The median of an exponential family and the normal law

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Abstract

Let $P$ be a probability on the real line generating a natural exponential family $(P_t)_{t \in \mathbb{R}}$. We show that the property that $t$ is a median of $P_t$ for all $t$ characterizes $P$ as the standard Gaussian law $N(0, 1)$.

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1 Introduction

Let $P$ be a probability on the real line and assume that

$$L(t) = \int_{-\infty}^{+\infty} e^{tx} P(dx) < \infty \quad \text{for} \quad t \in \mathbb{R}. \quad (1)$$

Such a probability generates the natural exponential family

$$\mathcal{F}_P = \{ P_t(dx) = \frac{e^{tx}}{L(t)} P(dx), \; t \in \mathbb{R} \}.$$ 

Then it might happen that the natural parameter $t$ of $\mathcal{F}_P$ is always a median of $P_t$, in the sense of

$$P_t((-\infty, t)) \leq \frac{1}{2} \leq P_t((-\infty, t]) \quad \text{for} \quad t \in \mathbb{R}. \quad (2)$$

In the sequel we denote by $\mathcal{P}$ the set of probabilities $P$ such that (1) and (2) are fulfilled. A noteworthy example of an element of $\mathcal{P}$ is the standard normal
distribution \( N(0, 1) \), for which \( L(t) = e^{t^2/2} \) and \( P_t = N(t, 1) \). It will turn out that it is the only one. The following preliminary lemmas simplify the study of \( P \).

**Lemma 1.** If \( P \in \mathcal{P} \), then \( P \) is absolutely continuous with respect to Lebesgue measure. As a consequence, we have equality throughout in (2).

**Lemma 2.** If \( P \in \mathcal{P} \), then its distribution function is strictly increasing.

If \( P \in \mathcal{P} \), then Lemma 1 allows us to write

\[
P(dx) = g(x)\varphi(x)dx,
\]

(3)

where \( g \) is some measurable non-negative function and \( \varphi(x) = e^{-x^2/2}/\sqrt{2\pi} \) denotes the standard normal density, and we will show that then \( g(x) = 1 \) a.e. to get:

**Theorem 1.** \( \mathcal{P} = \{ N(0, 1) \} \).

The proofs of the above results are contained in Section 2, followed by a conjecture and a further theorem.

## 2 Proofs

**Proof of Lemma 1.** The next paragraph shows that the distribution function of \( P \) is locally Lipschitz, and this implies the claimed absolute continuity, even with a locally bounded density, compare for example Royden and Fitzpatrick (2010, pp. 120–124).

For \( t \in \mathbb{R} \), multiplying in assumption (2) by \( L(t) \) yields

\[
h(t) := \int_{(-\infty, t]} e^{tx}P(dx) \geq \frac{1}{2} L(t) \geq \int_{(-\infty, t]} e^{tx}P(dx) = h(t^-).
\]

(4)

Hence, if \( A > 0 \) is given, then for \( s, t \) with \( -A \leq s < t \leq A \), we get

\[
P((s, t)) = \int_{(s,t)} e^{-tx}e^{tx}P(dx) \leq e^{A^2} \int_{(s,t)} e^{tx}P(dx)
\]

\[
= e^{A^2} \left( h(t^-) - h(s) + \int_{(-\infty, s]} (e^{sx} - e^{tx})P(dx) \right)
\]

\[
\leq e^{A^2} \left( \frac{1}{2}(L(t) - L(s)) + (t - s) \int_{\mathbb{R}} |x|e^{A|x|}P(dx) \right)
\]

\[
\leq c_A \cdot (t - s)
\]

for some finite constant \( c_A \). We have been using (4) and \( |e^u - e^v| \leq |u - v|e^w \) for \( |u|, |v| \leq w \) at the penultimate step. Using assumption (1), we rely at the ultimate step on local Lipschitzness of \( L \), due to its analyticity, and on finiteness of \( \int_{\mathbb{R}} |x|e^{A|x|}P(dx) \).

\( \square \)


**Proof of Lemma 2.** Assume to the contrary that there exist \( a, b \in \mathbb{R} \) with \( a < b \) and \( P((a,b)) = 0 \). Then, for \( t \in (a,b) \), Lemma 1 and (2) yield
\[
\int_{-\infty}^{a} e^{tx} P(dx) = \int_{-\infty}^{t} e^{tx} P(dx) = \int_{t}^{+\infty} e^{tx} P(dx) = \int_{b}^{+\infty} e^{tx} P(dx).
\]

Thus the two measures \( 1_{(-\infty,a]}(x)P(dx) \) and \( 1_{[b,+)P(dx)} \) have finite and identical Laplace transforms on some non-empty interval. Hence the two measures coincide, and hence \( P \) must be the zero measure, which is absurd. \( \square \)

**Proof of Theorem 1.** With the representation (3) for \( P \in \mathcal{P} \), assumption (2) is rewritten as
\[
\int_{-\infty}^{t} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} g(x) \, dx = 1 \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} g(x) \, dx. \tag{5}
\]

We multiply both sides by \( e^{-t^2/2} \):
\[
\int_{-\infty}^{t} e^{-\frac{(t-x)^2}{2}} \frac{1}{\sqrt{2\pi}} g(x) \, dx = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-\frac{(t-x)^2}{2}} \frac{1}{\sqrt{2\pi}} g(x) \, dx. \tag{6}
\]

In other terms the unknown function \( g \) satisfies
\[
\int_{-\infty}^{+\infty} \text{sign} (t-x) \varphi(t-x) g(x) \, dx = 0 \tag{7}
\]
for all \( t \in \mathbb{R} \). A formal derivation of (7) in \( t \), using the product rule under the integral, and with one derivative being twice a delta function, leads to the equation
\[
g(t) = \int_{-\infty}^{+\infty} q(t-x) g(x) \, dx \tag{8}
\]
a.e. in \( t \), where \( q(y) := \frac{1}{2} |y| e^{-\frac{y^2}{2}} \) is a probability density, but instead of justifying this formal differentiation, it seems easier to start by computing the derivative of
\[
h(t) := \int_{-\infty}^{t} e^{tx} P(dx).
\]

By Lemma 2 the distribution function \( F \) of \( P \) has a continuous inverse \( F^{-1} \). Using the quantile transform we have
\[
h(t) = \int_{0}^{1} 1_{(F^{-1}(u) \leq t)}(u) e^{tF^{-1}(u)} \, du = \int_{0}^{F(t)} e^{tF^{-1}(u)} \, du = H(F(t), t)
\]
with \( H(s,t) := \int_{0}^{s} e^{tF^{-1}(u)} \, du \) for \( s \in (0,1) \) and \( t \in \mathbb{R} \). Now \( H \) has continuous partial derivatives \( H_1(s,t) = e^{tF^{-1}(s)} \) and \( H_2(s,t) = \int_{0}^{s} F^{-1}(u)e^{tF^{-1}(u)} \, du \), due to the
continuity of $F^{-1}$, and hence $H$ is differentiable. Let $f$ be a Lebesgue density of $P$. Then, at every $t$ where $F'(t) = f(t)$, and hence at Lebesgue-a.e. $t$, the chain rule yields

$$h'(t) = H_1(F(t), t)f(t) + H_2(F(t), t) = e^{t^2}f(t) + \int_0^{F(t)} F^{-1}(u)e^{tF^{-1}(u)} \, du$$

$$= e^{t^2}f(t) + \int_{-\infty}^{t} xe^{tx}f(x) \, dx.$$

Thus differentiating the identity (5) and observing that $f(x) = g(x)\varphi(x)$ we obtain the following a.e.-identity

$$\frac{1}{\sqrt{2\pi}}e^{t^2/2}g(t) + \int_{-\infty}^{t} xe^{tx-x^2/2} \frac{1}{\sqrt{2\pi}}g(x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} xe^{tx-x^2/2} \frac{1}{\sqrt{2\pi}}g(x) \, dx,$$

and multiplying the latter by $\sqrt{2\pi}e^{-t^2/2}$ gives

$$g(t) = \frac{1}{2} \left( \int_{t}^{\infty} xe^{-(t-x)^2/2}g(x) \, dx - \int_{-\infty}^{t} xe^{-(t-x)^2/2}g(x) \, dx \right).$$

Adding to the right hand side above the quantity

$$0 = \frac{t}{2} \left( \int_{-\infty}^{t} e^{-(t-x)^2/2}g(x) \, dx - \int_{t}^{\infty} e^{-(t-x)^2/2}g(x) \, dx \right)$$

(recall (6)) yields the desired (8).

Next, with the (positive) Radon measures $\mu(dx) := g(x)dx$ and $\sigma(dx) := q(x)dx$, equation (8) can be rewritten as the so-called Choquet-Deny equation $\mu = \mu * \sigma$. Observe that $t \mapsto \int_{-\infty}^{\infty} e^{tx}\sigma(dx)$ is even and strictly convex, and is therefore equal to 1 only at $t = 0$. We can now use the results in section 6 of Deny (1960), where “$n > 1$” is evidently a misprint for “$n \geq 1$”, to conclude that $\mu$ has to be a positive scalar multiple of the Lebesgue measure. Since $g$ is a probability density with respect to a probability measure, we have $g = 1$ a.e., and the theorem is proved.

Finally, it is worthwhile to mention a natural conjecture about exponential families which seems harder to establish:

**Conjecture.** Suppose that the probability $P$ satisfies (1), and denote $m(t) := \int_{\mathbb{R}} xP_t(dx)$. If for all $t$ real $m(t)$ is a median of $P_t$, then $P = N(m, \sigma^2)$ for some $m$ and $\sigma$.

This conjecture, which is probably more meaningful from a methodological point of view than the result established in the paper, does not translate in a neat harmonic analysis statement as (7) and (8) and as such it seems harder to establish. The next simple result offers some support to the conjecture. A probability $Q$ on $\mathbb{R}^n$ is said to be symmetric if there exists some $m \in \mathbb{R}^n$ such that $X - m \sim m - X$ when $X \sim Q$. 

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Theorem 2. Let $P$ be a probability on $\mathbb{R}^n$ such that

$$L(t) = \int_{\mathbb{R}^n} e^{\langle t,x \rangle} P(dx)$$

is finite for all $t \in \mathbb{R}^n$. Assume that for all $t \in \mathbb{R}^n$ the probability $P_t(dx) = e^{\langle t,x \rangle} P(dx)/L(t)$ is symmetric. Then $P$ is normal.

Proof. Clearly $m(t) = \int_{\mathbb{R}^n} xP_t(dx) = L'(t)/L(t)$ exists and, since $P_t$ is symmetric, $X_t - m(t) \sim m(t) - X_t$ when $X_t \sim P_t$. Therefore its Laplace transform

$$s \mapsto \mathbb{E}(e^{\langle s,X_t-m(t) \rangle}) = e^{-\langle s,m(t) \rangle \frac{L(t+s)}{L(t)}}$$

does not change when we replace $s$ by $-s$. Considering the logarithm and taking the derivative in $s$ we get $2m(t) = m(t+s) + m(t-s)$. Taking again the derivative in $s$ we get $m'(t+s) = m'(t-s)$ for all $t, s \in \mathbb{R}^n$, which means that $m'$ is constant, hence log $L$ is polynomial of degree at most 2, and hence $P$ is normal.

3 References

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