Exact Quantum Dynamics, Shortcuts to Adiabaticity, and Quantum Quenches in Strongly-Correlated Many-Body Systems: The Time-Dependent Jastrow Ansatz

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The description of strongly-correlated quantum many-body systems far from equilibrium is intrinsically challenging due to the amount of information it requires. We introduce a generalization of the Jastrow ansatz for time-dependent wavefunctions, that provides an efficient and exact description of the time-evolution of a variety of systems exhibiting strong correlations. Exact solutions previously known are characterized by scale invariance, making the evolution of local correlations, such as the spatial density, self-similar. However, we find that a complex-valued time-dependent Jastrow ansatz (TDJA) is not restricted to scale-invariance and can describe processes lacking it. The associated time evolution is equivalent to the implementation of a shortcut to adiabaticity (STA) by counterdiabatic driving along a continuous manifold of quantum states described by a real-valued TDJA. Thus, our results provide the means to engineer exact STA in strongly-correlated many-body quantum systems lacking scale invariance. We illustrate our findings in systems with inverse-square interactions, such as the Calogero-Sutherland and the hyperbolic models, that are supplemented with pairwise logarithmic interactions. We further consider the dynamics of bosons subject to both contact and Coulomb interactions in one dimension, known as the long-range Lieb-Liniger model. Our results can be used to study the quench dynamics in all these models. Our findings provide a benchmark for numerical and quantum simulations of nonequilibrium strongly-correlated systems with continuous variables.

I. INTRODUCTION

The nonequilibrium dynamics of isolated quantum many-body systems is a source of rich and intriguing physics, including phenomena such as quantum chaos, many-body localization and thermalization [1, 2]. Their understanding is often limited in the presence of strong interactions, as perturbative calculations fail. Alternative numerical and analytical techniques are then needed, such as exact diagonalization, density-matrix renormalization group [3], and quantum Monte Carlo algorithms, to name some important examples. In one spatial dimension [4, 5], additional techniques are available to account for the behavior of strongly correlated systems. For instance, the Bethe ansatz and quantum-inverse scattering method render certain integrable many-body quantum systems exactly solvable [6–8]. However, even when the exact eigenstates are known, the evaluation of correlation functions remains challenging. This is the case for systems in thermal equilibrium, and specially out-of-equilibrium [9, 10]. Furthermore, numerical techniques, such as density-matrix renormalization group and quantum Monte Carlo algorithms encounter accuracy limitations when simulating long-time unitary dynamics, which become pronounced in systems with continuous variables.

Yet, with the recent progress in quantum simulation, it is possible to experimentally implement strongly interacting models following the vision of Feynman [11]. A paradigmatic platform for analog quantum simulation is offered by ultracold gases [12]. For instance, it was shown by Olshanii that ultracold bosons governed by s-wave scattering are described by the celebrated Lieb-Liniger (LL) model [13, 14] when confined in a tight waveguide [15]. In addition, the implementation of these models in ultracold atoms makes it possible to tune their interactions from the strongly attractive to the strongly repulsive regimes via Feshbach resonance. As a result, it has become feasible to study the quench dynamics of those many-body models in the lab, e.g., following an interaction quench or a change in the confinement [4, 10, 16, 17], a scenario in which strong correlations are hard to describe theoretically [18–24]. Further progress in quantum simulation has been spurred by the digital approach to quantum simulation, in which the dynamics of interest is approximated by a quantum circuit. This paradigm is currently under exhaustive investigation, with applications ranging from condensed matter [25, 26] to quantum field theory [27] and quantum chemistry [28]. Hybrid analog-digital approaches provide yet a different alternative to explore quantum nonequilibrium phenomena [29]. The progress on quantum simulation is confronted with the limited availability of analytical or exact results on the nonequilibrium dynamics of strongly-correlated many-body systems [10, 30]. The latter are not only desirable for their fundamental value but can be utilized as a benchmark for quantum simulation algorithms and shed light on confounding experiments.

Many interacting many-body systems of interest exhibit strong correlations in the ground state [31–33]. The simplest many-body wave function that captures the ground state correlation is the so-called (Bijl-Dingle-) Jastrow ansatz [34–36], which assumes the ground state wave function to be a product of pair functions that only depend on the interparticle spacing. The Jastrow ansatz is generally considered an approximate wavefunction in the study of many-body quantum systems. It proves useful in the description of perturbative expansions at small densities [37] and is ubiquitously used as a trial wavefunction in quantum Monte Carlo techniques. In addition, the Jastrow ansatz provides the exact description of

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the ground state of certain interacting many-body systems. To determine such models, one can assume the ground state wave function in the Jastrow form and search for the parent Hamiltonian according to the time-independent Schrödinger equation. This program was initiated by Calogero [38] and Sutherland [8, 39] and led to the discovery of the celebrated family of one-dimensional integrable Calogero-Sutherland (CS) models, that include particles with inverse-square interactions in unbounded space as well as inverse-square sinusoidal interactions with periodic boundary conditions [8, 40]. Similarly, the attractive LL gas has long been known to exhibit bright quantum soliton states with a Jastrow form [41, 42]. By now, the complete family of models of interacting identical particles and a ground-state of Jastrow form is known in any spatial dimension [43–45]. It is also worth noting that the occurrence of Jastrow ground state has deep implications and for spinless particles in one dimension, it suffices to establish the integrability of the system [46]. The family of parent Hamiltonians with a Jastrow ground state generally includes two-body and three-body interactions. While the latter is absent in the celebrated CS and LL models, it has been shown by a renormalization group calculation [47] that the three-body interactions is irrelevant for long-wavelength, low-temperature physics, further justifying the utility of the Jastrow ansatz in physical applications.

The use of the Jastrow ansatz in scenarios away from equilibrium is by contrast less explored. In this context, it is worth mentioning that the dynamics of the rational CS model, describing one-dimensional bosons subject to inverse-square interactions and confined in a harmonic trap, is exactly described by a time-dependent Jastrow ansatz at all times [48–50]. This model not only offers a precious testbed for nonequilibrium phenomena but it includes hard-core bosons in the Tonks-Girardeau (TG) regime [51–54] as a limiting case of relevance to ultracold atom experiments [55–57]. As a result, this model has motivated a plethora of studies covering applications such as nonexponential decay, quantum speed limits, Loschmidt echoes, and orthogonality catastrophe [58–62], finite-time quantum thermodynamics [63–67], quantum quenches [17, 68], and quantum control [69–72], among other examples.

While these applications help to illustrate the value of exact solutions in quantum dynamics, they are all characterized by scale-invariance. This dynamical symmetry is highly restrictive and makes the time-evolution self-similar in the sense that the spatial probability densities (absolute square value of the wavefunction in the coordinate representation) at any two different times are simply related by a scaling of the coordinates variables. For instance, in one spatial dimension, the time-dependence of a many-body quantum state $\Psi(t)$ satisfies $|\Psi(x_1, \ldots, x_N, t)|^2 = |\Psi(x_1/b(t), \ldots, x_N/b(t), t = 0)|^2 / b(t)^N$ where $b(t) > 0$ is the scaling factor. In this case, the many-body time evolution simplifies dramatically and can be reduced to the determination of the scaling factor, which obeys an ordinary differential equation known as the Ermakov equation. The latter was first applied to the time-dependent harmonic oscillator [73, 74], and together with its generalizations, it plays an important role in the study and control of Bose-Einstein condensates [75–77] and ultracold Fermi gases [78–83]. Without imposing scale-invariance, results are scarce, and time-dependent Jastrow ansätze have only recently been considered in numerical methods. For instance, its use integrated with quantum Monte Carlo algorithms has been applied to describe the quench dynamics of the LL model [23].

One may wonder whether the construction of the parent Hamiltonian used for the original (stationary) Jastrow ansatz can be promoted to a time-dependent setting, in arbitrary processes. However, a straightforward extension of this approach leads to a parent Hamiltonian that is not necessarily Hermitian. The underlying reason for the non-Hermiticity is that the dynamics implicitly implied by the time-dependent trial wave function may break unitarity. Note that recent progress of finding parent Hamiltonians [84] for time-dependent quantum states focuses on discrete spin systems, which does not apply to the scenario here, i.e., many-body systems of continuous variables.

In this work, we extend the program of parent Hamiltonian of Jastrow wavefunctions to the time-dependent case for one-dimensional quantum many-body systems in the continuum, i.e., with continuous variables. We derive consistency conditions for the one-body and two-body pair functions determining the Jastrow ansatz and apply them to the case where the system is embedded in a harmonic trap. The consistency conditions lead to the parent Hamiltonian of the time-dependent and complex-valued time-dependent Jastrow ansatz (TDJA) where the amplitude of the two-body trial wave functions bears a similar functional form to that in the ground state of CS, Hyperbolic and attractive LL models. The Ermakov equation, ubiquitous in the context of scale-invariant dynamics with certain interactions [58, 77, 78], also emerges in our approach, although, crucially, the dynamics implied by complex-valued TDJA is not necessarily scale-invariant. Application of these findings to the aforementioned three types of trial wave functions yields generalizations of the CS, Hyperbolic, and LL models with additional interactions.

As an application of our results, we demonstrate that the parent Hamiltonian of complex-valued TDJA can be applied to the engineering of Shortcut to Adiabaticity (STA) [85, 86], that provide fast control protocol to prepare a target state from a given initial eigenstate without requiring the long time scales necessary for adiabatic preparation. More specifically, we show that the parent Hamiltonian of the complex-valued TDJA can be viewed as an implementation of the adiabatic evolution of the real-valued TDJA. In addition, we show that our findings can be used to study the exact dynamics following a quantum quench of the the interparticle interactions as we illustrate in the long-range Calogero-Sutherland models with logarithmic interactions and the long-range Lieb-Liniger model with Coulomb interactions.

II. THE TIME-DEPENDENT JASTROW ANSATZ (TDJA)

The original Jastrow ansatz [34–36], constructed in terms of products of a pair function and a one-body function, is time-independent and real-valued (TIRV). Its use turned out
to be a fruitful approach, which led to the discovery of many integrable models, including the Calogero-Sutherland family of models [38, 39, 43–45]. More recently, it has been shown that in one spatial dimension the parent Hamiltonian of a real-valued time-independent Jastrow ansatz (TIJA) is integrable [46].

In this section, we focus on the generalization to the time-dependent case and introduce the time-dependent Jastrow wave function,

$$\Psi(t) = \frac{1}{\exp[N(t)]} \prod_{i<j} f_{ij}(t) \prod_k g_k(t), \quad (1)$$

where $\exp[N(t)]$ is the normalization of the wave function, $N(t)$ is a real-valued function that only depends on the time, and $f_{ij}(t) \equiv f(x_{ij}, t)$ and $g_k(t) \equiv g(x_k, t)$ are functions of the particles’ position and time. Throughout this work, we focus on bosonic wave functions exclusively so that $f(x, t) = f(-x, t)$. The TDJA (1) describes the exact solution to the time-dependent Schrödinger equation

$$i\hbar \dot{\Psi}(t) = \mathcal{H}(t)\Psi(t), \quad (2)$$

when the dynamics is generated by the many-body Hamiltonian

$$\mathcal{H}(t) = \sum_i \frac{p_i^2}{2m} + \frac{\hbar^2}{2m} \sum_i v_i^{(i)} + \frac{\hbar^2}{m} \sum_{i<j} v_{ij}^{(ij)} + \frac{\hbar^2}{m} \sum_{i<j<k} v_{ijk}^{(ijk)} - i\hbar N(t), \quad (3)$$

where

$$v_i^{(i)} = \frac{g_i''}{g_i} + \frac{2im}{\hbar} \frac{\dot{g}_i}{g_i}, \quad (4)$$

$$v_{ij}^{(ij)} = \frac{f_{ij}''}{f_{ij}} + \frac{(g_i' - g_j')}{g_i g_j} \frac{f_{ij}'}{f_{ij}} + \frac{im}{\hbar} \frac{\dot{f}_{ij}}{f_{ij}}, \quad (5)$$

$$v_{ijk}^{(ijk)} = -\left( \frac{f_{ij} f_{jk}'}{f_{ij} f_{jk}} + \frac{f_{ij} f_{jk}'}{f_{ij} f_{jk}} + \frac{f_{ij} f_{jk}'}{f_{ij} f_{jk}} \right), \quad (6)$$

are the normalized one-body, two-body, and three-body potentials bearing the dimension of inverse length squared. Throughout the work, the overdot denotes the time-derivative while the superscript ’ denotes the derivative with spatial coordinate.

We note that the many-body time-dependent potential in Eq. (3) is not Hermitian in general. Once the Hermiticity is guaranteed, the unitarity of the dynamics dictates that the norm of $\Psi$ must be constant in time. This, in turn, determines time-dependence of a dynamic multi-dimensional integral over the spatial coordinates up to a constant, i.e.,

$$\exp[2N(t)] \propto \int \cdots \int \prod_{m=1}^N dx_m \prod_{i<j} |f_{ij}(t)|^2 \prod_k |g_k(t)|^2. \quad (7)$$

However, we note that imposing a constant norm of the trial wave function alone is not sufficient to yield a Hermitian Hamiltonian. The lack of non-Hermiticity arises from the fact that although the evolution of the trial Jastrow wave function (1) preserves the norm, this may not be a true set of complete orthogonal many-body basis. A unitary evolution is not guaranteed unless additional restrictions are imposed. Thus, the Hermiticity of the Hamiltonian will introduce strong constraints on the functional forms of $f_{ij}(t)$ and $g_k(t)$, which we will discuss in Sec. III.

The family of Hamiltonians (3) provides a generalization to driven systems of the seminal result known in the stationary case, i.e., the family of parent Hamiltonians with stationary real-valued Jastrow ground state [8, 43–45]. Naturally, this family and the corresponding real-valued TDJA as the ground state can be recovered by choosing $f_{ij}$ and $g_k$ time-independent and real-valued in Eqs. (3) and (1), respectively.

Yet, we can consider two additional options for promoting these results to the time-dependent case.

The most general option is to make $f_{ij}(t)$ and $g_k(t)$ both time-dependent and complex-valued. Since any complex number can be represented in the polar form by a non-negative amplitude multiplied by a phase, without loss of generality, we take

$$f_{ij}(t) = e^{\Gamma_{ij}(t) + i\phi_{ij}(t)}, \quad g_k(t) = e^{\Lambda_k(t) + i\psi_k(t)}, \quad (8)$$

where we use the compact notation $\Gamma_{ij}(t) \equiv \Gamma(x_{ij}, t)$, $\Lambda_k(t) \equiv \Lambda(x_k, t)$, etc. A position-independent but time-dependent global phase $e^{i\tau(t)}$ can be incorporated into the phase factor $e^{i\psi_k(t)}$ by defining $\phi_k(t) \rightarrow \phi_k(t) + \tau(t)/N$.

We propose to reverse engineer the parent Hamiltonian using the time-dependent Schrödinger (2) equation with the Jastrow ansatz as an exact solution. Specifically, we shall solve for $\mathcal{H}(t)$ and demonstrate that it leads to a variety of applications, including the use of STA and quench dynamics for several one-dimensional many-body strongly-correlated quantum models.

Before doing so, we note that an alternative ansatz can be constructed by considering $f_{ij}$ and $g_k$ to be time-dependent, while keeping them real-valued, i.e.,

$$\Phi(t) = \frac{1}{\exp[N(t)]} \prod_{i<j} e^{\Gamma_{ij}(t)} \prod_k e^{\Lambda_k(t)}, \quad (9)$$

which we shall call real-valued TDJA. The parent Hamiltonian of the real-valued TDJA according to the time-dependent Schrödinger equation

$$i\hbar \dot{\Phi}(t) = \mathcal{H}'(t)\Phi(t), \quad (10)$$

can be also found analogously. Clearly,

$$\Psi(t) = \mathcal{W}(t)\Phi(t), \quad (11)$$

where the many-particle unitary operation is defined as

$$\mathcal{W}(t) \equiv \prod_{i<j} e^{i\phi_{ij}(t)} \prod_k e^{i\psi_k(t)}. \quad (12)$$

The corresponding Hamiltonians $\mathcal{H}'(t)$ and $\mathcal{H}(t)$ are similarly related

$$\mathcal{H}'(t) = \mathcal{W}'(t)\mathcal{H}(t)\mathcal{W}(t) - i\hbar \mathcal{W}'(t)\mathcal{W}(t). \quad (13)$$

In what follows, we shall find $\mathcal{H}(t)$ first and then $\mathcal{H}'(t)$ through Eq. (13). We will see that the many-body potential
in $H'(t)$ is non-local, i.e., it involves a term linear in the particles’ momenta. Nevertheless, one can still define a so-called pseudo parent Hamiltonian according to the instantaneous time-independent Schrödinger equation

$$H'_0(t)\Phi(t) = 0,$$  \hspace{1cm} (14)

i.e., $\Phi(t)$ is the instantaneous eigenstate of $H'_0(t)$ with zero eigenvalue (if $\Phi(t)$ has no node, then it is generally the ground state of $H'_0(t)$ provided $H'_0(t)$ is bounded from below). The procedure of finding $H'_0(t)$ leads to

$$H'_0(t) = \sum_i \frac{p_i^2}{2m} + \frac{\hbar^2}{2m} \sum_i (\Lambda'_i + \Lambda'^2_i) + \frac{\hbar^2}{m} \sum_{i<j} [\Gamma''_{ij} + \Gamma^2_{ij} + (\Lambda_i - \Lambda_j)\Gamma'_{ij}] - \frac{\hbar^2}{m} \sum_{i<j<k} (\Gamma'_{ij}\Gamma'_{jk} + \Gamma_{ij}\Gamma'_{ki} + \Gamma_{ki}\Gamma'_{jk}),$$ \hspace{1cm} (15)

which is the same as the parent Hamiltonian in the real-valued TIIA, except that $H'_0(t)$ is now time-dependent. Apart from the phases characterized by $H(t)$, the dynamics is said to be scale invariant if $\Psi(t)$ or $\Phi(t)$ can be expressed in terms of the corresponding wavefunction at $t = 0$ using a scaling transformation of the spatial coordinates. Scale-invariant dynamics has been extensively studied in ultracold atoms [77, 78, 87, 88]. However, we do not impose scale invariance here, although it will emerge naturally in this approach.

We conclude this section by noting that the trial wave functions (1) and (9) must be normalizable so that $\Psi$ and $\Phi$ do not blow up when particles are far apart.

III. CONSISTENCY CONDITIONS BY IMPOSING HERMITICITY

As already advanced, the Hermicity of $H(t)$ may introduce strong constraints on the functional forms of $f_j(t)$ and $g_k(t)$. To find such constraints, in principle one can rewrite $H(t)$ in terms of $\Gamma_{ij}(t), \theta_{ij}(t), \Lambda_3(t)$ and $\Phi_3(t)$ by substituting Eq. (8) into Eq. (3) and then impose that the imaginary part of $V$ vanishes. Specifically, we note that the two-body and three-body terms can by no means be reduced to a one-body potential unless it is independent of the particles’ positions. This dictates that $\text{Im}\omega_1^{(i)}$ must be a function of time only. Although in some cases the three-body interaction $\psi_3^{(ijk)}$ can reduce to two-body interactions [46, 89], for the sake of simplicity, we shall assume similar constraints on the two-body and three-body interactions, i.e., that $\text{Im}\omega_2^{(ijk)}$ and $\text{Im}\omega_3^{(ijk)}$ are functions of time only. Thus, we introduce

$$\dot{N}_s(t) = \text{Im}\omega_s, \hspace{0.5cm} s = 1, 2, 3,$$ \hspace{1cm} (16)

where $N_s$ is a real-valued function of time and again the over-dot denotes time derivation. Thus, the Hermicity of $H(t)$ imposes

$$\mathcal{N}(t) \sim \frac{Nh(N-1)}{2m} \mathcal{N}_1(t) + \frac{Nh(N-1)(N-2)}{6m} \mathcal{N}_2(t),$$ \hspace{1cm} (17)

where ~ denotes equivalence up to a constant independent of time and particles’ positions. This is our first result for considering the complex-valued TDJA. We will exemplify these results in several examples.

In general, Eq. (16) for $s = 3$ can lead to complicated conditions. In this work, we shall restrict our attention to the case where the three-body interactions are either real-valued while depending on both coordinates and time or complex-valued but depending on time only. The latter essentially boils the trial function down to three types of functions

$$e^{\Gamma_{ij}(t)} \propto \begin{cases} |x_{ij}|^{\eta(t)} \text{CS} \\ \sinh[c(t)x_{ij}]^{\eta(t)} \text{Hyperbolic} \\ \exp[c(t)x_{ij}] \text{LL} \end{cases},$$ \hspace{1cm} (18)

corresponding to the celebrated CS, hyperbolic, and LL models respectively in the case of TDJA [8, 44]. In all these cases, the following quantity

$$W_3(t) = - \sum_{i<j<k} (\Gamma'_{ij}\Gamma'_{jk} + \Gamma_{ij}\Gamma'_{ki} + \Gamma_{ki}\Gamma'_{jk}),$$ \hspace{1cm} (19)

is independent of the particles’ coordinates. In particular, one can find that $W_3(t)$ is independent of the coordinates for the trial wave functions in Eq. (18), i.e.,

$$W_3(t) = \begin{cases} 0 \text{CS} \\ N(N-1)(N-2)\lambda^2(t)c^2(t)/6 \text{Hyperbolic} \\ N(N-1)(N-2)\lambda^2(t)/6 \text{LL} \end{cases}.$$ \hspace{1cm} (20)

In Appendix A, we show that for these cases, one may take the two-body phase angle to be

$$\theta_{ij}(t) = \eta(t)\Gamma_{ij}(t),$$ \hspace{1cm} (21)

where $\eta(t)$ is any given real-valued function of time. Then the three-body term defined in Eq. (6) turns out to be

$$\sum_{i<j<k} \psi_3^{(ijk)} = W_3(t)[1 + i\eta(t)]^2,$$ \hspace{1cm} (22)

and as a result,

$$\frac{N(N-1)(N-2)}{6} \dot{N}_3(t) = 2\eta(t)W_3(t).$$ \hspace{1cm} (23)

Note that in the particular case when $\eta(t) = 0$, $W_3(t)$ does depend on the coordinates in general, but $\dot{N}_3(t) = 0$.

Next, we note that in general Eq. (16) with $s = 1, 2$ yields for these models consistency conditions between the trial functions, which are discussed in detail in Appendix A. For example, the two-body consistency condition corresponding to case $s = 2$ is given by

$$\eta(t)\Gamma''_{ij} + 2\eta(t)\Gamma'_{ij}^2 + \frac{m}{\hbar^2}\Gamma_{ij} - \frac{m}{\hbar^2}c_2(t)\Gamma_{ij}x_{ij} = \dot{N}_2(t),$$ \hspace{1cm} (24)
where $\mathcal{C}_2(t)$ is some function of time, which will be determined from the one-body consistency condition corresponding to the case $s = 1$.

Furthermore, we shall focus on the case where the particles are embedded in the harmonic trap, i.e., $\text{Re}v_i(t)$ is quadratic in $\chi_i$. As we argue in Appendix B, the one-body consistency condition in this case simplifies dramatically. We shall take $\Lambda_2(t) = -m\omega_0\chi_i^2/(2\hbar)$. The one-body consistency condition explicitly gives the functional form of $\mathcal{C}_2(t)$,

$$\mathcal{C}_2(t) \equiv \eta(t)\omega(t) + \frac{\dot{\omega}(t)}{2\omega(t)}, \quad (25)$$

and

$$N_1(t) = -\frac{m\dot{\omega}(t)}{2\hbar\omega(t)}. \quad (26)$$

The one-body phase angle is then found to be

$$\phi_k(t) = -\frac{m\dot{\omega}(t)}{4\hbar\omega(t)} \chi_i^2 + \tau(t). \quad (27)$$

The Hamiltonian (3) then becomes

$$\mathcal{H}(t) = \frac{1}{2m} \sum_{i} p_i^2 + \frac{1}{2} m\Omega^2(t) \sum_{i} \chi_i^2$$

$$+ \frac{\hbar^2}{m} \sum_{i<j} \left( \Gamma_{ij}^2 + [1 - \eta^2(t)] \Gamma_{ij}^2 \right)$$

$$- \hbar \sigma(t) \sum_{i<j} \Gamma_{ij}^2 \chi_i \chi_j - \hbar \sum_{i<j} \frac{d}{dt}[\eta(t)\Gamma_{ij}] + \mathcal{E}(t), \quad (28)$$

where the frequency of the trap $\Omega(t)$ is defined as

$$\Omega^2(t) \equiv \omega^2(t) + \frac{d}{dt} \left[ \frac{\dot{\omega}(t)}{2\omega(t)} \right] - \left[ \frac{\dot{\omega}(t)}{2\omega(t)} \right]^2, \quad (29)$$

and

$$\mathcal{E}(t) \equiv -\frac{1}{2} N\hbar \theta(t) + \frac{\hbar^2}{m} [1 - \eta^2(t)] W_3(t), \quad (30)$$

$$\sigma(t) \equiv \omega(t) \left[ 1 - \eta(t) \dot{\omega}(t) \right], \quad (31)$$

$$\theta(t) \equiv 2\tau(t) + \omega(t). \quad (32)$$

The generic time-dependent Jastrow ansatz (1) becomes

$$\Psi(t) = \frac{1}{\text{exp}[N(t)]} \prod_{i<j} \Gamma_{ij}(t)[1+i\eta(t)] \prod_k e^{-\frac{m\omega_0^2}{2\hbar} \chi_i^2 - \frac{m\omega_0^2}{4\hbar} \chi_i^2 + i\tau(t)}. \quad (33)$$

Remarkably, upon making the change of variables $\omega(t) = \omega_0 / b^2(t)$ with $\omega_0 \equiv \omega(0)$, one immediately observes that $\dot{\omega}(t)/[2\omega(t)] = -b(t)/b(t)$ and Eq. (29) reduces to the Ermakov equation

$$\ddot{b}(t) + \Omega^2(t)b(t) = \omega_0^2 b^{-3}(t), \quad (34)$$

with the initial condition

$$b(0) = 1. \quad (35)$$

The one-body phase angle Eq. (27) can thus be rewritten as

$$\phi_k(t) = \frac{m\dot{b}(t)\chi_i^2}{2\hbar b(t)} + \tau(t). \quad (36)$$

It is worth noting that Eq. (34) is the Ermakov equation governing the scaling factor in scale-invariant dynamics, emerging when the interactions of particles have given scaling properties [58, 77, 78]. Similarly, Eq. (36) is the corresponding phase angle under scale invariance. However, it is important to note two differences from the previous literature [58, 77, 78]: (i) First, previously an additional condition $b(0) = 0$ is introduced. Here we only impose the condition $b(0) = 1$ for the Ermakov equation and $b(0)$ can be arbitrary. (ii) Secondly, in the Ermakov equation discussed by the previous literature, it is assumed that $\Omega_0 \equiv \Omega(0)$ is always equal to $\omega_0 \equiv \omega(0)$. Here, such a constraint does not necessarily hold.

As we shall discuss in detail in Sec. IX, if we further impose $b(0) = 0$ and $\Omega_0 = \omega_0$, together with the conditions for $\eta(t)$ at $t = 0$, we obtain $\mathcal{H}(0)\Psi(0) = 0$, i.e., at time $t = 0$, the TDJA is also an eigenstate of the corresponding parent Hamiltonian, which is always a requirement for finding the counterdiabatic driving for scale-invariant dynamics, as discussed previously [88, 90]. We emphasize that in our discussion here, the initial time $t_0$ is not necessarily zero. In fact, both Eq. (33) and the interaction in Eq. (28) can break, in general, scale-invariance. As we shall see subsequently, both scale-invariant and non-scale-invariant dynamics follow naturally from our results.

To summarize, our central results are Eq. (28) and the parent Hamiltonian of the complex-valued TDJA (33). Physically, Eq. (33) describes the nonequilibrium dynamics of one-dimensional interacting bosons in a harmonic trap with interactions given by Eq. (28). Knowledge of the exact nonequilibrium dynamics of strongly-correlated quantum systems is rare and precious, and we shall explore its applications to counterdiabatic driving in STA [85, 86, 91–94] and quench dynamics subsequently.

### IV. SHORTCUTS TO ADIABATICITY FOR THE JASTROW ANSATZ

In a general setting, for a given reference Hamiltonian as $\mathcal{H}_0(t)$, with instantaneous eigenvectors $\Phi_n(t)$ and eigenvalues $\epsilon_n(t)$, one can consider the use of counterdiabatic control fields $\mathcal{H}_c(t)$ that are able to drive the dynamics through the adiabatic manifold of $\Phi_n(t)$ parametrized by $t$ [91–94]. This is the goal of counterdiabatic driving, also known as transitionless quantum driving, a technique that provides a universal way to construct STA protocols [85, 86]. The application of these techniques to quantum fluids has led to manifold applications ranging from quantum microscopy [69, 95] to the engineering of efficient friction-free quantum thermal machines [64, 82, 96]. Experimental progress has focused on the case of atomic clouds driven by time-dependent confinements [81, 82, 97–100].

Counterdiabatic driving aims at finding a global Hamiltonian

$$\mathcal{H}'(t) = \mathcal{H}_0(t) + \mathcal{H}_c(t), \quad (37)$$
where
\[ \mathcal{H}_0'(t) \Phi_n(t) = \varepsilon_n(t) \Phi_n(t). \] (38)

In general, finding the \( \mathcal{H}'(t) \) that satisfies Eq. (38) for a specific eigenstate may not be easy. Nevertheless, if Eq. (38) is valid for all eigenstates, it can be shown that \( \mathcal{H}'(t) = i \hbar \sum_n [\mathcal{V}_n(t)] \langle \Phi_n(t) | \Phi_n(t) \rangle \), so that
\[ \mathcal{H}'_1(t) = \sum_n [i \hbar \mathcal{V}_n(t) | \Phi_n(t) \rangle - \varepsilon_n | \Phi_n(t) \rangle \langle \Phi_n(t) |]. \] (39)

The construction of explicit constructions of \( \mathcal{H}'(t) \) are only restricted to particular cases, including harmonic oscillator [101], scale-invariant dynamics [88, 90], integrable systems [102], etc. In the engineering of STA for many-body systems, several difficulties may occur: (i) Computing \( \mathcal{H}'(t) \) requires knowledge of the spectrum of \( \mathcal{H}'_0 \) which is difficult to compute (ii) The potential in \( \mathcal{H}'_1 \) usually contain nonlocal terms, i.e., terms that combine both the position and momentum operators and cannot be associated with an external local potential or interaction. For scale-invariant invariant dynamics, it was shown that [88, 90] these two difficulties can be removed and the non-locality of the potential in \( \mathcal{H}'_1 \) can be gauged away.

However, there is little understanding of the exact counterdiabatic driving in many-particle systems beyond the scale-invariant dynamics. To the best of our knowledge, there is not yet an analytical finding of the counterdiabatic driving that is invariant dynamics. To the best of our knowledge, there is not yet an analytical finding of the counterdiabatic driving that is invariant dynamics. To the best of our knowledge, there is not yet an analytical finding of the counterdiabatic driving that is invariant dynamics. To the best of our knowledge, there is not yet an analytical finding of the counterdiabatic driving that is invariant dynamics. To the best of our knowledge, there is not yet an analytical finding of the counterdiabatic driving that is invariant dynamics.

As already mentioned in the last section, we shall focus on the harmonic trap, the case of \( \eta(t) = 0 \), and possible variants of CS, Hyperbolic, and LL models. The real-valued TDIA and the unitary operator \( \mathcal{U} \) are
\[ \mathcal{U}(t) = \prod_{i<j} e^{i \Omega(t) \mathcal{G}_{ij}(t)} \prod_i e^{- \frac{\omega(t)}{4 \hbar} \mathcal{Z}_{ij}(t)^2 + i \eta(t)}, \] (40)
\[ \Phi(t) = \frac{1}{\exp[\mathcal{N}(t)]} \prod_{i<j} e^{\mathcal{V}_{ij}(t)} \prod_i e^{- \frac{m \omega(t)}{2} \mathcal{S}_{ij}(t)}. \] (41)

The parent Hamiltonian of the real-valued TIJA, Eq. (15) now becomes
\[ \mathcal{H}_0'(t) = \frac{1}{2m} \sum_i \left(p_i^2 + \frac{1}{2} m \omega(t)^2 \sum_j x_j^2 + \frac{\hbar^2}{m} \sum_{i<j} (\Gamma_{ij} + \Gamma_{ji}^*) \right) \]
\[ - \hbar \omega(t) \sum_{i<j} \mathcal{G}_{ij} x_{ij} + \mathcal{E}_{ij}'(t), \] (42)
where
\[ \mathcal{E}_{ij}'(t) = - \frac{1}{2} \hbar \omega(t) + \frac{\hbar^2}{m} \mathcal{W}_3(t). \] (43)

Since \( \Phi(t) \) is the instantaneous eigenstate of \( \mathcal{H}_0'(t) \), keeping the adiabatic evolution along \( \Phi(t) \) requires infinitely slow driving of \( \mathcal{H}_0'(t) \). Using Eq. (13), we find the counterdiabatic driving \( \mathcal{H}'(t) \) is (see details in Appendix C)
\[ \mathcal{H}'(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \omega(t)^2 \sum_i x_i^2 + \frac{\hbar^2}{m} \sum_{i<j} (\Gamma_{ij} + \Gamma_{ji}^*) \]
\[ - \hbar \omega(t) \sum_{i<j} \mathcal{G}_{ij} x_{ij} + \mathcal{E}_{ij}'(t) + \mathcal{H}'_1(t), \] (44)
where \( \mathcal{H}'_1(t) \) contains the non-local terms
\[ \mathcal{H}'_1(t) \equiv \frac{\hbar \eta(t)}{2m} \sum_{i<j} \mathcal{G}_{ij}(t) p_{ij} - \frac{\dot{\omega}(t)}{4 \omega(t)} \sum_i \{ x_i, p_i \}, \] (45)
and \( p_{ij} \equiv p_i - p_j \). The counterdiabatic control term \( \mathcal{H}'_1(t) \) is thus the sum of the one-body squeezing operator, familiar from the study of driven scale-invariant systems [88, 90, 103, 104], and its two-body generalization, involving \( \{ \mathcal{G}_{ij}(t), p_{ij} \} \).

While \( \mathcal{H}'(t) \) contains non-local terms which may be difficult to implement in an experiment, as we have shown, the parent Hamiltonian of the complex-valued TDIA \( \mathcal{H}(t) \) provides a feasible alternative. Furthermore, it is straightforward to check that when \( \eta(t) = 0 \), \( \mathcal{H}'_1(t) \) bears the same form as \( \mathcal{H}(t) \), except for the fact that the driving in the former is \( \omega(t) \) while in the latter is \( \Omega(t) \), and the different expressions for the time-dependent constants \( \mathcal{E}_{ij}'(t) \) and \( \mathcal{E}(t) \). Similar observations are discussed in Ref. [88, 90], where the interactions have scaling properties. However, here we do require scaling interactions.

In what follows, we shall apply the analysis in this section and Sec. III to the cases of vanishing \( \eta(t) \). As we have discussed before, when \( \eta(t) = 0 \), the trial wave function is not necessarily restricted to Eq. (18), but can be generic as long as it respects the bosonic symmetry. In this case, \( W_3(t) \) is a three-body potential that depends on both coordinates and time and \( \mathcal{N}_3(t) = 0 \). The two-body consistency condition (24) simplifies to
\[ \frac{m}{\hbar^2} \mathcal{G}_{ij} + \frac{\dot{b}(t) m}{b(t)} \hbar \mathcal{G}_{ij} x_{ij} = \mathcal{N}_2(t). \] (46)
This condition can be satisfied if we take \( \mathcal{G}_{ij}(t) \) takes the form,
\[ \mathcal{G}_{ij}(t) = \Gamma \left( \frac{c_0 x_{ij}}{\bar{b}(t)} \right) \] (47)
The integrand is oscillating in the configuration space and can be accurately computed in the complex-valued TDJA (49), one can compute the survival probability for STA governed by scale-invariance in Case (i) satisfies the property of Eq. (47). As a result, Eq. (52) becomes

$$\langle \Psi(t) | \Psi(t_0) \rangle = \frac{e^{-iN[t(t)-t(t_0)]}}{[b(t)b(t_0)]^{N/2} \alpha^N(t, t_0)} \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dy_1 \cdots d y_N e^{-\pi \sum_{i} \left( \Gamma \left( \frac{\nu_i}{N/2} \right) \right) \nu_i^2} \Gamma \left( \frac{\nu_i}{N/2} \right) \Sigma_i y_i^2$$

where the integral contour $\mathcal{C} \equiv (-\infty + i \arg[\alpha(t, t_0)], + i \arg[\alpha(t, t_0)])$ denotes the argument of the complex variable $\alpha(t, t_0)$. Whenever $\Gamma \left( \frac{\nu_i}{N/2} \right) \nu_i$ is an analytic function with no poles on the complex hyperplane $\mathbb{C}^N$ (excluding the infinity) and the whole integrand decays fast enough as $\Sigma_i y_i^2 \to \infty$, one can analytically continue the integral from the hypercontour $\mathcal{C}^N$ to the real configuration space $\mathbb{R}^N$, resulting in

$$\langle \Psi(t) | \Psi(t_0) \rangle = \frac{e^{-iN[t(t)-t(t_0)]}}{[b(t)b(t_0)]^{N/2} \alpha^N(t, t_0)} \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dy_1 \cdots d y_N e^{-\pi \sum_{i} \left( \Gamma \left( \frac{\nu_i}{N/2} \right) \right) \nu_i^2} \Gamma \left( \frac{\nu_i}{N/2} \right) \Sigma_i y_i^2$$

The multi-dimensional integral in this form only involves a real-valued integrand and therefore can be easily carried out with various Monte Carlo algorithms. We note that a similar change of variables has been used to compute the survival probability for the scale-invariant dynamics in the Calogero-Sutherland model [58] and other scale-invariant systems, including the two-dimensional Bose gas with contact interactions and the three-dimensional isorotational unitary Fermi gas [62]. In what follows, using Eq. (55), we shall discuss the survival probability for the other models, including the logarithmic Calogero-Sutherland model and the long-range Lieb-Liniger model.

VI. THE TIME-DEPENDENT CALOGERO-SUTHERLAND MODEL

The CS model has manifold applications across different fields. As we have discussed, it describes ultracold atoms confined in tight waveguides in the Tonks-Girardeau limit, and its generalizations with inverse-square interactions. It also appears naturally in matrix models, random matrix theory and quantum chaos [105–107], even if the system is itself integrable. As we next show, our framework accounts for the dynamics of the CS model and its generalizations.

We start with the generic time-dependent Jastrow ansatz without assuming scale-invariance by taking $\Gamma_{ij}(t) = \lambda(t) \ln |x_{ij}|$ in Eq. (33) so that $\Gamma_{ij}(t) = \lambda(t) |x_{ij}|$. The two-body consistency equation (24) yields two nontrivial cases [108]: (i) $\eta(t) = 0$ and $\lambda(t) = \lambda_0$, where $\lambda_0$ is an arbitrary value independent of time. (ii) $\lambda(t) = 1/2$ with arbitrary $\eta(t)$. We now discuss case by case. One can readily observe that the shifting $\Gamma_{ij}(t)$ by a time-dependent function does not affect the satisfaction of Eq. (24). It only shifts the $N_2(t)$ by the same time-dependent function. Thus, shifting $\Gamma_{ij}(t)$ by $\ln |b(t)|$ for case (i) satisfies the property of Eq. (47).
A. Recovering the scaling-invariant dynamics.

The well-known dynamics in the literature for the Calogero-Sutherland model is the scale-invariant dynamics \([48-50, 58]\), i.e.,

\[
\Psi(t) = \frac{1}{[b(t)]^{\frac{N(N-1)}{2}}} \prod_{i < j} \frac{x_{ij}}{b(t)^{\frac{1}{2}}} \exp\left[\frac{\alpha}{4} \sum_{i < j} x_{ij}^2 / b(t) + \frac{\alpha_0}{2} \sum_{i < j} x_{ij}^2 + \ln b(t)\right], \tag{56}
\]

where the scaling factor \(b(t) > 0\) satisfying the Ermakov equation (34). Note that the norm of this many-body function is preserved in time upon making the change of variables \(x_i / b(t) \to x_i\). Comparing Eq. (56) with Eq. (33), one immediately recognizes that

\[
\mathcal{N}(t) = \frac{1}{2} [N + \lambda_0 N(N - 1)] \ln b(t). \tag{57}
\]

For case (i), it follows that the two-body phase angle \(\theta_{ij}\) vanishes. Moreover, the two-body consistency condition also determines

\[
\mathcal{N}_2(t) = \frac{m \omega_0(t) \lambda_0}{2 \hbar \omega(t)}. \tag{58}
\]

In addition, the three-body potential vanishes in this case, i.e. \(v_{3C}(t) = 0\), and Eq. (28) becomes

\[
\mathcal{H}(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \Omega^2(t) \sum_i x_i^2 - \frac{\lambda_0(N - 1)}{x_{ij}} + \mathcal{E}(t), \tag{59}
\]

where

\[
\mathcal{E}(t) = \mathcal{E}(t) - \frac{1}{2} N(N - 1) \hbar \omega(t)(t). \tag{60}
\]

We note that Eq. (17) yields

\[
\mathcal{N}(t) = -\frac{\ln \omega(t)}{4} [N + N(N - 1) \lambda_0], \tag{61}
\]

which agrees with the observation (57) up to a constant, upon identifying \(\omega(t) = \omega_0 / b^2(t)\). We will see that it recovers the scale-invariant dynamics for the Calogero model. Upon making \(\omega(t)\) time-independent and setting \(\tau(t) = 0\), Eq. (60) agrees with the one found from time-independent Jastrow ansatz \([38, 44]\). The position-independent phase angle \(\tau(t)\) can be determined by imposing \(\mathcal{E}(t) = 0\), leading to

\[
\tau(t) = -\frac{1}{2} [(N - 1) \lambda_0 + 1] \int_{0}^{\infty} \frac{\omega_0}{b^2(s)} ds. \tag{62}
\]

Finally, for the counter-diabatic driving \(\mathcal{H}'(t) = \mathcal{H}_0'(t) + \mathcal{H}_1'(t)\), the reference Hamiltonian Eq. (42) becomes

\[
\mathcal{H}_0'(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \Omega^2(t) \sum_i x_i^2 - \frac{\lambda_0(N - 1)}{x_{ij}} + \mathcal{E}_0'(t), \tag{63}
\]

where

\[
\mathcal{E}_0'(t) = \mathcal{E}'(t) - \frac{1}{2} N(N - 1) \hbar \omega(t)(t). \tag{64}
\]

The auxiliary control field is given by Eq. (50). We see that the counterdiabatic driving or the parent Hamiltonian of the real-valued TJIA, Eq. (44) is the same as the one found in Refs. \([88, 90]\), which takes the advantage of the scale-invariance of the inverse square potential in the Calogero-Sutherland model.

As we have mentioned, the implementation of \(\mathcal{H}'(t)\) is through the implementation of \(\mathcal{H}(t)\) where the nonlocal terms disappear. In this particular case, \(\mathcal{H}(t)\) is scale-invariant and hence \(\mathcal{H}'(t)\) can keep the adiabatic evolution of all the eigenstates of \(\mathcal{H}_0'(t)\) \([88, 90]\), not only the adiabatic manifold formed by \(\Phi(t)\).

For this case, one can get around the multi-dimensional integral in the evaluation of the survival probability by taking advantage of the scaling property of the two-body wave function \(|x_{ij}/b(t)|^{\lambda_0}\). The result was first computed exactly in Ref. \([58]\). Here we reproduce it with Eq. (55). Comparing Eq. (56) with Eq. (49), we identify \(\Gamma\left(x_{ij}/b(t)\right) = \lambda_0 \ln \left|x_{ij}/b(t)\right|\) with \(\epsilon_0 = 1\). Then we find

\[
\langle \Psi(t) \vert |\Psi(t)\rangle \rangle = \frac{1}{[b(t)b(t)]^{N(N-1)/2N}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\eta_1 \cdots d\eta_N \prod_{i < j} |\alpha(t, \eta_j)|^{2\lambda_0} |b(t)b(t)|^{-\lambda_0} e^{-\eta_j^2} \lambda_0. \tag{65}
\]

Thus, the survival probability (51) reads

\[
SP(t, t_0) = \left[\frac{m \omega_0}{\hbar b(t)b(t)} \right]^{N(N-1)/2} \tag{66}
\]

B. Logarithmic Calogero-Sutherland gas

For case (ii), where \(\Gamma_{ij} = \ln|x_{ij}|/2\), the complex-valued TDJA (33) becomes

\[
\Psi(t) = \frac{1}{[b(t)]^{N/2}} \prod_{i < j} \frac{x_{ij}}{b(t)^{1/2}} \left|x_{ij}i^{N/2}/2 \right| |\alpha(t, \eta_j)|^{2\lambda_0} \sum_{i < j} x_{ij}^2 / b(t)^{1/2} + \frac{\lambda_0}{2} \sum_{i < j} x_{ij}^2 + \ln b(t). \tag{67}
\]

The corresponding parent Hamiltonian is

\[
\mathcal{H}(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \Omega^2(t) \sum_i x_i^2 - \frac{\lambda_0(N - 1)}{x_{ij}} + \mathcal{E}(t)
- \frac{\hbar^2}{m} \sum_{i < j} \sum_i x_i^2 / 4 x_{ij}^2 - \frac{\hbar^2}{2} \sum_{i < j} \ln |x_{ij}|. \tag{68}
\]

where

\[
\mathcal{E}(t) = \mathcal{E}(t) - \frac{1}{2} N(N - 1) \hbar \omega(t)(t), \tag{69}
\]

and \(\sigma(t)\) is defined in Eq. (31).

When \(\dot{\eta}(t) \neq 0\), the inverse square interaction has a different scaling exponent from the logarithmic interaction. Nevertheless, as is clear from Eq. (67), the dynamics given by
the complex-valued TDJA still preserves the scale-invariance. Note that scale-invariant dynamics only requires the amplitude of the many-body wave function scales with $b(t)$, i.e., $|\Psi(x_1, \ldots, x_N, t)|^2 = |\Psi(x_1/b(t), \ldots, x_N/b(t), t = 0)|^2/b(t)^N$. There are no restrictions on the one-body and two-body phases.

Furthermore, the case of $\eta(t) = \eta_0 \neq 0$ corresponds to the CS model with interaction strength taking the threshold value 1/4, below which the thermodynamic limit does not exist. Nevertheless, this regime still makes sense as long as the number of particles remains finite. Our results indicate that in this regime the eigenstate of the CS is still of the Jastrow form but with a non-vanishing two-body phase, which cannot be described by the TIRV Jastrow ansatz.

Finally, we note that in this case the pseudo-parent Hamiltonian or the reference Hamiltonian $H'_0(t)$ is the same form as Eq. (63) with $\lambda_0$ set to be 1/2. The control field for STA now becomes

$$H'_1(t) \equiv -\frac{\hbar \eta(t)}{4m} \sum_{i<j} \left[ \frac{1}{x_{ij}} + \frac{\dot{b}(t)}{2b(t)} \sum_i \{ x_i, p_i \} \right].$$

(70)

We observe an interesting result: For $\lambda_0 = 1/2$, we actually give two different counterdiabatic Hamiltonians which can steer the same adiabatic evolution of $\Phi(t)$, whose control field is given by Eq. (50) and Eq. (70), respectively. As shown previously, the former satisfies Eq. (39) and the latter is a counterdiabatic driving that does not satisfy Eq. (39).

The non-vanishing two-body phase angle $\eta(t)$ presents a challenge in the exact evaluation of the survival probability, even when using the change of variables in Sec. V. However, one can find an upper bound to the survival probability by resorting to the following inequality

$$|\langle \Psi(t) | \Psi(t_0) \rangle| \leq \frac{1}{[b(t)b(t_0)]^{N/2} \alpha(t, t_0)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dy_1 \cdots dy_N \prod_{i<j} \frac{|v_{ij}|}{|\alpha(t, t_0)| |b(t)b(t_0)|^{1/2}} e^{-\sum_i v_i^2},$$

(71)

which is saturated at the initial time $t_0$ due to the cancellation of the two-body phase. Therefore, we obtain that the survival probability is upper bounded by

$$SP(t, t_0) \leq \left[ \frac{\mu \omega_0}{\hbar b(t_0)b(t_0)|\alpha(t, t_0)|^2} \right]^{(N+N(N-1)/2)}.$$

(72)

VII. THE TIME-DEPENDENT HYPERBOLIC MODELS

Hyperbolic models have been long studied and are characterized by both hard-core and finite-range interactions. In the limit of low densities, the finite-range contribution leads to exponentially decaying interparticle potential reminiscent of the Toda lattice, both in the homogeneous [8], and trapped cases [44]. Hyperbolic models also admit generalizations to higher spatial dimensions [45]. Let us thus focus on their exact quantum dynamics.

We take the two-body wave function $\Gamma_{ij}(t)$ to bear the similar form as the ground wave function in the hyperbolic models,

$$\Gamma_{ij}(t) = \lambda(t) \ln |\sinh[c(t)x_{ij}]|,$$

(73)

$$\Gamma_{ij}'(t) = \lambda(t)c(t) \coth[c(t)x_{ij}]$$

(74)

and

$$\Gamma_{ij}''(t) = -\lambda(t)c(t)^2 \csc^2[c(t)x_{ij}].$$

(75)

The two-body consistency condition (24) leads to the following condition: $\lambda(t) = \lambda_0, \dot{N}_2(t) = 0$, with either (i) $\eta(t) = 0$ or (ii) $\lambda_0 = 1/2$, where $c_0 \equiv c(0)$ and $\mathcal{C}(t)$ is defined in Eq. (25).

A. The time-dependent generalized hyperbolic models

For case (i), Eq. (77) simplifies to

$$c(t) = \frac{\hbar}{\mu} \sqrt{\frac{\omega(t)}{\omega(0)}} = \frac{c_0}{b(t)}.$$

(78)

Thus, $\Gamma_{ij}(t)$ for this case satisfies the property of Eq. (47). The complex-valued TDJA becomes

$$\Psi(t) = \frac{1}{\exp[N(t)]} \prod_{i<j} \left| \sinh \left[ \frac{c_0 x_{ij}}{b(t)} \right] \right|^{\lambda_0}$$

$$\times e^{-\frac{m}{2} \sum_i x_i^2 \frac{\omega(t)}{\omega(0)}} \sum_i x_i^{2N(t)}$$

(79)

where again we have set $\tau(t) = 0$ and

$$N(t) \sim -\frac{1}{4} N \ln \omega(t).$$

(80)

The corresponding parent Hamiltonian is

$$H(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \Omega^2(t) \sum_i x_i^2$$

$$+ \frac{\hbar^2}{m} \sum_{i<j} \lambda_0(\lambda_0 - 1)c_i^2(t)$$

$$- \frac{\hbar}{m} \lambda_0(\lambda_0 - 1)c_i^2(t) \sum_{i<j} \coth[c(t)x_{ij}] + \tilde{\delta}(t),$$

(81)

where $\Omega(t)$ is given by the Ermakov equation (34) and

$$\tilde{\delta}(t) = \delta(t) + \frac{N(N-1)\lambda_0^2 c_i^2(t)}{2m}.$$ 

(82)

Since in this case $\eta(t) = 0$, the reference Hamiltonian or the pseudo-parent Hamiltonian is given by Eq. (42), which has
the same functional form as Eq. (81), replacing \( \Omega(t) \) with \( \omega(t) \) and replacing \( \bar{\Psi}_0(t) \) with \( \bar{\Psi}_0'(t) \), where

\[
\bar{\Psi}_0'(t) = \bar{\Psi}_0(t) + \frac{N(N-1)\hbar^2 A_0^2 c^2(t)}{2m}.
\] (83)

It is straightforward to show that the control field is the same as in Eq. (50), according to the general expression (45).

The hyperbolic model reduces to the CS model for small values of \( c_0 \). In general, as \( c_0 \) increases the scale-invariance of both \( \mathcal{H}_0'(t) \) and the dynamics given by \( \Phi(t) \) or \( \Psi(t) \) is violated. Remarkably, we see that the control field which keeps the adiabatic evolution of the complex-valued TDIA does not change when varying \( a_0 \). It is worth noting that the dynamics given by Eq. (79) is scale-invariant, while for the hyperbolic interaction in Eq. (81) does not have any scaling properties.

In many situations, e.g., describing unrestricted expansions, the scaling factor \( b(t) \) satisfies

\[
\lim_{t \to \infty} b(t) = \infty, \quad \lim_{t \to \infty} \frac{b(t)}{b_0} = r_0 = \text{const.} \quad (84)
\]

This is the case in time-of-flight experiments in which particles are released after suddenly switching off a confining trap. It also applies to the quenching of the interparticle interactions, as we will see in Sec. IX. For \( t \to \infty \), one finds \( |a(t, t_0)| \) approaches a constant \( |a_\infty(t_0)| \) asymptotically, where

\[
|a_\infty(t_0)| \equiv \left( \frac{m_0 \hbar^2}{2} \right)^{1/2} \left[ \frac{1}{[b(t_0)]^{1/2}} + \frac{1}{\omega_0} \left( \frac{b(t_0)}{b(t_0) - r_0} \right)^{2} \right]^{1/4}. \quad (85)
\]

When Eq. (84) is satisfied, at late times, to the leading order of \( b(t) \), the two-body wave function can be linearized as

\[
\sinh \left[ \frac{c_0 x_{ij}}{b(t)} \right] \approx \left[ \frac{c_0 x_{ij}}{b(t)} \right]. \quad (86)
\]

Thus, at late times, the complex-valued TDIA for time-dependent hyperbolic model (79) degenerates to the one for the scale-invariant dynamics of the CS model. Thus, as \( t_0 \to \infty \) and \( t = t_0 \to \infty \), asymptotically the survival probability in the generalized hyperbolic model is given by

\[
\text{SP}(t, t_0) \sim \left[ \frac{m_0 \hbar}{b(t) b(t_0) |a_\infty(t_0)|^2} \right]^{[N+\ln(N(N-1))]}. \quad (87)
\]

**B. The time-dependent logarithmic-hyperbolic model**

For case (ii), the TDCV Jastrow ansatz becomes

\[
\Psi(t) = \frac{1}{\exp[N(t)]} \prod_{i<j} |\sinh[c(t) x_{ij}]|^{1+i\eta(t)}/2 
\times e^{-\frac{m_0}{2} \sum_i x_{ii}^2 + r(t) x_{ii}^2} \sum_i x_{ii}^N}[N(t)],
\]

where \( c(t) \) is given by Eq. (77) and

\[
N(t) \sim \frac{N \ln \omega(t)}{4} + \frac{hN(N-1)(N-2)}{12m} \int_0^t c^2(\tau)\eta(\tau)d\tau. \quad (88)
\]

The corresponding parent Hamiltonian is

\[
\mathcal{H}_1(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{\hbar^2}{2m} \sum_i \left( b^2(t) + \frac{\hbar^2}{m} \sum_{i<j} \frac{c^2(t)}{4 \sinh^2[c(t) x_{ij}]} \right) - \frac{1}{2} \hbar \omega(t) c(t) \sum_{i<j} \coth[c(t) x_{ij}] x_{ij} + \bar{\Psi}_0'(t) \\
- \frac{1}{2} \hbar \bar{\eta}(t) \sum_{i<j} \ln |\sinh[c(t) x_{ij}]|, \quad (90)
\]

where \( \mathcal{H}_1(t) \) is given by the Ernmarkov equation (34),

\[
\bar{\Psi}(t) = \bar{\Psi}(t) + \frac{N(N-1)\hbar^2 [1 - \eta(t)] c^2(t)}{8m}, \quad (91)
\]

and we have used the identity \( \sigma(t) c(t) + \eta(t) \bar{\eta}(t) = \omega(t) c(t) \) with \( \sigma(t) \) defined in Eq. (31). We note that for small value \( c(t) \), case (i) and (ii) reduces to the corresponding cases respectively in the CS model.

For non-vanishing \( \eta(t), c(t) \) cannot be brought to the scaling form, as is clear from Eq. (77). Thus the dynamics given by (88) is no longer scale-invariant. Finally, we note that in this case the pseudo-parent Hamiltonian or the reference Hamiltonian \( \mathcal{H}_1(t) \) is

\[
\mathcal{H}_1(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{\hbar^2}{2m} \sum_i x_{ii}^2 - \frac{\hbar^2}{m} \sum_{i<j} \frac{c^2(t)}{4 \sinh^2[c(t) x_{ij}]} \\
- \frac{1}{2} \hbar \omega(t) c(t) \sum_{i<j} \coth[c(t) x_{ij}] x_{ij} + \bar{\Psi}_0'(t), \quad (92)
\]

where \( \bar{\Psi}_0'(t) \) is the same as Eq. (83) except now \( c(t) \) is expressed by Eq. (77) and \( \lambda_0 \) is set to be 1/2. The control field is

\[
\mathcal{H}_1(t) = \frac{\hbar \eta(t) c(t)}{4m} \sum_{i<j} \{\coth[c(t) x_{ij}], p_{ij}\} + \frac{b(t)}{2 \hbar(t)} \sum_i \{x_i, p_i\}. \quad (93)
\]

Similar with previous discussions, to evaluate the survival probability, we would like to set \( c(t) \to 0 \) as \( t \to \infty \) so that at late times Eq. (88) degenerates to the complex-valued TDIA for the logarithmic CS model. To this end, in addition to Eq. (84), we further assume

\[
\int_0^\infty \frac{\eta(\tau) d\tau}{b^2(\tau)} \ll \frac{\omega_0^{-1}}{2}. \quad (94)
\]

Then as \( t \to \infty \), one can find \( c(t) \sim c_0/b(t) \)

\[
\Psi(t) \sim \Psi_{\text{logCS}}(t), \quad (95)
\]

where \( \Psi_{\text{logCS}}(t) \) is the complex-valued TDIA for the logarithmic CS model defined in Eq. (67). This situation is reminiscent of that in expansions of a LL gas, that lacking scale-invariance, approaches the scale-invariant strongly-interacting Tonks-Girardeau limit asymptotically [19, 20].
Using the same procedure for bounding the the survival probability of logarithmic CS model, we find as $t_0 \to \infty$ and $t - t_0 \to \infty$,
\[
\text{SP}(t, t_0) \leq \left[ \frac{m \omega_0}{\hbar b(t)b(t_0)|\alpha(t, t_0)|^2} \right]^{N^2(N-1)/2}.
\]

(96)

**VIII. THE TIME-DEPENDENT LONG-RANGE LL MODEL**

The conventional LL model describes ultracold bosons in one spatial dimension subject to contact interactions [13, 14], e.g., describing $s$-wave scattering in ultracold atoms [15]. The generalization of the LL model with additional one-dimensional Coulomb or gravitational interactions was introduced in Ref. [42, 44]. A study of its ground-state properties reveals a rich phase diagram including a trapped McGuire soliton in the attractive case, as well as an incompressible Laughlin-like fluid and Wigner-crystal behavior in the repulsive case. The model has also been generalized to higher spatial dimensions [45] and by changing the attractive and repulsive character of the interactions [109].

We next apply our results to discuss its time dependence exactly. To this end, we take
\[
\Gamma_{ij}(t) = c(t)|\chi_{ij}|, \Gamma'_{ij}(t) = c(t)\text{sgn}(\chi_{ij}),
\]
\[
\Gamma''_{ij}(t) = 2\epsilon(t)\delta_{ij}, \hat{\Gamma}_{ij}(t) = \hat{\epsilon}(t)|\chi_{ij}|
\]
(97)
(98)

Then, the only solution consistent with Eq. (24) is $\eta(t) = 0$, $\hat{N}_2(t) = 0$ and
\[
c(t) = c_0 \sqrt{\frac{\omega(t)}{\omega_0}} = \frac{c_0}{b(t)}.
\]
(99)

Then $\Gamma_{ij}(t)$ for case (i) satisfy the property of Eq. (47). Thus, the complex-valued TDIA is
\[
\Psi(t) = \left[ \frac{\omega_0}{b^4(t)} \right]^{N^4/4} \prod_{i<j} e^{i\theta_{ij}b(0)} e^{-\frac{m}{\omega_0} \sum_i \frac{v_i^2}{b(t)} + i \sum_i \frac{c_0}{b(t)} v_i + iN\tau(t)}.
\]
(100)

Note that as long as $\omega(t)$ is strictly positive, the sign of $c_0$ can be arbitrary. In the case where $\omega(t) = 0$, $c(t)$ must be negative so that $\Psi$ is still normalizable. According to Eq. (28), the corresponding parent Hamiltonian reduces to
\[
\mathcal{H}(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{\hbar}{2} \Omega^2(t) \sum_i x_i^2 + \frac{2c_0}{b(t)} \sum_{i<j} \delta_{ij}
\]
\[
- \frac{\hbar \omega_0 c_0}{b^4(t)} \sum_{i<j} |\chi_{ij}| + \hat{\epsilon}(t),
\]
(101)

where $\Omega(t)$ is given by the Ermakov equation (34) and
\[
\hat{\epsilon}(t) = \epsilon(t) + \frac{N(N-1)\hbar^2 c_0^2}{2mb^2(t)}.
\]
(102)

A few comments are in order. First, as already advanced, the time-independent version of Eq. (101) was found previously [44] in the context of the real-valued TIA, and is the long-range LL model, where the long-range interaction is the Coulomb repulsion in one dimension [42]. Secondly, it is worth noting that in the limit $c_0 \to \infty$ the long-range interaction term does not vanish and it does not reduce to the scale-invariant dynamics for Tonks-Girardeau gas [53]. Instead, it can be recovered by taking limit $\lambda_0 \to 1$ in the Calogero model discussed above (see the Supplemental Material of Ref. [46]). Finally, the dynamics is in this case still scale-invariant as one can see from Eq. (101), but the interaction in the parent Hamiltonian no longer has the scaling property.

When applying to STA, the reference Hamiltonian has the same form as Eq. (101) with $\Omega(t)$ replaced by $\omega(t)$ and $\hat{\epsilon}(t)$ replaced by $\hat{\epsilon}_0(t)$, where
\[
\hat{\epsilon}_0(t) = \hat{\epsilon}_0(t) + \frac{N(N-1)\hbar^2 c_0^2}{2mb^2(t)}.
\]
(103)

The control field is given by Eq. (50) and is thus the same as the one for the CS model.

For a scaling factor $b(t)$ that satisfies Eq. (84), according to Eq. (55), we conclude for fixed number of particles $N$, the asymptotic time-dependence of $|\langle \Psi(t)|\Psi(t_0)\rangle|$ is given by $1/[b(t)]^{N/2}$ so that
\[
\text{SP}(t, t_0) \propto \frac{1}{[b(t)]^N}, \text{ as } t \to \infty.
\]
(104)

We shall check this prediction against numerical calculations in next section for specific choices of $b(t)$.

**IX. QUENCH DYNAMICS**

Although in previous sections we have investigated the parent Hamiltonians corresponding to a time-dependent Jastrow ansatz, it is worth discussing the quench limit where some parameter(s) in the Hamiltonian change from one value to another suddenly. For the sake of illustration, we shall discuss the quench logarithmic CS model and LL model. A similar analysis is also applicable to the generalized hyperbolic model and the hyperbolic-logarithmic model.

As we have discussed in Sec. III, the Ermakov equation (34) is a second-order nonlinear differential equation and we only impose $b(0) = 1$ and $\hat{b}(0)$ is not specified. If we further impose the initial condition
\[
\hat{b}(0) = 0, \quad \eta(0) = 0, \quad \dot{\eta}(0) = 0,
\]
(105)

according to Eqs. (40, 45) we find
\[
\Psi(0) = e^{iN\tau(0)}, \quad \Psi(0) = \Phi(0)e^{iN\tau(0)}, \quad \mathcal{H}'(0) = 0.
\]
(106)

If we further impose
\[
\Omega_0 = \omega_0,
\]
(107)
i.e., $\Omega(t)$ and $\omega(t)$ coincide at $t = 0$, then according to Eqs. (28, 42), we find
\[
\mathcal{H}(0) = \mathcal{H}_0'(0).
\]
(108)
Since $\mathcal{H}_0^0(0)\Phi(0) = 0$, Eqs. (105, 107) would imply $\mathcal{H}(0)\Psi(0) = 0$. That is, the complex-valued TDJA not only satisfies the time-dependent Schrödinger equation, but its projection at time $t = 0$ is also an eigenstate of $\mathcal{H}(0)$.

However, we should keep in mind that the conditions specified Eqs. (105, 107) correspond only to one of the possible choices. In fact, by changing these conditions, one can account for the quench dynamics resulting from fast changes of $b(t)$ or $\eta(t)$. Below we shall further exemplify the previously discussed time-dependent exact dynamics by specifying the functional forms of $b(t)$ or $\eta(t)$.

A. Quench the logarithmic CS model

The TDJA leads to the scaling-invariant dynamics for the standard CS model, as we have discussed in Sec. VI, a feature extensively used in the previous literature [48–50, 58]. Quenches in this model of the trap frequency have been discussed in Ref. [48, 58]. Here we focus on the time-dependent logarithmic CS model (68), where the interaction between the particles is allowed to quench from one value to another. We assume $\Omega(t) = \Omega_0 = \omega_0$, implying $b(t) = 1$. We take

$$\eta(t) = \eta_0 e^{-\omega_0 t},$$

(109)

and Eq. (68) becomes

$$\mathcal{H}(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \Omega_0^2 \sum_i x_i^2 - \frac{\hbar^2}{4m} \sum_{i<j} \frac{1}{x_{ij}^2}$$

$$+ \left\{ \begin{array}{ll} \frac{\hbar^2}{4m} \sum_{i<j} \frac{\eta_0^2}{x_{ij}^2} & t = 0^- \\ \delta(t) \frac{\hbar}{2} \ln \sum_{i<j} \ln |x_{ij}| & t = 0 \\ 0 & t = 0^+ \end{array} \right.,$$

(110)

where we note that a sudden quench in $\eta(t)$ is now supplemented by the delta-kick at $t = 0$. Under such a quench, the complex-valued TDJA (67) reads

$$\Psi(t) = \prod_{i<j} [x_{ij}]^{1+i\eta_0 e^{-\omega_0 t}/2} e^{-\frac{\hbar^2}{2m} \sum_i x_i^2 + iN\tau(t)},$$

(111)

where $\tau(t)$ is determined by imposing $\bar{\mathcal{E}}(t) = 0$ and is found to be $\tau(t) = -\frac{1}{\Omega_0}(N+1)\omega_0 t$. We see that there is a phase slip in the two-body wave function.

B. Quench the long-range LL model

As a first example for the time-dependent long-range LL model (101), we assume the initial time $t_0 = 0$ and impose Eqs. (105, 107). We consider quenching the trap from initial frequency $\Omega_0$ to zero. In this case, one can readily find $b(t) = \sqrt{1 + \omega_0^2 t^2}$. The time-dependence of all the relevant parameters are shown in Fig. 1. Imposing $\bar{\mathcal{E}}(t) = 0$, we find

$$\tau(t) = \left[ \frac{\hbar(N^2 - 1)c_0^2}{6m} - \frac{1}{\Omega_0} \right] \int_0^t ds \frac{1}{b^2(s)}$$

$$= \left[ \frac{\hbar(N^2 - 1)c_0^2}{6mc_0} - \frac{1}{2} \right] \arctan(\omega_0 t),$$

(112)

The Hamiltonian (101) becomes

$$\mathcal{H}(t) = \frac{1}{2m} \sum_i p_i^2 + 2c_0 \sum_{i<j} \delta_{ij} - \frac{\hbar c_0}{b^2(t)} \sum_{i<j} |x_{ij}|$$

$$+ \left\{ \begin{array}{ll} \frac{\hbar}{2m} \sum_i x_i^2 & t = 0^- \\ 0 & t \geq 0 \end{array} \right.,$$

(113)

where TDJA is given by Eq. (100). The quasi-Monte Carlo calculation of survival probability according to Eq. (55) is displayed in Fig. 2, validating the asymptotics for the long-time decay given by Eq. (104).

As a second example, we consider the quench of interactions in the LL model. We first make change of variables $\beta(t) = 1/b(t)$, Eq. (101) becomes

$$\mathcal{H}(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \Omega(t)^2 \sum_i x_i^2 + 2c_0 \sum_{i<j} \delta_{ij}$$

$$- \frac{\hbar c_0}{b^2(t)} \sum_{i<j} |x_{ij}| + \bar{\mathcal{E}}(t),$$

(114)

and the Ermakov equation (34) becomes

$$\Omega^2(t) = \omega_0^2 \beta^2(t) - \frac{2\beta^2(t)}{\beta^2(t)} + \frac{\beta(t)}{\beta(t)},$$

(115)

Consider experimentally the interaction is quenched by setting

$$\beta(t) = \frac{2}{1 + e^{\kappa t}},$$

(116)

where $\kappa$ is sufficiently large so that $\beta(t)$ can be approximated by $\Theta(-\tau)$ and the factor of 2 is to satisfy the condition $\beta(0) = b(0) = 1$. 

![Figure 1. Engineered time-dependence of the control parameters as a function of $\omega_0 t$. The initial time $t_0 = 0$ and $a(t)$ is the strength of the long-range Coulomb interactions defined as $a(t) \equiv \hbar \omega_0 c_0/b^2(t)$.](image-url)
Instead of considering the initial time $t_0 = 0$, we shall take the initial time $t_0 < 0$ with $|t_0| \ll 1/\kappa$. Next, imposing $\hat{E}(t) = 0$, one can find the expression for $\tau(t)$. In particular, for $t \gg \kappa^{-1}$, we find

$$\tau(t) \approx \tau_{\infty} \equiv \left[ \frac{2 \omega_0 - \frac{2\hbar(N^2 - 1)c_0^2}{3m}}{\omega_0} \right] t_0, \quad (117)$$

where we have used the approximation $\int_{t_0}^t \frac{1}{(1 + e^{s})} ds \approx -t_0$ for $t \gg \kappa^{-1}$.

Figure 2. The numerical calculation versus the analytic prediction of the survival probability after a quench of the harmonic trap in the long-range LL model. The values of parameters: $\hbar = 1$, $N = 4$, $\omega_0 = 1$ and $c_0 = -1$. The quasi-Monte Carlo calculation is performed according to Eq. (55). The long-time decay is characterized by asymptotic expression in Eq. (104).

Figure 3. Engineered time-dependence of the control parameters as a function of $\kappa t$. For the plot of $\Omega^2(t)/\omega_c^2$, $\kappa/\omega_0 = 4$. The initial time is $t_0 = -50\kappa^{-1}$ and $a(t)$ is the strength of the long-range Coulomb interactions defined as $a(t) \equiv \hbar c_0 \omega_0 |\hat{E}(t)|$. For $\kappa t > 1$, the time dependence of all these parameters is well approximated by the Heaviside function and therefore leads to a sudden-quench protocol.
The values of parameters: $\hbar = 1$, $N = 4$, $\omega_0 = 1$, $\kappa = 5$ and $c_0 = -1$. The quasi-Monte Carlo calculation is performed according to Eq. (55). For this case, the survival probability is perfectly predicted by Eq. (104).

Therefore, according to Fig. 3 for $\kappa t \gg 1$, we find

$$\mathcal{H}(t) = \frac{1}{2m} \sum_i p_i^2 + \tilde{\delta}(t) + \begin{cases} +8m\omega_0^2 \sum_i x_i^2 + 4c_0 \sum_{i<j} \delta_{ij} & t = t_0 \\ -8\hbar \omega_0 c_0 \sum_{i<j} |x_{ij}| & t \gg 1/\kappa \end{cases}, \quad (118)$$

where we have used the approximation that $\kappa$ is large enough such that $\kappa t \gg 1$. The complex-valued TDJA is given by Eq. (100) with $b(t) = (1 + e^{it})/2$. In particular, we note that

$$\Psi(t) = \begin{cases} (4\omega_0)^{N/4} \prod_{i<j} e^{2\omega_0|x_{ij}|} e^{-\frac{2\omega_0}{\kappa} \sum_i x_i^2} & t = t_0 \\ (4\omega_0)^{N/4} e^{-\kappa \omega t/2} \prod_{i<j} e^{2\omega_0|x_{ij}|} e^{it} & t \gg 1/\kappa, \\ e^{-\frac{2\omega_0}{\kappa} \sum_i x_i^2} e^{it} + \prod_i x_i^{1+NR_m} \sum_i x_i^{1+NR_m} \quad (119) \end{cases}$$

where again we have used $\kappa t \gg 1$. In particular, we note that the case of $\omega_0 = 0$ with $c_0 < 0$ corresponds to the dynamics of quenching the attractive interactions in the LL model from the McGuire soliton state and then applying an inverted harmonic trap. The quasi-Monte Carlo calculation of the survival probability according to Eq. (55) is displayed in Fig. 4, validating the analytic prediction for the long-time decay given by Eq. (104).

X. DISCUSSION

We have shown how finding the parent Hamiltonian of a complex-valued TDJA leads to the systematic generation of exactly-solvable one-dimensional strongly-correlated systems away from equilibrium. While we have focused on spinless bosonic systems, in which the many-body wavefunction is permutation-symmetric under particle exchange, it is natural to consider generalizations to fermionic and anyonic systems. In the presence of hard-core interactions, a natural avenue to do so is via the Bose-Fermi mapping [51] and its generalization to anyonic systems [110]. Such dualities establish a map between bosonic states and states with other quantum exchange statistics, both in and out of equilibrium [53, 111, 112]. In the absence of hardcore interactions, as in the LL gas, one can still find as well documented in the LL gas. In a time-dependent setting, by considering generalizations of the Bose-Fermi mapping that hold at strong coupling for finite interactions, following Gaudin [7]. This approach has already been fruitful in the study of the time-dependent LL gas [19, 20, 113] and paves the way to find time-dependent fermionic and anyonic models dual to the long-range Lieb-Liniger model we have discussed. Yet a different approach is to rely on the exact mapping uncovered by Bethe-ansatz between models with different particle statistics [114–117].

Likewise, further studies can be envisioned by generalizing our results to higher spatial dimensions [43, 45, 118], mixtures [119, 120], truncated interactions [121, 122], and particles with spin or internal structure [123–125].

One may further wonder whether the exact evolutions discussed here admit an integrable structure analogous to that of time-dependent quantum Hamiltonians constructed from a zero-curvature representation in the space of parameters [126].

XI. CONCLUSION

In summary, we have introduced a family of one-dimensional time-dependent many-body Hamiltonians whose nonequilibrium strongly-correlated dynamics is exactly described by the complex-valued TDJA. To guarantee the Hermiticity of the parent Hamiltonian, we derive consistency conditions for the one-body and two-body functions determining the Jastrow ansatz. In doing so, the description of the dynamics is remarkably reduced and boils down to determining the flow of the coupling constants in these models, in terms of simple ordinary differential equations.

We illustrate our findings in four classes of examples, including the celebrated Calogero-Sutherland, hyperbolic, and Lieb-Liniger models in the presence of a driven harmonic trap. We show that these results can be applied to quantum control, and specifically, to the engineering shortcuts to adiabaticity, leading to protocols that steer the adiabatic evolution of the real-valued TDJA. The main findings are summarized in Table I. Finally, we show that our results allow us to describe the exact nonequilibrium dynamics following a quench of the interactions, as discussed in the generalized Calogero-Sutherland models with logarithmic interactions and the long-range Lieb-Liniger model describing bosons subject to contact and Coulomb interactions.

Our findings provide valuable analytic solutions for the dynamics of one-dimensional strongly-correlated quantum sys-
tems of continuous variables, making it possible to study in these systems the quantum work statistics, Loschmidt echos [58, 61], orthogonality catastrophe [59, 127], fidelity susceptibility [128], quantum speed limits [60, 62] and other bounds on quantum evolution [129]. We hope our results can inspire further experimental studies of non-equilibrium dynamics with strongly-correlated quantum matter and be used as a benchmark for numerical methods, quantum simulators, and quantum computers.

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Appendices

Appendix A: Details on the consistency conditions

1. The three-body potential $v_3^{(jk)}$

We introduce the prepotential [130]

$$w(x, t) = f'(x, t)/f(x, t) = \Gamma(x, t) + i\theta'(x, t).$$  \hspace{1cm} (A1)

Note that $w$, $\Gamma'$ and $\theta'$ are all odd functions of $x$. However, unlike the case of the TIJA wave function, the ansatz is a complex-valued function. The condition for the complex three-body potential term in Eq. (3) to reduce to two-body complex potential is the same as in the case of the TIJA wave function [8, 46, 89], i.e., $w$ must satisfy

$$w(x, t)w(y, t) + w(y, t)w(z, t) + w(z, t)w(x, t) = h(x, t) + h(y, t) + h(z, t),$$  \hspace{1cm} (A2)

where $h(x, t)$ is some even function in $x$. This leads to the following differential equation [89]

$$c_1(t)w'''(x, t) - 12c_2(t)w'(x, t) + 6[w(x, t)]^2 = c_0(t),$$  \hspace{1cm} (A3)

where $c_1(t), c_2(t)$, and $c(t)$ are functions of $t$ exclusively. Upon making the change of variables $w'(x, t) = -c_1(t)w(x, t) + c_2(t)$, we find

$$u''(x, t) = 6[u(x, t)]^2 - [6c_2(t) + c_0(t)]/c_1^2(t).$$  \hspace{1cm} (A4)

In the case where $w(x, t) = \int^t u(y, t)dy = \Gamma(x, t)$ is real, i.e., $\theta'(x, t) = 0$, its general solution can be expressed in terms of the Weierstrass zeta function. When

$$\Gamma(x, t) = \frac{A(t)}{x}, \quad \lambda(t)a(t)\coth[a(t)x], \quad c(t)\text{sgn}(x),$$  \hspace{1cm} (A5)

which corresponds to

$$e^{\Gamma(x, t)} \propto |x|^{A(t)} \cdot |\sinh[a(t)x]|^{A(t)} \cdot \exp[c(t)|x|].$$  \hspace{1cm} (A6)

the three-body potential is a constant. Note that replacing $x \rightarrow b(t)x$ in $e^{\Gamma(x, t)}$ will not changes the form of the $\Gamma(x, t)$.

One can show that for $e^{\Gamma(x, t)} \propto |x|^{A(t)}$

$$W_3(t) = 0.$$  \hspace{1cm} (A7)

For $e^{\Gamma(x, t)} \propto |\sinh[c(t)x]|^{A(t)},$

$$W_3(t) = \frac{N(N-1)(N-2)c^2(t)}{6}.$$  \hspace{1cm} (A8)

In addition, for $e^{\Gamma(x, t)} \propto \exp[c(t)|x|]$

$$W_3(t) = \frac{N(N-1)(N-2)c^2(t)}{6}.$$  \hspace{1cm} (A9)
In general, it may be complicated to find complex-valued functions \( w \) that satisfy Eq. (A2) in terms of elementary functions. However, we shall discuss a simple but non-trivial case where the real part of \( w(x, t) \) is proportional to its imaginary part. This results in

\[
\vartheta'(x, t) = \eta(t) \Gamma'(x, t). \tag{A10}
\]

Upon integration of both sides, we obtain

\[
\vartheta(x, t) = \eta(t) \Gamma(x, t), \tag{A11}
\]

\[
w(x, t) = \Gamma'(x, t)[1 + i\eta(t)]. \tag{A12}
\]

When \( e^{\Gamma(x, t)} \) takes the functional form of the ground state wave functions of the CS, Hyperbolic, and LL models, the three-body potential is then given by Eq. (22) in the main text. As discussed there, we have effectively absorbed a position-independent phase in \( \vartheta(x, t) \) into the definition of \( \phi(x, t) \).

2. Two-body potential \( v^{(ij)}_2 \)

It is straightforward to compute

\[
f'_{ij} = (\Gamma'_{ij} + i\omega'_{ij}) f_{ij} = (1 + i\eta) \Gamma'_{ij} f_{ij}, \tag{A13}
\]

\[
f''_{ij} = (1 + i\eta) \Gamma''_{ij} f_{ij} + (1 + i\eta)^2 \Gamma'_{ij}^2 f_{ij}, \tag{A14}
\]

\[
\dot{\theta}_{ij} = (\tilde{\Gamma}_{ij} + i\tilde{\eta}) f_{ij}, \tag{A15}
\]

\[
\dot{\epsilon}_{i} = (\Gamma'_{i} + i\phi'_{i}) \epsilon_{i}. \tag{A16}
\]

Thus, we find

\[
\text{Re} v^{(ij)}_2 = \Gamma''_{ij} + (1 - \eta^2) \Gamma'_{ij}^2 + \Gamma'_{ij}[(\Gamma'_{ij} - \Gamma'_{ij}^2) - \eta(\phi'_{ij} - \phi'_{ij})] - \frac{m}{\hbar} \tilde{\theta}_{ij}, \tag{A17}
\]

\[
\text{Im} v^{(ij)}_2 = \eta\Gamma''_{ij} + 2\eta\Gamma'_{ij}^2 + \Gamma'_{ij}[(\eta(\Gamma'_{ij} - \Gamma'_{ij}) + (\phi'_{ij} - \phi'_{ij})] + \frac{m}{\hbar} \tilde{\Gamma}_{ij}. \tag{A18}
\]

Note that according to Eq. (A11), it follows that

\[
\dot{\theta}_{ij} = \eta \Gamma_{ij} + \eta \dot{\Gamma}_{ij}. \tag{A19}
\]

As we have seen in the previous subsection, the three-body potential term is at most a complex-valued constant. The two-body terms, if depending on \( x_{ij} \), in any case, cannot be canceled by the one-body potential. Then the Hermiticity of the Hamiltonian requires that \( \text{Im} v^{(ij)}_2 \) is a function of time only and independent of \( x_{ij} \). Since \( \Gamma'_{ij}(x) \) is an odd function, we note that \( \Gamma''_{ij}, \Gamma'_{ij}^2 \) and \( \dot{\Gamma}_{ij} \) are even. This implies that

\[
\eta(\Gamma'_{ij} - \Gamma'_{ij}) + (\phi'_{ij} - \phi'_{ij}) = \frac{1}{\Gamma_{ij}} \left[ -\eta \Gamma''_{ij} - 2\eta \Gamma'_{ij}^2 - \frac{m}{\hbar} \tilde{\Gamma}_{ij} + \tilde{N}_2 \right] \tag{A20}
\]

must be an even function of \( x_{ij} \) only and independent of \( \bar{x}_{ij} = (x_{i} + x_{j})/2 \) and that \( \tilde{N}_2(t) \) is a function of time only.

**Lemma 1.** For a differentiable function \( F(x) \), the only possibility for \( F(x) - F(y) \) to be only dependent on \( x - y \) is that \( F(x) \) is a linear function of \( x \).

**Proof.** The proof is straightforward: We denote \( F(x) - F(y) = G(x - y) \). Performing a Taylor expansion, we find

\[
F'(x)\epsilon = G'(0)\epsilon, \tag{A21}
\]

which dictates the \( F'(x) \) must be a constant. \( \square \)

According to the above Lemma, we conclude that

\[
\eta(t) \Lambda'_{x}(t) + \phi'_{x}(t) = -\left[ \frac{m}{\hbar} \dot{\epsilon}^{(2)}(t) x_{k} + \frac{m^2}{\hbar^2} \dot{\varphi}^{(2)}(t) \right], \tag{A22}
\]
where $\mathcal{E}_2(t)$ and $\mathcal{D}_2(t)$ are real-valued functions of time. Integrating both sides of Eq. (A22) over $x$, yields

$$\phi_i(t) = -\left[\eta(t)\Lambda_i(t) + \frac{m}{2\hbar} \mathcal{E}_2(t)x_i^2 + \frac{m^2}{\hbar^2} \mathcal{D}_2(t)x_i\right] + \tau(t),$$

(A23)

where $\tau(t)$ is the overall phase angle.

Substituting Eq. (A22) into Eq. (A20) yields

$$\eta(t)\Gamma''_{ij} + 2\eta(t)\Gamma^2_{ij} + \frac{m}{\hbar} \mathcal{E}_2(t)\Gamma_{ij}x_{ij} = \dot{N}_2(t).$$

(A24)

With Eq. (A22), Eq. (A17) becomes

$$\text{Rev}_2^{(ij)} = \Gamma''_{ij} + [1 - \eta^2(t)]\Gamma^2_{ij} + [1 + \eta^2(t)]\Gamma'_{ij}(\Lambda'_j - \Lambda_j) + \frac{m}{\hbar} \eta(t)\mathcal{E}_2(t)\Gamma_{ij}x_{ij} - \frac{m}{\hbar} \frac{d}{dt}[\eta(t)\Gamma_{ij}].$$

(A25)

3. One-body potential

It is straightforward to compute

$$g_i'' = (\Lambda_i'' + i\phi_i'')g_i + (\Lambda_i + i\phi_i')^2g_i,$$

$$g_i' = (\Lambda_i + i\phi_i')g_i.$$  

(A26)

(A27)

Then, according to $v_i^{(i)} = g_i''/g_i + 2im\dot{g}_i/\hbar g_i$, we find

$$\text{Rev}_i^{(i)} = \Lambda_i'' + \Lambda_i^2 - \phi_i'^2 - \frac{2m}{\hbar} \phi_i,$$

(A28)

$$\text{Im}_i^{(i)} = \phi_i'' + 2\Lambda_i\phi_i' + \frac{2m}{\hbar} \Lambda_i.$$  

(A29)

The Hermiticity of the one-body potential dictates that $\text{Im}U_1^{(i)}$ is only a function of time and independent of $x_i$, i.e.

$$\phi_i'' + 2\Lambda_i\phi_i' + \frac{2m}{\hbar} \Lambda_i = \dot{N}_1.$$  

(A30)

Substituting Eq. (A22) into Eq. (A30) yields,

$$-\eta(t)\Lambda_i'' - 2\eta(t)\Lambda_i^2 - \frac{2m}{\hbar} \mathcal{E}_2(t)\Lambda_i'x_i - \frac{2m^2}{\hbar^2} \mathcal{D}_2(t)\Lambda_i + \frac{2m}{\hbar} \Lambda_i = \dot{N}_1(t) + \frac{m}{\hbar} \mathcal{E}_2(t).$$

(A31)

Similarly, substitution of Eq. (A22) into Eq. (A28) yields

$$\text{Rev}_i^{(i)} = \Lambda_i'' + \Lambda_i^2[1 - \eta^2(t)] - \frac{m^2}{\hbar^2} \mathcal{E}_2(t)x_i^2 - \frac{m^4}{\hbar^4} \mathcal{D}_2(t)$$

$$- \frac{2m^3}{\hbar^3} \frac{d}{dt}[\eta(t)\Lambda_i] + \frac{2m}{\hbar} \mathcal{E}_2(t)x_i^2 + \frac{m^2}{\hbar^2} \mathcal{D}_2(t)x_i - \dot{\tau}(t).$$

(A32)

4. The general consistency conditions

To summarize, Eqs. (A24, A31) are the generic consistency conditions for the parent Hamiltonian of time-dependent Jastrow ansatz when the trial wave function $e^{i\phi(t)}$ is restricted to the CS, hyperbolic, and LL models. Eqs. (A24, A31) can be further recast into

$$\eta(t)\Gamma''_{ij} + 2\eta(t)\Gamma^2_{ij} + \frac{m}{\hbar} \mathcal{E}_2(t)\Gamma_{ij}x_{ij} = \dot{N}_2(t),$$

(A33)

$$\eta(t)\Lambda''_i + 2\eta(t)\Lambda^2_i + \frac{2m}{\hbar} \mathcal{E}_2(t)\left(\Lambda'_i x_i + \frac{1}{2}\right) + \frac{2m^2}{\hbar^2} \mathcal{D}_2(t)\Lambda_i' - \frac{2m}{\hbar} \Lambda_i = -N_1(t).$$

(A34)
Appendix B: Consistency conditions for harmonic trap

For systems confined in a harmonic trap, we would like Eq. (A32) to be at most quadratic in \( x_k \). Thus, we take

\[
\Lambda_k = \frac{-m\omega(t)x_k^2}{2\hbar}.
\]

(B1)

In this case, the one-body consistency condition Eq. (A34) simplifies dramatically, giving \( \mathcal{D}_2(t) = 0 \) and

\[
\mathcal{E}_2(t) = \eta\omega(t) + \frac{\dot{\omega}(t)}{2\omega(t)},
\]

(B2)

\[
\mathcal{N}_1(t) = -\frac{m}{\hbar}[\mathcal{E}_2(t) - \eta(t)\omega(t)] = -\frac{m\dot{\omega}(t)}{4\hbar\omega(t)}.\]

(B3)

The one-body phase angle (A23) now becomes

\[
\phi_k(t) = -\frac{m}{2\hbar}[\mathcal{E}_2(t) - \eta(t)\omega(t)]x_k^2 + \tau(t) = -\frac{m\dot{\omega}(t)}{4\hbar\omega(t)}x_k^2 + \tau(t).
\]

(B4)

Therefore, substituting Eq. (B4) into Eq. (A28), the real part of the one-body potential becomes

\[
\text{Re}^{(i)} = \frac{m^2\Omega^2(t)}{\hbar^2}x_i^2 - \frac{m}{\hbar}\vartheta(t),
\]

(B5)

where \( \Omega(t) \) and \( \theta(t) \) are defined in Eq. (29) and Eq. (32) in the main text, respectively. One can perform a sanity check by substituting Eq. (B2) into Eq. (A32) and find that the real part of the one-body potential is indeed independent of \( \eta(t) \) and Eq. (29) is reproduced.

Eq. (A25) becomes

\[
\text{Re}^{(j)} = \Gamma_{ij}'' + [1 - \eta^2(t)]\Gamma_{ij}^2 - \frac{m\omega(t)}{\hbar}[1 + \eta^2(t)]\Gamma_{ij}\xi_{ij}
\]

\[
+ \frac{m}{\hbar}\eta(t)\left[\eta\omega(t) + \frac{\dot{\omega}(t)}{2\omega(t)}\right]\Gamma_{ij}\xi_{ij} - \frac{m}{\hbar}\frac{d}{dt}[\eta(t)\Gamma_{ij}]
\]

\[
= \Gamma_{ij}'' + [1 - \eta^2(t)]\Gamma_{ij}^2 - \frac{m\varpi(t)}{\hbar}\Gamma_{ij}\xi_{ij} - \frac{m}{\hbar}\frac{d}{dt}[\eta(t)\Gamma_{ij}],
\]

(B6)

where \( \varpi(t) \) is defined in Eq. (31) in the main text.

Appendix C: The parent Hamiltonian of the real-valued TDJA

One can rewrite \( \mathcal{U}(t) \) as

\[
\mathcal{U}(t) = \exp\left[i\left(\eta(t)\sum_{i<j}\Gamma_{ij} - \sum_k\frac{m\dot{\omega}(t)}{4\hbar\omega(t)}x_k^2 + N\tau(t)\right)\right]
\]

(C1)

\[
= \exp\left[i\left(\eta(t)\sum_{i\neq j}\Gamma_{ij} - \frac{m\dot{\omega}(t)}{4\hbar\omega(t)}x_i^2\right) + i\left(\eta(t)\sum_{k<i,j\neq i}\Gamma_{kl} - \frac{m\dot{\omega}(t)}{4\hbar\omega(t)}\sum_{l\neq i}x_l^2 + \tau(t)\right)\right].
\]

(C2)

With Eqs. (C1, C2), it is straightforward to calculate

\[
\mathcal{U}(t) = i\mathcal{U}(t)\left(\sum_{i<j}\frac{d}{dt}[\eta(t)\Gamma_{ij}] - \frac{1}{2\hbar}\frac{d}{dt}\left[\frac{m\dot{\omega}(t)}{2\omega(t)}\right]\sum_kx_k^2 + N\tau(t)\right).
\]

(C3)

\[
\mathcal{U}^{-1}(t)\mathcal{U}(t) = p_i - i[\eta(t)\sum_{l\neq i}\Gamma_{il} - \frac{m\dot{\omega}(t)}{4\hbar\omega(t)}x_l^2, p_i]
\]

\[
= p_i + \hbar\eta(t)\sum_{l\neq i}\Gamma_{il} - \frac{m\dot{\omega}(t)}{2\hbar\omega(t)}x_l.
\]

(C4)
Substituting Eqs. (C3, C4) into Eq. (13) in the main text, we find

\[
\mathcal{H}'(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{\hbar^2}{2m} \dot{\eta}(t) \sum_i \sum_{j \neq i} \Gamma_{ij}^r \left( \sum_k \Gamma_{ik}^r \right) \\
+ \frac{1}{2m} \sum_i \left\{ \eta_0(t) \sum_{j \neq i} \Gamma_{ij}^r - \frac{m \omega(t)}{2\omega(t)} x_i, p_i \right\} - \frac{\hbar \dot{\omega}(t) \eta(t)}{2\omega(t)} \sum_i \sum_{j \neq i} \Gamma_{ij}^r x_i \\
+ \frac{1}{2} m \omega(t)^2 \sum_k x_k^2 + \frac{\hbar^2}{m} \sum_i \sum_{j \neq i} \left( \Gamma_{ij}^r + [1 - \eta^2(t)] \Gamma_{ij}^2 \right) \\
- \hbar \sigma(t) \sum_{i < j} \Gamma_{ij}^r x_{ij} + \tilde{\delta}'(t), \tag{C5}
\]

where

\[
\tilde{\delta}'(t) \equiv \delta'(t) + N \hbar \dot{\tau}(t) = -\frac{1}{2} N \hbar \omega(t) + \frac{\hbar^2}{m} [1 - \eta^2(t)] W_3(t). \tag{C6}
\]

Remarkably, upon transforming \( \mathcal{H}(t) \to \mathcal{H}'(t) \), the frequency of the trap transforms as \( \Omega(t) \to \omega(t) \) [88]. Furthermore, the term \( \sum_{i < j} \frac{\hbar}{m} \eta(t) \Gamma_{ij} \) in \( \mathcal{H}(t) \) [see Eq. (28)] cancels. Finally, we note that

\[
\sum_i \left[ \sum_{j \neq i} \Gamma_{ij}^r \right] \left[ \sum_k \Gamma_{ik}^r \right] = \sum_{i \neq j \neq k} \Gamma_{ij}^r \Gamma_{jk}^r + \sum_{i \neq j} \Gamma_{ij}^2. \tag{C7}
\]

Furthermore, we find

\[
\sum_{i \neq j \neq k} \Gamma_{ij}^r \Gamma_{jk}^r = - \sum_{i \neq j \neq k} \Gamma_{ij}^r \Gamma_{jk}^r, \tag{C8}
\]

where we have changed the dummy indices \( i \) and \( j \). Upon changing the dummy indices \( j \) and \( k \), we find

\[
\sum_{i \neq j \neq k} \Gamma_{ij}^r \Gamma_{jk}^r = \sum_{i \neq j \neq k} \Gamma_{ik}^r \Gamma_{kj}^r = \sum_{i \neq j \neq k} \Gamma_{ki}^r \Gamma_{jk}^r. \tag{C9}
\]

Therefore, it follows that

\[
\sum_{i \neq j \neq k} \Gamma_{ij}^r \Gamma_{jk}^r = - \frac{1}{3} \sum_{i \neq j \neq k} \left( \Gamma_{ij}^r \Gamma_{jk}^r + \Gamma_{ij}^r \Gamma_{ki}^r + \Gamma_{ji}^r \Gamma_{jk}^r \right) \\
= -2 \sum_{i < j < k} \left( \Gamma_{ij}^r \Gamma_{jk}^r + \Gamma_{ij}^r \Gamma_{ki}^r + \Gamma_{ki}^r \Gamma_{jk}^r \right). \tag{C10}
\]

According to Eq. (19),

\[
\sum_i \left[ \sum_{j \neq i} \Gamma_{ij}^r \right] \left[ \sum_k \Gamma_{ik}^r \right] = -2 \sum_{i < j < k} \left( \Gamma_{ij}^r \Gamma_{jk}^r + \Gamma_{ij}^r \Gamma_{ki}^r + \Gamma_{ji}^r \Gamma_{jk}^r \right) + 2 \sum_{i < j} \Gamma_{ij}^2 \\
= 2 W_3(t) + 2 \sum_{i < j} \Gamma_{ij}^2. \tag{C10}
\]

Furthermore, we note that since \( \Gamma_{ij}^r \) is skew-symmetric in the indices \( i \) and \( j \),

\[
\sum_i \sum_{j \neq i} \Gamma_{ij}^r(t) x_i \equiv \frac{1}{2} \sum_i \sum_{j \neq i} \Gamma_{ij}^r(t) x_{ij} = \sum_{i < j} \Gamma_{ij}^r(t) x_{ij}. \tag{C11}
\]
Similarly,

\[
\sum_i \left\{ \sum_{j \neq i} \Gamma_{ij}^r(t), \ p_i \right\} = \sum_{i \neq j} \Gamma_{ij}^r(t)p_i + \sum_{i \neq j} p_i \Gamma_{ij}^r(t) \\
= \frac{1}{2} \sum_{i \neq j} \Gamma_{ij}^r(t)(p_i - p_j) + \frac{1}{2} \sum_{i \neq j} (p_i - p_j) \Gamma_{ij}^r(t) \\
= \frac{1}{2} \sum_{i \neq j} \Gamma_{ij}^r(t), \ p_{ij} \\
= \sum_{i < j} (\Gamma_{ij}^r(t), \ p_{ij}).
\]  

(C12)

Substituting Eqs. (C10, C11, C12) into Eq. (C5) yields

\[
\mathcal{H}^e(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} m \omega^2(t) \sum_k x_k^2 + \frac{\hbar^2}{m} \sum_{i < j} (\Gamma_{ij}^r + \Gamma_{ij}^0) \\
+ \frac{\hbar \eta(t)}{2m} \sum_{i < j} [\Gamma_{ij}(t), \ p_{ij}] - \frac{\dot{\omega}(t)}{4\omega(t)} \sum_i \{\dot{x}_i, \ p_i\} - \hbar \left[ \sigma(t) + \frac{\dot{\omega}(t) \eta(t)}{2\omega(t)} \right] \sum_{i < j} \Gamma_{ij} x_{ij} + \mathcal{E}_0^e(t).
\]  

(C13)

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The case $\lambda(t) = 0$ is trivial.