ON $H^*(BP\text{U}_n;\mathbb{Z})$ AND WEYL GROUP INVARIANTS

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Abstract. For the projective unitary group $PU_n$ with a maximal torus $T_{PU_n}$ and Weyl group $W$, we show that the integral restriction homomorphism

$$\rho_{PU_n} : H^*(BP\text{U}_n;\mathbb{Z}) \to H^*(BT_{PU_n};\mathbb{Z})^W$$

to the integral invariants of the Weyl group action is onto. We also present several rings naturally isomorphic to $H^*(BT_{PU_n};\mathbb{Z})^W$.

In addition we give general sufficient conditions for the restriction homomorphism $\rho_G$ to be onto for a connected compact Lie group $G$.

1. Introduction

Let $G$ be a compact connected Lie group. Its maximal tori are determined up to conjugation. Let $T_G$ be one of the maximal tori and $N_G$ its normalizer. Then the Weyl group $W_G := N_G/T_G$ acts by conjugation on $T_G$. When the group $G$ is clear from the context, we shall often omit the subscript $G$ from the notations.

Let $BH$ denote the classifying space of a topological group $H$, $H^*(BH; R)$ cohomology with coefficients in a commutative ring $R$. The conjugation action of $W$ on $T$ induces an action of $W$ on the cohomology ring $H^*(BT; R)$ and the inclusion $T \hookrightarrow G$ induces a homomorphism of cohomology rings

$$H^*(BG; R) \to H^*(BT; R),$$

whose image lies in $H^*(BT; R)^W$, the sub-ring of invariants under the $W$-action. Hence we obtain the restriction homomorphism

$$(1.1) \quad \rho_{G; R} : H^*(BG; R) \to H^*(BT; R)^W.$$ 

The study of $H^*(BG; R)$ via the homomorphism $\rho_{G; R}$ was pioneered by Borel and Hirzebruch [5] and determining the image of $\rho_{G; R}$ remains an important problem in algebraic topology. It is well known that $\rho_{G; \mathbb{Q}}$ is onto for all $G$: there are numerous proofs, such as the one in Chapter III of [11]. It is also well known for $G = U_n$, the unitary group, that $\rho_{PU_n; R}$ is onto for all co-efficient groups $R$. For the projective unitary group $G = PU_n = U_n/S^1$, where $S^1 \subseteq U_n$ is the center, the situation is less clear: Toda proved that $\rho_{PU_2; \mathbb{Z}/2}$ is onto for $n \equiv 2 \pmod{4}$ or $n = 4$ [14] and we discuss the integral case below.

In this paper we will focus on integral Weyl group invariants and set

$$H^*(-; \mathbb{Z}) := H^*(-; \mathbb{Z}), \quad \rho_G := \rho_{G; \mathbb{Z}}.$$ 

Feshbach [7] proved that $\rho_{SO_n}$ is onto for all $n$ but that $\rho_{Spin_{12}}$ is not onto. Later Benson and Wood [4] proved for all $n \geq 6$ that $\rho_{Spin_n}$ is onto if and only if $n$ is not congruent to 3, 4 or 5 modulo 8. Kameko and Mimura [13] studied more cases. For $G = PU_2$, we have $PU_2 \cong SO_3$ and so $\rho_{PU_2}$ is onto. Vezzosi showed that $\rho_{PU_3}$ is
onto [15] and this was later generalized by Vistoli to $\rho_{PU_n}$ for all odd primes $p$ [16].

The second author also proved for degrees $* \leq 12$, that $\rho_{PU_n}$ is onto for all $n$ [8].

Despite the progress discussed above, the question of whether $\rho_{PU_n}$ was onto remained opened in general. Recently, the work of Antieau and Williams [1] and the second author [9], [10] showed that the determination of $H^*(BU_n)$ is important input for studying the Topological Period-Index Problem [1] and this has renewed interest in the map $\rho_{PU_n}$. Our main result is the following

**Theorem 1.1.** For all $n$, the restriction homomorphism

$$\rho_{PU_n} : H^*(BU_n) \to H^*(BT_{PU_n})^W$$

is onto.

We present the outline of the proof of Theorem 1.1 in Section 1.1 after first discussing some of its consequences. Let $q_n : U_n \to PU_n := U_n/S^1$ denote quotient map defining $PU_n$ and consider the related fibration of classifying spaces

$$BU_n \xrightarrow{Bq_n} BU_n \to B2S^1. \tag{1.2}$$

As we explain in Section 4, Theorem 1.1 is equivalent to the following

**Corollary 1.2.** The image of $(Bq_n)^* : H^*(BU_n) \to H^*(BU_n)$ is a summand, which is naturally isomorphic to $H^*(BT_{PU_n})^W$.

We define four further rings naturally isomorphic to $H^*(BT_{PU_n})^W$.

(i) For $0 \leq j \leq n$, let $c_j \in H^2(BU_n)$ be $j$th Chern class. Define the derivation

$$\nabla_n : H^*(BU_n) \to H^{*-2}(BU_n)$$

by $\nabla_n(c_j) := (n-j+1)c_{j-1}$ and extending by the Leibnitz rule. Then we have $\text{Ker}(\nabla_n) \subset H^*(BU_n)$.

(ii) Let $\mu : BU_n \times BS^1 \to BU_n$ be the map which classifies the homomorphism $U_n \times S^1 \to U_n$, where $S^1 \subset U_n$ is the centre and define the subring of primitive elements

$$PH^*(BU_n) := \{c \mid \mu^*(c) = c \otimes 1\} \subset H^*(BU_n).$$

(iii) Let $d_3^* : H^*(BU_n) \to H^{*-3}(BU_n)$ denote the $d_3$-differential from the 0-column of the Leray-Serre spectral sequence of the fibration $\text{Ker}(d_3^*) \subset H^*(BU_n)$.

(iv) For a space $X$, set $FH^*(X) := H^*(X)/TH^*(X)$, where $TH^*(X) \subseteq H^*(X)$ is the torsion ideal and consider $FH^*(BU_n)$.

**Theorem 1.3.** There are natural isomorphisms

$$FH^*(BU_n) \cong H^*(BT_{PU_n})^W \cong \text{Im}((Bq_n)^*) = \text{Ker}(\nabla_n) = PH^*(BU_n) = \text{Ker}(d_3^*).$$

We briefly discuss the groups and maps in Theorem 1.3. The Weyl group invariants $H^*(BT_{PU_n})^W$ are described algebraically by $\text{Ker}(\nabla_n)$ and topologically by $PH^*(BU_n)$. The equality $\text{Im}((Bq_n)^*) = \text{Ker}(d_3^*)$ entails that all higher differentials, $d_r^*$ for $r > 3$, vanish in the Serre spectral sequence of the fibration in (1.2). As $FH^*(BU_n) \cong H^*(BT_{PU_n})^W$, there is a split short exact sequence of graded groups

$$0 \to TH^*(BU_n) \to H^*(BU_n) \xrightarrow{Bq_n} H^*(BT_{PU_n})^W \to 0 \tag{1.3}$$

and Vistoli [15] has constructed a splitting of the above sequence as graded rings in the case that $n$ is an odd prime. However, the situation for general $n$ is open.
Hence finding natural splittings of $\rho_{PU_n}$; i.e., finding natural constructions of characteristic classes in $H^*(BP\!\!U_n)$ which map onto the Weyl group invariants, remains an interesting open problem; see Remark 1.10.

1.1. General sufficient conditions for $\rho_G$ to be onto. We will prove Theorem 1.7 by establishing general sufficient conditions for $\rho_G$ to be onto and verifying that they hold for $G = PU_n$. Recall that $N$ is the normaliser of a maximal torus $T$ of $G$ and consider the inclusion $N \to G$. As $G$ and $N$ have the same maximal torus and Weyl group, the natural map $\nu: BN \to BG$ induces an isomorphism $\nu^*: H^*(BG; \mathbb{Q}) \to H^*(BN; \mathbb{Q})$. Since $G/N$ has Euler characteristic 1 (see [2, §6]), the composition

$$H^*(BG) \xrightarrow{\nu^*} H^*(BN) \xrightarrow{\pi^*} H^*(BG)$$

is the identity, where $\pi^*$ denotes the transfer in cohomology [2, Theorem 5.5]. Hence $\nu^*: H^*(BG) \to H^*(BN)$ is a split injection; see, for example, [2, Proposition A.1] and its proof. It follows that the induced map

$$FH^*(BG) \to FH^*(BN)$$

is an isomorphism. Since $H^*(BT)$ is torsion free, so is $H^*(BT)^W$ and so the natural map $H^*(BN) \to H^*(BT)^W$ factors over the induced map $FH^*(BN) \to H^*(BT)^W$. Hence we have proven

**Lemma 1.4.** The restriction homomorphism $\rho_G: H^*(BG) \to H^*(BT)^W$ is onto if and only if $\rho_N: H^*(BN) \to H^*(BT)^W$ is onto. $\square$

By definition, there is a short exact sequence $1 \to T \to N \to W \to 1$ and to investigate when $\rho_N$ is onto, we consider the more general situation, where we have a short exact sequence of topological groups,

$$1 \to A \to M \xrightarrow{\pi} V \to 1,$$

with $A \subseteq M$ a closed normal subgroup and $V$ a finite discrete group. Standard arguments give that $\pi: BA \to BM$, the map classifying the inclusion $A \to M$, has a model which is a $V$-covering space where $BM$ is a $CW$-complex. (We will make the meaning of a model more precise in Section 2) This induces a $V$-$CW$-structure on $BA$ where $BA//V := EV \times_V BA \to BV$ is a model for $BM \to BV$ and the induced map

$$\pi^*: Z^*(BM; R) \to Z^*(BA; R)$$

of cellular cocycles takes values in $Z^*(BA; R)^V$, the subgroup of $V$-invariant cocyles. Suppose, for argument’s sake, that the induced map

$$\pi^*: H^*(BM; R) \to H^*(BA; R)^V$$

is onto. Then for all $x \in H^*(BA; R)^V$, $x = \pi^*(y)$ for some $y \in H^*(BM; R)$ and we let $c_y \in Z^*(BM; R)$ be a cocyle representing $y$. Now $x = \pi^*(y)$ is represented by the $V$-invariant cocyle $\pi^*(c_y)$. Thus, we have shown that if the induced map $\pi^*: H^*(BM; R) \to H^*(BA; R)^V$ is onto, then $BA$ admits a $V$-$CW$ structure such that the natural map $BA//V \to BV$ is a model for $BM \to BV$ and the canonical map $Z^*(BA; R)^V \to H^*(BA; R)^V$ is onto. The following theorem states that after weakening the conditions on the $V$-action on a model for $BA$, the converse holds. Specifically, we define a weak $V$-$CW$-complex to be a $CW$-complex with a cellular $V$ action (but no further requirement that the action is trivial on cells with are fixed set-wise by the action).
Theorem 1.5. Let
\[ 1 \to A \to M \xrightarrow{\phi} V \to 1 \]
be a short exact sequence of topological groups, where \( A \) is a closed subgroup of \( A \) and \( V \) is discrete. Then the following are equivalent:

1. The natural map \( H^*(BM; R) \to H^*(BA; R)^V \) is onto;
2. There is a model for \( BA \) which is a weak \( V \)-CW-complex such that the natural map \( BA/V \to BV \) is a model for \( BM \to BV \) and the induced map of \( V \)-invariants
\[ Z^*(BA; R)^V \to H^*(BA; R)^V \]
is onto.

Hence the restriction homomorphism \( H^*(BM; R) \to H^*(BA; R)^W \) is onto if and only if \( BA \) has a \( V \)-CW-structure where \( BA/V \to BV \) models \( BM \to BW \) and every \( V \)-invariant class in \( H^*(BA; R) \) is represented by a \( V \)-invariant cocyle.

As a direct consequence of Lemma 1.4 and Theorem 1.5 we have Corollary 1.6.

Let \( G \) be a connected compact Lie group with \( N \) the normaliser of the maximal torus \( T \) and Weyl group \( W = N/T \). Then the restriction homomorphism \( \rho_G : H^*(BG) \to H^*(BT)^W \) is onto if there is a model for \( BT \) which is a weak \( W \)-CW-complex such that \( BT/W \to BW \) is a model for \( BN \to BW \) and the induced map of \( W \)-invariants
\[ Z^*(BT)^W \to H^*(BT)^W \]
is onto.

We will prove Theorem 1.1 by showing that \( BT_{PU_n} \) has a model satisfying the condition of Corollary 1.6; see Section 3.

As mentioned earlier, Benson and Wood [4] proved that Condition (1) of Theorem 1.5 fails for \( G = BSpin_n \), when \( n > 6 \) and \( n \equiv 3, 4, 5 \) (mod 8). Therefore, Theorem 1.5 imposes restrictions on possible \( W \)-equivariant CW-decompositions of the maximal torus of \( G \). More precisely, we have the following Corollary 1.7.

Let \( G = Spin_n \) for \( n > 6 \), \( n \equiv 3, 4, 5 \) (mod 8). There is no \( W \)-CW-complex model of \( BT \) such that \( BT/W \to BW \) is a model for \( BN \to BW \) and the homomorphism
\[ Z^*(BT)^W \to H^*(BT)^W \]
is surjective.

Curtis, Wiederhold and Williams [6] showed that for \( G = Spin_n \), the canonical surjection \( N \to W \) does not split. On the other hand, in all known cases where \( G \) is a connected compact Lie group and \( \rho_G : H^*(BG) \to H^*(BT)^W \) is surjective, including the case \( G = PU_n \), the canonical surjection \( N \to W \) does split; equivalently, \( N \cong T \times W \). Hence we ask the following Question 1.8.

If \( G \) is a connected compact Lie Group and \( N \cong T \times W \), is \( \rho_G \) onto? Equivalently, does \( BT \) admit a weak \( W \)-CW-structure where \( BT/W \to BW \) models \( BN \to BW \) and \( Z^*(BT)^W \to H^*(BT)^W \) is onto?

When \( G \) is not a connected compact Lie group, there are examples of Benson and Feshbach [5], where the short exact sequence
\[ 1 \to A \to M \xrightarrow{\phi} V \to 1 \]
splits and the conditions of Theorem 1.5 are not satisfied. Specifically, for each positive integer \( n \), Benson and Feshbach constructed a finite group \( M_n \) with a split extension

\[
1 \to A_n \to M_n \to \mathbb{Z}/2 \to 1,
\]

where \( A_n \) is a non-abelian group and where (for the discrete topologies on all groups) the canonical homomorphism

\[
H^*(BM_n; \mathbb{F}_2) \to H^*(BA_n; \mathbb{F}_2)^{\mathbb{Z}/2}
\]

is not surjective. As a consequence of Theorem 1.5, we have

**Corollary 1.9.** For the Benson-Feshbach groups \( M_n \cong A_n \rtimes \mathbb{Z}/2 \), there is no weak \((\mathbb{Z}/2)\)-CW-complex structure on \( BA_n \) such that \( BA_n/(\mathbb{Z}/2) \to B(\mathbb{Z}/2) \) is a model for \( BM_n \to B(\mathbb{Z}/2) \) and the canonical homomorphism

\[
Z^*(BA_n; \mathbb{F}_2)^{\mathbb{Z}/2} \to H^*(BA_n; \mathbb{F}_2)^{\mathbb{Z}/2}
\]

is surjective.

**Organisation.** In Section 2 we complete the proof of Theorem 1.5 by studying the Serre spectral sequence of the fibration \( BA \to BM \to BV \) and related spectral sequences. In Section 3 we present a \( WP_{U_n} \)-equivariant CW-complex structure on the classifying space \( BT_{PU_n} \) and prove Theorem 1.1. Section 4 discusses the Weyl group invariants \( H^*(TPU_n)^W \), proving Corollary 1.2 and Theorem 1.3. Finally, in Section 5 we briefly discuss the examples of Benson and Feshbach and prove Corollary 1.9.

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2. The Serre spectral sequence for a fibration of classifying spaces

In this section we prove Theorem 1.5. Fix an epimorphism of topological groups \( \phi : M \to V \), where \( V \) is a finite discrete group \( V \), and suppose that the kernel \( A \) is a closed subgroup of \( M \).

By definition, there is a short exact sequence

\[
1 \to A \to M \to V \to 1
\]

and the corresponding fibration sequence of classifying spaces

\[
BA \to BM \to BV.
\]

Let \( H \to G \) be a homomorphism of topological groups. We begin by defining what we mean for a map \( f : X \to Y \) to be a model for \( BH \to BG \); there should be a universal principal \( G \)-bundle \( \gamma \) over \( Y \), a universal principal \( H \)-bundle \( \delta \) over \( X \) and an isomorphism of principal \( G \)-bundles \( F_G(\delta) \to p^*(\gamma) \), where \( F_G(\delta) \) is the principal \( G \)-bundle associated to \( \delta \).

If \( p : X \to Y \) is a fibration and a model for \( BM \to BV \) then if \( W := p^{-1}(y) \) is a fibre of \( p \) then the inclusion \( W \to X \) is a model for \( BA \to BM \). This is an exercise in the universal properties of classifying spaces, which we leave to the reader.

We now consider the problem of determining when a \( V \)-action on \( X \), some model for \( BA \), is such that \( X/\!/V \to BV \) is a model for \( BM \to BV \). We first make the following simple observation.
Lemma 2.1. If \( f: X \to Y \) is a \( V \)-equivariant map between CW-complexes, which is a homotopy equivalence, then \( X//V \to BV \) is a model for \( BM \to BV \) if and only if \( Y//V \to BV \) is as well.

Proof. Since \( f \) is a \( W \)-equivariant map it induces a map of homotopy quotients \( f//W: X//W \to Y//W \), which fits into the following map of fibrations

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X//W & \xrightarrow{f//W} & Y//W \\
\downarrow & & \downarrow \\
BV & = & BV
\end{array}
\]

The 5-Lemma shows that \((f//W)_* : \pi_*(X//W) \to \pi_*(Y//W)\) is an isomorphism. Since both spaces have the homotopy type of CW-complexes, \( f//W \) is a homotopy equivalence, which commutes with the maps to \( BV \), and the lemma follows. \( \square \)

Next we give a construction of \( V \)-actions on \( X \) to model \( BM \to BV \).

Lemma 2.2. Suppose that the free right action of \( A \) on \( EA \) extends to a \( (\text{not necessarily free}) \) \( M \)-action. Then the induced \( V \)-action on \( BA = EA/A \) is such that \( BA//V \to BV \) is a model for \( BM \to BV \). Furthermore, the map \( BM \to BV \) is a fiber bundle with structural group \( V \) and fiber \( X \cong BA = EA/A \) with the \( V \)-action being the one passing from the \( M \)-action on \( EA \).

Proof. By definition, \( BA//V = BA \times_\pi EV = (BA \times EV)/V \) where \( V \) acts on \( BA \times EV \) by

\[
(BA \times EV) \times V \to BA \times EV, \quad ((b, z), v) \mapsto (bv, zv).
\]

where the \( V \)-action on \( BA \) is the one passing from the \( M \)-action on \( EA \).

Now the \( V \)-action on \( BA \) is the quotient of a \( V \)-action on \( EA \) and we can define the following free right \( M \)-action on \( EA \times EV \),

\[
(IA \times EV) \times M \to EA \times EV, \quad ((e, z), m) \mapsto (em, z\varphi(m)).
\]

Then the map

\[
(IA \times EV) \to (IA \times EV)/M = (BA \times EV)/V = BA//V
\]

is a universal principal \( M \)-bundle such that the induced \( V \)-bundle is identified with the pull-back of \( EV \to BV \) along \( BA//V \to BV \). Indeed, we have the following commutative diagram

\[
\begin{array}{ccc}
EV \times_{BV} (BA//V) & \xrightarrow{\cong} & (IA \times EV) \times M V \\
\downarrow & & \downarrow \\
BA//V & \xrightarrow{} & \end{array}
\]

where the horizontal arrow is a homeomorphism given by

\[
EV \times_{BV} (BA \times_\pi EV) \to (IA \times EV) \times M V, \\
(e_V, [e_A], e_V v) \mapsto (e_A, e_V, v)
\]

with \( e_V \in EV \) and \( e_A \in EA \), and \([e_A]\) denotes the orbit of \( e_A \) in \( BA = EA/A \).
Hence $BA\times V \to BV$ is a model for $BM \to BV$, and is a fiber bundle with structural group $V$ and fiber $X \simeq BA = EA/A$ with the $V$-action being the one passing from the $G$-action on $EA$. □

We next show that $BM \to BV$ is always modelled by $X\times V \to BV$ for some $V$-action on $X$, a model for $BA$. Let $EM \to BM$ be a universal principal $M$-bundle. In other words, we have a contractible space $EM$ with a free $M$-action $\phi$ on it. The action $\phi$ restricts to $A \subset M$ and we take $EM/A$ as a model for $BA$. The action of $M$ on $EM$ passes to a right action of $V = M/A$ on $BA = EM/A$ which we also denote by $\phi$. Hence by Lemma 2.2 we have

**Lemma 2.3.** With the above defined model of $X = EM/A$ and the right action $\phi$, $X\times V \to BV$ is a model for $BM \to BV$. Furthermore, the map $BM \to BV$ is a fiber bundle with structural group $V$ and fiber $X \simeq BA$ with the $V$-action $\phi$. □

Suppose there is a $V$-CW-complex $X$ and a $V$-equivariant homotopy equivalence $X \to BA$. Then for every $k \geq 1$, $V$ acts on $X^{(k-1)}$, the $(k-1)$-skeleton of $X$ and hence also on the quotient CW-complex $X_k := X/X^{(k-1)}$ and we have

**Lemma 2.4.** There is a homotopy equivalence

$$EV \times_V X \to EV \times_V BA \simeq BM$$

and for all $k \geq 1$, there are $V$-equivariant quotient maps

$$q_k: EV \times_V X \to EV \times_V X_k,$$

which fit into the following commutative diagram of fibrations over $BV$

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & BV \\
\downarrow & & \downarrow \\
X_k & \xrightarrow{\pi_k} & EV \times_V X_k
\end{array}
\]

Moreover, for $k \geq 1$, the $V$-action on $X_k$ has a fixed point and so $\pi_k$ has a section. □

**Proof of Theorem 1.5.** In the discussion preceding Theorem 1.5, we have shown that Condition (1) implies Condition (2). It remains to show that Condition (2) implies Condition (1).

We consider the Serre spectral sequence associated to (2.2), which we denote by $E_{\ast}^{s,t}$:

\[
\begin{align}
E_2^{s,t} & \cong H^s(BV; H^t(BA)_\phi) \Rightarrow H^{s+t}(BM), \\
d_r^{s,t} : E_r^{s,t} & \to E_r^{s+r-t,r+1}.
\end{align}
\]

Here $H^t(BA)_\phi$ is the local coefficient system given by the canonical action $\phi$ of $V = \pi_1(BV)$ on $H^*(BA)$.

On the $E_2$-page we have the groups $E_2^{0,k} \cong H^k(BA; R)^V$ and on the $E_\infty$-page we have that $E_\infty^{0,k} = \text{Im}(H^k(BM; R) \to H^k(BA; R)^V)$. Hence it suffices to show that all differentials coming out of the 0-column consisting of the groups $E_r^{0,k}$ column are trivial for all $r$ and all $k$.

For $k = 0$ all differentials leaving $E_2^{0,0}$ vanish for dimensional reasons. For $k \geq 1$, we consider the map of fibrations $q_{k+1}: EV \times_V X \to EV \times_V X_k$ and let $d_r^{s,t} E_{r+1}^{s,t}$ be
the Serre spectral sequence associated to the homotopy fiber sequence

\[(2.5) \quad X_k \to EV \times_V X_k \to BV.\]

The commutative diagram \[(2.3)\] induces a morphism of spectral sequences

\[E^{s,t}(q_k) : kE^{s,t} \to E^{s,t},\]

which on the \(E_2\)-page is given by the homomorphism

\[H^s(BV; H^t(X_k)) \to H^s(BV; H^t(X_k)).\]

In particular, for \((s, t) = (0, k)\), we have

\[(2.6) \quad E_0^{0,k}(q_k) = \phi^*: kE_0^{0,k} = H^k(X_k)^V \to H^k(X)^V = E_2^{0,k},\]

and by Condition (2) of Theorem \(1.5\), the homomorphism \((2.6)\) is a surjection. By construction, \(X_k\) is \((k-1)\)-connected. Hence in the spectral sequence \(kE^{s,t}\) we have

\[kE_0^{0,k} = kE_0^{0,k}.\]

On the other hand, the homotopy fiber sequence \((2.5)\) has a section by Lemma \(2.4\). Therefore the differential

\[k+1d_0^0 : kE_2^{0,k} = kE_2^{0,k} \to kE_2^{0,k+1}\]

is trivial and so

\[(2.7) \quad kE_2^{0,k} = kE_2^{0,k} = kE_\infty^{0,k}.\]

Since the homomorphism of \((2.6)\) is surjective, \((2.7)\) entails that

\[E_2^{0,k} = E_\infty^{0,k},\]

which completes the proof that Condition (2) implies Condition (1).

\[\square\]

**Remark 2.5.** We conclude this section with a discussion of the second equivalent condition of Theorem \(1.5\) which requires that \(BA\) has a weak \(V\)-CW model \(X\) in which the natural map \(Z^*(X; R)^V \to H^*(X; R)^V\) is onto. By definition, there is a short exact sequence of \(RV\)-modules

\[(2.8) \quad 0 \to B^i(X; R) \to Z^i(X; R) \to H^i(X; R) \to 0,\]

where \(B^i(X; R) = \partial(C^{i-1}(X; R)) \subseteq Z^i(X; R)\) is the subgroup of coboundaries. Writing \(B^i = B^i(X; R), Z^i = Z^i(X; R)\) and \(H^i = H^i(X; R)\) and regarding \((2.8)\) as a sequence of coefficient \(R[V]\)-modules, we have the the exact sequence

\[H^0(V; Z^i) \to H^0(V; H^i) \to \delta_X H^1(V; B^i) \to i_X H^1(V; Z^i).\]

Using the identifications \((Z^i)^V = H^0(V; Z^i)\) and \((H^i)^V = H^0(V; H^i)\) we see the second condition of Theorem \(1.5\) holds whenever there is weak \(V\)-CW model \(X\) for \(BA\) where \(\delta_X = 0\), or equivalently where \(i_X\) is injective.
3. An equivariant CW-decomposition of $BT_{PU_n}$

In this section we construct a CW-complex model for $BT_{PU_n}$ in such a way that Condition (2) of Theorem 1.5 is satisfied in the case $M = N_{PU_n}$, $A = T_{PU_n}$ and $V = W_{PU_n}$. Let $W = N_{PU_n}/T_{PU_n}$ be the Weyl group of the normalizer $N_{PU_n}$. Then $W$ is isomorphic to $W_{U_n}$, which is the symmetric group $S_n$. Recall that a weak $W$-complex is a CW-complex with a cellular $W$-action.

**Proposition 3.1.** There is a weak $W$-$CW$-structure on $BT_{PU_n} \cong BT^{n-1}$ such that $BT_{PU_n}/W \to BW$ is a model for $BN_{PU_n} \to BW$ and the induced homomorphism of $W$-invariants

$$Z^*(BT_{PU_n})^W \to H^*(BT_{PU_n})^W$$

is surjective. Here $Z^*(BT_{PU_n})$ is the graded abelian group of cocycles of the cellular cochain complex.

The proof of Proposition 3.1 requires the following three preliminary lemmas.

**Lemma 3.2.** Let $f_i : E_i \to B$, $i = 0, 1$ be fiber bundles such that there is a map $h : E_0 \to E_1$ satisfying $f_0 \simeq f_1 h$. Then $h$ is homotopic to some $h' : E_0 \to E_1$ satisfying $f_0 \simeq f_1 h'$. In other words, $h$ is homotopy equivalent to a bundle map.

**Proof.** Use the homotopy lifting property with respect to the homotopy $f_0 \simeq f_1 h$ and the fiber bundle $f_1 : E_1 \to B$. \hfill $\square$

**Lemma 3.3.** Let $G$ be a discrete group. For $i = 0, 1$, let $F_i$ be $G$-spaces, $p_i : E_i \to BG$ the fiber bundles with $E_i = EG \times_G F_i$, and $p_i$ the canonical projection. Let $f : E_0 \to E_1$ be a fiber map over $BG$, i.e., we have $p_0 = p_1 f$. Then there is a $G$-equivariant map $h : F_0 \to F_1$ such that we have the following commutative diagram,

$$
\begin{array}{ccc}
F_0 & \to & E_0 \\
\downarrow h & & \downarrow f \\
F_1 & \to & E_1,
\end{array}
$$

where the horizontal arrows are inclusions of fibers.

**Proof.** Let $p_G : EG \to BG$ be the universal covering space/universal $G$-bundle over $G$. For any $G$-space $F$ and the fiber bundle $p : E \to BG$ over $BG$ with $E = EG \times_G F$ and $p$ the canonical projection, we define the following action of $\pi_1(BG,x) \cong G$ on $p^{-1}(x) \cong F$.

Let $I$ be the unit interval $[0,1]$,

$$\sigma : (I, 0, 1) \to (BG, x, x)$$

a loop in $BG$ at the base point $x$ and $e$ a point in $p^{-1}_G(x)$. By the theory of universal coverings, there is a unique path

$$\tilde{\sigma}_e : I \to EG$$

satisfying $p_G \tilde{\sigma} = \sigma$ and $\tilde{\sigma}_e(0) = e$. Furthermore, $\tilde{\sigma}_e(1)$ depends only on the homotopy class $[\sigma] \in \pi_1(BG,x)$. We define

$$\langle [\sigma] : EG \times_G F \to EG \times_G F, (e, z) \mapsto (\tilde{\sigma}_e(1), z)$$

and the above action restricts to $F \cong p^{-1}_G(x)$. It is elementary to check that the action given by (3.1) is well-defined and compatible with the given $G$-action on $F$. 


Restricting \( f : E_0 \to E_1 \) to the fiber over a base point \( x \in B G \), we obtain a map

\[ h : F_1 \cong p_0^{-1}(x) \to p_1^{-1}(x) \cong F_2 \]

and it is straightforward to check that the above defined \( G \)-actions on \( p_0^{-1}(x) \) and \( p_1^{-1}(x) \) are compatible with \( h \). \( \square \)

For the next lemma, consider the space \( BT^n := (BS^1)^n \) with the action of \( W = S_n \) by permuting the coordinates.

**Lemma 3.4.** There is a model for \( BT^{n-1} \) with a \( W \) action such that

1. \( BT^{n-1}/W \) is of the homotopy type \( BN_{PU_n} \), and
2. there is a \( W \)-equivariant map \( Bq_n : BT^n \to BT^{n-1} \) which is a model for the map induced by the quotient homomorphism \( q_n : T^n \to T^{n-1} \), where \( T^{n-1} \) is identified as \( T^n \) modulo the diagonal.

In particular, regarding the homology of \( BT^{n-1} \) as

\[ H_*(BT^{n-1}) \cong P[t_1, t_2, \ldots, t_n]/(\sum_{i=1}^n t_i) \]

where \( P[t_1, t_2, \ldots, t_n] \) is the divided power algebra in \( t_1, \ldots, t_n \) over \( \mathbb{Z} \), the \( W \)-action is determined by permuting \( t_1, \ldots, t_n \). The induced homomorphism

\[ (Bq_n^*)_* : H_*(BT^n) \cong P[t_1, t_2, \ldots, t_n] \to H_*(BT^{n-1}) \cong P[t_1, t_2, \ldots, t_n]/(\sum_{i=1}^n t_i) \]

is the quotient homomorphism.

**Proof.** Let \( ES^1 \) be contractible space with a free \( S^1 \)-action and consider the product \( ET^n := (ES^1)^n \). Then the \( T^n \) action on \( ET^n \) extends to \( NU_n = T^n \times W \) by permuting the coordinates. By Lemma 2.2 there is a model for \( BN_{U_n} \to BW \) which is a fiber bundle with structure group \( W \) and fiber \( BT^n \) with \( W \) permuting the coordinates. In the rest of the proof, we let \( BN_{U_n} \to BW \) denote this particular model.

Consider the semi-direct product \( NU_n = T^{n-1} \rtimes W \). By Lemma 2.2 there is a model for \( BN_{PU_n} \to BW \) which is a fiber bundle with structural group \( W \) and fiber of the homotopy type \( BT^{n-1} \). We let \( BN_{PU_n} \to BW \) denote this particular model.

The commutative diagram of Lie groups

\[
\begin{array}{ccc}
T^n & \longrightarrow & NU_n \\
\downarrow & & \downarrow \\
T^{n-1} & \longrightarrow & NU_{U_n} \\
\end{array}
\]

induces a homotopy commutative diagram

\[
\begin{array}{ccc}
BT^n & \longrightarrow & BN_{U_n} \\
\downarrow & & \downarrow \\
BT^{n-1} & \longrightarrow & BN_{PU_n} \\
\end{array}
\]

where the horizontal arrows are the above mentioned models, which, in particular, are fiber bundles. By Lemma 3.2 we may assume that the diagram (3.2) without the dashed arrow is strictly commutative. Therefore, by Lemma 3.3 we have a
W-equivariant map of fibers: $Bq^n_k : BT^n \to BT^{n-1}$ with the W-action on $BT^{n-1}$ satisfying the lemma.

**Proof of Proposition 3.4** We first outline the strategy to construct the desired weak W-CW model for $BT^{n-1}$. We start from the map $Bq^n_n : BT^n \to BT^{n-1}$ of Lemma 3.4 (2). Here $BT^n = (BS^1)^n$ and W acts by permuting the co-ordinates. We give $BS^1$ a CW-structure which has one cell in every even dimension and no odd-dimensional cells and we equip $BT^n$ with the product CW-structure, so that $BT^n$ is a weak W-CW-complex and we use that W-equivariant map as a guide for adding W-free cells to $BT^n$ to create a weak W-CW-complex model $X$ for $BT^{n-1}$.

Starting from the weak W-CW-complex $BT^n$, we denote the $k$-skeleton of the relative W-CW-complex model for $BT^{n-1}$ by $X^{(k)} := BT^n \cup (\bigcup_{i=1}^{m} e^{k_i})$.

We add W-free cells inductively so that the map $Bq^n_k : BT^n \to BT^{n-1}$ constructed in Lemma 3.4 extends to a $k$-equivalence $(Bq^n_k)^{(k)} : X^{(k)} \to BT^{n-1}$.

For example, since $Bq^n_n$ is a 2-equivalence, we take $X^{(1)} = BT^n$. It suffices to prove that we can construct $X^{(k)}$ with the following properties:

1. There is a splitting of $ZW$-modules $H_k(X^{(k)}) = H_k(BT^n) \oplus \tilde{A}_k$, for some $ZW$-module $A_k$;
2. The attaching maps of the $(k+1)$-cells of $X^{(k+1)}$ have homological image either in $H_k(BT^n) \oplus \{0\}$ or $\{0\} \oplus \tilde{A}_k$;
3. We have isomorphisms of $ZW$-modules $H_i(X^{(k)}) \cong H_i(BT^{n-1})$ for $i < k$.

Given (1), we can surely attach the $(k+1)$-cells of $X^{(k+1)}$ as in (2). Hence it suffices to prove (1) for $i = k+1$, given that (2) and (3) hold for $i = k$. We proceed by induction. Both (2) and (3) are clearly true when $k = 1$, when $X^{(1)} = BT^n$.

Let $A_i$ denote free $ZW$-module on the $i$-cells attached to $X^{(i-1)}$ to form $X(i)$. By induction, for $i \leq k$, we can write $A_i = A_i' \oplus A_i''$, where the cells $e^{ki}$ forming a basis of $A_i'$ are attached to $BT^n$ and the cells $e^{ri}$ forming a basis of $A_i''$ have homological image in $\{0\} \oplus \tilde{A}_{i-1}$. In particular, since $BT^n$ only has cells in even dimensions, we may assume that $A_{2j}' = 0$ for all $j$. It follows that the $ZW$-chain complex of $X^{(k-1)}$ has the following form when $k = 2j$ is even:

$$
\begin{array}{cccc}
\{0\} & \oplus & \{0\} & \oplus & \{0\} \\
C_{2j}(BT^n) & \oplus & \{0\} & \oplus & A''_{2j} \\
\{0\} & \oplus & A_{2j-1}' & \oplus & A''_{2j-1}' \\
C_{2j-2}(BT^n) & \oplus & \{0\} & \oplus & A''_{2j-2}'
\end{array}
$$
Here the homomorphism $A^2_{2j-1} \to C_{2j-2}(BT^n)$ has cokernel isomorphic to the $ZW$-module $H_{2j-2}(BT^{n-1})$ and each sequence of two successive arrows is exact. Similarly, the $ZW$-chain complex of $X^{(k-1)}$ has the following form when $k = 2j+1$ is odd:

$$C_{2j+2}(BT^n) \oplus \{0\} \oplus \{0\}$$

$$\{0\} \oplus A^2_{2j+1} \oplus A''_{2j+1}$$

$$C_{2j}(BT^n) \oplus \{0\} \oplus A^2_{2j}$$

$$\{0\} \oplus A^2_{2j-1} \oplus A''_{2j-1}$$

Here the homomorphism $A_{2j+1} \to C_{2j}(BT^n)$ has cokernel isomorphic to the $ZW$-module $H_{2j}(BT^{n-1})$ and the sequence

$$A''_{2j+1} \to A''_{2j} \to A'_{2j-1} \oplus A''_{2j-2}$$

is exact.

Setting $A_{2j} := \text{Ker}(A''_{2j} \to A_{2j-1})$ and $A_{2j+1} := \text{Ker}(A_{2j+1} \to C_{2j}(BT^n) \oplus A''_{2j})$ we see both $k = 2j$ and $k = 2j+1$ that $H_k(X^{(k)}) = H_k(BT^n) \oplus \tilde{A}_k$, as required.

We now define $(X, BT^n)$ to be the the infinite relative $CW$-complex obtained by attaching free $W$-cells in all dimensions:

$$X := \bigcup_{k \to \infty} X^{(k)}$$

Finally, we show that the weak $W$-$CW$-complex $X$ is a model for $BT^{n-1}$ with its $W$ action; i.e. that $X/W$ is a model for $BN_{PU_n}$. By the construction of $X$, the $W$-equivariant map $Bq^T_n: BT^n \to BT^{n-1}$ factors as

$$BT^n \overset{q}{\to} X \overset{p}{\to} BT^{n-1},$$

where the first arrow is the canonical inclusion $q: BT^n \hookrightarrow X$ and $p$ is the union of the maps $Bq^T_n(k): X^{(k)} \to BT^{n-1}$. Since $p$ is the increasing union of $k$-equivalences and $X$ and $BT^{n-1}$ are $CW$-complexes, it follows that $p$ is a homotopy equivalence.

Thus $p: X \to BT^{n-1}$ is a $W$-equivariant map which is a homotopy equivalence. Since $(BT^{n-1}, W)$ models $BN_{PU_n} \to W$, then by Lemma 2.1 so does $(X, W)$.

\[ \square \]

**Proof of Theorem 1.1.** It follows from Proposition 3.1 that $PU_n$ satisfies the condition of Corollary 1.6. Therefore, Theorem 1.1 holds. \[ \square \]

4. **On the Weyl group invariants $H^*(BT_{PU_n})^W$**

In the last section we proved that $p_{PU_n}: H^*(BU_n) \to H^*(BT_{PU_n})^W$ is onto. In this section we present several descriptions of $H^*(BT_{PU_n})^W$ and $p_{PU_n}$, proving Corollary 1.2 and Theorem 1.3. First we summarise what is known rationally.
Let \( SU_n \subset U_n \) denote the special unitary group. There is a \( \mathbb{Z}/n \)-covering space \( \mathbb{Z}/n \to SU_n \xrightarrow{p_n} PU_n \) and a corresponding fibration sequence
\[
\mathbb{B} \mathbb{Z}/n \to BSU_n \xrightarrow{Bp_n} BPU_n.
\]
Since the fibre \( B\mathbb{Z}/n = K(\mathbb{Z}/n, 1) \) has trivial rational cohomology, the Serre spectral sequence of the above fibration shows that
\[
(Bp_n)^* : H^*(BU_n; \mathbb{Q}) \to H^*(BSU_n; \mathbb{Q}) = \mathbb{Q}[c_2, \ldots, c_n]
\]
is an isomorphism. Since \( H^*(BU_n; \mathbb{Q}) \to H^*(BU_n; \mathbb{Q})^W \) is an isomorphism, by, for example, [11, Chapter III, Lemma 1.1, Reduction 2], we have

**Proposition 4.1.** \( H^*(BU_n; \mathbb{Q})^W \cong H^*(BU_n; \mathbb{Q}) \cong \mathbb{Q}[c_2, \ldots, c_n] \). \( \square \)

4.1. The isomorphism \( FH^*(BU_n) \to H^*(BU_n)^W \). For any space \( X \), we let \( TH^*(X) \subseteq H^*(X) \) denote the ideal of torsion classes and define the free quotient of \( H^*(X) \) by \( FH^*(X) := H^*(X)/TH^*(X) \). Since \( H^*(BU_n)^W \) is torsion free, Theorem [11] implies that \( \rho_{PU_n} \) induces an isomorphism
\[
\overline{\rho}_{PU_n} : FH^*(BU_n) \to H^*(BU_n)^W.
\]
Of course in each degree \( i \) there are splittings
\[
H^i(BU_n) \cong TH^i(BU_n) \oplus FH^i(BU_n)
\]
but in general we do not know if there is a natural splitting of \( \overline{\rho}_{PU_n} \) or if there is an algebra isomorphism \( H^*(BU_n) \cong TH^*(BU_n) \oplus FH^*(BU_n) \). When \( n = 3 \), Vistolli [10], constructs a natural splitting of \( \overline{\rho}_{PU_3} \), completing the work of Vezzosi [15].

4.2. The isomorphism \( H^*(BU_n)^W \to \text{Ker}(\nabla_n) \). Recall the fibration sequence
\[
BS^1 \to BU_n \xrightarrow{Bq_n} BPU_n,
\]
which is obtained by applying the classifying space functor to the central extension
\[
S^1 \to U_n \xrightarrow{q_n} PU_n.
\]
We will investigate the image of \( (Bq_n)^* : H^*(BU_n) \to H^*(BU_n) \). Consider the restriction of \( q_n \) on the maximal torus of \( U_n \) of diagonal matrices
\[
q^n_T : TU_n \to TPU_n
\]
which may be identified with the quotient map \( T^n \to T^{n-1} = T^n/S^1 \), where \( S^1 \subset T^n \) is the diagonal circle, that the central extension (4.2) induces an isomorphism \( W_{U_n} \to W_{PU_n} \) of Weyl groups and that there is a canonical identification \( W_{U_n} = S_n \). We consider the following commutative diagram:

\[
\begin{array}{ccc}
H^*(BU_n) & \xrightarrow{\rho_{PU_n}} & H^*(BU_n)^W \\
\downarrow{(Bq_n)^*} & & \downarrow{(Bq^n_T)^*} \\
H^*(BU_n)^W & = & \mathbb{Z}[v_1, \ldots, v_n]^{S_n}
\end{array}
\]

Here \( B(q^n_T)(w_i) = v_{i+1} - v_i \) and \( \rho_{PU_n} \) is an isomorphism, which we shall regard as an identification. In order to characterise the image of \( (Bq^n_T)^* \) we define the
differential operator
\[ \nabla_n = \sum_{i=1}^{n} \frac{\partial}{\partial v^i} : \mathbb{Z}[v_1, \ldots, v_n] \to \mathbb{Z}[v_1, \ldots, v_n]. \]

Since \( \nabla_n \) commutes with the action of \( S_n \) on \( \mathbb{Z}[v_1, \ldots, v_n] \), it restricts to the symmetric polynomials and we use the same symbol to denote the restriction
\[ \nabla_n : \mathbb{Z}[v_1, \ldots, v_n]^{S_n} \to \mathbb{Z}[v_1, \ldots, v_n]^{S_n}. \]

Equivalently, \( \nabla_n \) can be regarded as a derivation,
\[ (4.4) \quad \nabla_n : H^*(BU_n) \to H^{*-2}(BU_n), \]
which was investigated by the second author in [8]. If \( c_k \in \mathbb{Z}[v_1, \ldots, v_n]^{S_n} \) denotes the \( k \)th elementary symmetric polynomial, equivalently the \( k \)th Chern class, then an elementary calculation gives
\[ \nabla_n(c_k) = (n-k+1)c_{k-1}. \]

Along with the Leibnitz rule, this identity determines \( \nabla_n \), and another elementary calculation gives

**Lemma 4.2** (Proposition 3.3, [8]).
\[ (Bq_n^T)^*(\mathbb{Z}[w_1, \ldots, w_{n-1}]^{S_n}) = \text{Ker}(\nabla_n) \subset \mathbb{Z}[v_1, \ldots, v_n]^{S_n}. \]

Since \( (Bq_n^T)^* \) is injective, Lemma 4.2 gives

**Lemma 4.3.** \( (Bq_n^T)^*: H^*(BT_{PU_n})^W \to \text{Ker}(\nabla_n) \) is an isomorphism. \( \square \)

Notice that \( \text{Ker}(\nabla_n) \subset H^*(BU_n) \) is a summand, since it is the kernel of a homomorphism to the torsion free group \( H^{*-2}(BU_n) \). From Lemma 4.3 and the diagram (4.3), we obtain the following algebraic characterisation of \( (Bq_n^T)^*(H^*(BP_{U_n})) \subset H^*(BU_n) \).

** Proposition 4.4.** \( (Bq_n^T)^*(H^*(BP_{U_n})) = \text{Ker}(\nabla_n) \) is a summand of \( H^*(BU_n) \).

Together, Proposition 4.4 and Lemma 4.3 prove Corollary 1.2.

**Remark 4.5.** From Proposition 4.4 and the preceding discussion, it was known that \( (Bq_n^T)^*(H^*(BP_{U_n})) \subset \text{Ker}(\nabla_n) \) was a subgroup of finite index in each degree. With Theorem 1.3 we now know that equality holds.

**Example 4.6.** In degree 2, \( \text{Ker}(\nabla_n) \subset H^2(BU_n) \) is trivial and in degree 4 we have the following generator for \( (Bq_n^T)^*(H^4(BP_{U_n})) \):
\[ \begin{cases} 2nc_2 - (n-1)c_1 & \text{even}, \\ nc_2 - \lfloor n/2 \rfloor c_1^2 & \text{odd}. \end{cases} \]

**Remark 4.7.** We consider the ring structure of \( \text{Ker}(\nabla_n) \). By Proposition 4.1 setting \( c_1 = 0 \) induces an isomorphism \( \text{Ker}(\nabla_n) \otimes \mathbb{Q} \cong \mathbb{Q}[c_2, \ldots, c_n] \) but integrally the situation can be more complex. In the special case \( n = 2 \), \( \text{Ker}(\nabla_2) = \mathbb{Z}[4c_2 - c_1^2] \) is a polynomial ring on one generator of degree 4. However, for \( n = 3 \), Vistoli has shown that \( \text{Ker}(\nabla_3) \) is not a polynomial ring [16, Theorem 3.7]. Indeed, there is a class \( \delta \in FH^{12}(BP_{U_3}) \) which is not a polynomial in lower dimensional classes, but 27\( \delta \) is. In general, it is not clear to the authors whether \( \text{Ker}(\nabla_n) \) is even finitely generated as a graded ring.
Since we have a ring isomorphism \( \text{Ker}(\nabla_n) \cong H^*(BT_{PU_n})^W \), the above amounts to discussing whether \( H^*(BT_{PU_n})^W \) is a polynomial ring over \( \mathbb{Z} \). For a general compact Lie group \( G \) and a commutative unital ring \( R \), it is interesting to consider whether \( H^*(BT; R)^W \) is a polynomial ring over \( R \). Ishiguro, Koba, Miyauchi and Takigawa [12] studied many cases.

4.3. The equality \( PH^*(BU_n) = \text{Ker}(\nabla_n) \). In the previous subsection we characterised

\[
(Bq_n)^*(H^*(BP_U n)) \subset H^*(BU_n)
\]

algebraically. In this subsection we characterise this subgroup topologically.

Recall the map \( \mu : BS^1 \times BU_n \to BU_n \) corresponding to the tensor product of the universal complex line bundle over \( S^1 \) and the universal complex \( n \)-vector bundle over \( BU_n \). Then \( \mu \) induces a homomorphism

\[
\mu^* : H^*(BU_n) \to H^*(BS^1) \otimes H^*(BU_n)
\]

which is a coalgebra structure of \( H^*(BU_n) \) over \( H^*(BS^1) \). A primitive element of this coalgebra is an element \( c \in H^*(BU_n) \) satisfying \( \mu^*(c) = 1 \otimes c \). The primitive elements of \( H^*(BU_n) \) form a subring

\[
PH^*(BU_n) := \{ c \in H^*(BU_n) \mid \mu^*(c) = 1 \otimes c \}.
\]

Since the map \( \mu : BS^1 \times BU_n \to BU_n \) also classifies the tensor product of the universal complex line bundle with the universal rank \( n \) complex bundle, it follows that the primitive subring of \( H^*(BU_n) \) can be described as the subring of those characteristic classes of rank \( n \) complex vector bundles \( E \) which remain unchanged after tensor product with any line bundle \( L \):

\[
PH^*(BU_n) := \{ c \in H^*(BU_n) \mid c(E) = c(L \otimes E) \}
\]

**Proposition 4.8.** \( PH^*(BU_n) = \text{Ker}(\nabla_n) \). \( \square \)

**Proof.** By Proposition 3.7 of Toda [14], \( \mu^* : H^*(BU_n) \to H^*(BS^1) \otimes H^*(BU_n) \) satisfies

\[
\mu^*(c_k) = \sum_{i=0}^{k} v^i \otimes \frac{\nabla^i}{i!}(c_k),
\]

where we have \( H^*(BS^1) = \mathbb{Z}[v] \) with \( v \) of degree 2. We will show that (4.5) holds for any class \( x \in H^*(BU_n) \), i.e., that we have

\[
\mu^*(x) = \sum_{i=0}^{k} v^i \otimes \frac{\nabla^i}{i!}(x).
\]

The proposition then follows directly.

For the proof of (4.6), without loss of generality, we assume that \( x \) is a monomial in the classes \( c_k \), and we proceed by induction on the number of factors of the form
ck in x. The base case is therefore (4.5). For the inductive argument, we consider
\[ \mu^*(xc_k) = \mu^*(x)\mu^*(c_k) \]
\[ = \left( \sum_i v^i \otimes \frac{\nabla^i_n}{i!}(x) \right) \left( \sum_j v^j \otimes \frac{\nabla^j_n}{j!}(c_k) \right) \]
\[ = \sum_i \sum_j v^{i+j} \otimes \frac{\nabla^i_n}{i!}(x) \frac{\nabla^j_n}{j!}(c_k) \]
\[ = \sum_i v^i \otimes [ \sum_{i+j=l} \frac{\nabla^i_n}{i!}(x) \frac{\nabla^j_n}{j!}(c_k) ]. \]

Since \( \nabla_n \) is a differential operator, we have
\[ \nabla^i_n(xc_k) = \sum_{i+j=l} \binom{l}{i} \nabla^i_n(x) \nabla^j_n(c_k). \]

We resume the computation (4.7) and obtain
\[ \mu^*(xc_k) = \mu^*(x)\mu^*(c_k) \]
\[ = \sum_i v^i \otimes \left( \sum_{i+j=l} \frac{\nabla^i_n}{i!}(x) \frac{\nabla^j_n}{j!}(c_k) \right) \]
\[ = \sum_i v^i \otimes \frac{\nabla^i_n}{i!}(xc_k) \]
and the inductive step is complete. \( \square \)

From Propositions 4.8 and Proposition 4.1 we obtain

**Proposition 4.9.** \((Bq_n)^*(H^*(BU_n)) = PH^*(BU_n).\) \( \square \)

**Remark 4.10.** From the isomorphism \( PH^*(BU_n) = \text{Ker}(\nabla_n) \) and Vistoli’s theorem [16, Theorem 3.7] (see Remark 4.7), we know that \( PH^*(BU_n) \) is not in general a polynomial algebra. Nonetheless, it would be interesting to identify a canonical minimal set of generators \( \overline{c}_{i,j} \in PH^{2i}(BU_n) \) of \( PH^*(BU_n) \) and then find lifts \( c_{i,j} \in H^*(BPU_n) \) of \( \overline{c}_{i,j}. \) If a natural construction of such classes \( c_{i,j} \) could be found, it would be tempting to call them *projective Chern classes*. For example, \( H^4(BPU_n) \cong \mathbb{Z} \) for all \( n \geq 2 \) and so is generated by one projective Chern class \( c_{2,0} \) which, up to sign, unique.

4.4. **The equality** \( \text{Ker}(d^0_3) = \text{Ker}(\nabla_n). \) One approach to the computation of \( H^*(BPU_n) \) is to use the Serre spectral sequence associated to the fiber sequence
\[ BU_n \xrightarrow{Bq_n} BPU_n \to K(\mathbb{Z}, 3). \]

We denote the cohomological Serre spectral sequence of this fiber sequence by and \( E^{s,t}_*: \)
\[ E^{s,t}_2 = H^s(K(\mathbb{Z}, 3); H^t(BU_n)) \Rightarrow H^{s+t}(BPU_n), \]
\[ d^{s,t}_r : E^{s,t}_r \to E^{s+r,t-r+1}_r. \]
Let \( x_1 \in H^3(K(\mathbb{Z}, 3)) \) be the fundamental class. In the spectral sequence \( E_\ast^\ast \) of [4.9], by [8, Corollary 3.4] we have

\[
d_3(\xi \eta) = \nabla_n(\xi) x_1 \eta
\]

for \( \xi \in H^\ast(BU_n) \) and \( \eta \in H^\ast(K(\mathbb{Z}, 3)) \). Hence \( \text{Ker}(d_3^0) = \text{Ker}(\nabla_n) \) and so by Propositions 4.4 we have

**Proposition 4.11.** \( \text{Ker}(d_3^0) = (Bq_n)^\ast(H^\ast(BPU_n)). \)

Since \( E_\infty^0 = (Bq_n)^\ast(H^\ast(BPU_n)) \), we obtain

**Theorem 4.12.** The Serre spectral sequence \( E_\ast^\ast \) above satisfies \( d_0^r = 0 \) for \( r > 3 \).

4.5. **Summary.** We conclude this section with a diagram summarising the description of \( H^\ast(BPU_n)^W \) which we have presented. Each isomorphism is labelled with the result most relevant to its proof:

\[
\begin{array}{ccccccc}
FH^\ast(BPU_n) & \cong & H^\ast(BTPU_n)^W & \cong & \text{Ker}(\nabla_n) & \cong & \text{Ker}(d_3^0) \\
\text{Im}((Bq_n)^\ast) & \cong & \text{Thm 1.1} & \cong & \text{Lem 4.3} & \cong & \text{Prop 4.7}
\end{array}
\]

The isomorphisms in Diagram (4.10) prove Theorem 1.3.

5. **The examples of Benson and Feshbach**

In Section 3 of [3], Benson and Feshbach construct for each positive integer \( n \) a finite group \( M_n \) with a split extension

\[
1 \to A_n \to M_n \to \mathbb{Z}/2 \to 1.
\]

The groups \( M_n \) are such that, in the Lyndon-Hochschild-Serre spectral sequence

\[
E_2^{s,t} = H^s(B\mathbb{Z}/2; H^t(BA_n; \mathbb{F}_2)^\phi) \Rightarrow H^{s+t}(BM_n; \mathbb{F}_2),
\]

\[
d_r^{s,t}: E_r^{s,t} \to E_r^{s+r, t-r+1},
\]

the differential \( d_n^{0,n-1} \) is nontrivial. Here \( H^t(BA_n; \mathbb{F}_2)^\phi \) denotes the local coefficient system given by the \( \mathbb{Z}/2 \)-action \( \phi \) on \( BA_n \). In the above spectral sequence, we have

\[
E_2^{0,t} = H^0(B\mathbb{Z}/2; H^t(BA_n; \mathbb{F}_2)^\phi) = H^t(BA_n; \mathbb{F}_2)^{\mathbb{Z}/2},
\]

where \( H^t(BA_n; \mathbb{F}_2)^{\mathbb{Z}/2} \) denotes the subgroup of invariants of the \( \mathbb{Z}/2 \)-action. Since the differential \( d_n^{0,n-1} \) is nontrivial, the induced homomorphism

\[
H^\ast(BM_n; \mathbb{F}_2) \to H^\ast(BA_n; \mathbb{F}_2)^{\mathbb{Z}/2}
\]

is not surjective. Corollary 1.9 now follows from Theorem 1.5.
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