HOCHSCHILD COHOMOLOGY OF SOME QUANTUM COMPLETE INTERSECTIONS

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Abstract. We compute the Hochschild cohomology ring of the algebras $A = k\langle X, Y \rangle/(X^a, XY - qYX, Y^a)$ over a field $k$ where $a \geq 2$ and where $q \in k$ is a primitive $a$-th root of unity. We find the the dimension of $\text{HH}^n(A)$ and show that it is independent of $a$. We compute explicitly the ring structure of the even part of the Hochschild cohomology modulo homogeneous nilpotent elements.

1. Introduction

Let $k$ be a field, and let $0 \neq q \in k$. Quantum complete intersections originate from work of Manin [8]. Here we focus on the algebras $A_q = k\langle X, Y \rangle/(X^a, XY - qYX, Y^a)$.

Such algebras have provided several examples giving answers to homological conjectures and questions. Perhaps most spectacular amongst these is Happel’s question. In [6] Happel asked whether an algebra whose Hochschild cohomology is finite-dimensional, must have finite global dimension. The main result of [3] gave a negative answer: It shows that the Hochschild cohomology of the quantum complete intersection $A_q$ as above, when $a = 2$ and $q$ not a root of unity, is finite-dimensional. However the algebra $A_q$ is selfinjective, hence has infinite global dimension. Already earlier, R. Schulz discovered unusual properties for these algebras $A_q$, see [11] and [10].

Furthermore, there is a theory of support varieties in terms of Hochschild cohomology provided the algebra satisfies suitable finite generation properties, known as condition (Fg) (see [5] and [13]). For $A_q$, this condition is satisfied precisely when $q$ is a root of unity. The general theory of these support varieties has now been well established in several papers. However, in order to actually compute the varieties over a given algebra, one needs to determine the ring structure of the Hochschild cohomology, or at least modulo homogeneous nilpotent elements.

The results in this paper will be a contribution towards this goal. We determine the ring structure of the even part of $\text{HH}^\ast(A_q)$ (which will be denoted $\text{HH}^{2\ast}(A_q)$) modulo the ideal of homogeneous nilpotent elements for $A_q$ when $q$ is a primitive $a$-th root of unity. The proofs are quite technical, but this illustrates the typical difficulties and computations one is faced with when trying to compute Hochschild cohomology.

First we present an unpublished result by P. Bergh and K. Erdmann which determines the dimensions of the Hochschild cohomology groups; this is done via exploiting Hochschild homology. Surprisingly, the answer is independent of $a$ (see Theorem 3.1 and Corollary 3.2). This suggests that perhaps also the ring structure might not depend too much on the parameter $a$. We determine explicit bases of $\text{HH}^{2\ast}(A)$ (see Section 5.2).

Furthermore, we compute the algebra structure of $\text{HH}^{2\ast}(A)$ modulo the largest homogeneous nilpotent ideal. We show that it is $\mathbb{Z}_2$-graded, with degree zero part isomorphic to the polynomial ring in two variables, generated in degree 2. The explicit description is given in 4.2 when $a = 2$, and in 5.4 when $a \geq 3$. 

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An explicit description when \( a = 2 \) was also given in [3, Section 3.4]. We include this case (in Section 4), as it shows that it is part of the general pattern.

S. Oppermann gave also a description of the Hochschild cohomology and homology of more general quantum complete intersections in [9]. The products are, however, not computed completely explicitly, though it discusses a more general setting. However, in this paper we calculate products explicitly by liftings along a minimal projective resolution (which will be discussed in Section 2). This illustrates techniques that might be of independent interest.

In a larger context, there is even more structure in some classes of Hochschild cohomology than the well known Gerstenhaber algebra structure. In [7] T. Lambre, G. Zhou and A. Zimmermann prove that the Hochschild cohomology ring of quantum complete intersections is a so called Batalin–Vilkovisky algebra (Corollary 5.8). Roughly speaking a Batalin–Vilkovisky algebra is a Gerstenhaber algebra with an additional operation \( \Delta : HH^{n} \rightarrow HH^{n-1} \) which squares to zero and which, together with the cup product, can express the Lie Bracket.

2. Preliminaries

More generally, let \( A \) be any finite-dimensional algebra over a field \( k \), and let \( A^e = A \otimes_k A^{\text{op}} \) denote the enveloping algebra. We view bimodules over \( A \) as left modules over \( A^e \). In this setting, the Hochschild cohomology of \( A \) can be taken as \( HH^*(A) = \text{Ext}^*_{A^e}(A, A) \), the \( n \)-th cohomology of the complex \( \text{Hom}_{A^e}(P, A) \), i.e.

\[
\text{Ext}^n_{A^e}(A, A) = \ker d^*_n + 1 / \text{im } d^*_n,
\]

where \( d^{*}_n = \text{Hom}_{A^e}(d_n, A) \) and where \( d_n \) are the maps in a minimal projective resolution:

\[
P : \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \rightarrow 0.
\]

Then the Hochschild cohomology

\[
HH^*(A) = \text{Ext}^*_{A^e}(A, A)
\]

is a \( k \)-algebra which is graded-commutative. There are various equivalent ways to define the product; here we will work with the Yoneda product.

We specialize now to the quantum complete intersections. Let \( a \) be an integer such that \( a \geq 2 \). We also let \( q \in k \) be a primitive \( a \)-th root of unity, and \( A \) is the \( k \)-algebra defined by

\[
A = k\langle X, Y \rangle/(X^a, XY - qYX, Y^a).
\]

We write \( x \) and \( y \) for the residue classes of \( X \) and \( Y \), respectively.

In [2], for arbitrary parameter \( q \neq 0 \), an explicit minimal projective bimodule resolution \( P \) as in (2.2) was constructed. The \( n \)th bimodule in \( P \) is

\[
P_n = \bigoplus_{i=0}^{n} A^e f^i_n,
\]

the free \( A^e \)-module of rank \( n + 1 \) having generators \( \{ f^0_n, f^1_n, \ldots, f^n_n \} \). For each \( s \geq 0 \) define the following four elements of \( A^e \):

\[
\tau_1(s) = q^s(1 \otimes x) - (x \otimes 1)
\]
\[
\tau_2(s) = (1 \otimes y) - q^s(y \otimes 1)
\]
\[
\gamma_1(s) = \sum_{j=0}^{a-1} q^{js}(x^{a-1-j} \otimes x^j)
\]
\[
\gamma_2(s) = \sum_{j=0}^{a-1} q^{js}(y^j \otimes y^{a-1-j})
\]
The maps $d_n : P_n \to P_{n-1}$ in $\mathbb{P}$ are given by

\begin{align}
\text{(2.10)} \quad d_{2t} : f_i^{2t} &\mapsto \begin{cases} 
\gamma_2 \left( \frac{ai}{2} \right) f_i^{2t-1} + \gamma_1 \left( \frac{2at-ai}{2} \right) f_i^{2t-1} & \text{for } i \text{ even} \\
-\gamma_2 \left( \frac{a-i+2}{2} \right) f_i^{2t-1} + \gamma_1 \left( \frac{2at-a-i+2}{2} \right) f_i^{2t-1} & \text{for } i \text{ odd}
\end{cases} \\
\text{(2.11)} \quad d_{2t+1} : f_i^{2t+1} &\mapsto \begin{cases} 
\gamma_2 \left( \frac{ai}{2} \right) f_i^{2t} + \gamma_1 \left( \frac{2at-ai+2}{2} \right) f_i^{2t} & \text{for } i \text{ even} \\
-\gamma_2 \left( \frac{a-i+2}{2} \right) f_i^{2t} + \gamma_1 \left( \frac{2at-a-i+2}{2} \right) f_i^{2t} & \text{for } i \text{ odd}
\end{cases}
\end{align}

where the convention $f_i^{n+1} = f_i^{n+1} = 0$ has been used. So far, $q$ is arbitrary. Later in our setting we will simplify these expressions.

We will wish to identify nilpotent elements of Hochschild cohomology. This can be done by exploiting the following result of N. Snashall and Ø. Solberg, see Proposition 4.4 in [12].

**Proposition 2.1.** Assume $k$ is a field and $A$ is a finite-dimensional $k$-algebra. Suppose $\eta$ is a map into $A$ representing an element of $\text{HH}^n(A)$. If $\text{im}(\eta)$ is in the radical of $A$ then $\eta$ is nilpotent in $\text{HH}^n(A)$.

### 3. Dimensions of Hochschild Cohomology Groups

We recall an unpublished result by Petter A. Bergh and Karin Erdmann which determines the dimensions.

By viewing $A$ as a left $A^e$-module, it follows from [4, p. VI.5.3] that $D(\text{HH}^*(A, A))$ is isomorphic to $\text{Tor}^A_n(D(A), A)$ as a vector space, where $D$ denotes the usual $k$-dual i.e. $D(-) := \text{Hom}_k(-, k)$. In particular, we see that $\dim \text{HH}^n(A) = \dim \text{Tor}^A_n(D(A), A)$ for all $n \geq 0$. Moreover, it follows from [2] that $A$ is a Frobenius algebra with Nakayama automorphism $\nu : A \to A$ defined by

$$\nu : \begin{cases} x \mapsto q^{1-a}x \\ y \mapsto q^{a-1}y. \end{cases}$$

The bimodules $D(A)$ and $\nu A_1$ are isomorphic; here the left action on $\nu A_1$ is taken as $a \cdot m := \nu(a)m$. Consequently the dimensions of the Hochschild cohomology of $A$ are given by

$$\text{dim } \text{HH}^n(A) = \dim \text{Tor}^A_n(\nu A_1, A)$$

for all $n \geq 0$.

To compute $\text{Tor}^A_n(\nu A_1, A)$, we tensor the deleted projective bimodule resolution $\mathbb{P}$ with the right $A^e$-module $\nu A_1$.

We then obtain an isomorphism

$$\begin{array}{cccccccc}
\cdots & \nu A_1 \otimes A^e P_{n+1} & \overset{1 \otimes d_{n+1}}{\longrightarrow} & \nu A_1 \otimes A^e P_n & \overset{1 \otimes d_n}{\longrightarrow} & \nu A_1 \otimes A^e P_{n-1} & \cdots \\ 
\cong & \cong & \cong \\
\cdots & \oplus_{i=0}^{n+1}(\nu A_1)e_i^{n+1} & \overset{\delta_{n+1}}{\longrightarrow} & \oplus_{i=0}^{n}(\nu A_1)e_i^n & \overset{\delta_n}{\longrightarrow} & \oplus_{i=0}^{n-1}(\nu A_1)e_i^{n-1} & \cdots
\end{array}$$

of complexes, where $\{e_0^n, e_1^n, \ldots, e_n^n\}$ is the standard generating set of $n + 1$ copies of $\nu A_1$. Now given an element $\alpha \in k$ and a positive integer $t$, define an element $K_t(\alpha) \in k$ by

$$K_t(\alpha) := \sum_{j=0}^{t-1} \alpha^j.$$
The map $\delta_n$ is then given by
\begin{equation}
\delta_{2i} : y^a x^e e_{2i}^t \mapsto \begin{cases}
qK_a(q^{a+1})y^{a+a-1}x^v e_{2i-1}^t + K_a(q^{a+1})y^{a}x^{v+a-1} e_{2i-1}^t & \text{for } i \text{ even} \\
[q^{a+1} - a^{a-1}]y^{a+1} x^v e_{2i-1}^t + [q^{a+2} - 1]y^a x^{v+1} e_{2i-1}^t & \text{for } i \text{ odd}
\end{cases}
\end{equation}
\begin{equation}
\delta_{2i+1} : y^a x^e e_{2i+1}^t \mapsto \begin{cases}
[q^{a-1} - q]y^{a+1} x^v e_{2i+1}^t + K_a(q^{a+2})y^{a}x^{v+a-1} e_{2i+1}^t & \text{for } i \text{ even} \\
-qK_a(q^{a+2})y^{a+a-1}x^v e_{2i+1}^t + [q^{a+1} - 1]y^a x^{v+1} e_{2i+1}^t & \text{for } i \text{ odd}
\end{cases}
\end{equation}

where we use the convention $c_{n+1}^n = c_{n+1}^n$ = 0. This was proved in [2] in a more general setting, and by specializing $q$ and using that $x, y$ have the same nilpotency index, we obtain the above formulae (correcting an unimportant sign error in [2]).

For the following result we use this complex to compute the Hochschild cohomology of our algebra $A$, in the case when $q$ is a primitive $a$-th root of unity. The result shows that the dimensions of the cohomology groups do not depend on the characteristic of the field, except that the characteristic of $k$ does not divide $a$ since $k$ contains a primitive $a$-th root of unity.

**Theorem 3.1.** If $q$ is a primitive $a$-th root of unity, then $\dim_k HH^n(A) = 2n + 2$ for all $n \geq 0$.

**Proof.** Since $HH^0(A)$ is isomorphic to the centre of $A$, we see immediately that $HH^0(A)$ is 2-dimensional. To find the dimension of $HH^n(A)$ for $n > 0$, we compute $\ker \delta_{2t}$ for $t \geq 1$ and $\ker \delta_{2t+1}$ for $t \geq 0$.

Since $k$ contains a primitive $q$-th root of unity, the characteristic of $k$ does not divide $a$. The equalities $0 = 1 - (q^m)^a = (1 - q^m)K_a(q^m)$, valid for any integer $m$, show that $K_a(q^m) = 0$ if and only if $m$ is not divisible by $a$. We will use this fact throughout.

We first compute $\ker \delta_{2t}$ for $t \geq 1$. By the previous observation, $K_a(q^{a+1}) = 0$ if and only if $0 \leq v \leq a - 2$, whereas $K_a(q^{a+1}) = 0$ if and only if $0 \leq u \leq a - 2$. Therefore
\begin{equation}
\delta_{2t}(y^a x^e e_{2t}^t) = 0 \iff \begin{cases}
u \in \{1, 2, \ldots, a - 1\}, v \in \{1, 2, \ldots, a - 1\}, i \text{ even}, & 0 \leq i \leq 2t \\
u \in \{0, 1, \ldots, a - 2\}, v = 0, i \text{ even}, & 0 \leq i \leq 2t \\
u = 0, v \in \{0, 1, \ldots, a - 2\}, i \text{ even}, & 0 \leq i \leq 2t \\
u = 0, v = a - 1, i = 2t \\
u = a - 1, v = 0, i = 0 \\
u = a - 2, v = a - 2, i \text{ odd}, & 1 \leq i \leq 2t - 1 \\
u = a - 1, v = a - 1, i \text{ odd}, & 1 \leq i \leq 2t - 1
\end{cases}
\end{equation}
and there are $a^2t + a^2$ such elements.

Let $B := \{y^a x^e e_{2t}^t : 0 \leq u, v \leq a - 1, 0 \leq i \leq 2t\}$, a basis for $\nu A_1 \otimes A_2 P_{2t}$. We split this basis into three parts. Let $X$ be the set of basis vectors which are in the kernel of $\delta_{2t}$, so that $X$ is given by the above list. Next, let
\begin{equation}
y := \{y^a x^e e_{2t}^t : i \text{ odd}, 0 \leq u, v < a - 2\}
\end{equation}
One checks directly that $\ker(\delta_{2t}) \cap \text{Sp}(y) = \{0\}$. Let $Z := B \setminus (X \cup y)$. We find that $Z$ is equal to
$$\{x^{a-1}e_{2t}^t : 0 \leq j < t\} \cup \{y^{a-1}e_{2t}^t : 0 < j \leq t\} \cup \{y^{a-2}x^{a-1}e_{2t+1}^t : 0 \leq j < t\} \cup \{y^{a-1}x^{a-2}e_{2t+1}^t : 0 \leq j < t\}.$$
Next we compute ker $\delta_{2t+1}$ for $t \geq 0$, recall that the characteristic of $k$ does not divide $a$. We see that
\begin{equation}
\delta_{2t+1}(y^u x^v e_i^{2t+1}) = 0 \iff \begin{cases}
u = a - 1, v \in \{0, \ldots, a - 1\}, & i \text{ arbitrary} \\
u \in \{0, \ldots, a - 2\}, v = a - 1, & i \text{ arbitrary}
\end{cases}
\end{equation}
and there are $(2a - 1)(2t + 2)$ such elements.

Let $B := \{y^u x^v e_i^{2t+1}: 0 \leq u, v \leq a - 1, 0 \leq i \leq 2t + 1\}$, a basis for $\nu A \otimes_A P_{2t+1}$. We split this basis into three parts. Let $X$ be the set of basis vectors which are in the kernel of $\delta_{2t+1}$, that is $X$ is given by the above list. Next, consider
\[
Y = \{y^u x^v e_i^{2t+1}: i \text{ even}, 0 \leq v \leq a - 2\} \cup \{y^u e_i^{2t+1}: i \text{ even}, 0 \leq u \leq a - 3\}
\cup \{y^u x^v e_i^{2t+1}: i \text{ odd}, 0 \leq u \leq a - 2\} \cup \{x^v e_i^{2t+1}: i \text{ odd}, 0 \leq v \leq a - 3\}.
\]
One checks that $\text{Sp}(Y) \cap \text{Ker}(\delta_{2t+1}) = \{0\}$. Now let $Z := B \setminus (X \cup Y)$. This is the disjoint union of two sets, $Z = Z_e \cup Z_o$ where
\[
Z_e := \{y^u x^v e_i^{2t+1}: i \text{ even}, 0 \leq u \leq a - 3, 1 \leq v \leq a - 2\}
\]
\[
Z_o := \{y^u x^v e_i^{2t+1}: i \text{ odd}, 1 \leq u \leq a - 2, 0 \leq v \leq a - 3\}
\]
both of size $(a - 2)^2(t + 1)$. Then $\delta_{2t+1}(k(Z_e)) = k(\tilde{Z}_e)$ and $\delta_{2t+1}(k(Z_o)) = k(\tilde{Z}_o)$ where
\[
\tilde{Z}_e := \{y^u x^v e_i^{2t}: j \text{ even}, 1 \leq u \leq a - 2, 1 \leq v \leq a - 2\}
\]
which also has size $(a - 2)^2(t + 1)$. By the rank-nullity formula, the kernel of $\delta_{2t+1}$ restricted to $Z$ has dimension $(a - 2)^2(t + 1)$. (Note that, if $a = 2$, then $Z = \emptyset$). In total we get that $\dim_k \ker \delta_{2t+1} = (a^2 + 2)(t + 1)$.

We have now computed ker $\delta_{2t}$ for $t \geq 1$ and ker $\delta_{2t+1}$ for $t \geq 0$. Using the equalities
\begin{equation}
\dim_k \ker \delta_n + \dim_k \ker \delta_n = \dim_k \bigoplus_{i=0}^{n} (\nu A_i) e_i^n = (n + 1)a^2,
\end{equation}
we see that $\dim_k \ker \delta_{2t+1} = \dim_k \ker \delta_{2t+2} = (a^2 - 2)(t + 1)$. Consequently
\begin{equation}
\dim_k \HH^{2t+1}(A) = \dim_k \ker \delta_{2t+1} - \dim_k \ker \delta_{2t+2} = 4t + 4
\end{equation}
\begin{equation}
\dim_k \HH^{2t+2}(A) = \dim_k \ker \delta_{2t+2} - \dim_k \ker \delta_{2t+3} = 4t + 6
\end{equation}
for $t \geq 0$, and the proof is complete. \hfill \Box

This result implies immediately the following:

**Corollary 3.2.** The dimension of the cohomology groups $\HH^n(A)$ is independent of $a$.

4. **Hochschild cohomology when $a = 2$**

In this section we let $a = 2$ and $q = -1$ (and char($k$) $\neq 2$), so we have that
\begin{equation}
A = k\langle X, Y \rangle/(X^2, XY + YX, Y^2).
\end{equation}
We write $x, y$ again for the images of $X, Y$ in $A$. We also mention related work by P. A. Bergh in [1] where the main objective is to compute the homology and cohomology of $A$ with coefficients in the twisted bimodule $\epsilon A \phi$ for any $k$-linear automorphism $\phi$ of the algebra $A$.

We will simplify the differentials of the minimal projective resolution, before studying the even part of cohomology ring for this case.
4.1. Minimal projective resolution when $a = 2$. We introduce the following notation:

\begin{align}
(4.2) \quad & \beta_y = (1 \otimes y) + (y \otimes 1) \\
(4.3) \quad & \alpha_y = (1 \otimes y) - (y \otimes 1)
\end{align}

Now we can rewrite the differentials for the minimal projective resolution $\mathcal{P}$ in Equation 2.10 and 2.11; we get:

\begin{align}
(4.4) \quad & d_n(f^n_i) = \begin{cases} 
(\delta_{xy})^n(\beta_y f^{n-1}_i + \beta_x f^{n-1}_{i-1}) & \text{when } n \text{ is even} \\
(\delta_{xy})^n(\alpha_y f^{n-1}_i - \alpha_x f^{n-1}_{i-1}) & \text{when } n \text{ is odd}.
\end{cases}
\end{align}

4.2. Description of cohomology groups. In Section 3 we have seen that $\dim \text{HH}^n(A) = 2n + 2$. Knowing this, we will determine a basis for $\text{HH}^n(A)$ for arbitrary even degrees $n$. We write $\delta_{ir}$ as usual for the Kronecker symbol.

**Lemma 4.1.** Let $n = 2t$. For $r = 0, 1, \ldots, 2t$ define maps $\xi_r, \eta_r : P_{2t} \to A$ as follows.

\begin{align}
(4.5) \quad & \xi_r(f^{2t}_i) = \delta_{ir} \cdot 1_A, \quad \eta_r(f^{2t}_i) = \delta_{ir} \cdot xy.
\end{align}

(a) The classes of these maps form a basis for $\text{HH}^{2t}(A)$.

(b) The classes of the $\eta_r$ give nilpotent elements in $\text{HH}^*(A)$.

**Proof.** Part (b) will follow from Proposition 2.1. We prove now part (a). Note that these are $2n + 2$ elements, so we only have to show that the maps are in the kernel of $d'_{2t+1}$, and that they are linearly independent modulo the image of $d'_{2t}$.

1. We apply $\xi_r$ to $d_{2t+1}(f^{2t+1}_i)$, this gives

\begin{align}
(4.6) \quad & \xi_r((-1)^i(\alpha_y f^{2t}_i - \alpha_x f^{2t}_{i-1})) = (-1)^i[\alpha_y(\delta_{ir} \cdot 1_A)] - \alpha_x[\delta_{i-1,r} \cdot 1_A] = 0
\end{align}

(we view $A$ as a left $A^e$ module, and $\alpha_y \cdot 1_A = 0 = \alpha_x \cdot 1_A$). Similarly we apply $\eta_r$ to $d_{2t+1}(f^{2t+1}_i)$ and get

\begin{align}
(4.7) \quad & \eta_r((-1)^i(\alpha_y f^{2t}_i - \alpha_x f^{2t}_{i-1})) = (-1)^i[\alpha_y(\delta_{ir} \cdot xy)] - \alpha_x[\delta_{i-1,r} \cdot xy] = 0
\end{align}

(since $xy$ is in the socle of $A$ we see that $\alpha_y \cdot xy = 0$ and $\alpha_x \cdot xy = 0$).

2. Let $c_r, d_r \in K$ and $\rho : P_{2t-1} \to A$ such that

\begin{align}
(4.8) \quad & \sum_{r=0}^{2t} c_r \xi_r + d_r \eta_r = \rho \circ d_{2t} \in \text{im}(d'_{2t}).
\end{align}

We must show that $c_r = 0 = d_r$ for all $r$. Write $\rho(f^{2t-1}_i) = p_i = a_i + b_i x + c_i y + d_i xy \in A$. Then we have

\begin{align}
(4.9) \quad & \rho \circ d_{2t}(f^{2t}_i) = (-1)^i[\beta_y p_i + \beta_x p_{i-1}] = (-1)^i[2a_i y + 2a_{i-1} x]
\end{align}

which are elements in $A$. On the other hand if we apply the map given by the sum to $f^{2t}_i$ then we get

\begin{align}
(4.10) \quad & c_i + d_i xy,
\end{align}

also elements in $A$. We assume these are equal, and it follows that all scalars are zero.

$\square$
4.3. Products in even degrees of $\text{HH}^*(A)$. Recall that the even part $\text{HH}^{2*}(A)$ is a subring of the Hochschild cohomology, and it is commutative. The aim of this section is to prove the following:

**Theorem 4.2.** Let $k$ be a field with char($k$) $\neq 2$, and let $A = k\langle X, Y \rangle/(X^2, XY + YX, Y^2)$. Assume

$$\text{(4.11)} \quad R = \text{Sp}\{\xi_t^{2i} : t \geq 0, \text{ and } 0 \leq i \leq 2t\}.$$ 

Then $R$ is a subalgebra of $\text{HH}^{2*}(A)$. It is $\mathbb{Z}_2$-graded, with

$$\text{(4.12)} \quad R_0 := k\{\xi_t^{2i} : i \text{ even }\}, \quad \text{ and } R_1 := k\{\xi_t^{2i} : i \text{ odd }\}.$$ 

We have $\xi_0^{2m}\xi_r^{2t} = \xi_{t+r}^{2m+2t}$. The subalgebra $R_0$ is isomorphic to the polynomial ring $k[z_0, z_1]$ where we identify $\xi_0^2$ with $z_0$ and $\xi_1^2$ with $z_1$. Moreover, $R_1 = R_0\xi_1^2$ and $\xi_1^2 \cdot \xi_1^2 = \xi_1^4$.

**Corollary 4.3.** Let $N$ be the largest homogeneous nilpotent ideal of $\text{HH}^{2*}(A)$. Then

$$\text{(4.13)} \quad \text{HH}^{2*}(A)/N \cong R.$$ 

We fix a degree $2t$, and we will compute the product of a general element $\xi$ of degree $2t$ with an element $\chi$ of degree $2m$, and we let $2m$ vary. We take representatives $\xi : P_{2t} \to A$ and $\chi : P_{2m} \to A$ which are $k$-linear combinations of the basis. Let

$$\text{(4.14)} \quad \xi(f_i^{2t}) = p_i \in A \quad \text{ with } 0 \leq i \leq 2t$$

$$\text{(4.15)} \quad \chi(f_i^{2m}) = \bar{p}_i \in A \quad \text{ with } 0 \leq i \leq 2m.$$ 

By (4.5), the elements $p_i$ and $\bar{p}_i$ are then in the centre of $A$, we will use this freely.

**Definition 4.4.** The Yoneda product $\chi \bullet \xi$ is the residue class of $\chi \circ h_{2m}$ where the family $(h_s)$ with $h_s : P_{2t+s} \to P_s$ is a lifting of $\xi$. That is, we have the following diagram:

$$\begin{array}{c}
P_{2t+t+s} \xrightarrow{d_{2t+s}} P_{2t+2m-1} \rightarrow \cdots \rightarrow P_{2t+s} \xrightarrow{d_{2t+s}} P_{2t+s-1} \rightarrow \cdots \rightarrow P_{2t+1} \xrightarrow{d_{2t+1}} P_{2t} \\
\downarrow h_{2m} \quad \downarrow h_{2m-1} \quad \vdots \quad \downarrow h_s \quad \downarrow h_{s+1} \quad \downarrow h_1 \quad \downarrow h_0 \quad \downarrow \xi \\
P_{2m} \xrightarrow{d_{2m}} P_{2m-1} \rightarrow \cdots \rightarrow P_s \xrightarrow{d_s} P_{s-1} \rightarrow \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \\
\downarrow \chi \quad \downarrow \vdots \quad \downarrow A \\
A
\end{array}$$

where $\xi = \mu \circ h_0$ and where all squares commute. We define maps $h_s$ $(0 \leq s \leq 2m)$, and will show that they are a lifting.

$$\text{(4.16)} \quad h_s(f_i^{2t+s}) = \begin{cases} 
\sum_{j=0}^{s} p_{i-j} f_j^s & \text{ when } s \text{ even} \\
(-1)^i \left( \sum_{j=0}^{s} (-1)^j p_{i-j} f_j^s \right) & \text{ when } s \text{ odd}
\end{cases}$$

**Proposition 4.5.** The maps $h_s$ for $0 \leq s \leq 2m$ make the lifting diagram commutative, that is $d_s \circ h_s = h_{s+1} \circ d_{2t+s}$.

**Proof.** When $2t$ is fixed the proof of this result is an examination when $s$ is even and when $s$ odd, and the result follows from explicit calculations.
In particular if we let \( \xi \) show that \( R \) we apply the lifting formula and obtain the first part directly. If we take \( f \) then \( p \) then \( \chi \)

\[
(4.21) \quad (d_{s} \circ h_{s})(f_{i}^{2t+s}) = d_{s} \left( \sum_{j=0}^{s} p_{i-j}f_{j}^{s} \right) = \sum_{j=0}^{s} p_{i-j}d_{s}(f_{j}^{s}) 
= \sum_{j=0}^{s} p_{i-j}(-1)^{j} (\beta_{y}f_{j}^{s-1} + \beta_{x}f_{j-1}^{s-1}).
\]

\[
(4.18) \quad (h_{s-1} \circ d_{2t+s})(f_{i}^{2t}) = h_{s-1} \left( (-1)^{i} (\beta_{y}f_{i}^{2t+s-1} + \beta_{x}f_{i-1}^{2t+s-1}) \right) 
= \beta_{y}h_{s-1}(f_{i}^{2t+s-1}) + \beta_{x}h_{s-1}(f_{i-1}^{2t+s-1}) 
= \sum_{j=0}^{s-1} (-1)^{j} \beta_{y}p_{i-j}f_{j}^{s-1} + \sum_{j=0}^{s} (-1)^{i-1} \beta_{x}p_{i-(j+1)}f_{j}^{s-1} 
= \sum_{j=0}^{s} (-1)^{j} p_{i-j}(\beta_{y}f_{j}^{s-1} + \beta_{x}f_{j-1}^{s-1}).
\]

We observe that the expressions are equal, which deals with the odd case. In total, these show that \( (h_{s})_{s \geq 0} \) defines a lifting map.

\[
(4.17) \quad (d_{s} \circ h_{s})(f_{i}^{2t+s}) = d_{s} \left( \sum_{j=0}^{s} p_{i-j}f_{j}^{s} \right) = \sum_{j=0}^{s} p_{i-j}d_{s}(f_{j}^{s}) 
= \sum_{j=0}^{s} p_{i-j}(-1)^{j} (\beta_{y}f_{j}^{s-1} + \beta_{x}f_{j-1}^{s-1}).
\]

\[
(4.18) \quad (h_{s-1} \circ d_{2t+s})(f_{i}^{2t}) = h_{s-1} \left( (-1)^{i} (\beta_{y}f_{i}^{2t+s-1} + \beta_{x}f_{i-1}^{2t+s-1}) \right) 
= \beta_{y}h_{s-1}(f_{i}^{2t+s-1}) + \beta_{x}h_{s-1}(f_{i-1}^{2t+s-1}) 
= \sum_{j=0}^{s-1} (-1)^{j} \beta_{y}p_{i-j}f_{j}^{s-1} + \sum_{j=0}^{s} (-1)^{i-1} \beta_{x}p_{i-(j+1)}f_{j}^{s-1} 
= \sum_{j=0}^{s} (-1)^{j} p_{i-j}(\beta_{y}f_{j}^{s-1} + \beta_{x}f_{j-1}^{s-1}).
\]

The expressions are equal, which deals with the odd case. In total, these show that \( (h_{s})_{s \geq 0} \) defines a lifting map.

\[
4.4. \text{Description of Yoneda products.} \quad \text{In Section 4.2 we have described a basis for} \quad HH^{2t+2m}(A). \quad \text{Now we compute the Yoneda product of} \quad \xi \in HH^{2t}(A) \quad \text{and} \quad \chi \in HH^{2m}(A).
\]

\[
\text{Corollary 4.6.} \quad \text{Let} \quad \xi(f_{2t}) = p_{r} \in A \quad \text{and} \quad \chi(f_{2m}) = \bar{p}_{r} \in A. \quad \text{Then}
\]

\[
(4.21) \quad \chi \circ h_{2m}(f_{i}^{2t+2m}) = \sum_{0 \leq l \leq 2m \text{ and } 0 \leq i-l \leq 2t} p_{i-l} \bar{p}_{l}.
\]

In particular if we let \( \xi_{u}^{2m} \) and \( \xi_{v}^{2t} \) denote the basis elements of Lemma 4.1 then we have

\[
(4.22) \quad \xi_{u}^{2m} \cdot \xi_{v}^{2t} = \xi_{u+v}^{2t+2m}
\]

showing that \( R \) is closed under multiplication.

\[
\text{Proof.} \quad \text{We apply the lifting formula and obtain the first part directly. If we take} \quad \chi = \xi_{u}^{2m} \quad \text{and} \quad \xi = \xi_{v}^{2t} \quad \text{then} \quad p_{v} = 1 \quad \text{and} \quad p_{r} = 0 \quad \text{for} \quad r \neq v \quad \text{and similarly} \quad \bar{p}_{u} = 1 \quad \text{and} \quad \bar{p}_{r} = 0 \quad \text{otherwise. So we get that the image of} \quad f_{v}^{2t+2m} \quad \text{is} \quad 1 \quad \text{if} \quad v = u + v \quad \text{and is zero otherwise. The last part follows.} \quad \square
\]
4.5. Completing the proof of Theorem 4.2. We are left to show that $R_0$ is isomorphic to the polynomial ring $k[z_0, z_1]$, the rest follows from (4.22). We define $z_0 \mapsto \xi_0^r$ and $z_1 \mapsto \xi_2^r$; this extends to an algebra map (recall that $R_0$ is commutative). This takes $z_0^r z_1^r$ to $(\xi_0^r)^r(\xi_2^r)^s = \xi_0^{r^2} \xi_2^{rs} = \xi_2^{r^2 + 2s}$. The map is bijective: namely a general basis vector $\xi_2^{2t}$, where we must have $r \leq t$, factorizes uniquely as

$$\xi_2^{2t} = \xi_0^{2(t-r)} \xi_2^r.$$

Corollary 4.3 is a direct consequence: The intersection of $R$ with $N$ is zero, and as we have observed, any element in the span of maps $\eta_r$ is in $N$. □

5. Cohomology for $a \geq 3$

Now we study the cohomology when $a \geq 3$. Still let $q$ be an a-th root of unity and assume the algebra is

$$A = k(X,Y)/(X^a,XY - qYX,Y^a).$$

We write again $x,y$ for the images of $X,Y$ in $A$.

5.1. Differentials. We assume $a \geq 3$, then we can simplify the differentials defined in 2.10 and 2.11. We observe that the elements in $A$ introduced in 2.6 to 2.9 depend only on the parity of $s$ modulo $a$, and the arguments in 2.10 and 2.11 make only use of the cases where $s \equiv 0$ or $s \equiv 1$ modulo $a$. Using this the differentials take the following form which we will use from now:

$$(5.2) \quad d_{2t} : f_{i}^{2t} \mapsto \begin{cases} \gamma_y(0)f_{i}^{2t-1} + \gamma_x(0)f_{i+1}^{2t-1} & \text{for } i \text{ even} \\ -\tau_y(1)f_{i}^{2t-1} + \tau_x(1)f_{i+1}^{2t-1} & \text{for } i \text{ odd} \end{cases}$$

$$(5.3) \quad d_{2t+1} : f_{i}^{2t+1} \mapsto \begin{cases} \tau_y(0)f_{i}^{2t} + \gamma_x(1)f_{i+1}^{2t-1} & \text{for } i \text{ even} \\ -\gamma_y(1)f_{i}^{2t} + \tau_x(0)f_{i+1}^{2t-1} & \text{for } i \text{ odd} \end{cases}$$

where we have replaced

$$(5.4) \quad \tau_1 = \tau_x \quad \tau_2 = \tau_y \quad \gamma_1 = \gamma_x \quad \gamma_2 = \gamma_y$$

5.2. A basis for $\text{HH}^2(A)$ for $a \geq 3$. As observed the dimension of the degree $2t$ part is always $4t + 2$ which is independent of $a$. We therefore expect that there should be a basis when $a \geq 3$ which is not so different from the one we had for $a = 2$.

**Definition 5.1.** Let $\zeta_j : P_{2t} \to A$ be the map

$$(5.5) \quad \zeta_j(f_{i}^{2t}) = \begin{cases} 1 & i = j \\ 0 & \text{else.} \end{cases}$$

Let $j$ be even, then define

$$(5.6) \quad \eta^+_j(f_{i}^{2t}) = \begin{cases} x^a \gamma^a & i = j \\ 0 & \text{else.} \end{cases}$$

Now let $j$ be odd, then define

$$(5.7) \quad \eta^-_j(f_{i}^{2t}) = \begin{cases} xy & i = j \\ 0 & \text{else.} \end{cases}$$

**Lemma 5.2.** We fix a degree $2t$.

(a) The classes of the elements $\zeta_j$ and $\eta^\pm_j$ as defined above form a basis of $\text{HH}^2(A)$.

(b) The classes of the elements $\eta^\pm_j$ give nilpotent elements of $\text{HH}^*(A)$. 
We consider a linear combination of the above elements and assume that it lies in the image of \( d_{2t}^2 \), and that they are linearly independent modulo the image of \( d_{2t}^2 \).

(1) Let \( \xi \) be one of these maps. We write \( \xi(f_i^{2t}) = p_i \in A \), so that \( p_i \) is either 0 or 1 or one of \( x^{a-1}y^{a-1} \) or \( xy \) depending on the parity of \( i \). We need to check that \( \xi(d_{2t+1}(f_i^{2t+1})) = 0 \).

(a) Assume \( i \) is even, then this is equal to

\[
\xi(d_{2t+1}(f_i^{2t+1})) = \xi(\tau_y(0)f_i^{2t} + \gamma_x(1)f_i^{2t}) = \tau_y(0)p_i + \gamma_x(1)p_{i-1}.
\]

This has to be calculated in \( A \) which is viewed as an \( A^e \) left module. We have

\[
\tau_y(0)p_i = p_iy - yp_i
\]

This is zero if \( p_i = 1 \). Otherwise since \( i \) is even we only need to consider \( p_i = x^{a-1}y^{a-1} \) and then \( p_iy = 0 \) and \( yp_i = 0 \). Next, if \( p_{i-1} = 1 \) then

\[
\gamma_x(1)p_{i-1} = \sum_{j=0}^{a-1} q^j x^{a-1-j} \cdot 1 \cdot x^j = (\sum_{j=0}^{a-1} q^j)x^{a-1}
\]

and this is zero, note that \( 1 + q + \ldots + q^{a-1} = 0 \) since \( q \) is an \( a \)-th root of 1. Otherwise \( p_{i-1} = xy \) and then

\[
\gamma_x(1)p_{i-1} = \sum_{j=0}^{a-1} q^j (x^{a-1-j} y x^j)
\]

and this is a scalar multiple of \( x^a y \) and hence is zero.

(b) Let \( i \) be odd, we get

\[
\xi(-\gamma_y(1)f_i^{2t} + \tau_x(0)f_i^{2t}) = -\gamma_y(1)p_i + \gamma_x(0)p_{i-1}.
\]

By calculations similar to part (a) we see that this is zero in all cases to be considered.

(2) We consider a linear combination of the above elements and assume that it lies in the image of \( d_{2t}^2 \). Explicitly let

\[
\sum_{j=0}^{2t} c_j \xi_j + \sum_{j \text{ even}} s_j^+ \eta_j^+ + \sum_{j \text{ odd}} s_j^- \eta_j^- = \xi \circ d_{2t}
\]

where \( \xi : P_{2t-1} \rightarrow A \), with \( c_j \) and \( s_j^\pm \) in \( k \). We must show that this is only possible, as \( \xi \) varies, with all \( c_j \) and \( s_j^\pm \) equal to zero.

(a) Apply the LHS to \( f_i^{2t} \) with \( i \) even, this gives

\[
c_i + s_i^+ x^{a-1} y^{a-1}.
\]

On the other hand,

\[
\xi \circ d_{2t}(f_i^{2t}) = \gamma_y(0)\xi(f_i^{2t-1}) + \gamma_x(0)\xi(f_i^{2t-1}).
\]

This is an element in \( A \) viewed as an \( A^e \) left module. For any element \( z \in A \), \( \gamma_x(0)z \) or \( \gamma_y(0)z \) can never have a non-zero constant term since \( \gamma_x(0) \) and \( \gamma_y(0) \) are in the radical of \( A^e \). Hence the Equation (5.15) does never have a non-zero constant term and \( c_i = 0 \).

We claim that we also cannot get a term which is a multiple of \( x^{a-1} y^{a-1} \). Namely if so this could only come from either \( \gamma_y(0)x^{a-1} \) or from \( \gamma_x(0)y^{a-1} \). Now,

\[
\gamma_y(0)x^{a-1} = \sum_{j=0}^{a-1} y^j x^{a-1-j} y^{a-1-j} = \sum_{j=0}^{a-1} (-q)^j x^{a-1-j} y^{a-1-j} = 0
\]

since \( \sum_{j=0}^{a-1} q^j = 0 \). Hence \( s_i^+ = 0 \). Similarly one sees that \( \gamma_x(0)y^{a-1} = 0 \).
(b) Apply the LHS to \( f_i^{2t} \) with \( i \) odd, this gives
\[
(5.17) \quad c_i + s_i^{-}(x y).
\]

On the other hand,
\[
(5.18) \quad \xi \circ d_{2t}(f_i^{2t}) = -\tau_y(1)\xi(f_i^{2t-1}) + \tau_x(1)\xi(f_i^{2t-1}).
\]

As before, since \( \tau_y(1) \) and \( \tau_x(1) \) are in the radical of \( A^e \), this cannot have non-zero constant terms. Hence \( c_i = 0 \).

We must check that we cannot get \( x y \). If \( x y \) should occur in \( \tau_y(1) \) this can only come from \( \tau_y(1)x \) but this is equal to \( x y - q y x = 0 \). Similarly \( \tau_x(1)y = 0 \) and we do not get \( x y \). Hence \( s_i^{-} = 0 \).

We have proved that the \( 4t + 2 \) maps are linearly independent modulo the image of \( d_2^{2t} \). By dimensions, they are a basis of \( HH^{2t}(A) \). \( \square \)

The aim of this section is to prove the following.

**Theorem 5.3.** Let \( k \) be a field, \( a \geq 3 \) an integer, \( q \in k \) a primitive \( a \)-th root of unity, and \( A \) the quantum complete intersection \( k\langle X, Y \rangle/(X^a, XY - qYX, Y^a) \). Assume
\[
(5.19) \quad \mathcal{R} := S_p\mathcal{L}(\zeta_i^{2t} : t \geq 0 \text{ and } 0 \leq i \leq 2t).
\]
Then \( R \) is a subalgebra of \( HH^{2*}(A) \). It is \( \mathbb{Z}_{2} \)-graded with \( R_0 := k\langle \zeta_i^{2t} : i \text{ even} \rangle \) and \( R_1 := k\langle \zeta_i^{2t} : i \text{ odd} \rangle \). Moreover
\[
(5.20) \quad \zeta_l^{2m} : \zeta_r^{2t} = \begin{cases} 
0 & \text{if } l, r \text{ odd} \\
\zeta_{l+m+2t} & \text{otherwise}.
\end{cases}
\]

As for the case \( a = 2 \) we can see:

**Corollary 5.4.** The even part \( R_0 \) of \( R \) is isomorphic to the polynomial ring in two variables.

**Corollary 5.5.** Assume \( A \) is as in the Theorem, and let \( N \) be the largest homogeneous nilpotent ideal of \( HH^{2*}(A) \). Then \( HH^{2*}(A)/N \) is isomorphic to \( R_0 \).

5.3. **Lifting.** We compute the Yoneda product \( \chi \bullet \xi \) where \( \chi, \xi \) are \( k \)-linear combinations of maps \( \zeta_j \) as in Definition 5.1.

For \( \xi \) in the span of the \( \zeta_j \), the values of \( \xi \) are scalars and therefore they commute with elements of \( A^e \). Luckily, we are only interested in the even Hochschild cohomology modulo homogeneous nilpotent elements.

Similar as for the case where \( a = 2 \) we use liftings along the minimal projective resolution to define the Yoneda products in the cohomology ring. Let \( \xi : P_{2t} \to A \) where
\[
(5.21) \quad \xi(f_i^{2t}) := p_i
\]
and we assume \( p_i \) is a scalar multiple of 1, for all \( i \). Consequently the values \( p_i \) commute with all elements in \( A^e \). As usual we set \( p_i = 0 \) if \( i > 2t \) or if \( i < 0 \).

The map \( h_0 : P_{2t} \to P_0 \) is defined by
\[
(5.22) \quad h_0(f_i^{2t}) := p_i f_i^0 \quad (0 \leq i \leq 2t).
\]

Moreover, we search explicit formulae for maps
\[
(5.23) \quad h_s : P_{2t+s} \to P_s.
\]
For \( s \geq 1 \) we require
\[
(5.24) \quad h_{s-1} \circ d_{2t+s} = d_s \circ h_s.
\]
If so, then \( (h_s)_{s \geq 0} \) lifts \( \xi \) along the minimal projective resolution.
5.3.1. Some formulae in $A^e$. In order to define such lifting maps $h_s$ for $s > 0$ we establish some formulae in $A^e$. Let

\[ c_i = 1 + q + \ldots + q^i \quad \text{for } 0 \leq i \leq a - 2. \] (5.25)

**Definition 5.6.** For an integer $s$ we define

\[ \beta_x(s) = \sum_{i=0}^{a-2} c_i q^i (x^{a-2-i} \otimes x^i) \] (5.26)

\[ \beta_y(s) = \sum_{i=0}^{a-2} c_i q^i (y^i \otimes y^{a-2-i}) \] (5.27)

Recall now the elements in $A^e$ which occur in the definition of the differentials:

\[ \gamma_y(s) = \sum_{j=0}^{a-1} q^j (y^j \otimes y^{a-1-j}) \] (5.28)

\[ \gamma_x(s) = \sum_{j=0}^{a-1} q^j (x^{a-1-j} \otimes x^j) \] (5.29)

At the end we will only need $s = 0$ and $s = 1$. Recall also

\[ \tau_y(1) = (1 \otimes y) - q(y \otimes 1) \] (5.30)

\[ \tau_x(1) = q(1 \otimes x) - (x \otimes 1) \] (5.31)

\[ \tau_y(0) = (1 \otimes y) - (y \otimes 1) \] (5.32)

\[ \tau_x(0) = (1 \otimes x) - (x \otimes 1). \] (5.33)

With this notation, we will define maps $h_s : P_{2t+s} \to P_s$, defined on the generators $f_{i}^{2t+s}$ of the free $A^e$ module $P_{2t+s}$, and we will show below that they lift $\xi$:

**Definition 5.7.**

Assume $s$ is even. For an integer $i$ we define the following elements in the algebra,

\[ \omega(j) = \begin{cases} 
\beta_x(-1)\beta_y(1) & j \text{ odd} \\
1 & j \text{ even}
\end{cases} \] (5.34)

With this, we define for $s$ even

\[ h_s(f_i^{2t+s}) := \begin{cases} 
\sum_{j=0}^{s} p_{i-j} \omega(j) f_j^s & i \text{ even} \\
\sum_{j=0}^{s} p_{i-j} f_j^s & i \text{ odd}
\end{cases} \] (5.35)

Now assume $s$ is odd. Here we need two parameters in $A^e$, one for $x$ and one for $y$. We set

\[ \varepsilon_x(j) = \begin{cases} 
-\beta_x(0) & j \text{ odd} \\
1 & j \text{ even}
\end{cases} \]

\[ \varepsilon_y(j) = \begin{cases} 
1 & j \text{ odd} \\
-\beta_y(0) & j \text{ even}
\end{cases} \] (5.36)

With these, we define for $s$ odd,

\[ h_s(f_i^{2t+s}) := \begin{cases} 
\sum_{j=0}^{s} p_{i-j} \varepsilon_x(j) f_j^s & i \text{ even} \\
\sum_{j=0}^{s} p_{i-j} \varepsilon_y(j) f_j^s & i \text{ odd}
\end{cases} \] (5.37)

We will show that $(h_s)_{s \geq 0}$ is a lifting for $\xi$. For this, we need some formulae.
Lemma 5.8. We have that the following relations hold:

(a) \( \beta_y(1) \tau_y(1) = \gamma_y(2) \)

(b) \( \beta_x(-1) \gamma_y(2) = \gamma_y(0) \beta_x(0) \)

(c) \( \beta_x(-1) \beta_y(1) = \beta_y(-1) \beta_x(1) \)

(d) \( \beta_x(1) \tau_x(1) = -\gamma_x(2) \)

(e) \( \beta_y(-1) \gamma_x(2) = \gamma_x(0) \beta_y(0) \)

(f) \( \beta_y(0) \tau_y(0) = \gamma_y(1) \)

(g) \( \beta_x(0) \tau_x(0) = -\gamma_x(1) \)

(h) \( \tau_y(1) \beta_y(0) = \gamma_y(0) \)

(i) \( \tau_x(0) \beta_x(-1) = -\gamma_x(-1) \)

(j) \( \gamma_x(-1) \beta_y(1) = \beta_y(0) \gamma_x(1) \)

(k) \( \tau_y(0) \beta_y(-1) = \gamma_y(-1) \)

(l) \( \tau_x(1) \beta_x(0) = -\gamma_x(0) \)

(m) \( \beta_x(0) \gamma_y(1) = \gamma_y(-1) \beta_x(1) \)

Proof. We prove (a) and (b), and the other relations follows from the same kind of reasoning. Start with (a), we have

\[
\beta_y(1) \tau_y(1) = \left( \sum_{i=0}^{a-2} c_i q^i (y^i \otimes y^{a-2-i}) \right) ((1 \otimes y) - q(y \otimes 1))
\]

\[
= \sum_{i=0}^{a-2} \left( c_i q^i (y^i \otimes y^{a-1-i}) - c_i q^{i+1} (y^{i+1} \otimes y^{a-2-i}) \right)
\]

\[
= c_0 (1 \otimes y^{a-1}) + c_1 q(y \otimes y^{a-2}) + \cdots + c_{a-2} q^{a-2} (y^{a-2} \otimes y) - c_0 q(y \otimes y^{a-2}) - \cdots - c_{a-2} q^{a-2} (y^{a-2} \otimes y) - c_{a-2} q^{a-1} (y^{a-1} \otimes 1)
\]

\[
= c_0 (1 \otimes y^{a-1}) + q(c_1 - c_0)(y \otimes y^{a-2}) + q^2(c_2 - c_1)(y^2 \otimes y^{a-3}) + \cdots + q^{a-2}(c_{a-2} - c_{a-3})(y^{a-2} \otimes y) - q^{a-1}c_{a-2}(y^{a-1} \otimes 1)
\]

where we have that \( c_0 = 1, \ c_1 - c_0 = 1 + q - 1 = q, \ldots \)

\[
c_{i+1} - c_i = (1 + q + \cdots + q^{i+1}) - (1 + q + \cdots + q^i) = q^{i+1}
\]

We also observe

\[
c_{a-2} = 1 + q + \cdots + q^{a-2} = -q^{a-1}
\]

since \( a \) is a root of unity and hence \( 1 + q + \cdots + q^{a-2} + q^{a-1} = 0 \). Then we have,

\[
\beta_y(1) \tau_y(1) = (1 \otimes y^{a-1}) + q^2(y \otimes y^{a-2}) + \cdots + q^{2(a-1)}(y^{a-1} \otimes 1) = \gamma_y(2)
\]

For the relation (b) we inspect a typical element in this sum:

\[
\beta_x(1) \tau_x(1) = (1 \otimes y^{a-1}) + q^2(y \otimes y^{a-2}) + \cdots + q^{2(a-1)}(y^{a-1} \otimes 1)
\]

(5.47)

where \( * \) denotes the multiplication in \( A^{op} \). Now we recall that \( xy = qyx \) and \( x \ast y = q^{-1}yx \) hence \( x^{a-2-i}y^j = q^{i(a-2-i)}y^jx^{a-2-i} \) and \( x^i \ast y^{a-1-j} = q^{-i(a-1-j)}y^{a-1-j}x^i \). We get

\[
c_i q^{-i}q^{2j}q^{i(a-2-i)}q^{i(a-1-j)}(y^i x^{a-2-i} \otimes y^{a-1-j} \ast x^i) = (y^i \otimes y^{a-2-j})c_i(x^{a-2-i} \otimes x^i)
\]

(5.48)

which is the most typical element in the sum \( \gamma_y(0) \beta_x(0) \). \( \square \)
The relations (a) to (m) in Lemma 5.8 can be used to prove that the maps $h_s$ are liftings for the given map $\xi$:

**Proposition 5.9.** The lifting formulas make the suggested squares commutative, that is $h_{s-1} \circ d_{2t+s} = d_s \circ h_s$ when $s \geq 1$ and $\xi = \mu \circ h_0$.

**Proof.** We give details when $s$ and $i$ are even, the other cases are similar. The strategy is to apply both sides to $f_j^{2t+s}$ and express the answer in terms of the basis $\{f_j^{s-1}\}$, with coefficients in $A^e$ and then show that the coefficients of the $f_j^{s-1}$ in the two expressions are equal.

We have

\begin{equation}
(d_s \circ h_s)(f_j^{2t+s}) = d_s \left( \sum_{j=0}^s p_{i-j}\omega(j)f_j^s \right)
\end{equation}

\begin{equation}
= \sum_{\text{even}, 0 \leq j \leq s} p_{i-j}\omega(j) \left[ \gamma_y(0)f_j^{s-1} + \gamma_x(0)f_j^{s-1} \right]
\end{equation}

\begin{equation}
+ \sum_{\text{odd}, 0 \leq j \leq s} p_{i-j}\omega(j) \left[ -\gamma_y(1)f_j^{s-1} + \tau_x(1)f_j^{s-1} \right]
\end{equation}

We must show that for each $j$ the coefficients of $f_j^{s-1}$ in (5.54) and in (5.57) are equal.

(a) Assume first $j$ is even. We require

\begin{equation}
p_{i-j}\gamma_y(0) + p_{i-j-1}\omega(j+1)\tau_x(1) = p_{i-j}\gamma_y(0)\varepsilon_x(j) + p_{i-j-1}\gamma_x(0)\varepsilon_y(j)
\end{equation}

For $j$ even, $\varepsilon_x(j) = 1$ and the first terms agree. The second terms agree provided

\begin{equation}
\omega(j+1)\tau_x(1) = \gamma_x(0)\varepsilon_y(j)
\end{equation}
Consider the LHS, by identities (c), (d) and (e) it is equal to
\begin{equation}
\beta_y(-1)\beta_x(1)\tau_y(1) = -\beta_y(-1)\gamma_x(2) = -\gamma_x(0)\beta_y(0) = \gamma_x(0)\epsilon_y(j)
\end{equation}

from the definition of \(\epsilon_y(j)\) in this case. Hence the second terms agree as well.

(b) Now assume \(j\) is odd. We require
\begin{equation}
p_{i-j-1}\gamma_x(0) - p_{i-j}\omega(j)\tau_y(1) = p_{i-j}\gamma_y(0)\epsilon_x(j) + p_{i-1-j}\gamma_x(0)\epsilon_y(j)
\end{equation}

For \(j\) odd, \(\epsilon_y(j) = 1\) and the terms with \(p_{i-j-1}\) agree. For the other two terms to agree we need
\begin{equation}
\gamma_y(0)\epsilon_x(j) = -\omega(j)\tau_y(1)
\end{equation}

We have using the definition and identities (a) and (b) that
\begin{equation}
-\omega(j)\tau_y(1) = -\beta_x(-1)\beta_y(1)\tau_y(1) = -\beta_x(-1)\gamma_y(2) = -\gamma_y(0)\beta_x(0) = \gamma_y(0)\epsilon_x(j)
\end{equation}
as required.

Similar as for the case \(a = 2\) we define the Yoneda product of the residue classes represented by \(\xi\) of degree \(2t\) and \(\chi\) of degree \(2m\) to be the residue class represented by the composition
\begin{equation}
\xi \cdot \chi = \chi \circ h_{2m}.
\end{equation}

5.4. Description of Yoneda products of basis elements when \(a \geq 3\). In the definition 5.7 of the lifting maps, we have the term \(\omega(j) = \beta_x(-1)\beta_y(1) \in \mathbb{A}^e\) (for \(j\) odd). When this is evaluated in \(A\), it becomes \(\omega(j) \cdot 1_A\). We claim that this is always zero, in fact \(\beta_y(1) \cdot 1_A = 0\).

Namely, we must view \(A\) as an \(\mathbb{A}^e\) bimodule and then
\begin{equation}
\beta_y(1) \cdot 1_A = \sum_{i=0}^{a-2} c_i q^i y^{a-2} = \left(\sum_{i=0}^{a-2} c_i q^i\right) y^{a-2}
\end{equation}
The following shows that this is zero:

**Lemma 5.10.** Let \(q\) be a primitive \(a\)-th root of unity for \(a \geq 3\). Let \(c_i = 1 + q + \ldots + q^i\) for \(i \geq 0\), then
\begin{equation}
\sum_{i=0}^{a-2} c_i q^i = 0
\end{equation}

**Proof.** Set also \(c_{-1} := 0\). Then we have for \(i \geq 0\) that \(c_i - c_{i-1} = q^i\). We get
\begin{equation}
\sum_{i=0}^{a-2} c_i q^i = \sum_{i} c_i (c_i - c_{i-1})
\end{equation}

Therefore (all summations are from \(i = 0\) to \(a - 2\))
\begin{align*}
(1 + q)(\sum_i c_i q^i) &= \sum_i c_i q^i + \sum_i c_i q^{i+1} \\
&= \sum_i c_i c_{i-1} + \sum_i c_i (c_{i+1} - c_i) \\
&= \sum_i (c_i c_{i+1} - c_{i+1} c_{i-1}) \\
&= c_{a-2} c_{a-1} - c_0 c_{-1} \\
&= 0
\end{align*}
since \(c_{a-1} = 1 + q + \ldots + q^{a-1} = 0\) and \(c_{-1} = 0\). But \(q \neq -1\), so we can cancel by \((1 + q)\) and get the claim. \(\square\)
We analyse now the products, and this will complete the proof of Theorem 5.3. Define

\[(5.68)\quad R = Sp\{\omega^t : t \geq 0, 0 \leq i \leq 2t\}.
\]

We compute products of elements in \(R\).

Let \(\chi\) be of degree \(2m\) and \(\xi\) of degree \(2t\), both in \(R\). Let \(\xi(f_j^{2t}) = p_l \in K\) for \(0 \leq i \leq 2t\) and \(\chi(f_j^{2m}) = \bar{p}_j \in K\) for \(0 \leq j \leq 2m\). As before we set \(p_l = 0\) for \(i < 0\) or \(i > 2t\), and similarly we define \(\bar{p}_j\) for any \(j \in \mathbb{Z}\).

Then \(\chi \circ \xi\) is the class of \(\chi \circ h_{2m}\), where \((h_s)\) is a lifting of \(\xi\), with the formula computed above. Note that we only need the case when \(s = 2m\) is even. We have

\[(5.69)\quad \chi \circ h_s(f_j^{2t+s}) = \chi(\sum_{j=0}^{s} p_i - j \omega(j) f_j^i) = \begin{cases} \sum_{i=0}^{s} p_i - j \omega(j) \bar{p}_j & \text{if } i \text{ even} \\ \sum_{i=0}^{s} p_i - j \bar{p}_j & \text{if } i \text{ odd.} \end{cases}
\]

Now assume \(\chi = \zeta_l\) for some \(0 \leq l \leq s\), so \(\bar{p}_l = 1\) and \(\bar{p}_l = 0\) otherwise. Then the above simplifies to

\[(5.70)\quad f_j^{2t+s} \mapsto \begin{cases} p_l - l \omega(l) \cdot 1 & \text{if } i \text{ even} \\ p_l - 1 \cdot 1 & \text{if } i \text{ odd.} \end{cases}
\]

Now take \(\xi = \zeta_r\) for some \(0 \leq r \leq 2t\). Then \(p_{l-r} = 1\) if \(i - l = r\), and \(0\) otherwise.

Note that \(\omega(l) \cdot 1_A\) is zero for \(l\) odd and is equal to \(1\) otherwise. The zero occurs precisely when \(l\) is odd and \(i = l + r\) is even, i.e. if both \(l, r\) are odd. So we get

\[(5.71)\quad \zeta_l^{2m} \cdot \zeta_r^{2t} = \begin{cases} \zeta_l^{2m+2t} & l, r \text{ not both odd} \\ 0 & l, r \text{ odd.} \end{cases}
\]

As for the case \(a = 2\) we see that \(R_0\) is isomorphic to the polynomial ring in two variables.

Furthermore, we see that elements in \(R_1\) are nilpotent. The subalgebra \(R_0\) intersects the largest homogeneous nilpotent ideal \(N\) trivially, and the span of the \(\eta^\pm\) is contained in \(N\).

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