THE VIOLATION OF THE LIPMAN–ZARISKI CONJECTURE IN POSITIVE CHARACTERISTIC

PATRICK GRAF

Abstract. We study the failure of the Lipman–Zariski conjecture in positive characteristic. For rational double points, the conjecture holds true except for a short finite list of exceptions. For log canonical surface singularities, the conjecture continues to hold with the same list of exceptions under an additional tameness hypothesis. In particular, among rational double points in characteristic \(p \geq 7\), Lipman’s counterexample is the only one, and the conjecture holds for all tame \(F\)-pure normal surface singularities.

1. Introduction

The LZ (= Lipman–Zariski) conjecture asserts that a complex algebraic variety \(X\) with locally free tangent sheaf \(\mathcal{T}_X\) is necessarily smooth [Lip65]. Here \(\mathcal{T}_X = \mathcal{H}om_{O_X}(\Omega_X^1, O_X)\) is the dual of the sheaf of Kähler differentials. The case where \(X\) is a normal surface is essentially the only open case left, due to Lipman [Lip65, Thm. 3], Becker [Bec78, Sec. 8, p. 519], and Flenner [Fle88, Corollary]. On the other hand, Lipman already observed that the conjecture fails in positive characteristic. To be more precise, he gave the following (series of) counterexamples: consider the surface \(X \subset \mathbb{A}^3_k\) over a perfect field \(k\) of characteristic \(p\) defined by the equation

\[xy - z^n = 0, \quad \text{where } p \text{ divides } n.\]

Then one can see that \(\mathcal{T}_X\) is freely generated by the two derivations \(\delta_1 = x\partial_x - y\partial_y\) and \(\delta_2 = \partial_z\), cf. [Lip65, p. 892].

The purpose of the present work is a more detailed study of the Lipman–Zariski conjecture on (normal) surfaces in positive characteristic. First, in Proposition 1.1 below we give sufficient conditions for it to hold in a given situation. We take our clue from the case \(k = \mathbb{C}\). In this case, the Lipman–Zariski conjecture is closely related to the problem whether 1-forms on the smooth locus of \(X\) extend to a (log) resolution of singularities \(\pi: Y \to X\) (with exceptional locus \(E\)). In fact, we have the following chain of implications (dim \(X\) arbitrary, \(k = \mathbb{C}\)):

\[
\begin{align*}
\text{The sheaf } \pi_*(\Omega^1_Y) & \text{ is reflexive} \\
& \quad \Downarrow \quad [\text{GK14, Thm. 3.1}] \\
\text{The sheaf } \pi_*\Omega^1_Y & \text{ is reflexive} \\
& \quad \Downarrow \quad [\text{SvS85, (1.6)}] \\
\text{The Lipman–Zariski conjecture holds for } X
\end{align*}
\]

Date: May 6, 2022.

2010 Mathematics Subject Classification. 13N15, 14J17.

Key words and phrases. Lipman–Zariski conjecture, positive characteristic, rational double points, log canonical surface singularities.
If $X$ is a surface over a field of positive characteristic, neither of the above implications holds. An example where the first implication fails is given in [Gra21, Ex. 10.2]. It has the bad behaviour that both $\pi_!\Omega^1_Y(\log E)$ and $\pi_!\omega_Y$ are reflexive, but $\pi_!\Omega^1_Y$ is not. However, in [Gra21, Thm. 1.3] we showed that the first implication remains valid if $X$ is tame (cf. Definition 2.1).

The problem with the second implication is that, unlike in characteristic zero, a derivation $\delta$ on a variety $X$ may not lift to any resolution of $X$, intuitively because it need not preserve the singular locus $X_{sg}$ – or the Jacobian ideal of $X$, for that matter. As an example, consider $\delta = \partial_z$ on Lipman’s example above. If the failure of this lifting property is “not too bad”, we say that the MGR (= minimal good resolution) is almost equivariant, cf. Definition 2.3. With this notation, our first result is as follows.

**Proposition 1.1** (Sufficient conditions for LZ to hold). Let $(0 \in X)$ be a normal surface singularity over an algebraically closed field of characteristic $p > 0$, with $\pi : Y \to X$ the MGR and $E = \text{Exc}(\pi)$. Assume that

1. $\pi_!\Omega^1_Y(\log E)$ is reflexive,
2. $(0 \in X)$ is tame, and
3. $\pi$ is almost equivariant.

Then $(0 \in X)$ satisfies the Lipman–Zariski conjecture. That is, if $\mathcal{T}_X$ is free then $(0 \in X)$ is smooth.

**Remark 1.2.** Of the above conditions, (1.1.1) is probably the most restrictive, but in [Gra21] we gave sufficient conditions for it to hold. Determining whether (1.1.2) holds is easy from the dual graph. The techniques of [Wah75, §2] in principle allow to compute equivariance of $\pi$, and that paper also has some sufficient conditions. Furthermore, [Hir19] has completely determined the almost equivariant rational double points.

With Proposition 1.1 at hand, we investigate the following (vague) question:

**Question 1.3.** Is Lipman’s counterexample in any sense unique, or does the failure of the Lipman–Zariski conjecture rather occur “generically”?

Since Lipman’s example is an RDP (= rational double point), we first concentrate on this class of singularities. Here we obtain the following:

**Theorem 1.4** (LZ conjecture for RDPs). Let $(0 \in X)$ be a two-dimensional RDP over an algebraically closed field $k$ of characteristic $p > 0$. Then $(0 \in X)$ satisfies the Lipman–Zariski conjecture, except in the following cases:

\begin{align*}
A_n, & \quad n \equiv -1 \mod p, \quad p \text{ arbitrary}, \\
D_n, & \quad n \geq 4, \quad p = 2, \\
E_6, & \quad p = 2, 3, \\
E_7, & \quad p = 2, 3, \\
E_8, & \quad p = 2, 3, 5.
\end{align*}

(1.4.1)

Conversely, in each of the above classes, there does exist an RDP that violates the Lipman–Zariski conjecture.

The first item in the above list is precisely Lipman’s example. We see in particular that his example is indeed unique at least among RDPs in characteristic $p \geq 7$, providing a partial answer to Question 1.3.

**Remark.** In Theorem 1.4, we do not claim that every RDP in (1.4.1) violates the Lipman–Zariski conjecture. For example in characteristic $p = 2$, the tangent sheaf
of $E_0^0$, $E_0^1$ and $E_2^2$ is free, while for $E_8^3$ and $E_8^5$ it needs four generators (cf. Appendix A for notation).

Remark 1.5 (Fundamental groups). The local fundamental groups of RDPs are known in all characteristics by [Dur79] and [Art77]. Therefore, it is reasonable to ask whether the counterexamples in Theorem 1.4 are exactly those for which the local fundamental groups are smaller than the corresponding characteristic zero group. This is indeed true for the $A_n$ singularities, but not in general. For example, all $E_8$ singularities ($p \leq 5$) have “too small” fundamental group, but some of them do satisfy the Lipman–Zariski conjecture. It is true in general that if the fundamental group is the same as over $\mathbb{C}$, then the Lipman–Zariski conjecture is satisfied. But in low characteristics, this applies only to a few cases.

The equations of all RDPs over an algebraically closed field are known by [Art77]. Therefore, Theorem 1.4 could also have been obtained by computer calculations and in fact, this is what we do in Appendix A. But the point of having a conceptual argument is that it can also be applied in situations beyond RDPs, such as log canonical singularities. This is what we are going to do next.

Theorem 1.6 (LZ conjecture for lc surfaces). Let $(0 \in X)$ be a log canonical surface singularity over an algebraically closed field of characteristic $p > 0$. Assume that $(0 \in X)$ is tame and that it is not an RDP contained in (1.4.1). Then $(0 \in X)$ satisfies the Lipman–Zariski conjecture.

Example 4.2 shows that the statement of Theorem 1.6 is sharp in the sense that the tameness assumption cannot be dropped, even if $X$ is $F$-pure. In terms of Question 1.3, this suggests that if we do not impose a tameness condition, then Lipman’s example (which itself is not tame) is very far from being unique.

Going back to the tame setting, by combining Theorem 1.6 with the calculations in Appendix A, we arrive at the following corollary.

Corollary 1.7 (LZ conjecture for tame lc surfaces). Tame log canonical surface singularities over an algebraically closed field satisfy the Lipman–Zariski conjecture, except for precisely seven RDPs of type $E_n$ in characteristic $p \leq 5$:
- $E_0^0, E_0^1, E_2^2, E_8^5$ (p = 2),
- $E_2^1, E_8^0$ (p = 3),
- $E_8^0$ (p = 5).

Since none of the above “exceptional” RDPs are $F$-pure, we also have:

Corollary 1.8 (LZ conjecture for tame $F$-pure surfaces). The Lipman–Zariski conjecture holds for tame $F$-pure normal surface singularities over an algebraically closed field of characteristic $p > 0$.

It would be nice to have a direct proof of Corollary 1.8 that does not rely on any case-by-case analysis of explicit equations. One might try to show that $F$-pure singularities satisfy the Logarithmic Extension Theorem and that they are almost equivariant, in order to apply Proposition 1.1. The first condition is probably true and could be proven using the classification results of [MS91] and [Har98]. Unfortunately, the latter condition fails: the rational double point $D_{2n-1}^n$ (p = 2) is $F$-pure, but not almost equivariant.

Acknowledgements. This work was begun, and completed to about 69%, while the author stayed at the University of Utah in 2018/19, funded by a research fellowship of the DFG (= Deutsche Forschungsgemeinschaft). I would like to thank the anonymous referee for her/his precise suggestions for improving the paper.
2. Basic definitions

By a (normal) surface singularity \((0 \in X)\) defined over a field \(k\) we mean a scheme of the form \(X = \text{Spec } R\), where \((R, m)\) is a two-dimensional excellent (normal) local ring containing a field isomorphic to \(k = R/m\). By [Lip78, Thm.], a normal surface singularity admits a resolution, and hence also a unique minimal good resolution.

Definition 2.1. Let \((0 \in X)\) be a normal surface singularity defined over a field \(k\). We say that \((0 \in X)\) is tame if for every resolution of singularities \(\pi: Y \to X\) with exceptional curves \(E_1, \ldots, E_\ell\), the determinant of the intersection matrix \((E_i \cdot E_j)\) is not divisible by \(\text{char } k\).

By the calculations in [Gra21, Sec. 8.A], it suffices to check tameness on a single resolution. We also recall the following notation from [Gra21] in the special case of surface singularities.

Definition 2.2 (Extension Theorems). Let \((0 \in X)\) be a normal surface singularity defined over a field \(k\).

- We say that \((0 \in X)\) satisfies the Regular Extension Theorem (for 1-forms) if for some/any log resolution \(\pi: Y \to X\) with exceptional divisor \(E = \text{Exc}(\pi)\), the natural inclusion
  \[\pi_* \Omega^1_{Y/k} \hookrightarrow \Omega^1_{X/k} \] is an isomorphism. Equivalently, the sheaf \(\pi_* \Omega^1_{Y/k}\) is reflexive.

- We say that \((0 \in X)\) satisfies the Logarithmic Extension Theorem (for 1-forms) if for some/any \(\pi\) as above, the natural inclusion
  \[\pi_* \Omega^1_{Y/k}(\log E) \hookrightarrow \Omega^1_{X/k}\] is an isomorphism. Equivalently, the sheaf \(\pi_* \Omega^1_{Y/k}(\log E)\) is reflexive.

Definition 2.3 (Almost equivariant MGR). Let \((0 \in X)\) be a normal surface singularity defined over a field \(k\), with minimal good resolution \(\pi: Y \to X\). Consider the natural short exact sequence of coherent sheaves

\[0 \to \pi_* \mathcal{R}_Y \to \mathcal{R}_X \to \mathcal{Q} \to 0.\]

We say that \(\pi\) is almost equivariant if \(\dim \mathcal{Q}_0 \leq 1\). Equivalently, and purely in terms of \(Y\),

\[\dim \mathbb{C} H^0(Y \setminus E, \mathcal{R}_Y) / H^0(Y, \mathcal{R}_Y) \leq 1,\]

where \(E \subset Y\) is the exceptional locus of \(\pi\).

The name stems from the case where \(\mathcal{Q} = 0\), or equivalently, when the sheaf \(\pi_* \mathcal{R}_Y\) is reflexive. In this case one says that \(\pi\) is equivariant [Wah75]. This terminology comes from the observation that if \(k = \mathbb{C}\) and \(X\) is compact, then the identity component of the automorphism group \(\text{Aut}^o(X)\) will act on \(Y\) in such a way that \(\pi\) is equivariant. For more information on this topic, see e.g. [GKK10, Sec. 4].

3. The Logarithmic Extension Theorem for rational double points

In this section, we determine which RDPs satisfy the Logarithmic Extension Theorem. This is necessary for the application of Proposition 1.1 in the proof of Theorem 1.4.
**Theorem 3.1** (Logarithmic Extension Theorem for RDPs). Let \((0 \in X)\) be a two-dimensional RDP over an algebraically closed field \(k\) of characteristic \(p > 0\). Then \((0 \in X)\) satisfies the Logarithmic Extension Theorem, except in the following cases:

\[
\begin{align*}
D_n, & \quad n \geq 4, \quad p = 2, \\
E_6, & \quad p = 2, 3, \\
E_7, & \quad p = 2, 3, \\
E_8, & \quad p = 2, 3, 5.
\end{align*}
\]

Conversely, in each of the above classes, there does exist an RDP that violates the Logarithmic Extension Theorem.

Note that not every singularity in (3.1.1) violates the Logarithmic Extension Theorem. E.g. by Example 3.3 below, the \((F\text{-pure})\) singularity \(D_{2n-1}^n\) in characteristic \(p = 2\) satisfies it.

**Proof of Theorem 3.1.** If \(p \geq 7\), the claim is a special case of [Gra21, Thm. 1.2]. Hence the only cases left are \(A_n\) \((p = 2, 3, 5)\), \(D_n\) \((p = 3, 5)\), and \(E_{6, 7}\) \((p = 5)\). For \(A_n\), logarithmic extension immediately follows from [Gra21, Thm. 6.1]. For \(D_n\) and \(E_{6, 7}\), the arguments from the proof of [Gra21, Thm. 1.2] still apply because for the primes in question there does exist a “tame resolution”, e.g. for \(D_n\) only \(p = 2\) needs to be excluded. Cf. in particular cases (7.8.6) and (7.8.7) in that proof.

It remains to check the second half of the statement. This is done in the following examples. □

**Example 3.2 \((D_n\) singularities).** Fix a field \(k\) of characteristic \(p = 2\). Then for the \(D_{2n}^n\) singularity \(X = \{ f = z^2 + x^2 y + x y^n = 0 \} \subset \mathbb{A}^3_k\), \(n \geq 2\) arbitrary, the Logarithmic Extension Theorem does not hold. More precisely, note that Kähler differentials on \(X\) satisfy the relation \(y^n dx + (x^2 + nxy^{n-1})dy = 0\) and hence we may consider the (a priori only rational) 1-form

\[
\sigma = y^{-n} dy = (x^2 + nxy^{n-1})^{-1} dx.
\]

As \(y\) and \(x^2 + nxy^{n-1}\) vanish simultaneously only at the origin, \(\sigma\) is in fact a regular 1-form on \(X \setminus \{0\} = X_{\text{reg}}\). In other words, \(\sigma \in H^0(X, \Omega_X^{[1]})\) is a reflexive differential form on \(X\). We blow up the origin \(n - 1\) times in a row, yielding a map \(\varphi : \mathbb{A}^3_k \to \mathbb{A}^3_k\).

In suitable coordinates, this map is given by

\[
\varphi(u, v, w) = (u^{n-1} v, v^{-1} w).
\]

We compute

\[
\varphi^* (f) = v^{2n-2} u^2 + u^2 v^{2n-1} + uv^{2n-1} = v^{2n-2} \cdot \left( \frac{w^2}{w(u+1)} \right) \quad \text{equation of strict transform } \tilde{X} \text{ of } X
\]

We see that \(\tilde{X}\) can be parametrized rationally by the \((u, w)\)-plane, namely by setting \(v = \frac{u^2}{u(u+1)}\). In this parametrization, the pullback of \(\sigma\) is given by

\[
\varphi^* (\sigma) = v^{-n} dv = \left( \frac{w^2}{u(u+1)} \right)^{-n} d \left( \frac{w^2}{u(u+1)} \right) = \cdots = (u(u+1))^{n-2} \frac{du}{u^{2n-2}}.
\]

This shows that the extension of \(\sigma\) to \(\tilde{X}\) does not have at most logarithmic poles. Even worse, as \(n\) goes to infinity, the pole orders become arbitrarily large.

Similar calculations for the \(D_{2n+1}^n\) singularities \(\{ z^2 + x^2 y + y^n z = 0 \} \) show that the reflexive form \(\sigma = y^{-n} dy = (x^2 + ny^{n-1} z)^{-1} dz\) does not extend logarithmically.
Example 3.3 (The $D_{2n}^{*}$ singularity). According to the SINGULAR program in Appendix B, for the $D_{2n}^{*}$ singularity $X = \{ z^2 + x^2y + xy^2 + xz^2 = 0 \}$ the tangent sheaf is freely generated by
\[
\begin{align*}
v_1 &= y \partial_y + (x + z + ny^{n-1}) \partial_z, \\
v_2 &= x \partial_x + (z + y^{n-1}) \partial_z.
\end{align*}
\]
Consider the 1-forms $\alpha_1 = d \log y := y^{-1} dy$ and $\alpha_2 = d \log x := x^{-1} dx$. Then we have $\alpha_1(v_1) = \delta_{ij}$ and hence $\{\alpha_1, \alpha_2\}$ is a basis of $\Omega_X^{[1]}$. (Note that e.g. $\alpha_1$ does not actually have a logarithmic pole along $\{y = 0\}$ because $y$ vanishes to order two along that divisor.) Clearly, the pullback of each $\alpha_i$ to any resolution has only logarithmic poles and so we see that $X$ satisfies the Logarithmic Extension Theorem.

Example 3.4 ($E_n$ singularities). Due to their similarity to Example 3.2, we omit the calculations and only write down in each case the defining equation and a reflexive form that does not extend logarithmically. We only treat $E_6$ and $E_7$ ($p = 2, 3$) since $E_8$ ($p = 2, 3, 5$) has already been done in [Gra21, Ex. 10.1].

- $E_6, p = 2$: $f = z^2 + x^3 + y^3$, $\sigma = y^{-1} dx = x^{-1} dz$,
- $E_7, p = 2$: $f = z^2 + x^3 + xy^2$, $\sigma = x^{-1} y^{-2} dx = (x^2 + y^3)^{-1} dy$,
- $E_6, p = 3$: $f = z^2 + x^3 + y^4$, $\sigma = z^{-1} dy = y^{-3} dz$,
- $E_7, p = 3$: $f = z^2 + x^3 + y^3$, $\sigma = z^{-1} dx = y^{-3} dz$.

4. Proof of main results

Proof of Proposition 1.1. Since $(0 \in X)$ is tame and satisfies the Logarithmic Extension Theorem, it also satisfies the Regular Extension Theorem by [Gra21, Thm. 1.3]. In other words, we have $H^0(X, \Omega_X^{[1]}) = H^0(Y, \Omega_Y^{[1]})$.

Assume now that $\mathcal{R}_X$ is freely generated by the two derivations $\{v_1, v_2\}$. Since $\pi: Y \to X$ is almost equivariant by assumption, the images of the $v_i$ in $\mathcal{R}_X/\pi_* \mathcal{R}_Y$ are linearly dependent, say $v_1 + \lambda v_2 = 0$ for some $\lambda \in \mathbb{C}$. After replacing $v_1$ by $v_1 + \lambda v_2$, we may thus assume that $v_1$ lifts to $\tilde{v}_1 \in H^0(Y, \mathcal{R}_Y)$. Furthermore, in view of the short exact sequence
\[
0 \to \mathcal{R}_Y(-\log E) \to \mathcal{R}_Y \to \bigoplus_{i=1}^{\ell} N_{E_i/Y} \to 0
\]
and the negativity of the self-intersections $E_i^2 < 0$, it is clear that $H^0(Y, \mathcal{R}_Y) = H^0(Y, \mathcal{R}_Y(-\log E))$.

Let $\{\alpha_1, \alpha_2\}$ be the basis of $\Omega_X^{[1]}$ dual to $\{v_1, v_2\}$. By the Regular Extension Theorem, we can lift $\alpha_i$ to $\tilde{\alpha}_i \in H^0(Y, \Omega_Y^{[1]})$. Then on $Y$, we still have $\tilde{\alpha}_1(\tilde{v}_1) \equiv 1$. In particular, $\tilde{v}_1$ has no zeroes. But for each $i$, the derivation $\tilde{v}_1$ restricts to a derivation $\tilde{v}_1|_{E_i}$ on $E_i$, and the latter vanishes at each point of $(E - E_i) \cap E_i$, cf. [Wah75, (1.10.2)]. Unless $E$ is empty, it follows that $\tilde{v}_1 = E_1$ is smooth, and therefore an elliptic curve, since it carries the nowhere vanishing derivation $\tilde{v}_1|_{E'}$. By [Wah75, Prop. 2.12], $\pi$ is actually equivariant. So we also have a lift $\tilde{v}_2 \in H^0(Y, \mathcal{R}_Y(-\log E))$, and it satisfies $\tilde{\alpha}_1(\tilde{v}_2) \equiv \delta_{ij}$ on $Y$.

Pick an arbitrary (smooth) point $p \in E$. Evaluating the above relation at $p$ shows that $\tilde{v}_1(p), \tilde{v}_2(p) \in T_p Y$ are linearly independent. However, they are both contained in the one-dimensional subspace $T_p E \subset T_p Y$. This contradiction shows that $E = \emptyset$, hence $\pi$ is an isomorphism and $(0 \in X)$ is smooth. \qed
Lemma 4.1 (Tame RDPs). Let $(0 \in X)$ be an RDP over an algebraically closed field $k$ of characteristic $p > 0$. The cases where $(0 \in X)$ is not tame are exactly the following:

\begin{align*}
A_n, & \quad n \equiv -1 \mod p, \ p \text{ arbitrary}, \\
D_n, & \quad n \geq 4, \ p = 2, \\
E_6, & \quad p = 3, \\
E_7, & \quad p = 2.
\end{align*}

Proof. The determinants of the intersection matrices of the exceptional curves in the minimal resolution are as follows:

\begin{align*}
|\det(A_n)| &= n + 1, \\
|\det(D_n)| &= 4, \\
|\det(E_n)| &= 9 - n.
\end{align*}

The lemma follows immediately. \hfill \blacksquare

Proof of Theorem 1.4. Let $(0 \in X)$ be an RDP not contained in (1.4.1). Then by Theorem 3.1 and Lemma 4.1, the singularity $(0 \in X)$ satisfies the Logarithmic Extension Theorem and it is tame. Also, the MGR of $(0 \in X)$ is equivariant by [Wah75, Thm. 5.17]. Now apply Proposition 1.1 to get the first part of the statement. Concerning the second part, e.g. the following singularities have free tangent sheaf:

- $A_n$, $n \equiv -1 \mod p$, $p$ arbitrary,
- $D_{2n}$ and $D_{2n+1}$, $p = 2$,
- $E_6^0$, $E_7^0$, $E_8^0$, $p = 2$,
- $E_6^0$, $E_7^0$, $E_8^0$, $p = 3$,
- $E_8^0$, $p = 5$.

For the verification of the above list, we refer to Appendix A. \hfill \blacksquare

Proof of Theorem 1.6. Let $(0 \in X)$ be a log canonical surface germ. The possibilities for such $X$ have been classified in [Kol13, Ch. 3], see also [Gra21, Sec. 7.B]. If we want to prove the Lipman–Zariski conjecture for a given $X$, this means that we are assuming $\mathcal{F}_X$ to be free. In particular, $K_X$ is Cartier and hence all discrepancies are integers. This allows us to exclude many cases in the classification.

More precisely, in the minimal resolution $f : X' \to X$, either all discrepancies $a_i = 0$, or all $a_i = -1$, by [KM98, Cor. 4.3] (whose proof works in any characteristic).

- If all $a_i = 0$, then $(0 \in X)$ is an RDP and we have already handled this case in Theorem 1.4.
- If all $a_i = -1$, then $(0 \in X)$ is either simple elliptic or a cusp. (In case $\text{Exc}(f)$ is a nodal rational curve, we need to blow up the node to get the minimal good resolution $\pi : Y \to X$. The discrepancy of the new $(-1)$-curve is still $-1$.)

In the simple elliptic case, $E = \text{Exc}(\pi)$ is an elliptic curve and in the cusp case, it is a cycle of smooth rational curves. In both cases, $-(K_Y + E)$ is $\pi$-nef and the Logarithmic Extension Theorem for $(0 \in X)$ follows from [Gra21, Thm. 6.1].

In the simple elliptic case, $\pi$ is equivariant by [Wah75, Prop. 2.12]. In the cusp case, we apply [Wah75, Cor. 2.16] instead (the first condition is vacuous since the set $\mathcal{B}$ of “rational boundary components” of $E$ is empty, and a component $E_j$ as in the second condition exists because the fundamental cycle $Z_0$ satisfies $Z_0^2 < 0$).
Since \( \{ 0 \in X \} \) is tame by assumption, Proposition 1.1 now applies to show that the Lipman–Zariski conjecture holds for \( \{ 0 \in X \} \). □

Example 4.2 (Sharpness of Theorem 1.6). Set \( k = \mathbb{F}_9 \), and let \( a \in k^\times \) be a generator with minimal polynomial \( X^2 - X - 1 \). Consider the surface \( X \subset \mathbb{P}^2_k \) defined by the equation \( y^2 z - x(x - az)(x + z) = 0 \). Thus \( X \) is the cone over the elliptic curve \( C \subset \mathbb{P}^2_k \) defined by the same equation. In particular, \( \{ 0 \in X \} \) is log canonical and by the proof of Theorem 1.6, it satisfies the Logarithmic Extension Theorem and the MGR is equivariant. Also, \( C \) is ordinary by \([\text{Har}, \text{IV, Prop. 4.21}]\) and therefore \( \{ 0 \in X \} \) is \( F \)-pure by \([\text{MS}, \text{Thm. 1.2}]\). (Alternatively, one may use Fedder’s criterion \([\text{Fedder}, 1983]\).) However, it is not tame because the exceptional curve \( E \cong C \) has \( E^2 = -3 \). By the SINGULAR program in Appendix B, the tangent sheaf \( \mathcal{T}_X \) is freely generated by

\[
x \partial_x + y \partial_y + z \partial_z \quad \text{and} \quad a^5 y \partial_x + (x + a^6 z) \partial_y.
\]

Hence \( \{ 0 \in X \} \) violates the Lipman–Zariski conjecture. It also follows that \( \{ 0 \in X \} \) does not satisfy the Regular Extension Theorem (otherwise we could apply the arguments from the last paragraph of the proof of Proposition 1.1 to conclude that \( X \) is smooth).

A. Computations on all the rational double points

In this appendix, we work through the list of all RDPs over an algebraically closed field \([\text{Art}, 1977]\). We determine which of them are \( F \)-pure and satisfy the Lipman–Zariski conjecture, using the program in Appendix B. The information about \( F \)-purity is in principle contained in \([\text{Har}, \text{Thm. 1.2 and Rem. 1.3}]\), albeit not in the exhaustive form presented here. We also include the information whether the MGR is almost equivariant, which can easily be extracted from \([\text{Hir}, \text{Thms. 4.1 and 5.1}]\). For the reader’s convenience, we first list the “classical” equations from characteristic zero, which are also valid in sufficiently high characteristic:

\[
\begin{align*}
A_n & : xy + z^{n+1} = 0 \quad (n \geq 1) \\
D_n & : z^2 + x^2 y + y^{n-1} = 0 \quad (n \geq 4) \\
E_6 & : z^2 + x^3 + y^4 = 0 \\
E_7 & : z^2 + x^3 + xy^3 = 0 \\
E_8 & : z^2 + x^3 + y^5 = 0
\end{align*}
\]

| Equation | Parameters | \( F \)-pure | almost equiv. | LZ holds |
|----------|------------|--------------|---------------|---------|
| \( A_n \) | classical | \( n \geq 1 \), \( n \equiv -1 \mod p \) | ✓ | ✓ | — |
|         |           | \( n \geq 1 \), \( n \equiv -1 \mod p \) | ✓ | ✓ | ✓ |

Table 1. The \( A_n \) singularities in characteristic \( p \geq 2 \)

B. Computing minimal generating sets

The following SINGULAR procedure takes as arguments a polynomial \( f \) and a prime number \( p \). It then computes whether the singularity \( \{ 0 \in X \} \) defined by \( f \) in characteristic \( p \) is \( F \)-pure and whether \( \mathcal{T}_X \) is free. It also outputs a minimal generating set for \( \mathcal{T}_X \).
THE VIOLATION OF THE LIPMAN–ZARISKI CONJECTURE

Equation Parameters $F$-pure almost equiv. LZ holds

|$D_{2n}^0$ | $z^r + x^r y + x^r y^r$ | $n \geq 2$ | — | — | — |
|$D_{2n}^r$ | $z^r + x^r y + x^r y^r + x^r y^{n-r}$ | $n \geq 2$, $1 \leq r \leq n-2$ | — | — | — |
|$D_{2n+1}^r$ | $z^r + x^r y + y^r z$ | $n \geq 2$, $r = n-1$ | — | — | — |

Table 2. The $D_n$ and $E_n$ singularities in characteristic $p = 2$

Equation Parameters $F$-pure almost equiv. LZ holds

|$D_n$ | classical | $n \geq 4$ | ✓ | ✓ | ✓ |

Table 3. The $D_n$ singularities in characteristic $p \geq 3$

Equation $F$-pure almost equiv. LZ holds

|$E_0^6$ | classical | — | — | — |
|$E_1^6$ | $z^2 + x^2 y + y^2 z$ | ✓ | ✓ | ✓ |
|$E_0^7$ | classical | — | — | — |
|$E_1^7$ | $z^2 + x^2 y + x^2 y^2$ | ✓ | ✓ | ✓ |
|$E_0^8$ | classical | — | — | — |
|$E_1^8$ | $z^2 + x^2 + x^2 y^2$ + $x^2 y^2$ | ✓ | ✓ | ✓ |

Table 4. The $E_n$ singularities in characteristic $p = 3$

```java
proc LipmanZariski(poly f, A, int p) {
    /** f: polynomial to be checked
     * A: ring of which f is an element
     * (should be char 0 polynomial ring)
     * p: characteristic to be checked
     * (needs to be a prime number) **/
```
For example, in order to check the $E_8^0$ singularity $z^2 + x^3 + y^5 = 0$ in characteristic two, one may use the following commands in an interactive SINGULAR session (assuming the above code is contained in the file funnylz.lib):

```plaintext
/* pass to char p polynomial ring */
ring P = p, (x, y, z), ds; // local monomial ordering
poly f = fetch(A, f);
printf("f = %s in characteristic p = %s", f, p);

/* check F-purity using Fedder's criterion */
ideal Fed = x^p, y^p, z^p;
if( reduce( f^(p-1), std(Fed) ) == 0 ) {
    print("X = { f = 0 } is not F-pure.");
} else {
    print("X = { f = 0 } is F-pure.");
}

/* pass to quotient ring by f */
qring R = std(f);
poly f = fetch(P, f);

/* compute minimal resolution of tangent sheaf */
matrix Jac[1][3] = jacob(f);
module T = syz(Jac);
resolution rs = mres(T, 3);
if( size(rs[1]) == 2 ) {
    print("T_X is free.");
} else {
    print("T_X is not free.");
}
printf("Minimal generating set for T_X:");
print(rs[1]);
```
LIB "funnylz.lib";
ring r = 0, (x, y, z), ds;
LipmanZariski(z^2 + x^3 + y^5, r, 2);

\[ f = z^2 + x^3 + y^5 \] in characteristic \( p = 2 \)
\[ \mathcal{X} = \{ f = 0 \} \] is not F-pure.
\( T_X \) is free.
Minimal generating set for \( T_X \):
0,\( y^4 \),
0,\( x^2 \),
1, 0

In usual notation, this means that \( \mathcal{X} \) is generated by the two derivations \( \partial_z \) and \( y^4 \partial_x + x^2 \partial_y \).

**References**

[Art77] M. Artin: Coverings of the rational double points in characteristic \( p \), Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 11–22. ↑ 3, 8

[Bec78] J. Becker: Higher derivations and integral closure, Amer. J. Math. 100 (1978), no. 3, 495–521. ↑ 1

[Dur79] A. H. Durfee: Fifteen characterizations of rational double points and simple critical points, Enseign. Math. (2) 25 (1979), no. 1-2, 131–163. ↑ 3

[Fed83] R. Fedder: F-purity and rational singularity, Trans. Amer. Math. Soc. 278 (1983), no. 2, 461–480. ↑ 8

[Fle88] H. Flenner: Extendability of differential forms on nonisolated singularities, Invent. Math. 94 (1988), no. 2, 317–326. ↑ 1

[Gra21] P. Graf: Differential forms on log canonical spaces in positive characteristic, J. London Math. Soc. 104 (2021), 2208–2239. ↑ 2, 4, 5, 6, 7

[GK14] P. Graf and S. J. Kovács: An optimal extension theorem for 1-forms and the Lipman-Zariski Conjecture, Documenta Math. 19 (2014), 815–830. ↑ 1

[KKK10] D. Greb, S. Kebekus, and S. J. Kovács: Extensions theorems for differential forms and Bogomolov–Sommese vanishing on log canonical varieties, Compositio Math. 146 (2010), 193–219. ↑ 4

[Har98] N. Hara: Classification of two-dimensional F-regular and F-pure singularities, Adv. Math. 133 (1998), no. 1, 33–53. ↑ 3, 8

[Har77] R. Hartshorne: Algebraic geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977. ↑ 8

[Hir19] M. Hironaka: Further evaluation of Wahl vanishing theorems for surface singularities in characteristic \( p \), Michigan Math. J. 68 (2019), no. 3, 621–636. ↑ 2, 8

[Kol13] J. Kollár: Singularities of the minimal model program, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, with a collaboration of Sándor Kovács. ↑ 7

[KM98] J. Kollár and S. Mori: Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. ↑ 7

[Lip65] J. Lipman: Free Derivation Modules on Algebraic Varieties, Amer. J. Math. 87 (1965), no. 4, 874–898. ↑ 1

[Lip78] J. Lipman: Desingularization of two-dimensional schemes, Ann. of Math. (2) 107 (1978), no. 1, 151–207. ↑ 4

[MS91] V. B. Mehta and V. Srinivas: Normal F-pure surface singularities, J. Algebra 143 (1991), no. 1, 130–143. ↑ 3, 8

[SvS85] J. H. M. Steenbrink and D. van Straten: Extendability of holomorphic differential forms near isolated hypersurface singularities, Abh. Math. Sem. Univ. Hamburg 55 (1985), 97–110. ↑ 1

[Wah75] J. M. Wahl: Vanishing theorems for resolutions of surface singularities, Invent. Math. 31 (1975), no. 1, 17–41. ↑ 2, 4, 6, 7

Lehrstuhl für Mathematik I, Universität Bayreuth, 95440 Bayreuth, Germany
Email address: patrick.graf@uni-bayreuth.de
URL: www.graficland.uni-bayreuth.de