Optimal Hardy-weights for elliptic operators with mixed boundary conditions

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Abstract

We construct families of optimal Hardy-weights for a subcritical linear second-order elliptic operator \((P, B)\) with degenerate mixed boundary conditions. By an optimal Hardy-weight for a subcritical operator we mean a nonzero nonnegative weight function \(W\) such that \((P - W, B)\) is critical, and null-critical with respect to \(W\). Our results rely on a recently developed criticality theory for positive solutions of the corresponding mixed boundary value problem.

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1 | INTRODUCTION

The classical Hardy inequality states that if \(n \geq 3\), then

\[
\int_{\mathbb{R}^n \setminus \{0\}} |\nabla \phi|^2 \, dx \geq \left( \frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n \setminus \{0\}} \frac{|\phi|^2}{|x|^2} \, dx \quad \forall \phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) .
\]

(1.1)

It is well-known that the constant \(( (n -2)/2)^2\) in (1.1) is sharp but not achieved in the appropriate Hilbert space. In the past four decades, Hardy-type inequalities were developed for vast classes of operators with different types of boundary conditions, see, for example, [4, 5, 8, 10, 12, 13], and references therein. In [4], the authors established a general method to generate an explicit optimal Dirichlet–Hardy-weight for a general (not necessarily symmetric) subcritical linear elliptic operators either in divergence form or in nondivergence form defined on a domain \(\Omega \subset \mathbb{R}^n\) or on a noncompact manifold. In some definite sense, an optimal Hardy-weight is “as large as possible” Hardy-weight.

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In this paper, we utilize the approach developed in [4, 20] to produce families of optimal Hardy-inequalities for a general linear, second-order, elliptic operator with degenerate mixed boundary conditions. Our approach relies on criticality theory of positive weak solutions for elliptic operators with mixed boundary conditions that has been recently established in [21]. Though our methods are similar to those in [4, 20], where (generalized) Dirichlet boundary conditions are considered, a more intricate treatment is required when mixed boundary conditions are considered.

We remark that in [13] (resp., [8]) a variational approach was taken in order to produce Hardy-inequalities in a Lipschitz (resp., $C^2$-convex) domain for the Laplace (resp., $p$-Laplace) operator with degenerate Robin boundary conditions on $\partial \Omega_{\text{Dir}}$. Our approach does not assume convexity of the domain, and provides, in particular, a nontrivial improvement of a certain (Robin) Hardy-inequality obtained in [13] for the Laplacian in the exterior of the unit ball (see Example 4.2). See also [6] for related results.

We recall some essential definitions and results that are discussed in detail in [21]. Let $\Omega$ be a second-order, linear, elliptic operator with real measurable coefficients that is defined on a domain $\Omega \subset \mathbb{R}^n$. We assume that $P$ is in the divergence form

$$Pu := -\text{div}[A(x)\nabla u + u\tilde{b}(x)] + \tilde{b}(x) \cdot \nabla u + c(x)u \quad x \in \Omega. \quad (1.2)$$

Let $\partial \Omega_{\text{Rob}}$ be a relatively open $C^1$-portion of $\partial \Omega$, and consider the oblique boundary operator

$$Bu := \beta(x)(A(x)\nabla u + u\tilde{b}(x)) \cdot \tilde{n}(x) + \gamma(x)u \quad x \in \partial \Omega_{\text{Rob}}, \quad (1.3)$$

where $\tilde{n}(x)$ is the outward unit normal vector to $\partial \Omega$ at $x \in \partial \Omega_{\text{Rob}}$, $\xi \cdot \eta$ denotes the Euclidean inner product of $\xi, \eta \in \mathbb{R}^n$, and $\beta, \gamma$ are real measurable functions defined on $\partial \Omega_{\text{Rob}}$ such that $\beta > 0$ on $\partial \Omega_{\text{Rob}}$. The boundary of $\Omega$ is then naturally decomposed to a disjoint union of its Robin part $\partial \Omega_{\text{Rob}}$, and its Dirichlet part $\partial \Omega_{\text{Dir}}$. That is, $\partial \Omega = \partial \Omega_{\text{Rob}} \cup \partial \Omega_{\text{Dir}}$, where $\partial \Omega_{\text{Rob}} \cap \partial \Omega_{\text{Dir}} = \emptyset$. If further $\tilde{b} = \bar{b}$ in $\Omega$, we say that $(P, B)$ is symmetric in $\Omega$.

**Definition 1.1.** Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^n$ and consider the operator $(P, B)$ defined on $\Omega \setminus \partial \Omega_{\text{Dir}}$, where $P$ and $B$ are of the forms (1.2) and (1.3), respectively.

- We say that $(P, B)$ is nonnegative in $\Omega$ (in short $(P, B) \geq 0$ in $\Omega$) if there exists a positive weak solution to the mixed boundary value problem

$$\begin{cases}
Pu = 0 & \text{in } \Omega, \\
Bu = 0 & \text{on } \partial \Omega_{\text{Rob}}.
\end{cases} \quad (P,B)$$

- We say that $W \geq 0$ is a Hardy-weight for $(P, B)$ in $\Omega$ if $(P - W, B) \geq 0$ in $\Omega$.
- A Hardy-weight $W$ is said to be a Dirichlet–Hardy-weight if $\partial \Omega_{\text{Rob}} = \emptyset$.
- A nonnegative operator $(P, B)$ in $\Omega$ is said to be subcritical (resp., critical) in $\Omega$ if $(P, B)$ admits (resp., does not admit) a Hardy-weight for $(P, B)$ in $\Omega$.

By [21, Theorem 5.19], $(P, B)$ is subcritical in $\Omega$ if and only if $(P, B)$ admits a positive minimal Green function $G_{\Omega,B}^{\Omega}(x,y)$ in $\Omega$. On the other hand, $(P, B)$ is critical in $\Omega$ if and only if the equation $(P, B)u = 0$ admits (up to a multiplicative constant) a unique positive supersolution $\phi$ in $\Omega$. 
In fact, \( \phi \) is a positive solution, called the \((\text{Agmon})\) ground state (for the definition of a ground state see Definition 2.27). Furthermore, \((P, B)\) is critical in \( \Omega \) if and only if \((P^*, B^*)\) is critical in \( \Omega \), where \((P^*, B^*)\) is the formal adjoint of \((P, B)\) in \( L^2(\Omega) \) [21, Corollary 5.20].

Next we recall the notion of optimal Hardy-weight and optimality at \( \infty \) [4].

**Definition 1.2.** Let \( W \) be a Hardy-weight for \((P, B)\) in \( \Omega \). A Hardy-weight \( W \) is said to be optimal in \( \Omega \) if \((P - W, B)\) is critical in \( \Omega \) and \( \int_\Omega \phi \phi^* W \, dx = \infty \), where \( \phi \) and \( \phi^* \) are, respectively, the ground states of \((P - W, B)\) and \((P^* - W, B^*)\) in \( \Omega \). In this case we say that \((P - W, B)\) is null-critical with respect to the weight \( W \).

**Definition 1.3.** We say that a Hardy-weight \( W \) is optimal at infinity in \( \Omega \) if for any \( K \subset \subset \Omega \), \( \partial K \cap \partial \Omega_{\text{Dir}} = \emptyset \), and \( \partial K \cap \partial \Omega_{\text{Rob}} \subset \partial \Omega_{\text{Rob}} \) with respect to the relative topology on \( \partial \Omega_{\text{Rob}} \) (in short, \( K \subset \subset R_{\Omega} \)), we have

\[
\sup\{\lambda \in \mathbb{R} \mid (P - \lambda W, B) \geq 0 \text{ in } \Omega \setminus K\} = 1.
\]

**Remark 1.4.** The definition of an optimal Hardy-weight in [4] includes the requirement that \( W \) should be optimal at infinity in \( \Omega \). But, for the case \( \partial \Omega_{\text{Rob}} = \emptyset \), it is proved in [14] that if \( P - W \) is null-critical with respect to \( W \) in \( \Omega \), then \( W \) is also optimal at infinity. The same proof applies to the more general case considered in the present paper, hence, in Definition 1.2 we avoid the requirement of optimality at infinity.

**Definition 1.5.** Let \((P, B)\) be a subcritical operator in \( \Omega \), and let \( G(x, y) := G_{\Omega, P, B}(x, y) \) the corresponding positive minimal Green function. Fix \( 0 \leq \varphi \in C^\infty_0(\Omega) \). The Green potential with a density \( \varphi \) is the function

\[
G_\varphi(x) := \int_\Omega G(x, y) \varphi(y) \, dy.
\]

Next, we define the notion of a function vanishing near the Dirichlet boundary part and at \( \infty \). We first need to define an exhaustion of \( \Omega \) subordinated to \( \Omega \setminus \partial \Omega_{\text{Dir}} \).

**Definition 1.6.** A sequence \( \{\Omega_k\}_{k \in \mathbb{N}} \subset \Omega \) is called an exhaustion of \( \Omega \setminus \partial \Omega_{\text{Dir}} \) if it is an increasing sequence of Lipschitz domains such that \( \Omega_k \subset \subset \Omega_{k+1} \subset \subset \Omega \), and \( \bigcup_{k \in \mathbb{N}} \Omega_k = \Omega \setminus \partial \Omega_{\text{Dir}} \). For \( k \geq 1 \) we denote:

\[
\partial \Omega_{k, \text{Rob}} := \text{int}(\partial \Omega_k \cap \partial \Omega_{\text{Rob}}), \quad \partial \Omega_{k, \text{Dir}} := \partial \Omega_k \setminus \partial \Omega_{k, \text{Rob}}, \quad \Omega_k^* := (\Omega \setminus \partial \Omega_{\text{Dir}}) \setminus \overline{\Omega_k}.
\]

**Remark 1.7.** For the existence of such an exhaustion, see the appendix in [21]. We note that the construction of \( \{\Omega_k\}_{k \in \mathbb{N}} \) ensures that \( \partial \Omega_{k, \text{Rob}}^* \) is Lipschitz as well.

**Definition 1.8.** Let \( K \subset \subset \Omega \) and \( f \in C((\overline{\Omega \setminus K}) \setminus \partial \Omega_{\text{Dir}}) \). We say that

\[
\lim_{x \to \infty_{\partial \Omega_{\text{Dir}}}} f(x) = 0
\]

if for any \( \varepsilon > 0 \) and any exhaustion \( \{\Omega_k\}_{k \in \mathbb{N}} \) of \( \overline{\Omega \setminus \partial \Omega_{\text{Dir}}} \), there exists \( k_0 \) such that \( |f(x)| < \varepsilon \) in \( \Omega \setminus \Omega_{k_0} \).
Remark 1.9.

(1) In the case $\partial \Omega_{\text{Rob}} = \emptyset$, (1.4) is equivalent to the condition $\lim_{x \to \infty} f(x) = 0$, where $\infty$ denotes the ideal point in the one-point compactification of $\Omega$.

(2) Let $K \subseteq \Omega$, then $f \in C((\Omega \setminus K) \setminus \partial \Omega_{\text{Dir}})$ satisfying (1.4) is bounded in $\Omega \setminus K$.

Next, we define the notion of nonnegativity of the operator $(P, B)$.

**Definition 1.10.** We denote the cone of all positive weak solutions and positive weak supersolutions of the equation $(P, B)u = 0$ in $\Omega$ by $\mathcal{H}^0_{P,B}(\Omega)$ and $\mathcal{S} \mathcal{H}^1_{P,B}(\Omega)$, respectively. The operator $(P, B)$ is said to be nonnegative in $\Omega$ (in short, $(P, B) \geq 0$ in $\Omega$) if $\mathcal{H}^0_{P,B}(\Omega) \neq \emptyset$.

The first main result of the paper provides an explicit family of optimal Hardy weights. It reads as follows:

**Theorem 1.11.** Let Assumptions 2.5 hold in a domain $\Omega \subset \mathbb{R}^n$, $(n \geq 2)$ containing $x_0$. Let $(P, B)$ be a subcritical operator in $\Omega$ and let $G(x) := G^0_{P,B}(x, x_0)$ be its minimal positive Green function with singularity at $x_0 \in \Omega$. Let $\varphi$ be the Green potential with a density $0 \leq \varphi \in C^\infty_0(\Omega)$. Assume that $(P, B)$ admits a positive solution $u \in \mathcal{H}^0_{P,B}(\Omega)$ satisfying Ancona’s condition

$$\lim_{x \to \infty \text{Dir}} \frac{G(x)}{u(x)} = 0. \quad (1.5)$$

Let

$$0 \leq a \leq \frac{1}{\sup_{\Omega} (G^\varphi/u)} \quad \text{and} \quad W := \frac{P(uf_w(G^\varphi/u))}{uf_w(G^\varphi/u)}.$$ 

Then $(P - W, B)$ is critical in $\Omega$, and

$$W = \frac{|\nabla (G^\varphi/u)|_A^2}{(G^\varphi/u)^2(2 - aG^\varphi/u)^2} \quad \text{in} \quad \Omega \setminus \text{supp}(\varphi). \quad (1.6)$$

Furthermore, assume that one of the following regularity conditions are satisfied.

(1) $(P, B)$ is symmetric, $A \in C^{0,1}_\text{loc}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)$, $b = \tilde{b} \in C^\alpha(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)$, $c \in L^\infty(\overline{\Omega} \setminus \partial \Omega_{\text{Rob}})$, and $\partial \Omega_{\text{Rob}} \in C^{1,\alpha}$.

(2) $\partial \Omega_{\text{Rob}}, \partial \Omega_{\text{Dir}}$ are both relatively open and closed sets, $\partial \Omega_{\text{Rob}}$ is bounded and admits a finite number of connected components, and the coefficients of $P$ are smooth functions in $\Omega$ (or more generally, $A \in C^{[(3n-2)/2],1}_\text{loc}(\Omega, \mathbb{R}^n)$, $b \in C^{[(3n-2)/2],1}_\text{loc}(\Omega, \mathbb{R}^n)$, $\tilde{b} \in C^{[(3n-2)/2]-1,1}_\text{loc}(\Omega, \mathbb{R}^n)$, $c \in C^{[(3n-2)/2]-1,1}_\text{loc}(\Omega)$).

Then $W$ is an optimal Hardy-weights for $(P, B)$ in $\Omega$.

**Remark 1.12.** Theorem 1.11 is proved in [4] for the specific (classical) case $\partial \Omega_{\text{Rob}} = \emptyset$ and $a = 0$. Moreover, Theorem 1.11 with $\partial \Omega_{\text{Rob}} = \emptyset$ and $a > 0$ was proved in [20], and provides greater
Hardy-weights than the optimal Hardy-weight $W_{\text{class}}$ given in [4], in the sense that

$$W = \frac{|\nabla (G_\varphi / u)|^2_A}{(G_\varphi / u)^2(2 - a(G_\varphi / u))^2} > W_{\text{class}} := \frac{|\nabla (G_\varphi / u)|^2_A}{4(G_\varphi / u)^2} \quad \text{in } \Omega \setminus (\text{supp} (\varphi) \cup \text{Crit}(G_\varphi / u)).$$

Clearly, the above inequality holds true also in the case $\partial \Omega_{\text{Rob}} \neq \emptyset$.

The second main result of the paper is a generalization of Theorem 1.11. We show that any optimal Dirichlet–Hardy-weights for the one-dimensional Laplacian in $\mathbb{R}_+$ provides an $n$-dimensional optimal Hardy weights for $(P, B)$ in $\Omega$ (cf. [20, Theorem 5.2]).

**Definition 1.13.** We say that $0 \leq w \in L^1_{\text{loc}}(\mathbb{R}_+)$ belongs to $\mathcal{W}(\mathbb{R}_+)$ if $w$ is an optimal Dirichlet–Hardy-weight of $Ly := -y''$ in $\mathbb{R}_+$.

For a characterization of optimal Dirichlet–Hardy-weights for Sturm–Liouville operators see [20, Proposition 3.1] (see also Proposition 3.8).

**Theorem 1.14.** Assume that the operator $(P, B)$, and the functions $G$, $u$, and $G_\varphi$ satisfy the assumptions of Theorem 1.11.

Let $w \in \mathcal{W}(\mathbb{R}_+)$, and let $\psi_w(t)$ be the corresponding ground state. Suppose further that $\psi_w' \geq 0$ on $\{t = G_\varphi(x)/u(x) \mid x \in \Omega\}$, and set

$$W := \frac{P(u\psi_w(G_\varphi/u))}{u\psi_w(G_\varphi/u)}.$$

Then, the following assertions are satisfied.

1. $W \geq 0$ in $\Omega$ and $W := |\nabla (G_\varphi / u)|^2_A w(G_\varphi / u)$, in $\Omega \setminus \text{supp}(\varphi)$.
2. $(P - W, B)$ is critical in $\Omega$ with ground state $u\psi_w(G_\varphi / u)$.
3. Assume that $\partial \Omega_{\text{Rob}}$, $\partial \Omega_{\text{Dir}}$ are both relatively open and closed sets, $\partial \Omega_{\text{Rob}}$ is bounded and admits a finite number of connected components, and the coefficients of $P$ are smooth functions in $\Omega$ (or, $A \in C_{\text{loc}}^{[(3n-2)/2],1}(\Omega, \mathbb{R}^n)$, $b \in C_{\text{loc}}^{[(3n-2)/2],1}(\Omega, \mathbb{R}^n)$, $\tilde{b} \in C_{\text{loc}}^{[(3n-2)/2]-1,1}(\Omega, \mathbb{R}^n)$, $c \in C_{\text{loc}}^{[(3n-2)/2]-1,1}(\Omega)$). Then $W$ is optimal at infinity.
4. If $(P, B)$ is symmetric, $A \in C_{\text{loc}}^{0,1}(\Omega \setminus \partial \Omega_{\text{Dir}})$, $\tilde{b} = b \in C^\infty(\Omega \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)$, and $c \in L_\text{loc}^\infty(\Omega \setminus \partial \Omega_{\text{Dir}})$, then $(P - W, B)$ is null-critical with respect to $W$, and therefore, $W$ is an optimal Hardy-weight for $(P, B)$ in $\Omega$.

The paper is organized as follows. In Section 2, we introduce the necessary notation and recall some previously obtained results needed in the present paper. We proceed in Section 3, with proving a Khas’minskii-type criterion for the mixed boundary value problem and prove our main results. Finally, in Section 4, we present a couple of examples.

## 2  PRELIMINARIES AND NOTATION

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $\partial \Omega = \partial \Omega_{\text{Rob}} \cup \partial \Omega_{\text{Dir}}$, where $\partial \Omega_{\text{Rob}} \cap \partial \Omega_{\text{Dir}} = \emptyset$. We assume that $\partial \Omega_{\text{Rob}}$, the Robin portion of $\partial \Omega$, is a relatively open subset of $\partial \Omega$, and $\partial \Omega_{\text{Dir}}$, the
Dirichlet part of $\partial \Omega$, is a closed set of $\partial \Omega$. Moreover, if $\Omega$ is a bounded domain, we further assume that in the relative topology of $\partial \Omega$ we have $\text{int}(\partial \Omega_{\text{Dir}}) \neq \emptyset$. Throughout the paper, we use the following notation and conventions.

- For any $\xi \in \mathbb{R}^n$ and a positive definite symmetric matrix $A \in \mathbb{R}^{n \times n}$, let $|\xi|_A := \sqrt{A\xi \cdot \xi}$, where $\cdot$ denotes the Euclidean inner product of $\xi, \eta \in \mathbb{R}^n$.
- From time to time, we use the Einstein summation convention.
- Let $g_1, g_2$ be two positive functions defined in $\Omega$. We use the notation $g_1 \precsim g_2$ in $\Omega$ if there exists a positive constant $C$ such that
  \[ C^{-1}g_2(x) \leq g_1(x) \leq Cg_2(x) \quad \text{for all } x \in \Omega. \]
- Let $g_1, g_2$ be two positive functions defined in $\Omega$, and let $\xi \in \overline{\Omega} \cup \{\infty\}$. We use the notation $g_1 \sim g_2$ near $x_0$ if there exists a positive constant $C$ such that
  \[ \lim_{x \to \xi} \frac{g_1(x)}{g_2(x)} = C. \]
- The gradient of a function $f$ will be denoted either by $\nabla f$ or $Df$.
- $\partial \Omega_{\text{Dir}}$ and $\partial \Omega_{\text{Rob}}$ denote the Dirichlet and Robin parts of $\partial \Omega$, respectively.
- We write $A_1 \Subset A_2$ if $A_1$ is a compact set, and $\overline{A_1} \subset A_2$.
- For a subdomain $\Omega' \subset \Omega$, let $\partial \Omega'_{\text{Rob}} := \text{int}(\partial \Omega' \cap \partial \Omega_{\text{Rob}})$, and $\partial \Omega'_{\text{Dir}} := \partial \Omega' \setminus \partial \Omega'_{\text{Rob}}$.
- Let $\Omega'$ be a subdomain of $\Omega$. We write $\Omega' \Subset \Omega$ if $\Omega' \Subset \overline{\Omega}, \partial \Omega' \cap \partial \Omega \Subset \partial \Omega_{\text{Rob}}$.
- $C$ refers to a positive constant that may vary from line to line.
- For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we use the notation $x = (x', x_n)$.
- For any $R > 0$ and $x \in \mathbb{R}^n$, $B_R(x)$ is the open ball of radius $R$ centered at $x$.
- For any real measurable function $u$ and a measurable set $\omega \subset \mathbb{R}^n$,
  \[ \inf_{\omega} u := \text{ess inf}_{\omega} u, \quad \sup_{\omega} u := \text{ess sup}_{\omega} u, \quad u^+ := \max(0, u), \quad u^- := \max(0, -u). \]
- For any real function $W$, we say that $W \geqslant 0$ in $\Omega$ if $W \geqslant 0$ in $\Omega$ and $\sup_{\Omega} W > 0$.
- $\mathcal{H}^l, 1 \leq l \leq n$, denotes the $l$-dimensional Hausdorff measure on $\mathbb{R}^n$.
- For a differentiable function $f$, $\text{Crit}(f)$ denotes the set of critical points of $f$.
  
Let $R, K > 0$, and let $w$ be a real-valued Lipschitz continuous function defined on $\{x' \in \mathbb{R}^{n-1} : |x'| < R\}$ with
  \[ |w(x') - w(y')| \leq K|x' - y'| \]
for all $x', y' \in B(x_0, R)$, that satisfy $w(0) = 0$. We define

\[
\begin{align*}
\Omega[R] &= \{x \in \mathbb{R}^n : x_n > w(x'), \ |x| < R\}, \\
\sigma[R] &= \{x \in \mathbb{R}^n : x_n \geq w(x'), \ |x| = R\}, \\
\Sigma[R] &= \{x \in \mathbb{R}^n : |x| < R, \ x_n = w(x')\}.
\end{align*}
\]
Definition 2.1 (Lipschitz and $C^1$ portions). Let $x_0 \in \partial \Omega$ and $R > 0$ such that $\Omega[x_0,R] := \Omega \cap B_R(x_0)$ is a Lipschitz (resp., $C^1$) domain. The set $\Sigma[x_0,R] := \partial \Omega \cap B_R(x_0)$ is called a Lipschitz (resp., $C^1$) portion of $\partial \Omega$.

Further, we introduce some functional spaces. Let $E \subset \partial \Omega$ be a closed subset in the relative topology of $\partial \Omega$. We define $D(\Omega,E) := C^0(\Omega \setminus E)$. So, $u \in D(\Omega,E)$ if $u$ has compact support, and $\text{supp } u := \{x \in \Omega \mid u(x) \neq 0\} \subset \Omega \setminus E$.

For $q \geq 1$, we define $W^{1,q}_E(\Omega)$ to be the closure of $D(\Omega,E)$ with respect to the Sobolev norm of $W^{1,q}(\Omega)$. We also consider the following spaces:

\[ L^q_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) := \{u \mid \forall x \in \Omega \cup \partial \Omega_{\text{Rob}}, \exists r_x > 0 \text{ s.t. } u \in L^q(\Omega \cap B_{r_x}(x))\}, \]

\[ W^{1,q}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) := \{u \mid \forall x \in \Omega \cup \partial \Omega_{\text{Rob}}, \exists r_x > 0 \text{ s.t. } u \in W^{1,q}(\Omega \cap B_{r_x}(x))\}. \]

If $q = 2$, we write $H^{1}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) := W^{1,2}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$.

Remark 2.2. For every Lipschitz subdomain $\Omega' \subset \subset \Omega$ and $1 < q < \infty$ the space $W^{1,q}(\Omega')$ is a reflexive Banach space and therefore, $W^{1,q}_{\partial \Omega_{\text{Dir}}}(\Omega')$ is reflexive as well.

The space $W^{1,q}_E(\Omega)$, can be characterized using the following result.

Lemma 2.3 [7, Theorem 2.1] and [23, Remark 2.1]. Let $\Omega$ be a bounded Lipschitz domain. Let $E \subset \partial \Omega$ be a compact subset with respect to the relative topology on $\partial \Omega$, and let $d_E(x) := \text{dist}(x,E)$, where $x \in \Omega$. Then, for $u \in H^1(\Omega)$ the following assertions are equivalent.

- $u \in H^1_E(\Omega)$.
- \( \int_{\Omega} \frac{|u|^2}{d_E^2} \, dx < \infty. \)
- \( \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x) \cap \Omega} |u| \, dx = 0 \text{ for } H^{n-1}\text{-almost every } x \in E. \)

Corollary 2.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain.

1. Assume that $v$ and $u$ belong to $H^1(\Omega)$, and $u \geq 0$. Assume further that $v \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega)$. Then, $(u - v)^- \leq |v|$ in $\Omega$ and $(u - v)^- \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega)$.

2. Let $u, v \in H^1(\Omega) \cap C(\overline{\Omega})$, and $u \leq v$ on $\partial \Omega_{\text{Dir}}$. Then $(v - u)^- \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega)$.

Proof. Let $E = \partial \Omega_{\text{Dir}}$.

1. Lemma 2.3 implies that

\[ v \in H^1_{\partial \Omega_{\text{Dir}}} \iff \int_{\Omega} \frac{|v|}{d_E} \, dx < \infty. \]

The inequality $(u - v)^- \leq |v|$ is trivial when $u \geq 0$, and therefore,

\[ \int_{\Omega} \frac{|(u - v)^-|^2}{d_E^2} \, dx \leq \int_{\Omega} \frac{|v|^2}{d_E^2} \, dx < \infty. \]
(2) The continuity of $u$ and $v$ implies that $(v - u)^- \text{ vanishes continuously on } E$. In particular, 
$$\lim_{r \to 0} \frac{1}{r} \int_{B_r(x) \cap \Omega} |(v - u)^-| \, dx = 0 \text{ for every } x \in E.$$ 

Consider an elliptic operator $P$ of the form (1.2) and a Robin boundary operator $B$ of the form (1.3). Throughout the paper, we assume the following regularity assumptions on $P$, $B$ and $\partial \Omega$:

Assumptions 2.5.

- $\partial \Omega = \partial \Omega_{\text{Rob}} \cup \partial \Omega_{\text{Dir}}$, where $\partial \Omega_{\text{Rob}} \cap \partial \Omega_{\text{Dir}} = \emptyset$, and $\partial \Omega_{\text{Rob}}$ is a relatively open subset of $\partial \Omega$.
- For each $x_0 \in \partial \Omega_{\text{Rob}}$, $\exists R > 0$ such that $\Sigma[x_0, R]$ is a $C^1$-portion of $\partial \Omega$.
- $A = (a_{ij})_{i,j=1}^n \in L^\infty(\Omega \setminus \partial \Omega_{\text{Dir}}; \mathbb{R}^{n \times n})$ is a symmetric positive definite matrix valued function that is locally uniformly elliptic in $\Omega \setminus \partial \Omega_{\text{Dir}}$, that is, for any compact $K \subset \Omega \setminus \partial \Omega_{\text{Dir}}$ there exists $\Theta_K > 0$ such that

$$\Theta_K^{-1} \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Theta_K \sum_{i=1}^n \xi_i^2 \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in K.$$ 

- $\bar{b}, \underline{b} \in L^p_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}; \mathbb{R}^n)$, and $c \in L^{p/2}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$ for some $p > n$.
- $\beta > 0$, and $\gamma/\beta \in L^\infty(\partial \Omega_{\text{Rob}})$.

In the sequel, we use the following terminology.

**Definition 2.6.** Let $u \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$. We say that $u \geq 0$ on $\partial \Omega_{\text{Dir}}$ if $u^- \in H^1(\Omega)$.

Next, we define (weak) solutions and supersolutions of the mixed boundary value problem

$$\begin{cases}
Pu = 0 \quad \text{in } \Omega, \\
Bu = 0 \quad \text{on } \partial \Omega_{\text{Rob}}.
\end{cases} \quad (P,B)$$

**Definition 2.7.** We say that $u \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$ is a weak solution (resp., supersolution) of $(P,B)$ in $\Omega$, if for any (resp., nonnegative) $\phi \in D(\Omega, \partial \Omega_{\text{Dir}})$ we have

$$B_P(u, \phi) := \int_{\Omega} \left[ \left( a_{ij} D_j u + u \bar{b}^i \right) D_i \phi + \left( \bar{b}^i D_i u + c u \right) \phi \right] \, dx + \int_{\partial \Omega_{\text{Rob}}} \frac{\gamma}{\beta} u \phi \, d\sigma = 0 \text{ (resp., } \geq 0),$$

where $d\sigma$ is the $(n-1)$-dimensional surface measure. In this case we write $(P,B)u = 0$ (resp., $(P,B)u \geq 0$). Also, $u$ is a subsolution of $(P,B)$ if $-u$ is a supersolution of $(P,B)$.

The above definition should be compared with the following standard definition of weak (super)solutions of the equation $Pu = 0$ in a domain $\Omega$. 

Definition 2.8. We say that $u \in H^1_{\text{loc}}(\Omega)$ is a weak solution (resp., supersolution) of the equation $Pu = 0$ in $\Omega$ if for any (resp., nonnegative) $\phi \in C^\infty_0(\Omega)$

$$B_p(u, \phi) := \int_\Omega \left[ \left( a^{ij} D_j u + u \bar{b}^i \right) D_i \phi + \left( \bar{b}^i D_i u + cu \right) \phi \right] dx = 0 \ (\text{resp.,} \geq 0).$$

Hence, any weak solution (resp., supersolution) of the equation $(P, B)u = 0$ in $\Omega$ is a weak solution (resp., supersolution) of $Pu = 0$ in $\Omega$. In the sequel, by a (super)solution of $(P, B)$ we always mean a weak (super)solution.

Remark 2.9. By [21, section 3.2], any positive solution of the problem $(P, B)v = 0$ in $\Omega$ is Hölder continuous in $\Omega \setminus \partial \Omega_{\text{Dir}}$. Furthermore, $u > 0$ on $\partial \Omega_{\text{Rob}}$.

The formal $L^2$-adjoint of the operator $(P, B)$ is given by the operator $\left(P^*, B^*\right)$

$$\begin{cases}
P^*u := -\text{div} \left[ A\nabla u + \bar{b}u \right] + \bar{b} \cdot \nabla u + cu, \\
B^*u := \beta (A\nabla u + u\bar{b}) \cdot \vec{n} + \gamma u.
\end{cases} \quad \left(P^*, B^*\right)$$

Indeed, if $\bar{b}, \bar{b}$ and $\partial \Omega$ are sufficiently smooth, then for any $\phi, \psi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})$ satisfying $B\psi = B^*\phi = 0$ on $\partial \Omega_{\text{Rob}}$ in the classical sense, we have

$$\int_\Omega P(\psi) \phi \, dx = \int_\Omega \left( -\text{div} \left( A\nabla \psi + \bar{b}\psi \right) + \bar{b} \cdot \nabla \psi + c\psi \right) \phi \, dx =$$

$$\int_{\partial \Omega_{\text{Rob}}} \psi \phi \left( \frac{\gamma}{\beta} + b \cdot \vec{n} \right) \, d\sigma + \int_\Omega (A\nabla \psi + \bar{b}\psi) \cdot \nabla \phi - \psi \nabla \cdot (\bar{b}\phi) + c\psi \phi \, dx =$$

$$\int_{\partial \Omega_{\text{Rob}}} \psi (-\text{div} \left( A\nabla \phi + \bar{b}\phi \right) + \bar{b} \cdot \nabla \phi + c\phi) \, d\sigma = \int_\Omega \psi P^*(\phi) \, dx.$$

The Fredholm alternative [21, section 2] implies:

Lemma 2.10. Assume that $(P, B) \geq 0$ in $\Omega$ and let $\Omega' \Subset_\rho \Omega$ be a Lipschitz subdomain of $\Omega$. For any $Y = g_0 + \text{div} \ g \in (H^1_{\partial \Omega'_{\text{Dir}}}(\Omega'))^*$ there exists a unique solution $u_Y \in H^1_{\partial \Omega'_{\text{Dir}}}(\Omega')$ to the problem

$$\begin{cases}
Pw = g_0 + \text{div} \ g & \text{in } \Omega', \\
Bw = -g \cdot \vec{n} & \text{on } \partial \Omega'_{\text{Rob}},
\end{cases} \quad (2.1)$$

(in short, $(P, B)w = g_0 + \text{div} \ g$ in $\Omega'$) in the following (weak) sense:

For all $\phi \in \mathcal{D}(\Omega', \partial \Omega'_{\text{Dir}})$,

$$\int_{\Omega'} \left[ A\nabla u_Y \cdot \nabla \phi + u_Y \bar{b} \cdot \nabla \phi + \bar{b} \cdot \nabla u_Y \phi + cu_Y \phi \right] \, dx + \int_{\partial \Omega'_{\text{Rob}}} \frac{\gamma}{\beta} u_Y \phi \, d\sigma = \int_{\Omega'} (g_0 \phi - g \cdot \nabla \phi) \, dx.$$

Remark 2.11.

(1) If $(P, B)$ is symmetric, $\partial \Omega_{\text{Rob}} \in C^{1,\alpha}$, $A \in C^{0,1}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^{n \times n})$, $\bar{b} = \bar{b} \in C^{\alpha}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)$, and $c \in L^\infty_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$, then any positive solution of the equation $(P, B)u = 0$ in $\Omega$ belongs to $C^{1,\alpha}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$ [16, Theorem 5.54].
(2) Assume that \((P, B)1 = 0\) in \(\Omega\), \(A \in C^0_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)\), \(\mathbf{b} \in C^0_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)\), \(\overline{\mathbf{b}} \in L^\infty(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)\) and \(c \in L^\infty(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\). Then, any solution of the equation \((P, B)u = 0\) in \(\Omega\) belongs to \(W^{2,2}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})\) (see \([16, \text{Theorems 5.29 and 5.54}]\), where \(\partial \Omega_{\text{Rob}} \in C^2\) is assumed.). In fact, using the “even” extension argument as in proof of \([21, \text{Lemma 5.9}]\), one obtains the above regularity result for the case \(\partial \Omega_{\text{Rob}} \in C^1\) (cf.\([11, \text{Theorem 8.8}]\)).

2.1 Criticality theory

The results in this subsection were recently proved in \([21]\).

**Definition 2.12.** Let \(\Omega\) be a domain in \(\mathbb{R}^n\), and let \(0 < V \in L^{p/2}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\), where \(p > n\). The **generalized principal eigenvalue** of \((P, B)\) in \(\Omega\) with respect to \(V\) is defined by

\[
\lambda_0 = \lambda_0(P, B, V, \Omega) := \sup\{\lambda \in \mathbb{R} : (P - \lambda V, B) \geq 0 \text{ in } \Omega\}.
\]

**Definition 2.13.** We say that the **generalized maximum principle** holds in a bounded domain \(\Omega\) if for any \(u \in H^1(\Omega)\) satisfying \((P, B)u \geq 0\) in \(\Omega\) and \(u^- \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega)\), we have \(u \geq 0\) in \(\Omega\).

Assume that \((P, B) \geq 0\) in a domain \(\Omega\). We recall the notion of the ground state transform that implies a generalized maximum principle. Set

\[
\mathcal{R}SH_{P, B}(\Omega) := \left\{ u \in SH_{P, B}(\Omega) \mid u, u^{-1}, Pu \in L^\infty(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}) \text{ and } \frac{Bu}{\beta} \in L^\infty(\partial \Omega_{\text{Rob}}) \right\} \neq \emptyset.
\]

A supersolution \(u \in \mathcal{R}SH_{P, B}(\Omega)\) is called a **regular positive supersolution** of \((P, B)\) in \(\Omega\).

**Definition 2.14** (Ground state transform). Assume that \((P, B) \geq 0\) in a domain \(\Omega\), and let \(u \in \mathcal{R}SH_{P, B}(\Omega)\). Consider the bilinear form

\[
B_{pu, Bu}(\phi, \psi) := B_{P, B}(u\phi, u\psi),
\]

where \(u\phi, u\psi \in D(\Omega, \partial \Omega_{\text{Dir}})\). The form \(B_{pu, Bu}\) corresponds to the elliptic operator

\[
P^u(w) := \frac{1}{u} P(uw), \quad \text{with the boundary operator } B^u(w) := \frac{1}{u} B(uw).
\]

The operator \(P^u\) is called the **ground state transform** of \(P\) with respect to \(u\). The operators \(P^u(w)\) and \(B^u(w)\) are given explicitly by

\[
P^u(w) = -\frac{1}{u^2} \text{div}(u^2 A(x) \nabla w) + [\mathbf{b} - \overline{\mathbf{b}}] \nabla w + \frac{Pu}{u} w, \quad \text{and } B^u(w) = \beta(A \nabla w, \vec{n}) + \frac{Bu}{u} w.
\]

We say that \(w\) is a weak (resp., super)solution of \((P^u, B^u)\) in \(\Omega\), if \(w \in H^1_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\) and for any (resp., nonnegative) \(\phi\) such that \(\phi u \in D(\Omega, \partial \Omega_{\text{Dir}})\) we have

\[
B_{pu, Bu}(w, \phi) = \int_{\Omega} (a^{ij} D_i w D_j \phi + (\mathbf{b}^i - \overline{\mathbf{b}}^i) D_i w \phi) u^2 dx + \int_{\Omega} u(Pu) w \phi dx + \int_{\partial \Omega_{\text{Rob}}} \frac{Bu}{\beta} w \phi d\sigma
\]

\[
= 0 \quad \text{(resp., } \geq 0). \quad (2.2)
\]
Remark 2.15. Note that $B_{pu,Bv}$ is defined on $L^2(\Omega, u^2 \, dx)$. Furthermore, if $u \in H^0_{P,B}(\Omega)$, then the adjoint operator of $(P^u,B^u)$ in $L^2(\Omega, u^2 \, dx)$ is given by

$$(P^u)^* w = \frac{P^*(uw)}{u}, \quad \text{and} \quad (B^u)^* w = \frac{B^*(uw)}{u}.$$  

Remark 2.16.

(1) Let $u,v \in H^0_{P,B}(\Omega)$ (resp., $u \in H^0_{P,B}(\Omega), v \in S H^1_{P,B}(\Omega)$). Assume that $v/u \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}, u^2 \, dx)$. Then $v/u$ is a weak positive (resp., super) solution of the equation $(P^u,B^u)w = 0$ in $\Omega$.

(2) If $u \in H^0_{P,B}(\Omega)$, then for any $w \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$ and $\phi \in D(\Omega, \partial \Omega_{\text{Dir}})$, we have

$$B_p u, B_v (w, \phi) = \int_{\Omega} \left( a_{ij} D_i w D_j \phi + (\bar{b}_i - \bar{b}'_i) D_i w \phi \right) u^2 \, dx.$$  

The following generalized maximum principle for a nonnegative operators $(P,B)$ is proved in [21, Lemma 3.19] and is a consequence of the ground state transform.

Lemma 2.17 (Generalized maximum principle). Assume that the operator $(P,B)$ is nonnegative in $\Omega$. Consider a Lipschitz bounded subdomain $\Omega' \Subset R \Omega$. If $v \in H^1(\Omega')$ is a supersolution of the equation $(P,B)u = 0$ in $\Omega'$ with $v^- \in H^1_{\text{loc}}(\Omega' \setminus \partial \Omega'_{\text{Dir}})$, then $v$ is nonnegative in $\Omega'$.

We proceed with Harnack convergence principle [20, Lemma 3.27].

Lemma 2.18 (Harnack convergence principle). Suppose that Assumptions 2.5 hold in $\Omega$, and let $\{\Omega_k\}_{k \in \mathbb{N}}$ be an exhaustion of $\Omega \setminus \partial \Omega_{\text{Dir}}$. Let $x_0 \in \Omega_1$ be a fixed reference point. For each $k \geq 1$, let $u_k \in H^1_{\text{loc}}(\Omega_k \cup \partial \Omega_k, \partial \Omega_{\text{Rob}})$ be a positive solution of the problem

$$\begin{cases} Pu = 0 & \text{in } \Omega_k, \\ Bu = 0 & \text{on } \partial \Omega_k, \partial \Omega_{\text{Rob}}, \end{cases} \tag{2.3}$$

satisfying $u_k(x_0) = 1$. Then the sequence $\{u_k\}_{k \in \mathbb{N}}$ admits a subsequence converging locally uniformly in $\Omega \setminus \partial \Omega_{\text{Dir}}$ to a positive solution $u \in H^0_{P,B}(\Omega)$.

Remark 2.19. Lemma 2.18 holds once $P$ is replaced by $P^k(u) = -\text{div} (A^k \nabla u + \bar{b}_k u) + \bar{b}_k \nabla u + c_k u$, where $A_k \in L^\infty_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$ is a sequence of symmetric and positive definite matrices converging to $A$ in $L^\infty_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$; $\bar{b}_k, \bar{b}_k \in L^p_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$, $\bar{b}_k \rightarrow \bar{b}$ and $\bar{b}_k \rightarrow \bar{b}$ in $L^p_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}, \mathbb{R}^n)$; and $c_k \in L^{p/2}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$, $c_k \rightarrow c$ in $L^{p/2}_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$.

The following characterization of $\lambda_0$ was proved in [21].

Theorem 2.20 [21, Theorem 4.1]. The following assertions are equivalent.

(1) $H^0_{P,B}(\Omega) \neq \emptyset$, and in particular, $\lambda_0(P,B,1,\Omega) \geq 0$.
(2) $R S H^1_{P,B}(\Omega) \neq \emptyset$.
(3) $\lambda_0(P,B,1,\Omega') > 0$ for any Lipschitz subdomain $\Omega' \Subset R \Omega$.
(4) $(P,B)$ satisfies the generalized maximum principle in any Lipschitz subdomain $\Omega' \Subset R \Omega$. 


Lemma 2.21 [21, Lemma 4.3]. Suppose that $H^0_{P,B}(\Omega) \neq \emptyset$, and let $\Omega' \Subset R \Omega$ be a bounded Lipschitz subdomain of $\Omega$. Let $K \Subset \Omega'$ be a Lipschitz subdomain. Then for any nonzero nonnegative function $f \in C^\infty_0(\Omega' \setminus K)$ there exists a unique positive weak solution $u \in H^1 \setminus \Omega_{\text{Dir}} \cup \Omega'_{\text{Rob}}$ to the problem:

\[
\begin{cases}
Pw = f & \text{in } \Omega' \setminus K, \\
Bw = 0 & \text{on } \partial \Omega'_{\text{Rob}}, \\
\text{Trace}(w) = 0 & \text{on } (\partial \Omega' \cup \partial K) \setminus \partial \Omega'_{\text{Rob}}.
\end{cases}
\]

(2.4)

Next, we introduce the notion of positive solution of minimal growth for $(P,B)$ (cf. [1, 19, 20]). In the sequel $\{\Omega_k\}_{k \in \mathbb{N}}$ is an exhaustion of $\Omega \setminus \partial \Omega_{\text{Dir}}$.

Definition 2.22. A function $u$ is said to be a positive solution of minimal growth in a neighborhood of infinity in $\Omega$ if $u \in H^0_{P,B}(\Omega^*_j)$ for some $j \geq 1$ and for any $l > j$ and $v \in C(\Omega^*_l \cup \partial \Omega_{l,\text{Dir}}) \cap H^0_{P,B}(\Omega^*_l)$, $u \leq v$ on $\partial \Omega_{l,\text{Dir}} \Rightarrow u \leq v$ on $\Omega^*_l$.

Lemma 2.23 [21, Lemma 4.5]. Suppose that $H^0_{P,B}(\Omega) \neq \emptyset$. Then, for any $x_0 \in \Omega$, the equation $(P,B)u = 0$ has (up to a multiplicative constant) a unique positive solution $v$ in $\Omega \setminus \{x_0\}$ of minimal growth in a neighborhood of infinity in $\Omega$.

Lemma 2.24 [21, Lemma 4.7]. Suppose that $H^0_{P,B}(\Omega) \neq \emptyset$. Then $(P,B)$ is critical in $\Omega$ if and only if there exists a unique $u \in RSH_{P,B}(\Omega)$ (up to a multiplicative constant).

Lemma 2.25 [21, section 5]. Suppose that $H^0_{P,B}(\Omega) \neq \emptyset$. Then $(P,B)$ is subcritical in $\Omega$ if and only if $(P,B)$ admits a unique positive minimal Green function $G_{P,B}^\Omega(x,y) \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}} \setminus \{y\})$ in $\Omega$ satisfying (in the sense of distributions) $(P,B)G_{P,B}^\Omega(\cdot,y) = \delta_y$ in $\Omega$, where $\delta_y$ is the Dirac measure supported in $\{y\}$. Moreover, $(P,B)$ is subcritical in $\Omega$ if and only if $(P^*,B^*)$ is subcritical in $\Omega$. In such a case, $G_{P,B}^\Omega(x,y) = G_{P^*,B^*}^\Omega(y,x)$ for all $(x,y) \in \Omega \times \Omega$ satisfying $x \neq y$.

The following elementary result concerning Green potentials is a consequence of [21, section 5.2].

Proposition 2.26. Assume that $(P,B)$ is subcritical in $\Omega$, and let $0 \leq \varphi \in C^\infty_0(\Omega)$. Then

1. $0 < G_{\varphi} \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) \cap C^\infty(\Omega \setminus \partial \Omega_{\text{Dir}})$,
2. $(P,B)G_{\varphi} = \varphi$ in $\Omega$,
3. for any $x_0 \in \Omega$, there exists $K \Subset R \Omega \setminus \partial \Omega_{\text{Dir}}$ such that $G_{\varphi} \asymp G_{P,B}^\Omega(x,x_0)$ in $\Omega \setminus K$.

Definition 2.27. A function $u \in H^0_{P,B}(\Omega)$ is called an (Agmon) ground state of $(P,B)$ in $\Omega$ if $u$ has minimal growth in a neighborhood of infinity in $\Omega$.

We conclude this section with the following lemma.

Lemma 2.28 [21, Lemma 4.10]. Suppose that $H^0_{P,B}(\Omega) \neq \emptyset$. Then $(P,B)$ admits a ground state in $\Omega$ if and only if $(P,B)$ is critical in $\Omega$. 

OPTIMAL HARDY INEQUALITIES FOR MIXED BOUNDARY VALUE PROBLEMS

The aim of this section is to prove Theorems 1.11 and 1.14 concerning the existence of families of optimal Hardy-weights (see Definition 1.3). The case $\partial \Omega_{\text{Rob}} = \emptyset$ (i.e., Dirichlet–Hardy-weights), has been studied in [4, 20]. We remark that all the results in this paper include the Dirichlet case, $\partial \Omega_{\text{Rob}} = \emptyset$.

First, we prove a Khas’minskii-type criterion for the criticality of the mixed boundary value problem $(P,B)$ (cf. [4, Proposition 6.1], and references therein).

**Lemma 3.1** (Khas’minskii-type criterion). Let $u_0 \in H^0_{P,B}(\Omega)$ and $u_1 \in SH_{P,B}(\Omega \setminus K) \cap C(\overline{\Omega} \setminus (\partial \Omega_{\text{Dir}} \cup K))$, where $K \in_R \Omega$ is a Lipschitz subdomain. Assume that

$$\lim_{x \to \partial \Omega_{\text{Dir}}} \frac{u_0(x)}{u_1(x)} = 0.$$ (3.1)

Then $u_0$ is a a ground state of $(P,B)w = 0$ in $\Omega$, and therefore, $(P,B)$ is critical in $\Omega$.

**Proof.** We need to prove that $u_0$ has minimal growth at infinity. Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be an exhaustion of $\Omega \setminus \partial \Omega_{\text{Dir}}$ and a Lipschitz subdomain $K' \in_R \Omega$ such that $K \in_R K' \in_R \Omega_1$, and $\partial (\Omega \setminus K') \cap \partial \Omega_{\text{Rob}}$ is Lipschitz (see Remark 1.7). Fix $x_0 \in K$ and let $G(x) \in H^0_{P,B}(\Omega \setminus \{x_0\})$ having minimal growth in a neighborhood of infinity in $\Omega$. Let $C > 1$ be fixed such that

$$C^{-1} G(x) \leq u_0(x) \leq CG(x) \quad \text{for all } x \in \overline{\Omega_1 \setminus K'}.$$ (3.2)

The minimal growth of $G(x)$ implies

$$C^{-1} G \leq u_0 \quad \text{in } \Omega \setminus K'.$$ (3.3)

Furthermore, (3.1) implies that for any $\varepsilon > 0$, the exists $k_\varepsilon$ such that for any $k \geq k_\varepsilon$

$$u_0 \leq \varepsilon u_1 \leq CG + \varepsilon u_1 \quad \text{on } \partial \Omega_k \cap \Omega.$$

Notice that $u_0$ and $CG + \varepsilon u_1$ belong to $C(\overline{\Omega} \setminus (\partial \Omega_{\text{Dir}} \cup K)) \cap SH(\Omega \setminus K)$. Consider the set $D_k := \Omega_k \setminus K'$. Then, Corollary 2.4 and (3.2) imply that $(CG + \varepsilon u_1 - u_0)^- \in H^1_{\partial D_k,\text{Dir}}(D_k)$. The generalized maximum principle (Lemma 2.17) in $D_k$ then implies

$$u_0 \leq CG + \varepsilon u_1 \quad \text{in } D_k.$$ (3.4)

Letting $k \to \infty$, we obtain

$$u_0 \leq CG + \varepsilon u_1 \quad \text{in } \Omega \setminus K'.$$

Letting $\varepsilon \to 0$ and (3.3) imply, $u_0 \asymp G$ in $\Omega \setminus K'$, namely, $u_0$ is a ground state, and therefore, $(P,B)$ is critical in $\Omega$. \qed

As a corollary of Lemma 3.1, we obtain the criticality claim in Theorem 1.11.

**Theorem 3.2.** Let Assumptions 2.5 hold in a domain $\Omega \subset \mathbb{R}^n$, $(n \geq 2)$. Let $(P,B)$ be a subcritical operator in $\Omega$ and let $G(x) := G^\Omega_{P,B}(x,x_0)$ be its minimal positive Green function with singularity
at \( x_0 \in \Omega \). Let \( G_\varphi \) be the corresponding Green potential with density \( 0 \leq \varphi \in C^\infty_0(\Omega) \). Assume that there exists \( u \in H^0_{P,B}(\Omega) \) satisfying Ancona’s condition

\[
\lim_{x \to \infty, \text{Dir}} \frac{G(x)}{u(x)} = 0. \tag{3.4}
\]

Let

\[
0 \leq a \leq \frac{1}{\sup_{\Omega} (G_\varphi / u)}, \quad f_w(t) := \sqrt{2t - at^2}, \quad W := \frac{P(u f_w(G_\varphi / u))}{u f_w(G_\varphi / u)}.
\]

Then \((P - W, B)\) is critical in \( \Omega \).

**Proof.** By Proposition 2.26 and Remark 1.9, we may assume without loss of generality that \( G_\varphi / u < 1 \) in \( \Omega \). Hence, \( f_w(G_\varphi / u) \) is well-defined and \( f_w'(G_\varphi / u) > 0 \) in \( \Omega \). Let \( w(t) := (2t - at^2)^{-2} \). It can be easily checked that the functions \( f_w \) and

\[
f_1(t) := f_w(t) \int_1^t \frac{1}{f_w^2(s)} \, ds
\]

are linearly independent solutions of the equation

\[-y'' - wy = 0 \quad \text{in } \mathbb{R}^+,\]

which is related to the Ermakov–Pinney equation \(-y'' = \frac{1}{y^3}\) (see [21]). Moreover, \( f_1 \) is positive for \( t < 1 \), negative for \( t > 1 \), and satisfies

\[
\lim_{t \to 0} \frac{f_w(t)}{f_1(t)} = \lim_{t \to \infty} \frac{f_w(t)}{f_1(t)} = 0. \tag{3.5}
\]

Consider the functions \( h, v : \Omega \to \mathbb{R} \) given by

\[
v(x) := u(x)f_w\left(\frac{G_\varphi(x)}{u(x)}\right), \quad h(x) := u(x)f_1\left(\frac{G_\varphi(x)}{u(x)}\right),
\]

and recall that \( G_\varphi \) is a positive solution of the problem

\[
\begin{aligned}
Pw &= \varphi \quad \text{in } \Omega, \\
Bw &= 0 \quad \text{on } \partial \Omega_{\text{Rob}}.
\end{aligned}
\]

A direct calculation (see [4, (4.13)]) shows that if \( 0 < G_\varphi / u < 1 \) in \( \Omega \), then we have

\[
Pv = P(u f_w(G_\varphi / u)) = -u f_w''(G_\varphi / u)|\nabla(G_\varphi / u)|^2_A + u f_w'(G_\varphi / u)P(G_\varphi) = u f_w(G_\varphi / u)w(G_\varphi / u)|\nabla(G_\varphi / u)|^2_A + u f_w'(G_\varphi / u)P(G_\varphi) = Wv \geq 0 \quad \text{in } \Omega, \tag{3.6}
\]

and

\[
Ph = P(u f_1(G_\varphi / u)) = -u f_1''(G_\varphi / u)|\nabla(G_\varphi / u)|^2_A + u f_1'(G_\varphi / u)P(G_\varphi) = u f_1(G_\varphi / u)w(G_\varphi / u)|\nabla(G_\varphi / u)|^2_A + u f_1'(G_\varphi / u)P(G_\varphi) \geq 0 \quad \text{in } \Omega. \tag{3.7}
\]
Moreover,

\[ \beta \frac{f'(G/\mu)}{\mu} \langle uAVG - GAVu, \vec{n} \rangle = f'_w(G/\mu) (\beta \langle A\nabla G + G\hat{b}, \vec{n} \rangle + \gamma G) = f'_w(g/\mu)BG = 0 \]

on \( \partial \Omega_{Rob} \) in the weak sense. Therefore,

\[ Bu = B(uf_w(G/\mu)) = \beta f_w(G/\mu) \langle A\nabla u, \vec{n} \rangle \]

\[ + \beta \frac{f'(G/\mu)}{\mu} \langle uAVG - GAVu, \vec{n} \rangle + \beta \langle uf_w(G/\mu)\hat{b}, \vec{n} \rangle + \gamma uf_w(G/\mu)u \]

\[ = f_w(G/\mu)Bu = 0. \quad \text{on } \partial \Omega_{Rob} \text{ in the weak sense.} \]

Similarly, one can verify that \( B(h) = 0 \) on \( \partial \Omega_{Rob} \) in the weak sense. As a result, we obtain that \( v \in \mathcal{H}^0_{P-W,B}(\Omega) \) and \( h \in \mathcal{S}\mathcal{H}_{P-W,B}(\Omega) \cap C(\overline{\Omega} \setminus \partial \Omega_{Dir}) \). Moreover, (3.4) and (3.5) imply

\[ \lim_{x \to \infty_{Dir}} \frac{v(x)}{h(x)} = \lim_{x \to \infty_{Dir}} \frac{u(x)f_w \left( \frac{G(x)}{u(x)} \right)}{u(x)f_1 \left( \frac{G(x)}{u(x)} \right)} = 0. \]

By Lemma 3.1, \( v(x) \) is a ground state of \((P-W,B)\) in \( \Omega \), and therefore, \((P-W,B)\) is critical in \( \Omega \).

**Remark 3.3.** Sufficient conditions for the existence of a function \( u \) satisfying Ancona’s condition (3.4) are known in the case \( \partial \Omega_{Rob} = \emptyset \). Indeed, Ancona proved in [2] that if \( P \) is symmetric (or more generally quasi-symmetric in the sense of Ancona) such a positive solution \( u \) exists. Moreover, Ancona gave a counter example of a nonsymmetric operator [2] that does not admit such a positive solution \( u \). It seems that Ancona’s approach of constructing such a solution \( u \) applies also to our setting (Ancona, Private Communication).

Clearly, in the nonsymmetric case, the existence of such a \( u \) is guaranteed if \( \Omega \) is a bounded Lipschitz domain, the coefficients of \((P,B)\) are up to the boundary regular enough, \((P,B)\mathbf{1} = 0 \) in \( \Omega \), and \( \partial \Omega_{Rob}, \partial \Omega_{Dir} \neq \emptyset \) are both relatively open and closed disjoint smooth bounded sets.

Note that \( uf_w(G/\mu) \), the ground state of \((P-W,B)\) in \( \Omega \), is Hölder continuous in \( \overline{\Omega} \setminus \partial \Omega_{Dir} \), and \( W \) in (1.6) belongs to \( L^1_{loc}(\overline{\Omega} \setminus \partial \Omega_{Dir}) \). The following lemma guarantees the Hölder continuity of the ground state of \((P^* - W, B^*)\), the adjoint operator of \((P-W,B)\).

**Lemma 3.4.** Let \( W \in L^1_{loc}(\overline{\Omega} \setminus \partial \Omega_{Dir}) \), and assume that \((P-W,B)\) is critical in \( \Omega \) with a ground state \( \psi \in \mathcal{H}^0_{P-W,B}(\Omega) \cap C(\overline{\Omega} \setminus \partial \Omega_{Dir}) \), and let \( \psi^* \in \mathcal{H}^1_{P-W,B}(\Omega) \) be the ground state of \((P - W)^*, (B^*)^* \) in \( \Omega \). Then \( \psi^* = g^* \psi \), where \( g^* \) is the ground state of \(((P - W)^*)^*, (B^*)^* \) in \( \Omega \).

In particular, \( \psi^* \in C^\alpha(\overline{\Omega} \setminus \partial \Omega_{Dir}) \).

**Proof.** As \( g^* \) is a weak solution of an operator satisfying Assumptions 2.5, \( g^* \in C^\alpha(\overline{\Omega} \setminus \partial \Omega_{Dir}) \). Consequently, Remark 2.15 implies that \( \psi^* = g^* \psi \in C^\alpha(\overline{\Omega} \setminus \partial \Omega_{Dir}) \).
We continue with the following elementary proposition that is needed later in the paper.

**Proposition 3.5** [20, Proposition 6.3]. Let \( w(t) = (2t - at^2)^{-2} \). Then for any \( \xi, M, a > 0 \), and \( t \in (0, 2/a) \) the function

\[
u(\xi) := \sqrt{2t - at^2} \cos \left( \frac{\xi}{2} \log \left( \frac{Mt}{2 - at} \right) \right)
\]  

(3.8)

satisfies the following properties.

1. \(-u'' - (1 + \xi^2)wu = 0\) in \(2/(Me^{\pi/\xi} + a), 2/(M + a))\).
2. The oblique boundary condition: \(u'(2/(M + a)) = \frac{M^2 - a^2}{4M} u(2/(M + a))\).
3. The Dirichlet boundary condition: \(u(2/(Me^{\pi/\xi} + a)) = u(2/(Me^{\pi/\xi} + a)) = 0\).
4. \(u\) converges pointwise to \(\sqrt{2t - at^2}\) as \(\xi \to 0\).
5. \(|u(\xi)| \leq \sqrt{2t - at^2}\).

**Theorem 3.6.** Assume that \((P, B), G, G_\varphi, W,\) and \(u\) satisfy the assumptions of Theorem 3.2.

Assume further that one of the following conditions is satisfied.

1. \((P, B)\) is symmetric, \(A \in C^{0, \alpha}_\text{loc}(\Omega \setminus \partial\Omega_{\text{Dir}, \mathbb{R}^n})\), \(b = \bar{b} \in C^\alpha_\text{loc}(\Omega \setminus \partial\Omega_{\text{Dir}, \mathbb{R}^n})\), \(c \in L^\infty_\text{loc}(\Omega \setminus \partial\Omega_{\text{Dir}})\), and \(\partial\Omega_{\text{Rob}} \in C^{1, \alpha}\).
2. \(\partial\Omega_{\text{Rob}}, \partial\Omega_{\text{Dir}}\) are both relatively open and closed sets, \(\partial\Omega_{\text{Rob}}\) is bounded and admits a finite number of connected components; the coefficients of \(P\) are smooth functions in \(\Omega\) (or more generally \(A \in C^{\lceil(3n-2)/2\rceil, 1}_\text{loc} (\Omega, \mathbb{R}^n)\)), \(b \in C^{\lceil(3n-2)/2\rceil, 1}_\text{loc} (\Omega, \mathbb{R}^n)\), \(\bar{b} \in C^{\lceil(3n-2)/2\rceil - 1, 1}_\text{loc} (\Omega, \mathbb{R}^n)\), \(c \in C^{\lceil(3n-2)/2\rceil - 1, 1}_\text{loc} (\Omega)\).

Then \((P - W, B)\) is null-critical with respect to \(W\).

**Proof.**

1. By Remark 2.11, \(u \in C^{1, \alpha}_\text{loc}(\Omega \setminus \partial\Omega_{\text{Dir}})\). Therefore, the ground state transformed operator, \((P^u, B^u)\) (Definition 2.14), satisfies the required regularity assumptions as well. As a result, we may assume without loss of generality that \((P, B)1 = 0\). Let \(\xi > 0\) be fixed and consider the set

\[
\Omega(\xi) := \left\{ x \in \Omega \middle| \frac{2}{M e^{\pi/\xi} + a} < G_\varphi(x) < \frac{2}{M + a} \right\}.
\]  

(3.9)

Letting \(M\) large enough, we may assume that

\[
\text{supp}(\varphi) \cap \Omega(\xi) = \emptyset, \quad \text{and} \quad \Omega(\xi) \subset \Omega \setminus \partial\Omega_{\text{Dir}}.
\]

By Remark 2.11, \(G_\varphi \in C^{1, \alpha}_\text{loc}(\Omega \setminus \partial\Omega_{\text{Dir}}) \cap W^{2, 2}_\text{loc}(\Omega \setminus \partial\Omega_{\text{Dir}})\). \(G_\varphi \in C^{1, \alpha}(\Omega)\) implies that \(\Omega(\xi)\) is a set of finite perimeter for almost every \(\xi\) [9, proof of Theorem 5.9]. Therefore, we may use the coarea formula (cf. [4, Lemma 9.2] and [9, Theorem 5.9]) to obtain

\[
\int_{\Omega(\xi)} W(f_w(G_\varphi))^2 \, dx = \int_{\Omega(\xi)} \frac{|V G_\varphi|^2}{2G_\varphi - aG_\varphi^2} \, dx = \int_{2/(M+\xi)}^{2/(M+a)} \frac{1}{2t - at^2} \, dt \int_{G_\varphi = t} A V G_\varphi \cdot \vec{n} \, d\sigma,
\]  

(3.10)
where for almost every \( t \), the vector \( \nabla G_{\varphi}(x) \) is parallel (in the metric \( | \cdot |_{A} \)) to the normal vector \( \vec{n}(x) \) for \( H^{n-1} \)-almost every \( x \) in the level set \( \{ G_{\varphi} = t \} \) [3]. Moreover, \( G_{\varphi} \in W^{2,2}_{\text{loc}}(\Omega \setminus \partial\Omega_{\text{Dir}}) \) is a strong solution to the equation \( Pu = \varphi \) in \( \Omega \) and satisfies \( A\nabla G_{\varphi} \cdot \vec{n} = 0 \) everywhere on \( \partial\Omega_{\text{Rob}} \). Therefore, we may use the divergence theorem for almost every \( t_{1}, t_{2} \), satisfying

\[
\frac{2}{Me^{\pi/\xi} + a} < t_{1} < t_{2} < \frac{2}{M + a},
\]

and obtain

\[
0 = -\int_{\{x \in \Omega: t_{1} < G_{\varphi} < t_{2}\}} \text{div}(A\nabla G_{\varphi}) \, dx = \int_{G_{\varphi}=t_{2}} A\nabla G_{\varphi} \cdot \vec{n} \, d\sigma - \int_{G_{\varphi}=t_{1}} A\nabla G_{\varphi} \cdot \vec{n} \, d\sigma
\]

(see [5, Proposition 3.1]). In particular,

\[
\int_{G_{\varphi}=t_{1}} A\nabla G_{\varphi} \cdot \vec{n} \, d\sigma = \int_{G_{\varphi}=t_{2}} A\nabla G_{\varphi} \cdot \vec{n} \, d\sigma
\]

is a nonzero constant. Therefore, letting \( \xi \to 0 \) in (3.10) implies that

\[
\int_{\Omega} W(f_{w}(G_{\varphi}))^{2} \, dx \geq \lim_{\xi \to 0} \int_{\Omega_{\xi}} W(f_{w}(G_{\varphi}))^{2} \, dx \geq \int_{0}^{1} \frac{1}{2t - at^{2}} \, dt = \infty.
\]

(2) The proof in the nonsymmetric case is identical to proof of [4, Theorem 8.2]. Indeed, our assumptions imply that \( \partial\Omega_{\xi} \cap \partial\Omega_{\text{Rob}} = \emptyset \), where \( \Omega_{\xi} \) is given by (3.9). Therefore, we may repeat almost literally the steps in proof of [4, Theorem 8.2].

**Proof of Theorem 1.11.** The theorem follows from the criticality and the null-criticality of \((P - W, B)\) with respect to \( W \) proved in Theorems 3.2 and 3.6, respectively. □

**Remark 3.7.**

(1) The parameter \( a \) in the proof of Theorem 1.11 was chosen such that \( G_{\varphi}/u < 1/a \) in \( \Omega \). In fact, we may choose any constant \( a \geq 0 \) satisfying \( G_{\varphi}/u \leq 1/a \) in \( \Omega \).

(2) In the nonsymmetric case of Theorem 3.6, the stated smoothness assumptions on the coefficients simplify the calculations in [4, Theorem 8.2]. In fact, these regularity assumptions can be further weakened.

The following proposition, a particular case of [20, Proposition 3.1], is a characterization of \( \mathcal{W}(\mathbb{R}+) \), the set of all optimal Dirichlet–Hardy-weights of the Laplacian in \( \mathbb{R}+ \).

**Proposition 3.8.** Let \( 0 \leq w \in L_{\text{loc}}^{1}(\mathbb{R}+) \). Then \( w \in \mathcal{W}(\mathbb{R}+) \) with a corresponding ground state \( \psi_{w} \) if and only if the following three conditions are satisfied.

1. \( \psi_{w} > 0 \) satisfies \( -\psi'' - w\psi = 0 \) in \( \mathbb{R}+ \).
2. \( \int_{0}^{1} \frac{1}{\psi_{w}^{2}} \, dt = \int_{1}^{\infty} \frac{1}{\psi_{w}^{2}} \, dt = \infty. \)
3. \( \int_{0}^{1} \psi_{w}^{2} \, dt = \int_{1}^{\infty} \psi_{w}^{2} \, dt = \infty. \)
Remark 3.9. An explicit example of $w \in \mathcal{W}(\mathbb{R}_+)$ is the classical Hardy-weight $w_c(x) = 1/(4x^2)$ with a ground state $\psi_{w_c} = \sqrt{x}$. In fact, infinitely many optimal Hardy-weights $w \in \mathcal{W}(\mathbb{R}_+)$ can be constructed as follows.

Let $w_1 \in C_0^\infty(\mathbb{R}_+)$ be such that $0 \leq w_1(x) \leq w_c(x)$ for all $x \in \mathbb{R}_+$, and consider the subcritical operator $Py := -y'' - w_c y + w_1 y$ in $\mathbb{R}_+$.

Let $0 \leq w_2 \neq w_1$ be a smooth nonnegative small perturbation of the operator $P$ in $\mathbb{R}_+$ (for the definition of small perturbation see [17, 18]). By the results in [17, 18], there exists $\alpha > 0$ such that $P - \alpha w_2$ is critical in $\mathbb{R}_+$. It follows that $w = w_c - w_1 + \alpha w_2$ is an optimal Hardy weight in $\mathbb{R}_+$, with a ground state $\psi_w \sim \sqrt{x}$.

We are now in a position to prove Theorem 1.14.

Proof of Theorem 1.14.

(1) Can be verified as in the proof of Theorem 3.2.

(2) Use word by word the proof of Theorem 3.2.

(3) The proof is identical to [20, Theorem 5.2], where $\partial \Omega_{Rob} = \emptyset$ is assumed.

(4) It remains to prove that in the symmetric case, $(P - W, B)$ is null-critical with respect to $W$.

Without loss of generality we may assume that $(P, B) \mathbf{1} = 0$ in $\Omega$. Take $\alpha > 0$ sufficiently small such that

$$\{x \in \Omega \mid 0 < G_\varphi(x) < \alpha \} \cap \text{supp } \varphi = \emptyset.$$ 

For any $0 < \varepsilon < \alpha$, the coarea formula (3.10) implies

$$\int_{\varepsilon < G_\varphi < \alpha} (\psi_w(G_\varphi))^2 W \, dx = C \int_{\varepsilon}^{\alpha} \psi_w^2(t)w(t) \, dt.$$ 

Recall that $w \in \mathcal{W}(\mathbb{R}_+)$ with a corresponding ground state $\psi_w(t)$. Therefore, letting $\varepsilon \to 0$, and using part (3) of Proposition 3.8 we obtain

$$\int_\Omega (\psi_w(G_\varphi))^2 W \, dx \geq C \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\alpha} \psi_w^2(t)w(t) \, dt = \infty.$$ 

□

4 | EXAMPLES

In this short section, we illustrate two examples for which Theorem 1.11 provides an optimal Hardy-weight.

Example 4.1. Let $n \geq 3$, and either

$$\Omega = B_1^+(0), \; \partial \Omega_{Rob} = \{x \in B_1(0) \mid x_n = 0\}, \text{ or } \Omega = \mathbb{R}_+^n, \; \partial \Omega_{Rob} = \{x \in \mathbb{R}_n \mid x_n = 0\}.$$ 

Consider the operator $Pu := -\Delta u$ together with the boundary operator $Bu = \nabla u \cdot \vec{n}$ on $\partial \Omega_{Rob}$. Clearly, $(P, B)$ is subcritical in $\Omega$, and $(P, B)\mathbf{1} = 0$ in $\Omega$. Indeed, for $x \in \Omega$ let $\hat{x} = (x', -x_n)$, then,
for each \( x, y \in \Omega \) with \( x \neq y \),

\[
G^\Omega_{P,B}(x,y) = \begin{cases} 
G^{B_1(0)}_P(x,y) + G^{B_1(0)}_P(\hat{x},y) & \Omega = B^+_1(0) \\
G^{\mathbb{R}^n}_P(x,y) + G^{\mathbb{R}^n}_P(\hat{x},y) & \Omega = \mathbb{R}^n_+,
\end{cases}
\]

where \( G^{B_1(0)}_P(x,y) \) (resp., \( G^{\mathbb{R}^n}_P(x,y) \)) is the Dirichlet–Green function of \( P \) in \( B_1(0) \) (resp., \( \mathbb{R}^n \)). Obviously, \( \lim_{x \to \infty_{\text{Dir}}} G_\varphi(x,y) = 0 \), and hence, Theorem 1.11 implies that the function \( W = \frac{P(f_w(G_\varphi))}{f_w(G_\varphi)} \) is an optimal weight for \((P,B)\) in \( \Omega \).

We note that in the case \( \Omega = B^+_1(0) \), \( W(x) \sim (2 \cdot \text{dist}(x,\partial \Omega_{\text{Dir}}))^{-2} \) as \( x \to \xi \), where \( \xi_n > 0 \) and \( |\xi| = 1 \) [15, Lemma 3.2].

On the other hand, in the case \( \Omega = \mathbb{R}^n_+ \), \( W \) is a continuous function in \( \Omega \) and \( W(x) \sim (n-2)^2 \frac{4}{|x|^{-2}} \) as \( x \to \infty \) such that \( x/|x| \to (\xi',\xi_n) \) with \( \xi_n > 0 \).

**Example 4.2.** Let \( n \geq 3 \), and \( \Omega = \{ x \in \mathbb{R}^n \mid |x| > 1 \} \) with \( \partial \Omega_{\text{Rob}} = \partial \Omega \). Assume that \( Pu = -\Delta u \) and \( Bu = \nabla u \cdot \vec{n} + \gamma(x)u \) on \( \partial \Omega_{\text{Rob}} \), where \( \gamma \in L^\infty(\partial \Omega_{\text{Rob}}) \) satisfies \( \gamma > (1-n)/2 \), and take \( \varepsilon > 0 \) such that \( \varepsilon(n+2\gamma-1) \geq 1 \) on \( \partial \Omega_{\text{Rob}} \). Then,

\[
v := \sqrt{|x| - 1 + \varepsilon} |x|^{1-n} \in RSH_{P,B}(\Omega)
\]

and

\[
\begin{cases}
-\Delta v - \frac{(n-1)(n-3)v}{4|x|^2} - \frac{v}{4(|x| - 1 + \varepsilon)^2} = 0 & \text{in } \Omega, \\
\nabla v \cdot \vec{n} + \gamma v = \frac{-1 + \varepsilon(n+2\gamma-1)}{2\sqrt{\varepsilon}} \geq 0 & \text{on } \partial \Omega_{\text{Rob}}.
\end{cases}
\]

Hence, the Agmon–Allegretto–Piepenbrink-type theorem [21, Theorem 6.1] implies the Hardy-type inequality

\[
\int_\Omega |\nabla \phi|^2 \, dx + \int_{\partial \Omega_{\text{Rob}}} \gamma \phi^2 \, d\sigma \geq \int_\Omega \left[\frac{(n-1)(n-3)}{4|x|^2} + \frac{1}{4(|x| - 1 + \varepsilon)^2}\right] \phi^2 \, dx \quad (4.1)
\]

for all \( \phi \in C_0^\infty(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}) \).

**Remark 4.3** (Improved Hardy-inequality in the exterior of the unit ball). Assume further that \( \gamma \geq 0 \) is constant. In [13, Theorem 5.1], (4.1) has been obtained for the case \( \varepsilon = (2\gamma)^{-1} \). Obviously, by letting \( \varepsilon = \varepsilon_\gamma := (n-1+2\gamma)^{-1} \) in (4.1), we obtain an improvement of the Hardy-type inequality in [13, Theorem 5.1]. In particular, the function

\[
u_\gamma := \sqrt{|x| - 1 + \varepsilon_\gamma} |x|^{1-n}
\]
is a positive solution of the equation
\[
\begin{cases}
-\Delta v - \frac{v(n-1)(n-3)}{4|x|^2} - \frac{v}{4(|x| - 1 + \varepsilon)^2} = 0 & \text{in } \Omega, \\
\nabla v \cdot \hat{n} + \gamma v = 0 & \text{on } \partial \Omega_{\text{Rob}}.
\end{cases}
\] (4.2)

Furthermore, the function \( w_\gamma := v_\gamma \log(|x| - 1 + \varepsilon) \) is a positive solution of the equation
\[
-\Delta v - \frac{v(n-1)(n-3)}{4|x|^2} - \frac{v}{4(|x| - 1 + \varepsilon)^2} = 0
\] in a neighborhood of \( \infty \), satisfying
\[
\lim_{|x| \to \infty} \frac{v_\gamma}{w_\gamma} = 0.
\]

By Lemma 3.1, any positive solution \( \phi \) of (4.3) having minimal growth at \( \infty \) satisfies \( \phi \asymp v_\gamma \) in a neighborhood of \( \infty \). As a result, \( v_\gamma \) is the ground state of (4.2). Moreover, it is easy to see that (4.2) is also null-critical, implying that the function
\[
W := \frac{(n-1)(n-3)}{4|x|^2} + \frac{1}{4(|x| - 1 + \varepsilon)^2}
\]
is an optimal Hardy-weight of \((P, B)\) in \( \Omega \). For an explicit formula for \( G_{P, B}^\Omega(x, y) \) in the cases \( \gamma = 0, (n - 2)/2 \), see [22].

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**JOURNAL INFORMATION**

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**REFERENCES**

1. S. Agmon, *On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds*, Methods of functional analysis and theory of elliptic equations, 1982, pp. 19–52.
2. A. Ancona, *Some results and examples about the behavior of harmonic functions and Green’s functions with respect to second order elliptic operators*, Nagoya Math. J. 165 (2002), 123–158.
3. G. Chen, M. Torres, and W. Ziemer, *Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws*, Comm. Pure Appl. Math. 62 (2009), 242–304.
4. B. Devyver, M. Fraas, and Y. Pinchover, *Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon*, J. Funct. Anal. 266 (2014), 4422–4489.
5. B. Devyver and Y. Pinchover, *Optimal \( L^p \) Hardy-type inequalities*, Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2016), 93–118.
6. M. Egert, R. Haller-Dintelmann, and J. Rehberg, *Hardy’s inequality for functions vanishing on a part of the boundary*, Potential Anal. **43** (2015), 49–78.

7. M. Egert and P. Tolksdorf, *Characterizations of Sobolev functions that vanish on a part of the boundary*, Discrete Contin. Dyn. Syst. Ser. S **10** (2017), 729–743.

8. T. Ekholm, H. Kovářík, and A. Laptev, *Hardy inequalities for p-Laplacians with Robin boundary conditions*, Nonlinear Anal. **128** (2015), 365–379.

9. L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in advanced mathematics, CRC Press, Boca Raton, FL, 1992.

10. S. Filippas and A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal. **192** (2002), 186–233.

11. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in mathematics, Springer, Berlin, 2001.

12. M. Keller, Y. Pinchover, and F. Pogorzelski, *Optimal Hardy inequalities for Schrödinger operators on graphs*, Comm. Math. Phys. **358** (2018), 767–790.

13. H. Kovářík and A. Laptev, *Hardy inequalities for Robin Laplacians*, J. Funct. Anal. **262** (2012), 4972–4985.

14. H. Kovářík and Y. Pinchover, *On minimal decay at infinity of Hardy-weights*, Commun. Contemp. Math. **22** (2019), 1950046, 18 pp.

15. P. D. Lamberti and Y. Pinchover, *$L^p$ Hardy inequality on $C^{1,\gamma}$ domains*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19** (2019), 1135–1159.

16. G. M. Lieberman, *Oblique derivative problems for elliptic equations*, World Scientific Publishing, Hackensack, NJ, 2013.

17. M. Murata, *Semismall perturbations in the Martin theory for elliptic equations*, Israel J. Math. **102** (1997), 29–60.

18. Y. Pinchover, *Criticality and ground states for second-order elliptic equations*, J. Differential Equations **80** (1989), 237–250.

19. Y. Pinchover and T. Saadon, *On positivity of solutions of degenerate boundary value problems for second-order elliptic equations*, Israel J. Math. **132** (2002), 125–168.

20. Y. Pinchover and I. Versano, *On families of optimal Hardy-weights for linear second-order elliptic operators*, J. Funct. Anal. **278** (2020), 108428.

21. Y. Pinchover and I. Versano, *On criticality theory for mixed boundary value problems for elliptic operators in divergence form*, Commun. Contemp. Math. **25** (2022), 2250051.

22. M. Sadybekov, B. Torebek, and B. Turmetov, *Representation of the Green’s function of the exterior Neumann problem for the Laplace operator*, Sibirsk. Mat. Zh. **58** (2017), 199–205; translation in Sib. Math. J. **58** (2017), 153–158.

23. T. Toro, *Analysis and geometry on non-smooth domains*, Rev. Acad. Colombiana Cienc. Exact. Fis. Natur. **41** (2017), 521–527.