DEFORMATIONS OF $\mathbb{Q}$-CALABI–YAU THREEFOLDS
AND $\mathbb{Q}$-FANO THREEFOLDS

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Abstract. We investigate some coboundary map associated to a 3-dimensional terminal singularity which is important in the study of deformations of singular 3-folds. We prove that this map vanishes only for quotient singularities and a $A_{1,2}/4$-singularity, that is, a terminal singularity analytically isomorphic to a $\mathbb{Z}/4$-quotient of the singularity $(x^4+y^5+z^3+u^2=0)$.

As an application, we prove that a $\mathbb{Q}$-Fano 3-fold with terminal singularities can be deformed to one with only quotient singularities and $A_{1,2}/4$-singularities. We also treat the $\mathbb{Q}$-smoothability problem on $\mathbb{Q}$-Calabi–Yau 3-folds.

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1. Introduction

We consider algebraic varieties over the complex number field $\mathbb{C}$.

1.1. $\mathbb{Q}$-smoothing of $\mathbb{Q}$-Fano 3-folds. In this paper, a $\mathbb{Q}$-Fano 3-fold means a 3-dimensional projective variety with only terminal singularities whose anticanonical divisor is ample. A $\mathbb{Q}$-Fano 3-fold is an important object in the classification theory of algebraic 3-folds. It is one of the end products of the Minimal Model Program. Toward the classification of $\mathbb{Q}$-Fano 3-folds, it is fundamental to study their deformations.

Locally, a 3-dimensional terminal singularity has a $\mathbb{Q}$-smoothing, that is, it can be deformed to a variety with only quotient singularities. In general, local deformations of singularities may not lift to a global deformation of a projective 3-fold as shown for Calabi–Yau 3-folds (cf. [8, Example 5.8]). Nevertheless, Altınok–Brown–Reid ([1, 4.8.3]) conjectured that a $\mathbb{Q}$-Fano 3-fold has a $\mathbb{Q}$-smoothing. This conjecture aims to reduce the classification of $\mathbb{Q}$-Fano 3-folds to those with only quotient singularities. For example, there are several papers (cf. [2], [14]) on the classification of certain $\mathbb{Q}$-Fano 3-folds with only quotient singularities.
Previously, deformations of $\mathbb{Q}$-Fano 3-folds are treated in several papers (cf. [9], [6], [15], [12]). In [12, Theorem 1.5], the author proved that a $\mathbb{Q}$-Fano 3-fold with only “ordinary” terminal singularity has a $\mathbb{Q}$-smoothing. (See Definition 2.1 for the ordinariness of the singularity.) In this article, we treat the remaining case, that is, a $\mathbb{Q}$-Fano 3-fold with non-ordinary terminal singularities. The following result almost gives an answer to the conjecture.

**Theorem 1.1.** A $\mathbb{Q}$-Fano 3-fold can be deformed to one with only quotient singularities and $A_{1,2}/4$-singularities.

Here, an $A_{1,2}/4$-singularity means a singularity analytically isomorphic to $0 \in (x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4(1,3,2,1)$, where $x, y, z, u$ are coordinates on $\mathbb{C}^4$ and $\mathbb{C}^4/\mathbb{Z}_4(1,3,2,1)$ is the quotient of $\mathbb{C}^4$ by an action of $\mathbb{Z}_4 = \langle \sigma \rangle$ as follows:

$$\sigma \cdot (x, y, z, u) = (\sqrt{-1}x, -\sqrt{-1}y, z, -\sqrt{-1}u).$$

If there exists an anticanonical element with only Du Val singularities on $X$, we can prove the $\mathbb{Q}$-smoothability of $X$. ([12, Theorem 1.9]) However, we do not know whether we can deform $A_{1,2}/4$-singularities on a general $\mathbb{Q}$-Fano 3-fold.

**1.2. Methods of the proof.** We use a method which is used in [12, Theorem 3.5]. Let $(U, p)$ be a germ of a 3-dimensional terminal singularity. The key tool of our method is the coboundary map $\phi_U$ associated to some local cohomology group on a birational modification $\hat{U} \to U$. (See [2] for the definition of $\phi_U$.) This map is used in several papers as [10], [6], [12] to find a smoothing or a $\mathbb{Q}$-smoothing of a projective 3-fold. The following purely local statement is the main result of Section 2.

**Theorem 1.2.** Let $(U, p)$ be a germ of a “non-ordinary” 3-dimensional terminal singularity. (See Definition 2.1)

(i) Assume that the germ $(U, p)$ is not the $A_{1,2}/4$-singularity. Then we have $\phi_U \neq 0$.

(ii) Assume that the germ $(U, p)$ is the $A_{1,2}/4$-singularity. Then $\phi_U = 0$.

The map $\phi_U$ is known to be nonzero when $(U, p)$ is Gorenstein ([10, Theorem 1.1]) or $(U, p)$ is an ordinary singularity ([6], [12]).

Let us mention about the proof of Theorem 1.2. Since a terminal singularity $(U, p)$ of index $r$ is a $\mathbb{Z}_r$-quotient of a hypersurface singularity $(V, q)$, the set $T^1_{(U, p)}$ of first order deformations of $(U, p)$ is the $\mathbb{Z}_r$-invariant part of $T^1_{(V, q)}$. The set $T^1_{(V, q)}$ can be written as $\mathcal{O}_{V, q}/J_{V, q}$ for the Jacobian ideal of $(V, q)$. We calculate the map $\phi_U$ by using this structure and the inequality [4] proved in [10].

By Theorem 1.2 (ii), the map $\phi_U$ vanishes for a neighborhood $U$ of an $A_{1,2}/4$-singularity. It seems that we need a new method to treat a $\mathbb{Q}$-Fano 3-fold with $A_{1,2}/4$-singularities. (See also Remark 3.1)

**1.3. $\mathbb{Q}$-smoothing of $\mathbb{Q}$-Calabi–Yau 3-folds.** As another corollary of Theorem 1.2, we obtain a similar result for $\mathbb{Q}$-Calabi–Yau 3-folds. Here, a $\mathbb{Q}$-Calabi–Yau 3-fold is a normal projective 3-fold with only terminal singularities whose canonical divisor is a torsion class. Let $r$ be the Gorenstein index of $X$, that is, the minimal positive integer such that $\mathcal{O}_X(rK_X) \simeq \mathcal{O}_X$. The isomorphism $\mathcal{O}_X(rK_X) \simeq \mathcal{O}_X$ determines the global index one cover $\pi : Y := \text{Spec} \oplus_{j=1}^{r-1} \mathcal{O}_X(jK_X) \to X$.

As a consequence of Theorem 1.2 and the proof of [6, Main Theorem 1], we obtain the following.
Theorem 1.3. Let $X$ be a $\mathbb{Q}$-Calabi–Yau 3-fold. Assume that the global index one cover $Y \to X$ is $\mathbb{Q}$-factorial.

Then a $\mathbb{Q}$-Calabi–Yau 3-fold $X$ can be deformed to one with only quotient singularities and $A_{1,2}/4$-singularities.

Remark 1.4. Namikawa studied another invariant for terminal singularities and $\mathbb{Q}$-smoothability of $\mathbb{Q}$-Calabi–Yau 3-folds in his unpublished note.

2. Calculation of coboundary maps

First, we introduce the coboundary map of local cohomology which is used in [12, 3.2] to find a $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold. (See also [10, Section 1], [6, Section 4].)

Let $(U, p)$ be a germ of a 3-dimensional terminal singularity. Let $\pi_U : (V, q) \to (U, p)$ be the index one cover. By the classification ([7], [11]), we see that $(V, q)$ is a hypersurface singularity and $\pi_U$ is étale outside $p$. Moreover, we have

$$(V, q) \simeq ((f = 0), 0) \subset (\mathbb{C}^4, 0)$$

for some $f \in \mathbb{C}[x, y, z, u]$, where $x, y, z, u$ are coordinate functions on $\mathbb{C}^4$ and $f$ satisfies $\sigma \cdot f = \zeta_\nu f$ for the generator $\sigma \in G := \text{Gal}(V/U) \simeq \mathbb{Z}_r$ and $\zeta_\nu = \pm 1$.

Definition 2.1. Let $(U, p)$ be a germ of a 3-dimensional terminal singularity. The germ $(U, p)$ is called ordinary (resp. non-ordinary) if $\zeta_\nu = 1$ (resp. $\zeta_\nu = -1$).

Assume that $(U, p)$ is non-ordinary. By the classification ([7], [11]), we have

$$(U, p) \simeq ((x^2 + y^2 + g(z, u) = 0), 0)/\mathbb{Z}_4 \subset (\mathbb{C}^4/\mathbb{Z}_4, 0),$$

where $g(z, u) \in \mathbb{Z}_4\{-1, 1\}$ is some $\mathbb{Z}_4$-semi-invariant polynomial in $z, u$ and $\sigma \in \mathbb{Z}_4$ acts on $\mathbb{C}^4$ by $\sigma \cdot (x, y, z, u) = (\sqrt{-1}x, -\sqrt{-1}y, -z, \sqrt{-1}u)$.

Let $\nu : \tilde{V} \to V$ be a $\mathbb{Z}_4$-equivariant resolution such that its exceptional divisor $F \subset \tilde{V}$ has SNC support and $\tilde{V} \setminus F \simeq V \setminus \{q\}$. Let $V' := V \setminus \{q\}$ and

$$\tau_V : H^1(V', \Omega^2_{\tilde{V}}(-K_{V'})) \to H^2_{\tilde{F}}(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V))$$

the coboundary map of the local cohomology. Note that the sheaf $\mathcal{O}_V(-K_V)$ is not isomorphic to the sheaf $\mathcal{O}_V$ as $\mathbb{Z}_4$-equivariant sheaves. Let $\tilde{\pi} : \tilde{V} \to U := \tilde{V}/\mathbb{Z}_4$ be the finite morphism induced by $\pi$ and $E \subset \tilde{U}$ the exceptional locus of the birational morphism $\mu : \tilde{U} \to U$ induced by $\nu$. Let $U' := U \setminus \{p\}$ and $\mathcal{F}^{(0)}_U$ the $\mathbb{Z}_4$-invariant part of $\tilde{\pi}_*\Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V)$. Then we have the coboundary map

$$\phi_U : H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_{\mathcal{F}^{(0)}_U}(U, \mathcal{F}^{(0)}_U)$$

which is the $\mathbb{Z}_4$-invariant part of $\tau_V$. We shall study these coboundary maps $\tau_V$ and $\phi_U$ in this section.

We have $H^2_{\mathcal{F}^{(0)}_U}(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) = 0$ by the proof of [13, Theorem 4]. We also have $H^2(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) = 0$ by the Guillén–Navarro Aznar–Puerta–Steenbrink vanishing theorem. Thus we have an exact sequence

$$0 \to H^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V)) \to H^1(V', \Omega^2_{V'}(-K_{V'})) \xrightarrow{\tau_V} H^2_{\mathcal{F}^{(0)}_U}(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V)) \to 0$$

We have the following inequality.
Proposition 2.2. We have
\[ \dim \text{Ker } \tau_V \leq \dim \text{Im } \tau_V. \]

Proof. This is proved in Remark after [10, Theorem (1.1)]. Let us recall the proof for the convenience of the reader.

By the exact sequence (2), it is enough to show that
\[ h^1(\tilde{V}, \Omega^2_V(\log F)(-F)) \leq h^1(\tilde{V}, \Omega^2_V(\log F)(-F)). \]

We have a surjection
\[ H^2_F(\tilde{V}, \Omega^2_V(\log F)(-F)) \to H^2_F(\tilde{V}, \Omega^2_V(\log F)) \]

since we have \( H^2_F(\tilde{V}, \Omega^2_V(\log F) \otimes \mathcal{O}_F) = \text{Gr}_F H^2_{(q)}(V, \mathbb{C}) = 0 \). By the local duality, we have
\[ H^2_F(\tilde{V}, \Omega^2_V(\log F))^* \simeq H^1(\tilde{V}, \Omega^1_V(\log F)(-F)). \]

Moreover we see that the differential homomorphism
\[ d: H^1(\tilde{V}, \Omega^1_V(\log F)(-F)) \to H^1(\tilde{V}, \Omega^2_V(\log F)(-F)) \]

is surjective by studying the spectral sequence
\[ H^q(\tilde{V}, \Omega^p_V(\log F)(-F)) \Rightarrow \mathbb{H}^{p+q}(\tilde{V}, \Omega^*(\log F)(-F)) = 0 \]

as in the proof of [10] Theorem (1.1)]. Thus we obtain relations
\[ h^2_F(\tilde{V}, \Omega^2_V(\log F)(-F)) \geq h^2_F(\tilde{V}, \Omega^2_V(\log F)) = h^1(\tilde{V}, \Omega^1_V(\log F)(-F)) \]
\[ \geq h^1(\tilde{V}, \Omega^2_V(\log F)(-F)) \]

and this implies (4).

Let \( T^1_{(V,q)} \), \( T^1_{(U,p)} \) be the sets of first order deformations of the germs \((V, q)\) and \((U, p)\) respectively. Recall that we have an isomorphism \( T^1_{(V,q)} \simeq \mathcal{O}_{V,q}/J_{V,q} \) of \( \mathcal{O}_{V,q} \)-modules for the Jacobian ideal \( J_{V,q} \subset \mathcal{O}_{V,q} \). Hence we have a surjective \( \mathcal{O}_{V,q} \)-module homomorphism \( \varepsilon : \mathcal{O}_{V,q} \to T^1_{(V,q)} \) which sends \( h \in \mathcal{O}_{V,q} \) to the corresponding deformation \( \varepsilon h \in T^1_{(V,q)} \). Also we have a commutative diagram
\[ T^1_{(U,p)} \xrightarrow{\sim} H^1(U', \Omega^2_{U'}(-K_{U'})) \]
\[ \downarrow \]
\[ T^1_{(V,q)} \xrightarrow{\sim} H^1(V', \Omega^2_{V'}(-K_{V'})), \]

where the horizontal isomorphisms are restrictions by open immersions and the upper terms inject into the lower terms as the \( \mathbb{Z}_4 \)-invariant parts. Thus we identify \( T^1_{(V,q)} \), \( T^1_{(U,p)} \) and \( H^1(V', \Omega^2_{V'}(-K_{V'})), H^1(U', \Omega^2_{U'}(-K_{U'})) \) respectively via these isomorphisms.

We use the following notion of right equivalence ([4, Definition 2.9]).

Definition 2.3. Let \( \mathbb{C}\{x_1, \ldots, x_n\} \) be the convergent power series ring of \( n \) variables. Let \( f, g \in \mathbb{C}\{x_1, \ldots, x_n\} \).

We say that \( f \) is right equivalent to \( g \) if there exists an automorphism \( \varphi \) of \( \mathbb{C}\{x_1, \ldots, x_n\} \) such that \( \varphi(f) = g \). We write this as \( f \sim g \).

By using these ingredients, we prove Theorem 1.2. We repeat the statement.
Theorem 2.4. Let \((U, p)\) be a germ of a non-ordinary 3-dimensional terminal singularity.

(i) Assume that the index one cover \((V, q) \not\cong ((x^2 + y^2 + z^3 + u^2 = 0), 0)\). Then we have \(\phi_V \neq 0\).

(ii) Assume that \((V, q) \cong ((x^2 + y^2 + z^3 + u^2 = 0), 0)\). Then \(\phi_V = 0\).

Proof. (i) Suppose that \(\phi_V = 0\). We show the claim by contradiction. We can write \(g(z, u) = \sum a_{i,j} z^i u^j \in \mathbb{C}[z, u]\) for some \(a_{i,j} \in \mathbb{C}\) for \(i, j \geq 0\). Since the generator \(\sigma \in \mathbb{Z}_4\) acts on \(g\) by \(\sigma \cdot g = -g\) and on \(z^i u^j\) by \(\sigma \cdot z^i u^j = \sqrt{-1} T^{i+j} z^i u^j\), we see that \(a_{i,j} \neq 0\) only if

\[
i, j \neq 0 \mod 4.
\]

Let \(J_g := \left(\frac{\partial g}{\partial z}, \frac{\partial g}{\partial u}\right) \subset \mathbb{C}[z, u]\) be the Jacobian ideal of the polynomial \(g\). Note that a monomial \(z^i u^j\) is an \(O_{V,q}\)-module homomorphism, we obtain a surjection \(C_{V,q} \twoheadrightarrow \text{Im}(\tau_V(\epsilon_{x} \epsilon_{y} = 0)) \in T_{(V,q)}^1\).

(Case 1) Assume that \(a_{0,2} \neq 0\). We can write

\[
g(z, u) = u^2(1 + h_1(z, u)) + h_2(z)
\]

for some polynomials \(h_1(z, u) \in (z, u) \subset \mathbb{C}[z, u]\) and \(h_2(z) \in (z) \subset \mathbb{C}[z]\). Thus \(g(z, u) \in O_{C,0}\) is right equivalent to \(u^2 + h_2(z)\). We see that \(h_2(z) \in O_{C,0}\) is right equivalent to \(z^{2i_0 + 1}\) for some positive integer \(i_0\) since \((g = 0)\) has an isolated singularity and by the condition \(6\). Thus we have

\[
(V, q) \cong ((x^2 + y^2 + z^{2i_0 + 1} + u^2 = 0), 0).
\]

If \(i_0 = 1\), it contradicts the assumption \((V, q) \not\cong ((x^2 + y^2 + z^3 + u^2 = 0), 0)\).

Hence we have \(i_0 \geq 2\). By calculating the partial derivatives of \(x^2 + y^2 + z^{2i_0 + 1} + u^2\), we see that \(\epsilon_1, \epsilon_2, \epsilon_3 \in T_{(V,q)}^1\) are linearly independent and

\[
\dim T_{(V,q)}^1 \geq 3.
\]

On the other hand, we see that \(\tau_V(\epsilon_{z}) = 0\) since we assumed \(\phi_V = 0\) and \(\epsilon_z \in T_{(U,p)}^1\).

By this and the fact that \(\tau_V\) is an \(O_{V,q}\)-module homomorphism, we obtain a surjection \(\mathbb{C}[z, u]\to \text{Im}(\tau_V)\) since \(\epsilon_u = 0\). By this surjection and \(\mathbb{C}[z, u]\to (z, u) \cong \mathbb{C}\), we obtain \(\dim \text{Im}(\tau_V) \leq 1\). By this and the inequality \(4\), we obtain an inequality

\[
\dim T_{(V,q)}^1 = \dim \text{Im}(\tau_V) + \dim \ker(\tau_V) \leq 1 + 1 = 2
\]

and it is a contradiction.

(Case 2) Assume that \(a_{0,2} = 0\). Then we see that \(a_{i,j} \neq 0\) only if \(2i + j \geq 6\) by \(6\). Note that a monomial \(z^i u^j\) with \(2i + j \geq 6\) is some multiple of either \(z^3, z^2 u^2, z^4 u^2\) or \(u^6\). By computing partial derivatives of these monomials, we see that \((g, J_g) \subset (z^2, z u^2, u^4)\). Thus we see that \(\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5, \epsilon_6 \in T_{(V,q)}^1\) are linearly independent and we obtain

\[
\dim T_{(V,q)}^1 \geq 6.
\]

On the other hand, by the assumption \(\phi_V = 0\), we have \(\tau_V(\epsilon_{z}) = 0, \tau_V(\epsilon_{u^2}) = 0\) since \(\epsilon_z, \epsilon_{u^2} \in T_{(U,p)}^1\). Thus we have a relation \((z, u^2) \subset \ker(\tau_V) \circ \epsilon \subset O_{V,q}\) and obtain a surjection \(\mathbb{C}[z, u]\to \text{Im}(\tau_V)\). This implies an inequality \(\dim \text{Im}(\tau_V) \leq \dim \mathbb{C}[z, u]/(z, u^2) = 2\). By this inequality and the inequality \(4\), we have an inequality

\[
\dim T_{(V,q)}^1 = \dim \ker(\tau_V) + \dim \text{Im}(\tau_V) \leq 2 + 2 = 4.
\]

This contradicts \(7\).
Hence we obtain \( \phi_U \neq 0 \) and finish the proof of (i).

(ii) For non-negative integers \( i, j \), we set
\[
b^{i,j} := \dim H^j(\tilde{V}, \Omega^i_V(\log F)(-F)),
\]
\[
t^{i,j} := \dim H^j(F, \Omega^i_V(\log F) \otimes \mathcal{O}_F).
\]

Let \( s_k(V, q) \) for \( k = 0, 1, 2, 3 \) be the Hodge number of the Milnor fiber of \((V, q)\) as in [13] Section 4. By [13] Theorem 6, we have \( s_0 = 0, s_1 = b^{1,1}, s_2 = b^{1,1} + t^{1,1} \) and \( s_3 = t^{0,2} \). We see that \( t^{0,2} = 0 \) by [13] Lemma 2. Since the sum \( \sum_{k=0}^3 s_k(V, q) \) is the Milnor number of \((V, q)\), we obtain \( 2b^{1,1} + t^{1,1} = 2 \). Since \( b^{1,1} \neq 0 \) by [10] Theorem 2.2, we obtain
\[
b^{1,1} = 1, \quad t^{1,1} = 0.
\]

There exists an exact sequence
\[
H^0(F, \Omega^1_V(\log F) \otimes \mathcal{O}_F) \to H^1(\tilde{V}, \Omega^1_V(\log F)(-F)) \to H^1(\tilde{V}, \Omega^1_V(\log F)) \to H^1(F, \Omega^1_V(\log F) \otimes \mathcal{O}_F).
\]

Since \( t^{1,0} = 0 \) by [13] Lemma 1], the both outer terms are zero and the homomorphism in the middle is an isomorphism. By this and (8), we have
\[
C \simeq H^1(\tilde{V}, \Omega^1_V(\log F)) \simeq H^2_F(\tilde{V}, \Omega^2_V(\log F)(-F))^*.
\]

Suppose that \( \tau_V(\varepsilon_z) \neq 0 \). Then \( \varepsilon_z \notin \text{Ker} \tau_V \). This implies that \( \text{Ker} \tau_V = 0 \) since \( T^1(V, q) \simeq \mathbb{C}[z]/(z^2) \) as \( \mathbb{C}[z] \)-modules. Thus \( \mathbb{C}^2 \simeq \text{Im} \tau_V \simeq H^2_F(\tilde{V}, \Omega^2_V(\log F)(-F)) \).

This contradicts [10].

Thus we obtain \( \tau_V(\varepsilon_z) = 0 \). Since \( T^1(U, p) \simeq \mathbb{C} \) is generated by \( \varepsilon_z \), we see that \( \phi_U = 0 \). Thus we finish the proof of (ii). \( \square \)

We have another coboundary map
\[
\tau_V : H^1(V', \Omega^2_{V'}(-K_{V'})) \to H^2_F(\tilde{V}, \Omega^2_V(-\nu^* K_V))
\]
and this fits in the commutative diagram
\[
\begin{array}{ccc}
H^1(V', \Omega^2_{V'}(-K_{V'})) & \xrightarrow{\tau_V} & H^2_F(\tilde{V}, \Omega^2_V(-\nu^* K_V)) \\
\downarrow{\tau_V} & & \downarrow{\tau_V} \\
H^2_F(\tilde{V}, \Omega^2_V(\log F)(-F - \nu^* K_V)),
\end{array}
\]

where the injectivity of \( \tau'_V \) is proved in the proof of [10] Theorem 1.1].

Let \( \tilde{F}^{(0)}_U := (\pi, \Omega^2_V(-\nu^* K_V))^Z_4 \) be the \( \mathbb{Z}_4 \)-invariant part. Let
\[
\tilde{\phi}_U : H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_F(\tilde{U}, \tilde{F}^{(0)}_U)
\]
be the coboundary map. It is the $\mathbb{Z}_4$-invariant part of $\bar{\tau}_V$. As the $\mathbb{Z}_4$-invariant part of the diagram (11), we obtain the following diagram:

$$
\begin{array}{c}
H^1(U', \Omega^2_{U'}(-K_{U'})) \xrightarrow{\phi_U} H^2_E(U, \mathcal{F}_U^{(0)}) \\
\downarrow \phi_U \\
H^2_E(U, \mathcal{F}_U^{(0)}),
\end{array}
$$

By these arguments, we obtain the following corollary of Theorem 2.4.

**Corollary 2.5.** Let $(U, p)$ be a germ of a non-ordinary 3-dimensional terminal singularity. Assume that $\bar{\phi}_U = 0$.

Then the germ $(U, p)$ is an $A_{1,2}/4$-singularity.

Let

$$
\nu_* : H^1(\tilde{V}, \Omega^2_{\tilde{V}}(-K_{\tilde{V}})) \to H^1(V', \Omega^2_{V'}(-K_{V'}))
$$

be the restriction homomorphism by the open immersion $V' \hookrightarrow \tilde{V}$. We use this notation since there is a commutative diagram

$$
\begin{array}{ccc}
H^1(\tilde{V}, \Omega^2_{\tilde{V}}(-K_{\tilde{V}})) & \xrightarrow{\nu_*} & H^1(V', \Omega^2_{V'}(-K_{V'})) \\
\downarrow \cong & & \downarrow \cong \\
T^1_{\tilde{V}} & \rightarrow & T^1_V,
\end{array}
$$

where the lower horizontal homomorphism is the blow-down homomorphism of deformations ([16]). We can prove the relation

(12) $\text{Im} \, \nu_* \subset \text{Ker} \, \tau_V = \text{Ker} \, \bar{\tau}_V$

by the same argument as in [12, Claim 3.7].

3. Application to $\mathbb{Q}$-smoothing problems

As an application of Theorem 2.4, we obtain a proof of Theorem 1.1 as follows.

**Proof of Theorem 1.1.** By [12, Theorem 3.2], we can deform a $\mathbb{Q}$-Fano 3-fold $X$ to one with only singularities $p_1, \ldots, p_l$ such that $\phi_{U_i} = 0$, where $U_i$ is a Stein neighborhood of $p_i$ for $i = 1, \ldots, l$. By Theorem 2.4 such a terminal singularity is either a quotient singularity or an $A_{1,2}/4$-singularity. Thus we finish the proof. □

**Remark 3.1.** We give a comment on a $\mathbb{Q}$-Fano 3-fold with $A_{1,2}/4$-singularities.

Let $X$ be a $\mathbb{Q}$-Fano 3-fold. The local-to-global spectral sequence of Ext groups induces an exact sequence

$$
\text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(X, \text{Ext}^1(\Omega^1_X, \mathcal{O}_X)) \to H^2(X, \Theta_X),
$$

where $\text{Ext}^1$ is a sheaf of Ext groups. Recall that $\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ and $H^0(X, \text{Ext}^1(\Omega^1_X, \mathcal{O}_X))$ are the sets of first order deformations of $X$ and the singularities on $X$, respectively. Thus, if we have $H^2(X, \Theta_X) = 0$, we see that $X$ is $\mathbb{Q}$-smoothable.

However, this approach does not work in general. We can construct an example of a $\mathbb{Q}$-Fano 3-fold $X$ with $A_{1,2}/4$-singularities such that $H^2(X, \Theta_X) \neq 0$ by taking certain global quotient of the example given in [9, Example 5]. However, we can deform the singularities on it. We exhibit this example in elsewhere.

Thus we do not know $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold with $A_{1,2}/4$-singularities.
As another application of Theorem 2.4 we obtain a proof of Theorem 1.3 as follows.

Proof of Theorem 1.3 The proof is a modification of the proof of [6, Main Theorem 1]. We sketch the proof for the convenience of the reader.

We can assume that $X$ has only quotient singularities and non-ordinary terminal singularities by [6, Main Theorem 1]. First we prepare notations to define the diagram (13).

Let $p_1, \ldots, p_l \in X$ be the non-ordinary singularities and $U_1, \ldots, U_l$ their Stein neighborhoods. Let $\nu : Y \to Y$ be a $\mathbb{Z}_r$-equivariant resolution such that its exceptional divisor $F$ is a SNC divisor and $Y \setminus F \simeq Y \setminus \nu^{-1}((p_1, \ldots, p_l))$. Let $\pi : X \to X$ be the quotient morphism and $\mu : X \to X$ the induced birational morphism with the exceptional divisor $E$.

Let $V_i := \pi^{-1}(U_i)$, $\tilde{V}_i := F \cap \tilde{V}_i$ and $\nu_i := \nu|_{\tilde{V}_i} : \tilde{V}_i \to V_i$ be the restrictions. Let $\tilde{U}_i := \nu^{-1}(U_i)$, $E_i := E \cap \tilde{U}_i$ and $\tilde{\pi}_i := \tilde{\pi}_{|\tilde{V}_i} : \tilde{V}_i \to \tilde{U}_i$ the induced finite morphism. Let $\tilde{F}(0) := (\tilde{\pi}_i \Omega Y^{-2}(-\nu^* K_Y))_{\mathbb{Z}_r}$ be the $\mathbb{Z}_r$-invariant part and $\tilde{F}_i(0) := \tilde{F}(0)_{\tilde{U}_i}$ its restriction.

Then we have the diagram

$$
\begin{aligned}
H^1(X', \Omega^2_{X'}(-K_{X'})) &\xrightarrow{\oplus \psi_i} \oplus_{i=1}^l H^2_{E_i}(X, \tilde{F}(0)) \xrightarrow{\oplus B_i} H^2(X, \tilde{F}(0)) \\
\oplus_{i=1}^l H^1(U_i, \Omega^2_{U_i}(-K_{U_i})) &\xrightarrow{\oplus \phi_i} \oplus_{i=1}^l H^2_{E_i}(U_i, \tilde{F}(0)),
\end{aligned}
$$

where $X' := X \setminus \{p_1, \ldots, p_l\}$ and $U'_i := U_i \cap X'$.

Let $V'_i := \pi^{-1}(U'_i)$. Note that $B_i \circ \varphi_i^{-1} \circ \tilde{\phi}_i$ is the $\mathbb{Z}_r$-invariant part of the composition

$$
\begin{aligned}
H^1(V'_i, \Omega^2_{V'_i}(-K_{V'_i})) &\to H^2_{E_i}(V'_i, \Omega^2_{V'_i}(-\nu^* K_{V'_i})) \to H^2(Y, \Omega^2_Y(-\nu^* K_Y)) \\
&\to H^2(Y, \Omega^2_Y(-\nu^* K_Y)).
\end{aligned}
$$

We see that this is zero by [10, Proposition 1.2] since we assumed that $Y$ is $\mathbb{Q}$-factorial. Thus we also see that $B_i \circ \varphi_i^{-1} \circ \tilde{\phi}_i = 0$.

There exists an element $\eta_i \in H^1(U'_i, \Omega^2_{U'_i}(-K_{U'_i}))$ such that $\tilde{\phi}_i(\eta_i) \neq 0$ by Theorem 1.2. Since $B_i \circ \varphi_i^{-1} \circ \tilde{\phi}_i(\eta_i) = 0$, there exists $\eta \in H^1(X', \Omega^2_X(-K_{X'}))$ such that $\psi_i(\eta) = \varphi_i^{-1}(\tilde{\phi}_i(\eta_i))$. By the relation (12) and $p_{U_i}(\eta) - \eta_i \in \ker \tilde{\phi}_i$, we see that $p_{U_i}(\eta) \notin \text{Im}(\psi_i)$, where we use the inclusion $H^1(U'_i, \Omega^2_{U'_i}(-K_{U'_i})) \subset H^1(V'_i, \Omega^2_{V'_i}(-K_{V'_i}))$. By arguing as in the proof of [12, Theorem 3.5], we can deform singularity $p_i \in U'_i$ as long as $\tilde{\phi}_i \neq 0$. By Corollary 2.3 we obtain a required deformation since the deformations of a $\mathbb{Q}$-Calabi–Yau 3-fold are unobstructed ([8, Theorem A]).

\[\square\]

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