Two-dimensional topological order of kinetically constrained quantum particles

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Motivated by recent experimental and theoretical work on driven optical lattices, we investigate how imposing kinetic restrictions on quantum particles that would otherwise hop freely on a two-dimensional lattice can lead to topologically ordered states. The kinetically constrained models introduced here are derived as a generalization of strongly interacting particle systems in which hoppings are given by Haldane-like Hamiltonians, and are pertinent to systems irradiated by circularly polarized light. After introducing a broad class of models, we focus on particular realizations and show numerically that they exhibit topological order, as witnessed by topological ground-state degeneracies and the quantization of corresponding invariants. These results have direct implications for the realization of topologically ordered phases in existing cold-atom setups and also demonstrate that the correlations responsible for fractional quantum Hall states in lattices can arise from processes other than density-density interactions.

The pursuit of topological states of matter has fueled substantial experimental and theoretical developments in physics. From the integer quantum Hall effect [1] to topological insulators [2, 3] and beyond, topological states possess remarkable properties of both fundamental and technological interest. For instance, certain correlated topologically ordered states, such as fractional quantum Hall (FQH) states [4], have the potential to be used for fault-tolerant quantum computation [5]. The functionality required for novel high-end technological applications, however, steers material design towards microscopic structures with increasingly complex features.

Recently, several steps have been taken in order to bridge the gap between theoretical descriptions of FQH-type topological order and experimentally accessible systems. A class of lattice models called Chern insulators [6] have been shown to develop FQH-like ground states, termed fractional Chern insulators (FCI) [7–9], upon introduction of a short-range repulsion between particles occupying a fractionally filled band, even in the absence of an external magnetic field (see Refs. 10, 11 for comprehensive reviews). Initially, these models were fine-tuned so that the lowest band of their energy dispersion imitates the lowest Landau level: it is almost perfectly flat and topological, i.e., characterized by a Chern number $C = \pm 1$ and hence named Chern band. Proposals for realizations of FCI states based on oxide heterostructures [12], layered multi-orbital systems [13, 14], optical lattices [15] and strained or irradiated graphene [16, 17] followed soon thereafter.

As bands in solids are generally not flat and interaction strengths vary, an obvious way to bring FQH-like states closer to reality is to relax the energetic conditions imposed on relevant systems in order to emulate Landau levels. It was recognized early on that interaction strengths may be allowed to increase beyond band gaps [8, 13, 18, 19] and subsequent results showed that a finite dispersion may actually favor certain FCI states [14, 20]. One can then venture into the strong-correlation regime by allowing for arbitrarily strong repulsion strengths. Surprisingly, one still finds robust FCI states — as well as more exotic, topologically ordered states [21] — even though interactions now mix bands with opposite Chern numbers [8, 22]. Strongly correlated materials without sharply defined bands may therefore be considered as candidates for the realization of FCI or similar states.

When particles repel their neighbors very strongly, it is reasonable to approximate the interaction as a hardcore constraint which does not allow particles to occupy neighboring sites. On two-dimensional lattices of corner- or side-sharing triangles, the configurations allowed by this constraint are identical to those of the so-called hard-hexagon (HH) model of classical lattice gases [23, 24]. In the quantum version of these HH models, transitions from one configuration to another are caused by hopping terms. When the hoppings considered give rise to Chern bands, imposing a hardcore constraint in partially filled lattices can lead to FQH-like ground states [22].

Here we explore the possibility of obtaining the same physics by replacing the hardcore constraint with a kinetic one. Instead of energetically penalizing or — in the infinite-interaction limit — removing configurations from the Fock space altogether, we start with all configurations being a priori equivalent. Constraints are then introduced only in the transitions between configurations. The resulting systems can be thought of as quantum versions of cooperative lattice gases [25]. In this Letter, we show that when the constrained kinetic terms are endowed with appropriate Haldane-like phases, the obtained ground states show definitive features of FQH topological order.

This result is far from being purely academic: current experiments on optical lattices can generate a partially filled Chern band in the laboratory [26, 27]. Some of the kinetic terms that model the experimental setting were recently shown to exhibit precisely the type of constraints studied here [28]. The same kinetic constraints are also an approximate limit of strong short-range repulsion and therefore relevant to the realization of FQH physics in strongly correlated materials.

To establish a general formalism, we consider $N$ quantum particles of a single species that hop on a two-dimensional lattice $\Lambda$. The discussion below will be limited to spinless fermions, but there is no fundamental obstacle to treating bosons in the same fashion. Distances between nearest-neighboring sites of $\Lambda$ are set to unity for convenience and
periodic boundary conditions are assumed in both spatial directions. The general form of the Hamiltonian we consider below reads

\[ \hat{H} = \sum_{i \neq j} \left( \hat{c}^\dagger_i F_{ij} \hat{c}_j + \text{H.c.} \right), \tag{1} \]

where \( \hat{c}_i (\hat{c}^\dagger_i) \) is a regular particle annihilation (creation) operator acting at position \( i \) of \( \Lambda \). An appropriate choice of the operator-valued function \( F_{ij} \) can result in any possible \( n \)-body interaction term. For example, setting \( F_{ij} = 1 \) yields all possible hopping terms with equal amplitude, whereas \( F_{ij} = \hat{r}_i \hat{r}_j \) yields all possible density-density interactions with equal magnitude.

In the following, we restrict the \( \hat{F}_{ij} \) to products of hole-density operators of the form \( (1 - \hat{n}_j) \) acting in the neighborhood of \( i \) and \( j \). Such terms emerge effectively from strong interactions \[\text{[29].}\] To see this, consider particles that hop around on the lattice and at the same time interact with each other via a nearest-neighbor repulsion of strength \( V \). In the infinite- \( V \) limit, particles cannot occupy nearest-neighbor sites. This suggests that states containing nearest-neighbor particles can be removed from the Fock space. Accordingly, for any site \( i \in \Lambda \) we define the projected operator \( \hat{c}_i^{\dagger} \) by demanding that its action on any state is to create a particle at \( i \) if and only if this site and all of its nearest neighbors are empty. Formally,

\[ \hat{c}_i^{\dagger} := \hat{c}_i^\dagger \prod_{\{j:|j-i|=1\}} (1 - \hat{n}_j), \tag{2} \]

where \( \hat{n}_j := \hat{c}_j^\dagger \hat{c}_j \). If we allow particles to hop while obeying the above hard-core condition, then the system can be described by Eq. (1) with

\[ \hat{F}_{ij}^{\text{HC}} = t_{ij} \prod_{\{l:|l-i|=|l-j|=1\}} (1 - \hat{n}_l), \tag{3} \]

where \( t_{ij} \) are generally complex-valued hopping amplitudes. The strong-repulsion limit can therefore be recast into a Hamiltonian of the form of Eq. (1), with \( \hat{F}_{ij} \) being a product of hole-density operators in the vicinity of \( i \) and \( j \). On two-dimensional lattices of corner- or edge-sharing triangles, the allowed states are exactly the configurations of classical HH models, widely studied in the context of glassiness. Due to this resemblance, we shall call the above Hamiltonians quantum hard-hexagon (QHH) models.

We now wish to reduce the above hardcore constraint to a new one that does not correspond to a density-density interaction. The main motivation for doing so is that density-hopping terms of the form \( \hat{c}^\dagger (1 - \hat{n}) \hat{c} \) have been shown to arise in the description of optical lattice experiments, where trapped atoms are periodically driven using circularly polarized light, or equivalent settings \[\text{[26, 27].}\] The theoretical treatment of relevant models describes how the driving, apart from affecting the preexisting hopping and density-density terms, introduces new frequency-dependent density-hopping terms \[\text{[28].}\]

FIG. 1. Illustration of vacancy-assisted hopping models on (a) the triangular lattice, with nearest and third-nearest neighbor hoppings as defined in Refs. 13 and 14, and (b) the kagome lattice, with nearest and second-nearest neighbor hoppings as defined in Ref. 31. Solid and dashed circles denote particles and vacancies, respectively. The hoppings (arrows) are only allowed if a vacancy is located at the position shown. Note that some of the hoppings are imaginary; see Eqs. (5) and (6).

Instead of treating the latter in conjunction with other processes in a complicated setting, here we isolate them and study their properties in a simpler context. As we shall show below, such terms can generate nontrivial behavior even on their own.

Notice that \( \hat{F}_{\text{HC}} \) is just a product of hole-density operators at different sites. In any partially filled system, products of \( (1 - \hat{n}) \) operators that act on different sites are more likely to vanish than single \( (1 - \hat{n}) \) operators. We can therefore attempt to restrict the product in Eq. (3) to only a few \( (1 - \hat{n}) \) operators and see whether this captures the same physics as the full product. To decide which terms to truncate, we draw inspiration from classical models of diffusion. As mentioned above, \( \hat{F}_{\text{HC}} \) is the quantum counterpart of the HH model on the triangular lattice. In the HH model, each particle can be visualized as a hard disc of radius \( 1/2 < r < 1 \), so that configurations with particles on nearest-neighboring sites are forbidden. If the disc radius is reduced to \( \sqrt{3}/4 < r < 1/2 \), then all particle configurations are a priori allowed. The particle motion, however, is still constrained if the hard discs move from site to site along straight lines: on lattices of edge- or corner-sharing equilateral triangles, they cannot move past one another (see Fig. 1). This is an example of a cooperative lattice-gas model \[\text{[30].}\]

A quantum analogue of the kinetically constrained model described above can be straightforwardly constructed by discarding all factors in Eq. (3) apart from those which pertain to common neighbors of \( i \) and \( j \). With this choice,

\[ \hat{F}_{ij}^{\text{KC}} = t_{ij} \prod_{\{l:|l-i|=|l-j|=1\}} (1 - \hat{n}_l), \tag{4} \]

Examples of the resulting terms are pictorially represented in Fig. 1 for two lattices. Following the classical terminology, we shall call the processes represented by \( \hat{F}_{ij}^{\text{KC}} \) vacancy-assisted hoppings (VAH).

We now ask whether the correlations induced by the vacancy-assisted hoppings described above can generate topological order. To answer this, we shall impose this kinetic con-
is defined as

\[ H = \sum_{j} \psi_{k,j} H \psi_{k,j} \]

and \( \mathbf{a} \) is given by

\[ \mathbf{a} = (1/2, \sqrt{3}/2, 0) \]

where \( \mathbf{a}_1 = (1, 0) \), \( \mathbf{a}_2 = \sqrt{3}/2 \), \( \mathbf{a}_3 = \mathbf{a}_2 - \mathbf{a}_1 \), \( \mathbf{b}_1 = \mathbf{a}_2 + \mathbf{a}_3 \), \( \mathbf{b}_2 = \mathbf{a}_3 - \mathbf{a}_1 \), \( \mathbf{b}_3 = \mathbf{a}_1 + \mathbf{a}_2 \) and \( \mathbf{\lambda} \equiv (\lambda_1, \lambda_2, \lambda_3) \), \( \mathbf{\lambda} \equiv (\lambda_1, \lambda_2, \lambda_3) \) are vectors of the Gell-Mann matrices

\[ \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

(7a)

\[ \tilde{\lambda}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}, \tilde{\lambda}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tilde{\lambda}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \tilde{\lambda}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}. \]

(7b)

We set \( t_3/t = 0.19 \) for the triangular lattice and \( t_2/t = 0.3, \delta/t = 0.28, \delta_2/t = 0.2 \) for the kagome lattice. These hoppings generate relatively flat lowest bands with Chern number \( C = -1 \). We have verified that all our results remain valid in a finite range around the values chosen here. We focus on average particle densities \( \rho = 1/6 \) for the triangular lattice and \( \rho = 1/9 \) for the kagome lattice, at which the lowest band of the pure hopping counterparts of the models we study here would be at filling \( \nu = 1/3 \). We then impose the constraint encapsulated in Eq. (4). The resulting vacancy-assisted hopping models are sketched in Fig. 1.

FIG. 3. (a) Berry curvature of the triangular-lattice model on a 42-site cluster with \( N = 7 \) as a function of the magnetic fluxes threading the two handles of the toroidal system; (b) error in the quantization of \( \sigma_H \) as a function of the number of subintervals in the partition of the Brillouin zone of fluxes.

To establish consistency with previous results [22], Fig. 2(a) shows the spectral flow of the QHH version of the triangular-lattice model, where simultaneous occupation of nearest-neighboring sites is prohibited but hoppings are unconstrained. In Fig. 2(c) we show that the same physics occurs in the kagome-lattice model. Apart from the characteristic FQH features in the energy spectra, the ground states obtained have a very precisely quantized Hall conductivity \( \sigma_H = e^2/(3h) \).
We now reduce the HH constraint of Eq. 3 to the kinetic constraint of Eq. 4, as outlined above. We notice that the energy spectra, shown on the right of Fig. 2, are only quantitatively altered. The symmetry sectors in which the quasi-degenerate ground states reside are the same as in the case of hardcore interactions and can be predicted by methods already in use for finite FQH and FCI systems [9, 32]. More importantly, the ground states of the VAH models have nontrivial topological characteristics. Their many-body Berry curvature is a smooth function that integrates to a precisely quantized Hall conductivity, even for the finite systems considered here. An example of this quantization is presented in Fig. 3 for the triangular-lattice model.

The FQH-like ground states of the VAH models are generally less gapped than those of their QHH counterparts (see Fig. 4). Nevertheless, there is a finite volume in parameter space in which one obtains gapped FQH-like ground states even in the VAH models for most clusters. Finite-size effects are sizable and do not allow for a definitive answer as to whether there is topological order in the ground state of the VAH models in the thermodynamic limit, although the gaps of the VAH models do seem to follow the same trend as those of their QHH counterparts, which remain well gapped up to the largest accessible system sizes (see also Ref. 22). Furthermore, we find no signatures of a charge instability, so the only evident competitor for the ground state is a compressible metallic state. The latter is clearly disfavored in previous detailed studies of FCIIs in the models defined by Eqs. (5) and (6) [14, 33].

Regardless of energetics, however, the key conclusion of this Letter is that imposing kinetic constraints in partially filled Chern bands can generate topologically ordered states, which in turn means that correlations crucial for FQH-like states can be generated by kinetic constraints. This may be a stepping stone for further intuition into the microscopics of the FQH effect itself. More hints in this direction can be found in recent work [34], where the intimate relation between FQH and FCI states gives rise to similar density-hopping terms. Finally, we briefly mention that similar tendencies towards topological ordering are to be expected when suitable kinetic constraints are introduced in bosonic versions of Haldane-like models [35].

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