On uniqueness in the inverse obstacle problem via the positive supersolutions of the Helmholtz equation

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Abstract

This paper is concerned with an inverse obstacle scattering problem of an acoustic wave for a single incident plane wave and a wave number. The Colton–Sleeman theorem states the unique recovery of sound-soft obstacles with a smooth boundary from the far-field pattern of the scattered wave for a single incident plane wave at a fixed wave number. The wave number has a bound given by the first Dirichlet eigenvalue of the negative Laplacian in an open ball that contains the obstacles. In this paper, another proof of the Colton–Sleeman theorem that works also for the case when we have a known unbounded set that contains obstacles is given. Unlike the original one, the proof given here is not based on the monotonicity of the first Dirichlet eigenvalue of the negative Laplacian. Instead, it relies on a positive supersolution of the Helmholtz equation in a known domain that contains obstacles. Some corollaries which are new and not covered by the Colton–Sleeman theorem are also given.

1. Introduction

This paper is concerned with an inverse obstacle scattering problem of an acoustic wave for a single incident plane wave and a wave number.

Let \( D \subset \mathbb{R}^m \), \( m = 2, 3 \), be a bounded domain with \( C^2 \) boundary such that \( \mathbb{R}^m \setminus \overline{D} \) is connected. The total wave field \( u \) outside a sound-soft obstacle \( D \) takes the form
\[
    u(x; d, k) = e^{ik \cdot d} + w(x)
\]
with \( k > 0 \), \( d \in S^{m-1} \) and satisfies
\[
\begin{align*}
& \Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^m \setminus \overline{D}, \\
& u = 0 \quad \text{on} \quad \partial D, \\
& \lim_{r \to \infty} r^{(m-1)/2} \left( \frac{\partial w}{\partial r} - ikw \right) = 0.
\end{align*}
\]
The last condition above is called the Sommerfeld radiation condition [3].
The scattered wave \( w = u - e^{ikr}d \) for fixed \( d \) and \( k \) has the asymptotic behavior
\[
 w(r) = \frac{e^{ikr}}{r^{(m-1)/2}} F_D(\varphi, d; k) + O\left(\frac{1}{r^{(m+1)/2}}\right), \quad r \to \infty
\]
uniformly for \( \varphi \in S^{m-1} \), and coefficient \( F_D(\varphi, d; k) \) is called the far-field pattern.

We consider the uniqueness issue of the inverse problem: determine \( D \) from \( F_D(\cdot, d; k) \)
for a fixed \( d \) and \( k \).

This is a well-known open problem, and its complete answer is yet unknown [5]. However,
there is a partial result with a constraint on the range of \( k \) depending on a priori information
about the location of \( D \) and not the shape. In this paper, we denote by \( \lambda_{j,m}(U) \) a bounded
connected open set \( U \subset \mathbb{R}^m \) the \( j \)th Dirichlet eigenvalue of \(-\Delta\) in \( U \).

1.1. A review of the Colton–Sleeman theorem and Gintides’s improvement

In [2], Colton and Sleeman have established the following theorem which we call the Colton–
Sleeman theorem.

**Theorem 1.1** ([2]). Assume that there exists an open ball \( B \) with radius \( R \) such that \( D \subset B \). If \( k^2 < \lambda_{1,m}(B) \), then \( D \) is uniquely determined by \( F_D(\cdot, d; k) \) for a fixed \( d \) and \( k \).

Note that \( \lambda_{1,1}(B) = (\pi / R)^2 \) and \( \lambda_{1,2}(B) = (\gamma_0 / R)^2 \), where \( \gamma_0 \) is the smallest positive
zero of the Bessel function \( J_0(z) \).

The assumption means that for fixed \( k \), the radius of \( B \) that contains \( D \) which is a priori
information about the location of \( D \) cannot be large. It should be pointed out that the optimal
case \( k^2 = \lambda_{1,m}(B) \) is excluded. Their proof does not work for this case.

Their assertion is as follows.

Let \( D_1 \) and \( D_2 \) be two obstacles, and \( u_1 \) and \( u_2 \) denote the corresponding total fields. Let
\( F_1 \) and \( F_2 \) be the corresponding far-field patterns for fixed \( d \) and \( k \). If \( F_1 = F_2 \), then \( D_1 = D_2 \).

They employ a contradiction argument which is a traditional one in proving several
uniqueness theorems in inverse obstacle scattering and goes back to Schiffer’s idea.

The proof can be divided into five steps.

**Step 1.** Assume that the conclusion is not true: \( D_1 \neq D_2 \).

**Step 2.** Showing \( u_1 = u_2 \) in \( D^\infty \) with the use of the Rellich lemma [3], where \( D^\infty \) denotes the
unbounded connected component of the set \( \mathbb{R}^m \setminus (\overline{D}_1 \cup \overline{D}_2) \).

**Step 3.** Showing the existence of a nonempty connected open set \( D_* \subset D^\infty \setminus \overline{D}_1 \) such that
\( u_1 = 0 \) on \( \partial D_* \), where \( D_* = \mathbb{R}^m \setminus \overline{D^\infty} \) and, if necessary, changing the role of \( u_1 \) and
\( u_2 \). See [5] for this point.

**Step 4.** Showing \( u_1|_{D_*} \in H^1_0(D_*) \). This means that there exists a sequence of smooth functions
\( \varphi_n \in C_0^\infty(D_*) \) such that \( \varphi_n \rightharpoonup u_1 \) in \( H^1_0(D_*) \). This is because of the general fact: if \( U \)
is an arbitrary bounded connected open set and \( \varphi \in H^1(U) \cap C^0(\overline{U}) \) satisfies \( \varphi = 0 \)
on \( \partial U \), then \( \varphi \in H^1_0(U) \) (cf corollary 3.28 of [6]). Note that the boundary of \( D_* \) can
be wild in general and thus one cannot use the characterization of \( H^1_0(D_*) \) by the trace
operator onto \( \partial D_* \). See also [3, 7] for this point.

**Step 5.** Showing \( k^2 \geq \lambda_{1,m}(D_*). \) This is because \( u_1 \neq 0 \) and \( \Delta u_1 + k^2 u_1 = 0 \) in \( D_* \).

**Step 6.** Showing \( \lambda_{1,m}(D_*) \geq \lambda_{1,m}(B). \) This is because \( D_* \subset B \) and the monotonicity of the
first Dirichlet eigenvalue with respect to the domain which is an implication of the
mini–max principle for eigenvalues (the Rayleigh–Ritz formula).

From the last step, we have \( k^2 \geq \lambda_{1,m}(B) \). Contradiction.

Note that in step 6 one cannot say more like \( \lambda_{1,m}(D_*) > \lambda_{1,m}(B) \). This is the reason why
the case \( k^2 = \lambda_{1,m}(B) \) is excluded.
Gintides [4] improved the restriction on $k$ in theorem 1.1 as

$$k^2 < \lambda_{2,m}(B).$$

His argument after step 4 is based on the following four facts.

- $\mathcal{P}_1$ also satisfies the same Helmholtz equation.
- $\mathcal{P}_1$ and $\eta_1$ are linearly independent.
- The dimension of the first Dirichlet eigenspace is 1. This is the Courant nodal theorem. See, e.g., p 133 of [6] for the proof for a bounded domain without any regularity assumption on the boundary just like $D_*$.
- The monotonicity of the second Dirichlet eigenvalue with respect to the domain.

From these, he concludes $k^2 \geq \lambda_{2,m}(D_*) \geq \lambda_{2,m}(B)$. Contradiction.

All the arguments stated above are based on the multiplicity of the eigenvalues and their monotonicity with respect to the domain.

Remark 1.1. Note also that, instead of the monotonicity $\lambda_{1,m}(D_*) \geq \lambda_{1,m}(B)$, Stefanov and Uhlmann in [7] used an implication of the Poincaré inequality, that is,

$$\omega_m \leq (\lambda_{1,m}(D_*))^{m/2} \text{Vol}(D_*),$$

where $\omega_m$ denotes the volume of the unit ball in $\mathbb{R}^m$. They proved a uniqueness theorem at fixed $k$ and $d$ provided $D$ contains a known obstacle $D_-$ and is contained in a known obstacle $D_+$ and $\text{Vol}(D_+ \setminus D_-) < \omega_m k^{-m}$.

1.2. Statement of the results

In this paper, we present another method which is based on a real-valued special function $v$, satisfying $\Delta v + k^2 v \leq 0$ in a domain that contains unknown obstacles.

Our main result is the following theorem.

Theorem 1.2. Let $\Omega$ be an open connected set with $\overline{\Omega} \subset \Omega$. Let $k_0 > 0$. Assume that there exists a real-valued function $v \in C^2(\Omega)$ such that $\Delta v + k_0^2 v \leq 0$ in $\Omega$ and $v(x) > 0$ for all $x \in \Omega$. If $k \leq k_0$, then $D$ is uniquely determined by $F_D(\cdot, d; k)$ for fixed $d$ and $k$.

Note that $\Omega$ can be unbounded; it is assumed that $\overline{\Omega} \subset \Omega$ not $D \subset \Omega$. The $v$ in theorem 1.2 should be called a supersolution of the Helmholtz equation in $\Omega$ at the wave number $k_0$ (cf [6] for the notion of the supersolution). Thus theorem 1.2 can be considered as an application of the supersolution in inverse obstacle scattering problems. The following corollary corresponds to theorem 1.1 including the case when $k^2 = \lambda_{1,m}(\Omega)$.

Corollary 1.1. Let $\Omega$ be a bounded open connected set with $\overline{\Omega} \subset \Omega$. If $k^2 \leq \lambda_{1,m}(\Omega)$, then $D$ is uniquely determined by $F_D(\cdot, d; k)$ for a fixed $d$ and $k$.

Proof. Let $\phi$ be the first Dirichlet eigenfunction for the negative Laplacian in $\Omega$. By the Courant nodal theorem, one may assume that $\phi(x) > 0$ for all $x \in \Omega$. Thus, from theorem 1.2 with $k_0^2 = \lambda_{1,m}(\Omega)$ and $v = \phi$, one obtains the desired uniqueness result.

However, this fact itself is not new since the result is a special case of the result by Gintides [4] as mentioned in subsection 1.1 under the condition $k^2 < \lambda_{2,m}(B)$ and $D \subset \Omega \equiv B$. 3
Example 1. Let $B = \{ x \in \mathbb{R}^m \mid |x| < R \}$. For $\Omega = B$ one can choose

$$
\phi(x) = \begin{cases} 
J_0(k_0|x|), & \text{if } m = 2, \\
\sin k_0|x|/|x|, & \text{if } m = 3,
\end{cases}
$$

where $k_0 = \lambda_{1,m}(B)$.

When $\Omega$ is bounded, one cannot find a positive supersolution of the Helmholtz equation in $\Omega$ at the wave number $k > \sqrt{\lambda_{1,m}(\Omega)}$. This is because of the following fact.

**Proposition 1.1.** Let $\Omega$ be a bounded open connected set of $\mathbb{R}^m$. There exists a real-valued function $v \in C^2(\Omega)$ such that $\Delta v + k^2 v \leq 0$ in $\Omega$ and $v(x) > 0$ for all $x \in \Omega$ if and only if $k^2 \leq \lambda_{1,m}(\Omega)$.

For the proof see the appendix. Thus theorem 1.2 does not yield a new result beyond the Colton–Sleeman theorem and Gintides’s result when $\Omega$ is bounded. However, when $\Omega$ is unbounded, there is a possibility of having a positive supersolution in $\Omega$. This is an advantage of theorem 1.2. The following two corollaries are new and not covered by the Colton–Sleeman theorem or Gintides’s result.

**Corollary 1.2.** Let $\Omega'$ be a bounded open connected set of $\Omega$. If $k^2 \leq \lambda_{1,2}(\Omega')$, then $D$ is uniquely determined by $D(\cdot, d; k)$ for a fixed $d$ and $k$.

**Proof.** Let $\phi'$ be the first positive Dirichlet eigenfunction for the negative Laplacian in $\Omega'$. Define $v(x_1, x_2, x_3) = \phi'(x_2, x_3)$ for $x \in \mathbb{R} \times \Omega'$. This $v$ satisfies $\Delta v + k_0^2 v = 0$ in $\mathbb{R} \times \Omega'$ with $k_0^2 = \lambda_{1,2}(\Omega')$ and $v(x) > 0$ for all $x \in \mathbb{R} \times \Omega'$.

**Example 2.** Let $\Omega' = ]-R, R[ \times ]-h, h[ \times \mathbb{R}$ with $h, R > 0$. Then

$$
\lambda_{1,2}(\Omega') = \left( \frac{\pi}{2} \right)^2 \left( \frac{1}{R^2} + \frac{1}{4h^2} \right)
$$

and

$$
\phi'(x_2, x_3) = \cos \frac{\pi}{2R} x_2 \cos \frac{\pi}{2h} x_3, \quad (x_2, x_3) \in ]-R, R[ \times ]-h, h[.
$$

Thus the condition $k^2 \leq \lambda_{1,2}(\Omega')$ is equivalent to

$$
k \leq \frac{\pi}{2} \sqrt{\frac{1}{R^2} + \frac{1}{4h^2}}.
$$

A similar idea yields

**Corollary 1.3.** Let $J$ be a bounded open interval of $\mathbb{R}$ with $J \subset \mathbb{R} \times J$. If $k^2 \leq \lambda_{1,1}(J)$, then $D$ is uniquely determined by $D(\cdot, d; k)$ for a fixed $d$ and $k$.

**Example 3.** Let $J = ]-h, h[ \times \mathbb{R}$ with $h > 0$. Then $\lambda_{1,1}(J) = (\pi/2h)^2$ and an associated positive Dirichlet eigenfunction for the negative Laplacian in $J$ is given by

$$
\phi(x_3) = \cos \frac{\pi}{2h} x_3, \quad |x_3| < h.
$$

The condition $k^2 \leq \lambda_{1,1}(J)$ is equivalent to

$$
k \leq \frac{\pi}{2h}.
$$

Examples 2 and 3 suggest that the larger the number of unbounded directions of the domain $\Omega$, the lower is the upper bound $k_0$.
2. Proof of theorem 1.2

We start with describing a well-known identity.

**Proposition 2.1.** Let $u$ and $v$ be arbitrary smooth functions on an open set $U$ and satisfy $v(x) \neq 0$ for all $x \in U$. Then we have

$$\nabla \cdot (v^2 \nabla \varphi) = v \Delta u - u \Delta v \quad \text{in } U,$$

where

$$\varphi = \frac{u}{v}.$$

Using this identity, we have the following lemma.

**Lemma 2.1.** Let $k_0 > 0$. Assume that there exists a real-valued function $v \in C^2(\Omega)$ such that $\Delta v + k_0^2 v \leq 0$ in $\Omega$ and $v(x) > 0$ for all $x \in \Omega$. Let $U$ be a bounded open connected set of $\Omega$ with $\overline{U} \subset \Omega$. If $k \leq k_0$ and $u \in C^2(U) \cap C^0(\overline{U})$ satisfies

$$\Delta u + k^2 u = 0 \quad \text{in } U,$$

$$u = 0 \quad \text{on } \partial U,$$

then $u = 0$ in $U$.

**Proof.** Define $\varphi = u/v$ in $U$. It follows from proposition 2.1 that

$$\nabla \cdot (v^2 \nabla \varphi) + (\Delta v + k_0^2 v)\varphi = 0 \quad \text{in } U.$$

Since $v^2$ has a positive uniform lower bound on $U$, $\varphi = 0$ on $\partial U$ and

$$(\Delta v + k_0^2 v)\varphi = (\Delta v + k_0^2 v)v - (k_0^2 - k^2)v^2 \leq 0 \quad \text{in } U,$$

the weak maximum principle \cite{6} yields $\varphi = 0$ in $U$ and thus $u = 0$ in $U$. \hfill \Box

The proof of theorem 1.2 starts with step 3. Applying lemma 2.1 with $u = u_1$ and $U = D_*$, we have $u_1 = 0$ in $D_*$. Then the unique continuation gives $u_1 = 0$ in $\mathbb{R}^m \setminus \overline{D}_1$ and this contradicts $u_1 \sim e^{ik \cdot d}$ as $|x| \to \infty$. Therefore it must hold that $D_1 = D_2$.

**Remark 2.1.** Note that if one starts with step 4 in subsection 1.1, then $\varphi = u_1/v|_{D_*} \in H^1_0(D_*)$. Using (2.1) and a sequence in $C^0(D_*)$ that converges to $\varphi$ in $H^1_0(D_*)$, we have

$$\int_{D_*} v^2 |\nabla \varphi|^2 \, dx \to \int_{D_*} (\Delta v + k^2) v |\varphi|^2 \, dx = 0.$$

This also yields the same conclusion as above. This avoids the use of the weak maximum principle, however, needs knowledge that $u_1|_{D_*} \in H^1_0(D_*)$.

**Remark 2.2.** The argument given in the proof of lemma 2.1 together with the use of proposition 2.1 is a well-known typical one in studying the maximum principle for general elliptic partial differential equations (e.g. \cite{6} and the introduction of \cite{1}). Here, we presented it just for the use of (2.1) in remark 2.1, i.e. its use in inverse obstacle scattering problems.

3. Conclusion

The previous known proofs of the Colton–Sleeman and Gintides’s improvement are based on some facts on the Dirichlet eigenvalues of the negative Laplacian in a domain and their monotone dependence on domains.

Our proof is extremely simple and uses a positive *supersolution* $v$ of the Helmholtz equation in a domain $\Omega$ that contains the closure of the unknown obstacle. The domain $\Omega$ in three dimensions can be *unbounded* for a *single direction* as shown in corollary 1.2 and *two directions* as in corollary 1.3 if the wave number has a bound depending on the size of the ‘bounded part’ of $\Omega$. 

5
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Appendix. Proof of proposition 1.1

First assume the existence of $v$ and that the conclusion is not true. Thus one has $k^2 > \lambda_{1,m}(\Omega)$. Let $\Omega_1, \Omega_2, \ldots$ be an exhaustion of $\Omega$ from below in the sense that each $\Omega_j$ is open connected and $\overline{\Omega}_{j-1} \subset \Omega_j \uparrow \Omega$. Then we have $\lambda_{1,m}(\Omega_j) \geq \lambda_{1,m}(\Omega)$ and $\lambda_{1,m}(\Omega_j) \downarrow \lambda_{1,m}(\Omega)$. The latter is also a well-known consequence of the Rayleigh–Ritz formula. Thus for a large $j$ we have $k^2 > \lambda_{1,m}(\Omega_j)$. Let $\phi_j$ denote the first Dirichlet eigenfunction for the negative Laplacian in $\Omega_j$. Define $\varphi = \phi_j/v$ in $\Omega_j$. Since $\Omega_j \subset \Omega$, $\varphi$ belongs to $C^2(\Omega_j) \cap C^0(\overline{\Omega_j})$. It follows from proposition 2.1 that

$$\nabla \cdot (v^2 \nabla \varphi) + (\Delta v + k_0^2 v) v \varphi = 0 \quad \text{in } \Omega_j,$$

(2.1)

where $k_0^2 = \lambda_{1,m}(\Omega_j)$. Since $\varphi = 0$ on $\partial \Omega_j$ and $(\Delta v + k_0^2 v) v = (\Delta v + k_2^2 v) v + (k_0^2 - k^2) v^2 \leq 0$ in $\Omega_j$, the maximum principle yields $\varphi = 0$ in $\Omega_j$; however, this is impossible by the Courant nodal theorem. Therefore it must hold that $k^2 \leq \lambda_{1,m}(\Omega)$. Conversely, if $k^2 \leq \lambda_{1,m}(\Omega)$, then choose the first positive Dirichlet eigenfunction $\varphi$ for the negative Laplacian in $\Omega$ and set $\varphi = \phi$. Then $\Delta v + k^2 v = \Delta \phi + k_0^2 \phi + (k^2 - k_0^2) \phi \leq 0$ in $\Omega$, where $k_0^2 = \lambda_{1,m}(\Omega)$.

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