Distributed Chernoff Test: Optimal decision systems over networks

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Abstract—In this work, we propose two different sequential and adaptive hypothesis tests, motivated from classic Chernoff’s test, for both decentralized and distributed setup of sensor networks. In the former setup, the sensors can communicate via central entity i.e. fusion center. On the other hand, in the latter setup, sensors are connected via communication link, and no central entity is present to facilitate the communication. We compare the performance of these tests with the optimal consistent sequential test in the sensor network. In decentralized setup, the proposed test achieves the same asymptotic optimality of the classic one, minimizing the expected cost required to reach a decision plus the expected cost of making a wrong decision, when the observation cost per unit time tends to zero. This test is also asymptotic optimal in the higher moments of decision time. The proposed test is parsimonious in terms of communications as the expected number of channel uses required by each sensor, in the regime of vanishing observation cost per unit time, to complete the test converges to four. In distributed setup, the proposed test is evaluated on the same performance measures as the test in decentralized setup. We also provide sufficient conditions for which the proposed test in distributed setup also achieves the same asymptotic optimality as the classic one. Like the proposed test in decentralized setup, under these sufficient conditions, the proposed test in distributed setup is also asymptotic optimal in the higher moments of time required to reach a decision in the sensor network. This test is parsimonious in terms of communications in comparison to the state of art schemes proposed in the literature for distributed hypothesis testing.

I. INTRODUCTION

Sensor networks have become popular in various inference systems. This is mainly due to the increasingly low cost of the sensors, the embedded computational capabilities of the sensors, inherent redundancy provided by the distributed structure of the network, and the availability of high-speed wireless communication channels [1]. The sensor networks are popularly used for detection. In detection, a set of hypotheses is tested based on the observations collected at the sensors, and the result of the test is used to choose future actions in the network. Applications that fall in this framework include intrusion and target detection, and object classification and recognition [2]–[6].

The architecture of sensor networks are broadly classified into three types: centralized setting, decentralized setting and distributed setting. In centralized setting, sensors send all observations to a central processor, where the inference task is performed. In decentralized setting, computational capabilities of the sensors are used to perform preliminary processing at the sensors. Thus, only limited amount of information, following preliminary processing, is communicated to the central processor. The key advantage of this setup is the reduction in the communication overhead compared to the centralized setting. However, it may lead to sub-optimal performance. Lastly, in distributed setting, sensors are connected to each other via communication links, typically forming a sparse network, and there is no central processing unit. Thus, the sensors need to perform computations locally, share their processed data with neighboring sensors, and collectively reach a decision. A natural question is what kind of local processing to perform at the sensor nodes, and what fusion scheme to adopt at the central processor (in decentralized setup) and sensors (in distributed setup), in order to reduce the communication burden while keeping a high level of detection performance. In this work, we address this question by proposing a statistical test for both decentralized and distributed sensor networks, and comparing their performance with the optimal test in sensor network.

Hypothesis testing techniques are broadly classified as sequential or non-sequential tests, and adaptive or non-adaptive tests. Our focus is on a sequential and adaptive test. In a sequential test the number of observations needed to make a decision is not fixed in advance, but depends on the specific realization of the observed data. The test proceeds to collect and process data until a decision with a prescribed level of reliability can be made, and an important performance figure — in addition to the probability of correct decision — is the average number of observations required to end the test. In an adaptive test, the sensors’ probing actions are chosen on the basis of the collected data in a causal manner. Hence, the sensors learn from the past, and adapt their future probing actions in a closed loop fashion.

We propose a Decentralized Chernoff Test (DCT) for the decentralized setup and a Consensus-based Chernoff Test (CCT) for the distributed setup of sensor network. The performance of these tests are compared with the optimal consistent sequential test in the sensor network. We provide an upper bound on the test performance in terms of expected risk, the expected cost required to reach a decision plus the expected cost of making a wrong decision. We also derive theoretical bounds on the higher order moments of time required to reach a decision. We provide a converse showing the best possible performance of a sequential test in the sensor network in terms of risk and
the higher moments of decision time. DCT is asymptotically optimal in terms of both the performance measures, risk and higher moments of decision time, as the observation cost per unit time vanishes. Additionally, for DCT, we show that, as the observation cost per unit time vanishes, the expected number of times each sensor node uses the communication channel tends to four. We also evaluate the performance of CCT on these performance measures, and provide sufficient conditions for which CCT retains the asymptotic optimality of Chernoff’s original solution, both in terms of risk and higher moments of decision time. In distributed hypothesis testing, the analysis of the existing techniques in terms of expected decision time, however, has been limited and CCT is an asymptotically optimal solution for this.

The rest of the paper is organized as follows: Section II discusses the related work, Section III formulates the problem; Section IV reviews the standard Chernoff test; Section V introduces decentralized Chernoff test; Section VI informally describes the main idea behind DCT; Section VII presents rigorous theoretical results on DCT; Section VIII introduces consensus based Chernoff test; Section IX informally describes the main idea behind CCT; Section X presents rigorous theoretical results on CCT; Section XI presents simulation results and Section XII concludes the work.

II. RELATED WORK

Sequential tests were first introduced by Wald in [7]. One of such tests, the Sequential Probability Ratio Test (SPRT) was established to be optimum for binary hypothesis testing in [8]. Among the feasible error exponents for sequential tests, SPRT achieves the best possible error exponent. The asymptotic optimality of SPRT was proved for multi-hypothesis testing in [9], [10]. Further, sequential tests can be combined with adaptive schemes to enhance the performance because these schemes adapt their choice of actions based on the past observations. In the case of sequential and adaptive tests, Chernoff provided the optimal test for binary composite hypotheses [11]. The asymptotic optimality of Chernoff test was proved for multi-hypothesis testing in [12], see also [13] and therein references. Later, the sequentiality and adaptivity gains for different classes of tests were studied for hypothesis testing, and it was established that sequential-adaptive tests outperforms other classes of tests [14]. The gains can vary from application to application [15]–[17]. All the above results were established in a centralized setup.

Various works discuss the extension of the results from the literature of hypothesis testing in a decentralized setup to sensor networks. The hypothesis testing was discussed for various configurations of the sensor networks in [18], [19]. Additionally, the techniques for combining the information at the central processor from various sensors are well studied in the literature [1], [20]–[22]. Asymptotically optimal sequential and non-adaptive test has been developed for hypothesis testing in sensor network [23], [24]. The measure of optimality in these works is same as the one used in our setup, however, these tests do not consider the realm of adaptive tests in sensor networks [23], [24].

Various gossiping protocols have been studied for distributed estimation of network parameters like mean, sum, minimum and maximum [25]–[30]. These protocols can be divides into two categories: consensus protocols and running consensus protocols. In consensus protocols, the protocol estimates the desirable parameter after the measurements are collected at the sensors [25], [26], [28], [29]. On the other hand, in running consensus protocols, the sensing of the environment and the estimation of parameters are performed simultaneously [27], [30]. Additionally, the necessary and sufficient conditions for the convergence of the former type of protocols are well studied in the literature [31].

Motivated by distributed estimation, the protocols for distribution detection are based on the computation of a belief about a hypothesis and its propagation over the network using the concepts from running consensus protocols. These works focus on different strategies to transmit and combine the belief vectors of the hypotheses over the network, and study the learning rate of these strategies [32]–[36]. [32] proposes on a non-Bayesian update rule where each node averages the belief about a hypothesis with the beliefs of its neighbors. In [32], the learning rate is characterized in terms of the total variational error across the network. Another strategy based on distributed dual averaging was proposed by [35] which utilizes the optimization algorithm from [37]. [36] introduces a bayesian strategy for updating and combining the beliefs about the hypothesis. The bounds on the asymptotic learning rate are presented in terms of KL-divergences and the centrality of the network nodes as $t \to \infty$. The analysis of these techniques in terms of expected decision time has been limited. Under the assumption that the log likelihood ratio is bounded, [34] presents finite time analysis on KL divergence cost > However, the bounds presented in this work are probabilistic and do not study the behavior in expectation. Under the same assumption, similar results have been presented for time varying graph [33], [39]. In our work, we relax this assumption and extend the horizon of evaluation for distributed hypothesis testing schemes. The analysis of these techniques in terms of expected decision time, however, has been limited and our work provides an asymptotically optimal solution for this.

III. PROBLEM FORMULATION

We consider both decentralized setup and distributed setup of sensor network.

In decentralized setup, we consider a sensor network composed of $L$ sensors and one fusion center. The sensors and the fusion center can communicate with each other, while no direct mode of communication between the sensors is allowed.

We consider a distributed sensor network with a fully-flat architecture without any fusion center. All the communication and the information processing tasks take place at the node level, and are fully distributed because nodes exploit only locally available information. The network is composed by $L$ sensors and is modeled as a graph $G(\mathcal{L}, \mathcal{E})$, where the set of
nodes \( \mathcal{L} = \{1, 2, \ldots, L\} \) represents the \( L \) sensors, and the elements of set \( \mathcal{E} \) are the edges, namely un-ordered pairs of different nodes \( \{(i, j)\} \), in which \((i, j)\) represents the communication link between sensors \( i \) and \( j \). \( i \neq j \). The inter-sensor communication is allowed only over the edges \( \mathcal{E} \) of \( \mathcal{G}(\mathcal{L}, \mathcal{E}) \). The diameter \( d^\mathcal{G} \) of the network is the maximum shortest distance between any pair of nodes of \( \mathcal{G}(\mathcal{L}, \mathcal{E}) \). We also denote by \( h^\mathcal{G} \) the shortest height of all possible spanning trees of \( \mathcal{G}(\mathcal{L}, \mathcal{E}) \). It is assumed that the network is strongly connected, namely, there exists a path between any two sensors \( i \) and \( j \). This ensures that \( d^\mathcal{G} \) and \( h^\mathcal{G} \) are both finite, and it follows trivially from the definition of these network parameters.

The state of nature to be detected is one of \( M \) exhaustive and mutually exclusive hypotheses \( \{h_i\}_{i \in [M]} \), where the short-cut notation \( [M] = \{1, 2, \ldots, M\} \) is used. At each time instant, each sensor takes a probing action, selected from a fixed set of actions \( S = \{u_i\}_{i \in [M]} \). We assume that sensors select their actions independently of each other, and that the cardinality of set \( S \) is equal to \( M \). Under this latter assumption, action \( u_i \) can be interpreted as the "best" action when the state of nature is \( h_i \), \( i \in [M] \). All the results of this paper can be extended to the more general case.

Suppose that the state of nature is \( h_i \), \( i \in [M] \), and consider sensor \( \ell \in \mathcal{L} \). Let \( u_k \), \( k \in [M] \), be the probing action taken by sensor \( \ell \) at a given time. Then, the probability distribution of the observation received at the sensor as a consequence of its probing action is denoted by \( p^{u_k}_{i,\ell} \). Given the true hypothesis \( H^* \), the observations received by any sensor are independent of the observations received by other sensors. On the other hand, for a given sensor, observations collected at different time instants are not independent, because the probing actions are observation-dependent. The sensor learns from the past and try to select the best action for the future.

The performance measure used in this work – the risk – is analogous to the one considered in [11] Under hypothesis \( h_i \), the risk \( R^\mathcal{G}_i \) of a sequential test \( \delta \) is defined as follows:

\[
R^\mathcal{G}_i = c E^\mathcal{G}_i[N] + \omega_i P^\mathcal{G}_i(\hat{H} \neq h_i),
\]

where \( E^\mathcal{G}_i[N] \) is the conditional expected time required to reach a global decision in the network, \( c \) is the observation cost per unit time, \( H \) is the decision made, \( P^\mathcal{G}_i(\hat{H} \neq h_i) \) is the conditional probability of making a wrong decision, and \( \omega_i \) is the cost of a wrong decision. Note that the risk is the sum of the expected cost required to reach a decision and the expected cost of making a wrong decision. Note also that the expectation and the probability operators in [11] are computed under the true state of nature \( h_i \).

This work proposes hypothesis tests for both decentralized and distributed setup of sensor network, and evaluates their performance in terms of risk and on the higher moments of the decision time \( N \) for all \( i \in [M] \), as \( c \to 0 \). In decentralized setting, we propose a test which is asymptotically optimal in terms of risk as well as the higher moments of decision time \( N \), as \( c \to 0 \). In distributed setting, we study the same performance measures for the proposed test, and provide sufficient conditions under which the proposed test is asymptotically optimal in these performance measures, as \( c \to 0 \).

We assume that observations corresponding to probing actions are instantly available at the sensors, the communication link between the sensors is noise free, and the information sent along this link is instantly available at the receiving end.

The KL-divergence between the hypotheses is assumed to be finite for the entire action set \( S \), namely, for all \( \ell \in [L] \) and \( i, j, k_1 \in [M] \), we have \( D(p^{u_{k_1}}_{i,\ell} || p^{u_{j}}_{j,\ell}) < \infty \). Also, for all \( \ell \in [L] \) and \( i, j \in [M] \), there exists an action \( u_{k_1} \), where \( k_1 \in [M] \), such that \( D(p^{u_{k_1}}_{i,\ell} || p^{u_{j}}_{j,\ell}) > 0 \). This assumption entails little loss of generality, rules out trivialities, and is commonly adopted in the literature, see e.g. [11]. Also, for all \( \ell \in [L] \) and \( i, j \in [M] \), we assume \( \mathbb{E}[\log(p^{u_{k_1}}_{i,\ell}(Y)) / \log(p^{u_{j}}_{j,\ell}(Y))]^2 < \infty \).

We shall use of the following notation: if \( v_1 = [v_{1,1}, v_{2,1}, \ldots, v_{k_1,1}] \) and \( v_2 = [v_{1,2}, v_{2,2}, \ldots, v_{k_2,2}] \) are two vectors of same dimension \( k \), then \( v_1 \preceq v_2 \) means \( v_{1,i} \leq v_{2,i} \) for all \( i \in [k] \), and \( \max(v_1) \) denotes the largest element of the vector \( v_1 \).

IV. STANDARD CHERNOFF TEST [11]

Let us consider sensor \( \ell \) alone, with no interactions with other sensors of the network. Chernoff test for this isolated sensor works as follows:

1) At step \( k - 1 \), a temporary decision is made, based on the maximum posterior probability of the hypotheses, given the past observations and actions. In formula, the temporary decision is in favor of \( h_{i_{k-1}} \) iff

\[
i_{k-1} = \arg \max_{i \in [M]} P(H^* = h_i | y_{k-1}^{\ell}, u_{k-1}^{\ell}), \quad (2)
\]

where \( H^* \) is the true hypothesis, \( y_{k-1}^{\ell} = \{y_{1,\ell}, y_{2,\ell}, \ldots, y_{k-1,\ell}\} \), where \( y_{1,\ell} \) is the observation collected at time index \( (\text{step } i) \), \( u_{k-1}^{\ell} = \{u_{1,\ell}, u_{2,\ell}, \ldots, u_{k-1,\ell}\} \), and \( u_{k,\ell} \) is the action made at step \( i \).

2) At step \( k \), the action \( u_{k,\ell} \) is randomly chosen among the elements of \( S \), according to the Probability Mass Function (PMF) \( Q_{i_{k-1}}^{\ell} \), where

\[
Q_{i_{k-1}}^{\ell} = \arg \max_{i \in [M]} \min_{j \in [M]} \sum_{u} q(u) D(p^{u}_{i,\ell} || p^{u}_{j,\ell}), \quad (3)
\]

in which \( Q \) denotes the set of all the possible PMFs over the alphabet \( [M] \) of \( S \), and \( M_{i_{k-1}} \) is \( [M] \setminus \{i_{k-1}\} \).

3) For all \( i \in [M] \), update the posterior probabilities \( P(H^* = h_i | y_k^{\ell}, u_k^{\ell}) \).

4) The test stops at step \( N \) if the worst case log-likelihood ratio crosses a prescribed fixed threshold \( \gamma \), i.e.,

\[
\log \frac{p_{i_{k-1}}(y_N^{\ell}, u_N^{\ell})}{\max_{j \neq i_{k-1}} p_{j,\ell}(y_N^{\ell}, u_N^{\ell})} \geq \gamma, \quad (4)
\]

where \( p_{i_{k-1}}(y_N^{\ell}, u_N^{\ell}) \) is the posterior probability \( P(H^* = h_{i_{k-1}}, y_N^{\ell}, u_N^{\ell}) \) at sensor \( \ell \). If the test stops at step \( N \), then the final decision is \( h_{i_{k-1}} \). Otherwise, \( k \leftarrow (k - 1) \), and the procedures continues from 1).
V. DECENTRALIZED CHERNOFF TEST

As the observation cost per unit time tends to zero, the probability of wrong detection for the standard Chernoff test tends to zero \([11]\). It follows that minimizing the risk in \([11]\) also corresponds to minimizing the expected number of samples required to reach a decision. When one sample is collected at each time step, minimizing the expected number of samples is obviously the same as minimizing the expected time for making a decision. However, this is not necessarily true in a decentralized setting.

To further illustrate this point, consider first minimizing the total expected number of samples collected by the \(L\) sensors to reach a decision, and assume that the amount of communication between sensors and fusion center is unconstrained. A straightforward design, which we call Fusion Center based Chernoff Test (FCT), is as follows. The action set \(S\) is modified to \(S'\) with cardinality \(ML\), where action \(a_{i,\ell} \in S'\) corresponds to the selection of \(u_i \in S\) and sensor \(\ell \in [M]\). Then, a Chernoff test is performed on \(S'\) at the fusion center where the selection of \(a_{i,\ell}\) corresponds to activating sensor \(\ell\), and enabling the activated sensor to use the probing action \(u_i\) in order to collect the corresponding observation, which is then delivered to the fusion center. It is not hard to see that, as the probability of wrong detection tends to zero, the FCT minimizes the total expected number of collected samples. The proof of this claim is similar to the one of the optimality of Chernoff test in \([11]\) and is thus omitted.

The FCT also minimizes the total number of probing actions performed by the sensors. However, there is only one active sensor, out of \(L\), per unit time, and all observations are communicated to the fusion center. Clearly, this is highly inefficient in terms of both communication overhead and decision time, and motivates introducing a different kind of test. Thus, we propose DCT for minimizing the risk \([11]\).

Our proposed DCT operates in two phases. In the initialization phase, each sensor \(\ell\) sends a vector \(v_i\) to the fusion center, where the elements of \(v_i\) are, for all \(i \in [M]\)

\[
v_{i,\ell} = \max_{q \in Q} \min_{j \neq \ell} \sum_u q(u) D(p_{u,i}^{n}||p_{u,j}^{n}). \tag{5}
\]

The quantity \(v_{i,\ell}\) is a measure of the capability of sensor \(\ell\) to detect hypothesis \(h_i\) (see \([11]\) for a discussion), and plays a critical role in designing the test. After receiving \(v_i\) from all sensors, the fusion center sends back to sensor \(\ell\) a response vector \(\rho_{i,\ell}\), whose \(L\) entries are the scalars

\[
\rho_{i,\ell} = v_{i,\ell}/I(i),
\]

where \(I(i) = \sum_{\ell=1}^{L} v_{i,\ell}\) is a measure of cumulative capability of the network to detect hypothesis \(h_i\).

At this point, the test phase begins. All sensors perform a Chernoff test, independently of each other, consisting of steps 1-4 described in Section IV with an important difference: if any time \(n\) and sensor \(\ell\) we have

\[
\log \frac{p_{u_{j},\ell}'(y_{l}^{n},u_{j}^{n})}{\max_j p_{u_{j},\ell}'(y_{l}^{n},u_{j}^{n})} \geq \rho_{i,\ell} \log e, \tag{7}
\]

then a local decision in favor of \(h_{i_{\ell}}\) is communicated to the fusion center. This is not a stopping criterion for the test at sensor \(\ell\), but only a triggering condition for the communication between sensor \(\ell\) and the fusion center. Thus, sensor \(\ell\) continues to run the test until the fusion center sends a halting message.

The final decision \(\hat{H}\) is made at the fusion center in favor of hypothesis \(h_j\) when the local decisions from all the sensors are in favor of \(h_j\). After the final decision is made, the fusion center sends a halting message to all the sensors.

Apart from the initialization phase, the proposed DCT only requires the communication of an index \(i \in [M]\) during the test phase. Thus, the communication resources required are considerably less compared to the FCT, where continuous random variables are sent continuously over the network at each step. In addition, our results show that, while maintaining the same asymptotic optimality of Chernoff’s test as \(c \to 0\), the oscillations in the local decision vanish, and each sensor tends to use the communication channel on average only four times: two in the initialization phase, one to communicate the local decision, and one to receive the halting message.

VI. INFORMAL DISCUSSION ON DCT

The key idea behind the proposed DCT is to determine the individual capabilities of the sensors for detecting the hypotheses. These capabilities — that depend on the true hypothesis \(H^*\) — are captured by the vector \(v_i\), whose \(i\)th element is a measure of sensor capability to detect the hypothesis \(h_i\). The fusion center gathers this information, and utilizes it to control the threshold at each sensor through the response vector \(\rho_{i,\ell}\).

At the fusion center, \(I(i)\) is the measure of the cumulative detection capability of the network for hypothesis \(h_i\), and \(\rho_{i,\ell}\) denotes the fraction of this capability contributed by sensor \(\ell\) for hypothesis \(h_i\). To minimize the expected time to reach a decision, it is desirable to determine the threshold for each sensor \(\ell\) such that all the sensors require roughly the same time to reach the triggering condition. This is analogous to dividing the task of hypothesis testing among the sensors based on their speed of performing the task, such that all the sensors finish their share of the task at roughly the same time.

VII. THEORETICAL RESULTS ON DCT

In the following theorems, \(N\) indicates the time required to make a decision, and \(C\) indicates the communication overhead, namely the number of times a sensor communicates with the fusion center. The superscripts \(C\) and \(d\) refer to the DCT and to a generic decentralized sequential test, respectively.

Part (i) of Theorem 1 states that the probability of making a wrong decision can be made as small as desired by an appropriate choice of \(c\). Part (ii) provides a bound on the expected time to reach the final decision, and part (iii) bounds the risk as an immediate consequence of parts (i) and (ii).

**Theorem 1.** (Direct). The following statements hold:

(i) For all \(c \in (0,1)\) and for all \(i \in [M]\), given that hypothesis
Using the above result of $r$ then we have

$$P^r_i(\hat{H} \neq h_i) \leq \min\{(M - 1)c, 1\}. \tag{i}$$

For all $i \in [M]$, given that hypothesis $h_i$ is true, the expected decision time is

$$E^r_i[N] \leq (1 + o(1))\frac{\log c}{I(i)}, \quad c \to 0. \tag{8}$$

Combining (i) and (ii), the risk defined in \[7\] verifies

$$E^r_i[N^r] \leq (1 + o(1))\frac{\log c}{I(i)}, \quad c \to 0. \tag{9}$$

For all $i \in [M]$, $i, j, k \in [M]$ and $r \geq 2$, if

$$E[\log \frac{p_{i,k}^{u_i,i}(Y)/p_{j,k}^{u_i,i}(Y)}{r}] < \infty,$$

then

$$E^r_i[N^r] \leq \left(1 + o(1)\right)\frac{c\log c}{I(i)}, \quad c \to 0. \tag{10}$$

The following theorem provides a matching converse result.

**Theorem 2.** (Converse) For any sequential test $\delta$, if for all $i \in [M]$ the probability of missed detection satisfies

$$P^\delta_i(\hat{H} \neq h_i) = O(c\log(c)), \quad c \to 0, \tag{11}$$

then we have

$$E^\delta_i[N^r] \geq \left(1 + o(1)\right)\frac{\log c}{I(i)}, \quad c \to 0. \tag{12}$$

Using the above result of $r = 1$, we have

$$E^\delta_i[N] \geq \left(1 + o(1)\right)\frac{\log c}{I(i)}, \quad c \to 0. \tag{13}$$

The following result is a consequence of Theorems 1 and 2. It shows the asymptotic optimality of the DCT, and presents the expected communication overhead, as $c \to 0$.

**Theorem 3.** Suppose for all $\ell \in [L]$, $i, j, k \in [M]$ and $r \geq 2$, $E[\log \frac{p_{i,j}^{u_i,i}(Y)/p_{j,k}^{u_i,i}(Y)}{r}] < \infty$. For the DCT, for all $i \in [M]$, we have

$$E^r_i[N^r] = \left(1 + o(1)\right)\frac{\log c}{I(i)}, \tag{14}$$

$$E^r_i[N] = \left(1 + o(1)\right)\frac{\log c}{I(i)}, \quad c \to 0. \tag{15}$$

$$\lim_{c \to 0} E^r_i[C] = 4. \tag{16}$$

The proofs of Theorem 1, 2, and 3 are in Appendix A, B, and C.

VIII. CONSENSUS BASED CHERNOFF TEST

One major advantage of sensor networks is their robustness to node failures and external attacks. From this viewpoint, an alternative solution where all the information processing is completely distributed and there is no central unit, is certainly desirable. In this section, we extend our test for decentralized, DCT, to a fully distributed architecture.

We propose a version of the above Chernoff test designed for fully-flat sensor networks, which is referred to as the consensus-based Chernoff test (CCT). The proposed CCT consists of three phases: consensus among the sensors regarding their cumulative capability to detect hypothesis $i \in [M]$ (defined later in \[17\]), performing a Chernoff test locally at each sensor, and consensus regarding the true hypothesis among the sensors. The first two phases of CCT can be performed in parallel while the last phase begins after completing the first two.

In the first phase, the goal of each sensor is to acquire knowledge about the cumulative capability of the network to detect hypothesis $h_i$, $i \in [M]$. For sensor $\ell$, the measure of the capability to detect $h_i$ is given by $v_{i,\ell}$. Thus, the cumulative capability of the network to detect hypothesis $h_i$ is

$$I(\ell) = \sum_{\ell = 1}^{L} v_{i,\ell}, \quad i \in [M], \tag{17}$$

and $I = [I(1), \ldots, I(M)]$ is the corresponding vector. Since there is no central entity to facilitate the computation of the quantities in \[17\], the sensors use local information and consensus techniques to acquire this knowledge. Consensus techniques allow to compute the arithmetic mean of remotely-collected observations by exploiting only information locally available to the sensors, see e.g., \[25\], \[26\], \[27\], \[28\]. Assuming that the number of sensors $L$ is known to all the sensors, we use a linear consensus technique to estimate the arithmetic mean $I/L$ of the cumulative capability, which multiplied by $L$ provides the desired estimate of $I$. The distributed linear consensus technique is of the form

$$I^{(n+1)}_\ell = w_{\ell,\ell} \cdot \hat{I}^{(n)}_\ell + \sum_{j \in N_\ell} w_{\ell,j} \cdot I^{(n)}_j, \tag{18}$$

where $\hat{I}^{(n)} = [\hat{I}^{(n)}(1), \ldots, \hat{I}^{(n)}(M)]$ is the vector of estimated cumulative capabilities for $M$ hypotheses at sensor $\ell$ and instance $n$, $w_{\ell,j}$ is the weight assigned by the sensor $\ell$ to the estimate of sensor $j$, and $N_\ell = \{j|\ell, j \in \mathcal{E}\}$ is the set of immediate neighbors of sensor $\ell$ in $\mathcal{G}(\mathcal{L}, \mathcal{E})$. At $n = 0$, estimated cumulative capabilities are initialized as $\hat{I}^{(0)} = [v_{1,\ell}, \ldots, v_{M,\ell}]$, and the initialization $\hat{I}^{(0)}$ can be computed locally at the sensors. Since the sensor $\ell$ does not communicate to sensors $\{j|j \notin N_\ell \cup \{\ell\}\}$, it follows that $w_{\ell,j} = 0$. Thus, for the network graph $\mathcal{G}(\mathcal{L}, \mathcal{E})$, the linear consensus technique \[18\] can be written as

$$\hat{I}^{(n+1)} = W \cdot \hat{I}^{(n)}, \tag{19}$$
where $\hat{T} = [\hat{T}_1, \ldots, \hat{T}_L]^T$ is $L \times M$ matrix of the estimate of the cumulative capability of the network at time instance $n$ at all sensors, and $W$ is an $L \times L$ matrix where an element $w_{\ell,j}$ denotes the weight that the sensor $\ell$ assigns to the estimate of sensor $j$. The matrix $W$ is constrained to

$$T = \{W \in \mathbb{R}^{L \times L} | w_{\ell,j} > 0 \text{ if } j \in \mathcal{N}_\ell \cup \{ \ell \} \text{ else } w_{\ell,j} = 0\}. \quad (20)$$

The consensus iteration (19) will converge to the mean of the cumulative capability of the network $I/L$ if and only if

$$\lim_{n \to \infty} W^n = \frac{1}{L} \cdot 1^T, \quad (21)$$

where $1$ is a vector of all ones with dimension $L \times 1$. The necessary and sufficient conditions for (21) to hold are (31), and can be determined using standard techniques from the literature.

Another challenge in the Phase 1 of CCT is the stopping rule of the consensus algorithm. Assuming the number of sensors in the network are known to all the sensors, localized stopping rule is proposed in (39). Using this rule, the first phase of CCT is summarized in Algorithm 1. Since $z_\ell > L + 1$, sensor $\ell$ sends a termination bit $m_\ell^1 = 1$ to inform its neighbors $\mathcal{N}_\ell$ that the consensus to $I/M$ has been reached. When sensor $j$ receives $m_\ell^1 = 1$, it halts the consensus, scale the estimate $\hat{I}_\ell(n)$ by $L$ to get an estimate of $I$, and forwards the termination bit corresponding to Phase 1 to $m_\ell^1$ to its neighbors $\mathcal{N}_\ell$. Using threshold $c/L^2$ at each sensor (see Algorithm 1), the stopping rule ensures that the network has reached an uniformly local $c/L^2$-consensus, i.e. for all $\ell \in [L]$ and $j \in \mathcal{N}_\ell$,

$$|\hat{I}_\ell(n) - \hat{I}_j(n)| \leq \frac{c}{L^2}. \quad (25)$$

Therefore, for all $\ell, j \in [L],$

$$|\hat{I}_\ell(n) - \hat{I}_j(n)| \leq \frac{c}{L^2}. \quad (25)$$

because the diameter $d^G$ of the network is bounded above by $L$. After scaling by $L$, at termination of Phase 1, for all $\ell, j \in [L],$ we have

$$|\hat{I}_\ell(n) - \hat{I}_j(n)| \leq c. \quad (26)$$

In the second phase of CCT, all sensors perform a Chernoff test independently of each other, and compute

$$\log p_{\hat{I}_\ell,\hat{I}_\ell}(y^n, u^n)/\max_{j \neq \ell} p_{\hat{I}_\ell,\hat{I}_\ell}(y^n, u^n)$$

Following the termination of Phase 1, if at time $n$ and sensor $\ell$

$$\log \frac{p_{\hat{I}_{\ell,\ell}}(y^n, u^n)}{\max_{j \neq \ell} p_{\hat{I}_{\ell,\ell}}(y^n, u^n)} \geq \hat{\rho}_{\ell,n}^{(n)} |\log c|, \quad (27)$$

where $\hat{\rho}_{\ell,n}^{(n)} = v_{\ell,\ell}(\hat{I}_\ell(n))$, then the local decision $\hat{H}_\ell(n)$ is updated in favor of hypothesis $h_{\ell,n}$, otherwise, it is set to null. Note that the Chernoff test and the computation of log-likelihood can be performed in parallel with the first phase. However, the local decisions of the sensors are updated only after the termination of Phase 1. This ensures $\hat{\rho}_{\ell,n}^{(n)}$ in (27) is a reliable estimate of $v_{\ell,\ell}(\hat{I}_\ell(n))$ before the update of the local decision $\hat{H}_\ell(n)$.

The third phase follows after the update of the local decision at the sensors. Each sensor $\ell$ communicates its local decision $\hat{H}_\ell(n)$, if any, to its neighbors $\mathcal{N}_\ell$. Additionally, it also communicates $d_\ell^{(n)}$ defined as

$$d_\ell^{(n)} = \min\{ \min_{j \in \mathcal{N}_\ell \cup \{ \ell \}} d_{j}^{(n-1)}, x_\ell^{(n-1)} \} + 1, \quad (28)$$

where

$$x_\ell^{(n)} = \begin{cases} x_\ell^{(n-1)} + 1 & \text{if, for all } j \in \mathcal{N}_\ell, \hat{H}_\ell(n) = \hat{H}_j(n) \\ 1 & \text{if, for all } j \in \mathcal{N}_\ell, \hat{H}_\ell(n) = \hat{H}_j(n) \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

Say $x_\ell^{(n)}$ is some constant $k$, then the decision of the neighbors of $\ell$ is same as the local decision of sensor $\ell$ $\hat{H}_\ell(n)$ for past $k$ time instances. $d_\ell^{(n)}$ is responsible for the percolation of this information along the sensor network. Using (29), if a sensor $j$ does not report its local decision, then $x_\ell^{(n)} = 0$ in the neighborhood of $j$. A sensor $\ell$ stops the test at time instance $N$ if $d_\ell^{(N)} > L + 1$. This ensures that there exists

**Algorithm 1 Phase 1 of CCT**

**Initialization:** $n = 0$; For all $\ell \in [L], \hat{I}_\ell(0) = \{v_{\ell,1}, \ldots, v_{\ell,M}\}$, $B_n = B$; $m_k = M_k$, $y_\ell = 0$ and $z_\ell = 0$

**while True do**

For all $\ell \in [L]$, broadcast local information $\hat{I}_\ell(n)$ and $z_\ell$.

**Update the local cumulative capability using (18).**

$z_\ell = \min\{y_\ell, \min_{j \in \mathcal{N}_\ell \cup \{ \ell \}} z_j \} + 1$

if $z_\ell > L + 1$ then

Sensor $\ell$ broadcasts $m_\ell^1 = 1$ and stop updating.

**Break While;**

**end if**

if $\max_{j \in \mathcal{N}_\ell} |\hat{I}_\ell(n) - \hat{I}_j(n)| \leq c/L^2$ then

$y_\ell = y_\ell + 1$

else

$y_\ell = 0$

**end if**

$n = n + 1$

**end while**
a time instance $k \leq N$ at which the local decision of all the sensors are same i.e. $\min_{j \in [L]} x_{i,k}^{(n)} \geq 1$ (Lemma 2 in Appendix D). Additionally, if $d_{i,N}^{(n)} > L + 1$, then sensor $\ell$ informs its neighbors $N_\ell$ that the final decision is $\hat{H}_N^{(n)}$, and sends a termination bit $m_{i,\ell}^3 = 1$ to terminate the test. When a sensor $j$ receives the termination bit $m_{j,\ell}^3 = 1$ and the final decision, it halts the test and forwards the termination bit $m_{j,\ell}^3$ along with the final decision to its neighbors $N_j$. All the sensors will receive $m_{i,\ell}^3 = 1$ after at most $d_{\ell}^0$ time instances.

In the literature, distributed hypothesis testing is performed while communicating the posterior probabilities for all hypotheses which are real valued vectors over the network [32]–[36]. On the other hand, CCT only communicates the local decision of the sensors. CCT requires communication of $I^{(n)}_i$, a real valued vector, in its first phase to find the cumulative capabilities of the network. However, since the termination time of Phase 1 is bounded (see (72), these communications are finite and bounded. Thus, CCT is parsimonious in terms of communication in comparison to the other schemes proposed in the literature [32]–[36].

IX. INFORMAL DISCUSSION ON CCT

The key idea behind the proposed CCT is to determine the individual capabilities of the sensors for detecting the hypotheses. These capabilities — that depend on the true hypothesis $H^*$ — are captured by $w_{i,j}$, which is a measure of sensor’s capability to detect the hypothesis $h_i$. All the sensors in the network determine their cumulative capabilities to detect any hypothesis. Since there is no central entity to facilitate the communication of this information, therefore, they use consensus algorithm, first phase of CCT, to acquire this information. If the consensus algorithm stops at time instance $N$, then $I(N)$ is the estimate of the cumulative capability of the sensor network at the sensors. Thus, $r^{(n)}_{i,\ell}$ denotes the estimated fraction of the capability contributed by sensor $\ell$ for hypothesis $h_i$. To minimize the expected time to reach a decision, it is desirable to determine the threshold for each sensor $\ell$ such that all the sensors require roughly the same time, following the termination of Phase 1, to reach the triggering condition (27). Given the estimate of cumulative capabilities $I^{(n)}$, this is analogous to dividing the task of hypothesis testing among the sensors based on their speed of performing the task, such that all the sensors finish their share of the task at roughly the same time. Phase 3 is a localized stopping criterion of the Chernoff test, and ensures the sensors stop the test as they reach the same decisions. $x_{i,k}^{(n)}$ and $d_{i,k}^{(n)}$ captures this information mathematically, and percolate it over the network.

X. THEORETICAL RESULTS ON CCT

In the following theorems, $N$ indicates the time required to make a decision. Let us define the ergodic coefficient of the matrix $W$ as

$$\eta(W) = \min_{i \neq j} \sum_{k=1}^L \min\{w_{i,k}, w_{j,k}\}.$$ 

The superscripts $CC$ and $\delta$ refer to the CCT and to a generic decentralized sequential test, respectively.

Part (i) of Theorem 5 states that the probability of making a wrong decision can be made as small as desired by an appropriate choice of $c$. Part (ii) provides a bound on the expected time to reach the final decision, and part (iii) bounds the risk as an immediate consequence of parts (i) and (ii). (iv) presents the bound on the higher moments of the decision time $N$ of CCT. Firstly, we present a lemma which is used in the proof of Theorem 5.

Lemma 4. As the network is strongly connected, $0 < \eta(W h^\delta) < 1$ where $h^\delta$ is the shortest height of all possible spanning trees of $\mathcal{G}(L, E)$ [40, Proposition 1].

Theorem 5. (Direct). The following statements hold:\n
(i) For all $c \in (0, 1)$ and for all $i \in [M]$, given that hypothesis $h_i$ is true, the probability that the CCT makes an incorrect decision is bounded as

$$\mathbb{P}^{CC}_i(\hat{H} \neq h_i) \leq \min\{(M-1)c, 1\}.$$ 

(ii) For all $i \in [M]$, given that hypothesis $h_i$ is true, the expected decision time is

$$\mathbb{E}^{CC}_i[N] \leq (1 + o(1)) \max \left\{ \frac{h^\delta \cdot \log(c/ \max_{j \in [L]} I(j))}{\log(1 - \eta(W h^\delta))}, \frac{|\log c|}{I(i) - c} \right\},$$

as $c \to 0$.

(iii) Combining (i) and (ii), the risk defined in (4) verifies

$$\mathbb{E}^{CC}_i \leq (1 + o(1)) \max \left\{ \frac{h^\delta \cdot |\log(1/ \max_{j \in [L]} I(j))|}{|\log(1 - \eta(W h^\delta))|}, \frac{1}{I(i) - c} \right\} \cdot c|\log c|,$$

as $c \to 0$.

(iv) For all $\ell \in [L], i, j, k_1 \in [M]$ and $r \geq 2$, if

$$\mathbb{E}[\log p_{i,j,k_1}(Y)/p_{i,k_1}^{\ell}(Y)]^r < \infty,$$

then

$$\mathbb{E}^{CC}_i[N^r] \leq \left(1 + o(1)\right) \max \left\{ \frac{h^\delta \cdot \log(c/ \max_{j \in [L]} I(j))}{\log(1 - \eta(W h^\delta))}, \frac{|\log c|}{I(i) - c} \right\}^r,$$

as $c \to 0$.

The following result is a consequence of Theorems 5 and 2. It provides a sufficient condition for the asymptotic optimality of the CCT w.r.t the centralized setup of sensor network, as $c \to 0$. 

The following statements hold:
**Theorem 6.** For the CCT, if max operation in (30) results in $|\log c|/I(i) < c$, then for all $i \in [M]$ we have

\[
\lim_{c \to 0} \frac{E_{\delta^*}[N]}{E_{\delta}[N]} = 1, \quad (33)
\]

\[
\lim_{c \to 0} \frac{E_{\delta^*}[N]}{E_{\delta}[N]} = 1. \quad (34)
\]

where $\delta^*$ is the optimal sequential test in the sensor network.

**Proof.** Combining Theorem 5 and 2, (34) and (33) follow immediately.

Corollary 6.1 provides sufficient conditions for which max operation in (30) results in $|\log c|/I(i) < c$.

**Corollary 6.1.** As $c \to 0$, following are the sufficient conditions for CCT to be asymptotically optimal

(i) For all $i \in [M]$, we have

\[
I(i) \log \left( \max_{j \in [M]} I(j) \right) < \left| \log \left( 1 - \eta(W^{h \delta^*}) \right) \right|. \quad (35)
\]

(ii) For all $i \in [M]$, we have

\[
I(i) \log \left( \max_{j \in [M]} I(j) \right) < \left| \log \left( 1 - \eta(W^{h \delta}) \right) \right|. \quad (36)
\]

(iii) For all $i \in [M]$, we have

\[
I(i) \log \left( \max_{j \in [M]} I(j) \right) < \left| \log \left( 1 - \eta(W) \right) \right|. \quad (37)
\]

Here we briefly discuss the physical significance of the sufficient conditions presented in Corollary 6.1. The consensus in Phase 1 of CCT should be reached before the stopping criterion (27) for Chernoff Test is satisfied. Thus, the Phase 1 should not become a bottleneck in the termination of Phase 2. In other words, consensus along the network should be faster than the time required by Chernoff test to accumulate sufficient information to make a decision.

**XI. Simulation Results**

We evaluate the performance of our proposed tests experimentally as well. The performance of these tests is evaluated for different sizes of sensor networks. For CCT, given the number of sensors $L$, $[L/2]$ sensors are connected to form a ring topology, and the remaining sensors are randomly connected to the sensors in the ring. An example of the sensor network for the performance evaluation of CCT can be seen in Figure 1. The number of hypothesis are three i.e. $M = 3$. The probability distribution $p_{i,k}$ is a Bernoulli distribution with parameter $p_i$, which is selected uniformly at random from $(0, 1/3), (1/3, 2/3)$ and $(2/3, 1)$ for $i = 1, 2$ and 3 respectively.

Figure 1 and 2 show the performance of CCT. The expected decision time increases as the observation cost per unit time $c$ reduces. This is because the threshold in the triggering condition (7) increases which ensures that the sensors have a greater confidence about their local decision. On the other hand, the expected decision time decreases as the number of sensors $L$ increases. The cumulative capability of the sensor network increases as the number of sensors increases. Also, for a given $c$, the task of hypothesis testing is divided among more sensors, therefore, the final decision can be reached quickly. Our observations are in accordance with the our theoretical results obtained for DCT.

Figure 3 shows that the expected decision time increases with the increase in sensors. However, the increase in the expected decision time with the number of sensors reduces eventually. There is a trade-off between two factors: consensus among the sensors and the cumulative capabilities of the sensors. As the number of sensors increases, the consensus scheme will require more time. On the other hand, the increase in number of sensors increases the cumulative capabilities of the sensors, thus, reducing the time required to reach
the local decision [27] Figure 3 shows that the consensus between the sensors in the first phase of CCT becomes the dominating factor in the decision time. This is in accordance with the theoretical bounds provided in Theorem 5. As the number of sensors $L$ increases, the cumulative capability $I(i)$ increases. The term corresponding to the time of first phase $N_e$ becomes dominant, and increases with $L$. Therefore, the expected decision time increases.

Another key observation from Figure 3 is that the expected decision time reduces with the increase in observation cost per unit time $c$. As $c$ increases, the stopping criterion of Phase 1 and the decision criterion in Phase 2 [27] are relaxed, and can be reached earlier. The observation is in accordance with the theoretical bounds provided in Theorem 5.

XII. CONCLUSION

For decentralized setup, we proposed DCT which is parsimonious in terms of communications, and is asymptotically optimal in terms of detection performance, when the observation cost per unit time vanishes.

One major advantage of sensor networks is their robustness to node failures and external attacks. From this viewpoint, we use the key ideas from DCT to provide an alternative solution CCT where all the information processing is completely distributed and there is no central unit. In CCT, the key quantity $I(i)$, computed in the initialization phase of CCT by the fusion center, can be obtained by gossip protocols using consensus techniques [25]–[27], [41]. Likewise, once all the sensors reach the triggering condition [27], equivalent to (7) in DCT, the final decision can also be easily computed in a distributed way.

There has been enormous work in the direction of distributed hypothesis testing, however, the schemes mainly focused on the communication of real-valued belief vectors, which is analogous to posterior probability of the hypotheses, over the sensor network. On contrary, CCT is parsimonious in terms of communication as the communication in the detection phases (Phase 2 and Phase 3 of CCT) is limited to local decisions and [28] which can be encoded in $2 \log_2(M) + 1$ bits. Although Phase 1 of CCT requires communication of real valued vectors, however, the time for this communication is bounded by a finite number [72]. Thus, CCT is efficient in terms of communication as compared to the other tests proposed in the literature for distributed hypothesis testing.

Additionally, in literature, the analysis of distributed hypothesis testing schemes is limited to the asymptotic learning rate as time $n \to \infty$. In our work, we study the probability of missed detection, expected decision time, and the higher moments of decision time for CCT. We also provide converse showing the lower bound on these performance measures for any sequential test in sensor network. Later, we provide sufficient conditions for which CCT is asymptotic optimal in terms of these performance measure, as the observation cost per unit time becomes vanishingly small.

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To illustrate the proof of Theorem 1, we need the following additional notation. We let $y_{k,\ell}^i = \{y_{1,\ell}, y_{2,\ell}, \ldots, y_{K,\ell}\}$, where $y_{i,\ell}$ is the $i^{th}$ observation sample at sensor $\ell$; $u_{k,\ell}^i = \{u_{1,\ell}, u_{2,\ell}, \ldots, u_{K,\ell}\}$, where $u_{i,\ell}$ is the $i^{th}$ action at sensor $\ell$. We also let $A_{n,\ell}$ be the set of sample paths where the decision made by the fusion center in favor of $h_1$ at the $n^{th}$ step, and we indicate a single sample path as $\{(u_{1,\ell}^1, y_{1,\ell}^1) \ldots (u_{L,\ell}^n, y_{L,\ell}^n)\}$. We indicate by $A_{n,\ell}$ the set of sample paths in $A_{n,\ell}$ corresponding to the $\ell^{th}$ sensor. Finally, we define

$$N_{i,\ell} = \inf \left\{ n : \sum_{k=1}^{n} \log \frac{P_i u_{k,\ell}^i(y_{k,\ell})}{\max_{j \neq i} P_j u_{k,\ell}^i(y_{k,\ell})} \geq \rho_{i,\ell} |\log(\epsilon)| \right\}.$$ 

The proof consists of two parts. First, we write $\mathbb{P}^\varphi_{\epsilon}(\hat{H} \neq h_i)$ as the probability of a countable union of disjoint sets of sample paths. An upper bound on this probability then follows from an upper bound on the probability of these disjoint sets, in conjunction with the union bound. Second, we upper bound $\mathbb{E}^\varphi_{\epsilon}[N]$ by the sum of the expected time required to reach the triggering condition $[\ell]$ for hypothesis $h_{\ell}$, and the expected delay between the time of triggering and the time when the final decision is taken in favor of hypothesis $h_1$ at the fusion center. We then show that these expectations are the same at all sensors, so that $[\ell]$ follows.
Consider the probability $\mathbb{P}_i^c(\hat{H} = h_j)$. This is same as the probability of the countable union of disjoint sets $A_{n,j}$. Thus, for $j \neq i$, we can write

$$
\mathbb{P}_i^c(A_{n,j}) = \int_{A_{n,j}} \prod_{\ell=1}^L \prod_{k=1}^n p_{i,\ell}^{uk_{\ell}}(y_{k,\ell}) \, dy_{1,\ell}(u_{1,\ell}) \ldots dy_{n,\ell}(u_{n,\ell})
$$

which is independent of $\ell$. Additionally, from \cite{42} eq. (19) we have, as $c \to 0$

$$
\text{Var}(N_{i,\ell}) = O(\|\log c\|).
$$

Hence, as $c \to 0$, and for all $\ell \in [L]$, we have

$$
\left( \mathbb{E}_i \left| N_{i,\ell} - (1 + o(1)) \frac{|\log c|}{I(i)} \right| \right)^2 \leq \mathbb{E}_i \left( N_{i,\ell} - (1 + o(1)) \frac{|\log c|}{I(i)} \right)^2 = \text{Var}(N_{i,\ell}) = O(\|\log c\|). \tag{42}
$$

The above yields,

$$
\mathbb{E}_i \left[ \max_{1 \leq \ell \leq L} N_{i,\ell} \right] = \frac{|\log c|}{I(i)} (1 + o(1)) + \mathbb{E}_i \left[ \max_{1 \leq \ell \leq L} N_{i,\ell} - (1 + o(1)) \frac{|\log c|}{I(i)} \right] \leq \frac{|\log c|}{I(i)} (1 + o(1)) + \sum_{\ell=1}^L \mathbb{E}_i \left| N_{i,\ell} - (1 + o(1)) \frac{|\log c|}{I(i)} \right| = \frac{|\log c|}{I(i)} (1 + o(1)) + O(\sqrt{|\log c|}), \tag{43}
$$

where the inequality follows by $\max_{\ell} N_{i,\ell} \leq \sum_{\ell} |N_{i,\ell}|$, and the last equality follows by \cite{42}.

Let $N^*$ be the time instance such that for all $n \geq N^*$, $\hat{H}_t = H^*$. Using \cite{11} Lemma 1, there exists a $K > 0$ and $b > 0$ such that

$$
\mathbb{P}_n(N^* \geq n) \leq k \exp(-bn). \tag{44}
$$

Since $\mathbb{P}_i(N^* < \infty) = 1$ (see \cite{44}) and for all $\ell \in [L]$ and $i,j,k_1,k_2 \in [M]$ $0 \leq D(p_{i,\ell}^{uk_{\ell}} || p_{j,\ell}^{uk_{\ell}})$, $\mathbb{E}[\log(p_{i,\ell}^{uk_{\ell}}(y_{k,\ell})/\max_{j \neq i} p_{j,\ell}^{uk_{\ell}}(y_{k,\ell}))]$ is strictly positive and finite. Then, a sensor $\ell$ following time $N^*$ selects the actions in an i.i.d fashion according to the probability mass function given by \cite{3}. Thus, the log-likelihood ratio following time $N^*$ $\sum_{k=1}^{N_{i,\ell}} \log(p_{i,\ell}^{uk_{\ell}}(y_{k,\ell})/\max_{j \neq i} p_{j,\ell}^{uk_{\ell}}(y_{k,\ell}))$ is the summation of i.i.d random variables with mean $v_{\ell}$. From our assumption there exists an action $u_k$, such that $D(p_{i,\ell}^{uk_{\ell}} || p_{j,\ell}^{uk_{\ell}}) > 0$, thus for all $i \in [M]$ $v_{\ell} > 0$.

Now,

$$
\mathbb{P}_i(N^* < \infty) = 1 \text{ (see } \cite{44} \text{),}
$$

$$
\mathbb{E}[\log(p_{i,\ell}^{uk_{\ell}}(y_{k,\ell})/\max_{j \neq i} p_{j,\ell}^{uk_{\ell}}(y_{k,\ell}))] = v_{\ell} > 0.
$$

Using the above two equations, $r^{th}$ moment of log-likelihood ratio is finite, Corollary \cite{5} and Lemma \cite{9} we have

$$
\mathbb{E}_i[\tau(N_{i,\ell})]^r < \infty. \tag{45}
$$

Since $\mathbb{E}[\log(p_{i,\ell}^{uk_{\ell}}(y_{k,\ell})/\max_{j \neq i} p_{j,\ell}^{uk_{\ell}}(y_{k,\ell}))]^2$ is finite, using \cite{45}, the term $\mathbb{E}[\tau(N_{i,\ell})]$ on the right-hand side of \cite{40} is finite and independent of $c$. Now, combining equation \cite{40}, \cite{43} and the finiteness of $\mathbb{E}_i[\tau(N_{i,\ell})]$, as $c \to 0$ we get \cite{48}. Thus, proving the (ii) of the theorem.
Using [11] Lemma 2, for all $0 < \epsilon < 1$ and $n \geq -(1 + \epsilon) \log(c)/I(i) - c$, there exists a $K'(\epsilon)$ and $b'(\epsilon)$ such that

$$\mathbb{P}_i(N_{i,\ell} > n) \leq K'(\epsilon) \cdot e^{-b'(\epsilon)n}.$$  

(46)

Therefore, using (46), we have

$$\mathbb{E}_i[(N_{i,\ell} - N^*)^r] \leq \left(1 + o(1)\right) \frac{\log(c)}{I(i)}.$$  

(47)

Now, using the same direction as in [42] eq. (19), as $c \to 0$, we have

$$\text{Var}\left(N^r_{i,\ell} - \left(1 + o(1)\right) \frac{\log(c)}{I(i)}\right)^r = O((\log c)^{2r-1}).$$  

(48)

The above yields,

$$\mathbb{E}_i \left[\max_{1 \leq \ell \leq L} N^r_{i,\ell}\right] = \left(1 + o(1)\right) \frac{\log(c)}{I(i)} + O((\log c)^{r-1/2})$$  

(49)

where the inequality follows by $\max_{1 \leq \ell \leq L} N_{i,\ell} \leq \sum_{\ell} N_{i,\ell}$, and the last equality follows by (48) and Jensen’s inequality. Now,

$$\mathbb{E}_i[N^r] \leq \mathbb{E}_i \left[\max_{1 \leq \ell \leq L} N_{i,\ell} + \sum_{\ell \in [L]} \tau(N_{i,\ell}) + 1\right]^r.$$  

(50)

The moment of $\sum_{\ell \in [L]} \tau(N_{i,\ell})$ is finite and independent of c. The dominant terms, depend on c, in the expansion of right hand side of (50) will be contributed by $\max_{1 \leq \ell \leq L} N_{i,\ell}$. Therefore, using (48) and (49), as $c \to 0$, we have

$$\mathbb{E}_i^{CC}[N^r] \leq \left(1 + o(1)\right) \frac{\log(c)}{I(i)}^r.$$  

(51)

### APPENDIX B

#### PROOF OF THEOREM 2

The proof of Theorem 2 consists of two parts. First, for all $m \neq i$ and $0 < \epsilon < 1$, we show that for the probability of error $\mathbb{P}_i(H \neq h_i) = O(-c \log(c))$ to be close to zero, the log-likelihood ratio should be in favor of $h_i$, and should be greater than $-(1 - \epsilon) \log(c)$ with high probability as $c \to 0$, i.e.

$$S^N(h_i, h_m) = \sum_{\ell=1}^{L} \sum_{k=1}^{N} \log \frac{p_{i,\ell}^{u_k}}{p_{m,\ell}^{u_k}}(y_k, \ell) \geq -(1 - \epsilon) \log(c),$$  

(52)

with high probability, as $c \to 0$. Second, we show that for all $0 < \epsilon < 1$ and $n < -(1 - \epsilon) \log(c)/I(i)$, it is unlikely that for all $m \neq i$, $S^N(h_i, h_m) \geq -(1 - \epsilon) \log(c)$.

We start by defining two hypothesis $h'_0 = \{h_k\}$ and $h'_1 = \{h_k\}_{k \neq i}$. By [11], both type I and type II errors of the hypothesis test $h'_0$ vs. $h'_1$ are $O(-c \log(c))$. Thus, by [11] Lemma 4, for all $m \neq i$ and $0 < \epsilon < 1$, we have

$$\mathbb{P}_i \left(S^N(h_i, h_m) \leq -(1 - \epsilon) \log(c)\right) = O(-c \log(c)).$$  

(53)

This concludes the first part of the proof.

Now, we show that for all $\epsilon > 0$, we have

$$\lim_{n \to \infty} \mathbb{P}(\max_{1 \leq n' \leq n'} S^N(h_i, h_m) \geq n'(I(i) + \epsilon)) = 0.$$  

(54)

We rewrite,

$$S^N(h_i, h_m) = \sum_{\ell=1}^{L} \sum_{k=1}^{n} \log \frac{p_{i,\ell}^{u_k}}{p_{m,\ell}^{u_k}}(y_k, \ell)$$  

(55)

$$- D(p_{i,\ell}^{u_k} || p_{m,\ell}^{u_k}).$$  

$$= M^N_1 + M^N_2,$$

where,

$$M^N_1 = \sum_{\ell=1}^{L} \sum_{k=1}^{n} \left(\log \frac{p_{i,\ell}^{u_k}}{p_{m,\ell}^{u_k}}(y_k, \ell) - D(p_{i,\ell}^{u_k} || p_{m,\ell}^{u_k})\right),$$

(56)

is a martingale with mean 0, and

$$M^N_2 = \sum_{\ell=1}^{L} \sum_{k=1}^{n} D(p_{i,\ell}^{u_k} || p_{m,\ell}^{u_k}).$$

Then, for all $1 \leq n \leq n'$

$$\min_{m \neq i} M^N_2 \geq \sum_{\ell=1}^{L} \sum_{k=1}^{n} D(p_{i,\ell}^{u_k} || p_{m,\ell}^{u_k})$$

(57)

where $(a)$ follows from the definition of $v_{t,i}$ in (5), $(b)$ follows from the definition of $I(i)$, and $(c)$ follows from $n \leq n'$.

If the event in (54) occurs for a fixed $n_1$, i.e.

$$\min_{m \neq i} (M^N_1 + M^N_2) \geq n'(I(i) + \epsilon),$$

then...
then there exists a hypothesis \( h_i \) such that \( M_i^{n_i} \geq n' \epsilon \). Thus, there exists a constant \( K' > 0 \) such that the probability in (54) becomes
\[
\mathbb{P} \left( \max_{1 \leq n \leq n'} \min_{m \neq i} S_n(h_i, h_m) \geq n'(I(i) + \epsilon) \right) \\
\leq \sum_{n \neq i} \mathbb{P} \left( \max_{1 \leq n \leq n'} M_i^n \geq n' \epsilon \right) \\
(a) \frac{(M-1)K'}{n' \epsilon^2},
\]
where (a) follows from the fact \( M_i^n \) is a martingale with mean zero and using the Doob-Kolmogorov extension of Chebyshev’s inequality. Thus, (54) follows. As discussed in [11, Theorem 2], if \( n_0 = -(1 - \epsilon) \log(c)/I(i) \), then
\[
\mathbb{P}(N \leq n_0) \\
\leq \mathbb{P} \left( \text{For all } m \neq i, N \leq n_0 \text{ and } S_N(h_i, h_m) \geq n_0(I(i) + \epsilon) \right) \\
+ \mathbb{P} \left( \text{there exists } m \neq i \text{ s.t. } S_N(h_i, h_m) \leq n_0(I(i) + \epsilon) \right) \\
\leq \mathbb{P} \left( \max_{1 \leq n \leq n_0} S_n(h_i, h_m) \geq n_0(I(i) + \epsilon) \right) \\
+ \mathbb{P} \left( \text{there exists } m \neq i \text{ s.t. } S_N(h_i, h_m) \leq n_0(I(i) + \epsilon) \right)
\]
(59)
It now follows by (54) and (53) that he first and second terms in the right hand side of (59) approaches zero. Now, using (59), we have
\[
\mathbb{P}(N^r \leq n_0) = \mathbb{P}(N \leq n_0) \rightarrow 0
\]
(60)
Thus, the proof follows.

APPENDIX C
PROOF OF THEOREM 3

Combining Theorem 1 and 2 (14) and (15) follow immediately. We then turn to the proof of (16).

For all \( \ell \in [L] \), given that hypothesis \( h_i \) is true, we have that as \( c \rightarrow 0 \) the probability of incorrect detection tends to zero. It follows that \( \hat{H} = h_i \) and
\[
\mathbb{E}^c[N] = (1 + o(1)) \frac{\log c}{I(i)} \\
= \mathbb{E}^c[N_{i,\ell}],
\]
where the last equality follows from (11). Thus, as \( c \rightarrow 0 \) all the sensors reach the same local decision on average at the same time, and the average number of messages that each sensor sends to the fusion center to communicate this local decision is one. It follows that as \( c \rightarrow 0 \), the total expected communication overhead is four: two in the initialization phase, one to communicate the local decision, and one to receive the halting message.
follows from an upper bound on the probability of these disjoint sets, in conjunction with the union bound. Second, \( P^C_i[N] \) is dependent on the time required to reach and detect the consensus during the first phase, time required to reach the triggering condition (27) for a hypothesis in second phase, and time required to reach and detect that the sensors have reached a common decision about a hypothesis in the third phase. Since Phase 1 and Phase 2 run in parallel followed by phase three, thus stopping time \( N \) can be bounded as below

\[
N \leq \max\{N^c, \max_{1 \leq i \leq L} (N_{i,t} + \tau(N_{i,t}))\} + N^s. \tag{63}
\]

Consider the probability \( P^C_i\{\hat{H} = h_j\} \). This is same as the probability of the countable union of disjoint sets \( B_{n,j} \). Thus, for \( j \neq i \), we can write

\[
P^C_i\{B_{n,j}\} = \int_{B_{n,j}} \prod_{\ell=1}^L \prod_{k=1}^n p_{n,\ell}^{u_k}(y_{k,\ell}) dy_{1,\ell}(y_{1,\ell}) \ldots dy_{n,\ell}(y_{n,\ell})
\]

\[
= \prod_{\ell=1}^L \int_{B_{n,j,\ell}} \prod_{k=1}^n p_{n,\ell}^{u_k}(y_{k,\ell}) dy_{1,\ell}(y_{1,\ell}) \ldots dy_{n,\ell}(y_{n,\ell})
\]

\[
= e^{I(i)/(I(i)-c)} \prod_{\ell=1}^L \prod_{k=1}^n p_{n,\ell}^{u_k}(y_{k,\ell}) dy_{1,\ell}(y_{1,\ell}) \ldots dy_{n,\ell}(y_{n,\ell})
\]

where (a) follows from the definition of \( B_{n,j,\ell}; \) (b) follows from the definition of \( N_{i,t}; \) (c) follows from \( \sum_{\ell=1}^L \rho_{i,\ell} \leq I(i)/I(i) - c\). Now, we can bound \( P^C_i\{\hat{H} \neq h_i\} \) as follows

\[
P^C_i\{\hat{H} \neq h_i\} = \sum_{j \neq i} p_{n,1}^{u_k}(y_{k,1}) dy_{1,1}(y_{1,1}) \ldots dy_{n,1}(y_{n,1})
\]

\[
= \sum_{j \neq i} e^{I(i)/(I(i)-c)} \prod_{\ell=1}^L \prod_{k=1}^n p_{n,\ell}^{u_k}(y_{k,\ell}) dy_{1,\ell}(y_{1,\ell}) \ldots dy_{n,\ell}(y_{n,\ell})
\]

Now, if \( k_0 \) be the time to reach uniformly local \( c/L^2 \)-consensus, then following the scaling by \( L \), for all \( \ell, j \in [L] \), \( e^{(k_0)}_{\ell,j} \leq c \) (using (60)). Thus, there exists a \( k' \in \mathbb{N} \) such that \( h^{k'} \cdot k' \leq k_0 \leq h^{k'}(k' + 1) \). Using (66), for all \( \ell, j \in [L] \), we have

\[
e^{(k_0)}_{\ell,j} \leq e^{(k_0)}_{\ell,j} \leq \left(1 - \eta(W^s)\right)^{k'} e^{(k)}_{\ell,j} \leq \left(1 - \eta(W^s)\right)^{k'} \frac{1}{I}
\]

where (a) follows from \( \hat{I}(h^{k'} \cdot k') = W^s \), and using Lemma 4 we have \( 0 < \eta(W^s) < 1 \), and (b) follows from, for all \( \ell, j \in [L] \), \( e^{(0)}_{\ell,j} \leq I \).

For all \( \ell, j \in [L] \), \( e^{(k_0)}_{\ell,j} \leq c \). Therefore, using (67), we have

\[
\left(1 - \eta(W^s)\right)^{k'} \leq \frac{\log(c/\max_{j \in [L]} I(j))}{\log(1 - \eta(W^s))} \leq c,
\]

(68)

\[
k' \leq \frac{1}{\log(c/\max_{j \in [L]} I(j))} - 1
\]

(69)

Since \( k_0 \leq h^s(k' + 1) \), therefore,

\[
k_0 \leq h^s \left(\frac{\log(c/\max_{j \in [L]} I(j))}{\log(1 - \eta(W^s))} + 1\right)
\]

(70)

Now, let \( k_d \) be the time to detect the consensus. From \( [39] \), we have

\[
k_d \leq h^s \left(\frac{-\log(d^s)}{\log(1 - \eta(W^s))} + 1\right) + L + 1
\]

(71)

Thus,

\[
N^c \leq k_0 + k_d
\]

\[
\leq h^s \left(\frac{\log(c/\max_{j \in [L]} I(j))}{\log(1 - \eta(W^s))} + 1\right) + h^s \left(\frac{-\log(d^s)}{\log(1 - \eta(W^s))} + 1\right) + L + 1
\]

(72)

The expected time of second phase is bounded above by

\[
E_t [\max_{1 \leq t \leq L} (N_{t,\ell} + \tau(N_{t,\ell}))]
\]

Using the result of Theorem 1 as \( c \to 0 \), we have

\[
E_t \left[\max_{1 \leq t \leq L} (N_{t,\ell} + \tau(N_{t,\ell}))\right] \leq \frac{\log(c)}{I(i) - c} (1 + o(1))
\]

(73)

Now we compute the time for the third phase. The network will reach the consensus about a hypothesis for all \( n > \max_{1 \leq i \leq L} \tau(N_{i,\ell} + k_r) \), where \( k_r \) is the time taken by termination message \( m_i^k \) to reach every sensor after its initiation at any sensor. Thus, the time of phase three \( N^s \) is bounded above as

\[
N^s \leq \max_{1 \leq i \leq L} \tau(N_{i,\ell} + k_r)
\]

Therefore,

\[
E[N^s] \leq \sum_{t=1}^L E[\tau(N_{t,\ell})] + E[k_r]
\]

(74)
Since \( \mathbb{E}[\log(p_{i,j}^{u,j}(y_{k,l})/\max_{j\neq i}p_{j,k}^{u,j}(y_{k,l}))]^2 \) is finite and 
\( \mathbb{E}[\log(p_{i,j}^{u,j}(y_{k,l})/\max_{j\neq i}p_{j,k}^{u,j}(y_{k,l}))] \) is strictly positive, using (45), the term \( \mathbb{E}[\tau(N_{i,l})] \) on the right-hand side of (74) is 
finite and independent of \( c \). Additionally, \( k_r < d^g + 1 \). Thus, \( \mathbb{E}[N^*] \) is finite and independent of \( c \).

Combining equation (72), (73) and the finiteness of \( \mathbb{E}[N^*] \), we get (30) as \( c \to 0 \). Thus, proving (ii) of the theorem.

Now we derive the bounds for higher moments of \( N \). We have
\[
N \leq \max\{N^c, \max_{1 \leq \ell \leq L} (N_{i,\ell} + \tau(N_{i,\ell}))\} + N^s,
\]
\[
\leq \max\{N^c, \max_{1 \leq \ell \leq L} (N_{i,\ell})\} + \max_{1 \leq \ell \leq L} \tau(N_{i,\ell}) + k_r,
\]
\[
\leq \max\{N^c, \max_{1 \leq \ell \leq L} (N_{i,\ell})\} + 2 \sum_{\ell \in [L]} \tau(N_{i,\ell}) + k_r.
\]
(75)

Now, we present the bound on the \( r \)-th moment of each term in
the right hand side of (75).

Using (72), \( N^c \) is bounded above by a constant. As \( c \to 0 \), we have
\[
(N^s)^r \leq \left(1 + o(1)\right)^r \frac{h^g \log(c) \max_{j \in [L]} I(j)}{\log(1 - \eta(W^{h^g}))}.
\]
(76)

Now, using (49), we have
\[
E_i \left[ \max_{1 \leq \ell \leq L} N^s_{i,\ell} \right] = \left(1 + o(1)\right)^r \frac{\log(c)}{I(i) - c} + O(\log(c)^{-r/2}).
\]
(77)

Using (45), the higher moments of the second term in the right hand side of (75) are finite and independent of \( c \) by
definition of \( \tau(N_{i,\ell}) \). Additionally, \( k_r \leq L + 1 < \infty \). Now,
\[
E_i [N^r] \leq E_i \left[ \max\{N^c, \max_{1 \leq \ell \leq L} (N_{i,\ell})\} + 2 \sum_{\ell \in [L]} \tau(N_{i,\ell}) + k_r \right]^r.
\]
(78)

The moments of \( \sum_{\ell \in [L]} \tau(N_{i,\ell}) + k_r \) are finite and independent of \( c \). The dominant terms, dependent on \( c \), in
the expansion of right hand side of (78) will be contributed by \( \max\{N^c, \max_{1 \leq \ell \leq L} (N_{i,\ell})\} \). Therefore, as \( c \to 0 \),
\[
E_i^C \left[ N^r \right] \leq \left(1 + o(1)\right)^r \frac{h^g \log(c) \max_{j \in [L]} I(j)}{\log(1 - \eta(W^{h^g}))} \frac{|\log(c)|}{I(i) - c}.
\]
(79)

APPENDIX G

PROOF OF MISCELLANEOUS RESULTS

In this section, we present results related to time \( \tilde{N}_{i,l} \), where
\[
\tilde{N}_{i,l} = \sup \left\{ n : \sum_{k=1}^{n} \log \frac{p_{i,k}^{u,i}(y_{k,l})}{\max_{j \neq i} p_{j,k}^{u,j}(y_{k,l})} > 0 \right\},
\]
(80)
and given \( H^* = h_i, y_{k,l} \) are independent and identically
distributed random variables. It is the last time at which
\[
S^n_\ell(h_{i,\ell}) = \sum_{k=1}^{n} \log \frac{p_{i,k}^{u,i}(y_{k,l})}{\max_{j \neq i} p_{j,k}^{u,j}(y_{k,l})} > 0.
\]
(81)

These results are used in deriving bounds on \( \tau(N_{i,l}) \) in
Theorem 1 and 5 (see (45)).

Lemma 8. For all \( \ell \in [L] \) and \( i, j, k_1 \in [M] \), if
\[
\mathbb{E}[\log(p_{i,k_1}^{u,i}(Y)/p_{j,k_1}^{u,j}(Y))]^r < \infty \text{ and } 0 < \epsilon_0 < \mathbb{E}[\log(p_{i,k}^{u,i}(Y)/p_{j,k}^{u,j}(Y))],
\]
then
\[
E_i[\tilde{N}_{i,l}]^r \leq \left(\frac{2}{\epsilon_0}\right)^r \sum_{k=1}^{\infty} k^{r-1} \mathbb{P}(S^k_{i}(h_{i,\ell}) + k\epsilon_0/2 > 0),
\]
(82)

where \( S^* = \max_{j > 1} S^i_j(h, l) \).

Proof. Proof of the lemma follows from some basic definitions
and bounds from probability theory.
Thus, (87) follows from (88).

For all $i \in E$ subset of the union of three events i.e. $X_i$ will follow. Event $X_i$ follows trivially from the definition of $N'$. Hence, $A \subset A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)}$, and

$$\mathbb{P}(A) \leq \mathbb{P}(A_n^{(1)}) + \mathbb{P}(A_n^{(2)}) + \mathbb{P}(A_n^{(3)}).$$

Therefore,

$$\sum_{n=1}^{\infty} n t^{-1} \mathbb{P}(A) \leq \sum_{n=1}^{\infty} n t^{-1} \mathbb{P}(A_n^{(1)}) + \sum_{n=1}^{\infty} n t^{-1} \mathbb{P}(A_n^{(2)}) + \sum_{n=1}^{\infty} n t^{-1} \mathbb{P}(A_n^{(3)}).$$

Now, we bound the probability of all three events in the right hand side of the above equation.

Let $a_i = \mathbb{P}(X_i \geq 2^i)$. We have

$$\sum_{i=0}^{\infty} 2^{ir} a_i \leq \sum_{i=0}^{\infty} 2^{ir} (a_i - a_{i+1})$$

where (a) follows from the fact that $\int_{y_1}^{y_2} y^r dy \geq \int_{y_1}^{y_2} y^r dy$, (b) follows trivially from the definition of $\mathbb{E}[X_i^r]$, and (c) follows from the assumption in the statement of the lemma. Thus, using (90), we have

$$\sum_{i=0}^{\infty} 2^{ir} a_i < \infty.\tag{91}$$

Now, we bound the probability of event $A_n^{(1)}$

$$\sum_{n=1}^{\infty} n t^{-1} \mathbb{P}(|X_k| \geq 2^i) \leq \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} 2^{(i+1)(r-1)} a_i$$

$$= \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} 2^{ir-i-1} a_i$$

$$= \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} 2^{(r-1)+r-1} a_i$$

$$< \infty, \tag{92}$$

where the first inequality follows from $\int_{y_1}^{y_2} y^r dy \leq \int_{y_1}^{y_2} y^r dy$, and the inequality follows from (91).
Since the $r^{th}$ moment is finite, therefore, for all $k \in \mathbb{N}$, there exists a finite $K$ such that

$$\mathbb{P}(|X_k| \geq u) \leq K/u^r. \quad (93)$$

Now, we bound the probability of event $A_{(2)}^n$

$$\mathbb{P}(A_{(2)}^n) \leq \sum_{1 \leq k_1 < k_2 \leq n} \mathbb{P}(|X_{k_1}| > n^{4/5} \text{ and } |X_{k_2}| > n^{4/5})$$

$$\leq n^2 \cdot \mathbb{P}(|X_1| > n^{4/5}) \mathbb{P}(|X_2| > n^{4/5})$$

$$\leq K^2 \cdot n^2 \cdot n^{-4r/5} \cdot n^{-4r/5} \quad (94)$$

where $(a)$ follows from the definition of the event and union bound, $(b)$ follows from the independence of random variables and number of possible combinations of $k_1$ and $k_2$, and $(c)$ follows from $(93)$. Therefore,

$$\sum_{n=1}^{\infty} n^{r-1} \mathbb{P}(A_{(2)}^n) \leq \sum_{n=1}^{\infty} K^2 \cdot n^{r-1} n^{r-1} \cdot n^2 \cdot n^{-4r/5} \cdot n^{-4r/5}$$

$$= \sum_{n=1}^{\infty} K^2 \cdot n^{-3r/5+1}$$

$$\leq \infty \quad (95)$$

where $(a)$ follows from $(94)$, and $(b)$ follows as $r \geq 2$.

Now, we bound the probability of event $A_{(3)}^n$. Let

$$X_k^+ = \begin{cases} X_k & |X_k| \leq n^{4/5} \\ 0 & \text{otherwise} \end{cases} \quad (96)$$

Now, let $\mathbb{E}[X_k^+] = \epsilon_n$ and $Y_k = X_k^+ - \epsilon_n$. Thus, $\mathbb{E}[Y_k] = 0$. Now, there exists a finite constant $K'$ such that

$$\mathbb{E}[\sum_{k=1}^{n} Y_k^{2r}] \leq K' \cdot n^{4r/5} n^{-r-1+0.5r/n} \quad (97)$$

where $(a)$ follows from the binomial expansion of $\left(\sum_{k=1}^{n} Y_k\right)^{2r}$ and independence of the random variables, $(b)$ follows from $(96)$, and $(c)$ follows from the fact that the largest binomial coefficient in the expansion of $\left(\sum_{k=1}^{n} Y_k\right)^{2r}$ is $O(n^{-r+0.5r/n})$, and the $r^{th}$ moment of $Y_k$ is finite. Thus, using $(97)$, we have

$$\mathbb{P}(\sum_{k=1}^{n} Y_k | > n/16) \leq K' \cdot n^{4r/5} n^{-r-1+0.5r/n} n^{2r} \quad (98)$$

Now, using $(96)$, as $n \rightarrow \infty$, $\epsilon_n \rightarrow 0$. Thus, there exists a $N_\epsilon$ such that for all $n > N_\epsilon$, $\epsilon_n < 1/16$. Therefore, for all $n > N_\epsilon$, $X_k^+ \leq Y_k + 1/16$. Hence, the probability of event $A_{(3)}^n$, for all $n > N_\epsilon$, is

$$\mathbb{P}(A_{(3)}^n) = \mathbb{P}(\sum_{k=1}^{n} X_k^+) > 2^{i-2}$$

$$\leq \mathbb{P}(\sum_{k=1}^{n} Y_k | > n/16)$$

$$\leq K' \cdot n^{4r/5} n^{-r-1+0.5r/n} n^{2r} \quad (99)$$

where the last inequality follows from $(98)$. Thus,

$$\sum_{n=1}^{\infty} n^{r-1} \mathbb{P}(A_{(3)}^n) \leq \sum_{n=1}^{\infty} \sum_{n_{(3)}} n^{r-1} K' \cdot n^{4r/5} n^{-r-1+0.5r/n} n^{2r}$$

$$\leq K' \sum_{n=N_\epsilon}^{\infty} K' \cdot n^{6r/5}$$

$$\leq \infty \quad (100)$$

where $(a)$ follows from $(99)$, $(b)$ uses the fact that the finite sum of finite numbers is finite and is denoted by the constant $K''$, and $n \geq 1$, $(c)$ follows from simplification of previous inequality, and $(d)$ follows from the fact that $r \geq 2$.

Thus, using $(92)$, $(95)$ and $(100)$, $(67)$ follows.