Certain subclass of harmonic univalent functions defined by 
$q$-differential operator

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Abstract

In this paper, we define certain subclass of harmonic univalent function in the unit disc 
$U = \{z \in \mathbb{C} : |z| < 1 \}$ by using $q$-differential operator. Also we obtain coefficient inequalities, growth and distortion theorems for this subclass.

2000 Mathematics Subject Classification: 30C45, 30C50

Keywords: Harmonic, Univalent, Salagean $q$-differential operator.

1 Introduction

Clunie and Sheil-Small [1] investigated the class $S_H$ as well as its geometric subclasses and established some coefficient bounds. Since then, there have been several related papers on $S_H$ and its subclasses. In fact, by introducing new subclasses, Silverman [11], Silverman and Silvia [12], Jahangiri [7], Sangle and Yadav [8], Dixit and Porwal [4], Singh and Porwal [13] and Ravindar et.al [14] etc. presented a systematic and unified study of harmonic univalent functions.

The concepts of $q$-calculus has many applications in subfields of science, some of them are $q$-difference equations and geometric function theory. Motivated by the research work done by Jahangiri [2, 3], Joshi and Sangle [5, 9], Purohit et al. [6], we define some subclasses of harmonic mappings using the Salagean $q$-differential operator.

Also, we determine extreme points and coefficient estimates of $\mathcal{S}_H^q(m, \alpha, u)$ and $\overline{\mathcal{S}}_H^q(m, \alpha, u)$. 

1
Let $A$ be family of analytic functions in unit disk $U$ and $A^0$ be the class of all normalized analytic functions. For $0 < q < 1$ and for positive integer $u$, the $q$-integer number is denoted by $[u]_q$ and also it is written as

$$[u]_q = \frac{1 - q^u}{1 - q} = \sum_{k \geq 0} q^k. \quad (1.1)$$

By making use of differential calculus, we can check that

$$\lim_{q \to 1^-} [u]_q = u$$

For $h \in A$, the $q$-difference operator [6] is specified as

$$\partial_q h(z) = \frac{h(z) - h(qz)}{(1 - q) z} \quad (1.2)$$

where

$$\lim_{q \to 1^-} \partial_q h(z) = h'(z).$$

Let the functions $h \in A$ be of the form

$$h(z) = z + \sum_{u \geq 2} a_u z^u. \quad (1.3)$$

J. M. Jahangiri [3] defined the Salagean $q$-differential operator for the above functions $h$ as

$$D_q^0 h(z) = h(z)$$

$$D_q^1 h(z) = z \partial_q h(z) = \frac{h(z) - h(qz)}{(1 - q) z}, ...$$

$$D_q^m h(z) = z \partial_q D_q^{m-1} h(z) = h(z) \ast \left( z + \sum_{u \geq 2} [u]_q^m z^u \right) = z + \sum_{u \geq 2} [u]_q^m a_u z^u \quad (1.4)$$

where $m$ is a positive integer. The operator $D_q^m$ is called Salagean $q$-differential operator. The complex-valued harmonic functions can be written as $f = h + \overline{g}$ in where $h$ and $g$ have the following power series expansions

$$h(z) = z + \sum_{u \geq 2} a_u z^u, \quad g(z) = \sum_{u \geq 1} b_u z^u, |b_1| < 1. \quad (1.5)$$

Clunie and Sheil-Small [1] defined the function of form $f = h + \overline{g}$ that are locally univalent, sense-preserving and harmonic in $U$. A sufficient condition for the harmonic functions $f$ to be univalent in $U$ is that $|h'(z)| \geq |g'(z)|$ in $U$.

J. M. Jahangiri [3] defined the Salagean $q$-differential operator for the harmonic functions $f$ by

$$D_q^m f(z) = D_q^m h(z) + (-1)^m D_q^m g(z) \quad (1.6)$$

where $D_q^m$ is defined by (1.4).

Now, for $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$ and $z \in U$, suppose that $S^q_H(m, \alpha, u)$ denote the family of harmonic univalent function $f$ of the form $f = h + \overline{g}$ such that

$$\text{Re}\left( \frac{D_q^m h(z) + D_q^m g(z)}{z} \right) > \alpha \quad (1.7)$$
where $D^m_f(z)$ is defined by J. M. Jahangiri [3].

Further let the subclass $S_{H}(m, \alpha, u)$ consisting harmonic functions $f = h + \overline{\eta}$ in $S_{H}(m, \alpha, u)$ so that $h$ and $g$ are of the form

$$h(z) = z - \sum_{u \geq 2} |a_u| z^u \quad \text{and} \quad g(z) = \sum_{u \geq 1} |b_u| z^u. \quad (1.8)$$

2 Main Results

**Theorem 2.1.** Let the function $f = h + \overline{\eta}$ be such that $h$ and $g$ are given by (1.5), Furthermore

$$\sum_{u \geq 2} |u|^m_q |a_u| + \sum_{u \geq 1} |u|^m_q |b_u| \leq (1 - \alpha) \quad (2.1)$$

where $0 \leq \alpha < 1$ and $m \in N_0$. Then $f$ is harmonic univalent, sense-preserving in $U$ and $f \in S_{H}(m, \alpha, u)$.

**Proof:** If $z_1 \neq z_2$ then,

$$\frac{|f(z_1) - f(z_2)|}{|h(z_1) - h(z_2)|} \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right|$$

$$= 1 - \left| \frac{\sum_{u \geq 1} b_u (z_1^u - z_2^u)}{z_1 - z_2 + \sum_{k \geq 2} a_u (z_1^u - z_2^u)} \right|$$

$$\geq 1 - \frac{\sum_{u \geq 1} |b_u|}{1 - \sum_{u \geq 1} |a_u|} \geq 1 - \left( \frac{\sum_{u \geq 1} |u|^m_q |b_u|}{1 - \sum_{u \geq 1} |u|^m_q |a_u|} \right) \geq 0.$$

Hence $f$ is univalent in $U$. 

3
Now, we show that $f \in \mathcal{S}_{H}^{m}(m, \alpha, u)$. Using the fact that $\text{Re}(w) > \alpha$ if and only if $|1 - \alpha + w| > |1 + \alpha - w|$, it suffices to show that

$$
\left| (1 - \alpha) + \frac{D_{q}^{m}h(z) + D_{q}^{m}g(z)}{z} \right| - \left| (1 + \alpha) - \frac{D_{q}^{m}h(z) + D_{q}^{m}g(z)}{z} \right| > 0 \tag{2.2}
$$

Substituting for $D_{q}^{m}h(z)$ and $D_{q}^{m}g(z)$ in (2.2), we obtain

$$
= \left| (2 - \alpha) + \sum_{u \geq 2}^{\infty} \frac{[u]_{q}^{m} a_{u}}{1 - \alpha} z^{u-1} + \sum_{u \geq 1}^{\infty} \frac{[u]_{q}^{m} b_{u}}{1 - \alpha} z^{u-1} \right| - \left| \alpha - \sum_{u \geq 2}^{\infty} \frac{[u]_{q}^{m} a_{u}}{1 - \alpha} z^{u-1} - \sum_{u \geq 1}^{\infty} \frac{[u]_{q}^{m} b_{u}}{1 - \alpha} z^{u-1} \right|
$$

$$
\geq 2(1 - \alpha) \left\{ \left| 1 - \sum_{u \geq 2}^{\infty} \frac{[u]_{q}^{m} a_{u}}{1 - \alpha} |z|^{u-1} \right| - \left| \sum_{u \geq 1}^{\infty} \frac{[u]_{q}^{m} b_{u}}{1 - \alpha} |z|^{u-1} \right| \right\}
$$

$$
> 2(1 - \alpha) \left\{ \sum_{u \geq 2}^{\infty} \frac{|u|}{1 - \alpha} |a_{u}| - \sum_{u \geq 1}^{\infty} \frac{|u|}{1 - \alpha} |b_{u}| \right\}
$$

The harmonic mappings

$$f(z) = z + \sum_{u \geq 2}^{\infty} \frac{1 - \alpha}{|u|_{q}^{m}} x_{u} z^{u} + \sum_{u \geq 1}^{\infty} \frac{1 - \alpha}{|u|_{q}^{m}} y_{u} z^{u},$$

where $\sum_{u \geq 2}^{\infty} |x_{u}| + \sum_{u \geq 1}^{\infty} |y_{u}| = 1$, show that coefficient bound given by (2.1) is sharp.

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f = h + \overline{g}$ where $h$ and $g$ are of the form (1.8).

**Theorem 2.2.** Let $f = h + \overline{g}$ be given by (1.8). Then $f \in \mathcal{S}_{H}^{m}(m, \alpha, u)$ if and only if

$$
\sum_{u \geq 2}^{\infty} \frac{|u|^{m}}{1 - \alpha} |a_{u}| + \sum_{u \geq 1}^{\infty} \frac{|u|^{m}}{1 - \alpha} |b_{u}| \leq 1 \tag{2.3}
$$

where $0 \leq \alpha < 1$ and $m \in N_{0}$. 

4
**Proof:** The if part follows from Theorem 2.1. For the only if part, we show that $f \in \overline{S}_H^q(m, \alpha, u)$ if the condition (2.3) holds. We notice that the condition

$$\text{Re}\left\{ \frac{D_q h(z) + D_q g(z)}{z} \right\} > \alpha$$

is equivalent to

$$\text{Re}\left\{ 1 - \sum_{u \geq 2} [u]_q^m |a_u| |z|^{u-1} - \sum_{u \geq 1} [u]_q^m |b_u| |z|^{u-1} \right\} > \alpha.$$  

The above required condition must hold for all values of $z$ in $U$. Taking the values of $z$ on the positive real axis, where $0 \leq |z| = r < 1$, we must have

$$1 - \sum_{u \geq 2} [u]_q^m |a_u| - \sum_{u \geq 1} [u]_q^m |b_u| \geq \alpha$$

which is precisely the assertion (2.3).

Next, we determine the extreme points of closed convex hulls of class $\overline{S}_H^q(m, \alpha, u)$.

**Theorem 2.3.** Let $f$ be given by (1.8). Then $\overline{S}_H^q(m, \alpha, u)$ if and only if

$$f(z) = \sum_{u=1}^{\infty} (x_u h_u(z) + y_u g_u(z)),$$

where $h_1(z) = z$,

$$h_k(z) = z - \frac{1 - \alpha}{[u]_q^m} z^u, \quad (u = 2, 3, 4, ...),$$

$$g_k(z) = z - \frac{1 - \alpha}{[u]_q^m} z^u, \quad (u = 1, 2, 3, 4, ...),$$

$x_u \geq 0, y_u \geq 0, \sum_{u=1}^{\infty} x_u + y_u = 1$. In particular the extreme points of $\overline{S}_H^q(m, \alpha)$ are $\{h_u\}$ and $\{g_u\}$.

The following theorem gives the bounds for functions in $\overline{S}_H^q(m, \alpha, u)$ which yields a covering result for this class.

**Theorem 2.4.** Let $f \in \overline{S}_H^q(m, \alpha, u)$. Then for $|z| = r < 1$, we have

$$|f(z)| \leq (1 + |b_1|) r + \frac{1}{2^m} (1 - |b_1| - \alpha) r^2, \quad |z| = r < 1$$

and

$$|f(z)| \geq (1 - |b_1|) r - \frac{1}{2^m} (1 - |b_1| - \alpha) r^2, \quad |z| = r < 1.$$  

**Proof:** Let $f \in \overline{S}_H^q(m, \alpha, u)$. Taking the absolute value of $f(z)$, we have

$$|f(z)| \leq (1 + |b_1|) r + \sum_{u \geq 2} (|a_u| + |b_u|) r^u$$

$$\leq (1 + |b_1|) r + \sum_{u \geq 2} (|a_u| + |b_u|) r^2$$

$$\leq (1 + |b_1|) r + \frac{1}{[2]_q} \sum_{u \geq 2} [u]_q^m (|a_u| + |b_u|) r^2$$

$$\leq (1 + |b_1|) r + \frac{1}{[2]_q} (1 - \alpha - |b_1|) r^2$$

5
and

\[ |f(z)| \geq (1 - |b_1|) r - \sum_{u \geq 2}^{\infty} (|a_u| + |b_u|) r^u \]

\[ \geq (1 - |b_1|) r - \sum_{u \geq 2}^{\infty} (|a_u| + |b_u|) r^2 \]

\[ \geq (1 - |b_1|) r - \frac{1}{|a_1|^w} \sum_{u \geq 2}^{\infty} [u]^m ([a_u] + [b_u]) r^2 \]

\[ \geq (1 - |b_1|) r - \frac{1}{|a_1|^w} (1 - \alpha - |b_1|) r^2 \]

The functions \( z + |b_1| \pi + \frac{1}{|a_1|^w} (1 - \alpha - |b_1|) \pi^2 \) and \( z - |b_1| z - \frac{1}{|a_1|^w} (1 - \alpha - |b_1|) z^2 \) for \( |b_1| \leq (1 - \alpha) \).
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