Bar Code and Janet-like division

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Abstract

Bar Codes are combinatorial objects encoding many properties of monomial ideals.
In this paper we employ these objects to study Janet-like divisions. Given a finite set of terms $U$, from its Bar Code we can compute the Janet-like nonmultiplicative power of its elements and detect completeness of the set. Some observation on the computation of Janet-like bases conclude the work.

Keywords: Janet-like division, Bar Code, multiplicative variables

1 Introduction

Bar Codes are combinatorial objects encoding many properties of monomial ideals. In [5], they have been employed to count zerodimensional (strongly) stable monomial ideals in 2 and 3 variables with affine Hilbert polynomial $p \in \mathbb{N}$, setting a bijection between such ideals and some particular partition of the integer $p$ and then counting these partitions using determinantal formulas.

In [7], instead, they are the main tool to compute the Groebner escalier of zerodimensional radical ideals given their variety, without passing through the (usually inefficient) Groebner bases’ computation.

In this paper, we show that Bar Codes can be successfully used as tools to study, describe and build Janet-like division, i.e. a divisibility relation on terms, introduced by Gerdt and Blinkov [18][17] to efficiently compute Groebner bases.

Janet-like division, though not being an involutive division [15][16][19], is strictly related to this concept, being a generalization of Janet division [20] and preserving most of its properties. As Janet division was based on the concept of multiplicative/nonmultiplicative variables of the elements of a finite set of terms (the leading terms of a generating set of an ideal, with respect to some term ordering), Janet-like division is based on the concept of nonmultiplicative power for the same terms. In the case of Janet division, a term $t$ was reducible by a generating polynomial $f$ if and only if $t = T(f)w$, where $T(f)$ was the leading term of $f$ and $w$ a product of powers of multiplicative variables of $T(f)$. The case of Janet-like division is analogous, but $w$ should be non-divisible by any nonmultiplicative power of $T(f)$.

We see in this paper that thanks to Bar Codes it is possible to detect nonmultiplicative
Given a semigroup ideal \( J \) \( T \setminus \) For all subsets \( G \) if \( T \subseteq G < \) which is a term ordering. Since we do not consider any term ordering other than Lex, \( T \) leading terms defined as \( \text{Gröbner escalier} < \) Fixed a term order \( i.e., \) roughly speaking, if given any \( t \) there exists a generator reducing it. If it does not happen, it is possible to update the generating set.

Note that the classical cases of Janet/Pommaret division can be easily treated analogously. Other applications of BarCode to Janet decomposition are discussed in [3].

More precisely, after setting the notation (Section 2) and giving a brief recap on Bar Codes (Section 3), we study Janet-like divisions by means of Bar Codes in Section 4 and we relate Janet nonmultiplicative powers to the concept of infinite corner (Section 5). In the last section, we give an overview on the potential future work on this topic.

### 2 Notation

Throughout this paper we mainly follow the notation of [27]. We denote by \( \mathcal{P} := k[x_1, ..., x_n] \) the ring of polynomials in \( n \) variables with coefficients in the field \( k \). The semigroup of terms, generated by the set \( \{x_1, ..., x_n\} \) is

\[
\mathcal{T} := \{x_1^{\gamma_1} \cdots x_n^{\gamma_n} \mid \gamma := (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\}.
\]

If \( t = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \), then \( \deg(t) = \sum_{i=1}^{n} \gamma_i \) is the degree of \( t \) and, for each \( h \in \{1, ..., n\} \), \( \deg_h(t) := \gamma_h \) is the \( h \)-degree of \( t \). A semigroup ordering \( < \) on \( \mathcal{T} \) is a total ordering such that \( t_1 < t_2 \Rightarrow s t_1 < s t_2 \), \( \forall s, t_1, t_2 \in \mathcal{T} \). For each semigroup ordering \( < \) on \( \mathcal{T} \), we can represent a polynomial \( f \in \mathcal{P} \) as a linear combination of terms arranged w.r.t. \(<\), with coefficients in the base field \( k \):

\[
f = \sum_{t \in \mathcal{T}} c(f, t) t = \sum_{i=1}^{s} c(f, t_i) t_i : c(f, t_i) \in k \setminus \{0\}, t_i \in \mathcal{T}, t_1 > ... > t_s,
\]

with \( \text{LT}(f) := t_1 \) the leading term of \( f \), \( \text{LC}(f) := c(f, t_1) \) the leading coefficient of \( f \) and \( \text{tail}(f) := f - c(f, \text{LT}(f)) \text{LT}(f) \) the tail of \( f \).

A term ordering is a semigroup ordering which is also a well ordering or, equivalently, such that 1 is lower than every variable.

In all paper, we consider the lexicographical ordering induced by \( x_1 < ... < x_n \), i.e:

\[
x_1^{\gamma_1} \cdots x_n^{\gamma_n} <_{\text{Lex}} x_1^{\delta_1} \cdots x_n^{\delta_n} \iff \exists j \mid \gamma_j < \delta_j, \gamma_i = \delta_i, \forall i > j,
\]

which is a term ordering. Since we do not consider any term ordering other than Lex, we drop the subscript and denote it by \(<\) instead of \(<_{\text{Lex}}\).

A subset \( J \subseteq \mathcal{T} \) is a semigroup ideal if \( t \in J \Rightarrow s t \in J \), \( \forall s \in \mathcal{T} \); a subset \( \mathcal{N} \subseteq \mathcal{T} \) is an order ideal if \( t \in \mathcal{N} \Rightarrow s \in \mathcal{N} \forall s t \in \mathcal{T} \). We have that \( \mathcal{N} \subseteq \mathcal{T} \) is an order ideal if and only if \( \mathcal{T} \setminus \mathcal{N} = J \) is a semigroup ideal.

Given a semigroup ideal \( J \subseteq \mathcal{T} \) we define \( \mathcal{N}(J) := \mathcal{T} \setminus J \). The minimal set of generators \( \mathcal{G}(J) \) of \( J \) is called monomial basis of \( J \).

For all subsets \( G \subseteq \mathcal{P} \), \( \mathcal{T}(G) := \{\text{LT}(g), g \in G\} \) and \( \mathcal{T}(G) \) is the semigroup ideal of leading terms defined as \( \mathcal{T}(G) := \{\text{LT}(g), t \in \mathcal{T}, g \in G\} \).

Fixed a term order \(<\), for any ideal \( I \subseteq \mathcal{P} \) the monomial basis of the semigroup ideal \( \mathcal{T}(I) = \mathcal{T}(J) \) is called monomial basis of \( I \) and denoted again by \( \mathcal{G}(I) \), whereas the ideal \( \mathcal{N}(I) := \{\text{LT}(J)\} \) is called initial ideal and the order ideal \( \mathcal{N}(I) := \mathcal{T} \setminus \mathcal{T}(I) \) is called Groebner escalier of \( I \).
3 Recap on Bar Codes

In this section, referring to [5][4], we summarize the main definitions and properties about Bar Codes, which will be used in what follows. First of all, we recall the general definition of Bar Code.

**Definition 1.** A Bar Code $\mathcal{B}$ is a picture composed by segments, called bars, superimposed in horizontal rows, which satisfies conditions a., b. below. Denote by

- $B_j^{(i)}$ the $j$-th bar (from left to right) of the $i$-th row (from top to bottom), $1 \leq i \leq n$, i.e. the $j$-th $i$-bar;
- $\mu(i)$ the number of bars of the $i$-th row
- $l_i(B_j^{(i)}) := 1$, $\forall j \in \{1, 2, ..., \mu(1]\}$ the (1-)length of the $1$-bars;
- $l_i(B_j^{(k)})$, $2 \leq k \leq n$, $1 \leq i \leq k-1$, $1 \leq j \leq \mu(k)$ the $i$-length of $B_j^{(k)}$, i.e. the number of $i$-bars lying over $B_j^{(k)}$

a. $\forall i, j$, $1 \leq i \leq n-1$, $1 \leq j \leq \mu(i)$, $\exists! \pi_j \in \{1, ..., \mu(i+1]\}$ s.t. $B_j^{(i+1)}$ lies under $B_j^{(i)}$

b. $\forall i_1, i_2 \in \{1, ..., n\}$, $\sum_{j=1}^{\mu(i_1)} l_{i_1}(B_{j_1}^{(i_1)}) = \sum_{j=1}^{\mu(i_2)} l_{i_2}(B_{j_2}^{(i_2)})$; we will then say that all the rows have the same length.

**Example 2.** An example of Bar Code $\mathcal{B}$ is

1 ________
2 ________
3 ________

The 1-bars have length 1. As regards the other rows, $l_1(B_1^{(2)}) = 2$, $l_1(B_2^{(2)}) = l_1(B_3^{(2)}) = l_1(B_4^{(2)}) = 1$, $l_2(B_1^{(3)}) = 1$, $l_2(B_2^{(3)}) = 2$, $l_2(B_3^{(3)}) = l_1(B_2^{(3)}) = 3$, so $\sum_{j=1}^{\mu(1)} l_1(B_{j_1}^{(1)}) = \sum_{j=1}^{\mu(2)} l_1(B_{j_1}^{(2)}) = \sum_{j=1}^{\mu(3)} l_1(B_{j_1}^{(3)}) = 5$.  

We outline now the construction of the Bar Code associated to a finite set of terms. For more details, see [4], while for an alternative construction, see [5].

First of all, given a term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in T \subset \mathbb{K}\{x_1, ..., x_n\}$, for each $i \in \{1, ..., n\}$, we take $\pi(t) := x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} \in T$. Taken a finite set of terms $M \subset T$, for each $i \in \{1, ..., n\}$, we define $M^{(i)} := \pi(M) := \{\pi(t) | t \in M\}$. We take $M \subset T$, with $|M| = m < \infty$ and we order its elements increasingly w.r.t. Lex, getting the list $\overline{M} = [t_1, ..., t_m]$. Then, we construct the sets $M^{(i)}$, and the corresponding lexicographically ordered lists $[\overline{M}^{(i)}]$, for $M^{(i)}$ cannot contain repeated terms, while the $\overline{M}^{(i)}$, for $1 < i \leq n$, can. In case some repeated terms occur in $\overline{M}^{(i)}$, $1 < i \leq n$, they clearly have to be adjacent in the list, due to the lexicographical ordering.
\[ i = 1, \ldots, n. \] We can now define the \( n \times m \) matrix of terms \( M \) s.t. its \( i \)-th row is \( \overline{M}_{i} \), \( i = 1, \ldots, n \), i.e.

\[
M := \begin{pmatrix}
\pi^1(t_1) & \ldots & \pi^1(t_m) \\
\pi^2(t_1) & \ldots & \pi^2(t_m) \\
\vdots & \ddots & \vdots \\
\pi^n(t_1) & \ldots & \pi^n(t_m)
\end{pmatrix}
\]

**Definition 3.** The Bar Code diagram \( B \) associated to \( M \) (or, equivalently, to \( \overline{M} \)) is a \( n \times m \) diagram, made by segments s.t. the \( i \)-th row of \( B \), \( 1 \leq i \leq n \), is constructed as follows:

1. take the \( i \)-th row of \( M \), i.e. \( \overline{M}_{i} \)

2. consider all the sublists of repeated terms, i.e. 
   \( [\pi^i(t_{j_1}), \pi^i(t_{j_1+1}), \ldots, \pi^i(t_{j_1+h})] \) s.t. \( \pi^i(t_{j_1}) = \pi^i(t_{j_1+1}) = \ldots = \pi^i(t_{j_1+h}) \), noticing that \( 0 \leq h < m \)

3. underline each sublist with a segment

4. delete the terms of \( \overline{M}_{i} \), leaving only the segments (i.e. the \( i \)-bars).

We usually label each \( 1 \)-bar \( B_j^{(1)} \), \( j \in \{1, \ldots, \mu(1) = m\} \), with the term \( t_j \in \overline{M} \).

A Bar Code diagram is a Bar Code in the sense of definition 1.

**Example 4.** Given \( M = \{x_1, x_1^2, x_2, x_3, x_1x_2^2x_3, x_1x_3^3, x_2^3x_3\} \subset k[x_1, x_2, x_3] \), we have the \( 3 \times 5 \) matrix \( M \) and the associated Bar Code displayed below:

\[
M := \begin{pmatrix}
x_1 & x_1^2 & x_2 & x_1 & x_1^2x_3 & x_3^3 \\
1 & 1 & x_2 & x_2 & x_1x_2x_3 & x_2x_3 \\
1 & 1 & x_3 & x_3 & x_3 & x_3 \\
x_1 & x_1 & x_2 & x_1 & x_1^2x_3 & x_3^3 \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6
\end{pmatrix}
\]

It is also possible to associate a finite set of terms \( M_B \) to a given Bar Code \( B \). In [5] we first give a more general procedure to do so and then we specialize it in order to have a unique set of terms for each Bar Code.

If we apply such specialized procedure to a Bar Code obtained as above from an order ideal \( M = N \), the unique set we get is exactly \( N \).

Here we give only the specialized version, so we follow the steps below:

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2 Clearly if a term \( \pi^i(\mathcal{P}) \) is not repeated in \( \overline{M} \), the sublist containing it will be only \( [\pi^i(\mathcal{P})] \), i.e. \( h = 0 \).
consider the $n$-th row, composed by the bars $B_j^{(n)}$, $B_{\mu(j)}^{(n)}$. Let $l_1(B_j^{(n)}) = \ell_j^{(n)}$, for $j \in \{1, ..., \mu(n)\}$. Label each bar $B_j^{(n)}$ with $\ell_j^{(n)}$ copies of $x_n^{j-1}$.

For each $i = 1, ..., n-1, 1 \leq j \leq \mu(n-i+1)$ consider the bar $B_j^{(n-i+1)}$ and suppose that it has been labelled by $\ell_j^{(n-i+1)}$ copies of a term $t$. Consider all the $(n-i)$-bars $B_j^{(n-i)}$, $B_{\mu(j)}^{(n-i)}$ lying immediately above $B_j^{(n-i+1)}$; note that $h$ satisfies $0 \leq h \leq \mu(n-i) - j$. Denote the $1$-lengths of $B_j^{(n-i)}$, $B_{\mu(j)}^{(n-i)}$ by $l_1(B_j^{(n-i)}) = \ell_j^{(n-i)}$, $l_1(B_{\mu(j)}^{(n-i)}) = \ell_{\mu(j)}^{(n-i)}$. For each $0 \leq k \leq h$, label $B_{\mu(j)}^{(n-i)}$ with $\ell_k^{(n-i)}$ copies of $tx^{k-1}$.

**Definition 5.** A Bar Code $B$ is admissible if the set $M$ obtained by applying $\mathfrak{V}1$ and $\mathfrak{V}2$ to $B$ is an order ideal.

We give now the definition of block in a Bar Code and of e-list associated to a 1-bar, which give a connection between the bars and the terms obtained from the rules $\mathfrak{V}1$ and $\mathfrak{V}2$ (see Remark 8).

**Definition 6.** Given a Bar Code $B$, for each $1 \leq l \leq n$, $1 \leq i \leq n$, $1 \leq j \leq \mu(i)$, an $l$-block associated to a bar $B_j^{(i)}$ of $B$ is the set containing $B_j^{(i)}$ itself and all the bars of the $(l-1)$ rows lying immediately above $B_j^{(i)}$.

**Definition 7.** Given a Bar Code $B$, let us consider a 1-bar $B_j^{(1)}$, with $j_1 \in \{1, ..., \mu(1)\}$. The e-list associated to $B_j^{(1)}$ is the $n$-tuple $e(B_j^{(1)}) := (b_{j_1, n}, ..., b_{j_1, 1})$, defined as follows:

- consider the $n$-bar $B_{j_1}^{(n)}$, lying under $B_j^{(1)}$. The number of $n$-bars on the left of $B_{j_1}^{(n)}$ is $b_{j_1, n}$.
- for each $i = 1, ..., n-1$, let $B_{j_{i+1}}^{(n-i+1)}$ and $B_{j_{i+1}}^{(n-i)}$ be the $(n-i+1)$-bar and the $(n-i)$-bar lying under $B_{j_i}^{(1)}$. Consider the $(n-i+1)$-block associated to $B_{j_{i+1}}^{(n-i+1)}$, i.e. $B_{j_{i+1}}^{(n-i+1)}$ and all the bars lying over it. The number of $(n-i)$-bars of the block, which lie on the left of $B_{j_{i+1}}^{(n-i)}$ is $b_{j_1, n-i}$.

**Remark 8.** Given a Bar Code $B$, fix a 1-bar $B_j^{(1)}$, with $j \in \{1, ..., \mu(1)\}$. Comparing definition 8 and the steps $\mathfrak{V}1$ and $\mathfrak{V}2$ described above, we can observe that the values of the e-list $e(B_j^{(1)}) := (b_{j, n}, ..., b_{j, 1})$ are exactly the exponents of the term labelling $B_j^{(1)}$, obtained applying $\mathfrak{V}1$ and $\mathfrak{V}2$ to $B$.

**Example 9.** For the Bar Code $B$

\[
\begin{array}{cccc}
0 & 1 & x_1 & x_2 & x_3 \\
1 & & & & \\
2 & & & & \\
3 & & & & \\
\end{array}
\]

the e-list of $B_j^{(1)}$ is $e(B_j^{(1)}) := (0, 1, 0)$; the bars involved in its computation as stated in Definition 7 are those highlighted in blue in the above picture.
\textbf{Proposition 10} (Admissibility criterion, \cite{5}). A Bar Code $B$ is admissible if and only if, for each 1-bar $B_j^{(i)}$, $j \in \{1, \ldots, \mu(1)\}$, the e-list $e(B_j^{(i)}) = (b_{j,n}, \ldots, b_{j,1})$ satisfies the following condition: $\forall k \in \{1, \ldots, n\}$ s.t. $b_{j,k} > 0$, $\exists j \in \{1, \ldots, \mu(1)\} \setminus \{j\}$ s.t.

$$e(B_j^{(i)}) = (b_{j,n}, \ldots, b_{j,k+1}, (b_{j,k}) - 1, b_{j,k-1}, \ldots, b_{j,1}).$$

\square

Consider the sets $\mathcal{A}_n := \{B \in \mathcal{B}_n \text{ s.t. } B \text{ admissible}\}$ and $\mathcal{N}_n := \{N \subset T, |N| < \infty \text{ s.t. } N \text{ is an order ideal}\}$. We can define the map

$$\eta : \mathcal{A}_n \to \mathcal{N}_n; \ B \mapsto N,$$

where $N$ is the order ideal obtained applying $\mathcal{B}1$ and $\mathcal{B}2$ to $B$, and it can be easily proved that $\eta$ is a bijection.

Up to this point, we have discussed the link between Bar Codes and order ideals, i.e. we focused on the link between Bar Codes and Groebner escalaris of monomial ideals.

We show now that, given an admissible Bar Code $B$ and the order ideal $N = \eta(B)$ it is possible to deduce a very specific generating set for the monomial ideal $I$ s.t. $N(I) = N$.

\textbf{Definition 11.} The star set of an order ideal $N$ and of its associated Bar Code $B = \eta^{-1}(N)$ is a set $\mathcal{F}_N$ constructed as follows:

a) $\forall 1 \leq i \leq n$, let $t_i$ be a term which labels a 1-bar lying over $B_{i\mu(i)}^{(i)}$, then $x_i^\pi(t_i) \in \mathcal{F}_N$;

b) $\forall 1 \leq i \leq n - 1$, $\forall 1 \leq j \leq \mu(i) - 1$ let $B_j^{(i)}$ and $B_{j+1}^{(i)}$ be two consecutive bars not lying over the same $(i + 1)$-bar and let $t_j^{(i)}$ be a term which labels a 1-bar lying over $B_j^{(i)}$, then $x_i^\pi(t_j^{(i)}) \in \mathcal{F}_N$.

We usually represent $\mathcal{F}_N$ within the associated Bar Code $B$, inserting each $t \in \mathcal{F}_N$ on the right of the bar from which it is deduced. Reading the terms from left to right and from the top to the bottom, $\mathcal{F}_N$ is ordered w.r.t. Lex.

\textbf{Example 12.} For $N = \{x_1, x_2, x_3\} \subset k[x_1, x_2, x_3]$, we have $\mathcal{F}_N = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}$; looking at Definition \cite{11} we can see that the terms $x_1x_3, x_2x_3, x_3^2$ come from a), while the terms $x_1^2, x_1x_2, x_2^2$ come from b).

\[
\begin{array}{cccc}
0 & 1 & x_1 & x_2 & x_3 \\
1 & & x_1^2 & x_1x_2 & x_1x_3 \\
2 & & & x_2^2 & x_2x_3 \\
3 & & & & x_3^2 \\
\end{array}
\]

\diamond

In \cite{8}, given a monomial ideal $I$, the authors define the following set, calling it \textit{star set}:

$$\mathcal{F}(I) = \left\{ x^\gamma \in \mathcal{T} \setminus N(I) \left| \frac{x^\gamma}{\min(x^\gamma)} \in N(I) \right. \right\}.$$
Proposition 13 (§4). With the above notation \( \mathcal{F}_N = \mathcal{F}(I) \).

The star set \( \mathcal{F}(I) \) of a monomial ideal \( I \) is strongly connected to Janet’s theory [20, 21, 22, 23] and to the notion of Pommaret basis [29, 30, 33, 32], as explicitly pointed out in [8]. In particular, for quasi-stable ideals, the star set is finite and coincides with their Pommaret basis.

4 Bar Code and Janet-like divisions

Janet division dates back to the 1920 paper by Janet [20] and it is first developed to study partial differential equations via algebraic methods, following and formalizing the approach by Riquier [31].

This division is defined, for each set of terms \( U \subset T \), as a divisibility relation on terms. In particular, each \( t \in U \) is equipped with a set \( M_J(t, U) \) of multiplicative variables, according to the following definition.

Definition 14. [20, ppg.75-9] Let \( U \subset T \) be a set of terms and \( t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) be an element of \( U \). A variable \( x_j \) is called multiplicative for \( t \) with respect to \( U \) if there is no term in \( U \) of the form \( t' = x_1^{\beta_1} \cdots x_j^{\beta_j+1} \cdots x_n^{\beta_n} \) with \( \beta_j > \alpha_j \). We denote by \( M_J(t, U) \) the set of multiplicative variables for \( t \) with respect to \( U \). The variables that are not multiplicative for \( t \) w.r.t. \( U \) are called non-multiplicative and we denote by \( M_J(t, U) \) the set containing them.

The divisibility relation is defined as follows: for each \( u \in T \), we say that a term \( t \in U \) Janet-divides \( u \) if \( u = tv \) and each \( x_j \mid v \), \( j \in \{1, ..., n\} \), belongs to \( M_J(t, U) \), i.e. \( v \) is a product of powers of multiplicative variables for \( t \). In this case, \( t \) is a Janet-divisor of \( u \) and \( u \) a Janet-multiple of \( t \). With the definition below, we group together all the Janet-multiples of any term \( t \in U \).

Definition 15. With the previous notation, the cone of \( t \) with respect to \( U \) is the set

\[
C_J(t, U) := \{ tx_1^{\lambda_1} \cdots x_n^{\lambda_n} | \text{where } \lambda_j \neq 0 \text{ only if } x_j \in M_J(t, U) \}.
\]

It can be proved [20] that each \( u \in T \) has at most a Janet-divisor, i.e. the cones are disjoint. A priori, it may happen that a term \( u \in T \) has no Janet-divisor; the notion of completeness characterizes the case in which this cannot happen.

Definition 16. A set \( U \subset T \) is complete if \( T(U) = \bigcup_{t \in U} C_J(t, U) \).

Janet division is employed to construct a special kind of Groebner basis for an ideal \( I = (G) \) called Janet basis. Roughly speaking, the complete set \( U \) is the set \( T(G) \) of all leading terms for the generators and any term \( u \in T \) is reduced by means of the polynomial \( f \in G \) such that \( t := T(f) \in U \) is the Janet-divisor of \( u \). Gerdt and Blinkov [15, 16, 19] give a generalization of Janet division and Janet bases, by defining involutive divisions and involutive bases [2, 1].

In [18, 17] they introduce Janet-like division and Janet-like bases, with the aim to
decrease the number of elements in the basis.

We recall now the definitions of non-multiplicative power and of Janet-like divisor from [18, 17].

**Definition 17.** Let \( U \subset T \) be a finite set of terms; for each \( u \in U \), \( 1 < i < n \) consider
\[
h_i(u, U) = \max\{\deg_j(v) : v \in U, \deg_j(v) = \deg_j(u), i + 1 \leq j \leq n\} - \deg(u) \in \mathbb{N}.
\]
If \( h_i(u, U) > 0 \), define \( k_i := \min\{\deg_i(v) - \deg_i(u) : \deg_j(v) = \deg_j(u), i + 1 \leq j \leq n, \deg_i(v) > \deg_i(u)\} \); then \( x_i^{k_i} \) is called non-multiplicative power of \( u \in U \). We denote by \( \text{NMP}(u, U) \) the set of nonmultiplicative powers for \( u \in U \).

**Definition 18.** Let \( U \subset T \) be a finite set of terms and \( u \in U \); the elements in the monoid ideal
\[
\text{NM}(u, U) = \{v \in T | \exists w \in \text{NMP}(u, U) : w | v\}
\]
are called Janet-like nonmultipliers for \( u \), whereas the elements in \( M(u, U) = T \setminus \text{NM}(u, U) \) are called Janet-like multipliers for \( u \).

A term \( u \in U \) is a Janet-like divisor of \( w \in T \) if \( w = uv \) with \( v \in M(u, U) \).

**Example 19.** Let us consider the set \( U = \{x_1^5, x_2^1, x_2^3, x_2^1x_1^2, x_2^2x_1x_2, x_2^3x_1^2, x_2^3, x_2^3x_1^2, x_2^3x_1^2, x_2^3x_1^2, x_2^3x_1^2, x_2^3x_1^2\} \subset T \) of \([19]\) and suppose \( x_1 < x_2 < x_3 \). The nonmultiplicative powers are summarized in the following table:

| \( t \) | \( \text{NMP}(t, U) \) |
|---|---|
| \( x_1^5 \) | \( x_2, x_1^2 \) |
| \( x_2^2x_1^2 \) | \( x_2^3, x_1^2 \) |
| \( x_2^1x_1^1 \) | \( x_2^3 \) |
| \( x_2^1x_2^3 \) | \( x_2, x_1^2 \) |
| \( x_2^1x_2^3x_1^1 \) | \( x_2^3, x_1^2 \) |
| \( x_2^1x_2^3x_1^1 \) | \( x_2^3 \) |
| \( x_2^3x_1^2 \) | \( \emptyset \) |

We remark that, though Janet-like divisions preserves many properties of Janet division, it is not an involutive division.

In what follows, we see that a Bar Code can be used as a tool for studying Janet and Janet-like division. The construction of a Bar Code can help to assign to each element \( t \) of a finite set of terms \( U \subset T \) its multiplicative variables, according to Janet’s Definition[4].

Let \( U \subset T \subset k[x_1, ..., x_n] \) be a finite set of terms and suppose \( x_1 < x_2 < ... < x_n \). As explained in section[3] we can associate a Bar Code \( B \) to it. Once \( B \) is constructed, even if \( B \) may be a non-admissible Bar Code, we can mimick on it the set up we generally perform to construct the star set[3]. In particular:

---

[3] We put a star symbol * in the diagram in the places where in the star set construction we would have placed the terms.

8
Moreover, Janet divisibility implies Janet-like divisibility, whereas the vice versa does not hold (see [18] for a proof of this fact).

Proposition 21. Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by $B_U$ its Bar Code. For each $t \in U$, $1 \leq i \leq n$, is multiplicative for $t$ if and only if, in $B_U$, the $i$-bar $B^{(i)}_t$, over which $t$ lies, is followed by a star.

Now, we state the following proposition, which connects the stars placed above with Janet multiplicative variables.

Proposition 20. [6 Prop. 19] Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by $B_U$ its Bar Code. For each $t \in U$, $x_i \subseteq T$ with $1 \leq i \leq n$, is Janet-multiplicative for $t$ if and only if, in $B_U$, the $i$-bar $B^{(i)}_t$, over which $t$ lies, is followed by a star.

Now, we start focusing on how to study Janet-like division using Bar Codes.

As remarked in [18], every nonmultiplicative power is nothing else than the power of Janet-nonmultiplicative variable.

Indeed, consider $u \in U$ and $1 \leq i \leq n$. Since $u \in \{v \in U : \deg_j(v) = \deg_j(u), i + 1 \leq j \leq n\}$, we can immediately deduce that $h_i(u, U) = 0$. In the case $h_i(u, U) = 0$, there is no term $v$ with $\deg_j(v) = \deg_j(u), i + 1 \leq j \leq n$ s.t. $\deg_j(v) > \deg_j(u)$, so, by Definition 14, $x_i$ is Janet-multiplicative for $u$. Otherwise, i.e. if $h_i(u, U) > 0$, then there is a term $v$ with $\deg_j(v) = \deg_j(u), i + 1 \leq j \leq n$ s.t. $\deg_j(v) > \deg_j(u)$, so, again by Definition 14, $x_i$ is Janet-nonmultiplicative for $u$. This reflects on the Bar Code associated to $U$, since trivially the absence of stars after some bar is equivalent to the presence of a non-multiplicative power of the corresponding variable for the terms over that bar. Moreover, Janet divisibility implies Janet-like divisibility, whereas the vice versa does not hold (see [18] for a proof of this fact).

We prove now the analogous of Proposition 20 for Janet-like division.

Proposition 21. Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by $B_U$ its Bar Code. Let $t \in U$, $x_i \in \text{NM}_j(t, U)$ a Janet-nonmultiplicative variable, $B^{(i)}_t$ the $i$-bar under $t$ and $t'$ any term over $B^{(i)}_{t+1}$, Then

$$k_i = \deg_i(t') - \deg_i(t).$$

Proof. We first remark that since $x_i \in \text{NM}_j(t, U)$, by Proposition 20, $B^{(i)}_t$ is not followed by a star, so again we will find nonmultiplicative powers only when there are no stars.

Now since $x_i \in \text{NM}_j(t, U)$, there is a term $v \in U$ such that $\deg_j(v) = \deg_j(t), i + 1 \leq j \leq n$ and $\deg_j(v) > \deg_j(t)$. In order to find the value $k_i$, we should find the minimal exponent of a term with the same $j$-degree as $t$, $i + 1 \leq j \leq n$, and bigger $i$-degree.

All terms over $B^{(i)}_t$ have the same $i$-degree as $t$, $i \leq i \leq n$; considering $B^{(i)}_{t+1}$, we have terms which have the same $i$-degree as $t$, $i + 1 \leq i \leq n$ (if $B^{(i)}_{t+1}$ would not be over the
same \((i + 1)\)-bar as \(B^{(i)}\), we would have a star after \(B^{(i)}\). Moreover, their \(i\)-degree is bigger than \(\deg_i(t)\) and it is the minimum with this property due to the Lex ordering of the terms in the Bar Code.

The concept of completeness w.r.t. Janet-like division is analogous to that defined for Janet division in Definition\[16\]

**Definition 22.** A set \(U \subset \mathcal{T}\) is called complete w.r.t. Janet-like division if for the sets

\[
C_J(U) := \{uv : u \in U, v \in M(u, U)\}
\]

and

\[
C(U) := \{uv : u \in U, v \in \mathcal{T}\}
\]

holds \(C(U) = C_J(U)\).

**Proposition 23.** A set \(U \subset \mathcal{T}\) is complete w.r.t. Janet-like division if and only if and only if

\[
\forall u \in U, \forall p \in \text{NMP}(u, U), \exists v \in U : v | up \text{ w.r.t. Janet-like division}.
\]

Bar Codes can help us to detect completeness of a finite set of terms, as it is shown in the theorem below.

**Theorem 24.** Let \(U \subset \mathcal{T}\) be a finite set of terms, \(B\) its Bar Code, \(t \in U, p = x^k_i \in \text{NMP}(t, U)\) a nonmultiplicative power and \(B^{(i)}\) the \(i\)-bar under \(t\). Let \(s \in U; s | tp \text{ w.r.t. Janet-like division}

1. \(s | pt\)
2. \(s \) lies over \(B^{(i)}_{j+1}\) and
3. \(\forall j' \text{ such that } x_{f'} | \frac{pt}{w}, \text{ either there is a star after the } j'\text{-bar under } s \text{ or the nonmultiplicative power } \text{w.r.t. } x_{f'} \text{ has greater degree } \deg_j(\frac{pt}{w}).\)

**Proof.** “\(\Rightarrow\)” It is an obvious consequence of proposition\[21\] indeed, by 1. \(s | pt\). Thanks to (3), \(\frac{pt}{w}\) is not divided by nonmultiplicative powers of any variable. Notice that \(x_i \not| w := \frac{pt}{w}, \text{ since } s \text{ lies over } B^{(i)}_{j+1}, \text{ so } \deg_i(s) = k_i + \deg_i(t) \text{ by the minimality of the nonmultiplicative power.}

So \(sw = pt\) and \(w\) does not contain nonmultiplicative powers for \(s\); therefore \(s | pt\) w.r.t. Janet-like division.

“\(\Leftarrow\)” Let \(s \in U, s | pt \text{ w.r.t. Janet-like division; } s | pt \text{ by definition of Janet-like division.}

If \(s\) would lie over \(B^{(i)}\), then \(\deg_i(s) = \deg_i(t)\) for \(l = i, \ldots, n, \text{ i.e. in } s \text{ and } t \) the variables \(x_i, \ldots, x_n\) appear with the same exponent. Then, being \(s | pt\) and \(\deg_i(s) = \deg_i(t)\), \(x_i^k | w := \frac{pt}{w}\), so either \(x_i\) is multiplicative for \(s\), or the nonmultiplicative power of \(x_i\) for \(s\) is greater than \(k_i\). Both these alternative are impossible: if \(x_i\) was multiplicative for \(s\) then there would be a star after \(B^{(i)}\), which is impossible by hypothesis, since \(p = x^k_i\) is a nonmultiplicative power for \(t\) and they lay over the same \(i\)-bar. It is also impossible
that the nonmultiplicative power of \( x_i \) for \( s \) is greater than \( k_i \) since \( \text{deg}_s(t) = \text{deg}_l(t) \) for \( l = i, \ldots, n \), and by the minimality of the nonmultiplicative power.

If \( s \) would lie over \( \mathcal{U}_l^0 \), \( l > j + 1 \), there exists \( h \in \{i, \ldots, n\} \) s.t. \( \text{deg}_s(x) > \text{deg}_h(pt) \) (remember that the nonmultiplicative power is minimal), so \( s \not| pt \), which is again a contradiction.

If \( s \) would lie over \( \mathcal{U}_l^0 \), \( l < j \), then \( s <_{lex} t \) and it cannot happen that \( \text{deg}_e(s) = \text{deg}_e(t) \) for \( e = i, \ldots, n \) (since otherwise \( s \) would have been over \( \mathcal{U}_l^0 \)). Let \( x_k := \max \{x_h, h = 1, \ldots, n\} \) \( \text{deg}_s(x) < \text{deg}_h(t) \); it is clear that \( k \geq i \). Since \( t \in \mathcal{U} \) and \( \text{deg}_s(t) = \text{deg}_s(s), \ldots, \text{deg}_s(e+1)(t) = \text{deg}_s(e+1)(s) \) and \( \text{deg}_s(t) > \text{deg}_s(s) \), \( x_k \) cannot be a multiplicative variable for \( s \). Now, let \( x_k^i \) the nonmultiplicative power of \( s \) w.r.t. the variable \( x_k \). Being \( \text{deg}_s(t) = \text{deg}_s(s), \ldots, \text{deg}_s(e+1)(t) = \text{deg}_s(e+1)(s) \) and \( \text{deg}_s(t) > \text{deg}_s(s), h_k \leq \text{deg}_s(t) - \text{deg}_s(s) \), so \( \text{deg}_s(x_k^i) \leq \text{deg}_s(t) \leq \text{deg}_s(pt), \) and this is again a contradiction.

Then \( s \) must lie over \( \mathcal{U}_j^{j+1} \).

For being \( s \not| pt \), all the variables appearing with nonzero exponent in \( \mathcal{U} \) must be multiplicative for \( s \) or with exponent of non-multiplicative variables smaller than non-multiplicative powers and this implies that (3) holds.

\[ \square \]

**Example 25.** Let us consider the set \( U = \{x_1, x_2 x_1^2, x_2^3 x_1^2, x_3^2 x_2^2 x_1, x_3^5\} \subset T \) of \([18]\) and suppose \( x_1 < x_2 < x_3 \). The associated Bar Code is displayed below:

| \( x_3^5 \) | \( x_2 x_1^2 \) | \( x_2^4 x_1 \) | \( x_2^3 x_1^2 \) | \( x_2^2 x_2 x_1 \) | \( x_3^5 \) |
| --- | --- | --- | --- | --- | --- |
| \( * \) | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) |
| \( * \) | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) |
| \( * \) | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) |

Let us consider the elements in \( U \) and identify their nonmultiplicative powers (in complete accordance with Table [1]):

- \( x_3^5 \): \( x_3 \) is multiplicative, the nonmultiplicative powers are \( x_2, x_3^3 \) since \( \text{deg}_2(x_3^5) = \text{deg}_3(x_3^5) = 1 \) and \( \text{deg}_2(x_3^3 x_1^2) = \text{deg}_3(x_3^3 x_2^2 x_1) = \text{deg}_3(x_3^5) = 2; \)
- \( x_2 x_1^2 \): \( x_1 \) is multiplicative, the nonmultiplicative powers are \( x_3^3, x_3^2 \) since \( \text{deg}_2(x_2 x_1^2) = \text{deg}_3(x_2 x_1^2) = 3 \) and \( \text{deg}_2(x_3^3 x_1^2) = \text{deg}_3(x_3^3 x_2^2 x_1) = \text{deg}_3(x_3^5) = 2; \)
- \( x_2^4 x_1 \): \( x_1, x_2 \) are multiplicative, the nonmultiplicative power is \( x_3^3 \) since \( \text{deg}_2(x_2^4 x_1) = \text{deg}_3(x_2^4 x_1) = \text{deg}_3(x_2^4 x_3 x_1) - \text{deg}_3(x_3^3 x_1) = 2; \)
- \( x_2^3 x_1^2 \): \( x_1, x_2 \) are multiplicative, the nonmultiplicative powers are \( x_3^3, x_3^2 \) since \( \text{deg}_2(x_2^3 x_1^2) = \text{deg}_3(x_2^3 x_1^2) = 2 \) and \( \text{deg}_3(x_3^3 x_1^2) = \text{deg}_3(x_3^3 x_2^2 x_1) = 3; \)
- \( x_2^2 x_2 x_1 \): \( x_1, x_2 \) are multiplicative, the nonmultiplicative power is \( x_3^3 \) since \( \text{deg}_2(x_2^2 x_2 x_1) = \text{deg}_3(x_2^2 x_2 x_1) = 3; \)
- \( x_3^5 \): all variables are multiplicative.

Now we show that \( U \) is a complete set, by multiplying any of its terms by its nonmultiplicative powers and showing that the conditions of Theorem [24] hold.
• $x_1^5$: its nonmultiplicative powers are $x_2, x_3^2$, so we consider $x_1^5 x_2$ and $x_1^5 x_3^2$:
  - $x_1^5 x_2 = (x_1^5 x_2) x_1^3$, so the Janet-like divisor is $x_1^3 x_2$;
  - $x_1^5 x_3^2 = (x_1^5 x_3^2) x_1^3$, so the Janet-like divisor is $x_1^3 x_3^2$.
• $x_2 x_1^7$: its nonmultiplicative powers are $x_2^5, x_2^3$, so we consider $x_2^5 x_1^3$ and $x_2 x_1^3$:
  - $x_2^5 x_1^3 = (x_2^5) x_1^3$, so the Janet-like divisor is $x_2^5 x_1^3$;
  - $x_2 x_1^3 = (x_2^5) x_1^3$, so the Janet-like divisor is $x_2 x_1^3$ (note that, in this case, $x_2$ is not Janet-multipicative for $x_2^5 x_1^3$, but the nonmultiplicative power is $x_2^5$).
• $x_3^5 x_1$: its nonmultiplicative power is $x_2^5$, so we have $x_3^5 x_1^2 = (x_3^5) x_1^2$, thus the Janet-like divisor is $x_3^5 x_1^2 x_1$;
• $x_3^5 x_1^2$: its nonmultiplicative powers are $x_2^5, x_3^2$, so we consider $x_2^5 x_1^2$ and $x_3^5 x_1^2 x_1$:
  - $x_2^5 x_1^2 = (x_2^5) x_1^2$, so the Janet-like divisor is $x_2^5 x_1$;
  - $x_3^5 x_1^2 x_1 = (x_3^5) x_1^2 x_1$, so the Janet-like divisor is $x_3^5 x_1^2 x_1$.
• $x_3^5 x_2 x_1$: its nonmultiplicative power is $x_3^5$ so we have $x_3^5 x_2 x_1 = (x_3^5) x_2 x_1$ thus the Janet-like divisor is $x_3^5$.
• $x_3^5$: all variables are multiplicative, so there is nothing to prove.

Note that, in complete accordance with Theorem 24, for each $t \in U$, the Janet-like divisor with respect to a nonmultiplicative power $x_t$ lies over the subsequent $i$-bar. ♦

5 An historical note

In this section, we set a connection between Janet-like multiplicative power and previous results on decomposition of ideals in irreducible primary components. The first result in this framework dates back to Macaulay [25], who gave an irreducible primary decomposition of a zero-dimensional ideal within a fixed coordinates’ system. Such a result has been generalized by Alonso, Marinari an Mora, who gave the definition of infinite corner [26].

Let $I$ be an ideal of $k[x_1, ..., x_n]$. If $I$ is zero-dimensional, its corner set is defined as $C(I) := \{ t \in N(I) : \forall 1 \leq i \leq n, x_i t \in T(I) \} \subset N(I)$. In the non 0-dimensional case, the corner set can be generalized [25] considering also elements $\tau = x_1^{a_1} \cdots x_n^{a_n}, a_i \in \mathbb{N} \cup \{ \infty \}$ and setting

$$\omega \mid \tau \iff \beta_i \leq a_i, \forall \omega = x_1^{b_1} \cdots x_n^{b_n}.$$  

It is then easy to see that there is a finite set

$$C^\infty(I) \subset \{ x_1^{a_1} \cdots x_n^{a_n}, a_i \in \mathbb{N} \cup \{ \infty \} \}$$

which satisfies

$$\omega \in N(I) \iff \exists \tau \in C^\infty(I) : \omega \mid \tau.$$
The ideas in [26] can be interpreted in the language by Gerdt and Blinkov in the sense that nonmultiplicative power arise from infinite corners. The idea behind this connection is to take a generating set \( U = \{ t_1, \ldots, t_m \} \subset T \) for a monomial ideal \( J \) and consider it ordered decreasing order with respect to Lex, so \( t_1 > t_2 > \ldots > t > m \). First of all we consider the term \( t_1 \); all multiples of \( t_1 \) are in \( J \) and all the variables are multiplicative for \( t_1 \) so we say that its infinite corner is \( x^\infty y^\infty \).

Taken then \( t_2 \), we want to consider all the multiples of \( t_2 \) not divided by \( t_1 \). The infinite corner of \( t_2 \) with respect to \( t_1 \) gives the nonmultiplicative powers of \( t_2 \). In particular, the nonmultiplicative powers are the finite exponents of the corresponding variables, while the infinite ones represent the multiplicative variables. Continuing in this fashion with \( t_3, \ldots, t_m \), we get all the nonmultiplicative powers. As a simple example, if \( U = \{ y^3, xy, x^2 \} \subset k[x, y] \), we have that the corner of \( xy \) with respect to \( y^3 \) is \( x^\infty y^2 \) and the corner of \( x^2 \) with respect to \( \{ y^3, xy \} \) is \( x^\infty y \), as shown in the following picture.

6 Perspectives: reduced Janet-like bases computation

In this section we give an overview on how to compute the Janet-like reduced basis for a zerodimensional radical ideal \( I := I(X) \subset k[x_1, \ldots, x_n] \), given its (finite) variety \( X = \{ P_1, \ldots, P_N \} \), in a Groebner-free fashion, following what stated first in [28, 24] and explicitly expressed and sponsored in the book [27, Vol.3.4.12.14.15]. This approach aims to avoid the computation of a Groebner basis of a (0-dimensional) ideal \( I \subset k[x_1, \ldots, x_n] \) in favour of combinatorial algorithms describing instead the structure of the quotient algebra \( k[x_1, \ldots, x_n]/I \).

In the paper [24], Lundqvist proposes four methods to compute the normal form of a polynomial with respect to \( I \), without passing through Groebner bases. In particular, we recall the following proposition.
Proposition 26 ([24]). Let \( X = \{P_1, \ldots, P_N\} \) be a finite set of points, \( I := I(X) \subseteq \mathbb{k}[x_1, \ldots, x_n] \) its ideal of points and \( N = \{t_1, \ldots, t_N\} \subseteq \mathbb{k}[x_1, \ldots, x_n] \) such that \( \{N\} = \{[t_1], \ldots, [t_N]\} \) is a basis for \( A := \mathbb{k}[x_1, \ldots, x_n]/I \). Then, for each \( f \in \mathbb{k}[x_1, \ldots, x_n] \) we have
\[
N(f, N) = (t_1, \ldots, t_N)(N[[X]])^{-1}j^j(f(P_1), \ldots, f(P_N))^j,
\]
where \( N(f, N) \) is the normal form of \( f \) w.r.t. \( N \) and \( N[[X]] \) is the matrix whose rows are the evaluations of the elements of \( N \) at all the points.

If we want to compute a reduced Janet-like basis for \( I \) given \( X \), we only need:

- the points in \( X \);
- a basis \( N \) for the quotient algebra \( A := \mathbb{k}[x_1, \ldots, x_n]/I \);
- a complete set \( U \) of terms w.r.t. Janet-like division, which generates \( T(I) \)

so that the basis is the set \( B = \{Nf(t, N) : t \in U\} \).

A very simple basis for \( A \) is the lexicographical Groebner escalier \( N(X) \) of \( I \) and it can be computed in a purely combinatorial way, without using Groebner bases (see [7, 11, 12, 13, 14, 24]). Once one has the escalier, it is a trivial task to find a generating set \( U \) for \( T(I) \).

Finally, one can construct the Bar Code associated to \( U \) and use Theorem 24 to update it dynamically by adding those terms of the form \( tv, t \in U, v \in NMP(t, U) \) such that it has no Janet-like divisors in \( U \). This way, we can get a completion of \( U \) and a simple application of Proposition 26 to the elements of the completion gives the desired basis, following the approach of [10, 9].

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