Abstract
We consider a family of quantum Hamiltonians $H_\hbar = -\hbar^2 (d^2/dx^2) + V(x)$, $x \in \mathbb{R}$, $\hbar > 0$, where $V(x) = i(x^3 - x)$ is an imaginary double well potential. We prove the existence of infinite eigenvalue crossings with the selection rules of the eigenvalue pairs taking part in a crossing. This is a semiclassical localization effect. The eigenvalues at the crossings accumulate at a critical energy for some of the Stokes lines.

1 Introduction and statement of the results
In this paper we consider some spectral properties of the cubic oscillator described by the family of closed Hamiltonians

$$H_\hbar = -\hbar^2 (d^2/dx^2) + V(x), \quad x \in \mathbb{R}, \quad \text{with} \quad V(x) = i(x^3 - x) \quad \text{and} \quad \hbar > 0 \quad (1.1)$$

defined on the domain $D_H = D(d^2/dx^2) \cap D(|x|^3)$. More precisely we prove the existence of an infinite number of crossings of its eigenvalues $E_n(\hbar)$ (or levels) and we specify the selection rules for the two level pairs involved in a given crossing. Actually, the crossing is possible only for a pair of levels because for large $\hbar$ only one of the nodes is unstable and the localization can be in one of only two wells. Since the spectrum is simple, the crossing of two levels implies the non holomorphy.
of the functions $E_n(h)$ involved. In particular, at the crossing there is a branch point called Bender-Wu singularity $[1–3]$. For real $h > 0$ the Hamiltonians (1.1), and in particular the potential, are $PT$-symmetric operators $[4–7]$. As $V(x)$ has two stationary points at $x_{\pm} = \pm1/\sqrt{3}$ we will speak of a $PT$-symmetric double well oscillator, with states that possibly localize at one of the wells or at both of them.

Anharmonic oscillators are basic non solvable models in quantum mechanics. They pose a summability problem analogous to the one encountered in quantum field theory and therefore they have been extensively investigated since a long time $[8–12]$. The cubic oscillator also has been studied by several approaches. We use here the nodal analysis $[13,14]$, namely the study of the confinement of zeros of the entire eigenfunctions $\psi_n(x) = \psi_n(x,h)$ (or states) and of their derivatives. For that, we use the Loeffel-Martin method for the control of the zeros $[8]$, the asymptotic behavior of Sibuya $[15]$, the semiclassical accumulation of the zeros in some of the Stokes lines $[16]$ and the exact semiclassical quantization. Moreover the results of perturbation theory $[12,14]$ and the unique summability of the perturbation series are also relevant. In such a way we believe that we can draw an exhaustive picture of the level crossing. We must however mention that the semiclassical theory $[1–3]$ has given good results for low values of the parameter $h$ up to the crossing value, slightly extended by the exact semiclassical theory $[17–19]$ to larger, although not very large values of $h$. A still different rigorous technique is found in $[20,21]$. Of course all these treatments are very useful and complementary to ours, which was presented in $[22]$ in a preliminary and not completely rigorous form in some parts. We recall that analytic families of self-adjoint Hamiltonians $[23]$ cannot present level crossings with Bender-Wu singularities. This is also the case of many families of $PT$-symmetric Hamiltonians with a single well potential $[2,4,5]$. In the following we also will take advantage of the results established for two more families of Hamiltonians, defined on the same domain $D_H$ as (1.1), with cubic potentials different from $V(x)$ but related to $H_h$ by changes of parametrization. The first one is an analytic family of type A $[23]$ single well complex cubic oscillators

$$\tilde{H}_\beta = -\left(\frac{d^2}{dx^2}\right) + x^2 + i\sqrt{\beta}x^3, \quad \beta \neq 0, \quad |\arg(\beta)| < \pi$$

(1.2)

which will be used to identify a semiclassical level as a continuation of a perturbative one. All the levels $\tilde{E}_n(\beta)$ of $\tilde{H}_\beta$ and the corresponding states $\tilde{\psi}_n(\beta)$ are perturbative with labels $n$ determined by the number of zeros which are stable at $\beta = 0$ (or nodes). The $n + 1$ stationary points of $\tilde{\psi}_n(\beta)$ stable at $\beta = 0$ will be called antinodes. Notice that $H_{\beta=0}$ reduces to an harmonic oscillator whose states are concentrated in the interval $[-\sqrt{E}, \sqrt{E}]$, namely about their antinodes. In the paper $[14]$ A. Martinez and one of us (V.G.) have extended to (1.2) the proof of the absence of crossings. For later use we show here the relationship between
We first make a translation of $H_h$ centered at each of the two wells letting $x = y + x_\pm = y \pm 1/\sqrt{3}$ and defining the two isospectral Hamiltonians

$$
H_h^\pm = -\hbar^2 \left( \frac{d^2}{dy^2} \right) + i(y^3 \pm \sqrt{3}y^2) \pm E_0, \quad E_0 = -ic, \quad c = \frac{2}{3\sqrt{3}} \quad (1.3)
$$

In order to use the perturbation theory [14] we make the dilatations [24]: 

$$
y = \lambda^\pm (\hbar) z \quad (1.4)
$$

we find isospectrality ($\sim$) between $\tilde{H}_\beta$ and $H_h$ in the form

$$
\tilde{H}_\beta^\pm (h) \sim (1/\hbar c^\pm) H_h^\pm \mp E_0, \quad \tilde{E}_n(\beta^\pm (h)) = (1/\hbar c^\pm) E_n^\pm (h) \mp E_0 \quad (1.5)
$$

It can be proved rather easily that for $h$ small enough the levels $E_n^\pm (h)$ are respectively obtained from $E_n(h \exp(\mp i\pi/4))$ by analytic continuations along paths in the complex plane of $h$ with $|h|$ fixed [12]. For small $h$ we also have that $E_n^\pm (h)$ are non-real and complex conjugated.

The necessity of level crossings is determined by comparing the behavior of the levels $E_n(h)$ for large values of $h$ with the behavior of $E_n^\pm (h)$ for small $h$. It is very useful to introduce a parametrization of the Hamiltonians (1.1) more suited when $h$ becomes large. Again on the domain $D_H$ we thus define the family

$$
\hat{H}_\alpha = -\left( \frac{d^2}{dx^2} \right) + i(x^3 + \alpha x), \quad \alpha \in \mathbb{C} \quad (1.6)
$$

with levels $\hat{E}_n(\alpha)$ and states $\hat{\psi}_n(\alpha)$. The simple regular dilation

$$
x \to \lambda x, \quad \lambda = \sqrt{-\alpha} = h^{-2/5} \quad (1.7)
$$

gives the relation

$$
\hat{E}_n(\alpha(h)) = h^{-6/5} E_n(h), \quad \alpha = -h^{-4/5} \leq 0 \quad , \quad (1.8)
$$

and, in particular, $h^{-6/5} E_n(h) \to \hat{E}_n(0)$ as $h \to +\infty$. The eigenvalues $\hat{E}_n(\alpha)$ are thus holomorphic in a neighborhood of the origin and on the sector

$$
\mathbb{C}_\alpha = \{ \alpha \in \mathbb{C}, \alpha \neq 0, |\arg(\alpha)| < 4\pi/5 \}. \quad (1.9)
$$

In view of (1.5) we have isospectrality between $\tilde{H}_\beta$ and $\hat{H}_\alpha$. Indeed the relations

$$
\tilde{H}_\beta \sim \beta^{1/5} \hat{H}_\alpha(\beta) - \frac{2}{27\beta}, \quad \tilde{E}_n(\beta) = \beta^{1/5} \hat{E}_n(\alpha(\beta)) - \frac{2}{27\beta}, \quad \alpha(\beta) = \frac{1}{3\beta^{4/5}} \quad (1.10)
$$
are easily obtained by composing the following analytic translation and the dilation
\[ x \rightarrow x + i/3\sqrt{\beta}, \quad x \rightarrow \beta^{-1/10}x \]  

Later on we will prove that the eigenvalues \( \hat{E}_n(\alpha) \) are real analytic for \( \alpha \in \mathbb{R} \) in a neighborhood of the origin. Through (1.8) and (1.10) the levels \( E_n(h) \) are analytic continuations of the perturbative levels \( \tilde{E}_n(\beta) \) and extensible as many-valued functions to the sector of the \( h \) complex plane
\[ \mathbb{C}^0 = \{ h \in \mathbb{C}; h \neq 0, \mid \arg(h) \mid < \pi/4 \}; \]  

We can now formulate in very simple terms the crossing selection rule. The two positive levels \( E_{m^\pm}(h) \), \( m^\pm = 2n + (1/2) \pm (1/2) \), defined for sufficiently high \( h \) undergo a crossing at a value \( h > h_n \) and become the two complex conjugate levels \( E_{\pm}^n(h) \) defined in (1.5) for \( h < h_n \). The corresponding states \( \psi_{m^\pm}(h) \) are \( PT \)-conjugated. We will refer to \( E_{c}^n > 0 \) as to the limit level at \( h = h_n \) and to \( \psi_{c}^n \) as to the corresponding \( PT \)-symmetric state.

This process is possible because of the instability of a single node of \( \psi_{m^\pm}(h) \) and the instability of the \( PT \)-symmetry of both the states \( \psi_{m^\pm}(h) \). More explicitly, the crossing rule is given in terms of the analytic continuations. The two functions \( E_{m^\pm}(h) \), holomorphic for large \( |h| \), are analytically continued along the positive semi-axis for decreasing \( h \) by passing above the singularity at \( h = h_n \) and becoming respectively the two levels \( E_{\pm}^n(h) \) for small \( h > 0 \). Thus the Bender-Wu singularities are square root branch points. As \( n \rightarrow \infty \), we have the limits \( h = h_n \rightarrow 0 \) and \( E_{c}^n \rightarrow E_{c}^\geq 0 \), where \( E_{c}^\geq \) is an instability point of the set of the Stokes lines supposed unique.

We give a brief summary of the content of the following sections where all the statements will be rigorously proved. In Sec. 2 we study the levels and the states for small parameter \( h \). In particular, we show the stability of the nodes in one of the half planes \( \pm \Re z > 0 \). In Sec. 3 we deal with the behavior of the levels and of the nodes for large values of \( h \). We prove a confinement of the nodes and the positivity of the spectrum, and the possible existence of only one imaginary node, which is the one becoming unstable at \( h_n \). We also prove the selection rules of the crossings. In Sec. 4 we show the semiclassical nature of the problem, we give the quantization rules and we prove the local boundedness of the levels. We then determine the crossing rules and we consider the Riemann surfaces of the levels in a neighborhood of the real axis of \( h \). In Sec. 5 we give the conclusions and suggest possible extensions to complex values of the parameter \( h \).
2 Confinement of the escape line at $E = \pm E_n(0)$ and the zeros of $\psi_n^\pm(z, \hbar)$ for small $\hbar$

Let us consider the Stokes complex in our case. For any energy $E \in \mathbb{C}$, a Stokes line is defined by a starting point, one of the turning points called $I_\pm$ and $I_0$. At a point $z$ the Stokes vector $dz(z)$ satisfies the condition

$$-p_0^2(E, z) \frac{dz^2(z)}{dz^2} > 0, \quad p_0^2(E, z) = V(z) - E \quad (2.1)$$

where the choice of the sign is consistently defined in all the complex plane. In such a way we give an orientation to the Stokes lines. We are interested in two particular Stokes lines: the internal one (hereafter called the oscillatory range $\rho(E)$) and the exceptional one (called the escape line $\eta(E)$) [16]. Their union $\rho(E) \cup \eta(E)$ is the union of the classical trajectories $\tau(E)$. For later use it is also convenient to introduce the following notations:

$$\mathbb{C}^\pm = \{ z, \Re(z) \gtrless 0 \} \quad \mathbb{C}_\pm = \{ z, \Im(z) \gtrless 0 \} \quad (2.2)$$

With the definitions (1.3) of $E_0$ and $c$ we prove

**Lemma 2.1.** The escape lines $\eta(E)$ at the levels $E = \mp E_0$, are in $\mathbb{C}^\pm$ respectively, and the $\rho(\mp E_0)$ are the stationary points $x_\pm \in \mathbb{C}^\mp$.

**Proof.** We fix $E = -E_0$. The case $E = E_0$ is completely analogous. The turning point $I_0 = 2/\sqrt{3} \in \mathbb{C}^+$ is the starting point of the oriented exceptional Stokes line. The two turning points $I_\pm$ coincide and we have $\rho(E) = I_+ = I_-$. Using the variable $w = y - ix$, the condition (2.1) for the Stokes field becomes

$$-p_0^2(E, w) \frac{dw^2}{dw^2} = (w^3 + w + E) \frac{dw^2}{dw^2} > 0 \quad (2.3)$$

For $w = -iI_0 + \delta$, at the first order in $\delta$ we have $p_0^2(-iI_0 + \delta) \delta^2 \sim 3\delta^3 < 0$. Hence $\delta^3 < 0$ and $\arg \delta = \pm \pi/3$. In the $z$ plane, $z = I_0 + i\delta$ with $\arg(i\delta) = (\pi/2) \pm \pi/3$.

By the choice $\arg(i\delta) = (\pi/2) + \pi/3 = 5\pi/6$, we obtain the oriented exceptional Stokes line $\eta(-E_0)$. We know that in the $z$ plane this line is asymptotic to the imaginary axis at $+i\infty$. For large $y > 0$ the behavior of the action integral is

$$S(w) = \int_{-iI_0}^w \sqrt{-p_0^2(E, w)} \ dw \sim S_\alpha(w) = (2/5)w^{5/2} + w^{1/2} + Ew^{-1/2} \quad (2.4)$$

Thus, if $w = w(y) = y - ix(y)$ and $x(y) \to 0$ as $y \to \infty$, we have

$$S_\alpha(w(y)) = (2/5) y^{5/2} + y^{1/2} - iy^{3/2}((y^2 + 1/2)x(y) - \Im E) \quad (2.5)$$
and $\Im S_a(w(y)) = 0$ implies
\[ x(y) \sim c/(y^2 + 1/2). \quad (2.6) \]

Therefore the escape line $\eta(-E_0)$ stays in $\mathbb{C}^+$. On the imaginary axis, the vectors $\pm dw(y)$ are determined by the condition
\[ p_0^2(E, y) \, dw^2(y) < 0, \quad p_0^2(y) = -(y^3 + y) + E. \quad (2.7) \]

We consider the regular field of velocities on the imaginary axis letting $dy > 0$. We make explicit the two conditions (2.7):

\[
\Re(p_0^2(E, y) \, dw^2(y)) = (y^3 + y) (dx^2 - dy^2) + 2 \, c \, dx \, dy < 0
\]
\[
\Im(p_0^2(E, y) \, dw^2(y)) = (y^3 + y) 2 \, dy \, dx - c (dx^2 - dy^2) = 0.
\]

Substituting the equality into the inequality we find
\[
(2/c) \left( (y^3 + y)^2 + c^2 \right) \, dx \, dy < 0,
\]

As $dy > 0$ we get $dx < 0$. Moreover, as the field of vectors $dw(y)$ is regular on the imaginary axis, the oriented line $\eta(E_0)$ could exit but not come back to $\mathbb{C}^+$, so that it stays always in $\mathbb{C}^+$.

The following useful result is a consequence of the Lemma 2.1 and the exact semi-classical theory (See [16], Theorem 1).

**Corollary 2.2.** Consider the zeros of $\psi^\pm_n(z, \hbar)$ with energy $E = E^\pm_n(\hbar)$ and small $\hbar$. In the limit $\hbar \to 0^+$, $E^\pm_n(\hbar) \to \pm E_0$, all the $n$ nodes go to $x_\pm \in \mathbb{C}^\pm$ and all the other zeros go to $\eta(\pm E_0) \in \mathbb{C}^\mp$. \hfill $\Box$

We now study the behavior of levels and states in the semiclassical limit.

**Lemma 2.3.** For any $n \in \mathbb{N}$, there exists $\hbar_n > 0$ such that for $0 < \hbar < \hbar_n$ there are complex conjugate levels $E^\pm_n(\hbar)$ whose corresponding states $\psi^\pm_n(\hbar, x)$ are $PT$-conjugated
\[
\psi^{-}_n(h) = PT \psi^{+}_n(h). \quad (2.8)
\]

Both the corresponding entire functions $\psi^\pm(z)$ have $n$ nodes respectively tending to the points $x_\pm \in \mathbb{C}^\pm$ as $\hbar \to 0^+$. Their kernels in $\mathbb{C}$ are $P_x$-conjugated, namely $\ker \psi^{-}_n(z) = P_x \ker \psi^{+}_n(z)$ where $P_x f(x + iy) = f(-x + iy)$. 

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Proof. The isospectrality of $H^+_h$ and $\hat{H}_{\beta}(h)$ has already established in \cite{13-15}. Let us only add that for positive $h$ the parameters $\beta^\pm(h)$ are not in in the complex plane cut along the negative axis, $\mathbb{C}_+ = \{ z \in \mathbb{C}; \; z \neq 0, \; |\arg z| < \pi \}$. This means that just the results of \cite{14} are not sufficient, but we also need some of the results of \cite{12}. In particular we use the fact that there exists a $b_n > 0$ such that the perturbative level $E_n(\beta)$ admits analytic continuations in the open disks of radius $b_n$ with centers at $\exp(\pm i\pi) b_n$ respectively. The perturbation theory yields that the semiclassical behavior of the levels is

$$E_n^\pm(h) = \pm E_0 + hc^\pm(2n + 1) + O(h^2). \tag{2.9}$$

We prove that the corresponding states $\psi_n^\pm(h)$ are $PT$–conjugated for a suitable choice of the phase factors. Indeed $PT$ is a bounded involution. Therefore, from the relation $H\psi^+ = E^+\psi^+$ we get $(PT\ H\ PT)\ (PT\psi^+) = H(PT\psi^+) = E^+(PT\psi^+) = E^-(PT\psi^+)$, which implies (2.8) since the spectrum is simple. Moreover, the set of the zeros of $\psi^-$ is the reflexion with respect of the imaginary axis of the set of zeros of $\psi^+$, or $\ker(\psi^-) = P_x \ker(\psi^+)$. It is relevant to notice that in the perturbation theory of $H_\beta$ \cite{12}, all the nodes of $\psi_n(\beta^\pm(h))$ have a semiclassical limit in $\rho(2n + 1) = [-\sqrt{2n + 1}, \sqrt{2n + 1}]$, while the corresponding nodes of the semiclassical functions $\psi_n^\pm(h)$ go to the stationary points $x_\pm$ respectively. \hfill \Box

We next prove that the zeros $\psi_n^\pm(h)$ are stably confined in $\mathbb{C}^\pm$ respectively, so that such zeros coincide with the nodes tending respectively to $x_\pm$ as $h \to 0$. This shows that no crossing exists between levels of the same set $\{E_n^-(h)\}$ or $\{E_n^+(h)\}$. Since $E_n^\pm(h)$ are complex conjugated for small $h$, the levels $E_n^-(h)$ and $E_n^+(h)$ will cross at $h_n$, where they become real.

Lemma 2.4. Let $E = E_n^\pm(h)$ be the non-real levels at $0 < h < h_n$. The corresponding states $\psi_n^\pm(z)$ are non vanishing on the imaginary axis.

Proof. Let $E$ be one of the non-real levels and $\psi(z)$ the corresponding state. Let $\phi(y) = \psi(iy)$, $y \in \mathbb{R}$, be the eigenfunction on the imaginary axis. With $w = y - ix$, for a fixed $x$ we define the Hamiltonian

$$H_h^x(x) = -\hbar^2(d^2/dy^2) - w^3 - w \tag{2.10}$$

having a level $-E$. The corresponding translated state, $\phi_\pm(y) = \phi(w)$, has the well known asymptotic behavior for large $y$ \cite{15}

$$\phi_\pm(y) = \frac{C(1 + O(y^{-1/2}))}{\sqrt{p_0(E,w)}} \cos\left(\frac{S_w(E,w)}{\hbar} + \theta\right), \tag{2.11}$$

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where $C > 0$, $\theta \in \mathbb{R}$, $p_0(E, w)$ is defined in (2.3) and $S_\alpha(w)$ in (2.4). We have
\begin{equation}
|\phi(y)|^2 = |\phi_0(y)|^2 = O(|y|^{-3/2}) \quad \text{as} \quad y \to +\infty.
\end{equation}

We now consider the Loeffel-Martin formula for $\phi(y)$, producing the same result of the law of the imaginary part of the shape resonances:
\begin{equation}
h^2 \Im(\phi(y)\phi'(y)) = -\Im \int_y^\infty |\phi(s)|^2 ds, \forall y \in \mathbb{R},
\end{equation}

Due to the asymptotic behavior (2.12) the integral exists and is finite. Therefore $\psi_n^\pm(z, h)$ is not vanishing for $z$ on the imaginary axis.

**Lemma 2.5.** Let $E = E_n^\pm(h)$ and $\psi_n^\pm(z)$ as above. The large zeros $Z_j^\pm$ of $\psi_n^\pm(z)$ are in the half planes $\mathbb{C}^\pm$ and their nodes are stable in $\mathbb{C}^\pm$ respectively.

**Proof.** For $y \to \infty$, $x(y) \to 0$, we get from (2.11) the asymptotic condition
\[ \Im(S_\alpha(w(y)) + h\theta) \sim -(y^{3/2} + 1/2) x(y) + \Im E + hy^{1/2} \Im \theta = 0 \]
If $\Im \theta = 0$, the large zero $Z_j^\pm = x(y) + iy$ has an asymptotic behavior with
\begin{equation}
x(y) \sim (\Im E_n^\pm + hy^{1/2} \Im \theta) / (y^2 + 1/2),
\end{equation}

These asymptotic behaviors are imposed by the continuity of the zeros and their impossibility of crossing the imaginary axis by Corollary 2.2. This proves the stability of the zeros (nodes) in $\mathbb{C}^\pm$ respectively. At the limit of $h \to h_n^-$ the energies $E_n^\pm(h)$ become positive and the large zeros $Z_j^\pm$ become imaginary. \hfill \Box

From [16], the Corollary 2.2 the continuity of the nodes and Lemma 2.4 describing the barrier on the imaginary axis, we have

**Corollary 2.6.** Let $E = E_n^\pm(h)$ and $\psi_n^\pm(z)$ as above. For small $h$ all the $n$ nodes are in a neighborhood $U_\pm \subset \mathbb{C}^\pm$ of $x_\pm$.

We finally have an analyticity result for the levels $E_n^\pm(h)$.

**Proposition 2.7.** The two functions $E_n^\pm(h)$ are analytic for $0 < h < h_n$. The two levels $E_n^\pm(h)$ and the two states $\psi_n^\pm(h)$ coincide at the crossing limit $h \to h_n^-$. The limit level $E_n^c$ is positive. The limit state $\psi_n^c(z)$ is $P_x T$-symmetric and has $2n$ non-imaginary zeros. The large zeros are imaginary.

**Proof.** According to Lemma 2.3 and Lemma 2.5 for $h < h_n$ the $n$ nodes of the two states $\psi_n^\pm(h)$, are the only zeros in $\mathbb{C}^\pm$, respectively. Since the states $\psi_n^\pm(h)$ are the only ones having $n$ zeros in $\mathbb{C}^\pm$ the function $E_n^\pm(h)$ are analytic. From the relation (2.8) for $h < h_n$ and the limit $\psi_n^\pm(h) \to \psi_n^c$ as $h \to h_n^-$ we get $\psi_n^c = PT\psi_n^c$. As the states $\psi_n^\pm(h)$ have only $n$ zeros in $\mathbb{C}^\pm$ and at the limit $h \to h_n^-$ these zeros cannot diverge or become imaginary, the limits of the $2n$ non-imaginary zeros of both the state $\psi(h)$ are all the non-imaginary zeros of the limit state $\psi_n^c$. \hfill \Box
3 Analysis of levels and nodes for large \( \hbar \)

We have already stated in the Introduction that level crossing comes from looking at the behavior of the levels for small and large \( \hbar \). We have also described in (1.6)-(1.11) the appropriate scaling for dealing with large values of \( \hbar \) or \( \beta \), corresponding to small values of \( \alpha \).

We are now going to prove a confinement of the zeros of \( \tilde{\psi}_m(\alpha) \), for small \( \alpha \), in two regions. We define nodes the zeros confined in one of these regions by identifying them with the nodes of the states \( \psi_n(\beta) \) for large \( \beta \). We find it useful to introduce the two disjoint sets

\[
C_\rho = \{ z = x + iy, \ y < 0, \ |x| < -\sqrt{3}y \} \subset \mathbb{C}_-
\]
\[
C_\eta = \{ z = x + iy, \ y > 0, \ |x| < \sqrt{3}y \} \subset \mathbb{C}_+
\]

(3.1)

**Lemma 3.1.** The \( m \) nodes of \( \tilde{\psi}_m(\alpha) \) for small \( |\alpha| \) are confined in \( C_\rho \) and correspond to the nodes of \( \tilde{\psi}_m(\beta) \) in \( \mathbb{C}_- \). The other zeros of \( \tilde{\psi}_m(\alpha) \) are in \( C_\eta \). The function \( \tilde{E}_m(\alpha(\beta)) \) is real analytic and coincides with \( \tilde{E}_m(\beta) \) by (1.10).

**Proof.** Define the translated operator \( \tilde{H}_{\alpha=0} \) by \( x \to x + iy \),

\[
\tilde{H}_{\alpha=0,y} = -(d^2/dx^2) + V_y(x) , \quad V_y(x) = y(y^2 - 3x^2) + ix(x^2 - 3y^2)
\]

(3.2)

Apply the Loeffel-Martin method [8] to the state \( \psi = \tilde{\psi}_m(\alpha = 0) \) with energy \( E = \tilde{E}_m(\alpha = 0) > 0 \), for \( \pm x \geq \sqrt{3}|y| \):

\[
-3 \Im [\bar{\psi}(x + iy) \partial_x \psi(x + iy)] = \int_x^\infty 3V_y(s)|\psi(s + iy)|^2 ds = \int_x^\infty (s^2 - 3y^2)s|\psi(s + iy)|^2 ds = - \int_{-\infty}^x (s^2 - 3y^2)s|\psi(s + iy)|^2 ds \neq 0
\]

For \( \alpha = 0 \) the nodes are thus rigorously confined in \( C_\rho \). The confinement extends to \( \alpha > 0 \). Since the \( m \) zeros of \( \tilde{\psi}_m(\alpha) \) on \( \mathbb{C}_- \) are stable for \( \alpha \to +\infty \), they are nodes by definition. By (1.11) the set \( \mathbb{C}_- \) is invariant for positive dilations in the limit of infinite \( \beta \) and for \( \beta > 0 \) the \( m \) nodes of \( \tilde{\psi}_m(\beta) \) are in \( \mathbb{C}_- \) [14]. By (1.10) the level \( \tilde{E}_m(\alpha) \) for small \( |\alpha| \) connects the perturbative level \( \tilde{E}_m(\beta) \) with the level \( E_m(h) \) for large positive \( \beta \) and \( h \) respectively. The state \( \tilde{\psi}_m(\alpha) \) is defined by the number \( m \) of its zeros in \( \mathbb{C}_- \) which, by Lemma [3.1] are identified as nodes. Since by continuity the number of nodes of \( \psi_m(\alpha) \) is stable for \( 0 < |\alpha| < \epsilon \) while, although unbounded, the scaling (1.7) is regular and phase preserving, the function \( \tilde{E}_m(\alpha) \) is real analytic for all \( \alpha > 0 \) due to the results of [14] and to (1.8). This property is stable for small \( -\alpha > 0 \), up to the first crossing and the function \( E_m(h) \) is real analytic by (1.8) and (1.10). Finally the \( m \) nodes of \( \psi_m(h) \) for large positive \( h \) are all its zeros in \( C_\rho \) as well as in \( \mathbb{C}_- \). \( \square \)
Corollary 3.2. For large $\hbar$ the level $E_m(\hbar)$ exists and is positive.

Proof. The analytic level $\hat{E}_m(\alpha)$, with corresponding normalized state $\hat{\psi}_m(\alpha)$, has positive real part due to the positivity of $-(d^2/dx^2)$:

$$\Re \hat{E}_m(\alpha) = \Re \langle \hat{\psi}_m(\alpha), \hat{H}_{\alpha} \hat{\psi}_m(\alpha) \rangle = \langle \hat{\psi}_m(\alpha), - (d^2/dx^2) \hat{\psi}_m(\alpha) \rangle > 0,$$

The result follows from (1.8).

We now establish some properties of the $PT$-symmetric states which will be useful later on. In analogy to (3.2) we consider the translated operator

$$H_{\hbar,y} = -\hbar^2 (d^2/dx^2) + V(x + iy)$$

(3.3)

with $V$ as in (1.1).

Lemma 3.3. A $PT$-symmetric state $\psi_y(x) = \psi(x + iy)$ of the translated operator $H_{\hbar,y}$ has even real part and odd imaginary part. In particular it satisfies at the origin the conditions

$$\Im \psi_y(0) = \Im \psi(iy) = 0, \quad \Re \psi'_y(0) = \Re \psi'(iy) = 0.$$  

(3.4)

Proof. Indeed if $\psi_y(x) = R(x) + iI(x)$ we have

$$PT(R(x) + iI(x)) = R(-x) - iI(-x) = R(x) + iI(x)$$

$$\square$$

Let us recall the criterion for the nodes in the case of a positive level $E_m(\hbar)$. A zero $Z_j(h)$ of $\psi_m(h,z)$ is a node if, continued to a parameter $h' > h$ large enough, $Z_j(h')$ belongs to $\mathbb{C}_-$, i.e. $\Im Z_j(h') < 0$. Let also recall that an imaginary zero $Z_j(h)$ of the state $\psi_m(h,z)$ stays imaginary for any $h' > h$, because of the $P_x$-symmetry of its kernel and the simplicity of the spectrum.

Lemma 3.4. The level $E_m = E_m(h)$ exists positive for $h$ large enough, with the corresponding $PT$-symmetric state $\psi_m(x) = \psi_m(x,h)$. There is an alternative:

(a) the absence of imaginary state $\psi_m(z)$,
(b) the existence of only one imaginary node of the function $\psi_m(z)$.

The second case is possible if and only if $m$ is odd.
Proof. For \( \hbar \) large enough the Hamiltonian (1.1) admits a positive level \( E_m = E_m(\hbar) \) with eigenfunction \( \psi_n(z) = \psi_n(z, \hbar) \). We consider the Hamiltonian (2.10) on the imaginary axis, \( x = 0 \) or \( w = y \), and we observe that \( H^x_h = H^y_h(x = 0) = -H_h \) is real. The eigenfunction \( \phi_m(y) = \psi_m(iy) \) of \( H^y_h \) is also real by the conditions (3.4) at the origin. \( H^x_h \) and \( -H_b \) have the same spectrum and \( -E_n = -E_n(\hbar) < 0 \) is one of its eigenvalues. Therefore for large positive \( y \) the solution \( \phi_m(y) \) is the function \( \phi_{x=0}(y) \) given in (2.11). For large \( -y > 0 \) the solution \( \phi_m(y) \) is a real combinations of the two fundamental solutions \( 15 \) and reads

\[
\phi_m(y) = \frac{C'}{\sqrt{p_0(E, y)}} \left( \exp\left(\frac{S_a(-y)}{\hbar}\right) + a \exp\left(-\frac{S_a(-y)}{\hbar}\right)\right),
\]

with \( C' > 0 \), \( a = a_m(\hbar) \in \mathbb{R} \) and \( p_0(E, y) = \sqrt{-y^2 - y + E} \).

For \([m/2] = n \in \mathbb{N} \) and \( \hbar \geq \hbar_n \) both functions \( \psi_m(z) \) have \( n \) nodes on both half-planes \( \mathbb{C}^\pm \). They are distinguished by the number of imaginary nodes. If we define the the complement of the escape line in the imaginary axis,

\[
\eta^c(E) = \{ z = iy, -\infty < y < y_0 \},
\]

where \( y_0 = -iI_0 \) and \( I_0 \) is the imaginary turning point, then the crossing process for \( \hbar \geq \hbar_n \) can be studied by looking at the behaviors \( \psi_m(z) \) with energy \( E_m \) on the semi-axis \( \eta^c(E_m) \). We therefore consider the behavior of the two states \( \phi_m(y) \) in an open interval \( A \subseteq \eta^c(E_m) \) for large \( \hbar \) and we see that a state is concave when it is positive and convex when negative. Since we can consider \( \phi_m(y) \) positive decreasing for \( y < y_0 \), only two possibilities are admitted:

(a) there exists a single zero on \( \eta^c(E_m) \);

(b) no zero exists on \( \eta^c(E_m) \).

According to Lemma 3.1 for large positive \( \hbar \) an imaginary node of a state \( \psi_n(z, \hbar) \) is in \( \mathbb{C}_- \). As \( y_0 > 0 \) when \( E > 0 \) an imaginary node should lie in the intersection of \( \mathbb{C}_- \) with the imaginary axis, contained in \( \eta^c(E_m) \). \( \square \)

Corollary 3.5. Assume \( \hbar > h_n \) and let \( \psi_m(\hbar) \) be a generic state tending to \( \psi_n^c \) for \( \hbar \to h_n^+ \). Then the non-imaginary zeros of \( \psi_m(\hbar) \) are exactly \( 2n \) and they are stable at the limit \( h_n^+ \). Since at most there exists one imaginary node, the number \( m \) of the nodes of \( \psi_m(\hbar) \) is not greater than \( m^+ = 2n + 1 \).

Proof By Lemma 3.4 when \( \hbar > h_n \) the levels \( E_m(\hbar) \) are positive and the states \( \psi_m(\hbar) \) are PT-symmetric. Since the spectrum is simple and the nodes are symmetric, a zero on the imaginary axis cannot leave it and a non imaginary zero cannot become purely imaginary. A non imaginary zero of the state \( \psi_m(\hbar) \), however, can go to infinity moving along a path having the imaginary axis as asymptote at infinity \( 14 \). Moreover at a fixed \( \hbar > h_n \) the state \( \psi_m(\hbar) \) has the behavior (2.11)
so that it is non-vanishing for large \( y \) and small \(|x| \neq 0\). We can therefore conclude that the large zeros are imaginary and the non-imaginary nodes are stable. The non-imaginary zeros of the two states \( \psi_m(h) \) as well as of the limiting state \( \psi_n \) are 2\( n \) and by Lemma 3.4 there exists at most one zero on the imaginary axis. Thus the number of nodes is \( 0 \leq m \leq 2n + 1 \). Due to the independence of the two states \( \psi_m(h) \) we actually have two different numbers \( m \), one not greater than \( 2n + 1 \) and the other not greater than \( 2n \).

Collecting all the previous results we can finally prove the

**Theorem 3.6.** For each \( n \in \mathbb{N} \), there exists a crossing parameter \( h_n \). Two levels \( E_{m^\pm}(h), m^\pm = 2n + (1 \pm 1)/2 \), are defined for \( h > h_n \) and two levels \( E_n^\pm(h_n) \) for \( h < h_n \). The two pairs cross at \( h_n \). Both the states \( \psi_{m^\pm}(z, h) \) have a \( P_\pi \)-symmetric set of \( 2n \) non-imaginary nodes. Only \( \psi_{m^+}(z, h) \) has an imaginary node.

**Proof** Before giving the proof an observation is in order. Actually we do not prove the uniqueness of this crossing and the possible existence of a next pair of anti-crossing and crossing is left open. It could happen that the two levels \( E_{m^\pm}(h) \), for \( h > h_n \), and the two levels \( E_n^\pm(h) \) for \( h > h_n^\prime \), cross at \( h_n^\prime \). Moreover \( E_{m^\pm}(h) \), for \( h > h_n^\prime \), and \( E_n^\pm(h_n) \), for \( h < h_n^\prime \), cross at \( h_n^\prime \). For simplicity, we disregard this possibility.

Let us now proceed with the proof of the theorem. The non-reality of \( E_n^\pm(h) \) for small \( h \) and positivity of \( E_m(h) \) for large \( h \) necessarily yield the existence of crossings. Seen from \( h \leq h_n \), the crossing occurs when the two levels \( E_n^\pm(h) \) become real. Considering the pairs of positive levels, of the kind \( E_m(h) \), obtained by the crossing at \( h_n \), only the pairs of numbers \( m \) equal to \( m^\pm \) are compatible with the uniqueness of such levels for large \( h \), namely only the sequence of pairs, \( \{(E_{2n}(h), E_{2n+1}(h))\}_{n=0,1,...} \) corresponds exactly to the sequence \( \{E_m(h)\}_{m=0,1,...} \) we have for large \( h \). As a consequence we also have that the non-imaginary zeros of the states \( \psi_m(h) \) are the non-imaginary nodes.

At the crossing, the imaginary node of the state \( \psi_{m^+}(h_n) \) coincides with the lowest imaginary zero of \( \psi_m^-(h_n) \). The crossing between the levels \( E_{m^\pm}(h) \) is possible because of the stability of the \( 2n \) non-imaginary nodes of both the entire functions \( \psi_{m^\pm}(z, h) \) and the instability of the imaginary node of the functions \( \psi_{m^+}(z, h) \).

Because of the \( P_\pi T \)-symmetry of both the states \( \psi_{m^\pm}(z, h) \), they have \( n \) nodes in both the half-planes. If we continue the state \( \psi_{m^+}(z, h) \) along a path in the \( h \) complex plane, coming and returning to a \( h > 0 \), large enough, turning around \( h_n \), at the end we get the state with \( m^- \) zeros in the half-plane \( \mathbb{C}_- \) of \( z \), without the imaginary one.

Finally, by continuing to \( h < h_n \) the \( n \) nodes of both the states \( \psi_m(z, h) \) in \( \mathbb{C}_+ (\mathbb{C}_-) \) we obtain the nodes of \( \psi_{m^+}(z, h) (\psi_{m^-}(z, h)) \).

**Remarks 3.7.** (i) For large \( h \) all the zeros in the upper half-plane are imaginary. This statement strengthens the confinement of the zeros of \( \psi_m(z, h) \) for large \( h \)
obtained above. It ensues from the result that all the non-imaginary zeros are nodes, and all the nodes are in the lower half-pane for large $\hbar$.

(ii) We have seen that the states $\tilde{\psi}_n = \tilde{\psi}_n(0)$ of $\tilde{H}_\beta$ at fixed $\beta = 0$, have definite parity: $P\tilde{\psi}_n = (-1)^n \tilde{\psi}_n$. This means that $|\tilde{\psi}_n|^2$ is $P$-symmetric, and the expectation value of the parity is $\langle \tilde{\psi}_n, P\tilde{\psi}_n \rangle = (-1)^n$. We recall that the state at the crossing, $\psi^c_n = \psi^c_n(h_n)$ has vanishing average value of the parity, $\langle \psi^c_n, P\psi^c_n \rangle = 0$, so that it is totally $P$-asymmetric in the sense that $\psi^c_n$ is orthogonal to $P\psi^c_n$.

4 Boundedness of the levels, quantization and selection rules

In this section we examine further properties of the levels in connection with the quantization and the selection rules.

The levels are always semiclassical and are given by some semiclassical quantization rules excluding the divergence of the level. The semiclassical nature of the problem is made clear using the dilations. By a regular scaling $x \to \lambda x$, $\lambda = 1/\sqrt{\delta} > 1$ we get the equivalent operator

$$\hat{H}_k(\delta) = -k^2 \left(d^2/dx^2\right) + i(x^3 - \delta x) \sim \delta^{3/2} \hat{H}_h, \quad k = h \delta^{5/4} \quad (4.1)$$

where the new parameter $k$ vanishes, for any fixed $h$, as $\delta \to 0$. In this new representation the energy is $\tilde{E}_m(k, \delta) = \delta^{3/2} E_m(h)$. Fixing $\delta = 0$ and setting $k = h$ we reproduce the well known semiclassical operator $\hat{H}_h(0) = -h^2 \left(d^2/dx^2\right) + ix^3 \quad (4.2)$

We recall that $\hat{H}_h(0)$ is related by a scaling to the operator $\hat{H}_{\alpha=0}$ studied above, its spectrum is positive and the Stokes lines have the trivial dependence $\tau(E) = E^{1/3} \tau(1)$ upon $E > 0$, so that there are no critical energies. Thus, we can consider a large enough scaling factor $\lambda$ in order to get the parameter $k$, replacing $h$, as small as we want.

Bounded levels $E^\pm_n(h)$, $E_m(h)$ are obtained by two different quantizations. We recall that the nodes of a state $\psi$ are confined in $\mathbb{C}^\pm$ according to whether its energy satisfies the condition $E \in \mathbb{C}^\pm$. Therefore for $h < h_n$ both levels $E^\pm_n(h)$ satisfy the unique conditions on the imaginary part and on the nodes of the states, have no crossing and are analytic. At $h = h_n$ the levels cross and become positive.

We have seen that there are two continuations of $E^\pm_n(h)$ from $h < h_n$ to $h > h_n$ and that the continuations of the corresponding states have $n$ nodes each one in
the half planes $\mathbb{C}^{\pm}$. There exist two regular regions $\Omega^{\pm} \subset \mathbb{C}^{\pm}$ large enough whose boundaries $\gamma^{\pm} = \partial\Omega^{\pm}$ satisfy $P_{x}\gamma^{\pm} = \gamma^{-}$ such that the exact quantization conditions are

$$J^{\pm}(E, h) = \frac{\hbar}{2i\pi} \oint_{\gamma^{\pm}} \frac{\psi'(z)}{\psi(z)} dz + \frac{\hbar}{2} = h \left( n + \frac{1}{2} \right), \quad (4.3)$$

with $E = E_n^\pm(h) \in \mathbb{C}^\pm$ and $\psi(z) = \psi_n^\pm(z, h)$. For small $h$ and bounded $nh$, from (4.3) we get the semiclassical quantization

$$J^{\pm}(E, h) = \frac{1}{2i\pi} \oint_{\gamma^{\pm}} \sqrt{V(z) - E} \, dz + O(h^2) = h \left( n + \frac{1}{2} \right), \quad (4.4)$$

where $\gamma^{\pm}$ squeeze along both the edges of $\rho(E)$. At the critical value $h = h_n$, the two quantizations (4.3) yield equal solutions $E_n^\pm$, $\psi_n^\pm$. When $h > h_n$, both (4.3) admit the two solutions $E_m(h), \psi_m(h)$, $[m/2] = n$ distinguished by the selection condition $E_{2n+1}(h) > E_{2n}(h)$ compatible with the order of the levels $E_{2n+1}(0) > E_{2n}(0)$. Thus, we have the boundedness and continuity of the functions $E_n^\pm(h), h \leq h_n$, becoming $E_n^\pm(h)$ for $h \geq h_n$ and both the functions $E_n^\pm(h)$ are analytic in the hypothesis of maximal analyticity.

We now consider the semiclassical regularity of the levels $E_m(h)$ for large $1/h$ and $m = 2n$ or $m = 2n + 1$. In particular, we expect to find positive semiclassical levels with $E > E_c = 0.352268$. [19] by a semiclassical quantization we consider here. If $\Omega_m \subset \mathbb{C}_-$ is large enough in order to contain all the $m$ nodes and $\Gamma_m = \partial\Omega_m$, for a fixed $h$, we have the exact quantization rules

$$J_2(E, h) = \frac{\hbar}{2i\pi} \oint_{\Gamma_m} \frac{\psi'(z)}{\psi(z)} dz + \frac{\hbar}{2} = h \left( m + \frac{1}{2} \right) \quad (4.5)$$

where $E = E_m(h), \psi(z) = \psi_m(h, z)$. For large $m$, small $h$, $mh$ bounded, we have

$$J_2(E, h) = \frac{1}{2i\pi} \oint_{\Gamma_m} \sqrt{V(z) - E} \, dz + O(h^2) = h \left( m + \frac{1}{2} \right). \quad (4.6)$$

We expect a bounded limit of both $2nh_n \to J_2$ and $E_n^c \to E_c$ as $n \to \infty$. The peculiarity of $E_c$ is the instability of $\tau(E)$ and the connection of $\rho(E)$ and $\eta(E)$ at this point. The quantization conditions (4.3) and (4.6) are compatible with the existence of $\lim_n E_n^c = E^c > 0$ for $n \to \infty$, $J_2 = J_2(E^c, 0) = 2J(E^c, 0)$ with $\Omega = \Omega^+ \cup \Omega^-$. The localization of the nodes near $\rho(E)$ for small $h$ and the localization of the other zeros near $\eta(E)$ implies this property of $\tau(E)$ at $E_c$.

We can now establish the local boundedness of the levels in the real axis.

**Lemma 4.1.** Each of the four continuous functions $E_n^\pm(h)$ for $h < h_n$ and $E_m^\pm(h)$ for $h > h_n$ is locally bounded.
Proof Let \( E(h) \) be one of the levels \( E_{n}^{\pm}(h) \) for \( h < h_n \) with one of its continuations \( E_{m}^{\pm}(h) \) for \( h > h_n \) and let \( \psi(z) \) be the corresponding state. Assume that the lemma is false and there is a divergence of \( E(h) \) at \( h^c \gg h_n \). We rescale the Hamiltonian \( H_h \) by \( x \to |E(h)|^{1/3} x \). Upon dividing by \( |E(h)| \) we get the operator

\[
-\hbar^2 \left( \frac{d^2}{dz^2} \right) + i z^3 - i |E|^{-2/3} z - E/|E|, \quad k = |E|^{-5/6} \hbar
\]

As \( h \to h^c \), so that \( k \to 0 \), and neglecting the linear term in \( z \), the semiclassical quantization reads

\[
\frac{1}{2\pi i} \oint_{\Gamma_m} \sqrt{iz^3 - E/|E|} \, dz = k \left( m + \frac{1}{2} \right) + O(k^2), \tag{4.7}
\]

where \( \Gamma_m \) is the boundary of a region \( \Omega_m \subset \mathbb{C}_- \) containing the \( m \) nodes of \( \psi(h^c) \). For \( k \to 0 \), (4.7) could be satisfied only if \( E/|E| \to 0 \), obviously absurd. \( \square \)

On the complex plane of the parameter \( h \) consider now the sector \( \mathbb{C}^0 \setminus \{1, 2\} \) where the functions \( E_{m}^{\pm}(h) \), are analytic with Riemann sheets \( \mathbb{C}^0_{m}^{\pm} \) having a square-root-type singularity and a cut \( \gamma_n = (0, h_n] \) on the real axis. We assume the inequality

\[
E_{m}^{+}(h) > E_{m}^{-}(h), \quad h > h_n, \tag{4.8}
\]

the only one compatible with the order of the levels \( \tilde{E}_{2n+1}(0) > \tilde{E}_{2n}(0) \). We prove the following:

**Theorem 4.2.** The positive analytic functions \( E_{m}^{\pm}(h) \) have the following behaviors at the edges of \( \gamma_n \)

\[
E_{m}^{-}(h \pm i0^+) = E_{m}^{\mp}(h), \quad E_{m}^{+}(h \pm i0^+) = E_{m}^{\mp}(h) \tag{4.9}
\]

where \( E_{n}^{\mp}(h) \to \pm E_0 \) as \( h \to 0 \).

Proof In the hypothesis of unicity of the crossing (see the proof of Theorem 3.6) for \( h > h_n \) we admit the inequality \( E_{2n+1}(h) > E_{2n}(h) \), the only one compatible with the order of the levels \( \tilde{E}_{2n+1}(0) > \tilde{E}_{2n}(0) \). Since both the functions \( E_{m}^{\pm}(h) \), have a square root singularity at \( h_n \) and \( E_{m}^{+}(h_n + \epsilon) - E_{m}^{-}(h_n + \epsilon) = O(\sqrt{\epsilon}) > 0 \) for small positive \( \epsilon \), then \( \pm \exists \left[ E_{m}^{+}(h_n + \exp(\pm i\pi)\epsilon) - E_{m}^{-}(h_n + \exp(\pm i\pi)\epsilon) \right] < 0 \) and \( \mp \exists E_{n}^{\pm}(h) > 0 \). We necessarily have

\[
E_{m}^{+}(h_n + \exp(\pm i\pi)\epsilon) = E_{n}^{\pm}(h_n - \epsilon), \quad E_{m}^{-}(h_n + \exp(\pm i\pi)\epsilon) = E_{n}^{\pm}(h_n - \epsilon)
\]

and the results extends to any \( \epsilon < h_n \). \( \square \)
Remarks 4.3. (i) We can look at the crossing process following a path which starts from $\hbar = 0^+$, encircles the singularity $\hbar_n$ and comes back to $\hbar = 0^+$. At the beginning of the path the state $\psi_n^-(z,0^+)$ is mainly localized around $x_+$ and at the end turns into $\psi_n^-(z,0^-)$, mainly localized around $x_-$. We can also look at a path beginning at a large $\hbar$, going around $\hbar_n$ and returning to the initial $\hbar$. If the initial state $\psi_{m^+}(\hbar)$ is odd at the end it becomes the even state $\psi_{m^-}(\hbar)$ and the imaginary node in the lower half plane is changed into the lowest zero on the positive imaginary axis.

(ii) It is possible that the Riemann sheet $C^0_\gamma$ of the fundamental level has only the square root branch point $h_0$ with the cut $\gamma_0 = [0, h_0]$ on the real axis [19]. From Theorem 4.2 the discontinuity on the cut $\gamma_0$ is

$$E_0(h + i0^+) - E_0(h - i0^+) = 2i \Im E_0^\pm(h).$$

The function $E_0(h)$, analytic for large $|h|$, if continued to small $|h|$ while keeping $\arg h = \pm \pi/4$ coincides by definition with $E_0^+(h)$, respectively. The absence of complex singularities is compatible with the identities on the edges of the cut $\gamma_0$

$$E_0(h \pm i0^+) = E_0^\pm(h \pm i0^+).$$

We finally prove the theorem

Theorem 4.4. There exists an instability point, $E^c \geq 0$, of $\rho(E)$. For $n \to \infty$, we have the limits $E_n^c \to E^c$, $2nh_n \to J_2$ where $J_2 = J_2(E^c, 0)$ as in [1.6].

Proof Since the eigenvalue problem of $H_{\hbar}$ is semiclassical, the existence of the infinite crossings is possible if it exists a critical point $E^c \geq 0$ of $\rho(E)$, which we assume to be unique. The existence of a critical point is due to the $P_x$-symmetry of $\rho(E_m(h))$ for $h > 0$ and $E_m(h) > 0$ large, together with the symmetry breaking for small $h$. Actually at the limit $h \to 0^+$ we have at $E_n^\pm \to \pm E_0$ and $\rho(\pm E_0)$ reduces to the points $x_\pm$, respectively. The breaking of $\rho(E)$ at $E^c \geq 0$ is possible only if $I_0(E^c) \in \rho(E^c)$. In particular, we have the $P_x$-symmetry breaking at $E = E^c$ where $\rho(E)$ is a line touching the turning points $I_\pm$ with $\Re I_\pm \neq 0$ and containing the point $I_0$. Thus, the symmetry breaking of $\rho(E)$ implies its breaking at $E^c$ and its redefinition as one half of it containing only a pair of turning points, $(I_0, I_\pm)$ or $(I_-, I_0)$. Our eigenvalue problem is always semiclassical and the change of the semiclassical regime is related to the instability of the nodes used for the semiclassical quantization. Since it is possible to change representation by the scaling (4.1), it is always possible to have a sequence of parameters $k_n \to 0$ together with a sequence $\delta_n$. If a subsequence $\delta_n(j)$ vanishes as $j \to \infty$, this is incompatible
with the absence of crossings of the levels $\hat{E}_m(\alpha)$ at $\alpha = 0$. Thus, we have the inequality of the sequence $\delta_n > \epsilon > 0$ for an $\epsilon > 0$ and $n > n_\epsilon$. This means that also the original sequence vanishes, namely $\hbar_n \to 0$. Moreover, it is necessary that $E^c_n \to E^c$ as $n \to \infty$, because in the semiclassical limit there is the instability of the nodes at $E = E^c$ only. Thus, the sequence $E^c_n$ has the limit $E^c_n \to E^c > 0$ as $n \to \infty$. But, due to the semiclassical quantizations rules, we also have a bounded limit of $2n\hbar_n \to J^c_2 > 0$ (4.6), where $J^c_2 = J_2(E^c, 0)$ as $n \to \infty$. □

5 Conclusions

We have proved the existence of a crossing for any pair of levels $E_m^{\pm}(\hbar)$, with $m^{\pm} = 2n + (1 \pm 1)/2$, giving the pair of levels $E_n^{\pm}(\hbar)$ for smaller $\hbar$. In this $PT$-symmetric model we see the competition of two different effects: the conservation of the symmetry and the semiclassical localization of the states. The semiclassical localization prevails for small $\hbar$. This semiclassical transition is impossible in families of selfadjoint Hamiltonians. Thus, these $PT$-symmetric models can be used to describe the appearance of the classical world in non isolated systems.

In order to understand the physical meaning of the classical trajectories $\tau(E)$, we consider $\hbar$ at the border of the sector $\mathbb{C}^0$. At $\arg \hbar = \pi/4$ we can factorize the imaginary unit $i$ and consider the real cubic oscillator

$$H_r(\hbar) = p^2 + V^r(x), \quad p = -i\hbar (d/dx) \quad V^r(x) = x^3 - x, \quad \hbar > 0 \quad (5.1)$$

for an energy value $E \in A_0 = (-c, c)$, with $c = 2/(3\sqrt{3})$. Here we are not concerned with the non completeness of the problem at $-\infty$. We get the classical Hamiltonian $H_r(p, x)$ by substituting in $H_r(\hbar)$ classical momentum $p$ to the operator $-i\hbar (d/dx)$. The union $\tau(E)$ of the classical trajectories at energy $E \in A_0$ consists of the oscillation range $\rho(E) = [I_-(E), I_+(E)]$ and the escape line

$$\eta = \eta(E) = (-\infty, I_0(E)], \quad I_0(E) < -1/\sqrt{3} < I_-(E) < I_+(E).$$

The real potential makes clear the meaning of our definitions of oscillatory range $\rho(E)$ and of escape line $\eta(E)$ previously used with complex potential. Notice that $\tau(E)$ is unstable at $E = c$, where $\rho(E)$ touches $\eta(E)$.

Going back to the Hamiltonians $H_\hbar$ at the limit cases $\arg \hbar = \pm i(\pi/4)^-$, we have the critical values of the energy

$$E^c(\arg \hbar = \pm (\pi/4)^-) = \pm ic = \mp E_0 = E^c_n(0).$$

Thus we expect the existence of an infinite set of crossings in the complex $\hbar$ plane with complex accumulation points $E^c(\theta)$ of the crossing energies for $|\hbar| \to 0$ along
the direction \( \arg(h) = \theta \neq 0 \). About the discussion on all the crossings in the \( \mathbb{C}^0 \) complex sector of the \( h \) variable, see [22]. We expect the generalized crossing rules in terms of the four limits

\[
E_{m\pm}(h(n-,n^+)+\epsilon)-E_{m\pm}^\pm(h(n-,n^+)-\epsilon) \to 0, \quad \text{as } \epsilon \to 0, \quad m^\pm = n^- + n^+ + (1\pm 1)/2,
\]

at \( h(n-,n^+) \), where \( h(n,n) = h_0 \). Thus, we expect as a general rule, the instability of one of the nodes of the state \( \psi_{m^+}(h) \) and the partition between the two states \( \psi_{n^\pm}^\pm(h) \) of the \( n^- + n^+ \) nodes.

The hypothesis of minimality about the singularities gives the following picture. The analytic function \( E_m(h) \) for large \( |h| \) has a sequence of singularities in \( \mathbb{C}^0 \) ordered by the increasing values of \( \Im h_{j,k} \):

\[
h_{m,0}, h_{m-1,0}, h_{m-1,1}, h_{m-1,2}, \ldots, h_{1,m-2}, h_{1,m-1}, h_{0,m-1}, h_{0,m}
\]

It is possible to divide \( \mathbb{C}^0 \), for small \( |h| \), by cuts going from 0 to the branch points in a sequence of stripes

\[
S_m^-, S_0^+, S_{m-1}^+, S_1^-, \ldots, S_{m-1}^-, S_0^+, S_m^+
\]

in which, for \( h \to 0 \), the level \( E_m(h) \) has the behaviors

\[
E_{m^-}(h), E_0^+(h), E_{m-1}^-(h), E_1^+(h), \ldots, E_1^-(h), E_{m-1}^+(h), E_0^-(h), E_m^+(h)
\]

respectively.

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