The double copy for heavy particles

Kays Haddad* and Andreas Helset†
Niels Bohr International Academy & Discovery Center, Niels Bohr Institute,
University of Copenhagen, Blegdamsvej 17, DK-2100, Copenhagen, Denmark
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We show how to double-copy Heavy Quark Effective Theory (HQET) to Heavy Black Hole Effective Theory (HBET) for spin $s \leq 1$. In particular, the double copy of spin-s HQET with scalar QCD produces spin-s HBET, while the double copy of spin-1/2 HQET with itself gives spin-1 HBET. Finally, we present novel all-order-in-mass Lagrangians for spin-1 heavy particles.

I. INTRODUCTION

An expanding family of field theories has been observed to obey double-copy relations [2–33]. In particular, scattering amplitudes of gravitational theories with massive matter can be calculated from the double copy of gauge theories with massive matter [34–44].

As Heavy Quark Effective Theory (HQET) [45] is derived from QCD and Heavy Black Hole Effective Theory (HBET) [46] is derived from gravity coupled to massive particles, the amplitudes of HBET should be obtainable as double-copies of HQET amplitudes. Indeed, this is the main result of this paper. We show through direct computation that the three-point and Compton amplitudes of HQET and HBET satisfy the schematic relations

\begin{align}
(QCD_{s=0}) \times (HQET_s) &= HBET_s, \\
(QCD_{s=1/2}) \times (HQET_{s=1/2}) &= HBET_{s=1},
\end{align}

for $s \leq 1$, where the spin-$s$ HQET and HBET matter states are equal in the free-field limit, and the spin-1 heavy polarization vectors are related to the heavy spinors through eq. (29). While we only show here the double copy for three-point and Compton amplitudes, invariance of the $S$-matrix under field redefinitions implies that eq. (1) holds more generally whenever QCD double-copies to gravitationally interacting matter. Equation (1) expands the double copy in powers of $\hbar$ since the operator expansion for heavy particles can be interpreted as an expansion in $\hbar$ [46]. The $\hbar \to 0$ limit of the double copy is currently of particular relevance [36, 39, 40].

We will begin in Section II with a brief review of the color-kinematics duality, and we will also discuss double-copying with effective matter fields. In Sections III to V we demonstrate the double copy at tree level for three-point and Compton amplitudes for spins 0, 1/2, and 1, respectively. We conclude in Section VI. The Lagrangians used to produce the amplitudes in this paper are presented in Appendix A. Among them are novel all-order-in-mass Lagrangians for spin-1 HQET and HBET given in eqs. (A6) and (A11).

II. COLOR-KINEMATICS DUALITY AND HEAVY FIELDS

An $n$-point gauge-theory amplitude, potentially with external matter, can be written as

\[ A_n = \sum_{i \in \Gamma} \frac{c_i n_i}{d_i}, \]

where $\Gamma$ is the set of all diagrams with only cubic vertices. Also, $c_i$ are color factors, $n_i$ encode the kinematic information, and $d_i$ are propagator denominators. A subset of the color factors satisfies the identity

\[ c_i + c_j + c_k = 0. \]

If the corresponding kinematic factors satisfy the analogous identity,

\[ n_i + n_j + n_k = 0, \]

and have the same anti-symmetry properties as the color factors, then the color and kinematic factors are dual. In this case, the color factors in eq. (2) can be replaced by kinematic factors to form the amplitude

\[ M_n = \sum_{i \in \Gamma} \frac{n_i' n_i}{d_i}, \]

which is a gravity amplitude with anti-symmetric tensor and dilaton contamination.\(^3\) In general, $n_i'$ and $n_i$ need not come from the same gauge theory, and only one of the sets must satisfy the color-kinematics duality.

In this paper we are interested in applying the double-copy procedure to HQET. A complicating factor to double-copying effective field theories (EFTs) is that Lagrangian descriptions of EFTs are not unique, as the Lagrangian can be altered by redefining one or more of the

\(^2\) We omit coupling constants for the sake of clarity. Reinstating them is straightforward: after double-copying the gauge theory coupling undergoes the replacement $g \to \sqrt{3}/2$.

\(^3\) For an amplitude of arbitrary multiplicity containing massive external states with an arbitrary spectrum, eq. (5) may not represent a physical amplitude [47]. However, for the cases under consideration in this paper, the application of the double copy will yield a well-defined gravitational amplitude.
fields. The LSZ procedure [48] guarantees the invariance of the S-matrix, and in particular eqs. (2) and (5), under such field redefinitions by accounting for wavefunction normalization factors (WNFs) $\mathcal{R}^{-1/2}$, which contribute to the on-shell residues of two-point functions. Under the double copy the WNFs from each matter copy combine in a spin-dependent manner, which complicates the matching of the double-copied amplitude to one derived from a gravitational Lagrangian.

In order to ease the double-copying of HQET to HBET, we would like to avoid having to compensate for the WNFs. This can be achieved by ensuring that HQET and HBET have the same WNFs – i.e. that the asymptotic states for the spin-s particles in HQET and HBET are equal – and double-copying HQET with QCD, which has a trivial WNF.

The asymptotic states – that is, the states in the free-field limit – of the canonically normalized theories (given by complex Klein-Gordon, Dirac, and symmetry-broken Proca actions) are related to their respective asymptotic heavy states (labelled by a velocity $v$) in position-space through

$$\varphi(x) = e^{-imv \cdot x} \left[ 1 - \frac{1}{2m + iv \cdot \partial + \alpha^2 / 2m} \right] \phi_v(x),$$  

$$\psi(x) = e^{-imv \cdot x} \left[ 1 + \frac{i}{2m + iv \cdot \partial} (\phi - v \cdot \partial) \right] \psi_v(x),$$  

$$A^\mu(x) = e^{-imv \cdot x} \left[ \delta^\mu_v - \frac{iv^\mu \partial_v - (\partial^\mu - \partial^\nu/2m) v^\nu/2m}{m + iv \cdot \partial/2} \right] B_v^\mu(x),$$

where $\alpha^\mu = a^\mu - \nu^\mu(v \cdot a)$ for a vector $a^\mu$. Here, the momentum is decomposed as $p^\mu = mv^\mu + k^\mu$ in the usual heavy-particle fashion. The Lagrangians for the heavy fields in eq. (7) are given in Appendix A. Converting to momentum space, eq. (7) gives the WNFs

$$R_{s=0}^{-1/2}(p) = \frac{1}{2m} \left[ 1 + \frac{k^2}{4m^2 + 2mv \cdot k - k^2} \right],$$  

$$R_{s=1/2}^{-1/2}(p) = 1 + \frac{1}{2m + v \cdot k} (k - v \cdot k),$$  

$$\left(R_{s=1}^{-1/2}(p)\right)^\nu_{\mu} = \frac{1}{2m} \left[ \delta^\nu_{\mu} - \frac{v^\mu k^\nu + k^\mu k^\nu/2m}{m + v \cdot k/2} \right].$$

We will demonstrate that spin-s HBET amplitudes can directly be obtained by double-copying spin-s HQET amplitudes with scalar QCD for spins $s \leq 1$. At $s = 1$ there is also the possibility to double-copy using two spin-1/2 amplitudes. We will discuss this point further below.

### III. Spin-0 Gravitational Amplitudes

We begin with the simplest case of spinless amplitudes. Consider first the three-point amplitude. For scalar HQET we have that

$$A^H_{s=0} = -T^{a}_{ij} \epsilon^{*\mu} _{q} \phi^a_v \left(1 + \frac{k^2_1 + k^2_2}{4m^2}\right) \phi_v$$

$$\times \left[ v^\mu + \frac{(k_1 + k_2)_{\mu}}{2m} \right] + \mathcal{O}(m^{-4}),$$  

where $k_2 = k_1 - q$. For scalar QCD the amplitude is

$$A^Q_{s=0} = -T^{a}_{ij} \epsilon^{*\mu} _{q} \left[2mv^\mu + (k_1 + k_2)_{\mu}\right].$$

Note that we have left the external heavy scalar factor $\phi_v$ explicit in the HQET amplitude. This is because, in contrast to the canonically normalized scalar fields, the heavy scalar factors are not equal to 1 in momentum space. Indeed, for the HQET amplitude to be equal to the QCD amplitude, the heavy scalar factor in momentum space must be equal to the inverse of eq. (8a). This will cancel the extra factor in round brackets in eq. (9).

The double copy at three-points is simply given by a product of amplitudes:

$$A^H_{s=0} A^Q_{s=0} = \epsilon^{*\mu}_v \epsilon^{*\nu}_v \phi^a_v \left(1 + \frac{k^2_1 + k^2_2}{4m^2}\right) \phi_v$$

$$\times 2m \left[ v^\mu v^\nu + v^\mu \frac{k_{1\nu} + k_{2\nu}}{m} + \frac{(k_1 + k_2)_{\mu}(k_1 + k_2)_{\nu}}{4m^2} \right] + \mathcal{O}(m^{-3}).$$

As the only massless particle in this process is external, we can easily eliminate the massless non-graviton degrees of freedom by identifying the outer product of gluon polarization vectors with the graviton polarization tensor. After doing so, eq. (11) agrees with the three-point amplitude derived from eq. (A9).

As another example, consider the Compton amplitude. The color decomposition for Compton scattering is

$$A^c_{i} = c_s n_s + c_t n_t + c_u n_u,$$

where

$$c_s = T^{a}_{ik} T^{b}_{kj}, \quad c_t = i f^{abc} T^{c}_{ij}, \quad c_u = T^{a}_{ik} T^{a}_{kj}.$$

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4. Note that $\mathcal{R}^{-1/2} = 1$ for canonically normalized fields. The WNF for an effective state $\xi$ can thus be determined by relating it to a canonically normalized state $\epsilon$ through $\epsilon = \mathcal{R}^{-1/2} \cdot \xi$. 

5. We have computed all Compton amplitudes using NRQCD propagators. It is also possible to perform the computations using HQET propagators: in that case, a comparison to the Compton amplitude for the emission of bi-adjoint scalars from heavy particles (described by the Lagrangians in eqs. (A1) to (A3)) – analogous to the treatment in ref. [19] – is necessary to identify kinematic numerators. Both methods produce the same results.
The kinematic numerators for scalar HQET are
\[ n_s^{H,s=0} = -2m\phi^*\epsilon_q^*\epsilon_{q_2}v_\mu v_\nu \left( \frac{1}{4m^2} + \frac{k_1^2 + k_2^2}{4m^2} \right) \phi_v, \] (13a)
\[ n_t^{H,s=0} = 0, \] (13b)
\[ n_u^{H,s=0} = n_u^{H,s=0}|_{q_1+q_2}, \] (13c)
where \( k_2 = k_1 - q_1 - q_2 \). Those for scalar QCD are
\[ n_s^{n=0} = -4m^2\epsilon_q^*\epsilon_{q_2}v_\mu v_\nu, \] (14a)
\[ n_t^{n=0} = 0, \] (14b)
\[ n_u^{n=0} = n_u^{n=0}|_{q_1+q_2}. \] (14c)

For brevity we have written the numerators under the conditions \( k_1 = q_1 \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0 \); the initial residual momentum can always be set to 0 by reparameterizing \( v \), and such a gauge exists for opposite helicity gluons. We have checked explicitly up to and including \( \mathcal{O}(m^{-2}) \) that the following results hold when relaxing all of these conditions.

Both the HQET and QCD numerators satisfy the color-kinematics duality in the form
\[ c_s - c_u = c_t \iff n_s - n_u = n_t. \] (15)
We can therefore replace the color factors in the HQET amplitude with the QCD kinematic numerators,
\[ \mathcal{M}_4^{H,s=0} = \frac{n_s^{H,s=0}}{d_s} + \frac{n_t^{H,s=0}}{d_t} + \frac{n_u^{H,s=0}}{d_u}. \] (16)

Identifying once again the outer products of gluon polarization vectors with graviton polarization tensors, we find that the Compton amplitude derived from eq. (A9) agrees with eq. (16).

To summarize, we have explicitly verified that
\[ (\text{QCD}_{s=0}) \times (\text{HQET}_{s=0}) = \text{HBET}_{s=0} \] (17)
for three-point and Compton amplitudes.

IV. SPIN-1/2 GRAVITATIONAL AMPLITUDES

We now move on to the double copy of spin-1/2 HQET with scalar QCD to obtain spin-1/2 HBET. The three-point spin-1/2 HQET amplitude is
\[ \mathcal{A}_3^{H,s=1/2} = -T_{ij}^{a\mu} \tilde{u}_v \epsilon_{q_1}^* \epsilon_{q_2}^* \mu \left( v_\mu + \frac{k_1^\mu}{m} + \frac{k_2^\mu - k_1 \cdot q}{4m^2} v_\mu \right) \]
\[ - \frac{i}{2m} T_{ij}^{a\mu} \tilde{u}_v \sigma^\alpha \beta_{\mu}^\cdot \epsilon_{q_1}^* \epsilon_{q_2}^* \mu \left( q_\alpha \eta_{\beta\mu} - \frac{1}{2m} q_\alpha k_1 \mu v_\mu \right) \]
\[ + \mathcal{O}(m^{-3}) \] (18)

Double-copying with scalar QCD, we find
\[ \mathcal{M}_3^{H,s=1/2} = \mathcal{A}_3^{H,s=1/2}, \] (19)

where \( \mathcal{M}_3^{H,s=1/2} \) is the amplitude derived from eq. (A10).

We turn now to Compton scattering. For brevity we write here the amplitudes in the case \( k_1 = q_1 \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0 \). We have checked explicitly that the results hold when these conditions are relaxed. Also, we have performed the calculation up to \( \mathcal{O}(m^{-2}) \) but only present the kinematic numerators up to \( \mathcal{O}(m^{-1}) \). They are
\[ n_s^{n=1/2} = -2m\bar{u}_v \epsilon_q^* \epsilon_{q_2} \]
\[ - \frac{i}{2m} \sigma_{\mu\nu}(\epsilon_q^* \epsilon_{q_1}^* \epsilon_{q_2}^* \mu \nu + \epsilon_{q_2}^* \epsilon_{q_1}^* \epsilon_q^* \mu \nu - q_{\alpha}^* \epsilon_{q_2}^* \epsilon_{q_1}^* \mu \nu) \] (20a)
\[ n_t^{n=1/2} = 0, \] (20b)
\[ n_u^{n=1/2} = n_n^{n=1/2}|_{q_1+q_2}. \] (20c)

In this case, the color-kinematic duality eq. (15) is violated at \( \mathcal{O}(m^{-2}) \). Nevertheless, since the scalar QCD kinematic numerators satisfy the duality we can use them to double copy the spin-1/2 Compton amplitude. Doing so we find
\[ \mathcal{M}_4^{n=1/2} = \frac{n_s^{n=0} n_n^{n=1/2}}{d_s} + \frac{n_t^{n=0} n_n^{n=1/2}}{d_t} + \frac{n_u^{n=0} n_n^{n=1/2}}{d_u}, \] (21)

where \( \mathcal{M}_4^{n=1/2} \) is the spin-1/2 HBET Compton amplitude derived from eq. (A10).

We have seen that
\[ (\text{QCD}_{s=0}) \times (\text{HQET}_{s=1/2}) = \text{HBET}_{s=1/2} \] (22)
for the three-point and Compton amplitudes.

V. SPIN-1 GRAVITATIONAL AMPLITUDES

Gravitational amplitudes with spin-1 matter can be obtained by double-copying two gauge theories with matter in two ways: spin-0 × spin-1 or spin-1/2 × spin-1/2 [41–43]. This fact also holds for heavy particles. We now show this in two examples by deriving the spin-1 gravitational three-point and Compton amplitudes using both double-copy procedures.

A. 0 × 1 Double Copy

The three-point spin-1 HQET amplitude is
\[ \mathcal{A}_3^{H,s=1} = T^{a\beta}_{\mu\nu} \epsilon_q^* \epsilon_{q_2} \]
\[ + \frac{1}{2m}(\eta_{\alpha\beta}(k_1 + k_2)_{\mu} - 2q_{\alpha}\eta_{\beta\mu} + 2q_{\alpha}\eta_{\beta\mu}) \]
\[ + \frac{1}{2m^2} \epsilon_q^* (k_1 \cdot q_2 + q_2 \cdot q_{\alpha} + q_{\beta} k_{1\alpha}), \] (23)
where \( k_{1\mu} = k_1^\mu - q_1^\mu \). Double-copying with scalar QCD we find
\[ \mathcal{M}_3^{H,s=1} = \mathcal{A}_3^{H,s=1}, \] (24)
where $M^H_{i=1}$ is the amplitude derived from eq. (A11) after applying the field redefinition in eq. (A12).

Compton scattering for spin-1 HQET is given by the kinematic numerators

$$n^H_{s=1} = 2m^2 \tilde{c}_v \left[ v \cdot \epsilon^*_{q_1} v \cdot \epsilon^*_{q_2} \eta_{\alpha \beta} + \frac{v_\mu}{m} \left( \eta_{\alpha \mu} \eta_{\beta \nu} - \eta_{\alpha \nu} \eta_{\beta \mu} \right) \left( \epsilon^*_{q_1 \mu} q^\nu_{q_2} + \epsilon^*_{q_2 \nu} q^\mu_{q_1} \right) - \frac{v \cdot q_2}{2m} \left( \epsilon^*_{q_1 \alpha} \epsilon^*_{q_2 \beta} - \epsilon^*_{q_2 \alpha} \epsilon^*_{q_1 \beta} \right) \right],$$

(25a)

$$n^H_{t=1} = 0,$$

(25b)

$$n^H_{u=1} = n^{H,s=1}_{s=1} |q_1 + q_2|,$$

(25c)

where, for brevity, we again write the numerators up to $O(m^{-1})$ and in the case where $k_1 = \epsilon_i \cdot \epsilon_j = q_i \cdot \epsilon_j = 0$. We have performed the calculation up to $O(m^{-2})$ and checked the general case explicitly. The double copy becomes

$$M^H_{s=1} = \frac{n^s=0 n^H_{s=1}}{d_s} + \frac{n^s=0 n^H_{t=1}}{d_t} + \frac{n^s=0 n^H_{u=1}}{d_u},$$

(26)

where $M^H_{s=1}$ is derived from eq. (A11) after applying the field redefinition in eq. (A12).

Thus, we find that

$$(\text{QCD}_{s=0}) \times (\text{HQET}_{s=1}) = \text{HBET}_{s=1}$$

(27)

for three-point and Compton amplitudes.

B. $\frac{1}{2} \times \frac{1}{2}$ Double Copy

The spin-1 gravitational amplitudes can also be obtained by double-copying the spin-1/2 HQET amplitudes. To do so, we use the on-shell heavy particle effective theory (HPET) variables of ref. [30] to modify eq. (2.11) of ref. [43] for the case of heavy particles. Using the fact that the on-shell HPET variables correspond to momenta $p^\mu_k = m_k v^\mu$ with mass $m_k = m(1 - k^2 / 4m^2)$, following the derivation of ref. [43] leads to

$$M^H_{i=\frac{1}{2} \times \frac{1}{2}} = \frac{m_k m_{k_2}}{m} \sum_{\alpha \beta} K_{\alpha \beta} \text{Tr}[\mathcal{A}^H_{\alpha \alpha} P_+ \mathcal{A}^H_{\beta \beta} P_- ^*],$$

(28)

where $P_\pm = (1 \pm \gamma_5) / 2$, $K_{\alpha \beta}$ is the KLT kernel, and $\alpha, \beta$ represent color orderings. Here $\mathcal{A}^H$ and $\mathcal{A}^H$ are amplitudes with the external states stripped, and $\mathcal{A}^H = -\gamma_5 (\mathcal{A}^H)^\dagger \gamma_5$. We have also adopted the convention that only the initial matter momentum is incoming. Converting to the on-shell HPET variables, it can be easily seen that

$$\varepsilon_{ij}^{\gamma_5} = \frac{1}{2\sqrt{2m_k}} \bar{u}_i^\dagger(p) \gamma_5 \gamma_{\mu} u_j^\dagger(p),$$

(29)

with $I, J$ being massive little group indices. Given the WNF for the heavy spinors, the WNF for the polarization vector can easily be computed by comparing eq. (29) to its canonical polarization vector analog. We find that it is indeed given by eq. (28).

Applying eq. (28) to eq. (18) with the three-point KLT kernel $K_3 = 1$, we immediately recover the left-hand side of eq. (24). For Compton scattering the KLT kernel is

$$K_4 = \frac{(s - m^2)(u - m^2)}{2q_1 \cdot q_2}.$$

(30)

Then, applying eq. (28) to the spin-1/2 HQET Compton amplitude with $k_1 \cdot \epsilon_j, \epsilon_i \cdot \epsilon_j \neq 0$ up to and including terms of order $O(m^{-2})$, we find eq. (26) up to $O(m^{-1})$. When imposing $k_1 = q_i \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0$, cancellations make the double copy valid up to $O(m^{-2})$. The extension to higher inverse powers of the mass amounts to simply including the contributions of higher-order operators in the HQET and HBET amplitudes.

Therefore, by using eq. (28) to convert heavy spinors in amplitudes to heavy polarization vectors, we have shown that

$$(\text{HQET}_{s=1/2}) \times (\text{HQET}_{s=1/2}) = \text{HBET}_{s=1}$$

(31)

for three-point and Compton amplitudes.

VI. CONCLUSION

We have shown that the three-point and Compton amplitudes derived from HQET can be double-copied to those of HBET for spins $s \leq 1$. As long as the matter states of HQET and HBET are related through the double copy, in the sense described in Section II, and as long as higher-point amplitudes obey the spectral condition of ref. [47], we see no obstacles to extending the double copy to higher-point amplitudes.

As mentioned in the introduction, due to the operator expansion of HPETs, the double-copy relation between HQET and HBET can be studied at each order in the $\hbar$ expansion, with the classical limit being of special interest. Studying the double copy of HPETs through this lens may provide some insight into the connection between the double copy with matter at the quantum and classical levels. We leave this study for future work.

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Appendix A: Lagrangians for heavy particles

We present Lagrangians for heavy particles coupled to bi-adjoint scalars, gluons, and gravitons. The heavy-particle Lagrangians were used to derive the scattering amplitudes in the paper. For clarity, we omit the subscript \( v \) for the heavy spin-1 fields.

### Bi-adjoint scalars and heavy particles

We couple the bi-adjoint scalars \( \Phi \) to heavy particles with spins \( s \leq 1 \). The spin-0 Lagrangian is

\[
L_{\text{bi-adjoint}}^{s=0} = \phi_v^* \left[ iv \cdot \partial - \frac{\partial^2}{2m} - y_s \frac{\partial^2 - y_s \Phi}{2m} \right] \phi_v .
\]

The spin-1/2 Lagrangian is

\[
L_{\text{bi-adjoint}}^{s=1/2} = \bar{Q}_v \left[ iv \cdot \partial + y_f \Phi + (i\partial_\perp) \frac{1}{2m + iv \cdot \partial - y_f \Phi} \right] Q_v .
\]

The spin-1 Lagrangian is

\[
L_{\text{bi-adjoint}}^{s=1} = -B_\mu^s (iv \cdot \partial) B^\mu - \frac{1}{4m} B_\mu^s B^{\mu\nu} + \frac{y_v}{2m} B_\mu^s \Phi B^\mu - (F_\mu^s B_\lambda^s) \frac{2}{m + \frac{1}{m} \partial_\perp^2} (F_\lambda^s B_\mu^s)
\]

where

\[
F_\mu^s = \left( \pm \frac{i}{2} \partial^\mu - \frac{1}{2m} \partial^\mu (v \cdot \partial) + \frac{y_v \Phi}{2m} \right) .
\]

The coupling constants between the bi-adjoint scalars and the heavy scalars, fermions, and vectors are \( y_s, y_f, \) and \( y_v \), respectively.

### Gluons and heavy particles

We couple gluons to heavy particles. The covariant derivative in this case is given by \( D_\mu = \partial_\mu + ig_s T^a A_\mu^a \). The scalar Lagrangian is

\[
L_{\text{gluon}}^{s=0} = \phi_v^* \left[ iv \cdot D - D_\perp^2 \frac{1}{2m} + \frac{D_\perp^2 \partial_\perp^2}{2m} \right] \phi_v .
\]

The spin-1/2 Lagrangian is

\[
L_{\text{gluon}}^{s=1/2} = \bar{Q}_v \left[ iv \cdot D + (i\partial_\perp) \frac{1}{2m + iv \cdot D} (i\partial_\perp) \right] Q_v .
\]

The spin-1 Lagrangian [51] with gyromagnetic ratio \( g = 2 \) can be written as

\[
L_{\text{gluon}}^{s=1} = -B_\mu^s (iv \cdot D) B^\mu - \frac{1}{4m} B_\mu^s B^{\mu\nu} + \frac{ig}{2m} F^{\mu\nu} B_\mu^s B_\nu - (E_\mu^s B_\lambda^s) \frac{2}{m + \frac{1}{m} \partial_\perp^2} (E_\lambda^s B_\mu^s)
\]

where

\[
E_\mu^s = \left( \pm \frac{i}{2} D^\mu - \frac{1}{2m} D^\mu (v \cdot D) \pm \frac{ig v^\mu F^{\nu\mu}}{2m} \right) .
\]

The heavy spin-1 states described by this Lagrangian are related to the canonical massive spin-1 states through

\[
A^\mu(x) = e^{-imv \cdot x} \left[ \delta^\mu_{\nu} - \frac{1}{1 + iv \cdot \partial/m} \frac{iv^\mu \partial_{\nu}}{m} \right] B^\nu(x) .
\]

To obtain the desired heavy spin-1 states we apply the field redefinition

\[
B_\mu \rightarrow \left[ \delta^\nu_{\mu} + \frac{1}{2m^2} (-v_\mu v \cdot D + D_\mu) D^\nu \right] B_\nu + O(m^{-3}) .
\]
Gravitons and heavy particles

We couple gravitons to heavy particles. The spin-0 Lagrangian is

$$\sqrt{-g} L_{\text{graviton}}^{s=0} = \sqrt{-g} \phi_0 \left[ A_1 + (A_2) - \frac{1}{2m + i (\nu^\mu \nabla_\mu + \nabla_\mu \nu^\mu) - A_1} (A_2^+) \right] \phi_0, \quad (A9a)$$

where

$$A_1 = \frac{1}{2} g^{\mu \nu} (v_\mu \nabla_\nu + \nabla_\mu v_\nu) + \frac{1}{2} m (g^{\mu \nu} - \eta^{\mu \nu}) v_\mu v_\nu - \frac{1}{2m} \nabla_\mu ((g^{\mu \nu} - \eta^{\mu \nu}) \nabla_\nu + \eta^{\mu \nu} \nabla_\perp), \quad (A9b)$$

$$A_{2\pm} = \frac{1}{2m} (i m v_\mu - \nabla_\mu) ((g^{\mu \nu} - \eta^{\mu \nu}) (-i m v_\nu + \nabla_\nu)) - \frac{1}{2m} \nabla_\mu (\eta^{\mu \nu} \nabla_\perp) \pm \frac{1}{2} i [\nabla_\mu, v_\nu], \quad (A9c)$$

with $v^\mu \equiv \eta^{\mu \nu} v_\nu$ and $\nabla_\perp \equiv \nabla_\mu - v_\mu (v^\nu \nabla_\nu)$. The spin-1/2 Lagrangian is

$$\sqrt{-g} L_{\text{graviton}}^{s=1/2} = \sqrt{-g} \bar{Q} v \left[ i \nabla + B + (i \nabla + B) P_+ \frac{1}{2m - (i \nabla + B) P_- (i \nabla + B)} \right] Q_v, \quad (A10a)$$

where $\nabla \equiv \delta^a_{\mu} \gamma^a \nabla_\mu$ and

$$B = (e^a_\mu - \delta^a_\mu) (i \gamma^a \nabla_\mu + m \gamma^a v_\mu). \quad (A10b)$$

The spin-1 Lagrangian can be written as

$$\sqrt{-g} L_{\text{graviton}}^{s=1} = \sqrt{-g} \left[ -\frac{m}{2} (v_\mu B^*_\rho) (v_\rho B_\sigma) ((g^{\sigma \rho} - \eta^{\sigma \rho}) g^{\nu \sigma} - (g^{\mu \rho} - \eta^{\mu \rho}) (g^{\nu \rho} - \eta^{\nu \rho})) \right. \nonumber$$

$$\left. + \frac{i}{2} \left[ (\nabla_\mu B^*_\rho) (v_\rho B_\sigma) - (v_\rho B^*_\rho) (\nabla_\rho B_\sigma) \right] (g^{\mu \rho} g^{\nu \sigma} - g^{\mu \sigma} g^{\rho \nu}) \right. \nonumber$$

$$\left. - \frac{1}{4m} B^{\mu \rho} B_{\rho \sigma} g^{\mu \rho} g^{\nu \sigma} - (C^a_{\mu} B^*_a) \frac{1}{D} \left( C^a_B B_a \right) \right], \quad (A11a)$$

where

$$C^a_{\pm} = - \frac{m}{2} (g^{\alpha \nu} - \eta^{\alpha \nu}) v_\nu \pm \frac{i}{2} v_\nu \left[ g^{\mu \rho} g^{\alpha \nu} - g^{\alpha \mu} g^{\rho \nu} \right] \nabla_\mu \left( v_\nu \pm \frac{i}{m} \nabla_\rho \right), \quad (A11b)$$

$$D = \frac{m}{2} (v_\mu v_\sigma g^{\rho \sigma} + \frac{1}{2m} v_\nu \left[ g^{\mu \rho} g^{\nu \sigma} - g^{\mu \sigma} g^{\nu \rho} \right] \nabla_\mu \nabla_\rho v_\nu \sigma. \quad (A11c)$$

Note that though the velocity four-vector is constant its covariant derivative does not vanish because of the metric connection. The heavy spin-1 states described by this Lagrangian are related to the canonical massive spin-1 states through eq. (A7). To obtain the desired heavy spin-1 states we apply the field redefinition

$$B_\mu \rightarrow \left[ \delta^\mu_\nu + \frac{1}{2m^2} \left( -g^{\alpha \beta} v_\alpha D_\beta v_\mu + D_\mu \right) g^{\nu \lambda} D_\lambda \right] B_\nu \nonumber$$

$$+ \mathcal{O}(m^{-3}). \quad (A12)$$

Extending this redefinition to higher orders in $1/m$ is straight-forward.

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