Texture Zeros and WB Transformations in the Quark Sector of the Standard Model

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Stimulated by the recent attention given to the texture zeros found in the quark mass matrices sector of the Standard Model, an analytical method for identifying (or to exclude) texture zeros models will be implemented here. We use the WB transformation process to find equivalent representations. It is shown that the number of non-equivalent quark mass matrix representations is finite. We give numerical results for parallel and non-parallel four-texture zeros models. We find that some five-texture zeros Ansätze are in agreement with all present experimental data. And we confirm definitely that six-texture zeros of Hermitian quark mass matrices are not viable models anymore.

I. INTRODUCTION

Although the gauge sector of the Standard Model (SM) with the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ symmetry is very successful, the Yukawa sector of the SM is still poorly understood. The origin of the fermion masses, the mixing angles and the CP violation remain as open problems in particle physics. There have been a lot of studies of possible fundamental symmetries in the Yukawa coupling matrices of the SM [1–3]. In the absence of a more fundamental theory of interactions, an independent phenomenological model approach to search for possible textures or symmetries in the fermion mass matrices is still playing an important role.

In the SM, the mass term is given by

$$ - \mathcal{L}_M = \bar{u}_R M_u u_L + \bar{d}_R M_d d_L + h.c, $$

(1.1)

where the mass matrices $M_u$ and $M_d$ are three-dimensional complex matrices. In the most general case, they contain 36 real parameters. A first simplification, without losing generality, is by making use of the polar decomposition theorem of matrix algebra, by which, one can always express a general mass matrix as a product of a hermitian and unitary matrix. Therefore, we can consider quark mass matrices to be hermitian as the unitary matrix can be absorbed in the right handed quark fields. This immediately brings down the number of free parameters from 36 to 18.

A simple and instructive ansatz of hermitian quark mass matrices with six-texture zeros was first proposed in reference [1]. An additional non-parallel six-texture zeros was given in [4]. Both textures are currently ruled out [2], because, among other things, they do not reproduce some entries of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix $V$. Specifically, in both cases, the magnitude of $|V_{ub}/V_{cb}|$ predicted by $\sqrt{m_u/m_c}$ is too low ($V_{ub}/V_{cb} \approx 0.06$ or smaller for reasonable values of the quark masses $m_u$ and $m_c$ [4, 5]) to agree with the present experimental result ($V_{ub}/V_{cb} \approx 0.09$ [9]). Because of this, some authors have highly recommended the use of four-texture zeros [3, 8, 9]. But it is shown in this work, that this point of view is not completely certain.

We would therefore present an analytical method to calculate models containing various texture zeros in the quark mass matrix sector, taking into account the latest experimental data provided [9]. We use simultaneously, in our research, two very common approach: one approach consists of placing zeros (called texture zeros) at certain entries of quark mass matrices that can predict self-consistent and experimentally-favored relations between quark masses and flavor mixing parameters [4, 10, 11] which is used in conjunction with the other approach, the WB transformation (weak basis transformation), that transforms the quark mass matrix representations into new equivalent ones [5].

This paper is organized as follows: in Sect. II we discuss some issues related to the WB transformation method and its utilities. We dedicate, in Sect. III, to obtain some numerical parallel and non-parallel four-texture zeros quark mass matrices using special techniques for that; which we use, in Sect. IV, to find five-texture zeros in quark mass matrices compatible with the present experimental data; this configuration, is studied from an analytical point of view, in Sect. V and our conclusions are presented in Sect. VI. In Appendix A a new theorem that shows two equivalent quark mass matrices are WB related is proved. And the method used extensively throughout this paper to find texture zeros is verified in Appendix B.

II. WB TRANSFORMATIONS

The most general WB transformation [5], that leaves the physical content invariant and the mass matrices Hermitian, is

$$ M_u \longrightarrow M'_u = U^\dagger M_u U, $$

$$ M_d \longrightarrow M'_d = U^\dagger M_d U, $$

(2.1)

where $U$ is an arbitrary unitary matrix. We say that the two representations $M_{u,d}$ and $M'_{u,d}$ are equivalent each other. Besides, it implies that the number of equivalent representations is infinity. This kind of transformation will be used extensively in calculations below. But, firstly, let us show that the WB transformation is
Hermitian quark mass matrices, given by

\[ (\text{Theorem 1}. \) In the Standard Model, any two pairs of Hermitian quark mass matrices, given by \((M_u, M_d)\) and \((M_u', M_d')\), with identical eigenvalues to a specific scale energy, they are related through a WB transformation \]

\[ M_d = U^\dagger M_d' U \quad \text{and} \quad M_u = U^\dagger M_u' U, \quad (2.2) \]

where \(U\) is an unitary matrix.

A proof of this theorem is given in the Appendix A.

The importance of the WB transformation, as calculation tool, can be appreciated from the following results.

A. The preliminary matrix representation

In the quark-family basis, it is more convenient to use the following quark mass matrix representation \[8, 12\]

\[ M_u = D_u = \begin{pmatrix} \lambda_{1u} & 0 & 0 \\ 0 & \lambda_{2u} & 0 \\ 0 & 0 & \lambda_{3u} \end{pmatrix}, \quad (2.3) \]

\[ M_d = V D_d V^\dagger, \]

which comes from a WB transformation, and we call it as the \textit{u-diagonal representation}. We call the other possibility

\[ M_u = V^\dagger D_u V; \]

\[ M_d = D_d = \begin{pmatrix} \lambda_{1d} & 0 & 0 \\ 0 & \lambda_{2d} & 0 \\ 0 & 0 & \lambda_{3d} \end{pmatrix}, \quad (2.4) \]

as the \textit{d-diagonal representation}. One advantage of using representations \((2.3)\) (or \((2.4)\)) is to be able to use simultaneously the CKM mixing matrix \(V\) and the quark mass eigenvalues \(|\lambda_{iu,d}| (i = 1, 2, 3)\). Where \(\lambda_{iu,d}\) may be either positive or negative and satisfy the hierarchy

\[ |\lambda_{1u,d}| \ll |\lambda_{2u,d}| \ll |\lambda_{3u,d}|. \quad (2.5) \]

The importance of these representations can be appreciated verifying the following result:

"The CKM matrix can be parameterized by three mixing angles and a CP-violating phase"

B. One phase and three angles in the CKM matrix

It is usually said that the CKM matrix is an arbitrary unitary matrix with five phases rotated away through the phase redefinition of the left handed up and down quark fields \[13\]. This can be shown by using the following unitary matrix

\[ \begin{pmatrix} e^{ix} & e^{iy} \\ e^{iy} & 1 \end{pmatrix} \]

in order to make a WB transformation on \((2.2)\). The up matrix

\[ M_u = \begin{pmatrix} e^{ix} & e^{iy} \\ e^{iy} & 1 \end{pmatrix} D_u \begin{pmatrix} e^{ix} & e^{iy} \\ e^{iy} & 1 \end{pmatrix}^\dagger = D_u, \quad (2.6) \]

remains equal, while the down matrix takes the form

\[ M_d = \begin{pmatrix} e^{ix} & e^{iy} \\ e^{iy} & 1 \end{pmatrix} (V D_d V^\dagger) \begin{pmatrix} e^{ix} & e^{iy} \\ e^{iy} & 1 \end{pmatrix}^\dagger, \quad (2.7) \]

\[ M_d = \begin{pmatrix} e^{ix} \\ e^{iy} \end{pmatrix} V \begin{pmatrix} e^{i\alpha_1} & e^{i\alpha_2} \\ e^{i\alpha_2} & e^{i\alpha_3} \end{pmatrix} D_d \]

\[ \times \begin{pmatrix} e^{ix} \\ e^{iy} \end{pmatrix} V^\dagger \begin{pmatrix} e^{i\alpha_1} & e^{i\alpha_2} \\ e^{i\alpha_2} & e^{i\alpha_3} \end{pmatrix}^\dagger, \quad (2.8) \]

where in the last step we have used the identity \((2.6)\) applied to the diagonal down mass matrix. The expression into the square brackets is precisely the most general way to write an unitary matrix \[13\].

In this representation, the matrix \(M_d\), in \((2.7)\), contains two free parameters \(x \text{ and } y\), which plays an important role to obtain texture zeros as we shall see later.

C. An unique negative eigenvalue

The Theorem 1 permits to use the u-diagonal representation \((2.3)\) (or the d-diagonal representation \((2.4)\)) as the starting point, to generate any other representation. If they exist, by this method, important texture zeros in mass matrix can be found.

Because some texture zeros must lie along the diagonal entries of both up and down Hermitian quark mass matrices, it implies that at least one and at most two of its eigenvalues be negative \[8\]. Furthermore, for the case of two negative eigenvalues, these mass matrices can be reduced to have only one negative eigenvalue, by factoring a minus sign out which can be included, for instance, into the mass matrix basis \((2.3)\). Thus, without loss of generality, the texture zeros models can be deduced by assuming

"that each one of quark mass matrices \(M_u\) and \(M_d\) contains exactly one negative eigenvalue." \(2.9\)
III. NUMERICAL FOUR-TEXTURE ZEROS

There are a wide variety of four-texture zeros representations. Using a specific approach, some non-parallel texture are easy to obtain. But more laborious methods are required in parallel cases. In our analysis we will use the next physical quantities.

\[
V = \begin{pmatrix}
c_{12} c_{13} & \bar{s}_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} \\
-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} \\
s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & c_{12} s_{23} - s_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} + s_{12} s_{23} s_{13} e^{i\delta}
\end{pmatrix},
\]

where \( s_{ij} = \sin \theta_{ij}, c_{ij} = \cos \theta_{ij}, \) and \( \delta \) is the phase responsible for all CP-violating phenomena in flavor-changing processes in the SM. The angles \( \theta_{ij} \) can be chosen to lie in the first quadrant, so \( s_{ij}, c_{ij} \geq 0. \)

It is known experimentally that \( s_{13} \ll s_{23} \ll s_{12} \ll 1, \) and it is convenient to exhibit this hierarchy using the Wolfenstein parametrization. We define \[17, 18\]

\[
s_{12} = \lambda, \quad s_{23} = A \lambda^2, \quad s_{13} e^{i\delta} = \frac{A \lambda^3 (\rho + i \eta)}{\sqrt{1 - A^2 \lambda^4}},
\]

(3.3)

The constraints implied by the unitarity of the three generation CKM matrix significantly reduce the allowed range of some of the CKM elements. The fit for the Wolfenstein parameters defined in Eq. \[5, 6\] gives

\[
\lambda = 0.2253 \pm 0.0007, \quad A = 0.808^{+0.012}_{-0.015}, \quad \rho = 0.132^{+0.012}_{-0.014}, \quad \eta = 0.341 \pm 0.013.
\]

(3.4)

These values are obtained using the method of Refs. \[17, 19\]. The fit results for the values of all four CKM elements are.

\[
V = \begin{pmatrix}
0.97428 & 0.22530 & 0.003469 e^{-1.2020} \\
0.22516 e^{-1.1410} & 0.97346 e^{-1.0089} & 0.040248 e^{-1.1235} \\
0.0086194 e^{-1.3737} & 0.040248 e^{-1.3737} & 0.99915
\end{pmatrix}
\]

(3.5)

and the Jarlskog invariant is

\[
J = (2.91^{+0.19}_{-0.11}) \times 10^{-5}.
\]

(3.6)

B. Non-parallel four-texture zeros

It is the most simple case. For instance, let us take the eigenvalues signs pattern as follow

\[
\lambda_{1u} = -m_u, \lambda_{2u} = m_c, \lambda_{3u} = m_t, \quad \lambda_{1d} = m_d, \lambda_{2d} = -m_s, \lambda_{3d} = m_b.
\]

(3.7)

A. Quark masses and CKM

Let us consider the quark masses (in MeV) in the \( \overline{MS} \) scheme \[7, 14\], given by

\[
\begin{align*}
m_u & = 2.5^{+0.6}_{-0.8}, \quad m_c = 1290^{+50}_{-110}, \quad m_t = 172900^{+600}_{-500}, \\
m_d & = 5.0^{+0.7}_{-0.9}, \quad m_s = 100^{+30}_{-20}, \quad m_b = 4190^{+180}_{-160}.
\end{align*}
\]

(3.1)

The Cabibbo-Kobayashi-Maskawa (CKM) matrix \[7, 14, 15\] is a \( 3 \times 3 \) unitary matrix. It can be parametrized by three mixing angles and the CP-violating KM phase \[15\]. Of the many possible conventions, a standard choice has become \[16\]

\[
\begin{align*}
M_u & = \begin{pmatrix} -2.5 & 1290 \\ 1290 & 172900 \end{pmatrix} \text{MeV}, \\
M_d & = \begin{pmatrix} -0.27947 & -22.814 - 0.55561i \\ -22.814 + 0.55561i & 6.1758 - 13.535i \\ 6.1758 + 13.535i & 175.61 + 0.074512i \end{pmatrix} \text{MeV},
\end{align*}
\]

(3.9)

where we have used the numerical CKM matrix \[3, 5\].

We observe that the entry \( M_d(1,1) \) in \[(3.9)\] is much smaller than the remaining entries. So, we can assume \( M_d(1,1) = 0, \) as was pointed out in Reference \[12\].

Making a WB transformation on \[(3.9)\] using the following unitary matrix

\[
U = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix},
\]

(3.10)

with \( \tan \theta = \sqrt{\frac{m_c}{m_t}}, \) the matrices \[(3.9)\] transform into a form, where the entries \( (1,1), (1,2) \) and \( (2,3) \) of matrix \( M_u \) becomes zero. Then, we have

\[
M_u = U D_u U^\dagger = \begin{pmatrix} 0 & 0 & 657.457 \\ 0 & 1290 & 0 \\ 657.457 & 0 & 172897 \end{pmatrix} \text{MeV},
\]

(3.11)

and

\[
M_d = U M_d U^\dagger = \begin{pmatrix} -0.17202 & -22.146 - 0.55532i & 22.082 - 13.535i \\ -22.146 + 0.55532i & 175.70 + 0.076642i & 4182.6 \\ 22.082 + 13.535i & 175.70 - 0.076642i & 4182.6 \end{pmatrix} \text{MeV},
\]

(3.12)
where the entry (1, 1) of \( M_d \) was assumed equal to zero. This result follows from the fact that the top quark mass is much greater than up quark mass.

We finally obtain a non-parallel four-texture zeros mass matrix representation.

\[
M_u = \begin{pmatrix}
0 & 0 & 657.457 \\
0 & 1290 & 0 \\
657.457 & 0 & 172897
\end{pmatrix} \text{MeV},
\]

\[
M_d = \begin{pmatrix}
22.153 e^{-3.1165 i} & 25.90 e^{-0.54990 i} \\
0 & 175.70 e^{0.0043612 i} & 4182.6
\end{pmatrix} \text{MeV}.
\]

(3.13)

New equivalent four-texture zeros representations can be obtained using the former representation. For example, if we use unitary matrices looking like

\[
U_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

(3.14)

and apply them to (3.13), it allows us to obtain new non-parallel four-texture zeros representations. For the case (3.14), we have

\[
M_u = \begin{pmatrix}
0 & 0 & 657.457 \\
0 & 1290 & 0 \\
657.457 & 0 & 172897
\end{pmatrix} \text{MeV},
\]

\[
M_d = \begin{pmatrix}
22.153 e^{-3.1165 i} & 25.90 e^{-0.54990 i} \\
0 & 175.70 e^{0.0043612 i} & 4182.6
\end{pmatrix} \text{MeV}.
\]

(3.15)

\[
O_u = \begin{pmatrix}
e^{ix} \rho \sqrt{\frac{\lambda_{1u} \lambda_{3u} (A_u - \lambda_{1u})}{A_u (A_u - \lambda_{2u}) (\lambda_{3u} - \lambda_{1u})}} & e^{iy} \sqrt{\frac{\lambda_{1u} \lambda_{3u} (\lambda_{2u} - A_u)}{A_u (A_u - \lambda_{1u}) (\lambda_{3u} - \lambda_{2u})}} \\
e^{-ix} \eta \sqrt{\frac{\lambda_{1u} (A_u - \lambda_{1u}) (\lambda_{3u} - \lambda_{1u})}{A_u (A_u - \lambda_{2u}) (\lambda_{3u} - \lambda_{1u})}} & e^{-iy} \eta \sqrt{\frac{\lambda_{1u} (A_u - \lambda_{1u}) (\lambda_{3u} - \lambda_{1u})}{A_u (A_u - \lambda_{2u}) (\lambda_{3u} - \lambda_{1u})}}
\end{pmatrix},
\]

(3.16)

where \( \eta \equiv \lambda_{2u}/m_c = +1 \) or \(-1 \), and \( \rho \equiv \lambda_{3u}/m_t = +1 \) or \(-1 \) corresponding to the possibility \((\lambda_{1u}, \lambda_{2u}, \lambda_{3u}) = (-m_u, -m_c, m_t) \) or \((\lambda_{1u}, \lambda_{2u}, \lambda_{3u}) = (m_u, m_c, -m_t) \). The pure complex phases in (3.14) were included, in order that given them appropriated values, the CKM matrix generated becomes compatible with the chosen convention [3.2].

Note that \( \tilde{B}_u, |B_u| \) and \( |C_u| \) can be expressed in terms of \( \lambda_{iu} \) \((i = 1, 2, 3) \) and \( A_u \) using invariant matrix functions

\[
\text{tr} M_u \Rightarrow \tilde{B}_u = \lambda_{1u} + \lambda_{2u} + \lambda_{3u} - A_u, \]

(3.17)

\[
|B_u|^2 = \frac{(A_u - \lambda_{1u})(A_u - \lambda_{2u})(\lambda_{3u} - A_u)}{A_u},
\]

(3.18)

\[
\text{det} M_u \Rightarrow |C_u| = \frac{-\lambda_{1u} \lambda_{2u} \lambda_{3u}}{A_u},
\]

(3.19)

where “tr” and “det” are the trace and the determinant respectively. The matrix \( O_u \) can be seen as the unitary matrix such that the WB transformation transforms the
where “Re” refers to the real part and “Im” the imaginary part of the function. In the process the following details have to be taken into account:

- The formulas (3.19) through (3.21) must be real numbers. Therefore, the parameter $A_u$ is restricted to lie into an interval. Let us see the different possibilities:
  - If $\lambda_{1u} = -m_u$, $\lambda_{2u} = m_c$, and $\lambda_{3u} = m_t$ then $m_c < A_u < m_t$. (3.25)
  - If $\lambda_{1u} = m_u$, $\lambda_{2u} = -m_c$, and $\lambda_{3u} = m_t$ then $m_u < A_u < m_t$. (3.26)
  - If $\lambda_{1u} = m_u$, $\lambda_{2u} = m_c$, and $\lambda_{3u} = -m_t$ then $m_u < A_u < m_c$. (3.27)
  
  where the hierarchy (2.4) was considered.

- The phases given in (3.18) could have been included initially in the transformation (2.4), instead to write them explicitly in the matrix $O_u$. The validity of this point of view is checked by observing that the matrix (3.18) can be decomposed as the product of two matrices, where the right hand side contains the pure complex phases as follows

$$O_u = O_u(z=0,y=0) \begin{pmatrix} e^{ix} & e^{iy} \\ 1 & 1 \end{pmatrix},$$

such that, after replacing this decomposition into (3.28) and comparing with (2.4), we conclude that both points of view concur.

In appendix [A] we will work a case previously studied in the paper [8] and replicate the results presented there by using the techniques implemented here.

### 1. Example 1: parallel four-texture zeros

We are mainly concerned to find four-texture zeros with the recent data given in Section III A. Let us take the following case

$$\lambda_{1u} = m_u, \lambda_{2u} = m_c, \lambda_{3u} = m_t, \quad (3.29)$$
$$\lambda_{1d} = m_d, \lambda_{2d} = m_s, \lambda_{3d} = m_b. \quad (3.30)$$

We have, in the u-diagonal representation, the following mass matrix representation.

$$M_u = \begin{pmatrix} 2.5 & 0 & 0 \\ 0 & 1290 & 0 \\ 0 & 0 & -17290 \end{pmatrix} \text{ MeV},$$
$$M_d = V D_d V^\dagger = \begin{pmatrix} 9.7717 & 20.62 + 0.55690 i & -6.1069 + 13.567 i \\ -20.62 - 0.55690 i & 87.967 & -175.63 + 0.067420 i \\ -6.1069 - 13.567 i & -175.63 - 0.067420 i & -4182.7 \end{pmatrix} \text{ MeV}. \quad (3.31)$$

Making a WB transformation on (3.31), using the unitary matrix $O_u$ (Eq. 3.18), the following conditions

$$M_d'(1,1)(A_u, x_1, x_2, y_1, y_2) = 0,$$
$$\text{Re} \left[ M_d'(1,3)(A_u, x_1, x_2, y_1, y_2) \right] = 0,$$
$$\text{Im} \left[ M_d'(1,3)(A_u, x_1, x_2, y_1, y_2) \right] = 0,$$

are established, in order to find vanishes values in the entries (1,1), (1,3) and (3,1) of the resulting matrix $M_d' = O_u M_d O_u^\dagger$, where the pure phases given in $O_u$ were defined as $e^{ix} = \cos x + i \sin x = x_1 + ix_2$ and $e^{iy} = \cos y + i \sin y = y_1 + iy_2$, such that

$$x_1^2 + x_2^2 = 1 \quad \text{and} \quad y_1^2 + y_2^2 = 1. \quad (3.33)$$

Eqs. (3.32) and (3.33) gives the following solution.

$$A_u = 12.311, \quad x_1 = 0.62571, \quad x_2 = 0.78006, \quad$$
$$y_1 = 0.59378, \quad y_2 = 0.80462. \quad (3.34)$$

Finally, we obtain an exact parallel four-texture zeros mass matrix representation.

$$M_u' = O_u M_u O_u^\dagger = \begin{pmatrix} 0 & 6729.91 & 0 \\ 6729.91 & -171620 & 0 \\ 0 & 13269 & 12.9114 \end{pmatrix} \text{ MeV}, \quad (3.35)$$
$$M_d' = O_u M_d O_u^\dagger = \begin{pmatrix} 196.732 & 196.732 + 66.746i \\ -196.732 - 66.746i & 424.424 + 124.56i \\ 0 & 424.424 - 124.56i & 48.539 \end{pmatrix} \text{ MeV}. \quad (3.36)$$

In the same way, we can find another non-equivalent parallel four-texture zeros representation. Let us look another case.

### 2. Example 2: another parallel four-texture zeros model

Another possibility that works well is

$$\lambda_{1u} = -m_u, \lambda_{2u} = m_c, \lambda_{3u} = m_t, \quad (3.37)$$
$$\lambda_{1d} = m_d, \lambda_{2d} = -m_s, \lambda_{3d} = m_b. \quad (3.38)$$
from which, we have $A_u = 89554, x_1 = 0.97542, x_2 = -0.22036, y_1 = -0.41069$ and $y_2 = -0.91178$. Therefore, the parallel four-texture zeros mass matrix representation is

\[ M'_u = O_u M_u O_u^\dagger = \begin{pmatrix} 0 & 78908 & 0 \\ 78908 & 84633 & 85771 \\ 0 & 85771 & 89554 \end{pmatrix} \text{MeV}, \]  \hspace{1cm} (3.39)

\[ M'_d = O_d M_d O_d^\dagger = \begin{pmatrix} 1.7292 + 30.945i & 0 & 1.7292 - 30.945i \\ 0 & 1914.0 & 2132.3 - 159.30i \\ 0 & 2132.3 + 159.30i & 2181.0 \end{pmatrix} \text{MeV}. \]  \hspace{1cm} (3.40)

**IV. NUMERICAL FIVE-TEXTURE ZEROS**

Now, let us try to find five-texture zeros for the quark mass matrix sector. If this cannot be achieved, we can conclude that five and six-texture zeros are not viable models. For that, we will use the mathematical tools previously implemented in Sect. III. We shall begin as usual by proposing a texture zeros configuration, in this case with three zeros for the up/down quark mass matrix, and see how many zeros can be reached for the down/up quark mass matrix. In principle, there are many possibilities, but many of them are equivalent ones. In total, there are two non-equivalent cases, depending on the number of zeros included in their diagonal entries. Therefore, we have only two possibilities: one-zero or two-zero in diagonal entries. Let us name them as one-zero family and two-zero family, respectively. With an appropriated unitary matrix and performing the corresponding WB transformation the other possibilities are obtained. In the table both families are indicated, which summarizes the equivalent possibilities for each case. Let us study each family.

**A. Two-zero family**

In what follows, we work the cases $u$-diagonal and $d$-diagonal simultaneously. The standard representation for the two-zero family is

\[ M_{u,d} = \begin{pmatrix} 0 & |C_{u,d}| & 0 \\ |C_{u,d}| & 0 & |B_{u,d}| \\ 0 & |B_{u,d}| & A_{u,d} \end{pmatrix}. \]  \hspace{1cm} (4.1)

and its diagonalization matrix satisfies the following relation

\[ O_{u,d}^\dagger M_{u,d} O_{u,d} = \begin{pmatrix} \lambda_{1u,d} & \lambda_{2u,d} \\ \lambda_{2u,d} & \lambda_{3u,d} \end{pmatrix}, \]  \hspace{1cm} (4.2)

\[ \lambda_{1u,d} > 0, \quad \lambda_{2u,d} < 0, \quad \lambda_{3u,d} \geq 0. \]  \hspace{1cm} (4.10)

| Unitary matrix $p_1$ | Two zero family $M_{u,d}$ | Family $p_2$ | One zero family $M_{u,d}$ |
|---------------------|------------------------|-------------|------------------------|
| $(p_1, M_{u,d}, p_1^\dagger)$ | $(p_2, M_{u,d}, p_2^\dagger)$ | $(p_1, M_{u,d}, p_1^\dagger)$ | $(p_2, M_{u,d}, p_2^\dagger)$ |
| $(1, 1)$ | $(0, |C_{u,d}|, 0)$ | $(0, 0, |B_{u,d}|)$ | $(0, 0, |B_{u,d}|)$ |
| $(0, |C_{u,d}|, 0)$ | $(0, 0, |B_{u,d}|, A_{u,d})$ | $(0, 0, 0, A_{u,d})$ | $(0, 0, 0, A_{u,d}, C_{u,d})$ |
| $(0, 0, |B_{u,d}|)$ | $(0, 0, 0, C_{u,d})$ | $(0, 0, 0, 0)$ | $(0, 0, 0, 0, 0)$ |
| $(0, 0, 0, A_{u,d})$ | $(0, 0, 0, 0, 0)$ | $(0, 0, 0, 0, 0)$ | $(0, 0, 0, 0, 0, 0)$ |

**TABLE I. One and two zero Family.**

where one and only one $\lambda_{u,d}$ is assumed to be a negative number. The invariant quantities “det” and “trace” applied on (4.1) and (4.2)

\[ \text{tr} M_{u,d} = A_{u,d} = \lambda_{1u,d} + \lambda_{2u,d} + \lambda_{3u,d}, \]  \hspace{1cm} (4.3)

\[ \text{tr} M_{u,d}^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \]  \hspace{1cm} (4.4)

\[ \lambda_1^2 = \lambda_{1u,d}^2 + \lambda_{2u,d}^2, \]  \hspace{1cm} (4.5)

allow us to express the parameters of (4.1) in terms of its eigenvalues

\[ A_{u,d} = \lambda_{1u,d} + \lambda_{2u,d} + \lambda_{3u,d}, \]  \hspace{1cm} (4.6)

\[ |B_{u,d}| = \lambda_{1u,d} \lambda_{2u,d} \lambda_{3u,d} A_{u,d}, \]  \hspace{1cm} (4.7)

\[ |C_{u,d}| = \sqrt{\lambda_{1u,d} \lambda_{2u,d} \lambda_{3u,d}}, \]  \hspace{1cm} (4.8)

From expression (4.3) together with (4.4), we have that

\[ A_{u,d} > 0, \]  \hspace{1cm} (4.9)

and using (4.7) and the hierarchy (4.8) we found that only one possibility is permitted

\[ \lambda_{1u,d}, \lambda_{3u,d} > 0 \text{ and } \lambda_{2u,d} < 0. \]  \hspace{1cm} (4.10)
For the u-diagonal case, the diagonalization matrix is

\[
O_u = \begin{pmatrix} 0.999032e^{ix} & -0.0439812e^{iy} & 0.0000283438 \\ 0.0438159e^{ix} & 0.995333e^{iy} & 0.0859736 \\ -0.00380943e^{ix} & -0.0858892e^{iy} & 0.996297 \end{pmatrix}.
\]

(4.11)

And for the d-diagonal case, the diagonalization matrix is given by

\[
O_d = \begin{pmatrix} 0.975887e^{ix} & -0.218277e^{iy} & 0.000803792 \\ 0.215727e^{ix} & 0.965034e^{iy} & 0.148899 \\ -0.0332769e^{ix} & -0.145135e^{iy} & 0.988852 \end{pmatrix}.
\]

(4.12)

As you can see, in both cases, we are treating with quasi diagonal matrices.

Performing the WB transformation using the unitary matrix \(O_{u,d}\) we have

\[
M'_{u,d} = O_{u,d} \begin{pmatrix} \lambda_{1u,d} & 0 & \lambda_{3u,d} \\ 0 & |C_{u,d}| & 0 \\ 0 & |B_{u,d}| & A_{u,d} \end{pmatrix} O^\dagger_{u,d},
\]

(4.13)

\[
M'_{d,u} = O_{d,u} M_{d,u} O^\dagger_{d,u},
\]

(4.15)

where the matrices

\[
M_d = V D_d V^\dagger \quad \text{and} \quad M_u = V^\dagger D_u V,
\]

(4.16)

depend on if we work with either the u-diagonal or the d-diagonal case.

In order to facilitate the calculus we define the following new variables

\[
e^{ix} = x_1 + ix_2, \quad x_1^2 + x_2^2 = 1,
\]

\[
e^{iy} = y_1 + iy_2, \quad y_1^2 + y_2^2 = 1,
\]

(4.17)

where their norms satisfy

\[
|x_1|, |x_2| \leq 1, \quad \text{and} \quad |y_1|, |y_2| \leq 1.
\]

(4.18)

With the former definitions, the elements of the matrix \(M'_{d,u}\) defined in (4.15) have now a polynomial form. The results are summarized in Tables (II) and (III).

1. **Analysis of “down” mass matrix.**

Table (II) summarizes the components of \(M'_d\) for the u-diagonal case. By simple inspection, using (4.18), shows that is not possible to find zeros at entries (2,2), (2,3) and (3,3). And not solutions were found for either

\[
\text{Re}[M'_d(1,2)] = 0, \quad \text{Im}[M'_d(1,2)] = 0, \quad \text{or}
\]

\[
\text{Re}[M'_d(1,3)] = 0, \quad \text{Im}[M'_d(1,3)] = 0,
\]

equations. Therefore, it is impossible to find two texture zeros into the down quark mass matrix coming from an u-diagonal representation for the two zero family case.

2. **Analysis of “up” mass matrix and a model with five-texture zeros.**

We consider the d-diagonal case. The entries of matrix \(M'_d\), after the WB transformation is made, are given in the Table (III). According to the table, only entries (1,2) and (1,3) deserve some attention. From which, only the case \(\lambda_{1u} = -m_u\) gives an acceptable solution.

\[
M'_u(1, 2) = 0,
\]

(4.19)

\[
M'_u(1, 1) \approx 0,
\]

(4.20)

with

\[
x_1 = 0.990715, \quad y_1 = 0.135959,
\]

\[
x_2 = -0.950378, \quad y_2 = -0.311096.
\]

(4.21)

The corresponding five-texture zeros representation obtained, is

\[
M'_u = \begin{pmatrix} 0 & 0 & -45.1398 + 268.508i \\ 0 & 7348.64 & 31603.6 + 2136.6i \\ -45.1398 - 268.508i & 31603.6 - 2136.6i & 166839 \end{pmatrix} \text{MeV},
\]

(4.22a)

\[
M'_u = \begin{pmatrix} 0 & 22.6186 & 0 \\ 22.6186 & 0 & 630.903 \\ 0 & 630.903 & 4935 \end{pmatrix} \text{MeV}.
\]

(4.22b)
The following matricial functions allow us to write the elements of $M_{u,d}$ in terms of its eigenvalues $\lambda_{u,d}$. They are

$$\text{tr}M_{u,d} = A_{u,d} + C_{u,d} = \lambda_{1u,d} + \lambda_{2u,d} + \lambda_{3u,d}, \quad (4.26)$$

$$\text{tr}M_{u,d}^2 = A_{u,d}^2 + 2|B_{u,d}|^2 + C_{u,d}^2,$$

$$= \lambda_{1u,d}^2 + \lambda_{2u,d}^2 + \lambda_{3u,d}^2, \quad (4.27)$$

$$\det M_{u,d} = -A_{u,d}|B_{u,d}|^2 = \lambda_{1u,d}\lambda_{2u,d}\lambda_{3u,d},$$

from which we have various solutions

$$a): \quad A_{u,d} = \lambda_{1u,d}, \quad |B_{u,d}| = \sqrt{-\lambda_{2u,d}\lambda_{3u,d}}, \quad (4.29)$$

$$C_{u,d} = \lambda_{2u,d} + \lambda_{3u,d},$$

$$\lambda_{1u,d} + \lambda_{2u,d} + \lambda_{3u,d}.$$
TABLE III. The d-diagonal representation: the “up” mass matrix entries for the two zero family case.

| $M'_u$ | Case 1. $\lambda_{1u} = -m_u$ (MeV) | Case 2. $\lambda_{2u} = -m_u$ (MeV) | Case 3. $\lambda_{3u} = -m_1$ (MeV) |
|--------|-----------------------------------|-----------------------------------|-----------------------------------|
| entries |                                   |                                   |                                   |
| $M'_{u}(1, 1)$ | 143.949 + 2.15605x_{1u} - 0.852947x_{2u} + 2.42129y_{1u} + 144.008x_{1y} - 9.68582x_{2y} + 0.0441168y_{2u} + 9.68582x_{1y} + 144.008x_{2y} | 118.674x_{1u} + 299.964x_{1y} + 12.646y_{1y} - 0.0182744x_{1y} - 0.272269x_{2y} + 673.866y_{2y} + 0.272269x_{1y} - 0.0182744x_{1y} | 97.0109 - 2.19334x_{1u} + 0.852947x_{2u} + 2.42129y_{1u} + 144.008x_{1y} - 9.68582x_{2y} + 0.0441168y_{2u} + 9.68582x_{1y} + 144.008x_{2y} |
| $M'_{u}(1, 2)$ | -279.836 + 199.966x_{1u} - 79.096x_{2u} + 218.914y_{1u} - 303.054x_{1y} + 20.3407y_{2u} + 0.39877y_{1y} - 9.78994x_{1y} + 199.462x_{2y} + 1.18375y_{1y} | 220.182y_{1u} + 203.860x_{1y} + 20.3407y_{2u} + 0.39877y_{1y} - 9.78994x_{1y} + 199.462x_{2y} + 1.18375y_{1y} | -207.624 - 203.405x_{1u} + 79.096x_{2u} - 222.182y_{1u} + 208.386x_{1y} + 20.3407y_{2u} + 0.39877y_{1y} - 9.78994x_{1y} + 199.462x_{2y} + 1.18375y_{1y} |
| $M'_{u}(1, 3)$ | 182.926 + 524.648x_{1u} - 143.189x_{2u} + 140.184y_{1u} + 45.166x_{1y} - 27.1517y_{2u} + 0.52467x_{1y} + 50.4368x_{2y} + 3.8526x_{1y} + 50.4368x_{2y} | 108.792 + 134.19x_{1u} + 524.648x_{2u} + 152.412y_{1u} - 30.6183x_{1y} + 27.1517y_{2u} + 0.52467x_{1y} + 50.4368x_{2y} + 3.8526x_{1y} + 50.4368x_{2y} | -109.118 - 134.19x_{1u} + 524.648x_{2u} + 152.412y_{1u} - 30.6183x_{1y} + 27.1517y_{2u} + 0.52467x_{1y} + 50.4368x_{2y} + 3.8526x_{1y} + 50.4368x_{2y} |
| $M'_{u}(2, 2)$ | 5299.6 + 289.023x_{1u} - 34.928x_{2u} + 1983.03y_{1u} - 141.037x_{1y} + 9.46621y_{2u} - 36.1361y_{1y} - 9.46621y_{2u} + 141.037x_{1y} | 2946.99 + 89.821x_{1u} - 34.928x_{2u} + 2012.64y_{1u} + 94.8473x_{1y} + 9.46621y_{2u} - 36.1361y_{1y} - 9.46621y_{2u} + 141.037x_{1y} | -2946.54 - 89.821x_{1u} - 34.928x_{2u} + 2012.64y_{1u} + 94.8473x_{1y} + 9.46621y_{2u} - 36.1361y_{1y} - 9.46621y_{2u} + 141.037x_{1y} |
| $M'_{u}(2, 3)$ | 25203.9 + 284.362x_{1u} - 113.287x_{2u} + 6435.63y_{1u} + 21.4833x_{1y} - 1.44194y_{2u} - 117.26y_{1y} - 1.44194y_{2u} - 117.26y_{1y} | 2554.66 + 291.328x_{1u} - 113.287x_{2u} + 6531.71y_{1u} - 14.4517x_{1y} - 117.26y_{1y} - 1.44194y_{2u} - 117.26y_{1y} | -2554.66 - 291.328x_{1u} - 113.287x_{2u} + 6531.71y_{1u} - 14.4517x_{1y} - 117.26y_{1y} - 1.44194y_{2u} - 117.26y_{1y} |
| $M'_{u}(3, 3)$ | 16968.3 + 158.84x_{1u} + 35.781x_{2u} + 190.61y_{1u} - 3.27191y_{1y} + 0.219607y_{2y} + 36.0875y_{1y} - 0.219607y_{2y} + 36.0875y_{1y} | 16875.8 - 92.0148x_{1u} + 35.7792x_{2u} + 2010.18y_{1u} + 2.20099x_{1y} + 0.22909x_{2y} + 36.0875y_{1y} - 0.219607y_{2y} + 36.0875y_{1y} | -16875.8 + 92.0148x_{1u} + 35.7792x_{2u} + 2010.18y_{1u} + 2.20099x_{1y} + 0.22909x_{2y} + 36.0875y_{1y} - 0.219607y_{2y} + 36.0875y_{1y} |

b): $A_{u,d} = \lambda_{2u,d}$, $|B_{u,d}| = \sqrt{-\lambda_{1u,d}\lambda_{3u,d}}$, $C_{u,d} = \lambda_{1u,d} + \lambda_{3u,d}$, $M'_{u} = P^\dagger \begin{pmatrix} 0 & 1 & \lambda_{u,d} |

\textbf{V. ANALYTICAL FIVE-TEXTURE ZEROS AND THE CKM MATRIX}

The five-texture zeros form of Eq. (4.22), derived under the conditions given in section [V A 2], is specially interesting because with the latest low energy data shows that it is a viable model, something not considered or rule out in papers like [3, 8, 20]. Let us assume the following five-texture zeros model

$$M'_{u} = P^\dagger \begin{pmatrix} 0 & 0 & |C_{u}| & |B_{u}| & |A_{u}| 
0 & 1 & \lambda_{u,d} & |C_{u}| & |B_{u}| 
1 & 0 & |C_{u}| & |B_{u}| & |A_{u}| 
\end{pmatrix}.$$

where up and down quark mass matrices are given in the most general way, $P = \text{diag}(e^{-i\phi_{u} x}, e^{-i\phi_{d} x}, 1)$ with $\phi_{u} = \arg(B_{u})$ and $\phi_{d} = \arg(C_{u})$, where the phases for $M'_{u}$ no were considered because they can be absorbed, through
where \( \mathbf{O} \) was considered.

\[
U_u = P^1 \cdot p_2 \cdot O_u \approx \begin{pmatrix}
\frac{\sqrt{|A_u - m_c|}}{\sqrt{m_c}} & \frac{\sqrt{|m_s - 2m_u|}}{\sqrt{m_s}} & \frac{\sqrt{|m_s - m_u|}}{\sqrt{m_s}} \\
\frac{\sqrt{|A_u - m_c|}}{\sqrt{m_c}} & 1 & \frac{\sqrt{|m_s - m_u|}}{\sqrt{m_s}} \\
\frac{\sqrt{|A_u - m_c|}}{\sqrt{m_c}} & \frac{\sqrt{|m_s - 2m_u|}}{\sqrt{m_s}} & 1
\end{pmatrix},
\]

where pure phases present in \( O_u \) (Eq. (3.13)) were omitted, because of (2.3). The \( 3 \times 3 \) matrix \( p_2 = [(1, 0, 0), (0, 0, 1), (0, 1, 0)] \) and the hierarchy (2.3) together with (5.25) were considered.

And the unitary matrix \( U_d \) which diagonalizes \( M_d \) is given by

\[
U_d \approx \begin{pmatrix}
1 & -\frac{\sqrt{m_d}}{\sqrt{m_s}} & \frac{\sqrt{m_d}}{\sqrt{m_s}} \\
\frac{\sqrt{m_d}}{\sqrt{m_s}} & 1 & -\frac{\sqrt{m_d}}{\sqrt{m_s}} \\
\frac{\sqrt{m_d}}{\sqrt{m_s}} & -\frac{\sqrt{m_d}}{\sqrt{m_s}} & 1
\end{pmatrix}.
\]

Now, we can easily find the CKM matrix \( V = U_d^\dagger U_u \). In particular, using the matrix form (5.4) and (5.5) for \( U_u, U_d \) respectively, can survive current experimental tests. To leading order, we obtain,

\[
|V_{ud}| \approx |V_{us}| \approx |V_{tb}| \approx 1,
\]

\[
|V_{us}| \approx |V_{cd}| \approx \sqrt{\frac{A_u - m_c}{A_u}} \frac{m_u}{m_c} - e^{\pm i\phi_{bu} - \phi_{cu}} \sqrt{|m_d|/m_s},
\]

\[
|V_{cb}| \approx |V_{ts}| \approx \sqrt{\frac{m_c}{m_b}} - e^{\pm i\phi_{tb} - \phi_{tb}} \sqrt{|A_u - m_c|/m_t},
\]

\[
\frac{|V_{ub}|}{|V_{cd}|} \approx \frac{m_u}{m_c} \sqrt{\frac{A_u - m_c}{m_t} - e^{\mp i\phi_{bu} - \phi_{cb}} \frac{m_u - m_c}{m_t}},
\]

\[
\frac{|V_{td}|}{|V_{cd}|} \approx \frac{m_u}{m_c} \frac{A_u - m_c}{m_t} - e^{\mp i\phi_{tu} - \phi_{cd}} \frac{m_u - m_c}{m_t},
\]

where we assume \( A_u \ll m_t \). The sign “+” for \( V_{us}, V_{cb} \) and “−” for \( V_{cd}, V_{ts} \). Note that if \( A_u \gg m_c \) then \( |V_{ub}| \approx \sqrt{m_u/m_c} \), but this is not our case.

It is obvious that Eqs. (5.6a), (5.6b) and (5.6c) are consistent with the previous results [5, 21]. A good fit of Eqs. (5.6) to the experimental data suggests

\[
A_u = 7348.64 \text{ MeV}, \quad \phi_{bu} = 0.0675036, \quad \phi_{cu} = 1.73735,
\]

Taking into account (4.9) and (4.10), for the down mass matrix we have that

\[
A_d = -m_s + m_d + m_b,
\]

\[
|B_d| = \frac{\sqrt{m_d + m_b - m_s}}{\sqrt{m_s + m_d + m_b}},
\]

\[
|C_d| = \frac{\sqrt{m_b}}{\sqrt{m_d + m_b}}.
\]

The unitary matrix \( U_u \) which diagonalizes \( M_u \) is given by

\[
\begin{pmatrix}
\frac{\sqrt{|A_u - m_c|}}{\sqrt{m_c}} & \frac{\sqrt{|m_s - 2m_u|}}{\sqrt{m_s}} & \frac{\sqrt{|m_s - m_u|}}{\sqrt{m_s}} \\
\frac{\sqrt{|A_u - m_c|}}{\sqrt{m_c}} & 1 & \frac{\sqrt{|m_s - m_u|}}{\sqrt{m_s}} \\
\frac{\sqrt{|A_u - m_c|}}{\sqrt{m_c}} & \frac{\sqrt{|m_s - 2m_u|}}{\sqrt{m_s}} & 1
\end{pmatrix}.
\]

that not differ very much from the values given in [5, 21], \( \phi_1 \approx \frac{\pi}{2} \sim \phi_{cu} - \phi_{bu} \), implying maximal CP-violation in the context of present mass matrices, and \( \phi_2 \approx 0 \sim \phi_{bu} \). The inner angles of the CKM unitarity triangle, \( V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \), are

\[
\beta = \arg \left( -\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right) = 19.8749^\circ,
\]

\[
\alpha = \arg \left( -\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} \right) = 90.595^\circ,
\]

\[
\gamma = \arg \left( -\frac{V_{ub}V_{ub}^*}{V_{cd}V_{cb}^*} \right) = 69.53^\circ,
\]

and the Jarlskog invariant is

\[
J = \text{Im}(V_{us}V_{ub}^*V_{cd}V_{cb}^*) = 2.90804 \times 10^{-5}.
\]

VI. CONCLUSIONS

Within the Standard Model framework, we have investigated texture zeros for quark mass matrices that reproduce the quark masses and experimental mixing parameters. To simplify the problem, without loss of generality, we consider that the quark mass matrices are Hermitian, since the right chirality fields are singlets under the gauge symmetry SU(2). So, for any model where the fields are right chiral singlet under the local gauge symmetry, we may consider that their mass matrices are Hermitian. Specific six-texture zeros in quark mass matrices, including the Fritzsch model [1] and others like [4], have already been discarded because they cannot adjust their results to the experimental data known at present. In Theorem [1] together with the definition of WB transformation, it is shown that the number of non-equivalent representations for the quark mass matrices is finite, which
greatly simplifies the problem. Through WB transformations was relatively easy to find non-parallel four-texture zeros mass matrices. More difficult, but feasible, was the case for parallel four-texture zeros mass matrices. Significant was the consistent five-texture zeros quark mass matrix found by us. Similarly, we show the impossibility, under any circumstances, to find mass matrices with six-texture zeros consistent with experimental data. This is a generalization of six-texture zeros mass matrices discarded by Fritzsch et al.

Throughout this letter, into the SM, we have used the fact that all WB are equivalent. The opposite case is valid too, i.e., two equivalent quark mass matrices representations must be related through a WB transformation. Which is condensed in Theorem I.

By making appropriated WB transformations, numerical parallel and non parallel four-texture zeros were found. An exhaustive deduction process allows us to find a five-texture zeros numerical structure compatible with the experimental data, Eq. (4.22). This representation was found in the two zero family case. Equivalent representations are given in Table II.

We have determined the impossibility to find quark mass matrices having a total of six-texture zeros which are consistent with the measured values of the quark masses and mixing angles. While, a consistent model with five-texture zeros were successful. The five texture zero Ansatz of Eq. (5.1), together with some assumptions which include appropriated values for $A_u, \phi_b$, and $\phi_c$, does lead to successful predictions for $V_{CKM}$ such as those of Eqs. (5.7), (5.8) and (5.9). One nice thing about five-texture zeros quark mass matrices (5.1) is that no hierarchies on quark masses is necessary to be imposed to make correct predictions, although, expressions (5.6) becomes a more complex notation.

Appendix A: Proof of Theorem I

Let us first consider the representation of Hermitian quark mass matrices indicated by $(M_u, M_d)$, and diagonalize them as follows

$$U^\dagger_u M_u U_u = D_u \quad \text{and} \quad U^\dagger_d M_d U_d = D_d. \quad (A1)$$

The CKM mixing matrix is given by

$$V_{ckm} = U^\dagger_u U_d. \quad (A2)$$

We assume the following case:

$$\lambda_{1u} = -m_u, \lambda_{2u} = m_e, \lambda_{3u} = m_t, \quad (B4)$$

$$\lambda_{1d} = -m_d, \lambda_{2d} = m_s, \lambda_{3d} = m_b. \quad (B5)$$

On the other hand, the prime representation $(M_u', M_d')$ gives

$$U^\dagger_u M_u' U_u' = D_u \quad \text{and} \quad U^\dagger_d M_d' U_d' = D_d, \quad (A3)$$

and

$$V_{ckm} = U^\dagger_u U_d'. \quad (A4)$$

Equating the expressions (A2) and (A4) yields

$$U^\dagger_u U_d = U^\dagger_u U_d' \Rightarrow U^\dagger_u U_u' = U^\dagger_d U_d'. \quad (A5)$$

And equating (A1) and (A3), gives respectively

$$U^\dagger_u M_u' U_u' = U^\dagger_u M_u U_u \quad \text{and} \quad U^\dagger_d M_d' U_d' = U^\dagger_d M_d U_d', \quad (A6)$$

where we find that the mass matrices $M_u$ and $M_d$ can be expressed in terms of the mass matrices $M_u'$ and $M_d'$ as follows

$$M_u = U_u U^\dagger_u M_u' U_u', \quad (A7)$$

$$M_d = U_d U^\dagger_d M_d' U_d'. \quad (A8)$$

Using (A5) into (A8), we have

$$M_u = U_u U^\dagger_u M_u' U_u', \quad (A9)$$

Noting the matrices (A7) and (A9), the theorem is proven by defining the unitary matrix $U = U_u U_u^\dagger$.

In this reasoning, we have assumed that both representations generates the same CKM mixing matrix $(V_{ckm})$, something valid due that a WB transformation makes them equal, as was shown in section (III).

Appendix B: Verification of the Method.

The paper [8] uses the following quark mass data:

$$m_u = 2.50 \text{ MeV}, \quad m_c = 600 \text{ MeV}, \quad m_t = 174000 \text{ MeV}, \quad (B1)$$

$$m_d = 4.00 \text{ MeV}, \quad m_s = 80 \text{ MeV}, \quad m_b = 3000 \text{ MeV}. \quad (B2)$$

and the numerical CKM matrix used is

$$V = \begin{pmatrix}
0.036195 + 0.97493i & -0.057798 + 0.21177i & 0.00037188 + 0.999769i \\
-0.21247 + 0.05447i & 0.97351 + 0.050582i & -0.004402 - 0.039760i \\
0.0043605 + 0.0083871i & 0.0086356 - 0.038067i & 0.99836 + 0.004693i
\end{pmatrix}. \quad (B3)$$

Then, the quark mass matrices (2.3) are

$$M_u = \begin{pmatrix}
-2.5 \\
600 \\
174000
\end{pmatrix} \text{ MeV}, \quad (B6)$$

$$M_d = \begin{pmatrix}
-0.086447 \\
-3.4055 - 17.6555i \\
-0.398385 + 10.774i
\end{pmatrix} \text{ MeV}. \quad (B7)$$
where the approximation $A_u \gg m_u$ was assumed because of the restriction (3.25). The matrix $O_u$ now plays the role of a unitary matrix to make the WB transformation.

$$O_u \approx 10^{-3} \begin{pmatrix} -997.92 & 0.15442\sqrt{A_u} \\ 0.15442\sqrt{A_u} & 174000 - A_u \end{pmatrix}$$

Let us use the diagonalization matrix (8.18) with $x = \pi$ and $y = \pi$.

$$M_d(1,3) = Y(A_u) \times (8144.2 - 42221i)\sqrt{174000 - A_u} + (95.463 + 25819i)\sqrt{A_u - 600} -$$

$$- (2174.0 + 17906i)(A_u - 600) \sqrt{A_u - 600} + (0.0013716 - 0.37097i)(174000 - A_u) \sqrt{A_u - 600}$$

whose solution is $A_u \approx 84621$ MeV, which agrees perfectly with the value given in the aforementioned paper. The quark mass matrices (B6) and (B7) take the form

$$M'_d = O_u M_u O_u^T \approx \begin{pmatrix} 0 & 55.537 & 0 \\ 55.537 & 89977 & 86660 \\ 0 & 86660 & 84621 \end{pmatrix} \text{MeV},$$

$$M'_d = O_u M_d O_u^T \approx \begin{pmatrix} 2.5792 - 25.325i & 0 \\ 2.5792 + 25.325i & 1600.5 \\ 0 & 1456.0 - 114.63i \end{pmatrix} \text{MeV}.\quad(8.19)$$

At the present stage we have not yet obtained the matrices given in (25) and (26) of paper [8]. But we can make an additional WB transformation using the following pure phase complex unitary matrix

$$P = \begin{pmatrix} 1 \\ e^{i4.4984} \\ e^{-0.063300} \end{pmatrix}.\quad(8.20)$$

We finally get the desired matrices

$$M''_u = P^T M'_d P = \begin{pmatrix} 0 & -11.794 - 54.270i \\ 0 & -11.794 - 54.270i \end{pmatrix} \text{MeV},$$

$$M''_d = P^T M'_d P = \begin{pmatrix} 24.199 - 7.8983i & 0 \\ 24.199 - 7.8983i & 1600.5 \end{pmatrix} \text{MeV}.\quad(8.21)$$

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