GENERALIZED HAMMING WEIGHTS OF PROJECTIVE REED–MULLER-TYPE CODES OVER GRAPHS

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Abstract. Let $G$ be a connected graph and let $X$ be the set of projective points defined by the column vectors of the incidence matrix of $G$ over a field $K$ of any characteristic. We determine the generalized Hamming weights of the Reed–Muller-type code over the set $X$ in terms of graph theoretic invariants. As an application to coding theory we show that if $G$ is non-bipartite and $K$ is a finite field of characteristic $p \neq 2$, then the $r$-th generalized Hamming weight of the linear code generated by the rows of the incidence matrix of $G$ is the $r$-th weak edge biparticity of $G$. If $\text{char}(K) = 2$ or $G$ is bipartite, we prove that the $r$-th generalized Hamming weight of that code is the $r$-th edge connectivity of $G$.

1. Introduction

In this work we study basic parameters of projective Reed–Muller-type codes over graphs using an algebraic geometric approach via graph theory and commutative algebra, and give some applications to linear codes whose generator matrices are incidence matrices of graphs.

Let $K$ be a field of characteristic $p \geq 0$, let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$, and let $t_1, \ldots, t_s$ and $f_1, \ldots, f_m$ be the vertices and edges of $G$, respectively.

The incidence matrix of $G$, over the field $K$, is the $s \times m$ matrix $A = (a_{ij})$ given by $a_{ij} = 1$ if $t_i \in f_j$ and $a_{ij} = 0$ otherwise. The edge biparticity of $G$, denoted $\varphi(G)$, is the minimum number of edges whose removal makes the graph bipartite. The $r$-th weak edge biparticity of $G$, denoted $\upsilon_r(G)$, is the minimum number of edges whose removal results in a graph with $r$ bipartite components. If $r = 1$, $\upsilon_1(G)$ is the weak edge biparticity of $G$ and is denoted by $\upsilon(G)$. The $r$-th edge connectivity of $G$, denoted $\lambda_r(G)$, is the minimum number of edges whose removal results in a graph with $r + 1$ connected components. If $r = 1$, $\lambda_1(G)$ is the edge connectivity of $G$ and is denoted by $\lambda(G)$. We will use these invariants to study the minimum distance and the Hamming weights of Reed-Muller-type codes over graphs.

The edge biparticity and the edge connectivity are well studied invariants of a graph [16, 33]. In Section 2 we give an algebraic method for computing the edge biparticity (Proposition 2.3). For a discussion of computational and algorithmic aspects of edge bipartization problems we refer to [25]. One has the following relationships [7, 16]:

$$\kappa(G) \leq \lambda(G) \leq \Delta(G) \text{ and } \max\{\upsilon(G), \lambda(G)\} \leq \varphi(G),$$

where $\kappa(G)$ is the vertex connectivity and $\Delta(G)$ is the minimum degree of the vertices of $G$.

The set of columns $\{P_1, \ldots, P_m\}$ of $A$ can be regarded as a set of points $X = \{[P_1], \ldots, [P_m]\}$ in a projective space $\mathbb{P}^{s-1}$ over the field $K$. Consider a polynomial ring $S = K[t_1, \ldots, t_s] = \bigoplus_{d=0}^{\infty} S_d$.
over the field $K$ with the standard grading. The vanishing ideal $I(\mathcal{X})$ of $\mathcal{X}$ is the graded ideal of $S$ generated by the homogeneous polynomials of $S$ that vanish at all points of $\mathcal{X}$. Fix integers $d \geq 1$ and $r \geq 1$. The aim of this work is to determine the following number in terms of the combinatorics of the graph $G$:

$$\delta_{\mathcal{X}}(d, r) := \min\{|X \setminus V_{\mathcal{X}}(F)| : F = \{f_i\}^r_{i=1} \subset S_d, \dim_K(\{\overline{f_i}\}^r_{i=1}) = r\},$$

where $V_{\mathcal{X}}(F)$ is the set of zeros or projective variety of $F$ in $\mathcal{X}$, and $\overline{f_i} = f_i + I(\mathcal{X})$ is the class of $f_i$ modulo $I(\mathcal{X})$. This is equivalent to determine

$$\text{hyp}_{\mathcal{X}}(d, r) := \max\{|V_{\mathcal{X}}(F)| : F = \{f_i\}^r_{i=1} \subset S_d, \dim_K(\{\overline{f_i}\}^r_{i=1}) = r\}$$

because $\delta_{\mathcal{X}}(d, r) = |X| - \text{hyp}_{\mathcal{X}}(d, r)$.

A projective Reed–Muller-type code of degree $d$ on $\mathcal{X}$ \cite{8 13}, denoted $C_{\mathcal{X}}(d)$, is the image of the following evaluation linear map

$$\text{ev}_d : S_d \to K^m, \quad f \mapsto (f(P_1), \ldots, f(P_m)).$$

The motivation to study $\delta_{\mathcal{X}}(d, r)$ comes from algebraic coding theory because—over a finite field—the $r$-th generalized Hamming weight of the Reed–Muller-type code $C_{\mathcal{X}}(d)$ of degree $d$ is equal to $\delta_{\mathcal{X}}(d, r)$ \cite[Lemma 4.3(iii)]{hyp}. Generalized Hamming weights were introduced by Wei \cite{18 21 30}. For convenience we recall this notion. Let $K = \mathbb{F}_q$ be a finite field and let $C$ be a $[m, k]$ linear code of length $m$ and dimension $k$, that is, $C$ is a linear subspace of $K^m$ with $k = \dim_K(C)$. Let $1 \leq r \leq k$ be an integer. Given a linear subspace $D$ of $C$, the support of $D$ is the set

$$\chi(D) := \{i \mid \exists (a_1, \ldots, a_m) \in D, a_i \neq 0\}.$$

The $r$-th generalized Hamming weight of $C$, denoted $\delta_r(C)$, is given by

$$\delta_r(C) := \min\{|\chi(D)| : D \text{ is a subspace of } C, \dim_K(D) = r\}.$$

The set $\{\delta_1(C), \ldots, \delta_k(C)\}$ is called the weight hierarchy of the code $C$. The following duality of Wei \cite{30} Theorem 3] is a classical result in this area that shows a strong relationship between the weight hierarchies of $C$ and its dual $C^\perp$:

$$\{\delta_i(C) | i = 1, \ldots, k\} = \{1, \ldots, m\} \setminus \{m + 1 - \delta_i(C^\perp) | i = 1, \ldots, m - k\}. $$

These numbers are a natural generalization of the notion of minimum distance and they have several applications from cryptography (codes for wire–tap channels of type II), $t$–resilient functions, trellis or branch complexity of linear codes, and shortening or puncturing structure of codes; see \cite{1 3 6 9 11 12 17 19 24 27 28 30 31 32} and the references therein. If $r = 1$, we obtain the minimum distance $\delta(C)$ of $C$ which is the most important parameter of a linear code. In this paper we give combinatorial formulas for the weight hierarchy of $C_{\mathcal{X}}(d)$ for $d \geq 1$.

Our main results are:

**Theorems 2.10, 2.11, 2.12** Let $G$ be a connected graph with $s$ vertices, $m$ edges, $r$-th weak edge biparticity $\upsilon_r(G)$, $r$-th edge connectivity $\lambda_r(G)$, and let $A$ be the incidence matrix of $G$ over a field $K$ of characteristic $p$. If $\mathcal{X}$ is the set of column vectors of $A$, then

$$\delta_{\mathcal{X}}(d, r) = \delta_r(C_{\mathcal{X}}(d)) = \begin{cases} \upsilon_r(G) & \text{if } d = 1, p \neq 2, G \text{ is non-bipartite}, 1 \leq r \leq s, \\ \lambda_r(G) & \text{if } d = 1, p = 2, 1 \leq r \leq s - 1, \\ \lambda_r(G) & \text{if } d = 1, G \text{ is bipartite}, 1 \leq r \leq s - 1, \\ r & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m. \end{cases}$$
Thus computing \( v_r(G) \) and \( \lambda_r(G) \) is equivalent to computing the \( r \)-th generalized Hamming weight of \( C_p(G) \) for \( K = \mathbb{F}_2 \) or \( K = \mathbb{F}_3 \). These are the only cases that matter.

The incidence matrix code of a graph \( G \) over a finite field \( K \) of characteristic \( p \), denoted \( C_p(G) \), is the linear code generated by the rows of the incidence matrix of \( G \). As an application to coding theory we obtain the following combinatorial formulas for the generalized Hamming weights of \( C_p(G) \) when \( G \) is connected (Corollary 2.13).

\[
\delta_r(C_p(G)) = \begin{cases} 
  v_r(G) & \text{if } p \neq 2, G \text{ is non-bipartite}, 1 \leq r \leq s, \\
  \lambda_r(G) & \text{if } p = 2, 1 \leq r \leq s-1, \\
  \lambda_r(G) & \text{if } G \text{ is bipartite}, 1 \leq r \leq s-1.
\end{cases}
\]

The minimum distance of the incidence matrix code of the graph \( G \) is defined as

\[
\delta(C_p(G)) := \min \{ \omega(a) : a \in C_p(G) \setminus \{0\} \},
\]

where \( \omega(a) \) is the Hamming weight of the vector \( a \), that is, the number of non-zero entries of \( a \). The minimum distance of \( C_p(G) \) is \( \delta_1(C_p(G)) \), the 1st Hamming weight of this code. Then we can recover the combinatorial formulas of Dankelmann, Key and Rodrigues [4, Theorems 1–3] for the minimum distance of \( C_p(G) \) in terms of the weak edge biparticity \( v(G) \) and the edge connectivity \( \lambda(G) \) of \( G \) (Corollary 2.14).

Using Macaulay 2 [14], SageMath [26], and Wei’s duality [30, Theorem 3], we can compute the weight hierarchy of \( C_p(G) \). In Sections 3 and 4 we illustrate this with some examples and procedures. There are algebraic methods that can be used to obtain lower bounds for \( \delta_r(C_p(G)) \) or equivalently for \( \lambda_r(G) \) and \( v_r(G) \) [11, Theorem 4.9].

2. Reed–Muller-type codes over connected graphs

In this section we present our main results. To avoid repetitions, we continue to employ the notations and definitions used in Section 1.

**Lemma 2.1.** Let \( G \) be a connected graph and let \( e_1, \ldots, e_r \) be a minimum set of edges whose removal makes the graph bipartite. Then there is \( \omega : V(G) \to \{+, -\} \) such that the edges of \( G \) whose vertices have positive sign or negative sign are precisely \( e_1, \ldots, e_r \).

**Proof.** If \( G \) is bipartite, there is nothing to prove. If \( G \) is non-bipartite, pick a bipartition \( V_1, V_2 \) of the graph \( G \setminus \{e_1, \ldots, e_r\} \). Setting \( \omega(v) = + \) if \( v \in V_1 \) and \( \omega(v) = - \) if \( v \in V_2 \), note that the vertices of each \( e_i \) have the same sign. Indeed if the vertices of \( e_i \) have different sign, then \( G \setminus \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r\} \) is bipartite, a contradiction. \( \square \)

The edge biparticity of a graph \( G \) can be easily expressed by considering all possible ways of making \( G \) a vertex-signed graph.

**Lemma 2.2.** Let \( G \) be a connected graph, let \( \mathcal{F} \) be the set of surjective maps \( \omega : V(G) \to \{+, -\} \), and let \( E_\omega \) be the set of edges of \( G \) whose vertices have the same sign. Then

\[
\varphi(G) = \min \{|E_\omega| : \omega \in \mathcal{F}\}.
\]

**Proof.** If \( G \) is bipartite, \( \varphi(G) = 0 \) and there is nothing to prove. Assume that \( G \) is non-bipartite. Then \( E_\omega \neq \emptyset \) for \( \omega \in \mathcal{F} \). By Lemma 2.1 there is \( \omega \in \mathcal{F} \) such that \( \varphi(G) = |E_\omega| \). Thus, one has the inequality \( \geq \). To show the reverse inequality take \( \omega \in \mathcal{F} \). It suffices to show that \( \varphi(G) \leq |E_\omega| \). The vertex set of \( G \) can be partitioned as \( V(G) = V^+ \cup V^- \), where \( V^+ \) (resp. \( V^- \)) is the set of vertices of \( G \) with positive (resp. negative) sign. Then \( G \setminus E_\omega \) is bipartite with bipartition \( V^+, V^- \). Thus \( \varphi(G) \leq |E_\omega| \). \( \square \)
This lemma can be used to compute $\varphi(G)$. Let $K$ be a field of $\text{char}(K) \neq 2$. Each $\omega$ in $\mathcal{F}$ defines a linear polynomial

$$h_\omega = \sum_{\omega(t_i)=+} t_i - \sum_{\omega(t_i)=-} t_i.$$ 

The number of points of $X$ where $h_\omega$ does not vanish is equal to $|E_\omega|$. As a consequence one obtains the following algebraic formula for the edge biparticity.

**Proposition 2.3.** Let $G$ be a connected non-bipartite graph over a field of $\text{char}(K) \neq 2$. Then $\varphi(G) = \min\{|X\setminus V_k(h_\omega)| : h = a_1t_1 + \cdots + a_st_s, a_i \in \{1,-1\}, \forall i\}.$

**Proof.** As $|E_\omega| = |X\setminus V_k(h_\omega)|$ for $\omega \in \mathcal{F}$, the result follows from Lemma 2.2.

This result can be used in practice to compute $\varphi(G)$ using Macaulay2 [14] (see the examples and procedures of Sections 3 and 4).

**Remark 2.4.** If we allow $a_1, \ldots, a_s$ to be in $\{0,1,-1\}$ such that not all of them are zero, we obtain the minimum distance of $C_p(G)$. This follows from [11] Lemma 4.3(iii)].

The following result is well known.

**Proposition 2.5.** [2] [15] [20] Let $G$ be a connected graph with $s$ vertices and let $A$ be its incidence matrix over a field $K$. Then

$$\text{rank}(A) = \begin{cases} s & \text{if char}(K) \neq 2 \text{ and } G \text{ is non-bipartite,} \\ s-1 & \text{if char}(K) = 2 \text{ or } G \text{ is bipartite.} \end{cases}$$

**Corollary 2.6.** Let $G$ be a connected graph with $s$ vertices and $m$ edges and let $C = C_p(G)$ (resp. $C^\perp$) be the code (resp. dual code) of $G$. Then

(a) $C$ (resp. $C^\perp$) is an $[m,s]$ (resp. $[m,m-s]$) code if $p \neq 2$ and $G$ is non-bipartite.

(b) $C$ (resp. $C^\perp$) is an $[m,s-1]$ (resp. $[m,m-s+1]$) code if $p = 2$ or $G$ is bipartite.

**Proof.** This follows from Proposition 2.5 noticing that $\text{dim}(C) + \text{dim}(C^\perp) = m.$

**Lemma 2.7.** Let $G$ be a connected graph and let $K$ be a field. The following hold.

(a) If $\text{char}(K) \neq 2$, $G$ is non-bipartite and $h$ is a linear form in $I(X)$, then $h = 0$.

(b) If $\text{char}(K) = 2$ and $h$ is a linear form in $I(X)$ in $s-1$ variables, then $h = 0$.

(c) If $\text{char}(K) = 2$ and $h \neq 0$ is a linear form in $I(X)$, then $h = c \sum_{i=1}^{s} t_i$, for some $c \in K$.

**Proof.** Let $\psi$ be the linear map $\psi: K^s \to K^m, x \mapsto xA^\top$, where $A^\top$ is the transpose of the incidence matrix of $G$. Fix a linear form $h = \sum_{i=1}^{s} a_it_i$ of $S_1$ and set $v_h = (a_1, \ldots, a_s)$. Then $v_h$ is in $\ker(\psi)$ if and only if $h \in I(X)$. For use below notice that $s = \text{dim}(\ker(\psi)) + \text{rank}(A)$.

(a): By Proposition 2.5 $\ker(\psi) = \{0\}$. Thus $v_h = 0$, that is, $h = 0$.

(b): Again by Proposition 2.5, $\ker(\psi)$ has dimension 1. Hence $\ker(\psi)$ is generated by the vector $1 = (1, \ldots, 1)$. As $v_h$ is in the kernel of $\psi$, it is a scalar multiple of $1$, and since at least one of the $a_i$’s is zero, we get $v_h = 0$, that is, $h = 0$.

(c): Since $\ker(\psi)$ is generated by $1 = (1, \ldots, 1)$ and $v_h \in \ker(\psi)$, the result follows.

**Lemma 2.8.** Let $G$ be a connected bipartite graph with bipartition $V_1, V_2$. The following hold.

(a) If $K$ is a field and $h \neq 0$ is a linear form of $S$ that vanishes at all points of $X$, then $h = c(\sum_{t_i \in V_1} t_i - \sum_{t_i \in V_2} t_i)$ for some $c \in K$. 
has exactly $r$ components of $H$. We set $F$ of $K$ linearly independent over $H$. Hence, $G$ is a field of $G$. Let $H$ be a connected non-bipartite graph with $s$ vertices and $m$ edges, let $K$ be a field of $K$ such that $\text{char}(K) \neq 2$, and let $A$ be the incidence matrix of $G$. If $X$ is the set of column vectors of $A$ and $\nu_r(G)$ is the $r$-th weak edge biparticity of $G$, then

$$\delta_X(d, r) = \begin{cases} \nu_r(G) & \text{if } d = 1 \text{ and } 1 \leq r \leq s = \dim_K(C_X(d)), \\ r & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m = \dim_K(C_X(d)). \end{cases}$$

Proof. Assume $d = 1$. First we show the inequality $\delta_X(1, r) \geq \nu_r(G)$. We proceed by contradiction assuming that $\nu_r(G) > \delta_X(1, r)$. Then $\nu_r(G) > |X \setminus V_X(F)|$ for some set $F$ consisting of $r$ linear forms $h_1, \ldots, h_r$, which are linearly independent modulo $I(X)$. Let $[P_1], \ldots, [P_s]$ be the points in $X \setminus V_X(F)$ and let $f_1, \ldots, f_\ell$ be the edges of $G$ corresponding to these points. Consider the graph $H = G \setminus \{f_1, \ldots, f_\ell\}$. Let $H_1, \ldots, H_n$ be the bipartite connected components of $H$. Since $\nu_r(G) > \ell$, $n$ is at most $r - 1$. Let $X_H$ be the set of points corresponding to the columns of the incidence matrix of $H$. Note that $h_i$ vanishes at all points of $X_H$ for $i = 1, \ldots, r$. Then, by Lemma 2.7, $h_1, \ldots, h_r$ are linear forms in the variables $V(H_1) \cup \cdots \cup V(H_n)$. For each $1 \leq j \leq n$, let $A_1^j, A_2^j$ be the bipartition of $H_j$ and set $g_j = \sum_{t_i \in A_1^j} t_i - \sum_{t_i \in A_2^j} t_i$. Then, by Lemma 2.8, $F = \{h_1, \ldots, h_r\}$ is in the $K$-linear space generated by $g_1, \ldots, g_n$, a contradiction because $F$ is linearly independent over $K$ and $n < r$.

Now we show the inequality $\delta_X(1, r) \leq \nu_r(G)$. Note that, by Lemma 2.7, it suffices to find a set $F = \{h_1, \ldots, h_r\}$ of linearly independent forms of degree 1 such that $\nu_r(G) = |X \setminus V_X(F)|$. We set $\ell = \nu_r(G)$. There are edges $f_1, \ldots, f_\ell$ of $G$ such that the graph

$$H = G \setminus \{f_1, \ldots, f_\ell\}$$

has exactly $r$ connected bipartite components (see Lemma 2.9). We denote the connected components of $H$ by $H_1, \ldots, H_n$, where $H_1, \ldots, H_r$ are bipartite. Consider a bipartition $A_1^j, A_2^j$ of $H_j$ for $j = 1, \ldots, r$ and set

$$h_j = \sum_{t_i \in A_1^j} t_i - \sum_{t_i \in A_2^j} t_i.$$

Let $P_i$ be the point in $P^{s-1}$ that corresponds to $f_i$ for $i = 1, \ldots, \ell$. To complete the proof of the case $d = 1$ we need only show the equality $\{|[P_1], \ldots, [P_s]| = |X \setminus V_X(F)|$. To show the
inclusion “⊂” fix an edge \( f_k \) with \( 1 \leq k \leq \ell \) and set

\[
H' = \bigcup_{i=1}^{r} H_i, \quad H'' = \bigcup_{i=r+1}^{n} H_i \quad \text{and} \quad G' = G \setminus \{f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_{\ell}\}.
\]

Note that \( f_k \not\subseteq V(H_j) \) for \( r < j \), otherwise \( G' \) has \( r \) bipartite components. As a consequence \( f_k \) intersects \( V(H') \), otherwise \( f_k \subseteq V(H'') \), \( f_k \) joins \( H_i \) and \( H_j \) for some \( r < i < j \), and the graph \( G' \) has a \( r \) bipartite components, a contradiction.

Case (1): \( f_k \subseteq V(H_j) \) for some \( 1 \leq j \leq r \). As \( V(H_j) = A_1^j \cup A_2^j \), either \( f_k \subseteq A_1^j \) or \( f_k \subseteq A_2^j \), otherwise the graph \( G' \) has \( r \) bipartite components, a contradiction. Hence, as \( \text{char}(K) \neq 2 \), we get that \( h_j(P_k) \neq 0 \). Thus \( [P_k] \in X \setminus V_X(F) \).

Case (2): \( f_k \cap V(H_i) \neq \emptyset \) and \( f_k \not\subseteq V(H_j) \) for some \( i < j \leq r \). Then using the bipartitions of \( H_i \) and \( H_j \) we get \( h_i(P_k) \neq 0 \) and \( h_j(P_k) \neq 0 \). Thus \( [P_k] \in X \setminus V_X(F) \).

Case (3): \( f_k \cap V(H_i) \neq \emptyset \) for some \( 1 \leq i \leq r \) and \( f_k \not\subseteq V(H'') \) \( \neq \emptyset \). Then using the bipartition of \( H_i \) we get \( h_i(P_k) \neq 0 \). Thus \( [P_k] \in X \setminus V_X(F) \).

To show the inclusion “⊃” take \( P \in X \setminus V_X(F) \) and denote by \( f \) its corresponding edge in \( G \). Then there is \( 1 \leq i \leq n \) such that \( h_i(P) \neq 0 \). We proceed by contradiction assuming \( P \notin \{(P_1), \ldots, (P_r)\} \), that is, \( f \neq f_i \) for \( i = 1, \ldots, \ell \). Then \( f \) is an edge of \( H \). Thus \( f \) is an edge of \( H_{k} \) for some \( 1 \leq k \leq n \). If \( r < k \), then \( h_i(P) = 0 \) for \( i = 1, \ldots, r \) by construction of the \( h_i \)’s, a contradiction. Thus \( 1 \leq k \leq r \). If \( f \subseteq A_1^k \) or \( f \subseteq A_2^k \), then \( H_k \) would not be bipartite, a contradiction. Hence \( f \) joins \( A_1^k \) with \( A_2^k \), and consequently \( h_i(P) = 0 \) for \( i = 1, \ldots, r \) by construction of the \( h_i \)’s, a contradiction. Thus \( P = P_j \) for some \( 1 \leq i \leq \ell \), as required.

Assume \( d \geq 2 \). We claim that \( \dim_K(S_d/I(X)_d) \) is equal to \( m = |E(G)| = |X| \), the number of edges of \( G \). The set of all squarefree monomials \( t_{i_1}t_{j_1} \) such that \( \{t_{i_1}, t_{j_1}\} \) is an edge of \( G \) is \( K \)-linearly independent modulo \( I(X) \). This follows using that the vanishing ideal of \( X \) is the intersection of the vanishing ideals of the points of \( X \) and using a well known formula for the vanishing ideal of a projective point [22, p. 398, Corollary 6.3.19]. Therefore \( \dim_K(S_d/I(X)_d) \geq |X| \).

As \( \dim_K(S_d/I(X)_d) \) is a non-decreasing function of \( d \) and it is bounded from above by the number of points of \( |X| \) (see [10]), the claim follows. Therefore, since \( S_d/I(X)_d \simeq C_X(d) \), one has \( C_X(d) = K^m \). Thus \( \delta_X(d, r) = r \) for \( 1 \leq r \leq m \). \( \square \)

We come to another of our main results.

**Theorem 2.11.** Let \( G \) be a connected graph with \( s \) vertices and \( m \) edges, let \( K \) be a field of \( \text{char}(K) = 2 \), and let \( A \) be the incidence matrix of \( G \). If \( X \) is the set of column vectors of \( A \) and \( \lambda_r(G) \) is the \( r \)-th edge connectivity of \( G \), then

\[
\delta_X(d, r) = \begin{cases} \lambda_r(G) & \text{if } d = 1 \text{ and } 1 \leq r \leq s-1 = \dim_K(C_X(d)), \\ r & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m = \dim_K(C_X(d)). \end{cases}
\]

**Proof.** Assume \( d = 1 \). First we show the inequality \( \delta_X(1, r) \geq \lambda_r(G) \). We proceed by contradiction assuming that \( \lambda_r(G) > \lambda_X(1, r) \). Then \( \lambda_r(G) > |X \setminus V_X(F)| \) for some set \( F \) consisting of \( r \) linear forms \( h_1, \ldots, h_r \) which are linearly independent modulo \( I(X) \). We set \( \ell = |X \setminus V_X(F)| \). Let \( [P_1], \ldots, [P_{\ell}] \) be the points in \( X \setminus V_X(F) \) and let \( f_1, \ldots, f_{\ell} \) be the edges of \( G \) corresponding to these points. Consider the graph \( H = G \setminus \{f_1, \ldots, f_{\ell}\} \) and denote by \( H_1, \ldots, H_{\ell} \) its connected components. Since \( \lambda_r(G) > \ell \), \( H \) cannot have \( r + 1 \) components, that is, \( n \leq r \). Let \( X_H \) be the set of points corresponding to the columns of the incidence matrix of \( H \). Note that \( h_i \) vanishes at all points of \( X_H \) for \( i = 1, \ldots, r \). Indeed, take a point \( [P] \in X_H \), then its corresponding edge \( f \) is in \( H_k \) for some \( k \), then \( f \neq f_j \) for \( j = 1, \ldots, \ell \) and \( [P] \notin X \setminus V_X(F) \), that is, \( h_i(P) = 0 \). We
set \( g_j = \sum_{t_i \in V(H_j)} t_i \) for \( j = 1, \ldots, n \). As \( h_i \in I(\Xi_H) \), by Lemma 2.7, \( h_i \) is a linear combination of \( g_1, \ldots, g_n \) for \( i = 1, \ldots, r \). Therefore

\[
Kh_1 \oplus \cdots \oplus Kh_r \subseteq Kg_1 \oplus \cdots \oplus Kg_n,
\]

and consequently \( r \leq n \). Thus \( r = n \) and the inclusion above is an equality. Therefore taking classes modulo \( I(\Xi) \), we get

\[
K\overline{h_1} \oplus \cdots \oplus K\overline{h_r} = K\overline{g_1} \oplus \cdots \oplus K\overline{g_n}.
\]

As \( \overline{h_1}, \ldots, \overline{h_r} \) are linearly independent, so are \( \overline{g_1}, \ldots, \overline{g_n} \) because \( r = n \), a contradiction because by construction of the \( g_i \)'s and since \( \text{char}(K) = 2 \), one has \( \sum_{i=1}^n \overline{g_i} = \sum_{i=1}^s \overline{t_i} = \overline{0} \).

Next we show the inequality \( \delta_\Xi(1, r) \leq \lambda_r(G) \). Note that by Lemma 2.7. It suffices to find a set \( F = \{h_1, \ldots, h_r\} \) of forms of degree 1 whose image \( \overline{F} = \{\overline{t_1}, \ldots, \overline{t_r}\} \) in \( S/I(\Xi) \) is linearly independent over \( K \) and \( \lambda_r(G) = |\Xi \setminus V_\Xi(F)| \). We set \( \ell = \lambda_r(G) \). There are edges \( f_1, \ldots, f_\ell \) of \( G \) such that the graph \( H = G \setminus \{f_1, \ldots, f_\ell\} \)

has exactly \( r + 1 \) connected components \( H_1, \ldots, H_{r+1} \). For \( j = 1, \ldots, r \), we set

\[
h_j = \sum_{t_i \in V(H_j)} t_i.
\]

Note that \( h_i \) and \( h_j \) have no common variables for \( i \neq j \) and any sum of the polynomials \( h_1, \ldots, h_r \) is a linear form in \( s - 1 \) variables. Hence, by Lemma 2.8, \( \overline{G} \) is linearly independent.

Let \( P_i \) be the point in \( \mathbb{P}^{s-1} \) that corresponds to \( f_i \) for \( i = 1, \ldots, \ell \). To complete the proof of the case \( d = 1 \) we need only show the equality \( \{[P_1], \ldots, [P_\ell]\} = |\Xi \setminus V_\Xi(F)| \). To show the inclusion “\( \supseteq \)” fix an edge \( f_k \) with \( 1 \leq k \leq \ell \) and set \( G' = G \setminus \{f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_\ell\} \).

Note that \( f_k \not\subset V(H_j) \) for \( j = 1, \ldots, r+1 \), otherwise \( G' \) has \( r + 1 \) components, a contradiction. As a consequence \( f_k \) joins \( H_i \) and \( H_j \) for some \( i < j \). Thus \( h_i(P_k) \neq 0 \) and \( P_k \in \Xi \setminus V_\Xi(F) \).

To show the inclusion “\( \subseteq \)” take \( [P] \in \Xi \setminus V_\Xi(F) \) and denote by \( f \) its corresponding edge in \( G \). Then there is \( 1 \leq j \leq r \) such that \( h_j(P) \neq 0 \). We proceed by contradiction assuming \( [P] \notin \{[P_1], \ldots, [P_\ell]\} \), that is, \( f \neq f_i \) for \( i = 1, \ldots, \ell \). Then \( f \) is an edge of \( H \). As \( \text{char}(K) = 2 \), we get \( h_i(P) = 0 \) for \( i = 1, \ldots, r \) by construction of \( h_i \), a contradiction.

If \( d \geq 2 \), the equality \( \delta_\Xi(d, r) = r \) for \( 1 \leq r \leq m = \dim_K(C_\Xi(d)) \) follows from the proof of Theorem 2.10.

The next result is a hybrid of Theorems 2.10 and 2.11 and is characteristic free.

**Theorem 2.12.** Let \( G \) be a connected bipartite graph with \( s \) vertices and \( m \) edges, let \( K \) be a field of any characteristic, and let \( A \) be the incidence matrix of \( G \). If \( \Xi \) is the set of column vectors of \( A \) and \( \lambda_r(G) \) is the \( r \)-th edge connectivity of \( G \), then

\[
\delta_\Xi(d, r) = \begin{cases} 
\lambda_r(G) & \text{if } d = 1 \text{ and } 1 \leq r \leq s - 1 = \dim_K(C_\Xi(d)), \\
r & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m = \dim_K(C_\Xi(d)).
\end{cases}
\]

**Proof.** Let \( V_1, V_2 \) be the bipartition of \( G \). Consider the set \( \Psi \) of all points \( [e_i - e_j] \) in \( \mathbb{P}^{s-1} \) such that \( \{t_i, t_j\} \) is an edge of \( G \) with \( t_i \in V_1 \) and \( t_j \in V_2 \), where \( e_i \) is the \( i \)-th unit vector in \( K^s \). Noticing that the polynomial \( h = t_1 + \cdots + t_s \) vanishes at all points of \( \Psi \) and the equality \( C_\Xi(1) = C_\Psi(1) \), the result follows adapting Lemma 2.7 and the proof of Theorem 2.11 with \( \Psi \) playing the role of \( \Xi \).

\( \square \)
Corollary 2.13. Let $C_p(G)$ be the code of a connected graph $G$ with $s$ vertices, $m$ edges, $r$-th weak edge biparticity $\nu_r(G)$, $r$-th edge connectivity $\lambda_r(G)$, over a finite field $K$ of $\text{char}(K) = p$. Then the $r$-th generalized Hamming weight of $C_p(G)$ is given by

$$\delta_r(C_p(G)) = \begin{cases} 
\nu_r(G) & \text{if } p \neq 2, G \text{ is non-bipartite}, 1 \leq r \leq s, \\
\lambda_r(G) & \text{if } p = 2, 1 \leq r \leq s - 1, \\
\lambda_r(G) & \text{if } G \text{ is bipartite}, 1 \leq r \leq s - 1.
\end{cases}$$

Proof. Note that the linear code $C_p(G)$ is the image of $S_1$—the vector space of linear forms of $S$—under the evaluation map $\text{ev}_1 : S_1 \rightarrow K^m$, $f \mapsto (f(P_1), \ldots, f(P_m))$. Note that the image of the linear function $t_i$, under the map $\text{ev}_1$, gives the $i$-th row of the incidence matrix of $G$. This means that $C_p(G)$ is the Reed–Muller-type code $C_X(1)$. Hence, the result follows using the equality $\delta_X(1, r) = \delta_r(C_X(1))$ [11, Lemma 4.3(iii)] and Theorems 2.10, 2.11, and 2.12. □

Corollary 2.14. [4, Theorems 1–3] Let $C_p(G)$ be the code of a connected graph $G$ with $s$ vertices, $m$ edges, weak edge biparticity $\nu(G)$, edge connectivity $\lambda(G)$, over a finite field $K$ of $\text{char}(K) = p$. Then the minimum distance of $C_p(G)$ is given by

$$\delta(C_p(G)) = \begin{cases} 
\nu(G) & \text{if } p \neq 2, G \text{ is non-bipartite}, 1 \leq r \leq s, \\
\lambda(G) & \text{if } p = 2, 1 \leq r \leq s - 1, \\
\lambda(G) & \text{if } G \text{ is bipartite}, 1 \leq r \leq s - 1.
\end{cases}$$

Proof. It follows from Corollary 2.13 making $r = 1$. □

3. Examples

Let $G$ be a connected graph and let $C_p(G)$ be the incidence matrix code of $G$ over a finite field $\mathbb{F}_q$ of characteristic $p$. Using Macaulay 2 [14], SageMath [26], and Wei’s duality [30] Theorem 3, we can compute the weight hierarchy of $C_p(G)$. We illustrate this with some examples.

Note that, by Theorem 2.13 we can compute the corresponding higher weak biparticity and edge connectivity numbers of the graph. Conversely any algorithm that computes these graph invariants can be used to compute the weight hierarchy of $C_p(G)$.

Example 3.1. Let $G$ be the graph of Figure 1. Recall that the dimension of $C_p(G)$ is 6 if $p = 3$ and is 5 if $p = 2$ (Corollary 2.6). For use below we denote the dual code by $C_p(G)^\perp$.

Using Procedure 4.11 together with, Wei’s duality [30] Theorem 3] we obtain the following table with the weight hierarchy of $C_p(G)$. The edge biparticity of this graph is 2, the weak edge biparticity is 2, and the edge connectivity is 3.
Example 3.2. Let $G$ be the Petersen graph of Figure 2. Recall that the dimension of $C_p(G)$ (resp. $C_p(G)^\perp$) is 9 (resp. 6) if $p = 2$, and the dimension of $C_p(G)$ (resp. $C_p(G)^\perp$) is 10 (resp. 5) if $p \neq 32$ (Corollary 2.6).

Using Procedure 4.2, together with, Wei’s duality [30, Theorem 3] we obtain the following table with the weight hierarchy of $C_p(G)$. The edge biparticity, the weak edge biparticity, and the edge connectivity of the Petersen graph are equal to 3.

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| $\delta_r(C_2(G))$ | 3 | 5 | 7 | 9 | 10 | 12 | 13 | 14 | 15 |    |
| $\delta_r(C_2(G)^\perp)$ | 5 | 8 | 10 | 12 | 14 | 15 |     |     |     |     |
| $\delta_r(C_3(G))$ | 3 | 5 | 7 | 8 | 9 | 11 | 12 | 13 | 14 | 15 |
| $\delta_r(C_3(G)^\perp)$ | 6 | 10 | 12 | 14 | 15 |     |     |     |     |     |

Table 2. Weight hierarchy of $C_p(G)$, $p = 2$, for the graph of Figure 2

4. Procedures for Macaulay2 and SageMath

Procedure 4.1. Computing the weight hierarchies using Macaulay2 [13], SageMath [26], and Wei’s duality [30]. This procedure corresponds to Example 3.1. It could be applied to any connected graph $G$ to obtain the generalized Hamming weights of $C_p(G)$. The next procedure for Macaulay2 uses the algorithms of [11] to compute generalized minimum distance functions.

--Procedure for Macaulay2
input "points.m2"
q=3, R = ZZ/q[t1,t2,t3,t4,t5,t6]--p=char(K)=3
A = transpose(matrix{{1,1,0,0,0,0},{0,1,1,0,0,0},{1,0,1,0,0,0}},}
\{0,0,1,1,0\}, \{0,0,0,0,1,1\}, \{0,0,0,1,0,1\}, \{1,0,0,1,0,0\},
\{0,1,0,0,0,1\}, \{0,0,1,0,1,0\}\}
I=\text{ideal}(\text{projectivePointsByIntersection}(A,R)), M=\text{coker gens gb I}
genmd\left(\text{degree } M, \text{max apply(apply(apply(apply(apply((hilbertFunction(d,M))
-\left(\text{set}\{\emptyset\}\right)^{\text{hilbertFunction(d,M))}, \text{toList}), x->\text{basis}(d,M)*\text{vector } x),
z->\text{ideal(\text{flatten entries } z)}), r), \text{ideal}), x-> \text{if } \#\text{set flatten entries}
\text{mingens ideal(leadTerm gens } x)==r \text{ and not quotient}(I,x)==1
\text{then degree}(I+x) \text{ else } 0)
\)--The following are the first two generalized Hamming weights
\text{genmd}(1,1), \text{genmd}(1,2)
#Procedure for SageMath
A = \text{transpose}(\text{matrix}(\text{GF}(3),[[1,1,0,0,0,0],[0,1,1,0,0,0],[1,0,1,0,0,0],
[0,0,0,1,1,0],[0,0,0,0,1,1],[0,0,0,1,0,1],[1,0,0,1,0,0],[0,1,0,0,0,1],[0,0,1,0,1,0])))
C = \text{codes.LinearCode}(A)
C.\text{parity_check_matrix}()
C.\text{generator_matrix}()
#the next line Gives the minimum distance of the dual code
C.\text{dual_code}.\text{minimum_distance}()

\textbf{Procedure 4.2.}\text{ Computing the weight hierarchies using \textit{Macaulay}2 [14], SageMath [26], and}
Wei's duality [30]. This procedure corresponds to Example 3.2. The next procedure for
\textit{Macaulay}2 uses the algorithms of [11] to compute generalized footprint functions. The foot-
print gives lower bounds for the generalized weights.

--Procedure for Macaulay2 for Petersen graph
input "points.m2"
R = \text{QQ}[t1,t2,t3,t4,t5,t6,t7,t8,t9,t10]
--Incidence matrix to compute the edge biparticity
A = \text{transpose} matrix\{\{1,1,0,0,0,0,0,0,0,0\}, \{0,1,1,0,0,0,0,0,0,0\},
\{0,0,1,1,0,0,0,0,0,0\}, \{0,0,0,1,1,0,0,0,0,0\}, \{1,0,0,0,1,0,0,0,0,0\},
\{1,0,0,0,0,1,0,0,0,0\}, \{0,1,0,0,0,0,1,0,0,0\}, \{0,0,0,0,1,0,0,0,1,0,0\},
\{0,0,0,0,0,0,1,0,1,0,0\}, \{0,0,0,0,0,1,0,0,0,1\}, \{0,0,0,0,0,1,0,0,1,0,0\},
\{0,0,0,0,0,0,1,0,1,0,0\}, \{0,0,0,0,0,1,0,0,0,1\}, \{0,0,0,0,0,0,1,0,0,1\},
\{0,0,0,0,0,0,0,1,0,1\})
q=2, R = \text{ZZ}/q[t1,t2,t3,t4,t5,t6,t7,t8,t9]
--Generator matrix computed with Sage to find Hamming weights.
A1=\text{matrix}(\{\{1,0,0,0,1,0,0,0,0,0,1,0,0,0,1\},
\{0,1,0,0,1,0,0,0,0,0,0,1,0,1,1,1\}, \{0,0,0,1,1,0,0,0,0,0,0,1,1,1,1,1\},
\{0,0,0,1,0,1,0,0,0,0,0,1,1,0,0,1\}, \{0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,1\},
\{0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0\}, \{0,0,0,0,0,0,1,0,0,0,0,1,0,0,1,0\},
\{0,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0\}, \{0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0\})
q=2, R = \text{ZZ}/q[t1,t2,t3,t4,t5,t6]
--Parity check matrix computed with Sage to find
--the Hamming weights of dual code
A2=\text{matrix}(\{\{1,1,0,0,0,0,0,0,0,0,0,1,1,1,1,1\},
\{0,1,0,0,0,0,0,0,0,0,0,1,1,0,0,0\}, \{0,0,0,1,0,0,0,0,0,0,0,1,1,1,1,1\},
\{0,0,0,1,0,0,0,0,0,0,0,1,1,0,0,0\}, \{0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0\},
\{0,0,0,0,0,0,1,0,0,0,0,1,0,0,1,0\}, \{0,0,0,0,0,0,0,1,0,0,0,1,0,1,0,0\},
\{0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,1\}, \{0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0\}))
\{0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1\})

--The following functions can be applied to A, A1, A2
--I=ideal(projectivePointsByIntersection(A,R)), M=coker gens gb I
--This function computes the edge biparticity of Petersen graph.
--using the incidence matrix over the rational numbers
genmd1=(d,r)->degree M-max apply(apply(apply(apply(toList (set(1,-1))^**(hilbertFunction(d,M))
-(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),r),ideal),x-> if #set flatten entries
mingens ideal(leadTerm gens x)==r and not quotient(I,x)==I
then degree(I+x) else 0)

--To compute the r-th Hamming weight of the dual code
--use genmd(1,r) of the previous procedure:
genmd(1,1),genmd(1,2),genmd(1,3),genmd(1,4),genmd(1,5)
--To compute the edge biparticity use genmd1(1,1)
init=ideal(leadTerm gens gb I),degree M
er=(x)-> if not quotient(init,x)==init then degree ideal(init,x) else 0
--This is the footprint function
fpr=(d,r)->degree M - max apply(apply(apply(subsets(flatten entries basis(d,M),r),toSequence),ideal),er)
--To find lower bounds for Hamming weights use the footprint:
fpr(1,1),fpr(1,2),fpr(1,3),fpr(1,4),fpr(1,5),fpr(1,6),fpr(1,7),fpr(1,8)

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