Thermodynamic origin of universal fluctuations and two-power laws

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We discuss universality of response functions in systems with excited degrees of freedom. We propose a unification of two existing phenomenologies, two-power law decay and deviation from power law due to non-extensivity. A universal curve is derived by maximizing entropy with a non-linear constraint. The same formalism can explain the universal fluctuation curves which have been discovered recently by Bramwell, Holdsworth, and Pinton.

Many experiments measure the probability \( p(t)\) that an event takes place within the time interval \([t, t + dt]\) after excitation of the system at \( t = 0\). Classical arguments predict exponential decay. We are interested here in a power law decay at large times

\[
p(t) \sim t^{-1/(1-\alpha)}
\]  
(1)

with \( 0 < \alpha < 1 \).

There is an extensive literature explaining non-exponential decay by means of fractal theory. In this domain of research it is accepted that a scaling law like (1) holds only for a limited range \( t_1 < t < t_2 \) of the argument \( t \). For \( t < t_1 \) collective effects presumably restore non-fractal behavior. For large \( t \) individual behavior on microscopic scale dominates and non-universal decay is expected. A nearly ideal example of this behavior has been observed recently in quantum dots [1], with a power law extending over a range of \( 10^5 \) of time scales. Many theoretical models explain power law decay as the correct asymptotics in the large time limit.

There is more and more evidence that also non-power law decay can be described by a universal probability density function (PDF). An early explanation [2] of non-exponential decay involves the notion of an activation density function (PDF). An early explanation [2] of non-power law decay can be described by a universal probability asymptotics in the large time limit.

The actual formula, which we will derive below, and which covers the whole fractal region, is (see Fig. 1)

\[
p(t) \sim e^{-(\omega t)^\gamma},
\]  
(3)

which is expected in case of processes with a scaling distribution of relaxation rates. Here, we restrict ourselves to the case of asymptotic decay with a single power law.

In dielectric relaxation one observes usually two regions of power law response (in our notations)

\[
p(t) \sim (\omega t)^{-1/(1-\alpha)} \quad \omega t \ll 1
\]

\[
\sim (\omega t)^{-1/(1-\rho)} \quad \omega t \gg 1.
\]

(4)

(see e.g. Weron and Jurlewicz [3]). The point of view of the present paper is that both behaviors, as described by (4) and (3), should be combined. More precisely, the generic behavior in the region \( \omega t \ll 1 \) should be described by (3) instead of (4). This substitution makes sense because (3) describes a curve which starts off linearly and bends over towards a power law decay with exponent \(-1/(1-\alpha)\).

FIG. 1. Log-log-plot of \( p(t) \) with \( a = 500 \) for different combinations of exponents \((\alpha, \rho)\): (0.1,0.1) dotted, (0.1,1) solid, (0.1,0.8) short-dashed, (0.1,1.08) long-dashed.

The actual formula, which we will derive below, and which covers the whole fractal region, is (see Fig. 1)

\[
p(t) \sim \left(1 - a + a (1 + (1 - \rho)\omega t)^{1/(1-\alpha)}\right)^{-1/(1-\alpha)}.
\]

(5)

Three regions can be distinguished. In regions 1 and 2 the decay is as described by (2). Regions 2 and 3 show power law decay according to (3). Note that \( \rho > 1 \) corresponds to a super-exponential decay.

For \( \rho = \alpha \) expression (5) reduces to (4). If \( a = 1 \) then it reduces to (2) with \( \alpha \) replaced by \( \rho \). In the limit \( \rho = 1 \) one obtains
\[ p(t) \sim (1 - a + a \exp((1 - \alpha)\omega t))^{-1/(1 - \alpha)}. \]

In the limit \( \alpha = 1 \) one obtains
\[ p(t) \sim (1 + (1 - \rho)\omega t)^{-\alpha/(1 - \rho)}. \]

In any case, in the limit \( \alpha = \rho = 1 \) the decay is exponential \( p(t) \sim \exp(-\omega t). \)

We found first evidence for (5) in the work of Tsallis, Bemski, and Mendes [6]. They reanalyze the experimental data of [2] and observe that the description using (5) improves significantly by using (6), formula which was introduced on an ad hoc basis. The full data set of [2] could be described with it. The universality of (6) is further supported by recent work of Montemurro [7] using a very convincing analysis of linguistic data. In both papers [6] and [7] there is evidence that the asymptotic behavior for large \( t \) still deviates from (6), and follows a power law instead of decaying exponentially. In the logic of the present paper, this is a cross-over from one power law to another, as described by (6). Hence (6) should be used with \( \rho \neq 1 \) instead of (5). In [7] an ad hoc modification of (6) was proposed, slightly different from (6). The same expression was used in [7] and fits the experimental data over the whole range of \( t \)-values. It is clear that (6) will fit as well, but has a number of advantages: 1) there is a general formalism, which is used to derive it; 2) the PDF has a workable algebraic expression, which is not the case for the formula proposed in [7]; 3) within the context of the formalism an average time \( \langle \omega t \rangle \) is defined using a nonlinear average – see (14) below. The average time characterizes the PDF \( p(t) \) together with the two exponents \( \alpha \) and \( \rho \), and the cross-over frequency \( \omega \). It replaces the fitting parameter \( \alpha \) which has a less clear physical meaning (see Fig. 2). In particular, in the case \( \rho = 1 \), we show [6] that \( \langle \cdot \rangle \) satisfies a property of additivity, which implies that it is meaningful to add average times of different experiments all having the same values of \( \alpha \) and \( \omega \).

Up to now we considered decay as a function of time. However the PDF (5) can also be considered as a generalization of the distribution of Boltzmann-Gibbs. Time \( t \) is then replaced by energy \( E \), frequency \( \omega \) by a fixed inverse temperature \( \beta \). With this interpretation (5) is the equilibrium distribution of non-extensive thermostatistics [9]. The entropic parameter \( q \) of this formalism, coincides with the exponent \( \alpha \) (the relation becomes \( q = 2 - \alpha \) in the recently modified formalism [10]). In the context of non-extensive thermostatistics (5) has numerous applications — for a review see [8] or [12]. In this context (5) becomes the equilibrium distribution for thermostatistics based on Rényi’s choice of Komogorov-Nagumo averages and \( \alpha \)-entropies – see [9].

Let us now discuss universal distributions of fluctuating quantities. Their universality has been discovered recently by Bramwell, Holdsworth, and Pinton (BHP) [13,14], based on earlier work [15]. The result received ample support in the literature [16–24]. The BHP density function is of the form
\[ g_{\text{BHP}}(\epsilon) \sim \exp(b(y - c\epsilon^\rho)) \] with \( y = c(\epsilon - u) \). Setting \( b = 1 \) one finds the famous Fisher-Tippett density appearing in statistics of extremes (see Gumbel [25]). However, the value which empirically best describes fluctuation spectra is \( b = \pi/2 \).

Using the standard central limit theorem (CLT) we show at the end of this Letter that our formalism implies (for \( \rho > 2/3 \))
\[ \tilde{g}(\epsilon) = \left( \exp_\rho [v + \epsilon] \right)^{\rho - \alpha} \times g \left( \frac{\sqrt{N}}{\sigma} [\phi_{\alpha\rho}(v + \epsilon) - \phi_{\alpha\rho}(v)] \right), \]
with \( \sigma^2 \) the variance of \( p(t) \), \( g(x) \) the normal density, and the functions \( \phi_{\alpha\rho} \) and \( \exp_\rho \) defined later in the text (for \( \rho < 2/3 \) the normal density function \( g(\epsilon) \) should be replaced by a Lévy-stable PDF).

Of particular interest is the limit \( \rho = 1 \) (corresponding to Rényi \( \alpha \)-entropies – see further on). Then (6) can be written as
\[ \tilde{g}(\epsilon) \sim \exp \left( y - \frac{1}{2} d^2(c \epsilon - 1)^2 \right) \]
with \( y = (1 - \alpha)\epsilon + \frac{1}{2} \) and \( d = \exp((1 - \alpha)\epsilon)/N^{1/2} \sigma(1 - \alpha) \). This expression differs in an essential way from (6). However, both PDFs have a similar shape (see Fig. 3). Note that (6) has an additional fitting parameter, the role of which is taken over here by the exponent \( \rho \). Very accurate data will be needed to distinguish the two PDFs on an experimental basis. An advantage of (10) is that in the limit \( \alpha = 1 \) it reduces to the normal density with width \( \sigma \), as expected. Note also that, if \( g(\epsilon) \) were replaced by the exponential density, and \( \rho = 1 \), then (6) reduces
to (8) with $b = 1$ and $c = 1 - \alpha$. However, the exponential density is not Lévy-stable. Therefore we do not expect it on thermodynamic grounds.

In the final part of this Letter we derive formulas (3) and (4) using thermodynamic arguments. Weron and Jurlewicz [4,5] express the response function $p(t)$ in terms of PDFs $p_i(t)$, describing relaxation of individual dipoles. To do this, they use statistics of extremes [25]. The underlying assumption is that the whole system relaxes as soon as one of the individual dipoles relaxes. Our assumptions are not so drastic. We start with a thermodynamic argument, and replace statistics of extremes by the usual CLT.

About 20 years ago Montroll and Shlesinger [26] proposed a thermodynamic derivation of Einstein’s diffusion law. It follows by optimizing entropy of the PDF $p(t)$ under the constraint of zero first moment and fixed second moment. They observed that anomalous diffusion can be explained along similar lines using an ad hoc constraint replacing the constraint on the second moment. Alemany and Zanette [27,28], followed by Tsallis et al [29,30], repeat these arguments, replacing Shannon’s entropy by the generalized entropy used in nonextensive thermodynamics [31,32]

$$S_{\alpha}(p) = \int dt \frac{p(t)^\alpha - p(t)}{1 - \alpha}, \quad \alpha > 0, \alpha \neq 1.$$  (11)

These authors introduce a suitably adapted constraint on the second moment. Recently [8], the present authors generalized nonextensive thermostatics by founding it on nonlinear Kolmogorov-Nagumo averages [31,32]

$$\langle \langle f \rangle \rangle = \phi^{-1} \left( \int dt p(t) \phi(f(t)) \right).$$  (12)

This average depends on a monotonically increasing function $\phi(x)$. In the present paper we choose $\phi = \phi_{\alpha \rho}$ with

$$\phi_{\alpha \rho}(x) = \ln_{\alpha}(\exp_{\rho}(x))$$

$$= \frac{1}{1 - \alpha} \left[ (1 + (1 - \rho)x)^{(1-\alpha)/(1-\rho)} - 1 \right].$$  (13)

The $\alpha$-deformed exponential and logarithmic functions are defined by [31,32]

$$\exp_{\alpha}(x) = [1 + (1 - \alpha)x]^ {1/(1-\alpha)} \quad \text{and} \quad \ln_{\alpha}(x) = \frac{x^{1-\alpha} - 1}{1 - \alpha}.$$  (14)

The inverse function is $\phi_{\alpha \rho}(x)$. Following [8], the corresponding definition of entropy is

$$S_{\alpha \rho}(p) = \phi_{\alpha \rho} \left( \int dt p(t) \ln_{\alpha}(1/p(t)) \right).$$  (15)

In the limit $\rho = 1$ this is the $\alpha$-entropy of Rényi [33,34],

$$S_{\alpha 1}(p) = \frac{1}{1 - \alpha} \ln \left( \int dt p(t)^\alpha \right).$$  (16)

On the other hand, one has $\phi_{\alpha \rho}(x) = x$ so that $S_{\alpha \rho}(p)$ coincides with (11). Let us now optimize entropy $S_{\alpha \rho}(p)$ under the constraint that the average decay time $\langle \omega(t) \rangle$ has a given value. A straightforward calculation using Lagrange parameters produces an implicit expression for the PDF $p(t)$. By introducing a free parameter $a$ the latter can be made explicit, resulting in (4) (see [8]).

In order to calculate the fluctuation spectrum assume that $\rho > 2/3$ (then the second moment of $p(t)$ exists and CLT holds). If $A(t)$ is an arbitrary random variable we will denote its linear (experimental and theoretical) averages by

$$\langle A \rangle = \sum_{k=1}^{N} A(t_k)/N, \quad \langle A \rangle = \int dt p(t)A(t).$$

Denote by $P(x)$ the probability of an event $x$, given the PDF $p(t)$. In this notation CLT can be formulated as

$$P\left( \langle A \rangle - \sigma \right) = F\left( \frac{\sqrt{N}}{\sigma} (z - \langle A \rangle) \right)$$  (17)

with $\sigma^2$ the variance of $p(t)$, $F$ the normal distribution function, and $N$ sufficiently large [33].

The nonlinear averages

$$\langle \langle A \rangle \rangle = \phi^{-1} \left( \langle \phi(A) \rangle \right), \quad \langle \langle A \rangle \rangle = \phi^{-1} \left( \langle \phi(A) \rangle \right)$$

are related via $\phi^{-1}$ to the linear ones of the random variable $B(t) = \phi(A(t))$. Since $\phi$ is monotonically increasing we can write

$$P\left( \langle \langle A \rangle \rangle \leq \langle A \rangle + \epsilon \right) = P\left( \langle \phi(A) \rangle \leq \phi(\langle A \rangle) + \epsilon \right)$$

and apply CLT to its right side. One finds

$$P\left( \langle \langle A \rangle \rangle \leq \langle A \rangle + \epsilon \right) \approx F\left( \frac{\sqrt{N}}{\sigma} \phi(\langle A \rangle) + \epsilon \right).$$  (18)
Introduce a limiting PDF $\tilde{g}(\epsilon)$ by

$$
\frac{d}{d\epsilon} P\left(\langle A \rangle_N \leq \langle A \rangle + \epsilon\right) \approx \frac{\sqrt{N}}{\sigma} \tilde{g}(\epsilon). \tag{19}
$$

Using $g(x) = dF(x)/dx$ and choosing $\phi = \phi_{\alpha P}$ we finally obtain

$$
\tilde{g}(\epsilon) = \left(\exp_{\rho}\left[\langle A \rangle + \epsilon\right]\right)^{\rho - \alpha} \times g\left(\frac{\sqrt{N}}{\sigma} \left[\phi\left(\langle A \rangle + \epsilon\right) - \phi\left(\langle A \rangle\right)\right]\right). \tag{20}
$$

It is clear that for $\alpha = \rho$ one gets $\tilde{g}(\epsilon) = g(\sqrt{N}\epsilon/\sigma)$.

Let us summarize the results. Applying the standard formalism of maximum entropy principles but formulated in terms of nonlinear averages we can explain a large class of response functions with asymptotic power law decay, including the ubiquitous two-power law. Moreover, using the same formalism and standard central limit theorem we obtain a universal behavior of fluctuation spectra. In both cases the universal laws we propose differ slightly from the known ones. To distinguish experimentally between them one needs more precise experiments.

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