Locally unitary principal series representations of $GL_{d+1}(F)$
by Elmar Grosse-Klönne

To Peter Schneider on the occasion of his 60th birthday

Abstract

For a local field $F$ we consider tamely ramified principal series representations $V$ of $G = GL_{d+1}(F)$ with coefficients in a finite extension $K$ of $\mathbb{Q}_p$. Let $I_0$ be a pro-$p$-Iwahori subgroup in $G$, let $\mathcal{H}(G, I_0)$ denote the corresponding pro-$p$-Iwahori Hecke algebra. If $V$ is locally unitary, i.e. if the $\mathcal{H}(G, I_0)$-module $V^{I_0}$ admits an integral structure, then such an integral structure can be chosen in a particularly well organized manner, in particular its modular reduction can be made completely explicit.

Contents

1 Introduction 1
2 Functions on symmetric groups 3
3 Hecke lattices in principal series representations I 9
4 Hecke lattices in principal series representations II 15
5 $\mathcal{H}(G, I_0)_k$-modules of $W$-type 18

1 Introduction

Let $F$ be a local non-Archimedean field with finite residue field $k_F$ of characteristic $p > 0$, let $G = GL_{d+1}(F)$ for some $d \in \mathbb{N}$. Let $K$ be another local field which is a finite extension of $\mathbb{Q}_p$, let $\mathfrak{o}$ denote its ring of integers, $\pi \in \mathfrak{o}$ a non-zero element in its maximal ideal and $k$ its residue field.

The general problem of deciding whether a given smooth (or, more generally, locally algebraic) $G$-representation $V$ over $K$ admits a $G$-invariant norm — or equivalently: a $G$-stable free $\mathfrak{o}$-sub module containing a $K$-basis of $V$ — is of great importance for the $p$-adic local Langlands program. It is not difficult to formulate a certain necessary condition for the existence of a $G$-invariant norm on $V$. This has been emphasized first by Vignéras, see also [2], [3], [6], [7]. If $V$ is a tamely ramified smooth principal series representation
and if \( d = 1 \) then this condition turns out to also be sufficient, see [8]. Unfortunately, if \( d > 1 \) it is unknown if this condition is sufficient. See however [4] for some recent progress.

In this note we consider tamely ramified smooth principal series representations \( V \) of \( G \) over \( K \) for general \( d \in \mathbb{N} \). More precisely, we fix a maximal split torus \( T \), a Borel subgroup \( P \) and a pro-\( p \)-Iwahori subgroup \( I_0 \) in \( G \) fixing a chamber in the apartment corresponding to \( T \). We then consider a smooth \( K \)-valued character \( \Theta \) of \( T \) which is trivial on \( T \cap I_0 \), view it as a character of \( P \) and form the smooth induction \( V = \text{Ind}_P^G \Theta \).

Let \( \mathcal{H}(G, I_0) \) denote the pro-\( p \)-Iwahori Hecke algebra with coefficients in \( \mathfrak{o} \) corresponding to \( I_0 \). The \( K \)-subspace \( V^{I_0} \) of \( I_0 \)-invariants in \( V \) is naturally a module over \( \mathcal{H}(G, I_0) \otimes_k K \). The said necessary condition for the existence of a \( G \)-invariant norm on \( V \) is now equivalent with the condition that the \( \mathcal{H}(G, I_0) \otimes_k K \)-module \( V^{I_0} \) admits an integral structure, i.e. an \( \mathfrak{o} \)-free \( \mathcal{H}(G, I_0) \)-sub module \( L \) containing a \( K \)-basis of \( V^{I_0} \). One might phrase this as the condition that \( V \) be locally integral, or locally unitary.

It is not difficult to directly read off from \( \Theta \) whether \( V \) is locally unitary. (Besides [2] Proposition 3.2 we mention the formulation in terms of Jacquet modules as propagated by Emerton ([3]), see also section 4 below.) We rederive this relationship here. However, the proper purpose of this paper is to provide explicit and particularly well structured \( \mathfrak{o} \)-lattices \( L_\nabla \) in \( V^{I_0} \) as above whenever \( V \) is locally unitary.

Our approach is completely elementary; for example, it does not make use of the integral Bernstein basis for \( \mathcal{H}(G, I_0) \) (e.g. [7]). It is merely based on the investigation of certain \( \mathbb{Z} \)-valued functions \( \nabla \) on the finite Weyl group \( W = N(T)/T \), and thus on combinatorics of \( W \). We consider the canonical \( K \)-basis \( \{f_w\}_{w \in W} \) of \( V^{I_0} \) where \( f_w \in V^{I_0} \) has support \( PwI_0 \) and satisfies \( f_w(w) = 1 \) (we realize \( W \) as a subgroup in \( G \)). We then ask for functions \( \nabla : \mathbb{Z} \to \mathbb{Z} \) such that \( L_\nabla = \bigoplus_{w \in W} (\pi)_{\nabla(w)} f_w \) is an \( \mathfrak{o} \)-lattice as desired. We show (Theorem 4.2) that whenever \( V \) is locally unitary, then \( V^{I_0} \) admits an \( \mathcal{H}(G, I_0) \)-stable \( \mathfrak{o} \)-lattice of this particular shape.

The structure of the \( \mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_k k \)-modules \( L_\nabla \otimes_k k \) so obtained is then encoded in combinatorics of the (finite) Coxeter group \( W \). Approaching them abstractly we suggest the notion of an \( \mathcal{H}(G, I_0)_k \)-module of \( W \)-type (or: a reduced standard \( \mathcal{H}(G, I_0)_k \)-module): This is an \( \mathcal{H}(G, I_0)_k \)-module \( M[\theta, \sigma, \epsilon_\bullet] \) with \( k \)-basis parametrized by \( W \) and whose \( \mathcal{H}(G, I_0)_k \)-structure is characterized, by means of some explicit formulae, through a set of data \( (\theta, \sigma, \epsilon_\bullet) \) as follows: \( \theta \) is a character of \( I/I_0 = (T \cap I)/(T \cap I_0) \) where \( I \supset I_0 \) is the corresponding Iwahori subgroup; \( \sigma \) is a function \( \{w \in W \mid \ell(ws_d) > \ell(w)\} \to \{-1, 0, 1\} \) where \( s_d \) is the simple reflection corresponding to an end in the Dynkin diagram, and \( \ell \) is the length function on \( W \); finally, \( \epsilon_\bullet = \{\epsilon_w \mid w \in W\} \) is a set of units in \( k \). (But not any such set of data \( (\theta, \sigma, \epsilon_\bullet) \) defines an \( \mathcal{H}(G, I_0)_k \)-module \( M[\theta, \sigma, \epsilon_\bullet] \).)

The explicit nature of \( L_\nabla \otimes_k k \), and more generally of an \( \mathcal{H}(G, I_0)_k \)-module of \( W \)-
type, is particularly well suited for computing its value under a certain functor from finite dimensional \( \mathcal{H}(G, I_0)_k \)-modules to \((\varphi, \Gamma)\)-modules (if \( F = \mathbb{Q}_p \)), see [5].

We intend to generalize the results of the present paper to other reductive groups in the future. Moreover, the relationship between \( \mathcal{H}(G, I_0)_k \)-modules of \( W \)-type (reduced standard \( \mathcal{H}(G, I_0)_k \)-modules) and standard \( \mathcal{H}(G, I_0)_k \)-modules should be clarified.

The outline is as follows. In section 2 we first introduce the notion of a balanced weight of length \( d + 1 \): a \((d + 1)\)-tuple of integers satisfying certain boundedness conditions which later on will turn out to precisely encode the condition (on \( \Theta \)) for \( V \) to be locally unitary. Given such a balanced weight, we show the existence of certain functions \( \nabla : W \to \mathbb{Z} \) ‘integrating’ it. In section 3 we introduce \( V = \text{Ind}_P^G \Theta \) and show that if a function \( \nabla \) ‘integrates’ the ‘weight’ associated with \( \Theta \), then \( L_{\nabla} \) is an \( \mathcal{H}(G, I_0) \)-stable \( \sigma \)-lattice as desired. In section 4 we put the results of sections 2 and 3 together. In section 5 we introduce \( \mathcal{H}(G, I_0)_k \)-modules of \( W \)-type.

Acknowledgements: I am very grateful to the referee for critical comments — they helped to significantly improve the exposition.

## 2 Functions on symmetric groups

For a finite subset \( I \) of \( \mathbb{Z}_{\geq 0} \) we put

\[
\Delta(I) = \sum_{i \in I} i - \frac{|I| \cdot (|I| - 1)}{2}.
\]

**Definition:** Let \( d, r \in \mathbb{N} \). We say that a sequence of integers \((n_i)_{0 \leq i \leq d} = (n_0, \ldots, n_d)\) is a balanced weight of length \( d + 1 \) and amplitude \( r \) if \( \sum_{i=0}^d n_i = 0 \) and if for each subset \( I \subset \{0, \ldots, d\} \) we have

\[
|r\Delta(I)| \geq \sum_{i \in I} n_i \geq -r\Delta(\{0, \ldots, d\} - I).
\]

**Lemma 2.1.** If \((n_i)_{0 \leq i \leq d}\) is a balanced weight of length \( d + 1 \) and amplitude \( r \), then so is \((-n_{d-i})_{0 \leq i \leq d}\).
Proof: For any $I \subset \{0, \ldots, d\}$ we compute
\[
\Delta(I) = \sum_{i \in I} i - \frac{|I| \cdot (|I| - 1)}{2}
\]
\[
= \sum_{i=0}^d i - \sum_{i \notin I} i - d|I| - \frac{|I|^2}{2} + \frac{(d+1)|I| + d|I|}{2}
\]
\[
= \frac{d(d+1)}{2} - \sum_{i \notin I} i - d|I| - \frac{|I|^2}{2} + \frac{(d+1)|I| + d|I|}{2}
\]
\[
= d(d+1 - |I|) - \sum_{i \notin I} i - \frac{(d+1 - |I|)(d - |I|)}{2}
\]
\[
= \sum_{i \notin I} (d - i) - \frac{(d+1 - |I|)(d - |I|)}{2}
\]
\[
= \Delta(\{d - i \mid i \in \{0, \ldots, d\} - I\}).
\]
Together with the assumption $\sum_{i=0}^d n_i = 0$ this shows that the set of inequalities (I) for $(n_i)_{0 \leq i \leq d}$ is equivalent with the same set of inequalities for $(-n_{d-i})_{0 \leq i \leq d}$. Namely, given $I \subset \{0, \ldots, d\}$, the inequalities (I) for $(n_i)_{0 \leq i \leq d}$ and $I$ are equivalent with the inequalities (I) for $(-n_{d-i})_{0 \leq i \leq d}$ and $\{d - i \mid i \in \{0, \ldots, d\} - I\}$.

Lemma 2.2. Let $(n_i)_{0 \leq i \leq d}$ be a balanced weight of length $d + 1$ and amplitude $r$.

(a) There is a balanced weight $(\tilde{n}_i)_{0 \leq i \leq d}$ of length $d + 1$ and amplitude $r$ such that $\tilde{n}_0 = 0$ and $0 \leq n_i - \tilde{n}_i \leq r$ for all $1 \leq i \leq d$.

(b) There is a balanced weight $(m_i)_{0 \leq i \leq d-1}$ of length $d$ and amplitude $r$ such that $0 \leq n_i - m_{i-1} \leq r$ for each $i = 1, \ldots, d$.

Proof: We first show that (b) follows from (a). Indeed, suppose we are given $(\tilde{n}_i)_{0 \leq i \leq d}$ as in (a). Then put $m_{i-1} = \tilde{n}_i$ for $i = 1, \ldots, d$. We clearly have $\sum_{i=0}^{d-1} m_i = 0$. Next, let $I \subset \{0, \ldots, d - 1\}$. Putting $I^+ = \{i + 1 \mid i \in I\}$ and $I_0^+ = I^+ \cup \{0\}$ we then find
\[
r\Delta(I) = r\left(\sum_{i \in I} i - \frac{|I|(|I| - 1)}{2}\right)
\]
\[
= r\left(\sum_{i \in I^+} i - |I| - \frac{|I|(|I| - 1)}{2}\right)
\]
\[
= r\left(\sum_{i \in I_0^+} i - \frac{|I_0^+|(|I_0^+| - 1)}{2}\right)
\]
\[
= r\Delta(I_0^+)
\]
\[
\sum_{i \in I_0^+} \tilde{n}_i = \sum_{i \in I} m_i
\]
where (i) holds true by assumption. Similarly, we find

\[-r\Delta(\{0, \ldots, d-1\} - I) = -r\left( \sum_{i \in \{0, \ldots, d-1\} - I} i - \frac{(d - |I|)(d - |I| - 1)}{2} \right) \]

\[= -r\left( \sum_{i \in \{0, \ldots, d\} - I^+} i - (d - |I|) - \frac{(d - |I|)(d - |I| - 1)}{2} \right) \]

\[= -r\left( \sum_{i \in \{0, \ldots, d\} - I^+} i - \frac{(d + 1 - |I^+|)(d - |I^+|)}{2} \right) \]

\[= -r\Delta(\{0, \ldots, d\} - I^+) \]

\[\leq \sum_{i \in I^+} \Delta_i = \sum_{i \in I} m_i \]

where (ii) holds true by assumption.

Now we prove statement (a) in three steps.

Step 1: For any sequence of integers \(t_1, \ldots, t_d\) satisfying

\[r|I|(d - \frac{1}{2}(|I| - 1)) \geq \sum_{i \in I} t_i \geq \frac{1}{2}r|I|(|I| - 1) \]

for each subset \(I \subset \{1, \ldots, d\}\), there exists another sequence of integers \(\tilde{t}_1, \ldots, \tilde{t}_d\), again satisfying formula (2) for each \(I' \subset \{1, \ldots, d\}\) and such that \(\sum_{i=1}^{d} \tilde{t}_i = \frac{1}{2}rd(d - 1)\) and \(0 \leq \tilde{t}_i - t_i \leq r\) for all \(1 \leq i \leq d\).

For a subset \(I \subset \{1, \ldots, d\}\) we write \(I^c = \{1, \ldots, d\} - I\). Put

\[\delta = \sum_{i=1}^{d} t_i - \frac{1}{2}rd(d - 1).\]

To construct \(\tilde{t}_1, \ldots, \tilde{t}_d\) as desired, we put \(s_i^{(0)} = t_i\) and define inductively sequences \(s_i^{(m)}, s_i^{(m)}\) for \(1 \leq m \leq \delta\) such that \(0 \leq t_i - s_i^{(m)} \leq r\), such that \(0 \leq s_i^{(m-1)} - s_i^{(m)} \leq 1\), such that \(\delta - m = \sum_{i=1}^{d} s_i^{(m)} - \frac{1}{2}d(d - 1)\) and such that for any fixed \(m\) the sequence \((s_i^{(m)})_i\) satisfies (2) for each subset \(I \subset \{1, \ldots, d\}\). Once all the \((s_i^{(m)})_i\) are constructed we may put \(\tilde{t}_i = s_i^{(\delta)}\).

Suppose \((s_i^{(m)})_i\) have been constructed for some \(m < \delta\). Let \(I_0 \subset \{1, \ldots, d\}\) be maximal such that \(\sum_{i \in I_0} s_i^{(m)} = \frac{1}{2}r|I_0|(|I_0| - 1)\). We have

\[s_i^{(m)} < s_k^{(m)}\]

for each \(i_o \in I_0\) and each \(k \in I_0^c\).
This follows from combining the three formulae
\[
\sum_{i \in I_0 \cup \{k\}} s_i^{(m)} \geq \frac{1}{2} r |I_0 \cup \{k\}|(|I_0 \cup \{k\}| - 1) = \frac{1}{2} r |I_0|(|I_0| - 1) + r |I_0|,
\]

\[
\sum_{i \in I_0} s_i^{(m)} = \frac{1}{2} r |I_0|(|I_0| - 1),
\]

\[
\sum_{i \in I_0 - \{i_0\}} s_i^{(m)} \geq \frac{1}{2} r |I_0 - \{i_0\}|(|I_0 - \{i_0\}| - 1) = \frac{1}{2} r |I_0|(|I_0| - 1) - r(|I_0| - 1)
\]

(the first one and the last one holding by hypothesis).

Claim: There is some \(k \in I_0^c\) such that \(s_k^{(m)} + r > t_k\).

Suppose that, on the contrary, \(s_k^{(m)} + r = t_k\) for all \(k \in I_0^c\). As \((t_i)_i\) satisfies (2) we then have
\[
r |I_0^c| (d - \frac{1}{2} (|I_0^c| - 1)) \geq \sum_{k \in I_0^c} s_k^{(m)} + r
\]
or equivalently
\[
r |I_0^c| (d - 1 - \frac{1}{2} (|I_0^c| - 1)) \geq \sum_{k \in I_0^c} s_k^{(m)}.
\]

On the other hand, as \(m < \delta\) we find
\[
\sum_{k \in I_0^c} s_k^{(m)} = (\sum_{k \in I_0} s_k^{(m)}) - \sum_{k \in I_0} s_k^{(m)}
\]
\[
> \frac{1}{2} r d (d - 1) - \frac{1}{2} r |I_0|(|I_0| - 1)
\]
\[
= r \sum_{n=|I_0|}^{d-1} n
\]
\[
= r |I_0^c| (d - 1 - \frac{1}{2} (|I_0^c| - 1)).
\]

Taken together this is a contradiction. The claim is proven.

We choose some \(k \in I_0^c\) such that \(s_k^{(m)} + r > t_k\) and put \(s_k^{(m+1)} = s_k^{(m)} - 1\) and \(s_i^{(m+1)} = s_i^{(m)}\) for \(i \in \{1, \ldots, d\} - \{k\}\).

Claim: \((s_i^{(m+1)})_i\) satisfies the inequality on the right hand side of (2) for each \(I \subset \{1, \ldots, d\}\).

If \(k \notin I\) this follows from the inequality on the right hand side of (2) for \(I\) and \((s_i^{(m)})_i\).

Similarly, if \(\sum_{i \in I} s_i^{(m)} > \frac{1}{2} r |I|(|I| - 1)\) the claim is obvious. Now assume that \(k \in I\) and \(\sum_{i \in I} s_i^{(m)} = \frac{1}{2} r |I|(|I| - 1)\). We then find some \(i_0 \in I_0\) with \(i_0 \notin I\), because otherwise \(I_0 \subset I\) and hence (since \(k \in I\) but \(k \notin I_0\)) even \(I_0 \subset I\), which would contradict the maximality of \(I_0\) as chosen above. Formula (3) gives \(s_k^{(m+1)} \geq s_{i_0}^{(m)}\), hence the inequality on the right hand side of (2) for \((I - \{k\}) \cup \{i_0\}\) and \((s_i^{(m)})_i\) implies the inequality on the right hand side of (2) for \(I\) and \((s_i^{(m+1)})_i\).
The claim is proven. All the other properties required of \((s_i^{(m+1)})_i\) are obvious from its construction.

Step 2: The sequence \(t_1,\ldots,t_d\) defined by \(t_i = n_i + r(d - i)\) satisfies formula (2) for each subset \(I \subset \{1,\ldots,d\}\).

Indeed, for each \(I \subset \{1,\ldots,d\}\) the formula (2) for \((t_i)_{1\leq i \leq d}\) is equivalently converted into the formula (1) for \((n_i)_{1\leq i \leq d}\) by means of the following equations:

\[
\sum_{i \in I} r(d - i) = r\Delta(I) + \sum_{i \in I} r(d - i),
\]

\[
\frac{1}{2} r\vert I \vert (\vert I \vert - 1) = -r\Delta(\{0,\ldots,d\} - I) + \sum_{i \in I} r(d - i).
\]

Step 3: If for the \(t_i\) as in step 2 we choose \(~w_i~\) as in step 1, then the sequence \((\tilde{w}_i)_{0 \leq i \leq d}\) defined by \(\tilde{w}_0 = 0\) and \(\tilde{w}_i = \tilde{w}_i - r(d - i)\) for \(1 \leq i \leq d\) satisfies the requirements of statement (a).

It is clear that \(\tilde{w}_0 = 0\) and \(0 \leq n_i - \tilde{w}_i \leq r\) for all \(1 \leq i \leq d\), as well as \(\sum_{i=0}^{d} \tilde{w}_i = 0\). It remains to see that \((\tilde{w}_i)_{0 \leq i \leq d}\) satisfies the inequalities (1) for any \(I \subset \{0,\ldots,d\}\). If \(0 \notin I\) then, using the same conversion formulae as in the proof of step 2, this follows from the fact that \((\tilde{t}_i)_{1\leq i \leq d}\) satisfies formula (1) for each \(I \subset \{1,\ldots,d\}\). If however \(0 \in I\) then we use the property \(\sum_{i=0}^{d} \tilde{w}_i = 0\): it implies that, for \((\tilde{w}_i)_{0 \leq i \leq d}\), the left hand (resp. right hand) side inequality of formula (1) for \(I\) is equivalent with the right hand (resp. left hand) side inequality of formula (1) for \(\{0,\ldots,d\} - I\), thus holds true because the latter holds true — as we just saw.

\(\square\)

Let \(W\) denote the finite Coxeter group of type \(A_d\). Thus, \(W\) contains a set \(S_0 = \{s_1,\ldots,s_d\}\) of Coxeter generators satisfying \(\text{ord}(s_is_{i+1}) = 3\) for \(1 \leq i \leq d - 1\) and \(\text{ord}(s_is_{j+1}) = 2\) for \(1 \leq i < j \leq d - 1\). Put \(\varpi = s_d \cdots s_1\). Let \(\ell : W \to \mathbb{Z}_{\geq 0}\) denote the length function.

It is convenient to realize \(W\) as the symmetric group of the set \(\{0,\ldots,d\}\) such that \(s_i = (i - 1, i)\) (transposition) for \(1 \leq i \leq d\). For \(w \in W\) and \(1 \leq i \leq d\) we then have

\[
\ell(ws_i) > \ell(w) \quad \text{if and only if} \quad w(i - 1) < w(i),
\]

see Proposition 1.5.3 in [1].

Let \(W'\) denote the subgroup of \(W\) generated by \(s_1,\ldots,s_{d-1}\). Any element \(w\) in \(W\) can be uniquely written as \(w = \varpi w'\) for some \(w' \in W'\), some \(0 \leq i \leq d\). We may thus define \(\mu(w) = i\); equivalently, \(\mu(w) \in \{0,\ldots,d\}\) is defined by asking \(\varpi^{-\mu(w)} w \in W'\).

**Theorem 2.3.** Let \((n_i)_{0 \leq i \leq d}\) be a balanced weight of length \(d + 1\) and amplitude \(r\). There exists a function \(\nabla : W \to \mathbb{Z}\) such that for all \(w \in W\) we have

\[
\nabla(w) - \nabla(w\varpi) = -\mu(w)
\]
and such that for all \( s \in S_0 \) and \( w \in W \) with \( \ell(ws) > \ell(w) \) we have

\[ \nabla(w) - r \leq \nabla(ws) \leq \nabla(w). \]

**Proof:** We argue by induction on \( d \). The case \( d = 1 \) is trivial. Now assume that \( d \geq 2 \) and that we know the result for \( d - 1 \). By Lemma 2.2 we find a balanced weight \((m_i)_{0 \leq i \leq d - 1}\) of length \( d \) and amplitude \( r \) such that \( 0 \leq n_i - m_{i-1} \leq r \) for each \( i = 1, \ldots, d \).

Put \( \overline{w} = s_{d-1} \cdots s_1 \). Define \( \mu' : W' \rightarrow \{0, \ldots, d-1\} \) by asking that for any \( w \in W' \) the element \((\overline{w})^{\mu'(w)}w\) of \( W' \) belongs to the subgroup generated by \( s_1, \ldots, s_{d-2} \). By induction hypothesis there is a function \( \nabla' : W' \rightarrow \mathbb{Z} \) with

\[ \nabla'(w) - \nabla'(\overline{w}) = -m_{\mu'(w)} \]

for all \( w \in W' \) and

\[ \nabla'(w) - r \leq \nabla'(ws) \leq \nabla'(w) \]

for all \( w \in W', s \in \{s_1, \ldots, s_{d-1}\} \) with \( \ell(ws) > \ell(w) \). Writing \( w \in W \) uniquely as \( w = w'\overline{w}^{d'} \) with \( w' \in W' \) and \( 0 \leq d' \leq d \) we define

\[ \nabla(w) = \nabla'(w') + \sum_{t=0}^{d'-1} n_{\mu(w'\overline{w}^{d'})}. \]

That this function \( \nabla \) satisfies condition (5) for all \( w \in W \) is obvious. We now show that it satisfies condition (6) for \( s = s_d \) and all \( w \in W \) with \( \ell(ws_d) > \ell(w) \). Write \( w = w'\overline{w}^{d} \) with \( w' \in W' \) and \( 0 \leq j \leq d \).

If \( j = d \) then \( w = \overline{w}^{d} = w_s s_1 \cdots s_d \) so that \( \ell(ws_d) \leq \ell(w) \) (since \( w' \in W' \)). Thus, for \( j = d \) there is nothing to prove.

Now assume \( 1 \leq j \leq d - 1 \). We then have \( ws_d = w'\overline{w}^{j}s_{d-j}\overline{w}^{j'} = w's_{d-j}\overline{w}^{j'} \) with \( w's_{d-j} \in W' \), and we claim that \( \ell(ws_d) > \ell(w) \) implies \( \ell(w's_{d-j}) > \ell(w') \). Indeed, \( \ell(ws_d) > \ell(w) \) means \( w(d-1) < w(d) \), by formula (4). As \( \overline{w}(d) = d - j \) and \( (\overline{w})^{j'}d - j \) \( d - 1 - j \) this implies \( w'(d-1-j) < w'(d-j) \), hence \( \ell(w's_{d-j}) > \ell(w') \), again by formula (4). The claim is proven.

Moreover, for \( 0 \leq t \leq j - 1 \) we have \( w's_{d-j}\overline{w}^{j}w = w'\overline{w}^{j}w_{d-j+t} \) with \( w_{d-j+t} \in W' \). This implies \( \mu(w's_{d-j}\overline{w}^{j}) = \mu(w'\overline{w}^{j}) \). Therefore the claim \( \nabla(w) - r \leq \nabla(ws_d) \leq \nabla(w) \) is reduced to the assumption \( \nabla'(w') - r \leq \nabla'(w's_{d-j}) \leq \nabla'(w') \).

Finally assume that \( j = 0 \), i.e. \( w = w' \in W' \). Then \( \nabla(w) = \nabla'(w) \) and

\[ \nabla(ws_d) = \nabla(w'\overline{w}^{d}) \]

\[ = \nabla'(w'\overline{w}^{d}) + \sum_{t=0}^{d-1} n_{\mu(w'\overline{w}^{d})}. \]
Here $\nabla'(w^t, \pi^t) = \nabla'(w) + m \mu'(w)$ by the assumption on $\nabla'$. On the other hand $\sum_{i=0}^{d-1} n_{\mu(ws^i d)} = -n_{\mu(ws_d)}$ as $\sum_{i=0}^{d} n_i = 0$. Now we claim that $\mu'(w) + 1 = \mu(ws_d)$. Indeed, we have $w(d) = d - \mu(w)$ and hence also $ws_d(d) = d - \mu(ws_d)$ for $w \in W$. Similarly, we have $w(d - 1) = d - 1 - \mu'(w)$ and hence also $ws_d(d - 1) = d - 1 - \mu'(w)$ for $w \in W'$, and the claim is proven.

Inserting all this transforms the assumption $0 \leq n_{\mu(ws_d)} - m_{\mu(ws_d)-1} \leq r$ into the condition (6) (for $s = s_d$).

We have proven condition (6) for all $s \in S_0$ and all $w \in W$ with $\ell(ws_d) > \ell(w)$. Condition (6) for all $s \in S_0$ and all $w \in W$ with $\ell(ws) > \ell(w)$ can be checked directly as well. However, alternatively one can argue as follows.

In the setting of section 3 (and in its notations) choose an arbitrary $F$ with residue field $\mathbb{F}_q$ (for an arbitrary $q$), and choose $K/\mathbb{Q}_p$ and $\pi \in K$ such that our present $r$ satisfies $\pi^r = q$. We use the elements $t_{w^i}$ of $T$ (explicitly given by formula (13)) to define the character $\Theta : T \to K^\times$ by asking that $\Theta(t_{w^i}) = \pi^{-n_{w^i}-1}$ and that $\Theta|_{T \cap I} = \theta$ be the trivial character. (This is well defined as $T$ is the direct product of $T \cap I$ and the free abelian group on the generators $t_{w^i}$ for $0 \leq i \leq d$.) The implication (iii)$\Rightarrow$(ii) in Lemma 3.5 applied to this $\Theta$, shows that what we have proven so far is enough. 

\[\square\]

3 Hecke lattices in principal series representations I

Fix a prime number $p$. Let $K/\mathbb{Q}_p$ be a finite extension field, $\mathfrak{o}$ its ring of integers and $k$ its residue field.

Let $F$ be a non-Archimedean locally compact field, $\mathcal{O}_F$ its ring of integers, $p_F \in \mathcal{O}_F$ a fixed prime element and $k_F = \mathbb{F}_q$ its residue field with $q = p^{\log_p q} \in p^N$ elements.

Let $G = \text{GL}_{d+1}(F)$ for some $d \in \mathbb{N}$. Let $T$ be a maximal split torus in $G$, let $N(T)$ be its normalizer. Let $P$ be a Borel subgroup of $G$ containing $T$, let $N$ be its unipotent radical.

Let $X$ be the Bruhat Tits building of $\text{PGL}_{d+1}(F)$, let $A \subset X$ be the apartment corresponding to $T$. Let $I$ be an Iwahori subgroup of $G$ fixing a chamber $C$ in $A$, let $I_0$ denote its maximal pro-$p$-subgroup. The (affine) reflections in the codimension-1-faces of $C$ form a set $S$ of Coxeter generators for the affine Weyl group. We view the latter as a subgroup of the extended affine Weyl group $N(T)/T \cap I$. There is an $s_0 \in S$ such that the image of $S_0 = S - \{s_0\}$ in the finite Weyl group $W = N(T)/T$ is the set of simple reflections.

We find elements $u, s_d \in N(T)$ such that $uC = C$ (equivalently, $uI = Iu$, or also
$uI_0 = I_0u$), such that $u^{d+1} \in \{p_F \cdot \text{id}, p_F^{-1} \cdot \text{id}\}$ and such that, setting

$$s_i = u^{d-i} s_d u^{i-d} \quad \text{for } 0 \leq i \leq d$$

the set $\{s_1, \ldots, s_d\}$ maps bijectively to $S_0$, while $\{s_0, s_1, \ldots, s_d\}$ maps bijectively to $S$; we henceforth regard these bijections as identifications. Let $u = s_d \cdots s_1 \in W \subset G$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function with respect to $S_0$.

For convenience one may realize all these data explicitly, e.g. according to the following choice: $T$ consists of the diagonal matrices, $P$ consists of the upper triangular matrices, $N$ consists of the unipotent upper triangular matrices (i.e. the elements of $P$ with all diagonal entries equal to 1). Then $W$ can be identified with the subgroup of permutation matrices in $G$. Its Coxeter generators $s_i$ for $i = 1, \ldots, d$ are the block diagonal matrices

$$s_i = \text{diag}(I_{i-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{d-i})$$

while $u$ is written in block form as

$$u = \begin{pmatrix} I_d \\ p_F \end{pmatrix}.$$ 

(Here $I_m$, for $m \geq 1$, always denotes the identity matrix in $\text{GL}_m$.) The Iwahori group $I$ consists of the elements of $\text{GL}_{d+1}(\mathcal{O}_F)$ mapping to upper triangular matrices in $\text{GL}_{d+1}(k_F)$, while $I_0$ consists of the elements of $I$ whose diagonal entries map to $1 \in k_F$.

For $s \in S_0$ let $\iota_s : \text{GL}_2(F) \to G$ denote the corresponding embedding. For $a \in F^\times$, $b \in F$ put

$$h_s(a) = \iota_s\left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right), \quad \nu_s(b) = \iota_s\left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right), \quad \delta_s = \iota_s\left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

We realize $W$ as a subgroup of $G$ in such a way that

$$\iota_s\left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = s$$

for all $s \in S_0$. Notice that $\text{Im}(\nu_s) \subset N$ for all $s \in S_0$.

**Lemma 3.1.** (a) For $s \in S_0$ and $a \in F^\times$ we have

$$s \nu_s(a)s = h_s(a^{-1}) \nu_s(a) \delta_s \nu_s(a^{-1}).$$

(b) For $w \in W$ and $s \in S_0$ with $\ell(ws) > \ell(w)$ and for $b \in F$ we have

$$w \nu_s(b) w^{-1} \in N.$$

10
PROOF: Statement (a) is a straightforward computation inside $\text{GL}_2(F)$. For statement (b) write $s = s_i$ for some $1 \leq i \leq d$. Then the matrix $ww_i(b)w^{-1}$ has entry $b$ at the $(w(i - 1), w(i))$-spot (and coincides with the identity matrix at all other spots). As $\ell(ws_i) > \ell(w)$ implies $w(i - 1) < w(i)$ by formula (4), this implies $ww_i(b)w^{-1} \in N$. □

Let $\text{ind}_{I_0}^G \mathfrak{1}_\sigma$ denote the $\sigma$-module of $\sigma$-valued compactly supported functions $f$ on $G$ such that $f(ig) = f(g)$ for all $g \in G$, all $i \in I_0$. It is a $G$-representation by means of the formula $(g'f)(g) = f(gg')$ for $g, g' \in G$. Let

$$\mathcal{H}(G, I_0) = \text{End}_{\mathfrak{g}[G]}(\text{ind}_{I_0}^G \mathfrak{1}_\sigma)^{\text{op}}$$

denote the corresponding pro-$p$-Iwahori Hecke algebra with coefficients in $\sigma$. Then $\text{ind}_{I_0}^G \mathfrak{1}_\sigma$ is naturally a right $\mathcal{H}(G, I_0)$-module. For a subset $H$ of $G$ we let $\chi_H$ denote the characteristic function of $H$. For $g \in G$ let $T_g \in \mathcal{H}(G, I_0)$ denote the Hecke operator corresponding to the double coset $I_0gI_0$. It sends $f : G \to \sigma$ to

$$T_g(f) : G \to \sigma, \quad h \mapsto \sum_{x \in I_0 \backslash G} \chi_{I_0gI_0}(hx^{-1}) f(x).$$

In particular we have

$$(10) \quad T_g(\chi_{I_0}) = \chi_{I_0g} = g^{-1} \chi_{I_0} \quad \text{if} \ gI_0 = I_0g.$$

Let $R$ be an $\mathfrak{g}$-algebra, let $V$ be a representation of $G$ on an $R$-module. The submodule of $V^{I_0}$ of $I_0$-invariants in $V$ carries a natural (left) action by the $R$-algebra $\mathcal{H}(G, I_0)_R = \mathcal{H}(G, I_0) \otimes_\sigma R$, resulting from the natural isomorphism $V^{I_0} \cong \text{Hom}_{\mathfrak{g}[G]}((\text{ind}_{I_0}^G \mathfrak{1}_\sigma) \otimes_\sigma R, V)$. Explicitly, for $g \in G$ and $v \in V^{I_0}$ the action of $T_g$ is given as follows: If the collection $\{g_j\}_j$ in $G$ is such that $I_0gI_0 = \bigsqcup_j I_0g_j$, then

$$(11) \quad T_g(v) = \sum_j g_j^{-1}v.$$

Let $\overline{T} = (I \cap T)/(I_0 \cap T) = I/I_0$.

Suppose we are given a character $\Theta : T \to K^\times$ whose restriction $\theta = \Theta|_{I \cap T}$ to $I \cap T$ factors through $\overline{T}$. As $\overline{T}$ is finite, $\theta$ takes values in $\mathfrak{g}^\times$, hence induces a character (denoted by the same symbol) $\theta : \overline{T} \to K^\times$. For any $w \in W$ it defines a homomorphism

$$\theta(wh_s(.)w^{-1}) : k_F^\times \to k_F^\times, \quad x \mapsto \theta(wh_s(x)w^{-1})$$

and it makes sense to compare it with the constant homomorphism $1$ taking all elements of $k_F^\times$ to $1 \in k_F^\times$. Notice in the following that $\theta(wh_s(.)w^{-1}) = 1$ if and only if $\theta(wh_s(.)sw^{-1}) = 1$. For $w \in W$ and $s \in S_0$ put

$$\kappa_{w,s} = \kappa_{w,s}(\theta) = \theta(wh_sw^{-1}) \in \{\pm 1\}.$$
Read \( \Theta \) as a character of \( P \) by means of the natural projection \( P \to T \) and consider the smooth principal series representation

\[
V = \text{Ind}_P^G \Theta = \{ f : G \to K \text{ locally constant} \mid f(pg) = \Theta(p)f(g) \text{ for } g \in G, p \in P \}
\]

with \( G \)-action \( (gf)(x) = f(xg) \). For \( w \in W \) let \( f_w \in V \) denote the unique \( I_0 \)-invariant function supported on \( PwI_0 \) and with \( f_w(w) = 1 \). It follows from the decomposition \( G = \bigsqcup_{w \in W} PwI_0 \) that the set \( \{f_w\}_{w \in W} \) is a \( K \)-basis of the \( \mathcal{H}(G, I_0)_K \)-module \( V^{I_0} \).

**Lemma 3.2.** Let \( w \in W \) and \( s \in S_0 \), let \( a \in \mathcal{O}_F \).

(a) If \( \ell(ws) > \ell(w) \) and \( a \notin (p_F) \) then \( ws a s \notin PwI_0 \).

(b) If \( \ell(ws) > \ell(w) \) then \( v_n a s \notin PwI_0 \) for all \( v \in W - \{ws\} \).

(c) \( v_n a s \notin PwI_0 \) for all \( v \in W - \{w, ws\} \).

**Proof:** We have \( v_n (\mathcal{O}_F) \subset I_0 \). Therefore all statements will follow from standard properties of the decomposition \( G = \bigsqcup_{w \in W} PwI_0 \), or rather the restriction of this decomposition to \( \text{GL}_{d+1}(\mathcal{O}_F) \); notice that this restriction projects to the usual Bruhat decomposition of \( \text{GL}_{d+1}(k_F) \).

(a) The assumption \( a \notin (p_F) \), i.e. \( a \in \mathcal{O}_F^* \), implies that \( ws a s \in wIsI \), by formula \( \text{[S]} \). The assumption \( \ell(ws) > \ell(w) \) implies \( wIsI \subset PwsI = PwsI_0 \) by standard properties of the Bruhat decomposition, hence \( wIsI \cap PwI_0 = \emptyset \).

(b) Standard properties of the Bruhat decomposition imply \( vI_0 s \subset PvsI_0 \cup PwI_0 \), as well as \( vI_0 s \subset PvsI_0 \) if \( \ell(vs) > \ell(v) \). As \( \ell(ws) > \ell(w) \) and \( v \neq ws \) statement (b) follows.

(c) The same argument as for (b).

\[ \square \]

**Lemma 3.3.** Let \( w \in W \) and \( s \in S_0 \). We have

\[
T_s(f_w) = \begin{cases} 
  f_{ws} & : \ell(ws) > \ell(w) \\
  qf_{ws} & : \ell(ws) < \ell(w) \text{ and } \theta(w h_s(.) w^{-1}) \neq 1 \\
  qf_{ws} + \kappa_{ws,s}(q-1)f_w & : \ell(ws) < \ell(w) \text{ and } \theta(w h_s(.) w^{-1}) = 1
\end{cases}
\]

**Proof:** We have \( I_0 s I_0 = \bigsqcup_a I_0 s v_n (a) \) where \( a \) runs through a set of representatives for \( k_F \) in \( \mathcal{O}_F \). For \( y \in G \) we therefore compute, using formula \( \text{[III]} \):

\[
(T_s(f_w))(y) = \left( \sum_a v_n (a) s f_w \right)(y) = \sum_a f_w(y v_n (a) s).
\]

(12)

Suppose first that \( \ell(ws) > \ell(w) \). For \( a \notin (p_F) \) we then have \( ws a s \notin PwI_0 \) by Lemma \( \text{[S2]} \) hence \( f_w(ws a s) = 0 \). On the other hand \( f_w(ws a s) = f_w(w) = 1 \).
Together we obtain \((T_s(f_w))(w) = 1\). For \(v \in W - \{ws\}\) and any \(a \in \mathcal{O}_F\) we have \(v \nu_s(a) \notin P_wI_0\) by Lemma 3.2 hence \((T_s(f_w))(v) = 0\). It follows that \(T_s(f_w) = f_{ws}\).

Now suppose that \(\ell(ws) < \ell(w)\). Then \(w \nu_s(a) w^{-1} \in N\) for any \(a\), by formula (9), hence \(f_w(w \nu_s(a)s) = \theta(w \nu_s(a)w^{-1}) f_w(w) = 1\). Summing up we get

\[(T_s(f_w))(ws) = \sum_a f_w(w \nu_s(a)s) = |k_F| = q.\]

To compute \((T_s(f_w))(w)\) we first notice that \(f_w(w \nu_s(0)s) = f_w(ws) = 0\). On the other hand, for \(a \notin (p_F)\) we find

\[f_w(w \nu_s(a)s) = f_w(w s \nu_s(a)s)\]

\[\overset{(i)}{=} f_w(w s h_s(a^{-1}) \nu_s(a) \delta_s \nu_s(a^{-1}))\]

\[= \theta(w s h_s(a^{-1}) \nu_s(a) \delta_s \nu_s(a^{-1})) f_w(w \nu_s(a^{-1}))\]

\[\overset{(ii)}{=} \theta(w s h_s(a^{-1}) \delta_s \nu_s(a^{-1}))\]

\[= \kappa_{ws,s} \theta(w s h_s(a^{-1}) \delta_s \nu_s(a^{-1})).\]

Here (i) uses formula (8) while (ii) uses \(f_w(w \nu_s(a^{-1})) = f_w(w) = 1\) as well as

\[(w s h_s(a^{-1}) \nu_s(a) \delta_s \nu_s(a^{-1})) (w s h_s(a^{-1}) \delta_s \nu_s(a^{-1}))^{-1} = w \nu_s(a^{-1}) w^{-1} \in N,\]

formula (9). Now

\[\sum_{a \notin (p_F)} \theta(w s h_s(a) w^{-1}) = \begin{cases} q - 1 & : \theta(w h_s(\cdot) w^{-1}) = 1 \\ 0 & : \theta(w h_s(\cdot) w^{-1}) \neq 1 \end{cases}\]

Thus \(\sum_{a \notin (p_F)} f_w(w \nu_s(a)s) = \kappa_{ws,s}(q - 1)\) if \(\theta(w h_s(\cdot) w^{-1}) = 1\), but \(\sum_{a \notin (p_F)} f_w(w \nu_s(a)s) = 0\) if \(\theta(w h_s(\cdot) w^{-1}) \neq 1\). We have shown that \((T_s(f_w))(w) = \kappa_{ws,s}(q - 1)\) if \(\theta(w h_s(\cdot) w^{-1}) = 1\), but \((T_s(f_w))(w) = 0\) if \(\theta(w h_s(\cdot) w^{-1}) \neq 1\). Finally, for \(v \in W - \{w, ws\}\) and \(a \in \mathcal{O}_F\) we have \(v \nu_s(a) \notin P_wI_0\) by Lemma 3.2 hence \((T_s(f_w))(v) = 0\). Summing up gives the formulae for \(T_s(f_w)\) in the case \(\ell(ws) < \ell(w)\). \(\square\)

As \(\overline{u}\) is the unique element in \(W \subset G\) lifting the image of \(u\) in \(W = N(T)/T\) we have \(\overline{u}^{-1} u \in T\). For \(w \in W\) we define

\[t_w = w \overline{u}^{-1} w^{-1} \in T.\]

We record the formulae

\[\overline{u}^{-1} u = t_{\overline{u}} = \text{diag}(p_F, I_d),\]

\[t_{\overline{u}} = \text{diag}(I_{d-i+1}, p_F, I_{i-1}) \quad \text{for } 1 \leq i \leq d,\]

(13)

In particular we notice that \(t_w = t_{ws, i}\) for \(2 \leq i \leq d\).
Lemma 3.4. For $w \in W$ we have

\begin{equation}
T_{u^{-1}}(f_w) = \Theta(t_w)f_{w\pi^{-1}} \quad \text{and} \quad T_u(f_w) = \Theta(t_w^{-1})f_{w\pi}.
\end{equation}

For $w \in W$ and $t \in T \cap I$ we have

\begin{equation}
T_t(f_w) = \theta(wt^{-1}w^{-1})f_w.
\end{equation}

**Proof:** We use formula (10) in both cases: First,

\[(T_{u^{-1}}(f_w))(w\bar{\pi}^{-1}) = (uf_w)(w\pi^{-1}) = f_w(w\bar{\pi}^{-1}u) = \Theta(t_w)f_w(w) = \Theta(t_w)\]

but

\[(T_{u^{-1}}(f_w))(v) = (uf_w)(v) = f_w(vu) = \Theta(vu\bar{\pi}^{-1}v^{-1})f_w(v\bar{\pi}) = 0\]

for $v \in W - \{w\bar{\pi}^{-1}\}$, hence the first one of the formulae in (14); the other one is equivalent with it (or alternatively: proven in the same way). Next,

\[(T_t(f_w))(w) = (t^{-1}f_w)(w) = f_w(wt^{-1}) = \theta(wt^{-1}w^{-1})f_w(w) = \theta(wt^{-1}w^{-1}),\]

but

\[(T_t(f_w))(v) = (t^{-1}f_w)(v) = f_w(vt^{-1}) = \theta(vt^{-1}v^{-1})f_w(v) = 0\]

for $v \in W - \{w\}$, hence formula (15). \hfill \square

We assume that there is some $r \in \mathbb{N}$ and some $\pi \in \mathfrak{o}$ such that $\pi^r = q$ and such that $\Theta$ takes values in the subgroup of $K^\times$ generated by $\pi$ and $\mathfrak{o}^\times$. Notice that, given an arbitrary $\Theta$, this can always be achieved after passing to a suitable finite extension of $K$. Let $\text{ord}_K : K \to \mathbb{Q}$ denote the order function normalized such that $\text{ord}_K(\pi) = 1$.

Suppose we are given a function $\nabla : W \to \mathbb{Z}$. For $w \in W$ we put $g_w = \pi^{\nabla(w)}f_w$ and consider the $\mathfrak{o}$-submodule

\[L_{\nabla} = L_{\nabla}(\Theta) = \bigoplus_{w \in W} \mathfrak{o}.g_w\]

of $V^{I_0}$ which is $\mathfrak{o}$-free with basis $\{g_w \mid w \in W\}$. We ask under which conditions on $\nabla$ it is stable under the action of $\mathcal{H}(G, I_0)$ on $V^{I_0}$. Consider the formulae

\begin{equation}
\nabla(w) - \nabla(w\pi) = \text{ord}_K(\Theta(t_w\pi)),
\end{equation}

\begin{equation}
\nabla(w) - r \leq \nabla(ws) \leq \nabla(w).
\end{equation}

**Lemma 3.5.** The following conditions (i), (ii), (iii) on $\nabla$ are equivalent:

(i) $L_{\nabla}$ is stable under the action of $\mathcal{H}(G, I_0)$ on $V^{I_0}$.

(ii) $\nabla$ satisfies formula (16) for any $w \in W$, and it satisfies formula (17) for any $s \in S_0$ and any $w \in W$ with $\ell(ws) > \ell(w)$.

(iii) $\nabla$ satisfies formula (16) for any $w \in W$, and it satisfies formula (17) for $s = s_d$ and any $w \in W$ with $\ell(ws_d) > \ell(w)$.

14
PROOF: For $t \in T \cap I$ and $w \in W$ it follows from Lemma 3.4 that

\begin{equation}
T_t(g_w) = \theta(w^{-1}t^{-1})g_w,
\end{equation}

(18)

\begin{equation}
T_{u^{-1}}(g_w) = \pi^{-1}g_w^{-1} \Theta(t_w)g_w^{-1},
\end{equation}

(19)

\begin{equation}
T_u(g_w) = \pi^{-1}g_w^{-1} \Theta(t_w^{-1})g_w^{-1}.
\end{equation}

(20)

For $w \in W$ and $s \in S_0$ it follows from Lemma 3.3 that

\begin{equation}
T_s(g_w) =
\begin{cases}
\pi^{-1}g_w^{-1} : & \ell(ws) > \ell(w) \\
\pi^{-1}g_w^{-1} : & \ell(ws) < \ell(w) \text{ and } \theta(ws^{-1})w^{-1} \neq 1 \\
\pi^{-1}g_w^{-1} + \kappa_{ws,s}(\pi^{-1}g_w^{-1}) & \ell(ws) < \ell(w) \text{ and } \theta(ws^{-1})w^{-1} = 1
\end{cases}
\end{equation}

(21)

From these formulae we immediately deduce that condition (i) implies both condition (ii) and condition (iii) on $\nabla$. Now it is known that $H(G, I_0)$ is generated as an $\mathfrak{o}$-algebra by the Hecke operators $T_t$ for $t \in T \cap I$ together with $T_{u^{-1}}, T_u$ and $T_{s_d}$. Thus, to show stability of $L_{\nabla}$ under $H(G, I_0)$ it is enough to show stability of $L_{\nabla}$ under these operators. The above formulae imply that this stability is ensured by condition (iii). Thus (i) is implied by (iii), and a fortiori by (ii). 

\[\square\]

4 Hecke lattices in principal series representations II

In Lemma 3.5 we saw that the (particularly nice) $H(G, I_0)$ stable $\mathfrak{o}$-lattices $L_{\nabla}$ in the $H(G, I_0)_{K}$-module $V^{I_0}$ for $V = \text{Ind}_{P}^{G} \Theta$ are obtained from functions $\nabla : W \to \mathbb{Z}$ satisfying the conditions stated there. We now want to explain that the existence of such a function $\nabla$ can be directly read off from $\Theta$. For $0 \leq i \leq d$ put

\[n_i = -\text{ord}_K(\Theta(t_{\pi^{i+1}})).\]

Corollary 4.1. If $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d + 1$ and amplitude $r$ then there exists a function $\nabla : W \to \mathbb{Z}$ such that $L_{\nabla}$ is stable under the action of $H(G, I_0)$ on $V^{I_0}$.

PROOF: By Theorem 2.3 there exists a function $\nabla : W \to \mathbb{Z}$ satisfying condition (iii) of Lemma 3.5. Thus we may conclude with that Lemma. 

\[\square\]
Thus we need to decide for which $\Theta$ the collection $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d + 1$ and amplitude $r$.

We now assume that $F \subset K$. We normalize the absolute value $|.| : K^\times \to \mathbb{Q}^\times \subset K^\times$ on $K$ (and hence its restriction to $F$) by requiring $|p_F| = q^{-1}$. Let $\delta : T \to F^\times$ denote the modulus character associated with $P$, i.e. $\delta = \prod_{\alpha \in \Phi^+} |\alpha|$ where $\Phi^+$ is the set of positive roots. Let $N_0 = N \cap I$ and

$$T_+ = \{ t \in T \mid t^{-1}N_0t \subset N_0 \}.$$  

The group $W$ acts on the group of characters $\text{Hom}(T, K^\times)$ through its action on $T$.

**Theorem 4.2.** Suppose that for all $w \in W$ and all $t \in T^+$ we have

$$|((w\Theta)(w\delta^{-1})\delta^{1/2})(t)| \leq 1$$  

and that the restriction of $\Theta$ to the center of $G$ is a unitary character. Then $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d + 1$ and amplitude $r$, and $L_\nabla$ is stable under the action of $\mathcal{H}(G, I_0)$ on $V_{I_0}$.

As the center of $G$ is generated by the element $\prod_{j=0}^d t_{\mathfrak{m}^j} = p_F I_{d+1}$ (cf. formula (13)) together with $\mathcal{O}_F^\times \cdot I_{d+1}$, the condition that the restriction of $\Theta$ to the center of $G$ be a unitary character is equivalent with the condition

$$\prod_{j=0}^d |\Theta(t_{\mathfrak{m}^j})| = 1.$$  

**Proof:** (of Theorem 4.2) Recall that, for convenience, we work with the following realization: $T$ is the group of diagonal matrices, $P$ is the group of upper triangular matrices, $s_i$ (for $1 \leq i \leq d$) is the $(i - 1, i)$-transposition matrix and $u = \overline{u} \cdot \text{diag}(p_F, 1, \ldots, 1)$. Thus $T_+$ is the subgroup of $T$ generated by all $t \in T$ (viewed as a subgroup of $T$ by means of the Teichmüller character), by the scalar diagonal matrices (the center of $G$), and by all the matrices of the form $\text{diag}(1, \ldots, 1, p_F, \ldots, p_F)$. The modulus character is

$$\delta : T \longrightarrow F^\times, \quad \text{diag}(\alpha_0, \ldots, \alpha_d) \mapsto \prod_{i=0}^d |\alpha_i|^{d-2i}.$$  

Write $\Theta = \text{diag}(\Theta_0, \ldots, \Theta_d)$ with characters $\Theta_j : F^\times \to K^\times$. Reading $W$ as the symmetric group of the set $\{0, \ldots, d\}$, formula (22) for $t = \text{diag}(\alpha_0, \ldots, \alpha_d)$ reads

$$\prod_{i=0}^d |\Theta_{r(i)}(\alpha_i)|^{|r(i) - i|} \leq 1.$$
for all permutations $\tau$ of \{0, \ldots, d\}. Asking formula (24) for all $\text{diag}(\alpha_0, \ldots, \alpha_d) \in T^+$ is certainly equivalent with asking it for all $\text{diag}(p^{-1}_F, \ldots, p^{-1}_F, 1 \ldots, 1)$ and for all $\text{diag}(1 \ldots, 1, p_F, \ldots, p_F)$ (and all $\tau$). This is equivalent with asking

$$|q|^\Delta(I) \leq |\prod_{j \in I} \Theta_j(p_F)| \leq |q|^{-\Delta(0, \ldots, d) - I} \quad (25)$$

for all $I \subset \{0, \ldots, d\}$. Indeed, the inequalities on the left hand side of (25) are the inequalities (24) for the $\text{diag}(p^{-1}_F, \ldots, p^{-1}_F, 1 \ldots, 1)$ and suitable $\tau$. The inequalities on the right hand side of (25) are the inequalities (24) for the $\text{diag}(1 \ldots, 1, p_F, \ldots, p_F)$ and suitable $\tau$. Now observe that $\Theta_j(p_F) = \Theta(t_{\pi^{d+1-j}})$ and hence $|\Theta_j(p_F)| = |\pi^{\text{ord}(\Theta(t_{\pi^{d+1-j}}))}| = |\pi^{-n_d-j}|$ for $0 \leq j \leq d$. We also have $|q| = |\pi|$. Together with Lemma 2.2 we recover formula (1). On the other hand, formula (23) is just the property $\sum_{i=0}^d n_i = 0$. We thus conclude with Corollary 4.1.

\[\square\]

Remarks: (1) We (formally) put $\chi = \Theta^{-\frac{1}{2}}$. Let $P \subset G$ denote the Borel subgroup opposite to $P$. The same arguments as in [3] page 10 show that (at least if $\chi$ is regular) for all $w \in W$ the action of $T$ on the Jacquet module $J_{\mathfrak{P}}(V)$ of $V$ (formed with respect to $\mathfrak{P}$) admits a non-zero eigenspace with character $(w\chi)\delta^\frac{1}{2}$, i.e. with character $(w\Theta)(w\delta^\frac{1}{2})\delta^\frac{1}{2}$. From [3] we then deduce that the conditions in Theorem 4.2 are a necessary criterion for the existence of an integral structure in $V$.

(2) This necessary criterion has also been obtained in [2]. Moreover, in loc. cit. it is shown (in a much more general context) that it implies the existence of an integral structure in the $\mathcal{H}(G, I_0)$-module $V^{I_0}$. The point of Theorem 4.2 is that it explicitly describes a particularly nice such integral structure.

(3) Consider the smooth dual $\text{Hom}_K(V, K)^{\text{sm}}$ of $V$; it is isomorphic with $\text{Ind}_G^H \Theta^{-\delta}.\delta$. Our conditions (22) and (23) for $\Theta$ are equivalent with the same conditions for $\Theta^{-\delta}$.

Remark: Suppose we are in the setting of Corollary 4.1 or Theorem 4.2. Let $H$ denote a maximal compact open subgroup of $G$ containing $I$. Abstractly, $H$ is isomorphic with $\text{GL}_{d+1}(O_F)$. Let $\mathfrak{o}[H].L_\mathfrak{V}$ denote the $\mathfrak{o}[H]$-sub module of $V$ generated by $L_\mathfrak{V}$, let $(\mathfrak{o}[H].L_\mathfrak{V})^{I_0}$ denote its $\mathfrak{o}$-sub module of $I_0$-invariants. Then one can show (we do not give the proof here) that the inclusion map $L_\mathfrak{V} \rightarrow (\mathfrak{o}[H].L_\mathfrak{V})^{I_0}$ is surjective (and hence bijective). On the one hand this may be helpful for deciding whether $V$ contains an integral structure, i.e. a $G$-stable free $\mathfrak{o}$-sub module containing a $K$-basis of $V$. On the other hand it implies (in fact: is equivalent with it) that the induced map

$$L_\mathfrak{V} \otimes_\mathfrak{o} k \rightarrow (\mathfrak{o}[H].L_\mathfrak{V}) \otimes_\mathfrak{o} k$$

is injective. This might be a useful observation about the $\mathcal{H}(G, I_0)_k$-module $L_\mathfrak{V} \otimes_\mathfrak{o} k$ (which we call an $\mathcal{H}(G, I_0)_k$-module of $W$-type in section 5).
5 $\mathcal{H}(G, I_0)_k$-modules of $W$-type

We return to the setting of section 3. For $w \in W$ we define

$$\epsilon_w = \epsilon_w(\Theta) = \pi^{-\text{ord}_K(\Theta(t_w))}\Theta(t_w).$$

Let us write $W^{s_d} = \{w \in W \mid \ell(ws_d) > \ell(w)\}$. For a function $\sigma : W^{s_d} \to \{-1, 0, 1\}$, for $w \in W$ and $i \in \{-1, 0, 1\}$ we understand the condition $\sigma(w) = i$ as a shorthand for the condition $[w \in W^{s_d}$ and $\sigma(w) = i]$.

For $w \in W$ we write $\kappa_w = \kappa_{ws_d,s_d}$.

Suppose that the function $\nabla : W \to \mathbb{Z}$ satisfies the equivalent conditions of Lemma 3.5. Define a function $\sigma : W^{s_d} \to \{-1, 0, 1\}$ by setting

$$\sigma(w) = \begin{cases} 1 : & \nabla(ws_d) = \nabla(w) \\ 0 : & \nabla(w) - r < \nabla(ws_d) < \nabla(w) \\ -1 : & \nabla(w) - r = \nabla(ws_d) \end{cases}$$

The action of $\mathcal{H}(G, I_0)$ on $L_\nabla$ induces an action of $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_\Theta k$ on $L_\nabla \otimes_\Theta k$. The $o$-basis $\{g_w \mid w \in W\}$ of $L_\nabla$ induces a $k$-basis $\{g_w \mid w \in W\}$ of $L_\nabla \otimes_\Theta k = L_\nabla(\Theta) \otimes_\Theta k$ (we use the same symbols $g_w$).

**Corollary 5.1.** The action of $\mathcal{H}(G, I_0)_k$ on $L_\nabla \otimes_\Theta k$ is characterized through the following formulae: For $t \in T \cap I$ and $w \in W$ we have

$$T_t(g_w) = \theta(wt^{-1}w^{-1})g_w,$$

(27)

$$T_{u^{-1}}(g_w) = \epsilon_w g_ww^{-1} \quad \text{and} \quad T_u(g_w) = \epsilon_w^{-1}g_ww,$$

(28)

$$T_{s_d}(g_w) = \begin{cases} g_{ws_d} : & [\sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}(.)w^{-1}) \neq 1] \text{ or } \sigma(w) = 1 \\ -\kappa_w g_w : & \sigma(ws_d) \in \{0, 1\} \text{ and } \theta(wh_{s_d}(.)w^{-1}) = 1 \\ g_{ws_d} - \kappa_w g_w : & \sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}(.)w^{-1}) = 1 \\ 0 : & \text{all other cases} \end{cases}$$

(29)

**Proof:** Formula (27) follows from formula (18). The assumption $\nabla(wu^{-1}) - \nabla(w) = \text{ord}_K(\theta(t_w))$ implies that the formulae in (28) follow from formulae (19) and (20). Finally, formula (29) follows from formula (21) by a case by case checking. $\square$

Forgetting their origin from some $\Theta$ and $\nabla$, we formalize the structure of $\mathcal{H}(G, I_0)_k$-modules met in Corollary 5.1 in an independent definition.
**Definition:** We say that an \( H(G, I_0)_k \)-module \( M \) is of \( W \)-type (or: a reduced standard module) if it is of the following form \( M = M(\theta, \sigma, \epsilon_\bullet) \). First, a \( k \)-vector space basis of \( M \) is the set of formal symbols \( g_w \) for \( w \in W \). The \( H(G, I_0)_k \)-action on \( M \) is characterized by a character \( \theta : \mathbb{T} \to k^\times \) (which we also read as a character of \( T \cap I \) by inflation), a map \( \sigma : W^{sd} \to \{-1, 0, 1\} \) and a set \( \epsilon_\bullet = \{\epsilon_w\}_{w \in W} \) of units \( \epsilon_w \in k^\times \). Namely, for \( w \in W \) we define \( \kappa_w = \kappa_w(\theta) = \theta(ws_d\delta ws_dw^{-1}) \in \{\pm 1\} \). Then it is required that for \( t \in T \cap I \) and \( w \in W \) formulae (27), (28) and (29) hold true.

Conversely we may begin with a character \( \theta : \mathbb{T} \to k^\times \), a map \( \sigma : W^{sd} \to \{-1, 0, 1\} \) and a set \( \epsilon_\bullet = \{\epsilon_w\}_{w \in W} \) of units \( \epsilon_w \in k^\times \) and ask:

**Question 1:** For which set of data \( \theta, \sigma, \epsilon_\bullet \) do formulae (27), (28) and (29) define an action of \( H(G, I_0)_k \) on \( \bigoplus_{w \in W} k : g_w \)?

**Question 2:** For which set of data \( \theta, \sigma, \epsilon_\bullet \) does there exist some \( H(G, I_0) \)-module \( L_\nabla(\Theta) \) as in Corollary 5.1 such that \( L_\nabla(\Theta) \otimes_k k \cong M(\theta, \sigma, \epsilon_\bullet) \) as an \( H(G, I_0)_k \)-module?

In question 2 we regard \( \theta \) as taking values in \( \sigma^\times \subset K^\times \) by means of the Teichmüller lifting. Clearly those \( \theta, \sigma, \epsilon_\bullet \) asked for in question 2 belong to those \( \theta, \sigma, \epsilon_\bullet \) asked for in question 1.

We do not consider question 1 in general, but provide a criterion for a positive answer to question 2. Suppose we are given a set of data \( \theta, \sigma, \epsilon_\bullet \) as above.

**Proposition 5.2.** Suppose that \( \epsilon_w = \epsilon_{w_{s_i}} \) for all \( 2 \leq i \leq d \) and that there exists a function \( \partial : W \to [-r, r] \cap \mathbb{Z} \) with the following properties:

\[
\sigma(w) = \begin{cases} 
1 & : \quad w \in W^{sd} \text{ and } \partial(w) = 0 \\
0 & : \quad w \in W^{sd} \text{ and } 0 < \partial(w) < r \\
-1 & : \quad w \in W^{sd} \text{ and } \partial(w) = r 
\end{cases}
\]

\[
\partial(ws_d) = -\partial(w)
\]

(30)

\[
\partial(w \bar{w}^{d-i}) + \partial(ws_i \bar{w}^{d-i}) = \partial(w \bar{w}^{d-j}) + \partial(ws_j \bar{w}^{d-i})
\]

for \( 1 \leq i < j - 1 < d \), and

(31)

\[
\partial(w \bar{w}^{d-i}) + \partial(ws_i \bar{w}^{d-i-1}) + \partial(ws_i s_{i+1} \bar{w}^{d-i}) = \partial(w \bar{w}^{d-i-1}) + \partial(ws_{i+1} \bar{w}^{d-i}) + \partial(ws_{i+1} s_i \bar{w}^{d-i-1})
\]

for \( 1 \leq i < d \).

Then there exists an extension \( \Theta : T \to K^\times \) of \( \theta \) and a function \( \nabla : W \to \mathbb{Z} \) as before such that we have an isomorphism of \( H(G, I_0)_k \)-modules \( L_\nabla(\Theta) \otimes_k k \cong M(\theta, \sigma, \epsilon_\bullet) \).
**Proof:** Step 1: Let \( w, v \in W \). Choose a (not necessarily reduced) expression \( v = s_{i_1} \cdots s_{i_r} \) (with \( i_m \in \{1, \ldots, d\} \)) and put

\[
\partial(w, v) = \sum_{m=1}^{r} \partial(ws_{i_1} \cdots s_{i_{m-1}}u^{d-i_m}).
\]

Claim: This definition does not depend on the chosen expression \( s_{i_1} \cdots s_{i_r} \) for \( v \).

Indeed, it follows from hypothesis (30) that for \( 1 \leq i < j - 1 < d \) we have \( \partial(w, s_is_j) = \partial(w, s_js_i) \) where on either side we use the expression of \( s_is_j = s_js_i \) as indicated. Similarly, it follows from hypothesis (31) that for \( 1 \leq i < d \) we have \( \partial(w, s_{i+1}s_i) = \partial(w, s_is_{i+1}s_i) \) where on either side we use the expression of \( s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \) as indicated. Finally, for \( 1 \leq i \leq d \) we have \( \partial(w, s_is_i) = 0 \) where we use the expression \( s_is_i = s_i^2 = 1 \in W \): this follows from the definition of \( \partial \) and from \( s_is_i = u_{d-i} = u^{d-i}s_d \). Thus we see that our definition of \( \partial(w, v) \) (viewed as a function in \( v \in W \), with fixed \( w \in W \)) respects the defining relations for the Coxeter group \( W \). Iterated application implies the stated claim.

Step 2: The definition of \( \partial(w, v) \) implies \( \partial(w, v) + \partial(wv, x) = \partial(w, vx) \) for \( v, w, x \in W \). Therefore there is a function \( \nabla : W \to \mathbb{Z} \), uniquely determined up to addition of a constant function \( W \to \mathbb{Z} \), such that

\[
\nabla(w) - \nabla(wv) = \partial(w, v) \quad \text{for all } v, w \in W.
\]

It has the following properties. First, it fulfills formula (26). Next, we have

\[
(32) \quad \nabla(w) - \nabla(w\overline{s}_i) = \nabla(ws_i) - \nabla(ws_i\overline{s}_i) \quad \text{for } w \in W \text{ and } 1 \leq i \leq d - 1.
\]

\[
(33) \quad \nabla(w\overline{s}_{i-1}) - \nabla(w) = \nabla(ws_i\overline{s}_{i-1}) - \nabla(ws_i) \quad \text{for } w \in W \text{ and } 2 \leq i \leq d.
\]

These formulae are equivalent, as \( s_is_i = \overline{s}_i = s_{i+1} \) for \( 1 \leq i \leq d - 1 \). To see that they hold true we compute

\[
\nabla(w) - \nabla(ws_i) = \partial(w, s_i)
= \partial(wu^{d-i})
= \partial(wu, s_{i+1})
= \nabla(wu) - \nabla(ws_iu)
\]

and formula (32) follows.

Step 3: For \( w \in W \) we define

\[
\Theta(t_w) = \pi^{\nabla(wu^{d-i} - \nabla(w)u)} \epsilon_w \in K^\times.
\]
Formula \([33]\) together with our assumption on the \(\epsilon_w\) implies that this is well defined, because for \(w, w' \in W\) we have \(t_w = t_{w'}\) if and only if \(w^{-1}w'\) belongs to the subgroup of \(W\) generated by \(s_2, \ldots, s_d\). As \(T/T \cap I\) is freely generated by the \(t_w\) this defines a character \(\Theta : T \to K^\times\) extending \(T \cap I \to T \to \theta \to k^\times \subset K^\times\), as desired. \(\square\)

**Corollary 5.3.** Assume that \(d \leq 2\). If we have \(\epsilon_w = \epsilon_{wsd}\) for all \(2 \leq i \leq d\) then there exists an extension \(\Theta : T \to K^\times\) of \(\theta\) and a function \(\nabla : W \to \mathbb{Z}\) such that we have an isomorphism of \(\mathcal{H}(G, I_0)\)-modules \(L_{\nabla}(\Theta) \otimes_o k \cong M(\theta, \sigma, \epsilon)\).

**Proof:** Choose a function \(\partial : W^{sd} \to [0, r] \cap \mathbb{Z}\) such that

\[
\partial(w) = 0 \text{ if } \sigma(w) = 1, \quad 0 < \partial(w) < r \text{ if } \sigma(w) = 0, \quad \partial(w) = r \text{ if } \sigma(w) = -1.
\]

Extend \(\partial\) to a function \(\partial : W \to [-r, r] \cap \mathbb{Z}\) by setting \(\partial(ws_d) = -\partial(w)\) for \(w \in W^{sd}\). Then, as we assume \(d \leq 2\), properties \([31]\) and \([31]\) are empty resp. fulfilled for trivial reasons. Therefore we conclude with Proposition 5.2. \(\square\)

**References**

[1] A. Björner, F. Brenti, Combinatorics of Coxeter Groups, Graduate Texts in Mathematics 231, Springer, New York (2005)

[2] J. F. Dat, Représentations lisses p-tempérées des groupes p-adiques, Amer. J. Math. 131, 227–255 (2009)

[3] M. Emerton, p-adic L-functions and unitary completions of representations of p-adic reductive groups, Duke Math. J. 130 (2005), no. 2, 353–392

[4] E. Grosse-Klönne, On the universal module of p-adic spherical Hecke algebras, preprint

[5] E. Grosse-Klönne, From pro-p-Iwahori Hecke modules to \((\varphi, \Gamma)\)-modules, preprint

[6] P. Schneider, J. Teitelbaum, Banach-Hecke algebras and p-adic Galois representations, Doc. Math., Extra Vol., 631–684 (2006)

[7] M. F. Vignéras, Algèbres de Hecke affines génériques, Representation Theory 10, 120 (2006)
[8] M. F. Vignéras, A criterion for integral structures and coefficient systems on the tree of $\text{PGL}(2,F)$, Pure Appl. Math. Q. 4, no. 4, Special Issue: In honor of Jean-Pierre Serre. Part 1, 1291–1316 (2008)