Cumulative jet generation in a plane parallel potential flow of the perfect incompressible fluid

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Abstract. This manuscript deals with the plane parallel unsteady potential flow of the perfect incompressible fluid limited by a free boundary. As the result of the free boundary deformation, cumulative jets may form. A cumulative jet generation in a problem of cylindrical cavity deformation is considered in this manuscript. A numerical algorithm for this problem is proposed. Based on the boundary elements method with no saturation, the algorithm has a high order of approximation and reasonable computation time. The distribution of points along the length of the cavity boundary is required to be constant at computation time. This requirement provides the numerical stability of the algorithm. To verify the correctness of the proposed algorithm, conservation laws are used.

1. Introduction
Cumulative jet generation is a particular case of a problem with a free boundary. Its numerical modelling is one of areas in computational methods and fluid mechanics. This manuscript studies numerically cumulative jets that are formed within cylindrical cavities under different potential flows. Only plane parallel flows are considered. Fluid is perfect and incompressible.

There are various semi-analytical and numerical methods for the problem. The most notable results have been obtained by Voinov in paper [1]. It proposes a numerical algorithm for one of the simplest cases of the problem, i.e. deformation of a circular cylindrical cavity with a unit radius in a plane parallel flow. Potential and stream functions for the problem at initial time are given by the equations

$$\Phi = \frac{x}{x^2 + y^2}, \quad \Psi = -\frac{y}{x^2 + y^2}. \quad (1)$$

(the initial condition for the problem of flow past the circular cylindrical cavity moving with a unit velocity). Calculation results have shown that a cumulative is formed under the flow and the cavity splits into two symmetrical parts.

This manuscript proposes a numerical algorithm for the problem. It is more stable rather than the one proposed in paper [1] and allows us to improve the previous calculation results.

Cumulative jets generation under asymmetrical plane parallel potential flows with circulation is also considered in this manuscript.

To verify the correctness of the proposed algorithm, conservation laws are used.
2. General equations

Since the proposed algorithm is based on the boundary elements method, all of the calculations are made on the cavity boundary $\partial S$. The basic equation is written in integral form and expresses the relationship between partial derivative of potential function $\Phi$ with respect to spatial coordinate and stream function $\Psi$

$$-2\pi \Psi = A \frac{\partial \Phi}{\partial s} + B \Psi,$$

where

$$(A \frac{\partial \Phi}{\partial s})(M, t) := \oint_{\partial S} G(M, M') \frac{\partial \Phi}{\partial s'}(M', t) \ ds',$$

$$(B \Psi)(M, t) := \oint_{\partial S} \frac{\partial G}{\partial n'}(M, M')(\Psi(M', t) - \Psi(M, t)) \ ds',$$

$G(M, M') := \ln(r(M, M')),$

$$r^2(M, M') := (x - x')^2 + (y - y')^2,$$

$M(x, y), M'(x', y') \in \partial S.$

Cylindrical cavity boundary in a plane parallel flow at any moment of time $t$ is one-dimensional and can be parameterized by a single variable. Let us denote it by $\zeta$. The following parameterization is used in the proposed algorithm:

$$ds = l(t) f(\zeta) d\zeta, \quad 0 \leq \zeta \leq 1, \quad \int_{0}^{1} f(\zeta) \ d\zeta = 1,$$

where $l(t)$ is the overall length of the boundary and $f(\zeta)$ is a positive function that is used to manage the distribution of points along the cavity boundary. Let us call it the density function. The reason for this parameterization is numerical stability. Points with fixed values of $\zeta$ will be used to approximate cavity boundary, therefore the distribution of points along the cavity boundary will be constant at computation time. Due to the choice of density function, we can arbitrary change the number of points that are used to approximate different segments of the boundary.

The normal component of the boundary velocity is equal to the fluid velocity and can be calculated as a partial derivative of the stream function $\Psi$ with respect to spatial coordinate $s$

$$V(\zeta, t) = \frac{\partial \Psi}{\partial s}(\zeta, t) = \frac{1}{l(t) f(\zeta)} \cdot \frac{\partial \Psi}{\partial \zeta}(\zeta, t).$$

Tangential and normal vectors are required for further calculations. They can be derived from the following equations:

$$\tau(\zeta, t) = \left( \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s} \right) = \frac{1}{l(t) f(\zeta)} \left( \frac{\partial x}{\partial \zeta}, \frac{\partial y}{\partial \zeta} \right),$$

$$n(\zeta, t) = \left( \frac{\partial y}{\partial s}, -\frac{\partial x}{\partial s} \right) = \frac{1}{l(t) f(\zeta)} \left( \frac{\partial y}{\partial \zeta}, -\frac{\partial x}{\partial \zeta} \right).$$

Using the equations above, we can calculate spatial coordinates $x$ and $y$ as follows:

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{1}{l(t) f(\zeta)} \left( U \frac{\partial x}{\partial \zeta} + V \frac{\partial y}{\partial \zeta} \right),$$
\[
\frac{\partial y}{\partial t}(\zeta, t) = \frac{1}{l(t)f(\zeta)} \left( U \frac{\partial y}{\partial \zeta} - V \frac{\partial x}{\partial \zeta} \right),
\]
where \(U\) is the tangential component of velocity. It is chosen to keep the distribution of points along the cavity boundary unchanged at computation time.

Finally, we will use Bernoulli’s equation:

\[
\frac{\partial \Phi}{\partial t} \bigg|_{x,y} + \frac{1}{2} \left( V^2 + \left( \frac{\partial \Phi}{\partial s} \right)^2 \right) + p = 0,
\]
where \(p(t)\) is an unknown pressure function. We can express the derivative of \(\Phi\) with respect to \(t\) at constant \(\zeta\) using the derivative of \(\Phi\) with respect to \(t\) at constant \(x, y\) by the following formula:

\[
\frac{\partial \Phi}{\partial t} \bigg|_{\zeta} = \frac{\partial \Phi}{\partial t} \bigg|_{x,y} + U \frac{\partial \Phi}{\partial s} + V^2.
\]

Therefore, we have

\[
\frac{\partial \Phi}{\partial t} \bigg|_{\zeta} + p = \frac{1}{2} V^2 + \frac{U}{l(t)f(\zeta)} \frac{\partial \Phi}{\partial \zeta} - \frac{1}{2} \left( \frac{1}{l(t)f(\zeta)} \frac{\partial \Phi}{\partial \zeta} \right)^2.
\]

Note that we need only partial derivatives of \(\Phi\) with respect to spatial coordinates in equation (2), therefore we can replace \(\Phi\) with the following function:

\[
\tilde{\Phi}(\zeta, t) := \Phi(\zeta, t) + \int_0^t p(t') dt'.
\]

The final form of equation (11) is

\[
\frac{\partial \tilde{\Phi}}{\partial t} \bigg|_{\zeta} = \frac{1}{2} V^2 + \frac{U}{l(t)f(\zeta)} \frac{\partial \tilde{\Phi}}{\partial \zeta} - \frac{1}{2} \left( \frac{1}{l(t)f(\zeta)} \frac{\partial \tilde{\Phi}}{\partial \zeta} \right)^2.
\]

3. Approximation
Let us choose \(N\) points along the length of the cavity boundary for numerical computations. Each point corresponds to the fixed value \(\zeta_i = \frac{i}{N}\), where \(i = 1, 2, ..., N\). In this case distribution of points will be constant at computation time. This requirement is used to calculate tangent components of velocity \(U_i\). To satisfy it numerically, we require that the distances between adjacent points change in proportion to the length of the boundary. It can be expressed by the following system of equations:

\[
\Delta s_i(t + \Delta t) = \frac{l(t + \Delta t)}{Nf(\zeta_i)} \left( \zeta_i - \frac{1}{2} \right),
\]

where \(i = 1, 2, ..., N\) and \(\Delta s_i(t + \Delta t)\) is the distance between points with indices \(i\) and \(i + 1\). Expressing \(\Delta s_i(t + \Delta t)\) in terms of spatial coordinates \(x_i, y_i\) and velocities \(V_i, U_i\) at the moment of \(t\), we derive a system of nonlinear equations for \(U_i\). Note that we still can arbitrary change one of the tangent components \(U_i\) because a position of initial point \((\zeta = 0)\) is not fixed. Iterative process is used to solve the system of equations.

The curvature of the boundary segment where cumulative jet is formed usually increases in course of time. To provide high accuracy of computations on the forward end of cumulative jet,
a high number of points should be used to approximate it. We can guarantee this because of the density function $f(\zeta)$ choice.

To approximate equation (2), we should replace integral terms with quadrature formulas. Integral terms are represented by $A$ and $B$ operators. Both of them have a point of singularity and should be approximated very accurate. The advantage of the proposed algorithm is that quadrature formulas with no saturation are used. These quadratures have been proposed in paper [2]. The idea behind them is to construct quadrature formulas that yield the exact result for the first $N^2$ partial sums of Fourier series. Since the error of approximation by Fourier series of infinitely differentiable periodic function decaying faster than $O(\frac{1}{N^n})$ for all $n > 0$, high order of approximation is granted by the quadrature formulas. Quadrature formulas for the problem have the form

$$\int_{0}^{1} G(\zeta, \zeta') \frac{\partial \Phi}{\partial \zeta'}(\zeta', t) \, d\zeta' \approx \frac{1}{N} \sum_{j=1}^{N} (\beta(|i - j|) + G_{ij}) \frac{\partial \Phi}{\partial \zeta'}(\zeta_j, t), \quad (17)$$

$$\beta(0) = \alpha(0), \quad \beta(m) = -\ln \left| \sin \frac{\pi m}{N} \right| + \alpha(m),$$

$$\alpha(m) = -\left( \ln 2 + \frac{(-1)^m}{N} + \sum_{k=1}^{N-1} \frac{1}{k} \cos \frac{2\pi km}{N} \right),$$

$$G_{ij} = G(\zeta_i, \zeta_j) \quad i \neq j,$$

$$G_{ii} = \lim_{\zeta \to \zeta_i} (G(\zeta_i, \zeta) - \ln |\sin(\pi(\zeta - \zeta_i))|),$$

$$\int_{0}^{1} \frac{\partial G}{\partial n'}(\zeta, \zeta')(\Psi(\zeta', t) - \Psi(\zeta, t)) \, d\zeta' \approx \frac{1}{N} \sum_{j=1}^{N} G^n_{ij}(\Psi(\zeta_j, t) - \Psi(\zeta_i, t)), \quad (18)$$

$$G^n_{ij} = \frac{x_j - x_i}{r_{ij}^2} \frac{\partial y}{\partial \zeta}(\zeta_j, t) - \frac{y_j - y_i}{r_{ij}^2} \frac{\partial x}{\partial \zeta}(\zeta_j, t) \quad i \neq j,$$

$$G^n_{ii} = 0.$$

Most of the differential equations contain partial derivatives of $\Phi, \Psi, x$ and $y$ with respect to spatial coordinates. These derivatives can be approximated sufficiently accurate using a cubic spline for periodic functions.

To calculate the potential function and spatial coordinates functions on the next time step, the Runge–Kutta method is applied to equations (9), (10) and (15).

4. Calculation results

The proposed algorithm was implemented and tested using the problem stated in [1], i.e. evolution of a circular cylindrical cavity with a unit radius in a plane parallel potential flow. 128 points were used to approximate the cavity boundary. Near the right edge of the cavity a cumulative jet is formed. The forward end of the jet has a high curvature, therefore a high number of points is required to approximate it. Density function

$$f(\zeta) = 1 - 0.99 \cos(2\pi \zeta). \quad (19)$$

was used by the algorithm. The distribution is symmetrical about the $x$ axis.

Cavity boundaries for eight different moments of time are shown on Figure 1, i.e. $t = 0, 0.17, 0.31, 0.43, 0.57, 0.71, 0.86$ and 0.93. All of them except the last one have been chosen to
demonstrate results in paper [1]. The proposed algorithm has shown similar results with those obtained in paper [1], but the cumulative jet calculated by the algorithm was closer to hitting the opposite side. The left-most point of the cumulative jet and the point on the opposite side velocities are shown on Figure 2. Note that cumulative jet velocity turns stable in course of time.

To verify correctness of the proposed algorithm, conservation of energy, momentum and area were checked. The following equations were used to calculate energy and momentum:

\[ E_c = -\frac{1}{2} \int_0^1 \Phi \frac{\partial \Psi}{\partial \zeta} d\zeta, \]  

\[ I_x = -\int_0^1 \Phi \frac{\partial y}{\partial \zeta} d\zeta. \]  

When approximating the cavity boundary with 128 points, overall errors for them did not exceed \( 3 \cdot 10^{-4} \).

Let us consider more complex example viz. cumulative jet generation under asymmetric plane parallel potential flow with circulation.

In this case potential function at initial time is given by the following equation:

\[ \Phi = \cos \varphi \frac{\Gamma}{r} + \frac{\Gamma}{2\pi} \varphi, \]  

where \( \varphi \) and \( r \) are angular and radial coordinates of the point on the cavity boundary and \( \Gamma \) is a circulation. Three different cases of \( \Gamma = 1, 5, 10 \) were considered. Calculation results are shown on Figures 3, 4, 5.

The results show that cumulative jet shifts from horizontal axis under the flow with circulation, therefore additional efforts are required to keep a high density of points on the jet at computation time. This difficulty was overcome by shifting points along the cavity boundary. Tangent component of velocity \( U_N \) was chosen in such a way to keep corresponding point on the
forward end of the jet. Experimentally it was obtained that $U_N$ can be set automatically using tangent component of the fluid velocity in this point. Tangent component of fluid velocity can be found as a partial derivative of potential function $\Phi$ with respect to spatial coordinate.

Note that $\Phi$ is a multivalued function because of circulation. It must be taken into account when deriving conservation laws. Conservation of linear momentum can be derived from equation

$$\frac{d}{dt} \int_S F n ds = \int_S \left( \frac{\partial F}{\partial t} n + v_n \nabla F \right) ds$$  \hspace{1cm} (23)$$

that is valid for all continuously differentiable single-valued functions $F$. Applying this equation
to single-valued function $F = \Phi - \frac{\Gamma}{2\pi} \varphi$ and transforming it, we can derive the following equations:

$$\frac{d}{dt} \int_{S} \left( \Phi - \frac{\Gamma}{2\pi} \varphi \right) n_x ds + \frac{\Gamma}{2\pi} \int_{S} v_n \frac{\partial \varphi}{\partial x} ds = 0,$$

(24)

$$\frac{d}{dt} \int_{S} \left( \Phi - \frac{\Gamma}{2\pi} \varphi \right) n_y ds + \frac{\Gamma}{2\pi} \int_{S} v_n \frac{\partial \varphi}{\partial y} ds = 0.$$

(25)

Using the equations above, we can derive:

$$I_x = \int_{0}^{1} \left( - \frac{\partial \Phi}{\partial \zeta} y + \frac{\Gamma}{2\pi} \left( \frac{\partial \varphi}{\partial \zeta} y + \ln r \frac{\partial x}{\partial \zeta} \right) \right) d\zeta,$$

(26)

$$I_y = \int_{0}^{1} \left( \frac{\partial \Phi}{\partial \zeta} x + \frac{\Gamma}{2\pi} \left( - \frac{\partial \varphi}{\partial \zeta} x + \ln r \frac{\partial y}{\partial \zeta} \right) \right) d\zeta.$$

(27)

Conservation laws (26) and (27) were checked at computation time. Their overall error did not exceed $1 \cdot 10^{-4}$.

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References

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