Diagonalization of a Hermitian matrix and its application to neutrino mass matrix

Chao-Shang Huang¹, Wen-Jun Li²

1.CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China
2.Physics school, Henan Normal University, Xinxiang 453007, China

Abstract
We carry out diagonalization of a 3×3 Hermitian matrix and apply it to Majorana neutrino mass matrix. It is shown in a model-independent way that \( \delta = \pm \pi/2 \) implies the maximal strength of CP violation in neutrino oscillations. The same in a model-independent way we obtain the prediction \( \cos(2\delta_{23}) = 0 \). It is shown that the Hermitian Majorana neutrino mass matrix \( M = M_L M_R \) has only five real parameters and furthermore, only one free real parameter (D or A) if using the measured values of three mixing angles and mass differences as input. The constraints, which come from our result and prediction, on theoretical models in neutrino physics published in e-prints in recent years are examined.

1. Introduction
Scores of years witness impressive progress in neutrino measurements. Massive data from solar, atmospheric, reactor, and accelerator neutrinos experiments imply neutrino with trivial mass and mixing. The LH flavour neutrino fields \( \nu_l(x) \) could written with the neutrino mixing matrix \( U_{PMNS} \) and the fields of neutrino \( \nu_i \)

\[
\nu_l(x) = \sum_{i} U_{li} \nu_i(x), \quad l = e, \mu, \tau.
\]

(1)

where \( \nu_l(x) \) is the LH component of the field of \( \nu_l \) possessing a mass \( m_l \) and \( U \) is the neutrino mixing matrix \( U_{PMNS} \). It could be expressed by [1]:

\[
U = \begin{pmatrix}
    c_{12}c_{13} & c_{12}s_{13} & s_{12} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{12}c_{13} \\
    s_{12}c_{23} - c_{12}s_{23}s_{13} & -c_{12}c_{23} - s_{12}s_{23}s_{13} & c_{12}c_{13}
\end{pmatrix}
\]

(2)

where \( c_{ij} \equiv \cos \theta_{ij}, s_{ij} \equiv \sin \theta_{ij}, (ij = 12, 13, 23), \delta \) is the Dirac CP violating phase and irrelevant Majorana phases have been assumed to be zero for the sake of simplicity.

The current experimental data of mixing angles and CP-violation phases are listed in the following Table:

| Normal mass ordering (\( m_1 < m_2 < m_3 \)) |
|-------------------------------|-----------------|-----------------|
| Capozzi et al[55] | Best fit | 1σ range | 3σ range |
| \( \sin^2 \theta_{23} / 10^{-1} \) | 3.04 | 2.91 | 3.18 | 2.65 | 3.46 |
| \( \sin^2 \theta_{31} / 10^{-2} \) | 2.14 | 2.07 | 2.23 | 1.90 | 2.39 |
| \( \sin^2 \theta_{32} / 10^{-1} \) | 5.51 | 4.81 | 5.70 | 4.30 | 6.02 |
| \( \delta_{CP} / \pi \) | 1.32 | 1.14 | 1.55 | 0.83 | 1.99 |

| Inverted mass ordering (\( m_3 < m_1 < m_2 \)) |
|-------------------------------|-----------------|-----------------|
| Esteban et al[86] | Best fit | 1σ range | 3σ range |
| \( \sin^2 \theta_{23} / 10^{-1} \) | 3.10 | 2.98 | 3.23 | 2.75 | 3.50 |
| \( \sin^2 \theta_{23} / 10^{-2} \) | 2.24 | 2.17 | 2.31 | 2.04 | 2.44 |
| \( \sin^2 \theta_{31} / 10^{-1} \) | 5.82 | 5.63 | 5.97 | 4.28 | 6.24 |
| \( \delta_{CP} / \pi \) | 1.21 | 1.05 | 1.43 | 0.75 | 2.03 |

3Recently in a review paper[3] by Z.Z. Xing the three flavor mixing angles and one CP-violating phase extracted from a global analysis of current neutrino oscillation data is listed in Fig 10. The Table here is copied from the paper.

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The deeper discussion on neutrino mixing matrix could give us more hints about unknown neutrino sector. There are two ways to diagonalize the Majorana neutrino mass matrix $M_n$. One way is directly to diagonalize $M_n$, the other is to diagonalize the Hermitian matrix $M = M_n M_n^*$. In this letter we shall use the latter way since a Hermitian matrix has advantages of less number of parameters. For two Hermitian matrices which can be diagonalized by the same unitary matrix $U$ simultaneously, they must satisfy a very interesting condition which was originally given in [5] and has been used in [6]. That is, $M_1 M_2^* - M_2 M_1^* = 0$. Using the condition, number of parameters will be decreased further. In the letter we carry out diagonalization of a Hermitian matrix by using the condition. Then we apply it to Majorana neutrino mass matrix Hermitian $M$.

2. Diagonalization of a Hermitian matrix

Consider a $3 \times 3$ Hermitian matrix of which has the general pattern as

$$H = \begin{pmatrix} a_1 + i d_1 & b_1 + i c_1 & b_2 + i c_2 \\ b_4 + i c_4 & a_2 + i d_2 & b_3 + i c_3 \\ b_5 + i c_5 & b_6 + i c_6 & a_3 + i d_3 \end{pmatrix}$$  (3)

$H$ contains eighteen real parameters such that all $a, b, c$ and $d$ are real. It is well known that the Hermitian condition leads that $d_j = 0$, $b_{j+3} = b_j$, and $c_{j+3} = -c_j$, $(j = 1, 2, 3)$. Thus, Eq.(3) now becomes $H = H_R + i H_I$, with

$$H_R = \begin{pmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{pmatrix}, \quad H_I = \begin{pmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{pmatrix}$$  (4)

That is, $H_R$ is symmetric, $H_I$ is anti-symmetric. So the number of parameters is reduced from eighteen down to nine.

For two Hermitian matrices which can be diagonalized by the same unitary matrix $U$ simultaneously, they must satisfy a very interesting condition which was originally given in [5] and has been used in [6]. That is,

$$M_1 M_2^* - M_2 M_1^* = 0.$$  (5)

Since the real component $H_R$ and the imaginary component $i H_I$ of a Hermitian matrix $H = H_R + i H_I$ must are also Hermitian, respectively, and it is easy to prove that they can be diagonalized by a same unitary matrix. So it follows that

$$H_R (i H_I)^* - (i H_I) H_R^* = 0. \tag{6}$$

Substituting Eq.(4) into Eq.(6), we will receive four equations

$$b_1 = c_2 c_3 (a_1 - a_2)/g, \quad b_2 = -c_1 c_3 (a_1 - a_2)/g, \quad b_3 = c_1 c_2 (a_1 - a_2)/g, \quad a_3 = a_2 + (a_1 - a_2) g_1/g.$$  (7-10)

with $g = c_2^2 - c_3^2, g_1 = c_2^2 - c_1^2$ which can be used to further reduce the parameter number from nine down to five. We come to the conclusion: a $3 \times 3$ Hermitian matrix has only five real parameters. The conclusion has been pointed out in the study on two-Higgs-doublet model (2HDM) without Flavor-Changing Neutral Currents at Tree-Level [6]. As a matter of convenience we make the following one choice: $a_1, a_2, c(i = 1, 2, 3)$ are five real parameters, the other four real parameters can be determined from Eqs.(7-10).

3. The general nature of neutrino mass matrix

As pointed in [2], the particular angles and phase in mixing matrix are parameterization convention dependent. In this letter we assume the parameterization in [1], i.e., the PMNS matrix which is explicitly given in Eq.(2) in Introduction.

The symmetric Majorana neutrino mass term can be parameterized by

$$M_n = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$  (3.1)

which has twelve real parameters. Although $M_n$ can be diagonalized by $U$ [4], a simpler method is to diagonalize the Hermitian matrix $M = M_n M_n^*$ [5]

$$M = \begin{pmatrix} A & B & C \\ B^* & D & E \\ C^* & E^* & F \end{pmatrix}$$  (3.2)

with

$$A = |a|^2 + |b|^2 + |c|^2, \quad B = a^* b + b^* d + c^* e, \quad C = a^* c + b^* e + c^* f, \quad D = |b|^2 + |d|^2 + |e|^2, \quad E = b^* c + d^* e + e^* f, \quad F = |c|^2 + |e|^2 + |f|^2.$$  (3.3)
As it is obvious, M has nine real parameters. Then we can use the method and results pointed out in last section. We diagonalize M by

\[ U_{PMNS}^\dagger M U_{PMNS} = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \]  

(3.4)

We get

\[ m_1^2 = \xi_1 - \chi t_{12}, \]  

(3.5)

\[ m_2^2 = \xi_2 + \chi t_{12} = \xi_1 + \frac{\chi}{t_{12}}, \]  

(3.6)

\[ m_3^2 = c_{13}^2 (\xi_3 + t_{13} A + t_{13} |y| 2 \cos \beta), \]  

(3.7)

where

\[ \xi_1 = A - |y| \cos \beta t_{13}, \]  

(3.8)

\[ \chi = \frac{c_{23} ReB - s_{23} ReC}{c_{13}}, \]  

(3.9)

\[ \xi_2 = D + ReE(-t_{23}), \]  

(3.10)

\[ \xi_3 = A + \frac{2|y|(\cos(2\theta_{13}) \cos \beta + i \sin \beta)}{\sin(2\theta_{13})}, \]  

(3.11)

with

\[ y = s_{23} B + c_{23} C = |y| e^{i\alpha}, \]  

\[ \beta = \alpha + \delta. \]  

(3.12)

\[ \xi_2 + \xi_3 = D + F. \]  

(3.13)

And

\[ ImB = t_{23} ImC, \]  

(3.14)

\[ ImE = s_{13} \chi \sin \delta, \]  

(3.15)

\[ \cos \delta = \frac{\sin(2\theta_{23}) \frac{F - D}{s_{13} \chi} - \cos(2\theta_{23}) ReE}{s_{13} \chi}, \]  

(3.16)

(3.7) means that \( \xi_3 \) must be real, so

\[ \sin \beta = 0, \beta = \pm(n\pi), n = 0, 1, 2, \ldots \]  

(3.17)

Furthermore, \( \xi_1, \xi_2 \) satisfies the constraint

\[ (\xi_2 - \xi_1) \mu g(2t_{12}) = 2 \chi. \]  

(3.18)

According to the analysis of the previous section, a 3 \times 3 Hermitian matrix has only five real parameters. That is, there are four constraints:

\[ ReB ImB = -ReC ImC = ReE ImE, \]  

(3.19)

\[ A - D = \frac{ReE ImC - ReC ImE}{ImB}, \]  

(3.20)

\[ F - D = \frac{ReC ImB + ReB ImC}{ImE}. \]  

(3.21)

We can easily obtain the five independent real parameters by using the above four constraints (so \( M \) has 8 real parameters) and transform them into two parameters plus three mixing angles. And we obtain two important conclusions when the four constraints are imposed.

From (3.14) and (3.19), one has

\[ ReC = -t_{23} ReB. \]  

(3.22)

Eq.(3.22) gives \( Rey = 0 \), i.e.,

\[ \cos \alpha = 0, \alpha = \pm(\frac{\pi}{2} + n\pi), n = 0, 1, 2, \ldots \]  

(3.23)

and

\[ Imy = \frac{ImC}{c_{23}}. \]  

(3.24)

Eqs.(3.23) and (3.17) lead to

\[ \cos \delta = 0, \delta = \pm(\frac{\pi}{2} + n\pi), n = 0, 1, 2, \ldots \]  

(3.25)

Therefore, Eq.(3.16) gives

\[ \frac{\sin(2\theta_{23}) \frac{F - D}{s_{13} \chi} - \cos(2\theta_{23}) ReE}{s_{13} \chi} = 0. \]  

(3.26)

From Eqs.(3.14, 3.15, 3.19, 3.22), one has

\[ F - D = \frac{\cos(2\theta_{23}) ImC}{c_{23} t_{13} \sin \delta}. \]  

(3.27)

Substituting (3.27) into (3.26), one has

\[ \cos(2\theta_{23}) \frac{\sin(2\theta_{23}) ImC}{2 c_{23} t_{13} \sin \delta} - ReE = 0, \]  

(3.28)

with \( s_{13} \chi \) is not zero. So,

a) \( \cos(2\theta_{23}) = 0, \)  

or

b) \( ReE = \frac{\sin(2\theta_{23}) ImC}{2 c_{23} t_{13} \sin \delta}, \)  

(3.29b)

or both factors are zeros.

Eqs.(3.19), (3.22), (3.9), (3.15) lead to

\[ ReE = \frac{ImC s_{23}}{t_{13} \sin \delta}. \]  

(3.30)

Eqs.(3.14, 3.15, 3.19, 3.22) lead to

\[ ReB = \frac{ImE c_{23}}{t_{13} \sin \delta}. \]  

(3.31)

\[ ReC = \frac{-ImE s_{23}}{t_{13} \sin \delta}. \]  

(3.32)

Eqs.(3.10, 3.11, 3.13, 3.27) lead to

\[ A - D = \frac{-2|y| \cos(2\theta_{13}) \cos \beta}{\sin(2\theta_{13})} + \frac{ReE}{t_{23}}. \]  

(3.33)
Eqs.(3.19,3.20,3.22,3.15) lead to
\[ g = (A - D) \frac{ImC(t_{13} \sin \delta)}{c_{23}}, \]  
(3.34)
where \( g = ImC^2 - ImE^2 \).

Eqs.(3.33,3.34) lead to
\[ ImE^2 = |\nu|^2(2|c_{23}|^2 - 1 + r_{13}^2). \]  
(3.35)
Thus, we can choose \( D, ImC \), and three mixing angles as five real parameters for specific. Eqs.(3.8), (3.10), (3.33) and (3.18) lead to the sum rule of mixing angles
\[ 2\sqrt{2|c_{23}|^2 - 1 + r_{13}^2} = \frac{c_{13}g(2\theta_{12})}{c_{23}} + \epsilon = 1, \]  
(3.36)
where \( \epsilon = \frac{c_{13}g}{c_{23}} \). Using data of the current global fits which are listed in the table in Introduction, it follows that the sum rule is in agreement with data at 1 sigma level for normal mass hierarchy and at 3 sigma level for both mass hierarchy.

In the case of \( \cos(2\theta_{23}) = 0 \), Eq.(3.27) leads \( F = D \). It is straightforward to obtain elements of the matrix \( M_r \), which are useful to compare with theoretical models and give constraints to them, in terms of above five real parameters by soving Eq.(3.3).

In summary of this section, we emphasize that \( \delta = \pm \frac{\pi}{2} \), (3.25) implies the maximal strength of CP violation in neutrino oscillations for given values of \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \) (i.e., the leptonic Jarlskog parameter is maximal in this case[8]), which is a robust result. The prediction \( \cos(2\theta_{23}) = 0 \), (3.29a)[6] is one of probabilities, i.e., both factors of (3.28) \( =0 \), since (3.29b) has been satisfied (see (3.30)).

4. The constraints to theoretical models

From Eqs.(3.5,3.6,3.7), we have
\[ \Delta m^2_{21} = |m^2_2 - m^2_1| = \frac{\chi(1 + r_{12}^2)}{t_{12}}, \]  
(4.1)
\[ \Delta m^2_{31} = |m^2_3 - m^2_1| = \frac{|\nu|^2\cos\beta}{\sin(2\theta_{13})} + \chi t_{12}. \]  
(4.2)
Then we can use the measured values of mass differences to get the parameter \( ImC \). Thus, there is only one parameter \( D \) in neutrino mass matrix \( M \).

The result \( \delta = \pm (\frac{\pi}{2} + n\pi), n = 0, 1, 2, ... \) should constrain theoretical models in neutrino physics. In the recent paper[9] a simple neutrino mass matrix model based on \( U(3) \times U(3)' \) family symmetry is proposed. Their predictions (see Eq.(25) and (26) in the ref.) deviate our result \( \delta = \pm (\frac{\pi}{2} + n\pi) \) and prediction \( \theta_{23} = 45^\circ \) by about 6% and 5% respectively.

In ref[10] authors apply the \( U(1)_{B-\Lambda,\alpha} (\alpha = e,\mu,\tau) \) gauge symmetry to the scotogenic model and propose three patterns of the one-zero-texture structures of \( M_r \).

The texture \( A \) in the case of normal mass hierarchy can in agreement with our prediction (see Fig.2). The texture \( B \) in the case of normal mass hierarchy can agree with our result \( \delta = \pm (\frac{\pi}{2} + n\pi), n = 0, 1, 2, ... \) but \( \theta_{23} > 45^\circ \) (see Fig.3). The texture \( B \) in the case of inverted mass hierarchy can in agreement with our prediction (see Fig.4). For the texture \( C \), our prediction support the case of normal mass hierarchy (see Figs.5,6).

In the ref[11] Dirac CP violating phase \( \delta_C \) is predicted to be \( 259.579^\circ \) which is excluded by our result.

Ref[12] examines seven cases of texture two zero mass matrices and gives the viable cases \( B_1 \) and \( B_3 \). The combined results favor the \( NO(m_1 < m_2 < m_3) \), \( \delta = 270^\circ \) and \( \theta_{23} > 45^\circ \) at higher CL. Our result \( \delta = \pm (\frac{\pi}{2} + n\pi), n = 0, 1, 2, ... \) supports these two viable cases but \( \theta_{23} > 45^\circ \) is not agreed with our prediction \( \theta_{23} = 45^\circ \).

Fig. 2 in ref[13] shows \( \delta \) may be \( \frac{\pi}{2} + \pi = 270^\circ \) which can agree with our result.

In the ref[14] the most general lepton flavor-dependent \( U(1) \) is introduced to the SM gauge sector and three minimal models consistent with the recent experiments are found. Look at the Fig.4, only the predictions of the model \( C^{\phi}(NO) \) in (a) of Fig.4 can agree with our predictions on \( \theta_{23} \) and \( \delta \).

In a benchmark model with \( \Gamma_3 = T' \) modular symmetry[15], \( \frac{\alpha}{\chi} = 1.313 \) (see Eq.(63) in the paper) is predicted. The value deviates from our result, \( \delta = \pm (\frac{\pi}{2} + n\pi), n = 0, 1, 2, ... \) by about 12%.

A new littlest seesaw model, using a new predictive neutrino mass model building scheme for the minimal seesaw model which is called the tri-direct CP approach, is proposed[16]. This model predicts that mixing parameters are to lie in the rather narrow regions which are given in Eq.(17) in the paper. Comparing the ranges of \( \sin(2\theta_{23}) \) and \( \delta \) in Eq.(17) with our predictions, it is obvious that the model should be excluded.

In ref[17], authors construct a supersymmetric \( S_4 \) flavor symmetry model with one of the trimaximal neutrino mixing patterns, the so-called \( TM_3 \). Fitting one of sum rules in \( TM_3 \) pattern (see Eq.(20) in the paper) with data, one obtains the best-fit value \( \cos \delta = 0.33 \), which

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4Because \( \cos(2\theta_{23}) = 0 \) is not a necessary condition for \( \cos \delta = 0 \), we call it a prediction, not a result.
is far away from our result $\cos \delta = 0$. However, the predictions of $TM_1$ can fall in preferred region of $\sin(2\theta_{23})$ and $\delta$ within 3$\sigma$ range (see Fig.1 in the paper).

An extended version of $\mu - \tau$ symmetric mass matrix which accommodate non-zero $\theta_{13}$ along with maximal CP violation is given in the [1]  

$$
\begin{pmatrix}
an & b & -b^* 
\ b & d & e
\ -b^* & e & d^*
\end{pmatrix},
\tag{4.3}
$$

here $a$ and $e$ are real. The mass matrix has been discussed in a number of works with discrete flavour symmetry models [18]. It corresponds to $\theta_{23} = \frac{\pi}{4}$ and $\delta = -\left(\frac{\pi}{4}\right)$, which are exactly our predictions. In other words, our predictions support the assumption of extended $\mu - \tau$ symmetric mass matrix.

In [19] the models of lepton masses and mixing based on residual symmetries resulting from the breaking of the $A_4$ modular symmetry have been constructed. In these models the $PMNS$ matrix [1] is predicted to be of the trimaximal mixing form [20]. In the case of which the charged lepton mass matrix is diagonal only the neutrino mass matrix contributes to the $PMNS$ matrix. The neutrino mass matrix in this case leads to the so called $TM_3$ mixing form of $PMNS$ matrix $U_{PMNS}$ [20] where the second column of $U_{PMNS}$ is Trimaximal. The trimaximal mixing pattern leads to the following sum rules for the solar neutrino mixing angle $\theta_{12}$ and for the Dirac phase $\sigma$ [21] (see also [21]):

$$
\sin^2 \theta_{12} = \frac{1}{3 \cos^2 \theta_{13}},
\tag{4.4}
$$

$$
\cos \delta = \frac{\cos(2\theta_{23}) \cos(2\theta_{13})}{\sin(2\theta_{23}) \sin(2\theta_{13}) (2 - 3 \sin^2 \theta_{13})}.
\tag{4.5}
$$

Eq.(4.4) is consistent with the current data at 3$\sigma$ level and Eq.(4.5) means $\cos \delta$ is proportional to $\cos(2\theta_{23})$, which is agreed with our predictions $\cos \delta = 0$ and $\cos(2\theta_{23}) = 0$.

A model independent Analysis of Dirac CP Violating Phase for some well known mixing scenarios is presented in [22]. Our result $\delta = \pm(\pi/2 + n\pi), n = 0, 1, 2, ..., \pi/4$, excludes all mixing scenarios given in Table 10 in the paper.

In [23] authors promote the idea of $\mu - \tau$ mixing symmetry which is a generalized version of the $\mu - \tau$ interchange symmetry. Their results show that most probable values of $\delta$ prefers maximal Dirac CP violation while the atmospheric mixing angle $\theta_{13}$ is not necessarily maximal. So their result on $\delta$ is agreed with ours but the result on $\theta_{13}$ is not agreed with our prediction.

For more examples, let us look at the models in [24, 25]. The tri-direct littlest seesaw model predicts $\delta = 279^\circ$ and $\theta_{23} = 47^\circ$ which are close to our predictions. In the ref.[25] based on the type-I seesaw mechanism, authors propose new lepton-mixing textures and seven models of neutrino Majorana mass matrix. Table 1 and Table 2 of the ref.[25] list the $3\sigma$ bounds for various observables in the models with normal neutrino mass ordering and inverted neutrino mass ordering respectively. Our result $\delta = \pm(\pi/2 + n\pi), n = 0, 1, 2, ..., \pi/4$, leads to the conclusion definitely: in Table 1 only model 2 ok and in Table 2 only model 7 (2nd solution) ok, and all other models should be excluded.

In an earlier paper [26] it is shown that the two textures of neutrino mass matrix in two phenomenologically appealing scenarios can lead to maximal atmospheric mixing angle $\theta_{23} = \pi/4$ and Dirac CP phase $\delta = -\frac{\pi}{4}$. Our prediction support the result in the ref.

As the finish of this section let us look at the review paper [27] by Z.Z. Xing in which the three flavor mixing angles and one CP-violating phase extracted from a global analysis of current neutrino oscillation data is listed in Fig.10. We can see from the figure that for the best fit value, our result on CP violation supports inverted mass ordering ($m_3 < m_1 < m_2$). For the values in 3$\sigma$ range, normal and inverted mass orderings both are ok.

5. Summary and Conclusions

We have carried out diagonalization of a $3 \times 3$ Hermitian matrix by using a very interesting condition which is shown by G. C. Branco et. al. [5] and shown an important conclusion that a $3 \times 3$ Hermitian matrix has only five real parameters which has been shown in the ref.[6]. We have applied the method to the Hermitian Majorana neutrino mass matrix M. Because the matrix elements of M have to satisfy four constraints we obtain in a model-independent way an important result, Dirac CP violation phase $\delta = \pm(\pi/2 + n\pi), n = 0, 1, 2, ..., \pi/4$, atmospheric mixing angle $\theta_{23} = \pi/4$, and some sum rules for mixing angles. We have shown obviously that the mass matrix M has only five real parameters and furthermore, only one free real parameter (D or A) if using the measured values of three mixing angles and mass differences as input. We have also examined constraints on theoretical models in neutrino physics published in e-prints in recent years.

In the ref.[27] authors examine the physics reach of the proposed medium baseline muon decay experiment MOMENT. To reach the precision of $\delta_{CP}$ at 10$^{-4}$ or better at 1$\sigma$ confidence level, they find it sufficient to combine the data of MOMENT, DUNE and T2HK. We believe
that the precision measurement of $\delta_{CP}$ in the near future will prove our result.

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