LOCAL-GLOBAL PRINCIPLE FOR 0-CYCLES ON FIBRATIONS
OVER RATIONALLY CONNECTED BASES

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ABSTRACT. We study the Brauer–Manin obstruction to the Hasse principle and to weak approximation for 0-cycles on algebraic varieties that possess a fibration structure. The exactness of the global-to-local sequence (E) of Chow groups of 0-cycles was known only for a fibration whose base is a curve or the projective space. In the present paper, we prove the exactness of (E) for fibrations having other bases: smooth quadrics, del Pezzo surfaces of degree at least 5, Châtelet surfaces, homogeneous varieties of linear algebraic groups. Generally speaking, the base can be a rationally connected variety whose weak approximation can be controlled by its Brauer group. We require that either all fibres are geometrically integral and most fibres satisfy weak approximation, or the generic fibre has a 0-cycle of degree 1 and weak approximation on most fibres can be controlled by their Brauer groups.

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1. INTRODUCTION
1.1. Background. Let $k$ be a number field and we consider algebraic varieties $X$ defined over $k$. It is conjectured by Colliot-Thélène–Sansuc and Kato–Saito that the Hasse principle and weak approximation for 0-cycles on proper smooth varieties can be controlled by their Brauer groups. To be precise, we are interested in the exactness of the following complex which will be explained with more details in §2

\[(E) \quad \varprojlim \frac{\text{CH}_0(X)}{n} \rightarrow \prod_{v \in \Omega_k} \varprojlim \frac{\text{CH}_0^0(X_v)}{n} \rightarrow \text{Hom}(\text{Br}X, \mathbb{Q}/\mathbb{Z}).\]

The exactness of (E) for $X = \text{Spec}(k)$ (or $X = \mathbb{P}^m$) is assured by global class field theory. The exactness of (E) for smooth projective curves was proved by Saito [Sai89] and [CT99] under the assumption of the finiteness of Tate–Shafarevich groups of their Jacobians. Concerning higher dimensional varieties, known results are only available for a few homogeneous spaces of linear algebraic groups and

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varieties possessing a fibration structure. We consider only fibrations \( X \to B \). Existing results have dealt with fibrations whose base \( B \) is

- either a curve \( C \) (assuming the finiteness of \( \text{III}(\text{Jac}_{C,k}) \)),
- or the projective space \( \mathbb{P}^m \) (or a product of \( \mathbb{P}^m \), or a \( k \)-rational variety) by induction reducing to \( m = 1 \).

Concerning analogue questions for rational points, the fibration method allows us to deal with more general bases other than those mentioned above, for example [Har94, Thm. 4.3.1] and [Har07, Thm. 3], however the arguments cannot be extended directly to 0-cycles. The main difficulty is that we do not know how to approximate topologically a family of effective local 0-cycles in good position by a single global closed point even if \( B \) satisfies weak approximation. This is trivial for rational points; and for 0-cycles this can be done by Salberger’s device [Sal88] if \( B \) is \( \mathbb{P}^1 \) and more generally by Colliot-Thélène’s argument [CT00] if \( B \) is a smooth projective curve. We are trying to surmount this difficulty at least for geometrically rationally connected varieties.

1.2. Our results. In the author’s Ph.D. dissertation defence, Per Salberger asked whether one can prove results for fibrations whose base is a Grassmannian. This work is a first try to answer this question, actually we prove much more general results which can certainly be applied when \( B \) is a Grassmannian. Roughly speaking, under several assumptions on fibres of \( X \to B \), we prove the exactness of (E) for \( X \), supposing that \( B \) is geometrically rationally connected and that the Brauer–Manin obstruction is the only obstruction to weak approximation. We require that

- either all fibres are geometrically integral and most fibres satisfy weak approximation,
- or the generic fibre has a 0-cycle of degree 1 and weak approximation on most fibres can be controlled by their Brauer groups.

We have several immediate applications for specific fibrations.

1.3. Organisation of the paper. Our main results Theorem 2.3, Theorem 2.5, and applications Corollary 2.8 are stated in §2 after some recall of terminologies. The detailed proofs are given in §3.

2. Main results and applications

We begin with fixing the notation. The base field \( k \) will be a number field unless otherwise stated. The set of its places is denoted by \( \Omega_k \) and \( S \) denotes always a finite subset of \( \Omega_k \). Let \( k' \) be a finite extension of \( k \), the set of places of \( k' \) lying above places in \( S \) will be denoted by \( S \otimes_k k' \). We will consider algebraic varieties (which mean separated schemes of finite type) \( X \) defined over \( k \). Its cohomological Brauer group is defined by \( \text{Br}_X = H^2_{\text{ét}}(X, \mathbb{G}_m) \), and its Chow group of 0-cycles is denoted by \( \text{CH}_0(X) \). The base change \( X \times_{\text{Spec}(k)} \text{Spec}(K) \) is written simply \( X_K \) for any extension \( K \) of \( k \). If \( K = k_v \) the local field at \( v \in \Omega_k \), we write \( X_v \) instead of \( X_{k_v} \). The modified local Chow group \( \text{CH}_0'(X_v) \) is defined to be the usual Chow group \( \text{CH}_0(X_v) \) if \( v \) is a non-archimedean place, and otherwise \( \text{Coker} \left[ N_{C/K} : \text{CH}_0(X_C) \to \text{CH}_0(X_R) \right] \), in particular it is 0 if \( v \) is a complex place.

Let \( A \) be an abelian group, for non-zero integer \( n \) we denote by \( A/n \) the cokernel \( \text{Coker} \left[ n : A \to A \right] \) of the multiplication by \( n \). Let \( L/K \) be a Galois extension of Galois group \( \text{Gal}_{L/K} \) and \( M \) be a continuous \( \text{Gal}_{L/K} \)-module, we denote its Galois
cohomologies by $H^n(L|K, M)$. If $L = \bar{K}$ is the separable closure of $K$ we will write $H^n(K, M)$ for short.

Before the statement of the main results, we recall several terminologies. Let $X$ be a proper smooth geometrically integral variety defined over $k$. By extending Manin’s well-known pairing $\text{Manin71}$

$$\prod_{v \in \Omega_k} X(k_v) \times \text{Br} X \to \mathbb{Q}/\mathbb{Z}$$

one can define a natural pairing by evaluation of cohomology classes at closed points of $X_v$.

$$\prod_{v \in \Omega_k} \text{CH}_0(X_v) \times \text{Br} X \to \mathbb{Q}/\mathbb{Z},$$

which factorizes through

$$\prod_{v \in \Omega_k} \text{CH}'_0(X_v) \times \text{Br} X \to \mathbb{Q}/\mathbb{Z},$$

see for example $\text{CT95}$ for a detailed definition. This induces a global-to-local complex

$$\text{CH}_0(X) \to \prod_{v \in \Omega_k} \text{CH}'_0(X_v) \to \text{Hom}(\text{Br} X, \mathbb{Q}/\mathbb{Z}).$$

Since $X$ is regular, its Brauer group is torsion, one deduces the complex mentioned in the introduction

$$\lim_{\longleftarrow} \frac{\text{CH}_0(X)}{n} \to \prod_{v \in \Omega_k} \lim_{\longleftarrow} \frac{\text{CH}'_0(X_v)}{n} \to \text{Hom}(\text{Br} X, \mathbb{Q}/\mathbb{Z}),$$

whose exactness means roughly that the obstruction to the local-global principle is controlled by the Brauer group $\text{Br} X$.

**Terminology.** Let $\delta$ be an integer. For a family of algebraic varieties, the *Hasse principle for $0$-cycles of degree $\delta$* says that the existence of local families of $0$-cycles of degree $\delta$ implies the existence of a global $0$-cycle of degree $\delta$. We say that the *Brauer–Manin obstruction is the only obstruction to the Hasse principle for $0$-cycles of degree $\delta$*, if the existence of local families of $0$-cycles of degree $\delta$ orthogonal to $\text{Br} X$ implies the existence of a global $0$-cycle of degree $\delta$.

We say that $X$ verifies *weak approximation for $0$-cycles of degree $\delta$* if the following statement is satisfied.

Given a family of local $0$-cycles $\{z_v\}_{v \in \Omega}$ of degree $\delta$, for any positive integer $n$ and for any finite subset $S$ of $\Omega_k$, there exists a global $0$-cycle $z = z_{n,S}$ of degree $\delta$ such that $z$ and $z_v$ have the same image in $\text{CH}_0(X_v)/n$ for all $v \in S$.

We say that the *Brauer–Manin obstruction is the only obstruction to weak approximation for $0$-cycles of degree $\delta$*, if the following statement is satisfied.

Given a family of local $0$-cycles $\{z_v\}_{v \in \Omega}$ of degree $\delta$ orthogonal to $\text{Br} X$, for any positive integer $n$ and for any finite subset $S$ of $\Omega_k$, there exists a global $0$-cycle $z = z_{n,S}$ of degree $\delta$ such that $z$ and $z_v$ have the same image in $\text{CH}_0(X_v)/n$ for all $v \in S$.

We say that $X$ verifies *weak approximation for rational points* if the diagonal image of $X(k)$ in $\prod_{v \in \Omega_k} X(k_v)$ is dense. We say that the *Brauer–Manin obstruction is the only obstruction to weak approximation for rational points* if the closure of
de diagonal image of $X(k)$ in $\prod_{v\in\Omega_k} X(k_v)$ equals to the subset consisting of local points that are orthogonal to $Br X$.

Remark 2.1.

(1) The exactness of (E) implies that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree 1, [Wit12, Rem. 1.1(iii)].

(2) The exactness of (E) implies that Brauer–Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree $\delta$ admitting the existence of a global 0-cycle of degree 1, [Lia13, Prop. 2.2.1].

(3) The exactness of (E) implies that Brauer–Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree 1, this follows from the previous two.

Definition 2.2. Let $B$ be a geometrically integral $k$-variety. A subset $\text{Hil}$ of closed points of $B$ is called a generalised Hilbertian subset if there exists a finite étale morphism $Z \rightarrow U \subset B$ where $Z$ is an integral $k$-variety and $U$ is a nonempty open subvariety of $B$ such that

$$\text{Hil} = \{\theta \in U | \theta \text{ is a closed point of } B \text{ such that the fibre } \rho^{-1}(\theta) \text{ is connected} \}.$$  

The main results of this paper concern the exactness of (E) for fibrations over a base of dimension $m > 1$ which is probably not birational to $\mathbb{P}^m$ over $k$.

Theorem 2.3. Let $f : X \rightarrow B$ be a flat dominant morphism between proper smooth geometrically integral varieties defined over a number field $k$.

Fix a finite extension $L$ of $k$ and denote by $\mathcal{K}_L$ the set of finite extensions $K$ of $k$ that are linearly disjoint with $L$.

1. all fibres are geometrically integral;
2. for each $K \in \mathcal{K}_L$, for all $K$-rational points $\theta$ in a certain non-empty open subset of $B_K$, the fibre $X_{K\theta} = f_K^{-1}(\theta)$ verifies weak approximation (resp. to the Hasse principle) for 0-cycles of degree 1;
3. $H^1(B_k, \mathcal{O}_{B_k}) = 0$, $H^2(B_k, \mathcal{O}_{B_k}) = 0$, and the Néron–Severi group $\text{NS}_{B_k}$ is torsion-free;
3.2. for all $K \in \mathcal{K}_L$ the Brauer–Manin obstruction is the only obstruction to weak approximation for $K$-rational points on $B_K$.

Then the Brauer–Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree $\delta$ (resp. to the Hasse principle for 0-cycles of degree $\delta$) on $X$. Moreover if

4. Pic$X_k$ is torsion-free, and deg $: \text{CH}_0(X \otimes k(X)) \rightarrow \mathbb{Z}$ is injective,

then the sequence (E) is exact for $X$.

Remark 2.4. To clarify the idea, we state Theorem 2.3 in such a version and we will give a proof of this version in [§3. Actually, the hypothesis 2 can be weaken: similarly to Theorem 2.3 stated afterwards, we only need to verify the property for $K$-rational points $\theta$ contained in a certain Hilbertian subset of $B_K$. The more general version does have applications, but we omit the proof, for which one needs to adapt the argument of generalised Hilbertian subsets as in the proof of Theorem 2.6.
Theorem 2.5. Let \( f : X \to B \) be a flat dominant morphism between proper smooth geometrically integral varieties defined over a number field \( k \).

Fix a finite extension \( L \) of \( k \) and denote by \( K_L \) the set of finite extensions \( K \) of \( k \) that are linearly disjoint with \( L \).

Suppose that

1.1, for the geometric generic fibre \( X_{\eta} = X_\eta \otimes k(B) \), the Néron–Severi group \( \text{NS}(X_{\eta}) \) is torsion-free, and \( H^1(X_{\eta}, \mathcal{O}_{X_\eta}) = 0 \), \( H^2(X_{\eta}, \mathcal{O}_{X_\eta}) = 0 \).

1.2, the generic fibre \( X_\eta \) viewed as a \( k(B) \)-variety possesses a 0-cycle of degree 1;

2, for each \( K \in K_L \) there exists a generalised Hilbertian subset \( \text{Hil}_K \) of \( B_K \) such that for all \( K \)-rational points \( \theta \) contained in \( \text{Hil}_K \) the Brauer–Manin obstruction is the only obstruction to weak approximation (resp. to the Hasse principle) for 0-cycles of degree 1 on the fibre \( X_{K\theta} = f^{-1}(\theta) \);

3.1, \( H^1(B_k, \mathcal{O}_{B_k}) = 0 \), \( H^2(B_k, \mathcal{O}_{B_k}) = 0 \), and the Néron–Severi group \( \text{NS}(B_k) \) is torsion-free;

3.2, for all \( K \in K_L \) the Brauer–Manin obstruction is the only obstruction to weak approximation for \( K \)-rational points on \( B_K \).

Then the Brauer–Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree \( \delta \) (resp. to the Hasse principle for 0-cycles of degree \( \delta \)) on \( X \). Moreover if

4, \( \text{Pic}(X_k) \) is torsion-free, and \( \text{deg} : \text{CH}_0(X \otimes k(X)) \to \mathbb{Z} \) is injective,

then the sequence (E) is exact for \( X \).

The proof will be given in \( \S 3 \) it is based on the proof of Theorem 2.3 combined with arguments for generalised Hilbertian subset to compare Brauer groups.

Remark 2.6. Let \( V \) be a variety defined over algebraically closed field of characteristic 0. If \( H^1(V, \mathcal{O}_V) = 0 \), \( H^2(V, \mathcal{O}_V) = 0 \) and if the Néron-Severi group \( \text{NS}(V) \) is torsion-free, then \( \text{Pic}(V) \) is torsion-free and \( \text{Br}(V) \) is finite, in particular it is the case if \( V \) is a rationally connected variety. Moreover if the base \( B \) and the generic fibre \( X_\eta \) are both geometrically rationally connected, then \( X \) is also the case, therefore hypotheses 1.1, 3.1, and 4, are automatically satisfied.

Remark 2.7. In both of the theorems, the arithmetic hypothesis concerning the Brauer–Manin obstruction of the base \( B \) is a property of rational points, it may not be replaced by the exactness of (E) for \( B \) (which is weaker by [Lia13, Thm. A, B]). However, the arithmetic hypothesis on fibres concerns the Hasse principle or weak approximation for 0-cycles of degree 1, it can be deduced from the exactness of (E) for fibres, which may be valid even though the similar question for rational points is still open (or solved conditionally).

As application, we establish the exactness of the sequence (E) for several varieties who have a fibration structure.

Corollary 2.8. The sequence (E) is exact for smooth proper varieties that are birationally equivalent to one of the following varieties \( X \).

- \( X \to B \) is a Severi–Brauer scheme over \( B \);
- \( X \to B \) is a smooth fibration of Châtelet surfaces (or more generally Châtelet \( p \)-folds, or even certain normic varieties) whose generic fibre is a Châtelet surface defined by degree 4 irreducible polynomial;
• $X \to B$ is a connected affine group scheme over $B$;
• the generic fibre of $X \to B$ is defined by $N_{L/k(B)}(x) = P(t)$ where $L/k(B)$ is a cyclic extension and $P(t) \in k(B)[t]$ is a separable polynomial of degree prime to $[L : k(B)]$;

where the base $B$ is one of the following varieties
• del Pezzo surfaces of degree at least 5;
• Châtelet surfaces;
• smooth quadrics;
• smooth compactifications of homogeneous spaces of connected linear algebraic groups with connected stabilizers;
• smooth compactifications homogeneous spaces of semisimple simply connected algebraic groups with abelian stabilizers;

3. PROOFS OF THE THEOREMS

3.1. Preliminaries for the proofs. For the convenience of the reader, we list some well-known statements in this subsection, they will be used in the proofs of the main results.

Lemma 3.1 (Moving lemma for 0-cycles). Let $X$ be a smooth connected variety defined over an infinite perfect field $k$. Let $X_0$ be a nonempty open subvariety of $X$. Then every 0-cycle $z$ of $X$ is rationally equivalent to a 0-cycle $z'$ supported in $X_0$.

Proof. This is well-known, for an elementary proof see for example [CT05, Complément §3].

Definition 3.2. A 0-cycle $z$ written as $z = \sum n_P P$ with distinct closed points $P$ is said to be separable if all nonzero multiplicities $n_P$ are either 1 or $-1$.

Effective separable 0-cycles on $\mathbb{A}^1_k$ correspond bijectively to separable polynomials over $k$ (up to scalar). Let $f : X \to Y$ be a proper morphism between algebraic varieties and let $z$ be a 0-cycle on $X$, then $f_*(z)$ is a separable 0-cycle on $Y$ implies that $z$ is separable.

Definition 3.3. Let $d$ be a positive integer and $X$ be an algebraic variety defined over a topological field $k$. Let $\text{Sym}^d X$ be the $d$-th symmetric product of $X$. Effective 0-cycles $z$ and $z'$ of degree $d$ on $X$ correspond to rational points $[z], [z'] \in \text{Sym}^d(X)(k)$. One says that $z$ is sufficiently close to $z'$ if $[z]$ is sufficiently close to $[z']$ with respect to the topology on $\text{Sym}^d(X)(k)$ induced by $k$.

We need this notion to compute the Brauer–Manin pairing and to apply the implicit function theorem. More explicitly, let $z' = \sum P'$ (with $P'$ distinct closed points) be an effective separable 0-cycle, which will always be the case in the proofs of theorems. We fix an algebraic closure $\bar{k}$ and an embedding $k(P') \hookrightarrow \bar{k}$ for each $P'$, we can view $P'$ as a $k(P')$-rational point of $X$. Then $z$ is sufficiently close to $z'$ implies that $z$ is separable and it can also be written as $z = \sum P$ (in a suitable order) with $P$ a closed point associated for each $P'$ appeared in the support of $z'$, moreover $P$ has residual field $k(P) = k(P')$, and it is sufficiently close to $P'$ viewed as a $k(P')$-rational point. For a (or finitely many) fixed element(s) $b \in \text{Br}(X)$, by the continuity of the evaluation of elements in Brauer group at rational points, we deduce that the evaluations $\langle z, b \rangle_k$ and $\langle z', b \rangle_k$ are equal as long as $z$ is sufficiently close to $z'$. 
If $k$ is $\mathbb{R}$, $\mathbb{C}$ or any finite extension of $\mathbb{Q}_p$, the map $\text{Sym}^d(X)(k) \to \text{CH}_0(X)/n$ associating an effective 0-cycle to its class is locally constant for a fixed integer $n > 0$, cf. [Wit12, Lem. 1.8]. In other words $z$ and $z'$ have the same image in $\text{CH}_0(X)/n$ if $z$ is sufficiently close to $z'$.

**Lemma 3.4** (Relative moving lemma). Let $\pi : X \to \mathbb{P}^1$ be a dominant morphism between algebraic varieties defined over $\mathbb{R}$, $\mathbb{C}$ or any finite extension of $\mathbb{Q}_p$. Suppose that $X$ is smooth.

Then for any 0-cycle $z'$ on $X$, there exists a 0-cycle $z$ on $X$ such that $\pi^*(z)$ is separable and such that $z$ is sufficiently close to $z'$.

**Proof.** Essentially, the statement follows from the implicit function theorem. We find detailed arguments in [CTSSD98, p.19] and [CTSD94, p.89]. □

**Lemma 3.5** (Hilbert’s irreducibility theorem for 0-cycles). Let $S$ be a nonempty finite subset of places of a number field $k$. Let $\text{Hil} \subset \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ be a generalised Hilbertian subset of $\mathbb{P}^1_k$. For each $v \in S$, let $z_v$ be a separable effective 0-cycle of degree $d > 0$ with support contained in $\mathbb{A}^1$.

Then there exists a closed point $\lambda$ of $\mathbb{P}^1$ such that

- $\lambda \in \text{Hil}$;
- as a 0-cycle $\lambda$ is sufficiently close to $z_v$ for any $v \in S$.

**Proof.** It is a particular case of [Lia12, Lem. 3.4]. □

**Lemma 3.6** (Harari [Har07, Lem. 1]). Let $B$ be a smooth geometrically integral variety defined over a number field $k$ such that $B(k_v) \neq \emptyset$ for all places $v \in \Omega_k$.

Suppose that for $k$-rational points $B$ verifies weak approximation outside a certain finite set $T$ of places. Let $\mathcal{H}$ be a (classical) Hilbertian subset of $B(k)$. Let $S \subset \Omega_k$ be a finite subset of places and $\{P_v\}_{v \in S}$ be a family of local points of $B$ such that: for every finite subset $S' \subset \Omega_k \setminus (S \cup T)$ and arbitrary family $\{P'_v\}_{v \in S'}$ of local rational points, the union $\{Q_v\}_{v \in S \cup S'}$ of $\{P_v\}_{v \in S}$ and $\{P'_v\}_{v \in S'}$ is in the closure of the diagonal image of $B(k)$.

Then there exists a rational point of $B$ contained in $\mathcal{H}$ arbitrarily close to $P_v$ for all $v \in S$.

We will apply this lemma only in the case where the Brauer–Manin obstruction is the only obstruction to weak approximation for rational points on a proper variety $B$ with finite $\text{Br}B/\text{Br}k$. Take $T \subset \Omega_k$ large enough such that representatives of $\text{Br}B/\text{Br}k$ have good reductions outside $T$. If $\{P_v\}_{v \in \Omega_k}$ is good orthogonal to $\text{Br}B$, then for arbitrary finite subset $S \subset \Omega_k$ the family $\{P_v\}_{v \in S}$ satisfies automatically the assumption of the lemma.

**Lemma 3.7** (Harari’s formal lemma [Har94, Cor. 2.6.1] [CTSSD98, Lem. 4.5]). Let $X$ be a smooth proper geometrically integral variety defined over a number field $k$. Let $X_0$ be a nonempty open subvariety of $X$ and $\Lambda \subset \text{Br}X_0 \subset \text{Br}k(X)$ be a finite set of elements of the Brauer group. We note by $B$ the intersection in $\text{Br}k(X)$ of $\text{Br}X$ and the subgroup generated by $\Lambda$.

Let $\delta$ be an integer. Suppose that for every $v \in \Omega_k$, there exists a 0-cycle $z_v$ on $X_v$ of degree $\delta$ supported in $X_{0v}$ such that the family $\{z_v\}_{v \in \Omega_k}$ is orthogonal to $B$.

Then for every finite subset $S \subset \Omega_k$, there exist a finite subset $S' \subset \Omega_k$ contained $S$ and for each $v \in S' \setminus S$ a 0-cycle $z'_v$ on $X$ of degree $\delta$ and supported
in \( X_0 \), such that
\[
\sum_{v \in S} \inv_v \langle z_v, b \rangle_v + \sum_{v \in S' \setminus S} \inv_v \langle z'_v, b \rangle_v = 0
\]
for all \( b \in \Lambda \).

3.2. Proof of Theorem 2.3

Proof of Theorem 2.3 We will only prove the exactness of (E), and the argument for weak approximation is already included. The statement for the Hasse principle can be proved similarly.

Since \( \Br X \to \Br X \times \mathbb{P}^1 \) is an isomorphism, it suffices to prove the exactness of (E) for \( X' = X \times \mathbb{P}^1 \). As \( \deg : \CH_0(X_0) \to \mathbb{Z} \) is injective, there exists a non-zero integer \( N \) such that for all extensions \( F \) of \( k \) the kernel \( \ker \deg : \CH_0(X_F) \to \mathbb{Z} \) and the cokernel \( \coker \deg : \CH_0(X_F) \to \mathbb{Z} \) are annihilated by \( N \), see the proof of [CT05, Prop. 11] for the kernel. Note that \( \Pic X_0 \) can be proved similarly.

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We apply [Lia13, Thm. 2.1] for a more general statement. We apply [Wit12, Prop. 3.1] (see also [Lia13, §3.5] for a simplified argument for this special case: a fibration over \( \mathbb{P}^1 \)) to the composition \( \pr_{\mathbb{P}^1} \circ F \) of \( F = f \times id : X' = X \times \mathbb{P}^1 \to B' = B \times \mathbb{P}^1 \) with \( \pr_{\mathbb{P}^1} : B' = B \times \mathbb{P}^1 \to \mathbb{P}^1 \). In order to prove the exactness of (E) for \( X' \), it suffices to verify the following property for all finite subset \( S \subset \Omega_k \) of places of \( k \).

\((P_S)\) Let \( \{ z_v \}_{v \in \Omega_k} \) be a family of 0-cycles of degree \( \delta \) on \( X' \). If it is orthogonal to \( \Br X' \), then for all integer \( n > 0 \), there exists a 0-cycle \( z \) of \( X' \) of degree \( \delta \), such that for all \( v \in S \) we have \( z = z_v \) in \( \CH_0(X'_v) \) if \( v \) is non-archimedean and \( z = z_v + N_{k_v/k}(u_v) \) in \( \CH_0(X'_v) \) for a certain \( u_v \in \CH_0(X'_v) \) if \( v \) is real.

The property \((P_S)\) is implied by the following property.

\((P'_S)\) Let \( \{ z_v \}_{v \in \Omega_k} \) be a family of 0-cycles of degree \( \delta \) on \( X' \). If it is orthogonal to \( \Br X' \), then for all integer \( n > 0 \), there exists a 0-cycle \( z \) of \( X' \) of degree \( \delta \), such that for all \( v \in S \) we have \( z = z_v \) in \( \CH_0(X'_v)/2n \).

For simplicity, we are going to prove \((P'_S)\) only for \( \delta = 1 \). For general \( \delta \), one needs to adapt the proof with an argument of generalised Hilbertian subset, see the proof of Theorem 2.5 for more details.

Note that \( X \) is geometrically integral, by enlarging \( S \) if necessary, according to Lang–Weil estimation and Hensel’s lemma, we may assume that \( X \) has \( k_v \)-rational points for all \( v \in \Omega_k \setminus S \). A fortiori, it is also the case for \( B \). As all fibres of \( F \) are geometrically integral, we may also assume that for every closed point \( \theta \in B' \) the fibre \( F^{-1}(\theta) \) possesses \( k(\theta)_{w} \)-rational points for any \( w \) lying above a place in \( \Omega_k \setminus S \).

Under the geometric hypothesis on \( B \) the group \( \Br B_k \) is finite and \( \Pic B_k \) is torsion-free, we deduce the finiteness of \( \Br B/Brk \) via the Leray spectral sequence. We fix a finite set of representatives \( \Gamma \subset \Br B \simeq \Br B' \). By the argument of good reduction, we may also assume that the finite set \( S \) is large enough such that local evaluations of elements \( b \in \Gamma \) are all 0 for \( v \notin S \). By augment \( L \) if necessary, according to [Lia13, Prop. 3.1.1] we may assume that for any \( K \in \mathcal{K}_L \) the base extension \( \Br B/Brk \to \Br B_K/BrK \) is an isomorphism.
We begin with a family \( \{z_v\}_{v \in \Omega_k} \) of 0-cycles of degree 1 orthogonal to \( \text{Br}X' \). We fix an (even) integer \( n \), and we are looking for a global 0-cycle \( z \) of degree 1 having the same image in \( \text{CH}_0(X'_n)/\langle \rangle \) for \( v \in S \). We have for any \( b \in \Gamma \)
\[
\sum_{v \in \Omega_k} \text{inv}_v((z_v,F^*(b))_v) = 0,
\]
then
\[
\sum_{v \in S} \text{inv}_v((z_v,F^*(b))_v) = 0.
\]

Let \( m \) be a positive integer who annihilates all elements in \( \Gamma \) and who is a multiple of \( [L : k] \). Fix a closed point \( P \) of \( X' \), denote by \( \delta_P = [k(P) : k] \) the degree of \( P \).

For each \( v \in S \), we write \( z_v = z_v^+ - z_v^- \) where \( z_v^+ \) and \( z_v^- \) are effective 0-cycles with disjoint supports. We pose \( z_v^+ = \psi \equiv 1 \mod \text{mn}\delta_P \) and they are all effective 0-cycles. We add to each \( z_v^1 \) a suitable multiple of \( \text{deg} 0 \)-cycle \( \text{mn}Z_v \) where \( Z_v = P \times_{\text{Spec}(k)} \text{Spec}(k) \) and we obtain \( z_v^2 \) of the same degree \( \Delta \equiv 1 \mod \text{mn}\delta_P \) for all \( v \in S \). According to the choice of \( m \), we have \( \langle z_v^2,F^*(b) \rangle_v = \langle z_v, F^*(b) \rangle_v \) for any \( v \in S \) and \( b \in \Gamma \). We apply Lemma 3.4 to \( \text{pr}_{P\cdot} \circ F : X' \rightarrow \mathbb{P}^1 \) and obtain an effective 0-cycle \( z_v^3 \) sufficiently close to \( z_v^2 \) such that \( (\text{pr}_{P\cdot} \circ F)_{\ast}(z_v^3) \) is separable. A fortiori, \( F_{\ast}(z_v^3) \) are also separable. By the continuity of the local evaluation, we have \( \langle z_v^3, F^*(b) \rangle_v = \langle z_v, F^*(b) \rangle_v \). We also check that \( z_v^3, z_v^1, z_v^2 \) and \( z_v^3 \) have the same image in \( \text{CH}_0(X'_n)/\langle \rangle \).

We choose a rational point \( \infty \in \mathbb{P}^1(k) \) outside the supports of \( (\text{pr}_{P\cdot} \circ F)_{\ast}(z_v^3) \) for all \( v \in S \). Then for each \( v \in S \) we can write \( (\text{pr}_{P\cdot} \circ F)_{\ast}(z_v^3) - \Delta \infty = \text{div}(\psi_v) \) with \( \psi_v \in k_v(\mathbb{P}^1(k))/k_v^0 \). Actually, each \( \psi_v \) is a separable polynomial (may be supposed to be monic) of degree \( \Delta \) since \( (\text{pr}_{P\cdot} \circ F)_{\ast}(z_v) \) is effective separable and of degree \( \Delta \). We choose a place \( v_0 \in \Omega_k \setminus S \), and choose \( \psi_{v_0} \) a monic irreducible polynomial of degree \( \Delta \) with coefficients in \( k_{v_0} \), for example, an Eisenstein polynomial. According to the weak approximation property for number fields, we obtain a monic polynomial \( \psi \) of degree \( \Delta \) with coefficients in \( k \) such that \( \psi \) is sufficiently close to \( \psi_v \) for all \( v \in S \). \( \cup \{v_0\} \).

The polynomial \( \psi \) is irreducible over \( k_{v_0} \) by Krasner’s lemma, a fortiori irreducible over \( k \), hence defines a closed point \( \lambda \) of \( \mathbb{P}^1 \) of degree \( \Delta \). It is sufficiently close to \( (\text{pr}_{P\cdot} \circ F)_{\ast}(z_v^3) \) for all \( v \in S \). More precisely, we write \( \lambda_v = \lambda \times_{\mathbb{P}^1} \mathbb{P}^1_b = \bigcup_{w|v,w \in \Omega_k} \text{Spec}(k(\lambda)_w) \) for \( v \in \Omega_k \), the image of \( \lambda \) in \( Z_0(\mathbb{P}^1_b) \) is written as
\[
\lambda_v = \sum_{w|v,w \in \Omega_k(\lambda)} P_w \quad \text{where} \quad P_w = \text{Spec}(k(\lambda)_w) \quad \text{is a closed point of} \quad \mathbb{P}^1_{k(\lambda)} \quad \text{of residual field} \quad k(\lambda).\]  
For each \( v \in S \), the 0-cycle \( \lambda_v \) is sufficiently close to \( (\text{pr}_{P\cdot} \circ F)_{\ast}(z_v^3) \), where the effective separable 0-cycle \( (\text{pr}_{P\cdot} \circ F)_{\ast}(z_v^3) \) is written as \( \sum_{w|v,w \in \Omega_k(\lambda)} Q_w \) with distinct \( Q_w \)’s. Then \( k(\lambda)_w = k_v(P_w) = k_v(Q_w) \) and \( P_w \) is sufficiently close to \( Q_w \in \mathbb{P}^1_{k(\lambda)}(k(\lambda)_w) \). And we know that \( z_v^3 \) is written as \( \sum_{w|v,w \in \Omega_k(\lambda)} M_w(\lambda) \) with \( k_v(M_w(\lambda) = k(\lambda)_w \) and \( M_w \in X'_n(k(\lambda)_w) \) is situated on the fiber of \( \text{pr}_{P\cdot} \circ F \) at the closed point \( Q_w \). The implicit function theorem implies that there exists a smooth \( k(\lambda)_w \)-point \( M_w \) on the fiber \( (\text{pr}_{P\cdot} \circ F)^{-1}(\lambda) \) sufficiently close to \( M_w^0 \) for every \( w \in S \). Then the closed point \( M_w \) and \( M_w^0 \) have the same image in \( \text{CH}_0(X'_n)/\langle \rangle \).

Since the degree \( [k(\lambda) : k] = \Delta \) is prime to \( [L : k] \), the residue field \( k(\lambda) \) is linearly disjoint from \( L \) (in order to obtain linearly disjointness for general \( \delta \) other than 1, one has to replace the previous two paragraphs by an argument with a generalised Hilbertian subset), hence \( k(\lambda) \) belongs to \( K_L \).
Note that for each $v \in S$, the local point $F(M_w)$ is sufficiently close to $F(M^0_w) \in B'(k(\lambda)_w)$. By continuity of the local evaluation, for any $b \in \Gamma$ we obtain the following equality on $B$

$$
\sum_{w \in S \otimes_k k(\lambda)} \text{inv}_w(\langle F(M_w), b \rangle_{k(\lambda)_w}) = \sum_{w \in S \otimes_k k(\lambda)} \text{inv}_w(\langle b(F(M_w)) \rangle) \\
= \sum_{w \in S \otimes_k k(\lambda)} \text{inv}_w(\langle b(F(M^0_w)) \rangle) = \sum_{v \in S \; | v} \sum_{w \in S \otimes_k k(\lambda)} \text{inv}_v(\langle b(F(M^0_w)) \rangle) = \sum_{v \in S \; | v} \sum_{w \in S} \text{inv}_v(\langle b(F(M^0_w)), b \rangle_{k_v}) = \sum_{v \in S} \sum_{w \in S} \text{inv}_v(\langle M^0_w, F^*(b) \rangle_{k_v}) = \sum_{v \in S} \sum_{w \in S} \text{inv}_v(\langle z_v^3, F^*(b) \rangle_{k_v}) = 0.
$$

On the $k(\lambda)$-variety $(pr_B \circ F)^{-1}(\lambda) = X \times \lambda \simeq X_{k(\lambda)}$, we fix a $k(\lambda)_w$-rational point $M_w$ for each $w \in \Omega_{k(\lambda)} \setminus S \otimes_k k(\lambda)$, this is possible by the choice of $S$. We set $N_w = F(M_w)$ for all $w \in \Omega_{k(\lambda)}$. Then we have the equality for all $b \in \Gamma$

$$
\sum_{w \in \Omega_{k(\lambda)}} \text{inv}_w(\langle N_w, b \rangle_{k(\lambda)_w}) = 0.
$$

By abuse of notation, here $b$ denotes also its image under the restriction $\text{Br}B' \to \text{Br}B \times \lambda$. Then by functoriality, the above equality is viewed as the Brauer–Manin pairing on $B \times \lambda$. Recall that the base extension $\text{Br}B/\text{Br}k \to \text{Br}B_K/\text{Br}k$ is an isomorphism for all $K \in \mathcal{K}_L$, the restriction of $\Gamma$ in $\text{Br}B \times \lambda$ generates $\text{Br}B \times \lambda/\text{Br}k(\lambda)$. Therefore $\{N_w\}_{w \in \Omega_{k(\lambda)}}$ is orthogonal to the Brauer group of $B \times \lambda$.

By hypothesis, we obtain a $k(\lambda)$-rational point $\theta$ on $B \times \lambda$ sufficiently close to $N_w$ for all $w \in S \otimes_k k(\lambda)$, we can also require that the fibre $F^{-1}(\theta)$ is smooth. By implicit function theorem $F^{-1}(\theta)$ has $k(\lambda)$-points $M^0_w$ sufficiently close to $M_w$ for $w \in S \otimes_k k(\lambda)$. It has local points for all other $w$ by the choice of $S$. By hypothesis, there exists a 0-cycle $z'$ of degree 1 on the $k(\lambda)$-variety $F^{-1}(\theta)$ such that it has the same image as $M^0_w$ in $\text{CH}_0(F^{-1}(\theta)_w)/n$ for all $w \in S \otimes_k k(\lambda)$. Viewed as a 0-cycle of $X'$, the 0-cycle $z'$ is of degree $\Delta \equiv 1 \mod mnd \rho$, by subtracting a suitable multiple of $P$, we obtain a 0-cycle $z$ of degree 1. We verify that $z$ and $z_v$ have the same image in $\text{CH}_0(X'_w)/n$ for all $v \in S$ which terminates the proof.

3.3. **Proof of Theorem 2.5** In order to prove Theorem 2.5 we need to establish some lemmas on generalised Hilbertian subsets to compare Brauer groups. Recall that if $k$ is a number field, deduced from Hilbert’s irreducibility theorem, generalised Hilbertian subsets of geometrically integral varieties are always nonempty. Let $\text{Hil}_1$ and $\text{Hil}_2$ be generalised Hilbertian subsets of an integral variety, then there exists a generalised Hilbertian subset $\text{Hil}$ contained in $\text{Hil}_1 \cap \text{Hil}_2$.

**Lemma 3.8.** Let $B$ be a geometrically integral variety defined over $k$. Let $\text{Hil}_B$ be a generalised Hilbertian subset of $B$. Consider the projections $pr_B : B \times \mathbb{P}^1 \to B$ and $pr_{\mathbb{P}^1} : B \times \mathbb{P}^1 \to \mathbb{P}^1$. Then there exists a generalised Hilbertian subset $\text{Hil}_{\mathbb{P}^1}$ of $\mathbb{P}^1$ satisfying the following property.

- For each $\lambda \in \text{Hil}_{\mathbb{P}^1}$, the $k(\lambda)$-variety $pr_{\mathbb{P}^1}^{-1}(\lambda) = B \times \lambda \simeq B_{k(\lambda)}$ contains a generalised Hilbertian subset $\text{Hil}_\lambda$ such that the image $pr_B(\text{Hil}_\lambda) \subset B$ is contained in $\text{Hil}_B$. 

Proof. The generalised Hilbertian subset $\text{Hil}_B$ is defined by a finite étale morphism $\rho : Z \to U \subset B$, where $U$ is a non-empty open subset of $B$ and $Z$ is an integral $k$-variety. The function field $k(Z)$ is a finitely generated field extension of $k$, let $k'$ be the algebraic closure of $k$ in $k(Z)$. The finite étale morphism $\mathbb{P}^1 \times_{\text{Spec}(k)} \text{Spec}(k') \to \mathbb{P}^1$ defines a generalised Hilbertian subset $\text{Hil}_{B^1}$ of $\mathbb{P}^1$.

For each $\lambda \in \text{Hil}_{B^1}$, the extension $k(\lambda)/k$ is linearly disjoint from $k'/k$, hence the base change $Z_{k(\lambda)}$ is still an integral variety over $k(\lambda)$. Let $\text{Hil}_\lambda$ be the generalised Hilbertian subset defined by $\rho_{k(\lambda)} : Z_{k(\lambda)} \to U_{k(\lambda)} \subset B_{k(\lambda)}$. For each $\theta \in \text{Hil}_\lambda$, the (connected) fibre $\rho_{k(\lambda)}^{-1}(\theta)$ is a closed point of $Z_{k(\lambda)}$. We can show by contradiction that the fibre of $\rho$ at the closed point $pr_B(\theta)$ of $B$ has also to be connected, which signifies that $pr_B(\theta) \in \text{Hil}_B$ and we have $pr_B(\text{Hil}_\lambda) \subset \text{Hil}_B$. □

Lemma 3.9 (Comparison of Brauer groups). Let $f : X \to B$ be a dominant morphism between proper smooth geometrically integral varieties defined over $k$. Assume that the generic fibre $X_\eta$ possesses a 0-cycle of degree 1. Suppose for the geometric generic fibre $X_{\eta_1}$, that the Néron–Severi group $\text{NS}_{X_\eta}$ is torsion-free, and $H^1(X_\eta, \mathcal{O}_{X_\eta}) = 0$, $H^2(X_\eta, \mathcal{O}_{X_\eta}) = 0$.

Then there exists a generalised Hilbertian subset $\text{Hil}_{B^1}$ of $\mathbb{P}^1$ satisfying the following property.

- For each $\lambda \in \text{Hil}_{B^1}$, the $k(\lambda)$-variety $pr_{\mathbb{P}^1}^{-1}(\lambda) = B \times \lambda \simeq B_{k(\lambda)}$ contains a generalised Hilbertian subset $\text{Hil}_\lambda$ such that for each $k(\lambda)$-rational point $\theta$ contained in $\text{Hil}_\lambda$ all homomorphisms in the following commutative diagram are isomorphisms

\[
\begin{array}{ccc}
\text{Br}X_\eta & \xrightarrow{i} & \text{Br}X_{pr_B(\theta)} \\
\downarrow & & \downarrow \\
\text{Br}(X \times \lambda)_\eta & \simeq & \text{Br}(X \times \lambda)_\eta \times_{\text{Br}k(\lambda)} \text{Br}(X \times \lambda)_\theta \\
\downarrow & & \downarrow \\
\text{Br}(B \times \lambda)_{\text{Br}k(B)} & \simeq & \text{Br}(B \times \lambda)_{\text{Br}k(\theta)}
\end{array}
\]

where $\eta_\lambda$ is the generic point of $B \times \lambda$ and where the horizontal homomorphisms are specializations and the vertical homomorphisms are base changes.

Proof. By applying [Lin13, Prop. 3.1.1], which holds for any base field of characteristic 0, to the generic fibre $X_\eta$, we can choose a finite extension $K'$ of $k(B)$ such that the left vertical homomorphism is an isomorphism once $k(\lambda)(B)$ is linearly disjoint from $K'$ over $k(B)$. The field $K'$ is a finitely generated extension of $k$, let $k'$ be the algebraic closure of $k$ in $K'$. Any finite extension $l$ of $k$ linearly disjoint from $k'/k$ is then linearly disjoint from $K'/k$, and therefore $l(B)$ is linearly disjoint from $K'$ over $k(B)$. The finite étale morphism $\mathbb{P}^1 \times_{\text{Spec}(k)} \text{Spec}(k') \to \mathbb{P}^1$ defines a generalised Hilbertian subset $\text{Hil}_1$ of $\mathbb{P}^1$ such that for each $\lambda \in \text{Hil}_1$ the left vertical homomorphism is an isomorphism.

It suffices to make sure that the two horizontal specializations are isomorphisms for $\theta$ and $\lambda$ in certain suitable generalised Hilbertian subsets that are going to be
chosen. First we show this for $Sp_{X \to B, pr_B(\theta)}$, and for the lower horizontal homomorphism it is exactly the same argument applied for $X \times \lambda$ instead of $X$. To simplify notation, we replace $pr_B(\theta) \in B$ by $\theta \in B$ in (and only in) the next paragraph.

By the restriction-corestriction argument, the existence of a 0-cycle of degree 1 on the generic fibre $X_\eta$ assures that $H^3(k(B), \mathbb{G}_m) \to H^3(X_\eta, \mathbb{G}_m)$ is injective. Then from the Leray spectral sequence

$$E_2^{p,q} = H^p(k(B), H^q(X_\eta, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$$

one deduces an exact sequence

$$0 \to H^1(k(B), Pic X_\eta) \to Br X_\eta/Brk(B) \to (Br X_\eta)^{Gal_{L/k(B)}} \to H^2(k(B), Pic X_\eta),$$

see for example [CTS13, Prop. 1.3] or [Har97, Prop. 2.2.1]. Take a finite Galois extension $L$ of $k(B)$ such that the generic fibre $X_\eta$ possesses a $L$-rational point, such that the absolute Galois group $Gal_{L/k}$ acts trivially on $Pic X_\eta$ and $Br X_\eta$ (both are finitely generated under the hypotheses on $X_\eta$), and such that the image of $(Br X_\eta)^{Gal_{L/k(B)}}$ in $H^2(k(B), Pic X_\eta)$ lies inside the subgroup $H^2(L/k(B), Pic X_\eta)$. Then the exact sequence becomes an exact sequence of finite abelian groups

$$0 \to H^1(L/k(B), Pic X_\eta) \to Br X_\eta/Brk(B) \to (Br X_\eta)^{Gal_{L/k(B)}} \to H^2(L/k(B), Pic X_\eta).$$

We denote $X_\lambda \times_{Spec(k(\theta))} Spec(k(\theta))$ by $X_\lambda$. By taking an integral model over a certain sufficiently small non-empty open subset $U$ of $B$, one can specialise each term of the above sequence at each point $\theta \in U$ and one obtains a commutative diagram

$$0 \to H^1(l/k(\theta), Pic X_\lambda) \to Br X_\lambda/Brk(\theta) \to (Br X_\lambda)^{Gal_{l/k(\theta)}} \to H^2(l/k(\theta), Pic X_\lambda),$$

where all the vertical maps are specializations at $\theta$. Moreover, if $\theta$ belongs to the generalised Hilbertian subset $Hil_B$ defined by the finite morphism whose generic fibre corresponds to the finite extension $L/k(B)$, then the specialization $Spec(l) \times_B \theta \to Spec(l)$ at $\theta$ is connected. This signifies that $l$ is a field and $Gal_{L/k(B)} \simeq Gal_{l/k(\theta)}$. Replacing by smaller $U$ if necessary, the specialization maps $Pic X_\eta \to Pic X_\lambda$ and $Br X_\eta \to Br X_\lambda$ are both isomorphisms, cf. [Har97, Prop. 3.4.2] and [Har97, Prop. 2.1.1]. The same argument as for the generic fibre shows that the second line of the diagram is also exact. The isomorphism between Galois groups implies that all the three unnamed vertical maps in the above diagram are isomorphisms. By diagram chasing we prove that $Sp_{X \to B, \theta}$ is also an isomorphism.

In the previous paragraph we have proved that the specialization $Sp_{X \to B, pr_B(\theta)}$ is an isomorphism as long as $pr_B(\theta)$ is contained in $Hil_B$. By Lemma 3.8 there exist a generalised Hilbertian subset $Hil_2$ of $\mathbb{P}^1$ and $Hil_\lambda \subset B \times \lambda$ such that for all $\lambda \in Hil_2$ we have $pr_B(Hil_\lambda) \subset Hil_B$, hence for such $\lambda$ and $\theta \in Hil_\lambda$ the specialization $Sp_{X \to B, pr_B(\theta)}$ is an isomorphism. Moreover, by the same argument as in the previous paragraph applied to the $k(\lambda)$-variety $B \times \lambda$, by shrinking $Hil_2$ if necessary, we may also assure that $Sp_{X \times \lambda \to B \times \lambda, \theta}$ is an isomorphism when $\theta \in Hil_\lambda$.

Now we take a generalised Hilbertian subset $Hil_{pr} \subset Hil_1 \cap Hil_2$ of $\mathbb{P}^1$, it verifies the desired property. \hfill $\square$

Right now we are ready to give a complete proof of Theorem 2.5.
Proof of Theorem 2.5. We will only prove the exactness of (E), the argument for weak approximation is included, and it is similar for the Hasse principle.

As in the proof of Theorem 2.3, we consider the fibration $F = f \times id : X' = X \times \mathbb{P}^1 \to B' = B \times \mathbb{P}^1$ and it suffices to prove the following property

($P'_S$) Let $\{z_v\}_{v \in \Omega_k}$ be a family of 0-cycles of degree $\delta$ on $X'$. If it is orthogonal to $BrX'$, then for all integer $n > 0$, there exists a 0-cycle $z$ of $X'$ of degree $\delta$, such that for all $v \in S$ we have $z = z_v$ in $\text{CH}_0(X'_v)/2n$.

We are going to prove ($P'_S$) for arbitrary degree $\delta$.

Note that $B$ is geometrically integral, by enlarging $S$ if necessary, according to Lang–Weil estimation and Hensel’s lemma, we may assume that $B$ has $k_v$-rational points for all $v \in \Omega_k \setminus S$. Under the geometric hypothesis on $B$ the group $\text{Br}B_k$ is finite and $\text{Pic}B_k$ is torsion-free, we deduce the finiteness of $\text{Br}B/\text{Br}k$ via the Leray spectral sequence. We fix a finite set of representatives $\Gamma_B \subset \text{Br}B \simeq \text{Br}B'$. By the argument of good reduction, we may also assume that the finite set $S$ is large enough such that local evaluations of elements $b_B \in \Gamma_B$ are all 0 for $v \notin S$.

Similarly, the quotient $\text{Br}X_\eta/\text{Br}k(B)$ is finite, we choose a finite set $\Gamma_X \subset \text{Br}X_\eta \subset \text{Br}k(X)$ of representatives. By hypothesis, let $z_0$ be a 0-cycle of degree 1 on $X_\eta$. We write $z_0 = \sum_j n_j R_j(\eta)$ where $R_j(\eta)$ is a closed point of $X_\eta$ of residual field $K_j$. Denote by $d_j$ the degree $[K_j : k(B)]$. There exist an nonempty open subset $U$ of $B$ and an étale morphism $Z_j \to U$ of degree $d_j$ for each $j$, where $Z_j$ is an integral closed subvariety of $X \times_B U$ of function field $K_j$. We pose $b_X^j = \sum_j n_j \text{cores}_{K_j[k(B)}(b_X(R_j(\eta))) \in \text{Br}k(B)$. Replacing $b_X$ by $b_X - b_X^j$ if necessary, we may assume that

\[
\sum_j n_j \text{cores}_{K_j[k(B)}(b_X(R_j(\eta))) = 0 \in \text{Br}k(B).
\]

Let $X_0$ be a nonempty open subset of $X$ such that $\Gamma_X \subset \text{Br}X_0 \subset \text{Br}k(X)$. By shrinking $U$ and $X_0$ if necessary, we may assume that $f(X_0) = U$.

By augment $L$ if necessary, according to [Liu13, Prop. 3.1.1] we may assume that for any $K \in K_L$ the base extension $\text{Br}B/\text{Br}k \to \text{Br}k/\text{Br}K$ is an isomorphism. The finite étale morphism $\mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1$ defines a generalised Hilbertian subset $\text{Hil}_0$ of $\mathbb{P}^1$. As long as $\lambda \in \text{Hil}_0$, the homomorphism $\text{Br}B/\text{Br}k \to \text{Br}B \times \lambda/\text{Br}k(\lambda)$ is an isomorphism and $\text{Hil}_{k(\lambda)} \subset B_{k(\lambda)}$ is defined as in the hypothesis.

We begin with a family $\{z_v\}_{v \in \Omega_k}$ of 0-cycles of degree $\delta$ orthogonal to $BrX'$. We fix an (even) integer $n$, and we are looking for a global 0-cycle $z$ of degree $\delta$ having the same image in $\text{CH}_0(X'_v)/n$ for $v \in S$. After Lemma 3.1, we may suppose that the 0-cycles $z_v$ are all supported in $X_0 \times \mathbb{P}^1$. Firstly, we have for any $b_B \in \Gamma_B$

\[
\sum_{v \in \Omega_k} \text{inv}_v(\langle z_v, F^*(b_B) \rangle_v) = 0,
\]

then

\[
\sum_{v \in S} \text{inv}_v(\langle z_v, F^*(b_B) \rangle_v) = 0.
\]

According to Harari’s formal lemma (Lemma 3.7), by augmenting $S$ (which will never change the above equality because by the choice of $S$ the evaluations at newly added places are always 0) we have

\[
\sum_{v \in S} \text{inv}_v(\langle z_v, b_X \rangle_v) = 0
\]
for all $b_X \in \Gamma_X$.

Let $m$ be a positive integer which annihilates all elements in $\Gamma_B$ and $\Gamma_X$. Fix a closed point $P$ of $X'$, denote by $\delta_P = [k(P) : k]$ the degree of $P$.

For each $v \in S$, we write $z_v = z_v^1 - z_v^2$ where $z_v^1$ and $z_v^2$ are effective 0-cycles with disjoint supports. We pose $z_v^1 = z_v + m\delta_P z_v^2 = z_v^1 + (m\delta_P - 1)z_v^2$, then $\deg(z_v^1) \equiv \delta \mod m\delta_P$ and they are all effective 0-cycles. We add to each $z_v^1$ a suitable multiple of $0$-cycle $m\delta_P z_v^2$ where $P_v = P \times \text{Spec}(k)$ $Spec(k_v)$ and we obtain $z_v^2$ of the same degree $\Delta \equiv \delta \mod m\delta_P$ for all $v \in S$. According to the choice of $m$, we have $(z_v^2, F^*(b))_v = (z_v, F^*(b))_v$ for any $v \in S$ and $b \in \Gamma_B$. We apply Lemma 3.3 to $pr_{p^1} \circ F : X' \to \mathbb{P}^1$ and obtain an effective 0-cycle $z_v^3$ sufficiently close to $z_v^2$ such that $(pr_{p^1} \circ F)_*(z_v^3)$ is separable. A fortiori, $F_*(z_v^3)$ are also separable. By the continuity of the local evaluation, we have $(z_v^3, F^*(b))_v = (z_v, F^*(b))_v$.

We also check that $z_v, z_v^1, z_v^2$ and $z_v^3$ have the same image in $\text{CH}_0(X'_v)/n$.

By Hilbert’s irreducibility for 0-cycles (Lemma 3.3) applied to $pr_{p^1} \circ F : X' \to \mathbb{P}^1$, we find a closed point $\lambda \in \text{Hil}_{\mathbb{P}^1}$ such that $\lambda$ is sufficiently close to $(pr_{p^1} \circ F)_*(z_v^3)$. More precisely, we write $\lambda_v = \lambda \times_{\mathbb{P}^1} \mathbb{P}^1_v = \bigcup_{u \mid \nu \in \Omega_k} \text{Spec}(k(\lambda)_w)$ for $v \in \Omega_k$, the image of $\lambda$ in $\mathbb{P}^1_v$ is written as $\lambda_v = \sum_{u \mid \nu \in \Omega_k} P_{w_v}$, where $P_w = \text{Spec}(k(\lambda)_w)$ is a closed point of $\mathbb{P}^1_v$ of residual field $k(\lambda)_w$. For each $v \in S$, the 0-cycle $\lambda_v$ is sufficiently close to $(pr_{p^1} \circ F)_*(z_v^3)$, where the effective separable 0-cycle $(pr_{p^1} \circ F)_*(z_v^3)$ is written as $\sum_{u \mid \nu \in \Omega_k} Q_w$ with distinct $Q_w$’s. Then $k(\lambda)_w = k_v(P_w) = k_v(Q_w)$, and $P_w$ is sufficiently close to $Q_w \in \mathbb{P}^1_v(k(\lambda)_w)$. And we know that $z_v^3$ is written as $\sum_{u \mid \nu \in \Omega_k} M_w^0$ with $k_v(M_w^0) = k(\lambda)_w$ and $M_w^0 \in X'_w(k(\lambda)_w)$ is situated on the fiber of $(pr_{p^1} \circ F)$ at the closed point $Q_w$. The implicit function theorem implies that there exists a smooth $k(\lambda)_w$-point $M_w$ on the fiber $(pr_{p^1} \circ F)^{-1}(-1)$ sufficiently close to $M_w^0$ for every $w \in S \otimes_k k(\lambda)$. Then the closed point $M_w$ and $M_w^0$ have the same image in $\text{CH}_0(X'_w)/n$.

As in the proof of Theorem 2.3, by continuity of the local evaluation, for any $b_B \in \Gamma_B$ we obtain on $B'$ the equality

$$\sum_{w \in S \otimes_k k(\lambda)} \text{inv}_w(\langle F(M_w), b_B \rangle_{k(\lambda)_w}) = 0.$$  

And similarly, on $X'$, for $b_X \in \Gamma_X$

$$\sum_{w \in S \otimes_k k(\lambda)} \text{inv}_w(\langle M_w, b_X \rangle_{k(\lambda)_w}) = 0.$$  

On the $k(\lambda)$-variety $pr_{p^1}^{-1}(\lambda) = B \times \lambda \simeq B_{k(\lambda)}$, we fix a $k(\lambda)_w$-rational point $N_w$ for each $w \in \Omega_k(\lambda) \setminus S \otimes_k k(\lambda)$, this is possible by the choice of $S$. We set $N_w = F(M_w)$ for $w \in S \otimes_k k(\lambda)$. Then we have the equality for all $b_B \in \Gamma_B$

$$\sum_{w \in \Omega_k(\lambda)} \text{inv}_w(\langle N_w, b_B \rangle_{k(\lambda)_w}) = 0.$$  

By abuse of notation, here $b_B$ denotes also its image under the restriction $\text{Br}B' \to \text{Br}B \times \lambda$. Then by functoriality, the above equality is viewed as the Brauer–Manin...
pairing on $B \times \lambda$. Recall that $\lambda \in \text{Hil}_{\mathbb{P}^1} \subset \text{Hil}_0$, the base extension $\text{Br}B/\text{Br}k \to \text{Br}B \times \lambda/\text{Br}k(\lambda)$ is an isomorphism. Therefore $\{N_w\}_{w \in \Omega_{k(\lambda)}}$ is orthogonal to the Brauer group of $B \times \lambda$.

By hypothesis, weak approximation gives us a $k(\lambda)$-rational point $\theta$ on $B \times \lambda$ sufficiently close to $N_w = F(M_w)$ for all $w \in S \otimes_k k(\lambda)$, we can also require that the fibre $F^{-1}(\theta)$ is smooth. Moreover, by Lemma 3.9 we can choose such a rational point $\theta$ contained in the (classic) Hilbertian subset $\text{Hil}_\lambda \cap (B \times \lambda)(k(\lambda))$. By implicit function theorem $F^{-1}(\theta)$ has $k(\lambda)$-points $M_w^{\theta}$ sufficiently close to $M_w$ for $w \in S \otimes_k k(\lambda)$. By continuity of the Brauer–Manin pairing, we have

$$\sum_{w \in S \otimes_k k(\theta)} \text{inv}_w((M_w^{\theta}, b \mathcal{X})_{k(\theta)_w}) = 0$$

for all $b \mathcal{X} \in \Gamma_X$. We consider the specialization $R_j(\theta) = (Z_j)_{k(\lambda)} \times u_{k(\lambda)} \theta = Z_j \times u \theta$ of $R_j(\eta)$ at the $k(\lambda)$-rational point $\theta$ of $B \times \lambda \simeq B(k(\lambda))$. Then $\sum_j n_j R_j(\theta)$ is a global 0-cycle of degree 1 on the fibre $F^{-1}(\theta) = (B \times \lambda)_\theta$. The equality (⋆) implies that

$$\sum_j n_j \text{cores}_{k(R_j(\theta))} b \mathcal{X}(R_j(\theta))) = 0 \in \text{Br}k(\theta).$$

For $w \in \Omega_{k(\lambda)} \setminus S \otimes_k k(\lambda)$, we set $M^{\theta}_w$ to be the local 0-cycle of degree 1

$$\sum_j n_j R_j(\theta) \times_{\text{Spec}(k(\theta))} \text{Spec}(k(\theta)_w)$$

then $\text{inv}_w((M^{\theta}_w, b \mathcal{X})_{k(\theta)_w}) = 0$, and finally we obtain

$$\sum_{w \in \Omega_{k(\theta)}} \text{inv}_w((M^{\theta}_w, b \mathcal{X})_{k(\theta)_w}) = 0.$$

By functoriality of the Brauer–Manin pairing, it can be viewed as an equality on the $k(\theta)$-variety $F^{-1}(\theta)$. Since $\theta \in \text{Hil}_\lambda$, by comparison of Brauer groups (Lemma 3.9) $\Gamma_X$ maps surjectively onto $\text{Br}F^{-1}(\theta)/\text{Br}k(\theta)$, and by hypothesis there exists a global 0-cycle $z'$ of degree 1 on the $k(\theta)$-variety $F^{-1}(\theta)$ having the same image as $M^{\theta}_w$ in $\text{CH}_0(F^{-1}(\theta)_w)/n$ for all $w \in S \otimes_k k(\theta)$. Viewed as a 0-cycle of $X'$, the 0-cycle $z'$ is of degree $\Delta \equiv \delta \mod mn\delta_P$, by subtracting a suitable multiple of $P$, we obtain a 0-cycle $z$ of degree $\delta$. We verify that $z$ and $z_v$ have the same image in $\text{CH}_0(X'_v)/n$ for all $v \in S$ which terminates the proof.

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