Notes on the formal degrees of supercuspidal unipotent representations and spectral transfer morphisms for affine Hecke algebras

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Abstract

The notion of spectral transfer morphisms of affine Hecke algebras, introduced by Opdam [15], is a useful tool in the study of formal degrees of unipotent discrete series representations (cf. [16, 8]). Based on the uniqueness result established in [8], Opdam proved that the unipotent discrete series representations of classical groups, can be classified by the associated formal degrees, in the same spirit as Reeder’s result [18] for split exceptional adjoint groups. Consequently, Lusztig’s classification of unipotent representations [10], appeared as an “arithmetic/geometric correspondence”, can be explained in terms of harmonic analysis.

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Let $k$ be a non-archimedean local field with a finite residue field $\mathbb{F}_q$. Fix a separable closure $k_s$ of $k$. Let $k_{nr} \subset k_s$ be the maximal unramified extension of $k$. Throughout this paper, assume that $G$ is a connected absolutely simple algebraic group of adjoint type defined over $k$. Let $G$ splits over $k_{nr}$. Lusztig [10] defined unipotent representations for $G(k)$ (the group of $k$-rational points of $G$). We recall this definition in Section 2. In particular, the irreducible summands of a representation $\text{c-Ind}_{P_k}^{G(k)} \sigma$ of $G(k)$, compactly induced from a cuspidal unipotent representation $\sigma$ of a parahoric subgroup $P_k \subset G(k)$, are supercuspidal unipotent representations, provided that $P_k$ is a maximal parahoric subgroups. If we normalise the Haar measure on $G(k)$ as $[6]$. Then, the volume of a parahoric subgroup $P_k$ equals to

$$\text{Vol}(P_k) = v^{-a/|P_k|},$$

where $v$ is the positive square root of $q$, $a$ is the rank of $G$ regarded as an algebraic group defined over a separable closure $k_s$ of $k$, and $P_k$ is the the reductive quotient of $P_k$ by its pro-unipotent radical $P_{k,+}$. So $P_k$ is a finite group of Lie type. Using the results in [4, §2.9] we can determine $|P_k|$, the order of $P_k$.

When $\pi$ is an irreducible summand of $\text{c-Ind}_{P_k}^{G(k)} \sigma$, under the above normalisation of Haar measure, the formal degree $\text{fdeg}(\pi)$ equals to $\text{Vol}(P_k)^{-1} \deg(\sigma)$, up to a rational constant (independent of $q$) and a power of $q$. So that $\text{fdeg}(\pi)$ is simply the denominator of the degree of $\sigma$ (up to a rational constant). In particular, from the list in [4, §13.7], we know that in this case $1/\text{fdeg}(\pi)$ is a product of cyclotomic polynomials in $q$. From here on we will regard $\text{fdeg}(\pi) = \text{fdeg}(\pi,q)$ as a rational function in variable $q$ with rational coefficients.

The first objective of this paper is to illustrate how to construct local Langlands correspondence of supercuspidal unipotent representations of non-split but quasi-split exceptional simple groups over $k$ by using the formal degrees. Reeder [18] has proved that for split exceptional groups of adjoint type over $p$-adic fields, one can partition their unipotent discrete series representations into $L$-packets such that if $\pi_1$ and $\pi_2$ belong to the same $L$-packet, then $\text{fdeg}(\pi_1,q) = \text{fdeg}(\pi_2,q)$ up to a rational constant. We will explain how to assign a Langlands parameter to supercuspidal unipotent representations in case of the other exceptional root systems, namely $^{3}D_{4}$ and $^{2}E_{6}$. This will be the main content of Section 2.

Hiraga, Ichino and Ikeda [HII] conjectured in general that for a discrete series representation of a connected reductive group defined over a local field, the $\text{fdeg}(\pi)$ equals to $|\gamma(\varphi_{\pi})|$ times a rational factor, where $\gamma(\varphi_{\pi})$ is the adjoint gamma factor associated with the Langlands parameter $\varphi_{\pi}$ corresponding to $\pi$. Reeder’s result in [18] is compatible with this conjecture (see [HII, §3.4]). Recently, Opdam [16, Thm. 4.11] proved this conjecture for unipotent discrete series representations of $G(k)$, for general $G$ satisfying our assumption. The main tool Opdam used is called Spectral Transfer Morphism for affine Hecke algebras which he developed in [15].

Here we use an easy example to explain the idea of spectral transfer morphisms. It is known that there is a bijection between the collection $\text{Irr}_{unp}(G(k), P_k, \sigma)$ of representations of $G(k)$ which

\(\text{Irr}_{unp}(G(k), P_k, \sigma)\)
are irreducible summands of some compact induction $\text{c-Ind}_P^{G(k)} \sigma$, and simple $H_{upt}$-modules, where $H_{upt}$ is an extension of an affine Hecke algebra $H(G(k), P_k, \sigma)$ whose parameters are explicitly determined by $G$, the parahoric subgroup $P_k$ and the cuspidal unipotent character $\sigma$. (The parameters can be computed directly using the method in [1, §1].) Suppose now $\pi$ belongs to $\text{Irr}_{upt}(G(k), P_k, \sigma)$ for some maximal parahoric subgroup $P_k$ and some cuspidal unipotent character $\sigma$ of $\overline{P}_k$. Then $\pi$ is supercuspidal and $H(G(k), P_k, \sigma)$ is simply $C$. Let $\varphi_\pi : \text{Frob}^Z \times \text{SL}_2(C) \to L^1 G$ be the unramified Langlands parameter associated with $\pi$. Then $\varphi_\pi$ is discrete, meaning that the centraliser in the complex dual group $G^\vee$ of the image of $\varphi_\pi$ is finite. Since $\varphi_\pi$ is determined by the images $s_0 := \varphi_\pi(\text{Frob}, \text{diag}(v, v^{-1}))$ and $u := \varphi_\pi(\text{id}, (\frac{1}{0} \frac{1}{1}))$, then $\varphi_\pi$ is discrete if and only if $u$ is a distinguished unipotent element in the connected centraliser $H := Z_{G^\vee}(s_0)^0$, where $s := \varphi_\pi(\text{Frob}, I_2)$. The adjoint gamma factor $\gamma(\varphi_\pi)$ can be computed using the $\mu$-function attached to the Iwahori-Hecke algebra $H^{IM}$ (cf. [III, Lem. 3.4]). In this case, the spectral transfer morphism from $C$ to $H^{IM}$ is given by a morphism $S : \text{Spm} C \to \text{Spm} Z(H^{IM})$, where $\text{Spm}$ is the maximal spectrum. Let $W$ be the finite Weyl group of the underlying root datum of $H^{IM}$. Denote by $W \mathfrak{r}$ the $W$-orbit of the image of $S$. We require that $W \mathfrak{r}$ is mapped to the $\text{Ad}(G^\vee)$-orbit of $s := \varphi_\pi(\text{Frob}, \text{diag}(v, v^{-1})) \in L^1 G$ by the bijection [1, Prop. 6.7]. This condition defines the morphism $S$ uniquely.

The above example can be generalised a little. The affine Hecke algebra $H(G(k), P_k, \sigma)$ attached to some $\pi \in \text{Irr}_{upt}(G(k), P_k, \sigma)$ can have positive rank. A spectral transfer morphism from $H(G(k), P_k, \sigma)$ to $H^{IM}$ is given by a finite morphism on the associated algebraic tori $S : \text{Spm} Z(H) \to \text{Spm} Z(H^{IM})$ satisfying some extra conditions (cf. [15, Def. 5.1]).

In the general setting for spectral transfer morphisms, a trace functional $\tau$ is defined in an affine Hecke algebra $H$. This makes $H$ a type I Hilbert algebra in the sense of [7]. The spectral measure of this trace functional is called the Plancherel measure. The density of this Plancherel measure is expressed by a function, called $\mu$-function, which is a rational function defined on the associated algebraic torus $T := \text{Spm} Z H$. A spectral transfer morphism between two pairs $(H_i, \tau_i), i = 1, 2$, is a map $(H_1, \tau_1) \sim (H_2, \tau_2)$, given by a finite morphism $S : T_1 \to T_2$ which preserves the associated Plancherel measure.

The second objective of this paper is to verify three maps are spectral transfer morphisms. In order to do this, we first describe the structure of the affine Hecke algebra $H(G(k), P_k, \sigma)$ and define the $\mu$-function in Section 1. A residual point is, by definition, a point $\mathfrak{r} \in T$ at which the number $P(\mu, \mathfrak{r}) - N(\mu, \mathfrak{r})$ attains the maximal possible value. Here $P(\mu, \mathfrak{r}), N(\mu, \mathfrak{r})$ denote the order of poles and zeroes of $\mu$ at $\mathfrak{r}$ respectively. The “residue” of the $\mu$-function at a residual point $\mathfrak{r}$ is defined. We will see that the adjoint gamma factor can be expressed as the kind of “residue” in Section 2. And hence the related residual point can be used to determine the Langlands parameter. After this, we will verify three maps $\phi, \psi$ and $\xi$ are spectral transfer morphisms in last section. The proof is computational.

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1 Affine Hecke algebras and $\mu$-functions

1.1 Generic affine Hecke algebras

Let $\mathcal{R} = (X, R_0, F_0, Y, R_0^\vee, F_0^\vee)$ be a reduced irreducible and semi-simple root datum, with a perfect pairing $(\cdot, \cdot) : X \times Y \to \mathbb{Z}$. Here $F_0$ is a base of $R_0$. Let $\mathbb{Z}R_0$ be the root lattice span by the root system $R_0$. Then $\Omega_X := X/\mathbb{Z}R_0$ is a finite abelian group. Let $W_0 = W_0(R_0)$ be the finite Weyl group of the root system $R_0$, with a set $S_0$ of distinguished generators, in bijection with $F_0$. We equip the vector space $E^* := \mathbb{R} \otimes \mathbb{Z} X$ with a Euclidean structure which is invariant under the action of $W_0$. (Observe that $R_0$ spans $E^*$ since $R_0$ is semi-simple.) Following the convention in [15], we call elements of the set $R := R_0^\vee \times \mathbb{Z}$ affine roots. If $a = \alpha^\vee + n \in R$, the reflection $r_a$ induced by $a$ is defined by $r_a(x) = x - a(x)\alpha$, where $a(x) = \langle x, \alpha^\vee \rangle + n$ for all $x \in E^*$. Such reflections are viewed as affine linear transformations on $E^*$.

The semi-direct product $\mathcal{W} := X \rtimes W_0$ is called the extended affine Weyl group. Inside $W$ sits a normal subgroup $\mathcal{W}_a \simeq \mathbb{Z}R_0 \rtimes W_0$ (the unextended affine Weyl group). $\mathcal{W}_a$ is an affine Coxeter group. Let $S_a$ be a set of distinguished generators of $\mathcal{W}_a$.

Let $C \subset E^*$ be the fundamental alcove for the action of $\mathcal{W}_a$ on $E^*$. The closure $\overline{C}$ of $C$ is a fundamental domain for the action of $\mathcal{W}_a$ on $E^*$. The stabiliser $\text{Stab}_W(C) := \{ w \in W : w(C) = C \}$ of $C$ in $W$ is isomorphic to the quotient $W/\mathcal{W}_a$, and hence isomorphic to $\Omega_X$.

On the affine Coxeter group $(\mathcal{W}_a, S_a)$ we have a canonical length function $l : \mathcal{W}_a \to \mathbb{N}$ such that every distinguished generator $s \in S_a$ has length one. We extend $l$ to $\mathcal{W}$ by defining that $l(\omega) = 0, \forall \omega \in \Omega_X$. Let $t_x \in W$ denote the translation corresponding to $x \in X$. Then $l$ satisfies

$$l(wt_x) = l(w) + \sum_{\alpha \in R_0^+} \langle x, \alpha^\vee \rangle$$

for all $w \in \mathcal{W}_a$ and all $x \in X^+ := \{ x \in X : \langle x, \alpha^\vee \rangle \geq 0, \forall \alpha \in F_0 \}$.

Let $C[v^\pm]$ be the Laurent polynomial ring in variables $v, v^{-1}$. It is the affine coordinate ring of $C^\times$. We define an associative algebra $\mathcal{H}(\mathcal{R}, m)$ over $C[v^\pm]$, with distinguished $C[v^\pm]$-basis $\{ N_w : w \in \mathcal{W} \}$ parametrized by $w \in \mathcal{W}$, satisfying the following relations:

(i) If $l(ww') = l(w) + l(w')$, then $N_w N_w' = N_{ww'}$.

(ii) $(N_s - v^{m(s)})(N_s + v^{-m(s)}) = 0$ for all $s \in S_a$.

Here the parameters $m(s) \in \mathbb{Z}$ are given by a function $m : \mathcal{W}/S_a \to \mathbb{Z}$, which is defined on the $W$-conjugacy classes of $S_a$. The algebra $\mathcal{H}(\mathcal{R}, m)$ is called the generic affine Hecke algebra (with parameter $m$). If there is no danger of ambiguity, we shall simply write $\mathcal{H}(\mathcal{R})$ for $\mathcal{H}(\mathcal{R}, m)$.

Remark 1.1. (i) Using the Bernstein presentation (see e.g. [15]) for $\mathcal{H}(\mathcal{R}, m)$, one can show that the centre $Z(\mathcal{H}(\mathcal{R}, m))$ of $\mathcal{H}(\mathcal{R}, m)$ is $(C[v^\pm][X])^{W_0}$.

(ii) When the roots in $R_0$ have more than one length, from the definition, we note that $\mathcal{H}(\mathcal{R}, m)$ is not sensitive to the orientation of the double or triple edges in the Dynkin diagram of $R_0$.

Define $\tau : \mathcal{H} \to C[v^\pm]$ by $\tau(\sum_w a_w N_w) = a_1$. Let $\mathcal{H}_v$ be the specialisation of $\mathcal{H}$ at $v = v \in \mathbb{R}_{>1}$. Then $\tau$ determines a family of $C$-valued linear functional $\mathbb{R}_{>1} \ni v \mapsto \tau_v$. 

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on the family of the algebras $\mathcal{H}_v$, ($v \in \mathbb{R}_{>1}$). Define an anti-involution $*$ on $\mathcal{H}_v$ by $(\sum_w a_w N_w)^* = \sum_w a_w N_{w^{-1}}$, then the functional $\tau_v$ is positive with respect to this anti-involution. Moreover, for all $v \in \mathbb{R}_{>1}$, the triple $(\mathcal{H}_v, \tau_v, *)$ form a type I Hilbert algebra in the sense of Dixmier [7].

The spectral measure associated with $\tau_v$ on $\mathcal{H}_v$ is called the Plancherel measure. Its density is given by a rational function called $\mu$-function which we will define in next section.

1.2 The $\mu$-function

Suppose $(X, R_0, Y, R'_0)$ is the root datum associated with our unipotent affine Hecke algebra.

We keep the notations in Section 1.1. Let $T$ be the complex torus whose character lattice is $X$. We define a function $m_R : R \to \mathbb{Z}$ by the rules that (i) $m_R$ is $W$-invariant and (ii) $m_R(\alpha) = m(s)$ if the simple reflection $s$ is conjugate to the reflection $r_\alpha$ induced by $\alpha \in R = R'_0 \times \mathbb{Z}$. This function $m_R$ gives rise to the functions $m_\pm : W_0R_0 \to \frac{1}{2}\mathbb{Z}$ defined as

$$m_+(\alpha) = \frac{m_R(\alpha^\vee) + m_R(1 + \alpha^\vee)}{2}, \quad m_-(\alpha) = \frac{m_R(\alpha^\vee) - m_R(1 + \alpha^\vee)}{2} \quad (1.2)$$

One observes that $m_-(\cdot) = 0$ if and only if the two reflections $r_{\alpha^\vee}$ and $r_{1+\alpha^\vee}$ are $W$-conjugate. So if $m_-(\cdot) \neq 0$, then $R_0$ must contain a component of type B.

For each $\alpha \in R_0$, define

$$c(m, \alpha) := \frac{(1 - v^{-2m_+(\alpha)}a^{-1})(1 + v^{-2m_-(\alpha)}a^{-1})}{1 - a^{-2}} \quad (1.3)$$

We put $c(m) := \prod_{\alpha \in R_0^+} c(m, \alpha)$. The Weyl group $W_0$ acts on $c(m, \alpha)$, and hence on $c(m)$, via its action on the roots $\alpha$. The $\mu$-function of $(\mathcal{H}(R, m, d))$ is defined as

$$\mu(R_0, v) = v^{-2m_W(w_0)} \frac{d(v)}{c(m)c(w_0)} \quad (1.4)$$

where $w_0 \in W_0$ is the longest element, $m_W : W \to \mathbb{Z}$ is defined by

$$m_W(w) = \sum_{a \in R^+ \cap w^{-1}(R^-)} m_R(a),$$

and the factor $d(v)$ is a rational function in variable $v$. Clearly, the $\mu$-function is $W_0$-invariant.

Note that $\mu(\emptyset, v) = d(v)$. The explicit expression of $d$ will be given in Section 2. Here we point out the following fact: let $e$ be the identity element of the affine Hecke algebra $\mathcal{H}(R, m)$, then the functional $N_w \mapsto \delta_{w,e} d(v)$ defines a trace $\tau'(\cdot) := d(v)\tau(\cdot)$ on $\mathcal{H}(R, m)$. Therefore, we shall regard $d(v)$ as a normalisation factor.

Let $T$ be the diagonalisable group scheme with character lattice $X \times \mathbb{Z}$. We view $T$ as a group scheme over $\text{Spec} \mathbb{C}[v^\pm]$ via the homomorphism $\mathbb{C}[v^\pm] \to \mathbb{C}[X \times \mathbb{Z}]$ defined by $v^n \mapsto (0, n)$. We write $T_v$ as the fibre at $v \in \mathbb{C}^\times$ of $T$. The function $\mu(R_0, v)$ is a rational function on $T$ with rational coefficient.
1.3 Residual cosets and residual points

Let $(\mathcal{H}_\nu(R, m), d)$ be an affine Hecke algebra with trace $d(v)\tau_\nu(\cdot)$, obtained as specialisation from a generic affine Hecke algebra at $\nu = v \in \mathbb{C}^\times$. Let $\mu(R_0, v)$ be the $\mu$-function associated with $(\mathcal{H}(R, m), d(v))$, and set $d := d(v)|_{\nu=v}$, $\mu_{m,d}(R_0, v) := \mu(R_0, v)|_{\nu=v}$. We call $\mu_{m,d}(R_0, v)$ the $\mu$-function associated with $(\mathcal{H}_\nu(R, m), d)$.

Let $T = T_0$ be the complex torus as above, so that the character lattice of $T$ is $X$. Given a coset $L \subset T$, we define (where $\epsilon = \pm$ is a sign)

$$p_\epsilon(L) = \{ \alpha \in R_0 : \epsilon\alpha|_L = v^{-2m(\alpha)} \}, \quad z_\epsilon(L) = \{ \alpha \in R_0 : \epsilon\alpha|_L = 1 \} \quad (1.5)$$

The coset $L \subset T$ is said to be residual if

$$|p_+(L)| + |p_-(L)| - |z_+(L)| - |z_-(L)| = \text{codim}(L). \quad (1.6)$$

Suppose $L$ is a residual coset for $\mu_{m,d}(R_0, v)$ with $p_\epsilon(L)$ and $z_\epsilon(L)$ as defined above, by the regularisation of $\mu$ along $L$ we mean the following expression

$$\mu_{m,d}^L(R_0, v) := d \frac{v^{-2m(\nu(\omega))} \prod_{\alpha \in R_0 \setminus z_-(L)} (1 + \alpha^{-1}) \prod_{\alpha \in R_0 \setminus z_+(L)} (1 - \alpha^{-1})}{\prod_{\alpha \in R_0 \setminus p_-(L)} (1 + v^{-2m(\alpha)} \alpha^{-1}) \prod_{\alpha \in R_0 \setminus p_+(L)} (1 - v^{-2m(\alpha)} \alpha^{-1})}. \quad (1.7)$$

When specialises at $v = v \neq 1$ (a condition automatically satisfied in our case), the regularisation $\mu_{m,d}^L(R_0, v)$ restricting onto $L$ defines a nonzero rational function $\mu_{m,d}^L(R_0, v)$ in $v$ over the residual coset $L$. In particular, if $L = \{ \nu \}$ is just a residual point, we call the expression $\mu_{m,d}^L(R_0, v)$ the “residue” of the $\mu$-function at the residual point $\nu$. (Compare with the computation of residue of a one-variable meromorphic function at a simple pole.)

Because $T = \text{Hom}(X, \mathbb{C}^\times)$, the polar decomposition of complex numbers gives rise to a polar decomposition $T = T_0 T_+$, where $T_0 := \text{Hom}(X, \mathbb{R}/\mathbb{Z})$ is a compact subgroup of $T$ and $T_+ := \text{Hom}(X, \mathbb{R}_{>0})$. For any point $r \in T$, we write $r = st \in T_0 T_+$ for the polar decomposition.

We conclude this section with the following important result, which is Theorem 4.8 in [15].

**Proposition 1.2.** Let $(\mathcal{H}(R, m), d(v))$ be a generic affine Hecke algebra with $\mu$-function $\mu(R_0, v)$. Let $\delta$ be a generic irreducible discrete series representations of $\mathcal{H}(R, m)$, with formal degree $\text{fdeg}(\delta, v)$. There exists a generic residual point $\nu(v)$ of $\mathcal{H}(R, m)$, such that for all $v \in \mathbb{R}_{>1}$, the specialisations $\mu_{m,d}^L(R_0, v)$ of $\mu_{m,d}^L(R_0, v)$ and $\text{fdeg}(\delta, v)$ at $v = v$, are equal up to a non-zero rational constant which only depends on $\delta$.

1.4 Parabolic induction associated to residual cosets

Suppose $L \subset T$ is a coset of a subtorus $T^L \subset T$. The natural projection $T \rightarrow T_L := T/T^L$ corresponds to a sublattice $X^*_L \subset X$. Then $L$ is a principal homogeneous space of $T^L$. Suppose $\nu$ is the base point of $L$, then we can write $L = \nu T^L$.

The intersection $R_L := R_0 \cap X_L$ is a parabolic root sub-system of $R_0$. If necessary, we use an appropriate element in $W_0$ to map $R_L$ to a standard position, then, we can choose a base $J$ of $R_L$ which is a subset of the base $F_0$ of $R_0$. We have a root datum $R_L := (X^*(T_L), R_L, X_+(T_L), R_+_L)$.
and a parabolic subgroup $W_0^L$ of $W_0$. (In fact, the root datum $R_L$ and the Weyl group $W_0^L$ only depend on the subset $J$ of $F_0$.) Consequently, we can decompose the $\mu$-function as

$$\mu_{m,d}^{\left(1\right)}\left(R\right) = \frac{\mu_{m,d}^{\left(1\right)}\left(R_\mathcal{L}\right)\prod_{a\in R_0,\lambda} c\left(m,\alpha\right)c\left(m,\alpha w\right)}{w^{-2m_\mathcal{W}_0\left(w\right)}},$$

(1.8)

where $w^L := w_0w_L^{-1} \in W^L \subset W$ is the longest element.

2 Unipotent representations

We will define unipotent representations for $G\left(k\right)$ and discuss the affine Hecke algebras derived from unipotent representations.

Firstly we will recall the assumptions we have put on the connected absolutely simple $G$ of adjoint type defined over $k$. We also assume that $G$ is $k$-quasi-split and split over $k_{nr}$. Here $k_{nr}$ is the maximal unramified extension of $k$, with residual field $\overline{F}_q$, an algebraic closure of $F_q$.

Recall that the Galois group $Gal\left(k_{nr}/k\right) \simeq Gal\left(\overline{F}_q/F_q\right) \simeq \hat{\mathbb{Z}}$. The geometric Frobenius element $Frob$, whose inverse induces the automorphism $x \mapsto x^q$ for any $x \in \overline{\mathbb{F}}_q$, is a topological generator of the Galois group.

Denote $G := G\left(k_{nr}\right)$. The $k$-rational structure of $G$ can be determined by applying the automorphism $F$ (say) of $G$ induced by Frobenius. In particular, the $F$-fixed points $G^F = G\left(k\right)$.

Let $S \subset G$ be a maximal $k$-torus of $G$ which is maximally $k$-split (recall that this means the largest $k$-split subtorus $S_d \subset S$ is maximal $k$-split in $G$). We have the associated root datum $\Sigma\left(G,S\right) := \left(X^*(S),\Sigma, X_*(S),\Sigma'\right)$, where $X^*(S), X_*(S)$ are the character and cocharacter lattices of $S$ respectively. Since $G$ is simple, the quotient group $\Omega = X_*(S)$ by the lattice of the dual root system $\Sigma'$, is a finite abelian group. Let $\mathcal{P} < G$ be a parahoric subgroup such that $F(\mathcal{P}) = \mathcal{P}$. Choose a base $\Sigma_0$ of the root system $\Sigma$ compatible with $\mathcal{P}$, in the sense that the alcove $C_\mathcal{P}$ defined by $\Sigma_0$ has the property that $\mathcal{P}$ corresponds to a facet $F_\mathcal{P}$ of $C_\mathcal{P}$. Corresponding to the whole $C_\mathcal{P}$ is an Iwahori subgroup $I \subset \mathcal{P}$. We have an isomorphism $\Omega \cong N_G(I)/I$, and hence $\Omega$ acts on the affine Dynkin diagram $\text{Dyn}(X_0^{(1)})$ of $G$ as diagram automorphisms.

Recall that the equivalence classes of inner forms of $G$ are parameterized by the Galois cohomology set $H^1\left(k, G\right)$. By a theorem of Steinberg, which says that $H^1\left(Gal\left(k_s/k_{nr}\right), G\right) = 1$, we have an isomorphism $H^1\left(k, G\right) \cong H^1\left(F,G\right) := H^1\left(Gal\left(k_{nr}/k\right), G\right)$. By a theorem of Kottwitz, we then have a bijection $H^1\left(F,G\right) \cong \Omega/(1-\theta)\Omega$. Let $z \in Z^1\left(F,G\right)$ and denote $u := z(Frob) \in G$. Let $F_u := \text{Ad}(u) \circ F$ be an inner twist of $F$ by $u$. Then $F_u$-action on $G$ defines a $k$-structure on $G$. If $G^u$ is the inner form of $G$ corresponding to $u$, then $G^u\left(k\right) = G^{F_u}$.

We can find a parahoric subgroup $\mathcal{P} \subset G$ such that $F_u(\mathcal{P}) = \mathcal{P}$. Let $\mathcal{P}_+$ be the pro-$p$-radical of $\mathcal{P}$. The quotient $\mathcal{P}/\mathcal{P}_+$ is reductive. Since $\mathcal{P}_+$ is connected, we can apply the pro-algebraic version of Lang’s theorem and obtain $\left(\mathcal{P}/\mathcal{P}_+\right)^F = \mathcal{P}^{F_u}/\mathcal{P}^{F_u}_+$, which is a finite reductive group. We shall denote this finite group by $\overline{\mathcal{P}}^{F_u}$.

A smooth representation $\pi$ of $G^{F_u}$ is called unipotent, if there exists an $F_u$-stable parahoric subgroup $\mathcal{P}$ of $G$, with a cuspidal unipotent representaion $\sigma$ of $\overline{\mathcal{P}}^{F_u}$, such that the $\mathcal{P}^{F_u}_+$-invariants of $\pi$ contain $\sigma$. (Here we lift $\sigma$ to a representation of $\mathcal{P}^{F_u}$ via the natural projection $\mathcal{P}^{F_u} \to \overline{\mathcal{P}}^{F_u}$.) The pair $s := (\mathcal{P}, \sigma)$ a type of $G$ in the sense of $\left[\right]$. We write $\text{Irr}(G^{F_u}, s)$ the totality of isomorphism classes of irreducible unipotent representations of $G^{F_u}$ associated with a unipotent type $s = (\mathcal{P}, \sigma)$ of $G$. 

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Given a unipotent type $s = (P, \sigma)$ of $G$ with $F_u(P) = P$, define a representation $c$-\text{Ind}_{F_u}^{G_{F_u}} \sigma$ of $G_{F_u}$. Let $V$ be the vector space of this representation. The endomorphism algebra $\text{End}(V)$ is isomorphic to the $(P_{F_u}, \sigma)$-spherical Hecke algebra of $G_{F_u}$. We denote this $(P^F, \sigma)$-spherical Hecke algebra by $H_s$. Let $\text{Irr}(H_s)$ be the set of isomorphism classes of (finite dimensional) simple $H_s$-modules. From the work of Morris [12] and Lusztig [10], we know that

**Proposition 2.1.** Given a unipotent type $s = (P, \sigma)$ of $G$ with $F_u(P) = P$,

(i) there is a natural bijection between the sets $\text{Irr}(G_{F_u}, s)$ and $\text{Irr}(H_s)$.

(ii) two sets $\text{Irr}(G_F, s)$ and $\text{Irr}(G_F, s')$ are either disjoint or identical. They are equal if and only if there exists an element $g \in G^F$ which conjugates $P$ to $P'$, and $\sigma$ to a representation isomorphic to $\sigma'$.

### 2.1 Unipotent affine Hecke algebras

We keep the notations in Prop. 2.1. We will describe the structure of $H_s$. Let $\bar{I}$ denote the set of all nodes of $\text{Dyn}(\Sigma^{(1)}_0)$, and suppose that $P$ corresponds to a subset $J \subset \bar{I}$. Denote by $\theta$ the diagram automorphism of $\text{Dyn}(\Sigma^{(1)}_0)$ induced by Frobenius (or, by $F$). We also have the twisted group $\theta_u$, induced by $F_u$. Since $P$ is $F_u$-stable, we see that $\theta_u$ preserves $J$. Furthermore, let $\Omega^{\theta_u}$ denotes the elements in $\Omega$ which commutes with $\theta_u$ as diagram automorphisms. Since $\Omega$ is abelian, we thus have $\Omega^{\theta_u} = \Omega^{\theta}$. The action of $\Omega$ on $\text{Dyn}(\Sigma^{(1)}_0)$ induces an action of $\Omega^{\theta}$ on the $\theta_u$-orbits of this Dynkin diagram. Denote by $\Omega^{\theta}(P)$ the isotropy group of $P$. Then $\Omega^{\theta}(P)$ acts on the set $(I - J)/\theta_u$ of $\theta_u$-orbits as permutations. Let $\Omega^{\theta}_1(P)$ be the pointwise stabilizer of $(I - J)/\theta_u$, and let $\Omega^{\theta}_2(P) := \Omega^{\theta}(P)/\Omega^{\theta}_1(P)$.

Lusztig [10, 1.18, 1.19, 1.20] gave an Iwahori-Matsumoto presentation of $H_s$. In particular, $H_s$ has a subalgebra in $Z(H_s)$ isomorphic to the group algebra of $\Omega^{\theta}_1(P)$. This commutative subalgebra decomposes into a direct sum of one-dimensional subalgebras corresponding to the irreducible characters $\psi : \Omega^{\theta}_1(P) \to C^\times$. This results in a direct sum decomposition of $H_s$ into two-sided ideals $H_s = H^{\psi} \times \Omega^{\theta}(P)$, with each $H^{\psi}$ corresponding to an irreducible character $\psi$ of $\Omega^{\theta}_1(P)$. He then constructed an isomorphism between $H^{\psi}$ and an affine Hecke algebra obtained as a generic affine Hecke algebra specialised at $v = q^{1/2}$, and thus gave the Iwahori-Matsumoto presentation of $H^{\psi}$. Associated to $H^{\psi}$ is an affine Weyl group $W_s$ with a set $S_s$ of distinguished generators. The structure of $W_s$ as well as the parameter function $m_s$ can be entirely determined by the data of $G$ and $s$ (cf. [13]). However, there is no canonical way to choose the set $S_s$. But different choices of $S_s$ give us algebras which are essentially the same, in the sense that there exist admissible isomorphisms (cf. [15, 2.1.7] and [16, text around Eq. (19)]) among them.

The root datum $R^{\psi} = (X, R_0, Y, R^\vee_0)$ of $H^{\psi}$ can be obtained by the following algorithm in [20, §15.6]. We start with the root datum dual to $\Sigma(G, S) = (X^*, \Sigma, X_s, \Sigma^\vee)$. Recall the maximally $k$-split $k$-subtorus $S_d \subset S$. The character and cocharacter lattices of $S_d$ will be denoted by $X^*(S_d), X_s(S_d)$ respectively. Note that $X^*(S_d)$ is a quotient of $X^* = X^*(S)$ by the annihilator of $X_s(S_d)$ in $X^*$. Let $W_0$ be the finite Weyl group associated with the root datum $\Sigma(G, S)$. We equip with real vector space $V := R \otimes \mathbb{Z} X^*$ a bilinear symmetric form $( , )$ which is $W_0$-invariant, and use this form to identify $V$ and its dual. Notice then that $V^\psi := R \otimes \mathbb{Z} X^*(S_d)$ is the orthogonal complement of $R \otimes \mathbb{Z} X_s(S_d)$. Let $pr : V \to V^\psi$ be the natural projection. Then the non-zero vectors in $pr(\Sigma) - \{0\}$, and define $R_0 \subset X_s(S_d)$ as the root system dual to
We are now able to explicitly define the normalisation factor
\[ d = |\Omega^F_u(P)|^{-1} \text{Vol}(P^{F_u})^{-1} \deg(\sigma). \tag{2.1} \]

\section{2.2 Supercuspidal unipotent representations}

We consider supercuspidal unipotent representations obtained as irreducible components of 
\[ \text{c-Ind}_{\bar{F}_u}^{G_u} \sigma, \]
where \( \bar{P} \) is a maximal parahoric subgroup of \( G \) such that \( F_u(\bar{P}) = \bar{P} \).
Lusztig classified unipotent representations by the \( G^\vee \)-orbits of \( (s,n,\rho) \) where \( s \) is a semisimple element in the complex group \( G^\vee \) dual to \( G \), \( n \) is a nilpotent element in the Lie algebra of \( G^\vee \) satisfying \( \text{Ad}(s)n = qn \), and \( \rho \) is an irreducible representation of the component group of the simultaneous centraliser in \( G^\vee \) of \( s \) and \( n \). We take Lusztig’s parameters \( (s,n,\rho) \) as Langlands parameters. Then, if
\[ \varphi_\pi : \text{Frob}^Z \times \text{SL}_2(\mathbb{C}) \to \text{^L}G := G^\vee \rtimes \langle \theta \rangle \]
is the Langlands parameter of an irreducible summand \( \pi \) of \( \text{c-Ind}_{\bar{F}_u}^{G_u} \sigma \), we have \( \varphi(\text{Frob}) = s\theta, \varphi(\frac{1}{1}) = u \), where \( u \in G^\vee \) is a unipotent element which is the exponential of some \( n \). Moreover, the centraliser of the image of \( \varphi \) in \( G^\vee \) is not contained in any proper Levi subgroup, and \( u \) is a \textit{distinguished} unipotent element\(^1\) in the connected centraliser \( Z_{G^\vee}(s)^0 \). Since we are

\(^1\)A unipotent element is called \textit{distinguished} if its centraliser does not contain any nontrivial torus.
working with an almost simple algebraic group, we deduce that \( s \) must be an isolated semi-simple element, i.e. the fixed-point algebra \( \text{Lie}(G^\vee)^s \) is semi-simple. The argument in Reeder [19, §3.8] shows that under the covering map \( \exp : \text{Lie}(T) \to T_u \), an isolated element \( s \in G^\vee \) is conjugate to some \( \exp(v) \), where \( v \) is a vertex of the fundamental alcove. Thus \( v \) corresponds to a node \( v_s \) (say) of the affine extended Dynkin diagram associated to \( R_0 \) (with an arbitrary chosen base \( F_0 \)).

The adjoint gamma factor \( \gamma(\varphi) \) can be computed using [HII, Lem. 3.4]. We see that \( \gamma(\varphi) \) is equal to the “residue” of the \( \mu \)-function associated to \( H_{\text{upt}} \) at certain residual point. As the example in the Introduction suggested, there is only one orbit of such a residual point under the finite Weyl group \( W_0 \) of \( R \). For split exceptional groups this is clear by Reeder [18]. For \( G \) of classical type, the uniqueness property proved in [8] confirms that for a given irreducible supercuspidal unipotent representation \( \pi \), there is only one Langlands parameter \( \varphi_\pi \) (up to the action by the group of irreducible characters of \( \Omega^{G} \)) such that \( \text{fdeg}(\pi, q) = \gamma(\varphi_\pi, q) \) as rational functions in \( q \) (up to a non-zero rational constant). We will verify this uniqueness property for \( G \) of type \( 3D_4 \) and \( 2E_6 \) in Sections 2.2.3, 2.2.4.

The relation between residual point and discrete unramified local Langlands parameters are discussed in [14, Appendix B] and [8].

### 2.2.1 Groups of type \( B_{l^2+1} \)

From [4, §13.7] we know that a group of type \( B_2 \) has no cuspidal unipotent character unless \( n = l^2 + l \) for some \( l \in \mathbb{Z}_{>0} \), and in this case, it has one cuspidal unipotent character.

We will look at the simplest case, namely, \( B_2 \). The special orthogonal group \( G = \text{SO}_5 \) defined over \( k \) is split and of adjoint type. We take the parahoric subgroup \( P = \text{SO}_5(\mathcal{O}_k) \). Then \( \overline{P} = \text{SO}_5(\mathcal{O}_q) \) has a cuspidal unipotent representation \( \pi \) (say). The compactly induced representation \( \pi = c\text{-Ind}_{\text{SO}_5(\mathcal{O}_k)}^{\text{SO}_5(\mathcal{O}_q)} \sigma \) is irreducible. The formal degree \( \text{fdeg}(\pi, q) \) is the product of a power of 2, a power of \( q \) and the \( q' \)-part \( \text{fdeg}_{q'}(\pi, q) := (q + 1)^{-2}(q^2 + 1)^{-1} \).

The complex group dual to \( \text{SO}_5(K) \) is \( \text{Sp}_4(C) \). Let \( T < \text{Sp}_4(C) \) be the maximal torus. Let \( (t_1, t_2, t_2^{-1}, t_1^{-1}) \) be the coordinates in \( T \), we can write the root system \( C_2 \) multiplicatively as \( \{(t_1t_2)^{\pm 1}, (t_1t_2^{-1})^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}\} \). We have the \( \mu \)-function as follow:

\[
\mu(C_2, v) := \frac{v^{-8}}{(v-v^{-1})^2} \prod_{i=1}^{2} \frac{(1-t_i^{-2})(1-t_i^2)}{(1-v^{-2}t_i^{-2})(1-v^{-2}t_i^2)} \times \frac{(1-t_1^{-1}t_2)(1-t_1t_2^{-1})(1-t_1^{-1}t_2^{-1})(1-t_1t_2)}{(1-v^{-2}t_1^{-1}t_2)(1-v^{-2}t_1t_2^{-1})(1-v^{-2}t_1^{-1}t_2^{-1})(1-v^{-2}t_1t_2)}.
\]

There are two orbits of residual points under the Weyl group associated to \( C_2 \) (cf. [17, §6.3]), with representatives \( r = (t_1, t_2, t_2^{-1}, t_1^{-1}) = (v^3, v, v^{-1}, v^{-3}) \) and \( r' = (-v, v, v^{-1}, -v^{-1}) \) in \( T \) respectively. Both \( r \) and \( r' \) are poles of \( \mu(C_2, v) \) of order 2, equal to \( \dim T \), so they are indeed residual points. We now compute the residues of \( \mu(C_2) \) at these two points in the following way.

\[
\mu(r)(C_2) = \frac{q(1-q^3)}{(1+q)(1-q^4)} = \frac{(v+1+v^{-1})(v-1+v^{-1})}{(v^2+v^{-2})(1+v^2)(1+v^{-2})}.
\]

This is the formal degree of the Steinberg representation (cf. [6, §5.2]).
We see that the $q'$-part of $\mu^{(r')}_{(G)}|_{r'}$ is exactly $\text{fdeg}_{q'}(\pi, q)$, while the $q'$-part of $\mu^{(r)}_{(G)}|_{r}$ is not equal to $\text{fdeg}_{q'}(\pi, q)$.

**Remark 2.3.** The factor $v^{-8} = q^{-4}$ is introduced according to our normalization of Haar measure. (See [6, §5.1]. The related factor is $|F_j|^{1/2}$.) Observe that the total degrees of $t_j$ ($j = 1, 2$) in $\mu$ are zero, and there are eight factors in the denominator contains the factor $v^{-2}$. As a consequence, the $\mu$-function is symmetric with respect to $v \leftrightarrow v^{-1}$.

### 2.2.2 Groups of type $G_2$

Let $G$ be the split $k$-group of type $G_2$ and let $G = G(k)$. Then $R_0 = G_2$. Let $F_0^{(1)} = \{\alpha_0, \alpha_1, \alpha_2\}$ be the base of affine simple roots, with $\alpha_1$ short and $\alpha_0 = 1 - 3\alpha_1 - 2\alpha_2$. The other positive roots are $\alpha_1 + \alpha_2$ (short), $2\alpha_1 + \alpha_2$ (short), $3\alpha_1 + \alpha_2$ (long) and $3\alpha_1 + 2\alpha_2$ (long). The highest (long) root is $h_r := 3\alpha_1 + 2\alpha_2$. Because $G$ is split, every $m^\vee = 1$. So the $\mu_{IM}$-function associated to the Iwahori-Hecke algebra $H_{IM}$ of split $G_2$ is

$$
\mu_{IM}(G_2) = \frac{1}{(1 - q)^2} \prod_{\alpha \in G_{2,+}} \frac{(1 - \alpha^{-1})(1 - \alpha)}{(1 - q^{-1}\alpha^{-1})(1 - q^{-1}\alpha)}
$$

(2.4)

with $d_{IM} = (1 - q)^{-2}$.

There are four residual orbits for $G_2$. Since we know that the unitary part $s$ of a residual point $r$ corresponds to a node $v_s$ in the extended affine Dynkin diagram $\text{Dyn}(R_0^{(1)})$, we will write $s = ?$ if we get a root system of type ? after deleting the node $v_s$. We tabulate the relevant data as below. In the first column, we use the notation $G(a_1)$ from the notations of distinguished unipotent orbits in [4, §5.9].

| $s$ | $r$ | $\mu_{IM}^{(r)}(\pi(v), v)$ | $U^0$ |
|-----|-----|-----------------------------|-------|
| $G_2$ | $(q, q)$ | $\Phi_2^{-2}\Phi_3^{-1}\Phi_5^{-1}$ | $G_2[1]$ |
| $G_2(a_1)$ | $(1, q)$ | $\Phi_2^{-2}\Phi_3^{-1}$ | $G_2[1]$ |
| $A_2$ | $(q^{-q^{-2}}, q)$ | $\Phi_2^{-2}\Phi_3^{-1}$ | $G_2[1]$ |
| $A_1A_1$ | $(-q^{-q^{-2}}, q)$ | $\Phi_2^{-2}\Phi_3^{-1}$ | $G_2[-1]$ |

The rightmost column of the table is the cuspidal unipotent characters for $G$ of type $G_2$, with the same notation as in Carter [4, §13.7]. (See also [10, 7.33–7.35].) Here $\vartheta$ is a primitive third root of unity. The two cuspidal unipotent representations $G_2[\vartheta]$ and $G_2[\vartheta^2] = G_2[\vartheta^{-1}]$ are isomorphic and have the same degree. It is clear that different residual orbits give different residues. Moreover, interpreting the adjoint gamma factor $\gamma(\varphi_{\pi})$ as residue of the $\mu_{IM}$-function at residual points, the uniqueness property is clear. (The residual point $r = (q, q)$ does not give any cuspidal unipotent degree, so the corresponding box is blank.)
Remark 2.4. As in the previous example, we can multiply a suitable power of \( v \) to \( \mu^{IM,(\{r\}),(\mathbf{r})) \) to make it invariant with respect to the variable change \( v \mapsto v^{-1} \). For example, for the residue \( \Phi_2^{-2}\Phi_3^{-1} \) we shall multiply with \( v^{-4} = q^{-2} \). Note that Reeder [18] includes this power of \( q \) in his results of formal degrees. However, to verify the uniqueness property, we only need the \( q' \)-part of the formal degree. Also note that our normalisation factor helps us to get rid of the cyclotomic polynomial \( \phi_1 \), while Reeder did not.

2.2.3 Groups of type \( ^3\!D_4 \)

Now consider the case that \( G \) is a \( k \)-group of type \( ^3\!D_4 \). Under the Frobenius action \( \theta \), we get a relative root system \( R_0' = G_2 \). The parameters of the \( \mu \)-functions are given as follows. Denote the long (resp. short) root in the base \( F_0' \subset R_0' \) as \( \alpha_1' \) (resp. \( \alpha_2' \)), and let \( \alpha_0' = 1 - 2\alpha_1' - 3\alpha_2' \). Then \( F_0^{(1)} = \{ \alpha_1', \alpha_2', \alpha_0' \} \). We see that \( m(s_{\alpha_i'}) = 1, m(s_{\alpha_2'}) = 3 \) and \( m(s_{\alpha_0'}) = 1 \) because \( \alpha_0' \) is \( W \)-conjugate to \( \alpha_1' \).

We know that \( R_0 = G_2 \). Let \( \alpha_1, \alpha_2 \in F_0 \) be the short root and long root respectively in the base \( F_0 \) of \( R_0 \). We have \( m_+(\alpha_1) = m_R(\alpha_1') = 1, m_+(\alpha_2) = m_R(\alpha_2') = 3 \), we deduce that the labels of the spectral diagram are as follows: \( m'(\alpha_1) = m_+(\alpha_1) = 1, m'(\alpha_2) = m_+(\alpha_2) = 3 \) and \( m'(\alpha_0) = 3 \). (Here \( \alpha_0 = 1 - 3\alpha_1 - 2\alpha_2 \) is the affine extended root in \( F_0^{(1)} - F_0 \). Note that \( \alpha_0 \) is conjugate to \( \alpha_2 \) under Weyl group action.)

\[
\begin{align*}
\Sigma_0 &= \text{^3\!D_4} \\
R_0' &= G_2 \\
R_0 &= G_2
\end{align*}
\]

Figure 1: Root systems \( \Sigma_0 = ^3\!D_4 \).

We denote the Iwahori-Hecke algebra by \( G_2(3,1)[q] \), with the \( \mu^{IM} \)-function given by

\[
\mu^{IM} = d^{IM} \prod_{\alpha \in R_0, +} \frac{(1 - \alpha(r))^2}{(1 - q_{\alpha'} \alpha(r))(1 - q_{\alpha'} \alpha(r)^{-1})}
\]

It is normalised by

\[
d^{IM} = \frac{v - v^{-1}}{v^3 - v^{-3}} \frac{1}{(v - v^{-1})^2} = \frac{1}{(v - v^{-1})(v^3 - v^{-3})},
\]

and the parameter \( q_{\alpha'} \) is given by the rule that if \( \alpha \in R_0 \) is a long root, then \( q_{\alpha'} = q^3 \), and if \( \alpha \in R_0 \) is a short root, then \( q_{\alpha'} = q \).

The algebra \( G_2(3,1)[q] \) has four orbits of residual points. They are \( s = G_2, r = (q^3, q) \), \( s = G_2(a_1), r = (1, q) \) (these two with real central characters), and the non-real orbits \( s = A_2, r = (q^3, \phi q^{-3}) \) and \( s = A_1 A_1, r = (-q^{-3}, q) \). By computing the residue of the \( \mu^{IM} \)-function at these points, we find that at \( s = G_2(a_1) \) we have a residue matching the degree of \( ^3\!D_4[1] \), and at \( s = A_1 A_1 \) matching the degree of \( ^3\!D_4[-1] \). These are the only residual orbits supporting cuspidal unipotent representations.
Table 2: Cuspidal unipotent characters of $^3D_4$.

| $s$   | $\mathbf{r}$  | $\mu_{q',\{\mathbf{r}\}}(\mathbf{r})$ | $\mathcal{U}^0$ |
|-------|----------------|----------------------------------|-----------------|
| $G_2$ | $(q^3,q)$      | $(\Phi_3\Phi_9)/(\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12})$ |                 |
| $G_2(a_1)$ | $(1,q)$    | $(\Phi_2^2\Phi_3^2\Phi_6^2)^{-1}$ | $^3D_4[1]$       |
| $A_2$  | $(q^3,\vartheta_3q^{-3})$ | $\Phi_3^{-2}\Phi_6^{-2}\Phi_9\Phi_{12}^{-1}$ |                 |
| $A_1A_1$ | $(-q^{-3},q)$ | $(\Phi_2^2\Phi_6^2\Phi_{12})^{-1}$ | $^3D_4[-1]$      |

2.2.4 Groups of type $^2E_6$

We now consider the $k$-group of type $^2E_6$. The Frobenius action $\theta$, the relative system $R_0^\vee$, and the parameters are given as below. This time $R_0^\vee = F_4 = R_0$. The parameter function $m_+$ takes value 2 (resp. 1) at long (resp. short) roots in $R_0^\vee = F_4$.

There are 24 positive roots in $F_4$. The normalisation factor is

$$d = (v - v^{-1})^{-2}(v^2 - v^{-2})^{-2} = q^{-3}(q - 1)^{-2}(q^2 - 1)^{-2},$$

and there are eight residual points. With the help of Maple, we obtain Table 3 below. We see that there are two (W$_0$-orbits of) residual points give the $q'$-parts of formal degrees of cuspidal unipotent representations, corresponding to three cuspidal unipotent characters of the quasi-split finite group of type $^2E_6$. Among these three characters, two of them have the same degree.

$$\Sigma_0 = ^2E_6$$

$$R_0^\vee = F_4$$

$$R_0 = F_4$$

Figure 2: Root systems $\Sigma_0 = ^2E_6$.

3 Spectral transfer morphisms

We turn to the spectral transfer morphisms. First we give the definition.

Given two normalised affine Hecke algebras $(\mathcal{H}(R_1,m_1),d_1)$ and $(\mathcal{H}(R_2,m_2),d_2)$. Let $T_1, T_2$ be the associated tori. Let $L = rT^L \subset T_2$ be a residual coset, with $r \in T_1$ a residual point (here $T^L$ is a subtorus of $T$). Let $W_{2,0}$ be the Weyl group associated with the root datum of $R_2$. Denote by $W_{2,0}(L)$ the quotient group of the stabiliser of $L$ in $W_{2,0}$ by the pointwise stabiliser. Note that $L$ is a $T^L$-torsor.
Consider a morphism \( \phi_T : T_1 \to T_2 \), whose range is a \( L \). We call \( \phi_T \) a spectral transfer morphism if it satisfies the following conditions:

(T1) \( \phi_T \) is a finite morphism (in the sense of algebraic geometry);

(T2) \( \phi_T \) maps the identity \( e \) of \( T_1 \) into \( L \cap T_L \), and if we declare \( \phi_T(e) \) to be the base point of \( L \), then \( \phi_T \) is a homomorphism of algebraic tori.

(T3) There exists an \( a \in \mathbb{Q}^\times \) such that \( \phi_T^*(\mu_{m_2,d_2}(R_2)) = a \mu_{m_1,d_1}(R_1) \).

**Remark 3.1.** There is an extra technical condition (T4). However for our current purpose we do not need it and hence we do not state it. In practice, the conditions (T1), (T2) and (T4) are usually easily verified. So the key point is the verification of (T3).

**Example 3.2.** Let \( \mathcal{W}_0 \) be the finite Weyl group associated to the root datum \( \mathcal{R} \). Every \( w \in \mathcal{W}_0 \) acts on the associate complex torus as automorphism. Since the \( \mu \)-function is \( \mathcal{W}_0 \)-invariant, this morphism clearly satisfies the condition (T3).

There are other kinds of spectral transfer morphisms inducing from the algebra structure of \( \mathcal{H}_r \), and presented by the identity map on the tori. See [15, §2.1.7, §7]. □

We now look at three morphisms are from unipotent affine Hecke algebras associated with classical groups. Opdam defined them in [16, 3.2.6] and claimed (without proof) there that these morphisms are spectral transfer morphisms. We now give their definitions and prove that they are indeed spectral transfer morphisms.

For a classical \( G \), let \( \mathcal{H}_{upt}(P, \sigma) \) be the unipotent affine Hecke algebra attached to some unipotent representation of \( G(k) \). Recall that at Section 2.1, we pointed out that the parameters \( m_\pm(\cdot) \) of \( \mathcal{H}_{upt}(P, \sigma) \) can of determined by the data \( (P, \sigma) \). Namely, using [16, 3.2.1] we can obtain two numbers \( m_\pm \in (1/4)\mathbb{Z} \) from the structure of \( P^{F_u} \subset G(k) = G^{F_u} \). These numbers lie in a parameter space \( V \), which is the disjoint union of six subsets (corresponding to six possibilities for \( m_\pm \). We describe them here. Firstly, we need some notations: if \( m \in \mathbb{Z} \pm 1/4 \),
we write

$$|m| = \kappa + \frac{2\epsilon - 1}{4},$$  \hspace{1cm} (3.1)

where \( \epsilon \in \{0,1\} \) and \( \kappa \in \mathbb{Z}_{\geq 0} \). The integer \( \kappa \) determines a number \( \delta \in \{0,1\} \) by the rule that \( \kappa - \delta \in 2\mathbb{Z} \). In other words, \( \delta \) indicates the parity of \( \kappa \). We regard \((m_-,m_+)\) as an unordered pair. The six classes of parameters \((m_-,m_+)\) are defined as follows.

\[
\begin{align*}
(m_-,m_+) \in V^I & \quad \text{iff } m_\pm \in \mathbb{Z}/2 \text{ and } m_- - m_+ \notin \mathbb{Z}, \\
(m_-,m_+) \in V^{II} & \quad \text{iff } m_\pm \in \mathbb{Z} + 1/2 \text{ and } m_- - m_+ \in \mathbb{Z}, \\
(m_-,m_+) \in V^{III} & \quad \text{iff } m_\pm \in \mathbb{Z} \text{ and } m_- - m_+ \notin 2\mathbb{Z}, \\
(m_-,m_+) \in V^{IV} & \quad \text{iff } m_\pm \in \mathbb{Z} \text{ and } m_- - m_+ = 2\mathbb{Z}, \\
(m_-,m_+) \in V^V & \quad \text{iff } m_\pm \in \mathbb{Z} \pm 1/4 \text{ and } \delta_+ = \delta_- = \delta, \\
(m_-,m_+) \in V^{VI} & \quad \text{iff } m_\pm \in \mathbb{Z} \pm 1/4 \text{ and } \delta_+ \neq \delta_-.
\end{align*}
\]  \hspace{1cm} (3.2)

Now we are ready to give the definitions of the morphisms. We write \( T^n \) to indicate that the rank of the algebraic torus \( T \) is \( n \).

1. For \( m_\pm \in \mathbb{Z} + 1/2 \), define a homomorphism \( \phi_{T,m_-,m_+} : T^n \to T^{n+m_-+1/2} \) of algebraic tori over \( \mathbb{C} \) by

\[
\phi_{T,m_-,m_+}(t_1, \ldots, t_n) = (t_1, \ldots, t_n, v^b, v^{3b}, \ldots, v^{2b(m_-+1)}).
\]  \hspace{1cm} (3.3)

2. For \( m_+ \in \mathbb{Z}_{\geq 0} \), define a homomorphism \( \psi_{T,m_-,m_+} : T^n \to T^{n+2m_+} \) of algebraic tori over \( \mathbb{C} \) by

\[
\psi_{T,m_-,m_+}(t_1, \ldots, t_n) = (t_1, \ldots, t_n, 1, q^b, q^{2b}, \ldots, q^{b(m_-+2)}, q^{b(m_+)}).
\]  \hspace{1cm} (3.4)

3. (The extra-special cases \( V^V \) and \( V^{VI} \)). For \( m_+ > 0 \), set

\[
l := \begin{cases} 
2n + (a/2)(a + 1) + 2b(b + 1) & \text{if } \mathcal{X} = V, \\
2n + (a/2)(a + 1) + 2b^2 - \delta_+ & \text{if } \mathcal{X} = \mathcal{SL}.
\end{cases}
\]  \hspace{1cm} (3.5)

Let \( \kappa_\pm, \epsilon_\pm \) be defined as in (3.1). If we let \( l_\pm := \kappa_\pm(\kappa_\pm + \epsilon_\pm - 1/2) \), then \( l = 2n + [l_-] + [l_+] \) in both parameter types \( V, \mathcal{SL} \). For \( m \in \mathbb{Z} \pm \frac{1}{4} \text{ and } m > 1 \), let

\[
\sigma_\epsilon(m) = (q^\delta, q^{\delta+1}, \ldots, q^{2m-\frac{3}{2}}).
\]

Define the residual points \( r_\epsilon(m) \) recursively by putting

\[
r_\epsilon\left(\frac{1}{4}\right) = r_\epsilon\left(\frac{3}{4}\right) := \emptyset; \quad r_\epsilon(m) = (\sigma_\epsilon(m); r_\epsilon(m-1)) \text{ if } m > 1.
\]

Finally we define the representing morphism \( \xi_{T,m_-,m_+} : T^n \to T^l \) of the extra-special STM by

\[
\xi_{T,m_-,m_+}(t_1, \ldots, t_n) = (-r_\epsilon(m_-), v^{-1}t_1, vt_1, \ldots, v^{-1}t_n, vt_n, r_\epsilon(m_+)).
\]  \hspace{1cm} (3.6)

### 3.1 Proofs

We will prove in this section that the morphisms \( \phi_{T,m_-,m_+} \) (3.3), \( \psi_{T,m_-,m_+} \) (3.4) and \( \xi_{T,m_-,m_+} \) (3.6) satisfy the condition (T3) in the definition of spectral transfer morphisms, and hence they are indeed spectral transfer morphisms (the conditions (T1), (T2) are easy to verified, and we
will not do it). A key idea to the proof is the factorisation of the \( \mu \)-function by a parabolic root sub-system (cf. Section 1.4). Namely, these morphisms are all from some \( T^n \) to some \( T^l \) with \( n < l \). Let us temporarily denote the associated affine extended root data as \( \Pi^n \) and \( \Pi^l \) respectively. Based on Lusztig’s classification of unipotent representations (cf. [10, §§82–5]), there is a parabolic root sub-system with base \( \Pi_J \subset \Pi^l \), such that we have a corresponding factorisation \( T^l = T^n \times T^{l-n} \) and the base root system \( \Pi^n \) can be mapped bijectively onto \( \Pi^l - \Pi_J \). We can think that these three morphisms mentioned above add extra coordinate components to \( T^n \), completing it as \( T^l \). And these extra coordinate components correspond to the parabolic root sub-system \( \Pi_J \).

We have some conventions before start. Given a nonzero rational number \( x \) we denote by \( \epsilon(x) = x/|x| \in \{ \pm 1 \} \) its sign. Recall that the normalisation factor \( d = d(v) \), regarded as a rational function in \( v \), satisfies \( d(v) = d(v^{-1}) \), and hence so does the \( \mu \)-function. Thus, we do not need to care about the minus signs. Moreover, we will neglect the powers of \( q \), because we are only interested in the \( q \)-part. Consequently, we can replace every expression \( 1 - q^n \) by \( 1 - q^a \). Accordingly, we can rewrite the \( \mu \)-function as we did before:

\[
\mu_{m_\pm,d}(R) = \prod_{t \in R_+} \frac{d(1 - \alpha(t)^2)}{(1 + v^{2m_\pm} - \alpha(t))(1 - v^{2m_\pm} - \alpha(t))(1 - v^{2m_\pm} + \alpha(t))} \tag{3.7}
\]

To simplify the notations, in the proofs below, we write \( \text{Res}(\mu, x) \) for \( \mu^{(1)}(x) \).

The involutions \( \eta_\pm : m_- \mapsto -m_+, m_- \mapsto m_-, \eta_- : m_+ \mapsto m_+, m_- \mapsto -m_- \) and \( \eta \) interchanging \( m_+ \) and \( m_- \) act on the \( \mu \)-functions, inducing isomorphisms of normalised affine Hecke algebras. The group \( \text{Iso} \) is isomorphic to the dihedral group of order eight, and it is called the spectral isomorphism group (cf. [15, §7]). By virtue of \( \text{Iso} \), it is enough to verify the condition (T3) \( m_+ \geq m_- \geq 0 \). If \( m_+ = 1/2 \) then \( \phi_{T,m_-1/2} = \eta_+ \). Hence we may and will assume that \( m_+ > 1/2 \) from now on.

The morphism \( \phi_{T,m_-m_+}^r : T^r \to T^{r+m_-1/2} \) (3.3). We first check that the image \( L \) of \( \phi_{T,m_-m_+}^r \) in \( T^{r+m_-1/2} \) is a residual coset. The subset \( R_L \subset R_0^{(1)} \) (the affine extension of \( R_0 \)) of roots which are constant on the image is the set

\[
R_L = \{ t_i^{\pm 1} : r < i \leq r + m_+ - 1/2 \} \cup \{ t_j t_j^{-1} : r < j \neq i \leq r + m_+ - 1/2 \},
\]

a root system of type \( B_{m_+ - 1/2} \). Consider the factorisation \( T^{r+m_-1/2} = T^r \times T^{m_-1/2} \). By [15, Thm. 3.7(ii)] we need to show that the point \( r_{m_-,m_+} \) in the torus \( \{ id \} \times T^{m_-1/2} \) defined by \( t_i(r_{m_-,m_+}) = v^{2i(r-r)-1} \) for \( i = r + 1, \ldots, r + m_+ - 1/2 \) is a residual point with respect to the \( \mu \)-function \( \mu_L := \mu_{m_-,m_+}^{m_-1/2} \) associated to the roots of \( R_L \). By definition of the residual points we need to check that there are precisely \( m_+ - 1/2 \) factors in the denominator of \( \mu_{m_-,m_+}^{m_-1/2} \) which are zero in this point. Indeed, these are the factors corresponding to the roots \( t_i t_i^{-1} \) (for \( i = r + 1, \ldots, r + m_+ - 3/2 \)) and the factor corresponding to \( t_{r+m_-1/2} \).

It is a straightforward computation to determine the factor \( \text{Res} (\mu_{m_-,d}(R_L), r_{m_-,m_+}) \), which is equal to \( d \cdot \text{Res} (\mu_{m_-,1}(R_L), r_{m_-,m_+}) \) (in the notation of (1.8), with \( d = d_{m_-,m_+} \)). We obtain

\[
\text{Res} (\mu_{m_-,1}(R_L), r_{m_-,m_+}) = -q^{(m_- - 1/2)(m_+ - 1/2)} \frac{(q - 1)^{m_+ - 1/2}}{\prod_{k=m_-m_+1}^{m_+} (q^k + 1)} \tag{3.8}
\]

In this computation it is not necessary to compute the exact power of \( q \), since \( a \) \textit{priori} we know that \( \text{Res} (\mu_{m_-,d}(R_L), r_{m_-,m_+}) \) is invariant with respect to the substitution \( v \mapsto v^{-1} \). This remark simplifies the computation considerably.
In order to establish (T3) for \( \phi_T \) we need to check firstly that
\[
d_{m_-, m_+} = \lambda \cdot [d_{m_-, m_+ - 1} \cdot \text{Res}(\mu_{m_L, 1}(R_L), r_{m_-, m_+ - 1})]
\]  
for some \( \lambda \in \mathbb{Q}^\times \). This is easy using (3.8). (Again, it is not necessary to compute the power of \( q \)).

Next, we need to establish (T3) for the non-constant part of \( \mu^L \), by considering the restriction to \( L \) of the roots in \( B_{r + m_- - 1/2} \) which are not constant on \( \{ id \} \times T^{m_- - 1/2} \). For the long roots this is trivial, and by the Weyl group symmetry of \( \mu \) we only need to consider the roots that restrict to one fixed short root and its opposite, let us say \( t_{r+1} \) and \( t_{r-1} \). It is straightforward to collect all such factors and check that, after cancellation of equal terms in numerator and denominator, we will obtain
\[
\frac{(1 - t_{r+1}^{-2})(1 - t_{r+1}^2)}{(1 + v^{-2m_-}t_{r+1}^{-1})(1 + v^{2m_-}t_{r+1})(1 - v^{-2m_-}t_{r+1})(1 - v^{2m_-}t_{r+1})}.
\]
This is a factor of \( \text{Res}(\mu_{m_L, 1}(R_L), r_{m_-, m_+ - 1}) \) with \( m_L = (m_-, m_+) \). In short, the pull-back of the STM \( \phi_{T, m_-, m_+} \) (3.3) on the \( \mu \)-function \( \mu_{m_-, m_+ - 1, d_{m_-, m_+ - 1}} \), is equal to the \( \mu \)-function \( \mu_{m_-, m_+, d_{m_-, m_+}} \) (up to a non-zero rational constant). As desired, finishing the proof.

Next, we discuss the case that \( m_+ \neq 0 \).

The morphism \( \psi_{T, m_-, m_+}^\varphi : T^r \rightarrow T^{r + 2(m_+ - 1)} \) (3.4). We will sketch a proof similar to the proof of \( \phi_{T, m_-, m_+} \). It is enough to consider the case \( m_+ > 0 \). Let us check that the image of \( \psi_{T, m_-, m_+}^\varphi \) is a residual coset. The parabolic subset \( R_L \) of roots which are constant on the image is the set
\[
R_L = \{ t_i^{\pm 1} : r < i \leq r + 2(m_+ - 1) \} \cup \{ t_i t_j^{-1} : r < i \neq j \leq r + 2(m_+ - 1) \},
\]
a root system of type \( B_2(m_+ - 1) \). Consider the factorization \( T^{r + 2(m_+ - 1)} = T^r \times T^{2(m_+ - 1)} \). Again by [15, Theorem 3.7(ii)] we need to show that the point \( r_{m_-, m_+ - 2} \) in the torus \( \{ id \} \times T^{2(m_+ - 1)} \) defined by \( t_i(r_{m_-, m_+ - 2}) = v^2((i-r)/2) \) for \( i = r + 1, \ldots, r + 2(m_+ - 1) \) is a residual point with respect to the \( \mu \)-function \( \mu_L := \mu^{r + 2(m_+ - 1)} \) associated to the roots of \( R_L \). Precisely \( 2(m_+ - 1) \) roots are equal to \( 1 \) on this point, so by the definition of residual points we need to check that there are precisely \( 2(m_+ - 1) + 2(m_+ - 1) = 4(m_+ - 1) \) factors in the denominator of \( \mu_{m_-, m_+ - 1} \) which are zero in this point. Indeed, there are \( 4(m_+ - 2) \) such factors corresponding to roots of the form \( t_i t_j^{-1} \), 2 such factors from the roots \( t_i \) with \( i = r + 2m_+ - 4 \) and \( i = r + 2m_+ - 3 \), and finally 2 such factors from the roots \( t_{r+1} t_{r+2} \) and \( t_{r+1} t_{r+3} \).

This time we have (in the notation of (1.8), with \( d = d_{m_-, m_+ - 2} \):
\[
\text{Res}(\mu_{m_L, 1}(R_L), r_{m_-, m_+ - 2}) = \frac{q^{m_-(m_+ - 1)}(q - 1)^{2(m_+ - 1)}}{\prod_{k=m_- - m_+ - 2}^{m_+ + m_+ - 2}(q^{k+1} + 1)^2(q^{m_+ - m_-})^1 + 1)(q^{m_+ + m_-})^1 + 1
\]
Again, in this computation it is not necessary to compute the exact power of \( q \).

In view of the factorisation (1.8) of \( \mu^L \), in order for (T3) to hold we need to check firstly that
\[
d_{m_-, m_+} = \lambda \cdot [d_{m_-, m_+ - 2} \cdot \text{Res}(\mu_{m_L, 1}(R_L), r_{m_-, m_+ - 2})]
\]  
for some \( \lambda \in \mathbb{Q}^\times \). This is easy using (3.8). (Again, it is not necessary to compute the exact power of \( q \)). The rest of the argument is an elementary computation as in the proof for \( \phi_{T, m_-, m_+} \), finishing the proof.
Now we turn to the most complicated case.

**Proposition 3.3.** The morphism $\xi_{T_m,m_+}^r$ defines a spectral transfer morphism

$$\xi_{T_m,m_+}^r : T^r \to T^l \quad (3.13)$$

with $l$ as defined in (3.5), which is called extra-special.

**Remark 3.4.** Note that the unipotent affine Hecke algebra $H_{m_m, m_+}$ associated with $T^r$ has parameter $q^2$, while the unipotent affine Hecke algebra $H_{\delta_-, \delta_+}$ associated with $T^l$ has parameter $q$.

We will use induction, and the following lemma will serve as the induction basis.

**Lemma 3.5.** Proposition 3.3 is true for $(m_m, m_+) = (1/4, 1/4), (3/4, 1/4), (1/4, 3/4)$ and $(3/4, 3/4)$.

**Proof of Lemma 3.5.** Suppose $m_{\pm} = 1/4$. Then $\delta_{\pm} = 0$, and $a = b = 0$, and $r_{1/4} = 0$. If moreover the rank $r = 0$ then we have nothing to verify because $d_{1/4,1/4}$ reduces to 1, as does $d_{0,0}$, and there is no root contributing the $\mu$-functions so that both $\mu$-functions reduce to 1 as well.

Now assume $r > 0$. Since $m_{\pm} = 1/4$, we note that this is of parameter type VI with $a = b = 0$. Hence $l = 2r$, so that $d_{1/4,1/4} = (v^2 - v^{-2})^{-r}$ (recall that we have parameter $q^2$ here) and $d_{0,0} = (v - v^{-1})^{-2r}$. The $\mu$-function $\mu_{2r,0}$ has only contributions from the type D roots:

$$\mu_{2r,0}^2 = (v - v^{-1})^{-2r} \prod_{1 \leq i < j \leq 2r} \frac{(1-t_it_j^{-1})^2(1-t_i^{-1}t_j^{-1})^2}{(1-qt_it_j^{-1})(1-qt_i^{-1}t_j^{-1})(1-qt_i^{-1}t_j)(1-q^{-1}t_j^{-1}t_i^{-1})}.$$ 

We compute the pull-back $\xi^*(\mu_{2r,0}^{2r})$, so we need to substitute $(t_1, t_2, \ldots, t_{2r-1}, t_{2r})$ by

$$(vs_1, v^{-1}s_1, vs_2, v^{-1}s_2, \ldots, vs_r, v^{-1}s_r)$$

in $\mu_{2r,0}^{2r}$, after regularising the expression along the image $L$ of $\xi$. Concerning the regularisation, observe that the parabolic subsystem of roots which is constant on the image is

$$\{(t_1t_2^{-1})_{\pm 1}, (t_3t_4^{-1})_{\pm 1}, \ldots, (t_{2r-1}t_{2r})_{\pm 1}\},$$

of type $A_r^+$. After dropping the singular factors, the remaining “constant factors” yield (including the normalisation factor $(v - v^{-1})^{-2r}$), up to a power of $v$ which is irrelevant for us (as explained before): $(v^2 - v^{-2})^{-r}$, which is indeed the normalisation factor $d_{1/4,1/4}$.

For the non-constant part, by Weyl group invariance, it is enough to check the contribution of one type B root in each $W_0$-orbit. First consider $s_1s_2$. The roots whose pull-back along $\xi$ equals a nonzero power of $s_1s_2$ times a power of $v$ are $t_1t_3, t_2, t_3, t_4, t_2t_4$ (together with their opposites). These yield $q s_1s_2, s_1s_2, s_1s_2$ and $q^{-1}s_1s_2$ respectively, so these root give us the factor (after some cancellations, and up to powers of $v$ and other characters of $T^r$, which are irrelevant anyhow):

$$\frac{(1 - s_1s_2)^2}{(1 - q^{-2}s_1s_2)(1 - q^2s_1s_2)} \quad (3.14)$$

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which shows that the pull back of $\mu_{0,0}^2$ yields a factor which is a type D $\mu$ function with base $q^2$, as desired for $\mu_{1/4,1/4}^r$.

The type D roots which pull back to a power of $v$ times a nonzero power of $s_1$ (a type $A_1$ root) are only $t_1t_2$ (and its opposite). Its pull back is $s_1^2$. This gives a factor in $\xi^r(\mu_{0,0}^2)^l$ of the form:

$$\frac{(1-s_1^2)(1+s_1^2)}{(1+v^{-1}s_1)(1+vs_1)(1-v^{-1}s_1)(1-vs_1)}$$

(3.15)

This second fraction is exactly the factor in $\mu_{1/4,1/4}^r$ (with parameter $q^2$) for this last remaining type of root in the type B root system. In other words, the condition (T3) is satisfied. Hence we have verified that $\xi^r_{T,1/4,1/4}$ represents an extra-special STM.

Now we turn to the proof of Proposition 3.3 itself.

Proof of Proposition 3.3. Recall that $m_\pm \in \mathbb{Z} \pm \frac{1}{4}$. Using spectral isomorphisms from $\text{Iso} = \langle \eta_+, \eta \rangle$, we can assume that $m_\pm > 0$. So we can write $m_- = \kappa_- + (2\kappa_- - 1)/4$ with $\kappa_\pm \in \mathbb{Z}_{\geq 0}$ and $\epsilon_\pm \in \{0,1\}$. Let $\delta_-$ be defined by $\kappa_- - \delta_- \in 2\mathbb{Z}$. In other words, $\delta_- = 0$ (resp. 1) if $\kappa_- = 0$ (resp. odd). Similarly we have $m_+ = \kappa_+ + (2\kappa_+ - 1)/4$ and $\delta_+$.

We will apply an inductive argument on $m_- + m_+$, where the induction base is provided by Lemma 3.5. Let us first assume that $m_+ > m_- > 0$ and that $m_+ > 1$. By induction we may now assume that $\xi^r_{m_-,m_+ - 1}$ represents an STM.

Note that $\delta_\pm, \epsilon_\pm \in \{0,1\}$. We define $\delta_\pm^c, \epsilon_\pm^c$ by the rules that $\delta_\pm^c + \delta_\pm^c = 1, \epsilon_\pm^c + \epsilon_\pm^c = 1$. Also we define $A(m) = 2m - \frac{1}{2}$ and write $A_\pm = A(m_\pm) = 2(\kappa_\pm - 1) + \epsilon_\pm^c \in \mathbb{Z}_{\geq 1}$. Observe that $A_+ \geq 1$ by our assumptions.

To proceed, we need more notations. The $\mu$-function associated to the source normalised Hecke algebra $(\mathcal{H}_{\mu_{m_-,m_+}}^r, r^d)$ of the alleged STM $\xi = \xi_{m_-^d, m_+}$ with parameters $m_\pm$ will be denoted by $\mu_{m_-^d, m_+}^r$, where we often omit the rank $r$ if there should be no confusion. When $d = d_{m_-, m_+}$ we will simply write $\mu_{m_-^d, m_+}^r$. The $\mu$-function of the target is denoted by $\mu_{\delta_-, \delta_+}^l$, with $l$ given as in (3.5). Recall that we have, up to irrelevant factors, $\mu_{\delta_-, \delta_+}^l = d_{\delta_-, \delta_+}^l \mu_{\delta_-, \delta_+}^{A,l} \mu_{\delta_-, \delta_+}^{D,l}$ with

$$d_{\delta_-, \delta_+}^l = (v - v^{-1})^{-l}(v + v^{-1})^{-\delta_- \delta_+}.$$  

(3.16)

The second factor arises in [16, Prop. 2.5] from the fact that the reductive quotient $\mathcal{F}$ of a minimal $F$-stable parahoric in the case $(\delta_-, \delta_+) = (1,1)$ is an $\mathbb{F}_q$-torus of split rank $l$ whose maximal $\mathbb{F}_q$-anisotropic subtorus has $q + 1$ rational points over $\mathbb{F}_q$. Furthermore,

$$\mu_{\delta_-, \delta_+}^{D,l} = \prod_{1 \leq i < j \leq l} \frac{(1 - t_it_j)^2}{(1 - q^n t_i^{-1}t_j^{-1})(1 - q^{-1}t_i^{-1}t_j^{-1})},$$

$$\mu_{\delta_-, \delta_+}^{A,l} = \prod_{i=1}^l \frac{(1 - t_i^2)^2}{(1 + q^{\delta_-} t_i)(1 + q^{-\delta_-} t_i)(1 - q^{\delta_+} t_i)(1 - q^{-\delta_+} t_i)}.$$  

(3.17)

On the other hand $\mu_{m_-^d, m_+}^r = d_{m_-^d, m_+}^r \mu_{m_-^d, m_+}^{D,r} (q^2)^{\mu_{m_-^d, m_+}^{A,r}}$ with

$$d_{m_-^d, m_+}^r = (v - v^{-1})^{-r} d_{m_-^d, m_+}^0,$$  

(3.18)

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and the normalisation factor $d^0_{m_-,m_+}$ is given by
\[
d^0_{m_-,m_+} = \prod_{i=1}^{[m_- - m_+] \over \rho_{2m_- - m_+ + 2j}} \prod_{j=1}^{[m_+ + m_+]} \frac{q^{2m_- - m_+ - 2j}}{1 + q^{2m_- - m_+ - 2j}}
\]
as in [16, Eq. (36)]. Here $\mu^{D,r}(q^2)$ is similar to $\mu^{D,l}$, only with rank $r$ instead of $l$ and with parameter $q^2$ instead of $q$. Finally:
\[
\mu^{A,r}_{m_-,m_+} = \prod_{i=1}^{r} \frac{(1 - s^2_i)^2}{(1 + v^{4m_+} + s_i)(1 + v^{-4m_+} + s_i)(1 - v^{4m_+} + s_i)}
\]
(3.18)
Recall that the morphism $\xi := \xi_{T,m_-,m_+} : \mathbb{T}_r \to \mathbb{T}_r$ is defined by
\[
\xi_{T,m_-,m_+}((s_1, \ldots, s_r)) = (-r_c(m_+), v^{-1}s_1, v^{-1}s_2, \ldots, v^{-1}s_r, v^{-1}s_r, r_c(m_+))
\]
(3.19)
where $r_c(\frac{1}{4}) = r_c(\frac{3}{4}) = 0, r_c(m) = (\sigma_c(m), r_c(m - 1))$ and
\[
\sigma_c(m) = (q^{\delta}, q^{\delta + 1}, \ldots, q^{2m - \frac{3}{4}}).
\]
We denote $\tau_0 := (-r_c(m_-), r_c(m_+))$. Note that if we re-order the coordinates of $\xi(s_1, \ldots, s_r)$ (or of $\tau_0$), or invert them, then the result lies in the same $W_{2,0}$-orbit. The $\mu$-function is invariant under the $W_{2,0}$-action, hence also invariant under such operation.

We now consider the rank 0 case, i.e. $r = 0$. This is the main challenge, as we will see.

We need to verify the condition (T3), assuming that $m_+ \geq m_- > 0$ and $m_+ > 1$, and (by the induction hypothesis) that $\xi^0_{T,m_-,m_+ - 1}$ represents an STM.

Write $l_0 = [l_-] + [l_+]$ for the rank of the target Hecke algebra if we use the parameters $(m_-, m_+)$, and $l'_0 = [l_-] + [l'_+]$ if we use the parameters $(m'_-, m'_+) := (m_-, m_+ - 1)$. Observe that $l'_+ = l_+$ and $l'_- = l'_+$. By our normalisations of the Hecke algebras it suffices to show the following identity for the ratio of residues:
\[
C_{m_-,m_+ - 1} := \frac{\text{Res}(\mu^{D,l_0}, \tau_0) \cdot \text{Res}(\mu^{A,l_0}, \tau_0)}{\text{Res}(\mu^{D,l'_0}, \tau'_0) \cdot \text{Res}(\mu^{A,l'_0}, \tau'_0)} = \frac{d^0_{\delta,\delta_+} \cdot \text{Res}(\mu^{l_0}_{\delta,\delta_+}, \tau_0)}{d^0_{\delta,\delta_+} \cdot \text{Res}(\mu^{l'_0}_{\delta,\delta_+}, \tau'_0)}.
\]
(3.20)
The last expression equals, up to powers of $v$ and rational constants,
\[
A(m_+) := \frac{d^0_{\delta,\delta_+} \cdot d^0_{m_- - m_+}}{d^0_{\delta,\delta_+} \cdot d^0_{m_-,m_+ - 1}} = (v - v^{-1})^{A_+ + \delta_+} (v + v^{-1})^{\delta_+ - \delta_-} \frac{d^0_{m_-,m_+}}{d^0_{m_-,m_+ - 1}}.
\]
The second equality is easy to check, using the relations $l_0 - l'_0 = A_+ + \delta_+ = 2m_+ - \frac{3}{4} - \delta_+$.

**Notation.** In the equations below we will simplify notations by omitting the references to the rank (if the arguments are given, the rank equals the number of coordinates of the argument so the explicit references to the ranks are superfluous), and we will simply write "reg" to indicate that we are using the regularisation of $\mu$-functions, i.e. omitting the factors that are identically 0 after evaluation at the argument. Finally, an expression like $\mu^{D,\text{reg}(\tau_1; \tau_2)}$ means that we only consider the product in the numerator and the denominator of those type D-roots $t_1^{1} t_2^{1}$ for
which \( t_i \) is a coordinate of \( \mathbf{r}_1 \) and \( t_j \) is a coordinate of \( \mathbf{r}_2 \), and only those factors which are not identically 0.

Since \( \mathbf{r}_0 = (\mathbf{r}_0', \sigma_e(m_+)) \), we see that \( C_{m_-, m_+ - 1} \) is equal to

\[
\frac{\mu^\text{A.reg}_{\delta_-, \delta_+}(\mathbf{r}_0')}{\mu^\text{A.reg}_{\delta_-, \delta_+}(\mathbf{r}_0')} \mu^\text{A.reg}_{\delta_-, \delta_+}(\sigma_e(m_+)) \times \\
\mu^\text{D.reg}(\sigma_e(m_+)) \mu^\text{D.reg}(-r_e(m_-); \sigma_e(m_+)) \mu^\text{D.reg}(\sigma_e(m_+); r_e(m_+ - 1))
\]

(3.21)

It is easy to see that

\[
\frac{\mu^\text{A.reg}_{\delta_-, \delta_+}(\mathbf{r}_0')}{\mu^\text{A.reg}_{\delta_-, \delta_+}(\mathbf{r}_0')} = \prod_{u_i \in \mathcal{r}_e(m_-)} \frac{(1 + q^{\delta_+} u_i)(1 + q^{-\delta_+} u_i)}{(1 + q^{\delta_+} u_i)(1 + q^{-\delta_+} u_i)} \prod_{w_j \in \mathcal{r}_e(m_+ - 1)} \frac{(1 - q^{\delta_+} w_j)(1 - q^{-\delta_+} w_j)}{(1 - q^{\delta_+} w_j)(1 - q^{-\delta_+} w_j)}
\]

(3.22)

while

\[
\mu^\text{A.reg}_{\delta_-, \delta_+}(\sigma_e(m_+)) = \prod_{t_k \in \sigma_e(m_+)} \frac{(1 - t_k^2)^2}{(1 + q^{2m_+ - 3/2})(1 + q^{2m_+ - 3/2})(1 + q^{2m_+ - 3/2})(1 + q^{2m_+ - 1/2})}
\]

\[
= \begin{cases} 
1 & (\delta_-, \delta_+) = (0, 0) \\
\frac{1 + q^{2m_+ - 3/2}}{1+q^{2m_+ - 1/2}} & (\delta_-, \delta_+) = (1, 0) \\
\frac{1 - q^{2m_+ - 3/2}(1-q)}{1+q^{2m_+ - 1/2}} & (\delta_-, \delta_+) = (0, 1) \\
\frac{1}{1+q^{2m_+ - 1/2}} & (\delta_-, \delta_+) = (1, 1) 
\end{cases}
\]

(3.23)

We denote this last expression by \( (\mathbf{P}1) = \mu^\text{A.reg}_{\delta_-, \delta_+}(\sigma_e(m_+)) \).

Next we consider \( \mu^\text{D.reg}(-r_e(m_-); \sigma_e(m_+)) \) and \( \mu^\text{D.reg}(\sigma_e(m_+); r_e(m_+ - 1)) \).

\[
\mu^\text{D.reg}(-r_e(m_-); \sigma_e(m_+)) = \\
\prod_{t_i \in \sigma_e(m_-), t_j \in \mathcal{r}_e(m_-)} \frac{(1 - t_i t_j^{-1})^2(1 - t_i t_j)^2}{(1 - q^{A_+} t_i^{-1})(1 - q^{A_+} t_j)(1 - q^{-1} t_i t_j)(1 - q^{-1} t_i t_j)}
\]

(3.24)

where \( A_+ = 2m_+ - 3/2 \). We can likewise obtain

\[
\mu^\text{D.reg}(\sigma_e(m_+), r_e(m_+ - 1)) = \\
\prod_{t_i \in \sigma_e(m_+), t_j \in \mathcal{r}_e(m_+ - 1)} \frac{(1 - t_i t_j^{-1})^2(1 - t_i t_j)^2}{(1 - q^{A_+} t_i^{-1})(1 - q^{A_+} t_j)(1 - q^{-1} t_i t_j)(1 - q^{-1} t_i t_j)}
\]

(3.25)
Observe that up to some power of \( v \), we can cancel some factors from (3.22), (3.24) and (3.25) and obtain

\[
\frac{\mu_{\delta_-, \delta_+}(x_0^t)}{\mu_{\delta_-, \delta_+}(x_0^t)} \times \mu(D, q, -r_e(m_-), \sigma_e(m_+)) \times \mu(D, q, \sigma_e(m_+), r_e(m_+ - 1)) = \prod_{t_j \in \epsilon_e(m_-)} \frac{(1 + q^{A_+} t_j^{-1})(1 + q^{A_+} t_j)}{(1 + q^{A_+ + 1} t_j^{-1})(1 + q^{A_+ + 1} t_j)} \prod_{t_j \in \epsilon_e(m_+-1)} \frac{(1 - q^{A_+} t_j^{-1})(1 - q^{A_+} t_j)}{(1 - q^{A_+ + 1} t_j^{-1})(1 - q^{A_+ + 1} t_j)}
\]

(3.26)

Denote

\[
(P2) = \prod_{t_j \in \epsilon_e(m_-)} \frac{(1 + q^{A_+} t_j^{-1})(1 + q^{A_+} t_j)}{(1 + q^{A_+ + 1} t_j^{-1})(1 + q^{A_+ + 1} t_j)}
\]

and

\[
(P3) = \prod_{t_j \in \epsilon_e(m_+-1)} \frac{(1 - q^{A_+} t_j^{-1})(1 - q^{A_+} t_j)}{(1 - q^{A_+ + 1} t_j^{-1})(1 - q^{A_+ + 1} t_j)}.
\]

They can be simplified further. Observe that

\[
r_e(m_-) = \left( \sigma_e(m_-), r_e(m_- - 1) \right) = \left( \sigma_e(m_-), \sigma_e(m_- - 1), \ldots, \sigma_e\left(\frac{7 - 2\epsilon_-}{4}\right) \right)
\]

The number of \( \sigma_e \)'s in \( r_e(m_-) \) is \( \kappa_- \). Recall that for \( g \in \mathbb{Z}_{>0} \) we defined:

\[
\sigma_e(g + \frac{2\epsilon_- - 1}{4}) = (q^{\tilde{g}}, q^{\tilde{g}+1}, \ldots, q^{2(g-1)+\epsilon_-})
\]

where \( \tilde{g} = 0 \) if \( g \) is even and \( \tilde{g} = 1 \) if \( g \) is odd. Therefore we can write (P2) as

\[
(P2) = \prod_{g=2-\epsilon_-}^{\kappa_-} \prod_{t_j \in \epsilon_e(g+\frac{2\epsilon_- - 1}{4})} \frac{(1 + q^{A_+} t_j^{-1})(1 + q^{A_+} t_j)}{(1 + q^{A_+ + 1} t_j^{-1})(1 + q^{A_+ + 1} t_j)}
\]

\[
= \prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A_+ - (2g-2+\epsilon_-)})(1 + q^{A_+ + \tilde{g}})}{(1 + q^{A_+ + 1 - \tilde{g}})(1 + q^{A_+ + (2g-1)+\epsilon_-})}\cdot \frac{(1 + q^{A_+ + \tilde{g}})}{(1 + q^{A_+ + (2g-1)+\epsilon_-})}
\]

(3.27)

Notice that \( \delta_- \) indicates the parity of \( \kappa_- \), so

\[
\prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A_+ + \tilde{g}})}{(1 + q^{A_+ + 1 - \tilde{g}})} = \begin{cases} 1 & \text{if } \epsilon_- \neq \delta_- \\ \frac{1 + q^{A_+}}{1 + q^{A_+ + 1}} & \text{if } \epsilon_- = \delta_- = 0 \\ \frac{1 + q^{A_+ + 1}}{1 + q^{A_+}} & \text{if } \epsilon_- = \delta_- = 1. \end{cases}
\]

So (P2) is equal to

\[
\left( \frac{1 + q^{A_+ + \epsilon_-}}{1 + q^{A_+ + 1 - \epsilon_-}} \right)^{\epsilon_- + \epsilon \cdots} \frac{(1 + q^{A_+ - 2 + \epsilon_-})(1 + q^{A_+ - 4 + \epsilon_-}) \cdots (1 + q^{A_+ - A_-})}{(1 + q^{A_+ + 3 - \epsilon_-})(1 + q^{A_+ + 5 - \epsilon_-}) \cdots (1 + q^{A_+ + A_+ - 1})},
\]

(3.28)

where \( A_- = 2m_- - 3/2 \). (Recall that \( m_+ \geq m_- > 0 \) and \( m_+ > 1 \), hence \( A_+ \geq A_- \geq -1 \) and \( A_+ \geq 1 \). Observe that (P2) = 1 if \( 0 < m_- < 1 \).
Similarly we can compute that (P3) is equal to

\[
\left( \frac{1 - q^{A_+ + \epsilon_+}}{1 - q^{A_+ + 1 - \epsilon_+}} \right)^{\epsilon_+ \delta_+^0 + \epsilon_+ \delta_+} \left( \frac{1 - q^2(1 - q^4) \cdots (1 - q^{A_+ + 2 + \epsilon_+})}{(1 - q^{2A_+ - 1})(1 - q^{2A_+ - 3}) \cdots (1 - q^{A_+ + 3 - \epsilon_+})} \right). \tag{3.29}
\]

The final term to compute is \( \mu_{D, \text{reg}}^\sigma(m_+) = \mu_{D_0, \text{reg}}(\sigma_m) \mu_{D_\neq0, \text{reg}}(\sigma_m) \). Here \( D_0 \) denotes the type D roots whose coordinates sum up to zero (this is a maximal proper parabolic root subsystem, irreducible of type A), and \( D_\neq0 \) denotes the remaining roots of type D (this is not a root subsystem).

Recall that \( \sigma_m(m_+) = (q^{\delta_+}, q^{\delta_+ + 1}, \ldots, q^{2m_+ - 3/2}) \). We can easily compute that

\[
\mu_{D_0, \text{reg}}(\sigma_m(m_+)) = \prod_{1 \leq i < j \leq A_+ - \delta_+} \left( \frac{1 - q^{1-i}}{1 - q^{j-1-i}} \right) = \frac{(1 - q)^{A_+ + 1 - \delta_+}}{(1 - q^{A_+ + 1 - \delta_+})} \tag{3.30}
\]

by considering the multiplicities of the range of \( j - i \in \{1, 2, \ldots, A_+ - \delta_+ - 1\} \). The same idea applies to computing \( \mu_{D_\neq0, \text{reg}}(\sigma_m(m_+)) \):

\[
\mu_{D_\neq0, \text{reg}}(\sigma_m(m_+)) = \prod_{1 \leq i < j \leq A_+ - \delta_+} \left( \frac{1 - q^{1+i+j+2(\delta_+ - 1)}}{(1 - q^{1+i+j+2\delta_+ - 1})(1 - q^{i+j+2\delta_+ - 3})} \right)
\]

\[
\begin{align*}
&= \frac{(1 - q^{1+2\delta_+})(1 - q^{3+2\delta_+}) \cdots (1 - q^{A_+ + 2\delta_+})}{(1 - q^{A_+ + 2\delta_+})(1 - q^{2A_+ - 1})(1 - q^{2A_+ - 3}) \cdots (1 - q^{A_+ + 3 - \epsilon_+})} \\
&\quad \text{if } \epsilon_+ = \delta_+ \tag{3.31} \\
&= \frac{(1 - q^{2(1+\delta_+)})(1 - q^{3+2\delta_+}) \cdots (1 - q^{A_+ + 1 + \epsilon_+})}{(1 - q^{A_+ + 1 + 2 + \epsilon_+}) \cdots (1 - q^{2A_+})} \\
&\quad \text{if } \epsilon_+ \neq \delta_+.
\end{align*}
\]

Here, if \( \delta_+ = 0 \) then the denominator starts with \( (1 - q^2) \).

We denote \( \mu_{D, \text{reg}}^\sigma(m_+), \mu_{D_0, \text{reg}}(\sigma_m(m_+)), \mu_{D_\neq0, \text{reg}}(\sigma_m(m_+)) \) respectively by (P4), (P4a) and (P4d).

Now we multiply (P1), (P2), (P3) and (P4). The 4 parameters \( \epsilon_+, \delta_-, \epsilon_-, \delta_- \) take values in \( \{0, 1\} \) independently. So basically we need to consider 16 cases (The parity of \( A_+ \) is the same as \( \epsilon_+ \)). But we spot a simplification when taking the product of (P3) and (P4d).

We see that:

\[
(P3) \times (P4d) = \frac{1 - q^{A_+ + \epsilon_+}}{1 - q^{A_+ + 1 - \epsilon_+}} \times \frac{(1 - q^2(1 - q^4) \cdots (1 - q^{A_+ + 2 + \epsilon_+})}{(1 - q^{2A_+ - 1})(1 - q^{2A_+ - 3}) \cdots (1 - q^{A_+ + 3 - \epsilon_+})} \times \frac{(1 - q^{1+2\delta_+})(1 - q^{3+2\delta_+}) \cdots (1 - q^{A_+ + 1 + \epsilon_+})}{(1 - q^{A_+ + 1 + 2 + \epsilon_+}) \cdots (1 - q^{2A_+})}.
\]

no matter if \( \epsilon_+ \) equals to \( \delta_+ \) or not. Here in the third and the fourth equations we insert in both the numerators and denominators the factors in square brackets to produce the factors \( 1 + q^x (x = 1, \ldots, A_+) \) in the denominator.
We proceed to combine with (P4a) and (P1) to obtain

\[
\frac{(1-q)^{A_+ \downarrow 1 - \delta_+}}{(1-q^{A_+ \downarrow 1 - 1})} \times \left( \frac{1 - q^{A_+ \downarrow 1}}{1 - q^{A_+ \downarrow 1 + 1}} \right)^{\delta_+} \times \left( \frac{1 + q^{A_+ \downarrow 1 + 1}}{1 + q^{A_+ \downarrow 1 + 1}} \right)^{-\delta_-} \times (1 + q)^{\epsilon_+ \downarrow (-1) \delta_+} (1 - q)^{\delta_-}
\]

\[
\times \frac{1}{(1+q)(1+q^2) \cdots (1+q^{A_+ \downarrow 1 + 1})} (1 - q)^{\delta_-} \times (1 + q^{A_+ \downarrow 1 + 1})^\delta_- = (P1) \times (P3) \times (P4),
\]

no matter the value of \( \delta_+ \).

Finally we multiply with (P2). Note first that

\[
\frac{1 + q^{A_+ \downarrow 1 - \epsilon_-}}{1 + q^{A_+ \downarrow 1 - 1 - \epsilon_-}} \times \frac{1 + q^{A_+ \downarrow 1 + 1}}{1 + q^{A_+ \downarrow 1 + 1}} \delta_- = 1 + q^{A_+ \downarrow 1 - \epsilon_-} / 1 + q^{A_+ \downarrow 1 + 1}
\]

and hence no matter the values of \( \epsilon_- \) and \( \delta_- \), the total product \( C_{m_- m_+ - 1} \) is equal to

\[
(1 + q)^{\epsilon_- \downarrow (-1) \delta_-} (1 - q)^{A_+ \downarrow 1 - \delta_+} (1 + q^{A_+ \downarrow 1 + 1 - \delta_+})(1 + q^{A_+ \downarrow 1 + 1 - \delta_-})(1 + q^{A_+ \downarrow 1 + 1 - \delta_-}) \cdots (1 + q^{A_+ \downarrow 1 + 1 - \delta_-})
\]

\[
\times \frac{1}{(1+q)(1+q^2) \cdots (1+q^{A_+ \downarrow 1 + 1})} (1 - q)^{\delta_-} \times (1 + q^{A_+ \downarrow 1 + 1})^\delta_- = C_{m_- m_+ - 1}.
\]

(The second and third factor of this product may be equal to 1, if \( A_+ \leq A_- + 1 \) or if \( A_- = -1 \) respectively). In view of the expression of \( d_{m_- m_+}^0 \) we now verify easily that \( C_{m_- m_+ - 1} = A_{(m_+)} \). Furthermore, we remark that if \( m_- > m_+ \) then we should compute \( C_{m_- m_+ - 1} \) which can be obtained by changing the subscripts \( \mu \) in (P1) to (P4) to \( \mu \) and similarly to obtain \( A_{(m_-)} \). One can likewise verify that \( C_{m_- - 1 m_+} = A_{(m_-)} \). To sum up, we have verified by induction on \( m_- + m_+ \) that in the rank 0 case of \( \xi_T \) indeed represents a spectral transfer morphism.

Now we consider the positive rank cases. Again, assume that \( m_+ \geq m_- \). Still we need to verify the condition (T5) for the \( \mu \)-functions. We put \( s = (s_1, \ldots, s_r) \). Let \( \xi(s) := \xi_{T, m_- m_+}^r (s) \) and \( \xi'(s) := \xi_{T, m_- m_+ - 1}^r (s) \) be given as in (3.19). Observe that \( \xi(s) = (\xi'(s), \sigma_m(s)) \). Let \( L' \) be the image of \( \xi' \), and let \( L' \) be the image of \( \xi' \). Observe that the parabolic root system \( R_0' \) of roots of \( R_0' = B_0 \) which restrict to constant functions on \( L \) is isomorphic to \( A_r^\uparrow \times B_0 \), while the roots of \( R_0' = B_0 \) restricting to constant roots on \( L' \) form a parabolic subsystem of type \( A_r^{\uparrow} \times B_0 \). Using (3.20) it suffices to prove that the ratio

\[
\frac{\mu_{\epsilon, \delta}^L \xi(s)}{\mu_{\epsilon, \delta}^L \xi'(s)} = \frac{(v + v^{-1})^{-\gamma} \text{Res} \left( \frac{\mu_{\epsilon, \delta}^L \xi(s)}{\mu_{\epsilon, \delta}^L \xi'(s)} ; R_0' \right) \cdot \mu_{\epsilon, \delta}^{L, R_0' \setminus R_0 \setminus \xi(s)} \cdot \mu_{\epsilon, \delta}^{L, R_0' \setminus R_0 \setminus \xi'(s)} \right)}{(v + v^{-1})^{-\gamma} \text{Res} \left( \frac{\mu_{\epsilon, \delta}^L \xi'(s)}{\mu_{\epsilon, \delta}^L \xi(s)} ; R_0' \right) \cdot \mu_{\epsilon, \delta}^{L, R_0' \setminus R_0 \setminus \xi'(s)} \cdot \mu_{\epsilon, \delta}^{L, R_0' \setminus R_0 \setminus \xi(s)} \right)}
\]

which we denoted as \( C_{m_- m_+ - 1}^r(s) \), is equal to \( C_{m_- m_+ - 1} \times \frac{\mu_{m_- m_+ - 1}^r(s)}{\mu_{m_- m_+ - 1}^r(s)} \) where \( C_{m_- m_+ - 1} \) is defined in (3.20), and

\[
\frac{\mu_{m_- m_+ - 1}^r(s)}{\mu_{m_- m_+ - 1}^r(s)} = \prod_{i=1}^{r} \frac{(1 - q^{-2(m_- + 1)} s_i) (1 - q^{-2(m_- + 1)} s_i)}{(1 - q^{-2m_+ s_i}) (1 - q^{-2m_+ s_i})}.
\]
Based on the result of rank 0 case, to prove that $C_{m-,m_+}(s) = C_{m-,m_+} \times \frac{\mu_{m-,m_+}^r(s)}{\mu_{m-,m_+}^{m_+}(s)}$, we just need to verify that

$$\frac{\mu_{\delta_-,\delta_+},R_0,RL}(\xi'(s),\sigma_{e}(m_+)) = \prod_{i=1}^{r} \frac{(1 - q^{-2(m_+ - 1)}s_i)(1 - q^{2(m_+ - 1)}s_i)}{(1 - q^{-2m_+}s_i)(1 - q^{2m_+}s_i)}.$$  (3.35)

It is enough to consider the contribution to the factors of the right hand side involving $s_1$. For this we need to consider the contribution in the numerator of the left hand side of the set of type D-roots $t_i^{\pm 1}t_j^{\pm 1}$ such that $t_i = v^{-1}s_1$ or $vs_1$, and $t_j$ is a coordinate of $\sigma_{e}(m_+)$. In addition, we need to consider the contribution in the numerator and denominator of the left hand side of the roots $t_1 = v^{-1}s_1$, $t_2 = vs_1$ and their opposites. Therefore, in this computation we may assume that $r = 1$.

The contribution from the type $A_1$ roots $t_1 = v^{-1}s_1$ and $t_2 = vs_1$ is:

$$\frac{\mu_{\delta_-,\delta_+}(v^{-1}s_1,vs_1)}{\mu_{\delta_-,\delta_+}(v^{-1}s_1,vs_1)} = \frac{(1 - q^{\delta_+ - 1/2}s_1)(1 - q^{\delta_+ + 1/2}s_1)(1 - q^{-\delta_+ - 1/2}s_1)(1 - q^{-\delta_+ + 1/2}s_1)}{(1 - q^{\delta_+ - 1/2}s_1)(1 - q^{\delta_+ + 1/2}s_1)(1 - q^{-\delta_+ - 1/2}s_1)(1 - q^{-\delta_+ + 1/2}s_1)} = \left(\frac{1 - q^{-3/2}s_1}{1 - q^{-3/2}s_1}\right)^{(-1)^{\delta_+}}.$$

Next, consider the type D-roots $t_1t_j^{\pm 1}$ and $t_2t_j^{\pm 1}$ in the numerator, with $t_1 = v^{-1}s_1$, $t_2 = vs_1$, and $t_j \in \sigma_{e}(m_+)$. These yield:

$$\prod_{s = \delta_+}^{2m_+ - 3/2} \frac{(1 - q^{-s+1/2}s_1)(1 - q^{-s-1/2}s_1)(1 - q^{s+1/2}s_1)(1 - q^{s-1/2}s_1)}{(1 - q^{-s+3/2}s_1)(1 - q^{-s-3/2}s_1)(1 - q^{s+3/2}s_1)(1 - q^{s-3/2}s_1)} = \frac{(1 - q^{-2m_+ + 2}s_1)(1 - q^{2m_+ - 2}s_1)(1 - q^{-\delta_+ - 1/2}s_1)(1 - q^{-\delta_+ + 1/2}s_1)}{(1 - q^{-\delta_+ + 3/2}s_1)(1 - q^{-\delta_+ - 3/2}s_1)(1 - q^{-2m_+ + s_1})(1 - q^{2m_+ + s_1})}.$$

Multiplying the contributions from the type A and type D roots, we are quickly led to (3.35). Using induction on $m_+ + m_+$ (as in the rank zero case), and the induction basis Lemma 3.5, we can finish the proof that $\xi_{T,m_-,m_+}$ represents an extra-special spectral transfer morphism. □

3.2 A conclusive remark

Compositions of $\phi_{T,m_-,m_+}$ and $\psi_{T,m_-,m_+}$ are obviously defined. By applying (the compositions of) $\phi_T, \psi_T$ and $\xi_T$, we obtain a spectral transfer morphism from a rank zero unipotent affine Hecke algebras to the one of maximal rank with parameters attain the minimum in the same parameter type. These three spectral transfer morphisms are necessary to prove the main result, Theorem 3.4 in [16], called the essential uniqueness of spectral transfer morphism.

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