Level-Crossing in the Instanton-Anti-Instanton Valley

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ABSTRACT: We study the level crossing of the fermion system described by
the euclidean Dirac Hamiltonian in the valley background. One chiral fermion level
is shown to cross twice the zero value in the case of well-separated instanton-anti-
instanton background. Below a critical separation, however, level crossings are absent.
The phenomenon can be interpreted as the transition to a gauge field configuration of
purely perturbative nature, below a critical instanton-anti-instanton separation. In
the context of high-energy electroweak interactions, our findings seem to definitely
invalidate some optimistic argument concerning the observability of baryon number
violation based on the use of the optical theorem in conjunction with the valley fields.
1. Introduction.

The purpose of the present paper is to gain a deeper insight in the behaviour of chiral fermions in non-Abelian gauge theories in the background of the so-called valley fields (i.e., a particular class of gauge backgrounds of instanton-anti-instanton type), by analyzing the spectral flow of the corresponding euclidean Dirac Hamiltonian.

The interest in this issue arose from the problem of the observability of the anomaly-induced fermion-number violation at high energies in the standard electroweak theory, from the study of the vacuum structure in QCD, and also from the questions related to the large-order behaviour of perturbation theory in non-Abelian gauge theories in general.

The problems involved, in particular the one related to unitarity and chiral anomaly, have been studied by two of the present authors in Ref.[1]. It was shown there that the forward elastic four-fermion amplitude computed in the valley backgrounds indeed satisfies unitarity by having an anomalous piece, proportional to the products of the standard fermion zero modes in spite of the fact that the relevant gauge field is a topologically (globally) trivial one. Furthermore, it was found that such an anomalous contribution is there as long as the instanton anti-instanton \((i - a)\) separation is large enough.

On the other hand, at small \(i - a\) separation the amplitude reduces to a perturbative one. In fact, on the basis of the behaviour of the integrated topological density,

\[
C(x_4) = - \int_{-\infty}^{x_4} d^3x \frac{g^2}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}_{\mu\nu} = N_{CS}(x_4) - N_{CS}(-\infty),
\]

where \(N_{CS}(x_4) = -\frac{g^2}{16\pi^2} \int d^3x \epsilon^{ijk} \text{tr}(F_{ij}A_k - \frac{2}{3}A_iA_jA_k)\) is the Chern-Simons number, the same authors argued that such a transition to purely perturbative amplitude should takes place abruptly at around the unit \(i - a\) separation, \(R/\rho \approx 1\). However, the studies of Ref.[1] left unresolved the question exactly at which value of the \(i - a\) separation this occurs, and did not quite clarify the nature of this transition.

In this note we close that gap by studying the zero level crossing (spectral flow) of the chiral fermion system described by the euclidean Dirac Hamiltonian in the valley

\(^1\)i.e. of pure instanton or of pure anti-instanton background. Recall that no four-dimensional zero modes exist in the valley background. [1]
background. Our numerical analysis clearly reveals a transition in the behaviour of the spectral flow at a critical value of separation, \( R = R_c \approx \sqrt{\frac{4}{3}} \rho \), \( \rho \) being the common size of the instantons. For \( R > R_c \) we find two level crossings, located near the positions of instanton and anti-instanton, while for \( R < R_c \) no level crosses zero.

2. Spectral flow

Let us consider an \( SU(2) \) gauge theory with \( N_F \) massless left-handed doublets. We consider in particular the fermions in a fixed external gauge field, \( A_{\mu}(valley) = -i(g_\mu \sigma \nu - \delta_{\mu \nu}) \left( \frac{(x - x_a)_\nu}{(x - x_a)^2 + \rho^2} \right) + \left( \frac{(x - x_i)_\nu \rho^2}{(x - x_i)^2((x - x_i)^2 + \rho^2)} \right) \). \( y^\mu = -R^\mu/(z - 1); \) \( z = (R^2 + 2\rho^2 + \sqrt{R^4 + 4\rho^2 R^2})/2\rho^2; \) \( R^\mu = (x_i - x_a)^\mu \).

\( N_F \) must be an even number in order for the theory to be well defined. These fermions can be rearranged (in an \( SU(2) \) gauge theory) to \( \frac{N_F}{2} \) Dirac fermions by appropriately combining pairs of left-handed fermions and their righthanded antifermions; the resulting Dirac fermions are coupled vectorially to the gauge boson. In this case the axial charge will be anomalous, corresponding to the fermion number violation of the original model.

For simplicity of notation, let us restrict below to \( N_F = 2 \), equivalent to the case of a single Dirac doublet. The generalisation to the case with \( N_F \) chiral fermions is straightforward. The euclidean Dirac operator is \( i\gamma_\mu D_\mu \), where \( D_\mu = (\partial - igA)_{\mu} \) is the usual covariant derivative. In the Lorentz gauge of Eq.(2.1) the gauge field can be expressed as:

\[
A_\mu = -i g (\sigma_\mu \sigma_\nu - \delta_{\mu \nu}) F_\nu = \eta^a_{\mu \nu} F_\nu \sigma^a
\]

\[
F_\mu(x) = \frac{1}{2} \partial_\mu \log L(x); \quad L(x) = \frac{(x - x_a)^2 + \rho^2}{(x - x_i)^2 + \rho^2} (x - x_i + y)^2.
\]
The instanton and anti-instanton location will be chosen at: \( x_i = (-\frac{R}{2}, 0, 0, 0) \) and \( x_a = (\frac{R}{2}, 0, 0, 0) \); as a consequence the vector \( y \) above (2.2) is in the time direction \( y = (y, 0, 0, 0) \).

Decomposing \( i\gamma_\mu D_\mu = i\gamma_0(D_0 + \mathcal{H}) \) as
\[
\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},
\]
and passing to a noncovariant formalism, we find:
\[
H_+ = -H_- = H
\tag{2.5}
\]
with
\[
H = +i\vec{\sigma} \cdot \vec{\nabla} - \vec{\sigma} \wedge \vec{\tau} \cdot \vec{F} - F_0\vec{\sigma} \cdot \vec{\tau}
\tag{2.6}
\]
where \( x_\mu \equiv (t, \vec{x}); \quad \vec{F} = \vec{x}f(r, t); \quad F_0 = F_0(r, t) \).

Clearly \( \mathcal{H} \) is a self-adjoint operator with entirely essential spectrum, that is, eventual (normalisable) eigenvalues are imbedded into the continuum, which covers the real axis of energies.

To have a picture of well separated levels, some of which crossing zero at certain values of the euclidean time\(^3\), one must consider the Dirac operator on a compactified space, \( S^4 \) (sphere) or \( T^4 \) (torus), but one must make sure that the compactification preserves the zero modes present in the continuum theory. We shall bypass the whole problem, just studying the eventual zero modes of \( \mathcal{H} \) for some values of \( t \) as the signal of level crossing, and disregarding the more complex problem of the full analysis of time evolution of levels.

The condition (2.5) implies that zero modes always appear in pair of different chirality. We assume that for each chirality zero modes are non degenerate, so that we must look for singlets of the total angular momentum (which is a symmetry of \( \mathcal{H} \)), \( \vec{J} = \vec{x} \wedge -i\vec{\nabla} + \vec{\sigma} + \vec{\tau} \). As noted in [4] the most general form of the singlet is:
\[
\eta(\vec{x}, t) = \sigma_2 S(r, t) - i\vec{x}\vec{\sigma}\sigma_2 T(r, t).
\tag{2.7}
\]

\(^2\) Usually the spectral flow is studied in the axial gauge where \( A_0 = 0 \). The Hamiltonian in that gauge would be related to (2.6) by a time-dependent gauge transformation of spatial components of the gauge field that however preserve the spectrum of the Hamiltonian and correspondingly spectral flow.

\(^3\) Note that \( \mathcal{H} \) depends explicitly on \( t \).
The equation \( H\eta = 0 \) then reads:

\[
3F_0 S - 3T - x \nabla T + 2\vec{F} \cdot \vec{x} = 0, \quad (2.8)
\]

\[
-F_0 r^2 + x \nabla S + 2\vec{F} S = 0. \quad (2.9)
\]

Making the substitution

\[
S = \frac{1}{L} \tilde{S}, \quad T = \frac{L}{r^3} \tilde{T}, \quad (2.10)
\]

we get a simplified system:

\[
\tilde{T}'(r; t) = 3F_0 \frac{r^2}{L^2} \tilde{S}; \quad \tilde{S}'(r; t) = F_0 \frac{L^2}{r^2} \tilde{T}, \quad (2.11)
\]

where the prime means differentiation with respect to \( r \) and the time \( t \) appears as a parameter. Normalisability of the solution implies:

\[
\int_0^\infty dr r^2 |S|^2 = \int_0^\infty \frac{r^2}{L^2} |\tilde{S}|^2; \quad \int_0^\infty dr r^4 |T|^2 = \int_0^\infty \frac{L^2}{r^2} |\tilde{T}|^2. \quad (2.12)
\]

Recall that with our choice of parameters we have:

\[
L \equiv \frac{t_a^2 + r^2 + \rho^2}{t_i^2 + r^2 + \rho^2} (t_y^2 + r^2) \quad (2.13)
\]

\[
F_0 \equiv \frac{t_a}{t_a^2 + r^2 + \rho^2} - \frac{t_i}{t_i^2 + r^2 + \rho^2} + \frac{t_y}{t_y^2 + r^2 + \rho^2} \quad (2.14)
\]

where \( t_a \equiv t + \frac{R}{2}; \quad t_i \equiv t - \frac{R}{2}; \quad t_y \equiv t - \frac{R}{2} + y \). We want to know for which range of the parameter \( R \) and for which values of \( t \) - if any - (the other parameter \( \rho \) just fixes the scale) the system Eq.(2.11) has a normalisable solution. Clearly, normalisability enforces the following initial condition (at a given \( t \)) \[
\tilde{S}(0) = 1; \quad \tilde{T}(0) = 0; \quad \text{if } t_y \neq 0 \quad (2.15)
\]

\[
\tilde{S}(0) = 0; \quad \tilde{T}(0) = 1; \quad \text{if } t_y = 0. \quad (2.16)
\]

The problem turns out to be too hard to be treated exactly: it just suffices to note that the system is equivalent to a second order linear equation for \( \tilde{T} \) with 12 regular

\[
^4 \text{We take into account the fact that the only spatial vector is } \vec{x}.
\]

\[
^5 \text{Note the qualitative change in the behaviour of } L \text{ which occurs at } t_y = 0.
\]
fuchsian points! We thus proceeded to a numerical method for finding solutions of the problem (2.11), (2.12) and (2.13) or (2.16).

In passing we recall [3] that for the case of a pure anti-instanton (or an instanton) a normalisable solution can be found easily. For instance, in the case of an anti-instanton (centered at $x_{\mu} = 0$) the gauge field has the same form as (2.3), with $F_{\nu} = x_{\nu}/(x^2 + \rho^2)$ so that

$$F_0 = \frac{t}{t^2 + r^2 + \rho^2}, \quad L = t^2 + r^2 + \rho^2.$$  \hfill (2.17)

The normalisation condition (2.12) requires in this case that

$$\tilde{S}(0) = 1; \quad \tilde{T}(0) = 0.$$  \hfill (2.18)

It is clear that for $t = 0$ (corresponding to the time position of the anti-instanton) $F_0 = 0$ for all $r$ and the system decouples. It follows that $\tilde{S}(r) = 1; \quad \tilde{T}(r) = 0$, i.e.,

$$S(r,0) = \frac{1}{r^2 + \rho^2}; \quad T(r,0) = 0$$

is the desired normalisable solution. Furthermore, it can be shown that $t = 0$ is the only value of the parameter for which such a solution exists.

3. Numerical solution of the system.

Solving the system by a power series in $r$ (method of Frobenius), we find the asymptotic behaviour of two independent solutions of (2.11), which is reported in Table 1 for $r \sim 0$ and in Table 2 for $r \to \infty$. We treated separately the special cases, $t = \frac{R}{2} - y$ (corresponding to $t_y = 0$), $t = t_0$ (where $t_0$ is by definition a zero of $F_0(r = 0)$), $t = t_1$ (where $t_1 = -\frac{R}{2} - y$).

Clearly the solution satisfying (2.15) or (2.16) can be easily selected near $r = 0$; the problem is how this solution evolves at $r \to \infty$. To see this an accurate numerical analysis is required. We have made such a numerical analysis, passing to a compactified variable $p = r/(1 + r)$, and using as initial conditions at a point very near $p = 0$ (i.e. $r = 0$) the values of $\tilde{S}$ and $\tilde{T}$ which are obtained as a power-series.

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6Disregarding the numerical coefficients.

7Note that the coefficients of the system are singular at $r = 0$, so that numerical evolution must start a little after zero, typically $r = 10^{-9}$ (in $\rho$ units).
approximation of the adequate solution of Table 1. The common normalisation of $\tilde{S}$ and $\tilde{T}$ at this point can be arbitrarily set to unity. Then we studied whether the solution evolves towards $r = \infty$ in such a way that the normalisation condition (2.12) is satisfied.

First, we find that for $t_y = 0$ the solution satisfying the initial condition (2.16), is not normalisable.

For $t_y \neq 0$, we ask whether $\tilde{T}(r = \infty) = 0$, i.e., whether the solution of the initial condition problem is normalisable. In Fig. 1a $\tilde{T}(r = \infty)$ is plotted as a function of $t$ for $R = 10$: it is seen that the curve intersects zero twice. Note that the two values of $t$ at which the solution is normalisable are very close to the (time) positions of the instanton and anti-instanton.

The singularity in Fig. 1a is located at $t_y = 0$ ($t \sim 5.1$), and is due to the fact that there are no solutions of the system for this value of $t$ that satisfy the condition (2.15): see Table 1.

We find furthermore that the situation is similar for all values of the instanton-anti-instanton separation above a critical value, i.e., $R > R_c$. (See also Fig. 1b where $R = 1.5$). The critical separation turns out to be \[ \frac{R_c}{\rho} \simeq 1.15470\ldots \simeq \sqrt[4]{\frac{4}{3}}. \] (3.1)

At $R = R_c$ the two level-crossing points coalesce into one.

Below the critical distance, $R < R_c$, on the contrary, no values of $t$ are found for which the system Eq.(2.11) has a normalisable solution. (See Fig. 1c that reports the situation just below the threshold, for $R = 1.0$).

For large instanton anti-instanton separations, our calculations thus indicates the presence of level crossings near the positions of instanton and anti-instanton centers, that seems to reproduce locally the situation mentioned above for a single (anti)-instanton field. This means that the lowest level $H_+$ (in a compactified space), for instance, starts from a positive value at $t = -\infty$ and crosses zero to negative, then crosses zero back to positive, and returns to the original value at $t = \infty$. A level of opposite chirality also crosses zero twice but in the specular manner with respect to the $E = 0$ line.

\footnote{The value $\sqrt[4]{\frac{4}{3}}$ is a fit to our numerical result.}
At the separation below the critical value $R_c$ there is a qualitative change of behaviour: no levels cross zero. The valley field thus appears to lose the topologically non-trivial aspect and degenerate into a field configuration of purely perturbative nature. (At $R = 0$ the valley field Eq.(2.1) is indeed a gauge transform of $A_\mu = 0$.) In other words, the instanton-anti-instanton pair ”melt” at $R \simeq \sqrt{(4/3)\rho}$.

Before closing this section we observe that the critical value of $R$, $R = R_c \simeq \sqrt{\frac{4}{3}} \rho$ corresponds to the value of the conformally invariant variable $z$ (see Eq.2.2)), very close to $z = 3$. And this corresponds to the maximum of the integrated topological density, equal to $C(x_4 = 0) = \frac{1}{2}$ (see Eq. (4.3) of [4] for the explicit expression in terms of $z$). The situation is very similar to what happens in the Schwinger model on the cylinder (QED on $R^1 \otimes S^1$) (see [4]), where the level crossings disappear at the same value of the analogous topological quantity. It is interesting that this value of $C(x_4 = 0)$ just corresponds to the top of the hill separating the two adjacent minima of the gauge field action.

4. Discussion

We propose here a physical interpretation of the mathematical analysis made above, on the level crossing of the euclidean Dirac Hamiltonian.

Consider the (euclidean) time evolution of a given state from $t = -\infty$ to $t = \infty$. For definiteness first consider the vacuum-to-vacuum transition. In the first quantized approach, the wave function is given at $t = -\infty$ by the Dirac sea with all negative energy levels filled. Then we follow its (euclidean) time evolution in the Schrödinger picture. Suppose that the time evolution is adiabatic. This means that an eigenfunction of $H(t)$ with eigenvalue $E_n(t)$ evolves at a successive instant $t'$ into an eigenfunction of $H(t')$ with eigenvalue $E_n(t')$, corresponding to the original one. If one level crosses 0 during the evolution, for example from negative to positive, the state we started with will no longer be the vacuum of the theory at the instant $t'$ but corresponds to the state with one positive energy level filled, that is to a one-particle...

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9 We thank P.Provero for pointing this out to us.

10 The adiabatic approximation suppresses the unavoidable perturbative creation of particle-anti-particle pairs of same chirality (hence with no net chirality change) that is present for every time-dependent background, allowing us to concentrate on the nonperturbative phenomenon related to the level crossings.
state (where "particle" means here "quantum of energy", see [6] for a possible, more accurate, definition). Analogous situation for creation of anti-particles, corresponding to empty negative levels (holes) with respect to the new vacuum.

In Section 3 we found that in the valley background with $R > R_c$ one left level crosses zero twice (and so does a right level). This suggests \[\text{that during the time evolution every quantum state acquires an extra lefthanded fermion and an antiparticle of the righthanded fermion near the anti-instanton position, hence with the net chirality change, } -2.\[\] An analogous net chirality change $+2$ occurs near the instanton center. In the case of vacuum-to-vacuum transition this would mean the propagation of two lefthanded fermions between the instanton centers, signalling the instability of the vacuum. The vacuum-to-vacuum amplitude would get an imaginary part, corresponding to intermediate states with $Q_5 = 2$ or $Q_5 = -2$.

Although the physical interpretation of the level-crossing similar to this is commonly used ([7],[4]), a rigorous universally accepted interpretation seems to be lacking in mathematical physics literature [8]: we do not pretend to go beyond our simple picture here.

In the case of elastic four fermion amplitude (two fermion- two fermion transition) as the one considered in Ref.[1] (relevant to the problem of high energy fermion number violation in the electroweak theory), the presence of the level-crossings for the backgrounds with well-separated instanton and anti-instanton ($R > R_c$) precisely corresponds to the anomalous term in the amplitude,[1] with an imaginary part associated with intermediate states without the initial fermions.

The disappearance of the level crossings at and below the critical $i - a$ separation $R = R_c \simeq \sqrt{\frac{4}{3}} \rho$, on the other hand, means that the anomalous, nonperturbative contribution to the amplitude is absent in these backgrounds. This confirms the idea that the valley background Eq.(2.1) ceases to be topologically significant long before $R$ reaches 0 where $A_\mu$ is the vacuum field.[3]

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11 We assume on physical grounds that in the valley the level crossing near the instanton location is such that a left (right) positive (negative) level crosses zero downwards (upwards) as in the case of a single instanton field; and an opposite situation near the anti-instanton.

12 A more precise statement is $\Delta Q_5 \equiv \{Q_R - Q_L\}_{t_{a+}} - \{Q_R - Q_L\}_{t_{a-}} = -2$: whether this means the creation of two lefthanded particles, the annihilation of two righthanded particles, or e.g., the creation of one left particle and the annihilation of a right particle, depends on the particular process considered and is a matter of no importance.

13 As is well known, the form of the valley field is not unique, and depends on the choice of the
It is important, though it might appear surprising at first sight, that the gauge background, Eq. (2.1), which is perfectly smooth as a function of $R$ and $\rho$, leads to a discontinuous physical result at $R = R_c$. This conclusion is also corroborated by (and independently implied by) the known behaviour of the action [9] and of the integrated topological density [1] $C(0)$, Eq. (1.1), as functions of $R$. Indeed, both of them behave as $\sim R^2$ for $R/\rho \ll 1$, which is clearly and simply a reflection of the perturbative, quadratic fluctuations around $A_\mu = 0$.

In the physics context of Tev-region electroweak interactions, the outcome of our analysis is that the valley configuration at $R < R_c$, hence with action $S < S_c = \frac{16\pi^2}{g^2}(0.5960...)$, has no relation at all to the instanton-induced fermion number violating processes. The argument made in some literature [9] that such processes become unsuppressed, on the basis of the behaviour of the valley field at $R = 0$, thus seems to be unfounded. The valley field is relevant only up to an energy where the cross section is still exponentially suppressed ($\sigma_c = \exp -\frac{16\pi^2}{g^2}(0.3184...)$).

As for the problem of the QCD vacuum, a series of works in Ref. [10] have led to the picture of the physical vacuum (with broken chiral symmetry) as a sort of instanton liquid, with the average instanton-anti-instanton distance about three times their mean sizes. Our result of the minimum separation $R_c/\rho \simeq \sqrt{4/3}$ for the instanton anti-instanton configuration to be non-perturbative, seems to support their assumptions.

Eventual implications of our work to the study of the large-order behaviour of perturbative series in non-Abelian gauge theories, are still to be worked out.

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14This conclusion is not really surprising. As is well known from the theory of phase transitions, physical results in a system with an infinite number of degrees of freedom can have a nonanalytic dependence on the parameters of the system, even if the Hamiltonian is analytic in these. A relativistic system of fermions we are concerned with here, is just such a system.

15This is evident in the "clever" gauge used in [1] in which $A_\mu \propto R$ for small $R$.

16We agree on this point with a conjecture made in [3].
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Figure Captions

Fig. 1. Behaviour of $\bar{T}(r = \infty)$ in function of time, for a) $R/\rho = 10$, b) $R/\rho = 1.5$, c) $R/\rho = 1.0$. 
\[
t_{\text{generic}} = t_0 = t_1 = R_2 - y \quad \tilde{S}_1 = 1 + r^2 \quad 1 + r^4 \quad 1 + r^2 \quad 1 + r^2 \\
\tilde{T}_1 = r^3 \quad r^5 \quad r^3 \quad r^{-1} \\
\tilde{S}_2 = r^{-1} \quad r \quad r^{-1} \quad r^3 \\
\tilde{T}_2 = 1 + r^2 \quad 1 + r^6 \quad 1 + r^2 \quad 1 + r^2
\]

**Table 1**: Behaviour of the two linearly independent solutions near \( r = 0 \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
& t_{\text{generic}} & t = t_0 & t = t_1 & t = \frac{R}{2} - y \\
\hline
\tilde{S}_3 & r & r & r^{-1} & r \\
\tilde{T}_3 & 1 + r^{-2} & 1 + r^{-2} & 1 + r^{-6} & 1 + r^{-2} \\
\tilde{S}_4 & 1 + r^{-2} & 1 + r^{-2} & 1 + r^{-6} & 1 \\
\tilde{T}_4 & r^{-3} & r^{-3} & r^{-5} & r^{-3} \\
\hline
\end{array}
\]

**Table 2**: Behaviour of the two linearly independent solutions, at \( r \to \infty \).