Fluid properties of the nonlinear light systems and the Bose-Einstein condensates (BEC) have been noticed and scrutinised over the past two or three decades. These studies have been particularly well advanced for systems described by the Gross-Pitaevskii (nonlinear Schrödinger) equation for which the Madelung transformation (the amplitude-phase decomposition of the complex field) allows to bring this equation to the fluid mass and momentum conservation form. Among these studies, works devoted to turbulence in optical and BEC systems are most interesting and important.

However, there are also optical systems, which clearly exhibit turbulent behaviour, but their dynamical equations are not fully known or too complicated to be analysed directly mathematically or even numerically. Yet, as we will show in this Letter using the example of a magneto-optical trap (MOT) experiment, a lot can be said about such systems using the standard characterizations from the turbulence theory.

In this Letter, we will analyze the data obtained in an unstable MOT experiment described further below in the text. Unstable MOTs have been studied in various groups and with different approaches, ranging from atomic physics, non linear dynamics to plasma physics and astrophysics. The goal is to study the experimental images using methods from the field of turbulence. These images at hand visually differ from each other in being more or less turbulent, as well as approximately statistically homogeneous and/or isotropic. We stress that images that are visually non-turbulent may, in reality, correspond to small-scale turbulent fluctuations masked by presence of a strong mean large-scale background. Thus, a quantitative tool is needed to be able to distinguish between various turbulent, laminar and transitional states.

In turbulence one distinguishes the scale-invariant “mean field” behaviour, usually associated with the Kolmogorov-Obukhov-Corsin (KOC) theory, and intermittency that is seen in deviations from the KOC scaling in high-order structure functions (see below) which are sensitive to presence of coherent quasi-singular structures, such as e.g. shocks, sharp peaks, solitons etc. Presence of a large-scale mean motion is also a factor that alters the behaviour of small-scale turbulence in a substantial way. We will compare the MOT data with the scalings expected for smooth/laminar fields, KOC turbulence, shocks and features arising when projecting 2D surfaces in 3D space onto a 2D plane.

Note that our analysis in purely formal, because we do not use the underlying dynamical equations, and we are not exploring the underlying physics leading to the MOT dynamics under consideration. However, even tough similarities between the obtained scalings with the considered reference theories might be coincidental, we will see that the results of our analysis are strongly correlated with visual differences observed in the MOT images.

To collect the data analyzed in this Letter, we use a MOT setup able to trap and cool a large number \( N \) of \(^{87}\text{Rb} \) atoms (up to \( 1.5 \times 10^{11} \), see [20]). The resulting cloud of atoms is centimeter-sized, with a temperature around 200 \( \mu \text{K} \). For such large \( N \) values, the MOT is known to exhibit spatio-temporal instabilities when the trapping laser frequency is brought sufficiently close to the atomic transition frequency, namely when \( |\delta| \ll |\delta_{cr}| \). Here, the negative-valued laser detuning is \( \delta = \omega_L - \omega_\text{at} \) where \( \omega_L \) and \( \omega_\text{at} \) are the laser’s and atomic frequencies respectively, and the threshold detuning \( \delta_{cr} \) depends on MOT parameters (such as magnetic field gradient \( \nabla B \) and laser intensity) and on atom number \( N \).

Our analysis of MOT turbulence is based on a set of 50 fluorescence images recorded by a fast camera (one image every ms) to resolve fast dynamics of the shape of the MOT. The nine different sets of data presented in this Letter correspond to different values of the pair of MOT parameters \( (\delta, \nabla B) \): (1) \( (\delta = -\Gamma, \nabla B = 2.4 \text{ G/cm}) \); (2) \( (\delta = -\Gamma, \nabla B = 1.2 \text{ G/cm}) \); (3) \( (\delta = -4\Gamma, \nabla B = 1.2 \text{ G/cm}) \); (4) \( (\delta = -\Gamma, \nabla B = 4.8 \text{ G/cm}) \); (5) \( (\delta = -\Gamma, \nabla B = 12 \text{ G/cm}) \); (6) \( (\delta = -\Gamma, \nabla B = 9.6 \text{ G/cm}) \); (7) \( (\delta = -\Gamma, \nabla B = 7.2 \text{ G/cm}) \); (8) \( (\delta = -4\Gamma, \nabla B = 7.2 \text{ G/cm}) \); (9) \( (\delta = -\Gamma, \nabla B = 2.4 \text{ G/cm}) \). Indeed, we have shown that qualitatively different unstable MOT regimes can be observed when varying these parameters [9]. The 50 images are randomly picked from a larger set, to probe the full dynamics of the cloud. The fluorescence images are recorded at the MOT operation detuning \( \delta \), with the camera’s line of sight corresponding roughly to the strong axis of the magnetic field gradient. The images...
that we record thus correspond to 3D light intensity distributions integrated along this line of sight (see discussion below). Note that because the employed detuning values are rather small, the distribution of fluorescence light intensity may differ from the actual atomic density distribution, due to multiple scattering of light within the cloud.

The structure functions are by far the most frequently used objects in analysing turbulence [15]. The structure function of order $p$ for the two-point increment of a field $\rho(x)$ is defined as:

$$S_p(x, \ell) = \langle |\rho(x) - \rho(x + \ell)|^p \rangle.$$  \hspace{1cm} (1)

We will be analysing 2D black-and-white images, so $\rho$ will be the light intensity on the image on the 2D physical coordinate $x = (x, y)$. Correspondingly, $\ell = (\ell_x, \ell_y) = (\ell \cos(\theta), \ell \sin(\theta))$ is a 2D coordinate increment. The averaging is performed over the ensemble or time (assuming ergodicity, these two averages give the same result). However, in statistically homogeneous systems one can use the space averaging (e.g. over the the position $x$), and assuming statistical isotropy one can average over $\theta$. In the latter case, the structure function is independent of $x$ and $\theta$, i.e. $S_p(x, \ell) \equiv S_p(\ell)$.

In turbulence, the structure function exhibits a scaling behaviour, i.e. it behaves as a power-law $S_p(\ell) \propto \ell^\xi_p$ in a wide range of scales $\ell$. Often this range is referred to as the inertial range. The quantities $\xi_p$ are called the structure function exponents. They contain a significant information about the turbulence statistics, typical processes (e.g. turbulent cascades) and presence of coherent and/or singular structures causing intermittency of the turbulent signal (e.g. shocks or sharp spikes).

Let us summarise several types of the scalings of the structure functions which we will use as reference in our analysis later.

\textbf{Differentiable field} (often called smooth ramps): $\xi_p = p$. This result trivially follows the leading order Taylor expansion of the structure function in small $\ell$.

\textbf{Shocks}: $\xi_p = \text{const}$. This result relies on the fact that for sufficiently small $\ell$ the main contribution to $S_p(\ell)$ comes from the pairs of points $x$ and $x + \ell$ located at the different sides with respect to the jump.

\textbf{Burgulence} (random field governed by the Burgers equation). Here smooth ramps coexist with shocks. Hence, a bifractal behaviour is observed: ramp scaling is seen for $0 < p < 1$ and shocks for $1 < p < \infty$.

\textbf{Passive scalar advected by Kolmogorov turbulence}. In atmospheric turbulence the temperature field could be seen as approximately passively advected. The mean-field theory of Kolmogorov-Oboukhov-Corsin (KOC) predicts $\xi_p = p/3$ for passive scalar fields advected by the classical Kolmogorov turbulence. This prediction is made for the inertial range of scales $\ell_d < \ell < \ell_E$, where $\ell_d$ and $\ell_E$ are the dissipative and the energy-containing (integral) scales respectively. The scaling is derived under the assumption that the energy dissipation rate is the only quantity defining the statistical properties in this range [16] [17].

\textbf{Intermittent turbulence}. This is what is most typical for real turbulent systems: Kolmogorov type scaling at low values of $p$ is replaced by a more shallow slope for higher $p$’s due to presence of coherent quasi-singular structures/events [19] [21].

To create the structure function we randomly (with a uniform distribution) choose a pixel $(x, y)$ from within the box $[0.25N, 0.75N]^2$, then choose the second point by selecting a random $\ell$ pixels from $[0, N/2]$ and $\theta$ from $[0, 2\pi)$. With these $\ell$ and $\theta$, we read the value of $\rho$ corresponding to the pixel into which the second point falls, so that we can calculate calculate the structure function of Eq. (1). Additional averaging is done in time by repeating the process for the whole dataset which consists of 50 frames.

Note that choosing random $(x, y)$ and random $\theta$ (while keeping the $\ell$-bin constant when computing the structure function) amounts to the space and direction averaging which make sense only for systems which are approximately statistically uniform and isotropic in the sampling volume. In principle, the angle averaging could be dropped if one is interested in studying the anisotropy effects.

We have studied the effect of detector pixelation using an image with fixed Gaussian profile and established that the pixelation causes an error less than 3% for $\ell > 10$. Thus, in our analysis we take $\ell > 10$.

We have analysed images from the nine experimental datasets described above, ordered from visually most turbulent dataset (1) to most laminar dataset (9). The two extreme limits are presented in Fig. 3 where the upper and the lower rows correspond to experiments (1) and (9) respectively. On the left, we show structure functions $S_p(\ell)$ in log-log scale and a power law fits in the ranges of low, medium and high separations $\ell$. The right figures plot the exponents of the latter power law fits of $S_p(\ell)$. There is a common feature on all the plots of $S_p(\ell)$: they decrease at $\ell \gtrsim 200$ which is close to the radius of the cloud. But we can also see that the results for the most turbulent and the most laminar cases are drastically different in other respects. The most laminar case is characterised by a nearly linear behaviour of the structure function exponents $\xi(p)$ as a function of $p$ which is almost independent on the fitting range in $\ell$ (low, medium or high). This behaviour is typical for the smooth fields which, as we know, should exhibit $\xi(p) = p$. The slope of $\xi(p)$ for our most laminar case is not as high as 1(it is $\sim 0.8$), which indicates that the corresponding field is not fully laminar/smooth. Indeed, the set of MOT parameters for the dataset (9) ($\delta = -4\Gamma, V B = 2.4 \ G/cm$) corresponds to a stable MOT, with a density distribution quite stationary and smooth. In contrast, in the most turbulent experiment the curves $\xi(p)$ are concave up which is typical for intermittent turbulence. These curves are different for different fitting ranges: the slopes
FIG. 1. Top figures correspond to dataset (1) and bottom—to dataset (9). The left figures are the structure functions with different values of $p$ plotted in different colors. Each structure function is overlaid with three lines corresponding to the low-$\ell$, medium-$\ell$ and high-$\ell$ power-law fits. The slopes of these fits are presented as a function of $p$ in the right figure with blue corresponding to the low-$\ell$, green— to the middle-$\ell$ and red— to the high-$\ell$ fits. These are further overlaid with the slopes at low and high $p$.

are steepest for the low-$\ell$ fitting, which is indicative of presence of a dissipation at small $\ell$’s, making the fields smoother at these scales. The slopes at the low $p$ side are consistent with the KOC scaling ($\xi(p) = p/3$), whereas much smaller slopes are seen at the high $p$ side which indicates presence of quasi-singular structures such as shocks (which would result in zero slope) or “cliffs” as in the passive scalar turbulence described in [21].

To summarise and classify the data of all nine experiments, we present Fig. 2 which shows the results for the slopes $\frac{d}{dp}\xi(p)$ for three different fitting ranges. The plots are overlayed with respective single-shot images for visual comparison. We see that the smoothness of the insets matches closely the change in the slopes $\frac{d}{dp}\xi(p)$: with the more turbulent cases corresponding to smaller slopes. It is interesting, however, that the low-$\ell$, low-$p$ curve is the least sensitive to the variations in the different experiments, with the value of $\frac{d}{dp}\xi(p) \sim 0.6 - 0.7$ for all the images from most turbulent to most laminar. This points at presence of small-scale fluctuations which are not noticeable to eye on the visibly laminar flow images.

Finally, we will also mention another standard quantification of turbulence in terms of the spectra of the light intensity which are shown in Figure 3 for the most turbulent and the most laminar image. Here we see that the spectrum for the more turbulent case has a scaling range where the spectrum approximately follows a power law with exponent -3. There is no such scaling range in the most laminar case: the corresponding spectrum decays much faster than in the case of the turbulent case which corresponds to weakness of the small-scale fluctuations. Thus, the spectra provide another useful diagnostic tool for distinguishing between the cases with different levels of turbulence. However, the information provided by
of the intensity field provide an effective tool for classification of the states and the level of turbulence in self-oscillating MOT clouds. In particular, for most turbulent MOT states, they allow to observe features typical for intermittent turbulence of passive scalars, with KOC-type scaling for the lower-order structure functions and cliff-like coherent structures captured by the high-order structure functions. Most laminar states are exhibiting scalings corresponding to smooth large-scale flows with superimposed small-scale fluctuations. For the intensity field spectra, we have observed a power-law with exponent $-3$ in the most turbulent case and a non-universal steeply decaying spectrum in the least turbulent case. These observations could be used to discriminate between different models aiming at explaining the complex physics of unstable MOTs.

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