Connection between functionals of the field-theory equations and state functionals of the mathematical physics equations

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Abstract. It is well known that the field-theory equations are equations for such functionals as the action functional, entropy, Pointing’s vector, Einstein’s tensor, wave function, and others. It turns out that the equations of mathematical physics describing material systems (material media) such as the thermodynamic, gas-dynamic and cosmic systems as well as the systems of charged particles and others can be expressed in terms of the same functionals. Such a duality of functionals points out to a connection between the field-theory equations, which describe physical fields, and the mathematical physics equation, which describe material media. This discloses a connection of physical fields and material media.

1. Introduction
As it is known, the field-theory equations are equations for such functional as the action functional, entropy, Pointing’s vector, Einstein’s tensor, wave function, and others. It has been shown that a relation for such functionals can be obtained from the mathematical physics equations, which describe material media. Investigation of the relation for functionals, which is evolutionary and nonidentical, enables one to disclose specific properties and physical meaning of functionals. This investigation showed that there exists a correspondence between evolutionary relation for functionals obtained from the mathematical physics equations and the field-theory equations. This fact points out to a connection between the field-theory equations and the mathematical physics equations. The present investigation bases on the properties of conservation laws, which lie at the basis of the field-theory equations and the mathematical physics equations. A peculiarity consists in the fact that physical fields described by the field-theory equations and material media described by the mathematical physics equations are subject to different conservation laws. The conservation laws for physical fields mean a statement on an existence of conservative quantities or objects. Such conservation laws (which can be referred to as exact ones) are described by closed skew-symmetric forms. Whereas the conservation laws of material media are conservation laws for energy, momentum, angular momentum, and mass that are described by differential equations. It turns out that there exists a connection between conservation laws for physical fields and material media, and these points out to a connection of the field-theory equations with the mathematical physics equations. It appears to be possible to obtain the results of present paper due to using the mathematical apparatus of skew-symmetric forms, which can describe conservation laws [1-2]. The peculiarity of such investigation consisted in the fact that, in addition to exterior forms, skew-symmetric forms, which basis, as opposed to the basis of exterior forms, is nonintegrable manifolds, were used. Such skew-symmetric forms which are obtained from differential equations and are evolutionary ones [1], possess a unique property, namely, they can generate closed exterior forms that are invariants and corresponds
to conservation laws for physical fields (exact conservation laws). Such properties of skew-symmetric forms defined on nonintegrable manifolds (non of existing mathematical formalism possesses these properties) enable to describe an advent of various structures, quantum transitions and so on.

2. Peculiarities of the mathematical physics equations for material media

The equations of mathematical physics, which describe material media, consist of the equations of conservation laws for energy, linear momentum, angular momentum, and mass [3-5]. Commonly the equations of mathematical physics are applied for a description of a change of physical quantities (such as energy, pressure, density) that characterize material media. However, it turns out that such equations possess properties that also give a possibility to describe the processes of emergence of various structures. The physical structures from which physical fields are formatted may be such structures. This discloses a connection between the field-equation equations and the equations of mathematical physics. Below it will be shown that such a connection is caused by the properties of corresponding functionals. However, such properties of the mathematical physics equations and potentialities caused by that appear to be hidden ones. They do not directly follow from differential equations and reveal only in describing evolutionary processes. This result follows from the properties of conservation laws and reveals when the consistency of the conservation law equations, from which the mathematical physics equations are made up, are accounted for.

2.1. Investigation of the consistency of the conservation law equations. Evolutionary relation for the state functionals

For investigation of the consistency of the conservation law equations it is necessary to use two nonequivalent frames of reference. Usually the equations of mathematical physics are written in the inertial frame of reference (the Euler frame of reference is an example of such frame).

In the case under consideration, in addition to the equations in inertial frame of reference, the equations obtained after transition from inertial frame of reference to accompanying one were used. The accompanying frame of reference is a frame of reference connected with the manifold made up by the trajectories of the material media elements (the Lagrange frame of reference is an example of such a frame).

A specific feature of present investigation is the point that the conservation law equations are transformed into equations expressed in the terms of state functionals. Such a possibility relates to the properties of desired physical quantities of material media. Since the physical quantities (like temperature, energy, pressure, density) relates to a single material medium, a connection between them should exist. Such a connection is described by state functionals. The functionals such as wave function, entropy, the action functional, the Pointing vector, the Einstein tensor and so on, which are the field-equation functionals, are also functionals of the equations describing material media [6]. (As it will be shown below, such functionals describe a material medium state, that is, they are state functionals).

Let us analyze the correlation of the equations that describe the conservation laws for energy and linear momentum.

In the inertial frame of reference the energy equation can be reduced to the following form:

\[
\frac{D\psi}{Dt} = A_t,
\]

where \( D / Dt \) is the total derivative with respect to time, \( A_t \) is a quantity that depends on specific features of material system (material medium) and on external energy actions onto the system, \( d\psi \) is the functional that specifies a material system (and which is a state functional). The action functional, entropy, wave function, Einstein’s tensor, Pointing’s vector and others can be regarded as examples of the functional \( \psi \).
In the accompanying frame of reference, which is tied to the manifold made up by the trajectories of elements (particles) of material system the total derivative with respect to time is a derivative along the trajectory. For this reason, in the accompanying frame of reference the equation of energy is written in the form

$$\frac{\partial \psi}{\partial \varepsilon^i} = A_i.$$ \hspace{1cm} (2)

Here $\varepsilon^i$ are the coordinates along the trajectory.

Thus, the equation for energy expressed in terms of the action functional $S$ has a similar form:

$$\frac{\partial S}{\partial \varepsilon^i} = DS / Dt = L.$$ And the equation for the energy of ideal gas can be presented in the form [3]:

$$\frac{\partial s}{\partial \varepsilon^i} = Ds / Dt = 0,$$ where $s$ is the entropy.

In a similar way, in the accompanying frame of reference the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial \psi}{\partial \varepsilon^\nu} = A_\nu, \hspace{1cm} \nu = 2,.. \hspace{1cm} (3)$$

where $\varepsilon^\nu$ are the coordinates in the direction normal to the trajectory, $A_\nu$ are the quantities that depend on the specific features of material system and on force actions.

Here it should be called attention to a certain peculiarity, namely, for a single quantity (the functional $\psi$ ) on has two equations.

How to study such redefined set of equations?

Since equations (2) and (3) are expressions for derivatives along different directions, they can be convoluted into the relation

$$d\psi = A_\mu d\varepsilon^\mu, \hspace{1cm} \mu = 1, \nu.$$ \hspace{1cm} (4)

Relation (4) can be rewritten as

$$d\psi = \omega,$$ \hspace{1cm} (5)

here $\omega = A_\mu d\varepsilon^\mu$ is a skew-symmetric differential form of the first degree. (A summing over repeated indices is carried out.)

In the general case (for energy, linear momentum, angular momentum and mass) this relation will be the written as

$$d\psi = \omega^p,$$ \hspace{1cm} (6)

where $\omega^p$ is the form degree $p$ ($p$ takes the values $p = 0,1,2,3$). (The relation for $p = 0$ is an analog to that in the differential forms, and it has been obtained from the interaction of energy and time.)

Since the conservation law equations are evolutionary ones, the relations obtained are also evolutionary relations, and the skew-symmetric form $\omega^p$ is evolutionary ones.

In this case $p = 2$ the functional is Pointing's vector [2]. The relation for Einstein's tensor is obtained when integrating the evolutionary relation for $p = 3$.

The evolutionary relation obtained from the equations of the conservation law possesses the properties which describe the connection of the field-theory equations with the equations of mathematical physics.

2.2. Specific properties of evolutionary relation

The evolutionary relation possesses a peculiarity, namely, it appears to be nonidentical. The evolutionary relation was obtained in the accompanying frame of reference, which is connected with the manifold built up by the trajectories of the material system elements. Such a manifold is a deforming nonintegrable one. The skew-symmetric form defined on nonintegrable manifold cannot be closed since the commutator of skew-symmetric form defined on such manifold includes an additional term, namely, the commutator of metric form, which is nonzero (because the metric form of
nonintegrable manifold is not closed one). Since the evolutionary form is unclosed and is not a differential, the evolutionary relation (6) turns out to be nonidentical.

The evolutionary relation possesses one more peculiarity, namely, this relation is a self-varying relation.

The evolutionary nonidentical relation is a self-varying one, because, firstly, it is a nonidentical, namely, it contains two objects one of which appears to be unmeasurable, and, secondly, it is an evolutionary relation, that is, the variation of any object of the relation in some process leads to a variation of another object; and, in turn, the variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot terminate.

The evolutionary relation obtained from the equations of mathematical physics discloses the properties of the solutions of mathematical physics equations and physical properties of these solutions that enable one to describe the processes of emergence of physical structures. And this is connected with the properties of the evolutionary relation functionals.

2.3. Properties of solutions to the mathematical physics equations

From evolutionary relation it follows that the mathematical physics equations has a double solutions, namely, the solutions that are not functions (their derivatives do not made up a differential) and the solutions that are discrete functions. These are precisely the properties of the mathematical physics equations that enable one to describe evolutionary processes and the processes of emergence of various structures.

2.4. Inexact solutions to mathematical physics equations (which are not the functions)

The nonidentical evolutionary relation \( d\psi = \omega^\rho \) cannot be integrated directly since its right-hand side contains unclosed skew-symmetric form which is not a differential. This means that the original equations of mathematical physics prove to be nonintegrable (they cannot be convoluted into identical relation for differentials and be integrated). In this case solutions to the initial equations of mathematical physics are not functions (their derivatives do not made up a differential). These solutions will depend on the commutator of the evolutionary skew-symmetric form \( \omega^\rho \), which is nonzero. If the commutator be equal to zero, the evolutionary relation would be identical and this would point out to integrability of original equation.

However, it appears that the identical relation can be obtained from nonidentical evolutionary relation. This relation can be integrated, and this fact will point out to a realization of the solutions that are functions.

2.5. Exact solutions to equations of mathematical physics (discrete functions)

From the evolutionary nonidentical relation one can obtain an identical relation composed of differentials that can be integrated. This will point out to a realization of integrability of the initial mathematical physics equation. This can happen if a closed exterior form will be obtained from the evolutionary skew-symmetric form in the right-hand side of evolutionary relation. However, since the evolutionary form differential is nonzero (because the evolutionary form is unclosed), whereas the differential of closed exterior form equals to zero, the transition from evolutionary form to closed exterior form is possible only under degenerate transformation, namely, under a transformation that does not preserve a differential.

Such degenerate transformations can take place under additional conditions, which are due degrees of freedom. The vanishing of such functional expressions as determinants, Jacobians, Poisson’s brackets, residues, and others corresponds to these additional conditions. These conditions can be realized (spontaneously) under a change of nonidentical evolutionary relation, which, as it was noted, appears to be a self-varying relation. Under degenerate transformation from the unclosed evolutionary form \( \omega^\rho \)
(see evolutionary relation (6)) with non-vanishing differential \(d\omega^p \neq 0\), one can obtain a closed inexact (only on some pseudostructure) exterior form with vanishing (interior) differential. That is, it is realized the transition 
\[
d\omega^p \neq 0 \rightarrow (\text{degenerate transformation}) \rightarrow d_x \omega^p = 0, \quad d_x^* \omega^p = 0.
\]
The realization of the conditions \(d_x^* \omega^p = 0\) and \(d_x \omega^p = 0\), means that it is realized the closed dual form \(\omega^p\), which describes some structure \(\pi\) (this is a pseudostructure with respect to its metric properties), and the closed exterior (inexact) form \(\omega^p\), which basis is a pseudostructure, is obtained. The realization of closed inexact exterior form \(\omega^p\) leads to the fact that on a pseudostructure from evolutionary relation (6) it follows the relation 
\[
d\psi_x = \omega^p_x, \quad (7)
\]
that occurs to be an identical one, since the form \(\omega^p_x\) is a differential.
Since the identical relation can be integrated (because it contains only of differentials), this means that on the pseudostructure the equations of mathematical physics become locally integrable (only on pseudostructure). In this case the pseudostructure is an integrable structure. The solutions to the mathematical physics equations on integrable structures are generalized solutions, which are discrete functions, since they are realized only under additional conditions (on the integrable structures). Due to the fact that generalized solutions are defined only on integral structures, the derivatives of generalized solutions have a break in the direction normal to integral structure.

3. Description of evolutionary processes in material media and the processes of physical structure emergence
It turns out that inexact solution describes nonequilibrium state of material media, whereas exact solution describes locally-equilibrium state. This follows from the evolutionary relation and is connected with the property of the functional \(\psi\).

3.1. Nonequilibrium state of material media
The functional \(\psi\) in the left-hand side of the evolutionary relation \(d\psi = \omega^p\) possesses a unique property, namely, this functional specifies a state of material media.
The differential of functional availability means that there exists a state function, and this fact points out to equilibrium state of material medium. However, in this case there is a certain delicate point related to the nonidentity of evolutionary relation. Since the evolutionary relation turns out to be not identical, from this relation one cannot get the differential \(d\psi\) that could point out to the equilibrium state of material medium. The absence of the differential \(d\psi\) means that the state of material medium is nonequilibrium. And this means that an internal force acts in material medium. It is evident that the internal force originates at the expense of some quantity described by the evolutionary form commutator. (If the evolutionary form commutator be zero, the evolutionary relation would be identical, and this would point to the equilibrium state, i.e. the absence of internal forces.) Everything that gives a contribution into the evolutionary form commutator leads to emergence of internal force.
No equilibrium state of material medium is caused by the properties of conservation laws, namely, its noncommutativity. From the nonidentity of the evolutionary relation it follows that the equations of balance conservation laws turn out to be inconsistent. And this points to a noncommutativity of balance conservation laws. The reason of noncommutativity of balance conservation laws is the fact that the actions of different nature (energetic, force and others), which are inconsistent with a nature of material medium itself, act to material medium (that is, these actions are non-potential), and hence these actions cannot be directly transformed into physical quantities of material medium system itself.
They are accumulated as a certain nonmeasurable quantity. The evolutionary form commutator just describes such nonmeasurable quantity.

Self variation of the nonidentical evolutionary relation points to the fact that the nonequilibrium state of material medium turns out to be selfvarying. In this case the state of material medium changes but remains to be nonequilibrium since under selfvariation the evolutionary relation remains to be nonidentical.

3.2. Transition of material medium into locally equilibrium state. Advent of observable formations

From the evolutionary relation one found that inexact solutions to the mathematical physics equations describe nonequilibrium state of material medium.

It turns out that the generalized solutions, which are discrete functions, describe a locally-equilibrium state of material medium. And this follows from the physical properties of identical relation obtained.

From the identical relation \( \text{d}\psi = \omega \text{d}x \) one can obtain the differential of the state functional \( \text{d}\psi \), and this points out to a presence of the state function and the transition of material medium from nonequilibrium state into equilibrium one. However, such a state of material medium turns out to be realized only locally due to the fact that the differential of state functional obtained is an interior differential (only on pseudostucture). And yet the total state of material medium remains to be a non-equilibrium state because the evolutionary relation, which describes a material medium state, remains to be nonidentical one. (That is, there exists a duality. Nonidentical evolutionary relation goes on to act simultaneously with identical relation.)

It can be noted that these results point out to the fact that the functionals of evolutionary relation, i.e., functionals of the mathematical physics equations, are really state functionals, that is, they characterize a state of material medium. Under realization of the degenerate transformation condition, which can be caused by any degrees of freedom, and realization of identical relation they locally convert into state function.

The transition from non-equilibrium state to locally-equilibrium state means that unmeasurable quantity, which is described by the evolutionary form commutator and act as internal force, converts into a measurable quantity of material medium.

The transition of unmeasurable quantity into a measurable quantity of material medium reveals in emergence in material medium of some observed formations. Waves, vortices, fluctuations, turbulent pulsations and so on are examples of such formations. The intensity of such formations is controlled by a quantity accumulated by the evolutionary form commutator. (This discloses a mechanism of such processes like an origin of vortices and turbulence [7].)

Such emerged formations are described by generalized solutions to the equations of mathematical physics. The functions that correspond to generalized solutions, as it is known, are discrete functions that have breaks of functions itself or its derivatives [8].

3.3. Emergence of physical structures

It was shown that the transition of material medium from nonequilibrium state to the locally-equilibrium one is accompanied by the emergence of observable formations in material medium.

The advent of observable formations relates to emergence of physical structures.

The transition from nonidentical relation to identical one, which describes a transition of material medium from nonequilibrium state to a locally-equilibrium one, and an advent of observable formations, occurs at realization of closed inexact exterior and dual forms. This points out to emergence of a pseudostructure (closed dual form) with conservative quantity (closed inexact exterior form), that is, emergence of a physical structure (a pseudostructure on which exact conservation law fulfills). Massless particles, structures made up by eiconal surfaces and wave fronts, and so on are examples of physical structures.

Duality of physical structures and observed formations is described by identical relation (7). The left-hand side of this relation includes the differential, which specifies material medium and whose availability points to the locally-equilibrium state of material medium and an advent of observable
measurable formations. And the right-hand side includes a closed inexact form, which is a characteristic of physical structures.

Thus, one can see that the emergence of physical structures in evolutionary process (to which the realization of closed inexact exterior form points out) in material system reveals as an advent of certain observable formations, which develop spontaneously. It appears those physical structures and the observable formations of material media are a manifestation of the same phenomena. The light is an example of such a duality, namely, as a massless particle (photon) and as a wave.

However, physical structures and observable formations are not identical objects. Whereas the wave is an observable formation, the element of wave front made up the physical structure in the process of its motion.

The generation of physical structures by material media discloses a connection between the field-theory equations and the equations of mathematical physics.

The physical structures are structures on which exact conservation laws are fulfilled, that is, there exist conservative quantities or objects.

It is evident that physical fields, which obey exact conservation laws, are formatted by such physical structures on which exact conservation law for physical fields are fulfilled. To this it also points the fact that there exists a correspondence between the evolutionary relation, from which follows closed exterior forms describing physical structures, and the field-theory equations. Closed exterior forms also obtained from evolutionary relation are solutions to the field-theory equations. And besides, the field-theory equations, as opposed to usual differential equations, have the form of nonidentical relations (or its tensor and differential analogs).

4. State functionals of equations of mathematical physics. Connection between functionals of the field-theory equations and functionals of the mathematical physics equations

From a description of the processes of emerging physical structures and observable formations one can see a role of functionals of the mathematical physics equations in a description of these processes. It is evident that functionals of the mathematical physics equations are state functionals. They characterize a state of material medium. Such a role of functionals of the mathematical physics equations relates to a specific peculiarity of functionals. They can at once be either functionals and state functions or potentials. As functionals they describe a nonequilibrium state of material medium, and as state functions they point out to a locally-equilibrium state of material medium. The transition from functionals to state functions describe a transition of material medium from nonequilibrium state to locally-equilibrium one which is accompanied by emergence of physical structures. That is, the transition from functionals to state functions describes the mechanism of physical structure origination. This discloses physical meaning of functionals both as functionals of mathematical physics equations and as field-theory functionals. This demonstrates a connection between field-theory functionals and functionals of mathematical physics. Such a duality of functionals point out to a connection between the field-theory equations, which describe physical fields, and the mathematical physics equations, which describe material media. This discloses a connection of physical fields with material media. As it was shown, from the mathematical physics equations, which describe material media, it follows the evolutionary relation, which describes the process of emergence of physical structures, from which physical fields are formatted. This points out to the fact that physical fields are generated by material media.

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