Abelianized structures in spherically symmetric hypersurface deformations: Inconsistency of a quantum notion of covariance in models of loop quantum gravity

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Abstract

In canonical gravity, general covariance is implemented by hypersurface-deformation symmetries on phase space. The different versions of hypersurface deformations required for full covariance have complicated interplays with one another, governed by non-Abelian brackets with structure functions. For spherically symmetric space-times, it is possible to identify a certain Abelian substructure within general hypersurface deformations, which suggests a simplified realization as a Lie algebra. The generators of this substructure can be quantized more easily than full hypersurface deformations, but the symmetries they generate do not directly correspond to hypersurface deformations. The availability of consistent quantizations therefore does not guarantee general covariance or a meaningful quantum notion thereof. In addition to placing the Abelian substructure within the full context of spherically symmetric hypersurface deformation, this paper points out several subtleties relevant for attempted applications in quantized space-time structures. In particular, it follows that recent constructions by Gambini, Olmedo and Pullin in an Abelianized setting fail to address the covariance crisis of loop quantum gravity.

1 Introduction

Canonical gravity describes the 4-dimensional, generally covariant structure of space-time by canonical fields defined on the slices of a spatial foliation. Evolution of these fields in time as well as transformations between different foliations are described by the geometrical structure of hypersurface deformations. In a canonical theory, these transformations are generated by certain phase-space functions, the diffeomorphism and Hamiltonian constraints. In spherically symmetric models, which will be considered here, the full set of constraints can be written as $D[M]$ and $H[N]$ with arbitrary spatial functions $M$ (of density weight $-1$) and $N$. The constraint equations $D[M] = 0$ and $H[N] = 0$, valid for any $M$ and $N$, restrict the phase-space degrees of freedom, given by the spatial metric and its momentum related to extrinsic curvature.
At the same time, the constraints generate (i) time evolution,

\[ \mathcal{L}_{(N,M)} f = \{ f, H[N] + D[M] \} \quad (1) \]

for a phase-space function \( f \) along a time-evolution vector field \( t^a = Nn^a + Ms^a \) in space-time with the unit normal \( n^a \) to a spatial slice and the tangent vector field \( s^a = (\partial/\partial x)^a \) within the radial manifold (with coordinate \( x \)) of a spatial slice, and (ii) gauge transformations

\[ \delta_{\xi(\eta,\epsilon)} f = \{ f, H[\eta] + D[\epsilon] \} \quad (2) \]

along a space-time vector field

\[ \xi^a = \eta n^a + \epsilon s^a \quad (3) \]

where \( \epsilon \), like \( M \), has density weight \(-1\).

The reference to normal and tangential directions relative to a foliation implies crucial differences between the mathematical formulation of hypersurface deformations in canonical gravity and the more common formulation of general covariance in terms of space-time tensors. In space-time, vector components \( \xi^a \) transform, by definition, in such a way that \( \xi^a \partial/\partial x^a \) determines a unique direction independent of coordinate choices. Similarly, the spatial vector \( \epsilon s^a = \epsilon \partial/\partial x^a \) defines a coordinate-independent direction because a scalar of density weight \(-1\) in one dimension transforms like a 1-form dual to \( \partial/\partial x \). The normal deformation, however, cannot be introduced in this way because the canonical setting does not provide a time coordinate or the corresponding \( \partial/\partial t \). Moreover, even if such a coordinate could be introduced by hand, for instance by using \( t \) merely as a parameter as it also appears in Hamilton’s equations, it would be impossible to endow \( \eta \) with a density weight \(-1\) in the time direction because, canonically, there is no time manifold. The only alternative is given by the procedure that has been used since [1, 2] and formalized in [3]: The normalization of \( n^a \) as a unit vector (with respect to the spatial metric on a slice, which is available in the canonical setting) associates a unique normal displacement to any given function \( \eta \) (without density weight).

The normal can be made unit only by reference to the metric on a spatial slice, which is one of the canonical degrees of freedom. The geometrical meaning of normal hypersurface deformations and their commutators therefore depend on the metric, resulting in structure functions in the canonical bracket relations. As a consequence, the canonical symmetries do not form a Lie algebra. This property is responsible for several complications well-known in attempts of canonical quantizations of the theory, starting with [4]. It also makes it harder to develop suitable mathematical structures of the transformations generated by the constraints, in particular in an off-shell manner when one does not insist on solving the constraint equations. In [3], for instance, it was shown that a direct composition of transformations generated by the constraints is meaningful in the sense of path independence (a notion introduced in there) only on-shell.

The full structure of transformations is nevertheless required for general covariance to be implemented properly in the solutions of a canonical theory of gravity, in particular one that has been quantized, modified or deformed by new physical effects. While the restricted
on-shell behavior may be easier to handle, the off-shell structure is important to make sure that the theory has a well-defined space-time structure, independently of the dynamics. Only in this case can the theory be considered a *geometrical* effective theory of some deeper and as yet unknown quantum space-time, just as different dynamical versions of gravity given by higher-curvature effective actions make use of the same Riemannian form of space-time. Because of its importance for covariance and the classification of meaningful effective theories, we will review the structure of hypersurface deformations in the beginning of our first section below, combining classic results from gravitational physics with more recent mathematical developments [5, 6].

We will focus on aspects of hypersurface deformations of importance for a suggested simplification of the hypersurface-deformation brackets in spherically symmetric models, given by a partial Abelianization [7], but our statements will apply also to a variety of other reformulations. Analyzing a partial Abelianization in the context of hypersurface deformations, we will show that this construction captures only a certain subset of these transformations and, upon modification or quantization, does not guarantee that invariance under hypersurface deformations or general covariance are still realized. This conclusion may be surprising because, at first sight, a partial Abelianization appears to implement the same number of symmetry generators as standard hypersurface deformations and uses only a linear redefinition of the generators. However, the coefficients of these linear redefinitions are phase-space dependent, complicating their mathematical description [5, 6]. (Heuristically, phase-space dependent linear redefinitions of the generators introduce new structure functions or modify existing ones.) It is then a non-trivial question whether the redefinitions can be inverted. If they cannot be inverted, the redefined theory is not invariant under full hypersurface deformations and its solutions violate general covariance. An additional construction is therefore needed in a partially Abelianized model (or other reformulations of standard hypersurface deformations) in order to recover all space-time transformations. As shown by explicit examples, this is not always possible if the generators have been modified by quantum corrections.

A recent paper [8] claims that it may be possible to realize general covariance in partial Abelianizations of spherically symmetric models with different types of quantum modifications, such as a spatial discretization. The claim is not accompanied by a successful reconstruction of hypersurface deformations and instead relies on a technical and so far incomplete case-by-case study of quantities that should be invariant in a covariant theory. Using our results about general hypersurface deformation structures, we will explain why the covariance claims of [8] cannot hold.

# 2 Hypersurface deformations

Space-time vector fields with their standard Lie bracket generate the Lie algebra of diffeomorphisms. Similarly, the transformations generated by the canonical constraints form an algebraic structure. They are labeled by the components $\eta$ and $\epsilon$ of a vector field $\xi$ used in (3) in a basis $(n^a, s^a)$ adapted to a spatial foliation, rather than a coordinate basis. Their
are determined by Poisson brackets \(\{H[\eta_1] + D[\epsilon_1], H[\eta_2] + D[\epsilon_2]\}\) of the constraints (using the Jacobi identity). Because the unit normal \(n^a\) is normalized by using a spatial metric \(q_{ab}\) on a slice, the brackets of two canonical gauge transformations \([9, 1, 2]\) turn out to depend on the metric. In spherically symmetric models, in which the radial part of the metric is determined by a single function, \(q\) (of density weight 2), we have

\[
\{H[\eta_1] + D[\epsilon_1], H[\eta_2] + D[\epsilon_2]\} = H[\epsilon_1 \eta_2' - \epsilon_2 \eta_1' + D[\epsilon_1 \epsilon_2' - \epsilon_2 \epsilon_1'] + q^{-1}(\eta_1 \eta_2' - \eta_2 \eta_1')] .
\] (5)

In general, the metric components are spatial functions independent of the components \(\eta\) and \(\epsilon\) that label different gauge transformations. Unlike the Lie bracket of two space-time vector fields, the bracket of two pairs \(\delta\xi_i, i = 1, 2\), implied by the Poisson bracket \((5)\) does not form a Lie algebra because coefficients determined by spatial fields \(q_{ab}\) or \(q\) cannot be considered structure constants.

### 2.1 Algebroids

Instead, the brackets have structure functions or, in a suitable mathematical formulation, form the higher algebraic structure of an \(L_\infty\)-algebroid rather than a Lie algebra \([10, 11, 12]\). An \(L_\infty\)-algebroid is defined as a vector bundle over a base manifold \(M\) with fiber \(F\) and bracket relations on bundle sections together with suitable anchor maps that map bundle sections to objects in the tangent bundle of \(M\). A Lie algebroid, for instance, has a Lie bracket \([\cdot, \cdot]\) on its sections and an anchor \(\rho\) that maps (as a homomorphism) bundle sections to vector fields on the base manifold. The latter is used to define a Leibniz rule

\[
[s_1, fs_2] = f[s_1, s_2] + s_2 \mathcal{L}_\rho(s_1) f
\] (6)

where \(s_1\) and \(s_2\) are sections and \(f\) is a function on the base manifold. The anchor brings abstract algebraic relations on bundle sections in correspondence with geometrical transformations as vector fields on the base manifold. While an anchor that maps any section to the zero vector field is always consistent with the Lie-algebroid axioms (in which case the Lie algebroid is a bundle of Lie algebras given by the fibers), non-trivial transformations on the base require a larger image of the anchor. A Lie algebroid with a non-trivial anchor generalizes bundles of Lie algebras. Yet more generally, and in particular in the case of structure functions, the brackets of bundle sections obey the axioms of an \(L_\infty\)-algebra, a generalized form of a Lie algebra in which the Jacobi identity is not required to hold strictly.

The introduction of the base manifold makes it possible to formalize brackets with structure functions. In particular for gravity, the base manifold contains the manifold of
spatial metrics. (It is a suitable extension of the phase space of canonical gravity \[6\].) The fiber of the hypersurface-deformation algebroid is given by the components \(\eta\) and \(\epsilon\) of a gauge transformation. A section is then an assignment of spatial functions \(\eta\) and \(\epsilon\) to any metric (or a pair of a metric and its momentum). In this way, the \(q\)-dependent structure function in \[5\] finds a natural home in the algebroid as a bracket of sections over the space of metrics (and momenta).

For constant sections, given by pairs of \(\eta\) and \(\epsilon\) that are functions on space but do not depend on the phase-space degrees of freedom, the \(L_\infty\) structure can be reduced to a Lie algebroid \[5\], in which the bracket on sections is a Lie bracket \[13\]. In general, however, the formulation as an algebroid over (an extended version of) the phase space as base manifold should allow for non-constant sections in which the functions \(\eta\) and \(\epsilon\) depend not only on space but also on phase space. This general case of non-constant sections, discussed in \[6\], either violates some of the Lie-algebra relations on sections (in the controlled way of a specific \(L_\infty\)-structure, as it follows from a BV-BFV extension of general relativity \[14, 15\]) or requires an extension of the Lie algebroid to a Lie-Rinehart algebra \[16\] in which functions on the base manifold are replaced with a suitable commutative algebra.

Phase-space dependent functions \(\eta\) and \(\epsilon\) are also important for physics. They are often considered in specific gravitational applications, as in the simple case of cosmological evolution written in conformal time where the lapse function equals the scale factor, a metric component. More importantly for our purposes, the partial Abelianization of \[7\] relies on an application of phase-space dependent \(\epsilon\) and \(\eta\). Hypersurface deformations with such non-constant sections form a Lie algebroid only on-shell \[6\] when the constraints are solved. The partial Abelianization is therefore able to describe the solution space to all constraints and its covariance transformations, but it is not guaranteed that it correctly captures off-shell transformations which are relevant for general covariance.

Since the standard derivation of the brackets \[5\] assumes that \(\eta\) and \(\epsilon\) are not phase-space dependent, the general brackets must be extended by additional terms that, heuristically, result from Poisson brackets of constraints with phase-space dependent \(\eta\) and \(\epsilon\).

The Poisson bracket of two diffeomorphism constraints, for instance, can still be written in the compact form

\[
\{D[\epsilon_1], D[\epsilon_2]\} = D[\epsilon_2 \epsilon_1' - \epsilon_1 \epsilon_2']
\]

but with an application of the chain rule in the derivatives. Similarly, the mixed Poisson bracket of a Hamiltonian and a diffeomorphism constraint in general form reads

\[
\{H[\eta], D[\epsilon]\} = H[-\epsilon \eta'] + D[\eta L_n \epsilon]
\]

where the normal derivative \(L_n\) of a spatial function is defined by the Poisson bracket with the Hamiltonian constraint, \(\eta_1 L_n \eta_2 = \{H[\eta_1], \eta_2\}\). For two Hamiltonian constraints, we have the Poisson bracket

\[
\{H[\eta_1], H[\eta_2]\} = D[q^{-1}(\eta_1 \eta_2' - \eta_2 \eta_1')] + H[\eta_1 L_n \eta_2 - \eta_2 L_n \eta_1].
\]

In general, the extra terms implied by phase-space dependent \(\eta\) and \(\epsilon\), such as those in \(\epsilon' = \partial_x \epsilon + (\partial_x q_i)(\partial_q \epsilon) + (\partial_x k_i)(\partial_k \epsilon)\) summing over the two independent components \(q_i\),
i = 1, 2, of a spherically symmetric spatial metric as well as two components \( k_i \) of extrinsic curvature, introduce further structure functions, such as \( \partial_x q_i \) and \( \partial_x k_i \), that depend on the metric as well as its momenta.

While these Poisson brackets illustrate the additional complications encountered with phase-space dependent \( \epsilon \) and \( \eta \), they do not immediately show the algebraic nature of general non-constant sections of hypersurface deformations. In particular, Poisson brackets do not directly mirror relevant \( L_\infty \)-structures. In our following discussion, we will not need the full algebraic structure and instead perform a comparison of different versions of constant and non-constant sections in gravitational applications.

2.2 Partial Abelianization

As noticed in [7], certain linear combinations of \( H[\eta] \) and \( D[\epsilon] \) have vanishing Poisson brackets in spherically symmetric models. In order to specify these combinations, we have to refer to explicit variables that determine the spatial metric and its momenta. Following [17, 18], this is conveniently done in triad variables \((E^x, E^\varphi)\) such that the spatial metric is given by the line element

\[
ds^2 = \frac{(E^x)^2}{E^x} dx^2 + E^x (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)
\]

in standard spherical coordinates. (For our purposes, it is sufficient to assume \( E^x > 0 \), fixing the orientation of the triad.) The triad components are canonically conjugate (up to constant factors) to components of extrinsic curvature, \((K_x, K_\varphi)\), such that

\[
\{K_x(x), E^x(y)\} = G\delta(x, y) \quad , \quad \{K_\varphi(x), E^\varphi(y)\} = \frac{1}{2} G\delta(x, y)
\]

with Newton’s constant \( G \). The delta functions disappear in Poisson brackets of integrated (smeared) expressions, resulting in well-defined brackets. In particular, the diffeomorphism constraint

\[
D[M] = \frac{1}{G} \int dx M(x) \left( -\frac{1}{2} (E^x)'K_x + K_\varphi E^\varphi \right),
\]

and Hamiltonian constraint

\[
H[N] = -\frac{1}{2G} \int dx N(x) \left( |E^x|^{-1/2}E^\varphi K_\varphi^2 + 2|E^x|^{1/2} K_\varphi K_x + |E^x|^{-1/2}(1 - \Gamma_\varphi^2)E^\varphi + 2\Gamma_\varphi |E^x|^{1/2} \right)
\]

where \( \Gamma_\varphi = -(E^\varphi)'/(2E^\varphi) \) have Poisson brackets

\[
\{D[M_1], D[M_2]\} = D[M_1 M_2']
\]

\[
\{H[N], D[M]\} = -H[M N']
\]

\[
\{H[N_1], H[N_2]\} = D[E^x(E^\varphi)^{-2}(N_1 N_2' - N_2 N_1')]
\]

(for spatial functions \( M_i \) and \( N_i, i = 1, 2 \), that do not depend on the phase-space variables) of the correct form for hypersurface deformations in spherically symmetric space-times.
Simple algebra and integration by parts shows that the linear combinations

\[ C[L] = H[(E^x)'(E^x)^{-1} \int E^x L dx] - 2D[K_\varphi \sqrt{E^x}(E^x)^{-1} \int E^x L dx], \]

(17)

where \( \int E^x L dx \) is understood as a function of \( x \) obtained by integrating \( E^x L \) from a fixed starting point up to \( x \), have zero Poisson brackets with one another for different \( L \):

\[ \{C[L_1], C[L_2]\} = 0 \]

(18)

for all functions \( L_1 \) and \( L_2 \) on a spatial slice. To see this, it is sufficient to notice that the combination eliminates any dependence on \( K_x \) and on spatial derivatives of \( E^x \). The antisymmetric nature of the Poisson bracket then implies that it must vanish. Explicitly, the new combination of constraints takes the form

\[ C[L] = -\frac{1}{G} \int dx L(x)E^x \left( \sqrt{E^x} \left( 1 + K_\varphi^2 - \Gamma_\varphi^2 \right) + \text{const.} \right). \]

(19)

A free constant appears because a constant \( \int E^x L dx \) implies a non-vanishing lapse function in (17), and therefore a non-trivial constraint, but corresponds to a vanishing \( E^x L \) in (19). The new constraint \( C[L] \) therefore constrains one degree of freedom less than the original \( H[N] \). The free constant in (19) can be determined through boundary conditions, which would also restrict the lapse functions allowed in gauge transformations.

At first sight, it seems that the partial Abelianization eliminates structure functions from the brackets and may simplify quantization and the preservation of symmetries and therefore covariance. However, the importance of metric-dependent structure functions in the standard brackets, which make sure that deformations are defined with respect to a unit normal that is in fact normalized, raises the question of whether an elimination of these structure functions and their metric dependence by redefined generators can still capture the full picture of general covariance. To answer this question, it is instructive to place the partial Abelianization of the brackets in the context of the hypersurface-deformation algebroid. Several features of the full construction of the algebroid are then relevant.

First, the integration of \( E^x L \) required to define \( C[L] \) as a combination of \( H[N] \) and \( D[M] \) may seem unusual, but while this means that the relevant \( N \) and \( M \) are non-local in space, they are local within the algebroid, both its fiber (spatial functions \( N \) and \( M \)) and its base (an extension of the gravitational phase space with independent functions \( E^x, E^\varphi, K_x \) and \( K_\varphi \)). The combination (17) therefore defines an admissible set of sections in the algebroid.

Secondly, while the section defined by (17) makes use of phase-space dependent \( N \) and \( M \) in the Hamiltonian and diffeomorphism constraints, which are therefore not constant over the base manifold, an Abelian bracket (18) is obtained only for functions \( L_1 \) and \( L_2 \) that do not have the full phase-space dependence allowed for general sections. In particular, if \( L_1 \) or \( L_2 \) are allowed to depend on \( (E^x)' \) or \( K_x \), the bracket \( \{C[L_1], C[L_2]\} \) no longer vanishes, and it can then have structure functions. Partial Abelianization is therefore obtained for a restricted class of sections, defined such that \( L \) does not depend on \( (E^x)' \) and \( K_x \) (while it may still have an unrestricted spatial dependence). If \( L \) does not depend
on \((E^x)'\) and \(K_x\) but on the other independent phase-space variables, \(K_\varphi\) as well as \(E^x\) or on \(E^x\) but not its derivatives, the bracket \(\{C[L_1], C[L_2]\}\) remains zero, but there are then structure functions in the bracket of \(C[L]\) with the diffeomorphism constraint, analogously to (8). Therefore, structure functions are eliminated from the brackets only for a restricted class of sections. This observation raises the question whether full covariance can still be realized.

A restriction to constant sections over the base manifold is not unusual, for certain purposes. A similar assumption is made in the standard form (14)–(16) of hypersurface-deformation brackets, in which case the original \(N\) and \(M\) are often assumed to be constant over the base (while their spatial dependence remains unrestricted). There is, however, a crucial difference between assuming constant \(N\) and \(M\) over the base and assuming constant \(L\) over the base: In the former case, allowing for non-constant sections produces additional terms in the brackets, shown in (7), (9) and (8), that follow directly from an application of the product rule of Poisson brackets. The partial Abelianization, however, relies on cancellations between different structure functions in the original brackets that are no longer realized once non-constant sections with phase-space dependent \(L\) are allowed.

In particular, allowing for phase-space dependent \(L\) and \(M\) in the \((D[M], C[L])\) system makes the transformation from \((N, M)\) to \((M, L)\) invertible. It is then possible to write the original \(H[N]\) as a combination of \(D[M]\) and \(C[L]\) in the partial Abelianization, regaining the full non-Abelian brackets with metric-dependent structure functions. Restricting the system to phase-space independent \(L\), by contrast, implies that the transformation from the original hypersurface-deformation algebroid to the brackets of \(D[M]\) and \(C[L]\) is not invertible. It is then unclear whether hypersurface deformations and general covariance can be recovered from a partial Abelianization, in particular if the latter has been modified by quantum corrections.

### 2.3 Modified deformations

It has been known for some time [19, 20, 21] that spherically symmetric hypersurface deformations can be modified consistently, maintaining closed brackets while modifying the structure functions. The dependence on \(K_\varphi\) in (13) can be modified to

\[
H[N] = \frac{-1}{2G} \int dx N(x) \left( |E^x|^{-1/2} E^x f_1(K_\varphi) + 2 |E^x|^{1/2} f_2(K_\varphi) K_x + |E^x|^{-1/2} (1 - \Gamma_\varphi^2) E^\varphi + 2 \Gamma_\varphi^\prime E^\varphi |E^x|^{1/2} \right)
\]

where \(f_1\) and \(f_2\) are functions of \(K_\varphi\) related by

\[
f_2(K_\varphi) = \frac{1}{2} \frac{df_1(K_\varphi)}{dK_\varphi}.
\]

If this equation is satisfied, the bracket of two Hamiltonian constraints is still closed,

\[
\{H[N_1], H[N_2]\} = D[\beta(K_\varphi) E^x (E^\varphi)^{-2} (N_1 N_2' - N_2 N_1')]
\]
for phase-space independent $N_1$ and $N_2$. In this bracket, $D[M]$ is the unmodified diffeomorphism constraint, but the structure function is multiplied by a new factor of

$$\beta(K_ϕ) = \frac{df_2(K_ϕ)}{dK_ϕ} = \frac{1}{2} \frac{d^2 f_1(K_ϕ)}{dK_ϕ^2}. \quad (23)$$

Additional terms in the bracket for non-constant sections follow immediately from the product rule for Poisson brackets.

Similarly, the Abelianized constraint $C[L]$ can be modified in its dependence on $K_ϕ$, using the same function $f_1$ as before:

$$C[L] = -\frac{1}{G} \int dx L(x) E^ϕ \left( \sqrt{|E^ϕ|} \left( 1 + f_1(K_ϕ) - \Gamma_ϕ^2 \right) + \text{const.} \right). \quad (24)$$

Its brackets remain Abelian for phase-space independent $L$. There is no obvious term in $C[L]$ where the second function $f_2$ might appear or the important consistency condition (21). It therefore seems easier to modify (or quantize) the constraint $C[L]$ compared with $H[N]$. However, for full hypersurface deformations and covariance to be realized in the modified setting, we still have to make sure that the transformation from $(N, M)$ to $(L, M)$ can be inverted. As shown in [22], this is possible only if we also modify the transformation (17) to

$$C[L] = H[(E^ϕ)'E^ϕ] - 2D[f_2(K_ϕ)\sqrt{|E^ϕ|}E^ϕLdx] \quad (25)$$

where $f_2$ obeys the same consistency condition with $f_1$, (21), as derived from the modified Hamiltonian constraint. The partial Abelianization and the original form of hypersurface deformations therefore imply equivalent results, provided one makes sure that the transformation of sections can be inverted. Only then can access to full hypersurface deformations and covariance be realized.

3 Non-covariant modifications of Abelianized brackets

A recent paper [8] by Gambini, Olmedo and Pullin (GOP) argues that general covariance can be realized in modified versions of spherically symmetric models, for which a partial Abelianization of the brackets plays a crucial role: As the abstract claims, “We show explicitly that the resulting space-times, obtained from Dirac observables of the quantum theory, are covariant in the usual sense of the way — they preserve the quantum line element — for any gauge that is stationary (in the exterior, if there is a horizon). The construction depends crucially on the details of the Abelianized quantization considered, the satisfaction of the quantum constraints and the recovery of standard general relativity in the classical limit and suggests that more informal polymerization constructions of possible semi-classical approximations to the theory can indeed have covariance problems.”
These claims raise several questions. For instance, how can the construction depend "crucially on the details of the Abelianized quantization considered" if a partial Abelianization is either completely equivalent to the non-Abelian original version of hypersurface deformations (if the transformation is made sure to be invertible) or gives access to only a subset of hypersurface deformations (if the transformation is not invertible owing to a restriction to a subset of sections)?

A closer inspection of technical calculations performed by GOP shows that spherically symmetric hypersurface deformations are, in fact, violated in the construction. GOP use two different kinds of modifications, a modified dependence of $C[L]$ on $K_\varphi$ of the form (24), and a spatial discretization of phase-space functions and their derivatives. Because the authors use a certain combination of solutions to the constraints and gauge-fixing conditions, it turns out that only the latter modification survives in the final expressions for line elements that are supposed to be invariant.

However, also the former (a modified dependence on $K_\varphi$) is relevant because, as we have seen, the correct form of a modification must appear in two different places, in the constraint $C[L]$ and in the transformation back to unrestricted hypersurface deformations. These two appearances are clear but somewhat implicit in [8]: The modified $C[L]$ is implied by the modified solutions in equation (14) in [8] (or, equivalently, (21) there, referring to the preprint version) where $f_1(K_\varphi) = \sin^2(\rho K_\varphi)/\rho^2$ with a spatial function $\rho$. The modified transformation back to unrestricted hypersurface deformations is implied by equation (20) in [8] which in our notation amounts to replacing $K_\varphi$ in (17) with $\sqrt{f_1(K_\varphi)}$. Using the same function $f_1(K_\varphi)$ is crucial for the constructions in [8] because the partial gauge fixing employed there replaces $\sqrt{f_1(K_\varphi)}$ with a fixed function on space (rather than phase space). The same gauge-fixing function is then used in both places, in the constraint $C[L]$ or its solutions and in the transformation back to unrestricted hypersurface deformations from which a line element can be constructed. However, this construction, which is equivalent to assuming $f_2(K_\varphi) = \sqrt{f_1(K_\varphi)}$ in (25), violates the condition (21) required for unrestricted hypersurface deformations to follow for the modified constraint. (For the specific $f_1(K_\varphi)$ considered by GOP, $f_2$ should have an additional cosine factor, or equivalently have a doubled argument of the sine function.) The constructions of [8] therefore violate hypersurface deformations.

How can GOP then claim to have performed crucial steps toward demonstrating general covariance in this setting? Unfortunately, much of the constructions are obscured by an application of incompletely defined mixtures of gauge fixings and idiosyncratic notions of observables. Here, it suffices to highlight only a few of the shortcomings found in the GOP analysis. Continuing with the replacement of $\sqrt{f_1(K_\varphi)}$ by a gauge-fixing function that depends only on space, GOP replace any appearance of $\sqrt{f_1(K_\varphi)}$ with the classical value of $K_\varphi$ in two sample space-time slicings. Through the gauge fixing, the procedure therefore replaces $f_1(K_\varphi)$ with $K_\varphi$ and thereby removes this modification in an evaluation of the modified constraints. It is then not surprising that certain classical transformations between slicings can be extended to the modified case, in spite of a violation of (21), because only the spatial discretization remains in the system. However, this procedure cannot be
correct because it would imply that the classical constraint (19) and the modification (24) always have equivalent solutions, even though they generate inequivalent equations of motion and inequivalent gauge transformations. GOP are not analyzing covariance of the modified constraint (24), as they claim, but rather of a discretized constraint with the classical dependence on $K_\varphi$.

The gauge fixing is problematic for various other reasons as well. In particular, after solving the constraint $C[L] = 0$ for $E^x$ as a function of $K_\varphi$ and $E^x$, the authors claim that this solution is an observable even though it clearly does not Poisson commute with $C[L]$. It commutes only after the gauge fixing has been implemented, eliminating $K_\varphi$ in favor of a spatial function, but this replacement does not turn a phase-space function into an observable. (If this were the case, any phase-space function would be an observable.)

Another misidentified observable is the function $E^x$. GOP arrive at this formal claim by applying a procedure from loop quantum gravity [23, 24] in which phase-space variables are discretized in space and represented on spin-network states. In spherically symmetric models [25, 17], the networks are 1-dimensional with labels on vertices (used as eigenvalues for an operator $\hat{E}^e$) and on connecting links (used as eigenvalues for an operator $\hat{E}^x$). The diffeomorphism constraint is then implemented not as a quantization of the phase-space function (12) but rather in a finite version that shifts the vertices by finite amounts according to a spatial diffeomorphism. The vertex labels are invariant under this transformation, and $E^x$ commutes with the Abelianized constraint $C[L]$ or its quantum version. In this construction, $\hat{E}^x$ may therefore be considered a Dirac observable.

However, there is an unresolved contradiction with the fact that the classical $E^x$, simply related to a specific metric component, is not space-time diffeomorphism invariant. In canonical gravity, it is not a Dirac observable and does not commute with the diffeomorphism constraint. GOP acknowledge this tension only implicitly when they use, later on in the paper, additional gauge-fixing conditions for the eigenvalues $k$ of $\hat{E}^x$, even though this operator is supposed to be a Dirac observable that should be gauge invariant. (See “gauge fixing such that $x_j^2 = \ell_{\text{Planck}}^2 k_j$” with the vertex position $j$ and $\hat{E}^x$-eigenvalues $k$ just before equation (10) in [8] and “in the gauge $E_j^x = \text{sig}(j) x_j^2$ just before equation (34) there.”

Finally, GOP construct only a limited set of transformations for a line element, which does not suffice to indicate general covariance. It is not at all clear that the same transformation between discretized spatial slices can be used for all possible quantities that must be invariant in a covariant theory. GOP mention that “We have also studied the covariance of several curvature scalars: the Ricci and the Kretschmann scalars, and the scalar obtained by contracting the Weyl tensor with itself. We checked that in the approximation where $x_j$ is treated as a continuous variable, which allows to use derivatives instead of finite differences, these scalars do not depend on the choice of the gauge function $F(x)$. This gives robustness to our model regarding its covariance.” However, far from implying robustness, the sentence describes a trivial observation because this limit removes the only non-classical effect (discretization) used in the explicit calculations, after an earlier modification of the $K_\varphi$-dependence has been eliminated by an unclear gauge fixing.
4 Conclusions

Our discussion of non-constant sections in the hypersurface-deformation algebroid has clarified a previously puzzling issue of partial Abelianizations in spherically symmetric models: Is it possible for partial Abelianizations to simplify the construction of quantum modifications of hypersurface deformation generators and, at the same time, retain full access to all transformations required for general covariance? We have shown that the answer is negative. A simplified construction of modified generators is based on the absence of structure functions in partially Abelianized brackets obtained for a specific choice of phase-space dependent gauge generators (lapse and shift functions). However, the partial Abelianization is maintained only if the new generators are then restricted to be phase-space independent. This condition renders the transformation from hypersurface-deformation brackets to partially Abelian brackets non-invertible, such that access to unrestricted hypersurface deformations and general covariance is lost in a partially Abelianized setting. Consistent modifications of the partially Abelian brackets then do not necessarily imply consistent realizations of general covariance.

A recent paper [8] by Gambini, Olmedo and Pullin has implicitly recognized this shortcoming and instead proposed to test general covariance in a tedious case-by-case study of presumed invariants, beginning with a discretized version of the line element. We have pointed out a specific place (the choice of modification functions $f_1$ and $f_2$) where hypersurface deformations are treated inconsistently in these constructions, which may perhaps lead to improved versions of the transformations considered by GOP. However, correcting this inconsistency requires an analysis of unrestricted hypersurface deformations even in the partially Abelian setting, making sure that the transformation between these two versions of the brackets can be inverted. It is therefore impossible to analyze covariance in isolation from general hypersurface deformations, as proposed by GOP. No-go results [26] for covariance in models of loop quantum gravity, partially based on various analyses of modified hypersurface deformations, therefore cannot evaded by the constructions of GOP.

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