Dynamical exponents of an even-parity-conserving contact process with diffusion

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We provide finite-size scaling estimates for the dynamical critical exponent of the even-parity-conserving universality class of critical behavior through exact numerical diagonalizations of the time evolution operator of an even-parity-conserving contact process. Our data seem to indicate that upon the introduction of a small diffusion rate in the process its critical behavior crosses over to that of the directed percolation universality class. A brief discussion of the many-sector decomposability of parity-conserving contact processes is presented.

I. INTRODUCTION

One of the most important theoretical programs in modern statistical mechanics is that of the understanding and classification of the critical behavior of nonequilibrium interacting particle systems. A first major step towards this program was given by the so-called directed percolation (DP) conjecture \( \mathcal{DP} \), which in its original formulation stated that all dynamically driven continuous phase transitions about a single absorbing state in single-component systems with a scalar order parameter in the absence of internal symmetries are in the DP universality class of critical behavior. In this form the conjecture has been confirmed in a host of model systems, among others the basic contact process, Schlögl’s models for autocatalytic chemical reactions and a phenomenological Euclidean field theory of high energy hadronic collision processes (Reggeon field theory). Further investigation revealed that the DP universality class accommodates more general models, with more than a single component as well as with multiple, in some cases infinitely many absorbing states. Actually, even some non-equilibrium models without absorbing states at all were found to share some of the DP exponents.

It has been found, however, that not all phase transitions involving an absorbing state fall into the DP universality class. Examples of models exhibiting non-DP critical behavior range from probabilistic cellular automata and two-temperature kinetic Ising models to interacting monomer-dimer and monomer-monomer models, branching and annihilating random walks with even number of offspring, and a class of parity-conserving contact processes that is the subject of this paper. The common feature of all these models is that the number of interacting particles, whether they are actually particles or are domain walls, is locally conserved modulo 2, i.e., their local dynamical rules conserve parity. It thus appeared at first that local conservation laws were affecting the critical behavior of nonequilibrium systems, as one may have guessed from his knowledge of equilibrium critical phenomena. The new universality class that emerged became known as the parity conserving (PC) universality class. Controversy arose when some parity-conserving models were shown to belong to the DP universality class. In, it was noticed that when one adds a parity-conserving external field to the dynamics of a certain monomer-dimer model with two equivalent absorbing states for which the number of domain walls is conserved modulo 2, the universality class of the model crosses from the PC class to the DP class. This kind of behavior was then subsequently observed in other models, and nowadays it is believed that it is not the symmetry of parity conservation that determines the critical behavior in these systems, but that it is the presence of two equivalent, \( Z_2 \)-symmetric absorbing states that matters. This has led some people to refer to the new universality class as the directed Ising universality class, in allusion to the fact that the Ising model has two \( Z_2 \)-symmetric equivalent ground states. Whether the PC class is as robust as the DP class remains largely an open question, and there is lot of room left to research; for a review, see.

In this paper we provide finite-size scaling estimates for the dynamical critical exponent \( z = v_\| / v_\perp \) of the PC universality class of critical behavior through exact numerical diagonalizations of the master operator of a certain parity-conserving contact process in one dimension. We show that upon the introduction of diffusion in the model its critical behavior crosses over to another universality class that seem to be characterized by DP exponents, although our data are not very good in this case. We also present a brief discussion of the many-sector decomposability of the state space of a class of parity-conserving contact processes, and how diffusion breaks this structure, making the models with diffusion strong candidates to show new types of dynamical critical behavior.
II. PARITY-CONSERVING CONTACT PROCESSES WITH DIFFUSION

Contact processes may be viewed as models for the spread of an epidemic among individuals living in a d-dimensional lattice. We consider a parity-conserving generalization of the basic contact process first introduced in [23]. In this class of processes, called CP(m), an array of m adjacent healthy individuals surrounded by k infected individuals X becomes infected at rate kλ, while an array of m adjacent infected individuals becomes healthy at unit rate. Pictorially, in one dimension we have for CP(1) the elementary processes $X\emptyset \rightarrow XX\emptyset$, $\emptyset X \rightarrow \emptyset XX$, $X\emptyset X \rightarrow X X X$, and $X \rightarrow \emptyset$, corresponding to the basic usual contact process, while for CP(2) we have the elementary processes $X\emptyset \emptyset \rightarrow X XX \emptyset$, $\emptyset \emptyset X \rightarrow \emptyset X XX$, $X\emptyset \emptyset X \rightarrow X X XX$, and $XX \rightarrow \emptyset \emptyset$, and analogously for $m > 2$. We clearly see that CP(m) processes conserve the number of particles modulo m.

For finite systems with an absorbing state, the steady state coincides with it. For the CP(1) process, it is simply given by the completely empty lattice $\emptyset = (0, 0, \ldots, 0)$. CP(2) has, besides $\emptyset$, the two other absorbing states $(0, X, \emptyset, X, \emptyset)$ and $(X, \emptyset, X, \emptyset, 0)$. Under periodic boundary conditions these two configurations are the same; let us call it $\emptyset$. Whether $\emptyset$ belongs to the space of allowed configurations depends on the parity of the lattice size $N$ and on the parity sector $N \mod 2$, where $N$ is the number of particles of the initial configuration. If $L$ is odd, then the only absorbing state for CP(2) is $\emptyset$, since it is impossible to fill the lattice with a repetition of $\emptyset$’s only. For a lattice of even size $L$, the state $\emptyset$ has $L/2$ particles, and this number has to be compatible with the parity of $N \mod 2$: if $L = 4l$, with $l \in \mathbb{N}$, and $N \mod 2 = 0$, then $\emptyset$ is an absorbing state of the process; if otherwise $L = 4l + 2$, then $\emptyset$ is an absorbing state only if $N \mod 2 = 1$. In this work, for definiteness we carried out our calculations on lattices of odd size $L$ and on the $N \mod 2 = 0$ parity sector, such that the only steady state of the process is $\emptyset$. Our methods do of course equally apply to the other sectors of the dynamics as well.

The existence of more than one absorbing state for CP(2) already signals a general feature of CP(m) processes, that the number of absorbing states of these models grows with $m$. Actually, for $m \geq 3$ the number of absorbing states grows exponentially with the system size $L$, a property called many-sector decomposability. For example, in a class of adsorption-desorption processes of $m$-mers on the lattice $\mathbb{Z}$.

The number $I_m(L)$ of absorbing, fully jammed states in these systems grows asymptotically as $I_m(L) \sim 2^\phi L^2$, where $\phi$ is the largest real root of $\phi^m = 2\phi^{m-1} - 1$; for $m = 3, \phi = \frac{1}{2} (1 + \sqrt{5}) \approx 1.618$, the golden mean, while for $m = 4, \phi \approx 1.839$. Notice that $\phi < 2$ for all $m < \infty$. This many-sector structure of the phase space can, however, be broken by adding diffusion $X\emptyset := \emptyset X$ at a finite rate $\mu$ to the processes, since it allows the otherwise jammed states to evolve, eventually leading to a state with an array of $m$ adjacent healthy or infected sites that can then react according to the CP(m) rules. Diffusion thus reduces the number of absorbing states of CP(m) from $I_m(L) \sim 2^\phi L^2$ to $I_m(L) = m$, corresponding to the number of $m$-parity equivalent absorbing states. While it is well known that diffusion, as long as the diffusion constant remains finite, is an irrelevant perturbation for the basic contact process CP(1) [36], for systems with more than one absorbing state diffusion may become a highly relevant perturbation, changing the critical behavior of the process, as recently verified in the pair contact process with diffusion [37,38].

III. FINITE-SIZE SCALING

As is well known [33,41], we may write the master equation for reaction-diffusion processes on the lattice as a Schrödinger-like equation in Euclidean time,

$$\frac{d}{dt} |P(t)\rangle = -H |P(t)\rangle,$$

with $|P(t)\rangle$ the generating vector of the probabilities $P(n, t) = \langle n | P(t) \rangle$ of observing the particular configuration $n = (n_1, n_2, \ldots, n_L) \in \{0, X\}^L$ at instant $t$, and with $H$ the infinitesimal generator of the Markov semi-group. The lowest gap $E_L^{(1)} - E_L^{(0)} = E_L^{(1)}$ in the spectrum of $H$ may be used to perform a finite-size scaling analysis in the same way as one does in equilibrium problems [62]. Around the critical point $\lambda \sim \lambda^*$, the correlation lengths of the infinite system behave like

$$\xi || L \sim (\lambda - \lambda^*)^{-\nu ||} \sim (\lambda - \lambda^*)^{-\nu || L},$$

where $\xi ||$ and $\xi \perp$ are the correlation lengths respectively in the time and space directions, $\nu ||$ and $\nu \perp$ are the corresponding critical exponents, and $z = \nu || / \nu \perp$ is the dynamical critical exponent. For finite systems of size $L$, we expect that

$$\xi^{-1} ||, L = L^{-zL} \Phi \left( |\lambda - \lambda^L||L^{1/\nu \perp}, L \right),$$

where $z_L$ and $\nu \perp, L$ are the finite versions of $z$ and $\nu \perp$, and $\Phi(u)$ is a scaling function with $\Phi(u \gg 1) \sim u^0$. On general grounds one expects $\lim_{L \rightarrow \infty} \xi ||, L, \nu \perp, L = \nu \perp, z, \nu \perp$. From Eqs. (2) and (3) we obtain

$$\ln \frac{\left[ \xi ||, L \lambda^L \right]}{\xi \perp, L \lambda^L} = \frac{\left[ \xi \perp, L \lambda^L \right]}{\ln (L/L')} = \frac{zL}{\ln (L'/L)},$$

which through the comparison of three different system sizes $L' < L < L''$ furnishes simultaneously $\lambda^L_L$ and $z_L$. Of course, $\xi ||, L$ and the gap $E_L^{(1)}$ of $H$ are related by $\xi^{-1} ||, L = \text{Re} \{E_L^{(1)}\}$. 

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IV. DYNAMICAL CRITICAL EXponents

We calculated the gaps of $H$ through the power method, which requires only matrix-by-vector multiplications that can be carried out efficiently, and does not require a diagonalization in the usual, $QR$ sense, a step that may lessen the quality of the data. The particular implementation of the power method we use takes full advantage of the presence of absorbing states in the process, and is also suitable for the investigation of time dependent properties of finite-state Markov chains [43].

Our results for $\lambda^*$ and $z$ for the CP(2) both without diffusion and with symmetric diffusion $X\emptyset = 0X$ at rate $\mu = 0.05$ are summarized in Table I. The choice $\mu = 0.05$ is arbitrary, except that it represents only a small perturbation to the main coupling $\lambda$ near the critical points, but not so small as to render too much unbalanced, ‘stiff’ matrices that may become difficult to diagonalize. We were able to diagonalize systems with up to $L = 25$ sites in a reasonable amount of time and memory space, comparable to recent density-matrix renormalization group studies of similar systems [57]. The extrapolated values for the $\mu = 0$ case were obtained through a Bulirsch-Stoer extrapolation [44], while those for the $\mu \neq 0$ case were obtained through a least-squares fit to the curve $y_L = a_0 + a_1 L^{-1} + a_2 L^{-2}$, and can be seen in Fig. I. The uncertainties associated with our extrapolated numbers are mainly due to finite-size effects and corrections to scaling, as well as to the extrapolation procedures itself. We obtained in the diffusionless case $\lambda^* = 0.88 \pm 0.01$ ($\omega_{\text{BST}} = 0.924$) and $z = 1.75 \pm 0.01$ ($\omega_{\text{BST}} = 1.802$), and for the diffusive case $\lambda^* = 0.25 \pm 0.05$ and $z = 1.3 \pm 0.1$. (A linear fit $y_L = a_0 + a_1 L^{-1}$ to the last four points of the data with diffusion furnishes $\lambda^* = 0.38 \pm 0.04$ with a correlation coefficient $\gamma = 0.989$, and $z = 1.36 \pm 0.06$ with a correlation coefficient $\gamma = 0.989$.) We thus see that the pure CP(2) without diffusion has a critical behavior governed by the dynamical critical exponent of the parity-conserving universality class, for which the best known value to date is given by $z_{\text{PC}} = 1.750 \pm 0.005$ [24]. The value of the critical point agrees well with the value $\lambda^* = 0.8935 \pm 0.0004$ found in [24]. Upon the introduction of diffusion, however, the number of steady states of CP(2) reduces from two to one, and according to our data its critical behavior crosses over to that of another universality class. In a first moment the critical exponent seem to be converging to $z \simeq 2$. Data for larger lattice sizes, however, point toward a lower value of $z$, definitely different from $z = 2$, at least as far as our finite-size data go. We also observed a non-monotonic behavior of the data, that unfortunately seems to be more common in this type of calculation than it would be desirable. This non-monotonic behavior is probably related with the existence of a whole critical (in the present case, diffusion-like) phase to the right of the critical point, and has already been observed in other studies of systems with extended critical phases [8, 37]. Our extrapolation gives $z = 1.3 \pm 0.1$, closer to the exponent of the DP universality class, namely $z_{\text{DP}} \approx 1.5807$ [8, 45], than to other known values. A DP critical behavior is what one could have expected on the basis of the DP conjecture, since the process with diffusion has a single absorbing state. Intuition, however, has proved not to be reliable in guessing the critical behavior of nonequilibrium systems with local conservation laws or many absorbing states, as one infers from the recent history in the field, and the presence of an additional symmetry might well have driven the critical exponents of CP(2) with diffusion to those of the PC universality class. Diffusion is then seen to be a highly relevant perturbation for systems with more than one absorbing state.

| System sizes | $\lambda^*$ | $z_L$ | Extrapolated | $\mu = 0$ | $\mu = 0.05$ |
|--------------|-------------|-------|--------------|-----------|-------------|
| $L, L', L''$ | $\lambda^*$ | $z_L$ | $\lambda^*$ | $z_L$ | $\lambda^*$ | $z_L$ |
| 7, 9, 11     | 0.584505    | 1.916330 | 0.705564     | 1.975081 |
| 9, 11, 13    | 0.607946    | 1.898353 | 0.779294     | 2.013974 |
| 11, 13, 15   | 0.625558    | 1.884819 | 0.791092     | 2.046900 |
| 13, 15, 17   | 0.641431    | 1.872620 | 0.798318     | 2.056315 |
| 15, 17, 19   | 0.656023    | 1.861476 | 0.783969     | 2.034662 |
| 17, 19, 21   | 0.669350    | 1.851398 | 0.753899     | 1.985477 |
| 19, 21, 23   | 0.681446    | 1.842353 | 0.715444     | 1.921154 |
| 21, 23, 25   | 0.692390    | 1.834263 | 0.675905     | 1.856800 |
| Extrapolated | 0.88(1)     | 1.75(1) | 0.25(5)      | 1.3(1)    |

V. SUMMARY AND CONCLUSIONS

In summary, we have conducted a finite-size scaling analysis of the dynamical critical exponent of an even-parity-conserving contact process through exact numerical diagonalizations of its time evolution operator. We showed that the critical behavior of the ‘pure’ model with two absorbing configurations belongs to the parity-conserving universality class. In the presence of symmetric diffusion, however, the number of absorbing configurations in the model reduces from two to one, and its critical behavior crosses over to that of another universality class that seem to be that of the directed percolation process. This behavior is in accordance with what one expects on the basis of the directed percolation conjecture, at least for small diffusion rates.

It would be interesting to perform time-dependent Monte Carlo simulations of the CP(2) model both with and without diffusion in order to determine its critical exponents more precisely, as well as to investigate the critical behavior of other members of the CP($m$) class of processes. In particular, it would be very interesting to investigate the CP(3), since it was argued on the basis of
Monte Carlo simulations and Padé approximants analyses that this process does not suffer an ordinary second order phase transition [28]. Moreover, the CP(3) process has the many-sector decomposability property, and it would be interesting to see how diffusion changes the scenario for the phase transitions, if any, in this model.

It seems that the question as to the roles of symmetries and many-sector decomposability on the critical behavior of nonequilibrium interacting particle systems is far from being answered, and more numerical and analytical work has to be done before a consistent scenario emerges.

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