Deformations of Scalar-Flat Anti-Self-Dual metrics and Quotients of Enriques Surfaces

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Abstract

In this article, we prove that a quotient of a $K3$ surface by a free $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action does not admit any metric of positive scalar curvature. This shows that the scalar flat anti self-dual metrics (SF-ASD) on this manifold can not be obtained from a family of metrics for which the scalar curvature changes sign, contrary to the previously known constructions of this kind of metrics on manifolds of $b^+ = 0$

1 Introduction

One of the most interesting features of the space of anti-self-dual or self-dual (ASD/SD) metrics on a manifold is that the scalar curvature can change sign on a connected component. That means, one can possibly join two ASD metrics of scalar curvatures of opposite signs by a 1-parameter family of ASD metrics. Whereas, this is not the case, for example for the space of Einstein metrics. There, each connected component has a fixed sign for the scalar curvature.

As a consequence, contrary to the Einstein case, most of the examples of SF-ASD metrics are constructed by first constructing a family of ASD metrics. Then showing that there are metrics of positive and negative scalar curvature in the family, and guaranteeing that there is a scalar-flat member in this family. In the $b^+ = 0$ case actually this is the only way known to construct SF-ASD metrics on a 4-manifolds. This paper presents an example of a SF-ASD Riemannian 4-manifold which is impossible to obtain by this kind of techniques since it does not have a positive scalar curvature deformation.

§2 reviews the known examples of ASD metrics constructed by a deformation changing the sign of the scalar curvature, §3 introduces an action on the $K3$ surface and furnish the quotient manifold with a SF-ASD metric, §4 shows that the smooth manifold defined in §3 does not admit any positive scalar curvature (PSC) or PSC-ASD metric, finally §5 includes some related examples and remarks.

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2 Constructions of SF-ASD metrics

Here we review some of the constructions for SF-ASD metrics on 4-manifolds. We begin with

**Theorem 2.1** (LeBrun [LeOM]). For all integers \( k \geq 6 \), the manifold
\[
k\mathbb{CP}^2 = \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2
\]

admits a 1-parameter family of real analytic ASD conformal metrics \([g_t]\) for \( t \in [0,1] \) such that \([g_0]\) contains a metric of \( s > 0 \) on the other hand \([g_1]\) contains a metric of \( s < 0 \).\]

**Corollary 2.2** (LeBrun [LeOM]). For all integers \( k \geq 6 \), the connected sum \( k\mathbb{CP}^2 \) admits scalar-flat anti-self-dual (SF-ASD) metrics.

**Proof.** Let \( h_t \in [g_t] \) be a smooth family of metrics representing the smooth family of conformal classes \([g_t]\) constructed in Lebrun [LeOM].

We know that the smallest eigenvalue \( \lambda_t \) of the Yamabe Laplacian \((\Delta + s/6)\) of the metric \( h_t \) exists, and is a continuous function of \( t \). Which measures the sign of the conformally equivalent constant scalar curvature metric [LP].

But the theorem (2.1) tells us that \( \lambda_0 \) and \( \lambda_1 \) has opposite signs. Then there is some \( c \in [0,1] \) for which \( \lambda_c = 0 \). Let \( u \) be the eigenfunction corresponding to the eigenvalue 0, for the Yamabe Laplacian of \( h_c \). Rescale it by a constant so that it has unit integral.

Since \( u \) is a continuous function on the compact manifold, it has a minimum say at \( m \). Choose the normal coordinates around there, so that \( \Delta u(m) = -\sum_{k=1}^4 \partial^2 u(m) \). Second partial derivatives are greater than or equal to zero, \( \Delta u(m) \leq 0 \) so \( u(m) = -\frac{2}{3} \Delta u(m) \geq 0 \). Assume \( u(m) = 0 \). Then the maximum of \( -u \) is attained and it is nonnegative with \((\Delta - s/6)(-u) = 0 \geq 0 \). So the strong maximum principle is applicable and \( -u \equiv 0 \). Which is not an eigenfunction. So \( u \) is a positive function. For a conformally equivalent metric, the change in the scalar curvature is
\[
\tilde{s} = 6u^{-3}(\Delta + s/6)u
\]

Thus \( g = u^2 h_c \) is a scalar-flat anti-self-dual metric on \( k\mathbb{CP}^2 \) for any \( k \geq 6 \).\]

Another construction tells us

**Theorem 2.3** ([Kim]). There exist a continuous family of self-dual metrics on a connected component of the moduli space of self-dual metrics on
\[
l(S^3 \times S^1)\# m\mathbb{CP}^2 \quad \text{for any } m \geq 1 \text{ and for some } l \geq 2
\]

which changes the sign of the scalar curvature

\footnote{Quite recently, LeBrun and Maskit announced that they have extended this result to the case \( k = 5 \) with similar techniques, which is the minimal number for these type of connected sums according to [LeSD].}
3 SF-ASD metric on the quotient of Enriques Surface

In this section we are going to describe what we mean by \(K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2\), and the scalar-flat anti-self-dual(SF-ASD) metric on it.

Let \(A\) and \(B\) be real \(3 \times 3\) matrices. For \(x, y \in \mathbb{C}^3\), consider the algebraic variety \(V_{2,2,2} \subset \mathbb{CP}_5\) given by the equations

\[
\sum_j A^i_j x^2_j + B^i_j y^2_j = 0, \quad i = 1, 2, 3
\]

more precisely

\[
\begin{align*}
A^1_1 x^2_1 + A^2_1 x^2_2 + A^3_1 x^2_3 + B^1_1 x^2_1 + B^2_1 x^2_2 + B^3_1 x^2_3 &= 0 \\
A^1_2 x^2_1 + A^2_2 x^2_2 + A^3_2 x^2_3 + B^1_2 x^2_1 + B^2_2 x^2_2 + B^3_2 x^2_3 &= 0 \\
A^1_3 x^2_1 + A^2_3 x^2_2 + A^3_3 x^2_3 + B^1_3 x^2_1 + B^2_3 x^2_2 + B^3_3 x^2_3 &= 0
\end{align*}
\]

For generic \(A\) and \(B\), this is a complete intersection of three nonsingular quadric hypersurfaces. By the Lefschetz hyperplane theorem, it is simply connected, and

\[
K_{V_2} = K_{\mathbb{P}^5} \otimes [V^p_2] = \mathcal{O}(-6) \otimes \mathcal{O}(1)^{\otimes 2} = \mathcal{O}(-4)
\]

since \([V_2]_h = 2[H]_h\) and taking Poincare duals, similarly

\[
\begin{align*}
K_{V_{2,2}} &= K_{V_2} \otimes [V^p_{2,2}] = \mathcal{O}(-4) \otimes \mathcal{O}(2) = \mathcal{O}(-2) \\
K_{V_{2,2,2}} &= K_{V_{2,2}} \otimes [V^p_{2,2,2}] = \mathcal{O}(-2) \otimes \mathcal{O}(2) = \mathcal{O}
\end{align*}
\]

finally. So the canonical bundle is trivial. \(V\) is a \(K3\) Surface. We define the commuting involutions \(\sigma^\pm\) by

\[\sigma^+(x, y) = (x, -y) \quad \text{and} \quad \sigma^-(x, y) = (\bar{x}, \bar{y})\]

and since we arranged \(A\) and \(B\) to be real, \(\sigma^\pm\) both act on \(V\).

At a fixed point of \(\sigma^+\) on \(V\), we have \(y_j = -y_j = 0\), so \(\sum_j A^i_j x^2_j = 0\) implying \(\sum_j B^i_j x^2_j = 0\), too. So if we take \(A\) and \(B\) to be invertible, these conditions are only satisfied for \(x_j = y_j = 0\) which does not correspond to a point, so \(\sigma^+\) is free and holomorphic. At a fixed point of \(\sigma^-\) on \(V\), \(x_j\)’s and \(y_j\)’s are all real. If \(A^i_j, B^i_j > 0\) for all \(j\) then \(\sum_j A^i_j x^2_j + B^i_j y^2_j = 0\) forces \(x_j = y_j = 0\) making \(\sigma^-\) free. At a fixed point \(\sigma^- \sigma^+\) on \(V\), \(x_j = \bar{x}_j\) and \(y_j = -\bar{y}_j\), so \(x_j\)’s are real and \(y_j\)’s are purely imaginary. Then \(y_j^2\) is a negative real number. So if we choose \(A^i_j > 0\) and \(B^i_j < 0\), this forces \(x_j = y_j = 0\), again we obtain a free action for \(\sigma^- \sigma^+\). Thus choosing \(A\) and \(B\) within these circumstances \(\sigma^\pm\) generate a free \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) action and we define \(K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2\) to be the quotient of \(K3\) by this free action. We have

\[
\chi = \sum_{k=0}^4 (-1)^k b_k = 2 - 2b_1 + b_2 = 2 + (2b^+ - \tau) \quad \text{hence} \quad b^+ = (\chi + \tau - 2)/2
\]

so, \(b^+(K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2) = (24/4 - 16/4 - 2)/2 = 0\), a special feature of this manifold.

Next we are going to furnish this quotient manifold with a Riemannian metric. For that purpose, there is a crucial observation [HitLin] that, for any free involution on \(K3\), there exists a complex structure on \(K3\) making this involution holomorphic, so the quotient is a complex manifold. We begin by stating the
Theorem 3.1 (Calabi-Yau[Ca, Yau, GHJ, Joyce]). Let $(M, \omega)$ be a compact Kähler n-manifold. Given a $(1,1)$-form $\rho$ belonging to the class $2\pi c_1(M)$ so that it is closed. Then, there exists a unique Kähler metric with form $\omega'$ which is in the same class as in $\omega$, whose Ricci form is $\rho$.

Intuitively, you can slide the Kähler form $\omega$ in its cohomology class and obtain any desired reasonable Ricci form $\rho$.

Since $c_1(K3) = 0$ in our case, taking $\rho \equiv 0$ gives us a Ricci-Flat (RF) metric on the K3 surface, the Calabi-Yau metric. This metric is hyperkahler since, the holonomy group of Kähler manifolds are a subgroup of $U_2$, but Ricci-Flatness causes a reduction in the holonomy for harmonic forms are parallel because of the Weitzenböck Formula for spin manifolds (4.3). Scalar flatness and non triviality of $b^+$ is to be checked. $b_1(K3) = 0$ implying $b^+(K3) = (24 - 16 - 2)/2 = 3$, which is nonzero. Actually $b^+$ is nontrivial for any Kähler surface since the Kähler form is harmonic & self-dual. So we have the reduction because there are some harmonic parallel forms, the holonomy group supposed to fix these forms causing a shrinking, the next possible option is $SU_2$ which is equal to $Sp_1$ in this dimension, hence the Calabi-Yau metric is hyperkähler.

So we have at least three almost complex structures $I, J, K$, parallel with respect to the Riemannian connection. By duality we regard these as three linearly independent self-dual 2-forms, parallelizing $\Lambda_+^2$. So any parallel $\Lambda_+^2$ form on $K3$ defines a complex structure after normalizing. In other words $aI + bJ + cK$ defines a complex structure for the constants satisfying $a^2 + b^2 + c^2 = 1$, i.e the normalized linear combination. On the other hand

$$b_1(K3/\mathbb{Z}_2) = b_1(K3) = 0, \quad b^+(K3/\mathbb{Z}_2) = (12 - 8 - 2)/2 = 1$$

Since the pullback of harmonic forms stay harmonic, the generating harmonic 1-form on $K3/\mathbb{Z}_2$ is coming from the universal cover, so is fixed by the $\mathbb{Z}_2$ action. It is also a parallel self-dual form so its normalization is then a complex structure left fixed by $\mathbb{Z}_2$. So the quotient is a complex surface with $b_1 = 0$ and $2c_1 = 0$ implying that it is an Enriques Surface.

Now consider another metric on $K3$ : the restriction of the Fubini-Study metric on $\mathbb{CP}^5$ obtained from the Kähler form

$$\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log \left| (x_1, x_2, x_3, y_1, y_2, y_3) \right|^2$$

We also denote the restriction metric by $g_{FS}$. It is clear that $\sigma^\pm$ leaves this form invariant, hence they are isometries of $g_{FS}$. Hence the Fubini-Study metric projects down to the metrics $g_{FS}^\pm$ on $K3/\mathbb{Z}_2^\pm$. Let $h^\pm$ be the Calabi-Yau metric (3.1) on $K3/\mathbb{Z}_2^\pm$ with Kähler form cohomologous to that of $g_{FS}^\pm$. To remedy the ambiguity in the negative side, keep in mind that, $\sigma^-$ fixes the metric and the form on $K3$, though the quotient is not a Kähler manifold initially since it is not a complex manifold, it is locally Kähler. We arrange the complex structure of $K3$ to provide a complex structure to the form, so the quotient
manifold is Kähler. Now we have two Kähler metrics on the quotient (for different complex structures) but we do not know much about their curvatures, but we want to make the curvature Ricci-Flat, so we use the Calabi-Yau argument. Since $c_1(K^3/\mathbb{Z}_2) = 0$ with real coefficients, we pass to the Calabi-Yau metric for $\rho = 0$. $\pi^\pm$ denoting the quotient maps, the pullback metrics $\pi^\pm h^\pm$ are both Ricci-Flat-Kähler (RFK) metrics on $K^3$ with Kähler forms cohomologous to that of $g_{FS}$. Their Ricci forms are both 0. By the uniqueness of the Yau metric we have $\pi^+ h^+ = \pi^- h^-$. Hence this is a Ricci-Flat Kähler metric on $K^3$ on which both $\sigma^\pm$ act isometrically. This metric therefore projects down to a Ricci-Flat metric on our manifold $K^3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. It is the scalar-flat, and being locally Kähler implies locally scalar-flat anti-self-dual, hence a SF-ASD metric.

4 Weitzenböck Formulas

Now we are going to show that the smooth manifold $K^3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any positive scalar curvature metric. For that purpose we state the Weitzenböck Formula for the Dirac Operator on spin manifolds. Before that we introduce some notation together with some ingredients of the formula.

The Levi-Civita connection is going to be the linear map we denote by $\nabla : \Gamma(E) \to \Gamma(\text{Hom}(TM, E))$ for any vector bundle $E$ over a Riemannian Manifold $M$. Then we get the adjoint $\nabla^* : \Gamma(\text{Hom}(TM, E)) \to \Gamma(E)$ defined implicitly by

$$\int_M \langle \nabla^* S, s \rangle dV = \int_M \langle S, \nabla s \rangle dV$$

and we define the connection Laplacian of a section $s \in \Gamma(E)$ by their composition $\nabla^* \nabla s$. Notice that the harmonic sections are parallel for this operator. Using the metric, we can express its action as :

**Proposition 4.1** ([Pet] p179). Let $(M, g)$ be an oriented Riemannian manifold, $E \to M$ is a vector bundle with an inner product and compatible connection. Then

$$\nabla^* \nabla s = -tr \nabla^2 s$$

for all compactly supported sections of $E$

**Proof.** First we need to mention the second covariant derivatives and then the integral of the divergence.

We set

$$\nabla^2 K(X, Y) = (\nabla \nabla K)(X, Y) = (\nabla_X \nabla K)(Y)$$

Then using the fact that $\nabla_X$ is a derivation commuting with every contraction: [KN1] p124

$$\nabla_X \nabla_Y K = \nabla_X C(Y \otimes \nabla K) = C(\nabla_X Y \otimes \nabla K) = C(\nabla_X Y \otimes \nabla K + Y \otimes \nabla_X \nabla K) = \nabla_{\nabla_X Y} K + (\nabla_X \nabla K)(Y) = \nabla_{\nabla_X Y} K + \nabla^2 K(X, Y)$$

hence $\nabla^2 K(X, Y) = \nabla_X \nabla_Y K - \nabla_{\nabla_X Y} K$ for any tensor $K$. That is how the second covariant derivative is defined. Higher covariant derivatives are defined inductively.

For the divergence, remember that

$$(\text{div} X) dV_g = \mathcal{L}_X dV_g$$
which is taken as a definition sometimes [KNM] p.281. Combining this with the Cartan’s Formula: \( \mathcal{L}_XdV = di_XdV + i_X(dV) = di_XdV \). Then the Stokes’ Theorem yields that 
\[
\int_M (\text{div} X) dV = \int_M \mathcal{L}_X dV = \int_M d(i_XdV) = \int_{\partial M} i_X dV = 0
\]
for a compact manifold without boundary. This is actually valid even for a noncompact manifold together with a compactly supported vector field.

Now take an open set on \( M \) with an orthonormal basis \( \{E_i\}_{i=1}^n \). Let \( s_1 \) and \( s_2 \) be two sections of \( E \) compactly supported on the open set. We reduce the left-hand side by multiplying by \( s_2 \) as follows:

\[
(\nabla^* \nabla s_1, s_2)_{L^2} = \int_M (\nabla^* \nabla s_1, s_2) dV = \int_M (\nabla s_1, \nabla s_2) dV = \int_M \text{tr}((\nabla s_1)^* \nabla s_2) dV
\]

\[
= \sum_{i=1}^n \int_M ((\nabla s_1)^* \nabla s_2(E_i), E_i) dV = \sum \int_M ((\nabla s_1)^* \nabla E_i, E_i) dV
\]

\[
= \sum \int_M (\nabla E_i, s_2, \nabla s_1(E_i)) dV = \sum \int_M (\nabla E_i, s_1, \nabla s_2) dV
\]

Define a vector field \( X \) by \( g(X,Y) = (\nabla_Y s_1, s_2) \). Divergence of this vector field is

\[
\text{div} X = -d^* (X^0) = \text{tr} \nabla X = \sum_{i=1}^n (\nabla E_i, X, E_i) = \sum (E_i (X, E_i) - (X, \nabla E_i, E_i))
\]

\[
= \sum (E_i (\nabla E_i, s_1, s_2) - (\nabla \nabla E_i, E_i, s_1, s_2))
\]

We know that its integral is zero, so our expression continues to evolve as

\[
\sum \int_M (\nabla E_i, s_1, \nabla E_i, s_2) dV - \int_M (\text{div} X) dV
\]

\[
= \sum \int_M (\nabla E_i, s_1, \nabla E_i, s_2) dV - \int_M (\nabla E_i, s_1, s_2, \nabla E_i, s_2) dV
\]

\[
= \sum \int_M (- (\nabla E_i, \nabla E_i, s_1, s_2) + (\nabla \nabla E_i, E_i, s_1, s_2)) dV
\]

\[
= \sum \int_M (-\nabla^2 s_1(E_i, E_i), s_2) dV = - \int_M (\sum \nabla^2 s_1(E_i), E_i, s_2) dV
\]

\[
= \int_M (-\text{tr} \nabla^2 s_1, s_2) dV = (-\text{tr} \nabla^2 s_1, s_2)_{L^2}
\]

So we established that \( \nabla^* \nabla s_1 = -\text{tr} \nabla^2 s_1 \) for compactly supported sections in an open set.

\[\square\]

**Theorem 4.2** (Atiyah-Singer Index Theorem [LM] p.256, [MoSW] p.47). Let \( M \) be a compact spin manifold of dimension \( n = 2m \). Then, the index of the Dirac operator is given by

\[
\text{ind}(\not D^+) = \hat{A}(M) = \hat{A}(M)[M]
\]

More generally, if \( E \) is any complex vector bundle over \( M \), the index of \( \not D^+_E : \Gamma(S_+ \otimes E) \to \Gamma(S_+ \otimes E) \) is given by

\[
\text{ind}(\not D^+_E) = \{\text{ch}(E) \cdot \hat{A}(M)\}[M]
\]

For \( n = 4 \), \( \hat{A}(M) = 1 - p_1/24 \) and the first formula reduces to

\[
\text{ind}(\not D^+) = \hat{A}(M) = \int_M \frac{p_1(M)}{24} = -\frac{\tau(M)}{8}
\]

by the Hirzebruch signature Theorem.
Let us explain the ingredients beginning by the cohomology class $\hat{A}(M)$. Consider the power series of the following function:

$$\frac{t/2}{\sinh t/2} = \frac{t}{e^{t/2} - e^{-t/2}} = 1 + A_2 t^2 + A_4 t^4 + \ldots$$

where we compute the coefficients as

$$A_2 = -\frac{1}{24}, \quad A_4 = \frac{7}{10 \cdot 24 \cdot 24} = \frac{7}{5760}$$

Consider the Pontrjagin classes $p_1, \ldots, p_k$ of $M^{4k}$. Represent these as the elementary symmetric functions in the squares of the formal variables $x_1^2 \cdots x_k^2$:

$$x_1^2 + \cdots + x_k^2 = p_1, \quad \cdots, \quad x_1^2 x_2^2 \cdots x_k^2 = p_k$$

Then $\prod_{i=1}^k \frac{x_i/2}{\sinh x_i/2}$ is a symmetric power series in the variables $x_1^2 \cdots x_k^2$, hence defines a polynomial in the Pontrjagin classes. We call this polynomial as $\hat{A}(M)$

$$\hat{A}(M) = \prod_{i=1}^k \frac{x_i/2}{\sinh x_i/2}$$

In lower dimensions we have

$$\hat{A}(M^4) = 1 - \frac{1}{24} p_1, \quad \hat{A}(M^8) = 1 - \frac{1}{24} p_1 + \frac{7}{5760} p_1^2 - \frac{1}{1740} p_2$$

If the manifold has dimension $n = 4k + 2$, again it has $k$ Pontrjagin classes, and we define the polynomial $\hat{A}(M^{4k+2})$ by the same formulas.

Secondly, we know that $\mathcal{D}^+ : \Gamma(S_+) \to \Gamma(S_-)$ is an elliptic operator, so its kernel is finite dimensional and its image is a closed subspace of finite codimension. The index of an elliptic operator is defined to be $\dim \ker \mathcal{D}^+ - \dim \ker \mathcal{D}^-$. Actually in our case $\mathcal{D}^+$ and $\mathcal{D}^-$ are formal adjoints of each other: $(\mathcal{D} \psi, \eta)_{L^2} = (\psi, \mathcal{D} \eta)_{L^2}$ for $\psi, \eta$ compactly supported sections. Consequently the index becomes $\dim \ker \mathcal{D}^+ - \dim \ker \mathcal{D}^-$. This index is computed from the symbol in the following way. Consider the pullback of $S_\pm$ to the cotangent bundle $T^*M$. The symbol induces a bundle isomorphism between these bundles over the complement of the zero section of $T^*M$. In this way the symbol provides an element in the relative $K$-theory of $(T^*M, T^*M - M)$. The Atiyah-Singer Index Theorem computes the index from this element in the relative $K$-theory. In the case of the Dirac operator the index is $\hat{A}(M)$, the so-called $A$-hat genus of $M$.

Now we state our main tool

**Theorem 4.3** (Weitzenböck Formula). On a spin Riemannian manifold, consider the Dirac operator $\mathcal{D} : \Gamma(S_\pm) \to \Gamma(S_\mp)$. The Dirac Laplacian might be expressed in terms of the connection/rough Laplacian as

$$\mathcal{D}^2 = \nabla^* \nabla + \frac{s}{4}$$

where $\nabla$ is the Riemannian connection.
Finally we state and prove our main result:

**Theorem 4.4.** The smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any metric of positive scalar curvature (PSC)

**Proof.** If $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ admits a metric of PSC then $K3$ is also going to admit such a metric because one pulls back the metric of the quotient, and obtain a locally identical metric on which the PSC survives.

So we are going to show that the $K3$ surface does not admit any metric of PSC. First of all the canonical bundle of $K3$ is trivial so that $c_1(K3) = 0 = w_2(K3)$ implying that it is a spin manifold.

By the Atiyah-Singer Index Theorem 4.2

$$\text{ind}\mathcal{D}^\pm = \hat{A}(M)[M] = -\frac{\tau(M)}{8} = 2$$

for the $K3$ Surface. Since it is equal to $\dim \ker - \dim \text{coker}$, this implies that the $\dim \ker \mathcal{D}^+ \geq 2$.

Let $\varphi \in \ker \mathcal{D}^+$. Then $\mathcal{D}^2 \varphi = 0$ since $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$. So by the spin Weitzenböck Formula 4.3

$$0 = \nabla^* \nabla \varphi + \frac{s}{4} \varphi$$

Taking the inner product with $\varphi$, so integrating over the manifold yields

$$0 = (\nabla^* \nabla \varphi, \varphi)_{L^2} + \left(\frac{s}{4} \varphi, \varphi\right)_{L^2} = (\nabla \varphi, \nabla \varphi)_{L^2} + \left(\frac{s}{4} \varphi, \varphi\right)_{L^2} = \int_M (|\nabla \varphi|^2 + \frac{s}{4} |\varphi|^2) dV$$

and $s > 0$ implies that $|\nabla \varphi| = |\varphi| = 0$ everywhere, hence $\varphi = 0$. So $\ker \mathcal{D}^+ = 0$, which is not the case.

Notice that $s \geq 0$ and $s(p) > 0$ for some point is also enough for the conclusion because then $\varphi$ would be parallel and zero at some point implies zero everywhere.

Alternatively, we could use the Weitzenböck Formula for the Hodge/modern Laplacian to show that there are no PSC anti-self-dual (ASD) metrics on $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This is a weaker conclusion though sufficient for our purposes.

**Theorem 4.5** (Weitzenböck Formula 2 [LeOM]). On a Riemannian manifold, the Hodge/modern Laplacian might be expressed in terms of the connection/rough Laplacian as

$$(d + d^*)^2 = \nabla^* \nabla - 2W + \frac{s}{3}$$

where $\nabla$ is the Riemannian connection and $W$ is the Weyl curvature tensor.

**Theorem 4.6.** The smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any anti-self-dual (ASD) metric of positive scalar curvature (PSC)

**Proof.** Again we are going to show this only for $K3$ as in 4.4. Anti-self-duality reduces our Weitzenbock Formula 4.5 to the form

$$(d + d^*)^2 = \nabla^* \nabla - 2W_- + \frac{s}{3}$$
because $W = W_-$ or $W_+ = 0$.

We have already explained that $b_2^+$ of the K3 surface is nonzero. So take a harmonic self-dual 2-form $\varphi$. $W_+ \Gamma(\lambda^-) \to \Gamma(\lambda^-)$ only acts on anti-self-dual forms, so it takes $\varphi$ to zero. Applying the formula

$$0 = \nabla^* \nabla \varphi + \frac{s}{3} \varphi$$

taking the inner product with $\varphi$, so integrating over the manifold yields similarly

$$0 = (\nabla^* \nabla \varphi, \varphi)_{L^2} + \frac{s}{3} (\varphi, \varphi)_{L^2} = (\nabla \varphi, \nabla \varphi)_{L^2} + \frac{s}{3} (\varphi, \varphi)_{L^2} = \int_M (|\nabla \varphi|^2 + \frac{s}{3} |\varphi|^2) dV$$

and $s > 0$ implies that $|\nabla \varphi| = |\varphi| = 0$ everywhere, hence $\varphi = 0$. So $\ker(d + d^*) = 0$, which is not the case since the space spanned by the harmonic representatives are already contained.

## 5 Other Examples

In this section, we will go through some examples. We begin with the case $b^+ = 1$.

**Theorem 5.1 ([KimLePon], [RS-SFK]).** For all integers $k \geq 10$, the connected sum $\mathbb{C}P^2 \# k \mathbb{C}P^2$ admits scalar-flat-Kähler(SFK) metric.

The case $k \geq 14$ is achieved in [KimLePon]. They start with blow ups of $\mathbb{C}P_1 \times \Sigma$ the cartesian product of rational curve and genus-2 curve, which already have a SFK metric via the hyperbolic ansatz of [LeExp]. After applying an isometric involution, they got a SFK orbifold, which has isolated singularities modelled on $\mathbb{C}^2/\mathbb{Z}_2$. Replacing these singular models with smooth ones, they obtain the desired metric.

For the case $k = 10$, Rollin and Singer first construct a related SFK orbifold with isolated and cyclic singularities of which the algebra $a_0$ of non-parallel holomorphic vector fields is zero. This is done by an argument analogous to that of [Burns-Bart]. The target manifold is the minimal resolution of this orbifold. To obtain the target metric, they glue some suitable local models of SFK metrics to the orbifold. These models are asymptotically locally Euclidean(ALE) scalar flat Kähler metrics constructed in [Cal-Sing].

When a metric is Kähler, from the decomposition of the Riemann Curvature operator, scalar-flatness turns out to be equivalent to being anti-self-dual. So these metrics are SF-ASD.

Since these manifolds have $b^+ \neq 0$ Weitzenböck Formulas apply as in section §4, so automatically the scalar curvature can not change sign. These examples show why the case $b^+ = 0$ we focussed on, is interesting.

A second type of example is

**Example 5.2.** Let $\Sigma_g$ be the genus-$g$ surface with Kähler metric of constant curvature $\kappa = -1$, and $S^2$ be the 2-sphere with the round $\kappa = +1$ metric. Consider the product metric on $S^2 \times \Sigma_g$ which is Kähler with zero scalar curvature. So it is anti-self-dual. Then we have fixed point free, orientation reversing, isometric involutions of both surfaces obtained

It is a curious fact that $k = 10$ is the minimal number for these type of metrics(SF-ASD) on $\mathbb{C}P^2 \# k \mathbb{C}P^2$, known by [LeSD] long before these constructions made. See [LeOM] for a survey.

Thanks to the referee for pointing out this example and the remark.
by antipodal maps. Combination of these involutions yield an isometric involution on the product and the metric falls down to a metric on \((S^2 \times \Sigma_g)/\mathbb{Z}_2\) which is SF-ASD as these properties are preserved under isometry. This is an example with all the key properties where the metric is completely explicit. Note that this manifold is non-orientable.

**Remark 5.3.** The other side of the story discussed here is that we have ASD, conformally flat deformations to negative scalar curvature metrics. It is obtained by deforming the

\[
\rho : \pi_1(M) \rightarrow SL(2, \mathbb{H})
\]

the representation of the fundamental group in \(SL(2, \mathbb{H})\).

Also, by further investigation, it is possible to get examples which are doubly covered by e.g. the simply connected examples of [KimLePon].
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