Noncommutative Extension of AdS/CFT and Holographic Superconductors

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Abstract: In this work, we consider a Non-Commutative (NC) extension of AdS-CFT correspondence and its effects on a Holographic Superconductor. NC corrections enter the model from two very different sectors: the NC generalization of Schwarzschild-AdS Black Hole metric and from NC extension of the abelian gauge theory. We study the NC effects on the relations connecting the charge density, critical temperature and condensation operator of the Holographic Superconductor. Our results suggest that generically, NC effects lower the critical temperature of the holographic superconductor.

1 Introduction

In recent years AdS/CFT correspondence, proposed by Maldacena [1], has captured the attention of both High Energy and Condensed Matter theorists since it can address issues in strongly interacting systems in the latter, (that are otherwise intractable in conventional Condensed Matter framework), by exploiting results obtained in weakly coupled systems in the former. In particular, there exists explicit mapping between relevant operators and parameters of a field theory in the bulk AdS space-time to those of a Conformal Field Theory living in the (one dimension lowered) boundary. Armed with the AdS/CFT dictionary, in its simplest form, it was possible to have glimpses of the unique features of high \(T_c\) superconductors (referred to as Holographic Superconductors) from the study of scalar electrodynamics in a Schwarzschild-AdS background, as shown by Gubser [2].

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More specifically, the model describing phase transitions in high $T_c$ superconductors consists of a charged scalar field minimally coupled to an Abelian gauge field in an AdS Black Hole background. The Black Hole admits scalar hair at a temperature $T$ below a certain critical temperature $T_c$ by the mechanism of breaking of a local $U(1)$ symmetry near the event horizon of the Black Hole. The emergence of a hairy AdS Black Hole implies the formation of a charged scalar condensate in the dual CFTs by the AdS/CFT correspondence. The minimal model has been generalized in different directions in the AdS sector to study its effects on the CFT. An interesting extension was discussed in [4] that considered non-linear Born-Infeld model instead of the usual Maxwell lagrangian. The analysis showed that the critical temperature of the Holographic Superconductor decreased with increase of the non-linear coupling. In the present article we study a completely different type of generalization of the gravity-field theory sector - Non-Commutative (NC) extension of the space-time. Our results indicate that NC effects are small but can lower the critical temperature of the Holographic Superconductor. ratio of NC parameter $\theta$ to Black Hole mass is within a certain ratio. Our results smoothly reduce to the conventional results for $\theta = 0$.

Noncommutativity in spacetime was introduced long ago by Snyder [5] in the hope of removing short distance singularities in quantum field theory but it was not successful. Later NC field theory was resurrected by Seiberg and Witten [6] who demonstrated that in the weak energy limit open strings attached to $D$-branes induced noncommutativity in the $D$-branes. In [6] rules were provided for extending QFTs to NC QFTs where normal products between local fields were replaced by $*$-products so that NC QFTs can be studied perturbatively for small NC parameter $\theta$. Furthermore, NC gauge theories had to be treated in a special way by incorporating the Seiberg-Witten map [6]. (For a review see [7].) Later Nicolini, Smailagic and Spalucci [8] were able to construct an NC extension of Schwarzschild-AdS metric by directly solving the Einstein equation with a smeared matter source that takes into account the NC effect. The Black Hole singularity was successfully removed in this scenario. NC effects on salient Black Hole properties, such as Hawking radiation, have been studied using this $\theta$ -corrected metric [9].

In the present paper we aim to study the bulk NC effect on Holographic Superconductors. Interestingly, the NC generalization in the bulk requires both Seiberg-Witten mapped extension [6, 7] for the gauge sector and Nicolini-Spalucci extension [8] for the Black Hole sector. As far as we are aware of, this form of generalized NC extension has not been studid before. We will
show there are some subtleties involved in the lowest order approximation in the NC parameter $\theta$. It is well known that asymptotic behavior of the scalar and gauge fields in the bulk dictate properties of the boundary CFT. It is perhaps significant that, (at least to the lowest non-trivial order of $\theta$), NC effect does not change the asymptotic behavior of bulk fields qualitatively, that is, the functional forms remain unchanged but the numerical parameters undergo NC corrections. This allows us to use the same AdS/CFT dictionary in order to compute the $\theta$-corrected relation between the critical temperature and charge density of the Holographic Superconductor and the condensate-temperature relation.

The paper is organized as follows: In Section 2, we introduce the AdS Black Hole metric and define the action for an Abelian gauge field (coupled with a scalar) in this noncommutative background spacetime. In Section 3, we study the asymptotic behavior of the gauge and scalar field. Then we proceed to study the relation between critical temperature and charge density in Section 4. In following Section 5, we calculate the critical exponents and condensation values. Finally we discuss and summarize our findings and conclude in Section 6.

2 Noncommutative AdS Black Hole in charged background

We construct the action of NC charged scalar electrodynamics [6] in NC AdS Black Hole background [8]. We consider the Lorentz covariant form

$$[x^\mu, x^\nu] = i \theta^{\mu\nu} \quad \mu, \nu = 0, ..., n.$$  \hspace{1cm} (1)

Here $\theta^{\mu\nu}$, an antisymmetric Lorentz tensor, is represented in terms of a block diagonal form

$$\theta^{\mu\nu} = diag \left( \hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_{n/2} \right) \quad \mu, \nu = 0, ..., n$$ \hspace{1cm} (2)

where $\hat{\theta}_i = \theta_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus to maintain the covariance of (1), a foliation of spacetime into noncommutative planes has been done, as defined by (2). But Lorentz invariance and unitarity implies that noncommutativity does not privilege any of such planes, which enforces a unique noncommutative parameter $\theta_1 = \theta_2 = ... = \theta_{n/2} = \theta$.

In this approach [8] one can interpret the conventional coordinates as mean values of coordinate operators subject to (1), to take into account the quantum geometrical fluctuations of the spacetime manifold. Mean values are calculated over coherent states, which results eigenstates of ladder
operator, built with noncommutative coordinates only. In the absence of common eigenstates, coherent states induce the states of minimal uncertainty and provide the best resolution of the position over a noncommutative manifold. In other words, some smeared position on the manifold is introduced instead of the concept of a point. At working level, the delocalization of fields is realized by deforming the source term of their equations of motion, namely substituting Dirac delta distribution (local source) with Gaussian distribution (nonlocal source) of width $\sqrt{\theta}$. As a result, the ultraviolet behavior of classical and quantum fields is cured. In physical terms, if we define $\sqrt{\theta}$ to be the average magnitude of an element of $\theta^{ij}$, then $1/\sqrt{\theta}$ corresponds to the energy scale beyond which the conventional differential spacetime manifold turns out to be a noncommutative one. For this reason, there is a general consensus about the appearance of noncommutative phenomenology at intermediate energies between the scale of the Standard Model of particle physics and the Planck scale, which at least we can hope to be experimentally verified in near future.

Thus the noncommutative Black Hole metric is considered in the form \[8\]
\[
  ds^2 = -f_1(r)dt^2 + \frac{dr^2}{f_1(r)} + r^2(dx^2 + dy^2) \tag{3}
\]
where
\[
  f_1(r) = K - \frac{2M}{r} + \frac{r^2}{L^2} + \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{r^2}{4\theta}}. \tag{4}
\]
For small $\theta$ the exponential damping reduces the metric to its original form. With this metric \[3\], the gauge field action is constructed as
\[
  S = \int d^4x \sqrt{-g} \left( R + \frac{6}{L^2} - \frac{1}{4} \dot{F}_{\mu\nu} \dot{F}^{\mu\nu} - |\partial_{\mu} \ast \psi - iq\dot{A}_\mu \ast \psi|^2 - m^2|\psi|^2 \right) \tag{5}
\]
where, to $O(\theta)$, star product is defined as
\[
  f \ast g = fg + \frac{i\theta^{\mu\nu}}{2}(\partial_\mu f)(\partial_\nu g). \tag{6}
\]
The NC extension of gauge fields needs extra care and Seiberg-Witten map \[6\] has to be employed. The deformed gauge potential to $O(\theta)$ is
\[
  \dot{A}_\mu = A_\mu + \frac{1}{2} \theta^{\alpha\beta} A_\alpha (2\partial_\beta A_\mu - \partial_\mu A_\beta), \tag{7}
\]
by virtue of which the electromagnetic tensor becomes
\[
  \dot{F}_{\mu\nu} = (\partial_\mu + i\dot{A}_\mu) \ast \dot{A}_\nu - (\partial_\nu + i\dot{A}_\nu) \ast \dot{A}_\mu = F_{\mu\nu} + \theta^{\alpha\beta} (A_\alpha \partial_\beta F_{\mu\nu} - F_{\mu\alpha} F_{\nu\beta}). \tag{8}
\]
Notice that the NC Abelian electromagnetic tensor has a non-Abelian flavor. Using (6) and (7), the fourth term in the action (5) becomes

\[ |\partial_\mu \psi - iqA_\mu \psi|^2 = g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi^* + ig g^{\mu\nu} A_\mu J_\nu + \frac{iq}{2} g^{\mu\nu} \theta^{\alpha\beta} A_\alpha (2 \partial_\beta A_\nu - \partial_\nu A_\beta) J_\mu \]

\[ + \frac{iq^2}{2} g^{\mu\nu} \theta^{\alpha\beta} (\partial_\alpha A_\nu) J_\beta A_\mu + q^2 g^{\mu\nu} \theta^{\alpha\beta} \partial_\alpha A_\nu (\partial_\beta \psi \partial_\mu \psi^* + \partial_\beta \psi^* \partial_\mu \psi) \]

\[ + q^2 g^{\mu\nu} \theta^{\alpha\beta} A_\alpha (2 \partial_\beta A_\nu - \partial_\nu A_\beta) A_\mu |\psi|^2 + q^2 g^{\mu\nu} A_\mu A_\nu |\psi|^2 \] (9)

As is customary in this context, we consider gauge field \( A_\mu \) to have only temporal component \([2,3]\), i.e. \( A_\mu = (\phi(r), 0, 0, 0) \) and \( \psi = \psi(r) \). Our action (5) becomes

\[ S = \int d^4x \left( R + \frac{6}{L^2} - \frac{1}{2} g^{rr} g^{tt} \partial_r \phi \partial_t \phi - g^{rr} g^{tt} (\partial_r \phi)^2 \theta - g^{rr} g^{tt} \phi (\partial_r \phi) (\partial_r \phi) \theta \right. \]

\[ - g^{rr} (\partial_r \psi) (\partial_r \psi^*) - g^{tt} q^2 \phi^2 |\psi|^2 - 2 g^{tt} q^2 \phi^2 (\partial_r \phi) |\psi|^2 \theta - m^2 |\psi|^2 \] (10)

Lagrange’s equation of motion for scalar field \( \psi \),

\[ \frac{1}{\sqrt{-g}} \left( \partial_\tau \left( \frac{\partial L}{\partial (\partial_\tau \psi)} \right) \right) - \frac{\partial L}{\partial \psi} = 0, \] (11)

yields

\[ \psi'' + \left( \frac{f'(r)}{f_1(r)} + \frac{2}{r} \right) \psi' - \frac{m^2}{f_1(r)} \psi + \frac{q^2 \phi^2}{f_1^2(r)} \psi + \frac{\partial q^2 \phi^2 (\partial_r \phi) f_1}{f_1^2(r)} \psi = 0 \] (12)

The Lagrange’s equation for the gauge potential \( \phi \) is given by

\[ \frac{1}{\sqrt{-g}} \left( \partial_\tau^2 \left( \frac{\partial L}{\partial (\partial_\tau^2 \phi)} \right) \right) - \frac{1}{\sqrt{-g}} \left( \partial_\tau \left( \frac{\partial L}{\partial (\partial_\tau \phi)} \right) \right) + \frac{\partial L}{\partial \phi} = 0 \] (13)

from which we get

\[ \phi'' + \frac{2}{r} \phi' - 2 q^2 |\psi|^2 \frac{f_1}{f_1(r)} \phi \]

\[ + \theta \left( 3 \phi' \phi'' + \frac{2}{r} \phi'^2 - \frac{2}{r^2} \phi \phi' - \frac{4q^2 |\psi|^2}{f_1(r)} \phi \phi' + \frac{4q^2 |\psi|^2}{rf_1(r)} \phi^2 \right) + 2q^2 \theta \partial_r \left( \frac{|\psi|^2 \phi^2}{f_1(r)} \right) = 0. \] (14)

Equations (12) and (2) are the governing equations of our model.

### 3 Asymptotic behavior of \( \psi \) and \( \phi \)

Let us find the asymptotic behavior of (12) and (2). We restrict ourselves to the lowest non-trivial effects of \( \theta \). Note that the NC extension in Black Hole metric does not allow a naive \( \theta \)-expansion and whereas for the NC-gauge sector it is permissible. Hence we expand the differential equation
for electric field $\phi$ i.e. (2) to the order $O(\theta)$ and $O\left(\frac{2M}{\sqrt{\pi\theta}}e^{-\frac{r^2}{4\theta}}\right)$ [9]. The second, third, fourth and fifth term of [12] can be written to the first order of $O\left(\frac{2M}{\sqrt{\pi\theta}}e^{-\frac{r^2}{4\theta}}\right)$ as

$$\frac{f'(r)}{f(r)} + \frac{2}{r} \psi = \left(\frac{f'(r)}{f(r)} + \frac{2}{r}\right) \psi - \left(\frac{f'(r)}{f^2(r)} + \frac{r}{2\theta f(r)}\right) \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \psi \quad (15)$$

$$\frac{m^2}{f_1(r)} \psi = m^2 \frac{2M}{f^2(r)} \sqrt{\pi\theta} e^{-\frac{r^2}{4\theta}} \psi \quad (16)$$

$$\frac{q^2 \phi^2}{f^2(r)} \psi = \frac{q^2 \phi^2}{f^2(r)} \psi - \frac{2q^2 \phi^2}{2M} \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \psi \quad (17)$$

$$\frac{\theta q^2 \phi^2 \partial_r \phi}{f_1(r)^2} \psi = \frac{\theta q^2 \phi^2 \partial_r \phi}{f^2(r)} \psi - \frac{2q^2 \phi^2}{2M} \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \psi, \quad (18)$$

where $f(r) = K - \frac{2M}{r} + \frac{r^2}{r^2}$. This approximation is justified as we discussed in the previous section about how the noncommutative parameter $\theta$ is related to the physical energy scale beyond which one cannot probe.

Similarly the third, eighth, ninth and tenth term of [2] can be written as

$$\frac{2q^2|\psi|^2}{f_1(r)} \phi = \frac{2q^2|\psi|^2}{f(r)} \phi - \frac{2q^2|\psi|^2}{f^2(r)} \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \phi \quad (19)$$

$$\theta \frac{4q^2|\psi|^2}{f_1(r)} \phi = \theta \frac{4q^2|\psi|^2}{f(r)} \phi - \theta \frac{4q^2|\psi|^2}{f^2(r)} \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \phi \quad (20)$$

$$\theta \frac{4q^2|\psi|^2}{f(r)} \phi^2 = \theta \frac{4q^2|\psi|^2}{f(r)} \phi^2 - \theta \frac{4q^2|\psi|^2}{f^2(r)} \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \phi^2 \quad (21)$$

$$2\theta q^2 \partial_r \left(\frac{|\psi|^2 \phi^2}{f_1(r)}\right) = 2\theta q^2 \partial_r \left(\frac{|\psi|^2 \phi^2}{f(r)} - \frac{|\psi|^2 \phi^2}{f^2(r)} \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \right). \quad (22)$$

With the above approximation, the equation for $\psi$ and $\phi$ can be written as

$$\psi'' + \left(\frac{f'(r)}{f(r)} + \frac{2}{r}\right) \psi - \frac{m^2}{f^2(r)} \psi + \frac{q^2 \phi^2}{f^2(r)} \psi + \theta \frac{q^2 \phi^2 \partial_r \phi}{f^2(r)} \psi$$

$$- \left(\frac{f'(r)}{f^2(r)} + \frac{r}{2\theta f(r)}\right) \psi - \frac{m^2}{f^2(r)} \psi + \frac{2q^2 \phi^2}{f^2(r)} \psi + \theta \frac{q^2 \phi^2 \partial_r \phi}{f^2(r)} \psi$$

$$\frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} = 0 \quad (23)$$

and

$$\phi'' + \frac{2}{r} \phi' - \frac{2q^2|\psi|^2}{f(r)} \phi + \theta \left(3\phi'' + \frac{2}{r} (\phi')^2 - \frac{2}{r} \phi \phi'' - \frac{2}{r} \phi \phi' - \frac{4q^2|\psi|^2}{f(r)} \phi \phi' + \frac{4q^2|\psi|^2}{f(r)} \phi^2 \right)$$

$$+ \left(\frac{2q^2|\psi|^2}{f^2(r)} \phi + \theta \frac{4q^2|\psi|^2}{f^2(r)} \phi \phi' - \theta \frac{4q^2|\psi|^2}{f^2(r)} \phi^2 \right) \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} + 2\theta q^2 \partial_r \left(\frac{|\psi|^2 \phi^2}{f(r)} - 2\theta q^2 \partial_r \left(\frac{|\psi|^2 \phi^2}{f(r)} \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \right) = 0 \quad (24)$$
Since \( f(r) = K - \frac{2M}{r} + r^2 \) we have \( \frac{1}{f(r)} \approx \frac{L^2}{2r^2} \) (considering terms up to \( \frac{1}{r^2} \) only). Replacing all \( f(r) \) by \( \frac{1}{r^2} \), the above equation for \( \psi \) can be written (keeping terms only up to \( \frac{1}{r^2} \)) as

\[
\psi'' + \frac{4}{r} \psi' + \frac{2}{r^2} \psi - \left( \frac{L^2}{2\theta r} \psi' \right) \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} = 0
\]  

(25)

It is known that \( \psi = \frac{C}{r} + \frac{D}{r^2} \) is a solution of

\[
\psi'' + \frac{4}{r} \psi' + \frac{2}{r^2} \psi = 0.
\]  

(26)

Therefore we assume that the solution of (25) to be of the form

\[
\psi = \frac{C}{r} + \frac{D}{r^2} + \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \psi_1
\]  

(27)

Substituting (27) into (25) we see that the differential equation for \( \psi_1 \) becomes

\[
\psi_1'' + \left( \frac{4}{r} - \frac{r}{\theta} \right) \psi_1' + \left( \frac{r^2}{4\theta^2} - \frac{5}{2\theta} + \frac{2}{r^2} \right) \psi_1 = \frac{L^2}{2\theta r} \psi_1' = -\frac{L^2}{2\theta r} \left( \frac{C}{r^2} + \frac{D}{r^3} \right) \approx 0,
\]  

(28)

where we considered terms up to \( O(\frac{1}{r^2}) \).

The solution of (28) is given by

\[
\psi_1 = \left( \frac{E}{r} + \frac{F}{r^2} \right) e^{\frac{r^2}{4\theta}}
\]  

(29)

Therefore the asymptotic behavior of \( \psi \) can be expressed as

\[
\psi = \frac{C}{r} + \frac{D}{r^2} + \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}} \psi_1 = \frac{C}{r} + \frac{D}{r^2} + \frac{2M}{\sqrt{\pi\theta}} \left( \frac{E}{r} + \frac{F}{r^2} \right)
\]  

(30)

As mentioned earlier, the \( \theta \)-contribution has the same structure as the original one. Substituting \( f(r) = \frac{r^2}{2} \) and (30) in (3) and considering terms up to \( \frac{1}{r^2} \) we have

\[
\phi'' + \frac{2}{r} \phi' + \theta \left( 3\phi' \phi'' + \frac{2}{r} (\phi')^2 - \frac{2}{r} \phi \phi'' - \frac{2}{r^2} \phi \phi' \right) = 0
\]  

(31)

The solution of

\[
\phi'' + \frac{2}{r} \phi' = 0
\]  

(32)

is known to be

\[
\phi = a - \frac{b}{r}
\]  

(33)

So, let us consider the solution of (31) in the form

\[
\phi = a - \frac{b}{r} + \theta \phi_1.
\]  

(34)
Substituting this form in (31) we have the differential equation for $\phi_1$ as

$$\phi_1'' + \frac{2}{r}\phi_1' + \text{Terms containing either } O(\theta) \text{ or } O\left(\frac{1}{r^3}\right) \text{ or both} = 0.$$  

(35)

Neglecting the last terms, as we have considered only first order term of $\theta$, we arrive at

$$\phi_1'' + \frac{2}{r}\phi_1' = 0 \quad \Rightarrow \quad \phi_1 = a - \frac{b}{r}.$$  

(36)

Therefore we finally have the solution

$$\phi = a - \frac{b}{r} + \theta \left(a - \frac{b}{r}\right).$$  

(37)

Once again the $\theta$-contribution has the same structure as the original one. This asymptotic behavior can be written in the form

$$\phi = \mu - \frac{\rho}{r} \quad \text{where} \quad \mu = a(1 + \theta) \text{ and } \rho = b(1 + \theta).$$  

(38)

From (38), it can be clearly observed the the effect of noncommutativity enters through the very definition of the charge density $\rho$ and the chemical potential $\mu$.

### 4 Relation between critical temperature and charge density

In order to obtain the relation between the critical temperature $T_c$ and charge density ($\rho$), we follow the technique used in [4]. As explained earlier, we expand the differential equation for electric field $\phi$ i.e. (2) to the order $O(\theta)$ and $O\left(\frac{2M}{\sqrt{\pi \theta}} e^{-\frac{2}{\sqrt{\pi \theta}} r^2}\right)$

as

$$\phi'' + \frac{2}{r}\phi' - \frac{2q^2}{f(r)}|\psi|^2\phi + \theta \left(3\phi'\phi'' + \frac{2}{r}(\phi')^2 - \frac{2}{r^2}\phi\phi' - \frac{4q^2|\psi|^2}{rf(r)}\phi\phi' + \frac{4q^2|\psi|^2}{rf(r)}\phi^2\right) + \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{2}{\sqrt{\pi \theta}} r^2} \left(\frac{2q^2|\psi|^2}{f(r)^2}\phi + \theta \frac{4q^2|\psi|^2}{rf(r)^2}\phi' - \frac{4q^2|\psi|^2}{rf(r)^2}\phi^2\right) + 2\theta q^2 \partial_r \left(\frac{|\psi|^2\phi^2}{f(r)}\right) - 2\theta q^2 \partial_r \left(\frac{|\psi|^2\phi^2}{f(r)^2} \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{2}{\sqrt{\pi \theta}} r^2}\right) = 0.$$  

(39)

At the critical temperature $T = T_c$ the scalar field $\psi$ vanishes, $\psi = 0$. By virtue of this the above equation (40) becomes

$$\phi'' + \frac{2}{r}\phi' + \theta \left(3\phi'\phi'' + \frac{2}{r}(\phi')^2 - \frac{2}{r^2}\phi\phi' - \frac{2}{r^2}\phi\phi'\right) = 0.$$  

(40)

The solution of (40) is

$$\phi = a_1(1 + \theta) - \frac{b_1(1 + \theta)}{r^2} - \theta \frac{a_1 b_1}{r^2} + \theta \frac{b_1^2}{r^3}.$$  

(41)
The Horizon $r = r_+$ is the solution of $f_1(r) = 0$. We are now going to change our solution region from $r_+ \leq r < \infty$ to $1 \geq z > 0$ by the transformation $z = \frac{r_+}{r}$. We rewrite the asymptotic behavior of $\phi$ i.e. (37) and to first order $O(\theta)$ and $O\left(\frac{2M}{\sqrt{\pi\theta}}e^{-\frac{z^2}{4\theta}}\right)$ the solution of $\phi$ i.e. (31) respectively, at the critical temperature $T = T_c$ as

$$\phi = a(1 + \theta) - \frac{b(1 + \theta)}{r_{+c}}z$$

and

$$\phi = a_1(1 + \theta) - \frac{b_1(1 + \theta)}{r_{+c}}z - \theta \frac{a_1b_1}{r_{+c}^2}z^2 + \theta \frac{b_1^2}{r_{+c}^3}z^3.$$  

Using the transformation $z = \frac{r_+}{r}$ the horizon comes to at $z = 1$ and asymptotic boundary behavior becomes at $z = 0$. Since (42) represents asymptotic behavior of $\phi$, then at $z = 0$ equation (42) and (43) must be same. Substituting $z = 0$ in the above two equation we have $a_1 = a$. Again differentiating (43) and (42) w.r.to “$z$” and then equating them at the asymptotic boundary $z = 0$, we have $b_1 = b$. Another boundary condition at the horizon $z = 1$ i.e. $\phi(z = 1) = 0$ gives from (43) to the first order of $O(\theta)$

$$a_1 = \frac{b_1}{r_{+c}}.$$  

Therefore the solution (43) of scalar potential $\phi$ at the critical temperature $T = T_c$ to the lowest non-trivial order of $\theta$ can be written as

$$\phi = \frac{b(1 + \theta)}{r_{+c}}(1 - z)\left(1 - \frac{\theta}{(1 + \theta)^2}\frac{b(1 + \theta)}{r_{+c}^2}z^2\right) \approx \lambda r_{+c}(1 - z)(1 - \theta \lambda z^2)$$

where $\lambda = \frac{b(1 + \theta)}{r_{+c}^2} = \frac{\theta}{r_{+c}^2}$.

We are now going to investigate the boundary behavior of the scalar field $\psi$ as $T \to T_c$. To the first order of $O(\theta)$ and $O\left(\frac{2M}{\sqrt{\pi\theta}}e^{-\frac{z^2}{4\theta}}\right)$ the differential equation for scalar field $\psi$ (12), after replacing $z = \frac{r_+}{r}$ and using the solution (45) (as $T \to T_c$) becomes

$$\psi'' + \left(\frac{f''(z)}{f(z)} + \left(-\frac{f'(z)}{f(z)^2} + \frac{r_+^2}{2z^3f(z)}\right)\frac{2M}{\sqrt{\pi\theta}}e^{-\frac{z^2}{4\theta}}\right)\psi' - \frac{r_+^2m^2}{z^4f(z)}\left(1 - \frac{2M}{\sqrt{\pi\theta}}\frac{2}{f(z)}e^{-\frac{r_+^2}{4\theta}}\right)\psi$$

$$= -\lambda^2 \frac{g^2r_+^4}{z^4f(z)^2}(1 - z)^2(1 - \theta \lambda z^2)\left(1 - \frac{2M}{\sqrt{\pi\theta}}\frac{2}{f(z)}e^{-\frac{r_+^2}{4\theta}}\right)\psi.$$  

where $f(z) = K - \frac{2M}{r_+}z + \frac{r_+^2}{L^2}$. Near the asymptotic boundary (as $z \to 0$) we define

$$\psi(z) = \frac{\langle J \rangle}{\sqrt{2r_+}}zF(z).$$
where $F(z)$ satisfies $F(0) = 1$ and $F'(0) = 0$. Using (47) the above equation (41) becomes

$$F''(z) + \left( \frac{2}{z} + \frac{f'(z)}{f(z)} + \left( -\frac{f'(z)}{f(z)^2} + \frac{r_+^2}{2\theta z^2 f(z)} \right) \frac{2M}{\sqrt{\pi \theta}} e^{-r_+^2/4r_+^2} \right) F'(z)$$

$$+ \left( \frac{f'(z)}{zf(z)} - \frac{r_+^2 m^2}{z^4 f(z)} + \left( -\frac{f'(z)}{zf(z)^2} + \frac{r_+^2}{2\theta z^4 f(z)} + \frac{r_+^2 m^2}{z^4 f(z)^2} \right) \frac{2M}{\sqrt{\pi \theta}} e^{-r_+^2/4r_+^2} \right) F(z)$$

$$+ \lambda^2 \frac{q^2 r_+^4}{z^2 f(z)^2} (1 - z)^2 (1 - \theta \lambda z^2) \left( 1 - \frac{2M}{\sqrt{\pi \theta}} \frac{2}{f(z)} e^{-r_+^2/4r_+^2} \right) F(z) = 0. \quad (48)$$

Now it is straightforward to cast (41) into a Sturm-Liouville (SL) eigenvalue problem of the generic form (4)

$$(T(z)F'(z)) - Q(z)F(z) + \lambda^2 P(z)F(z) = 0 \quad (49)$$

where

$$T(z) = r_+ L^2 z^2 f(z) e^{2M/r_+ z} e^{-r_+^2/4r_+^2}, \quad (50)$$

$$Q(z) = -r_+ L^2 e^{2M/r_+ z} e^{-r_+^2/4r_+^2} \left( -\frac{2M}{r_+} - \frac{2M}{r_+} \frac{z}{f(z)} + \frac{r_+^2}{2\theta z^2} \right) \frac{2M}{\sqrt{\pi \theta}} e^{-r_+^2/4r_+^2}, \quad (51)$$

$$P(z) = \frac{q^2 r_+^4 L^2}{z^2 f(z)} (1 - z)^2 (1 - \theta \lambda z^2) \left( 1 - \frac{2M}{\sqrt{\pi \theta}} \frac{2}{f(z)} e^{-r_+^2/4r_+^2} \right) \frac{2M}{\sqrt{\pi \theta}} e^{-r_+^2/4r_+^2}. \quad (52)$$

where $f(z) = K - \frac{2Mz}{r_+} + \frac{r_+^2}{2L^2 z^2}$. $F(z)$ is an arbitrary function that will be fixed later. The Hawking temperature (11) $T$ is related to $r_+$ by the relation $T = \frac{3}{4\pi} r_+$ and $\lambda$ is related to charge density $\rho$ by the relation $\lambda = \frac{\rho}{r_+^2}$, where $\rho = b(1 + \theta)$. Therefore the relation between critical temperature $T_c$ and charge density $\rho$ becomes

$$T_c = \frac{3}{4\pi \sqrt{\lambda_{\min}}} \sqrt{\rho} \equiv \zeta \sqrt{\rho}. \quad (53)$$

Note that both $\rho$ and $\lambda_{\min}$ contain $\theta$-corrections. Explicitly $\lambda_{\min}$ is found from minimizing the expression

$$\lambda^2 = \frac{\int_0^1 \left( T(z) \{F'(z)\}^2 + Q(z) \{F(z)\}^2 \right) dz}{\int_0^1 P(z) \{F(z)\}^2 dz}, \quad (54)$$

for a given choice of $F(z)$. We may take $F(z) = 1 - cz^2$ as a trial solution (3) (10) which satisfy $F(0) = 1$ and $F'(0) = 0$, where $c$ is the minimizing parameter, in order to find $\lambda$ from (54). To evaluate $\lambda$ for different values of $\theta$ we may simplify the integration in the denominator of (54) by considering $\lambda \theta = \theta(\lambda|_{\theta=0}) + O(\theta^2)$, where $\lambda|_{\theta=0}$ is the value of $\lambda$ for $\theta = 0$ (4). We would like to
mention a crucial point here: For the usual AdS scenario without noncommutativity, \( \theta = 0 \), we have

\[
f(r) = K - \frac{2M}{r} + \frac{r^2}{L^2}.
\]

From (55), horizon radius \( r_+ \) can be calculated as \( f(r = r_+) = 0 \) which gives

\[
r_+^3 = 2M,
\]

considering the usual convention \( L = 1 \), for the flat case \( K = 0 \). One can see here that using (55) and \( r_+ \) the expression for \( \lambda \) becomes

\[
\lambda^2 = \frac{\int_0^1 \left[ (1 - z^3) (-2cz)^2 + z (1 - c z^2)^2 \right] \, dz}{\int_0^1 z^2 \left( 1 - c z^2 \right)^2 \, dz}.
\]

From (57) one notes that \( M \) cancels out from the numerator and the denominator and \( r_+ \) does not appear explicitly. Thus the expression (57) is independent of the choice of \( M \) though \( M \) is related to \( r_+ \) through (56). However, for the noncommutative AdS model which we have considered in this work, we have

\[
f_1(r) = K - \frac{2M}{r} + \frac{r^2}{L^2} + \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{r^2}{4\theta}}
\]

from which the horizon radius can be determined by \( f_1(r = r_+) = 0 \) which gives (for \( K = 0, L = 1 \))

\[
r_+^3 = 2M \left( 1 - \frac{r_+}{\sqrt{\pi \theta}} e^{-\frac{r_+^2}{4\theta}} \right).
\]

Clearly the second term in r.h.s. of (59) destroys the simple explicit relation (56). Using the form of \( f(z) = -\frac{2Mz}{r_+} + \frac{r_+^2}{z^2} \) and the relation (59) the expression for eigenvalue \( \lambda \) stands out to be

\[
\lambda^2 = \frac{\int_0^1 \left[ \left( 1 - z^3 - \frac{r_+}{\sqrt{\pi \theta}} e^{-\frac{r_+^2}{4\theta}} + \frac{r_+}{\sqrt{\pi \theta}} e^{-\frac{r_+^2}{4\theta}} z \right) (-2cz)^2 + z \left( 1 - \frac{r_+^2}{2\theta z^2} + \frac{1}{\sqrt{\pi \theta}} e^{-\frac{r_+^2}{4\theta z^2}} \right) (1 - c z^2)^2 \right] \, dz}{\int_0^1 \frac{q^2}{1 + z^2} \left( 1 - 2\theta z^2 \right) \left[ 1 - z + \frac{r_+}{\sqrt{\pi \theta}} \frac{2z^3 - 1}{1 + z^2} e^{-\frac{r_+^2}{4\theta z^2}} - \frac{r_+}{\sqrt{\pi \theta}} \frac{z^2}{1 + z^2} e^{-\frac{r_+^2}{4\theta z^2}} \right] \left( 1 - c z^2 \right)^2 \, dz}.
\]

Thus the eigenvalue \( \lambda \) depends on both \( M \) and \( \theta \) through \( r_+ \). Subsequent results, such as \( \zeta \), (the coefficient of \( \sqrt{\rho} \), in the relation of critical temperature \( T_c \) and charge density \( \rho \) becomes \( \theta \) as well as \( M \) dependent. This is a highly non-trivial feature of the noncommutative AdS model considered here. We speculate that it hints at a generalized form of AdS/CFT duality where the Holographic Superconductor will have other parameters, besides the charge density and chemical potential one generally associates it with. From (60), one can now determine numerical value of \( \zeta \) in (53), as studied in [10, 4]. This is shown in Tables 1, 2, 3 and 4. We will discuss the implications in the last section.
5 Critical exponent and condensation value

In this section we are going to construct the condensation value of the condensation operator \( J \) near the critical temperature \( T = T_c \). In order to do that we substitute \( z = \frac{r}{r^+} \) in the differential equation for \( \phi \) i.e. \( \phi'' \), which after using the relation \( (\ref{eq:relation}) \) becomes

\[
\phi'' - \theta \left( \frac{3z^2}{r^+} \phi' \phi'' + \frac{4z}{r^+} \phi'^2 + \frac{2z}{r^+} \phi \phi'' + \frac{2}{r^+} \phi' \right) = \frac{\langle J \rangle^2}{r^+_2} B(z) \phi(z) \tag{61}
\]

where

\[
B(z) = \frac{q^2 r^+_2 F^2(z)}{z^2 f(z)} - \theta \frac{2q^2 r^+_2 F^2(z)}{zf(z)} \phi - \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{r^2}{4z^2}} \left( \frac{q^2 r^+_2 F^2(z)}{zf^2(z)} - \theta \frac{2q^2 r^+_2 F^2(z)}{zf^2(z)} \phi \right)
\]

\[
+ \theta \frac{q^2 r^+_2}{z^2} \partial_z \left( \frac{z^2 F^2(z)}{f(z)} \right) + \theta \frac{q^2 r^+_2}{z^2} \partial_z \left( \frac{z^2 F^2(z)}{f^2(z)} \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{r^2}{4z^2}} \right) \phi
\]

and \( f(z) = K - \frac{2M}{r^+} z + \frac{r^2}{r^+} \). At \( T = T_c \), \( \psi = 0 \) and \( \phi = \lambda r^+(1 - z)(1 - \lambda \theta z^2) \) is a solution of \( \phi'' - \theta \left( \frac{3z^2}{r^+} \phi' \phi'' + \frac{4z}{r^+} \phi'^2 + \frac{2z}{r^+} \phi \phi'' + \frac{2}{r^+} \phi' \right) = 0 \). Then for temperatures \( T \) close to \( T_c \) we can consider the solution of \( (\ref{eq:61}) \) to be of the form

\[
\frac{\phi}{r^+} = \lambda (1 - z)(1 - \lambda \theta z^2) + \frac{\langle J \rangle^2}{r^+_2} \chi(z). \tag{62}
\]

where the parameter \( \frac{\langle J \rangle^2}{r^+_2} \) is small and \( \chi(z) \) satisfies the boundary condition \( \chi'(1) = 0 \). Again since \( \( (\ref{eq:phi}) \) \) is the asymptotic behavior of \( \phi \) then

\[
\frac{\mu}{r^+} - \frac{\rho}{r^+_2} z = \lambda(1 - z)(1 - \lambda \theta z^2) + \frac{\langle J \rangle^2}{r^+_2} \chi(z) = \lambda(1 - z)(1 - \lambda \theta z^2) + \frac{\langle J \rangle^2}{r^+_2} (\chi(0) + z \chi'(0) + ...). \tag{63}
\]

where we have expanded \( \chi(z) \) about \( z = 0 \). Comparing the coefficient of \( z \) from both side we have

\[
- \frac{\rho}{r^+_2} = -\lambda + \frac{\langle J \rangle^2}{r^+_2} \chi'(0). \tag{64}
\]

Now substituting \( (\ref{eq:62}) \) into \( (\ref{eq:61}) \) and considering first order terms of \( O(\theta) \) and the small parameter \( O\left(\frac{\langle J \rangle^2}{r^+_2}\right) \), we get differential equation for \( \chi \) as

\[
\chi''(z)(1 - 2\theta \lambda z + 5\theta \lambda z^2) - 2\theta \lambda(1 - 5z) \chi'(z) + 2\theta \lambda \chi(z)
\]

\[
= \lambda q^2 r^+_2 (1 - z) \left( \frac{F^2(z)}{z^2 f(z)} (1 - 2\theta \lambda z + \theta \lambda z^2) \left(1 - \frac{2M}{\sqrt{\pi \theta}} \frac{1}{f(z)} e^{-\frac{r^2}{4z^2}} \right) \right)
\]

\[
+ \theta \lambda \frac{(1 - z)}{z^2} \partial_z \left( \frac{z^2 F^2(z)}{f(z)} \right) - \theta \lambda \frac{(1 - z)}{z^2} \partial_z \left( \frac{z^2 F^2(z)}{f^2(z)} \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{r^2}{4z^2}} \right)
\]. \tag{65}
Now (5) can be written as
\[
(\chi'(z)(1 - 2\theta \lambda z + 5\theta \lambda z^2))' + 2\theta \lambda \chi(z) = \lambda q^2 r_+^2 (1-z) \left( F^2(z) \left( 1 - 2\theta \lambda z + \theta \lambda z^2 \right) \left( 1 - \frac{2M}{\sqrt{\pi \theta}} \frac{1}{r_+^2} e^{-\frac{r_+^2}{4\theta z^2}} \right) \right.
\]
\[
+ \theta \lambda \frac{(1-z)}{z^2} \partial_z \left( \frac{z^2 F^2(z)}{f(z)} \right) - \theta \lambda \frac{(1-z)}{z^2} \partial_z \left( \frac{z^2 F^2(z) \frac{2M}{\sqrt{\pi \theta}} e^{-\frac{r_+^2}{4\theta z^2}}}{f(z)} \right) \right).
\]  
(66)

To find \( \chi'(0) \) we integrate (5) between 0 to 1 and using the boundary condition \( \chi'(1) = 0 \) we have
\[
\chi'(0) = -\lambda q^2 L^2 \int_0^1 \left[ \frac{(1-z)(1-cz^2)^2(1-2\theta \lambda z + \theta \lambda z^2)}{(1 + K L_0^2 z^2 - 2 M L_0^2 z^3)} \left( 1 - \frac{2M}{\sqrt{\pi \theta}} \frac{L_0^2 z^2}{r_+^2} \right) e^{-\frac{r_+^2}{4\theta z^2}} \right.
\]
\[
\left. + \theta \lambda \frac{(1-z)^2}{z^2} \partial_z \left( \frac{z^4(1-cz^2)^2}{(1 + K L_0^2 z^2 - 2 M L_0^2 z^3)} \right) \right] dz + 2\theta \lambda \int_0^1 \chi(z) dz.
\]  
(67)

Applying integration by parts, the last term of (5) can be written as
\[
2\theta \lambda \int_0^1 \chi(z) dz = -2\theta \lambda \int_0^1 z \chi'(z) dz = \theta \lambda \int_0^1 z^2 \chi''(z) dz
\]  
(68)

where we have used the boundary condition \( \chi'(1) = \chi(1) = 0 \). Now substituting \( \chi'' \) from (50) into (68) and considering terms up to order \( O(\lambda \theta) \) we have
\[
2\theta \lambda \int_0^1 \chi(z) dz = \lambda q^2 L^2 \int_0^1 \frac{\lambda z^2 (1-z)(1-cz^2)^2}{(1 + K L_0^2 z^2 - 2 M L_0^2 z^3)} dz
\]  
(69)

Using this result the final form of \( \chi'(0) \) becomes
\[
\chi'(0) = -\lambda q^2 L^2 \int_0^1 \left[ \frac{(1-z)(1-cz^2)^2(1-2\theta \lambda z + \theta \lambda z^2)}{(1 + K L_0^2 z^2 - 2 M L_0^2 z^3)} \left( 1 - \frac{2M}{\sqrt{\pi \theta}} \frac{L_0^2 z^2}{r_+^2} \right) e^{-\frac{r_+^2}{4\theta z^2}} \right.
\]
\[
\left. + \theta \lambda \frac{(1-z)^2}{z^2} \partial_z \left( \frac{z^4(1-cz^2)^2}{(1 + K L_0^2 z^2 - 2 M L_0^2 z^3)} \right) \right] dz
\]
\[
= -\lambda A \quad (say).
\]  
(70)

Substituting \( \chi'(0) \) from (50) into (64) and using the relations \( \lambda = \frac{\theta}{r_+^2} \) and \( T = \frac{3r_+}{4\theta} \), we finally obtain the expression for condensation operator \( J \), as \( T \to T_c \)
\[
\langle J \rangle = \gamma T_c \sqrt{1 - \frac{T}{T_c}}
\]  
(71)

where
\[
\gamma = \frac{4\sqrt{2} \pi}{3\sqrt{\lambda A}}.
\]

This relation (71) is crucial for further study of other properties of the noncommutative holographic superconductor, considered here. Numerical estimates of \( \gamma \) are provided in Tables 1, 2, 3 and 4 given below:
Table 1:

| \( \theta \) | \( M \) | \( r_+ \) | \( c \) | \( \lambda^2 \) | \( \zeta \) | \( \gamma \) |
|---|---|---|---|---|---|---|
| 0.5 | \( \frac{10^5 \sqrt{\theta}}{G} \) | 52.10007 | 0.247345 | 1.3514 | 0.2214 | 8.090 |
| 0.1 | \( \frac{10^5 \sqrt{\theta}}{G} \) | 39.8422 | 0.240512 | 1.28414 | 0.2243 | 8.077 |
| 0.01 | \( \frac{10^5 \sqrt{\theta}}{G} \) | 27.1442 | 0.239061 | 1.26989 | 0.2249 | 8.075 |

Table 2:

| \( \theta \) | \( M \) | \( r_+ \) | \( c \) | \( \lambda^2 \) | \( \zeta \) | \( \gamma \) |
|---|---|---|---|---|---|---|
| 0.5 | \( \frac{100 \sqrt{\theta}}{G} \) | 5.209997 | 0.247343 | 1.3514 | 0.2214 | 8.09 |
| 0.5 | \( \frac{40 \sqrt{\theta}}{G} \) | 3.836269 | 0.24659 | 1.34991 | 0.2215 | 8.087 |
| 0.5 | \( \frac{5 \sqrt{\theta}}{G} \) | 1.683034 | 0.107880 | 1.02643 | 0.2372 | 7.648 |
| 0.5 | \( \frac{2.75 \sqrt{\theta}}{G} \) | 1.290122 | 0.09991 | 0.91208 | 0.2443 | 7.6639 |
| 0.5 | \( \frac{0.5 \sqrt{\theta}}{G} \) | 0.731424 | 0.261094 | 1.22616 | 0.2269 | 8.130 |
| 0.5 | \( \frac{0.2 \sqrt{\theta}}{G} \) | 0.559243 | 0.289856 | 1.37356 | 0.2205 | 8.227 |
| 0.5 | \( \frac{0.005 \sqrt{\theta}}{G} \) | 0.182310 | 0.276193 | 1.4804 | 0.2164 | 8.199 |

Table 3:

| \( \theta \) | \( M \) | \( r_+ \) | \( c \) | \( \lambda^2 \) | \( \zeta \) | \( \gamma \) |
|---|---|---|---|---|---|---|
| 0.1 | \( \frac{100 \sqrt{\theta}}{G} \) | 3.984220 | 0.24051 | 1.28414 | 0.224 | 8.077 |
| 0.1 | \( \frac{8 \sqrt{\theta}}{G} \) | 1.715632 | 0.23973 | 1.28264 | 0.224 | 8.073 |
| 0.1 | \( \frac{5 \sqrt{\theta}}{G} \) | 0.752676 | 0.09557 | 0.95898 | 0.241 | 7.623 |
| 0.1 | \( \frac{0.55 \sqrt{\theta}}{G} \) | 0.576960 | 0.08612 | 0.84343 | 0.2491 | 7.937 |
| 0.1 | \( \frac{0.1 \sqrt{\theta}}{G} \) | 0.327102 | 0.24875 | 1.14163 | 0.2310 | 8.106 |
| 0.1 | \( \frac{0.04 \sqrt{\theta}}{G} \) | 0.250101 | 0.27845 | 1.28611 | 0.2242 | 8.205 |
| 0.1 | \( \frac{0.001 \sqrt{\theta}}{G} \) | 0.081532 | 0.26754 | 1.40119 | 0.2194 | 8.182 |
In this paper, we have considered a noncommutative charged AdS Black Hole background and a scalar field coupled to gravity, thus introducing a hairy Black Hole. We have used the AdS/CFT correspondence to study the effects of this noncommutative background on the properties of holographic high $T_c$ superconductors. First we studied the asymptotic behavior of the gauge and scalar field and explicitly show the effects of noncommutativity on the physical parameters like charge density and chemical potential. Then we proceeded to analyse the modified relation between critical temperature and charge density. We also calculated the modified expressions for critical exponents and condensation values for this noncommutative scenario.

We have provided some numerical estimates in Table 1. We consider the established lower bound of $\theta$ to be $\theta \leq (10 \text{ TeV})^{-2}$ [12]. In [8] the Black Hole mass $M$ is related to $\theta$ by $M \approx \sqrt{\theta}/G$ where the Newton’s constant $G$ has been reinstated. This yields $M \approx 10^{33}\text{GeV}$, which, however, is far below the mass of the astrophysical Black Holes. In Table 1 we have taken values of $\theta$ to be lower than the above bound [12] and the corresponding Black Hole masses to be considerably larger than the [8]. We find that larger values of $\theta$ tend to lower the critical temperature of the holographic superconductor. Expectedly for very small $\theta$ the relation $T_c = 0.225\sqrt{\rho}$ [10] is recovered.

However, for smaller mass Black Holes, (i.e. less than [8]), and same values of $\theta$, we find that the critical temperature rises above the $\theta = 0$ value. Interestingly, from Table 2, we find that for $\theta = 0.5$ and $M = 2.75 \sqrt{(\theta)/G}$ (which is close to the mass $M = 2.4 \sqrt{\theta}/G$ considered in [8]) the value of $\zeta = 0.2443$ which is appreciably larger than the $\theta = 0$ result of $\zeta = 0.225$ [10] indicating a larger critical temperature. But once again for larger $M$, $\zeta$ stabilises to 0.2214, which is less than

| $\theta$ | $\frac{\sqrt{\theta}}{G}$ | $r_+$ | $c$ | $\lambda^2$ | $\zeta$ | $\gamma$ |
|-----------|------------------|-------|-----|-------------|-------|-------|
| 0.01      | $\frac{10\sqrt{\theta}}{G}$ | 1.259921 | 0.23906 | 1.26989 | 0.2250 | 8.074 |
| 0.01      | $\frac{0.8\sqrt{\theta}}{G}$ | 0.542530 | 0.23827 | 1.26839 | 0.2250 | 8.070 |
| 0.01      | $\frac{0.1\sqrt{\theta}}{G}$ | 0.238017 | 0.09300 | 0.94495 | 0.2421 | 7.6181 |
| 0.01      | $\frac{0.05\sqrt{\theta}}{G}$ | 0.182451 | 0.08327 | 0.82931 | 0.2502 | 7.6315 |
| 0.01      | $\frac{0.01\sqrt{\theta}}{G}$ | 0.103439 | 0.24618 | 1.12409 | 0.2319 | 8.102 |
| 0.01      | $\frac{0.001\sqrt{\theta}}{G}$ | 0.079089 | 0.27606 | 1.26786 | 0.2250 | 8.200 |
|          | $\frac{0.0001\sqrt{\theta}}{G}$ | 0.025783 | 0.26571 | 1.38447 | 0.2201 | 8.179 |

Table 4:

6 Summary and Discussion

In this paper, we have considered a noncommutative charged AdS Black Hole background and a scalar field coupled to gravity, thus introducing a hairy Black Hole. We have used the AdS/CFT correspondence to study the effects of this noncommutative background on the properties of holographic high $T_c$ superconductors. First we studied the asymptotic behavior of the gauge and scalar field and explicitly show the effects of noncommutativity on the physical parameters like charge density and chemical potential. Then we proceeded to analyse the modified relation between critical temperature and charge density. We also calculated the modified expressions for critical exponents and condensation values for this noncommutative scenario.

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However, for smaller mass Black Holes, (i.e. less than [8]), and same values of $\theta$, we find that the critical temperature rises above the $\theta = 0$ value. Interestingly, from Table 2, we find that for $\theta = 0.5$ and $M = 2.75 \sqrt{(\theta)/G}$ (which is close to the mass $M = 2.4 \sqrt{\theta}/G$ considered in [8]) the value of $\zeta = 0.2443$ which is appreciably larger than the $\theta = 0$ result of $\zeta = 0.225$ [10] indicating a larger critical temperature. But once again for larger $M$, $\zeta$ stabilises to 0.2214, which is less than
It will be interesting to consider non-zero vector potential in the non-commutative framework to study conductivity and other properties of the holographic superconductors. Furthermore, in the noncommutative extension considered here, other parameters of the bulk theory such as the noncommutative parameter $\theta$ and the Black Hole mass are involved. Mapping of these parameters to the boundary theory may lead to generalized forms of holographic superconductors.

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