**f(R) gravity with torsion: a geometric approach within the J-bundles framework**

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We discuss the f(R)-theories of gravity with torsion in the framework of J-bundles. Such an approach is particularly useful since the components of the torsion and curvature tensors can be chosen as fiber J-coordinates on the bundles and then the symmetries and the conservation laws of the theory can be easily achieved. Field equations of f(R)-gravity are studied in empty space and in presence of various forms of matter as Dirac fields, Yang–Mills fields and spin perfect fluid. Such fields enlarge the jet bundles framework and characterize the dynamics. Finally we give some cosmological applications and discuss the relations between f(R)-gravity and scalar-tensor theories.

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\section{I. INTRODUCTION}

In some previous papers\textsuperscript{[1, 2, 3]}, a new geometric approach for Gauge Theories and General Relativity (GR), in the tetrad–affine formulation, has been proposed. It is called the J-bundles framework (from now on the J-bundles).

The starting point for the construction of such a framework is that several field Lagrangians have corresponding Lagrangian densities which depend on the fields derivatives only through suitable antisymmetric combinations. This is the case of the Einstein–Hilbert Lagrangian which, in the tetrad–affine formulation, depends on the antisymmetric derivatives of the spin–connection through the curvature.

In view of this fact, the basic idea developed in\textsuperscript{[1, 2, 3]} consists in defining a suitable quotient space of the first jet–bundle, making equivalent two sections which have a first order contact with respect to the exterior differentiation (or, equivalently, with respect to the exterior covariant differentiation), instead of the whole set of derivatives. The resulting fiber coordinates of the so defined new spaces are exactly the antisymmetric combinations appearing in the Lagrangian densities.

For GR, it has been shown that the fiber coordinates of the quotient space can be identified with the components of the torsion and curvature tensors and the approach results particularly useful in the gauge treatment of gravity (see\textsuperscript{[4]} for a general discussion).

The aim of this paper is to extend the mathematical machinery developed in\textsuperscript{[3]} to the f(R)-theories of gravity with torsion\textsuperscript{[5]} in order to recast such theories in the J-bundle formalism. As we will see, such an approach is particularly useful to put in evidence the peculiar geometric structures of the theories, as symmetries and conservation laws.

Before starting with this program, we want to recall what are the physical motivations to enlarge the Hilbert–Einstein approach to more general theories.

The basic considerations are related to cosmology and quantum field theory since several shortcomings of GR have induced to investigate whether such a theory is the only fundamental scheme capable of explaining the gravitational interaction (see\textsuperscript{[6, 7, 8]} for a review). Other motivations to modify GR come from the old issue to construct a theory capable of recovering the Mach principle.\textsuperscript{[9, 10, 11]}

One of the most fruitful approaches has been that of \textit{Extended Theories of Gravity} (ETGs) which have become a sort of paradigm in the study of gravitational interaction. It is based on taking into account physically motivated corrections and enlargements of the Hilbert–Einstein action. In particular, the f(R)–gravity is the simplest extension consisting in relaxing the very stringent hypothesis that the only gravitational action is that linear in the Ricci scalar \(R\): in f(R)-gravity, generic functions of the Ricci scalar are taken into account, but the approach can be extended to include any curvature invariant and/or any scalar fields because, from a conceptual point of view, there are no \textit{a priori} reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar \(R\), minimally coupled with matter\textsuperscript{[12]}

However ETGs are not the "full quantum gravity" but they are needed as working schemes toward it. In fact, every unification scheme, as Superstrings, Supergravity and Grand Unified Theories, takes into account effective actions where non–minimal couplings or higher–order terms in the curvature invariants come out. Specifically, this scheme has been adopted in order to deal with the quantization on curved spacetimes and the result has been that the interactions between quantum fields and background geometry (or the gravity self–interactions) yield correction terms in the Einstein–Hilbert Lagrangian\textsuperscript{[13, 14]}. Moreover, it has been realized that such corrections are inescapable, if
we want to obtain the effective action of quantum gravity on scales closed to the Planck length \[13\]; finally the idea that there are no “exact” law of physics but that the effective interactions can be described by “stochastic” functions with local gauge invariance (i.e. conservation laws) has been recently taken into serious consideration \[10\].

A part the fundamental physics motivations, all these theories have acquired a huge interest in early and late time cosmology due to the fact that they “naturally” exhibit inflationary eras able to overcome the shortcomings of Standard Cosmological Model \[17, 18\] and well fit the Dark Energy issues of the today observed accelerated behavior \[19\]. The related cosmological models seem realistic and capable, in principle, of matching with the observations \[20, 21\].

In all these approaches, the problem of reducing general theories to the Einstein standard form plus scalar fields has been extensively treated; one can see that, through a Legendre transformation on the metric, higher–order theories, under suitable regularity conditions on the Lagrangian, take the form of the Einstein one in which a scalar field (or more than one) is the source of the gravitational field; on the other side, it has been studied the equivalence between models with variable gravitational coupling with the Einstein gravity through a suitable conformal transformation. In any case, the debate on the physical meaning of conformal transformations has not been solved up to now (see \[22, 23\] and references therein for a comprehensive review). Several authors claim for a true physical difference between Jordan frame (higher-order theories and/or variable gravitational coupling) since there are experimental and observational evidences which point out that the Jordan frame is suitable to match solutions with data. Others state that the true physical frame is the Einstein one according to the energy theorems \[22\]. In any case, the discussion is open and no definite statement has been achieved up to now. The problem can be faced from a more general viewpoint and the Palatini approach to gravity could be useful to this goal. The Palatini approach was firstly introduced and analyzed by Einstein himself \[25\]. It was however called the Palatini approach as a consequence of historical misunderstandings \[26\]. The fundamental idea at the bases of the Palatini formalism is to consider the connection \(\Gamma\), entering the definition of the Ricci tensor, to be independent of the metric \(g\), defined on the spacetime \(\mathcal{M}\). The Palatini formalism for the standard Hilbert–Einstein torsion–less theory results to be equivalent to the purely metric theory: this follows from the fact that the field equations for the connection fields result exactly the same connection \(\Gamma\), firstly considered independent: it is the Levi-Civita connection of the metric \(g\). There is, consequently, no reason to impose the Palatini variational principle in the standard Hilbert-Einstein theory instead of the metric (Einstein) variational principle.

The situation, however, completely changes when we consider the case of EFTs depending on analytic functions of curvature invariants, as \(f(R)\), or non-minimally coupled scalar fields. In these cases, the Palatini and the metric variational principle provide different field equations \[27\].

From a physical viewpoint, considering the metric \(g\) and the connection \(\Gamma\) as independent fields is somehow equivalent to decouple the metric structure of spacetime and its geodesic structure (i.e. the connection is not the Levi-Civita connection of \(g\)), governing respectively the chronological structure of spacetime and the trajectories of particles, moving in it. This decoupling enriches the geometric structure of spacetime and generalizes the purely metric formalism. This metric-affine structure of spacetime (here, we simply mean that a connection \(\Gamma\) and a metric \(g\) are involved) is naturally translated, by means of the same (Palatini) field equations, into a bi-metric structure of spacetime. Beside the physical metric \(g\), another metric \(h\) has to be considered. This new metric, at least in the \(f(R)\) theories, is simply related to the connection. As a matter of facts, the connection \(\Gamma\) results to be the Levi-Civita connection of \(h\) and thus provides the geodesic structure of spacetime.

A further ingredient to generalize this metric–affine formalism is considering also torsion in \(f(R)\)-gravity. In \[3\], we have discussed this issue showing that the torsion field plays a fundamental role into dynamics with remarkable applications in cosmology. Here we want to develop \(f(R)\)-gravity with torsion in the framework of the \(J\)-bundles. The paper is organized as follows. In Sect. II, we briefly sketch the main features of the metric-affine approach to \(f(R)\)-gravity with torsion as discussed in \[3\].

In Sect. III, we review the construction of \(J\)-bundles and their main geometric properties. We recall the definition of the Poincaré–Cartan form associated with a given Lagrangian and derive field equation from a variational principle built on the new space.

Sect. IV is devoted to the application of \(J\)-bundles geometry to the tetrad–affine formulation of \(f(R)\)-gravity with torsion. We derive field equations in vacuum and in presence of matter. We study explicitly the coupling with Dirac fields, Yang–Mills fields and spin fluids, giving some applications to cosmological models. Finally, we discuss the equivalence between the \(f(R)\)-gravity and scalar-tensor gravity with torsion. Conclusions are drawn in Sect.V.

### II. \(f(R)\)-GRavity With Torsion: Preliminaries

For convenience of the reader, we briefly sketch the theory discussed in \[3\]. In \(f(R)\)-gravity with torsion, the dynamical fields are the pairs \((g, \Gamma)\) consisting of a pseudo–Riemannian metric \(g\) and a metric compatible linear
connection $\Gamma$ on the space–time manifold $\mathcal{M}$. Such a theory is based on the action functional

$$\mathcal{A}(g, \Gamma) = \int \sqrt{|g|} f(R) \, ds$$

(1)

where $f(R)$ is a real function, $R(g, \Gamma) = g^{ij} R_{ij}$ (with $R_{ij} := R^h_{\alpha ij}$) is the curvature scalar associated with the connection $\Gamma$ and $ds := dx^1 \wedge \cdots \wedge dx^4$. Throughout the paper, we use the index notation

$$R^h_{\alpha ij} = \frac{\partial \Gamma^h_{\alpha ij}}{\partial x^\alpha} - \frac{\partial \Gamma^h_{\alpha ij}}{\partial x^\beta} + \Gamma^h_{\alpha p} \Gamma^p_{\beta ij} - \Gamma^h_{\alpha p} \Gamma^p_{\beta ij}$$

(2)

for the curvature tensor and

$$\nabla_{\alpha} \frac{\partial}{\partial x^\beta} = \Gamma^h_{\alpha ij} \frac{\partial}{\partial x^h}$$

(3)

for the connection coefficients.

As it is well known, given a metric tensor $g_{ij}$, every $g$-metric compatible connection $\Gamma$ can be represented as

$$\Gamma^h_{ij} = \tilde{\Gamma}^h_{ij} - K^h_{ij}$$

(4)

where (in the holonomic basis $\{ \frac{\partial}{\partial x^i}, dx^i \}$) $\tilde{\Gamma}^h_{ij}$ denote the coefficients of the Levi–Civita connection associated with the metric $g_{ij}$ and $K^h_{ij}$ indicate the components of the contortion tensor \[28\]. Therefore, the degrees of freedom of the theory may be identified with the (independent) components of the tensors $g_{ij}$ and $K^h_{ij}$ (the contortion tensor satisfies the antisymmetry property $K^j_{ih} = -K^i_{jh}$).

Variations with respect to the metric and the connection (contortion) give rise to the field equations \[5\]

$$f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} = 0$$

(5a)

$$T^h_{ij} = -\frac{1}{2 f'} \frac{\partial f'}{\partial x^p} \left( \delta^p_i \delta^h_j - \delta^p_j \delta^h_i \right)$$

(5b)

where $T^h_{ij} := \Gamma^h_{ij} - \Gamma^h_{ji}$ denotes the torsion tensor. The presence of matter is embodied in the action functional \[11\] by adding to the gravitational Lagrangian a suitable matter Lagrangian density $\mathcal{L}_m$, namely

$$\mathcal{A}(g, \Gamma) = \int \left( \sqrt{|g|} f(R) + \mathcal{L}_m \right) \, ds$$

(6)

In \[3\], we have supposed the matter Lagrangian being independent of the connection. In this case the field equations result to be

$$f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} = \Sigma_{ij}$$

(7a)

$$T^h_{ij} = -\frac{1}{2 f'(R)} \frac{\partial f'(R)}{\partial x^p} \left( \delta^p_i \delta^h_j - \delta^p_j \delta^h_i \right)$$

(7b)

where $\Sigma_{ij} := -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}}$ plays the role of the energy–momentum tensor.

### III. THE $J$-BUNDLES FORMALISM

#### A. The geometric framework

Let $\mathcal{M}$ be a 4-dimensional orientable space–time manifold, with a metric tensor $g$ of signature $\eta = (1, 3) = (-1, 1, 1, 1)$. Let us denote by $\mathcal{E}$ the co–frame bundle of $\mathcal{M}$. Moreover, let $P \to \mathcal{M}$ be a principal fiber bundle over
\[ M, \text{with structural group } G = SO(1,3). \text{ We denote by } \mathcal{C} := J_1(P)/SO(1,3) \text{ the space of principal connections over } P. \text{ We refer } \mathcal{E} \text{ and } \mathcal{C} \text{ to local coordinates } x^i, e^\mu_i (i, \mu = 1, \ldots, 4) \text{ and } x^i, \omega^{\mu \nu}_i (\mu < \nu) \text{ respectively.}

The configuration space of the theory is the fiber product \( \mathcal{E} \times_M \mathcal{C} (\mathcal{E} \times \mathcal{C} \text{ for short}) \) over \( M \). The dynamical fields are (local) sections of \( \mathcal{E} \times \mathcal{C} \), namely pairs formed by a (local) tetrad field \( e(x) = e^\mu_i(x) dx^i \) and a principal connection 1-form \( \omega(x) = \omega^{\mu \nu}_i(x) dx^i \). We notice that the connection \( \omega(x) \) is automatically metric–compatible with the metric \( g(x) = \eta_{\mu \nu} e^\mu_i(x) \otimes e^\nu_i(x) \) (\( \eta_{\mu \nu} := \text{diag}(-1,1,1,1) \)), induced on \( M \) by the tetrad field \( e^\mu(x) \) itself.

We consider the first \( J \)-bundle \( J(\mathcal{E} \times \mathcal{C}) \) (see \[2\]) associated with the fibration \( \mathcal{E} \times \mathcal{C} \rightarrow M \). It is built similarly to an ordinary \( J \)-bundle, but the first order contact between sections is calculated with respect to exterior (or exterior covariant) differentials. \( J \)-bundles have been recently used to provide new geometric formulations of gauge theories and GR \[1,2,5,29,31,32\].

For convenience of the reader, we briefly recall the construction of the bundle \( J(\mathcal{E} \times \mathcal{C}) \). Let \( J_1(\mathcal{E} \times \mathcal{C}) \) be the first \( J \)-bundle associated with \( \mathcal{E} \times \mathcal{C} \rightarrow M \), referred to local jet–coordinates \( x^i, e^\mu_i, \omega^{\mu \nu}_i, e^\mu_{ij}, \omega^{\mu \nu}_{ij} \). We introduce on \( J_1(\mathcal{E} \times \mathcal{C}) \) the following equivalence relation. Let \( z = (x^i, e^\mu_i, \omega^{\mu \nu}_i, e^\mu_{ij}, \omega^{\mu \nu}_{ij}) \) and \( \hat{z} = (x^i, e^\mu_i, \hat{\omega}^{\mu \nu}_i, e^\mu_{ij}, \hat{\omega}^{\mu \nu}_{ij}) \) be two elements of \( J_1(\mathcal{E} \times \mathcal{C}) \), having the same projection \( x \) over \( M \). Denoting by \( \{e^\mu_i(x), \omega^{\mu \nu}_i(x)\} \) and \( \{\hat{e}^\mu_i(x), \hat{\omega}^{\mu \nu}_i(x)\} \) two different sections of the bundle \( \mathcal{E} \times \mathcal{C} \rightarrow M \), respectively chosen among the representatives of the equivalence classes \( z \) and \( \hat{z} \), we say that \( z \) is equivalent to \( \hat{z} \) if and only if

\[
e^\mu_i(x) = \hat{e}^\mu_i(x), \quad \omega^{\mu \nu}_i(x) = \hat{\omega}^{\mu \nu}_i(x)
\]

and

\[
de^\mu_i(x) = d\hat{e}^\mu_i(x), \quad D\omega^{\mu \nu}_i(x) = D\hat{\omega}^{\mu \nu}_i(x)
\]

where \( D \) is the covariant differential induced by the connection. In local coordinates, it is easily seen that \( z \sim \hat{z} \) if and only if the following identities hold

\[
e^\mu_i = \hat{e}^\mu_i, \quad \omega^{\mu \nu}_i = \hat{\omega}^{\mu \nu}_i
\]

\[
(e^\mu_{ij} - e^\mu_{\hat{j} \hat{i}}) = (\hat{e}^\mu_{ij} - \hat{e}^\mu_{\hat{j} \hat{i}}), \quad (\omega^{\mu \nu}_{ij} - \omega^{\mu \nu}_{\hat{j} \hat{i}}) = (\hat{\omega}^{\mu \nu}_{ij} - \hat{\omega}^{\mu \nu}_{\hat{j} \hat{i}})
\]

We denote by \( J(\mathcal{E} \times \mathcal{C}) \) the quotient space \( J_1(\mathcal{E} \times \mathcal{C})/ \sim \) and by \( \rho : J_1(\mathcal{E} \times \mathcal{C}) \rightarrow J(\mathcal{E} \times \mathcal{C}) \) the corresponding canonical projection. A system of local fiber coordinates on the bundle \( J(\mathcal{E} \times \mathcal{C}) \) is provided by \( x^i, e^\mu_i, \omega^{\mu \nu}_i, E^\mu_{ij} := \frac{1}{2} (e^\mu_{ij} - e^\mu_{\hat{j} \hat{i}}), \Omega^{\mu \nu}_i : = \frac{1}{2} (\omega^{\mu \nu}_i - \omega^{\mu \nu}_{\hat{i} \hat{j}}) \) (\( i < j \)).

The geometry of \( J \)-bundles has been thoroughly examined in Refs. \[1,2,5\]. As a matter of fact, the quotient projection \( \rho \) endows the bundle \( J(\mathcal{E} \times \mathcal{C}) \) with most of the standard features of jet–bundles geometry (\( J \)-extension of sections, contact forms, \( J \)-prolongation of morphisms and vector fields), which are needed to implement variational calculus on \( J(\mathcal{E} \times \mathcal{C}) \).

Referring the reader to \[1,2,5\] for a detailed discussion on \( J \)-bundles geometry, the relevant fact we need to recall here is that the components of the torsion and curvature tensors can be chosen as fiber \( J \)-coordinates on \( J(\mathcal{E} \times \mathcal{C}) \). In fact, the following relations

\[
T^\mu_{ij} = 2E^\mu_{ij} + \omega^{\mu \lambda}_i e^\lambda_j - \omega^{\mu \lambda}_j e^\lambda_i
\]

\[
R^{\mu \nu}_{ij} = 2\Omega^{\mu \nu}_j + \omega^{\mu \lambda}_i \omega^{\lambda \nu}_j - \omega^{\mu \lambda}_j \omega^{\lambda \nu}_i
\]

can be regarded as fiber coordinate transformations on \( J(\mathcal{E} \times \mathcal{C}) \), allowing to refer the bundle \( J(\mathcal{E} \times \mathcal{C}) \) to local coordinates \( x^i, e^\mu_i, \omega^{\mu \nu}_i, T^\mu_{ij} \) (\( i < j \)), \( R^{\mu \nu}_{ij} \) (\( i < j, \mu < \nu \)). In such coordinates, local sections \( \gamma : M \rightarrow J(\mathcal{E} \times \mathcal{C}) \) are expressed as

\[
\gamma : x \rightarrow (x^i, e^\mu_i(x), \omega^{\mu \nu}_i(x), T^\mu_{ij}(x), R^{\mu \nu}_{ij}(x))
\]

In particular, a section \( \gamma \) is said holonomic if it is the \( J \)-extension \( \gamma = J \sigma \) of a section \( \sigma : M \rightarrow \mathcal{E} \times \mathcal{C} \). In local coordinates, a section is holonomic if it satisfies the relations \[8\]

\[
T^\mu_{ij}(x) = \frac{\partial e^\mu_j(x)}{\partial x^i} - \frac{\partial e^\mu_i(x)}{\partial x^j} + \omega^{\mu \lambda}_i e^\lambda_j(x) - \omega^{\mu \lambda}_j e^\lambda_i(x)
\]

\[
R^{\mu \nu}_{ij}(x) = \frac{\partial \omega^{\mu \nu}_j(x)}{\partial x^i} - \frac{\partial \omega^{\mu \nu}_i(x)}{\partial x^j} + \omega^{\mu \lambda}_i \omega^{\lambda \nu}_j(x) - \omega^{\mu \lambda}_j \omega^{\lambda \nu}_i(x)
\]
namely if the quantities $T_{ij}^\mu(x)$ and $R_{ij}^{\mu\nu}(x)$ are the components of torsion and curvature tensors associated with the tetrad $e_i^\mu(x)$ and the connection $\omega_{i}^{\mu\nu}(x)$, in turn, represents the section $\sigma$.

We also recall that the bundle $J(\mathcal{E} \times \mathcal{C})$ is endowed with a suitable contact bundle. The latter is locally spanned by the following 2-forms

$$\theta^\mu = de_i^\mu \wedge dx^i + E_{ij}^\mu dx^i \wedge dx^j$$

(13a)

$$\theta^{\mu\nu} = d\omega_i^{\mu\nu} \wedge dx^i + \Omega_{ij}^{\mu\nu} dx^i \wedge dx^j$$

(13b)

It is easily seen that a section $\gamma: \mathcal{M} \to J(\mathcal{E} \times \mathcal{C})$ is holonomic if and only if it satisfies the condition $\gamma^*(\theta^\mu) = \gamma^*(\theta^{\mu\nu}) = 0 \forall \mu, \nu = 1, \ldots, 4$. Moreover, in the local coordinates $\{x, e, \omega, T, R\}$, the 2-forms (13) can be expressed as

$$\theta^\mu = r^\mu - T^\mu$$

and

$$\theta^{\mu\nu} = \rho^{\mu\nu} - R^{\mu\nu}$$

(14)

being $r^\mu = de_i^\mu \wedge dx^i + \omega_{i}^{\mu\nu} e_i^\nu dx^i \wedge dx^i$, $T^\mu = \frac{1}{2} T_{ij}^\mu dx^i \wedge dx^j$, $\rho^{\mu\nu} = d\omega_i^{\mu\nu} \wedge dx^i + \frac{1}{2} \left( \omega_j^{\mu\nu} e_i^\lambda \omega_i^{\lambda\nu} - \omega_j^{\mu\nu} \omega_i^{\lambda\nu} \right) dx^i \wedge dx^j$ and $R^{\mu\nu} = \frac{1}{2} R_{ij}^\mu dx^i \wedge dx^j$.

### B. The field equations

We call a Lagrangian on $J(\mathcal{E} \times \mathcal{C})$ any horizontal 4-form, locally expressed as

$$L = \mathcal{L}(x^i, e_i^\mu, \omega_i^{\mu\nu}, T_{ij}^\mu, R_{ij}^{\mu\nu}) \, ds$$

(15)

Associated with any such a Lagrangian there is a corresponding Poincaré–Cartan 4-form, having local expression (see [3])

$$\Theta = \mathcal{L} \, ds - \frac{1}{2} \frac{\partial \mathcal{L}}{\partial T_{hk}^\alpha} \theta^\alpha \wedge ds_{hk} - \frac{1}{4} \frac{\partial \mathcal{L}}{\partial R_{hk}^{\alpha\beta}} \theta^{\alpha\beta} \wedge ds_{hk}$$

(16)

where $ds_{hk} := \frac{\partial}{\partial x^h} \int \frac{\partial}{\partial x^k} \, ds$. Taking the identities $dx^i \wedge ds_{ij} = -\delta^i_j \, ds_i + \delta^i_j \, ds_j$ and $dx^p \wedge dx^i \wedge ds_{ij} = - (\delta^p_i \delta^i_j - \delta^p_j \delta^i_i) \, ds$ into account, it is easily seen that the 4-form (16) may be expressed as

$$\Theta = \mathcal{L} \, ds - \frac{\partial \mathcal{L}}{\partial T_{hk}^\alpha} \left( de_h^\alpha \wedge ds_k - \omega_h^{\alpha\nu} e_k^\nu \, ds + \frac{1}{2} T_{hk}^\alpha \, ds \right)$$

$$- \frac{1}{2} \frac{\partial \mathcal{L}}{\partial R_{hk}^{\alpha\beta}} \left( d\omega_h^{\alpha\beta} \wedge ds_k - \omega_h^{\alpha\lambda} \omega_k^{\lambda\beta} \, ds + \frac{1}{2} R_{hk}^{\alpha\beta} \, ds \right)$$

(17)

The field equations are derived from the variational principle

$$\mathcal{A}(\sigma) = \int J \sigma^*(\Theta) = \int J \sigma^*(\mathcal{L} \, ds)$$

(18)

where $\sigma: \mathcal{M} \to \mathcal{E} \times \mathcal{C}$ denotes any section and $J \sigma: \mathcal{M} \to J(\mathcal{E} \times \mathcal{C})$ its $J$-extension satisfying eqs. (12).

Referring the reader to [3] for a detailed discussion, we recall here that the corresponding Euler–Lagrange equations can be expressed as

$$J \sigma^*(J(X) \, \mathcal{L} \, ds) = 0$$

(19)

for all $J$-prolongable vector fields $X$ on $\mathcal{E} \times \mathcal{C}$. Moreover, we notice that the expression of $J$-prolongable vector fields and their $J$-prolongations, involved in eq. (19), is not needed here. In order to make explicit eq. (19), we calculate
Due to the arbitrariness of $\theta$, that is

$$d\Theta = d\mathcal{L} \wedge ds - d \left( \frac{\partial \mathcal{L}}{\partial T^k_{\alpha \beta}} \right) \wedge \left( de^\alpha_k \wedge ds_k - \omega^\alpha_k \omega^\alpha_k ds + \frac{1}{2} T^\alpha_{hk} ds \right)$$

$$- \frac{1}{2} \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \left( -e^\alpha_k d\omega^\alpha_k \wedge ds_k - \omega^\alpha_k \omega^\alpha_k \wedge ds + \frac{1}{2} dT^\alpha_{hk} \wedge ds \right)$$

$$- \frac{1}{2} \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \left( d\omega^\alpha_k \wedge ds_k - \omega^\alpha_k \lambda \omega^\lambda_k \wedge ds + \frac{1}{2} R^\alpha_{hk} ds \right)$$

$$- \frac{1}{2} \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \left( -2\omega^\alpha_k \lambda d\omega^\lambda_k \wedge ds + \frac{1}{2} dR^\alpha_{hk} \wedge ds \right)$$

$$\left. \right) = \frac{\partial \mathcal{L}}{\partial e^\mu_q} ds + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \omega^\mu_{\alpha \beta}} d\omega^\alpha_k \wedge ds + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \wedge ds$$

Choosing infinitesimal deformations $X$ of the special form

$$X = G^\mu_q(x) \frac{\partial}{\partial e^\mu_q} + \frac{1}{2} G^\mu_{q \nu}(x) \frac{\partial}{\partial \omega^\mu_{\alpha \beta}}$$

we have then (20)

$$\mathcal{J}(X) \int d\Theta = \left[ \frac{\partial \mathcal{L}}{\partial e^\mu_q} ds + d \left( \frac{\partial \mathcal{L}}{\partial T^k_{\alpha \beta}} \right) \wedge ds_k + \frac{\partial \mathcal{L}}{\partial T^k_{\alpha \beta}} \omega^\alpha_k \mu ds \right] G^\mu_q$$

$$+ \left[ \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \omega^\mu_{\alpha \beta}} ds + \frac{\partial \mathcal{L}}{\partial T^k_{\alpha \beta}} \eta_{\alpha \beta} ds + \frac{1}{2} d \left( \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \right) \wedge ds_k + \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \omega^\alpha_k \mu ds \right] G^\mu_{\alpha \beta}$$

$$- \mathcal{J}(X) \int d \left( \frac{\partial \mathcal{L}}{\partial T^k_{\alpha \beta}} \right) \wedge \left( de^\alpha_k \wedge ds_k - \omega^\alpha_k \lambda \omega^\lambda_k \wedge ds + \frac{1}{2} R^\alpha_{hk} ds \right)$$

Due to the arbitrariness of $X$ and the holonomy of the $\mathcal{J}$-extension $\mathcal{J}\sigma$ (compare with eqs. (23)), the requirement yields two sets of final field equations

$$\mathcal{J}\sigma^* \left( \frac{\partial \mathcal{L}}{\partial e^\mu_q} + \frac{\partial \mathcal{L}}{\partial T^k_{\alpha \beta}} \omega^\alpha_k \mu \right) - \frac{\partial}{\partial x^k} \left( \mathcal{J}\sigma^* \left( \frac{\partial \mathcal{L}}{\partial T^\mu_k} \right) \right) = 0$$

(23a)

and

$$\mathcal{J}\sigma^* \left( \frac{\partial \mathcal{L}}{\partial \omega^\mu_{\alpha \beta}} + \frac{\partial \mathcal{L}}{\partial T^k_{\alpha \beta}} \omega^\alpha_k \mu \right) + \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \omega^\alpha_k \mu + \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \omega^\alpha_k \mu$$

$$+ \frac{\partial}{\partial x^k} \left( \mathcal{J}\sigma^* \left( \frac{\partial \mathcal{L}}{\partial R^k_{\alpha \beta}} \right) \right) = 0$$

(23b)

To conclude, it is worth noticing that all the restrictions about the vector fields $\mathcal{J}(X)$ in eq. (20) may be removed. In fact, it is easily seen that eq. (19) automatically implies

$$\mathcal{J}\sigma^*(X \int d\Theta) = 0, \quad \forall X \in D^1(\mathcal{J}(E \times C))$$

(24)
IV. $f(R)$-GRAVITY WITHIN THE $J$-BUNDLE FRAMEWORK

A. Field equations in empty space

Let us now apply the above formalism to the $f(R)$ theories of gravity. The Lagrangian densities which we are going to consider are of the specific kind $\mathcal{L} = ef(R)$, with $e = \det (e_i^\mu)$ and $R = R_{ij}^{\mu\nu} e_i^\mu e_j^\nu$. Therefore, taking the identities $\partial e_i^\mu / \partial e_i^\nu = e_i^\mu$ and $\partial e_i^\mu / \partial e_i^\nu = -e_i^\nu e_i^\mu$ into account, we have

$$\frac{\partial \mathcal{L}}{\partial e_i^\mu} = e_i^\mu f(R) - 2e f'(R) R_{\mu\sigma}^{\lambda\nu} e_\lambda^\nu$$

(25a)

$$\frac{\partial \mathcal{L}}{\partial R_{\mu\nu}} = 2e f'(R) [e_i^\nu e_i^\mu - e_i^\mu e_i^\nu]$$

(25b)

In view of this, eqs. (28) become

$$e_i^\mu f(R) - 2f'(R) R_{\mu\nu}^{\lambda\sigma} e_\lambda^\nu = 0$$

(26a)

and

$$\frac{\partial}{\partial x^k} \left[ 2e f'(R) (e_i^\nu e_i^\mu - e_i^\mu e_i^\nu) - \omega_i^\lambda \left( 2e f'(R) (e_i^\nu e_i^\mu - e_i^\mu e_i^\nu) \right) \right] = 0$$

(26b)

After some calculations, eqs. (26b) may be rewritten in the form

$$ef''(R) \frac{\partial R}{\partial x^k} e_i^\alpha - ef''(R) \frac{\partial R}{\partial x^k} e_i^\alpha - ef'(R) (T^\alpha_{ts} - T^\sigma_{t\sigma} e_s^\alpha + T^\sigma_{ts} e_i^\alpha) = 0$$

(27)

where $T^\alpha_{ts} = \frac{\partial e_i^\alpha}{\partial x^s} - \frac{\partial e_i^\alpha}{\partial x^t} + \omega_i^\alpha \lambda e_s^\lambda - \omega_i^\alpha \lambda e_t^\lambda$ are the torsion coefficients of the connection $\omega_i^{\mu\nu}(x)$.

Recalling the relationships $R_{ij}^{\mu\nu} = R_{ij}^{\mu\nu} e_i^\nu e_j^\nu$ and $T_{ij}^{h} = T_{ij}^{h} e_i^\nu e_j^\nu$ among the quantities related to the spin connection $\omega$ and the associated linear connection $\Gamma$, that is $\Gamma_{ij}^{h} = e_i^h \left( \frac{\partial e_i^\mu}{\partial x^j} + \omega_i^\mu \lambda e_j^\lambda \right)$, it is straightforward to see that eqs. (26a) and (27) are equivalent to eqs. (5) obtained in the metric–affine formalism.

At this point, the same considerations made in [6] hold. In particular, let us take into account the trace of the equation (26a), namely

$$2f(R) - f'(R)R = 0$$

(28)

This is identically satisfied by all possible values of $R$ only in the special case $f(R) = kR^2$. In all the other cases, equation (28) represents a constraint on the scalar curvature $R$. As a conclusion, it follows that, if $f(R) \neq kR^2$, the scalar curvature $R$ has to be a constant (at least on connected domains) and coincides with a given solution value of $25$. In such a circumstance, equations (27) imply that the torsion $T^\alpha_{ij}$ has to be zero and the theory reduces to a $f(R)$-theory without torsion, thus leading to Einstein equations with a cosmological constant.

In particular, we it is worth noticing that:

• in the case $f(R) = R$, eq. (28) yields $R = 0$ and therefore eqs. (26a) are equivalent to Einstein’s equations in empty space;

• if we assume $f(R) = kR^2$, by replacing eq. (28) into eq. (26), we obtain final field equations of the form

$$\frac{1}{4} e_i^\mu R - R_{\mu\lambda}^\lambda e_i^\lambda = 0$$

(29a)

$$\frac{1}{R} \frac{\partial R}{\partial x^k} e_i^\alpha - \frac{1}{R} \frac{\partial R}{\partial x^k} e_i^\alpha - (T^\alpha_{ts} - T^\sigma_{t\sigma} e_s^\alpha + T^\sigma_{ts} e_i^\alpha) = 0$$

(29b)

After some straightforward calculations, eq. (29b) can be put in normal form with respect to the torsion, namely

$$T^\alpha_{ts} = -\frac{1}{2R} \frac{\partial R}{\partial x^t} e_s^\alpha + \frac{1}{2R} \frac{\partial R}{\partial x^s} e_i^\alpha$$

(30)
B. Symmetries and conserved quantities

The Poincaré-Cartan formulation of the field equations turns out to be especially useful in the study of symmetries and conserved quantities. To see this point, we recall the following.

Definition IV.1 A vector field \( Z \) on \( J(\mathcal{E} \times \mathcal{C}) \) is called a generalized infinitesimal Lagrangian symmetry if it satisfies the requirement

\[
L_Z(\mathcal{L} ds) = d\alpha
\]

for some 3-form \( \alpha \) on \( J(\mathcal{E} \times \mathcal{C}) \).

Definition IV.2 A vector field \( Z \) on \( J(\mathcal{E} \times \mathcal{C}) \) is called a Noether vector field if it satisfies the condition

\[
L_Z\Theta = \omega + d\alpha
\]

where \( \omega \) is a 4-form belonging to the ideal generated by the contact forms and \( \alpha \) is any 3-form on \( J(\mathcal{E} \times \mathcal{C}) \).

Proposition IV.1 If a generalized infinitesimal Lagrangian symmetry \( Z \) is a \( J \)-prolongation, then it is a Noether vector field.

Proposition IV.2 If a Noether vector field \( Z \) is a \( J \)-prolongation, then it is an infinitesimal dynamical symmetry.

We can associate with any Noether vector field \( Z \) a corresponding conserved current. In fact, given \( Z \) satisfying eq. (35) and a critical section \( \sigma : \mathcal{M} \rightarrow \mathcal{E} \times \mathcal{C} \) we have

\[
dJ\sigma^*(Z \mathcal{J} \Theta - \alpha) = J\sigma^*(\omega - Z \mathcal{J} d\Theta) = 0
\]

showing that the current \( J\sigma^*(Z \mathcal{J} \Theta - \alpha) \) is conserved on shell.

As it is well known, diffeomorphisms and Lorentz transformations (for tetrad and connection) have to be dynamical symmetries for the theory: let us prove it.

To start with, let \( Y = \xi^i \frac{\partial}{\partial x^i} \) be the generator of a (local) one parameter group of diffeomorphisms on \( \mathcal{M} \). The vector field \( Y \) may be “lifted” to a vector field \( X \) on \( \mathcal{E} \times \mathcal{C} \) by setting

\[
X = \xi^i \frac{\partial}{\partial x^i} - \frac{\partial \xi^k}{\partial x^j} e^j_k \frac{\partial}{\partial e^k} - \frac{1}{2} \frac{\partial \xi^k}{\partial x^j} \omega^i_k \mu^\nu \frac{\partial}{\partial \omega^i_{\mu \nu}}
\]

The vector fields are \( J \)-prolongable and their \( J \)-prolongations are expressed as

\[
J(X) = \xi^i \frac{\partial}{\partial x^i} - \frac{\partial \xi^k}{\partial x^j} e^j_k \frac{\partial}{\partial e^k} - \frac{1}{2} \frac{\partial \xi^k}{\partial x^j} \omega^i_k \mu^\nu \frac{\partial}{\partial \omega^i_{\mu \nu}} + T^i_{jk} \frac{\partial}{\partial x^j} \frac{\partial}{\partial \omega^i_{\mu \nu}} + \frac{1}{2} R^i_{jk} \mu^\nu \frac{\partial}{\partial \omega^i_{\mu \nu}}
\]

A direct calculation shows that the vector fields satisfy \( L_{J(X)}(ef(R) ds) = 0 \), so proving that they are infinitesimal Lagrangian symmetries for generic \( f(R) \)-models. Due to Propositions IV.1 and IV.2, we conclude that the vector fields are Noether vector fields and thus infinitesimal dynamical symmetries. There are no associated conserved quantities, the inner product \( \mathcal{J}(X) \mathcal{J} \Theta \) consisting in an exact term plus a term vanishing identically when pulled-back under critical section. We have indeed

\[
\mathcal{J}(X) \mathcal{J} \Theta = \epsilon \xi^i \left( f(R) \delta^i_j - 2f'(R) R^i_{jk} \right) ds_{hk} - \frac{1}{4} \epsilon^{\alpha \beta} \wedge \left( \mathcal{J}(X) \frac{\partial}{\partial R^i_{jk}} \alpha \beta ds_{hk} \right) + \frac{1}{4} \epsilon \omega^j_{\alpha \beta} \frac{\partial}{\partial \omega^j_{\alpha \beta}} ds_{hk} - \frac{1}{4} \epsilon \omega^j_{\alpha \beta} D \left( \frac{\partial}{\partial \omega^j_{\alpha \beta}} \right) \wedge ds_{hk}
\]

where \( D \left( \frac{\partial}{\partial R^i_{jk}} \alpha \beta \right) = D \left( \frac{\partial}{\partial R^i_{jk}} \alpha \beta \right) - \frac{\partial}{\partial R^i_{jk}} \omega^l_{\alpha \beta} \wedge dx^l - \frac{\partial}{\partial R^i_{jk}} \omega^l_{\alpha \beta} \wedge dx^l \).

Infinitesimal Lorentz transformations are represented by vector fields on \( J(\mathcal{E} \times \mathcal{C}) \) of the form

\[
Y = A^i \epsilon^i \sigma^\alpha \frac{\partial}{\partial e^i} - \frac{1}{2} D_q A^\mu \frac{\partial}{\partial \omega^\mu}
\]
where $A^\mu\nu(x) = -A^{\nu\mu}(x)$ is a tensor-valued function on $\mathcal{M}$ in the Lie algebra of $SO(3,1)$, and $D_q A^\mu\nu = \frac{\partial A^\mu\nu}{\partial x^q} + \omega^\mu_q \sigma A^{\sigma\nu} + \omega^\nu_q \sigma A^{\mu\sigma}$. As above, the vector fields $J_1$ are $\mathcal{J}$-prolongable and their $\mathcal{J}$-prolongations are expressed as

$$
\mathcal{J}(Y) = A^\gamma e^\sigma_\gamma \frac{\partial}{\partial e^\sigma_q} - \frac{1}{2} D_q A^\mu\nu \frac{\partial}{\partial \omega^\mu_q} + \frac{1}{2} A^\mu T^\sigma_{ij} \frac{\partial}{\partial T^\sigma_{ij}} + \frac{1}{2} A^\mu R_{ij} \frac{\partial}{\partial R_{ij}}
$$

(38)

It is straightforward to verify that the vector fields $\mathcal{J}_1$ obey the condition $L_{\mathcal{J}(Y)}(ef(R)ds) = 0$. Once again, the conclusion follows that they are infinitesimal dynamical symmetries. Moreover, it is easily seen that

$$
\mathcal{J}(Y) \Theta = -\frac{1}{4} d \left( A^{\alpha\beta} \frac{\partial L}{\partial R_{hk}} ds_{hk} \right) + \frac{1}{4} A^{\alpha\beta} D \left( \frac{\partial L}{\partial R_{hk}} \right) \wedge ds_{hk}
$$

(39)

Therefore, as above, since the inner product $\mathcal{J}(X) \Theta$ consists in an exact term plus a term vanishing identically when pulled–back under critical section, there are no associated conserved quantities.

C. Field equations in presence of matter

In presence of matter, the configuration space–time of the theory results to be the fiber product $\mathcal{E} \times \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ over $\mathcal{M}$, where $\mathcal{E} \times \mathcal{C}$ and the bundle $F \to \mathcal{M}$ where the matter fields $\psi^A$ take their values. The field equations are derived from a variational problem built on the manifold $\mathcal{J}(\mathcal{E} \times \mathcal{C}) \times \mathcal{M} J_1(F)$, where $J_1(F)$ indicates the standard first $\mathcal{J}$-bundle associated with the fibration $F \to \mathcal{M}$.

The total Lagrangian density of the theory is obtained by adding to the gravitational one a suitable matter Lagrangian density $L_m$. Throughout the paper, we shall consider matter Lagrangians of the kind $L_m = L_m(e, \omega, \psi, \psi^A)$. The corresponding Poincaré–Cartan form is given by the sum $\Theta + \theta_m$, where $\theta_m = L_m ds + \frac{\partial L_m}{\partial \psi^A} \theta^A \wedge ds_i$ is the standard Poincaré–Cartan form associated with the matter density $L_m$, being $\theta^A = d\psi^A - \psi^A_t ds_i$ the usual contact 1-forms of the bundle $J_1(F)$.

In such a circumstance, the Euler–Lagrange equations (23) assume the local expression

$$
f'(R) R_{\mu\nu}^{\lambda\sigma} e^\lambda_\mu - \frac{1}{2} e^\mu_\mu f'(R) = \Sigma^i_{\mu}
$$

(40a)

and

$$
f'(R) (T^\alpha_{is} - T^\sigma_{is} e^\alpha_\sigma + T^\sigma_{is} e^\alpha_i) = \frac{\partial f'(R)}{\partial e^\lambda_s} e^\lambda_s - \frac{\partial f'(R)}{\partial e^\lambda_i} e^\lambda_i + S^\alpha_{ts}
$$

(40b)

where $\Sigma^i_{\mu} := \frac{1}{2e} \frac{\partial L_m}{\partial \omega^{\mu\nu}} e^\nu_i$ and $S^\alpha_{ts} := -\frac{1}{2e} \frac{\partial L_m}{\partial \omega^{\mu\nu}} e^\nu_i e^\alpha_i$ play the role of energy–momentum and spin density tensors respectively. In particular, from eqs. (40b), we obtain

$$
f'T^\alpha_{is} = -\frac{3}{2} \frac{\partial f'}{\partial x^i} - \frac{1}{2} S^\alpha_{ts}
$$

(41)

Then, substituting eqs. (41) into eqs. (40b), we find the expression for the torsion

$$
T^\alpha_{ts} = -\frac{1}{2f'} \left( \frac{\partial f'}{\partial x^s} + S^\sigma_{ps} \right) (\delta^\alpha_s e^\sigma_i - \delta^\sigma_i e^\alpha_s) + \frac{1}{f'} S^\alpha_{ts}
$$

(42)

Eqs. (42) tell us that, in presence of $\omega$-dependent matter, there are two sources of torsion: the spin density $S^\alpha_{ts}$ and the nonlinearity of the gravitational Lagrangian. It is important to stress that this feature is not present in standard GR.

Now, by considering the trace of eqs. (40a) we obtain a relation between the scalar curvature $R$ and the trace $\Sigma$ of the energy–momentum tensor given by

$$
f'(R) R - 2f(R) = \Sigma
$$

(43)
When the trace $\Sigma$ is allowed to assume only a constant value, the present theory amounts to an Einstein–like (if $S_i^a = 0$, i.e. $\omega$-independent matter) or an Einstein–Cartan–like theory (if $S_i^a \neq 0$, i.e. $\omega$-dependent matter) with cosmological constant. In fact, in such a circumstance, eq. (43) implies that the scalar curvature $R$ also is constant. As a consequence, eqs. (40a) and (42) can be expressed as

$$R^i_{\mu} - \frac{1}{2} (R + \Lambda) e_i^\mu = k \Sigma^i_{\mu}$$

(44a)

$$T^a_{ts} = \frac{k}{2} (2S^a_{ts} - S^a_\tau e^\tau_s + S^a_\sigma e^\sigma_t)$$

(44b)

where $\Lambda = kf(R) - R$ and $k = \frac{1}{f'(R)}$, $R$ being the constant value determined by eq. (13), provided that $f'(R) \neq 0$. The previous discussion holds with the exception of the particular case $\Sigma = 0$ and $f(R) = \alpha R^2$. Indeed, under these conditions, eq. (13) is a trivial identity which imposes no restriction on the scalar curvature $R$.

From now on, we shall suppose that $\Sigma$ is not forced to be a constant when the matter field equations are satisfied. Besides, we shall suppose that the relation (43) is invertible so that the scalar curvature can be thought as a suitable function of $\Sigma$, namely

$$R = F(\Sigma)$$

(45)

With this assumption in mind, defining the tensors $R^i_j := R_{\mu\sigma} e^\mu_i e^\sigma_j$, $\Sigma^i_j := \Sigma^i_\mu e^\mu_j$, $T^a_{ij} := T^a_{ij} e^a_t$ and $S^a_{ij} := S^a_{ij} e^a_t$, we rewrite eqs. (40a) and (42) in the equivalent form

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{1}{f'(F(\Sigma))} \left( \Sigma_{ij} - \frac{1}{4} S g_{ij} \right) - \frac{1}{4} F(\Sigma) g_{ij}$$

(46a)

$$T^a_{ij} = -\frac{1}{2f'(F(\Sigma))} \left( \partial_{\xi^p} F(\Sigma) \right) \left( S^a_{ij} - \frac{1}{2} S g_{ij} \right)$$

(46b)

In eqs. (46a) one has to distinguish the order of the indexes since, in general, the tensors $R_{ij}$ and $\Sigma_{ij}$ are not symmetric.

Moreover, following [5], in the l.h.s. of eqs. (46a), we can distinguish the contribution due to the Christoffel terms from that due to the torsion dependent terms. To see this point, from eqs. (2) and (4), we first get the following representation for the contracted curvature tensor

$$R_{ij} = \hat{R}_{ij} + \hat{\nabla}_j K^h_{ih} - \hat{\nabla}_h K^j_{ih} + K^p_{ij} K^h_{ph} - K^p_{hi} K^j_{ph}$$

(47)

where $\hat{R}_{ij}$ is the Ricci tensor of the Levi–Civita connection $\hat{\nabla}$ associated with the metric $g_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j$, and $\hat{\nabla}$ denotes the Levi–Civita covariant derivative. Then, recalling the expression of the contortion tensor [28]

$$K^h_{ij} = \frac{1}{2} \left( -T^h_{ij} + T^h_{ji} - T^h_{ij} \right)$$

(48)

and using the second set of field equations (46b), we obtain the following representations

$$K^h_{ij} = \hat{K}^h_{ij} + \hat{S}^h_{ij}$$

(49a)

$$\hat{S}^h_{ij} := \frac{1}{2f'} \left( -S^h_{ij} + S^h_{ji} - S^h_{ij} \right)$$

(49b)

$$\hat{K}^h_{ij} := -T^h_{ji} + T^h_{ip} g^{ph} g_{ij}$$

(49c)

$$\hat{T}^h_{ji} := \frac{1}{2f'} \left( \partial_{\xi^p} S^h_{ji} \right)$$

(49d)

Inserting eqs. (49) in eq. (47) we end up with the final expression for $R_{ij}$

$$R_{ij} = \hat{R}_{ij} + \hat{\nabla}_j \hat{K}^h_{ih} + \hat{\nabla}_h \hat{S}^h_{ij} - \hat{\nabla}_h \hat{K}^j_{ih} - \hat{\nabla}_h \hat{S}^h_{ij} + \hat{K}^j_{ij} K^h_{ph} + \hat{K}^j_{pi} \hat{S}^h_{ij} + \hat{S}^j_{pi} \hat{K}^h_{ph} - \hat{K}^j_{pi} \hat{S}^h_{ij} - \hat{\nabla}_h \hat{S}^h_{ij} + \hat{S}^j_{pi} \hat{K}^h_{ph} - \hat{S}^j_{pi} \hat{S}^h_{ij}$$

(50)

The last step is the substitution of eqs. (50) into eqs. (46a). Explicit examples of the described procedure are given in the next subsections. We will discuss specific cases of fields coupled with gravity acting as matter sources.
D. The case of Dirac fields

As a first example of matter, we consider the case of Dirac fields $\psi$. The matter Lagrangian density is given by

$$\mathcal{L}_D = \frac{i}{2} \left[ \bar{\psi} \gamma^i D_i \psi - D_i \bar{\psi} \gamma^i \psi \right] - m \bar{\psi} \psi$$

(51)

where $D_i \psi = \frac{\partial \psi}{\partial x^i} + \omega_{i\mu} S_{\mu \nu} \psi$ and $D_i \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^i} - \bar{\psi} \omega_{i\mu} S_{\mu \nu}$ are the covariant derivatives of the Dirac fields, $S_{\mu \nu} = \frac{1}{8} [\gamma_\mu, \gamma_\nu]$, $\gamma^i = \gamma^\mu e^i_{\mu}$; $\gamma^\mu$ denotes the Dirac matrices.

The field equations for the Dirac fields are

$$i \gamma^h D_h \psi - m \psi = 0, \quad i D_h \bar{\psi} \gamma^h + m \bar{\psi} = 0$$

(52)

Since the Lagrangian (51) vanishes for $\psi$ and $\bar{\psi}$ satisfying the equation (52), the corresponding energy–momentum and spin density tensors are expressed, respectively, as

$$\Sigma_{ij} = \frac{i}{4} \left[ \bar{\psi} \gamma_i D_j \psi - (D_j \bar{\psi}) \gamma_i \psi \right]$$

(53)

and

$$S^{i}_h = -\frac{i}{2} \bar{\psi} \{ \gamma^h, S_{ij} \} \psi$$

(54)

with $S_{ij} = \frac{1}{8} [\gamma_i, \gamma_j]$. Now, using the properties of the Dirac matrices, it is easily seen that $\{ \gamma^h, S_{ij} \} = \frac{1}{2} \gamma^{[i} \gamma^j \gamma^h]$. This fact implies the total antisymmetry of the spin density tensor $S^{i}_h$. As a consequence, the contracted curvature and scalar curvature assume the simplified expressions (compare with eq. (50))

$$R_{ij} = \tilde{R}_{ij} - 2 \nabla_j \tilde{T}_i - \nabla_i \tilde{T}^h g_{ij} + 2 \tilde{T}_i \tilde{T}_j - 2 \tilde{T}_h \tilde{T}^h g_{ij} - \nabla_i \tilde{S}_j^h - \tilde{S}_h^p \tilde{S}_j^p$$

(55a)

and

$$R = \tilde{R} - 6 \nabla_i \tilde{T}^i - 6 \tilde{T}_i \tilde{T}^i - \tilde{S}_h^p \tilde{S}_j^p$$

(55b)

where now $\tilde{T}_i = \frac{1}{2f} \frac{\partial f}{\partial x^i}$ and $\tilde{S}_i^h := -\frac{1}{2f} S_{ij}^h$. Inserting eqs. (53) into eqs. (46a) and using of the above expression for $\tilde{T}_i$, we obtain the final Einstein–like equations

$$\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{1}{\varphi^2} \left( -\frac{3}{4} \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \varphi \tilde{\nabla}^k g_{ij} + \frac{3}{4} \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \varphi \tilde{\nabla}^k g_{ij} \right) - \varphi \tilde{\nabla}^h \varphi \frac{\partial \varphi}{\partial x^h} g_{ij} + V(\varphi) g_{ij} + \tilde{S}_h^p \tilde{S}_j^p - \frac{1}{2} \tilde{S}_h^p \tilde{S}_j^p \varphi \tilde{\nabla}^h \varphi g_{ij}$$

(56)

where we have defined the scalar field

$$\varphi := f'(F(\Sigma))$$

(57)

and the effective potential

$$V(\varphi) := \frac{1}{4} \left[ \varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2 (f')^{-1}(\varphi) \right]$$

(58)

To conclude, we notice that eqs. (55) can be simplified by performing a conformal transformation. Indeed, setting $\tilde{g}_{ij} := \varphi \eta_{ij} e^i e^j$, eqs. (55) can be rewritten in the easier form

$$\tilde{R}_{ij} - \frac{1}{2} \tilde{R} \tilde{g}_{ij} = \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^2} V(\varphi) \tilde{g}_{ij} + \varphi \tilde{\nabla}^h \tilde{S}_j^h + \tilde{S}_h^p \tilde{S}_j^p - \frac{1}{2\varphi} \tilde{S}_h^p \tilde{S}_j^p \varphi g_{ij}$$

(59)

where $\tilde{R}_{ij}$ and $\tilde{R}$ are respectively the Ricci tensor and the Ricci scalar curvature associated with the conformal metric $\tilde{g}_{ij}$. 


E. The case of Yang–Mills fields

As it has been shown in some previous works [1, 2, 3, 22], also gauge theories can be formulated within the framework of \( \mathcal{J} \)-bundles. Therefore, we can describe \( f(R) \)-gravity coupled with Yang–Mills fields in the new geometric setting.

To see this point, let \( Q \rightarrow \mathcal{M} \) be a principal fiber bundle over space–time, with structural group a semisimple Lie group \( G \). We consider the affine bundle \( J_1Q/G \rightarrow \mathcal{M} \) (the space of principal connections of \( Q \rightarrow \mathcal{M} \)) and refer it to local coordinates \( x^i, a^A_i, A = 1, \ldots, r = \dim G \).

In a combined theory of \( f(R) \)-gravity and Yang–Mills fields, additional dynamical fields are principal connections of \( Q \) represented by sections of the bundle \( J_1Q/G \rightarrow \mathcal{M} \). The extended configuration space of the theory is then the fiber product \( \mathcal{E} \times \mathcal{C} \times J_1Q/G \rightarrow \mathcal{M} \).

Following the approach illustrated in Sec. II, we may construct the quotient space \( \mathcal{J}(\mathcal{E} \times \mathcal{C} \times J_1Q/G) \) (see [8] for details). As additional \( \mathcal{J} \)-coordinates the latter admits the components \( F_{ij}^A \) of the curvature tensors of the principal connections of \( Q \rightarrow \mathcal{M} \). Holonomic sections of the bundle \( \mathcal{J}(\mathcal{E} \times \mathcal{C} \times J_1Q/G) \) are of the form (12) together with

\[
F_{ij}^A(x) = \frac{\partial a^A_i(x)}{\partial x^j} - \frac{\partial a^A_j(x)}{\partial x^i} + a^B_i(x)a^C_j(x)C_{CB}^A
\]

where \( C_{CB}^A \) are the structure coefficients of the Lie algebra of \( G \).

The Poincaré–Cartan 4-form associated with a Lagrangian on \( \mathcal{J}(\mathcal{E} \times \mathcal{C} \times J_1Q/G) \) of the form \( L = \mathcal{L}(x^i, e^\gamma_i, a^A_i, R_{ijkl}^{\mu\nu}, F_{ij}^A) \) is

\[
\Theta = \mathcal{L} \, ds - \frac{1}{4} \partial \mathcal{L} \frac{\partial \mathcal{L}}{\partial R_{ik}} \theta^{ik} \wedge ds_{nk} - \frac{1}{2} \partial \mathcal{L} \frac{\partial \mathcal{L}}{\partial F_{ik}} \theta^i \wedge ds_{jk}
\]

where \( \theta^A = \Phi^A - F^A \), being \( F^A := \frac{1}{2} F_{ij}^A \, dx^i \wedge dx^j \) and \( \Phi^A := da^A_i \wedge dx^i + \frac{1}{2} a^B_i a^C_j C_{CB}^A \, dx^i \wedge dx^j \).

Variational field equations are still of the form

\[
\mathcal{J} \, \sigma^* (\mathcal{J}(X) \, \mathcal{J} \, d\Theta) = 0
\]

for all \( \mathcal{J} \)-prolongable vector fields \( X \) on \( \mathcal{E} \times \mathcal{C} \times J_1Q/G \). Due to the arbitrariness of the infinitesimal deformations \( X \), eq. (62) splits into three sets of final equations, respectively given by eqs. (63) together with

\[
\mathcal{J} \, \sigma^* \left( \frac{\partial \mathcal{L}}{\partial a^A_i} - D_k \frac{\partial \mathcal{L}}{\partial F_{ik}^A} \right) = 0
\]

In particular, if the Lagrangian density is

\[
\mathcal{L} = e \left( f(R) - \frac{1}{4} F_{ij}^A F_j^{B\gamma} \rho^{\mu\nu} \epsilon^\gamma_{\mu\nu} \epsilon^\lambda_{\sigma} \epsilon^e_{\alpha} \right)
\]

expressing \( f(R) \)-gravity coupled with a free Yang–Mills field, eqs. (64) and (65) assume the explicit form

\[
e^i_\mu f(R) - 2 f'(R) R^{\mu\lambda} e^i_\lambda = \frac{1}{4} F_{ik}^A e^i_\mu - F_{ik}^{ij} F_{Ak}^{ij} e^k_\mu
\]

\[
\frac{\partial}{\partial x^k} \left[ 2e f'(R) (e^k_\mu e^i_\nu - e^i_\mu e^k_\nu) \right] - \omega^e_\mu^\lambda \left[ 2e f'(R) (e^k_\mu e^i_\lambda - e^i_\mu e^k_\lambda) \right] = 0
\]

\[
D_k (e F_{ik}^A) = 0
\]

where

\[
D_k (e F_{ik}^A) = \frac{\partial (e F_{ik}^A)}{\partial x^k} - a^B_k (e F_{ik}^C) C_{CB}^A.
\]

Since the trace of the energy–impulse tensor \( T^{\mu}_\mu := \frac{1}{4} F_{jk}^A F_{ij}^{A\mu} e^i_\mu - F_{ij}^{A\mu} F_{Ak}^{ij} e^k_\mu \) vanishes identically, we have again:

- if \( f(R) = kR \) we recover the Einstein–Yang–Mills theory;
- if \( f(R) \neq kR^2 \) the torsion is necessarily zero and we recover a \( f(R) \)-theory without torsion coupled with a Yang–Mills field;
- We can have non–vanishing torsion only in the case \( f(R) = kR^2 \).
F. The case of spin fluid matter

As a last example of matter source, we consider the case of a semiclassical spin fluid. This is characterized by an energy–momentum tensor of the form
\[
\Sigma^{ij} = (\rho + p)U^iU^j + p g^{ij}
\] (66a)
and a spin density tensor given by
\[
S^{ih} = S_{ij} U^h
\] (66b)
where \( U^i \) and \( S_{ij} \) denote, respectively, the 4-velocity and the spin density of the fluid (see, for example, [33] and references therein). However, the constraint \( U^i U^i = -1 \) must hold. Other models of spin fluids are possible, where, due to the treatment of spin as a thermodynamical variable, different expressions for the energy–momentum tensor may be taken into account [34, 35, 36].

The 4-velocity and the spin density satisfy by the so called convective condition
\[
S^{ij} U^j = 0
\] (67)
It is easily seen that the relations (67) imply the identities
\[
\hat{S}^{ih} = -\hat{S}^{hi}
\] (68)
obtained inserting eq. (66b) in eq. (49b) and using (67). Making use of (68) as well as of eqs. (49b), (50), (66b) and (67), we can express the Ricci curvature tensor and scalar respectively as
\[
R_{ij} = \tilde{R}_{ij} - \frac{1}{2f'} \tilde{\nabla}^h g_{ij} + 2 \tilde{T}^i \tilde{T}^j - 2 \tilde{T}^h \tilde{T}^j - 1 \frac{1}{f'} \tilde{T}^h S^h_{ij} U_i
\] (69a)
and
\[
R = \tilde{R} - 6 \tilde{\nabla}^i \tilde{T}^i - 6 \tilde{T}^i \tilde{\nabla}^i - \frac{1}{4(f')} S^{pq} S_{pq} U^i U^j
\] (69b)
where \( \tilde{T}^i = \frac{1}{2f'} \frac{\partial f'}{\partial x^i} \). In view of this, substituting eqs. (69) in eqs. (46a), and using the definitions (57) and (58), we obtain Einstein–like equations of the form
\[
\tilde{R}_{ij} - \frac{1}{2f'} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{1}{\varphi^2} \left( -3 \frac{\partial \varphi}{2} \frac{\partial \varphi}{\partial x^j} + \varphi \tilde{\nabla}^j \frac{\partial \varphi}{\partial x^i} + 2 \frac{\partial \varphi}{f'} \frac{\partial \varphi}{\partial x^j} + 3 \frac{\partial \varphi}{f'} \frac{\partial \varphi}{\partial x^j} g^{hk} g_{ij} \right)
\]
\[
+ \frac{1}{\varphi^2} \tilde{T}^h S^h_{ij} U_i + \frac{1}{2\varphi} \tilde{\nabla}^h \left( S_{ij} U^h + S^h_{ij} U_j - S^h_{ji} U_i \right)
\]
\[
- \frac{1}{4\varphi^2} S^{pq} S_{pq} U^i U^j - \frac{1}{8\varphi^2} S^{pq} S_{pq} g_{ij}
\] (70)
Eqs. (70) are the ”microscopic” field equations for \( f(R) \) gravity with torsion, coupled with a semiclassical spin fluid. In this form, all the source contributions are put in evidence and their role is clearly defined into dynamics.

G. Cosmological applications

In order to apply the above considerations to Friedmann-Robertson-Walker(FRW) cosmological models, let us consider an isotropic and homogeneous universe filled with a cosmological spin fluid. Eqs. (70) are valid in the microscopic domain of matter. Cosmological equations can be derived by a suitable space–time averaging of (70). In this situation, the simplest cosmological scenario is achieved by supposing that the cosmological fluid is unpolarized. In fact, being the spin randomly oriented, we can assume that the average of the
spin and its gradient vanish, but the same is not true for the spin–squared terms as \(<S^{pq}S_{pq}>\). As conclusion, it follows that, after averaging, eqs. (70) reduce to

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{1}{\varphi^2} \left( -3 \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + \varphi \tilde{\nabla}_j \frac{\partial \varphi}{\partial x^i} + 3 \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^k} \tilde{g}^{jk} g_{ij} \right) - \varphi \tilde{\nabla}_j \frac{\partial \varphi}{\partial x^i} + V(\varphi) g_{ij} - \frac{1}{2\varphi^2} \tilde{s}^2 U_i U_j - \frac{1}{4\varphi^3} \tilde{s}^2 g_{ij}
\]

(71)

where we have introduced the notation \(s^2 = 2S^{pq}S_{pq}\) (see [33]).

As for the case of Dirac fields, eqs. (71) can be simplified by performing a conformal transformations. In view of this, we can suppose that \(\varphi > 0\) where a sufficient condition to satisfy this request is \(f' > 0\). Moreover, following the line illustrated in [6], we introduce the vector field \(\tilde{U}^i := \frac{U^i}{\sqrt{\varphi}}\) representing the four velocity of the fluid with respect to the conformal metric \(\tilde{g}_{ij} := \varphi g_{ij}\) \((\tilde{e}^\mu := \sqrt{\varphi} e^\mu)\). The 1-form \(\tilde{U}_i := \sqrt{\varphi} U_i\) denotes the corresponding covariant relation.

Then, performing the conformal transformation \(\tilde{e}^\mu := \sqrt{\varphi} e^\mu\), from (71) we obtain a set of equivalent Einstein–like equations for the barred tetrad, expressed as

\[
\tilde{G}_{ij} = \frac{1}{\varphi^2} \Sigma_{ij} - \frac{1}{\varphi^3} V(\varphi) g_{ij} - \frac{1}{2\varphi^2} \tilde{s}^2 \tilde{U}_i \tilde{U}_j - \frac{1}{4\varphi^3} \tilde{s}^2 g_{ij}
\]

(72)

where \(\tilde{G}_{ij}\) is the Einstein tensor in the barred metric and \(\tilde{\Sigma}_{ij} := (\rho + p) \tilde{U}_i \tilde{U}_j + p \tilde{g}_{ij}\) the new stress-energy tensor for the perfect fluid. Now, looking for a FRW solution

\[
e^0 = dt, \quad e^1 = a(t) d\psi, \quad e^2 = a(t) \chi d\theta, \quad e^3 = a(t) \chi \sin \theta d\phi
\]

(73)

for eqs. (72), we get the Friedmann–like equations of the form

\[
3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} = \frac{\rho}{\varphi^2} + \frac{V(\varphi)}{\varphi^3} - \frac{1}{2\varphi^3} s^2 + \frac{1}{4\varphi^3} s^2
\]

(74a)

and

\[
-2 \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 = \frac{k}{a^2} = \frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3} - \frac{1}{4\varphi^3} s^2
\]

(74b)

In conclusion, we notice that, once a solution \(\tilde{e}^\mu\) is found, also the conformal tetrad \(e^\mu = \frac{1}{\sqrt{\varphi}} \tilde{e}^\mu\) (solution of (71)) gives rise to a FRW metric. In fact, by performing the time variable transformation

\[
d\tau := \frac{1}{\sqrt{\varphi(t)}} dt
\]

(75)

we can express the tetrad \(e^\mu\) as

\[
e^0 = d\tau, \quad e^1 = A(\tau) d\psi, \quad e^2 = A(\tau) \chi d\theta, \quad e^3 = A(\tau) \chi \sin \theta d\phi
\]

(76)

where \(A := \frac{a}{\sqrt{\varphi}}\).

H. Equivalence with scalar–tensor theories

The results of the last subsection lead to take into account the analogies between \(f(R)\)-gravity with torsion and scalar-tensor theories with torsion, as discussed, for example, in [37, 38]. To this end, let us consider a Lagrangian density of the form

\[
\mathcal{L} = \varphi eR - eU(\varphi) + \mathcal{L}_m
\]

(77)

where \(\varphi\) is a scalar field, \(U(\varphi)\) is a suitable potential and \(\mathcal{L}_m\) is a matter Lagrangian density.
The Euler–Lagrange equations applied to the Lagrangian density, yield the corresponding field equations

\[ R_{\mu\sigma}^{\lambda\sigma} e^\lambda_i - \frac{1}{2} R e^\mu_i = \frac{1}{\varphi} \Sigma^i - \frac{1}{2\varphi} U(\varphi) e^i \]  

(78a)

\[ \varphi (T^a_{ts} - T^a_{ts} e^a_s + T^a_{ts} e^a_t) = \frac{\partial \varphi}{\partial x^s} e^a_s - \frac{\partial \varphi}{\partial x^t} e^a_t + S^a_{ts} \]  

(78b)

while the Euler–Lagrange equation for the scalar field is given by

\[ R = U'(\varphi) \]  

(78c)

Inserting eq. (78c) in the trace of eqs. (78a), we obtain an algebraic relation between the matter trace \( \Sigma \) and the scalar field \( \varphi \) expressed as

\[ \Sigma - 2U(\varphi) + \varphi U'(\varphi) = 0 \]  

(79)

Now, under the conditions \( U(\varphi) = \frac{2}{\varphi} V(\varphi) \), where \( V(\varphi) \) is defined as in eq. (58) and \( f'' \neq 0 \). It is easily seen that the relation represents exactly the inverse of (57). In fact, from the definition of the effective potential (58) and the expression \( F^{-1}(X) = f'(X)X - 2f(X) \) we have

\[ U(\varphi) = \frac{2}{\varphi} V(\varphi) = \frac{1}{2} \left[ F^{-1}((f')^{-1}(\varphi)) + \varphi(f')^{-1}(\varphi) \right] = \left[ \varphi(f')^{-1}(\varphi) - f((f')^{-1}(\varphi)) \right] \]  

(80)

so that

\[ U'(\varphi) = (f')^{-1}(\varphi) + \frac{\varphi}{f''((f')^{-1}(\varphi))} - \frac{\varphi}{f''((f')^{-1}(\varphi))} = (f')^{-1}(\varphi) \]  

(81)

and then

\[ \Sigma = -\varphi U'(\varphi) + 2U(\varphi) = f'((f')^{-1}(\varphi))((f')^{-1}(\varphi) - 2f((f')^{-1}(\varphi)) = F^{-1}((f')^{-1}(\varphi)) \]  

(82)

In view of the latter relation, eqs. (78) result to be equivalent to eqs. (45) and (46). This fact proves the equivalence between \( f(R) \)-gravity and scalar-tensor theories with torsion obtained also in the \( J \)-bundles framework.

V. CONCLUSIONS

In this paper, we have discussed the \( f(R) \)-theories of gravity with torsion in the \( J \)-bundles framework.

This formalism gives rise to a new geometric picture which allows to put in evidence several features of the theories, in particular their symmetries and conservation laws. In particular, due to the fact that the components of the torsion and curvature tensors can be chosen as fiber \( J \)-coordinates on \( J(\mathcal{E} \times \mathcal{C}) \), the field equations can be easily obtained in suitable forms where the role of the geometry and the sources is clearly defined.

Furthermore, such a representation allows to classify the couplings with respect to the various matter fields whose global effect is that to enlarge and characterize the \( J \)-bundle.

We have given specific examples of couplings taking into account Dirac fields, Yang-Mills fields and spin fluids. In every case, the \( J \)-vector fields allow to write the \( f(R) \)-field equations in such a way that curvature, torsion and matter components have a clear and distinct role into dynamics.

This result is particularly useful in cosmology where the role of sources is crucial to define dynamics and, in some sense, to coherently match the observations. In fact, the passage from the "microscopic" domain of field equations and a suitable average domain for cosmology is extremely relevant to define self-consistent cosmological models having well-founded theoretical bases. For example, in the present state of cosmology, one of the main shortcomings is due to the fact that a huge amount of models explain the "same" data and, in our opinion, the degeneracy could be due to the fact that the relation between the microscopic and the macroscopic descriptions are often not well defined, a part the very crucial issue to have homogeneous and reliable data at any redshift. In this sense, giving a priori a straightforward classification of curvature, torsion and matter components is extremely useful.
In a forthcoming paper, we will show how all these considerations could contribute to give a geometric, well-founded view of the dark side of the universe.

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