Frequency-Hopping Sequence Sets With Low Average and Maximum Hamming Correlation
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Abstract—In frequency-hopping multiple-access (FHMA) systems, the average Hamming correlation (AHC) among frequency-hopping sequences (FHSs) as well as the maximum Hamming correlation (MHC) is an important performance measure. Therefore, it is a challenging problem to design FHS sets with good AHC and MHC properties for application. In this paper, we analyze the AHC properties of an FHS set, and present new constructions for FHS sets with optimal AHC. We first calculate the AHC of some known FHS sets with optimal MHC, and check their optimality. We then prove that any uniformly distributed FHS set has optimal AHC. We also present two constructions of FHS sets with optimal AHC based on cyclotomy. Finally, we show that if an FHS set is obtained from another FHS set with optimal AHC by an interleaving, it has optimal AHC.

Index Terms—Average Hamming correlation, maximum Hamming correlation, frequency-hopping multiple-access, frequency-hopping sequences.

I. INTRODUCTION

In multiple-access communication systems, the receiver is confronted with the interference caused by undesired signals when it attempts to demodulate one of the signals sent from several transmitters. For frequency-hopping multiple-access (FHMA) systems, such a multiple-access interference (MAI) arise mainly from the hits of frequencies assigned to users in each time slot. It is possible to reduce the MAI in multiple-access systems by employing frequency-hopping sequence (FHS) sets with low Hamming correlation. There are two measures on the Hamming correlation of an FHS set used in FHMA systems. The average Hamming correlation (AHC) among FHSs measures its average performance, while the maximum Hamming correlation (MHC) represents its worst-case performance. Therefore, AHC as well as MHC is an important performance measure for an FHS set.

In general, it is desirable that an FHS set should have a large set size and a low AHC or MHC value, when its length and the number of available frequencies are fixed. There are several known constructions [7]–[13] for FHS sets having optimal MHC with respect to the Peng-Fan bound [6]. On the other hand, only a few constructions for FHS sets with optimal AHC have been known because AHC has been recently considered [14]. Peng et al. in [14] established a bound on the AHC of an FHS set and presented some FHS sets with optimal AHC, which are based on cubic polynomials. By using the theory of cyclotomy, [15], Liu et al. also constructed FHS sets with optimal AHC [16]. Unfortunately, some previously known FHS sets with optimal AHC do not have good MHC properties. Therefore, it is a challenging problem to design FHS sets with optimal AHC and low MHC.

In this paper, we deal with FHS sets having optimal AHC and low MHC. We check the relation between optimal MHC and AHC by calculating the AHC values of some known optimal FHS sets with respect to the Peng-Fan bound. We also show that any “uniformly distributed” FHS set has optimal AHC, and present some examples with low MHC. We then construct some FHS sets with optimal AHC and low MHC based on cyclotomy, which have lengths $p$ or $p^r-1$ for a prime $p$ and a positive integer $r$. Finally, we analyze the optimality of FHS sets constructed by interleaving techniques, and give some new interleaved FHS sets with optimal AHC and low MHC.

The outline of the paper is as follows. In Section II, some preliminaries on AHC and MHC are presented. We prove that a uniformly distributed FHS set has optimal AHC and give some examples in Section III. In Section IV, we present new constructions for FHS sets with optimal AHC based on cyclotomy. The AHC properties of FHS sets constructed by interleaving techniques are analyzed in Section V. Finally, we give some concluding remarks in Section VI.

II. AVERAGE HAMMING CORRELATION OF FHS SETS

Throughout the paper, we denote by $\lfloor x \rfloor$ the smallest integer greater than or equal to $x$. We also denote by $\langle x \rangle$ the least nonnegative residue of $x$ modulo $y$ for an integer $x$ and a positive integer $y$. For an integer $m$, we denote by $\mathbb{Z}_m$ the set of integers modulo $m$.

A. Maximum Hamming Correlation

Let $\mathcal{F} = \{f_0, f_1, \ldots, f_{M-1}\}$ be a set of available frequencies. A sequence $X = \{X(t)\}_{t=0}^{N-1}$ is called an FHS of length $N$ over $\mathcal{F}$ if $X(t) \in \mathcal{F}$ for all $0 \leq t \leq N - 1$. For two FHSs $X$ and $Y$ of length $N$ over $\mathcal{F}$, the periodic Hamming correlation between $X$ and $Y$ is defined as

$$H_{X,Y}(\tau) = \sum_{t=0}^{N-1} h[X(t), Y((t + \tau) \pmod N)], \quad 0 \leq \tau \leq N - 1$$

where

$$h[x, y] = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

If $X = Y$, $H_{X,Y}(\tau)$ is called the Hamming autocorrelation of $X$, denoted by $H_X(\tau)$. The maximum out-of-phase Hamming correlation
autocorrelation of $X$ is defined as

$$H(X) = \max_{1 \leq \tau \leq N-1} \{H_X(\tau)\}.$$ 

A well-known bound on it, called the Lempel-Greenberger bound [7], is given in the following lemma.

**Lemma 1** ([7]). For any FHS $X$ of length $N$ over $\mathcal{F}$ with $|\mathcal{F}| = M$,

$$H(X) \geq \left(\frac{(N-b)(N+b-M)}{M(N-1)}\right)$$

where $b = \langle N \rangle_M$.

Let $\mathcal{U}$ be an $(N, M, L)$-FHS set, that is, an FHS set consisting of $L$ FHSs of length $N$ over $\mathcal{F}$. For any two distinct FHSs $X$ and $Y$ in $\mathcal{U}$, let

$$H(X, Y) = \max_{0 \leq \tau \leq N-1} \{H_{X,Y}(\tau)\}.$$ 

The maximum out-of-phase Hamming autocorrelation $H_a(\mathcal{U})$ and the maximum Hamming crosscorrelation $H_c(\mathcal{U})$ of $\mathcal{U}$ are defined as

$$H_a(\mathcal{U}) = \max_{X \in \mathcal{U}} \{H(X)\},$$

$$H_c(\mathcal{U}) = \max_{X,Y \in \mathcal{U}, X \neq Y} \{H(X,Y)\},$$

respectively. The maximum Hamming correlation of $\mathcal{U}$ is also defined as

$$H(\mathcal{U}) = \max\{H_a(\mathcal{U}), H_c(\mathcal{U})\}.$$ 

Peng and Fan established some bounds on the maximum out-of-phase Hamming autocorrelation and the maximum Hamming crosscorrelation of an FHS set in terms of frequency set size, length, and the number of FHSs [6].

**Lemma 2** ([6]). Let $\mathcal{U}$ be an $(N, M, L)$-FHS set. Then

$$M(N-1)H_a(\mathcal{U}) + NM(N-1)H_c(\mathcal{U}) \geq N(NL-M).$$

(2)

An FHS set $\mathcal{U}$ is said to have optimal MHC if $(H_a(\mathcal{U}), H_c(\mathcal{U})) = (\lambda_a, \lambda_c)$, where the integer pair $(\lambda_a, \lambda_c)$ satisfies (2), but $(\lambda_a - \delta, \lambda_c - \delta)$ does not satisfy (2) for any positive integer $\delta$ [13]. The set $\mathcal{U}$ is said to have near-optimal MHC if $(H_a(\mathcal{U}), H_c(\mathcal{U})) = (\lambda_a + 1, \lambda_c + 1)$.

**B. Average Hamming Correlation**

Let $\mathcal{U}$ be an $(N, M, L)$-FHS set. For our convenience, let $S_a(\mathcal{U})$ be the sum of all out-of-phase Hamming autocorrelation values in $\mathcal{U}$, that is,

$$S_a(\mathcal{U}) = \sum_{X \in \mathcal{U}} \sum_{\tau=1}^{N-1} H_X(\tau).$$

Similarly, let $S_c(\mathcal{U})$ be the sum of all Hamming crosscorrelation values in $\mathcal{U}$, given by

$$S_c(\mathcal{U}) = \sum_{X,Y \in \mathcal{U}, X \neq Y} \sum_{\tau=0}^{N-1} H_{X,Y}(\tau).$$

Then, the average Hamming autocorrelation and crosscorrelation of $\mathcal{U}$ are defined by

$$A_a(\mathcal{U}) = \frac{S_a(\mathcal{U})}{L(N-1)}$$

and

$$A_c(\mathcal{U}) = \frac{S_c(\mathcal{U})}{N(N-1)(L-1)},$$

respectively. Peng et al. established a bound on the AHC of an FHS set [14].

**Lemma 3** ([14]). Let $\mathcal{U}$ be an $(N, M, L)$-FHS set. Then

$$A_a(\mathcal{U}) + A_c(\mathcal{U}) \geq \frac{NL-M}{M(N-1)(L-1)}.$$  

(3)

An FHS set $\mathcal{U}$ will be said to have optimal AHC if the pair $(A_a(\mathcal{U}), A_c(\mathcal{U}))$ satisfies (3) with equality. Note that an optimal pair satisfying the bound in (3) may consist of rational numbers, while every optimal pair satisfying the Peng-Fan bound is an integer pair. Moreover, AHC is not directly related to MHC from the viewpoints of their definitions. In fact, an optimal FHS set with respect to the Peng-Fan bound is not necessarily optimal with respect to the AHC bound in (3). Therefore, it is interesting to investigate the AHC properties of known FHS sets with optimal MHC. The AHC values of some known FHS sets with optimal MHC are calculated and summarized in Table I.

**Remark:** In Theorem 2 of [14], Peng et al. mentioned without a detailed proof that any FHS set with optimal MHC has optimal AHC, assuming that $(A_a(\mathcal{U}), A_c(\mathcal{U}))$ is an integer pair. However, their argument is not true in general since the assumption is not always valid, as discussed above.

**III. AVERAGE HAMMING CORRELATION OF UNIFORMLY DISTRIBUTED FHS SETS**

Balancedness is one of the major randomness measures for deterministically generated sequences, since it is closely related to unpredictability [17]. Hence, it is very important to design balanced sequences and analyze their correlation properties for their application to communication systems. In this section, we investigate the AHC properties of a special class of FHS sets, called ‘uniformly distributed’ FHS sets, and present some examples of them which also have low MHC.

Given an FHS $X = \{X(t)\}_{t=0}^{N-1}$ over $\mathcal{F}$, let

$$N_X(a) = |\{t : X(t) = a, 0 \leq t \leq N - 1\}|$$

for $a \in \mathcal{F}$. When $|N_X(a) - N_X(b)| \leq 1$ for any $a, b \in \mathcal{F}$, we call $X$ a balanced FHS. In particular, $X$ will be referred to as a perfectly balanced FHS if $|N_X(a) - N_X(b)| = 0$ for any $a, b \in \mathcal{F}$. Note that $N$ should be a multiple of the size of $\mathcal{F}$ if $X$ is perfectly balanced. An FHS set consisting of (perfectly) balanced FHSs is called a (perfectly) balanced FHS set.

The following well-known identity gives a relationship between the distribution of frequencies and the sum of Hamming correlation values between two FHSs. It is very useful in
deriving some bounds on the MHC or AHC of an FHS set, including the Peng-Fan bound [4].

**Lemma 4.** Let \( X = \{X(t)\}_{t=0}^{N-1} \) and \( Y = \{Y(t)\}_{t=0}^{N-1} \) be two FHS sets over \( F \). Then
\[
\sum_{\tau=0}^{N-1} H_{X,Y}(\tau) = \sum_{a \in F} N_X(a) N_Y(a).
\]

In particular,
\[
\sum_{\tau=0}^{N-1} H_X(\tau) = \sum_{a \in F} N_X(a)^2.
\]

Given an \((N, M, L)\)-FHS set \( \mathcal{X} = \{X_i \mid 0 \leq i \leq L-1\}\) over \( F \), define
\[
N_X(a) = \sum_{i=0}^{L-1} N_{X_i}(a)
\]
for \( a \in F \). The FHS set \( \mathcal{X} \) is called a uniformly distributed FHS set if \( |N_X(a) - N_X(b)| = 0 \) for any \( a, b \in F \). In this case, it is required that \( M \mid NL \). Clearly, an FHS set is uniformly distributed if it is perfectly balanced. The following lemma gives a relation between the sums of Hamming correlation values of \( \mathcal{X} \) and the numbers \( N_X(a), a \in F \).

**Lemma 5.** Let \( \mathcal{X} \) be an \((N, M, L)\)-FHS set over \( F \). Then
\[
S_a(\mathcal{X}) + S_{\bar{a}}(\mathcal{X}) = \sum_{a \in F} N_X(a) \cdot (N_X(a) - 1).
\]

**Proof.** Note that
\[
S_a(\mathcal{X}) + S_{\bar{a}}(\mathcal{X}) = \sum_{0 \leq i, j \leq L-1} \sum_{\tau=0}^{N-1} H_{X_i, X_j}(\tau) - \sum_{0 \leq i \leq L-1} H_{X_i}(0).
\]

By applying Lemma 4, the fact that \( H_{X_i}(0) = \sum_{a \in F} N_{X_i}(a) \), we obtain
\[
S_a(\mathcal{X}) + S_{\bar{a}}(\mathcal{X}) = \sum_{0 \leq i, j \leq L-1} \sum_{a \in F} N_{X_i}(a) N_{X_j}(a) - \sum_{0 \leq i \leq L-1} N_{X_i}(a) = \sum_{a \in F} \left( \sum_{0 \leq i \leq L-1} N_{X_i}(a) \right) \left( \sum_{0 \leq j \leq L-1} N_{X_j}(a) \right) - \sum_{0 \leq i \leq L-1} N_{X_i}(a) = \sum_{a \in F} \left( \sum_{0 \leq i \leq L-1} N_{X_i}(a) - 1 \right) = \sum_{a \in F} N_X(a) \cdot (N_X(a) - 1),
\]
where the last equality directly comes from the definition of \( N_X(a) \).

By Lemma 5 it is possible to prove the optimality of a uniformly distributed FHS set.

**Theorem 6.** Let \( \mathcal{X} \) be a uniformly distributed \((N, M, L)\)-FHS set. Then \( \mathcal{X} \) has optimal AHC.

**Proof.** Note that \( M \mid NL \) and \( N_X(a) = \frac{NL}{M} \) for all \( a \in F \) since \( \mathcal{X} \) is a uniformly distributed FHS set. By Lemma 5, we have
\[
S_a(\mathcal{X}) + S_{\bar{a}}(\mathcal{X}) = M \cdot \frac{NL}{M} \cdot \left( \frac{NL}{M} - 1 \right) = \frac{NL(NL-M)}{M}.
\]

Therefore, the left-hand side (LHS) and the right-hand side (RHS) of (3) are given by
\[
\text{LHS} = \frac{NL(NL-1)(L-1)}{M(N-1)(L-1)} = \frac{NL-M}{M(N-1)(L-1)},
\]
and
\[
\text{RHS} = \frac{NL-M}{M(N-1)(L-1)},
\]
respectively.

**Corollary 7.** Let \( \mathcal{X} \) be a perfectly balanced \((N, M, L)\)-FHS set. Then \( \mathcal{X} \) has optimal AHC.

Theorem 6 and Corollary 7 tell us that any uniformly distributed or perfectly balanced FHS set is optimal with respect to the bound in (3). However, such an FHS set is also required to have good MHC properties if it is applicable to FHMA systems. In the following, we will give three examples of uniformly distributed or perfectly balanced FHS sets with low MHC.

**Example 8.** Let \( p \) be an odd prime. For \( 0 \leq t \leq p^2 - 1 \), let \( t = t_0 p + t_1 \), where \( 0 \leq t_0, t_1 \leq p - 1 \). Let \( \mathcal{X}_1 \) be the \((p^2, p, p)\)-FHS set defined as \( \mathcal{X}_1 = \{X_i \mid 0 \leq i \leq p - 1\} \) where \( X_i = \{X_i(t)\}_{t=0}^{p^2-1} \) is the FHS over \( p \mathbb{Z}_p \), given by
\[
X_i(t) = (p t_0 t_1 + p t_1^2) \mod p.
\]

It was proved by Kumar [5] that \( \mathcal{X}_1 \) is optimal with respect to the Peng-Fan bound. Note that \( \mathcal{X}_1 \) is a uniformly distributed FHS set, since \( N_{X_1}(a) = p^2 \) for any \( a \in p \mathbb{Z}_p \). Therefore, \( \mathcal{X}_1 \) has optimal AHC by Theorem 6.

**Example 9.** Let \( k \) and \( N \) be two positive integers such that \( N \geq 3 \) and \( 2 \leq k < N \). Assume that \( N = Ld + r \), where \( L \) is a positive integer, \( 0 \leq r < d \), and \( 1 \leq d < \frac{N}{2} \). For an integer \( 0 \leq i \leq L-1 \), let \( \mathcal{X}_i = \{X_i(t)\}_{t=0}^{N-1} \) be the FHS over \( \mathbb{Z}_k \times \mathbb{Z}_N \) defined as
\[
X_i(kt_1 + t_0) = \begin{cases} 
(t_0, t_1 + id), & \text{if } t_0 = 0, 1, \ldots, \left\lfloor \frac{r}{d} \right\rfloor \\
(t_0, t_1 + (L - 1 - i)d), & \text{if } t_0 = \left\lfloor \frac{r}{d} \right\rfloor + 1, \ldots, k - 1
\end{cases}
\]
where $0 \leq t_0 \leq k - 1$ and $0 \leq t_1 \leq N - 1$. It was shown by Chung et al. [18] that the $(kN, kN, L)$-FHS set $X_2 = \{X_1 | 0 \leq i \leq L - 1\}$ has zero Hamming autocorrelation for any $0 < |\tau| < d - 1$, and zero Hamming crosscorrelation for any $0 \leq |\tau| < d - 1$, that is, it is a no-hit-zone FHS set. In particular, it is optimal with respect to the bound given in [19]. Note that $X_2$ is a perfectly balanced FHS set, since $N_{X_i}(a, b) = 1$ for any $0 \leq i \leq L - 1$ and any $(a, b) \in \mathbb{Z}_k \times \mathbb{Z}_N$. Therefore, $X_2$ has optimal AHC by Corollary 7.

Example 10. Let $p$ be an odd prime. For $0 \leq t \leq p^2 - p - 1$, let $t_0 = (t)_{p-1}$ and $t_1 = (t)_p$. Let $X_3$ be the $(p^2 - p, p)$-FHS set defined as $X_3 = \{X_1 | 0 \leq i \leq p - 1\}$ where $X_1 = \{X_i(t)\}_{t=0}^{p^2-p-1}$ and

$$X_i(t) = \langle(t_0 + 1) \cdot t_1 + i \rangle_p.$$

It was proved in [20] that $X_3$ is optimal with respect to the Peng-Fan bound. Note that $X_3$ is a perfectly balanced FHS set, since $N_{X_i}(a) = p - 1$ for any $0 \leq i \leq p - 1$ and any $a \in \mathbb{Z}_p$. Therefore, $X_3$ has optimal AHC by Corollary 7.

IV. FHS SETS WITH OPTIMAL AVERAGE HAMMING CORRELATION BASED ON CYCLOMOTY

Let $\mathbb{F}_q$ be the finite field of $q = p^n$ elements and $\mathbb{F}_q^*$ the set of nonzero elements in $\mathbb{F}_q$ where $p$ is a prime and $n$ is a positive integer. For some positive integers $M$ and $f$, let $q = Mf + 1$. For a primitive element $\alpha$ of $\mathbb{F}_q$, $\mathbb{F}_q^*$ is decomposed into $M$ disjoint subsets

$$C_r = \{\alpha^M + r | 0 \leq l \leq f - 1\}, \quad r = 0, 1, \ldots, M - 1$$

which are called the cyclomatic classes of $\mathbb{F}_q$ of order $M$. For two integers $i$ and $j$ in $\mathbb{Z}_M$, the number defined by

$$(i, j)_M := |(C_i + 1) \cap C_j|$$

is called a cyclomatic number of $\mathbb{F}_q$ of order $M$ [21].

The result in the following lemma was first proven by Sze et al. in [22] and was rediscovered by Chu and Colburn in [9].

Lemma 11 [22]. Let $q = Mf + 1$ be a prime power. Then we have

$$\sum_{i=0}^{M-1} (i + j)_M = \begin{cases} f - 1, & \text{if } j \equiv 0 \mod M \\ f, & \text{if } j \not\equiv 0 \mod M. \end{cases}$$

In [9], Chu and Colburn gave optimal FHSs over $\mathbb{Z}_M$ or $\mathbb{Z}_M \cup \{\infty\}$ with respect to the Lempel-Greenenger bound as well as an optimal FHS set over $\mathbb{Z}_M \cup \{\infty\}$ with respect to the Peng-Fan bound. Although the FHS set has optimal MHC, it does not have optimal AHC with respect to the bound in [3]. In order to get an FHS set with optimal AHC and near-optimal MHC, we modify their construction as follows:

Construction A: Let $p = Mf + 1$ be an odd prime for some positive integers $M$ and $f$. For $0 \leq i \leq M - 1$, define the FHS $X_i = \{X_i(t)\}_{t=0}^{p^2-1}$ as

$$X_i(0) = i$$

and

$$X_i(t) = (r + i)_M \quad \text{if } t \in C_r.$$

where $C_r$ is a cyclotomic class of $\mathbb{F}_p$ of order $M$. Let $X_4$ be the $(p, M, M)$-FHS set defined by

$$X_4 = \{X_i | 0 \leq i \leq M - 1\}$$

By using the theory of cyclotomy [21], it is possible to calculate the AHC and MHC values of $X_4$ and check its optimality.

Theorem 12. The set $X_4$ in Construction A has optimal AHC with respect to the bound in [3]. Moreover, it is near-optimal with respect to the Peng-Fan bound.

Proof. Clearly, the set $X_4$ is a uniformly distributed FHS set, and so it has optimal AHC by Theorem 6. Let $H_{i,j}(\tau)$ be the Hamming correlation between $X_i$ and $X_j$. It is obvious that $H_{i,j}(0) = p$ if $i = j$, and $H_{i,j}(0) = 0$, otherwise. For $1 \leq \tau \leq p - 1$, we have

$$H_{i,j}(\tau) = \sum_{\tau \in \mathbb{Z}_p \setminus (-\tau, 0)} h[X_i(t), X_j(t + \tau)] + I(\tau \in C_{i-j}) + I(-\tau \in C_{j-i})$$

$$= \sum_{r=0}^{M-1} (r, r + (j - i))_M + |\{\tau \in C_{i-j}\} + |\{\tau \in C_{j-i}\}.$$

Hence,

$$H_a(X_4) = f + 1$$

and

$$H_c(X_4) = f + 2$$

by Lemma 11. It is easily checked that $X_4$ has near-optimal MHC.

Remark: The AHC values of $X_4$ may be easily computed by applying Lemma 4 to $S_a(X_4)$ and $S_c(X_4)$. The average Hamming autocorrelation of $X_4$ is calculated as follows:

$$A_a(X_4) = \frac{1}{M(p-1)} \sum_{X \in X_4} \sum_{\tau=1}^{p-1} H_X(\tau)$$

$$= \frac{1}{M^2 f} \sum_{X \in X_4} \sum_{\alpha \in \mathbb{F}_q} \left[N_X(a)^2 - N_X(0)\right]$$

$$= \frac{M ((f + 1)^2 + (M - 1)f^2 - (Mf + 1))}{M^2 f}$$

$$= f - 1 + \frac{2}{M}.$$

Similarly, the average Hamming crosscorrelation is given by

$$A_{c}(X_4) = \frac{1}{M(M - 1)p} \sum_{X, Y \in X_4, X \neq Y} \sum_{\tau=0}^{p-1} H_{X,Y}(\tau)$$

$$= \frac{1}{M(M - 1)p} \sum_{X, Y \in X_4, X \neq Y} \sum_{\alpha \in \mathbb{F}_q} N_X(a)N_Y(\alpha)$$

$$= \frac{M(M - 1)(2f(f + 1) + (M - 2)f^2)}{M(M - 1)(Mf + 1)}$$

$$= \frac{Mf^2 + 2f}{Mf + 1}.$$
It is easily checked that the pair \((A_5(X_4), A_c(X_4))\) satisfies the bound in (3) with equality.

**Remark:** Each FHS in \(X_3\) is optimal with respect to the Lempel-Greenberger bound \([7]\).

In \([10]\), Ding and Yin gave FHS sets of length \(q-1\) over \(\mathbb{Z}_M\) or \(\mathbb{Z}_M \cup \{\infty\}\) based on the discrete logarithm in \(\mathbb{F}_q\) for a prime power \(q\). Han and Yang observed in \([11]\) that these FHS sets are closely related to Sidel'nikov sequences and some of the results in \([23]\) and \([10]\) are not correct, and made corrections to them. The FHS set over \(\mathbb{Z}_M\) by Ding and Yin \([10]\) can be equivalently represented as follows:

**Construction B:** For a prime power \(q = p^n\) such that there exist two integers \(M\) and \(f\) such that \(q = Mf + 1\), let \(C_r, 0 \leq r \leq M - 1\) be the cyclotomic class of order \(M\) of \(\mathbb{F}_q\).

Let \(X_5\) be the \((q-1, M, M)\)-FHS set over \(\mathbb{Z}_M\) given by

\[
X_5 = \left\{ X_i \mid X_i = \{ X_i(t) \}_{t=0}^{q-2}, \ 0 \leq i \leq M - 1 \right\}
\]

where

\[
X_i(t) = \left\{ \begin{array}{ll}
\langle r + i \rangle_M, & \text{if } \alpha^t + 1 \in C_r \\
i, & \text{if } \alpha^t + 1 = 0.
\end{array} \right.
\]

**Theorem 13.** The set \(X_5\) in Construction B has optimal AHC and \(H(X_5) \leq f + 2\). In particular, \(X_5\) has near-optimal MHC if \((2l, l)_M = 0\) for all \(l \in \mathbb{Z}_M\).

**Proof.** Clearly, the set \(X_5\) is a perfectly balanced FHS set, and so it has optimal AHC by Theorem \([6]\) Let \(H_{i,j}(\tau)\) be the Hamming correlation between \(X_i\) and \(X_j\) in \(X_5\). It is obvious that \(H_{i,j}(0) = q - 1\) if \(i = j\), and \(H_{i,j}(0) = 0\), otherwise. For \(1 \leq \tau \leq q - 2\), \(H_{i,j}(\tau)\) is given by

\[
H_{i,j}(\tau) = \sum_{r=0}^{M-1} (j + \tau, i) + I(1 - \alpha^\tau \in C_{i-j}) + I(-\alpha^{-\tau} (1 - \alpha^\tau \in C_{j-i})).
\]

After some calculation, it is checked that \(X_5\) is a near-optimal FHS set with respect to the Peng-Fan bound when \(|C_l \cap (C_{2l} + 1)| = 0\), that is, \((2l, l)_M = 0\) for all \(l \in \mathbb{Z}_M\). \(\square\)

**V. AVERAGE HAMMING CORRELATION OF FHS SETS BASED ON INTERLEAVING TECHNIQUES**

Interleaving techniques are used to construct a sequence of length \(kN\) from \(k\) sequences of length \(N\), which are not necessarily distinct for some positive integers \(k\) and \(N\) \([24]\). They have been widely employed in the construction of sequences with low correlation \([24], [25], [26]\). In particular, Chung et al. \([12]\) firstly applied the interleaving techniques to the design of FHSs, and presented several FHS sets with optimal MHC. A similar approach was given in \([13]\) to construct no-hit-zone FHS sets. However, these previous works dealt only with the MHC of FHS sets constructed by interleaving techniques. In this section, we will focus on the AHC of such FHS sets. First of all, we will show that the sum of Hamming correlation values and the optimality on the AHC of an FHS set are preserved under any interleaving in the following theorem.

**Theorem 14.** Let \(X \triangleq \{ X_i \mid 0 \leq i \leq L - 1 \}\) be an \((N, M, L)\)-FHS set over \(\mathbb{F}\) and \(Y \triangleq \{ Y_j \mid 0 \leq j \leq L' - 1 \}\) an \((N', M, L')\)-FHS set obtained by interleaving \(X\) such that \(NL = N'L'\) and the FHS \(Y_j \triangleq \{ Y_j(s) \}_{s=0}^{N'-1}, 0 \leq j \leq L'-1\), is defined as

\[
Y_j(s) = X_i(t)
\]

for some \(0 \leq i \leq L - 1\) and \(0 \leq t \leq N - 1\). Assume that \(i_1 = i_2\) and \(t_1 = t_2\) if and only if \(j_1 = j_2\) and \(s_1 = s_2\), when \(Y_j(s_1) = X_i(t_1)\) and \(Y_j(s_2) = X_i(t_2)\). Then, we have

\[
S_a(X) + S_c(X) = S_a(Y) + S_c(Y).
\]

Furthermore, \(X\) has optimal AHC if and only if \(Y\) has optimal AHC.

**Proof.** By the assumption on \(X\) and \(Y\), we have \(N_a(X) = N_a(Y)\) for all \(a \in \mathbb{F}\). Hence, \(S_a(X) + S_c(X) = S_a(Y) + S_c(Y)\) by Lemma \([8]\).

Let \(LHS_X\) (resp. \(LHS_Y\)) be the left-hand side of (3) for the FHS set \(X\) (resp. \(Y\)), and \(RHS_X\) (resp. \(RHS_Y\)) the right-hand side of (3) for \(X\) (resp. \(Y\)). By (4) and the assumption that \(NL = N'L'\), we have

\[
\begin{align*}
LHS_Y &= \frac{S_a(Y) + S_c(Y)}{N'L'(N'-1)(L'-1)} \\
&= \frac{S_a(X) + S_c(X)}{NL(N'-1)(L'-1)} \\
&= RHS_X \cdot \frac{(N-1)(L-1)}{(N'-1)(L'-1)}.
\end{align*}
\]

and

\[
\begin{align*}
RHS_Y &= \frac{NL - M}{M(N'-1)(L'-1)} \\
&= RHS_X \cdot \frac{(N-1)(L-1)}{(N'-1)(L'-1)}.
\end{align*}
\]

Therefore, \(X\) has optimal AHC if and only if \(Y\) has optimal AHC. \(\square\)

**Remark:** The assumption on the indices \(i, j, s, t\) and \(t\) in Theorem \([14]\) guarantees that \(X_i(t)\) appears exactly once in \(Y\) through interleaving for any \(0 \leq i \leq L - 1\) and \(0 \leq t \leq N - 1\).

The most common interleaving is applied to an FHS set in the following construction.

**Construction C:** Let \(X = \{ X_0, \ldots, X_{L-1} \}\) be an \((N, M, L)\)-FHS set. Let \(k\) be a positive integer such that \(2 \leq k \leq L\) and \(k \mid L\). For \(0 \leq i \leq L/k - 1\), the FHS \(Y_i = \{ Y_i(t) \}_{t=0}^{kN-1}\) is defined as

\[
Y_i(kt_1 + t_0) = X_{ki+t_0}(t_1)
\]

where \(0 \leq t_0 \leq k - 1\) and \(0 \leq t_1 \leq N - 1\). The set \(Y \triangleq \{ Y_i \mid 0 \leq i \leq L/k - 1 \}\) is an \((kN, M, L/k)\)-FHS set.

**Corollary 15.** Let \(X\) be an FHS set with optimal AHC. Then, \(Y\) in Construction C is an FHS set with optimal AHC.
Several FHS sets with low MHC were constructed by interleaving techniques in [12]. Theorem 14 tells us that interleaving techniques may also be a good design tool for optimal AHC. We give a construction example by applying Construction C to $X_4$ in Construction A as follows.

Corollary 16. Let $p = Mf + 1$ be an odd prime for a positive integer $M$ and an odd integer $f$. Let $X_4 = \{X_i \mid 0 \leq i \leq M - 1\}$ be the $(p, M, M)$-FHS set given in Construction A. For $0 \leq i \leq M/2 - 1$, let $Y_i = \{Y_i(t)\}_{t=0}^{p-1}$ be the FHS defined as

$$Y_i(2t_1 + t_0) = X_{2t_1+t_0}(t_1)$$

where $0 \leq t_0 \leq 1$, $0 \leq t_1 \leq p - 1$. Then $Y_4 \triangleq \{Y_i \mid 0 \leq i \leq M/2 - 1\}$ is a $(2p, M, M/2)$-FHS set with optimal AHC.

By extending Construction B1 in [12], it is also possible to obtain an FHS set with optimal AHC and MHC.

Theorem 17. Let $N = p_1^{e_1} \cdots p_r^{e_r}$ where $r \geq 1$, $e_i \geq 1$ for all $1 \leq i \leq r$, and $p_1 < \cdots < p_r$ are odd primes. Define $\mathcal{X} \triangleq \{X_i \mid 0 \leq i \leq p_1 - 1\}$ as the $(N, N, p_1 - 1)$-FHS set over $\mathbb{Z}_N$, where

$$X_i(t) = (i + 1)^t N.$$  

for $0 \leq t \leq N - 1$. For a positive divisor $k$ of $p_1 - 1$, let $\mathcal{Y} \triangleq \{Y_i \mid 0 \leq i \leq (p_1 - 1)/k - 1\}$ be defined as

$$Y_i(kt_1 + t_0) = X_{kt_1+t_0}(t_1)$$

where $0 \leq t_0 \leq k - 1$ and $0 \leq t_1 \leq N - 1$. Then, $\mathcal{Y}$ is a $(kN, N, (p_1 - 1)/k)$-FHS set with optimal AHC and MHC.

Proof. Note that $\mathcal{X}$ is perfectly balanced, since $N_{X_i}(a) = 1$ for all $a \in \mathbb{Z}_N$ and all $0 \leq i \leq p_1 - 2$. This implies that $\mathcal{X}$ has optimal AHC by Corollary 7. Hence, $\mathcal{Y}$ also has optimal AHC by Corollary 15. The MHC of $\mathcal{Y}$ can be derived from the results of [27] and [12]. In [27], it was shown that

$$H_{X_i, X_j}(\tau) = \begin{cases} N, & \text{if } i = j \text{ and } \tau = 0 \\ 0, & \text{if } i = j \text{ and } \tau \neq 0 \\ 1, & \text{if } i \neq j. \end{cases}$$

By extending the Proof of Theorem 15 in [12], we obtain $H_{X_i, X_j}(\mathcal{Y}) = H_{X_i}(\mathcal{Y}) = k$. Therefore, $\mathcal{Y}$ is optimal with respect to the Peng-Fan bound.

VI. CONCLUSION

Some known FHS sets with optimal MHC were classified by their AHC properties. It was shown that any uniformly distributed FHS set has optimal AHC. Two FHS constructions with optimal AHC and near-optimal MHC were presented by using the theory of cyclotomy. The optimality of FHS sets obtained by interleaving techniques was analyzed. These results motivate us to find more FHS sets with optimal AHC and low MHC. Furthermore, it may also be a challenging problem to find a necessary and sufficient condition that an FHS set has optimal AHC.

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