.** The hypotheses imply that \( D \) is a square (étale locally) at every point where
\( x = z = 0 \) and \( w_1 \cdots w_r \neq 0 \). So there is an étale neighbourhood of \( a \) in which
every irreducible factor of \( D \) occurs to even power, except for those factors which are
nonvanishing outside \({w_1 \cdots w_r = 0}\). \( \square \)

**Lemma 3.9.** Assume that \( f \) satisfies the hypotheses of Lemma \( \ref{lem:3.8} \) and that \( D \)
factors in an étale coordinate neighbourhood of \( a = 0 \) as in \( \ref{eq:3.5} \). Then, after a
finite number of blowings-up with centres of the form \({z = x = w_j = 0}\), for some
\( j \), we can assume that \( D(v_1^2, \ldots, v_r^2, u, x) \) is a square.

**Proof.** We can assume that \( a_1 = 0 \) in \( \ref{eq:3.4} \) (by completing the \( k \)th power). According
to Lemma \( \ref{lem:3.8} \)
\[
\Psi(w, u, x) = \xi(u, w) + x_1 \theta_1(u, w, x) + \cdots + x_{k-1} \theta_{k-1}(u, w, x),
\]
where \( \xi(u, w) \) does not vanish outside \({w_1 \cdots w_r = 0}\); i.e., the zero set of \( \xi \) is a
subset of \({w_1 \cdots w_r = 0}\), so that \( \xi(u, w) = w^\alpha \eta(u, w) \), where \( w^\alpha = w_1^{\alpha_1} \cdots w_r^{\alpha_r} \)
is a monomial and \( \eta(u, w) \) is a unit. If \( \alpha = 0 \), then \( D \) is already a square, so we can
assume that \( \alpha \neq 0 \).

Consider the blowing-up \( \sigma \) with centre \({z = x = w_j = 0}\), for some \( j \) such that
\( \alpha_j \neq 0 \). The subspace \({z = x = 0}\) lifts to the \( w_j \)-chart of \( \sigma \), given by substituting
\((w, u, w_j x, w_j z)\) for \((w, u, x, z)\), and we have
\[
f'(w, u, x, z) := w_j^{-k} f(w, u, w_j x, w_j z) = z^k + a_1'(w, u, x)z^{k-2} + \cdots + a_k'(w, u, x),
\]
where each \( a_i'(w, u, x) = w_j^{-1} a_i(w, u, w_j x) \). Since \( D \in (x)^{k(k-1)} \), \( f'(w, u, x, z) \) has
discriminant
\[
D' = w_j^{-k(k-1)} D \circ \sigma = (\Phi')^2 \cdot \Psi \circ \sigma,
\]
and
\[
(\Psi \circ \sigma)(w, u, x) = w_j \left( w^\alpha' \eta'(w) + \theta'(w, x) \right),
\]
where \( w^\alpha' = w_1^{\alpha_1} \cdots w_r^{\alpha_r} \), \( \eta' \) is a unit and \( \theta' \in (x) \).

It follows that, after \( \alpha_1 + \cdots + \alpha_r \) blowings-up with centres of the form \({z = x = w_j = 0}\), for some \( j \), \( D(v_1^2, \ldots, v_r^2, u, x) \) is a square. \( \square \)
Remark 3.10. In Section 3.3 we will deal with an embedded hypersurface $X \hookrightarrow Z$ together with a simple normal crossings divisor $E$, and we will apply Lemma 3.9 and Theorem 1.13 (proved in §3 following) to a function $g(y_1, \ldots, y_r, w, u, x, z) = y_1 \cdot \cdots \cdot y_r f(y, w, u, x, z)$, where the $y_j$ are local generators of the components of $E$, $f$ is as in (3.4) with coefficients $a_i = a_i(y, w, u, x)$, and $f$ satisfies the hypotheses of Lemma 3.9. The latter holds in this case with centres $\{z = x = y = w_j = 0\}$, and the proof is the same.

3.3. Limits of nc($k$) in $k + 1$ variables. In this subsection, we prove Theorem 1.13. See also Lemma 3.5 and Remark 3.6. The statement of Theorem 1.13 means, more precisely, that, after finitely many blowings-up of 0, the strict transform of $x$ splits in $C$ and to an operation on $C$, we also denote $\psi$ for $\psi_1$,

Proof of Theorem 1.13 Let us change notation and write $x = (x_1, \ldots, x_{k-1}, x_k)$, where $x_k = w$. Given any field $K$, we write $K((t^{1/q}))$ to denote the field of Puiseux Laurent series in $t^{1/q}$, where $q$ is a positive integer. Let $SL^+_{\text{lex}}(k, Z)$ denote the multiplicative subsemigroup of $SL(k, Z)$ consisting of upper-triangular matrices

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1k} \\ 0 & 1 & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where the $a_{ij}$ are nonnegative integers. Clearly, $SL^+_{\text{lex}}(k, Z)$ acts on monomials $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ by $x^\alpha \mapsto x^\alpha A$, $A \in SL^+_{\text{lex}}(k, Z)$, where

$$\alpha A := (\alpha_1, \ldots, \alpha_k) \cdot \begin{pmatrix} 1 & a_{12} & \cdots & a_{1k} \\ 0 & 1 & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Write $\psi_A(x^\alpha) := x^\alpha A$. Of course, $\psi_A$ extends to an operation on $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_k]$, and to an operation on $\mathbb{C}[x][z]$ (by the preceding operation on coefficients), which we also denote $\psi_A$, in each case. Since $\psi_A$ extends and to $\mathbb{C}[w][x_1, \ldots, x_{k-1}]$, or on $\mathbb{C}(w)[x_1, \ldots, x_{k-1}]$. By the theorem of Soto and Vicente [15], there exists a positive integer $q$ such that $f$ splits in $\mathbb{C}((x_1^{1/q}) \cdots (x_k^{1/q}))[z]$ and, moreover, there exists $A \in SL^+_{\text{lex}}(k, Z)$ such that $\psi_A(f)$ splits in $\mathbb{C}[x_1^{1/q}, \ldots, x_k^{1/q}][z]$. Let $c_i \in \mathbb{C}[x_1^{1/q}, \ldots, x_{k-1}^{1/q}, w^{1/q}]$, $i = 1, \ldots, k$, denote the roots of $\psi_A(f)$.

By Lemma 3.1, $f$ splits in $\mathbb{C}(w)[x_1, \ldots, x_{k-1}][z]$. Let $b_i \in \mathbb{C}(w)[x_1, \ldots, x_{k-1}] \subset \mathbb{C}((w))[x_1, \ldots, x_{k-1}]$, $i = 1, \ldots, k$, denote the roots of $f$.

By the uniqueness of formal expansion, the set $\{c_i\}$ of roots of $\psi_A(f)$ coincides with the set $\{\psi_A(b_i)\}$; i.e., each $c_i \in \mathbb{C}[x_1, \ldots, x_{k-1}, w^{1/q}]$. 

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Note that, given any monomial $x_1^{\alpha_1} \cdots x_k^{\alpha_k-1}$, $w = x_k$ appears in $\psi_A(x_1^{\alpha_1} \cdots x_k^{\alpha_k-1})$ to the power $\sum_{j=1}^{k-1} \alpha_j \alpha_k$: i.e., to a power at most $d \mu$, where $d$ is the degree $\alpha_1 + \cdots + \alpha_k - 1$, and $\mu = \max \{ \alpha_k \}$.

It follows that blowing up the origin $\mu$ times will clear all denominators in the roots $b_i$. \hfill \Box

3.4. Normality. The purpose of this subsection is to give an elementary proof that circulant singularities have smooth normalization (Proposition 3.13), and that, in Theorem 1.13, we get smooth normalization after finitely many blowings-up of the origin (see Corollary 3.14). As usual, $\mathbb{C}$ can be replaced by any algebraically closed field in the results following.

Proposition 3.11. Let $f \in \mathbb{C}[w, x][z] = \mathbb{C}[w, x_1, \ldots, x_n][z]$ denote an irreducible monic polynomial of degree $k$ in $z$ with coefficients in $\mathbb{C}[w, x]$; i.e.,

$$f(w, x, z) = z^k + a_1(w, x)z^{k-1} + \cdots + a_k(w, x),$$

where each $a_i(w, x) \in \mathbb{C}[w, y]$. Let $R := \mathbb{C}[w, x][z]/(f)$, and let $R'$ denote the integral closure of $R$ in its field of fractions $\text{Frac}(R)$. Then the following are equivalent:

1. $f(w, x, z)$ splits into $k$ factors in $\mathbb{C}[w^{1/k}, x][z]$;
2. There exists $u \in R'$ such that $u^k = w$.

Proof (due to Pierre Lairez). Let $A := \mathbb{C}[w, x]$ and let $K$ denote the field of fractions of $A$. The polynomial $p(w, y) := y^k - w$ is irreducible in $A[y]$ or in $K[y]$, by Eisenstein’s criterion. In particular, $(p)$ is a maximal ideal in $K[y]$, so that $K[w^{1/k}] = K[y]/(p)$ is a field, and, therefore, $K[w^{1/k}]$ is the field of fractions of $A[w^{1/k}] = A[y]/(p)$.

The ring $A[w^{1/k}]$ is a unique factorization domain, so that $f$ splits over $A[w^{1/k}]$ (i.e., $f$ splits in $A[w^{1/k}][z]$) if and only if $f$ splits over its field of fractions $K[w^{1/k}]$.

By hypothesis, $f$ is irreducible in $A[z]$, and therefore in $K[z]$. By Lemma 3.13 following, $f$ splits over the field $K[y]/(p) = K[w^{1/k}]$ if and only if $p$ splits over $K[z]/(f)$. Now, $K[z]/(f)$ is isomorphic to the field of fractions $\text{Frac}(R)$ of $R$, and, of course, $p(w, y)$ splits over $\text{Frac}(R)$ if and only if $p(w, y)$ has a root in $\text{Frac}(R)$; since $p(w, y)$ is monic, such a root would belong to $R'$.

Remark 3.12. We recall that, if $f(x)$ is an irreducible polynomial (of a single variable $x$) with coefficients in a field $K$, then the ideal $(f(x)) \subset K[x]$ is maximal, so that $L_f := K[x]/(f(x))$ is field ($L_f$ is the splitting field of $f(x)$ over $K$). In general, if $A$ is a reduced Noetherian ring, let $Q(A)$ denote the total quotient algebra $Q(A) := S^{-1}A$, where $S$ is the multiplicative subset of $A$ consisting of non-zerodivisors. Then $Q(A)$ is the product of the fields of fractions $Q(A/p_i)$, where $p_1, \ldots, p_r$ are the minimal primes of $A$. In particular, if $p(x) \in L[x]$ is a polynomial with coefficients in a field $L$, and with $r$ distinct irreducible factors, then $Q(L[x]/(p(x))$ is a product of $r$ fields that are uniquely determined by $p(x)$.

Lemma 3.13. Let $f(x) \in K[x]$ and $g(y) \in K[y]$ both denote irreducible polynomials of a single variable over a field $K$. Then the number of irreducible factors of $f(x)$ over the field $L_g = K[y]/(g(y))$ equals the number of irreducible factors of $g(y)$ over the field $L_f = K[x]/(f(x))$.

Proof. The algebra $K[x, y]/(f(x), g(y))$ can be identified with both $L_f[y]/(g(y))$ and $L_g[x]/(f(x))$, so the assertion is an immediate consequence of Remark 3.12.
As an alternative argument, we can use that fact that the irreducible factors of \( f(x) \) over \( \mathcal{L}_g \) are in one-to-one correspondence with the irreducible components of \( \text{Spec}(\mathcal{L}_g[x]/(f(x))) = \text{Spec}(K[x, y]/(f(x), g(y))) \), and likewise for the irreducible factors of \( g(y) \).

\[ \square \]

**Corollary 3.14.** With the hypotheses of Proposition 3.11, if either of the (equivalent) conditions (1), (2) of the proposition holds, then \( R' \) is regular.

**Proof.** By condition (1) of Proposition 3.11 we can write

\[ f(y^k, x, z) = \prod_{i=0}^{k-1} g(\epsilon^i y, x, z), \]

where \( g \in \mathbb{C}[y, x][z] \) and \( \epsilon = e^{2\pi i/k} \). Of course,

\[ \prod_{i=0}^{k-1} g(\epsilon^i y, x, z) = f(w, x, z) = (y^k - w)h(y^k, w, x, z), \]

where \( h \in \mathbb{C}[w, x][z] \).

By condition (2), \( y^k - w \) has a root \( u \) in \( R' \subset \text{Frac}(R) \), and the homomorphism from \( R[y] \) onto \( R[u] \subset R' \) induced by \( y \mapsto u \) has kernel \( (g(\epsilon^i y, x, z), y^k - w) \), for some \( i = 0, \ldots, k-1 \), by (3.6). Therefore,

\[ R[u] \cong \frac{R[y]}{(g(\epsilon^i y, x, z), y^k - w)} \cong \frac{\mathbb{C}[y, w, x][z]}{(g(\epsilon^i y, x, z), y^k - w)} \cong \mathbb{C}[y, z]. \]

In particular, \( R[u] \subset R' \) is regular and hence already integrally closed, and, therefore, coincides with \( R' \), as required. \[ \square \]

**Example 3.15.** The variety \( X := \{ z^2 - w_1 w_2 x^2 \} \) has singular normalization \( \{ z^2 - w_1 w_2 = 0 \} \). Let \( w_1 = y_1^2 \), \( w_2 = y_2^2 \). Then \( z^2 - w_1 w_2 x^2 \) splits as \( (z - y_1 y_2 x)(z + y_1 y_2 x) \). The mapping to \( X \) from each irreducible component \( \{ z \pm y_1 y_2 x = 0 \} \) is generically 2-to-1.

4. LIMITS OF \( k \)-FOLD NORMAL CROSSINGS IN \( k + 1 \) VARIABLES

In this section, we prove Theorem 1.16. The proof consists of two parts. The first part is formulated as Theorem 4.1 below. Theorem 4.1 begins with the hypotheses of Theorem 1.16 and the proof provides a construction in étale (or analytic) local coordinates that proves the assertion of Theorem 1.16 although it may not be evident a priori that the blowings-up involved are global admissible smooth blowings-up.

There are actually two sequences of blowings-up involved in the proof of Theorem 1.16 or Theorem 4.1. First there is a sequence of cleaning blow-ups, following [6, Section 2], after which \( X \) can be described locally at a limit of \( \text{nc}(k) \) points, by a certain pre-circulant normal form. In the irreducible case, for example, the latter means an equation of the following form in suitable local étale (or analytic) coordinates:

\[ \Delta_k \left( z, w^{n_1+1/k} x_1, \ldots, w^{n_{k-1}+1/(k-1)/k} x_{k-1} \right) = 0 \]

(where we can be more precise about the integers \( n_j \)—see Remark 4.3).

Once we have such local coordinates, there is a second blowing-up sequence which we use to reduce each \( n_j \) to zero, to get circulant normal form. For example,
given \( j \), we can reduce \( n_j \) to zero by blowings-up with centre \( \{ z = w = 0, x_\ell = 0, \text{ for all } \ell \neq j \} \). We will show in a simple fashion how to choose local coordinates for (4.1) so that these centres of blowing up make sense in a global combinatorial way. A similar idea will be used in the proof of Proposition 5.11 as well as in \S\S 6.2 (B)(II) and 6.3.1.

The purpose of the second part of the proof of Theorem 1.16 is to describe the first sequence of blowings-up above in an invariant global way that is independent of the local construction in the proof of Theorem 4.1, these blowings-up are called cleaning blow-ups, following [6, Section 2]. Similar ideas have been developed by Kollár [11] and Abramovich, Temkin and Wlodarczyk [1].

The proof of Theorem 4.1 requires little knowledge of the technical details of the desingularization algorithm, except for a very basic understanding of maximal contact and the coefficient ideal. A reader unfamiliar with the technology of desingularization can safely read the rest of the article without the details of the second part of the proof of Theorem 1.16. At the same time, the proof of Theorem 4.1 introduces some of this technology by explicit local computation that we hope may be helpful in understanding the cleaning procedure, as described in \S 4.3. Some basic details of the desingularization algorithm and the invariant inv are recalled in \S 4.2 and will be needed also in Section 6. The reader is again referred to [5] and the Crash course on the desingularization invariant [6, Appendix] for all of the notions from resolution of singularities that we use.

### 4.1. Circulant normal form.

**Theorem 4.1.** Assume that (after an inv-admissible sequence of blowings-up; cf. [4.2 below]) \( f(w, x_1, \ldots, x_{k-1}, z) \) satisfies the hypotheses of Theorem 1.15 and that

\[ \text{inv}(0) = \text{inv(nc}(k)) \]

Then there is a finite sequence of admissible blowings-up (in fact, admissible for the truncated invariant \( \text{inv}_1 \); see [4.2] that are isomorphisms over the nc(k) locus, after which the only singularities that may occur as limits of nc(k) points, are products of circulant singularities.

More precisely, assume that \( f = f_1 \cdots f_s \), where each \( f_i \) is an irreducible polynomial

\[ f_i(w, x, z) = z_{k_i} + \sum_{j=1}^{k_i} a_{ij}(w, x) z_{k_i-j} \]

with regular or algebraic (or analytic) coefficients, and \( k_1 + \cdots + k_s = k \). Then there is a finite sequence of admissible blowings-up that are isomorphisms over the nc(k) locus, after which, at the limit of the nc(k) points, there is a local étale (or analytic) coordinate system \( (w, (y_\ell, y_{1\ell}, \ldots, y_{k_1-1\ell}, \ldots, y_{k_s-1\ell})) \) in which the strict transform of \( f(w, x, z) = 0 \) is given by

\[ \prod_{i=1}^{s} \Delta_{k_i} \left( y_{i0}, w^{1/k_i} y_{i1}, \ldots, w^{(k_i-1)/k_i} y_{i, k_i-1} \right) = 0. \]

**Remark 4.2.** In Section 6, where we treat \( X \) together with a simple normal crossings divisor \( E \), we will apply Theorem 4.1 to a product of local generators of the ideals of \( X \) and the components of \( E \); see also Remark 3.10. The circulant normal form
(4.2) then becomes
\[ y_1 \cdots y_r \prod_{i=1}^{s} \Delta_{k_i} \left(y_0, w^{1/k_i} y_{i1}, \ldots, w^{(k_i-1)/k_i} y_{i,k_i-1} \right) = 0, \]
where the \( \{y_j = 0\} \) are the strict transforms of the components of \( E \).

**Proof of Theorem 4.1.** By Theorem 1.13 after a finite number of blowings-up of the origin, we can assume that \( f \) splits in \( \mathbb{C}[w^{1/p}, x][z] \), for some positive integer \( p \).

We will first consider the case that \( f \) is irreducible (so we get \( \text{cp}(k) \) as limit), and then handle the general case.

**Irreducible case.** By Lemma 3.5(2) and Remark 3.6 we can take \( p = k \); i.e., \( f \) splits in \( \mathbb{C}[w^{1/k}, x][z] \), and we can write
\[ f(v^k, x, z) = \prod_{\ell=0}^{k-1} (z + b(\varepsilon^\ell v, x)), \]
where \( b(v, x) \in \mathbb{C}[v, x] \) and \( \varepsilon = e^{2\pi i/k} \).

We can assume that \( a_1(w, x) = 0 \) (by the Tschirnhausen transformation; i.e., completion of the \( k \)th power). Set
\[ Y_\ell := z + b(\varepsilon^\ell v, x), \quad \ell = 0, \ldots, k-1, \]
and define \( X_0, \ldots, X_{k-1} \) by (2.7); i.e.,
\[ X_0 = \frac{1}{k} \sum_{\ell=0}^{k-1} Y_\ell = z, \]
\[ X_\ell = \frac{1}{k} \sum_{\ell=0}^{k-1} \varepsilon^{\ell(k-\ell)} Y_\ell = \frac{1}{k} \sum_{\ell=0}^{k-1} \varepsilon^{\ell(k-\ell)} b(\varepsilon^\ell v, x), \quad \ell = 1, \ldots, k-1. \]

It is easy to check that, for each \( \ell = 1, \ldots, k-1 \), \( v^{k-\ell} X_\ell \) is invariant under the action of the group \( \mathbb{Z}_k \) of \( k \)th roots of unity (where the operation of \( \varepsilon \) on functions of \( (v, x) \) is induced by \( (v, x) \mapsto (\varepsilon v, x) \)). In other words,
\[ v^{k-\ell} X_\ell = \eta_\ell(v^k, x), \]
where \( \eta_\ell(w, x) \in \mathbb{C}[w, x] \), \( \ell = 1, \ldots, k-1 \). Since each \( \eta_\ell \) must, therefore, be divisible by \( v^k \), we can write
\[ X_\ell = v^{km_\ell + \ell} \zeta_\ell(v^k, x) = w^{m_\ell + \ell/k} \zeta_\ell(w, x), \]
where \( m_\ell \) is a nonnegative integer and \( \zeta_\ell(w, x) \in \mathbb{C}[w, x] \) is not divisible by \( w \), \( \ell = 1, \ldots, k-1 \).

Since the \( X_1, \ldots, X_{k-1} \) are given by an invertible linear combination of \( b(\varepsilon v, x), \ldots, b(\varepsilon^{k-1} v, x) \) (recall that \( \sum_{\ell=0}^{k-1} b(\varepsilon^\ell v, x) = 0 \)), it follows that the coefficient ideal of the marked ideal \( (f, k) \) is equivalent to the marked ideal
\[ \mathcal{I}^1 := (w^{km_\ell + \ell} \zeta_\ell), k \]
on the maximal contact subspace \( N^1 := \{z = 0\} \) (cf. [3] Example A.13). Set
\[ \alpha_1 := \min_{1 \leq \ell \leq k-1} (km_\ell + \ell), \]
and let \( \ell_1 \) denote the (unique) corresponding \( \ell \) (realizing the minimum), and \( \xi_\ell_1 \) the corresponding \( \zeta_\ell \). Then \( \xi_\ell_1 \) has order \( 1 \), since \( \text{inv}(a) = (k, 0, 1, \ldots) \). The monomial
w^{\alpha_1} generates the monomial part of the coefficient ideal $C^k$ (cf. §A.6). Set $p_1 := m_{\ell_1}$.

It follows that the second coefficient ideal $C^2$ (still with marked or associated order $k$), on the second maximal contact subspace $N^2 := \{z = \xi_{\ell_1} = 0\}$, is generated by

$$w^{km_\ell + \ell - \alpha_1}_{\xi_{\ell_1}}|_{z = \xi_{\ell_1} = 0}, \quad \ell \neq \ell_1.$$  

Therefore, for each $\ell \neq \ell_1$, there is a nonnegative integer $\bar{m}_\ell$ such that

$$\zeta_\ell = \eta_{\ell_1}^1 + w^{\bar{m}_\ell}_{\xi_{\ell_1}},$$

where $\eta_{\ell_1}^1$ is in the ideal generated by $\xi_{\ell_1}$, and $\xi_{\ell_1}|_{z = \xi_{\ell_1} = 0}$ is not divisible by $w$. Let

$$\alpha_2 := \min_{\ell \neq \ell_1} \left( k(m_\ell + \bar{m}_\ell) + \ell - \alpha_1 \right),$$

and let $\ell_2 \neq \ell_1$ denote the (unique) corresponding $\ell$. Then $\xi_{\ell_2}$ has order 1, since $\text{inv}(a) = (k, 0, 1, 0, 1, \ldots)$. The monomial $w^{\alpha_2}$ generates the monomial part of the coefficient ideal $C^2$. Clearly, $\alpha_2 = kp_2 + h_{\ell_2}$, where $p_2, h_{\ell_2}$ are nonnegative integers and $1 \leq h_{\ell_2} < k$.

We repeat this construction for each $j = 3, \ldots, k - 1$.

We now apply cleaning blow-ups, with centre $N^j \cap \{w = 0\}$, to successively reduce each $p_j$ to zero, $j = k - 1, k - 2, \ldots, 1$. In particular, for each $j$, we reduce $\bar{m}_{\ell_j}$ to 0 since $\eta_{\ell_j}^{j-1}$ is in the ideal generated by $\xi_{\ell_j}, \ldots, \xi_{\ell_{j-1}}$.

We can then make a formal (or étale) coordinate change

$$y_{\ell_1} := \xi_{\ell_1},$$

$$y_{\ell_j} := \eta_{\ell_j}^{j-1} + \xi_{\ell_j}, \quad j = 2, \ldots, k - 1,$$

to reduce each $X_{\ell}$ to $w^{n_{\ell} + h_{\ell}/k} y_{\ell}, \ell = 1, \ldots, k - 1$, where \{$h_1, \ldots, h_{k-1}$} = \{1, \ldots, k - 1\}, each $n_{\ell}$ is a nonnegative integer, and $n_{\ell_1} = 0$. (After re-ordering indices if necessary, and relabelling variables, this means that we have reduced the equation $f(w, x, z) = 0$ to a pre-circulant singularity \ref{eq:4.1}).

**Remark 4.3.** The cleaning computation above shows that \ref{eq:4.1} can be written in the following way with minimal choice of $n_{\ell}$:

$$\Delta_k \left( z, w^{h_1/k} x_1, w^{n_2+h_2/k} x_2, \ldots, w^{n_{k-1}+h_{k-1}/k} x_{k-1} \right) = 0,$$

where \{$h_1, \ldots, h_{k-1}$} = \{1, \ldots, k - 1\} and each $n_{\ell} = \min\{m \in \mathbb{N} : m + h_{\ell}/k > n_{\ell-1} + h_{\ell-1}/k\}$, $n_1 = 0$.

**Example 4.4.** The equation

$$\Delta_3(z, w^{2/3} x_1, w^{4/3} x_2) = 0$$

is in (minimal) pre-circulant normal form, but not in circulant normal form.

Now, beginning with pre-circulant normal form \ref{eq:4.1}, we can blow up to reduce each $n_{\ell}$ to 0, as described in the introduction to this section. For later reference, we write the argument as the following remark.

**Remark 4.5.** To make the blowings-up described in a global way, we take advantage of the normal form \ref{eq:4.1} to introduce a small trick or *astuce* that will be repeated in the proof of Proposition 5.11 as well as in $\S 6.2$ (B)(II) and $\S 6.3$. We first make a single blowing-up $\sigma$ with centre \{0\}. This blowing-up does not change \ref{eq:4.1}, which
is transformed to the same equation in the \( w \)-chart of the blowing-up \( \sigma \), given by substituting \((w, wx, wz)\) for the original variables \((w, x, z)\).

But \( \{w = 0\} \) is now the exceptional divisor \( D_1 \) of \( \sigma \), and, in the new coordinates \((w, x, z)\) (after the substitution above), the centre of blowing up \( \{z = w = 0, x_j = 0, \text{ for all } j \text{ where } n_j = 0\} \), needed to decrease all positive \( n_j \), for example, extends to a global smooth subvariety of \( D_1 \); more precisely, the blowing-up in the \( w \)-chart extends to a global admissible blowing-up which can be described in an explicit way in every coordinate chart of \( \sigma \). We can continue, to decrease each \( n_j \) to 0.

Remark 4.6. It is not necessary to assume that \( a_1 = 0 \) in the proof for the irreducible case above (i.e., in the hypotheses of Theorem 1.13); the Tschirnhausen transformation will appear naturally in the construction (see (4.5) below). This is, in fact, more convenient in the general case following.

**General case.** Consider \( f = f_1 \cdots f_s \) as in the statement of the theorem. By Lemma 3.5(2) and Remark 3.6, for each \( i = 1, \ldots, s \), we can write

\[
f_i(v^{k_i}, x, z) = \prod_{\ell=0}^{k_i-1} (z + b_i(\varepsilon_{i\ell}^v, v, x)),
\]

where \( b_i(v, x) \in \mathbb{C}[v, x] \) and \( \varepsilon_i = e^{2\pi i/k_i} \).

For each \( i = 1, \ldots, s \) and \( \ell = 0, \ldots, k_i - 1 \), set

\[
Y_{i\ell} := z + b_i(\varepsilon_{i\ell}^v, v, x),
\]

and define \( X_{i\ell} \) as in (2.7); i.e.,

\[
X_{i\ell} = \frac{1}{k_i} \sum_{j=0}^{k_i-1} \varepsilon_{i\ell}^{(k_i-j)} Y_{ij}.
\]

Then, for each \( i \),

\[
X_{i0} = z + \frac{1}{k_i} a_{i1}(v^{k_i}, x),
\]

\[
X_{i\ell} = \frac{1}{k_i} \sum_{j=0}^{k_i-1} \varepsilon_{i\ell}^{(k_i-j)} b_i(\varepsilon_{i\ell}^v, v, x), \quad \ell = 1, \ldots, k_i - 1;
\]

hence,

\[
X_{i0}(w, x, z) = z + \frac{1}{k_i} a_{i1}(w, x),
\]

\[
X_{i\ell}(w, x) = w^{m_{i\ell} + \ell/k_i} \zeta_{i\ell}(w, x), \quad \ell = 1, \ldots, k_i - 1,
\]

where each \( m_{i\ell} \) is a nonnegative integer and each \( \zeta_{i\ell}(w, x) \in \mathbb{C}[w, x] \) is not divisible by \( w \).

We can use \( \{X_{i0}(w, x, z) = 0\} \), for any choice of \( i \), as the first maximal contact subspace \( N^1 \); let us take

\[
N^1 := \{X_{i0} = 0\} = \{z + \frac{1}{k_i} a_{i1}(w, x) = 0\}.
\]
The coefficient ideal of \((f, k)\) is equivalent to the marked ideal on \(N^1\) given by (the restriction to \(N^1\) of)
\[
\sum_{i=1}^{s} \left( \begin{pmatrix} X_{id}^{k_i} \\ 0 \leq \ell \leq k_i - 1 \end{pmatrix}, k_i \right)
\]
(sum of marked ideals; see [12 3.3], [6 Definition A.8]), or by
\[
\mathcal{I}_1 := \left( \begin{pmatrix} X_{K/k}^{k_i} \\ 0 \leq \ell \leq k_i - 1 \end{pmatrix}, K \right),
\]
where \(K\) denotes the least common multiple (or any given common multiple) of \(k_1, \ldots, k_q\).

We carry out the construction as in the irreducible case, for each \(j = 1, \ldots, k - 1\). For example, the monomial part of the coefficient ideal \(\mathcal{I}_1\) is generated by a monomial \(y_i K^{\alpha_1/k_i}\), where
\[
\alpha_1 = k_i m_i + \ell_1,
\]
as in [4,3]; etc. The construction involves successive nonnegative integers \(m_i, \ell_j\), each associated to some \(X_{id} = X_{ij,\ell_j}\) (except for \(i = 1, \ell = 0\), as above).

We can now apply cleaning blow-ups as above, to successively reduce each \(m_i, \ell_j\), to 0. We can then introduce new formal (or étale) coordinates \(y_{i0} := X_0\) and \(y_{i\ell}\), \((i, \ell) \neq (1, 0)\), as above. The effect is to reduce each \(f_i(w, x, z)\) to a pre-circulant singularity
\[
\Delta_{k_i} \left( y_{i0}, w^{n_{i1} + 1/k_i}; y_{i1}, \ldots, w^{n_{i,k_i - 1} + (k_i - 1)/k_i}; y_{i,k_i - 1} \right);
\]
i.e., to reduce \(f(w, x, z)\) to the product of pre-circulant singularities
\[
\prod_{i=1}^{s} \Delta_{k_i} \left( y_{i0}, w^{n_{i1} + 1/k_i}; y_{i1}, \ldots, w^{n_{i,k_i - 1} + (k_i - 1)/k_i}; y_{i,k_i - 1} \right).
\]

We can now proceed to reduce each \(n_{ij}\) to 0 by global admissible blowings-up, as in Remark 4.3. This completes the proof of Theorem 4.1.

\[\square\]

4.2. Recall on the desingularization invariant. Let \(X \hookrightarrow Z\) denote an embedded variety (\(Z\) smooth), and let \(E \subset Z\) denote an snc divisor.

Let \(\sigma : Z' \to Z\) denote a blowing-up with centre \(C\). We say that \(\sigma\) is admissible (for \((X, E)\)) if \(C\) is smooth and snc with respect to \(E\), and the Hilbert-Samuel function \(H_{X,C}\) is locally constant (as a function of \(x\)) on \(C\). In the case that \(X\) is a hypersurface (i.e., \(\dim X = n - 1\), where \(n = \dim Z\)), the latter property is equivalent to the condition that the order \(\text{ord}_z X\) is locally constant on \(C\).

Given a sequence of admissible blowings-up
\[
(4.6) \quad Z = Z_0 \leftarrow \cdots \leftarrow Z_1 \leftarrow \cdots \leftarrow Z_t,
\]
we consider successive transforms \((X_q, E_q)\) of \((X_0, E_0) := (X, E)\): for each \(q\), \(X_{q+1}\) denotes the strict transform of \(X_q\) by \(\sigma_{q+1}\), and \(E_{q+1}\) denotes the divisor whose components are the strict transforms of all components of \(E_q\), together with the exceptional divisor \(\sigma_{q+1}^{-1}(C_q)\) of \(\sigma_{q+1}\). We sometimes also call \(E_q\) the exceptional divisor, in a given year \(q\).

The desingularization invariant \(\text{inv}\) is defined step-by-step over a sequence of blowings-up \((4.6)\), where each successive blowing-up is inv-admissible (meaning that \(\text{inv}\) is locally constant on the centre of each blowing-up \(\sigma_{q+1}\)).
Assume that $X$ is an embedded hypersurface; i.e., $X \hookrightarrow Z$, where $\dim Z = \dim X + 1$.

Let $a \in X_0$. Then $\text{inv}(a)$ has the form
\begin{equation}
\text{inv}(a) = (\nu_1(a), s_1(a), \ldots, \nu_t(a), s_t(a), \nu_{p+1}(a)),
\end{equation}
where each $\nu_j(a)$ is a positive rational number (residual order) if $j \leq p$, each $s_j(a)$ is a nonnegative integer (which counts certain components of $E_q$), and $\nu_{p+1}(a)$ is either 0 or $\infty$. The successive pairs $(\nu_j(a), s_j(a))$ can be defined iteratively over maximal contact subvarieties of increasing codimension.

For example, in year zero (i.e., if $q = 0$), then $\nu_1(a) = \text{ord}_a X$ and $s_1(a) = \# E(a)$ (the number of components of $E$ at $a$).

The invariant $\text{inv}$ is upper-semicontinuous on each $X_q$ (where sequences of the form \((4.7)\) are ordered lexicographically), and infinitesimally upper-semicontinuous in the sense that $\text{inv}$ can only decrease after blowing up with $\text{inv}$-admissible centre.

For any positive integer $j$, let $\text{inv}_j$ denote the truncation of $\text{inv}$ after the $j$th pair $(\nu_j, s_j)$; i.e., $\text{inv}_j(a) = \text{inv}(a)_j$, where the latter means $\text{inv}(a)$ truncated after the pair $(\nu_j(a), s_j(a))$ (and $\text{inv}_j(a) := \text{inv}(a)$ if $j > p$).

The truncated invariant $\text{inv}_j$ can, in fact, be defined step-by-step over a sequence of blowings-up \((4.6)\), where each successive blowing-up is $\text{inv}_j$-admissible. Moreover, $\text{inv}_j$ is upper-semicontinuous on each $X_q$ and infinitesimally upper-semicontinuous. Blowings-up that are admissible for $\text{inv}_j$ are not necessarily admissible for $\text{inv}$, but they are admissible for $X$ in the sense of Definition \(1.6\) or for $(X, E)$ in the more general sense of Section \(6\).

Remark 4.7. If $I$ denotes the maximum value of $\text{inv}$ on a given open set $U$ in $Z_q$, then the truncation $I_j$ is the maximum value of the truncated invariant $\text{inv}_j$ on $U$ (because of the lexicographic order).

Remark 4.8. As stated above, $\text{inv}$ is defined recursively over a sequence of $\text{inv}$-admissible blowings-up; more precisely, $\text{inv}$ in year $q$ depends on the previous blowings-up $\sigma_1, \ldots, \sigma_q$. This memory, or dependence on the previous history, is encoded by the $s_j$ entries in $\text{inv}$, which count the number of components of $E_q$ in certain subblocks of the latter.

In articles on the desingularization algorithm, the notation $\text{inv} = \text{inv}_{X,E}$ is used for $\text{inv}$ as defined recursively over the particular sequence of $\text{inv}$-admissible blowings-up used in the desingularization algorithm, where the data in any year $q$ depends ultimately only on the year zero data $(X_0, E_0) = (X, E)$.

4.3. Cleaning algorithm. An algorithm for resolution of singularities as in \(4\), \(5\) \S5 involves factoring an ideal into its monomial part $\mathcal{M}$, generated locally by a monomial in components of the exceptional divisor, and residual part $\mathcal{R}$, divisible by no such component; the residual ideal is resolved first, to reduce to the monomial case where there is a simple combinatorial version of resolution of singularities.

The cleaning algorithm following reverses this process in a certain sense, resolving the monomial part directly to obtain a simpler clean ideal or singularity.

Proof of Theorem \(1.16\). By Theorem \(1.13\) after blowing up the discrete set of non-nc($k$) points of $S$ finitely many times, we can assume that the ideal of $X$ is generated locally at each non-nc($k$) point of $S$ by a function \(1.7\) satisfying the conclusion of Theorem \(1.13\).
We will show that the conclusion of Theorem 1.16 holds after a finite sequence of cleaning blow-ups of \( U \), followed by the additional blowings-up of Remark 4.5.

Take \( j < k \). Let \( T_j \) denote the locus \( \{ \text{inv}_j = \text{inv}(\text{nc}(k)) \} \) of points \( a \in U \) where \( \text{inv}_j(a) = \text{inv}(\text{nc}(k)) \). Then \( T_j \) is a closed subset of \( U \), by Remark 4.7. Following the proof of the desingularization theorem in [5], [6, Appendix], \( T_j \) is locally the cosupport of a marked ideal \( \mathcal{C}^j = (\mathcal{D}^j, d^j) \) on a maximal contact subvariety \( N^j \) of codimension \( j \) in \( U \). (Here \( \mathcal{D}' \) is an ideal in \( O_{N^j} \), \( d^j \) is a positive integer, and \( \text{cosupp} \mathcal{C}^j := \{ a \in U : \text{ord}_a \mathcal{D}^j \geq d^j \} \). See the preceding references for a detailed exposition of all these notions.)

Let \( \mathcal{D}^j = \mathcal{M}(\mathcal{C}^j) \cdot \mathcal{R}(\mathcal{C}^j) \) denote the factorization of \( \mathcal{D}^j \) into its monomial and residual parts: the monomial part \( \mathcal{M}(\mathcal{C}^j) \) is an ideal generated locally by a monomial in components of the exceptional divisor transverse to \( N^j \), and the residual ideal \( \mathcal{R}(\mathcal{C}^j) \) is divisible by no such component. Let \( \mathcal{M}(\mathcal{C}^j) \) denote the marked ideal \( (\mathcal{M}(\mathcal{C}^j), d^j) \). The \( \text{cosupp} \mathcal{M}(\mathcal{C}^j) \subset \text{cosupp} \mathcal{C}^j \), and any sequence of blowings-up that is admissible for the marked ideal \( \mathcal{M}(\mathcal{C}^j) \) is also admissible for \( \mathcal{C}^j \) (where a blowing-up is admissible for a marked ideal if its centre lies in the cosupport and is snc with the exceptional divisor).

The exponents (each divided by \( d_j \)) of a local monomial generator of \( \mathcal{M}(\mathcal{C}^j) \) are invariants of \( (X, E) \) (in particular, independent of the choice of a local maximal contact subvariety), by [5, Thm 6.2]. By combinatorial resolution of singularities in the monomial case, there is an invariantly defined inv\(_j\)-admissible sequence of blowings-up, after which \( \text{cosupp} \mathcal{M}(\mathcal{C}^j) = \emptyset \).

We call the blowings-up in such a sequence cleaning blow-ups. The centres of these cleaning blow-ups are invariably defined closed subsets of \( \{ \text{inv}_j = \text{inv}(\text{nc}(k)) \} \).

To complete the proof of Theorem 1.16 we apply the preceding cleaning algorithm successively, for each \( j = k - 1, \ldots, 1 \), and afterwards make the additional blowings-up described in Remark 4.5. The cleaning blow-ups involved coincide with those described locally in the proof of Theorem 1.11. So Theorem 1.16 follows from Theorem 1.11.

\textbf{Definition 4.9.} We say that the coefficient ideal \( \mathcal{C}^j \) above is clean at a point \( a \) if \( a \notin \text{cosupp} \mathcal{M}(\mathcal{C}^j) \). We say also that \( X \) is clean at \( a \) if \( \mathcal{C}^j \) is clean at \( a \), for \( j = 1, \ldots, k - 1 \).

\textbf{Remark 4.10.} Normal crossings singularities, for example, are clean, and circulant singularities are clean, according to the proof of Theorem 1.11.

Theorem 1.16 will be applied in Section 6 to an open set \( U \), where \( U \) is the complement of a closed algebraic (or analytic) set \( \Sigma \), and \( X \) is clean in a neighbourhood of \( \Sigma \) in \( U \). In this situation, the centres of the blowings-up involved in Theorem 1.16 are smooth closed subsets of \( X \) containing no clean points.

\textbf{Remark 4.11.} The transform \( \mathcal{M}(\mathcal{C}^j)' \) of \( \mathcal{M}(\mathcal{C}^j) \) by a cleaning blow-up does not, in general, coincide with the monomial part \( \mathcal{M}(\mathcal{C}^j)' \) of the transform of \( \mathcal{C}^j \) because the exceptional divisor may factor from the pull-back of \( \mathcal{R}(\mathcal{C}^j) \). So monomial desingularization of \( \mathcal{M}(\mathcal{C}^j) \) does not guarantee that \( \mathcal{C}^j \) becomes clean.

The cleaning lemma [6, Lemma 2.1] provides simple sufficient conditions (which are satisfied in Theorem 1.16) for desingularization of \( \mathcal{M}(\mathcal{C}^j) \) to lead to a clean ideal \( \mathcal{C}^j \). We do not need the cleaning lemma in the proof of Theorem 1.16 because the explicit local computation in the proof of Theorem 1.11 shows that all \( \mathcal{C}^j \) become clean. So we do not go into the details of the preceding paragraph.
5. Limits of triple normal crossings

A proof of our main Conjecture 1.5 following the approach of this article requires an analogue of Theorem 1.16 for any $k < n$. We provide this here for $k \leq 3$; see Theorem 5.1 following. The analogue of Theorem 5.1 in the case $k = 2$ is much simpler, and is also proved in [6]. In particular, in the case that $n = 5$, these results together with Theorem 1.16 give the required analogue of the latter for any $k < n$. We will use this in Section 6 to prove Theorems 1.7 and 1.8.

**Theorem 5.1.** Consider an embedded hypersurface $X \hookrightarrow Z$. Let $U$ denote an open subset of $Z$. Assume that (after an inv-admissible sequence of blowings-up) the maximum value of inv on $U$ is $\text{inv}(\text{nc}(3)) = (3, 0, 1, 0, 1, 0, \infty)$, so that the stratum $S := \{ \text{inv} = \text{inv}(\text{nc}(3)) \}$ is a smooth subvariety of dimension $n - 3$ in $U$, where $n = \dim Z$. Suppose $X$ is generically nc(3) on $S$. Then there is a finite sequence of inv$_1$-admissible blowings-up of $U$, preserving the nc(3)-locus, after which $X$ is a product of circulant singularities at every point of the strict transform of $S$.

**Remark 5.2.** Under the hypotheses of Theorem 5.1, the non-nc(3) points of $X$ in $S$ form a proper closed algebraic (or analytic) subset of $S$ (by Lemma 3.7). It follows from resolution of singularities of this subset that, after a finite number of inv-admissible blowings-up, we can assume that every non-nc(3) point $a$ of $S$ has an étale (or analytic) neighbourhood in $Z$ with coordinates $(w_1, w_2, x_1, x_2, z)$ in which $\{ w_i = 0 \}$, $i = 1, \ldots, r$, are the components of $E$ at $a = 0$, $S = \{ z = x = 0 \}$, $X$ is nc(3) on $S \setminus \{ w_1 \cdots w_r = 0 \}$, and the ideal of $X$ is generated by a function

$$(5.1) \quad f(w, u, x, z) = z^3 - 3B(w, u, x)z + C(w, u, x),$$

where the coefficients $B, C$ are regular (or analytic) functions, $f$ is in the ideal generated by $x_1, x_2, z$, and $f$ splits formally (into three factors of order 1) at every point where $z = x = 0$ and $w_1 \cdots w_r \neq 0$.

Theorem 5.1 then follows from Proposition 5.3 below, which is an analogue of Theorem 4.1. Proposition 5.3 will be stated using local hypotheses as in Theorem 4.1 with globalization via the cleaning algorithm, as in [4.3] and the analogue of Remark 4.5.

**5.1. Splitting.**

**Proposition 5.3.** Let $f$ denote a function as in (5.1) (satisfying the conditions given in Remark 5.2). Assume, moreover, that

$$(5.2) \quad \text{inv}(0) = \text{inv}(\text{nc}(3)) = (3, 0, 1, 0, 1, 0, \infty),$$

and that $\{w_1 \cdots w_r = 0\}$ is the exceptional divisor. Then, after a finite number of blowings-up with (inv-admissible) centres of the form $\{ z = x = w_j = 0 \}$, for some $j$, we can assume that $f(v^0_1, \ldots, v^0_n, u, x, z)$ splits.

**Remark 5.4.** More precisely, there is a finite sequence of admissible blowings-up of $U$, as indicated, after which the conclusion of Proposition 5.3 holds at every point of $S \cap E$, where $f$ is a local generator of the ideal of $X \subset Z$: this is a consequence of the fact that the centres of blowing up involved in Lemma 3.9 (as used in the proof following) are each given by the intersection of $S$ with a component of the exceptional divisor.
Remark 5.5. The proof of Proposition 5.3 uses only \( \text{inv}(0) = (3, 0, 1, \ldots) \), instead of all the information given by (5.2).

Remark 5.6. In Section 6, we will use a version of Proposition 5.3 for a product of \( f \) as above with generators \( y_1, \ldots, y_r \) of the ideals of the components of \( E \) at \( a \). See Remarks 3.10 and 4.2. In this context, splitting as in Proposition 5.3 holds after a finite number of inv-admissible blowings-up with centres of the form \( \{ z = x = y = w_j = 0 \} \).

It is not difficult to extend Proposition 5.3 to a product of functions \( f \), each of order \( \leq 3 \); in this article, we will need only the preceding situation, or the simpler version for \( f \) of order 2 and the components of \( E \).

Proof of Proposition 5.3. The discriminant \( D \) of \( f \) is given by

\[
D = -\frac{1}{27} \left( C^2 - 4B^3 \right).
\]

By Lemma 3.9 after a finite number of blowings-up with centres of the form \( \{ z = x = w_j = 0 \} \), for some \( j \), we can assume that \( \Delta(v_1^2, \ldots, v_r^2, u, x) \) is a square. The result follows from Lemma 5.7 below.

The following two lemmas are essentially Lemmas 3.3 and 3.6 in [3]. We will use the fact that the first coefficient (marked) ideal of the marked ideal \((f, 3)\) is

\[
I := ((B^3, C^2), 6) = ((C^2, D), 6).
\]

Since \( \text{inv}(a) = (3, 0, 1, \ldots) \), we have \( I = w^\gamma \tilde{I} \), where \( w^\gamma \) is a monomial and \( \tilde{I} \) has order 6 at 0.

Set

\[
R := \mathbb{C}[w, u, x], \quad S := \overline{\mathbb{C}[[w]]}[u, x].
\]

Then \( f \) splits in \( S[z] \); say,

\[
f = (z + b_0)(z + b_1)(z + b_2).
\]

Moreover, each \( b_j \) belongs to the ideal \((x)\). Define

\[
\eta_i := \frac{1}{3} \sum_{j=0}^{2} \varepsilon^{ij}(z + b_j), \quad i = 0, 1, 2,
\]

where \( \varepsilon = e^{2\pi i/3} \). Then \( \eta_0 = z \) and

\[
(5.3) \quad f = \prod_{i=0}^{2} (z + \varepsilon^i \eta_1 + \varepsilon^{2i} \eta_2)
\]

in \( S[z] \). In particular,

\[
B = \eta_1 \eta_2, \quad C = \eta_1^3 + \eta_2^3, \quad D = -\frac{1}{27} (\eta_1^3 - \eta_2^3)^2
\]

in \( S \).

Lemma 5.7. Assume that \( D \) is a square in \( R \). Then \( f(v_1^3, \ldots, v_r^3, u, x, z) \) splits.
Proof. Write \( D = A^2 \in R \); we can take \( A = \eta_1^3 - \eta_2^3 \). Recall that \( I = (B^3, C^2) = (D, C^2) = w^\gamma \tilde{I} \), as above. Then \( w^\gamma \) is the monomial in \( w \) of largest exponent which factors from both \( A^2, C^2 \). Therefore, each \( \gamma_k \) is even; say \( \gamma = 2\alpha \).

We have \( 4B^3 = (C - A)(C + A) \).

We claim that \( w^{-\alpha}C \) and \( w^{-\alpha}A \) are relatively prime in \( R \). Indeed, it is easy to check they are relatively prime in \( S \) since \( A = \eta_1^3 - \eta_2^3 \), \( C = \eta_1^3 + \eta_2^3 \), and the ideal \( (\eta_1, \eta_2) = (x_1, x_2) \) in \( S \). Since \( \tilde{I} \) has order 6, either \( \text{ord } w^{-\gamma}D = \text{ord } w^{-\gamma}D \) or \( \text{ord } w^{-\gamma}C^2 = \text{ord } w^{-\gamma}C^2 \). In either case, we can use Lemma 5.8 following to conclude that \( w^{-\alpha}C, w^{-\alpha}A \) are relatively prime in \( R \).

Therefore, \( w^{-\delta}(C - A) = 2w^{-\delta}\eta_2^3 \) and \( w^{-\delta}(C + A) = 2w^{-\delta}\eta_1^3 \) are relatively prime in \( R \), where \( \delta \) denotes the largest exponent of a monomial in \( w \) that divides \( C - A \) and \( C + A \). Moreover, the product \( C^2 - A^2 = 4w^{-2\delta}B^3 \) is a cube times a monomial \( w \) in \( R \). Hence both \( \eta_1^3 \) and \( \eta_2^3 \) are cubes (times monomials in \( w \)) in \( R \).

By (5.3), \( f(v_1^3, \ldots, v_r^3, u, x, z) \) splits in \( \mathbb{C}[v, u, x][z] \) and the result follows. \( \square \)

Lemma 5.8. Let \( G \in R. \) Suppose that \( \text{ord } G = \text{ord } G \). Let \( \theta \in R \) be a nonunit which divides \( G \). Then \( \theta \) is also a nonunit in \( S \).

Proof. Consider a decomposition of \( G \) into irreducible factors in \( R, G = \prod \theta_i^{m_i} \), where the \( m_i \) are positive integers. For all \( i, \text{ord } \theta_i = \text{ord } \theta_i \). By the hypothesis, \( \sum m_i \text{ord } \theta_i = \sum m_i \text{ord } \theta_i \). Therefore, \( \text{ord } \theta_i = \text{ord } \theta_i \), for all \( i \). The result follows. \( \square \)

5.2. Splitting exponents. Consider \( f(w, u, x, z) \) as in (5.1). Assume that \( f \) is irreducible and that \( f(v_1^p, \ldots, v_r^p, u, x, z) \) splits, for some \( p \). Let \( S_3 \) denote the group of permutations of the the roots of \( f(v_1^p, \ldots, v_r^p, u, x, z) \). Then \( (\\mathbb{Z}_p)^r \) maps onto a subgroup of \( S_3 \) which acts transitively on the roots (since \( f \) is irreducible). It follows (as in the proof of Lemma 5.5(2)) that \( f(v_1^p, \ldots, v_r^p, u, x, z) \) splits, where, for each \( i = 1, \ldots, r, q_i \leq 3 \) and the group \( \mathbb{Z}_q^{(i)} := \{1\}^{r-1} \times \mathbb{Z}_{q_i} \times \{1\}^{r-i} \) maps onto a cyclic subgroup \( \mathbb{Z}_{q_i} \) of \( S_3 \); moreover, \( \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_r} \) maps onto a subgroup \( G \) of \( S_3 \) that acts transitively on the roots of \( f(v_1^{q_1}, \ldots, v_r^{q_r}, u, x, z) \).

Lemma 5.9. Suppose \( f(w, u, x, z) \) is irreducible and \( f(v_1^{q_1}, \ldots, v_r^{q_r}, u, x, z) \) splits, where \( q_1, \ldots, q_r \) are chosen as above. Then \( q_i = 1 \) or \( q_i = 3 \), for each \( i = 1, \ldots, r \).

Proof. For each \( i \), the translates of the subgroup \( \mathbb{Z}_{q_i} \) of \( S_3 \) above by the elements of \( G \) provide a partition of the set of roots of \( f(v_1^{q_i}, \ldots, v_r^{q_i}, u, x, z) \) into subsets of \( q_i \) elements; therefore, \( q_i \) divides 3. \( \square \)

Example 5.10. Consider the splitting

\[
z^2 - w_1 w_2 x^2 = \left( z - w_1^{1/2} w_2^{1/2} x \right) \cdot \left( z + w_1^{1/2} w_2^{1/2} x \right).
\]

Both \( q_1 = 2 \) and \( q_2 = 2 \) are needed for a splitting, although each \( \mathbb{Z}_q^{(i)} \) maps onto the cyclic group \( \mathbb{Z}_2 = S_2 \) (which acts transitively on the roots).

5.3. Circulant normal form. Theorem 5.1 is a consequence of the following result, which we prove in this subsection.

Proposition 5.11. Assume that (after an inv-admissible sequence of blowings-up) \( X \subset Z \) is defined locally at a given point by a function

\[
f(w, u, x, z) = f(w_1, \ldots, w_r, u_1, \ldots, u_q, x_1, x_2, z)
\]
as in (5.11), where \( f \) is \( nc(3) \) on \( \{ z = x = 0, w_1 \cdots w_r \neq 0 \} \), \( \{ w_1 \cdots w_r = 0 \} \) is the exceptional divisor, and \( \text{inv}(0) = \text{inv}(nc(3)) \). Then there is a finite sequence of \( \text{inv}_1 \)-admissible blowings-up that are isomorphisms over the \( nc \) locus, after which the only non-\( nc(3) \) singularities in the stratum \( S \) given by the closure of the \( nc(3) \) points are products of circulant singularities.

In particular, if \( f \) is irreducible, then we reduce to the case that the only non-
\( nc(3) \) singularities in \( S \) are circulant singularities \( \Delta_{3}(z, w_1^{1/3}y_1, w_2^{1/3}y_2) \), for some \( i = 1, \ldots, r \).

Remark 5.12. There is again a more general statement involving products of circulant singularities as in Proposition 5.11 with generators of the ideals of the component of a simple normal crossings divisor \( E \). See Remarks 4.2 and 5.6.

Proof. We will prove the result for \( f \) irreducible, and make a remark at the end about the general case.

Irreducible case. We follow the outline of the proof of Theorem 4.1. By Lemma 5.9 (and Remark 5.4), we can assume that the function \( f \rangle = \) irreducible, and make a remark at the end about the general case.

For each \( \ell \), \( v_1^{3-\ell}X_\ell \) is invariant under the action of the group \( \mathbb{Z}_3 \) of cube roots of unity, induced by \( (v, t, u, x) \mapsto (\varepsilon v_1, v_2, \ldots, v_s, t, u, x) \), so that

\[
Y_\ell := z + b(\varepsilon^\ell v_1, v_2, \ldots, v_s, t, u, x), \quad \ell = 0, 1, 2,
\]

and define \( X_0, X_1, X_2 \) by (2.7); i.e.,

\[
X_0 = \frac{1}{3} \sum_{j=0}^{2} Y_j = z,
\]

\[
X_\ell = \frac{1}{3} \sum_{j=0}^{2} \varepsilon^{3(3-j)} Y_j = \frac{1}{3} \sum_{j=0}^{2} \varepsilon^{3(3-j)} b(\varepsilon^j v_1, v_2, \ldots, v_s, t, u, x), \quad \ell = 1, 2.
\]

For each \( \ell = 1, 2 \), \( v_1^{3-\ell}X_\ell \) is invariant under the action of the group \( \mathbb{Z}_3 \) of cube roots of unity, induced by \( (v, t, u, x) \mapsto (\varepsilon v_1, v_2, \ldots, v_s, t, u, x) \), so that

\[
v_1^{3-\ell}X_\ell = \eta_\ell(v_1, v_2, \ldots, v_s, t, u, x),
\]

where \( \eta_\ell(w_1, v_2, \ldots, v_s, t, u, x) \in \mathbb{C}[w_1, v_2, \ldots, v_s, t, u, x] \), \( \ell = 1, 2 \). Since each \( \eta_\ell \)

must, therefore, be divisible by \( v_1^3 \), we can write

\[
X_\ell = v_1^{3m_1+\ell} \zeta_\ell(v_1, v_2, \ldots, v_s, t, u, x) = w_1^{m_1+\ell/3} \zeta_\ell(1)(w_1, v_2, \ldots, v_s, t, u, x),
\]

where \( m_1 \) is a nonnegative integer and \( \zeta_\ell(1)(w_1, v_2, \ldots, v_s, t, u, x) \in \mathbb{C}[w_1, v_2, \ldots, v_s, t, u, x] \) is not divisible by \( w_1, \ell = 1, 2 \).

Likewise, the roots of \( f(v_1^3, v_2^3, v_3^3, \ldots, v_s^3, t, u, x, z) = 0 \) are permuted by the action of \( \mathbb{Z}_3 \) induced by \( (v, t, u, x) \mapsto (\varepsilon v_1, \varepsilon v_2, \varepsilon v_3, \ldots, v_s, t, u, x) \), and it follows that \( X_\ell \) can be written

\[
X_\ell = w_1^{m_1+\ell/3} \zeta_\ell(2)(w_1, w_2, v_3, \ldots, v_s, t, u, x), \quad \ell = 1, 2,
\]

where \( \zeta_\ell(2) \) is divisible by neither \( w_1 \) nor \( w_2 \), and \( \{q_21, q_22\} = \{1, 2\} \).

We repeat this process for \( w_3, \ldots, w_s \), and conclude that

\[
X_\ell = w_1^{m_1+\ell/3} \zeta_\ell(3)(w, u, x) = w_1^{m_1+\ell/3} \zeta_\ell(w, u, x), \quad \ell = 1, 2,
\]
where \( t^{nt} \) is a monomial in \( t = (w_{s+1}, \ldots, w_r) \) (with integral exponents), \( \zeta_\ell \) is divisible by no \( w_i \), \( i = 1, \ldots, r \), and each \( \{q_{11}, q_{12}\} = \{1, 2\} \).

As in the proof of Theorem 4.1, the coefficient ideal of the marked ideal \((f, 3)\) is equivalent to the marked ideal
\[
\mathfrak{C}^1 := \left( \left( u_1^{3m_1+1} w_2^{3m_2+q_{21}} \cdots w_s^{3m_s+q_{12}} t^{3n_1},
\quad u_1^{3m_1+2} w_2^{3m_2+q_{22}} \cdots w_s^{3m_s+q_{12}} t^{3n_1} \zeta_1^3 \right) \right)
\]
on the maximal contact subspace \( N^1 := \{ z = 0 \} \). Since \( \text{inv}(0) = (3, 0, 1, \ldots) \), it follows that the exponent \( r \)-tuple of one of the two monomials in \( w \) in \( \mathfrak{C}^1 \) (which we denote \( w^{1} \)) is less than the other (denoted \( w^{1+} \)), and the \( \zeta_\ell \) corresponding to the first (say, \( \zeta_{\ell_1} \), where \( \ell_1 = 1 \) or \( 2 \)) has order 1.

We can then apply a cleaning procedure in the proof of Theorem 4.1. We first blow up with combinatorial centres \( \{w_{i_1} = \cdots = w_{i_p} = 0\} \), where \( p \leq 3 \), in the maximal contact subspace \( N^2 := \{ z = \zeta_{\ell_1} = 0 \} \) to reduce to \( \delta = (\delta_1, \ldots, \delta_s, 0, \ldots, 0) \) with \( |\delta| = \delta_1 + \cdots + \delta_s < 3 \). Note that \( |\delta| < 3 \) implies that \( w^{\delta} \) depends on at most two variables \( w_i \). Moreover, using the fact that each \( \{q_{11}, q_{12} = \{1, 2\} \) above, it is easy to see this implies also that we have modified our expression for \( \mathfrak{C}^1 \) in such a way that now \( s \leq 2 \).

For the second cleaning step, we can now blow up with codimension one centres \( \{w_i = 0\} \) in \( N^1 := \{ z = 0 \} \) (which preserve \( |\delta| < 3 \)) to get also \( \gamma = (\gamma_1, \ldots, \gamma_s, 0, \ldots, 0) \), where \( s \leq 2 \) and \( |\gamma| < 3 \). Set \( \alpha := \gamma/3, \beta := \delta/3 \).

We conclude that, after cleaning, \( f \) can be written as
\[
\Delta_3 \left( z, w^\alpha y_1, w^{\alpha+\beta} y_2 \right),
\]
where \( y_1, y_2 \) are suitable étale (or analytic) coordinates (as in the proof of Theorem 4.1), \( w^\alpha \) and \( w^{\beta} \) are each monomials in \( w_1^{1/3}, \ldots, w_s^{1/3} \) of order < 1, and
\[
\Delta_3 \left( z, w^\alpha y_1, w^{\alpha+\beta} y_2 \right) = z^3 + w^{2\alpha} y_1^3 + w^{3(\alpha+\beta)} y_2^3 - 3w^{2\alpha+\beta} y_1 y_2 z
\]
is a polynomial in \( (w, y, z) \); i.e., \( w^{2\alpha+\beta} \) has integral exponents (cf. (2.5)). The only possibilities for \( w^\alpha \), \( w^{\beta} \) satisfying these conditions are
\[
\begin{align*}
    w^\alpha &= 1, \quad w_1^{1/3}, \quad w_1^{2/3} \quad \text{or} \quad w_1^{1/3} w_2^{1/3}
\end{align*}
\]
(after reordering the \( w_i \) if necessary), and then
\[
\begin{align*}
    w^{\beta} &= 1, \quad w_1^{1/3}, \quad w_1^{2/3} \quad \text{or} \quad w_1^{1/3} w_2^{1/3} \quad \text{(respectively)}.
\end{align*}
\]

In other words, after cleaning, we reduce to four possible cases:
\[
\begin{align*}
    \Delta_3(z, y_1, y_2) : & \quad \text{nc(3)}, \\
    \Delta_3(z, w_1^{1/3} y_1, w_1^{2/3} y_2) : & \quad \text{cp(3)}, \\
    \Delta_3(z, w_1^{2/3} y_1, w_1^{4/3} y_2) \quad \text{(cf. Example 4.4).} & \\
    \Delta_3(z, w_1^{1/3} w_2^{1/3} y_1, w_1^{2/3} w_2^{2/3} y_2). &
\end{align*}
\]

In particular, we have either \( \text{cp(3)} \) or one of the pre-circulant third and fourth cases at every point of \( S \cap E \), where \( S \) denotes the closure of the \( \text{nc(3)} \)-locus. The third and fourth cases can be handled as in Remark 4.3. For both of these cases, we first blow up with centre given by the non-cp(3) points of \( S \cap E \).
In the third case, this means that (locally) we first blow up with centre \( \{ z = y_1 = y_2 = w_1 = 0 \} \) to introduce the divisor \( D_1 \). Afterwards, we blow up with centre given by \( \{ z = y_1 = w_1 = 0 \} \) in the \( w_1 \)-chart—this extends to a global smooth centre in \( D_1 \) given by a component of the intersection of \( D_1 \) with the locus of points of order 3 of \( f \) or \( X \)—and we thereby reduce to \( \text{cp}(3) \).

In the fourth case, let \( E_i \) denote the component \( \{ w_i = 0 \} \) of \( E \), \( i = 1, 2 \). The first blowing-up above means that (locally) we introduce \( \{ z = y_1 = y_2 = w_1 = w_2 = 0 \} \). Then \( \Delta_3(z, w_1^{1/3} w_2^{1/3} y_1, w_1^{2/3} w_2^{2/3} y_2) \) transforms to
\[
\Delta_3(z, w_1^{2/3} w_2^{1/3} y_1, w_1^{4/3} w_2^{2/3} y_2)
\]
in the \( w_1 \)-chart, and a symmetric expression in the \( w_2 \)-chart. We now blow up with centre given by \( \{ z = y_1 = w_1 = 0 \} \) in the \( w_1 \)-chart and by \( \{ z = y_1 = w_2 = 0 \} \) in the \( w_2 \)-chart; again this extends globally to a smooth centre given by a component of the intersection of \( D_1 \) with the order 3 locus of \( X \). (More precisely, the latter intersection is \( \{ z = w_1 = y_1 w_2 = 0 \} \) in the \( w_1 \)-chart, for example, and we are blowing up the irreducible component not contained in \( D_1 \cap E_2 \).) In the new \( w_1 \)-chart of the latter blowing-up of \( (5.4) \), we get \( \Delta_3(z, w_1^{2/3} w_2^{1/3} y_1, w_1^{1/3} w_2^{2/3} y_2) \). After a further blowing-up with centre \( \{ z = w_1 = w_2 = 0 \} \) (globally, \( X \cap D_1 \cap E_2 \)), we have only \( \text{cp}(3) \) points.

**General case.** In the case that \( f \) is not irreducible, we can also follow the proof of Theorem 4.1. The result of cleaning in this case is to reduce \( f \) already to a product of circulant singularities; i.e., to either \( \text{nc}(3) \) or \( \text{smooth} \times \text{cp}(2) \) :

\[
y_{10} \Delta_2(y_{20}, w_{1/2} y_{21}) = y_{10} (y_{20}^2 - w y_{21}^2).
\]

\( \square \)

## 6. Partial desingularization algorithm

Let \( X \hookrightarrow Z \) denote an embedded variety (\( Z \) smooth), and let \( E \subset Z \) denote an \( \text{nc} \) divisor. We say that \( (X, E) \) is normal crossings (\( \text{nc} \)) at a point \( a \) if \( X \cup E \) is \( \text{nc} \) at \( a \).

Let \( \sigma : Z' \rightarrow Z \) denote a blowing-up with centre \( C \). We say that \( \sigma \) is admissible (for \( (X, E) \)) if \( C \) is smooth and \( \text{nc} \) with respect to \( E \), and the Hilbert-Samuel function \( H_{X,x} \) is locally constant (as a function of \( x \)) on \( C \) (cf. Definitions 1.1, 1.6). In the case that \( X \) is a hypersurface (i.e., \( \dim X = n - 1 \), where \( n = \dim Z \)), the latter property is equivalent to the condition that the order \( \text{ord}_x X \) is locally constant on \( C \).

Given a sequence of admissible blowings-up
\[
Z = Z_0 \xleftarrow{\sigma_1} Z_1 \leftarrow \cdots \leftarrow \xleftarrow{\sigma_t} Z_t,
\]
we consider successive transforms \( (X_j, E_j) \) of \( (X_0, E_0) := (X, E) \), as in \( \ref{1.2} \). In a given year \( j \), it will often be convenient to drop the index \( j \) and simply write \( M, X, E \) instead of \( M_j, X_j, E_j \).

**Theorem 6.1.** Assume that \( \dim X \leq 4 \). Then there is a finite sequence of admissible blowings-up \( \ref{6.1} \) such that every \( \sigma_j \) is an isomorphism over the \( \text{nc} \) locus of \( (X_0, E_0) = (X, E) \), and \( X_j \) has smooth normalization.

Theorem 6.1 is a corollary of the following more precise result.
Theorem 6.2. Assume that \( \dim X \leq 4 \). Then there is a finite sequence of admissible blowings-up \((6.1)\) such that every \( \sigma_j \) is an isomorphism over the nc locus of \((X_0, E_0) = (X, E)\), and \((X_t, E_t)\) has only minimal singularities.

There is an analogous version of Theorem 1.8 for a pair \((X, E)\), where we preserve normal crossings singularities of \((X, E)\), i.e., of \(X \cup E\), of order at most three, in any dimension. See Theorem 6.14 below.

Normal crossings singularities and, more generally, minimal singularities, are hypersurface singularities. The class of minimal singularities denotes the class of products of circulant singularities (as given by Theorems 1.16, 4.1) and their neighbours. A neighbour of a circulant singularity means either a singularity that occurs in a small neighbourhood of the latter, or a limit of singularities in a neighbourhood which cannot be eliminated. (See §6.3.2.) There are finitely many minimal singularities (up to étale isomorphism) in Theorem 6.2.

The class of minimal singularities of \((X,E)\) means the class of minimal singularities of \(X \cup E\).

Minimal singularities for \( \dim X \leq 4 \) have smooth normalization (see §6.3.2, Remark 6.10), so that Theorem 6.1 is an immediate consequence of Theorem 6.2.

This section is devoted mainly to a proof of Theorem 6.2, though the first steps below apply to any dimension. In the case of Theorem 6.14, the entire argument follows parts of the proof of Theorem 6.2 that apply to any dimension, and we will add detail in §6.4.

In general dimension, we can reduce the theorems to the case that \( X \) is an embedded hypersurface (i.e., \( X \hookrightarrow Z \), where \( n := \dim Z = \dim X + 1 \)) using the standard desingularization algorithm. Indeed, the Hilbert-Samuel function \( H_{X,x} \) determines the local minimal embedding dimension \( e_{X,x} = H_{X,x}(1) - 1 \), so that the desingularization algorithm first eliminates points of embedding codimension > 1 without modifying nc points.

So from now on, we assume that \( X \) is an embedded hypersurface.

6.1. Invariant for a normal crossings singularity. Let \( a \in X \). Set \( p := \text{ord}_a X \) and \( r := \# E(a) \) (the number of components of \( E \) at \( a \)). We will call \((p, r)\) the order of \((X, E)\) at \( a \). The order \((p, r)\), as a function of the point \( a \), is upper-semicontinuous with respect to the lexicographic ordering of pairs \((p, r)\).

If \((X, E)\) has order \((p, r)\) and is nc at \( a \), then the desingularization invariant

\[
\text{inv}(a) = \text{inv}_{X,E}(a) = (p, r, 1, 0, \ldots, 1, 0, \infty),
\]

where there are \(p+r\) pairs (before \(\infty\)). Note that \(\text{inv}_{X,E}\) here is the desingularization invariant in year zero (before we begin blowing up; see §4.2 and also [6 §A.2]). The condition \((6.2)\) does not, in general, imply that \((X, E)\) is nc at \( a \), as explained by an example in §1.3. See [2, Thm. 3.4] for a more precise statement about the invariant at an nc point. (Note that nc is snc in an étale neighbourhood.)

Let \(\text{inv}_{p,r}\) denote the right-hand side of \((6.2)\), so that, in particular, \(\text{inv}_{p,0} = \text{inv}(\text{nc}(p))\). Given a sequence of inv-admissible blowings-up \((6.1)\) and a pair of nonnegative integers \((p, r)\), let \(S_{p,r}\) denote the \(\text{inv}_{p,r}\)-stratum in a given year \( j \); i.e., the locus of points where \(\text{inv} = \text{inv}_{p,r}\) in year \( j \).

Note that, at a point \( a \in S_{p,r} \) in a year \( j > 0 \), \(\text{ord}_a X_j = p\), but the order of \((X_j, E_j)\) may be greater than \((p, r)\) because the order of \((X_j, E_j)\) counts all components of \(E_j\) at \( a \), while \( r \) in \(\text{inv}_{p,r}\) counts only old components of \(E_j\) (see [6 Remark A.18]).
Remarks 6.3. (1) If $p + r > n = \dim Z$, then $S_{p,r} = \emptyset$. If $p + r = n$, then $S_{p,r}$ is a discrete subset of $X \cup E$.

(2) In a year $j > 0$, $(X_j, E_j)$ (or $(X, E)$, in our shorthand language above) need not be nc at a point of a stratum $S_{p,r}$ even if $(X_j, E_j)$ is generically nc on $S_{p,r}$; e.g., circulant singularities may occur. On the other hand, $(X_j, E_j)$ is nc at every point of $S_{1,r}$.

(3) Theorems 6.1 and 6.2 preserve normal crossings points of $(X, E) = (X_0, E_0)$, but not necessarily normal crossings points of $(X_j, E_j)$, $j \geq 1$.

6.2. Overview of the proof. Given $n$, let $I_n$ denote the finite lexicographic sequence of pairs $(p, r)$, where $p + r \leq n$. Our proof of Theorem 6.2 (in the case $n = 5$, say) will be presented as a recursive or iterative algorithm involving successive modification of non-nc points of the strata $S_{p,r}$, $(p, r) \in I_n$, in decreasing order; i.e., beginning with $(p, r) = (5, 0)$ and terminating with the base case $(p, r) = (1, 0)$.

To prove Conjecture 1.5 in general, we would need the corresponding argument by induction over the sequence $I_n$, for any $n$, beginning with the base case $(1, 0)$. The inductive claim can be formulated as follows.

Claim 6.4. Given $(p, r) \in I_n$, there is an admissible sequence of blowings-up $(6.1)$, satisfying the following conditions:

1. each blowing-up is an isomorphism over the locus of normal crossings points of $(X_0, E_0)$ of order at most $(p, r)$;
2. over any open subset $U$ where $(X, E)|_U$ is normal crossings, $(6.1)$ coincides with the blow-up sequence given by the desingularization algorithm, stopped when $\operatorname{inv} \leq \operatorname{inv}_{p,r}$;
3. $(X_t, E_t)$ has only minimal singularities.

We will make a concluding remark on a strategy to prove Claim 6.4 by induction on $(p, r) \in I_n$, in §6.5 below.

Claim 6.4 in the base case $(p, r) = (1, 0)$ is an immediate consequence of resolution of singularities. We will need to consider item (2) of the claim with the following caveat: In the desingularization algorithm, each blowing-up has centre given by a smooth subspace which may have several components. We allow to replace this blowing-up by the finite number of blowings-up of the components, one at a time. Of course, the resulting morphisms are the same. See §6.4.

Conjecture 1.5 follows from Claim 6.4 in the case $(p, r) = (n, 0)$, for general $n$; in this case, all blowing-up are isomorphisms over the nc locus of $(X_0, E_0)$. Theorem 6.2 covers $n \leq 5$. In §6.4 we prove the claim for general $n$ and $(p, r) = (3, 0)$; this gives Theorem 6.14.

In Theorem 6.2 as well as in Theorem 6.14 each step of the iterative procedure involves (A) an application of the standard desingularization algorithm, followed by (B) modification of the non-nc points of a stratum $S_{p,r}$ using four additional blow-up sequences (B1)–(B4) based on Sections 3, 4 and 5 above, and §§6.3, 6.4 below. We concentrate on Theorem 6.2 here, and deal with Theorem 6.14 in detail in §6.4.

6.2.1. First steps. Let $X$ denote an embedded hypersurface (i.e., $X \hookrightarrow Z$, where $Z$ is smooth and $n := \dim Z = \dim X + 1$), and let $E \subset Z$ denote an snc divisor.

The sequence $I$ begins $(n, 0), (n - 1, 1), (n - 1, 0), (n - 2, 2), \ldots$. 
To begin the iterative process, we use the standard desingularization algorithm to blow up until the maximal value of inv is (at most) \( \text{inv}_{n,0} \). Then the strata \( S_{n,0} \) and \( S_{n-1,1} \) are discrete, so we can blow up non-nc points of these strata to reduce to the case that \( (X, E) \) is nc in a neighbourhood of \( S_{n,0} \cup S_{n-1,1} \). Set \( T_{n-1,1} := S_{n,0} \cup S_{n-1,1} \), \( D_{n-1,1} := \emptyset \subset E \) and \( \Sigma_{n-1,1} := T_{n-1,1} \cup D_{n-1,1} \).

We can now apply the desingularization algorithm in the complement of \( \Sigma_{n-1,1} \), 
resetting the current year to year zero, and blowing up with inv-admissible centres 
\( S \).

Suppose the stratum \( X, E \) reduce to the case that \( (X, E) \) is nc in a neighbourhood of \( S_{n-1,0} \). Then the centres of blowing up involved are closed in \( S \).

We can blow up to eliminate any component of \( S = S_{n-1,0} \) that is not generically normal crossings of order \( n - 1 \).

Now, since the stratum \( S \) (where \( \text{inv}_{n-1,0} \) is constant) is a smooth curve, the non-nc points of \( S \) form a discrete subset, given by the intersection of \( S \) with the exceptional divisor (each non-nc point is the intersection with a single component of the exceptional divisor).

**Remark 6.5.** (B1) In the general iterative step, there is an inv-admissible sequence of blowings-up over the non-nc locus in \( S_{p,r} \), after which the non-nc locus in \( S_{p,r} \) lies in \( E' \subset E \), where \( E' \) is transverse to \( S_{p,r} \) (see Lemma 3.7 and Remark 5.2). In the case that \( S \) is a curve, (B1) is void and \( E' = E \). We proceed to the following.

(B2) **Splitting.** We apply Theorem 1.13 in \( S_{n-1,0} \cap E' \), where we continue to write \( E' \) for the appropriate transform of \( E' \) above. The blowings-up involved are again inv-admissible.

(B3) **Cleaning, to get circulant normal form.** We apply Theorem 1.16 to obtain circulant normal form in \( S_{n-1,0} \cap E' \). The blowings-up involved are inv\(_1\)-admissible. See Remark 6.6 below.

The preceding applies to any dimension.

6.2.2. **Continuation in the case \( n = 5 \).** In this case, we proceed to modify the non-nc points of \( S_{4,0} \) to determine the neighbours of the circulant singularities given by (B3) above, as follows.

(B4) **Moving away.** We make a single blowing up of any non-nc point \( a \) of \( S_{4,0} \) to introduce a distinguished component \( D_1 \) of \( E \), throughout which \( (X, E) \) is described by equations that are transformed from the circulant normal form of Theorem 1.16 at \( a \). The reason for this blowing-up is that the singularities in a neighbourhood of a circulant point cannot be eliminated, but we do not a priori have good control over the limits of the neighbouring singularities arising from the previous blowings-up. These limiting singularities will be moved away from \( D_1 \). (Compare with Remark 1.5)

More precisely, for each such non-nc point \( a \), we make a further sequence of blowings-up with centres in \( X \cap D_1 \), following 6.3 below, after which \( \text{ord} X \leq 3 \) and \( (X, E) \) is nc, in a neighbourhood of \( S_{4,0} \cup D_1 \). The neighbours of the given singularity in circulant normal form at \( a \) are the nearby nc singularities of \( (X, E) \), together with the singularities of \( (X, E) \) that live in the corresponding \( D_1 \).
Let $D$ denote the union of the divisors $D_4$ above. We define $D_{4,0}$ by adjoining $D$ to $D_{4,1}$. Let $T_{4,0}$ denote the union of (the strict transforms of) $T_{4,1}$ and $S_{4,0}$, and $\Sigma_{4,0} := T_{4,0} \cup D_{4,0}$. These objects satisfy the following properties.

1. $T_{4,0} \cap D_{4,0}$ is the set of non-nc points of $(X, E)$ in $T_{4,0}$, and $T_{4,0} \setminus D_{4,0}$ contains all nc points of order $\geq (4,0)$ of $(X_0, E_0)$.

2. $(X, E)$ has only minimal singularities in $D_{4,0}$ (in particular, they have smooth normalization).

3. There is a neighbourhood $U_{4,0}$ of $\Sigma_{4,0}$ such that $(X, E)$ is normal crossings and $X$ has order $< 4$ in $U_{4,0} \setminus \Sigma_{4,0}$.

Remark 6.6. Given $(p, r) \in I$, $p \leq n - 1$, let $(p, r)^+ = (p^+, r^+)$ denote the lexicographic successor of $(p, r)$; i.e., the smallest $(q, s) > (p, r)$ in the lexicographic order of pairs. When we apply the desingularization algorithm in the complement of $\Sigma_{p^+, r^+}$, for some $(p, r)$, as above, we first reset to year zero, and then blow up with centres given by (A), stopping when the maximal value of inv becomes $\leq \inv_{p, r}$. In this step, the blowings-up are inv-admissible for the reset desingularization invariant (see §4.2). It is important that the blowings-up involved in (B1) and (B2) are also inv-admissible because the following procedure (B3) is based on Theorems 1.10 and 1.11 which have hypotheses involving inv. On the other hand, the blowings-up involved in (B3) and (B4) are admissible (see Definition 1.6), but not necessarily inv-admissible. This is the reason that we have to reset to year zero in the next iterative step.

When we apply (B3) or (B4), or proceed to the next steps, we will continue to use the notation $S_{p, r}$ for the successive strict transforms of the latter (following our convention for the strict transforms of $X$); likewise for $T_{p, r}$ and $\Sigma_{p, r}$.

We now continue to the next step. The stratum $S_{3,2}$ is discrete, so we treat it like $S_{5,0}, S_{4,1}$ above. Then we set $T_{3,2} := T_{4,0} \cup S_{3,2}, D_{3,2} := D_{4,0}$ and $S_{3,2} := T_{3,2} \cup D_{3,2}$, and we proceed to the stratum $S_{3,1}$, repeating the process above:

We first reset to year zero, and apply the standard desingularization algorithm in the complement of $S_{3,2}$, stopping when the maximum value of inv $\leq \inv_{3,1}$. Note that all centres of blowing up are closed in $X$ (in fact, the desingularization morphism is the identity over $U_{4,0} \setminus \Sigma_{4,0}$), because of property (3) above. The procedures (B1)–(B4) are repeated, where Proposition 5.3 and Theorem 5.1 now play the role of Theorems 1.13 and 1.16 respectively, above. For details of (B4), see §§6.3.3, 6.3.4.

We get $D_{3,1}, T_{3,1} := T_{3,2} \cup S_{3,1}$ and $\Sigma_{3,1} := T_{3,1} \cup D_{3,1}$, as before.

A new element in the proof appears, however, when we pass from $(p, r) = (3, 1)$ to $(3, 0)$ (or, in general, when we pass from $(p, r)$, where $r > 1$, to $(p, r - 1)$). As in property (3) above, there is a neighbourhood $U_{3,1}$ of $\Sigma_{3,1}$ such that, in $U_{3,1} \setminus \Sigma_{3,1}$, $(X, E)$ is normal crossings, but now only ord $X \leq 3$. It will no longer be true, when we reset to year zero and apply the desingularization algorithm in the complement of $\Sigma_{3,1}$, stopping when the maximum value of inv $= \inv_{3,0}$, that the centres of blowing up involved will be closed in $X$—they may have limit points in $\Sigma_{3,1}$ (but not in $\Sigma_{4,0}$ or $\Sigma_{3,2}$, nor at nc points). Nevertheless, the centres of blowing up extend to admissible centres of blowing up for $(X, E)$, and the blowings-up preserve the minimal singularities at the limit points. For details, see Remark 6.11.

Remark 6.7. Since $X$ is a hypersurface, when $r > 0$ and we apply the desingularization algorithm after resetting to year zero, as above, the desingularization
blow-up sequence for \((X, E)\) is the same as that for the ideal given by the product of the ideal \(I_X\) of \(X\) in \(O_Z\), and the ideals \(I_H\) of all components \(H\) of \(E\). On the level of the desingularization invariant, \(\text{inv}_{p+\tau,0}\) for the product ideal replaces \(\text{inv}_{p,r}\) for \((X, E)\). The implication for the subsequent splitting and cleaning steps \((B2)\) and \((B3)\) is that Theorem 1.13 or Proposition 5.3 for \((B2)\), or Theorem 1.16 or Proposition 5.11 for \((B3)\) are simply applied with \(f\) given by a local generator of the product ideal. It is therefore not necessary to rewrite the statements of these results to explicitly mention the case \(r > 0\). See Remarks 3.10, 4.2, 5.6 and 5.12.

The resulting circulant normal form from Theorems 1.16 or 5.1 for \((X, E)\) at a point of the stratum \(S_{p,r}\) will be a product of circulant singularities—the local normal form for \(I_X\) times the \(r\) smooth factors corresponding to the components of \(E\). So, if \(I_X\) is \(\text{cp}(2)\), for example, we will write \(\text{exc}^r \times \text{cp}(2)\) for the local normal form of \((X, E)\).

In the case \(n = 5\), after dealing with the stratum \(S_{4,0}\) (in a manner analogous to but simpler than \(S_{3,0}\); see \((6.3.2)\), we still have to treat the strata \(S_{2,r}, r \leq 3\), and \(S_{1,r}, r \leq 4\), to complete the proof of Theorem 6.2. For Theorem 6.14 we will need to treat \(S_{2,r}, r \leq n - 2\), followed by \(S_{1,r}, r \leq n - 1\), in any dimension \(n\). Details will be provided in \((6.4)\). This will complete the proofs of Theorems 6.2 and 6.14.

6.3. **Minimal singularities in five variables.** This subsection provides details of the blow-up procedure \((B4)\) for the strata \(S_{p,r}\), \((p, r) = (4, 0)\) or \((3, 1)\), in five variables, as well as for \((p, r) = (3, 0)\) or \((2, 0)\), in arbitrary ambient dimension \(n\). The cases \((2, r), r \geq 1\), are treated in \((6.4)\).

We assume that \(X\) has normal form given by Theorem 1.16 at a non-nc point \(a\) of \(S = S_{4,0}\); i.e., by a product of circulant singularities—either \(\text{cp}(4)\), smooth \(\times \text{cp}(3)\), \(\text{cp}(2) \times \text{cp}(2)\), or smooth \(\times\) smooth \(\times \text{cp}(2)\). The case \(\text{cp}(4)\) is the most intricate, and we carry it out in detail.

**Remark 6.8.** In all cases below, the distinguished divisor \(D_1\) plays an important part, as explained in \((6.2.2)\) above. The use of several divisors in place of \(D_1\) provides more flexibility, in general; for example, we can get an analogue of \((6.3.1)\) following for \(\text{cp}(k)\), using \(2(k - 1)\) blowings-up (in particular, 6 blowings-up for \(\text{cp}(4)\), as opposed to 9 below). Since the use of a single distinguished divisor suffices for all results in this article, we leave the more general approach to a subsequent work.

6.3.1. **Circulant point \(\text{cp}(4)\).** Let us write \(\Delta := \Delta_4\). There are étale coordinates \((w, x, z) = (w, x_1, x_2, x_3, z)\) at \(a = 0\) in which \(X\) is the vanishing locus of

\[
\Delta \left( z, w^{1/4} x_1, w^{2/4} x_2, w^{3/4} x_3 \right) = \prod_{\ell = 0}^3 \left( z + \varepsilon^\ell w^{1/4} x_1 + \varepsilon^{2\ell} w^{2/4} x_2 + \varepsilon^{3\ell} w^{3/4} x_3 \right),
\]

where \(\varepsilon = e^{2\pi i / 4}\) and \(\{ w = 0 \}\) is a component of the exceptional divisor. \(\Delta\) will call \(\{ w = 0 \}\) the *old exceptional divisor \(D_{\text{old}}\).* Let us enumerate the singularities of \(X\) in \(\{ z = w = 0 \}\). In this 3-dimensional subspace, \(X\) is smooth at a point where \(x_1 \neq 0\) (with tangential exceptional divisor \(D_{\text{old}}\)). In \(\{ z = w = x_1 = 0 \}\):

1. at any nonzero point of the \(x_3\)-axis, \(X\) has order 3, and is given by the vanishing locus of

\[
\Delta \left( z, w^{1/4} x_1, w^{2/4} x_2, w^{3/4} \right),
\]

after a change of variable to absorb the unit \(x_3\);
(2) at any nonzero point of the $x_2$-axis, $X$ has order 2, and is given by the vanishing locus of
$$\Delta \left( z, w^{1/4}x_1, w^{2/4}, w^{3/4}x_3 \right),$$
after a change of variable to absorb $x_2$;
(3) at any point where $z = w = x_1 = 0, x_2 \neq 0, x_3 \neq 0$, $X$ also has order 2,
and is given by the vanishing locus of
$$\Delta \left( z, w^{1/4}x_1, w^{2/4}, w^{3/4} \right).$$

Let us explain why $X$ has isomorphic singularities at any two points in \{ $z = w = x_1 = 0, x_2 \neq 0, x_3 \neq 0$ \}; i.e., in (3) above. Note that $\Delta$ is homogeneous with respect to $(x,z)$, but also weighted homogeneous with respect to $(w,x,z)$; i.e.,
$$\Delta(t \cdot (w,x,z)) = t^4 \Delta(w,x,z),$$
where
$$t \cdot (w,x,z) := (tw, t^{3/4}x_1, t^{2/4}x_2, t^{1/4}x_3, tz).$$

By homogeneity, $X$ has isomorphic singularities at any points of a curve (parametrized by $t$) coming from either notion of homogeneity. But the families of curves coming from either notion of homogeneity each foliate \{ $z = w = x_1 = 0, x_2 \neq 0, x_3 \neq 0$ \}, and any pair of curves, one from each family, intersect.

As an essentially equivalent explanation,
$$\frac{1}{x_2^3} \Delta = \Delta \left( \frac{x_3}{x_2}, w^{1/4} \frac{x_1}{x_2}, w^{2/4}, w^{3/4} \frac{x_3}{x_2} \right),$$
so that
$$\left( \frac{x_3}{x_2} \right)^8 \frac{1}{x_2^2} \Delta = \Delta \left( \left( \frac{x_3}{x_2} \right)^2 \frac{x_3}{x_2}, w^{1/4} \left( \frac{x_3}{x_2} \right)^2 \frac{x_1}{x_2}, w^{2/4} \left( \frac{x_3}{x_2} \right)^2 \frac{x_3}{x_2}, w^{3/4} \left( \frac{x_3}{x_2} \right)^3 \right).$$

We can now absorb units into $w, x_1, z$ to get the normal form of item (3) above.

We will now give the remainder of the procedure (B4) of the minimal singularities algorithm for $\text{cp}(4)$; i.e., we give a finite sequence of blowings-up needed to obtain a finite collection of minimal singularities occurring as neighbours of $\text{cp}(4)$ (i.e., occurring in a small neighbourhood of $\text{cp}(4)$ or as a limit of singularities in a neighbourhood). The neighbours of $\text{cp}(4)$ are the three singularities (1), (2), (3) above, together with a variant (2') of (2), all of which are listed in \S 6.3.2 below.

Blow-up 1. Introduction of a distinguished exceptional divisor $D_1$. Centre $= \text{cp}(4) = 0$ in the coordinate chart above. The blowing-up is covered by 5 coordinate charts, in each of which we will retain the same notation $(w, x_1, x_2, x_3, z)$ for the coordinates, using the following convention.

z-chart. We substitute $(wz, x_1z, x_2z, x_3z, z)$ for the original coordinates, and factor $z^4$ to obtain the strict transform of $X$ as the vanishing locus of
$$\Delta \left( 1, w^{1/4}z^{1/4}x_1, w^{2/4}z^{2/4}x_2, w^{3/4}z^{3/4}x_3 \right).$$
We do not need to examine this chart because the strict transform of $X$ lies entirely in the remaining charts, following.
\textit{w-chart.} We substitute \((w, wx, wx_2, wx_3, wz)\) to get
\[
\Delta \left( z, w^{1/4}x_1, w^{2/4}x_2, w^{3/4}x_3 \right)
\]
for the strict transform. This is the same as the original formula, but the meaning of \(w\) has changed—here \(\{w = 0\}\) is the new exceptional divisor \(D_1\) (the inverse image of the centre of blowing up), and \(D_{\text{old}}\) has been moved away. Subsequent blowings-up will have centres in \(D_1\) or its successive strict transforms (which we continue to label as \(D_1\)).

\textit{x_1-chart.} The substitution \((wx, x_1, x_1x_2, x_1x_3, x_1z)\) gives
\[
\Delta \left( z, w^{1/4}x_1^{1/4}, w^{2/4}x_2^{2/4}, w^{3/4}x_3^{3/4} \right), \quad D_1 = \{x_1 = 0\}.
\]
and \(D_1 = \{x_1 = 0\}\). (In the remaining charts, we do not write the substitution explicitly; it will follow the same pattern, and we will describe only the strict transform and exceptional divisor. In each chart, \(D_{\text{old}}\) is present as \(\{w = 0\}\), unless \(\{w = 0\}\) represents another component of \(E\) as indicated, in which case \(D_{\text{old}}\) does not intersect the chart.)

\textit{x_2-chart.} \(\Delta \left( z, w^{1/4}x_2^{1/4}, w^{2/4}x_2^{2/4}, w^{3/4}x_3^{3/4} \right), \quad D_1 = \{x_2 = 0\}\).

\textit{x_3-chart.} \(\Delta \left( z, w^{1/4}x_3^{1/4}, w^{2/4}x_3^{2/4}, w^{3/4}x_3^{3/4} \right), \quad D_1 = \{x_3 = 0\}\).

\textbf{Blow-up 2.} Centre = points of order 4 outside \(\text{cp}(4)\); this centre of blowing up is given by \(D_1 \cap \{z = w = x_1 = 0\}\) in the \(x_2\)- and \(x_3\)-charts above. The effect of this blowing-up is to separate the \(w\)- and \(x_3\)-axes in the \(x_3\)-chart, or the \(w\)- and \(x_2\)-axes in the \(x_2\)-chart.

Over the \(x_2\)-chart, we will have 4 charts which we label as the \(x_2z\)-, \(x_2w\)-, \(x_2x_1\)-, \(x_3x_2\)-charts, following the pattern above. We need not consider either the \(x_2z\)-chart (like the \(z\)-chart above) or the \(x_2x_2\)-chart, which does not intersect (the strict transform of) \(D_1\). Likewise, we do not have to consider the \(x_3z\)- or \(x_3x_3\)-charts. Let us describe the strict transform of \(X\) along with \(D_1\) and the new exceptional divisor \(D_2\) in the four remaining charts.

\textit{x_3\text{-}\textit{w-chart.}} This is obtained from the substitution \((wx, w, wx_1, x_2, wx_3)\) with respect to the coordinates of the \(x_3\)-chart, so we have
\[
\Delta \left( z, w^{2/4}x_3^{1/4}x_1, x_3^{2/4}x_2, w^{2/4}x_3^{3/4} \right), \quad D_1 = \{x_3 = 0\}, \quad D_2 = \{w = 0\}.
\]

\textit{x_3\text{-}\textit{x_1-chart.}}
\[
\Delta \left( z, w^{1/4}x_1^{1/4}x_3, x_1^{2/4}x_3, w^{2/4}x_2^{3/4}x_3 \right),
\]
\[
D_1 = \{x_3 = 0\}, \quad D_2 = \{x_1 = 0\}.
\]

\textit{x_2\text{-}\textit{w-chart.}}
\[
\Delta \left( z, w^{2/4}x_2^{1/4}x_1, x_2^{2/4}x_3, w^{2/4}x_2^{3/4}x_3 \right), \quad D_1 = \{x_2 = 0\}, \quad D_2 = \{w = 0\}.
\]

\textit{x_2\text{-}\textit{x_1-chart.}}
\[
\Delta \left( z, w^{1/4}x_1^{1/4}x_3, x_1^{2/4}x_3, w^{2/4}x_2^{3/4}x_3 \right),
\]
\[
D_1 = \{x_2 = 0\}, \quad D_2 = \{x_1 = 0\}.
\]
Blow-up 3. Centre = 0 in the $x_3w$-chart—an isolated point of order 4. As above, we need to give the strict transform of $X$ only in the $x_3ww$, $x_3wx_2$- and $x_3wx_1$-charts.

**$x_3ww$-chart.**

\[
\Delta \left( z, w^{3/4}x_3^{1/4}x_1, w^{2/4}x_2^{3/4}w_2, w^{1/4}x_3^{3/4} \right),
\]

\[D_1 = \{ x_3 = 0 \}, \quad D_2 = \{ w = 0 \};\]

$D_2$ has been moved away.

**$x_3wx_2$-chart.**

\[
\Delta \left( z, w^{2/4}x_2^{3/4}x_3^{1/4}x_1, w^{2/4}x_2^{1/4}x_3^{3/4} \right),
\]

\[D_1 = \{ x_3 = 0 \}, \quad D_2 = \{ w = 0 \}, \quad D_3 = \{ x_2 = 0 \}.
\]

**$x_3wx_1$-chart.**

\[
\Delta \left( z, w^{-2/4}x_1^{3/4}x_3^{1/4}x_1, w^{2/4}x_2^{1/4}x_3^{3/4} \right),
\]

\[D_1 = \{ x_3 = 0 \}, \quad D_2 = \{ w = 0 \}, \quad D_3 = \{ x_1 = 0 \}.
\]

**Blow-up 4.** Centre = points of order 4 given by $D_{old} \cap D_1 \cap D_2 \cap \{ z = 0 \}$, in the $x_3x_1$- and $x_2x_1$- charts. We need only consider the following:

**$x_3x_1$-chart.**

\[
\Delta \left( z, x_1^{2/4}x_3^{1/4}x_1, x_2^{2/4}x_2, w^{2/4}x_1^{3/4}x_3 \right),
\]

\[D_1 = \{ x_3 = 0 \}, \quad D_2 = \{ x_1 = 0 \}, \quad D_4 = \{ w = 0 \}.
\]

**$x_3x_1x_1$-chart.**

\[
\Delta \left( z, w^{1/4}x_3^{1/4}, x_2^{2/4}x_2, w^{2/4}x_1^{3/4}x_3 \right),
\]

\[D_1 = \{ x_3 = 0 \}, \quad D_4 = \{ x_1 = 0 \}.
\]

**$x_2x_1$-chart.**

\[
\Delta \left( z, x_1^{2/4}x_2^{1/4}x_2, x_2^{2/4}x_2, w^{2/4}x_1^{3/4}x_3 \right),
\]

\[D_1 = \{ x_3 = 0 \}, \quad D_2 = \{ x_1 = 0 \}, \quad D_4 = \{ w = 0 \}.
\]

**$x_2x_1x_1$-chart.**

\[
\Delta \left( z, w^{1/4}x_2^{1/4}x_2, w^{2/4}x_2^{2/4}x_2, w^{3/4}x_1^{3/4}x_3 \right),
\]

\[D_1 = \{ x_3 = 0 \}, \quad D_4 = \{ x_1 = 0 \}.
\]

**Blow-up 5.** Centre = points of order 4 given by $D_1 \cap D_3 \cap \{ z = 0 \}$, appearing in the three charts of blow-up 3; i.e., in the $x_3ww$, $x_3wx_2$- and $x_3wx_1$-charts. We need only consider a single chart in each case.

**$x_3ww$-chart.**

\[
\Delta \left( z, x_3^{1/4}x_1, x_3^{2/4}x_2, x_3^{3/4} \right), \quad D_1 = \{ x_3 = 0 \}, \quad D_5 = \{ w = 0 \}.
\]

This singularity is a neighbour (1) of cp(4).
\(x_3 w x_2 x_2\)-chart.
\[
\Delta \left( z, w^{2/4} x_3^{1/4} x_1, x_3^{2/4}, w^{2/4} x_3^{3/4} \right),
\]
\(D_1 = \{ x_3 = 0 \}, D_2 = \{ w = 0 \}, D_5 = \{ x_2 = 0 \}.
\]
\(x_3 w x_1 x_1\)-chart.
\[
\Delta \left( z, w^{2/4} x_3^{1/4} x_1, x_3^{2/4} x_2, w^{2/4} x_3^{3/4} \right),
\]
\(D_1 = \{ x_3 = 0 \}, D_2 = \{ w = 0 \}, D_5 = \{ x_1 = 0 \}.
\]

**Blow-up 6.** Centre = \(D_1 \cap D_5 \cap \{ z = 0 \}\), generically of order 2, appearing in the \(x_2 w\)-, \(x_3 x_1 w\)-, \(x_2 x_1 w\)-, \(x_3 w x_2 x_2\)- and \(x_3 w x_1 x_1\)-charts. We need to consider the following.

\(x_2 w z\)-chart.
\[
\Delta \left( z^{2/4}, z^{1/4} w^{2/4} x_2^{1/4} x_1, x_2^{2/4}, z^{3/4} w^{2/4} x_2^{3/4} x_3 \right),
\]
\(D_1 = \{ x_2 = 0 \}, D_2 = \{ w = 0 \}, D_6 = \{ z = 0 \}.
\]

\(x_2 w w\)-chart.
\[
\Delta \left( w^{2/4} z, w^{1/4} x_2^{1/4} x_1, x_2^{2/4}, w^{3/4} x_2^{3/4} x_3 \right),
\]
\(D_1 = \{ x_2 = 0 \}, D_6 = \{ w = 0 \}.
\]

\(x_3 x_1 w z\)-chart.
\[
\Delta \left( z^{2/4}, z^{1/4} w^{2/4} x_1^{1/4} x_3, x_1^{2/4} x_2, w^{3/4} x_1^{2/4} x_2^{3/4} x_3 \right),
\]
\(D_1 = \{ x_3 = 0 \}, D_2 = \{ x_1 = 0 \}, D_4 = \{ w = 0 \}, D_6 = \{ z = 0 \}.
\]

\(x_3 x_1 x_1\)-chart.
\[
\Delta \left( x_1^{2/4} z, x_1^{1/4} x_3^{1/4} x_3^{3/4} x_2, w x_1^{3/4} x_3^{3/4} \right),
\]
\(D_1 = \{ x_3 = 0 \}, D_4 = \{ w = 0 \}, D_6 = \{ x_1 = 0 \}.
\]

\(x_2 x_1 x_1\)-chart.
\[
\Delta \left( z^{2/4}, z^{1/4} w^{2/4} x_2^{1/4} x_1, x_2^{2/4}, w^{3/4} x_2^{2/4} x_3 \right),
\]
\(D_1 = \{ x_2 = 0 \}, D_4 = \{ w = 0 \}, D_6 = \{ z = 0 \}.
\]

\(x_2 x_1 x_1\)-chart.
\[
\Delta \left( x_1^{2/4} z, x_1^{1/4} x_2^{1/4} x_2^{3/4} x_2, w x_1^{3/4} x_2^{3/4} x_3 \right),
\]
\(D_1 = \{ x_2 = 0 \}, D_4 = \{ w = 0 \}, D_6 = \{ x_1 = 0 \}.
\]

\(x_3 w x_2 w w\)-chart.
\[
\Delta \left( w^{2/4} z, w^{1/4} x_3^{1/4} x_1, x_3^{2/4}, w^{3/4} x_3^{3/4} \right),
\]
\(D_1 = \{ x_3 = 0 \}, D_5 = \{ x_2 = 0 \}, D_6 = \{ w = 0 \}.
\]

\(x_3 w x_1 x_1 w\)-chart.
\[
\Delta \left( w^{2/4} z, w^{1/4} x_3^{1/4} x_3^{3/4} x_2, w^{3/4} x_3^{3/4} x_3 \right),
\]
\(D_1 = \{ x_3 = 0 \}, D_5 = \{ x_1 = 0 \}, D_6 = \{ w = 0 \}.
\]

**Blow-up 7.** Centre = \(D_1 \cap D_6\). We need to consider only the following charts, over each of the preceding.
\(x_2 w w\)-chart.
\[
\Delta \left( z, x_2^{1/4} x_1, x_2^{2/4}, w x_2^{3/4} x_3 \right), \quad D_1 = \{ x_2 = 0 \}, \quad D_7 = \{ w = 0 \}.
\]
This is a neighbour of \(c p(4)\) (with smooth normalization); see (2') in 6.3.2 below.

\(x_3 x_1 w x_1\)-chart.
\[
\Delta \left( z, x_3^{1/4}, x_3^{2/4} x_2, w x_3^{3/4} x_3 \right), \quad D_1 = \{ x_3 = 0 \}, \quad D_4 = \{ w = 0 \}, \quad D_7 = \{ x_1 = 0 \}.
\]
This is smooth.

\(x_2 x_1 w x_1\)-chart.
\[
\Delta \left( z, x_2^{1/4}, x_2^{2/4}, w x_1 x_2^{3/4} x_3 \right), \quad D_1 = \{ x_2 = 0 \}, \quad D_4 = \{ w = 0 \}, \quad D_7 = \{ x_1 = 0 \}.
\]
Smooth again.

\(x_3 x_2 x_2 w w\)-chart.
\[
\Delta \left( z, x_3^{1/4}, x_2^{2/4}, w x_2^{3/4} x_3 \right), \quad D_1 = \{ x_3 = 0 \}, \quad D_5 = \{ x_2 = 0 \}, \quad D_7 = \{ w = 0 \}.
\]
A neighbour (2) of \(c p(4)\).

\(x_3 x_1 w x_1\)-chart.
\[
\Delta \left( z, x_3^{1/4}, x_3^{2/4} x_2, w x_3^{3/4} x_3 \right), \quad D_1 = \{ x_3 = 0 \}, \quad D_5 = \{ x_1 = 0 \}, \quad D_7 = \{ w = 0 \}.
\]
Smooth again.

**Blow-up 8.** Centre = \(D_{\text{old}} \cap D_1 \cap \{ z = 0 \}\), appearing in the \(x_1\)-, \(x_3 x_1 x_1\)- and \(x_2 x_1 x_1\)-charts. Over these three charts, we need to consider the following.

\(x_1 z\)-chart.
\[
\Delta \left( w^2 z x_1^{1/4}, w^{2/4} x_1^{2/4} x_2, w x_1^{3/4} x_3 \right), \quad D_1 = \{ x_1 = 0 \}, \quad D_8 = \{ z = 0 \}.
\]

\(x_1 w\)-chart.
\[
\Delta \left( w^2 z, x_1^{1/4}, w^{2/4} x_1^{2/4} x_2, w x_1^{3/4} x_3 \right), \quad D_1 = \{ x_1 = 0 \}, \quad D_8 = \{ w = 0 \}.
\]
This is smooth.

\(x_3 x_1 x_1 z\)-chart.
\[
\Delta \left( w^2 z, x_3^{1/4}, w^{2/4} x_3^{2/4} x_2, z w x_3^{3/4} x_3 \right), \quad D_1 = \{ x_3 = 0 \}, \quad D_4 = \{ x_1 = 0 \}, \quad D_8 = \{ z = 0 \}.
\]

\(x_3 x_1 x_1 w\)-chart.
\[
\Delta \left( w^2 z, x_3^{1/4}, w^{2/4} x_3^{2/4} x_2, w x_1 x_3^{3/4} \right), \quad D_1 = \{ x_3 = 0 \}, \quad D_4 = \{ x_1 = 0 \}, \quad D_8 = \{ w = 0 \}.
\]
Smooth again.
\(x_2x_1x_1z\)-chart.
\[
\Delta \left( z^{2/4}, w^{1/4}x_2^{1/4}, z^{2/4}w^{2/4}x_2^{2/4}, zw^{3/4}x_2x_3^{3/4} \right),
\]
\(D_1 = \{x_2 = 0\}, \ D_4 = \{x_1 = 0\}, \ D_8 = \{z = 0\}\).

\(x_2x_1x_1w\)-chart.
\[
\Delta \left( w^{2/4}z, x_2^{1/4}, w^{2/4}x_2^{2/4}, wx_2x_3^{3/4} \right),
\]
\(D_1 = \{x_2 = 0\}, \ D_4 = \{x_1 = 0\}, \ D_8 = \{w = 0\}\).

Smooth again.

**Blow-up 9.** Centre = \(D_{\text{old}} \cap D_1 \cap D_8\), appearing in the \(x_1z\)-, \(x_3x_1x_1z\)- and \(x_2x_1x_1z\)-charts. Over these three charts, we need only consider the following, and they are all smooth.

\(x_1zw\)-chart.
\[
\Delta \left( \frac{z}{2}, x_1^{1/4}, \frac{z}{2}w^{1/4}x_1^{1/4}, \frac{z}{2}w^{2/4}x_1^{2/4}, \frac{z}{2}w^{3/4}x_1x_3^{3/4} \right),
\]
\(D_1 = \{x_1 = 0\}, \ D_8 = \{z = 0\}, \ D_9 = \{w = 0\}\).

\(x_3x_1x_1zw\)-chart.
\[
\Delta \left( \frac{z}{2}, x_3^{1/4}, \frac{z}{2}w^{1/4}x_3^{1/4}, \frac{z}{2}w^{2/4}x_3^{2/4}, \frac{z}{2}w^{3/4}x_3x_3^{3/4} \right),
\]
\(D_1 = \{x_3 = 0\}, \ D_4 = \{x_1 = 0\}, \ D_8 = \{z = 0\}, \ D_9 = \{w = 0\}\).

\(x_2x_1x_1zw\)-chart.
\[
\Delta \left( \frac{z}{2}, x_2^{1/4}, \frac{z}{2}w^{1/4}x_2^{1/4}, \frac{z}{2}w^{2/4}x_2^{2/4}, \frac{z}{2}w^{3/4}x_2x_3^{3/4} \right),
\]
\(D_1 = \{x_2 = 0\}, \ D_4 = \{x_1 = 0\}, \ D_8 = \{z = 0\}, \ D_9 = \{w = 0\}\).

There is of course some flexibility in the choice of the preceding blowings-up; for example, blow-ups 8, 9 could have been performed before blow-ups 6, 7, and we may switch the order of 3 and 4, or of 4 and 5.

6.3.2. Summary of the \(cp(4)\) case. After the preceding sequence of blowings-up, only singularities \(\{\Delta_1 = 0\}\) of the following kind appear in the exceptional divisor \(D_1\). (Here we have re-labelled coordinates to be consistent with the normal forms (1)-(3) above.)

1. \(\Delta_4 \left( z, w^{1/4}x_1, w^{2/4}x_2, w^{3/4} \right)\),
2. \(\Delta_4 \left( z, w^{1/4}x_1, w^{2/4}x_2, w^{3/4}x_3 \right)\),
3. \(\Delta_4 \left( z, w^{1/4}x_1, w^{2/4}, w^{3/4}x_2x_3 \right)\),
4. \(\Delta_4 \left( z, w^{1/4}x_1, w^{2/4}, w^{3/4} \right)\).

These singularities are the neighbours of \(cp(4)\). In (2), \(x_3\) may or may not represent an exceptional divisor, and in (2'), \(x_2\) represents an exceptional divisor. In (1) or (2), moreover, there may be an additional exceptional divisor \(x_3\) or \(x_2\) (respectively).

Note that the nearby singularities outside \(D_1\) are only normal crossings singularities because, in the order 2 cases (2), (2'), (3) (respectively, in the order 3 case (1)), the gradients of any two factors (respectively, of any three factors) of \(\Delta = \Delta_4\) are linearly independent at such a nearby point. Moreover, \(X\) and the exceptional divisor \(E\) are simultaneously normal crossings at nearby points outside \(D_1\).
We summarize these results in the following lemma (where, as usual, we use the same notation \((X, E)\), etc., for the transforms of our objects after a sequence of blowings-up).

**Lemma 6.9.** After first blowing up a \text{cp}(4) point to introduce a new exceptional divisor \(D_1\), there is a sequence of seven admissible blowings-up with centres in \(D_1\), after which

1. \(X\) has only minimal non-\text{nc} singularities as above (besides the \text{cp}(4) point), and therefore smooth normalization, at points of \(D_1\);
2. there is a neighbourhood \(U\) of \(D_1\) such that \((X, E)\) has only \text{nc} singularities in \(U \setminus D_1\), which are of order \(< (4,0)\) outside \(S_{4,0}\).

**Remark 6.10.** Circulant singularities of lower order. The cases

\[
\text{cp}(3) \quad \Delta_3 \left( z, w^{1/3} x_1, w^{2/3} x_2 \right),
\]

\[
\text{cp}(2) \quad \Delta_2 \left( z, w^{1/2} x \right) \quad \text{(pinch point)}
\]

are much simpler versions of the \text{cp}(4) case above (see [3]). In particular, \text{cp}(3) has only one singular neighbour

\[
\Delta_3 \left( z, w^{1/3} x_1, w^{2/3} x_2 \right)
\]

in the exceptional divisor \(D_1 = \{ w = 0 \}\) (this singularity was called a degenerate pinch point in [3]), and \text{cp}(2) has only a smooth neighbour in \(D_1\). After the first blowing-up to introduce \(D_1\), only three additional blowings-up are needed for \text{cp}(3), and only one for \text{cp}(2).

Moreover, following Theorem 5.1 and Proposition 5.11 in the case of an irreducible limit of \text{nc}(3), in any dimension (and the simpler version for limits of \text{nc}(2); cf. [6]), we get the preceding normal forms of \text{cp}(3) and \text{cp}(2) (independent of the remaining variables), and we obtain the neighbours above, by global blowings-up. See also Remark 6.13 below.

In five variables, apart from \text{cp}(4), singularities of the following three kinds may occur at an isolated point of the stratum \(S_{4,0}\).

6.3.3. Smooth \times \text{cp}(3). Let us now write \(\Delta = \Delta_3\). There are étale coordinates \((w, x_1, x_2, y, z)\) in which \(X\) is the vanishing locus of

\[
y \Delta \left( z, w^{1/3} x_1, w^{2/3} x_2 \right).
\]

**Blow-up 1.** Introduction of \(D_1\). Centre \(= 0\) in the coordinate chart above. The blowing-up is covered by 5 coordinate charts, with the following transforms of the ideal of \(X\) and exceptional divisor \(D_1\).

- **z-chart.** \(y\) (smooth), \(D_1 = \{ z = 0 \}\).
- **w-chart.** \(y \Delta \left( z, w^{1/3} x_1, w^{2/3} x_2 \right), \ D_1 = \{ w = 0 \}\).
- **x_1-chart.** \(y \Delta \left( z, w^{1/3} x_1^{1/3}, w^{2/3} x_2^{2/3} x_1 \right), \ D_1 = \{ x_1 = 0 \}\).
- **x_2-chart.** \(y \Delta \left( z, w^{1/3} x_2^{1/3} x_1, w^{2/3} x_2^{2/3} x_2 \right), \ D_1 = \{ x_2 = 0 \}\).
- **y-chart.** \(\Delta \left( z, w^{1/3} y^{1/3} x_1, w^{2/3} y^{2/3} x_2 \right), \ D_1 = \{ y = 0 \}\).
We now make three further blowings-up, which are essentially the three blowings-up needed for $cp(3)$ after the introduction of $D_1$ (see Remark 6.10). Over the $w$-, $x_1$- and $x_2$-charts, in fact, these are simply the blowings-up for $cp(3)$ in the presence of the additional variable $y$; after each blowing-up, we get $y \times$ the transform of the blowing-up for $cp(3)$. So we leave the computation to the reader, and describe only the transforms over the $y$-chart above (where the centre of blowing up extends, in any case, to the centre needed over the $w$-, $x_1$- and $x_2$-charts).

**Blow-up 2.** Centre = points of order 3, $D_{old} \cap D_1 \cap \{z = x_1 = 0\}$ in the $y$-chart (and the $x_2$-chart). Over the $y$-chart, we need consider only the following.

**$yx_1$-chart.**

$\Delta \left( z, w^{1/3}y^{1/3}x_1^{2/3}, w^{2/3}y^{2/3}x_1^{1/3}x_2 \right)$, $D_1 = \{y = 0\}$, $D_2 = \{x_1 = 0\}$.

**$yw$-chart.**

$\Delta \left( z, w^{2/3}y^{1/3}x_1, w^{1/3}y^{2/3}x_2 \right)$, $D_1 = \{y = 0\}$, $D_2 = \{w = 0\}$.

**Blow-up 3.** Centre = points of order 3, $D_1 \cap D_2 \cap \{z = 0\}$. We need consider only the following.

**$yx_1x_1$-chart.**

$\Delta \left( z, w^{1/3}y^{1/3}, w^{2/3}y^{2/3}x_2 \right)$, $D_1 = \{y = 0\}$, $D_3 = \{x_1 = 0\}$.

**$yw$-chart.**

$\Delta \left( z, y^{1/3}x_1, y^{2/3}x_2 \right)$, $D_1 = \{y = 0\}$, $D_3 = \{w = 0\}$.

This is $cp(3)$.

**Blow-up 4.** Centre = order 2 points, $D_{old} \cap D_1 \cap \{z = 0\}$. We have to consider only the $yx_1x_1w$-chart, where $X$ becomes smooth.

6.3.4. **Summary of the smooth $\times$ cp(3) case.** We get the following as non-nc singular neighbours of smooth $\times$ cp(3), in $D_1$.

1. $cp(3)$,
2. $y\Delta_3 \left( z, w^{1/3}x_1, w^{2/3} \right)$,
3. $\Delta_3 \left( z, w^{1/3}x_1, w^{2/3} \right)$.

These occur already in a small neighbourhood of smooth $\times$ cp(3). Moreover, after the four blowings-up above, there is a neighbourhood $U$ of $D_1$ such that $(X, E)$ has only normal crossings singularities of order $\leq (3, 1)$ in $U \setminus (D_1 \cup S_{4,0})$. It follows that the desingularization invariant, where we reset the current year to year zero, is $\leq inv_{3,1}$ in $U \setminus (D_1 \cup S_{4,0})$. We will formulate a summary lemma analogous to Lemma 6.9 covering all three cases smooth $\times$ cp(3), $cp(2) \times cp(2)$ and smooth $\times$ smooth $\times$ cp(2); see Lemma 6.12.

Remark 6.11. We treat the case $\exp \times cp(3)$, which appears when dealing with the stratum $S_{3,1}$, using the same blow-up sequence as for smooth $\times$ cp(3), where $\exp = \{y = 0\}$ (see Remark 6.7). After the four blowings-up above, there is a neighbourhood $U$ of $D_1$ such that $(X, E)$ has only normal crossings singularities of order $\leq (3, 1)$ in $U \setminus (D_1 \cup S_{4,1}) = U \setminus \Sigma_{3,1}$.

In order to continue to the stratum $S_{3,0}$, as in §6.2 we have to consider the desingularization morphism $\sigma$ over the complement of $\Sigma_{3,1}$, where we first reset to
year zero, and stop when $\text{inv} \leq \text{inv}_{3,0}$. Note that the centres of blowing-up involved in $\sigma$ lie over only the $ywv$-chart of \[6.3.3\] in fact, $\sigma$ consists of a single blowing-up with centre given in the $ywv$-chart by the smooth curve $C = \{z = x_1 = x_2 = w = 0\} \subset D_3$; this curve does not intersect the strict transform of $S_{3,1}$, which lies in the $w$-chart, but it does intersect $D_1$. After blowing up with centre $C$, we already have $\text{inv} \leq \text{inv}_{3,0}$, and the class of minimal singularities in $\Sigma_{3,1}$ is preserved.

6.3.5. $\text{cp}(2) \times \text{cp}(2)$. $X$ is given by

$$\tag{6.4} (z_1^2 - wx_1^2)(z_2^2 - wx_2^2) = 0.$$ 

The non-nc singularities in a small neighbourhood of the origin are the following.

1. $(z_1^2 - w)(z_2^2 - wx_2^2)$,
2. $(z_1^2 - w)(z_2^2 - w)$,
3. $\text{cp}(2)$.

We again blow up the origin to introduce $D_1$. Then we get the same equation \[6.4\] in the $w$-chart. We get the following in the $x_1$-chart:

$$(z_1^2 - wx_1)(z_2^2 - wx_1 x_2^2) = 0, \quad D_{\text{old}} = \{w = 0\}, \quad D_1 = \{x_1 = 0\},$$

and a symmetric description in the $x_2$ chart. Also, in the $z_1$-chart, we get

$$z_2^2 - wz_1 x_2^2 = 0, \quad D_{\text{old}} = \{w = 0\}, \quad D_1 = \{z_1 = 0\},$$

and we get a symmetric description in the $z_2$-chart. There is a further sequence of blowings-up with centres in $D_1$, after which we have only the preceding non-nc singularities in $D_1$, and only nc singularities of order $\leq (3,1)$ in $U \setminus (D_1 \cup S_{4,0})$, where $U$ is a neighbourhood of $D_1$. We leave the full blowing-up computation to the reader.

6.3.6. Smooth $\times$ smooth $\times$ $\text{cp}(2)$. The non-nc singular neighbours are smooth $\times$ $\text{cp}(2)$ and $\text{cp}(2)$, and we get a statement similar to that in \[6.3.3\] (cf. \[6.3.3\] above, as well as [3]).

6.3.7. Summary lemma for the stratum $S_{4,0}$.

**Lemma 6.12.** In each case smooth $\times \text{cp}(3)$, $\text{cp}(2) \times \text{cp}(2)$ or smooth $\times$ smooth $\times \text{cp}(2)$, after first blowing up the point to introduce an exceptional divisor $D_1$, there is a sequence of admissible blowings-up with centres in $D_1$, after which

1. $X$ has only minimal non-nc singularities as listed respectively in \[6.3.3\] or \[6.3.4\] and therefore smooth normalization, at points of $D_1$;
2. there is a neighbourhood $U$ of $D_1$ such that $(X, E)$ has only nc singularities in $U \setminus D_1$, which are of order $< (4, 0)$ outside $S_{4,0}$.

**Remark 6.13.** In the simpler cases $\text{cp}(3)$ and smooth $\times \text{cp}(2)$ analogous to $\text{cp}(4)$ and smooth $\times \text{cp}(3)$, respectively, the analogues of Lemmas 6.9 and 6.12 hold in any dimension; see Remark 6.10.

6.4. Minimal singularities of order at most 3. This section provides details of the blow-up procedure (B4) in the remaining cases, not already covered in \[6.3\] i.e., for the strata $S_{2,r}$ and $S_{1,r}$. We need to treat the strata $S_{3,0}$, $S_{2,r}$ and $S_{1,r}$, in any number of variables, in order that the results complete the proof not only of Theorem 6.2 but also of the following more precise version of Theorem 1.8, for a pair $(X, E)$. 
**Theorem 6.14.** Given \((X, E)\) in arbitrary dimension, there is a finite sequence of admissible blowings-up \([6.1]\) such that every \(\sigma_j\) is an isomorphism over the locus of normal crossings points of \((X_0, E_0) = (X, E)\) of order at most \((3, 0)\), and \((X_1, E_1)\) has only minimal singularities.

For the stratum \(S_3,0\), \((B4)\) has been covered in Remarks \([6.10\, 6.13]\). Singularities \(\exp \times \text{cp}(2)\) in the stratum \(S_{2,1}\) are analogous to smooth \(\times \text{cp}(2)\) in \(S_{3,0}\) (cf. Remark \([6.11]\)). We will carry out the details of \(\exp^r \times \text{cp}(2)\), \(0 < r \leq n - 2\), for general \(n\).

Consider an \(\exp^r \times \text{cp}(2)\) point \(a\) in the stratum \(S_{2,r}\). In this case, the exceptional divisor at \(a\) can be separated into two parts: \(E_{\text{old}}\) corresponding to \(\exp^r\), and \(E_{\text{new}}\) given by the components of \(E \setminus E_{\text{old}}\) at \(a\) (introduced in the previous modifications of \(S_{2,r}\)).

There are \(\text{étale}\) coordinates

\[
(w, v, y, u, x, z) = (w, v_1, \ldots, v_s, y_1, \ldots, y_r, u_1, \ldots, u_t, x, z)
\]

at \(a = 0\), in which

\[
X = \{z^2 - wx^2 = 0\}, \quad E_{\text{old}} = \{y_1 \cdots y_r = 0\}, \quad E_{\text{new}} = \{wv_1 \cdots v_s = 0\}
\]

In these coordinates, \(S_{2, r} = \{z = x = y = 0\}\). Let us write \(D_w := \{w = 0\}\).

**Blow-up 1.** Centre \(S_{2,r} \cap D_w\), to introduce \(D_1\). The blowing-up is covered by \(r+3\) charts, including \(r\) symmetric \(y_j\)-charts, so we can consider only the following.

\(-\text{chart.} \quad X \cap D_1 = 0.\)

\(-x\text{-chart.} \quad X = \{z^2 - wx = 0\}, \quad D_1 = \{x = 0\}.\)

\(-w\text{-chart.} \quad z^2 - wx^2, \quad E_{\text{old}} = \{y_1 \cdots y_r = 0\}, \quad E_{\text{new}} = \{wv_1 \cdots v_s = 0\},\)

where \(D_1 = \{w = 0\}\). Here, \(\{z = x = y = 0\}\) is (the strict transform of) \(S_{2,r}\).

\(-y_1\text{-chart.} \quad z^2 - y_1wx^2, \quad E_{\text{old}} = \{y_2 \cdots y_r = 0\}, \quad E_{\text{new}} = \{wv_1 \cdots v_s y_1 = 0\},\)

where \(D_1 = \{y_1 = 0\}\).

**Blow-up 2.** Centre = \(\{\text{order 2 points of } X\} \cap D_1 \cap D_w\).

\(-xw\text{-chart.} \quad X\) is smooth.

\(-y_1w\text{-chart.} \quad z^2 - y_1x^2, \quad E_{\text{old}} = \{y_2 \cdots y_r = 0\}, \quad E_{\text{new}} = \{wv_1 \cdots v_s y_1 = 0\},\)

where \(D_1 = \{y_1 = 0\}\). Note that \((X, E)\) has only \(nc\)-singularities outside \(D_1\).

We now reset to year zero, and use the desingularization algorithm to blow up outside \(\Sigma_{2,r}\) (i.e., outside \(S_{2,r} \cup D_1\) in the coordinate charts above), stopping when the maximum value of \(\text{inv}\) is \(\text{inv}_{2,r-1}\). There is a neighbourhood \(U_{2,r}\) of \(\Sigma_{2,r}\) such that \((X, E)\) is \(nc\) in \(U_{2,r} \setminus D_2\) (see \([6.2]\)), so the resolution process is essentially combinatorial over \(U_{2,r} \setminus \Sigma_{2,r}\).

Each centre of blowing up may have limits at \(\text{cp}(2)\) points of \(X\) in \(D_{2,r}\), and the centre of blowing up may have several disjoint components with the same limit point. In suitable local coordinates at such a point, \(X\) is given by \(\{z^2 - yx^2 = 0\}\), where \(\{y = 0\}\) is a component of \(D_{2,r} \setminus D_{3,0}\), and each component of the centre of blowing up is given by the intersection of \(\{z = x = 0\}\) and at least \(r\) components of \(E\); therefore, at least one of these components belongs to \(E_{\text{new}}\), so the blowings-up
do not modify the nc locus of \((X_0, E_0)\). Clearly, each of the components extends to a closed smooth subspace of \(X\), as required, and we can blow up one at a time. Since none of the components of \(E\) defining the centre of blowing-up is \(\{y = 0\}\), the limiting cp(2) singularity is preserved.

For example, in the \(y_1 w\)-chart above, the first centre of blowing up is \(\{z = x = y_2 = \cdots y_r = v = w = 0\}\).

The blow-up sequence leads to \(\text{inv} \leq \text{inv}_{2, r-1}\) in the complement of \(\Sigma_{2, r}\), to complete the step.

Methods similar to the preceding are used in [6, Proofs of Theorems 3.4, 1.18].

Finally, we can handle the stratum \(S_{2,0}\) as in Remark 6.10 and then deal with the strata \(S_{1, r}\) in a similar way to \(S_{2, r}\), to complete the proofs of Theorems 6.2 and 6.14. The \(S_{1, r}\) case has much in common with problem of partial desingularization preserving snc, discussed in Section 1.

6.5. **Concluding remark.** As remarked in §6.2, Conjecture 1.5 follows from Claim 6.4. We propose to prove the claim by induction on \((p, r) \in I_n\), in the following way (see also §1.2). Given \((p, r)\), we first apply (A) the standard desingularization algorithm to blow up until \(\text{inv} \leq \text{inv}_{p, r}\). Secondly, we modify the stratum \(S_{p, r}\) using the four blowing up procedures (B1)–(B4), generalized to arbitrary dimension. And finally, we apply the inductive assumption for the predecessor of \((p, r)\) in \(I_n\), to the complement of \(\Sigma_{p, r} := S_{p, r} \cup D_{p, r}\), where \(D_{p, r}\) denotes the union of the special divisors \(D_1\) introduced in (B4) (and where the centres of blowing up in the complement are extended as in §6.4 in the case that \(r > 0\); see also Remark 6.8).

In view of Remark 1.17, the main challenge remaining for partial desingularization preserving normal crossings in arbitrary dimension is a generalization of the moving away techniques (B4) of this section.

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(A. Belotto da Silva) Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, F-75013 Paris, France

Email address: andre.belotto@imj-prg.fr

(E. Bierstone) University of Toronto, Department of Mathematics, 40 St. George Street, Toronto, ON, Canada M5S 2E4

Email address: bierston@math.utoronto.ca

(R. Ronzon Lavie) University of Toronto, Department of Mathematics, 40 St. George Street, Toronto, ON, Canada M5S 2E4

Email address: ramon.ronzon@mail.utoronto.ca