Abstract

This comparative survey explores three formal approaches to reasoning with partly true statements and degrees of truth, within the family of Lukasiewicz logic. These approaches are represented by infinite-valued Lukasiewicz logic (L), Rational Pavelka logic (RPL) and a logic with graded formulas that we refer to as Graded Rational Pavelka logic (GRPL). Truth constants for all rationals between 0 and 1 are used as a technical means to calibrate degrees of truth. Lukasiewicz logic ostensibly features no truth constants except 0 and 1; Rational Pavelka logic includes constants in the basic language, with suitable axioms; Graded Rational Pavelka logic works with graded formulas and proofs, following the original intent of Pavelka, inspired by Goguen’s work. Historically, Pavelka’s papers preceded the definition of GRPL, which in turn precedes RPL; retrieving these steps, we discuss how these formal systems naturally evolve from each other, and we also recall how this process has been a somewhat contentious issue in the realm of Lukasiewicz logic. This work can also be read as a case study in logics, their fragments, and the relationship of the fragments to a logic.

1 Introduction

Fuzzy logics with (rational) constants present a sui generis research area which arises from the preference for greater expressivity of the propositional language. The inclusion of constants in the language was an ingenious move that, in retrospect, seems quite natural: once it is admitted that propositions can take many different truth values, and the idea is embraced that the truth values form an algebra, the next step—identifying the term-definable functions in the algebra, and introducing more connectives for some of the functions that are not term-definable in it—proposes itself. Propositional constants, being nullary connectives, thus present one conceptually easy move in this enterprise.

But to be fair, let us take an upfront look at some of the pitfalls of such a move. The previous paragraph mentions without further specification that the truth values form “an algebra”. A typical member of the family of fuzzy logics (say, an extension of the logic MTL) will be algebraizable, and the quasivariety that forms its equivalent algebraic semantics will

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1It is only with the benefit of hindsight that any such conceptual picture could be outlined. To forestall an (erroneous) impression that the historical development would have followed the arity of the expanding connectives, let us recall the early and sophisticated system of Takeuti and Titani [59], published as early as 1992, employing several families of propositional connectives, including all rational constants, the Lukasiewicz connectives, the product conjunction, etc.
have complicated structure, with many subclasses of independent interest. This is, indeed, the case of Łukasiewicz logic, whose equivalent algebraic semantics is given by the variety $\mathbf{MV}$ of MV-algebras, which is $\mathbb{Q}$-universal: that is, for every quasivariety $\mathbb{K}$ of algebras in a finite language, the lattice of all subquasivarieties of $\mathbb{K}$ is a homomorphic image of a sublattice of the lattice of all quasivarieties of $\mathbf{MV}$—hence the subquasivariety lattice is “as complicated as it can be” in a well-defined sense. By selecting one such algebra—namely, the MV-algebra on the rationals in the interval $[0, 1]$ with the usual order, introducing a propositional constant for each element of this algebra, and introducing axioms that capture the behaviour of the MV-operations on all the elements (such axioms are often referred to as the *bookkeeping axioms* in our *milieu*), one obtains a logic that is still algebraizable but the class forming the equivalent algebraic semantics is reduced significantly. In particular, adding the constants for rationals in $[0, 1]$ with the bookkeeping axioms to Ł leads to the logic RPL: the equivalent algebraic semantics of RPL is the variety of RPL-algebras, each nontrivial member of which contains an isomorphic copy of the MV-algebra on the rationals in $[0, 1]$, and the class has no nontrivial subquasivarieties.\(^2\)

Moreover, the inclusion of constants (or, more generally, any significant expansion of language) turns out to be costly when it comes to the possibility of comparing the investigated logic to other systems as to logical strength. The adherence to logics that rely on the language of classical propositional logic (CPC), only enable the formal derivation of *fully true* statements, and are algebraizable with each nontrivial element of the algebraic semantics containing an isomorphic copy of the two-element Boolean algebra, places each such logic in question automatically in a broad family of logics that are comparable in strength to classical logic, or to other significant systems (be it, for example the family of substructural logics, where indeed many fuzzy logics can smoothly be conceived). If the language of a logic is conspicuously broader that that of CPC or other well-known systems, then one compares *fragments* of that logic to CPC or other systems. As an example of this phenomenon, linear logic with its rich language is often compared to other logics via its fragments (see, e.g., \[^{19}\] for details). Logics with rational constants are ranked naturally by their fragment without constants.

These reservations notwithstanding, fuzzy logics with constants do hold a clear philosophical appeal. To appreciate it, it is profitable to begin with J. A. Goguen’s two papers—now classics in the area—namely *L-Fuzzy Sets* \[^{22}\] and *The logic of inexact concepts* \[^{23}\]. Both these works tailed closely Zadeh’s work *On fuzzy sets* \[^{62}\], pinpointing the kind of uncertainty they aim to address (Goguen in *L-Fuzzy Sets* speaks of *ambiguity* and seeks to differentiate it from probability), and developed the idea of algebra of truth values, starting from the notion of a complete lattice and adding a multiplication $\odot$ that left- and right-distributes over arbitrary joins (possibly with additional properties, such as commutativity for $\odot$ or distributivity of the lattice operations). Goguen’s considerations thus lead to the notion of a *residuated lattice*, first considered by Ward and Dilworth \[^{61}\]. Thus he significantly contributed to the

\[^{2}\] But notice that the situation is quite different e.g. for product logic, where the product algebra on the rationals in $[0, 1]$ is non-simple, namely has the two-element Boolean algebra as its homomorphic image; hence any set of axioms for constants that are valid in the product algebra on the rationals is also valid in the two-element Boolean algebra. Thus each product algebra can be expanded to a model of product logic with rational constants, such that the constant indexed with zero is interpreted with 0 and each constant with nonzero index is interpreted with 1 (cf. \[^{57}\] Definition 3.7). An analogous situation is not possible in Łukasiewicz logic, where the MV-algebra on the rationals is simple.

\[^{3}\] Notice, for example, that \[^{26}\] Theorem 2.1.8] refers to the residuum in the standard product algebra as *Goguen implication*. \[^{2}\]
development of algebraic fuzzy logic, which was later on pursued by other researchers\footnote{In particular, see \cite[Section 2.2]{4} for a discussion of the outstanding merit of Goguen’s analysis.}, see the three volumes of the Handbook of Mathematical Fuzzy Logic \cite{14,12} and the references therein for general overview, and in particular \cite{51,25,45,24,26,27,10,47,4,29,30} for expansions of Łukasiewicz logic with constants; more generally \cite{18,11} for expansions of wider classes of logics with propositional constants. Furthermore the survey \cite{18} and the references therein complete the picture for logics with graded formulas.

Taking into account the eventual impact of Goguen’s work, we can say that fuzzy logic was historically born with the capability of formally processing partly true statements and plausibly estimating the validity of derivations of such statements. Logics with rational constants can naturally be viewed as continuing this research line and enhancing this ability, within the framework of truth preserving logics; see \cite[Section 2]{18} and references therein for a discussion of logics preserving degrees of truth, another and perhaps more radical departure from the tradition of deriving true conclusions from true axioms.

Goguen advocated algebraic computations with truth degrees, but he did not use explicit truth constants. This was first done by Pavelka \cite{51}, who continued the general line of building the logic (taken as a structural consequence relation) of a complete residuated lattice, and exemplified his general treatment with building the calculus of the infinite-valued Łukasiewicz logic with constants for all the reals. After authoring the series of three papers (which formed the subject matter of his dissertation), Pavelka did not continue this research. It was taken up by Hájek and by Novák, in two rather different attitudes, the evolution of both of which will be discussed in Section 2.

While this survey, at various points, considers many different systems of fuzzy logic with constants, it aims to compare three particular systems, taken as three representatives of logics of comparative truth. These three systems, introduced in sufficient detail below, are as follows:

- infinite-valued Łukasiewicz logic $\mathbb{L}$ as in \cite{39,55} and in the monographs \cite{9,44};
- Rational Pavelka logic RPL as in \cite{25,26};
- Graded Rational Pavelka logic GRPL as in \cite{24}.

The evolution of the logics is outlined in sufficient detail in Section 2, while Section 3 gives a technical background to make this survey reasonably self-contained. Section 4 provides ways of obtaining information on the provability relation of one of the logics from the provability relation of another one. It was in order to enable such a comparison that we chose two logics (RPL and GRPL) each of which expands the infinite-valued Łukasiewicz logic with constants for the rationals rather than the reals, despite the fact that the research school that investigates the logics with graded formulas puts more emphasis on languages where the constants represent all the reals. Such a logic in turn would correspond to a Real Pavelka logic in place of our RPL, for the sake of a comparison. Finally Section 5 discussed some views on the presented material, including that of the author, and Section 6 concludes the survey.

The technical aspects of the relationship between $\mathbb{L}$, RPL, and GRPL are either published or implicit in existing literature. Technical presentations of various authors may even casually throw in a remark on the author’s preference of one logic over the others, even though the logics have existed side by side for decades. Our aim here is to highlight the very tight relationship...
of the three systems, but perhaps especially of RPL and GRPL, the broader impact of which
does not appear to have been made explicit. Since the argument that inspired this work is
about propositional logic, we remain throughout on the propositional level, where the ideas
and solutions can clearly be exposed.

2 Evolution of the three logics

Łukasiewicz infinite-valued logic was first considered by Łukasiewicz and Tarski in their paper
[39], published in 1930. This logic, along with its finite-valued counterpart, prominently
the three-valued system introduced by Łukasiewicz already in 1920 [37], sought to capture
some types of uncertainty in the semantics of natural language that users have little difficulty
internalizing and employing on a daily basis, but that appeared difficult to model within—in
fact appeared inconsistent with—classical logic.

The pioneering work of Łukasiewicz gives some leeway in the interpretation of a third truth
value, as introduced in [37]; it is described as indifferent in [38]. The latter retrospective work
aims at recording Łukasiewicz’s early account of his deep-rooted doubt on the validity of the
law of the excluded middle, and can also be read as prolegomena to his later systems of modal
logic, as well as (or indeed primarily as) to his formal, truth-functional systems where the law
of the excluded middle fails. These initial considerations developed into one of several grand
avenues toward formal many-valued and fuzzy logic (with other such considerations provided
by Zadeh and Goguen, Gödel, Heyting, or Post; see [26, Section 10.1]) and a fortiori also to,
on the one hand, advanced theory of Łukasiewicz logic and MV-algebras—see [9, 44, 15] for
overviews, and on the other, applications and (re-) interpretations that the formal system of
fuzzy logic has to offer in the area of reasoning, broadly conceived: here we cannot aim for a
complete picture, see e.g. [36, 5, 13] for some recent developments.

Taking →, →, and (definable) ∨ as basic connectives, Łukasiewicz and Tarski listed a
set of axioms, conjectured earlier by Łukasiewicz to be complete w.r.t. the infinite-valued
semantics (when modus ponens is taken as the only rule of deduction). Rose and Rosser [55]
proved the standard completeness of this axiomatization, while Hay [32] extended it both to
a propositional finite strong standard completeness proof, and to a first-order axiomatization
with an infinitary rule (the set of first-order tautologies is not recursively enumerable [58] and
is actually Π²-complete [54]).

A powerful impulse for the development of Łukasiewicz logic came with the works of
Chang (see [7, 8]) on the algebraic semantics provided by the class of MV-algebras. This
work provided what nowadays would be referred to as the equivalent algebraic semantics for
the logic, and MV-algebras have since gathered a voluminous and manifold body of research
work (cf. the already mentioned [9, 44, 15]; proof theory of Łukasiewicz logic has been covered
in [11]).

Goguen’s main contribution to the evolution of fuzzy logic consists in adopting an ax-
iomatic approach to establishing an algebraic structure on the range of Zadeh’s fuzzy sets
(represented as, or downright identified with, their characteristic functions), and on the col-
lection of fuzzy sets as such, and quickly proceeding to identifying an appropriate class of

5 which emerge, nowadays, as axiomatic extensions of the (weaker) infinite-valued logic
6 Interestingly, Łukasiewicz’s views on the very core tenets of philosophy of logic apparently underwent a
change, engendered at least in part by his work in many-valued and modal logic; e.g., towards the end of his
career he would no longer admit a distinction between empirical and a priori sentences. See [56] for details.
algebras for this enterprise. In [22, p. 147] he writes: “We have used the axiomatic method, in the sense that our underlying assumptions, especially about \( \mathcal{L} \), are abstract; it can thus be ascertained to what extent our results apply to some new problem.” The subsequent [23, p. 355] continues on this note: “Our general method will be to consider a fixed but arbitrary truth set \( \mathcal{L} \), and to deduce properties of the logic using only the closg axioms. This has two advantages: generality; and invariance. Anything which follows from the closg axioms is true for any particular closg; and also results will be invariant under closg automorphisms.” In particular, in these two papers the acronym closg stands for a complete lattice-ordered semigroup: since the semigroup operation is required to distribute over infinite joins, this gives left and right residuation. In [23], Goguen adopts an axiomatics that in modern parlance would amount to a completely ordered \( FL_w \)-algebra. Both Goguen’s papers maintain a non-technical facet with a plethora of practical examples that make clear the width of the author’s scope, ranging from engineering considerations to semantical analysis; and yet both papers read as cornerstones in the philosophy of vagueness.

Furthemore, Goguen [23] advocates computations on truth degrees of formulas as a counterpart of formula derivation, allowing to estimate algebraically the validity of conclusions in chains of formal derivations. This is Goguen’s blueprint for truth constants, elaborated and deployed subsequently by Pavelka. On p. 365 he remarks on his approach as follows: “A traditional way to develop a logical system is through its tautologies. […] but no list of tautologies can encompass the entire system because we want to perform calculations with degrees of validity between 0 and 1. In this sense, the logic of inexact concepts does not have a purely syntactic form. Semantics, in the form of specific truth values of certain assertions, is sometimes required.” He suggests that proofs in his logic capturing vagueness may be of the form \([A \Rightarrow B] \geq a\), with \(A, B\) well-formed formulas of a fixed language and \(a\) a value from a fixed algebra, representing a lower bound on the validity of the derivation \(A \Rightarrow B\); the latter can be computed from a given derivation. The operator \([\cdot]\) assigns to each proposition \(P\) its truth value \([P]\) in a closg (cf. page 333 of the paper). However, the expression \([P]\) is not, in Goguen’s work, an element of the syntax. These two papers of Goguen are both broader in scope and richer in detail than the brief account here suggests, but hopefully it does do enough to recall how his work posed the challenge to develop a formal calculus for propositions that naturally admit truth values other than the boolean ones, and at the same time indicated a way towards such a calculus.

Pavelka starts by extending the notion of a consequence relation to fuzzy theories (that is, fuzzy sets of formulas; again the range is assumed to be a complete lattice \( \mathcal{L} \), possibly expanded with other operations) and then embarks on an investigation of conditions under which one can obtain a corresponding syntactical consequence relation. This starts from the definition of an \( \mathcal{L} \)-syntax, consisting of an \( \mathcal{L} \)-fuzzy set of axioms and \( \mathcal{L} \)-rules of inference; this explicitly involves names for elements of \( \mathcal{L} \), although in the general form (as presented in [50], for example) it is by no means clear that these names, which take explicit part in syntactic derivations, need also to feature in the set of formulas \( F \) as nullary connectives.

Part II of his work [52] is a detailed study of the properties of algebraic structures that provide the truth values of his logic. Here, Pavelka uses the notion residuated lattice in the sense of a \( FL_{ew} \)-algebra, following Ward and Dilworth [61] and Goguen [23]. In the last paper of the series [53], Pavelka deduces that the only eligible candidate for a complete axiomatization

\[ \text{The commutativity of the semigroup operation is not strictly imposed, but it is implicitly used throughout the paper.} \]
of the fuzzy consequence relation, considering the residuated lattices on \([0, 1]\), is the pair of \(\text{Łukasiewicz logic}\) and the algebra \([0, 1]_L\), and proves the famous Pavelka completeness. Still, a major part of the paper \[53\] the construction is kept quite general (e.g. considering also additional operations) and only for the final part the exposition mentions concrete algebras, such as the standard \(\text{MV-algebra}\) or its finite subalgebras. See also Bělohlávek’s account in \[3\] Section 3.

Pavelka’s work inspired the research area of logics with graded syntax, whose detailed inspection is beyond the scope of this paper and we limit the exposition to a few brief remarks; see \[48\] for a full account. Similarly as in the case of the logic \(L\) above, our intent here is not to capture every work ever published on the system, but to outline relevant aspects of its evolution. Novák established completeness for first-order version of Pavelka’s system in \[45\] (this seems to have been anticipated at the end of \[53\]); the logic introduces constants for the reals into the system of infinite-valued \(\text{Łukasiewicz logic}\). One milestone of the development of graded syntax logics was the publication \[49\]. Thereafter, some advanced topics, such as a fuzzy type theory \[46\] were pursued.

From a model-theoretic point of view, there is nothing extraordinary in adding constant elements, the \(\text{name}\)s for each element of a structure, to a formal language; the move is familiar from the construction of the diagram of a structure. It is equally clear that going beyond countably many constants, while necessitated by the choice of the structure, can be viewed as an aberration from what is commonly understood by the term \(\text{reasoning}\), since it makes the latter non-finitary and thus in particular, beyond the reach of commonplace reasoners—humans, or even machines.

The gap between logics with constants for the reals (or for elements of other complete residuated lattices) and those logics that introduce constants for the rationals or other countable subdomains of \([0, 1]\) was first bridged in Hájek’s early paper \[24\], which keeps the reals as truth values of propositions, but reduces the set of propositional constants, the set of grades occurring in graded formulas, and grades of membership of formulas in fuzzy theories to the rationals, while still proving Pavelka completeness (see below). This paper keeps the graded elements in the syntax; but the year 1995 when it was published saw Hájek introduce, in a plenary lecture at the SOFSEM conference and the attending paper \[25\] in the proceedings, the system \(RPL\) for the first time. It was later presented in more detail and with proofs in \[26\].

### 3 Introducing the logics formally

This section provides a brief exposition of the Hilbert deduction systems available for the three logics \(L, RPL,\) and \(GRPL,\) along with their algebraic semantics. The material is selected with the prospect of a comparative study. Comprehensive surveys of \(\text{Łukasiewicz logic}\) include the volumes \[9, 44\] and the chapters \[15\] and \[26\] Chapter 3; see also the references therein. \(\text{Łukasiewicz logic}\) is semantically motivated; as already remarked, the expansion with constants for rationals adds to its dependence on the semantics. Accordingly, we provide the rudiments of the theory of \(\text{MV-algebras},\) the equivalent algebraic semantics of \(L.\)

\(\text{Łukasiewicz logic}\) is arguably quite strong. One hallmark of this is the fact that its connectives are interdefinable, whereby it is possible to opt for only a few basic ones and define the remaining ones, out of the usual set of connectives considered for this logic, which comprises

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8On countable sets of grades in the graded syntax milieu, see \[47\].
constants $0$ and $1$, unary $\neg$ (negation), and binary $\odot$ (strong conjunction or multiplication), $\oplus$ (strong disjunction), $\land$ and $\lor$ (lattice conjunction and disjunction), $\to$ (implication), and $\equiv$ (equivalence). One tradition, following the Hilbert formal system as introduced by Łukasiewicz and Tarski, takes the set $\{\neg, \to\}$ for basic connectives. We shall refer to this language as $L(L)$. Then one defines other connectives as shortcuts: $\alpha \lor \beta$ as $(\alpha \to \beta) \to \beta$; $\alpha \oplus \beta$ as $\neg \alpha \to \beta$; $\alpha \odot \beta$ as $\neg (\alpha \oplus \neg \beta)$; $\alpha \land \beta$ as $\alpha \odot (\alpha \to \beta)$; and $\alpha \equiv \beta$ as $(\alpha \to \beta) \land (\beta \to \alpha)$.

By recursive application of these metarules, each formula in the full language $L(L)$ can be rewritten as a formula in the basic language. Further shortcuts may include $\varphi^n$, standing for $n$ times $\varphi \odot \ldots \odot \varphi$, and $n \cdot \varphi$, standing for $\varphi \oplus \ldots \oplus \varphi$.

The algebraic tradition usually opts for $\{\neg, \oplus\}$ as the set of basic function symbols\(^9\) on the ground of a tight connection between $L$-algebras and lattice-ordered abelian groups (see below). Chang’s $MV$-algebras \([7, 8]\) were introduced as a tool for algebraic study of Łukasiewicz logic and can in fact be shown to be the equivalent algebraic semantics of $L$ in the sense of Blok and Pigozzi \([6]\). Our definition presents $MV$-algebras as a subclass of $FL_{ew}$-algebras. An $MV$-algebra is an algebraic structure of the form $A = \langle A, \odot, \to, \land, \lor, 0, 1 \rangle$ where $\langle A, \land, \lor, 0, 1 \rangle$ is a bounded lattice, $\langle A, \odot, 1 \rangle$ is a commutative monoid with unit $1$, $\to$ and $\land$ form a residuated pair ($x \odot y \leq z$ if and only if $x \leq y \to z$ for $x, y, z \in A$), and moreover for $x, y \in A$ we have $(x \to y) \lor (y \to x) = 1$ (semilinearity), $x \land y = x \odot (x \to y)$ (divisibility), and $(x \to 0) \to 0 = x$ (involutiveness). $MV$-algebras can be shown to form a variety. The top and bottom element of any nontrivial $MV$-algebra $A$ form a two-element Boolean subalgebra of $A$; in particular, the variety of Boolean algebras is a subvariety of $MV$.

A totally ordered $MV$-algebra is called a $\textit{chain}$; it can be shown that every $MV$-algebra is a subdirect product of chains, and hence, the lattice reduct is distributive. Another fact that contributes to the informal claim on strength of $L$ is the extant complete description of the lattice of its axiomatic extensions \([35]\), which is dually isomorphic to the lattice of subvarieties of $MV$.

Examples of $MV$-algebras can be obtained from $\ell$-groups (lattice-ordered abelian groups). Let $G = \{G, \land, \lor, +, -, 0\}$ be an $\ell$-group. Let $u$ be a positive element of $G$. For each $x, y \in [0, u]$, define $x \oplus y = u \land (x + y)$ and $-x = u - x$ (the operations on the right side of equations are the $\ell$-group operations). Then $\langle [0, u], \oplus, -, 0\rangle$ is an $MV$-algebra, denoted $\Gamma(G, u)$. In fact, there is a categorical equivalence between the category of $MV$-algebras and the category of $\ell$-groups with a strong unit, obtained by Mundici \([12]\). Prior to that, Chang \([8]\) established a correspondence between totally ordered $\ell$-groups and $MV$-chains.

The intended semantics for $L$, often referred to as the $\textit{standard}$ $MV$-algebra and denoted $[0,1]_L$, can be simply introduced as $\Gamma(R, 1)$ (where $R$ stands for the additive $\ell$-group on the real numbers). Its subalgebra on the rational numbers will be denoted $Q_L$. Moreover, the finite subalgebra of $[0,1]_L$ (and of $Q_L$) on the domain $\{0, 1/n, \ldots, n/n\}$ is denoted $L_n$; for $n = 1$, one just obtains the two-element Boolean algebra.

A function $f: [0,1]^n \to [0,1]$ is a McNaughton function provided that it is continuous, piecewise linear (there are finitely many linear polynomials $\{p_i\}_{i \in I}$, with $p_i(\bar{x}) = \sum_{j=1}^{n} a_{ij} x_j + b_i$, such that for any $\bar{x} \in [0,1]^n$ there is an $i \in I$ with $f(\bar{x}) = p_i(\bar{x})$), and the polynomials $p_i$ have integer coefficients $a_{ij}, b_i$. The paper \([40]\) established what is known as McNaughton theorem: namely, McNaughton functions $\textit{concede}$ with term-definable functions in $[0,1]_L$. It is clear from this characterization that no constant function is term-definable except for the

\(^9\)Then $\to$ is introduced as $\neg \alpha \oplus \beta$ and the remaining definitions are as before.
constant 0 and the constant 1.

The following are axiom schemata of Łukasiewicz logic presented in [39]:

(A1) \( \varphi \rightarrow (\psi \rightarrow \varphi) \);

(A2) \( (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \);

(A3) \( ((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi) \);

(A4) \( (\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi) \).

The deduction rule is modus ponens: \( \varphi, \varphi \rightarrow \psi \vdash \psi \). Łukasiewicz logic, then, is formally identified with the provability relation \( \vdash_L \) for these axioms and rule (it follows that the logic is finitary).

The finite strong standard completeness theorem for \( \vdash_L \) states that if \( \Gamma \) is a finite set of formulas and \( \varphi \) a formula of the language \( L(\vdash_L) \), then \( \Gamma \vdash_L \varphi \) if and only if \( \Gamma \models_{[0,1]_L} \varphi \) (this result is implicit in [8]; see also [32, Lm. B], the discussion in [21], and [26, Lm. 3.2.11]). The result is equivalent to stating that the variety of MV-algebras is generated by its member \( [0,1]_L \) as a quasivariety.

Rational Pavelka logic (RPL) was first presented in [25], and subsequently in much more detail in [26, Section 3.3], as a simplified variant of Pavelka’s system [51]. RPL expands the language \( L(\vdash_L) \) with a set \( Q = \{ \bar{q} \mid q \in Q \cap [0, 1] \} \) of constants; this expanded language will be referred to as the language of RPL (\( L(RPL) \)). The bookkeeping axioms are the formulas

- \( q \rightarrow \tau \equiv q \rightarrow^L r \);
- \( \neg q \equiv \neg^L q \);
- \( \top \equiv 1 \);
- \( 0 \equiv 0 \)

for all rationals \( q \) and \( r \); on the right-hand side, \( \rightarrow^L \) denotes the residuum operation of \( [0, 1]_L \) and \( \neg^L \) denotes the negation thereof, for the sake of clarity; thus, e.g., the display \( q \rightarrow^L r \) denotes the rational obtained by applying the binary operation \( \rightarrow^L \) to the rational numbers \( q \) and \( r \) in \( [0, 1] \), and \( q \rightarrow^L r \) denotes the constant indexed by that rational.

Notice that the bookkeeping axioms are obtained in a uniform way, given the set of basic connectives of the presentation of the logic \( L \) (here in particular, for the language \( L(L) = \{ \rightarrow, \neg \} \)), provided that such a set of connectives guarantees that each McNaughton function is term definable. In general, the set of bookkeeping axioms consists of all formulas \( \circ(q_1, \ldots, q_n) \equiv q_{\circ(q_1, \ldots, q_n)} \) for each basic connective \( \circ \). Given these axioms, analogous bookkeeping formulas can then be proved also for any of the defined connectives.

Again RPL is an algebraizable logic (see, e.g., the discussion in [16]) with the the class of RPL-algebras forming its equivalent algebraic semantics: an RPL-algebra \( \mathcal{A} \) is an algebra in the language \( L(RPL) \) (i.e., interpreting the language of MV-algebras and all the rational constants from \( Q \)), such that the \( L(L) \) reduct of \( \mathcal{A} \) is an MV-algebra, and all of the bookkeeping axioms are valid in \( \mathcal{A} \). Moreover, \( [0, 1]_L^Q \) denotes the expansion of \( [0, 1]_L \) with constants from \( Q \) where each rational constant \( \tau \) is interpreted with \( r \).

The original set included also the axiom \( (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi) \), derivable from the rest.
The usual strong completeness theorem w.r.t. chains holds: for a set \( \Gamma \cup \{ \varphi \} \) of formulas of \( \mathcal{L}(\text{RPL}) \) we have \( \Gamma \vdash_{\text{RPL}} \varphi \) if and only if, for each RPL-chain \( A \), \( \Gamma \models_{A} \varphi \) (see, e.g., [16]). Also the following theorem has been obtained:

**Fact 3.1. (Finite strong standard completeness theorem for RPL)** [26, Theorem 3.3.14] If \( \Gamma \cup \{ \varphi \} \) is a finite set of formulas in the language \( \mathcal{L}(\text{RPL}) \), then \( \Gamma \vdash_{\text{RPL}} \varphi \) if and only if \( \Gamma \models_{[0,1]_{\mathbb{Q}}} \varphi \).

In the algebraization of RPL, the translation of propositional formulas to algebraic identities (denoted \( \tau \) in [19]), given the axiom \( \overline{q} \rightarrow \overline{r} \equiv \overline{q} \rightarrow \overline{r}^{L} \), yields \( \overline{q} \rightarrow \overline{r} \approx 1 \), which is equivalent in each RPL-algebra to \( \overline{r} \rightarrow \overline{r} \approx 1 \). Analogously we obtain \( \overline{-q} \approx \overline{q} \) and \( \overline{1} \approx 1 \) and \( \overline{0} \approx 0 \) from the other types of axioms of RPL. This set of identities entails the *equational diagram* of the algebra \( Q_{L} \) (see e.g. [33]). Indeed, our expansion of the language \( L_{L} \) to the language of RPL is exactly with constants for the elements of \( Q_{L} \), and it is not difficult to see that the bookkeeping axioms imply all atomic sentences and their negations involving only rational constants that are true in \( Q_{L} \), and hence in \( [0,1]_{L} \). For identities this is easily observed using induction on term structure, for negated identities it is useful to realize that any model of the bookkeeping axioms contains a homomorphic image of \( Q_{L} \), and since this algebra is simple, it contains an isomorphic copy of \( Q_{L} \). As remarked in Section 1, this can be seen as a considerable drawback of RPL: the expansion with constants, aimed at increasing expressivity, restricts severely the class that forms its equivalent algebraic semantics, compared to the logic \( L \).

The apex of Pavelka’s treatment of degrees of truth is a completeness theorem for partial truth. For any set of formulas \( \Gamma \cup \{ \varphi \} \) in the language \( \mathcal{L}(\text{RPL}) \), the provability degree of \( \varphi \) over \( \Gamma \) is defined as

\[
|\varphi|_{\Gamma} = \sup\{ r \mid \Gamma \vdash_{\text{RPL}} \overline{r} \rightarrow \varphi \}
\]

whereas, in the same setting, the validity degree is

\[
\| \varphi \|_{\Gamma} = \inf\{ v(\varphi) \mid v(\psi) = 1 \text{ for each } \psi \in \Gamma \}
\]

where \( v \) denotes assignments in \( [0,1]_{\mathbb{Q}} \).

**Fact 3.2. (Pavelka completeness theorem for RPL)** [26, Theorem 3.3.5] For \( \Gamma \cup \{ \varphi \} \) a set of formulas in the language \( \mathcal{L}(\text{RPL}) \),

\[
|\varphi|_{\Gamma} = \| \varphi \|_{\Gamma}.
\]

A system that will be called *Graded Rational Pavelka Logic* (GRPL) for the scope of this paper was introduced in [24]; we follow that presentation (see also the references therein). This system can be viewed as an early step, taken by Hájek, in further developing Pavelka’s work, taking into account also the work of Novák. Notice in particular that, contrary to our presentation here, GRPL precedes RPL chronologically. GRPL simplifies considerably the works of Pavelka in that only rational truth constants are used while still obtaining Pavelka completeness. Moreover, Hájek also simplifies the axioms, pointing out, among other things, that Pavelka was presumably not familiar with the work of Rose and Rosser [55]. The system GRPL admits graded formulas and fuzzy theories with graded proofs, iconic of Pavelka’s work and (at least ostensibly) absent from RPL as presented in [26].
The language of GRPL coincides with that of RPL\footnote{As a matter of fact Hájek in \cite{pavelka1979} starts out with constants for all reals but very soon switches to employing rational constants only. This appears to be a precondition for some of the results in his paper \cite{pavelka1979}, which are computational: determining the position of GRPL in the arithmetical hierarchy is the main technical achievement. It is therefore essential that the strings representing the formulas, constants included, be finite, and hence amenable to algorithmic considerations.}. A fuzzy theory $\Gamma$ is a fuzzy set of formulas\footnote{This definition, via Pavelka, draws on Zadeh’s paper on fuzzy sets; also the ontology offered there for fuzzy sets is maintained—e.g., a fuzzy set $S$ is a function (here, on the domain of formulas), so the membership of an object $m$ in the set $S$ is denoted $S(m)$.} in the language of RPL: that is, each formula $\psi \in \mathcal{L}(\text{RPL})$ belongs to $\Gamma$ in some grade, which is a rational number, denoted $\Gamma(\psi)$. An assignment $v$ respects $\Gamma$ provided that $\Gamma(\alpha) \leq v(\alpha)$ for each $\alpha \in \mathcal{L}(\text{RPL})$. The validity degree of $\varphi$ in $\Gamma$ is $\inf\{v(\varphi) \mid v \text{ respects } \Gamma\}$; if $r$ is the validity degree of $\varphi$ in $\Gamma$, one can write $\Gamma \models_r \varphi$.

A graded formula is a pair $\langle r, \varphi \rangle$, with $r$ a rational (called the grade of $\varphi$) and $\varphi$ any formula in $\mathcal{L}(\text{RPL})$. The following list specifies the fuzzy set of axioms of GRPL:

\begin{enumerate}
\item[(A)] Instances of the axioms of $\mathcal{L}$ in grade 1;
\item[(B)] the constant $q$ in grade $q$, for each rational $q \in [0,1]$;
\item[(C)] $\neg q \equiv \overline{q}$ and $q \rightarrow \top \equiv \overline{q} \cdot \overline{r}$ in grade 1, for all rational $q,r \in [0,1]$ \textup{(bookkeeping)}
\item[(D)] all other formulas of $\mathcal{L}(\text{RPL})$ in grade 0.
\end{enumerate}

The rules of deduction are

\begin{enumerate}
\item[(E)] $\frac{\langle q, \varphi \rangle \langle r, \varphi \rightarrow \psi \rangle}{\langle q \cdot L, r, \psi \rangle}$ for all rational $q,r \in [0,1]$ \textup{(graded modus ponens)}\footnote{see \cite{pavelka1979}}
\item[(F)] $\frac{\langle q, \varphi \rangle}{\langle r \rightarrow L, q, \top \rightarrow \varphi \rangle}$ for all rational $q,r \in [0,1]$ \textup{(lifting)}.
\end{enumerate}

A graded proof in the logic GRPL is a finite sequence of graded formulas such that each element is either an axiom (A) – (D) or follows using the rules (E), (F) from preceding elements in the sequence. Analogously, a graded proof from a fuzzy theory $\Gamma$ is a finite sequence where each element is an axiom (A) – (D), or an axiom from the theory $\Gamma$ (a graded formula $\langle q, \varphi \rangle$ where $\Gamma(\varphi) = q$), or follows from its predecessors using the rules (E), (F). Clearly, GRPL is a logic of graded formulas only; there is no way to derive a non-graded formula. In case $\langle r, \varphi \rangle$ is the last element of some proof (possibly from a theory $\Gamma$), we say that the formula $\varphi$ has a proof of value $r$ (from $\Gamma$); we write $\Gamma \vdash \langle r, \varphi \rangle$ (or just $\vdash \langle r, \varphi \rangle$ in case $\Gamma$ is empty). Finally, $\Gamma \vdash_r \varphi$ provided that $r = \sup\{a \mid \Gamma \vdash \langle a, \varphi \rangle\}$; this is the provability degree of $\varphi$ in $\Gamma$.

**Theorem 3.3. (Pavelka completeness for GRPL)** \cite{pavelka1979} For any fuzzy theory $\Gamma$ and any formula $\varphi$, the provability degree of $\varphi$ from $\Gamma$ in GRPL coincides with its validity degree under $\Gamma$.

Hájek’s paper \cite{pavelka1979} only presents the system GRPL briefly, on its way to an undecidability result (the paper comprises five pages of two-column print). It is fair to conjecture that if the system were developed in more detail, it would also have included rules allowing to switch between graded and non-graded formulas. Indeed, while Pavelka considers $\mathcal{L}$-consequence...
relations, where \( \mathcal{L} \) stands for a complete lattice (in part I) or a complete \( \text{FL}_{ew} \)-algebra\(^{14} \) (in part II), and speaks about \( \mathcal{L} \)-consequence operators (p. 46) and \( \mathcal{L} \)-deductive systems, he does also consider usual formulas in the \( \text{FL}_{ew} \)-language with constants for elements in some complete \( \text{FL}_{ew} \)-algebra, and obtains the free algebra from the term algebra, as usual in algebraic logic.

4 Relationships between the logics GRPL, RPL, and \( \mathcal{L} \)

RPL is an expansion\(^{15} \) of \( \mathcal{L} \): namely, RPL is obtained from a presentation of \( \mathcal{L} \) by adding the bookkeeping axioms (for all basic connectives used in that presentation). Notice that whenever a logic \( \mathcal{L}' \) expands a logic \( \mathcal{L} \), this entails conservativity for \( \mathcal{L}' \) over \( \mathcal{L} \). On the propositional level, the conservativity of RPL over \( \mathcal{L} \) is an immediate consequence of the finitarity of both logics, the local deduction theorem, soundness w.r.t. the standard semantics, and standard completeness theorem for \( \mathcal{L} \); see [17, Prop. 9] for an analogous consideration in another logic with constants. On the other hand, it is nontrivial to prove that first-order Rational Pavelka logic is conservative over first-order Łukasiewicz logic, where standard completeness is not available; see [27, Corollary 2.5] for the result. Thus, \( \mathcal{L} \) is a fragment of RPL, obtained by considering the language without rational constants (except the constants 0 and 1, either of which are either present in the basic set of connectives, or term definable from them).

It is clear that RPL cannot, in turn, be viewed as a fragment of \( \mathcal{L} \), since RPL defines functions that \( \mathcal{L} \) does not: in particular, if \( q \) is a rational number in the interval \((0, 1)\), then there is a term \( t \) in the language of RPL such that the interpretation of \( t \) in the standard RPL-algebra is the constant function \( q \): namely, \( t \) is the term \( q \). No such term is available in \( \mathcal{L} \), due to McNaughton theorem.

Still, for many purposes, one can use implicit definability of rationals in the standard MV-algebra \([0, 1]_L \) and in its rational subalgebra \( Q_L \), to translate between provable consecutions in RPL and provable consecutions in \( \mathcal{L} \), and thus effectively “read” the former from the latter. This is based on the account given in [20], which we proceed to explain.

Consider the algebra \([0, 1]_L \). Let \( a \in [0, 1] \), let \( \varphi(x_1, \ldots, x_n) \) be a formula in the language \( \mathcal{L}(L) \) and consider an \( i \) s.t. \( 1 \leq i \leq n \). The formula \( \varphi \) implicitly defines the element \( a \) in variable \( x_i \) within \([0, 1]_L \) provided that \( \varphi \) is satisfiable in \([0, 1]_L \) (that is, there is an assignment \( u \) such that \( u(\varphi) = 1 \) in \([0, 1]_L \)) and moreover, for each assignment \( u \) s.t. \( u(\varphi) = 1 \), we have \( u(x_i) = a \). An element \( a \) is (implicitly) definable in \([0, 1]_L \) provided that there is a formula that defines it there.

In other words, an element of the real interval \([0, 1] \) is implicitly definable provided that it is the only solution, in a given variable, to some finite system of equations in the language of MV-algebras within \([0, 1]_L \). Now given a rational \( p/q \) and an \( n \in \mathbb{N} \setminus \{0\} \), and \( i \leq n \), it is easy to define an \( n \)-variable McNaughton function \( f(x_1, \ldots, x_n) \) that yields the value 1 only

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\(^{14}\)Pavelka speaks about complete residuated lattices and this reflects a mismatch in terminology: “residuated lattice” may imply integrality and commutativity (as it did to Ward and Dilworth, who coined the term in [61]), along with boundedness (as it did to Pavelka); or none of these (it is used in this general sense in [19]).

\(^{15}\)A logic \( \vdash' \) expands a logic \( \vdash \) provided that \( \mathcal{L}(\vdash') \supseteq \mathcal{L}(\vdash) \) and for every set of formulas \( T \cup \{ \varphi \} \) of \( \mathcal{L}(\vdash) \), we have \( T \vdash \varphi \) if and only if \( T \vdash' \varphi \).
if $x_i = p/q$: namely

$$f(x_1, \ldots, x_n) = \begin{cases} 
0 & \text{if } x_i \leq (p-1)/q \\
q x_i - p + 1 & \text{if } x_i \in [(p-1)/q, p/q) \\
-q x_i + p + 1 & \text{if } x_i \in [p/q, (p+1)/q) \\
0 & \text{if } x_i \geq (p+1)/q.
\end{cases}$$

The function $f$ is term-definable in $[0, 1]$, so let $\varphi_f$ be the formula that defines this function. Then $\varphi_f$ implicitly defines the rational number $p/q$ in variable $x_i$, within $[0, 1]$.

Each researcher working in Łukasiewicz logic or MV-algebras may have their favourite way of defining the rationals in the standard MV-algebra $[0, 1]$. Here we rely on a result of Torrens: [60, Section 2] implies that the equation $a^{n-1} = -a$ only has one solution in $[0, 1]$, namely $(n-1)/n$, and hence the equation $a = (-a)^{n-1}$ has the unique solution $1/n$. Under this implicit definition, to define $m/n$ for $m < n$, it is sufficient to consider the term $m \cdot 1/n$.

We mention in passing that Łukasiewicz logic does not have the Beth property [34]. Indeed the equivalence $p \equiv (-p)^{n-1}$ implicitly defines $1/n$, while this constant is not explicitly definable (i.e., term-definable) in $[0, 1]$ (and consequently, in $\mathbb{L}$); see also the discussion in [29, Section 3.3].

On the other hand, if a formula in the language of $\mathcal{L}(\mathbb{L})$ is satisfiable in $[0, 1]$, then it is satisfied by an assignment in rational numbers (with small denominators) [13, 31, 2, 28]. It follows that no irrational number is implicitly definable in $[0, 1]$.

Thus one can implicitly define precisely each of the rationals in $[0, 1]$. Hájek uses this fact as a technical device to prove finite strong standard completeness of RPL: see [26, 3.3.11 – 3.3.14] [16]. Moreover, as pointed out in [29], a straightforward way to obtain an implicit definition of any rational number $p/q$ is via the bookkeeping axioms: to implicitly define $p/q$, take a version of the bookkeeping axioms and select all such axioms that only use the constants $0, 1/q, \ldots, (q-1)/q$, $\top$; call this set $B_q$. Notice that this set is finite. Let $B_q^*$ be obtained from $B_q$ by replacing each constant $p/q$ with a variable $z_{p/q}$, distinct from ordinary propositional variables, and in such a way that for $p \neq p'$ the variables $z_{p/q}$ and $z_{p'/q}$ are distinct. Notice that $B_q^*$ features no propositional constants except possibly 0 and 1. Then the theory $B_q^*$ is consistent and (i.e., term-definable) in the rationals $0, 1/q, \ldots, (q-1)/q, 1$. In particular, $B_q^*$ proves the Torrens formula $z_{1/q} \equiv (-z_{1/q})^{q-1}$ in $\mathbb{L}$. If $v$ is a model of $B_q^*$, we have $v(z_{p/q}) = p/q$ for each $q \in \mathbb{Q}$. Finally, this construction is independent of the choice of the set of bookkeeping axioms (which, in turn, depends on the set of basic connectives of $\mathbb{L}$).

It follows that the only expansion of the algebra $[0, 1]_L$ with rational constants is the canonical one, i.e., the one where the algebraic constant 7 is interpreted as $q \in [0, 1]$. This was proved in [16] (see the remark under Theorem 26 of that paper). This is in tight relation

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16The latter equation was also used in [31, Section 6]. With a slight abuse of language, we use equations instead of implicit definitions, without further mention.

17In infinitary Łukasiewicz logic, the notion of implicit definability naturally extends to infinite theories, whereby each irrational number becomes implicitly definable; see [29, Section 3.4].

18The proof is based on the fact that, given a finite set $T \cup \{\varphi\}$ of formulas of $\mathcal{L}(\mathbb{L})$, if $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ are all the rational constants in $T \cup \{\varphi\}$ and if $\alpha_1, \ldots, \alpha_n$ are formulas that implicitly define $q_1, \ldots, q_n$ in $[0, 1]_L$ (assuming that the variables occurring in $\alpha_i$ are disjoint from the ordinary propositional variables and that the variables in $\alpha_i$ and $\alpha_j$ are disjoint whenever $i \neq j$), then $T \models [0, 1]_L \varphi$ if and only if $\exists t \in [0, 1]_L \varphi^*$, where $\varphi^*$ are obtained from $T$, $\varphi$ when each constant $q_i$ is replaced with the variable in $\alpha_i$ that implicitly defines it—say, a $z_i$. 

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to the fact that the standard MV-algebra has no nontrivial automorphisms [9 Corollary 7.2.6]. This fact may render the notion “canonical” rather superfluous in RPL; but the notion becomes important in, e.g., the expansion of product logic with constants, where not all interpretations of constants in the standard product algebra are canonical; see [57].

We now proceed to examine the relationship between the logics RPL and GRPL. Given how Hájek’s works around these topics (namely, the papers [24, 25] and finally, section 3.3 in [26]) are presented, it can be assumed that the series of lemmas here were known to him. Technically the statements are very easy to derive provided only one is familiar with both the systems RPL and GRPL. We present them here so that the reader may appreciate some of their implications, which bear on the discussion of the respective merits of fuzzy logics with graded syntax vs. those with classical syntax.

**Lemma 4.1.** The following are derivable rules in GRPL: \( \langle q, \varphi \rangle \vdash (\langle 1, \overline{q} \rangle \to \varphi). \)

*Proof.* Use graded modus ponens for right-to-left direction; lifting for the converse one.  

**Lemma 4.2.** (GRPL embeds in RPL) Let \( T \) be a set of graded formulas and \( \langle q, \varphi \rangle \) a graded formula. Then \( \vdash_{\text{GRPL}} \langle q, \varphi \rangle \) if and only if \( \vdash_{\text{RPL}} (\psi \to \varphi \mid \langle r, \psi \rangle \in T) \).

*Proof.* First assume \( \vdash_{\text{GRPL}} \langle q, \varphi \rangle \): there is a sequence of graded formulas such that each element is an instance of an axiom of GRPL, a \( T \)-axiom, or obtained from earlier elements using graded modus ponens or lifting, and the last element is \( \langle q, \varphi \rangle \). From this sequence, the desired proof in RPL can easily be obtained with the help of the following observations:
- if \( \langle q, \varphi \rangle \) is an axiom instance of GRPL, then \( \overline{q} \to \varphi \) is a theorem of RPL;
- the rules \( \overline{r} \to \varphi \overline{s} \to \psi \) and \( \overline{q} \to \varphi \) are derivable in GRPL.

The rule on the left is Lemma 3.3.2 in [26] (still named “graded modus ponens” in that text). The one on the right is provable thanks to the provability of the implication \( (\overline{q} \to \varphi) \to ((\overline{r} \to \overline{q}) \to (\overline{r} \to \varphi)) \) (an instance of the axiom \( \text{(A1)} \) of Hájek’s Basic Logic BL, cf. [26]) in the language of RPL; then swap \( \overline{r} \to \overline{\varphi} \) for \( \overline{r} \to \overline{\varphi} \) using bookkeeping.

Assume now that the sequence \( A = \alpha_1, \ldots, \alpha_n \) is an RPL-proof of \( \overline{q} \to \varphi \) from \( \{ \overline{r} \to \varphi \mid \langle r, \psi \rangle \in T \} \): each \( \alpha_i \) is an axiom instance of RPL, a formula \( \overline{r} \to \varphi \) for \( \langle r, \psi \rangle \) an element of \( T \), or derived from earlier elements of \( A \) using modus ponens, while \( \overline{\alpha_n} \to \varphi \). Take the sequence of graded formulas \( A' = \alpha'_1, \ldots, \alpha'_m \) with \( \alpha'_i = \langle 1, \alpha_i \rangle \). Notice that if \( \alpha_i \) is provable from some \( \alpha_j \) and \( \alpha_k \) (with \( j, k < i \)) in RPL, then \( \alpha'_i \) is provable from \( \alpha'_j \) and \( \alpha'_k \) using graded modus ponens in GRPL. If \( \alpha_i \) is an axiom instance of RPL, then \( \langle 1, \alpha_i \rangle \) is an axiom instance of GRPL, and if \( \alpha_i \) is \( \overline{r} \to \varphi \) for \( \langle r, \psi \rangle \in T \), then \( \langle 1, \alpha_i \rangle \) and \( \langle r, \psi \rangle \) form an invertible rule in GRPL, by Lemma 4.1. Hence we have a proof of \( \alpha'_n \) in GRPL from \( T \); using again Lemma 4.1 to get \( \langle q, \varphi \rangle \) from \( \alpha'_n \) in GRPL, one obtains the desired GRPL-proof.  

The following lemma is immediate:

**Lemma 4.3.** Let \( T \) be a set, and \( \varphi_1, \varphi_2, \ldots, \varphi_n \) a sequence, of formulas in \( \mathcal{L}(\text{RPL}) \). Then \( \varphi_1, \varphi_2, \ldots, \varphi_n \) is a proof in RPL from \( T \) if and only if the sequence \( \{ \langle 1, \varphi_1 \rangle, \langle 1, \varphi_2 \rangle, \ldots, \langle 1, \varphi_n \rangle \} \) (all grades 1) is a proof in GRPL from \( \{ \langle 1, \psi \rangle \mid \psi \in T \} \).

**Lemma 4.4.** Let \( T_0 \) be a set of formulas in \( \mathcal{L}(\text{RPL}) \), \( T \) the set of graded formulas \( \{ \langle 1, \varphi \rangle \mid \varphi \in T_0 \} \), and \( \langle 1, \varphi \rangle \) a graded formula. Assume \( \vdash_{\text{GRPL}} \langle 1, \varphi \rangle \). Then there is a GRPL-proof \( P \) of \( \langle 1, \varphi \rangle \) from \( T \) such that all formulas in \( P \) have the grade 1.
Proof. Let \( P_0 \) be any GRPL-proof of \( \langle 1, \varphi \rangle \) from \( T \): a sequence \( \langle q_1, \varphi_1 \rangle , \ldots , \langle q_n, \varphi_n \rangle \), such that \( q_n \) is 1 and \( \varphi_n \) is \( \varphi \).

Define a sequence \( P_1 \) of \( n \) graded formulas: if \( \langle q_i, \varphi_i \rangle \) is the \( i \)-th element in \( P_0 \), let \( \langle 1, q_i \rightarrow \varphi_i \rangle \) be the \( i \)-th element in \( P_1 \).

Now recall that if \( \langle r, \varphi \rangle \) is an axiom of GRPL, then \( \langle 1, r \rightarrow \varphi \rangle \) is provable in GRPL; and moreover, analogously to the observations in proof of Lemma 4.2, the rules

\[
\frac{\langle 1, r \rightarrow \varphi \rangle}{\langle 1, r \rightarrow \psi \rangle}
\]

\[
\frac{\langle 1, (\varphi \rightarrow \psi) \rangle}{\langle 1, (r \rightarrow \varphi) \rightarrow (s \rightarrow \psi) \rangle}
\]

are derivable in GRPL.

But \( P_1 \) is not a proof from \( T \): if \( \langle 1, \psi \rangle \) is an element of the sequence \( P_0 \) for some formula \( \psi \in T_0 \), this has been replaced in \( P_1 \) by the formula \( \langle 1, 1 \rightarrow \psi \rangle \). But these two graded formulas are interderivable by Lemma 4.1 so one can formally prefix the sequence \( P_1 \) with the finitely many formulas \( \langle 1, \psi \rangle \) from \( T \) that are used in the proof. This (possibly longer) sequence \( P_2 \) is indeed a GRPL-proof from \( T \). Finally, one derives \( \langle 1, \varphi \rangle \) from the last line \( \langle 1, T \rightarrow \varphi \rangle \) of \( P_2 \), using Lemma 4.1. This yields the sequence \( P \). \( \square \)

Corollary 4.5. (RPL embeds in GRPL.) Let \( T \cup \{ \varphi \} \) be a set of formulas of the language of RPL. Then \( T \vdash_{\text{RPL}} \varphi \) if and only if \( \{ \langle 1, \psi \rangle \mid \psi \in T \} \vdash_{\text{GRPL}} \langle 1, \varphi \rangle \).

Proof. Left to right: use Lemma 4.3. Right to left: use Lemma 4.3 then again apply Lemma 4.3 \( \square \)

Corollary 4.6. (GRPL embeds in GRPL.) Let \( T \) be a set of graded formulas and \( \langle q, \varphi \rangle \) a graded formula. Then \( T \vdash_{\text{GRPL}} \langle q, \varphi \rangle \) if and only if \( \{ \langle 1, r \rightarrow \psi \rangle \mid (r, \psi) \in T \} \vdash_{\text{GRPL}} \langle 1, q \rightarrow \varphi \rangle \). Moreover, the proof whose existence is asserted on the right can be chosen in such a way that all graded formulas in the proof have the grade 1.

In plain words, the provability relation in GRPL is fully captured by those formulas and proofs in the logic GRPL whose degree is 1. This is witnessed by an embedding that can be carried out with minimal overhead.

5 Discussion

Several authors, on various occasions, voiced their views on the respective similarities and differences between the three types of logics of our interest. An early example among such remarks is the already quoted one (see Sections 112 of Goguen [23, p. 365] on semantics making its way into the syntax; on this issue, recall our discussion of RPL-algebras as expansions of MV-algebras in Section 3 pointing out that the introduction of constants and the bookkeeping axioms significantly restrict the range of algebras that form the equivalent algebraic semantics. This is peculiar to MV-algebras, for some other classes (such as product algebras) the situation may be different.

Interestingly, Hájek offered subsequently two comments that may indicate some evolution of his view. The earlier one can be found in [25] and pertains to RPL [24]. “Logics of partial truth were studied, in a very general manner, as early as in the seventies by the Czech mathematician Jan Pavelka [51] and since then have been substantially simplified; I refer to [24] but here we describe a still simpler version. It is very different from the original Pavelka’s

19The numbering of the references in this and subsequent quotations is replaced with the numbering of the same works in our references.
version and looks as an ‘innocent’ extension of Lukasiewicz’s $L$; but the main completeness result of Pavelka still holds.” The paper [25] is a survey of various systems of fuzzy logic, presented at the SOFSEM conference to an audience of varied background; it does not give any proofs. Nevertheless, it is within this paper that Hájek first introduces RPL in the form in which it will have, some years later, been presented in [26] in full. Compared to the presentation of [24], it omits graded formulas altogether, while both papers employ constant only for the rational elements, and both simplify the axioms. We stress that the comparison in [25] is made between RPL on the one hand, and (not GRPL but) Pavelka’s logic from [51] on the other; thus indeed there are many differences. But our view is that the “innocence” mentioned in the quotation refers predominantly to the absence of graded formulas in RPL, rather than the other (notable) differences such as the countability of language of RPL and simpler axiomatization for the $L$-fragment. By the time his monograph on fuzzy logic was finished, though, Hájek had settled into the following view [26, Section 3.3]: “A graded formula is a pair $(\varphi, r)$ where $\varphi$ is a formula and $r$ a rational element of $[0,1]$; it is just another notation for the formula $(7 \rightarrow \varphi)$.” In other words, Hájek takes the embedding we spelled out in Lemma 4.2 for granted, and considers the difference between graded formulas and implications to be merely notational.

Some authors do not really distinguish between the logics that employ graded formulas explicitly (here represented by the logic GRPL) those that merely employ constants (such as RPL): in [18, p. 628] Esteva, Godo, and Marchioni write: “On the other hand, in some situations one might be also interested in explicitly representing and reasoning with intermediate degrees of truth. A way to do so, while keeping the truth preserving framework, is to introduce truth-constants into the language. This approach actually goes back to Pavelka [75], who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Lukasiewicz logic obtained by adding to the language a truth-constant $r$ for each real $r \in [0,1]$, together with some additional axioms.” Here, of course, the two equivalent systems both feature uncountable set of constants, but the difference between them is of the same nature as with GRPL and RPL. The equivalence in this quotation is rendered in Section 3 in Lemmas 4.2 and 4.5.

Finally, let us recall the opinion Hájek, Paris, and Shepherdson [27], the paper which provided a cue for the title of our survey. “From our conservation results it could be argued that Rational Pavelka logic is the preferred system to use in that it has the best of both worlds. It does not extend Lukasiewicz logic (Theorem 2), even for statements involving partial truth (Theorem 2.6 and Section 3), yet adding truth constants for all real truth values in $[0,1]$ (to give the original Pavelka logic) only produces a conservative extension (Theorem 2.7). [On the other hand one might also argue that since the two logics are equivalent, Lukasiewicz logic is to be preferred because of its greater syntactical simplicity.]” Here, the two worlds that the quotation refers to appear to be Lukasiewicz logic on the one hand and Pavelka logic on the other; thus RPL is neither, and is viewed as a suitable compromise between the two. The equivalence stipulated within the brackets is accounted for by Theorem 2.6 of the paper.

These views of Hájek and other authors certainly have met with considerable dissent. It turns out that it has been the use of graded formulas, theories, and proofs, rather than a mere presence of rational (or real) constants, what many researchers have taken to be the genuine insignia of fuzzy logic. And moreover, since the graded logic of Pavelka, developed in the late

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20 The numbers of theorems in the quotation refer to the numbering in [27], and the text in square brackets is a part of the original quotation, rather than an insertion of the current author.
1970’s, preceded the tumultuous development of mathematical fuzzy logic at the turn of the century, it has been the involvement of Hájek that has been perceived as a dissent.

One such stance was expressed in the work of Bělohlávek, while analyzing the wake of Pavelka’s work, in his survey paper [1]: “This RPL [GRPL in our sense] is still a kind of abstract fuzzy logic with the notion of degree of provability defined as by Pavelka.// He [Hájek] later presented RPL in his [26]. This RPL, however, is conceptually different from the RPL of [24] [GRPL in our sense]. Namely, it is not an abstract fuzzy logic but rather an expansion of the ordinary Lukasiewicz infinitely-valued logic by truth constants for rationals with extra axioms regarding truth degrees (essentially same as those described above). In this logic, the genuine notions of abstract fuzzy logic, such as that of degree of provability, are ‘simulated’ by the ordinary notions.” Bělohlávek makes a direct comparison of the logics RPL and GRPL, pinpointing the perceived difference (which rests neither in the cardinality of the set of constants, nor in the axiomatization of the L-fragment, but in the presence of graded formulas and graded proofs and the natural way in which they yield the notion of provability degree.

Still more dismissive views on Hájek’s simplifications of Pavelka logic, which eventually yielded RPL, have been put forward by Novák [48, p. 1100]: “Then the provability degree of a formula $A$ is defined […] Unfortunately, Hájek did not provide a sound justification of this notion and introduced it only as an additional and not really organic concept. RPL thus became a special extension of Lukasiewicz logic, for which it is unclear why it should be studied.”

In this survey, we have taken some trouble to spell out simple technical lemmas that other authors may gloss over in pursuit of more interesting mathematics. The technicalities hopefully help to make the following simple point. The transition from GRPL to RPL begins by admitting the validity of our Corollary 4.6: the logic GRPL embeds into itself in such a way that all formulas in the range of the embedding have degree 1. Moreover, each GRPL-provable consecution on a set of premises with all grades 1 and a conclusion with grade 1 has a GRPL-proof in which each formula also has grade 1. The embedding is obtained invoking the lifting rule, present both in GRPL and in Pavelka logic.

Even if the logic RPL had never been conceived and developed, as a next step in the investigation of fuzzy logics with constants, it would still be clear, from the metamathematical considerations on GRPL alone, that extant grades for all formulas that the formalism employs are an explicit formulation of what is implicit already in the fragment of GRPL that only uses the grade 1: this includes provability degrees and (a version of) Pavelka completeness. Put yet differently, GRPL simulates itself (in the same sense in which Bělohlávek claims RPL simplates GRPL) with graded formulas where all grades are 1.

Given the above, our inquiry is not inconclusive: we argue the view that graded syntax systems, such as GRPL, merged into the traditional ones, such as RPL, due to an application of Occam’s razor, which yielded an ostensibly simpler, algebraizable logic, with fewer axioms, with the usual syntax and a few truth constants thrown in. The fact that the provability relation in GRPL can be fully encoded in formulas and proofs with all grades 1 makes the need for graded formulas questionable. In particular, researchers championing graded syntax admit that it hardly makes sense to explicitly employ grades if they are all 1: “It should be noted that traditional syntax can also be taken as evaluated [graded in our sense], but the only evaluations of formulas are 1 or 0. This is an extreme situation and of course, it can hardly have any sense to write explicitly the evaluation 1 with each formula.” (see [48, p. 1063]) The introduction of the logic RPL thus did precisely this: dropped the degree 1 and
made more explicit the above mentioned embedding.

We pointed out that RPL restricts the class of algebras that forms the equivalent algebraic semantics in comparison to L: only the trivial MV-algebra or an MV-algebra that contains an isomorphic copy of the MV-algebra on the rationals in $[0, 1]$ with the usual order as a subalgebra can be expanded to an RPL-algebra. Despite this fact, are there still reasons to prefer RPL over L? It is useful to realize here that already the MV-algebra on the rationals in $[0, 1]$ with the usual order has the desiderata for graded reasoning in a finite setting (which covers most real-life situations, except those where infinite models of appear, for some reason, preferable): bounded, dense, totally ordered. In fact for such reasoning in a finite setting, a finite MV-chain is sufficient, and the MV-chain on the rationals is the smallest MV-chain among those MV-algebras into which all finite MV-chains embed. To conclude, we borrow an elegant quotation from [27, p. 677]: “...even for partial truth, Rational Pavelka logic deals with exactly the same logic as Lukasiewicz logic—but in a very much more convenient way.” The elegance is a cognitive one—we remark in passing, for example, that it is computationally hard to decide which rational value (if any) is implicitly defined by a particular formula of Lukasiewicz logic [29, Lemma 4.6].

6 Concluding remarks

The material presented in this survey can be read as a contribution, within the family of Lukasiewicz logic, to the study of fragments of logics, embeddings between logics, and the merit of studying these phenomena. It is clear that either of the embeddings between RPL and GRPL is computationally feasible, in fact almost trivial. The same is true—but less obviously—about translating the finite provability relation of RPL into the finite provability relation in L: this was shown in [26, 27], moreover by [29, Theorem 4.3] it can be done via a function operating in polynomial time in the size of the input.

One direction for a future research in the area of expanded languages is indicated by the close connection between MV-algebras and lattice-ordered abelian groups: since the variety is generated by the multiplicative group on the positive rationals, it is natural to wonder about the relationship between the theory of this group and the theory of its expansion with constants for rationals.

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