HYDRODYNAMIC REDUCTIONS OF
DISPERSIONLESS HARRY DYM HIERARCHY

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ABSTRACT. We investigate the reductions of dispersionless Harry Dym hierarchy to systems of finitely many partial differential equations. These equations must satisfy the compatibility condition and they are diagonalizable and semi-Hamiltonian. By imposing a further constraint, the compatibility is reduced to a system of algebraic equations, whose solutions are described.

Key Words: Dispersionless Harry Dym Hierarchy, Hydrodynamic Reductions, Semi-Hamiltonian.
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1. Introduction
Dispersionless integrable equations arise in various contexts and have attracted people to investigate it from different point of view, such as topological field theory (WDVV equation) [3, 13, 14, 15, 30], matrix models [6, 7], conformal maps and interface dynamics [32, 40, 41], Einstein-Weyl space [16, 17]. The hydrodynamic reductions are the most developed method to find the exact solutions of the dispersionless integrable equations [5, 8, 12, 21, 22, 28, 29, 30]. From the hydrodynamic reductions, one can construct the Riemann’s invariants and the corresponding characteristic speeds satisfy the semi-Hamiltonian property or Tsarev’s condition [38]. Hence the generalized hodograph method can be used to find the exact solutions. Also, the solutions
of dispersionless integrable equations can be found by nonlinear Beltrami equation [26, 27], slightly different from the generalized hodograph method.

Let’s recall dispersionless non-standard Lax hierarchy [33, 39]. Suppose that $\lambda$ is an algebra of Laurent series of the form

$$
\lambda = \{ A | A = \sum_{i=-\infty}^{N} a_i p^i \},
$$

with coefficients $a_i$ depending on an infinite set of variables $t_1 \equiv x, t_2, t_3, \cdots$. We can define a Lie-Bracket associated with $\lambda$ as follows:

$$
\{ A, B \} = \frac{\partial A}{\partial x} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial x}, \quad A, B \in \lambda
$$

which can be regarded as the Poisson bracket defined in the 2-dimensional phase space $(x, p)$. The algebra $\lambda$ can be decomposed into the Lie subalgebras as

$$
\lambda = \lambda_{\geq k} \oplus \lambda_{< k}, \quad (k = 0, 1, 2)
$$

where

$$
\lambda_{\geq k} = \{ A \in \Lambda | A = \sum_{i\geq k} a_i p^i \}
$$

$$
\lambda_{< k} = \{ A \in \Lambda | A = \sum_{i<k} a_i p^i \}.
$$

Based on this, the Lax formulation of the dispersionless integrable hierarchy can be formally defined as

$$
\frac{\partial \lambda}{\partial t_n} = \{ (\lambda_n^{\geq k}), \lambda \}.
$$

- For $k = 0$, it’s called dispersionless KP hierarchy (dKP) [39]
- For $k = 1$, it’s called dispersionless modified KP (dmKP) hierarchy [11, 34]
- For $k = 2$, it’s called dispersionless Harry Dym (dDym) hierarchy [11, 12, 34]. It’s the purpose of this paper.

We define the dispersionless Harry Dym (dDym) hydrodynamic systems as follows. The Lax operator of dDym has the form

$$
\lambda(p) = A_{-1} p + A_0 + A_1 p^{-1} + A_2 p^{-2} + A_3 p^{-3} + \cdots.
$$

Then the dDym hydrodynamic system is [11, 34]

$$
\frac{\partial \lambda}{\partial t_n} = \{ \lambda, \Omega_n(p) \},
$$

where

$$
\Omega_n(p) = (\lambda(p)^n)_{\geq 2}.
$$
Here \( (\cdot)_{\geq 2} \) denote the projection of Laurent series onto a linear combination of \( \lambda(p)^n \) with \( n \geq 2 \). From the zero curvature equation \((t_3 = t \text{ and } t_2 = y)\)

\[
\frac{\partial \Omega_2(p)}{\partial t} - \frac{\partial \Omega_3(p)}{\partial y} = \{\Omega_2(p), \Omega_3(p)\},
\]

where

\[
\Omega_2(p) = A_{-1}^2 p^2 \quad \text{and} \quad \Omega_3(p) = A_{-1}^3 p^3 + 3 A_{-1}^2 A_0 p^2,
\]

one can get the dispersionless Harry Dym Equation \[11\]

\[
\frac{\partial A_{-1}}{\partial t} = \frac{3}{4} \frac{1}{A_{-1}} \left[ A_{-1}^2 \frac{\partial}{\partial x} \left( \frac{A_{-1}}{A_{-1}^2} \right) \right] y.
\]

Now, considering the \( y \)-flow, we have

\[
(2) \quad \lambda_y = \{\lambda, A_{-1}^2 p^2\} = 2 p A_{-1}^2 \lambda_x - (A_{-1}^2)_x p^2 \lambda_p,
\]

or

\[
(3) \quad A_{-1y} = 2 A_{-1}^2 A_{0x} \\
A_{0y} = 2 A_{-1}^2 A_{1x} + (A_{-1}^2)_x A_1 \\
\vdots
\]

where \( n = -1, 0, 1, 2, 3 \cdots \). Comparing it with the Benney moment chain \[23, 35, 44\], one calls \[3\] the dDym moment chain. It’s the subject of this paper.

The paper is organized as follows. In the next section, one considers the reduction problems and obtains finitely many partial differential equations. In section 3, we prove the semi-Hamiltonian property when using Riemann invariants. In section 4, by imposing a further constraint, one gets some particular reductions. In the final section, we discuss some problems to be investigated.

2. The Compatibility Conditions

In this section, we consider the hydrodynamic reduction problems following \[1, 21, 22\]. For non-hydrodynamic reductions, one refers to \[2\].

We assume that the moments \( A_i \) are functions of only \( N \) independent variables \( u_i \). If the \( A_i \) satisfy \[3\], then it’s straightforward to show that the mapping

\[
(u_{-1}, u_0, u_1, u_2, \cdots, u_{N-2}) \rightarrow (A_{-1}, A_0, A_1, A_2, \cdots, A_{N-2})
\]
is non-degenerate. Hence without loss of generality we set $u_{-1} = A_{-1}, u_0 = A_0, u_1 = A_1, \cdots, u_{N-2} = A_{N-2}$. The first $N$-moments are the independent variables, while the higher moments are functions of them, i.e.,

$$A_k = A_k(A_{-1}, A_0, \cdots, A_{N-2}), \quad k \geq N - 1.$$ The equations of motion for $A_{-1}, A_0, \cdots, A_{N-2}$ become

$$\frac{\partial A_j}{\partial y} = 2A_{-1}^2A_{j+1,x} + (j + 1)(A_{-1}^2)xA_{j+1}, \quad j \leq N - 3$$

$$\frac{\partial A_{N-2}}{\partial y} = 2A_{-1}^2A_{N-1,x} + (N - 1)(A_{-1}^2)xA_{N-1}$$

$$= 2A_{-1}^2 \frac{\partial A_{N-1}}{\partial A_j} \frac{\partial A_j}{\partial x} + (N - 1)(A_{-1}^2)xA_{N-1},$$

while each higher moment ($A_{N-1}, \cdots$) must satisfy the overdetermined system ($k \geq N - 1$) using (4)(5)

$$\frac{\partial A_k}{\partial y} = \sum_{j=1}^{N-3} \frac{\partial A_k}{\partial A_j}[2A_{-1}^2A_{j+1,x} + (j + 1)(A_{-1}^2)xA_{j+1}]$$

$$+ \frac{\partial A_k}{\partial A_{N-2}}[2A_{-1}^2 \sum_{j=1}^{N-2} \frac{\partial A_{N-1}}{\partial A_j} \frac{\partial A_j}{\partial x} + (N - 1)(A_{-1}^2)xA_{N-1}]$$

$$= 2A_{-1}^2 \sum_{j=1}^{N-2} \frac{\partial A_{k+1}}{\partial A_j} \frac{\partial A_j}{\partial x} + (k + 1)(A_{-1}^2)xA_{k+1}.$$

Comparing the coefficients of $\frac{\partial A_k}{\partial x}(j = -1, 0, 1, \cdots, N - 2)$, one has

$$\frac{\partial A_{k+1}}{\partial A_j} = \frac{\partial A_k}{\partial A_{j-1}} + \frac{\partial A_k}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial A_j}, \quad 0 \leq j \leq N - 2$$

$$A_{-1} \frac{\partial A_{k+1}}{\partial A_{-1}} = \sum_{j=1}^{N-2} \frac{\partial A_k}{\partial A_j}(j + 1)A_{j+1} + A_{-1} \frac{\partial A_{k-1}}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial A_{-1}}$$

$$- (k + 1)A_{k+1}.$$

Now, letting $k = N - 1$ and defining $r = \log A_{-1}$, one has

$$\frac{\partial A_N}{\partial A_j} = \frac{\partial A_{N-1}}{\partial A_{j-1}} + \frac{\partial A_{N-1}}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial A_j}, \quad 0 \leq j \leq N - 2$$

$$\frac{\partial A_N}{\partial r} = \sum_{j=1}^{N-2} \frac{\partial A_{N-1}}{\partial A_j}(j + 1)A_{j+1} + \frac{\partial A_{N-1}}{\partial A_{N-2}} \frac{\partial A_{N-1}}{\partial r}$$

$$- NA_N.$$
The compatibility of (8) and (9) gives a system $\Gamma$ of $\frac{N(N-1)}{2}$ non-linear second order equation for the single known $A_{N-1}(A_{-1}, A_0, A_1, \cdots, A_{N-2})$. One can show that by induction if $\Gamma$ is satisfied then the analogous compatibility for $A_k (k \geq N)$ is also derived. Let’s investigate the case $N=2$ in more details. Then we have ($A_0 = s$)

$$
\begin{align*}
\frac{\partial A_2}{\partial s} &= A_1 r \exp(-r) + A_1^2 s \\
\frac{\partial A_2}{\partial r} &= A_1 A_1 + A_1 s A_1 r - 2A_2.
\end{align*}
$$

Cross-differentiating, one gets the quasi-linear second differential equation

$$
A_{1rr} \exp(-r) + A_{1s} A_{1rs} - (A_1 + A_{1rs}) A_{1ss} + A_{1r} \exp(-r) + (A_{1s})^2 = 0.
$$

Letting $A_1 = a, A_{1r} = b, A_{1s} = c,$ one can express (10) as the degenerate non-homogeneous hydrodynamic system

$$
\begin{pmatrix} a \\ b \\ c \end{pmatrix}_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -c \exp r & (a+b) \exp r \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_s \\
+ \begin{pmatrix} b \\ -b - c^2 \exp r \\ 0 \end{pmatrix}
$$

Define the characteristic speeds as

$$
u = \frac{-c \exp r + \sqrt{c^2 \exp(2r) + 4(a+b) \exp r}}{2},$$
$$v = \frac{-c \exp r + \sqrt{c^2 \exp(2r) - 4(a+b) \exp r}}{2}.$$

A simple calculation yields,

$$
\begin{pmatrix} u \\ v \end{pmatrix}_r = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_s + \begin{pmatrix} \frac{uv}{u-v} \\ \frac{uv}{u-v} \end{pmatrix},
$$

there being no $a$-term! It is of non-homogeneous hydrodynamic systems of Tsarev-Gibbons type and it has one obvious hydrodynamic type conserved density ($u + v$). Moreover, according to the theory of
Poisson commuting Hamiltonians \[18\], one can also find a conserved density of first derivatives:

\[(u - v)[\left(\frac{u_s}{u}\right)^2 - \left(\frac{v_s}{v}\right)^2].\]

For a given affinor \(v^i_j(u)\) of a hydrodynamic system

\[
\frac{\partial u_i}{\partial y} = v^i_j(u) \frac{\partial u_j}{\partial x},
\]

one can define the Nijenhuis tensor \(N^k_{ij}(u)\) \[35\]

\[
N^k_{ij} = v^s_i \frac{\partial v^k_j}{\partial u_s} - v^s_j \frac{\partial v^k_i}{\partial u_s} + v^k_s \frac{\partial v^i_j}{\partial u_s} - v^k_s \frac{\partial v^j_i}{\partial u_s}.
\]

Then we can define the corresponding Haantjes tensor \(H^i_{jk}(u)\) \[24\]

\[
H^i_{jk} = (N^q_{qp} v^i_k - N^q_{kp} v^i_q) v^p_j - (N^p_{qj} v^i_k - N^p_{kj} v^i_q) v^q_p.
\]

The system \[13\] is diagonalized if and only if the Haantjes tensor \(H_{jk}^i(u)\) vanishes identically and all the eigenvalues of the affinor \(v^i_j(u)\) are real and distinct. That is, there exist \(N\)-functions \(\lambda_n\) (Riemann invariants), depending on the variables \(u_i\), in which the equation \[13\] is diagonalized

\[
\frac{\partial \lambda_n}{\partial y} = V_n \frac{\partial \lambda_n}{\partial x},
\]

where \(V_n\) are the eigenvalues of the matrix \(v^i_j(u)\), called characteristic speed. For the hydrodynamic system \[14\] and \[5\] of the reduced dDym, the Haantjes tensor \(H_{jk}^i(u)\) vanishes identically whenever the system \[8\] and \[9\] is satisfied. Therefore, any consistent reduction of dDym is diagonalizable and then we can use Riemann invariants to discuss problem further.

Finally, one remarks that as in the case of dKP hierarchy \[28\], a similar argument shows that for the reduced dDym hierarchy \[14\] we also have the Kodama-Gibbons formulation: the Riemann invariants are

\[
\lambda_n = \lambda(q_n), \quad \text{where} \quad \frac{\partial \lambda}{\partial p}(q_n) = 0, \quad n = 1, 2, \ldots, N
\]

and then the hierarchy \[14\] can be expressed as

\[
\frac{\partial \lambda_n}{\partial \hat{t}_m} = \hat{\Omega}_m(\hat{V}_n) \frac{\partial \lambda_n}{\partial x},
\]

where \(\hat{V}_n = (q_1, q_2, \ldots, q_N)\) and \(\hat{\Omega}_m(\hat{V}_n) = \frac{d\lambda_m(p)}{dp}|_{p=\hat{V}_n} \hat{\hat{V}}_n\).
3. SEMI-HAMILTONIAN PROPERTY

In this section, one will prove the semi-Hamiltonian property (Tsarev’s condition) of the reduced dDym hierarchy using Riemann invariants. Suppose that each moments $A_n$ can be expressed as the Riemann’s invariants $\lambda_i$, which satisfy the equation ($t_2 = y$)

$$\frac{\partial \lambda_i}{\partial y} = V_i \frac{\partial \lambda_i}{\partial x},$$

where

$$V_i = 2A^2_1 g_i, \quad i = 1, 2, 3 \ldots, N.$$  

Then the moment equations (3) can be written as

$$V_i A_{n, \lambda_i} = 2A^2_1 A_{n+1, \lambda_i} + (n + 1)A_{n+1}(A^2_1)_{\lambda_i},$$

As indicated in (16), we also have, equivalent to (17),

$$\frac{\partial \lambda}{\partial y} = \frac{\partial \lambda}{\partial p} (A^2_1)p^2 - 2 \frac{\partial \lambda}{\partial x} A^2_1 p$$

and then, using (16), one obtains, after reshuffling terms,

$$\frac{\partial \lambda}{\partial \lambda_i} = p^2 \frac{\partial \lambda}{\partial p} (A^2_1)_{\lambda_i} \frac{1}{V_i + 2A^2_1 p}.$$ 

Cross-differentiating, one has ($\phi = A^2_1$)

$$\phi_{\lambda_i \lambda_j} (V_i + 2\phi p)(V_j + 2\phi p)^2 - \phi_{\lambda_i \lambda_j} 2\phi p^2 (V_i + 2\phi p) - (V_j + 2\phi p)^2 \phi_{\lambda_i \lambda_j} \frac{\partial V_i}{\partial \lambda_j}$$

$$-2 \phi_{\lambda_i \lambda_j} (V_j + 2\phi p)^2$$

$$= \phi_{\lambda_i \lambda_j} (V_j + 2\phi p)(V_i + 2\phi p)^2 - \phi_{\lambda_i \lambda_j} 2\phi p^2 (V_j + 2\phi p) - (V_i + 2\phi p)^2 \phi_{\lambda_i \lambda_j} \frac{\partial V_j}{\partial \lambda_i}$$

$$-2 \phi_{\lambda_i \lambda_j} (V_i + 2\phi p)^2.$$ 

Letting $p = -\frac{V_j}{2\phi}$, we obtain

$$\frac{\partial V_j}{\partial \lambda_i} = \frac{1}{2} (\ln \phi)_{\lambda_i} \left[ \frac{V_i V_j}{V_i - V_j} + V_j \right], \quad i \neq j.$$ 

On the other hand, comparing the coefficients of p-power and using (19), we get the only equation

$$\phi_{\lambda_i \lambda_j} = \phi \frac{\phi_{\lambda_i \lambda_j}}{\phi} \left[ \frac{V_i V_j}{(V_j - V_i)^2} + 1 \right], \quad i \neq j.$$ 

The higher moments $A_n$, with $n \geq 0$ can be solved recursively using (17). These equations (19) (20) are compatible and their solutions are parameterized by $2N$ functions of a single variable.
A direct calculation, using MAPLE, confirms that the reduced equation (16) is semi-Hamiltonian, that is,
\[ \frac{\partial}{\partial \lambda_k} \left( \frac{\partial V_j}{\partial \lambda_i} \right) \left( \frac{1}{2} \left( \ln \phi \right)_{\lambda_i} \left[ 1 - \left( \frac{V_i}{V_j - V_k} \right)^2 \right] \right) = \frac{\partial}{\partial \lambda_i} \left( \frac{\partial V_k}{\partial \lambda_k} \right), \]
for \( i, j, k \) all distinct. Then the reduced equations (16) are thus integrable by the generalized hodograph transformation (18).

4. Algebraic Equations and Special Reductions

To investigate the reduction problems, we introduce \( A = 2 \ln A_1 \) to put (19) and (20) in a more compact form (\( i \neq j \))
\[ \frac{\partial V_j}{\partial \lambda_i} = \frac{1}{2} A \left[ \frac{V_i V_j}{V_i - V_j} + V_j \right], \]
\[ A_{\lambda_i \lambda_j} = A_{\lambda_i} A_{\lambda_j} \left( \frac{V_i V_j}{V_i - V_j} \right)^2, \]
or, noting that \( V_i = 2 A_{i-1} q_i \),
\[ \frac{\partial q_j}{\partial \lambda_i} = \frac{1}{2} A \left[ \frac{q_i q_j}{q_i - q_j} - q_j \right], \]
\[ A_{\lambda_i \lambda_j} = A_{\lambda_i} A_{\lambda_j} \left( \frac{q_i q_j}{q_i - q_j} \right)^2, \]
Then as in (21) for the dKP case, we impose two further restrictions on the reduced system (21) and (22). First, from the form (21), we require that the reduced system is translation-invariant in the sense that
\[ \lambda_i \rightarrow \lambda_i + c \Rightarrow q_i \rightarrow q_i, \quad A \rightarrow A \]
or, equivalently,
\[ \delta q_i = 0, \quad \delta A = 0, \]
where \( \delta = \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \). Secondly, we require the homogeneity of the functions \( A \) and \( q_j \) in the variables \( \lambda_i \). \( A \) should be of weight 0, and the \( q_j \) of weight \(-1\). Hence
\[ R q_i = -\frac{1}{\kappa} \frac{\partial A}{\partial \lambda_i}, \]
\[ R \frac{\partial A}{\partial \lambda_i} = -\frac{\partial A}{\partial \lambda_i}, \]
where \( \kappa \) is a positive integer and \( R = \sum_{i=1}^{N} \lambda_i \frac{\partial}{\partial \lambda_i} \). Plugging (22) into (24), (26) and eliminating the second derivative, we obtain

\[
-\frac{\partial A}{\partial \lambda_i} = R \frac{\partial A}{\partial \lambda_i} = \sum_{j=1, j \neq i}^{N} \lambda_j \frac{\partial^2 A}{\partial \lambda_i \lambda_j} + \lambda_i \frac{\partial^2 A}{\partial \lambda_i^2}
\]

\[
= \sum_{j=1, j \neq i}^{N} \lambda_j \frac{\partial A}{\partial \lambda_j} + \lambda_i (N - \sum_{j=1, j \neq i}^{N} \frac{\partial^2 A}{\partial \lambda_i \lambda_j})
\]

\[
= \sum_{j=1, j \neq i}^{N} (\lambda_j - \lambda_i) A\lambda_j A\lambda_i \frac{q_j q_j}{(q_j - q_i)^2}.
\]

Hence, either \( A\lambda_i = 0 \) or

\[
(27) \quad \sum_{j=1, j \neq i}^{N} (\lambda_j - \lambda_i) A\lambda_j A\lambda_i \frac{q_j q_j}{(q_j - q_i)^2} = -1.
\]

Similarly, plugging (21) into (23) and (25), we get

\[
-\frac{1}{\kappa} q_i = R q_i = \sum_{j=1, j \neq i}^{N} \lambda_j \frac{\partial q_i}{\partial \lambda_j} + \lambda_i \frac{\partial q_i}{\partial \lambda_i}
\]

\[
= \sum_{j=1, j \neq i}^{N} \lambda_j \frac{\partial q_i}{\partial \lambda_j} - \lambda_i \sum_{j=1, j \neq i}^{N} \frac{\partial q_i}{\partial \lambda_j}
\]

\[
= \sum_{j=1, j \neq i}^{N} (\lambda_j - \lambda_i) \frac{\partial q_i}{\partial \lambda_j}
\]

\[
= \frac{1}{2} \sum_{j=1, j \neq i}^{N} (\lambda_j - \lambda_i) \frac{\partial A}{\partial \lambda_j} [q_j q_j - q_j q_i - q_i] \frac{q_j q_j}{(q_j - q_i)^2}.
\]

The systems (27) and (28) form a \( 2N \) algebraic equations for \( 2N \) unknowns, the \( q_i \) and \( \frac{\partial A}{\partial \lambda_i} \). The solutions of the system are essentially unique. With two Riemann invariants and \( \kappa = 1 \), a simple calculation can yield by carefully choosing the integration constants

\[
(29) \quad q_1 = -q_2 = \frac{4}{\lambda_1 - \lambda_2}, \quad A_{-1} = \frac{(\lambda_1 - \lambda_2)^2}{16}.
\]

Hence from (16) we have

\[
(30) \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_y = \begin{pmatrix} \frac{(\lambda_1 - \lambda_2)^3}{32} & 0 \\ 0 & -\frac{(\lambda_1 - \lambda_2)^3}{32} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_x.
\]
This equation (30) will correspond to the reduction of Lax operator constructed by Riemann-Hilbert method [11, 12]

\(\lambda(p) = A_{-1}p + A_0 + \frac{1}{p}.\)

If we consider more general Lax reductions of the form for positive integers \(m\) and \(\kappa\) [11, 13]

\(\lambda(p)^m = A_{-1}^m p^m + w_{m-1} p^{m-1} + w_{m-2} p^{m-2} + \cdots \)

\[+ w_{-\kappa+1} p^{-\kappa+1} + p^{-\kappa}, \quad m + \kappa = N,\]

then from (15) it’s not difficult to see that (32) satisfies the conditions (23)-(24)-(25)-(26). Hence its corresponding characteristic speeds and \(A_{-1}\) will be one of the solution of the \(2N\) algebraic equation (27) and (28). It would be interesting to know whether any other solutions exist.

Next, one generalizes the operator \(R\) in (25) or (26) to the following forms

\[\hat{R} = \sum_{k=1}^{N} \hat{g}_k(\vec{\lambda}) \partial_{\lambda_k} \hat{h}_k(\vec{\lambda}),\]

\(\hat{R}V_i = -\frac{1}{\kappa}V_i\)

and

\[\tilde{R} = \sum_{k=1}^{N} \tilde{g}_k(\vec{\lambda}) \partial_{\lambda_k} \tilde{h}_k(\vec{\lambda}),\]

\[\tilde{R} \partial A \partial_{\lambda_i} = -\partial A \partial_{\lambda_i},\]

where \(\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)\) and \((\hat{g}_k, \hat{h}_k), (\tilde{g}_k, \tilde{h}_k)\) are arbitrary functions satisfying the conditions compatible with (23)-(24)

\(\delta \hat{g}_k = \delta \hat{h}_k = \delta \tilde{g}_k = \delta \tilde{h}_k = 0.\)

Then a similar calculation can get the following \(2N\) algebraic equations generalizing (27)-(28)

\[\sum_{j=1,j\neq i}^{N} \left( \hat{g}_j \hat{h}_j - \hat{g}_i \hat{h}_i \right) A_{\lambda_j} \left[ \frac{q_j q_i}{q_j - q_i} + q_i \right] = -\frac{1}{\kappa}V_i (1 + \kappa \sum_{j=1}^{N} \hat{g}_j \partial \hat{h}_j)\]

\[\sum_{j=1,j\neq i}^{N} \left( \tilde{g}_j \tilde{h}_j - \tilde{g}_i \tilde{h}_i \right) A_{\lambda_j} \left[ \frac{q_j q_j}{q_j - q_i} \right]^2 = -(1 + \sum_{j=1}^{N} \hat{g}_j \partial \hat{h}_j).\]

For \(N = 2\) and \(\kappa = 1\), simple calculations can show that a suitable choice of \((\hat{g}_1, \hat{h}_1), (\hat{g}_2, \hat{h}_2), (\tilde{g}_1, \tilde{h}_1), (\tilde{g}_2, \tilde{h}_2)\), not unique, can also obtain
Finally, one notices that if we let 
\[ h_j = h_j = 1 \]
for \( j = 1, 2, \ldots, N \), then we get the weaker condition than \( \delta \hat{g}_k = \delta \tilde{g}_k \).

Hence the equations \( \delta \hat{g}_k \) reduce to
\[
\sum_{j=1, j \neq i}^{N} (\hat{g}_j - \hat{g}_i) A_{\lambda_j} \left[ \frac{q_j q_i}{q_j - q_i} + q_i \right] = -\frac{1}{\kappa} V_i
\]
\[
\sum_{j=1, j \neq i}^{N} (\tilde{g}_j - \tilde{g}_i) A_{\lambda_j} \frac{q_j q_i}{(q_j - q_i)^2} = -1
\]
If \( \hat{g}_j = \tilde{g}_j = \lambda_j \), then we can obtain \( \text{(27)} \) and \( \text{(28)} \).

5. CONCLUDING REMARKS

We prove the semi-Hamiltonian property of reductions for the d-Dym and find some particular solutions invariant under translation and homogeneity. In spite of the results obtained, there are some interesting issues deserving investigations.

- The integrability and solution structure of the equation \( \text{(12)} \) (or \( \text{(10)} \)) is unclear \[5, 18, 19, 20\]. It is not difficult to see that \( \text{(12)} \) can be extended to:
\[
\frac{\partial u_i}{\partial r} = r_i \frac{\partial u_i}{\partial s} + \frac{u_1 u_2 \cdots u_N}{\Pi_{k \neq i}(u_k - u_i)}, \text{where } r_i = \left( \sum_{k=1}^{N} u_k \right) - u_i.
\]

One hopes issue these problems elsewhere.

- In \[31\], the algebraic reductions for dKP are found and in \[25, 43\], the waterbag reduction (non-algebraic) for dKP is also found. If we define
\[
A_n = \int_{-\infty}^{\infty} q^n f(q, x, y) dq, \quad n = -1, 0, 1, 2, \ldots
\]
then we obtain
\[
\lambda = p^2 (P \int_{-\infty}^{\infty} \frac{f / q}{p - q} dq), \quad f = f(q, x, y),
\]
where \( P \int \) denotes the Cauchy principal value of the integral. Also, from \[38\], the ”distribution” function \( f(x, y, q) \) must satisfy the Vlasov-like equation \[23, 37, 41\]
\[
f_y = 2 A_{-1}^2 q f_x - (A_{-1}^2) q^2 f_q = \{ f, A_{-1}^2 q^2 \}_{x, q}.
\]
Comparing (39) and (2), we can assume $f = F(\lambda)$ for any function $F$. Hence the Lax operator $\lambda$ will satisfy the non-linear singular integral equation

$$\lambda = p^2(P \int_{-\infty}^{\infty} \frac{F(\lambda)}{q} dq).$$

This equation will help us find the (non-)algebraic reductions [22, 34], study the initial value problem for d-Dym as in the case of dKP [42, 43] and it needs further investigations.

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