The Lagrange–Poincaré equations for a mechanical system with symmetry on the principal fiber bundle over the base represented by the bundle space of the associated bundle

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Abstract

The Lagrange–Poincaré equations for a mechanical system which describes the interaction of two scalar particles that move on a special Riemannian manifold, consisting of the product of two manifolds, the total space of a principal fiber bundle and the vector space, are obtained. The derivation of equations is performed by using the variational principle developed by Poincaré for the mechanical systems with a symmetry. The obtained equations are written in terms of the dependent variables which, as in gauge theories, are implicitly determined by means of equations representing the local sections of the principal fiber bundle.

1 Introduction

Full research of mechanical systems suggests a finding of all critical points belonging to the systems, together with the behavior of phase curves in their vicinities. The knowledge of these points allows one to reconstruct the evolution of the mechanical systems. At the equilibrium (the point which is

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fixed in time under the dynamics), the evolution can be described, for instance, with the help of the normal forms method proposed by Poincaré and Birkhoff.

In the case of dynamical systems with symmetry, we are also interested in finding the "fixed points". But now they characterize the steady motions (or relative equilibria) of the original systems. These fixed points correspond to the equilibrium points of the reduced mechanical systems [1][2].

The points of the relative equilibria are defined by the equations that follow from the Lagrange-Poincaré equations. These last equations are represented by a system consisting of two equations. Usually they are referred to as horizontal and vertical equations [1].

Notice that such a description of the evolution is a consequence of the choice of the special coordinate basis for the total space of the principal fiber bundle which can be associated with the original mechanical system with symmetry.

At present, the Lagrange-Poincaré equations were obtained for many mechanical systems (see e.g. [5][6] and references therein). The cited papers are dealing mainly with the systems that are invariant under the free and proper action of a Lie groups. The case of a non-free action was also considered. We refer to [7] were the necessary references may also be found.

The finite-dimensional mechanical systems with symmetry are of interest for us not only by themselves. Thanks to their properties, they can be also used as model systems in studies that are carried out in gauge theories.

For example, an intrinsic geometry of the mechanical system, which describes the motion of the scalar particle on a Riemannian manifold with a free and proper action of a group Lie, is similar to the geometry of the gauge theory for pure Yang–Mills fields [8].

Due to the symmetry, the original motion of this particle can be viewed as occurring on the total space of the corresponding principal fiber bundle. In addition, as in gauge theories, the reduction process leads to the “true motion” given on the reduced space – the base space of the principal fiber bundle [1].

But in order to be able to use the mechanical systems with symmetry as a model systems for the gauge theories, it is necessary to have an appropriate description of the evolution of these mechanical systems.

In gauge theories, one usually deals with constrained variables, i.e. with the variables that are not independent but satisfy some functional equations. These equations define (locally) the gauge surface in the space of gauge fields. The gauge surface, in turn, determines the section of the corresponding triv-

\footnote{These equations are also known as Wong’s equations [3][4].}
tial principal bundle. Hence the gauge surface is used for the coordinatization of the bundle space. This means that in gauge theories, in the case of “unresolved gauges”, that is when we are not able to find explicit solutions of the equations defining the gauge surface, we have to use dependent variables\(^2\) (or locally, the dependent coordinates) for description of the evolution. It is these variables must be used in the Lagrange-Poincaré equations for model mechanical systems with symmetry, so that it would be possible to consider obtained equations as appropriate analogues for the corresponding equations of the gauge field theories.

This approach has been employed in our works [9, 10], where we have derived the Lagrange-Poincaré equations for the above discussed mechanical system with a symmetry. This enabled us to obtain the equations for Yang–Mills fields having the same structure as the equations for the mechanical system. As follows from the Lagrange-Poincaré equations for Yang–Mills fields [11], the defining equations for the relative equilibria are based on the special spectral problem and therefore may have an infinite number of solutions representing the possible relative equilibria.

We note that in the previously mentioned works on the mechanical systems [5–7], as well as in [12], where the Lagrange-Poincaré equations were obtained for the field theories, the questions related to the description of the evolution in terms of dependent variables have been left untouched.

Next an important task in the approach based on using dependent variables, is to determine the relative equilibria in the system consisting of the gauge field which interacts with a scalar field. Before proceeding to its solution, and as a first step, it would be useful to examine an appropriate dynamical system in mechanics. This is also necessary for verifying the correctness of the corresponding equations in gauge theory.

In the present paper we consider a mechanical system which describes a motion of two interacting scalar particles on a special configuration space – the product of two spaces. The first space in this product is a finite-dimensional Riemannian manifold (without boundary), the second space is a finite-dimensional vector space. Moreover, we assume that there is a free, proper, isometric and smooth action of the compact Lie group on these spaces and, therefore, on the original manifold as a whole. It can be shown that such an action on the first space, given by the Riemannian manifold, leads to the principal fiber bundle in which this Riemannian manifold is a total space.

As a result of a group action on the whole space, we also come to the principal fiber bundle. For this bundle, the manifold, representing the origi-

\(^2\)They are the solutions of these equations.
nal configuration space of the whole system, can be regarded as a total space. But the base of the bundle, i.e. the orbit space of the group action, now is the bundle space of the associated vector bundle.

We see, that the obtained principal bundle is exactly the same which is used in the construction leading to the associated vector bundle in geometry. So the coordinates in the principal bundle are usually introduced by taking into account this fact.

The coordinates in this principal bundle are determined in a standard way, i.e. with the help of the local sections. The method, which we use for the coordinatization of this principal bundle, is typical for Yang–Mills fields with interactions. In [13], for example, it was used in the study of the quantization procedure in the scalar electrodynamics. In this method, an important role belongs to the sections of the principal fiber bundle related to the Riemannian manifold representing the first space in our configuration space.

The purpose of the present paper is to obtain the Lagrange-Poincaré equation for the model mechanical system which has been described above. It will be made, as in [10], with the help of the Poincaré variational principle [14–16]. This variational principle was developed by Poincaré for mechanical systems with a symmetry.

The paper will be organized as follows. In Section 2 we will introduce the principal fiber bundle coordinates on our original Riemannian manifold and get a new representation for the metric tensor of this manifold. In Section 3 we will change the coordinate basis of our manifold for the horizontal lift basis and also consider the corresponding transformation of the Lagrangian arising from it. In Section 4, after brief recalling the Poincaré approach to the calculus of variations, we will derive the differential relations between the variations and the quasi-velocities. These relations are necessary for the Poincaré method. In Section 5 we will obtain the Lagrange-Poincaré equations. The first of two Appendices will be dedicated to the derivation of the differential relation between the variations and the quasi-velocities associated with the group variables. The second one will contain the properties of the projection operators and Killing relations for the horizontal metric. In the last section will be given concluding remarks.
2 Principal fiber bundle coordinates on the configuration space

The configuration space of our mechanical system, describing the interaction of two scalar particles, is represented by the product manifold $\mathcal{P} \times \mathcal{V}$, where $\mathcal{P}$ is a smooth finite-dimensional Riemannian manifold (without the boundary) and $\mathcal{V}$ is a finite-dimensional vector space.

Let $(Q^A, f^n)$, $A = 1, \ldots, N_P$ and $n = 1, \ldots, N_V$ be the coordinates of a point $(p, v)$ given on a chart $(\mathcal{U}, \varphi^A)$ of the original product manifold. We assume that open sets $\mathcal{U}$ of the charts are chosen to be equal $(\mathcal{U}_P \times \mathcal{U}_V)$, and the coordinate functions for these sets are given by $\varphi^A = (\varphi^A, \varphi^n)$. So, we have $Q^A = \varphi^A(p)$ and $f^n = \varphi^n(v)$.

In these coordinates, the Riemannian metric of the manifold is written as follows:

$$ds^2 = G_{AB}(Q)dQ^AdQ^B + G_{mn}df^mdf^n,$$

where the first term represents the Riemannian metric of the manifold $\mathcal{P}$ and the second one, with matrix $G_{mn}$ consisting of some fixed constant elements, is used as the metric of the inner product in $\mathcal{V}$. It is admitted that $G_{mn}$ may be non diagonal. In the paper we assume that we are given a free, proper, smooth and isometric action of the compact Lie group $\mathcal{G}$ on the original manifold. Also we assume that the group acts on the manifold from the right: $(p, v)g = (pg, g^{-1}v)$. Being written in coordinates, this action is given as follows:

$$\tilde{Q}^A = F^A(Q, g), \quad \tilde{f}^n = \tilde{D}^n_m(g)f^m.$$  

Here $\tilde{D}^n_m(g) \equiv D^m_n(g^{-1})$, and by $D^m_n(g)$ we denote the matrix of the finite dimensional representation of the group $\mathcal{G}$ acting on the vector space $\mathcal{V}$.

For the right action of the group $\mathcal{G}$ on the point $p$ with the coordinates $Q$, we have

$$F(F(Q, g_1), g_2) = F(Q, \hat{\Phi}(g_1, g_2)),$$

where the function $\hat{\Phi}$ determines the group multiplication law in the space of the group parameters.

Note that since the group $\mathcal{G}$ acts on $\mathcal{P}$ isometrically, the metric tensor $G_{AB}$ must satisfy the following relation:

$$G_{AB}(Q) = G_{DC}(F(Q, g))F^D_A(Q, g)F^C_B(Q, g),$$

3In the sequel, the expressions having the capital indices with the tilde mark will be treated in a similar way.
with \( F^B_{\alpha}(Q, g) = \frac{\partial F_{\alpha}(Q, g)}{\partial Q^A} \). A similar relation for the tensor \( G_{mn} \):

\[
G_{pq} = G_{mn}\bar{D}^m_p(g)\bar{D}^n_q(g),
\]

(3)
can be derived from the linear isometrical action of the group \( G \) on the vector space \( V \).

The Killing vector fields for the product metric of the original manifold are given by the corresponding vector fields for the manifolds \( P \) and \( V \):

\[
K^A_{\alpha}(Q) \frac{\partial}{\partial Q^A} \bigg|_{a=e} \quad \text{and} \quad K^n_{\alpha}(f) \frac{\partial}{\partial f^n} \bigg|_{a=e} = \frac{\partial \tilde{D}^m_{\alpha}(a)}{\partial a^m} \bigg|_{a=e} = (\tilde{J}_m)^{\alpha}_{\alpha} f^m.
\]
The generators \( \tilde{J}_\alpha \) of the representation \( \bar{D}^m_{\alpha}(a) \) satisfy the commutation relation \([\tilde{J}_\alpha, \tilde{J}_\beta] = \bar{c}^{\gamma}_{\alpha\beta} \tilde{J}_\gamma\), where the structure constants \( \bar{c}^{\gamma}_{\alpha\beta} = -c^{\gamma}_{\alpha\beta} \).

Using the condensed notation, in which \( \tilde{A} \equiv (A, p) \), we can rewrite the components of the Killing vector fields as

\[
K^\tilde{A}_\mu = (K^A_{\mu}, K^p_\mu).
\]

We know from the general theory [1] that the action of a group \( G \), provided that this action satisfies the same requirements as we have assumed in our paper, leads to the orbit-fibering of the original manifold \( P \times V \). Therefore this manifold can be regarded as the total space of the principal fiber bundle where the orbit space manifold \( P \) is the base space. This means that it is possible to introduce the principal fiber bundle coordinates in each local neighborhood of the original manifold and then to express the coordinates \((Q^A, f^n)\) of the point \((p, v)\) in terms of the bundle coordinates. We briefly recall this well-known procedure [13, 17–22].

First we note that the action of a group \( G \) on \( P \) results in the principal fiber bundle \( P(M, G) \) having the base space \( M = P/G \). The coordinates of the points on this bundle will be used as the part of new coordinates on \( P \times V \).

The total space \( P \) of the principal fiber bundle \( P(M, G) \) has the following local representation \( \pi^{-1}(\mathcal{U}_\epsilon) \sim \mathcal{U}_\epsilon \times G \), where \( \mathcal{U}_\epsilon \) is an open neighborhood of the point \( x = \pi(p) \) which belongs to the chart \( (\mathcal{U}_\epsilon, \varphi_\epsilon) \) of this bundle. It follows that the principal fiber bundle coordinates of the point \( p \) are usually given by the coordinates \((x^i, a^\alpha)\), \( i = 1, \ldots, N_{\mathcal{M}}, N_{\mathcal{M}} = \dim \mathcal{M}, \alpha = 1, \ldots, N_{\mathcal{G}}, N_{\mathcal{G}} = \dim G \) \((N_P = N_{\mathcal{M}} + N_{\mathcal{G}})\).

As the coordinates on \( P(M, G) \), we take in the paper not the coordinates \((x^i, a^\alpha)\), since often it is difficult to find the interrelation between the new coordinates \( x^i \) and the initial coordinates \( Q^A \) of the point \( p \in P \), but the constrained (or dependent) coordinates. In gauge theories, such coordinates...
are determined with the help of the gauge constraints and known also as adapted coordinates \[13\]. The same approach can be used in our case. We suppose that in each sufficiently small neighbourhood of a point \( p \in \mathcal{P} \) it is possible to determine such a local submanifold \( \Sigma \) of the manifold \( \mathcal{P} \), which has a transversal intersection with the orbits. This property enables the existence of the local sections of the principal fiber bundle \[23\]. As a rule the submanifold \( \Sigma \) is given by the equations \( \{ \chi^\alpha(Q) = 0, \alpha = 1, \ldots, N_G \} \). In gauge theories, the corresponding (functional) equations are known as the gauge constraints. Hence, the submanifold \( \Sigma \), which defines the local section of the principal fiber bundle, plays the same role as the gauge fixing surface.

The coordinates of the points on the local submanifold \( \Sigma \) will be denoted by \( Q^*^\alpha \). Since they satisfy the equations \( \{ \chi^\alpha(Q^*) = 0 \} \), they are called the dependent (or constrained) coordinates. We will use these coordinates, together with the group coordinates \( a^\alpha \), as the principal bundle coordinates for the points given on the total space \( \mathcal{P} \) of the principal fiber bundle \( \mathbb{P} (\mathcal{M}, \mathcal{G}) \).

The constrained coordinates were exploited in many works devoted to the dynamical systems with a symmetry (see e.g. \[17, 18, 24–27\] and others). The use of these coordinates for the coordinatization of the principal fiber bundles was considered in \[27–29\]. We will mainly follow these works where the explanation of this procedure was given. Although as the objects of the study of these works were chosen the gauge field theories, the approach, developed there, can be also applied to our case. But, of course, it can be done only after the proper adaptation of this approach to the mechanical systems.

As it is now well-known, the introduction of the coordinates in the principal fiber bundles is based on two statements. The first statement is concerned with the existence of the special locally finite open covering \( \{ U_i \} \) of the base manifold \( \mathcal{M} \), which is necessary for coordinate definition of the principal fiber bundle. It is assumed that this open covering is constructed with the help of the set \( \{ \Sigma_i \} \) which is formed by the given local submanifolds (surfaces) \( \Sigma_i \) of the total space \( \mathcal{P} \). Moreover, it is required that a family local sections \( \{ \sigma_i \} \) of the principal fiber bundle \( \pi : \mathcal{P} \to \mathcal{M} \) can be determined by these local surfaces \( \Sigma_i \): the section \( \sigma_i \) is the map \( \sigma_i : U_i \to \Sigma_i \) such that \( \pi_{\Sigma_i} \cdot \sigma_i = \text{id}_{U_i} \).

The second statement is related to the definition of the coordinate functions of the bundle charts by which the coordinate principal fiber bundle is given. In a standard case, these functions are defined by means of the rule which establishes the local isomorphism between \( U_i \times \mathcal{G} \) and \( \pi^{-1}(U_i) \). In \[28\] and \[18\], the coordinate functions were obtained with the help of the local sections determined by means of parametrically given local surfaces \( \Sigma_i \). In these works it was supposed that the equations \( \chi^\alpha(Q) = 0 \) had the following solution: \( Q^A = Q^*^A(x), \ x \in \mathcal{M} \).
But in many cases such a representation for the solution is impossible. This is the reason of using the dependent coordinates. So, to define the coordinate functions of the principal fiber bundle arising in these cases one should use the existing local isomorphism between trivial principal bundle \( \Sigma_i \times G \to \Sigma_i \) and \( P(M, G) \) \[^{13, 29}\]. And now the coordinate functions of a bundle chart \((U_i, \varphi_i)\) perform the following isomorphism:

\[
\varphi_i : \Sigma_i \times G \to \pi^{-1}(U_i).
\]

In coordinates, this map is written as

\[
\varphi_i : (Q^*B, a^\alpha) \to Q^A = F^A(Q^*B, a^\alpha),
\]

where \( Q^*B \) are the coordinates of a point given on the local surface \( \Sigma_i \) and \( a^\alpha \) – the coordinates of an arbitrary group element \( a \). This element carries the point, taken on \( \Sigma_i \), to the point \( p \in P \) which has the coordinates \( Q^A \).

The inverse map \( \varphi_i^{-1} \),

\[
\varphi_i^{-1} : \pi^{-1}(U_i) \to \Sigma_i \times G,
\]

has the following coordinate representation:

\[
\varphi_i^{-1} : Q^A \to (Q^*B(Q), a^\alpha(Q)).
\]

Here the group coordinates \( a^\alpha(Q) \) of a point \( p \) are the coordinates of the group element which connects, by means of its action on \( p \), the surface \( \Sigma_i \) and the point \( p \in P \). These group coordinates are given by the solutions of the following equation:

\[
\chi^\beta(F^A(Q, a^{-1}(Q))) = 0. \tag{4}
\]

The coordinates \( Q^*B \) are defined by the equation

\[
Q^*B = F^B(Q, a^{-1}(Q)). \tag{5}
\]

We see that the map \( \varphi_i^{-1} \), thus defined, enables one to find the principal bundle coordinates \( (Q^*B, a^\alpha) \) of the point \( p \) from the known initial coordinates \( Q^A \) of this point given on the manifold \( P \).

We note that the bundle coordinates of \( p \in P \) were determined for the bundle chart \((U_i, \varphi_i)\) related to the local surface \( \Sigma_i \). The relationship of these coordinates of the point \( p \) with the coordinates obtained for the chart \((U_j, \varphi_j)\) is given by the transition function \( \varphi_{ji} = \varphi_j^{-1}\varphi_i \) \[^{13}\]:

\[
\varphi_{ji} : (\Sigma_i \cap \pi^{-1}(U_j)) \times G \to (\Sigma_j \cap \pi^{-1}(U_j)) \times G.
\]
In coordinates, this map is written as
\[ \varphi_{ji} : (Q^*, a) \rightarrow \left( F(Q^*, a^{-1}_j(Q^*)), \hat{\Phi}(a_j(Q^*), a) \right), \]
where by \( Q^* \) we denote the coordinates of the point belonging to \( \Sigma_i \), \( a_j(Q^*) \) – the coordinates of the group element defined by means of the local surface \( \Sigma_j \) and by \( a \) was denoted the coordinates of an arbitrary group element.

It is not difficult to check that these transition functions satisfy the cocycle relation \( \varphi_{ji} \varphi_{ik} = \varphi_{jk} \). It can be done by using the following formulae of the coordinate transformations:
\[ g^\alpha(F(Q, a)) = \hat{\Phi}^\alpha(g(Q), a), \quad g^{-1}(F(Q, a)) = \hat{\Phi}(a^{-1}, g^{-1}(Q)), \]
together with the general formulae for a group action: \( F(F(Q, g_1), g_2) = F(Q, \hat{\Phi}(g_1, g_2)) \) and \( \hat{\Phi}(g, \hat{\Phi}(g, h)) = \hat{\Phi}(\hat{\Phi}(g, a), h) \).

For the principal fiber bundle \( P(P \times_G V, G) \), adapted coordinates can be defined by the same method as it was done for \( P(M, G) \). Now \( \pi : P \times V \rightarrow P \times_G V \) means that locally we have a map \( \pi : (p, v) \rightarrow [p, v] \), where \([p, v]\) is the equivalence class formed by the equivalence relation \((pg, g^{-1}v)\).

The local section \( \tilde{\sigma}_i \) of this bundle, \( \pi \cdot \tilde{\sigma}_i = \text{id} \), is the map which sends \([p, v]\) to some element \((\tilde{p}, \tilde{v}) \) \( \in P \times V \). The section \( \tilde{\sigma}_i \) is given by
\[ \tilde{\sigma}_i([p, v]) = (\sigma_i(x), a(p)v), \]
where \( \sigma_i \) is a local section of \( P(M, G) \), \( \sigma_i : U_i \rightarrow \pi^{-1}_P(U_i), x = \pi_P(p) \) and \( a(p) \) is the group element defined by \( p = \sigma_i(x)a(p) \). Since
\[ (\sigma_i(x), a(p)v) = (pa^{-1}(p), a(p)v) = (p, v)a^{-1}(p), \]
we get
\[ \tilde{\sigma}_i([p, v]) = (p, v)a^{-1}(p). \]

We see that the image of \( \tilde{\sigma}_i \), the local surface \( \tilde{\Sigma}_i \), consists of the elements that are obtained in a similar way as the elements of “gauge fixing surface” \( \Sigma_i \in P \) for the principal fiber bundle \( P(M, G) \).

For a properly chosen family of sections \( \{\tilde{\sigma}_i\} \), and, respectively, the family of \( \{\tilde{\Sigma}_i\} \), the local isomorphisms of the principal fiber bundle \( P(P \times_G V, G) \) and the trivial principal bundles \( \tilde{\Sigma}_i \times G \rightarrow \tilde{\Sigma}_i \) enables one to introduce a new atlas on \( P(P \times_G V, G) \) with charts that are related to the submanifolds \( \{\tilde{\Sigma}_i\} \).

The coordinate functions of these charts \((\tilde{U}_i, \tilde{\varphi}_i)\), where \( \tilde{U}_i \) is an open neighborhood of the point \([p, v]\) given on the base space \( P \times_G V \), are such that
\[ \tilde{\varphi}_i^{-1} : \pi^{-1}(\tilde{U}_i) \rightarrow \tilde{\Sigma}_i \times G, \]
or in coordinates,
Here $Q^A$ and $f^m$ are the coordinates of a point $(p, v) \in \mathcal{P} \times V$, $Q^*(Q)$ is given by (5) and

$$\tilde{f}^n(Q) = D^m_n(a(Q)) f^m,$$

$a(Q)$ is defined by (4), and we have used the following property: $\bar{D}^m_n(a^{-1}) \equiv D^m_n(a)$. The coordinates $Q^*, \tilde{f}^n$, representing a point given on a local surface $\Sigma_i$, satisfy the constraints: $\chi(Q^*) = 0$. That is, they are dependent coordinates.

The coordinate function $\tilde{\varphi}_i$ maps $\tilde{\Sigma}_i \times G \to \pi^{-1}(U_i)$:

$$\tilde{\varphi}_i : (Q^*B, \tilde{f}^n, a^\alpha) \to (F^A(Q^*, a), \tilde{D}^m_n(a) \tilde{f}^n).$$

Thus, we have defined the special local bundle coordinates $(Q^*, \tilde{f}^n, a^\alpha)$, also named as adapted coordinates, on the principal fiber bundle $\pi : \mathcal{P} \times V \to \mathcal{P} \times G V$.

In the sequel we will deal, in fact, only with the local expressions that are given on a separate chart. This case may be proper regarded, and also treated, by supposing that the principal fiber bundle $\mathcal{P}(M, \mathcal{G})$ is trivial. It takes place, for example, when the local submanifolds $\{\chi^\alpha = 0\}$ form the global submanifold of the manifold $\mathcal{P}$. Note that $\mathcal{P}(\mathcal{P} \times G V, \mathcal{G})$ will be also trivial. For simplicity of further consideration, it will be assumed in the paper that such a restriction, imposed on the considered principal fiber bundles, is fulfilled.

As a consequence, we come to a local isomorphism of the trivial principal fiber bundle $\mathcal{P}(M, \mathcal{G})$ and the trivial principal bundle $\pi_\Sigma : \Sigma \times G \to \Sigma [13][28]$. Therefore, the charts of the total space $\mathcal{P}$ are expressed through the charts of the global submanifold $\Sigma$. And constrained global variables, defined on $\Sigma$, can be used as the coordinate functions of these charts. It follows that in this case, for the trivial principal fiber bundle $\mathcal{P}(\mathcal{P} \times G V, \mathcal{G})$, we have a bundle isomorphism $\tilde{\varphi} : \Sigma \times G \to \mathcal{P} \times V$ which enables us to define the charts with adapted coordinates on this bundle.

It is not difficult to obtain the representation for the Riemannian metric given on $\mathcal{P} \times V$ in terms of the principal bundle coordinates $(Q^*, \tilde{f}^n, a^\alpha)$ which we have just introduced on the principal fiber bundle. The replacement of the coordinates $(Q^A, f^m)$ of a point $(p, v) \in \mathcal{P} \times V$ for a new coordinates

$$Q^A = F^A(Q^*, a^\alpha), \quad f^m = \tilde{D}_n^m(a) \tilde{f}^n$$

leads to the following transformation of the local coordinate vector fields:

$$\frac{\partial}{\partial f^m} = D^m_n(a) \frac{\partial}{\partial f^m},$$

Here $Q^A$ and $f^m$ are the coordinates of a point $(p, v) \in \mathcal{P} \times V$, $Q^*(Q)$ is given by (5) and

$$\tilde{f}^n(Q) = D^m_n(a(Q)) f^m,$$

$a(Q)$ is defined by (4), and we have used the following property: $\bar{D}^m_n(a^{-1}) \equiv D^m_n(a)$. The coordinates $Q^*, \tilde{f}^n$, representing a point given on a local surface $\Sigma_i$, satisfy the constraints: $\chi(Q^*) = 0$. That is, they are dependent coordinates.

The coordinate function $\tilde{\varphi}_i$ maps $\tilde{\Sigma}_i \times G \to \pi^{-1}(U_i)$:

$$\tilde{\varphi}_i : (Q^*B, \tilde{f}^n, a^\alpha) \to (F^A(Q^*, a), \tilde{D}^m_n(a) \tilde{f}^n).$$

Thus, we have defined the special local bundle coordinates $(Q^*, \tilde{f}^n, a^\alpha)$, also named as adapted coordinates, on the principal fiber bundle $\pi : \mathcal{P} \times V \to \mathcal{P} \times G V$.
\[
\frac{\partial}{\partial Q^B} = \frac{\partial Q^A}{\partial Q^B} \frac{\partial}{\partial Q^A} + \frac{\partial \alpha}{\partial Q^B} \frac{\partial}{\partial \alpha} + \frac{\partial \tilde{f}^n}{\partial Q^B} \frac{\partial}{\partial f^n} \\
= F_B^C \left( N_C^A(Q^\ast) \frac{\partial}{\partial Q^A} + \chi_C^\mu(\Phi^{-1})^\beta_{\mu}(a) \frac{\partial}{\partial a^\mu} - \chi_C^\mu(\Phi^{-1})^\beta_{\mu}(J_\nu)^a \tilde{f}_\mu^a \frac{\partial}{\partial f^\nu} \right) . \tag{7}
\]

Here \( F_B^C \equiv F_B^C(F(Q^\ast, a), a^{-1}) \) is an inverse matrix to the matrix \( F_B^A(Q^\ast, a) \), \( \chi_C^\mu \equiv \frac{\partial \chi^\mu(Q)}{\partial Q^a} \bigg|_{Q=Q^\ast} \), \((\Phi^{-1})^\beta_{\mu} \equiv (\Phi^{-1})^\beta_{\mu}(Q^\ast) \) – the matrix which is inverse to the Faddeev–Popov matrix:

\[
(\Phi)^\beta_{\mu}(Q) = K_A^\mu(Q) \frac{\partial \chi^\beta(Q)}{\partial Q^A} ,
\]

the matrix \( \tilde{v}_\beta^\mu(a) \) is inverse of the matrix \( \tilde{u}_\beta^\mu(a) \). \(^4\)

The operator \( N_C^A \), defined as

\[
N_C^A(Q) = \delta_C^A - K_A^\alpha(Q)(\Phi^{-1})^\alpha_{\mu}(Q)\chi_C^\mu(Q),
\]

is the projection operator \((N_B^A N_C^B = N_C^A)\) onto the subspace which is orthogonal to the Killing vector field \( K_A^\alpha(Q) \frac{\partial}{\partial Q^\alpha} \). \( N_C^A(Q^\ast) \) is the restriction of \( N_C^A(Q) \) to the submanifold \( \Sigma \):

\[
N_C^A(Q^\ast) \equiv N_C^A(F(Q^\ast, e)) \quad N_C^A(Q^\ast) = F_B^C(Q^\ast, a) N_B^M(F(Q^\ast, a)) F_M^A(Q^\ast, a)
\]

\( e \) is the unity element of the group.

We note also that formula (7) is a generalization of an analogous formula from [19, 21].

As an operator, the vector field \( \frac{\partial}{\partial Q^A} \varphi(Q^\ast) \) is determined by means of the following rule:

\[
\frac{\partial}{\partial Q^A} \varphi(Q^\ast) = (P_\perp)_A^D(Q^\ast) \frac{\partial \varphi(Q)}{\partial Q^D} \bigg|_{Q=Q^\ast}
\]

where the projection operator \((P_\perp)_B^A\) on the tangent plane to the submanifold \( \Sigma \) is given by

\[
(P_\perp)_B^A = \delta_B^A - \chi_B^\alpha (\chi^\alpha)^{-1} \beta_a (\chi^\alpha)^{\beta}_a.
\]

In this formula, \((\chi^\alpha)^\beta_a\) is a transposed matrix to the matrix \(\chi^\beta_a\):

\[
(\chi^\alpha)^\beta_a = G^{aB} \gamma_{\mu\nu} \chi_B^\nu \quad \gamma_{\mu\nu} = K_A^\mu G^{AB} K_B^\nu .
\]

Using the above explicit expression for the projection operators, it is easy to derive their multiplication properties:

\[
(P_\perp)_B^A N_C^B = (P_\perp)_B^C, \quad N_B^A(P_\perp)_A^C = N_B^C.
\]

\(^4\) \( \det \tilde{u}_\beta^\mu(a) \) is the density of the right-invariant measure given on the group \( G \).
In the new coordinate basis \((\partial/\partial Q^A, \partial/\partial \hat{f}^m, \partial/\partial a^a)\) the metric \(G\) of the original manifold \(P \times V\) can be rewritten as follows:

\[
\hat{G}_{AB}(Q^*, \hat{f}, a) = \begin{pmatrix}
G_{CD}(P_\perp)\delta^A_B(P_\perp) & 0 & G_{CD}(P_\perp)\Lambda^B_C K^D_F \bar{u}_\alpha^D \\
0 & G_{mn} & G_{mp} K^F_P \bar{u}_\nu^p \\
G_{BC} K^C_F \bar{u}_\beta^C & G_{np} K^p_F \bar{u}_\nu^p & d_{\mu \nu} \bar{u}_\alpha^\mu \bar{u}_\beta^\nu
\end{pmatrix}
\]  

(8)

where \(G_{CD}(Q^*) \equiv G_{CD}(F(Q^*, e))\):

\[
G_{CD}(Q^*) = F^M_C(Q^*, a) F^N_D(Q^*, a) G_{MN}(F(Q^*, a)),
\]

the projection operators \(P_\perp\) and the components \(K^A_F\) of the Killing vector fields depend on \(Q^*\), \(\bar{u}_\beta^\mu(a)\), \(K^\nu_F(\hat{f})\), \(d_{\mu \nu}(Q^*, \hat{f}) \bar{u}_\alpha^\mu(a) \bar{u}_\beta^\nu(a)\) is the metric on \(\mathcal{G}\)-orbit through the point \((p, v)\). The components \(d_{\mu \nu}\) of this metric are given by

\[
d_{\mu \nu}(Q^*, \hat{f}) = K^A_F(Q^*) G_{AB}(Q^*) K^B_F(Q^*) + K^m_m(\hat{f}) G_{mn} K^n_F(\hat{f})
\]

\[
\equiv \gamma_{\mu \nu}(Q^*) + \gamma'_{\mu \nu}(Q^*).
\]

Also we note that when we made the change of the coordinates in the differential \(df\):

\[
df = D^n_m(a) df^m + \frac{\partial D^n_m(a)}{\partial a^\alpha} \hat{f}^m da^\alpha,
\]

we have used the following transformations:

\[
\frac{\partial D^n_m(a)}{\partial a^\nu} \hat{f}^m = (\hat{J}_\mu)_m \hat{D}^n_m(a) \bar{u}_\beta^\nu(a) \hat{f}^m = K^1_\beta(\hat{f}) \hat{D}^n_m(a) \bar{u}_\mu^\beta(a).
\]

The last equality is due to the identity \(D^k_\beta(a) (\hat{J}_\beta)_k \hat{D}^n_m(a) = \rho_\beta^k(a) (\hat{J}_\beta)_k\), in which \(\rho_\beta^k(a) = \bar{u}_\gamma^\beta(a) v_\alpha^k(a)\) is the matrix of the adjoint representation of the group \(\mathcal{G}\).

The pseudoinverse matrix \(\tilde{G}^{AB}(Q^*, \hat{f}, a)\) to matrix \(G\) is as follows:

\[
\begin{pmatrix}
G_{EF} N^A_E N^B_F \\
-G_{EF} N^A_E \Lambda^\nu_F K^m_m \\
G_{EF} N^A_E \Lambda^\beta_F \nu^\alpha_eta
\end{pmatrix}
-\begin{pmatrix}
-G_{EF} N^A_E \Lambda^\nu_F K^m_m \\
G_{mn} + G_{EF} \Lambda^\nu_F \Lambda^\mu_F K^m_m K^m_k \\
-G_{EF} \Lambda^\beta_F \nu^\alpha_\beta
\end{pmatrix}
-\begin{pmatrix}
G_{EF} N^A_E \Lambda^\beta_F \nu^\alpha_\beta
\end{pmatrix}
\]  

(9)

Here \(\Lambda^\nu_E \equiv (\Phi^{-1})_\mu^\nu(Q^*) \chi^\mu_E(Q^*)\).

The pseudoinversion of \(\tilde{G}_{AB}\) means that

\[
\tilde{G}^{\hat{A} \hat{B}} \tilde{G}_{\hat{A} \hat{B}} = \begin{pmatrix}
(P_{\perp})_\perp & 0 & 0 \\
0 & \delta^\alpha_\beta & 0 \\
0 & 0 & \delta^\alpha_\beta
\end{pmatrix}
\]  

(12)
3 Transformation of the Lagrangian

In terms of initial local coordinates defined on the original manifold $P \times V$, the Lagrangian for the considered mechanical system can be written as follows:

$$\mathcal{L} = \frac{1}{2} G_{AB} (Q) \dot{Q}^A \dot{Q}^B + \frac{1}{2} G_{mn} \dot{f}^m \dot{f}^n - V(Q, f).$$

(10)

By our assumption, the potential $V(Q, f)$ is a $G$-invariant function: $V(Q, f) = V(F(Q, a), D(a)f)$. So the whole Lagrangian is also invariant.

As the configuration space $P \times V$ of our mechanical system is a total space of the principal fiber bundle $P(P \times G, G_V)$, the evolution of the system may be equally represented by using the bundle coordinates. In particular, in the Lagrangian (10), new coordinates are introduced by using the replacement of the local coordinates (6). As a result, we get

$$\mathcal{L} = \frac{1}{2} G_{CD} \left( \frac{dQ^C}{dt} + K^C_{\mu} \tilde{u}_{\mu}(a) \frac{da^\alpha}{dt} \right) \left( \frac{dQ^D}{dt} + K^D_{\nu} \tilde{u}_{\nu}(a) \frac{da^\beta}{dt} \right)$$

$$+ \frac{1}{2} G_{mn} \left( \frac{df^m}{dt} + K^m_{\beta} \tilde{u}_{\beta}(a) \frac{da^\alpha}{dt} \right) \left( \frac{df^n}{dt} + K^n_{\nu} \tilde{u}_{\nu}(a) \frac{da^\mu}{dt} \right) - V,$$

(11)

where now $G_{CD}, K^C_{\mu}$ depend on $Q^*, K^m_{\beta} = K^m_{\beta}(\tilde{f})$, and $V = V(Q^*, \tilde{f})$.

We note that transformation of the velocities $\dot{Q}^A(t)$ in (11), for $Q^A(t) = F^A(Q^*^D(t), a^\alpha(t))$, was made as follows:

$$\dot{Q}^A(t) = \frac{dQ^A}{dt} = F^A_C \left( P_{\perp}^C \frac{dQ^D}{dt} + F^A_{\alpha} \frac{da^\alpha}{dt} \right)$$

$$= F^A_C \left( \frac{dQ^C}{dt} + K^C_{\beta}(Q^*) \tilde{u}_{\beta}(a) \frac{da^\alpha}{dt} \right),$$

where $F^A_C = \frac{\delta F^A(Q^*, a)}{\delta a^\alpha} = F^A_C K^C_{\beta} \tilde{u}_{\beta}$. Besides, we have used the identity $(P_{\perp})^C_D \frac{dQ^D}{dt} = \frac{dQ^C}{dt}$, which is valid since the velocity $dQ^D/\dot{f}$ belongs to the tangent space $T \Sigma$ of the gauge fixing surface $\Sigma = \{Q^*^D : \chi^\alpha(Q^*) = 0\}$.

Our next task is to introduce a special coordinate basis (the horizontal lift basis) on the total space of the principal fiber bundle. This basis is needed for derivation of the Lagrange-Poincaré equations in the considered problem. We note that coordinate vector fields of this basis do not commute between themselves. Sometimes, mainly in a physical literature, such bases are called the nonholonomic. In our works [9][10], an analogous basis was constructed for $P(M, G)$.

Since the new basis consists of the horizontal and vertical vector fields, this means that there is a connection on the considered principal fiber bundle.
In case of the reduction of mechanical systems, it is this case we study here, there exists [1] a natural connection called the "mechanical connection". So, it is quite natural that this connection takes part in the process of the separation of vector fields into two orthogonal sets. We briefly recall how one can introduce these vector fields.

The one-form $\tilde{\omega}^\alpha$ on the principal fiber bundle $P(\mathcal{P} \times G, \mathcal{G})$ the connection form, is given by the following formula written in terms of the initial local coordinate given on the total space $\mathcal{P} \times V$:

$$\tilde{\omega}^\alpha(Q, f) = d^{\alpha\beta}(Q, f) \left( K_B^\beta(Q)G_{BA}(Q)dQ^A + K_D^\beta(f)G_{pq}df^q \right). \tag{12}$$

To rewrite this connection form in terms of the principal fiber bundle coordinates $(Q^A, \tilde{f}^\alpha, a^\alpha)$, one should perform the corresponding transformations of all terms in this expression. It can be made as follows:

$$dQ^A = F^A_\lambda((P_\perp)_D^\mu dQ^{\mu D} + K^\lambda_\mu(Q^*)\tilde{u}_\alpha(a)da^\alpha),$$

with $(P_\perp)_D^\mu dQ^{\mu D} = dQ^{\mu A}$, $df^\alpha = \tilde{D}_n^\alpha(a) (d\tilde{f}^\alpha + K_n^\alpha(\tilde{f})\tilde{u}_\alpha(a)da^\alpha)$.

$$K_B^\beta(F(Q^*, a)) = \rho_{\beta}(a)K_D^\beta(Q^*)F_D^\beta(Q^*, a),$$

and

$$K_D^\beta(D(a)\tilde{f}) = \rho_{\beta}(a)D^\beta(\tilde{f}).$$

To transform $G_{BA}(Q)$ and $G_{pq}$ one must takes into account the relations (2) and (3). The above transformations leads to

$$\tilde{\omega}^\alpha = \tilde{\rho}^\alpha_{\mu}(a) \left( d^{\alpha\mu}K_D^\beta(Q^*)G_{DA}(Q^*)dQ^{\mu A} + d^{\alpha\mu}K_B^\mu(\tilde{f})G_{pq}df^q \right) + u^\alpha_\beta(a)da^\alpha,$$

where now $d^{\alpha\mu} = d^{\alpha\mu}(Q^*, \tilde{f})$. The obtained expression for $\tilde{\omega}^\alpha$ may be rewritten as follows:

$$\tilde{\omega}^\alpha = \mathcal{A}^\alpha_B(Q^*, \tilde{f}, a)dQ^A + \mathcal{A}^\alpha_m(Q^*, \tilde{f}, a)df^m + u^\alpha_\beta(a)da^\alpha, \tag{13}$$

where we have introduced the (gauge) potentials $\mathcal{A}^\alpha_B$, and $\mathcal{A}^\alpha_m$, together with a new notation: $\mathcal{A}^\alpha_B = \tilde{\rho}^\alpha_{\mu}(a)\mathcal{A}^\alpha_B(Q^*, \tilde{f})$.

The same may be written in the condensed notation:

$$\tilde{\omega}^\alpha = \tilde{\mathcal{A}}^\alpha_B(Q^*, \tilde{f}, a)dQ^A + u^\alpha_\beta(a)da^\alpha,$$

The one-form $\tilde{\omega}$ with the value in the Lie algebra of the group Lie $\mathcal{G}$ is $\tilde{\omega} = \tilde{\omega}^\alpha \otimes \lambda_\alpha$. 

14
implying that the index represented by the capital Latin letter with the tilde mark has two components: $\tilde{B} = (B, p)$ and, respectively, the variables are given as $Q^{\ast \tilde{B}} = (Q^{\ast B}, \tilde{f}^p)$. The condensed notation will be also used in the sequel.

We note that analogous transformations convert the Killing vector field $K_\alpha(Q, f)$, the vertical vector field,

$$K_\alpha(Q, f) = K^B_\alpha(Q) \frac{\partial}{\partial Q^B} + K^p_\alpha \frac{\partial}{\partial f^p},$$

into the vector field $L_\alpha = v^\nu_\alpha(a) \frac{\partial}{\partial a^\nu}$ which is the left-invariant vector field given on the group manifold $G$.

In order to define the horizontal vector fields, we need the horizontal projection operators. These operators must extract the direction which is normal to the orbit: $\Pi^A \tilde{E}_K \tilde{E}_\alpha = 0$. They are defined as follows:

$$\Pi^A \tilde{B} = \delta^A \tilde{B} - K^A_\alpha d^{\alpha \beta} K^D_\beta G_{DB}.$$

By $\Pi^A \tilde{B}$, we denote the four component operator:

$$\Pi^A \tilde{B} = (\Pi^A, \Pi^A_m, \Pi^m_A, \Pi^m_n).$$

The components are given by the following formulae:

$$\Pi^A_B = \delta^A_B - K^A_\alpha d^{\alpha \beta} K^D_\beta G_{DB},$$

$$\Pi^A_m = - K^A_\mu d^{\mu \nu} K^p_\nu G_{pm},$$

$$\Pi^m_A = - K^m_\nu d^{\mu \nu} K^D_\mu G_{DA},$$

$$\Pi^m_n = \delta^m_n - K^m_\mu d^{\mu \nu} K^r_\nu G_{rn}.$$

The horizontal vector fields are defined as follows:

$$H_A(Q, f) = \Pi^R_A \frac{\partial}{\partial Q^R} + \Pi^q_A \frac{\partial}{\partial f^q},$$

$$H_p(Q, f) = \Pi^R_p \frac{\partial}{\partial Q^R} + \Pi^m_p \frac{\partial}{\partial f^m}. (15)$$

For the connection form $\hat{\omega}^\alpha$ from (12), one can easily check the fulfilment of the following equalities:

$$\hat{\omega}^\alpha(H_A) = 0, \quad \hat{\omega}^\alpha(H_p) = 0, \quad \hat{\omega}^\alpha(K_\beta) = \delta^\alpha_\beta.$$
This means that $H_A$ and $H_p$ are the horizontal vector fields. The Killing vector field $K_β$ is the vertical one.

With formulae (7) and the properties of the projection operator $Π^A_B$ that are given in Appendix, we may express the horizontal vector fields (14) and (15) in terms of the principal fiber bundle coordinates. They are given as follows:

$$H_A(F^B(Q^*, a), D^r_p(a) \tilde{f}^p) = \tilde{F}^M_A H_M(Q^*, \tilde{f}, a),$$

where

$$H_M(Q^*, \tilde{f}, a) = \left[ N^T_M \left( \frac{∂}{∂Q^*} - ς^T_{\alpha L} L_α \right) + N^m_M \left( \frac{∂}{∂f^m} - ς^m_{\alpha L} L_α \right) \right], \quad (16)$$

and

$$H_p(F^B(Q^*, a), D^r_p(a) \tilde{f}^p) = D^m_p(a) H_m(Q^*, \tilde{f}, a),$$

where

$$H_m(Q^*, \tilde{f}, a) = \left( \frac{∂}{∂f^m} - ς^m_{\alpha L} L_α \right).$$

In equation (16), we use the components of the projection operator $N^A_C$:

$$N^A_C = (n^A_C, n^A_m, n^m_A, n^m_p).$$

Besides of $N^A_C$, which was already defined, the components are

$$N^A_m = 0, \quad N^m_A = -K^m_{\alpha}(\Phi^{-1})_\alpha^\mu \lambda^\mu_A = -K^m_\alpha \Lambda^\alpha_A, \quad N^m_p = \delta^m_p.$$

The operator $N^A_B$ satisfy the following properties:

$$N^A_B N^B_C = N^A_C, \quad Π^L_B N^A_L = N^A_B, \quad Π^L_A N^A_C = Π^A_C.$$

The horizontal vector fields that are defined by the formulae (11) and (17), together with the left-invariant vector field $L_α$, represent the new local coordinate basis for our principal fiber bundle. The horizontal coordinate vector fields of this basis do not commute between themselves. They commutation relations are as follows:

$$[H_A, H_B] = C^T_{AB} H_T + C^p_{AB} H_p + C^\alpha_{AB} L_α, \quad (18)$$

where the “structure constants” are given by

$$C^T_{AB} = (∆^A_B N^R_B - ∆^A_B N^R_A) K^T_{R},$$

$$C^p_{AB} = -N^R_A (Λ^α_{R,D} - Λ^α_{D,R}) K^p_α - c_αβ γ A B K^p_α, \quad \alpha, \beta = 1, \ldots, r.$$
\[ C_{AB} = -N_A^SN_B^P \tilde{F}_{SP}^\alpha - (N_A^SN_B^P - N_B^SN_A^P) \tilde{F}_{Ep}^\alpha + N_A^mN_B^p \tilde{F}_{pm}^\alpha. \]

In \( \mathbb{C}^T_{AB} \), we denote the partial derivative of \( K^T_{\gamma} \) with respect to \( Q^* \) by \( K^T_{\gamma} \).

In \( \mathbb{C}^\alpha_{AB} \), the curvature tensor \( \tilde{F}_{SP}^\alpha \) of the connection \( \tilde{A}^\alpha_{SP} \) is given by

\[ \tilde{F}_{SP}^\alpha = \frac{\partial}{\partial Q^*S} \tilde{F}_{SP}^\alpha - \frac{\partial}{\partial Q^*P} \tilde{F}_{SP}^\alpha + c^\alpha_{\nu\sigma} \tilde{F}_{\nu S}^\alpha \tilde{F}_{\sigma P}^\alpha, \]

\[ (\tilde{F}_{SP}^\alpha(Q^*, a) = \tilde{\rho}^\alpha_{\mu}(a) F_{SP}^\mu(Q^*)). \]

The tensors \( \tilde{F}_{Ep}^\alpha \) and \( \tilde{F}_{pm}^\alpha \) are defined in a similar way.

Next commutation relations are

\[ [H_A, H_p] = C_{Ap}^m H_m + C_{Ap}^\alpha L_\alpha \]

with

\[ C_{Ap}^m = (J_\alpha)_p^m \Lambda_A^\alpha, \quad C_{Ap}^\alpha = -N_A^F \tilde{F}_{Ep}^\alpha - N_A^m \tilde{F}_{mp}^\alpha, \]

and

\[ [H_p, H_q] = C_{pq}^\alpha L_\alpha \]

with

\[ C_{pq}^\alpha = -\tilde{F}_{pq}^\alpha. \]

We notice that the left-invariant vector fields \( L_\alpha \) of the new basis commute with the coordinate horizontal vector fields:

\[ [H_A, L_\alpha] = 0, \quad [H_p, L_\alpha] = 0. \]

Also, for \( L_\alpha \) we have \[ L_\alpha, L_\beta = c_{\alpha\beta}^\gamma L_\gamma. \]

In the new coordinate basis \((H_A, H_p, L_\alpha)\), the metric \( \tilde{G}_{AB} \) can be written as the metric \( \tilde{G}_{AB} \) with following components:

\[ \tilde{G}_{AB}(Q^*, f, a) = \begin{pmatrix} \tilde{G}_{AB}^H & \tilde{G}_{Am}^H & 0 \\ \tilde{G}_{nB}^H & \tilde{G}_{nm}^H & 0 \\ 0 & 0 & \tilde{d}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \tilde{G}_{AB}^H & 0 \\ 0 & \tilde{d}_{\alpha\beta} \end{pmatrix}, \]

where \( \tilde{d}_{\alpha\beta} = \rho^\alpha_{\alpha'} \rho^\beta_{\beta'} d_{\alpha'\beta'} \). The components of the “horizontal metric” \( \tilde{G}_{AB}^H \) depending on \((Q^*A, \tilde{f}^m)\) are defined as follows:

\[ \tilde{G}_{AB}^H = \Pi_A^A \Pi_B^B G_{AB} - G_{AD} K_{\alpha}^D a^{\alpha\beta} K_{\beta}^R G_{RB}. \]

because of \( \Pi_A^A \Pi_B^B G_{CD} = \Pi_A^C \Pi_B^D G_{CD} + \Pi_A^B \Pi_B^p G_{qp} \).
\[ \tilde{G}_{Am} = -G_{AB} K^B_{\alpha} d^{\alpha \beta} K^p_{\beta} \tilde{G}_{pm}. \]

Notice that \( \tilde{G}_{Am} \) is equal to
\[ \tilde{G}_{mA} = -G_{mq} K^q_{\mu} d^{\mu \nu} K^D_{\nu} \tilde{G}_{DA}. \]

\[ \tilde{G}_{mn} = \Pi_m \Pi_n G, \quad \text{or} \quad \Pi \tilde{C}_m \Pi \tilde{D}_n G \tilde{C} \tilde{D} = \Pi C_m \Pi D_n G_{CD} + \Pi_r \Pi_q G_{rq} = G_{mn} - G_{mq} K^q_{\mu} d^{\mu \nu} K^D_{\nu} \tilde{G}_{DA}. \]

It worth to note that metric \( \tilde{G}_{AB} \) is given on the local surface \( \tilde{\Sigma} \) and gives rise the metric on the orbit space \( \mathcal{P} \times g V \) provided that the submanifold \( \tilde{\Sigma} \) is given parametrically.

The pseudoinverse matrix \( \tilde{G}^{AB} \) to the matrix \( (21) \) is represented as
\[ \tilde{G}^{AB} = \begin{pmatrix} G_{EF} N^A_E N^B_F & G_{EF} N^q_E N^p_F & 0 \\ G_{FP} N^p_E N^A_F & G_{FP} N^q_E N^p_F & 0 \\ 0 & 0 & \tilde{d}^{\alpha \beta} \end{pmatrix}. \quad (22) \]

This matrix is defined from the following orthogonality condition:
\[ \tilde{G}^{AB} \tilde{G}_{BE} = \begin{pmatrix} N^A_D & 0 & 0 \\ N^p_D & \delta^p_m & 0 \\ 0 & 0 & \delta^\alpha_\beta \end{pmatrix} = \begin{pmatrix} N^\alpha_D & 0 & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix}, \]

where
\[ N^A_D = \begin{pmatrix} N^A_m \\ N^p_m \\ N^p_m \end{pmatrix}, \quad (N^A_m = 0, N^p_m = \delta^p_m). \]

In the local coordinates of the basis \( (H_A, H_p, L_\alpha) \), the expression \( (11) \) of the Lagrangian \( L \) is transformed into
\[ \dot{L} = \frac{1}{2} (\tilde{G}^{AB}_{AB} \omega^A \omega^B + \tilde{G}^{AH}_{AP} \omega^A \omega^p + \tilde{G}^{HI}_{DA} \omega^D \omega^A + \tilde{G}^{HI}_{pq} \omega^p \omega^q + \tilde{d}_{\mu \nu} \omega^\mu \omega^\nu) - V, \quad (23) \]

where we have introduced the new time-dependent variables \( \omega^A, \omega^p \) and \( \omega^\alpha \) that are related to the velocities:
\[ \begin{align*}
\omega^A &= (P^A_B \frac{dQ^B}{dt}) \frac{d\dot{Q}^A}{dt}, \quad \omega^p = \frac{d\tilde{f}^p}{dt} \\
\omega^\alpha &= u_\mu^A \frac{d\dot{u}^\mu}{dt} + \omega^E_{\alpha} \frac{d\dot{Q}^E}{dt} + \omega^m_{\alpha} \frac{d\tilde{f}^m}{dt}. \quad (24)
\end{align*} \]
4 Differential relations between variations and quasi-velocities

The main peculiarity of the Poincaré variational principle is that it exploits the special variations. These variations are connected with the independent vector fields, which, unlike of the usual case, may not commute between themselves. If these vector fields \( v_1, \ldots, v_n \), given on a some smooth manifold, form a basis, then, in general, their commutator is as follows:

\[
[v_i, v_j] = c_{ij}^k(q) v_k,
\]

where the “structure constants” are represented by the functions on this manifold.

For a smooth path \( q^i(t) \), defined on a manifold, the time derivative of the function \( f \) taken on this path is determined as

\[
\frac{df(q(t))}{dt} = \frac{\partial f}{\partial q^i} \frac{dq^i}{dt} = \sum_i v_i(f) \omega^i,
\]

(25)

where \( v_i(f) \) is the directional derivative of \( f \) along the vector field \( v_i \). The variables \( \omega^i \) are called the quasi-velocities. They are linear functions of the velocities \( \dot{q}^i(t) \).

In the approach of Poincaré, the deformation of the path \( q(t) \), which we will denote by \( q(u, t) \), such that \( q(0, t) = q(t) \), have the standard properties. We refer to [16], where for the variations with the fixed ends, these properties were considered.

But as for the derivative of the function \( f(q(u, t)) \) with respect to the deformation parameter \( u \), it is given by the following expression:

\[
\frac{\partial f(q(u, t))}{\partial u} = \sum_i v_i(f) w^i(u, t).
\]

(26)

Also, it is required that the variations \( w^i(u, t) \) are independent within the time interval \([t_1, t_2]\) of the considered variational problem. And at the ends of the time interval, they satisfy the following conditions: \( w^k(u, t_1) = 0 \) and \( w^k(u, t_2) = 0 \).

The variation of the functional \( F(q(t)) \) in this variational calculus is obtained by the usual method:

\[
\delta F = \left. \frac{dF(q(u, t))}{du} \right|_{u=0}.
\]

(27)

We will apply the Poincaré variational principle to the action functional

\[
S = \int_{t_1}^{t_2} \dot{\mathcal{L}} \, dt,
\]

(28)
with the Lagrangian \( [23] \).

But first of all, we have to obtain the differential relations between the quasi-velocities and the variations. These relations are necessary in order to calculate the variations of the functionals by the Poincaré method.

In our case we deal with the vector fields of the local coordinate basis \((H_A, H_p, L_\alpha)\). It can be shown that the time derivative of the function \( \varphi \) which is given on the path \((Q^*A(t), \tilde{f}^p(t), a^\alpha(t))\) is calculated in accordance with \([25]\):

\[
\frac{d\varphi}{dt}(Q^*A(t), \tilde{f}^p(t), a^\alpha(t)) = \omega^E H_E^A(\varphi) + \omega^p H_p(\varphi) + \omega^\alpha L_\alpha(\varphi),
\]

where quasi-velocities \(\omega^E, \omega^p\) and \(\omega^\alpha\) are defined by \([24]\). By \(H_E(\varphi)\) and \(H_p(\varphi)\), we denote the action of the vector fields \(H_E\) and \(H_p\) on the function \(\varphi\). A similar notation is used for \(L_\alpha(\varphi)\).

First we consider the differential relation between \(\omega^A\) and \(w^A\). Using \([29]\) for \(Q^*A(t)\), we get

\[
\frac{dQ^*A(t)}{dt} = \omega^E H_E^A(Q^*(t)),
\]

where we write \(H_E^A(Q^*)\) for \(H_E^A(Q^*A(t))\) which is equal to \(N_E^A(Q^*)\).

The obtained equality for the time derivative of \(Q^*(t)\) can be extended to the corresponding equality for the deformed paths \(Q^*A(u, t)\):

\[
\frac{\partial Q^*A(u, t)}{\partial t} = H_E(Q^*A(u, t)) \omega^E(u, t),
\]

where \(\omega^E(u, t)\) is given by the linear function of the velocities that are defined for \(Q^*E(u, t)\).

On the other hand, for the partial derivative of \(Q^*A(u, t)\) with respect to \(u\), we have, as supposed by the Poincaré method, the following equation:

\[
\frac{\partial Q^*A(u, t)}{\partial u} = H_E(Q^*A(u, t)) w^E(u, t).
\]

Note that introduced variations \(w^E(u, t)\) are independent inside of the time interval \((t_1, t_2)\) and vanish on its boundary, i.e. \(w^E(u, t_1) = w^E(u, t_2) = 0\). As functions, these variations depend (functionally) on deformations of the paths \(Q^*A(u, t)\).

Now taking the partial derivative of \([30]\) with respect to \(u\), we obtain

\[
\frac{\partial}{\partial u} \frac{\partial Q^*A(u, t)}{\partial t} = \frac{\partial H_E^A(Q^*)}{\partial Q^B} \frac{\partial Q^B}{\partial u} \omega^E + H_E^A \frac{\partial \omega^E}{\partial u} = \frac{\partial H_E^A}{\partial Q^B} H_B^P w^P \omega^E + H_E^A \frac{\partial \omega^E}{\partial u}.
\]
While, differentiation of (31) with respect to $t$ yields

$$\frac{\partial}{\partial t} \frac{\partial Q^A(u, t)}{\partial u} = \frac{\partial H^A_E(Q^*)}{\partial Q^B} H^B_E \omega^P w^E + H^A_E \frac{\partial w^E}{\partial t}. \quad (33)$$

Subtracting (33) from (32), we obtain

$$\left( \frac{\partial H^A_E}{\partial Q^*} H^B_P - \frac{\partial H^A_P}{\partial Q^*} H^B_E \right) \omega^E w^P + H^A_E \frac{\partial \omega^E}{\partial u} - H^A_E \frac{\partial w^E}{\partial t} = 0.$$ 

Taking into account the commutation relation (18) and the used notation, by which $H^A_R(Q^*) = N^A_R(Q^*)$, we come to the following differential relation:

$$N^A_R \left( \frac{\partial \omega^R}{\partial u} - \frac{\partial w^R}{\partial t} + C^R_{PE} \omega^E w^P \right) = 0, \quad (34)$$

with $C^R_{PE} = (\Lambda^R_P N^S_E - \Lambda^R_E N^S_P) K^{R, \gamma_S}.$

Next we derive the differential relation between the quasi-velocity $\omega^p$ and the variation $w^p$. It is done as in the previous case.

For the path $\tilde{f}^p(t)$, equation (29) is written as follows:

$$\frac{d\tilde{f}^p(t)}{dt} = H^p_E(\tilde{f}^p(t)) \omega^p + H^m_p(\tilde{f}^p(t)) \omega^m = N^p_E(\tilde{f}^p(t)) \omega^E + \omega^p.$$

Provided that the deformations of the paths $Q^A(u, t)$ and $\tilde{f}(u, t)$ are properly chosen, we may obtain an analogous representation for time derivative of $\tilde{f}(u, t)$:

$$\frac{\partial \tilde{f}^p(u, t)}{\partial t} = N^p_E(\tilde{f}(u, t)) \omega^E(u, t) + \omega^p(u, t), \quad (35)$$

where now each $\omega^p(u, t)$ is a linear function of the velocities defined for the deformed paths $\tilde{f}^m(u, t)$.

For the partial derivative of $\tilde{f}(u, t)$ with respect to $u$, we take the representation which is similar in form with (35), but where the quasi-velocities of the deformed paths are replaced by the corresponding deformations:

$$\frac{\partial \tilde{f}^p(u, t)}{\partial u} = H^p_R(\tilde{f}^p(u, t)) w^R(u, t) + H^q_p(\tilde{f}^p(u, t)) w^q(u, t) = N^p_R(\tilde{f}(u, t)) w^R(u, t) + w^p(u, t). \quad (36)$$

Taking the partial derivative of (35) with respect to $u$, and then the partial derivative of (36) with respect to $t$, we get two equal expression:

$$\frac{\partial^2 \tilde{f}^p}{\partial u \partial t} = H_R(N^p_E) w^R \omega^E + H_n(N^p_E) w^n \omega^E + N^p_E \frac{\partial \omega^E}{\partial u} + \frac{\partial \omega^p}{\partial u}$$

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and
\[ \frac{\partial^2 \dot{f}^p}{\partial t \partial u} = H_E(N_R^p) w^R \omega^E + H_n(N_R^p) w^n \omega_R + N_E^p \frac{\partial w^E}{\partial t} + \frac{\partial w^p}{\partial t}. \]

Subtracting the second expression from the first, we obtain
\[ (H_R(N_E^p) - H_E(N_R^p)) w^R \omega^E + H_q(N_E^p) w^q \omega^E - H_q(N_R^p) \omega^q w_R \]
\[ + N_E^p \left( \frac{\partial \omega^E}{\partial u} - \frac{\partial w^E}{\partial t} \right) + \left( \frac{\partial \omega^p}{\partial u} - \frac{\partial w^p}{\partial t} \right) = 0. \]

Now because of \( H_E(\tilde{f}^p) = N_E^p(\tilde{f}) \), where \( N_E^p(\tilde{f}) = -K^p_{\alpha}(\tilde{f}) \Lambda^\alpha_E \) and \( K^p_{\alpha}(\tilde{f}) = (J_{\alpha})_m^p \tilde{f}^m \), the obtained equation can be rewritten as follows:
\[ [H_R, H_E](f^p) w^R \omega^E + H_n(N_E^p) w^n \omega^E - H_n(N_R^p) \omega^n w_R \]
\[ + N_E^p \left( \frac{\partial \omega^E}{\partial u} - \frac{\partial w^E}{\partial t} \right) + \left( \frac{\partial \omega^p}{\partial u} - \frac{\partial w^p}{\partial t} \right) = 0. \]

Then, replacing the commutator by its explicit expression from (18), we get the following differential relation:
\[ N_T^p \left( \frac{\partial \omega^T}{\partial u} - \frac{\partial w^T}{\partial t} - C^T_{ER} \omega^E w^R \right) \]
\[ + \left( \frac{\partial \omega^p}{\partial u} - \frac{\partial w^p}{\partial t} - C^p_{ER} \omega^E w^R - C^p_{Eq} \omega^E w^q - C^p_{qR} \omega^q w^R \right) = 0. \] (37)

We note that this differential relation can be written by using the condensed notation for indices:
\[ \tilde{N}^{\hat{A}}_{\hat{B}} \frac{\partial \omega^{\hat{B}}}{\partial u} = \tilde{N}^{\hat{A}}_{\hat{B}} \frac{\partial w^{\hat{B}}}{\partial t} + \tilde{N}^{\hat{A}}_{\hat{B}} C_{ER} \omega \dot{w}^R. \]

The last differential relation dealing with the partial derivatives of \( \omega^\alpha \) and \( w^\alpha \) is obtained in Appendix A. The result is given by equation (A.3):
\[ \frac{\partial \omega^\beta}{\partial u} - \frac{\partial w^\beta}{\partial t} + C_{RE}^\beta w^R \omega^E - C_{Ep}^\beta w^p \omega^E + C_{Rm}^\beta w^R \omega^m \]
\[ + C_{pm}^\beta w^p \omega^m + C_{p\nu}^\beta w^\nu \omega^\mu = 0. \]

Now we can proceed to derivation of the Lagrange-Poincaré equations.
5 The Lagrange-Poincaré equations

We make use of the Poincaré variational principle to derive the Lagrange-Poincaré equations. The variational integral, which we take for this purpose, is given by the functional \( \delta S \). The variation \( \delta S \) of this functional is defined by \( \delta S = \int_{t_1}^{t_2} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \omega^C} \frac{\partial \omega^C}{\partial u} + \frac{\partial \tilde{\mathcal{L}}}{\partial \omega^p} \frac{\partial \omega^p}{\partial u} + \frac{\partial \tilde{\mathcal{L}}}{\partial \omega^a} \frac{\partial \omega^a}{\partial u} + \frac{\partial \tilde{\mathcal{L}}}{\partial Q^B} \frac{\partial Q^B}{\partial u} \right) dt. \) (38)

In order to perform the integration by parts in the integral (38), we have to transform the terms with the derivatives of the quasi-velocities of the integrand into new terms with the time derivatives of the variations. This can be made with the help of the obtained differential relations between the quasi-velocities and deformations. We begin with the transformation of the first two terms of the integrand.

Since in our case \( \tilde{G}_{Ap}^H = \tilde{G}_{pA}^H \), these terms can be rewritten as follows:

\[
\frac{\partial \tilde{\mathcal{L}}}{\partial \omega^C} \frac{\partial \omega^C}{\partial u} + \frac{\partial \tilde{\mathcal{L}}}{\partial \omega^p} \frac{\partial \omega^p}{\partial u} = \tilde{G}_{CD}^H \frac{\partial \omega^C}{\partial u} \omega^D + \hat{G}_{Ap}^H \frac{\partial \omega^A}{\partial u} \omega^p + \hat{G}_{Ap}^H \frac{\partial \omega^A}{\partial u} \omega^p + \hat{G}_{pq}^H \frac{\partial \omega^q}{\partial u}. \]  

Denoting temporarily the differential relation \( (34) \) by \( (A) \), and \( (37) \) by \( (B) \), let us consider their linear combination \( \tilde{G}_{BA}^H \cdot (A) + \hat{G}_{Bm}^H \cdot (B) = 0 \). Using the identity

\[
\tilde{G}_{BA}^H N_T^A + \hat{G}_{Bm}^H N_T^m = \tilde{G}_{BT}^H \quad \text{(or)} \quad \tilde{G}_{BA}^H N_T^A = \tilde{G}_{BT}^H,
\]

we get the following differential relation:

\( (A') \)

\[
\tilde{G}_{BT}^H \frac{\partial \omega^T}{\partial u} = \tilde{G}_{BT}^H \left( \frac{\partial w^T}{\partial t} + \mathcal{C}_{ER} \omega^E w^R \right)
- \hat{G}_{Bm}^H \left( \frac{\partial \omega^m}{\partial u} - \frac{\partial w^m}{\partial t} - \mathcal{C}_{EF} \omega^F w^F - \mathcal{C}_{Eg} \omega^F w^q - \mathcal{C}_{qR} \omega^q w^R \right).
\]
Similarly, considering the linear combination \( \tilde{G}_{pA}^H \cdot (A) + \tilde{G}_{pq}^H \cdot (B) = 0 \), and taking into account
\[
\tilde{G}_{pA}^H N_T^A + \tilde{G}_{pq}^H N_T^q = \tilde{G}_{pT}^H \quad (\text{or} \quad \tilde{G}_{pB}^H N_T^B = \tilde{G}_{pT}^H),
\]
we obtain
\[
(B')
\]
\[
\tilde{G}_{pT}^H \frac{\partial \omega^T}{\partial u} = \tilde{G}_{pT}^H \left( \frac{\partial w^T}{\partial t} + C_{ER}^T \omega^E w^R \right)
\]
\[
-\tilde{G}_{pm}^H \left( \frac{\partial \omega^m}{\partial u} - \frac{\partial w^m}{\partial t} - C_{EF}^m \omega^E w^F - C_{Eq}^m \omega^E w^q - C_{qR}^m \omega^q w^R \right).
\]

Multiplying \((A')\) by \( \omega^B \) and using the result of the multiplication in the right hand side of \((39)\), we get for it the following expression:
\[
\tilde{G}_{BT}^H \omega^B \left( \frac{\partial w^T}{\partial t} + C_{ER}^T \omega^E w^R \right)
\]
\[
-\tilde{G}_{Bm}^H \omega^B \left( \frac{\partial \omega^m}{\partial u} - \frac{\partial w^m}{\partial t} - C_{EF}^m \omega^E w^F - C_{Eq}^m \omega^E w^q - C_{qR}^m \omega^q w^R \right)
\]
\[
+\tilde{G}_{Ap}^H \frac{\partial \omega^A}{\partial u} \omega^p + \tilde{G}_{Ap}^H \omega^A \frac{\partial \omega^p}{\partial u} + \tilde{G}_{pq}^H \omega^p \frac{\partial \omega^q}{\partial u}.
\]

We see that underlined terms are cancelled. Next we insert the result of the multiplication \((B')\) by \( \omega^p \) in just obtained expression. As a consequence, we come to the following expression for the right hand side of \((39)\):
\[
\tilde{G}_{BT}^H \omega^B \left( \frac{\partial w^T}{\partial t} + C_{ER}^T \omega^E w^R \right)
\]
\[
-\tilde{G}_{Bm}^H \omega^B \left( \frac{\partial \omega^m}{\partial u} - \frac{\partial w^m}{\partial t} - C_{EF}^m \omega^E w^F - C_{Eq}^m \omega^E w^q - C_{qR}^m \omega^q w^R \right)
\]
\[
+\tilde{G}_{pT}^H \omega^p \left( \frac{\partial w^T}{\partial t} + C_{ER}^T \omega^E w^R \right) - \tilde{G}_{pm}^H \omega^p \left( \frac{\partial \omega^m}{\partial u} - \frac{\partial w^m}{\partial t} \right)
\]
\[
-\tilde{G}_{pq}^H \omega^p \frac{\partial \omega^q}{\partial u}.
\]

Since the underlined terms in this expression are cancelled, now we can perform the integration by parts of this expression.

Before writing out the result of the integration, it worth to note that one can rewrite the obtained expression in a compact form by making use of the
In our case $C$ in the integral (39). This lead us to condensed notation in which $Q^{*\hat{A}}$ means $(Q^{*A}, \hat{f}^{\nu})$:

\[
\hat{G}^{H}_{BT} \omega^{B} \left( \frac{\partial w^{T}}{\partial t} + C^{T}_{ER} \omega^{F} w^{R} \right) + \tilde{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial w^{m}}{\partial t} + C^{m}_{EF} \omega^{E} w^{g} + C^{m}_{qR} \omega^{q} w^{R} \right) = \hat{G}^{H}_{BT} \omega^{B} \left( \frac{\partial w^{T}}{\partial t} + C^{T}_{ER} \omega^{F} w^{R} \right).
\]

The right hand side of the obtained expression can be rewritten as

\[
\int_{t_{1}}^{t_{2}} \left( \frac{d}{dt} \left( \hat{G}^{H}_{BT} \omega^{B} w^{T} + \tilde{G}^{H}_{BT} \omega^{B} w^{T} \right) \right) dt = \left( \hat{G}^{H}_{BT} \omega^{B} w^{T} + \tilde{G}^{H}_{BT} \omega^{B} w^{T} \right) \bigg|_{t_{1}}^{t_{2}}
\]

\[
- \int_{t_{1}}^{t_{2}} \left( \frac{d}{dt} \left( \hat{G}^{H}_{BT} \omega^{B} + \tilde{G}^{H}_{BT} \omega^{B} \right) + \hat{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{F} E w^{F} \right) - \hat{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{E} w^{F} \right) + \tilde{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{E} w^{F} \right) + \tilde{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{E} w^{F} \right) \right) dt.
\]

The result of the integration by parts of (40) is given as follows:

\[
\int_{t_{1}}^{t_{2}} \left( \frac{d}{dt} \left( \hat{G}^{H}_{Bm} \omega^{B} + \tilde{G}^{H}_{Bm} \omega^{B} \right) + \hat{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{E} w^{F} \right) - \hat{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{E} w^{F} \right) + \tilde{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{E} w^{F} \right) + \tilde{G}^{H}_{Bm} \omega^{B} \left( \frac{\partial}{\partial t} \omega^{E} w^{F} \right) \right) dt.
\]

In a similar manner we can carry out the integration of the following term in the integral (39). This lead us to

\[
\int_{t_{1}}^{t_{2}} \left( \frac{d}{dt} \left( \hat{G}^{H}_{Bm} \omega^{B} + \tilde{G}^{H}_{Bm} \omega^{B} \right) \right) dt = \left( \hat{G}^{H}_{Bm} \omega^{B} + \tilde{G}^{H}_{Bm} \omega^{B} \right) \bigg|_{t_{1}}^{t_{2}}
\]

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\[-\int_{t_1}^{t_2} \left( \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \omega^e} \right) \right) w^e - \left( \frac{\partial \hat{L}}{\partial \omega^e} \right) \left( -C^e_{RE} \omega^E - C^e_{m} \omega^m \right) w^R \]
\[-\left( \frac{\partial \hat{L}}{\partial \omega^e} \right) \left( C^e_{Ep} \omega^E - C^e_{pm} \omega^m \right) w^p + \left( \frac{\partial \hat{L}}{\partial \omega^e} \right) c^e_{v\mu} \omega^\mu w^\nu \right) dt.

The remaining terms of the integrand in the integral (38) can be transformed as
\[
\begin{align*}
\frac{\partial \hat{L}}{\partial Q^B} \frac{\partial Q^B}{\partial u} + \frac{\partial \hat{L}}{\partial f^p} \frac{\partial f^p}{\partial u} + \frac{\partial \hat{L}}{\partial a^\alpha} \frac{\partial a^\alpha}{\partial u} &= \frac{\partial \hat{L}}{\partial Q^B} N^B_E \omega^E + \frac{\partial \hat{L}}{\partial f^p} N^p_E \omega^E + \frac{\partial \hat{L}}{\partial f^p} w^p \\
+ \frac{\partial \hat{L}}{\partial a^\alpha} \left( -N_{E}^E \omega^E v^E_{\beta} w^E - N_{E}^{p} \omega^p v^p_{\alpha} w^E - \omega^E_{p} v^p_{\alpha} w^p + v^p_{\beta} w^\beta \right) &= H_E(\hat{L}) w^E + H_p(\hat{L}) w^p + L_\alpha(\hat{L}) w^\alpha.
\end{align*}
\]

Since the variations \(w^E\), \(w^p\) and \(w^\alpha\) are independent between themselves and satisfy the standard boundary conditions, this enables us to obtain the following system of the Lagrange-Poincaré equations:
\[
\begin{align*}
-\frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^C} \right) C^T_{CE} \omega^C + \left( \frac{\partial \hat{L}}{\partial \omega^p} \right) \left( C^p_{CE} \omega^C + C^p_{qE} \omega^q \right) \\
+ \left( \frac{\partial \hat{L}}{\partial \omega^e} \right) \left( C^a_{CE} \omega^C + C^a_{mE} \omega^m \right) + H_E(\hat{L}) = 0 \quad (41)
\end{align*}
\]
\[
\begin{align*}
-\frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \omega^m} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^E} \right) C^p_{Em} \omega^E \\
+ \left( \frac{\partial \hat{L}}{\partial \omega^m} \right) \left( C^p_{Em} \omega^E + C^m_{pm} \omega^p \right) + H_m(\hat{L}) = 0 \quad (42)
\end{align*}
\]
\[
\begin{align*}
-\frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \omega^\beta} \right) + \left( \frac{\partial \hat{L}}{\partial \omega^\mu} \right) c^\beta_{\mu\alpha} \omega^\mu + L_\alpha(\hat{L}) = 0. \quad (43)
\end{align*}
\]

The first two equations of this system are the horizontal equations, and the last equation, describing the motion of the group variable, is the vertical one.

6 Concluding remarks

In the paper, we have obtained the Lagrange-Poincaré equations using the dependent variables, determined on a global surface \(\tilde{\Sigma}\). This means that the principal fiber bundle related to our original mechanical system is trivial.
one. One meets with a similar case in gauge theories, where it is not possible in general to obtain a set of gauge-invariant variables that are globally determined on the orbit space of the principal fiber bundle.

We note that obtained equations may be also used for description of the local evolution (in terms of dependent variables) given on an appropriate domain of the orbit space of the non-trivial principal bundle. But, as in gauge theories, the problem of “gluing” these evolutions into the global one is not yet settled.

We remark that our horizontal equations are analogous in form with that ones from [4]. But the “structure constants” in our case are calculated for the horizontal lift basis and differ from the structure constants of the cited work.

Note also that the horizontal Lagrange-Poincaré equations of the present case can be derived from the similar equations of our paper [10] by considering them as if they were written in terms of the condensed notations, that have been used in this paper. However, the “structure constants” should be taken those as in (18), (19) and (20).

Appendix A

Differential relation between $\omega^\alpha$ and $w^\alpha$

By the general formula (29) applied to the path $a^\alpha(t)$, the velocity $da^\alpha(t)/dt$ is decomposed as follows:

$$\frac{da^\alpha(t)}{dt} = H_E(a^\alpha(t))\omega^E(t) + H_m(a^\alpha(t))\omega^m(t) + L_\mu(a^\alpha(t))\omega^\mu(t),$$

where each of the quasi-velocities $\omega^E(t)$, $\omega^m(t)$ and $\omega^\mu(t)$ is a linear function of the velocities. Taking sufficient small variations of the paths, such a representation for $da^\alpha(t)/dt$ can be extended to the representation which determines the decomposition of the velocity $da^\alpha(u,t)/dt$: \[ \text{(A.1)} \]

$$\frac{\partial a^\alpha(u,t)}{\partial t} = H_E(a^\alpha(u,t))\omega^E(u,t) + H_m(a^\alpha(u,t))\omega^m(u,t) + L_\mu(a^\alpha(u,t))\omega^\mu(u,t).$$

In this representation, the quasi-velocities $\omega^E(u,t)$, $\omega^m(u,t)$ and $\omega^\mu(u,t)$ denote the linear functions of the velocities of the deformed paths.

---

$\text{\#}a^\alpha(u,t)$ is a deformation of the original path $a^\alpha(t)$, i.e., for instance, as $a^\alpha(u,t) = a^\alpha(t) + uW^\alpha(t)$. 

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In the considered calculus of variation, it is supposed that \( \frac{\partial a^{\alpha}(u,t)}{\partial u} \) is defined similarly to (A.1), where, however, quasi-velocities are replaced by variations:

\[
\frac{\partial a^{\alpha}(u,t)}{\partial u} = H_{R}(a^{\alpha}(u,t))w^{R}(u,t) + H_{p}(a^{\alpha}(u,t))w^{p}(u,t) + L_{\nu}(a^{\alpha}(u,t))w^{\nu}(u,t).
\] (A.2)

Taking the partial derivative of (A.1) with respect to \( u \), we get

\[
\frac{\partial^{2} a^{\alpha}}{\partial u \partial t} = H_{R}H_{E}(a^{\alpha})w^{R}w^{E} + H_{R}H_{m}(a^{\alpha})w^{R}w^{m} + H_{R}L_{\mu}(a^{\alpha})w^{R}w^{\mu}
\]

\[
+ H_{p}H_{E}(a^{\alpha})w^{p}w^{E} + H_{p}H_{m}(a^{\alpha})w^{p}w^{m} + H_{p}L_{\mu}(a^{\alpha})w^{p}w^{\mu} + L_{\nu}H_{E}(a^{\alpha})w^{\nu}w^{E}
\]

\[
+ L_{\nu}H_{m}(a^{\alpha})w^{\nu}w^{m} + L_{\nu}L_{\mu}(a^{\alpha})w^{\nu}w^{\mu} + H_{E}(a^{\alpha})\frac{\partial w^{E}}{\partial u} + H_{m}(a^{\alpha})\frac{\partial w^{m}}{\partial u}.
\]

Similarly, the differentiation of (A.2) with respect to \( t \) yields

\[
\frac{\partial^{2} a^{\alpha}}{\partial t \partial u} = H_{E}H_{R}(a^{\alpha})w^{E}w^{R} + H_{E}H_{p}(a^{\alpha})w^{E}w^{p} + H_{E}L_{\nu}(a^{\alpha})w^{E}w^{\nu}
\]

\[
+ H_{m}H_{R}(a^{\alpha})w^{m}w^{R} + H_{m}H_{p}(a^{\alpha})w^{m}w^{p} + H_{m}L_{\nu}(a^{\alpha})w^{m}w^{\nu} + L_{\mu}H_{R}(a^{\alpha})w^{\mu}w^{R}
\]

\[
+ L_{\mu}H_{p}(a^{\alpha})w^{\mu}w^{R} + L_{\mu}L_{\nu}(a^{\alpha})w^{\mu}w^{\nu} + H_{R}(a^{\alpha})\frac{\partial w^{R}}{\partial t} + H_{p}(a^{\alpha})\frac{\partial w^{p}}{\partial t} + L_{\nu}(a^{\alpha})\frac{\partial w^{\nu}}{\partial t}.
\]

Subtracting this expression from the expression given above, we obtain:

\[
[H_{R}, H_{E}](a^{\alpha})w^{R}w^{E} + [H_{p}, H_{E}](a^{\alpha})w^{p}w^{E} + [H_{R}, H_{m}](a^{\alpha})w^{R}w^{m}
\]

\[
+ [H_{p}, H_{m}](a^{\alpha})w^{p}w^{m} + [H_{p}, L_{\mu}](a^{\alpha})w^{p}w^{\mu} + [H_{R}, L_{\mu}](a^{\alpha})w^{R}w^{\mu}
\]

\[
+ [L_{\nu}, H_{E}](a^{\alpha})w^{\nu}w^{E} + [L_{\nu}, H_{m}](a^{\alpha})w^{\nu}w^{m} + [L_{\mu}, L_{\nu}](a^{\alpha})w^{\nu}w^{\mu}
\]

\[
+ H_{E}(a^{\alpha})\left(\frac{\partial w^{E}}{\partial u} - \frac{\partial w^{E}}{\partial t}\right) + H_{m}(a^{\alpha})\left(\frac{\partial w^{m}}{\partial u} - \frac{\partial w^{m}}{\partial t}\right) + L_{\mu}(a^{\alpha})\left(\frac{\partial w^{\mu}}{\partial u} - \frac{\partial w^{\mu}}{\partial t}\right) = 0.
\]

By making use of the commutation relations (18), (19), (20), together with the commutation relations for the left-invariant vector fields \( L_{\alpha} \), we rewrite the obtained equation as follows:

\[
(C_{RE}^{T}H_{T}(a^{\alpha}) + C_{RE}^{p}H_{p}(a^{\alpha}) + C_{RE}^{\gamma}L_{\gamma}(a^{\alpha}))w^{R}w^{E}
\]

\[
- (C_{Ep}H_{m}(a^{\alpha}) + C_{Ep}L_{\gamma}(a^{\alpha}))w^{p}w^{E} + C_{pm}L_{\gamma}(a^{\alpha})w^{p}w^{m}
\]

\[
+ (C_{Rm}H_{q}(a^{\alpha}) + C_{Rm}L_{\gamma}(a^{\alpha}))w^{R}w^{m} + C_{\mu\nu}L_{\gamma}(a^{\alpha})w^{\mu}w^{\nu}.
\]
\[ H_E(a^\alpha) \left( \frac{\partial \omega^E}{\partial u} - \frac{\partial w^E}{\partial t} \right) + H_m(a^\alpha) \left( \frac{\partial \omega^m}{\partial u} - \frac{\partial w^m}{\partial t} \right) + L_\mu(a^\alpha) \left( \frac{\partial \omega^\mu}{\partial u} - \frac{\partial w^\mu}{\partial t} \right) = 0. \]

We note that
\[ H_T(a^\alpha) = -N_D \tilde{\omega}_D^\mu L_\mu(a^\alpha) - N_m \tilde{\omega}_m^\mu L_\mu(a^\alpha) = -(N_D \tilde{\omega}_D^\mu + N_m \tilde{\omega}_m^\mu) \nu_\mu(a). \]

and \( H_p(a^\alpha) = -\tilde{\omega}_p^\mu \nu_\mu(a). \) Therefore, multiplying the equation by \( u^{\beta \alpha}(a), \) we get rid off the common multiplier \( v_\alpha^\mu(a). \) As a result, we arrive at
\[ [\tilde{C}_D^E \omega_D^\beta + \tilde{C}_m^E \omega_m^\beta - C_p^E \omega_p^\beta + C_p^E \omega_p^\beta] w^R \omega^E - (C_p^E \omega_p^\beta + C_p^E \omega_p^\beta) w^R \omega^m + C_p^E \omega_p^\beta w^p \omega^m - (N_D \tilde{\omega}_D^\beta + N_m \tilde{\omega}_m^\beta) \left( \frac{\partial \omega^E}{\partial u} - \frac{\partial w^E}{\partial t} \right) - \tilde{\omega}_m^\beta (\frac{\partial \omega^m}{\partial u} - \frac{\partial w^m}{\partial t}) + (\frac{\partial \omega^\beta}{\partial u} - \frac{\partial w^\beta}{\partial t}) = 0. \]

First we observe that in just obtained equation, the sum of the terms with common multiplier \( N_D^T \) has to vanish because of the differential relation (34) for \( \omega_A. \) Hence, we deal, in fact, with the equation which looks as follows: \( \omega^\beta(\ldots) + (\ldots) = 0. \) But in this equation the first summand also has to vanish because of the second differential identity (37) for \( \omega^m. \) It follows that final equation representing the differential relation for \( \omega^\beta \) is given by
\[ \frac{\partial \omega^\beta}{\partial u} - \frac{\partial w^\beta}{\partial t} + C_p^E w^p \omega^E - C_p^E w^p \omega^E + C_p^E w^R \omega^m + C_p^E w^p \omega^m + C^\beta_p w^\nu \omega^\mu = 0. \]  

(A.3)

Appendix B

Projectors, their properties and some identities

The horizontal projector \( \Pi_B^{\hat{A}} \)

By definition
\[ \Pi_B^{\hat{A}} = \delta_B^\alpha - K_\beta^\alpha \omega^\beta K_D^B D^B. \]

Its components are
\[ \Pi_B^{\hat{A}} = (\Pi_B^A, \Pi_B^n, \Pi_B^m, \Pi_n^m). \]
\[ \Pi_B^A = \delta_B^A - K^A_{\alpha} d^{\alpha \beta} K^{D}_{\beta} G_{DB}, \quad \Pi_n^A = -K^A_{\mu} d^{\mu \nu} K^{p}_{\nu} G_{pn}, \]

\[ \Pi_B^m = -K^m_{\mu} d^{\mu \nu} K^{D}_{\nu} G_{DB}, \quad \Pi_n^m = \delta_n^m - K^m_{\mu} d^{\mu \nu} K^{p}_{\nu} G_{rn}. \]

The main properties:

\[ \Pi_B^A \Pi_B^C = \Pi_C^A, \quad \Pi_B^L N_C^L = N_C^L, \quad \Pi_B^F N_L^F = \Pi_C^F. \]

Under the transformation

\[ Q^A = F^A(Q^*, a), \quad F_C^A(Q^*, a) \equiv \frac{\partial F^A(Q, a)}{\partial Q^C}|_{Q=Q^*}, \quad F_A^C \equiv F_A^C(F(Q^*, a), a^{-1}) \]

and because of

\[ K^B_{\alpha}(F(Q^*, a)) = \rho^B_{\alpha}(a) K^{D}_{\mu}(Q^*) F^B_{D}(Q^*, a), \quad K^{p}_{\alpha}(\tilde{D}(a) \tilde{f}) = \rho^{p}_{\alpha}(a) K^{p}_{\mu}(\tilde{f}) \tilde{D}^{p}_{q}(a), \]

\[ G_{AB}(F(Q^*, a)) = G_{DC}(Q^*) F_{A B}^C, \quad G_{pq} = G_{mn} \tilde{D}^{m}_{p}(a) \tilde{D}^{n}_{q}(a), \]

\[ d_{\alpha \beta}(Q, f) = \rho^{\mu}_{\alpha}(a) \rho^{\nu}_{\beta}(a) d_{\mu \nu}(Q^*, \tilde{f}), \]

it follows that

\[ \Pi_B^A(Q, f) = F_C^A \Pi_D^C(Q^*, \tilde{f}) \tilde{F}^{D}_{B}, \quad \Pi_n^A(Q, f) = F_B^A \Pi_p^B(Q^*, \tilde{f}) D^{p}_{n}(a), \]

\[ \Pi_B^m(Q, f) = \tilde{F}_C^C \Pi_q^q(Q^*, \tilde{f}) \tilde{D}^{q}_{q}(a), \quad \Pi_n^m(Q, f) = \tilde{D}^{m}_{q}(a) \Pi_s^q(Q^*, \tilde{f}) D^{q}_{n}(a). \]

The projector \( N_B^L \)

Its components:

\[ N_C^A = (N_B^A, N_n^A, N_B^m, N_n^m), \]

\[ N_B^A = \delta_B^A - K^A_{\mu}(\Phi^{-1})^{\mu}_{\nu} \chi^{\nu}_B, \quad N_n^A = 0, \quad N_B^m = -K^m_{\mu}(\Phi^{-1})^{\mu}_{\nu} \chi^{\nu}_B, \quad N_n^m = \delta_n^m. \]

The main properties:

\[ N_B^A N_C^L = N_C^A, \quad (P_{\perp})_{B}^L N_C^L = (P_{\perp})_{C}^L, \quad N_B^A (P_{\perp})_{A}^C = N_B^C. \]

Transformations

\[ N_C^A(Q^*) = F_C^A(Q^*, a) N_B^M(F(Q^*, a)) \tilde{F}_M^A(Q^*, a), \quad N_C^A(Q^*) \equiv N_C^A(F(Q^*, e)), \]

\( e \) is the unity element of the group.
The projector \((P_\perp)^A_B\)

Its components:

\[
(P_\perp)^A_B = \left( (P_\perp)^A_A, (P_\perp)^A_B, (P_\perp)_B^m, (P_\perp)_B^n \right),
\]

\[
(P_\perp)_B^A = \delta_B^A - \chi_B (\chi^\top)^{-1} \chi_B \gamma, \quad \chi_B^\top = G^A_B \gamma_B, \quad \gamma = K^A_B G^R_B = 0,
\]

\[
(P_\perp)_B^m = 0, \quad (P_\perp)_B^n = \delta^n_m.
\]

Some identities derived from \(K^R_G G^{H}_{RA} = 0\)

1. \(\hat{A} \rightarrow A\)

\[
K^R_G G^{H}_{RA} + K^\gamma_G \tilde{G}^H_{p\lambda} = 0 \quad \text{or} \quad K^\gamma_G \tilde{G}^H_{RA} = 0
\]

(A) \(K^R_G \tilde{G}^H_{RA} + K^\gamma_G \tilde{G}^H_{p\lambda} = 0.\)

(D) \(\tilde{G}^H_{AR,\gamma} K^\gamma_{\gamma} + \tilde{G}^H_{Ap,\gamma} K^\gamma_{\gamma} + \tilde{G}^H_{Ap,\gamma} = 0.\)

2. \(\hat{A} \rightarrow p\)

\[
\tilde{G}^H_{pq} K^\mu_{\mu} + \tilde{G}^H_{p\lambda} K^A_{\mu} = 0 \quad \text{or} \quad \tilde{G}^H_{p\lambda} K^A_{\mu} = 0
\]

(B) \(\tilde{G}^H_{pq} K^p_{\mu} + \tilde{G}^H_{p\lambda} K^p_{\mu} + \tilde{G}^H_{p\lambda} = 0.\)

(C) \(\tilde{G}^H_{p\lambda} K^p_{\mu} + \tilde{G}^H_{p\lambda} K^p_{\mu} + \tilde{G}^H_{p\lambda} = 0.\)

These relations are obtained as a result of the differentiations.

Killing relations for the horizontal metric \(G^{H}_{AB}\)

\[
\tilde{G}^H_{AB,\beta} K^\beta_{\alpha} + \tilde{G}^H_{RB} K^\beta_{\alpha} + \tilde{G}^H_{AR} K^\beta_{\alpha} = 0
\]

\(\hat{A} \rightarrow A, \quad \hat{B} \rightarrow B\)

I. \(\tilde{G}^H_{AB,\beta} K^\beta_{\alpha} + \tilde{G}^H_{AB} K^\beta_{\alpha} + \tilde{G}^H_{RB} K^\beta_{\alpha} + \tilde{G}^H_{AR} K^\beta_{\alpha} = 0.\)

\(\hat{A} \rightarrow p, \quad \hat{B} \rightarrow q\)

II. \(\tilde{G}^H_{pq,\beta} K^\beta_{\alpha} + \tilde{G}^H_{q\beta} K^\beta_{\alpha} + \tilde{G}^H_{r\beta} K^\beta_{\alpha} + \tilde{G}^H_{p\beta} K^\beta_{\alpha} = 0.\)

\(\hat{A} \rightarrow p, \quad \hat{B} \rightarrow B\)

III. \(\tilde{G}^H_{pB,\beta} K^\beta_{\alpha} + \tilde{G}^H_{pB} K^\beta_{\alpha} + \tilde{G}^H_{pB} K^\beta_{\alpha} + \tilde{G}^H_{pB} K^\beta_{\alpha} = 0.\)

\(\hat{A} \rightarrow B, \quad \hat{B} \rightarrow p\)

IV. \(\tilde{G}^H_{pB,\beta} K^\beta_{\alpha} + \tilde{G}^H_{pB} K^\beta_{\alpha} + \tilde{G}^H_{pB} K^\beta_{\alpha} + \tilde{G}^H_{pB} K^\beta_{\alpha} = 0.\)

IV = III

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