An Introduction to Clifford Supermodules

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Abstract

We analyze the Clifford action on superspaces with a view on generalized Dirac fields taking values in some Clifford supermodule. The stress here is on two principles: complexification and polarisation. For applications in field theory, the underlying vector space may carry either a Euclidean or a Minkowskian structure.

1 Introduction

This is the first part of a series of articles in which we aim to present those ideas and methods which can now be effectively used by physicists working in the theory of fermions. The reason for studying the subject is the observation that generalized (multi-component) Dirac fields are instances of Clifford supermodules [1,2]. The particular point of view presented here will help to better understand the role of the Higgs field and Yukawa interactions in gauge theories, seemingly the essential features of the Standard Model. In pursuing this idea, we analyze the mathematical structure underlying the modern theory of Dirac operators on manifolds [3] and their relation to superconnections on $\mathbb{Z}_2$-graded vector bundles with an outlook on possible applications in particle physics. We introduce only tools appropriate for a consistent and clear formulation of field theoretical models and begin by discussing the algebraic fundament which is the structure of Clifford algebras [4,5,6] and their action...
on superspaces. The emphasis will be on two important principles, *complexification and polarisation*, and on two structural aspects, *skew tensor products and twisting*, thereby stressing the superalgebra point of view. But to keep both feet on the ground we complement the discussion by studying specific examples for illustration. The ideas, though presented here in an abstract setting, will come alive when applied to supersymmetric Dirac theory [7] and the gauge theory of quarks and leptons [2].

While spinors belong to the standard repertoire of physicists since Dirac’s seminal work on spin $\frac{1}{2}$ particles, in the mathematics community, the use of spinors has been spreading slowly. It is only recent that spinor modules and the associated Dirac operators have developed into a fundamental tool in differential geometry of Riemannian (spin) manifolds. With same emphasis, Penrose [8], Witten [9], and Connes [10] place spinors at the fundament of their constructions in geometry, be it commutative or non-commutative.

## 2 Clifford Algebras

There are various ways to naturally associate an algebra to some $n$-dimensional real vector space $V$. Each one is characterized by a universal property and obtained explicitly from the tensor algebra $T(V)$ by taking the quotient with respect to some two-sided ideal $I$. The Clifford algebra provides a prominent example. In more detail: let $q(v)$ denote a (possibly degenerate) quadratic form in $V$, or, equivalently, let there be given a bilinear symmetric form $(,)$ in $V$ such that $q(v) = (v, v)$. Let $I$ be the ideal in $T(V)$ generated by the elements $v \otimes v + q(v)1$ ($v \in V$) where $1$ is the unit in $T(V)$. Then the Clifford algebra $C(V)$ over $V$ is the quotient $T(V)/I$. The embedding $V \to C(V)$ allows us to identify $v \in V$ with $v \in C(V)$ without confusion. By construction, the *Clifford relation*

$$v \otimes v + q(v)1 = 0 \quad (v \in V) \quad (1)$$

holds in in any Clifford algebra $C(V)$. In addition, it has the following universal property. Any linear map $c : V \to A$ into an associative algebra $A$ with unit satisfying

$$c(v)^2 + q(v)1 = 0 \quad (2)$$

extends to an algebraic homomorphism $c : C(V) \to A$. A linear map $c$ satisfying (2) is called a *Clifford map*.

The first thing to be noticed about this construction is that the $\mathbb{Z}$-grading of the tensor algebra $T(V)$ is lost when the quotient $T(V)/I$ is formed. Nevertheless, there remains a $\mathbb{Z}_2$-graded structure given by the involution
τ(v) = −v in V (leaving the ideal I invariant). With respect to the eigenvalues ±1 of τ we may write

\[ C(V) = C^+(V) \oplus C^-(V). \]

One remark is in order. We shall often use the parities ±1 instead of the numbers 0, 1 ∈ \( \mathbb{Z}_2 \) to indicate the grading, because it seems more convenient. Also, parities are more intuitive from the physics point of view. The grading gives \( C(V) \) the structure of a superalgebra as the following properties demonstrate:

\[ C^\pm(V)C^+(V) \subset C^\pm(V), \quad C^\pm(V)C^-(V) \subset C^\mp(V). \]

As with any superalgebra, the positive part \( C^+(V) \) is a subalgebra.

Second, from (1) we immediately obtain the anticommutation relation

\[ vv' + v'v + 2(v, v')1 = 0 \quad (v, v' \in V) \tag{3} \]

valid in the Clifford algebra \( C(V) \). For a basis \( \{e_i\}_{i=1}^n \) in \( V \) we tacitly assume that \( (e_i, e_k) = 0 \) if \( i \neq k \). Of course, this condition would be empty on a subspace where the bilinear form vanishes. Now, if \( I \) runs over all subsets of \( \{1, \ldots, n\} \), then (3) can be used to show that the ordered products

\[ e_I = e_{i_1} \cdots e_{i_k} \quad (i_1 < \cdots < i_k), \tag{4} \]

where \( I = \{i_1, \ldots, i_k\} \), form a basis in \( C(V) \). For the empty set, one puts \( e_\emptyset = 1 \). This then demonstrates that

\[ \dim C^+(V) = \sum_{k=\text{even}} \binom{n}{k} = 2^{n-1} \]

\[ \dim C^-(V) = \sum_{k=\text{odd}} \binom{n}{k} = 2^{n-1}. \]

Note that the dimension \( 2^{n-1} \) of \( C^\pm(V) \) is independent of the choice of the bilinear form \( (, ) \), be it degenerate or not.

Third, if the vector space \( V \) has not been equipped with a bilinear form, we express this by saying that \( (, ) \) vanishes everywhere in \( V \). The Clifford algebra \( C(V) \) then coincides with the exterior algebra \( \Lambda V \). In this limiting case, the exterior algebra inherits the \( \mathbb{Z}_2 \)-graded structure of the tensor algebra \( T(V) \). Elements \( a \in \Lambda^k V \) are said to be homogeneous of degree \( k \) or are simply called \( k \)-vectors. We shall however be mainly concerned with the \( \mathbb{Z}_2 \)-grading of \( \Lambda V \):

\[ \Lambda^+V = \sum_{k=\text{even}} \Lambda^k V, \quad \Lambda^-V = \sum_{k=\text{odd}} \Lambda^k V. \]
It is common practice to prefer the notation $a \land b$ over $ab$ for the product of two elements $a, b \in \wedge V$. We shall not always adhere to this convention and warn the reader. As we shall see lateron, there is a close relationship between Clifford algebras and their associated exterior algebras. Thus, to avoid confusion it seems wise to follow the tradition and to distinguish between the Clifford product $ab$ and the exterior product $a \land b$.

3 Clifford Modules and Supermodules

One example, intensively studied and applied by physicists, is provided by the four-dimensional Minkowski space $M_4$ with the Lorentz metric. Dirac’s idea of representing the Clifford algebra $C(M_4)$ by complex $4 \times 4$ matrices can be generalized and leads us to the concept of a Clifford module. To start with, we introduce the concept of a real Clifford module, which is a real vector space $E$ together with an algebraic homomorphism

$$c : C(V) \to \text{End} E.$$  \hfill (5)

By the universal property of $C(V)$, it suffices to assume that there be given a linear map $c : V \to \text{End} E$ satisfying the Clifford relation $c(v)^2 + q(v)1 = 0$.

Of particular interest is the case where $E$ is $\mathbb{Z}_2$-graded (hence is a super-space) giving $\text{End} E$ the structure of a superalgebra:

$$\text{End}^{\pm} E = \text{Hom}(E^{\pm}, E^{\pm}) \oplus \text{Hom}(E^{\mp}, E^{\mp})$$
$$\text{End} E = \text{End}^{+} E \oplus \text{End}^{-} E.$$

We would then require that the map (5) is in fact a homomorphism between superalgebras (respecting the grading):

$$c : C^{\pm}(V) \to \text{End}^{\pm} E.$$

We say that $E$ is a Clifford supermodule. For this to be the case it suffices to assume that $c(v) \in \text{End}^{-} E$ (all $v \in V$), i.e., that $c(v)$ changes the parity of vectors in $E$:

$$c(v) : E^{\pm} \to E^{\mp}.$$

There are further conditions that may be imposed on Clifford modules. We mention two of them.

(1) If the Clifford algebra acts irreducibly on $E$, i.e., if $c : C(V) \to \text{End} E$ is an isomorphism, $E$ is called an irreducible Clifford module. The module $E$ is reducible if it contains an invariant (proper) subspace, completely reducible if it is a direct sum of invariant subspaces.
(2) There is a linear operation on $C(V)$ that respects the grading and mimics the notion of passing to the adjoint, familiar from operator algebras. In fact, we can extend the involution $v^* = -v$ in $V$ to all of $C(V)$ so as to satisfy the axioms of a *algebra:

$$
\begin{align*}
1^* &= 1 \\
(a^*)^* &= a \\
(ab)^* &= b^*a^*, \quad a, b \in C(V).
\end{align*}
$$

Frequently, $\text{End} E$ is a *algebra, in which case it seems natural to require that $c(a)^* = c(a^*)$. The Clifford module $E$ is then said to be selfadjoint. This entails that there is some non-degenerate symmetric bilinear form $(,)$ in the Clifford module $E$ and, for any $A \in \text{End} E$, one defines the adjoint $A^*$ with respect to $(,)$. Supposing moreover that $E$ is a Clifford supermodul, we would have to add the condition $(\text{End}^\pm E)^* = \text{End}^\pm E$. The following remark is useful in applications. Given a Clifford map $c : V \to \text{End} E$. In order to establish selfadjointness of the extension $c : C(V) \to \text{End} E$ it suffices to check that the Clifford map satisfies $c(v)^* + c(v) = 0$.

4 The Exterior Algebra $\wedge V$ as Clifford Supermodul

As before, $V$ is assumed to be an $n$-dimensional real vector space. With $V^*$ its dual, there is a canonical quadratic form $q$ in $V^* \oplus V$ given by

$$q(u, v) = u(v), \quad u \in V^*, v \in V$$

and a Clifford algebra $C(V^* \oplus V)$ with the canonical embedding

$$V^* \oplus V \to C(V^* \oplus V), \quad (u, v) \mapsto \epsilon(v) - i(u). \quad (6)$$

We have thus introduced $\epsilon(v)$ and $i(u)$ as (odd) generators of the Clifford algebra. They satisfy the relations

$$\{i(u), \epsilon(v)\} = u(v)1, \quad \{i(u), i(u')\} = 0, \quad \{\epsilon(v), \epsilon(v')\} = 0 \quad (7)$$

known in physics as canonical anticommutation relations (CAR).

Conversely, suppose $V$ has dimension $2n$ and subspaces $V_\pm$ of dimension $n$, dual to each other in the sense that $V_\pm \cong V^*_\mp$, such that

$$V \cong V_+^* \oplus V_+. \quad (8)$$
We say that the isomorphism (8) provides a (real) polarisation of $V$ if the quadratic form $q$ in $V$ corresponds to the canonical quadratic form in $V^*_+ \oplus V_+$ up to a factor $\frac{1}{2}$, i.e.,

$$(v_-, v_+) = \frac{1}{2}q(v), \quad v = v_+ + v_- \quad (v_\pm \in V_\pm).$$

The factor $\frac{1}{2}$ is automatic. It follows from $(v_\pm, v_\pm) = 0$ and

$$q(v) = (v_+ + v_-, v_+ + v_-) = (v_+, v_-) + (v_-, v_+) = 2(v_-, v_+).$$

**Example.** The two-dimensional Minkowski space, $M_2$, admits a polarisation because the Lorentz metric may be written

$$(x, x) = (x^0 - x^1)(x^0 + x^1).$$

The subspaces are $V_\pm = \{ x \in M_2 \mid x^0 \pm x^1 = 0 \} \cong \mathbb{R}$.

The intimate connection between the concepts of a Clifford and an exterior algebra becomes apparent when we now show that there is a natural isomorphism between superalgebras,

$$C(V^* \oplus V) \cong \text{End} \bigwedge V,$$  

(9)

or phrased differently, that the above CAR algebra is irreducibly represented on the superspace $\bigwedge V$.

The proof proceeds in steps. First, the linear map $\epsilon : V \rightarrow C(V^* \oplus V)$ satisfying $\epsilon(v)^2 = 0$ extends to an embedding of algebras,

$$\epsilon : \bigwedge V \rightarrow C(V^* \oplus V),$$

which allows us to identify $\bigwedge V$ with $\epsilon(\bigwedge V)1$. This assigns to $\epsilon(v)$ the role of a multiplication operator on $\bigwedge V$,

$$\epsilon(v)a = v \wedge a \quad (v \in V, \ a \in \bigwedge V),$$

and to $\iota(u)$ the role of a contraction operator, uniquely characterized by the conditions

$$\iota(u)1 = 0$$

$$\iota(u)(v \wedge a) = u(v)a - v \wedge \iota(u)a \quad (u \in V^*, \ v \in V, \ a \in \bigwedge V).$$

Second, the algebra $\text{End} \bigwedge V$ is easily shown to be generated by the operators $\epsilon(v), \iota(u)$ and hence by the operators $\epsilon(v) - \iota(u)$. But (9) says that the elements $\epsilon(v) - \iota(u)$ also generate the Clifford algebra $C(V^* \oplus V)$ and so the Clifford map

$$V^* \oplus V \rightarrow \text{End} \bigwedge V, \quad (u, v) \mapsto \epsilon(v) - \iota(u),$$

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extends to an isomorphism \( \Phi \). Since both \( \epsilon(v) \) and \( \iota(u) \) are operators on \( \Lambda V \) of odd parity, it is guaranteed that the isomorphism respects the \( \mathbb{Z}_2 \)-grading. To summarize, the exterior algebra \( \Lambda V \) is an irreducible Clifford supermodule for the Clifford algebra \( C(V^* \oplus V) \).

Let us assume that the bilinear form \( (, ) \) on \( V \) is non-degenerate. Then the linear map

\[
V \to V^*, \ v \mapsto v^#, \quad v^#(v') = (v, v')
\]

establishes a natural isomorphism between the vector space \( V \) and its dual. In this situation we need no longer distinguish between the two spaces, \( V \) and \( V^* \), or distinguish between \( v \) and \( v^# \), i.e., we shall write \( \iota(v) \) where we really mean \( \iota(v^#) \).

Consider the following Clifford map:

\[
c : V \to \text{End} \Lambda V, \quad c(v) = \epsilon(v) - \iota(v).
\]

Its extension \( c: C(V) \to \text{End} \Lambda V \) gives the exterior algebra \( \Lambda V \) the structure of a selfadjoint \( C(V) \) supermodul. It is always reducible. To prove selfadjointness we must first extend \( (, ) \) on \( V \) to a bilinear form on \( \Lambda V \). This is done in a standard fashion:

\[
(1, 1) = 1, \quad (\Lambda^k V, \Lambda^l V) = 0 \quad (k \neq l),
\]

\[
(v_1 \wedge \ldots \wedge v_k, v'_1 \wedge \ldots \wedge v'_k) = \det(v_i, v'_j)_{i,j=1}^k \quad (k = 1, \ldots, n).
\]

Laplace’s expansion formula for determinants is simply stated as \( \epsilon(v)^* = \iota(v) \).

Therefore, \( \iota(v)^* = \epsilon(v) \) and

\[
c(v)^* = \left( (\epsilon(v) - \iota(v))^* \right) = -c(v) \quad (v \in V)
\]

which suffices to establish selfadjointness of the Clifford modul.

It helps the physical intuition to compare the formalism with that of the CAR algebra used in the theory of fermions. For it is clear that the Clifford action on \( \Lambda V \), the ‘Fock space’ in physics, is fully determined by the conditions:

\[
\{ \iota(v_1), \epsilon(v_2) \} = (v_1, v_2), \quad \iota(v) 1 = 0
\]

\[
\{ \iota(v_1), \iota(v_2) \} = 0 = \{ \epsilon(v_1), \epsilon(v_2) \}.
\]

Thus, \( \iota(v) \) and \( \epsilon(v) \) may be viewed as annihilation and creation operators respectively, while the unit \( 1 \in \Lambda V \) serves as the ‘vacuum’.
The $C(V)$ module structure of $\Lambda V$ now allows us to set up an isomorphism between vector spaces,

$$\sigma : C(V) \to \Lambda V, \quad \sigma(a) = c(a)1,$$

known as the symbol map. Since $c$ (and hence $\sigma$) respects the $\mathbb{Z}_2$-grading, $\sigma$ is in fact an isomorphism between superspaces though not an algebraic isomorphism. For example, with $v_i \in V$ we have

$$\sigma(1) = 1$$
$$\sigma(v_1) = v_1$$
$$\sigma(v_1v_2) = v_1 \wedge v_2 - (v_1, v_2)1$$
$$\sigma(v_1v_2v_3) = v_1 \wedge v_2 \wedge v_3 - (v_1, v_2)v_3 + (v_3, v_1)v_2 - (v_2, v_3)v_1.$$

If $e_i$ is a basis in $V$, assuming that $(e_i, e_k) = 0$ for $i \neq k$, let $e_I$ be the induced basis in $C(V)$ given by Eq.(4). Then the calculation (using an inductive argument and the fact that contractions are absent)

$$\sigma(e_I) = \sigma(e_{i_1} \ldots e_{i_k}) = c(e_{i_1} \ldots e_{i_k})1 = c(e_{i_1}) \cdot \ldots \cdot c(e_{i_k})1 = c(e_{i_1}) \cdot \ldots \cdot c(e_{i_{k-1}})e_{i_k} = c(e_{i_1}) \cdot \ldots \cdot c(e_{i_{k-2}})(e_{i_{k-1}} \wedge e_{i_k}) = \ldots = e_{i_1} \wedge \ldots \wedge e_{i_k}$$

reveals that the basis in $C(V)$ is mapped onto the corresponding basis in $\Lambda V$.

A frequently used map is the inverse $\sigma^{-1} : \Lambda V \to C(V)$. It is referred to as the quantization map because, intuitively, one likes to think of the Clifford algebra $C(V)$ as a quantum deformation of the ‘classical’ (supercommutative) algebra $\Lambda V$. The quantum deformation is visible in contractions terms involving $(v_i, v_k)$ and disappears when $(,)$ vanishes identically on $V$, i.e., when $\sigma : \Lambda V \to \Lambda V$ reduces to the identity map.

It may be instructive to see the quantum analogues of $k$-vectors:

$$\sigma^{-1}(v_1 \wedge \ldots \wedge v_k) = \frac{1}{k!} \sum_\pi \text{sign}(\pi) v_{\pi(1)}v_{\pi(2)} \ldots v_{\pi(k)}.$$
The quantization map can now be used to carry the \( \mathbb{Z} \)-grading of the exterior algebra \( \bigwedge V = \sum_k \bigwedge^k V \) to the Clifford algebra \( C(V) \):

\[
C(V) = \sum_k C^k(V) \quad \text{where} \quad C^k(V) := \sigma^{-1}(\bigwedge^k V).
\]

Of course, the sum has only \( n + 1 \) terms since \( C^k(V) = \{0\} \) if either \( k < 0 \) or \( k > n \). Note that the \( \mathbb{Z} \)-graded vector space \( C(V) \) does not satisfy the conditions of \( \mathbb{Z} \)-graded algebra since the quantization map \( \sigma^{-1} \) fails to be an algebraic isomorphism. For instance,

\[
v_1v_2 + (v_1, v_2)1 \in C^2(V) \quad (v_i \in V)
\]

and hence the upper index \( k \) in \( C^k(V) \) should be interpreted and used with great care. Nevertheless, the \( \mathbb{Z} \)-grading (as vector space) is consistent with the \( \mathbb{Z}_2 \)-grading (as algebra):

\[
C^+(V) = \sum_{k=\text{even}} C^k(V), \quad C^-(V) = \sum_{k=\text{odd}} C^k(V).
\]

Note in particular that \( C^0(V) = \mathbb{R} \) and \( C^1(V) = V \). Also,

\[
\dim C^k(V) = \dim \bigwedge^k V = \binom{n}{k}.
\]

In applications it is important to realize that the map \( a \mapsto a^* \), that is taking adjoints \( C(V) \), respects the \( \mathbb{Z} \)-grading. In fact, for a basis \( e_I \) in \( C^k(V) \) with \( |I| = k \), we get

\[
e^*_I = (e_{i_1} \cdots e_{i_k})^* = (-1)^k e_{i_k} \cdots e_{i_1} = (-1)^k (-1)^{k(k-1)/2} e_I = (-1)^{k(k+1)/2} e_I
\]

and thus \( a^* = (-1)^{k(k+1)/2} a \) for all \( a \in C^k(V) \).

## 5 The Spin Group

In dealing with a superalgebra \( A \) the notion of the supercommutator of two of its elements will be important:

\[
[a, b] = \begin{cases} 
ab + ba & \text{if } a, b \in A^- \\
ab - ba & \text{otherwise}.
\end{cases}
\]

Supercommutators \( [a, .] \) with respect to some fixed element \( a \in A \) may then be viewed as inner derivations of the algebra \( A \) owing to the formula

\[
[a, bc] = \begin{cases} 
[a, b] c - b [a, c] & \text{if } a, b \in A^- \\
[a, b] c + b [a, c] & \text{otherwise}.
\end{cases}
\]
There are two remarkable properties of the subspaces $C^k(V) \subset C(V)$:

$$\left[ C^1(V), C^k(V) \right] \subset C^{k-1}(V) \quad (11)$$

$$\left[ C^2(V), C^k(V) \right] \subset C^k(V). \quad (12)$$

These inclusions are consequences of the fact that, for any $v \in V$, the following diagram commutes:

$$
\begin{array}{ccc}
C(V) & \xrightarrow{[\cdot, \cdot]} & C(V) \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
\wedge V & \xrightarrow{-2\iota(v)} & \wedge V
\end{array}
$$

The isomorphism $\sigma$ makes the supercommutator and the contraction correspond, or phrased differently, the formula

$$-\frac{1}{2} \left[ v, a \right] = \sigma^{-1} \left( \iota(v)\sigma(a) \right) \quad (v \in V, a \in C(V)) \quad (13)$$

extends the Clifford relation $\left[ v, v' \right] = -2(v, v')1$. To prove Eq.(13) it suffices to show its validity for a basis in $C(V)$. Let $e_I$ be the basis elements given by Eq.(4). Setting $a = e_I$ with $|I| = k$ we obtain:

$$\left[ v, e_I \right] = \left[ v, e_{i_1} \cdots e_{i_k} \right] = \left[ v, e_{i_1} \right] \left[ e_{i_2} \cdots e_{i_k} \right] - e_{i_1} \left[ v, e_{i_2} \cdots e_{i_k} \right] = -2(v, e_{i_1})e_{i_2} \cdots e_{i_k} - e_{i_1} \left[ v, e_{i_2} \cdots e_{i_k} \right] = \ldots = -2 \sum_{l=1}^{k} (-1)^{l+1} (v, e_{i_l})e_{i_1} \cdots \hat{e}_{i_l} \cdots e_{i_k} = -2\sigma^{-1} \left( \iota(v)(e_{i_1} \wedge \ldots \wedge e_{i_k}) \right) = -2\sigma^{-1} \left( \iota(v)\sigma(e_I) \right).$$

Having proved Eq.(13) we see that the statement (11) is an easy consequence, simply because the contraction operator $\iota(v)$ has degree $-1$, that is, it maps $\wedge^k V$ into $\wedge^{k-1} V$. The second statement (12) is more involved. Working with a basis, we first find that, for $i \neq j$,

$$\left[ e_i e_j, e_I \right] = e_i \left[ e_j, e_I \right] + (-1)^{|I|} \left[ e_i, e_I \right] e_j.$$

The first term on the right hand side describes a replacement of $j \in I$ by $i$, the second of $i \in I$ by $j$. The result, of course, could also be zero: for the first term if $i \in I$ or $j \notin I$, for the second term if $j \in I$ or $i \notin I$. In the case $i, j \in I$ one uses $\left[ e_i e_j, e_I \right] = 0$ to show that $\left[ e_i e_j, e_I \right] = 0$. In any case, the result is in $C^k(V)$. 

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Eq. (12) plays a decisive role in a number of applications, because the supercommutator, if suitably restricted, preserves the $\mathbb{Z}$-grading. Note that, in the case of Eq. (12), the supercommutator is in fact the commutator. Our interest lies in the case $k = 2$,

$$[\ , \ ] : C^2(V) \times C^2(V) \to C^2(V),$$

but also in the case $k = 1$:

$$[C^2(V), V] \subset V.$$

Thus, $C^2(V)$ is a Lie algebra operating on $V$ via the adjoint representation:

$$C^2(V) \xrightarrow{\text{ad}} \text{End } V$$

$$a \mapsto [a, \cdot].$$

Obviously, $\text{ad} [a, b] = [\text{ad} a, \text{ad} b]$. Note also the relation

$$(\text{ad} (a)v, v') + (v, \text{ad} (a)v') = 0 \quad (v, v' \in V)$$

which states that $\text{ad} (a)$ is an element of $\text{so}(V)$, the Lie algebra of $\text{SO}(V)$. Since

$$\dim \text{so}(V) = \dim C^2(V) = \binom{n}{2}, \quad \ker \text{ad} = 0,$$

the map $\text{ad} : C^2(V) \to \text{so}(V)$ turns out to be a Lie isomorphism. It remains to prove Eq. (14). The argument runs as follows:

$$(\text{ad} (a)v, v') + (v, \text{ad} (a)v') = -\frac{1}{2}(\ [a, v], v' + v, [a, v'])$$

$$= -\frac{1}{2} [a, [v, v']]$$

$$= [a, (v, v')1] = 0.$$

In the first and the third step, we used the Clifford relation while in the second step we applied the generalized Jacobi identity,

$$[a, [b, c] ] - [ [a, b], c ] = \begin{cases} -[b, [a, c] ] & \text{if } a, b \in A^- \\ [b, [a, c] ] & \text{otherwise}, \end{cases}$$

valid in any superalgebra $A$.

To pass from a Lie algebra of operators to a Lie group is a straightforward procedure known as the exponential mapping. The resulting group

$$\text{Spin} (V) = \exp C^2(V) \subset C^+(V)$$
is called the spin group. By the above construction, this group is seen to act on the vector space $V$ via the adjoint representation $\text{Ad} = \exp \circ \text{ad}$. Hausdorff’s formula then provides a more explicit description:

$$\text{Ad}(e^a)v = e^{\text{ad}(a)}v = e^a v e^{-a}.$$  

Notice that the product on the right hand side has to be taken within $C(V)$. The above analysis guarantees that the result will again be in $V \subset C(V)$.

If $V$ is Euclidean and $\dim V > 1$ or if the bilinear form $(,)$ is nondegenerate and $\dim V > 2$, then, setting $\mathbb{Z}_2 = \{1, -1\}$, the following diagram has exact sequences as its rows:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & C^2(V) & \xrightarrow{\text{ad}} & \text{so}(V) & \longrightarrow & 0 \\
& & \big| \exp & \big| \exp & & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \text{Spin}(V) & \xrightarrow{\text{Ad}} & \text{SO}(V) & \longrightarrow & 1
\end{array}
$$

Typically, the spin group is a double covering of $SO(V)$. For the convenience of the reader, we include the proof. The exactness of the top sequence has already been demonstrated. As for the bottom sequence, we need only show that $\ker \text{Ad} = \mathbb{Z}_2$. This is done in three steps.

1. If $g \in \ker \text{Ad}$ (i.e., $\text{Ad}(g) = 1$), then $gvg^{-1} = v$ for all $v \in V$ or $[v, g] = 0$ since $g \in C^+$. From Eq. (13) we infer that $\iota(v)\sigma(g) = 0$ implying $\sigma(g) \in \bigwedge^0 V$ or $g \in C^0(V) \cong \mathbb{R}$.

2. Under the above assumptions on $V$ we may find two vectors, $v$ and $w$, such that $(v, w) = 0$ and $(v, v) = (w, w) = \pm 1$. Since $vw \in C^2(V)$ and $(vw)^2 = vvwv = -vvww = -(v, v)(w, w) = -1$,

$$\exp(tvw) = (\cos t)1 + (\sin t)vw \in \text{Spin}(V) \quad (t \in \mathbb{R}).$$

In particular, for $t = \pi$, we learn that $-1 \in \text{Spin}(V)$ and hence $\mathbb{Z}_2 \subset \text{Spin}(V)$. Obviously, $\text{Ad}(-1) = 1$ and thus $\mathbb{Z}_2 \subset \ker \text{Ad}$.

3. To demonstrate the equality $\mathbb{Z}_2 = \ker \text{Ad}$ we recall that the map $a \mapsto a^*$ leaves $C^k(V)$ invariant and $a^* = -a$ for $a \in C^2(V)$. Thus, any group element $g = \exp(a) \in \text{Spin}(V)$ satisfies $gg^* = 1$, because

$$g^* = \exp(a^*) = \exp(-a) = g^{-1},$$

and moreover, if $g \in C^0(V)$, then also $g^2 = 1$ and hence $g \in \{1, -1\}$ which completes the proof.

If $E$ is a (real) Clifford module, the Clifford action $c$ automatically induces
a representation $c$ of the spin group on $E$. If the module is selfadjoint, then $c(g)^* = c(g^{-1})$ and the representation is orthogonal in the sense that $c(g)^* = c(g^{-1})$, i.e., $c(g) \in SO(E)$. If $E$ is a supermodule, the representation is reducible: the subspaces $E^\pm$ turn out to be invariant since $\text{Spin}(V) \subset C^+(V)$.

To illustrate the foregoing discussion we will study two examples relevant for physics.

**Example 1.** Let $V$ be the Euclidean space $E_3$. With respect to some (orthonormal) basis $(e_i)_{i=1}^3$ in $E_3$, the Clifford algebra $C(E_3)$ of dimension 8 is defined through the relations $e_i e_j + e_j e_i + 2 \delta_{ij} 1 = 0$. The Lie algebra $C^2(E_3)$ is 3-dimensional with basis $a_i := \frac{1}{4} \epsilon_{ijk} e_j e_k (i = 1, 2, 3)$. The commutator relations

$$[a_i, a_j] = \epsilon_{ijk} a_k$$

are those of the Lie algebra $su(2)$. We may prove now that the adjoint action on $E_3$ is given by

$$\text{ad} (a_i) = A_i, \quad (A_i)_{jk} := -\epsilon_{ijk}.$$ 

Indeed, for $v \in E_3$,

$$\text{ad} (a_i) v = [a_i, v] = \frac{1}{4} \epsilon_{ijk} [e_j e_k, v]$$

$$= \frac{1}{4} \epsilon_{ijk} (e_j [e_k, v] - [e_j, v] e_k)$$

$$= \frac{1}{4} \epsilon_{ijk} (-2e_j (e_k, v) + 2(e_j, v) e_k)$$

$$= -\epsilon_{ijk} (e_k,v) e_j = (A_i)_{jk} (e_k,v) e_j = A_i v$$

and hence the adjoint representation of the group $\text{Spin}(E_3)$ on $E_3$ is given by

$$\text{Ad}(e^a) v = e^A v, \quad v \in E_3, \quad a \in C^2(E_3), \quad A = \text{ad} (a)$$

is but the familiar action of the group $SO(3)$. The spin group itself, a double cover of $SO(3)$, is thus seen to be isomorphic to the unitary group $SU(2)$.

**Example 2.** Let $V$ be the 4-dimensional Minkowski space $M_4$ with metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ ($\mu, \nu = 0, \ldots, 3$) and standard basis vectors $e_\mu$. The Clifford algebra $C(M_4)$ of dimension 16 is defined through the relations $e_\mu e_\nu + e_\nu e_\mu + 2g_{\mu\nu} 1 = 0$. The Lie algebra $C^2(M_4)$ is 6-dimensional with basis $m_{\mu\nu} := -\frac{1}{2} \epsilon_{\mu\nu} (\mu < \nu)$. It is convenient to regard $m_{\mu\nu}$ as an antisymmetric tensor. The commutation relations

$$[m_{\mu\nu}, m_{\sigma\tau}] = g_{\tau\mu} m_{\nu\sigma} + g_{\mu\nu} m_{\tau\sigma} + g_{\tau\nu} m_{\sigma\mu} + g_{\sigma\mu} m_{\tau\nu}$$
are those of the Lie algebra $sl(2,\mathbb{C})$, and the adjoint action on $M_4$ is given by
\[ \text{ad}(m_{\mu\nu}) = M_{\mu\nu}, \quad (M_{\mu\nu})_{\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha} \]
implying that the adjoint action $\text{Ad} = \exp \circ \text{ad}$ of the spin group $\text{Spin}(M_4) \cong SL(2,\mathbb{C})$ on $M_4$ coincides with the action of the Lorentz group (the identity component thereof, strictly speaking). As is well known, the group $SL(2,\mathbb{C})$ is a double covering of the Lorentz group.

We cannot, however, incorporate Dirac’s $\gamma$ matrices in the present framework unless we are willing to complexify the Clifford algebra and to study complex Clifford modules. For it is clear that, with regard to the last example, $\gamma_\mu$ corresponds to $i e_\mu$, where $i$ is the imaginary unit, and the spinor space, on which the $\gamma$’s act, is isomorphic to $\mathbb{C}^4$. Hence, with respect to some basis in $\mathbb{C}^4$, the $\gamma$’s are represented by complex matrices. The necessary steps to deal with this problem of complex extension are the subject of the next sections.

6 The Spinor Module

When dealing with the real vector space $V$ of dimension $n$, the complexified Clifford algebra of (complex) dimension $2^n$ is the tensor product $C(V) \otimes \mathbb{C}$. The concept of a complex Clifford module is then obvious: it is some complex vector space $E$ together with an action of $C(V) \otimes \mathbb{C}$ respecting the complex linear structure. The goal now is to construct, for suitable $V$’s, a canonical irreducible complex Clifford module, $S$, termed the spinor module, i.e., we want to construct an isomorphism
\[ c : C(V) \otimes \mathbb{C} \rightarrow \text{End} \, S. \] (16)
This gives $\text{End} \, S$ the dimension $2^n$ and consequently $S$ the dimension $2^{n/2}$ which makes sense provided $n = \text{even}$. In Minkowski’s model of spacetime, where $V = M_4$, we are fortunate to encounter an even dimension, $n = 4$, and so $\dim S = 2^2 = 4$ which is the dimension of Dirac spinors, the elements of the spin module.

To construct the spinor module in general we pass to the complex space $V \otimes \mathbb{C}$ first and then extend the quadratic form $q(v)$ in $V$ to a quadratic form $q(w)$ in $V \otimes \mathbb{C}$. Such an extension is unique. Moreover, there is a complex-bilinear form $(,)$ in $V \otimes \mathbb{C}$ extending the real-bilinear form in $V$ and satisfying $q(w) = (w, w)$. Supposing there are complimentary subspaces $V_{\pm}$ of $V \otimes \mathbb{C}$, dual to each other in the sense that $V_{\pm}^* \cong V_{\mp}$, then the isomorphism
\[ V \otimes \mathbb{C} \cong V_{+}^* \oplus V_{+} \]
is said to provide a complex polarisation of $V$ if the quadratic form $q$ in $V \otimes \mathbb{C}$ corresponds to the canonical quadratic form in $V^* \oplus V$, that is to say, if
\[ \frac{1}{2} q(w) = w^\#(w) = (w_-, w_+) \]
for $w_\pm \in V_\pm$ such that $w = w_+ + w_-$. The factor $\frac{1}{2}$ is inevitable because $(w_+, w_+) = 0$ and $q(w) = (w_+ + w_-, w_+ + w_-) = 2(w_-, w_+)$. 

**Example 1.** The four-dimensional Minkowski space, $M_4$, admits a polarisation because the Lorentz metric, extended to $M_4 \otimes \mathbb{C}$, may be written in a polarized form:
\[ (x, x) = (x^1 + ix^2)(-x^1 + ix^2) + (x^0 + x^3)(x^0 - x^3) \quad (x^\mu \in \mathbb{C}). \]
Thus, a possible choice of the subspaces is
\[ V_\pm = \{ x \in M_4 \otimes \mathbb{C} \mid \pm x_1 + ix_2 = x^0 \pm x^3 = 0 \} \cong \mathbb{C}^2. \]

**Example 2.** The four-dimensional Euclidean space, $E_4$, admits a polarisation because the Euclidean metric, extended to $E_4 \otimes \mathbb{C}$, may be written in a polarized form:
\[ (x, x) = (x^1 + ix^2)(x^1 - ix^2) + (x^3 + ix^4)(x^3 - ix^4) \quad (x^\mu \in \mathbb{C}). \]
A possible choice of the subspaces is
\[ V_\pm = \{ x \in E_4 \otimes \mathbb{C} \mid x^1 \pm ix^2 = x^3 \pm ix^4 = 0 \} \cong \mathbb{C}^2. \]

The two previous examples are typical in that they demonstrate a general fact: any real pseudo-Euclidean space of even dimension admits a complex polarisation. The space $V$ is said to be pseudo-Euclidean if there is a bilinear form preserving isomorphism
\[ V \otimes \mathbb{C} \cong E_n \otimes \mathbb{C} \]
where $E_n$ denotes the $n$-dimensional Euclidean space. Equivalently stated, one can find a basis $e_j$ in $V \otimes \mathbb{C}$ such that
\[ (e_j, e_k) = \delta_{jk} \quad (j, k = 1, \ldots, n). \]
Such a basis will be called orthonormal. If $V$ happens to be Euclidean, any orthonormal basis in $V$ would do.

Note, there are many cases falling into the category of pseudo-Euclidean spaces, such as the Minkowski space $M_4$, the deSitter (dS), and the Anti-deSitter (AdS) space. However, with the present emphasis on bilinear forms...
rather than on scalar products, the distinction between Euclidean and pseudo-
Euclidean spaces disappears in the complex domain. From now on we shall
always assume that $V$ is (pseudo-)Euclidean and even-dimensional.

It is not difficult to demonstrate that $V$ admits a polarisation. The rea-
son is this. Given an orthonormal basis $(e_i)_{i=1}^n$ in $V \otimes \mathbb{C}$, we can define
complementary subspaces by

$$V_\pm = \text{span}\left\{ e_{2k-1} \pm ie_{2k} \mid k = 1, \ldots, n/2 \right\}$$

and set up isomorphisms $V_\pm \rightarrow V_\mp^*$, $w \mapsto w^\flat$, by

$$w^\flat(w') = (w, w'), \quad w' \in V_\mp.$$ 

It remains to prove that

$$(w_-, w_+) = \frac{1}{2} q(w) \quad (w = w_+ + w_-, \ w_\pm \in V_\pm). \quad (17)$$

To this end we write $w = x^k e_k \in V \otimes \mathbb{C}$ with coordinates $x^k \in \mathbb{C}$ so that
$q(w) = \sum_k (x^k)^2$ and $w = w_+ + w_-$ with

$$w_\pm = \frac{1}{2} \sum_{k=1}^{n/2} (x^{2k-1} \mp i x^{2k}) (e_{2k-1} \pm ie_{2k}).$$

Then

$$(w_-, w_+) = \frac{1}{2} \sum (x^{2k-1} + i x^{2k})(x^{2k-1} - i x^{2k})$$

$$= \frac{1}{2} \sum (x^k)^2 = \frac{1}{2} q(w).$$

Consider now the exterior algebra

$$S = \bigwedge V_+.$$ 

As complex vector space, $S$ has the dimension $2^{n/2}$. To give $S$ the structure
of a Clifford module, we need only know the action of $v \in V$ on $S$. We define
$c : V \rightarrow \text{End} \ S$ by

$$c(v) = \sqrt{2} \left( \epsilon(w_+) - \iota(w_-) \right), \quad v = w_+ + w_-, \quad w_\pm \in V_\pm$$

and immediately verify that $c$ is a Clifford map:

$$c(v)^2 = -2 \left[ \epsilon(w_+), \iota(w_-) \right] = -2(w_+, w_-) = -q(v).$$
Its extension to $V \otimes \mathbb{C}$ and $C(V) \otimes \mathbb{C}$ is straightforward. As is well known and will be shown later, the Clifford algebra is simple. Irreducibility of a Clifford module $E$ then is equivalent to the map $C(V) \otimes \mathbb{C} \cong \text{End} \ E$ being an isomorphism. The proof that the spinor module $S$ is irreducible offers no problem and will be skipped.

We now show that the spinor module $S$ is in fact a supermodule. Namely, with respect to an orthonormal basis in $V \otimes \mathbb{C}$ we define the chirality operator $\Gamma \in C(V) \otimes \mathbb{C}$ by

$$\Gamma = i^{n/2}e_1e_2\cdots e_n.$$  

The factor in front has been chosen so as to guarantee that $\Gamma^2 = 1$. In physics where $V = M_4$, the corresponding operator $c(\Gamma)$ on $S$ is known as the $\gamma_5$ matrix.

The question arises whether the definition of $\Gamma$ depends on our choice of the basis or ‘frame’. Suppose that $e'_j$ is another orthogonal basis in $V \otimes \mathbb{C}$. Then $e'_j = A_{jk}e_k$ for some complex matrix $A$ with $A^T A = 1$ so as to preserve orthogonality of the basis. Consequently, $\det A = \pm 1$. It is clear now that there are precisely two classes of frames or orientations, the two frames $(e_k)$ and $(e'_k)$ have the same orientation if $\det A = 1$.

From the definition of the quantization map we infer that

$$e'_1 \cdots e'_n = \sigma^{-1}(e'_1 \wedge \cdots \wedge e'_n) = \det A \sigma^{-1}(e_1 \wedge \cdots \wedge e_n) = \pm e_1 \cdots e_n$$

which tells us that, given some orientation, there will be no ambiguity in the definition of $\Gamma$ because then $\det A = 1$ always. For the sake of consistency, we define both $S$ and $\Gamma$ with respect to some orthogonal and oriented basis in $V$.

Since $n$ is even, $\Gamma a = \pm a \Gamma$ for $a \in C^\pm(V) \otimes \mathbb{C}$. Therefore, if we define the projection operators

$$p^\pm = \frac{1}{2}(1 \pm c(\Gamma)) \in \text{End} \ S,$$

we get a grading of the spinor module respected by the Clifford action:

$$S = S^+ \oplus S^-,$$

$$S^\pm = p^\pm S.$$  

There seem to be two different $\mathbb{Z}_2$-gradings on $S$, both of them respected by the Clifford action: one given by the chirality operator and another one given by the structure of $S$ as an exterior algebra: $S^\pm = \bigwedge^\pm V_+$. It is important to realize that these two gradings coincide. The proof is facilitated by choosing a suitable basis,

$$e_\pm^k = \frac{1}{\sqrt{2}}(e_{2k-1} \pm ie_{2k}) \in V_\pm.$$
and related multiplication and contraction operators,
\[
\iota_k = \iota(e_k^-), \quad \epsilon_k = \epsilon(e_k^+),
\]
so that
\[
\left[ \epsilon_k, \iota_l \right] = (\epsilon_k^-, \epsilon_l^+) = \delta_{kl}.
\]
We introduce operators \(a_k\) which commute:
\[
a_k := \frac{1}{2}(e_k^+e_k^- - e_k^-e_k^+) = ie_{2k-1}e_{2k} \in C^+(V) \otimes \mathbb{C}.
\]
It is immediately clear that the chirality operator assumes the form
\[
\Gamma = a_1a_2 \cdots a_{n/2}
\] (18)
while \(c(a_k)\) may be written in terms of multiplication and contraction operators:
\[
c(a_k) = \iota_k\epsilon_k - \epsilon_k\iota_k = 1 - 2\epsilon_k\iota_k.
\]
For any \(k\), the product \(\epsilon_k\iota_k\) has eigenvalues 0, 1 and, therefore, \(c(a_k)\) has eigenvalues \(\pm 1\). Namely, on the basis
\[
e_I = e_{i_1}^+ \wedge \ldots \wedge e_{i_p}^+ \in \bigwedge^p S
\]
we obtain
\[
c(a_k)e_I = \begin{cases} e_I & \text{if } k \notin I \\ -e_I & \text{if } k \in I \end{cases}
\]
and hence \(c(\Gamma)e_I = (-1)^{|I|}e_I\) which completes the proof.
In passing we remark that the operators \(a_k\) are elements of the complex Lie algebra \(C^2(V) \otimes \mathbb{C}\). Moreover,
\[
ia_k \in \text{Spin}_\mathbb{C} V = \exp \left( C^2(V) \otimes \mathbb{C} \right)
\]
as can be seen from \(a_k^2 = 1\) and
\[
\exp \left( i\frac{\pi}{2}a_k \right) = ia_k.
\]
By Eq.(18), the chirality operator \(\Gamma\) is in \(\text{Spin}_\mathbb{C} V\). In the Euclidean case, choosing a basis \(e_j \in E_n\), we have \(ia_k \in C^2(E_n)\), but also \(ia_k \in \text{Spin } E_n\). By Eq.(18), \(\Gamma \in \text{Spin } E_n\) iff \(i^{n/2} \in \mathbb{R}\), i.e., iff \(n = 0 \pmod{4}\).
7 Selfadjointness for Complex Modules

We want to extend the concept of selfadjointness to complex modules when there is an isomorphism $V \otimes \mathbb{C} \cong E_n \otimes \mathbb{C}$. We may thus take any orthogonal basis in $E_n$ and regard it as orthogonal basis in $V \otimes \mathbb{C}$. The implied polarisation has the property
\[ V_\pm = V_\mp \]
where $w \mapsto \bar{w}$ means complex conjugation in $E_n \otimes \mathbb{C}$. To the previously listed properties, satisfied by the $\ast$ operation, we have to add only its antilinearity,
\[ (\lambda a)\ast = \bar{\lambda}a\ast \]
Then the standard $\ast$ operation in the real Clifford algebra $C(V)$ has a unique extension to its complex counterpart, the algebra $C(V) \otimes \mathbb{C}$. In particular,
\[ w\ast = -\bar{w}, \quad w \in V \otimes \mathbb{C}. \]

If $E$ is some complex Clifford module, then in order to give $\text{End} E$ the structure of a $\ast$ algebra we need to have a Hermitian structure on $E$. For, if $\langle, \rangle$ is a scalar product in $E$, then the adjoint of $A \in \text{End} E$ is given by
\[ \langle A^\ast x, y \rangle = \langle x, Ay \rangle \quad (x, y \in E). \]

A complex Clifford module $E$ with Clifford action $c : C(V) \otimes \mathbb{C} \to \text{End} E$ satisfying
\[ c(a\ast) = c(a)\ast \]
is said to be selfadjoint. In particular, the relation
\[ c(w)\ast + c(\bar{w}) = 0 \]
holds which, as we know, suffices to establish selfadjointness.

To demonstrate that the spinor module $S$ is selfadjoint, we need to specify a scalar product in $V \otimes \mathbb{C}$,
\[ \langle w, w' \rangle = \langle \bar{w}, w' \rangle, \]
restrict it to $V_+$, subspace of $V \otimes \mathbb{C}$, and then to extend $\langle, \rangle$ to all of $\bigwedge V_+$:
\[
\begin{align*}
\langle 1, 1 \rangle &= 1 \\
\langle \bigwedge^k V_+, \bigwedge^l V_+ \rangle &= 0 \quad (k \neq l) \\
(\wedge w_k, \wedge w'_k) &= \det(\langle w_i, w'_j \rangle)_{i,j=1}^{k} \quad (k = 1, \ldots, n/2).
\end{align*}
\]

It is easy to see that $\langle, \rangle$ is indeed a scalar product on $V \otimes \mathbb{C}$: if $w = x^k e_k$, then $\bar{w} = \bar{x}^k e_k$ and $\langle w, w \rangle = (\bar{w}, w) = \sum |x^k|^2$. Moreover, $V_+$ and $V_-$ are orthogonal subspaces.
To summarize, we have passed from the bilinear form $(, )$ in $V \otimes \mathbb{C}$ to the scalar product $\langle , \rangle$. It seems natural to change the definition of the contraction operator accordingly,

$$\bar{\iota}(w) = \iota(\bar{w}) \quad (w \in V_+),$$

so that $\bar{\iota}(w)(w' \wedge a) = \langle w, w' \rangle a - w' \wedge \bar{\iota}(w)a$ in $\wedge V_+$. Under the $^*$ operation in $\text{End} \wedge V_+$ the behavior of the multiplication and contraction operators (which generate the endomorphism algebra) is as follows:

$$\epsilon(w)^* = \bar{\iota}(w), \quad \bar{\iota}(w)^* = \epsilon(w) \quad (w \in V_+).$$

Consequently, for $w = w_1 + \bar{w}_2$ with $w_j \in V_+$ (such decomposition is unique),

$$c(w)^* = \sqrt{2}(\epsilon(w_1) - \bar{\iota}(w_2))^*$$

$$= \sqrt{2}(\bar{\iota}(w_1) - \epsilon(w_2)) = -c(\bar{w}) = c(w^*)$$

thereby proving that the spinor module $S$ is selfadjoint.

The case of a pseudo-Euclidean vector space $V$ is in no way different from the case of a Euclidean vector space, except that ‘complex conjugation’ receives a different meaning: it is defined in $E_n \otimes \mathbb{C}$ rather than in $V \otimes \mathbb{C}$. Therefore, $v \in V$ is not a ‘real’ element of $E_4 \otimes \mathbb{C}$ unless $V \cong E_n$. As a consequence, we lose the property $c(v)^* + c(v) = 0$. This observation may be phrased as follows: as $C(V)$ modul, the spinor module $S$ is not selfadjoint unless $V$ is Euclidean. So the lesson is: complex modules ought to be regarded as $C(V) \otimes \mathbb{C}$ modules.

**Example.** In Dirac’s relativistic theory of the electron, the Hilbert space is $L^2(\mathbb{R}^3) \otimes S$ where $S$ is the spinor module with scalar product $\langle , \rangle$ as constructed above. The Clifford algebra is $C(M_4^*) \otimes \mathbb{C}$ with $M_4^*$ the so-called momentum space. It is dual to the Minkowski space $M_4$. If $e^\mu$ is the standard basis in $M_4$, we let $e^\mu$ denote the dual basis in $M_4^*$. Hence, any $p \in M_4^*$ is of the form $p = p_\mu e^\mu$ with $p_\mu \in \mathbb{R}$. To make effective use of the isomorphism

$$M_4^* \otimes \mathbb{C} \cong E_4 \otimes \mathbb{C}$$

we need to introduce complex momenta, too. There is an orthogonal basis in $E_4$ which, under the above isomorphism, corresponds to the vectors

$$e^0, \ i e^1, \ i e^2, \ i e^3$$

and the preferred way to expand complex momenta is:

$$p = p_0 e^0 + p_1 i e^1 + p_2 i e^2 + p_3 i e^3 \quad (p_\mu \in \mathbb{C}).$$
If \( p \) has complex coordinates \( p_\mu \), then \( \bar{p} \) has the complex conjugate coordinates \( \bar{p}_\mu \). Another way of stating this peculiar property is:

\[
e^0 = e^0, \quad e^k = -e^k, \quad k = 1, 2, 3.
\]

Restricted to \( M^*_4 \), the operation \( p \mapsto \bar{p} \) is thus seen to coincide with the reflection in 3-space (or parity operation). The chirality operator is

\[
\Gamma = i^2 e^0 (ie^1)(ie^2)(ie^3) = ie^0 e^1 e^2 e^3 \in C(M^*_4) \otimes \mathbb{C}.
\]

To make contact with Dirac’s theory, we define

\[
\gamma^\mu = ic(e^\mu), \quad \gamma_5 = c(\Gamma)
\]

so that \( \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \). The *operation sending \( p \) to \( p^* = -\bar{p} \) acts on the basis vectors as follows:

\[
e^{0*} = -e^0, \quad e^{k*} = e^k, \quad k = 1, 2, 3.
\]

From the fact that the spinor module is selfadjoint we infer:

\[
\gamma^{0*} = \gamma^0, \quad \gamma^{5*} = \gamma_5, \quad \gamma^{k*} = -\gamma^k, \quad k = 1, 2, 3.
\]

There are various matrix representations of the \( \gamma \)'s used in physics. All of them respect these relations. Finally, \( S \) is a supermodule. With respect to the grading \( S^+ \oplus S^- \), the \( \gamma \)'s may be represented in block form:

\[
\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k = 1, 2, 3).
\]

As these matrices indicate, the \( \gamma \)'s are odd operators, i.e., they map \( S^\pm \) into \( S^\mp \), while \( \gamma_5 \) is diagonal:

\[
\gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.
\]

At the same time, the subspaces \( S^\pm \) reduce the representation of the Spin group \( SL(2, \mathbb{C}) \). The two subrepresentations (so-called fundamental representations) are irreducible and inequivalent. Note, however, that the group \( SL(2, \mathbb{C}) \cong Spin M^*_4 \) is the real spin group, and its representation on \( S \) lacks unitarity. It would be more appropriate to pass to the complex spin group,

\[
Spin_{\mathbb{C}} M^*_4 = \exp \left( C^2(M^*_4) \otimes \mathbb{C} \right) \cong SL(2, \mathbb{C}) \times SL(2, \mathbb{C}),
\]

whose adjoint action on \( M^*_4 \otimes \mathbb{C} \) is known as the complex Lorentz group and whose (complex) Lie algebra is

\[
C^2(M^*_4) \otimes \mathbb{C} \cong (su(2) \oplus su(2)) \otimes \mathbb{C},
\]
and regard the spinor module \( S \) as representation space of this larger group. As both \((ie_1)(ie_2)\) and \((ie_3)e_0\) are elements in

\[
\text{Spin} E_4 < \text{Spin}_\mathbb{C} M_4^*,
\]

so is their product, the chirality operator \( \Gamma \). Sure enough, \( \Gamma \) is not an element of \( \text{Spin} M_4^* \) which means that a helicity change \( S^\pm \to S'^\pm \) cannot be effected by some element in \( SL(2,\mathbb{C}) \), but can be effected by some element in

\[
SU(2) \times SU(2) < SL(2,\mathbb{C}) \times SL(2,\mathbb{C}),
\]

that is, by extension into the complex domain.

It is no surprise that the complex Lorentz group has played a prominent role in the Wightman formulation of field theory, especially in the proof of the PCT theorem [11]. Moreover, passage to the complex Lie algebra has been important for the study of irreducible representations of the group \( SL(2,\mathbb{C}) \).

### 8 The Relation between Spinors and Vectors

As part of a general folklore, spinors are thought of as “square roots of vectors”. In his book [4] Chevalley gave this idea a precise meaning. Namely, in a complex setting one may argue that the tensor product \( S \otimes S^* \) recovers the vector space \( \bigwedge V \otimes \mathbb{C} \). In other words, complex \( k \)-vectors, i.e., elements of \( \bigwedge^k V \otimes \mathbb{C} \), are sums of products of spinors and their duals.

The relation between spinors and vectors follows immediately from our route of constructing the spinor module: use complexification as the first step and polarization as the second. In fact, the isomorphism, we want to draw attention to, is already there and part of a large commutative diagram:

\[
\begin{array}{ccc}
C(V) \otimes \mathbb{C} & \overset{\sigma}{\longrightarrow} & \bigwedge V \otimes \mathbb{C} \\
\downarrow c & & \downarrow \\
\text{End} S & \longrightarrow & S \otimes S^* = \bigwedge V_+ \otimes \bigwedge V^*_+
\end{array}
\]

All objects of this diagram are regarded as \( \mathbb{Z}_2 \)-graded vector spaces while the arrows indicate isomorphisms. Typical arrows of this diagram have previously been considered: the symbol map \( \sigma \) and the Clifford action \( c \). All other arrows relate to standard constructions in multilinear algebra [5].

The point of view taken by practitioners in linear algebra is that \( k \)-vectors (elements of \( \bigwedge^k V \)) are more fundamental than spinors (elements of \( S \)). Likewise in physics, \( k \)-vectors are considered to be more ‘classical’ than spinors. One may, however, advocate the opposite point of view, namely, that everything (vectors, tensors, operators etc.) should be built from spinors. Such a
radical chance of viewpoint has led Penrose [8] to introduce twistors to re-
construct spacetime in General Relativity. This example suggests to think of
spinors in more geometric terms. It also marks the birth of Spin Geometry.
Another seminal work, that started a new line of research, is Witten’s proof
[9] of the Positive-Energy Conjecture in General Relativity where he uses
spinor fields in a classical context. Another subject, termed non-commutative
geometry, has been introduced by Connes [10] with possible applications in
particle physics, not to mention the impressive use of Dirac operators on
general spin manifolds, initiated by Atiyah and Singer. Thanks to these de-
velopments we came to acknowledge the fact that the use of spinors is not
only within the domain of quantum physics.

Last not least we want to point out that in Dirac theory the intimate
relationship between $k$-vectors and elements of $\text{End } S$ has always been recog-
nized and made use of. For, if $A$ is some operator on the spinor module $S$, it may be decomposed as $A = \sum A_k$ where $A_k$ ($k = 0, \cdots, n$) are operators
of degree $k$ and parity $(-1)^k$, obtained from complex antisymmetric tensors $a_{\mu_1 \cdots \mu_k}$:

$$A_k = \frac{1}{k!} a_{\mu_1 \cdots \mu_k} \sigma^{\mu_1 \cdots \mu_k}$$

where

$$\sigma^{\mu_1 \cdots \mu_k} = c \left( \sigma^{-1}(e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}) \right),$$

In the Dirac theory, we thus obtain

$$\sigma^{\mu_1 \cdots \mu_k} = \frac{i^{-k}}{k!} \sum_\pi \text{sign}(\pi) \gamma^{\mu_{\pi(1)}} \cdots \gamma^{\mu_{\pi(k)}}.$$

In short, there is a 1:1 correspondence between $a \in \bigwedge V \otimes \mathbb{C}$ and $A \in \text{End } S$
given by $A = c(\sigma^{-1}(a))$. The parity of $a$ is compatible with the parity of $A$,
i.e., if $a \in \bigwedge^k V$ and $(-1)^k = \pm 1$, then $A \in \text{End} \mathbb{C}^V$.

9 Universality of the Spinor Modul

Having constructed the spinor module $S$, we immediately see that it is a
universal object within the category of complex irreducible Clifford modules.
That is to say, any irreducible Clifford action $C(V) \otimes \mathbb{C} \rightarrow \text{End } E$ factorizes,

$$C(V) \otimes \mathbb{C} \xrightarrow{c} \text{End } S \xrightarrow{r} \text{End } E,$$

the algebraic isomorphism $r$ being induced by some vector space isomor-
phism:

$$S \xrightarrow{g} E, \quad r(b) = gbg^{-1}, \quad b \in \text{End } S.$$
Therefore, the spinor module is, up to isomorphism, the only irreducible complex Clifford module.

The problem of determining all possible Clifford modules \( E \) reduces to the study of representations

\[
r : \text{End} \, S \to \text{End} \, E.
\]

(19)

Unless \( \dim S = 1 \), one cannot construct a trivial representation (some homomorphism \( \text{End} \, S \to \mathbb{C} \)) of the algebra \( \text{End} \, S \). Since \( \dim S = 2^{n/2} \geq 2 \), trivial representations of the Clifford algebra do not occur. More is true. Simple facts about endomorphism algebras teach us that any representation (19) is completely reducible and decomposes into subrepresentations, each one of them being isomorphic to the fundamental representation \((\text{id} : \text{End} \, S \to \text{End} \, S)\). A representation of an algebra (or group) is called primary, if it is a multiple of a single irreducible representation. Summarizing, any Clifford module is primary or, equivalently, assumes the form

\[
E \cong W \otimes S
\]

where it is understood that \( c(a) \) acts trivially on the vector space \( W \):

\[
c(a)(w \otimes s) = w \otimes as \quad (w \in W, s \in S, a \in C(V) \otimes \mathbb{C}).
\]

Modules with this structure are called twisted spinor modules. The space \( W \) is termed twisting space. Abstractly, the space \( W \) can be identified with \( \text{Hom}_{\text{Cl}}(S, E) \), i.e., with the vector space of linear maps \( w : S \to E \) that commute with the Clifford action. The isomorphism

\[
\text{Hom}_{\text{Cl}}(S, E) \otimes S \to E, \quad w \otimes s \mapsto ws
\]

is then obvious.

Conversely, let \( W \) be any vector space. Then the tensor product \( W \otimes S \) is a Clifford module (with trivial Clifford action on \( W \)). In physics this construction is used to incorporate further degrees of freedom beyond those of the spin polarization. These extra degrees may describe the momentum of a particle (see Dirac’s theory of the electron for example) or be related to internal symmetries as is the case in gauge theories. The above result “Clifford modules are twisted spin modules” imposes severe restrictions on model building.

Whatever the twisting space \( W \), the module \( E = W \otimes S \) is \( \mathbb{Z}_2 \)-graded, hence a supermodule:

\[
E = E^+ \oplus E^- \quad (E^\pm = W \otimes S^\pm).
\]
This gives \( \text{End} \, E \) the structure of a superalgebra. We will, however, also deal with cases where the twisting space carries a \( \mathbb{Z}_2 \)-grading. Then

\[
E^\pm = (W^+ \otimes S^\pm) \oplus (W^- \otimes S^\mp)
\]

and even(odd) operators on \( S \) extend to even(odd) operators on \( E \). This shows that in a variety of situations the twisted spinor module is indeed a supermodule.

Recall that we have previously constructed a canonical scalar product in the spinor module \( S \). The existence of another scalar product in the twisting space \( W \) would turn the module \( E \) into a Hermitian space with a selfadjoint Clifford action, a situation we normally encounter in physical applications.

## 10 Supercommuting Endomorphisms

This section is devoted to studying the structure of \( \text{End} \, E \) when \( E \) is some complex \( C(V) \otimes \mathbb{C} \) supermodule and \( V \) is pseudo-Euclidean. To facilitate the discussion we use a shorthand for the complexified Clifford algebra:

\[
A = C(V) \otimes \mathbb{C}.
\]

The main result will be the isomorphism

\[
\text{End} \, E \cong A \hat{\otimes} \text{End}_A E.
\]

(20)

Since both \( A \) and \( \text{End}_A E \) are superalgebras, we have to exercise some care:

1. The algebra \( \text{End}_A E \) has endomorphisms \( b \) as its elements that supercommute with the Clifford action: \([a, b] = 0, \, a \in A\).

2. The tensor product \( \hat{\otimes} \) is special for \( \mathbb{Z}_2 \)-graded algebras (often called the skew tensor product):

\[
(a \hat{\otimes} b)(a' \hat{\otimes} b') = \begin{cases} 
-aa' \hat{\otimes} bb' & \text{if } a, b \text{ are odd} \\
 aa' \hat{\otimes} bb' & \text{otherwise}.
\end{cases}
\]

Sure enough, as tensor product of vector spaces there would be no difference and hence no confusion.

The proof of (20) runs as follows. Choose some orthogonal basis \((e_i)_{i=1}^n\) in \( V \otimes \mathbb{C} \), and consider the induced basis \( e_I \) in \( A \). For \( k = 0, 1, \ldots, n \) and
I ⊂ \{1, \ldots, k\} we define projection operators \(P^{(k)}_I \in \text{End } A\) recursively. If \(k = 0\) we put \(P^{(0)}_\emptyset = \text{id}\) and, for \(k \geq 1,\)

\[
P^{(k)}_I a = \begin{cases} 
-\frac{1}{2} \left[ e_k, e_k P^{(k-1)}_I a \right] & \text{if } k \not\in I \\
-\frac{1}{2} e_k \left[ e_k, P^{(k-1)}_I a \right] & \text{if } k \in I
\end{cases}
\]  

\(a \in A\). \quad (21)

Using the quantization map (of Section 4) one sees that each supercommutator \(-\frac{1}{2} \left[ e_k, \cdot \right]\) acts like an annihilation (contraction) operator while each multiplication from the right by \(e_k\) acts like a creation operator. So one obviously deals here with the product of these two operators, either in normal or in reversed order. In the physics literature, such products occur as ‘number operators’ having eigenvalues 0 and 1 only, so they may justly be called projection operators, too. The correspondence with Fermi operators helps to establish the following results. For \(J \subset \{1, \ldots, n\},\)

\[
P^{(k)}_I e_J = n_k(I, J)e_J, \quad n_k(I, J) = \begin{cases} 
1 & \text{if } \{1, \ldots, k\} \subset (I \cap J) \cup (I^c \cap J^c) \\
0 & \text{otherwise.}
\end{cases}
\]

\((I^c = \text{compliment of } I \text{ in } \{1, \ldots, n\})\). For \(P_I = P^{(n)}_I\) we have

\[
P_I = e_I Q_I, \quad Q_I e_I = \delta_{IJ}1, \quad \sum_I P_I = \text{id}
\]  

(22)

with operators \(Q_I \in \text{End } A\), serving as left inverses of the multiplication by \(e_I\). Explicitly, for a suitable choice of \(\sigma_I = \pm 1,\)

\[
Q_I a = (-\frac{1}{2})^n \sigma_I \left[ c_n, \ldots \left[ c_2, \left[ c_1, e_I \cdot a \right] \right] \ldots \right] \quad (a \in A).
\]

Owing to the existence of these operators, the Clifford algebra is seen to be simple. Namely, by virtue of the relations (22), any nontrivial ideal must contain the unit 1 \(\in A\) and hence coincide with the whole of \(A\). As a consequence, Clifford actions are always injective.

Consider now a Clifford module \(E\). Under the algebraic homomorphism \(c : A \rightarrow \text{End } E\), the operators \(P^{(k)}_I\) and \(Q_I\) have images \(\hat{P}^{(k)}_I\) and \(\hat{Q}_I\) acting linearly on \(\text{End } E\). By analogy to (21), setting \(c_k = c(e_k)\) and \(\hat{P}^{(0)}_\emptyset = \text{id}\), we recursively define

\[
\hat{P}^{(k)}_I b = \begin{cases} 
-\frac{1}{2} \left[ c_k, c_k \hat{P}^{(k-1)}_I b \right] & \text{if } k \not\in I \\
-\frac{1}{2} c_k \left[ c_k, \hat{P}^{(k-1)}_I b \right] & \text{if } k \in I
\end{cases}
\]  

\((b \in \text{End } E)\). \quad (23)

Similarly, setting \(c_I = c(e_I)\), we also define

\[
\hat{Q}_I b = (-\frac{1}{2})^n \sigma_I \left[ c_n, \ldots \left[ c_2, \left[ c_1, e_I \cdot b \right] \right] \ldots \right]
\]

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so that the operator identities $\hat{P}_I = c_I \hat{Q}_I$ hold. We show now the validity of the completeness relation

$$\sum_I \hat{P}_I b = b \quad (b \in \text{End } E).$$

Though, by construction, this relation is certainly valid for $b = c(a)$ and $a \in A$ arbitrary, we cannot assert that the relation extends \textit{automatically} to all of End $E$. But in fact it does. To make the proof more transparent, let us introduce auxiliary elements

$$b_k = \sum \hat{P}_I^{(k)} b \in \text{End } E \quad (k = 0, 1, \ldots, n)$$

where the sum is over $I \subset \{1, \ldots, k\}$. The assertion may now be stated as $b_k = b$, and what is known may be stated as $b_0 = \hat{P}_0^{(0)} b = b$. Therefore, the proof would be complete, once we have shown that $b_k$ is independent of $k$.

The key equation is

$$-\frac{1}{2} \left[ c_k, c_k b \right] - \frac{1}{2} c_k \left[ c_k, b \right] = b \quad (b \in \text{End } E) \quad (24)$$

which follows from the generalized Jacobi identity and

$$\left[ c_k, c_k \right] b = c(\left[ e_k, e_k \right]) b = -2c(1)b = -2b.$$  

From (23) and (24) we infer

$$\hat{P}_I^{(k)} b + \hat{P}_{I \cup \{k\}}^{(k)} b = \hat{P}_I^{(k-1)} b, \quad I \subset \{1, \ldots, k-1\}.$$

Summing both sides over $I$ yields $b_k = b_{k-1}$ which completes the proof.

Any $b \in \text{End } E$ can now be decomposed as

$$b = \sum_I \hat{P}_I b = \sum_I c(e_I) \hat{Q}_I b = \sum_I e_I \hat{\otimes} \hat{Q}_I b.$$

This way we have explicitly constructed the decomposition in $A \hat{\otimes} \text{End}_A E$. It remains to demonstrate the property

$$\left[ c(a), \hat{Q}_I b \right] = 0 \quad (a \in A) \quad (25)$$

stating that $\hat{Q}_I b$ is some element of $\text{End}_A E$ and also that the tensor product is ‘skew’. Reason: skewness implies that, for arbitrary $b \in \text{End}_A E$,

$$\left[ c(a), b \right] = (a \hat{\otimes} 1)(1 \hat{\otimes} b) \pm (1 \hat{\otimes} b)(a \hat{\otimes} 1) = 0$$
with a plus sign if both \( a \) and \( b \) are odd and a minus sign otherwise. It suffices to prove the property (25) for the special case \( a = e_i \). To see the strategy of the proof, take \( a = e_n \) first:

\[
[c_n, \hat{Q} b] \sim [c_n, [c_n, \ldots, [c_1, c_{Ic}] b] \ldots]
\]

\[
= [ [c_n, c_n], [c_{n-1}, \ldots] ] - [c_n, [c_n, [c_{n-1}, \ldots] ]]
\]

\[
= -[c_n, [c_n, \ldots, [c_1, c_{Ic} b] \ldots] ] = 0.
\]

To pass from the first to the second line we have used the generalized Jacobi identity. To obtain the third line we have used \(-\frac{1}{2} [c_n, c_n] = \text{id}\). Suppose now that we run the same calculation with \( a = e_i \). We would also apply the Jacobi identity and the property \(-\frac{1}{2} [c(w), c(w')] = (w, w')\text{id}\), valid for all \( w, w' \in V \otimes \mathbb{C} \). In effect, we make use of the formula

\[
[c(w), [c(w'), \cdot ] ] + [c(w), [c(w'), \cdot ] ] = 0
\]

to carry out the proof along the same lines. So far, Eq.(25) is seen to be correct for the generators \( a = e_i \) only. But the general assertion (for \( a = e_I \)) is an immediate consequence.

Last not least we draw attention to the fact that all algebras involved in the foregoing discussion are \( \mathbb{Z}_2 \)-graded. It is easily checked that the decomposition (20) conforms to these structures, i.e., we have

\[
\text{End}^\pm E = (A^+ \otimes \text{End}^+_A E) \oplus (A^- \otimes \text{End}^-_A E).
\]

where \( A = C(V) \otimes \mathbb{C} \) as before.

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