A Cosmic Lattice as the Substratum of Quantum Fields

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Abstract

A cosmology inspired structure for phase space is introduced, which leads to finitization and lattice-like discretization of position and momentum eigenvalues in a preferred, cosmic frame. Lorentz invariance is broken at very high energies, at present inaccessible. The divergent perturbation terms in quantum electrodynamics become finite and small; this could become a requirement leading to model restrictions in other perturbative theories. So the very success of the usual renormalization procedures is simply explained by their finitization, and is viewed as indicating the reality of the lattice.

I. THE BOUNDLESS BUT FINITE COSMIC LATTICE

The idea of assigning a fundamental length to physical space has a long history. A large part of the literature can be traced from the work of Gudder and Naroditsky [1]. This notion arises from dissatisfaction with the way renormalization operations are performed in quantum field theory - both the nonrigorous and the too-rigorous methods. A lattice structure for space or spacetime is an obvious remedy for the worst problem, ultraviolet divergences, but people have been discouraged by the prospect of Lorentz and rotational non-invariance. Recently some work has appeared, which faces just this possibility. Thus Nielsen and collaborators have developed both a quantum electrodynamics (QED) and a Yang-Mills theory that violate Lorentz invariance [2], and Zee [3] has gone so far as suggesting experimental verification of such a breakdown. Some time ago Wheeler [4] pointed out the possibility of a breakdown of the spacetime concept itself on the scale of Planck’s length.

In this paper I introduce upper limits for both measurable length and measurable momentum, as a means of reaching a kind of lattice structure for phase space. The conclusion will be that the perturbative series of QED works so well because its terms become, in a quasi-invariant sense to be defined, convergent to small numbers. The idea has a cosmological inspiration: a Friedmann-Robertson-Walker (FRW) model with flat space sections [5] has the local metric $ds^2 = c^2 dt^2 - b^2(t) dx^2$. Its global spatial topology may be assumed to be that of a flat torus $T^3$, which is an identification space isometric to the quotient space $E^3/\Gamma$, where $E^3$ is Euclidean space and $\Gamma$ is a group generated by finite translations in the
three directions. For the development and motivations of this idea in cosmology see the list of refs. [6], in particular Ellis and Schreiber’s recent work.

The expansion factor in this metric is \( b(t) = (t/t_0)^{2/3} \), where \( t_0 \approx 10^{10} \text{ yr} \) is the age of the of the universe. Since \((db/dt)(t_0) \approx 10^{-18} \text{ sec}^{-1}\), let us put \( b(t) = 1 \) in \( ds^2 \) for the discussion of laboratory physics (adiabatic approximation). We are left with a locally Minkowskian spacetime, whose \( T^3 \) spatial sections may be obtained from a cube of side \( L \), by the identification (“gluing”) of opposite faces. Therefore \( \sqrt{3}L \) will be a maximum distance in space, and I take \( L = c/H \), where \( H \approx 75 \text{ km sec}^{-1}\text{Mpc}^{-1} \) is Hubble’s constant.

Now I make the crucial assumption that momentum space (relative to the above cosmic frame) has the same \( T^3 \) topology as configuration space, so that the corresponding phase space is the product manifold \( T^3 \times T^3 \). Besides providing an upper limit for momentum, this sort of duality appears to the author as a more reasonable way of assuring discretization of position than just drawing from crystal structure analogy, with its “neo-ether” connotation. The flat torus for momentum is obtained by identifying opposite faces of a cube of side \( P = 2\pi h/a \), where \( a = (Gh/c^3)^{1/2} = 1.61 \times 10^{-33} \text{ cm} \) is Planck’s length. Incidentally, a cutoff for momentum is quite reasonable, for otherwise the energy of a single virtual particle can exceed the total mass of the observable universe. It is also in agreement with Wheeler’s idea cited above: if spacetime breaks down at Planck’s length, so must the validity of momenta greater than \( P \). (Of course the \( T^3 \times T^3 \) topology postulate may come to be seen as a phenomenological assumption, if and when future developments lead to a more general theoretical framework, where phase space would be perceived as an approximation suitable for 1988 physics - like the idea of orbit that, from a quantum mechanical viewpoint, is seen as an approximation suitable for the classical limit.)

In quantum mechanics we are used to box quantization with periodic boundary conditions imposed by convenience. But in the above defined cosmic tori these conditions are just the natural ones. So we get eigenvalues \( x^k = n^k a \) and \( p^k = n^k(\pi h/Na) \), \( k = 1, 2, 3 \). Thus space becomes a very large box, which is both finite and boundless, with discrete eigenvalues for particle position and momentum. This suggests that we call it a cosmic lattice (CL), but note it is an abstract, not a granulated, crystal-like lattice.

II. LORENTZ QUASI-INVARIENCE

What about Lorentz invariance? First, the preferred status of the CL frame should not cause much surprise. It is the home frame of our cosmos, similar to the comoving system of Einstein-de Sitter’s cosmology, the 2.7 K radiation providing its concrete realization (except for spatial orientation; see Gott [6] for an explanation on how the apparent isotropy of cosmic observations is preserved despite the loss of global invariance for rotations). Second, let us define the composition of 4-momenta \( p^0_1 \) and \( p^0_2 \) in the CL system. Setting \( \hbar = c = 1 \), if \( |p_1^k + p_2^k| \leq \pi/a \), then energy-momenta are composed as usually: \( p^\mu = p_1^\mu + p_2^\mu \). If \( |p_1^k + p_2^k| > \pi/a \), then we add or subtract \( 2\pi/a \), so as to obtain \( p^k \) in the allowed range. This \( p^k \) is defined to be the resultant. With \( s \equiv (p_1^0 + p_2^0)^2 - (p_1^0 + p_2^0)^2 \), the resultant energy is \( p^0 = (s + p^2)^{1/2} \leq p_1^0 + p_2^0 \). Therefore energy-momentum conservation is only conserved in collisions if \( p_1 + p_2 \) is a CL momentum eigenvalue. But this is hardly a constraint, since laboratory momenta are far from our limit, \( \sqrt{3\pi/a} \approx 10^{20} \text{ GeV}/c \).
Thirdly, Lorentz transformations, including rotations, are performed as usually, but the limits of $p^k$ in an arbitrary frame are derived from those in the CL frame. Thus if a system has velocity $\beta = (v, 0, 0), v > 0$, and no rotation with respect to the CL, then

$$p_{\text{max}}^1 = \gamma \left[ \frac{\pi}{a} - v \left( m^2 + \frac{\pi^2}{a^2} \right)^{1/2} \right] \approx \frac{\pi}{2a\gamma}$$

and

$$p_{\text{min}}^1 = \gamma \left[ -\frac{\pi}{a} - v \left( m^2 + \frac{\pi^2}{a^2} \right)^{1/2} \right] \approx -\frac{2\gamma\pi}{a},$$

for large $\gamma$. The limits for $p^2, p^3$ remain unaltered. If we now rotate this frame, the physics will be invariant if all relevant momenta are smaller than $\pi/2a\gamma$. Again, in laboratory situations we do not have to worry about these limits. Summarizing, Lorentz and rotational invariance are preserved if we restrict ourselves to laboratory energies and to a theoretical range of frames - say, $\gamma < 10^{10}$ with respect to the CL, which guarantees invariance up to $\sim 10^9$ GeV in the moving frame. I shall refer to this restricted meaning as Lorentz quasi-invariance.

### III. FINITE RENORMALIZATION

The practice of renormalization on QED has been so strongly associated with the removal of infinities that the fundamental meaning of the former became blurred. See, for example, Schweber’s [7] warning against this tendency. Actually the aim of renormalization is to combine some unobservable parameters of a basic model into observable ones, so that the renormalized model is expressed in terms of the latter. The advantage of this process is obvious when one considers that the purpose of theoretical models is to represent experimental facts. Consider mass renormalization in QED: we write $m = m_0 + \delta m$, and say that $m$ is the experimental mass of the electron. But the underlying formalism suggests that we also interpret $m_0$ and $\delta m$ anyway, as bare mass and the effect of virtual photons always surrounding the electron respectively. It seems to the author that if $m_0$ and $\delta m$ can be made finite, so much the better: their interpretation is reinforced, and we might even think of making them observable - as when one tries to assign mass differences in isospin multiplets to electromagnetic interactions [8]. (See, however, [9].)

Therefore my program is not to abandon renormalization, but rather to make it step-by-step finite [10]. I will essentially follow the established formalism with a few adaptations: (a) integrals are in principle replaced by sums over the CL, but in practice the latter are approximated by integrals that formally resemble the original ones but are now finite and small (and justifiably so); (b) the calculations are preferably performed in the CL reference frame, which is the natural system in this context, just like a Sun-centered system is natural for planetary astronomy (this naturalness can of course be formalized); (c) restricted Lorentz transformations, as discussed in Sec. II, are seen to hold for the results of calculations.

Let us examine some problems of perturbative field theory in terms of the above ideas. The great success of the usual formalism suggests that we try to keep its analytical basis,
rather than for example switching to difference equations [1], [11]. This is physically reasonable, since the scale of Planck’s length is so much finer than that of currently observable processes. Therefore I will here assume minimum departures from established analytical expressions. For comparison with standard results I will rely on Itzykson and Zuber’s textbook [12], henceforth referred to as (IZn), where n is the page number.

The infrared catastrophes will be transformed into finite contributions (since the minimum energy of a massless particle is \( \pi/Na \), not zero), and if these are still too large they may be dealt with as usually, e.g. as in (IZ334).

Consider now charge renormalization in QED. The notation below is adapted from (IZ319ff). The one-loop contribution to vacuum polarization, after use of Feynman’s trick

\[
(ab)^{-1} = \int_0^1 dz [az + b(1 - z)]^{-2},
\]

is [13]

\[
\omega_{\rho\nu}(k^2) = -4e^2 \int_0^1 dz (z - z^2) \sum_{q \in CL} \int_{-\infty}^{\infty} \frac{dq_0 (g_{\rho\nu}k^2 - 2k_\rho k_\nu)(z - z^2) - g_{\rho\nu}(q^2/2 - m^2)}{[q^2 + k^2(z - z^2) - m^2 + i\epsilon]^2}.
\] (1)

This expression is well defined, so it can be safely simplified by gauge invariance, which leads to

\[
\omega_{\rho\nu}(k^2) = -i(g_{\rho\nu}k^2 - k_\rho k_\nu)\omega(k^2),
\] (2)

with

\[
\omega(k^2) = 2e^2 \int_0^1 dz (z - z^2) \sum_{q \in CL} |q^2 + m^2(z - z^2)|^{-3/2}.
\]

Eq. (2) is Lorentz quasi-invariant, in the above defined sense. Although calculated in the CL frame, \( \omega(k^2) \) is an invariant - like, say, the contribution of vibrational energy to the mass of a crystal. Approximating [14] the sum over the cosmic box by an integral over a ball of radius \( \pi/a \), and neglecting positive powers of \( ma \), I obtained for \( k^2 < 4m^2 \),

\[
\omega(k^2) = \frac{\alpha}{3\pi} \left[ \ln \left( \frac{2\pi}{ma} \right)^2 - 2 + \frac{k^2}{5m^2} \ldots \right].
\]

Hence \( Z_3 = [1 + \omega(0)]^{-1} = 0.925 \) and \( e = 0.962e_0 \). The Uehling term is the same as in (IZ327).

Similarly, I got for mass renormalization, to the same order,

\[
\frac{m}{m_0} = 1 + \frac{3\alpha}{4\pi} \left[ \ln \left( \frac{2\pi}{ma} \right)^2 - \frac{1}{3} \right] = 1.185,
\]

and, in Feynman’s gauge,

\[
Z_2^{-1} - 1 = \frac{\alpha}{4\pi} \left[ \ln \left( \frac{2\pi}{ma} \right)^2 + 2 \ln \left( \frac{\pi}{Nma} \right)^2 + 3 \right],
\]

hence \( Z_2 = 1.160 \). If we compare the above results with their counterparts in (IZ325,334,335), we see that the logarithmic terms in the former can be obtained from
those in the latter if we replace $\Lambda$ by $2\pi/a$ and $\mu$ by $\pi/Na$. The author hopes to derive similar results for higher order terms in QED, and possibly for other theories of fundamental processes. The important immediate consequence of the achieved finitization is that the terms of the perturbative series, which are normally understood in a context of formal procedures, to “extract sensible results from apparently ill-defined expressions” (IZ318), become legitimized as ordinary finite terms. Theories satisfying this condition could be called \textit{perturbatively renormalizable}, and this property might be a further guide for model building. (So, for example, the perturbative treatment of $\lambda\phi^4$ models might be deemed unacceptable, because of its quadratic divergence in mass renormalization.) As bonuses, the calculations become less difficult - the “naive prescription” (IZ374) of cutting off large momenta becomes the natural one - and the meaning of the renormalized Lagrangian gets a numerical foundation - compare (IZ345,346). Interpreted in this light, the fact that renormalization theory has been so successful can be invoked as an argument for the physicality of the CL (or some related concept). It remains to be seen whether this notion will be tested, for example in proton decay, as suggested by Zee [3].

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[10] One of the discoverers of QED renormalization has written that the rules for eliminating the infinities are like “sweeping the dirt under the rug”(cf. [8], p. 37). These rules can perhaps be justified by sophisticated mathematics. Here instead I am suggesting a mathematically simple and physically sensible way to avoid the occurrence of infinities.

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[13] This trick implies the translation $q \rightarrow q - k(1 - z)$ in the virtual momenta. Hence the sum will no longer be strictly symmetrical in the CL. But since $|k| << \pi/a$, this asymmetry is disregarded.

[14] A better approximation would be to integrate over a cube of side $2\pi/a$, but for this preliminary work I adopted the simpler integration over a ball.