Global Stability of Fractional Order Coupled Systems with Impulses via a Graphic Approach

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Abstract: Based on the graph theory and stability theory of dynamical system, this paper studies the stability of the trivial solution of a coupled fractional-order system. Some sufficient conditions are obtained to guarantee the global stability of the trivial solution. Finally, a comparison between fractional-order system and integer-order system ends the paper.

Keywords: neural networks; global stability; impulse

1. Introduction

Due to the great significance in applied science (e.g., signal and image processing, artificial intelligence, pattern classification), the neural networks have attracted many scholars’ attention. There are a large amount of scientific research results on the stability and synchronization of both integer-order and fractional-order differential equations. For examples, one can refer to [1–15]. Besides, there are many results about fractional equations such as [16–22]. However, in the real world, at certain moments, many behaviors in neural networks may experience a sudden change. They are affected by short-term perturbations whose duration is particularly short comparing to the process with no change. We can use impulsive differential equations to describe the phenomena. Some works considered the impulsive effects on the neural networks (e.g., see [23–28]). It is worthwhile to mention that the fractional-order impulsive differential equations were studied recently (see e.g., [29–36]). Among them, Stamov and Stamova [31–34] studied the almost periodicity of the fractional-order impulsive differential equations. It is difficult to get less conservative conditions to guarantee the global stability of a system. Recently, a new powerful tool is to apply graph theory to study the stability and synchronization of neural networks (see e.g., [37–42]). Inspired by the previous works, we consider the global stability of fractional-order coupled systems with impulses on digraph $G$.

\[
\begin{align*}
D^\mu x_p &= -w_p x_p + \sum_{q=1}^n a_{pq} f_q(x_q(t)) + \sum_{q=1}^n a_{pq} (x_p(t) - x_q(t)), \quad t \geq 0, \quad t \neq t_k, \\
\Delta x_p(t_k) &= I_k(x_p(t_k)), \\
x(t_k^+) &= x(t_k), \quad k = 1, 2, \cdots,
\end{align*}
\]

where $\Delta x_p(t_k) = x_p(t_k^+) - x_p(t_k^-)$ are the impulses at moments $t_k$ and $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$ as $k \to \infty$ (see e.g., [30–32,43–46]). $I_k : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and $I_k = 0$ when the impulses are absent. For the fractional order systems, the criteria to determine the stability for the integer order differential systems may not be applicable because fractional derivative may not maintain the properties of the integer derivative. (e.g., see [47,48]). The difficulty comes from the following facts.
1. For the integer derivative, the sign of the first order derivative implies the monotonicity of a function. However, this is not valid for the fractional derivative (see [47]). This difference results in great difficulties to deal with the impulses at moment \( t_k \).

2. For the integer-order system \( \frac{dx}{dt} = f(x,t) \), the first derivative \( \frac{dV(x)}{dt} \leq -\omega(x) < 0 \) implies the asymptotically stability in the sense of Lyapunov. However, this classical Lyapunov stability result is not valid for fractional-order system. The derivative \( D^\alpha V(x) \leq -\omega(x) < 0 \) does not imply the asymptotically stability (see Lemma 2 in next section). It can only guarantee the stability.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, main results of this paper is presented by employing graph theory. In Section 4, an example and its simulations are presented to verify the feasibility of the obtained results. Finally, Conclusions and Discussion end the paper.

2. Preliminaries

There are a lot of different definitions of fractional derivative (e.g., Riemann-Liouville, Caputo, the conformable fractional derivative, [49–51]). In this paper, we employ Caputo fractional integral and derivative.

**Definition 1.** [50] The fractional integral with noninteger order \( \mu > 0 \) for a function \( x(t) \) is defined as

\[
I^\mu x(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} (t - \tau)^{\mu-1} x(\tau) \, d\tau,
\]

where \( t \geq t_0, t_0 \) is the initial time, \( \Gamma(\cdot) \) is the gamma function, given by \( \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt \).

**Definition 2.** [50] The Caputo fractional derivative of order \( \mu \) for a function \( x(t) \) is defined as

\[
D^\mu x(t) = \frac{1}{\Gamma(n - \mu)} \int_{t_0}^{t} (t - \tau)^{n-\mu-1} x^{(n)}(\tau) \, d\tau,
\]

in which \( t \geq t_0, t_0 \) is the initial time, \( n - 1 < \mu < n \in \mathbb{Z}^+ \).

**Lemma 1.** [52] Suppose that \( x(t) \in \mathbb{R} \) is a continuous and differentiable vector-value function. Then for any time instant \( t \geq t_0 \), we have

\[
\frac{1}{2} D^\alpha x^2(t) \leq x(t) D^\alpha x(t)
\]

when \( 0 < \alpha < 1 \).

**Lemma 2.** [47] Consider system \( D^\alpha x = f(x,t) \), where \( 0 < \alpha \leq 1, f : (D \subset \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n \). Let \( V(x,t) \) be a continuously differentiable and positive definite function. Let \( \omega(x) \) be a positive definite function continuous at \( x = 0 \) such that in the ball \( B(r) \subseteq D \) around \( x = 0 \) with \( x_0 \in B(r) \) we have

\[
D^\alpha V(x,t) \leq -\omega(x) \leq 0.
\]

Then \( \lim_{t \to \infty} \|x\| = 0 \) and \( x = 0 \) is stable at \( t = 0 \). In particular, \( x = 0 \in \bigcap_{t \in B(r)} \Omega(x) \). For \( \alpha = 1 \), \( \lim_{t \to \infty} \|x\| = 0 \) (\( x = 0 \) is asymptotically stable at \( t = 0 \)).

Then in what follows, we recall some basic knowledge of graph theory [40,53].

A directed graph or digraph \( G = (V,E) \) contains a vertex set \( V = \{1,2,\ldots,n\} \) and a set \( E \) of arcs \((p,q)\) from \( p \) to \( q \). \( \mathcal{H} \subseteq G \) is said to be spanning if the vertex set of \( \mathcal{H} \) is the same as \( G \). If each \((p,q)\) is assigned a positive weight \( a_{pq} \), then we say graph \( G \) is weighted. In our convention, \( a_{pq} > 0 \) if and only if there is an arc from \( p \) to \( q \). The weight of a subgraph \( \mathcal{H} \) is the product of the weight of each arc.
A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with vertices $\{p_1, p_2, \ldots, p_m\}$ such that its set of arcs is $\{(p_k, p_{k+1}) : k = 1, 2, \ldots, m - 1\}$. If the arc $(p_m, p_1)$ exists, then we call $\mathcal{P}$ a directed cycle. If there does not exist any cycle in the connected subgraph $\mathcal{T}$, then we call $\mathcal{T}$ a tree. For a tree $\mathcal{T}$, if there does not exist any arc to vertex $p$, then $\mathcal{T}$ is rooted at vertex $p$. If a subgraph $\mathcal{Q}$ is a disjoint union of some rooted trees and the roots of these trees can form a directed cycle, then we say $\mathcal{Q}$ is unicyclic.

For a given weighted digraph $\mathcal{G}$ with $n$ vertices, $A = (a_{pq})_{n \times n}$ is the weight matrix whose entry $a_{pq}$ is the weight of $(p, q)$ if it exists, and 0 otherwise. For our purpose, we write a weighted digraph as $(\mathcal{G}, A)$. If for any pair of vertices there exists a directed arc from one to the other, then $\mathcal{G}$ is strongly connected. The we define the Laplacian matrix of $(\mathcal{G}, A)$ as

$$L = \begin{pmatrix}
\sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk}
\end{pmatrix}. \tag{1}
$$

Let $c_p$ be the cofactor of the $p$-th diagonal element of $L$. Then we have the following results.

**Lemma 3.** [40] Assume $n \geq 2$. Then

$$c_p = \sum_{\mathcal{T} \in \mathcal{T}_p} w(\mathcal{T}), \quad p = 1, 2, \ldots, n, \tag{2}$$

where $\mathcal{T}_p$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ that are rooted at vertex $p$, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_p > 0$ for $1 \leq p \leq n$.

For the coupled system on a directed graph $\mathcal{G}$:

$$D^a u_p = f_p(t, u_p) + \sum_{q=1}^{n} g_{pq}(t, u_p, u_q), \quad p = 1, 2, \ldots, n, \tag{3}$$

where $u_p \in \mathbb{R}^{m_p}$, $f_p : \mathbb{R} \times \mathbb{R}^{m_p} \to \mathbb{R}^{m_p}$, $g_{pq} : \mathbb{R} \times \mathbb{R}^{m_p} \times \mathbb{R}^{m_q} \to \mathbb{R}^{m_p}$ represent the influence from vertex $p$ to vertex $q$, and $g_{pq} = 0$ if there does not exist arc from $p$ to $q$ in $\mathcal{G}$.

Motivated by Theorem 3.4 in [40], for fractional-order systems, we have the following theorem.

**Theorem 1.** Assume that the following assumptions hold.

(i) For the Lyapunov function $V_p(t, u_p)$ on each vertex. There exist $F_{pq}(t, u_p, u_q)$, $a_{pq} \geq 0$, and $b_p \geq 0$ such that

$$D^a V_p(t, u_p) \leq -b_p V_p(t, u_p) + \sum_{q=1}^{n} a_{pq} F_{pq}(t, u_p, u_q), \quad t > 0, \quad u_p \in D_p, \quad 1 \leq p \leq n$$

holds.

(ii) Along each directed cycle $\mathcal{C}$ in the weighted digraph $(\mathcal{G}, A)$, $A = (a_{pq})$,

$$\sum_{(i,j) \in E(\mathcal{C})} F_{rs}(t, u_r, u_s) \leq 0, \quad t > 0, \quad u_r \in D_r, \quad u_s \in D_s.$$

(iii) $c_p$ are constants which are given in Lemma 3.

Then $V(t, u) = \sum_{p=1}^{m} c_p V_p(t, u_p)$ satisfies

$$D^a V(t, u) \leq -b V(t, u), \quad t > 0, \quad u \in D,$$
where \( b = \min \{ b_1, b_2, \ldots, b_n \} \).

\textbf{Proof.} For a spanning tree \( T \) (see Figure 1) rooted at \( q \), by adding an arc \((p, q)\) from \( p \) to \( q \), we obtain a unicyclic graph \( Q \) (see Figure 2).

![Figure 1. A rooted tree \( T \).](image1)

![Figure 2. A unicyclic graph \( Q \).](image2)

According to the definition for the weight of a graph, we have \( w(Q) = w(T) a_{pq} \). As a result, \( w(T) a_{pq} F_{pq}(t, u_p, u_q) = w(Q) F_{pq}(t, u_p, u_q), \ (q, p) \in E(C_Q) \). Here \( F_{pq}(t, u_p, u_q), 1 \leq p, q \leq n \) are arbitrary functions, \( C_Q \) denotes the directed cycle of \( Q \).

When we do this operation to all rooted spanning trees in digraph \( G \) in all possible ways, we will derive all unicyclic graphs in \( G \). Then we get

\[
\sum_{p,q=1}^{n} c_p a_{pq} F_{pq}(t, u_p, u_q) = \sum_{Q \in Q} w(Q) \sum_{(s,r) \in E(C_Q)} F_{rs}(t, u_r, u_s),
\]

where \( Q \) is a set which includes all spanning unicyclic graphs of \((G, A)\).

Based on the definition of the Caputo fractional order derivative, we know that \( D^a[lx(t) + my(t)] = lD^a x(t) + mD^a y(t) \) easily. Thus, for \( V(t, u) = \sum_{p=1}^{n} c_p V_p(t, u_p) \), we have

\[
D^a V(t, u) = D^a \sum_{p=1}^{n} c_p V_p(t, u_p) \\
= \sum_{p=1}^{n} c_p D^a V_p(t, u_p) \\
\leq \sum_{p=1}^{n} c_p [-b_p V_p(t, u_p) + \sum_{q=1}^{n} a_{pq} F_{pq}(t, u_p, u_q)] \\
= -\sum_{p=1}^{n} b_p c_p V_p(t, u_p) + \sum_{p,q=1}^{n} c_p a_{pq} F_{pq}(t, u_p, u_q) \\
= -\sum_{p=1}^{n} b_p c_p V_p(t, u_p) + \sum_{Q \in Q} w(Q) \sum_{(s,r) \in E(C_Q)} F_{rs}(t, u_r, u_s).
\]
In view of the condition (ii) and \( w(\mathcal{Q}) > 0 \), we have
\[
D^\alpha V(t, u) \leq - \sum_{p=1}^{n} b_p c_p V_p(t, u_p) \leq - \sum_{p=1}^{n} b_c p V_p(t, u_p) = -b V(t, u),
\]
here \( b = \min\{b_1, b_2, \ldots, b_n\} \).

**Remark 1.** To study the stability of the coupled systems, constructing a proper Lyapunov function is of great importance. Theorem 1 reveals that a global Lyapunov function for (3) can be the combination of the Lyapunov function \( V_i \) of each vertex system, which decreases the difficulty for us.

### 3. Main Results

Given a network represented by a digraph \( G \) with \( n \) vertices. Assume that the dynamic of each vertex is described by the following impulsive differential equation:
\[
\begin{cases}
D^\mu x_p = -w_p x_p + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)), & t \geq 0, \quad t \neq t_k, \\
\Delta x_p(t_k) = I_k(x_p(t_k)), & k = 1, 2, \cdots, \\
x(t_k^-) = x(t_k^+), & k = 1, 2, \cdots.
\end{cases}
\]
\( p, q = 1, 2, \ldots, n \), where \( 0 < \mu < 1 \), \( w_p > 0 \) is the self-regulating parameters of the \( p \)-th vertex, \( a_{pq} \) represents the weight of the arc from vertex \( p \) to \( q \). \( f_q(x) \) is the neuron activation function satisfying Lipschitz condition: for all \( x, y \in \mathbb{R} \), there exists a Lipschitz constant \( l_j > 0 \) such that \( |f_q(x) - f_q(y)| \leq l_q |x - y| \). In addition, \( f_q(0) = 0 \).

Now we consider the following impulsive coupled system on digraph \( G \):
\[
\begin{cases}
D^\mu x_p = -w_p x_p + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)) + \sum_{q=1}^{n} a_{pq}(x_p(t) - x_q(t)), & t \geq 0, \quad t \neq t_k, \\
\Delta x_p(t_k) = I_k(x_p(t_k)), & k = 1, 2, \cdots, \\
x(t_k^-) = x(t_k^+), & k = 1, 2, \cdots.
\end{cases}
\]

**Theorem 2.** Assume \((G, A)\) is strongly connected. If the following conditions hold:

1. \( b = \min_{1 \leq p \leq n} \left( 2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp} \right) > 0 \);
2. \( I_k(x_p(t_k)) = \delta_p x_p(t_k), \) where \(-1 < \delta_p < 0\);
3. In each interval, \( x_p(t) \) satisfies \( |x_p(t_k)| < |x_p(t_{k-1}^+)| \).

Then the trivial solution of (5) is globally stable.
**Proof.** Construct a Lyapunov function \( V_p = \frac{x_p^2}{2} \) and calculate the \( \mu \)-order derivative of \( V_p \) along (5), we have
\[
D^\mu V_p = D^\mu \frac{x_p^2}{2} \leq x_p D^\mu x_p \\
= x_p [-w_p x_p + \sum_{q=1}^{n} a_{pq} f_q(x_q) + \sum_{q=1}^{n} a_{pq} (x_p - x_q)] \\
\leq -w_p x_p^2 + \sum_{q=1}^{n} |x_p| a_{pq} |f_q(x_q)| + \sum_{q=1}^{n} a_{pq} (x_p - x_q) x_p \\
\leq -w_p x_p^2 + \sum_{q=1}^{n} |x_p| a_{pq} l_q |x_q| + \sum_{q=1}^{n} a_{pq} (x_p - x_q) x_p \\
\leq -w_p x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} (x_p^2 + x_q^2) + \sum_{q=1}^{n} a_{pq} (x_p - x_q) x_p \\
= -w_p x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} x_q^2 + \sum_{q=1}^{n} a_{pq} (x_p - x_q) x_p \\
= -w_p x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} x_p^2 + \frac{1}{2} \sum_{q=1}^{n} l_q a_{pq} x_q^2 + \sum_{q=1}^{n} a_{pq} (x_p - x_q) x_p \\
= -\frac{1}{2} (2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp}) x_p^2 + \sum_{q=1}^{n} a_{pq} (x_p - x_q) x_p \\
\leq -(2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp}) V_p + \sum_{q=1}^{n} a_{pq} \left(-\frac{1}{2} (x_p - x_q)^2 + \frac{1}{2} (x_q^2 - x_p^2)\right) \\
\leq -(2w_p - \sum_{q=1}^{n} l_q a_{pq} - \sum_{q=1}^{n} l_p a_{qp}) V_p + \sum_{q=1}^{n} a_{pq} \left[-\frac{1}{2} (x_q^2 - x_p^2)\right], \quad t \neq t_k.
\]

Let \( F_{pq} = \frac{1}{2} (x_q^2 - x_p^2) \), along every directed cycle \( C \) of the weighted digraph \((G, A)\) we have \( \sum_{(s, r) \in E(C)} F_{rs}(x_r, x_s) = \sum_{(s, r) \in E(C)} \frac{1}{2} (x_s^2 - x_r^2) = 0. \)

Let \( b_p = 2w_p - \sum_{q=1}^{n} l_q |a_{pq}| - \sum_{q=1}^{n} l_p |a_{qp}| \), \( V = \sum_{q=1}^{n} c_p V_p \). In view of Theorem 1, we obtain
\[
D^\mu V(t, x) \leq -b V(t, x) \quad t > 0, \quad t \neq t_k,
\]
where \( b = \min\{b_1, b_2, \ldots, b_n\} \). Now we select \( \omega = b V(t, x) \), then \( \omega \) is a positive definite function.

From lemma 2, we know that the trivial solution is globally stable when \( t \neq t_k \).

When \( t = t_k \), \( \Delta x_p(t_k) = I_k(x_p(t_k)) = \delta_{pk} x_p(t_k) \). Besides, \( \Delta x_p(t_k) = x_p(t_k^+ - t_k^-) = x_p(t_k^+ - x_p(t_k)) \), then we can obtain
\[
x_p(t_k^+) = (1 + \delta_{pk}) x_p(t_k).
\]

Due to \(-1 < \delta_{pk} < 0\), then \( |x_p(t_k^+)| \leq |x_p(t_k)| \). In view of the third condition of this theorem, we derive
\[
|x_p(t_k^+)| \leq |x_p(t_k)| < |x_p(t_k^-)| \leq |x_p(t_k-1)|.
\]

As a consequence, in each interval, we get \( V(x_p(t_k^+)) < V(x_p(t_k-1)) \). In view of \( 0 < t_1 < t_2 < \cdots < t_k < \cdots, t_k \to \infty \) as \( k \to \infty \), then \( V(x_p(t_k)) \to 0 \) as \( k \to \infty \).

This ends the proof. \( \square \)
4. Example and Numerical Simulation

In this section, we study the following fractional impulsive system on a digraph with two vertices.

\[
\begin{cases}
\frac{D^\mu}{dt}x_1 = -w_1 x_1 + \sum_{j=1}^2 a_{1j} f_j(x_j(t)) + \sum_{j=1}^2 a_{1j} f_j(x_1(t) - x_j(t)), & t \geq 0, \ t \neq 5, 10, 15, \ldots, \\
\frac{D^\mu}{dt}x_2 = -w_2 x_2 + \sum_{j=1}^2 a_{2j} f_j(x_j(t)) + \sum_{j=1}^2 a_{2j} f_j(x_2(t) - x_j(t)), & t \geq 0, \ t \neq 5, 10, 15, \ldots, \\
\Delta x_i(t_k) = \delta_{ik} (x_i(t_k)), \\
x(t_{k-}) = x(t_k), & t_k = 5, 10, 15, \ldots,
\end{cases}
\]

(6)

When \( \mu = 0.92, \delta_{ik} = -\frac{1}{2}, w_1 = w_2 = 5, a_{11} = a_{22} = 4, a_{12} = a_{21} = 0, f_i(s) = \tanh(s) \). Obviously, we can take the Lipschitz constant \( l_i = 1 \). The initial conditions are assumed that \( x_1(t) \) and \( x_2(t) \) are \( x_1(0) = 2 \) and \( x_2(0) = -2 \). The simulation result for the above system is shown in Figure 3.

![Figure 3. Dynamical behaviors of states \( x_1(t) \) and \( x_2(t) \) under above parameters.](image3)

When \( \mu = \frac{1}{2}, \delta_{ik} = -\frac{1}{2}, w_1 = 15, w_2 = 14, a_{11} = a_{22} = 1, a_{12} = a_{21} = 0, f_i(s) = \tanh(s) \). Obviously, we can take the Lipschitz constant \( l_i = 1 \). The initial conditions are assumed that \( x_1(t) \) and \( x_2(t) \) are \( x_1(0) = 1 \) and \( x_2(0) = -1 \). The simulation result for the above system is shown in Figure 4.

![Figure 4. Dynamical behaviors of states \( x_1(t) \) and \( x_2(t) \) under above parameters.](image4)
5. Conclusions and Discussions

In this paper, we apply the graph theory and stability theory of dynamical system to study the stability of a coupled fractional-order system. This method can be extended to the other complex networks or multi-layer networks. In fact, many classical results for the integer-order system are not valid for the fractional-order system. We summarize the differences between fractional derivative and integer derivative as follows.

1. For the integer derivative, the sign of the first order derivative implies the monotonicity of a function. However, this is not valid for the fractional derivative (see [47]). This difference raises great difficulties for us to deal with the impulses at moment $t_k$. In order to ensure the stability of the trivial solution of (5), we have to add the condition $|x_p(t_k)| < |x_p(t_k^{(k-1)})|.

2. For the integer-order system $\frac{dx}{dt} = f(x,t)$, the first derivative $\frac{dV(x)}{dt} \leq -\omega(x) < 0$ implies the asymptotically stability in the sense of Lyapunov. However, this classical Lyapunov stability result is not valid for fractional-order system. The derivative $D^\alpha V(x) \leq -\omega(x) < 0$ does not imply the asymptotically stability in view of Lemma 2. It can only guarantee the stability.

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