On stability of Abrikosov lattices

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August 25, 2013

Abstract
We consider Abrikosov-type vortex lattice solutions of the Ginzburg-Landau equations of superconductivity, for magnetic fields close to the second critical magnetic field and for superconductors filling the entire space. We study stability of such solutions within the context of the time-dependent Ginzburg-Landau equations - the Gorkov-Eliashberg-Schmid equations. For arbitrary lattice shapes, we prove that there exists a modular function depending on the lattice shape such that Abrikosov vortex lattice solutions are asymptotically stable under finite energy perturbations (defined precisely in the text), provided the superconductor is of Type II and this function is positive, and unstable otherwise.

1 Introduction
1.1 Problem and results
The macroscopic theory of superconductivity, by now a classical theory presented in any book on superconductivity and solid state or condensed matter physics, was developed by Ginzburg and Landau along the lines of Landau’s theory of the second order phase transitions before the microscopic theory was discovered. At the foundation of this theory lie the celebrated Ginzburg-Landau equations,
\[
\begin{aligned}
-\Delta A \Psi &= \kappa^2 (1 - |\Psi|^2) \Psi, \\
\text{curl}^* \text{curl} A &= \text{Im}(\bar{\Psi} \nabla A \Psi)
\end{aligned}
\] (1.1)
which describe superconductors in thermodynamic equilibrium. Here \(\Psi\) is a complex-valued function, called the order parameter, \(A\) is a vector field (the magnetic potential), \(\kappa\) is a positive constant, called the Ginzburg-Landau parameter, \(\nabla_A = \nabla - iA\) and \(\Delta_A = \nabla_A \cdot \nabla_A\) are the covariant gradient and Laplacian. Physically, \(|\Psi|^2\) gives the (local) density of superconducting electrons (Cooper pairs), \(B = \text{curl} A\) is the magnetic field. The second equation is Ampère’s law with \(J_S = \text{Im}(\bar{\Psi} \nabla A \Psi)\) being the supercurrent associated to the electrons having formed Cooper pairs.

We assume, as common, that superconductors fill in all of \(\mathbb{R}^2\) (the cylindrical geometry in \(\mathbb{R}^3\)). In this case, \(\text{curl} A := \partial_{x_1} A_2 - \partial_{x_2} A_1\) and \(\text{curl}^* f = (\partial_{x_2} f, -\partial_{x_1} f)\).

By far, the most important and celebrated solutions of the Ginzburg-Landau equations are magnetic vortex lattices, discovered by Abrikosov (II), and known as Abrikosov (vortex) lattice solutions or simply Abrikosov lattices. Among other things understanding these solutions is important for maintaining the superconducting current in Type II superconductors, i.e., for \(\kappa > \frac{1}{\sqrt{2}}\).

Abrikosov lattices have been extensively studied in the physics literature. Among many rigorous results, we mention that the existence of these solutions was proven rigorously in [38, 10, 13, 7, 55, 47]. Moreover, important and fairly detailed results on asymptotic behaviour of solutions, for \(\kappa \to \infty\) and applied magnetic fields, \(h\), satisfying \(h \leq \frac{1}{2} \ln \kappa + \text{const}\) (the London limit), were obtained in [55] (see this paper and the book [43] for references to earlier work). Further extensions to the Ginzburg-Landau equations for anisotropic and high temperature superconductors in the \(\kappa \to \infty\) regime can be found in [4, 5]. (See [27, 45] for reviews.)
In this paper we are interested in dynamics of the Abrikosov lattices, as described by the time-dependent generalization of the Ginzburg-Landau equations proposed by Schmid (14) and Gorkov and Eliashberg (21) (earlier versions are due to Bardeen and Stephen and Anderson, Luttinger and Werthamer). These equations are of the form

\[
\begin{align*}
\gamma \partial_t \Phi &= \Delta \Phi + \kappa^2 (1 - |\Phi|^2) \Phi, \\
\sigma \partial_t \Phi &= - \text{curl}^* \text{curl} A + \text{Im}(\Phi \nabla \Phi).
\end{align*}
\]  

(1.2)

Here \( \Phi \) is the scalar (electric) potential, \( \gamma \) a complex number, and \( \sigma \) a two-tensor, and \( \partial_t \Phi \) is the covariant time derivative \( \partial_{t,\Phi}(\Phi, A) = ((\partial_t + i \Phi) \Phi, \partial_t A + \nabla \Phi) \). The second equation is Ampère’s law, \( \text{curl} B = J \), with \( J = J_N + J_S \), where \( J_N = -\sigma(\partial_t A + \nabla \Phi) \) (using Ohm’s law) is the normal current associated to the electrons not having formed Cooper pairs, and \( J_S = \text{Im}(\Phi \nabla A \Phi) \), the supercurrent. We use the the gauge transformation \( T_\eta \), with \( \eta(x, t) = \int_0^t \Phi(x, s) ds \), to achieve the gauge

\[ \Phi(x, t) = 0, \]  

(1.3)

which we assume from now on.

Eqs (1.2), which we call Gorkov-Eliashberg-Schmid equations (they are also known as the Gorkov-Eliashberg or the time-dependent Ginzburg-Landau equations), have a much narrower range of applicability than the Ginzburg-Landau equations (53) and many refinements have been proposed. However, though improvements of these equations are, at least notationally, rather cumbersome, they do not alter the mathematics involved in an essential way.

The Abrikosov lattices are defined as solutions, \( (\Psi, A) \), to (1.1), whose physical characteristics, \( |\Psi|^2, \text{curl} A, \text{curl} \Psi \) and \( J_S = \text{Im}(\Phi \nabla A \Phi) \) are double-periodic w.r. to a lattice \( \mathcal{L} \subset \mathbb{R}^2 \). They are static solutions to (1.2) and their stability w.r. to the dynamics induced by these equations is an important issue.

In [47], we considered the stability of the Abrikosov lattices under the simplest perturbations, namely those having the same (gauge-) periodicity as the underlying Abrikosov lattices (we call such perturbations gauge-periodic). We proved for a lattice \( \mathcal{L} \) of arbitrary shape, \([\tau]\), and for the average magnetic field, \( b \), per lattice cell close to either the second or first critical magnetic field, that, under gauge-periodic perturbations,

(i) *Abrikosov vortex lattice solutions are asymptotically stable for \( \kappa^2 > \kappa_c(\tau) \);*

(ii) *Abrikosov vortex lattice solutions are unstable for \( \kappa^2 < \kappa_c(\tau) \).*

Here, by the lattice shape we understand the class \([\tau]\) of lattices, equivalent under rotations and dilatations, parametrized by points \( \tau \) in the fundamental domain, \( \Pi^+ / \text{SL}(2, \mathbb{Z}) \), of the modular group \( \text{SL}(2, \mathbb{Z}) \) acting on the Poincaré half-plane \( \Pi^+ \) (see Supplement I), and

\[ \kappa_c(\tau) := \sqrt{\frac{1}{2} \left( 1 - \frac{1}{\beta(\tau)} \right)}, \]  

(1.4)

where \( \beta(\tau) \) is the Abrikosov ‘constant’, defined in Remark 2) below. (We assume always that the co-ordinate origin is placed at one of the vertices of the lattice \( \mathcal{L} \).) For the definitions of the average magnetic field, \( b \), and various stability notions see Subsections 1.4 and 1.5.

This result belies the common belief among physicists and mathematicians that Abrikosov-type vortex lattice solutions are stable only for triangular lattices and \( \kappa > \frac{1}{\sqrt{2}} \), and it seems this is the first time the threshold (1.4) has been isolated.

Gauge-periodic perturbations are not a common type of perturbations occurring in superconductivity. In this paper we address the problem of the stability of Abrikosov lattices under local or finite-energy perturbations (defined in Subsections 1.5 below). We consider Abrikosov lattices of arbitrary shape, with the average magnetic fields per lattice cell close to the second critical magnetic field \( h_{c2} = \kappa^2 \) (due to the flux quantization - see Subsection 1.4 this means that \( |\mathcal{T}| \), where \( \mathcal{T} := \mathbb{R}^2 / \mathcal{L} \), is close to \( \frac{2\pi}{\kappa^2} \)). We formulate informally our main results:

- There exists a continuous function \( \gamma(\tau) \) depending on the lattice shape parameter \( \tau \in \Pi^+ / \text{SL}(2, \mathbb{Z}) \), such that, under finite-energy perturbations, and *Abrikosov lattice solution for a lattice \( \mathcal{L} \in [\tau] \) is asymptotically stable for \( \kappa > \frac{1}{\sqrt{2}} \) and \( \gamma(\tau) > 0 \) and unstable otherwise.*
• The function $\gamma(\tau)$, on lattice shape parameters, $\tau$, is defined as

$$
\gamma(\tau) := \inf_{\chi \in \mathcal{L}_\tau} \gamma_\chi(\tau), \quad \text{where } \gamma_\chi(\tau) := 2 \langle |\phi_0|^2 |\phi_\chi|^2 \rangle_{\tau} + \langle |\phi_0^2\phi_\chi\phi_{\chi-1}| \rangle_{\tau} - \langle |\phi_0|^4 \rangle_{\tau}.
$$

(1.5)

Here $\mathcal{L}_\tau$ stands for the dual group of the group of lattice translations, $\mathcal{L}_\tau = \sqrt{\text{area}} \mathcal{L}$, $\mathcal{T}_\tau := \mathbb{R}^2 / \mathcal{L}_\tau$, $(f)_\Omega = \frac{1}{|\Omega|} \int_{\Omega} f$, and the functions $\phi_\chi$, $\chi \in \mathcal{L}_\tau$, are unique solutions of the equations

$$
(-\Delta_0 - 1)\phi = 0, \quad \phi(x + s) = e^{i\frac{\sqrt{3}}{2}s \cdot Jx} \chi(s) \phi(x), \quad \forall s \in \mathcal{L}_\tau,
$$

(1.6)

with $a^0(x) := -\frac{1}{2}Jx$ and $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, normalized as $\langle |\phi_\chi|^2 \rangle_{\tau} = 1$.

By the rotational covariance of (1.5) - (1.6) (see (2.11) below) and normalization of $\mathcal{L}_\tau$, the function $\gamma_\chi(\tau)$ and therefore also $\gamma(\tau)$, do not depend on the choice of the representative $\mathcal{L} \in [\tau]$ in (1.5) - (1.6).

Various properties of the function $\gamma(\tau)$ are summarized in Section 1.6. Moreover, calculations of Appendix B suggest that $\gamma(\tau)$ has a unique global maximum at $\tau = e^{\frac{2\pi}{3}}$ and a saddle point at $\tau = e^{\frac{2\pi}{\sqrt{3}}}$ and show that $\gamma(\tau) > 0$ for all equilateral lattices, $|\tau| = 1$, and is negative for $|\tau| \geq 1.3$. These calculations are based on the explicit expression for the functions $\gamma_\chi(\tau)$, $\chi \in \mathcal{L}_\tau$, which is described in Subsection 1.6.

Though Abrikosov lattices are not as rigid under finite energy perturbations, as under gauge-periodic ones, they are still surprisingly stable.

Finally, we address the following seeming contradiction of our results with the fact that, for $\kappa > 1/\sqrt{2}$, the triangular lattice has the lowest energy (see Theorem 1.1 below), which seems to suggest that other lattices should be unstable. The reason that this energetics does not affect the stability under local perturbations can be gleaned from investigating the zero mode of the Hessian of the energy functional associated with different lattice shapes, $\tau$. This mode is obtained by differentiating the Abrikosov lattice solutions w.r.t. $\tau$, which shows that it grows linearly in $|\tau|$. To rearrange a non-triangular Abrikosov lattice into the triangular one, one would have to activate this mode and hence to apply a perturbation, growing at infinity (at the same rate).

This also explains why the Abrikosov ‘constant’ $\beta(\tau)$ mentioned above, which plays a crucial role in understanding the energetics of the Abrikosov solutions, is not directly related to the stability under local perturbations, the latter is governed by $\gamma(\tau)$.

Remarks. 1) The fundamental domain $\Pi^+ / \text{SL}(2, \mathbb{Z})$ is given explicitly as $\Pi^+ / \text{SL}(2, \mathbb{Z}) = \{ \tau \in \mathbb{C} : \text{Im} \tau > 0, \ |\tau| \geq 1, \ -\frac{1}{2} < \text{Re} \tau \leq \frac{1}{2} \}$, see Fig. 1

![Figure 1: Fundamental domain of $\gamma(\tau)$.](image)

2) The the Abrikosov constant, $\beta(\tau)$, arises as $\beta(\tau) = \langle |\phi_0|^4 \rangle_{\tau} = \frac{1}{2}\gamma_0(\tau)$. The term Abrikosov constant comes from the physics literature, where one often considers only equilateral triangular or square lattices.
3) The way we defined the Abrikosov constant \( \beta(\tau) \), it is manifestly independent of \( b \). Our definition differs from the standard one by rescaling: the standard definition uses the function \( \phi_b(x) = \phi_0(\sqrt{b}x) \), instead of \( \phi_0(x) \).

4) We think of \( \gamma_\tau(\tau) \) as the ‘Abrikosov beta function with characteristic’ (while \( \beta(\tau) \) is defined in terms of the standard theta function, \( \gamma_\tau(\tau) \) is defined in terms of theta functions with finite characteristics, see below).

The methods we develop are fairly robust and can be extended - at the expense of significantly more technicalities - to substantially wider classes of perturbation, which will be done elsewhere. Moreover, the same techniques could be used in other problems of pattern formation, which are ubiquitous in applications.

In the rest of this section we introduce some basic definitions (stemming from properties of the Ginzburg-Landau and Gorkov-Eliashberg-Schmidt equations), present our results and sketch the approach and possible extensions.

### 1.2 Ginzburg-Landau energy

The Ginzburg-Landau equations are the Euler-Lagrange equations for the Ginzburg-Landau energy functional

\[
E_Q(\Psi, A) = \frac{1}{2} \int_Q \left\{ |\nabla_A \Psi|^2 + |\text{curl} A|^2 + \kappa^2 \frac{1}{2} (1 - |\Psi|^2)^2 \right\},
\]

(1.7)

where \( Q \) is any domain in \( \mathbb{R}^2 \). (\( E_Q \) is the difference in (Helmholtz) free energy between the superconducting and normal states.)

The Gorkov-Eliashberg-Schmidt equations have the structure of a gradient-flow equation for \( E_Q(\Psi, A) \). Indeed, they can be put in the form

\[
\partial_t \Psi(\Psi, A) = -\lambda \nabla E'/E(\Psi, A),
\]

(1.8)

where \( \lambda := \text{diag}(\gamma^{-1}, \sigma^{-1}) \), \( \partial_t \Psi(\Psi, A) := (\partial_t + i\Phi)\Psi, \partial_x A + \nabla \Phi \) and \( E'/E \) is the \( L^2 \)–gradient of \( E(\Psi, A) \equiv E_{\mathbb{R}^2}(\Psi, A) \), defined as \( \langle v, E'(u) \rangle_{\mathbb{R}^2} = dE(u)v \), with \( d \) being the Gâteaux derivative, \( dF(u)v := \left. \frac{\partial}{\partial s} F(u + sv) \right|_{s=0} \). This definition implies that

\[
E'(\Psi, A) = (-\Delta A \Psi - \kappa^2 (1 - |\Psi|^2) \Psi, \text{curl}^* \text{curl} A - \text{Im}(\bar{\Psi} \nabla A \Psi)).
\]

(1.9)

We note that the symmetries above restrict to symmetries of the Ginzburg-Landau equations by considering time-independent transformations.

### 1.3 Symmetries

The Gorkov-Eliashberg-Schmidt equations \([1, 2]\) admit several symmetries, that is, transformations which map solutions to solutions.

- **Gauge symmetry:** for any sufficiently regular function \( \gamma : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R} \),

\[
T^\text{gauge}_\gamma : (\Psi(t, x), A(t, x), \Phi(t, x)) \mapsto (e^{i\gamma(t, x)} \Psi(t, x), A(t, x) + \nabla \gamma(t, x), \Phi(t, x) - \partial_t \gamma(t, x));
\]

(1.10)

- **Translation symmetry:** for any \( h \in \mathbb{R}^2 \),

\[
T^\text{trans}_h : (\Psi(t, x), A(t, x), \Phi(t, x)) \mapsto (\Psi(t, x + h), A(t, x + h), \Phi(t, x + h));
\]

(1.11)

- **Rotation symmetry:** for any \( \rho \in SO(2) \),

\[
T^\text{rot}_\rho : (\Psi(t, x), A(t, x), \Phi(t, x)) \mapsto (\Psi(t, \rho^{-1}x), \rho^{-1}A(t, (\rho^{-1})^T x), \Phi(t, \rho^{-1}x)),
\]

(1.12)

- **Reflection symmetry:**

\[
T^\text{refl} : (\Psi(t, x), A(t, x), \Phi(t, x)) \mapsto (\Psi(t, -x), -A(t, -x), \Phi(t, -x)).
\]

(1.13)
1.4 Abrikosov lattices

As was mentioned above, Abrikosov vortex lattices (or just Abrikosov lattices), are solutions, whose physical characteristics, density of Cooper pairs, $|\Psi|^2$, the magnetic field, $\text{curl} A$, and the supercurrent, $J_S = \text{Im}(\bar{\Psi}\nabla A \Psi)$, are double-periodic w.r. to a lattice $L \subset \mathbb{R}^2$.

We note that the symmetries of the previous subsection map Abrikosov lattices to Abrikosov lattices. Moreover, for Abrikosov states, for $(\Psi, A)$, the magnetic flux, $\int_{\Omega} \text{curl} A$, through a lattice cell, $\Omega$, is quantized,

$$\int_{\Omega} \text{curl} A = 2\pi n,$$

for some integer $n$. Indeed, the periodicity of $n_s = |\Psi|^2$ and $J = \text{Im}(\bar{\Psi}\nabla A \Psi)$ imply that $\nabla \theta - A$, where $\Psi = |\Psi|e^{i\theta}$, is periodic, provided $\Psi \neq 0$ on $\partial \Omega$. This, together with Stokes’s theorem, $\int_{\Omega} \text{curl} A = \int_{\partial \Omega} A = \int_{\partial \Omega} \nabla \theta$ and the single-valuedness of $\Psi$, implies (1.14). Using the reflection symmetry of the problem, one can easily check that we can always assume $n \geq 0$.

Equation (1.14) implies the relation between the average magnetic flux, $b$, per lattice cell, $b = \frac{1}{|\Omega|} \int_{\Omega} \text{curl} A$, and the area of a fundamental cell

$$b = \frac{2\pi n}{|\Omega|}.$$

Due to the quantization relation (1.15), the parameters $\tau$, $b$, and $n$ determine the lattice $L$ up to a rotation and a translation. As the equations are invariant under rotations and translations, we will say that a gauge-periodic pair $(\Psi, A)$ is of type $(\tau, b, n)$, if the underlying lattice has shape parameter $\tau$, the average magnetic flux per lattice cell is equal to $b$, and there are $n$ quanta of magnetic flux per lattice cell. Recall the definition of the Ginzburg-Landau parameter threshold $\kappa_c(\tau)$ given in (1.4). We have the following existence theorem.

**Theorem 1.1.** For any $\tau \in \mathbb{C}$, $\text{Im} \tau > 0$, and for any $b$ such that $\kappa^2 - b$ is sufficiently small and satisfying

$$\text{if } \kappa > \kappa_c(\tau), \text{ then } b < \kappa^2, \text{ if } \kappa < \kappa_c(\tau), \text{ then } b > \kappa^2,$$

there exists a smooth Abrikosov lattice solution $u_\omega = (\Psi_\omega, A_\omega)$ of type $\omega = (\tau, b, 1)$.

More detailed properties of these solutions are given in Subsection 4.2 below. As we deal only with the case $n = 1$, we now assume that this is so and drop $n$ from the notation.

1.5 Finite-energy ($H^1$) perturbations

We now wish to study the stability of these Abrikosov lattice solutions under a class of perturbations that have finite-energy. More precisely, we fix an Abrikosov lattice solution $u_\omega$ and consider perturbations $v : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$ that satisfy

$$\Lambda_{u_\omega}(v) = \lim_{Q \to R^2} \left( \mathcal{E}_Q(u_\omega + v) - \mathcal{E}_Q(u_\omega) \right) < \infty.$$

Clearly, $\Lambda_{u_\omega}(v) < \infty$, for all vectors of the form $v = T_{(\tau, b)} u_{(\xi, \alpha)}$, where $\gamma \in H^2(\mathbb{R}^2; \mathbb{R})$.

In fact, we will be dealing with the smaller class, $H^1_{\text{cov}}$, of perturbations, where $H^1_{\text{cov}}$ is the Sobolev space of order 1 defined by the covariant derivatives, i.e., $H^1_{\text{cov}} := \{ v \in L^2(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2) \mid ||v||_{H^1} < \infty \}$, where the norm $||v||_{H^1}$ is determined by the covariant inner product

$$\langle v, v' \rangle_{H^1} = \text{Re} \int \xi \xi' + \nabla_{\alpha} \xi \cdot \nabla_{\alpha'} \xi' + \alpha \cdot \alpha' + \sum_{k=1}^2 \nabla \alpha_k \cdot \nabla \alpha'_k,$$

where $v = (\xi, \alpha), \ v' = (\xi', \alpha')$, while the $L^2$-norm is given by

$$\langle v, v' \rangle_{L^2} = \text{Re} \int \xi \xi' + \alpha \cdot \alpha'.$$
In Lemmas 3.8 and 3.9 in Subsection 3.5 below, we will find an explicit representation of \( \Lambda_{u_\omega}(v) \) and show that \( \Lambda_{u_\omega}(v) < \infty \) for \( v \in H^1_{cov} \).

To formulate the notion of asymptotically stability we define the manifold

\[
\mathcal{M}_\omega = \{ T^{\text{gauge}} u_\omega : \gamma \in H^1(\mathbb{R}^2, \mathbb{R}) \}
\]

of gauge equivalent Abrikosov lattices and the \( H^1 \)-distance, \( \text{dist}_{H^1} \), to this manifold.

**Definition 1.2.** We say that the Abrikosov lattice \( u_\omega \) is asymptotically stable under \( H^1_{cov} \) perturbations, if there is \( \delta > 0 \) s.t. for any initial condition \( u_0 \) satisfying \( \text{dist}_{H^1}(u_0, \mathcal{M}_\omega) \leq \delta \) there exists \( g(t) \in H^1 \), s.t. the solution \( u(t) \) of \( (1.6) \) satisfies \( \lim_{t \to \infty} \| u(t) - T^{\text{gauge}} u_\omega \|_{H^1} = 0 \). We say that \( u_\omega \) is energetically unstable if \( \inf_{v \in H^1_{cov}} (\mathcal{E}(u_\omega) v) < 0 \) for the hessian, \( \mathcal{E}''(u_\omega) \), of \( \mathcal{E}(u) \) at \( u_\omega \).

The hessian, \( \mathcal{E}''(u) \) of the energy functional \( \mathcal{E} \), - at \( u \in u_\omega + H^1_{cov} \) - is defined as \( \mathcal{E}''(u) = d\mathcal{E}'(u) \) (the Gâteaux derivative of the \( L^2 \)-gradient map), where \( d \) and \( \cdot \) are the Gâteaux derivative and \( L^2 \)-gradient map defined in the paragraph preceding (1.9). Although \( \mathcal{E}(u) \) is infinite on \( u_\omega + H^1_{cov} \), the hessian \( \mathcal{E}''(u) \) is well defined as a differential operator explicitly and is given in Appendix \( C \). We restrict the initial conditions \((\Psi_0, A_0)\) for (1.2) satisfying

\[
T^{\text{refl}}(\Psi_0, A_0) = (\Psi_0, A_0).
\]

Note that, by uniqueness, the Abrikosov lattice solutions \( u_\omega = (\Psi_\omega, A_\omega) \) satisfy \( T^{\text{refl}} u_\omega = u_\omega \) and therefore so are the perturbations, \( v_0 := u_0 - u_\omega \), where \( u_0 := (\Psi_0, A_0) \):

\[
T^{\text{refl}} v_0 = v_0.
\]

### 1.6 Main results

Recall that \( \beta(\tau) \) is the Abrikosov 'constant', \( \beta(\tau) = \langle |\phi_0|^4 \rangle_{T_\tau} \), where \( \phi_0 \) is a unique (see e.g. \[25\] and also Proposition 2.1 of Subsection 2.1) solution of the equation (1.5), and let \( b \) be sufficiently close to \( \kappa^2 \), in the sense that

\[
|\kappa^2 - b| < \kappa^2((2\kappa^2 - 1)\beta(\tau) + 1].
\]

Then, under \( H^1 \)-perturbations, satisfying (1.19), the Abrikosov lattice \( u_\omega \) is

- asymptotically stable for all \( (\tau, \kappa) \) s.t. \( \kappa^2 > \frac{1}{2} \) and \( \gamma(\tau) > 0 \);
- energetically unstable otherwise.

Concerning the function \( \gamma(\tau) \), we have the following

**Proposition 1.4.**

- \( \gamma(\tau) \), \( \tau \in \Pi^+/SL(2, \mathbb{Z}) \), is symmetric w.r.to the imaginary axis.
- \( \gamma(\tau) \) has critical points at \( \tau = e^{i\pi/2} \) and \( \tau = e^{i\pi/3} \), provided it is differentiable at these points.

The first property implies that it suffices to consider \( \gamma(\tau) \) on the \( \text{Re} \tau \geq 0 \) half of the fundamental domain, \( \Pi^+/SL(2, \mathbb{Z}) \), (the heavily shaded area on the Fig. 1). The function \( \gamma(\tau) \) is studied numerically in Appendix \[13\] where it is shown that it becomes negative for \( \text{Im} \tau \geq 1.81 \). The explicit representation of \( \gamma(\tau) \) below and the numerics suggest also that for fixed \( \text{Re} \tau \in [0, 1/2] \), \( \gamma(\tau) \) is a decreasing function of \( \text{Im} \tau \). Moreover, it is computed that

\[
|\gamma(\tau) - c| \leq 7.5 \cdot 10^{-3} \quad \text{where} \quad c = 0.64 \quad \text{for} \quad \tau = e^{i\pi/3} \quad \text{and} \quad c = 0.4 \quad \text{for} \quad \tau = e^{i\pi/2}.
\]

The numerics mentioned above are based on the explicit expression for the functions \( \chi(\tau) \), \( \chi \in \hat{\mathcal{L}}_\tau \), which we describe now. We identify the dual group, \( \hat{\mathcal{L}} \), with a fundamental cell, \( \Omega^* \), of the dual lattice \( \mathcal{L}^* \). (The identification given explicitly by \( \chi(s) \to \chi(s) = e^{ik \cdot s} \leftrightarrow k \). The dual, or reciprocal, lattice, \( \mathcal{L}^* \), of \( \mathcal{L} \)
consists of all vectors $s^* \in \mathbb{R}^2$ such that $s^* \cdot s \in 2\pi \mathbb{Z}$, for all $s \in \mathcal{L}$.) We chose $\Omega^*$ so that it is invariant under reflections, $k \rightarrow -k$ (then $\chi^{-1}(s) \rightarrow \chi^{-1}(s) = e^{i\pi s^*} \leftrightarrow -k$).

We identify $\mathbb{R}^2$ with $\mathbb{C}$, via the map $(x_1, x_2) \rightarrow x_1 + ix_2$, and let $\gamma_k(\tau) = \gamma_{k}\chi(\tau)$, $\chi(k) = e^{ik}s$, $k \in \Omega^*$, where $\Omega^*$ is an elementary cell of the dual lattice $\mathcal{L}^*$. Finally, recalling that $\mathcal{L}_{\tau} = \sqrt{\frac{2\pi}{|\tau|}} \mathcal{L}$, with $\mathcal{L} \in [\tau]$, we take $\mathcal{L}_{\tau} := \sqrt{\frac{2\pi}{|\tau|}} (\mathbb{Z} + \tau \mathbb{Z})$ (see Supplement I). Then we have the following explicit representation of the function $\gamma_k(\tau)$, as a fast convergent series (cf [2, 37]).

**Theorem 1.5.** For the functions $\gamma_k(\tau)$, $\text{Im} \tau > 0, k \in \Omega^*$, defined in (1.5), have the explicit representation

$$\gamma_k(\tau) = 2 \sum_{\ell \in \mathcal{L}^*_{\tau}} e^{-\frac{1}{2}|\ell|^2} \cos[\text{Im}(\ell k)] + \left| \sum_{\ell \in \mathcal{L}^*_{\tau}} e^{-\frac{1}{2}|\ell+k|^2+i\text{Im}(\ell k)} \right| - \sum_{\ell \in \mathcal{L}^*_{\tau}} e^{-\frac{1}{2}|\ell|^2}. \quad (1.23)$$

This theorem follows from Proposition 2.1 and Proposition A.1 of Appendix A (cf [2, 37]). It is easy to see that $k = 0$ is a point of maximum of $\gamma_k(\tau)$ in $k \in \Omega^*$. Also, our computations show that

- $\gamma_k(\tau)$ is minimized at $k \approx \sqrt{\frac{2\pi}{|\tau|}} (\frac{1}{2} - \frac{1}{2\sqrt{3}})$ at the point $\tau = e^{i\pi/3}$, and a value of $k \approx \sqrt{\frac{2\pi}{|\tau|}} (\frac{1}{2} + \frac{1}{2\sqrt{3}})$ for $\tau = e^{i\pi/2}$, which corresponds to vertices of the corresponding Wigner-Seitz cells.

Interestingly, in Proposition 2.3 below, we show that the points $k \in \frac{1}{2} \mathcal{L}^*_{\tau}$ are critical points of the function $\gamma_k(\tau)$ in $k$.

**Remark.** $\mathcal{L}$ is the group of characters, $\chi : \mathcal{L} \rightarrow U(1)$, By using (1.5) and (1.6), with $\chi(s) = e^{ik}s$, one can define the function $\gamma_k(\tau)$, and therefore $\gamma(\tau)$, on the entire Poincaré half-plane $\Pi^+$, rather than just the fundamental domain $\Pi^+/SL(2, \mathbb{Z})$. In this case, $\gamma_k(\tau)$, and therefore $\gamma(\tau)$, are modular functions on $\Pi^+$.

### 1.7 The key ideas of approach

First step is to realize that a state $(\Psi, A)$ is an Abrikosov lattice if and only if $(\Psi, A)$ is gauge-periodic or gauge-equivariant (with respect to the lattice $\mathcal{L}$) in the sense that there exist (possibly multivalued) functions $g_s : \mathbb{R}^2 \rightarrow \mathbb{R}, s \in \mathcal{L}$, such that

$$T_{\text{trans}}^s(\Psi, A) = T_{\text{gauge}}^s(\Psi, A). \quad (1.24)$$

Indeed, if state $(\Psi, A)$ satisfies (1.24), then all associated physical quantities are $\mathcal{L}$-periodic, i.e. $(\Psi, A)$ is an Abrikosov lattice. In the opposite direction, if $(\Psi, A)$ is an Abrikosov lattice, then curl $A(x)$ is periodic w.r.to $\mathcal{L}$, and therefore $A(x+s) = A(x) + \nabla g_s(x)$, for some functions $g_s(x)$. Next, we write $\Psi(x) = |\Psi(x)|e^{i\phi(x)}$. Since $|\Psi(x)|$ and $J(x) = |\Psi(x)|^2(\nabla \phi(x) - A(x))$ are periodic w.r.to $\mathcal{L}$, we have that $\nabla \phi(x+s) = \nabla \phi(x) + \nabla g_s(x)$, which implies that $\phi(x+s) = \phi(x) + g_s(x)$, where $g_s(x) = \tilde{g}_s(x) + c_s$, for some constants $c_s$.

Since $T_{\text{trans}}^s$ is a commutative group, we see that the family of functions $g_s$ has the important cocycle property

$$g_{s+t}(x) - g_s(x+t) - g_t(x) \in 2\pi \mathbb{Z}. \quad (1.25)$$

This can be seen by evaluating the effect of translation by $t + s$ in two different ways. We call $g_s(x)$ the *gauge exponent*.

As usual, the stability of the static solution $u_\omega = (\Psi_\omega, A_\omega)$, $\omega := (\tau, b, 1)$, is decided by the sign of the infimum $\mu(\omega; k) := \inf_{v \in H^1} \langle v, L_\omega v \rangle_{L^2}/\|v\|^2_{L^2}$, where, recall, $L_\omega := \mathcal{E}''(u_\omega)$ is the Hessian of the energy functional $\mathcal{E}$ at $u_\omega$, on the space orthogonal to symmetry zero modes of $L_\omega$. The key idea of the proof of the first part of Theorem 1.3 stems from the observation that since the Abrikosov lattice solution $u_\omega = (\Psi_\omega, A_\omega)$ is gauge periodic (or equivariant) w.r.to the lattice $\mathcal{L}$, the linearized map $L_\omega$ commutes with magnetic translations,

$$\rho_s = T_{\text{mag-trans}}^s \oplus T_{\text{trans}}^s, \quad \forall s \in \mathcal{L}_\omega, \quad (1.26)$$

where $\mathcal{L}_\omega$ is the linearized lattice of $L_\omega$. Finally, recalling that $\mathcal{L}_{\tau} = \sqrt{\frac{2\pi}{|\tau|}} (\mathbb{Z} + \tau \mathbb{Z})$ (see Supplement I). Then we have the following explicit representation of the function $\gamma_k(\tau)$, as a fast convergent series (cf [2, 37]).
where $T_{\text{trans}} = (T^{\text{trans}})^{-1} T^{\text{trans}}$ is the group of magnetic translations and, recall, $T^{\text{trans}}$ denotes translation by $s$, which, due to \cite{125}, give a unitary group representation of $\mathcal{L}_\omega$. (Note that \cite{124} implies that $u_\omega$ is invariant under the magnetic translations, $\mathcal{L}^{\text{mag-trans}} u_\omega = u_\omega$.) Therefore $L_\omega$ is unitary equivalent to a fiber integral over the dual group, $\mathcal{L}_\omega$, of the group of lattice translations, $\mathcal{L}_\omega$.

$$L_\omega \approx \int_{\mathcal{L}_\omega} L_\omega \xi d\xi$$ acting on \( \int_{\mathcal{L}_\omega} \mathcal{H} \xi d\xi \),

where $d\xi$ is the usual Lebesgue measure on $\mathcal{L}_\omega$ normalized so that $\int_{\mathcal{L}_\omega} d\xi = 1$, $L_\omega \xi$ is the restriction of $L_\omega$ to $\mathcal{H} \xi$, $\xi \in \mathcal{L}_\omega$, and $\mathcal{H} \xi$ is the set of all functions, $\xi \chi$, from $L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$, which are gauge-periodic,

$$\rho_s \xi \chi(x) = \chi(s) \xi \chi(x), \quad \forall s \in \mathcal{L}_\omega, \quad (1.27)$$

with $\chi(s) v = (\chi(s) \xi, \chi(s) \alpha)$. The inner product in $\mathcal{H} \xi$ is given by $\langle \xi, \xi' \rangle_{L^2} = \frac{1}{|\mathcal{L}_\omega|} \int_{\mathcal{L}_\omega} \text{Re} \xi \xi' + \alpha \alpha'$, where $v = (\xi, \alpha)$, $v' = (\xi', \alpha')$ and $\mathcal{T}_\omega := \mathbb{R}^2 / \mathcal{L}_\omega$, and in $\int_{\mathcal{L}_\omega} \mathcal{H} \xi d\xi$, by $\langle \xi, \xi' \rangle_{\mathcal{H}} := \frac{1}{|\mathcal{L}_\omega|} \int_{\mathcal{L}_\omega} \langle \xi, \xi' \rangle_{\mathcal{H}} d\xi$. (The normalization used will be useful later on.)

By the formula above, the smallest spectral point, $\mu(\omega, \kappa)$, of $L_\omega$ is given by $\mu(\omega, \kappa) = \inf_{\chi \in \mathcal{L}_\omega} \mu_\omega(\omega, \kappa)$, where $\mu_\chi(\omega, \kappa)$ are the smallest eigenvalues of $L_\omega \chi$. The spectral analysis of fibers $L_\omega \chi$, $\chi \in \mathcal{L}_\omega$, gives an expression of $\mu_\chi(\omega, \kappa)$ in terms of the function $\gamma(\tau)$ defined in \cite{124}.

The linear result above gives the linearized (energetic) stability of $u_\omega$, if $\mu_{\omega, \chi}(\kappa) > 0$, and the instability, if $\mu_{\omega, \chi}(\kappa) < 0$. To lift the stability part to the (nonlinear) asymptotic stability, we use the functional $\Lambda_{u_\omega}(v)$, given in \cite{117}. Using an explicit expression for $\Lambda_{u_\omega}(v)$ given in Lemmas \ref{3.8} and \ref{3.9} in Subsection \ref{3.5} below, we will find appropriate differential inequalities for $\Lambda_{u_\omega}(v)$, which imply the asymptotic stability. \( \square \)

Finally, we mention that the following properties of $g_s$ play an important role in the proofs:

(a) If $(\Psi, A)$ satisfies \cite{124} with $g_s(x)$, then $T^{\text{gaugge}}(\Psi, A)$ satisfies \cite{124} with $g_s(x) \rightarrow g'_s(x)$, where

$$g'_s(x) = g_s(x) + \gamma (x + s) - \gamma(x). \quad (1.28)$$

(b) The functions $g_s(x) = \frac{1}{2} s \cdot x + c_s$, where $b$ satisfies $|b| \in 2\pi \mathbb{Z}$ and $c_s$ are numbers satisfying $c_{t+s} - c_s - c_t - \frac{1}{2} bs \in 2\pi \mathbb{Z}$, satisfies \cite{125}.

(c) Every exponential $g_s$ satisfying the cocycle condition \cite{125} is equivalent to $\frac{1}{2} s \cdot x + c_s$, for some $b$ and $c_s$ satisfying $c_{t+s} - c_s - c_t - \frac{1}{2} bs \in 2\pi \mathbb{Z}$.

Indeed, the first and second statements are straightforward. For the third property, see e.g. \cite{13, 35, 52, 50}, though in these papers it is formulated differently. In the present formulation it was shown by A. Weil and generalized in \cite{22}.

**Remark.** Relation \cite{125} for Abrikosov lattices was isolated in \cite{17}, where it played an important role. This condition is well known in algebraic geometry and number theory (see e.g. \cite{23}). However, there the associated vector potential (connection on the corresponding principal bundle) $A$ is not considered there.

### 1.8 Possible extensions

The next step would be to extend the results to more general perturbations. Firstly, one would like to remove the restrictive condition \cite{119}. This condition rules out a gapless branch of the spectrum of $L_\omega$ starting at 0. Indeed, since the solution $u_\omega$ of \cite{111} breaks the translational invariance, the operator $L_\omega$ has the translational zero mode

$$S_{h'} = ((h' \cdot \nabla A_{\omega})(\nabla A_{\omega})(J_{h'}), \quad h' \in \mathbb{R}^2, \quad (1.29)$$

i.e., $L_\omega S_{h'} = 0$ (see Subsection \ref{3.1} and Supplement II). Since $S_{h'}$ is only bounded and not $L^2$, we have that $0 \in \sigma_{gs}(L_\omega)$, Moreover, $e^{ik \cdot x} S_{h'}$, $k \in \Omega_\omega$, for $|k|$ small, are almost zero (generalized or Bloch) modes of $L_\omega$: $e^{-ik \cdot x} L_\omega e^{ik \cdot x} = L_\omega + O(|k|)$ and therefore $L_\omega e^{ik \cdot x} S_{h'} = O(|k|) e^{ik \cdot x} S_{h'}$. Hence there is a positive
(gapless) branch of the spectrum of $L_\omega$ on $\mathcal{K}$, starting at $0$, corresponding to translations of the lattice (see Supplement II).

(Since the solution $u_\omega$ breaks also the rotational invariance the operator $L_\omega$ has the rotational zero mode, $R_{\omega'}$, i.e., $L_\omega R_{\omega'} = 0$, - see Subsection 3.1 - but this mode is growing at infinity.)

Though removing condition (1.19) would be technically cumbersome, we expect this would not change the result above.

Secondly, one would like to consider non-local perturbations, say, perturbations of the form $T_g u_\omega + v - u_\omega$, with $v \in L^\infty(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2)$, where $G$ is the full symmetry group

\[ G = H^2(\mathbb{R}^2; \mathbb{R}) \times \mathbb{R}^2 \times SO(2) \]  

and $T_g = T_{\gamma} g_T h_T^\text{trans} T_{\rho}^\text{rot}$ is the action of $G$ on pairs $u = (\Psi, A)$. (1.30) is a semi-direct product, with elements $g = (\gamma, h, \rho) \in G$, and the composition law given by $gg' = (\gamma + T_{\gamma} g_T h_T^\text{trans} T_{\psi} \gamma', \rho^{-1}(\rho^{-1} h + h'), \rho \rho')$. For such perturbations we would have to generalize the notion of asymptotic stability by replacing $T_{\gamma}$ by $T_g$.

Specifically,

Definition 1.6. We say that the Abrikosov lattice $u_\omega$ is asymptotically stable under finite-energy perturbations if there is $\delta > 0$ s.t. for any initial condition $u_0$, whose $H^1$- distance to the infinite-dimensional manifold $\mathcal{M} = \{ T_g u_\omega : g \in G \}$ is $\leq \delta$, the solution $u(t)$ of (1.2) satisfies $\| T_{g(t)}^{-1} u(t) - u_\omega \|_{L^\infty} \to 0$ as $t \to \infty$, for some path, $g(t)$, in $G$.

1.9 Organization of the paper

We prove Proposition 1.4 and Theorem 1.5 in Section 2 and Appendix A. Theorem 1.3 is proven in Section 3, with technical results proven in Section 4 and Appendices B - E. Numerical investigation of functions $\gamma(\tau)$ and $\gamma(\tau)$ quoted in Subsections 1.1 and 1.6 is done in Appendix B by Daniel Ginsberg.

Acknowledgement. The first author is grateful to Jürg Fröhlich, Dan Ginsberg, Gian Michele Graf, Lev Kapitanski, Yuri Ovchinnikov, Peter Sarnak, and Tom Spencer for useful discussions.

2 Proof of Proposition 1.4 and Theorem 1.5

2.1 Functions $\phi_k(x)$, $\theta_q(z, \tau)$ and all that

In this subsection we investigate the functions $\phi_k(x)$ solving (1.6), used in the definition (1.5) of $\gamma(\tau)$. It is convenient to define $\phi_k(x)$, for $\tau$ in the entire Poincaré half-plane $\Pi^+$, rather than just for the fundamental domain $\Pi^+ / SL(2, \mathbb{Z})$. To this end, we identify the dual group, $\mathcal{L}_\tau$, with a fundamental cell, $\Omega^*_\tau$, of the dual lattice $\mathcal{L}^*_\tau$, and the torus $T_\tau := \mathbb{R}^2 / \mathcal{L}_\tau$, with a fundamental cell, $\Omega_\tau$, of $\mathcal{L}_\tau$, and consider equation (1.6), with $\chi(s) = e^{i k \cdot s}$, which becomes

\[ (-\Delta_\rho - 1) \phi = 0, \quad \phi(x + s) = e^{i \frac{1}{2} s \cdot J_s} e^{i k \cdot s} \phi(x), \quad \forall s \in \mathcal{L}_\tau, \quad k \in \Omega^*_\tau, \]  

(2.1)

with $\phi^0(x)$ the same as before. We denote the solution of this equation by $\phi_k(x)$, $k \in \Omega^*_\tau$. Recall that we take $\mathcal{L}_\tau = \sqrt{2 \pi \Im \tau} (\mathbb{Z} + \mathbb{Z} \tau)$. An explicit form of the functions $\phi_k(x)$, $k \in \Omega^*_\tau$, is described in the following

Proposition 2.1. The functions $\phi_k$, $k \in \Omega^*_\tau$, (normalized as $\langle |\phi_k|^2 \rangle_{\Omega^*_\tau} = 1$) satisfy the relations (2.1) if and only if they are given by

\[ \phi_k(x) = c_0 e^{\frac{2 \pi i}{\Im \tau} (z - x)^2} e^{i k \cdot x} \theta_q(z, \tau), \quad \text{with} \quad x_1 + i x_2 = \sqrt{\frac{2 \pi}{\Im \tau}} z \quad \text{and} \quad k = \sqrt{\frac{2 \pi}{\Im \tau}} i q, \]  

(2.2)

where $c_0$ is such that $\langle |\phi_k|^2 \rangle_{\Omega^*_\tau} = 1$, and $\theta_q$ are entire functions (i.e. they solve $\bar{\partial} \theta_q = 0$) and satisfy the periodicity conditions

\[ \theta_q(z + 1, \tau) = e^{-2 \pi i a} \theta_q(z, \tau), \]  

(2.3)

\[ \theta_q(z + \tau, \tau) = e^{-2 \pi i b} e^{-i \pi \tau - 2 \pi i z} \theta_q(z, \tau), \]  

(2.4)
where \( a, b \) are real numbers defied by \( q = -a\tau + b \). Consequently,

\[
\theta_q(z, \tau) := e^{\pi i(a^2 \tau - 2ab - 2az)} \sum_{m=-\infty}^{\infty} e^{2\pi iqm} e^{\pi im^2 \tau} e^{2\pi imz}.
\] (2.5)

(In this section, \( b \) stands for a component of \( q \), and the average magnetic flux per cell.)

**Proof.** We prove only that if the functions \( \phi_k \) satisfy (2.1), then they are given by (2.2) - (2.5). Standard methods again show that the operator \(-\Delta_{\phi} \) on \( L^2(\Omega; \mathbb{C}) \) with boundary conditions in (2.1) is positive self-adjoint with discrete spectrum. To find its eigenvalues, we define the harmonic oscillator annihilation and creation operators, \( c \) and \( c^* \), with

\[
c = -\partial_{\alpha^0} = \partial_x + i\partial_{x_2} + \frac{1}{2}(x_1 + ix_2).
\] (2.6)

These operators satisfy the relations

1. \([c, c^*] = 2\);
2. \(-\Delta_{\phi} - 1 = c^* c\).

The representation \(-\Delta_{\phi} - 1 = c^* c\) implies that \( \text{Null}(-\Delta_{\phi} - 1) = \text{Null} c \) and so we study the latter. Thus we consider the equation

\[
c\phi_k = 0.
\] (2.7)

The relations (2.2) and (2.7) imply the Riemann-Cauchy equation \( \partial \theta_q(z, \tau) = 0 \), i.e. \( \theta_q(z, \tau) \) are entire functions.

Next, it is straightforward to verify (see Corollary I.3 of Supplement I) that the periodicity relation in (2.1) implies that the functions

\[
\varphi_q(z) = \phi_k(x), \quad x_1 + ix_2 = \sqrt{\frac{2\pi}{\text{Im} \tau}} z \quad \text{and} \quad k = \sqrt{\frac{2\pi}{\text{Im} \tau}} iq,
\] (2.8)

satisfy the periodicity relations

\[
\varphi(z + s) = e^{\frac{k}{\text{Im} \tau}(\text{Im}(sz) + 2\text{Im}(sq))} \varphi(x), \quad \forall s \in \mathcal{L}.
\] (2.9)

Hence by Corollary I.3 of Supplement I, \( \theta_q(z, \tau) \) satisfy the periodicity conditions (2.3) - (2.4). This first statement and Lemma I.2 of Supplement I show that \( \theta_q(z, \tau) \) is given by (2.5).

**Proof of Theorem 1.5.** Now, let \( \Omega_\tau := \sqrt{\frac{\text{Im} \tau}{2\pi}} \Omega_\tau \) and rewrite the functions \( \gamma_k(\tau) = \gamma_k(\tau), \chi(s) = e^{ikx}, \) defined in (1.3), in terms of the functions \( \varphi_q \), introduced in (2.8):

\[
\gamma_k(\tau) := 2(|\varphi_0|^2 |\varphi_q|^2)_{\Omega_\tau} + |\langle \varphi_0, \varphi q \rangle_{\Omega_\tau} - \langle |\varphi_0|^2 \rangle_{\Omega_\tau}|.
\] (2.10)

Theorem 1.5 follows from (2.10), Proposition 2.1 and Proposition A.1 of Appendix A (the latter computes the integrals in (2.10)), and fact that \( |\Omega_\tau| = \text{Im} \tau \).

**Corollary 2.2.** For each \( k \in \Omega_\tau^* \), the solution \( \phi_k(x) = \phi_k^G(x) \) of the problem (2.1) (displaying the dependence of \( \phi_k \) on the lattice \( \mathcal{L} \)) is unique and satisfies

\[
\phi_k^G(gx) = \phi_k^{-1}^G(x), \quad \text{for any} \quad g \in O(2).
\] (2.11)

To prove (2.11), we observe that, as can be easily verified, the function \( \psi_k(x) = \phi_k^{\mathcal{L}}(gx) \) satisfies (2.1). Hence the uniqueness for (2.1) gives (2.11). Since \( g : x \to -x \) and \( g : x \to \sigma x \), where \( \sigma(x_1, x_2) = (-x_1, x_2) \), leave the lattice \( \mathcal{L} \) invariant, the equation (2.11) implies

\[
\phi_k(-x) = \phi_{-k}(x), \quad \text{and} \quad \phi_k(\sigma x) = \phi_{\sigma k}(x).
\] (2.12)
2.2 Symmetries of $\gamma_k(\tau)$

It is convenient to think of $\gamma_k(\tau)$, as functions on the entire Poincaré half-plane $\Pi^+$, rather than just the fundamental domain $\Pi^+/\SL(2,\mathbb{Z})$, appearing in the number theory, while

$\phi$ functions are discussed briefly in Supplement I.

The relation (2.13) implies that $\gamma(\tau) = e^{\pi i/2} \gamma(\tau)$, appearing in the number theory, while

$\theta$ functions are discussed briefly in Supplement I. 2) Proposition 2.1 implies that the products $|\phi_k|^2$, $\delta\phi_k\bar{\phi}_{-k}$ are periodic on $\Omega_\tau$.

3) The equations (2.2) and (2.5) imply (by an inspection) (2.12).

Proof. We will use that, due to (2.14), $\gamma(tk) = \gamma_k(\tau)$, for any $t \in \mathbb{Z}$. Differentiating this relation w.r. to $Re$ and $Im$ at $k \in \frac{1}{2} \mathbb{Z}^*_+$ and using that the points $k \in \frac{1}{2} \mathbb{Z}^*_+$ are fixed points under the maps $k \rightarrow t - k$, where $t \in \mathbb{Z}^*_+$, we find $\partial_{Re} k \gamma_k(\tau) = 0$ and $\partial_{Im} k \gamma_k(\tau) = 0$ for $k \in \frac{1}{2} \mathbb{Z}^*_+$.

2.3 Proof of Proposition 1.4

The relation (2.13) implies that $\gamma(\tau)$, $\tau \in \Pi^+$, is a modular function, symmetric w.r.t. to the imaginary axis, specifically,

$$\gamma(\tau + 1) = \gamma(\tau), \quad \gamma(-\tau^{-1}) = \gamma(\tau), \quad \gamma(-\bar{\tau}) = \gamma(\tau).$$  

(2.16)

To prove the second statement, we will use that the points $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$ are fixed points under the maps $\tau \rightarrow -\bar{\tau}$, $\tau \rightarrow -\tau^{-1}$ and $\tau \rightarrow 1 - \bar{\tau}$, $\tau \rightarrow 1 - \tau^{-1}$, respectively. By the first and third
relations in (2.16), we have that $\gamma(n - \tau) = \gamma(\tau)$, for any integer $n$. Remembering the definition $\tau = \tau_1 + i\tau_2$, differentiating the relation $\gamma(n - \tau) = \gamma(\tau)$ w.r. to $\tau_1$, and using that the points $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$ are fixed points under the maps $\tau \to -\tau$ and $\tau \to 1 - \tau$, respectively, we find $\partial_{\tau_1} \gamma(\tau) = 0$ for $\tau = e^{i\pi/2}$ ($n = 0$) and for $\tau = e^{i\pi/3}$ ($n = 1$).

Next, we find the derivatives w.r. to $\tau_2$. We consider the function $\gamma(\tau)$ as a function of two variables, $\gamma(\tau_1, \tau_2)$. Then the relation $\gamma(\tau) = \gamma(n - \tau^{-1})$, where $n$ is an integer, which follows from the first two relations in (2.16), can be rewritten as $\gamma(\tau_1, \tau_2) = \gamma(n - \frac{\tau_1}{|\tau|^2}, \frac{\tau_2}{|\tau|^2})$. Differentiating the latter relation w.r. to $\tau_2$, we find

$$(\partial_{\tau_2} \gamma)(\tau_1, \tau_2) = 2 \frac{\tau_1 \tau_2}{|\tau|^4} (\partial_{\tau_1} \gamma)(n - \frac{\tau_1}{|\tau|^2}, \frac{\tau_2}{|\tau|^2}) + 2 \frac{\tau_1^2 - \tau_2^2}{|\tau|^4} (\partial_{\tau_2} \gamma)(n - \frac{\tau_1}{|\tau|^2}, \frac{\tau_2}{|\tau|^2}).$$

(2.17)

Since $\partial_{\tau_1} \gamma(\tau) = 0$ for $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$ and since the points $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$ are fixed points under the maps $\tau \to -\tau^{-1}$ and $\tau \to 1 - \tau^{-1}$, respectively, this gives $(\partial_{\tau_2} \gamma)(0, 1) = 0$ for $\tau = e^{i\pi/2}$ ($n = 0$) and for $\tau = e^{i\pi/3}$ ($n = 1$).

3 Proof of Theorem 1.3

Recall that we work in the gauge $\Phi(x, t) = 0$ (see (1.3)). As was mentioned in Subsection 1.7, we look for solutions of (1.2), satisfying condition (1.24), or explicitly as, for $s \in \mathcal{L}$,

$$
\begin{align*}
\Psi(x + s) &= e^{ig_s(x)} \Psi(x), \\
A(x + s) &= A(x) + \nabla g_s(x),
\end{align*}
$$

(3.1)

where $g_s$ satisfies (1.25). By property (c) of $g_s$ given in Subsection 1.7 it can be taken to be

$$g_s(x) = \frac{b}{2} s \wedge x + c_s,$$

(3.2)

where $b$ is the average magnetic flux, $b = \frac{1}{4\pi} \int_{\Omega} \text{curl} A$ (note that $bs \wedge t \in 2\pi\mathbb{Z}$), and the $c_s$ satisfy

$$c_{t+s} - c_s - c_t - \frac{1}{2} bs \wedge t \in 2\pi\mathbb{Z}.$$  

(3.3)

Let $(\nu_1, \nu_2)$ be a basis in $\mathcal{L}$. The relation (3.3) and the property $bs \wedge t \in 2\pi\mathbb{Z}$ imply that

$$c_s = mc_{\nu_1} + nc_{\nu_1} + b\pi mn,$$

(3.4)

modulo $2\pi\mathbb{Z}$, where $s = mv_1 + nv_2 \in \mathcal{L}$. From now on, we assume that $g_s$ is given by (3.2) with $c_s$ satisfying (3.3) and is given by (3.4). In what follows we take the particular $c_s$ determined by $c_{\nu_1} = 0$, $c_{\nu_2} = 0$ and drop it from the notation.

3.1 Zero modes

Since the solution $u_\omega$ of (1.1) breaks the gauge invariance, we have

**Lemma 3.1.** The operator $L_\omega$ has the gauge zero modes, $L_\omega G_{\gamma'} = 0$, where

$$G_{\gamma'} := (i\gamma' \Psi_\omega, \nabla \gamma').$$

(3.5)

**Proof.** Substitute $T^{\text{gauge}}_{\gamma'} u_\omega$ into the Ginzburg-Landau equations (1.1) to obtain $\mathcal{E}'(T^{\text{gauge}}_{\gamma'} u_\omega) = 0$, where $\mathcal{E}'(u)$ is the r.h.s. of (1.1), given explicitly in (1.9). Then we differentiate the result w.r.to $\gamma$ and use the definition of $L_\omega$ in (3.20) to obtain $L_\omega G_{\gamma'} = 0$. \qed
To solve this equation, we use the Implicit Function Theorem. From the definition, it is clear that $f$, $g$, $T$, $V$, and $W$ are tangent vectors to the group $H^2(\mathbb{R}^2, \mathbb{R})$, be the generator (Gâteaux derivatives) of the transformation (1.10), defined, for $u = (\Psi, A)$, as

$$V_{\gamma'}u := (i\gamma'\Psi, \nabla\gamma').$$

The tangent space, $T_{u_0}\mathcal{M}$ at $u_0 \in \mathcal{M}$, is spanned by the gauge zero modes $G_{\gamma'}$, given by $G_{\gamma'} = V_{\gamma'}u_0$, or, explicitly, in (3.5).

Below we consider orthogonal complements the tangent spaces, $T_{u_0}\mathcal{M}$, where the orthogonality is understood in the sense of the inner product (1.18). We define, for $\delta > 0$, its tubular neighbourhood,

$$U_\delta = \{T_{\gamma'}^{\text{gauge}}(u_0 + v) : \gamma = H^1(\mathbb{R}^2, \mathbb{R}), v \in H^1_{\text{cov}}, \left\|v\right\|_{H^1} = \delta\}.$$  

and prove the following decomposition for $u$ close to the manifold $\mathcal{M}$.

**Proposition 3.2.** There exist $\delta_0 > 0$ (depending on $\epsilon$) and a map $\eta : U_\delta \to H^2(\mathbb{R}^2, \mathbb{R})$ such that $v := (T_{\eta(u)})^{-1}u - u_0 \perp G_{\gamma'}$ for all $\gamma' \in H^1(\mathbb{R}^2, \mathbb{R})$.

**Proof.** We omit the superindex "gauge" in $T_{\gamma'}^{\text{gauge}}$. Our goal is to solve the equation $\langle G_{\gamma'}, u - T_{\gamma'}^{\text{sym}}u_0 \rangle_{L^2} = 0$, $\forall \gamma' \in H^1(\mathbb{R}^2, \mathbb{R})$, for $\gamma$ in terms of $u$. Define the affine space $X = u_0 + H^1_{\text{cov}}$ and let $G_{\gamma'} := G_{\gamma'}$. Then, $\langle G_{\gamma'}, u - T_{\gamma'}^{\text{sym}}u_0 \rangle_{L^2} = \langle \gamma', \Gamma^*(u - T_{\gamma'}^{\text{sym}}u_0) \rangle_{L^2} = 0$, $\forall \gamma' \in H^1(\mathbb{R}^2, \mathbb{R})$, is equivalent to $\Gamma^*T_{\gamma'}(u - T_{\gamma'}u_0) = 0$. Hence our problem can be reformulated as solving the equation $f(\gamma, u) = 0$ for $\gamma$ in terms of $u$, where the map $f$ is given by

$$f(\gamma, u) = \Gamma^*T_{\gamma'}(u - T_{\gamma'}u_0).$$

To solve this equation, we use the Implicit Function Theorem. From the definition, it is clear that $f : H^2(\mathbb{R}^2, \mathbb{R}) \times X \to L^2(\mathbb{R}^2, \mathbb{R})$, is a $C^1$ map and $f(\gamma, T_{\gamma}u_0) = 0$. Finally, we calculate the linearized map:

$$d_uf(\gamma, u)|_{u=T_{\gamma}u_0} = \Gamma^*T_{\gamma'}d_uf_{T_{\gamma}u_0}\gamma'$$

and $d_uf_{T_{\gamma}u_0}\gamma' = T_{\gamma}g_{T_{\gamma}u_0} = T_{\gamma}\Gamma\gamma'$, where $\gamma' \in H^1(\mathbb{R}^2, \mathbb{R})$, which gives $d_uf(\gamma, u)|_{u=T_{\gamma}u_0} = \Gamma^*\Gamma\gamma'$. Since $\Gamma*v = -i\Psi\omega\xi - \text{div} \alpha$, we have $\Gamma^*\Gamma = -\Delta + |\Psi\omega|^2$. The last two relations give

$$d_uf(\gamma, u)|_{u=T_{\gamma}u_0} = \Delta - |\Psi\omega|^2.$$

For $|\Psi\omega|^2$ periodic, $-\Delta + |\Psi\omega|^2$ is self-adjoint and, as easy to see using uncertainty principle near zeros of $|\Psi\omega|^2$, is strictly positive, $-\Delta + |\Psi\omega|^2 \geq \delta > 0$, with $\delta$ depending on $\epsilon$. Therefore it is invertible. The Implicit Function Theorem then gives us a neighbourhood $V$ of $u_0$ in $X$ and a neighbourhood $W$ of $0$ in $G$ and a map $H : V \to W$ such that $f(\gamma, u) = 0$ for $(\gamma, u) \in W \times V$ if and only if $\gamma = H(u)$. We can always assume that $V$ is a ball of radius $\delta_0$.

We can now define the map $\eta$ on $U_\delta$ for $\delta < \delta_0$ as follows. Given $u \in U_\delta$, choose $\gamma$ such that $u = T_{\gamma}(u_0 + v)$ with $u_0 + v \in V$. We define $\eta(u) = \gamma H(u_0 + v)$. To show that $\eta$ is well defined, we first show that if $\eta$ is sufficiently close to the identity, then $H(T_{\eta}u) = \gamma H(u)$. To begin with, we note for all $\gamma$, $T_{\gamma}(V) \subset V$. One can easily verify, by the definition of $f$, that $f(\gamma, T_{\gamma}u) = f(\delta, u)$. Indeed, we have $f(\gamma, T_{\gamma}u) = \Gamma^*T_{\gamma}\delta(T_{\gamma}u - T_{\gamma}\delta u_0) = \Gamma^*T_{\gamma}(u - T_{\gamma}\delta u_0) = f(\delta, u)$. Hence $f(\gamma H(u), T_{\gamma}u) = f(H(u), u) = 0$, and
therefore by the uniqueness of $H$, it suffices to show that $\gamma H(u) \in W$, but this can easily be done by taking $V$ to be smaller if necessary.

Suppose now that we have also $u = T_\gamma (u_\omega + v')$. Then $T_{\gamma^{-1}}(u_\omega + v') = u_\omega + v$. Therefore, by the relation $H(T_\gamma u) = \gamma H(u)$, we have

$$\gamma H(u_\omega + v) = \gamma H(T_{\gamma^{-1}}(u_\omega + v')) = \gamma \gamma^{-1} \gamma' H(u_\omega + v') = \gamma' H(u_\omega + v'),$$

so $\eta$ is well-defined and the proof is complete.

**Remark.** It is straightforward to show that an element $v = (\xi, \alpha) \in H^1_{\text{cov}}$ is orthogonal to all $G_\gamma$ if and only if

$$\text{Im}(\tilde{\Psi}_t \xi) - \text{div} \alpha = 0. \quad (3.10)$$

### 3.3 GES equations in the moving frame

**Proposition 3.3.** If $(\Psi, A, \Phi)$ is a solution to (1.2), $u = (\Psi, A)$, and $\tilde{u} = T_{\gamma^{-1}} u$, then $\tilde{u}$ satisfies the equation

$$\partial_t + \Phi + \tilde{u} = G(\tilde{u}), \quad (3.11)$$

where, according the definition of $\partial_t, \partial_t \Phi + \gamma = \partial_t + \Phi + V_\gamma$.

**Proof.** We write the equations (1.2) in the form $\partial_t u = G(u)$, where $\partial_t u := (\partial_t \Psi, \partial_t A + \nabla \Phi)$, for $u = (\Psi, A)$, and the map $G(u)$ is given by the r.h.s. of (1.9). The map $G(u)$ is covariant under gauge transformations (1.10), in the sense that

$$G(T_{\gamma} \Psi, A) = T_{\gamma} G(u), \quad (3.12)$$

for every $\gamma$, where $T_{\gamma} \Psi, A$ is applied to the pairs $u = (\Psi, A)$ rather than to the triples $(\Psi, A, \Phi)$ and

$$T_{\gamma} G : (\Psi, A) \mapsto (e^{i \gamma} \Psi, A).$$

(Clearly, $G$ is also covariant under translations and rotations, (1.11) - (1.12).) We see also that

$$\partial_t T_{\gamma} \Psi, A = T_{\gamma} \Phi + \partial_t \Psi + g_\gamma u. \quad (3.13)$$

Now, use (3.12), (3.13) and the definition $\tilde{u} = T_{\gamma^{-1}} u$, introduced earlier, to obtain the equation (3.11). \qed

We reformulate the result of Proposition 3.2 as

$$u = T_\gamma (u_\omega + v), \quad \text{with} \quad v \perp G_{\gamma'} \forall \gamma', \quad \text{and some } \gamma. \quad (3.14)$$

Plugging $\tilde{u} := T_{\gamma^{-1}} u = u_\omega + v$ into (3.11), expanding $G(u_\omega + v)$ in $v$ and using (3.8) gives

**Proposition 3.4.** If $(\Psi, A, \Phi)$ is a solution to (1.2) and $u = (\Psi, A)$, then $\eta(u) = -\gamma$ and $v$, defined in Proposition 3.2, satisfy the equation

$$\partial_t v = -L_\omega v + V_\gamma v + N_\omega(v) + G_\gamma, \quad (3.15)$$

where $L_\omega = E''(u_\omega)$ is the Gâteaux derivative of the gradient map, $E'(u)$ (see (1.9)) at $u_\omega$ (see (3.20)), $g_\gamma$ is the map defined in (3.8) and $N_\omega(v)$ is the $v-$ nonlinearity,

$$N_\omega(v) = E'(u_\omega + v) - E'(u_\omega) - E''(u_\omega)v. \quad (3.16)$$

The terms $L_\omega$ and $N_\omega(v)$ are given explicitly by expressions (C.1) and (C.2) of Appendix C.

Eq (3.15) is for the unknowns $v$ and $\gamma$ and is supplemented by the conditions $v \perp G_{\gamma'} \forall \gamma'$. Projecting it onto the tangent vectors, $G_\gamma \forall \gamma'$ (see (3.5)), we find the equation for $\gamma$:
\textbf{Lemma 3.5.} Let \( N_\omega(v) = (N_\xi(v), N_\alpha(v)) \). We have the following equation for \( \gamma \)

\[
(-2\Delta + |\Psi_\omega|^2 + \text{Re}(\overline{\Psi}_\omega \xi))\gamma = -\text{Im}(\overline{\Psi}_\omega N_\xi(v)) + \text{div} \, N_\alpha(v).
\]

\( (3.17) \)

\textbf{Proof.} Multiplying \( (3.15) \) scalarly by \( G_x = (i\chi \Psi_\omega, \nabla \chi) \) and using \( \langle G_x, v \rangle = 0 \), \( L_\omega G_x = 0 \), we find

\[
(G_x, V_\gamma v + N_\omega(v) + G_\gamma) = 0.
\]

\( (3.18) \)

Now, remembering the definitions \( (3.8), (3.16) \), and the gauge condition \( \Phi = 0 \), and using that \( \langle G_x, g_\gamma v \rangle = \langle \chi, (-\Delta + \text{Re}(\overline{\Psi}_\omega \xi))\gamma \rangle \) and \( \langle G_x, N_\omega(v) \rangle = \langle \chi, \text{Im}(\overline{\Psi}_\omega N_\xi(v)) - \text{div} \, N_\alpha(v) \rangle \) and

\[
(G_x, G_\gamma) = \langle \chi, (\Delta + |\Psi_\omega|^2)\gamma \rangle,
\]

we see that \( (3.18) \) can be rewritten in the form \( \langle \chi, f \rangle = 0 \), where \( f := (-2\Delta + |\Psi_\omega|^2 + \text{Re}(\overline{\Psi}_\omega \xi))\gamma + \text{Im}(\overline{\Psi}_\omega N_\xi(v)) - \text{div} \, N_\alpha(v) \), which, since \( \chi \in H^2 \) is arbitrary and \( \text{div} \, A_\omega = 0 \), implies the equation \( (3.17) \). \( \square \)

\textbf{Remark.} The projection operator onto the space spanned by the gauge modes is given by \( \Pi := -\Gamma h^{-1} \Gamma^* \), where \( \Gamma \gamma' := G_{\gamma'} \) and \( \Gamma^* v = -\text{Re}(i\overline{\Psi}_\omega \xi) - \text{div} \, \alpha \) (see the proof of Proposition \( 3.2 \)) and \( h := -\Delta + |\Psi_\omega|^2 \). The latter operators satisfy \( \Gamma^* \Gamma = h \) and \( \Gamma G_x = G_{h \chi} \) and therefore \( \Pi G_x = G_x \).

\subsection*{3.4 Hessian}

The chief tool in the proof of the stability result is the analysis of the linearization of the map on the r.h.s. of \( (1.2) \) - or the Hessian, \( L_\omega := \mathcal{E}''(u_\omega) \) of the energy functional \( \mathcal{E}(u) \) at \( u_\omega \), which, recall, is a real-linear operator defined as

\[
\mathcal{E}''(u) = d\mathcal{E}'(u),
\]

\( (3.20) \)

where \( d \) and \( \cdot \) are the Gâteaux derivative and \( L^2 \)-gradient map (for \( \mathcal{E}' \) see \( (1.9) \)), and explicitly given by \( (C.1) \) of Appendix \( C \). Its quadratic form can be defined directly by the equation

\[
\langle v, \mathcal{E}''(u)w \rangle_{L^2} = \frac{\partial^2}{\partial \delta \partial \delta} \mathcal{E}(u + \delta v + \epsilon w)|_{\delta = \epsilon = 0},
\]

for \( u, v, w \), s.t. \( \mathcal{E}(u + \delta v + \epsilon w) \) is finite for all \( \delta, \epsilon \geq 0 \), and then extended to a larger class of functions.

The main theorem that we need concerns the positivity of its quadratic form. Observe that by Lemma \( 3.1 \), the space \( T_{u_\omega} \mathcal{M}_\omega \) is spanned by the gauge zero modes, \( T_{u_\omega} \mathcal{M}_\omega = \{ G_{\gamma'}; \gamma' \in H^1(\mathbb{R}^2, \mathbb{R}) \} \). We denote by \( H_1^1 \) the set of all vectors \( v \) from the \( L^2 \)-orthogonal complement of \( T_{u_\omega} \mathcal{M}_\omega \) in \( H^1_{\text{conv}} \), satisfying \( T_{\text{refl}} v = v \). We define the perturbation parameter

\[
\epsilon = \sqrt{\frac{\kappa^2 - b}{\kappa^2(2\kappa^2 - 1)\beta(\tau) + 1}}.
\]

\( (3.21) \)

The term \( (2\kappa^2 - 1)\beta(\tau) + 1 \) in the denominator of \( (3.21) \) is necessary in order to have a positive expression under the square root and to regulate the size of the perturbation domain.

\textbf{Theorem 3.6.} Suppose that \( b \) is sufficiently close to \( \kappa^2 \) in the sense of \( (1.21) \). Then

\[
\inf_{v \in H_1^1} \langle v, L_\omega v \rangle_{L^2}/\|v\|^2_{L^2} = \mu(\omega, \kappa),
\]

\( (3.22) \)

with \( \mu(\omega, \kappa) \) of the form

\[
\mu(\omega, \kappa) = b(\kappa^2 - \frac{1}{2})\gamma(\tau)\epsilon^2 + O(\epsilon^3),
\]

\( (3.23) \)

with \( \gamma(\tau) \) and \( \epsilon \) defined in \( (1.5) \) and \( (3.21) \).
Clearly, for $\epsilon$ sufficiently small, $\mu(\omega, \kappa)$ is positive/negative if and only if $(\kappa^2 - \frac{1}{2}) \gamma(\tau)$ is positive/negative. Proof of Theorem 3.6 is given in Section 4. We now upgrade the lower bound (3.22) to the one involving the $H^1_{\text{cov}}$ norm on the r.h.s..

**Corollary 3.7.** Assume $\mu(\omega, \kappa) > 0$. Then there is $c > 0$ s.t. for all $v \in H^1_\perp$,

$$
\langle v, L_\omega v \rangle_{L^2} \geq c \mu(\omega, \kappa) \| v \|_{H^1}^2.
$$

(3.24)

**Proof.** To begin with we show that for $v \in H^1_\perp$,

$$
\langle v, L_\omega v \rangle_{L^2} \geq \| v \|_{H^1}^2 - C \| v \|_{L^2}^2,
$$

(3.25)

for some positive constant $C$. Indeed, we write $v = (\xi, \alpha)$. Integration by parts and (3.10) gives

$$
\langle v, L_\omega v \rangle_{L^2} = \int |\nabla_\omega \xi|^2 + \kappa^2 (2|\Psi_\omega|^2 - 1)|\xi|^2 + \kappa^2 \Re(\Psi_\omega \xi^2)
$$

$$
+ (\text{curl } \alpha)^2 + |\Psi_\omega|^2 |\alpha|^2 + 2(\text{div } \alpha)^2 - 4\alpha \cdot \text{Im}(\xi \nabla_\omega \Psi_\omega),
$$

and therefore

$$
\langle v, L_\omega v \rangle_{L^2} \geq \| \nabla_\omega \xi \|_{L^2}^2 + \| \nabla \alpha \|_{L^2}^2 - C \| \alpha \|_{L^2}^2 - C \| \xi \|_{L^2}^2
$$

and (3.25) now follows.

Now let $\delta \in [0,1]$ be arbitrary and let $\mu = \mu(\omega, \kappa)$. We combine (3.25) with the bound

$$
\langle v, L_\omega v \rangle_{L^2} \geq \mu \| v \|_{L^2}^2,
$$

(3.26)

which follows from (3.22), to obtain (here we omit the argument in $\mu(\omega, \kappa)$)

$$
\langle v, L_\omega v \rangle_{L^2} = (1 - \delta) \langle v, L_\omega v \rangle_{L^2} + \delta \langle v, L_\omega v \rangle_{L^2}
$$

$$
\geq (1 - \delta) \mu \| v \|_{L^2}^2 + \delta \left( \frac{1}{2} \| v \|_{H^1}^2 - C \| v \|_{L^2}^2 \right)
$$

$$
= ((1 - \delta) \mu - \delta C) \| v \|_{L^2}^2 + \frac{\delta}{2} \| v \|_{H^1}^2.
$$

(3.24) now follows by choosing $\delta = \frac{\mu}{\frac{1}{2} \mu + \ov{c} \ov{c}}$.

We also mention the estimate

$$
\| \langle v, L_\omega v \rangle_{L^2} \| \lesssim \| v \|_{H^1}^2,
$$

(3.27)

which is obtained by using the explicit form (C.1) of $L_\omega$ given in Appendix C, integrating by parts and using the fact that $\Psi_\omega$, as a smooth gauge-periodic function, is bounded together with the Cauchy-Schwarz inequality.

### 3.5 Asymptotic stability

We now consider the Gorkov-Eliashberg-Schmidt equations (1.2) (in the gauge $\Phi(x, t) = 0$) with an initial condition $(\Psi_0, A_0) \in U_\delta_0$, with $\delta_0 < \delta_*$, where $\delta_*$ is given in Proposition 3.2, satisfying (1.19). Then, by the local existence there $T > 0$ s.t. (1.2) has a solution, $u(t) = (\Psi(t), A(t)) \in C^1([0,T]; U_\delta)$ for some $\delta < \delta_*$, and, by the uniqueness, this solution satisfies

$$
T^{\text{ref}} u(t) = u(t).
$$

(3.28)

Since $u(t) = (\Psi(t), A(t)) \in C^1([0,T]; U_\delta)$ for some $\delta < \delta_*$, by Proposition 3.2 and (3.28), there are $\gamma : [0,T] \to H^1(\mathbb{R}^2, \mathbb{R})$ and $v(t) \in H^1_{\text{cov}}$ s.t. (3.14) holds. We define the Lyapunov functional

$$
\Lambda_\omega(v) = \frac{1}{2} \langle v, L_\omega v \rangle_{L^2} + R_\omega(v),
$$

(3.29)
where $R_\omega(v)$ satisfies $R'_\omega(v) = N_\omega(v)$, where $N_\omega(v)$ is defined in (3.16), and is given explicitly by

$$R_\omega(v) = \int (|\alpha|^2 + \kappa^2|\xi|^2) \text{Re}(\bar{\Psi}_\omega \xi) - \alpha \cdot \text{Im}(\bar{\xi} \nabla A_\omega \xi) + \frac{1}{2} (|\alpha|^2 + \frac{\kappa^2}{2} |\xi|^2)|\xi|^2. \quad (3.30)$$

Lemma 3.8. For all $v \in H^1_{\text{cov}}$, we have

$$\lim_{Q \to \mathbb{R}^2} (\mathcal{E}_Q(u_\omega + v) - \mathcal{E}_Q(u_\omega)) = \Lambda_\omega(v). \quad (3.31)$$

Moreover, $R_\omega(v)$ satisfies $R'_\omega(v) = N_\omega(v)$, where $N_\omega(v)$ is defined in (3.16).

Proof. We first consider smooth $v$ with compact support. Choose any $Q \subset \mathbb{R}^2$ that is bounded and contains the support of $v$. Using definition (1.1), we expand $\mathcal{E}_Q(u_\omega + v)$ in $v$, collecting terms that are linear in $v$,

$$\frac{1}{2} \int_Q 2 \text{Re}(\nabla_\omega \xi \cdot \nabla_\omega \Psi_\omega + i\alpha \cdot \bar{\Psi}_\omega \nabla_\omega \Psi_\omega + 2(\text{curl} A_\omega)(\text{curl} \alpha) - 2\kappa^2(1 - |\Psi_\omega|^2) \text{Re}(\bar{\xi} \Psi_\omega)), \quad (3.29)$$

and integrate by parts (boundary terms vanish due to $v$) to see that they are equal to

$$\text{Re} \int_Q \bar{\xi} (-\Delta_{\omega} \Psi_\omega - \kappa^2 (1 - |\Psi_\omega|^2) \Psi_\omega) + \alpha \cdot (\text{curl}^* \text{curl} A_\omega - \text{Im}(\bar{\Psi}_\omega \nabla_\omega \Psi_\omega)) = 0,$$

since $(\Psi_\omega, A_\omega)$ is a solution of the Ginzburg-Landau equations. For the quadratic terms we again integrate by parts and use the fact that the terms vanish outside $Q$ to see that they give us exactly $\frac{1}{2} \langle v, L_\omega v \rangle_{L^2}$. Similarly the higher order terms give us $R_\omega(v)$. We thus have

$$\mathcal{E}_Q(u) - \mathcal{E}_Q(u_\omega) = \frac{1}{2} \langle v, L_\omega v \rangle_{L^2} + R_\omega(v).$$

The right hand side is independent of $Q$, and therefore taking the limit proves (3.31) for smooth compactly supported $v$. The general result follows from the fact that $\Lambda_\omega$ is a continuous functional on the space $H^1_{\text{cov}}$. \hfill \Box

The explicit expression above together with the inequality (3.7) and Sobolev embedding theorems implies

Lemma 3.9. There are $c, C > 0$, s. t. for all $v \in H^1_{\text{cov}}$, we have

$$c\mu(\omega, \kappa)\|v\|^2_{H^1} - C\|v\|^2_{H^1} \leq \Lambda_\omega(v) \lesssim \sum_{k=2}^4 \|v\|^k_{H^1}. \quad (3.32)$$

We derive a differential inequality for $\Lambda_\omega(v)$. Using the relations (3.29), we compute $\partial_t \Lambda_\omega = \langle \partial_t v, L_\omega v \rangle + \langle R'_\omega(v), \partial_t v \rangle$. Now using the equation (3.15) to express $\partial_t v$ and using that $R'_\omega(v) = N_\omega(v)$, we obtain

$$\partial_t \Lambda_\omega = \langle L_\omega v, -L_\omega v + V_\gamma v + N_\omega(v) + G_\gamma \rangle + \langle N_\omega(v), -L_\omega v + V_\gamma v + N_\omega(v) + G_\gamma \rangle.$$

Finally, using the fact that, by Proposition 3.2 $\langle G_\gamma, L_\omega v \rangle = 0$, we obtain

$$\partial_t \Lambda_\omega = -\|L_\omega v\|^2 + \langle L_\omega v, V_\gamma v \rangle + \|N_\omega(v), V_\gamma v \| + \|N_\omega(v)\|^2 + \langle N_\omega(v), G_\gamma \rangle. \quad (3.33)$$

The lower bound (3.24) implies that $L_\omega > 0$ on $H^1_{\text{cov}}$. Moreover, if $v \in H^1_{\text{cov}}$, then $\frac{1}{2} \langle v, L_\omega^2 v \rangle \in H^1_{\text{cov}}$. Therefore (3.22) gives

$$\langle L_\omega v, L_\omega v \rangle = \langle L_\omega^2 v, \frac{1}{2} L_\omega v \rangle \geq \mu \langle L_\omega^2 v, L_\omega^2 v \rangle = 2\mu \langle \Lambda_\omega(v) - R_\omega(v) \rangle. \quad (3.34)$$

where we use the shorthand $\mu = \mu(\omega, \kappa)$. The last two relations imply

$$\partial_t \Lambda_\omega(v) \leq -\mu \Lambda_\omega(v) - \frac{1}{2} \|L_\omega v\|^2 + \|N_\omega(v)\|^2 - 2\mu R_\omega(v)$$

$$+ \langle L_\omega v, V_\gamma v \rangle + \langle N_\omega(v), V_\gamma v \rangle + \langle N_\omega(v), G_\gamma \rangle. \quad (3.35)$$
This estimate, together with the bounds (D.1) - (D.2) and (E.1) - (E.3) of Appendices D and E and the estimates
\[
\|v\|_{H^2} \lesssim \|L\omega v\|_{L^2} + \|v\|_{L^2},
\] (3.36)
and (3.24), give, for some \(C > 0\),
\[
\partial_t \Lambda_\omega(v) \leq -c_\mu(\omega, \kappa)\Lambda_\omega(v) - \left( \frac{1}{2} - C \sum_2^4 \|v\|_{H^1}^2 \right)\|L\omega v\|_{L^2} + C \sum_3^4 \Lambda_\omega(v)^{n/2}.
\] (3.37)

Lemma 3.10. There is \(\epsilon > 0\) such that if \(\|v(t)\|_{H^1} < \epsilon\) for all \(t \in [0, T]\), then \(\|v(t)\|_{H^1}^2 \lesssim \epsilon^{-\frac{1}{2}c_\mu} \Lambda_\omega(v_0)\) for all \(t \in [0, T]\), where \(\mu = \mu(\omega, \kappa)\).

Proof. We pick \(\epsilon\) so that \(C \sum_1^4 \epsilon^n \leq \frac{1}{2}\), where \(C\) is the same as in (3.37). Then relation (3.37) implies that \(\partial_t \Lambda_\omega(v) \leq -c_\mu \Lambda_\omega(v) + C \sum_3^6 \Lambda_\omega(v)^{n/2}\). Let \(C''\) be such that \(\Lambda_\omega(v) \leq C'\|v\|_{H^1}^2\). We decrease \(\epsilon\) furthermore, if necessary, so that \(C(C'\|v\|_{H^1}^2)^{\frac{1}{2} - 1} \leq \frac{1}{2}c_\mu\). This gives \(\partial_t \Lambda_\omega(v) \leq -\frac{1}{2}c_\mu \Lambda_\omega(v)\), which implies \(\Lambda_\omega(v) \lesssim \epsilon^{-\frac{1}{2}c_\mu} \Lambda_\omega(v_0)\), where \(v_0 = \lim_{t \to 0} v(t)\), and therefore, by the estimate \(\|v\|_{H^1}^2 \lesssim \Lambda_\omega(v)\) and a standard argument, \(\|v\|_{H^1}^2 \lesssim \epsilon^{-\frac{1}{2}c_\mu} \Lambda_\omega(v_0)\).

Proof of asymptotic stability. By the LWP of (3.15) in \(H^1\), there is \(T > 0\) s.t. if \(\|v_0\|_{H^1} < \epsilon/2\), then \(\|v(t)\|_{H^1} < \epsilon\) for all \(t \in [0, T]\). Then by Lemma 3.10, we have \(\|v(t)\|_{H^1}^2 \lesssim \epsilon^{-\frac{1}{2}c_\mu} \Lambda_\omega(v_0)\) for all \(t \in [0, T]\), so we can iterate to obtain \(\|v(t)\|_{H^1}^2 \lesssim \epsilon^{-\frac{1}{2}c_\mu} \Lambda_\omega(v_0)\) for all \(t \in [0, \infty)\). Since \(u = T_{-g} u_\omega + v\), the last lemma and a standard bootstrap imply \(\|u(t) - T_{-g(u(t))} u_\omega\|_{H^1} \lesssim \epsilon^{-\frac{1}{2}c_\mu} \|u_0 - T_{-g_0} u_\omega\|_{H^1}\), which yields the asymptotic stability.

Remark. We can also consider higher the Lyapunov functionals \(\Lambda_n(v) = \frac{1}{2} \langle v, L_n^\omega v \rangle + \tilde{R}_\omega(v)\) for an appropriate \(\tilde{R}_\omega(v)\).

3.6 Instability
By the definition of the energetic stability/instability, the instability of \(u_\omega\) for all \((\tau, \kappa)\) s.t. \(\theta(\tau, \kappa) > 0\) follows directly from Theorem 3.6.

4 Estimates on Hessian
In this section we prove Theorem 3.6 concerning the Hessian \(L_\omega\).

4.1 Shifted Hessian
Instead of the Hessian \(L_\omega\) it will be more convenient to consider a shifted Hessian \(\tilde{L}_\omega\) which induces the same quadratic form as \(L_\omega\) on \(H^1_\perp\). Let \(\tilde{\Gamma} v := G_\pi(v), \) with \(\pi(v) := \text{Im}(\bar{\Psi}_\omega \xi) - \text{div} \xi \) for \(v = (\xi, \alpha)\). We define \(\tilde{L}_\omega\) as
\[
\tilde{L}_\omega = L_\omega + \tilde{\Gamma}.
\] (4.1)
The explicit expression for \(\tilde{L}\) (as well as for \(L\)) is given in Appendix E. Note that the quadratic forms of \(\tilde{L}_\omega\) and \(L_\omega\) are related as
\[
\langle \tilde{L}_\omega v, v \rangle_{L^2} = \langle L_\omega v, v \rangle_{L^2} + \int \pi(v)^2,
\] (4.2)
and that \(\tilde{L}_\omega G_{\gamma} = G_{h\gamma}\), where \(h := -\Delta + |\Psi_\omega|^2\), which shows that
\[
\inf_{\gamma, ||G_{\gamma}||_{L^2} = 1} \langle \tilde{L}_\omega G_{\gamma}, G_{\gamma} \rangle_{L^2} = \inf h.
\] (4.3)
It follows from (3.10) that the two induced quadratic forms do indeed agree on the subspace \(H^1_\perp\).
4.2 Rescaling

As we treat $b$ as a perturbation parameter (see (3.21)), it is convenient to rescale the problem so that the resulting lattice is $b$-independent. Given a pair $(\Psi, A)$ of type $\omega = (\tau, b, 1)$, we set $\sigma := \frac{1}{\sqrt{b}}$, and introduce the rescaling $U_\sigma : (\Psi(x), A(x)) \mapsto (\sigma \Psi(\sigma x), \sigma A(\sigma x))$. This has the effect that the rescaled state $(\psi, a) = U_\sigma(\Psi, A)$ is of type $\omega' = (\tau, 1, 1)$.

We note that the rescaled Abrikosov lattice solution $u_{\omega'} := (\psi_{\omega'}, a_{\omega'}) := U_\sigma(\Psi_{\omega'}, A_{\omega})$ satisfies the rescaled Ginzburg-Landau equations

$$
\begin{cases}
(-\Delta - \lambda_{\omega}) \psi + \kappa^2 \vert \psi \vert^2 \psi = 0, \\
\text{curl}^* \text{curl} a - \text{Im}(\bar{\psi} \nabla_a \psi) = 0,
\end{cases}
$$

(4.4)

where $\lambda_{\omega} = \frac{\kappa^2}{\tau}$, with the rescaled periodicity conditions (see (3.1) - (3.2) and Remark 3.4)

$$
\begin{cases}
\psi(x + s) = e^{i4\pi \alpha x} \psi(x), \\
a(x + s) = a(x) + \nabla g_a(x),
\end{cases}
$$

(4.5)

for $s \in L_{\omega'} = \frac{1}{\sqrt{\tau}} L_\omega$, $\omega' = (\tau, 1, 1)$, the rescaled lattice corresponding to $b = 1$.

We now define the rescaled Hessian to be $L_{\omega'}^{\text{resc}} := \frac{1}{\tau} L_\omega L_\sigma^{-1}$. It is explicitly given in Appendix F.2.

In what follows, we omit the subindex $\omega'$ from the notation for operators, lattices and their fundamental domains and their dual and write $L^{\text{resc}}$, $L$, $\Omega$, $L^*$, $\Omega^*$ for $L_{\omega'}^{\text{resc}}$, $L_{\omega'}$, $\Omega_{\omega'}$, $L_{\omega'}^*$, $\Omega_{\omega'}^*$.

4.3 Complexification

We now pass from the real-linear operator $L^{\text{resc}}$ to a complex-linear one: we complexify the space $H^1_{\text{cov}}$ and extend the operator $L_{\omega'}^{\text{resc}}$ to the new spaces. We first identify $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with the function $\alpha^C = \alpha_1 - i \alpha_2 : \mathbb{R}^2 \to \mathbb{C}$. (Whenever it does not cause confusion we drop the $^C$ superscript from the notation.) We note that $\alpha \cdot \alpha' = \text{Re}(\bar{\alpha} \alpha'^C)$. We also introduce the differential operator $\partial = \partial_{x_1} - i \partial_{x_2}$. We note that $\bar{\partial} \alpha^C = \text{div} \alpha - i \text{curl} \alpha$, where the $\bar{\partial}$ denotes the complex conjugate operator. In general, for an operator $A$, $A^C := CA\bar{C}$, where $\bar{C}$ denotes complex conjugation. Remember that the objects below are specified by a triple $\omega'$, which is not displayed in their notation.

We now define the complex Hilbert space $\mathcal{K} = L^2(\mathbb{R}^2; \mathbb{C})^4$, with the usual $L^2$ inner product (different from (1.18))

$$
(v, v')_{L^2} = \int \bar{\xi} \bar{\phi} + \bar{\phi} \phi' + \bar{\alpha} \alpha' + \bar{\omega} \omega',
$$

where $v = (\xi, \phi, \alpha, \omega)$, $v' = (\xi', \phi', \alpha', \omega') \in L^2(\mathbb{R}^2; \mathbb{C})^4$. The space $H^1_{\text{cov}}$ is embedded in $\mathcal{K}$ via the injection

$$
\pi : (\xi, \alpha) \mapsto (\xi, \alpha)^C := \frac{1}{\sqrt{2}}(\xi, \bar{\xi}, \alpha, \bar{\alpha}).
$$

(4.6)

This embedding transfers the operator $L^{\text{resc}}$ to a subspace of $\mathcal{K}$. We want to extend the resulting operator to all of $\mathcal{K}$ and call the extension $K$. To this end it is convenient to rewrite the operator $K$ in complex notation, which is done explicitly in Appendix F.3.

There is a simple relation between $L^{\text{resc}}$ and its complexification, $K$, which plays an important role in our analysis:

$$
(v, L^{\text{resc}} v)_{L^2} = (v^C, K v^C)_{L^2}.
$$

(4.7)

We compute the rescaled, complexified gauge zero modes. Introducing the notation $\partial_{\alpha^C} = \partial - i \alpha^C$, we obtain

$$
\tilde{G}_\gamma = (i \gamma \psi_{\omega'}, -i \gamma \bar{\psi}_{\omega'}, \partial_{\gamma}, \bar{\partial}_{\gamma}).
$$

(4.8)
Now, we define the subspace $\mathcal{K}_\perp$ to be the subspace of $\mathcal{K}$ consisting of those vectors orthogonal to these zero modes, $G_\gamma$, and denote by $\mathcal{K}_\perp^s$ the corresponding Sobolev spaces. (Note that since $G_\gamma$ depend on $\omega$, so do $\mathcal{K}_\perp, \mathcal{K}_\perp^2$.) It is straightforward to verify that $v \in H_\perp^1$ if and only if $v^C \in \mathcal{K}_\perp^1$ and in particular, $\pi H_\perp^1 \subset \mathcal{K}_\perp^1$. Hence we have

$$\inf_{v^C \in \pi H_\perp^1} \langle v^C, K v^C \rangle = \inf_{v \in H_\perp^1} \langle v, L^{\text{resc}} v \rangle. \quad (4.9)$$

Though going from $L^{\text{resc}}$ to $K$ we doubled the size of the matrix, the latter is symmetric with respect to interchange of $\xi, \alpha$ and $\bar{\xi}, \bar{\alpha}$, which allows us to split the space $\mathcal{K}$ on which it is defined into two disjoint, invariant real vector subspaces $V_{\pm}, \mathcal{K} = V_+ \oplus V_-$, where $V_{\pm}$ are defined by

$$V_{\pm} : = \{ (\xi, \pm \bar{\xi}, \alpha, \pm \bar{\alpha}) : (\xi, \alpha) \in L^2(\mathbb{R}^2; \mathbb{C})^2 \}. \quad (4.10)$$

Indeed, the subspaces $V_{\pm}$ span the entire space $\mathcal{K}$: any vector in $\mathcal{K}$ can be written as a sum of such two orthogonal eigenvectors:

$$(\xi, \phi, \alpha, \omega) = \frac{1}{2} (\xi + \phi, \xi + \phi, \alpha + \bar{\omega}, \bar{\alpha} + \omega) + \frac{1}{2} (\xi - \phi, -\bar{\xi} + \phi, \alpha - \bar{\omega}, -\bar{\alpha} + \omega). \quad (4.11)$$

By the definition, $V_+ = \text{Ran} \pi$.

### 4.4 Bloch decomposition for gauge-periodic operators

The key tool in analyzing the Hessian is to exploit the gauge-periodicity of the Abrikosov lattice in a similar way as in the Bloch theory (or Floquet theory) of Schrödinger operators with periodic potentials (see [11][17]). The basic idea of the analysis is to decompose the space $\mathcal{K}$ as the direct integral of spaces on a compact domain in such a way that the operator $K$ is decomposed as the direct integral of operators on these spaces.

Let functions $g_t : \mathbb{R}^2 \rightarrow \mathbb{R}, t \in \mathcal{L}$, satisfy the cocycle condition (1.25): $g_{t+s}(x) - g_s(x + t) - g_t(x) \in 2\pi \mathbb{Z}$. Using these functions we define for each $t \in \mathcal{L}$ the magnetic translation operator $T_t$ on $L^2(\mathbb{R}^2; \mathbb{C})$ to act as

$$T_t \xi(x) = e^{-ig_t(x)} \xi(x + t). \quad (4.12)$$

Recall the notation $\hat{\mathcal{A}} := \text{CAC}$ and recall that $T_t^{\text{trans}}$ denotes translation by $t$. We now let $\rho_t$ be the operator on $\mathcal{K}$ defined by

$$\rho_t = T_t \oplus \bar{T}_t \oplus T_t^{\text{trans}} \oplus T_t^{\text{trans}}. \quad (4.13)$$

**Proposition 4.1.** $\rho$ is a unitary group representation of $\mathcal{L}$ in $\mathcal{K}$ (i.e. $\rho_t$ is unitary and $\rho_t \rho_s = \rho_{t+s}$, for all $s, t \in \mathcal{L}$) and, if functions $g_t : \mathbb{R}^2 \rightarrow \mathbb{R}, t \in \mathcal{L}$, in (4.12) are the same as those entering the gauge-periodicity condition (4.5) for the Abrikosov lattice solution $v_{\omega'}$, then $\rho$ commutes with the operator $K = K_{\omega'}$ (i.e. $\rho_t K = K \rho_t$, for all $t \in \mathcal{L}$).

**Proof.** Clearly, the operators $S_t$ and $T_t$ are unitary. To show that $T_t$ satisfy $T_t T_s = T_{t+s}$ we use the cocycle condition to see that

$$T_t T_s \xi(x) = T_t e^{-ig_s(x)} \xi(x + s) = e^{-ig_t(x)} e^{-ig_s(x + t)} \xi(x + s + t) = e^{-ig_{t+s}(x)} \xi(x + s + t) = T_{t+s} \xi(x).$$

Thus $T_t$ is homomorphism from $\mathcal{L}$ to the group of unitary operators on $L^2(\mathbb{R}; \mathbb{C})$. The fact that $\rho$ commutes with the operator $K$ follows by a simple verification. \qed

Recall that $\hat{\mathcal{L}}$ denotes the dual group of $\mathcal{L}$, i.e. $\hat{\mathcal{L}}$ is the group of all continuous homomorphisms from $\mathcal{L}$ to $U(1)$. We identify $\hat{\mathcal{L}}$ with the fundamental cell $\Omega^*$ of the dual lattice $\hat{\mathcal{L}}^*$, as $k \in \Omega^* \leftrightarrow \chi_k \in \hat{\mathcal{L}}$, where $\chi_k(s) = e^{ik \cdot s}$. We extend $\chi_k(s)$ to act on $v = (\xi, \phi, \alpha, \omega)$ as the multiplication operator

$$\chi_k(s) v = (\chi_k(s) \xi, \chi_k(s) \phi, \chi_k(s) \alpha, \chi_k(s) \omega).$$
Note that the subspace \( \text{Ran} \pi = \{ (\xi, \bar{\xi}, \alpha, \bar{\alpha}) \in L^2(\mathbb{R}^2; \mathbb{C})^4 \} \), we started with, is not invariant under this operator.

We now define the Hilbert space \( \mathcal{H} \) to be the direct integral \( \mathcal{H} = \int_{\Omega^*} \mathcal{H}_k \, d\hat{k} \), where \( \hat{d}k \) is the usual Lebesgue measure on \( \Omega^* \), divided by \( |\Omega^*| \), and \( \mathcal{H}_k \) is the set of functions, \( v \), from \( L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{C})^4 \), satisfying

\[
\rho_k v_k(x) = \chi_k(s) v_k(x), \quad \forall s \in \mathcal{L},
\]
a.e. and endowed with the inner product

\[
\langle v, v' \rangle_{L^2} = \int_{\Omega} \xi' + \bar{\phi}' + \bar{\alpha}' + \bar{\omega}', \quad \forall \Omega \text{ is the fundamental cell of the lattice } \mathcal{L}, \text{ identified with } \mathcal{T} := \mathbb{R}^2 / \mathcal{L},
\]

where \( \mathcal{T} \) is the usual Lebesgue measure on \( \Omega^* \), divided by \( |\Omega^*| \), and \( \mathcal{H}_k \) is the set of functions, \( v \), from \( L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{C})^4 \), satisfying

\[
\rho_k v_k(x) = \chi_k(s) v_k(x), \quad \forall s \in \mathcal{L},
\]
a.e. and endowed with the inner product

\[
\langle v, v' \rangle_{L^2} = \int_{\Omega} \xi' + \bar{\phi}' + \bar{\alpha}' + \bar{\omega}', \quad \forall \Omega \text{ is the fundamental cell of the lattice } \mathcal{L}, \text{ identified with } \mathcal{T} := \mathbb{R}^2 / \mathcal{L},
\]

where \( \Omega \) is the fundamental cell of the lattice \( \mathcal{L} \), identified with \( \mathcal{T} := \mathbb{R}^2 / \mathcal{L} \), and \( v = (\xi, \phi, \alpha, \omega), \quad v' = (\xi', \phi', \alpha', \omega') \). The inner product in \( \mathcal{H} \) is given by \( \langle v, w \rangle_{\mathcal{H}} := \int_{\mathcal{L}} \langle v_k, w_k \rangle_{\mathcal{H}_k} \, d\hat{k} \). We write \( f = \int_{\Omega^*} f_k \, d\hat{k} \), where \( f_k \) is the \( k \)-component of \( f \), and by the symbol \( \int_{\Omega^*} T_k \, d\hat{k} \) we understand the operator \( T \) acting on \( \mathcal{H} \) as

\[
Tf = \int_{\Omega^*} T_k f_k \, d\hat{k}.
\]

For \( k \in \Omega^* \), let \( K_k \) be the operator \( K \) acting on \( \mathcal{H}_k \) with the \( \mathcal{H}_k \cap L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{C})^4 \). It is easy to check that the operator \( K \) leaves these boundary conditions invariant. Note also that \( \mathcal{C} K_k = K_{-k} \mathcal{C} \). We have (cf. [40])

**Proposition 4.2.** Define \( U : \mathcal{K} \to \mathcal{H} \) on smooth functions with compact domain by the formula

\[
(Uv)_k(x) = \sum_{t \in \mathcal{L}} \chi^{-1}_k(t) \rho_t v(x)
\]

Then \( U \) extends uniquely to a unitary operator satisfying

\[
UKU^{-1} = \int_{\Omega^*} K_k \, d\hat{k}.
\]

Each \( K_{-k} \) is a self-adjoint operator with compact resolvent (and therefore purely discrete spectrum), and

\[
\sigma(K) = \bigcup_{k \in \Omega^*} \sigma(K_k).
\]

**Proof.** We begin by showing that \( U \) is an isometry on smooth functions with compact domain. Using Fubini’s theorem and the property \( |\Omega^*| \cdot |\Omega^*| = 1 \), we calculate

\[
\|U v\|_{\mathcal{H}}^2 = \int_{\Omega^*} \| (Uv)_k\|_{L^2}^2 \, d\hat{k} = \int_{\Omega^*} \int_{\mathcal{L}} \int_{\mathcal{L}} \chi^{-1}_k(t) \rho_t v(x) \, dx \, d\hat{k}
\]

\[
= \int_{\Omega^*} \left( \sum_{t \in \mathcal{L}} \rho_t v(x) \, dx \int_{\mathcal{L}} \chi^{-1}_k(t) \chi_k(s) \, d\hat{k} \right) \, dx.
\]

We compute, after writing \( k = s + t \) in the coordinate form, \( \int_{\Omega^*} \chi^{-1}_k(t) \chi_k(s) \, d\hat{k} = \delta_{s,t} \). Using this, we obtain furthermore

\[
\|U v\|_{\mathcal{H}}^2 = \int_{\Omega^*} \sum_{t \in \mathcal{L}} |\rho_t v(x)|^2 \, dx = \int_{\mathbb{R}^2} |v(x)|^2 \, dx = \|v\|_{K}^2.
\]

Therefore \( U \) extends to an isometry on all of \( \mathcal{K} \). To show that \( U \) is in fact a unitary operator we define \( U^* : \mathcal{H} \to \mathcal{K} \) by the formula

\[
(U^* g)_k(x) = \int_{\Omega^*} \chi_k(t) (\rho^*_k g_k)(x) \, d\hat{k},
\]
for any $x \in \Omega + t$ and $t \in \mathcal{L}$. Straightforward calculations show that $U^*$ is the adjoint of $U$ and that it too is an isometry, proving that $U$ is a unitary operator.

For completeness we show that $UU^* = 1$. Using that the definition of implies that $\rho_t(U^*g)(x) = \int_{\Omega^*} \chi_k(t)g_k(x)dk$, we compute

$$(UU^*)_k(x) = \sum_{t \in \mathcal{L}} \chi_k^{-1}(t) \int_{\Omega^*} \chi_{k'}(t)g_{k'}(x)\hat{dk}'. $$

Furthermore, using the Poisson summation formula,

$$\sum_{t \in \mathcal{L}} \chi_k^{-1}(t) = |\Omega^*| \sum_{t' \in \mathcal{L}^*} \delta(t' - k), \quad (4.19)$$

we find

$$(UU^*)_k(x) = \int_{\Omega^*} \sum_{t' \in \mathcal{L}^*} \delta(t' - k)g_{k'}(x)\hat{dk}' = g_k(x).$$

Next, we show that $(Uv)_k$ satisfies the boundary conditions $\{4.14\}:

$$\rho_t(Uv)_k(x) = \sum_{s \in \mathcal{L}} \chi_s^{-1}(s)\rho_s v(x) = \sum_{s \in \mathcal{L}} \chi_s^{-1}(s)\rho_{t+s} v(x) = \chi_k(t)\sum_{s \in \mathcal{L}} \chi_s^{-1}(t+s)\rho_{t+s} v(x),$$

which gives that

$$\rho_t(Uv)_k(x) = \chi_k(t)(Uv)_k(x).$$

We now have that

$$(K(Uv)_k)(x) = \sum_{t \in \mathcal{L}} \chi_k^{-1}(t)K\rho_t v(x) = \sum_{t \in \mathcal{L}} \chi_k^{-1}(t)\rho_t K v(x)$$

and therefore

$$(K(Uv)_k)(x) = (UKv)_k(x),$$

which establishes $\{4.17\}$.

The self-adjointness of the operators $K_k$ and the compactness of their resolvents follow by standard arguments. We now turn to the relation $\{4.18\}$. We first prove the $\supset$ inclusion. Suppose that $\lambda \in \sigma(K_{\omega k})$ for some $k \in \Omega^*$. Then there exists a smooth eigenfunction $v \in \mathcal{D}(K_k)$ solving $K_k v = \lambda v$. By the definition of $K_k$, the function $v$ solves the equation $Kv = \lambda v$. Therefore, by Schnol-Simon theorem (see e.g. [25]), $\lambda$ must be in the essential spectrum of $K$.

As for the $\subseteq$ inclusion, suppose that $\lambda \notin \bigcup_{k \in \Omega^*} \sigma(K_k)$. Then the operators $(K_k - \lambda)^{-1}$ are uniformly bounded, and therefore $(K - \lambda)^{-1} = \int_{\Omega^*} (K_k - \lambda)^{-1} dk$ is also bounded and therefore $\lambda \notin \sigma(K)$. \hfill $\square$

The definition of $U$ and the fact that $\chi_k^{-1}(t) = \chi_k^{-1}(t)$ imply

$$(Uv)_{k+s^*}(x) = (Uv)_k(x), \forall s^* \in \mathcal{L}^*.$$  \quad (4.20)

**Remark.** We can think of $\mathcal{H}$, as the set of functions, $v_k(x)$, from $L^2_{loc}(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{C})$, satisfying $\{4.14\}$ and $v_{k+s^*}(x) = v_k(x), \forall s^* \in \mathcal{L}^*$, a.e. and endowed with the $L^2(\Omega \times \Omega^*; \mathbb{C})^4$ inner product.

Let $U_k: K \to \mathcal{H}_k$ be the fibre map, defined by $U_k v := (Uv)_k$. Define $\mathcal{H}_{k+}$ and $\mathcal{H}_{k-}$ to be the images of $\mathcal{K}_{k+}$ and $\mathcal{K}_{k-} \cap V_{\perp}$ under the map, $U_k$, so that $\mathcal{H}_{k+} = \mathcal{K}_{k+} \oplus \mathcal{H}_{k-}$ (the orthogonality here is meant w.r.t to the real inner product, $\text{Re}<v, v'>$, cf \{4.18\}) and $\mathcal{K}_{k+} \cap V_{\perp} = \int_{\Omega^*} \mathcal{H}_{k-} dk$. (Remember that $\mathcal{H}_{k\pm}$ are real-linear spaces.)
It is not hard to see that the real-vector spaces $\mathcal{H}_{k\perp\pm}$ are invariant under $K_{\omega k}$. Now, since each operator $K_k$ has purely discrete, real spectrum ($K_k$ is a self-adjoint) and since the real-vector spaces $\mathcal{H}_{k\perp\pm}$ are invariant under $K_k$, one can choose a basis of eigenvectors which belong to either $\mathcal{H}_{k\perp+}$ or $\mathcal{H}_{k\perp-}$. Denote the set of the corresponding eigenvalues by $\sigma(K_k|_{\mathcal{H}_{k\perp\pm}})$. Then we have that $\inf_{v\in\mathcal{H}_{k\perp\pm}} \langle v, K_k v \rangle = \inf_{k\in\Omega} \sigma(K_k|_{\mathcal{H}_{k\perp\pm}})$ and, by relations (4.9) (or (4.7)) and (4.18),

$$\inf_{v\in\mathcal{H}_{k\perp+}} \langle v, L^{\text{esc}} v \rangle = \inf_{v\in\mathcal{H}_{k\perp+}} \langle v^C, K v^C \rangle = \inf_{k\in\Omega} \sigma(K_k|_{\mathcal{H}_{k\perp+}\cap\mathcal{V}_+}).$$

As a side remark, we mention that the fiber map $U_k$ acts as $U_k = \hat{\chi}_k U_0(\hat{\chi}_k)^{-1}$, where $(U_0 v)(x) = \sum_{s\in\mathcal{L}} \rho_s v(x) \in \mathcal{H}_0$ and $\hat{\chi}_k : v \to e^{ik_x v}$.

### 4.5 Ground state energy of $K_k$

We now turn to analysis of the spectrum of the fibre operators $K_k$. Recall that $\mathcal{H}_{k\perp}$ denotes the image of $K_k$ under the fibre map. The following proposition is the main result of this section.

**Proposition 4.3.** The two lowest eigenvalues, $\mu_{\omega n k\pm}$, of the operator $K_k$ on the subspace $\mathcal{H}_{k\perp}$ are of the form

$$\mu_{\omega n k\pm} = \mu_{\tau k\pm} \epsilon^2 + O(\epsilon^3),$$

where $\epsilon$ defined in (3.21) and $\mu_{\tau k\pm}$ are given by

$$\mu_{\tau k\pm} = (\kappa^2 - \frac{1}{2} \{2|\langle \phi_0 | \phi_k \rangle_\Omega^2 \mp |\langle \phi_0^2 \phi_{-k} \phi_k \rangle_\Omega - \beta(\tau)\} + \delta_{k,0},$$

with the functions $\phi_k$ described in Proposition 2.1 (i.e. $\phi_k$ satisfy (2.1)) and normalized as $|\langle \phi_k |^2 \rangle_\Omega = 1$. Moreover, the corresponding eigenfunctions of the operator $K_k$ belong to $\mathcal{H}_{k\perp\pm}$ and converge, as $\epsilon \to 0$, to

$$\psi_{k\pm}^0 = (\sigma^{-1/2} \phi_k, \pm \sigma^{1/2} \bar{\phi}_{-k}, 0, 0),$$

where $\sigma := \frac{\langle \phi_0^2 \phi_{-k} \phi_k \rangle_\Omega}{|\langle \phi_0 | \phi_k \rangle_\Omega^2 |}$. (Note that $\sigma^{-1/2} = \sigma^{1/2}$.)

**Proof.** We begin with the expansion of lattice states $(\psi_{\omega'}, a_{\omega'})$ in powers of the parameter $\epsilon$ introduced in (3.21). It is shown in [55] that for each $\tau$ there is $\epsilon_0 > 0$, such that the solution branch $(\psi_{\omega'}, a_{\omega'})$ has the following expansion

$$\begin{cases}
\psi_{\omega'} = e \psi^0 + e^3 \psi^1 + O(\epsilon^5), \\
a_{\omega'} = a^0 + e^2 a^1 + O(\epsilon^4), \\
\lambda_{\omega'} = 1 + e^2 \lambda^1 + O(\epsilon^4),
\end{cases}$$

where $a^0 := \frac{1}{2} Jx$, $\psi^0$ satisfies (2.1) with $k = 0$ and is normalized as $|\langle \psi^0 |^2 \rangle_\Omega = 1$, so that $\psi^0 = \phi_0$ (see Proposition 2.1), and $a^1$ satisfies

$$i \partial a^1 = \frac{1}{2} \langle \phi_0 |^2 \rangle_\Omega - |\phi_0|^2, \quad a^1(x + s) = a^1(x), \quad \forall s \in \mathcal{L},$$

$$\Delta a^1 = \frac{i}{2} \phi_0 \partial_\omega \phi_0,$$

and $\lambda_1$ is given by

$$\lambda_1 = \left[ 1 + (\kappa^2 - \frac{1}{2}) \beta(\tau) \right] |\langle \phi_0 |^2 \rangle_\Omega.$$
Both sets consist of eigenvalues of multiplicity the orthonormal basis in $L^2(\mathbb{R}; \mathbb{C})$ equivalent to the multiplication operator by $\lambda^\pm$. Lemma 4.4.

The operators $\Delta a^\sigma$, introduced in (2.6) and satisfying the commutation relations $[c, c^*] = 2$, the latter operators are self-adjoint, and have discrete spectrum, and therefore so is and does the operator $K^0$. Corollary 4.5. $K^0 \geq 0$ and $\text{Null} K^0$ is spanned by functions $\phi_k$, where $\phi_k$ are described in Proposition 2.7.

We begin with the operator $K^0 := K|_{\epsilon = 0}$. It is the fiber integral of the unperturbed operators $K^0_k := K_k|_{\epsilon = 0}$, which are of the form (4.29), but defined on $L^2(\mathbb{R}; \mathbb{C})^4$, with the periodicity conditions (4.14). It is a direct sum of the operators $\Delta_{\omega} - 1, -\Delta_{\omega} - 1, -\Delta - \Delta$ on $L^2(\mathbb{R}; \mathbb{C})$, with the periodicity conditions $e^{-i s^\omega} \phi(x + s) = e^{i k^\omega} \phi(x)$ and $\alpha(x + t) = e^{it} \alpha(x)$, respectively. By a standard theory (see e.g. [1], [20]), the latter operators are self-adjoint, and have discrete spectrum, and therefore so is and does the operator $K^0_k$. The relation between the spectra of $K^0_k$ and $\Delta_{\omega}$ and $\Delta$ is given by $\sigma(K^0_k) = \sigma(-\Delta_{\omega} - 1) \cup \sigma(-\Delta)$. In fact, the spectra of the operators $\Delta_{\omega}$ and $\Delta$ and therefore of $K^0_k$ can be found explicitly:

Lemma 4.4. The operators $\Delta_{\omega}$ and $\Delta$ are self-adjoint with discrete spectrum given by $\sigma(-\Delta_{\omega} - 1) = \{0, 2, 4, \ldots\}$, $\sigma(-\Delta) = \{|k + t|^2 : t^* \in \mathbb{L}^*\}$. Both sets consist of eigenvalues of multiplicity 2. Moreover, $\text{Null}(\Delta_{\omega} - 1)$ is spanned by functions $\phi_k$, where $\phi_k$ are described in Proposition 2.7.

Proof. To describe the spectra of $-\Delta_{\omega}$ and $-\Delta$, we first consider the operator $-\Delta_{\omega}$ on $L^2(\Omega; \mathbb{C})$ with boundary conditions, $e^{-i s^\omega} \phi(x + s) = e^{i k^\omega} \phi(x)$ (see (2.1)). In Subsection 3.4, we obtained the representation $-\Delta_{\omega} - 1 = c c^*$, where $c$ and $c^*$ are the harmonic oscillator annihilation and creation operators, introduced in (2.6) and satisfying the commutation relations $[c, c^*] = 2$. This representation implies that $\sigma(-\Delta_{\omega}) = \{1, 3, 5, \ldots\}$. We now turn to the operator $-\Delta$ acting on on $L^2(\Omega; \mathbb{C})$, which is $L^2(\mathbb{R}; \mathbb{C})$, with the periodicity $\alpha(x + t) = e^{it} \alpha(x)$. Standard methods show that this is a positive self-adjoint operator with discrete spectrum. Using the orthonormal basis in $L^2(\mathbb{R}; \mathbb{C})$, given by $e_s(x) = e^{i(k^\omega + s) x}$, $s \in \mathbb{L}^*$, one can show that $-\Delta$ is unitarily equivalent to the multiplication operator by $|k + s|^2$ on the space $L^2(\mathbb{L}^*; \mathbb{C})$.

Corollary 4.5. $K^0_k \geq 0$ and $\text{Null} K^0_k$ is spanned by the vectors $v_{1k}^0 := (\phi_k, 0, 0, 0)$, $v_{2k}^0 := (0, \tilde{\phi}_{-k}, 0, 0)$.
where \( \phi_k \in \text{Null}(-\Delta, 0 - 1) \) are described in Proposition 2.1, if \( k \neq 0 \), and by the vectors \( v_{k_1}^0, v_{k_2}^0 \) and
\[
w_{10} := (0, 0, 1, 0), \quad w_{20} := (0, 0, 0, 1),
\]
if \( k = 0 \). Moreover, for \( k \neq 0 \), there are, in addition, almost zero eigenvectors, as \( k \to 0 \), due to the operator \(-\Delta\), namely the vectors
\[
w_{1k} := (0, 0, e^{ik\cdot x}, 0), \quad w_{2k} := (0, 0, 0, e^{ik\cdot x}),
\]
with the eigenvalues \(|k|^2\).

Note that the eigenvectors \( w_{1k} \) and \( w_{2k} \) are ruled out by the condition (1.19), which implies (3.28), which in turn gives \( T^\text{eff} v = v \). (See Supplement II for detailed discussion.)

By standard perturbation theory (see for example \[11, 23, 25\]) the spectrum of \( K_{\omega k} \) consists of eigenvalues which cluster in \( \epsilon \)-neighbourhoods of the eigenvalues of \( K_{k_0}^0 \) and each cluster has the same total multiplicity as the eigenvalue of \( K_{k_0}^0 \) it originates from. Denote by \( \mu_{\omega k_{\rho k}}, \rho = \pm \), the two lowest branches. To find them, we use the Feshbach-Schur map argument (see e.g. \[9, 26\] and Supplement III). This argument says that given an operator \( K \) and a projection \( P \),
\[
\lambda \in \sigma(K) \quad \text{if and only if} \quad \lambda \in \sigma(\mathcal{F}_P(\lambda)),
\]
where, with \( \bar{P} = 1 - P \),
\[
\mathcal{F}_P(\lambda) := [PKP - \bar{P}K\bar{P}(\bar{P}K\bar{P} - \lambda)^{-1}\bar{P}KP]_{\text{Ran} \bar{P}},
\]
provided the operator \( \bar{P}K\bar{P} - \lambda \) is invertible on \( \text{Ran} \bar{P} \) and the operators \( PKP \) and \( P\bar{P} \) are bounded. (The latter conditions suffice for the right hand side of (4.37) to be well defined. The proof of the above statement is elementary and is given in Supplement III.)

We use the Feshbach-Schur map argument for the operator \( K_k \). As the projection \( P \) we take the orthogonal projection onto \( \text{Null} K_k^0 \). By above, we have to check that the operator \( PK_k\bar{P} - \lambda \) is invertible on \( \text{Ran} \bar{P} \) and the operators \( PK_kP \) and \( P\bar{P} \) are bounded. Due to the relation \( PK_kP = \bar{P}(K_k - K_k^0)P \) and the explicit form of \( K_k - K_k^0 \), we see that the operator \( PK_kP \) is bounded (in fact, \( \|K_k - K_k^0\| \lesssim \epsilon \)). We know that \( \sigma(PK_k^0\bar{P}) \subset [\nu_0, \infty) \) for some \( \nu_0 > 0 \) and therefore, by standard perturbation theory (since \( \|K_k - K_k^0\| \lesssim \epsilon \)), we have that
\[
\sigma(\bar{P}K_k\bar{P}|_{\text{Ran} \bar{P}}) \subset [c, \infty),
\]
with \( c = \nu_0 + O(\epsilon) \). Hence the self-adjoint operator \( \bar{P}K_k\bar{P} - \lambda \) is invertible on \( \text{Ran} \bar{P} \), provided \( \lambda < c \) and \( \|\bar{P}K_k\bar{P} - \lambda\|^{-1} \leq c^{-1} \) (again provided \( \lambda < c \)). Hence (4.40) is well defined for \( \lambda < c \). By above, we conclude that the Feshbach - Schur argument is applicable and implies that
\[
\lambda \in \sigma(K_k) \quad \text{if and only if} \quad \lambda \text{ solves } \mathcal{F}_k(\lambda) - \lambda = 0,
\]
where
\[
\mathcal{F}_k(\lambda) := [PK_{\omega k}P - PK_k\bar{P}(PK_k\bar{P} - \lambda)^{-1}PK_kP]_{\text{Ran} \bar{P}}.
\]

We also use the relation \( K_k^0P = PK_k^0 = 0 \) and the facts \( \|PK_k\bar{P}\| = O(\epsilon) \) and \( \|(PK_k\bar{P} - \lambda)^{-1}\| \lesssim 1 \), provided \( \lambda < c \) (by (4.38)). Since we are studying the eigenvalue in \( O(\epsilon) - \) neighbourhood of 0, we can assume that \( \lambda = O(\epsilon) \). As an orthonormal basis in \( \text{Null} K_k^0 \) we take the vectors (4.33), with the normalization \( \langle |v_k^0|^2 \rangle_{\Omega} = 1 \). Using this, we obtain
\[
\mathcal{F}_1(\epsilon) = \epsilon^2 F_1 + \epsilon^2 F_2 + O(\epsilon^3),
\]
where
\[
F_1 := \langle v_k^0, W^1 v_k^0 \rangle
\]
and
\[
F_2 := \frac{1}{2} \langle v_k^0, W^2 v_k^0 - v_k^0, W^1 \bar{P}(PK_k^0\bar{P})^{-1}PW^1 v_k^0 \rangle.
\]
The operator, $W^1$ is explicitly given by (4.30). This expression shows that $W^1$ switches the first two entries of the vectors it acts on with the last two. Since the last two entries of $v_{ki}^0$ are zero, this implies that $\langle v_{ki}^0, W^1 v_{ki}^0 \rangle = 0$ and therefore 

$$\mathcal{F}_1 = 0.$$  

(4.43)

We now turn to the $\epsilon^2$ order operator, $\mathcal{F}_2$.

**Lemma 4.6.** We have 

$$\mathcal{F}_2 := (\kappa^2 - \frac{1}{2}) \left( \begin{array}{c} \Omega - \beta(\tau) \\ \Omega - \beta(\tau) \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \delta_{k,0}. \quad (4.44)$$

**Proof.** We begin with $\langle v_{ki}^0, W^2 v_{ki}^0 \rangle$. We first note that $W^2$ and $v_{ki}^0$ are explicitly given by (4.31) and (4.33). Using the fact that $\partial_{\alpha^0}^* v_{ki} = 0$, the normalization $\langle | \phi_k^0 |^2 \rangle = 1$ and (4.28), we calculate that 

$$\langle v_{ki}^0, W^2 v_{ki}^0 \rangle = -\lambda^2 \langle | \phi_k^0 |^2 \rangle + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

$$= -\frac{1}{2} + (\kappa^2 - \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

$$= -\frac{1}{2} + (\kappa^2 - \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

$$= -\frac{1}{2} + (\kappa^2 - \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

We integrate by parts and use the fact that $\partial_{\alpha^0}^* v_{ki} = 0$, together with the identity (4.26), to obtain 

$$\langle v_{ki}^0, W^2 v_{ki}^0 \rangle = -\lambda^2 \langle | \phi_k^0 |^2 \rangle + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

$$\quad = -\frac{1}{2} + (\kappa^2 - \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

Similarly, using (4.25) and (4.33) and (4.28), we calculate 

$$\langle v_{ki}^0, W^2 v_{ki}^0 \rangle = -\lambda^2 \langle | \phi_k^0 |^2 \rangle + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

The equations (4.45), (4.48) and (4.29) and the normalization $\langle | \phi_k^0 |^2 \rangle = 1$ give 

$$\langle v_{ki}^0, W^2 v_{ki}^0 \rangle = -\lambda^2 \langle | \phi_k^0 |^2 \rangle + (2\kappa^2 + \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle$$

$$\quad = -\frac{1}{2} + (\kappa^2 - \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

To compute the second term in $\mathcal{F}_2$ we note that $PW^1 P = W^1 P$, and use (4.29) and (4.30) to calculate 

$$\langle v_{ki}^0, W^2 v_{ki}^0 \rangle = (\bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle = -\lambda^2 \langle | \phi_k^0 |^2 \rangle + (2\kappa^2 + \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle$$

$$\quad = -\frac{1}{2} + (\kappa^2 - \frac{1}{2}) \beta(\tau) + (2\kappa^2 + \frac{1}{2}) \langle | \phi_k^0 |^2 \rangle + \langle \bar{v}_k (i\alpha \partial_{\alpha^0}^* - i\alpha \partial_{\alpha^0}^*) v_{ki} \rangle$$

Now note that by (2.7), $\partial_{\alpha^0} v_{ki} = -\frac{1}{2} (x_1 + i\epsilon x_2) v_{ki} = i\alpha^0 v_{ki}$, and therefore 

$$\partial_{\alpha^0} \phi_{\alpha^0} = \bar{\phi}_{\alpha^0} \partial_{\alpha^0} \phi + \phi_{\alpha^0} \partial_{\alpha^0} \bar{\phi} = i\alpha^0 \phi_{\alpha^0} \phi + \phi_{\alpha^0} \partial_{\alpha^0} \bar{\phi} = \phi_{\alpha^0} (\partial_{\alpha^0} \bar{\phi}).$$

This gives 

$$\text{Re}(\bar{\phi}_{\alpha^0} (\partial_{\alpha^0} \bar{\phi})) = \text{Re}(\phi_{\alpha^0} \partial_{\alpha^0} \bar{\phi} \Delta^{-1} \partial_{\alpha^0} \bar{\phi} \Delta^{-1}\bar{\phi}_{\alpha^0} \phi_{\alpha^0} \phi).$$

(4.53)

Since $\partial_{\alpha^0} \bar{\phi} = -\Delta$, we have 

$$\partial_{\alpha^0} \bar{\phi} \Delta^{-1} \partial_{\alpha^0} \bar{\phi} = -\partial_{\alpha^0} \bar{\phi},$$

(4.54)

where $\partial_{\alpha^0} \bar{\phi}$ is the orthogonal projection onto the orthogonal complement of $\text{Null} \partial_{\alpha^0} \bar{\phi}$, where $X_k$ is the subspace of $H^1(\Omega)$ consisting of functions satisfying the boundary condition $f(x + s) = e^{ikx} f(x)$. Thus $\partial_{\alpha^0} \bar{\phi} := 1 - P_0$,
with $p_0 := \frac{1}{|k^2|}|1\rangle\langle 1|$, for $k = 0$, and $\tilde{p}_0 = 1$ for $k \neq 0$. This and the fact that the function $f := (\tilde{\phi}_k)$ satisfies $f(x+s) = e^{ikx}f(x)$, yield

$$
(\tilde{\phi}_k(\bar{\partial}_\theta \psi_0) \Delta^{-1}(\phi_+ (\bar{\partial}_\theta \psi)))_\Omega = -\langle |\psi_0|^2 | \phi_k |^2 \rangle_\Omega + |\phi_k |^2 \delta_{k,0}.
$$

(4.55)

Similarly we have $\langle \phi_{-k}(\bar{\partial}_\theta \phi_0) \Delta^{-1}(\bar{\partial}_\theta \phi) \rangle_\Omega = -\langle |\psi_0|^2 | \phi_k |^2 \rangle_\Omega + |\phi_k |^2 \delta_{k,0}$. The fact that $\langle |\phi_k |^2 \rangle_\Omega = 0$ now gives

$$
-(\langle \psi_0^0, W^1 \bar{P} (\bar{P} K^0 \bar{P})^{-1} \bar{P} W^1 \psi_0^0 \rangle_{k,1} = -\langle |\psi_0|^2 | \phi_k |^2 \rangle_\Omega + \delta_{k,0}.
$$

(4.56)

$$
-(\langle \psi_0^0, W^1 \bar{P} (\bar{P} K^0 \bar{P})^{-1} \bar{P} W^1 \psi_0^0 \rangle_{k,2} = -\langle |\psi_0|^2 | \phi_k |^2 \rangle_\Omega + \delta_{k,0}.
$$

(4.57)

Collecting the results above we obtain (4.44). □

The matrix (4.44) has the eigenvalues (4.23) with the eigenvectors $(\sigma^{-1/2}, \pm \sigma^{1/2})$, where $\sigma := \frac{(\tilde{\phi}_k \Delta^{-1} \phi_k)^2_\Omega}{|\phi_k |^2 \delta_{k,0}}$, in the orthonormal basis (4.33) in Null $K^0_\tau$, i.e. with the eigenfunctions (4.24). Equations (4.39), (4.41), (4.43) and (4.44) imply the first part of Proposition 4.3. By Theorem III. 1 Supplement III, the eigenvectors (4.24) of $K_k$ can be lifted to the eigenfunctions $v_{k,\pm} := Q v_{0,k}$, where $Q := P - \bar{P} (\bar{P} K_k \bar{P} - \lambda)^{-1} \bar{P} K_k P$, of the operator $K_k$, which converge to (4.24) in the limit $\epsilon \to 0$. Since $P$ and $K_k$ commute with the real-linear operator $\gamma$ defined in Subsection 4.3, so does $Q$. Since $V_{\pm}$ are eigenspaces of $\gamma$ and the eigenvectors (4.24) belong to $\mathcal{H}_{\pm} \cap V_{\pm}$, then so do $v_{k,\pm} = v_{k,\pm} \in \mathcal{H}_{\pm} \cap V_{\pm}$. This proves the second part of the proposition. □

4.6 Proof of Theorem 3.6

Proposition 4.3 and the relation (4.21) and the notation $\mu_{\omega nk} := \mu_{\omega nk}$ imply Theorem 3.6 □

A Proof of the representation (1.23)

In this appendix we compute the integrals entering (2.10), completing this way the proof of the explicit representation (1.23) of the functions $\gamma_k(\tau)$. Recall that $\gamma_k(\tau)$ in (2.10) are expressed in terms of the functions $\varphi_k(z)$, defined in (2.8), and that, by Proposition 2.1, the latter are related to the theta-functions $\theta_k(z, \tau)$ as

$$
\varphi_k(z) = c_0 e^{\sqrt{-1} \pi m z (z^2 - |z|^2)} \theta_k(z, \tau).
$$

(A.1)

(Here $c_0$ is such that $\langle |\varphi_k|^2 \rangle_{\Omega, \tau} = 1$.) To compute the integrals entering (2.10), we use the relation (A.1) and the explicit series representation (2.5) for $\theta_k(z, \tau)$.

**Proposition A.1.** Recall $q = -a\tau + b$ and $\text{Re } \tau = \tau_1$, $\text{Im } \tau = \tau_2$, and and let $\Omega := \bar{\Omega}_\tau := \{ b + \tau a : 0 \leq b, a \leq 1 \}$. We have

$$
\int_{\Omega} |\varphi_k|^2 dz = c \sqrt{\tau_2},
$$

(A.2)

$$
\int_{\Omega} |\varphi_k|^2 |\varphi_k|^2 dz = c^2 \sum_{p,n=-\infty} e^{-\sqrt{\tau_2} |n-p|^2} \cos 2\pi (bp + na),
$$

(A.3)

$$
\int_{\Omega} \varphi_k^* \varphi_k \varphi_k^* \varphi_k dz = c^2 e^{-2\pi i b} \sum_{p,n=-\infty} e^{-\sqrt{\tau_1} |n-(p+a)\tau+b|^2 - 2\pi i (bp-na)},
$$

(A.4)

where $c = \frac{\sqrt{\tau_2}}{\sqrt{\tau_1}}$, with the constant $c_0$ is given in (2.8).

**Proof.** The functions $|\varphi_0|^2$ and $|\varphi_k|^2$ are periodic functions w.r. to the lattice $\mathcal{L}$. To convert this to standard periodicity (w.r. to the square lattice), we write $z = z_1 + iz_2 = u_1 + u_2 \tau : 0 \leq u_i \leq 1$, $i = 1, 2$, to obtain

$$
z_1 = u_1 + \tau u_2 \quad \text{and} \quad z_2 = \tau_2 u_2.
$$

(A.5)
Then the functions $|\varphi_0|^2$ and $|\varphi_q|^2$ are periodic functions w.r.t. $u_i$ with the period 1. Let $c_{n_1,n_2}(f)$ denote the Fourier coefficients of a function $f$ w.r.t. $u_i$. Passing to the variables $u_i$ with the Jacobian det \( \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} = \tau_2 \) and using the Plancherel theorem, we find

\[
\int_{\Omega} |\varphi_0|^2 |\varphi_q|^2 dz = \tau_2 \int_{[0,1]^2} |\varphi_0|^2 |\varphi_q|^2 du = \tau_2 \sum_{n_1,n_2=-\infty}^{\infty} c_{n_1,n_2}(|\varphi_0|^2) c_{n_1,n_2}(|\varphi_q|^2).
\]

To compute the Fourier coefficients $c_{n_1,n_2}(|\varphi_q|^2)$, we compute the FTs of $|\varphi_q|^2$ on the entire $\mathbb{R}^2$ and use that for any function $f(u)$ of the period 1, we have

\[
\int_{\mathbb{R}^2} f(u)e^{2\pi i\xi \cdot u} du = \sum_{n \in \mathbb{Z}^2} c_n(f) \delta(\xi_1 - n_1) \delta(\xi_1 - n_1).
\]

Using the series representation \(2.8\) - \(2.5\) for $\varphi_q(z)$, we obtain

\[
|\varphi_q|^2 = c_0^2 \sum_{m,m'=\infty}^{\infty} e^{2\pi i\alpha_{m,m'}(u)},
\]

where

\[
\alpha_{m,m'}(u) = \frac{1}{\tau_2} \frac{z_2^2}{2} + 2a z_2 - a^2 \tau_2 + i(qm - \tilde{q}m' + \frac{1}{2}(m^2 \tau - m'^2 \tau) + mz - m' \tilde{z}).
\]

Now, we use the relations

\[
mz - m' \tilde{z} = (m - m')z_2 + i(m + m')z_2 = (m - m')(u_1 + \tau_1 u_2) + i(m + m')\tau_2 u_2,
\]

and \(A.9\) to derive

\[
- \frac{1}{\tau_2} \frac{z_2^2}{2} - ia(z - \tilde{z}) + i(mz - m' \tilde{z}) = -\tau_2 u_2^2 + 2a \tau_2 u_2 + i(m - m')(u_1 + \tau_1 u_2) - (m + m')\tau_2 u_2
\]

\[
= -\tau_2[u_2^2 - 2au_2 + (m + m')u_2] + i(m - m')(u_1 + \tau_1 u_2)
\]

\[
= -\tau_2[(u_2 - a + \frac{1}{2} (m + m'))^2 - a^2 + a(m + m') - \frac{1}{4}(m + m')^2] + i(m - m')(u_1 + \tau_1 u_2)
\]

\[
= -\tau_2(u_2 - a + \frac{1}{2} (m + m'))^2 + i(m - m')(u_1 + \tau_1 u_2) + \tau_2 \frac{1}{4} (m + m')^2 + \tau_2 a(a - m - m').
\]

Next, we use that

\[
i(qm - \tilde{q}m') = i(-a\tau_1 + b)(m - m') + a\tau_2(m + m'),
\]

\[
i(m^2 \tau - m'^2 \tilde{\tau}) = i(m^2 - m'^2)\tau_1 - (m^2 + m'^2)\tau_2,
\]

\[
- \frac{1}{2}(m^2 + m'^2) + \frac{1}{4}(m + m')^2 = -\frac{1}{4}(m - m')^2.
\]

The last four relations together with \(A.8\) and the notation $p = m - m'$ and $m' = m - p$ give

\[
\alpha_{m,m'}(u) = -\tau_2(u_2 - a + m - \frac{1}{2} b)^2 + ip(u_1 + \tau_1 u_2) + i(-a\tau_1 + b)p
\]

\[
+ a\tau_2(2m - p) + ip(m - \frac{1}{2} p)\tau_1 - \frac{1}{4} b^2 \tau_2 + \tau_2 a(a - 2m + p)
\]

\[
= -\tau_2(u_2 - a + m - \frac{1}{2} b)^2 + ipu_1 + ip\tau_1(u_2 - a + m - \frac{1}{2} p) + \beta_p,
\]

where $\beta_p := ibp - \frac{1}{4} b^2 \tau_2$. Now, the Fourier transform of $|\varphi_q|^2$ is given by

\[
FT(|\varphi_q|^2) = \frac{c_0^2}{\tau_2} \sum_{m,m'=-\infty}^{\infty} \int_{\mathbb{R}^2} du e^{2\pi i(u_1 + \tau_1 u_2 + \alpha_{m,m'}(u))},
\]
Using the change of variables and the Plancherel theorem, we find relation (A.7) again.

To compute the Fourier coefficients $c_p, \ell$ this expression and the relations (A.14), we obtain

$$\int_{\mathbb{R}} dy_1 e^{2\pi i \xi y_1} = \delta(\xi),$$

and changing $p$ to $-p$, we obtain

$$\int_{\mathbb{R}} dy_2 e^{2\pi i (-\tau y_2^2 + i n y_2)} = \frac{1}{\sqrt{2\tau_2}} e^{-2\pi \frac{|n|}{\sqrt{\tau_2}}}.$$  

Using the standard formulæ, the first of which is the Poisson summation formula,

$$\sum_{m=-\infty}^{\infty} e^{-2\pi i m \xi_2} = \sum_{n=-\infty}^{\infty} \delta(\xi_2 - n),$$

which together with (A.14) gives, after passing to the new variables $y_1 = u_1$ and $y_2 = u_2 - a + m - \frac{1}{2} p$,

$$\text{FT}(|\varphi_q|^2) = c_0^2 \sum_{p, n, \ell = -\infty}^{\infty} \int_{\mathbb{R}^2} dy e^{2\pi i (\xi_1 y_1 + \xi_2 (y_2 + a - m + \frac{1}{2} p) - \tau_2 y_2^2 + ipy_1 + ip\tau_1 y_2 + \beta_\ell)}. \tag{A.15}$$

Now, using the standard formulæ, the first of which is the Poisson summation formula,

$$\sum_{m=-\infty}^{\infty} e^{-2\pi i m \xi_2} = \sum_{n=-\infty}^{\infty} \delta(\xi_2 - n), \tag{A.16}$$

we obtain

$$\int_{\mathbb{R}} dy_1 e^{2\pi i \xi y_1} = \delta(\xi) \tag{A.17}$$

and changing $p$ to $-p$, we obtain

$$\int_{\mathbb{R}} dy_2 e^{2\pi i (-\tau y_2^2 + i n y_2)} = \frac{1}{\sqrt{2\tau_2}} e^{-2\pi \frac{|n|}{\sqrt{\tau_2}}}.$$  

Since $\beta_p := -ibp - \frac{1}{2} p^2 \tau_2$, we have $\frac{1}{4\tau_2} (n - pr_1)^2 - \beta_p = \frac{1}{4\tau_2} |n - pr|^2 + ibp$. Due to (A.7), this gives the Fourier coefficients

$$c_{p, n}(|\varphi_q|^2) = c_0^2 \frac{1}{\sqrt{2\tau_2}} e^{-2\pi(\frac{|n|}{\sqrt{\tau_2}})}.$$

This expression and the relations $\int_{\Omega} |\varphi_q|^2 dz = \tau_2 \int_{[0,1]^2} |\varphi_q|^2 du = \tau_2 c_{00}$ and (A.6) yield (A.2) and

$$\int_{\Omega} |\varphi_0|^2 |\varphi_q|^2 dz = c_0^2 \frac{1}{2} \sum_{p, n, \ell = -\infty}^{\infty} e^{-2\pi(\frac{|n|}{\sqrt{\tau_2}})}.$$  

Separating the summation over positive and negative $p$, $\ell$, we see that this expression is real and gives (A.3).

We compute now the integral $\int_{\Omega} \varphi_0^* \varphi_q \varphi_{-q} dz = \int_{\Omega} f_{-q} f_q dz$, where $f_q := e^{\frac{2\pi i}{\tau_2} \text{Im}(\bar{q}x)} \bar{\varphi}_0 \varphi_q$. The functions $f_q$ are periodic functions w.r.t. the lattice $\mathcal{L}$. As above, we convert this to standard periodicity (w.r.t. the square lattice), by using (A.5), so that the functions $f_q$ are periodic functions w.r.t. $u_i$ with the period 1. Using the change of variables and the Plancherel theorem, we find

$$\int_{\Omega} \varphi_0^* \varphi_{-q} dz = \tau_2 \sum_{n_1, n_2 = -\infty}^{\infty} c_{-n_1, n_2} (f_{-q}) c_{n_1, n_2} (f_q). \tag{A.23}$$

To compute the Fourier coefficients $c_{n_1, n_2}(f_q)$, we compute the FTs of $f_q$ on the entire $\mathbb{R}^2$ and use the relation (A.7) again.

Using this, the series representation (2.8) - (2.5) for $\varphi_q(x)$, we obtain

$$f_q := e^{\frac{2\pi i}{\tau_2} \text{Im}(\bar{q}x)} \bar{\varphi}_0 \varphi_q = \sum_{m, m' = -\infty}^{\infty} e^{2\pi i m_{m'}(u)},$$
where

\[
\alpha_{m,m'}(u) = \frac{i}{\tau_2} \text{Im}(\bar{q}z) - \frac{1}{\tau_2} \bar{z}_2^2 - iaz + iqm + \frac{1}{2}i(m^2\tau - m'^2\bar{\tau}) + i(mz - m'\bar{z}) + \frac{1}{2}ia^2\tau - iab.
\]  

(A.24)

We use that \( \tau := \tau_1 + i\tau_2 \) and \( q = -a\tau + b \) and (A.5), to obtain

\[
i\frac{i}{\tau_2} \text{Im}(\bar{q}z) - iaz = \frac{i}{\tau_2}((-a\tau_1 + b)z_2 + a\tau_2z_1) - a(iz_1 - z_2) = \frac{1}{\tau_2}(i(-a\tau_1 + b) + a\tau_2)z_2 = i\frac{1}{\tau_2}qz_2 = iqu_2.
\]  

(A.25)

Next, we use the relations (A.5) and (A.9) and the notation \( p = m - m' \) and \( m' = m - p \) to obtain

\[
i\frac{i}{\tau_2} \text{Im}(\bar{q}z) - \frac{1}{\tau_2}z_2^2 - iaz + i(mz - m'z) = -\tau_2u_2^2 + a\tau_2u_2 + i(-a\tau_1 + b)u_2 + ip(u_1 + \tau_1u_2) - (2m - p)\tau_2u_2
\]

\[= -\tau_2[u_2^2 - au_2 + \ell u_2] + i[p - a]\tau_1 + bu_2
\]

\[= -\tau_2[u_2^2 - \frac{1}{2}a + m - \frac{1}{2}p] - (2m - p)\tau_2u_2 + i((p - a)\tau_1 + b)u_2
\]

\[= -\tau_2(u_2^2 - \frac{1}{2}a + m - \frac{1}{2}p)^2 + ipu_1 + i((p - a)\tau_1 + b)u_2 + \frac{1}{4}(2m - p - a)^2.
\]  

(A.26)

Now, we use that (see (A.12))

\[
qm = [i(-a\tau_1 + b) + a\tau_2]m,
\]

(A.27)

\[
\frac{1}{2}i(m^2\tau - (m')^2\bar{\tau}) = ip(m - \frac{1}{2}p)\tau_1 - \frac{1}{2}(m^2 - mp + \frac{1}{2}p^2)\tau_2.
\]  

(A.28)

The last three relations together with (A.24) give

\[
\alpha_{m,m'}(u) = -\tau_2(u_2^2 - \frac{1}{2}a + m - \frac{1}{2}p)^2 + i[p - a]\tau_1 + b(u_2^2 - \frac{1}{2}a + m - \frac{1}{2}p) + ipu_1 + \beta_{p,m},
\]  

(A.29)

where

\[
\beta_{p,m} = -ip(p - a)\tau_1 + b\frac{1}{2}(-a + 2m - p) + \tau_2\frac{1}{4}(-a + 2m - p)^2
\]

\[+ i(-a\tau_1 + b) + a\tau_2|m + ip(m - \frac{1}{2}p)\tau_1 - \frac{1}{4}(p^2 + (2m - p)^2)\tau_2 + \frac{1}{2}ia^2\tau - iab
\]

\[= i\frac{1}{2}b(p - a) - \frac{1}{4}(p - a)^2\tau_2.
\]  

(A.30)

Now, the Fourier transform of \( f_q \) is given by

\[
\text{FT}(f_q) = \sum_{m,m'} \int_{\mathbb{R}^2} du d\xi e^{2\pi i(\xi_1u_1 + \xi_2u_2 + \alpha_{m,m'}(u))}.
\]

Using (A.29) and passing to the new variables \( y_1 = u_1 \) and \( y_2 = u_2 - \frac{1}{2}a + m - \frac{1}{2}p \), we find

\[
i\xi_1y_1 + i\xi_2y_2 + \alpha_{m,m'}(u) = i\xi_1y_1 + i\xi_2(y_2 + \frac{1}{2}a - \frac{1}{2}\ell) - \tau_2y_2^2 + i(p - a)\tau_1y_2 + iby_2 + ipy_1 + \beta_{p,\ell}
\]

\[= -i\xi_2m + \frac{1}{2}i\xi_2(p + a) + i(\xi_1 + p)y_1 - \tau_2y_2^2 + i(\xi_2 + (p - a)\tau_1 + b)y_2 + \beta_{p,m}.
\]  

(A.32)
Now, changing $p$ to $-p$, using the Poisson summation formula (A.16) and the standard formulae (A.17)-(A.18), we obtain

\[
\text{FT}(f_q) = \sum_{p,n=-\infty}^{\infty} \int_{\mathbb{R}^2} dy e^{2\pi i (\xi_1 - p)y_1 - \gamma y_2^2 + i(\xi_2 -(p+a)\tau_1 + b) y_2 + i \frac{1}{2} \xi_1(p-a) + \beta_{-p,m}} \delta(\xi_2 - n)
\]

\[
= \sum_{p,n=-\infty}^{\infty} \int_{\mathbb{R}} dy e^{2\pi i (\xi_2 -(p+a)\tau_1 + b) y_2 - i \xi_2 (p-a) + \beta_{-p,m}} \delta(\xi_2 - n) \delta(\xi_1 - p)
\]

\[
= \frac{1}{\sqrt{2\pi}} \sum_{p,n=-\infty}^{\infty} e^{-2\pi i \left( \frac{1}{4\tau_2} \left[ n-(p+a)\tau_1 + b \right]^2 + i\frac{1}{2} \left[ n(p-a) + b(p+a) \right] + \frac{1}{4} (p+a)^2 \tau_2 \right)} \delta(\xi_2 - n) \delta(\xi_1 - p).
\]

(A.33)

This expression together with the computations

\[
\frac{1}{4\tau_2} \left[ n-(p+a)\tau_1 + b \right]^2 + i\frac{1}{2} \left[ n(p-a) + b(p+a) \right] + \frac{1}{4} (p+a)^2 \tau_2
\]

\[
= \frac{1}{4\tau_2} \left[ n-(p+a)\tau_1 + b \right]^2 + i\frac{1}{2} \left[ n(p-a) + b(p+a) \right] + \frac{1}{4} (p+a)^2 \tau_2
\]

\[
= \frac{1}{4\tau_2} \left[ n-(p+a)\tau_1 + b \right]^2 + i\frac{1}{2} \left[ np - na + bp + ab \right]
\]

(A.34)

and (A.7), gives the Fourier coefficients for $f_q$

\[
e_p,n(f_q) = \frac{1}{\sqrt{2\pi}} e^{-\pi \left( \frac{1}{4\tau_2} \left[ n-(p+a)\tau_1 + b \right]^2 + i\left[ np - na + bp + ab \right] \right)}.
\]

(A.35)

Since $e^{-2\pi i np} = 1$, this formula, together with (A.23), yields

\[
\int_{\Omega} f_q dz = \sum_{p,n=-\infty}^{\infty} e^{-2\pi i \left( \frac{1}{4\tau_2} \left[ n-(p+a)\tau_1 + b \right]^2 + i\left[ np - na + bp + ab \right] \right)}
\]

(A.36)

which in turn gives (A.4).

\[
\square
\]

B  Numerical investigation of $\gamma(\tau)$ and $\gamma_k(\tau)$ (by D. Ginsberg)

In this appendix, we include the results of some numerical experiments which provide evidence for the conjectures made in Sections 1.1 and 1.6, namely that $\gamma(\tau)$, defined in (1.5) on the Poincaré strip $\{ \tau \in \mathbb{C}, \text{Re} \tau \in [0,1/2], \text{Im} \tau \geq 0, \ |\tau| \geq 1 \}$,

- achieves its unique global maximum at $\tau = e^{i\pi/3}$;
- has a saddle point at $\tau = e^{i\pi/2}$; more precisely, $\tau = e^{i\pi/2}$ appears to be a local maximum in the direction of the Im $\tau$ axis, but a cusp along the Re $\tau$ axis (see Figure 4 for a plot of the contours and gradient field of the approximation $\gamma_{appr}(\tau)$ defined in (B.11));
- for fixed $\text{Re} \tau \in [0,1/2]$, $\gamma(\tau)$ is a decreasing function of $\text{Im} \tau$;
- $\gamma(\tau)$ is positive for $\text{Im} \tau \leq 1.81$ and negative otherwise and takes values (approximately) 0.64 and 0.4 at $\tau = e^{i\pi/3}$ and $\tau = e^{i\pi/2}$, respectively.

Recall that the function $\gamma(\tau)$, $\text{Im} \tau > 0$, on lattice shapes $\tau$, is defined as $\gamma(\tau) := \inf_{k \in \Omega} \kappa(\tau)$, where the functions $\kappa(\tau)$, $\text{Im} \tau > 0$, $k \in \Omega^*$, admit the explicit representation (1.23). Let $k = \sqrt{\frac{2\pi}{\text{Im} \tau}} iq$, with $q = b - a \tau$, $\frac{1}{2} \leq a, b \leq \frac{1}{2}$. Our computations show that

- $\kappa(\tau)$ is minimized at $q = \frac{1}{2} - \frac{1}{2\sqrt{3}} i$ at the point $\tau = e^{i\pi/3}$, and a value of $q \approx \frac{1}{2} + i \frac{1}{2}$ for $\tau = e^{i\pi/2}$, which corresponds to one of the vertices of the corresponding Wigner-Seitz cells.
Figure 2: Plots of the function $\gamma^{approx}(\tau)$. Computed in Matlab on a uniform grid with step size 0.01. The plot on the right is the function plotted only on the Poincaré strip. The circled points are $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$.

Figure 3: Plot of the gradient of $\gamma^{approx}(\tau)$, computed on the same grid as in the previous figure. The two marks indicate the points $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$.

See Figure 2 for the result of solving the problem $\gamma(\tau) := \inf_{k \in \Omega^*_\tau} \gamma_k(\tau)$ (see (1.5)) in Matlab, using the default Nelder-Mead algorithm.

The points $q \approx \frac{1}{2} - \frac{1}{2\sqrt{3}}i$ at the point $\tau = e^{i\pi/3}$, and $q \approx \frac{1}{2} + i\frac{1}{2}$ for $\tau = e^{i\pi/2}$ correspond to one of the vertices of the corresponding Wigner-Seitz cells. Recall that the Wigner-Seitz cell, which we denote it by $\Omega^*_W$, is defined as the set of all points in $\mathbb{R}^2$ closer to a given vertex of $\mathcal{L}$- say, at the origin - than to any other point of $\mathcal{L}$. (In physics literature, the Wigner-Seitz cell $\Omega^*_\tau$ is called the Brillouin zone of $\mathcal{L}$.) Its advantage is that it is independent of the choice of the basis and reveals symmetries of the underlying lattice. Note that for $\tau = e^{i\pi/2}$, the Wigner-Seitz cell coincides with a usual fundamental cell.

The minimization over $\Omega^*_\tau$ is equivalent to the minimization over $\Omega^*_{WS}$. To see that this, we show that there is a map $\varphi : \Omega^*_WS \to \Omega^*_\tau$, s.t. $\varphi(x) - x \in \mathcal{L}^*_\tau$. To construct such a map, we let $\tilde{\Omega}^*_\tau = \frac{2\pi}{\Im \tau} i \Omega^*_\tau$ and $\tilde{\Omega}^*_{WS} = \sqrt{\frac{2\pi}{\Im \tau}} i \Omega^*_WS$ and observe that $\tilde{\Omega}^*_{WS}$ intersects four fundamental cells, $\tilde{\Omega}^*_\tau + (1/2)(1 + i)$, $\tilde{\Omega}^*_\tau - (1/2)(1 + i)$, $\tilde{\Omega}^*_\tau + (1/2)(1 - i)$, $\tilde{\Omega}^*_\tau - (1/2)(1 - i)$. It is easy to see by translating three of these intersections, by an element of $\mathcal{L}^*_\tau$, into the cell of the fourth one, i.e. consolidating all the intersections in a single cell, that
these translations tile up the fundamental cell, $\hat{\Omega}_\tau^\ast$. This maps $\hat{\Omega}_{\tau WS}^\ast$ onto $\hat{\Omega}_\tau^\ast$ and shows that, as expected, $\inf$ over the Wigner-Seitz cell $= \inf$ over a fundamental cell.

The vertices of the Wigner-Seitz cell $\Omega_{\tau WS}^\ast$. are preimages under the above map of the orbit of following the point of $\hat{\Omega}_\tau^\ast$:

\begin{equation}
(1/2) - (1/2\sqrt{3})i \tag{B.1}
\end{equation}

under rotations by $\pi/3$ and multiples of it. (This is the point on the intersection of the diagonal from the lower left to the upper right vertex and the perpendicular at the midpoint to the upper side of the parallelogram $\hat{\Omega}_\tau^\ast$.) Of course, because of the discrete rotational symmetry, each of these points has the same minimum, so it suffices to check only one of them.

Now we explain our computations. We rewrite the functions $\gamma_k(\tau)$, $\Im \tau > 0$, $k \in \Omega_{\tau WS}^\ast$, given explicitly by (1.23), as

\begin{equation}
\gamma_k(\tau) = 2\gamma_{q1}(\tau) + |\gamma_{q2}(\tau)| − \gamma_{01}(\tau) \tag{B.2}
\end{equation}

where the variable $q = b - a\tau$, $\frac{1}{2} \leq a, b \leq \frac{1}{2}$, is related to the variable $k$ as $k = \sqrt{\frac{2\pi}{\Im \tau}} iq$ and

\begin{align}
\gamma_{q1}(\tau) &= \sum_{m,n=-\infty}^{\infty} e^{-\frac{\pi}{2}|n-m\tau|^2} \cos[2\pi(bm-an)] \tag{B.3} \\
\gamma_{q2}(\tau) &= \sum_{m,n=-\infty}^{\infty} e^{-\frac{\pi}{2}|n-m\tau+q|^2} e^{-2\pi i[mn-an]} \tag{B.4} \\
\gamma_{01}(\tau) &= \sum_{m,n=-\infty}^{\infty} e^{-\frac{\pi}{2}|n-m\tau|^2}. \tag{B.5}
\end{align}

We note that since $\gamma_{-k}(\tau) = \gamma_k(\tau)$ (see (2.14)), it suffices to consider $q = -a\tau + b$, with $b > 0$.

To estimate of $\gamma_{q1}(\tau)$, we observe that the exponentials in (B.3) decrease very fast with $m$ and $n$ increasing and rewrite $\gamma_{qi}(\tau)$ as $\gamma_{qi}(\tau) = \gamma_{qi}^{appr}(\tau) + \text{rem}_i$, $i = 1, 2$, where

\begin{equation}
\gamma_{q1}^{appr}(\tau) = 1 + 2(e^{-\frac{\pi}{2}|\tau|^2} \cos[2\pi a] + e^{-\frac{\pi}{2}|\tau|^2} \cos[2\pi b] − e^{-\frac{\pi}{2}|1-\tau|^2} \cos[2\pi(b-a)]), \tag{B.6}
\end{equation}
\[ \gamma_{appr}^{q_2}(\tau) := \sum_{m,n=-2}^{2} e^{\frac{\pi}{2}[n-m|\tau|^2-2\pi i|bm-an]} \]
\[ = e^{-\frac{\pi}{2}|\tau|^2} + e^{-\frac{\pi}{2}|\tau|^2+2\pi i} + e^{-\frac{\pi}{2}|\tau|^2-2\pi i} + e^{-\frac{\pi}{2}|\tau|^2-2\pi i} + e^{-\frac{\pi}{2}|\tau|^2+2\pi i} \]

(remember that |\tau| \geq 1), and the \text{rem}_1 is defined by these expressions.

We estimate the \text{rem}_1. It is given explicitly as
\[ \text{rem}_1 = 2e^{-\frac{\pi}{2}|\tau|^2} \cos[2\pi(b+a)] + \text{rem}', \quad \text{rem}' := 2 \sum_{m \geq 2} (e^{-\frac{\pi}{2}m|\tau|^2} \cos[2\pi bm] + e^{-\frac{\pi}{2}m^2} \cos[2\pi am]) \]
\[ + 2 \sum_{m,n \geq 2} (e^{-\frac{\pi}{2}|n-m|\tau|^2} \cos[2\pi(bm-na)] + e^{-\frac{\pi}{2}|n+m|\tau|^2} \cos[2\pi(bm+na)]). \]

We estimate the \text{rem}_1. For Re \\tau \in [0,1/2] and Im \\tau \geq \frac{\sqrt{2}}{2}, the first term in (B.8) can be estimated as
\[ 2e^{-\frac{\pi}{2}|\tau|^2} \cos[2\pi(b+a)] \leq 2e^{-\frac{\pi}{2}} \leq 2 \cdot 10^{-3}. \]

On the other hand, for \text{rem}', we have
\[ |\text{rem}'| \leq 2 \sum_{m \geq 2} (e^{-\frac{\pi}{2}m^2|\tau|^2} + e^{-\frac{\pi}{2}m^2}) + 2 \sum_{m,n \geq 2} (e^{-\frac{\pi}{2}|n-m|\tau|^2} + e^{-\frac{\pi}{2}|n+m|\tau|^2}). \]

For Re \\tau \in [0,1/2] and |\tau| \geq 1, we have that |n-m| \geq n^2+m^2-nm \geq \frac{1}{2}(n^2+m^2) and |n+m| \geq n^2+m^2, which leads to
\[ |\text{rem}'| \leq 4 \sum_{m \geq 2} e^{-\frac{\pi}{2}m^2} + 2 \sum_{m,n \geq 2} (e^{-\frac{\pi}{2}(n^2+m^2)} + e^{-\frac{\pi}{2}(n^2+m^2)}) \]
\[ = 4 \sum_{m \geq 2} e^{-\frac{\pi}{2}m^2} + 2(\sum_{m \geq 2} e^{-\frac{\pi}{2}m^2})^2 + 2(\sum_{m \geq 2} e^{-\frac{\pi}{2}m^2})^2. \]

For Im \\tau \geq \frac{\sqrt{2}}{2}, this implies |\text{rem}''| \leq 10^{-5} and therefore
\[ |\text{rem}_1| \leq 2.5 \cdot 10^{-3}. \]

Similarly to (B.9), we have |\text{rem}_2| \leq 2.5 \cdot 10^{-3}.

The above implies that
\[ |\gamma_k(\tau) - \gamma_k^{appr}(\tau)| \leq 7.5 \cdot 10^{-3}. \]

where
\[ \gamma_k^{appr}(\tau) = 2\gamma_{q_1}^{appr}(\tau) + |\gamma_{q_2}^{appr}(\tau)| - \gamma_{0_1}^{appr}(\tau). \]

Now, numerical computation of \( \gamma_k^{appr}(\tau) \) using MatLab leads to the conclusions above.

\section{Explicit expressions for \( L_\omega \) and \( N_\omega(v) \)}

In this appendix we present the explicit expressions for the linear operator \( L_\omega \) and the \( v \)– nonlinearity \( N_\omega(v) \) in the equation \( (3.15) \):
\[ L_\omega v = \left( -\Delta A_\omega \xi + \kappa^2(2|\Psi_\omega|^2 - 1)\xi + \kappa^2\Psi_\omega^2\xi + 2i(\nabla A_\omega \Psi_\omega) \cdot \alpha + i\Psi_\omega \text{div} \alpha \right), \text{curl} \xi \text{curl} \alpha + |\Psi_\omega|^2 \alpha - \text{Im}(\xi \nabla A_\omega \Psi_\omega + \Psi_\omega \nabla A_\omega \xi) \right), \]
where \( v = (\xi, \alpha) \),
\[ N_\omega(v) = \left( -2i\alpha \cdot \nabla A_\omega \xi - |\alpha|^2\Psi_\omega - |\alpha|^2\xi - i\xi \text{div} \alpha - 2\kappa^2 \text{Re}(\Psi_\omega \xi) \xi - \kappa^2\Psi_\omega |\xi|^2 - \kappa^2|\xi|^2\xi \right). \]
D Estimates of remainder $R_\omega$ and nonlinearity $N_\omega$

In this appendix we obtain the bounds on the remainder $R_\omega$ and the nonlinearity $N_\omega$ used in Section 3.5.

We have

Proposition D.1. We have the estimates

$$\begin{align*}
|R_\omega(v)| &\lesssim \|v\|_{H^1}^2 + \|v\|_{H^1}^4, \quad (D.1) \\
\|N_\omega(v)\|^2 &\lesssim (\|v\|_{H^1}^2 + \|v\|_{H^1}^5)\|v\|_{H^2}, \quad (D.2)
\end{align*}$$

Proof. We first note that, due to the d-magnetic inequality for $A \in L^2_{loc}(\mathbb{R}^2)$ (see [33]), $|\nabla f| \leq |\nabla A f|$, and by the standard Sobolev embedding theorem $H^1_{cov}$ is continuously embedded in $L^p(\mathbb{R}^2; C \times \mathbb{R}^2)$ for all $p \in [2, \infty)$. We also note that since $\Psi_\omega$ is a gauge-periodic smooth function, it is bounded. So we have

$$\int |\xi|^2 \text{Re}(\bar{\Psi}_\omega \xi) \leq \int |\xi|^3 |\bar{\Psi}_\omega| \lesssim \int |\xi|^3 \lesssim \|v\|_{H^1}^3.$$ 

We also have

$$\int |\alpha| \text{Im}(\bar{\xi} \nabla_\omega \xi) \leq \left( \int |\alpha|^4 \right)^{\frac{1}{4}} \left( \int |\xi|^4 \right)^{\frac{1}{4}} \left( \int |\nabla_\omega \xi|^2 \right)^{\frac{1}{4}} \lesssim \|v\|_{H^1}^4.$$ 

The other terms of $R_\omega$ are handled similarly.

To establish the bound (D.2), we consider the worst term $\bar{\xi} \nabla \xi$ in (C.2) (coming from $\bar{\xi} \nabla_\omega \xi$). Then

$$\|\bar{\xi} \nabla \xi\|^2 \leq \|\xi\|_p^2 \|\nabla \xi\|_p \|\nabla \xi\|_2,$$

with $p^{-1} + (2q)^{-1} = 4^{-1}$. Now, as long as $p, q < \infty$, by the Sobolev embedding theorem in dimension 2, we have $\|\xi\|_p \lesssim \|\xi\|_{H^1}$ and $\|\nabla \xi\|_q \lesssim \|\xi\|_{H^2}$ and therefore

$$\|\bar{\xi} \nabla \xi\|^2 \lesssim \|\xi\|_{H^1}^2 \|\xi\|_{H^2}.$$ 

The remaining terms are simpler and treated similarly. □

E Estimates of terms involving $V_\gamma$ and $F_\omega$

In this appendix we obtain the bounds on the terms in (3.35) involving $V_\gamma$ and $F_\omega$ used in Section 3.5.

Proposition E.1. We have the estimates

$$\begin{align*}
\langle L_\omega v, V_\gamma v \rangle &\lesssim \epsilon^2 (\|v\|_{H^1}^2 + \|v\|_{H^1}^4)\|v\|_{H^2}^2, \quad (E.1) \\
\langle N_\omega(v), V_\gamma v \rangle &\lesssim \epsilon^2 (\|v\|_{H^1}^3 + \|v\|_{H^1}^5)\|v\|_{H^2}^2, \quad (E.2) \\
\|\langle N_\omega(v), F_\omega \rangle\| &\lesssim \epsilon^2 (\|v\|_{H^1}^2 + \|v\|_{H^1}^4)\|v\|_{H^2}^2. \quad (E.3)
\end{align*}$$

Proof. We begin with an estimate $|\dot{\gamma}|$.

Lemma E.2. We have the estimates

$$\|\dot{\gamma}\|_{H^2} \lesssim \epsilon^2 (1 + \|v\|_{H^1}^4)\|v\|_{H^2}^2, \quad (E.4)$$

Proof. We use the equation (3.17) for $\dot{\gamma}$ and therefore first we have to show that the operator $\Delta + |\Psi_\omega|^2 + \text{Re}(\bar{\Psi}_\omega \xi)$ in (3.17), considered from $H^{s+2}$ to $H^s$, is invertible, for $\|\xi\|_{L^2}$ sufficiently small. The latter fact follows from the bound $-\Delta + |\Psi_\omega|^2 \geq c\epsilon^2$ for some constant $c > 0$ independent of $\epsilon$ and the condition $\|\xi\|_{L^2} \ll \epsilon^2$. This implies that $\|(-\Delta + |\Psi_\omega|^2 + \text{Re}(\bar{\Psi}_\omega \xi))^{-1}\|_{L^2 \rightarrow H^2} \lesssim \epsilon^{-2}$, which due to (3.17), gives that $\|\dot{\gamma}\|_{H^2} \lesssim \epsilon^{-2} (\|\text{Im}(\bar{\Psi}_\omega N_\omega(v)) - \text{div} N_\alpha(v)\|_{L^2}) \lesssim \epsilon^{-2} (\|\bar{\xi} N_\omega(v)\|_{L^2} + \|N_\alpha(v)\|_{H^1})$. Remembering the definitions $N_\omega(v) = (N_\xi(v), N_\alpha(v))$ and (C.2) and using the Sobolev embedding theorem as in the proof of (D.2), we obtain $\|N_\alpha(v)\|_{H^1} \lesssim \|v\|_{H^1}^2 + \|v\|_{H^1}^4\|v\|_{H^2}^2$, which implies (E.4). □
The definitions of $N_\omega(v)$ and $F_\omega$ and integration by parts give

$$
\langle L_\omega v, V_\omega v \rangle \lesssim \|v\|_{L^2}^2,
$$

(E.5)

$$
\langle N_\omega(v), V_\omega v \rangle \lesssim \|v\|_{H^2}^2 + \|v\|_{H^1}^4,
$$

(E.6)

$$
\|\langle N_\omega(v), F_\omega \rangle\| \lesssim \|\gamma\|_{H^2}^2 + \|v\|_{H^1}^3.
$$

(E.7)

(E.5) - (E.7) follow from (E.5) - (E.7) and Lemma E.2.

\[ \square \]

## Explicit expressions of various hessians

In this appendix we present the explicit expressions for various hessians derived from the hessian $L_\omega$, given explicitly in (4.1), and used in the main text. Let $v = (\xi, \alpha)$. It is straightforward to show that the shifted $\tilde{L}_\omega$, defined by (4.2), is given explicitly as

$$
\tilde{L}_\omega v = \left( -\Delta_\omega \xi - \kappa^2 \xi + (2\kappa^2 + \frac{1}{2})|\Psi_\omega|^2 \xi + (\kappa^2 - \frac{1}{2})\Psi_\omega \bar{\xi} + 2i(\bar{\nabla}_\omega \Psi_\omega) \cdot \alpha \right).
$$

(F.1)

The rescaled hessian defined as $L_{\omega}^{\text{resc}} := \sigma^2 U_\sigma L_\omega U_\sigma^{-1}$. It is explicitly given by

$$
L_{\omega}^{\text{resc}} v = \left( -\Delta_\omega \xi - \lambda_\omega \xi + (2\kappa^2 + \frac{1}{2})|\psi_\omega|^2 \xi + (\kappa^2 - \frac{1}{2})\psi_\omega \bar{\xi} + 2i\alpha \cdot \nabla_\omega \psi_\omega \right).
$$

(F.2)

We now derive the explicit expression for the complexified hessian $K$, introduced in Section 4.3. We introduce the notation $\partial_{ac} = \partial - ia^C$. Straightforward calculations show that

$$
2\alpha \cdot \nabla_\omega \psi = -i(\partial^a_\omega \psi) \alpha^C + i(\partial_{ac} \psi) \bar{a}^C,
$$

and that

$$
-\Im(\xi \nabla_\omega N_\omega)^C = \frac{i}{2}(\partial_\omega^a \overline{\psi}) \xi + \frac{i}{2}(\partial_{ac} \psi) \bar{\xi}.
$$

Using the above relations $K_\omega$ is explicitly given by

$$
K = \begin{pmatrix}
    h_\omega - \lambda_\omega & (\kappa^2 - \frac{1}{2})\psi_\omega^2 & -i(\partial^a_\omega \psi_\omega) & i(\partial_{ac} \psi_\omega) \\
    (\kappa^2 - \frac{1}{2})\overline{\psi}_\omega^2 & h_\omega - \lambda_\omega & -i(\partial^a_\omega \overline{\psi}_\omega) & i(\partial_{ac} \overline{\psi}_\omega) \\
    i(\overline{\partial}^a_\omega \psi_\omega) & i(\partial_\omega^a \psi_\omega) & -\Delta + |\psi_\omega|^2 & 0 \\
    -i(\overline{\partial}^a_\omega \overline{\psi}_\omega) & i(\partial_{ac} \overline{\psi}_\omega) & 0 & -\Delta + |\overline{\psi}_\omega|^2
\end{pmatrix}.
$$

(F.3)

where $h_\omega := -\Delta_\omega + (2\kappa^2 + \frac{1}{2})|\psi_\omega|^2$. It is not hard to check that $K$ restricted to vectors on the right hand side of (4.6) gives $L_{\omega}^{\text{resc}}$.

Now, using the expansion (4.25), we easily obtain $K = K^0 + \epsilon W^1 + \epsilon^2 W^2 + o(\epsilon^3)$, with (4.29) and (4.30) and (4.31).

Finally, we note that the subspaces $V_\omega$ are spectral eigenspaces of a real-linear operator commuting with $K_\omega$. Indeed, we define the real-linear operator $\gamma = CS$, where $S$ is the operator given by

$$
S = \begin{pmatrix}
    0 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0
\end{pmatrix}
$$

(F.4)

and the facts that for all $b$,

(a) $[K_\omega, \gamma] = 0$,

(b) $\gamma^2 = 1$,
(c) $\gamma$ leaves the space $K$ invariant,

(d) $\gamma$ has eigenvalues $\pm 1$ and the corresponding subspaces span the entire space $K$.

To see the last property we note that the vectors of the form $(\xi, \pm \xi, \alpha, \pm \bar{\alpha})$ are eigenvectors of the operator $\gamma$ with the eigenvalues $\pm 1$ and that any vector in $K$ can be written as a sum of such eigenvectors as shown in (4.11). The eigenspaces of $\gamma$ on $K$ corresponding to the eigenvalues $\pm 1$ are invariant under the operator $K$ and are exactly the spaces $V_\rho$, $\rho = \pm$, defined above. Hence it suffices to study the restrictions $K_\rho$ of $K$ to these invariant subspaces, $V_\rho$, $\rho = \pm$.

**Supplement I. Refined theta functions**

This supplement contains a description of parameterization of lattices and several elementary computations related to modified theta functions.

**Parametrization of classes of lattices.** Here we present standard results on parametrization of lattices. Recall that we identify $\mathbb{R}^2$ with $\mathbb{C}$, via the map $(x_1, x_2) \rightarrow x_1 + ix_2$. Given a basis $(\nu_1, \nu_2)$ in $\mathcal{L}$ (so that $(\nu_1, \nu_2)$ as $\mathcal{L} = \mathbb{Z}\nu_1 + \mathbb{Z}\nu_2$), define the complex number

$$\tau = \frac{\nu_1 + \nu_2 + \gamma}{\nu_1 + \nu_2 + \delta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, and $\alpha \delta - \beta \gamma = 1$ (with the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

an element of the modular group $SL(2, \mathbb{Z})$). Under this map, the shape parameter $\tau = \nu_1/\nu_2$ is being mapped into $\tau' = \nu'_1/\nu'_2$ as $\tau \rightarrow \tau' = g\tau$, where $g^t := \frac{\alpha \tau + \beta}{\gamma \tau + \delta}$. Hence, if we care only about the lattice and not about a particular choice of a basis in it, it suffices to restrict $\tau$ to the fundamental domain $\Pi^+ / SL(2, \mathbb{Z})$. This uniquely parametrizes the lattices up to rotations and dilatations, and therefore $\tau \in \Pi^+ / SL(2, \mathbb{Z})$ is in one-to-one correspondence with the equivalence classes $[\mathcal{L}]$.

We can associate the equivalence class $[\tau]$ with the class $[\mathbb{Z} + \tau \mathbb{Z}]$.

**Theta functions.** In this paragraph we prove the following

**Lemma I.2.** The functions $\theta_q(z, \tau)$ are given by (2.5), if and only if they are entire functions (i.e. they solve $\partial \theta_q = 0$) and satisfy the periodicity conditions (2.3) - (2.4).

**Proof.** It is easy to see that the functions given by (2.5) are entire functions satisfying the periodicity conditions (2.3) - (2.4). Now we show the converse. The relation (2.3) shows that the function $e^{2\pi i az} \theta_q(z, \tau)$ has the periodicity properties of $\theta(z, \tau)$ and therefore it has an absolutely convergent Fourier expansion

$$e^{2\pi ia z} \theta_q(z, \tau) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi imz}, \quad (I.1)$$

while the relation (2.4), on the other hand, leads

$$e^{2\pi i k + \pi i \tau} e^{-2\pi i a \tau} \sum_{m=-\infty}^{\infty} c_m e^{2\pi im(z+\tau)} = \sum_{m=-\infty}^{\infty} c_m e^{2\pi imz-2\pi i z} = \sum_{m=-\infty}^{\infty} c_{m+1} e^{2\pi imz},$$

which implies $c_{m+1} = e^{2\pi i k + \pi i \tau} e^{-2\pi i a \tau} e^{2\pi i m \tau} c_m = e^{2\pi i q \tau} e^{2\pi i m \tau} c_m$. Iterating this relation and using $\sum_{m}^{m} n = \frac{1}{2} m (m + 1)$, we find $c_m = e^{2\pi i q + \pi im \tau}$, which implies the series representation for $\theta_q$ given by (2.5). □

**Corollary I.3.** The functions $\varphi_q$, defined as $\varphi_q(z) = \phi_k(x)$, where, recall, $x_1 + ix_2 = \sqrt{\frac{2\pi}{\Im \tau}} z \; \text{and} \; \xi = \sqrt{\frac{2\pi}{\Im \tau}} iq$, satisfy the periodicity relation (2.9). In the opposite direction, (2.9) implies (2.3) - (2.4).
Proof. The relations (2.8) - (2.5) and (2.3) imply \( \varphi_q(z+1, \tau) = e^{\alpha_1} \varphi_q(z, \tau) \), where \( \alpha_1 := \frac{\pi}{\text{Im} \tau} (z - z_1) - 2 \pi i a = \frac{\pi}{\text{Im} \tau} (\text{Im} z + 2 \text{Im} q) = \frac{\pi}{\text{Im} \tau} (\text{Im}(\bar{s} z) + 2 \text{Im}(\bar{s} q)) \), for \( s = 1 \). Similarly, the relations (2.8) and (2.4) imply \( \varphi_q(z + \tau, \tau) = e^{\alpha_2} \varphi_q(z, \tau) \), where

\[
\alpha_2 := \frac{\pi}{\tau_2} (z \tau + \frac{1}{2} \tau_2^2 - z_1 \tau - \frac{1}{2} \tau_1^2 - z_2 \tau - \frac{1}{2} \tau_2^2) - \pi (2ib + i \tau + 2i z)
\]

for \( s = \tau \). The latter relation implies (2.3) and (2.4).

In the opposite direction, (2.9) implies \( \theta_q(z + s, \tau) = e^{\beta_3} \theta_q(z, \tau) \), where

\[
\beta_3 = \frac{\pi}{\tau_2} (i z_2 s_1 + i z_1 s_2 - 2 s_2 s_2 + i s_1 s_2 - s_2^2) - i(z_1 s_2 - s_2 s_1) - i 2 \text{Im}(\bar{s} q))
\]

\[
= \frac{2 \pi i}{\tau_2} (z s_2 + \frac{1}{2} s s_2 - \text{Im}(\bar{s} q))
\]

\[
= \frac{2 \pi i}{\tau_2} \left[ \text{Im}(\bar{s} s) - (\text{Im} s) (z + \frac{1}{2} s) \right].
\]

The latter relation implies (2.3) and (2.4). □

Remarks. 1) Alternatively, one can define the refined theta function \( \theta_q(z, \tau) \) as an entire function satisfying the gauge-periodicity conditions (2.3) - (2.4).

2) It is easy to verify directly that the function \( e^{\frac{\pi i \text{Im}(\bar{s} z)}{\text{Im} \tau}} \theta_q(z, \tau) \) has the periodicity properties of \( \theta(z, \tau) \).

3) In the terminology of Sect 13.19, eqs 10-13 of [19], our theta function \( \theta(z, \tau) \) is called \( \theta_3(z, \tau) \). The choice of the original theta function determines the location of zeros of \( \phi_3(z) \): The zeros of \( \theta_3(z, \tau) \) are located at the points of \( \mathbb{Z} + \tau \mathbb{Z} + \frac{1}{2} + \frac{1}{2} \tau \), while the zeros of \( \theta_3(z, \tau) \) (in the terminology of [19]) are located at the points of \( \mathbb{Z} + \tau \mathbb{Z} \). To compare, \( \theta_1(z, \tau) \) is defined as \( \theta_1(z, \tau) := \sum_{m=-\infty}^{\infty} e^{\pi im} e^{\pi i(m - \frac{1}{2} + \frac{1}{2} \tau)} e^{\pi i(2m - 1)z} \).

4) One can easily show that the refined and standard theta functions, \( \theta_q \) and \( \theta \), are related as \( \theta_q = T_a S_b \theta \), where \( q = -a \tau + b \) and \( (S_b f)(z) := f(z + b) \) and \( (T_a f)(z) := e^{-2 \pi i a z} f(z - a \tau) \).

Supplement II. Translational zero modes and spectrum

In this appendix we consider the translational zero modes and their relation to the spectrum of \( K \). We begin by writing out the rescaled gauge and translation modes

\[
G_{\gamma \gamma}^{\text{resc}} = (i \gamma' \psi_{\omega'}, \nabla \gamma'), \quad S_j^{\text{resc}} = \left( (\nabla_{a \omega'}) \psi_{\omega'}, (\text{curl} a_{\omega'} ) J e_j \right),
\]

and observing how the shifted and rescaled operator, \( L_{\omega'}^{\text{resc}} \), acts on them as

\[
L_{\omega'}^{\text{resc}} G_{\gamma \gamma}^{\text{resc}} = G_{h \gamma \gamma}^{\text{resc}}, \quad \text{where} \quad h := -\Delta + |\psi_{\omega'}|^2,
\]

and

\[
L_{\omega'}^{\text{resc}} S_j^{\text{resc}} = G_{\gamma_j \gamma_j}^{\text{resc}}, \quad \text{where} \quad \gamma_j := \text{Im}(\bar{s} \psi_{\omega'} \nabla_{a \omega'} \psi_{\omega'}) - \text{div}(\text{curl} a_{\omega'} J e_j).
\]

Though \( S_j^{\text{resc}} \) are not zero modes of \( L_{\omega'}^{\text{resc}} \), the functions

\[
T_j^{\text{resc}} := S_j^{\text{resc}} - G_{h - \gamma_j}^{\text{resc}}
\]

are.
are (generalized) eigenfunctions with the eigenvalue 0. Indeed, using (II.2) and (II.4), we obtain \( L_{\omega}^\text{resc} (\gamma_\omega^\text{resc} - G_{\gamma_\omega}^\text{resc}) = G_{\gamma_\omega}^\text{resc} - G_{\gamma_\omega}^\text{resc} = 0 \) and therefore we still have 0 \( \in \sigma_{\text{ess}}(L_{\omega}^\text{resc}) \). Now, using \( \langle \gamma_\omega^\text{resc}, G_{\gamma_\omega}^\text{resc} \rangle = \gamma_\omega \) and \( \langle G_{\gamma_\omega}^\text{resc}, \gamma_\omega \rangle = \langle h^{-1} \gamma_\omega \rangle \), we compute

\[
\langle T^\text{resc}_2, G_{\gamma_\omega}^\text{resc} \rangle = 0. \tag{II.5}
\]

The relations \( L_{\omega}^\text{resc} G_{\gamma_\omega}^\text{resc} = G_{\gamma_\omega}^\text{resc} \) and \( \langle G_{\gamma_\omega}^\text{resc}, G_{\gamma'_\omega}^\text{resc} \rangle = \langle \gamma_\omega, h \gamma'_\omega \rangle \) show that

\[
\inf_{\gamma, \|G_{\gamma_\omega}^\text{resc}\| = 1} \langle L_{\omega}^\text{resc} G_{\gamma_\omega}^\text{resc}, G_{\gamma_\omega}^\text{resc} \rangle_{L^2} = \inf h. \tag{II.6}
\]

Moreover, using (II.1) and (II.3), we compute

\[
\langle T^\text{resc}_2, L_{\omega}^\text{resc} S_{\gamma_\omega}^\text{resc} \rangle_{L^2(\Omega)} = \langle \text{Im}(\tilde{\psi}_{\omega} \nabla a_{\omega,j} \psi_{\omega}) \gamma_j - \text{div}(\text{curl} a_{\omega,j} E_j) \gamma_j \rangle_{\Omega} = \langle \gamma_j^2 \rangle_{\Omega}. \tag{II.7}
\]

To find the asymptotics of \( \gamma_j \), we use the expansions (4.26) below to obtain

\[
\gamma_j := \epsilon^2 [\text{Im}(\tilde{\psi}_{\omega} \nabla a_{\omega,j} \psi_{\omega}) - \text{div}(\text{curl} a^1 E_j)] + O(\epsilon^4). \tag{II.8}
\]

Now we complexify the translational modes discussed above. To this end we recall the notation \( \partial_a = \partial - ia \) and use the relation \( \partial_a = -i \text{curl} a + \text{div} a \) (here \( a = a^2 \) are the complexified vector fields) to write the complexified version of (3.6) as

\[
\tilde{S}_1 = (\partial_{a}, \psi_{\omega}, -\partial_a \psi_{\omega}, \partial_a \omega, \partial_a \omega), \quad \tilde{S}_2 = (\partial_{a}, \psi_{\omega}', -\partial_a \psi_{\omega}', -i \partial_a \omega, i \partial_a \omega). \tag{II.9}
\]

One can easily check that, as in (II.3), \( \tilde{S}_1 \) are not zero modes of \( K \). However, the complexifications, \( \tilde{T}_i \), of \( T_i^\text{resc} \), given in (II.4), are zero modes of \( K \). Since the vectors \( \tilde{T}_i \) are periodic w.r.t \( \mathcal{L} \) and therefore belong to \( K_0 \), they are zero modes of \( K_0 \). Moreover, \( e^{ik \cdot x} \tilde{T}_i \), for \( |k| \) small, are almost zero modes of \( K_0 \). Indeed, \( e^{-ik \cdot x} Ke^{ik \cdot x} = K + O(|k|) \) and therefore \( Ke^{ik \cdot x} \tilde{T}_i = O(|k|) e^{ik \cdot x} \tilde{T}_i \). Hence there is a positive branch of the spectrum of \( K \) on \( \mathcal{K} \), starting at 0, corresponding to translations of the lattice. This branch is ruled out by the condition (3.28).

Using the expansion \( \psi_{\omega} = O(\epsilon) \), \( a_\omega = a^0 + O(\epsilon^2) \), where \( a^0 := \frac{1}{2} J x \) and the computation \(-Ja^0 + Jx \cdot \nabla a^0 = 0\), the tangent vectors (II.1) can be expanded as \( G_{\gamma_\omega}^\text{resc} = G_{\gamma_\omega}^0 + O(\epsilon), \quad S_{\gamma_\omega}^\text{resc} = S_{\gamma_\omega}^0 + O(\epsilon) \), where \( G_{\gamma_\omega}^0 = (0, \nabla \gamma), \quad S_{\gamma_\omega}^0 = (0, J \gamma') \), or \( \tilde{S}_j = \tilde{S}_j^0 + O(\epsilon) \), where

\[
\tilde{S}_1^0 = (0, 0, 1, 1), \quad \tilde{S}_2^0 = (0, 0, i, -i). \tag{II.10}
\]

Hence they are related to the eigenvectors \( w_0^0 \), \( w_2^0 \) described in Corollary 4.5. \( \tilde{S}_1^0 = w_0^0 \) and \( \tilde{S}_2^0 = i w_0^0 - i w_2^0 \) and the spectrum mentioned above originates from the subspace

\[
\{ f_k w_0^0 + g_k w_2^0 : f_k, g_k \in L^2(\Omega^\ast) \} = \{ (f_k w_0^0 + \tilde{f}_k w_2^0) + i (g_k w_2^0 - \tilde{g}_k w_0^0) : f_k, g_k \in L^2(\Omega^\ast) \}. \tag{II.11}
\]

**Supplement III. Feshbach-Schur perturbation theory**

In this appendix we present for the reader’s convenience the main result of the Feshbach-Schur perturbation theory. Let \( P \) and \( \overline{P} \) be orthogonal projections on a separable Hilbert space \( X \), satisfying \( P + \overline{P} = 1 \). Let \( H \) be a self-adjoint operator on \( X \). We assume that \( \text{Ran} P \subset D(H) \), that \( H_{\overline{P}} := \overline{P} H \overline{P} \mid_{\text{Ran} \overline{P}} \) is invertible, and

\[
\| R_{\overline{P}} \| < \infty, \quad \| P H R_{\overline{P}} \| < \infty \quad \text{and} \quad \| R_{\overline{P}} H P \| < \infty, \tag{III.1}
\]

where \( R_{\overline{P}} = \overline{P} H_{\overline{P}}^{-1} \overline{P} \). We define the operator

\[
F_P(H) := P(H - H R_{\overline{P}} H) P \mid_{\text{Ran} \ P}. \tag{III.2}
\]
The key result for us is the following:

**Theorem III.1.** Assume (III.1) hold. Then

\[ 0 \in \sigma(H) \iff 0 \in \sigma(F_P(H)) \tag{III.3} \]

and

\[ H\psi = 0 \iff F_P(H)\phi = 0, \tag{III.4} \]

where \( \psi \) and \( \phi \) are related by \( \phi = P\psi \) and \( \psi = Q\phi \), with the (bounded) operator \( Q \) given by

\[ Q = Q(H) := P - R_{\mathcal{F}}HP. \tag{III.5} \]

**Proof.** Both relations are proven similarly so we prove only the second one which suffices for us. First, in addition to (III.5), we define the operator

\[ Q^\# = Q^\#(H) := P - PHR_{\mathcal{F}}. \tag{III.6} \]

The operators \( P, Q \) and \( Q^\# \) satisfy

\[ HQ = PH', \quad Q^\# H = H'P, \tag{III.7} \]

where \( H' = F_P(H) \). Indeed, using the definition of \( Q \), we transform

\[
HQ = PH - HPHP_{\mathcal{F}}^{-1}PHP \\
= PHP + PHP_{\mathcal{F}}^{-1}PHP - PHHP_{\mathcal{F}}^{-1}PHP \\
= PHP - PHHP_{\mathcal{F}}^{-1}PHP \\
= PF_P(H). 
\]

Next, we have

\[
Q^\# H = PH - PPH_{\mathcal{F}}^{-1}PH \\
= PHP + PHP_{\mathcal{F}} - PHPHP_{\mathcal{F}}^{-1}PHP - PHPHP_{\mathcal{F}}^{-1}PHP \\
= PHP - PHPHP_{\mathcal{F}}^{-1}PHP \\
= F_P(H)P. 
\]

This completes the proof of (III.7).

Now, we show

\[ \text{Null } Q \cap \text{Null } H' = \{0\} \quad \text{and} \quad \text{Null } P \cap \text{Null } H = \{0\}, \tag{III.9} \]

The first relation in (III.9) follows from the fact that the projections \( P \) and \( \mathcal{F} \) are orthogonal, which implies the inequality

\[ \|Qu\|^2 = \|Pu\|^2 + \|R_{\mathcal{F}}HPu\|^2 \geq \|Pu\|^2, \]

and the relation \( \text{Null } P \subset \text{Null } H' \), which follows from the definition of \( H' \). To prove the second relation in (III.9) we use the equation \( P + \mathcal{F} = 1 \) and the definitions \( H_{\mathcal{F}} = \mathcal{F}H\mathcal{F} \) and (III.5) to obtain

\[ 1 = QP + R_{\mathcal{F}}H, \tag{III.10} \]

which, in turn, is implies the second relation in (III.9). Indeed, applying (III.10) to a vector \( \phi \in \text{Null } P \cap \text{Null } H \), we obtain

\[ \phi = QP\phi + R_{\mathcal{F}}H\phi = 0. \]

Now the statement (III.4) follows from relations (III.7) and (III.9). \( \square \)
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