Fundamental limits on the speed of evolution of quantum states

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Abstract
This paper reports on some new inequalities of Margolus–Levitin–Mandelstam–Tamm type involving the speed of quantum evolution between two orthogonal pure states. The clear determinant of the qualitative behavior of this time scale is the statistics of the energy spectrum.

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1. Motivation
As quantum information processors evolve in architectural complexity, it will be increasingly important to develop qualitative, architecture-independent methods that can answer fundamental questions about processor properties as a function of growing complexity without recourse to detailed, large-scale simulations, and independently of the details of processor architecture. The key questions will be speed of quantum evolution (analogous to clock speed), decoherence rates and stability through control of decoherence.

Basic limits on the speed of quantum evolution can be deduced directly from the Schrödinger equation [1–7]. Consider, for example, a system with Hamiltonian $H$ in a pure quantum state evolving in time according to

$$|\psi_t\rangle = e^{-iHt}|\psi_0\rangle,$$

where we rescaled the time parameter $t$ in inverse-energy units as $t \equiv \text{time}/\hbar$ (or, equivalently, set $\hbar = 1$). Expanding in the energy eigenbasis $|E_n\rangle$ of the Hamiltonian we obtain

$$\Phi(t) \equiv \langle \psi_0 | \psi_t \rangle = e^{-iE_0 t} \sum_{n=0}^{\infty} |c_n|^2 e^{-i(E_n - E_0)t},$$

where $|E_0\rangle$ is the lowest energy (ground) state of $H$ (so $E_0 \geq 0 \forall n$). We let the notation $\langle F | = \sum_{n=0}^{\infty} |c_n|^2 F_n = \langle \psi_0 | F | \psi_0 \rangle$ denote the expectation value of an observable $F$ in the initial state $|\psi_0\rangle$. If $T_0$ denotes the first zero of the overlap $\Phi(t)$ (i.e. the earliest time $t$ at which $|\psi_0\rangle$ evolves to an orthogonal state), it is shown in [4–7] that, after restoring to natural time units, equations (1) and (2) lead to the inequality

$$T_0 \geq \frac{\pi \hbar}{2(H - E_0)}.$$

On the other hand, the Schrödinger equation in the Heisenberg form for an operator $A$,

$$\frac{dA}{dt} = \frac{1}{i\hbar}[H, A]$$

combined with the uncertainty inequality (for any state vector $|s\rangle$)

$$(\Delta A)^2 (\Delta H)^2 \geq \frac{1}{4} |\langle s | [A, H] | s \rangle|^2,$$

where $|\langle s | A | s \rangle|^2 \equiv \langle s | A^2 | s \rangle - \langle s | A | s \rangle^2$, gives rise to the Mandelstam–Tamm inequality [8] (see also [9])

$$T_0 \geq \frac{\pi \hbar}{2 (\Delta H)}.$$

Lower bounds on the first zero $T_0$ probe the ‘long-time’ behavior of the overlap $\Phi(t) = \langle \psi_0 | \psi_t \rangle$, and can be interpreted as ‘speed limits’ on the rate of quantum evolution [10]. On the other hand, the short-time behavior of $\Phi(t)$ is completely determined by the variance $\Delta H$ since it can be shown easily that

$$|\Phi(t)|^2 = 1 - \frac{(\Delta H)^2}{\hbar^2} t^2 + O(t^3).$$
As a quick application, equation (7) implies that the quantum Zeno effect [11] for the initial state \(|\psi_0\rangle\), which requires the condition \(\lim_{n \to \infty} |\Phi(t/n)|^{2n} = 1\), is in principle always present as long as the variance \(\Delta H\) is finite. More precisely, if \(n\) projective measurements of the operator \(|\psi_0\rangle\langle\psi_0|\) are performed successively at equal intervals \(t/n\), the enhanced probability of finding the system in the initial state \(|\psi_0\rangle\) at time \(t\) is given by
\[
|\Phi(t/n)|^{2n} \approx e^{-((\Delta H)^2/n)(t^2/n)}. \tag{8}
\]
Consequently, this 'stasis' probability can be made close to unity provided
\[
n \gtrsim \frac{(\Delta H)^2}{\hbar^2} t^2 \gtrsim \left( \frac{t}{T_0} \right)^2. \tag{9}
\]

While the Zeno effect is significant for applications involving quantum control of decoherence, the long-time behavior of the overlap \(\Phi(t)\) has significant for exploring ultimate limits on the speed of quantum information processing.

2. New inequalities

A key observation about the overlap function \(\Phi(t)\) (equation (2)) is that it can be expressed as the Fourier transform (or 'characteristic function') of a positive probability distribution function \(\rho(E)\), characterizing the energy spectrum of the system in the initial quantum state \(|\psi_0\rangle\): \[
\Phi(t) = \int e^{-iEt} \rho(E) \, dE, \tag{10}
\]
where again we used the rescaled time parameter \(t \equiv \text{time}/\hbar\). For example, if the Hamiltonian \(H\) has a discrete spectrum \(|E_i\rangle\), the state \(|\psi_0\rangle\) can be expanded as
\[
|\psi_0\rangle = \sum_j c_j |E_j\rangle, \tag{11}
\]
and the distribution function and the overlap have the discrete forms
\[
\rho(E) = \sum_j |c_j|^2 \delta(E - E_j), \quad \Phi(t) = \sum_j |c_j|^2 e^{-iE_j t}. \tag{12}
\]
More generally, using the spectral theorem [12], the Hamiltonian \(H\) can be expressed in the form
\[
H = \int E \, dP(E), \tag{13}
\]
where \(dP(E)\) denotes integration with respect to a projection-valued measure on the Hilbert space, and the energy probability distribution ('density of states') \(\rho(E)\) can be defined via the identity
\[
\rho(E) \, dE = \langle \psi_0 | dP(E) | \psi_0 \rangle. \tag{14}
\]
From a practical point of view, the distribution function \(\rho(E)\) is a key design parameter, since it is relatively easy to manipulate. This distribution is related to the overlap function \(\Phi(t)\) via the inverse Fourier transform
\[
\rho(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iEt} \Phi(t) \, dt. \tag{15}
\]

It is useful to extend the function \(\Phi(t)\) to the complex domain; in fact, we will assume a Paley–Wiener [13] condition on the distribution \(\rho(E)\) such that \(\Phi(t)\) as defined by equation (10) is an entire analytic function on the complex plane \(t \in \mathbb{C}\). For example, this is the case if \(\rho(E)\) is of compact support, or, more generally, falls off faster than any exponential as \(E \to \pm \infty\). In either case, one can reasonably expect such Paley–Wiener conditions to hold for most physical systems.

Denoting complex time with the commonly used symbol \(z\), equation (10) implies that the analytic function \(\Phi(z)\) has the power series expansion
\[
\Phi(z) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n \langle E^n \rangle}{n!} z^n, \tag{16}
\]
which is a rephrasing of the generating-function identities
\[
\langle E^n \rangle = i^n \left. \frac{\partial^n \Phi(z)}{\partial z^n} \right|_{z=0} = i^n \Phi^{(n)}(0). \tag{17}
\]
Consequently, the function \(\log \Phi(z)\) can be expanded in a power series around \(z = 0\):
\[
\log \Phi(z) = \sum_{n=1}^{\infty} \gamma_n z^n, \tag{18}
\]
where
\[
\gamma_n \equiv \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left. \sum_{l_k \geq \ldots \geq l_1 \geq 0} \frac{(-i)^{l_k} \langle E^{l_k} \rangle}{l_k!} \cdots \frac{(-i)^{l_1} \langle E^{l_1} \rangle}{l_1!} \right|_{z=0}.
\]
Clearly, \(T_0\) cannot be less than the radius of convergence of the power series equation (18), which gives us our first new inequality
\[
T_0 \geq \frac{\hbar}{\lim_{n \to \infty} |\gamma_n|^{1/n}}, \tag{20}
\]
where
\[
\gamma_n = (-i)^n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left. \sum_{l_k \geq \ldots \geq l_1 \geq 0} \frac{\langle E^{l_k} \rangle \cdots \langle E^{l_1} \rangle}{l_k! \cdots l_1!} \right|_{z=0}. \tag{21}
\]
For our next set of new inequalities, we will rely on Bochner’s Theorem from real analysis [14]. First, a
complex-valued function $f$ on $\mathbb{R}$ is called positive definite if for any choice of complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$ and for any $x_1, x_2, \ldots, x_r \in \mathbb{R}$, we have

$$\sum_{i,j=1}^{r} \alpha_i \overline{\alpha_j} f(x_i - x_j) \geq 0. \tag{22}$$

Observe that consequences of positive definiteness are: (i) $f(0) \geq 0$ (derived from equation (22) with $r = 1$), and (ii) $f(-x) = f(x)$ and $f(0) \geq |f(x)| \forall x \in \mathbb{R}$ (derived from equation (22) with $r = 2$). It turns out that positive definiteness, along with the normalization condition $\Phi(0) = 1$, completely characterizes functions $\Phi$ that are expressible as the Fourier transform of a positive density of states $\rho(E)$ as in equation (10).

**Bochner’s theorem.** A continuous complex-valued function $f : \mathbb{R} \rightarrow \mathbb{C}$ with $f(0) = 1$ is the Fourier transform $f(x) = \int e^{-i x E} \rho(E) \, dE$ of a finite, normalized, positive Borel measure $\rho(E)$ if and only if $f$ is positive definite.

Since Bochner’s theorem completely characterizes a general overlap function $\Phi(t)$, the (in general infinite) class of inequalities

$$\sum_{i,j=1}^{r} \alpha_i \overline{\alpha_j} \Phi(t_i - t_j) \geq 0, \forall \alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{C},$$

$$t_1, t_2, \ldots, t_r \in \mathbb{R}, \tag{23}$$

are the most general, universal set of inequalities constraining an overlap function $\Phi(t)$ defined as in equation (2) for a pure state. Consequently, every inequality of the Margolus–Levitin–Mandelstam–Tamm type constraining the first zero $T_0$ of $\Phi(t)$ is necessarily a consequence of equations (23).

To derive some new examples of such inequalities for the first zero-crossing time $T_0$ from equations (23), observe that if $\Phi(t)$ is an overlap function (hence is positive definite), then equations (10) and (17) combined with Bochner’s theorem imply that for any positive integer $n$,

$$f(t) \equiv (-1)^n \frac{\Phi^{(2n)}}{E^{2n}}, \tag{24}$$

is also a positive-definite function satisfying $f(0) = 1$. For ease of calculation, let us redefine the Hamiltonian as

$$H \rightarrow H - \langle H \rangle, \tag{25}$$

so that the average (first moment) of $H$ vanishes: $\langle H \rangle = \langle E \rangle = 0$. Taking $n = 1$ in equation (24) and applying the basic consequence $1 = f(0) \geq |f(t)|$ of equation (22) gives

$$- \Phi'(t) \leq \langle E^2 \rangle. \tag{26}$$

Integrating equation (26) once from $t = 0$ to $t$ and using $\Phi(0) = -i \langle E \rangle = 0$, we obtain the inequality

$$- \Phi'(t) \leq \langle E^2 \rangle \cdot t. \tag{27}$$

Integrating equation (27) once more from $t = 0$ to $t = T_0$ (the first zero-crossing) gives

$$1 \leq \langle E^2 \rangle \frac{T_0^2}{2}. \tag{28}$$

After restoring to natural time units and undoing the rescaling equation (25), equation (28) becomes the inequality

$$T_0 \geq \frac{\sqrt{2 \hbar}}{\sqrt{\langle (E - \langle E \rangle)^2 \rangle}}. \tag{29}$$

Applying an entirely parallel stream of arguments and starting from $n = 2, 3, \ldots$, in equation (24), we obtain the following infinite series of new inequalities involving the first zero-crossing time $T_0$:

$$\langle (E - \langle E \rangle)^2 \rangle \frac{T_0^4}{24 \hbar^4} \geq 1,$$

$$\langle (E - \langle E \rangle)^4 \rangle \frac{T_0^8}{24^2 \hbar^8} \geq \langle (E - \langle E \rangle)^2 \rangle \frac{T_0^2}{2 \hbar^2} - 1,$$

$$\langle (E - \langle E \rangle)^6 \rangle \frac{T_0^{12}}{24^3 \hbar^{12}} \geq \langle (E - \langle E \rangle)^4 \rangle \frac{T_0^4}{24^2 \hbar^4} - \langle (E - \langle E \rangle)^2 \rangle \frac{T_0^2}{2 \hbar^2} + 1,$$

$$\cdots$$

$$\langle (E - \langle E \rangle)^{2n} \rangle \frac{T_0^{2n}}{(2n)! \hbar^{2n}} \geq \sum_{s=1}^n (-1)^s \langle (E - \langle E \rangle)^{2(n-s)} \rangle \times \frac{T_0^{2(n-s)}}{(2(n-s))! \hbar^{2(n-s)}}. \tag{30}$$

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