Quantum projective planes finite over their centers

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Abstract. For a three-dimensional quantum polynomial algebra \( A = \mathcal{A}(E, \sigma) \), Artin, Tate, and Van den Bergh showed that \( A \) is finite over its center if and only if \( |\sigma| < \infty \). Moreover, Artin showed that if \( A \) is finite over its center and \( E \neq \mathbb{P}^2 \), then \( A \) has a fat point module, which plays an important role in noncommutative algebraic geometry; however, the converse is not true in general. In this paper, we will show that if \( E \neq \mathbb{P}^2 \), then \( A \) has a fat point module if and only if the quantum projective plane \( \text{Proj} \mathcal{A} \) is finite over its center in the sense of this paper if and only if \( |\nu^*\sigma^3| < \infty \) where \( \nu \) is the Nakayama automorphism of \( A \). In particular, we will show that if the second Hessian of \( E \) is zero, then \( A \) has no fat point module.

1 Introduction

A quantum polynomial algebra is a noncommutative analogue of a commutative polynomial algebra, and a quantum projective space is the noncommutative projective scheme associated to a quantum polynomial algebra, so they are the most basic objects to study in noncommutative algebraic geometry. In fact, the starting point of the subject noncommutative algebraic geometry is the paper [3] by Artin, Tate, and Van den Bergh, showing that there exists a nice correspondence between three-dimensional quantum polynomial algebras \( A \) and geometric pairs \( (E, \sigma) \) where \( E = \mathbb{P}^2 \) or a cubic divisor in \( \mathbb{P}^2 \), and \( \sigma \in \text{Aut} E \), so the classification of three-dimensional quantum polynomial algebras reduces to the classification of “regular” geometric pairs. Write \( A = \mathcal{A}(E, \sigma) \) for a three-dimensional quantum polynomial algebra corresponding to the geometric pair \( (E, \sigma) \). The geometric properties of the geometric pair \( (E, \sigma) \) provide some algebraic properties of \( A = \mathcal{A}(E, \sigma) \). One of the most striking results of such is in the companion paper [4].

**Theorem 1.1** [4, Theorem 7.1] Let \( A = \mathcal{A}(E, \sigma) \) be a three-dimensional quantum polynomial algebra. Then \( |\sigma| < \infty \) if and only if \( A \) is finite over its center.

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Let $A = A(E, \sigma)$ be a three-dimensional quantum polynomial algebra. To prove the above theorem, fat points of the quantum projective plane $\text{Proj}_{nc} A$ play an essential role. By Artin [2], if $A$ is finite over its center and $E \neq \mathbb{P}^2$, then $\text{Proj}_{nc} A$ has a fat point; however, the converse is not true. To check the existence of a fat point, there is a more important notion than $|\sigma|$, namely,

$$\|\sigma\| := \inf\{ i \in \mathbb{N}^+ \mid \sigma^i = \phi|_E \text{ for some } \phi \in \text{Aut}\mathbb{P}^2 \}.$$ 

In fact, $\text{Proj}_{nc} A$ has a fat point if and only if $1 < \|\sigma\| < \infty$ by [2].

In [13], the notion that $\text{Proj}_{nc} A$ is finite over its center was introduced, and the following result was proved.

**Theorem 1.2** [13, Theorem 4.17] Let $A = A(E, \sigma)$ be a three-dimensional quantum polynomial algebra such that $E \subset \mathbb{P}^2$ is a triangle. Then $\|\sigma\| < \infty$ if and only if $\text{Proj}_{nc} A$ is finite over its center.

The purpose of this paper is to extend the above theorem to all three-dimensional quantum polynomial algebras. In fact, the following is our main result.

**Theorem 1.3** (Theorem 3.6 and Corollary 4.1) Let $A = A(E, \sigma)$ be a three-dimensional quantum polynomial algebra such that $E \neq \mathbb{P}^2$, and $\nu \in \text{Aut} A$ the Nakayama automorphism of $A$. Then the following are equivalent:

1. $|\nu^* \sigma^3| < \infty$.
2. $\|\sigma\| < \infty$.
3. $\text{Proj}_{nc} A$ is finite over its center.
4. $\text{Proj}_{nc} A$ has a fat point.

Note that if $E = \mathbb{P}^2$, then $\|\sigma\| = 1$, but $\text{Proj}_{nc} A$ has no fat point (see Lemma 2.14).

As a biproduct, we have the following corollary.

**Corollary 1.4** Let $A = A(E, \sigma)$ be a three-dimensional quantum polynomial algebra. If the second Hessian of $E$ is zero, then $A$ is never finite over its center.

These results are important to study representation theory of the Beilinson algebra $\nabla A$, which is a typical example of a 2-representation infinite algebra defined in [6]. This was the original motivation of the paper [13].

## 2 Preliminaries

Throughout this paper, we fix an algebraically closed field $k$ of characteristic 0. All algebras and (noncommutative) schemes are defined over $k$. We further assume that all (graded) algebras are finitely generated (in degree 1) over $k$, that is, algebras of the form $k(x_1, \ldots, x_n)/I$ for some (homogeneous) ideal $I \subset k(x_1, \ldots, x_n)$ (where $\deg x_i = 1$ for every $i = 1, \ldots, n$).
2.1 Geometric quantum polynomial algebras

In this subsection, we define geometric algebras and quantum polynomial algebras.

**Definition 2.1** [12, Definition 4.3] A geometric pair \((E, \sigma)\) consists of a projective scheme \(E \subset \mathbb{P}^{n-1}\) and \(\sigma \in \text{Aut}_k E\). For a quadratic algebra \(A = k(x_1, \ldots, x_n)/I\) where \(I \triangleleft k(x_1, \ldots, x_n)\) is a homogeneous ideal generated by elements of degree 2, we define

\[
\mathcal{V}(I_2) := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0 \text{ for any } f \in I_2\}.
\]

1. We say that \(A\) satisfies (G1) if there exists a geometric pair \((E, \sigma)\) such that \(\mathcal{V}(I_2) = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}\).

In this case, we write \(\mathcal{P}(A) = (E, \sigma)\), and call \(E\) the point scheme of \(A\).

2. We say that \(A\) satisfies (G2) if there exists a geometric pair \((E, \sigma)\) such that \(I_2 = \{f \in k(x_1, \ldots, x_n) \mid f(p, \sigma(p)) = 0 \text{ for any } p \in E\}\).

In this case, we write \(A = A(E, \sigma)\).

3. A quadratic algebra \(A\) is called geometric if \(A\) satisfies both (G1) and (G2) with \(A = A(\mathcal{P}(A))\).

**Definition 2.2** A right Noetherian graded algebra \(A\) is called a \(d\)-dimensional quantum polynomial algebra if

1. \(\operatorname{gldim} A = d\),
2. \(\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}\) and
3. \(H_A(t) := \sum_{i=0}^{\infty} (\dim_k A_i) t^i = (1 - t)^{-d}\).

Note that a three-dimensional quantum polynomial algebra is exactly the same as a three-dimensional quadratic AS-regular algebra, so we have the following result.

**Theorem 2.1** [3] Every three-dimensional quantum polynomial algebra is a geometric algebra where the point scheme is either \(\mathbb{P}^2\) or a cubic divisor in \(\mathbb{P}^2\).

**Remark 2.2** There exists a four-dimensional quantum polynomial algebra which is not a geometric algebra; however, as far as we know, there exists no example of a quantum polynomial algebra which does not satisfy (G1).

We define the type of a three-dimensional quantum polynomial algebra \(A = A(E, \sigma)\) in terms of the point scheme \(E \subset \mathbb{P}^2\).

- **Type P** \(E\) is \(\mathbb{P}^2\).
- **Type S** \(E\) is a triangle.
- **Type S'** \(E\) is a union of a line and a conic meeting at two points.
Type $T E$ is a union of three lines meeting at one point.
Type $T' E$ is a union of a line and a conic meeting at one point.
Type $NC E$ is a nodal cubic curve.
Type $CC E$ is a cuspidal cubic curve.
Type $TL E$ is a triple line.
Type $WL E$ is a union of a double line and a line.
Type $EC E$ is an elliptic curve.

2.2 Quantum projective spaces finite over their centers

Definition 2.3 A noncommutative scheme (over $k$) is a pair $X = (\text{mod} X, \mathcal{O}_X)$ consisting of a $k$-linear abelian category $\text{mod} X$ and an object $\mathcal{O}_X \in \text{mod} X$. We say that two noncommutative schemes $X = (\text{mod} X, \mathcal{O}_X)$ and $Y = (\text{mod} Y, \mathcal{O}_Y)$ are isomorphic, denoted by $X \cong Y$, if there exists an equivalence functor $F : \text{mod} X \to \text{mod} Y$ such that $F(\mathcal{O}_X) \cong \mathcal{O}_Y$.

If $X$ is a commutative Noetherian scheme, then we view $X$ as a noncommutative scheme by $(\text{mod} X, \mathcal{O}_X)$ where $\text{mod} X$ is the category of coherent sheaves on $X$ and $\mathcal{O}_X$ is the structure sheaf on $X$.
Noncommutative affine and projective schemes are defined in [5].

Definition 2.4 If $R$ is a right Noetherian algebra, then we define the noncommutative affine scheme associated to $R$ by $\text{Spec}_{nc} R = (\text{mod} R, R)$ where $\text{mod} R$ is the category of finitely generated right $R$-modules and $R \in \text{mod} R$ is the regular right module.

Note that if $R$ is commutative, then $\text{Spec}_{nc} R \cong \text{Spec} R$.

Definition 2.5 If $A$ is a right Noetherian graded algebra, $\text{grmod} A$ is the category of finitely generated graded right $A$-modules, and $\text{tors} A$ is the full subcategory of $\text{grmod} A$ consisting of finite-dimensional modules over $k$, then we define the noncommutative projective scheme associated to $A$ by $\text{Proj}_{nc} A = (\text{tails} A, \pi A)$ where $\text{tails} A : = \text{grmod} A/\text{tors} A$ is the quotient category, $\pi : \text{grmod} A \to \text{tails} A$ is the quotient functor, and $A \in \text{grmod} A$ is the regular graded right module. If $A$ is a $d$-dimensional quantum polynomial algebra, then we call $\text{Proj}_{nc} A$ a quantum $\mathbb{P}^{d-1}$. In particular, if $d = 3$, then we call $\text{Proj}_{nc} A$ a quantum projective plane.

Note that if $A$ is commutative, then $\text{Proj}_{nc} A \cong \text{Proj} A$. It is known that if $A$ is a two-dimensional quantum polynomial algebra, then $\text{Proj}_{nc} A \cong \mathbb{P}^1$.

For a three-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$, we have the following geometric characterization when $A$ is finite over its center.

Theorem 2.3 [4, Theorem 7.1] Let $A = \mathcal{A}(E, \sigma)$ be a three-dimensional quantum polynomial algebra. Then the following are equivalent:

1. $|\sigma| < \infty$.
2. $A$ is finite over its center.
Since the property that $A$ is finite over its center is not preserved under isomorphisms of noncommutative projective schemes $\text{Proj}_{nc} A$, we will make the following rather ad hoc definition.

**Definition 2.6** Let $A$ be a $d$-dimensional quantum polynomial algebra. We say that $\text{Proj}_{nc} A$ is **finite over its center** if there exists a $d$-dimensional quantum polynomial algebra $A'$ finite over its center such that $\text{Proj}_{nc} A \cong \text{Proj}_{nc} A'$.

For a three-dimensional quantum polynomial algebra, the above definition coincides with [13, Definition 4.14] by the following result.

**Lemma 2.4** [1, Corollary A.10] Let $A$ and $A'$ be three-dimensional quantum polynomial algebras. Then $\text{grmod} A \cong \text{grmod} A'$ if and only if $\text{Proj}_{nc} A \cong \text{Proj}_{nc} A'$.

To characterize “geometric” quantum projective spaces finite over their centers, we will introduce the following notion.

**Definition 2.7** [13, Definition 4.6] For a geometric pair $(E, \sigma)$ where $E \subset \mathbb{P}^{n-1}$ and $\sigma \in \text{Aut}_k E$, we define

$$\text{Aut}_k(\mathbb{P}^{n-1}, E) := \{ \phi|_E \in \text{Aut}_k E \mid \phi \in \text{Aut}_k \mathbb{P}^{n-1} \},$$

and

$$\|\sigma\| := \inf\{ i \in \mathbb{N}^+ \mid \sigma^i \in \text{Aut}_k(\mathbb{P}^{n-1}, E) \}.$$  

For a geometric pair $(E, \sigma)$, clearly $\|\sigma\| \leq |\sigma|$. The following are the basic properties of $\|\sigma\|$.

**Lemma 2.5** [13, Lemma 4.16(1)], [14, Lemma 2.5] Let $A$ and $A'$ be $d$-dimensional quantum polynomial algebras satisfying (G1) with $\mathcal{P}(A) = (E, \sigma)$ and $\mathcal{P}(A') = (E', \sigma')$.

1. If $A \cong A'$, then $E \cong E'$ and $|\sigma| = |\sigma'|$.
2. If $\text{grmod} A \cong \text{grmod} A'$, then $E \cong E'$ and $\|\sigma\| = \|\sigma'\|$.

In particular, if $A$ and $A'$ are three-dimensional quantum polynomial algebras such that $\text{Proj}_{nc} A \cong \text{Proj}_{nc} A'$, then $E \cong E'$ (that is, $A$ and $A'$ are of the same type) and $|\sigma| = \|\sigma'\|$.

For a three-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ of Type S, we have the following geometric characterization when a quantum projective plane $\text{Proj}_{nc} A$ is finite over its center.

**Theorem 2.6** [13, Theorem 4.17] Let $A = \mathcal{A}(E, \sigma)$ be a three-dimensional quantum polynomial algebra of Type S. Then the following are equivalent:

1. $\|\sigma\| < \infty$.
2. $\text{Proj}_{nc} A$ is finite over its center.

The purpose of this paper is to extend the above theorem to all types.
2.3 Points of a noncommutative scheme

\textbf{Definition 2.8} Let \( R \) be an algebra. A point of \( \text{Spec}_{nc} R \) is an isomorphism class of a simple right \( R \)-module \( M \in \text{mod} R \) such that \( \dim_k M < \infty \). A point \( M \) is called \textit{fat} if \( \dim_k M > 1 \).

\textbf{Remark 2.7} If \( R \) is a commutative algebra and \( p \in \text{Spec} A \) is a closed point, then \( A/m_p \in \text{mod} R \) is a point where \( m_p \) is the maximal ideal of \( R \) corresponding to \( p \). In fact, this gives a bijection between the set of closed points of \( \text{Spec} R \) and the set of points of \( \text{Spec}_{nc} R \). In this commutative case, there exists no fat point.

\textbf{Remark 2.8} Fat points are not preserved under Morita equivalences. For example, \( \text{mod} k \cong \text{mod} M_2(k) \), but it is easy to see that \( \text{Spec}_{nc} k \) has no fat point while \( \text{Spec}_{nc} M_2(k) \) has a fat point. However, since \( \text{Spec}_{nc} R \cong \text{Spec}_{nc} R' \) if and only if \( R \cong R' \), fat points are preserved under isomorphisms of \( \text{Spec}_{nc} R \).

\textbf{Example 2.9} If \( R = k(u,v)/(uv - vu - 1) \) is the first Weyl algebra, then it is well known that there exists no finite-dimensional right \( R \)-module, so \( \text{Spec}_{nc} R \) has no point at all.

\textbf{Example 2.10} (cf. [15]) If \( R = k(u,v)/(vu - uv - u) \) is the enveloping algebra of a two-dimensional nonabelian Lie algebra, then the set of points of \( \text{Spec}_{nc} R \) is given by \( \{ R/\mu R \}_{\mu \in k} \), so \( \text{Spec}_{nc} R \) has no fat point. In fact, the linear map \( \delta : k[u] \to k[u] \) defined by \( \delta(f(u)) = uf'(u) \) is a derivation of \( k[u] \) such that \( R = k[u][v; \delta] \) is the Ore extension, so that \( v f(u) = f(u)v + uf'(u) \). If \( M \) is a finite-dimensional right \( R \)-module, then there exists \( f(u) = a_d u^d + \cdots + a_1 u + a_0 \in k[u] \subset R \) of the minimal degree \( \deg f(u) = d \geq 1 \) such that \( Mf(u) = 0 \). Since \( uf'(u) = v f(u) - f(u) v, M(df(u) - uf'(u)) = 0 \) such that \( \deg(df(u) - uf'(u)) < \deg f(u), df(u) = uf'(u) \) by minimality of \( \deg f(u) = d \geq 1 \), but this is possible only if \( f(u) = a_1 u \), so \( Mu = 0 \). It follows that \( M \) can be viewed as an \( R/(u) \)-module, a point of \( \text{Spec}_{nc}(R/(u)) \cong \text{Spec}_{nc} k[v] \), so \( M \cong R/\mu R + (v - \mu)R \) for some \( \mu \in k \). Since \( \text{Spec}_{nc}(R/(u)) \cong \text{Spec}_{nc} k[v] \) is a commutative scheme, \( \text{Spec}_{nc} R \) has no fat point.

\textbf{Example 2.11} [13, Lemma 4.19] If \( R = k(u,v)/(uv + vu) \) is a two-dimensional (ungraded) quantum polynomial algebra, then the set of points of \( \text{Spec}_{nc} R \) is given by

\[ \{ R/(u - \lambda)R + vR \}_{\lambda \in k} \cup \{ R/\mu R + (v - \mu)R \}_{\mu \in k} \]

\[ \cup \{ R/(x^2 - \lambda)R + (\sqrt{\mu}x + \sqrt{-\lambda} y)R + (y^2 - \mu)R \}_{\lambda, \mu \in k}. \]

Among them, \( \{ R/(x^2 - \lambda)R + (\sqrt{\mu}x + \sqrt{-\lambda} y)R + (y^2 - \mu)R \}_{\lambda, \mu \in k} \) is the set of fat points of \( \text{Spec}_{nc} R \).

\textbf{Definition 2.9} Let \( A \) be a graded algebra. A point of \( \text{Proj}_{nc} A \) is an isomorphism class of a simple object of the form \( \pi M \in \text{tails} A \) where \( M \in \text{grmod} A \) is a graded right
A-module such that \( \lim_{i \to \infty} \dim_k M_i < \infty \). A point \( \pi M \) is called *fat* if \( \lim_{i \to \infty} \dim_k M_i > 1 \), and, in this case, \( M \) is called a *fat point module over* \( A \).

**Remark 2.12** If \( A \) is a graded commutative algebra and \( p \in \text{Proj}A \) is a closed point, then \( \pi (A/m_p) \in \text{tails}A \) is a point where \( m_p \) is the homogeneous maximal ideal of \( A \) corresponding to \( p \). In fact, this gives a bijection between the set of closed points of \( \text{Proj}A \) and the set of points of \( \text{Proj}_{nc}A \). In this commutative case, there exists no fat point.

**Remark 2.13** It is unclear that fat points are preserved under isomorphisms of \( \text{Proj}_{nc}A \) in general. However, fat point modules are preserved under graded Morita equivalences, so if \( A \) and \( A' \) are both three-dimensional quantum polynomial algebras such that \( \text{Proj}_{nc}A \cong \text{Proj}_{nc}A' \), then there exists a natural bijection between the set of fat points of \( \text{Proj}_{nc}A \) and that of \( \text{Proj}_{nc}A' \) by Lemma 2.4.

The following facts will be used to prove our main results.

**Lemma 2.14** [2, 13] Let \( A = \mathcal{A}(E, \sigma) \) be a three-dimensional quantum polynomial algebra.

1. \( \| \sigma \| = 1 \) if and only if \( E = \mathbb{P}^2 \).
2. \( 1 < \| \sigma \| < \infty \) if and only if \( \text{Proj}_{nc}A \) has a fat point.

**Theorem 2.15** [13, Theorem 4.20] If \( A \) is a quantum polynomial algebra and \( x \in A \) is a homogeneous normal element of positive degree, then there exists a bijection between the set of points of \( \text{Proj}_{nc}A \) and the disjoint union of the set of points of \( \text{Proj}_{nc}A/(x) \) and the set of points of \( \text{Spec}_{nc}A[x^{-1}]_0 \). In this bijection, fat points correspond to fat points.

## 3 Main results

In this section, we will state and prove our main results.

Let \( A \) be a graded algebra and \( v \in \text{Aut}A \) a graded algebra automorphism. For a graded \( A-A \)-bimodule \( M \), we define a new graded \( A-A \) bimodule \( M_v = M \) as a graded vector space with the new actions \( a * m * b := amv(b) \) for \( a, b \in A, m \in M \). Let \( A \) be a \( d \)-dimensional quantum polynomial algebra. **The canonical module of** \( A \) is defined by \( \omega_A := \lim_{i \to \infty} \text{Ext}^d_A(A/A_{\geq i}, A) \), which has a natural graded \( A-A \) bimodule structure. It is known that there exists \( v \in \text{Aut}A \) such that \( \omega_A \cong A_{v^{-1}}(-d) \) as graded \( A-A \) bimodules. We call \( v \) the Nakayama automorphism of \( A \). Since \( A_0 = k \), the Nakayama automorphism \( v \) is uniquely determined by \( A \). Among quantum polynomial algebras, Calabi–Yau quantum polynomial algebras defined below are easier to handle.

**Definition 3.1** A quantum polynomial algebra \( A \) is called **Calabi–Yau** if the Nakayama automorphism of \( A \) is the identity.

The following theorem plays an essential role to prove our main results, claiming that every quantum projective plane has a three-dimensional Calabi–Yau quantum polynomial algebra as a homogeneous coordinate ring.
Theorem 3.1 [8, Theorem 4.4] For every three-dimensional quantum polynomial algebra $A$, there exists a three-dimensional Calabi–Yau quantum polynomial algebra $A'$ such that $\text{grmod}A \cong \text{grmod}A'$, so that $\text{Proj}_{nc}A \cong \text{Proj}_{nc}A'$.

By the above theorem, the proofs of our main results reduce to the Calabi–Yau case.

3.1 Calabi–Yau case

Let $E = \mathbb{V}(x^3 + y^3 + z^3 - \lambda xyz) \subset \mathbb{P}^2$, $\lambda \in k$, $\lambda^3 \neq 27$ be an elliptic curve in the Hesse form. We fix a group structure with the identity element $o := (1, -1, 0) \in E$, and write $E[n] := \{ p \in E \mid np = o \}$ the set of $n$-torsion points. We also denote by $\sigma_p \in \text{Aut}_k E$ the translation automorphism by a point $p \in E$. It is known that $\sigma_p \in \text{Aut}_k(\mathbb{P}^2, E)$ if and only if $p \in E[3]$ (cf. [12, Lemma 5.3]).

Lemma 3.2 Denote a three-dimensional Calabi–Yau quantum polynomial algebra as

$$A = k(x, y, z)/(f_1, f_2, f_3) = \mathcal{A}(E, \sigma).$$

Then Table 1 gives a list of defining relations $f_1, f_2, f_3$ and the corresponding geometric pairs $(E, \sigma)$ for such algebras up to isomorphism. In Table 1, we remark that:

1. Type S and Type T are further divided into Type $S_1$ and Type $S_3$, and Type $T_1$ and Type $T_3$, respectively, in terms of the form of $\sigma$.
2. The point scheme $E$ may consist of several irreducible components, and, in this case, $\sigma$ is described on each component.
3. For Type NC and Type CC, $\sigma$ in Table 1 is defined except for the unique singular point $(0, 0, 1) \in E$, which is preserved by $\sigma$.
4. For Type TL and Type WL, $E$ is nonreduced, and the description of $\sigma$ is omitted.

Proof The list of the defining relations $f_1, f_2, f_3$ is given in [7, Theorem 3.3] and [9, Corollary 4.3]. It is not difficult to calculate their corresponding geometric pairs $(E, \sigma)$ using the condition (G1) (see, for example, [16, proof of Theorem 3.1] for Type $P, S_1, S_3, S'$, and [14, proof of Theorem 3.6] for Type $T_1, T'$). We only give some calculations to check that $(E, \sigma)$ in Table 1 is correct for Type CC.

Let $A = k(x, y, z)/(f_1, f_2, f_3)$ be a three-dimensional Calabi–Yau quantum polynomial algebra of Type CC where

$$f_1 = yz - zy + y^2 + 3x^2, \quad f_2 = zx - xz + xy + xy - yz - zy, \quad f_3 = xy - yx - y^2,$$

and let $E = \mathbb{V}(x^3 - y^2z)$, and

$$\sigma(a, b, c) = \begin{cases} \left( a - b, b, -\frac{a^2}{b} + 3a - b + c \right) & \text{if } (a, b, c) \neq (0, 0, 1), \\ (0, 0, 1) & \text{if } (a, b, c) = (0, 0, 1), \end{cases}$$
Table 1: List of defining relations and the corresponding geometric pairs.

| Type | $f_1, f_2, f_3$                                      | $E$     | $\sigma$                      |
|------|---------------------------------------------------|---------|-------------------------------|
| P    | $yz - axy$                                        | $x = 1$ | $\sigma(a, b, c) = (a, ab, a^2c)$ |
|      | $zx - axz$                                        |         |                               |
|      | $xy - ayx$                                        |         |                               |
|      | $yz - axy$                                        | $V(x)$  | $\sigma(0, b, c) = (0, b, ac)$ |
|      | $zx - axz$                                        | $\cup V(y)$ | $\sigma(a, 0, c) = (aa, 0, c)$ |
|      | $xy - ayx$                                        | $\cup V(z)$ | $\sigma(a, b, 0) = (a, ab, 0)$ |
| S<sub>1</sub> | $zy - ax^2$                                      | $V(x)$  | $\sigma(0, b, c) = (ac, 0, b)$ |
|      | $xz - ay^2$                                       | $\cup V(y)$ | $\sigma(a, 0, c) = (c, aa, 0)$ |
|      | $yx - az^2$                                       | $\cup V(z)$ | $\sigma(a, b, 0) = (0, a, ab)$ |
| S<sub>3</sub> | $yz - axy + x^2$                                 | $V(x)$  | $\sigma(0, b, c) = (0, b, ac)$ |
|      | $zx - axz$                                        | $\cup V(x^2 - \lambda yz)$ | $\sigma(a, b, c) = (a, ab, a^{-1}c)$ |
|      | $xy - ayx$                                        | $\lambda = \frac{a^3 - 1}{a}$             |                               |
| T<sub>1</sub> | $yz - xy - yx + y^2$                             | $V(x)$  | $\sigma(0, b, c) = (b, 0, b + c)$ |
|      | $-xz - zx + x^2$                                  | $\cup V(y)$ | $\sigma(a, 0, c) = (0, a, a + c)$ |
|      | $-zy - yz - y^2$                                  | $\cup V(x - y)$ | $\sigma(a, a, c) = (a, a, -a + c)$ |
| T<sub>3</sub> | $yz - zy + xy + yx$                              | $V(y)$  | $\sigma(0, b, c) = (b, 0, b + c)$ |
|      | $zx - xz + x^2$                                   | $\cup V(x^2 - yz)$ | $\sigma(a, 0, c) = (a, a, -c)$ |
|      | $-zy - yz + y^2$                                  |            | $\sigma(a, a, c) = (0, a, -c)$ |
| T<sub>3</sub>' | $yz - zy + xy + yx$                             | $V(y)$  | $\sigma(0, b, c) = (b, 0, b + c)$ |
|      | $zx - xz + x^2$                                   | $\cup V(x^2 - yz)$ | $\sigma(a, 0, c) = (a, a, -c)$ |
|      | $-zy - yz + y^2$                                  |            | $\sigma(a, a, c) = (0, a, -c)$ |
| NC   | $yz - axy + x^2$                                  | $V(x^3 + y^3)$ | $\sigma(a, b, c) = (a - b, b, -2a + b + c)$ |
|      | $zx - axz + y^2$                                 | $-\lambda yz$ | $\sigma(a, b, c) = (a - b, b, -2a + b + c)$ |
|      | $xy - ayx$                                        | $\lambda = \frac{a^3 - 1}{a}$             |                               |
| CC   | $yz - zy + y^2 + 3x^2$                            | $V(x^3 - y^2z)$ | $\sigma(a, b, c) = (a - b, b, -3a - b + c)$ |
|      | $zx - xz + xy + xy$                               |            |                               |
|      | $-yz - yz + y^2$                                  |            |                               |
|      | $xy - yx - y^2$                                   |            |                               |
| TL   | $yz - axy - x^2$                                  | $V(x^3)$  | omitted                       |
|      | $zx - axz$                                        |            |                               |
|      | $xy - ayx$                                        |            |                               |
as in Table 1. If \( p = (a, b, c) \in E \), then \( a^3 - b^2c = 0 \), so

\[
f_1(p, \sigma(p)) = f_1((a, b, c), \left( a - b, b, -\frac{3a^2}{b} + 3a - b + c \right)) = b \left( -\frac{3a^2}{b} + 3a - b + c \right) - cb + b^2 + 3a(a - b) = -3a^2 + 3ab - b^2 + bc - bc + b^2 + 3a^2 - 3ab = 0,
\]

\[
f_2(p, \sigma(p)) = f_2((a, b, c), \left( a - b, b, -\frac{3a^2}{b} + 3a - b + c \right)) = c(a - b) - a \left( -\frac{3a^2}{b} + 3a - b + c \right) + b(a - b) + ab - b \left( -\frac{3a^2}{b} + 3a - b + c \right) - cb = ac - bc + \frac{3a^3}{b} - 3a^2 + ab - ac + ab - b^2 + ab + 3a^2 - 3ab + b^2 - bc - bc = \frac{3}{b} (a^3 - b^2c) = 0,
\]

\[
f_3(p, \sigma(p)) = f_3((a, b, c), \left( a - b, b, -\frac{3a^2}{b} + 3a - b + c \right)) = ab - b(a - b) - b^2 = ab - ab + b^2 - b^2 = 0,
\]

hence \( \{(p, \sigma(p)) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid p \in E\} \subset V(f_1, f_2, f_3) \). Since \( E \subset \mathbb{P}^2 \) is a cuspidal cubic curve (and we know that the point scheme of \( A \) is not \( \mathbb{P}^2 \)), \( E \) is the point scheme of \( A \), so \( \mathcal{P}(A) = (E, \sigma) \).

**Theorem 3.3** If \( A = \mathcal{A}(E, \sigma) \) is a three-dimensional Calabi–Yau quantum polynomial algebra, then \( ||\sigma|| = |\sigma^3| \), so the following are equivalent:

1. \( |\sigma| < \infty \).
2. \( ||\sigma|| < \infty \).
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(3) A is finite over its center.
(4) \( \text{Proj}_{nc}A \) is finite over its center.

**Proof** First, we will show that \( ||\sigma|| = |\sigma^3| \) for each type using the defining relations \( f_1, f_2, f_3 \) and geometric pairs \( (E, \sigma) \) given in Lemma 3.2. Recall that \( \sigma^i \in \text{Aut}_k(\mathbb{P}^2, E) \) if and only if it is represented by a matrix in \( \text{PGL}_3(k) \cong \text{Aut}_k \mathbb{P}^2 \).

**Type P** Since \( \sigma^3 = \text{id}, ||\sigma|| = 1 = |\sigma^3| \).

**Type S** Since

\[
\begin{align*}
\sigma^i(0, b, c) &= (0, b, \alpha^i c), \\
\sigma^i(a, 0, c) &= (\alpha^i a, 0, c) = (\alpha^{2i} a, 0, \alpha^i c), \\
\sigma^i(a, b, 0) &= (a, \alpha^i b, 0) = (\alpha^{2i} a, \alpha^{3i} b, 0),
\end{align*}
\]

\( \sigma^i \in \text{Aut}_k(\mathbb{P}^2, E) \) if and only if \( \alpha^{3i} = 1 \), so \( ||\sigma|| = |\alpha^3| = |\sigma^3| \).

**Type S’** Since

\[
\begin{align*}
\sigma^i(0, b, c) &= (0, b, \alpha^i c), \\
\sigma^i(a, 0, c) &= (\alpha^i a, 0, c) = (\alpha^{2i} a, 0, \alpha^i c), \\
\sigma^i(a, b, 0) &= (a, \alpha^i b, 0) = (\alpha^{2i} a, \alpha^{3i} b, 0),
\end{align*}
\]

\( \sigma^i \in \text{Aut}_k(\mathbb{P}^2, E) \) if and only if \( \alpha^{3i} = 1 \), so \( ||\sigma|| = |\alpha^3| = |\sigma^3| \).

**Type T** Since

\[
\begin{align*}
\sigma^i(0, b, c) &= (0, b, \alpha^i c), \\
\sigma^i(a, 0, c) &= (\alpha^i a, 0, c) = (\alpha^{2i} a, 0, \alpha^i c), \\
\sigma^i(a, b, 0) &= (a, \alpha^i b, 0) = (\alpha^{2i} a, \alpha^{3i} b, 0),
\end{align*}
\]

\( \sigma^i \notin \text{Aut}_k(\mathbb{P}^2, E) \) for every \( i \geq 1 \), so \( ||\sigma|| = \infty = |\sigma^3| \).
Since
\[ \sigma^3(0, b, c) = (0, b, ib + c), \]
\[ \sigma^3(a, 0, c) = (a, 0, ia + c), \]
\[ \sigma^3(a, a, c) = (a, a, -ia + c), \]
\[ \sigma^3 \notin \text{Aut}_k(\mathbb{P}^2, E) \text{ for every } i \geq 1, \text{ so } ||\sigma|| = \infty = |\sigma^3|. \]

Type T’ Since
\[ \begin{align*}
\sigma^i(a, 0, c) &= (a, 0, ia + c), \\
\sigma^i(a, b, c) &= (a - ib, b, -2ia + i^2b + c),
\end{align*} \]
\[ \sigma^i \notin \text{Aut}_k(\mathbb{P}^2, E) \text{ for every } i \geq 1, \text{ so } ||\sigma|| = \infty = |\sigma^3|. \]

Type NC Since
\[ \sigma^i(a, b, c) = \left( a, a^i b, -\frac{a^{3i} - 1}{a^{i-1}(a^{3i} - 1)} \frac{a^2}{b} + \alpha^2i^i c \right), \]
\[ \sigma^i \in \text{Aut}_k(\mathbb{P}^2, E) \text{ if and only if } a^{3i} = 1, \text{ so } ||\sigma|| = |\alpha^3| = |\sigma^3|. \]

Type CC Since
\[ \sigma^i(a, b, c) = \left( a - ib, b, -3i \frac{a^2}{b} + 3i^2a - i^3b + c \right), \]
\[ \sigma^i \notin \text{Aut}(\mathbb{P}^2, E) \text{ for every } i \geq 1, \text{ so } ||\sigma|| = \infty = |\sigma^3|. \]

Type TL Since \( A = k(x, y, z)/(yz - zy - x^2, zx - xz, xy - xy) \), \( a^3 = 1 \), we see that \( x \in A_1 \) is a regular normal element. Since \( A/(x) \cong k(y, z)/(yz - zy) \) is a two-dimensional quantum polynomial algebra, \( \text{Proj}_\text{nc} A/(x) \cong \mathbb{P}^1 \) has no fat point. Since \( A[x^{-1}]_0 \cong k(u, v)/(uv - vu - a) \) where \( u = xy^{-1}, v = zx^{-1} \) is isomorphic to the first Weyl algebra, \( \text{Spec}_{\text{nc}} A[x^{-1}]_0 \) has no (fat) point by Example 2.9. By Theorem 2.15, \( \text{Proj}_\text{nc} A \) has no fat point. Since \( E \neq \mathbb{P}^2 \), \( ||\sigma|| = \infty = |\sigma^3| \) by Lemma 2.14.

Type WL Since \( A = k(x, y, z)/(yz - zy - (1/3)y^2, zx - xz - (1/3)(yx + xy), xy - xy) \), we see that \( y \in A_1 \) is a regular normal element. Since \( A/(y) \cong k[x, z] \) is a two-dimensional (quantum) polynomial algebra, \( \text{Proj}_\text{nc} A/(y) = \mathbb{P}^1 \) has no fat point. Since \( A[y^{-1}]_0 \cong k[u, v]/(vu - uv - u) \) where \( u = xy^{-1}, v = zy^{-1} \) is isomorphic to the enveloping algebra of a two-dimensional nonabelian Lie algebra, \( \text{Spec}_{\text{nc}} A[y^{-1}]_0 \) has no fat point by Example 2.10. By Theorem 2.15, \( \text{Proj}_\text{nc} A \) has no fat point. Since \( E \neq \mathbb{P}^2 \), \( ||\sigma|| = \infty = |\sigma^3| \) by Lemma 2.14.

Type EC Since \( \sigma_p = \sigma_{ip} \in \text{Aut}_k(\mathbb{P}^2, E) \) if and only if \( ip \in E[3] \) if and only if \( 3ip = a, ||\sigma_p|| = |3p| = |\sigma_p^3| \).

Next, we will show the equivalences \( 1 \iff 2 \iff 3 \iff 4 \). Since \( ||\sigma|| = |\sigma^3| \) for every type, \( 1 \iff 2 \). By Theorem 2.3, \( 1 \iff 3 \). By definition, \( 3 \Rightarrow 4 \), so it is enough to show that \( 4 \Rightarrow 2 \). Indeed, if \( \text{Proj}_\text{nc} A \) is finite over its center, then there
exists a three-dimensional quantum polynomial algebra \( A' = \mathcal{A}(E', \sigma') \) which is finite over its center such that \( \text{Proj}_{nc} A \cong \text{Proj}_{nc} A' \) by Definition 2.6, so \( \| \sigma \| = \| \sigma' \| \leq |\sigma'| < \infty \) by Lemma 2.5 and Theorem 2.3.

### 3.2 General case

**Definition 3.2** [14, Definition 3.2] For a \( d \)-dimensional geometric quantum polynomial algebra \( A = \mathcal{A}(E, \sigma) \) with the Nakayama automorphism \( \nu \in \text{Aut} A \), we define a new graded algebra \( A := \mathcal{A}(E, \nu^* \sigma^3) \) satisfying (G2).

**Lemma 3.4** [14, Theorem 3.5] Let \( A \) and \( A' \) be geometric quantum polynomial algebras. If \( \text{grmod} A \cong \text{grmod} A' \), then \( A \cong A' \).

**Remark 3.5** If \( A \) and \( A' \) are both three-dimensional quantum polynomial algebras of the same Type P, S, S', T, T', then the converse of the above lemma was proved in [14, Theorem 3.6].

**Theorem 3.6** If \( A = \mathcal{A}(E, \sigma) \) is a three-dimensional quantum polynomial algebra with the Nakayama automorphism \( \nu \in \text{Aut} A \), then \( |\sigma| = |\sigma^3| \), so the following are equivalent:

1. \( |\nu^* \sigma^3| < \infty \).
2. \( |\sigma| < \infty \).
3. \( \text{Proj}_{nc} A \) is finite over its center.

Moreover, if \( A \) is of Type T, T', CC, TL, WL, then \( A \) is never finite over its center.

**Proof** For every three-dimensional quantum polynomial algebra \( A = \mathcal{A}(E, \sigma) \), there exists a three-dimensional Calabi–Yau quantum polynomial algebra \( A' = \mathcal{A}(E', \sigma') \) such that \( \text{grmod} A \cong \text{grmod} A' \) by Theorem 3.1. Since the Nakayama automorphism of \( A' \) is the identity, \( \mathcal{A}(E, \nu^* \sigma^3) = \mathcal{A}(E', \sigma^3) \) by Lemma 3.4, so

\[
|\sigma| = |\sigma'| = |\sigma^3| = |\nu^* \sigma^3|
\]

by Lemma 2.5 and Theorem 3.3. Since \( \text{Proj}_{nc} A \) is finite over its center if and only if \( \text{Proj}_{nc} A' \) is finite over its center if and only if \( |\sigma'| < \infty \) by Theorem 3.3, we have the equivalences (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3).

If \( A \) is a three-dimensional quantum polynomial algebra of Type T, T', CC, TL, WL, then \( A' \) is of the same type by Lemma 2.5, so \( |\sigma| = |\sigma'| = \infty \) by the proof of Theorem 3.3. It follows that \( |\sigma| = \infty \), so \( A \) is not finite over its center by Theorem 2.3.

### 4 An application to Beilinson algebras

We finally apply our results to representation theory of finite-dimensional algebras.

**Definition 4.1** [6, Definition 2.7] Let \( R \) be a finite-dimensional algebra of \( \text{gldim} R = d < \infty \). We define an autoequivalence \( \nu_d \in \text{Aut}^b(\text{mod} R) \) by \( \nu_d(M) := M \otimes_R DR[-d] \) where \( D^b(\text{mod} R) \) is the bounded derived category of \( \text{mod} R \) and...
\[ DR := \text{Hom}_k(R, k) \]. We say that \( R \) is \textit{d-representation infinite} if \( v_d^i(R) \in \text{mod}R \) for all \( i \in \mathbb{N} \). In this case, we say that a module \( M \in \text{mod}R \) is \textit{d-regular} if \( v_d^i(M) \in \text{mod}R \) for all \( i \in \mathbb{Z} \).

By [10], a 1-representation infinite algebra is exactly the same as a finite-dimensional hereditary algebra of infinite representation type. For representation theory of such an algebra, regular modules play an essential role.

For a \( d \)-dimensional quantum polynomial algebra \( A \), we define the Beilinson algebra of \( A \) by

\[
\nabla A := \begin{pmatrix}
A_0 & A_1 & \cdots & A_{d-1} \\
0 & A_0 & \cdots & A_{d-2} \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & A_0
\end{pmatrix}.
\]

The Beilinson algebra is a typical example of a \((d-1)\)-representation infinite algebra by [11, Theorem 4.12]. To investigate representation theory of such an algebra, it is important to classify simple \((d-1)\)-regular modules.

**Corollary 4.1** Let \( A = A(E, \sigma) \) be a three-dimensional quantum polynomial algebra with the Nakayama automorphism \( \nu \in \text{Aut}A \). Then the following are equivalent:

1. \(|\nu^* \sigma^3| = 1 \) or \( \infty \).
2. \( \text{Proj}_{nc}A \) has no fat point.
3. The isomorphism classes of simple 2-regular modules over \( \nabla A \) are parameterized by the set of closed points of \( E \subset P^2 \).

In particular, if \( A \) is of \( P, T, T', CC, TL, WL \), then \( A \) satisfies all of the above conditions.

**Proof** (1) \( \Leftrightarrow \) (2): This follow from Theorem 3.6 and Lemma 2.14.

(2) \( \Leftrightarrow \) (3): By [13, Theorem 3.6], isomorphism classes of simple 2-regular modules over \( \nabla A \) are parameterized by the set of points of \( \text{Proj}_{nc}A \). On the other hand, it is well known that the points of \( \text{Proj}_{nc}A \) which are not fat (called ordinary points in [13]) are parameterized by the set of closed points of \( E \) (see [13, Proposition 4.4]); hence, the result holds.

**Remark 4.2** We have the following characterization of Type \( P, T, T', CC, TL, WL \). Let \( A = A(E, \sigma) \) be a three-dimensional quantum polynomial algebra. Write \( E = \mathcal{V}(f) \subset P^2 \) where \( f \in k[x, y, z]_3 \). Recall that the Hessian of \( f \) is defined by \( H(f) := \det\begin{pmatrix}
 f_{xx} & f_{xy} & f_{xz} \\
 f_{yx} & f_{yy} & f_{yz} \\
 f_{zx} & f_{zy} & f_{zz}
\end{pmatrix} \in k[x, y, z]_3 \). Then \( A \) is of Type \( P, T, T', CC, TL, WL \) if and only if \( \text{Hess}(f) := H(H(f)) = 0 \).

**Remark 4.3** If \( A \) is a two-dimensional quantum polynomial algebra, then \( \nabla A \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix} \cong k( \begin{array}{c} \bullet \\ \bullet \end{array} \rightarrow \begin{array}{c} \bullet \\ \bullet \end{array} ) \), so \( \nabla A \) is a finite-dimensional hereditary algebra of tame
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representation type. It is known that the isomorphism classes of simple regular modules over $\nabla A$ are parameterized by $\mathbb{P}^1$ (cf. [13, Theorem 3.19]). For a three-dimensional quantum polynomial algebra $A$, we expect that the following are equivalent:

1. $\text{Proj}_{\text{inf}} A$ is finite over its center.
2. $\nabla A$ is 2-representation tame in the sense of [6].
3. The isomorphism classes of simple 2-regular modules over $\nabla A$ are parameterized by $\mathbb{P}^2$.

These equivalences are shown for Type S in [13, Theorems 4.17 and 4.21].

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