Teachers’ conceptual understanding of fraction operations: results from a national sample of elementary school teachers

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Abstract
Teachers’ understanding of the concepts they teach affects the quality of instruction and students’ learning. This study used a sample of 303 teachers from across the USA to examine elementary school mathematics teachers’ knowledge of key concepts underlying fraction arithmetic. Teachers’ explanations were coded based on the accuracy of their explanations and the kinds of concepts and representations they used in their responses. The results showed that teachers’ understanding of fraction arithmetic was limited, especially for fraction division, yet a moderate relationship was found between teachers’ understanding of fraction addition and division. Furthermore, more experienced teachers seemed to have a deeper understanding of fraction arithmetic, whereas special education teachers had a substantially limited understanding.

Keywords Teacher knowledge · Mathematical knowledge for teaching · Fraction operations · Content knowledge for teaching mathematics

Fractions are one of the most important topics in students’ progressions through school mathematics. Competence with fractions is associated with mathematics achievement (e.g., Siegler et al., 2012; Siegler, Thompson, & Schneider, 2011) and is critical for learning more advanced mathematical concepts (National Mathematics Advisory Panel [NMAP], 2008). Furthermore, proficiency in fraction arithmetic is a major goal of K-8 students’ mathematics education (NMAP, 2008).

With the importance of fractional understanding known, children in the USA are expected to receive substantial instruction in fractions beginning in the third grade (e.g., Common Core State Standards Initiative, 2010). However, even with a substantial amount of instruction, students continue to struggle understanding fraction concepts and arithmetic (e.g., Provasnik, Dogan, Erberber, & Zheng, 2020; Siegler & Lortie-Forgues, 2015). For example, 39% of US

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fourth-grade students added both numerators and denominators when they were asked to add fractions (National Assessment of Educational Progress [NAEP], 2013). Additionally, teachers are aware of such challenges, evidenced by a national sample of secondary mathematics teachers in public schools who reported that rational numbers and operations involving rational numbers was the second poorest area of preparation among incoming students, after solving word problems (Hoffer, Venkataraman, Hedberg, & Shagle, 2007).

Research suggests that not only students struggle with understanding fractions, but so do prospective and in-service elementary school teachers (e.g., Ma, 2010; Newton, 2008; Olanoff, Lo, & Tobias, 2014). This understanding of current and prospective math teachers’ knowledge is critical, given that teachers’ knowledge of the mathematics they are expected to teach has an impact on the quality of their instruction (e.g., Borko et al., 1992; Copur-Gencturk, 2015; Hill et al., 2008). Moreover, much of the research on teachers has shown that they struggle with understanding fraction concepts, especially fraction arithmetic (e.g., for a review, see Olanoff et al., 2014). More importantly, these studies point out that although preservice and in-service teachers at the elementary level know how to perform algorithms such as invert and multiply (i.e., procedural knowledge), they struggle to understand why, when, or where to apply these procedures (i.e., conceptual understanding; e.g., Ball, 1991; Ma, 2010; Newton, 2008; Son & Crespo, 2009; Tirosh, 2000). The lack of a deep and flexible knowledge base has critical implications and needs to be further understood.

The present study continues the work in the area of teacher knowledge about fractions and explores teachers’ understanding of two fraction operations, contributing to the field in three major ways. First, most studies focusing on teachers’ understanding of fraction arithmetic have been conducted with preservice teachers (e.g., Ball, 1990; Newton, 2008; Simon, 1993; Son & Crespo, 2009; for a review, see Olanoff et al., 2014); thus, we know little about the extent to which in-service teachers understand the mathematical underpinnings of fraction arithmetic. This gap in the literature creates a potential for in-service teachers’ fraction understanding to be drastically different from preservice teachers, since they might have developed a deeper understanding of these concepts over time from teaching these concepts, interacting with students and the curriculum materials, and participating in professional development. Therefore, exploring in-service teachers’ understanding of fraction arithmetic could help us ascertain why students might be struggling with fraction arithmetic, as well as best practices to support teachers’ understanding of fractions.

Second, studies in general, and especially ones conducted with in-service teachers, have not focused on teachers’ understanding of the mathematical concepts behind fraction arithmetic. In fact, in the majority of these studies, teachers were asked to write a story problem presenting fraction division, a task intended to capture their understanding of when to use this fraction operation or how to model fraction arithmetic (e.g., Ball, 1990; Li & Kulm, 2008; Ma, 2010; Simon, 1993; Son & Crespo, 2009; for a review, see Olanoff et al., 2014). Although these studies have revealed that teachers seem to mix which operation to use, multiplication and division more so than addition and subtraction, they have provided little insight into how teachers understand the operation itself. It seems more plausible that by directly focusing on teachers’ understanding of the underlying meaning and concepts of fraction arithmetic, we could better understand the link between teacher knowledge and why students struggle with fraction operations. Additionally, although asking teachers to use a particular method, such as
using visuals, could provide insights into teachers’ knowledge of the key ideas involved in an operation (e.g., the referent whole in fraction division), restricting teachers to using a particular method might also limit researchers’ ability to assess whether teachers know these concepts accurately. It may be that teachers who may not be able to model an operation may know how to explain the underlying mathematical ideas by using real-world examples, and allowing teachers to use any representation they wish and specifically asking them to explain the mathematical ideas, as in this study, allows us to capture their understanding of and struggles with these concepts. This, in turn, helps us discover potential reasons why students continue to make certain errors related to fraction arithmetic.

Third, much of the earlier work on teachers’ understanding of fraction algorithms has been conducted with a small number of teachers (e.g., Borko et al., 1992; Ma, 2010). This important work has contributed to our understanding of teachers’ struggles with their conceptual understanding of fraction arithmetic but lacks evidence for how common these struggles are across teachers. Using a national sample of in-service teachers, with varying degrees of educational preparation and experience, to investigate teachers’ understanding of the division algorithm could uncover common patterns of understanding among in-service teachers and make broader claims about mathematics teachers.

This study aimed to address the gap in the literature by collecting data from a national sample of 303 upper elementary school mathematics teachers who were teaching in grades where fraction concepts and arithmetic are introduced and taught. It is my contention that understanding teachers’ in-depth understanding of a school subject will allow us to better address teachers’ needs in teacher education and professional development programs and will help us to better understand what learning opportunities teachers are providing to students to make sense of school subjects. In the following sections, I justify the importance of studying teachers’ conceptual understanding of concepts taught in school and review prior work on teachers’ conceptual understanding of fraction arithmetic.

1 Teachers’ conceptual understanding of the concepts they are expected to teach

Teachers’ knowledge of the subject matter, along with their content-specific pedagogical skills, has been the focus of numerous studies (e.g., Baumert et al., 2010; Blomeke, Busse, Kaiser, Konig, & Suhl, 2016; Campbell et al., 2014; Copur-Gencturk, 2015; Copur-Gencturk, Tolar, Jacobson, & Fan, 2019; Hill, Rowan, & Ball, 2005; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012; Ma, 2010; Sadler, Sonnert, Coyle, Cook-Smith, & Miller, 2013). In fact, in the last three decades, much more progress has been made toward identifying the components of the knowledge teachers need to teach school subjects effectively (Ball, Thames, & Phelps, 2008; Grossman, 1990).

One of the primary knowledge domains for effective teaching as agreed by almost everyone studying teacher knowledge is subject matter knowledge (Ball, 1991; Ball et al., 2008; Grossman, 1990; Leinhardt & Smith, 1985; Shulman, 1986). Logically, teachers cannot teach things they themselves do not know (Ball, 1991). Thus, the most easily recognized component of subject matter knowledge is the knowledge of facts, rules, and concepts (Ball, 1991; Shulman, 1986). However, subject matter knowledge involves more than knowing the key
facts, rules, or procedures (e.g., Ball, 1991; Grossman, 1990; Shulman, 1986), and it also involves having an explicit conceptual understanding of underlying procedures and knowing why such rules and facts are warranted (e.g., Ball, 1991; Ball et al., 2008).

Although teachers’ conceptual understanding alone may not guarantee effective teaching, teachers’ lack of understanding makes it impossible to create a learning environment in which students can build a meaningful understanding of the concepts they are learning (Ball, 1991; NMAP, 2008). For instance, a teacher cannot explain to students the principles underlying fraction addition if they do not conceptually understand why a common denominator is needed when adding fractions with unlike denominators. In turn, this can lead students to develop incorrect strategies, such as adding across numerators and denominators, because they rely on their prior knowledge of working with whole numbers. Furthermore, the level of teachers’ conceptual understanding affects the pedagogical resources teachers employ in their practice (e.g., Borko et al., 1992; Eisenhart et al., 1993). Thus, the way teachers understand the rules and algorithms shapes students’ opportunities to learn these concepts.

Several documents published by mathematics education organizations explicitly acknowledge the importance of teachers’ deep understanding of mathematics by setting that teachers’ robust understanding of school mathematics as a standard for teaching mathematics (e.g., Bezuk et al., 2017; National Council of Teachers of Mathematics, 1991). Today’s student is not only expected to know the rules and how to execute procedures but also to conceptually understand what the mathematical procedures mean and the connections among mathematical concepts (Common Core State Standards Initiative, 2010). Therefore, a teacher’s own understanding of the mathematical concepts underlying the rules and procedures plays a vital role in supporting students’ development of a robust understanding of the concepts that meets the expectation of the standards.

The importance of teachers’ knowledge or lack of knowledge in instruction has also been supported by prior empirical work. In fact, several studies have documented that teachers’ deep understanding of concepts is linked to effective instruction and student learning (e.g., Charalambous, 2010; Copur-Gencturk, 2015; Hill et al., 2008; Kersting et al., 2012; Tchoshanov, 2011). Specifically, evidence suggests that teachers’ lack of understanding is associated with mathematical errors in instruction (e.g., Borko et al., 1992; Hill et al., 2008) and that strong mathematical knowledge is associated with a higher quality of mathematics instruction, such as making key mathematical ideas more explicit in teaching (Copur-Gencturk, 2015). Although research showing a direct relationship between teachers’ understanding of the subject matter and students’ learning is mixed (cf. Hill et al., 2005; Kersting et al., 2012), several studies have shown that teachers’ subject matter knowledge has an indirect impact on students’ learning through instruction (Baumert et al., 2010; Kersting et al., 2012).

2 Prior literature on teachers’ knowledge of fraction arithmetic

An accumulating body of research has documented that although a significant portion of teachers can perform fraction operations, a smaller portion of the same teachers understand where to use which fraction operations, especially division (e.g., Lo & Luo, 2012; Ma, 2010; Newton, 2008). For instance, Son and Crespo (2009) found that all 17 US prospective elementary teachers and 17 US prospective secondary teachers were able to solve a fraction division arithmetic problem (2/9 ÷ 1/3).
However, none of the preservice elementary teachers and only 35% of the secondary teachers in their study were able to create a story problem that would illustrate $2/9 \div 1/3$.

Newton’s (2008) examination of the fraction knowledge of 85 preservice teachers enrolled in a mathematics course in an elementary education program provided further insight into preservice teachers’ struggles with the underpinnings of fraction arithmetic. Participants were asked to compute 3–5 fraction arithmetic problems in each operation (e.g., $2^{\frac{1}{3}} \div 9$ and $2/4 - 3/6$). Analysis of the errors prospective teachers made in computing fraction arithmetic suggested they had a weak understanding of the conceptual underpinnings of fractions and fraction arithmetic, such as the role of the denominator. Specifically, participating teachers used different processes when the denominators of fractions were the same in multiplication and division questions.

Although much of the work examining teachers’ conceptual understanding of fraction arithmetic has been conducted with preservice teachers, studies conducted with in-service teachers have also painted a picture of U.S. teachers as having an underdeveloped understanding of fraction arithmetic (e.g., Izsák, Jacobson, & Bradshaw, 2019; Ma, 2010). As an example, in an international comparison study of Chinese and US teachers, Ma (2010) interviewed 23 US in-service elementary school teachers who were asked to solve four mathematical tasks, one of which focused on fraction division. She documented that 48% of US teachers in her study were correctly able to divide $1\frac{3}{4}$ by $\frac{1}{2}$, and none of the teachers were able to come up with a real-word situation or a story problem that would meaningfully represent the division of the two fractions.

When looking at previous studies on fraction arithmetic, one issue stands out: much of the prior work has focused on proxies of teachers’ conceptual understanding of fraction arithmetic, such as writing a word problem or drawing or selecting a model to illustrate a situation (e.g., Son & Crespo, 2009). Although these studies have delineated problems with teachers’ performance, they have failed to detect why teachers have provided these incorrect responses. Explicitly capturing teachers’ understanding of an algorithm could reveal potential reasons for their struggles. A case where this was captured is in a study by Borko et al. (1992). Their work with a teacher candidate, Ms. Daniels, found that right after the class computed the answer to a fraction division problem ($\frac{3}{4} \div \frac{1}{2}$), a student in her student teaching placement asked her to explain why the invert-and-multiply rule for fraction division worked. Ms. Daniels created a fraction multiplication situation and then realized her model was incorrect. It was clear that the teacher lacked an understanding of the rule that had implications for her practice and student learning. Such in-the-moment struggles (i.e., not being able to correctly model or create a real-life example to illustrate the mathematical situation) have been reported in other studies where similar methods have been used (i.e., asking participants to model or create a real-world problem).

Yet, because Borko et al. (1992) collected data from Ms. Daniels regarding her understanding of the invert-and-multiply algorithm, they were better able to gain insights to her struggle. Respectively, when Ms. Daniels was asked to explain why the invert-and-multiply algorithm worked during an interview, before taking a methods course on teaching mathematics, her response focused on multiplication being an inverse operation of division:
you turn [it] into a multiplication problem and since multiplication is the inverse operation of division, then you have to take that second number you see or your divisor and turn it over because you’re doing the inverse to it as you would with the division sign. (p. 208, Borko et al., 1992).

Even though Ms. Daniels had completed several courses in advanced mathematics and was enrolled in an elementary mathematics methods course, she did not seem to understand why the algorithm worked. Yet when she was reviewing the algorithm with students in her placement, she explained the algorithm the same way she had explained it during the interview. Thus, I contend that asking teachers why an algorithm works may reveal how they explain it to their students.

Nevertheless, revealing teachers’ understanding of a concept by asking why a procedure works is not an easy task because of the linguistic difficulty surrounding the meaning of why sayings like “invert-and-multiply” work for fraction division algorithms (Borko et al., 1992). One way of addressing such a difficulty is to capture teachers’ understanding of the key concepts underlying the algorithms for fraction arithmetic. For example, given that the algorithm for fraction addition and subtraction is based on the idea of partitioning a whole into equal-sized pieces, capturing teachers’ understanding of the role of the denominator throughout the process of adding or subtracting fractions could provide insight into their more nuanced understanding of the algorithm.

Similarly, attending to the units or wholes the fractions refer to in fraction multiplication and division situations is vital for understanding fraction multiplication and division. For instance, conceptualizing fraction division by using the divisor as a referent unit (whole) and making sense of division as the number of groups that can be made by the dividend could help teachers develop a foundation for fraction operations. Thus, a fraction arithmetic problem such as $3/2 \div 1/4$ could be conceptualized as the number of groups of $1/4$ that can be made from $3/2$. Generating a fraction equivalent to $3/2$, such as $6/4$ (i.e., creating a common denominator) is also needed to see how many groups of $1/4$ can be made from $3/2$. Thus, the solution to $3/2 \div 1/4$ is the same as the solution to $6/4 \div 1/4$, which makes it easier to see that there are six groups of $1/4$ in $3/2$. Focusing on the measurement meaning of division and referent units can be applied to any fraction division problem (i.e., $a/b \div c/d$) and provides a conceptual explanation for the invert-and-multiply algorithm (see Fig. 1). Specifically, $a/b \div c/d$ can be conceptualized as the number of groups of the divisor, $c/d$ that can be made from the dividend, $a/b$. Because fractional representations depend on equal-size pieces, creating a common denominator for these two fractions, $a/b$ and $c/d$, is needed for the fractional representation of the quotient. As shown by the blue rectangle in Fig. 1c, one group of a divisor, $c/d$, can be made by $c \times b$ equal-sized parts. The dividend, $a/b$, has $a \times d$ same-sized parts (see the green rectangle in Fig. 1c). The quotient (i.e., the number of groups of the divisor) can then be found by dividing the total number of parts, $a \times d$, by the number of parts in one referent unit, $b \times c$. This approach provides a conceptual explanation for the division algorithm in that $a/b \div c/d = (a \times d)/(c \times b)$, and by focusing on the fact that the referent unit for the quotient is the divisor, it also creates a meaningful explanation for the denominator of the quotient.
Thus, the measurement interpretation of division along with the use of the common denominator approach could conceptually explain the division algorithm. That being said, there are other mathematical explanations for the division algorithm (e.g., Beckmann, 2012; Tirosh, 2000). For instance, using the same example, the division problem of $\frac{3}{2} \div \frac{1}{4}$ can be found by using the knowledge that division and multiplication are inverse operations. The division of $\frac{3}{2}$ by $\frac{1}{4}$ can be thought of as an unknown factor multiplication problem (i.e., $\frac{3}{2} \div \frac{1}{4} = q$ is equivalent to $q \times \frac{1}{4} = \frac{3}{2}$). Using the fact that the product of a number and its reciprocal is 1, multiplying both sides of the equation ($q \times \frac{1}{4} = \frac{3}{2}$) by the reciprocal of $\frac{1}{4}$ (i.e., $4/1$) will then lead to $\frac{3}{2} \times 4/1 = q$. Note that both expressions, $\frac{3}{2} \div \frac{1}{4}$ and $\frac{3}{2} \times 4/1$, equal $q$. Thus, the division of $\frac{3}{2}$ by $\frac{1}{4}$ can be found by multiplying the dividend by the reciprocal of the divisor. This approach is valid for the division of any fraction in that:

![Fig. 1](image)
Similarly, the knowledge of complex fractions in which the denominator, numerator, or both can be fractions, the knowledge of equivalent fractions, and the knowledge that the product of a number by its reciprocal equals 1 can be used to explain how division of the fraction algorithm works. Using the same division example, \( \frac{3}{2} \div \frac{1}{4} \) can also be thought of as a complex fraction \( \frac{3}{2} \). A fraction equivalent to \( \frac{3}{2} \) can be created by multiplying both the numerator (i.e., \( \frac{3}{2} \)) and the denominator (i.e., \( \frac{1}{4} \)) by the reciprocal of the divisor (i.e., \( 4/1 \)). This results in \( \frac{3}{2} \times \frac{4}{1} \). Because the product of a number by its reciprocal equals 1 and the division of a number by 1 is equal to itself, the expression is \( \frac{3}{2} \times \frac{4}{1} = \frac{3}{2} \times 4 \).

This method could also be used to prove the division algorithm for the division of any fraction pair:

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c} = \frac{a}{b} \times \frac{d}{c}
\]

Understanding the key concepts behind these rules is essential not only for developing a robust understanding of fraction arithmetic, but also for understanding fraction concepts. For instance, Izsák, Orrill, Cohen, and Brown (2010) collected data from a convenience sample of 201 middle-grade teachers through multiple-choice items targeting fraction knowledge. As a result of using a mixture-Rasch model to analyze the data, they identified two latent groups of teachers that differed in their identification of the appropriate referent units in problem situations (Izsák, Jacobson, de Araujo, & Orrill, 2012). Furthermore, Copur-Gencturk and Olmez (2020) found that teachers’ understanding of referent units was associated with their overall performance on other fraction concepts. Thus, capturing teachers’ knowledge of the conceptual underpinnings of fraction arithmetic could delineate their understanding of other fraction concepts.

Building off the work of other scholars, this study captures teachers’ understanding of the mathematical underpinnings of two fraction operations, fraction addition and fraction division, to investigate the extent of their conceptual understanding of each operation as well as

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1 I chose these two operations because the algorithms for the addition and subtraction of fractions are identical, and the conceptual underpinnings of multiplication and division are similar. That is, both algorithms can be conceptually explained by attending to referent units. The multiplication algorithm can be conceptually explained by the fact that the multiplier and product are referring to the same referent unit, and the multiplicand refers to the multiplier as a referent whole. The division algorithm can be explained by the fact that the dividend and divisor are referring to the same referent whole, whereas the quotient refers to the divisor as the referent unit.
what concepts and representations they used in their explanations. Specifically, this study aimed to answer the following four research questions:

1. What is the accuracy of teachers’ explanations of fraction algorithms? Is there a difference in the mathematical accuracy of their explanations for the addition and division algorithms?
2. What is the relationship between teachers’ explanations for fraction addition and division?
3. What are the characteristics of teachers’ explanations? What concepts and representations do they use to explain fraction addition and division?
4. What aspects of teachers’ professional background are related to the accuracy of their explanations?

3 Methods

3.1 Study context

The data for this study were collected from teachers who were teaching fourth- or fifth-grade mathematics during the 2017-2018 academic year, the data collection period. Participants were recruited through an education survey company and professional development organizations. Given that recruitment was done through an intermediary and data were collected through an online survey, several precautionary steps were taken to ensure that the data were being collected from the targeted population in an appropriate manner. Specifically, teachers received an e-mail with a link to a screening survey, which began with general questions regarding the participant’s career and were followed by specific questions about their time working as a classroom teacher. Following the screener survey, those who were eligible to participate (i.e., teachers currently teaching mathematics in grades 4 and 5) were allowed to respond to the items used in the study.

3.2 Analytic sample

The analytic sample included mathematics teachers who provided background information, and at least one of the items used in the analysis (N = 303). As shown in Table 1, the study sample was similar to a nationwide sample of elementary school teachers in terms of race, gender, and certification (Snyder, de Brey, & Dillow, 2019). However, the analytic sample seemed to include more teachers with 3 to 9 years of teaching experience and fewer teachers with more than 20 years of teaching experience. In general, teachers in the study sample had, on average, 9.8 years of mathematics teaching experience (SD = 7.58); 66.7% held multiple subject teaching credentials, 16.2% held a mathematics teaching credential, and 12.9% held a special education teaching credential. Although the participating teachers were currently teaching mathematics in Grades 4 and 5, more than one-third of them (34.7%) ever taught mathematics in Grade 6 or higher.

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1 Teachers who completed the survey received an online gift card for their participation.
3.3 Tasks

Two tasks were developed to capture teachers’ conceptual understanding of two fraction operations: addition and division (see Appendix for the two tasks). Because this study aimed to capture teachers’ understanding of the mathematical ideas behind these two operations, the wording of the tasks used in the study specifically focused on revealing their understanding of the key concepts.

Item development followed several iterative processes. First, the two tasks are adapted from two problems developed by Van de Walle, Karp, Bay-Williams, and Wray (2010), one of the most widely used elementary teacher education resources. The problems were then shared with mathematics education scholars for feedback, and revised accordingly. Next, in-service teachers in Grades 4 and 5 were interviewed using these tasks. During this process, both tasks were then revised to capture teachers’ understanding of the mathematics behind these procedures. Finally, the final versions of the tasks were tested by another 10 teachers to ensure that teachers’ understanding of these operations were able to be captured accurately.

3.4 Analysis

Data were coded in two major phases. Teachers’ responses were first coded according to the correctness of their explanations (i.e., incorrect/no, partially, or correct). Then, subcodes (Saldana, 2013) were developed to identify common representations the teachers used (e.g., real-world examples or visuals) or the kinds of explanations they gave in their responses (e.g., the measurement meaning of division).

In the first phase, teachers’ explanations were evaluated based on the extent to which their explanations focused on key ideas underlying the algorithms. The correctness of teachers’ explanations was rated using three categories: incorrect/no explanation, partially correct, and correct explanations (Table 2). The incorrect explanations category included incorrect responses or responses that focused on an algorithm. For instance, one teacher said, “We need a common denominator to make the problem easier to solve.” This response was coded as an incorrect explanation because the mathematical reasoning underlying fraction addition is not to, “…make the problem easier to solve.” Second, responses coded as partially correct explanations consisted of correct answers that omitted key ideas such as the meaning of division.

Table 1  Background characteristics of teachers in the present sample compared with a nationwide sample

|                          | Study sample | Nationwide sample of elementary school teachers (2015–2016) |
|--------------------------|--------------|-------------------------------------------------------------|
| Female                   | 87.1         | 89.3                                                        |
| White                    | 82.5         | 80.2                                                        |
| Regular certification    | 94.4         | 91.1                                                        |
| Years of teaching experience |            |                                                             |
| Fewer than 3             | 10.9         | 10.1                                                        |
| 3 to 9                   | 37.0         | 28.3                                                        |
| 10 to 20                 | 39.3         | 39.3                                                        |
| More than 20             | 12.9         | 22.3                                                        |

Source: Digest of Education Statistics, https://nces.ed.gov/programs/digest/d17/tables/dt17_209.22.asp?current=yes.
of responses in which the key mathematical ideas underlying these procedures were implied, but not made explicit. For example, a teacher said, “It’s difficult to add fractions that are not equal pieces. So, one must ‘cut up’ the pieces into pieces that are multiples of both. Can’t add apples + oranges.” This statement provides a mathematical rationale for creating a common denominator; however, it does not explicitly articulate the mathematical idea behind the operation, and the example provided is not completely appropriate. Although the teacher mentioned that the pieces are not equal, she or he did not explain why an equal denominator is needed (e.g., fractional representation is based on the number of equal-sized pieces that make up the whole). Additionally, examples using apples and oranges could be problematic in that when adding and subtracting fractions, the same referent whole is used; therefore, using two different fruits could potentially be misleading. The final category, correct explanations, included responses that explicitly focused on the underlying key ideas. Responses such as the following were coded in this manner:

This is done because it will become harder to compute and understand in case you are adding or subtracting. The denominator of a fraction tells you the relative size of the pieces. For instance, \( \frac{1}{2} \) is bigger than \( \frac{1}{4} \) because it only takes 2 pieces to make a whole, as opposed to 4 pieces to make the whole. One might connect the need for a common denominator to the need for having common units before adding and subtracting (you wouldn’t add 12 inches to 12 feet and get 24 for an answer). Therefore, the reason fractions need a common denominator before adding or subtracting is so that the number of pieces you are adding/subtracting are all the same size. Note that the numerator for a fraction just tells you how many pieces you have of that size.

As shown in this response, the key mathematical idea is to create equal-sized pieces because a denominator tells the number of equal-size pieces needed to make the whole. Furthermore, the example chosen to illustrate the point is mathematically appropriate in that both inches and feet have the same referent unit.

After finalizing an initial version of a rubric to use with the responses teachers provided from the task, a second rater was trained. Together, we coded several responses to establish interrater reliability and refined the criteria for each category. After finalizing the rubric criteria and gaining confidence in using the rubric to code the responses reliably, we independently coded the responses.

Table 2  Scoring rubric used to categorize teachers’ explanations

| Category            | Operation | Description                                                  |
|---------------------|-----------|--------------------------------------------------------------|
| Incorrect/no explanations | Addition | The response was either incorrect or teachers simply stated the steps in the algorithm. Teachers who reported they did not know how to explain were also coded in this category. |
|                     | Division  | The response was either incorrect or teachers simply stated the steps for the algorithm. Teachers who reported they did not know how to explain were also coded in this category. |
| Partially correct explanations | Addition | Either the key idea (same size or equal partitioning) was not explicitly stated or the examples or visuals provided were not accurate. |
|                     | Division  | Either the key idea (why the invert-and-multiply algorithm works or the need to attend to the referent units) was not explicitly stated or the examples or visuals provided were not accurate. |
| Correct explanations | Addition  | The key idea (same size or equal partitioning) was explicitly stated. |
|                     | Division  | Why the invert-and-multiply algorithm works was explained conceptually or the teacher attended to the referent whole, or both. |
from 10 teachers and discussed our ratings, working through any disagreements and noting exemplars. We continued rating 10 teachers’ responses at a time, until reaching 90% exact agreement. Upon reaching the 90% threshold, we coded separately and held a final meeting to review the ratings and settle any disagreements in our ratings.

In the second phase, the co-rater and I developed low-inference subtopics to capture noticing from the data. An example of a low-inference subtopic is how, in both tasks, teachers were often using visuals or real-world problems to explain these procedures. Specifically, for the division problem, it was noticed that teachers used certain interpretations of division problems, or it was found that they mixed fraction multiplication with division. These codes required less interpretation than the codes in the first phase because the topics were explicitly derived from the teachers’ responses. The co-rater and I then discussed the subcodes and began coding the dataset again with these subcodes in mind. Similar to the first phase, we coded data together until reaching 90% interrater reliability and then coded the remaining teacher responses separately while meeting to discuss and settle any disagreements.

In sum, each individual response was coded in two ways, by two independent raters. Once to measure the correctness of the teachers’ explanations and a second time based on subtopic codes for the concepts or representations they used in their explanations.

Using this rated and coded data, quantitative analyses were utilized to address the four research questions (RQ) guiding this study. To investigate the extent to which teachers correctly explained the conceptual underpinnings of the fraction addition and division algorithms (RQ-1a), frequencies of responses for both tasks are reported separately. Additionally, to test whether teachers had a better understanding of the fraction addition algorithm than the division algorithm (RQ-1b), a paired-sample t test was employed. To examine the relationship between the correctness levels of teachers’ explanations for both procedures (RQ-2), a $3 \times 3$ contingency table was used for a chi-square test of independence to examine the relationship between teachers’ explanations for these two operations. This analysis allowed me to examine whether teachers who understood one algorithm were likely to understand the other. To answer the extent to which the teachers explained the underpinnings of these operations and what concepts and representations they used in their explanations (RQ-3), I summarized characteristics of the explanations for each correctness level along with sample responses and reported the frequency of the subtopics and representations used for each operation. Finally, to further investigate how teachers’ educational background was related to the correctness of their explanations (RQ-4), an ordered logistic regression,\(^4\) in which teachers’ correctness level was predicted by their years of mathematics teaching experience (standardized), credential type (generalist, teaching mathematics, or other, with generalist being the reference category), highest grade level of mathematics being taught (ranging from 4 to 9), and whether they held a regular teaching certificate. The tasks were added as fixed effects, and standard errors were clustered around teachers in this analysis.

4 Results

4.1 Mathematical accuracy of teachers’ explanations of fraction addition and division

The purpose of this study was to investigate teachers’ conceptual understanding of fraction addition and division. As shown in Fig. 2, 19.8% of teachers provided an incorrect or no

\(^4\) I used ordered logistic regression because the explanatory categories were hierarchical in that partially correct responses were more accurate than incorrect responses but less accurate than correct responses.
explanation for the fraction addition procedure whereas this rate was 58.1% for fraction division. About half (55.6%) of the teachers explained the need for a common denominator mathematically and included correct explanations for adding fractions with unlike denominators, whereas only 26.1% of the teachers provided a conceptual explanation for the division procedure. The difference in mathematical accuracy of teachers’ responses was statistically significant ($M_{\text{difference}} = .67$, $SD = 1.03$, $t(275) = 10.9$, $p < .001$), indicating teachers were able to provide a more accurate explanation for the fraction addition algorithm than the division algorithm.

4.2 Relationships among teachers’ explanations for fraction addition and division

There was a statistically significant and moderate relationship between teachers’ understanding of the conceptual underpinning of both operations ($\chi^2(4, 276) = 15.85$, $p = .003$; $\text{Gamma} = .34$). Specifically, 15.2% of the teachers provided incorrect explanations for both operations, and 18.9% provided correct explanations for both operations (see Table 3). Half of the teachers who could explain the addition procedure provided incorrect explanations for the division procedure, whereas only 7.4% of the teachers who failed to explain the addition procedure were able to explain the division procedure.

Table 3 Number of teachers in the explanation categories for division and addition

|          | Division | Addition | Incorrect/No | Partially correct | Correct |
|----------|----------|----------|--------------|-------------------|---------|
| Incorrect/no |          |          | 42           | 8                 | 4       |
| Partially correct |    |          | 38           | 12                | 17      |
| Correct     |          |          | 78           | 25                | 52      |
4.3 Teachers’ explanations for fraction addition and division

Here, teachers’ explanations of the addition and division algorithms are delineated by overall patterns along with sample responses. For both operations, teachers whose responses were incorrect either reported that they did not know why the procedure worked or they simply stated the steps of the operation (see Fig. 3 for sample incorrect explanations).

The teachers who provided partially correct explanations for the addition algorithm (24.7%) implicitly focused on the key concept underlying the fraction addition procedure (i.e., equal partitioning); however, either their responses did not make the concept explicit or the examples they provided and the visuals they used were not accurate. For instance, as shown in the left panel of Fig. 4, the teacher did not state explicitly that equivalent fractions made the fraction pieces equal, but his or her drawing used the same size whole and equal-sized pieces. For the division operation, 15.8% of the teachers’ explanations focused on the key ideas involved in the fraction division procedure, such as making the number of groups of the divisor ($\frac{2}{3}$), but they failed to show it accurately. Again, as shown in the right panel of Fig. 4, the teacher attempted to provide an explanation for the division procedure but did not explicitly state how the procedure worked.

Among those who provided correct explanations, about half (55.6%) of the teachers explained the need for a common denominator mathematically and included correct explanations for adding fractions with unlike denominators, whereas only 26.1% of these teachers provided a conceptual explanation for the division procedure (see Fig. 5). Additionally, for the division algorithm, 11.0% of the teachers provided an explanation for how the division algorithm worked and how to make sense of the quotient.

![Fig. 3](image1.png) Sample responses for incorrect explanations

![Fig. 4](image2.png) Sample responses for partially correct responses
4.4 Representations and concepts used in teachers’ explanations

Teachers heavily used visuals and real-world examples in their explanations (see Fig. 6). In their responses, 74.0% of the teachers used drawings to explain the addition procedure, whereas 52.6% used visuals in their explanations for fraction division. Additionally, teachers used more real-world examples to explain the addition procedure than the division procedure (27.1% vs. 7.6%), and teachers who provided real-world examples generally used examples such as “you cannot add apples and oranges” to explain why a common denominator would be needed. As mentioned, this example could be problematic, given that apples and oranges are different fruits, yet addition and subtraction can be done if both fractions refer to the same unit.

In addition to teachers’ use of visuals, three patterns emerged in teachers’ explanations of fraction division (see Fig. 6). First, teachers used easier fraction pairs, such as a whole number divided by a unit fraction, to explain fraction division (6.2%). In fact, 21.7% of the partially correct responses included easier fractions (e.g., 2÷1/2). Second, the measurement meaning of division (how many groups of a divisor could be made with the dividend) seemed to be used for more than half of the partially correct and correct explanations (52.2% and 54.0%).
respectively). Third, 8.3% of the teachers mixed division with multiplication. In fact, 79.2% of these teachers’ drawings were modeling fraction multiplication rather than fraction division.

### 4.5 Linking teachers’ educational background to their explanations

Since the teachers’ explanations were coded using a 3-point scale, in which levels were hierarchical, an ordered logistic regression was used to investigate the relationships among teachers’ professional background characteristics and the level of correctness of their explanations to the fraction addition and division tasks. Of teachers’ professional background indicators, only teaching credentials and years of mathematics teaching experience were associated with the mathematical accuracy of their explanations (see Table 4). Specifically, for teachers holding credentials in other subjects (i.e., special education or science), the odds of providing fully correct explanations versus not providing fully correct explanations (i.e., partial or incorrect explanations) were .53 times those of teachers holding multiple subject teaching credentials (i.e., generalist teachers), \( p = .003 \). However, teachers who held a mathematics teaching credential did not seem to provide statistically more accurate explanations than those who held multiple subject credentials. Additionally, for a 1 \( SD \) increase in teachers’ mathematics teaching experience when all

| Teachers’ educational background indicators | Odds ratios (SE) |
|--------------------------------------------|------------------|
| Full certification                         | 1.18 (.44)       |
| Highest grade of mathematics taught        | 1.02 (.08)       |
| Credential (mathematics)                   | .98 (.26)        |
| Credential (others)                        | .53** (.12)      |
| Mathematics teaching experience (standardized) | 1.21* (.11)     |

The operation is included as a fixed effect in the model. Numbers in parentheses are robust standards errors

\*\( p < 0.05 \); **\( p < 0.01 \)

*One of the assumptions underlying ordered logistic regression is that the relationship between each pair of outcome groups is the same. Thus, the coefficients that describe the relationship between incorrect explanations versus all higher categories of the response variable (i.e., partially correct or correct explanations) are the same as those that describe the relationship between partially correct and correct explanations combined versus incorrect explanations. Because the relationship between all pairs of groups is the same, each predictor in the model has only one set of coefficients. Thus, given that all the other predictors are held constant in the model, the odds ratios could be interpreted as either that (1) for a 1 \( SD \) increase in teachers’ mathematics teaching experience, the odds of providing a correct explanation versus the combined categories of partially correct and incorrect explanations are 1.21 times or (2) for a 1 \( SD \) increase in teachers’ mathematics teaching experience, the odds of providing a correct explanation or a partially correct explanation are 1.21 times those of providing an incorrect explanation.*
other variables were held constant, the odds of explaining the procedure fully correctly were 1.21 times those of explaining it partially correctly or incorrectly. The highest grade-level at which teachers taught mathematics or whether they held a regular teaching certificate was not linked to the correctness level of their explanations.

5 Discussion

Teachers’ in-depth understanding of the concepts they are expected to teach has important consequences for the quality of instruction they can provide as well as the kinds of understandings students can develop. Thus, examining teachers’ conceptual underpinning of fraction arithmetic has important implications for student learning, research, and teacher education. The present study contributes to the existing literature by investigating trends in what a national sample of fourth- and fifth-grade mathematics teachers knows about mathematical underpinnings of fraction operations.

The results of this study are consistent with earlier work showing that teachers have limited understanding of fraction operations (e.g., Ma, 2010; Son & Crespo, 2009), but extend prior work by providing insights into why teachers struggle with these concepts. An important result from this study is that, even for the relatively less complex algorithm (i.e., the fraction addition procedure), only half of the teachers provided explanations targeting the mathematical underpinnings of the operation. Further examination of these results illustrates that teachers do not seem to thoroughly understand the role the denominator plays in fraction representations (i.e., the number of equal-sized pieces that make up a whole). This result deserves attention, given that one of the most common problems students have when carrying out the addition of two fractions is adding across the numerator and denominator (e.g., NAEP, 2013; Schumacher & Malone, 2017). Given the teacher’s role in student learning, it is likely that a teachers’ lack of understanding regarding what a denominator represents and what creating a common denominator means when adding fractions contributes to students’ struggles with fraction addition. Teachers seem to need more targeted opportunities to learn the conceptual underpinnings of fraction algorithms in teacher education and profession development programs.

In line with the results of prior studies, teachers in this study seemed to struggle with fraction division (e.g., Siegler & Lortie-Forgues, 2015). Yet this study suggests that teachers’ understanding of the measurement meaning of division (i.e., how many groups of a divisor can be made by the divided) seemed to help them make sense of fraction division to some extent. Such an interpretation of division has been part of the methods courses in regular teacher education programs (Borko et al., 1992; Newton, 2008; Son & Crespo, 2009); thus, it is important that teachers in different preparation routes are given opportunities to learn this interpretation of fraction division.

It is also important to call attention to the fact that teachers provided more conceptually meaningful explanations when they were modeling division by creating common denominators (i.e., $2/3 \div 1/2$ could be solved by creating fractions with common denominators, $4/6 \div 3/6$). I argue that by doing so, the invert-and-multiply algorithm could be seen visually as well as how the referent unit was changing during the process of dividing fractions (i.e., you can make $4/3$ groups of $3/6$ with $4/6$). The teachers who provided partially correct explanations were also using visuals and the measurement meaning of division, yet they were using easier fractions to avoid explicitly dealing with the referent units. Thus, accompanying the measurement meaning of division with a common denominator strategy could be a promising approach to help teachers make sense of the fraction division algorithm.
In relation to the prior point, it is important to highlight that a deep understanding of the referent unit is an important concept required for fraction operations (Izsák et al., 2019), and that teachers struggle with attending to the referent unit when working with fraction operations (Copur-Gencturk & Olmez, 2020). This study provided further evidence that teachers struggle with referent units in fraction division. Only 14% of the teachers correctly commented on the referent units in their explanations of fraction division. Therefore, teachers need learning opportunities, explicitly targeting the concept of referent units, which could create the conditions needed to overcome the limitations in teachers’ knowledge of the division of fractions and could help them develop the conceptual information they need to use it in teaching.

Although teachers’ knowledge of fraction division appears more concerning than their knowledge of fraction addition, the results suggest a moderate relationship between the two. Teachers who knew the conceptual underpinnings of one of the operations seemed to understand the other as well and vice-versa. This could partly be because one of the underlying concepts for both operations is the referent unit; thus, teachers who have a developed understanding of the referent unit seemed to successfully apply this knowledge to both operations. By presenting the two fraction procedures together, this study provided evidence for how an understanding of the one concept was associated with the other.

It is also important to point out that many prior studies with preservice teachers (e.g., Ball, 1990; Simon, 1993) and in-service teachers (e.g., Ma, 2010) in the last three decades have reported that very few or almost no teachers at the elementary school level were able to create a situation illustrating a division problem. This study provides a relatively more positive picture. This could also be because the data for this study were collected from teachers across the USA, which may have captured the variation in teachers’ understanding compared with those studies that focused on teachers from the same institution or location. It could also be that in-service teachers are different from prospective teachers in terms of their knowledge, given that they may be learning from their teaching, the curriculum materials, and their professional development. Alternatively, these key concepts may have begun to be covered in teacher education and professional development programs in recent years. Unfortunately, this study design does not allow us to conclude why the results are different. Therefore, further research is needed to understand potential differences in the knowledge of preservice and in-service teachers by collecting the same types of data from representative samples of both in-service and preservice teachers.

Finally, the findings suggest that teachers who had been teaching mathematics longer seemed to have a deeper knowledge of fraction arithmetic. This finding is in line with the findings of prior studies showing that mathematical knowledge for teaching is associated with years of teaching experience (Hill, 2010). Although scholars have shown that teachers in secondary teacher education programs were able to construct a story problem to illustrate fraction arithmetic or provide an appropriate representation (Ball, 1990; Son & Crespo, 2009), in this study, teachers holding a credential in teaching mathematics did not seem to provide mathematically more sound explanations than those holding multiple-subject credentials. Given that teachers holding mathematics teaching credentials and generalists usually differ in the number of mathematics content courses they have completed, this result is in line with arguments made by some mathematics education scholars that teachers’ completing courses on content they are not expected to teach may not be the most efficient use of their time in teacher education programs (e.g., Ball et al., 2008). Again, more research is needed to better understand how teachers’ mathematics background is related to their understanding of the content they are expected to teach in school. Note that teachers who held a credential in other areas, such as special education, seemed to not to have robust knowledge of fraction arithmetic.
arithmetic. This could again be related to the learning opportunities provided to teachers in their teacher preparation programs. I argue that a thorough analysis of what mathematics content is covered in these programs is vital to understanding why special education teachers appeared to struggle with fractions. Regardless, the findings underscore the importance of professional learning opportunities for teachers, especially those holding special education credentials, to help them develop a conceptual understanding of fraction arithmetic.

6 Conclusion

In conclusion, the study findings indicate that although in-service teachers seemed to have relatively more knowledge of the conceptual underpinnings of fraction arithmetic than did the preservice teachers depicted in prior studies, a significant portion of the teachers still did not seem to be able to showcase a deep understanding of fraction procedures. Thus, teachers need more learning opportunities to develop a robust understanding of why the rules and procedures make sense. This study also calls attention to the need to more precisely capture teachers’ robust understanding of the facts and procedures of the subject matter they are expected to teach so that we can better explore the role that teachers’ conceptual understanding plays in the development of students’ knowledge and their misconceptions.

Appendix. Tasks used in the study

Mrs. Johnson is planning a lesson on fraction addition. She understands that she needs to find a common denominator when adding two fractions. For example, when she adds \( \frac{1}{3} + \frac{3}{4} \), she finds a common denominator of 12: \( \frac{4}{12} + \frac{9}{12} = \frac{13}{12} \). However, she does not understand why she needs a common denominator or how to interpret the need for a common denominator when adding fractions. Can you explain to Mrs. Johnson why we need to use a common denominator to add fractions? Please feel free to use any method to explain the key concepts (e.g., visual representations, real-world examples, etc.).

Ms. Bryson is planning a lesson on fraction division. She understands that sometimes when we divide fractions, the answer includes a fraction that has a different denominator than either the divisor or the dividend. For example, \( \frac{5}{4} \div \frac{2}{3} = \frac{15}{8} \). She knows the denominator of the answer is different because of the invert-and-multiply algorithm \( \left( \frac{5}{4} \times \frac{3}{2} = \frac{15}{8} \right) \), but she does not understand why it happens (e.g., where does \( \frac{1}{8} \) come from?) or how to interpret the denominator of the answer (what does \( \frac{15}{8} \) mean?). Can you explain to Ms. Bryson why the denominator of the answer is different? Please feel free to use any method to explain the key concepts (e.g., visual representations, real-world examples, etc.).

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