Decentralized Federated Learning via Non-Coherent Over-the-Air Consensus

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Abstract—This paper presents NCOTA-DGD, a Decentralized Gradient Descent (DGD) algorithm that combines local gradient descent with a novel Non-Coherent Over-The-Air (NCOTA) consensus scheme to solve distributed machine-learning problems over wirelessly-connected systems. NCOTA-DGD leverages the waveform superposition properties of the wireless channels: it enables simultaneous transmissions under half-duplex constraints, by mapping local optimization signals to a mixture of preamble sequences, and consensus via non-coherent combining at the receivers. NCOTA-DGD operates without channel state information at transmitters and receivers, and leverages the average channel pathloss to mix signals, without explicit knowledge of the mixing weights (typically known in consensus-based optimization algorithms). It is shown both theoretically and numerically that, for smooth and strongly-convex problems with fixed consensus and learning stepsizes, the updates of NCOTA-DGD converge in Euclidean distance to the global optimum with rate $\mathcal{O}(K^{-1/4})$ for a target of $K$ iterations. NCOTA-DGD is evaluated numerically over a logistic regression problem, showing faster convergence vis-à-vis running time than implementations of the classical DGD algorithm over digital and analog orthogonal channels.

I. INTRODUCTION

Federated learning (FL) [2] has emerged as a new paradigm to alleviate the communication burden and privacy concerns associated with the transmission of raw data to a ML server, by leveraging decentralized computational and communication resources at the edge of the network. Typically, it aims to solve

$$
\mathbf{w}^* = \arg\min_{\mathbf{w} \in \mathbb{R}^d} F(\mathbf{w}) \triangleq \frac{1}{N} \sum_{i=1}^{N} f_i(\mathbf{w}) \quad (P)
$$

among $N$ edge devices, where $f_i(\mathbf{w})$ is the empirical loss based on the local dataset of node $i$ (known to $i$ alone), and $\mathbf{w}$ is a $d$-dimensional parameter vector, so that $F(\mathbf{w})$ represents the empirical loss over the network. Conventional FL solves (P) based on a client-server architecture, where the $N$ nodes interact with a parameter server (PS, such as a base station) over multiple rounds [3]: in each round, edge devices compute local gradients based on a global model broadcast by the PS, and transmit them to the PS; the latter aggregates the local gradients and updates the global model via gradient descent.

Yet, in many important scenarios, a PS may be lacking [4], or direct communication with the PS may be challenging due to severe channel propagation conditions. In these cases, a decentralized learning architecture may be more attractive, in which the edge devices communicate with each other without the aid of a PS [4]. A renowned algorithm to solve (P) in this setting is Decentralized Gradient Descent (DGD) [5]: at iteration $k$, each node $(i)$ updates its local optimization signal $\mathbf{w}_{i,k}$ by combining a consensus signal $\mathbf{c}_{i,k}$ (weighted sum of the neighbors’ signals) with a local gradient step, as

$$
\mathbf{w}_{i,k+1} = \mathbf{w}_{i,k} + \sum_{j=1}^{N} \omega_{i,j}(\mathbf{w}_{j,k} - \mathbf{w}_{i,k}) - \eta \nabla f_i(\mathbf{w}_{i,k}); \quad (1)
$$

here, $\omega_{i,j}$ are non-negative mixing weights with $\omega_{i,i} = 0$ and $\sum_{j=1}^{N} \omega_{i,j} = 1$ (if $i$ and $j$ do not communicate directly).

Nevertheless, (1) relies on communications over orthogonal, noise- and error-free links, and on knowledge of the mixing weights to aggregate the incoming signals. In many practical scenarios (e.g., swarms of UAVs), communications occur over wireless links: interference from simultaneous transmissions, fading and noise may preclude the ability to receive error-free signals. Mitigating these sources of errors typically requires: 1) centralized coordination of network scheduling and interference management operations; 2) channel state information (CSI) to compensate signal fluctuations and link outages due to fading. Yet, such coordination may be non-trivial in wireless decentralized systems, and CSI acquisition may be impaired by pilot contamination and a source of severe overhead.

This calls for the design of decentralized optimization schemes that operate over (and leverage properties of) wireless channels. In this paper, we present NCOTA-DGD, an implementation of (1) over wireless channels subject to noise, fading, and interference. Its main feature is a Non-Coherent (NC)-Over-The-Air (OTA) consensus step that leverages the waveform superposition properties of the wireless channels, without the need for CSI at transmitters and receivers, and without explicit knowledge of the mixing weights. We show theoretically and numerically that, for smooth and strongly-convex problems with suitable constant consensus and learning stepsizes, the error (Euclidean distance) between the local models and the solution of (P) converges to zero with rate $\mathcal{O}(K^{-1/4})$ for a target of $K$ iterations.

Recent OTA-based schemes under client-server [6]–[10] and decentralized [11]–[13] FL rely on accurate CSI and careful power control to compensate signal fluctuations due to fading. Focusing on decentralized FL, the works [11]–[13] use graph coloring to break down the network into smaller non-interfering subgraphs, in which one device operates as the PS. This expedient enables the use of techniques developed for client-server FL (including channel inversion) coupled with a suitable consensus enforcing step. Yet, these schemes rely on CSI and power control to invert channels, scheduling operations (e.g., graph coloring), and knowledge of the
network structure and mixing weights $\omega_{i,j}$ for consensus. In contrast, NCOTA-DGD overcomes this need by using a set of orthogonal preamble sequences to encode signals, coupled with non-coherent combining at the receivers; it leverages the channel pathloss to mix signals, without explicit knowledge of the mixing weights. The paper [14] studies consensus over a shared multipath channel. While it assumes the channels to be noiseless and static, our work focuses on decentralized optimization over noisy, time-varying fading channels.

The preamble-based technique developed in this paper is inspired by the preamble-based random access scheme developed in [15] to encode local gradients in client-server FL. The scheme therein relies on noise-free downlink and inversion of the average pathloss at the transmitters. Differently from [15]: 1) NCOTA-DGD operates in decentralized settings, with all links subject to fading and noise; 2) instead of random access-based preamble selection of [15], NCOTA-DGD maps local signals deterministically to a suitable linear combination of preamble sequences; 3) rather than inverting channels, NCOTA-DGD leverages the average pathloss to mix signals.

This paper is organized as follows. In Sec. II, we describe NCOTA-DGD, followed by its convergence analysis in Sec. III. In Sec. IV, we present numerical results, followed by concluding remarks in Sec. V. An extended version of this work appears in [1], including complete proofs, a more general and non-trivial analysis with decreasing stepsizes, more general fading models and frequency selective channels, stochastic gradient descent updates and extensive numerical evaluations.

### Notation:
All vectors are in column form. For vector $a$ (boldface), $[a]_i$ is its $i$th component, and $\|a\| = \sqrt{\text{E} \{a^T a\}}$ its Euclidean norm. For random vector $a$, we define $\|a\|_E = \sqrt{\text{E} \{\|a\|^2\}}$ and $\|a\|_{E[A]}$ when the expectation is conditional on event $A$. $e_m$ is the standard basis vector with $m$th component equal to 1 and 0 otherwise; 1 and 0 are vectors of 1’s and 0’s; their dimension is deduced from the context. $I_n$ is the $n \times n$ identity matrix. $[A]$ is the indicator of event $A$. $A \otimes B$ is the Kronecker product of matrices $A$ and $B$.

### II. System Model and NCOTA-DGD
We consider $N$ wirelessly-connected edge devices, solving (P) via a noisy version of DGD in (1). We assume that the optimizer of (P), $w^*$, lies in a $d$-dimensional space $\mathcal{W}$ of radius $R$, within which the optimization is restricted (without loss of generality, centered at 0). For instance, for strongly-convex $F(\cdot)$ with strong-convexity parameter $\mu$, as assumed in the convergence analysis of Sec. III, since $\nabla F(w^*) = 0$, it holds $\|\nabla F(0)\| = \|\nabla F(0) - \nabla F(w^*)\| \geq \mu \|0-w^*\|$, hence $\forall w \in \mathbb{R}^d: \|w\| \leq R$, where $R = \frac{1}{\mu} \|\nabla F(0)\|$ may be initially estimated via a consensus phase.

To solve (P) iteratively, we divide time into frames of fixed duration $T$. In frame $k$, node $i$ generates the transmission signal $x_{i,k}$ using the Preamble-based Encoding procedure of Sec. II-A, and transmits it over the wireless channel. Upon receiving the signal from the other nodes in the network (Sec. II-B), it then computes a consensus signal using the Non-Coherent Over-the-Air procedure of Sec. II-C; finally, it updates the local optimization variable by combining the consensus signal with the local gradient descent (Sec. II-D). Due to randomness of noise and fading, this procedure induces a stochastic process defined on a proper probability space; we denote by $\mathcal{F}_k$ the $\sigma$-algebra consisting of all signals generated up to frame $k$ excluded, along with $w_{i,k}$.

#### A. Preamble-based Encoding
Let $Z \equiv \{z_m \in \mathbb{R}^d: m = 1, \ldots, M\}$ be a codebook of $M = d+1$ codewords and $Z = [z_1, \ldots, z_M]$ be the $d \times M$ matrix with $m$th column equal to $z_m$. These are defined as $z_{d+1} = -R1$ and $z_m = 2Rde_m - R1, m = 1, \ldots, d$. With this choice, any $w \in \mathcal{W}$ may be represented as a convex combination of $Z$. To see this, define the convex combination vector $p \in \mathbb{R}^M$ as
\[
[p]_m = \frac{|w|_m + R}{2Rd}, \quad \forall m = 1, \ldots, d, \quad [p]_{d+1} = 1 - \sum_{m=1}^d [p]_m.
\]

Since $w \in \mathcal{W}$ (i.e., $\|w\| \leq R$ and $|w|_m \geq 0, \forall m$), and
\[
w = \sum_{m=1}^M [p]_m z_m = Z \cdot p.
\]

Hence, $p$ in (2) defines the desired convex combination.

Let $U = \{u_m \in \mathbb{C}^M: m = 1, \ldots, M\}$ be a set of $M$ orthogonal preamble sequences, defined as $u_m = \sqrt{M}e_m$. Then, given its local optimization signal $w_{i,k}$, node $i$ generates the combination vector $p_{i,k}$ via (2), and the transmission signal
\[
x_{i,k} = \sqrt{E} \sum_{m=1}^M \sqrt{|p_{i,k}|_m} u_m,
\]
with average energy per sample $E = \|x_{i,k}\|^2/M$.

#### B. Transmission over the wireless channel
Each node then transmits its signal $x_{i,k}$ over the wireless channel. We assume Rayleigh flat fading channels $h_{i,j}^k \sim \mathcal{CN}(0, \sigma^2 I)$ between transmitting node $j$ and receiving node $i$ in frame $k$, where $h_{i,j}$ is the large-scale pathloss. We assume $\Lambda_{i,j} = \Lambda_{j,i}$ (channel reciprocity). Furthermore, $h_{i,j}^k$ is i.i.d. over $k$, and independent across $i,j$.

We assume that the nodes operate under a half-duplex constraint. We thus divide each frame of duration $T$ into 2 slots; each node is assigned to transmit in only one of the 2 slots, and operates in receive mode in the other slot. This assignment is kept fixed during the entire optimization session, and may be done randomly by each node. Let $N_i$ be the set of nodes that transmit when node $i$ is in receive mode. Node $i$ thus receives the signal
\[
y_{i,k} = \sum_{j \in N_i} h_{i,j}^k x_{j,k} + n_{i,k},
\]
where $n_{i,k} \sim \mathcal{CN}(0, \sigma^2 I_M)$ is AWGN noise with variance $\sigma^2$. $y_{i,k}$ is then correlated with the $M$ preamble sequences as
\[
x_{i,m}^k = \frac{u_m^H y_{i,k}}{\sqrt{E} \|u_m\|^2} = \sum_{j \in N_i} h_{i,j}^k \sqrt{|p|_{j,m}} + n_{i,m}^k;
\]
\[
n_{i,m}^k \triangleq u_m^H n_{i,k} / \sqrt{E} \|u_m\|^2 \sim \mathcal{CN}(0, \sigma^2/(M \cdot E))
\]
is the equivalent noise, i.i.d. over $i,k,m$, due to the orthogonality
of preamble sequences. Since \( h^{k}_{i,j} \sim \mathcal{CN}(0, \Lambda_{i,j}) \), one can see that \( \mathbb{E}[|x_{i,m}^{k}|^2 | F_k] \) has exponential distribution with mean
\[
\mathbb{E}[|x_{i,m}^{k}|^2 | F_k] = \sum_{m \in \mathcal{N}} \Lambda_{i,j} |p_{j,k}|^2 m + \frac{\sigma^2}{M \cdot E},
\]
a fact exploited to build the consensus signal in the next step.

C. Non-Coherent Over-the-Air Consensus

At the end of frame \( k \), node \( i \) computes the consensus signal
\[
d_{i,k} = \sum_{m=1}^{M} \left( |x_{i,m}^{k}|^2 - \frac{\sigma^2}{M \cdot E} \right) (z_m - w_{i,k}).
\]
Using (7), it is straightforward to see that
\[
\mathbb{E}[d_{i,k} | F_k] = \sum_{j \in \mathcal{N}} \Lambda_{i,j} \sum_{m=1}^{M} |p_{j,k}|^2 m (z_m - w_{i,k}).
\]
Furthermore, let \( \Lambda \triangleq \max_{j \in \mathcal{N}} \Lambda_{i,j} \), and define the mixing weights \( \omega_{i,j} = \frac{\Lambda_{i,j}}{\Lambda} \) for \( i \neq j \) and \( \omega_{i,i} = 1 - \sum_{j \neq i} \omega_{i,j} \).

D. Local optimization state update

Finally, node \( i \) updates \( w_{i,k} \) by combining the consensus signal with a local gradient descent (computed in parallel with transmission and reception), followed by a projection onto \( \mathcal{W} \), yielding the NCOTA-DGD update
\[
w_{i,k+1} = \Pi[w_{i,k} + \gamma d_{i,k} - \eta \nabla f_i(w_{i,k})],
\]
where \( \gamma > 0 \) and \( \eta > 0 \) are consensus and learning stepsizes, respectively. As shown in Sec. III, these need to be chosen suitably, in order to mitigate the impact of fading and noise. The projection operator \( \Pi[a] \) onto the sphere \( \mathcal{W} \) is defined as
\[
\Pi[a] = \arg \min_{w \in \mathcal{W}} \| w - a \| = \begin{cases} a, & \text{if } \| a \| \leq R, \\ \frac{R}{\| a \|} a, & \text{otherwise}, \end{cases}
\]
guarantees that \( w_{i,k} \in \mathcal{W}, \forall i, \forall k \). The process described in Sec. II-A to Sec. II-D is then repeated in frame \( k+1 \) with the new local optimization variable \( w_{i,k+1} \), and so on. A sketch of the overall NCOTA-DGD algorithm is shown below:

Algorithm 1 NCOTA-DGD

1: Initialization: \( w_{i,0} \in \mathcal{W}, \) transmit slot assignment for each node \( i = 1, \ldots, N \); 2: for \( k = 0, 1, \ldots \) each node do
3: \indent procedure (at node \( i \), given \( w_{i,k} \));
4: \indent Compute convex combination \( p_{i,k} \) via (2);
5: \indent Generate transmission signal \( x_{i,k} \) via (4);
6: \indent Transmit \( x_{i,k} \) on the assigned transmit slot;
7: \indent Receive \( y_{i,k} \) on the assigned receive slot (see (5));
8: \indent Compute \( r_{i,m}^{k} = \sum_{j \in \mathcal{N}} \rho_{i,j} |p_{j,k}|^2 / \sqrt{\mathbb{E}[|x_m|^4]}, \forall m = 1, \ldots, M \); 9: \indent and consensus signal \( d_{i,k} \), as in (8);
10: \indent Update \( w_{i,k+1} = \Pi[w_{i,k} + \gamma d_{i,k} - \eta \nabla f_i(w_{i,k})] \) as in (11); 11: \indent end procedure;
12: \indent \( k \leftarrow k + 1 \);
13: \indent end for.

We now express NCOTA-DGD as a noisy version of DGD in (1). Let \( e_{i,k} = d_{i,k} - \mathbb{E}[d_{i,k} | F_k] \) be the error due to fading and noise. Using (9), we can then rewrite (11) as
\[
w_{i,k+1} = \Pi \left[ w_{i,k} + \gamma \Lambda \sum_{j=1}^{N} \omega_{i,j} (w_{j,k} - w_{i,k}) - \eta \nabla f_i(w_{i,k}) + \gamma e_{i,k} \right].
\]

A few observations are in order:

1) The mixing weights satisfy \( \omega_{i,j} \geq 0, \forall i,j \) and \( \omega_{i,i} = \omega_{j,j} \), since \( \Lambda_{i,j} = \Lambda_{j,i} \) (channel reciprocity) and \( \{ j \in \mathcal{N} \} \Rightarrow \{ i \in \mathcal{N} \} \). Hence, \( \Omega_{i,j} \triangleq \omega_{i,j} \) is a symmetric, doubly-stochastic mixing matrix (as commonly assumed in consensus-based optimization [5]) induced by the large-scale propagation conditions of the channel.

2) When \( \gamma = (\Lambda^*)^{-1}, e_{i,k} = 0 \), and neglecting the projection operation, (12) reduces to the DGD updates (1). Hence, NCOTA-DGD can be interpreted as a projected DGD with noisy consensus. The consensus stepsize \( \gamma \) helps mitigate the detrimental effect of error propagation due to noise and fading. Remarkably, unlike (1), no explicit knowledge of the mixing weights is required in NCOTA-DGD.

Since each frame includes 2 slots, and in each slot \( M = d + 1 \) samples are transmitted, the frame duration of NCOTA-DGD is \( T = \frac{2(d+1)}{W_{tot}}, \) where \( W_{tot} \) is the bandwidth of the system.

III. CONVERGENCE ANALYSIS

Let \( w_k = \sum_{i=1}^{N} e_i \otimes w_{i,k} \) be the \( w_{i,k} \)-signals stacked over the network; similarly, let \( e_k = \sum_{i=1}^{N} e_i \otimes e_{i,k} \) be the error signals due to fading and noise, stacked over the network. Let
\[
f(W) = \sum_{i=1}^{N} f_i(w_i) \text{ and } \Omega = \Omega \otimes I_d.
\]
We then stack the updates (12) as
\[
w_{k+1} = \Pi^N [w_k + \gamma \Lambda^* (\hat{\Omega} - I_{Nd}) W_k - \eta \nabla f(W_k) + \gamma e_k],
\]
where \( \Pi^N[A] = \arg \min_{W \in \mathcal{W}^N} \| W - A \| \) is the projection operator, stacked over the network. Similarly to [5] for the analysis of DGD, we interpret this update as a noisy centralized projected gradient descent step with stepsize \( \eta \) (see (13)), based on the Lyapunov function
\[
G(W) \triangleq f(W) + \frac{\gamma^*}{2\eta} W^\top (I_{Nd} - \Omega) W,
\]
where the second term incentivizes consensus (in fact, it equals zero when \( w_i = w_j, \forall i,j \)). We can then rewrite
\[
w_{k+1} = \Pi^N [w_k - \eta \nabla G(W_k) + \gamma e_k].
\]

We study the convergence of NCOTA-DGD under the following standard assumptions.

Assumption 1. \( f_i(w) \) are \( \mu \)-strongly convex and \( L \)-smooth.

Assumption 2. \( \zeta \triangleq R \cdot \|w^*\| > 0 \) (\( w^* \) is in the interior of \( \mathcal{W} \)).

Since \( \Omega \) is symmetric and doubly-stochastic, its eigenvalues (\( \rho_i \) for the \( i \)th one) are real-valued and 1=\( \rho_1 \geq \rho_2 \geq \ldots \geq \rho_N \geq -1 \). We make the following standard assumption on \( \rho_2 \).

Assumption 3. \( \rho_2 < 1 \).
We are now ready to present the convergence properties of NCOTA-DGD. The main idea is to decompose the error between the local optimization variables and the global optimum into: 1) the error between W_k and the minimizer of the Lyapunov function G, defined as
\[
W^{(G)} = \arg \min_{W \in \mathbb{W}} G(W); \tag{14}
\]
and 2) the error between the latter and the global optimum w*. Accordingly, we bound via the triangle inequality
\[
\left( \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \|w_{i,k} - w^*\|^2 \right] \right)^{1/2} = \frac{1}{\sqrt{N}} \|W_k - 1 \otimes w^*\|_\mathbb{E} \leq \frac{1}{\sqrt{N}} \|W_k - W^{(G)}\|_\mathbb{E} + \frac{1}{\sqrt{N}} \|W^{(G)} - 1 \otimes w^*\|. \tag{15}
\]
These terms are individually bounded in Theorem 1. A sketch of its proof is provided in Appendix. Theorem 1. Assume: (C1) \(\eta(\mu + L) + \lambda \geq (1 - \rho_N) \leq 2\); (C2) \(\eta \leq \sqrt{\frac{d}{\psi_{\max}}}, \) with \(Z \triangleq (1 - \rho_N) \otimes 1\). Then,
\[
\frac{1}{\sqrt{N}} \|W_k - W^{(G)}\|_\mathbb{E} \leq 2R \left( \sqrt{2d} \frac{\tilde{\eta}^2}{\mu} \left( \lambda + \frac{\sigma^2}{\sqrt{\mu}} \right) \gamma + e^{-\mu \eta k} \right), \tag{16}
\]
and
\[
\frac{1}{\sqrt{N}} \|W_k - W^{(G)} - 1 \otimes w^*\|_\mathbb{E} \leq \frac{\sqrt{\max \eta}}{Z} \gamma. \tag{17}
\]
To minimize these errors, \(\gamma/\sqrt{\eta}\) and \(\tilde{\eta}/\gamma\) need both be small, while \(\eta\) should be large to make \(e^{-\mu \eta k}\) small, yielding a tradeoff between the tuning of \(\eta\) and \(\gamma\). To further investigate the convergence properties, let us consider a target timeframe \(K\) at which the algorithm stops. It appears then reasonable to choose \(\eta \approx a \cdot \tilde{K}^{-x}\) and \(\gamma \approx b \cdot \tilde{K}^{-y}\) for suitable \(a, b, x, y > 0\). Under this choice, (16)-(17) specialize as
\[
\frac{1}{\sqrt{N}} \|W_k - W^{(G)}\|_\mathbb{E} \leq 2R \left( \sqrt{2d} \frac{\tilde{\eta}^2}{\mu} \left( \lambda + \frac{\sigma^2}{\sqrt{\mu}} \right) \gamma + e^{-\mu \eta k} \right),
\]
and
\[
\frac{1}{\sqrt{N}} \|W_k - W^{(G)} - 1 \otimes w^*\|_\mathbb{E} \leq \frac{\sqrt{\max \eta}}{Z} \gamma.
\]
The exponential term requires \(x < 1\) to converge when \(K \to \infty\), while the other two terms are of order \(O(K^{(x-1)/3})\). Hence, \(\max \{y - x, x/2 - y\}\) should be minimized subject to \(x < 1\), yielding the following corollary.

Corollary 1. Let \(0 < \epsilon < 1\). With \(\eta \approx \tilde{K}^{-(1-\epsilon)}\) and \(\gamma \approx \tilde{K}^{-3(1-\epsilon)}\), we have \(e^{-\mu \eta k} = O(K^{-(1/4-\epsilon)})\) and
\[
\left( \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \|w_{i,k} - w^*\|^2 \right] \right)^{1/2} = O(K^{-(1/4)}).
\]
When \(\epsilon \to 0\), we can see that the error scales as \(O(K^{-1/4})\), which is also validated numerically in the next section.

IV. NUMERICAL RESULTS

We solve the ‘0 versus 1’ task based on the MNIST dataset [16]: the goal is to distinguish images of digits ‘0’ and ‘1’.

Network deployment: We consider \(N = 200\) nodes, spread uniformly at random over a region of 3km radius. The nodes communicate over a bandwidth of \(W_{\text{tot}} = 1\)MHz, carrier frequency \(f_c = 3\)GHz, with a fixed transmission power of \(P_{tx} = 5dBm\). The noise power spectral density at the receivers is \(N_0 = -169dBm\)W/Hz. The average pathloss \(\Lambda_{i,j}\) between node \(i\) and \(j\) follows Friis’ free space equation.

Data deployment: Each node has a local dataset with a single 28x28 pixels image: 100 nodes have digit ‘0’, the remainder have digit ‘1’. Node \(i\)’s image is converted into a 50-dimensional real feature vector \(d_i\), representing the components (out of a total of \(28 \times 28 = 784\)) with largest mean energy across the dataset, and then normalized to \(\|d_i\| = 1\). We define the label \(\ell_i = 1\) if node \(i\)’s image is of digit 0, otherwise \(\ell_i = -1\).

Optimization problem formulation: We solve the task via regularized logistic regression, with loss function
\[
f_i(w) = \frac{0.01}{2} \|w\|^2 + \ln(1 + \exp(-\ell_i d_i^T w)), \tag{18}
\]
where \(w \in \mathbb{R}^d\) is a \(d = 50\)-dimensional parameter vector. It can be shown that \(f_i(w)\), hence the global function \(F(w) = \frac{1}{N} \sum_{i=1}^{N} f_i(w)\), are all strongly-convex with parameter \(\mu = 0.01\), and smooth with parameter \(L = \mu + 1/4\).

Wireless distributed optimization algorithms: We implement the following algorithms, all initialized as \(w_{i,0} = 0, \forall i\):

- NCOTA-DGD (proposed): To enforce half-duplex constraints, 100 nodes, selected randomly, transmit in slot one, the others transmit in slot two. The frame duration is \(T = 102\mu s\).

We also compare the proposed NCOTA-DGD with implementations of DGD over orthogonal digital (OD-DGD) and analog (OA-DGD) channels. Both follow the updates
\[
w_{i,k+1} = \Pi_c(w_{i,k} - \eta \nabla f_i(w_{i,k})), \tag{19}
\]
where \(c_{i,k}\) is a reconstruction of \(\sum_{j=1}^{N} \omega_{i,j} w_{j,k}\), but differ in how signals are encoded and transmitted, and \(c_{i,k}\) is computed:

- Orthogonal Digital DGD (OD-DGD): each node scales \(w_{i,k}\) by the largest magnitude of its components, \(\|w_{i,k}\|_\infty\); each component of \(w_{i,k}/\|w_{i,k}\|_\infty\) (each \(\in [-1, 1]\)) is then quantized using 9 quantization levels uniformly spaced in the interval \([-1, 1]\). We use dithered quantization: letting \(\hat{w}_{i,k}\) be the quantized signal, it is such that \(\mathbb{E}[\hat{w}_{i,k} | w_{i,k}] = w_{i,k}\). With \(\|w_{i,k}\|_\text{loc}\) encoded using machine precision (64 bits), the data payload is thus \(L = 64 + d \log_{2}(9) \approx 223\)bits to encode the \(d\)-dimensional signal \(w_{i,k}\). Such payload is then transmitted by each node over orthogonal channels (via TDMA), using capacity achieving codes with rate \(R\). With the fading channel \(h_{i,j} \sim \text{CN}(0, \Lambda_{i,j})\) between transmitting node \(i\) and receiving node \(j\), and assuming CSI at the receiver, the probability of successful decoding is \(P_{\text{succ}} \approx \exp(- \frac{\gamma}{E_{\text{th}}}(2R - 1))\). \(R\) is chosen to guarantee a minimum 90% success probability for nodes within a 500m radius from the transmitting node, yielding \(R \approx 2\)bits/s/Hz. The resulting frame duration is \(T \approx 22.67\)ms. At the end of the \(N\) transmissions, node \(i\) computes
\[
c_{i,k} = w_{i,k} + \frac{1}{\max_{j \neq i} \sum_{j \neq i} \sum_{n_{ij}} \sum_{k_{ij}} (w_{j,k} - w_{i,k})},
\]
where \(\omega_{i,j} = 1 \{ R < \log_{2}(1 + |h_{i,j}|^2 E/\sigma^2) \}\) indicates a successful reception of \(w_{j,k}\) at node \(i\). With this choice of \(c_{i,k}\), the updates (19) represent a noisy version of (1) with weights \(\omega_{i,j} = \max_{n_{ij}} \sum_{n_{ij}} \sum_{k_{ij}} (w_{j,k} - w_{i,k})\), and the additional projection step (as seen by computing \(\mathbb{E}[c_{i,k} | F_k]\)).
Fig. 1: Optimality error (a), test error (b), vs running time. Optimal

- Orthogonal Analog DGD (OA-DGD): \( \mathbf{w}_{i,k} \) is first nor-

mized to unit norm; the first (respectively, second) half of

the normalized vector, \( \frac{[\mathbf{w}_{i,k}]_{1:d/2}}{[\mathbf{w}_{i,k}]_{d/2+1:d}} \), is mapped to

the real (imaginary) part of the baseband transmitted signal as

\[
x_{i,k} = \sqrt{\frac{E}{3}} \left[ \frac{[\mathbf{w}_{i,k}]_1 \mathbf{h}_{i,k} + [\mathbf{w}_{i,k}]_{d+1} \mathbf{h}_{i,k}}{[\mathbf{w}_{i,k}]_3} \right] ;
\]

note that \( x_{i,k} \) includes the norm \( \| \mathbf{w}_{i,k} \| \) (penultimate sample) and a pilot signal (last sample) to estimate the channel

at the receiver. This constitutes a \((d/2+2)\)-dimensional complex-valued signal, whose energy per sample satisfies

\[
\frac{1}{d/2+2} \| \mathbf{x}_{i,k} \|^2 \leq E ,
\]

consistent with the power constraint.

The signal is then transmitted by each node over orthogonal

channels (via TDMA), yielding the frame duration \( T = 5.4 \text{ms} \).

With the received signal \( y_{i,k} = h_{i,k}^* x_{i,k} + n_{i,k} \), node \( i \) first estimates \( h_{i,k}^* \) via maximum likelihood from the last sample, followed by the estimation of \( \| \mathbf{w}_{i,k} \| \) from the penultimate sample; it then estimates \( \mathbf{w}_{i,k} / \| \mathbf{w}_{i,k} \| \) from the first \( d/2 \)

samples. After receiving the signals from all nodes, and using the reconstructions \( \hat{\mathbf{w}}_{j,k} \) of \( \mathbf{w}_{j,k} \), node \( i \) then computes

\[
\mathbf{c}_{i,k} = \mathbf{w}_{i,k} + \sum_{j \neq i} \frac{\Lambda_{i,j}}{\max_n \sum_{j \neq i} \Lambda_{n,j}} \left( \hat{\mathbf{w}}_{j,k} - \mathbf{w}_{i,k} \right) ,
\]

so that signals are mixed proportionally to the average pathloss.

With this choice of \( \mathbf{c}_{i,k} \), the updates (19) represent a noisy version of (1) with weights \( \omega_{i,j} = \max_{n,j} \Lambda_{i,j} \), \( j \neq i \) and \( \omega_{i,i} = 1 - \sum_{j \neq i} \omega_{i,j} \), and the additional projection step.

Note that OD-DGD requires CSI at the receiver and knowledge

of \( \mathbf{D}_{\text{baseband}} \); OA-DGD requires knowledge of the average

pathloss \( \Lambda_{i,j} \). In this simulation, we idealistically assume that such information is available at no cost. In contrast, the proposed NCOTA-DGD does not require such knowledge.

Evaluations and Discussion: We evaluate: (1) the optimality

error \( \sqrt{\mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N \| \mathbf{w}_{i,k} - \mathbf{w}^* \|^2 \right] } \), measuring the deviation of the local models from the solution of (P) (bounded in expectation in Theorem 1); (2) the average test error \( \text{TEST}_{\text{k}} = \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N \text{TEST}_{i,k} \right] \), where \( \text{TEST}_{i,k} \) is the test error for node \( i \) at frame \( k \). This is computed on a test set of 100 ‘0’s and 100 ‘1’s; the label associated to feature vector \( \mathbf{d} \) is predicted as ‘0’ if \( \hat{\mathbf{w}}_{i,k} \mathbf{d} > 0 \), and ‘1’ otherwise. Here, \( \mathbb{E} [ \cdot ] \) denotes a sample average of 10 trajectories generated by the algorithms, over independent realizations of fading and noise.

In Fig. 1a, we plot the best optimality error vis-à-vis running time: all algorithms are evaluated using a set of fixed stepsizes; for each time instance in the \( x \)-axis, we plot only the optimality error evaluated on the best performing stepsize choice at that time. For NCOTA-DGD, we also show the curves for 3 representative stepsize choices. NCOTA-DGD achieves the best performance, followed by OD-DGD and OA-

DGD, thanks to its fast updates: during 500ms, NCOTA-DGD performs 4900 iterations, versus only 22 of OD-DGD and 93 of OA-DGD, which are both limited by the use of orthogonal channels. Yet, OD-DGD bridges the gap to NCOTA-DGD over time: this is due to its better noise-suppression capabilities—especially beneficial when approaching convergence to \( \mathbf{w}^* \).

This behavior suggests that a mixed analog-digital strategy may further improve performance—a study left for future work. OA-DGD performs the worst: it does not enjoy the noise suppression capabilities of OD-DGD, and its updates are 53× slower than NCOTA-DGD. In Fig. 1b, we plot the test error under the same best stepsize choice, vis-à-vis running time. It follows a similar trend as Fig. 1a. Both NCOTA-DGD and OD-DGD converge to the test error under the optimal \( \mathbf{w}^* \).

In Fig. 1c, we plot the best stepsizes \( \eta \) (first row), \( \gamma \) (second row) and best optimality error (third row) of NCOTA-DGD, for two different phases: initial phase, corresponding to the first ~5k iterations (500ms, left side); asymptotic phase, after 26k iterations (right side). For each phase, we also fit the data points to the theoretical stepsize tuning and convergence behavior found in Corollary 1: solid lines correspond to \( \eta \propto (K+\delta)^{-1}, \gamma \propto (K+\delta)^{-3/4} \) and "Opt. error" \( \propto (K+\delta)^{-1/4} \), where the scaling factors \( \propto \) and \( \delta \) are fit to the data points. We note that, in the initial phase, the optimal \( \eta \) and \( \gamma \) do not match the theoretical behavior. In fact, in this regime, the optimality error is dominated by (17), and decreases quicker than \( O(K^{-1/4}) \) (bottom left). Conversely, in the asymptotic phase, the optimal \( \eta \) and \( \gamma \) more closely match the theoretical scaling, and the optimality error decays as \( \propto (K+\delta)^{-1/4} \), as predicted (with \( \delta = 4227 \)). This is in line with Corollary 1 when \( \epsilon \rightarrow 0 \), and corresponds to a regime when both error terms (16)-(17) are equally dominant.

V. CONCLUSIONS

We presented NCOTA-DGD, an implementation of DGD that combines local gradient descent with a novel Non-Coherent Over-The-Air consensus scheme to solve distributed machine-learning problems over wirelessly-connected sys-
tems. NCOTA-DGD enables simultaneous transmissions by mapping local optimization signals to a mixture of preamble sequences, and consensus by correlating the received signals with the preamble sequences via non-coherent combining, without explicit knowledge of the mixing weights, nor channel state information. We proved its convergence properties, both theoretically and numerically, and showed superior performance than implementations of DGD over digital and analog orthogonal channels.

**APPENDIX**

**Proof sketch of Theorem 1.** To prove (16), we use the fixed-point optimality condition $W^{(G)} = \Pi_N W^{(G)} - \eta \nabla G(W^{(G)})$, the non-expansive property of projections [17], and take the expectation conditional on $F_k$, yielding $\|W_{k+1} - W^{(G)}\|_{F,F_k} \leq \|W_k - W^{(G)}\|_{F,F_k} - \eta \nabla G(W^{(G)})\|^2 + \eta^2 \Sigma$, where $\|W_{k+1}\|_{F,F_k} \leq \Sigma$ (Lemma 1 at the end of the Appendix). Assumption 1 implies that $G$ is $\mu$-strongly convex and $L_G \triangleq L + \lambda(1 - \rho \gamma)/\eta$-smooth. Then, using [18, Theorem 2.1.12], the first term above is further bounded as $\leq (1 - \mu \eta)^2 \|W_k - W^{(G)}\|^2$ as long as $\eta \leq 2/(\mu + L_G)$ (equivalent to C1), yielding, after the unconditional expectation, $\|W_{k+1} - W^{(G)}\|_{F,F_k} \leq (1 - \mu \eta)^2 \|W_k - W^{(G)}\|_{F,F_k} + \eta^2 \Sigma$.

The result (16) follows after solving the induction, noting that $\|W_0 - W^{(G)}\|_{F,F_k} \leq \sqrt{2R}$, using the expression of $\Sigma$ in Lemma 1, $\mu \eta \leq 1$ (implied by C1), $(1 - \mu \eta)^2 \leq e^{-2\mu \eta}$, and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$.

Next, we prove (17). Consider the unconstrained minimizer $\tilde{W} \triangleq \arg\min_{W \in \mathcal{W}} G(W)$. Hence, $0 = -\nabla G(W) = -\nabla f(W) = \nabla f(\tilde{W}) + \frac{\eta}{\mu} \lambda^* (\mathbb{I}_N d - \Omega) \tilde{W}$. From the multivariate mean value theorem, there exists $A \in \mathbb{A}$ such that $\nabla f(W) = \nabla f(\tilde{W}) + A (W - \tilde{W})$. Combining it with $V G(W) = 0$ yields $\|W - \tilde{W}\|^2 = \|B \| f(W^*)\|^2 = 0$, which implies $\max_{W} \nabla f(1 \otimes w^*) \perp \mathbb{I}_d$; hence we further bound $\|W - 1 \otimes w^*\|^2 \leq \eta \nabla f(1 \otimes w^*)\| f(1 \otimes w^*) \|_{1,\mathbb{I}_d} = \|B^*\|_1$.

Furthermore, $\|\nabla f(1 \otimes w^*)\| \triangleq \nabla f(1 \otimes w^*) \leq \sqrt{V_{\mathbb{W},N}}$, and it can be proved (not shown due to space constraints, see [1]) that $\max_{v, \perp \mathbb{I}_d} \|B^* v\| = \frac{\sqrt{1 + L + \mu \eta}}{\lambda^* (1 - \rho \gamma)} \frac{\eta}{\gamma}$, yielding $\|W - 1 \otimes w^*\| \leq \sqrt{\|V_{\mathbb{W},N}\| \frac{\eta}{\gamma}}$. Next, we show that $W \in \mathbb{W}^N$, hence it coincides with $W^{(G)}$ solution of the constrained problem. Since $w^*$ is at distance $\zeta$ from the boundary of $W$ (Assumption 2), it suffices to show that $w^*$ is closer to $w$ than to the boundary of $\mathbb{W}$, i.e., $\|w - w^*\| / \zeta \leq \zeta, \forall i$. This is a direct consequence of C2: $\|w - w^*\| \leq \|W - 1 \otimes w^*\| \leq \sqrt{\|V_{\mathbb{W},N}\| \frac{2}{\gamma}}$, hence $W = W^{(G)}$ and (17) follows.

**Lemma 1.** $\|\epsilon_{i,k}\|_{F,F_k} \leq 8N |Rd(\lambda^* + \alpha^2/E)|^2 \triangleq \Sigma$.

**Proof sketch.** Using (8), we rewrite $\epsilon_{i,k}$ as $s_k = \sum_{m=1}^{M} (\|r_{k,m}\|^2 - \mathbb{E} \|r_{k,m}\|^2 | F_k) (z_m - w_{i,k})$.

Using the triangle inequality, we bound

$$\|\epsilon_{i,k}\|_{F,F_k} \leq \sum_{m=1}^{M} \mathbb{E} \|r_{k,m}\|^2 | F_k) \| z_m - w_{i,k} \|,$$

where $\mathbb{E} \|r_{k,m}\|^2 | F_k) \| z_m - w_{i,k} \|$ is the standard deviation of $\|r_{k,m}\|^2 \| z_m - w_{i,k} \|$, conditional on $F_k$; since $\|r_{k,m}\|^2$ is exponentially distributed, it equals $\mathbb{E} \|r_{k,m}\|^2 | F_k) \| z_m - w_{i,k} \|$. The result directly follows after using $\sum_{i,j \in \mathbb{N}} \alpha_{ij} \leq \alpha^*$, squaring both sides and adding over $i$ ($\|\epsilon_{i,k}\|_{F,F_k} \leq 8N |Rd| \sum_{i,j \in \mathbb{N}} \alpha_{ij} + \alpha^2/E |\|$, which would provide an upper bound on $\|\epsilon_{i,k}\|_{F,F_k}$.

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