ANALYTIC REGULARITY OF A FREE BOUNDARY PROBLEM

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Abstract. In this paper, we consider a free boundary problem with volume constraint. We show that positive minimizer is locally Lipschitz and the free boundary is analytic away from a singular set with Hausdorff dimension at most \( n - 8 \).

1. Introduction

Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^n \), \( n \geq 2 \). We use \( \mathcal{M}_\Omega \) to denote the collection of all pairs of \((A,u)\) such that \( A \subset \Omega \) is a set of finite perimeter and \( u \in H^1(\Omega) \) satisfies

\[
u(x) = 0 \text{ a.e. } x \in A.
\]

We consider the energy functional

\[
(1.1) \quad E_\Omega(A,u) = \int_\Omega |\nabla u|^2 + P_\Omega(A),
\]

defined on \( \mathcal{M}_\Omega \), where \( P_\Omega(A) \) denotes the perimeter of \( A \) inside \( \Omega \) in the sense of De Giorgi, i.e.,

\[
P_\Omega(A) = \mathcal{H}^{n-1}(\partial^* A \cap \Omega),
\]

where \( \partial^* A \) is the reduced boundary of \( A \) and \( \mathcal{H}^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure.

A pair \((A,u)\) is said to be a local minimizer of (1.1) in its volume class if for any \((\tilde{A},\tilde{u})\) in \( \mathcal{M}_\Omega \) such that \((\tilde{A},\tilde{u})\) agrees with \((A,u)\) away from a compact set and satisfies the volume constraint \(|\tilde{A}| = |A|\), we have

\[
E_\Omega(A,u) \leq E_\Omega(\tilde{A},\tilde{u}).
\]

And we say \((A,u)\) is a nonnegative local minimizer of (1.1) in its volume class if in addition \( u \) is nonnegative.

This free boundary problem is a special case of what was considered in [5] where \( u \) maps \( \Omega \) to \( \mathbb{R}^p \), \( p \geq 1 \) and \( u(A) \subset \Sigma \) where \( \Sigma \) is a smooth submanifold in \( \mathbb{R}^p \). Hence, all the results in [5] hold. Especially, \( \partial^* A \) satisfies the so called mass ratio lower bound, i.e., given \( K \subset \subset \Omega \), there exists \( r_K > 0 \) and \( \lambda_K > 0 \), such that for any \( x \in K \cap \partial^* A \), and for any \( r < r_K \),

\[
\mathcal{H}^{n-1}(\partial^* A \cap B_r(x)) \geq \lambda_K r^{n-1}.
\]

Let

\[
A^* = \{ x \in \Omega \setminus \partial^* A : |B_r(x) \cap A| = |B_r(x)| \text{ for some } r > 0 \},
\]

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a consequence of mass ratio lower bound of \( \partial^* A \) is that the symmetric difference \( A^* A \) has \( \mathcal{H}^n \)-measure zero, i.e., the open set \( A^* \) is equivalent to \( A \) as a set of finite perimeter. So we can assume \( A = A^* \). Let \( \partial A \) be the topological boundary of the open set \( A \), mass ratio lower bound of \( \partial^* A \) now implies

\[
\mathcal{H}^{n-1} ((\partial A \setminus \partial^* A) \cap \Omega) = 0.
\]

Now we can state our main result:

**Theorem 1.1.** Let \( (A, u) \) be a nonnegative local minimizer of \( (1.1) \) in its volume class. Then \( u \) is locally Lipschitz in \( \Omega \) and \( \partial^* A \cap \Omega \) is an analytic hypersurface with \( \mathcal{H}^s ((\partial A \setminus \partial^* A) \cap \Omega) = 0 \) for any \( s > (n - 8) \).

If we drop the nonnegative assumption in the above theorem, it is proved in [5] that \( u \in C^{\frac{1}{2}} (\Omega) \), and the proof of Theorem 1.1 implies that \( \partial^* A \cap \Omega \) is an analytic hypersurface away from the set where \( u \) changes sign. On the other hand, when the space dimension is two, the full regularity in the sign changing case has recently been obtained in a joint work with C. Larsen [4] where a totally different blow up argument was used.

We also remark that the volume constraint is not essential for our regularity results. Locally, volume constraint is of higher order than \( P_\Omega (A) \) so it disappears after blowing up. If we drop the volume constraint or if we instead add a volume term \( c |A| \) in the energy, the results of Theorem 1.1 still hold.

A related problem was considered in [2] by I. Athanasopoulos, L. A. Caffarelli, C. Kenig and S. Salsa. Given \( g \in H^1 (\Omega) \), they were interested in the minimizer of \( E_\Omega (A, u) \) where \( u \in H^1 (\Omega) \) satisfies the boundary condition \( u - g \in H^1_0 (\Omega) \) and \( A \) is the set such that \( u \geq 0 \) in \( A \) and \( u \leq 0 \) in \( \Omega \setminus A \). The free boundary problem we are considering is quite different from theirs. Nonetheless, their techniques in proving the Lipschitz continuity of \( u \) still work for our nonnegative local minimizer.

The paper is organized in the following way: First, we collect some results proved in [5] and deduce the positive density property of the free boundary. In section 3 we prove the Lipschitz continuity of \( u \) following the arguments in [2]. Finally, we show \( \partial^* A \cap \Omega \) is analytic away from a singular set with Hausdorff dimension at most \( n - 8 \) by deriving the Euler-Lagrange equation of the free boundary using domain variation.

### 2. Preliminaries

To prove the regularity of free boundary using a variational approach, we need to construct good candidates to compare with. The volume constrain adds difficulty to such construction, luckily, we can ignore the volume constraint as long as we are willing to pay some penalty. More precisely, let \( (A, u) \) be a local minimizer of \( (1.1) \) in its volume class, we have

**Lemma 2.1.** There exists \( r_0 > 0 \), such that for any

\[
x \in \Omega, \quad r < \min \{ r_0, \text{dist} (x, \partial \Omega) \}
\]

and for any pair \( (A_1, u_1) \) which agrees with \( (A, u) \) away from \( B_r (x) \), there exists \( (A_2, u_2) \) which agrees with \( (A_1, u_1) \) in \( B_r (x) \) and agrees with \( (A, u) \) away from a precompact subset of \( \Omega \) such that \( |A_2| = |A|, \quad u_2 (x) = 0 \) a.e. \( x \in A_2 \) and

\[
E_\Omega (A_2, u_2) \leq E_\Omega (A_1, u_1) + C ||A| - |A_1||
\]

for some positive constant \( C \) independent of \( x \) and \( r \).
Before going to the proof of Lemma 2.1, let’s first recall a deformation lemma. We write any point $x \in \mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let $\xi$ be a smooth function on $[0, 1)$ such that $0 \leq \xi(r) \leq 1$, $\xi(r) \equiv 1$ if $0 \leq r \leq \frac{1}{3}$, $\xi(r) \equiv 0$ if $r \geq \frac{1}{3}$ and $|\xi'(r)| \leq 8$ for any $r \in [0, 1)$. For any $\varepsilon \in \mathbb{R}$, we introduce a map $f_\varepsilon$ from $B_1(0)$ into $\mathbb{R}^n$ defined by

$$f_\varepsilon(x) = f_\varepsilon(x', x_n) = (x', x_n + \varepsilon\xi(|x'|)\xi(|x_n|)),$$

then we have

**Lemma 2.2.** There exists a positive constant $\varepsilon_0 = \varepsilon_0(n)$ such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $f_\varepsilon$ is a diffeomorphism from $B_1(0) \subset \mathbb{R}^n$ into itself satisfying the following estimates:

1. For any $u \in H^1(B_1(0))$, we have

$$(1 - c_1|\varepsilon|) \int_{B_1(0)} \|\nabla u\|^2 \leq \int_{B_1(0)} |\nabla u_\varepsilon|^2 \leq (1 + c_1|\varepsilon|) \int_{B_1(0)} \|\nabla u\|^2$$

where $u_\varepsilon = u \circ f_\varepsilon$ and $c_1$ is a positive constant depending only on $n$.

2. Let $A \subset B_1(0)$ be a set of finite perimeter, then we have

$$|\mathcal{H}^{n-1}(\partial^* A_\varepsilon \cap B_1(0)) - \mathcal{H}^{n-1}(\partial^* A \cap B_1(0))| \leq c_2|\varepsilon|\mathcal{H}^{n-1}(\partial^* A \cap B_1(0)),$$

where $A_\varepsilon = f_\varepsilon(A)$ and $c_2$ is a positive constant depending only on $n$.

3. Let $A \subset B_1(0)$ be such that the symmetric difference $A \triangle B^+_1(0)$ satisfying

$$|A \triangle B^+_1(0)| \leq \delta_0$$

where $B^+_1(0) = \{x \in B_1(0) : x_n > 0\}$ and $\delta_0$ is a positive constant depending only on $n$. Then for some positive constant $c_3$ depending only on $n$, we have

$$|A_\varepsilon| \leq (1 - c_3|\varepsilon|)|A|$$

if $\varepsilon > 0$ and

$$|A_\varepsilon| \geq (1 + c_3|\varepsilon|)|A|$$

if $\varepsilon < 0$.

We refer the readers to [5][6][7] for its proof.

**Proof of Lemma 2.2.** Since $A$ is a set of finite perimeter, by a theorem of De Giorgi, $\partial^* A$ is $(n-1)$-rectifiable, and for every $x \in \partial^* A$, there is a hyperplane $\Pi$ passing $x$ such that, if we denote $H^\pm$, the two half spaces in $\mathbb{R}^n$ separated by $\Pi$, then as $r \to 0^+$

$$r^{-n} \|\chi_{H^+} - \chi_A\|_{L^1(B_r(x))} \to 0.$$  

After a rotation if necessary, we can always assume

$$\Pi = H_0 = \{x \in \mathbb{R}^n : x_n = 0\}.$$  

Let $r_0$ be sufficiently small, there exist $r_1 > 0$ and finite number of balls

$$B_{r_k}(x_k) \subset \Omega, \ 1 \leq k \leq K$$

such that $x_k \in \partial^* A$, and for any $B_r(x) \subset \Omega, r < r_0$, there exists $k$, such that

$$B_{r_k}(x_k) \cap B_r(x) = \emptyset.$$  

\[(2.2)\]
After a scaling if necessary, we can assume \( r_1 = 1 \), and with respect to the tangent plane \( \Pi_k \) of \( \partial^* A \) at \( x_k \),

\[
\left\| \chi_{H_k^+} - \chi_A \right\|_{L^1(B_1(x_k))} \leq \delta_0,
\]

where \( \delta_0 \) is defined in Lemma 2.2. And we further assume \( r_0 \) is so small such that

\[
r_0^n \leq c \varepsilon_0
\]

for some small number \( c \) depending on \( n \). Now let \((A_1, u_1)\) be a pair which agrees with \((A, u)\) away from \( B_r(x) \subset \Omega \) with \( r < r_0 \) and \( B_1(x_k) \) be the ball such that (2.2) holds. We define the new pair \((A_2, u_2)\) so that it agrees with \((A_1, u_1)\) away from \( B_1(x_k) \), and we can deform inside \( B_1(x_k) \) to meet the volume constraint from the estimates in the third part of Lemma 2.2. Finally, (2.1) follows from the estimates in the first two parts of Lemma 2.2.

□

Next, we recall the Hölder continuity of \( u \) and the mass ratio lower bound of \( \partial^* A \) proved in [5]:

Proposition 2.3. \( u \in C^1(\Omega) \) and \( \partial^* A \) satisfies mass ratio lower bound. i.e., for any \( K \subset\subset \Omega \), there are constants \( r_K, \lambda_K > 0 \), such that for all \( x \in \partial^* A \cap B_r(x) \),

\[
H_n^{-1}(\partial^* A \cap B_r(x)) \geq \lambda_K r^{n-1}.
\]

Mass ratio lower bound of \( \partial^* A \) implies Lemma 2.4.

Lemma 2.4.

\[
H_n^{-1}( (\partial^* A \setminus \partial^* A) \cap \Omega ) = 0.
\]

Proof. Since \( \partial^* A \cap \Omega \) is \( H_n^{-1} \) measurable and \( H_n^{-1}(\partial^* A \cap \Omega) < \infty \), standard density lemma implies that

\[
\lim_{r \to 0^+} \frac{H_n^{-1}(\partial^* A \cap B_r(x))}{\omega_{n-1} r^{n-1}} = 0
\]

holds for \( H_n^{-1} \) a.e. \( x \in \Omega \setminus \partial^* A \). On the other hand, Lemma 2.3 implies that for any \( x \in \partial^* A \cap \Omega \),

\[
\liminf_{r \to 0^+} \frac{H_n^{-1}(\partial^* A \cap B_r(x))}{\omega_{n-1} r^{n-1}} > 0,
\]

hence we conclude

\[
H_n^{-1}( (\partial^* A \setminus \partial^* A) \cap \Omega ) = 0.
\]

Define

\[
(2.3) \quad A^* = \left\{ x \in \Omega \setminus \overline{\partial^* A} : \lim_{r \to 0^+} \frac{|B_r(x) \cap A|}{|B_r(x)|} = 1 \right\},
\]

then \( A^* \) is an open set with

\[
\partial A^* \cap \Omega \subset \overline{\partial^* A} \cap \Omega
\]

where \( \partial A^* \) is the topological boundary of \( A^* \).

Lemma 2.5. The symmetric difference \( A^* \triangle A \) has \( H^n \)-measure zero, hence \( A^* \) and \( A \) are equivalent as sets of finite perimeter.
Proof. Let $O$ be any connected open set such that $\partial^* A \cap O$ is empty, then it is well known that either $|O \setminus A| = 0$ or $|O \cap A| = 0$. Apply this observation to each component of the open set $A^*$, we conclude $|A^* \setminus A| = 0$. Similarly, $B^* = \left\{ x \in \Omega \setminus \overline{\partial A} : \lim_{r \to 0^+} \frac{|B_r(x) \cap A|}{|B_r(x)|} = 0 \right\}$ is also an open set such that $\partial^* A \cap B^* = \emptyset$, and we can deduce $|B^* \cap A| = 0$. From Lemma 2.4, $\overline{\partial A} \cap \overline{B^*} \cap \Omega$ has finite $\mathcal{H}^{n-1}$-measure and hence zero $\mathcal{H}^n$-measure. Since $\Omega$ is the disjoint union of $A^*$, $B^*$ and $\partial A^* \cap \Omega$, we have

$$|A^* \triangle A| = |A^* \setminus A| + |A \setminus A^*| = |A^* \setminus A| + \partial A^* \cap \Omega = 0.$$  

□

From now on, we always assume that $A$ is the open set defined by 2.3. Let $\partial A$ be the topological boundary of $A$, then it is easy to verify that $\partial A \cap \Omega = \overline{\partial A} \cap \Omega$, hence

$$\mathcal{H}^{n-1}(\partial A \setminus \partial^* A) \cap \Omega = 0.$$  

Another application of mass ratio lower bound is the following positive density lemma which we will use in the proof of Lipschitz continuity of $u$:

Lemma 2.6. For any closed set $K \subset \subset \Omega$, there exists a constant $\lambda_K > 0$, such that for any $x \in \partial A \cap K$, and for any $r \leq \frac{1}{2} \text{dist}(K, \partial \Omega)$, we have

$$|A \cap B_r(x)| \geq \lambda_K r^n.$$  

Proof. If it is not true, then there would be a sequence $B_{r_k}(x_k)$, such that $x_k \in \partial A \cap K$, $r_k \leq \frac{1}{2} \text{dist}(K, \partial \Omega)$ while

$$|A \cap B_{r_k}(x_k)| \leq \frac{1}{k} r_k^n.$$  

First we claim

$$\lim_{k \to \infty} r_k = 0.$$  

Otherwise, since $\partial A \cap K$ is a compact set, extracting a subsequence if necessary, we can assume

$$\lim_{k \to \infty} x_k = x_0 \in \partial A \cap K$$

and

$$\lim_{k \to \infty} r_k = r_0 > 0.$$  

From (2.4), we also have

$$|A \cap B_{r_0}(x_0)| = 0$$

which contradicts to $x \in \partial A$.

Next, we choose $\rho_k \in (\frac{r_k}{2}, r_k)$, such that

$$\mathcal{H}^{n-1}(\partial^*(A \setminus B_{\rho_k}(x_k)) \cap \partial B_{\rho_k}(x_k)) \leq c(n) \frac{1}{k} r_k^{n-1}.$$  

Let $A_k = A \setminus B_{\rho_k}$, since $(A, u)$ is minimizing, when $k$ is large, we have $r_k$ is small, applying Lemma 2.4, we have,

$$E_{\Omega}(A, u) \leq E_{\Omega}(A_k, u) + C \cdot |A| - |A_k| = E_{\Omega}(A_k, u) + C |A \cap B_{\rho_k}(x_k)|,$$
where the last term came from the penalty for volume constraint. Hence

\[
\mathcal{H}^{n-1} \left( \partial^* A \cap B_{\rho_k} (x_k) \right) \\
\leq \mathcal{H}^{n-1} \left( \partial^* (A \setminus B_{\rho_k} (x_k)) \cap \partial B_{\rho_k} (x_k) \right) + C |A \cap B_{\rho_k} (x_k)| \\
\leq c(n) \frac{1}{k} r^{n-1} + C \frac{1}{k} r^n
\]

which contradicts the mass ratio lower bound when \( k \) is chosen sufficiently large. \( \square \)

### 3. Lipschitz Continuity of \( u \)

In this section, we will show that \( u \) is locally Lipschitz continuous in \( \Omega \) using the approach in \( \cite{2} \). Let \((A, u)\) be a nonnegative local minimizer of \( \cite{14} \) in its volume class. Our first step is to show that \( u \) grows at most linearly near the free boundary.

**Lemma 3.1.** Let \( B_{r^*} (x^*) \subset \Omega \) be such that \( \partial A \cap B_{r^*} (x^*) \neq \emptyset \), then for some positive constant \( C \),

\[
u (x) \leq C \text{dist} (x, \partial A)
\]

holds for any \( x \in B_{\frac{r^*}{2}} (x^*) \setminus A \).

**Proof.** Let \( \varphi \in C_0^\infty (\mathbb{R}^n) \) be a cutoff function such that \( 0 \leq \varphi (x) \leq 1 \) for any \( x \in \mathbb{R}^n \), \( \varphi \equiv 1 \) in \( B_{\frac{3}{2} r^*} (x^*) \), \( \varphi \equiv 0 \) outside \( B_{r^*} (x^*) \). For any \( \varepsilon \in \left( 0, \| u \|_{L^\infty (B_{r^*} (x^*))} \right) \),

we define

\[
w = (u - \varepsilon)^+,
\]

which is a continuous function in \( B_{r^*} (x^*) \). Now we consider

\[
M = \sup_{x \in B_{r^*} (x^*) \setminus A} \frac{w (x) \varphi (x)}{d (x)}
\]

where \( d (x) = \text{dist} (x, \partial A) \). It is easy to see that \( M \) is finite and it is achieved at some point \( x_0 \in B_{r^*} (x^*) \), i.e.,

\[
Md (x_0) = w (x_0) \varphi (x_0),
\]

and since \( \partial A \cap B_{r^*} (x^*) \neq \emptyset \), we have \( d (x_0) = |x_0 - y_0| < 2r^* \) for some \( y_0 \in \partial A \cap B_{3r^*} (x^*) \). By a rotation and translation if necessary, we may assume that \( y_0 = 0 \) and \( x_0 = d (x_0) e_1 \), and we also write

\[
x = (x_1, x').
\]

Let

\[
Q (x - x_0) = \frac{1}{2} (x - x_0) D^2 (w \varphi) (x_0) (x - x_0)^T,
\]

where \( D^2 \) is the Hessian matrix, and

\[
\nabla Q (x') = Q (d (x_0), x').
\]

Using the maximality of \( \frac{w (x) \varphi (x)}{d (x)} \) at \( x_0 \), we can show

\[
\Delta x_1 \nabla Q (x') \geq - \frac{CM}{\varphi (x_0)}
\]

and on the hyperplane \( x_1 = d (x_0) \),

\[
d (x) \geq d (x_0) + \frac{Q (x')}{M} + O \left( \frac{|x'|^3}{M} \right).
\]

\[\text{\( \square \)}\]
We refer the readers to the proof of (4.3), (4.4) in [2] for more details. Hence, near the origin, the free boundary $\partial A$ is below the surface

$$S = \left\{ (x_1, x') : x_1 = \psi (x') = -\frac{Q(x')}{M} + \frac{C|x'|^3}{M} \right\}.$$ 

Let $\kappa_S$ be the mean curvature of $S$, positive if convex with respect to $e_1$, we have

$$\kappa_S (0) = -\frac{1}{n-1} \Delta \psi (0) \leq \frac{C}{\varphi (x_0)}.$$ 

hence for any $x = (\psi (x'), x') \in S$ with $|x'|$ small, we have

$$\kappa_S (x) \leq \frac{C}{\varphi (x_0)} + O (|x'|).$$

Next, we define two families of surfaces

$$S^- = \left\{ (x_1, x') : x_1 = \psi^- (x') = \psi (x') - \frac{\delta_0}{\varphi (x_0)} |x'|^2 - t \right\},$$

and

$$S^+ = \left\{ (x_1, x') : x_1 = \psi^+ (x') = \psi (x') + t \right\}$$

where $t > 0$, $\delta_0 > 0$ both are small. Denote by $Z_t$ the lens-shaped domain between $S^+_t$ and $S^-_t$, i.e.,

$$Z_t = \left\{ \psi^- (x') < x_1 < \psi^+ (x') \right\},$$

then $Z_t \subset B_{3r^*} (x^*)$ when $t$ is sufficiently small. Let

$$V_t = A \cap \left\{ \psi^- (x') < x_1 < \psi (x') \right\}.$$ 

We define a competing pair $(A_t, u_t)$ such that inside $B_{3r^*} (x^*)$, $A_t = A \setminus Z_t$ and

$$u_t = \begin{cases} u & \text{in } B_{3r^*} \setminus Z_t, \\ v_t & \text{in } Z_t \end{cases}$$

where $v_t$ is the harmonic extension of $u$ in $Z_t$, i.e., $v_t$ is harmonic in $Z_t$ and $v_t = u$ on $\partial Z_t$. We also apply Lemma 2.1 away from $B_{3r^*} (x^*)$ to keep the volume constraint which produces an extra energy of size at most $C |V_t|$. Hence, since $(A, u)$ is a minimizer, we have

\[(3.3) \quad \int_{Z_t} |\nabla u|^2 + P (A, B_{3r^*} (x^*)) \leq \int_{Z_t} |\nabla v_t|^2 + P (A_t, B_{3r^*} (x^*)) + C |V_t|.
\]

Next, we claim that near the origin,

\[(3.4) \quad u (x) \geq \frac{cM}{\varphi (x_0)} x_1 + o (|x|).
\]

In fact, $u$ is positive and harmonic in $B_{d(x_0)} (x_0)$ and

$$u (x_0) \geq \varepsilon + \frac{Md (x_0)}{\varphi (x_0)}.$$ 

By the Harnack inequality, we have

$$u (x) \geq c \left( \varepsilon + \frac{Md (x_0)}{\varphi (x_0)} \right).$$
Let $v$ be the harmonic function in $B_{d(x_0)}(x_0) \setminus B_{d(x_0)}(x_0)$ such that $v = 0$ on $\partial B_{d(x_0)}(x_0)$ and $v = \frac{cMd(x_0)}{\varphi(x_0)}$ on $\partial B_{d(x_0)}(x_0)$, i.e.,

$$v(x) = \begin{cases} 
\frac{cMd(x_0)}{\varphi(x_0)^{2-n}} \left( \left( \frac{|x-x_0|}{d(x_0)} \right)^{2-n} - 1 \right) & \text{if } n \geq 3, \\
\frac{cMd(x_0)}{\varphi(x_0)^{2-n}} \ln \frac{|x-x_0|}{d(x_0)} & \text{if } n = 2.
\end{cases}$$

It is easy to verify that for $x \in B_{d(x_0)}(x_0)$,

$$v(x) \geq \frac{cM}{\varphi(x_0)} x_1 + o(|x|).$$

Hence we have, near the origin,

$$u(x) \geq v(x) \geq \frac{cM}{\varphi(x_0)} x_1 + o(|x|).$$

in $B_{d(x_0)}(x_0) \setminus B_{d(x_0)}(x_0)$. And for $x \notin B_{d(x_0)}(x_0)$, we have

$$x_1 \leq \frac{|x|^2}{2d(x_0)}.$$

and (3.4) follows from $u(x) \geq 0$. Now similar arguments as [2] imply

$$|\mathcal{P}(A_t, B_{3r^*}(x^*)) - \mathcal{P}(A, B_{3r^*}(x^*))| \leq \frac{C}{\varphi(x_0)} |V_t|$$

and

$$\int_{Z_t} |\nabla u|^2 - \int_{Z_t} |\nabla v|^2 \geq \frac{cM^2}{\varphi(x_0)^2} |V_t|^2.$$

Summarize, we have shown that for $t$ sufficiently small,

$$\frac{cM^2}{\varphi(x_0)^2} |V_t|^2 \leq \frac{C}{\varphi(x_0)} |V_t| + C|V_t|.$$

Finally, since $0 \in \partial A$, Lemma 2.6 implies

$$|V_t| \geq ct^n$$

for some constant $c > 0$, which guarantees the existence of a sequence of positive numbers $\{t_j\}_{j=1}^\infty$, $t_j \to 0$, such that

$$|V_{t_j}| \leq 2^{2n} |V_{t_j}|.$$

Let $t = t_j$ be sufficiently small, we have

$$\frac{cM^2}{\varphi(x_0)^2} \leq 2^{2n} \left( \frac{C}{\varphi(x_0)} + C \right),$$

hence $M \leq C$ for some constant independent of $\varepsilon$. The conclusion of Lemma follows by letting $\varepsilon \to 0$. \qed

By the standard covering argument, we have

**Corollary 3.2.** Let $K \subset \subset \Omega$, then for some positive constant $C_K$,

$$u(x) \leq C_K \text{dist}(x, \partial A)$$

holds for any $x \in K$. 

The sublinear growth of $u$ near the free boundary implies the local Lipschitz continuity of $u$:

**Theorem 3.3.** $u$ is locally Lipschitz continuous in $\Omega$.

**Proof.** We only need to consider the Lipschitz continuity of $u$ in any ball $B_{r_0}(x_0)$ such that $x_0 \in \partial A$ and $B_{2r_0}(x_0) \subset \subset \Omega$.

Let $K = B_{2r_0}(x_0)$, from Corollary 3.2, we have, for any $x \in B_{2r_0}(x_0) \setminus A$,

$$ u(x) \leq c_K \text{dist}(x, \partial A). $$

Let $x_1, x_2 \in B_{r_0}(x_0)$. If $x_1, x_2 \in A$, then we have

$$ |u(x_1) - u(x_2)| = |0 - 0| = 0. $$

If $x_1 \in A, x_2 \in B_{r_0}(x_0) \setminus A$, then

$$ |u(x_1) - u(x_2)| = u(x_2) \leq c_K \text{dist}(x_2, \partial A) \leq c_K |x_1 - x_2|. $$

Similarly,

$$ |u(x_1) - u(x_2)| \leq c_K |x_1 - x_2| $$

if $x_2 \in A, x_1 \in B_{r_0}(x_0) \setminus A$. So we only need to consider the remaining case $x_1, x_2 \in B_{r_0}(x_0) \setminus A$, if

$$ |x_1 - x_2| \geq \frac{1}{2} \min\{\text{dist}(x_1, \partial A), \text{dist}(x_2, \partial A)\}, $$

without loss of generality, we assume

$$ |x_1 - x_2| \geq \frac{1}{2} \text{dist}(x_1, \partial A), $$

then we have

$$ |u(x_1) - u(x_2)| \leq u(x_1) \leq c_K \text{dist}(x_1, \partial A) \leq 2c_K |x_1 - x_2|. $$

On the other hand, if

$$ |x_1 - x_2| \leq \frac{1}{2} \min\{\text{dist}(x_1, \partial A), \text{dist}(x_2, \partial A)\}, $$

then let

$$ r_1 = \text{dist}(x_1, \partial A) \leq |x_1 - x_0| < r_0, $$

we have

$$ x_2 \in B_{\frac{1}{2}r_1}(x_1) \subset B_{r_1}(x_1) \subset B_{2r}(x) \setminus A, $$

since $u$ is harmonic in $B_{r_1}(x_1) \setminus A$, we have

$$ \|\nabla u\|_{L^\infty(B_{\frac{1}{2}r_1}(x_1))} \leq \frac{c(n)}{r_1} \|u\|_{L^\infty(B_{2r}(x_1))} \leq \frac{c(n)}{2} c_K, $$

here we have applied Corollary 3.2 in the last inequality. Combining all the possibility, we have

$$ |u(x_1) - u(x_2)| \leq c_1(n) c_K $$

for any $x_1, x_2 \in B_{r_0}(x_0)$, hence $u$ is locally Lipschitz. \qed
4. Analyticity of the Reduced Boundary

The Lipschitz continuity of $u$ implies that $\partial A \cap \Omega$ is almost area-minimizing in the sense introduced by F. J. Almgren [1].

More precisely, we have

**Lemma 4.1.** For any $K \subset \subset \Omega$, there exists $C > 0$, such that

\begin{equation}
P(A, B_r(x)) \leq P(\tilde{A}, B_r(x)) + Cr^n
\end{equation}

holds for every $x \in K$, every $r < \frac{1}{3} \text{dist}(K, \partial \Omega)$, and every $\tilde{A}$ which agrees with $A$ away from $B_r(x)$.

**Proof.** Given $K \subset \subset \Omega$, let $x_0 \in K$, $r_0 < \frac{1}{3} \text{dist}(K, \partial \Omega)$ and $\tilde{A}$ be any set which agrees with $A$ away from $B_{r_0}(x_0)$. We can assume $P(A, B_{r_0}(x_0)) > 0$, or else (4.1) holds trivially. Since $u$ is Lipschitz continuous and $u(x) = 0$ for $x \in \partial A$, we have

$u(x) \leq C r_0$

for any $x \in B_{2r_0}(x_0)$. Now let $\varphi \in C^\infty(\mathbb{R}^n)$ be a function such that $\varphi \equiv 0$ in $B_{r_0}(x_0)$, $\varphi \equiv 1$ away from $B_{2r_0}(x_0)$ and

$$
\|\varphi\|_{L^\infty} = 1, \|\nabla \varphi\|_{L^\infty} \leq \frac{c(n)}{r_0}.
$$

Let

$$
\tilde{u} = \varphi u,
$$

then

$\tilde{u}(x) = 0$ for any $x \in \tilde{A}$.

Hence, the minimality of $(A, u)$ implies

$$
\int_{B_{2r_0}} |\nabla u|^2 + P(A, B_{r_0}(x)) \leq \int_{B_{2r_0}} |\nabla \tilde{u}|^2 + P(\tilde{A}, B_{r_0}(x)) + C \left( |\tilde{A}| - |A| \right).
$$

Now

$$
\int_{B_{2r_0}} |\nabla \tilde{u}|^2 = \int_{B_{2r_0}} |\nabla \varphi u + \varphi \nabla u|^2 \\
\leq |B_{2r_0}| \left( \|u\|_{L^\infty(\mathbb{R}^n)} \cdot \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} + \|\varphi\|_{L^\infty(\mathbb{R}^n)} \cdot \|\nabla u\|_{L^\infty(\mathbb{R}^n)} \right) \\
\leq C r_0^n,
$$

and

$$
C \left( |\tilde{A}| - |A| \right) \leq C r_0^n,
$$

hence, we have

$$
P(A, B_{r_0}(x)) \leq P(\tilde{A}, B_{r_0}(x)) + C r_0^n.
$$

\[\Box\]

Now the regularity result on almost area-minimizing boundaries implies [1, 3]:

**Theorem 4.2.** The reduced boundary $\partial^* A \cap \Omega$ is a $C^{1, \frac{1}{2}}$ hypersurface and the singular set $(\partial A \setminus \partial^* A) \cap \Omega$ has Hausdorff dimension at most $n - 8$. 


To obtain higher order regularity of regular part of the free boundary, we consider its Euler-Lagrange equation. Since \( \partial^* A \cap \Omega \) is a \( C^{1,\frac{1}{2}} \) hypersurface, elliptic regularity theory implies that \( \nabla u \in C^{1,\frac{1}{2}}(\Omega \setminus \overline{A}) \) and we use \( \nabla u^+ \) to denote its trace on \( \partial^* A \cap \Omega \), then \( \nabla u^+ \) is \( C^{1,\frac{1}{2}} \) on \( \partial^* A \cap \Omega \).

**Lemma 4.3.** Free boundary equation

\[
\kappa = |\nabla u^+|^2 + C
\]

is satisfied weakly along \( \partial^* A \cap \Omega \), where \( C \) is constant in any component of \( \partial^* A \cap \Omega \).

**Proof.** We use the technique of domain variation. Without loss of generality, we assume

\[
0 \in \partial^* A \cap \Omega, \quad B_r(0) \subset \Omega,
\]

and \( B_r(0) \cap \partial A \) is the graph of \( C^{1,\frac{1}{2}} \) function defined on a hyperplane passing through the origin. Let \( \varphi \in C^\infty_0(B^{n-1}_r(0), \mathbb{R}^n) \), we consider the mapping

\[
T_\varepsilon : B_r(0) \to \mathbb{R}^n
\]
given by

\[
y = T_\varepsilon(x) = x + \varepsilon \varphi.
\]

Let \( \varepsilon < \text{dist}(K, \partial B_r(0)) \) be sufficiently small, where \( K \) is the support of \( \varphi \), then

\[
T_\varepsilon : B_r(0) \to B_r(0)
\]
is a diffeomorphism. We define \((A_\varepsilon, u_\varepsilon)\) such that

\[
A_\varepsilon \cap B_r(0) = T_\varepsilon(A \cap B_r(0)),
\]

and

\[
u_\varepsilon(x) = u(T_\varepsilon^{-1}x) \quad \text{for any } x \in B_r(0).
\]

So from the minimality of \((A, u)\), we have

\[
\int_{B_r} |\nabla u|^2 \, dx + H^{n-1}(\partial A \cap B_r) \\
\leq \int_{B_r} |\nabla u_\varepsilon|^2 \, dx + H^{n-1}(\partial A_\varepsilon \cap B_r) + C |A_\varepsilon| - |A|,
\]

where the last term comes from volume constraint.

First, we have

\[
\lim_{\varepsilon \to 0} \frac{\int_{B_r} |\nabla u_\varepsilon|^2 \, dx - \int_{B_r} |\nabla u|^2 \, dx}{\varepsilon} \\
= \int_{B_r} |\nabla u|^2 \text{ div } \varphi - 2\nabla u \nabla \varphi \cdot (\nabla u)^T \\
= \int_{B_r \setminus \overline{A}} |\nabla u|^2 \text{ div } \varphi - 2\nabla u \nabla \varphi \cdot (\nabla u)^T.
\]
Integration by parts, using the fact that $\triangle u = 0$ in $B_r \setminus A$ and $u = 0$ on $\partial A$, we have
\[
\int_{B_r \setminus A} |\nabla u|^2 \text{div } \varphi - 2 \nabla u \nabla \varphi (\nabla u)^T
= \int_{\partial A} |\nabla u^+|^2 \varphi \cdot \nu - 2 (\varphi \cdot \nabla u^+) (\nu \cdot \nabla u^+)
= -\int_{\partial A} \varphi \cdot \nu |\nabla u^+|^2.
\]
Next,
\[
\lim_{\varepsilon \to 0} \frac{P_\Omega (A_\varepsilon) - P_\Omega (A)}{\varepsilon} = \int_{\partial A} \kappa \varphi \cdot \nu,
\]
here $\int_{\partial A} \kappa \varphi \cdot \nu$ is well defined in weak sense because the divergence structure of mean curvature, a precise formulation can be given using local coordinates. Finally, we have
\[
\lim_{\varepsilon \to 0} \frac{|A_\varepsilon| - |A|}{\varepsilon} = \int_{\partial A} \varphi \cdot \nu.
\]
Hence, we deduce
\[
-\int_{\partial A} \varphi \cdot \nu |\nabla u^+|^2 + \int_{\partial A} \kappa \varphi \cdot \nu + C \int_{\partial A} \varphi \cdot \nu \geq 0.
\]
If we change $\varphi$ to $-\varphi$, we have
\[
-\int_{\partial A} \varphi \cdot \nu |\nabla u^+|^2 + \int_{\partial A} \kappa \varphi \cdot \nu - C \int_{\partial A} \varphi \cdot \nu \leq 0.
\]
Hence for any $\varphi \in C^0_0 (B_r \setminus A)$ such that
\[
\int_{\partial A} \varphi \cdot \nu = 0,
\]
we have
\[
-\int_{\partial A} \varphi \cdot \nu |\nabla u^+|^2 + \int_{\partial A} \kappa \varphi \cdot \nu = 0.
\]
And we deduce
\[
-|\nabla u^+|^2 + \kappa = C
\]
is satisfied weakly on $\partial A \cap B_r$, here
\[
C = -\int_{\partial A} \varphi^* \cdot \nu |\nabla u^+|^2 + \int_{\partial A} \kappa \varphi^* \cdot \nu
\]
for any $\varphi^*$ satisfying
\[
\int_{\partial A} \varphi^* \cdot \nu = 0.
\]

Now standard elliptic regularity theory implies $\partial^* A$ is a $C^{2,\frac{1}{2}}$ hypersurface and the analyticity of $\partial^* A$ follows from a standard bootstrapping argument.

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