A criterion of graded coherentness of tensor algebras and its application to higher dimensional Auslander-Reiten theory.

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Abstract

Even if a ring $A$ is coherent, the polynomial ring $A[X]$ in one variable could fail to be coherent. In this note we show that $A[X]$ is graded coherent with the grading $\deg X = 1$. More generally, we give a criterion of graded coherentness of the tensor algebra $T_A(\sigma)$ of a certain class of bi-module $\sigma$.

As an application of the criterion, we show that there is a relationship between higher dimensional Auslander-Reiten theory and graded coherentness of higher preprojective algebras.

1 Introduction

In noncommutative projective geometry there is the important procedure which construct a graded ring from certain data. The resultant graded ring is called the twisted homogeneous coordinate ring and plays an important role in noncommutative projective geometry and representation theory (11, 12, 13, 14, 15). This construction is similar to the procedure to take the tensor algebra $T_A(\sigma)$ over a ring $A$ of an $A$--$A$ bi-module $\sigma$ equipped with the grading $\deg \sigma := 1$. In some case twisted homogeneous coordinate rings are actually tensor algebras. Although these are natural procedures to obtain graded rings, it is recognized that in general we can’t expect that the resultant graded algebras are graded Noetherian and that only we can hope that these are graded coherent. For more details see [6] Introduction, which suggest us that it is inevitable that we study graded coherentness more deeply.

The condition of (graded) coherentness is similar but weaker than that of (graded) Noetherianness. It is known that many theorems about finitely generated modules over (graded) Noetherian rings can be extended to finitely presented modules over (graded) coherent rings. However there is a significant difference between them. It is standard that if a ring $A$ is right Noetherian, then the polynomial ring $A[X]$ in one variable is again right Noetherian. Contrary to this, it is well-known for specialists that even if a ring $A$ is right coherent, the polynomial ring $A[t]$ in one variable could fail to be coherent. In fact, Soublin [7] showed that the ring $A = \mathbb{Q}[[x, y]]^{\text{finite}}$ has such a property.

In this note we show that if we take gradings into account, then the polynomial ring $A[X]$ become graded coherent for any coherent ring $A$. Namely we have

Theorem 1.1 (Corollary 2.4). If a ring $A$ is right coherent, then the polynomial ring $A[X]$ with the grading $\deg X = 1$ is graded right coherent.

More generally we give a criterion of graded coherentness of the tensor algebra $T_A(\sigma)$ of a certain class of an $A$--$A$ bi-module $\sigma$. The condition on an $A$--$A$ bi-module $\sigma$ is of homological nature. Therefore the tensor algebra of a bi-modules satisfying the condition is well-behaved and contains an important example: higher preprojective algebra, which is studied in Section 3.

In Section 3, we apply the criterion to show that there is a relationship between higher dimensional Auslander-Reiten theory of $n$-representation infinite algebra and graded coherentness of higher preprojective algebras. (for the back ground and the definitions of these, please see the beginning of Section 3.)

Notations and conventions

Unless otherwise noted all modules and all graded modules considered are right modules and graded modules. For a graded module $M := \bigoplus_{n \in \mathbb{Z}} M_n$, we denote by $M(i)$ the $i$-degree shift, that is, $M(i)_n := M_{n+i}$. We denote by $\otimes$ the graded tensor product.

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2 A criterion of graded coherentness

First we recall the definitions of coherent modules and coherent rings.

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1. An $A$-module (resp. graded $A$-module) $M$ is called coherent (resp. graded coherent), if it satisfies the following two conditions:
   (i) $M$ is finitely generated over $A$;
   (ii) for every $A$-homomorphism (resp. graded $A$-homomorphism) $f : P \to M$ with $P$ a finitely generated free $A$-module (resp. a finitely generated graded free $A$-module), the kernel $\ker(f)$ of is finitely generated over $A$.

2. $A$ is called right coherent (resp. graded right coherent) if the regular $A$-module $A_A$ is coherent (resp. graded coherent).

It is known that the full subcategory $\text{Coh} A$ of coherent modules is an extension closed abelian subcategory of the category $\text{Mod} A$ of $A$-modules. A coherent $A$-module is finitely presented over $A$. If a ring $A$ is right coherent. Then a $A$-module $M$ is coherent if and only if it is finitely presented over $A$. (See e.g. [6].)

Let $A$ be a ring. A complex $M^\bullet$ of $A$-modules is called pure if its cohomology group concentrates in degree 0, i.e., $H^i(M^\bullet) = 0$ for $i \neq 0$.

**Theorem 2.2.** Let $A$ be a right coherent ring and $\sigma$ an $A$-$A$-bimodule such that $\sigma$ is finitely presented as right $A$-modules and the iterated derived tensor product $\sigma^{\otimes_A}_{\leq n}$ of $\sigma$ is pure for $n \geq 1$. We denote $\sigma^{\otimes_A}_{\leq n}$ by $\sigma^n$. Let $T$ be the tensor algebra $T_A(\sigma)$ of $\sigma$ over $A$:

$$T := A \oplus \sigma \oplus \sigma^2 \oplus \sigma^3 \oplus \cdots$$

with the grading $\deg \sigma := 1$.

1. Assume that for any finitely presented $A$-module $M$ there is a natural number $m$ such that $(M \otimes_A \sigma^m) \otimes_A \sigma^n$ is pure for $n \geq 0$. Then $T$ is graded right coherent.

2. If $T$ has finite right graded global dimension, then the converse holds. Namely, if $T$ is graded right coherent, then for any finitely presented $A$-module $M$ there is a natural number $m$ such that $(M \otimes_A \sigma^m) \otimes_A \sigma^n$ is pure for $n \geq 0$.

Before giving a proof, we fix notations. For graded $T$-module $X := \oplus_{n \in \mathbb{Z}} X_n$, we denote by $\mu_{X,m,n}$ the $A$-homomorphism $X_m \otimes_A \sigma^n \to X_{m+n}$ induced by the multiplication $X \otimes_A T \to X, x \otimes t \mapsto xt$. Note that this $A$-homomorphism has the following compatibility with graded $T$-homomorphisms: let $f : X \to Y$ be a graded $T$-homomorphism. Then we have $f_{m+n} \circ \mu_{X,m,n} = \mu_{Y,m,n} \circ (f_m \otimes \text{id}_{\sigma^n})$.

$$\begin{array}{ccc}
X_m \otimes_A \sigma^n & \xrightarrow{f_m \otimes \text{id}_{\sigma^n}} & Y_m \otimes_A \sigma^n \\
\downarrow \mu_{X,m,n} & & \downarrow \mu_{Y,m,n} \\
X_{m+n} & \xrightarrow{f_{m+n}} & Y_{m+n}
\end{array}$$

**Proof.** 1. First note that we can easily check that $\sigma^n$ is coherent as a right $A$-module by induction on $n$.

Let $P,Q$ be finitely generated graded free $T$-modules. We prove that the kernel $K := \ker(f)$ of a graded $T$-homomorphism $f : P \to Q$ is finitely generated. Set $C := \text{coker}(f)$ and $I := \text{im}(f)$. Then for any $s \in \mathbb{Z}$, we have $K_s = \ker(f_s), C_s = \text{coker}(f_s)$ and $I_s = \text{im}(f_s)$. In other words, looking at degree $s$-part we obtain the exact sequences of $A$-modules:

$$(A_s) : \quad 0 \to I_s \to Q_s \to C_s \to 0,$$

$$(B_s) : \quad 0 \to K_s \to P_s \to I_s \to 0.$$

Applying the functor $- \otimes_A \sigma^m$ to the exact sequence $(A_s)$, we obtain the isomorphisms

$$(C_s) : \quad \text{Tor}_1^A(I_s, \sigma^m) \cong \text{Tor}_{i+1}^A(C_s, \sigma^m) \text{ for } i \geq 1$$

and the following commutative diagram $(D_s)$:

$$\begin{array}{cccccc}
0 & \xrightarrow{ \mu_{I,s,m} } & \text{Tor}_1^A(C_s, \sigma^m) & \xrightarrow{ \mu_{Q,s,m} } & I_s \otimes_A \sigma^m & \xrightarrow{ \mu_{C,s,m} } & \text{Tor}_1^A(I_s, \sigma^m) & \xrightarrow{ \mu_{Q,s,m} } & 0 \\
0 & \xrightarrow{ \mu_{I,s,m} } & I_{s+m} & \xrightarrow{ \mu_{Q,s,m} } & Q_{s+m} & \xrightarrow{ \mu_{C,s,m} } & C_{s+m} & \xrightarrow{ \mu_{Q,s,m} } & 0.
\end{array}$$

where the top and the bottom rows are exact.
We may assume that \( P, Q \) are of form

\[
P = \oplus_{p \leq i \leq q} T(-i)^{ \oplus a_i}, \quad Q = \oplus_{p \leq i \leq q} T(-i)^{ \oplus b_i}.
\]

Then for \( s \geq q \) we have \( P_s = \oplus_{p \leq i \leq q} (\sigma^{s-i})^{ \oplus a_i} \) and \( Q_s = \oplus_{p \leq i \leq q} (\sigma^{s-i})^{ \oplus b_i} \). Therefore for \( s \geq q \) and \( n \geq 0 \) the morphisms \( \mu_{P,s,n} \) and \( \mu_{Q,s,n} \) are isomorphisms. Hence by the right exactness of the functor \( - \otimes_A \sigma^n \), we see that the morphism \( \mu_{C,s,n} : C_s \otimes_A \sigma^n \rightarrow C_{s+n} \) is an isomorphism for \( s \geq q \) and \( n \geq 0 \).

By the first result \( \mu \) and \( \eta \) are coherent \( A \)-modules. Hence the cokernel \( C_0 = \mu_{\eta,A} \) is a coherent \( A \)-module. Let \( n \) be a natural number. Then \( (C_0)^{\otimes n} \) is a coherent \( T \)-module. Therefore for \( \mu \) and \( \eta \) are coherent \( A \)-modules, hence the cokernel \( C_0 \) is a coherent \( A \)-module.

In particular \( A \) is a graded coherent. In particular the polynomial algebra \( A = K \langle x_1, \ldots, x_n \rangle \) is generated by \( x_1, \ldots, x_n \). Since \( (C_0)^{\otimes n} \) is a graded coherent, hence the cokernel \( C_0 \) is a graded coherent.

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3 Graded coherent-ness of $n+1$-preprojective algebra and its relationship to higher dimensional Auslander-Reiten Theory

In [4] $n$-Fano algebras and its variants are introduced from a noncommutative algebro-geometric point of view. Among these variants, $n$-quasi-extremely Fano algebra (qe Fano algebra for short) is defined by the simplest condition. In [2] Herschend Iyama and Oppermann called $n$-qe Fano algebra $n$-representation infinite algebra and showed that $n$-qe Fano algebra plays the role in higher dimensional Auslander-Reiten theory as representation infinite algebra in classical Auslander-Reiten theory of path algebras of quivers. Actually they introduced the notion of $n$-hereditary algebra and proved the dichotomy that an $n$-hereditary algebra is either $n$-representation finite (which is defined in [3]) or $n$-representation infinite. Moreover they studied the representation theory of $n$-representation infinite algebras and introduced the notions of $n$-preprojective, $n$-preinjective and $n$-regular modules for $n$-representation infinite algebras, which are shown to be counterparts of preprojective, preinjective and regular modules in the representation theory of hereditary algebras of infinite representation type.

The Auslander correspondence tells us that the representation theory of a representation finite algebra $A$ is equivalent to the ring theory of the Auslander algebra $\text{Aus}(A)$ of $A$. In higher dimensional Auslander-Reiten theory for $n$-representation infinite algebras, the object which is expected to be the counterpart of the Auslander algebra is the $n+1$-preprojective algebra $\Pi_{n+1}(L)$. Therefore it is expected that the representation theory of $L$ has close relationship to the ring theory of $\Pi_{n+1}(L)$. We will observe such a phenomenon in Proposition 3.2, which suggest that graded coherentness of $n+1$-preprojective algebra has an importance in higher dimensional Auslander-Reiten theory.

We recall the definition of $n$-representation infinite algebras (or $n$-qe Fano algebras) and the $n+1$ preprojective algebra $\Pi_{n+1}(L)$. Let $k$ be a field.

**Definition 3.1.** Let $n$ be a natural number. A finite dimensional $k$-algebra $L$ is called an $n$-representation infinite algebras if the following conditions are satisfied:

1. the algebra $L$ is of finite global dimension,

2. Let $D(L) := \text{Hom}_k(L, k)$ be the $k$-dual of $L$ equipped with the natural bi-module structure. We define a complex $θ$ of $L$-bi-modules to be $θ := R\text{Hom}(D(L), L)[n]$. Then the complex $θ^\otimes L^s$ is pure for $s ≥ 1$.

(Note that by the second condition for $s = 1$, we may consider the complex $θ$ as the module $\text{Ext}^1_L(D(L), L)$.)

We denote $θ^\otimes L^s$ by $θ^s$. The $n+1$-preprojective algebra $\Pi_{n+1}(L)$ is defined to be the tensor algebra $T^L(θ)$.

$$\Pi_{n+1}(L) = L ⊕ θ ⊕ θ^2 ⊕ θ^3 ⊕ \cdots$$

We equip $\Pi_{n+1}(L)$ with the grading $\text{deg} θ := 1$.

We define the $n$-Auslander-Reiten translations by

$$τ_n := − \otimes_L θ$$ and $$τ_n^− := \text{Hom}_L(θ, −) : \text{mod} L \to \text{mod} L.$$

Note that the functors $τ_n$ and $τ_n^−$ are not quasi-inverse to each other but only an adjoint pair. In other words the unite morphism

$$η_M : M \to τ_n^−τ_nM$$

of the adjunction $τ_n^− \dashv τ_n$ is not an isomorphisms for general $M \in \text{mod} L$.

For a $L$-module $M$ and a natural number $s$, we denote by $η_{M,s}$ the $L$-homomorphism

$$η_{M,s} : τ_n^sτ_n^−M \to τ_n^{s+1}τ_n^−(s+1)M$$

which is induced by the adjunction $τ_n^− \dashv τ_n$. We can easily check that if $M$ is either $n$-preprojective, $n$-preinjective or $n$-regular, then $η_{M,s}$ is an isomorphism for $s ≥ 0$. For a general $L$-module $M$, we don’t know that $η_{M,s}$ become an isomorphism for $s ≫ 0$. The following proposition tells us that this problem relates to the graded coherentness of the $n+1$-preprojective algebra.

**Proposition 3.2.** We retain the above notations.

1. If $\Pi_{n+1}(L)$ is right graded coherent, then for each finitely generated $L$-module $M$ there is $s_0$ such that $η_{M,s}$ is an isomorphism for $s ≥ s_0$.

2. In the case when $n ≤ 2$, then the converse holds. Namely if for each finitely generated $L$-module $M$ there is $s_0$ such that $η_{M,s}$ is an isomorphism for $s ≥ s_0$, then $\Pi_n(L)$ is graded coherent.
Proof. (1) By [5, Theorem 4.2] the right graded global dimension of \( \Pi_{n+1}(L) \) is \( n+1 \). In particular it is finite. Now the verification is an easy application of Theorem 2.2.2.

(2) First note that we have an isomorphism \( \tau_{t-s}^n M \cong \text{Hom}_L(\theta^s, M \otimes_L \theta^n) \). For \( t \geq 0 \) we have

\[
(M \otimes_L \theta^s) \otimes_L \theta^t \cong \text{Hom}(\theta^s, M \otimes_L \theta^s) \otimes_L \theta^{s+t} \cong \text{Hom}(\theta^{s+t}, M \otimes_L \theta^{s+t}) \otimes_L \theta^{t+s} \cong M \otimes_L \theta^{s+t}
\]

where the first quasi-isomorphism follows from the following Lemma 3.3, the second quasi-isomorphism follows from the assumption and the third quasi-isomorphism follows from Lemma 3.3. Hence we see that \( (M \otimes_L \theta^s) \otimes_L \theta^t \) is pure for \( t \geq 0 \). By Theorem 2.2.1 we verify the claim.

**Lemma 3.3.** Let \( A \) be a ring of \( \text{gldim} \ A \leq 2 \). Let \( \sigma \) be an \( A \)-\( A \)-bi-module such that the complex \( \mathbb{R} \text{Hom}(\sigma, \sigma) \) is pure. Then for a finitely presented \( A \)-module \( M \), the complex \( \mathbb{R} \text{Hom}(\sigma, M \otimes_A \sigma) \) is pure.

**Proof.** Take a finite presentation \( P \to Q \to M \to 0 \) and compute \( \text{Ext}_A^{1,2}(\sigma, M \otimes_A \sigma) \). Standard homological algebra shows the claim.

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