Radiative transfer along rays in curved space–times

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ABSTRACT

Radiative transfer in curved space–times has become increasingly important to understanding high-energy astrophysical phenomena and testing general relativity in the strong field limit. The equations of radiative transfer are physically equivalent to the Boltzmann equation, where the latter has the virtue of being covariant. We show that by a judicious choice of the basis of the phase space, it is generally possible to make the momentum derivatives in the Boltzmann equation vanish along an arbitrary (including non-geodesic) path, thus reducing the problem of radiative transfer along a ray to a path integral in coordinate space.

Key words: black hole physics – radiative transfer – relativity.

1 INTRODUCTION

Radiative transfer in general relativistic environments is required to understand many high-energy astrophysical phenomena. Despite this, the approaches utilized either are ad hoc or use complicated transfer functions (see e.g. Cunningham 1975), moment formalisms (see e.g. Thorne 1981) or Monte Carlo methods (see e.g. Janka & Hillebrandt 1989), or the Boltzmann equation is solved in its entirety (see e.g. Lindquist 1966; Liebendörfer et al. 2005, and references therein). Many researchers who are interested in producing images or light curves of compact objects have tacitly assumed that the problem of radiative transfer is reduced to performing an integral along the line of sight (see e.g. Jaroszynski & Kurpiewski 1997; Broderick & Blandford 2004). While being intuitively obvious (radiative transfer is a local process), we have been unable to find an explicit justification for this in the literature. Here we show that, by Fermi–Walker propagating a basis along the ray, the Boltzmann equation may indeed be reduced to a path integral along the ray (as long as scattering is ignored).

2 BOLTZMANN EQUATION

The Boltzmann equation is given by

\[
\frac{dx}{d\tau} \frac{\partial N}{\partial x^\mu} + \frac{dp}{d\tau} \frac{\partial N}{\partial p^\mu} = S(x^\mu, p^\mu, N),
\]

where \(\tau\) is an affine parametrization of the ray, \(p^\mu\) is the tangent to the ray, \(N\) is the distribution function of the particles under consideration, and \(S\) contains the source terms (Lindquist 1966).\(^1\) The particular form of \(S\) and that it is a Lorentz scalar are discussed in Lindquist (1966) and Broderick & Blandford (2004). As shown in Lindquist (1966), equation (1) can be written in terms of an arbitrary tangent-space basis \(\{e^a_\mu\}\) as

\[
\frac{dx}{d\tau} \frac{\partial N}{\partial x^\mu} + \frac{dp}{d\tau} \frac{\partial N}{\partial p^\mu} = S(x^\mu, p^\mu, N),
\]

where \(p_a = p_\mu e^\mu_a\). The momentum terms will generally vanish if \(dp_a/d\tau = 0\), which is true by definition if \(\{e^\mu_a\}\) are Fermi–Walker transported\(^2\) along the ray (see e.g. Misner, Thorne & Wheeler 1973, section 6.5), resulting in

\[
\frac{dx}{d\tau} \frac{\partial N}{\partial x^\mu} = S(x^\mu, p^\mu, N).
\]

If the path is a geodesic, Fermi–Walker transport simply reduces to parallel transport. It is worth noting that the condition \(dp_a/d\tau = 0\) will also be satisfied if \(e^\mu_a\) is a Killing vector.

3 EXAMPLES OF APPLICATIONS

The formulation of radiative transfer as a path integral along the ray in coordinate space (as opposed to in the entire phase space) is especially convenient, e.g. for modelling high-resolution images and light curves of compact objects (see e.g. Broderick & Loeb 2005a,b,c) and ray-casting radiative transfer codes (see e.g. Razoumov & Scott 1999; Sokasian, Abel & Hernquist 2001). In the former, typically a congruence of rays is traced from a distant observer backwards in time towards an emitting region. In the latter an isotropic distribution of rays is traced outward from each grid-point. Using the radiative transfer equation in the form of equation (1) generally requires that the complete distribution function be constructed

\(^1\) In general, it is also possible to write this in terms of the conjugate coordinate–momentum pair \(x^\mu\) and \(p_\mu\), with identical results.

\(^2\) Fermi–Walker transport is defined by \(u^\alpha \nabla_\alpha f^\mu = (u^\alpha a^\mu - a^\mu u^\alpha) f^\mu\), where \(u^\mu\) is the unit tangent vector \((u^\alpha u_\alpha = -1)\) of the ray and \(a^\mu \equiv u^\alpha \nabla_\alpha u^\mu\). If the ray is a geodesic, \(a^\mu = 0\) and this trivially reduces to parallel transport.
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at each point along each ray. In contrast, equation (3) is trivially integrated along the ray at a single observing frequency (which for imaging purposes is usually the situation of interest). This is true even when the rays are not geodesics, e.g. in the presence of strong refraction (see e.g. Broderick & Blandford 2003).

For vacuum photon propagation in the Kerr space–time (that which is most likely to be astrophysically relevant), and in Petrov type D space–times more generally, the transported basis \( \{ e^u_\mu \} \) may be constructed algebraically. This is done by first noting that the tangent vector to the ray is already propagated in parallel by definition, and thus choosing \( e^0_\mu \propto dx^\mu/d\tau \). Two fiducial vectors, \( e^1_\mu \) and \( e^2_\mu \), orthogonal to each other and \( e^0_\mu \), are then chosen at some position along the ray. These may be transported along the ray algebraically by using the orthogonality relations, a normalization condition, and the complex constant first described by Walker & Penrose (1970) (although see Chandrasekhar 1992, as well).3

Finally, the fourth basis-vector is then obtained from the conditions

\[
e^4_\mu e^\nu_\mu = 1, \quad e^4_\mu e^\nu_\mu = e^4_\mu e^4_\mu = 0.
\]

If desired, an orthonormal basis can be created from \( \{ e^\nu_\mu \} \) by making linear combinations of \( e^0_\mu \) and \( e^2_\mu \).

However, for many applications, the explicit construction of \( \{ e^\nu_\mu \} \) is unnecessary. For example, consider the case when the source terms \( S \) depend only upon \( \mathcal{N} \), position \( [x^\nu(\tau)] \) and the momentum of photons moving along the ray \( [p^\mu(\tau) \propto dx^\nu/d\tau] \), i.e. anisotropic scattering into and out of the ray may be neglected. Thus only vector \( e^0_\mu \), as defined above, is required. However, this is presumably already available from the ray construction.

An astrophysical example in which this is the case is self-absorbed synchrotron emission. In this case \( S = j - a\mathcal{N} \) where, for a power-law electron distribution,

\[
j = j_0 n_e \alpha_B \sin \theta \frac{(\omega_B \sin \theta)}{\omega} a^{a+1},
\]

\[
a = a_0 n_e \alpha_B \omega_B \sin \theta \frac{(\omega_B \sin \theta)}{\omega} a^{a+3/2},
\]

where \( j_0 \) and \( a_0 \) are constants, \( \alpha \) is the spectral index, \( n_e \) is the proper electron number density, \( \omega \) and \( \omega_B \) are the photon and cyclotron frequency in the plasma rest frame, respectively, and \( \theta \) is the angle between the magnetic field and the ray in the plasma rest frame. Clearly, along the ray \( n_e \) and \( \omega_B \) are functions of \( x^\nu(\tau) \) only. The photon frequency as measured in the rest frame is given by \( u^\mu g_{\mu\nu} \) (where \( u^\mu \) is the plasma four-velocity) and thus is a function of \( x^\nu(\tau) \) only. Finally, the angle \( \theta \) may be defined by its cosine, which is in turn given by \( \phi^\mu u^\nu F_{\mu\nu}/\omega \), where \( F_{\mu\nu} \) is the dual of the electromagnetic field tensor. Thus, as claimed \( S \) is a function solely of \( \mathcal{N} \), \( x^\nu(\tau) \) and \( p^\mu(\tau) \). This may then be inserted into equation (3), which can now be trivially integrated to yield \( \mathcal{N}(\psi_0^u) \), where \( \psi_0^u \) enters via the initial conditions of the ray (presumably at \( \tau = 0 \)). An image produced by this procedure is shown in Fig. 1, and additional more complex examples can be found elsewhere in the literature (see e.g. Jaroszynski & Kurpiewski 1997; Falcke, Melia & Agol 2000; Broderick & Blandford 2003, 2004; Broderick & Loeb 2005a,b,c).

Figure 1. The intensity map of a stationary, non-thermal, self-absorbed synchrotron-emitting plasma surrounding a non-rotating black hole as seen at infinity for a single frequency. The contrast is linear, ranging from maximum intensity to vanishing intensity (black). The density (\( \propto r^{-3/2} \)) and magnetic field strength (\( \propto r^{-5/4} \)) are spherically symmetric and the latter is assumed to be randomly oriented (i.e. the emission and absorption have been averaged over field orientations). The spectral index of the emitting electrons is taken to be 2. For reference, the white ring shows the size of the horizon.

4 DISCUSSION

The simplification of the Boltzmann equation has a straightforward physical interpretation. Since radiative transfer is a local process, curvature enters only in relating the tangent space at different points of the space–time. For the observer moving along a ray, by definition, this is naturally accounted for by Fermi–Walker transporting a tangent space basis, thus reducing the problem to its flat-space analogue (for which the momentum derivative terms can always be made to vanish).

While it is possible to define the basis \( \{ e^\nu_\mu \} \) on any three-dimensional hypersurface of the space–time, unfortunately, it is not generally possible to produce a well-defined basis in the full space–time as a result of ray intersections. This does not mean, of course, that it is not possible to integrate the Boltzmann equation along any single ray. Rather, the phase-space bases for different rays that intersect will in general produce a different basis at the point of crossing, and thus cannot be compared directly (although they may be rotated).

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3 For the Kerr space–time, in Boyer–Lindquist coordinates the Penrose–Walker constant is given by (e.g. for \( e^4_\mu \)) \( P_{W} = (r - i a \cos \theta) \left( (e^4_\mu e^\nu_\mu - e^\nu_\mu e^4_\mu) + a(e^\nu_\mu e^4_\mu - e^4_\mu e^\nu_\mu) \sin^2 \theta - i(r^2 + a^2)(e^\nu_\mu e^\nu_\mu - e^\nu_\mu e^\nu_\mu) \right) \sin \theta).
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