ELLIPIC CURVES WITH A LOWER BOUND ON 2-SELMER RANKS OF QUADRATIC TWISTS

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Abstract. For any number field $K$ with a complex place, we present an infinite family of elliptic curves defined over $K$ such that $\dim_\mathbb{F}_2 \Sel_2(E^F/K) \geq \dim_\mathbb{F}_2 E^F(K)[2] + r_2$ for every quadratic twist $E^F$ of every curve $E$ in this family, where $r_2$ is the number of complex places of $K$. This provides a counterexample to a conjecture appearing in work of Mazur and Rubin.

1. Introduction

1.1. Distributions of Selmer ranks. Let $E$ be an elliptic curve defined over a number field $K$ and let $\Sel_2(E/K)$ be its 2-Selmer group (see Section 2 for its definition). The 2-Selmer rank of $E$, denoted $d_2(E/K)$, is defined as

$$d_2(E/K) = \dim_\mathbb{F}_2 \Sel_2(E/K) - \dim_\mathbb{F}_2 E(K)[2].$$

For a given elliptic curve and positive integer $r$, we are able to ask whether $E$ has a quadratic twist with 2-Selmer rank equal to $r$. A single restriction on which $r$ can appear as a 2-Selmer rank within the quadratic twist family of a given curve $E$ is previously known. Using root numbers, Dokchitser and Dokchitser identified a phenomenon called constant 2-Selmer parity where $d_2(E^F/K) \equiv d_2(E/K)$ (mod 2) for every quadratic twist $E^F$ of $E$ and showed that $E$ has constant 2-Selmer parity if and only if $K$ is totally imaginary and $E$ acquires everywhere good reduction over an abelian extension of $K$ [2].

In this paper, we show the existence of an additional obstruction to small $r$ appearing as 2-Selmer ranks within the quadratic twist family of $E$. We prove that there are curves having this obstruction over any number field $K$ with a complex place. Specifically:

**Theorem 1.** For any number field $K$, there exist infinitely many elliptic curves $E$ defined over $K$ such that $d_2(E^F/K) \geq r_2$ for every quadratic $F/K$. Moreover, these curves do not have constant 2-Selmer parity and none of them become isomorphic over $K$.

This result disproves a conjecture appearing in [7], which predicted that subject only to the restriction of constant 2-Selmer parity, the set of twists of $E$ having 2-Selmer rank $r$ has positive density within the set of all twists of $E$ for every $r \geq 0$. 

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We prove Theorem 1 by presenting a family of elliptic curves defined over \( \mathbb{Q} \) for which each curve in the family has the appropriate property when viewed over \( K \). For \( n \in \mathbb{N} \), let \( E_{(n)} \) be the elliptic curve defined by the equation
\[
E_{(n)}: y^2 + xy = x^3 - 128n^2x^2 - 48n^2x - 4n^2
\]
and define \( \mathcal{F} = \{ E_{(n)} : n \in \mathbb{N}, 1 + 256n^2 \notin (K^\times)^2 \} \). Each curve \( E \in \mathcal{F} \) has a single point of order 2 in \( E(K) \) and a cyclic 4-isogeny defined over \( K(E[2]) \) but not \( K \). Let \( \phi : E \to E' \) be the isogeny whose kernel is \( C = E(K)[2] \). Our results are obtained by using local calculations combined with a Tamagawa ratio of Cassels to establish a lower bound on the rank of the Selmer group associated to \( \phi \) (to be defined in Section 2).

Although curves \( E \in \mathcal{F} \) have the property that \( d_2(E^F/K) \geq r_2 \) for every quadratic \( F/K \), this does not hold in general for curves \( E \) with \( E(K)[2] \simeq \mathbb{Z}/2\mathbb{Z} \) that have a cyclic 4-isogeny defined over \( K(E[2]) \) but not over \( K \). In particular, the forthcoming work of this author can be used to show that every \( r \geq 0 \) appears infinitely often as a 2-Selmer rank within the quadratic twist family of \( E' \) for every \( E \in \mathcal{F} \) [4].

\section{Selmer groups}

We begin by briefly recalling the constructions of the 2-Selmer and \( \phi \)-Selmer groups along with some of the standard descent machinery. A more detailed explanation can be found in Section X.4 of [8].

If \( E \) is an elliptic curve defined over a field \( K \), then the Kummer map \( \delta_{[2]} \) maps \( E(K)/2(K) \) into \( H^1(K, E[2]) \). If \( K \) is a number field, then for each place \( v \) of \( K \) we define a distinguished local subgroup \( H^1_v(K_v, E[2]) \subset H^1(K_v, E[2]) \) by
\[
\text{Image}(\delta_{[2]} : E(K_v)/2E(K_v) \to H^1(K_v, E[2])).
\]

We define the 2-Selmer group of \( E \), denoted \( \text{Sel}_2(E/K) \), by
\[
\text{Sel}_2(E/K) = \ker \left( H^1(K, E[2]) \xrightarrow{\sum \text{res}_v} \bigoplus_{v \text{ of } K} H^1(K_v, E[2])/H^1_v(K_v, E[2]) \right).
\]

If \( E^F \) is the quadratic twist of \( E \) by \( F/K \) where \( F \) is given by \( F = K(\sqrt{d}) \), then there is an isomorphism \( E \to E^F \) given by \( (x, y) \mapsto (dx, d^{3/2}y) \) defined over \( F \). Restricted to \( E[2] \), this map gives a canonical \( G_K \) isomorphism \( E[2] \to E^F[2] \), allowing us to view \( H^1_v(K_v, E^F[2]) \) as sitting inside \( H^1(K_v, E[2]) \). The following lemma due to Kramer describes the connection between \( H^1_v(K_v E[2]) \) and \( H^1_v(K_v, E^F[2]) \).

Given a place \( w \) of \( F \) above a place \( v \) of \( K \), we get a norm map \( E(F_w) \to E(K_v) \), the image of which we denote by \( E_N(K_v) \).

**Lemma 2.1.** Viewing \( H^1_v(K_v, E^F[2]) \) as sitting inside \( H^1(K_v, E[2]) \), we have
\[
H^1_v(K_v, E[2]) \cap H^1_v(K_v, E^F[2]) \simeq E_N(K_v)/2E(K_v)
\]

**Proof.** This is Proposition 7 in [5] and Proposition 5.2 in [6]. The proof in [6] works even at places above 2 and \( \infty \). \( \square \)

If \( E(K)[2] \simeq \mathbb{Z}/2\mathbb{Z} \), then there is an isogeny \( \phi : E \to E' \) with kernel \( C = E(K)[2] \) that gives rise to a \( \phi \)-Selmer group, \( \text{Sel}_\phi(E/K) \). There is a connecting map arising from Galois cohomology, \( \delta_\phi : E'(K)/\phi(E(K)) \to H^1(K, C) \), taking the coset of
Q ∈ E′(K) to the coset defined by the cocycle c(σ) = σ(R) − R where R is any point on E(\overline{K}) with φ(R) = Q. Identifying C with \( \mu_2 \), we can view \( H^1(K, C) \) as \( K^\times/(K^\times)^2 \) and under this identification, \( \delta_\phi(C) = \langle \Delta_E \rangle \), where \( \Delta_E \) is the discriminant of (any model of) E. The map \( \delta_\phi \) can be defined locally as well and for each place \( v \) of K, we define a distinguished local subgroup \( H_1^1(K_v, C) \cap H_1^1(K_v, \phi(E(K_v))) \) under \( \delta_\phi \). We define the **φ-Selmer group** of E, denoted \( \text{Sel}_\phi(E/K) \), as

\[
\text{Sel}_\phi(E/K) = \ker \left( H^1(K, C) \rightarrow \bigoplus_{v \text{ of } K} H^1(K_v, C) / H^1_\phi(K_v, C) \right).
\]

The isogeny \( \phi \) on \( E \) gives rise to a dual isogeny \( \hat{\phi} \) on \( E' \) whose kernel is \( C' = \phi(E[2]) \). Exchanging the roles of \( (E, C, \phi) \) and \( (E', C', \hat{\phi}) \) in the above defines the **\( \hat{\phi} \)-Selmer group**, \( \text{Sel}_{\hat{\phi}}(E'/K) \), as a subgroup of \( H^1(K, C') \). The local conditions \( H^1_\phi(K_v, C) \) and \( H^1_\phi(K_v, C') \) are connected via the following exact sequence.

**Proposition 2.2.** The sequence

\[
0 \rightarrow C'/\phi(E(K_v)[2]) \xrightarrow{\delta_\phi} H^1_\phi(K_v, C) \xrightarrow{\iota} H^1_\phi(K_v, E[2]) \xrightarrow{\phi} H^1_\phi(K_v, C') \rightarrow 0
\]

is exact.

**Proof.** This well-known result follows from the sequence of kernels and cokernels arising from the composition \( \hat{\phi} \circ \phi = [2]_E \). See Remark X.4.7 in [8] for example. \( \square \)

The following two theorems allow us to compare the \( \phi \)-Selmer group, the \( \hat{\phi} \)-Selmer group and the 2-Selmer group.

**Theorem 2.3.** The \( \phi \)-Selmer group, the \( \hat{\phi} \)-Selmer group, and the 2-Selmer group sit inside the exact sequence

\[
0 \rightarrow E'(K)[2]/\phi(E(K)[2]) \xrightarrow{\delta_\phi} \text{Sel}_\phi(E/K) \rightarrow \text{Sel}_2(E/K) \xrightarrow{\phi} \text{Sel}_{\hat{\phi}}(E'/K).
\]

**Proof.** This is a diagram chase based on the exactness of (2.1). See Lemma 2 in [3] for example. \( \square \)

**Theorem 2.4** (Cassels). The **Tamagawa ratio**, defined as \( T(E/E') = \frac{|\text{Sel}_\phi(E/K)|}{|\text{Sel}_{\hat{\phi}}(E'/K)|} \), is given by a local product formula

\[
T(E/E') = \prod_{v \text{ of } K} \frac{|H^1_\phi(K_v, C)|}{2}.
\]

**Proof.** This is a combination of Theorem 1.1 and equations (1.22) and (3.4) in [1]. This product converges since \( H^1_\phi(K_v, C) \) equals the unramified local subgroup \( H^1_u(K_v, C) \) for all \( v \nmid 2\Delta_E \). \( \square \)
3. Local conditions for curves in $\mathcal{F}$

The goal of this section is to prove the following proposition.

**Proposition 3.1.** Let $E = E_{(n)} \in \mathcal{F}$. Then $\dim_{\mathbb{F}_2} H^1_{\phi}(K_v, C^F) \geq H^1(K_v, C) - 1$ for every place $v$ of $K$, where $C^F = E^F(K)[2]$.

Let $E = E_{(n)} \in \mathcal{F}$. The point $P = \left(-\frac{1}{4}, \frac{1}{8}\right)$ on $E$ has order 2 and $E' = E/\langle P \rangle$ can be given by a model $y^2 + xy = x^3 + 64n^2x^2 + 4n^2(1 + 256n^2)x$. The discriminants of the model (1.1) for $E$ and this model for $E'$ are given by $\Delta_E = 4n^2(1 + 256n^2)^3$ and by $\Delta_{E'} = 16n^4(1 + 256n^2)^3$, respectively. As $1 + 256n^2 \not\in (K^\times)^2$, we have $E(K)[2] = \langle P \rangle$.

Since $\Delta_E$ and $\Delta_{E'}$ differ by a square, we get that $K(E[2]) = K(E'[2])$ and it follows that $\dim_{\mathbb{F}_2} E(K_v)[2] = \dim_{\mathbb{F}_2} E'(K_v)[2]$ for every place $v$ of $K$. Proposition 3.1 will follow from some results applicable to all curves that have $K(E[2]) = K(E'[2])$ and some results that are specific to curves in $\mathcal{F}$.

**Remark 3.2.** Forthcoming work of this author shows if $E(K)[2] \simeq \mathbb{Z}/2\mathbb{Z}$, then $E$ does not have a cyclic 4-isogeny defined over $K$ but acquires one over $K(E[2])$ if and only if $K(E[2]) = K(E'[2])$. See Section 4 of [4] for more details.

**Lemma 3.3.** Let $E$ be an elliptic curve with $E(K)[2] \simeq \mathbb{Z}/2\mathbb{Z}$ and suppose further that $K(E[2]) = K(E'[2])$. If $E$ has additive reduction at a place $v \nmid 2$, then $\dim_{\mathbb{F}_2} H^1_{\phi}(K_v, C) = 1$.

**Proof.** Let $E_0(K_v)$ be the group of points on $E(K_v)$ with non-singular reduction, $E_1(K_v)$ the subgroup of points with trivial reduction, and $\mathbb{F}_v$ the residue field of $K_v$. The formal group structure on $E_1(K_v)$ shows that $E_1(K_v)$ is uniquely divisible by 2 and since $E_0(K_v)/E_1(K_v) \simeq \mathbb{F}_v^\times$, $E_0(K_v)$ is uniquely 2-divisible as well. Since $E(K_v)$ has a point of order 2, Tate’s algorithm then shows that $E(K_v)/E_0(K_v)$ is cyclic and therefore $E(K_v)[2^\infty]$ is either a direct sum of $\mathbb{Z}/2\mathbb{Z}$ or is cyclic of order 4.

Therefore, if $E(K_v)$ has a point $R$ of order 4, then $2R \in C$. It follows that $\phi(R) \in E'(K_v)[2] - C'$ and $E'(K_v)[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This contradicts the fact that $\dim_{\mathbb{F}_2} E(K_v)[2] = \dim_{\mathbb{F}_2} E'(K_v)[2]$ since the 2-part of $E(K_v)$ is cyclic. This shows that $E(K_v)$ has no points of order 4 and similar logic shows that the same is true for $E'(K_v)$. It then follows that $\dim_{\mathbb{F}_2} E'(K_v)/\phi(E(K_v)) = 1$ since $\dim_{\mathbb{F}_2} E(K_v)[2] = \dim_{\mathbb{F}_2} E'(K_v)[2]$ and $\phi$ has degree 2. \hfill $\square$

**Lemma 3.4.** Let $E$ be an elliptic curve with $E(K)[2] \simeq \mathbb{Z}/2\mathbb{Z}$ and suppose that $K(E[2]) = K(E'[2])$. If $E$ has split multiplicative reduction at a place $v$ where the Kodaira symbols of $E$ and $E'$ are $I_n$ and $I_2n$, respectively, then $H^1_{\phi}(K_v, C) = H^1(K_v, C)$.

Further, if $F/K$ is a quadratic extension in which $v$ does not split, then $\dim_{\mathbb{F}_2} H^1_{\phi}(K_v, C^F) = \dim_{\mathbb{F}_2} H^1(K_v, C) - 1$ and $H^1_{\phi}(K_v, C^F) = N_{F_{\infty}/K_v} F_{\infty} K_v^\times / (K_v^\times)^2$, where $\infty$ is the place of $F$ above $v$.

**Proof.** Since $E$ and $E'$ have split multiplicative reduction at $v$, $E/K_v$ and $E'/K_v$ are $G_{K_v}$-isomorphic to Tate curves $E_q$ and $E_{q'}$, respectively. By the condition on the Kodaira symbols, $|q|^2 = |q'|$. Observe that $E_q$ can be two-isogenous to three different curves: $E_{q^2}$, $E_{\sqrt{q}}$, and $E_{-\sqrt{q}}$. The curve $E_{q^2}$ must therefore be one of these possibilities and the only possibility with $|q|^2 = |q'|$ is $q' = q^2$. We therefore get $G_{K_v}$...
isomorphisms $\overline{K_v}^\times /q^2 \to E(\overline{K_v})$ and $\overline{K_v}^\times /q^{2Z} \to E'(\overline{K_v})$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\overline{K_v}^\times /q^2 & \xrightarrow{x \mapsto x^2} & \overline{K_v}^\times /q^{2Z} \\
\downarrow & & \downarrow \\
E(\overline{K_v}) & \xrightarrow{\phi} & E'(\overline{K_v})
\end{array}
$$

Since the maps in this diagram are $G_{K_v}$ equivariant, we can restrict to $K_v$ giving the following diagram, where the vertical arrows are isomorphisms.

$$
\begin{array}{ccc}
K_v^\times /q^2 & \xrightarrow{x \mapsto x^2} & K_v^\times /q^{2Z} \\
\downarrow & & \downarrow \\
E(K_v) & \xrightarrow{\phi} & E'(K_v)
\end{array}
$$

We therefore get a sequence of $G_K$-isomorphisms

$$H^1_\phi(K_v, C) \simeq E'(K_v)/\phi(E(K_v)) \simeq (K_v^\times /q^{2Z})/(K_v^\times /q^2)^2 \simeq K_v^\times /(K_v^\times)^2 \simeq H^1(K_v, C)$$

and that $H^1_\phi(K_v, C') = 0$ proving the first part of the lemma.

Further, by the exactness of (2.1), the map $i : H^1(K_v, C) \to H^1_\phi(K_v, E[2])$ is surjective. Because $E'(K_v) \simeq K_v^\times /q^{2Z}$, we see that $E'(K_v)[2] = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $K(E[2]) = K(E'[2])$, we then see that $E(K_v)[2] = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as well. The exactness of (2.1) then shows that $i$ is injective. We therefore get that the restriction $\tilde{i} : H^1_\phi(K_v, C') \to H^1_\phi(K_v, E[2]) \cap H^1_\phi(K_v, E^F[2])$ is also injective.

Let $c \in H^1_\phi(K_v, E[2]) \cap H^1_\phi(K_v, E^F[2])$. As $H^1_\phi(K_v, C) = 0$, $c$ maps trivially into $H^1_\phi(K_v, C')$ under the map $\phi$ in (2.1). It follows from Proposition 2.2 that $c$ is in the image of $H^1_\phi(K_v, C')$ and that $\tilde{i} : H^1_\phi(K_v, C') \to H^1_\phi(K_v, E[2]) \cap H^1_\phi(K_v, E^F[2])$ is surjective. Therefore $\tilde{i}$ is an isomorphism.

By Lemma 2.1, $H^1_\phi(K_v, E[2]) \cap H^1_\phi(K_v, E^F[2]) = N_{F_w/K_v}E(F_w)/2E(K_v)$. The elliptic curve norm map $N_{F_w/K_v} : E(F_w) \to E(K_v)$ translates into the usual field norm $N_{F_w/K_v} : F_w^\times /q^2 \to K_v^\times /q^{2Z}$, so $H^1_\phi(K_v, E[2]) \cap H^1_\phi(K_v, E^F[2])$ can be identified with

$$
(N_{F_w/K_v}F_w^\times /q^{2Z})/(K_v^\times /q^2)^2 \simeq N_{F_w/K_v}F_w^\times /(K_v^\times)^2.
$$

The isomorphism $E^F(K_v)/\phi(E^F(K_v)) \to E(K_v)/2E(K_v) \cap E^F(K_v)/2E^F(K_v)$ is given by $\hat{\phi}$. As $\hat{\phi}$ is given by $x \mapsto x$ in the above diagram, the identification of $H^1_\phi(K_v, E[2]) \cap H^1_\phi(K_v, E^F[2])$ with $N_{F_w/K_v}F_w^\times /(K_v^\times)^2$ identifies $H^1_\phi(K_v, C')$ with $N_{F_w/K_v}F_w^\times /(K_v^\times)^2$. Standard results from the theory of local fields then give that $\dim_{\mathbb{F}_2}H^1_\phi(K_v, C') = \dim_{\mathbb{F}_2}H^1(K_v, C) - 1$. 

\[\Box\]

**Lemma 3.5.** If $E = E_{(n)} \in \mathcal{F}$, then $E$ has multiplicative reduction at primes $p | 2n$. Further, if $k = \ord_p 2n$, then $E$ has Kodaira symbol $I_{2k}$ at $p$ and $E'$ has Kodaira symbol $I_{4k}$ at $p$. 

Proof. If \( p \mid 2n \), then the model (1.1) is minimal at \( p \). The reduction of (1.1) mod \( p \) has a node so \( E \) has multiplicative reduction at \( p \). We can then read the Kodaira symbols for \( E \) and \( E' \) off of the denominators of their j-invariants, which are \( j(E) = \frac{(1+1024n^2)^3}{4n^2} \) and \( j(E') = \frac{(1+64n^2)^3}{16n^4} \) respectively. \( \square \)

Proof of Proposition 3.1. Lemma 3.5 combined with Lemma 3.4 show that the proposition is true for all places \( v \mid 2n \). The \( j \)-invariant of \( E \) shows that these are the only places where \( E^F \) can have multiplicative reduction and the result then follows from Proposition 3.3. \( \square \)

4. Proof of main theorem

We begin by relating \( d_2(E/K) \) to the 2-adic valuation of \( T(E/E') \).

Proposition 4.1. If \( E(K)[2] \cong \mathbb{Z}/2\mathbb{Z} \) and \( K(E[2]) = K(E'[2]) \), then

\[
d_2(E/K) \geq \text{ord}_2 T(E/E').
\]

Proof. From the definition, we have

\[(4.1) \quad \text{ord}_2 T(E/E') = \dim_{\mathbb{F}_2} \text{Sel}_\phi(E/K) - \dim_{\mathbb{F}_2} \text{Sel}_\phi(E'/K).\]

Since \( E(K)[2] \cong \mathbb{Z}/2\mathbb{Z} \) and \( K(E[2]) = K(E'[2]) \), we get that \( E'(K)[2] \cong \mathbb{Z}/2\mathbb{Z} \) as well. It then follows from Theorem 2.3 that \( \dim_{\mathbb{F}_2} \text{Sel}_\phi(E'/K) \geq 1 \) and that the map of \( \text{Sel}_\phi(E/K) \) into \( \text{Sel}_2(E/K) \) is 2-to-1. Combined with (4.1), we get that the image of \( \text{Sel}_\phi(E/K) \) in \( \text{Sel}_2(E/K) \) has \( \mathbb{F}_2 \)-dimension at least \( \text{ord}_2 T(E/E') \).

Let \( P \) generate \( E(K)[2] \) and let \( c \in \text{Sel}_2(E/K) \) be the image of \( P \) in \( \text{Sel}_2(E/K) \). We can represent \( c \) by a cocycle \( \hat{\phi} : G_K \to E[2] \) given by \( \hat{\phi}(\gamma) = \gamma(R) - R \) for some \( R \in E(K)[4] \) with \( 2R = P \). Observe that since \( 2R = P \), it must be that \( \phi(R) \in E'[2] - C' \). If \( \sigma(R) - R \in C \) for every \( \sigma \in G_K \), then \( \hat{\phi}(C') = 0 \) and \( \hat{\phi}(\sigma(R) - R) = \sigma(\hat{\phi}(R)) - R \) for \( \sigma \in G_K \). Since this would contradict \( E'(K)[2] \cong \mathbb{Z}/2\mathbb{Z} \), it must be that \( \sigma(R) - R \notin C \) for some \( \sigma \in G_K \) and \( c \) therefore does not come from \( H^1(K,C) \). We therefore get that \( d_2(E/K) \geq \text{ord}_2 T(E/E') \). \( \square \)

Theorem 1 now follows easily from Proposition 3.1.

Proof of Theorem 1. Let \( E = E(n) \in \mathcal{F} \) and \( F/K \) quadratic.

By Lemma 2.4, \( \text{ord}_2 T(E^F/E'^F) \) is given by

\[
\text{ord}_2 T(E^F/E'^F) = \sum_{v \mid K} (\dim_{\mathbb{F}_2} H^1_{\phi}(K_v, C^F) - 1).
\]

By Proposition 3.1, we get that \( \dim_{\mathbb{F}_2} H^1_{\phi}(K_v, C^F) - 1 \geq 0 \) for all places \( v \mid 2\infty \). This yields

\[
\text{ord}_2 T(E^F/E'^F) \geq -(r_1 + r_2) + \sum_{v \mid 2} (\dim_{\mathbb{F}_2} H^1_{\phi}(K_v, C^F) - 1)
\]

\[
\geq -(r_1 + r_2) + \sum_{v \mid 2} (\dim_{\mathbb{F}_2} H^1(K_v, C) - 2),
\]

with the second inequality following from Proposition 3.1 as well.
As $H^1(K_v, C) \simeq K_v^\times/(K_v^\times)^2$, we get that $\dim_{F_2} H^1(K_v, C) = 2 + [K_v : \mathbb{Q}_2]$ for places $v | 2$. We therefore have
\[
\text{ord}_2 T(E^F/E'^F) \geq -(r_1 + r_2) + \sum_{v | 2} [K_v : \mathbb{Q}_2] = -(r_1 + r_2) + [K : \mathbb{Q}] = r_2.
\]
Proposition 4.1 then shows that $d_2(E^F/K) \geq r_2$.

The family $\mathcal{F}$ is infinite since every number field $K$ has infinitely many $n$ with $1 + 256n^2 \not\in (K^\times)^2$. The curves $E_n$ have distinct $j$-invariants and therefore are not isomorphic over $\overline{K}$. Since all of the $E_n$ have multiplicative reduction at all places above 2, work of Mazur and Rubin in [7] shows that none of them have constant 2-Selmer parity. □

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