Geometric Transitions, Flops and Non-Kähler Manifolds: I

Melanie Becker\(^1\), Keshav Dasgupta\(^2\), Anke Knauf\(^1,3\), Radu Tatar\(^4\)

\(^1\) Department of Physics, University of Maryland, College Park, MD 20742
melanieb@physics.umd.edu, anke@umd.edu

\(^2\) Department of Physics, Stanford University, Stanford CA 94305
keshav@itp.stanford.edu

\(^3\) II. Institut für Theoretische Physik, Universität Hamburg
Luruper Chaussee 149, 22761 Hamburg, Germany

\(^4\) Theoretical Physics Group, LBL Berkeley, CA 94720
rtatar@socrates.Berkeley.EDU

Abstract

We construct a duality cycle which provides a complete supergravity description of geometric transitions in type II theories via a flop in M-theory. This cycle connects the different supergravity descriptions before and after the geometric transitions. Our construction reproduces many of the known phenomena studied earlier in the literature and allows us to describe some new and interesting aspects in a simple and elegant fashion. A precise supergravity description of new torsional manifolds that appear on the type IIA side with branes and fluxes and the corresponding geometric transition are obtained. A local description of new \(G_2\) manifolds that are circle fibrations over non-Kähler manifolds is presented.
1. Introduction

During the last years there has been tremendous progress toward constructing the string theory dual descriptions of large N gauge theories. The first steps in this direction were made by considering the conformal $\mathcal{N} = 4$, $D = 4$ super Yang-Mills theory [1] and later on more realistic field theories with $\mathcal{N} = 1$ supersymmetry and confinement were described in [2],[3],[4] and [5]. In a slightly different context, the connection between gauge theories and topological string theory was discussed in [6] for type IIA strings and in [7] for type IIB strings, the latter leading to the powerful Dijkgraaf-Vafa conjecture by which non-perturbative computations in field theories can be performed using perturbative expansions in matrix models.

The seminal work of Vafa [5] very clearly showed how to embed the topological duality of [3] into the framework of the AdS/CFT correspondence. The basic idea of [5]
was to consider the $\mathcal{N} = 1$ theory resulting from type IIA superstring theory in the deformed conifold background $T^*S^3$ in the presence of $N$ D6 branes wrapped around the Lagrangian $S^3$ cycle and filling the external space and compute the corresponding superpotential of this theory. On the other hand it was known that the superpotential for the field theory living in the four noncompact directions of the D6 branes is described by a Chern-Simons gauge theory on $S^3$, whose superpotential can be computed in terms of topological field theory amplitudes. Therefore, a connection between large $N$ Chern-Simons theory/topological string duality to ordinary superstring theory and the AdS/CFT correspondence was established. The idea was soon extended to many more rather interesting models in [9],[10],[11],[12],[13],[14],[15] and [16].

In all the above mentioned models the superpotentials computed by topological strings are mapped by the AdS/CFT like correspondences to geometries generated by fluxes on the dual closed string side. Since we start with D6 branes, we expect to have RR two-form fluxes in the dual closed string solution. These fluxes thread through a holomorphic $P^1$ cycle inside a blow up of a conifold. The superpotential is a product between the RR two-form fluxes and the Kähler form associated with the four-cycle that is Hodge dual to $P^1$. It turns out [3] that open/closed string duality requires another term in the superpotential. This term originates from the field theory gluino condensate, as topological open string computations imply the existence of a term linear in the gluino condensate [17], which gets mapped into the size of the holomorphic two-cycle. Therefore one has to have a term linear in the size of the holomorphic two-cycle which can be obtained if this is multiplied by a four-form. Furthermore, this four-form should be of NS type.

The quest for this four-form has been the subject of intense scrutiny in the last years. As there is no D-brane which can create it, the flux originates from changes in the geometry. The best way to understand its appearance is to go to the mirror type IIB picture and consider the superpotential [18]

$$ W = \int \Omega \wedge (H_{RR} + \varphi H_{NS}), \quad \varphi = \chi + i e^{-\Phi}, \quad (1.1) $$

where $\chi$ is the axion, $\Phi$ is the dilaton, $H_{RR}$ is the RR three-form flux and $H_{NS}$ is the NS three-form flux. From here we can go to the mirror type IIA picture and the fluxes map into the RR two-form flux and an NS four-form flux respectively. Of course, the main question is why would the NS fluxes appear when one starts with brane configurations involving only D branes? A partial answer to this question was given in [19], where the
origin of the NS four-form $F_4^{NS}$ was related to the fact that the $(3,0)$ form $\Omega = \Omega^+ + i\Omega^-$ is not closed for the type IIA compactification, and therefore $d\Omega^+ \sim F_4^{NS}$. The fact that $d\Omega \neq 0$ allows an extra term in the superpotential $\int d\Omega \wedge J$, $J$ being the fundamental two-form. Manifolds with the property that the real part of the $(3,0)$ form is not closed, while the imaginary part satisfies $d\Omega^- = 0 = d(J \wedge J)$ are called half flat manifolds. Half flat manifolds are examples of non-Kähler manifolds.

At this point one might wonder, why would a non-Kähler manifold appear as a result of a geometric transition from a brane configuration with D6 branes wrapped on a cycle of a Calabi Yau manifold? One of the goals of this paper is to give an answer to this question. In short the answer is this: the fluxes live on a non-Kähler manifold because the D6 branes actually are themselves wrapped on a cycle inside a non-Kähler manifold and the geometric transition is a flop inside a $G_2$ manifold with torsion.

To arrive to the non-Kähler geometry where the D branes are wrapped we start with the type IIB solution corresponding to D5 branes wrapped on the resolution cycle $P^1$ of the resolved conifold. The supergravity solution involves, besides the RR three-form, a NS three-form and a RR five-form. These fields are required by the string equations of motion. Similarly as in [9] the presence of the NS three-form before the transition is a signal that a NS three-form should also exist after the geometric transition has taken place.

As the resolved conifold is a toric manifold we can easily identify three $S^1$ coordinates and take a T-duality in these directions. By applying Buscher’s T-duality formulas we observe that the NS three-forms transforms into the type IIA metric in such a way that the

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1 Notice that this term also plays a crucial role in the S-duality conjecture of [8].
2 The manifold that we will eventually get in type IIA side later in this paper, will however be more general than the half-flat manifold in the sense that both $d\Omega^\pm \neq 0$.
3 The outcome of these T-dualities is different from the ones of [21], [22] and [23] where one T-duality takes a type IIB picture to a type IIA brane configuration.
4 There have been previous attempts to relate the resolved conifold and the deformed conifold by starting with the deformed conifold [24]. As the deformed conifold does not admit a $T^3$ fibration using this manifold as a starting point may seem more problematic, although, we have been informed that there are some papers that overcome this problem [23].
resulting manifold is not Kähler. The D5 branes get mapped into D6 branes which are wrapped on a three-cycle inside a non-Kähler deformation of a deformed conifold.

The non-Kähler deformation of a deformed conifold is then locally lifted to M theory. The lift leads to a $G_2$ manifold which is a deformation of the manifold constructed in \cite{28}, \cite{29}, and as such is described in terms of some left invariant one forms. These one forms reduce to the ones of \cite{28}, \cite{29} in the absence of B fields and they can be exchanged giving rise to a flop transformation. The resulting type IIA geometry describes a non-Kähler deformation of a resolved conifold. The non-Kähler deformation has a non-closed $(3,0)$ form whose derivative can be used to construct the additional contribution to the superpotential $\int d\Omega \wedge J$. These non-Kähler deformations of resolved and deformed conifolds, and the $G_2$ manifolds resulting from their lift to M-theory are, to the best of our knowledge, first concrete examples. In earlier literature these manifolds were anticipated as solutions of type II and M-theories although no concrete examples were presented.

To summarize, we propose a new geometric transition in the type IIA theory which relates D-branes wrapped on cycles of non-Kähler manifolds and fluxes on other non-Kähler manifolds. When lifted to M theory this transition describes a flop inside a $G_2$ manifold with torsion. This would lead to a deeper understanding of non-Kähler geometries. These geometries have only recently been discussed in some detail in the context of heterotic strings in \cite{32}, \cite{33}, \cite{34}, \cite{35}, \cite{36}, \cite{37} and \cite{38} and also of type II and M theories in \cite{39}, \cite{40}, \cite{41}, \cite{26}, \cite{42}, \cite{43}.

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5 This is a generalization of the notion of the mirror symmetry, where the B field in the type IIB picture is traded for a non Kählerity in type IIA. A similar observation has been made in \cite{26} where the generalized mirror symmetry exchange was between a non closed $J + iB$ and a nonclosed $\Omega$.

6 By this we mean a manifold endowed with an almost $G_2$ structure. The holonomy however is contained in $G_2$. For more details on the almost $G_2$ structure the reader may want to refer to the work of \cite{27} and the references given in Appendix 2.

7 For an earlier discussion of special holonomy spaces like $spin(7)$ and the corresponding one-forms, the reader may want to see \cite{30}.

8 This is related to the recent results of \cite{31}. This paper discusses a topological string model concluding that the non-integrability of the complex structure is related to the existence of Lagrangian NS branes called “NS two-branes” in \cite{3}. In our language, the non-Kählerity condition will be related the existence of the “NS two-brane”. It would be extremely interesting to relate this non-Kählerity appearing in the supergravity description to a corresponding effect in the Chern-Simons theory.
Our new geometric transition would enrich the “landscape picture” advocated in [14], [15], [16] in the sense of extra identifications between branes on cycles of non-Kähler geometries and fluxes on cycles of related non-Kähler geometries.

This paper is organized as follows: In section 2 we give a very brief review on the subject of geometric transitions and outline of the calculation that we will perform in this paper. Our starting point is the type IIB metric describing $D5$ branes wrapping a $P^1$ of a resolved conifold and through a series of T-duality transformations and a flop we shall be able to describe the geometric transition taking place in the type IIA mirror in great detail. In section 3 we give an alternative way to derive the metric using fourfold compactifications in M-theory in the presence of fluxes. Section 4 discusses the mirror formulas that we will use to get the full background in the type IIA theory. In the absence of fluxes, it is known that the mirror type IIA picture involves $D6$ branes wrapping an $S^3$ of a deformed conifold [5]. In section 4.4 we write the metric of the deformed conifold in a simpler way by making a coordinate transformation. We will discuss the reason why a deformed conifold may not have a $T^3$ fibration. Section 5 begins the study of the mirror manifold. We will present an explicit way to get the mirror manifold in the type IIA theory. We will show that the naive $T^3$ direction of the resolved conifold does not lead to the right mirror metric, which can nevertheless be determined by a set of restricted coordinate transformations. These aspects will be discussed in sections 5.1 and 5.2. In section 5.3 we will determine the $B$ field background and the metric for the mirror manifold will appear as eqn. (5.64) (and later as a final metric in eqn. (6.23)). In section 6 we begin our ascent to M-theory. In the absence of any fluxes in the type IIB picture, we expect a manifold with $G_2$ holonomy after lifting to M-theory. In the presence of fluxes, we will also get a seven dimensional manifold which now has a $G_2$ structure and torsion [9]. The generic study of $G_2$ holonomy manifolds has been done earlier using left invariant one-forms [49]. For our case we will also have one forms that are appropriately shifted by the background $B$ fields of the type IIA theory. Locally these one forms look exactly like the ones without fluxes. However, globally the system is much more involved, as we do not have any underlying $SU(2)$ symmetry. The M-theory lift of the mirror type IIA manifold is given as eqn (6.24), (6.25).

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9 In this paper we will interchangeably use $G_2$ structure and $G_2$ holonomy. The seven dimensional manifold that we construct will have an almost $G_2$ structure, although we will not check the holonomy here. More detailed discussions will be relegated to part II of this paper. We thank K. Behrndt and G. Dall’Agata for correspondences on this issue. See also [17], [48].
After lifting to M-theory we shall discuss the flop taking place in the resulting \( G_2 \) manifold. This will be studied in section 7 using the one forms that we devised earlier. We shall show that for torsional \( G_2 \) manifolds the flop is a little subtle. We discuss this in detail and compute the form of the metric after the flop. The result for the metric is given as eqn. (7.17) of section 7.1. Knowing the M-theory metric after the flop, helps us to get the corresponding type IIA metric easily by dimensional reduction. The resulting manifold in the type IIA theory is non-Kähler and the metric is given in eqn. (7.18). We show that the metric is basically a non-Kähler deformation of the resolved conifold. Interestingly, this is a rather similar situation as in the type IIA manifold before the geometric transition has taken place, whose metric is a non-Kähler deformation of the deformed conifold. In this section we further study some properties of these manifolds like non-Kählerity and the underlying superpotential. We leave a detailed discussion on various aspects of the whole duality chain for part II of this paper. We end with a discussion in section 8.

2. Geometric Transitions, Fluxes and Gauge Theories

We will summarize here some useful facts about geometric transitions, fluxes and field theory results.

Geometric transitions are examples of generalised AdS/CFT correspondence which relate D-branes in the open string picture and fluxes in the closed string picture. There are several types of geometric transitions depending on the framework in which we formulate them. The type IIB geometric transition, which starts with \( D5 \) branes wrapping a \( P^1 \) of a resolved conifold, has a parallel counterpart in which \( D5 \) branes wrap a vanishing two cycle of a conifold. This is the Klebanov-Strassler model \([2]\). In fact, these two models have identical behaviors in the IR of the corresponding four-dimensional \( \mathcal{N} = 1 \) gauge theories. However the UV behaviors are different. In the UV the geometric transition models give rise to six-dimensional gauge theories, whereas the Klebanov-Strassler model remains four dimensional. Both these models show cascading behavior. The cascading behavior in the geometric transition models manifest as an infinite sequence of flop transitions \([12]\). The corresponding brane constructions for these models have been developed earlier in \([21]\). The precise equivalences between the two models were shown in \([22]\) using (a) T-dual brane constructions, and (b) M-theory four-fold compactifications. We will not go into the details of this in the present paper but instead delve directly to the supergravity aspects of geometric transitions in both type II and M-theories, with the starting point being \( D5 \)
wrapped on resolution $P^1$ cycle of a resolved conifold. (For the Klebanov-Strassler model, the duality chain is not obvious). Readers interested in the details of the equivalence should look up the above mentioned references. Some related work has also been done in [50].

The figure below gives an overview of the geometric transitions and flops that will be discussed in this paper:

Let us elaborate this figure. The type IIB theory, depicted at the bottom left level, is the most studied one in different approaches [4], [2] and [3]. Here one starts with the field theory living on D5 branes wrapped on the resolution $P^1$ cycle of a resolved conifold. In the strong coupling limit of the field theory the $P^1$ cycle shrinks but the theory avoids the singularity by opening up an $S^3$ cycle inside a deformed conifold. In the figure, this is given by a dotted line pointing to the box on the lower right side of the picture, where a geometric transition has taken place. The deformed geometry encodes the information about the strongly coupled field theory as the size of the $S^3$ cycle is identified with the gluino condensate in the field theory [3].

There is an analog version of this process for the type IIA string which is depicted in the middle line of the figure. This time one starts with the field theory living on D6 branes wrapped on the $S^3$ cycle inside a deformed conifold. In the strong coupling limit
of the field theory the $S^3$ cycle shrinks but the theory avoids the singularity by opening up a $P^1$ cycle inside a resolved conifold (given by the next dotted line). As before, the complexified volume of the $P^1$ cycle is identified with the gluino condensate appearing in the field theory.

The transition looks mysterious if seen from ten dimensions but its understanding can be simplified by going up to eleven dimensions where it appears as a flop transition inside a $G_2$ manifold \cite{49} as seen in the upper line of the figure. For the above mentioned case of D6 branes wrapped on an $S^3$ cycle, the $G_2$ manifold appears as a cone over a $Z_N$ quotient of $S^3 \times \tilde{S}^3$ and the flop switches the two $S^3$ cycles. The type IIB transition has also been lifted to an M theory picture involving a warped fourfold compactification \cite{22}. Alternatively, by using one T-duality, the geometric transition has also been discussed in the brane configuration language in \cite{21} and \cite{23}.

Even though there is a compelling evidence in favor of using geometric transitions to describe strongly coupled field theories, there are still some unanswered questions. One of them is related to the form of the superpotential (1.1). Intuitively one would think that if we start with D branes, the supergravity solution should involve only RR fluxes; but, as it turns out, we need NS fluxes, too. In the language of the Klebanov-Strassler model \cite{2}, both RR and NS fluxes appear naturally as we have fractional D branes but this is not obvious in the language of \cite{5}.

One goal of this paper is to fill this gap and clarify the presence of the NS fluxes. We shall start with a known solution for the type IIB configuration with wrapped D5 branes on the resolved conifold by including both NS and RR fluxes. This is depicted in the lower left box of the figure. We first go up one step by using three T-dualities and as a result we get a non-Kähler type IIA geometry located in the middle left box of the figure. Then we go up to M theory, obtaining a $G_2$ manifold with torsion. We follow the upper arrow by performing a flop inside the $G_2$ manifold and then descend to obtain another non-Kähler type IIA geometry. We shall leave the last step and a detailed discussion on various aspects of the duality chain, for a future publication as we are mostly concerned here with the type IIA geometric transition.

In the figure the dark arrows represent the directions that we will be following in this paper. The dotted arrows represent the connection between two geometries that are related by a geometric transition. The duality cycle will therefore be powerful enough to give the precise supergravity background for all the examples studied in the literature so far. Notice also the fact that the key difference between the work presented here and some of the related
work \cite{24} is that our starting point involves $D5$ branes wrapping a resolved conifold and not a deformed conifold with fluxes. We believe that a deformed conifold with fluxes does not have any obvious $T^3$ fibration and so a simple Strominger, Yau and Zaslow (SYZ) \cite{51} analysis may not be easy to perform (see however \cite{25}). We will elaborate more on this as we go along.

We will begin by describing the first box in this picture (located on the lower left part of the figure): the type IIB background.

3. The Type IIB Background From M-Theory Dual

The type IIB background with D5 wrapping an $S^2$ of a resolved conifold has been discussed in detail in \cite{20}. The metric of the system is shown to follow from the standard D3 brane metric at a point on the non-compact manifold. In this section we will give an alternative derivation from M-theory with fluxes that at least reproduces locally some aspects of \cite{20}. One of the major advantages of using M-theory as opposed to the type IIB theory is the drastic reduction of the field content. The bosonic field content of M-theory consists of only the metric and four-form fluxes. Furthermore preserving supersymmetry in lower dimensions puts some constraints on the fluxes. The constraint equations are generically linear and therefore one can avoid the complicated second order equations that we would get by solving equations of motion.

To be more specific, we shall consider a non-compact fourfold in M-theory with G-fluxes. The non-compact fourfold is a $T^2$ fibration over a resolved conifold base. The $T^2$ fibration will be trivial (for the time being) and therefore the manifold is almost a product manifold. At this point one might get a little worried by the fact that the trivial fibration may force the Euler characteristic to vanish and therefore may disallow fluxes. Of course by having a non-compact manifold we may still have the possibility of non-zero fluxes, but we can get non-zero Euler number in this framework simply by allowing the $T^2$ fiber to degenerate far away, i.e not in the local neighborhood. There is one immediate advantage of having such a fibration. The $T^2$ torus doesn’t degenerate locally and therefore when we shrink the fiber torus to zero size to go to the type IIB framework, there will be no seven branes (and possibly orientifold seven planes) in our local neighborhood. This will simplify the subsequent analysis. Also, having a non-compact manifold allows us to put as many branes in the setup as we like. For the compact case, the number of branes (and fluxes) is constrained by an anomaly cancellation rule \cite{52}, \cite{53}. 

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Let us therefore consider a fourfold that is locally of the form $\mathcal{M}_6 \times T^2$, where $\mathcal{M}_6$ denotes the resolved conifold which is oriented along $x^{4,5,6,7,8,9}$, with one of the $S^2$ along $x^{4,5}$ and the other $S^2$ along $x^{8,9}$. The second $S^2$ degenerates at the radial distance $x^7 = 0$, whereas the other $S^2$ has a finite size. The coordinate $x^6$ is the usual $U(1)$ fibration. Using angular coordinates in terms of which the metric is generically written, the two $S^2$’s have coordinates $\theta_1, \phi_1$ and $\theta_2, \phi_2$. The radial coordinate is $r \equiv x^7$ and the $U(1)$ coordinate is $\psi \equiv x^6$, the latter being non-trivially fibered over the two $S^2$’s. The product torus $T^2$ is oriented along $x^3$ and $x^{11}$, with $x^{11} \equiv x^a$ being the M-theory direction $^{10}$.

In the presence of G-fluxes with components $G_{457a}$ and $G_{3689}$, the backreaction on the metric has been worked out in $^{52}$. The metric picks up a warp factor $\Delta$ which is a function of the internal (radial) coordinate only. The generic form of the metric is
\[
\text{ds}^2 = \Delta^{-1}\text{ds}_{012}^2 + \Delta^{1/2}\text{ds}_{\mathcal{M}_6 \times T^2}^2,
\]
where $\text{ds}_{012}^2$ denotes the Minkowski directions. In case that two covariantly constant spinors of definite chirality on the internal space can be found, supersymmetry requires the internal G-fluxes to be primitive and hence self-dual in the eight dimensional sense $\text{II}$.$^{11}$. Finally, there is also a spacetime component $G_{012m}$, where $x^m$ is one of the internal space directions. This component of the flux is given in terms of warp factor as $G_{012m} = \partial_m \Delta^{-3/2}$. The warp factor, $\Delta$, in turn satisfies the equation $\square \Delta^{3/2} = \text{sources}$.

Let us replace all the fluxes with M2 branes. This situation has been considered earlier in section 4 of $^{36}$. The metric of the system is the metric of $N$ M2 branes at a point on the fourfold $\mathcal{M}_6 \times T^2$
\[
\text{ds}^2 = H_2^{-2/3}\text{ds}_{012}^2 + H_2^{1/3}\text{ds}_{\mathcal{M}_6 \times T^2}^2,
\]
where $H_2$ is the harmonic function of the M2 branes. As discussed in $^{36}$, there is a one-to-one connection between the two pictures: the harmonic function in the M2 brane framework is related to the warp factor describing the flux via the relation $H_2 = \Delta^{3/2}$. One can easily show that the source equations work out correctly using the above identification $^{36}$.

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$^{10}$ As we shall soon see, the coordinates $(\psi, \theta_i, \phi_i, r)$ will not be the right coordinates to express the local metric. The correct set of coordinate system will be provided in sec. 5.

$^{11}$ This is the case we shall be interested in. The generalization to non-chiral spinors on the internal space was worked out in $^{57}$, $^{53}$ and $^{56}$. In this case the primitivity condition is replaced by a more general equation.
Imagine now that instead of M2 branes we have M5 branes in our framework. These M5 branes wrap three-cycles inside the fourfold. The three-cycles are an $S^1$ product over an $S^2$ base. We will assume that the M5 branes wrap the directions $x^{a,4,5}$ inside the fourfold. The metric ansatz for the wrapped M5 brane is

$$ds^2 = H_5^\alpha ds_{012}^2 + H_5^\beta ds_{S^2 \times S^1}^2 + H_5^\gamma ds_{36789}^2,$$

(3.3)

where $H_5$ is the harmonic function of the wrapped M5 branes, and $\alpha, \beta, \gamma$ are constants. Observe that generically $\beta \neq \gamma$ and therefore the metric of wrapped M5 branes is warped differently along the $S^2 \times S^1$ and the remaining $x^{3,6,7,8,9}$ directions.

The above discussion was in the absence of any fluxes. Let us switch on three-form potentials $C_{a45}$ and $C_{378}$. In the presence of these potentials the M5 branes will contain the following world-volume term [57]

$$S = -\frac{1}{4} \int \Gamma^{il} \Gamma^{jm} \Gamma^{kn} (F_{ijk} - C_{ijk})(F_{lmn} - C_{lmn}),$$

(3.4)

in addition to the usual source term that contributes a five dimensional delta function in the supergravity equations of motion. We have also denoted the field strength of the self-dual two form propagating on the M5 branes as $F_{ijk}$. In the absence of three-form sources this term would be absent and the equations of motion will only contain a five dimensional delta function source related to the M5 branes. In the presence of $C$ fields, the above action will induce an M2 brane charge and therefore the delta function contribution to the action will become eight dimensional. Therefore the background configuration will be the usual M2 brane background, implying that $\beta = \gamma$. When reduced to the type IIB theory by shrinking the fiber torus to zero size, the metric will locally resemble the D3 brane metric obtained in [20]. Here we have provided a derivation of this metric from M-theory. For the M2 brane background the warp factor will satisfy the usual equation [52], [53]

$$*\square H_5 - 4\pi^2 X_8 + \frac{1}{2} G \wedge G = -4\pi^2 \sum_{i=1}^{n} \delta^8(y - y_i),$$

(3.5)

where the Hodge duality is over the unwarped metric, $n$ is the number of fractional M2 branes situated at points $y_i$ in the fourfold and $X_8$ is a polynomial in powers of the curvature that contains information about the seven-branes.

From the M-theory point of view, our choice of G-fluxes immediately reduces to the $H_{NS}$ and $H_{RR}$ fluxes in the type IIB theory. The fact that supersymmetry requires the...
G-fluxes to be primitive, implies that the NS and the RR fluxes should be dual to each other. This duality gives rise to linear equations in the type IIB theory. As expected, notice that the primitivity of G-fluxes in M-theory implies that the form should be of type (2,2) [52]. This means that the NS and the RR fluxes when combined to form a three-form \( G \equiv H_{NS} + \varphi H_{RR} \) with \( \varphi \) being the usual axion-dilaton scalar, will be a (2,1) form. This is where we face a problem. The (2,1) nature of the fluxes is in agreement with the analysis of [2] but unfortunately not with [20] (see also [58]). This means that the global geometry of [20] breaks susy, and therefore our background derived from M-theory should be the right global description and not [20]. From the way we derived the background, the metric would locally resemble the metric of [20] but globally there will be extra seven branes (see [59] for the full story).

To summarize, starting with M-theory we have rederived the form of the metric describing wrapped D5 metric in the type IIB theory. The choice of fluxes fixes the complex structure to some particular value \( \tau \). Using this we define a one form \( dz = dx^3 + \tau dx^a \).

The final background can be presented in a compact form using the notation of [20] (with \( \alpha = -2/3, \beta = \gamma = 1/3 \))

\[
ds^2 = H_5^{-2/3} ds^2_{012} + H_5^{1/3} ds^2_{M_6 \times T^2}, \tag{3.6}
\]

\[
G = e_{\theta_1} \wedge e_{\phi_1} \wedge G_1 + e_{\theta_2} \wedge e_{\phi_2} \wedge G_2,
\]

where \( e_{\theta_i} \) and \( e_{\phi_i} \) with \( i = 1, 2 \) are defined in [20]. The forms \( G_1 \) and \( G_2 \) also have an explicit representation. They can be defined in terms of the remaining forms \( dz, dr \) and \( e_\psi \) (defined in [20]) as

\[
G_1 = \bar{\tau} f' dz \wedge dr + \bar{\tau} dz \wedge e_\psi - c.c,
\]

\[
G_2 = \bar{\tau} g' dz \wedge dr - \bar{\tau} dz \wedge e_\psi - c.c, \tag{3.7}
\]

where \( f \) and \( g \) are solutions to the linear equations following from supersymmetry. Observe also the fact that we haven’t yet fixed the value of \( \tau \), the complex structure of the \( x^{3,a} \) torus [32]. It is not too difficult to see that the complex structure can be fixed to \( \tau = i \) so that we end up with a square torus. The analysis follows closely to the one discussed in [32], so we will not repeat it here. The fact that fluxes are not constant here (as opposed to the constant fluxes in [32]) does not alter the result for the complex structure.

\[12\] The complex structure of the base will be discussed later.
4. Mirror Formulas using Three T-dualities

In this section we will determine the formulas for the metric and fluxes \( B_{NS} \) and \( B_{RR} \) of a general type IIA manifold that is the mirror of a six dimensional type IIB manifold that is a \( T^3 \)-fibration over a three dimensional base. According to Strominger, Yau and Zaslow \([51]\) the mirror manifold can be determined by performing three T-dualities on the fiber. We shall be using the T-duality formulas of \([60]\). Later on we will use our general result for the particular case of the resolved conifold.

4.1. Metric Components

We will start by determining the metric components of the mirror manifold. Let us call the \( T^3 \) directions of the lagrangian \( T^3 \) fibered manifold with which we start as \( x, y \) and \( z \). We will be performing three T-dualities \([60]\) along these directions in the order \( x, y, z \).

The starting metric in the type IIB theory has the following components

\[
d s^2 = j_{\mu\nu} d x^\mu \, d x^\nu + j_{x\mu} d x \, d x^\mu + j_{y\mu} d y \, d x^\mu + j_{z\mu} d z \, d x^\mu + j_{xy} d x \, d y + j_{xz} d x \, d z + j_{yz} d y \, d z + j_{xx} d x^2 + j_{yy} d y^2 + j_{zz} d z^2
\]

where \( \mu, \nu \neq x, y, z \), and the \( j \)'s are for now arbitrary. After a straightforward calculation we obtain the form of the metric of the mirror manifold

\[
\begin{align*}
d s^2 &= \left(G_{\mu\nu} - \frac{G_{z\mu}G_{z\nu} - B_{z\mu}B_{z\nu}}{G_{zz}}\right) d x^\mu \, d x^\nu + 2 \left(G_{x\nu} - \frac{G_{zz}G_{x\nu} - B_{zz}B_{z\nu}}{G_{zz}}\right) d x \, d x^\nu \\
&+ 2 \left(G_{y\nu} - \frac{G_{zz}G_{y\nu} - B_{zz}B_{z\nu}}{G_{zz}}\right) d y \, d x^\nu + 2 \left(G_{xy} - \frac{G_{zz}G_{xy} - B_{zz}B_{zy}}{G_{zz}}\right) d x \, d y \\
&+ \frac{d z^2}{G_{zz}} + 2 \frac{B_{xz}}{G_{zz}} d x^\mu \, d z + 2 \frac{B_{yz}}{G_{zz}} d x \, d z + 2 \frac{B_{yy}}{G_{zz}} d y \, d z \\
&+ \left(G_{xx} - \frac{G_{zz}^2 - B_{zz}^2}{G_{zz}}\right) d x^2 + \left(G_{yy} - \frac{G_{zz}^2 - B_{zz}^2}{G_{zz}}\right) d y^2.
\end{align*}
\]

The various components of the metric can be written as

\[
G_{\mu\nu} = \frac{j_{\mu\nu} j_{xx} - j_{x\mu} j_{x\nu} + b_{x\mu} b_{x\nu}}{j_{xx}} - \frac{(j_{y\mu} j_{xx} - j_{x\mu} j_{y\nu} + b_{x\mu} b_{y\nu})(j_{y\nu} j_{xx} - j_{x\nu} j_{y\mu} + b_{x\nu} b_{y\mu})}{j_{xx}(j_{yy} j_{xx} - j_{x}^2 + b_{x}^2)} \\
+ \frac{(b_{y\mu} j_{xx} - j_{x\mu} b_{y\nu} + b_{x\mu} b_{y\nu})(b_{x\nu} j_{xx} - j_{x\nu} b_{x\mu} + b_{x\mu} b_{x\nu})}{j_{xx}(j_{yy} j_{xx} - j_{x}^2 + b_{x}^2)},
\]

(4.3)
\[
G_{\mu z} = \frac{j_{\mu z} j_{xx} - j_{\mu x} j_{xz} + b_{x\mu} b_{xz}}{j_{xx}} - \frac{(j_{y\mu} j_{xx} - j_{y x} j_{x x} + b_{x y} b_{x x})(j_{y z} j_{xx} - j_{y x} j_{x z} + b_{x y} b_{x z})}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)} \\
+ \frac{(b_{y \mu} j_{xx} - j_{x y} b_{x \mu} + b_{x y} j_{x \mu})(b_{y z} j_{xx} - j_{x y} b_{x z} + b_{x y} j_{x z})}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)},
\]
\[
G_{zz} = \frac{j_{zz} j_{xx} - j_{z x}^2 + b_{z x}^2}{j_{xx}} - \frac{(j_{z y} j_{xx} - j_{x y} j_{z x} + b_{x y} b_{z x})^2}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)} \\
+ \frac{(b_{y z} j_{xx} - j_{x y} b_{x z} + b_{x y} j_{x z})^2}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)},
\]
\[
G_{y\mu} = -\frac{b_{y \mu} j_{xx} - b_{x \mu} j_{xy} + b_{x y} j_{x \mu}}{j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2}, \quad G_{yz} = -\frac{b_{y z} j_{xx} - b_{z x} j_{xy} + b_{x y} j_{x z}}{j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2},
\]
\[
G_{yy} = \frac{j_{xx}}{j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2}, \quad G_{xx} = \frac{j_{yy}}{j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2}, \quad G_{xy} = \frac{-j_{xy}}{j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2},
\]
\[
G_{\mu x} = \frac{b_{\mu x}}{j_{xx}} + \frac{(j_{y \mu} j_{xx} - j_{y x} j_{x \mu} + b_{x y} b_{x \mu}) b_{x y}}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)} + \frac{(b_{y \mu} j_{xx} - j_{x y} b_{x \mu} + b_{x y} j_{x \mu}) j_{xy}}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)},
\]
\[
G_{zx} = \frac{b_{zx}}{j_{xx}} + \frac{(j_{z y} j_{xx} - j_{x y} j_{z x} + b_{x y} b_{z x}) b_{x y}}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)} + \frac{(b_{y z} j_{xx} - j_{x y} b_{x z} + b_{x y} j_{x z}) j_{xy}}{j_{xx} (j_{y y} j_{xx} - j_{x y}^2 + b_{x y}^2)}.
\]

In the above formulae we have denoted the type IIB \(B\) fields as \(b_{mn}\), whose explicit form will be computed in the next section. We will use this more general formula for the particular case of the resolved conifold a little later. Our next goal is to determine the NS fluxes on the mirror manifold for the most general case.

4.2. \(B_{NS}\) Components

For the generic case we will switch on all the components of the \(B\) field

\[
b = b_{\mu \nu} \, dx^\mu \wedge dx^\nu + b_{x \mu} dx \wedge dx^\mu + b_{y \mu} dy \wedge dx^\mu + b_{z \mu} dz \wedge dx^\mu \\
+ b_{x y} \, dx \wedge dy + b_{x z} \, dx \wedge dz + b_{z y} \, dz \wedge dy.
\]
In the later sections we will concentrate on the special components that describe a resolved conifold with branes. After applying again the T-dualities, the NS component of the \( B \) field in the mirror set-up will take the form

\[
\tilde{B} = \left( B_{\mu\nu} + \frac{2B_{z[\mu}G_{\nu]z}}{G_{zz}} \right) dx^\mu \wedge dx^\nu + \left( B_{\mu x} + \frac{2B_{z[\mu}G_{x]z}}{G_{zz}} \right) dx^\mu \wedge dx^x + \left( B_{\mu y} + \frac{2B_{z[\mu}G_{y]z}}{G_{zz}} \right) dx^\mu \wedge dy + \left( B_{\mu z} + \frac{2B_{z[\mu}G_{zz}}{G_{zz}} \right) dx^\mu \wedge dz + \frac{G_{z\mu}}{G_{zz}} dx^\mu \wedge dx + \frac{G_{zy}}{G_{zz}} dy \wedge dz.
\] (4.11)

Here the \( G_{mn} \) components have been given above, and the various \( \mathcal{B} \) components can now be written as

\[
B_{\mu\nu} = \frac{b_{\mu\nu}j_{xx} + b_{x\mu}j_{yx} - b_{x\nu}j_{xx}}{j_{xx}},
\]

\[
B_{\mu x} = \frac{b_{\mu x}j_{xx} + b_{x\mu}j_{xx} - b_{xx}j_{\mu x}}{j_{xx}},
\]

\[
B_{\mu y} = \frac{j_{yy}j_{xx} - j_{xy}j_{x\mu} + b_{xy}b_{x\mu}}{j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2}, \quad B_{\mu z} = \frac{j_{yy}j_{xx} - j_{xy}j_{x\mu} + b_{xy}b_{x\mu}}{j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2},
\]

\[
B_{\mu x} = \frac{j_{xx} - j_{yy}j_{xx} - j_{xy}j_{x\mu} + b_{xy}b_{x\mu}}{j_{xx}(j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2)} + \frac{b_{xy}(b_{x\mu}j_{xy} - b_{x\mu}j_{xx} - b_{x\mu}j_{xz})}{j_{xx}(j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2)},
\]

\[
B_{\mu z} = \frac{j_{xx} - j_{yy}j_{xx} - j_{xy}j_{x\mu} + b_{xy}b_{x\mu}}{j_{xx}(j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2)} + \frac{b_{xy}(b_{z\mu}j_{xy} - b_{z\mu}j_{xx} - b_{z\mu}j_{xz})}{j_{xx}(j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2)},
\]

\[
B_{xy} = \frac{-b_{xy}}{j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2}.
\] (4.17)
In the above analysis, there is one subtlety related to the compactness of the $x, y, z$ directions. The type IIB $B$ fields defined wholly along these directions, i.e. $B_{yz}, B_{zx}$ and $B_{xy}$, should be *periodic*. This would mean, for example, if we specify a value of $B_{yz}$ as (say) $\alpha_{yz}$ then this should also be equal to $-\alpha_{yz}$ because of periodicity. This implies that the values of $B_{yz}, B_{zx}$ and $B_{xy}$ found in (4.14), (4.16) and (4.17) are *ambiguous* up to a possible sign. Later on we shall use consistency conditions to fix the sign.

Furthermore, observe that we haven’t yet discussed how the RR $B$ fields look like in the mirror set-up. We will eventually compute the form of these fields when we perform the M-theory lift of the type IIA mirror. We also need to see how the fermions transform under mirror symmetry. This will be important in order to understand the complex structure of the mirror manifold and to check whether it is integrable or not. This will be discussed in the sequel to this paper. The string coupling constant in the type IIB theory and the one in the type IIA mirror are related in the following way

$$g_A = \frac{g_B}{\sqrt{(j_{xx}j_{yy} - j_{zy}^2 + b_{zxy}^2)} G_{zz}}, \quad (4.18)$$

where we have defined $G_{zz}$ in (4.5). This coupling constant is in general a function of the internal coordinates.

### 4.3. Background Simplifications

The background given in the above set of formulas can be written in a *compact* form which will be helpful to see the fibration structure more clearly

$$ds^2 = \frac{1}{G_{zz}}(dz + B_{\mu z} dx^\mu + B_{\nu z} dx^\nu + B_{y z} dy)^2 - \frac{1}{G_{zz}}(G_{\mu z} dx^\mu + G_{xx} dx + G_{zy} dy)^2$$

$$+ G_{\mu \nu} dx^\mu dx^\nu + 2G_{x \nu} dx dx^\nu + 2G_{y \nu} dy dx^\nu + 2G_{x y} dx dy + G_{xx} dx^2 + G_{yy} dy^2. \quad (4.19)$$

The above compact form can be simplified even further for the particular example we are interested in. More concretely the $G_{mn}$ components ($m, n = \mu, x, y, z$) become rather simple if one assumes the following choices of $j_{mn}, b_{mn}$

$$j_{\mu x} = j_{\mu y} = j_{\mu z} = 0; \quad b_{xy} = b_{zx} = b_{zy} = 0; \quad b_{\mu \nu} = 0. \quad (4.20)$$

In this case the negative components in the metric vanish. The above assumption implies that the type IIB metric of a D5 wrapping an $S^2$ of the resolved conifold has no off-diagonal
components. This can be easily checked and we shall elaborate this further later on. The 
type IIB $b$ field choice tells us that off-diagonal components are allowed but the cross terms 
vanish. This can also be verified easily. With this choice of type IIB metric the metric 
components of the mirror take the following form

\[
G_{\mu\nu} = j_{\mu\nu} + \frac{b_{x\mu}b_{x\nu}}{\dot{j}_{xx}} + \frac{(b_{yy}j_{xx} - b_{xy}j_{xy})(b_{yy}j_{xx} - b_{xy}j_{xy})}{\dot{j}_{xx}(j_{yy}j_{xx} - \dot{j}_{xy}^2)},
\]

\[
G_{\mu z} = G_{zx} = G_{zy} = 0,
\]

\[
G_{x x} = \frac{\dot{j}_{yy}}{j_{yy}j_{xx} - \dot{j}_{xy}^2}, \quad G_{y y} = \frac{\dot{j}_{xx}}{j_{yy}j_{xx} - \dot{j}_{xy}^2}, \quad G_{z z} = j_{zz} - \frac{\dot{j}_{zz}^2}{\dot{j}_{xx}} - \frac{(j_{yz}j_{xx} - j_{xy}j_{zz})^2}{\dot{j}_{xx}(j_{yy}j_{xx} - \dot{j}_{xy}^2)},
\]

\[
G_{x y} = -\frac{\dot{j}_{xy}}{j_{yy}j_{xx} - \dot{j}_{xy}^2},
\]

\[
G_{\mu x} = \frac{b_{\mu x}}{\dot{j}_{xx}} + \frac{(b_{yy}j_{xx} - b_{xy}j_{xy})j_{xx}}{\dot{j}_{xx}(j_{yy}j_{xx} - \dot{j}_{xy}^2)}, \quad G_{y \mu} = -\frac{b_{yy}j_{xx} - b_{xy}j_{xy}}{j_{yy}j_{xx} - \dot{j}_{xy}^2}.
\]

On the other hand, the $B$ fields appearing in the metric and fluxes (not to be confused 
with the $\tilde{B}$ fields in the type IIA picture) take the form

\[
B_{\mu\nu} = B_{\nu\mu} = B_{x\mu} = B_{x\nu} = 0,
\]

\[
B_{\mu z} = b_{\mu z} + \frac{b_{x\mu}j_{xx}}{\dot{j}_{xx}} - \frac{(j_{yy}j_{xx} - j_{xy}j_{xz})(b_{yy}j_{xx} - b_{xy}j_{xy})}{\dot{j}_{xx}(j_{yy}j_{xx} - \dot{j}_{xy}^2)},
\]

\[
B_{z x} = \frac{j_{xx}}{\dot{j}_{xx}} - \frac{j_{xy}(j_{yy}j_{xx} - j_{xy}j_{xz})}{\dot{j}_{xx}(j_{yy}j_{xx} - \dot{j}_{xy}^2)}, \quad B_{y z} = -\frac{j_{yy}j_{xx} - j_{xy}j_{xx}}{j_{yy}j_{xx} - \dot{j}_{xy}^2}.
\]

Using the above choices of $G_{mn}$ and $B_{mn}$ one can easily show that all components of the 
mirror NS flux vanish, $\tilde{B}_{mn} = 0$.

Naively one would expect that the background of the mirror manifold that we just 
derived corresponds to a deformed conifold of $[2]$ in the presence of fluxes. In order to see 
the relation to the deformed conifold our mirror background can be simplified further. But 
before doing so, let us rewrite the deformed conifold background of $[2]$ in a suggestive way 
so that a comparison can be made.
4.4. Rewriting the Deformed Conifold Background

The metric of a D6 brane wrapping a three cycle of a deformed conifold has been discussed earlier in [61]. Let us recapitulate the result. To obtain the metric, one defines a Kähler potential $\mathcal{F}$ as a function of $\rho^2 \equiv \text{tr}(W^\dagger W)$ with $W$ being a complex $2 \times 2$ matrix satisfying $\det W = -\epsilon^2/2$. The quantity $\rho$ is basically the radial parameter and $\epsilon$ is a real number. The generic form of the Ricci flat Kähler background is determined from (see [62] for details)

$$
\frac{ds^2}{\rho^4 - \epsilon^4} = \mathcal{F}' \text{tr}(dW^\dagger dW) + \mathcal{F}'' |\text{tr}(W^\dagger dW)|^2,
$$

(4.29)

where the primes are defined as $\mathcal{F}' = \frac{d\mathcal{F}}{d\rho^2}$ and the determinant of the deformed conifold metric is given by $\epsilon^{-8}(\rho^4 - \epsilon^4)^2$. For a Ricci flat metric $\mathcal{F}'$ becomes equal to

$$
\frac{(\sqrt{2}\epsilon)^{-\frac{4}{3}} (2\epsilon^2 \rho^2 \sqrt{\rho^4 - \epsilon^4} - 2\epsilon^6 \text{ch}^{-1}(\rho^2/\epsilon^2))^{1/3}}{\sqrt{\rho^4 - \epsilon^4}}.
$$

(4.30)

In this form it is not too difficult to write the metric of a bunch of D6 branes wrapping the three cycle of a deformed conifold. If we take the limit $\rho \to \epsilon$ the metric becomes the metric of an $S^3$ space. The generic form of the wrapped D6 metric is

$$
\frac{ds^2}{\rho^4 - \epsilon^4} = A_0 \, ds^2_{0123} + A_1 \, d\rho^2 + A_2 \, ds^2_1 + A_3 \, ds^2_2 + A_4 \, ds^2_3 + A_5 \, ds^2_4
$$

(4.31)

where $A_i$ are some specific functions of the radial coordinate $\rho^2$ and the metric components $ds_i$ are given by

$$
\begin{align*}
\frac{ds^2_1}{\rho^4 - \epsilon^4} &= (d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2)^2, \\
\frac{ds^2_2}{\rho^4 - \epsilon^4} &= d\theta_1^2 + \sin^2 \theta_1 \, d\phi_1^2, \quad \frac{ds^2_3}{\rho^4 - \epsilon^4} = d\theta_2^2 + \sin^2 \theta_2 \, d\phi_2^2, \\
\frac{ds^2_4}{\rho^4 - \epsilon^4} &= 2 \sin \psi \, (d\phi_1 d\theta_2 \, \sin \theta_1 + d\phi_2 d\theta_1 \, \sin \theta_2) + 2 \cos \psi \, (d\theta_1 d\theta_2 - d\phi_1 d\phi_2 \, \sin \theta_1 \sin \theta_2).
\end{align*}
$$

(4.32)

The appearance of $\sin \psi$ and $\cos \psi$ in the above metric is a little disconcerting as the expected $U(1)$ symmetry acting on $\psi$ as $\psi \to \psi + \epsilon$ is not present [24]. This means that the deformed conifold cannot be written as a simple $T^3$ fibration over a three dimensional base. An immediate consequence of this is that the usual SYZ technique cannot be applied. On the other hand, D5 branes wrapped on a resolved conifold do have the required $U(1)$ isometries related to constant shifts in $\psi, \phi_1, \phi_2$, so that we can perform three T-dualities and obtain the mirror manifold, as we are doing in this paper. The $T^3$ fibration corresponds
to the $\psi,\phi_1,\phi_2$ torus. In the notations of the previous subsection, $(\psi,\phi_1,\phi_2) \to (z,x,y)$ for the type IIB resolved conifold (we will give a more precise mapping soon).

In order to compare the deformed conifold metric with the metric obtained for our mirror manifold, let us assume that we fix the value of $\psi$ in $ds_3$ in (4.32) as $\psi = \psi_0$. It is then convenient to perform the following change of coordinates $\theta_2, \phi_2$:

$$
\begin{pmatrix}
\sin \theta_2 \\ d\phi_2
\end{pmatrix}
\to
\begin{pmatrix}
\cos \psi_0 & \sin \psi_0 \\ -\sin \psi_0 & \cos \psi_0
\end{pmatrix}
\begin{pmatrix}
\sin \theta_2 \\ d\phi_2
\end{pmatrix},
$$

(4.33)

with the other coordinates $\theta_1$ and $\phi_1$ remaining unchanged. Although identical in spirit, the above transformation is different from equation (2.2) of [24]. Under the transformation (4.33), the metric component $ds_3^2$ changes to

$$
ds_3^2 \to 2d\theta_1 
\theta_2 - 2\sin \theta_1 \sin \theta_2 
\phi_1 
\phi_2,
$$

(4.34)

so that the $\psi_0$ dependence is completely removed. Although this may mean that we regain the $U(1)$ isometry but this is only because of the obvious delocalisation procedure. In general, for non-constant $\psi$ in (4.33), the $\psi$ dependence would enter into $ds_1^2$ in such a way that a shift in the other coordinates would fail to remove it. Observe that the metric components $ds_2^2$ and $ds_4^2$ remain unaltered.

To summarize: the above observation tells us that a simple $T^3$ fibration of a deformed conifold does not exist. In a new coordinate system (which is discussed in [24]) it might be possible to regain some of the $U(1)$ isometries, although a SYZ description of the corresponding mirror appears futile (see [25] for some proposals to reconcile this were given). Nevertheless, the above change of coordinates will be useful to understand the type IIA mirror background and compare the metric with the expected deformed conifold metric.

In the following we will rewrite the metric (4.31) in such a way that it can be mapped to the mirror manifold obtained by using three T-dualities. The readers interested in the type IIA mirror background may want to skip this part and go directly to the next section. If we define a warp factor $h \equiv h(\rho)$ then the generic metric of D6 branes wrapping a three cycle of a deformed conifold can be written as

$$
ds^2 = h^\alpha 
ds_{0123}^2 + h^\beta \ dr^2 + h^\gamma \ ds_1^2 + h^\delta \ (ds_2^2 + ds_4^2) + h^\rho \ ds_3^2,
$$

(4.35)

where $\alpha, \beta, \gamma, \rho$ are the various numerical powers for the wrapped D6 branes (compare with the previous formula (4.31)) and $ds_i$ $(i = 1, 2, 3)$ are defined earlier. We have also taken a
slightly simplified case where corresponding to \( A_3 = A_4 \) in our earlier notation, as this is the case we are interested in. The above metric can now be written in a more suggestive way.

\[
ds^2 \equiv h^n \, ds_{0123}^2 + (h^{\beta/2} \, dr)^2 + h^\gamma \left( d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2 \right)^2 + \\
+ (h^\delta - h^\rho) \left( \sin \psi \sin \theta_1 \, d\phi_1 + \cos \psi \, d\theta_1 - d\theta_2 \right)^2 + \\
+ (h^\delta - h^\rho) \left( \cos \psi \sin \theta_1 \, d\phi_1 - \sin \psi \, d\theta_1 + \sin \theta_2 \, d\phi_2 \right)^2 + \\
+ (h^\delta + h^\rho) \left( \sin \psi \sin \theta_1 \, d\phi_1 + \cos \psi \, d\theta_1 + d\theta_2 \right)^2 + \\
+ (h^\delta + h^\rho) \left( \cos \psi \sin \theta_1 \, d\phi_1 - \sin \psi \, d\theta_1 - \sin \theta_2 \, d\phi_2 \right)^2.
\]

The above metric cannot be the full global picture, as we know that the corresponding type IIB manifold has extra seven branes. Thus we have to add extra six branes in this scenario to complete the picture. However we can still study the local metric using different set of coordinates that are more suited for this case. Our next goal therefore is to rewrite this metric in terms of the T-dual coordinates \( x, y, z \) that we used earlier to get the fields of the mirror manifold, by using the identification \((\psi, \phi_1, \phi_2) \rightarrow (z, x, y)\) and \(\mu, \nu = \theta_1, \theta_2\).

Omitting the \( r \) and the \( ds_{0123}^2 \) term, we might expect the metric (4.36) can be written as

\[
ds^2 = h^n \left( dz + f^1_{xx} \, dx + f^2_{xy} \, dy \right)^2 + (h^\delta - h^\rho) \left( f^3_{xx} \, dx + f^4_{x\mu} \, dx^\mu \right)^2 + \\
+ \left( f^5_{yy} \, dy + f^6_{xy} \, dx + f^7_{x\mu} \, dx^\mu \right)^2 + (h^\delta + h^\rho) \left( f^8_{xx} \, dx + f^9_{x\mu} \, dx^\mu \right)^2 + \\
+ \left( f^{10}_{yy} \, dy + f^{11}_{xy} \, dx + f^{12}_{x\mu} \, dx^\mu \right)^2,
\]

where \( f^i_{mn}, \ i = 1, 2, ..., 12, \ m, n = x, y, z \) can be easily related to the coefficients in (4.36) once the precise relation between \((\psi, \phi_1, \phi_2)\) and \((z, x, y)\) is spelled out. Having written the metric in the form of (4.36) still does not tell us that D6 branes wrapped on \( S^3 \) of a deformed conifold should have a \( T^3 \) fibration, because of the appearance of \(\sin \psi\) and \(\cos \psi\) in the product. On the other hand writing the metric in the form (4.37) will help us to relate it to the mirror metric that we derived in the previous section. We will do this in the next section.

Let us comment a little more on the transformation (4.33). As mentioned earlier, this transformation with non-constant \( \psi \) would remove the \( \psi \) dependence in \( ds_3 \) and bring it to the form (4.34). However the \( d\psi \) fibration structure will now change because \( \cos \theta_2 \, d\phi_2 \) changes under (4.33). The change will generically introduce some terms proportional to \( d\theta_2 \) in the \( d\psi \) fibration structure. The precise change will be

\[
cot \theta_2 \, dy \rightarrow \cot \theta_2 \left( \cos \psi \, dy + \sin \psi \, d\theta_2 \right),
\]

(4.38)
where $\bar{\theta}$ is the change in $\theta$ under the transformation (4.33). Now the change (4.38) explicitly introduces the $\psi$ dependence in the fibration structure but removes it from the other parts of the metric. In the delocalized limit, the $\psi$ values are basically constant and therefore can be approximated by constants. This is the only assumption that we will consider at this stage. Under this assumption the $d\theta$ dependent term appearing in the fibration structure (4.38) can be absorbed by a shift in $d\psi$ as $d\psi \to d(\psi - a \ln \sin \theta_2)$, where we have approximated $\bar{\theta}$ by $\theta$ and $a$ is a constant. Under this transformation and in the delocalization limit the $d\psi$ fibration structure does not change too much from its original value. In this way we can recover a simplified form of the deformed conifold metric. Observe that this doesn’t mean that we generate a $U(1)$ isometry in a theory that didn’t have an isometry before transformation. We can only get the metric with $\psi$ isometry in the delocalization limit.

Thus to summarise, we can get rid of the $\psi$ dependences in $ds_3$ by restricting to a specific value of $\psi$, i.e $\psi = \psi_0 \equiv \langle \psi \rangle$. This choice of $\psi$ can be easily obtained from the mirror map (that we are going to discuss in the next section). The mirror can be determined from performing three T-dualities along the $z, x$ and $y$ directions. T-dualities require that we delocalize the $z, x$ and $y$ directions. Since the resolved conifold metric is already independent of these directions, delocalizing simply amounts to setting $\psi = \psi_0 \equiv \langle \psi \rangle$ in (4.33) when we do the transformation. A somewhat related discussion on the transformation of $ds_3^2$ has been given in [63]. Therefore we will consider the specific delocalized limit of the deformed conifold where the $\psi_0$ dependences in $ds_3$ will appear from (4.34) by applying (4.33). Later on we will argue how generic value of $\psi$ can appear in the metric.

For completeness and since we will need these expressions for later comparison we will list the expressions for the vielbeins describing $D6$ wrapped on an $S^3$ of a deformed conifold without any additional six-branes as:

\[
\begin{align*}
e_{\phi_1}^1 &= \sqrt{h_+} \cos \psi \sin \theta_1, & e_{\theta_1}^1 &= -\sqrt{h_+} \sin \psi, & e_{\phi_2}^1 &= \sqrt{h_+} \sin \theta_2 \\
e_{\phi_1}^2 &= \sqrt{h_+} \sin \psi \sin \theta_1, & e_{\theta_1}^2 &= \sqrt{h_+} \cos \psi, & e_{\phi_2}^2 &= \sqrt{h_+} \\
e_{\phi_1}^3 &= \sqrt{h_-} \cos \psi \sin \theta_1, & e_{\theta_1}^3 &= -\sqrt{h_-} \sin \psi, & e_{\phi_2}^3 &= \sqrt{h_-} \sin \theta_2 \\
e_{\phi_1}^4 &= \sqrt{h_-} \sin \psi \sin \theta_1, & e_{\theta_1}^4 &= \sqrt{h_-} \cos \psi, & e_{\phi_2}^4 &= -\sqrt{h_-} \\
e_{\psi}^5 &= \sqrt{h_{\gamma}}, & e_{\phi_1}^5 &= \sqrt{h_{\gamma}} \cos \theta_1, & e_{\phi_2}^5 &= \sqrt{h_{\gamma}} \cos \theta_2, & e_{r}^5 &= \sqrt{h^5}
\end{align*}
\]

where $h_+ = h^\delta + h^\rho$ and $h_- = h^\delta - h^\rho$. Once we express these vielbeins using our local coordinates, many useful properties of the background such as the fundamental form, holomorphic three form etc., can be easily extracted.
5. Chain 1: The Type IIA Mirror Background

In this section we will determine the exact form of the mirror manifold and apply our
generic formulas to the special case $D5$ branes wrapping a resolved conifold in the type
IIB theory. We will find that the manifold is not quite a deformed conifold in the presence
of fluxes as one would have naively expected, rather it will turn out to be a non-K"ahler
manifold that could even be non-complex.

To determine the precise form of the manifold let us first present the metric for $D5$
branes wrapped on an $S^2$ of a resolved conifold. It is given in [20] in the following form

$$
\begin{align*}
\text{ds}^2 = h^{-1/2}ds_{S^{0123}}^2 + h^{1/2}\left[\gamma'/dr^2 + \frac{1}{4}\gamma'/r^2(d\psi + \cos \theta_1d\phi_1 + \cos \theta_2d\phi_2)^2 + \\
+ \frac{1}{4}(\gamma'\theta_1^2 + \sin^2 \theta_1d\phi_1^2) + \frac{1}{4}(\gamma + 4\gamma'^2)(\theta_2^2 + \sin^2 \theta_2d\phi_2^2)\right],
\end{align*}
$$

(5.1)

where we have used the notations of [20] and $\gamma$ is defined as a function of $r^2$ only. The
presence of wrapped $D5$ branes in the metric is signalled by the harmonic function $h$
whose functional form can be extracted from [20]. Observe that the parameter $a$
creates
an asymmetry between the two spheres denoted by $\theta_1, \phi_1$ and $\theta_2, \phi_2$. As discussed in [20],
for small $r$ (the radial coordinate) the $S^3$ denoted by $\psi, \theta_1, \phi_1$ shrinks to zero size whereas
the other sphere remains finite with radius $a$. This is the resolving parameter. As can
be easily seen, when the resolving parameter goes to zero size, the manifold becomes a
conifold, and the metric works out correctly. Furthermore, the curvature remains regular
all through. Notice also that the metric (5.1) has three isometries related to constant
shifts in $\psi, \phi_1$ and $\phi_2$ as $\psi \to \psi + c_1, \phi_1 \to \phi_1 + c_2, \phi_2 \to \phi_2 + c_3$. But there are no
isometries along $r, \theta_1$ and $\theta_2$ directions because of the warp factors and the $d\psi$
fibration structure. Therefore there is a natural $T^3$ structure associated with $\psi, \phi_1, \phi_2$ directions.
This $T^3$ could be a special lagrangian submanifold if the global metric of [20] preserve
supersymmetry (see also [64]). A direct way to see this would be to evaluate the condition
required for a cycle to be lagrangian. Alternatively, one can see that the $\phi_1, \phi_2$ directions
lead to a brane-box configuration after two T-dualities [64]. This configuration preserves
susy when the size of the resolution circle is zero. Furthermore, there a T-duality along
$\psi$ direction has been shown earlier to lead to a susy preserving configuration [65]. Thus

\footnote{We have also defined the radial coordinate $r$ and $\gamma'$ in the following way: $\gamma' \equiv \frac{d\gamma}{dr^2} = \frac{2}{3}\frac{\gamma + 6a^2}{\gamma + 4a^2}$ and the Ricci flatness condition gives rise to the equality $r = \sqrt{\gamma\sqrt{\gamma + 6a^2}}$.}
ψ, φ₁, φ₂ lead to a lagrangian submanifold that preserves susy after three T-dualities in the conifold limit. Unfortunately the global metric of [20] do not preserve susy [58], but our M-theory configuration does. In sec. 3 we showed that the fourfold would preserve susy with primitive fluxes. We took a fourfold that locally looks like a product manifold of a $T^2$ fiber and a resolved conifold base. It is now clear [59] that globally the manifold will preserve susy when the fiber degenerates over the base. This means that we would require seven branes, and the base will be Kähler instead of a Calabi-Yau manifold. The product structure of sec. 3 should then be regarded as though we have moved the seven branes far away. In fact this is exactly how we can study pure $\mathcal{N} = 1$ $SU(N)$ gauge theory! This way one might expect the three cycle $(\psi, \phi_1, \phi_2)$ would form a Lagrangian submanifold on which we can do T-dualities. However, as we will see below, the T-duality directions are not the naively expected isometry directions. The T-dualities in this scenario are a little subtle as we now elaborate.

To begin, we need to first convert $(\psi, \phi_1, \phi_2)$ into suitable coordinate system by which we can express our local metric and also perform the mirror map, as the original coordinate choice do not suffice (see our recent work [59] for details). Since the global metric of [20] break susy, we can only trust the local metric; and then add seven branes to make the system supersymmetric. The full global picture is now understood in [59]. Thus to write the local metric, we can use the T-duality coordinates $(z, x, y)$ that we referred to in the previous section. Therefore we shall use the following mapping to relate the above metric to the one presented earlier

\[
(x, y, z) \to (\phi_1, \phi_2, \psi),
\]

\[
(dx, dy, dz) = \left( \frac{1}{2} \sqrt{h^{1/2} \gamma} \sin \langle \theta_1 \rangle \ d\phi_1, \frac{1}{2} \sqrt{h^{1/2} (\gamma + 4a^2)} \sin \langle \theta_2 \rangle \ d\phi_2, \frac{1}{2} r_0 \sqrt{\gamma' h^{1/2}} \ d\psi \right),
\]

where we have picked a point $(r_0, \langle \theta_i \rangle, \langle \phi_i \rangle, \psi_0)$ to define (5.2). To avoid clutter $\gamma \equiv \gamma(r_0), \gamma' \equiv \gamma'(r_0), h \equiv h(r_0)$ henceforth unless mentioned otherwise. The physical meaning of $x, y, z$ can be given as follows: under a single T-duality along $\psi$, the system maps to an intersecting brane configuration [21], [22], [63]; $x, y$ and $z$ form the coordinates of the branes. More precisely, we are in fact converting the two spheres with coordinates $(\phi_1, \theta_1)$ and $(\phi_2, \theta_2)$ to tori with coordinates $(x, \theta_1)$ and $(y, \theta_2)$ respectively. Recall that a sphere is topologically the same as a tori with a degenerating cycle (i.e. if we shrink one of the cycles of the $T^2$ to zero size then this would be topologically the same as a sphere) and therefore this mapping would be locally indistinguishable (but will not have the full [0,
Furthermore, this mapping will be particularly useful to perform many simplifying manipulations later in the paper which are otherwise difficult in the absence of the full global metric. The local metric will become [53]:

\[ ds^2 = dr^2 + \left( dz + \sqrt{\frac{\gamma'}{\gamma}} r_0 \cot \theta_1 \, dx + \sqrt{\frac{\gamma'}{\gamma + 4a^2}} r_0 \cot \theta_2 \, dy \right)^2 + \]
\[ + \left[ \frac{\gamma h}{4} \, d\theta_1^2 + dx^2 \right] + \left[ \frac{(\gamma + a^2)h}{4} \, d\theta_2^2 + dy^2 \right] + \ldots \]  

where we see that the two tori are square tori. We have to soon modify this further, but before that let us find the effect of the seven branes in this scenario. Locally, if we keep the seven branes very far away, then the metric will be (5.3). When the seven branes are somewhat nearby, but still far away so that we can study pure \( \mathcal{N} = 1 \) SYM, we can keep the radial direction delocalized, but \( \theta_i \) arbitrary, i.e \((r_0, \langle \theta_i \rangle) \to (r_0, \theta_i)\) in (5.3). With this map, we can now write the various components of the wrapped D5 metric:

\[ j_{zz} = 1, \quad j_{xx} = 1 + \frac{\gamma'}{\gamma} r_0^2 \cot^2 \theta_1, \]
\[ j_{yy} = 1 + \frac{\gamma'}{\gamma + 4a^2} r_0^2 \cot^2 \theta_2, \]
\[ j_{zx} = \sqrt{\frac{\gamma'}{\gamma}} r_0 \cot \theta_1, \quad j_{zy} = \sqrt{\frac{\gamma'}{\gamma + 4a^2}} r_0 \cot \theta_2, \]
\[ j_{xy} = \frac{\gamma'}{\sqrt{\gamma(\gamma + 4a^2)}} r_0^2 \cot \theta_1 \cot \theta_2, \quad j_{rr} = \gamma' h^{1/2}, \]
\[ j_{\theta_1 \theta_1} = \frac{1}{4} \gamma h^{-1/2}, \quad j_{\theta_2 \theta_2} = \frac{1}{4} h^{1/2}(\gamma + 4a^2) \]  

with the rest of the components zero. The \( B_{NS} \) fields on the other hand have the following components (see also sec. 4 of [20]):

\[ b = J_1 \, d\theta_1 \wedge dx + J_2 \, d\theta_2 \wedge dy \]  

with the rest of the components zero and \( J_i \) are now functions of the radial and the angular coordinates \( \textit{globally} \), i.e \((r, \theta_1, \theta_2)\) although we have used local coordinate differentials to write \( b \) over a given coordinate patch. In [20] the \( B \) field was only function of the radial coordinates. Here since we converted all the spheres in the metric to tori, we will keep \( B \) as a generic function of \((r, \theta_1, \theta_2)\) to preserve supersymmetry globally. Notice also that
the choice of the $B$ field and the metric is consistent with the assumptions that we made in the previous section, namely $b_{xy} = b_{yz} = b_{zx} = 0$ and the cross term $j_{(x,y,z)\mu} = 0$. The small and the large radius behavior of $\gamma$ is \[20\]:

$$\gamma_{r_0 \to 0} = \frac{1}{\sqrt{6a}} r_0^2 - \frac{1}{72a^4} r_0^4 + {\mathcal{O}}(r_0^6), \quad \gamma_{r_0 \to \infty} = r_0^{4/3} - 2a^2 + {\mathcal{O}}(r_0^{-4/3}) \tag{5.6}$$

and so we can use the first relation to define $\gamma(r_0)$. In the above set of components (5.4), if we ignore the overall $h^{1/2}$ dependences, observe that for small $r$ (which we will concentrate on mostly) $\gamma$ is a small quantity, and thus terms like $j$ (which we will encounter soon) could be expanded in powers of $\gamma$ (or $r$), because the $\theta_i$ dependences can be made generically small. We will however try to avoid making approximations and concentrate on the exact values as far as possible.

Before moving ahead one comment is in order. The metric of $D5$ wrapping an $S^2$ of a resolved conifold has no $j_{\theta_1 \theta_2}$ component, i.e. no $d\theta_1 d\theta_2$ cross term. However, our anticipation will be to have such a cross term in the mirror, see e.g. (4.32). We know that T-dualities cannot generate such terms (in the absence of $B$ fields). In the presence of $B$ fields, as we show below, cross term of the form $d\theta_1 d\theta_2$ do get generated. However these cross terms combine together with $dx$ and $dy$ terms (as will become obvious soon) and therefore do not generate the single $d\theta_1 d\theta_2$ term. We will discuss a way to generate this later.

The expected mirror manifold will have the following form of the metric (4.19):

$$ds^2 = \frac{1}{G_{zz}} (dz + B_{\mu z} dx^\mu + B_{xz} dx + B_{yz} dy)^2 + G_{\mu \nu} dx^\mu dx^\nu + 2G_{x\nu} dy dx^\nu + 2G_{y\nu} ddy dx^\nu + 2G_{xy} dxdy + G_{xx} dx^2 + G_{yy} dy^2. \tag{5.7}$$

The $dz$ fibration structure is more or less consistent in form, so lets check whether the components work out fine. By denoting

$$\alpha^{-1} = j_{xx} j_{yy} - j_{xy}^2 + b_{xx}^2 = j_{xx} j_{yy} - j_{xy}^2, \tag{5.8}$$

we write:

$$B_{xz} = -\alpha j_{xz} = -\sqrt{\frac{\gamma'}{\gamma}} \alpha r_0 \cot \theta_1$$

$$B_{yz} = -\alpha j_{yz} = -\sqrt{\frac{\gamma'}{\gamma + 4a^2}} \alpha r_0 \cot \theta_2 \tag{5.9}$$

$$B_{\mu z} = b_{\mu z} + \alpha (b_{x\mu} j_{xz} + b_{y\mu} j_{yz}).$$
This can combine together with (5.9) to give the following fibration structure:

\[
(dz - b_{z\mu} \, dx^\mu) - \alpha \, j_{xz}(dx - b_{x\theta_1} \, d\theta_1) - \alpha \, j_{yz}(dy - b_{y\theta_2} \, d\theta_2)
\]  

(5.10)

where we have kept the \( B \) field component \( b_{z\mu} \). The above form of the fibration is highly encouraging because it looks similar to (4.36). And since \( \alpha = 1 + \) higher orders, up to those terms we seem to be getting the fibration structure in somewhat expected form.

Let us now look at other terms.

\[
G_{xx} = \alpha \, j_{yy}, \quad G_{yy} = \alpha \, j_{xx} \\
G_{\mu\nu} = j_{\mu\nu} + \alpha(j_{yy} \, b_{x\mu} \, b_{x\nu} + j_{xx} \, b_{y\mu} \, b_{y\nu} - j_{xy}(b_{y\mu} \, b_{x\nu} + b_{x\mu} \, b_{y\nu}))
\]

(5.11)

The existence of cross terms in the above formula is very important. This tells us that we can have components like \( G_{\theta_1 \theta_2} \). Such terms do exist in the usual deformed conifold metric and are absent in the resolved conifold metric. The other terms will be:

\[
G_{xy} = \frac{-j_{xy}}{j_{yy}j_{xx} - j_{xy}^2 + b_{xy}^2} = -\alpha \, j_{xy} \\
G_{x\mu} = \alpha(j_{yy} \, b_{\mu x} + j_{xx} b_{y\mu}), \quad G_{y\mu} = \alpha(j_{xx} b_{\mu y} + j_{xy} b_{x\mu})
\]

(5.12)

Again, the existence of cross term is important, they will give rise to components like \( G_{x\theta_2} \) and \( G_{y\theta_1} \). Such terms are not present in the resolved conifold setting, but do exist in the deformed conifold metric! Finally there is the zz component

\[
G_{zz} = \alpha
\]

(5.13)

The above term is again of order one. Let us furthermore introduce the shorthand notation

\[
j_{xz} = A = \Delta_1 \cot \theta_1, \quad j_{yz} = B = \Delta_2 \cot \theta_2,
\]

(5.14)

with \( \Delta_1(r_0) \) and \( \Delta_2(r_0) \) being warp factors. Now combining everything together we get the following mirror manifold:

\[
ds^2 = g_1 \left[ (dz - b_{z\mu} \, dx^\mu) - \alpha \, \Delta_1 \cot \theta_1 (dx - b_{x\theta_1} \, d\theta_1) - \alpha \, \Delta_2 \cot \theta_2 (dy - b_{y\theta_2} \, d\theta_2) + .. \right]^2 + \\
+ g_2 \, d\theta_1^2 + g_3 \, d\theta_2^2 + g_4 \, (dx - b_{x\theta_1} \, d\theta_1)^2 + \\
+ g_5 \, (dy - b_{y\theta_2} \, d\theta_2)^2 - g_7 \, (dx - b_{x\theta_1} \, d\theta_1)(dy - b_{y\theta_2} \, d\theta_2)
\]

(5.15)
where $g_i \equiv g_i(r = r_0, \theta_1, \theta_2)$ are some functions of $r_0, \theta_1, \theta_2$ coordinates and can be easily determined from our analysis above. They are given as

$$
\begin{align*}
&g_1 = \alpha^{-1}, \\
&g_2 = \frac{\gamma \sqrt{h}}{4}, \\
&g_3 = \frac{(\gamma + 4a^2) \sqrt{h}}{4}, \\
&g_4 = \alpha j_{yy}, \\
&g_5 = \alpha j_{xx}, \\
&g_7 = 2\alpha j_{xy}.
\end{align*}
\tag{5.16}
$$

At this point let us compare our metric (5.15) to the metric of the wrapped D6-branes on $S^3$ of a deformed conifold. The generic form of that metric is given by (we take the delocalized metric in (4.32))

$$
\begin{align*}
ds^2 &= \tilde{g}_1 (dz + \tilde{\Delta}_1 \cot \theta_1 \, dx + \tilde{\Delta}_2 \cot \theta_2 \, dy + \ldots)^2 + \\
&\quad \tilde{g}_2 [d\theta_1^2 + dx^2] + \tilde{g}_3 [d\theta_2^2 + dy^2] + \tilde{g}_4 [d\theta_1 \, d\theta_2 - dx \, dy]
\end{align*}
\tag{5.17}
$$

where $\tilde{g}_i$ are again some functions of $r_0, \theta_1, \theta_2$ that could be easily evaluated. Let us now compare the two metrics:

- As a general rule, everywhere where we would expect $dx$ or $dy$, they are replaced in (5.15) by the appropriate $B$-dependent fibration structure $(dx - b_x \theta_1 \, d\theta_1)$ or $(dy - b_y \theta_2 \, d\theta_2)$, respectively. In fact, this non–trivial fibration will be responsible for making the manifold (5.15) a non-Kähler space, as we will show later. If we define $\tilde{d}x = dx - b_x \theta_1 \, d\theta_1$ and $\tilde{d}y = dy - b_y \theta_2 \, d\theta_2$ for constant $b_x \theta_1$ and $b_y \theta_2$, we find agreement between (5.15) and (5.17) in all terms involving $dx$ and $dy$, up to differing warp factors.
- We also see that the $d\theta_1 \, d\theta_2$ cross term is now entirely absorbed in the fibration structure and there is no single $d\theta_1 \, d\theta_2$ term in (5.15). Whatever $d\theta_1 \, d\theta_2$ terms are generated actually combine with $dx$ and $dy$ to give us the $B$-dependent fibration, and therefore no extra $d\theta_1 \, d\theta_2$ term appears.
- Apart from the $dx$– and $dy$–fibration mentioned above the $dz$ fibration structure of both the metrics have similar form modulo some warp factors and relative signs. The relative signs between the $dx, dy$ terms in (5.15) and (5.17) can be fixed if we fix $B_{yz}, B_{zx}$ and $B_{xy}$ in (4.14), (4.16), and (4.17) as minus of themselves. In the following we will assume that we have fixed the signs of the $B$ fields. This way the fibration structures of (5.15) and (5.17) would tally.
- From the transformation (4.33) we expect the coefficients $g_3$ and $g_5$ to be the same. But a careful analysis (5.16) shows that they are in fact different. The coefficients $g_2$ and $g_4$ are also different, which could in principle be because (4.33) do not act on them. But $g_3, g_5$ should be the same if we hope to recover the deformed conifold scenario.
In the following we will try to argue a possible way to generate the \( \theta \) cross terms. We will see that the T-duality directions are slightly different from the naively expected directions (which we called \( x, y, z \)). In the process we will also fix the coefficients \( g_i \). We begin with the search for the \( d\theta_1 \, d\theta_2 \) term in the metric.

5.1. Searching for the \( d\theta_1 d\theta_2 \) term

The absence of the \( d\theta_1 d\theta_2 \) term in (5.15) is a near miss. As one can see that the metric that we get in (5.15) is almost the metric of a deformed conifold (in the delocalized limit) when we switch off the \( B \) fields except that we are missing the \( d\theta_1 d\theta_2 \) term. Furthermore this term should come in the metric with the precise coefficient \( \alpha j_{xy} \).

The absence of this term however raises some doubts about the directions that we made our T-dualities. But since we almost reproduced the correct form of the metric, we cannot be too far from the right choice of the isometry directions. Now whatever new isometry directions we choose in the resolved conifold side should keep the present form of the mirror metric intact. This is a strong restriction because we cannot change the \( dz, dx \) and \( dy \) fibration structure any more as they are already in the expected format. The only things that we could fiddle with are the \( d\theta_i \) terms in the mirror side. Therefore the question is: what changes in the resolved side are we allowed to perform that would only affect the \( d\theta_i \) parts of the mirror manifold?

An immediate guess would be to change the \( \theta_i \) terms so as to generate a \( j_{\theta_1 \theta_2} \) directly in the resolved conifold setup in type IIB theory (5.1). A way to get this would be to go to a new coordinate system given by:

\[
\theta_1 \rightarrow \theta_1 + \gamma \theta_2, \quad \theta_2 \rightarrow \theta_2 + \beta \theta_2,
\]

where \( \gamma, \beta \) are small integers. This will give us the necessary cross term and will change the other terms to

\[
cot \theta_1 \, dx \rightarrow (\gamma \theta_2 + \cot \theta_1 + ...) \, dx, \quad dx \rightarrow (1 + \gamma \theta_2 \cot \theta_1 + ...) \, dx,
\]

and similar changes to the \( dy \) terms. In other words the warp factors in front of the \( dx, dy \) terms will change and the metric will have a \( j_{\theta_1 \theta_2} \) term.

---

14 We have been a little sloppy here. By resolved conifold we will always mean \( D5 \) wrapped on \( T^2 \) of our local geometry unless mentioned otherwise.
Making a mirror transformation now to the resolved conifold metric will generate the requisite \( d\theta_1 d\theta_2 \) term term, but the coefficient of this is an arbitrary number. Therefore will not explain the \( \alpha j_{xy} \) coefficient that we require. Instead of this, we can perform the following infinitesimal rotation on the sphere coordinates of the type IIB metric (5.1):

\[
\begin{pmatrix}
    dx \\
    d\theta_1
\end{pmatrix} \rightarrow \begin{pmatrix}
    1 & \epsilon_1 \\
    -\epsilon_1 & 1
\end{pmatrix}
\begin{pmatrix}
    dx \\
    d\theta_1
\end{pmatrix}, \quad \begin{pmatrix}
    dy \\
    d\theta_2
\end{pmatrix} \rightarrow \begin{pmatrix}
    1 & \epsilon_2 \\
    -\epsilon_2 & 1
\end{pmatrix}
\begin{pmatrix}
    dy \\
    d\theta_2
\end{pmatrix}.
\] (5.20)

This will keep the metric of the two tori \((\theta_1, x)\) and \((\theta_2, y)\) invariant, but will generate \( j_{z\mu}, j_{x\mu}, j_{y\mu} \) and \( j_{\mu\nu} \) components. Interestingly, one can easily verify from the T-duality rules that this changes only the \( G_{\mu\nu} \) and \( G_{z\mu} \) components with all other metric components \( G_{mn} \) invariant. The change in \( G_{\mu\nu} \) can be written as

\[
G_{\mu\nu} = j_{\mu\nu} + \alpha [j_{yy} b_{x\mu} b_{x\nu} + j_{xx} b_{y\mu} b_{y\nu} - j_{xy} (b_{y\mu} b_{x\nu} + b_{x\mu} b_{y\nu})] + \\
- \alpha [j_{yy} j_{x\mu} j_{x\nu} + j_{xx} j_{y\mu} j_{y\nu} - j_{xy} (j_{y\mu} j_{x\nu} + j_{x\mu} j_{y\nu})].
\] (5.21)

The second line is by replacing \( b \leftrightarrow j \). Observe that only the second line in (5.21), which we shall call \( G_{\mu\nu}^{\text{new}} \), introduces new components in the metric. Similarly, \( G_{z\mu} \) can be written as:

\[
G_{\mu z} = j_{\mu z} - \alpha [j_{yy} j_{x\mu} j_{xz} + j_{xx} j_{y\mu} j_{yz} - j_{xy} (j_{y\mu} j_{xz} + j_{x\mu} j_{yz})].
\] (5.22)

Both these components will contribute to \( d\theta_1^2, d\theta_2^2 \) and \( d\theta_1 d\theta_2 \) (by keeping only the \( j \) components) in addition to the terms that we already have in (5.15), as:

\[
ds^2 \rightarrow ds^2 + G_{\mu\nu}^{\text{new}} dx^\mu dx^\nu - \frac{G_{z\mu} G_{z\nu}}{G_{z z}} dx^\mu dx^\nu.
\] (5.23)

This is almost what we might require, but again the coefficient is arbitrary. And there seems no compelling reason for a particular coefficient to show up in the metric. Both the above analysis have failed to provide a specific reason for the \( \alpha j_{xy} \) coefficient in the metric. Therefore it is time now to look at other possibilities as we have exploited the transformations on \( x, y \) and \( \theta_i \), but haven’t yet considered the possibilities of a \( z \) transformation. Can we change the \( dz \) terms without spoiling the consistency of the mirror manifold?

A little thought will tell us that the allowed changes should be done directly to the line element, so that we can define distances properly in both type IIB as well as the mirror type IIA. We can start defining some transformation on \( x, y, z \) and \( \theta_i \) using some (as yet) unknown functions. However, this procedure is rather involved, because eventually we
have to determine $dx, dy, dz$ and $d\theta_i$ thereby giving rise to set of PDE’s. Therefore it will be easier if we make transformations directly on $dx, dy, dz$ and $d\theta_i$. Transformations on the infinitesimal shifts are, on the other hand, not always integrable. Therefore to avoid this problem, let us start by making transformations on finite shifts $\delta x, \delta y, \delta z, \delta \theta_i$. We will later integrate these expressions to get the transformations on the coordinates $x, y, z$ and $\theta_i$. Furthermore as we are at a fixed point in the radial direction, i.e $r_0$, we will consider shifts only for a fixed $r$.

For finite shifts $\delta x, \delta y, \delta z, \delta \theta_i$ of the coordinates of the resolved conifold a typical distance $d$ on the resolved conifold can be written in terms of distances $d_1, d_2$ on the two tori with coordinates $(x, \theta_1)$ and $(y, \theta_2)$ as:

$$\|d\| = \sqrt{d_1^2 + d_2^2 + (\delta z + \Delta_1 \cot \theta_1 \delta x + \Delta_2 \cot \theta_2 \delta y)^2}$$

(5.24)

where we have ignored the contributions from the $dr^2$ term in (5.3) just for simplicity. We have also defined the distances along the two tori as:

$$d_1 = \frac{1}{2} \sqrt{\gamma \sqrt{h} (\delta \theta_1)^2 + 4(\delta x)^2}, \quad d_2 = \frac{1}{2} \sqrt{(\gamma + 4a^2) \sqrt{h} (\delta \theta_2)^2 + 4(\delta y)^2}.$$  

(5.25)

Now the allowed change in the resolved side that would affect only the $\delta \theta_i$ parts will be to change $\delta z$ to

$$\delta z \rightarrow \delta z + \rho_1 \delta \theta_1 + \rho_2 \delta \theta_2.$$  

(5.26)

This typically means that we are slanting the $z$ direction along the $\theta_i$ directions. This would convert the line element in (5.24) to

$$\|d\|^2 = d_1^2 + d_2^2 + \left[\delta \tilde{z} + \sum_{i=1}^{2} (\Delta_i \cot \theta_i \delta x_i + \rho_i \delta \theta_i)\right]^2$$

(5.27)

with $\tilde{z}$ being the new $z$ direction and $\rho_i$ are generic functions of $(r = r_0, \theta_i)$. We have also defined $x_1 = x$ and $x_2 = y$. Now without a loss of generality we can write the $\rho_i$’s as:

\footnote{This can be motivated as follows. The generic shift between two nearby points can be written as: $z_{(1)} - z_{(2)} = (\tilde{z}_{(1)} - \tilde{z}_{(2)}) + (\rho_{1(1)} \theta_{1(1)} - \rho_{1(2)} \theta_{1(2)}) + (\rho_{2(1)} \rho_{2(1)} - \alpha_{2(2)} \theta_{2(2)}) \theta_{2(2)}).$ Now defining $\rho_{j(1)} \theta_{j(1)} - \rho_{j(2)} \theta_{j(2)} = \rho_{j} (\theta_{j(1)} - \theta_{j(2)})$, we get the corresponding equation. Also since $\delta \theta_i = \theta_{i(1)} - \theta_{i(2)}$, there is no problem with the periodicity here. This will be clear later when we integrate these shifts and write the final result in terms of $\cos \theta_i$ and $\sin \theta_i$, which are periodic variables.}
\[ \rho_1 = f_1 \Delta_1 \cot \theta_1 \text{ and } \rho_2 = f_2 \Delta_2 \cot \theta_2 \] where \( f_i \) are now generic functions of \((r = r_0, \theta_i)\) that we have to determine. Using this, the line element will take the final form

\[ \| d \|^2 = d_1^2 + d_2^2 + \left[ \delta\tilde{z} + \sum_{i=1}^{2} (\Delta_i \cot \theta_1 \delta x_i + f_i \Delta_i \cot \theta_1 \delta \theta_i) \right]^2 \quad (5.28) \]

The above changes could also be viewed as having generated the following new components of the metric in the resolved side: \( \tilde{j}_{\tilde{z}\theta_1}, \tilde{j}_{\tilde{z}\theta_2}, \tilde{j}_{x\theta_1}, \tilde{j}_{y\theta_2} \) and \( \tilde{j}_{\theta_1\theta_2} \). As we discussed before these components will only change the \( \delta \theta_i \) part of the metric, i.e. the \((\delta \theta_1)^2 + (\delta \theta_2)^2\) part and will keep all the fibrations in the mirror picture intact. The changes in the mirror metric can therefore be calculated from (5.23).

Until now the arguments have been more or less parallel to the arguments that we provided earlier for the changes in the \( \delta x, \delta y \) or \( \delta \theta_i \) terms. However, as we show below, the changes in the \( \delta z \) part actually allows us to fix the form of the functions \( f_i \). To see how this is possible, make the following changes in \( \delta x, \delta y \) coordinates:

\[ \delta x \rightarrow \delta x - f_1 \delta \theta_1, \quad \delta y \rightarrow \delta y - f_2 \delta \theta_2. \quad (5.29) \]

The effect of this change is rather immediately obvious: it removes the effect of the changes made earlier by the \( \delta z \) transformation. However this change in \( \delta x, \delta y \) can also be assumed as though \( \tilde{j}_{x\theta_1} \) and \( \tilde{j}_{y\theta_2} \) cross terms have been added in the metric. Taking into account all the above changes, and also allowing a possible finite shift along the radial direction \( \delta r \), the line element on the resolved conifold will take the following form:

\[ \| d \|^2 = h^{1/2} \gamma'(\delta r)^2 + (\delta\tilde{z} + \Delta_1 \cot \theta_1 \delta x + \Delta_2 \cot \theta_2 \delta y)^2 + (\delta\tilde{x})^2 + (\delta\tilde{y})^2 + \\
+ \frac{1}{4} (\gamma \sqrt{h} + 4f_1^2) (\delta \theta_1)^2 - 2f_1 \delta\tilde{x} \delta \theta_1 + \frac{1}{4} (\gamma \sqrt{h} + 4a^2 \sqrt{h} + 4f_2^2) (\delta \theta_2)^2 - 2f_2 \delta\tilde{y} \delta \theta_2. \quad (5.30) \]

Observe that in the above line element cross components have developed and the distances along the \( \theta_i \) directions have changed by warp factors. Both these changes are given in terms of the unknown functions \( f_i \) (to be determined soon).

It is now important to consider some special limits of the functions \( f_i \), as one can easily show that for small and finite values of \( f_1 \) and \( f_2 \), a \( \delta \theta_1 \delta \theta_2 \) term does not get generated in

\[ \text{The physical meaning of } f_i \text{ will be discussed in the next subsection. For the time being we will view } f_i \text{ as being a consequence of generic coordinate transformations, whose integral form will be presented later in this section.} \]
the mirror picture. Assuming that the \( f_i \) are large, one could use a special regularization scheme that would generate this term. The coordinates \( \tilde{z}, \tilde{x}_i, \theta_i \) are now the coordinates in which the line element of the mirror manifold is in a known format. Therefore the above transformation from \((z, x, y) \rightarrow (\tilde{z}, \tilde{x}, \tilde{y})\) for the resolved conifold means that we are writing the line element in the coordinates of the mirror. To simplify the ensuing calculations, let us use the notation introduced in (5.14). With this definition, we can re-express the line element (5.30) as new components for the resolved conifold metric. The various components can now be written as:

\[
\begin{align*}
    j_{\tilde{z}\tilde{z}} &= 1 + A^2, \quad j_{\tilde{y}\tilde{y}} = 1 + B^2, \quad j_{\tilde{z}\theta_1} = -f_1, \quad j_{\tilde{y}\theta_2} = -f_2 \\
    j_{\theta_1\theta_1} &= \frac{1}{4}(4f_1^2 + \gamma \sqrt{h}), \quad j_{\theta_2\theta_2} = \frac{1}{4}(4f_2^2 + \gamma \sqrt{h} + 4a^2 \sqrt{h}) \\
    j_{\tilde{x}\tilde{x}} &= A, \quad j_{\tilde{y}\tilde{z}} = B, \quad j_{\tilde{z}\tilde{z}} = 1 - \epsilon, \quad j_{\tilde{x}\tilde{y}} = AB
\end{align*}
\]  

(5.31)

with the radial and the spacetime components remaining the same as earlier. Observe that we have shifted the \( j_{\tilde{z}\tilde{z}} \) component by a small amount \( \epsilon \) in (5.31). Letting \( \epsilon \rightarrow 0 \) and \( f_i \rightarrow \infty \) will result in a finite \( d\theta_1 d\theta_2 \) term in the mirror metric. This is our regularization scheme, so to speak. Ergo, (5.31) is the correct IIB starting metric, which we T–dualize along \( \tilde{x}, \tilde{y} \) and \( \tilde{z} \) to obtain the mirror manifold. It can also be easily verified that both the \( B_{NS} \) and the \( H_{RR} \) (given in the next section) remain completely unchanged in forms because of their wedge structures and antisymmetrisations. The only change there is that now everything is written by tilde-coordinates.

Let us now see the possible additional metric components that we can get after we make a mirror transformation. From the T-duality rules we see that the new components are:

\[
\begin{align*}
    G_{\tilde{z}\theta_1} &= -\alpha j_{\tilde{z}\theta_1} [j_{\tilde{z}\tilde{z}} j_{\tilde{y}\tilde{y}} - j_{\tilde{z}\tilde{y}} j_{\tilde{y}\tilde{z}}] = \alpha \ f_1 \ A \\
    G_{\tilde{z}\theta_2} &= -\alpha j_{\tilde{y}\theta_2} [j_{\tilde{y}\tilde{z}} j_{\tilde{x}\tilde{z}} - j_{\tilde{y}\tilde{y}} j_{\tilde{x}\tilde{y}}] = \alpha \ f_2 \ B \\
    G_{\theta_1\theta_1} &= j_{\theta_1\theta_1} + \alpha j_{\tilde{y}\theta_1} b_{\tilde{z}\theta_1}^2 - j_{\tilde{z}\theta_1}^2 = \frac{\gamma \sqrt{h}}{4} + \alpha \ (1 + B^2) \ b_{\tilde{z}\theta_1}^2 + \alpha \ f_1^2 A^2 \\
    G_{\theta_2\theta_2} &= j_{\theta_2\theta_2} + \alpha j_{\tilde{y}\theta_2} b_{\tilde{y}\theta_2}^2 - j_{\tilde{z}\theta_2}^2 = \frac{(\gamma + 4a^2) \sqrt{h}}{4} + \alpha \ (1 + A^2) \ b_{\tilde{y}\theta_2}^2 + \alpha \ f_2^2 B^2 \\
    G_{\theta_1\theta_2} &= -\alpha j_{\tilde{z}\tilde{y}} [b_{\tilde{z}\theta_1} b_{\tilde{y}\theta_2} - j_{\tilde{z}\theta_1} j_{\tilde{y}\theta_2}] = -\alpha \ AB \ b_{\tilde{z}\theta_1} b_{\tilde{y}\theta_2} + \alpha \ f_1 f_2 \ AB \\
    G_{\tilde{z}\tilde{z}} &= \alpha (j_{\tilde{x}\tilde{x}} j_{\tilde{y}\tilde{y}} j_{\tilde{z}\tilde{z}} - j_{\tilde{x}\tilde{z}} j_{\tilde{y}\tilde{z}}^2 - j_{\tilde{y}\tilde{z}} j_{\tilde{x}\tilde{z}}^2 - j_{\tilde{z}\tilde{z}} j_{\tilde{z}\tilde{y}}^2 + 2 j_{\tilde{x}\tilde{y}} j_{\tilde{y}\tilde{z}} j_{\tilde{z}\tilde{x}}) = \alpha - \epsilon.
\end{align*}
\]  

(5.32)

The \( B \) field dependent terms in (5.32) would reorganize themselves according to the fibration structure that we discussed earlier. What remains now is to see whether the
additional terms (which depend on $f_i$) can be used effectively. The distance along the new $\theta_1\theta_2$ directions will be (5.23):

$$ds^2_{\theta_1,\theta_2} = 2 \left( G^{\text{new}}_{\theta_1,\theta_2} - \frac{G_{\tilde{z}\theta_1}G_{\tilde{z}\theta_2}}{G_{\tilde{z}\tilde{z}}} \right) \delta \theta_1 \delta \theta_2$$

$$= - 2\alpha f_1 f_2 j_{\tilde{x}\tilde{y}} \left[ \frac{\epsilon}{\alpha - \epsilon} \right] \delta \theta_1 \delta \theta_2 = - 2f_1 f_2 j_{\tilde{x}\tilde{y}} \epsilon \delta \theta_1 \delta \theta_2. \quad (5.33)$$

At this point we can use our freedom to define the functions $f_1$ and $f_2$. As discussed earlier, we see that for finite values of the functions $f_i$ the above metric component is identically zero in the limit $\epsilon \to 0$. However if we define $f_i \equiv \epsilon^{-1/2} \beta_i$ such that $\beta_1 \beta_2 = -\alpha$, then (5.33) implies

$$ds^2_{\theta_1,\theta_2} = 2\alpha j_{\tilde{x}\tilde{y}} \delta \theta_1 \delta \theta_2 \quad (5.34)$$

which is what we require. In this way we recover the elusive $\theta_1\theta_2$ component of the metric.

At this point one might ask whether different choices for $f_1f_2$ could be entertained. From the mirror metric (5.15) we observed that — in the absence of $B$-fields — the metric resembles the deformed conifold in the delocalized limit (of course with the $d\theta_1d\theta_2$ absent). When we restore back this term (via the above analysis) we should recover the exact deformed conifold setup. This is possible, if in the absence of $B$ fields, the product $f_1f_2$ is proportional to $\alpha$. Therefore, in the presence of fluxes, we believe that this will continue to hold.

To show that the choice of $\beta_1 \beta_2 = -\alpha$ is consistent, we have to determine $f_i$ individually. To see this let us first bring the metric for the $\theta^2$ terms in (5.32) into a more canonical form:

$$\delta \theta_1 \to \frac{2}{\sqrt{h^{1/2} \gamma}} \delta \tilde{\theta}_1, \quad \delta \theta_2 \to \frac{2}{\sqrt{h^{1/2} (\gamma + 4a^2)}} \delta \tilde{\theta}_2. \quad (5.35)$$

Assuming now that the above equation can be integrated to give a relation between $\theta_i$ and $\tilde{\theta}_i$, the change in all terms with a $b$–fibration can easily be absorbed in (5.5) as:

$$b = \frac{2J_1}{\sqrt{h^{1/2} \gamma}} d\tilde{\theta}_1 \wedge d\tilde{x} + \frac{2J_2}{\sqrt{h^{1/2} (\gamma + 4a^2)}} d\tilde{\theta}_2 \wedge d\tilde{y}$$

$$= \tilde{b}_{\theta_1x} d\tilde{\theta}_1 \wedge d\tilde{x} + \tilde{b}_{\theta_2y} d\tilde{\theta}_2 \wedge d\tilde{y}, \quad (5.36)$$
where we have taken the infinitesimal limit to write \( b \) with one forms \( d\theta_i \). The relation (5.36) implies for the \( \theta \) dependent metric components:

\[
G_{\tilde{\theta}_1 \tilde{\theta}_2} = -\alpha AB \tilde{b}_{x\theta_1} \tilde{b}_{y\theta_2} + \alpha \frac{4AB f_1 f_2}{h^{1/2} \sqrt{\gamma(\gamma + 4a^2)}}
\]

\[
G_{\tilde{\theta}_1 \tilde{\theta}_1} = 1 + \alpha(1 + B^2) \tilde{b}_{x\theta_1}^2 + \alpha \frac{4}{h^{1/2} \gamma} f_1^2 A^2,
\]

\[
G_{\tilde{\theta}_2 \tilde{\theta}_2} = 1 + \alpha(1 + A^2) \tilde{b}_{y\theta_2}^2 + \alpha \frac{4}{h^{1/2}(\gamma + 4a^2)} f_2^2 B^2.
\]

Note, that this changes (5.33) to

\[
ds_{\tilde{\theta}_1 \tilde{\theta}_2}^2 = -2 \frac{4f_1 f_2 AB}{h^{1/2} \sqrt{\gamma(\gamma + 4a^2)}} \epsilon \tilde{\theta}_1 \tilde{\theta}_2,
\]

so we now want to require

\[
\beta_1 \beta_2 = -\frac{\alpha}{4} h^{1/2} \sqrt{\gamma(\gamma + 4a^2)}.
\]

To find out if this is consistent with the other metric components, take a look at

\[
ds_{\tilde{\theta}_2 \tilde{\theta}_2}^2 = \left( G_{\tilde{\theta}_2 \tilde{\theta}_2} - \frac{G_{\tilde{\theta}_1 \tilde{\theta}_2} G_{\tilde{\theta}_1 \tilde{\theta}_2}}{G_{\tilde{\theta}_1 \tilde{\theta}_1}} \right) (\delta \tilde{\theta}_2)^2
\]

\[
= \left( 1 + \alpha(1 + A^2) \tilde{b}_{y\theta_2}^2 - \frac{4B^2 \beta_2^2}{h^{1/2}(\gamma + 4a^2)} \right) (\delta \tilde{\theta}_2)^2.
\]

From the above analysis we see that the \( \tilde{b}_{y\theta_2} \) term will join \( \delta \tilde{y} \) in the fibration as shown in (5.15), and the rest of the term (which depends on \( \beta_2 \)) will act as a warp factor. The \( \delta \tilde{y} \) term has warp factor \( g_4 \). Can we use this to determine \( \beta_2 \)?

At first this may seem impossible as we can have any coefficients in front of \((\delta \tilde{\theta}_2)^2\) and \((\delta \tilde{y} - \tilde{b}_{y\theta_2} \delta \tilde{\theta}_2)^2\) or \((\delta \tilde{\theta}_1)^2\) and the corresponding \((\delta \tilde{x} - \tilde{b}_{x\theta_1} \delta \tilde{\theta}_1)^2\). A little thought will tell us that this is not quite true. Of course we are allowed to have any coefficients in front of \((\delta \tilde{\theta}_1)^2\) and \((\delta \tilde{x} - \tilde{b}_{x\theta_1} \delta \tilde{\theta}_1)^2\), but the case for \((\delta \tilde{\theta}_2)^2\) and \((\delta \tilde{y} - \tilde{b}_{y\theta_2} \delta \tilde{\theta}_2)^2\) is different. This is because of the transformation (4.33). Under this transformation (in the absence of fluxes) the coefficients of \( \delta \theta_2 \) and \( \delta y \) should be same so that under (4.33) the line element does not change. Now we expect similar thing when we switch on fluxes if we denote \( \delta \tilde{y} = \delta \tilde{y} - \tilde{b}_{y\theta_2} \delta \tilde{\theta}_2 \), and define an equivalent transformation between \((\delta \tilde{y}, \delta \tilde{\theta}_2)\). Notice that
there is no such constraint on the $\delta \tilde{x}, \delta \tilde{\theta}_1$ term. Therefore from the above argument and (5.16) we have the following equality:

$$1 - \frac{4}{h^{1/2}(\gamma + 4a^2)} \beta_2^2 B^2 = \alpha (1 + A^2) \Rightarrow \beta_2 = \pm \frac{1}{2} \sqrt{\alpha h^{1/2}(\gamma + 4a^2)}. \tag{5.41}$$

Thus we fix $\beta_2$ or equivalently $f_2$. As expected the line element for the $\theta_1^2$ component can now be written in terms of $\beta_1$ as

$$ds^2_{\theta_1} = \left(1 + \alpha (1 + B^2) \tilde{b}_{x\theta_1}^2 - \frac{4A^2 \beta_1^2}{h^{1/2}\gamma}\right) (\delta \tilde{\theta}_1)^2. \tag{5.42}$$

The $b$ dependent term can be equivalently absorbed in the $\delta \tilde{x}$ fibration structure as $\delta \tilde{x} \equiv \delta \tilde{x} + \tilde{b}_{x\theta_1} \delta \tilde{\theta}_1$. The rest of the remaining term serve as warp factor for the $(\delta \tilde{\theta}_1)^2$ term. What happens now if we argue an equality between the coefficients of the $\delta \tilde{\theta}_1$ and $\delta \tilde{x}$ terms? This would imply:

$$1 - \beta_1^2 \frac{4}{h^{1/2}\gamma} A^2 = \alpha (1 + B^2) \Rightarrow \beta_1 = \pm \frac{1}{2} \sqrt{\alpha h^{1/2}\gamma}. \tag{5.43}$$

From (5.41) and (5.43) we see that we can indeed choose the signs in a way that fulfills (5.39)! This is consistent with our assumption, implying that in this setup we will see the tori metrics appear with one unique warp factor. This is again expected in the case without fluxes. What we see here is that, this remains true for the case with fluxes also.

To summarize, we see that the T-duality directions are $\tilde{z}, \tilde{x}, \tilde{y}$ on the resolved side, and the line element is more or less the same as (5.24) with additional cross components (5.30). To go from (5.24) to (5.30) we have performed some transformations on the finite shifts $\delta z, \delta x, \delta y$ and $\delta \theta_i$. Defining three vectors $V_i$ as

$$V_1 = \begin{pmatrix} \delta z \\ \delta \theta_1 \\ \delta \theta_2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \delta x \\ \delta \theta_1 \\ \delta \theta_2 \end{pmatrix}, \quad V_3 = \begin{pmatrix} \delta y \\ \delta \theta_1 \\ \delta \theta_2 \end{pmatrix} \tag{5.44}$$

and three matrices as:

$$M_1 = \begin{pmatrix} 1 & \Delta_1 f_1 \cot \theta_1 & \Delta_2 f_2 \cot \theta_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -f_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} \cos \psi & 0 & \sin \psi - f_2 \cos \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi + f_2 \sin \psi \end{pmatrix} \tag{5.45}$$
we can convert the line element (5.24) to (5.30) via the transformations:

\[ V_1 \rightarrow M_1 V_1, \quad V_2 \rightarrow M_2 V_2, \quad V_3 \rightarrow M_3 V_3. \tag{5.46} \]

This will generate the T-duality directions \( \tilde{z}, \tilde{x}, \tilde{y} \), as mentioned above.

We can now try to go to the infinitesimal shifts given by the forms \( dx_i, d\theta_i \). At this point we therefore assume that we can replace

\[ (\delta \tilde{x}_i, \delta \tilde{\theta}_i, \delta \tilde{z}) \rightarrow (d\tilde{x}_i, d\tilde{\theta}_i, d\tilde{z}) \tag{5.47} \]

in the final line element. This would imply that the line element with finite shifts can serve as the metric for the mirror manifold. This is somewhat strong assumption as the transformations that we made on the finite shifts \( \delta x_i, \delta z, \delta \theta_i \) may not always be extrapolated to transformations on infinitesimal shifts. It turns out that the transformation that we made can be extrapolated if we assume that these transformations are restricted on the plane \( (x, y, z, \theta_i) \) at \( r = r_0 \) which is, of course, consistent with all our earlier assumptions.

To see the effects of the finite shifts, first view the \( z \) coordinate to be determined in terms of \( \psi_1 \) and \( \psi_2 \) as \( z \equiv \psi_1 - \psi_2 \). In fact, as we will soon encounter in the M-theory section, the coordinate \( z \) which will eventually be the \( z \) coordinate of the type IIA mirror manifold, and the M-theory eleventh direction \( x_{11} \) can be written in terms of \( \psi_1 \) and \( \psi_2 \) as:

\[ dz \equiv d\psi_1 - d\psi_2, \quad dx_{11} \equiv d\psi_1 + d\psi_2. \tag{5.48} \]

This means that in the type IIB picture we can distribute the coordinates on two different \( S^3 \)'s parametrized by: \( (\psi_1, x, \theta_1) \) and \( (\psi_2, y, \theta_2) \). Existence of two \( S^3 \) here is just for book keeping, and will eventually be related to the real \( S^3 \)'s of the mirror manifold\(^{17}\).

Let us now consider one \( S^3 \) with coordinates \( (\psi_1, x, \theta_1) \). This \( S^3 \) is already at a fixed \( r \) and now also at fixed \( (\psi_2, y, \theta_2) \). From the analysis done above, we see that the transformations generically lead to an integral of the form (compare e.g. to (5.28)):

\[ \int (b^2 + a^2 \cot^2 \theta_1)^{\frac{m}{2}} \cot^n \theta_1 \, d\theta_1 \tag{5.49} \]

\(^{17}\) In fact as will be clear from sec. 6, M-theory will allow metric fibrations that are no longer constrained at \( \psi = \psi_0 \).
where \( a \) and \( b \) are some constants on the sphere \( S^3, n = 0,1 \) and \( m = 1, -1 \). It turns out that the shifts \( \delta \psi_1, \delta x \) etc. can be integrated to yield the following transformations on the coordinates \( \psi_1, x, \theta_1 \):

\[
\psi_1 \rightarrow \psi_1 - \frac{1 - \left( \frac{a^2 - b^2}{a^2} \right) \sin^2 \theta_1}{\sqrt{\epsilon} \sin \theta_1} - \frac{\sin^{-1} \left( \frac{a^2 - b^2}{a^2} \right) \sin \theta_1}{\sqrt{\epsilon}} \cos \theta_1 + \sqrt{1 - \left( \frac{a^2 - b^2}{a^2} \right) \sin^2 \theta_1} \]
\]

\[
x \rightarrow x - \frac{\left( a^2 - b^2 \right)}{\sqrt{\epsilon}} \ln \frac{\left( \frac{b^2 - a^2}{a^2} \right) \sin \theta_1 + \sqrt{1 + \left( \frac{b^2 - a^2}{a^2} \right) \sin^2 \theta_1}}{\frac{a^2}{(b^2 - a^2) \sin \theta_1}} \]
\]

\[
(5.50)
\]

with \( \theta_1 \) transforming as \( \theta_1 \rightarrow c \cdot \theta_1 \) where \( c \) is another constant. (Observe that these expressions have the required periodicity). In the above transformations we have inherently assumed that \( a > b \). What happens for the case \( a < b \)?

This case turns out to be rather involved, but nevertheless do-able. The transformation can again be integrated for both \( m = \pm 1 \) to give us the following results (here we present the result only for \( m = 1 \)):

\[
\psi_1 \rightarrow \psi_1 + \left( \frac{b^2 - a^2}{a} \right) \sin \theta_1 + \sqrt{1 + \left( \frac{b^2 - a^2}{a^2} \right) \sin^2 \theta_1} \]
\]

\[
x \rightarrow x - \frac{a}{2\sqrt{\epsilon}} \ln \left[ \frac{\left( \frac{b^2 - a^2}{a^2} \right) \sin \theta_1 + \cos \theta_1}{\frac{a^2}{(b^2 - a^2) \sin \theta_1}} + \left( \frac{b^2 - a^2}{a^2} \right) \cos \theta_1 \right] + \frac{\left( b^2 - a^2 \right)}{\sqrt{1 + \left( \frac{b^2 - a^2}{a^2} \right)^2}} \]
\]

\[
(5.51)
\]

with \( \theta_1 \) transformation remaining the same. For the other \( S^3 \) the \( y \) and \( \theta_2 \) transformation would look similar to the above transformations on \( x \) and \( \theta_2 \) transformations respectively. However \( \psi_2 \) transformation will differ by relative signs. More details on the effect of these transformations on the mirror metric is given in Appendix 1.

\[18\] In the notations of Appendix 1, the constants \( a, b \) can be extracted from \( \langle \alpha \rangle_{1}^{\pm \frac{1}{2}} \) measured at a constant radius. There is also an overall constant related to the expectation value \( \langle \alpha \rangle \) that we ignore here. Replacing \( \theta_1 \) by \( \theta_2 \) the constants \( a, b \) should now be extracted from \( \langle \alpha \rangle_{2}^{\pm \frac{1}{2}} \).
Taking the above transformations into account, and then performing the three T-duality transformations we get the final mirror manifold (in the delocalised limit) with the following form of the metric written with $dx, dy, dz$ and $d\theta_i$:

$$ds^2 = g_1 \left[ (dz - b_{z\mu} dx^\mu) + \hat{\Delta}_1 \cot \hat{\theta}_1 (dx - b_{x\theta_1} d\theta_1) + \hat{\Delta}_2 \cot \hat{\theta}_2 (dy - b_{y\theta_2} d\theta_2) + ... \right]^2 +$$

$$+ g_2 \left[ d\theta_1^2 + (dx - b_{x\theta_1} d\theta_1)^2 \right] + g_3 \left[ d\theta_2^2 + (dy - b_{y\theta_2} d\theta_2)^2 \right] +$$

$$+ g_4 \left[ d\theta_1 d\theta_2 - (dx - b_{x\theta_1} d\theta_1)(dy - b_{y\theta_2} d\theta_2) \right]$$

(5.52)

where we have used un-tilded coordinates to avoid clutter (we will continue using this coordinates in the rest of the paper unless mentioned otherwise). The dotted part in the $dz$ fibration are the corrections to $\theta_i$ terms from the scaling etc. We have written $\cot \hat{\theta}_i$ instead of $\cot \theta_i$ to emphasize the change in $\theta_i$. Its is interesting to note that (as we saw earlier) this is the only change in $\theta$ because of (5.35). All other changes due to scalings etc. have been completely incorporated! Observe also that we now require only four warp factors $g_1, g_2, g_3$ and $g_4$ instead of six that we had earlier in (5.16). The precise warp factors can now be written explicitly as:

$$g_1 = \alpha^{-1}, \quad g_2 = \alpha j_{yy}, \quad g_3 = \alpha j_{xx}, \quad g_4 = 2\alpha j_{xy}$$

$$\hat{\Delta}_1 = \sqrt{\gamma' r_0} \alpha, \quad \hat{\Delta}_2 = \sqrt{\frac{\gamma'}{\gamma + 4a^2}} r_0 \alpha$$

(5.53)

where $\alpha$ and $j$ are defined in (5.8) and (5.4) (with $\cot \theta_i$ changed to $\cot \hat{\theta}_i$), respectively, and the $b$ fields have been rescaled according to (5.36). With these values the metric (5.52) can be compared to (5.17).

5.2. Physical meaning of $f_1$ and $f_2$

The transformations that we performed in the previous subsection to bring the metric in the form (5.30) using the functions $f_1$ and $f_2$ can be given some physical meaning\footnote{The discussion in this section is motivated from the conversations that one of us (R.T) had with the UPenn group, especially V. Braun and M. Cvetic.}. As discussed earlier, the conversion from $(\phi_1, \theta_1)$ and $(\phi_2, \theta_2)$ to $(x, \theta_1)$ and $(y, \theta_2)$ coordinates respectively is to write the metric as the metric of tori. Now the metric of the tori can be generically written in terms of complex structures $\tau_i$ where $i = 1, 2$ represent the two tori. We can define

$$dz_1 = dx - \tau_1 \, d\theta_1, \quad dz_2 = dy - \tau_2 \, d\theta_2$$

(5.54)
as the two coordinates of the two tori. The metric (5.24) can therefore be written in terms of $dz_i$ as:

$$ds^2 = (dz + \Delta_1 \cot \theta_1 \, dx + \Delta_2 \cot \theta_2 \, dy)^2 + |dz_1|^2 + |dz_2|^2,$$

with the complex structure not yet specified. The transformation that we performed in the previous section, would therefore correspond to the following choice of the complex structure of the base tori:

$$
\tau_1 = f_1 + \frac{i}{2} \sqrt{\gamma \sqrt{h}}, \quad \tau_2 = f_2 + \frac{i}{2} \sqrt{\gamma + 4a^2} \sqrt{h}
$$

where we have already defined $h(r_0), \gamma(r_0), a$ in earlier sections. Observe that when $f_1 = 0 = f_2$ then the base is a torus with complex structure

$$
\tau_1 = \frac{i}{2} \sqrt{\gamma \sqrt{h}}, \quad \tau_2 = \frac{i}{2} \sqrt{\gamma + 4a^2} \sqrt{h}
$$

In this limit the metric has no cross terms. This is basically the metric that we started off with. The transformations in the previous subsection are therefore to convert $\tau_i \rightarrow \tau_i + f_i$ via $SL(2, R)$ transformations on the two tori\footnote{For example using local $SL(2, R)$ matrices $\begin{pmatrix} 1 & f_i \\ 0 & 1 \end{pmatrix}$.}. In the limit when $f_i$ are very large, the base of the six dimensional manifold $(\theta_1, \theta_2, r = r_0)$ is very large compared to the $T^3$ fiber $(x, y, z)$. This situation is consistent with the fact that the generalised SYZ transformations require similar condition \footnote{We thank the referee for pointing this out.} for mirror rules to work properly (see also \footnote{\cite{I}). On the other hand, this limit is precisely the opposite to the one where the geometric transition takes place. This is one reason why we have to go through non-trivial manipulations to get to the final metric of the mirror manifold\footnote{\cite{II}.}

5.3. B Fields in the Mirror Setup

There is another possibility that we haven’t entertained yet. This is to allow new $B$ field components in the resolved side. Observe that the analysis presented above was done from the resolved conifold setup when we only had $B_{NS}$ fields with components $b_{x\theta_1}, b_{y\theta_2}$ and $b_{z\mu}$. What happens if we switch on a cross component $b_{xy}$ that has legs on both the spheres in the type IIB resolved conifold setup? Can this generate a $d\theta_1 d\theta_2$ term? First, of course this will not convert to a component of the metric under a mirror transformation and
will appear in the type IIA framework as a $B$ field. This $B$ field will become a threeform field in M-theory. We will discuss this later. Second, the mirror metric will change. This change can be easily evaluated — and we shall do this below — but before that lets see whether it is indeed possible to switch on such component in the type IIB setup.

To analyze this, we go back to our fourfold scenario that we had in section 3. In the fourfold setup, no matter what choice we make for the components of the $G$ fluxes, the metric will retain its warped form and the only thing that could change will be the exact value of the warp factor. Therefore we can choose an additional component, say $G_{587a}$, and get the corresponding $b_{xy}$ flux in type IIB.

Now under this choice of $B$ field, the metric will have the additional term $-\frac{1}{G_{xx}} (G_{z\mu} dx^\mu + G_{zx} dx + G_{zy} dy)^2$. One can show that the form of the $dz$ fibration structure remains the same, although the values will differ by additional $b_{xy}$ terms. In the notations of the earlier sections, let us assume the form of the metric to be:

\[
ds^2 = (dz + \text{fibration})^2 + G_{xx} dx^2 + G_{yy} dy^2 + G_{\theta,\theta_1} d\theta_1^2 + G_{\theta,\theta_2} d\theta_2^2 + 2G_{xy} dx dy + 2G_{x\theta_1} dx d\theta_1 + 2G_{y\theta_2} dy d\theta_2 + 2G_{x\theta_2} dx d\theta_2 + 2G_{y\theta_1} dy d\theta_1 + 2G_{\theta_1,\theta_2} d\theta_1 d\theta_2.
\]

(5.58)

Using the mirror rules given earlier, one can work out all these components. Since $b_{xy}$ is non-zero, the analysis gets a little involved. If we define again $\alpha^{-1} = j_{xx} j_{yy} - j_{xy}^2 + b_{xy}^2$, this time $b_{xy}$ being different from zero, we arrive exactly at the form (5.15):

\[
\begin{align*}
G_{xx} dx^2 + 2G_{x\theta_1} dx d\theta_1 + G_{\theta,\theta_1} d\theta_1^2 &= \alpha j_{yy} [d\theta_1^2 + (dx - b_{x\theta_1} d\theta_1)^2] \\
G_{yy} dy^2 + 2G_{y\theta_2} dy d\theta_2 + G_{\theta,\theta_2} d\theta_2^2 &= \alpha j_{xx} [d\theta_2^2 + (dy - b_{y\theta_2} d\theta_2)^2] \\
G_{xy} dx dy + G_{\theta,\theta_2} d\theta_1 d\theta_2 + G_{x\theta_2} dx d\theta_2 + G_{y\theta_1} dy d\theta_1 &= \\
- \alpha j_{xy} (dx - b_{x\theta_1} d\theta_1) (dy - b_{y\theta_2} d\theta_2).
\end{align*}
\]

(5.59)

Where we have given the precise warp factors of every terms. Therefore, introducing a new component of the $B$ field has not changed the form of the metric and has failed to generate the $d\theta_1 d\theta_2$ term. What changes, in the final mirror picture, is that we will now have a non zero $B_{NS}$ flux in type IIA setup. We will comment on this later.

The above choice of $B$ fields in the resolved conifold side is highly unnatural (although it may be allowed from the supergravity analysis). A more natural way to generate a $B$ field in the mirror side has already been taken into account when we switched to the tilde-coordinates. In fact the cross terms in the resolved conifold metric (5.30) i.e. the $j_{x\theta_1}$ and the $j_{y\theta_2}$ terms will be responsible to give a non-zero $B$ field in the mirror picture.
This way we can give another physical meaning to the shifts that we performed in (5.29). The background $B$ field in the type IIA background can now be written in terms of the deformed conifold coordinates as:

\[
\tilde{B} = \frac{2 f_1}{\sqrt{h^{1/2}} \gamma} d\tilde{x} \wedge d\tilde{\theta}_1 + \frac{2 f_2}{\sqrt{h^{1/2}(\gamma + 4a^2)}} d\tilde{y} \wedge d\tilde{\theta}_2
\]

\[
+ \left( \frac{2 Af_1}{\sqrt{h^{1/2} \gamma}} d\tilde{\theta}_1 + \frac{2 B f_2}{\sqrt{h^{1/2}(\gamma + 4a^2)}} d\tilde{\theta}_2 \right) \wedge d\tilde{z}
\]

with all other components vanishing in the limit $\epsilon \to 0$. These $B$ fields are in general large, because the transformations (5.29) that we performed in the resolved conifold setup is large. In the limit where we define $\tilde{B} \equiv \epsilon^{-1/2} \hat{B}$, the finite part $\hat{B}$ will be given by:

\[
\frac{\hat{B}}{\sqrt{\alpha}} = dx \wedge d\theta_1 - dy \wedge d\theta_2 + \left( A \sqrt{\langle \alpha \rangle_1} d\theta_1 - B \sqrt{\langle \alpha \rangle_2} d\theta_2 \right) \wedge dz
\]

modulo an overall sign if we employ the opposite choice in (5.41) and (5.43). Again, we have omitted the tildes in the final expression. Notice also the following interesting facts:

- We can replace $dx$ and $dy$ by the corresponding one forms $D\tilde{x}$ and $D\tilde{y}$ because of the wedge structure (at constant $b$). We will soon use this property (in the next sub-section) to get another form of the $B$ fields that is more adapted to our mirror set-up.

- When we use an integrable complex structure for the two tori in (5.56), i.e when we use $\langle \alpha \rangle_1, \langle \alpha \rangle_2$ instead of $\alpha$ (see Appendix 1), the $B_{NS}$ field takes the following form:

\[
\hat{B} = \sqrt{\langle \alpha \rangle_1} dx \wedge d\theta_1 - \sqrt{\langle \alpha \rangle_2} dy \wedge d\theta_2 + \left( A \sqrt{\langle \alpha \rangle_1} d\theta_1 - B \sqrt{\langle \alpha \rangle_2} d\theta_2 \right) \wedge dz.
\]

As one can easily see, this is a pure gauge! Thus even though we have large $B$ field in this scenario, the effect of this is nothing as it is a gauge artifact. More on this will appear in forthcoming papers [66], [59].

5.4. The Mirror Manifold

From the detailed analysis in the above two subsections, we can summarize the following: our metric of the mirror manifold has strong resemblance to the metric of D6 wrapped on deformed conifold, but they differ because of non-trivial $B$-dependent fibration of some of the terms. As we see, this is the key difference between the two metrics (apart from the non-trivial warp factors). The manifold (5.52) is generically non-Kähler whereas the metric (5.17) could be Kähler in some limit. The metric evaluated in (5.52) is actually
after we perform a coordinate transformation and therefore we see no \( \psi_0 \) dependence in the final picture. In the usual coordinate system, our ansatz therefore, for the exact metric in type IIA will be to take the “usual” \( D6 \) brane wrapped on the deformed conifold (i.e. eq. (5.17)) and replace the \((dz, dx, dy, \tilde{y})\) by

\[
\begin{align*}
 dz &\rightarrow dz - b_{z\mu} \, dx^\mu \\
 dx &\rightarrow dx - b_{x\theta_1} \, d\theta_1 \\
 dy &\rightarrow dy - b_{y\theta_2} \, d\theta_2 \\
 \tilde{g}_i(r_0, \theta_1, \theta_2) &\rightarrow g_i(r_0, \theta_1, \theta_2)
\end{align*}
\]

with the remaining terms unchanged. Observe that before this replacement (i.e in the absence of fluxes) (5.17) is exactly (4.32) up to \( \cos \psi \) and \( \sin \psi \) dependences. We believe, as discussed above, this has to do with the delocalization of the \( \psi \) coordinate. In the presence of fluxes, the final answer for the type IIA metric therefore will be to convert (5.52) into

\[
ds_{IIA}^2 = g_1 \left[ (dz - b_{z\mu} \, dx^\mu) + \Delta_1 \cot \hat{\theta}_1 (dx - b_{x\theta_1} \, d\theta_1) + \Delta_2 \cot \hat{\theta}_2 (dy - b_{y\theta_2} \, d\theta_2) + \ldots \right]^2
\]

\[
+ g_2 \left[ d\theta_1^2 + (dx - b_{x\theta_1} \, d\theta_1)^2 \right] + g_3 \left[ d\theta_2^2 + (dy - b_{y\theta_2} \, d\theta_2)^2 \right]
\]

\[
+ g_4 \sin \psi_0 \left[ (dx - b_{x\theta_1} \, d\theta_1) \, d\theta_2 + (dy - b_{y\theta_2} \, d\theta_2) \, d\theta_1 \right]
\]

\[
+ g_4 \cos \psi_0 \left[ d\theta_1 \, d\theta_2 - (dx - b_{x\theta_1} \, d\theta_1)(dy - b_{y\theta_2} \, d\theta_2) \right],
\]

(5.64)

where \( g_i \) are again given by (5.53). Similarly the finite part of the background \( B \) field can be transformed from (5.61) to the following form involving the \( \sin \psi_0 \) and \( \cos \psi_0 \) dependences as:

\[
\hat{B} = dx \wedge d\theta_1 - dy \wedge d\theta_2 + A \, d\theta_1 \wedge dz - B \left( \sin \psi_0 \, dy - \cos \psi_0 \, d\theta_2 \right) \wedge dz.
\]

(5.65)

\[^{22}\text{This ansatz can actually be given a little more rigorous derivation. To see this from (5.52), perform the following local transformation}
\]

\[
\begin{pmatrix}
D\hat{y} \\
d\theta_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cos \psi_0 & -\sin \psi_0 \\
\sin \psi_0 & \cos \psi_0
\end{pmatrix}
\begin{pmatrix}
D\hat{y} \\
d\theta_2
\end{pmatrix}
\]

where \( D\hat{y} \equiv dy + b_{y\theta_2} \, d\theta_2 \). The change in the \( dz \)-fibration structure will be in such a way as to restore back \( \cot \theta_2 dy \) via the reverse transformation a-la (4.38).
The type IIA coupling on the other hand can no longer be constant even though in the type IIB side we start with a constant coupling. The constant coupling on the type IIB side is generically fixed by RR and NS fluxes via a superpotential (though not always). If we start with a type IIB coupling $g_B$ (constant or non-constant), the type IIA theory is given by a non-constant coupling $g_A$, that depend on the coordinates of the internal space as

$$g_A = \frac{g_B}{\sqrt{1 - \frac{2}{\alpha}}}.$$  \hspace{1cm} (5.66)

Observe that a small coupling in the type IIB side implies a small coupling on the mirror manifold. Therefore any perturbative calculation in type IIB side will have a corresponding perturbative dual in the mirror side. This is another advantage that we get from the mirror manifolds.

The above analysis more or less gives the complete background for the mirror case (the RR background will be dealt with shortly). For later comparison we would however need the three form NSNS field strength defined as $H = d\hat{B}$. This is basically the finite part of the three form, and is given by \cite{23}:

$$H = -\sqrt{\alpha^3} A (A dA + B dB) \wedge d\theta_1 \wedge dz + \sqrt{\alpha^3} (A dA + B dB) \wedge dy \wedge d\theta_2$$
$$+ \sqrt{\alpha} A dA \wedge d\theta_1 \wedge dz + \sqrt{\alpha^3} B (A dA + B dB) \wedge (\sin \psi_0 \, dy - \cos \psi_0 \, d\theta_2) \wedge dz$$
$$- \sqrt{\alpha^3} (A dA + B dB) \wedge dx \wedge d\theta_1 - \sqrt{\alpha} dB \wedge (\sin \psi_0 \, dy - \cos \psi_0 \, d\theta_2) \wedge dz,$$

where we have used the following simplifying definitions in the above form of $H$:

$$dA = \partial_\theta A \, d\theta_i + \partial_r A \, dr = \partial_\theta A \, d\theta_i, \quad d\sqrt{\alpha} = -\sqrt{\alpha^3} (A dA + B dB)$$  \hspace{1cm} (5.68)

with similar definition for $dB$. Notice that $H$ involves all components of the mirror manifold and therefore will be spread over the whole space.

We now need to show the following things:

(1) The manifold is explicitly non-Kähler i.e. $dJ \neq 0$, where $J$ is the fundamental two form. The manifold should also be non-Ricci flat and have an $SU(3)$ structure. Recall that $SU(3)$ structure doesn’t imply Ricci-flatness.

\footnote{Written in terms of $\sqrt{\langle \alpha \rangle}_1$ and $\sqrt{\langle \alpha \rangle}_2$ this is exactly zero, and therefore serves as a gauge artifact.}
(2) The complex structure should in general be non-integrable. Therefore the manifold should be non-complex and non-Kähler. The properties of such manifolds have been discussed earlier in [19], [41].

(3) We have to calculate the superpotential and show that the holomorphic three form $\Omega$ is in general not closed.

Some of these details will be addressed in later sections of this paper. We will leave a more elaborate discussion for part II. We now go to the M-Theory analysis.

6. Chain 2: The M-theory Description of the Mirror

Now that we have obtained the mirror metric, it is time to go to the second chain of fig. 1 and lift the type IIA configuration to M-theory. Initial studies have been done in [49]. We will use their ideas to go to another $G_2$ holonomy manifold which is related to the previous one by a flop. But before moving ahead, we will require some geometric details of the background. These geometric details will help us to formulate the background in a way so that the procedure of flop will be simple to see.

6.1. One Forms in M-theory

We first need to define one forms in M-theory. These one forms are different from the ones presented in say [41], [28], [29] as they have contributions from the $B$ fields in the resolved conifold side. These $B$ fields are in general periodic variables and therefore let us denote them by angular coordinates $\lambda_1, \lambda_2$ as $\tan \lambda_1 \equiv a_1 b_{x\theta_1}, \tan \lambda_2 = a_2 b_{y\theta_2}$, with $a_1, a_2$ constants. These one-forms are however only defined locally because the $B$ fields that we will use in the definition are not globally defined variables. The existence of these one forms can be argued from the consistency of the metric $\sigma$. They are given by:

$$
\begin{align*}
\sigma_1 &= \sin \psi_1 \, dX + \sec \lambda_1 \cos (\psi_1 + \lambda_1) \, d\Theta_1 \\
\sigma_2 &= \cos \psi_1 \, dX - \sec \lambda_1 \sin (\psi_1 + \lambda_1) \, d\Theta_1 \\
\sigma_3 &= d\psi_1 + n_1 \cot \Theta_1 \, dX - n_2 \tan \lambda_1 \cot \Theta_1 \, d\Theta_1 
\end{align*}
$$

Observe that for $r_0 \approx 0$ the metric (5.64) is exactly the metric of $D6$ branes wrapping an $S^3$ of a deformed conifold (because $(b_{x\theta_1}, b_{y\theta_2}) \to 0$ at IR), and therefore will have similar one-forms as in [49], [28], [29] by which we can express the metric.
where we have defined new coordinates $\psi_1, \psi_2, X, \Theta_1, \hat{\Theta}_1$. Their relation to $z, x, \theta_1, \hat{\theta}_1$ will be determined as we proceed with our calculation. We have also included two functions $n_1$ and $n_2$ in the definition of $\sigma_3$. These are functions of all the coordinates, and will be analyzed later. Using these, we can define another set of one forms with $X, \Theta_1$ etc. replaced by $Y, \Theta_2$ etc. as:

\[
\begin{align*}
\Sigma_1 &= -\sin \psi_2 \, dY + \sec \lambda_2 \cos (\psi_2 - \lambda_2) \, d\Theta_2 \\
\Sigma_2 &= -\cos \psi_2 \, dY - \sec \lambda_2 \sin (\psi_2 - \lambda_2) \, d\Theta_2 \\
\Sigma_3 &= d\psi_2 - n_3 \cot \hat{\Theta}_2 \, dY + n_4 \tan \lambda_2 \cot \hat{\Theta}_2 \, d\Theta_2.
\end{align*}
\] (6.2)

We have chosen the respective signs with the foresight of making a simple identification with our original variables possible. The above set of one forms will suffice to define the corresponding seven dimensional manifolds in M-theory. Although, not important for our work here, we can make some interesting simplifications. The quantities $b_{x\theta_1}$ and $b_{y\theta_2}$, as discussed above, are basically periodic variables, and for small $\lambda_i$ we can define another angular coordinates $\beta_1$ and $\beta_2$ that modify the original $\psi_1, \psi_2$ as

\[
\beta_1 = \psi_1 - b_{x\theta_1}, \quad \beta_2 = \psi_2 - b_{y\theta_2}.
\] (6.3)

With these choices of angles, we can define another set of one forms in M-theory in the following way:

\[
\begin{align*}
\tilde{\sigma}_1 &= \sin \psi_1 \, dX + \cos \beta_1 \, d\Theta_1 \quad \tilde{\Sigma}_1 = \sin \psi_2 \, dY + \cos \beta_2 \, d\Theta_2 \\
\tilde{\sigma}_2 &= \cos \psi_1 \, dX + \sin \beta_1 \, d\Theta_1 \quad \tilde{\Sigma}_2 = \cos \psi_2 \, dY + \sin \beta_2 \, d\Theta_2,
\end{align*}
\] (6.4)

with $\tilde{\sigma}_3$ and $\tilde{\Sigma}_3$ being identical to $\sigma_3$ and $\Sigma_3$, respectively. This way of writing the one-forms helps us to compare them to the one-forms given in [49], [28], and [29]. Observe that for small background values of $b_{x\theta_1}$ and $b_{y\theta_2}$, the above set (6.4) is the same as (6.1) and (6.2). Furthermore, the field strength vanishes locally, and these one forms satisfy the $SU(2)$ algebra. This is true because over a small patch the $b_{x\theta_1}$ and $b_{y\theta_2}$ values are constants. Therefore we can approximate

\[
D\tilde{x} \equiv dx - b_{x\theta_1} \, d\theta_1 = d(x - b_{x\theta_1} \, \theta_1), \quad D\tilde{y} \equiv dy - b_{y\theta_2} \, d\theta_2 = d(y - b_{y\theta_2} \, \theta_2)
\] (6.5)

as exact one forms. In this way (6.1) and (6.2) appear like the usual one forms for the $G_2$ manifold and satisfy an $SU(2) \times SU(2)$ symmetry. Globally, there is no $SU(2) \times SU(2)$
symmetry because our manifold is no longer a Kähler manifold. This is also clear from the one-forms (6.1) and (6.2). We will use this identification many times to compare our results to the ones from literature. Also, having an exact form for $D\hat{x}$ and $D\hat{y}$ locally means vanishing type IIB $B$ fields. In this way we will be able to extend our results to the case with torsion.

6.2. M-theory Lift of the Mirror IIA Background

To perform the M-theory lift, we need the field strength $F_{mn}$ which comes from the mirror dual of the three-form $H_{RR} \equiv \mathcal{H}$ and five-form $F_5$ in type IIB theory with $D5$ on a resolved conifold. The five form appears because a $D5$ wrapped on an $S^2$ with $B_{NS}$ fluxes gives rise to a $D3$ brane source as we discussed in the beginning of this paper. The background value of the RR potential is given in [20] as:

$$\mathcal{H} = c_1 (dz \wedge d\theta_2 \wedge dy - dz \wedge d\theta_1 \wedge dx) + c_2 \cot \hat{\theta}_1 \, dx \wedge d\theta_2 \wedge dy - c_3 \cot \hat{\theta}_2 \, dy \wedge d\theta_1 \wedge dx$$
$$F_5 = K(r) \, (1 + \ast) \, dx \wedge dy \wedge dz \wedge d\theta_1 \wedge d\theta_2,$$

(6.6)

with $c_i$ is a constant coefficient with $K(r)$ being a function of the global transverse coordinate $r$ (we could as well define $F_5$ with $K(r_0)$ locally, as we did for $J_i$ in (5.5)). This means that we have the following components of the RR three form: $\mathcal{H}_{\mu\nu z}$, $\mathcal{H}_{\mu\nu y}$ and $\mathcal{H}_{\mu\nu x}$. The T-duality rules for the RR field strengths with components along the T-dual directions are given in [67]:

$$\tilde{F}^{(n)}_{ijk\ldots} = F^{(n+1)}_{xijk\ldots} - n B_{x[i} F^{(n-1)}_{jk\ldots] \
\tilde{F}^{(n)}_{xij\ldots} = F^{(n-1)}_{ij\ldots} - (n - 1) j^{-1}_{xx} B_{x[i} F^{(n-1)}_{xjk\ldots]} \]

(6.7)

where $n$ denote the rank of the form, $x$ is the T-duality direction and $B$ is the NS field. Notice also that in the above relation, the duality direction $x$ is inert under antisymmetrization. Under a mirror transformation, the RR three-form will give rise to the following gauge potentials in type IIA theory:

$$F_{z\theta_1} = \mathcal{H}_{xy\theta_1}, \quad F_{z\theta_2} = \mathcal{H}_{xy\theta_2}$$
$$F_{y\theta_1} = -\mathcal{H}_{xz\theta_1} + \mathcal{H}_{xy\theta_1} \left[ \frac{J_{yz} J_{xx} - J_{xy} J_{zx}}{J_{yy} J_{xx} - J_{zy}^2} - \mathcal{B}_{zy} \right] \]$$
$$F_{x\theta_2} = \mathcal{H}_{yz\theta_2} - \mathcal{H}_{xy\theta_2} \left[ \frac{J_{xy} (J_{yz} J_{xx} - J_{xy} J_{zx})}{J_{xx} (J_{yy} J_{xx} - J_{zy}^2)} + \mathcal{B}_{zx} - \frac{j_{xx}}{j_{xx}} \right]$$

(6.8)
which simplifies, after choosing the signs of $B_{mn}$ in a way that makes the fibration structure in the mirror consistent, as:

\[
\begin{align*}
F_{\theta_1} &= -c_3 \cot \theta_2, \\
F_{\theta_2} &= -c_2 \cot \theta_1, \\
F_\theta &= c_1 - 2c_3 \alpha B \cot \theta_2, \\
F_x &= c_1 - 2c_2 \alpha A \cot \theta_1.
\end{align*}
\] (6.9)

From above we will eventually extract gauge potentials $A_x$ and $A_y$ (the $A_z$ potential can be absorbed in the definition of $dx_{11}$)

The analysis done above only gave us some of the gauge fields in type IIA theory. To get the other components, we need to get the mirror dual of the five form $F_5$. One can easily show that the $F_5$ part contributes to $F_{\theta_1 \theta_2}$. The precise value turns out to be

\[
F_{\theta_1 \theta_2} = K(r)_{r \to r_0} - b_{x\theta_1} \mathcal{H}_{yz\theta_2} - b_{y\theta_2} \mathcal{H}_{xz\theta_1} - \mathcal{B}_{z\theta_2} \mathcal{H}_{y\theta_1} - \mathcal{B}_{z\theta_1} \mathcal{H}_{x\theta_2}
\]

\[
= K(r)_{r \to r_0} - c_1 (b_{x\theta_1} - b_{y\theta_2}) - 2c_3 \alpha B b_{y\theta_2} \cot \theta_2 + 2c_2 \alpha A b_{x\theta_1} \cot \theta_1
\] (6.10)

where again the sign in $\mathcal{B}_{z\theta_1}$ and $\mathcal{B}_{z\theta_2}$ have been chosen opposite to the definitions employed in sec. 4.2. The above mentioned gauge potentials will eventually appear as metric components in M-theory. As is well known, the M-theory metric components will typically look like

\[
G^M_{\mu\nu} = e^{-4\phi} g_{\mu\nu}^{IIA} - e^{4\phi} A_\mu A_\nu, \quad G^M_{\mu 11} = -e^{4\phi} A_\mu
\] (6.11)

where $A_\mu$ are the gauge fields and $\phi$ is the type IIA dilaton. In fact, we can use the above definitions to absorb some terms in $dx_{11}$ when we define the gauge fields. Up to some warp factors our ansätze for the gauge fields will therefore be:

\[
A_x = \Delta_3 \cot \theta_1, \quad A_{\theta_1} = -\Delta_3 b_{x\theta_1} \cot \theta_1
\]

\[
A_y = -\Delta_4 \cot \theta_2, \quad A_{\theta_2} = \Delta_4 b_{y\theta_2} \cot \theta_2
\] (6.12)

where $\Delta_{3,4}$ are some specific functions of $\theta_1, x, y$ and $z$. Now combining everything together we can write the part of the M-theory metric originating from the gauge fields as:

\[
A \cdot dX \equiv \Delta_3 \cot \theta_1 (dx - b_{x\theta_1} d\theta_1) - \Delta_4 \cot \theta_2 (dy - b_{y\theta_2} d\theta_2)
\] (6.13)

---

25 There is a simple reason for this. We had earlier defined $dz = d\psi_1 - d\psi_2$ and $dx_{11} = d\psi_1 + d\psi_2$. Thus $dx_{11} = d\psi_1 - d\psi_2 + 2d\psi_2 = dz + 2d\psi_2$. Therefore, any additional $dz$ dependent terms should be absorbed in $dx_{11}$ by changing the coefficient in front of $dz$ in $dx_{11}$. For example $dx_{11} + A_z dz = (1 + A_z) dz + 2d\psi_2 = d\tilde{x}_{11}$ when $A_z$ is a pure gauge.
where we remove any \( z \) dependences. The above potentials (6.13) are basically the wrapped \( D6 \) brane sources that have been converted to geometry giving rise to the M-theory metric

\[
ds^2 = e^{-\frac{2\phi}{3}} (h^{-1/2} \, ds^2_{0123} + h^{1/2} \gamma' \, dr^2) + e^{-\frac{2\phi}{3}} \, ds^2_{IIA} + e^{\frac{4\phi}{3}} (dx_{11} + \Delta_3 \cot \hat{\theta}_1 \, d\hat{x} - \Delta_4 \cot \hat{\theta}_2 \, d\hat{y})^2 \tag{6.14}
\]

with \( x_{11} \) being the eleventh direction, and \( ds^2_{IIA} \) is the metric given in (5.64). We have also used the definition of \( \hat{x} \) and \( \hat{y} \) (introduced in section 5) to write the fibration structure in a compact form. Recall also that we are using the un-tilded coordinates henceforth.

The metric (6.14) is basically the M-theory metric that we are looking for. To see how the \( G_2 \) structure appears from this, we need to write the metric using the one forms that we gave in the previous section. This will also help us to perform a flop in the metric.

From the M-theory metric (6.14) we see that the total fibration structure is

\[
g_1 \, e^{-\frac{2\phi}{3}} (dz + \Delta_1 \cot \hat{\theta}_1 \, d\hat{x} + \Delta_2 \cot \hat{\theta}_2 \, d\hat{y})^2 + e^{\frac{4\phi}{3}} (dx_{11} + \Delta_3 \cot \hat{\theta}_1 \, d\hat{x} - \Delta_4 \cot \hat{\theta}_2 \, d\hat{y})^2. \tag{6.15}
\]

In writing this, the careful reader might notice that we have used \( \cot \hat{\theta}_i \) in (6.6). However, we have two options here: we can either absorb the scaling etc. in the definition of warp factors or keep the warp factors as they are and change \( \theta \) in \( \cot \theta \) to accommodate this. Furthermore, the scaling of \( \theta \), which only affects this term, is actually of \( \mathcal{O}(1) \) and could be ignored in the subsequent calculations. We will however not assume any approximations and continue using \( \hat{\theta}_i \) in the fibration. Observe that all other one forms are defined wrt \( \theta \).

If we now identify \( dz \equiv d\psi_1 - d\psi_2 \) and \( dx_{11} \equiv d\psi_1 + d\psi_2 \), then one can easily see that the above fibration can be written in terms of the one forms that we devised earlier in (6.1) and (6.2), as

\[
\alpha_3^2 \left( \sigma_3 + \Sigma_3 \right)^2 + \alpha_4^2 \left( \sigma_3 - \Sigma_3 \right)^2 \tag{6.16}
\]

with \( \alpha_i \) being the relevant warp factors; and we also identify \((X,Y,\Theta_1,\Theta_2)\) to \((x,y,\theta_1,\theta_2)\). The coefficients \( a_i \) are simply the identity, so \( \tan \lambda_1 = b_{x\theta_1} \) and \( \tan \lambda_2 = b_{y\theta_2} \). This way of writing the \( z \)- and \( x_{11} \)- fibration also forces us to set \( n_1 = n_2 = \Delta_1 \) and \( n_3 = n_4 = \Delta_2 \). (6.16) is consistent with the expected form for the M-theory lift of the \( D6 \) configuration.

Let us now look at the other possible combinations of the one forms.

The first combination is to consider the sum of the squares of the difference between the one forms. In other words, we will consider:

\[
(\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2. \tag{6.17}
\]
This give rise to the following algebra:

\[
\begin{align*}
&\text{(sin } \psi_1 \text{ } dX + \sec \lambda_1 \cos (\psi_1 + \lambda_1) \text{ } d\Theta_1 + \sin \psi_2 \text{ } dY - \sec \lambda_2 \cos (\psi_2 - \lambda_2) \text{ } d\Theta_2)^2 + \\
&\text{(cos } \psi_1 \text{ } dX - \sec \lambda_1 \sin (\psi_1 + \lambda_1) \text{ } d\Theta_1 + \cos \psi_2 \text{ } dY + \sec \lambda_2 \sin (\psi_2 - \lambda_2) \text{ } d\Theta_2)^2 = \\
&d\Theta_1^2 + d\Theta_2^2 + (dX - \tan \lambda_1 \text{ } d\Theta_1)^2 + (dY - \tan \lambda_2 \text{ } d\Theta_2)^2 + \\
&- 2 \cos (\psi_1 - \psi_2) [d\Theta_1 \text{ } d\Theta_2 - (dX - \tan \lambda_1 \text{ } d\Theta_1) (dY - \tan \lambda_2 \text{ } d\Theta_2)] + \\
&- 2 \sin (\psi_1 - \psi_2) [d\Theta_1 (dY - \tan \lambda_2 \text{ } d\Theta_2) + d\Theta_2 (dX - \tan \lambda_1 \text{ } d\Theta_1)].
\end{align*}
\]

(6.18)

This is more or less the expected form, but differs from (5.64) by some relative signs. To fix the signs we need to evaluate the other possible combination of one forms, i.e. \((\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2\). This time the algebra will yield:

\[
\begin{align*}
&\text{(sin } \psi_1 \text{ } dX + \sec \lambda_1 \cos (\psi_1 + \lambda_1) \text{ } d\Theta_1 + \sin \psi_2 \text{ } dY + \sec \lambda_2 \cos (\psi_2 + \lambda_2) \text{ } d\Theta_2)^2 + \\
&\text{(cos } \psi_1 \text{ } dX - \sec \lambda_1 \sin (\psi_1 + \lambda_1) \text{ } d\Theta_1 + \cos \psi_2 \text{ } dY - \sec \lambda_2 \sin (\psi_2 + \lambda_2) \text{ } d\Theta_2)^2 = \\
&d\Theta_1^2 + d\Theta_2^2 + (dX - \tan \lambda_1 \text{ } d\Theta_1)^2 + (dY - \tan \lambda_2 \text{ } d\Theta_2)^2 + \\
&+ 2 \cos (\psi_1 - \psi_2) [d\Theta_1 \text{ } d\Theta_2 + (dX - \tan \lambda_1 \text{ } d\Theta_1) (dY - \tan \lambda_2 \text{ } d\Theta_2)] + \\
&+ 2 \sin (\psi_1 - \psi_2) [-d\Theta_1 (dY - \tan \lambda_2 \text{ } d\Theta_2) + d\Theta_2 (dX - \tan \lambda_1 \text{ } d\Theta_1)].
\end{align*}
\]

(6.19)

It differs from (6.18) only by overall minus signs in the \(\cos \psi\) and \(\sin \psi\) terms. To get the exact form of the metric that we have in (6.14), we write

\[
ds^2 = \alpha_1^2 \sum_{a=1}^{2} (\sigma_a + \xi \Sigma_a)^2 + \alpha_2^2 \sum_{a=1}^{2} (\sigma_a - \xi \Sigma_a)^2 + \alpha_3^2 (\sigma_3 + \Sigma_3)^2 + \alpha_4^2 (\sigma_3 - \Sigma_3)^2 + \alpha_5^2 \text{ } dr^2,
\]

(6.20)

where we have introduced the factor \(\xi\) for \(\Sigma_1\) and \(\Sigma_2\) to account for the different warp factors that the directions \((x, \theta_1)\) and \((y, \theta_2)\) have\(^{24}\). By comparison with (5.64) and (6.14) we determine the warp factors \(\alpha_i\) with \(\xi = \sqrt{g_3/g_2}\) to be:

\[
\begin{align*}
\alpha_1 &= \frac{1}{2} e^{-\frac{\phi}{2}} \sqrt{\xi^{-1} g_4 + 2g_2}, & \alpha_2 &= \frac{1}{2} e^{-\frac{\phi}{2}} \sqrt{2g_2 - \xi^{-1} g_4}, \\
\alpha_3 &= e^{\frac{2\phi}{h}}, & \alpha_4 &= e^{-\frac{\phi}{2}} \sqrt{g_1}, & \alpha_5 &= e^{-\frac{\phi}{2}} \sqrt{g' h}.
\end{align*}
\]

(6.21)

\(^{24}\) In the notations that we used here, they are tori of course. But it would be easy to get to the sphere case once we know the global type IIA metric. Observe also that when we switch off the \(\lambda_i\) the metric reduces to the well known \(G_2\) form.
Observe also that, writing the metric as (6.20), one can easily identify it to the metric presented in [49], [28], [29] where the vielbeins in terms of $\sigma_a$ and $\Sigma_a$ are written (see for example eq. 2.7 of [28]). This will be useful in the following.

Before moving ahead let us pause for a second to reflect on the metrics (6.20) and the type IIA metric (5.64). In (5.64) we have kept the $dz$ fibration structure distinct from the $\psi$ rotation. In fact the rotation generates constant warp factors of $\sin \psi_0$ and $\cos \psi_0$ in (5.64). But in (6.20) we see that the one forms (6.1) and (6.2) can in fact be used to write the M-theory metric with arbitrary $\psi$! Indeed now the relation between the non-constant $\psi$ and $z$ will become

$$z = \psi_1 - \psi_2 = a_s \psi$$  \hspace{1cm} (6.22)

where $a_s = a_s(r_0)$ is the constant of proportionality. With this in mind, now we see that the M-theory metric allows a type IIA solution which takes us away from the $\psi = \psi_0$ point and allow non constant $\sin \psi$ and $\cos \psi$ in (5.64). Thus the local metric in type IIA is delocalised only along the $r$ direction (i.e defined at $r = r_0$) and is now expressed as

$$ds^2_{IIA} = ds^2_{0123} + g_1 \left( D\tilde{z} + \Delta_1 \cot \hat{\theta}_1 D\tilde{x} + \Delta_2 \cot \hat{\theta}_2 D\tilde{y} + \ldots \right)^2 +$$

$$+ g_0 \, dr^2 + g_2 \left( d\theta_1^2 + D\hat{\theta}_1^2 \right) + g_3 \left( d\theta_2^2 + D\hat{\theta}_2^2 \right) +$$

$$+ g_4 \sin \left( za_s^{-1} \right) \left( D\tilde{x} \cdot d\theta_2 + D\tilde{y} \cdot d\theta_1 \right) + g_4 \cos \left( za_s^{-1} \right) \left( d\theta_1 \cdot d\theta_2 - D\tilde{x} \cdot D\tilde{y} \right)$$  \hspace{1cm} (6.23)

where we have already defined $D\tilde{z}$, $D\tilde{y}$ and $D\tilde{x}$ earlier. The effect of the additional six branes that are mirror dual to the type IIB seven branes is already accounted for in the local picture above. The full global story is presented in [59] where things get pretty involved because of the presence of orientifold six-planes, ungauged localised two form fluxes and non-trivial three form fluxes, along with the additional D6 branes. We will not discuss these issues further here, and the readers can find more details in [59].

To extend our earlier analysis, we will now require all the components of the seven dimensional metric. We denote the M-theory metric as $ds^2 = G_{mn} \, dx^m \, dx^n$, where $m, n =$

\footnote{From (5.2) we see that $a_s^2 = r_0^2 \sqrt{h} \frac{\partial \gamma}{\partial r} |_{r=r_0}$. Using now the small $r$ behavior of $\gamma$ (5.6), we see that $\frac{\partial \gamma}{\partial r} |_{r=r_0} = \frac{1}{\sqrt{6a}} + O(r_0^2)$ and $h$ is of $O(1)$ so that $a_s \sim r_0$. This will be useful later when we try to study the scaling behavior of our M-theory metric.}

50
$x, y, z, \theta_1, \theta_2, r, a$ and $x^a \equiv x^{11}$. The various components are now given by:

\begin{align*}
(1) \quad G_{xx} &= g_1 e^{-\frac{2\phi}{3}} \Delta_1^2 \cot^2 \theta_1 + g_2 e^{-\frac{2\phi}{3}} + e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \theta_1 \\
(2) \quad G_{yy} &= g_1 e^{-\frac{2\phi}{3}} \Delta_2^2 \cot^2 \theta_2 + g_3 e^{-\frac{2\phi}{3}} + e^{\frac{4\phi}{3}} \Delta_4^2 \cot^2 \theta_2 \\
(3) \quad G_{\theta_1 \theta_1} &= e^{-\frac{2\phi}{3}} (g_1 \Delta_1^2 \cot^2 \theta_1 + g_2) b_{x\theta_1}^2 + g_2 e^{-\frac{2\phi}{3}} + b_{x\theta_1}^2 e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \theta_1 \\
(4) \quad G_{\theta_2 \theta_2} &= e^{-\frac{2\phi}{3}} (g_1 \Delta_2^2 \cot^2 \theta_2 + g_3) b_{y\theta_2}^2 + g_3 e^{-\frac{2\phi}{3}} + b_{y\theta_2}^2 e^{\frac{4\phi}{3}} \Delta_4^2 \cot^2 \theta_2 \\
(5) \quad G_{x\theta_1} &= -e^{-\frac{2\phi}{3}} (g_1 \Delta_1^2 \cot^2 \theta_1 + g_2) b_{x\theta_1} - b_{x\theta_1} e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \theta_1 \\
(6) \quad G_{y\theta_2} &= -e^{-\frac{2\phi}{3}} (g_1 \Delta_2^2 \cot^2 \theta_2 + g_3) b_{y\theta_2} - b_{y\theta_2} e^{\frac{4\phi}{3}} \Delta_4^2 \cot^2 \theta_2 \\
(7) \quad G_{x\theta_2} &= -e^{-\frac{2\phi}{3}} \left( g_1 \Delta_1 \Delta_2 \cot \theta_1 \cot \theta_2 - \frac{g_4}{2} \cos \psi \right) b_{y\theta_2} + \frac{g_4}{2} e^{-\frac{2\phi}{3}} \sin \psi + e^{\frac{4\phi}{3}} \Delta_3 \Delta_4 \cot \theta_1 \cot \theta_2 b_{y\theta_2} \\
(8) \quad G_{y\theta_1} &= -e^{-\frac{2\phi}{3}} \left( g_1 \Delta_1 \Delta_2 \cot \theta_1 \cot \theta_2 - \frac{g_4}{2} \cos \psi \right) b_{x\theta_1} + \frac{g_4}{2} e^{-\frac{2\phi}{3}} \sin \psi + e^{\frac{4\phi}{3}} \Delta_3 \Delta_4 \cot \theta_1 \cot \theta_2 b_{x\theta_1} \\
(9) \quad G_{\theta_1 \theta_2} &= e^{-\frac{2\phi}{3}} \left( g_1 \Delta_1 \Delta_2 \cot \theta_1 \cot \theta_2 - \frac{g_4}{2} \cos \psi \right) b_{x\theta_1} b_{y\theta_2} + \frac{g_4}{2} e^{-\frac{2\phi}{3}} \cos \psi + e^{\frac{4\phi}{3}} \Delta_3 \Delta_4 \cot \theta_1 \cot \theta_2 b_{x\theta_1} b_{y\theta_2} \\
(10) \quad G_{xy} &= e^{-\frac{2\phi}{3}} \left( g_1 \Delta_1 \Delta_2 \cot \theta_1 \cot \theta_2 - \frac{g_4}{2} \cos \psi \right) - \frac{g_4}{2} e^{-\frac{2\phi}{3}} \sin \psi (b_{x\theta_1} + b_{y\theta_2}) - e^{\frac{4\phi}{3}} \Delta_3 \Delta_4 \cot \theta_1 \cot \theta_2 b_{x\theta_1} b_{y\theta_2} \\
(11) \quad G_{xx} &= g_1 e^{-\frac{2\phi}{3}} \Delta_1 \cot \theta_1 \\
(12) \quad G_{yy} &= g_1 e^{-\frac{2\phi}{3}} \Delta_2 \cot \theta_2 \\
(13) \quad G_{zz} &= g_1 e^{-\frac{2\phi}{3}} \\
(14) \quad G_{z\theta_1} &= -g_1 e^{-\frac{2\phi}{3}} \Delta_1 \cot \theta_1 b_{x\theta_1} \\
(15) \quad G_{z\theta_2} &= -g_1 e^{-\frac{2\phi}{3}} \Delta_2 \cot \theta_2 b_{y\theta_2} \\
(16) \quad G_{rr} &= e^{\frac{2\phi}{3}} \gamma' \sqrt{h} \\
(17) \quad G_{ax} &= \Delta_3 e^{\frac{4\phi}{3}} \cot \theta_1 \\
(18) \quad G_{ay} &= -\Delta_4 e^{\frac{4\phi}{3}} \cot \theta_2 \\
(19) \quad G_{a\theta_1} &= -\Delta_3 e^{\frac{4\phi}{3}} \cot \theta_1 b_{x\theta_1} \\
(20) \quad G_{a\theta_2} &= \Delta_4 e^{\frac{4\phi}{3}} \cot \theta_2 b_{y\theta_2} \\
(21) \quad G_{aa} &= e^{\frac{4\phi}{3}} \\
(6.24)
\end{align*}

The remaining seven components are all vanishing for this specific case:

\begin{align*}
G_{rx} &= G_{ry} = G_{rz} = G_{r\theta_1} = G_{r\theta_2} = G_{ra} = G_{az} = 0 \quad (6.25)
\end{align*}

with the spacetime metric $e^{-\frac{2\phi}{3}} h^{-\frac{1}{2}}$. Therefore (6.24) and (6.25) are basically the 28 components of our metric.

Having gotten the precise components and the one forms that describe our background, we can now get back to some of the questions that we raised earlier in the type IIA section. Our first question was to verify the non-Kähler nature of the type IIA picture (6.23). To do this we need the vielbeins. They can be calculated from the one forms (6.1) and (6.2).
With the assumption

\[ n_1 = n_2 = \Delta_1 = \Delta_3, \quad n_3 = n_4 = \Delta_2 = \Delta_4 \]  

(6.26)

the vielbeins are now easy to determine. They can be extracted from the metric components (6.24), (6.25) and the one forms (6.1), (6.2) if we put \( \phi = 0 \) in (6.24), and are defined as

\[ e^a = e^a_\mu \; dx^\mu = e^a_x \; dx + e^a_y \; dy + e^a_z \; dz + e^a_{\theta_1} \; d\theta_1 + e^a_{\theta_2} \; d\theta_2 + e^a_r \; dr, \]  

(6.27)

where \( a = 1, \ldots, 6 \) and \( e^a_\mu \) are given by (recall that \( \xi = \sqrt{g_3/g_2} \)):

\[
\begin{align*}
\frac{e_x^1}{\sqrt{2g_2 - g_4/\xi}} & = \frac{1}{2} \sin \psi_1 = \frac{e_x^3}{\sqrt{g_4/\xi + 2g_2}}, \\
\frac{e_y^1}{\sqrt{2g_2 - g_4/\xi}} & = \frac{1}{2} \sin \psi_2 = \frac{e_y^3}{\sqrt{g_4/\xi + 2g_2}}, \\
\frac{e_x^2}{\sqrt{2g_2 - g_4/\xi}} & = \frac{1}{2} \cos \psi_1 = \frac{e_x^4}{\sqrt{g_4/\xi + 2g_2}}, \\
\frac{e_y^2}{\sqrt{2g_2 - g_4/\xi}} & = \frac{1}{2} \cos \psi_2 = \frac{e_y^4}{\sqrt{g_4/\xi + 2g_2}}, \\
e^5_x & = \sqrt{g_1} \Delta_1 \cot \hat{\theta}_1, \quad e^5_y = \sqrt{g_1} \Delta_2 \cot \hat{\theta}_2, \quad e^5_{\theta_1} = -\sqrt{g_1} \Delta_1 \tan \lambda_1 \cot \hat{\theta}_1, \quad e^5_{\theta_2} = \sqrt{g_1} \Delta_2 \tan \lambda_2 \cot \hat{\theta}_2,
\end{align*}
\]

(6.28)

with the remaining components all vanishing except \( e^6_r \) given by \( e^6_r = \sqrt{\gamma'/\xi} \), where we have defined \( \gamma' \) in the beginning of section 5. These are the local vielbeins for our case, and the metric (6.23) can also be written in terms of (6.28). In fact, as is well known, both the metric and the fundamental two form \( J \) for our case can be written using the vielbeins (6.28) as:

\[ ds^2 = \sum_{a=1}^{6} e^a \otimes e^a, \quad J = \sum_{a,b} e^a \wedge e^b. \]  

(6.29)

One may also define complex vielbeins as \( (e^1 + ie^2), \; (e^3 + ie^4) \) and \( (e^5 + ie^6) \), then we get \( J = (e^1 \wedge e^2) + (e^3 \wedge e^4) + (e^5 \wedge e^6) \). This reads in components as:

\[
\begin{align*}
J & = (\alpha_2^2 - \alpha_1^2) \sin \psi \, dx \wedge dy - (\alpha_2^1 + \alpha_2^2) \, dx \wedge d\theta_1 + (\alpha_1^1 + \alpha_2^2) \, dy \wedge d\theta_2 \\
& \quad + (\alpha_2^1 - \alpha_1^2) \left[ \cos \psi - \sin \psi \tan \lambda_2 \right] \, dx \wedge d\theta_2 - (\alpha_2^2 - \alpha_1^2) \left[ \cos \psi - \sin \psi \tan \lambda_1 \right] \, dy \wedge d\theta_1 \\
& \quad - (\alpha_2^2 - \alpha_1^1) \left[ \cos \psi \left( \tan \lambda_1 + \tan \lambda_2 \right) + \sin \psi \left( 1 - \tan \lambda_1 \tan \lambda_2 \right) \right] \, d\theta_1 \wedge d\theta_2 \\
& \quad + \alpha_4 \alpha_5 \Delta_1 \cot \hat{\theta}_1 \, dx \wedge dr + \alpha_4 \alpha_5 \Delta_2 \cot \hat{\theta}_2 \, dy \wedge dr + \alpha_4 \alpha_5 \, dz \wedge dr \\
& \quad - \alpha_4 \alpha_5 \Delta_1 \cot \hat{\theta}_1 \tan \lambda_1 \, d\theta_1 \wedge dr - \alpha_4 \alpha_5 \Delta_2 \cot \hat{\theta}_2 \tan \lambda_2 \, d\theta_2 \wedge dr.
\end{align*}
\]

(6.30)
From here one can easily compute $dJ$ and show that in general $dJ \neq 0$ because of terms like $d\lambda_i$. This implies that the metric (6.23) is in general not Kähler.

Having reached the explicit form of the background, it is now time to pause again a little to discuss some of the expected mathematical properties of these backgrounds. The six dimensional manifold that we gave in (6.23) has an $SU(3)$ structure, is non-Kähler and in general could be non-complex. Mathematical properties of such manifolds have been discussed in some details in [27], [68], [33], [19] though an explicit example have never been given before. To our knowledge, the examples that we gave in this paper, are probably the first ones.

The holonomy of these manifolds are measured not wrt to the usual Riemannian connection, but wrt so-called torsional connection. Earlier concrete examples of this were given in [69], [32], [70], [34], [36]. These examples dealt mostly with heterotic theory and were non-Kähler but compact with integrable complex structures. The examples that we gave here are mostly non-compact, though compact examples could also be constructed with some effort. Both the heterotic cases and the type II cases are examples of torsional manifolds. As discussed in [27], [19] and [33], the torsional manifolds are classified by torsion-classes $W_i$, with $i = 1, 2, ..., 5$. In fact, the torsion $\mathcal{T}$ belongs to the five classes:

$\mathcal{T} \in W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5$ (6.31)

which in turn is related to the $SU(3)$ irreducible representations. To determine the torsion classes for our case, we need the $(3,0)$ form $\Omega$:

$$\Omega \equiv \Omega_+ + i\Omega_- = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)$$ (6.32)

where $e^i$ are computed in (6.28). From the definition of the fundamental two form $J$ in (6.29) it can be observed that both $\Omega_\pm$ are annihilated by $J$ and $\Omega_+ \wedge \Omega_- = \frac{2}{3} J \wedge J \wedge J$. For our case,

$$\Omega_+ = e^1 \wedge e^3 \wedge e^5 - e^2 \wedge e^4 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6$$

$$\Omega_- = e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^3 \wedge e^5 + e^1 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^6$$ (6.33)

Using the three forms, and the fundamental two form $J$ all the torsion classes can be constructed. The fundamental two form $J$ and the holomorphic $(3,0)$ form $\Omega$ are not covariantly constant, as we had observed earlier in [71], [32], [34], [36], and they obey

$$D_m J_{np} = \nabla_m J_{np} - T_{mn}^\ r J_{rp} - T_{mp}^\ r J_{nr} = 0$$

$$D_m \Omega_{nmp} = \nabla_m \Omega_{nmp} - T_{mn}^\ r \Omega_{rpq} - T_{mp}^\ r \Omega_{nrq} - T_{mq}^\ r \Omega_{npr} = 0.$$ (6.34)
The fact that the complex structure is also not integrable might be argued from the definition of the complex vielbeins: \( e^i + ie^{i+1} \), that we gave earlier. A more detailed analysis of the mathematical discussion that we gave here will be relegated to part II. The fact that the fundamental two form is not covariantly constant implies that \( dJ \) involves \((3,0)\) and \((2,1) +c.c\) pieces. The definition of \( d \) becomes:

\[
d\omega^{(p,q)} = d\omega^{(p-1,q+2)} + d\omega^{(p,q+1)} + d\omega^{(p+1,q)} + d\omega^{(p+2,q-1)} \tag{6.35}
\]

which, for an integrable complex structures will not have the \( d\omega^{(p-1,q+2)} + d\omega^{(p+2,q-1)} \) pieces.

The lift of these six dimensional manifolds in M-theory gives rise to \( G_2 \) manifolds in M-theory equipped with a \( G_2 \) structure. The \( G_2 \) structure is specified by a three form \( \hat{\Omega} \) that could be easily evaluated from the vielbeins. We have already defined six vielbeins earlier. Let us define \( e^7 = \sigma_3 + \Sigma_3 \). Using this we can use the \( SU(3) \) structure to determine a \( G_2 \) structure on the seven manifold as:

\[
\hat{\Omega} = J \wedge e^7 + \Omega_+ \\
= e^1 \wedge e^3 \wedge e^5 - e^2 \wedge e^4 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 \\
+ e^1 \wedge e^2 \wedge e^7 + e^3 \wedge e^4 \wedge e^7 + e^5 \wedge e^6 \wedge e^7. \tag{6.36}
\]

This determines the \( G_2 \) structure induced from the \( SU(3) \) structure of (6.23). The \( G \) fluxes will give rise to a three form \( G_3 = \ast_7 G \) on the seven manifold. This \( G_3 \) is not related to the torsion, the torsion three form being given by

\[
\tau = - \ast d\hat{\Omega} - \ast \left[ \frac{1}{3 \ast} (\ast d\hat{\Omega} \wedge \hat{\Omega}) \wedge \hat{\Omega} \right]. \tag{6.37}
\]

To see how the connections in both the theories behave, the reader may want to look into [72]. The equivalent to the torsion classes are now the four modules [73]:

\[
\mathcal{T} \in \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4 \tag{6.38}
\]

where \( \chi_i \) are the \( 1, 14, 27, 7 \) of \( SO(7) \). For our case we do not have a closed three form and therefore the manifold will have a \( G_2 \) structure. A \( G_2 \) structure of the type \( \chi_1 \oplus \chi_3 \oplus \chi_4 \) is in general integrable with a Dolbeault cohomology given in [74]. For further details see Appendix 2.
There are many questions that arise now regarding the seven dimensional manifold that we presented. They will be tackled in the sequel to this paper. To continue, we will assume that the manifold has an explicit $G_2$ structure. It goes without saying, of course that, since we followed strict mirror rules the background that we get should always preserve the set of conditions required.

7. Chain 3: M-theory Flop and Type IIA Reduction

Having the $G_2$ manifold, we can perform a flop transition and reduce to the corresponding type IIA picture. Before we perform the flop, we revisit the one-forms of the previous section. We will continue following the steps laid out in [28], [29], since the local behavior of our metric is almost that of [28], [29].

To proceed further, first define a set of $2 \times 2$ matrices $N_1$ and $N_2$ in the following way:

$$
N^\lambda_1 \equiv \left( \begin{array}{cc}
\sigma_3 & e^{i\psi_1} \left[ e^{-i\lambda_1} \sec \lambda_1 \, d\theta_1 - i \, dx \right] \\
-\sigma_3 & e^{i\psi_1} \left[ e^{i\lambda_1} \sec \lambda_1 \, d\theta_1 - i \, dx \right]
\end{array} \right) \tag{7.1}
$$

$$
N^\lambda_2 \equiv \left( \begin{array}{cc}
\Sigma_3 & \xi e^{i\psi_2} \left[ e^{-i\lambda_2} \sec \lambda_2 \, d\theta_2 + i \, dy \right] \\
-\Sigma_3 & \xi e^{i\psi_2} \left[ e^{i\lambda_2} \sec \lambda_2 \, d\theta_2 - i \, dy \right]
\end{array} \right) \tag{7.2}
$$

where $\sigma_3$ and $\Sigma_3$ are the one form appearing in (6.1) and (6.2), and we have again introduced the factor $\xi$ to account for the asymmetry in the $\hat{x}$ and $\hat{y}$ terms. The off–diagonal terms in the matrices are contributions from the torsional part of the metric in the type IIA theory. Observe that, in the absence of $B$ fields (in the original type IIB theory), these matrices will take the following known form:

$$
N^0_1 \equiv \left( \begin{array}{cc}
d\psi_1 + \Delta_1 \cot \hat{\theta}_1 \, dx \\
e^{-i\psi_1}(d\theta_1 - i \, dx)
\end{array} \right) \begin{array}{c} e^{i\psi_1}(d\theta_1 - i \, dx) \\
-d\psi_1 - \Delta_1 \cot \hat{\theta}_1 \, dx
\end{array} \tag{7.3}
$$

$$
N^0_2 \equiv \left( \begin{array}{cc}
d\psi_2 - \Delta_2 \cot \hat{\theta}_2 \, dy \\
\xi e^{-i\psi_2}(d\theta_2 + i \, dy)
\end{array} \right) \begin{array}{c} \xi e^{i\psi_2}(d\theta_2 + i \, dy) \\
-d\psi_2 + \Delta_2 \cot \hat{\theta}_2 \, dy
\end{array} \tag{7.3}
$$

which can be easily derived from the one forms given in [28], [29].

The matrices (7.1) and (7.2) can be combined in various ways to create new $2 \times 2$ matrices for our space. A generic combination will be

$$
N^{\lambda_1,\lambda_2}_{[a,b]} = a \, N^{\lambda_1}_1 - b \, N^{\lambda_2}_2 \tag{7.4}
$$

with integral $a, b$. Using various choices of $a, b$ we can express our eleven dimensional metric. Locally however, one can show, that there are two choices given by $N^{\lambda_1,\lambda_2}_{[1,1]}$ and
that are specifically useful to write the M-theory metric. In the absence of $B$ fields (in the original type IIB picture) these matrices are related to the left invariant one forms $\omega$ and $\tilde{\omega}$ for the $G_2$ spaces, i.e.

$$N_{[0,1]}^{\lambda_1 \lambda_2} \rightarrow \tilde{\omega}, \quad N_{[0,1]}^{00} \rightarrow \omega$$

(7.5)

as an equality up to $SU(2)$ group elements. Of course, since for our case there is no underlying $SU(2)$ symmetry globally (our manifold being non-Kähler from the start i.e. directly in the type IIA case) these relations are only in local sense. In the presence of $B$ fields – or non trivial fibrations in the mirror/M-theory set-up – the metric can still be written in terms of $N_{[a,b]}^{\lambda_1 \lambda_2}$ even though they are no longer related to $\omega$ and $\tilde{\omega}$. In fact, if we remove the restriction on $a, b$ in (7.4) as simple constants and allow more generic values for them, we can express our M-theory metric completely in terms of (7.4) in the following way:

$$ds^2 = -\left[ \det N_{[\beta, \beta]}^{\lambda_1 \lambda_2} - \text{Tr}^2 (N_{[\gamma, \gamma]}^{\lambda_1 \lambda_2} \cdot \Gamma_3) \right] - \left[ \det N_{[\delta, -\delta]}^{\lambda_1 \lambda_2} - \text{Tr}^2 (N_{[\epsilon, -\epsilon]}^{\lambda_1 \lambda_2} \cdot \Gamma_3) \right]$$

(7.6)

where $\Gamma_3$ is the third Pauli matrix, and $\beta, \gamma, \delta$ and $\epsilon$ can be extracted from (6.21) as:

$$\beta = \frac{e^{-\phi}}{2} \sqrt{2g_2 - g_4/\xi}, \quad \gamma = \frac{e^{-\phi}}{4} \sqrt{4g_1 - 2g_2 + g_4/\xi}$$

$$\delta = \frac{e^{-\phi}}{2} \sqrt{2g_2 + g_4/\xi}, \quad \epsilon = \frac{e^{-\phi}}{4} \sqrt{4e^{2\phi} - 2g_2 - g_4/\xi}$$

(7.7)

In the absence of fluxes i.e. $\lambda_i = 0$, on the other hand, it is conjectured that the M-theory metric can be written in terms of $N_{[a,b]}^{\lambda_1 \lambda_2}$ as [14]

$$ds^2 = -\det N_{[A,A]}^{00} - \det N_{[B,-B]}^{00}$$

(7.8)

where the values of $A$ and $B$ are defined at the radial distances $r = r_0$. We see that this type of metric is not realized in out set-up because (7.6) do not reduce to (7.8) by making $\lambda_i = 0$. To compare our result (7.6) to the one without fluxes, we need the metric of the $G_2$ manifold with terms of the form $\det N^{00} - \text{Tr}^2 (N^{00} \cdot \Gamma_3)$. Taking this limit is of course possible and an example of this has been given in [28], [29]. Our manifold would therefore resemble this scenario, although one has to be careful here. The identification is only local where the $\lambda_i$ values are approximately constant. The matrices $N_i$ in the presence and in the absence of fluxes differ by $\lambda_i$ terms, giving rise to one forms that do not have any
underlying SU(2) symmetry. To summarize the situation, our metric (7.6) in the presence of fluxes gives rise to new $G_2$ holonomy manifolds that have not been studied before. In the absence of fluxes, (7.6) gives rise to one of the examples studied in [28] (see e.g. equation (3.5), and discussion) which in our notation would look like:

$$
\text{ds}^2 = - \left[ \det N^{00}_{[a,a]} - \text{Tr}^2 (N^{00}_{[b,b]} \cdot \Gamma_3) \right] - \left[ \det N^{00}_{[c,-c]} - \text{Tr}^2 (N^{00}_{[d,-d]} \cdot \Gamma_3) \right] \quad (7.9)
$$

with $a, b, c$ and $d$ are defined at the radial coordinate $r = r_0$ as:

$$
a = \sqrt{\frac{4r_0^2 + 12r_0 - 27}{48}}, \quad b = \sqrt{\frac{36r_0 - 4r_0^2 - 81}{24}}, \quad c = \sqrt{\frac{4r_0^2 - 12r_0 - 27}{48}}
$$

$$
d = \sqrt{\frac{16r_0^4 - 48r_0^3 - 148r_0^2 + 108r_0 + 324}{48(4r_0^2 - 9)}} \quad (7.10)
$$

As mentioned in [28], [29], there are alternative expressions for the values of $a, b, c$ and $d$ obtained by scaling the metric. Then the scale factor will appear in the set of relations (7.10). This will be useful for us to get rid of $\epsilon^{-1}$ factors in the type IIA three form $d\tilde{B}$ with $\tilde{B}$ given earlier in (5.60). We will discuss this soon. For more details the readers can see sec. 3 of [28]. This is the point where our conclusions would differ from the results of [49] where (7.8) is presented as the M-theory lift of the type IIA configuration.

To proceed further, we need to perform the flop transition and go to the corresponding type IIA picture. On the other hand, if we followed [49], we would see from the one forms (6.1) and (6.2) that we now have two possible directions along which we can compactify and come down to type IIA: $dx$ and $d\theta_1$ (or correspondingly $dy$ and $d\theta_2$). But the $\theta_1$ direction is not globally defined as it comes with the corresponding type IIA $b_2, \theta_1$ field. So the compactification to type IIA should rather be performed along $dx$. To see this, let us first write the M-theory metric in the following suggestive way:

$$
\text{ds}^2 = e^{-\frac{2\phi}{3}} g_1 (dz + \gamma_1 d\theta_1 + \gamma_2 D\tilde{y})^2 + e^{\frac{4\phi}{3}} (dx_{11} + \gamma_3 d\theta_1 + \gamma_4 D\tilde{y})^2 + \\
+ e^{-\frac{2\phi}{3}} g_3 \left( d\theta_2^2 + D\tilde{y}^2 \right) + e^{-\frac{2\phi}{3}} g_2 \cot^2 \lambda_1 d\theta_1^2 + g_5 (dx + A)^2 - g_5 A^2 + \\
+ e^{-\frac{2\phi}{3}} g_4 \cos (\psi + \lambda_1) \sec \lambda_1 d\theta_1 d\theta_2 + e^{-\frac{2\phi}{3}} g_4 \sin (\psi + \lambda_1) \sec \lambda_1 D\tilde{y} d\theta_1 \quad (7.11)
$$

where we have already defined $D\tilde{y} \equiv dy - \tan \lambda_2 d\theta_2$ earlier, and the $dx$ fibration structure is represented as $dx + A$, $A$ being the corresponding one form that will appear in type IIA
as gauge fields. The other variables appearing in (7.11) can be defined as follows:

\[
\begin{align*}
\gamma_1 &= -\Delta_1 \cot \hat{\theta}_1 \tan \lambda_1, \\
\gamma_2 &= \Delta_2 \cot \hat{\theta}_2, \\
\gamma_3 &= -\Delta_3 \cot \hat{\theta}_1 \tan \lambda_1 \\
\gamma_4 &= -\Delta_4 \cot \hat{\theta}_2, \\
g_5 &\equiv e^{\frac{4\phi}{3}} = (g_1 \Delta_1^2 \cot^2 \hat{\theta}_1 + g_2) e^{-\frac{2\phi}{3}} + e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \hat{\theta}_1 \\
A &= A_1 \, dz + A_2 \, d\theta_1 + A_3 \, d\theta_2 + A_4 \, D\hat{y} + A_5 \, dx_{11}
\end{align*}
\]

where \( \Phi \) is the type IIA dilaton and \( A_i \) are the components of the gauge fields defined in the following way:

\[
\begin{align*}
A_1 &= \frac{e^{-\frac{2\phi}{3}} g_1 \Delta_1 \cot \hat{\theta}_1}{(g_2 + g_1 \Delta_1^2 \cot^2 \hat{\theta}_1) e^{-\frac{2\phi}{3}} + e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \hat{\theta}_1}, \\
A_2 &= -\tan \lambda_1, \\
A_3 &= \frac{\frac{1}{2} e^{-\frac{2\phi}{3}} g_4 \sin \psi}{(g_2 + g_1 \Delta_1^2 \cot^2 \hat{\theta}_1) e^{-\frac{2\phi}{3}} + e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \hat{\theta}_1}, \\
A_5 &= \frac{e^{\frac{4\phi}{3}} \Delta_3 \cot \hat{\theta}_1}{(g_2 + g_1 \Delta_1^2 \cot^2 \hat{\theta}_1) e^{-\frac{2\phi}{3}} + e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \hat{\theta}_1}, \\
A_4 &= \frac{\cot \hat{\theta}_1 \cot \hat{\theta}_2 (e^{-\frac{2\phi}{3}} g_1 \Delta_1 \Delta_2 - e^{\frac{4\phi}{3}} \Delta_3 \Delta_4) - \frac{1}{2} e^{-\frac{2\phi}{3}} g_4 \cos \psi}{(g_2 + g_1 \Delta_1^2 \cot^2 \hat{\theta}_1) e^{-\frac{2\phi}{3}} + e^{\frac{4\phi}{3}} \Delta_3^2 \cot^2 \hat{\theta}_1}
\end{align*}
\]

The metric (7.12) would basically be our answer, but if we look closely we see that (7.12) do not resemble the expected resolved conifold metric in the absence of \( b_{x\theta_1}, b_{y\theta_2} \). Therefore instead of reducing along \( dx \), which would not give us the expected resolved conifold metric in the absence of \( b_{x\theta_1}, b_{y\theta_2} \) without further coordinate transformation, we will consider performing a flop in M-Theory and then reduce along \( dx_{11} \). To consider the flop and the subsequent change in the metric we have to do a transformation to our M-theory metric (6.20). Before moving ahead, let us clarify one minor thing regarding the scalings of the metric (6.20). As discussed in [28, 29], scaling the metric from \( G \rightarrow a^2 \, G \) keeps the \( G_2 \) structure intact. We can use this freedom of rescaling the metric to remove the \( e^{-\frac{\phi}{3}} \) factor in (5.60). The coordinates of the \( G_2 \) manifold are given in terms of \( x, y, z, \theta_1, \theta_2 \) and \( x_{11} \) at a point \( r = r_0 \). Let us scale the radial coordinate \( r_0 \) as \( r_0 \rightarrow \epsilon^{\frac{1}{3}} r_0 \). We also want to scale \( dz \) as \( \epsilon^{\frac{1}{3}} \, dz \) so that \( d\psi \) would not scale. This is important, since now the \( \sin \psi \) and \( \cos \psi \) in the metric (6.23) will remain unchanged. If we now rescale

\[
x, y, z, \theta_i, x_{11} \rightarrow \epsilon^{\frac{1}{3}} \, x, \epsilon^{\frac{1}{3}} \, y, \epsilon^{\frac{1}{3}} \, z, \epsilon^{\frac{1}{3}} \, \theta_i, \epsilon^{\frac{1}{3}} \, x_{11}
\]

the metric (6.23) or the complete M-theory metric (6.20) will have an overall scale of \( \epsilon^{\frac{1}{3}} \) (provided, of course, that we also scale the \( B \) field \( \tilde{b}_{mn} \) in the fibration accordingly). This
metric preserves $G_2$ structure as before, but now the three form flux $C_3$ coming from the
two form $B_{NS}$ in (5.65) as $C_3 = \hat{B} \wedge dx_{11}$ will be finite. Thus, we can make the background
finite using the rescaling freedom, and therefore this gives us confidence in considering only
the finite part of the background (5.65) and the metric (6.23) without any $\epsilon$ dependences
anywhere.

After this detour, it is now time to consider the issue of flop on the M-theory metric
(6.20). This way we will be able to connect the final answer, after dimensional reduction,
to the type IIA metric implied above in (7.11). A simple way to guess the answer would be
to restore the case without any torsion. In the absence of torsion the type IIA reduction
should be a resolved conifold with fluxes and no $D6$ branes. This would imply that the
metric looks like two tori. If we now use our one forms (6.1) and (6.2), one way to generate
this would be if we consider the transformation on $N^{[a,b]}_{\lambda_1 \lambda_2}$ as:

$$N^{[1,0]}_{\lambda_1 \lambda_2} \rightarrow N^{[\frac{1}{1-f}, \frac{1}{1-f}]}_{\lambda_1 \lambda_2}, \quad N^{[0,1]}_{\lambda_1 \lambda_2} \rightarrow N^{[\frac{1}{1-f}, \frac{1}{1-f}]}_{\lambda_1 \lambda_2}$$

upto possible conjugations. Locally, the above relation will imply a similar relation in the
absence of type IIB fluxes. We have also kept a parameter $f$ in (7.14). Therefore, the
transformation (7.14) will convert our case (7.6) to:

$$ds^2_{\text{Flop}} = -\left[ \det N^{[\beta,0]}_{\lambda_1 \lambda_2} - \text{Tr}^2 \left( N^{[\gamma,0]}_{\lambda_1 \lambda_2} \cdot \Gamma_3 \right) \right] - \left[ \det N^{[f\delta,\delta]}_{\lambda_1 \lambda_2} - \text{Tr}^2 \left( N^{[f\epsilon,\epsilon]}_{\lambda_1 \lambda_2} \cdot \Gamma_3 \right) \right]$$

upto possible rescaling, and $\beta, \gamma, \delta$ and $\epsilon$ have already been given in (7.7).

In the limit $f \rightarrow 0$ the above metric gives the right tori parts but fails to give the
fibration structure correctly. This implies that a global definition a-la (7.14) may not be
possible here. What went wrong? A careful study of (7.15) reveals that in the summation
of the one forms $\Sigma_a - f \sigma_a, \ a = 1, 2, 3$ we had used the same $f$ for all the three terms. In
the limit where $f \rightarrow 0$ this gives the right torus metric but wrong fibration. A way out of
this can be immediately guessed by having a different factor in the third term, i.e. having
$\Sigma_3 - g \sigma_3$, and the rest with $f$. Locally, this is exactly the one predicted by [75], and
therefore, using out patch argument, we can extend this to all other patches. This implies
the following metric after we make a flop in M-theory:

$$ds^2_{\text{Flop}} = -\left[ \det N^{[\beta,0]}_{\lambda_1 \lambda_2} - \text{Tr}^2 \left( N^{[\gamma,0]}_{\lambda_1 \lambda_2} \cdot \Gamma_3 \right) \right]$$

$$- \left[ \det N^{[f\delta,\delta]}_{\lambda_1 \lambda_2} + \text{Tr}^2 \left( N^{[f\epsilon,\epsilon]}_{\lambda_1 \lambda_2} \cdot \Gamma_3 \right) - \text{Tr}^2 \left( N^{[g\alpha,\alpha]}_{\lambda_1 \lambda_2} \cdot \Gamma_3 \right) \right]$$

$$g \rightarrow 1, \quad f \rightarrow 0$$

(7.16)

where the variables have been defined in (7.7) and (6.21). Thus, before flop the metric is
given by (7.6) and after flop it is given by (7.16).
7.1. The Type IIA Background

To obtain the type IIA theory we can reduce either via $dz$ or via $dx_{11}$. This will not lead us back to the type IIA theory we started with because of the change induced in the metric by the flop. To have a one to one correspondence with the type IIA picture before flop, let us reduce along direction $dx_{11}$. The new $G_2$ metric can now be written in the following suggestive way:

$$
\begin{align*}
    ds^2 &= e^{\frac{4\phi}{3}} \left[ dz + \Delta_1 \cot \hat{\theta}_1 (dx - b_{x\theta_1} \, d\theta_1) + \Delta_2 \cot \hat{\theta}_2 (dy - b_{y\theta_2} \, d\theta_2) \right]^2 \\
    &\quad + e^{-\frac{2\phi}{3}} \left( \frac{g_2}{2} - \frac{g_4}{4\xi} \right) \left[ d\theta_1^2 + (dx - b_{x\theta_1} \, d\theta_1)^2 \right] + e^{-\frac{2\phi}{3}} \left( \frac{g_2}{2} + \frac{g_4}{4\xi} \right) \left[ d\theta_2^2 + (dy - b_{y\theta_2} \, d\theta_2)^2 \right] \\
    &\quad + \frac{1}{4} e^{-\frac{2\phi}{3}} g_1 \left[ dx_{11} + 2\Delta_1 \cot \hat{\theta}_1 (dx - b_{x\theta_1} \, d\theta_1) \right]^2
\end{align*}
$$

(7.17)

which clearly shows that the base is locally a resolved conifold. Note, that we have again used the freedom to absorb $A_z$ into $dx_{11}$. The $dx_{11}$ term in (7.17) is basically the fibration over which we have to reduce to get to type IIA theory. As mentioned above, we can also reduce along $dz$, as the $dx_{11}$ and $dz$ directions can be easily exchanged among each other. The metric (7.17) is thus the right $G_2$ metric after flop and could be compared to (7.11). A redefinition of the coordinates of (7.11) and some coordinate transformation would relate (7.11) to (7.17) and would also simplify the form of the gauge potential given earlier in (7.13). The final type IIA metric after a dimensional reduction turns out to be:

$$
\begin{align*}
    ds^2 &= \frac{1}{4} \left( 2g_2 - \frac{g_4}{\xi} \right) \left[ d\theta_1^2 + (dx - b_{x\theta_1} \, d\theta_1)^2 \right] + \frac{1}{4} \left( 2g_2 + \frac{g_4}{\xi} \right) \left[ d\theta_2^2 + (dy - b_{y\theta_2} \, d\theta_2)^2 \right] \\
    &\quad + e^{2\phi} \left[ dz + \Delta_1 \cot \hat{\theta}_1 (dx - b_{x\theta_1} \, d\theta_1) + \Delta_2 \cot \hat{\theta}_2 (dy - b_{y\theta_2} \, d\theta_2) \right]^2
\end{align*}
$$

(7.18)

which is precisely the metric of a resolved conifold when we switch off $b_{x\theta_1}$ and $b_{y\theta_2}$ (or consider it locally over a patch where $b_{x\theta_1}$ and $b_{y\theta_2}$ are constants). In the presence of $b_{x\theta_1}$ and $b_{y\theta_2}$ we get the “usual” metric but shifted by the generic ansatz that we proposed in (5.63). Therefore, we can now make a precise statement: the metric before geometric transition is given by (6.23), and after the transition is given by (7.18). The type IIA metric has two tori whose radii are proportional to

$$
    r_1 = \frac{1}{2} \sqrt{2g_2 - g_4 \sqrt{g_3 g_2^{-1}}}, \quad r_2 = \frac{1}{2} \sqrt{2g_2 + g_4 \sqrt{g_3 g_2^{-1}}}.
$$

(7.19)
One of them would shrink to zero size while the other doesn’t when we approach the origin. The type IIA coupling is now given by

\[ g_A = 2^{-\frac{7}{2}} e^{-\frac{\pi}{2} \alpha} \alpha^{-\frac{1}{2}} \]  

(7.20)

where \( \alpha \) is defined in (5.8). Observe that the coupling is not a constant but is a function of the internal coordinates, and it doesn’t blow up anywhere in the internal space. This background is the expected background after we perform a geometric transition on (6.23). This means that the D6 branes in (6.23) should completely disappear and should be replaced by fluxes in the type IIA picture. From the \( G_2 \) manifold that we had in (7.17), we see that this is indeed the case, and the gauge fluxes are given by:

\[ A \cdot dX = 2\Delta_1 \cot \hat{\theta}_1 (dx - b_x \theta_1 \, d\theta_1) \]  

(7.21)

which, as one can easily check, looks like the remnant of D6 brane sources modified appropriately by our ansätze (5.63). There are also \( B_{NS} \) fields that originate from the dimensional reduction of the three form fields in M-theory. Since we are reducing along the direction \( dx_{11} \) they would be the same \( \hat{B} \) field that we had in (5.65). The only difference will be that the finite part (which is of course \( \hat{B} \) itself) is now the exact solution as we had removed the \( \epsilon^{-1/2} \) dependence by scaling our \( G_2 \) manifold before flop. The \( B_{NS} \) can be written down directly from (5.65) as:

\[ \frac{B}{\sqrt{\alpha}} = dx \wedge d\theta_1 - dy \wedge d\theta_2 + A \, d\theta_1 \wedge dz - B (\sin \psi \, dy - \cos \psi \, d\theta_2) \wedge dz, \]  

(7.22)

which will again be a pure gauge artifact. Combining (7.18), (7.20), (7.21) and (7.22), we recover the precise background after geometric transition in type IIA picture.

7.2. Analysis of Type IIA Background and Superpotential

In this section we will try to verify the non-Kähler nature of our background and the corresponding superpotential. Other detail aspects, for example non integrability of complex structure, torsion classes etc., will be left for part II of this paper. To check the non-Kählerity of this background we will have to determine the corresponding vielbeins.
They can be easily extracted from (7.18), and are given by:

\[
e = \begin{pmatrix}
e^1_e & e^1_y & e^1_z & e^1_{\theta_1} & e^1_{\theta_2} & e^1_r \\
e^2_e & e^2_y & e^2_z & e^2_{\theta_1} & e^2_{\theta_2} & e^2_r \\
e^3_e & e^3_y & e^3_z & e^3_{\theta_1} & e^3_{\theta_2} & e^3_r \\
e^4_e & e^4_y & e^4_z & e^4_{\theta_1} & e^4_{\theta_2} & e^4_r \\
e^5_e & e^5_y & e^5_z & e^5_{\theta_1} & e^5_{\theta_2} & e^5_r \\
e^6_e & e^6_y & e^6_z & e^6_{\theta_1} & e^6_{\theta_2} & e^6_r 
\end{pmatrix}
\]

\[
e = \begin{pmatrix}
0 & 0 & 0 & r_1 & 0 & 0 \\
r_1 & 0 & 0 & -r_1 b_{x\theta_1} & 0 & 0 \\
0 & 0 & 0 & 0 & r_2 & 0 \\
0 & r_2 & 0 & 0 & -r_2 b_{y\theta_2} & 0 \\
e^\phi \Delta_1 \cot \hat{\theta}_1 & e^\phi \Delta_2 \cot \hat{\theta}_2 & e^\phi & -e^\phi \Delta_1 \cot \hat{\theta}_1 b_{x\theta_1} & -e^\phi \Delta_2 \cot \hat{\theta}_2 b_{y\theta_2} & 0 \\
0 & 0 & 0 & 0 & 0 & e^6_r 
\end{pmatrix}
\]

(7.23)

where \( r_1 \) and \( r_2 \) are the radii of the two tori as defined earlier. We have kept \( e^6_r \) undefined here. But this can also be easily seen to be the usual vielbein for the resolved conifold case in the type IIB picture. Now to check the non-Kählerity we have to construct the fundamental two form \( \mathcal{J} \) using these vielbeins. Before evaluating this, observe that in the absence of \( b_{x\theta_1} \) and \( b_{y\theta_2} \) the manifold should be Kähler with a Kähler form \( J \). In the presence of \( b_{x\theta_1} \) and \( b_{y\theta_2} \) the fundamental form \( \mathcal{J} \) can be written as a linear combination of the usual Kähler form \( J \) and additional \( b_{x\theta_1} \) and \( b_{y\theta_2} \) dependent terms, as

\[
\mathcal{J} = J + e^\phi \left( \Delta_1 \cot \hat{\theta}_1 b_{x\theta_1} e^6_r \wedge d\theta_1 + \Delta_2 \cot \hat{\theta}_2 b_{y\theta_2} e^6_r \wedge d\theta_2 \right)
\]

(7.24)

where \( dJ = 0 \). From above it is easy to see that \( d\mathcal{J} \neq 0 \) in general because of non-zero \( db_{x\theta_1} \) and \( db_{y\theta_2} \). Therefore the manifold (7.18) is a non-Kähler manifold\(^{28}\). For

\(^{28}\) As this point one might wonder about the global behavior of (7.18). Before geometric transition the global type IIA picture had extra six branes and other defects. After geometric transition we would still expect some of the six branes and possibly other defects to reappear. One has to carefully do the flop operation in the presence of these objects to see how many of them would survive in the type IIA side. More details on this will be presented elsewhere.
completeness, let us also write down all the components of the type IIA metric:

\[
g = \begin{pmatrix}
g_{xx} & g_{xy} & g_{xz} & g_{x\theta_1} & g_{x\theta_2} \\
g_{xy} & g_{yy} & g_{yz} & g_{y\theta_1} & g_{y\theta_2} \\
g_{xz} & g_{yz} & g_{zz} & g_{z\theta_1} & g_{z\theta_2} \\
g_{x\theta_1} & g_{y\theta_1} & g_{z\theta_1} & g_{\theta_1 \theta_2} \\
g_{x\theta_2} & g_{y\theta_2} & g_{z\theta_2} & g_{\theta_1 \theta_2} & g_{\theta_2 \theta_2}
\end{pmatrix}
\]

where \( A \) and \( B \) have been defined earlier in (5.14). The other variables appearing in (7.25) can be defined as follows:

\[
C_1 = C + A^2 e^{2\phi}, \quad D_1 = D + B^2 e^{2\phi}, \quad C = \frac{g_2}{2} - \frac{g_4}{4\xi}, \quad D = \frac{g_2}{2} + \frac{g_4}{4\xi}
\]

Thus, (7.25) is the final answer for the type IIA background without any \( D6 \) branes and with two– and three–form field strengths. There are many questions that arise from the explicit background that we have in (7.18) and (7.25). Let us elaborate them:

- The first issue is related to the choice of complex structure for our manifold. The complex structure is written in terms of the fermions, and therefore we have to see how the fermions transform under three T-dualities. From the generic analysis of [76], we see that the T-dual fermions give rise to a complex structures that is in general not integrable (in other words, the Nijenhaus tensor does not vanish). Therefore we will get a non-complex manifold.
- The next issue is related to the non-Kählerity of our manifold. The naive expectation (also from the results of [19]) would be that the manifold we get in type IIA will be half-flat. This comes from the fact that (7.18) is non-Kähler and also non-complex. Half-flat manifolds are classified by torsion classes. For our case all these can be explicitly derived from the metric. Below we will show that the naive expectation is not realized in string theory and our manifold will be more general than a half-flat manifold.
- The third issue is the asymptotic behavior of our metric. We haven’t yet checked whether the metric that we derived above is non-degenerate and non-singular. Although unrelated,
a similar metric with identical $B$ dependent fibration structure found in [39], [34], [36], showed a good asymptotic behavior and was non-degenerate and non-singular. For the present case however, we don’t know the full global metric as our type IIB starting point was the local metric that ignored the seven branes. Once the full global story becomes clear, we should study the singularity behavior of the metric.

• Last but not the least, we need to determine the superpotential that governs our type IIA background. Before doing so, let us start by recalling which the fields are present in M–theory. The NS field (5.65) is lifted to a three-form $C$ by adding a leg in the $x^{11}$ direction. Its derivative is $G = dC$. For the compactification on a $G_2$ manifold $X$ with the invariant 3-form $\tilde{\Omega}$ defined earlier in (6.36), the form of the superpotential was first proposed in [77] as:

$$W = \int_{X} \tilde{\Omega} \wedge G.$$  \hspace{1cm} (7.27)

This form of the superpotential has been corrected in [78], [79] in order to make the right hand side a complex quantity. The superpotential becomes:

$$W = \int_{X} (\tilde{\Omega} + iC) \wedge G.$$  \hspace{1cm} (7.28)

Then, when reducing from 11 dimensions to 10 dimensions as in [49], [75], instead of just obtaining the volume of the resolved conifold $J$, we get the complexified volume $J + iB$, and this is the quantity that enters in the 10 dimensional potential to give

$$\tilde{W}_1 = \int_{X_6} (J + iB) \wedge dH_3.$$  \hspace{1cm} (7.29)

This is true because $\tilde{\Omega}$ descends to $J$ as it loses one leg but $G$ descends as an RR 4-form. The RR 4-form should originate from D4 branes. Since we know that our brane configurations did not contain any D4 branes, there is no contribution from $dH_3$ in the superpotential.

But this is not the full story because of the properties of the compactification manifold. By considering the dimensional reduction on the manifold (7.18), we have to use the fact that the manifold does not have a closed (3,0) form. Let us first recall the results for the
case without torsion [81], [20]. In that case the condition that the (3,0) form is closed is related to a differential equation for a function depending on the radial coordinate. The result was a one parameter family of Calabi-Yau metrics on the resolved conifold.

In our case, the situation is different. We consider equation (7.18) and we read off the vielbeins from (7.23). The holomorphic 3-form is built as before as:

$$\Omega = (e_1 + ie_2) \wedge (e_3 + ie_4) \wedge (e_5 + ie_6). \quad (7.30)$$

From (7.30) we see that the condition for the (3,0) form to be closed implies a differential equation which involves the functions $b_{x\theta_1}, b_{y\theta_2}$, as they appear in $\hat{x}, \hat{y}, \hat{z}$. As the functions $b_{x\theta_1}, b_{y\theta_2}$ are arbitrary, the differential equation will not have a solution for generic values of $b_{x\theta_1}, b_{y\theta_2}$, so our situation is different from the one of [81]. It also differs from the situation of [20] in the sense that our manifold is not Ricci flat because the same differential equation does not have a solution for generic values of $b_{x\theta_1}, b_{y\theta_2}$. For our case one can explicitly evaluate the three-forms. Using the definitions of $D\hat{x}$ and $D\hat{y}$ we can express our result as:

$$\frac{\Omega_+}{\sqrt{4g_2^2 - g_4^2 \xi^{-2}}} = e_r^6 d\theta_1 \wedge d\theta_2 \wedge dr - e^\phi d\theta_1 \wedge D\hat{y} \wedge (dz + \Delta_1 \cot \hat{\theta}_1 D\hat{x}) +$$

$$- e_r^6 D\hat{x} \wedge D\hat{y} \wedge dr + e^\phi d\theta_2 \wedge D\hat{x} \wedge (dz + \Delta_2 \cot \hat{\theta}_2 D\hat{y})$$

$$\frac{\Omega_-}{\sqrt{4g_2^2 - g_4^2 \xi^{-2}}} = e^\phi d\theta_1 \wedge d\theta_2 \wedge (dz + \Delta_1 \cot \hat{\theta}_1 D\hat{x} + \Delta_2 \cot \hat{\theta}_2 D\hat{y}) +$$

$$+ e_r^6 (d\theta_1 \wedge D\hat{y} \wedge dr + D\hat{x} \wedge d\theta_2 \wedge dr) - e^\phi D\hat{x} \wedge D\hat{y} \wedge dz \quad (7.31)$$

where $e_r^6$ is the associated vielbein for the $r$ direction. From above we see that both $d\Omega_+$ and $d\Omega_-$ will not vanish for the background that we have. The existence of $b_{x\theta_1}, b_{y\theta_2}$ implies the non-closeness of the holomorphic 3-form $\Omega$. Therefore our manifold is a specific non-complex, non-Kähler manifold that is not half-flat. Manifolds with an $\Omega$ which is not closed have been studied in [13] where four forms $F^{2,2} \propto (d\Omega)^{2,2}$ correspond to harmonic forms measuring flux, and they can be expanded in some basis.\(^31\) The four forms can

\(^{30}\) This statement is equivalent to saying that the manifold is Ricci flat.

\(^{31}\) For integrable complex structures one could expand in $h^{1,1}$ basis, although if the manifold is simultaneously non-Kähler this would be tricky. Recall also that we are using $d\Omega \equiv d [\Omega^{(3,0)}] = d\Omega^{(2,2)} + d\Omega^{(3,1)} + d\Omega^{(4,0)}$ for the non-complex manifold.
then be combined with the independent holomorphic 2 forms to give contributions to the superpotential as

$$(J + iB) \wedge d\Omega.$$  \hspace{1cm} (7.32)

This way we encounter a first concrete example where the superpotential gets an extra piece from the non closed holomorphic 3-form. The case in [19] involved an $\Omega$ with only the real part non– closed and the manifold was a half flat manifold. Our case is more general, as both $d\Omega_+$ and $d\Omega_-$ can be non zero as functions of $b_{x\theta_1}, b_{y\theta_2}$.

8. Discussion and future directions

The subject of the present work was to clarify issues concerning NS fluxes in geometric transitions and the non-Kähler geometries arising in the mirror pictures. Our starting point was the observation of Vafa [5] that the closed string dual to D6 branes wrapped on a deformed conifold is not a Kähler geometry. To obtain this mysterious departure from Kählerity, we started\textsuperscript{32} from a IIB picture with D5 branes wrapped on a $P^1$ cycle inside the resolved conifold and went to the mirror picture by performing three T-dualities on the fiber $T^3$. The result was a non-Kähler geometry whose metric could be given precisely as a non-Kähler deformation of a deformed conifold. We then lifted this to M theory where the result was a new $G_2$ manifold with torsion. A flop inside the $G_2$ manifold and a reduction to type IIA brought us to the closed IIA picture with a non-Kähler deformation of the resolved conifold. The latter non-Kählerity can then be traced back to the existence of the NS flux in the initial type IIB picture. Our final result is not quite a half-flat manifold as anticipated earlier [19], but is actually more general because the imaginary part of the holomorphic 3-form is also not closed.

On the way we also solved some puzzles regarding the T-duality between branes wrapped on the deformed and resolved conifold. Previous attempts started from the deformed conifold and the problems encountered were related to the fact that this is not a toric variety. We started with the resolved conifold which is a toric variety and identified the $T^3$ fibration.

\textsuperscript{32} In terms of the dual $N = 1$ gauge theory this is the IR of the gauge theory. In terms of geometry this is the region where $r$, the radial parameter, is small.
8.1. Future directions

There are many unanswered questions that we left for future work. A sample of them are as follows:

- In the figure we have drawn, there is an extra step which should be covered, the mirror symmetry that goes from the closed IIA to closed type IIB and it would be interesting to see whether this would give rise to a Kähler geometry with NS flux or to a non-Kähler geometry. Unfortunately, there is an immediate problem that one would face while doing the mirror transformation. The background that we have has lost the isometry along the $z$ direction. Recall that the type IIB background that we started with in the beginning of the duality chain had complete isometries along the $x, y$ and $z$ directions. In the final type IIA picture the metric does surprisingly have all the isometries, but the $B$ field breaks it. Maybe a transformation of the form (4.33) could be used here to get the mirror metric.

- To get the cross terms in the type IIA mirror metric (on the $D6$ brane side) we had used only a set of restricted coordinate transformations which only lie on the two $S^3$ directions of the corresponding type IIB picture. It will now be interesting to see whether this could be generalized for the case where we could consider $\delta r$ variations. In particular, for a particular $S^3$ parametrized by $(\psi_1, \theta_1, x)$, one should now consider both $\delta r$ and $\delta \theta_2$ variations for a given variation of $x, \theta_1$ and $\psi_1$. Similar discussion should be done for the other $S^3$ parametrized by $(\psi_2, \theta_2, y)$.

- In the type IIA mirror background we have $B$ fields both before and after geometric transitions. On the $D6$ brane side, the presence of a $B$ field amounts to having non-commutativity on the world volume of $D6$ branes. However, this $B$ field is in general not a constant, and therefore may not have such a simple interpretation. This is somewhat related to a discussion on $C$-deformation in [82]. The $C$-deformations in general violates Lorentz invariance. It will be interesting to see if there is any connection to our result, or if the precise background that we propose does indeed realize the Lorentz violation and $C$-deformations.

- As discussed above, the manifold that we have in 11 dimensions has a $G_2$ structure. However we haven’t evaluated the holonomy of the manifold and it should be interesting to do so in order to check that the supersymmetry is preserved. One immediate thing to check would be whether the manifold could become complex. In other words whether the complex structure is integrable or not. This is an interesting question and can only be answered after we trace the behavior of fermions when we do the mirror transformation.
Generic studies done earlier have shown that in general these mirror manifolds do not have an integrable complex structure. In addition to this, there is also the question of the choice of the complex structure. Recall that in the type IIB theory which we started out with, the background fluxes generate a superpotential that fixes the complex structure. This would imply that the Kähler structures are all fixed in the type IIA picture. On the other hand, if we start with a type IIB framework with, say, $h^{1,1} = 1$ one might also be able to fix the complex structure in the mirror just by fixing the Kähler structure in the type IIB side via (non-perturbative) corrections to the type IIB superpotential. Thus the choice of complex structure in type IIA will be uniquely fixed. It will be important to see if the choice of complex structure that we made here is consistent with the value fixed by the superpotential.

- There is another important aspect of geometric transition that one needs to carefully verify. This has to do with the disappearance of $D6$ branes when we perform the geometric transition. Since $D6$ branes support gauge fluxes, the disappearance of $D6$ branes would imply that after geometric transition there cannot be any localized gauge fluxes. To show this aspect, the M-theory lift will be very useful. Recall that the world volume couplings (and interactions) of $D6$ branes can be extracted from M-theory lagrangian using the normalizable harmonic (1,1) form of the corresponding Taub-NUT space $[83]$. Now that we have fluxes and also a background non-Kähler geometry in type IIA theory (or in other words a torsional $G_2$ manifold in M-theory) the analysis of the normalizable harmonic form is much more complicated. In the presence of fluxes this has been considered in $[84]$. It was found there that the harmonic forms themselves change by the backreaction of the fluxes on geometry. It will now be important to evaluate this harmonic form and show that it is normalizable. This would allow a localized gauge flux to appear in the type IIA scenario, proving that the $G_2$ metric is indeed the lift of the $D6$ brane on a non-Kähler geometry. This harmonic form should either vanish or become non normalizable after we do a flop. This will show that we do not expect a localized gauge flux in type IIA theory and therefore the $D6$ branes have completely disappeared! This will confirm Vafa’s scenario. However as discussed in $[28]$, the issue of normalisable form is subtle here. In the absence of torsion, the usual form is badly divergent even before the transition. In the presence of torsion (or fluxes in the type IIB theory) the back reaction of fluxes on geometry might make this norm well behaved, as has been observed in $[84]$ in a different context. This will be useful for comparison.
• The $G_2$ manifold is explicitly non-Kähler (because it is odd dimensional), and appears from a fibration over another six-dimensional non-Kähler manifold. As we saw earlier, this manifold is also neither complex nor half-flat. Thus we have a new $G_2$ manifold whose slice is a specific six dimensional non-Kähler space. Therefore one should be able to use Hitchin flow equations \[85\] to construct the $G_2$ manifold from the six dimensional non-Kähler space. When the base is a half-flat manifold, this has already been done in the work of Hitchin \[85\]. The flow equations take the following form:

$$dJ = \frac{\partial \Omega_+}{\partial t}, \quad d\Omega_- = -J \wedge \frac{\partial J}{\partial t}$$

(8.1)

where $t$ is a real parameter that determines the $SU(3)$ structure of the base and is related to the vielben $e_7$. For our case, since we know almost every detail of the metric, it will be interesting to check whether the $G_2$ manifold follows from the flow equations.\[33\]

• The non-Kähler manifold that we get in type IIA theory both before and after geometric transitions should fit in the classification of torsion classes \[27\]. Since they are more generic than the half-flat manifolds, the torsion classes will be less constrained. It will also be interesting to find the full $G_2$ structure for the M-theory manifolds.

• The analysis that we performed in this paper starting with $D_5$ wrapped on resolution $P^1$ cycle of a resolved conifold is only the IR description of the corresponding gauge theory. The full analysis should involve additional $D_3$ branes in the type IIB picture, so that the cascading behavior can be captured. However the cascade being an infinite sequence of flop transitions, makes simple supergravity description of the full theory a little more involved. It will be interesting to pursue this direction to see how the type IIA mirror phenomena works. We hope to address this issue in near future.

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\[33\] For some more details on Hitchin’s flow equations related to a generic lift of six-manifolds to seven-manifolds, the readers may want to consult \[11\]. We thank G. Dall’Agata for correspondence on this issue.
9. Appendix 1: Algebra of $\alpha$

In the earlier sections we have defined $\alpha$ as $\alpha = \frac{1}{1 + A^2 + B^2}$ where $A, B$ are given in (5.14). This quantity $\alpha$ is a very crucial quantity as the finite transformation depends on it. While integrating the finite shifts we saw that we need to approximate the transformation on a particular sphere, parametrised by $(\psi_i, x_i, \theta_i)$, as though the other sphere components are constants. In other words, to study the transformation on, say, sphere 1 we take the $\theta_2$ terms as a constant (or fixed at an average value). In other words, any generic $\theta_i$ will be denoted as

$$\theta_1 = \langle \theta_1 \rangle + \vartheta_1, \quad \theta_2 = \langle \theta_2 \rangle + \vartheta_2$$

(9.1)

where $\langle \theta_{1,2} \rangle$ are the average values of the $\theta$ coordinates. To see how the $\alpha$ factor responds to this, let us first define three useful quantities:

$$\langle \alpha \rangle_1 = \frac{1}{1 + \Delta_1^2 \cot^2 \theta_1 + \Delta_2^2 \cot^2 \langle \theta_2 \rangle}, \quad \langle \alpha \rangle_2 = \frac{1}{1 + \Delta_1^2 \cot^2 \langle \theta_1 \rangle + \Delta_2^2 \cot^2 \theta_2}$$

$$\langle \alpha \rangle = \frac{1}{1 + \Delta_1^2 \cot^2 \langle \theta_1 \rangle + \Delta_2^2 \cot^2 \langle \theta_2 \rangle}$$

(9.2)

Using the above definitions, one can easily show that $\sqrt{\alpha}$ has the following expansion:

$$\sqrt{\alpha} = \sqrt{\langle \alpha \rangle_1} \left(1 + x \frac{\Delta_2^2}{\Delta_1^2} \cot^2 \langle \theta_2 \rangle \right)^{-\frac{1}{2}}$$

(9.3)

where we have kept the radial variations as constant as before, and the quantity $x$ appearing above being given by the following exact expression:

$$x = -\frac{4 \cosec 2\langle \theta_2 \rangle \tan \vartheta_2 \left(1 + \cot 2\langle \theta_2 \rangle \tan \vartheta_2 \right)}{1 + \cot \langle \theta_2 \rangle \tan \vartheta_2 \left(2 + \cot \langle \theta_2 \rangle \tan \vartheta_2 \right)}.$$  

(9.4)

If the warp factor $\Delta_2$ is chosen in such a way that in (9.3) the quantity in the bracket is always small, then $\alpha$ will have the following expansions at all points in the internal space:

$$\sqrt{\alpha} = \sqrt{\langle \alpha \rangle_1} - \frac{1}{2} x \Delta_2^2 \cot^2 \langle \theta_2 \rangle \langle \alpha \rangle_1^{3/2} + ....$$

(9.5)
and therefore could be approximated simply as $\sqrt{\langle \alpha \rangle_1}$. Similar argument will go through for $\langle \alpha \rangle_2$ for the other sphere. Another alternative way to write $\alpha$ is

$$\sqrt{\alpha} = \frac{\langle \alpha \rangle}{\sqrt{\langle \alpha \rangle_1}} + ... = \frac{\langle \alpha \rangle}{\sqrt{\langle \alpha \rangle_2}} + ... \quad (9.6)$$

where again the dotted terms can be easily determined for different spheres. The above two pair of expressions: $\sqrt{\alpha} = \sqrt{\langle \alpha \rangle_{1,2}}$ and $\sqrt{\alpha} = \frac{\langle \alpha \rangle}{\sqrt{\langle \alpha \rangle_{1,2}}}$ are responsible for the two different set of coordinate transformations in (5.50) and (5.51) with $m = \pm 1$. Observe also that under the above approximations, some of the components of the deformed conifold metric will now look like:

$$ds^2_{\theta_1 \theta_2} = -2 \sqrt{\langle \alpha \rangle_{1,2}} j_{xy} \, d\theta_1 \, d\theta_2,$$
$$ds^2_{\theta_1 \theta_2} = \langle \alpha \rangle_2 (1 + A^2_1) \, d\theta_2^2,$$
$$ds^2_{\theta_1 \theta_1} = \langle \alpha \rangle_1 (1 + B^2_1) \, d\theta_1^2 \quad (9.7)$$

where $A_1$ and $B_1$ are the values of $A$ and $B$ at the average values of $\theta_1$ and $\theta_2$ respectively.

10. Appendix 2: Details on $G_2$ structures

The threeform $\Omega$ that we described earlier in the context of type IIA manifold can be fixed by a subgroup $SU(3)$ of $SO(6)$. In fact both $J$, the fundamental form and $\Omega$ can be fixed simultaneously by $SU(3)$. The torsion classes that we mentioned earlier are basically the measure of the non-closedness of $\nabla J$ and $\nabla \Omega$. Detailed discussions on this are in [86], [27]. As we saw earlier, the type IIA manifold is neither Kähler nor complex. Therefore the torsion is generic. When the complex structure becomes integrable the torsional connection is known as Bismut connection [87].

Similarly the threeform $\tilde{\Omega}$ can be fixed by a subgroup of $GL_7$. This is the exceptional Lie group $G_2$ which is a compact simple Lie subgroup of $SO(7)$ of dimension 14. The existence of a $G_2$ structure is equivalent to the existence of the fundamental three form $\tilde{\Omega}$. The following interesting cases have been studied in the literature for the torsion free $G_2$ case (for a more detailed review on this the reader may look into the last reference of [72]. A short selection on $G_2$ manifolds are in [88], [48], [89], [90]):

- When $\nabla \tilde{\Omega} = 0$, then holonomy is contained in $G_2$ with a Ricci flat $G_2$ metric [91].
• When $d\tilde{\Omega} = \epsilon \tilde{\Omega} = 0$ then the fundamental form is harmonic and the corresponding $G_2$ manifold is called parallel. First examples were constructed in [88]. Later on, first compact examples were given in [90], [92].

• When $\varphi \equiv \epsilon (\tilde{\Omega} \wedge \epsilon d\tilde{\Omega}) = 0$ then the $G_2$ structure is known to be balanced, where $\varphi$ is known as the Lee form. If the Lee form is closed, then the $G_2$ structure is locally conformally equivalent to a balanced one. When $d\epsilon \tilde{\Omega} = \varphi \wedge \tilde{\Omega}$ then the $G_2$ structure is called integrable.

In the presence of torsion we have already described the changes that one would expect from the above mentioned conditions. In terms of the torsion three form $\tau$ given in (6.37), the connection shifts from $\nabla$ mentioned above to the usual expected form $\nabla + \frac{1}{2} \tau$. Now the manifold is no longer Ricci flat and the curvature tensor takes the following form:

$$R_{ijkl} = R_{ijkl} - \frac{1}{2} \tau_{ijm} \tau_{klm} - \frac{1}{4} \tau_{jkm} \tau_{ilm} - \frac{1}{4} \tau_{kim} \tau_{jlm}$$

(10.1)

where $R$ measures the curvature wrt the Riemannian connection. For the case when the base is half-flat, a construction of $G_2$ manifolds satisfying some of the features mentioned above is given in [72]. For our case we do not have a half-flat base, and therefore the manifold that we presented in sections 6 and 7 are new examples of $G_2$ manifolds with torsion that satisfy all the string equations of motion.
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