A Short Proof of Euler–Poincaré Formula

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Abstract

"V − E + F = 2", the famous Euler’s polyhedral formula, has a natural generalization to convex polytopes in every finite dimension, also known as the Euler–Poincaré Formula. We provide another short inductive proof of the general formula. Our proof is self-contained and it does not use shellability of polytopes.

Keywords: Euler–Poincaré formula, polytopes, discharging

1 Introduction

In this paper we follow the standard terminology of polytopes theory, such as Ziegler [7]. We consider convex polytopes, defined as a convex hull of finitely many points, in the d-dimensional Euclidean space for an arbitrary $d \in \mathbb{N}$, $d \geq 1$. We shortly say a polytope to mean a convex polytope. A landmark discovery in the history of combinatorial investigation of polytopes was famous Euler’s formula, stating that for any 3-dimensional polytope with $v$ vertices, $e$ edges and $f$ faces, $v - e + f = 2$ holds. This finding was later generalized, in every dimension $d$, to what is nowadays known as (generalized) Euler’s relation or Euler–Poincaré formula, as follows.

For instance, in dimension $d = 1$ we have $v = 2$, which can be rewritten as $v - 1 = 1$, and in dimension $d = 2$ we have got $v - e = 0$ or $v - e + 1 = 1$. Similarly, the $d = 3$ case can be rewritten as $v - e + f - 1 = 1$. Note that the ‘1’ left of ‘=’ stands in these expressions for the polytope itself. In general, the following holds:

Theorem 1 (“Euler–Poincaré formula”; Schläfli [5] 1852). Let $P$ be a convex polytope in $\mathbb{R}^d$, and denote by $f^c$, $c \in \{0, 1, \ldots, d\}$, the numbers of faces of $P$ of dimension $c$. Then

$$f^0 - f^1 + f^2 - \cdots + (-1)^d f^d = 1. \quad (1)$$

We refer to classical textbooks of Grünbaum [3] and Ziegler [7] for a closer discussion of the interesting history of this formula and of the difficulties associated with its proof. Here we just briefly remark that all the historical attempts to prove the formula in a combinatorial way, starting from Schläfli, implicitly assumed validity of a special property called shellability of a polytope, however, the shellability of any polytope was formally established only in 1971 by Bruggesser and Mani [1].

We provide a new self-contained inductive proof of (1) which does not assume shellability. Our proof has been in parts inspired by a proof of 3-dimensional Euler’s formula via angles [2] “Proof 8: Sum of Angles”, and by Welzl’s probabilistic proof [6] of Gram’s equation – although, the resulting exposition of the proof does not resemble either of those. In fact, we give two different expositions of the new proof, one in Section 2 and the other in Section 3.
2 New Self-contained Proof

Our new proof of Theorem 1 proceeds by induction on the dimension \( d \geq 1 \). Note that validity of (1) is trivial for \( d = 1, 2 \), and hence it is enough to show the following:

**Lemma 2.** Let \( k \geq 2 \) and \( P \) be a polytope of dimension \( k + 1 \). Assume that (1) holds for any polytope of dimension \( d \in \{k - 1, k\} \). Then (1) holds for \( P \) (with \( d = k + 1 \)).

**Proof.** Recall that \( f_c, c \in \{0, 1, \ldots, k + 1\} \), denote the numbers of faces of \( P \) of dimension \( c \). Our goal is to prove

\[
f^0 - f^1 + f^2 - \ldots + (-1)^{k-1} f^{k-1} + (-1)^k f^k + (-1)^{k+1} f^{k+1} = 1,
\]

or equivalently, since \( f^{k+1} = 1 \),

\[
f^0 - f^1 + f^2 - \ldots + (-1)^{k-1} f^{k-1} = 1 + (-1)^k (1 - f^k).
\]

Let \( \mathcal{R} \) denote a \( k \)-dimensional polyhedral complex which is the Schlegel diagram of \( P \) with respect to its facet \( T \) (i.e., \( \mathcal{R} \) results from \( P \) by projecting it to \( T \) from a point outside of \( P \) but sufficiently close to some interior point of \( T \); see Figure 1). Let the \((k\text{-dimensional})\) facets of \( \mathcal{R} \), which are the projections of the facets of \( P \) other than \( T \), be denoted by \( R_1, \ldots, R_a \) where \( a := f^k - 1 \), in any order, and let \( R_0 := T \). Clearly, for each \( c \in \{0, 1, \ldots, k - 1\} \), the number of faces of dimension \( c \) in the complex \( \mathcal{R} \) is exactly \( f^c \).

The high level idea of our proof is to double-count certain objects (signed flags) distributed to all the faces of \( \mathcal{R} \) of dimensions from 0 to \( k - 1 \).

We choose any straight line \( q \) which is in *general position* with respect to \( \mathcal{R} \). In particular, \( q \) is not parallel to any face of \( \mathcal{R} \). For every face \( F \) of \( \mathcal{R} \) of dimension \( 0 \leq c \leq k - 1 \), we select a point \( x_F \) in the relative interior of \( F \), and we add from \( x_F \) two “flags” in the opposite directions of line \( q \). Formally, these *flags* are the two line segments \( s_1, s_2 \) with the common end \( x_F, s_1 \cap s_2 = \{x_F\} \), which are parallel to \( q \) and of length \( \varepsilon > 0 \). We call \( x_F \) the *base point* and \( F \) the *base face* of the flags \( s_1, s_2 \). The length \( \varepsilon \) is chosen to be smaller than the minimum distance between two disjoint faces of \( \mathcal{R} \). Each flag whose base face is of dimension \( c \) gets the value of \( \frac{1}{2}(-1)^c \).

It is now clear from the definition that the total value of all flags distributed in \( \mathcal{R} \) equals the left-hand side of (2). Another easy observation is that, since \( q \) has been chosen in general
direction, every flag \( s \) is either contained in a unique facet of \( R \), or \( s \) except its base point is disjoint from the point set of \( R \) (in our case, actually, \( \text{set}(R) = R_0 \)). Furthermore, no two flags sharing the same base point are contained in the same facet of \( R \), by convexity.

Our next task is to sum the total value of the flags in each one facet \( R_i \) of \( R \) (see (1)), and also of the flags that are outside of \( R_0 \) (see (5)). Consider a facet \( R_i \) of \( R \), \( i \in \{1, \ldots, a\} \) and a face \( F \) of \( R_i \) of dimension \( c \). The following observation follows readily from the assumption of a general position of \( q \) and from convexity:

\[ 3 \text{ If } c = k - 1, \text{ then always one of the two flags with the base face } F \text{ belongs to } R_i. \text{ If } 0 \leq c \leq k - 2, \text{ then one of the two flags with the base } F \text{ belongs to } R_i \text{ if, and only if, the image of } F \text{ is not a face in the orthogonal projection of } R_i \text{ along } q. \]

We let \( S_i, i \in \{0, 1, \ldots, a\}, \) denote the \((k - 1)\)-dimensional image of \( R_i \) in the orthogonal projection along \( q \). Furthermore, let \( f^c_i, \) where \( c \in \{0, 1, \ldots, k\} \) and \( i \in \{0, 1, \ldots, a\}, \) denote the number of faces of \( R_i \) of dimension \( c \), and \( g^c_i, c \in \{0, 1, \ldots, k - 1\}, \) denote the number of faces of \( S_i \) of dimension \( c \). Now by (3), the number of flags with a base face of dimension \( c \) which are contained in \( R_i \), is the following: \( f^c_i \) if \( c = k - 1 \), and \( f^c_i - g^c_i \) if \( 0 \leq c \leq k - 2 \). Hence the overall value of all the flags contained in \( R_i \) is

\[
\frac{1}{2} (-1)^{k-1} f^k_{i-1} + \frac{1}{2} \sum_{c=0}^{k-2} (-1)^c (f^c_i - g^c_i) = \frac{1}{2} \sum_{c=0}^{k-1} (-1)^c f^c_i - \frac{1}{2} \sum_{c=0}^{k-2} (-1)^c g^c_i 
\]

(4)

where the latter (4) holds true by applying (1) to the polytopes \( R_i \) and \( S_i \) of dimensions \( d = k \) and \( k - 1 \), respectively, and by obvious \( f^k_i = g^{k-1}_i = 1 \).

The overall value of all the remaining flags, which are not in \( R_0 \), is similarly equal to

\[
\frac{1}{2} \sum_{c=0}^{k-1} (-1)^c f^c_0 + \frac{1}{2} \sum_{c=0}^{k-2} (-1)^c g^c_0 = \frac{1}{2} \left( 1 - (-1)^k f^k_0 \right) + \frac{1}{2} \left( 1 - (-1)^{k-1} g^{k-1}_0 \right) = 1. \]

(5)

Since we have been counting twice the same set of flags, we get that (the left-hand side of) (2) must equal the sum of (4) over \( i = 1, \ldots, a \), and of (5), leading to

\[ f^0 - f^1 + f^2 - \cdots + (-1)^{k-1} f^{k-1} = (-1)^{k-1} a + 1 = (-1)^k (1 - f^k) + 1, \]

and thus finishing the proof of (2) for \( P \) in dimension \( k + 1 \). \( \square \)

3 Another Point of View

Interestingly, there is another and quite different exposition of the proof ideas from Section 2 which deals directly with the polytope \( P \) instead of its Schlegel diagram, and which uses only very elementary and apparent combinatorial and geometric arguments. This exposition has also been published (in full) in the proceedings [4].

Alternative proof of Lemma 2. Recall that \( P \) is a polytope in dimension \( k + 1 \), and \( f^c, c \in \{0, 1, \ldots, k + 1\} \), denote the numbers of faces of \( P \) of dimension \( c \). Again, we assume validity of (1) in dimensions \( d \in \{k - 1, k\} \), and our goal is to prove

\[
f^0 - f^1 + f^2 - \cdots + (-1)^k f^k + (-1)^{k+1} f^{k+1} = 1. \]

(6)
The face \( (7) \) with the choice of \( T \) method in combinatorics. To every face \( P \) assigned to all faces of \( P \)

indeed, if \( x \in N \) Then \( \sum q \) coplanar with \( q \) which the plane of \( t \) sends to \( i \) for \( i \). In both cases, every side \( T \) not sending to \( i \)

for \( i \geq 3 \), we have that the vertices \( b, c, d \) send charge of \( \frac{1}{2} \) to \( T_1 \) by the rule \((7)\), while \( a, e \) are not sending to \( T_1 \). On the right \( (t \in T_i, i = 1, 2) \), all the vertices \( a, b, c, d, e \) send charge of \( \frac{1}{2} \) to \( T_1 \). Consequently, on the left \( T_i \) ends up with charge 0 (compare to \((9)\)), while on the right with charge 1 (cf.\((8)\)).

Figure 2: Proof of Lemma 2: a facet in a 3-dimensional polytope \( P \) \((k = 2)\). Each vertex and facet of \( P \) initially get charge of 1 and each edge \(-1\). Consider, e.g., a facet \( T_i \) of \( P \) which is a pentagon with vertices \( a, b, c, d, e \) and sides (edges) \( A, B, C, D, E \) in order. Let \( t_i \) be the point in which the plane of \( T_i \) intersects the line \( q \) (see in the proof). On the left of the picture \( (t \notin T_i, \text{ for } i \geq 3) \), we have that the vertices \( b, c, d \) send charge of \( \frac{1}{2} \) to \( T_1 \) by the rule \((7)\), while \( a, e \) are not sending to \( T_1 \). On the right \( (t \in T_i, i = 1, 2) \), all the vertices \( a, b, c, d, e \) send charge of \( \frac{1}{2} \) to \( T_1 \). Consequently, on the left \( T_i \) ends up with charge 0 (compare to \((9)\)), while on the right with charge 1 (cf.\((8)\)).

We choose arbitrary two facets \( T_1, T_2 \) of \( P \) (distinct, but not necessarily disjoint) and two points \( t_1 \in T_1 \) and \( t_2 \in T_2 \) in their relative interior, such that the straight line \( q \) determined by \( t_1, t_2 \) is in a general position with respect to \( P \). In particular, we demand that no nontrivial line segment lying in a face of \( P \) of dimension \( \leq k - 1 \) is coplanar with \( q \). We also denote by \( T_3, \ldots, T_j \) the remaining facets of \( P \), in any order. As in Section 2 for every face \( F \) of \( P \) of dimension \( 0 \leq c \leq k - 1 \), we select a point \( x_F \) in the relative interior of \( F \) (note that \( x_v = v \) if \( v \) is a vertex of \( P \)).

In the proof we use a discharging argument, an advanced variant of the double-counting method in combinatorics. To every face \( F \) of \( P \) of dimension \( 0 \leq c \leq k + 1 \), we assign charge of value \((-1)^c \) (note that charge applies also to whole \( F = P \)). Hence the total change initially assigned to all faces of \( P \) equals the left-hand side of \((6)\).

Now we discharge all the assigned charge from faces of dimension \( c \leq k - 1 \) to the facets of \( P \). The discharging rule is only one and quite simple. Consider a facet \( T_i \) of \( P \), \( 1 \leq i \leq f^k \). Let \( t_i \in q \) denote the unique point which is the intersection of the line \( q \) with the support hyperplane of \( T_i \). This is a sound definition of \( t_i \) according to a general position of \( q \), and it is consistent with the choice of \( t_1, t_2 \) above. For any proper face \( F \) of \( T_i \) (so \( F \) is a face of \( P \) as well and is of dimension \( 0 \leq c \leq k - 1 \)), the discharging rule reads (see in Figure 2):

\[(7) \text{ The face } F \text{ sends half of its initial charge, i.e. } \frac{1}{2}(-1)^c, \text{ to the facet } T_i \text{ if, and only if, the straight line passing through } x_F \text{ and } t_i \text{ intersects the relative interior of } T_i.\]

Note that we will be finished if we prove that, after applying the discharging rule, (i) every face of \( P \) of dimension \( \leq k - 1 \) ends up with charge 0, and (ii) the total charge of the facets of \( P \) and of \( P \) itself sums up to the right-hand side of \((6)\).

For the task (i), consider any face \( F \) of \( P \) of dimension \( c \leq k - 1 \) and the point \( x_F \) chosen in \( F \) above. Let \( L \) denote the plane determined by the line \( q = t_1 t_2 \) and the point \( x_F \notin q \). Then \( N := P \cap L \) is a convex polygon. See Figure 3. We claim that \( x_F \) must be a vertex of \( N \): indeed, if \( x_F \) belonged to a relative interior of a side \( A_0 \) of \( N \), then \( A_0 \subseteq F \) and \( A_0 \) would be coplanar with \( q \), contradicting our assumption of a general position of \( q \). Consequently, \( x_F \) is
Furthermore, by convexity, a face $F$ holds: every proper face of dimension $k$ incident to $x_F$ determine the two unique facets $T_{i1}, T_{i2}$ of $P$ that $F$ sends charge to.

incident to two sides $A_1, A_2$ of $N$, and there exist facets $T_{i1}, T_{i2}$ of $P$, $1 \leq i_1 \neq i_2 \leq f^k$, such that $A_j = T_{i_j} \cap L$ for $j = 1, 2$. Observe that the support line of $A_j$ intersects $q$ precisely in $t_{i_j}$ (which has been defined as the intersection of the support hyperplane of $T_{i_j}$ with $q$).

Moreover, since $A_j$ is coplanar with $q$, by our assumption of a general position of $q$ it cannot happen that $A_j$ is contained in a face of dimension $\leq k - 1$. Consequently, $A_j$ (except its ends) belongs to the relative interior of $T_{i_j}$, and $T_{i_j}$ is a unique such face for $A_j$. Hence, taking this argument for $j = 1, 2$, we see that $F$ sends away by (7) exactly two halves of its initial charge, ending up with charge 0.

For the task (ii), let $f_i^c$, where $c \in \{0, 1, \ldots, k\}$ and $i \in \{1, \ldots, f^k\}$, denote the number of faces of $T_i$ of dimension $c$. We first look at the two special facets $T_i$, $i = 1, 2$ (Figure 2 right). Since $t_i \in T_i$ in this case, by (7) $T_i$ receives (half) charge from every of its proper faces. Finally, $T_1$ starts with charge $(-1)^k$ which we halve in the computation. Using (1) for $T_i$, which is of dimension $k$ and so $f_i^k = 1$, we thus get that the total charge $T_i$ ends up with, is

$$\frac{1}{2} \left( f_0^0 - f_1^1 + \cdots + (-1)^{k-1} f_1^k + (-1)^k f_i^k \right) + \frac{1}{2} (-1)^k = \frac{1}{2} + \frac{1}{2} (-1)^k. \quad (8)$$

Second, consider a facet $T_i$ where $i \geq 3$. Let $H_i$ be the support hyperplane of $T_i$. Then $\{t_i\} = H_i \cap q$ and $t_i \notin T_i$. We restrict ourselves to the affine space formed by $H_i$, and denote by $S_i$ a projection of $T_i$ from the point $t_i$ onto a suitable hyperplane within $H_i$. Since $t_i$ is in a general position with respect to $T_i$ (which is implied by a general position of $q$), the following holds: every proper face of $S_i$ is the image of an equivalent face of $T_i$ (of the same dimension). Furthermore, by convexity, a face $F$ of $T_i$ has no image among the faces of $S_i$ if, and only if, the line through $x_F$ and $t_i$ intersects the relative interior of $T_i$. See also Figure 2 left.

Consequently, as directed by (7), $T_i$ receives charge precisely from those of its faces $F$ which do not have an image among the proper faces of $S_i$. Denote by $g_i^c$ the number of faces of $S_i$ of dimension $c \leq k - 1$, and notice that $f_i^k = g_i^{k-1} + 1$. Hence, precisely, $T_i$ receives $\frac{1}{2} (-1)^{k-1}$ of charge from each of its $f_i^{k-1}$ faces of dimension $k - 1$, and $\frac{1}{2} (-1)^c$ from $f_i^c - g_i^c$ of its faces of dimension $0 \leq c \leq k - 2$. Recall also that $T_i$ itself starts with charge $(-1)^k$. Summing together,
and using (1) for $T_i$ (of dimension $k$) and for $S_i$ (of dimension $k - 1$), we get

$$\frac{1}{2}(-1)^{k-1}f_i^{k-1} + \frac{1}{2} \sum_{c=0}^{k-2} (-1)^c (f_i^c - g_i^c) + \frac{1}{2}(-1)^k + \frac{1}{2}(-1)^k$$

$$= \frac{1}{2} \sum_{c=0}^{k-1} (-1)^c f_i^c - \frac{1}{2} \sum_{c=0}^{k-2} (-1)^c g_i^c + \frac{1}{2}(-1)^k - \frac{1}{2}(-1)^{k-1}$$

$$= \frac{1}{2} \sum_{c=0}^{k} (-1)^c f_i^c - \frac{1}{2} \sum_{c=0}^{k-1} (-1)^c g_i^c = \frac{1}{2} - \frac{1}{2} = 0. \tag{9}$$

Since the total charge is not changed, we get that the left-hand side of (6) must equal the sum of the charge of $P$, of (8) over $i = 1, 2$ and of (9) over remaining facets, leading to

$$(-1)^{k+1} + \frac{1}{2} + \frac{1}{2}(-1)^k + \frac{1}{2} + \frac{1}{2}(-1)^k + 0 = 1,$$

and thus finishing the proof of (6) for $P$. \hfill \square

References

[1] Heinz Bruggesser and Peter Mani. Shellable decompositions of cells and spheres. *Math. Scand.*, 29:197–205, 1971.

[2] David Eppstein. The geometry junkyard: Twenty proofs of Euler’s formula. https://www.ics.uci.edu/~eppstein/junkyard/euler/, 2016.

[3] Branko Grünbaum. *Convex polytopes*. Graduate texts in mathematics. Springer, New York, Berlin, London, 2003.

[4] Petr Hliněný. A short proof of Euler–Poincaré formula. In Jaroslav Nešetřil, Guillem Perarnau, Juanjo Rüé, and Oriol Serra, editors, *Extended Abstracts EuroComb 2021*, pages 92–96, Cham, 2021. Springer International Publishing.

[5] Ludwig Schläfli. *Theorie der vielfachen Kontinuität* [1852]. (in German). First publication 1901 Graf, J. H., ed. Republished by Cornell University Library historical math monographs, Zürich, Basel: Georg & Co., 2010.

[6] Emo Welzl. Gram’s equation - A probabilistic proof. In *Results and Trends in Theoretical Computer Science*, volume 812 of *Lecture Notes in Computer Science*, pages 422–424. Springer, 1994.

[7] Günter M. Ziegler. *Lectures on polytopes*. Graduate texts in mathematics. Springer, New York, 1995.