Abstract—This paper systematically compares two mathematical foundations for multitarget tracking: labeled random finite sets (LRFS’s) and trajectory random finite sets (TRFS’s).

I. INTRODUCTION

As an engineering discipline, multitarget tracking dates back to the seminal paper by Reid [1] and earlier—see [2] for an overview. The last 20 years of multitarget tracking research have differed from the 20 years preceding them in that there has been an increasing emphasis on theoretically rigorous statistical foundations that also facilitate the development of practical multitarget tracking algorithms.

Probably the most notable such algorithm is the generalized labeled multi-Bernoulli (GLMB) filter, which arose from the random finite set (RFS) [3] and labeled random finite set (LRFS) [4], [5] paradigms. Its most recent implementations—made possible by the application of advanced Gibbs statistical sampling techniques to the LRFS multitarget posterior density $f_k(X|Z_{1:k})$ on labeled random finite sets $X$—can simultaneously track over one million targets in significant clutter in real time using off-the-shelf computing equipment [6].

In the LRFS approach, the state of a multitarget system at time $t_k$ is modeled as a labeled finite subset $X = \{(x_1, \ell_1), ..., (x_n, \ell_n)\} \subseteq X_0 \times \mathcal{L}$ where: $X_0$ is the kinematic state space and $\mathcal{L}$ is a countable set of “track labels”; $X_0 = \{f_1, ..., f_T\}$ is the set of labels of the elements of $X$; and $|X| = |X_0|$ where $|S|$ denotes the cardinality of a finite set $S$. That is, $X$ is “labeled” if the targets in $X$ have unique labels. Given this, a time-evolving multitarget population is represented as a time-sequence $X_1, ..., X_k, ...$ of multitarget labeled state-sets at the measurement-collection times $t_1, ..., t_k, ...$

Since 2014, however, it has repeatedly been asserted that the LRFS framework is seriously deficient because LRFS labels (i.e., the unique identifiers that real-world targets actually possess) signify actual physical phenomena—i.e., the unique identities that real-world targets actually possess. Stated more forthrightly:

- Jettisoning the signifier (the label and the information that it carries) effectively throws the baby (the unique target that it signifies) out with the bathwater.

The purpose of this paper is to substantiate the following claims:

1) These criticisms of the LRFS framework are mistaken;
2) the TRFS framework itself has no direct and unambiguous connection to the physical phenomena under observation; and
3) it itself seriously increases uncertainty in the tracking problem.

The analysis that follows will begin with phenomenological and epistemological basics—specifically, with the basic semiotic concepts of signified (an entity) versus signifier (a name for that entity in some symbolic language). It will be shown, step-by-step, that the LRFS framework arises inevitably from the phenomenological requirements of the multitarget tracking application. It will then be shown that the TRFS framework arises by stripping from the LRFS framework the signifiers (labels) that signify actual physical phenomena—i.e., the unique identities that real-world targets actually possess. Stated more forthrightly:

II. MATHEMATICAL REPRESENTATION OF TARGETS

The section is organized as follows: mathematical representation of target kinematics (Section II-A); mathematical representation of target identity (Section II-B); and mathematical representation of multitarget states (Section II-C).

A. Mathematical Representation of Target Kinematics

A target is a macroscopic physical entity. This means in particular that it has a position, which is typically mathematically idealized as a unique point in three-dimensional Euclidean space. A point position must be designated by some sort of symbolic identifier—a “signifier,” in the patois of semiotics. This signifier is not—indeed, cannot be—unique. It typically
consists of a triplet of real numbers (continuous state variables) in some coordinate system (Cartesian, cylindrical, spherical, etc.) centered at some origin point (geocentric, etc.), each variable having a unit of measurement (metric, English, etc.), and the value of each variable being expressed in terms of some number-system base (decimal, binary, hexadecimal, etc.).

Two points should be emphasized:

1) The arbitrariness of a position signifier does not negate the reality, physicality, or uniqueness of the position that it signifies; and the same is true of any other kinematic state variable.

2) The existence of an infinitude of possible signifiers for a position does not contravene the usual requirement that there be a one-to-one correspondence between physical states and their mathematical representations. For position (or any other kinematic state variable), it is enough that this correspondence exists once the following have been selected: number base, measurement units, coordinate system origin, and coordinate system.

B. Mathematical Representation of Target Identity

As a macroscopic physical entity, a target has a discrete-valued state variable—its unique identity—which must be assigned some appropriate signifier. As with position, this signifier cannot be unique. For example, it can be (in the case of people) a social security number or a name such as “Bob Aardvark” or “Sue Zebra”; or (in the case of aircraft) a tail number or a manufacturer’s serial number.

The arbitrariness of this signifier does not preclude the reality, physicality, or uniqueness of the target identity which it signifies. And the infinitude of possible identity-signifiers does not violate the one-to-one correspondence principle. Since the number of target identities is finite but usually unknown, it is enough that such a correspondence exists once a countable set of identity-signifiers has been chosen.

When the actual identity of a physical object is unknown, it can be assigned a tentative identifier—for example, a “track label.” Neither the provisioanal nor the infinitude of possible such labels means that they are “artificial variables...added to the target states” ([9], p. 1884). Rather, they are stand-ins for the unique identities of actual physical objects and, as such, are no more “artificial” than any other signifier for those objects. To claim otherwise is to implicitly claim that (for example) target-classifier algorithms—which estimate the types or even identities of physical targets—are “non-physical.”

Remark 1: “Target types” or “target classes” are signifiers that are less accurate than identities but typically more accurate than labels. As noted in [8], p. 259, Remark 33, they—unlike identities—are not necessarily time-invariant because some targets can have multiple, distinct phenomenological “modes.” Typical examples include: variable swept-wing aircraft (extended-wing vs. delta-wing); mobile missile launchers (launch-ready vs. transiting); and diesel-electric submarines (surfaced vs. submerged). This complication can be addressed within the RFS/LRFS framework but will not be further addressed in this paper.

These points having been made, we are now in a position to address the theoretically and phenomenologically correct mathematical representation of time-evolving multitarget populations.

C. Mathematical Representation of Multitarget States

As noted earlier, the state of a point target at a particular instant has the mathematical form \( \xi = (x, \ell) \in X \times L \) where \( x \in X_0 \) is its kinematic state (position, velocity, orientation, etc.) and \( \ell \in L \) is a provisional identifying label drawn (without replacement) from a countable set \( L \) of such labels.

This, in turn, means that it is impossible for a target population to contain pairs \((x_1, \ell), (y, \ell)\) with \( x \neq y \) since, then (for example) “Bob” could be in two different places simultaneously. Nor can it contain pairs of (for example) “Bob” could exist simultaneously. Consequently, the state of a target population must be a collection \( \{\ell_1, \ell_2, \ldots, \ell_n\} = \{x_1, \ell_1\}, \ldots, \{x_n, \ell_n\} \) where the kinematic states have distinct labels—i.e.,

\[ |\{\{x_1, \ell_1\}, \ldots, \{x_n, \ell_n\}\}| = |\{\ell_1, \ldots, \ell_n\}| = n. \quad (2) \]

This collection does not have a natural ordering—e.g., is “Bob” greater or less than “Sue”? Thus \( \ell_1, \ldots, \ell_n \) must have the mathematical form \( \{\ell_1, \ldots, \ell_n\} \subseteq X_0 \times L \)—i.e., a finite set—and not \( \{\ell_1, \ldots, \ell_n\} \subseteq (X_0 \times L)^n \)—i.e., a vector.

There are three additional reasons why vectors are theoretically inappropriate representations of multitarget populations ([10], Section 2.4).

1) Statistical Bias. Imposing nonphysical information on a physical system can create a statistical bias. The most obvious such biases are deliberate—e.g., forcing a multitarget tracker-classifier to prioritize targets of interest (ToI’s), e.g., “Bob” is tactically more threatening than “Sue.” The concept of ToI is subjective and contextual, not physical. Forcing it upon a target population results in a nonphysical statistical bias in favor of more tactically significant ToI’s. Similarly, physical targets do not have a natural ordering. Forcing one upon a population (as with vector representation) runs the risk of introducing an unknown statistical bias. (In the TRFS framework, target ordering is also viewed as “non-physical,” see [7], Abstract.)

2) Uniqueness of States. There should be a one-to-one correspondence between physical states and their mathematical representations. Because physical states have no inherent ordering, there are \( n! \) possible vector representations \( \bar{\xi}_n = (\xi_{\pi_1}, \ldots, \xi_{\pi_n}) \) of the same multitarget population \( \xi_1, \ldots, \xi_n \), for permutations \( \pi \) on \( 1, \ldots, n \). There is no way to choose one particular permutation as a representative, without implicitly assuming that the targets have a specific ordering.

3) Performance Evaluation. Multitarget tracking algorithms are multitarget state estimators. Performance evaluation of such algorithms requires the existence of a mathematical metric on multitarget states, in order to measure the distance between the ground truth state and
any given estimated state. However, no such distance metric exists for vector representation. Assume the contrary: a metric \( d(\xi_1, \xi_2) \). Then if \( \pi \) is not the identity permutation, \( \xi_{\pi} \neq \xi \) and yet \( d(\xi_{\pi}, \xi) = d(\xi, \xi) = 0 \), which contradicts the definition of a mathematical metric on multistate sets.

Remark 2: In objection to Item 1, it could be argued that the GLMB filter’s labeling scheme is vulnerable to possible statistical bias. This is not the case. As was noted in Section III-A this scheme assigns labels of the form \( \ell_1 = (k, 1), \ldots, \ell_{n_k} = (k, n_k) \) where \( t_k \), the time that an ensemble of \( n_k \) targets appeared and where the natural numbers \( 1, 2, \ldots, n_k \) distinguish those targets from each other. But here is the crucial point: It is not assumed that the list \( 1, \ldots, n \) (and therefore also \( \ell_1, \ldots, \ell_{n_k} \)) is ordered. The theoretical foundation of modern mathematics is Cantor’s set theory, whose primitive concept is the set. Vectors must therefore be defined in terms of sets. The simplest such definition is as follows: \( (x_1, \ldots, x_n) \) is a finite set \( \{x_1, 1, \ldots, (x_n, n)\} \) that has been endowed with the ordering relation \( (x_i, i) < (x_j, j) \) if and only if \( i < j \). If no such relation has been stipulated (as in the GLMB scheme) then the list \( 1, \ldots, n \) (and therefore also \( \ell_1, \ldots, \ell_{n_k} \)) has no inherent ordering. Indeed, Greek letters, or any other arbitrary symbols in any order, could have been used instead. A vector representation, on the other hand, is not possible unless precisely this ordering relation has been explicitly imposed on the collection \( x_1, \ldots, x_n \).

III. MATHEMATICAL REPRESENTATION OF TRAJECTORIES

As noted in the Introduction, two mathematical representations of evolving multitarget populations have been proposed: labeled RFS’s (LRFS’s) and trajectory RFS’s (TRFS’s). The purpose of this section is to summarize and contrast the two. It is organized as follows: LRFS’s (Section III-A); TRFS’s (Section III-B); comparison of LRFS’s and TRFS’s (Section III-C); counterexamples (Section III-D); and the trajectory PHD/CPHD filters (Section III-E).

A. Labeled RFS’s

From its inception in 1997, the RFS approach has included uniquely identifying target identities or labels as target state variables—see [11], pp.135,196-197 and [3], pp. 505-507. Because of computational considerations, however, the first implementations of RFS filters mostly did not address track labeling; and, when they did, employed computationally expensive techniques such as track-to-track association. Later implementations (for example, those based on Gaussian mixture or particle-system approximation) addressed trajectories in a computationally tractable manner via heuristic label-propagation schemes—see, e.g., [8], p. 244-250.

The labeled RFS (LRFS) theory of B.-T. Vo and B.-N. Vo [5], introduced in 2011 [4], is the first systematic, theoretically rigorous formulation of true multitarget tracking. Their provably Bayes-optimal GLMB filter is the currently most sophisticated LRFS tracking algorithm. As previously noted, its latest implementations can simultaneously track over one million targets in significant clutter in real time using off-the-shelf computing equipment [6].

This section summarizes LRFS theory, following the discussion in Chapter 15 of [8].

As in [11], pp.135,196-197, the state of a point target is \((x, \ell) \in X = X_0 \times \mathcal{L}\) where \(X_0\) is the kinematic state space and \(\mathcal{L}\) is a countable space of labels \(\ell\). As was noted in Remark 2 GLMB filter labels have the form \((k, i)\) where \(k \geq 0\) denotes the time \(t_k\) that the target first appeared and where \(i \geq 1\) distinguishes it from all others born at the same time. As noted in [12], labels can be extended to contain target-type or target-identity information, thus permitting simultaneous target tracking and classification/identification.

A finite subset \(X = \{(x_1, \ell_1), \ldots, (x_n, \ell_n)\} \subseteq X\) is said to be labeled if \(|X_L| = |X|\) where \(X_L = \{\ell_1, \ldots, \ell_n\}\) is its set of labels. That is: every target is assumed to have a unique identifying label. If all instantiations of an RFS \(\Xi \subseteq X\) are labeled, then \(\Xi\) is an LRFS.

Remark 3: Note that labeled sets are allowed to contain pairs of the form \((x, \ell_1), (x, \ell_2)\) with \(\ell_1 \neq \ell_2\). This is not a conceptual flaw. It is physically possible, for example, for two or more targets traveling in formation to be so close together that, within the resolution of the sensor, they appear to have identical kinematic states over an extended duration of time. Additionally, because point targets are mathematical idealizations that do not have a physical extent, it is possible for two different point targets to (for example) have the same position.

Bayes-optimal multitarget tracking is accomplished via the labeled multitarget recursive Bayes filter

\[
\ldots \to f_{k-1}(X|Z_{1:k-1}) \to f_k(X|Z_{1:k-1}) \to f_k(X|Z_{1:k}) \to \ldots
\]

where \(Z_{1:k} : Z_1, \ldots, Z_k, \ldots\) is the time-sequence of measurement-sets collected by a single sensor at times \(t_1, \ldots, t_k, \ldots\) and \(f_k(X|Z_{1:k})\) is a probability distribution on \(X \subseteq X_0 \times \mathcal{L}\). It must be the case that \(f_k(X|Z_{1:k}) = 0\) if \(X\) is not labeled—i.e., if \(X\) is physically impossible. Stated differently, \(f_k(X|Z_{1:k})\) is an “LRFS distribution.” Since targets evolve in four-dimensional space-time, their states have the form \((x, t, \ell) \in X = X_0 \times \mathcal{L} \times \mathbb{R}^+\) where \(t\) is a known constant. Since time is assumed to belong to a discrete sequence \(t_1, \ldots, t_k, \ldots\), abbreviate \((x, t, \ell)\) as \((x, \ell, k)\) in \(X_0 \times \mathcal{L} \times \mathbb{N}\).

The labeled multi-Bernoulli (LMB) distribution is a simple example of an LRFS distribution. It has the form

\[
f_J(X) = \delta_{|X|,|X_L|} \left( \prod_{\ell \in J \setminus X_L} (1 - q_\ell) \right) \left( \prod_{(x, \ell) \in X} 1_{J(\ell)} \cdot q_\ell s_\ell(x) \right)
\]

where \(J\) is a finite subset of \(\mathcal{L}\); and where \(q_\ell\) and \(s_\ell(x)\) are, respectively, the existence probability and spatial distribution of the target with label \(\ell \in J\). Note that \(f_J(X) = 0\) unless \(|X| = |X_L|\) and \(X_L \subseteq J\).

At time \(t_k\), the multitarget state-set is estimated from \(f_k(X|Z_{1:k})\) using a Bayes-optimal multitarget state estimator,
e.g., the joint multitarget (JoM) or marginal multitarget (MaM) estimator—see \[3\]. If
\[
\hat{X}^k = \{ (\hat{x}_1^k, \hat{\ell}_1^k, k), ..., (\hat{x}_n^k, \hat{\ell}_n^k, k) \}
\]
with \(\hat{X}^k = \hat{n}_k\) then, at time \(t_k\), \(\hat{n}_k\) is the estimated number of targets and \(\hat{x}_1^k, ..., \hat{x}_n^k\) are their estimated labels. If \(\hat{X}_1, ..., \hat{X}_k\) is the time-sequence of multitarget state-estimates from time \(t_1\) to time \(t_k\), define
\[
\tilde{X}^{i}_{\ell} = \begin{cases} \{(x, \ell, i) \} & \text{if } (x, \ell, i) \in \hat{X}^{i} \text{ for some } x \in \mathbb{X}_0 \, \vline \, \ell \in \{1, ..., L\} \\ \emptyset & \text{otherwise} \end{cases}
\]
The \(\ell\)-trajectory (a.k.a. \(\ell\)-track) is the time-sequence \(\tilde{X}^{i}_{\ell}, ..., \tilde{X}^{i}_{\ell+1}\) of singleton or empty sets (which thereby accounts for the acquisition, dropouts, and reacquisitions of the target with label \(\ell\)). The time-consecutive nonempty subsequences of an \(\ell\)-trajectory are its \(\ell\)-segments.

It follows that, in the LRFS framework, a track segment can be represented as a vector of the form
\[
( (x^1, \ell, k), (x^2, \ell, k+1), ..., (x^i, \ell, k+i-1) )
\]
where \(t_k\) is the segment’s initial time, \(i \geq 1\) is its length, and \(x^1, ..., x^i\) are its kinematic states at times \(t_k, ..., t_{k+i-1}\).

Now let “\(A \approx B\)” abbreviate the phrase “\(A\) is notionally equivalent to \(B\)” (in the sense that \(A\) and \(B\) are characterized by the same parameters). Then we can successively re-annotate the track segment as follows:
\[
( (x^1, \ell, k), (x^2, \ell, k+1), ..., (x^i, \ell, k+i-1) ) \approx (\ell, (x^1, \ell), (x^2, \ell+1), ..., (x^i, \ell+i-1) ) \approx (\ell, k, (x^1), (x^2), ..., (x^i) ) = (\ell, k, x^{1;i})
\]

That is: the track segment can be equivalently notated as \(T_\ell = (\ell, k, x^{1;i})\) where \(0 \leq k \leq k_{\text{max}}\) and \(1 \leq i \leq k_{\text{max}} - k + 1\); and where \(t_{k_{\text{max}}}\) is the end-time of the scenario.

**B. Trajectory Random Finite Sets**

The TRFS framework was introduced in 2014 in \[7\] and subsequently elaborated in \[13, 14, 15, 16, 17\]. There it was claimed that LRFS labels are deficient because:

1. they are “non-physical” (\[7\], Abstract);
2. they have “...no direct and unambiguous connection to the physical phenomena under observation...” (\[7\], p. 2, column 2);
3. they “...do not represent an underlying physical reality...” (\[7\], p. 3, top of column 1); and
4. a “...multitude of labeling [sic] can be developed...[but] there is no general way of distinguishing the merits of a given scheme...” which “...contravene[s]...the usual notion of there being a ‘one-to-one correspondence between physical states and their mathematical representations’...” (\[7\], p. 2, column 2), here citing p. 405 of \[3\].

In addition, it was claimed in \[7\] that the example in Figure 2 of four slightly different scenarios with “...two targets approaching on the real line, pausing and then separating illustrates the appeal of using RFS trajectories [rather than labels]...” (\[7\], column 1).

“Trajectory random finite sets” (TRFS’s) were therefore proposed as a replacement for LRFS’s. As described in pp. 1-2 of \[13\], a “trajectory” is what we earlier called a track segment. It has the form \(T = (k, x^{1;k})\) where \(0 \leq k \leq k_{\text{max}}\) is the trajectory’s initial time, \(1 \leq i \leq k_{\text{max}} - k + 1\) is its length, and \(x^1, ..., x^i\) are its kinematic states at times \(t_k, ..., t_{k+i-1}\). A finite set of trajectories (SoT) is, therefore, \(T = \{T_1, ..., T_n\}\) where \(T_1, ..., T_n\) are trajectories. A TRFS is a random variable on the class of SoT’s.

It was also claimed that the TRFS approach subsumes the LRFS approach as a special case because \(T = \{(k, x_1), ..., (k, x_n)\}\) is an equivalent representation of a set \(X_k = \{x_1, ..., x_n\}\) of targets at time \(t_k\).

The TRFS framework subsequently provided the basis for the trajectory PHD (TPHD) filter \[13\]; the trajectory CPHD (TCPHD) filter \[14\]; and the “trajectory Poisson multi-Bernoulli mixture trajectory filter” \[16, 17\]. The trajectory PHD and CPHD filters will be discussed in detail in Section III-E.

**C. Comparison of the LRFS and TRFS Frameworks**

The four enumerated criticisms of the LRFS framework in the previous section have already been rebutted in Section II-B. To reiterate: Neither the provisionality nor the infinitude of track labels means that they are “non-physical.” Rather, they are stand-ins for the unique identities of actual physical objects. As such, they are no more non-physical than any other signifier for those objects.

As for the argument relating to Figure 2 of \[7\], it appears to involve an apples-with-oranges comparison. It was implicitly assumed there that the four trajectories were all being considered as entities—that is, the four scenarios ended before measurement processing commenced. However, it is well known that a smoother algorithm will typically outperform a filter algorithm, because the former processes both past and future measurements whereas the latter can process past measurements only. Stated differently, the smoother is a batch-processing algorithm whereas the filter cannot discern the future and thus is, in this sense, “real-time.” TRFS algorithms, such as the TPHD and TCPHD filters discussed in Section III-E, are also of the smoother type since, at any given time \(t_k\), all trajectories prior to \(t_k\) must be taken into consideration.

It might thereby be argued that the TRFS framework is still valid within the context of multitarget smoothing. But as we will shortly see, this is not the case.

Let us now turn to a more detailed comparison of the LRFS and TRFS frameworks. In the former, a “trajectory” (track segment) has the form \(T_\ell = (\ell, k, x^{1;i})\); whereas in the latter it has the form \(T = (k, x^{1;i})\). Since a SoT has the form \(T = \{T_1, ..., T_n\}\), it must therefore be the case that \(T_j = (k_j, x_j^{1;i_j})\) for some \(k_j, i_j\), and \(x_j^{1}, ..., x_j^{i_j}\).

That is: the trajectory \(T_j\) has been implicitly assigned the integer label \(j\)—which is then ignored (“stripped off”). But as was noted in Section II-B when the signifier of an
entity is jettisoned, critical information about it is jettisoned as well. As the following subsection demonstrates, this loss of information results in numerous and serious difficulties.

D. Counterexamples

Counterexample 1: The LRFS framework is not a special case of the TRFS framework. Let $T = \{(k, x_1), \ldots, (k, x_n)\}$ with $x_1, \ldots, x_n$ distinct. Then this is mathematically equivalent to the LRFS representation $X_k = \{(1, k, x_1), \ldots, (n, k, x_n)\}$. From this one might conclude that the TRFS framework includes the LRFS framework as a special case. But this is not so. In Remark 3 it was noted that a labeled set of the form $X = \{(x, \ell_1), (x, \ell_2)\}$ with $\ell_1 \neq \ell_2$ is possible. Thus consider the case $x_1 = \ldots = x_n = x$, in which case $T = \{(k, x)\}$ whereas $X_k = \{(1, k, x), \ldots, (n, k, x)\}$ is a valid labeled finite set. That is: there is no way to represent such labeled multitarget states in the TRFS framework. And this is not the only possible such anomaly—see Counterexample 3.

Counterexample 2: Contrary to claim, the TRFS framework contravenes the usual notion of there being a one-to-one correspondence between physical states and their mathematical representations. Consider $T_0 = \{T_0\}$ and $T_1 = \{T_1, T_2, T_3\}$ where $T_0 = (k, x, x^1, x^2)$, $T_1 = (k, x)$, $T_2 = (k + 1, x^1)$, $T_3 = (k + 2, x^2)$. Then because $(k, x, x^1, x^2)$ is an abbreviation of $((k, x), (k+1, x^1), (k+2, x^2))$, it follows that $T_0$ and $T_1$ are mathematically distinct representations of the same physical trajectory $(k, x), (k + 1, x^1), (k + 2, x^2)$. Now restore the implicit labels that have been stripped off: $T_0 = (0, k, x, x^1, x)$, $T_1 = (1, k, x)$, $T_2 = (2, k + 1, x^1)$, $T_3 = (3, k + 2, x^2)$. Then the difficulty vanishes because $T_0$ represents the trajectory of a single physical target with label 0; whereas $T_1$ represents the trajectories of three successively appearing and disappearing targets with respective labels 1, 2, 3.

Counterexample 3: Impossible scenarios cannot be represented in the TRFS framework even while valid ones can. Consider $T_2 = \{T_1, T_2\}$ where $T_1 = (k, x, x^1)$ and $T_2 = (k, x, x^2)$ with $x, x^1, x^2$ distinct. Then $T_2$ is physically impossible since a single target $x$ at time $t_k$ cannot evolve to two different states $x^1$ and $x^2$ at time $t_{k+1}$. Now restore the stripped labels: $T_1 = (1, k, x, x^1)$, $T_2 = (2, k, x, x^2)$. Then $T_2$ represents a target separation (e.g., a target-spawning event): targets 1, 2 had identical states $x$ at time $t_k$, at which point they separated and evolved respectively to $x^1$ and $x^2$ at time $t_{k+1}$.

Counterexample 4: Contrary to claim, stripping labels from states increases tracking uncertainty. Consider $T_3 = \{T_1, T_2\}$ where $T_1 = (k, x; 1:5)$ and $T_2 = (k + 10, y; 1:5)$. Then there is an ambiguity: Does $T_3$ represent a single dropped and then reacquired target, or two successively appearing and disappearing tracks? This ambiguity is resolved if we restore stripped labels: either $T_1 = (1, k, x; 1:5)$ and $T_2 = (1, k + 10, y; 1:5)$ (if restored to a single reacquired track) or $T_1 = (1, k, x; 1:5)$ and $T_2 = (2, k + 10, y; 1:5)$ (if restored to two consecutive tracks).

E. Trajectory PHD and CPHD Filters

These are not the only difficulties with the TRFS framework. As noted earlier, the TPHD/TCPHD filters were introduced in [13] and [15], [14], respectively. The purpose of this subsection is to demonstrate that they are theoretically problematic. It is organized as follows: CPHD filters (Section III-E1); multitarget state estimation for CPHD filters (Section III-E2); the TPHD filter (Section III-E3); multitarget state estimation for the TPHD filter (Section III-E4); and the theoretical basis of the TPHD filter (Section III-E5).

1) The CPHD Filters: The conventional PHD filter [3] has the form

$$\ldots \rightarrow D_{k-1}(x|Z_{1:k-1}) \rightarrow D_k(x|Z_{1:k-1}) \rightarrow D_k(x|Z_{1:k}) \rightarrow \ldots$$

Here, $D_k(x|Z_{1:k})$ with $x \in X_0$ is the first-order statistical moment (a.k.a. PHD or “intensity density”) of $f_k(X|Z_{1:k})$ with $X \subseteq \hat{X}_0$, which in turn is assumed to be the distribution of a Poisson RFS with PHD $D_k(x|Z_{1:k})$.

The CPHD filter [3] propagates the cardinality distribution $p_k(n|Z_{1:k})$ (i.e., the probability that there are $n$ targets present at time $t_k$) in addition to the PHD $D_k(x|Z_{1:k})$. It is based on i.i.d.c. RFS’s (which are generalizations of Poisson RFS’s).

Remark 4: Poisson RFS’s on $X_0 \times \Sigma$ are not LRFS’s because their distributions have the form

$$f(X) = e^{-\sum_i f(x, \ell) dx} \prod_{(x, \ell) \in X} D(x, \ell).$$

Suppose that $Y = \{(x_1, \ell), (x_2, \ell)\}$ for some $x_1 \neq x_2$ such that $D(x_1, \ell) > 0$, $D(x_2, \ell) > 0$. Then $Y$ is physically impossible since $|Y| = 2 \neq 1 = |Y_0|$. Yet $f(Y) \propto D(x_1, \ell), D(x_2, \ell) \neq 0$—i.e., $f(X)$ is not an LRFS distribution.

2) Multitarget State Estimation for the PHD Filter: This proceeds as follows. First, compute

$$\hat{N}_k = \int D_k(x|Z_{1:k}) dx,$$

which is the expected number of targets in the multitarget population at time $t_k$. Second, round $\hat{N}_k$ off to the nearest integer $\tilde{n}_k$. Third, determine the states $\hat{x}_1, \ldots, \hat{x}_{\tilde{n}_k}$ such that $D_k(\hat{x}_1|Z_{1:k}), \ldots, D_k(\hat{x}_{\tilde{n}_k}|Z_{1:k})$ are the heights of the $\tilde{n}_k$ tallest “peaks” of the graph of the function $D_k(x|Z_{1:k})$.

State estimation for the CPHD filter differs only in that

$$\tilde{n}_{k|k} = \arg \sup_{n \geq 0} p_k(n|Z_{1:k}).$$

3) The TPHD Filter: In [13] it was claimed that the PHD filter can be directly generalized from RFS’s to TRFS’s. This “trajectory PHD filter” (TPHD filter) has the form

$$\ldots \rightarrow \hat{D}_{k-1}(T|Z_{1:k-1}) \rightarrow \hat{D}_k(T|Z_{1:k-1}) \rightarrow \hat{D}_k(T|Z_{1:k}) \rightarrow \ldots$$

where (in the case of $\hat{D}_{k}(T|Z_{1:k}))$ $T = (k', x; 1:5)$ for all $0 \leq k' < k$, $1 \leq i \leq k' - k + 1$, and $x^{1:i} \in \Sigma$. This means that all trajectories prior to time $t_k$—i.e., all $(k', x; 1:5)$ with $0 \leq k' < k - 1$ and $1 \leq i \leq k - k'$—must be taken into consideration. Consequently, calculation of the TPHD

$$\hat{D}_k(T|Z_{1:k})$$

for all $T$ is significantly more computationally intensive than calculation of the corresponding PHD $D_k(x|Z_{1:k})$ for all $x$. 
4) Multitarget State Estimation for the TPHD Filter: This is claimed to have exactly the same form as before, i.e.: (a) compute \((13)\), Eq. (1))
\[
\hat{N}_k = \int \tilde{D}_k(T|Z_{1:k})dT
\]
\[
= \sum_{k'=0}^{k-k'+1} \sum_{i=1}^{k_i} \tilde{D}_k(k',x^{1:i}|Z_{1:k})d\bar{x}^{1:i};
\]  
(b) round off \(\hat{N}_k\) to \(n_k\); and (c) determine the locations \(T_1,...,T_{\hat{n}_k}\) of the \(n_k\) largest peaks of the graph of \(\tilde{D}_k(T|Z_{1:k})\).

This TPHD/TPHD estimation procedure is mathematically undefined. For, let \(\Upsilon\) be the unit of measurement of the single-target state space \(\mathcal{X}\). Then the unit of measurement of a trajectory \((k',x^{1:i})\) is \(\Upsilon\) and so the unit of \(\tilde{D}_k(k',x^{1:i}|Z_{1:k})\) is \(\Upsilon^{-1}\). Because it is impossible to numerically compare \(\tilde{D}_k(k',x^{1:i}|Z_{1:k})\) to \(\tilde{D}_k(k'',x^{1:i}|Z_{1:k})\) when \(i \neq i'\), it is impossible to determine \(T_1,...,T_{\hat{n}_k}\).

This remains true when the TPHD/TPHD filters are implemented using Gaussian mixture approximations. These have the form \((13)\), Eqs. (16,17)):
\[
\tilde{D}_k(k',x^{1:i}) \equiv \sum_{j=1}^{N_k} w_{k',j} \cdot \tilde{N}_{k',j}(k',x^{1:i};(k'_j,x^{1:i},j))
\]
where \(w_{k',j} \geq 0\); where
\[
\tilde{N}_{k',0}(k',x^{1:i});(k_0,x^{1:i})) = \delta_{k',k_0} \delta_{i,0} \cdot \tilde{N}_{k',0}(x^{1:i} - x_0^{1:i})
\]
is a probability distribution on trajectories \(T = (k',x^{1:i});\) and where \(\tilde{N}_{k',0}(x^{1:i})\) denotes a zero-mean Gaussian distribution in the variable \(x^{1:i} = (x^{1},...,x^{i})\) with covariance matrix \(\bar{P}_{k',0}^{i}\).

We may assume that \(w_{k',j} > 0\) for all \(j\), which forces \(i_{k',j} = i\) for all \(j\) if the summation in \((10)\) is to be mathematically well-defined. Thus at its most general, the GM in \((10)\) must have the form
\[
\tilde{D}_k(k',x^{1:i}) \equiv \hat{N}_k \sum_{j=1}^{N_k} w_{k',j} \cdot \tilde{N}_{k',j}(x^{1:i} - x_{k',j}^{1:i})
\]
where \(\hat{N}_k > 0\) and where \(w_{k',j} > 0\) are such that
\[
\sum_{k'=0}^{k-k'+1} \sum_{j=1}^{N_k} w_{k',j} = 1,
\]
from which follows \(\int \tilde{D}_k(T)dT \equiv \hat{N}_k\). Since \(\tilde{D}_k(k',x^{1:i})\) and \(\tilde{D}_k(k'',x^{1:i})\) are numerically incommensurable when \(i \neq i'\), it follows that the inequality \(w_{k',j} > w_{k'',j'}\) cannot compel us to conclude that GM component \(x_{k',j}^{1:i}\) has a higher “peak” than GM component \(x_{k'',j'}^{1:i}\). That is: it is not true that “...the estimated set of trajectories corresponds to...the components with highest weights...” (as was asserted in regard to Eq. (21) of \((13)\)).

5) Theoretical Basis of the TPHD Filter: The TPHD filter is inherently theoretically erroneous because it requires “Poisson TRFS’s”—which, like Poisson LRFS’s, do not exist. For, according to \((13)\), Eq. (7)) the distribution of such a TRFS has the form
\[
f(T) = e^{-f(T)dT} \prod_{T \in T} \tilde{D}(T).
\]
Let \(T_0 = T_0 \cup T_1\) where \(T_0,T_1\) were defined in Counterexample 2 of Section \((13)\) Then \(T_0\) is a physically impossible SoT since it contains two distinct instances of the same physical trajectory. If \(f(T)\) is a TRFS distribution then it must vanish on impossible SoT’s—in particular, it must be the case that \(f(T_0) = 0\). But
\[
f(T_0) \propto \tilde{D}(T_0) \cdot \tilde{D}(T_1) \cdot \tilde{D}(T_2) \cdot \tilde{D}(T_3) \neq 0,
\]
where the four factors on the right can be nonzero since \(T_0,T_1,T_2,T_3\) are, individually, valid trajectories.

The same comments apply to the trajectory Poisson multi-Bernoulli mixture trajectory filter \((15)\), \((17)\) since it also requires non-existent Poisson TRFS’s. The TCHP filter is similarly erroneous since it requires i.i.d.c. TRFS’s—which do not exist for the same reason that Poisson TRFS’s do not exist.

IV. Conclusions

This paper described and compared two proposed mathematical representations of multiple-target trajectories: the LRFS vs. TRFS frameworks. It was shown that the latter is questionable in multiple and serious respects, whereas the claimed deficiencies of the former are mistaken. This was accomplished via a systematic, step-by-step analysis beginning with phenomenological and epistemological basics—in particular, the semiotic concepts of signified vs. signifier.

References

[1] D. B. Reid, “An algorithm for tracking multiple targets,” IEEE Trans. Auto. Cont., vol. AC-24 no. 6, pp. 843-854, 1979.
[2] B.-N. Vo, M. Mallick, Y. Bar-Shalom, S. Coraluppi, R. Osborne III, R. Maher, and B.-T. Vo, “Multitarget Tracking,” in J. Webster (ed.), Wiley Encyclopedia of Electrical and Electronics Engineering, Wiley, New York, 2015.
[3] R. Maher, Statistical Multisource-Multitarget Information Fusion, Artech House, Norwood, MA, 2007.
[4] B.-T. Vo and B.-N. Vo, “A random finite set conjugate prior and application to multi-target tracking,” Proc. 2011 Int’l Conf. on Intelligent Sensors, Sensor Networks, and Information Processing (ISSNIP2011), Adelaide, Australia, Dec. 6-9, 2011.
[5] B.-T. Vo and V.-N. Vo, “Labeled random finite sets and multi-object conjugate priors,” IEEE Trans. Sign. Proc., vol. 61, No. 13, pp. 3460-3475, 2013.
[6] M. Beard, B.-T. Vo, and B.-N. Vo, “A solution for large-scale multi-object tracking,” IEEE Trans. Sign. Proc., 68: 2754-2769, 2020.
[7] L. Svensson and M. Morelande, “Target tracking based on estimation of sets of trajectories,” Proc. 17th Int’l Conf. on Information Fusion, Salamanca, Spain, July 7-10, 2014.
[8] R. Maher, Advances in Statistical Multisource-Multitarget Information Fusion, Artech House, Norwood, MA, 2014.
[9] Á. García-Fernández, J. Williams, K. Granström, and L. Svensson, “Poisson multi-Bernoulli mixture filter: Direct derivation and implementation,” IEEE Trans. Aerospace & Electronic Systems, vol. 54, no. 4, pp. 1883-1901, 2018.
[10] R. Maher, “Statistics 103” for multitarget tracking,” Sensors, 19(1) 202, 2019, open source: https://doi.org/10.3390/s19010202.
[11] I. R. Goodman, R. P. S. Maher, and H. T. Nguyen, Mathematics of Data Fusion, Kluwer Academic Publishers, New York, 1997.
[12] B.-T. Vo and B.-N. Vo, “Tracking, identification and classification with random finite sets,” *Proc. SPIE*, Vol. 8745, 2013.

[13] Á. García-Fernández and L. Svensson, “Trajectory probability hypothesis density filter,” *Proc. 21st Int’l Conf. on Information Fusion*, Cambridge, UK, pp. 1442-1449, 2018.

[14] Á. García-Fernández and L. Svensson, “Trajectory PHD and CPHD filters,” *IEEE Trans. Sign. Proc.*, 67(22): ???, 2019.

[15] Á. García-Fernández and L. Svensson, “Trajectory PHD and CPHD filters,” arXiv, version 3, 2019, 1811.08820v3.

[16] K. Granström, L. Svensson, J. Williams, and Á. García-Fernández, “Poisson multi-Bernoulli mixture trackers: continuity through random finite sets of trajectories,” *Proc. 21st Int’l Conf. on Information Fusion*, Cambridge, UK, pp. 986-994, 2018.

[17] Xuxuan Xia, K. Granström, L. Svensson, and Á. García-Fernández, “An implementation of the Poisson multi-Bernoulli mixture trajectory filter via dual decomposition,” *Proc. 21st Int’l Conf. on Information Fusion*, Cambridge, UK, pp. 2457-2464, 2018.