Constraining non-dissipative transport coefficients in global equilibrium

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The fluid in global equilibrium must fulfill some constraints. These constraints can be derived from quantum statistical theory or kinetic theory. In this paper we will show that how these constraints can be applied to determine the non-dissipative transport coefficients for chiral systems along with the energy-momentum conservation, chiral anomaly for charge current and trace anomaly in energy-momentum tensor.

I. INTRODUCTION

The charge currents associated with chiral anomaly exhibit peculiar properties which normal currents do not possess, such as the famous chiral magnetic effect [1,2] and chiral vortical effect [3,4]. These currents are all non-dissipative and could exist even in global equilibrium. These anomalous currents can be derived from various approaches such as gauge/gravity duality [6–11], principle of entropy increase [12–15], Kubo formula from quantum field theory [16–21] and quantum kinetic equation [22–29]. In this paper, we would provide another novel method to determine or constrain these non-dissipative transport coefficients in anomalous chiral fluids. Since these currents are non-dissipative, they could exist even in global equilibrium. However in order to arrive at global equilibrium, the system must satisfy some specific constraints especially when the electromagnetic field is present. Thanks to these constraints along with the energy-momentum conservation law, trace anomaly for energy-momentum tensor and chiral anomaly for charge current, we can determine or constrain the non-dissipative transport coefficients up to the second order. In Sec. II, we will provide another novel method to determine or constraine the non-dissipative transport coefficients in anomalous chiral fluids. Since these currents are non-dissipative, they could exist even in global equilibrium. However in order to arrive at global equilibrium, the system must satisfy some specific constraints especially when the electromagnetic field is present. Thanks to these constraints along with the energy-momentum conservation law, trace anomaly for energy-momentum tensor and chiral anomaly for charge current, we can determine or constrain the non-dissipative transport coefficients up to the second order. In Sec. II, we first review how the constraint in global equilibrium can be derived from either quantum statistical theory or kinetic theory when electromagnetic field is imposed. In Sec. III, we will show how to determine the energy-momentum tensor and charge current from the conservation laws and chiral anomaly. We summarize our results in Sec. IV.

II. GLOBAL EQUILIBRIUM CONSTRAINTS

When a fluid is in global equilibrium without external fields [30,31], the fluid four-velocity $u^\mu$ with $u^2 = 1$ should be expansion-free and shear-free, and that the thermal potential $\bar{\mu} = \mu/T$ which is defined as the chemical potential $\mu$ divided by the temperature $T$ should be constant, i.e.,

$$\Delta^{\mu\rho}\Delta^{\nu\sigma}(\partial_\rho u_\sigma + \partial_\sigma u_\rho) = 0, \quad \partial_\mu \bar{\mu} = 0$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ denotes the spatial projection tensor. In conjunction with the ideal hydrodynamical equation, it is easy to verify that these above conditions are equivalent to the following equations

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad \partial_\mu \bar{\mu} = 0$$

where $\beta^\mu = u^\mu/T$ can be referred to as thermal velocity similar to the thermal potential for chemical potential. These are just the constraint conditions which should be obeyed by the fluid in global equilibrium without external fields. When an external electromagnetic field tensor $F_{\mu\nu}$ is present, the constraint conditions are generalized to

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad \partial_\mu \bar{\mu} = -F_{\mu\nu}\beta^\nu$$

where the electromagnetic field should be static so as to be able to arrive at the global equilibrium. The second equation above indicates that the external electromagnetic field is balanced by the gradient of the thermal potential. In this paper, we will assume further that the electromagnetic field is also homogeneous which means that $F_{\mu\nu}$ must be constant, i.e., $\partial_\lambda F_{\mu\nu} = 0$.

Now we first review how these constraint conditions can be derived from more underlying theories. The derivation from quantum statistical theory is based on global thermodynamic equilibrium density operator which had been given
The general covariant form of the local thermodynamic equilibrium density operator is given by

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\sum d\Sigma \left(\hat{T}^{\mu\nu} \beta_\nu - \bar{\mu} \bar{\beta}_\mu\right)\right].$$  \hspace{1cm} (4)

where \(\hat{T}^{\mu\nu}\) is the symmetric energy-momentum tensor operator, \(\hat{j}^\mu\) the conserved current operator, \(Z\) is the normalization factor such that \(\text{tr} \hat{\rho} = 1\), and \(\Sigma\) is a spacelike 3-D hypersurface. In global equilibrium, the integrand should be time independent

$$\int_{\Sigma(\tau)} d\Sigma \left(\hat{T}^{\mu\nu} \beta_\nu - \bar{\mu} \bar{\beta}_\mu\right) - \int_{\Sigma(\tau+\Delta\tau)} d\Sigma \left(\hat{T}^{\mu\nu} \beta_\nu - \bar{\mu} \bar{\beta}_\mu\right) = 0$$  \hspace{1cm} (5)

and will not depend on the hypersurface \(\Sigma\) any more. With the assumption that the field \(\beta^\mu\) and \(\bar{\mu}\) vanish at the timelike boundary which connects two spacelike hypersurface \(\Sigma(\tau)\) and \(\Sigma(\tau+\Delta\tau)\) and according to Gauss’s theorem, the above equation implies that the integrand is divergenceless:

$$\partial_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \bar{\mu} \bar{\beta}_\mu\right) = \left(\partial_\mu \hat{T}^{\mu\nu}\right) \beta_\nu + \hat{T}^{\mu\nu} \partial_\mu \beta_\nu - (\partial_\mu \bar{\mu}) \hat{j}^\mu - \bar{\mu} \left(\partial_\mu \hat{j}^\mu\right) = 0$$  \hspace{1cm} (6)

Using the conservation equations for energy-momentum tensor \(\partial_\mu \hat{T}^{\mu\nu} = F^{\mu\nu} \hat{j}_\mu\) and charge current \(\partial_\mu \hat{j}^\mu = 0\) and the fact that the energy-momentum tensor is symmetric, we can obtain

$$\frac{1}{2} \hat{T}^{\mu\nu} \left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu\right) - (\partial_\mu \bar{\mu} + F_{\nu\mu} \beta^\nu) \hat{j}^\mu = 0$$  \hspace{1cm} (7)

It is obvious that this equation always holds if the constraint condition \(\hat{\Omega}_\mu\) is satisfied.

The global equilibrium condition can also be derived from kinetic theory [32, 33]. In equilibrium, the collision terms in the Boltzmann equation will vanish due to detailed balancing principle and the kinetic equation will reduce to Vlasov equation:

$$\delta(p^2 - m^2) p^\mu \left(\frac{\partial}{\partial x^\mu} - F_{\mu\nu} \frac{\partial}{\partial p_\nu}\right) f(x,p) = 0.$$  \hspace{1cm} (8)

where \(p^\mu\) denotes four-momentum of the particle with mass \(m\) and we have written Vlasov equation in Lorentz covariant form. In equilibrium, the distribution function \(f(x,p)\) should depend on \(x,p\) through the argument \(\beta \cdot p - \bar{\mu}\)

$$f(x,p) = g(y), \quad y = \beta \cdot p - \bar{\mu}.  \hspace{1cm} (9)$$

Then the kinetic equation \(\hat{\Omega}_\mu\) can be expressed as

$$\delta(p^2 - m^2) \left[\frac{1}{2} p^\mu p^\nu \left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu\right) - p^\mu \partial_\mu \bar{\mu} - p^\nu F_{\nu\mu} \beta^\mu\right] \frac{dg}{dy} = 0,$$ \hspace{1cm} (10)

Obviously, the kinetic equation always hold if the equilibrium conditions \(\hat{\Omega}_\mu\) are satisfied.

Now let us consider the constraint conditions listed above in more details. We can solve the first condition directly \(\hat{\Omega}_\mu\) and the general solution is given by

$$\beta_\mu = b_\mu - \Omega_{\mu\nu} x^\nu$$ \hspace{1cm} (11)

where \(b_\mu\) is a constant vector and \(\Omega_{\mu\nu}\) is a constant antisymmetric tensor. Actually \(\Omega_{\mu\nu}\) is just the thermal vorticity tensor of the fluid (there is a minus sign difference from usual definition)

$$\Omega_{\mu\nu} = \frac{1}{2} \left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right).$$ \hspace{1cm} (12)

The second condition in \(\hat{\Omega}_\mu\) has a solution only if the integrability condition is fulfilled. It can be obtained by differentiating both sides of second equation in Eq. \(\Omega_\mu\) with \(\partial_\nu\) and using the commutating property of ordinary partial derivatives

$$\partial_\nu \partial_\mu \bar{\mu} = \partial_\mu \partial_\nu \bar{\mu} = -F_{\mu\lambda} \partial_\nu \beta^\lambda = -F_{\nu\lambda} \partial_\mu \beta^\lambda,  \hspace{1cm} (13)$$

Together with Eq. \(\hat{\Omega}_\mu\), the above equation can written as

$$F_\lambda \mu \Omega^{\nu\lambda} - F_\lambda \nu \Omega^{\mu\lambda} = 0,$$ \hspace{1cm} (14)
The general solution under this integrability condition is given by
\[ \bar{\mu} = -\frac{1}{2} F^{\mu\lambda} x^\lambda \Omega_{\mu\nu} x^\nu + c \] (15)

We can decompose the antisymmetric tensors \( F_{\mu\nu} \) and \( \Omega_{\mu\nu} \) with the fluid velocity \( u_\mu \) as
\[ F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\rho} u^\rho B^\sigma, \] (16)
\[ \Omega_{\mu\nu} = \frac{1}{T} \left( \epsilon_{\mu\nu\rho} u_\rho - \epsilon_{\nu\mu\rho} u_\rho + \epsilon_{\mu\rho\sigma} u^\rho \omega^\sigma \right), \] (17)

where the electric field \( E^\mu \), magnetic field \( B^\mu \), acceleration vector \( \varepsilon^\mu \) and vorticity vector \( \omega^\mu \) are given, respectively,
\[ E^\mu = F^\mu_{\nu} u^\nu, \quad B^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta} u_\alpha F_{\beta\gamma}, \] (18)
\[ \varepsilon^\mu = T \Omega^\mu_{\nu} u^\nu, \quad \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta. \] (19)

With this decomposition, it is easy to verify that the integrability condition (14) is equivalent to
\[ E^\mu \omega^\nu - E^\nu \omega^\mu = -B^\mu \varepsilon^\nu + B^\nu \varepsilon^\mu, \quad E^\mu \varepsilon^\nu - E^\nu \varepsilon^\mu = B^\mu \omega^\nu - B^\nu \omega^\mu. \] (20)

We will show that these relations play an important role to determine the possible forms of the non-dissipative terms in energy-momentum tensor and charge current in global equilibrium.

III. NON-DISSIPATIVE TRANSPORT COEFFICIENTS

In this section, we will apply the conservation laws and trace anomaly to constrain the possible anomalous transport coefficients in a chiral system in which only right-hand or left-hand Weyl fermions are involved. These conservation laws and trace anomaly are given by
\[ \partial_\mu T^{\mu\nu} = F^{\nu\mu} j_\mu, \quad \partial_\mu j^\mu = CE \cdot B, \quad g_{\mu\nu} T^{\mu\nu} = \tilde{C}(E^2 - B^2) \] (21)

We will expand the energy-momentum tensor and charge current in powers of \( F^{\mu\nu} \) and \( \Omega^{\mu\nu} \) or equivalently in terms of \( B^\mu, E^\mu, \omega^\mu \) and \( \varepsilon^\mu \). Since \( F^{\mu\nu} \) and \( \Omega^{\mu\nu} \) are both constant, it is unnecessary to consider \( \partial_\mu T \), \( \partial_\mu u_\nu \) and \( \partial_\mu \bar{\mu} \) because all these derivatives can be expressed as the linear combination of \( E^\mu, \omega^\mu \) and \( \varepsilon^\mu \) by using the constraint condition (18), e.g.,
\[ \partial_\mu T = -T \varepsilon_\mu, \quad \partial_\mu u_\nu = -u_\mu \varepsilon_\nu + \epsilon_{\mu\nu\alpha\beta} u^\alpha \omega^\beta, \quad \partial_\mu \bar{\mu} = -\frac{E_\mu}{T} \] (22)

We take \( u^\mu, T \) and \( \bar{\mu} \) to be of the zeroth order, \( F^{\mu\nu} \) and \( \Omega^{\mu\nu} \) to be of the first order and so on.

Let us start with the zeroth-order \( T^{\mu\nu} \) and \( j^\mu \). They are just the well-known ideal hydrodynamical results:
\[ T^{(0)\mu\nu} = \rho u^\mu u^\nu - P \Delta^{\mu\nu}, \quad j^{(0)\mu} = n u^\mu \] (23)

where \( \rho \) is the energy density, \( P \) the pressure and \( n \) the charge density. It is easy to verify that
\[ \partial_\mu T^{(0)\mu\nu} = (\rho + P) u^\mu \partial_\mu u^\nu - \partial^\nu P = -\rho \varepsilon^\nu + T \partial_\nu \frac{P}{T} \] (24)

Using the thermal identity
\[ d \frac{P}{T} = n d \bar{\mu} - \rho d \frac{1}{T} \] (25)

and the last equation in (22), we obtain
\[ \partial_\mu T^{(0)\mu\nu} = E^\nu n = F^{\nu\mu} j^{(0)\mu}_{\mu} \] (26)

which indicates that the energy-momentum conservation law holds automatically. It is trivial to show that at zeroth-order charge current is also conserved automatically
\[ \partial_\mu j^{(0)\mu} = 0 \] (27)
There is no chiral anomaly at zeroth order as it should be. For the massless fermions, the conformal symmetry holds at the zeroth order and the trace of energy-momentum tensor must vanish which results in the well-known relation
\[ \rho = 3P. \] (28)

When we go beyond the zeroth-order, we need first pin down which frame we choose for the fluid velocity \( u^\mu \). In our work, we will use the \( \beta \) frame introduced in [36]. In this frame, the non-dissipative coefficients in global equilibrium would take more elegant form. We will assume the interactions which controls the chiral system keep charge, parity and time reversal invariance. Then at first order, the general expressions for the energy-momentum tensor and charge current take the following form
\[
\begin{align*}
T^{(1)}_{\mu\nu} &= \lambda^\nu (u^\mu \omega^\nu + u^\nu \omega^\mu) + \lambda^B (u^\mu B^\nu + u^\nu B^\mu), \\
j^{(1)}_\mu &= \xi \omega^\mu + \xi^B B^\mu
\end{align*}
\] (29, 30)

With this expression, the divergence of the current reads
\[
\partial_\mu j^{(1)}_\mu = \frac{\partial \xi}{\partial T} \partial_\mu T \omega^\mu + \xi \partial_\mu T \partial_\mu \omega^\mu + \frac{\partial \xi B}{\partial T} \partial_\mu T B^\mu + \frac{\partial \xi^B}{\partial T} \partial_\mu \bar{B}^\mu + \xi B \partial_\mu B^\mu
\] (31)

Using the relations (22) and the derived relations below
\[
\begin{align*}
\partial_\mu \omega_\nu &= \epsilon^\alpha \omega g^\mu_\nu - 2\epsilon^\mu \omega_\nu, \\
\partial_\mu B_\nu &= -E_{\mu \nu} + \epsilon \cdot B u^\mu u_\nu + \omega \cdot E \Delta^{\mu \nu} - (u^\mu \epsilon_{\nu \lambda \rho \sigma} + u_\nu \epsilon_{\mu \lambda \rho \sigma}) \mu^\lambda \epsilon^\rho E^\sigma
\end{align*}
\] (32, 33)

the equation (31) can be written as
\[
\partial_\mu j^{(1)}_\mu = \left(2\xi - T \frac{\partial \xi}{\partial T}\right) \epsilon \cdot \omega + \left(2\xi B - \frac{1}{T} \frac{\partial \xi B}{\partial T}\right) E \cdot \omega + \left(\xi B - T \frac{\partial \xi B}{\partial T}\right) \epsilon \cdot B - \frac{1}{T} \frac{\partial \xi B}{\partial T} E \cdot B
\] (34)

The fact that this result should equal to the anomalous term \( CE \cdot B \) from the second equation in (24) lead to the following equations:
\[
\begin{align*}
2\xi - T \frac{\partial \xi}{\partial T} &= 0, \quad 2\xi B - \frac{1}{T} \frac{\partial \xi B}{\partial T} = 0, \quad \xi B - T \frac{\partial \xi B}{\partial T} = 0, \quad -\frac{1}{T} \frac{\partial \xi B}{\partial T} = C.
\end{align*}
\] (35)

The general solution for this set of equations are easy to obtain
\[
\begin{align*}
\xi B &= -CT \bar{u} + bT = -C \mu + bT, \\
\xi &= -CT^2 \bar{\mu} + 2bT^2 \bar{\mu} + aT^2 = -C \mu^2 + 2bT \mu + aT^2
\end{align*}
\] (36, 37)

where \( a \) and \( b \) are both integral constants. It should be noted that the temperature dependence derived from the differential equations are consistent with the direct dimension analysis. Actually it is more convenient to determine the temperature power from dimension analysis. These results had been derived from the anomalous hydrodynamics by using the principle of entropy increase[12–15]. However it seems as if our method given here involve much less calculations. Similarly, the divergence of the energy-momentum tensor can be expressed as
\[
\partial_\mu T^{(1)}_{\mu\nu} = \frac{\partial \lambda}{\partial T} \partial_\mu T (u_\nu \omega^\mu + u^\nu \omega^\mu) + \frac{\partial \lambda B}{\partial T} \partial_\mu T (u_\nu B^\mu + u^\nu B^\mu) + \lambda \partial_\mu (u_\nu \omega^\mu + u^\nu \omega^\mu) + \lambda B \partial_\mu (u_\nu B^\mu + u^\nu B^\mu) + \lambda \partial_\mu (u_\nu \omega^\mu + u^\nu \omega^\mu) + \lambda B \partial_\mu (u_\nu B^\mu + u^\nu B^\mu) + \lambda \partial_\mu (u_\nu \omega^\mu + u^\nu \omega^\mu) + \lambda B \partial_\mu (u_\nu B^\mu + u^\nu B^\mu)
\]
\[
\frac{3\lambda - T \frac{\partial \lambda}{\partial T} \epsilon \cdot \omega + (2\lambda B - \frac{1}{T} \frac{\partial \lambda B}{\partial T}) E \cdot \omega + (2\lambda B - \frac{1}{T} \frac{\partial \lambda B}{\partial T}) \epsilon \cdot B - \frac{1}{T} \frac{\partial \lambda B}{\partial T} E \cdot B}{u^\nu}
\]
\[-2\lambda^B \epsilon^{\nu \alpha \beta \gamma} u_\alpha \omega_\beta B_\gamma
\] (38)

where we have used the second identity in Eq.[20]. The righthand of the energy-momentum conservation at first order in Eq.(21) is given by
\[
F^{\nu \mu} j^{(1)}_\mu = -\xi (E \cdot \omega) u_\nu - \xi B (E \cdot B) u_\nu - \xi \epsilon^{\nu \alpha \beta \gamma} u_\alpha \omega_\beta B_\gamma
\] (39)
Then the conservation law $\partial_\mu T^{(1)\mu\nu} = F^{\mu\nu} j^{(1)}_\mu$ requires
\[ 3\lambda - T \frac{\partial \lambda}{\partial T} = 0, \quad 2\lambda^B - \frac{1}{T} \frac{\partial \lambda^B}{\partial \mu} = -\xi, \quad 2\lambda^B - T \frac{\partial \lambda^B}{\partial T} = 0, \quad 1 \frac{\partial \lambda^B}{\partial \mu} = \xi^B, \quad 2\lambda^B = \xi \]  
(40)

From the last equation, we note that the coefficient $\lambda^B$ has been totally determined by the coefficient $\xi$ in the charge current. It is trivial to verify that both second and third last equations hold automatically with the result of $\xi$ in Eq. (37). Substituting the result of $\lambda^B$ into the first and second equations, we can obtain the general expression for $\lambda$. We list the solution for $\lambda^B$ and $\lambda$ in the following:
\[ \lambda^B = \frac{1}{2} \xi = \frac{1}{2} (-C_\mu^2 + 2b_\mu + a) T^2, \]  
(41)
\[ \lambda = \frac{2}{3} (-C_\mu^3 + 3b_\mu^2 + a_\mu + c) T^3 \]  
(42)

where $c$ is another integral constant. Similarly, the temperature dependence can also be obtained from direct dimension analysis. It is obvious that energy-momentum tensor at first order is traceless automatically.

Now let us move on to consider the second-order case. The charge current and energy-momentum tensor at second order take the general form
\[ j^{(2)\mu} = (\xi^{\mu\nu} \epsilon^{2} + \xi^{\omega\nu} \epsilon^{2} + \xi^{E\epsilon} \cdot E + \xi^{\omega B} \omega \cdot B + \xi^{E E} E^2 + \xi^{B B} B^2) u_\mu \]  
\[ + \xi^{\omega\nu} \epsilon^{\mu\nu\rho\sigma} u_\mu \sigma \omega_\rho + \xi^{E E} \epsilon^{\mu\nu\rho\sigma} u_\mu \sigma E_\rho \omega_\sigma + \xi^{E B} \epsilon^{\mu\nu\rho\sigma} u_\mu \sigma B_\rho \omega_\sigma, \]  
(43)
\[ T_s^{(2)\mu\nu} = (\lambda^{\mu\nu} \epsilon^{2} + \lambda^{\omega\nu} \epsilon^{2} + \lambda^{E\epsilon} \cdot E + \lambda^{\omega B} \omega \cdot B + \lambda^{E E} E^2 + \lambda^{B B} B^2) u_\mu u_\nu \]  
\[ + (\lambda^{\mu\nu} \epsilon^{\mu\nu\rho\sigma} u_\mu \sigma \omega_\rho + \lambda^{E E} \epsilon^{\mu\nu\rho\sigma} u_\mu \sigma E_\rho \omega_\sigma + \lambda^{E B} \epsilon^{\mu\nu\rho\sigma} u_\mu \sigma B_\rho \omega_\sigma) \]  
(44)

In order to calculate the divergence of these quantities, we need other useful relations:
\[ \partial_\mu \epsilon_\nu = \omega_\nu \omega_\mu - \epsilon_\mu \epsilon_\nu - \epsilon^{2} u_\mu u_\nu - \omega^2 \Delta_\mu \nu + (u_\mu \epsilon_\nu \omega_\sigma + u_\nu \epsilon_\mu \omega_\sigma) \omega^\epsilon \omega^\sigma, \]  
(45)
\[ \partial_\mu E_\nu = B_\mu \omega_\nu + \epsilon_\nu \cdot E u_\mu \sigma - \omega \cdot B \Delta_\mu \nu + (u_\mu \epsilon_\nu \omega_\sigma + u_\nu \epsilon_\mu \omega_\sigma) \omega^\epsilon \omega^\sigma, \]  
(46)
\[ 0 = \epsilon^{\alpha\beta\gamma\epsilon} \omega_\alpha \omega_\beta \omega_\gamma \omega_\epsilon = \epsilon^{\mu\nu\gamma} \omega_\alpha \omega_\beta \omega_\gamma E_\gamma = \epsilon^{\mu\nu\beta\gamma} \omega_\alpha \omega_\beta \omega_\gamma B_\gamma = \epsilon^{\mu\nu\beta\gamma} B_\alpha \omega_\beta \omega_\gamma \omega_\epsilon. \]  
(47)

All these relations can be derived from the first-order relations [22]. It is easy to verify that the conservation law for the charge current $\partial_\mu j^{(2)\mu}$ is satisfied automatically. Although we can not constrain any coefficients appearing in the second-order current $j^{(2)\mu}$, we still can relate the coefficients in second-order energy-momentum tensor $T_s^{(2)\mu\nu}$ to the ones in $j^{(2)\mu}$ through the energy-momentum conservation. Following the same step as we did at first order, the divergence of the energy-momentum tensor reads
\[ \partial_\mu T^{(2)\mu\nu} = X_1 \epsilon^{\mu\nu} + X_2 \omega_\mu \omega_\nu + X_3 \cdot \omega_\mu \omega_\nu \]  
\[ + X_4 \cdot B_\mu \omega_\nu + X_5 \cdot B_\mu \omega_\nu + X_6 \cdot E_\mu \omega_\nu + X_7 \epsilon^{\mu\nu} \omega_\mu \omega_\nu + X_8 \epsilon^{\mu\nu} \omega_\mu \omega_\nu \]  
\[ + X_9 \epsilon^{\mu\nu} \omega_\mu \omega_\nu + X_{10} \epsilon^{\mu\nu} \omega_\mu \omega_\nu + X_{11} \epsilon^{\mu\nu} \omega_\mu \omega_\nu + X_{12} \epsilon^{\mu\nu} \omega_\mu \omega_\nu + X_{13} \cdot B_\mu \omega_\nu \]  
\[ + X_{14} \epsilon^{\mu\nu} \omega_\mu \omega_\nu + X_{15} \epsilon^{\mu\nu} \omega_\mu \omega_\nu + X_{16} \epsilon^{\mu\nu} \omega_\mu \omega_\nu \]  
(48)

where the coefficients $X_1$, $X_2$ and $X_3$ which are irrelevant to electromagnetic field reads
\[ X_1 = -\lambda^{\mu\nu} + \lambda^{\nu\mu} - \lambda^{\mu\nu} - T \frac{\partial \lambda^{\mu\nu}}{\partial T} - T \frac{\partial \lambda^{\nu\mu}}{\partial T}, \]  
\[ X_2 = -\lambda^{\mu\nu} - 2\lambda^{\mu\nu} + 2\lambda^{\mu\nu} - 3\lambda^{\mu\nu} - 3\lambda^{\mu\nu} - T \frac{\partial \lambda^{\mu\nu}}{\partial T}, \]  
\[ X_3 = 2\lambda^{\mu\nu} + 2\lambda^{\mu\nu} + \lambda^{\mu\nu} - \lambda^{\mu\nu} - 2\lambda^{\mu\nu} - T \frac{\partial \lambda^{\mu\nu}}{\partial T}, \]  
(49)

the coefficients from $X_4$ to $X_8$ with linear dependence on electromagnetic field are given by
\[ X_4 = \lambda^{\epsilon E} - 3\lambda^{E} - \lambda^{B B} + 3\lambda^{B B} + T \frac{\partial \lambda^{\epsilon E}}{\partial T} + T \frac{\partial \lambda^{\epsilon E}}{\partial T} - T \frac{\partial \lambda^{B B}}{\partial T} - T \frac{\partial \lambda^{B B}}{\partial T} + \frac{1}{T} \frac{\partial \lambda^{\epsilon E}}{\partial T} \]
It should be noted that in order to arrive at the final result above (48), we have used the following identities:

$$X_5 = -\lambda^{EE} + \tilde{\lambda}^{EE} - \lambda^{BB} - T \frac{\partial \tilde{\lambda}^{EE}}{\partial T} - T \frac{\partial \lambda^{EE}}{\partial T} - T \frac{\partial \tilde{\lambda}^{\omega B}}{\partial T} - T \frac{\partial \lambda^{\omega B}}{\partial T} - \frac{1}{T} \frac{\partial \tilde{\lambda}^{\omega \omega}}{\partial \tilde{\mu}}$$

$$X_6 = \lambda^{EE} + 2\tilde{\lambda}^{EE} + \lambda^{BB} + 2\lambda^{\omega B} + 5\tilde{\lambda}^{\omega B} - 2\lambda^{\omega E} + T \frac{\partial \tilde{\lambda}^{EE}}{\partial T} + T \frac{\partial \lambda^{EE}}{\partial T} - T \frac{\partial \tilde{\lambda}^{\omega B}}{\partial T} - T \frac{\partial \lambda^{\omega B}}{\partial T} + \frac{1}{T} \frac{\partial \tilde{\lambda}^{\omega \omega}}{\partial \tilde{\mu}}$$

$$X_7 = -\lambda^{EE} - T \frac{\partial \tilde{\lambda}^{EE}}{\partial T} - 2T \frac{\partial \lambda^{EE}}{\partial T} - \frac{1}{T} \frac{\partial \tilde{\lambda}^{\omega E}}{\partial \tilde{\mu}} - \frac{1}{T} \frac{\partial \lambda^{\omega E}}{\partial \tilde{\mu}}$$

$$X_8 = -\lambda^{EE} - 2\tilde{\lambda}^{EE} - 3\lambda^{BB} - 2\lambda^{\omega B} + 2\lambda^{\omega E} + T \frac{\partial \tilde{\lambda}^{EE}}{\partial T} - T \frac{\partial \lambda^{EE}}{\partial T} + \frac{1}{T} \frac{\partial \tilde{\lambda}^{\omega B}}{\partial \tilde{\mu}} - \frac{1}{T} \frac{\partial \lambda^{\omega B}}{\partial \tilde{\mu}}$$

$$(50)$$

the coefficients with double linear dependence on electromagnetic field are

$$X_9 = -\tilde{\lambda}^{EE} - \lambda^{EE} - 2\lambda^{BB} - 3\tilde{\lambda}^{BB} + 2\lambda^{\omega B} - T \frac{\partial \tilde{\lambda}^{EE}}{\partial T} - \frac{1}{T} \frac{\partial \tilde{\lambda}^{\omega E}}{\partial \tilde{\mu}} - \frac{1}{T} \frac{\partial \lambda^{\omega B}}{\partial \tilde{\mu}}$$

$$X_{10} = \tilde{\chi}^{BB} + \lambda^{BB} - \lambda^{BB} - T \frac{\partial \tilde{\chi}^{BB}}{\partial T} - T \frac{\partial \lambda^{BB}}{\partial T}$$

$$X_{11} = 2\tilde{\lambda}^{EE} + 2\lambda^{BB} + \tilde{\lambda}^{EE} + 3\tilde{\lambda}^{BB} + 2\lambda^{\omega B} - 2\lambda^{\omega B} - T \frac{\partial \tilde{\lambda}^{EE}}{\partial T} - 2 \frac{\partial \lambda^{\omega B}}{\partial T}$$

$$X_{12} = 2\tilde{\lambda}^{EE} + 2\lambda^{BB} + \tilde{\lambda}^{EE} + 3\tilde{\lambda}^{BB} - 2\lambda^{\omega B} - 2\lambda^{\omega B} - T \frac{\partial \tilde{\lambda}^{EE}}{\partial T} - \frac{1}{T} \frac{\partial \tilde{\lambda}^{\omega B}}{\partial \tilde{\mu}} - 1 \frac{\partial \lambda^{\omega B}}{\partial \tilde{\mu}} - \frac{1}{T} \frac{\partial \tilde{\lambda}^{\omega \omega}}{\partial T} - T \frac{\partial \lambda^{\omega \omega}}{\partial T}$$

$$X_{13} = 2\lambda^{BB} - 2\tilde{\lambda}^{EE} - 2\lambda^{BB} - 3\lambda^{EE} - 3\lambda^{BB} - \lambda^{BB} + T \frac{\partial \lambda^{BB}}{\partial T} - \frac{1}{T} \frac{\partial \lambda^{\omega B}}{\partial \mu}$$

$$(51)$$

and the coefficients with triple linear dependence on electromagnetic field are given by

$$X_{14} = -\frac{1}{T} \left( \frac{\partial \tilde{\lambda}^{EE}}{\partial \tilde{\mu}} + \frac{\partial \lambda^{EE}}{\partial \tilde{\mu}} \right), \quad X_{15} = \frac{1}{T} \frac{\partial \tilde{\lambda}^{BB}}{\partial \tilde{\mu}}, \quad X_{16} = \frac{1}{T} \frac{\partial \tilde{\lambda}^{BB}}{\partial \mu} \cdot \tilde{E} \cdot \tilde{B}$$

$$(52)$$

It should be noted that in order to arrive at the final result above (48), we have used the following identities

$$\varepsilon \cdot \omega B^\nu = \varepsilon \cdot B \omega^\nu + \varepsilon^\nu E^\nu - \varepsilon \cdot E \varepsilon^\nu,$$

$$\omega \cdot EB^\nu = E \cdot B \omega^\nu + \varepsilon \cdot E \varepsilon^\nu - E^\nu \varepsilon^\nu,$$

$$\varepsilon \cdot BB^\nu = B^\nu \varepsilon^\nu - \omega \cdot BE^\nu + E \cdot B \omega^\nu,$$

$$\varepsilon \cdot \varepsilon^\nu = \varepsilon \cdot B \omega^\nu + \varepsilon^\nu E^\nu + \omega^\nu E^\nu - \omega \cdot B \varepsilon^\nu - \omega \cdot E \omega^\nu$$

$$(53)$$

which can be derived directly from the constraint (20). With these identities, we express the final result as the linear combination of independent terms. The source contribution from the coupling between the electromagnetic field and charge current is given by

$$F^{\mu \nu \lambda \mu}_{(2)} = -\xi^{\varepsilon \omega} (\varepsilon \cdot B) \varepsilon^\nu + \xi^{\varepsilon \omega} (\varepsilon \cdot B) \omega^\nu + \xi^{\varepsilon \varepsilon \omega} \varepsilon^2 E^\nu + \xi^{\omega \omega \omega} \omega^2 E^\nu + + \xi^{\varepsilon \omega} (E \cdot B) \omega^\nu - \xi^{\omega} (E \cdot B) \omega^\nu + \xi^{EE} E^2 E^\nu$$

$$+ \xi^{\varepsilon} (E \cdot B) \omega^\nu + \xi^{EE} (E \cdot B) B^2 E^\nu + \xi^{EE} (E \cdot B) B^2 E^\nu$$

$$(54)$$

Then from the conservation law $\partial_{\mu} T^{(2)}_{\mu \nu} = F^{\nu \mu \lambda \mu}_{(2)}$, we obtain the equations that could determine or constrain these coefficients. It is convenient to decompose these equations into three groups: The group I includes the coefficients for the pure $\varepsilon^\mu$ and $\omega^\mu$ term in energy-momentum tensor,

$$X_1 = 0, \quad X_2 = 0, \quad X_3 = 0$$

$$(55)$$

the group II contains the mixed terms between electromagnetic field and vorticity field in energy-momentum tensor

$$X_4 = -\xi^{\varepsilon \omega}, \quad X_5 = \xi^{\varepsilon \omega}, \quad X_6 = 0, \quad X_7 = \xi^{\varepsilon \omega}, \quad X_8 = \xi^{\omega \omega}$$

$$(56)$$
and the group III involves the pure electromagnetic terms in energy-momentum tensor,

\[ \begin{align*}
\mathcal{X}_9 &= 0, & \mathcal{X}_{10} &= 0, & \mathcal{X}_{11} &= \xi^{\omega E}, & \mathcal{X}_{12} &= -\xi^{E E}, & \mathcal{X}_{13} &= \xi^{\omega B} - \xi^{E E}, \\
\mathcal{X}_{14} &= \xi^{E E}, & \mathcal{X}_{15} &= \xi^{BB} - \xi^{EB}, & \mathcal{X}_{16} &= \xi^{EB}
\end{align*} \tag{57} \]

We note that if we know the coefficients in the energy-momentum tensor, we can directly obtain the coefficients in the charge current from the group II or the group III. At second order for the chiral fermions, the energy-momentum tensor would include trace anomaly which can lead to extra constraint identities referred as the group IV

\[ \begin{align*}
0 &= \lambda^{\varepsilon \varepsilon} + 3 \lambda^{\varepsilon \varepsilon} + \lambda^{\varepsilon \varepsilon}, \\
0 &= \lambda^{\omega \omega} + 3 \lambda^{\omega \omega} + \lambda^{\omega \omega}, \\
0 &= \lambda^{E E} + 3 \lambda^{E E} + 2 \lambda^{E E}, \\
0 &= \lambda^{\omega B} + 3 \lambda^{\omega B} + 2 \lambda^{\omega B}, \\
\hat{C} &= \lambda^{EE} + 3 \lambda^{EE} + \lambda^{EE}, \\
-\hat{C} &= \lambda^{BB} + 3 \lambda^{BB} + \lambda^{BB}
\end{align*} \tag{58-63} \]

From the group I together with the first two equations in the group IV, we note that only three coefficients are independent. From the naive dimension analysis, we know that these coefficients in group I must take the form of \( T^2 \). Choosing \( \lambda^{E E}, \lambda^{\omega \omega} \) and \( \lambda^{\omega \omega} \) as independent variables, we can obtain

\[ \begin{align*}
\hat{\lambda}^{E E} &= 0, \\
\hat{\lambda}^{\omega \omega} &= -\frac{1}{3} \lambda^{\omega \omega}, \\
\hat{\lambda}^{\omega \omega} &= -\frac{1}{3} \lambda^{\omega \omega} - 3 \lambda^{\omega \omega}, \\
\lambda^{\omega \omega} &= -\frac{1}{3} \lambda^{\omega \omega} + \frac{1}{2} \lambda^{\omega \omega} + \frac{5}{2} \lambda^{\omega \omega}
\end{align*} \tag{64-67} \]

Once these coefficients have been already known, from the group II and group IV together with the naive dimension analysis \( \lambda^{E E}, \lambda^{\varepsilon \varepsilon}, \lambda^{E E}, \lambda^{BB}, \lambda^{BB}, \lambda^{BB} \propto T \), we find that \( \xi^{E E}, \xi^{\omega \omega}, \xi^{\varepsilon \varepsilon} \) in \( j^{(2)} \mu \) satisfy the following constraint

\[ \xi^{E E} - \xi^{\omega \omega} - \xi^{\varepsilon \varepsilon} = \frac{1}{T} \lambda^{\varepsilon \varepsilon} + \frac{1}{T} \lambda^{\omega \omega} + \frac{1}{T} \lambda^{\varepsilon \varepsilon}, \]

which indicates that only two of \( \xi^{E E}, \xi^{\omega \omega}, \xi^{\varepsilon \varepsilon} \) are independent. Still from the group II with known \( \xi^{E E}, \) we have

\[ \hat{\lambda}^{\omega \omega} = \frac{1}{2} \left( \xi^{\omega \omega} - 2 \hat{\lambda}^{\omega \omega} \right), \tag{70} \]

which further leads to

\[ \hat{\lambda}^{BB} = \frac{1}{2} \left( \xi^{BB} - 2 \hat{\lambda}^{BB} \right), \tag{71} \]

Among the other transport coefficients for the mixed terms in energy-momentum tensor, we find only one transport coefficient is independent. We will choose \( \lambda^{BB} \) as the independent one and from the group II and the middle two equations in the trace constraint equations, we can express other coefficients as the following

\[ \begin{align*}
\hat{\lambda}^{\omega \omega} &= \frac{1}{2} \left( \lambda^{BB} + 3 \lambda^{\omega \omega} \right), \\
\hat{\lambda}^{E E} &= -\frac{1}{2} \left( \lambda^{BB} + \lambda^{\omega \omega} \right), \\
\lambda^{E E} &= -\left( 3 \lambda^{E E} + 2 \lambda^{\varepsilon \varepsilon} \right), \\
\lambda^{\omega E} &= \frac{1}{2} \left( 2 \lambda^{\omega B} + 4 \lambda^{\omega B} - \frac{1}{T} \lambda^{\omega \omega} \right)
\end{align*} \tag{72-75} \]
From the last equations in the group III, it is straightforward to obtain

$$\tilde{\lambda}_s^{BB} = - \int T \xi_s^{EB} d\tilde{\mu}_s,$$

$$\tilde{\lambda}_s^{BB} = - \int T (\xi_s^{BB} - \xi_s^{EB}) d\tilde{\mu}_s,$$

(76) (77)

where $\int T \xi^{XX} d\tilde{\mu}$ denotes the undetermined integral and possibly includes arbitrary functions with temperature dependence. Then from the group III together with the trace anomaly in group IV, the other coefficients can be totally determined by

$$\lambda_s^{BB} = - \tilde{C} - 3 \tilde{\lambda}_s^{BB} - \tilde{\lambda}_s^{BB},$$

$$\tilde{\lambda}_s^{EE} = - \frac{1}{2} \left( \xi_s^{EB} - 2 \tilde{\lambda}_s^{BB} + \frac{1}{T} \partial T \partial \tilde{\mu}_s + \frac{2}{T} \partial T \partial \tilde{\mu}_s \right),$$

$$\tilde{\lambda}_s^{EE} = - \tilde{\lambda}_s^{EE} - \int T \xi_s^{EE} d\tilde{\mu}_s,$$

$$\lambda_s^{EE} = \tilde{C} - 3 \tilde{\lambda}_s^{EE} - \tilde{\lambda}_s^{EE},$$

$$\lambda_s^{EB} = \frac{1}{4} \left( \xi_s^{EB} - 2 \xi_s^{EE} + 4 \tilde{\lambda}_s^{EE} + 4 \tilde{\lambda}_s^{BB} + 4 \tilde{\lambda}_s^{BB} - 2T \frac{\partial T \lambda_s^{BB}}{T} - 1 \frac{\partial T \lambda_s^{EB}}{T} - \frac{2}{T} \partial T \lambda_s^{EB} \right).$$

(78) (79) (80) (81) (82)

Three independent equations have not been used and remained as the constraint conditions:

$$0 = \tilde{\lambda}_s^{EE} + \lambda_s^{EE} + 2 \lambda_s^{BB} + 3 \tilde{\lambda}_s^{BB} - 2 \tilde{\lambda}_s^{EB} + T \frac{\partial T \lambda_s^{EE}}{T},$$

$$0 = \tilde{\lambda}_s^{BB} + \tilde{\lambda}_s^{BB} - \lambda_s^{BB} - T \frac{\partial T \lambda_s^{BB}}{T} - T \frac{\partial T \lambda_s^{BB}}{T},$$

$$\xi_s^{EE} = - 2 \tilde{\lambda}_s^{EE} - 2 \tilde{\lambda}_s^{BB} - 3 \tilde{\lambda}_s^{BB} + 2 \tilde{\lambda}_s^{EB} + T \frac{\partial T \lambda_s^{EE}}{T} + \frac{1}{T} \frac{\partial T \lambda_s^{EB}}{T} + \frac{1}{T} \frac{\partial T \lambda_s^{EB}}{T} + \frac{1}{T} \frac{\partial T \lambda_s^{EB}}{T}.$$

(83) (84) (85)

It should be noted that we have eliminated the partial derivative on temperature from the naive dimension analysis for the pure $\varepsilon, \omega$ terms and mixed terms between $\varepsilon, \omega$ and $E, B$ while we kept the partial derivative for the pure $E, B$ terms in Eqs. [32–33]. This is because the pure $E, B$ terms in energy-momentum tensor could include another regularization scale due to ultraviolet divergence and the naive dimension analysis would be broken while there is no such complexity for the pure $\varepsilon, \omega$ terms and mixed terms. This point had been demonstrated by the direct calculation given in [57]. We have checked that all these second-order results are totally consistent with the results which had been obtained from other approaches [37, 39].

IV. SUMMARY

When a system is in global equilibrium under electromagnetic field, only constant vorticity tensor is allowed when there is no gravity field involved. The electromagnetic and vorticity field must fulfill some constraint conditions. It turns out that these constraint conditions can be applied to determine the non-dissipative anomalous coefficients together with the energy-momentum conservation, chiral anomaly and trace anomaly.

At zeroth order, we find that the energy-momentum conservation and charge conservation hold automatically and trace vanishing leads to the well-known relation between the energy density and pressure. At first order, from the chiral anomaly and energy-momentum conservation, all the coefficients can be totally determined up to some integral constants, which is as well as what the hydrodynamic method had achieved from the second law of thermodynamics. The trace of the energy-momentum tensor always vanishes at first order. At second order, we find that the charge conservation holds automatically and we cannot say anything about the transport coefficients relevant to the charge current. However we can relate these transport coefficients in charge current to the ones in energy-momentum tensor by using the energy-momentum conservation law and find that once we obtain the coefficients in energy-momentum tensor, the coefficients for charge current could be derived directly. We find that among the coefficients relevant to the pure vorticity tensor in energy-momentum tensor there are only three coefficients are independent and the other four coefficients can be expressed as the linear combination of these three coefficients. We present the formulas which express the coefficients in the mixed terms from the electromagnetic and vorticity field as the ones associated with
the pure vorticity terms in energy-momentum tensor and charge current. Further we can determine the coefficients relevant to the pure electromagnetic field in the energy-momentum tensor from the charge current associated with electromagnetic field and the energy-momentum tensor associated with vorticity field. All these results do not depend on any specific interactions and are very general. They are supposed to be very helpful to determine the second-order anomalous transport coefficients in various chiral systems.

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