ON SEVERAL PROBLEMS ABOUT AUTOMORPHISMS OF THE FREE GROUP OF RANK TWO

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Abstract. Let $F_n$ be a free group of rank $n$. In this paper we discuss three algorithmic problems related to automorphisms of $F_2$.

A word $u$ of $F_n$ is called positive if $u$ does not have negative exponents. A word $u$ in $F_n$ is called potentially positive if $\phi(u)$ is positive for some automorphism $\phi$ of $F_n$. We prove that there is an algorithm to decide whether or not a given word in $F_2$ is potentially positive, which gives an affirmative solution to problem F34a in [1] for the case of $F_2$.

Two elements $u$ and $v$ in $F_n$ are said to be boundedly translation equivalent if the ratio of the cyclic lengths of $\phi(u)$ and $\phi(v)$ is bounded away from 0 and from $\infty$ for every automorphism $\phi$ of $F_n$. We provide an algorithm to determine whether or not two given elements of $F_2$ are boundedly translation equivalent, thus answering question F38c in the online version of [1] for the case of $F_2$.

We further prove that there exists an algorithm to decide whether or not a given finitely generated subgroup of $F_2$ is the fixed point group of some automorphism of $F_2$, which settles problem F1b in [1] in the affirmative for the case of $F_2$.

1. Introduction

Let $F_n$ be the free group of rank $n \geq 2$ with basis $\Sigma$. In particular, if $n = 2$, we let $\Sigma = \{a, b\}$, namely, $F_2$ is the free group with basis $\{a, b\}$. A word $v$ in $F_n$ is called cyclically reduced if all its cyclic permutations are reduced. A cyclic word is defined to be the set of all cyclic permutations of a cyclically reduced word. By $[v]$ we denote the cyclic word associated with a word $v$. Also by $\|v\|$ we mean the length of the cyclic word $[v]$ associated with $v$, that is, the number of cyclic permutations of a cyclically reduced word which is conjugate to $v$. The length $\|v\|$ is called the cyclic length of $v$. For two automorphisms $\phi$ and $\psi$ of $F_n$, by writing $\phi \equiv \psi$ we mean the equality of $\phi$ and $\psi$ over all cyclic words in $F_n$, that is, $\phi(w) = \psi(w)$ for every cyclic word $w$ in $F_n$.

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Recall that a Whitehead automorphism $\alpha$ of $F_n$ is defined to be an automorphism of one of the following two types (cf. [7]):

(W1) $\alpha$ permutes elements in $\Sigma^\pm$.
(W2) $\alpha$ is defined by a letter $x \in \Sigma^\pm$ and a set $S \subset \Sigma^\pm \setminus \{x, x^{-1}\}$ in such a way that if $c \in \Sigma^\pm$ then (a) $\alpha(c) = cx$ provided $c \in S$ and $c^{-1} \notin S$; (b) $\alpha(c) = x^{-1}cx$ provided both $c, c^{-1} \in S$; (c) $\alpha(c) = c$ provided both $c, c^{-1} \notin S$.

If $\alpha$ is of type (W2), we write $\alpha = (S, x)$. Note that in the expression of $\alpha = (S, x)$ it is conventional to include the defining letter $x$ in the defining set $S$, but for the sake of brevity of notation we will omit $a$ from $S$ as defined above.

Throughout the present paper, we let $\sigma = (\{a\}, b)$, $\tau = (\{b\}, a)$ be Whitehead automorphisms of type (W2) of $F_2$. Recently the author [7] proved that every automorphism of $F_2$ can represented in one of two particular types over all cyclic words of $F_2$ as follows:

**Lemma 1.1.** ([Lemma 2.3, 6]) For every automorphism $\phi$ of $F_2$, $\phi$ can be represented as $\phi \equiv \beta \phi'$, where $\beta$ is a Whitehead automorphism of $F_2$ of type (W1) and $\phi'$ is a chain of one of the forms

(C1) $\phi' \equiv \tau_{m_k} \sigma_{l_k} \ldots \tau_{m_1} \sigma_{l_1}$
(C2) $\phi' \equiv \tau^{-m_k} \sigma^{-l_k} \ldots \tau^{-m_1} \sigma^{-l_1}$

with $k \in \mathbb{N}$ and both $l_i, m_i \geq 0$ for every $i = 1, \ldots, k$.

With the notation of Lemma 1.1 we define the length of an automorphism $\phi$ of $F_2$ as $\sum_{i=1}^{k}(m_i + l_i)$, which is denoted by $|\phi|$. Then obviously $|\phi| = |\phi'|$.

In the present paper, with the help of Lemma 1.1, we resolve three algorithmic problems related to automorphisms of $F_2$. Indeed, the description of automorphisms $\phi$ of $F_2$ in the statement of Lemma 1.1 provides us with a very useful computational tool that facilitates inductive arguments on $|\phi|$ in the proofs of the problems.

The first problem we deal with is about potential positivity of elements in a free group the notion of which was first introduced by Khan [5].

**Definition 1.2.** A word $u$ of $F_n$ is called positive if $u$ does not have negative exponents. A word $u$ in $F_n$ is called potentially positive if $\phi(u)$ is positive for some automorphism $\phi$ of $F_n$. 

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It was shown by Khan [5] and independently by Meakin-Weil [8] that the Hanna Neumann conjecture is satisfied if one of the subgroups is generated by positive elements.

In Section 2, we shall describe an algorithm to decide whether or not a given word in $F_2$ is potentially positive, which gives an affirmative solution to problem F34a in [1] for the case of $F_2$.

The second problem we discuss here is related to the notion of bounded translation equivalence which is one of generalizations of the notion of translation equivalence, due to Kapovich-Levitt-Schupp-Shpilrain [4].

**Definition 1.3.** Two elements $u$ and $v$ in $F_n$ are called translation equivalent in $F_n$ if $\|\phi(u)\| = \|\phi(v)\|$ for every automorphism $\phi$ of $F_n$.

Several different sources of translation equivalence in free groups were provided by Kapovich-Levitt-Schupp-Shpilrain [4] and the author [6]. In another paper of the author [7], it is proved that there exists an algorithm to decide whether or not two given elements $u$ and $v$ of $F_2$ are translation equivalent.

In contrast with the notion of translation equivalence, bounded translation equivalence is defined as follows:

**Definition 1.4.** Two elements $u$ and $v$ in $F_n$ are said to be boundedly translation equivalent in $F_n$ if there is $C > 0$ such that

$$\frac{1}{C} \leq \frac{\|\phi(u)\|}{\|\phi(v)\|} \leq C$$

for every automorphism $\phi$ of $F_2$.

Clearly every pair of translation equivalent elements in $F_n$ are boundedly translation equivalent in $F_n$, but not vice versa. As one of specific examples of volume equivalence, we mention that two elements $a$ and $a[a, b]$ are boundedly translation equivalent in $F_2$. Indeed, if $u = a$ and $v = a[a, b]$, then we have, in view of Lemma 1.1 that

$$\frac{1}{5} \leq \frac{\|\phi(u)\|}{\|\phi(v)\|} \leq 1$$

for every automorphism $\phi$ of $F_2$.

In Section 3, developing further the technique used in [7], we shall demonstrate that there exists an algorithm to determine whether or not two given elements of $F_2$ are boundedly translation equivalent, thus affirmatively answering question F38c in the online version of [1] for the case of $F_2$.

Our last problem is concerned with the notion of fixed point groups of automorphisms of free groups.
Definition 1.5. A subgroup $H$ of $F_n$ is called the fixed point group of an automorphism $\phi$ of $F_n$ if $H$ is precisely the set of elements of $F_n$ which are fixed by $\phi$.

Due to Bestvina-Handel [2], a subgroup of rank bigger than $n$ cannot possibly be the fixed point group of an automorphism of $F_n$. Recently Martino-Ventura [9] provided an explicit description for the fixed point groups of automorphisms of $F_n$, generalizing the maximal rank case studied by Collins-Turner [3]. However, this description is not a complete characterization of all fixed point groups of automorphisms of $F_n$. On the other hand, Maslakova [10] proved that, given an automorphism $\phi$ of $F_n$, it is possible to effectively find a finite set of generators of the fixed point group of $\phi$.

In Section 4, we shall present an algorithm to decide whether or not a given finitely generated subgroup of $F_2$ is the fixed point group of some automorphism of $F_2$, which settles problem F1b in [1] in the affirmative for the case of $F_2$.

2. Potential positivity in $F_2$

Recall that $F_2$ denotes the free group with basis $\Sigma = \{a, b\}$, and that $\sigma$ and $\tau$ denote Whitehead automorphisms

$$\sigma = (\{a\},b), \quad \tau = (\{b\},a)$$

of $F_2$ of type (W2). We also recall from [7] the definition of trivial or nontrivial cancellation. For a cyclic word $w$ in $F_2$ and a Whitehead automorphism, say $\sigma$, of $F_2$, a subword of the form $ab^r a^{-1}$ ($r \neq 0$), if any, in $w$ is invariant in passing from $w$ to $\sigma(w)$, although there occurs cancellation in $\sigma(ab^r a^{-1})$ (note that $\sigma(ab^r a^{-1}) = ab \cdot b^r \cdot b^{-1} a^{-1} = ab^r a^{-1}$). Such cancellation is called trivial cancellation. And cancellation which is not trivial cancellation is called proper cancellation. For example, a subword $ab^{-r} a$ ($r \geq 1$), if any, in $w$ is transformed to $ab^{-r+1} ab$ by applying $\sigma$, and thus the cancellation occurring in $\sigma(ab^{-r} a)$ is proper cancellation.

The following lemma from [7] will play a fundamental role throughout the present paper.

Lemma 2.1. (Lemma 2.4 in [7]) Let $u$ be a cyclic word in $F_2$, and let $\psi$ be a chain of type (C1) (or (C2)). If $\psi$ contains at least $\|u\|$ factors of $\sigma$ (or $\sigma^{-1}$), then there cannot occur proper cancellation in passing from $\psi(u)$ to $\sigma \psi(u)$ (or $\psi(u)$ to $\sigma^{-1} \psi(u)$). Also if $\psi$ contains at least $\|u\|$ factors of $\tau$ (or $\tau^{-1}$), then there cannot occur proper cancellation in passing from $\psi(u)$ to $\tau \psi(u)$ (or $\psi(u)$ to $\tau^{-1} \psi(u)$).
The main result of this section is

**Theorem 2.2.** Let \( u \) be an element in \( F_2 \), and let \( \Omega \) be the set of all chains of type (C1) or (C2) of length less than or equal to \( 2 \|u\| + 3 \). Suppose that the cyclic word \( [\phi(u)] \) is positive for some automorphism \( \phi \) of \( F_2 \). Then there exists \( \psi \in \Omega \) and a Whitehead automorphism \( \beta \) of \( F_2 \) of type (W1) such that the cyclic word \( [\beta \psi(u)] \) is positive (which is obviously equivalent to saying that there exists \( c \in F_2 \) such that \( \pi_c \beta \psi(u) \) is positive, where \( \pi_c \) is the inner automorphism of \( F_2 \) induced by \( c \)).

Once this theorem is proved, an algorithm to decide whether or not a given word in \( F_2 \) is potentially positive is naturally derived as follows.

**Algorithm 2.3.** Let \( u \) be an element in \( F_2 \), and let \( \Omega \) be defined as in the statement of Theorem 2.2. Clearly \( \Omega \) is a finite set. Check if there is \( \psi \in \Omega \) and a Whitehead automorphism \( \beta \) of \( F_2 \) of type (W1) for which the cyclic word \( [\beta \psi(u)] \) is positive. If so, conclude that \( u \) is potentially positive; otherwise conclude that \( u \) is not potentially positive.

**Proof of Theorem 2.2.** By Lemma 1.1, \( \phi \) can be expressed as

\[
\phi \equiv \beta \phi',
\]

where \( \beta \) is a Whitehead automorphism of \( F_2 \) of type (W1) and \( \phi' \) is a chain of type (C1) or (C2). By the hypothesis of the theorem,

\[
[\phi(u)] = [\beta \phi'(u)] \text{ is positive.}
\]

If \( |\phi'| \leq 2 \|u\| + 3 \), then there is nothing to prove. So suppose that \( |\phi'| > 2 \|u\| + 3 \). We proceed with the proof by induction on \( |\phi'| \). Assume that \( \phi' \) is a chain of type (C1) which ends in \( \tau \) (the other cases are analogous). Write

\[
\phi' = \tau \phi_1,
\]

where \( \phi_1 \) is a chain of type (C1). Since \( |\phi_1| \geq 2 \|u\| + 3 \), \( \phi_1 \) must contain at least \( \|u\| + 2 \) factors of \( \sigma \) or \( \tau \). We consider two cases separately.

**Case 1.** \( \sigma \) occurs at least \( \|u\| + 2 \) times in \( \phi_1 \).

Write

\[
\phi_1 = \tau^{m_t} \sigma^{\ell_t} \cdots \tau^{m_1} \sigma^{\ell_1},
\]

where all \( m_i, \ell_i > 0 \) but \( \ell_1 \) and \( m_t \) may be zero.

**Case 1.1.** \( m_t \geq 1 \).
In this case, put

$$\phi_1 = \tau^{m_t} \phi_2,$$

where $\phi_2$ is a chain of type (C1). By Lemma 2.1 no proper cancellation can occur in passing from $[\sigma^{k-1} \cdots \tau^{m_t} \sigma^1(u)]$ to $[\phi_2(u)]$, and hence the cyclic word $[\phi_2(u)]$ does not contain a subword of the form $a^2$ or $a^{-2}$. From this fact and the assumption $m_t \geq 1$, we can observe that no proper cancellation occurs in passing from $[\phi_1(u)]$ to $[\tau \phi_1(u)] = [\phi'(u)]$. This implies from (1) that the cyclic word $[\beta \phi_1(u)]$ is positive, and thus induction completes the case.

**Case 1.2.** $m_t = 0$.

In this case, we may put

$$\phi_1 = \sigma \phi_3,$$

where $\phi_3$ is a chain of type (C1). Again by Lemma 2.1 no proper cancellation can occur in passing from $[\phi_3(u)]$ to $[\sigma \phi_3(u)] = [\phi_1(u)]$. Additionally, the proof of Theorem 1.2 of [7] shows that proper cancellation occurs in passing from $[\phi_3(u)]$ to $[\tau \phi_3(u)] = [\phi'(u)]$. Therefore, by (1), the cyclic word $[\beta \tau \phi_3(u)]$ is positive. Since $|\tau \phi_3| = |\phi'| - 1$, we are done by induction.

**Case 2.** $\tau$ occurs at least $\|u\| + 2$ times in $\phi_1$.

In this case, also by Lemma 2.1 no proper cancellation can occur in passing from $[\phi_1(u)]$ to $[\tau \phi_1(u)] = [\phi'(u)]$. It then follows from (1) that the cyclic word $[\beta \phi_1(u)]$ is positive; hence the required result follows by induction. \qed

3. **Bounded translation equivalence in $F_2$**

We begin this section by fixing notation. Following [4], if $w$ is a cyclic word in $F_2$ and $x, y \in \{a, b\}^{\pm 1}$, we use $n(w; x, y)$ to denote the total number of occurrences of the subwords $xy$ and $y^{-1}x^{-1}$ in $w$. Then clearly $n(w; x, y) = n(w; y^{-1}, x^{-1})$. Similarly we denote by $n(w; x)$ the total number of occurrences of $x$ and $x^{-1}$ in $w$. Again clearly $n(w; x) = n(w; x^{-1})$.

In this section, we shall prove that there exists an algorithm to determine bounded translation equivalence in $F_2$. Let $u \in F_2$. We first establish four preliminary lemmas which demonstrate the difference between $\|\sigma \psi(u)\|$ or $\|\tau \psi(u)\|$ and $\|\psi(u)\|$, and which describe the situation when this difference becomes zero, in the case where $\psi$ is a chain of type (C1) that contains a number of factors of $\sigma$. We remark that similar statements to the lemmas also hold if $\sigma$ and $\tau$ are interchanged with each other, or (C1) is replaced by (C2) and $\sigma$ and $\tau$ are replaced by $\sigma^{-1}$ and $\tau^{-1}$, respectively.
Lemma 3.1. Let \( u \in F_2 \). Suppose that \( \psi \) is a chain of type \((C1)\) which contains at least \( \|u\| + 2 \) factors of \( \sigma \). We may write \( \psi = \tau^m \sigma \psi_1 \), where \( m \geq 0 \) and \( \psi_1 \) is a chain of type \((C1)\). Then

(i) \( \| \sigma \psi(u) \| - \| \psi(u) \| = \| \sigma \tau^m \psi_1(u) \| - \| \tau^m \psi_1(u) \| + m(\| \sigma \psi_1(u) \| - \| \psi_1(u) \|) \);

(ii) \( \| \tau \psi(u) \| - \| \psi(u) \| = \| \tau \tau^m \psi_1(u) \| - \| \tau^m \psi_1(u) \| + \| \sigma \psi_1(u) \| - \| \psi_1(u) \|. \)

Proof. By the proof of Case 1 of Theorem 1.2 in [7], we see that

\[
(2) \quad n([\tau^i \sigma \psi_1(u)]; b, a^{-1}) = n([\tau^i \psi_1(u)]; b, a^{-1})
\]

for every \( i \geq 0 \), because \( \psi_1 \) contains at least \( \|u\| + 1 \) factors of \( \sigma \). In particular,

\[
(3) \quad n([\psi(u)]; b, a^{-1}) = n([\tau^m \psi_1(u)]; b, a^{-1}),
\]

for \( \psi = \tau^m \sigma \psi_1 \). Since only \( a \) or \( a^{-1} \) can possibly cancel or newly occur in the process of applying \( \tau \), the number of \( b \) and \( b^{-1} \) remains unchanged if \( \tau \) is applied. Thus

\[
(4) \quad n([\tau^i \sigma \psi_1(u)]; b) = n([\sigma \psi_1(u)]; b);
\]

\[
(5) \quad n([\tau^i \psi_1(u)]; b) = n([\psi_1(u)]; b)
\]

for every \( i \geq 0 \). Also since only \( b \) or \( b^{-1} \) can possibly cancel or newly occur in the process of applying \( \sigma \), we get

\[
(6) \quad n([\sigma \psi_1(u)]; b) = n([\psi_1(u)]; b) + \| \sigma \psi_1(u) \| - \| \psi_1(u) \|.
\]

By (1), this equality can be rewritten as

\[
(7) \quad n([\tau^i \sigma \psi_1(u)]; b) = n([\tau^i \sigma \psi_1(u)]; b, a^{-1})
\]

for every \( i \geq 0 \). In particular,

\[
(8) \quad n([\psi(u)]; b) = n([\tau^m \psi_1(u)]; b) + \| \sigma \psi_1(u) \| - \| \psi_1(u) \|,
\]

for \( \psi = \tau^m \sigma \psi_1 \).

Equality (5) together with (2) yields that

\[
(9) \quad n([\tau^i \sigma \psi_1(u)]; b) - n([\tau^i \sigma \psi_1(u)]; b, a^{-1})
\]

\[
= n([\tau^i \psi_1(u)]; b) - n([\tau^i \psi_1(u)]; b, a^{-1}) + \| \sigma \psi_1(u) \| - \| \psi_1(u) \|
\]

for every \( i \geq 0 \). Here, since

\[
\| \tau^{i+1} \sigma \psi_1(u) \| - \| \tau^i \sigma \psi_1(u) \| = n([\tau^i \sigma \psi_1(u)]; b) - 2n([\tau^i \sigma \psi_1(u)]; b, a^{-1});
\]

\[
\| \tau^{i+1} \psi_1(u) \| - \| \tau^i \psi_1(u) \| = n([\tau^i \psi_1(u)]; b) - 2n([\tau^i \psi_1(u)]; b, a^{-1}),
\]

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equality (7) can be rephrased as
\[ \|\tau^{i+1}\sigma\psi_1(u)\| - \|\tau^i\sigma\psi_1(u)\| = \|\tau^{i+1}\psi_1(u)\| - \|\tau^i\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\| \]
for every \( i \geq 0 \). By summing up both sides of these equalities changing \( i \) from 0 to \( m - 1 \), we have
\[ \|\tau^m\sigma\psi_1(u)\| - \|\sigma\psi_1(u)\| = \|\tau^m\psi_1(u)\| - \|\psi_1(u)\| + m(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|), \]
so that
\[ (8) \quad \|\tau^m\sigma\psi_1(u)\| - \|\tau^m\psi_1(u)\| = (m + 1)(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|). \]
Since \( \psi = \tau^m\sigma\psi_1 \), equality (8) can be rephrased as
\[ (9) \quad \|\psi(u)\| - \|\tau^m\psi_1(u)\| = (m + 1)(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|). \]
Clearly
\[ n([\psi(u)]; a) = \|\psi(u)\| - n([\psi(u)]; b); \]
\[ n([\tau^m\psi_1(u)]; a) = \|\tau^m\psi_1(u)\| - n([\tau^m\psi_1(u)]; b). \]
These equalities together with (6) and (9) yield that
\[ (10) \quad n([\psi(u)]; a) = n([\tau^m\psi_1(u)]; a) + m(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|). \]
It then follows from
\[ \|\sigma\psi(u)\| - \|\psi(u)\| = n([\psi(u)]; a) - 2n([\psi(u)]; a, b^{-1}); \]
\[ \|\sigma\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| = n([\tau^m\psi_1(u)]; a) - 2n([\tau^m\psi_1(u)]; a, b^{-1}) \]
together with (6) and (10) that
\[ \|\sigma\psi(u)\| - \|\psi(u)\| = \|\sigma\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| + m(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|), \]
thus proving the first assertion of the lemma.

On the other hand, we deduce from
\[ \|\tau\psi(u)\| - \|\psi(u)\| = n([\psi(u)]; b) - 2n([\psi(u)]; b, a^{-1}); \]
\[ \|\tau\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| = n([\tau^m\psi_1(u)]; b) - 2n([\tau^m\psi_1(u)]; b, a^{-1}) \]
together with (6) and (6) that
\[ \|\tau\psi(u)\| - \|\psi(u)\| = \|\tau\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|, \]
which proves the second assertion of the lemma. \( \square \)

**Lemma 3.2.** Let \( u \in F_2 \). Suppose that \( \psi \) is a chain of type \( (C1) \) which contains at least \( \|u\| \) factors of \( \sigma \). Then
\( (i) \quad \|\sigma\psi(u)\| - \|\psi(u)\| \geq 0; \)
\( (ii) \quad \|\tau\psi(u)\| - \|\psi(u)\| \geq 0. \)
Proof. Clearly

\[ ||\sigma \psi(u)|| - ||\psi(u)|| = n([\psi(u); a]) - 2n([\psi(u); a, b^{-1}]) = n([\psi(u); a, a]) + n([\psi(u); a, b]) - n([\psi(u); a, b^{-1}]). \]

Since \( \psi \) contains at least \( ||u|| \) factors of \( \sigma \), by Lemma 2.1 there cannot occur proper cancellation in passing from \([\psi(u)]\) to \([\sigma \psi(u)]\). Hence every subword of \([\psi(u)]\) of the form \( ab^{-1} \) or \( ba^{-1} \) is necessarily part of a subword of the form \( ab^{-r}a^{-1} \) or \( ab^ra^{-1} \) \((r > 0)\), respectively. This implies that

\[ n([\psi(u); a, b]) \geq n([\psi(u); a, b^{-1}])), \]

so that, from (11),

\[ ||\sigma \psi(u)|| - ||\psi(u)|| \geq n([\psi(u); a, a]) \geq 0, \]

thus proving (i).

On the other hand, clearly

\[ ||\tau \psi(u)|| - ||\psi(u)|| = n([\psi(u); b]) - 2n([\psi(u); b, a^{-1}) = n([\psi(u); b, b]) + n([\psi(u); b, a]) - n([\psi(u); b, a^{-1}]). \]

As above, every subword of \([\psi(u)]\) of the form \( ab^{-1} \) or \( ba^{-1} \) is necessarily part of a subword of the form \( ab^{-r}a^{-1} \) or \( ab^ra^{-1} \) \((r > 0)\), respectively. Observe that a subword of \([\psi(u)]\) of the form \( ab^ra^{-1} \) is actually part of either a subword of the form \( ba^sb^ra^{-1} \) or a subword of the form \( a^{-1}b^{-t}a^sb^ra^{-1} (s, t > 0) \). This implies that

\[ n([\psi(u); b, a]) \geq n([\psi(u); b, a^{-1}]), \]

so that, from (12),

\[ ||\tau \psi(u)|| - ||\psi(u)|| \geq n([\psi(u); b, b]) \geq 0, \]

thus proving (ii). □

Lemma 3.3. Let \( u \in F_2 \). Suppose that \( \psi \) is a chain of type (C1) which contains at least \( ||u|| + 1 \) factors of \( \sigma \). Then

(i) if \( ||\sigma \psi(u)|| = ||\psi(u)|| \), then \( ||\sigma^{i+1} \psi(u)|| = ||\sigma^i \psi(u)|| \) for every \( i \geq 0 \);

(ii) if \( ||\sigma^{i+1} \psi(u)|| = ||\sigma^j \psi(u)|| \) for some \( j \geq 0 \), then \( ||\sigma \psi(u)|| = ||\psi(u)|| \).

Proof. For (i), assume that \( ||\sigma \psi(u)|| = ||\psi(u)|| \). We shall prove \( ||\sigma^{i+1} \psi(u)|| = ||\sigma^i \psi(u)|| \) by induction on \( i \geq 0 \). The case where \( i = 0 \) is clear. So let \( i \geq 1 \).

By Lemma 3.1 (i) with \( m = 0 \), we have

\[ ||\sigma^{i+1} \psi(u)|| - ||\sigma^i \psi(u)|| = ||\sigma^i \psi(u)|| - ||\sigma^{i-1} \psi(u)||. \]
It follows from the induction hypothesis that
\[ \|\sigma^{i+1}\psi(u)\| = \|\sigma^i\psi(u)\|, \]
so proving (i).

For (ii), assume that \( \|\sigma^{j+1}\psi(u)\| = \|\sigma^j\psi(u)\| \) for some \( j \geq 0 \). We use induction on \( j \geq 0 \). If \( j = 0 \), then there is nothing to prove. So let \( j \geq 1 \). It follows from Lemma 3.1 (i) with \( m = 1 \) that
\[ 0 = \|\sigma^{j+1}\psi(u)\| - \|\sigma^j\psi(u)\| = \|\sigma^j\psi(u)\| - \|\sigma^{j-1}\psi(u)\|, \]
so that
\[ \|\sigma^j\psi(u)\| = \|\sigma^{j-1}\psi(u)\|. \]
Then by the induction hypothesis, we get the required result. \( \Box \)

**Lemma 3.4.** Let \( u \in F_2 \), and let \( \psi = \sigma\psi_1 \), where \( \psi_1 \) is a chain of type (C1) which contains at least \( \|u\| + 1 \) factors of \( \sigma \). Suppose that \( \|\tau\psi(u)\| = \|\psi(u)\| \). Then \( \|\sigma^{i+1}\psi_1(u)\| = \|\sigma^i\psi_1(u)\| \) for every \( i \geq 0 \).

**Proof.** By Lemma 3.1 (ii) with \( m = 0 \), we have
\[ 0 = \|\tau\psi(u)\| - \|\psi(u)\| = \|\tau\psi_1(u)\| - \|\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|. \]
Here, by Lemma 3.2 (ii), \( \|\tau\psi_1(u)\| - \|\psi_1(u)\| \geq 0 \). Also by Lemma 3.2 (i), \( \|\sigma\psi_1(u)\| - \|\psi_1(u)\| \geq 0 \). Hence we must have
\[ \|\tau\psi_1(u)\| = \|\psi_1(u)\| \text{ and } \|\sigma\psi_1(u)\| = \|\psi_1(u)\|. \]
The second equality \( \|\sigma\psi_1(u)\| = \|\psi_1(u)\| \) yields from Lemma 3.3 (i) that
\[ \|\sigma^{i+1}\psi_1(u)\| = \|\sigma^i\psi_1(u)\| \]
for every \( i \geq 0 \), thus proving the assertion. \( \Box \)

For the proof of the main result of the present section, we need the following two technical corollaries of Lemmas 3.1–3.4. We remark that similar statements to the corollaries also hold if \( \sigma \) and \( \tau \) are interchanged with each other, or (C1) is replaced by (C2) and \( \sigma \) and \( \tau \) are replaced by \( \sigma^{-1} \) and \( \tau^{-1} \), respectively.

**Corollary 3.5.** Let \( u, v \in F_2 \) with \( \|u\| \geq \|v\| \), and let \( \psi \) be a chain of type (C1) with \( |\psi| \geq 2\|u\| + 3 \). Put \( k = \|u\| + 1 \). Suppose that \( u \) and \( v \) have the property that
\[ \|\sigma^{k+1}\psi'(u)\| = \|\sigma^k\psi'(u)\| \text{ if and only if } \|\sigma^{k+1}\psi'(v)\| = \|\sigma^k\psi'(v)\|; \]
\[ \|\tau^{k+1}\psi'(u)\| = \|\tau^k\psi'(u)\| \text{ if and only if } \|\tau^{k+1}\psi'(v)\| = \|\tau^k\psi'(v)\|, \]
for every \( \psi \).
for every chain \( \psi' \) of type (C1) with \(|\psi'| < |\psi|\). Then we have

\[
(i) \quad \|\sigma^{k+1}\psi(u)\| = \|\sigma^k\psi(u)\| \text{ if and only if } \|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\|;
(ii) \quad \|\tau^{k+1}\psi(u)\| = \|\tau^k\psi(u)\| \text{ if and only if } \|\tau^{k+1}\psi(v)\| = \|\tau^k\psi(v)\|.
\]

**Proof.** Suppose that \( \psi \) ends in \( \tau \) (the case where \( \psi \) ends in \( \sigma \) is analogous). Since \(|\psi| \geq 2\|u\| + 3\), either \( \sigma \) or \( \tau \) occurs at least \( \|u\| + 2 \) times in \( \psi \). We consider two cases separately.

**Case 1.** \( \sigma \) occurs at least \( \|u\| + 2 \) times in \( \psi \).

First we shall prove (i). Suppose that \( \|\sigma^{k+1}\psi(u)\| = \|\sigma^k\psi(u)\| \). By Lemma 3.3 (i), we have

\[
\|\sigma\psi(u)\| = \|\psi(u)\|.
\]

Write

\[
\psi = \tau^\ell\sigma\psi_1,
\]

where \( \ell \geq 1 \) and \( \psi_1 \) is a chain of type (C1). Clearly \( \psi_1 \) contains at least \( \|u\| + 1 \) factors of \( \sigma \). By Lemma 3.1 (i), we have

\[
0 = \|\sigma\psi(u)\| - \|\psi(u)\| = \|\sigma\tau^\ell\psi_1(u)\| - \|\tau^\ell\psi_1(u)\| + \ell(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|).
\]

Here, since \( \|\sigma\tau^\ell\psi_1(u)\| - \|\tau^\ell\psi_1(u)\| \geq 0 \) and \( \|\sigma\psi_1(u)\| - \|\psi_1(u)\| \geq 0 \) by Lemma 3.2 (i), the only possibility is that

\[
\|\sigma\tau^\ell\psi_1(u)\| = \|\tau^\ell\psi_1(u)\| \text{ and } \|\sigma\psi_1(u)\| = \|\psi_1(u)\|.
\]

These equalities together with Lemma 3.3 (i) yield that

\[
\|\sigma^{k+1}\tau^\ell\psi_1(u)\| = \|\sigma^k\tau^\ell\psi_1(u)\| \text{ and } \|\sigma^{k+1}\psi_1(u)\| = \|\sigma^k\psi_1(u)\|.
\]

Since \(|\tau^\ell\psi_1| < |\psi|\) and \(|\psi_1| < |\psi|\), by the hypothesis of the corollary, we get

\[
\|\sigma^{k+1}\tau^\ell\psi_1(v)\| = \|\sigma^k\tau^\ell\psi_1(v)\| \text{ and } \|\sigma^{k+1}\psi_1(v)\| = \|\sigma^k\psi_1(v)\|.
\]

Again by Lemma 3.3 (ii), we have

\[
\|\sigma\tau^\ell\psi_1(v)\| = \|\tau^\ell\psi_1(v)\| \text{ and } \|\sigma\psi_1(v)\| = \|\psi_1(v)\|.
\]

Therefore, by Lemma 3.1 (i),

\[
\|\sigma\psi(v)\| - \|\psi(v)\| = \|\sigma\tau^\ell\psi_1(v)\| - \|\tau^\ell\psi_1(v)\| + \ell(\|\sigma\psi_1(v)\| - \|\psi_1(v)\|) = 0,
\]

namely, \( \|\sigma\psi(v)\| = \|\psi(v)\| \). Then the desired equality \( \|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\| \) follows from Lemma 3.3 (i).

Conversely, if \( \|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\| \), we can deduce, in the same way as above, that \( \|\sigma^{k+1}\psi(u)\| = \|\sigma^k\psi(u)\| \).
Next we shall prove (ii). Assume that \( \|\tau^{k+1}\psi(u)\| = \|\tau^k\psi(u)\| \). Apply Lemma 3.1 (ii) to get

\[
0 = \|\tau^{k+1}\psi(u)\| - \|\tau^k\psi(u)\| = \|\tau^{k+1}\tau\psi_1(u)\| - \|\tau^k\tau\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|,
\]

Here, since \( \|\tau^{k+1}\tau\psi_1(u)\| - \|\tau^k\tau\psi_1(u)\| \geq 0 \) by Lemma 3.2 (ii), and since \( \|\sigma\psi_1(u)\| - \|\psi_1(u)\| \geq 0 \) by Lemma 3.2 (i), we must have

\[
\|\tau^{k+1}\tau\psi_1(u)\| = \|\tau^k\tau\psi_1(u)\| \quad \text{and} \quad \|\sigma\psi_1(u)\| = \|\psi_1(u)\|.
\]

Since \( |\tau\psi_1| < |\psi| \), by the hypothesis of the corollary, the first equality of (16) implies that

\[
\|\tau^{k+1}\tau\psi_1(v)\| = \|\tau^k\tau\psi_1(v)\|.
\]

Also, from the second equality of (16), arguing as above, we deduce that

\[
\|\sigma\psi_1(v)\| = \|\psi_1(v)\|.
\]

Therefore, by Lemma 3.1 (ii),

\[
\|\tau^{k+1}\psi(v)\| - \|\tau^k\psi(v)\| = \|\tau^{k+1}\tau\psi_1(v)\| - \|\tau^k\tau\psi_1(v)\| + \|\sigma\psi_1(v)\| - \|\psi_1(v)\| = 0,
\]

that is, \( \|\tau^{k+1}\psi(v)\| = \|\tau^k\psi(v)\| \), as required.

It is clear that the converse is also true.

**Case 2.** \( \tau \) occurs at least \( \|u\| + 2 \) times in \( \psi \).

Since \( \psi \) is assumed to end in \( \tau \), we may write

\[
\psi = \tau\psi_2,
\]

where \( \psi_2 \) is a chain of type (C1) that contains at least \( \|u\| + 1 \) factors of \( \tau \).

First we shall prove (i). Suppose that \( \|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(u)\| \). By Lemma 3.1 (ii) with \( \sigma, \tau \) interchanged, we have

\[
0 = \|\sigma^{k+1}\psi(u)\| - \|\sigma^k\psi(u)\| = \|\sigma^{k+1}\psi_2(u)\| - \|\sigma^k\psi_2(u)\| + \|\tau\psi_2(u)\| - \|\psi_2(u)\|.
\]

This is a similar situation to (15) with \( \sigma, \tau \) interchanged. So arguing as in Case 1, we get the desired equality \( \|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\| \). Clearly the converse also holds.

Next we shall prove (ii). Suppose that \( \|\tau^{k+1}\psi(u)\| = \|\tau^k\psi(u)\| \). By Lemma 3.1 (i) with \( \sigma, \tau \) interchanged and \( m = 0 \), we have

\[
0 = \|\tau^{k+1}\psi(u)\| - \|\tau^k\psi(u)\| = \|\tau^k\psi(u)\| - \|\tau^{k-1}\psi(u)\|.
\]

So

\[
\|\tau^k\psi(u)\| = \|\tau^{k-1}\psi(u)\|.
\]
Thus, by Lemma 3.1 (i) with $\sigma, \tau$, that is,

$$(C1). \text{ Put } k = \|\psi\|,$$

namely,

$$\|\tau^{k+1}\psi_2(u)\| = \|\tau^k\psi_2(u)\|,$$

because $\psi = \tau\psi_2$. Since $|\psi_2| < |\psi|$, by the hypothesis of the corollary,

$$\|\tau^{k+1}\psi_2(v)\| = \|\tau^k\psi_2(v)\|,$$

that is,

$$\|\tau^k\psi(v)\| = \|\tau^{k-1}\psi(v)\|.$$

Thus, by Lemma 3.1 (i) with $\sigma, \tau$ interchanged and $m = 0$, we obtain

$$\|\tau^{k+1}\psi(v)\| - \|\tau^k\psi(v)\| = \|\tau^k\psi(v)\| - \|\tau^{k-1}\psi(v)\| = 0,$$

namely, $\|\tau^{k+1}\psi(v)\| = \|\tau^k\psi(v)\|$, as required. Obviously the converse is also true.

**Corollary 3.6.** Let $u, v \in F_2$ with $\|u\| \geq \|v\|$, and let $\psi$ be a chain of type (C1). Put $k = \|u\| + 1$. Suppose that $u$ and $v$ have the property that

$$\|\sigma^{k+1}\psi'(u)\| = \|\sigma^k\psi'(u)\| \text{ if and only if } \|\sigma^{k+1}\psi'(v)\| = \|\sigma^k\psi'(v)\|;$$

$$\|\tau^{k+1}\psi'(u)\| = \|\tau^k\psi'(u)\| \text{ if and only if } \|\tau^{k+1}\psi'(v)\| = \|\tau^k\psi'(v)\|,$$

for every chain $\psi'$ of type (C1) with $|\psi'| \leq |\psi|$. Then we have

(i) if $\psi$ contains at least $\|u\| + 1$ factors of $\sigma$, then

$$\|\sigma\psi(u)\| = \|\psi(u)\| \text{ if and only if } \|\sigma\psi(v)\| = \|\psi(v)\|;$$

(ii) if $\|\tau\psi(u)\| = \|\psi(u)\|$ or $\|\tau\psi(v)\| = \|\psi(v)\|$, and $\psi = \sigma\psi_1$, where $\psi_1$ is a chain of type (C1) which contains at least $\|u\| + 1$ factors of $\sigma$, then

$$\|\sigma\psi_1(u)\| = \|\psi_1(u)\| \text{ and } \|\sigma\psi_1(v)\| = \|\psi_1(v)\|;$$

(iii) if $\psi$ contains at least $\|u\| + 2$ factors of $\sigma$ and ends in $\tau$, then

$$\|\tau\psi(u)\| = \|\psi(u)\| \text{ if and only if } \|\tau\psi(v)\| = \|\psi(v)\|.$$

**Proof.** For (i), let $\psi$ contain at least $\|u\| + 1$ factors of $\sigma$, and suppose that $\|\sigma\psi(u)\| = \|\psi(u)\|$. By Lemma 3.3 (i), we have $\|\sigma^{k+1}\psi(u)\| = \|\sigma^k\psi(u)\|$. Then by the hypothesis of the corollary, $\|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\|$. Finally by Lemma 3.3 (ii), we get $\|\sigma\psi(v)\| = \|\psi(v)\|$. The converse also holds.

For (ii), let $\psi = \sigma\psi_1$, where $\psi_1$ is a chain of type (C1) containing at least $\|u\| + 1$ factors of $\sigma$, and suppose that $\|\tau\psi(u)\| = \|\psi(u)\|$. By Lemma 3.4 we have $\|\sigma\psi_1(u)\| = \|\psi_1(u)\|$. Then, by (i) of the corollary, $\|\sigma\psi_1(v)\| = \|\psi_1(v)\|$. The converse is proved similarly.
For (iii), let $\psi$ contain at least $\|u\| + 2$ factors of $\sigma$, and let $\psi$ end in $\tau$. Assume that $\|\tau\psi(u)\| = \|\psi(u)\|$. Write

$$\psi = \tau^\ell \sigma \psi_2,$$

where $\ell \geq 1$ and $\psi_2$ is a chain of type (C1). By Lemma 3.1 (ii), we have

$$0 = \|\tau\psi(u)\| - \|\psi(u)\| = \|\tau^\ell \psi_2(u)\| - \|\tau^\ell \psi_2(u)\| + \|\sigma \psi_2(u)\| - \|\psi_2(u)\|.$$

Here, since $\|\tau^\ell \psi_2(u)\| - \|\tau^\ell \psi_2(u)\| \geq 0$ by Lemma 3.2 (ii) and $\|\sigma \psi_2(u)\| - \|\psi_2(u)\| \geq 0$ by Lemma 3.2 (i), we must have

$$\|\tau^\ell \psi_2(u)\| = \|\tau^\ell \psi_2(u)\| \quad \text{and} \quad \|\sigma \psi_2(u)\| = \|\psi_2(u)\|.$$

Since $\|\sigma \psi_2(u)\| = \|\psi_2(u)\|$, by (i) of the corollary, $\|\sigma \psi_2(u)\| = \|\psi_2(u)\|$. Also, the following claim shows that $\|\tau^\ell \psi_2(u)\| = \|\tau^\ell \psi_2(u)\|$. Then by Lemma 3.1 (ii), we have $\|\tau\psi(u)\| = \|\psi(u)\|$, as required.

**Claim.** $\|\tau^\ell \psi_2(v)\| = \|\tau^\ell \psi_2(v)\|$

**Proof of the Claim.** Since $\|\tau\psi(u)\| = \|\psi(u)\|$, in view of (12), (13) and (14) in the proof of Lemma 3.2 we must have

$$n([\psi(u)]; b, a) = n([\psi(u)]; b, a^{-1}) \quad \text{and} \quad n([\psi(u)]; b, b) = 0.$$

Since the chain $\psi_2$ contains at least $\|u\| + 1$ factors of $\sigma$, by Lemma 2.1 no proper cancellation occurs in passing from $[\psi_2(u)]$ to $[\sigma \psi_2(u)]$. This yields that

$$a^2 \quad \text{or} \quad a^{-2} \quad \text{cannot occur in} \quad [\sigma \psi_2(u)] \quad \text{as a subword.}$$

From this, we see that, since $\ell \geq 1$,

$$\|\tau^\ell \psi_2(u)\| = \|\tau^\ell \psi_2(u)\| \quad \text{for every} \quad i \geq 0,$$

so that

$$\|\tau^i \psi_2(u)\| = \|\psi_2(u)\|$$

for every $i \geq 0$. Then, by applying $\sigma^{-1} \tau^{-\ell}$ to $[\psi(u)]$, we deduce that

$$[\psi_2(u)] = [\psi(u)] = [a^\epsilon b a^{-\epsilon} b^{-1} \cdots a^\epsilon b a^{-\epsilon} b^{-1}],$$

where either $\epsilon = 1$ or $\epsilon = -1$. Then, by applying $\sigma^{-1} \tau^{-\ell}$ to $[\psi(u)]$, we deduce that

$$[\psi_2(u)] = [\psi(u)] = [a^\epsilon b a^{-\epsilon} b^{-1} \cdots a^\epsilon b a^{-\epsilon} b^{-1}].$$

It then follows that

$$[\tau^i \psi_2(u)] = [\psi_2(u)]$$

for every $i \geq 0$, so that

$$\|\tau^{i+1} \psi_2(u)\| = \|\tau^i \psi_2(u)\|$$
for every \( i \geq 0 \). In particular,
\[
\|\tau^{k+1}\psi_2(u)\| = \|\tau^k\psi_2(u)\|.
\]

So by the hypothesis of the corollary,
\[
(21) \quad \|\tau^{k+1}\psi_2(v)\| = \|\tau^k\psi_2(v)\|.
\]

Then in the same way as obtaining (17), we get
\[
(22) \quad n([\tau^k\psi_2(v)]; b, a) = n([\tau^k\psi_2(v)]; b, a^{-1}) \quad \text{and} \quad n([\tau^k\psi_2(v)]; b, b) = 0.
\]

Since the chain \( \tau^k\psi_2 \) contains at least \( \|v\| + 1 \) factors of \( \tau \), by Lemma 21, no proper cancellation may occur in passing from \([\tau^k\psi_2(v)]\) to \([\tau^{k+1}\psi_2(v)]\). This together with (22) yields that
\[
[\tau^k\psi_2(v)] = [a^{s_1}ba^{t_1}1 \cdots a^{s_r}ba^{t_r}b^{-1}],
\]

where every \( s_j, t_j \) is a nonzero integer. Then, by applying \( \tau^{-k} \) to \([\tau^k\psi_2(v)]\), we deduce that
\[
[\psi_2(v)] = [a^{s_1}ba^{t_1}1 \cdots a^{s_r}ba^{t_r}b^{-1}].
\]

Thus it follows that
\[
[\tau^i\psi_2(v)] = [\psi_2(v)]
\]

for every \( i \geq 0 \), so that
\[
\|\tau^{i+1}\psi_2(v)\| = \|\tau^i\psi_2(v)\|
\]

for every \( i \geq 0 \). In particular, \( \|\tau\tau^i\psi_2(v)\| = \|\tau^i\psi_2(v)\| \), as required. \( \square \)

The proof of the corollary is now completed. \( \square \)

For a Whitehead automorphism \( \beta \) of \( F_2 \), a chain \( \psi \) of Whitehead automorphisms of \( F_2 \) and an element \( w \) in \( F_2 \), we let \( \|\beta : \psi : w\| \) denote the maximum of 1 and \( \|\beta\psi(w)\| - \|\psi(w)\| \), that is,
\[
\|\beta : \psi : w\| := \max\{1, \|\beta\psi(w)\| - \|\psi(w)\|\}.
\]

Now we are ready to establish the main result of the present section as follows.

**Theorem 3.7.** Let \( u, v \in F_2 \) with \( \|u\| \geq \|v\| \), and let \( \Omega \) be the set of all chains of type \( (C1) \) or \( (C2) \) of length less than or equal to \( 2\|u\| + 5 \). Let \( \Omega_1 \) be the subset of \( \Omega \) consisting of all chains of type \( (C1) \), and let \( \Omega_2 \) be the subset of \( \Omega \) consisting of all chains of type \( (C2) \). Put \( k = \|u\| + 1 \). Suppose that \( u \) and \( v \) have the property that
\[
\|\sigma^{k+1}\psi_1(u)\| = \|\sigma^k\psi_1(u)\| \quad \text{if and only if} \quad \|\sigma^{k+1}\psi_1(v)\| = \|\sigma^k\psi_1(v)\|;
\]
\[
\|\tau^{k+1}\psi_1(u)\| = \|\tau^k\psi_1(u)\| \quad \text{if and only if} \quad \|\tau^{k+1}\psi_1(v)\| = \|\tau^k\psi_1(v)\|,
\]

Then...
for every $\psi_1 \in \Omega_1$, and that
\[
\|\sigma^{-k-1}\psi_2(u)\| = \|\sigma^{-k}\psi_2(u)\| \text{ if and only if } \|\sigma^{-k-1}\psi_2(v)\| = \|\sigma^{-k}\psi_2(v)\|;
\]
\[
\|\tau^{-k-1}\psi_2(u)\| = \|\tau^{-k}\psi_2(u)\| \text{ if and only if } \|\tau^{-k-1}\psi_2(v)\| = \|\tau^{-k}\psi_2(v)\|,
\]
for every $\psi_2 \in \Omega_2$. Then $u$ and $v$ are boundedly translation equivalent in $F_2$.

More specifically,
\[
\min \Delta \leq \left(\begin{array}{ll}
\frac{\|\phi(u)\|}{\|\phi(v)\|} & \leq \max \Delta
\end{array}\right)
\]
for every automorphism $\phi$ of $F_2$, where
\[
\Delta := \left\{ \psi, \frac{\|\alpha : \psi_1 : u\| \|\alpha^{-1} : \psi_2 : u\|}{\|\psi(v)\| \|\alpha : \psi_1 : v\| \|\alpha^{-1} : \psi_2 : v\|} \mid \psi \in \Omega, \psi_1 \in \Omega_1, \alpha = \sigma \text{ or } \tau \right\}.
\]
(Obviously, $\Delta$ is a finite set consisting of positive real numbers.)

Proof. Let $\phi$ be an automorphism of $F_2$. By Lemma 1.1, $\phi$ can be represented as
\[
\phi \equiv \beta \phi',
\]
where $\beta$ is a Whitehead automorphism of $F_2$ of type (W1) and $\phi'$ is of type either (C1) or (C2). We proceed with the proof of the theorem by induction on $|\phi'|$. Letting $\phi'$ be a chain of type (C1) with $|\phi'| > 2\|u\| + 5$ (the case for (C2) is similar), assume that
\[
\|\sigma^{k+1}\psi(u)\| = \|\sigma^{k}\psi(u)\| \text{ if and only if } \|\sigma^{k+1}\psi(v)\| = \|\sigma^{k}\psi(v)\|;
\]
\[
\|\tau^{k+1}\psi(u)\| = \|\tau^{k}\psi(u)\| \text{ if and only if } \|\tau^{k+1}\psi(v)\| = \|\tau^{k}\psi(v)\|,
\]
and that
\[
\min \Delta \leq \frac{\|\psi(u)\|}{\|\psi(v)\|} \frac{\|\sigma : \psi : u\| \|\tau : \psi : u\|}{\|\sigma : \psi : v\| \|\tau : \psi : v\|} \leq \max \Delta,
\]
for every chain $\psi$ of type (C1) with $|\psi| < |\phi'|$.

By Corollary 3.5, it is easy to get
\[
\|\sigma^{k+1}\phi'(u)\| = \|\sigma^{k}\phi'(u)\| \text{ if and only if } \|\sigma^{k+1}\phi'(v)\| = \|\sigma^{k}\phi'(v)\|;
\]
\[
\|\tau^{k+1}\phi'(u)\| = \|\tau^{k}\phi'(u)\| \text{ if and only if } \|\tau^{k+1}\phi'(v)\| = \|\tau^{k}\phi'(v)\|.
\]
In the following Claims A, B and C, we shall prove that
\[
\min \Delta \leq \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \frac{\|\sigma : \phi : u\| \|\tau : \phi : u\|}{\|\sigma : \phi : v\| \|\tau : \phi : v\|} \leq \max \Delta,
\]
which is clearly equivalent to showing that
\[
\min \Delta \leq \frac{\|\phi(u)\|}{\|\phi(v)\|} \frac{\|\sigma : \phi : u\| \|\tau : \phi : u\|}{\|\sigma : \phi : v\| \|\tau : \phi : v\|} \leq \max \Delta.
\]
Suppose that $\phi'$ ends in $\tau$ (the case where $\phi'$ ends in $\sigma$ is analogous).

**Claim A.**

$$\min \Delta \leq \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leq \max \Delta$$

**Proof of Claim A.** Since $\phi'$ ends in $\tau$, we may write

$$\phi' = \tau \phi_1,$$

where $\phi_1$ is a chain of type (C1). Then obviously

\begin{align*}
\|\phi'(u)\| &= \|\tau \phi_1(u)\| - \|\phi_1(u)\| + \|\phi_1(u)\|; \\
\|\phi'(v)\| &= \|\tau \phi_1(v)\| - \|\phi_1(v)\| + \|\phi_1(v)\|.
\end{align*}

If both $\|\tau \phi_1(u)\| \neq \|\phi_1(u)\|$ and $\|\tau \phi_1(v)\| \neq \|\phi_1(v)\|$, then equalities (23) can be rephrased as

\begin{align*}
\|\phi'(u)\| &= \|\tau : \phi_1 : u\| + \|\phi_1(u)\|; \\
\|\phi'(v)\| &= \|\tau : \phi_1 : v\| + \|\phi_1(v)\|.
\end{align*}

Since

$$\min \Delta \leq \frac{\|\phi_1(u)\|}{\|\phi_1(v)\|}, \frac{\|\tau : \phi_1 : u\|}{\|\tau : \phi_1 : v\|} \leq \max \Delta$$

by the induction hypothesis, we obtain

$$\min \Delta \leq \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leq \max \Delta,$$

as required.

So assume that

$$\|\tau \phi_1(u)\| = \|\phi_1(u)\| \text{ or } \|\tau \phi_1(v)\| = \|\phi_1(v)\|.$$

Clearly the chain $\phi_1$ has length $|\phi_1| = |\phi'| - 1 \geq 2||u|| + 5$. Hence either $\sigma$ or $\tau$ occurs at least $||u|| + 3$ times in $\phi_1$. We consider two cases accordingly.

**Case A.1.** $\sigma$ occurs at least $||u|| + 3$ times in $\phi_1$.

Since $\phi_1$ is a chain of type (C1), $\phi_1$ ends in either $\sigma$ or $\tau$.

**Case A.1.1.** $\phi_1$ ends in $\sigma$.

Write

$$\phi_1 = \sigma \phi_2,$$
where $\phi_2$ is a chain of type (C1). In view of Corollary 3.6 (ii), our assumption (25) yields that

\begin{equation}
\|\sigma \phi_2(u)\| = \|\phi_2(u)\| \quad \text{and} \quad \|\sigma \phi_2(v)\| = \|\phi_2(v)\|.
\end{equation}

This together with Lemma 3.1 (ii) implies that

\begin{equation}
\|\tau \phi_1(u)\| - \|\phi_1(u)\| = \|\tau \phi_2(u)\| - \|\phi_2(u)\|;
\end{equation}

\begin{equation}
\|\tau \phi_1(v)\| - \|\phi_1(v)\| = \|\tau \phi_2(v)\| - \|\phi_2(v)\|.
\end{equation}

Since $\phi_1 = \sigma \phi_2$, we obtain from (26) that $\|\phi_1(u)\| = \|\phi_2(u)\|$ and $\|\phi_1(v)\| = \|\phi_2(v)\|$, so that, from (27),

\begin{equation}
\|\tau \phi_1(u)\| = \|\tau \phi_2(u)\|;
\end{equation}

\begin{equation}
\|\tau \phi_1(v)\| = \|\tau \phi_2(v)\|.
\end{equation}

Since $\phi' = \tau \phi_1$, (28) implies that

\begin{equation*}
\frac{\|\phi'(u)\|}{\|\phi'(v)\|} = \frac{\|\tau \phi_2(u)\|}{\|\tau \phi_2(v)\|},
\end{equation*}

and thus, by the induction hypothesis,

\[ \min \Delta \leq \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leq \max \Delta, \]

as desired.

**Case A.1.2.** $\phi_1$ ends in $\tau$.

In view of Corollary 3.6 (iii), our assumption (25) yields that both $\|\tau \phi_1(u)\| = \|\phi_1(u)\|$ and $\|\tau \phi_1(v)\| = \|\phi_1(v)\|$. We then have from (23) that

\[ \frac{\|\phi'(u)\|}{\|\phi'(v)\|} = \frac{\|\phi_1(u)\|}{\|\phi_1(v)\|}, \]

so that, by the induction hypothesis,

\[ \min \Delta \leq \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leq \max \Delta, \]

as required.

**Case A.2.** $\tau$ occurs at least $\|u\| + 3$ times in $\phi_1$.

In view of Corollary 3.6 (i) with $\tau$ in place of $\sigma$, we have from (25) both $\|\tau \phi_1(u)\| = \|\phi_1(u)\|$ and $\|\tau \phi_1(v)\| = \|\phi_1(v)\|$. It then follows from (23) that

\[ \frac{\|\phi'(u)\|}{\|\phi'(v)\|} = \frac{\|\phi_1(u)\|}{\|\phi_1(v)\|}, \]

as required.
so that, by the induction hypothesis,
\[ \min \Delta \leq \frac{\| \phi'(u) \|}{\| \phi'(v) \|} \leq \max \Delta, \]
as desired. \hfill \qed

**Claim B.**

\[ \min \Delta \leq \frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} \leq \max \Delta \]

**Proof of Claim B.** As in the proof of Claim A, writing
\[ \phi' = \tau \phi_1, \]
where \( \phi_1 \) is a chain of type (C1), we consider two cases separately.

**Case B.1.** \( \sigma \) occurs at least \( \|u\| + 3 \) times in \( \phi_1 \).

In this case, write
\[ \phi_1 = \tau^{m-1} \sigma \phi_2, \]
where \( m \geq 1 \) and \( \phi_2 \) is a chain of type (C1). Since \( \phi' = \tau \phi_1 \),
\[ \phi' = \tau^m \sigma \phi_2. \]

Then by Lemma 3.1 (i), we have
\begin{align*}
\| \sigma \phi'(u) \| - \| \phi'(u) \| &= \| \sigma \tau^m \phi_2(u) \| - \| \tau^m \phi_2(u) \| + m(\| \sigma \phi_2(u) \| - \| \phi_2(u) \|); \\
\| \sigma \phi'(v) \| - \| \phi'(v) \| &= \| \sigma \tau^m \phi_2(v) \| - \| \tau^m \phi_2(v) \| + m(\| \sigma \phi_2(v) \| - \| \phi_2(v) \|). 
\end{align*}

Here, since \( \phi_2 \) is a chain of type (C1) which contains at least \( \|u\| + 2 \) factors of \( \sigma \), Corollary 3.6 (i) yields that \( \| \sigma \tau^m \phi_2(u) \| = \| \tau^m \phi_2(u) \| \) if and only if \( \| \sigma \tau^m \phi_2(v) \| = \| \tau^m \phi_2(v) \| \). So if \( \| \sigma \tau^m \phi_2(u) \| = \| \tau^m \phi_2(u) \| \) or \( \| \sigma \tau^m \phi_2(v) \| = \| \tau^m \phi_2(v) \| \), then we get from (29) that
\begin{align*}
\| \sigma \phi'(u) \| - \| \phi'(u) \| &= m(\| \sigma \phi_2(u) \| - \| \phi_2(u) \|); \\
\| \sigma \phi'(v) \| - \| \phi'(v) \| &= m(\| \sigma \phi_2(v) \| - \| \phi_2(v) \|).
\end{align*}

This gives us
\[ \frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} = \frac{\| \sigma : \phi_2 : u \|}{\| \sigma : \phi_2 : v \|}, \]
and hence the desired inequalities
\[ \min \Delta \leq \frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} \leq \max \Delta \]
follow by the induction hypothesis.
Now let us assume that
\[ \| \sigma \tau^m \phi_2(u) \| \neq \| \tau^m \phi_2(u) \| \text{ and } \| \sigma \tau^m \phi_2(v) \| \neq \| \tau^m \phi_2(v) \|. \]
Again by Corollary 3.6 (i), we have \( \| \sigma \phi_2(u) \| = \| \phi_2(u) \| \) if and only if \( \| \sigma \phi_2(v) \| = \| \phi_2(v) \| \). Hence if \( \| \sigma \phi_2(u) \| = \| \phi_2(u) \| \) or \( \| \sigma \phi_2(v) \| = \| \phi_2(v) \| \), then, from (29),
\[
\begin{align*}
\| \sigma \phi'\(u\)\| - \| \phi'\(u\)\| &= \| \sigma \tau^m \phi_2\(u\)\| - \| \tau^m \phi_2\(u\)\|; \\
\| \sigma \phi'\(v\)\| - \| \phi'\(v\)\| &= \| \sigma \tau^m \phi_2\(v\)\| - \| \tau^m \phi_2\(v\)\|.
\end{align*}
\]
This yields
\[
\frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} = \frac{\| \sigma : \tau^m \phi_2 : u \|}{\| \sigma : \tau^m \phi_2 : v \|},
\]
which gives us
\[
\min \Delta \leq \frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} \leq \max \Delta
\]
by the induction hypothesis.

So let us further assume that
\[ \| \sigma \phi_2(u) \| \neq \| \phi_2(u) \| \text{ and } \| \sigma \phi_2(v) \| \neq \| \phi_2(v) \|. \]
It then follows from (29) that
\[
\begin{align*}
\| \sigma \phi'\(u\)\| - \| \phi'\(u\)\| &= \| \sigma : \tau^m \phi_2 : u \| + m \| \sigma : \phi_2 : u \|; \\
\| \sigma \phi'\(v\)\| - \| \phi'\(v\)\| &= \| \sigma : \tau^m \phi_2 : v \| + m \| \sigma : \phi_2 : v \|.
\end{align*}
\]
Since
\[
\min \Delta \leq \frac{\| \sigma : \tau^m \phi_2 : u \|}{\| \sigma : \tau^m \phi_2 : v \|}, \ \frac{\| \sigma : \phi_2 : u \|}{\| \sigma : \phi_2 : v \|} \leq \max \Delta
\]
by the induction hypothesis, we have from (30) that
\[
\min \Delta \leq \frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} \leq \max \Delta,
\]
as required.

**Case B.2.** \( \tau \) occurs at least \( \| u \| + 3 \) times in \( \phi_1 \).

In this case, it follows from Lemma 3.2 (ii) with \( \sigma, \tau \) interchanged and \( m = 0 \) that
\[
\begin{align*}
\| \sigma \phi'\(u\)\| - \| \phi'\(u\)\| &= \| \sigma \phi_1\(u\)\| - \| \phi_1\(u\)\| + \| \tau \phi_1\(u\)\| - \| \phi_1\(u\)\|; \\
\| \sigma \phi'\(v\)\| - \| \phi'\(v\)\| &= \| \sigma \phi_1\(v\)\| - \| \phi_1\(v\)\| + \| \tau \phi_1\(v\)\| - \| \phi_1\(v\)\|.
\end{align*}
\]
Here, by Corollary 3.6 (i) with \( \tau \) in place of \( \sigma \), we have \( \| \tau \phi_1(u) \| = \| \phi_1(u) \| \) if and only if \( \| \tau \phi_1(v) \| = \| \phi_1(v) \| \). Hence if \( \| \tau \phi_1(u) \| = \| \phi_1(u) \| \) or \( \| \tau \phi_1(v) \| = \| \phi_1(v) \| \), then, by (31),
\[
\| \sigma \phi'(u) \| - \| \phi'(u) \| = \| \sigma \phi_1(u) \| - \| \phi_1(u) \| ;
\]
\[
\| \sigma \phi'(v) \| - \| \phi'(v) \| = \| \sigma \phi_1(v) \| - \| \phi_1(v) \| ,
\]
and thus
\[
\| \sigma : \phi' : u \| = \| \sigma : \phi_1 : u \| ;
\]
\[
\| \sigma : \phi' : v \| = \| \sigma : \phi_1 : v \| .
\]
Then by the induction hypothesis,
\[
\min \Delta \leq \frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} \leq \max \Delta,
\]
as desired.

Now assume that
\[
\| \tau \phi_1(u) \| \neq \| \phi_1(u) \| \text{ and } \| \tau \phi_1(v) \| \neq \| \phi_1(v) \|. 
\]
We shall show that \( \| \sigma \phi_1(u) \| = \| \phi_1(u) \| \) if and only if \( \| \sigma \phi_1(v) \| = \| \phi_1(v) \| \). Let \( \| \sigma \phi_1(u) \| = \| \phi_1(u) \| \). If \( \phi_1 \) ends in \( \sigma \), then, by Corollary 3.6 (i) with \( \sigma, \tau \) interchanged, we have \( \| \sigma \phi_1(v) \| = \| \phi_1(v) \| \). On the other hand, if \( \phi_1 \) ends in \( \tau \), then, by Corollary 3.6 (ii) with \( \sigma, \tau \) interchanged, we get \( \| \tau \phi_2(u) \| = \| \phi_2(u) \| \), where \( \phi_1 = \tau \phi_2 \). But then from Lemma 3.3 (i) with \( \sigma, \tau \) interchanged, it follows that \( \| \tau^2 \phi_2(u) \| = \| \tau \phi_2(u) \| \), namely, \( \| \tau \phi_1(u) \| = \| \phi_1(u) \| \), which contradicts our assumption \( \| \tau \phi_1(u) \| \neq \| \phi_1(u) \| \). Therefore, we must have \( \| \sigma \phi_1(v) \| = \| \phi_1(v) \| \). Conversely, if \( \| \sigma \phi_1(v) \| = \| \phi_1(v) \| \), then, for a similar reason, it must follow that \( \| \sigma \phi_1(u) \| = \| \phi_1(u) \| \).

Thus if \( \| \sigma \phi_1(u) \| = \| \phi_1(u) \| \) or \( \| \sigma \phi_1(v) \| = \| \phi_1(v) \| \), then, from (31),
\[
\| \sigma \phi'(u) \| - \| \phi'(u) \| = \| \tau \phi_1(u) \| - \| \phi_1(u) \| ;
\]
\[
\| \sigma \phi'(v) \| - \| \phi'(v) \| = \| \tau \phi_1(v) \| - \| \phi_1(v) \| ,
\]
and so
\[
\| \sigma : \phi' : u \| = \| \tau : \phi_1 : u \| ;
\]
\[
\| \sigma : \phi' : v \| = \| \tau : \phi_1 : v \|. 
\]
Then by the induction hypothesis,
\[
\min \Delta \leq \frac{\| \sigma : \phi' : u \|}{\| \sigma : \phi' : v \|} \leq \max \Delta,
\]
as required.
So assume further that 
\[ \|\sigma \phi_1(u)\| \neq \|\phi_1(u)\| \text{ and } \|\sigma \phi_1(v)\| \neq \|\phi_1(v)\|. \]
It follows from (31) that 
\begin{align*}
\|\sigma \phi'(u)\| - \|\phi'(u)\| &= \|\sigma : \phi_1 : u\| + \|\tau : \phi_1 : u\|; \\
\|\sigma \phi'(v)\| - \|\phi'(v)\| &= \|\sigma : \phi_1 : v\| + \|\tau : \phi_1 : v\|.
\end{align*}
(32)
Since 
\[ \min \Delta \leq \frac{\|\tau : \phi_1 : u\|}{\|\tau : \phi_1 : v\|} \leq \max \Delta \]
by the induction hypothesis, we obtain from (32) that 
\[ \min \Delta \leq \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leq \max \Delta, \]
as desired. \hfill \Box

Claim C.
\[ \min \Delta \leq \frac{\|\tau : \phi' : u\|}{\|\tau : \phi' : v\|} \leq \max \Delta \]

Proof of Claim C. As in the proof of Claims A and B, writing 
\[ \phi' = \tau \phi_1, \]
where \( \phi_1 \) is a chain of type (C1), we consider two cases separately.

Case C.1. \( \sigma \) occurs at least \( \|u\| + 3 \) times in \( \phi_1 \).

As in Case B.1, write 
\[ \phi_1 = \tau^{m-1} \sigma \phi_2, \]
where \( m \geq 1 \) and \( \phi_2 \) is a chain of type (C1). Since \( \phi' = \tau \phi_1 \), 
\[ \phi' = \tau^m \sigma \phi_2. \]
It then follows from Lemma 3.1 (ii) that 
\begin{align*}
\|\tau \phi'(u)\| - \|\phi'(u)\| &= \|\tau \tau^m \phi_2(u)\| - \|\tau^m \phi_2(u)\| + \|\sigma \phi_2(u)\| - \|\phi_2(u)\|; \\
\|\tau \phi'(v)\| - \|\phi'(v)\| &= \|\tau \tau^m \phi_2(v)\| - \|\tau^m \phi_2(v)\| + \|\sigma \phi_2(v)\| - \|\phi_2(v)\|.
\end{align*}
(33)
By Corollary 3.6 (i), we have \( \|\sigma \phi_2(u)\| = \|\phi_2(u)\| \) if and only if \( \|\sigma \phi_2(v)\| = \|\phi_2(v)\| \). Also by Corollary 3.6 (iii), we get \( \|\tau \tau^m \phi_2(u)\| = \|\tau^m \phi_2(u)\| \) if and
only if \(\|\tau \tau' \sigma (v)\| = \|\tau' \sigma (v)\|\). Hence we can apply a similar argument as in Cases B.1 and B.2 to obtain the desired inequalities
\[
\min \Delta \leq \frac{\|\tau : \phi' : u\|}{\|\tau : \phi' : v\|} \leq \max \Delta.
\]

**Case C.2.** \(\tau\) occurs at least \(\|u\| + 3\) times in \(\phi_1\).

By Lemma 3.1 (i) with \(\sigma, \tau\) interchanged and \(m = 0\), we have
\[
\|\tau \phi'(u)\| - \|\phi'(u)\| = \|\tau \phi_1(u)\| - \|\phi_1(u)\|;
\]
\[
\|\tau \phi'(v)\| - \|\phi'(v)\| = \|\tau \phi_1(v)\| - \|\phi_1(v)\|.
\]

It then follows that
\[
\|\tau : \phi' : u\| = \|\tau : \phi_1 : u\|;
\]
\[
\|\tau : \phi' : v\| = \|\tau : \phi_1 : v\|,
\]
so that
\[
\min \Delta \leq \frac{\|\tau : \phi' : u\|}{\|\tau : \phi' : v\|} \leq \max \Delta
\]
by the induction hypothesis. This completes the proof of Claim C. \(\square\)

Now the theorem is completely proved. \(\square\)

The following theorem is the converse of Theorem 3.7.

**Theorem 3.8.** Let \(u, v \in F_2\) with \(\|u\| \geq \|v\|\), and \(\Omega, \Omega_1\) and \(\Omega_2\) be defined as in the statement of Theorem 3.7. Put \(k = \|u\| + 1\). Suppose that \(u\) and \(v\) are boundedly translation equivalent in \(F_2\). Then
\[
\|\sigma^{k+1} \psi_1(u)\| = \|\sigma^k \psi_1(u)\| \text{ if and only if } \|\sigma^{k+1} \psi_1(v)\| = \|\sigma^k \psi_1(v)\|;
\]
\[
\|\tau^{k+1} \psi_1(u)\| = \|\tau^k \psi_1(u)\| \text{ if and only if } \|\tau^{k+1} \psi_1(v)\| = \|\tau^k \psi_1(v)\|,
\]
for every \(\psi_1 \in \Omega_1\), and
\[
\|\sigma^{-k-1} \psi_2(u)\| = \|\sigma^{-k} \psi_2(u)\| \text{ if and only if } \|\sigma^{-k-1} \psi_2(v)\| = \|\sigma^{-k} \psi_2(v)\|;
\]
\[
\|\tau^{-k-1} \psi_2(u)\| = \|\tau^{-k} \psi_2(u)\| \text{ if and only if } \|\tau^{-k-1} \psi_2(v)\| = \|\tau^{-k} \psi_2(v)\|,
\]
for every \(\psi_2 \in \Omega_2\).

**Proof.** Suppose on the contrary that
\[
\|\sigma^{k+1} \psi_1(u)\| = \|\sigma^k \psi_1(u)\| \text{ but } \|\sigma^{k+1} \psi_1(v)\| \neq \|\sigma^k \psi_1(v)\|
\]
for some \(\psi_1 \in \Omega_1\). (The treatment of the other cases is similar.) Put
\[
K = \|\sigma^{k+1} \psi_1(v)\| - \|\sigma^k \psi_1(v)\|.
\]
By Lemma 3.2 (i) and the second inequality of (34), we have \( K \geq 1 \). By repeatedly applying Lemma 3.1 (i), we deduce that
\[
\| \sigma_{i+1} \psi_1 (u) \| = \| \sigma_{i+1} \psi_1 (v) \|
\]
for every \( i \geq k \); \( \| \sigma_{i+1} \psi_1 (v) \| = \| \sigma_k \psi_1 (v) \| + Ki + 1 - k \) for every \( i \geq k \).

Hence
\[
\frac{\| \sigma_{i+1} \psi_1 (u) \|}{\| \sigma_{i+1} \psi_1 (v) \|} = \frac{\| \sigma_k \psi_1 (u) \|}{\| \sigma_k \psi_1 (v) \| + Ki + 1 - k}
\]
for every \( i \geq k \), and thus
\[
\lim_{i \to \infty} \frac{\| \sigma_{i+1} \psi_1 (u) \|}{\| \sigma_{i+1} \psi_1 (v) \|} = 0.
\]
This contradiction to the hypothesis that \( u \) and \( v \) are boundedly translation equivalent in \( F_2 \) completes the proof. □

Consequently, in view of Theorems 3.7 and 3.8, we obtain the following algorithm to determine bounded translation equivalence in \( F_2 \).

**Algorithm 3.9.** Let \( u, v \in F_2 \) with \( \| u \| \geq \| v \| \), and let \( \Omega, \Omega_1 \) and \( \Omega_2 \) be defined as in the statement of Theorem 3.7. Put \( k = \| u \| + 1 \). Check if it is true that
\[
\| \sigma^{k+1} \psi_1 (u) \| = \| \sigma^k \psi_1 (u) \| \text{ if and only if } \| \sigma^{k+1} \psi_1 (v) \| = \| \sigma^k \psi_1 (v) \|;
\]
\[
\| \tau^{k+1} \psi_1 (u) \| = \| \tau^k \psi_1 (u) \| \text{ if and only if } \| \tau^{k+1} \psi_1 (v) \| = \| \tau^k \psi_1 (v) \|,
\]
for each \( \psi_1 \in \Omega_1 \), and if it is true that
\[
\| \sigma^{-k-1} \psi_2 (u) \| = \| \sigma^{-k} \psi_2 (u) \| \text{ if and only if } \| \sigma^{-k-1} \psi_2 (v) \| = \| \sigma^{-k} \psi_2 (v) \|;
\]
\[
\| \tau^{-k-1} \psi_2 (u) \| = \| \tau^{-k} \psi_2 (u) \| \text{ if and only if } \| \tau^{-k-1} \psi_2 (v) \| = \| \tau^{-k} \psi_2 (v) \|,
\]
for each \( \psi_2 \in \Omega_2 \). If so, conclude that \( u \) and \( v \) are boundedly translation equivalent in \( F_2 \); otherwise conclude that \( u \) and \( v \) are not boundedly translation equivalent in \( F_2 \).

4. Fixed point groups of automorphisms of \( F_2 \)

In this section, we shall demonstrate that there exists an algorithm to decide whether or not a given finitely generated subgroup of \( F_2 \) is the fixed point group of some automorphism of \( F_2 \). If \( H = \langle u_1, \ldots, u_k \rangle \) is a finitely generated subgroup of \( F_2 \), then we define
\[
|H| := \max_{1 \leq i \leq k} |u_i|.
\]
Clearly \( \| u_i \| \leq |u_i| \leq |H| \) for every \( i = 1, \ldots, k \).
Theorem 4.1. Let $H = \langle u_1, \ldots, u_k \rangle$ be a finitely generated subgroup of $F_2$. Suppose that $\phi$ is a chain of type (C1) with $|\phi| \geq 4|H| + 5$ such that $\|\phi(u_i)\| = \|u_i\|$ for every $i = 1, \ldots, k$. Then there exists a chain $\psi$ of type (C1) with $|\psi| < |\phi|$ such that $[\psi(u_i)] = [\phi(u_i)]$ for every $i = 1, \ldots, k$.

Proof. Since $\phi$ is a chain of type (C1) with $|\phi| \geq 4|H| + 5$, $\phi$ contains at least $2|H| + 3$ factors of $\sigma$ or $\tau$. Suppose that $\phi$ contains at least $2|H| + 3$ factors of $\sigma$ (the other case is similar). We may write

$$\phi = \tau^{m_t} \sigma^{\ell_1} \ldots \tau^{m_1} \sigma^{\ell_1} \phi',$$

where all $\ell_i, m_i > 0$ but $\ell_1$ and $m_t$ may be zero, and $\phi'$ is a chain of type (C1) which contains exactly $|H| + 2$ factors of $\sigma$.

Suppose that there exists $u_j$ ($1 \leq j \leq k$) such that $\|\sigma \phi'(u_j)\| \neq \|\phi'(u_j)\|$. Put

$$K = \|\sigma \phi'(u_j)\| - \|\phi'(u_j)\|.$$

Since $\phi'$ contains at least $\|u_j\| + 2$ factors of $\sigma$, by Lemma 3.2 (i), $K \geq 1$. Furthermore, since $\phi$ contains at least $2|H| + 3$ factors of $\sigma$ and $\phi'$ contains exactly $|H| + 2$ factors of $\sigma$,

$$\sum_{i=1}^{t} \ell_i \geq |H| + 1 \geq \|u_j\| + 1.$$

From the following claim, we shall obtain a contradiction.

Claim. $\|\phi(u_j)\| - \|\phi'(u_j)\| \geq \|u_j\| + 1$.

Proof of the Claim. First assume that $m_1 = 0$ in (35). Then $\phi = \sigma^{\ell_1} \phi'$, and so, from (36), $\ell_1 \geq \|u_j\| + 1$. By repeatedly applying Lemma 3.1 (i), we have

$$\|\phi(u_j)\| - \|\phi'(u_j)\| = \ell_1 K.$$

Since $K \geq 1$, it follows that

$$\|\phi(u_j)\| - \|\phi'(u_j)\| \geq \ell_1 \geq \|u_j\| + 1,$$

as desired.
Next assume that $m_1 > 0$ in (35). In view of Lemmas 3.1 and 3.2, we can observe that

$$\|\sigma \ell_1 \phi'(u_j)\| - \|\phi'(u_j)\| = \ell_1 K;$$

$$\|\tau m_1 \sigma \ell_1 \phi'(u_j)\| - \|\sigma \ell_1 \phi'(u_j)\| \geq m_1 K;$$

$$\ldots$$

$$\|\sigma \ell_t \ldots \tau m_1 \sigma \ell_1 \phi'(u_j)\| - \|\tau m_{t-1} \ldots \tau m_1 \sigma \ell_1 \phi'(u_j)\| \geq \ell_t K;$$

$$\|\tau m_1 \sigma \ell_t \ldots \tau m_1 \sigma \ell_1 \phi'(u_j)\| - \|\sigma \ell_t \ldots \tau m_1 \sigma \ell_1 \phi'(u_j)\| \geq m_t K.$$

Summing up all of these inequalities together with (36) yields

$$\|\phi(u_j)\| - \|\phi'(u_j)\| \geq \sum_{i=1}^{t} (\ell_i + m_i) K$$

$$\geq \left( \sum_{i=1}^{t} \ell_i \right) K$$

$$\geq \sum_{i=1}^{t} \ell_i$$

$$\geq \|u_j\| + 1,$$

as required. This completes the proof of the claim. \hfill \Box

It then follows from the claim that

$$\|\phi(u_j)\| \geq \|\phi'(u_j)\| + \|u_j\| + 1 \geq \|u_j\| + 1.$$

But this yields a contradiction to the hypothesis that $\|\phi(u_j)\| = \|u_j\|$. Therefore, we must have $\|\sigma \phi'(u_i)\| = \|\phi'(u_i)\|$ for every $i = 1, \ldots, k$. Then for each $i = 1, \ldots, k$,

(37) 

$$0 = \|\sigma \phi'(u_i)\| - \|\phi'(u_i)\| = n([\phi'(u_i)]; a) - 2n([\phi'(u_i)]; a, b^{-1})$$

$$= n([\phi'(u_i)]; a, a) + n([\phi'(u_i)]; a, b) - n([\phi'(u_i)]; a, b^{-1}).$$

Here, since $\phi'$ contains at least $\|u_i\| + 2$ factors of $\sigma$, by Lemma 2.1, there cannot occur proper cancellation in passing from $[\phi'(u_i)]$ to $[\sigma \phi'(u_i)]$, and so every subword of $[\phi'(u_i)]$ of the form $ab^{-1}$ or $ba^{-1}$ is necessarily part of a subword of the form $ab^{-r}a^{-1}$ or $ab^ra^{-1}$ ($r > 0$), respectively. This implies that

$$n([\phi'(u_i)]; a, b) \geq n([\phi'(u_i)]; a, b^{-1}),$$
so that, from (37),

\[ n([\phi'(u_i]); a, b) = n(([\phi'(u_i)]; a, b^{-1}) \text{ and } n([\phi'(u_i)]; a, a) = 0. \]

From the fact that no proper cancellation can occur in passing from \([\phi'(u_i)]\) to \([\sigma \phi'(u_i)]\) together with (38), each cyclic word \([\phi'(u_i)]\) must have the form

\[ [\phi'(u_i)] = [b^{s_{i1}}ab^{t_{i1}}a^{-1} \cdots b^{s_{ir}}ab^{t_{ir}}a^{-1}], \]

where every \(s_{ij}, t_{ij}\) is a nonzero integer, and hence

\[ [\sigma \phi'(u_i)] = [\phi'(u_i)] \]

for every \(i = 1, \ldots, t.\)

Thus letting

\[ \psi = \tau^{m_1}\sigma^{t_{i1}} \cdots \tau^{m_t}\sigma^{t_{it}} \phi', \]

we finally have

\[ [\psi(u_i)] = [\phi(u_i)] \]

for every \(i = 1, \ldots, t.\) Obviously \(|\psi| < |\phi|\), and so the proof of the theorem is completed.

We remark that Theorem 4.1 also holds if (C1) is replaced by (C2). From now on, let

\[ \delta_1 = (\{a^{\pm 1}\}, b), \ \delta_2 = (\{a^{\pm 1}\}, b^{-1}), \ \delta_3 = (\{b^{\pm 1}\}, a), \ \delta_4 = (\{b^{\pm 1}\}, a^{-1}) \]

be Whitehead automorphisms of \(F_2\) of type (W2).

**Lemma 4.2.** Let \(\alpha\) be a Whitehead automorphism of \(F_2\) of type (W2). Then \(\alpha\) can be expressed as a composition of \(\sigma^{\pm 1}, \tau^{\pm 1}\) and \(\delta_i\)'s.

**Proof.** If \(\alpha\) is not one of \(\sigma^{\pm 1}, \tau^{\pm 1}\) and \(\delta_i\)'s, then \(\alpha\) must be one of \((\{a^{-1}\}, b), (\{a^{-1}\}, b^{-1}), (\{b^{-1}\}, a)\) and \((\{b^{-1}\}, a^{-1})\). Then the following easy identities

\[ (\{a^{-1}\}, b) = \delta_1 \sigma^{-1}; \quad (\{a^{-1}\}, b^{-1}) = \delta_2 \sigma; \]
\[ (\{b^{-1}\}, a) = \delta_3 \tau^{-1}; \quad (\{b^{-1}\}, a^{-1}) = \delta_4 \tau \]

imply the required result.

The following two technical lemmas can be easily proved by direct calculations.

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Lemma 4.3. The following identities hold.
\[ \sigma \delta_1 = \delta_1 \sigma; \quad \sigma \delta_2 = \delta_2 \sigma; \quad \sigma \delta_3 = \delta_1 \delta_3 \sigma; \quad \sigma \delta_4 = \delta_4 \delta_2 \sigma; \]
\[ \tau \delta_1 = \delta_3 \tau; \quad \tau \delta_2 = \delta_4 \tau; \quad \tau \delta_3 = \delta_3 \tau; \quad \tau \delta_4 = \delta_4 \tau; \]
\[ \sigma^{-1} \delta_1 = \delta_1 \sigma^{-1}; \quad \sigma^{-1} \delta_2 = \delta_2 \sigma^{-1}; \quad \sigma^{-1} \delta_3 = \delta_2 \delta_3 \sigma^{-1}; \quad \sigma^{-1} \delta_4 = \delta_4 \delta_1 \sigma^{-1}; \]
\[ \tau^{-1} \delta_1 = \delta_4 \tau^{-1}; \quad \tau^{-1} \delta_2 = \delta_3 \tau^{-1}; \quad \tau^{-1} \delta_3 = \delta_4 \tau^{-1}; \quad \tau^{-1} \delta_4 = \delta_4 \tau^{-1}. \]

Lemma 4.4. The following identities hold.
\[ \sigma \tau^{-1} = \pi \delta_1 \sigma^{-1}; \quad \sigma^{-1} \tau = \pi^{-1} \delta_3 \sigma; \quad \tau \sigma^{-1} = \pi^{-1} \delta_3 \tau^{-1}; \quad \tau^{-1} \sigma = \pi \delta_1 \tau; \]
\[ \sigma \pi = \pi \delta_3 \tau^{-1}; \quad \sigma \pi = \pi \delta_4 \tau; \quad \sigma^{-1} \pi = \pi \delta_4 \tau; \quad \sigma^{-1} \pi^{-1} = \pi^{-1} \tau; \]
\[ \tau \pi = \pi \sigma^{-1}; \quad \tau \pi = \pi^{-1} \delta_3 \sigma; \quad \tau^{-1} \pi = \pi \sigma; \quad \tau^{-1} \pi^{-1} = \pi^{-1} \delta_2 \sigma, \]
where \( \pi \) is a Whitehead automorphism of \( F_2 \) of type (W1) that sends \( a \) to \( b \) and \( b \) to \( a^{-1} \).

The following corollary gives a nice description of automorphisms of \( F_2 \).

Corollary 4.5. Every automorphism \( \phi \) of \( F_2 \) can be represented as
\[ \phi = \beta \delta \phi', \]
where \( \beta \) is a Whitehead automorphism of \( F_2 \) of type (W1), \( \delta \) is a composition of \( \delta_i \)'s, and \( \phi' \) is a chain of type (C1) or (C2).

Proof. By Whitehead’s Theorem (cf. [II]) together with Lemmas 4.2 and 4.3 an automorphism \( \phi \) of \( F_2 \) can be expressed as
\[ \phi = \beta' \phi \tau_{q_1} \sigma_{p_1} \cdots \tau_{q_t} \sigma_{p_t}, \]
where \( \beta' \) is a Whitehead automorphism of \( F_2 \) of type (W1), \( \phi' \) is a composition of \( \delta_i \)'s, and both \( p_j, q_j \) are (not necessarily positive) integers for every \( j = 1, \ldots, t \). If not every \( p_j \) and \( q_j \) has the same sign (including 0), apply repeatedly Lemma 4.3 to the chain on the right-hand side of (39) to obtain that either \( \phi = \beta' \pi \tau_{r} \sigma_{m_k} \sigma_{l_k} \cdots \tau_{m_1} \sigma_{l_1} \) or \( \phi = \beta' \pi \tau_{r} \sigma_{m_k} \sigma_{l_k} \cdots \tau_{m_1} \sigma_{l_1} \), where \( \pi \) is as in Lemma 4.4 \( r \in \mathbb{Z} \), \( \delta \) is a composition of \( \delta_i \)'s, and both \( l_j, m_j \geq 0 \) for every \( j = 1, \ldots, k \). Putting \( \beta = \beta' \pi \), we obtain the required result.

The following is the main result of this section.

Theorem 4.6. Let \( H = \langle u_1, \ldots, u_k \rangle \) be a finitely generated subgroup of \( F_2 \). Suppose that \( H \) is the fixed point group of an automorphism \( \phi \) of \( F_2 \). Let \( \Omega_1 \) be the set of all chains of type (C1) or (C2) of length less than or equal to
4|H| + 4, and let \( \Omega_2 \) be the set of all compositions of \( \delta_i \)'s of length less than or equal to \( (2^{4|H|+4} + 1)|H| \). Put
\[
\Omega = \{ \beta \delta' \psi' \mid \psi' \in \Omega_1, \delta' \in \Omega_2, \text{ and } \beta \text{ is a Whitehead auto of } F_2 \text{ of type (W1)} \}.
\]
Then there exists \( \psi \in \Omega \) of which \( H \) is the fixed point group.

**Proof.** By Corollary 4.5, \( \phi \) can be written as
\[
\phi = \beta \delta \psi',
\]
where \( \beta, \delta \) and \( \phi' \) are indicated as in the statement of Corollary 4.5.

Since \( \phi(u_i) = u_i \) for every \( i = 1, \ldots, k \), it is easy to see that
\[
\| \phi'(u_i) \| = \| u_i \|
\]
for every \( i = 1, \ldots, k \). Then apply Theorem 4.1 continuously to obtain \( \psi' \in \Omega_1 \) such that
\[
[\psi'(u_i)] = [\phi'(u_i)]
\]
for every \( i = 1, \ldots, k \). Since \( |\delta \phi'(u_i)| = |\phi(u_i)| = |u_i| \leq |H| \) and \( |\psi'(u_i)| \leq 2^{4|H|+4}|u_i| \leq 2^{4|H|+4}|H| \) for every \( i = 1, \ldots, k \), we must have \( \delta' \in \Omega_2 \) such that
\[
\delta' \psi'(u_i) = \delta \phi'(u_i)
\]
for every \( i = 1, \ldots, k \), and hence
\[
\beta \delta' \psi'(u_i) = \beta \delta \phi'(u_i) = u_i
\]
for every \( i = 1, \ldots, k \). Therefore, letting
\[
\psi = \beta \delta' \psi',
\]
we finally have \( \psi \in \Omega \) and that \( H \) is the fixed point subgroup of \( \psi \). This completes the proof of the theorem. \( \Box \)

In conclusion, we naturally derive from Theorem 4.6 the following algorithm to decide whether or not a given finitely generated subgroup of \( F_2 \) is the fixed point group of some automorphism of \( F_2 \).

**Algorithm 4.7.** Let \( H = \langle u_1, \ldots, u_k \rangle \) be a finitely generated subgroup of \( F_2 \). Let \( \Omega_1, \Omega_2 \) and \( \Omega \) be defined as in the statement of Theorem 4.6. Clearly \( \Omega \) is a finite set. Check if there is \( \psi \in \Omega \) for which \( \psi(u_i) = u_i \) holds for every \( i = 1, \ldots, k \). If so, conclude that \( H \) is the fixed point group of some automorphism of \( F_2 \); otherwise conclude that \( H \) is not the fixed point group of any automorphism of \( F_2 \).
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