The uniqueness of elementary embeddings

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Outline

1. Conjecture: an elementary embedding of $V$ is uniquely determined by its target model.
   - Proofs (from various hypotheses):
     - Anti-large cardinal assumptions.
     - Properties of HOD.
     - Cardinal arithmetic.
     - The Ultrapower Axiom.

2. Embeddings between levels of the cumulative hierarchy.
   - The HOD Conjecture as a uniqueness principle.
   - The failure of the HOD Conjecture as an embedding principle.
   - A new class of HOD-ultrafilters.
Embeddings of $K$

**Theorem (Dodd-Jensen-Koepke-Mitchell)**

Assume $0^{\mathfrak{U}}$ does not exist. Then any elementary embedding from the core model $K$ into an inner model is a normal iteration of $K$.

Since there is at most one normal iteration from $K$ into a given inner model, the theorem has a fine structure-free consequence:

**Corollary**

Assume $0^{\mathfrak{U}}$ does not exist. If $j_0, j_1 : V \rightarrow M$ are elementary embeddings, then $j_0 \restriction \text{Ord} = j_1 \restriction \text{Ord}$. 
Embeddings of $V$

Conjecture

If $j_0, j_1 : V \rightarrow M$ are elementary embeddings, $j_0 \upharpoonright \text{Ord} = j_1 \upharpoonright \text{Ord}$.

- Equivalently, $j_0 \upharpoonright \text{HOD} = j_1 \upharpoonright \text{HOD}$, so one can think of uniqueness of embeddings as a $K$-like property of HOD.

- To what extent does the unconstrained large cardinal structure of $V$ differ from the special case when $K$ exists?
  - This sort of question turns out to hinge on the structure of HOD under large cardinal hypotheses, and in particular, on Woodin’s HOD Conjecture.
Definable embeddings

**Theorem (Woodin)**

Suppose $j_0, j_1 : V \to M$ are elementary embeddings that are definable from parameters. Then $j_0 \upharpoonright \text{Ord} = j_1 \upharpoonright \text{Ord}$.

**Proof.**

- Assume the scheme fails for $\Sigma_n$-definable embeddings.
- Consider the least ordinal $\alpha$ such that there exist elementary embeddings $i_0, i_1 : V \to N$ that are $\Sigma_n$-definable from parameters with $i_0(\alpha) \neq i_1(\alpha)$.
- $\alpha$ is definable without parameters! Say by a formula $\varphi$.
- So $i_0(\alpha) = i_1(\alpha)$, since both are the unique ordinal satisfying $\varphi$ in $M$. Contradiction.
Two embeddings into a single model

**Proposition**

*It is consistent that there are distinct elementary embeddings from the universe of sets into a common inner model.*

**Sketch.**

- Take \( U \) normal measure on \( \kappa \), \( j : V \rightarrow M \) the ultrapower.

- \( \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha \leq \kappa} \) Easton iteration adding a Cohen subset to every non-Mahlo inaccessible \( \delta \leq \kappa \). Take \( V \)-generic \( G \subseteq \mathbb{P}_\kappa \).

- For any \( M[G] \)-generic \( H \) for the tail forcing \( (\mathbb{P}_\kappa, j(\kappa))^M \), \( j \) extends to \( i : V[G] \rightarrow M[G \ast H] \) such that \( i(G) = G \ast H \).

- Assume GCH in \( V \), so \( \left| P(\mathbb{P}_\kappa, j(\kappa)) \cap M[G] \right|^{V[G]} = \kappa^+ \). Build \( M[G] \)-generics \( H_0 \neq H_1 \subseteq \mathbb{P}_\kappa, j(\kappa) \) with \( M[G \ast H_0] = M[G \ast H_1] \). Yields distinct \( i_0, i_1 : V[G] \rightarrow N \).
The Ultrapower Axiom

An embedding $i : M \to N$ between transitive models is an *ultrapower* if $i$ is the embedding associated to an ultrafilter of $M$.

**Ultrapower Axiom (UA)**

For any ultrapowers $j_0 : V \to M_0$ and $j_1 : V \to M_1$, there are ultrapowers $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ with $i_0 \circ j_0 = i_1 \circ j_1$.
The uniqueness of ultrapowers

**Theorem (UA)**

*Suppose $j_0, j_1 : V \rightarrow M$ are ultrapower embeddings. Then $j_0 = j_1$.***

**Proof.**

- Fix ultrapowers $i_0, i_1 : M \rightarrow N$ with $i_0 \circ j_0 = i_1 \circ j_1$.
- $M = H^M(j_n[V] \cup \text{Ord})$ so $i_n[M] = H^N(i_n \circ j_n[V] \cup i_n[\text{Ord}])$.
  - $i_0, i_1 : M \rightarrow N$ are definable over $M$, so $i_0 \upharpoonright \text{Ord} = i_1 \upharpoonright \text{Ord}$.
  - Thus $i_0 \circ j_0[V] \cup i_0[\text{Ord}] = i_1 \circ j_1[V] \cup i_1[\text{Ord}]$.
  - Therefore $i_0[M] = i_1[M]$, and the uncollapse map is $i_0 = i_1$.
- Since $i_0 \circ j_0 = i_1 \circ j_1$ and $i_0 = i_1, j_0 = j_1$. $lacksquare$
HOD Hypotheses

- $S^\delta_\omega$ denotes the set of countable cofinality ordinals below $\delta$.
- A cardinal $\kappa$ is “not $\omega$-strongly measurable in HOD” if for all $\gamma < \kappa$, one of the following holds:
  - $(2^\gamma)^{\text{HOD}} \geq \kappa$.
  - There is an OD partition of $S^\kappa_\omega$ into $\gamma$-many stationary sets.

**Definition (HOD Hypothesis)**

There is a proper class of cardinals that are not $\omega$-strongly measurable in HOD.

**Definition (Strong HOD Hypothesis)**

For every singular strong limit cardinal $\lambda$ of uncountable cofinality, $\lambda^+$ is not $\omega$-strongly measurable in HOD.
HOD Conjectures

The HOD Hypothesis is true if $V = \text{HOD}$ (by the Solovay splitting lemma), so the hypothesis is consistent with all large cardinals. The real question is whether it is provable:

**Definition (HOD Conjecture)**

ZFC proves the HOD Hypothesis.

Perhaps a more plausible conjecture is that ZFC + some large cardinal axiom implies the HOD Conjecture.
Solovay’s Lemma

Lemma (Solovay)

If $j : V \rightarrow M$ is elementary, $\delta$ is regular, $\langle S_\xi \rangle_{\xi < \gamma}$ is a stationary partition of $S^\delta_\omega$, $\langle T_\xi \rangle_{\xi < j(\gamma)} = j(\langle S_\xi \rangle_{\xi < \gamma})$, and $\delta_* = \sup j[\delta]$, then

$$j[\gamma] = \{\xi < j(\gamma) : T_\xi \cap \delta_* \text{ is stationary}\}$$

Sketch.

▶ $j : \delta \rightarrow \delta_*$ induces a Rudin-Keisler equivalence between the $\omega$-club filters $C^\delta_\omega$ and $C^{\delta_*}_\omega$ since $j$ is one-to-one and $j[\delta] \in C^{\delta_*}_\omega$.

▶ Stationary subsets of $S^\kappa_\omega$ are positive sets for $C^\kappa_\omega$.

▶ So $T \subseteq \delta_* \cap S^{j(\delta)}_\omega$ is stationary iff $j^{-1}[T]$ is stationary.

▶ But $j^{-1}[T_\xi]$ is stationary iff $\xi = j(\bar{\xi})$ for some $\bar{\xi} < \gamma$:

▶ If $\xi = j(\bar{\xi})$, $j^{-1}[T_\xi] = S_{\bar{\xi}}$. If $\xi \notin j[\gamma]$, $j^{-1}[T_\xi] = \emptyset$. 

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Uniqueness of embeddings and the HOD Conjecture, I

Theorem (Woodin)

Assume that the Strong HOD Hypothesis holds. Then for any $j_0, j_1 : V \rightarrow M$, $j_0 \upharpoonright \text{Ord} = j_1 \upharpoonright \text{Ord}$.

Proof.

- Take strong limit singular $\lambda$ fixed by $j_0$ and $j_1$, $\text{cf}(\lambda) > \omega$.
- Fix $\gamma < \lambda$. Let $\langle S_\xi \rangle_{\xi < \gamma}$ be the $\text{<}_{\text{OD}}$-least stationary partition of $S^\lambda_\omega$. Note that $j_0(\vec{S}) = j_1(\vec{S})$. Let $\vec{T} = j_n(\vec{S})$.
- $j_n[\gamma] = \{\xi < \gamma : T_\xi \text{ is stationary}\}$ by Solovay’s Lemma.

Corollary

If $V = \text{HOD}$, then for any $j_0, j_1 : V \rightarrow M$, $j_0 = j_1$. 

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The uniqueness of elementary embeddings
The uniqueness of embeddings under SCH

Theorem (SCH)

Suppose \( j_0, j_1 : V \to M \) are elementary embeddings. Then \( j_0 \upharpoonright \text{Ord} = j_1 \upharpoonright \text{Ord} \).

Idea.

- Consider the Skolem hull \( X = H^M(j_0[V] \cup j_1[V]) \).
  - Let \( \pi : X \to N \) be the transitive collapse.
  - Let \( i_0 = \pi \circ j_0, \ i_1 = \pi \circ j_1 \).
  - Suffices to show that \( i_0 \upharpoonright \text{Ord} = i_1 \upharpoonright \text{Ord} \).
- Using the SCH hypothesis, one can show that \( i_0 \) and \( i_1 \) are ultrapowers.
- So \( i_0 \) and \( i_1 \) are definable from parameters, and hence they agree on the ordinals by Woodin's theorem.
The uniqueness of embeddings under UA

**Theorem**

*Assume the Ultrapower Axiom. Suppose \( j_0, j_1 : V \to M \) are elementary embeddings. Then \( j_0 = j_1 \).*

No restriction to the ordinals, no restriction to ultrapowers.

**Proof assuming SCH.**

By the proof of the previous theorem, the theorem reduces to the case that \( j_0 \) and \( j_1 \) are ultrapower embeddings, which we dispensed with earlier.

One eliminates the SCH hypothesis using heavier machinery from the analysis of strongly compact cardinals under UA.
Uniqueness of embeddings and the HOD Conjecture, II

Assuming large cardinal axioms, the HOD Hypothesis is equivalent to a different uniqueness property of elementary embeddings:

**Theorem**

*Suppose* $\kappa$ *is extendible. The following are equivalent:

- *The HOD Hypothesis.*
- *For all regular* $\delta \geq \kappa$ *and all sufficiently large* $\lambda \geq \delta$, *if* $j_0, j_1 : V_\lambda \rightarrow V_\lambda'$ *are elementary embeddings with*
  $\sup j_0[\delta] = \sup j_1[\delta]$ *and* $j_0(\delta) = j_1(\delta)$, *then* $j_0 \upharpoonright \delta = j_1 \upharpoonright \delta$.*

In the forward direction, one uses Solovay’s Lemma and the existence of ordinal definable stationary partitions. In the opposite direction, one uses uniqueness of embeddings to show roughly that HOD inherits too much of the large cardinal structure of $V$ to be a small model.
Choiceless cardinals and HOD

Theorem (Woodin, ZF)

*If there is a Reinhardt cardinal and an extendible cardinal, then the HOD Conjecture is false.*

So large cardinals beyond choice imply the failure of the HOD Conjecture. The question roughly is whether this means:

1. The HOD Hypothesis is consistently false.
2. Berkeley cardinals are inconsistent, without assuming AC.

Rest of talk shows the HOD Hypothesis has consequences similar to those of the choiceless large cardinals.
Embeddings into HOD

- Let $T_2$ denote the $\Sigma_2$-theory of $V$ with ordinal parameters.
- The structure $(\text{HOD}, T_2)$ computes the theory of $V$.

**Theorem**

*For any definable embeddings $j_0, j_1 : (\text{HOD}, T_2) \rightarrow (M, S)$, $j_0 = j_1$."

A variant of the Kunen inconsistency states that if $\lambda$ is inaccessible, there is no $\omega$-continuous embedding $j : M \rightarrow V_\lambda$.

**Theorem**

*Assume $\kappa$ is extendible. Then exactly one of the following holds:* 
  
  - *The HOD Hypothesis.*
  - *For every inaccessible $\lambda > \kappa$, there is a transitive structure $(M, S)$ of height $\lambda$, admitting a nontrivial $\omega$-continuous elementary embedding $j : (M, S) \rightarrow (\text{HOD} \cap V_\lambda, T_2 \cap V_\lambda)$.*
Equalizers

Suppose $V_\gamma \prec \Sigma_2 V$, $j_0, j_1 : V_\gamma \rightarrow V_{\gamma'}$ are elementary embeddings, $\delta$ is regular, $\text{sup } j_0[\delta] = \text{sup } j_1[\delta]$, and $j_0(\delta) = j_1(\delta)$.

**Definition**

The *equalizer of $j_0$ and $j_1$* is $\text{Eq}(j_0, j_1) = \{x \in V_\gamma : j_0(x) = j_1(x)\}$.

- In $V_\gamma$:
  - $\text{Eq}(j_0, j_1)$ is definably closed.
  - $\text{Eq}(j_0, j_1)$ is closed under $\omega$-sequences, and in fact, under $<\kappa$-sequences where $\kappa = \text{crit}(j)$.
- In particular, $\text{Eq}(j_0, j_1) \cap \text{HOD} \prec (\text{HOD} \cap V_\gamma, T_2 \cap V_\gamma)$.
  - Let $j : (M, S) \rightarrow (\text{HOD} \cap V_\gamma, T_2 \cap V_\gamma)$ be the inverse of the transitive collapse of $\text{Eq}(j_0, j_1) \cap \text{HOD}$.
  - Then $j(\delta) = \delta$ since $\text{sup } j_0[\delta] = \text{sup } j_1[\delta]$ and $j_0(\delta) = j_1(\delta)$.
  - If $j_0 \upharpoonright \delta \neq j_1 \upharpoonright \delta$, then $j$ is a nontrivial elementary embedding.
The fixed point filter

Under choiceless large cardinal axioms, a filter called the *fixed point filter* is often more useful than the club filter.

**Definition**

Suppose $\kappa \leq \delta \leq \nu$ are ordinals below some cardinal $\gamma$. The *fixed point filter* associated to these ordinals is the filter generated by sets of the form

$$\text{Fix}(j_\alpha)_{\alpha<\eta} = \{\xi < \delta : \forall \alpha < \eta \ j_\alpha(\xi) = \xi\}$$

where $\eta < \kappa$ and for all $\alpha < \eta$, $j_\alpha : V_\gamma \rightarrow V_\gamma$ is an elementary embedding fixing $\nu$. 

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The uniqueness of elementary embeddings
The fixed point filter, continued

A cardinal $\lambda$ is a *revised Reinhardt cardinal* if for some elementary $j : V \rightarrow V$, $\lambda$ is the least ordinal above $\text{crit}(j)$ with $j(\lambda) = \lambda$.

**Theorem (NBG)**

*Suppose $\lambda$ is a revised Reinhardt cardinal and $\lambda$-DC holds. Then every $\lambda^+$-complete ultrafilter on an ordinal is of the form $\mathcal{F} \upharpoonright A$ for some fixed point filter $\mathcal{F}$.***

This is used to prove the following theorem:

**Theorem (NBG)**

*Suppose $\lambda$ is a revised Reinhardt cardinal and $\lambda$-DC holds. Then every $\lambda^+$-complete filter on an ordinal extends to a $\lambda^+$-complete ultrafilter.*
The equalizer filter

Assume $\kappa$ is extendible, $\gamma \geq \kappa^+$, $j : V_\gamma \rightarrow V_{\gamma'}$ is elementary, $p$ is a nonempty finite set of regular cardinals in the interval than $(\kappa, \gamma)$.

- $\mathcal{K}_{j,p}$ denotes the set of elementary embeddings $k : V_\gamma \rightarrow V'_{\gamma}$ such that for all $\delta \in p$, $\sup k[\delta] = \sup j[\delta]$ and $k(\delta) = j(\delta)$.
- The equalizer filter constrained by $j$ on $p$, denoted $\mathcal{E}\mathcal{F}_{j,p}$, is the filter generated by sets of the form

$$\text{Eq}(\sigma) = \{\xi < \max(p) : \forall k, \ell \in \sigma \ k(\xi) = \ell(\xi)\}$$

for all $\sigma \subseteq \mathcal{K}_{j,p}$ with $|\sigma| < \min p$.

Theorem (in this context)

Assume the HOD Hypothesis is false. Then there is a ordinal definable partition $\mathcal{P}$ of $\max(p)$ of size less than $\min(p)$ such that for all $A \in \mathcal{P}$, $(\mathcal{E}\mathcal{F}_{j,p} \cap \text{HOD}) \upharpoonright A$ is an ultrafilter in HOD.
Thanks!