ON CHAINS ASSOCIATED WITH ABSTRACT KEY POLYNOMIALS

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Abstract. In this paper, for a henselian valued field \((K, v)\) of arbitrary rank and an extension \(w\) of \(v\) to \(K(X)\), we use abstract key polynomials for \(w\) to give a connection between complete sets, saturated distinguished chains and Okutsu frames. Further, for a valued field \((K, v)\), we also obtain a close connection between complete set of ABKPs for \(w\) and Maclane-Vaquié chains of \(w\).

1. Introduction

Let \((K, v)\) be a henselian valued field and \(w\) be an extension of \(v\) to \(K(X)\). In this paper, we first give some conditions under which a saturated distinguished chain leads to the notion of a complete set of abstract key polynomials for a valuation-transcendental extension \(w\). Recall that a valuation-transcendental extension of \(v\) to \(K(X)\) is either value-transcendental or residually transcendental and they are well studied using abstract key polynomials (see [10], [16]-[19]).

In 1982, Okutsu associated to a monic irreducible polynomial \(F \in K[X]\) a family of monic irreducible polynomials, \(F_1, \ldots, F_r\), called the primitive divisor polynomials of \(F\) [20], later these polynomials were studied in papers [6], [7], [12] and [13], and they called the chain of such polynomials \([F_1, \ldots, F_r]\), an Okutsu frame for \(F\). Moreover, they proved that Okutsu frames, saturated distinguished chains and optimal Maclane chains are closely related. In this paper, we also establish a similar connection between saturated distinguished chains and Okutsu frames, however, our proof is elementary.

Next, for a valued field \((K, v)\), we give some conditions under which a complete set of ABKPs for a valuation \(w\) of \(K(X)\) give rise to an optimal Maclane chain of \(w\) and conversely. It is also observed that over a residually transcendental extension, the notion of saturated distinguished chains,

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Okutsu frames, optimal Maclane chains and complete set of ABKPs (under certain conditions) are equivalent.

In 1936, Maclane [9], proved that an extension $w$ of a discrete rank one valuation $v$ to $K[X]$ can be obtained as a chain of augmentations

$$w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_n, \gamma_n} w_n \xrightarrow{\phi_{n+1}, \gamma_{n+1}} \cdots$$

for some suitable key polynomials $\phi_i \in K[X]$ for intermediate valuations and elements $\gamma_i$ in some totally ordered abelian group containing the value group of $v$ as an ordered subgroup. Later, Vaquié generalized Maclane’s theory to arbitrary valued fields (see [23]). More recently, Nart gave a survey of generalized Maclane-Vaquié theory in [14] and [15]. Starting with a valuation $w_0$ which admit key polynomials of degree one, Nart also introduced Maclane-Vaquié chains, consisting of a mixture of ordinary and limit augmentations satisfying some conditions (see Definition 2.37). The main result Theorem 4.3 of [15] says that all extensions $w$ of $v$ to $K[X]$ falls exactly in one of the following category:

(i) It is the last valuation of a complete finite Maclane-Vaquié chain, i.e., after a finite number $r$ of augmentation steps, we get $w_r = w$.

(ii) After a finite number $r$ of augmentation steps, it is the stable limit of a continuous family of augmentations of $w_r$ defined by key polynomials of constant degree.

(iii) It is the stable limit of a complete infinite Maclane-Vaquié chain.

In this paper, we study Maclane-Vaquié chains of the first type and prove that a precise complete finite Maclane-Vaquié chain can be obtained using a given complete set $\{Q_i\}_{i \in \Delta}$ of ABKPs for $w$ such that $\Delta$ has a maximal element. Conversely, if $w$ is the last valuation of some complete finite Maclane-Vaquié chain, then there exists a complete set $\{Q_i\}_{i \in \Delta}$ of ABKPs for $w$ such that $\Delta$ has a maximal element.

To state the main results of the paper, we first recall some notations, definitions and preliminary results.

## 2. Notations, Definitions and Statements of Main Results

Throughout the paper, $(K, v)$ denote a valued field of arbitrary rank with value group $\Gamma_v$, valuation ring $O_v$ having a unique maximal ideal $M_v$, and residue field $k_v$. Let $\bar{v}$ be an extension of $v$ to a fixed algebraic closure $\overline{K}$ of $K$ with value group $\Gamma_{\bar{v}}$ and residue field $k_{\bar{v}}$. Let $w$ be an extension of $v$ to the simple transcendental extension $K(X)$ of $K$ with value group $\Gamma_w$ and residue field $k_w$. 
All extensions of \( v \) to \( K(X) \) are classified as follows:

**Definition 2.1.** The extension \( w \) of \( v \) to \( K(X) \) is said to be **valuation-algebraic** if \( \Gamma_w \) is a torsion group and \( k_w \) is algebraic over \( k_v \). The extension \( w \) is said to be **value-transcendental** if \( \Gamma_w \) is a torsion-free group and \( k_w \) is algebraic over \( k_v \).

If \( L \) is an extension field of \( K \), then an extension \( v_L \) of \( v \) to \( L \) is called residually transcendental (abbreviated as r. t.) if the corresponding residue field extension \( k_{v_L}/k_v \) is transcendental.

**Definition 2.2.** The extension \( w \) of \( v \) to \( K(X) \) is called **valuation-transcendental** if \( w \) is either value-transcendental or is residually transcendental.

Let \( \bar{w} \) be a common extension of \( w \) and \( \bar{v} \) to \( K(X) \). Then for a pair \( (\alpha, \delta) \in K \times \Gamma_{\bar{w}} \), the map \( \bar{w}_{\alpha,\delta} : K[X] \rightarrow \Gamma_{\bar{w}} \), given by

\[
\bar{w}_{\alpha,\delta} \left( \sum_{i \geq 0} c_i (X - \alpha)^i \right) := \min \{ \bar{v}(c_i) + i\delta \}, \quad c_i \in \overline{K},
\]

is a valuation on \( \overline{K}[X] \) and can be uniquely extended to \( \overline{K}(X) \) (cf. [5, Theorem 2.2.1]). Such a valuation is said to be defined by \( \min, \bar{v}, \alpha \) and \( \delta \). If \( \bar{w} = \bar{w}_{\alpha,\delta} \), then we say that \( (\alpha, \delta) \) is a pair of definition for \( w \).

**Definition 2.3.** A pair \( (\alpha, \delta) \) in \( K \times \Gamma_{\bar{w}} \) is called a **minimal pair** of definition for \( w \) if \( \bar{w} = \bar{w}_{\alpha,\delta} \) and for every \( \beta \) in \( \overline{K} \), satisfying \( \bar{v}(\alpha - \beta) \geq \delta \), we have \( \deg \beta \geq \deg \alpha \), where by \( \deg \alpha \) we mean the degree of the extension \( K(\alpha)/K \).

**Remark 2.4.** In the above definition, if \( \Gamma_{\bar{w}} = \Gamma_{\bar{v}} \), then the minimal pair \( (\alpha, \delta) \) is called a **(K,v)-minimal pair**.

Let \( (K, v) \) be a henselian valued field. If \( \theta \in K \), then \( (\theta, \delta) \) is a minimal pair for each \( \delta \in \Gamma_{\bar{v}} \) and it is immediate from the definition that a pair \( (\theta, \delta) \in (\overline{K} \setminus K) \times \Gamma_{\bar{v}} \) is minimal if and only if \( \delta \) is strictly greater than each element of the set \( M(\theta, K) \) defined by

\[
M(\theta, K) := \{ \bar{v}(\theta - \beta) \mid \beta \in \overline{K}, \deg \beta < \deg \theta \}.
\]

This led to the notion of main invariant

\[
\delta_K(\theta) := \max M(\theta, K)
\]
defined for those \( \theta \in \overline{K} \setminus K \) for which the set \( M(\theta, K) \) contains a maximum. In general, this maximum value may not exist. However, in 2002, Aghigh and Khanduja gave some necessary and sufficient condition under which the set \( M(\theta, K) \) has a maximum element for every \( \theta \in \overline{K} \setminus K \) (see [1, Theorem 1.1]). We now recall the notion of distinguished pairs which was introduced by Popescu and Zaharescu [22], for local fields in 1995 and was later generalized to arbitrary henselian valued fields ([1] and [3]).

**Definition 2.5 (Distinguished pairs).** A pair \((\theta, \alpha)\) of elements of \( \overline{K} \) is called a \((K, v)\)-distinguished pair if the following conditions are satisfied:

(i) \( \deg \theta > \deg \alpha \),
(ii) \( \bar{v}(\theta - \alpha) = \max \{ \bar{v}(\theta - \beta) \mid \beta \in \overline{K}, \deg \beta < \deg \theta \} = \delta_K(\theta) \),
(iii) if \( \eta \in \overline{K} \) be such that \( \deg \eta < \deg \alpha \), then \( \bar{v}(\theta - \eta) < \bar{v}(\theta - \alpha) \).

Equivalently, we say that \((\theta, \alpha)\) is a distinguished pair, if \( \alpha \) is an element in \( \overline{K} \) of minimal degree over \( K \) such that

\[
\bar{v}(\theta - \alpha) = \delta_K(\theta).
\]

Clearly (iii) implies that \((\alpha, \bar{v}(\theta - \alpha))\) is a \((K, v)\)-minimal pair. Also for any two monic irreducible polynomials \( f \) and \( g \) over \( K \), we call \((g, f)\) a distinguished pair, if there exists a root \( \theta \) of \( g \) and a root \( \alpha \) of \( f \) such that \((\theta, \alpha)\) is a \((K, v)\)-distinguished pair. Distinguished pairs give rise to distinguished chains in a natural manner. A chain \( \theta = \theta_r, \theta_{r-1}, \ldots, \theta_0 \) of elements of \( \overline{K} \) is called a saturated distinguished chain for \( \theta \) of length \( r \), if \((\theta_i+1, \theta_i)\) is a \((K, v)\)-distinguished pair for every \( 0 \leq i \leq r - 1 \) and \( \theta_0 \in K \).

**Definition 2.6.** Let \( w \) be a valuation of \( K(X) \) and \( \bar{w} \) a fixed common extension of \( w \) and \( \bar{v} \) to \( \overline{K}(X) \). For any polynomial \( f \) in \( K[X] \), we call a root \( \alpha \) of \( f \) in \( \overline{K} \) an optimizing root of \( f \) if

\[
\bar{w}(X - \alpha) = \max \{ \bar{w}(X - \alpha') \mid f(\alpha') = 0 \} = \delta(f).
\]

We call \( \delta(f) \) the optimal value of \( f \) with respect to \( \bar{w} \).

**Definition 2.7 (Abstract key polynomials).** A monic polynomial \( Q \) in \( K[X] \) is said to be an abstract key polynomial (abbreviated as ABKP) for \( w \) if for each polynomial \( f \) in \( K[X] \) with \( \deg f < \deg Q \) we have \( \delta(f) < \delta(Q) \).

It is immediate from the definition that all monic linear polynomials are ABKPs for \( w \). Also an ABKP for \( w \) is an irreducible polynomial (see [16, Proposition 2.4]).
Definition 2.8. For a polynomial $Q$ in $K[X]$ the $Q$-truncation of $w$ is a map $w_Q : K[X] \to \Gamma_w$ defined by

$$w_Q(f) := \min_{0 \leq i \leq n} \{ w(f_i^Q_i) \},$$

where $\sum_{i=0}^n f_i^Q_i$, $\deg f_i < \deg Q$, is the $Q$-expansion of $f$.

The $Q$-truncation $w_Q$ of $w$ need not be a valuation [16, Example 2.5]. However, if $Q$ is an ABKP for $w$, then $w_Q$ is a valuation on $K(X)$ (see [16, Proposition 2.6]). Also any ABKP, $Q$ for $w$, is also an ABKP for the truncation valuation $w_Q$. For an ABKP, $Q$ in $K[X]$, for $w$, we set

$$\alpha(Q) := \min \{ \deg f \mid f \in K[X], w_Q(f) < w(f) \}, \quad \text{(if } w_Q = w, \text{ then } \alpha(Q) := \infty)$$

and

$$\psi(Q) := \{ f \in K[X] \mid f \text{ is monic, } w_Q(f) < w(f) \text{ and } \deg f = \alpha(Q) \}.$$

Clearly $\alpha(Q) \geq \deg Q$. Also, observe that $w_Q$ is a proper truncation of $w$, (i.e., $w_Q < w$) if and only if $\psi(Q) \neq \emptyset$.

Lemma 2.9 (Lemma 2.11, [16]). If $Q$ is an ABKP for $w$, then every element $F \in \psi(Q)$ is also an ABKP for $w$ and $\delta(Q) < \delta(F)$.

Let $Q$ be an ABKP for $w$ and suppose that $Q$ has a saturated distinguished chain. In the following result we prove that each member of this chain is also an ABKP for $w$, further, we also observe that optimal values and main invariants associated with a saturated distinguished chain of polynomials are closely related.

Proposition 2.10. Let $(K,v)$ be a henselian valued field and $\bar{v}$ a unique extension of $v$ to $\overline{K}$. Let $Q$ be an ABKP for a valuation $w$ of $K(X)$. If $(Q_r = Q, Q_{r-1}, \ldots, Q_0)$ is a saturated distinguished chain for $Q$. Then

(i) Each $Q_i$, $0 \leq i \leq r-1$, is an ABKP for $w$.

(ii) $\delta(Q_r) > \delta(Q_{r-1}) > \cdots > \delta(Q_0)$ and for optimizing roots $\theta_i$ of $Q_i$, $0 \leq i \leq r$,

$$\delta_K(\theta_i) = \delta(Q_{i-1}) \quad \forall \ 1 \leq i \leq r.$$

The following result can be easily deduced from Theorem 1.1 of [19] and Theorem 1.1 of [17], and gives a characterization of valuation-transcendental extensions.

Theorem 2.11. An extension $w$ of $v$ to $K(X)$ is valuation-transcendental if and only if $w = w_Q$, for some ABKP, $Q$ for $w$. Moreover, if $\alpha$ is an optimizing root of $Q$, then $(\alpha, \delta(Q))$ is a minimal pair of definition for $w$ and $w = w_Q = w_{\alpha, \delta(Q)}|_{K(X)}$. 
It is known that if \( w \) is a valuation-transcendental extension of \( v \) to \( K(X) \) defined by some minimal pair \((\theta, \delta) \in \overline{K} \times \Gamma_\mathcal{M} \), then the minimal polynomial of \( \theta \) over \( K \) is an ABKP for \( w \) (see [19]). Therefore, keeping this in mind the next result follows immediately from Proposition 2.10.

**Corollary 2.12.** Let \((K, v)\) be a henselian valued field and \( w \) be a valuation-transcendental extension of \( v \) to \( K(X) \) defined by some minimal pair \((\theta, \delta) \in (\overline{K} \setminus K) \times \Gamma_\mathcal{M} \), and let \((\theta = \theta_r, \theta_{r-1}, \ldots, \theta_0)\) be a saturated distinguished chain for \( \theta \). Then the minimal polynomials \( Q_i \) of \( \theta_i \) over \( K \), \( 0 \leq i \leq r \), are ABKPs for \( w \).

**Definition 2.13.** A family \( \Lambda = \{Q_i\}_{i \in \Delta} \) of ABKPs for \( w \) is said to be a complete set of ABKPs for \( w \) if the following conditions are satisfied:

(i) \( \delta(Q_i) \neq \delta(Q_j) \) for every \( i \neq j \in \Delta \).

(ii) \( \Lambda \) is well-ordered with respect to the ordering given by \( Q_i < Q_j \) if and only if \( \delta(Q_i) < \delta(Q_j) \) for every \( i < j \in \Delta \).

(iii) For any \( f \in K[X] \), there exists some \( Q_i \in \Lambda \) such that \( w_{Q_i}(f) = w(f) \).

It is known that [16, Theorem 1.1], every valuation \( w \) on \( K(X) \) admits a complete set of ABKPs. Moreover, there is a complete set \( \Lambda = \{Q_i\}_{i \in \Delta} \) of ABKPs for \( w \) having the following properties (cf. [10, Remark 4.6] and [16, proof of Theorem 1.1]).

**Remark 2.14.** (i) \( \Delta = \bigcup_{j \in I} \Delta_j \) with \( I = \{0, \ldots, N\} \) or \( \mathbb{N} \cup \{0\} \), and for each \( j \in I \) we have \( \Delta_j = \{j\} \cup \vartheta_j \), where \( \vartheta_j \) is an ordered set without a maximal element or is empty.

(ii) \( Q_0 = X \).

(iii) For all \( j \in I \setminus \{0\} \) we have \( j-1 < i < j \), for all \( i \in \vartheta_{j-1} \).

(iv) All polynomials \( Q_i \) with \( i \in \Delta_j \) have the same degree and have degree strictly less than the degree of the polynomials \( Q_{i'} \) for every \( i' \in \Delta_{j+1} \).

(v) For each \( i < i' \in \Delta \) we have \( w(Q_i) < w(Q_{i'}) \) and \( \delta(Q_i) < \delta(Q_{i'}) \).

(vi) Even though the set \( \{Q_i\}_{i \in \Delta} \) of ABKPs for \( w \) is not unique, the cardinality of \( I \) and the degree of an abstract key polynomial \( Q_i \) for each \( i \in I \) are uniquely determined by \( w \).

(vii) The ordered set \( \Delta \) has a maximal element if and only if the following holds:

(a) the set \( I = \{0, \ldots, N\} \) is finite;

(b) \( \Delta_N = \{N\} \), i.e., \( \vartheta_N = \emptyset \).

From now on, we assume that all complete set of ABKPs in this paper satisfy the properties of Remark 2.14.
Keeping in mind the above notations for a complete set \( \Lambda = \{Q_i\}_{i \in \Delta} \) of ABKPs for \( w \) we have:

**Definition 2.15 (Limit key polynomials).** For an element \( i \in \Delta \), we say that \( Q_i \) is a limit key polynomial if the following conditions hold:

(i) \( i \in I \setminus \{0\} \).

(ii) \( \vartheta_i - 1 \neq \emptyset \).

In Theorem 1.23 of [11], it is proved that if \( \{Q_i\}_{i \in \Delta} \) is a complete set of ABKPs for \( w \) with \( \vartheta_i = \emptyset \), for every \( i \in I \), then there exist some \( n \in I \setminus \{0\} \) such that \( Q_n \) has a saturated distinguished chain. In the next result, we show that the converse of this result also holds for a valuation-transcendental extension.

**Theorem 2.16.** Let \((K,v)\) be a henselian valued field and \( w = w_Q \) a valuation-transcendental extension of \( v \) to \( K(X) \). If \( (Q_r = Q, Q_{r-1}, \ldots, Q_0) \) is a saturated distinguished chain for \( Q \), then \( \Lambda = \{Q_0\} \cup \{Q_1\} \cup \cdots \cup \{Q_r\} \) is a complete set of ABKPs for \( w \).

The notion of Okutsu frame was introduced by Okutsu in 1982 for local fields [20], and then studied by Nart in [6], which was later generalized to henselian valued field of arbitrary rank in [12] and [13].

To define Okutsu frames, we first recall some notations. Let \((K,v)\) be a henselian valued field of arbitrary rank. Let \( F \) in \( K[X] \) be a monic irreducible polynomial of degree \( n \) and \( \theta \in \overline{K} \) be a root of \( F \). Consider the sequences:

\[ m_0 = 1 < m_1 < \cdots < m_r = n \]

\[ \mu_0 < \mu_1 < \cdots < \mu_r = \infty, \]

defined in the following recurrent way:

\[ \mu_i := \max\{\bar{v}(\theta - \eta) \mid \eta \in \overline{K}, \deg \eta = m_i\} \quad \text{for every } 0 \leq i \leq r - 1,\]

\[ m_i := \min\{\deg \eta \mid \eta \in \overline{K}, \bar{v}(\theta - \eta) > \mu_{i-1}\} \quad \text{for every } 1 \leq i \leq r - 1.\]

Since \((K,v)\) is henselian, so these values does not depend upon the choice of the root \( \theta \) of \( F \).

**Definition 2.17 (Okutsu frames).** For a monic irreducible polynomial \( F \in K[X] \) having a root \( \theta \in \overline{K} \), let \( \theta_i \in \overline{K} \) be such that \( \deg \theta_i = m_i, \bar{v}(\theta - \theta_i) = \mu_i \), for every \( 0 \leq i \leq r - 1 \). If \( F_i \) is the minimal polynomial of \( \theta_i \) over \( K \), then the chain \([F_0, F_1, \ldots, F_{r-1}]\) of monic irreducible polynomials is called an Okutsu frame of \( F \).
Remark 2.18. It can be shown that the above definition of an Okutsu frame is equivalent to the one given in [12].

In the next result we give a connection between saturated distinguished chains and Okutsu frames. It may be pointed that a similar result is also proved in Theorem 2.6 of [13] and Corollary 3.5 of [7] but our proof is elementary.

**Theorem 2.19.** Let $(K, v)$ be a henselian valued field and $F$ in $K[X]$ be a monic irreducible polynomial having a root $\theta$ in $\overline{K}$. Then $(F = F_r, F_{r-1}, \ldots, F_0)$ is a saturated distinguished chain for $F$ if and only if $[F_0, F_1, \ldots, F_{r-1}]$ is an Okutsu frame of $F$.

We now recall the notion of key polynomials which was first introduced by Maclane in 1936 and later generalized by Vaquié in 2007 (see [9] and [23]).

**Definition 2.20 (Key polynomials).** For a valuation $w$ on $K(X)$ and polynomials $f, g$ in $K[X]$, we say that

(i) $f$ and $g$ are $w$-equivalent and write $f \sim_w g$ if $w(f - g) > w(f) = w(g)$.

(ii) $g$ is $w$-divisible by $f$ or $f$ $w$-divides $g$ (denoted by $f |_w g$) if there exist some polynomial $h \in K[X]$ such that $g \sim_w fh$.

(iii) $f$ is $w$-irreducible, if for any $h, q \in K[X]$, whenever $f |_w hq$, then either $f |_w h$ or $f |_w q$.

(iv) $f$ is $w$-minimal, if for every polynomial $h \in K[X]$, whenever $f |_w h$, then $\deg h \geq \deg f$.

(v) Any monic polynomial $f$ satisfying (iii) and (iv) is called a key polynomial for $w$.

In view of Proposition 2.10 of [4], any ABKP, $Q$ for $w$ is a key polynomial for $w_Q$ of minimal degree. Let $KP(w)$ denote the set of all key polynomials for valuation $w$. For any $\phi \in KP(w)$ we denote by $[\phi]_w$ the set of all key polynomials which are $w$-equivalent to $\phi$.

**Definition 2.21 (Ordinary augmentation).** Let $\phi$ be a key polynomial for a valuation $w$ and $\gamma > w(\phi)$ be an element of a totally ordered abelian group $\Gamma$ containing $\Gamma_w$ as an ordered subgroup. A map $w' : K[X] \rightarrow \Gamma \cup \{\infty\}$ defined by $$w'(f) = \min\{w(f_i) + i\gamma\},$$ where $\sum_{i \geq 0} f_i \phi^i$, $\deg f_i < \deg \phi$, is the $\phi$-expansion of $f \in K[X]$, gives a valuation on $K(X)$ (see [9] Theorem 4.2) called the ordinary augmentation of $w$ (or an augmented valuation) and is denoted by $w' = [w; \phi, \gamma]$. 
Clearly, $w(\phi) < w'(\phi)$ and the polynomial $\phi$ is a key polynomial of minimal degree for the augmented valuation $w'$ (see [14, Corollary 7.3]).

**Definition 2.22.** An extension $w$ of $v$ to $K(X)$ is called *commensurable* if $\Gamma_w / \Gamma_v$ is a torsion group; otherwise it is called *incommensurable*.

Note that, if $\Gamma_w / \Gamma_v$ is a torsion group, then we have a canonical embedding $\Gamma_w \hookrightarrow \Gamma_v \otimes \mathbb{Q}$.

It is known that any incommensurable extension $w$ of $v$, is value-transcendental (cf. [14, Theorem 4.2]). Moreover, if $\phi \in K[X]$ is a monic polynomial of minimal degree such that $w(\phi) \notin \Gamma_v \otimes \mathbb{Q}$, then $\phi$ is a key polynomial for $w$ and the set of all key polynomials for $w$ is given by

$$\{ \phi + g \mid g \in K[X], \deg g < \deg \phi, w(g) > w(\phi) \} = [\phi]_w.$$

In particular, every key polynomial for a value-transcendental extension have the same degree. On the other hand if $w$ is any commensurable extension of $v$ to $K(X)$, then $w$ is either valuation-algebraic or is residually transcendental. In fact any commensurable extension which admits key polynomials are always residually transcendental (see [21, Theorem 4.6]). Hence the set of all key polynomials for a valuation-algebraic extension $w$ is an empty set, i.e., $KP(w) = \emptyset$.

Let $w$ be a valuation on $K(X)$, with value group $\Gamma_w$, which admits key polynomials. If $\phi$ is a key polynomial for $w$ of minimal degree, then we define

$$\deg(w) := \deg \phi.$$

For any valuation $w'$ on $K(X)$ taking values in a subgroup of $\Gamma_w$, we say that

$$w' \leq w \text{ if and only if } w'(f) \leq w(f) \forall f \in K[X].$$

Suppose that $w' < w$ and consider the set

$$\overline{\Phi}_{w',w} := \{ f \in K[X] \mid w'(f) < w(f) \}.$$

If $d$ is the smallest degree of a polynomial in $\overline{\Phi}_{w',w}$, then we define

$$\Phi_{w',w} := \{ g \in K[X] \mid g \text{ is monic and } \deg g = d \},$$

i.e., the set of all monic polynomials $g \in K[X]$ of minimal degree such that $w'(g) < w(g)$.

**Theorem 2.23** (Theorem 1.15, [23]). Let $w$ be a valuation on $K(X)$ and $w' < w$. Then any $\phi \in \Phi_{w',w}$ is a key polynomial for $w'$ and

$$w' < [w'; \phi, w(\phi)] \leq w.$$
For any non-zero polynomial $f \in K[X]$, the equality $w'(f) = w(f)$ holds if and only if $\phi \vdash_w f$.

**Corollary 2.24** (Corollary 2.5, [15]). Let $w' < w$ be as above. Then

(i) $\Phi_{w',w} = [\phi]_{w'}$ for all $\phi \in \Phi_{w',w}$.

(ii) If $w' < \mu < w$ is a chain of valuations, then $\Phi_{w',w} = \Phi_{w',\mu}$. In particular,

$$w'(f) = w(f) \iff w'(f) = \mu(f), \forall f \in K[X].$$

Keeping in mind the above results, we can now define

$$\deg(\Phi_{w',w}) := \deg(\phi) \forall \phi \in \Phi_{w',w}.$$

**Remark 2.25.** Let $w$ be a valuation of $K(X)$. If $w' = w_{Q'}$, for some ABKP, $Q'$ for $w$, then $\Phi_{w',w} = \psi(Q')$.

Consider the group $\mathbb{Z} \times (\Gamma_v \otimes \mathbb{Q})$ equipped with the lexicographical ordering containing $\mathbb{Z} \times \Gamma_v$ as an ordered subgroup. Let $w_{-\infty} : K[X] \rightarrow (\mathbb{Z} \times \Gamma_v) \cup \{\infty\}$ be the valuation defined by

$$w_{-\infty}(f) = (-\deg f, \nu(a_n)),$$

where $a_n$ is the leading coefficient of the polynomial $f \in K[X]$. Since the value group, $\mathbb{Z} \times \Gamma_v$ is torsion free over $\Gamma_v$, so the extension $w_{-\infty}$ of $\nu$ is incommensurable and in view of Lemma 4.1 of [14], the set of all key polynomials for $w_{-\infty}$ is

$$KP(w_{-\infty}) = \{X + a \mid a \in K\} = [X]_{w_{-\infty}}.$$

Fix an order-preserving embedding $\Gamma_v \otimes \mathbb{Q} \hookrightarrow \mathbb{Z} \times (\Gamma_v \otimes \mathbb{Q})$ of ordered abelian groups, mapping $\gamma \mapsto (0, \gamma)$. If $w$ is any incommensurable extension of $\nu$ to $K(X)$, then from the above embedding we have $w_{-\infty} < w$, and $w_{-\infty}$ is called the minimal extension of $\nu$ to $K(X)$. Moreover the augmentation of $w_{-\infty}$ with respect to the key polynomial $\phi_0 = X + a$ for some $a \in K$, defined in a natural way, is denoted by $w_0$ (see [12, Subsection 2.2]).

**Definition 2.26.** A valuation $w$ is said to be inductive if it is attained after a finite number of augmentation steps starting with the minimal valuation:

$$(2.1) \quad w_{-\infty} \xrightarrow{\phi_0, \gamma_0} w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_{r-1}, \gamma_{r-1}} w_r = w,$$

where $\gamma_0, \gamma_1, \ldots, \gamma_r \in \Gamma_v \otimes \mathbb{Q}$ and $w_i = [w_{i-1}, \phi_i, \gamma_i], 1 \leq i \leq r$.

The minimal valuation $w_{-\infty}$ is not considered an inductive valuation and thus, inductive valuations are commensurable which admits key polynomials. In view of Corollary 7.3 of [14], $\phi_i$ is a key polynomial for $w_i$ of minimal degree.
Definition 2.27 (Optimal Maclane chain). A chain of augmentations of the form (2.1) such that,

\[ 1 = m_0 \mid m_1 \mid \cdots \mid m_r, \quad m_0 < m_1 < \cdots < m_r, \]

where \( m_i = \deg \phi_i, \quad 0 \leq i \leq r \), is called the optimal Maclane chain of \( w \).

It is known that all inductive valuations admit an optimal Maclane chain \[12\]. These chains are not unique, but

- the intermediate valuations \( w_0, w_1, \ldots, w_{r-1} \),
- the degrees \( m_0, m_1, \ldots, m_r \) of the key polynomials,
- \( \gamma_0, \gamma_1, \ldots, \gamma_r \) satisfies \( \gamma_i = w_i(\phi_i) = w(\phi_i) \) for all \( 0 \leq i \leq r \),
- \( \lambda_0 = \gamma_0 = w(\phi_0), \quad \lambda_i = w_i(\phi_i) - w_{i-1}(\phi_i) > 0, \quad 1 \leq i \leq r. \)

are some intrinsic invariants of \( w \).

We now give a relation between optimal Maclane chains and complete set of ABKPs for a valuation \( w \). We first give a precise complete set \( \{Q_i\}_{i \in \Delta} \) of ABKPs using a given optimal Maclane chain.

Theorem 2.28. Let \((K, v)\) be a valued field and \( w \) be an extension of \( v \) to \( K(X) \). If \( w \) has an optimal Maclane chain of the form (2.1), then \( \{\phi_0\} \cup \{\phi_1\} \cup \cdots \cup \{\phi_r\} \) is a complete set of ABKPs for \( w \).

In the next result we gives some necessary conditions under which an optimal Maclane chain is obtained using a complete set of ABKPs for \( w \).

Theorem 2.29. Let \((K, v)\) and \((K(X), w)\) be as in the above theorem. Let \( \{Q_i\}_{i \in \Delta} \) be a complete set of ABKPs for \( w \) such that \( \Delta \) has maximal element, say, \( N \) and \( \vartheta_i = \emptyset \) for every \( 0 \leq i \leq N \). Then \( w \) has an optimal Maclane chain, if \( w(Q_N) \in \Gamma_v \otimes \mathbb{Q} \).

Let \((K, v)\) be a valued field and \( w \) be an extension of \( v \) to \( K(X) \). We now recall the definition of a continuous family of augmentations of \( w \) \cite{23, 14}.

Definition 2.30. Let \( w \) be a valuation on \( K(X) \) admitting key polynomials. Then a continuous family of augmentations of \( w \) is a family of ordinary augmentations of \( w \)

\[ \mathcal{W} = (\rho_i = [w; \chi_i, \gamma_i])_{i \in A}, \]

indexed by a set \( A \), satisfying the following conditions:

(i) The set \( A \) is totally ordered and has no maximal element.
(ii) All key polynomials \( \chi_i \in KP(w) \) have the same degree.
(iii) For all $i < j$ in $A$, $\chi_j$ is a key polynomial for $\rho_i$ and satisfies:

$$\chi_j \not\sim \rho_i \chi_i \text{ and } \rho_j = [\rho_i; \chi_j, \gamma_j].$$

The common degree $m = \deg \chi_i$, for all $i$, is called the stable degree of the family $W$ and is denoted by $\deg(W)$.

A polynomial $f$ in $K[X]$ is said to be stable with respect to the family $W = (\rho_i)_{i \in A}$ (or is $W$-stable) if

$$\rho_i(f) = \rho_{i_0}(f), \text{ for every } i \geq i_0$$

for some index $i_0 \in A$. This stable value is denoted by $\rho_W(f)$. By Corollary 2.24 (ii), a polynomial $f \in K[X]$ is $W$-unstable if and only if

$$\rho_i(f) < \rho_j(f) \forall i < j.$$ 

The minimal degree of an $W$-unstable polynomial is denoted by $m_\infty$. If all polynomials are $W$-stable, then we set $m_\infty = \infty$.

**Remark 2.31.** The following properties hold for any continuous family $W = (\rho_i)_{i \in A}$ of augmentations (see [15], p. 9):

(i) The mapping defined by $i \to \gamma_i$ and $i \to \rho_i$ are isomorphisms of ordered sets between $A$ and $\{\gamma_i \mid i \in A\}$, $\{\rho_i \mid i \in A\}$, respectively.

(ii) For all $i \in A$, $\chi_i$ is a key polynomial for $\rho_i$ of minimal degree.

(iii) For all $i, j \in A$, $\rho_i(\chi_j) = \min\{\gamma_i, \gamma_j\}$. Hence, all the polynomials $\chi_i$ are stable.

(iv) $\Phi_{\rho_i, \rho_j} = [\chi_j]_{\rho_i} \forall i < j \in A$.

(v) All valuations $\rho_i$ are residually transcendental.

(vi) All the value groups $\Gamma_{\rho_i}$ coincide and the common value group is denoted by $\Gamma_W$.

**Definition 2.32 (Maclane-Vaquié limit key polynomials).** Let $W$ be a continuous family of augmentations of a valuation $w$. A monic $W$-unstable polynomial of minimal degree is called Maclane-Vaquié limit key polynomial (abbreviated as MLV) for $W$.

We denote by $KP_\infty(W)$ the set of all MLV limit key polynomials. Since the product of stable polynomials is stable, so all MLV limit key polynomials are irreducible in $K[X]$.

**Definition 2.33.** We say that $W$ is an essential continuous family of augmentations if $m < m_\infty < \infty$.

**Remark 2.34.** All essential continuous family of augmentations admit MLV limit key polynomials.
Let \( W \) be an essential continuous family of augmentations of a valuation \( w \) and \( Q \in KP_\infty(W) \) be any MLV limit key polynomial. Then any polynomial \( f \) in \( K[X] \) with \( \deg f < \deg Q \) is \( W \)-stable.

**Definition 2.35 (Limit augmentation).** Let \( Q \) be any MLV limit key polynomial for an essential continuous family of augmentations \( W = (\rho_i)_{i \in \mathbf{A}} \) and \( \gamma > \rho_i(Q) \), for all \( i \in \mathbf{A} \), be an element of a totally ordered abelian group \( \Gamma \cup \{\infty\} \) containing \( \Gamma_W \) as an ordered subgroup. Then a map \( w' : K[X] \rightarrow K \) defined by

\[
    w'(f) = \min_{i \geq 0} \{\rho_W(f_i) + i\gamma\},
\]

where \( \sum_{i \geq 0} f_i Q^i \), \( \deg f_i < \deg Q \), is the \( Q \)-expansion of \( f \in K[X] \), gives a valuation on \( K(X) \) and is called the limit augmentation of \( W \).

Note that \( w'(Q) = \gamma \) and \( \rho_i < w' \) for all \( i \in \mathbf{A} \). If \( \gamma < \infty \), then \( Q \) is a key polynomial for \( w' \) of minimal degree [14, Corollary 7.13].

We now recall the definition of Maclane-Vaquié chains given by Nart in [15]. For this, we first consider a finite, or countably infinite, chain of mixed augmentations

\[
\begin{array}{cccccccc}
    w_0 & \phi_1,\gamma_1 & w_1 & \phi_2,\gamma_2 & \ldots & w_n & \phi_{n+1},\gamma_{n+1} & w_{n+1} & \ldots \\
\end{array}
\]

in which every valuation is an augmentation of the previous one and is of one of the following type:

- **Ordinary augmentation:** \( w_{n+1} = [w_n; \phi_{n+1},\gamma_{n+1}] \), for some \( \phi_{n+1} \in KP(w_n) \).
- **Limit augmentation:** \( w_{n+1} = [W_n; \phi_{n+1},\gamma_{n+1}] \), for some \( \phi_{n+1} \in KP_\infty(W_n) \),

where \( W_n \) is an essential continuous family of augmentations of \( w_n \).

Let \( \phi_0 \in KP(w_0) \) be a key polynomial of minimal degree and let \( \gamma_0 = w_0(\phi_0) \). Then, in view of Theorem 2.23, Proposition 6.3 of [14], Proposition 2.1, 3.5 of [14] and Corollary 2.24, we have the following properties of a chain (2.2) of augmentations.

**Remark 2.36.** (i) \( \gamma_n = w_n(\phi_n) < \gamma_{n+1} \).

(ii) For all \( n \geq 0 \) for which \( \gamma_n < \infty \), the polynomial \( \phi_n \) is a key polynomial for \( w_n \) of minimal degree and therefore

\[
    \deg(w_n) = \deg \phi_n \text{ divides } \deg(\Phi_{w_n,w_{n+1}}).
\]

(iii)

\[
\Phi_{w_n,w_{n+1}} = \begin{cases} 
    [\phi_{n+1}]_{w_n}, & \text{if } w_n \rightarrow w_{n+1} \text{ is ordinary augmentation} \\
    \Phi_{w_n,W_n} = [X_1]_{w_n}, & \text{if } w_n \rightarrow w_{n+1} \text{ is limit augmentation} 
\end{cases}
\]
Let $a \in K$, $\gamma \in \Gamma \cup \{\infty\}$. Then the valuation defined by the pair $(a, \gamma)$ is called a depth zero valuation.

**Definition 2.37** (Maclane-Vaquié chains). A finite, or countably infinite chain of mixed augmentations as in (2.2) is called a Maclane-Vaquié chain (abbreviated as MLV), if every augmentation step satisfies:

- if $w_n \to w_{n+1}$ is ordinary augmentation, then $\deg(w_n) < \deg(\Phi_{w_n, w_{n+1}})$.
- if $w_n \to w_{n+1}$ is limit augmentation, then $\deg(w_n) = \deg(\Phi_{w_n, w_{n+1}})$ and $\phi_n \notin \Phi_{w_n, w_{n+1}}$.

A Maclane-Vaquié chain is said to be complete if $w_0$ is a depth zero valuation.

In the following result, using complete set of ABKPs for a valuation $w$, we give a construction of a complete finite MLV chain whose last valuation is $w$.

**Theorem 2.38.** Let $(K, v)$ be a valued field and let $w$ be an extension of $v$ to $K(X)$. If $\{Q_i\}_{i \in \Delta}$ is a complete set of ABKPs for $w$ such that $N$ is the last element of $\Delta$, then

$$w_0 \xrightarrow{Q_1, \gamma_1} w_1 \xrightarrow{Q_2, \gamma_2} \cdots \xrightarrow{Q_{N-1}, \gamma_{N-1}} w_N = w,$$

is a complete finite MLV chain of $w$ such that

(i) if $\vartheta_j = \emptyset$, then $w_j \to w_{j+1}$ is an ordinary augmentation. Further, $w_{j+1} = w_{Q_j+1}$, and $\gamma_{j+1} = w(Q_{j+1}).$

(ii) if $\vartheta_j \neq \emptyset$, then $w_j \to w_{j+1}$ is a limit augmentation. Further, $w_{j+1} = w_{Q_{j+1}}$ and $\gamma_{j+1} = w(Q_{j+1}).$

The converse of the above result also holds.

**Theorem 2.39.** Let $(K, v)$ be a valued field and let $w$ be an extension of $v$ to $K(X)$. If

$$w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_{N-1}, \gamma_{N-1}} w_N = w,$$

is a complete finite MLV chain, then $\{\phi_i\}_{i \in \Delta}$ forms a complete set of ABKPs for $w$ such that

(i) $N$ is the last element of $\Delta$.

(ii) if $w_j \to w_{j+1}$ is an ordinary augmentation, then $\phi_{j+1} \in \psi(\phi_j).$
(iii) if $w_j \rightarrow w_{j+1}$ is a limit augmentation, then $\phi_{j+1}$ is a limit key polynomial.

Note that if $w_j \rightarrow w_{j+1}$ is an ordinary augmentation for every $0 \leq j \leq N$ and $\gamma_N \in \Gamma_v \otimes \mathbb{Q}$, then the above MLV chain is nothing but an optimal Maclane chain and Theorem 2.39 is an immediate consequence of Theorem 2.28. On the other hand, if $\vartheta_j = \emptyset$ for every $j$ and $w(Q_N) \in \Gamma_v \otimes \mathbb{Q}$, then Theorem 2.38 follows from Theorem 2.29.

It is known that if $\{Q_i\}_{i \in \Delta}$ is a complete set of ABKPs for $w$, then $w$ is a valuation-transcendental extension of $v$ to $K(X)$ if and only if $\Delta$ has a maximal element, say, $N$, and then $w = w_{Q_N}$ (see [10, Theorem 5.6]). Therefore, as an immediate consequence of Theorems 2.38 and 2.39, we have the following result.

**Corollary 2.40.** Let $(K, v)$ and $(K(X), w)$ be as above. Then the following are equivalent:

(i) The extension $w$ is valuation-transcendental.

(ii) There exist a complete set $\{Q_i\}_{i \in \Delta}$ of ABKPs for $w$ such that $\Delta$ has a maximal element.

(iii) The extension $w$ is the last valuation of a complete finite MLV chain.

3. Preliminaries

Let $(K, v)$ be a valued field and $(\overline{K}, \overline{v})$ be as before. Let $w$ be an extension of $v$ to $K(X)$ and $\overline{w}$ be a common extension of $w$ and $\overline{v}$ to $\overline{K}(X)$. In this section we give some preliminary results which will be used to prove the main results.

We first recall some basic properties of ABKPs for $w$ (see Proposition 2.16 of [11], Proposition 3.8, Corollary 3.13 and Theorem 6.1 of [18]).

**Proposition 3.1.** For ABKPs, $Q$ and $Q'$ for $w$ the following holds:

(i) If $w_Q < w$, then $w_Q$ is an r. t. extension.

(ii) If $\delta(Q) < \delta(Q')$, then $w_Q(Q') < w(Q')$.

(iii) If $\deg Q = \deg Q'$, then

$$w(Q) < w(Q') \iff w_Q(Q') < w(Q') \iff \delta(Q) < \delta(Q').$$

(iv) Let $\delta(Q) < \delta(Q')$. For any polynomial $f \in K[X]$, if $w_{Q'}(f) < w(f)$, then $w_Q(f) < w_{Q'}(f)$.

(v) If $Q' \in \psi(Q)$, then $Q$ and $Q'$ are key polynomials for $w_Q$. Moreover, $w_{Q'} = [w_Q; Q', w_{Q'}(Q') = w(Q')]$. 


The following result gives a comparison between key polynomials and ABKPs.

**Theorem 3.2** (Theorem 2.17, [4]). Suppose that $w' < w$ and $Q$ is a key polynomial for $w'$. Then $Q$ is an ABKP polynomial for $w$ if and only if it satisfies one of the following two conditions:

(i) $Q \in \Phi_{w', w}$,
(ii) $Q \notin \Phi_{w', w}$ and $\deg Q = \deg w'$.

In the first case $w_Q = [w'; Q, w(Q)]$. In the second case $w_Q = w$.

The next result relates ABKPs with distinguished pairs.

**Lemma 3.3.** Let $(K, v)$ be a henselian valued field and $w$ be an extension of $v$ to $K(X)$. Let $F$ be an ABKP for $w$ and $Q$ be any polynomial such that $(F, Q)$ is a distinguished pair. Then the following holds:

(i) If $\theta$ and $\alpha$ are optimizing roots of $F$ and $Q$ respectively, then $(\theta, \alpha)$ is a $(K, v)$-distinguished pair.
(ii) The polynomial $Q$ is an ABKP for $w$. Moreover $w_Q < w$.

**Proof.** (i) The proof follows from Lemma 2.1 of [11].
(ii) Since $\deg Q < \deg F$ and $F$ is an ABKP for $w$, so $\delta(Q) < \delta(F)$, i.e.,

$$\nu(X - \alpha) = \delta(Q) < \delta(F) = \nu(X - \theta),$$

where $\theta$ and $\alpha$ are optimizing roots of $F$ and $Q$ respectively, which in view of strong triangle law implies that

$$\nu(\theta - \alpha) = \delta(Q) < \nu(X - \theta). \tag{3.1}$$

Let $g$ in $K[X]$ be any polynomial with $\deg g < \deg Q$. Then to prove that $Q$ is an ABKP for $w$ we need to show that $\delta(g) < \delta(Q)$. As $(F, Q)$ is a distinguished pair, so by (i), $(\theta, \alpha)$ is a $(K, v)$-distinguished pair. Now for an optimizing root $\beta$ of $g$ we have $\deg \beta < \deg \alpha$, which in view of the fact that $(\theta, \alpha)$ is a $(K, v)$-distinguished pair implies that

$$\nu(\theta - \beta) < \nu(\theta - \alpha) = \delta(Q).$$

From (3.1) and the above inequality, we have that

$$\delta(g) = \nu(X - \beta) = \min\{\nu(X - \theta), \nu(\theta - \beta)\} = \nu(\theta - \beta) < \delta(Q).$$

Hence $Q$ is an ABKP for $w$. As $\delta(Q) < \delta(F)$, so by Proposition 3.1 (ii), $w_Q(F) < w(F)$, i.e., $w_Q < w$. \[\square\]

The following result gives some necessary and sufficient conditions under which an ABKP for $w$ has a saturated distinguished chain.
Corollary 3.4 (Corollary 1.18, [11]). Let \( w \) be an extension of \( v \) to \( K(X) \) and \( Q \) be an ABKP for \( w \). Then \( Q \) has a saturated distinguished chain of ABKPs if and only if there exists ABKPs, \( Q_0, Q_1, \ldots, Q_r = Q \) for \( w \), such that \( \deg Q_0 = 1 \), \( \deg Q_{i-1} < \deg Q_i \) and \( Q_i \in \psi(Q_{i-1}) \) for each \( i, 1 \leq i \leq r \).

Lemma 3.5 (Lemma 5.1, [2]). Let \((K, v)\) be henselian valued field. If \((\theta, \theta_1)\) and \((\theta_1, \theta_2)\) are two distinguished pairs of elements of \( K \), then

\[
\delta_K(\theta) > \delta_K(\theta_1) = \bar{v}(\theta_1 - \theta_2) = \bar{v}(\theta - \theta_2).
\]

4. Proof of Main Results

Proof of Proposition 2.10 (i) Since \((Q_r = Q, Q_{r-1}, \ldots, Q_0)\) is a saturated distinguished chain for \( Q \), so \((Q_i, Q_{i-1})\) is a distinguished pair for each \( 1 \leq i \leq r \). In particular, for \( i = r \) we have that \((Q, Q_{r-1})\) is a distinguished pair and as \( Q \) is an ABKP for \( w \), so by Lemma 3.3 (ii), \( Q_{r-1} \) is an ABKP for \( w \). Arguing similarly we get that each \( Q_i, i \in \{0, \ldots, r-2\} \) is an ABKP for \( w \).

(ii) As \( \theta_i, 0 \leq i \leq r \), is an optimizing root of \( Q_i \), so by Lemma 3.3 (i), \((\theta = \theta_r, \theta_{r-1}, \ldots, \theta_0)\) is a saturated distinguished chain for \( \theta \) and therefore \((\theta_i, \theta_{i-1})\), \((\theta_{i-1}, \theta_{i-2})\) are distinguished pairs. Then by Lemma 3.5

\[
(4.1) \quad \bar{v}(\theta_i - \theta_{i-1}) = \delta_K(\theta_i) > \delta_K(\theta_{i-1}) = \bar{v}(\theta_{i-1} - \theta_{i-2}).
\]

Since \( Q \) is an ABKP for \( w \), so by (i), each \( Q_i, 0 \leq i \leq r-1 \), is also an ABKP for \( w \) and as \( \deg Q_{i-1} < \deg Q_i \), therefore

\[
\bar{w}(X - \theta_{i-1}) = \delta(Q_{i-1}) < \delta(Q_i) = \bar{w}(X - \theta_i),
\]

which in view of strong triangle law implies that

\[
\delta(Q_{i-1}) = \bar{v}(\theta_i - \theta_{i-1}).
\]

The above equality together with (4.1) gives

\[
\delta_K(\theta_i) = \delta(Q_{i-1}).
\]

Now \( \delta(Q_r) > \delta(Q_{r-1}) > \cdots > \delta(Q_0) \) follows from the definition of an ABKP.

\[\square\]

Proof of Theorem 2.16. By Proposition 2.10 (i), each \( Q_i, 0 \leq i \leq r \) is an ABKP for \( w \). Since \( \deg Q_{i-1} < \deg Q_i \), so \( \delta(Q_{i-1}) < \delta(Q_i) \) and from Corollary 3.4 it follows that \( Q_{i-1} \in \psi(Q_i) \), for each \( 1 \leq i \leq r \). Now let \( f \in K[X] \) be any polynomial. If \( \deg f < \deg Q_i \), for some \( 0 \leq i \leq r \), then \( w_{Q_i}(f) = w(f) \). On the other hand, if \( \deg f \geq \deg Q_i \), for every \( 0 \leq i \leq r \), then by definition of \( w \), \( w_{Q_i}(f) = w(f) \). Hence \( \Lambda = \{Q_0\} \cup \{Q_1\} \cup \cdots \cup \{Q_r\} \) is a complete set of ABKPs for \( w \).

\[\square\]
Therefore from the above inequality and (4.2), Lemma 3.5, we have that which in view of strong triangle law implies that

\[ \delta_K(\theta) = \bar{\delta}(\theta - \theta_{r-1}) \geq \delta_K(\theta_{r-1}) = \bar{\delta}(\theta_{r-1} - \theta_{r-2}) = \bar{\delta}(\theta - \theta_{r-2}). \]

Again on applying Lemma 3.5, for distinguished pairs \((\theta_{r-1}, \theta_{r-2})\), and \((\theta_{r-2}, \theta_{r-3})\), we get that

\[ \delta_K(\theta_{r-1}) = \bar{\delta}(\theta_{r-1} - \theta_{r-2}) = \bar{\delta}(\theta - \theta_{r-2}) \geq \delta_K(\theta_{r-2}) = \bar{\delta}(\theta - \theta_{r-3}) \]

which in view of strong triangle law implies that

\[ \delta_K(\theta_{r-1}) > \delta_K(\theta_{r-2}) = \bar{\delta}(\theta - \theta_{r-3}). \]

On continuing in the similar manner, for every \(1 \leq i \leq r - 1\), we have that

\[ \delta_K(\theta_{i+1}) = \bar{\delta}(\theta_{i+1} - \theta_i) = \bar{\delta}(\theta - \theta_i) > \delta_K(\theta_i) \]

Therefore from the above inequality and (4.2), \(\theta_i\) is of minimal degree over \(K\) such that

\[ \bar{\delta}(\theta - \theta_i) = \max\{\bar{\delta}(\theta_{i+1} - \eta) \mid \eta \in \overline{K}, \ deg \eta < deg \theta_{i+1}\} \quad \forall \ 0 \leq i \leq r - 1. \]

For any \(\eta \in \overline{K}\) with \(deg \eta < deg \theta_{i+1}\), we now claim that \(\bar{\delta}(\theta_{i+1} - \eta) = \bar{\delta}(\theta - \eta)\). For \(i = r - 1\), this holds trivially. Let \(0 \leq i \leq r - 2\), then as \(deg \eta < deg \theta_{i+1}\) and \((\theta_{i+1}, \theta_i)\) is a distinguished pair, so by (4.3)

\[ \bar{\delta}(\theta_{i+1} - \eta) \leq \bar{\delta}(\theta_{i+1} - \theta_i) = \bar{\delta}(\theta - \theta_i) \leq \bar{\delta}(\theta - \theta_{i+1}), \]

which in view of the strong triangle law implies that

\[ \bar{\delta}(\theta_{i+1} - \eta) = \bar{\delta}(\theta - \eta). \]

It now follows from (4.4) and the claim that

\[ \delta_K(\theta_{i+1}) = \bar{\delta}(\theta - \theta_i) = \max\{\bar{\delta}(\theta_{i+1} - \eta) \mid \eta \in \overline{K}, \ deg \eta < deg \theta_{i+1}\} = \max\{\bar{\delta}(\theta - \eta) \mid \eta \in \overline{K}, \ deg \eta < deg \theta_{i+1}\}. \]

The above equality immediately implies that

\[ \bar{\delta}(\theta - \theta_i) = \max\{\bar{\delta}(\theta - \eta) \mid \eta \in \overline{K}, \ deg \eta = deg \theta_i\}, \ for \ all \ 0 \leq i \leq r - 1. \]
In order to prove that \([F_0, F_1, \ldots, F_{r-1}]\) is an Okutsu frame for \(F\), in view of the above equality, and the fact that \(\deg \theta_0 = 1\), it only remains to show that for every \(1 \leq i \leq r - 1\),

\[
\deg \theta_i = \min \{ \deg \eta \mid \eta \in \overline{K}, \, v(\theta - \eta) > v(\theta - \theta_{i-1}) \}.
\]  

(4.6)

Let \(\eta \in \overline{K}\) be such that \(\deg \eta < \deg \theta_i\), then as \((\theta_i, \theta_{i-1})\) is a distinguished pair, so

\[
v(\theta_i - \eta) \leq v(\theta_i - \theta_{i-1}) = v(\theta - \theta_{i-1}) < v(\theta - \theta_i),
\]

which together with strong triangle law implies that

\[
v(\theta - \eta) = v(\theta_i - \eta) \leq v(\theta - \theta_{i-1}).
\]

Hence (4.6) follows.

Conversely, let \([F_0, F_1, \ldots, F_{r-1}]\) be an Okutsu frame for \(F\). Then there exist some root \(\theta_i\) of \(F_i\) such that

\[
v(\theta - \theta_i) = \max \{ v(\theta - \eta) \mid \eta \in \overline{K}, \, \deg \eta = \deg \theta_i, \, 0 \leq i \leq r - 1, \, \deg \theta_i = \min \{ \deg \eta \mid \eta \in \overline{K}, \, v(\theta - \eta) > v(\theta - \theta_{i-1}) \}, \quad 1 \leq i \leq r - 1,
\]

and

\[
1 = \deg \theta_0 < \cdots < \deg \theta_i < \deg \theta_{i+1} < \cdots < \deg \theta_r = \deg \theta
\]

(4.7) \(v(\theta - \theta_0) < \cdots < v(\theta - \theta_i) < v(\theta - \theta_{i+1}) < \cdots < v(\theta - \theta_{r-1}) < \infty\).

(4.8)

In order to prove that \((F, F_{r-1}, \ldots, F_0)\) is a saturated distinguished chain for \(F\), it is enough to show that \((\theta = \theta_r, \theta_{r-1}, \ldots, \theta_0)\) is a saturated distinguished chain for \(\theta\). From (4.8), on using strong triangle law we get that

\[
v(\theta - \theta_i) = v(\theta_{i+1} - \theta_i), \quad 0 \leq i \leq r - 1.
\]

(4.9)

Now for any \(\eta \in \overline{K}\) with \(\deg \eta < \deg \theta_{i+1}\), we show that

\[
v(\theta - \eta) = v(\theta_{i+1} - \eta) \text{ and } v(\theta_{i+1} - \theta_i) \geq v(\theta_{i+1} - \eta).
\]

For \(i = r - 1\), first equality holds trivially. If \(\deg \eta = \deg \theta_{r-1}\), then by (4.7),

\[
v(\theta - \eta) \leq v(\theta - \theta_{r-1}), \quad \text{on the other hand, if } \deg \eta \neq \deg \theta_{r-1}, \text{ then by definition of } \deg \theta_{r-1} \text{ and } (4.8) \text{ we have that}
\]

\[
v(\theta - \eta) \leq v(\theta - \theta_{r-2}) < v(\theta - \theta_{r-1}).
\]

Let \(0 \leq i \leq r - 2\). Since \(\deg \eta < \deg \theta_{i+1}\), so by definition of \(\deg \theta_{i+1}\) and (4.8), we get

\[
v(\theta - \eta) \leq v(\theta - \theta_i) < v(\theta - \theta_{i+1})
\]

which in view of strong triangle law and (4.9) implies that

\[
v(\theta_{i+1} - \eta) = v(\theta - \eta) \leq v(\theta - \theta_i) = v(\theta_{i+1} - \theta_i).
\]
Hence
\[ \tilde{v}(\theta_{i+1} - \theta_i) = \max\{\tilde{v}(\theta_{i+1} - \eta) \mid \eta \in \overline{K}, \ \deg \eta < \deg \theta_{i+1}\} = \delta_K(\theta_{i+1}) \]
and \( \deg \theta_i \) is minimal with this property, because if there exist some \( \beta \in \overline{K} \) with \( \deg \beta < \deg \theta_i \), then by definition of \( \deg \theta_i \) and (1.8),
\[ \tilde{v}(\theta - \beta) \leq \tilde{v}(\theta - \theta_{i-1}) < \tilde{v}(\theta - \theta_i) = \tilde{v}(\theta_{i+1} - \theta_i), \]
which on using strong triangle law gives
\[ \tilde{v}(\theta_{i+1} - \beta) = \min\{\tilde{v}(\theta_{i+1} - \theta_i), \tilde{v}(\theta_i - \theta), \tilde{v}(\theta - \beta)\} = \tilde{v}(\theta - \beta) < \tilde{v}(\theta - \theta_i) = \tilde{v}(\theta_{i+1} - \theta_i) = \delta_K(\theta_{i+1}), \]
i.e., \( \tilde{v}(\theta_{i+1} - \beta) < \delta_K(\theta_{i+1}) \). As \( \deg \theta_0 = 1 \), so \( \theta_0 \in K \). Hence \( \theta = (\theta_r, \theta_{r-1}, \ldots, \theta_0) \) is a saturated distinguished chain for \( \theta \).

Remark 4.1. From proof of the above theorem, we can conclude that \( [F_0, F_1, \ldots, F_{r-1}] \)
is an Okutsu frame for a monic irreducible polynomial \( F \in K[X] \), if there exist some root \( \theta_i \) of \( F_i \) such that
\[ \tilde{v}(\theta - \theta_i) = \max\{\tilde{v}(\theta - \eta) \mid \eta \in \overline{K}, \ \deg \eta < \deg \theta_{i+1}\} \text{ for all } 0 \leq i \leq r - 1, \]
de\( \deg \theta_i \) is minimal with this property and \( \deg \theta_0 = 1 \).

Proof of Theorem 2.28. Let
\[ w_{-\infty} \xrightarrow{\phi_0, \gamma_0} w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_{r-1}, \gamma_{r-1}} w_r = w, \]
be an optimal Maclane chain of \( w \). If \( r = 0 \), then \( w_{-\infty} \xrightarrow{\phi_0, \gamma_0} w_0 = w \) is an optimal Maclane chain of \( w \). Since \( w_0 = [w_{-\infty}; \phi_0, \gamma_0] \) and \( \gamma_0 = w_0(\phi_0) = w(\phi_0) \), so for any polynomial \( f \in K[X] \), with \( \phi_0 \)-expansion \( \sum_{i \geq 0} a_i \phi_0^i \), we have
\[ w_{\phi_0}(f) = \min\{w(a_i \phi_0^i)\} = \min\{v(a_i) + iw(\phi_0)\} = w_0(f) = w(f). \]
Therefore, \( \{\phi_0\} \) is a complete set of ABKP for \( w_0 = w \). Assume now that \( r \geq 1 \). Since \( \phi_i \) is a key polynomial of minimal degree for \( w_i \), i.e., \( \deg \phi_i = \deg w_i \), and \( w_i(\phi_i) = w(\phi_i) \), so \( \phi_i \notin \Phi_{w_i, w} \) which in view of Theorem 3.2 implies that \( \phi_i \) is an ABKP for \( w \). Moreover,
\[ (4.10) \]
\[ w_i = w_{\phi_i}, \ 0 \leq i \leq r. \]
As \( \deg \phi_{i-1} < \deg \phi_i \), and \( \phi_i \) is an ABKP for \( w \), so
\[ \delta(\phi_{i-1}) < \delta(\phi_i). \]
Since \( w_{\phi_{i-1}} < w \), \( \phi_i \) is a key polynomial for \( w_{\phi_{i-1}} \) which is also an ABKP for \( w \), and \( \deg(w_{\phi_{i-1}}) = \deg \phi_{i-1} < \deg \phi_i \), so by Theorem 3.2 we have that
\[ \phi_i \in \Phi_{w_{\psi_i-1}}, w. \] Therefore, by Remark 2.25 we get that
\[ (4.11) \quad \phi_i \in \psi(\phi_{i-1}), \ 1 \leq i \leq r. \]

Let \( f \) in \( K[X] \) be any polynomial. If \( \deg f < \deg \phi_i \) for some \( i \), then \( w_{\phi_i}(f) = w(f) \). Otherwise, if \( \deg f \geq \deg \phi_i \) for all \( 0 \leq i \leq r \), then from (4.10) and the fact that \( w_r = w \), we have \( w_{\phi_i} = w \). Hence \( w_{\phi_i}(f) = w(f) \). Thus \( \Lambda = \{ \phi_0 \} \cup \{ \phi_1 \} \cup \cdots \cup \{ \phi_r \} \) is a complete set of ABKPs for \( w \) such that \( \psi_i = \emptyset \), for every \( 0 \leq i \leq r \) (by 4.11) and \( r \) is the maximal element of \( \Delta \).

**Proof of Theorem 2.29.** Suppose \( \Lambda = \{ Q_i \}_{i \in \Delta} \) is a complete set of ABKPs for \( w \) such that \( N \) is the maximal element of \( \Delta \). If \( N = 0 \), then \( w \) is a depth zero valuation \( w_{Q_0} = w \) and by the hypothesis that \( w(Q_0) \in \Gamma_v \otimes \mathbb{Q} \), the extension \( w \) is commensurable. Since \( Q_0 \) is a monic polynomial of degree one, so \( Q_0 \) is a key polynomial for \( w_{-\infty} \), and by definition of \( w_{-\infty} \) we have that \( w_{-\infty}(Q_0) < w_{Q_0}(Q_0) = w(Q_0) \). Therefore \( w_{Q_0} = [w_{-\infty}, Q_0, w(Q_0)] \) is the augmentation of \( w_{-\infty} \) and the result holds in this case. Assume now that \( N \geq 1 \). Since each \( \psi_i = \emptyset \), so \( Q_i \in \psi(Q_{i-1}) \) for every \( 1 \leq i \leq N \), which in view of Proposition 3.1 (v) implies that \( Q_{i-1}, Q_i \) are key polynomials for \( w_{Q_{i-1}} \) and

\[ w_{Q_i} = [w_{Q_{i-1}}; Q_i, w_{Q_i}(Q_i) = w(Q_i)] \]

is the augmentation of \( w_{Q_{i-1}} \). Now from Proposition 3.1 (i), and the assumption that \( w(Q_N) \in \Gamma_v \otimes \mathbb{Q} \), we get that

\[ \gamma_i = w_{Q_i}(Q_i) = w(Q_i) \in \Gamma_v \otimes \mathbb{Q}, \ 0 \leq i \leq N. \]

By Remark 2.14 (iv), \( \deg Q_{i-1} < \deg Q_i \) for every \( 1 \leq i \leq N \) and as \( Q_i \in \psi(Q_{i-1}) \), so by [11] Theorem 1.12 (ii), we have that \( \deg Q_{i-1} | \deg Q_i \). Arguing similarly as in the case \( N = 0 \), we have that \( w_{Q_0} = [w_{-\infty}, Q_0, w(Q_0)] \) is the augmentation of \( w_{-\infty} \). Hence

\[ w_{-\infty} \overset{Q_0}{\rightarrow} w_{Q_0} \overset{Q_1}{\rightarrow} w_{Q_1} \overset{Q_2}{\rightarrow} \cdots \overset{Q_N}{\rightarrow} w_{Q_N} = w \]

is an optimal Maclane chain of \( w \).

**Proof of Theorem 2.38.** Let \( \{ Q_i \}_{i \in \Delta} \) be a complete set of ABKPs for \( w \) with \( N \) the maximal element of \( \Delta \). If \( N = 0 \), then \( w = w_{Q_0} \) is a depth zero valuation and result holds trivially. Assume now that \( N \geq 1 \). Then by Remark 2.14 (i), \( \Delta = \bigcup_{j=0}^{N} \Delta_j \), where \( \Delta_j = \{ j \} \cup \{ \psi_j \} \) and \( \psi_j \) is either empty or an ordered set without a maximal element. Since for each \( i \in \Delta \), \( Q_i \) is an ABKP for \( w \), so \( w_{Q_i} \) is a valuation on \( K(X) \) and we denote it by \( w_i \).

Suppose first that \( \psi_j = \emptyset \) for some \( 0 \leq j \leq N \), then \( Q_{j+1} \) is not a limit key polynomial, i.e., \( Q_{j+1} \in \psi(Q_{j}) \) which in view of Proposition 3.1 (v), implies
that $Q_{j+1}$ and $Q_j$ are key polynomials for $w_j$ and

$$w_{Q_{j+1}}(= w_{j+1}) = [w_j; Q_{j+1}, w_{Q_{j+1}}(Q_{j+1}) = w(Q_{j+1})]$$

is an ordinary augmentation of $w_j$ and $Q_{j+1}$, i.e., $w_j \rightarrow w_{j+1}$ is an ordinary augmentation. In fact $Q_j$ is a key polynomial of minimal degree for $w_j$, i.e.,

$$\deg Q_j = \deg(w_j)$$

and as $\Phi_{w_j, w_{j+1}} = [Q_{j+1}]w_j$, so

$$\deg w_j < \deg Q_{j+1} = \deg(\Phi_{w_j, w_{j+1}}).$$

Assume now that $\vartheta_j \neq \emptyset$ for some $0 \leq j \leq N$. Then by Remark 2.14, for each $i \in \vartheta_j$ there exists an ABKP, $Q_i$ for $w$ such that $Q_i \in \psi(Q_j)$ and

$$\deg Q_i = \deg Q_j = m_j \text{ (say)},$$

where $Q_j$ is the ABKP corresponding to $\{j\}$. From Proposition 3.1 (v), it follows that each $Q_i$ is a key polynomial for $w_j$ and $w_Q$, is an ordinary augmentation of $w_j$ with respect to $w(Q_i)$, i.e.,

$$w_{Q_i}(= w_i) = [w_j; Q_i, w(Q_i)].$$

Therefore, for each $i \in \vartheta_j$, $w_i$ is an ordinary augmentation of $w_j$. Now for any $i < i' \in \vartheta_j$, since $\delta(Q_i) < \delta(Q_{i'})$, so by Proposition 3.1 (ii), we have that $w_i(Q_{i'}) < w(Q_{i'})$, which together with $\deg Q_i = \deg Q_{i'}$ implies that $Q_{i'} \in \psi(Q_i)$. On using Proposition 3.1 (v), we get that $Q_{i'}$ is a key polynomial for $w_i$, and

$$w_{i'} = [w_i; Q_{i'}, w(Q_{i'})],$$

i.e., $w_{i'}$ is an ordinary augmentation of $w_i$ with respect to $w(Q_{i'})$ for every $i < i' \in \vartheta_j$. From Corollary 2.24 and Remark 2.25, we have $\psi(Q_i) = \Phi_{w_i, w} = \Phi_{w_i, w', \vartheta}$, and as $Q_i \notin \psi(Q_i)$, so

\begin{equation}
(4.12) \quad w_i(Q_i) = w_{i'}(Q_i), \quad \forall \ i' > i \in \vartheta_j,
\end{equation}

which in view of Theorem 2.23 implies that $Q_{i'} \not\in w_i Q_i$. Hence $W_j := \{w_i\}_{i \in \vartheta_j}$ is a continuous family of augmentations such that for each $i \in \vartheta_j$, $w_i$ is an ordinary augmentation of $w_j$ with respect to $w(Q_i)$ and from (4.12), $Q_i$ is $W_j$-stable with stability degree $m_j$. As $\vartheta_j \neq \emptyset$, so by Definition 2.15, $Q_{j+1}$ is a limit key polynomial. Since $\delta(Q_i) < \delta(Q_{i'})$, for every $i < i' \in \vartheta_j$, and $w_{i'}(Q_{j+1}) < w(Q_{j+1})$ for every $i' \in \vartheta_j$, (because $i' < j + 1 \in \Delta$), so in view of Proposition 3.1 (iv), we have that

$$w_i(Q_{j+1}) < w_{i'}(Q_{j+1}) \text{ for every } i < i' \in \vartheta_j.$$
Let \( \gamma_{j+1} \) denotes the valuation \( w(Q_{j+1}) \). Clearly, \( \gamma_{j+1} > w_i(Q_{j+1}) \), for all \( i \in \vartheta_j \). Let \( f \) be any polynomial in \( K[X] \) with \( Q_{j+1} \)-expansion \( \sum_{s \geq 0} f_s Q_{j+1}^s \). As \( \deg f_s < \deg Q_{j+1} = m_{j+1} \), so all coefficients are \( \mathcal{W}_j \)-stable, i.e., \( w_i(f_s) = w_i(f_s) \) for all \( i' > i \) in \( \vartheta_j \) and we denote these stable values by \( \rho_{\mathcal{W}_j}(f_s) \). Then

\[ \rho_{j+1}(f) = \min_{s \geq 0} \{ \rho_{\mathcal{W}_j}(f_s) + s\gamma_{j+1} \} \]

is a valuation on \( K[X] \), which implies that \( \rho_{j+1} = [\mathcal{W}_j; Q_{j+1}, \gamma_{j+1}] \) is a limit augmentation of an essential continuous family of augmentations of \( w_j \), or \( w_j \to \rho_{j+1} \) is a limit augmentation. Therefore \( \rho_{j+1}(Q_{j+1}) = \gamma_{j+1} = w(Q_{j+1}) \), i.e., \( Q_{j+1} \notin \Phi_{\rho_{j+1}, w} \). Since \( Q_{j+1} \) is a key polynomial of minimal degree for \( \rho_{j+1} \), so \( \deg Q_{j+1} = \deg(\rho_{j+1}) \) and hence in view of Theorem 3.2, we have that

\[ \rho_{j+1} = w_{Q_{j+1}} (= w_{j+1}). \]

As \( w_j < w_{j+1} \leq w \), so if \( w_{j+1} = w \), then by Remark 2.25, \( \Phi_{w_j, w_{j+1}} = \Phi_{w_j, w} = \psi(Q_j) \), otherwise this equality holds in view of Corollary 2.24 (ii) and Remark 2.25. By Remark 2.31 (ii), we have that \( \deg(w_j) = \deg(Q_j) = \deg(\psi(Q_j)) = \deg(\Phi_{w_j, w_{j+1}}) \). In fact, \( Q_{j+1} \notin \Phi_{w_j, w_{j+1}} = \psi(Q_j) \), because \( \deg Q_j < \deg Q_{j+1} \).

Clearly, \( w_{Q_N} = w \), for if there exist some polynomial \( f \in K[X] \) such that \( w_{Q_N}(f) < w(f) \), then as \( \Lambda \) is a complete set, so \( w_{Q_i}(f) = w(f) \) for some \( 0 \leq i < N \). But this will imply that \( w(f) = w_{Q_i}(f) \leq w_{Q_N}(f) < w(f) \).

Thus from the above arguments it follows that

\[ w_0 \xrightarrow{Q_1, \gamma_1} w_1 \xrightarrow{Q_2, \gamma_2} \cdots \xrightarrow{Q_{n-1}, \gamma_{n-1}} w_{N-1} \xrightarrow{Q_N, \gamma_N} w_N = w, \]

where \( w_i = w_{Q_i} \), \( \gamma_i = w(Q_i) \) for every \( 0 \leq i \leq N \), is a MLV chain whose last valuation is \( w_N = w \), such that:

- if \( \vartheta_j = \emptyset \), then \( w_j \to w_{j+1} \) is an ordinary augmentation.
- if \( \vartheta_j \neq \emptyset \), then \( w_j \to w_{j+1} \) is a limit augmentation of an essential continuous family of augmentations of \( w_j \).

Finally the chain is complete because \( w_0 = w_{Q_0} \), where \( Q_0 = X \), is defined by the pair \( (0, w_0(X)) \) and is a depth zero valuation.

\[ \square \]

**Proof of Theorem 2.39.** Let

\[ w_0 \xrightarrow{\phi_1, \gamma_1} w_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_{N-1}, \gamma_{N-1}} w_{N-1} \xrightarrow{\phi_N, \gamma_N} w_N = w, \]

be a complete finite MLV chain of \( w \). If \( N = 0 \), then \( w_0 = w \) is a depth zero valuation and the result holds trivially. Assume now that \( N \geq 1 \). Then each \( \phi_j \) is a key polynomial for \( w_j \) of minimal degree, i.e., \( \deg \phi_j = \deg(w_j) \), and...
Suppose first that $w_j \rightarrow w_{j+1}$ is an ordinary augmentation. Then by definition of MLV chain of $w$, we have that $\phi_{j+1} \in \Phi_{w_j, w_{j+1}}$. As $w_j < w_{j+1} \leq w$, so if $w_{j+1} < w$, then by Corollary 2.21, $\Phi_{w_j, w_{j+1}} = \Phi_{w_j, w}$, i.e., $w_j(\phi_{j+1}) < w(\phi_{j+1})$, otherwise this holds trivially. Now from (4.13), we get that $w_{\phi_j}(\phi_{j+1}) < w(\phi_{j+1})$ which together with the minimality of $\deg \phi_{j+1}$, implies that $\phi_{j+1} \notin \psi(\phi_j)$ and hence from Lemma 2.9 it follows that
\[
\delta(\phi_j) < \delta(\phi_{j+1}).
\]

Assume now that $w_j \rightarrow w_{j+1}$ is a limit augmentation. Then $\phi_{j+1}$ is a MLV limit key polynomial for an essential continuous family (say) $W_j$ of augmentations of $w_j$. Let $W_j = \{\rho_i\}_{i \in A_j}$, where $A_j$ is some totally ordered set without a maximal element and for each $i \in A_j$, $\rho_i = [w_j, \phi_i, \gamma_i]$ is an ordinary augmentation of $w_j$ with stability degree, (say) $m_j = \deg \phi_j = \deg \phi_i$. Also, for all $i < i' \in A_j$, $\phi_{i'}$ is a key polynomial for $\rho_i$ such that
\[
\phi_{i'} \not\in \rho_i, \phi_i, \text{ i.e., } \phi_{i'} \not\in \rho_i, \phi_i \text{ and } \rho_{i'} = [\rho_{i}, \phi_{i'}, \gamma_{i'}].
\]

Since $\rho_i < w$ and $\phi_{i'} \not\in \rho_i$, so by Theorem 2.21 $\rho_i(\phi_i) = w(\phi_i)$, which together with Corollary 2.21 gives
\[
(4.14) \quad \phi_i \notin \Phi_{\rho_i, w} = \Phi_{\rho_i, \rho_{i'}} = [\phi_{i'}]_{\rho_i}.
\]

Now by Remark 2.31 (ii), for each $i \in A_j$, $\phi_i$ is a key polynomial for $\rho_i$ of minimal degree, i.e., $\deg \phi_i = \deg \rho_i$, therefore keeping in mind that $\rho_i < w$, equation (4.14) in view of Theorem 3.2 (ii), implies that each $\phi_i$ is an ABKP for $w$ and
\[
\rho_i = w_{\phi_i}, \text{ for all } i \in A_j.
\]

Hence for each $i < i' \in A_j$, $\phi_i$ and $\phi_{i'}$ are ABKPs for $w$ such that
\[
w(\phi_i) = w_{\phi_i}(\phi_i) < w_{\phi_i}(\phi_{i'}) = w(\phi_{i'}) \text{ and } \deg \phi_i = \deg \phi_{i'},
\]
which in view of Proposition 3.1 (iii), implies that $w_{\phi_i}(\phi_{i'}) < w(\phi_{i'})$. Therefore, by Lemma 2.9 we get that
\[
\phi_{i'} \in \psi(\phi_i) \text{ and } \delta(\phi_i) < \delta(\phi_{i'}) \text{ for every } i < i' \in A_j.
\]

As $\deg \phi_j = \deg \phi_i$, for every $i \in A_j$ and $w(\phi_j) < w(\phi_i)$, so again by Proposition 3.1 (iii), we have that $\phi_i \in \psi(\phi_j)$ and then
\[
\delta(\phi_j) < \delta(\phi_i) \forall \ i \in A_j,
\]
follows from Lemma 2.9. Since $W_j$ is essential, so $\deg \phi_j = \deg \phi_i < \deg \phi_{j+1}$, for every $i \in A_j$, this together with the fact that $\phi_{j+1}$ is an ABKP for $w$, implies that

$$\delta(\phi_i) < \delta(\phi_{j+1}), \delta(\phi_i) < \delta(\phi_{j+1}) \text{ and } \phi_{j+1} \notin \psi(\phi_j).$$

For every $0 \leq j \leq N$, let $\Delta_j = \{j\} \cup A_j$, and $\Delta = \bigcup_{j=0}^N \Delta_j$. We now show that $\Lambda = \{\phi_i\}_{i \in \Delta}$ is a complete set of ABKPs for $w$. Clearly, as shown above for every $i < i' \in \Delta$, we have $\delta(\phi_i) < \delta(\phi_{i'})$. Therefore the set $\Lambda$ is well-ordered with respect to the ordering given by $\phi_i < \phi_{i'}$ if and only if $\delta(\phi_i) < \delta(\phi_{i'})$ for every $i < i' \in \Delta$. It only remains to prove that for any polynomial $f \in K[X]$, there exist some $i \in \Delta$ such that $w_{\phi_i}(f) = w(f)$. If $\deg f < \deg \phi_i$ for some $i \in \Delta$, then $w_{\phi_i}(f) = w(f)$. On the other hand, if $\deg f \geq \deg \phi_i$ for all $i \in \Delta$, then using the fact that $w_N = w$ and (4.13), we get $w_{\phi_N} = w$. Hence $w_{\phi_N}(f) = w(f)$. Thus $\{\phi_i\}_{i \in \Delta}$ is a complete set of ABKPs for $w$ such that

- if $w_j \rightarrow w_{j+1}$ is an ordinary augmentation, then $\phi_{j+1} \in \psi(\phi_j)$,
- if $w_j \rightarrow w_{j+1}$ is a limit augmentation, then $\phi_{j+1} \notin \psi(\phi_j)$ and therefore, $A_j \neq \emptyset$ implies that $\phi_{j+1}$ is a limit key polynomial.

\[\square\]

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