A New Method of Classification of Pure Tripartite Quantum States

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The classification of the multipartite entanglement is an important problem in quantum information theory. We propose a class of two qubit mixed states \( \sigma_{AB} = p|\chi_1\rangle\langle\chi_1| \otimes \rho_1 + (1-p)|\chi_2\rangle\langle\chi_2| \otimes \rho_2 \), where \( |\chi_1\rangle = \alpha|0\rangle + \beta|1\rangle \), \( |\chi_2\rangle = \beta|0\rangle + (-1)^n\alpha|1\rangle \). We have shown that the state \( \sigma_{AB} \) represent a classical state when \( n \) is odd while it represent a non-classical state when \( n \) is even. The purification of the state \( \sigma_{AB} \) is studied and found that the purification is possible if the spectral decomposition of the density matrices \( \rho_1 \) and \( \rho_2 \) represent pure states. We have established a relationship between three tangle, which measures the amount of entanglement in three qubit system and the quantity \( \langle \chi_1|\chi_2 \rangle \), which identifies whether the two qubit mixed state is classical or non-classical. The three qubit purified state is then classified as a separable or biseparable or W-type or GHZ-type state using the quantum correlation, measured by geometric discord, of its reduced two qubit density matrix.

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I. INTRODUCTION

Quantum Entanglement [1, 2] is an unavoidable part of quantum information theory and is used in many quantum information processing task such as teleportation [3], superdense coding [4], cryptography [5], remote state preparation [6]. The characterization and quantification of pure as well as mixed two qubit entanglement are well studied [7].

A general notion of entanglement in non-classical two-qubit systems is proposed by Ollivier and Zurek [8]. This measure of entanglement in non-classical system captured by quantum discord. For bipartite system, quantum discord is defined as the difference between the total mutual information and classical information [9, 10].

A generalization for multipartite systems is facilitated by the formulation of discord as a distance measure coded by the difference in relative entropy of the quantum state in question with that of the nearest classical state [11]. In a similar spirit the notion of normalised geometric discord has also been introduced [12], which is defined as

\[
D_G(\rho_{AB}) = 2\min_{\chi_{AB} \in T} \left\| \rho_{AB} - \chi_{AB} \right\|_{HS}^2
\]

(1)

where \( T \) denote the set of zero discord states and \( \| \cdot \|_{HS} \) denote Hilbert Schmidt norm. A common noise processes can be used to realize discordant states experimentally [13].

Unlike two qubit state, the structure of three qubit system is not simple. In three qubit system, there are three kind of states: separable, biseparable and genuine entangled states. It is very important to classify these three type of states. Dur et.al. [14] have shown that there are six different equivalence classes of three qubit pure states in terms of stochastic local operation and classical communication. In addition, they found that GHZ-class and W-class, where

\[
\langle GHZ \rangle = \lambda|000\rangle + \mu|111\rangle, \lambda^2 + \mu^2 = 1 \quad (2)
\]

\[
|W\rangle = \gamma|100\rangle + \delta|010\rangle + \nu|001\rangle, \gamma^2 + \delta^2 + \nu^2 = 1 \quad (3)
\]

are two inequivalent class of three qubit entangled states. W state is robust against loss of one qubit i.e. if one of the three qubits is lost, the state of the remaining 2-qubit system is still entangled whereas GHZ state is fully separable after loss of one qubit. Y. Gao et.al. [15] have studied the method of preparation of maximally entangled multipartite GHZ and W states. For three qubit states, there exist Bell inequalities that can be violated by W states and not violated by GHZ states and vice versa [16].

Recently, Gour and Wallach provided a classification of multipartite entanglement in terms of equivalence classes of states under stochastic local operations and classical communication (SLOCC) [17]. A method is proposed to characterize the genuine multisite entanglement in isotropic square spin-1/2 lattices [18]. R. Qu et.al. classified equivalence classes in the set of hypergraph states of three qubits using different entanglement measures [19].

One of the measure to quantify three qubit entanglement is residual tangle originally introduced in [20]. Another measure is called three-tangle which is defined...
as the square root of the residual tangle. The three-tangle is defined for the pure three qubit state \(|\psi\rangle = \sum_{i,j,k=0}^{1} a_{ijk} |ijk\rangle\) as \([21]\)

\[
\tau_3(|\psi\rangle) = 2\sqrt{|d_1 - 2d_2 + 4d_3|} \quad (4)
\]

where

\[

d_1 = a_{000}^2a_{111}^2 + a_{001}^2a_{010}^2 + a_{010}^2a_{011}^2 + a_{100}^2a_{011}^2 \\
\quad + a_{001}a_{110}(a_{010}a_{101} + a_{011}a_{010}) \\
\quad + a_{100}a_{010}a_{011}a_{101} \\
\quad + a_{000}a_{110}a_{010}a_{011} + a_{111}a_{001}a_{010}a_{100} \\
\quad \text{and } d_2 = \text{terms involving } a_{101}a_{010}a_{011}a_{101} \quad (5)
\]

It can be easily shown that the three-tangle of separable and biseparable states is zero. The three-tangle for GHZ state is \(2|\lambda|\). When \(\lambda = \mu = \sqrt{2}\tau_3(|GHZ\rangle) = 1\). While the three-tangle for W state is zero.

The aim of this work is to classify pure three qubit entanglement, mainly, separable, W class and GHZ class of states by reducing the three qubit pure states into two qubit mixed states. By analyzing the quantum correlation of the two qubit mixed state, we can make definite conclusion on the classification of pure three qubit states.

The paper is organised as follows: In section-II, we introduced a special type of two qubit mixed state and studied the quantum correlation, measured by geometric discord, of the proposed special two qubit mixed state. In section-III, We study the purification of the introduced state in section-II and found the condition of the possibility of the purification. In section-IV, we calculate the three-tangle of the purified state and then classify the three qubit purified state. We conclude with a summary of our result in section-V.

II. GEOMETRIC DISCORD OF SPECIAL TYPE OF TWO-QUBIT MIXED STATES

In this section we have proposed a special type of two qubit bipartite mixed states \(\sigma_{AB}\). The form of \(\sigma_{AB}\) is given by

\[
\sigma_{AB} = p|x_1\rangle\langle x_1| \otimes \rho_1 + (1-p)|x_2\rangle\langle x_2| \otimes \rho_2 \quad (6)
\]

where \(|x_1\rangle = \alpha|0\rangle + \beta|1\rangle\), \(|x_2\rangle = \beta|0\rangle + (-1)^n\alpha|1\rangle\), \(\rho_1 = \frac{1}{3}[I_2 + \vec{r} \cdot \vec{s}]\), \(\rho_2 = \frac{1}{3}[I_2 + \vec{s} \cdot \vec{r}]\), \(I_2\) represent identity matrix of order 2; \(\vec{r} = (r_x, r_y, r_z)\) and \(\vec{s} = (s_x, s_y, s_z)\) represent bloch vectors, \(n\) may take odd or even values, and \(\alpha^2 + \beta^2 = 1\).

Theorem: (i) If \(n\) is odd, then \(\langle x_1|x_2\rangle = 0\) and thus \(\sigma_{AB}\) represent a zero discord state and hence a classical state.

(ii) If \(n\) is even, then \(\langle x_1|x_2\rangle \neq 0\) and thus \(\sigma_{AB}\) represent a non-zero discord state and hence a non-classical state.

Proof: The state \(\sigma_{AB}\) given in (6) can be expanded in terms of pauli-matrices as

\[
\sigma_{AB} = \frac{1}{4}\left(I \otimes I + \sum_{i=x,y,z} (pr_i + s_i(1-p))I \otimes \sigma_i \\
\quad + [(2p-1)(\alpha^2 - \beta^2)\sigma_x + 2\alpha\beta(p + (-1)^n(1-p))\sigma_z] \otimes I + p[(\alpha^2 - \beta^2)\sigma_x + 2\alpha\beta\sigma_z] \otimes \vec{r} \cdot \vec{s} \\
\quad - (1-p)[(\alpha^2 - \beta^2)\sigma_x - (1)^n(2\alpha\beta\sigma_z)] \otimes \vec{s} \cdot \vec{r}\right) \quad (7)
\]

In the above Bloch sphere representation \((\vec{r}, \vec{s}, T)\) of \(\sigma_{AB}\), the Bloch vectors \(\vec{x}, \vec{y}\) is given by

\[
\vec{x} = (2\alpha\beta(p + (-1)^n(1-p)), 0, (2p-1)(\alpha^2 - \beta^2)), \\
\vec{y} = (pr_1 + (1-p)s_1, pr_2 + (1-p)s_2, pr_3 + (1-p)s_3) \quad (8)
\]

and the elements of correlation matrix \(T\) is given by

\[
t_{11} = 2\alpha\beta[pr_1 + (-1)^n(1-p)s_1], \\
t_{12} = 2\alpha\beta[pr_2 + (-1)^n(1-p)s_2], \\
t_{13} = 2\alpha\beta[pr_3 + (-1)^n(1-p)s_3], \\
t_{21} = 0, t_{22} = 0, t_{23} = 0; \\
t_{31} = (\alpha^2 - \beta^2)[pr_1 - (1-p)s_1], \\
t_{32} = (\alpha^2 - \beta^2)[pr_2 - (1-p)s_2], \\
t_{33} = (\alpha^2 - \beta^2)[pr_3 - (1-p)s_3] \quad (9)
\]

The normalised geometric discord of the state \(\sigma_{AB}\) is given by

\[
D_G(\sigma_{AB}) = \frac{1}{2}||\vec{x}||^2 + \frac{1}{2}||T||^2 - \lambda_{max}(\vec{x}^T + TT^T) \quad (10)
\]

where \(\vec{x}^T\) and \(TT^T\) denotes the transposition of \(\vec{x}\) and the correlation matrix \(T\) respectively. \(||\vec{x}||^2\) and \(||T||^2\) are given by

\[
||\vec{x}||^2 = 4(1-p)^2\langle x_1|x_2\rangle^2 + 8\alpha\beta(2p - 1)(1-p)^2\langle x_1|x_2\rangle + (2p-1)^2 \quad (11)
\]

\[
||T||^2 = \sum_{i=x,y,z} (pr_i - (1-p)s_i)^2 + 4\langle x_1|x_2\rangle^2(1-p)^2 \times \sum_{i=x,y,z} s_i^2 + 8\langle x_1|x_2\rangle(1-p)\alpha\beta \times \sum_{i=x,y,z} s_i(pr_i - (1-p)s_i) \quad (12)
\]
The symmetric matrix $xx^T + TT^T$ will be of the form

$$xx^T + TT^T = \begin{bmatrix} E & 0 & F \\ 0 & 0 & 0 \\ F & 0 & G \end{bmatrix}$$

(13)

where

$$E = 4\alpha^2\beta^2 \sum_{i=x,y,z} (pr_i -(1-p)s_i)^2 + (2p-1)^2$$

$$+ \frac{8}{(x^2+y^2+z^2)^2} \bigg[ \sum_{i=x,y,z} s_i (pr_i -(1-p)s_i) igg]$$

$$+ (2p-1)$$

$$F = 2\alpha\beta (\alpha^2 - \beta^2) \sum_{i=x,y,z} (pr_i -(1-p)s_i)^2 + (2p-1)^2$$

$$+ 2(\alpha^2 - \beta^2) \chi(x,y,z)^2 \sum_{i=x,y,z} s_i (pr_i -(1-p)s_i)$$

$$+ (2p-1)$$

$$G = (\alpha^2 - \beta^2)^2 \sum_{i=x,y,z} (pr_i -(1-p)s_i)^2$$

(14)

$$+ (2p-1)^2$$

(15)

A simple calculation gives us

$$E + G = \|x\|^2 + \|T\|^2$$

(16)

The maximum eigenvalue of the matrix $xx^T + TT^T$ is given by

$$\lambda_{\text{max}}(xx^T + TT^T) = \frac{1}{2} \left[ (E + G) + \sqrt{(E - G)^2 + 4F^2} \right]$$

(17)

Now we consider the following two cases:

Case-I: When $n$ is odd, $\langle \chi_1 | \chi_2 \rangle = 0$. This reduces the maximum eigenvalue of the matrix $xx^T + TT^T$ to $\|x\|^2 + \|T\|^2$ and it leads the geometric discord $D_G$ to zero. Thus if $\langle \chi_1 | \chi_2 \rangle = 0$, geometric discord of $\sigma_{AB}$ is also equal to zero. Hence the state $\sigma_{AB}$ is considered as a classical state.

Case-II: When $n$ is even, $\langle \chi_1 | \chi_2 \rangle \neq 0$ and hence in this case it can be easily shown that geometric discord of $\sigma_{AB}$ is not equal to zero and thus the state $\sigma_{AB}$ is a non-classical state.

### III. PURIFICATION OF $\sigma_{AB}$

Purification is a mathematical procedure which associate a $(n+1)$- qubit pure state with a $n$-qubit mixed state. An additional system is needed in purification procedure and it is known as reference system or ancillary system [22].

**Theorem:** Let us consider a state $\sigma_{AB}$ given in (10). There exist a purification of $\sigma_{AB}$ if and only if the density matrices described by $\rho_1$ and $\rho_2$ are pure.

**Proof:** The spectral decomposition of the density matrices $\rho_1$ and $\rho_2$ is given by

$$\rho_1 = \lambda^{(1)}_1 |\psi_1\rangle \langle \psi_1 | + \lambda^{(1)}_2 |\psi_2\rangle \langle \psi_2 |$$

$$\rho_2 = \lambda^{(2)}_1 |\psi'_1\rangle \langle \psi'_1 | + \lambda^{(2)}_2 |\psi'_2\rangle \langle \psi'_2 |$$

(19)

$$\lambda^{(1)}_1 = \frac{(1-\sqrt{r_x^2+r_y^2+r_z^2})}{2}, \lambda^{(1)}_2 = \frac{(1+\sqrt{r_x^2+r_y^2+r_z^2})}{2}$$

are the eigenvalues of $\rho_1$ and $|\psi_1\rangle = \frac{1}{\sqrt{N_1}} \left[ s_x + \sqrt{s_x^2+s_y^2+s_z^2} |0\rangle + |1\rangle \right]$ are corresponding orthonormal eigenvectors. $\lambda^{(2)}_1 = \frac{(1+\sqrt{s_x^2+s_y^2+s_z^2})}{2}, \lambda^{(2)}_2 = \frac{(1-\sqrt{s_x^2+s_y^2+s_z^2})}{2}$ are the eigenvalues of $\rho_2$ and $|\psi'_1\rangle = \frac{1}{\sqrt{N_1'}} \left[ s_x + \sqrt{s_x^2+s_y^2+s_z^2} |0\rangle + |1\rangle \right]$ are corresponding orthonormal eigenvectors. The normalization constants are given by $N = 1 + (r_x^2+r_y^2+r_z^2)^{1/2}, N_1 = 1 + (r_x^2+r_y^2+r_z^2)^{1/2}, N_1' = 1 + (s_x^2+s_y^2+s_z^2)^{1/2}$.

Let a purification of $\sigma_{AB}$ be

$$|\xi\rangle_{ABC} = \sqrt{p} |\chi_1\rangle_A \otimes (\sqrt{\lambda^{(1)}_1} |\psi_1\rangle_B \otimes |0\rangle_C$$

$$+ \sqrt{\lambda^{(1)}_2} |\psi_2\rangle_B \otimes |1\rangle_C) + \sqrt{1-p} |\chi_2\rangle_A$$

$$\otimes (\sqrt{\lambda^{(2)}_1} |\psi'_1\rangle_B \otimes |0\rangle_C + \sqrt{\lambda^{(2)}_2} |\psi'_2\rangle_B \otimes |1\rangle_C)$$

(20)

where the qubit $C$ represent the ancilla.

If we trace out the ancilla qubit then the reduced density
The three qubit states of \( |\psi_{1}\rangle \) and \( |\psi_{2}\rangle \) represent bloch vectors satisfying \( \vec{r} \) and \( \vec{r}' \) of type state and the amount of entanglement is given by

\[
\tau_{3}=\frac{2|\langle\psi_{1}|\psi_{2}\rangle|}{\alpha \beta^{2}} \tag{27}
\]

where \( \Delta = \sqrt{(s_{x} - r_{x})^{2} + (s_{y} - r_{y})^{2}} \).

**Case-I:** If \( ae = 0 \) holds then \( \tau_{3}(|\xi_{ABC}\rangle) = 0 \) for any non-zero values of \( \langle\chi_{1}|\chi_{2}\rangle \). In this case the state \( |\xi_{ABC}\rangle \) is a biseparable state.

**Case-II:** If \( \Delta = 0 \Rightarrow \frac{s_{x}}{1+r_{x}} = \frac{s_{y}}{1+r_{y}} \) and \( ae = 0 \) then the three tangle \( \tau_{3}(|\xi_{ABC}\rangle) = 0 \) for any non-zero values of \( \langle\chi_{1}|\chi_{2}\rangle \). Hence the state \( |\xi_{ABC}\rangle \) is a biseparable state.

**Case-III:** If \( \Delta = 0 \) and \( ae \neq 0 \) then the three tangle \( \tau_{3}(|\xi_{ABC}\rangle) = 0 \) for any non-zero values of \( \langle\chi_{1}|\chi_{2}\rangle \). Thus the state \( |\xi_{ABC}\rangle \) is a W-type state.

**Case-IV:** If \( \Delta \neq 0 \), \( ae \neq 0 \) and \( \langle\chi_{1}|\chi_{2}\rangle \neq \frac{a}{b} \) (i.e. \( \alpha \neq \frac{a}{b} \)) then the three qubit state \( |\xi_{ABC}\rangle \) represent GHZ-type state and the amount of entanglement is given by

\[
\tau_{3}(|\xi_{ABC}\rangle) = \frac{2|\langle\psi_{1}|\psi_{2}\rangle|}{\alpha \beta^{2}} \tag{27}
\]
V. CONCLUSION

In this work, we have investigated the different types of entanglement classification of purified tripartite state $|\xi\rangle_{ABC}$ of the two qubit mixed state $\varsigma_{AB}$ based on geometric discord of $\varsigma_{AB}$. We find that the three qubit state of the type biseparable, W-type and GHZ-type state can be generated from non-classical two qubit mixed state. We can also quantify the amount of entanglement in the generated GHZ-type state.

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