Wilson loops in SYM theory: from weak to strong coupling

Gordon W. Semenoff\textsuperscript{1} and K. Zarembo\textsuperscript{2}

\textsuperscript{1} The Niels Bohr Institute
Blegdamsvej 17, DK2100 Copenhagen \O, Denmark

\textsuperscript{2} Department of Theoretical Physics
Uppsala University, Box 803, S-751 08 Uppsala, Sweden

Abstract

We review Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with emphasis on the exact results. The implications are discussed in the context of the AdS/CFT correspondence.

This is a review compiled from presentations given by the authors at: Sapporo Winter School, Sapporo, Japan, January, 2002; APCTP/KIAS Winter School, Seoul/Pohang, Korea, December 2001; 14th Nordic Network Meeting, Stockholm, November 2001; “Light-cone Physics: Particles and Strings”, Trento, Italy, September, 2001; “Particles, Fields and Strings”, Burnaby, British Columbia, July 2001; Tohwa Symposium, Fukuoka, Japan, July, 2001; APS DPF Northwest Meeting, Seattle, May 2001; MRST Meeting, London, Ontario, May, 2001; Symposium on Gauge Theory, Jena, Germany, February, 2001; Lake Louise Winter School, February, 2001; Canadian Institute for Advanced Research Meeting, Banff, Alberta, February, 2001.

\textsuperscript{*}semenoff@nbi.dk Permanent address: Department of Physics and Astronomy, University of British Columbia, 6224 Agricultural Road, Vancouver, B.C. V6T 1Z1 Canada. Work supported by NSERC of Canada and SNF of Denmark.

\textsuperscript{†}Konstantin.Zarembo@teorfys.uu.se Also at ITEP, B. Cheremushkinskaja, 25, 117259 Moscow, Russia. Work supported by the Royal Swedish Academy of Sciences and by STINT grant IG 2001-062.
1 The AdS/CFT correspondence

One of the most remarkable aspects of string theory is the existence of a
dual description of D-branes. In perturbative string theory D-branes are
D+1-dimensional hypersurfaces in spacetime where open strings are allowed
to begin and end. On the other hand, they are also identified with the
solitonic black brane solutions of supergravity or type II superstring theory.
This gives two alternative formulations of their dynamics. In the first, the
low energy degrees of freedom are described by gauge fields which are the
lowest energy excitations of the open strings. The dynamics is that of a
supersymmetric Yang-Mills gauge theory living on the world-volume of the
branes. In the second, the low energy dynamics is supergravity which is
the low energy limit of closed string theory. The degrees of freedom are
fluctuations of the supergravity fields about the black brane background and
they live in the bulk of ten dimensional spacetime.

There are some situations where these two descriptions have an overlap-
ning domain of validity. In those cases, the same physical system is described
by two different theories which must therefore be dual to each other. Because
the degrees of freedom in these theories live on spaces of different dimensions,
this has been called holographic duality, and is often viewed as an explicit
realization of old ideas about the degrees of freedom in quantum gravity
[1][2]. The application of holographic duality to study the relationship be-
tween gauge fields and gravity is known as the AdS/CFT correspondence
[3][10].

The most precise statement of holographic duality is contained in the
Maldacena conjecture [11]. In this conjecture, on the gravity side, the asymp-
totically flat exterior of an extremal black D3-brane is replaced by its near-
horizon geometry which is a product of 5-dimensional anti-de Sitter (AdS)
space and the 5-sphere, AdS$^5 \times S^5$. The conjecture in its strongest form
then asserts an exact duality between type IIB superstring theory on this
background and four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills the-
ory (SYM) on flat 4-dimensional space. The gauge group of the Yang-Mills
theory is $SU(N)$ and there are $N$ units of Ramond-Ramond (RR) 4-form
flux in the string theory. This duality includes a prescription for identifying
correlation functions in the two theories [12][13].

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory has vanishing beta function
and is a conformal field theory. Its degrees of freedom are a gauge field $A_\mu$,
six scalars $\Phi_i$ and four Majorana spinors $\Psi$. All fields transform in the adjoint
representation of the gauge group. The Lagrangian (in Euclidean space) is

\[ \mathcal{L} = \frac{1}{g_{YM}^2} \text{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_i)^2 - \sum_{i<j} [\Phi_i, \Phi_j]^2 + i \bar{\Psi} \Gamma^\mu D_\mu \Psi + i \bar{\Psi} \Gamma^i [\Phi_i, \Psi] \right\} \]  

This action can be obtained as a dimensional reduction of ten-dimensional \( \mathcal{N} = 1 \) supersymmetric Yang-Mills theory to four dimensions. This is reflected in our notation where we assemble the fermions into a single ten-dimensional 16-component Majorana-Weyl spinor \( \Psi \) with \( (\Gamma^\mu, \Gamma^i) \) the ten dimensional Dirac matrices in the Majorana-Weyl representation.

The dual supergravity background is the near-horizon geometry of a black D3-brane which has \( N \) units of RR-flux. This is the string theory state with \( N \) coinciding D3-branes. The metric can be written with coordinates \( (x^\mu, y^i) \), \( \mu = 1, ..., 4 \), \( i = 1, ..., 6 \) in the form

\[ ds^2 = R^2 \frac{dx^\mu dx^\mu + dy^i dy^i}{y^2}. \]  

The unit 6-vector \( y^i/y \) parameterizes \( S^5 \) and \( x^\mu, y \) are the coordinates of \( AdS^5 \). The \( AdS^5 \) and \( S^5 \) have equal radii of curvature, \( R \). The boundary of the space is at \( y = 0 \) and the AdS horizon is at \( y = \infty \). The metric written explicitly in product form is

\[ ds^2 = R^2 \frac{dx^2_i + dy^2}{y^2} + R^2 d\Omega^2_{S^5}. \]  

In the AdS/CFT correspondence, the radius \( R \) is related to the Yang-Mills coupling by

\[ R = \sqrt{\alpha'} \left( g_{YM}^2 N \right)^{1/4} \]  

The line-element (1.2) is invariant under coordinate transformations which form the AdS group \( SO(2, 4) \). The rest of the isometry group of (1.2) is the symmetry group of \( S^5 \), which is \( SO(6) \sim SU(4) \). Together with the supersymmetry, these form the super-group \( SU(2, 2|4) \).

On the gauge theory side, the bosonic symmetries are manifest as the \( SO(2, 4) \) conformal symmetry and the \( SU(4) \) R-symmetry of SYM theory. In fact, the \( SO(2, 4) \) transformations which leave the AdS metric (1.2) invariant reduce to conformal transformations on the boundary of \( AdS^5 \) where the SYM observables are defined. The radial coordinate \( y \) is associated with the
scale in the SYM theory \cite{14, 15} – larger objects on the boundary probe larger distances in $AdS^5$.

The string theory on the background metric (1.2) is a sigma model with coupling constant given by the inverse of the effective string tension,

$$T = R^2 / 2\pi \alpha' = \sqrt{g_{YM}^2 N / 2\pi}$$  \hspace{1cm} (1.5)

which is a dimensionless quantity.

Furthermore, the string coupling $g_s$ and the Yang-Mills coupling $g_{YM}$ are related by

$$g_s = 4\pi g_{YM}^2$$  \hspace{1cm} (1.6)

This relation can be understood from the fact that the gauge theory action, in front of which the gauge coupling should appear as the factor $1/g_{YM}^2$, is obtained from the disc amplitude in string perturbation theory which is of order $1/g_s$.

With these identifications, the string theory and the SYM theory are conjectured to be exactly equivalent. This equivalence is a remarkable and extremely non-trivial fact. However, it is hard to work out its consequences in the general setting, when both sides of the duality correspond to complicated strongly interacting systems. Weaker and computationally more useful versions of the AdS/CFT duality are obtained by taking limits (table 1). The 't Hooft limit of the gauge theory \cite{16} takes $g_{YM} \to 0$ and $N \to \infty$ with the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$ held fixed. In the string theory, this coincides with the classical limit where $g_s \to 0$ and the radius of curvature of the background space, $R$, is held constant. In this limit, large $N$ Yang-Mills theory is dual to classical string theory on the $AdS^5 \times S^5$ background.

The $g_s \to 0$ limit of string theory in AdS space is still a complicated dynamical theory. The limit projects the string path integral onto an integration over world-sheets of minimal genus. In this limit, the string sigma model is still a highly non-linear two dimensional conformal field theory. This sigma model simplifies in its weak coupling limit, which coincides with the limit where the string tension $T$ and hence the radius of curvature of the space in string units is taken to be large. When the string tension is large, only massless states of the string are important. Other states become infinitely massive and decouple from low energy physics. Thus, the limit of the type IIB string theory which takes the string tension to infinity is approximated by type IIB supergravity on the background $AdS^5 \times S^5$. In the gauge theory, this corresponds to the limit of large 't Hooft coupling $\lambda \to \infty$. Thus,
Table 1: Different limits of the AdS/CFT correspondence

| $N = 4$ SYM | String theory in AdS$^5 \times S^5$ |
|-------------|-----------------------------------|
| Yang-Mills coupling: $g_{YM}$ | String coupling: $g_s$ |
| Number of colors: $N$ | String tension: $T$ |

**Level 1: Exact equivalence**

$g_s = g_{YM}^2 / 4\pi$, $T = \sqrt{g_{YM}^2 N / 2\pi}$

**Level 2: Equivalence in the 't Hooft limit**

$N \to \infty$, $\lambda = g_{YM}^2 N$-fixed (planar limit)  
$g_s \to 0$, $T$-fixed (non-interacting strings)

**Level 3: Equivalence at strong coupling**

$N \to \infty$, $\lambda \gg 1$  
$g_s \to 0$, $T \gg 1$ (classical supergravity)

the strongly coupled large $N$ limit of Yang-Mills theory should coincide with IIB supergravity on the background $AdS^5 \times S^5$.

Even the last, weakest version of this duality has profound consequences. Previous to it, the main quantitative tool which could be used for super-Yang-Mills theory was perturbation theory in $g_{YM}^2$, the Yang-Mills coupling constant. This is limited to the regime where $g_{YM}^2$ and $\lambda$ are both small. The conjectured duality allows one to do concrete computations in a new regime, the limit where $g_{YM}^2$ is small and $N$ and $\lambda$ are both large [12, 13].

The large $N$ expansion of gauge theory has long been thought to be related to some sort of weakly coupled string theory [16]. Development of this idea has been limited by the fact that, although some qualitative features of the large $N$ limit are known, it is not possible to solve the infinite $N$ limit explicitly. Maldacena’s conjecture now gives one explicit example where a string theory is dual to a gauge theory. Moreover, the string theory can be
used to solve the large $N$ and large $\lambda$ limit of the gauge theory.

The best evidence in support the AdS/CFT correspondence comes from symmetry. The global symmetries on the both sides of the correspondence combine into the super-group $SU(2, 2 \mid 4)$. Not only are the global symmetries the same, but some of those objects which carry the representations of the symmetry group — the spectrum of chiral operators in the field theory and the fields in supergravity theory can be matched \[13\]. Furthermore, both theories are conjectured to have a Montonen-Olive $SL(2, Z)$ duality.

The super-conformal symmetry of $\mathcal{N} = 4$ super-Yang-Mills theory severely restricts the form of correlation functions and in some cases it protects them from radiative corrections so that they have only a trivial dependence on the coupling constant. A number of these have been computed using the AdS/CFT correspondence and have been found to agree with their free field limit. This can be viewed as a simultaneous confirmation of supersymmetric non-renormalization theorems and the prediction of AdS/CFT. Examples are the two- and three-point functions of chiral primary operators \[17\].

However, because AdS/CFT and perturbation theory compute different limits, it is difficult to obtain an explicit check of the AdS/CFT correspondence for a quantity which has a non-trivial dependence on the coupling constant. An example of such a quantity is the free energy of Yang-Mills theory heated to temperature $T$ which, because of conformal invariance, must be of the form

$$ F = -f(\lambda, N) \frac{\pi^2}{6} N^2 T^4 V $$

When computed perturbatively in the large $N$ limit,

$$ f(\lambda, N) = 1 - 3\lambda/2\pi^2 + \ldots $$

The gravitational dual of SYM at non-zero temperature is the AdS black hole with Hawking temperature $T$. Its free energy can be deduced from its Beckenstein-Hawking entropy. There are also stringy corrections computed in \[18, 19\]. The result is (1.7) with

$$ f(\lambda, N) = \frac{3}{4} + \frac{45}{32} \zeta(3)\lambda^{-\frac{3}{2}} + \ldots $$

The first computation is an expansion in $\lambda$ whereas the second is an expansion in $1/\lambda^{1/2}$. Though it is not known in the intermediate regime, it has been conjectured that $f(\lambda)$ is a smooth function that interpolates monotonically
between 1 and $\frac{3}{4}$ as $\lambda$ goes from 0 to $\infty$. The corrections on both sides go in the right direction and are consistent with monotonicity of the transition from weak to strong coupling.

There are now a few examples of quantities which are non-trivial functions of the coupling constant and whose large $N$ limit is computable and is thought to be known to all orders in perturbation theory in planar diagrams [20, 21, 22]. All of these quantities involve Wilson loops, which play an important role in the AdS/CFT correspondence for several reasons. Apart from allowing one to obtain exact results in certain cases, Wilson loops are the objects in $\mathcal{N} = 4$ SYM theory whose string theory dual is a source for strings. Thus, they probe string theory directly. This is true even in the supergravity regime where the string that is induced by a Wilson loop source behaves as a classical object. A review of Wilson loops in $\mathcal{N} = 4$ SYM theory is the central theme of this Paper. This review is not comprehensive. Our main emphasis will be on the exact results and we will omit several interesting issues which are discussed extensively elsewhere. Notable omissions are computation of quantum corrections to Wilson loops due string fluctuations [23, 24, 25, 26, 27, 29], the instanton contribution to Wilson loop expectation values [28, 29], and extensions to less supersymmetric and non-conformal examples of gauge theory / gravity correspondence. Wilson loops in that context are reviewed in [30].

2 Wilson loops at strong coupling

The Wilson loop operator in $\mathcal{N} = 4$ super-Yang-Mills theory is associated with the holonomy of a heavy W-Boson. This W-Boson arises when the $SU(N + 1)$ gauge symmetry is broken to $SU(N) \times U(1)$ and the symmetry breaking condensate is sent to infinity. The phase factor in the path-integral representation of the W-Boson propagator involves not only gauge fields, but also scalars:

$$W(C) = \frac{1}{N} \text{tr} P \exp \left[ \oint_C d\tau \left( iA_\mu(x)\dot{x}^\mu + \Phi_i(x)\theta^i |\dot{x}| \right) \right]. \quad (2.1)$$

Here, $C$ is a closed curve parameterized by $x^\mu(\tau)$ and $\theta_i$ is a unit 6-vector, $\theta^2 = 1$, in the direction of the symmetry breaking condensate.

This operator plays more important role in the AdS/CFT correspondence than the usual Wilson loop for several reasons. One of the most important of
them is supersymmetry. The supersymmetry transformations of gauge and scalar fields are
\[ \delta \xi A_\mu (x) = \bar{\Psi} \Gamma_\mu \epsilon, \quad \delta \xi \Phi_i (x) = \bar{\Psi} (x) \Gamma_i \epsilon \] (2.2)
Under the infinitesimal supersymmetry transformation, the exponent in the Wilson loop changes by
\[ \bar{\Psi} \left( i \Gamma_\mu \dot{x}^\mu (\tau) - \Gamma_i \theta^i |\dot{x}(\tau)| \right) \epsilon. \]
The linear combination of Dirac matrices \((i \Gamma_\mu \dot{x}^\mu (\tau) - \Gamma_i \theta^i |\dot{x}(\tau)|)\) squares to zero and has eight zero eigenfunctions. When these eigenfunctions are \(\tau\)-independent, the loop retains half of the supersymmetry. This occurs only when \(\dot{x}^\mu (\tau)\) is a constant, that is, when \(C\) is a straight line. In that case \(W(C)\) is a BPS operator that commutes with half of the supercharges. Consistent with this property, it seems to be protected from radiative corrections. Indeed, in the leading orders of perturbation theory and also in the strong coupling limit which is computed by the AdS/CFT correspondence, it is independent of the coupling constant and
\[ \langle W(\text{straight line}) \rangle = 1 \] (2.3)
A Wilson loop which is not a straight line but is a smooth curve still has local supersymmetry and has better ultraviolet properties than the conventional loop which does not have the scalar field.

The AdS/CFT correspondence can be used to compute the expectation value of a Wilson loop in the large \(\lambda\), large \(N\) limit. In Yang-Mills theory, the amplitude for a heavy W-boson to traverse a closed curve \(C\) of length \(L(C)\) is given by the vacuum expectation value of the Wilson loop accompanied by an exponential factor which is associated with the mass of the W-Boson:
\[ \mathcal{A} = e^{-M L(C)} \langle W(C) \rangle, \] (2.4)
where \(M\) is the mass and this formula is accurate when \(M \to \infty\).

According to the AdS/CFT correspondence, this amplitude can also be computed using string theory. The strings propagate in the bulk of \(AdS^5 \times S^5\) and we should consider those whose worldsheets have boundary on the loop \(C\) [31, 32]:
\[ \mathcal{A} = \int DX^\mu DY^i D\sigma D\phi^a \exp \left( -\frac{\sqrt{\lambda}}{4\pi} \int_D d^2 \sigma \sqrt{h} h_{ab}^{ab} \partial_a X^\mu \partial_b X^\mu + \partial_a Y^i \partial_b Y^i \frac{Y^2}{Y^2} \right. \]
\[ \left. + \text{fermions} \right), \] (2.5)
where $\vartheta^\alpha$ are anticommuting coordinates on the superspace whose bosonic part is $AdS^5 \times S^5$. The fermion piece of the world sheet action, which makes it supersymmetric, is known \[33, 34, 35\] and takes a reasonably simple form in a suitable gauge \[36, 37\], but we will not need its explicit form here. The contour $C$ is located on the boundary of $AdS^5$, and the string partition function is supplemented by the following boundary conditions:

$$X^\mu|_{\partial D} = x^\mu(\tau), \quad Y^i|_{\partial D} = \theta^i Y|_{\partial D}, \quad Y|_{\partial D} = 0. \quad (2.6)$$

The string partition function (2.3) defines a complicated 2 dimensional sigma model which cannot be solved exactly. It simplifies considerably in the large 't Hooft coupling limit where the string tension, $T = \sqrt{\lambda}/2\pi$, becomes large and suppresses string fluctuations. The superstring path integral is then dominated by the bosonic action at its saddle-point. The saddle-point corresponds to a minimal surface in $AdS^5 \times S^5$. Because of the $O(6)$ symmetry of the boundary condition (2.6), the minimal surface is embedded in $AdS^5$ and sits at a particular point, $\theta^i$ on $S^5$.

The string action at the saddle-point is obtained by minimizing the Nambu-Goto action, that is classically equivalent to the Polyakov action in (2.5):

$$\text{Area}(C) = \int d^2 \sigma \frac{1}{\sqrt{2}} \sqrt{\det (\partial_a X^\mu \partial_b X^\mu + \partial_a Y \partial_b Y)}. \quad (2.7)$$

If we equate the two vacuum amplitudes (2.4) and (2.5) and solve for the Wilson loop we get

$$-\ln \langle W(C) \rangle = \frac{\sqrt{\lambda}}{2\pi} \text{Area}(C) - ML(C). \quad (2.7)$$

The area of a surface whose boundary is $C$ is infinite. This infinite part should cancel between the terms on the right-hand-side of (2.7) when we take $M$ to infinity. We shall discuss the reason for this cancellation shortly. The infinite part of the area can be regularized by letting the curve $C = (x^\mu(\tau), y^i(\tau))$ lie in the bulk of $AdS^5 \times S^5$ and later projecting it onto the boundary by taking $y^i(\tau) \to 0$. Let us show that the divergence that arises in this limit is always proportional to the perimeter of $C$. Take, for simplicity, $y^i(\tau) = \theta^i \varepsilon$. Then, it is straightforward to solve for the minimal surface near the boundary. In appropriate coordinates:

$$Y^i(\tau, y) = y \theta^i, \quad X^\mu(\tau, y) = x^\mu(\tau) + O(y^2), \quad (2.8)$$
\[
\text{Area}(C) = \int d\tau \int_\varepsilon dy \frac{1}{y^2} \sqrt{\dot{X}^2 + \dot{X}'^2} - (\dot{X} \cdot \dot{X}')^2 \\
= \int d\tau \int_\varepsilon \frac{dy}{y^2} \left( \sqrt{x^2} + O(y^2) \right) = \frac{1}{\varepsilon} L(C) + \text{finite.} \quad (2.9)
\]

This is the divergent part of the area which should cancel the term with the mass of the W-Boson in (2.7). Since the divergent piece is inversely proportional to the distance from the boundary, when we take the minimal area to be a functional of the boundary curve, the divergent part can be identified using the operator

\[
-\oint_C d\tau y^i(\tau) \frac{\delta}{\delta y^i(\tau)}.
\]

The finite part of the area determines the Wilson loop expectation value:

\[
-\ln \langle W(C) \rangle = \frac{\sqrt{\lambda}}{2\pi} \lim_{|y| \to 0} \left( 1 + \oint_C d\tau y^i(\tau) \frac{\delta}{\delta y^i(\tau)} \right) \text{Area}(C) \equiv \frac{\sqrt{\lambda}}{2\pi} \hat{A}(C) \quad (2.10)
\]

This is a Legendre transform with respect to the variable \(y^i/y^2\) which was noticed and given an interpretation in terms of T-duality in [38]. If one defines the momentum variable

\[
\pi^i = -y^2 \frac{\delta \text{Area}[x^\mu, y^i]}{\delta y^i(\tau)},
\]

then the above equation states that

\[
-\ln \langle W(C) \rangle = \frac{\sqrt{\lambda}}{2\pi} \hat{A}[x^i, \pi^i]
\]

is a function of the coordinates \(x^i\) and momenta \(\pi^i\). The latter should be specified in such a way that the position of world sheet boundary, which is obtained from it by the functional derivative

\[
\frac{y^i}{y^2} = -\frac{\delta \hat{A}}{\delta \pi^i(\tau)}
\]

is at the boundary of the AdS space. Of course, the equations of motion for the variational problem with area \(\hat{A}(C)\) are identical to those for \(\text{Area}(C)\) and the boundary condition is usually easily implemented once \(\hat{A}(C)\) is identified.
Let us also clarify why it is legitimate, at least on the qualitative level, to identify \(1/\varepsilon\) with the mass of the W-Boson. In type II string theory, \(\mathcal{N} = 4\) super-Yang-Mills theory describes the low energy limit of \(N\) parallel D3-branes stacked on top of each other. A W-boson appears in the Higgs phase when the \(SU(N + 1)\) symmetry is broken to \(SU(N) \times U(1)\) by a condensate \(\langle \Phi_i \rangle\). This corresponds to the state where one of the D3-branes is separated from the remaining stack. The W-boson is the lowest energy excitation of the superstring which connects the separated brane and the stack. Its mass is given by the string’s minimal length divided by \(\alpha'\). In the full D3-brane solution of type IIB supergravity, the near-horizon geometry, which is \(AdS^5 \times S^5\), is glued to the asymptotically flat region at the boundary of AdS space. The infinite mass of the W-boson is proportional to the distance from the horizon to the boundary. The subtraction in (2.7) is a regulated version of this distance times the length of the contour, \(C\). Indeed, the area of the surface

\[
Y^i = y \theta^i, \quad X^\mu = x^\mu(\tau),
\]

where \(y\) runs from \(\varepsilon\) to infinity is exactly \(1/\varepsilon\). The divergence in (2.9) is then equal to the mass of the W-boson times \(L(C)\). Thus, there is exact cancellation of the subtracted term and the W-boson mass.

By definition, the minimal surface has the smallest area for given boundary conditions. The area of the surface (2.11), to be subtracted for the sake of regularization, is always larger. Consequently, the renormalized area is always negative. Thus, there are three universal predictions of the AdS/CFT correspondence: in the strong ’t Hooft coupling limit the Wilson loop expectation value exponentiates, the exponent is proportional to \(\sqrt{\lambda}\) and the co-efficient is positive,

\[
\langle W(C) \rangle = \exp \left( \sqrt{\lambda} \times \text{positive number} \right).
\]

Corrections to the Wilson loop in the large \(\lambda\) limit come from the string fluctuations and are suppressed when \(\lambda\) is large. An expansion which includes them perturbatively is an ordinary \(\alpha'\) expansion of the world-sheet sigma model and, for AdS string, goes in powers of \(1/\sqrt{\lambda}\). There is also an overall factor associated with zero modes that arises upon gauge fixing in the integral over internal metrics. The number of zero modes is equal to three times the Euler character of the world sheet, and the path integral over each zero mode contributes a factor of \(\lambda^{1/4}\). For the disk amplitude, which determines the Wilson loop expectation value in the \(g_s \to 0\) limit, since the
Euler character of the disk is $-1$, this gives a factor of $\lambda^{-3/4}$. Hence, a general form of the strong-coupling expansion for a Wilson loop expectation value is

$$\langle W(C) \rangle = \lambda^{-3/4} e^{-\frac{\sqrt{\lambda}}{2\pi} \hat{A}(C)} \sum_{n=0}^{\infty} c_n \lambda^{-n/2},$$

(2.13)

where $c_n$ are numerical coefficients that depend on the contour $C$.

There are several cases of curve $C$ for which the minimal area can be calculated explicitly. As a warm-up exercise, we could try to produce the conjectured expectation value of the straight-line Wilson loop $\langle \tau, 0, 0, 0 \rangle$.

By symmetry, we expect that the minimal surface which has this boundary is an infinite plane which is perpendicular to the boundary of AdS space,

$$X^\mu(\sigma, \tau) = (\tau, 0, 0, 0) \quad Y^i(\sigma, \tau) = \sigma \theta^i$$

(2.14)

Indeed, it is easy to see that this surface solves the equations for a minimal surface which are obtained from the area using a variational principle. The induced metric is

$$ds^2 = d\tau d\tau + \frac{d\sigma d\sigma}{\sigma^2}$$

which is that of the space $AdS^2$. The area element is

$$dA = \frac{1}{\sigma^2} d\sigma d\tau$$

which, to compute the area should be integrated over $\tau \in (-\infty, \infty)$ and $\sigma \in [0, \infty)$. The integration has two sources of divergence, one coming from the infinite length of the line, which is $L = \int d\tau$, and the other coming from the expected singular behavior of the area element near the boundary of AdS space which we cut off according to our prescription of replacing 0 in the lower limit of the integral over $\sigma$ with $\epsilon$. Then, the area is

$$A(\text{straight line}) = \frac{L}{\epsilon}$$

The subtracted area is

$$\hat{A}(\text{straight line}) = \left(1 + \epsilon \frac{\partial}{\partial \epsilon}\right) A(\text{straight line}) = 0$$

(2.15)
which vanishes. Exponentiation gives (2.3), which is the expected unit expectation value of the straight line Wilson loop.

Another important example is the rectangular Wilson loop. The interaction potential for a $W - \bar{W}$ static pair of $W$-boson sources can be extracted from the expectation value of a rectangular Wilson loop with length $T$ and width $L$ by taking the limit

$$V(L) = -\lim_{T \to \infty} \frac{1}{T} \ln \langle W(C_{L\times T}) \rangle.$$  \hspace{1cm} (2.16)

Because of scale invariance, the expectation value of a rectangular loop can depend only on the ratio $T/L$. Then, dimensional analysis implies that $V(L) \sim 1/L$ which is the scale invariant Coulomb interaction. This is what is expected to occur in a conformally invariant gauge theory. Indeed, solving for the minimal surface and evaluating its area one finds [31, 32]:

$$V(L) = -\frac{4\pi^2 \sqrt{\lambda}}{\Gamma^4(1/4) L}.$$  \hspace{1cm} (2.17)

The effective Coulomb charge turns out to be proportional to $\sqrt{\lambda}$, which is smaller than one would expect from the naive extrapolation of the weak-coupling $O(\lambda)$ behavior. This can be interpreted as a screening effect of the processes corresponding to the sum of all planar Feynman diagrams.

### 2.1 Circular loop

Another example where the minimal area can be easily found is that of a circular loop.

The minimal surface whose boundary at $y = 0$ is a circle of radius $a$ is very simple [40, 38]. It is the solution of the quadratic equation

$$x_1^2 + x_2^2 + y^2 = a^2.$$  \hspace{1cm} (2.18)

The induced metric of this minimal surface is

$$ds^2 = \frac{a^2}{y^2(a^2 - y^2)} dy^2 + \frac{a^2-y^2}{y^2} d\varphi^2,$$

where we parameterize the surface by the AdS scale $y$ and the polar angle in the $(x_1, x_2)$ plain $\varphi$. The area element is:

$$dA = \frac{a}{y^2} dy d\varphi.$$
and the regularized minimal area is readily computed:

$$\hat{A}(\text{circle}) = \left(1 + \varepsilon \frac{\partial}{\partial \varepsilon}\right) 2\pi a \int_{\varepsilon}^{a} \frac{dy}{y^2} = -2\pi.$$  \hfill (2.19)

For the expectation value of the circular loop we get:

$$W(\text{circle}) = e^{\sqrt{\lambda}}.$$  \hfill (2.20)

This result does not look suspicious, unless one wonders how it was originally derived. The easiest way to solve for the minimal surface is to use the conformal invariance \cite{40}: the inversion transformation $x_\mu \rightarrow x_\mu / x^2$ maps the circle onto a straight line, for which the minimal surface in (2.14) is really simple. This transformation is conformal and can be extended to an isometry of AdS space:

$$x_\mu \rightarrow \frac{x_\mu}{x^2 + y^2},$$

$$y \rightarrow \frac{y}{x^2 + y^2}.$$  \hfill (2.21)

The minimal surface bounded by a straight line is a half-plane which extends to the horizon and has a geometry of AdS$^2$. The combination of the inversion and the translation by $a$ in $x_1$ maps the half-plane $x_3 = 0$, $x_1 = 1/2a$ onto the hemisphere (2.18).

The minimal area for the straight line (2.15), after the divergence is removed, is zero. This differs from the result for the circle (2.19). What is surprising is that the expectation values for the circle and for the straight line are not the same, in apparent contradiction with the conformal symmetry. Since the expectation values are different for conformally equivalent operators, conformal invariance has been violated.

The violation of conformal symmetry obviously stems from the necessity of regularizing the area. Any regularization explicitly breaks conformal invariance. There is the question of whether conformal symmetry is restored once the infinity is subtracted and the regularization is removed. When properly defined, the area is finite, but renormalization amounts to subtraction of a linearly divergent constant and this leaves a finite effect that breaks conformal invariance. In this respect, the difference between the Wilson line and the circular Wilson loop is reminiscent of the usual conformal anomaly.
The simplest regularization, used in (2.19), moves the boundary of AdS space from $y = 0$ to $y = \varepsilon$. The transformation (2.21) maps the true boundary $y = 0$ on itself and acts on it as the inversion. But the shifted boundary $y = \varepsilon$ gets mapped onto a sphere of a very large radius:

$$x^2 + \left( y - \frac{1}{2\varepsilon} \right)^2 = \frac{2}{4\varepsilon^2}. \quad (2.22)$$

Therefore, the conformal transformation changes the regularization prescription, fig. 1. The ”regularized” AdS space is now the interior of this sphere. If we want to calculate the minimal area for the circle by first mapping it onto a straight line, we must use this unusual regularization. The regularized minimal surface is then the interior of a circle

$$x^2 + \left( y - \frac{1}{2\varepsilon} \right)^2 = \frac{1}{4\varepsilon^2} - \frac{1}{4a^2}$$

in AdS$_2$. The discrepancy between the circle and the straight line derives from the difference in regularization prescriptions. This difference becomes even more evident after the rescaling $(x, y) \to (x/2\varepsilon, y/2\varepsilon)$, which is an isometry of AdS$_2$. The radius of the circle then becomes finite:

$$x^2 + (y - 1)^2 = 1 - \varepsilon^2/a^2. \quad (2.23)$$
The area of its interior is
\[
\int \frac{dxdy}{y^2} = 2 \int_{-\sqrt{1-\varepsilon^2/a^2}}^{\sqrt{1-\varepsilon^2/a^2}} dx \frac{\sqrt{1-\varepsilon^2/a^2 - x^2}}{x^2 + \varepsilon^2/a^2} = \frac{2\pi a}{\varepsilon} - 2\pi,
\]
(2.24)
in agreement with (2.19).

The difference between the Wilson loop expectation values has the classic form of an anomaly. In both cases there is a linear divergence that must be subtracted according to some prescription. The subtraction is what ruins the formal symmetry which would otherwise relate them. However, the area anomaly affects only extended objects such as Wilson loops and should not be confused with ordinary conformal anomaly which affects local operators and which is absent in \( \mathcal{N} = 4 \) SYM in flat Euclidean space.

We have already noted that it could be expected that the straight line Wilson loop has cancelling radiative corrections due to the fact that it is a BPS operator, i.e. it commutes with half of the supercharges. In the super-conformal algebra, besides the 16 supercharges, there are also 16 superconformal charges. The circular Wilson loop commutes with half of the super-conformal charges. In order to regulate the theory, it is necessary to introduce an ultraviolet cutoff. It is possible to cut off in a way that does not break the supersymmetry [20] and thereby preserve the algebra of supercharges in the cut off theory. However, the algebra of conformal supercharges cannot be preserved since the conformal invariance is broken by a cut off. Thus, one might expect that the straight line Wilson loop is more protected by supersymmetry than the circular Wilson loop. This leaves open the possibility that the circular loop can get quantum corrections.

### 2.2 Operator product expansion (OPE)

When probed from a distance much larger than the size of the loop, the Wilson loop should behave effectively as a local operator. More precisely, it can be expanded in a series of local operators [41, 40]:
\[
W(C) = \langle W(C) \rangle \sum C_A R^\Delta A \mathcal{O}^A(0)
\]
(2.25)
where \( \mathcal{O}^A(0) \) is an operator evaluated at the center of the loop, \( \Delta_A \) is the conformal dimension of \( \mathcal{O}^A(x) \), \( R \) is the radius of the loop, and \( C_A \) are OPE coefficients.
The OPE coefficients can be read off from the correlation functions of the Wilson loop with local operators. We can choose the basis of unit normalized primary operators (those which have the lowest dimension in a given representation of the conformal group):

\[ \langle \mathcal{O}^A(x)\mathcal{O}^B(y) \rangle = \frac{\delta^{AB}}{|x-y|^{\Delta_A + \Delta_B}} \quad (2.26) \]

Their OPE coefficients can be extracted from the large distance behavior of the connected two-point correlator:

\[ \frac{\langle W(C) \mathcal{O}^A(L) \rangle}{\langle W(C) \rangle} = \mathcal{C}_A \frac{R^{\Delta_A}}{L^{2\Delta_A}} + \ldots \quad (2.27) \]

where \( L \gg R \). The omitted terms correspond to descendants and are of higher order in \( R/L \).

![Figure 2: (a) A correlation function of the Wilson loop with a local operator is determined by an exchange of the supergravity mode between the classical string world sheet and the point of operator insertion on the boundary of AdS5. (b) At large distances the correlator factorizes, and the OPE coefficient is given by an integral of the appropriate vertex operator over the world sheet.](image)

The strong-coupling evaluation of the OPE coefficients [40] involves a hybrid of the string and the supergravity calculations: The classical string world sheet created by the Wilson loop absorbs the supergravity mode emitted at the point of operator insertion:

\[ \frac{\langle W(C) \mathcal{O}^A(L) \rangle}{\langle W(C) \rangle} = \int d^2\sigma \sqrt{\gamma} V_A(X, \partial/\partial X) G_A(X, L), \quad (2.28) \]

where \( V_A(X, \partial/\partial X) \) is the vertex operator of the supergravity mode associated with \( \mathcal{O}^A \), \( G_A(X, L) \) is bulk-to-boundary propagator, and the integral
runs over the classical string world sheet. The propagator factorizes at large separation and gives a factor $1/L^{2\Delta}$ (fig. 2). Indeed, the scalar bulk-to-boundary propagator associated with a dimension-$\Delta$ operator behaves at large distances as:\[\]

$$G(x, y; L) = \sqrt{\frac{\Delta - 1}{2\pi^2}} \left[ \frac{y}{y^2 + (L - x)^2} \right]^\Delta \to \sqrt{\frac{\Delta - 1}{2\pi^2}} \frac{y^\Delta}{L^{2\Delta}}.$$ \hspace{1cm} (2.29)

The OPE coefficient of a scalar operator is thus given by an integral of the vertex operator over the string world sheet:

$$C_A = R^{-\Delta_A} \sqrt{\frac{\Delta_A - 1}{2\pi^2}} \int d^2 \sigma \sqrt{h} V_A(X, \partial/\partial X)Y \Delta_A.$$ \hspace{1cm} (2.30)

Explicit calculations for a number of chiral operators can be found in Ref. [40].

A chiral primary operator (CPO) is a primary operator which commutes with half of the supercharges and therefore lies in a short representation of the super-conformal algebra. This particularly interesting set of operators are traces of the scalar fields,

$$O^I_k = \frac{(8\pi^2)^{k/2}}{\sqrt{k \lambda^{k/2}}} C^I_{i_1...i_k} \text{tr} \Phi^{i_1}...\Phi^{i_k},$$ \hspace{1cm} (2.31)

where $C^I_{i_1...i_k}$ are totally symmetric traceless tensors which are normalized as

$$C^I_{i_1...i_k} C^J_{i_1...i_k} = \delta^{IJ}.$$ \hspace{1cm} (2.32)

Here, we are following the convention of refs. [17, 40]. The first of the CPOs,

$$O^{ij} = \frac{8\pi}{\sqrt{2 \lambda}} \text{tr} \left( \Phi^i \Phi^j - \frac{1}{6} \delta^{ij} \Phi^2 \right),$$ \hspace{1cm} (2.33)

has lowest possible conformal dimension, $\Delta = 2$, and in this sense is the most important operator in $\mathcal{N} = 4$ SYM theory.

The overall coefficient in the definition of CPOs has been chosen to unit normalize their two-point functions. The two-point correlators of CPOs are protected by supersymmetry and do not receive radiative corrections. This insures that they have the correct normalization to all orders of perturbation.
theory once the normalization is set at weak coupling. This will be important when we will compare perturbative calculations to the supergravity predictions for strong coupling behavior.

The AdS duals of CPOs are particular linear combinations of spin-zero Kaluza-Klein modes on $S^5$ of the metric and the anti-symmetric two form. Each CPO thus is associated with a spherical function:

$$Y^I(\theta) = C^I_{i_1...i_k} \theta^{i_1} \ldots \theta^{i_k}. \quad (2.34)$$

The OPE coefficients of a Wilson loop must be proportional to $Y^I(\theta)$. An explicit calculation for the circular contour gives the large $\lambda$ limit of the correlator, $[40]$:

$$\frac{\langle W(C) \mathcal{O}_{kl} \rangle}{\langle W(C) \rangle} = 2^{k/2-1} \sqrt{k\lambda} \frac{R^k}{L^{2k}} Y^I(\theta) \quad (\lambda \to \infty). \quad (2.35)$$

### 2.3 Wilson loop correlator

The two-point correlator of Wilson loops in the regime when the distance between the loops is large compared to their sizes is one of the cases in which the use of OPE expansion is justified. For identical loops of opposite orientation separated by distance $L$,

$$\frac{\langle W(C_1)W(C_2) \rangle_c}{\langle W(C_1) \rangle \langle W(C_2) \rangle} = \sum |C_A|^2 \left( \frac{R}{L} \right)^{2\Delta_A}. \quad (2.36)$$

This representation of the Wilson loop correlator imposes certain constraints on the OPE coefficients. Since the number of operators of a given conformal dimension grows exponentially with the increase of the dimension, the sum over all operators in intermediate states in (2.36) will diverge at distances comparable to the size of the loops $R \sim L$, unless operators of large dimensions are strongly suppressed (stronger than exponentially). If there is no suppression, the correlator of Wilson loops will undergo a phase transition at some $L \propto R$.

Suppression of operators with large quantum numbers (such as conformal dimensions, spins, etc.) is quite a general statement, which applies to confining theories as well [42]. Indeed, consider the spectral representation for the Wilson loop correlator:

$$\langle W(C_1)W(C_2) \rangle_c = \int_0^\infty dE \rho_C(E) e^{-EL}, \quad (2.37)$$

19
where
\[ \rho_C(E) = \sum_{n \neq 0} \delta(E - E_n) |\langle 0|W(C)|n \rangle|^2. \quad (2.38) \]

The density of states is expected to grow exponentially as \( \exp(E/T_H) \), where \( T_H \) is the Hagedorn temperature. The form-factor of the Wilson loop must suppress this growth. Otherwise, the correlator will undergo a phase transition at \( L = 1/T_H \), which is similar to the Hagedorn transition at finite temperature. Such phase transitions in correlation functions are expected in quantum gravity \[43\], but not in gauge theories. Consequently, the form-factor of the Wilson loop must suppress highly excited states, either in \( \mathcal{N} = 4 \) SYM or in confining gauge theories.

The operators of large scaling dimension are indeed suppressed at weak coupling. Consider perturbative calculation of the OPE coefficients as defined by eq. (2.27); say, for chiral primary operators (2.31). The lowest order diagrams for the dimension-\( k \) operator contain at least \( k \) scalar propagators that go from the operator insertion to the Wilson loop and require an expansion of the loop to at least \( k \)-th order in the scalar fields. Since Wilson loop is an exponential, the OPE coefficient will be suppressed by \( 1/k! \). The same is obviously true at weak coupling for any other operator that has large scaling dimension or spin.

Can we see this suppression on the supergravity side of the AdS/CFT correspondence? The answer to this question seems to be negative. The OPE coefficients for the dimension-\( k \) chiral primary actually grow with \( k \) at strong coupling! This follows from the AdS/CFT prediction, eq. (2.35). Does this mean that, if the coupling is strong enough, the pair correlator of Wilson loops indeed diverges at short distances and there is a phase transition at some critical separation between the loops? We will argue later that growth of OPE coefficients with dimension is an artifact of taking the strong-coupling limit. Exact OPE coefficients rapidly decrease with \( k \) if we carefully take the limit \( k \to \infty \) at any large but finite \( \lambda \).

But there is indeed a phase transition in the Wilson loop correlator at strong coupling. However, it is associated with another phenomenon, the string breaking. The string breaking is a consequence of the area law, and is specific to string theory. At short distances, the correlator of two Wilson loops is saturated by the string stretched between the contours. When the separation between the loops grows, the area of the string world sheet evidently grows too. Since the string has tension, eventually the world sheet breaks into two minimal surfaces that span each of the contours separately.
In between the surfaces, the string world sheet degenerates into an infinitely thin tube which describes propagation of individual supergravity modes. The OPE expansion (2.36) then becomes a good approximation. The two regimes are separated by the Gross-Ooguri phase transition, and the correlator is not analytic in the distance between the loops [12, 13, 14]. As an example, we plot the logarithm of the correlator of two circular loops as a function of the distance $L$ between them in fig. 4. The first derivative of the correlator is discontinuous at the critical separation, so Gross-Ooguri transition is first order in this case.

The Gross-Ooguri transition in an inherently stringy phenomenon and looks rather counterintuitive from the field theory perspective. Indeed, any Feynman diagram that contributes to the Wilson loop correlator is an analytic function of the distance between the loops. Of course, one has to sum an infinite series of all planar diagrams to reach the strong-coupling limit on the field theory side. Surprisingly, even partial resummation that takes into account only planar graphs without internal vertices reveals the Gross-Ooguri transition at strong coupling [17]. It is also possible to see how the Gross-Ooguri transition disappears as one gradually decrease the coupling on the string side of the correspondence [12]. The string fluctuations, that should be taken into account beyond the strong-coupling limit, make the transition smooth, it becomes a crossover at finite $\lambda$ and is completely washed out at weak coupling.
Figure 4: $\ln \langle W(C_1)W(C_2) \rangle$ vs. the distance between the loops $C_1$ and $C_2$ for concentric circles of radius $R$ [13]. The Gross-Ooguri phase transition is of the first order and takes place at $L_c = 0.91R$ [12].

### 3 Wilson loops in perturbation theory

To the leading order in perturbation theory,

$$\langle W(C) \rangle = 1 + \frac{\lambda}{16\pi^2} \oint d\tau_1 d\tau_2 \frac{|\dot{x}(\tau_1)||\dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{|x(\tau_1) - x(\tau_2)|^2} + \cdots. \quad (3.1)$$

The first term in the integral comes from the scalars and the second comes from vector exchange. For a loop without cusps or self-intersections, their sum is finite. This cancellation occurs because of local supersymmetry of the Wilson loop operator. Cusps and self-intersections of the contour lead to divergences as discussed in [38]. An expectation value for a smooth contour is known to be finite at two [20] and three [48] loops. The cancellations are likely to persist to higher orders of perturbation theory, though no rigorous proof of the finiteness to all orders has been given.

The integrand in (3.1) is non-negative by triangle inequality. The extremal case is the straight line, for which the correction is strictly zero, as it should be for a BPS operator. For any other contour,

$$\ln \langle W(C) \rangle = \lambda \times \text{positive number}. \quad (3.2)$$

This is a general prediction of perturbation theory. Comparing to the string theory prediction (2.12), we see that a Wilson loop expectation value interpolates between linear and square-root scaling with $\lambda$ as we go from weak
to strong coupling. One would expect that this interpolation is smooth. In particular, higher-order perturbative corrections should decrease $\ln \langle W(C) \rangle$. Explicit calculations indeed demonstrate that next-to-leading order corrections go in the right direction for simplest contours. For instance, the first perturbative correction to the static potential is repulsive:

$$V(L) = -\left( \frac{\lambda}{4\pi} - \frac{\lambda^2}{8\pi^3} \ln \frac{1}{\lambda} + \ldots \right) \frac{1}{L}. \quad (3.3)$$

The non-analytic dependence on $\lambda$ is a consequence of an IR divergence in the rectangular Wilson loop in the limit when its temporal extent becomes infinite [49]. Careful treatment of this divergence requires infinite resummation of Feynman diagrams, which removes the IR singularity, but makes the static potential non-perturbative beyond the leading order of weak coupling expansion [50, 20].

Another example is the circular loop, for which

$$\ln \langle W(\text{circle}) \rangle = \frac{\lambda}{8} - \frac{\lambda^2}{384} + \ldots. \quad (3.4)$$

Again perturbative series is sign-alternating. Diagram calculations that lead to this formula can be generalized to include a particular class of diagrams to all orders of perturbation theory, namely diagrams without internal vertices (rainbow graphs). The sum of these diagrams is believed to give a large-$N$ exact result for the circular Wilson loop.

4 Exact results for circular Wilson loop

As we discussed before, the circular Wilson loop is almost a BPS operator. The circular loop and the straight line, which is exactly BPS, are conformally equivalent. This equivalence is spoiled by an anomaly and the circular loop gains an expectation value, which is a non-trivial function of the ’t Hooft coupling. Still, one can anticipate that supersymmetry leads to many cancellations among quantum correction for the circle. It was argued [21] that rainbow diagrams exhaust all correction that survive supersymmetry cancellations.
4.1 Expectation value to all orders in perturbation theory

In this section, we consider a circular Wilson loop, whose radius we can assume to be unity. A convenient parameterization of this loop is \( x(\tau) = (\cos \tau, \sin \tau, 0, 0) \).

We will sum all planar diagrams which have no internal vertices. It is instructive to consider first the lowest order of perturbation theory (3.1). For the circular loop, that expression greatly simplifies, because

\[
|x(\tau_1) - x(\tau_2)|^2 = 2 - 2x(\tau_1) \cdot x(\tau_2) = 2(1 - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)),
\]

and, consequently,

\[
\frac{|\dot{x}(\tau_1)| \dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{|x(\tau_1) - x(\tau_2)|^2} = \frac{1}{2},
\]

independently of \( \tau_1 \) and \( \tau_2 \). The contour integrals in (3.1) are trivial and just give an overall factor of \((2\pi)^2\). Computation of the first term in perturbative series (3.4) turns out very simple. The only complication we encounter at higher orders is path ordering and necessity to keep only planar diagrams. In virtue of (4.1), the gluon and the scalar propagators, whose ends lie on the same circle, always combine to a constant. This observation makes the problem of resummation of rainbow diagrams essentially zero-dimensional. In fact, we can express the circular loop in terms of a correlator in a zero-dimensional field theory:

\[
\langle W(\text{circle}) \rangle = \left\langle \frac{1}{N} \text{tr} \exp M \right\rangle_M, \tag{4.2}
\]

where the "path integral" is defined by the partition function

\[
Z = \int dN^2 M \exp \left( -\frac{8\pi^2}{\lambda} N\text{tr}M^2 \right). \tag{4.3}
\]

Averaging over \( M \) correctly accounts for the combinatorics of rainbow diagrams and the measure is chosen to reproduce the field-theory propagator.

It is now straightforward to compute the expectation value of the circular loop using classic results in random matrix theory \[51\]. The eigenvalues of the Gaussian random matrix \( M \) have a continuous distribution with finite
support in the large-$N$ limit. The distribution of eigenvalues obeys the semi-circle law:
\[ \left\langle \frac{1}{N} \mathrm{tr} \, f(M) \right\rangle_M = \frac{2}{\pi} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} dm \, \sqrt{\lambda - m^2} f(m). \] (4.4)
Substituting \( f(m) = e^m \), we find:
\[ \left\langle W(\text{circle}) \right\rangle = \frac{2}{\sqrt{\lambda}} I_1 \left( \sqrt{\lambda} \right), \] (4.5)
where \( I_1 \) is modified Bessel function.

We can compare this result with the prediction of AdS/CFT correspondence by taking the large-$\lambda$ limit:
\[ \left\langle W(\text{circle}) \right\rangle = \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}} \quad (\lambda \to \infty). \] (4.6)
The prediction of the string theory, eq. (2.13), has exactly the same form. Recalling that the area of minimal surface associated with the circle is equal to $-2\pi$, we find the complete agreement with string theory prediction! The exact expression (4.5) smoothly interpolate between perturbative series in $\lambda$ and the strong coupling regime, where the natural expansion parameter is $1/\sqrt{\lambda}$. This latter expansion is to be identified with $\alpha'$ expansion of the world-sheet sigma model.

The summation of rainbow diagrams for the circular Wilson loop can be extended to all orders of $1/N^2$ expansion. In agreement with expectations from string theory, each order contains the same exponential factor multiplied by an overall power of $\lambda^{1/4}$ at strong coupling [21]:
\[ \left\langle W(\text{circle}) \right\rangle = \sqrt{\frac{2}{\pi}} \sum_{g=0}^{\infty} \frac{1}{N^{2g} g!} \lambda^{(6g-3)/4} e^{\sqrt{\lambda}} \quad (\lambda \to \infty). \] (4.7)
As was explained by Drukker and Gross [21], the power of $\lambda^{1/4}$ at $g$-th order of $1/N^2$ expansion correctly counts the number of zero modes at $g$-th order of string perturbation theory.

### 4.2 OPE coefficients for chiral primary operators

At weak coupling, the OPE coefficient of the circular Wilson loop for dimension-$k$ CPO (2.31) is proportional to $\lambda^{k/2}$:
\[ \frac{\left\langle W(\text{circle}) \mathcal{O}^I_k \right\rangle}{\left\langle W(\text{circle}) \right\rangle} = 2^{-k/2} \frac{\sqrt{k}}{k!} \lambda^{k/2} \frac{R^k}{L^{2k}} Y^I(\theta) + \ldots \quad (\lambda \to 0). \] (4.8)
Comparing this with the AdS/CFT prediction (2.35) we see that OPE coefficients are non-trivial functions of the ’t Hooft coupling.

Again, appealing to supersymmetry and conformal invariance, we argue that correlators of the circular Wilson loop with chiral operators are saturated by free fields. Therefore, calculation of these correlators amounts in resummation of all planar rainbow diagrams of the kind shown in fig. 5. This is a rather lengthy exercise for arbitrary $k$ which involves the use of loop equations [52, 53, 54] in the matrix model (4.3). The details may be found in the original reference [22]. Here, we only quote the result:

$$\langle W(\text{circle}) \mathcal{O}_k \rangle = 2^{k/2} I_k \left( \sqrt{\lambda} \right) \frac{R^k}{L^{2k}} Y^1(\theta).$$  \hspace{1cm} (4.9)

We expect that this expression is exact in the large $N$ limit. Its perturbative series expansion starts with (4.8).

Figure 5: A typical diagram that contributes to the correlator of the circular Wilson loop with CPO.

At strong coupling we expect to reproduce the AdS/CFT prediction (2.35), and this is indeed the case, because all modified Bessel functions have the same asymptotics at infinity. This provides an infinite series of correlation functions, for which resummed perturbative series allow to trace an interpolation between weak coupling regime and the strong-coupling prediction of string theory in Anti-de-Sitter space.
The non-perturbative expression (4.9) resolves the puzzle mentioned in sec. 2.3, where we argued that OPE coefficients must be small for operators of large dimension and noticed that this is not the case if we use the supergravity prediction for OPE coefficients. However, expanding the Bessel function \( I_k(\sqrt{\lambda}) \) in \( \lambda \) one can check that the smallness parameter of perturbation theory for \( \langle W(C) \mathcal{O}_k^I \rangle \) is not \( \lambda \), but \( \lambda/k \), so large-\( k \) limit is always perturbative. If we take the limit \( k \to \infty \) before \( \lambda \to \infty \), we can keep only the first term of perturbation series, which is indeed suppressed by \( 1/k! \).

\section{Remarks}

One of the many achievements made possible by the discovery of the AdS/CFT correspondence is a systematic way to do computations in the interacting field theory at strong coupling. These computation are done by methods that are quite unusual and sometimes counterintuitive from the field theory perspective. It is therefore very important that non-trivial predictions of string theory and supergravity can be reproduced by more or less ordinary techniques of planar perturbation theory. Of course, this is possible only in special cases and depends on symmetries of \( \mathcal{N} = 4 \) SYM theory, but the very fact that it is possible is quite surprising. It is also important that exact field-theory calculations can be done for Wilson loops which probe string theory directly and therefore allow to test the AdS/CFT correspondence in its strongest form.

The current status of this subject leaves many questions unanswered. Some of the immediate questions are

\begin{itemize}
  \item The straight line Wilson loop appears to have unit expectation value. This is a prediction of the supergravity computation for the strong coupling limit and it also seems to be so for perturbative computations to a reasonably high order in the Feynman gauge. In other gauges which are related by conformal transformation with the Feynman gauge, the leading perturbative corrections need not vanish but can reproduce the perturbative limit of the circle Wilson loop. It is clear that a deeper understanding of the gauge dependence of this object is needed. It would be interesting to extend the arguments for non-renormalization of correlators of local BPS operators to the case of the Wilson line, which is a non-local operator.
  \item There should be a more rigorous proof that radiative corrections to the results in this paper actually cancel. One approach which was suggested in
\end{itemize}
is to show that the result for the circle comes from a conformal anomaly. Establishing this at a rigorous level would be an important step in the right direction.

• It should be possible to study other kinds of Wilson loops [50, 53, 48, 56].

• Most desirable would be to obtain some results for non-conformally invariant gauge theories. At this point this appears to be very difficult as most of the analytic computations that have been done so far depend heavily on conformal invariance.

References

[1] G. ’t Hooft, “Dimensional Reduction In Quantum Gravity,” arXiv:gr-qc/9310026.

[2] L. Susskind, “The World as a hologram,” J. Math. Phys. 36, 6377 (1995) arXiv:hep-th/9409089.

[3] J. L. Petersen, “Introduction to the Maldacena conjecture on AdS/CFT,” Int. J. Mod. Phys. A 14, 3597 (1999) arXiv:hep-th/9902131.

[4] P. Di Vecchia, “An introduction to AdS/CFT correspondence,” Fortsch. Phys. 48, 87 (2000) arXiv:hep-th/9903007.

[5] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183 arXiv:hep-th/9905111.

[6] E. T. Akhmedov, “Introduction to the AdS/CFT correspondence,” arXiv:hep-th/9911095.

[7] I. R. Klebanov, “TASI lectures: Introduction to the AdS/CFT correspondence,” arXiv:hep-th/0009139.

[8] S. Förste, “Strings, branes and extra dimensions,” arXiv:hep-th/0110054.

[9] D. Z. Freedman and P. Henry-Labordere, “Field theory insight from the AdS/CFT correspondence,” arXiv:hep-th/0011086.

[10] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence,” arXiv:hep-th/0201253.
[11] J. Maldacena, “The large N limit of super-conformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1998)] [hep-th/9711200].

[12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428 (1998) 105 [arXiv:hep-th/9802109].

[13] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[14] E. T. Akhmedov, “A remark on the AdS/CFT correspondence and the renormalization group flow,” Phys. Lett. B 442 (1998) 152 [arXiv:hep-th/9806217].

[15] A. W. Peet and J. Polchinski, “UV/IR relations in AdS dynamics,” Phys. Rev. D 59 (1999) 065011 [arXiv:hep-th/9809022].

[16] G. ’t Hooft, “A Planar Diagram Theory For Strong Interactions,” Nucl. Phys. B 72, 461 (1974).

[17] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, “Three-point functions of chiral operators in D = 4, N = 4 SYM at large N,” Adv. Theor. Math. Phys. 2, 697 (1998) [hep-th/9806074].

[18] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, “Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory,” Nucl. Phys. B 534, 202 (1998) [arXiv:hep-th/9805150].

[19] J. Pawelczyk and S. Theisen, “AdS(5) x S(5) black hole metric at O(α’3),” JHEP 9809, 010 (1998) [arXiv:hep-th/9808120].

[20] J. K. Erickson, G. W. Semenoff and K. Zarembo, “Wilson loops in N = 4 supersymmetric Yang-Mills theory,” Nucl. Phys. B 582, 155 (2000) [hep-th/0003053].

[21] N. Drukker and D. J. Gross, “An exact prediction of N = 4 SUSYM theory for string theory,” J. Math. Phys. 42 (2001) 2896 [arXiv:hep-th/0010274].

[22] G. W. Semenoff and K. Zarembo, “More exact predictions of SUSYM for string theory,” Nucl. Phys. B 616 (2001) 34 [arXiv:hep-th/0106015].

[23] N. Drukker, D. J. Gross and A. Tseytlin, “Green-Schwarz string in AdS(5) x S(5): Semiclassical partition function,” JHEP 0004, 021 (2000) [hep-th/0001204].
[24] S. Förste, D. Ghoshal and S. Theisen, “Stringy corrections to the Wilson loop in N = 4 super Yang-Mills theory,” JHEP 9908, 013 (1999) [hep-th/9903042].

[25] J. Greensite and P. Olesen, “Remarks on the heavy quark potential in the supergravity approach,” JHEP 9808, 009 (1998) [hep-th/9806233].

[26] S. Naik, “Improved heavy quark potential at finite temperature from anti-de Sitter supergravity,” Phys. Lett. B 464, 73 (1999) [hep-th/9904147].

[27] Y. Kinar, E. Schreiber, J. Sonnenschein and N. Weiss, “Quantum fluctuations of Wilson loops from string models,” Nucl. Phys. B 583, 76 (2000) [hep-th/9911123].

[28] M. Bianchi, M. B. Green and S. Kovacs, “Instantons and BPS Wilson loops,” arXiv:hep-th/0107028.

[29] M. Bianchi, M. B. Green and S. Kovacs, “Instanton corrections to circular Wilson loops in N = 4 supersymmetric Yang-Mills,” arXiv:hep-th/0202003.

[30] J. Sonnenschein, “What does the string / gauge correspondence teach us about Wilson loops?,” arXiv:hep-th/0003032.

[31] J. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80, 4859 (1998) [hep-th/9803002].

[32] S. J. Rey and J. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C 22 (2001) 379 [arXiv:hep-th/9803001].

[33] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in AdS(5) x S(5) background,” Nucl. Phys. B 533 (1998) 109 [arXiv:hep-th/9805028].

[34] R. Kallosh, J. Rahmfeld and A. Rajaraman, “Near horizon superspace,” JHEP 9809 (1998) 002 [arXiv:hep-th/9805217].

[35] I. Pesando, “A kappa gauge fixed type IIB superstring action on AdS(5) x S(5),” JHEP 9811 (1998) 002 [arXiv:hep-th/9808020].

[36] R. Kallosh and J. Rahmfeld, “The GS string action on AdS(5) x S(5),” Phys. Lett. B 443 (1998) 143 [arXiv:hep-th/9808038].

[37] R. Kallosh and A. A. Tseytlin, “Simplifying superstring action on AdS(5) x S(5),” JHEP 9810 (1998) 016 [arXiv:hep-th/9808088].
[38] N. Drukker, D. J. Gross and H. Ooguri, “Wilson loops and minimal surfaces,” Phys. Rev. D 60, 125006 (1999) [hep-th/9904191].

[39] O. Alvarez, “Theory Of Strings With Boundaries: Fluctuations, Topology, And Quantum Geometry,” Nucl. Phys. B 216 (1983) 125.

[40] D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large N limit,” Phys. Rev. D 59, 105023 (1999) [hep-th/9809188].

[41] M. A. Shifman, “Wilson Loop In Vacuum Fields,” Nucl. Phys. B 173 (1980) 13.

[42] K. Zarembo, “Wilson loop correlator in the AdS/CFT correspondence,” Phys. Lett. B 459 (1999) 527 [arXiv:hep-th/9904149].

[43] O. Aharony and T. Banks, “Note on the quantum mechanics of M theory,” JHEP 9903 (1999) 016 [arXiv:hep-th/9812237].

[44] D. J. Gross and H. Ooguri, “Aspects of large N gauge theory dynamics as seen by string theory,” Phys. Rev. D 58 (1998) 106002 [arXiv:hep-th/9805129].

[45] P. Olesen and K. Zarembo, “Phase transition in Wilson loop correlator from AdS/CFT correspondence,” arXiv:hep-th/0009210.

[46] H. Kim, D. K. Park, S. Tamarian and H. J. Muller-Kirsten, “Gross-Ooguri phase transition at zero and finite temperature: Two circular Wilson loop case,” JHEP 0103 (2001) 003 [arXiv:hep-th/0101233].

[47] K. Zarembo, “String breaking from ladder diagrams in SYM theory,” JHEP 0103, 042 (2001) [hep-th/0103058].

[48] J. Plefka and M. Staudacher, “Two loops to two loops in N = 4 supersymmetric Yang-Mills theory,” JHEP 0109, 031 (2001) [arXiv:hep-th/0108182].

[49] T. Appelquist, M. Dine and I. J. Muzinich, “The Static Limit Of Quantum Chromodynamics,” Phys. Rev. D 17 (1978) 2074.

[50] J. K. Erickson, G. W. Semenoff, R. J. Szabo and K. Zarembo, “Static potential in N = 4 supersymmetric Yang-Mills theory,” Phys. Rev. D 61, 105006 (2000) [hep-th/9911088].

[51] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” Commun. Math. Phys. 59 (1978) 35.
[52] A. A. Migdal, “Loop Equations And 1/N Expansion,” Phys. Rept. 102, 199 (1983).

[53] Y. Makeenko, “Loop Equations In Matrix Models And In 2-D Quantum Gravity,” Mod. Phys. Lett. A 6 (1991) 1901.

[54] G. Akemann and P. H. Damgaard, “Wilson loops in N = 4 supersymmetric Yang-Mills theory from random matrix theory,” Phys. Lett. B 513 (2001) 179 [Erratum-ibid. B 524 (2001) 400] [arXiv:hep-th/0101225].

[55] J. Erickson, G. W. Semenoff and K. Zarembo “BPS vs. non-BPS Wilson loops in N = 4 supersymmetric Yang-Mills theory,” Phys. Lett. B 466, 239 (1999) [hep-th/9906211].

[56] G. Arutyunov, J. Plefka and M. Staudacher, “Limiting geometries of two circular Maldacena-Wilson loop operators,” arXiv:hep-th/0111290.