Are best approximations really better than Chebyshev?

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Abstract

Best and Chebyshev approximations play an important role in approximation theory. From the viewpoint of measuring approximation error in the maximum norm, it is evident that best approximations are better than their Chebyshev counterparts. However, the situation may be reversed if we compare the approximation quality from the viewpoint of either the rate of pointwise convergence or the accuracy of spectral differentiation. We show that when the underlying function has an algebraic singularity, the Chebyshev projection of degree $n$ converges one power of $n$ faster than its best counterpart at each point away from the singularity and both converge at the same rate at the singularity. This gives a complete explanation for the phenomenon that the accuracy of Chebyshev projections is much better than that of best approximations except in a small neighborhood of the singularity. Extensions to superconvergence points, spectral differentiation, Chebyshev interpolants and other orthogonal projections are also discussed.

Keywords: Chebyshev projections, best approximations, Chebyshev interpolants, superconvergence points, Legendre projections

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1 Introduction

Spectral approximations, such as Chebyshev and Legendre projections and interpolants, are invaluable and powerful methods and they play an important role in numerous practical applications, including approximation theory, Gauss and Chebyshev type quadrature, rootfinding and spectral methods for differential and integral equations (see, e.g., [5, 6, 8, 13, 17, 19, 21, 22, 23, 25]). Of particular importance are Chebyshev approximations, which have several remarkable advantages including: (i) they are near-best approximations, that is, their approximation quality in the maximum norm is close to that of best approximations [14, 17, 19, 25, 29]; (ii) the discrete Chebyshev transforms,
i.e., the transforms between the values at the Chebyshev points and the Chebyshev expansion coefficients, can be achieved rapidly by means of the fast Fourier transform (FFT) \[17, 21\]; (iii) the evaluation of Chebyshev interpolants can be achieved stably and rapidly using the barycentric formula \[2, 12\]. Due to these attractive advantages, Chebyshev approximations are widely used in many branches of numerical analysis. We refer the interested reader to \[9, 17, 19, 25\] for more extensive overview.

Let $\Omega := [-1, 1]$ and let $T_k(x)$ be the Chebyshev polynomial of the first kind of degree $k$, i.e., $T_k(x) = \cos(k \arccos(x))$. It is well known that the Chebyshev polynomials $\{T_k(x)\}$ are orthogonal with respect to the weight function $(1 - x^2)^{-1/2}$ on $\Omega$. If $f$ satisfies the Dini-Lipschitz continuous on $\Omega$, then it has the following uniformly convergent Chebyshev series \[17, \text{Theorem 5.7}\]

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \quad a_k = \frac{2}{\pi} \int_{\Omega} f(x) T_k(x) \sqrt{1 - x^2} \, dx,$$  

(1.1)

where the prime indicates that the first term of the sum should be halved. Truncating the above infinite series after the first $n+1$ terms, we obtain the Chebyshev projection of degree $n$ of $f$, i.e.,

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x).$$  

(1.2)

Let $\Pi_n$ denote the space of polynomials of degree at most $n$ and let $p_n^*$ denote the best approximation in $\Pi_n$ to $f$ in the maximum norm, i.e., $\|f - p_n^*\|_{L^\infty(\Omega)} = \min_{h \in \Pi_n} \|f - h\|_{L^\infty(\Omega)}$. It is well known that $p_n^*$ exists and is unique whenever $f \in C(\Omega)$. We are interested in the comparison of $f_n$ and $p_n^*$. From the viewpoint of minimizing the maximum error of approximants, it is evident that $p_n^*$ is better than $f_n$. From the viewpoint of practical convenience, however, it is evident that $f_n$ is much more preferable than $p_n^*$ since $f_n$ depends linearly on $f$ and its implementations can be achieved efficiently by means of the FFT. In contrast, $p_n^*$ depends nonlinearly on $f$ and its implementation must resort to iterative methods, which results in an expensive computational cost, especially when $n$ is large. Now, we are interested in the difference in approximation quality between $p_n^*$ and $f_n$. A classical result associated with this issue is the following inequalities \[19, \text{Theorem 3.3}\]

$$\|f - p_n^*\|_{L^\infty(\Omega)} \leq \|f - f_n\|_{L^\infty(\Omega)} < \left( 4 + \frac{4}{\pi^2} \log n \right) \|f - p_n^*\|_{L^\infty(\Omega)},$$  

(1.3)

from which we see that the maximum error of $f_n$ is inferior than that of $p_n^*$ by at most a logarithmic factor. Direct calculation shows that $\|f - f_n\|_{L^\infty(\Omega)} < 9.6\|f - p_n^*\|_{L^\infty(\Omega)}$ for $n \leq 10^6$, which means that the accuracy of $p_n^*$ will never be better than that of $f_n$ by one digit even if the degree of both methods is one million. Regarding (1.3), we further ask the following question: Is $p_n^*$ really better than $f_n$ by a logarithmic factor? At first glance, this question has already been answered by the inequalities above. However, they only answer this question from the viewpoint of minimizing the maximum error of both
approximants. From the viewpoint of the rate of pointwise convergence, the inequalities above might be one-sided and misleading. To illustrate this, we plot the error curves of $f_n$ and $p_n^*$ for the test function $f(x) = |x - 1/2|^\alpha$ and we test two different values of $\alpha$ and $n$, respectively. Throughout the paper, $p_n^*$ is calculated by the Remez algorithm in Chebfun (see [9]). The pointwise error curves are depicted in Figure 1 which suggest that:

(i) The maximum errors of $f_n$ are always attained at a small neighborhood of $x = \xi$ even for small values of $n$. Moreover, the larger $n$, the narrower the neighborhood that the pointwise error curves attain their maximum;

(ii) The accuracy of $p_n^*$ is better than that of $f_n$ only in a small neighborhood of the singularity. Otherwise, the accuracy of $p_n^*$ is much worse than that of $f_n$.

![Figure 1: Pointwise error curves of $f_n$ and $p_n^*$ for $n = 10$ (left) and $n = 100$ (right). The top row corresponds to $\alpha = 1$ and the bottom row corresponds to $\alpha = 3/2$.](image)

We are not the first to notice these observations. In fact, Trefethen in [21] observed a similar phenomenon from the pointwise error curve of Chebyshev interpolant. He further
made the following comments:

“Which approximation would be more useful in an application? I think the only reasonable answer is, it depends. Sometimes one really does need a guarantee about worstcase behavior. In other situations, it would be wasteful to sacrifice so much accuracy over 95% of the range just to gain one bit of accuracy in a small subinterval.”

However, no theoretical justification was given to explain these observations.

In this work, we aim to justify the above observations and present a detailed analysis of the pointwise error estimates of Chebyshev projections. For ease of presentation, we restrict our analysis to the following model function

\[ f(x) = |x - \xi|^\alpha g(x), \quad (1.4) \]

where \( \xi \in \Omega \) and \( \alpha > 0 \) is not an even integer whenever \( \xi \in (-1, 1) \) and is not an integer whenever \( \xi = \pm 1 \), and \( g(x) \) is sufficiently smooth on \( \Omega \). We show that the key ingredient for explaining the pointwise error behavior of Chebyshev projections is to understand the asymptotic behavior of the following two functions

\[ \Phi_C(x, n) = \sum_{k=n+1}^{\infty} \frac{\cos(kx)}{k^{\nu+1}}, \quad \Phi_S(x, n) = \sum_{k=n+1}^{\infty} \frac{\sin(kx)}{k^{\nu+1}}, \quad (1.5) \]

where \( x \in \mathbb{R}, \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( \nu > 0 \) whenever \( x(\text{mod } 2\pi) \neq 0 \) and \( \nu > 1 \) whenever \( x(\text{mod } 2\pi) = 0 \). As will be shown later, these two functions are intimately related to the Lerch’s transcendent function and its special case Hurwitz zeta function (see, e.g., [18, Chapter 25]) and their asymptotic behavior has a crucial distinction between \( x(\text{mod } 2\pi) \neq 0 \) and \( x(\text{mod } 2\pi) = 0 \). Based on this finding, we are able to explain clearly why Chebyshev projections exhibit better performance than their best approximations except in the small neighborhood of the singularity (as displayed in Figure 4). Furthermore, this finding also allows us to explain why Chebyshev projections exhibit much better performance than their best approximations when applying them for spectral differentiation.

The paper is organized as follows. In section 2, we present a thorough analysis of the pointwise error estimates of Chebyshev projections. In section 3, we extend our analysis to a more general setting, including superconvergence points of Chebyshev projections, Chebyshev spectral differentiation, Chebyshev interpolants and other orthogonal projections. In section 4, we finish this paper with some concluding remarks.

### 2 Pointwise error estimates of Chebyshev projections

In this section, we present pointwise error estimates of Chebyshev projections for the model function defined in (1.4). Our main result will provide a thorough understanding
of the pointwise error behavior of Chebyshev projections for functions with a singularity. For notational simplicity, we introduce

$$\varphi^\pm_\xi(x) = \frac{\arccos(x) \pm \arccos(\xi)}{2}, \quad (2.1)$$

and we will drop the argument $x$ in $\varphi^\pm_\xi(x)$ whenever there is no ambiguity.

Before stating our main result, we present two preliminary results that will be used in the proof of our main result.

**Lemma 2.1.** Let $f$ be the function defined in (1.4) and let $a_k$ be its $k$th Chebyshev coefficient.

(i) If $\xi \in (-1, 1)$, then for $k \gg 1$, we have

$$a_k = I_1(\alpha, \xi) \frac{T_k(\xi)}{k^{\alpha+1}} + I_2(\alpha, \xi) \frac{U_{k-1}(\xi)}{k^{\alpha+2}} + O(k^{-\alpha-3}) \quad (2.2)$$

where $U_k(x)$ is the Chebyshev polynomial of the second kind of degree $k$ and

$$I_1(\alpha, \xi) = -\frac{4(1 - \xi^2)^{\alpha/2}\Gamma(\alpha + 1)}{\pi} \sin\left(\frac{\alpha \pi}{2}\right) g(\xi),$$

$$I_2(\alpha, \xi) = -\frac{4(1 - \xi^2)^{\alpha/2}\Gamma(\alpha + 2)}{\pi} \sin\left(\frac{\alpha \pi}{2}\right) \left[(1 - \xi^2)g'(\xi) - \frac{\alpha \xi}{2} g(\xi)\right].$$

(ii) If $\xi = \pm 1$, then for $k \gg 1$, we have

$$a_k = B(\alpha, \xi) \frac{T_k(\xi)}{k^{2\alpha+1}} + O(k^{-2\alpha-3}), \quad B(\alpha, \xi) = -\frac{\sin(\alpha \pi)\Gamma(2\alpha + 1)}{2^{2\alpha-1}\pi} g(\xi) \quad (2.3)$$

**Proof.** As for (i), it follows from Theorem 2.1 in [13]. As for (ii), it follows from Theorem 2.2 in [28].

**Lemma 2.2.** Let $\Psi^C_\nu(x, n)$ and $\Psi^S_\nu(x, n)$ be the functions defined in (1.5). For $n \gg 1$, the following two statements hold:

(i) When $x \pmod{2\pi} \neq 0$ and $\nu > 0$, we have

$$\Psi^C_\nu(x, n) = -\frac{\sin((2n+1)x/2)}{2\sin(x/2)} n^{-\nu-1} + O(n^{-\nu-2}), \quad (2.4)$$

and

$$\Psi^S_\nu(x, n) = -\frac{\cos((2n+1)x/2)}{2\sin(x/2)} n^{-\nu-1} + O(n^{-\nu-2}). \quad (2.5)$$

(ii) When $x \pmod{2\pi} = 0$ and $\nu > 1$, we have

$$\Psi^C_\nu(x, n) = \frac{1}{\nu} n^{-\nu} - \frac{1}{2} n^{-\nu-1} + O(n^{-\nu-2}). \quad (2.6)$$
Proof. We notice that
\[ \Psi_C^\nu(x, n) + i \Psi_S^\nu(x, n) = e^{i(n+1)x} \Phi(e^{ix}, \nu + 1, n + 1), \]
where \( i \) is the imaginary unit and \( \Phi(z, s, a) \) is the Lerch’s transcendent function defined by [18 §25.14(i)]
\[ \Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}, \tag{2.7} \]
and where \( |z| = 1, \Re(s) > 1 \) and \( a \neq 0, -1, \ldots \), and \( \Phi(z, s, a) \) is defined by analytic continuation for other values of \( z \). Moreover, the Lerch’s transcendent function reduces to the Hurwitz zeta functions whenever \( z = 1 \), i.e.,
\[ \Phi(1, s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} := \zeta(s, a), \tag{2.8} \]
where \( \zeta(s, a) \) is the Hurwitz zeta functions [18 §25.11(i)]. As for (i), we recall from [10, Theorem 1] that \( \Phi(z, \nu, n) = (1 - z)^{-1} n^{-\nu} + O(n^{-\nu-1}) \) for \( z \in \mathbb{C} \setminus [1, \infty) \), and thus
\[ \Psi_C^\nu(x, n) + i \Psi_S^\nu(x, n) = \frac{e^{i(n+1)x}}{1 - e^{ix}} n^{-\nu-1} + O(n^{-\nu-2}). \]

The results (2.4) and (2.5) follow from the above equation by taking real and imaginary parts, respectively. As for (ii), from [18 Equation (25.11.3)] we know that \( \Psi_C^\nu(x, n) = \Phi(1, n+1, \nu+1) = \zeta(\nu+1, n+1) - n^{-\nu-1} \). The result (2.6) then follows from the asymptotic expansion of \( \zeta(\nu, n) \) in [18 Equation (25.11.43)]. This completes the proof. \( \square \)

Several remarks are in order.

Remark 2.3. The magnitude of the Lerch’s transcendent function \( \Phi(z, s, a) \) has a peak near \( z = 1 \); see Figure 2 for an illustration. When \( |z| = 1 \) and \( a \to \infty \), we notice that \( \Phi(z, s, a) \) behaves like \( O(n^{-s}) \) whenever \( z \neq 1 \) and \( s > 0 \) and behaves like \( O(n^{1-s}) \) whenever \( z = 1 \) and \( s > 1 \). This explains why the asymptotic behavior of \( \Psi_C^\nu(x, n) \) has an obvious distinction between \( x(\text{mod} 2\pi) \neq 0 \) and \( x(\text{mod} 2\pi) = 0 \).

Remark 2.4. For any \( k \in \mathbb{Z} \) and \( n \gg 1 \), we have
\[ \Psi_C^\nu(k\pi, n) = \begin{cases} O(n^{-\nu}), & k \text{ is even}, \\ O(n^{-\nu-1}), & k \text{ is odd}, \end{cases} \quad \Psi_S^\nu(k\pi, n) = 0. \]

Our main result is stated in the following theorem.

Theorem 2.5. Let \( f \) be the function defined in (1.4) and let \( f_n \) be its Chebyshev projection of degree \( n \). As \( n \to \infty \), the following statements are true.

\[ \]
Figure 2: Modulus of the Lerch’s transcendent function $\Phi(z, s, a)$ with $s = 2.5$ and $a = 10$ in the complex plane.

(i) If $\xi \in (-1, 1)$, we have for $x \neq \xi$ that

$$f(x) - f_n(x) = -\frac{I_1(\alpha, \xi)}{4} \left[ U_{2n}(\cos(\varphi^2)) + U_{2n}(\cos(\varphi^2)) \right] n^{-\alpha - 1} + O(n^{-\alpha - 2}),$$

and for $x = \xi$

$$f(x) - f_n(x) = \frac{I_1(\alpha, \xi)}{2\alpha} n^{-\alpha} - \frac{I_1(\alpha, \xi)}{4} U_{2n}(\xi) + \frac{1}{4} n^{-\alpha - 1} + O(n^{-\alpha - 2}).$$

(ii) If $\xi = \pm 1$, there holds

$$f(x) - f_n(x) = B(\alpha, \xi) \begin{cases} \frac{1}{2\alpha} n^{-2\alpha} - \frac{1}{2} n^{-2\alpha - 1} + O(n^{-2\alpha - 2}), & x = \xi, \\ - \frac{U_{2n}(\cos(\varphi^2))}{2} n^{-2\alpha - 1} + O(n^{-2\alpha - 2}), & x \neq \xi. \end{cases}$$

Here $I_1(\alpha, \xi)$ and $B(\alpha, \xi)$ are defined as in Lemma 2.7.

Proof. First of all, it is easily seen that $f$ is Hölder continuous with exponent $\alpha$ whenever $\alpha \in (0, 1)$ and is absolutely continuous on $\Omega$ whenever $\alpha \geq 1$, and hence the Chebyshev projection $f_n$ converges uniformly to $f$ as $n \to \infty$. We now consider the error estimate
of \( f_n \) and we start with the case \( \xi \in (-1, 1) \). Due to the uniform convergence of \( f_n \) to \( f \), it follows from Lemma 2.1 that

\[
f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x)
\]

\[
= I_1(\alpha, \xi) \sum_{k=n+1}^{\infty} \frac{T_k(\xi)T_k(x)}{k^{\alpha+1}} + I_2(\alpha, \xi) \sum_{k=n+1}^{\infty} \frac{U_{k-1}(\xi)T_k(x)}{k^{\alpha+2}} + O(n^{-\alpha-2})
\]

Furthermore, by Lemma 2.2, the error of \( f_n \) can be rewritten as

\[
f(x) - f_n(x) = \frac{I_1(\alpha, \xi)}{2} \left[ \Psi^C_{\alpha}(2\varphi^+, n) + \Psi^C_{\alpha}(2\varphi^-, n) \right]

+ \frac{I_2(\alpha, \xi)}{2\sqrt{1 - \xi^2}} \left[ \Psi^S_{\alpha+1}(2\varphi^+, n) - \Psi^S_{\alpha+1}(2\varphi^-, n) \right] + O(n^{-\alpha-2}), \tag{2.9}
\]

where \( \Psi^C_{\alpha}(z, n) \) and \( \Psi^S_{\alpha+1}(z, n) \) are defined in (1.3). Let us now consider the pointwise error estimates of \( f_n \) for the case \( x \neq \xi \), it is easy to check that \( \varphi^+_\xi \in (0, \pi) \) and \( \varphi^-_\xi \in (-\pi/2, 0) \cup (0, \pi/2) \) for all \( x \in [-1, 1] \). Applying Lemma 2.2 to (2.9) gives

\[
f(x) - f_n(x) = - \frac{I_1(\alpha, \xi)}{4} \left[ \frac{\sin((2n+1)\varphi^+_\xi)}{\sin(\varphi^+_\xi)} + \frac{\sin((2n+1)\varphi^-_\xi)}{\sin(\varphi^-_\xi)} \right] n^{-\alpha-1} + O(n^{-\alpha-2})
\]

\[
= - \frac{I_1(\alpha, \xi)}{4} \left[ U_{2n}(\cos(\varphi^+_\xi)) + U_{2n}(\cos(\varphi^-_\xi)) \right] n^{-\alpha-1} + O(n^{-\alpha-2}). \tag{2.10}
\]

Now we turn to the case where \( x = \xi \). It is easily seen that \( \varphi^+_\xi = \arccos(\xi) \) and \( \varphi^-_\xi = 0 \). Hence, applying Lemma 2.2 to (2.9) again, we obtain that

\[
f(x) - f_n(x) = \frac{I_1(\alpha, \xi)}{2} \left[ \Psi^C_{\alpha}(2\arccos(\xi), n) + \Psi^C_{\alpha}(0, n) \right] + O(n^{-\alpha-2})
\]

\[
= \frac{I_1(\alpha, \xi)}{2\alpha} n^{-\alpha} - I_1(\alpha, \xi) \frac{U_{2n}(\xi) + 1}{4} n^{-\alpha-1} + O(n^{-\alpha-2}). \tag{2.11}
\]

This proves (i). The pointwise error estimates of \( f_n \) for the case \( \xi = \pm 1 \) can be analyzed in a similar way. In the case where \( \xi = 1 \), we see that \( \varphi^+_\xi = \varphi^-_\xi = \arccos(x)/2 \in [0, \pi/2] \).
Applying Lemma 2.2 again we obtain
\[
 f(x) - f_n(x) = B(\alpha, \xi) \begin{cases}
 U_{2n}(\cos(\varphi_{\xi}^{\pm})) & n^{-2\alpha-1} + O(n^{-2\alpha-2}), \ x \neq \xi, \\
 \frac{1}{2\alpha} n^{-2\alpha} - \frac{1}{2} n^{-2\alpha-1} + O(n^{-2\alpha-2}), & x = \xi.
\end{cases}
\] (2.12)

In the case where \( \xi = -1 \), we see that \( \varphi_{\xi}^{\pm} = (\arccos(x) \pm \pi)/2 \) and \( \varphi_{\xi}^{\pm} \in [\pi/2, \pi] \) and \( \varphi_{\xi}^{\pm} \in [-\pi/2, 0] \), and therefore
\[
 f(x) - f_n(x) = B(\alpha, \xi) \begin{cases}
 U_{2n}(\cos(\varphi_{\xi}^{-})) & n^{-2\alpha-1} + O(n^{-2\alpha-2}), \ x \neq \xi, \\
 \frac{1}{2\alpha} n^{-2\alpha} - \frac{1}{2} n^{-2\alpha-1} + O(n^{-2\alpha-2}), & x = \xi.
\end{cases}
\] (2.13)

This proves (ii) and the proof of Theorem 2.5 is complete.

Remark 2.6. In case of \( \xi \in (-1, 1) \), from (2.10) and (2.11) we see that the rate of convergence of \( f_n \) at \( x \neq \xi \) is \( O(n^{-\alpha-1}) \), which is one power of \( n \) faster than the rate of convergence of \( f_n \) at \( x = \xi \). This gives a clear explanation for the error curves of Chebyshev projections displayed in Figure 1. On the other hand, it is well known that the rate of convergence of \( p_n^* \) in this case is \( O(n^{-\alpha}) \) and, owing to the Chebyshev equioscillation Theorem, this can be attained at least at \( n + 2 \) points on \( \Omega \). Hence, we can deduce that \( f_n \) and \( p_n^* \) converge at the same rate whenever \( x \) is close to \( \xi \) and \( f_n \) converges one power of \( n \) faster than \( p_n^* \) whenever \( x \) is away from \( \xi \). Moreover, similar conclusions also hold for the case \( \xi = \pm 1 \).

Remark 2.7. Bernstein in [3, 4] initiated the study of the error of the best approximation to the function \( f(x) = \left| x \right|^\alpha \) and he proved that the limit
\[
 \lim_{n \to \infty} n^\alpha \|f - p_n^*\|_{L^\infty(\Omega)} := \mu(\alpha)
\]
exists for each \( \alpha > 0 \) and \( \alpha \neq 2, 4, 6, \ldots \). However, an explicit formula for \( \mu(\alpha) \) is still unknown. In the case \( \alpha = 1 \), it has been shown in [26] that \( \mu(\alpha) = 0.280169499 \ldots \) by using high-precision calculations. An immediate question that comes to mind is what is the corresponding result for Chebyshev projections. To address this, from Theorem 2.5 we can deduce immediately that
\[
 \lim_{n \to \infty} n^\alpha \|f - f_n\|_{L^\infty(\Omega)} = \frac{2\Gamma(\alpha)}{\pi} \left| \sin \left( \frac{\alpha\pi}{2} \right) \right|. \quad (2.14)
\]
In the case \( \alpha = 1 \), i.e., \( f(x) = |x| \), we see that \( \lim_{n \to \infty} n^\alpha \|f - f_n\|_{L^\infty(\Omega)} = 2/\pi \), and hence the maximum error of \( p_n^* \) is better than that of \( f_n \) by a factor of about 2.27 as \( n \to \infty \).
3 Extensions

In this section we present several extensions of our previous analysis which might be interesting either in theory or in practice.

3.1 Superconvergence points of Chebyshev projections

Superconvergence theory has received much attention in diverse areas ranging from h-version of finite element methods to collocation methods for Volterra integral equations (see, e.g., [7, 27]). In the case of spectral methods, however, only a few studies had been conducted in the literature. For instance, superconvergence points of spectral interpolation for functions analytic in a region containing Ω have been studied in [31, 34] and it was shown that the rate of convergence of Chebyshev and Jacobi-type interpolants can be improved by some algebraic factors at their superconvergence points. For functions with limited regularity, however, the rate of convergence of these spectral interpolants at their superconvergence points can not be improved anymore. In the following, we shall extend our discussion to the superconvergence points of Chebyshev projections. We show that superconvergence points of Chebyshev projections exist for the model function (1.4) and the rate of convergence of Chebyshev projections at their superconvergence points can be improved significantly.

![Graphs showing pointwise error curves and the error at the superconvergent points](image)

Figure 3: The pointwise error curves and the error at the superconvergent points (dots) for $n=10$ (left) and $n=50$ (right). Here $f(x) = |x - 0.25|^{5/2}e^x$.

We start from the case $\xi \in (-1, 1)$. Firstly, from Theorem [25] we know that the maximum error of $f_n$ on $\Omega$ satisfies $\|f - f_n\|_{L^\infty(\Omega)} = O(n^{-\alpha})$ as $n \to \infty$ and this error estimate is attained by the error at the singularity $x = \xi$. To obtain superconvergence points of Chebyshev projections, we impose the following condition

$$U_{2n}(\cos(\varphi_\xi^+)) + U_{2n}(\cos(\varphi_\xi^-)) = 0,$$

3.1
where, using elementary algebraic manipulations,

\[
\cos(\phi \pm \xi) = \sqrt{(1 + x)(1 + \xi)} \pm \sqrt{(1 - x)(1 - \xi)}
\]

It is easily verified that the left-hand side of (3.1) is a polynomial of degree \(n\), from which we can obtain a set of superconvergence points. Numerical experiments show that the equation (3.1) has \(n\) or \(n - 1\) zeros in \(\Omega\). In Figure 3 we illustrate the error curves of \(f_n\) and the errors at the superconvergent points. We can see that the error of \(f_n\) at the superconvergence points are smaller than the maximum error of \(f_n\), especially when the superconvergence points are away from the singularity. Moreover, we also observe that the larger \(n\), the more superconvergence points distributed in the neighborhood of the singularity \(x = \xi\). Recall that the maximum error of \(f_n\) is always attained at the singularity for moderate and large values of \(n\). Hence, to guarantee the superconvergence property, we introduce a small parameter \(\varepsilon\) which is independent of \(n\) and \(0 < \varepsilon \ll 1\).

Let \(S = \{y_j(\xi)\}\) be the solution set of (3.1), then we define the following subset

\[
S_{\varepsilon} = \{y_j(\xi) \in S \cap \Omega, |y_j(\xi) - \xi| \geq \varepsilon\}.
\]

As a consequence of Theorem 2.5, the rate of convergence of \(f_n\) at the superconvergence points in \(S_{\varepsilon}\) is \(O(n^{-\alpha-2})\), which is much faster than that of \(f_n\) in the maximum norm. In Figure 4 we illustrate the maximum errors of \(f_n\) on \(\Omega\) and the maximum errors of \(f_n\) at the set \(S_{\varepsilon}\) for two values of \(\alpha\). Clearly, we see that the numerical results are in good agreement with our theoretical analysis.

![Figure 4](image.png)

Figure 4: Maximum errors of \(f_n\) (dots) on \(\Omega\) and maximum errors of \(f_n\) at the set \(S_{\varepsilon}\) (circles) for \(\alpha = 2.5\) (left) and \(\alpha = 3.5\) (right). Here \(\xi = 0.25\), \(g(x) = e^x\) and \(\varepsilon = 0.1\).

Now we turn to the case \(\xi = \pm 1\). It is known that the maximum error of \(f_n\) on \(\Omega\) satisfies \(\|f - f_n\|_{L^\infty(\Omega)} = O(n^{-2\alpha})\) and this error estimate is attained by the error at \(x = \xi\). In this case, the superconvergence points can be obtained by imposing the condition
\(U_{2n}(\cos(\varphi_{\xi})) = 0\) whenever \(\xi = -1\) and by imposing the condition \(U_{2n}(\cos(\varphi_{\xi}^+)) = 0\) whenever \(\xi = 1\), from which one can deduce that
\[
y_j(\xi) = \xi \cos \left( \frac{2\pi j}{2n + 1} \right), \quad j = 1, \ldots, n.
\] (3.3)

Moreover, it is easily verified that these points are zeros of \(V_n(x)\) whenever \(\xi = -1\) and of \(W_n(x)\) whenever \(\xi = 1\), where \(V_n(x)\) and \(W_n(x)\) are Chebyshev polynomials of the third and fourth kind of degree \(n\), respectively (see, e.g., [17]). Let \(\varepsilon\) be a parameter independent of \(n\) and \(0 < \varepsilon \ll 1\), we choose the superconvergence points which satisfy
\[
|y_j(\xi) - \xi| \geq \varepsilon \implies j \geq \left\lfloor \frac{2n + 1}{2\pi} \arccos(1 - \varepsilon) \right\rfloor := n_{\varepsilon},
\]
where \(\lfloor \cdot \rfloor\) denotes the integer part. Let \(S_\varepsilon = \{y_j(\xi), \ j = n_\varepsilon, \ldots, n\}\) where \(y_j(\xi)\) is defined in (3.3) and \(n_\varepsilon\) is defined in the above equation. From Theorem 2.5 we can deduce that the rate of convergence of \(f_n\) at the superconvergence points in \(S_\varepsilon\) is \(O(n^{-2a - 2})\), which is also much faster than that of \(f_n\) in the maximum norm. In Figure 5 we illustrate the maximum errors of \(f_n\) on \(\Omega\) and the maximum errors of \(f_n\) at these superconvergence points \(\{y_{n_\varepsilon}(\xi), \ldots, y_n(\xi)\}\) for two different values of \(\alpha\). Clearly, we see that numerical results are in accordance with our theoretical analysis.

![Figure 5: Maximum errors of \(f_n\) (dots) on \(\Omega\) and maximum errors of \(f_n\) at the superconvergence points \(\{y_{n_\varepsilon}(\xi), \ldots, y_n(\xi)\}\) (circles) for two different values of \(\alpha\). Here \(\xi = -1, g(x) = 1/(3 - x)\) and \(\varepsilon = 0.01\).](image)

3.2 Chebyshev interpolants

In this subsection we consider pointwise error estimates of Chebyshev interpolants, i.e., polynomial interpolants at the zeros or extrema of Chebyshev polynomials. As will be shown below, the results are analogous to those of Theorem 2.5 but their derivation is a little more involved.
3.2.1 Chebyshev points of the first kind

Let \( \{ x_j \}_{j=0}^n \) be the Chebyshev points of the first kind, i.e.,
\[
x_j = \cos\left( \frac{j + 1/2}{n+1} \pi \right),
\]
and let \( p_n^I \) be the unique polynomial of degree \( n \) which interpolates \( f \) at \( \{ x_j \}_{j=0}^n \). For \( k = 0, \ldots, n \), by straightforward calculations, one can verify that
\[
\sum_{i=0}^n T_j(x_i)T_k(x_i) = (-1)^{\ell} \begin{cases} 
  n+1, & k = 0, \ j = 2\ell(n+1) + \ell, \ \ell \in \mathbb{N}_0, \\
  \frac{n+1}{2}, & k \in \mathcal{U}_n, \ j = 2\ell(n+1) + k, \ \ell \in \mathbb{N}_0, \\
  \frac{n+1}{2}, & k \in \mathcal{U}_n, \ j = 2\ell(n+1) - k, \ \ell \in \mathbb{N}, \\
  0, & \text{otherwise},
\end{cases}
\]
where \( \mathcal{U}_n = \{1, \ldots, n\} \). Based on the discrete orthogonality, the Chebyshev interpolant \( p_n^I(x) \) can be written explicitly as
\[
p_n^I(x) = \sum_{k=0}^n b_k T_k(x), \quad b_k = \frac{2}{n+1} \sum_{j=0}^n f(x_j) T_k(x_j).
\]

It is known that the coefficients \( \{b_0, \ldots, b_n\} \) can be computed rapidly by using the FFT.

We now turn to the task of analyzing the pointwise error estimate of \( p_n^I(x) \) and the key ingredient of our analysis is the aliasing formula of Chebyshev coefficients. Specifically, plugging the Chebyshev series (1.1) into \( b_k \) and applying (3.4), we obtain
\[
b_k = \frac{2}{n+1} \sum_{\ell=0}^\infty a_{\ell} \sum_{j=0}^n T_{\ell}(x_j) T_k(x_j) = a_k + \sum_{\ell=1}^\infty (-1)^{\ell} \left( a_{2\ell(n+1)-k} + a_{2\ell(n+1)+k} \right).
\]
Combining this with (1.1) and (3.5), we get
\[
f(x) - p_n^I(x) = \frac{a_0 - b_0}{2} + \sum_{k=1}^n (a_k - b_k) T_k(x) + \sum_{k=n+1}^\infty a_k T_k(x)
= \sum_{k=n+1}^\infty a_k \left( T_k(x) + \nu(k) T_{\eta(k)}(x) \right),
\]
where \( \eta(k) = |(k + n)(\text{mod } 2(n+1)) - n| \) and
\[
\nu(k) = \begin{cases} 
  0, & k = (2\ell + 1)(n+1), \ \ell \in \mathbb{N}_0, \\
  (-1)^{\lfloor(k-n-1)/(2n+2)\rfloor}, & \text{otherwise}.
\end{cases}
\]
It is easily seen that \( \{\eta(k)\} \) is an \((2n+2)\)-periodic sequence, i.e., \( \eta(k) = \eta(k + 2n + 2) \) for each \( k \geq n + 2 \). In the following, for simplicity of notation, we set \( \theta = \arccos(x) \in \frac{\pi}{2} \).
By using the periodic property of \( \{ \eta(k) \} \), and after somewhat tedious but quite elementary calculations, we arrive at

\[
f(x) - p_n^1(x) = \sum_{k=n+1}^{\infty} a_k \cos(k\theta) + \sum_{\ell=1}^{\infty} \cos(2(2\ell - 1)(n + 1)\theta) \sum_{k=2(2\ell - 1)(n+1)-n}^{(2\ell - 1)(n+1)+n} a_k \cos(k\theta) \\
+ \sum_{\ell=1}^{\infty} \sin(2(2\ell - 1)(n + 1)\theta) \sum_{k=2(2\ell - 1)(n+1)-n}^{(2\ell - 1)(n+1)+n} a_k \sin(k\theta) \\
- \sum_{\ell=1}^{\infty} \cos(4\ell(n+1)\theta) \sum_{k=4\ell(n+1)-n}^{4\ell(n+1)+n} a_k \cos(k\theta) \\
- \sum_{\ell=1}^{\infty} \sin(4\ell(n+1)\theta) \sum_{k=4\ell(n+1)-n}^{4\ell(n+1)+n} a_k \sin(k\theta).
\] (3.7)

Notice that the first sum on the right-hand side of (3.7) is exactly the remainder term of \( f_n \), its estimate follows immediately from Theorem 2.5. For the last four sums on the right-hand side of (3.7), their asymptotic estimates follow by means of Lemma 2.1 and Lemma 2.2. We omit the details of the derivation here since it is lengthy but similar to that of Theorem 2.5, and give the final results below: Let \( \epsilon > 0 \) be sufficiently small.

(i) If \( \xi \in (-1, 1) \), then for \( x \neq x_j \) we have

\[
|f(x) - p_n^1(x)| = \left\{ \begin{array}{ll}
O(n^{-\alpha}), & \text{if } |x - \xi| < \epsilon, \\
O(n^{-\alpha-1}), & \text{otherwise}.
\end{array} \right.
\] (3.8)

(ii) If \( \xi = \pm 1 \), then for \( x \neq x_j \) we have

\[
|f(x) - p_n^1(x)| = \left\{ \begin{array}{ll}
O(n^{-2\alpha}), & \text{if } |x - \xi| < \epsilon, \\
O(n^{-2\alpha-1}), & \text{otherwise}.
\end{array} \right.
\] (3.9)

Here we remark that the leading coefficients of the error estimates of \( p_n^1(x) \) involve rather lengthy expression and therefore are omitted. From the above error estimates, we can see clearly that the pointwise error estimates of \( p_n^1(x) \) are quite similar to that of \( f_n \); see Figure 6 for an illustration.

**Remark 3.1.** The accuracy of \( p_n^1(x) \) and \( p_n^* \) was compared in [14] from the point of view of measuring their maximum errors and it was shown that \( p_n^* \) is better than \( p_n^1(x) \) by only a fractional bit in computing elementary functions. Here, our results (3.8) and (3.9) provide a more thorough insight into the comparison of \( p_n^1(x) \) and \( p_n^* \), that is, \( p_n^1(x) \) is actually better than \( p_n^* \) by one power of \( n \) except at the singularity.
3.2.2 Chebyshev points of the second kind

Let \( \{x_j\}_{j=0}^n \) be the set of Chebyshev points of the second kind (also known as Chebyshev-Lobatto points or Clenshaw-Curtis points), i.e., \( x_j = \cos(j \pi / n) \), and let \( p_n^{II}(x) \) be the polynomial of degree \( n \) which interpolates \( f(x) \) at the Chebyshev points of the second kind. It is straightforward to verify that

\[
\sum_{i=0}^{n} w T_j(x_i) T_k(x_i) = \begin{cases} 
  n, & (j + k) \in \mathcal{V}_n, \ |j - k| \in \mathcal{V}_n, \\
  n/2, & (j + k) \not\in \mathcal{V}_n, \ |j - k| \in \mathcal{V}_n, \\
  n/2, & (j + k) \in \mathcal{V}_n, \ |j - k| \not\in \mathcal{V}_n, \\
  0, & (j + k) \not\in \mathcal{V}_n, \ |j - k| \not\in \mathcal{V}_n,
\end{cases}
\]  

(3.10)

where the double prime indicates that both the first and last terms of the summation are to be halved and \( \mathcal{V}_n = \{ k : k \in \mathbb{N}_0 \text{ and } k(\text{mod } 2n) = 0 \} \), the Chebyshev interpolant \( p_n^{II}(x) \) can be written explicitly as

\[
p_n^{II}(x) = \sum_{k=0}^{n} c_k T_k(x), \quad c_k = \frac{2}{n} \sum_{j=0}^{n} w f(x_j) T_k(x_j).
\]  

(3.11)

Moreover, it is well known that these coefficients \( \{c_0, \ldots, c_n\} \) can be computed rapidly by using the FFT in only \( O(n \log n) \) operations. Now we turn to the error estimate of \( p_n^{II}(x) \) and the derivation is similar to that of \( p_n^{I}(x) \). Plugging the Chebyshev series into \( c_k \) and applying the discrete orthogonality (3.10), we obtain

\[
c_k = \frac{2}{n} \sum_{\ell=0}^{\infty} a_\ell \sum_{j=0}^{n} w T_\ell(x_j) T_k(x_j) = a_k + \sum_{\ell=1}^{\infty} (a_{2\ell n - k} + a_{2\ell n + k}).
\]

The combination of the last equality with (3.11) gives

\[
f(x) - p_n^{II}(x) = \sum_{k=0}^{\infty} a_k T_k(x) - \sum_{k=0}^{n} c_k T_k(x) = \sum_{k=n+1}^{\infty} a_k \left( T_k(x) - T_{\psi(k)}(x) \right),
\]  

(3.12)

where \( \psi(k) = |(k + n - 1)(\text{mod } 2n) - (n - 1)| \). Moreover, it is easy to verify that \( \{\psi(k)\} \) is an \( 2n \)-periodic sequence, i.e., \( \psi(k) = \psi(k + 2n) \) for each \( k \geq n + 1 \). Combining the last equality with (3.12), and after some lengthy calculations, we arrive at

\[
f(x) - p_n^{II}(x) = \sum_{k=n+1}^{\infty} a_k \cos(k \theta) - \sum_{\ell=1}^{\infty} \cos(2\ell n \theta) \sum_{k=(2\ell-1)n+1}^{(2\ell+1)n} a_k \cos(k \theta)
\]

\[
- \sum_{\ell=1}^{\infty} \sin(2\ell n \theta) \sum_{k=(2\ell-1)n+1}^{(2\ell+1)n} a_k \sin(k \theta),
\]  

(3.13)
where $\theta = \arccos(x)$. Notice that the first sum is exactly the remainder term of $f_n$, its estimate follows immediately from Theorem 2.5. For the last two sums on the right-hand side of (3.13), their asymptotic estimates can be obtained using Lemma 2.1 and Lemma 2.2. Here we omit the details of the derivation and give the final results below: Let $\epsilon > 0$ be sufficiently small.

(i) If $\xi \in (-1, 1)$, then we have for $x \neq x_j$ that
\[
|f(x) - p^{II}_n(x)| = \begin{cases} 
O(n^{-\alpha}), & \text{if } |x - \xi| < \epsilon, \\
O(n^{-\alpha - 1}), & \text{otherwise}.
\end{cases}
\] (3.14)

(ii) If $\xi = \pm 1$, then we have for $x \neq x_j$ that
\[
|f(x) - p^{II}_n(x)| = \begin{cases} 
O(n^{-2\alpha}), & \text{if } |x - \xi| < \epsilon \text{ and } x \neq \xi, \\
O(n^{-2\alpha - 1}), & \text{otherwise}.
\end{cases}
\] (3.15)

Similar to the case of $p^{I}_n(x)$, the leading coefficients of the error estimates of $p^{II}_n(x)$ also involve rather lengthy expression and therefore are omitted. Moreover, from the above error estimates, we see that the pointwise error estimates of $p^{II}_n(x)$ are similar to that of $f_n$ and $p^{I}_n(x)$; see Figure 6. As a final remark, we point out that our results (3.14) and (3.15) give a clear theoretical explanation for the error behavior of $p^{II}_n(x)$, which was observed by Trefethen in [24, Myth 3].

Figure 6: Pointwise error curves of $p^{I}_n(x)$, $p^{II}_n(x)$ and $p^{*}_n(x)$ for $f(x) = |x + 0.5|^3 e^x$ with $n = 10$ (left) and $n = 100$ (right).

### 3.3 Chebyshev spectral differentiation

In this subsection, we extend our discussion to the pointwise error estimate of Chebyshev spectral differentiation. Throughout this subsection, we always assume that $\alpha > 1$. 

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We start from the case of $\xi \in (-1, 1)$. Recalling the fact that $T_k'(x) = kU_{k-1}(x)$ for $k \in \mathbb{N}$ and using Lemma 2.1, we obtain for $x \in (-1, 1)$ that
\[
f'(x) - f_n'(x) = I_1(\alpha, \xi) \sum_{k=n+1}^{\infty} \frac{T_k(\xi)U_{k-1}(x)}{k^\alpha} + \ldots
\]
\[
= \frac{I_1(\alpha, \xi)}{2\sqrt{1-x^2}} \sum_{k=n+1}^{\infty} \frac{\sin(2k\varphi_+^\xi) + \sin(2k\varphi_-^\xi)}{k^\alpha} + \ldots,
\]
where $\varphi_+^\xi$ and $\varphi_-^\xi$ are defined as in (2.1). Applying Lemma 2.2 to the leading term on the right-hand side we immediately deduce that $f'(x) - f_n'(x) = O(n^{-\alpha})$ for each $x \in (-1, 1)$. For $x = \pm 1$, recalling the fact that $U_k(\pm 1) = (\pm 1)^k(k + 1)$ for all $k \in \mathbb{N}_0$ we obtain
\[
f'(\pm 1) - f_n'(\pm 1) = \frac{I_1(\alpha, \xi)}{2\sqrt{1-x^2}} \sum_{k=n+1}^{\infty} \frac{(\pm 1)^{k-1}T_k(\xi)}{k^{\alpha-1}} + \ldots = O(n^{1-\alpha}),
\]
where we have used Lemma 2.2 in the last step. We summarize the final results on the pointwise error estimates of Chebyshev spectral differentiation below
\[
|f'(x) - f_n'(x)| = \begin{cases} O(n^{-\alpha}), & x \in (-1, 1), \\ O(n^{1-\alpha}), & x = \pm 1. \end{cases}
\]
(3.16)
The pointwise error estimates for the case $\xi = \pm 1$ can be performed in a similar fashion. Here we omit the details of the derivation, giving only the final pointwise error estimates:
\[
|f'(x) - f_n'(x)| = \begin{cases} O(n^{-2\alpha}), & x \in (-1, 1), \\ O(n^{1-2\alpha}), & x = -\xi, \\ O(n^{2-2\alpha}), & x = \xi. \end{cases}
\]
(3.17)
From (3.16) and (3.17) we can conclude that the rate of convergence of Chebyshev spectral differentiation in the maximum norm is $\|f' - f_n'\|_{L^\infty(\Omega)} = O(n^{1-\alpha})$ whenever $\xi \in (-1, 1)$ and is $\|f' - f_n'\|_{L^\infty(\Omega)} = O(n^{2-2\alpha})$ whenever $\xi = \pm 1$. Moreover, the maximum error of Chebyshev spectral differentiation is attained at one of the endpoints whenever $\xi \in (-1, 1)$ and at the singularity $\xi$ whenever $\xi = \pm 1$. In Figure 7 we illustrate the pointwise absolute errors of differentiation using $f_n$ and $p_n^*$ and the rates of convergence of spectral differentiation using $f_n$ and $p_n^*$ in the maximum norm. Clearly, we see that numerical results of Chebyshev spectral differentiation are in consistent with our analysis. Moreover, concerning the accuracy of spectral differentiation using $f_n$ and $p_n^*$ in the maximum norm, we see that the former is much better than the latter. On the other hand, concerning the rates of pointwise convergence of spectral differentiation using $f_n$ and $p_n^*$, we see that the former is also better than the latter except in a small neighborhood of the singularity.
Figure 7: Top row shows $|f' - f_n'|$ and $|f' - p_n'*|$ for $f(x) = |x - 0.25|^3 e^{|x|}$ with $n = 20$ (left) and $n = 40$ (right), and these points indicate the errors of $f_n'$ and $p_n'$ at both endpoints. Bottom row shows $\|f' - f_n'\|_{L^\infty(\Omega)}$ (dots) and $\|f' - p_n'\|_{L^\infty(\Omega)}$ (circles) for $f(x) = |x - 0.25|^\alpha e^{|x|}$ with $\alpha = 3$ (left) and $\alpha = 5$ (right). The dash lines indicate $O(n^{2-\alpha})$ (upper) and $O(n^{1-\alpha})$ (bottom).

3.4 Other orthogonal projections

The error estimate of other orthogonal projections such as Legendre, Gegenbauer and Jacobi projections has attracted attention in recent years (see, e.g., [1, 16, 29, 30, 32, 33]). In [29, Figure 3] it has been observed that the error curves of Legendre projections illustrate similar character as that of Chebyshev projections. In the case of Gegenbauer and Jacobi projections, however, the situation is slightly more complicated since the errors at both endpoints may dominate the maximum error (see [30]). In the following, we restrict our attention to the case of Legendre projections and show the intrinsic connection between the error of Legendre projections and the two functions $\Phi_C^\nu(x, n)$ and $\Phi_S^\nu(x, n)$ defined in (1.5).

Let $f$ be the model function defined in (1.4) and let $P_n(x)$ be its Legendre projection
of degree $n$, i.e.,
\[ P_n(x) = \sum_{k=0}^{n} a_k^f P_k(x), \quad a_k^f = \frac{2k + 1}{2} \int_{-1}^{1} f(x)P_k(x)dx, \tag{3.18} \]
where $P_k(x)$ is the Legendre polynomial of degree $k$. By taking the Taylor series of $g(x)$ at $x = \xi$ and using \[11\] Equation (7.232.3), we obtain for the asymptotic expansion of the Legendre coefficients
\[ a_k^f = \mathcal{E}(\alpha, \xi)\frac{\lambda_1(\xi)T_k(\xi) + \lambda_2(\xi)(1 - \xi^2)^{1/2}U_{k-1}(\xi)}{\kappa^{\alpha+1/2}} + O(k^{-\alpha-3/2}), \tag{3.19} \]
where $\lambda_j(\xi) = (1 + \xi)^{1/2} + (\xi)_{j+1}^{-1}(1 - \xi)^{1/2}$ and
\[ \mathcal{E}(\alpha, \xi) = -\sqrt{\frac{2}{\pi}}(1 - \xi^2)\alpha/2 + 1/4 \sin \left( \frac{\alpha\pi}{2} \right) \Gamma(\alpha + 1)g(\xi). \]

On the other hand, recall from \[20\] Theorem 8.21.2 that
\[ P_k(x) = \sqrt{\frac{2}{\pi}}k^{-1/4}(1 - x^2)^{-1/4} \cos \left( \frac{2k + 1}{2} \arccos(x) - \frac{\pi}{4} \right) + O(k^{-3/2}) \]
\[ = \sqrt{\frac{1}{2\pi k}} \left[ \lambda_1(\xi)T_k(\xi) + \lambda_2(\xi)(1 - x^2)^{1/2}U_{k-1}(\xi) \right] + O(k^{-3/2}), \tag{3.20} \]
where the asymptotic formula on the right-hand side of (3.20) holds uniformly on the interval $[-1 + \varepsilon, 1 - \varepsilon]$, where $\varepsilon > 0$ is small. Let us now consider the pointwise error estimate of $P_n(x)$ and we restrict the analysis to the case where $\xi \in (-1, 1)$ since the case $\xi = \pm 1$ can be treated in a similar way. Combining (3.19) and (3.20) and after some elementary calculations, we obtain for $x \in (-1, 1)$ that
\[ f(x) - P_n(x) = \frac{\mathcal{E}(\alpha, \xi)}{\sqrt{2\pi}(1 - x^2)^{1/4}} \left[ \frac{\lambda_1(\xi)\lambda_1(\xi)}{2} \sum_{k=n+1}^{\infty} \frac{\cos(2k\varphi^+_\xi) + \cos(2k\varphi^-_\xi)}{k^{\alpha+1}} ight. \\
+ \frac{\lambda_1(\xi)\lambda_2(\xi)}{2} \sum_{k=n+1}^{\infty} \frac{\sin(2k\varphi^+_\xi) + \sin(2k\varphi^-_\xi)}{k^{\alpha+1}} \\
+ \frac{\lambda_1(\xi)\lambda_2(\xi)}{2} \sum_{k=n+1}^{\infty} \frac{\sin(2k\varphi^-_\xi) - \sin(2k\varphi^+_\xi)}{k^{\alpha+1}} + O(n^{-\alpha-2}). \tag{3.21} \]

Clearly, we see that the leading term of the error of $P_n(x)$ can be represented as a linear combination of $\Phi^C_n(x, n)$ and $\Phi^S_n(x, n)$. Furthermore, applying Lemma \[22\] to (3.21), we can deduce that the term inside the square bracket on the right-hand side of the last
equality behaves like $O(n^{-\alpha})$ whenever $x$ is away from $\xi$ and behaves like $O(n^{-\alpha-1})$ whenever $x$ is close to $\xi$. For $x = \pm 1$, combining the fact that $P_k(\pm 1) = (\pm 1)^k$ for all $k \in \mathbb{N}_0$ as well as Lemma 2.2 and (3.19) yields

$$f(\pm 1) - P_n(\pm 1) = \sum_{k=n+1}^{\infty} (\pm 1)^k a_k^L = \mathcal{E}(\alpha, \xi) \left( \lambda_1(\xi) \sum_{k=n+1}^{\infty} \frac{(\pm 1)^k T_k(\xi)}{k^{\alpha+1/2}} + \lambda_2(\xi) \sum_{k=n+1}^{\infty} \frac{(\pm 1)^k (1 - \xi^2)^{1/2} U_{k-1}(\xi)}{k^{\alpha+1/2}} \right) + \cdots. \quad (3.22)$$

We see that the leading term on the right-hand side of the last equality can also be represented by $\Phi_C^\nu(x, n)$ and $\Phi_S^\nu(x, n)$, and using Lemma 2.2 we easily obtain $f(\pm 1) - P_n(\pm 1) = O(n^{-\alpha-1/2})$. Hence, we can deduce that the pointwise error estimate of $P_n(x)$ is

$$|f(x) - P_n(x)| = \begin{cases} 
O(n^{-\alpha}), & \text{if } |x - \xi| < \epsilon, \\
O(n^{-\alpha-1/2}), & \text{if } |x \pm 1| < \epsilon, \\
O(n^{-\alpha-1}), & \text{otherwise},
\end{cases} \quad (3.23)$$

where $\epsilon > 0$ is small. Comparing the rate of pointwise convergence, we see that $P_n(x)$ is better than $p_n^*(x)$ except at a small neighborhood of the singularity. On the other hand, if we compare the rate of convergence of $P_n(x)$ and $p_n^*(x)$ in the maximum norm, it is easily seen that both methods converge at the same rate and this has been discussed in [29].

4 Concluding remarks

This work presents a thorough analysis of the pointwise error estimate of Chebyshev projections for functions with an algebraic singularity. A key finding is that there exists an intrinsic connection between the error of classical orthogonal projections (Chebyshev, Legendre and Jacobi projections) and the two functions $\Phi_C^\nu(x, n)$ and $\Phi_S^\nu(x, n)$ defined in (1.5). This connection allows us to understand more thoroughly the error behavior of classical orthogonal projections, such as why the accuracy of Chebyshev projections is better than that of best approximations except for a small neighborhood of the singularity.

We also extend our discussion to superconvergence points of Chebyshev projections and the pointwise error estimate of Chebyshev spectral differentiation. In the latter case, we show that Chebyshev spectral differentiation is actually much better than spectral differentiation using best approximations whenever we are concerned with the maximum error of spectral differentiation. If we concern the pointwise error of spectral differentiation, the performance of Chebyshev spectral differentiation is still better than its best
counterpart except at a small neighborhood of the singularity. Our results provide new theoretical justification for the use of Chebyshev, Legendre and other spectral approximations in practice.

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