Renewal dynamical approach for non-minimal quasi-stationary distributions of one-dimensional diffusions

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Abstract

We consider quasi-stationary distributions for one-dimensional diffusions via the renewal dynamical approach. We show that convergence of the iterative renewal transform to quasi-stationary distributions is equivalent to a condition on the moment growth rate of the lifetime, which is at the same time a necessary condition for the existence of Yaglom limits.

1 Introduction

Let $X$ be an irreducible one-dimensional diffusion on the half-line $(0, \infty)$ stopped at 0 which hits 0 with probability one: $\mathbb{P}_x[T_0 < \infty] = 1 \ (x > 0)$. Here $T_0$ denotes the first hitting time of 0 and $\mathbb{P}_x$ denotes the underlying probability measure of $X$ starting from $x$. A probability measure $\nu \ (0, \infty)$ is called a quasi-stationary distribution of $X$ when the following holds:

$$\mathbb{P}_\nu[X_t \in dx \mid T_0 > t] = \nu(dx) \quad (t > 0),$$

and $\nu$ is called a Yaglom limit of a probability measure $\mu$ on $(0, \infty)$ when

$$\mu_t(dx) := \mathbb{P}_\mu[X_t \in dx \mid T_0 > t] \underset{t \to \infty}{\longrightarrow} \nu(dx).$$

Here and hereafter all the convergence of probability distributions is in the sense of the weak convergence. For a probability measure $\mu$ on $(0, \infty)$, we introduce the renewal transform of $\mu$ by

$$\Phi_\mu(dx) := \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty \mathbb{P}_\mu[X_t \in dx, T_0 > t]dt,$$

which is the 0-potential measure of $X$ normalized to be a probability measure under the initial distribution $\mu$, and can be defined when $\mathbb{E}_\mu T_0 < \infty$. The distribution $\Phi_\mu$ can be interpreted as the limit distribution of the conservative stochastic process which behaves as $X$ until $T_0$ and as soon as it hits 0, it jumps into a random point in $(0, \infty)$ according to the probability $\mu$ and starts afresh (see e.g., Ben-Ari and Pinsky [1]). The
renewal transform has a close relation to quasi-stationary distributions since every quasi-stationary distribution \( \nu \) of \( X \) is a fixed point of the renewal transform. Indeed, by the Markov property, it holds

\[
\Phi \nu(dx) := \frac{1}{\mathbb{E}_\nu T_0} \int_0^\infty \mathbb{P}_\nu[X_t \in dx \mid T_0 > t] \mathbb{P}_\nu[T_0 > t] dt
\]

(1.4)

\[
\nu(dx) = \frac{\nu(dx)}{\mathbb{E}_\nu T_0} \int_0^\infty \mathbb{P}_\nu[T_0 > t] dt = \nu(dx).
\]

(1.5)

This fact leads us to study Yaglom limits through the iterative renewal transform.

For some diffusions, it is known that there exist infinitely many quasi-stationary distributions, and in the case, the set of quasi-stationary distributions is totally ordered by a usual stochastic order (see Proposition 2.5 for the precise description). The minimum element is called the \textit{minimal quasi-stationary distribution}.

Our main objective in the present paper is to investigate the convergence of the iterative renewal transform to non-minimal quasi-stationary distributions. The reason is that as we will see in Theorem 1.3, for a probability measures \( \mu \) and \( \nu \) the convergence of the iterative renewal transform

\[
\Phi^n \mu \xrightarrow{n \to \infty} \nu
\]

(1.6)

is a necessary condition for the convergence (1.2), and there have been many detailed studies on the convergence (1.2) to the minimal quasi-stationary distribution. Thus, we focus on the convergence to non-minimal quasi-stationary distributions. In contrast to the minimal case, there are few studies on the convergence (1.2) to non-minimal quasi-stationary distributions. We review the previous studies in more detail in Section 1.2.

1.1 Main results

To state our main results, we prepare some notation. For a subset \( S \) of \( \mathbb{R} \), we denote the set of probability measures on \( S \) by \( \mathcal{P}(S) \) or \( \mathcal{P}_S \). Let us fix an irreducible \( \frac{d}{dm} \frac{d}{ds} \)-diffusion \( X \) on \((0, \infty)\) stopped at 0 with \( \mathbb{P}_x[T_0 < \infty] = 1 \) for every \( x > 0 \). Define

\[
\mathcal{P}_\Phi := \{ \mu \in \mathcal{P}(0, \infty) \mid \mathbb{E}_\mu T_0^n < \infty \ (n \geq 1) \},
\]

(1.7)

where \( \mathbb{P}_\mu := \int \mathbb{P}_x \mu(dx) \). We will see in Proposition 3.1 that the renewal transform \( \Phi \) is well-defined as the map on \( \mathcal{P}_\Phi \): \( \Phi : \mathcal{P}_\Phi \to \mathcal{P}_\Phi \). We denote the normalized \( \alpha \)-th moment of \( T_0 \) by \( m_\alpha^\mu \) and a density function of \( \Phi^n \mu \) w.r.t. \( dm \) by \( f_n^\mu \):

\[
m_\alpha^\mu := \frac{1}{\alpha!} \mathbb{E}_\mu T_0^\alpha \ (\alpha \in [1, \infty)), \quad f_n^\mu(x) dm(x) = \Phi^n \mu(dx),
\]

(1.8)

where \( \alpha! = \Gamma(\alpha + 1) \) for the Gamma function \( \Gamma \). The existence of the density \( f_n^\mu \) will be proven in Proposition 3.2. Let \( \lambda_0 \geq 0 \) be the bottom of the \( L^2 \)-spectrum of the generator
\[-\frac{d}{dm} \frac{d}{ds}\] with Dirichlet boundary condition at 0 and Neumann boundary condition at \(\infty\) if the boundary \(\infty\) is regular (see Section 2.1 for the boundary classification).

One of our main results is to give a sufficient condition for the convergence (1.6). Note that a necessary and sufficient condition for existence of infinitely many quasi-stationary distributions and the positivity of \(\lambda_0\) will be given in Theorem 2.3.

**Theorem 1.1.** Let \(\mu \in \mathcal{P}_\Phi\). Assume that there exist infinitely many quasi-stationary distributions, and the following holds:

\[
\lim_{n \to \infty} \frac{m_{n-1}^\mu}{m_n^\mu} = \lambda \in (0, \lambda_0].
\]

Then we have

\[
\lim_{n \to \infty} f_n^\mu(x) = \lambda \psi_{-\lambda}(x) \quad (x > 0),
\]

\[
\Phi_n^\mu \to \nu_\lambda,
\]

where a function \(u = \psi_\lambda\) \((\lambda \in \mathbb{C})\) is the unique solution of the following equation:

\[
\frac{d}{dm} \frac{d^+}{ds} u(x) = \lambda u(x), \quad \lim_{x \to +0} u(x) = 0, \quad \lim_{x \to +0} \frac{d^+}{ds} u(x) = 1 \quad (x \in (0, \infty)),
\]

and the probability measure \(\nu_\lambda\) \((\lambda \in (0, \lambda_0])\) is a quasi-stationary distribution defined by

\[
\nu_\lambda(dx) := \lambda \psi_{-\lambda}(x) dm(x).
\]

**Remark 1.2.** The function \(\psi_\lambda\) always exists since the process \(X\) is irreducible and the boundary 0 is regular or exit.

The other main result is to give a hierarchical structure of necessary conditions for the convergence (1.6).

**Theorem 1.3.** Let \(\mu \in \mathcal{P}_\Phi\). Assume that there exist infinitely many quasi-stationary distributions. Let us consider the following conditions for \(\lambda \in (0, \lambda_0]\):

\((i)\) \(\mu_t \xrightarrow{t \to \infty} \nu_\lambda\).

\((ii)\) \(\lim_{t \to \infty} \mathbb{P}_\mu[T_0 > t + s]/\mathbb{P}_\mu[T_0 > t] = e^{-\lambda s} \quad (s > 0).\)

\((iii)\) \(\lim_{n \to \infty} m_{n-1}^\mu/m_n^\mu = \lambda.\)

\((iv)\) \(\Phi_n^\mu \xrightarrow{n \to \infty} \nu_\lambda.\)

Then the following implications hold: \((i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Leftrightarrow (iv).\)
1.2 Previous studies

The renewal dynamical approach was intensively investigated in Ferrari, Kesten, Martínez and Picco [6] to show the existence of the minimal quasi-stationary distribution for Markov chains on $\mathbb{N}$. Under mild assumptions, their results ensure for wide range of Markov chains existence of the minimal quasi-stationary distribution. The present paper is strongly motivated by their study though we need a different approach since their arguments focus on the minimal quasi-stationary distribution. We also mention Takeda [20] as a general study on existence of the minimal quasi-stationary distribution for symmetric Markov process with the tightness property.

Yaglom limits of one-dimensional diffusions have long been studied. Mandl [16] gave a first remarkable result on this subject under the assumption of existence of a natural boundary. He gave a sufficient condition for the convergence to the minimal quasi-stationary distribution. His result has been extended and strengthened by many authors e.g., Collet, Martínez and San Martín [4], Hening and Kolb [7], Kolb and Steinsaltz [12], Martínez and San Martín [17], Littin [14] and Cattiaux, Collet, Lambert, Martínez, Méléard and San Martín [3]. These results show that under mild assumptions, convergence to the minimal quasi-stationary distribution follows for all compactly supported initial distributions.

In contrast, there are few studies on Yaglom limits to non-minimal quasi-stationary distributions. In Martínez, Picco and San Martín [18], they have considered Brownian motion with negative constant drifts: $X_t = B_t - ct$ ($c > 0$), where $B$ is a standard Brownian motion, and they gave a set of initial distributions whose Yaglom limit is a non-minimal quasi-stationary distribution. In Lladser and San Martín, they studied Ornstein-Uhlenbeck processes: $dX_t = dB_t - cX_tdt$ ($c > 0$), and obtained the similar results. In [21], a general approach for Yaglom limits to non-minimal quasi-stationary distributions was studied. One of the main results in [21] was to reduce the convergence (1.2) to the tail behavior of $T_0$ through the first hitting uniqueness. For a set $\mathcal{P} \subset \mathcal{P}(0, \infty)$ of probability measures, we say the first hitting uniqueness holds on $\mathcal{P}$, when the following map is injective.

$$\mathcal{P} \ni \mu \mapsto \mathbb{P}_\mu[T_0 \in dt].$$  (1.14)

The following theorem gives the reduction under the first hitting uniqueness on

$$\mathcal{P}_{\exp} := \{\mu \in \mathcal{P}(0, \infty) \mid \mathbb{P}_\mu[T_0 \in dt] = \lambda e^{-\lambda t}dt \text{ for some } \lambda > 0\}.$$  (1.15)

Theorem 1.4 ([21, Theorem 1.1]). Assume the first hitting uniqueness holds on $\mathcal{P}_{\exp}$ and there exist $\lambda > 0$ and $\nu \in \mathcal{P}(0, \infty)$ such that $\mathbb{P}_\nu[T_0 \in dt] = \lambda e^{-\lambda t}dt$. Then for $\mu \in \mathcal{P}(0, \infty)$, the following are equivalent:

(i) $\lim_{t \to \infty} \mathbb{P}_\mu[T_0 > t + s]/\mathbb{P}_\mu[T_0 > t] = e^{-\lambda s}$ ($s > 0$).

(ii) $\mathbb{P}_\mu[T_0 \in ds] \xrightarrow{t \to \infty} \lambda e^{-\lambda s}ds$. 

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(iii) $\mu_t \xrightarrow{t \to \infty} \nu$.

Applying this theorem, in [21, Theorem 1.2], for Kummer diffusions with negative drifts, which are diffusions including the processes treated in [18] and [15], a set of initial distributions whose Yaglom limit is non-minimal quasi-stationary distributions were given.

Outline of the paper

The remainder of the present paper is organized as follows. In Section 2, we will recall several known results on one-dimensional diffusions and quasi-stationary distributions. In Section 3, we will prove Theorem 1.1 and Theorem 1.3.

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2 Preliminaries

In this section, we recall several known results on one-dimensional diffusions and their quasi-stationary distributions.

2.1 Feller’s canonical form of second-order differential operators

Let $(X, \mathbb{P}_x)_{x \in I}$ be a one-dimensional diffusion on $I = [0, \ell)$ or $[0, \ell]$ $(0 < \ell \leq \infty)$, that is, the process $X$ is a time-homogeneous strong Markov process on $I$ which has a continuous path up to its lifetime. Throughout the present paper, we always assume an irreducibility in the following sense:

$$\mathbb{P}_x[T_y < \infty] > 0 \quad (x \in I \setminus \{0\}, \ y \in [0, \ell]),$$

where $T_y$ denotes the first hitting time of $y$. In addition, we assume $X$ certainly hits 0 and the point 0 is a trap;

$$\mathbb{P}_x[T_0 < \infty] = 1 \quad (x \in I \setminus \{0\}), \quad X_t = 0 \quad \text{for } t \geq T_0,$$
Let us recall Feller’s canonical form of the generator (see e.g., Itô [8, p.139]). There exist a Radon measure \( m \) on \( I \setminus \{0\} \) with full support and a strictly increasing continuous function \( s \) on \((0, \ell)\), and the local generator \( L \) on \((0, \ell)\) is represented by
\[
L = \frac{d}{dm} \frac{d}{ds}.
\] (2.3)
The measure \( m \) is called the speed measure and \( s \) is called the scale function of \( X \). We say \( X \) is a \( \frac{d}{dm} \frac{d}{ds} \)-diffusion. For a given second-order ordinary differential operator
\[
G = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \quad (x \in (0, \ell))
\] (2.4)
with \( a(x) > 0 \) on \((0, \ell)\), we can give, for example, its speed measure and its scale function by
\[
dm(x) := \frac{1}{a(x)} \exp \left( \int_c^x \frac{b(y)}{a(y)} \, dy \right) \, dx, \quad ds(x) := \exp \left( - \int_c^x \frac{b(y)}{a(y)} \, dy \right) \, dx
\] (2.5)
for an arbitrary taken \( c \in (0, \ell) \).

By using the speed measure \( m \) and the scale function \( s \), the boundaries of \( I \) are classified. Let \( \Delta = 0 \) or \( \ell \) and take \( c \in (0, \ell) \) and set
\[
I(\Delta) = \int_\Delta^c ds(x) \int_\Delta^x dm(y), \quad J(c) = \int_\Delta^c dm(x) \int_\Delta^x ds(y).
\] (2.6)
The boundary \( \Delta \) is classified as follows:
\[
\Delta \text{ is}\begin{cases}
\text{regular} & \text{when } I(\Delta) < \infty, \ J(\Delta) < \infty, \\
\text{exit} & \text{when } I(\Delta) = \infty, \ J(\Delta) < \infty, \\
\text{entrance} & \text{when } I(\Delta) < \infty, \ J(\Delta) = \infty, \\
\text{natural} & \text{when } I(\Delta) = \infty, \ J(\Delta) = \infty.
\end{cases}
\] (2.7)
From (2.1), the boundary \( 0 \) is always regular or exit in our setting.

2.2 Quasi-stationary distributions

We recall a necessary and sufficient condition for existence of infinitely many quasi-stationary distributions.

Define a function \( u = \psi_\lambda \ (\lambda \in \mathbb{C}) \) as the unique solution of the following equation:
\[
\frac{d}{dm} \frac{d^+}{ds} u(x) = \lambda u(x), \quad \lim_{x \to +0} u(x) = 0, \quad \lim_{x \to -0} \frac{d^+}{ds} u(x) = 1 \quad (x \in (0, \ell)),
\] (2.8)
where \( \frac{d^+}{ds} \) denotes the right-differential operator by the scale function \( s \). Note that from the assumption that the boundary \( 0 \) is regular or exit, the function \( \psi_\lambda \) always exists. The
operator $L = -\frac{d}{dm} \frac{d}{ds}$ defines a non-negative definite self-adjoint operator on $L^2(I, dm) := \{ f : I \to \mathbb{R} \mid \int_I |f|^2 dm < \infty \}$. Here we assume the Dirichlet boundary condition at 0 and the Neumann boundary condition at $\ell$ if the boundary $\ell$ is regular. We denote the infimum of the spectrum of $L$ by $\lambda_0 \geq 0$.

When the boundary $\ell$ is not natural, there exists a unique quasi-stationary distribution. Note that from the assumption (2.2), the boundary $\ell$ cannot be exit.

**Theorem 2.1** (see e.g., [14] Lemma 2.2, Theorem 4.1). Assume the boundary $\ell$ is regular or entrance. Then $\lambda_0 > 0$ and the function $\psi_{-\lambda_0}$ is strictly positive and integrable w.r.t. $dm$ and, there is a unique quasi-stationary distribution

$$\nu_{\lambda_0}(dx) := \lambda \psi_{-\lambda_0}(x) dm(x)$$

with $\mathbb{P}_{\nu_{\lambda_0}}[T_0 \in dt] = \lambda_0 e^{-\lambda_0 t} dt$.

Recall the following property of the function $\psi_{-\lambda}$ ($\lambda > 0$).

**Proposition 2.2** ([5] Lemma 6.7, Lemma 6.18). Suppose $\lambda_0 > 0$, the boundary $\ell$ is natural and $s(\ell) = \infty$. Then for $\lambda > 0$ the following are equivalent:

(i) $\lambda \in (0, \lambda_0]$.

(ii) The function $\psi_{-\lambda}$ is non-negative on $[0, \ell]$.

(iii) The function $\psi_{-\lambda}$ is strictly increasing on $[0, \ell]$.

(iv) The function $\psi_{-\lambda}$ is strictly positive and integrable on $(0, \ell)$.

Moreover, if one of these conditions holds, it holds

$$1 = \lambda \int_0^\ell \psi_{-\lambda}(x) dm(x).$$

The following is a necessary and sufficient condition for existence of infinitely many quasi-stationary distributions.

**Theorem 2.3.** Suppose the boundary $\ell$ is natural. Then infinitely many quasi-stationary distributions exist if and only if

$$\lambda_0 > 0 \quad \text{and} \quad s(\ell) = \infty.$$  

The condition (2.11) is equivalent to

$$m(c, \ell) < \infty \quad \text{for some } c \in (0, \ell) \quad \text{and} \quad \limsup_{x \to \ell} s(x)m(x, \ell) < \infty.$$  

In this case, the set of quasi-stationary distributions is $\{ \nu_{\lambda} \}_{\lambda \in (0, \lambda_0]}$ for

$$\nu_{\lambda}(dx) := \lambda \psi_{-\lambda}(x) dm(x),$$

and it holds $\mathbb{P}_{\nu_{\lambda}}[T_0 \in dt] = \lambda e^{-\lambda t} dt$. 

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Though Theorem 2.3 can be shown by a combination of known results, we prove for completeness.

Proof. We may assume without loss of generality that \( s(0) = 0 \). The equivalence between (2.11) and (2.12) follows from [13, Theorem 3 (ii), Appendix I]. The fact that \( \nu_\lambda (\lambda \in (0, \lambda_0]) \) is a quasi-stationary distribution can be shown by the same argument in [5, Lemma 6.18]. Thus, we only show that every quasi-stationary distribution is given by \( \nu_\lambda \) for some \( \lambda \in (0, \lambda_0] \).

Let \( \mu \) be a quasi-stationary distribution with

\[
\mathbb{P}_\mu[T_0 > t] = e^{-\lambda t} \quad (\lambda > 0).
\] (2.14)

Recalling that one-dimensional diffusions have a transition density w.r.t. its speed measure, that is, there exists a jointly-continuous function \( p(t,x,y) \) on \((0, \infty) \times (0, \ell)^2\) such that

\[
\mathbb{P}_x[X_t \in dx, T_0 > t] = p(t,x,y)dm(y)
\] (2.15)

(see e.g., McKean [9]). Since it holds

\[
e^{-\lambda t} \mu(A) = \mathbb{P}_\mu[X_t \in A, T_0 > t] = \int_0^\ell 1_A(y)dm(y) \int_0^\ell p(t,x,y)\mu(dx),
\] (2.16)

the probability measure \( \mu \) is absolutely continuous w.r.t. \( dm \), we denote the density by \( \rho \).

Applying the well-known formula for non-negative measurable function \( f \):

\[
\mathbb{E}_x \left[ \int_0^{T_0} f(X_t)1\{T_0 > t\}dt \right] = \int_0^\ell (s(x) \wedge s(y))f(y)dm(y)
\] (2.17)

(see e.g., [19, Theorem 49.1] and [10, Lemma 23.10]), we have for a measurable set \( A \subset (0, \ell) \)

\[
\Phi_\mu(A) = \lambda \int_0^\ell \rho(x)dm(x) \int_0^\ell \left(s(x) \wedge s(y)\right)1_A(y)dm(y)
\] (2.18)

\[
= \lambda \int_0^\ell \left( \int_0^x ds(y) \int_y^\ell \rho(z)dm(z) \right)1_A(x)dm(x).
\] (2.19)

Since it holds \( \Phi_\mu = \mu \) from (1.5), we obtain

\[
\rho(x) = \lambda \int_0^x ds(y) \int_y^\ell \rho(z)dm(z) \quad dm\text{-a.e.}
\] (2.20)

Since \( \int_0^\ell \rho(x)dm(x) = 1 \), the function \( u = \rho \) is the solution of the following equation

\[
u(x) = \lambda s(x) - \lambda \int_0^x ds(y) \int_y^\ell u(z)dm(z),
\] (2.21)

and hence \( \rho(x) = \lambda \psi_{-\lambda}(x) \) dm-a.e. From Proposition 2.2 it holds \( \lambda \in (0, \lambda_0] \) and therefore \( \mu = \nu_\lambda \).
From the proof of Theorem 2.3, we can see the set of quasi-stationary distributions coincides with the set of fixed points of \( \Phi \).

**Corollary 2.4.** Assume there exists infinitely many quasi-stationary distributions. For \( \mu \in P_\Phi \), \( \Phi \mu = \mu \) if and only if \( \mu = \nu_\lambda \) for some \( \lambda \in (0, \lambda_0) \).

The set \( \{ \nu_\lambda \}_{\lambda \in (0, \lambda_0)} \) is totally ordered by a usual stochastic order, and \( \nu_{\lambda_0} \) is the minimal element. This is why we call \( \nu_{\lambda_0} \) the minimal quasi-stationary distribution.

**Proposition 2.5** ([21, Proposition 2.3]). Suppose there exist infinitely many quasi-stationary distributions \( \{ \nu_\lambda \}_{\lambda \in (0, \lambda_0)} \). Then it holds

\[
\nu_\lambda (x, \ell) \leq \nu_{\lambda'} (x, \ell) \quad (x \in (0, \ell), \; 0 < \lambda' \leq \lambda \leq \lambda_0).
\]

(2.22)

### 3 Proof of the main results

For every \( \frac{d}{ds} \frac{d}{dn} \)-diffusion \( X \) on \((0, \ell) \) \((0 < \ell \leq \infty)\) satisfying (2.11), the diffusion \( s(X) \) is \( \frac{d}{ds} \frac{d}{dn} \)-diffusion on \((0, \infty)\) for \( dm(x) = d\tilde{m}(s^{-1}(x)) \). Thus, we may assume without loss of generality that the diffusion is under the natural scale: \( s(x) = x \). In this section, we only consider such natural scale diffusions on \((0, \infty)\).

At first, we check that \( \Phi \) preserves \( P_\Phi \).

**Proposition 3.1.** Let \( \mu \in P_\Phi \). For \( \alpha \in [1, \infty) \), we have

\[
\mathbb{E}_{\Phi \mu} T_0^\alpha = \frac{\mathbb{E}_\mu T_0^{\alpha+1}}{(\alpha + 1) \mathbb{E}_\mu T_0}.
\]

(3.1)

More generally, we have for \( 0 \leq k \leq m \)

\[
\mathbb{E}_{\Phi^m \mu} T_0^\alpha = \left( \frac{\alpha + k}{\alpha} \right)^{-1} \cdot \frac{\mathbb{E}_{\Phi^{m-k} \mu} T_0^{\alpha+k}}{\mathbb{E}_{\Phi^{m-k} \mu} T_0^k}.
\]

(3.2)

**Proof.** Since (3.2) follows from (3.1) by induction, we only prove (3.1). From the Markov property, it holds

\[
\mathbb{E}_{\Phi \mu} T_0^\alpha = \alpha \int_0^\infty t^{\alpha-1} \mathbb{P}_{\Phi \mu} [T_0 > t] dt
\]

(3.3)

\[
= \frac{\alpha}{\mathbb{E}_\mu T_0} \int_0^\infty t^{\alpha-1} dt \int_0^\infty ds \int_0^\infty \mathbb{P}_x [T_0 > t] \mathbb{P}_\mu [X_s \in dx]
\]

(3.4)

\[
= \frac{\alpha}{\mathbb{E}_\mu T_0} \int_0^\infty t^{\alpha-1} dt \int_0^\infty \mathbb{P}_\mu [T_0 > s] ds
\]

(3.5)

\[
= \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty s^\alpha \mathbb{P}_\mu [T_0 > s] ds
\]

(3.6)

\[
= \frac{\mathbb{E}_\mu T_0^{\alpha+1}}{(\alpha + 1) \mathbb{E}_\mu T_0}.
\]

(3.7)
For the proof of Theorem 1.1 we need some preparation. For a function \(g: (0, \infty) \to \mathbb{R}\) with
\[
\int_0^R dy \int_y^\infty |g(z)|dm(z) < \infty \quad \text{for every } R > 0,
\] (3.8)
define an integral operator \(K\) by
\[
Kg(x) := \int_0^x dy \int_y^\infty g(z)dm(z) = \int_0^\infty (x \land y)g(y)dm(y) \quad (x > 0).
\] (3.9)

Let us recall the formula (2.17). Then for a function \(g\) with (3.8), it holds
\[
\mathbb{E}_\mu \int_0^T g(X_t)dt = \int_0^\infty \mu(dx) \int_0^\infty (x \land y)g(y)dm(y) = \int_0^\infty Kg(x)\mu(dx).
\] (3.10)
Applying (3.10), we obtain the density function of \(\Phi^\mu\).

**Proposition 3.2.** For \(\mu \in \mathcal{P}_\Phi\) and \(n \geq 1\), there exists a density \(f_n^\mu\) of \(\Phi^\mu\) w.r.t. \(dm\); \(\Phi^\mu(dx) = f_n^\mu(x)dm(x)\). It is given by
\[
f_n^\mu(x) = \frac{1}{m_n} K^{n-1}G^\mu(x) \quad \text{dm-a.e.,}
\] (3.11)
where \(G^\mu(x) := \int_0^x \mu(y, \infty)dy\) and we denote \(K^\ell g := K(K^{\ell-1}g) \ (\ell \geq 1)\).

**Proof.** From (3.10), we have for a bounded measurable function \(g\) with compact support on \((0, \infty)\),
\[
\int_0^\infty g(y)\Phi^\mu(dx) = \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty \mu(dx) \int_0^\infty (x \land y)g(y)dm(y)
\] (3.12)
\[
= \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty G^\mu(y)g(y)dm(y).
\] (3.13)
Since it holds that
\[
G^\mu(x) = \int_0^x \Phi^\mu(y, \infty)dy
\] (3.14)
\[
= \frac{1}{\mathbb{E}_\mu T_0} \int_0^x dy \int_y^\infty G^\mu(z)dm(z)
\] (3.15)
\[
= \frac{1}{\mathbb{E}_\mu T_0} KG^\mu(x)
\] (3.16)
it follows from Proposition 3.1 and the (formal) self-adjointness of $K$ under $dm$ that

$$
\int_0^\infty g(y)\Phi^n\mu(dy) = \frac{1}{E\Phi_0-1} \int_0^\infty G\Phi_0-1 g(y) dm(y) \tag{3.17}
$$

$$
= \frac{1}{(E\Phi_0-1)(E\Phi_0-2)} \int_0^\infty KG\Phi_0-2 g(y) dm(y) \tag{3.18}
$$

$$
= \frac{1}{(E\Phi_0-1)(E\Phi_0-2)} \int_0^\infty G\Phi_0-2 K g(y) dm(y) \tag{3.19}
$$

$$
= \ldots \tag{3.20}
$$

$$
= \frac{1}{(E\Phi_0-1)(E\Phi_0-2)\ldots(E\Phi_0-n)(E\Phi_0-n)} \int_0^\infty K^{n-1} G\Phi_0-n g(y) dm(y) \tag{3.21}
$$

$$
= \frac{n!}{E\Phi_0^n} \int_0^\infty K^{n-1} G\Phi_0-n g(y) dm(y). \tag{3.22}
$$

The proof is complete. \[ \square \]

Now we prove Theorem 1.1.

Proof of Theorem 1.1. For a function $g$ with

$$
\int_0^\infty dy \int_0^y |g(z)| dm(z) < \infty \tag{3.24}
$$

for every $R > 0$, define an integral operator $I$ by

$$
Ig(x) := \int_0^x dy \int_0^y g(z) dm(z) \quad (x > 0). \tag{3.25}
$$

Let $g \in L^1((0, \infty), dm)$. We have

$$
Kg(x) = \int_0^x dy \int_y^\infty g(z) dm(z) \tag{3.26}
$$

$$
= x \int_0^\infty g(z) dm(z) - Ig(x). \tag{3.27}
$$

Then it follows that

$$
K^{n-1} G_\mu(x) = x \int_0^\infty K^{n-2} G_\mu(y) dm(y) - IK^{n-2} G_\mu(x) \tag{3.28}
$$

$$
= m_{n-1}^\mu x - IK^{n-2} G_\mu(x) \tag{3.29}
$$

$$
= m_{n-1}^\mu x - I(m_{n-2}^\mu x - IK^{n-3} G_\mu(x)) \tag{3.30}
$$

$$
= m_{n-1}^\mu x - m_{n-2}^\mu Ix + I^2 K^{n-3} G_\mu(x) \tag{3.31}
$$

$$
= \ldots \tag{3.32}
$$

$$
= \sum_{k=1}^{n-1} (-1)^{k-1} m_{n-k}^\mu I^{k-1} x + (-1)^{n-1} I^{n-1} G_\mu(x). \tag{3.33}
$$

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Then we have

\[ f_n^\mu(x) = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{m_{n-k}^\mu}{m_n^\mu} x^{k-1} + (-1)^{n-1} \frac{1}{m_n^\mu} I^{n-1} G_\mu(x). \]  

(3.34)

From (1.9) we have

\[ M := \sup_{n \geq 0} \frac{m_n^\mu}{m_n^\mu} < \infty, \]

where we denote \( m_0^\mu = 1. \) Then it follows that

\[ \sum_{k=1}^{n-1} \frac{m_{n-k}^\mu}{m_n^\mu} I^{k-1} x \leq \sum_{n=1}^{\infty} M^n I^{n-1} x = M \psi_M(x) < \infty. \]  

(3.35)

Next we show the second term in the RHS of (3.34) vanishes as \( n \to \infty. \) It is not difficult to check that

\[ I^n G_\mu(x) \leq \frac{1}{n!} x \left( \int_0^x y dm(y) \right)^n, \]  

(3.36)

and since \( m_n^\mu = \prod_{1 \leq i \leq n} (m_i^\mu / m_i^{\mu-1}) \geq M^{-n}, \) we obtain for every \( R > 0 \)

\[ \lim_{n \to \infty} \sup_{x \in [0, R]} \frac{1}{m_n^\mu} I^{n-1} |G_\mu(x)| = 0. \]  

(3.37)

Then from (3.34) and the dominated convergence theorem, we have

\[ \lim_{n \to \infty} f_n^\mu(x) = \lambda \psi_{\lambda}(x). \]  

(3.38)

From (3.35) and (3.36), we have

\[ f_n^\mu(x) \leq M \psi_M(x) + \frac{M^n}{(n-1)!} x \left( \int_0^x y dm(y) \right)^{n-1} \leq M \psi_M(x) + M x e^M \int_0^x y dm(y) \]  

(3.39)

and, it is obvious that

\[ \int_0^R \left( M \psi_M(x) + M x e^M \int_0^x y dm(y) \right) dm(x) < \infty \]  

(3.40)

for every \( R > 0. \) Hence, from the dominated convergence theorem, it follows that

\[ \Phi^n \mu \stackrel{n \to \infty}{\longrightarrow} \nu_\lambda. \]  

(3.41)

For the proof of Theorem 1.3 we prepare a continuity result for the transform \( \Phi. \)

**Proposition 3.3.** Let \( \mu_n, \mu \in \mathcal{P}(0, \infty) \) such that \( \mathbb{E}_{\mu_n} T_0, \mathbb{E}_\mu T_0 < \infty. \) Suppose \( \mu_n \xrightarrow{n \to \infty} \mu. \)

Then the following are equivalent:

(i) \( \mathbb{E}_{\mu_n} T_0 \xrightarrow{n \to \infty} \mathbb{E}_\mu T_0. \)
(ii) $\Phi \mu_n \xrightarrow{n \to \infty} \Phi \mu$.

**Proof.** Let $f$ be a continuous function with a compact support on $(0, \infty)$. From (2.17) it holds

$$
(E_{\mu_n T_0}) \Phi \mu_n (f) = \int_0^\infty \mathbb{P}_{\mu_n} [f(X_t) 1\{T_0 > t\}] dt
$$

(3.42)

$$
= \int_0^\infty \mu_n(dx) \int_0^\infty f(y)(x \wedge y) dm(y).
$$

(3.43)

Since the function $f$ is compactly supported, the function $K f(x) := \int_0^\infty f(y)(x \wedge y) dm(y)$ ($x > 0$) is bounded continuous. The rest of the proof is obvious.

We prove Theorem 1.3.

**Proof of Theorem 1.3.** The implication (i) $\implies$ (ii) is obvious. We first show (ii) $\implies$ (iii).

Note that when we set $g(t) := \mathbb{P}_\mu[T_0 > \log t]$, the condition (ii) is equivalent to

$$
\lim_{t \to \infty} g(st) = s^{-\lambda} \quad (s > 0),
$$

(3.44)

the regular variation of the function $g$ at $\infty$ of order $-\lambda$. Then from Karamata’s theorem [2, Proposition 1.5.10, Theorem 1.6.1], the condition (ii) is equivalent to the following:

$$
h(t) := \frac{1}{\mathbb{P}_\mu[T_0 > t]} \int_t^\infty \mathbb{P}_\mu[T_0 > s] ds \xrightarrow{t \to \infty} \frac{1}{\lambda}.
$$

(3.45)

From Fubini’s theorem, it holds for $n \geq 2$

$$
m_n^\mu = \frac{1}{(n-2)!} \int_0^\infty t^{n-2} \mathbb{P}_\mu[T_0 > t] h(t) dt.
$$

(3.46)

For $R > 0$ it is not difficult to see

$$
\frac{\int_0^R t^{n-2} \mathbb{P}_\mu[T_0 > t] h(t) dt}{\int_R^\infty t^{n-2} \mathbb{P}_\mu[T_0 > t] h(t) dt} \xrightarrow{n \to \infty} 0.
$$

(3.47)

Thus, from (3.45) it follows

$$
\lim_{n \to \infty} \frac{m_n^\mu}{m_{n-1}^\mu} = \lim_{n \to \infty} \frac{\int_0^\infty t^{n-2} \mathbb{P}_\mu[T_0 > t] h(t) dt}{\int_0^\infty t^{n-2} \mathbb{P}_\mu[T_0 > t] h(t) dt} = \frac{1}{\lambda}.
$$

(3.48)

Since we have already shown (iii) $\implies$ (iv) in Theorem 1.1, we finally show (iv) $\implies$ (iii).

Set $\mu_n := \Phi_n^\mu$. Since it holds $\mu_n \xrightarrow{n \to \infty} \nu_\lambda$ and $\Phi \mu_n = \mu_{n+1} \xrightarrow{n \to \infty} \Phi \nu_\lambda = \nu_\lambda$, we have from Proposition 3.3 and Proposition 3.1

$$
m_{n+1}^\mu = \mathbb{E}_{\mu_n} T_0 \xrightarrow{n \to \infty} \frac{1}{\lambda}.
$$

(3.49)
Remark 3.4. The condition (iii) in Theorem 1.3 implies

$$\lim_{t \to \infty} \frac{1}{t} \log\mathbb{P}_\mu[T_0 > t] = -\lambda. \quad (3.50)$$

Indeed, under the condition (iii), we see from Proposition 3.1 that

$$\lim_{n \to \infty} m_j^{\Phi_n^\mu} = j! / \lambda^j \quad \text{for } j \geq 1,$$

which implies

$$\mathbb{P}_{\Phi_n^\mu}[T_0 \in dt] \xrightarrow{n \to \infty} \lambda e^{-\lambda t} dt \quad \text{and} \quad m_\alpha^\mu / m_\alpha^\mu+1 \xrightarrow{a \to \infty} \lambda.$$ Then it follows

$$\frac{1}{\alpha} \log m_\alpha^\mu = \frac{1}{\alpha} \sum_{0 \leq j < \lfloor \alpha \rfloor} \log(m_\alpha^\mu - j)/m_\alpha^\mu - j \xrightarrow{\alpha \to \infty} -\log \lambda. \quad (3.51)$$

Then applying a Tauberian theorem of exponential type [11, Theorem 1] to the function

$$F(\alpha) := \int_0^\infty e^{\alpha \log(t/(\alpha)^{1/\alpha})} \mathbb{P}_\mu[T_0 \in dt] \quad (= m_\alpha^\mu), \quad (3.52)$$

we see (3.50) and (3.51) are equivalent.

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