Schwarzian derivative related to modules of differential operators on a locally projective manifold

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Abstract

We introduce a 1-cocycle on the group of diffeomorphisms Diff(M) of a smooth manifold M endowed with a projective connection. This cocycle represents a non-trivial cohomology class of Diff(M) related to the Diff(M)-modules of second order linear differential operators on M. In the one-dimensional case, this cocycle coincides with the Schwarzian derivative, while, in the multi-dimensional case, it represents its natural and new generalization. This work is a continuation of [3] where the same problems have been treated in one-dimensional case.

1 Introduction

1.1 The classical Schwarzian derivative. Consider the group Diff(S1) of diffeomorphisms of the circle preserving its orientation. Identifying S1 with RP1, fix an affine parameter x on S1 such that the natural PSL(2, R)-action is given by the linear-fractional transformations:

\[ x \rightarrow \frac{ax + b}{cx + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}). \]  

The classical Schwarzian derivative is then given by:

\[ S(f) = \left( \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 \right) dx^2, \]  

where f ∈ Diff(S1).

1.2 The Schwarzian derivative as a 1-cocycle. It is well known that the Schwarzian derivative can be intrinsically defined as the unique 1-cocycle on Diff(S1) with values in the space of quadratic differentials on S1, equivariant with respect to the Möbius group PSL(2, R) ⊂ Diff(S1), cf. [2, 3]. That means, the map (1.2) satisfies the following two conditions:

\[ S(f \circ g) = g^* S(f) + S(g), \]  

where \( f^* \) is the natural Diff(S1)-action on the space of quadratic differentials and

\[ S(f) = S(g), \quad g(x) = \frac{af(x) + b}{cf(x) + d}. \]  

Moreover, the Schwarzian derivative is characterized by (1.3) and (1.4).

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1.3 Relation to the module of second order differential operators. The Schwarzian derivative appeared in the classical literature in closed relation with differential operators. More precisely, consider the space of Sturm-Liouville operators: 

\[ A u = -2 \left( \frac{d}{dx} \right)^2 + u(x), \]

where \( u(x) \in C^\infty(S^1) \), the action of \( \text{Diff}(S^1) \) on this space is given by \( f(A u) = A_v \) with

\[
v = u \circ f^{-1} \cdot (f^{-1})^2 + \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2
\]

(see e.g. [16, ?]).

It, therefore, seems to be clear that the natural approach to understanding of multi-dimensional analogues of the Schwarzian derivative should be based on the relation with modules of differential operators.

1.4 The contents of this paper. In this paper we introduce a multi-dimensional analogue of the Schwarzian derivative related to the \( \text{Diff}(M) \)-modules of differential operators on \( M \).

Following [4] and [10], the module of differential operators \( D_{\lambda, \mu} \) will be viewed as a deformation of the module of symmetric contravariant tensor fields on \( M \). This approach leads to \( \text{Diff}(M) \)-cohomology first evoked in [4]. The corresponding cohomology of the Lie algebra of vector fields \( \text{Vect}(M) \) has been calculated in [10] for a manifold \( M \) endowed with a flat projective structure. We use these results to determine the projectively equivariant cohomology of \( \text{Diff}(M) \) arising in this context.

Note that multi-dimensional analogues of the Schwarzian derivative is a subject already considered in the literature. We will refer [1, 7, 11, 12, 13, 15, 14] for various versions of multi-dimensional Schwarzians in projective, conformal, symplectic and non-commutative geometry.

2 Projective connections

Let \( M \) be a smooth (or complex) manifold of dimension \( n \). There exists a notion of projective connection on \( M \), due to E. Cartan. Let us recall here the simplest (and naive) way to define a projective connection as an equivalence class of standard (affine) connections.

2.1 Symbols of projective connections

Definition. A projective connection on \( M \) is the class of affine connections corresponding to the same expressions

\[
\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n+1} \left( \delta^k_i \Gamma^l_{jl} + \delta^k_j \Gamma^l_{il} \right), \tag{2.1}
\]

where \( \Gamma^k_{ij} \) are the Christoffel symbols and we have assumed a summation over repeated indices.

The symbols (2.1) naturally appear if one considers projective connections as a particular case of so-called Cartan normal connection, see [8].
Remarks.
(a) The definition is correct (i.e. does not depend on the choice of local coordinates on $M$).
(b) The formula (2.1) defines a natural projection to the space of trace-less (2,1)-tensors, one has: $\Pi_{ik} = 0$.

2.2 Flat projective connections and projective structures

A manifold $M$ is said to be locally projective (or endowed with a flat projective structure) if there exists an atlas on $M$ with linear-fractional coordinate changes:

$$x^i = \frac{a^i_j x^j + b^i}{c^i_j x^j + d^i}$$

(2.2)

A projective connection on $M$ is called flat if in a neighborhood of each point, there exists a local coordinate system $(x^1, \ldots, x^n)$ such that the symbols $\Pi^k_{ij}$ are identically zero (see [8] for a geometric definition). Every flat projective connection defines a projective structure on $M$.

2.3 A projectively invariant 1-cocycle on $\text{Diff}(M)$

A common way of producing nontrivial cocycles on $\text{Diff}(M)$ using affine connections on $M$ is as follows. The map: $((f^*\Gamma)^k_{ij} - \Gamma^k_{ij})$ is a 1-cocycle on $\text{Diff}(M)$ with values in the space of symmetric (2,1)-tensor fields. It is, therefore, clear that a projective connection on $M$ leads to the following 1-cocycle on $\text{Diff}(M)$:

$$\ell(f) = \left( (f^*\Pi)^k_{ij} - \Pi^k_{ij} \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$$

(2.3)

vanishing on (locally) projective diffeomorphisms.

Remarks.
(a) The expression (2.3) is well defined (does not depend on the choice of local coordinates). This follows from a well-known fact, that the difference of two (projective) connections defines a (2,1)-tensor field.
(b) Already the formula (2.3) implies that the map $f \mapsto \ell(f)$ is, indeed, a 1-cocycle, that is, it satisfies the relation $\ell(f \circ g) = g^*\ell(f) + \ell(g)$.
(c) It is clear that the cocycle $\ell$ is nontrivial (cf. [10]), otherwise it would depend only on the first jet of the diffeomorphism $f$. Note that the formula (2.3) looks as a coboundary, however, the symbols $\Pi^k_{ij}$ do not transform as components of a (2,1)-tensor field (but as symbols of a projective connection).

Example. In the case of a smooth manifold endowed with a flat projective connection, (with symbols (2.1) identically zero) or, equivalently, with a projective structure, the cocycle (2.3) obviously takes the form:

$$\ell(f, x) = \left( \frac{\partial^2 f^l}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial f^l} - \frac{1}{n+1} \left( \delta^k_j \frac{\partial \log J_f}{\partial x^i} + \delta^k_i \frac{\partial \log J_f}{\partial x^j} \right) \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$$

(2.4)
where \( f(x^1, \ldots, x^n) = (f^1(x), \ldots, f^n(x)) \) and \( J_f = \det \left( \frac{\partial f^i}{\partial x^j} \right) \) is the Jacobian. This expression is globally defined and vanishes if \( f \) is given (in the local coordinates of the projective structure) as a linear-fractional transformation \([2.2]\).

The cocycle \([2.3,2.4]\) was introduced in \([13,11]\) as a multi-dimensional projective analogue of the Schwarzian derivative. However, in contradistinction with the Schwarzian derivative \([1.2]\), this map \([2.4]\) depends on the second-order jets of diffeomorphisms. Moreover, in the one-dimensional case \((n = 1)\), the expression \([2.3,2.4]\) is identically zero.

### 3 Introducing the Schwarzian derivative

Assume that \( \dim M \geq 2 \). Let \( S^k(M) \) (or \( S^k \) for short) be the space of \( k \)-th order symmetric contravariant tensor fields on \( M \).

#### 3.1 Operator symbols of a projective connection

For an arbitrary system of local coordinates fix the following linear differential operator \( T : S^2 \to C^\infty(M) \) given for every \( a \in S^2 \) by \( T(a) = T_{ij}(a^{ij}) \) with

\[
T_{ij} = \Pi^k_{ij} \frac{\partial}{\partial x^k} - \frac{2}{n-1} \left( \frac{\partial \Pi^k_{ij}}{\partial x^k} - \frac{n+1}{2} \Pi^k_{il} \Pi^l_{kj} \right),
\]

where \( \Pi^k_{ij} \) are the symbols of a projective connection \([2.1]\) on \( M \).

It is clear that the differential operator \([3.1]\) is not intrinsically defined, indeed, already its principal symbol, \( \Pi^k_{ij} \), is not a tensor field. In the same spirit that the difference of two projective connections \( \bar{\Pi}^k_{ij} - \Pi^k_{ij} \) is a well-defined tensor field, we have the following

**Theorem 3.1** Given arbitrary projective connections \( \bar{\Pi}^k_{ij} \) and \( \Pi^k_{ij} \), the difference

\[
\mathcal{T} = \bar{T} - T
\]

is a linear differential operator from \( S^2 \) to \( C^\infty(M) \) well defined (globally) on \( M \) (i.e., it does not depend on the choice of local coordinates).

**Proof.** To prove that the expression \([3.2]\) is, indeed a well-defined differential operator from \( S^2 \) into \( C^\infty(M) \), we need an explicit formula of coordinate transformation for such kind of operators.

**Lemma 3.2** The coefficients of a first-order linear differential operator \( A : S^2 \to C^\infty(M) \)

\[
A(a) = \left( t^k_{ij} \partial_k + u_{ij} \right) a^{ij}
\]

transform under coordinate changes as follows:

\[
t^k_{ij}(y) = t^*_{ab}(x) \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \frac{\partial y^k}{\partial x^c}
\]

\[
u_{ij}(y) = u_{ab}(x) \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} - 2t^*_{ab}(x) \frac{\partial^2 y^k}{\partial x^c \partial x^l} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \frac{\partial x^l}{\partial y^j}
\]

where round brackets mean symmetrization.

**Proof of the lemma:** straightforward.
Consider the following expression:

\[ \mathcal{T}(\alpha, \beta)_{ij} = \left( \Pi_{ij}^k - \Pi_{ij}^k \right) \partial_k + \alpha \partial_k \left( \tilde{\Pi}_{ij}^k - \Pi_{ij}^k \right) + \beta \left( \tilde{\Pi}_{il}^k \tilde{\Pi}_{jk}^l - \Pi_{il}^k \Pi_{jk}^l \right) \]

From the definition (3.1,3.2) for

\[ \alpha = -\frac{2}{n-1}, \quad \beta = \frac{n+1}{n-1} \]  

one gets \( \mathcal{T}(\alpha, \beta) = \mathcal{T} \).

Now, it follows immediately from the fact that \( \tilde{\Pi}_{ij}^k - \Pi_{ij}^k \) is a well-defined \((2,1)\)-tensor field on \( M \), that the condition (3.3) for the principal symbol of \( \mathcal{T}(\alpha, \beta) \) is satisfied.

The transformation law for the symbols of a projective connection reads:

\[ \Pi_{ij}^k(y) = \Pi_{ab}^c(x) \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} + \ell(y, x), \]

where \( \ell(y, x) \) is given by (2.4). Let \( u(\alpha, \beta)_{ij} \) be the zero-order term in \( \mathcal{T}(\alpha, \beta)_{ij} \), one readily gets:

\[ u(\alpha, \beta)(y)_{ij} = u(\alpha, \beta)(x)_{ab} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} - 2(\alpha + \beta) \left( \tilde{\Pi}_{ab}^c(x) - \Pi_{ab}^c(x) \right) \frac{\partial^2 y^k}{\partial x^c \partial x^l} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \frac{\partial x^l}{\partial y^j} \]

\[ + (\alpha + \frac{2\beta}{n+1}) \left( \tilde{\Pi}_{ab}^c(x) - \Pi_{ab}^c(x) \right) \frac{\partial \log J_y}{\partial x^c} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j}. \]

The transformation law (3.4) for \( u(\alpha, \beta)_{ij} \) is satisfied if and only if \( \alpha \) and \( \beta \) are given by (3.5). Theorem 3.1 is proven.

We call \( \Pi_{ij}^k \) given by (3.1) the **operator symbols** of a projective connection. This notion is the main tool of this paper.

**Remark.** The scalar term of (3.1) looks similar to the symbols \( \Pi_{ij} = -\partial \Pi_{ij}^k / \partial x^k + \Pi_{il}^k \Pi_{jk}^l \), which together with \( \Pi_{ij}^k \) characterise the normal Cartan projective connection (see §). We will show that the operator symbols \( T_{ij} \), and not the symbols of the normal projective connection, lead to a natural notion of multi-dimensional Scharzian derivative. However, the geometric meaning of (3.1) is still mysterious for us.

### 3.2 The main definition

Consider a manifold \( M \) endowed with a projective connection. The expression

\[ S(f) = f^*(T) - T, \]  

where \( T \) is the (locally defined) operator (3.1), is a linear differential operator well defined (globally) on \( M \).
**Proposition 3.3** the map $f \mapsto S(f)$ is a nontrivial 1-cocycle on $\text{Diff}(M)$ with values in $\text{Hom}(S^2, C^\infty(M))$.

**Proof.** The cocycle property for $S(f)$ follows directly from the definition (3.6). This cocycle is not a coboundary. Indeed, every coboundary $dB$ on $\text{Diff}(M)$ with values in the space $\text{Hom}(S^2, C^\infty(M))$ is of the form $B(f)(a) = f^*(B) - B$, where $B \in \text{Hom}(S^2, C^\infty(M))$. Since $S(f)$ is a first-order differential operator, the coboundary condition $S = dB$ would imply that $B$ is also a first-order differential operator and so, $dB$ depends at most on the second jet of $f$. But, $S(f)$ depends on the third jet of $f$. This contradiction proves that the cocycle (3.6) is nontrivial.  

The cocycle (3.6) will be called the *projectively equivariant Schwarzian derivative*. It is clear that the kernel of $S$ is precisely the subgroup of $\text{Diff}(M)$ preserving the projective connection.

**Example.** In the projectively flat case, $\Pi_{ij}^k \equiv 0$, the cocycle (3.6) takes the form:

$$S(f)_{ij} = \ell(f)^{k}_{ij} \frac{\partial}{\partial x^k} - 2 \frac{\partial}{\partial x^k} \left( \ell(f)^{k}_{ij} \right) + \frac{n+1}{n-1} \ell(f)^{m}_{im} \ell(f)^n_{kj},$$

(3.7)

where $\ell(f)^{k}_{ij}$ are the components of the cocycle (2.3) with values in symmetric $(2,1)$-tensor fields. The cocycle (3.7) vanishes if and only if $f$ is a linear-fractional transformation.

It is easy to compute this expression in local coordinates:

$$S(f)_{ij} = \ell(f)^{k}_{ij} \frac{\partial}{\partial x^k} + \frac{\partial^3 f^k}{\partial x^i \partial x^j \partial x^k} \frac{\partial x^l}{\partial x^k} = \frac{n+3}{n+1} \frac{\partial^2 J_f}{\partial x^i \partial x^j} J_f^{-1} + \frac{n+2}{n+1} \frac{\partial J_f}{\partial x^i} \frac{\partial J_f}{\partial x^j} J_f^{-2}.$$  

(3.8)

To obtain this formula from (3.7), one uses the relation:

$$\frac{\partial^3 f^k}{\partial x^i \partial x^j \partial x^k} \frac{\partial x^l}{\partial x^k} = \frac{\partial^2 f^k}{\partial x^i \partial x^m} \frac{\partial x^n}{\partial x^m} \frac{\partial x^s}{\partial x^k} = \frac{\partial^2 J_f}{\partial x^i \partial x^j} J_f^{-1} + \frac{\partial J_f}{\partial x^i} \frac{\partial J_f}{\partial x^j} J_f^{-2}.$$  

We observe that, in the one-dimensional case ($n = 1$), the expression (3.8) is precisely $-S(f)$, where $S$ is the classical Schwarzian derivative. (Recall that in this case $\ell(f) \equiv 0$.)

**Remarks.**

(a) The infinitesimal analogue of the cocycle (3.7) has been introduced in [10].

(b) We will show in Section 4.3, that the analogue of the operator (3.6) in the one-dimensional case, is, in fact, the operator of multiplication by the Schwarzian derivative.

### 3.3 A remark on the projectively equivariant cohomology

Consider the standard $\text{sl}(n+1, \mathbb{R})$-action on $\mathbb{R}^n$ (by infinitesimal projective transformations). The first group of differential cohomology of $\text{Vect}(\mathbb{R}^n)$, vanishing on the subalgebra $\text{sl}(n+1, \mathbb{R})$, with coefficients in the space $\mathcal{D}(S^k, S^\ell)$ of linear differential operators from $S^k$ to $S^\ell$, was calculated in [10]. For $n \geq 2$ the result is as follows:

$$H^1(\text{Vect}(\mathbb{R}^n), \text{sl}(n+1, \mathbb{R}); \mathcal{D}(S^k, S^\ell)) = \begin{cases} \mathbb{R}, & k - \ell = 2, \\ \mathbb{R}, & k - \ell = 1, \ell \neq 0, \\ 0, & \text{otherwise} \end{cases}$$
The cocycle (3.7) is, in fact, corresponds to the nontrivial cohomology class in the case \( k = 2, \ell = 0 \) integrated to the group \( \text{Diff}(\mathbb{R}^n) \), while the nontrivial cohomology class in the case \( k - \ell = 1 \) is given by the operator of contraction with the tensor field (2.4).

For any locally projective manifold \( M \) it follows that the cocycle (3.6) generates the unique nontrivial class of the cohomology of \( \text{Diff}(M) \) with coefficients in \( \mathcal{D}(S^2, C^\infty(M)) \), vanishing on the (pseudo)group of (locally defined) projective transformations. The same fact is true for the cocycle (2.3).

4 Relation to the modules of differential operators

Consider, for simplicity, a smooth oriented manifold \( M \). Denote \( \mathcal{D}(M) \) the space of scalar linear differential operators \( A : C^\infty(M) \to C^\infty(M) \). There exists a two-parameter family of \( \text{Diff}(M) \)-module structures on \( \mathcal{D}(M) \). To define it, one identifies the arguments of differential operators with tensor densities on \( M \) of degree \( \lambda \) and their values with tensor densities on \( M \) of degree \( \mu \).

4.1 Differential operators acting on tensor densities

Consider the the space \( \mathcal{F}_\lambda \) of tensor densities on \( M \), that mean, of sections of the line bundle \( (\Lambda^n T^* M)^\lambda \). It is clear that \( \mathcal{F}_\lambda \) is naturally a \( \text{Diff}(M) \)-module.

Since \( M \) is oriented, \( \mathcal{F}_\lambda \) can be identified with \( C^\infty(M) \) as a vector space. The \( \text{Diff}(M) \)-module structures are, however, different.

**Definition.** We consider the differential operators acting on tensor densities, namely,

\[
A : \mathcal{F}_\lambda \to \mathcal{F}_\mu.
\]

The \( \text{Diff}(M) \)-action on \( \mathcal{D}(M) \), depending on two parameters \( \lambda \) and \( \mu \), is defined by the usual formula:

\[
f_{\lambda,\mu}(A) = f^*^{-1} \circ A \circ f^*,
\]

where \( f^* \) is the natural \( \text{Diff}(M) \)-action on \( \mathcal{F}_\lambda \).

**Notation.** The \( \text{Diff}(M) \)-module of differential operators on \( M \) with the action (4.2) is denoted \( \mathcal{D}_{\lambda,\mu} \). For every \( k \), the space of differential operators of order \( \leq k \) is a \( \text{Diff}(M) \)-submodule of \( \mathcal{D}_{\lambda,\mu} \), denoted \( \mathcal{D}^k_{\lambda,\mu} \).

In this paper we will only deal with the special case \( \lambda = \mu \) and use the notation \( \mathcal{D}_\lambda \) for \( \mathcal{D}_{\lambda,\lambda} \) and \( f_\lambda \) for \( f_{\lambda,\lambda} \).

The modules \( \mathcal{D}_{\lambda,\mu} \) have already been considered in classical works (see [16]) and systematically studied in a series of recent papers (see [4, 9, 10, 3, 5] and references therein).

4.2 Projectively equivariant symbol map

From now on, we suppose that the manifold \( M \) is endowed with a projective structure. It was shown in [10] that there exists a (unique up to normalization) *projectively equivariant symbol map*, that is, a linear bijection \( \sigma_\lambda \) identifying the space \( \mathcal{D}(M) \) with the space of symmetric contravariant tensor fields on \( M \).
Let us give here the explicit formula of $\sigma_\lambda$ in the case of second order differential operators. In coordinates of the projective structure, $\sigma_\lambda$ associates to a differential operator

$$A = a_2^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + a_1^i \frac{\partial}{\partial x^i} + a_0,$$

(4.3)

where $a_\ell^{i_1 \ldots i_\ell} \in C^\infty(M)$ with $\ell = 0, 1, 2$, the tensor field:

$$\sigma_\lambda(A) = \bar{a}_2^{ij} \partial_i \otimes \partial_j + \bar{a}_1^i \partial_i + \bar{a}_0,$$

(4.4)

and is given by

$$\bar{a}_2^{ij} = a_2^{ij},$$

$$\bar{a}_1^i = a_1^i - \frac{2(n+1)\lambda + 1}{n+3} a_2^{ij} \frac{\partial a_2^{ij}}{\partial x^j},$$

(4.5)

$$\bar{a}_0 = a_0 - \lambda \frac{\partial a_1^i}{\partial x^i} + \frac{(n+1)\lambda + 1}{n+2} \frac{\partial^2 a_2^{ij}}{\partial x^i \partial x^j}.$$

The main property of the symbol map $\sigma_\lambda$ is that it commutes with (locally defined) $\text{SL}(n+1, \mathbb{R})$-action. In other words, the formula (4.5) does not change under linear-fractional coordinate changes (2.2).

### 4.3 Diff($M$)-module of second order differential operators

In this section we will compute the Diff($M$)-action $f_\mu$ given by (4.2) with $\lambda = \mu$ on the space $D^2_\lambda$ (of second order differential operators (4.3) acting on $\lambda$-densities).

Let us give here the explicit formula of Diff($M$)-action in terms of the projectively invariant symbol $\sigma^\lambda$. Namely, we are looking for the operator $\bar{f}_\lambda = \sigma_\lambda \circ f_\lambda \circ (\sigma_\lambda)^{-1}$ (such that the diagram below is commutative):

$$D^2_\lambda \xrightarrow{f_\lambda} D^2_\lambda \xrightarrow{\sigma_\lambda} \mathcal{S}^2 \oplus \mathcal{S}^1 \oplus \mathcal{S}^0 \xrightarrow{\bar{f}_\lambda} \mathcal{S}^2 \oplus \mathcal{S}^1 \oplus \mathcal{S}^0$$

(4.6)

where $\mathcal{S}^2 \oplus \mathcal{S}^1 \oplus \mathcal{S}^0$ is the space of second order contravariant tensor fields (4.4) on $M$.

The following statement, whose proof is straightforward, shows how the cocycles (2.3) and (3.6) are related to the module of second-order differential operators.

**Proposition 4.1** If $\dim M \geq 2$, the action of Diff($M$) on the space of the space $D^2_\lambda$ of second-order differential operators, defined by (4.2), (4.4) is as follows:

$$\bar{f}_\lambda \bar{a}_2^{ij} = (f^* \bar{a}_2)^{ij}$$

$$\bar{f}_\lambda \bar{a}_1^i = (f^* \bar{a}_1)^i + (2\lambda - 1) \frac{n+1}{n+3} \bar{f}_\lambda \bar{a}_2^{ij} \frac{\partial a_2^{ij}}{\partial x^j}$$

$$\bar{f}_\lambda \bar{a}_0 = f^* \bar{a}_0 - \frac{2\lambda(n-1)}{n+2} \mathcal{S}_kl(f^{-1}a_2)(f^* \bar{a}_2)^{kl}$$

(4.7)

where $f^*$ is the natural action of $f$ on the symmetric contravariant tensor fields.
Remark. In the one-dimensional case, the formula (4.7) holds true, recall that $\ell(f) \equiv 0$ and $S_{kl}(f^{-1})(f^*\bar{a}_2)^{kl} = S(f^{-1})f^*\bar{a}_2$ with the operator of multiplication by the classical Schwarzian derivative in the right hand side (cf. [3]). This shows that the cocycle (3.6) is, indeed, its natural generalization.

Note also, that the formula (1.5) is a particular case of (4.7).

4.4 Module of differential operator as a deformation

The space of differential operators $D^2_\lambda$ as a module over the Lie algebra of vector fields $\text{Vect}(M)$ was first studied in [4], it was shown that this module can be naturally considered as a deformation of the module of tensor fields on $M$. Proposition 1.1 extends this result to the level of the diffeomorphism group $\text{Diff}(M)$. The formula (1.7) shows that the $\text{Diff}(M)$-module of second order differential operators on $M$ $D^2_\lambda$ is a nontrivial deformation of the module of tensor fields $T^2$ generated by the cocycles (2.3) and (3.6).

In the one-dimensional case, the $\text{Diff}(S^1)$-modules of differential operators and the related higher order analogues of the Schwarzian derivative was studied in [3].

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