GLOBAL PRYM-TORELLI FOR DOUBLE COVERINGS RAMIFIED IN AT LEAST 6 POINTS

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Abstract. We prove that the ramified Prym map \( \mathcal{P}_{g,r} \) which sends a covering \( \pi : D \rightarrow C \) ramified in \( r \) points to the Prym variety \( P(\pi) := \text{Ker}(\text{Nm}_\pi) \) is an embedding for all \( r \geq 6 \) and for all \( g(C) > 0 \). Moreover, by studying the restriction to the locus of coverings of hyperelliptic curves, we show that \( \mathcal{P}_{g,2} \) and \( \mathcal{P}_{g,4} \) have positive dimensional fibers.

1. Introduction

The ramified Prym map \( \mathcal{P}_{g,r} \) assigns to a degree 2 morphism \( \pi : D \rightarrow C \) of smooth complex irreducible curves ramified in \( r > 0 \) points, a polarized abelian variety \( P(\pi) \) of dimension \( g - 1 + \frac{r}{2} \), where \( g \) is the genus of \( C \). We assume that \( g > 0 \) throughout the paper.

The variety \( P(\pi) \) is called the Prym variety of \( \pi \) and is defined as the kernel of the norm map \( \text{Nm}_\pi : JD \rightarrow JC \). Hence, by denoting by \( \mathcal{R}_{g,r} \) the moduli space of isomorphism classes of the morphisms \( \pi \), we have maps:

\[
\mathcal{P}_{g,r} : \mathcal{R}_{g,r} \rightarrow \mathcal{A}_{g-1+\frac{r}{2}}^\delta,
\]

to the moduli space of abelian varieties of dimension \( g - 1 + \frac{r}{2} \) with polarization type \( \delta := (1, \ldots, 1, 2, \ldots, 2) \), with 2 repeated \( g \) times.

This is the analogous version of the classical (or “unramified”) Prym maps \( (r = 0) \) and has attracted considerable attention in the last years. Combining the results contained in [11], [10], [15] and [16] we know that the ramified Prym map is generically injective as far as the dimension of \( \mathcal{R}_{g,r} \) is less or equal to the dimension of the dimension of \( \mathcal{A}_{g-1+\frac{r}{2}}^\delta \) with only one exception: when \( r = 4, g = 3 \) the map has degree 3 (see [16], [1]).

One could expect that, like in the unramified case, generalized tetragonal constructions (as defined by Donagi in [4]) or other procedures (as, for example, in [7], [9]) would give examples of non injectivity for these maps. In fact, the computation of the degree of \( \mathcal{P}_{3,4} \) in [16] is based on a tetragonal construction for coverings ramified in 4 points, such that the branch divisor belongs to the tetragonal linear series. For this reason, the recent work of Ikeda (see [4]) is somehow unexpected: he proves that \( \mathcal{P}_{1,r} \) is injective for \( r \geq 6 \). Hence it is natural to explore if, contrary to the initial intuition inspired by the classical case, a global Torelli theorem holds for \( \mathcal{P}_{g,r} \) for some
other values of $g$ and $r$. The aim of this article is to answer completely this question by proving the following results:

**Theorem 1.1.** The Prym map $P_{g,r}$ is an embedding for all $r \geq 6$ and all $g > 0$.

**Theorem 1.2.** Let $P_{g,r}^h$ be the restriction of $P_{g,r}$ to the locus of coverings of hyperelliptic curves of genus $g$ ramified in $r$ points. Then

a) The generic fiber of the map $P_{g,2}^h$ is birational to a projective plane.

b) The generic fiber of the map $P_{g,4}^h$ is birational to an elliptic curve.

Thus all the ramified Prym maps with either $r = 2$ or $r = 4$ have positive dimensional fibers.

Observe that the maps $P_{1,2}$ and $P_{1,4}$ have positive dimensional fibers by dimensional reasons, hence we will assume during the proof that $g \geq 2$.

The paper is organized as follows: In section 2 we give a review of some results and we give a detailed analysis of the injectivity of the differential of the Prym map $P_{g,r}$. We obtain that $dP_{g,r}(\pi)$ is injective in all the elements $\pi : D \to C$ for $r \geq 6$. The proof of the first theorem is completed in section 3. The main idea is that the covering $\pi$ can be recovered from the base locus of the linear system $|\Xi|$ of the theta divisor of the Prym variety $P(\pi)$. In section 4 we investigate the restriction of the Prym map to the locus of coverings of hyperelliptic curves. By means of the bigonal construction (see [3] and [17]) we prove Theorem (1.2) showing in particular the existence of positive dimensional fibers for $r = 2$ and $r = 4$ and all genus $g > 0$.

### 2. Preliminaries

#### 2.1. The differential of the Prym map and positive dimensional fibers.**

By the theory of double coverings, the moduli space $R_{g,r}$ can be alternatively described as the following moduli space of triples $(C, \eta, B)$:

$$R_{g,r} = \{(C, \eta, B) \mid |C| \in M_g, \eta \in \text{Pic}^0(C), B \text{ reduced divisor in } |\eta \otimes 2|/ \cong \}.$$  

The codifferential of $P_{g,r}$ at a point $[(C, \eta, B)] \in R_{g,r}$ is given by the multiplication map $(\mathfrak{S})$:

$$dP_{g,r}^*(C, \eta, B) : \text{Sym}^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^2 \otimes \mathcal{O}(B)).$$

Assume that this map is not surjective. Let us recall Theorem 1 in [3]: given $L$ very ample on a curve $C$ of genus at least 2, if

$$\deg(L) \geq 2g + 1 - 2h^1(C, L) - \text{Cliff}(C),$$

then $\text{Sym}^m H^0(C, L) \to H^0(C, L^{\otimes m})$ is surjective for all $m$. In our case $L = \omega_C \otimes \eta$ has degree $2g - 2 + \frac{r}{2}$. We get that, either $L$ is not very ample, or

$$\deg(L) < 2g + 1 - 2h^1(L) - \text{Cliff}(C).$$
Since $h^1(L) = h^0(\eta^{-1}) = 0$ this simply says that
\[ \text{Cliff}(C) < 3 - r/2. \]
Hence, since $r > 0$, the only possibilities are:

a) $L = \omega_C \otimes \eta$ is not very ample. In particular, $\deg L = 2g - 2 + \frac{r}{2} \leq 2g$, hence $r \leq 4$.

b) $r = 2$ and $\text{Cliff}(C) = 1$.

c) $r = 2, 4$ and $\text{Cliff}(C) = 0$.

Therefore we have

**Proposition 2.1.** If $r \geq 6$ then the differential of $P_{g,r}$ is injective in all the elements of $R_{g,r}$.

**Remark 2.2.** It is natural to ask whether (as in [14]) it is possible to characterize all positive dimensional fibers of $P_{g,r}$ by using this differential. More precisely, assume that there is a covering $[\pi : D \to C] \in R_{g,r}$ belonging to a positive dimensional fiber. Then we have seen that one of the cases a), b) or c) above occurs. Case a) is equivalent to, either $r = 2$ and $\eta = OC(x + y - z)$ for some points $x, y, z \in C$, or $r = 4$ and $h^0(C, \eta) > 0$. The cases b) and c) correspond to the coverings of hyperelliptic curves (with $r = 2, 4$), or trigonal curves (with $r = 2$) or quintic plane curves (with $r = 2$).

We prove that there are positive dimensional fibers in the hyperelliptic case, with $r = 2, 4$ (Theorem 1.2) and in the case of coverings over trigonal curves of genus 5 (see Proposition 2.4 below).

### 2.2. Other examples of non-injectivity.

As mentioned in the Introduction, we find instances of positive dimensional fibers of the ramified Prym map by studying ramified coverings of hyperelliptic curves and they appear only for the values $r = 2$ and $r = 4$. The non-injectivity of $P_{g,4}$ for $g \geq 3$ has been proved in [16], where Nagaraj and Ramanan extend the tetragonal construction (see [16, Proposition 9.9]) to double coverings over tetragonal curves, whose branch divisor is in the tetragonal linear series. They show that this construction, like in the unramified case, provides two other ramified coverings (over tetragonal curves) having the same Prym variety. They prove more:

**Theorem 2.3.** (Nagaraj-Ramanan, [16, Corollary 9.12]) For $g \geq 3$ the Prym map $P_{g,4}$, restricted to the locus of the coverings of tetragonal curves has degree 3.

We want to point out the existence of other positive dimensional fibers for $g = 5, r = 2$ which do not arise from hyperelliptic curves. In this case there is a nice relation with the geometry of cubic threefolds. Indeed, in [15] we show how $P_{5,2}$ can be identified with the restriction of the (compactified) unramified Prym map $\mathcal{P}_5 : \mathcal{R}_5 \to \mathcal{A}_5$ to the divisor $\Delta^n$ of the closure of the set of admissible coverings of nodal curves. Then:

$$ P_{5,2} : \mathcal{R}_{5,2} \hookrightarrow \Delta^n \subset \mathcal{R}_5 \xrightarrow{\mathcal{P}_5} \mathcal{A}_5. $$
The image of \( \mathcal{R}_{5,2} \) in \( \Delta^n \) is the open set consisting of the double coverings \( D_0 \rightarrow C_0 \) such that both curves are irreducible with only one node; the node in \( D_0 \) maps to the node in \( C_0 \), away of the nodes the covering is unramified, and \( p_a(C_0) = 6 \). The locus \( \mathcal{C} \) of the intermediate Jacobians of smooth cubic threefolds is contained in \( \mathcal{P}_{5,2}(\mathcal{R}_{5,2}) \). Let \( V \) be a generic smooth cubic threefold with Fano variety of lines \( F(V) \). Then the fiber of the unramified map \( \mathcal{P}_{5,2}^{-1}(V) \) is \( F(V) \) and \( \mathcal{P}_{5,2}^{-1}(V) = F(V) \cap \mathcal{R}_{5,2} \) is an irreducible curve for a general \( V \) (see [15], section 2 for the details). In particular, the fiber is positive dimensional. Since the trigonal curves of genus 5 are normalizations of nodal quintic plane curves, we have proved:

**Proposition 2.4.** The fibers of the restriction of \( \mathcal{P}_{5,2} \) to \( \mathcal{R}_{5,2}^{\text{tri}} \) are all positive dimensional.

### 2.3. Dual polarizations.

In this subsection we borrow the main result on dual polarizations from [2] adapted to our situation. Let \((A, L)\) be a polarized abelian variety of dimension \( g - 1 + \frac{r}{2} \) and polarization type \( \delta = (1, \ldots, 1, 2, \ldots, 2) \), where 2 is repeated \( g \) times. Then there is a natural polarization \( \widehat{L} \) in the dual abelian variety \( \widehat{A} \) characterized by the property that the polarization maps:

\[
\lambda_L : A \rightarrow \widehat{A}, \quad \lambda_{\widehat{L}} : \widehat{A} \rightarrow \widehat{\widehat{A}} \cong A
\]

satisfy \( \lambda_L \circ \lambda_L = 2_A \) and \( \lambda_L \circ \lambda_{\widehat{L}} = 2_{\widehat{A}} \). The type of \( \widehat{L} \) is \( \delta' = (1, \ldots, 1, 2, \ldots, 2) \) where now 1 is repeated \( g \) times (and therefore 2 is repeated \( \frac{2}{5} - 1 \) times). This construction can be done in families giving the following result:

**Theorem 2.5.** ([2, Theorem 1.1]) There is a canonical isomorphism of coarse moduli spaces

\[
A_{g-1+\frac{r}{2}}^\delta \rightarrow A_{g-1+\frac{r}{2}}^{\delta'}
\]

sending a polarized abelian variety of type \( \delta \) to its polarized dual abelian variety.

### 3. Global Torelli: Injectivity of \( \mathcal{P}_{g,r} \) for \( r \geq 6 \) and \( g \geq 1 \)

The aim of this section is to prove that the ramified Prym map \( \mathcal{P}_{g,r} \) for \( r \geq 6 \) and \( g \geq 1 \) is injective. This together with Proposition 2.1 prove our main Theorem 1.1. Given any Prym variety \((P, \Xi)\) in the image of \( \mathcal{P}_{g,r} \) we will recover the corresponding map \( \pi : D \rightarrow C \) from the base locus of \( |\Xi| \). The proof uses a particular description of the base locus of \( \Xi \) given in [15]. For sake of completeness, we recall briefly some results from [15] and we refer to this article for the details.

Let \( \pi : D \rightarrow C \) be the double covering ramified in \( r \geq 6 \) distinct points attached to the data \((C, \eta, B)\). According to [13] (Proposition p. 334) the base locus of \( |\Xi| \) can be identified canonically in

\[
\mathcal{P}^{\text{can}} = \text{Nm}_\pi^{-1}(\omega_C \otimes \eta) \subset \text{Pic}^{2g-2+\frac{r}{5}}(D) = \text{Pic}^{g(D)-1}(D)
\]
with the set
\[ \tilde{B}_s = \{ L \in P^{\text{can}} \mid \pi^*(JC) \subset \Theta^\text{can}_{D-L} \}, \]
where \( \Theta^\text{can}_D \) is the canonical presentation \( W^0_{g(D)-1}(D) \) of the theta divisor of \( JD \) and \( \Theta^\text{can}_{D-L} \) is the translation by the element \( L \in \text{Pic}^g(D)^{-1}(D) \) (see [15, Proposition 1.3] for the details). Next we define
\[ B_0 := \{ L = \pi^*(A)(p_1 + \cdots + p_{2g-2}) \mid A \in \text{Pic}^g(C), \ p_i \in D, \ Nm \pi L \cong \omega_C \otimes \eta \}. \]
In [15, Proposition 1.6] the following is proved:

**Proposition 3.1.** The equality \( B_0 = \tilde{B}_s \) holds.

Note that this result works for any triple in \( \mathcal{R}_{g,r} \) with \( r \geq 6 \). For \( g \geq 2 \), the idea of the proof is that the elements \( L \in P^{\text{can}} \) in the base locus, coincide with the line bundles on \( D \) whose push-forward \( \pi_* L \) are rank two vector bundles on \( C \) which are not semistable. The equality is then deduced from the condition of \( \pi_* L \) being unstable and a short exact sequence in [13] (proof of the Proposition in p. 338) involving \( \pi_* L \). In the case \( g = 1 \) the proposition is proven directly from this short exact sequence.

**Theorem 3.2.** The map \( \mathcal{P}_{g,r} \) is injective for all \( r \geq 6 \) and \( g \geq 1 \).

**Proof.** In order to reconstruct the covering \( [\pi : D \rightarrow C] \in \mathcal{P}_{g,r} \) from the base locus we study the birational class of \( B_0 \). We define
\[ T := \{ (A, M) \in \text{Pic}^g(C) \times \text{Pic}^{(g^2/2-2)}(D) \mid A^{g^2} \otimes \text{Nm}_M M \cong \omega_C \otimes \eta, \ \text{and} \ \text{h}^0(D, M) > 0 \} \]
and the projection \( T \rightarrow W^0_{g^2/2-2}(D) \) is surjective. The fiber at \( M \in W^0_{g^2/2-2}(D) \) is the finite set (of cardinality \( 2^{2g} \)) consisting of \( A \in \text{Pic}^g(C) \) such that
\[ A^{g^2} \cong \omega_C \otimes \eta \otimes \text{Nm}_M (M^{-1}) \in \text{Pic}^{2g^2-2+2g^2-2+2g^2-2/2}(C) = \text{Pic}^{2g^2}(C). \]
We claim that the generic element \( M \in W^0_{g^2/2-2}(D) \) satisfies \( \text{h}^0(D, M) = 1 \) and is “\( \pi \)-simple”, that is the effective divisor representing \( M \) does not contain fibers of \( \pi \). Indeed, since \( g(D) = 2g - 1 + \frac{r}{2} \) we have that \( \frac{r}{2} - 2 = g(D) - 2g - 1 < g(D) - 2 \); therefore the generic element \( M \) does not belong to \( W^1_{g^2/2-2}(D) \). On the other hand by dimensional reasons the map
\[ C \times W^0_{g^2/2-4}(D) \rightarrow W^0_{g^2/2-2}(D); \quad (x, M_0) \mapsto \pi^*(\mathcal{O}_C(x)) \otimes M_0 \]
cannot be surjective which implies that the generic element is \( \pi \)-simple. Now we look to the natural morphism \( \tau : T \rightarrow B_0 \) defined by
\[ \tau(A, M) = \pi^*(A) \otimes M. \]
We claim that this map is birational. Indeed, it is obviously surjective. Assume that \( (A, M), (A', M') \in T \) have the same image under \( \tau \) and that \( M \) is \( \pi \)-simple. Then
\[ \pi^*(A \otimes A'^{-1}) \otimes M \cong M'. \]
For any $\alpha \in \text{Pic}(C)$ and for any $\pi$-simple $M$ we have (see the proof of the Proposition in p. 338 \[13\]):

\[
0 \longrightarrow \alpha \longrightarrow \pi^*(\alpha \otimes M) \longrightarrow \alpha \otimes \text{Nm}_\pi(M) \otimes \eta^{-1} \longrightarrow 0.
\]

In our case this translates to

\[
0 \longrightarrow A \otimes A'^{-1} \longrightarrow \pi^*M' \longrightarrow A \otimes A'^{-1} \otimes \text{Nm}_\pi(M) \otimes \eta^{-1} \longrightarrow 0.
\]

Notice that the degree of the last term is $\frac{r}{2} - 2 - \deg(\eta) = -2$. Thus

\[
h^0(C, A \otimes A'^{-1}) = h^0(C, \pi_*M) = h^0(D, M) > 0.
\]

Since $\deg(A \otimes A'^{-1}) = 0$ this is only possible if $A \cong A'$ and then $M \cong M'$. This proves the claim.

Recall that $\pi^*(JC[2]) \cong JC[2]$ equals the kernel of the polarization map $\lambda_\Xi : P \longrightarrow P^\vee$ and it acts on $B_0$, respectively on $T$, by

\[
\alpha \cdot (\pi^*A(p_1 + \cdots + p_{\frac{r}{2} - 2})) = \pi^*(A \otimes \alpha)(p_1 + \cdots + p_{\frac{r}{2} - 2}), \quad \alpha \cdot (A, M) = (A \otimes \alpha, M),
\]

for $\alpha \in JC[2]$. The map $\tau$ is equivariant with respect to this action and $T/JC[2] = W^0_{\frac{r}{2} - 2}(D)$. In other words, we have recovered the locus $W^0_{\frac{r}{2} - 2}(D)$ up to birational equivalence. Moreover, the natural involution $L \mapsto \omega_D \otimes L^{-1}$ on $\tilde{B}s$ and $B_0$ corresponds to an involution $\iota$ on $T$. More precisely, if $\sigma$ denotes the involution on $D$ associated to the double covering $\pi$, $\iota$ is defined by

\[
\iota(A, M) = (A, \sigma^*M).
\]

**Lemma 3.3.** The equality $\iota \circ \tau = \tau \circ \iota$ holds.

**Proof.** Let $(A, M) \in T$. Since $\omega_D = \pi^*\omega_C \otimes \mathcal{O}_D(R)$, with $R$ the ramification divisor of $\pi$, we have

\[
\iota(\pi^*A \otimes M) = \pi^*(\omega_C \otimes A^{-1}) \otimes \mathcal{O}_D(R) \otimes M^{-1}
\]

On the other hand, $\tau(A, \sigma^*M) = \pi^*A \otimes \sigma^*M$, which equals \[2\] since

\[
\pi^*(\omega_C \otimes A^{-2}) = \pi^*(\text{Nm} M \otimes \eta^{-1}) = M \otimes \sigma^*M \otimes \mathcal{O}_D(-R).
\]

Observe that $\iota$ commutes with the action of $JC[2]$, therefore $\iota$ descends to an involution on the quotient $W^0_{\frac{r}{2} - 2}(D)$.

By a generalized Torelli Theorem due to Martens (see \[13\], \[12\]), the Brill-Noether locus $W^0_d(D)$ determines the curve $D$ for $d \leq g(D) - 1$. Furthermore, if $\Phi : W^0_d(D_1) \longrightarrow W^0_d(D_2)$ is a birational correspondence between two curves $D_1$ and $D_2$ of genus $g(D)$, with $d \leq g(D) - 2$, then $\Phi$ is induced by a birational map $\varphi : D_1 \longrightarrow D_2$. In our situation, this result implies that the involution on $W^0_{\frac{r}{2} - 2}(D)$ is completely determined by the involution on the double covering $\pi : D \longrightarrow C$. This finishes the proof. \(\square\)
4. THE Prym map restricted to the coverings of hyperelliptic curves

We study in this section the Prym map restricted to the locus of coverings of hyperelliptic curves. We denote by $RH_{g,r} \subset R_{g,r}$ the sublocus of the classes of coverings $D \rightarrow C$, with $C$ a hyperelliptic curve of genus $g \geq 2$, and by $P^h_{g,r}$ the restriction of $P^h_{g,r}$ to $RH_{g,r}$.

The bigonal construction (see [3, Section 2]) transforms general elements of $RH_{g,r}$ into elements of $RH_{r^2 - 1, 2g + 2}$. For convenience of the reader we remind some details of this construction. Given a "tower" of curves $D \rightarrow C \rightarrow P^1$, where $\pi$ and $f$ have degree 2 we define a new "tower"

$$D' \rightarrow C' \rightarrow P^1,$$

in the following way: consider $P^1$ embedded in $C^{(2)}$ by sending a point $p$ to its fiber $f^{-1}(p)$. Then we have a fiber product diagram

$$
\begin{array}{ccc}
D' & \rightarrow & D^{(2)} \\
4:1 & \downarrow & \downarrow \pi^{(2)} \\
\mathbb{P}^1 & \rightarrow & C^{(2)}.
\end{array}
$$

The natural involution on $D^{(2)}$ restricts to an involution on $D'$ and $C'$ is the quotient curve by this involution. Let $\pi' : D' \rightarrow C'$ denote the quotient map. If the branch locus of $\pi$ is disjoint from the ramification locus of $f$ then $D'$ and $C'$ are smooth and irreducible. The following result is a consequence of [4, Lemma 2.7]:

**Lemma 4.1.** Under the hypothesis above, we have

a) The bigonal construction applied to the tower $D' \rightarrow C' \rightarrow P^1$ gives the initial tower $D \rightarrow C \rightarrow P^1$.

b) The equalities $g(C) + g(C') = g(D) = g(D')$ hold.

Pantazis proves in [17] that the Prym variety of $\pi'$ is isomorphic, as polarized abelian varieties, to the dual of the Prym variety of $\pi$.

We apply the bigonal construction to $\pi : D \rightarrow C$, being $C$ hyperelliptic of genus $g \geq 2$ and $f : C \rightarrow \mathbb{P}^1$ the unique $g^1_2$ on $C$ (up to automorphisms on $\mathbb{P}^1$). We obtain a covering $D' \rightarrow C'$ where $g(C') = \frac{r}{2} - 1$. Notice that the construction provides a $g^1_2$ linear series on $C'$. This is irrelevant if $\frac{r}{2} - 1 \geq 2$ (that is, if $r \geq 6$), but has interesting consequences if $r = 2$ or $r = 4$. Let $\widetilde{\mathcal{R}}_{1,2g+2}$ be the isomorphism classes of $(\pi : D \rightarrow E, A)$, where $E$ is an elliptic curve, $\pi$ is a double covering ramified in $2g + 2$ points and $A \in \text{Pic}^2(E)$. Then, for $g \geq 2$, we have the following diagram:
where $\beta$ is the birational map provided by the bigonal construction, $\varphi$ is the forgetful map and the bottom horizontal isomorphism maps a polarized abelian variety to its polarized dual (see 2.3). The commutativity of the diagram (3) is the content of the main theorem of Pantazis in [17]. Since $P_{1,2g+2}$ is injective (use [6] or Theorem (1.1) above), the fiber of $P_{g,4}$ can be identified (birationally) with the fiber of $\varphi$, which is an elliptic curve.

In the case $r = 2$ the bigonal construction gives a birational map from $\mathcal{R}H_{g,2}$ to an open set of the moduli space $\tilde{\mathcal{R}}_{0,2g+2}$ parametrizing pairs $(\pi : D \to \mathbb{P}^1, f : \mathbb{P}^1 \overset{2:1}{\to} \mathbb{P}^1)$ of double coverings with $\pi$ ramified in $2g + 2$ points and $f$ ramified in two points. Hence, the forgetful map $\varphi : \tilde{\mathcal{R}}_{0,2g+2} \to \mathcal{R}_{0,2g+2}$ has fibers isomorphic to $\mathbb{P}^2$. The corresponding commutative diagram is the following:

where the bottom map is the isomorphism sending a principally polarized abelian variety to its dual polarized variety. This finishes the proof of Theorem (1.2).

Notice that for $r \geq 6$ (and still $g \geq 2$) the diagram above becomes simply (there is no longer forgetful map):

for some convenient polarization types $\delta$ and $\delta'$ (see subsection 2.3).
References

[1] F. Bardelli, C. Ciliberto and A. Verra, Curves of minimal genus on a general abelian variety, Compositio Math. 96 (1995), no. 2, 115–147.
[2] C. Birkenhake; H. Lange, An isomorphism between moduli spaces of abelian varieties, Math. Nachr. 253 (2003), 3–7.
[3] R. Donagi, The fibers of the Prym map, Curves, Jacobians, and abelian varieties (Amherst, MA, 1990), 55–125, Contemp. Math., 136, Amer. Math. Soc., 1992.
[4] R. Donagi, The tetrogonal construction, Bull. Amer. Math. Soc. 4 (1981), 181–185.
[5] M. Green, R. Lazarsfeld, On the projective normality of complete linear series on an algebraic curve, Invent. Math. 83 (1986), 73–90.
[6] A. Ikeda, Global Prym-Torelli for double coverings of elliptic curves, Algebr. Geom. 7 (2020), 544–560.
[7] E. Izadi; H. Lange, Counter-examples of high Clifford index to Prym-Torelli, J. Algebraic Geom. 21 (2012), no. 4, 769-787.
[8] H. Lange; A. Ortega, Prym varieties of cyclic coverings, Geom. Dedicata 150 (2011), 39-403.
[9] H. Lange; A. Ortega, The Trigonal construction in the ramified case, arXiv:1902.00251.
[10] V. Marcucci, J.C. Naranjo, Prym varieties of double coverings of elliptic curves, Int. Math. Res. Notices 6 (2014), 1689-1698.
[11] V. Marcucci, G.P. Pirola, Generic Torelli for Prym varieties of ramified coverings, Compositio Mathematica 148 (2012), 1147-1170.
[12] H. Martens, An Extended Torelli Theorem, Amer. J. Math. 87 (1965), 257-261.
[13] D. Mumford, Prym varieties, Contributions to analysis (a collection of papers dedicated to Lipman Bers), 325–350. Academic Press, New York, 1974.
[14] J.C. Naranjo, The positive dimensional fibers of the Prym map, Pacific J. Math. 172 (1996), 223-226.
[15] J.C. Naranjo, A. Ortega, Generic injectivity of the Prym map for double ramified coverings, Trans. Amer. Math. Soc. 371 (2019), 3627-3646.
[16] D.S. Nagaraj, S. Ramanan, Polarisations of type $(1, 2, \ldots, 2)$ on abelian varieties, Duke Math. J. 80 (1995), 157-194.
[17] S. Pantazis, Prym varieties and the geodesic flow on $SO(n)$, Math. Ann. 273 (1986), 297-315.
[18] Z. Ran, On a theorem of Martens, Rend. Sem. Mat. Univ. Politec. Torino 44 (1986), 287-291.

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