Counting isolated singularities in germs of applications $\mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0 \ n < p$

V. H. Jorge Pérez*

Abstract

In this paper we give a formula for counting the number of isolated stable singularities of a stable perturbation of corank 1 germs $f : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0$ with $n < p$ that appear in the image $f(\mathbb{C}^n)$. We also define a set of $A$-invariants and show that their finiteness is a necessary and sufficient condition for the $A$-finiteness of the germ $f$.

Key words: Isolated stable singularities, Finite determinacy.

1 Introduction

Let $f : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0$, be a germ of a map with a finite number of isolated stable singularities or zero-dimensional stable types in the discriminant of $f$. For example, if $n = 2, p = 3$, the isolated stable singularities in the hypersurface $f(\mathbb{C}^2)$ are the cross-caps and triple points. The main problem is to find an algebraic formula for counting the number of isolated singularities of $f$.

When $n = 2, p = 3$, D. Mond shows in [10] that the number of cross-cap ($C(f)$) and the number of triple points ($T(f)$) are given by the dimensions of local algebras associated to $f$.

When $n = 2, p = 2$ J. Rieger [12] shows that the number of cusps is given by the dimension of a local algebra associate to $f$ in the case where $f$ is of corank 1. T. Gaffney and D. Mond [2] give formulae for both the number of cusps and the number of double points for a general finitely-determined map-germ $\mathbb{C}^2, 0 \rightarrow \mathbb{C}^2, 0$.

When $n = p$ W. Marar, J. Montaldi and M.Ruas [8] give formulae for calculating the isolated stable singularities associated to $f$ in the case when $f$ is weighted homogeneous and of corank 1.

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In this work we consider the analogous problem for map-germs \( f : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0 \) with \( n < p \) and give a formula for counting the number of all isolated stable singularities. In particular, if \( f \) is weighted homogeneous, we give a formula for these numbers in terms of the weights of the variables and the degrees of each component of \( f \).

We also define a set of \( \mathcal{A} \)-invariants and show that their finiteness is a necessary and sufficient conditions for the \( \mathcal{A} \)-finiteness of the germ \( f \).

## 2 Stable types

Our notation are standard in singularity theory. We denote by \( \mathcal{A} \) the group \( \text{Diff}(\mathbb{C}^n, 0) \times \text{Diff}(\mathbb{C}^p, 0) \); this acts on \( \mathcal{O}(n, p) \) the space of germs \( \mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0 \), by composition on the right and on the left.

A \( d \)-parameter unfolding of a map-germ \( f_0 \in \mathcal{O}(n, p) \) is a germ \( F \in \mathcal{O}(n+d, p+d) \) of the form \( F(x, u) = (f(x, u), u) \), with \( f(x, 0) = f_0 \). A \( c \)-parameter unfolding \( F' \) de \( f_0 \) is induced from a \( d \)-parameter unfolding \( F \) by a germ \( h : \mathbb{C}^c, 0 \rightarrow \mathbb{C}^d, 0 \) if \( (f'(x, v), v) = (f(x, h(v)), v) \). An unfolding \( F \) of \( f_0 \) is \( \mathcal{A} \)-versal if every other unfolding of \( f_0 \) is \( \mathcal{A} \)-equivalent to an unfolding induced from \( F \). An \( \mathcal{A} \)-versal unfolding of \( f_0 \) contains, up to \( \mathcal{A} \)-equivalence, every other unfolding of \( f_0 \).

A map-germ \( f : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0 \) is \( k \)-\( \mathcal{A} \)-determined if, whenever \( g : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0 \) and \( j^k f(0) = j^k g(0) \), \( g \) is \( \mathcal{A} \)-equivalent to \( f \), and \( \mathcal{A} \)-finite if it is \( k \)-\( \mathcal{A} \)-determined for some \( k < \infty \). In this paper we shall refer almost exclusively to \( \mathcal{A} \)-finite. We say \( f \) is a stable germ if every nearby germ is \( \mathcal{A} \)-equivalent to \( f \).

Let \( F \) be versal unfolding of \( f \). We say that a stable type \( \mathcal{Q} \) appears in \( F \) if, for any representative \( F = (id, f_s) \) of \( F \), there exists a point \( (s, y) \in \mathbb{C}^s \times \mathbb{C}^p \) such that the germ \( f_s : \mathbb{C}^n, S \rightarrow \mathbb{C}^p, y \) is a stable germ of type \( \mathcal{Q} \), where \( S = f^{-1}(y) \cap \Sigma(f_s) \). We call \( (s, y) \) and the points \( (s, x) \) with \( x \in S \) points of stable type \( \mathcal{Q} \) in the target and in the source, respectively. If \( f \) is stable, we denote the set of points in \( \mathbb{C}^s \times \mathbb{C}^p \) of type \( \mathcal{Q} \) by \( \mathcal{Q}(f) \) and set \( \mathcal{Q}_S(f) = f^{-1}(\mathcal{Q}(f)) - \mathcal{Q}_\Sigma(f) \), where \( \mathcal{Q}_\Sigma(f) \) denotes \( f^{-1}(\mathcal{Q}(f)) \cap \Sigma(f) \).

If \( f \) is \( \mathcal{A} \)-finite, we denote \( \overline{\mathcal{Q}(f)} = (\{0\} \times \mathbb{C}^p) \cap \overline{\mathcal{Q}(F)} \) and \( \overline{\mathcal{Q}_S(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_S(F)} \).

We say that \( \mathcal{Q} \) is a zero-dimensional stable type or isolated stable singularity for the pair \( (n, p) \) if \( \mathcal{Q}(f) \) has dimension 0 where \( f \) is a representative of the stable type \( \mathcal{Q} \). We observe that \( \mathcal{Q}(F) \) is a close analytic set since \( \overline{\mathcal{Q}(F)} = \cap F((p^{p+1})F^{-1}(\overline{\mathcal{A}z_i})) \), where \( z_i \) is the \( p+1 \) jet of the stable type \( \mathcal{Q} \) and \( \mathcal{A}z_i \) is the \( \mathcal{A} \)-orbit of \( z_i \).
An $\mathcal{A}$-finite germ $f$ has a discrete stable type if there exist a versal unfolding $F$ of $f$ in which only a finite number of stable types appear. An $\mathcal{A}$-finite germ $f$ has a discrete stable type if the pair $(n, p)$ is in the nice dimensions (\cite{[7]})

In the next sections we define explicitly the zero-dimensional stable types or stable isolated singularities, that is, we define the ideal that defines each zero dimensional stable type of $f$.

3 Stable isolated singularities

Consider a versal unfolding $F$ of $f$ with base $\mathbb{C}^d, 0$,

$$F : \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^d, 0 \to \mathbb{C}^p \times \mathbb{C}^d, 0$$

$$F(x, y, u) = (x, h_1(x, y, u), ..., h_{p-n+1}(x, y, u), u)$$

For each $f$ we associate a partition $\mathcal{P} = (r_1, ..., r_\ell)$ of $k$ where $p - k(p - n + 1) + \ell = 0$ and consider $\tilde{V}(\mathcal{P}) \subset \mathbb{C}^{n-1} \times \mathbb{C}^\ell \times \mathbb{C}^d, 0$ defined by

$$\tilde{V}(\mathcal{P}) = \text{clos} \left\{ (x, \mathbf{z}, u) \in \mathbb{C}^{n-1} \times \mathbb{C}^\ell \times \mathbb{C}^d \mid \begin{array}{ll}
\mathbf{z} = (z_1, \cdots, z_\ell), & z_i \neq z_j \\
F(x, z_i, u) = F(x, z_j, u) \\
F_u \text{ has a } \mathcal{Q} \text{ 0-dimensinal} \\
\text{stable singularity at}(x, z_j)
\end{array} \right\}$$

where 'clos' means the analytic closure in $\mathbb{C}^{n-1} \times \mathbb{C}^\ell \times \mathbb{C}^d$. The varieties $\tilde{V}(\mathcal{P})$ are called zero-schemes and are related to the 0-stable types (see \cite{[3]}).

Let $\pi_\mathcal{P} : \tilde{V}(\mathcal{P}) \to \mathbb{C}^d$ be the restriction to $\tilde{V}(\mathcal{P})$ of the cartesian projection $\mathbb{C}^{n-1} \times \mathbb{C}^\ell \times \mathbb{C}^d \to \mathbb{C}^d$. For a generic $u \in \mathbb{C}^d$, the fiber $\pi_\mathcal{P}^{-1}(u)$ consists of the multiple points where $F_u$ has a $\mathcal{Q}(f, \mathcal{P})$ multi-germ that defines a zero dimensional stable type. We are thus interested in the degree of $\pi_\mathcal{P}$.

Let $\mathcal{P} = (r_1, \cdots, r_\ell)$ be a partition of $m \leq n$ with $r_1 \geq r_2 \geq \cdots r_\ell \geq 1$. Define $N(\mathcal{P})$ to be the order of the sub group of $S_\ell$ which fixes $\mathcal{P}$. Here $S_\ell$ acts on $\mathbb{R}^\ell$ by permuting the coordinates. For example, if $\mathcal{P} = (4, 4, 4, 2, 2, 1, 1)$ we have $N(\mathcal{P}) = (3!)^2 2!$.

**Proposition 3.1.** If $\mathcal{P} = (r_1, ..., r_\ell)$ is a partition of $k$, then

$$\sharp \mathcal{Q}(f, \mathcal{P}) = \frac{1}{N(\mathcal{P})} \deg(\pi_\mathcal{P}).$$
Proof Let \( y = (x, y_1, ..., y_\ell, u) \in \tilde{V}(\mathcal{P}) \) and \( \sigma \in S_\ell \). We have 
\[ y_\sigma = (y_{\sigma(1)}, ..., y_{\sigma(\ell)}, u) \in \tilde{V}(\mathcal{P}) \]
if and only if \( r_{\sigma(j)} = r_j \) for each \( j = 1, ..., \ell \). There are \( N(\mathcal{P}) \) such \( \sigma \). The points \( y \) and \( y_\sigma \) are distinct, but the corresponding sets \( \{x, y_1, ..., y_\ell\} \) are the same, and it is the contribution of the multiple points that are counted in \( \sharp \mathbb{Q}(f, \mathcal{P}) \).

Let \( I(\tilde{f}, \mathcal{P}) \) be the ideal in \( \mathcal{O}_n^{-1+\ell+d} \) that defines \( \tilde{V}(\mathcal{P}) \), and let 
\[ I(f, \mathcal{P}) = \frac{(I(\tilde{f}, \mathcal{P}) + <u_1, ..., u_d>)}{<u_1, ..., u_d>} \subset \mathcal{O}_n^{-1+\ell} \]
be the ideal corresponding to the intersection of \( \tilde{V}(\mathcal{P}) \) with \( \mathbb{C}^{n-1+\ell} \times \{0\} \).

It follows from the definition of \( I(\tilde{f}, \mathcal{P}) \) that, at generic points of \( \tilde{V}(\mathcal{P}) \) with \( z_i \neq z_j \), \( I(\tilde{f}, \mathcal{P}) = ((\{\frac{\partial f}{\partial z_i} \circ \pi_i(\mathcal{P})\} | j = 1, ..., p-n+1, 1 \leq s \leq r_i-1, 1 \leq i \leq m\}) + ((\{\tilde{f}_j \circ \pi_i(\mathcal{P}) - \tilde{f}_j \circ \pi_i(\mathcal{P})\} | j = 1, ..., p-n+1, 2 \leq i \leq m\}) \) in \( \mathcal{O}_n^{-1+\ell+d} \), \((x, z)\), where \( \pi_i(\mathcal{P}) : \mathbb{C}^{n-1+\ell} \to \mathbb{C}^{n}, 1 \leq i \leq m \), are given by \( \pi_i(\mathcal{P})(x, z_1, ..., z_\ell) = (x, z_{r_1} + ... + z_{r_i}+1) \). In particular, at generic point of \( \tilde{V}(\mathcal{P}) \), we have 
\[ \frac{\mathcal{O}_n^{-1+\ell+d}}{(m_d, I(f, \mathcal{P}))} \cong \frac{\mathcal{O}_n^{-1+\ell}}{I(f, \mathcal{P})} \]
where \( m_d \) is the maximal ideal of \( \mathbb{C}^d \).

Proposition 3.2. Suppose that \( \tilde{V}(\mathcal{P}) \) is non-empty. Then

1. \( \tilde{V}(\mathcal{P}) \) is smooth of dimension \( d \);
2. \( \pi_\mathcal{P} : \tilde{V}(\mathcal{P}) \to \mathbb{C}^d \) is finite and \( \pi_\mathcal{P}^{-1}(\pi_\mathcal{P}(0)) = \{0\} \);
3. The degree of \( \pi_\mathcal{P} \) coincides with \( \dim_{\mathbb{C}} \frac{\mathcal{O}_n^{-1+\ell}}{I(f, \mathcal{P})} \).

Proof 1. Since \( F \) is a versal unfolding, it is stable, and the proof follows by Propositions 2.13 in \([\text{II}]\).

2. The projection \( \pi_\mathcal{P} : \tilde{V}(\mathcal{P}) \to \mathbb{C}^d \) is finite. In fact, for \( u \in \mathbb{C}^d \) generic, the fiber \( \pi_\mathcal{P}^{-1}(u) \) is finite and consists of those multi-points where \( f_u \) has a \( \mathbb{Q}(f_u, \mathcal{P}) \) multi germ. The germ \( f_0 = f \) is \( \mathcal{A} \)-finite, so using the geometric criterion of Mather-Gaffney (\([\text{II}], [\text{II}3]\)), it is stable away from zero. Thus \( \pi_\mathcal{P}^{-1}(\pi_\mathcal{P}(0)) = \{0\} \).

3. Since \( \tilde{V}(\mathcal{P}) \) is smooth and it is Cohen-Macaulay at zero, the degree of \( \pi_\mathcal{P} \) coincides with \( \dim_{\mathbb{C}} \frac{\mathcal{O}_n^{-1+\ell}}{I(f, \mathcal{P})} \) (see Proposition 5.12 in \([\text{II}]\)). \( \square \)
Propositions 3.1 and 3.2 (3) give a formula for computing the multiplicities of \( \mathcal{Q}(f, \mathcal{P}) \) even in the case when \( f \) is not weighted homogeneous. We have

\[
\sharp \mathcal{Q}(f, \mathcal{P}) = \frac{1}{\dim \mathcal{P}} \dim_c \frac{\mathcal{O}_{n-1+\ell}}{I(f, \mathcal{P})}.
\]

The \( \dim_c \frac{\mathcal{O}_{n-1+\ell}}{I(f, \mathcal{P})} \) is not difficult to calculate, when \( I(f, \mathcal{P}) \) can be computed. The calculation can be done using computer algebra package such as Singular [5] or Macaulay.

**Example 3.3.** Let \( f : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0 \) be the \( \mathcal{A} \)-finite germ given by

\[
f(x, y) = (x, x^2 y + y^6 + y^7, xy^2 + y^4 + y^6 + y^9).
\]

We choose the partition \( \mathcal{P} \) of \( k \) such that \( k = \frac{3+\ell}{2} \in \mathbb{Z}^+ \). Then \( \mathcal{P} \) has two elements \((1, 1, 1)\) and \((2)\). We have

\[
I(f, (2)) = (x^2 + 6y^5 + 7y^6, 2xy + 4y^3 + 6y^5 + 9y^8) \quad \text{and} \quad I(f, (1, 1, 1)) = (-5z_1^6 + x^2 - 4z_1^2 z_3^2 - 4z_1^4 z_3^4 - 2z_1 z_3^3 - 8z_1^2 z_3^3 - 8z_1^4 z_3^3 - 4z_1^5 - 6z_1^3 z_3^2 - 2z_1 z_3^3 - 10z_3 z_1^5 + 3z_1^2 z_3^2 + 3z_1^4 z_3^4 + 2z_1 z_3^3 + 5z^4 z_3 + 2z_1 z_3^5 + 6z_1^3 z_3^3 + 4z_1^5 z_3^3 + z_3^3 + 5z_1^4 + z_3^3 - 2z_1^4 + 3z_1 - 7z_1^3 - 6z_1^5 z_3^3 - 10z_3^2 z_1^5 - 8z_1^3 z_3^4 - 4z_1^3 z_3^3 - 2z_1 z_3^3 - 8z_1^4 z_3^3 - 4z_1^5 z_3^3 - 2z_1 z_3^2 - 14z_3 z_1^4 - 4z_1^3 z_3^2 - 4z_1^5 z_3^2 - 12z_3 z_1^5 z_3 + x + 8z_1^5 + 4z_1^3 + 4z_1^5 + 3z_1^3 + z_3^2 + 2z_1 z_3^6 + 7z_3 z_1^6 + 2z_1 z_3^6 + 4z_3 z_3^3 + 6z_3 z_3^3 + 2z_1 z_3 + 3z_1^3 + z_3^3 + 5z_3 z_3^4 + z_3^3 + 3z_1^4 + 5z_1^3 + z_3^4).
\]

Using Theorem 3.2 and Singular we have \( \sharp \mathcal{Q}(f, (2)) = 6 \) and \( \sharp \mathcal{Q}(f, (1, 1, 1)) = 14 \).

## 4 Multiple points

We need the following definition. Given a continuous mapping \( f : X \to Y \) between analytic spaces, we define the \( k^{th} \) multiple point space of \( f \) as

\[
D_k(f) = \text{clos}\{(x_1, x_2, ..., x_k) \in X^k : f(x_1) = ... = f(x_k) \text{ for } x_i \neq x_j, i \neq j\}.
\]

Suppose \( f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0 \) is of corank 1 and is given in the form

\[
f(x_1, ..., x_{n-1}, z) = (x_1, ..., x_{n-1}, h_1(x, z), ..., h_{p-n+1}(x, z)).
\]

If \( g : \mathbb{C}^n, 0 \to \mathbb{C}, 0 \) is an analytic function then we define \( V_k^i(g) : \mathbb{C}^{n+k-1}, 0 \to \mathbb{C}, 0 \) to be

\[
\begin{pmatrix}
1 & z_1 & \cdots & z_1^{i-1} & g(x, z_1) & z_1^{i+1} & \cdots & z_1^{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & z_k & \cdots & z_k^{i-1} & g(x, z_k) & z_k^{i+1} & \cdots & z_k^{k-1}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & z_1 & \cdots & z_1^{i-1} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & z_k & \cdots & z_k^{i-1}
\end{pmatrix}
\]
Theorem 4.1. \((\mathbb{R})\) \(D^k(f, \mathcal{P}) = D^k(f)\) is defined in \(\mathbb{C}^{n+k-1}\) by the ideal \(\mathcal{I}^k(f)\) generated by \(V^i(\pi_j(x, z))\) for all \(i = 1, \ldots, k-1\), \(\mathcal{P} = (1, \ldots, 1)\)-\(k\) times and \(j = 1, \ldots, p-n+1\).

In what follows we take coordinates in \(\mathbb{C}^{n+k-1} = \mathbb{C}^{n-1} \times \mathbb{C}^k\) to be \((x, z) = (x_1, \ldots, x_{n-1}, z_1, \ldots, z_k)\).

Example 4.2. For a corank 1 map-germ \(f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0\), \(D^2(f)\) is defined by \(\frac{h_i(x, z_1) - h_i(x, z_2)}{z_1 - z_2}\), \(i = 1, \ldots, p-n+1\).

Definition 4.3. Let \(\mathcal{P} = (r_1, r_2, \ldots, r_\ell)\) be a partition of \(k\) where \(p-k(p-n+1) + \ell = 0\), that is, \(r_1 + r_2 + \ldots + r_\ell = k\). Let \(\mathcal{I}(\mathcal{P})\) be the ideal in \(\mathcal{O}_{n-1+k}\) generated by the \(k-\ell\) elements \(z_i - z_{i+1}\) for \(r_1 + r_2 + \ldots + r_j - 1, 1 \leq j \leq \ell\), and let \(\Delta(\mathcal{P}) = V(\mathcal{I}(\mathcal{P}))\). Define

\[
\mathcal{J}_\Delta(f, \mathcal{P}) = \mathcal{I}^k(f) + \mathcal{I}(\mathcal{P}) \text{ and } D^\ell(f, \mathcal{P}) = V(\mathcal{J}_\Delta(f, \mathcal{P})),
\]

equipped with the sheaf structure in \(\mathcal{O}_{n-1+k}/\mathcal{J}_\Delta(f, \mathcal{P})\).

Example 4.4. For an \(A\)-finite of corank 1 map germ \(f \in \mathcal{O}(2, 3)\), the stable types of \(f(\mathbb{C}^2)\) are \(D^2_1(f, (1, 1)) = D^2(f)\) (the set of double points), \(f(D^1_1(f, (2)))\) (the set of cross-cap points) and \(D^3(f, (1, 1, 1)) = D^3(f)\) (the set of triple points).

The geometric significance of \(D^\ell(f, \mathcal{P})\) is given in Lemma 2.7 [6] by W. Marar and D. Mond. Given a partition \(\mathcal{P} = (r_1, \ldots, r_\ell)\) of \(k\), define the projections \(\pi_i(\mathcal{P}) : \mathbb{C}^{n-1+k} \to \mathbb{C}^n\), for \(1 \leq i \leq m\), by \(\pi_i(\mathcal{P})(x, z_1, \ldots, z_k) = (x, z_{r_1+\ldots+r_{i-1}+1})\).

Lemma 4.5. \((\mathbb{R})\) Let \(\mathcal{P} = (r_1, \ldots, r_\ell)\) be a partition of \(k\). At a generic point \((x, z)\) of \(\Delta(\mathcal{P})\) we have

\[
\mathcal{J}_\Delta(f, \mathcal{P}) = \mathcal{I}(\mathcal{P}) + \left\{\frac{\partial^s f_j}{\partial z_i^s} \circ \pi_i(\mathcal{P}) | j = 1, \ldots, p-n+1, 1 \leq s \leq r_i - 1, 1 \leq i \leq \ell\right\} + \left\{f_j \circ \pi_1(\mathcal{P}) - f_j \circ \pi_i(\mathcal{P}) | j = 1, \ldots, p-n+1, 2 \leq i \leq \ell\right\},
\]

in \(\mathcal{O}_{n-1+k}(x, z)\).

In view of Lemma 4.3, a generic point of \(V(\mathcal{J}_\Delta(f, \mathcal{P}))\) is of the form \((x, z_1, \ldots, z_1, z_2, \ldots, z_2, \ldots, z_\ell, \ldots, z_\ell)\) with \(x \in \mathbb{C}^{n-1}\), \(z^i \in \mathbb{C}, z^i\) iterated \(r_i + 1\) times, \(z^i \neq z^j\) for \(i \neq j\), \(f(x, z^1) = \cdots = f(x, z^\ell)\). The local algebra of \(f\) at \((x, z^i)\) is isomorphic to \(\mathbb{C}[z^i]/(z^i)^{r_i+1}\).

In Corollary 2.15 in [6] is obtained the following result. If \(f\) is \(A\)-finite then for each partition \(\mathcal{P} = (r_1, \ldots, r_\ell)\) of \(k\) satisfying \(p-k(p-n+1) + \ell \geq 0\), the germ of \(D^k(f, \mathcal{P})\) at 0 is either an ICIS of dimension \(p-k(p-n+1) + \ell\) or is empty.
Proposition 4.6. Let \( F = (u, f) \) be a versal unfolding of an \( A \)-finite germ of corank 1. Then for each partition \( P \) of \( k \) where \( p - k(p - n + 1) + \ell = 0 \), we have the following.

1. \( D^\ell(F, P) \) is smooth of dimension \( s \) or is empty.
2. \( J_\Delta(f, P) \) is an ICIS.
3. Let \( j_\ell : C^{n-1} \times C^\ell \to C^{n-1} \times C^k \) be the embedding with image \( \Delta(P) \). Then the surjection \( j_\ell^* : O_{n-1+k} \to O_{n-1+\ell} \) satisfies \( j_\ell^*(J_\Delta(f, P)) = I(f, P) \) and consequently induces an isomorphism

\[
j_\ell^* : \frac{O_{n-1+k}}{J_\Delta(f, P)} \to \frac{O_{n-1+\ell}}{I(f, P)}.
\]

Proof The items 1 and 2 follow from \([6]\).

3. It follows from Lemma 2.7 \([6]\) (see also Lemma 4.5 above) that at generic points of \( \Delta(P) \) one has,

\[
J_\Delta(f, P) = I(P) + \{ \frac{\partial f_j}{\partial z_s} \circ \pi_i(P) | j = 1, \ldots, p - n + 1, 1 \leq s \leq r_i - 1, 1 \leq i \leq \ell \} + \{ f_j \circ \pi_1(P) - f_j \circ \pi_i(P) | j = 1, \ldots, p - n + 1, 2 \leq i \leq \ell \}.
\]

So generically \( j_\ell^*(J_\Delta(f, P)) = I(f, P) \) and as the two are reduced complete intersection ideals coincide we have,

\[
\frac{O_{n-1+k}}{J_\Delta(f, P)} \cong \frac{O_{n-1+\ell}}{I(f, P)}.
\]

\( \square \)

5 The weighted homogeneous case

In what follow we consider weighted homogeneous germs \( f : C^n, 0 \to C^p, 0 \). and write \( f = (f_1, f_2, \ldots, f_p) \) The germ \( f \) is weighted homogeneous if there exist positive integers \( w_1, w_2, \ldots, w_n \), (the weights) and positive integers \( d_1, d_2, \ldots, d_p \) (the degrees) such that for each \( f_i \) we have

\[
f_i(t^{w_1}x_1, \ldots, t^{w_n}x_n) = t^{d_i} f_i(x_1, \ldots, x_n),
\]

or equivalently

\[
\sum_{j=1}^n w_j \alpha_j = d_i,
\]

for each monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) of \( f_i \).

We give below a formula for calculating the number of isolate singularities in the case of a weighted homogeneous germ of corank 1.
Theorem 5.1. Let $f = (x, f_1, \ldots, f_{p-n+1}) : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be an $\mathcal{A}$–finite germ de
corank 1, with weighted $w_i$ and degree $d_i$ of $f_i$. Then

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n-1+\ell}} = \frac{1}{w^\ell w_n} \prod_{j=1}^{p-n+1} \prod_{i=1}^{r_i-1} (d_j - iw_n) \prod_{m=2}^{\ell} \prod_{j=1}^{p-n+1} (d_j - mw_n)$$

where $w = w_1, w_2, \ldots, w_{n-1}$.

Proof According to Proposition 4.6 (3) and Lemma 4.5 it is enough to compute

$$\dim_{\mathcal{J}_k(f, \mathcal{P})}$$

using Bezout’s theorem since $f$ is weighted homogeneous, this dimension can be computed

The generators of $\mathcal{J}_k(f, \mathcal{P})$ are $h_{ij} = \frac{\partial f_j}{\partial x_i} \circ \pi_i(\mathcal{P})$, $z_i - z_{i+1}$ and $g_{ij} = f_j \circ \pi_1(\mathcal{P}) - f_j \circ \pi_1(\mathcal{P})$ for each $j = 1, \ldots, n-p+1, i = 1, \ldots, \ell$ and $s = 1, \ldots, r_i - 1$. The degree of $h_{ij}$ is $d_j - sw_n$, the degree of $z_i - z_{i+1} = w_n$ for all $i$ and the degree of $g_{ij}$ is $d_j - mw_n$, where $m = 2, \ldots, \ell$. The product of all the degrees of the generators is therefore

$$\prod_{j=1}^{n-p+1} \prod_{i=1}^{r_i-1} (d_j - iw_n) \prod_{m=2}^{\ell} \prod_{j=1}^{p-n+1} (d_j - mw_n).w_n^{k-\ell}.$$

Since $\mathcal{J}_k(f, \mathcal{P})$ is a weighted homogeneous complete intersection, we can apply

Bezout’s Theorem, hence its colength is given by

$$\frac{1}{w^k \cdot w} \prod_{j=1}^{p-n+1} \prod_{i=1}^{r_i-1} (d_j - iw_n) \prod_{m=2}^{\ell} \prod_{j=1}^{p-n+1} (d_j - mw_n).w_n^{k-\ell}$$

as required.

Example 5.2. Let $f : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ be an $\mathcal{A}$–finite germ given by $f(x, y) =
(x, xy + y^3, y^4)$ (see [10]). Then the partition $\mathcal{P}$ is $(1, 1, 1)$ and $(2)$ and the weights and
degrees are $d_1 = 3, d_2 = 4$ and $w_1 = 2, w_2 = 1$. Therefore $\sharp \mathcal{Q}(f, (1, 1, 1)) = 1,$
$\sharp \mathcal{Q}(f, (2)) = 2$, that is $f$ has 1 triple point and 2 cross caps (see [10]).

Let $f : \mathbb{C}^4, 0 \to \mathbb{C}^4, 0$ be an $\mathcal{A}$–finite germ given by $f(x, y, z) = (x, y, yz + z^4, xz + z^3)$. The partition $\mathcal{P}$ is $(1, 1, 1)$ and $(2, 1)$ and the weights and degrees are $d_1 = 4, d_2 = 3$ and $w_1 = 2, w_2 = 3, w_3 = 1$. Therefore

$$\sharp \mathcal{Q}(f, (2, 1)) = \frac{(4 - 3)(3 - 1)(4 - 1)(3 - 1)}{2.3.1^2} = 2$$

and $\sharp \mathcal{Q}(f, (1, 1, 1, 1)) = 0$, that is $f$ has 2 singularities of type $\mathcal{Q}(f, (2, 1))$ and
has not quadruple points. In particular applying Theorem 5.1 or the method of
Example 3.3 all corank-1 simple germs \( f : \mathbb{C}^3, 0 \to \mathbb{C}^4, 0 \) classified by Houston and Kirk in \([9]\) satisfy

\[ \sharp \mathcal{Q}(f, (1, 1, 1, 1)) = 0. \]

Observe that if \( \mathcal{P} = (2, 1) \) and \( f : \mathbb{C}^3, 0 \to \mathbb{C}^4, 0, V(\mathcal{J}_\Delta(f, (2, 1))) \subset \mathbb{C}^5 \) and \( V(I(f, (2, 1))) \subset \mathbb{C}^4 \) are isomorphic. Hence \( \mu(V(\mathcal{J}_\Delta(f, (2, 1)))) = \mu(V(I^3(f) + \mathcal{I}(2, 1))) = \mu(D^3(f)|H) \) where \( H = V(\mathcal{I}(2, 1)) \), therefore as \( V(\mathcal{J}_\Delta(f, (2, 1))) \) is ICIS zero-dimensional, we have

\[ \mu(D^3(f)|H) = \dim_\mathbb{C} \frac{\mathcal{O}_5}{(I^3(f) + \mathcal{I}(2, 1))} - 1 = \dim_\mathbb{C} \frac{\mathcal{O}_4}{I(f, (2, 1))} - 1 = \sharp \mathcal{Q}(f, (2, 1)) - 1, \]

as Houston and Kirk compute \( \mu(D^3(f)|H) \), we calculate \( \sharp \mathcal{Q}(f, (2, 1)) \) in the table below.

| Label | Singularity                          | \( \mu(D^3(f)|H) \) | \( \sharp \mathcal{Q}(f, (2, 1)) \) |
|-------|--------------------------------------|----------------------|----------------------------------|
| \( P_1 \) | \((x, y, yz + z^4, xz + z^3)\) | 1                    | 2                                |
| \( P_2 \) | \((x, y, yz + z^5, xz + z^3)\) | 2                    | 3                                |
| \( P_3^k \) | \((x, y, yz + z^6 + z^{3k+2}, xz + z^3)\) | 3                    | 4                                |
| \( P_4^1 \) | \((x, y, yz + z^7 + z^8, xz + z^3)\) | 4                    | 5                                |
| \( P_4 \) | \((x, y, yz + z^7, xz + z^3)\) | 4                    | 5                                |
| \( Q_k \) | \((x, y, xz + y^2 z^2, y^k z + z^3)\) | 1                    | 2                                |
| \( R_k \) | \((x, y, xz + z^3, yz^2 + z^4 + z^{2k-1})\) | 2                    | 3                                |
| \( S_{j,k} \) | \((x, y, xz + y^j z^2 + z^{3j+2}, z^3 + y^k z)\) | 3                    | 4                                |

6 Necessary and sufficient conditions for \( \mathcal{A} \)-finiteness of map-germs

In this section we define new numerical invariants associated to each partition \( \mathcal{P} \) of \( k \) with \( p - k(p - n + 1) + \ell \geq 0 \). We show that a germ \( f \) is \( \mathcal{A} \)-finite if and only if these invariants are finite.

For any partition \( \mathcal{P} \) of \( k \), \( p - k(p - n + 1) + \ell \geq 0 \), denote by \( H_{\mathcal{P}} \) the map defined by

\[ H_{\mathcal{P}} : \mathbb{C}^{n-1+\ell} \to \mathbb{C}^{(p-n+1)(k-\ell)} \]

with components \( \frac{\partial f_j}{\partial z_s} \circ \pi_i(\mathcal{P}) \) for \( j = 1, \ldots, p - n + 1, 1 \leq s \leq r_i - 1, 1 \leq i \leq \ell \) and

\[ G_{\mathcal{P}} : \mathbb{C}^{n-1+\ell} \to \mathbb{C}^{(p-n+1)(\ell-1)} \]

with components \( f_j \circ \pi_1(\mathcal{P}) - f_j \circ \pi_i(\mathcal{P}) \) for \( j = 1, \ldots, p - n + 1, 2 \leq i \leq \ell \).
Let 

$$F_P = (G_P, H_P) : \mathbb{C}^{n-1+\ell} \to \mathbb{C}^{(p-n+1)(k-1)},$$

and define the following number

$$N(f, \mathcal{P}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{F_P^*(\mathcal{M}_{(p-n+1)(k-1)}) + J(F_P)}$$

where $J(F_P)$ denotes the ideal of $(p-n+1)(k-1) \times (p-n+1)(k-1)$ minors and $\mathcal{M}_{(p-n+1)(k-1)}$ is the maximal ideal in $\mathcal{O}_{(p-n+1)(k-1)}$.

We first show that $N(f, \mathcal{P})$ is $\mathcal{A}$-invariant.

**Proposition 6.1.** Let $f_1$ and $f_2$ be corank 1, $\mathcal{A}$-finite and $\mathcal{A}$-equivalent maps germs. Then $N(f_1, \mathcal{P}) = N(f_2, \mathcal{P})$.

**Proof** It is sufficient to prove that if $f_1$ is $\mathcal{A}$-equivalent to $f_2$, then the germs defined above, $F_{1P}$ associated to $f_1$ and $F_{2P}$ associated to $f_2$, are $\mathcal{K}$-equivalent. The last statement follows from [10].

**Theorem 6.2.** Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0, n < p$ be a corank 1 germ. Then following statements are equivalent:

1. $f$ is an $\mathcal{A}$-finite.

2. $N(f, \mathcal{P}) < \infty$, where $\mathcal{P}$ is $(1, 1, \ldots, 1, 1)$-times and $\sharp \mathcal{Q}(f, \mathcal{P}) < \infty$ for each partition $\mathcal{P}$ of $k$ where $p - k(p - n + 1) + \ell \geq 0$.

**Proof** (1) $\implies$ (2) $f$ is $\mathcal{A}$-finite if and only if for any representative of $f$ there exist neighborhoods $U$ of 0 in $\mathbb{C}^n$ and $V$ of 0 in $\mathbb{C}^p$, with $f(U) \subset V$, such that for all $y \neq 0$ in $V$, the multi germ $f : (U, f^{-1}(y) \cap \Sigma(f)) \to (V, y)$ is stable. If follows from Proposition 2.13 in [6] that if $f$ is of type $\Sigma^1_{r, 0}$ in $(x, y_i)$, with $y_i \in f^{-1}(y) \cap \Sigma(f)$ then $F_P$ defining $V(J_{\Delta}(f, \mathcal{P}))$ is a submersion at $(x, y_1, \ldots, y_k)$. Then at every point of $V(J_{\Delta}(f, \mathcal{P}))$, distinct from 0, the $(p-n+1)(k-1)$ functions generating $J_{\Delta}(f, \mathcal{P})$ define a submersion, and so $V(J_{\Delta}(f, \mathcal{P}))$ is ICIS. Therefore $V(F_P^*(\mathcal{M}_{(p-n+1)(k-1)}) + J(F_P))$, for each partition $k$ where $p - k(p - n + 1) + \ell \geq 0$, is zero in $\mathbb{C}^{n-1+\ell}$. Hence $V(F_P^*(\mathcal{M}_{(p-n+1)(k-1)}) + J(F_P)) \subset \{0\}$ and by Nullstellensatz we have $N(f, \mathcal{P}) < \infty$.

(1) $\iff$ (2) if $N(f, \mathcal{P}) < \infty$ we have $V(F_P^*(\mathcal{M}_{(p-n+1)(k-1)}) + J(F_P)) \subset \{0\}$, that is $V(F_P^*(\mathcal{M}_{(p-n+1)(k-1)}))$ have isolated singularities in $\{0\}$ for each partition $\mathcal{P}$, in particular for $(1, 1, \ldots, 1, 1)-k - 1$ times and for all partition $\mathcal{P}$ of $k$ where, $p - k(p - n + 1) + \ell = 0$. We choose representative $f : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^p,$
such that the representatives induced by the germs of $F_{\mathcal{P}}^{-1}(0)$ are differentiable. With this we have the multi germ $f : (U, S) \to (V, z)$, where $S = f^{-1}(z)$, which is stable for $z \neq 0$. The result follows then by the geometric criterion of Mather and Gaffney [4]. 

Example 6.3. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ be given by

$$f(x, y, z) = (x, y, xz + z^3, yz^2 + g(z))$$

where $g(z) = \Sigma_{i=1}^{k} a_i z^i$. Then $f$ is $A$-finite for all $a_i$. The partitions $\mathcal{P}$ are $(1, 1, 1, 1), (1, 1, 1), (1, 1), (2, 1)$ and (2) and the stable types of $f$ are: $D^1(f, (1, 1, 1, 1))$ which is empty, the triple points curve $D^3(f, (1, 1, 1))$, the types 2-dimensional $D^2(f, (1, 1))$, the types 1-dimensional $D^1(f, (2))$ and the zero-dimensional types $D^2(f, (2, 1))$. We compute the maps $F_{\mathcal{P}}$ for all partitions $\mathcal{P}$ using Maple.

If $\mathcal{P} = (1, 1, 1)$, then $F_{\mathcal{P}}(x, y, z, v, w) = (x - (zw + zv + vw), zv + w, V_1^3(g), y + V_2^3(g))$ where $V_i^k(g)$ is computed using the Definition given in the section [4]

If $\mathcal{P} = (1, 1)$, then $F_{\mathcal{P}}(x, y, z, v) = (x + z^2 + zv + v^2, y(z + v) + V_1^2(g))$.

Using the software Singular [5] we have

$$N(f, (1, 1, 1)) = \dim_{C}\frac{\mathcal{O}_{\mathbb{C}^3}(\{x-(zw+zw+vw), z+v+w, V_1^3(g), y+V_2^3(g), F(P)\})}{0 \text{ if } a_1 = a_2 = 0 \quad 2 \text{ if } \ell(g) = 3 \quad 5 \text{ if } \ell(g) = 4 \quad 20 \text{ if } \ell(g) = 5 \quad 17 \text{ if } \ell(g) = 6 \quad 26 \text{ if } \ell(g) = 7}$$

where $\ell(g)$ denote the lowest degree in $g$.

$$N(f, (1, 1)) = \begin{cases} \ell(g) - 1 & \text{if } \ell(g) \text{ is odd} \\ \ell(g) & \text{if } \ell(g) \text{ is even.} \end{cases}$$

The ideal that defines $D^2(f, (2, 1))$ is given by $\mathcal{I}^3(f) + \mathcal{I}(\mathcal{P})$ where $\mathcal{I}^3(f)$ defines $D^3(f) = D^3(f, (1, 1, 1))$ and $\mathcal{I}(\mathcal{P}) = v - w$. Then $D^2(f, (2, 1)) = D^3(f, (1, 1, 1)) \cap H$, and $D^2(f, (2, 1)) = V(x - (zv + zw + vw), z + v + w, V_1^3(g), y + V_2^3(g), v - w) \subset \mathbb{C}^2 \times \mathbb{C}^3$. If $\mathcal{P} = (2, 1)$, we have $F_{\mathcal{P}}(x, y, z, v) = (x - (zv + zv + v^2), z + 2v, V_1^3(g)(z, v), y + V_2^3(g)(z, v))$, then $N(f, (2, 1)) < \infty$.

The ideal $\mathcal{I}^2(f) + \mathcal{I}(\mathcal{P})$ defines $D^1(f, (2))$ where $\mathcal{I}^2(f)$ defines $D^2(f) = D^2(f, (1, 1))$ and $\mathcal{I}(\mathcal{P}) = v - z$. We have $D^1(f, (2)) = D^2(f, (1, 1)) \cap H$, where $H = V(v - w) \subset \mathbb{C}^2 \times \mathbb{C}^2$ is a hyperplane. Therefore $D^1(f, (2)) = V(x + z^2 + zv + v^2, y(z + v) + \ldots$
\[ V^2_1(g), v - z \subset \mathbb{C}^2 \times \mathbb{C}^2. \] If \( \mathcal{P} = (2) \) we have \( F_\mathcal{P}(x, y, z) = (x + 3z^2, 2zy + V^2_1(g)(z, z)) \), and therefore \( N(f, (2)) < \infty \) for all \( a_i \).

**Remark 6.4.** Observe that if \( g(z) = z^2 + z^7 \) and \( g(z) = z^5 + z^6 + z^7 \), then the germs

\[
\begin{align*}
  f_1(x, y, z) &= (x, y, xz + z^3, yz^2 + z^2 + z^7) \\
  f_2(x, y, z) &= (x, y, xz + z^3, yz^2 + z^5 + z^6 + z^7)
\end{align*}
\]

are not \( \mathcal{A} \)-equivalent as \( N(f_1, (1, 1)) \neq N(f_2, (1, 1)) \).

The number of invariants above involved depends on the dimensions \((n, p)\), and this number is large when \( n \) and \( p \) are large. It is then natural to ask: Fixing a pair \((n, p)\), what is the minimum number of invariants \( N(f, \mathcal{P}) \) are necessary and sufficient to ensure \( \mathcal{A} \)-finiteness of the germ \( f \)? Then we have the following.

**Problem:** Fixing a pair \((n, p)\), The numbers of invariants \( N(f, \mathcal{P}) \) where \( \mathcal{P} \) is the partition of \( k \) such that \( p - k(p - n + 1) + \ell = 0 \), are necessary and sufficient to ensure \( \mathcal{A} \)-finiteness of the germ \( f \)?

This question has been answered for the cases:

**Remark 6.5.** when \( n = 2 \) and \( p = 3 \), the answer is given by D. Mond in [10]. Then \( f \) is \( \mathcal{A} \)-finite if and only if the number of cross caps \( C(f) \), the number triple points \( T(f) \) and \( N(f) < \infty \) are finites. It turns out that \( N(f) = N(f, (1, 1)) \).

In the case \( n = p = 2 \) Gaffney and Mond [1] showed that \( f \) is \( \mathcal{A} \)-finite if and only if the number of cusps number \( c(f) \), the number double folds points \( d(f) \) are finite.

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Address: Instituto de Ciências Matemática e de Computação ICMC-USP
Av. do Trabalhador São-Carlene, 400 - Centro - Cx. Postal 668
São Carlos - São Paulo - Brasil CEP 13560-970
E-mail: vhjperez@icmc.usp.br