Convex Synthesis of Accelerated Gradient Algorithms for Optimization and Saddle Point Problems using Lyapunov functions

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Abstract—This paper considers the problem of designing accelerated gradient-based algorithms for optimization and saddle-point problems. The class of objective functions is defined by a generalized sector condition. This class of functions contains strongly convex functions with Lipschitz gradients but also non-convex functions, which allows not only to address optimization problems but also saddle-point problems. The proposed design procedure relies on a suitable class of Lyapunov functions and on convex semi-definite programming. The proposed synthesis allows the design of algorithms that reach the performance of state-of-the-art accelerated gradient methods and beyond.

I. INTRODUCTION

Gradient-based optimization algorithms are a standard tool in science and engineering. Many of these algorithms take the form of feedback interconnection between a time-discrete linear system and the gradient of the objective function. In case of a convex objective function, the corresponding gradient satisfies a certain sector condition. Hence such a feedback configuration falls in the class of so called Lur’e systems [7], which have been extensively studied in control theory. In recent years, results from Lur’e systems and techniques from robust control theory have been exploited to analyze convergence rates and robustness of known optimization algorithms and to design novel algorithms. Some of those new publications rely on IQCs (integral quadratic constraints) from robust control to generate convergence results. For example, IQCs were used in [5] to find upper bounds for the convergence rates of existing algorithms. This work was later extended to synthesis of algorithms in [6]. These IQC-based approaches gave rise to the development of the Triple Momentum Method [14]. This method has the fastest known upper convergence bound for strongly convex functions with Lipschitz gradients. Other related work that analyzes optimization algorithms from a dynamical systems perspective is for example given in [3] and [8], where also Lyapunov function techniques and robust control theory are employed, or in [15], where discrete-time algorithms are analyzed based on continuous-time counterparts. In addition, semi-definite programming formulations have been proposed in [2] and [13] to analyze the convergence properties of first order optimization methods. Further related results are discussed in the recent paper [9], where the design of robust algorithms for structured objective functions based on IQC theory is considered.

In this paper, we address convex design (convex synthesis) of gradient-based algorithms for optimization and saddle point problems, where the class of objective functions is defined by a generalized sector condition. In particular, the contributions of this paper are as follows. First we consider classes of functions that are more general than the classes of strongly convex functions usually considered in the literature. In particular, the classes under consideration also contain non-convex functions, which we utilize in our procedure to design algorithms capable of searching for saddle points instead of minima. For example, the ability to search for saddle points allows us to apply the design method to optimization problems with equality constraints. Second, based on a rather general class of Lyapunov functions, we derive convex synthesis conditions for algorithm design in the form of linear matrix inequalities. Specifically, we provide a non-conservative convexification in the sense that the analysis matrix inequalities (when algorithm parameters are given) are feasible if and only if the synthesis matrix inequalities (when algorithm parameters are decision variables) are feasible, i.e. our design procedure is not more conservative than the corresponding analysis. This is in contrast to many other results in the literature, where the step from convex analysis to convex synthesis is only possible by imposing additional assumptions (such as fixed IQC multipliers or quadratic Lyapunov functions). In the case of strongly convex functions, our design procedure reaches the same convergence rates as the Triple Momentum Method and it allows to incorporate additional structural properties of the objective function to design tailored algorithms with even faster convergence rates, as demonstrated in the paper.

II. PROBLEM STATEMENT AND PRELIMINARY RESULTS

A. Notation

By $\|v\|$, we denote the Euclidean norm of a vector $v \in \mathbb{R}^n$ and by $\|A\|$ the spectral norm of a matrix $A \in \mathbb{R}^{n \times n}$. The spectrum of a matrix will be denoted by $\sigma(A)$ and for the spectral radius we will write $\rho(A)$. We will also often use the notation $\|v\|_A = v^T A v$ for the semi-norm defined by a positive semi-definite matrix $A$, which is a full norm whenever $A$ is positive definite. If $A_1, A_2$ are two symmetric matrices of the same dimensions, then we write $A_1 \succ A_2$ ($\succeq$) if $A_1 - A_2$ is positive (semi-) definite and $A_1 \prec A_2$ ($\preceq$) if $A_1 - A_2$ is negative (semi-) definite. With $A^T$, we will denote the Moore-Penrose pseudo inverse of a matrix, while $A^T$ will denote its transpose. The orthogonal projection
matrix the kernel of a matrix $A$ will be denoted by $\Pi_{\ker A}$. In large matrix equations, we will sometimes write $A^T B \ast$. In that case, $(\ast)$ is to be understood as a copy of the matrix $A$.

B. Problem statement

Consider the gradient based algorithm defined by

$$x_{k+1} = Ax_k + B\nabla f(Cx_k),$$

where $x_k \in \mathbb{R}^n$ and the matrices $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times d}, C \in \mathbb{R}^{d \times n}$ are the algorithm parameters such that for any $x \in M$

the objective function $f \in C^1(\mathbb{R}^d)$ is assumed to satisfy the following generalized sector condition for all $z_1, z_2 \in \mathbb{R}^d$:

$$\frac{1}{2} \|z_1 - z_2\|_M^2 \leq f(z_2) - f(z_1) + (\nabla f(z_1))^T (z_1 - z_2) \leq \frac{1}{2} \|z_1 - z_2\|^2,$$

(2)

where $M \preceq L \in \mathbb{R}^{d \times d}$ are given symmetric matrices. In the following, $S(M, L)$ denotes the set of all $C^1$ functions that satisfy (2). Note that $S(mI_d, lI_d), m < l$, is a set of strongly convex functions, as typically found in the literature. In the case $f \in C^2(\mathbb{R}^d)$, (2) is equivalent to $M \preceq H_f(z) \preceq L$ for all $z \in \mathbb{R}^d$, where $H_f$ denotes the Hessian of $f$.

The algorithm design problem addressed in this paper is formally stated as:

**Problem 1.** For given $n \geq d, M \preceq L$, and convergence rate $\rho \in [0, 1[$, we aim to design matrices $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times d} \times \mathbb{R}^{d \times n}$ such that for any $f \in S(M, L)$ there exist $x_0 \in \mathbb{R}^n$ and $c_f \geq 0$ such that

$$\nabla f(z_f) = 0 \text{ for } z_f := Cx_f^*,$$

and the iterates $x_k$ of (1) satisfy

$$\|x_f - x_k\| \leq c_f \rho^k \|x_f^* - x_0\|,$$

for any $x_0 \in \mathbb{R}^n$, $k \in \mathbb{N}_0$.

In our setting, design (synthesis) refers to computing the algorithm parameters $(A, B, C)$ by solving a convex optimization problem, i.e., a semi-definite program.

Our goal is solving Problem 1. The following Problem 2 is similar to Problem 1 with the slight modification that all the functions $f$ under consideration have their critical points in $z_f = 0$. This is favourable for the application of tools from robust control theory, which are often formulated for fixed-points in zero.

**Problem 2.** For given $n \geq d, M \preceq L$, and $\rho \in [0, 1[$, design matrices $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times d} \times \mathbb{R}^{d \times n}$ satisfying the constraint

$$\overline{C}(\overline{A} - I_n)^{-1} \overline{B}M = I_d$$

(3)

such that for any $f \in S_0(0, \overline{L}) := \{ f \in S(0, \overline{L}) : \nabla f(0) = 0 \}$ there exists $c_f \in \mathbb{R}_{\geq 0}$ such that the iterates of (1) satisfy

$$\|x_k\| \leq c_f \rho^k \|x_0\|$$

for any $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$.

The subsequent theorem states that the two problems are equivalent.

**Theorem 1.** Let symmetric matrices $M \preceq L$ be given, set $\overline{L} := L - M$ and fix $\rho \in [0, 1[$. Then the matrices $(\overline{A}, \overline{B}, \overline{C})$ solve Problem 1 if and only if the matrices $(\overline{A}, \overline{B}, \overline{C})$ solve Problem 2 where $A = A + BMC, B = \overline{B}, C = C$.

This theorem justifies that we can solve Problem 2 instead of Problem 1.

C. Properties of the class $S(M, L)$

This subsection serves the purpose of introducing some important properties of $S(M, L)$. The first result gives some equivalent characterizations for when $f \in S(0, L)$ holds true. Note, that these conditions can be applied to any class $S(M, L)$ by using the fact $f \in S(M, L) \iff (z \mapsto f(z) - \frac{1}{2}z^T M z) \in S(0, L - M)$.

**Lemma 2** (Characterizations for $f \in S(0, L)$). Let $L \succeq 0$ and $f \in C^1(\mathbb{R}^d)$. All conditions below, holding for all $z_1, z_2 \in \mathbb{R}^n$, are equivalent to $f \in S(0, L)$:

1) $0 \leq f(z_2) - f(z_1) - (\nabla f(z_1))^T (z_2 - z_1) \leq \frac{1}{2} \|z_1 - z_2\|^2_L$,

2) $0 \leq (\nabla f(z_1) - \nabla f(z_2))^T (z_1 - z_2) \leq \|z_1 - z_2\|^2_L$.

3) $\frac{1}{2} \|\nabla f(z_1) - \nabla f(z_2)\|^2_{L_1} \leq f(z_2) - f(z_1) + (\nabla f(z_1))^T (z_2 - z_1)$ and $\Pi_{\ker L}(\nabla f(z_1) - \nabla f(z_2)) = 0$.

4) $\|\nabla f(z_1) - \nabla f(z_2)\|^2_{L_1} \leq \|\nabla f(z_1) - \nabla f(z_2)\|^T (z_2 - z_1)$

and $\Pi_{\ker L}(\nabla f(z_1) - \nabla f(z_2)) = 0$.

Not all possible variations of matrices $M \preceq L$ should be considered for optimization. For example, if there exists a singular matrix $Q$ such that $M \preceq Q \preceq L$, then the function $f$ defined by $f(z) = \frac{1}{2}z^T Q z + v^T z$, where $v$ is not in the range of $Q$, would be an element of $S(M, L)$ without any critical point. Therefore this set $S(M, L)$ would not make sense as a set of objective functions, since we cannot solve Problem 1 for it. The following Lemma characterizes when such cases can be avoided.

**Lemma 3** (Well-posed pairs $M, L$). Let $M, L \in \mathbb{R}^{d \times d}$ be symmetric matrices with $M \preceq L$. Then the following five statements are equivalent:

1) The matrices $M$ and $L$ have the same numbers of positive and negative, and no zero eigenvalues.

2) Any symmetric matrix $Q \in \mathbb{R}^{n \times n}$ with $M \preceq Q \preceq L$ is non-singular.

3) $L + M$ is non-singular and the spectral radius of $(L + M)^{-1}(L - M)$ is smaller than one.

4) $M$ is non-singular and $M^{-1} L$ has only positive eigenvalues.

5) $M$ and $L$ are non-singular and congruent, i.e., there exists a non-singular matrix $T \in \mathbb{R}^{d \times d}$ with $M = T^T LT$. 
Remark 4. In Lemma 3, statement 1) serves the purpose of giving the reader a good intuition for the property under consideration. Statement 2) and 3) will be useful in later proofs. Note that in particular 2) prevents the counter-example we constructed in the motivation of this lemma. Statement 4) offers the most efficiently verifiable test of the considered property, by the fact that the verification whether a matrix has positive eigenvalues can be done by solving a Lyapunov equation.

Because of the importance of this property we define a new notation for matrices \( M, L \) fulfilling one and thus all conditions in Lemma 3.

Definition 5 (Loewner-congruence ordering on symmetric matrices). For symmetric matrices \( M, L, L \in \mathbb{R}^{d\times d} \), we introduce the partial ordering

\[
L \succeq_c M \iff \begin{cases} 
L - M \text{ is positive semi-definite} \\
L \text{ and } M \text{ are congruent}
\end{cases}
\]

Under the Loewner-congruence ordering, a critical point exists, is unique, and a simple gradient method converges to the critical point, as stated in the following results.

Proposition 6 (A simple gradient method). Let \( L \succeq_c M \) be non-singular. Then for any convergence rate \( \rho > \rho^\text{grad} := \rho \left( (L + M)^{-1}(L - M) \right) \) there exists \( r \in \mathbb{R}_{>0} \) such that

\[
z \mapsto z - 2(M + L)^{-1}\nabla f(z)
\]

is a contraction for all \( f \in S(M, L) \) with contraction constant \( \rho \) on the Banach space \( (\mathbb{R}^d, \| \cdot \|_p) \), where \( P = (L + M)(L - M)^\dagger + r\Pi_{\text{vec}(L-M)}(L + M) \).

Remark 7. For \( M \preceq_c L \), the optimizer defined by \((A, B, C)\) with \( A = C = I_d \) and \( B = -2(L + M)^{-1} \) realizes the contraction in Proposition 6. As a consequence of the Banach fixed-point theorem, it converges faster than any convergence rate \( \rho > \rho^\text{grad} := \rho \left( (L + M)^{-1}(L - M) \right) \) and converges monotonically in the norm \( \| \cdot \|_p \) to the unique critical point. Finally, notice that in the case \( L = M \) is singular, the infimal convergence rate may not be attained, since \( r \) can go towards infinity if \( \rho \) goes towards \( \rho^\text{grad} \). However, if \( L - M \) is non-singular, then \( r \) disappears from the equation and the constructed gradient method converges at the rate \( \rho^\text{grad} \).

Theorem 8 (Existence and uniqueness of critical points). Let \( M, L \in \mathbb{R}^{d\times d} \) be given symmetric matrices. Then the following three statements are equivalent:

1) The matrices \( M, L \) are non-singular and satisfy \( M \preceq_c L \).
2) \( S(M, L) \) is not empty and for all \( f \in S(M, L) \) there exists at least one \( z_f^* \in \mathbb{R}^d \) with \( \nabla f(z_f^*) = 0 \).
3) \( S(M, L) \) is not empty and for all \( f \in S(M, L) \) there exists at most one \( z_f^* \in \mathbb{R}^d \) with \( \nabla f(z_f^*) = 0 \).

Remark 9. Theorem 8 shows that if we aim to design algorithms that are convergent for the whole class \( S(M, L) \), we must necessarily require \( M \preceq_c L \), because otherwise there would be elements of \( S(M, L) \) without critical points. Hence the introduced partial ordering plays a key role in our results. Note that it is no coincidence that in Theorem 8 the existence of critical points for all functions in \( S(M, L) \) and the uniqueness of critical points are two separate, equivalent statements. Similar to solutions of linear equation systems, here a solution for the equation \( \nabla f(z) = 0 \) exists for all \( f \in S(M, L) \) if and only if the solution is unique for all \( f \in S(M, L) \).

III. MAIN RESULTS

In this section, a convex synthesis approach of optimizer parameters \((A, B, C)\) for the set of objective functions \( S(M, L) \) and for a given convergence rate is provided. By Theorem 1 the design for the class \( S(M, L) \) reduces to designing algorithms for the class \( S_0(0, L) = \{ f \in S(0, L) | \nabla f(0) = 0 \} \) with \( L = L - M \). Hence we consider Problem 2 instead of Problem 1.

A. A Class of Lyapunov functions

To design the algorithm parameters \((A, B, C)\) with a prespecified convergence rate, we propose the following class of (non-quadratic) Lyapunov function candidates

\[
V_f(x) = \frac{1}{2} x^T (P_{11} P_{21} + P_{12} P_{22}) x + f(Cx) - f(0) - \frac{1}{2} \nabla f(Cx)^\dagger \nabla f(Cx)
\]

with parameter \( 0 < P = P^T \in \mathbb{R}^{n + d \times n + d} \). (Recall, that in Problem 2 \( L \) was defined as \( L = M \).) Similar Lyapunov functions have already been applied to Lur’e systems in continuous-time. Those Lyapunov functions share the first term, which is quadratic in the state \( x \) and the static non-linearity \( \nabla f(z) \). They have been proposed by Yakubovic for the case \( d = 1 \) in [16] and have been employed e.g. in [4], [11] and [12].

Our design approach, for a given convergence rate \( \rho \), is based on finding simultaneously a Lyapunov function \( P > 0 \) and algorithm parameters by semi-definite programming such that the Lyapunov conditions in the next theorem are satisfied.

Theorem 10 (Lyapunov function and convergence rate for the algorithms). Let \((A, B, C)\) be parameters of Algorithm 1 for the set of objective functions \( S \). If there exists a family of function \( V_f : \mathbb{R}^n \to [0, \infty] \) satisfying quadratic bounds

\[
\alpha_f \|x - x_f^*\|^2 \leq V_f(x) \leq \beta_f \|x - x_f^*\|^2 \quad \forall x \in \mathbb{R}^n, f \in S
\]

for some fixed \( \alpha_f, \beta_f \in \mathbb{R}_{>0} \) and the \( \rho \)-weighted increment bound

\[
V_f(x^+) - \rho^2 V_f(x) \leq 0 \quad \forall x \in \mathbb{R}^d,
\]

where \( x^+ = Ax + B\nabla f(Cx) \), then the optimizer defined by 1 is convergent with rate \( \rho \).
The following two lemmas provide useful bounds for the considered class of Lyapunov functions and their increments and imply as by-product the positive definiteness of $V_f$.

**Lemma 11** (Quadratic bounds on $V_f$). Let $f \in S_0(0, \bar{L})$. Then the Lyapunov function candidates $V_f$ fulfill the quadratic bounds

$$\alpha_f \|x\|^2 \leq V_f(x) \leq \beta_f \|x\|^2$$

with the constants $\alpha_f := \lambda_{\min}(P)$ and $\beta_f := \lambda_{\max}(P)(1 + \|\bar{L}\|^2\|C\|^2) + \|\bar{L}\|\|C\|^2$.

**Lemma 12** (Upper bound on the Lyapunov increment of $V_f$). Assume $f \in S_0(0, \bar{L})$. Then, the weighted increment of $V_f$ from (6) is upper bounded as follows:

$$V_f(x^+) - \rho^2V_f(x) \leq \left( \begin{array}{c} x \\ w \end{array} \right)^T \left( \begin{array}{cc} -\rho^2P_{11} & -\rho^2P_{12} \\ -\rho^2P_{21} & -\rho^2P_{22} \end{array} \right) \left( \begin{array}{c} x \\ w \end{array} \right) + \left( \begin{array}{c} x \\ w \end{array} \right)^T \left( \begin{array}{cc} 0 & 0 \\ -\rho^2C & 0 \end{array} \right) \left( \begin{array}{c} x \\ w \end{array} \right),$$

where $w = \nabla f(Cx)$, $w^T = \nabla f(Cx^+)$ and $x^+ = Ax + Bw$, for arbitrary $\lambda \in [0, \rho^2]$. 

**B. Convex synthesis of algorithms.**

The following theorem reformulates the condition (6) in Theorem 10 using the established bound in Lemma 12.

**Theorem 13** (Analysis Inequalities). Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times d}$ and $C \in \mathbb{R}^{d \times n}$ be given. Set $\hat{A} = A + BMC$. Then the gradient-based algorithm (1) solves Problem 1 and has convergence rate $\rho \in [0, 1]$, if there exist $P = P^T > 0$, $\lambda \in [0, \rho^2]$ and $r \in \mathbb{R}$ such that the constraint (3), i.e. $I_d = C(\hat{A} - I) - 1BM$, is satisfied and

$$\left( \begin{array}{c} I_n \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ I_d \end{array} \right)$$

$$\left( \begin{array}{c} \hat{A} \\ B \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

$$\left( \begin{array}{c} 0 \\ I_d \end{array} \right)$$

and

$$\left( \begin{array}{c} \hat{A} \\ B \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

$$\left( \begin{array}{c} 0 \\ I_d \end{array} \right)$$

$$\left( \begin{array}{c} 0 \\ I_d \end{array} \right)$$

$$\left( \begin{array}{c} 0 \\ I_d \end{array} \right)$$

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$$\left( \begin{array}{c} 0 \\ I_d \end{array} \right)$$

$$\left( \begin{array}{c} 0 \\ I_d \end{array} \right)$$

is satisfied, where $\bar{L} = L - M$ and $\Pi = \Pi_{\ker(L-M)}$.

**Theorem 14** (Synthesis Inequalities). Let $n \geq 3d$. Then there exist matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times d}$, $C \in \mathbb{R}^{d \times n}$, which render the conditions (7) and (3) in Theorem 13 for a given convergence rate $\rho$ feasible, if and only if there exist $\hat{A} = A + BMC$, such that

$$\begin{bmatrix} \rho^2P_{11} & 0 \\ 0 & \rho^2P_{22} \end{bmatrix} - \begin{bmatrix} \rho^2P_{12} & \rho^2P_{13} \\ \rho^2P_{23} & \rho^2P_{24} \end{bmatrix} \begin{bmatrix} \rho^2P_{11} & \rho^2P_{12} \\ \rho^2P_{21} & \rho^2P_{22} \end{bmatrix}$$

are satisfied, where $J_1 = (I_d, 0)$, $J_2 = (J_1, I_d)$, $J_3 = (0, 0)$. The algorithm parameter $A, B, C$ can be obtained by

$$B = P_{11}^{-1}B, \quad A = P_{11}^{-1}\hat{A} - BMC.$$

Finally, we want to show that the above matrix inequalities are always feasible, by analyzing or designing gradient algorithms.

**Theorem 15** (Existence of Solutions). The following four statements are equivalent:

i) The matrices $M, L$ are non-singular and satisfy $M \preceq_c L$.

ii) The gradient method defined by (A, B, C) with $A = I_d$, $B = -2(L + M)^{-1}$, $C = I_d$ fulfills the conditions (7) and (3) of Theorem 13 for any $\rho \in [\rho_{\text{grad}}, 1]$, where $\rho_{\text{grad}} = \rho((L + M)^{-1}(L - M))$.

iii) For all $n \geq d$, there exists an algorithm (A, B, C), $f \in S(M, L)$ and $\rho \in [0, 1]$ such that the conditions (7) and (3) in Theorem 13 are satisfied.

iv) For all $n \geq 3d$ there exists a solution to (8) and (9) in Theorem 13 for some $\rho \in [0, 1]$.

If one optimizes simultaneously over $A, B, C$ and $\rho$ in Theorem 13 or Theorem 14 then $\rho_{\text{grad}}$ is usually not the optimal rate. Often, there exist faster algorithms. However, it is the optimal rate for the gradient method from Proposition 6.

**C. Comparison to IQC based approaches**

As already mentioned in the introduction, there exist quite some publications on the application of methods from robust control theory to gradient-based optimization. Some of these approaches use a technique called IQCs (integral quadratic...
constraints). The goal of this subsection is to explain the relation between IQC based approaches (such as in [5], [9]) and the Lyapunov based approach as in this paper. For this purpose, we will restrict ourselves to the special case of \( S(m, l) \) with scalar \( m, l \in \mathbb{R}_{\geq 0} \), which is usually considered in the literature.

The main steps of IQC based approaches are summarized in the subsequent paragraphs:

(a) Let \( y \in \ell_2, \rho \) be signals with associated \( z \)-transforms \( \hat{y}(z) \) and \( \hat{u}(z) \). Then these signals are said to satisfy the \( \rho \)-IQC defined by a Hermitian complex-valued function \( \Pi \) if

\[
\int_0^{2\pi} \left( \hat{y}(pe^{i\omega}) \Pi(pe^{i\omega}) \hat{u}(pe^{i\omega}) \right) \, d\omega \geq 0. \tag{10}
\]

A bounded causal operator \( \Delta \) satisfies the \( \rho \)-IQC defined by \( \Pi \) if (10) holds for all \( y \in \ell_2 \) and \( u = \Delta(y) \).

IQC(\( \Pi, \rho \)) denotes the set of all \( \Delta \) that satisfy the \( \rho \)-IQC defined by \( \Pi \).

(b) Next view the gradient-based algorithm from Problem 2 as an interconnection of the linear system defined by the transfer function \( G(z) = C(zI_n - A)^{-1}B \) and the static nonlinearity defined by \( (y_k)_{k \in \mathbb{N}_0} \rightarrow (\nabla f(y_k))_{k \in \mathbb{N}_0} =: \Delta(y) \). It is well-known (see for example [5]) that \( \Delta \) satisfies the IQC, i.e. an operator (system) \( \Delta : \ell_2^d \rightarrow \ell_2^d \) which is static and slope restricted in the sector \([m, l]\) satisfies the IQC defined by the multiplier

\[
\Pi(z) = \Psi^*(z)R\Psi(z), \tag{11}
\]

where the factorization is given by

\[
\Psi(z) = \left( (I - \rho) - l - \rho z \right) - 1, R = \begin{bmatrix} I_d & 0 \\ -2I_d & I_d \end{bmatrix}. \]

(c) Finally, the following result from IQC theory (see e.g. [1], [9]), which is based on the exponential weighting operators \( \rho_\tau : \ell_2, (\mathbb{N}_0) \rightarrow \ell_2, (\mathbb{N}_0), (u_k) \rightarrow (\rho^k u_k) \), is invoked to verify convergence of the algorithm with rate \( \rho \).

**Theorem 16** (Exponential stability with IQCs). Fix \( \rho \in [0, 1] \). Let \( G \) be a stable, causal linear dynamical system with transfer function \( G(z) \) of all poles of \( G \) contained in \( \mathbb{C}_{|z|<\rho} \). Let further \( \Delta \) be a stable, causal dynamical system such that \( \Delta' = \rho_- \circ \Delta \circ \rho_+ \) is a bounded operator. Suppose that:

i) for all \( \tau \in [0, 1] \), the interconnection of \( G \) and \( \tau \Delta \) is well posed,

ii) for all \( \tau \in [0, 1] \), we have \( \tau \Delta \in \text{IQC}((\Pi, \rho)) \),

iii) the following frequency domain inequality (FDI) holds:

\[
\begin{bmatrix} G(z) \\ I \end{bmatrix}^* \Pi(z) \begin{bmatrix} G(z) \\ I \end{bmatrix} < 0, \quad \forall z \in \mathbb{C}_{|z|=\rho}. \tag{12}
\]

Then, the feedback interconnection of \( G \) and \( \Delta \) is exponentially stable with rate \( \rho \).

The connection to the proposed Lyapunov based approach can now be established by applying Theorem 16 to Problem 2.

- The interconnection of \( G \) and \( \nabla f \) is always well posed, because \( G \) is strictly proper and \( \nabla f \) has relative degree zero.
- Condition ii) is always satisfied for \( \nabla f \) with \( f \in S_0(0, lI_d - mI_d) \). (Note, that ii) is not satisfied for \( f \in S(mI_d, I_d) \).
- Finally, boundedness of \( \Delta' \) is a consequence of the Lipschitz continuity of \( \nabla f \).

The following Lemma states the relation between the frequency domain inequality (12) in iii) and the matrix inequality (7) from Theorem 13.

**Lemma 17** (Relation between IQC and Lyapunov-based approach). The FDI (12) and \( \text{FDD} \) hold if and only if (7) is feasible for some \( \lambda \in [0, \rho^2] \).

All together, it is possible to prove Theorem 13 using Theorem 16 and Lemma 17.

IV. EXAMPLES AND NUMERICAL RESULTS

A. Convergence rates

To demonstrate the performance of our synthesis, we apply it to the class \( S(m, l) \) of strongly convex functions, which is often considered in the literature (for example in [5], [9]). The algorithm parameters \((A, B, C)\) are computed by solving (8) and (9) in Theorem 14 for \( \lambda = \rho^2 \), where \( \rho \) is optimized using a bisection search. Here, setting \( \lambda \) equal to \( \rho^2 \) is motivated by the proof of Lemma 12 where \( \lambda = \rho^2 \) gives the sharpest estimate on the increment of the Lyapunov function. The obtained convergence rates are shown in Figure 1 where they are compared to the convergence rates of the Triple Momentum method from [14] and the theoretical lower bound on the convergence rates obtained by Nesterov. As can be observed, our synthesized algorithm has the same convergence rates as the Triple Momentum method. A result, that is also obtained in [6] using an IQC based approach.

Strictly speaking, the synthesis with Theorem 14 is not an LMI synthesis if we consider \( \lambda \) as a decision variable. This parameter could possibly be optimized using a line search algorithm, which we did in the first place. However, in our empirical experiments, we found that in the case \( M \gg 0 \) the value \( \lambda = \rho^2 \) was always the optimal one.

B. Structured objective functions

The following (academic) example shall demonstrate the possible benefits of including additional properties of the objective function into algorithm design compared to the design for \( S(m, l) \). Consider the class of functions \( S(M, L) \) with

\[
M = \begin{bmatrix} l - m + \frac{m^2}{2} & 0 \\ 0 & m \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 0 & 2m - \frac{m^2}{2} \end{bmatrix}, \quad S = \begin{bmatrix} \sqrt{1 - \left( \frac{m}{2} \right)^2} & -\frac{m}{2} \\ \frac{m}{2} & \sqrt{1 - \left( \frac{m}{2} \right)^2} \end{bmatrix}.
\]

These matrices fulfill \( mI \leq M \preceq L \leq lI \). Moreover, the largest eigenvalue of \( L \) is \( l \) and the smallest eigenvalue
Proposition 6 and by synthesis with Theorem 14 for Fig. 1. The convergence rate guarantees achieved by designing algorithms using Theorem 14 are plotted over the condition number \( l/m \) and compared to the rate bound of Triple Momentum \( \rho = 1 - \frac{\sqrt{m}}{\sqrt{m} + \sqrt{m}} \) and the theoretical lower bound \( \rho = \frac{\sqrt{m}}{\sqrt{m} + \sqrt{m}} \) from [10].

One can recognize that in this example the structured method and has a convergence rate \( \rho \) using Theorem 14 is fulfilled, then \( S(M, L) \) is a class of functions with unique critical (saddle) points. One particular saddle point problem can be obtained in the context of convex constrained optimization. If one aims to solve the (linearly) constrained optimization problem

\[
\begin{align*}
\text{minimize} & \quad f(x), \\
\text{subject to} & \quad x \in \mathbb{R}^d, \quad A_{eq} x = b_{eq},
\end{align*}
\]

where \( f \in S(M, L), A_{eq} \in \mathbb{R}^{d_x \times d} \) and \( 0 < M \prec L \) holds (such that \( f \) is strictly convex), then a solution can be found by solving the saddle point problem

\[
\sup_{\lambda \in \mathbb{R}^{d_2}} \inf_{x \in \mathbb{R}^d} f(x) + \lambda^T (A_{eq} x - b_{eq}).
\]

Here, the Lagrangian function \( L(x, \lambda) = f(x) + \lambda^T (A_{eq} x - b_{eq}) \) is an element of \( S(M_L, L_L) \), where

\[
M_L = \begin{pmatrix} M & A_{eq} \\ A_{eq} & 0 \end{pmatrix}, \quad L_L = \begin{pmatrix} L & A_{eq}^T \\ A_{eq} & 0 \end{pmatrix}.
\]

If \( M_L \preceq_c L_L \) is satisfied, then our design procedure can be applied to design a gradient based algorithm for \( L \), which solves the constrained optimization problem. The following lemma shows under rather mild conditions that this is possible.

**Lemma 18.** Let \( A_{eq} \in \mathbb{R}^{d_x \times d} \) and symmetric matrices \( M, L \in \mathbb{R}^{d \times d} \) be given. Consider \( M_L, L_L \) defined in (14) and assume that \( M \preceq_c L \) holds with \( M, L \) being non-singular and that \( A_{eq} \) has full row rank. Then \( M_L \preceq_c L_L \) holds, and \( M_L \) and \( L_L \) are non-singular.

As an academic example, consider the constrained optimization problem (13) with \( f \in S(m I_2, I_2) \) and \( A_{eq} = (1 \ 1) \). As described above, matrices \( M_L \preceq_c L_L \) can be constructed such that the Lagrangian \( L \) of this problem is in \( S(M_L, L_L) \). This enables algorithms of the form \( x_{k+1} = A x_k + B \nabla L(C x_k) \) to be designed. The algorithm
parameters $A, B, C$ can be designed by solving the matrix inequality from Theorem 14. The results are presented in Figure 3. For the sake of comparison, we added the rates of the descent algorithm from Proposition 6. Interestingly, the convergence rates are exactly equal to the convergence rates for the unconstrained optimization problems. In general, we observed in our experiments that the convergence rates for linearly constrained optimization problems were often faster than those for unconstrained problems, but never slower. Notice that we have the condition $n \geq 3d$ in Theorem 14, hence the algorithm with one equality constraint has at least dimension 9. However, it is often possible to reduce the dimension of the algorithm as outlined below. For example, we consider the algorithm parameters $A, B, C$ for $m = 1, l = 15$ designed using Theorem 14. The original matrices had dimension $n = 9$. We observed that the last three modes usually do not contribute much to the dynamics of the algorithm. Hence, it is possible to eliminate them using balanced truncation. In our example, we obtained the reduced parameters:

$$
A = \begin{pmatrix}
1 & 0 & 0 & 0.0135 & -0.0258 & -0.0017 \\
0 & 1 & 0 & 0.0135 & 0.0258 & -0.0017 \\
0 & 0 & 1 & -0.0676 & -0.0036 & -0.0363 \\
0 & 0 & 0 & -0.3097 & -0.0042 & -0.0474 \\
0 & 0 & 0 & -0.0039 & 0.3909 & -0.0002 \\
0 & 0 & 0 & 1.1631 & 0.0070 & 0.5255
\end{pmatrix}
$$

$$
B = \begin{pmatrix}
-0.0846 & 0.0707 & -0.1978 \\
0.0707 & -0.0846 & 0.1978 \\
-0.2758 & -0.2758 & -3.2399 \\
0.0860 & 0.0940 & -4.7039 \\
0.6738 & -0.6727 & -0.0264 \\
0.0896 & 0.0000 & 6.3240
\end{pmatrix}
$$

$$
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

We used Theorem 13 to check that the reduced algorithm still converges for $S(m = 1, l = 15)$. The reduced algorithm achieves a convergence rate of at least 0.7422, which is faster than the rate of gradient descent, which is 0.8750 and exactly as fast as the unreduced algorithm. In this example, we have chosen a specific representation in which the first $d$ columns of $A$ are the first $d$ unit vectors in $\mathbb{R}^n$ and $C$ takes the form of an identity matrix concatenated with a zero block. The existence of such a representation is guaranteed by (3). From this specific representation, it can be extracted that $A$ will always have $d$ eigenvalues at one. The modes with one eigenvalues play the role of a memory for the current best guess of the optimization algorithm and are therefore necessary.

V. Conclusion

We presented a convex synthesis procedure to design gradient-based algorithms based on a general class of Lur’e Lyapunov functions and linear matrix inequalities. The class of objective functions, which was considered, generalizes the class of strongly convex functions and offers the possibility to incorporate additional information into the algorithm design. It should be emphasized that this class of functions also includes non-convex functions - in particular functions with saddle points. The usefulness of our novel function class was demonstrated, firstly, by showing that additional information about the objective function can boost the convergence rate of algorithms considerably and, secondly, by showing that it can be used to design algorithms for solving optimization problems with linear equality constraints.

Open future research questions are for example the design of distributed algorithms or the design of optimization algorithms for problems with inequality constraints.

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APPENDIX

A. Projections and pseudo inverses

The pseudo inverse $L^\dagger$ and projection matrix $\Pi_{\ker L}/\Pi_{\im L}$ onto the kernel/image of a symmetric matrix $L$ are used at several places in the proofs of this paper. Hence, some important formulas are summarized below. Let $A = U^T \Sigma V$ be the singular value decomposition of a matrix $A$, then

$$A^\dagger = \begin{pmatrix} V_1/V_2 \\ \Sigma \end{pmatrix}^T \begin{pmatrix} \sigma_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r^{-1} \end{pmatrix} \begin{pmatrix} U_1/U_2 \\ \Sigma \end{pmatrix}$$

and $\Pi_{\ker A} = V_2^T V_2$, $\Pi_{\im A} = U_1 U_1^T$. We will particularly be interested in the following four identities for the projectors and pseudo inverses of a symmetric positive semidefinite matrix $L$, $r \neq 0$:

1. $\Pi_{\im L} = LL^\dagger = L^\dagger L$, (15)
2. $I_d = \Pi_{\im L} + \Pi_{\ker L}$, (16)
3. $(L + r \Pi_{\ker L})^{-1} = L^\dagger + \frac{1}{r} \Pi_{\ker L}$, (17)
4. $(L^\dagger + r \Pi_{\ker L})^{-1} = L + \frac{1}{r} \Pi_{\ker L}$. (18)

These identities follow from the singular value decomposition as shown above.

B. Proof of Theorem 7

Step 1. First assume that $(A, B, C)$ solves Problem 1. We prove that $(\tilde{A}, B, C)$ solves Problem 2. Let $g \in S_0(0, L) = S_0(0, L - M)$ be an arbitrary function. Then

$$f(z) = g(z) + \frac{1}{2} z^T M z$$

is an element of $S(M, L)$ with $\nabla f(0) = 0$. Now consider the iterates of algorithm (1) with the parameters $(\tilde{A}, B, C)$ for the objective functions $g:

$$x_{k+1} = \tilde{A} x_k + B \nabla g(C x_k)$$

$$= Ax_k + B (\nabla g(C x_k) + MC x_k)$$

$$= Ax_k + B \nabla f(C x_k).$$

Since those are the iterates of the algorithm defined by $(A, B, C)$ for $f \in S(M, L)$, we know that $x_k$ converges to $x^*_f$ at rate $\rho$ for any $x_0 \in \mathbb{R}^d$. Notice that $x^*_f$ must be zero in this case because zero is a fixed-point of the considered iteration (since $A0 + B \nabla f(C0) = 0$ by $\nabla f(0) = 0$) and hence, if $x^*_f$ were not zero, then the iterates for $x_0 = 0$ would not converge to $x^*_f$. It remains to show satisfaction of the constraint (3). For this purpose define $f \in S(M, L)$ as

$$f(z) = \frac{1}{2} (z - z^*_f)^T M (z - z^*_f)$$

for some $z^*_f \in \mathbb{R}^d$ and check that it satisfies $\nabla f(z^*_f) = 0$. By assumption, Problem 1 is solved, meaning that the iterates

$$x_{k+1} = A x_k + B M (C x_k - z^*_f)$$

$$= A x_k - B M z^*_f$$

of algorithm (1) converge to $x^*_f$ for any $x_0$. This implies, that $x^*_f$ is a solution of the fixed point equation

$$x^*_f = \tilde{A} x^*_f - B M z^*_f.$$

The convergence for arbitrary initial value implies that $\tilde{A}$ is Schur and, hence, $\tilde{A} - I_n$ must be non-singular. Then, the fixed point equation can be solved for $x^*_f$:

$$x^*_f = (\tilde{A} - I_n)^{-1} B M z^*_f.$$

By assumption, we have in addition

$$z^*_f = C x^*_f = C (\tilde{A} - I_n)^{-1} B M z^*_f.$$

Since $z^*_f$ is arbitrary, the constraint $C (\tilde{A} - I_n)^{-1} B M = I_d$ must hold.

Step 2. Now assume that $(\tilde{A}, B, C)$ is a solution of Problem 2. We prove that $(A, B, C)$ solves Problem 1. For that, we first consider all functions $f \in S(M, L)$ for which there exists a critical point $z^*_f$. Let $f \in S(M, L)$ be given such that there exists $z^*_f$ with $\nabla f(z^*_f) = 0$. Then $g$ defined by $g(z) = f(z + z^*_f) - \frac{1}{2} z^T M z$ is an element of $S_0(0, L - M) = S_0(0, \tilde{L})$. Hence, the iterative scheme

$$x_{k+1} = \tilde{A} x_k + B \nabla g(C x_k)$$

converges to zero at rate $\rho$ for any $x_0 \in \mathbb{R}^d$. Now add $x^*_f := (\tilde{A} - I_n)^{-1} B M z^*_f$ on both sides of the above equation and consider the new sequence $x_k := \tilde{x}_k + x^*_f$:

$$x_{k+1} = \tilde{x}_{k+1} + x^*_f$$

$$\tilde{\tilde{x}}_{k+1} = \tilde{A} (\tilde{x}_k + x^*_f) + B \nabla g(C \tilde{x}_k) + x^*_f - \tilde{A} x^*_f$$

$$= A x_k + B M C x_k + B \nabla g(C \tilde{x}_k) - B M z^*_f$$

$$= A x_k + B \nabla f(C \tilde{x}_k) + M C \tilde{x}_k.\right)$$

This is the equation for the iterates $x_k$ of the algorithm defined by $(A, B, C)$ and $f$. Since $\tilde{x}_k$ goes to zero at rate $\rho$, so does $x_k$ go to $x^*_f$. Finally, we argue that there cannot be an element of $S(M, L)$ with no critical point: If there were an $f \in S(M, L)$ with two critical points, then the above arguments would prove convergence of algorithm (1) to both critical points, which cannot be true. Hence, there exists no such function in $S(M, L)$. Consequently, Theorem 8 guarantees that any function in $S(M, L)$ has a critical point and thus $(A, B, C)$ solve Problem 1. (At this point the forward reference to Theorem 8 can only be avoided by considerable effort. Also note that the proof of Theorem 8 does in no way require Theorem 1.)
\textbf{C. Proof of Theorem 2}\]

2) \(\implies\) 1): The key to prove this statement is that the second term in inequality 1) can be written as the following integral:

\[
\int_0^1 (\nabla f(z_1 + t(z_2 - z_1)) - \nabla f(z_1))^T (z_2 - z_1) \, dt
\]

= \(f(z_2) - f(z_1) - (\nabla f(z_1))^T (z_2 - z_1)\).

Using 2), the integrand can be upper and lower bounded as follows

\[
0 \leq (\nabla f(z_1 + t(z_2 - z_1)) - \nabla f(z_1))^T (z_2 - z_1)
\]

\[
\leq \frac{1}{t} \|\nabla f(z_1 - z_2)\|_L^2 = \|\nabla f(z_1)\|_L^2, t \leq \|z_1 - z_2\|_L^2
\]

which implies 1).

1) \(\implies\) 3): Let \(f \in C^1(\mathbb{R}^d)\) fulfill 1). Define \(g(t) = f(z) - (\nabla f(z_1))^T z\). Then \(g \in S(0, L)\) and \(\nabla g(z_1) = 0\). Thus \(z_1\) is a minimizer of \(g\) and we have

\[
g(z_1) - g(z_2) \leq g(z_2 - A\nabla g(z_2)) - g(z_2)
\]

\[
\leq \frac{1}{2} \|A\nabla g(z_2)\|_L^2 - \nabla g(z_2)^T A\nabla g(z_2),
\]

for any matrix \(A \in \mathbb{R}^{d \times d}\) or equivalently

\[
\nabla g(z_2)^T A\nabla g(z_2) - \frac{1}{2} \|A\nabla g(z_2)\|_L^2 \leq g(z_2) - g(z_1).
\]

Now, we substitute \(g(z_2) = f(z_2) - (\nabla f(z_1))^T z_2\):

\[
(\nabla f(z_1) - \nabla f(z_2))^T A(\nabla f(z_1) - \nabla f(z_2))
\]

\[
\leq \frac{1}{2} \|A(\nabla f(z_1) - \nabla f(z_2))\|_L^2
\]

\[
\leq f(z_2) - f(z_1) + (\nabla f(z_1))^T (z_1 - z_2).
\]

For \(A = L^\dagger\), this is equivalent to

\[
\frac{1}{2} \|\nabla f(z_1) - \nabla f(z_2)\|_L^2 + f(z_2) - f(z_1) + (\nabla f(z_1))^T (z_1 - z_2).
\]

In the case \(A = r\Pi_{ker} L\), the result is

\[
r(\nabla f(z_1) - \nabla f(z_2))^T \Pi_{ker} L(\nabla f(z_1) - \nabla f(z_2))
\]

\[
\leq f(z_2) - f(z_1) + (\nabla f(z_1))^T (z_1 - z_2),
\]

which implies \(\Pi_{ker} L(\nabla f(z_1) - \nabla f(z_2)) = 0\), because \(r\) can be chosen arbitrarily large.

3) \(\implies\) 4): Adding the following inequalities \(\frac{1}{t} \|\nabla f(z_1) - \nabla f(z_2)\|_L^2 \leq f(z_2) - f(z_1) + (\nabla f(z_1))^T (z_2 - z_1)\) and \(\frac{1}{2} \|\nabla f(z_1) - \nabla f(z_2)\|_L^2 \leq f(z_1) - f(z_2) + (\nabla f(z_2))^T (z_2 - z_1)\) yields inequality in 4).

4) \(\implies\) 2): Let \(f \in C^1(\mathbb{R}^d)\) fulfill 4). Then

\[
\sqrt{L} \sqrt{L}(\nabla f(z_1) - \nabla f(z_2)) = (\nabla f(z_1) - \nabla f(z_2))
\]

holds for all \(z_1, z_2 \in \mathbb{R}^d\), because \(\Pi_{ker} L(\nabla f(z_1) - \nabla f(z_2)) = 0\) implies, that \(\nabla f(z_1) - \nabla f(z_2)\) is in the image of \(L\). This observation can be used to derive the bound using the Cauchy-Schwarz-Inequality (CSI)

\[
\|\nabla f(z_1) - \nabla f(z_2)\|_L^2 \leq (\nabla f(z_1) - \nabla f(z_2))^T (z_1 - z_2)
\]

\[
\leq (\nabla f(z_1) - \nabla f(z_2))^T \sqrt{L} \sqrt{L}(z_1 - z_2)
\]

\[
\leq \|\nabla f(z_1) - \nabla f(z_2)\|_L \|z_1 - z_2\|_L.
\]

This implies \(\|\nabla f(z_1) - \nabla f(z_2)\|_L \leq \|z_1 - z_2\|_L\). Now, \(f\) fulfills 2), because

\[
(\nabla f(z_1) - \nabla f(z_2))^T (z_1 - z_2)
\]

\[
\leq \|\nabla f(z_1) - \nabla f(z_2)\|_L \|z_1 - z_2\|_L
\]

\[
\leq \|z_1 - z_2\|_L.
\]

\[
\square
\]

\textbf{D. Proof of Lemma 3}\]

1) \(\implies\) 2): Let \(Q\) with \(M \preceq Q \preceq L\) be given and let \((\lambda_1^{(M)})_{i=1}^d, (\lambda_1^{(Q)})_{i=1}^d, (\lambda_1^{(L)})_{i=1}^d\) be the eigenvalues of those matrices in ascending order. It follows from \(M \preceq Q \preceq L\) and the theorem of Courant-Fischer that

\[
\lambda_1^{(M)} \leq \lambda_1^{(Q)} \leq \ldots \leq \lambda_d^{(M)} \leq \lambda_d^{(Q)} \leq \lambda_d^{(L)}
\]

holds. Since \(\lambda_i^{(M)}\) and \(\lambda_i^{(L)}\) always have the same sign and are not equal to zero by assumption, the values \(\lambda_i^{(Q)}\) cannot be zero for any \(i\). Hence, no eigenvalue of \(Q\) can be zero and hence, \(Q\) is invertible.

2) \(\implies\) 3): To show the first statement, consider the case \(Q = \frac{1}{2}(M + L)\). Then, it holds that \(M \preceq Q \preceq L\) and hence, \(Q = \frac{1}{2}(M + L)\) is invertible. To show the second statement, consider the case \(Q = \frac{1}{2}(M + L) + \frac{1}{2}(L - M)\). For \(\alpha \in [-1, 1]\), it holds that \(M \preceq Q \preceq L\) and thus

\[
0 \neq \det \left( \frac{1}{2}(M + L) + \alpha \frac{1}{2}(L - M) \right) \quad \forall \alpha \in [-1, 1].
\]

By non-singularity of \((M + L)\), the factor \(\frac{1}{2}(M + L)\) can be pulled out of the above expression, which gives

\[
0 \neq \det \left( \frac{1}{2}(M + L) \right) \det (I + \alpha(M + L)^{-1}(L - M))
\]

and consequently

\[
0 \neq \det (I + \alpha(M + L)^{-1}(L - M)) \quad \forall \alpha \in [-1, 1].
\]

This implies, that \((M + L)^{-1}(L - M)\) cannot have an eigenvalue in \(\mathbb{R} \setminus [-1, 1]\). However, because \((M + L)^{-1}(L - M)\) is similar to the symmetric matrix \(\sqrt{L - M}(M + L)^{-1}\sqrt{L - M}\), all of its eigenvalues have to be real. (Note that \(\sqrt{L - M}\) exists because \(L - M\) is positive semidefinite.) Hence, all eigenvalues of \((M + L)^{-1}(L - M)\) have to be in \([-1, 1]\) and thus also \(\rho((M + L)^{-1}(L - M)) < 1\) holds.

3) \(\implies\) 4): Suppose that \(M\) is not invertible, i.e. there exists a vector \(z \in \mathbb{R}^d \setminus \{0\}\) with \(Mz = 0\). Then

\[
(L + M)z = (L - M)z \implies z = (L + M)^{-1}(L - M)z
\]

implies that \(z\) is an eigenvector to the eigenvalue 1 of \((L + M)^{-1}(L - M)\), which contradicts \(\rho((L + M)^{-1}(L - M)) < 1\). Hence \(M\) is non-singular.

Next we show \(\sigma(M^{-1} L) \subseteq \mathbb{R} \setminus \{0\}\). Consider the identity

\[
(L + M)^{-1}(L - M) = I - 2(L + M)^{-1}M
\]

\[
= I - 2(M^{-1}L + I)
\]

Suppose, that \(M^{-1}L\) has an eigenvalue \(\lambda\) with associated eigenvector \(v\). Then \(M^{-1}L + I\) has eigenvalue \(\lambda + 1\) with
eigenvector $v$ and $(M^{-1}L + I)^{-1}$ has eigenvalue $\frac{\lambda - 1}{\lambda + 1}$ with eigenvector $v$. Thus

$$(L + M)^{-1}(L - M)v = (I - 2(M^{-1}L + I)^{-1})v = v - \frac{2}{\lambda + 1}v = \frac{\lambda - 1}{\lambda + 1}v.$$ 

Hence, $\frac{\lambda - 1}{\lambda + 1}$ is an eigenvalue of $(L + M)^{-1}(L - M)$ and thus it is in $]-1, 1[$. This implies $\lambda \in \mathbb{R}_{>0}$. Hence $\sigma(M^{-1}L) \subseteq \mathbb{R}_{>0}$ holds true.

4) $\Rightarrow$ 5): Suppose, that $LM^{-1}$ has only positive eigenvalues. Then there exists a symmetric positive definite matrix $P \in \mathbb{R}^{d \times d}$ such that the Lyapunov inequality

$$PLM^{-1} + M^{-1}LP > 0$$

is satisfied. A congruence transform with $M$ gives

$$MPL + LPM > 0.$$ 

By Lemma 20 we can infer that $M$ and $L$ are congruent.

5) $\Rightarrow$ 1): By Sylvester’s Law of Inertia, matrices have the same eigenvalue signature, if and only if they are congruent.

E. Proof of Proposition 6

We prove the contraction property of the map $\phi : z \mapsto z - 2(M + L)^{-1}\nabla f(z)$, by using the norm $\|z\|_P^2 = z^TPz$,

where $P = (L + M)((L - M)^\dagger + rI_{ker(L - M)})(L + M)$.

In a first step, rewrite $\phi$ as:

$$\phi(z) = (L + M)^{-1}(L + M)z - 2(L + M)^{-1}\nabla f(z) = (L + M)^{-1}((L - M)z - 2(\nabla f(z) - Mz))$$

with $g \in S(0, L - M)$ defined by $g(z) := f(z) - \frac{1}{2}z^TMz$. Consider now

$$\|\phi(z) - \phi(z_2)\|_P^2$$

$$= \|((L - M)^\dagger + rI_{ker(L - M)})^\frac{1}{2}(L + M)(L + M)^{-1}((L - M)(z_1 - z_2) - 2(\nabla g(z_1) - \nabla g(z_2)))\|_P^2$$

$$= \|\sqrt{L - M}(z_1 - z_2) - 2\sqrt{L - M}((\nabla g(z_1) - \nabla g(z_2)))\|_P^2,$$

where $g(z) := f(z) - \frac{1}{2}z^TMz$.

Concerning $(*)$ notice, that the kernel projector has no contribution, since the products are all zero and the underbraced expression being non-positive follows from Lemma 2. Finally, by Lemma 13 we know that for any $\rho > \rho_{grad}$ there exists some $r \in \mathbb{R}_{>0}$ such that $L - M \leq \rho^2P$ holds. Hence, we can overestimate $\|z_1 - z_2\|_{L - M}^2$ by $\rho^2\|z_1 - z_2\|_P^2$ (by choosing a sufficient value of $r$) resulting in the final estimate:

$$\|\phi(z) - \phi(z_2)\|_P^2 \leq \|z_1 - z_2\|_{L - M}^2 \leq \rho^2\|z_1 - z_2\|_P^2.$$ 

F. Proof of Theorem 8

Non-emptiness of $S(M, L)$ is equivalent to $M \preceq L$. It remains to show that the three statements in the theorem are equivalent under the condition $M \preceq L$.

1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3):

Assume $M \preceq L$ are non-singular. Let $f \in S(M, L)$ be given. Then, by Proposition 4 the mapping

$$\phi : z \mapsto z - 2(M + L)^{-1}\nabla f(z)$$

is a contraction on $\mathbb{R}^d$ and $(M + L)$ is non-singular. By the Banach fixed point theorem the mapping $\phi$ has exactly one fixed point $z^*_f$ with $\phi(z^*_f) = z^*_f \iff \nabla f(z^*_f) = 0$. This implies 2) and 3).

2) $\Rightarrow$ 1) and 2) $\Rightarrow$ 3):

Suppose that $M \preceq L$ does not hold or that either $M$ or $L$ or both are singular, but $M \preceq L$ holds (such that $S(M, L)$ is non-empty). Then there exists $Q = Q^\dagger \in \mathbb{R}^{d \times d}$ with $M \preceq Q \preceq L$ and det $Q = 0$ by Lemma 3. Let $v \in \mathbb{R}^d \setminus \{0\}$ be an element of the kernel of $Q$. Then the function $f_1 \in S(M, L)$ defined by $f_1(z) = \frac{1}{2}z^TQz + v^Tv$ has no critical point with $\nabla f(z) = 0$, because otherwise

$$v^T\nabla f_1(z) = v^T(Qz + v) = v^TQz + v^Tv = \|v\|^2$$

would have to be zero. At the same time, the function $f_2 \in S(M, L)$ defined by $f_2(z) = \frac{1}{2}z^TQz$ has infinitely many critical points with $\nabla f_2(z) = 0$, because any point $z = rv$ with $r \in \mathbb{R}$ is a critical point of $f$ by $\nabla f_2(z) = rQv = 0$.

G. Proof of Lemma 17

Step 1 (lower bound). The term $f(Cx) - f(0) - \frac{1}{2}\nabla f(Cx)^T\tilde{L}^\dagger\nabla f(Cx)$ can be lower bounded by the estimate

$$0 \leq f(Cx) - f(0) - \frac{1}{2}\|\nabla f(Cx)\|_L^2 - (\nabla f(0))^T\tilde{L}^\dagger\nabla f(Cx).$$

where the inequality sign follows from Lemma 2 and the equality sign follows from the fact $\nabla f(0) = 0$. This allows now the following lower bound on $V_f$:

$$V_f(x) = \left(\begin{array}{c} x \\ \nabla f(Cx) \end{array}\right)^T \left(\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array}\right) \left(\begin{array}{c} x \\ \nabla f(Cx) \end{array}\right)$$

$$+ f(Cx) - f(0) - \frac{1}{2}\nabla f(Cx)^T\tilde{L}^\dagger\nabla f(Cx)$$

$$\geq \left(\begin{array}{c} x \\ \nabla f(Cx) \end{array}\right)^T \left(\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array}\right) \left(\begin{array}{c} x \\ \nabla f(Cx) \end{array}\right)$$

$$\geq \lambda_{\min}(P) \left\|\left(\begin{array}{c} x \\ \nabla f(Cx) \end{array}\right)\right\|^2$$

$$\geq \lambda_{\min}(P) \|x\|^2.$$
Step 2 (upper bound). The term $f(Cx) - f(0) - \frac{1}{2} \nabla f(Cx)^T \bar{L} \nabla f(Cx)$ can be upper bounded by the following estimates:

$$f(Cx) - f(0) - \frac{1}{2} \nabla f(Cx)^T \bar{L} \nabla f(Cx) \leq 0$$

$$\leq f(Cx) - f(0) - \frac{1}{2} \nabla f(0)^T (Cx - 0)$$

$$\leq \frac{\|L - M\|}{2} \|Cx - 0\|^2$$

Note, that in (*) the term $(\nabla f(0))^T (Cx - 0)$ can be added because $\nabla f(0) = 0$. This allows now the following upper bound on $V_f$:

$$V_f(x) = \left( x \nabla f(Cx) \right)^T \left( \begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right) \left( x \nabla f(Cx) \right) + f(Cx) - f(0) - \frac{1}{2} \nabla f(Cx)^T \bar{L} \nabla f(Cx)$$

$$\leq \lambda_{\max}(P) \left( \left\| x \nabla f(Cx) \right\|^2 + \frac{\|L\|}{2} \|Cx\|^2 \right)$$

$$\leq \lambda_{\max}(P)(\|x\|^2 + \|\nabla f(Cx)\|^2) + \frac{\|L\|}{2} \|Cx\|^2$$

$$\leq \left( \lambda_{\max}(P)(1 + \|L\|^2 ||C||^2) + \frac{\|L\||C||^2}{2} \right) \|x\|^2$$

$$= \beta_f \|x\|^2. \quad \Box$$

H. Proof of Lemma 12

We define the abbreviations $w = \nabla f(Cx)$, $w^+ = \nabla f(Cx^+)$, $x^+ = Ax + Bw$ and $\bar{L} = L - M$. With that the $\rho$-weighted increment of the Lyapunov function is

$$V_f(x^+) - \rho^2 V_f(x) =$$

$$= \left( x^+ \right)^T \left( \begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right) \left( x^+ \right) - \rho^2 \left( x^+ \right)^T \left( \begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right) \left( x^+ \right)$$

$$\leq \lambda_{\max}(P) \left( \left\| x^+ \nabla f(Cx) \right\|^2 + \frac{\|L\|}{2} \|Cx\|^2 \right)$$

$$\leq \left( \lambda_{\max}(P)(1 + \|L\|^2 ||C||^2) + \frac{\|L\||C||^2}{2} \right) \|x\|^2$$

To upper bound expression $I$, we use the estimate

$$\frac{\|x\|^2}{2} \leq -\lambda \left( f(Cx) - f(0) - \frac{1}{2} \|w\|^2 \right)$$

which we can use to obtain

$$I \leq (1 - \lambda) \left( f(Cx^+) - f(0) + \frac{1}{2} \|w^+\|^2 \right)$$

$$+ \lambda \left( f(Cx^+) - f(Cx) + \frac{1}{2} \|w^+ - w\|^2 \right)$$

Now, this estimate for expression $I$ can be used to upper bound $V_f(x^+) - \rho^2 V_f(x)$ as follows:

$$V(x^+) - \rho^2 V(x) \leq \left( \begin{array}{cc} x^+ \w^+ \\ w^+ \end{array} \right)^T \left( \begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right) \left( \begin{array}{cc} x^+ \w^+ \\ w^+ \end{array} \right)$$

$$- \rho^2 \left( \begin{array}{cc} x^+ \w^+ \\ w^+ \end{array} \right)^T \left( \begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right) \left( \begin{array}{cc} x^+ \w^+ \\ w^+ \end{array} \right)$$

$$\leq (w^+)^T \left( Cx^+ - \lambda Cx - \bar{L}^\dagger (w^+ - \lambda w) \right),$$

which corresponds to the inequality in Lemma 12 \[ \Box \]

I. Proof of Theorem 13

First remember that Theorem 1 shows that an algorithm with parameters $(A, B, C)$ has convergence rate $\rho$ for $S(M, L)$ if an algorithm with parameters $(A, B, C)$, which satisfy the constraint (3), has convergence rate $\rho$ for $S_0(0, \bar{L})$. Hence, in the following we show convergence for $(A, B, C)$ and $S_0(0, \bar{L})$. By Theorem 10 an algorithm defined by $(A, B, C)$ is asymptotically stable and has convergence rate $\rho$, if there exists a Lyapunov function $V_f : \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$\alpha_f \|x - x_f\|^2 \leq V_f(x) \leq \beta_f \|x - x_f\|^2,$$

$$V_f(x^+) - \rho^2 V_f(x) \leq 0$$

holds for all $x \in \mathbb{R}^n$ and $f \in S_0(0, \bar{L})$ with $\beta_f \geq \alpha_f > 0$. The considered class of Lyapunov function candidates fulfills these requirements by Lemma 11 and Lemma 12 if

$$\left( \begin{array}{c} x \\ w \end{array} \right)^T \left( \begin{array}{ccc} -\rho^2 P_{11} & -\rho^2 P_{12} \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} x \\ w \end{array} \right) + \left( \begin{array}{c} x \\ w \end{array} \right)^T \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} x \\ w \end{array} \right)$$

$$= \left( \begin{array}{c} x \\ w \end{array} \right)^T \left( \begin{array}{ccc} -\rho^2 C & 0 \\ 0 & \frac{\|C\|^2}{2} \end{array} \right) \left( \begin{array}{c} x \\ w \end{array} \right).$$
is smaller than zero for all $x \in \mathbb{R}^n$, $w = \nabla f(Cx)$, $w^+ = \nabla f(Cx^+)$ and $x^+ = Ax + Bw$. At this point we can even improve the estimate by the observation that due to Lemma 2

$$0 = \Pi_{\ker \bar{L}} \nabla f(Cx) = \Pi_{\ker \bar{L}} w,$$

$$0 = \Pi_{\ker \bar{L}} \nabla f(Cx^+) = \Pi_{\ker \bar{L}} w^+$$

hold true. This implies, that the term

$$\begin{pmatrix} x \\ w \\ x^+ \\ w^+ \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r \Pi_{\ker \bar{L}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \Pi \end{pmatrix} \begin{pmatrix} x \\ w \\ x^+ \\ w^+ \end{pmatrix}$$

is zero for all $r \in \mathbb{R}$ and can hence be added (as an additional multiplier) to the estimate. Since the quantities $x, w, x^+, w^+$ are given by

$$\begin{pmatrix} x \\ w \\ x^+ \\ w^+ \end{pmatrix} = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & I_d \end{pmatrix} \begin{pmatrix} x \\ w \\ x^+ \\ w^+ \end{pmatrix}$$

negativity of $V_f(x^+) - \rho^2 V_f(x)$ follows now from inequality (7). Hence, (7) implies that the weighted increment of the Lyapunov function is negative definite and, as a consequence, that the algorithm defined by $(\bar{A}, \bar{B}, \bar{C})$ has convergence rate $\rho$ for $S_0(0, L - M)$.

\section*{I. Proof of Theorem 14}

We need to show that the matrix inequality (8) in the transformed variables $\bar{A}, \bar{B}, \bar{C}, \bar{P}$ is equivalent to (7). The proof of this theorem works in two steps. The first step is to apply the Schur complement to (7). The second (key) step is to define a linearizing change of variables.

Step 1. First, define $\tilde{Z}$ as follows

$$\begin{pmatrix} I_n & 0 & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & I_d \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \bar{C}^T \\ 0 & 0 & -\frac{1}{2} \bar{L}^T \\ -\frac{1}{2} C \bar{A} - \frac{1}{2} C \bar{B} + \frac{1}{2} \bar{L}^T \end{pmatrix} (\star)$$

and

$$\begin{pmatrix} I_n & 0 & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & I_d \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r \Pi \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & I_d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r \Pi \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: Z.$$

With $Z$, (7) becomes

$$\begin{pmatrix} I_n & 0 & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & I_d \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -\rho^2 P_{11} & -\rho^2 P_{12} & 0 \\ -\rho^2 P_{21} & -\rho^2 P_{22} & 0 \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & I_d \end{pmatrix} + Z = \begin{pmatrix} I_n & 0 & 0 & 0 \\ P_{11} \bar{A} & P_{11} \bar{B} & P_{11} \bar{P}_1 \\ P_{21} \bar{A} & P_{21} \bar{B} & P_{21} \bar{P}_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\rho^2 P & \rho^{-1} \end{pmatrix} (\star) + Z < 0.$$

The matrix $P$ is positive definite, by assumption of Theorem 13 and as a consequence of the matrix inequality from Theorem 14. Hence, this algebraic manipulation allows to apply the Schur complement, which states that the above inequality is equivalent to

$$\begin{pmatrix} -\rho^2 P_{11} & -\rho^2 P_{12} & 0 \\ 0 & -\rho^2 P_{21} & -\rho^2 P_{22} & 0 \\ \bar{C} \bar{A} - \frac{1}{2} \bar{C} \bar{B} + \frac{1}{2} \bar{L}^T & \bar{C} \bar{B} - \frac{1}{2} \bar{L}^T & 0 \end{pmatrix}$$

being negative definite.

Step 2. If we have a solution $(\bar{A}, \bar{B}, \ldots)$ of (8) and constraint (9), then we can just substitute $\bar{A} = P_{11}^{-1} \bar{A}, \bar{B} = P_{11}^{-1} \bar{B}$ into (8) and we see that we obtain the above inequality and hence a solution of (7). This solution also satisfies constraint (3) since

$$C(\bar{A} - I_n)^{-1} \bar{B} M = C(\bar{A} - I_n)^{-1} P_{11}^{-1} \bar{B} M$$

$$= C \bar{A} P_{11}^{-1} \bar{B} M$$

$$= C \bar{A} (P_{11}^{-1} \bar{B}) M$$

On the other hand, if we are given a solution of (7), (3) with $P > 0$ and we want to construct a solution of (8) by substituting $\bar{B} = P_{11} \bar{B}, \bar{A} = P_{11}^{-1} \bar{A}$ and by expressing all the nonlinear expressions $C \bar{A}, P_{21} \bar{A}, P_{11}^{-1} \bar{A}, C \bar{B}, P_{21} \bar{B}, P_{11}^{-1} \bar{B}$ in terms of $\bar{A}$ and $\bar{B}$, we cannot guarantee that (9) holds. However, in the following we show that there exists a state transformation of the algorithm such that this can be indeed guaranteed. Hence, any solution of (7), (3) is a solution of (8), (9) by an appropriate coordinate transformation.

If there exists a transformation (non-singular) matrix $T$ such that the transformed variables $\tilde{A} = T^{-1} \bar{A} T, \bar{B} = T^{-1} \bar{B}, C' = CT, P_{11}^\prime = T^T P_{11} T, P_{12}^\prime = T^T P_{12}, P_{21}^\prime = P_{21} T, P_{22}^\prime = P_{22}$ fulfill

$$(\tilde{A} - I_n) J_{11}^T = B' M, \quad J_2 P_{11}' = C', \quad J_3 P_{11}' = P_{21}'$$

then we have

$$\begin{pmatrix} C \tilde{A} \\ P_{21}^\prime \tilde{A} \\ P_{11}^\prime \tilde{A} \end{pmatrix} = \begin{pmatrix} J_2 \\ J_3 \end{pmatrix} \begin{pmatrix} C' \\ P_{21}' \\ P_{11}' \end{pmatrix} = \begin{pmatrix} J_2 \\ J_3 \end{pmatrix} \tilde{B},$$

and the transformed variables still form a solution of inequality (7). The arguments from Step 1 show that in this case $\tilde{A}'$, $\bar{B}'$, $C'$ and $P'$ form also a solution of (8) and by substituting $\tilde{A}'$ and $\bar{B}'$ for the nonlinear terms it becomes clear that there exists a solution to (8), (9) from Theorem 14.

Such a transformation $T$ must now fulfill the constraints

$$J_2 T^T P_{11} T = C T, \quad J_3 T^T P_{11} T = P_{21} T = P_{21}^\prime,$$

$$T^{-1} (\tilde{A} - I_n) T = T^{-1} \bar{B} M.$$
For the choice $J_1 = (I_d 0), J_2 = (0_d I_d 0), J_3 = (0_d 0_d I_d 0)$, these equations have the solution

$$T = \left( (A - I_n)^{-1}BM \enskip P_{11}^T C^T \enskip P_{11}^T P_{21}^T \enskip T_4 \right),$$

provided, that $n \geq 3d$. It remains to show that the transformation is non-singular. Notice that $(A - I_n)^{-1}BM$ and $C$ must have full rank because of $C(A - I_n)^{-1}BM = I_d$. Moreover $P_{11}, P_{21}$ can be slightly perturbed (without violating the strict definiteness of $P$ and the matrix inequality (7)), such that $(A - I_n)^{-1}BM \enskip P_{11}^T C^T \enskip P_{11}^T P_{21}^T$ has full rank too. Finally, $T_4 \in \mathbb{R}^{n \times n-3d}$ can be chosen such that $T$ is non-singular. Hence, all constraints of Theorem 14 are satisfied by construction of $T$, where $CJ_1 = I_d$ is implied by (3).

Consequently, it is possible to construct solutions related to Theorem 14 from solutions related to Theorem 13 and vice versa.

K. Proof of Theorem 15

Again, we introduce the abbreviations $\tilde{L} = L - M$ and $\Pi = \Pi_{\ker L-M}$. In this proof, it will be necessary to find explicit solutions for the matrix inequality (7) from Theorem 13. Therefore, it is purposeful to multiply out the matrix products in this inequality for $\lambda = 0$, which gives:

$$
\begin{align*}
\begin{pmatrix}
\tilde{A}^T P_{11} \tilde{A} - \rho^2 P_{12} & \tilde{A}^T P_{11} B - \rho^2 P_{12} \\
P_{21}^T \tilde{A} - \rho^2 P_{22} & P_{21}^T B - \rho^2 P_{22} - r \Pi
\end{pmatrix}
\end{align*}
\begin{pmatrix}
\frac{1}{2}C & \frac{1}{2}C \\
\frac{1}{2}C & \frac{1}{2}C
\end{pmatrix}
\begin{pmatrix}
\tilde{A} & P_{11} B - \rho^2 P_{12} \\
P_{22} - \tilde{L} - r \Pi
\end{pmatrix}
$$

i) $\Rightarrow$ ii): This step will be quite lengthy. We will show, that the matrices $(\tilde{A}, B, C)$ given by

$$\tilde{A} = A + BMC = I_d - 2(L + M)^{-1}M = (L + M)^{-1} (L - M)$$

$$B = -2(L + M)^{-1}$$

$$C = I_d$$

fulfill all the convergence rate conditions of Theorem 13 for an arbitrary given $\rho \in \rho_{\text{grad}}, 1$. Here, the matrix $\tilde{A}$ fulfills the Lyapunov inequality

$$\tilde{A}^T \tilde{P} \tilde{A} - \rho^2 \tilde{P} < 0$$

for $\tilde{P} := (L + M) ((L - M)^t + r \Pi) (L + M)$ and large enough $r \in \mathbb{R}_{>0}$ by Lemma 19 since $\tilde{A}^T \tilde{P} \tilde{A} = L - M$. To show, that the convergence conditions from Theorem 13 are met we choose $\lambda = 0$ and the following value for $P$:

$$
\begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix}
\begin{pmatrix}
\tilde{P} - \varepsilon(L + M)^2 & -\frac{1}{2}I_d \\
-\frac{1}{2}I_d & \tilde{L} + r \Pi - \frac{1}{2}I_d
\end{pmatrix}
$$

where $\varepsilon > 0$, and $r \in \mathbb{R}$ is the same as above. There are three things to show:

1) The constraint (3) of Theorem 13 is satisfied for $\tilde{A}, B, C$.

2) For large enough $r$ and small enough $\varepsilon$, $P$ solves the matrix inequality (7) of Theorem 13

3) For large enough $r$ and small enough $\varepsilon$, $P$ is positive definite.

Verifying 1) can be done by a simple calculation of formulas in the constraint.

We will now show 2). Note that $(P_{21} + \frac{1}{2}C) = \frac{1}{2}(I_d - I_d) = 0$ holds, which is why (7) from Theorem 13 simplifies to

$$
\begin{pmatrix}
\tilde{A}^T P_{11} \tilde{A} - \rho^2 P_{12} & \tilde{A}^T P_{11} B - \rho^2 P_{12} \\
B P_{21} \tilde{A} - \rho^2 P_{22} & B^2 P_{21} B - \rho^2 P_{22} - r \Pi
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & -\frac{1}{2}I_d
\end{pmatrix}
$$

Here it is left to show, that the left upper $2 \times 2$ block can be made negative definite by choosing $r$ big and $\varepsilon$ small. This is done by dividing the matrix inequality by $\tilde{A}^T$ and calculating the entries of the left upper blocks:

The first block is

$$
\frac{4}{\rho^2} \tilde{A}^T (\tilde{A}^T P_{11} B - \rho^2 P_{12})
$$

$$= \tilde{A}^T \left( \tilde{P} - \frac{\varepsilon}{2}(L + M)^2 \right) B
$$

$$= \tilde{A}^T \tilde{P} - \rho^2 \tilde{P} - \varepsilon \left( (L - M)^2 - \rho^2 (L + M)^2 \right)
$$

The second block is

$$
\frac{4}{\rho^2} (\tilde{A}^T P_{11} B - \rho^2 P_{12})
$$

$$= 2I_d + \tilde{A}^T \left( \tilde{P} - \varepsilon(L + M)^2 \right) B
$$

$$= 2I_d - 2(L - M) (L - M)^{-1} r \Pi - 2\varepsilon(L - M)
$$

$$= 2I_d - 2(L - M)(L - M)^{-1} 2r (L - M)\Pi - 2\varepsilon(L - M)
$$

= $2I_d - 2\Pi_{\ker(L-M)} - 2\varepsilon(L - M)$.

The third block is:

$$
\frac{4}{\rho^2} \tilde{A}^T (\tilde{B}^t P_{11} B - \rho^2 P_{22} - r \Pi)
$$

$$= \tilde{A}^T \tilde{B}^t P B - 4\varepsilon I_d - 4 \left( \tilde{L}^t + r \Pi - \frac{1}{4}I_d \right)
$$

= $4(\tilde{L}^t + r \Pi - 4\varepsilon I_d - 4 \left( \tilde{L}^t + r \Pi - \frac{1}{4}I_d \right)

= $4 \tilde{L}^t + r \Pi - 4\varepsilon I_d - 4 \rho^2 r \Pi

The calculation of these blocks reveals, that the upper $2 \times 2$ block is

$$
\tilde{A}^T \tilde{P} \tilde{A} - \rho^2 \tilde{P} - \varepsilon \left( (L - M)^2 - \rho^2 (L + M)^2 \right)
$$

$$= 2I_d - 2\varepsilon(L - M)
$$

$$= 2\Pi_{\ker(L-M)} - 2\varepsilon(L - M)
$$

which is negative definite for $\varepsilon > 0$ small enough and $r$ big enough.

Now it is left to show 3), namely that $P$ is positive definite for small enough $\varepsilon$ and large enough $r$. Therefore, we can show that $P$ is positive definite for $\varepsilon = 0$. Then it will also be positive definite for the small perturbation with $\varepsilon > 0$. By the Schur complement, the matrix $P$ for $\varepsilon = 0$ is positive.
Since $\rho > \rho_{\text{grad}}$, the matrix $\rho^2 P - (L - M)$ is positive definite by Lemma 19 and the matrix $\bar{L} + r\Pi$ is positive definite by construction. Thus, $\frac{\rho^2}{4} P - \frac{1}{4}(L - M) - \frac{1}{4}r\Pi$ is positive definite for large values of $r$. Hence, we only have to make $\varepsilon$ small enough and $r$ big enough, such that $P$ becomes positive definite.

ii) $\Rightarrow$ iii): From ii) it is clear that we have a special solution for $n = d$. Let $\bar{A}^{(d)}, B^{(d)}, C^{(d)}, P^{(d)}$ be this special solution. This solution can be extended to a solution for arbitrary dimension $n \geq d$ by setting

$$
\bar{A} = \begin{pmatrix}
A^{(d)} & 0_{d \times n - d} \\
0_{d \times n - d} & 0_{n - d \times n - d}
\end{pmatrix},
B = \begin{pmatrix}B^{(d)} \\
0_{n - d \times d}
\end{pmatrix},
C = \begin{pmatrix}C^{(d)} \\
0_{d \times n - d}
\end{pmatrix},
P_{22} = P^{(d)},
P_{11} = \begin{pmatrix}P_{11}^{(d)} & 0_{d \times n - d} \\
0_{n - d \times d} & I_{n - d}
\end{pmatrix},
P_{12} = \begin{pmatrix}P_{12}^{(d)} \\
0_{n - d \times d}
\end{pmatrix}.
$$

Showing that these values satisfy the constraints and the LMI of Theorem 13 is straightforward.

iii) $\Rightarrow$ iv): As stated in Theorem 14 the constraints and matrix inequality of this theorem are equivalent to the matrix inequality of Theorem 13 in the case $n \geq 3d$.

iv) $\Rightarrow$ i): If Theorem 14 admits a solution, then there exists an optimizer which satisfies the conditions of Theorem 13 and thus a solution to Problem 2. By Theorem 1 this solution would also solve Problem 1 which can only be solved, if any function $f \in S(M, L)$ has a fixed point. If any function $f \in S(M, L)$ has a fixed point, then holds $M \preceq_c L$ and $M$ and $L$ are non-singular by Theorem 8.

L. Proof of Lemma 17

First, notice that $\sigma(\bar{A}) \subseteq \mathbb{C}_{0} \setminus \rho$ is implied by the Lyapunov inequality $\bar{A}^T P_{11} \bar{A} - \rho^2 P_{11} \prec 0$, which is the left upper block of the matrix inequality (7). In this proof, we show the equivalence of the FDI (12) of Theorem 16 and the matrix inequality (7). Therefore, notice, the multiplier $\Pi$ from (10) can be factorized into $\Pi(z) = \Psi(z)^* R \Psi(z)$ with

$$
\Psi(z) = \begin{pmatrix}(l - m)(1 - \lambda z^{-1}) & z^{-1}I_d \\
0 & I_d
\end{pmatrix},
R = \begin{pmatrix}0 & I_d \\
I_d & -2I_d
\end{pmatrix}.
$$

The goal is to apply the discrete-time KYP-Lemma to (12). Therefore, a realization of the following concatenation of $G$

$$
\Psi(z) \begin{pmatrix}G(z) \\
I_d
\end{pmatrix} = \begin{pmatrix}(l - m)(1 - \lambda z^{-1}) & z^{-1}I_d \\
0 & I_d
\end{pmatrix} \begin{pmatrix}G(z) \\
I_d
\end{pmatrix} = \begin{pmatrix}z^{-1}I_d + (l - m)(1 - \lambda z^{-1})G(z) \\
0 & I_d
\end{pmatrix} = \begin{pmatrix}\lambda I_d + (l - m)(z - \lambda)G(z) \\
0 & I_d
\end{pmatrix}
$$

and $\Psi$ is needed:

$$
\Psi(z) \begin{pmatrix}G(z) \\
I_d
\end{pmatrix} = \begin{pmatrix}(l - m)(1 - \lambda z^{-1}) & z^{-1}I_d \\
0 & I_d
\end{pmatrix} \begin{pmatrix}G(z) \\
I_d
\end{pmatrix} = \begin{pmatrix}z^{-1}I_d + (l - m)(1 - \lambda z^{-1})G(z) \\
0 & I_d
\end{pmatrix} = \begin{pmatrix}\lambda I_d + (l - m)(z - \lambda)G(z) \\
0 & I_d
\end{pmatrix}
$$

Here, $H_1$ is realizable, because $(z - \lambda)G(z)$ is realizable, because $G$ has a relative degree of at least one. A realization of $H_1$ is

$$
\bar{A} = \begin{pmatrix}0 & I_d \\
0 & 0
\end{pmatrix}
$$

and a realization of $H_2$ is

$$
\begin{pmatrix}0 & I_d \\
I_d & 0
\end{pmatrix}
$$

With these realizations, the following is a realization of the chaining of $H_1$ and $H_2$:

$$
\begin{pmatrix}0 & I_d \\
I_d & 0
\end{pmatrix}
$$

This is now also a realization of $\Psi(I_d G)$. Satisfaction of the FDI

$$
\left(\begin{array}{c}I_d \\
G(z)
\end{array}\right)^* \Psi(z) R \Psi(z) \left(\begin{array}{c}I_d \\
G(z)
\end{array}\right) \prec 0 \quad \forall z \in \mathbb{C}_{|z| = \rho}
$$

is by the Kalman Yakubovic Popov Lemma (Corollary 13 of [11]) equivalent to existence of $P = P^T$ such that

$$
\begin{pmatrix}I_d & 0 & 0 \\
0 & P_{11} & -\rho^2 P_{12} \\
0 & 0 & P_{11}
\end{pmatrix} \prec 0
$$

is negative definite. A quick reformulation of the above terms reveals that they are nothing but inequality (7). It can be checked that inequality (7) can only have positive definite solutions $P$.

M. Proof of Lemma 18

We have to check whether inequality (2) holds for the Lagrangian function $L \in C^1$. Let therefore arbitrary values...
$z_1, z_2 \in \mathbb{R}^d$ and $\lambda_1, \lambda_2 \in \mathbb{R}^{d_2}$ be given. The lower bound in inequality follows from

$$\frac{1}{2} (z_1 - z_2)^T \begin{pmatrix} M & A_{eq}^T \vspace{1mm} \\ A_{eq} & 0 \end{pmatrix} \begin{pmatrix} z_1 - z_2 \vspace{1mm} \\ \lambda_1 - \lambda_2 \end{pmatrix} = \frac{1}{2} (z_1 - z_2)^T M (z_1 - z_2) + (\lambda_1 - \lambda_2)^T A_{eq} (z_1 - z_2) \leq f(z_2) - f(z_1) + (\nabla f(z_1))^T (z_1 - z_2) + (\lambda_1 - \lambda_2)^T A_{eq} (z_1 - z_2) = f(z_2) + \lambda_1^T (A_{eq} z_2 - b_{eq}) - (f(z_1) + \lambda_1^T (A_{eq} z_1 - b_{eq})) + (\nabla f(z_1) + A_{eq} \lambda_1)^T (z_1 - z_2) + (\lambda_1 - \lambda_2)^T (A_{eq} z_1 - b_{eq}) .$$

The upper bound can be shown analogously.

\[ \Box \]

N. Auxiliary results

**Lemma 19.** Let $M \preceq_c L$ be non-singular, symmetric matrices. Then for any $\rho > \rho_0 ( (M + L)^{-1} (L - M))$ there exists an $r_0 \in \mathbb{R}_{>0}$ such that for all $r \geq r_0$ $L - M \prec \rho^2 (L + M) \left((L - M)^\dagger + r \Pi_{ker L - M}\right) (L + M).$

**Proof.** Let $\rho > \rho_0 ( (M + L)^{-1} (L - M))$ be given. Define $\Pi := \Pi_{ker L - M}$ and

$$\hat{\rho} := \rho \left( (L + M)^{-1} \left( L - M + \frac{1}{r} \Pi \right) \right) \overset{(*)}{=} \rho \left( \sqrt{L - M + \frac{1}{r} \Pi} (L + M)^{-1} \sqrt{L - M + \frac{1}{r} \Pi} \right) = \left( \sqrt{L - M + \frac{1}{r} \Pi} (L + M)^{-1} \sqrt{L - M + \frac{1}{r} \Pi} \right)^2 .$$

Here, the equality $ (*)$ holds by a similarity transform with $\sqrt{L - M + \frac{1}{r} \Pi}$. This definition of $\hat{\rho}$ implies the matrix inequality

$$\hat{\rho}^2 I_d \preceq \left( \sqrt{L - M + \frac{1}{r} \Pi} (L + M)^{-1} \sqrt{L - M + \frac{1}{r} \Pi} \right)^2 .$$

A congruence transform with $(L - M + \frac{1}{r} \Pi)^{-\gamma} (L + M)$ yields

$$\hat{\rho}^2 (L + M) \left((L - M)^\dagger + r \Pi\right) (L + M) \preceq L - M + \frac{1}{r} \Pi ,$$

since $(L - M + \frac{1}{r} \Pi)^{-\gamma} \overset{(\text{**})}{=} \sqrt{L - M}^\dagger + r \Pi$. By the expression of $\hat{\rho}$ through the spectral norm and the continuity of the norm, it is clear that $\hat{\rho}$ converges to $\rho ( (M + L)^{-1} (L - M))$ for $r \to \infty$. Hence, we can choose $r$ large enough, such that $\hat{\rho}$ is smaller than $\rho$ and thus,

$$L - M \preceq \hat{\rho}^2 (L + M) \left((L - M)^\dagger + r \Pi_{ker L - M}\right) (L + M) < \rho^2 (L + M) \left((L - M)^\dagger + r \Pi_{ker L - M}\right) (L + M) .$$

Since increasing $r$ corresponds to adding a positive definite term to the right hand side of this inequality, the inequality remains valid for larger values of $r$. \[ \Box \]

**Lemma 20 (Congruence Lemma).** Let $M, L \in \mathbb{R}^{d \times d}$ be two symmetric matrices such that there exists a positive definite matrix $P = P^T \in \mathbb{R}^{d \times d}$ with

$$MP + LP \succ 0 .$$

Then $M$ and $L$ are congruent, i.e. there exists a non-singular matrix $T$ such that $T^T MT = L.$

**Proof.** By $P$ being positive definite, there exists a symmetric positive definite matrix $\sqrt{P} \in \mathbb{R}^{d \times d}$ with $\sqrt{P} = P$. A congruence transform with $\sqrt{P}$ yields

$$\sqrt{PM} \sqrt{P} \sqrt{PL} \sqrt{P} + \sqrt{PL} \sqrt{P} \sqrt{PM} \sqrt{P} \succ 0 . \quad (19)$$

The matrices $\tilde{M} := \sqrt{PM} \sqrt{P}$ and $\tilde{L} := \sqrt{PL} \sqrt{P}$ are congruent to $M$ and $L.$ Hence, it is sufficient to show that the matrices $\tilde{M}$ and $\tilde{L}$ are congruent.

Therefore, let $T$ be an orthogonal matrix, such that

$$T^T \tilde{M} T = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} ,$$

where $D_1$ is the diagonal matrix of all positive eigenvalues of $\tilde{M}$ and $D_2$ is the matrix of all negative eigenvalues of $\tilde{M}.$ Now, a congruence transform with $T$ can be applied to $\tilde{M}$ and $\tilde{L}$:

$$0 \prec T^T \tilde{L} \tilde{M} T + T^T \tilde{M} T \tilde{L} T .$$

From this inequality, one can read off

$$E_{11} D_1 + D_1 E_{11} > 0 , \quad E_{22} D_2 + D_2 E_{22} > 0$$

from the diagonal blocks. Hence, by the Lyapunov inequality, $E_{11} > 0$ and $E_{22} < 0.$ Now, $E$ is positive definite on the subspace corresponding to $E_{11}$ and negative definite on the subspace corresponding to $E_{22}.$ Consequently, $E$ has exactly $\dim E_{11} = \dim D_1$ positive and exactly $\dim E_{22} = \dim D_2$ negative eigenvalues according to Sylvester’s law of inertia. Thus the matrices $\tilde{M}$ and $\tilde{L}$, which are congruent to $D$ and $E$, are congruent to each other. \[ \Box \]