Quasilocal energy in modified gravity

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Abstract

A new generalization of the Hawking–Hayward quasilocal energy to scalar–tensor gravity is proposed without assuming symmetries, asymptotic flatness, or special spacetime metrics. The procedure followed is simple but powerful and consists of writing the scalar–tensor field equations as effective Einstein equations and then applying the standard definition of quasilocal mass. An alternative procedure using the Einstein frame representation leads to the same result in vacuo.

Keywords: Scalar–tensor gravity, quasilocal energy, Hawking–Hayward mass

(Some figures may appear in colour only in the online journal)

1. Introduction

There is little doubt that Einstein’s general relativity (GR) is not the final theory of gravity. GR breaks down at spacetime singularities and cannot be quantized. All the attempts to merge GR with quantum mechanics provide, in their low-energy limits, corrections to GR in the form of higher derivative equations or extra fields with explicit couplings to the spacetime curvature or to matter. A particularly compelling motivation for studying alternative theories of gravity comes from cosmology: the standard cosmological model based on GR, the Λ-cold dark matter model, can only explain the present acceleration of the universe discovered with type Ia supernovae by invoking a completely ad hoc dark energy [1]. Perhaps we are already observing deviations from GR in the cosmic acceleration that the Λ-cold dark matter model tries to fit into GR. Scalar–tensor theories of gravity and, in particular, the subclass known as $f(\mathcal{R})$ gravity have enjoyed enormous popularity in the last decade [2, 3], which only adds to previous motivation from string theory. In fact, the low-energy limit of string theories contains a dilaton very similar to the Brans–Dicke field of scalar–tensor gravity (the bosonic string theory reduces, in this limit, to a Brans–Dicke theory [4]). There is currently much interest in probing gravity at all scales to detect or constrain deviations from GR (which could
assume several forms in cosmology, black holes, or stellar objects [5, 6]), including the search for scalar hair [7].

The notion of mass of a relativistic gravitating system has been the subject of intense research in GR. Because of the equivalence principle, gravitational energy cannot be localized. The next best thing is a quasilocal notion of energy, i.e., the energy contained in a compact 2-surface in spacetime, and several definitions of quasilocal energy have been introduced over the years (see [8] for a review). It seems that the relativity community settled on the Hawking–Hayward quasilocal construct [9, 10], which we employ here, but other quasilocal energies could be used as well.

The concept of mass is not only important in principle and for its obvious applications to gravitating systems, but also because it appears in the first law of thermodynamics for gravity. Much literature has been devoted to black hole thermodynamics and the thermodynamics of gravity and spacetime (e.g., [11]), but this is still an active area of theoretical research.

Given the significance of modified gravity [5–7] it would be important to know whether the quasilocal energy can somehow be extended to these theories, beginning with the simplest and most popular alternative, scalar–tensor gravity (see [12] for the case of n-dimensional Lovelock gravity). Thus far, discordant prescriptions for a quasilocal mass have been given [13–16] but they are subject to important restrictions: (1) only \( f(R) \) gravity, which is a subclass of scalar–tensor theories, has been examined; (2) only spherical symmetry, and sometimes only special spacetime geometries, have been considered. These prescriptions have been obtained using spacetime thermodynamics and a first law [13–16]. However, the expressions of the other four quantities used in the first law of thermodynamics (temperature, entropy, work density, and heat supply vector, respectively) are not established beyond doubt, which introduces some ambiguity in the definition of quasilocal mass obtained by assuming a certain form for the first law. Additionally, the concept of horizon temperature requires quantum considerations that are highly nontrivial in curved spacetime, where it is difficult to complete quantum field theory calculations unambiguously. While we remain agnostic on these approaches in this paper, we propose to bypass these conceptual difficulties by introducing a quasilocal mass in scalar–tensor and \( f(R) \) gravity via considerations that are purely classical and independent of the thermodynamics of gravity. An advantage of this approach is that the generalization of the Hawking–Hayward mass to scalar–tensor gravity thus obtained is not restricted to the \( f(R) \) subclass nor to special metrics, and it does not require spherical symmetry or asymptotic flatness. To be concrete, we derive a quasilocal mass by writing the scalar–tensor field equations as effective Einstein equations and using the geometric derivation of the Hawking–Hayward mass in this ‘effective GR’ context.

Let us first review the basics of scalar–tensor gravity used here. The (Jordan frame) action is

\[
S_{ST} = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi} \left( \phi \mathcal{R} - \frac{\omega}{\phi} \mathcal{G}^{ab} \nabla_a \phi \nabla_b \phi \right) - V(\phi) \right\} + \mathcal{L}_{(m)},
\]

where \( \mathcal{R} \) is the Ricci curvature of the spacetime metric \( g_{ab} \) with determinant \( g \), \( \phi \) is the Brans–Dicke-like scalar field (the inverse of the effective gravitational coupling strength \( G_{eff} \), which is varying in these theories), \( V(\phi) \) is a scalar field potential, and \( \mathcal{L}_{(m)} \) is the matter Lagrangian density. The field equations of scalar–tensor theory are
\[ R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi}{\phi} T_{ab} + \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{V}{2\phi} g_{ab}, \] (2)

\[ \Box \phi = \frac{1}{2\omega + 3} \left( 8\pi T - \frac{d\omega}{d\phi} \nabla_c \phi \nabla^c \phi + \phi \frac{dV}{d\phi} - 2V \right), \] (3)

where \( T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{(\mu\nu)}} \left( \sqrt{-g} \mathcal{L}_{\text{em}} \right) \) is the stress-energy tensor of matter and \( T \equiv T^a_a \). \( f(R) \) theories \cite{3} are a subclass of scalar–tensor theories of gravity described by the action

\[ S = \int d^4 x \sqrt{-g} f(R) + S_{\text{em}}, \] (4)

where \( f(R) \) is a nonlinear function of the Ricci scalar. By setting \( f = f'(R) \) and

\[ V(\phi) = \phi f(\phi) - f(\phi), \] (5)

the action can be shown to be equivalent to the scalar–tensor one \cite{3}

\[ S = \int d^4 x \sqrt{-g} \left[ f R - V(\phi) \right] + S_{\text{em}}, \] (6)

a Brans–Dicke action with vanishing Brans–Dicke parameter \( \omega \) and potential \( V \) for the Brans–Dicke scalar \( \phi \).

### 2. Scalar–tensor quasilocal mass

The Hawking–Hayward quasilocal mass is defined as follows \cite{9,10}: let \( S \) be an embedded spacelike, compact, and orientable 2-surface with induced 2-metric \( h_{ab} \) and induced Ricci scalar \( \mathcal{R}^{(b)} \). Consider ingoing \((-)\) and outgoing \((+)\) null geodesic congruences from \( S \). Let \( \theta_{\pm} \) and \( \sigma_{ab}^{\pm} \) be the expansions and shear tensors of these congruences, respectively, and \( \omega^a \) be the projection onto \( S \) of the commutator of the null normal vectors to \( S \) (the anholonomicity \cite{10}). \( \mu \) is the volume 2-form on the surface \( S \) of area \( A \). The Hawking–Hayward quasilocal energy is \cite{9,10}

\[ M = \frac{1}{8\pi G} \sqrt{\frac{A}{16\pi}} \int_S \mu \left( \mathcal{R}^{(b)} + \theta_{(+)} \theta_{(-)} - \frac{1}{2} \sigma_{ab}^{(+)} \sigma_{ab}^{(-)} - 2\omega^a \omega^a \right). \] (7)

The contracted Gauss equation \cite{10}

\[ \mathcal{R}^{(b)} + \theta_{(+)} \theta_{(-)} - \frac{1}{2} \sigma_{ab}^{(+)} \sigma_{ab}^{(-)} = h^{ac} h^{bd} R_{abcd} \] (8)

can be used to compute the first three terms in the integral. The usual splitting of the Riemann tensor into a Weyl tensor and a Ricci part

\[ R_{abcd} = C_{abcd} + g_{a[c} R_{d]b} - g_{b[c} R_{d]a} - \frac{R}{3} g_{a[c} s_{d]b}, \] (9)

and the effective Einstein equation (2) yield

\[ h^{ac} h^{bd} R_{abcd} = h^{ac} h^{bd} C_{abcd} \]

\[ + \frac{8\pi}{\phi} h^{ac} h^{bd} \left[ g_{a[c} T_{d]b} - g_{b[c} T_{d]a} - \frac{T}{2} \left( g_{a[c} s_{d]b} - g_{b[c} s_{d]a} \right) \right] \]
By computing the individual terms

\[
h^{ab}h^{bd}\left( g_{a[c} \nabla_d \phi - g_{b[c} \nabla_d \phi \right)
\]

and putting them together in equation (10), one obtains

\[
M_{ST} = \frac{1}{8\pi} \sqrt{A} \int_{S} \mu \int_{S} \mu \left[ \frac{h^{ab}h^{bd}C_{abcd} - 2\omega \omega + \frac{8\pi}{3\phi} h^{ab}T_{ab} - \frac{16\pi T}{3\phi}}{\phi} \right] + \frac{h^{ab}\nabla_a \phi \nabla_b \phi}{\phi} + \frac{\omega}{\phi^2} \left( h^{ab} \nabla_a \phi \nabla_b \phi - \frac{1}{3} \nabla^c \phi \nabla_c \phi \right) + \frac{V}{3\phi},
\]

where the \( \phi \) factor in the first term on the right-hand side is introduced by the replacement \( G \to G_{eff} \). Note that we moved \( 1/G \) inside the integral in equation (7) before replacing \( G \) with \( G_{eff} \), because otherwise a factor \( \phi \) replacing \( 1/G \) would appear outside the integral in equation (16), making \( M_{ST} \) a function on the surface \( S \) instead of a number specified once this surface is assigned.

The factor \( 1/\phi = G_{eff} \) does not multiply all the terms in square brackets in the integrand of (16) which compose the Hawking–Hayward mass in Einstein theory, but only the two terms containing \( T_{ab} \) and its trace. Therefore, in general one does not expect to isolate the entire GR integrand divided by \( \phi \) in the integral. However, equation (16) reduces to the standard GR expression \([9, 10]\) in the GR limit in which \( \phi \) becomes constant. If \( T_{ab} \) describes a perfect fluid, \( T_{ab} = (P + \rho)u_a u_b + P g_{ab} \), with the fluid 4-velocity \( u^a \) normal to the 2-surface \( S \) (i.e., \( h_{ab} u^b = 0 \)), then it is

\[
\frac{8\pi}{\phi} \int T_{ab} - \frac{16\pi T}{3\phi} = \frac{16\pi \rho}{3\phi}
\]

and the quasilocal mass does not depend explicitly on the pressure (as remarked in \([10, 17]\) in the spherical case).
3. Spherical symmetry

Let us specialize now to spherical symmetry, in which the Hawking–Hayward quasilocal mass construct reduces \([17]\) to the better known Misner-Sharp-Hernandez mass \([18]\), which is defined by

\[
M_{\text{MSH}} = \frac{R}{2G} (1 - \nabla^a R \nabla_a R),
\]

(18)

where \(R\) is the areal radius. Let \(S\) be now a 2-sphere of symmetry with induced metric \(h_{ab}\) and write the line element as

\[
dx^2 = g_{00} dt^2 + g_{11} dr^2 + R^2 d\Omega^2_{(2)} = I_{ab} dx^a dx^b + h_{ab} dx^a dx^b
\]

(19)

in coordinates \((t, R, \theta, \varphi)\), where \(I_{ab} = \text{diag}(g_{00}, g_{11})\), \(h_{ab} = \text{diag}(R^2, R^2 \sin^2 \theta)\), and \(d\Omega^2_{(2)} = d\theta^2 + \sin^2 \theta d\varphi^2\) is the metric on the unit 2-sphere equation \((16)\) becomes

\[
M_{\text{ST}} = \frac{\phi R^3}{4} \left[ h^{ab} h^{cd} C_{abcd} + \frac{8\pi}{\phi} h^{ab} T_{ab} - \frac{16\pi T}{3\phi} \right]
\]

\[
+ \frac{\omega}{\phi^2} \left( h^{ab} \nabla_a \phi \nabla_b \phi - \frac{1}{3} \nabla^\rho \phi \nabla_\rho \phi \right) + \frac{h^{ab} \nabla_a \phi \nabla_b \phi}{\phi} + \frac{V}{3\phi}. \quad (20)
\]

Let us specialize now to cosmological metrics and then to the subclass of \(f(R)\) theories of gravity.

3.1. Friedmann–Lemaitre–Robertson–Walker (FLRW) geometry

Consider as an application the spatially flat FLRW space sourced by a perfect fluid. The line element is

\[
dx^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2_{(2)} \right)
\]

(21)

and the areal radius is \(R(t, r) = a(t) r\). One easily obtains

\[
M_{\text{ST}} = \frac{4\pi R^3}{3} \rho + \frac{\phi R^3}{4} \left( \frac{\omega}{3} \frac{\phi^2}{\phi^2} - \frac{2H \phi}{\phi} + \frac{V}{3\phi} \right). \quad (22)
\]

By using the fact that \(\frac{g}{2}(1 - g^{ab} \nabla_a R \nabla_b R) = H^2 R^3/2\), \(h^{ab} \nabla_a \phi = -2H \phi\), and the Hamiltonian constraint

\[
H^2 = \frac{8\pi \rho}{3\phi} - H \frac{\dot{\phi}}{\phi} + \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{V}{6\phi} \equiv \frac{8\pi (\rho + \rho_\phi)}{3\phi},
\]

(23)

replacing \(G\) with \(G_{\text{eff}} = \phi^{-1}\) leads to

\[
M_{\text{ST}} = \frac{H^2 R^3 \phi}{2} = \frac{4\pi R^3}{3} \left( \rho + \rho_\phi \right) = \frac{R}{2} (1 - \nabla^\rho R \nabla_\rho \phi). \quad (24)
\]

This is nothing but the expression of the Misner–Sharp–Hernandez mass \((18)\) with the replacement \(G \rightarrow G_{\text{eff}}\).

3.2. FLRW space in \(f(R)\) gravity

In metric \(f(R)\) gravity it is \(\phi = f'(R)\), \(\omega = 0\), \(V(\phi) = f'(R)R - f(R)\), and using the analogue of equation \((23)\)” [3]
\[ H^2 = \frac{1}{3f'} \left[ 8\pi \rho + \frac{\mathcal{R} f'}{2} - f - 3 \mathcal{H} (f') \right] , \]  

one obtains\(^1\)

\[ M_{f(R)} = \frac{H^2 \mathcal{R} \phi}{2} = \frac{4\pi R^3}{3} \rho + \frac{R^3}{2} \left( \frac{\mathcal{R} f'}{6} - \mathcal{H} \mathcal{R} \right) . \]  

4. Einstein frame

It is also possible to derive the result (16) in vacuo with an independent procedure using the Einstein frame representation of scalar–tensor gravity, although the derivation relies heavily on a technical result about the transformation of the quasilocal mass under conformal rescalings, which was obtained only recently \[19].

As is well known, under the conformal rescaling of the metric

\[ g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \Omega = \sqrt{\phi} \]  

and the nonlinear scalar field redefinition \( \phi \rightarrow \tilde{\phi}(\phi) \) with

\[ d\tilde{\phi} = \frac{2\omega + 3}{16\pi} \frac{d\phi}{\phi} , \]  

the action (1) assumes the Einstein frame form

\[ S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\mathcal{R}}{16\pi} - \frac{1}{2} \tilde{g}^{ab} \nabla_a \tilde{\phi} \nabla_b \tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}_{(m)}}{\tilde{\phi}^2} \right] , \]  

where

\[ U(\tilde{\phi}) = V \left[ \frac{\phi(\tilde{\phi})}{\tilde{\phi}} \right] \left( \frac{\phi(\tilde{\phi})}{\tilde{\phi}} \right)^2 . \]

In the Einstein conformal frame with tilded variables \( (\tilde{g}_{ab}, \tilde{\phi}) \) the ‘new’ scalar field \( \tilde{\phi} \) has canonical kinetic energy density and couples minimally with gravity but non-minimally with matter, hence the theory in vacuo is formally GR and the Hawking–Hayward quasilocal mass is well defined. The Einstein frame scalar field \( \tilde{\phi} \) has the canonical energy–momentum tensor

\[ \tilde{T}_{ab} = \nabla_a \tilde{\phi} \nabla_b \tilde{\phi} - \frac{1}{2} \tilde{g}_{ab} \tilde{g}^{cd} \nabla_c \tilde{\phi} \nabla_d \tilde{\phi} - U(\tilde{\phi}) \tilde{g}_{ab} . \]  

Using equations (27) and (28), this stress-energy tensor is written in terms of Jordan frame quantities as

\[ \tilde{T}_{ab} = \frac{2\omega + 3}{16\pi \tilde{\phi}^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} \tilde{g}_{ab} \nabla^c \phi \nabla_c \phi \right) - \frac{V}{16\pi} \tilde{g}_{ab} \]  

\[ \]  

\[^1\] This expression disagrees with that of [14], which contains extra terms and in which the sign of the last two terms on the right-hand side of our equation (26) is the opposite of ours, which means that in [14] the density \( \rho_\phi \), defined in equation (23) is subtracted, instead of being added, to \( M_{f(R)} \).
and its Einstein frame trace is

\[ g^{ac}T_{ac}^{(\phi)} = -\left(\frac{2\omega + 3}{16\pi\phi^4}\right)\nabla^c\phi\nabla_\phi - \frac{V}{4\pi\phi^2}. \]  

Regarding the scalar–tensor theory in the Einstein frame formally as GR (with the exception of the anomalous coupling of the scalar \( \phi \) to matter which, as we shall see below, has some consequences), we can see matter in the Einstein frame as being described by the total energy–momentum tensor

\[ T_{ab} = \tilde{T}^{(m)}_{ab} + \tilde{T}^{(\phi)}_{ab} = \frac{T_{ab}^{(m)}}{\phi^2} + \tilde{T}^{(\phi)}_{ab}. \]  

The Hawking–Hayward quasilocal mass in the Einstein frame is then given by

\[ M_{\text{GR}} = \frac{1}{8\pi} \sqrt{\frac{\bar{A}}{16\pi}} \int_S \left[ \frac{1}{G} \left( \tilde{h}^{ac} \tilde{h}^{bd} \tilde{C}_{abcd} + 8\pi G \tilde{T}^{ab,abcd} - \frac{16\pi G}{3} \tilde{g}^{abcd} \tilde{T}_{abcd} \right) + 8\pi G \tilde{h}^{ab,abcd} \tilde{T}^{ab,abcd} - \frac{16\pi G}{3} \tilde{g}^{abcd} \tilde{T}_{abcd} \right], \]

being mindful of writing Newton’s constant inside the integral in view of the discussion of the previous section. Since

\[ \tilde{h}^{ac} \tilde{h}^{bd} \tilde{C}_{abcd} = h^{ac} h^{bd} C_{abcd}/\phi, \]

\[ 8\pi G \tilde{h}^{ab,abcd} \tilde{T}_{abcd} = 8\pi G h^{ab} \tilde{T}^{ab} /\phi, \]

\[ -\frac{16\pi G}{3} \tilde{g}^{abcd} \tilde{T}_{abcd} = -\frac{16\pi G}{3} \tilde{T}^{(m)}, \]

\[ 8\pi G \tilde{h}^{ab,abcd} \tilde{T}_{abcd} = \left( \frac{2\omega + 3}{2\phi^3} \right) h^{ab} \nabla_a\phi \nabla_b\phi - \left( \frac{2\omega + 3}{2\phi^3} \right) \nabla^a \phi \nabla_a \phi - \frac{V}{\phi^2}, \]

\[ -\frac{16\pi G}{3} \tilde{g}^{abcd} \tilde{T}_{abcd} = \left( \frac{2\omega + 3}{2\phi^3} \right) \nabla^a \phi \nabla_a \phi + \frac{4V}{3\phi^2}, \]

equation (35) becomes

\[ M_{\text{GR}} = \frac{1}{8\pi} \sqrt{\frac{\bar{A}}{16\pi}} \int_S \left[ h^{ac} h^{bd} C_{abcd} + 8\pi h^{ab} \tilde{T}_{abcd} - \frac{16\pi G}{3} \tilde{T}_{abcd} \right] \]

\[ + \left( \frac{2\omega + 3}{2\phi^2} \right) h^{ab} \nabla_a\phi \nabla_b\phi - \left( \frac{2\omega + 3}{6\phi^2} \right) \nabla^a \phi \nabla_a \phi + \frac{V}{3\phi^2} - 2\omega\phi^{\alpha\beta} \right]. \]

We can now impose that, under a conformal transformation from the Jordan frame to the Einstein frame, the quasilocal mass transforms as it does under conformal rescalings in GR and we define the quasilocal mass in the Jordan frame of scalar–tensor gravity according to this rule. This approach is independent of that of the previous section. The transformation rule of the quasilocal mass under conformal rescalings \( g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab} \) in GR was obtained recently in [19] and is
In the special case of spherical symmetry, this formula reduces to the transformation property of the Misner–Sharp–Hernandez mass reported in [20]. Here we identify \( \tilde{M}_{\text{HH}} \) with the quantity \( M_{\text{GR}} \) of equation (41) and \( M_{\text{HH}} \) with the sought-for quasilocal mass in Jordan frame scalar–tensor gravity \( M_{\text{ST}} \). Then one has

\[
M_{\text{ST}} = \frac{\mathcal{A}}{\mathcal{A}} M_{\text{GR}} - \frac{1}{4\pi} \frac{\mathcal{A}}{16\pi} \int_S \mu \left[ \mathcal{K}^{ab} \left( \frac{2\nabla_a \Omega \nabla_b \Omega}{\Omega^2} - \frac{\nabla_a \nabla_b \Omega}{\Omega} \right) - \frac{\nabla^c \Omega \nabla_c \Omega}{\Omega^2} \right].
\]

or, using \( \Omega = \sqrt{\phi} \),

\[
M_{\text{ST}} = \frac{\mathcal{A}}{\mathcal{A}} M_{\text{GR}} - \frac{1}{4\pi} \frac{\mathcal{A}}{16\pi} \int_S \mu \left[ \mathcal{K}^{ab} \left( \frac{3\nabla_a \phi \nabla_b \phi}{2\phi} - \nabla_a \nabla_b \phi \right) - \frac{\nabla^c \Omega \nabla_c \Omega}{\Omega^2} \right].
\]

equation (41) then gives

\[
M_{\text{ST}} = \frac{1}{8\pi} \frac{\mathcal{A}}{16\pi} \int \mu \left[ \mathcal{K}^{ab} \mathcal{K}^{cd} - 2\omega \nabla^a \omega^a + 8\pi \mathcal{K}^{ab} T_{ab}^{(m)} - \frac{16\pi}{3} T_{ab}^{(m)} \right]
\]

\[
+ \frac{(2\omega + 3)}{2\phi^2} \mathcal{K}^{ab} \nabla_a \phi \nabla_b \phi + \frac{V}{3\phi} - \frac{(2\omega + 3)}{6\phi^2} \nabla^c \phi \nabla_c \phi
\]

\[
- \frac{3}{2\phi^2} \mathcal{K}^{ab} \nabla_a \phi \nabla_b \phi + \frac{h^{ab}}{\phi} \nabla_a \nabla_b \phi + \frac{\nabla^c \phi \nabla_c \phi}{2\phi^2} \right].
\]

In the Jordan frame one replaces \( G \) with \( G_{\text{eff}} = \phi^{-1} \), which yields

\[
M_{\text{ST}} = \frac{1}{8\pi} \frac{\mathcal{A}}{16\pi} \int \mu \left[ \mathcal{K}^{ab} \mathcal{K}^{cd} - 2\omega \nabla^a \omega^a + 8\pi \mathcal{K}^{ab} T_{ab}^{(m)} - \frac{16\pi}{3} T_{ab}^{(m)} \right]
\]

\[
+ \frac{\omega}{\phi^2} \mathcal{K}^{ab} \nabla_a \phi \nabla_b \phi + \frac{h^{ab}}{\phi^2} \nabla_a \nabla_b \phi + \frac{\omega}{3\phi^2} \nabla^c \phi \nabla_c \phi + \frac{V}{3\phi} \right].
\]

In vacuo, this equation coincides with the result (16) of the previous section, but in the presence of matter, \( T_{ab}^{(m)} = T_{ab}^{(m)} / \phi^2 \) appears instead of \( T_{ab}^{(m)} \). A possible explanation for this incomplete match between the two results (46) and (16) is that, formally, Einstein frame scalar–tensor gravity is not exactly GR because of the nonminimal coupling of the transformed Brans–Dicke-like scalar \( \phi \) to matter, and therefore the definition of Hawking–Hayward mass is not completely appropriate, which leaves a memory in the translation of the mass \( M_{\text{GR}} \) to the Jordan frame. In other words, the Einstein frame method is not fully applicable. However, in the absence of matter, Einstein frame scalar–tensor theory is formally GR with an ordinary scalar field minimally coupled and with canonical kinetic energy density, and the method does work. It should be added that, in any case, the Einstein frame

\[\text{2 There is no prescription for the conformal transformation of the quantity } \omega \nabla \omega, \text{ which needs to be redefined in each conformal frame according to the normalization chosen for the 4-tangents to the null geodesics congruences—see the discussion in [19].}\]
method requires the additional result (42) of [19], the derivation of which is highly non-trivial.

5. Conclusions

We have derived a new formula for a quasilocal mass in general scalar–tensor gravity without assuming symmetries or asymptotic flatness and without restricting to a subclass of theories or to specific spacetime metrics. The avenue followed is simply to rewrite the scalar–tensor field equations as effective Einstein equations by regarding the $\phi$-dependent terms as an extra effective stress-energy tensor in their right-hand side and by replacing Newton’s constant $G$ with the varying coupling $G_{\text{eff}} = 1/\phi$, as is familiar in scalar–tensor gravity. This straightforward but powerful approach (which has been used successfully, for example, in cosmological perturbation theory [21] or in the initial value problem [22]) is completely independent of thermodynamical considerations and has the advantage that one does not need to guess, or derive expressions for, the thermodynamical quantities appearing in the first law, which are subject to a certain degree of ambiguity (see, e.g., the discussion in [23]). A second approach using the Einstein frame representation of scalar–tensor gravity, and relying on a previous result on the conformal transformation of the Hawking–Hayward mass in GR, reproduces the same result in vacuo but there is a difference in the presence of matter, probably because in this case Einstein frame scalar–tensor gravity is not formally GR.

Here we remain agnostic on the thermodynamic approach to the quasilocal mass in scalar–tensor gravity. However, when specialized to spherical symmetry and to FLRW space, our quasilocal mass proposal differs from previous prescriptions derived from a first law of thermodynamics for gravity (which also differ between themselves). This disagreement provides an independent approach to revisit the first law, which will be the subject of future work.

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