Borehole radiation and reception responses for azimuthal shear-wave reflection imaging with an off-centred dipole acoustic tool

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Abstract

Radiation and reception responses of a dipole acoustic logging tool placed eccentrically in borehole fluid are an interesting and important topic in acoustic reflection imaging. Herein, we present a thorough research study on these responses. We treat the wave incidence from the reflector as the radiation from a virtual source and use the cylindrical-wave expansion method to solve both the wave radiation and reception problems for the off-centred tool, which, by using the steepest-descent method, yields asymptotic solutions for modelling the radiation and reception wavefield characteristics. The modelling results from the analytical solution and the 3D finite-difference method were in good agreement. Specifically, we analysed the radiation directivity of an eccentric dipole source in a fluid-filled borehole. The results revealed that the radiation pattern was asymmetric with respect to the borehole, and the asymmetry was determined by the eccentric distance, source frequency and formation properties. In particular, for the typical 3 kHz dipole logging frequency, the radiation was stronger in the off-centred direction than in the opposite direction. The asymmetry of the eccentric radiation resulted in a significant amplitude difference relative to its centred counterpart, which provided a potential method for addressing the 180° azimuth ambiguity of the dipole source. We used a theoretical waveform modelling example to demonstrate this advantage. Therefore, the results of this study provide a theoretical foundation for the development and application of dipole shear-wave imaging technology.

Keywords: borehole dipole reflection imaging, far-field radiation patterns, azimuthal reception response, tool eccentricity, azimuth ambiguity

1. Introduction

In recent years, dipole shear-wave imaging has made significant progress detecting geological structures outside the well, such as fractures, faults, and dissolution caves (Tang & Patterson 2009; Bradley et al. 2011; Tang et al. 2014; Patterson et al. 2016; Lee et al. 2019; Wang et al. 2020). It has also been used to delineate an existing open or cased borehole from a nearby well (Tang et al. 2016; Gu et al. 2021). The use of a dipole source for acoustic reflection imaging was first proposed by Tang (2004). Dipole shear imaging technology has deep penetration, sensitive orientation and fracture identification compared to a traditional monopole source (Hornby 1989; Meredith 1990). Recent developments have focused on analysing the azimuthal
response of the dipole tool to resolve the 180° ambiguity (Gong et al. 2015, 2018; Bennett 2019; Xu et al. 2019; Ben et al. 2020; Li et al. 2020; Hirabayashi 2021; Li et al. 2021a; Li et al. 2021b), and on the effects of casing and cement to facilitate imaging through the casing (Tang et al. 2020; Gu et al. 2021).

An important and challenging problem is the effect of tool off-centring on imaging applications, as is commonly encountered in downhole logging, especially in high-angle and/or horizontal wells. For acoustic modelling with an off-centred dipole tool, existing studies have investigated the wavefield inside a borehole for acoustic logging applications (Leslie & Randall 1990; Schmitt 1993; Byun & Toksöz 2006; Pardo et al. 2013; Kayama et al. 2021). However, we have not yet seen such analyses and results in the literature for the wavefield outside the borehole relevant to reflection imaging applications.

Therefore, the goal of this study was to extend previous analyses to determine the response characteristics of radiation and reception for the borehole-eccentred tool configuration. In detail, we first derived the far-field displacement asymptotic solution and radiation directivity of an eccentric dipole source. Then, combining the imaginary-source, cylindrical-wave expansion and steepest-descent methods, an efficient algorithm to model the reception responses of the azimuthally placed receivers was derived for the eccentric dipole tool case. The accuracy of our approach was validated using a 3D finite-difference method. Using these solutions, the effects of the off-centred distance, frequency and formation properties on the far-field radiation pattern of an eccentric dipole source in borehole fluid were analysed. Finally, we demonstrated an essential application of eccentric dipole reflection measurement in eliminating azimuth ambiguity through a theoretical synthetic example.

2. Theoretical analysis

2.1. Radiation from an off-centred dipole source inside the borehole

A borehole model with an off-centred tool for analysing azimuthal shear-wave reflection imaging is shown in figure 1. To obtain the analytical wavefield solution for this case, two coordinate systems \((X, Y, Z)\) and \((x, y, z)\) were used, with the origin of the first (second) system located at the tool (borehole) centre. The tool was immersed in the borehole fluid and was parallel to the borehole. In logging-while-drilling (LWD) measurements (Wang et al. 2015a, 2015b, 2017; Ji et al. 2021a, 2021b; Fu & Gou 2022), the effect of the drill collar should be considered because it occupies most of the borehole. However, in wireline
logging measurements, the tool size was significantly smaller than in the LWD case. In practice, the logging tool has notches in the transmitting and receiving parts, where the steel shell is very thin and is connected by soft materials such as rubber. This results in a significantly reduced effect of the tool body on field wireline measurements. Therefore, this study mainly considered the effects of the sources and receivers in the tool-eccentricity scenario, and the effect of the tool body was not considered.

The model in figure 1 depicts three wave processes: eccentric-source radiation, wave reflection and borehole reception of the reflected wave. We first analyse the source radiation along the X- and Y-directions, where the X (Y) source consists of two eccentric point sources, X1 (Y1) and X2 (Y2), of the same amplitude but opposite polarity, as shown in figure 2a. For the borehole cylindrical coordinates \((r, \varphi, z)\), the off-centring of the source system is derived in two steps. First, the centred system (figure 2a) is offset along the direction \(\varphi_o\) by a distance \(D_o\) (figure 2b). Then, it was rotated by the angle \(\varphi\), (figure 2c), which allowed us to simulate an arbitrary eccentric case of a dipole source. For the centred cross-dipole source (figure 2a), the coordinates of the four point sources were \(X1(r_0, 0, 0)\), \(Y1(r_0, \pi/2, 0)\), \(X2(r_0, \pi, 0)\) and \(Y2(r_0, 3\pi/2, 0)\), respectively. The symbol \(r_0\) is the dipole-source radius. After offsetting and rotating, the four point sources in figure 2c were located at \(X1(r_{x1}, \varphi_{x1}, 0)\), \(Y1(r_{y1}, \varphi_{y1}, 0)\), \(X2(r_{x2}, \varphi_{x2}, 0)\) and \(Y2(r_{y2}, \varphi_{y2}, 0)\), respectively. The method for determining the coordinates of the eccentric-source locations in figure 2c is given in Appendix A.

For the eccentric point source \(X1(r_{x1}, \varphi_{x1}, 0)\) in figure 2c, the displacement potential in the borehole fluid can be written as (Tang & Cheng 2004)

\[
\phi^d_j = -\frac{e^{ik FR}}{4\pi R},
\]

where \(R = \sqrt{r^2 + r_{x1}^2 - 2rr_{x1}\cos(\varphi - \varphi_{x1}) + z^2}\) denotes the distance between the field point and the source; \((r, \varphi, z)\) is the field point location in the borehole cylindrical coordinate system; \(k_j = \omega/\alpha_j\) stands for the borehole-fluid wavenumber, \(\alpha_j\) is the borehole-fluid velocity and \(\omega\) denotes the circular frequency. The spherical-wave propagation factor \(e^{ikFR}/R\) can be expanded in the form of the multipole superposition of cylindrical waves (Tang & Cheng 2004)

\[
\phi^d_j = -\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \epsilon_m \cos[m(\varphi - \varphi_{x1})] \left\{ \begin{array}{ll} I_m(fr_{x1})K_m(fr) & (r > r_{x1}) \\ I_m(fr)K_m(fr_{x1}) & (r < r_{x1}) \end{array} \right\} e^{ikR} dk,
\]

where \(k\) stands for the axial wavenumber of both the fluid and formation; \(f = \sqrt{k^2 - k_j^2}\) stands for the radial borehole-fluid wavenumber; \(I_m\) \((K_m)\) represents the modified Bessel function of the first (second) kind with order \(m\) and \(\epsilon_m\) represents Neumann’s factor, which is defined as 1 for \(m = 0\) and 2 for \(m \neq 0\). The waves interacting with the borehole induce the reflected wavefield in the fluid inside and radiated wavefield in the formation outside the borehole. Using the Helmholtz theorem, the particle displacements \(u_j\) and \(u_s\) in the borehole fluid and elastic formation, respectively, are defined as

\[
\begin{align*}
\mathbf{u}_j &= \nabla \left( \phi_j^d + \phi^r_j \right) \\
\mathbf{u}_s &= \nabla \phi + \nabla \times (\chi \mathbf{z}) + \nabla \times \nabla \times (\Gamma_{z}\mathbf{z}),
\end{align*}
\]

Figure 2. Construction of an eccentric cross-dipole-source system from the centred position (a) to the off-centre position (c) by offsetting (b) and rotating (c) the system.
where \( \mathbf{z} \) is the unit vector along the \( z \)-axis; \( \phi_j^d \) and \( \phi_j^r \) represent the direct and reflected potentials in the borehole fluid and \( \phi \), \( \chi \) and \( \Gamma \) represent the radiated \( P \)-, \( SH \)- and \( SV \)-wave potentials in the formation, respectively. To match the functional forms in equation (2), the general solutions of these displacement potentials in equation (3) are expressed as

\[
\begin{align*}
\phi_j^r &= \sum_{m=0}^{+\infty} A_m I_m(r) I_m(f r X_1) \cos \left[ m (\varphi - \varphi X_1) \right] \quad (0 < r < r_b) \\
\phi &= \sum_{m=0}^{+\infty} B_m K_m(r) I_m(f r X_1) \cos \left[ m (\varphi - \varphi X_1) \right] \\
\chi &= \sum_{m=0}^{+\infty} D_m K_m(r) I_m(f r X_1) \sin \left[ m (\varphi - \varphi X_1) \right] \\
\Gamma &= \sum_{m=0}^{+\infty} F_m K_m(r) I_m(f r X_1) \cos \left[ m (\varphi - \varphi X_1) \right] 
\end{align*}
\]

where \( r_b \) is the radius of the borehole; \( p = \sqrt{k^2 - (\omega/\alpha)^2} \) is the radial wavenumber of the formation \( P \)-wave where \( \alpha \) is the velocity of the formation \( P \)-wave; \( s = \sqrt{k^2 - (\omega/\beta)^2} \) is the radial wavenumber of the formation \( S \)-wave where \( \beta \) is the velocity of the formation \( S \)-wave. For different orders \( m \), the wave amplitude coefficients \( A_m \sim F_m \) are obtained by applying the boundary conditions at the wall of the well \( (r = r_b) \), where the radial displacement and radial stress are continuous, and the azimuthal and axial stresses vanish, as follows

\[
\begin{align*}
\left( u_r(r_b^+) - u_r(r_b^-) \right) &= 0 \\
\left( \sigma_{rr}(r_b^+) - \sigma_{rr}(r_b^-) \right) &= 0 \\
\left( \sigma_{rp}(r_b^+) \right) &= 0 \\
\left( \sigma_{rz}(r_b^+) \right) &= 0
\end{align*}
\]

The displacements in the borehole fluid and formation were determined using equations (2)–(4). Substituting the formation displacements into the strain-displacement relation to calculate the strains, the strains, in turn, are used to calculate the stresses using Hooke’s law. To obtain the unknown amplitude coefficients in equation (4), substituting these displacements and stresses into equation (5) yields the following matrix equation

\[
\mathbf{M} \times \left[ A_m, B_m, D_m, F_m \right]^T = \mathbf{b},
\]

where \( \mathbf{M} \) is a 4x4 matrix and \( \mathbf{b} \) is a 4x1 vector, whose expressions are given in Appendix B. Solving the coefficients from equation (6), the radiated waves can be obtained with discrete wavenumber integration (Tang & Cheng 2004). For the reflection imaging survey, the conditions of \( |pr| \gg 1 \) and \( |sr| \gg 1 \) are satisfied when the wavelength is less than the radiation distance, which allows us to determine the far-field solutions of the radiated waves by replacing this wavenumber integration with the steepest-descent solution (Aki & Richards 1980), as follows

\[
\begin{align*}
\phi &= -\frac{e^{i\omega R/\pi \cdot \pi/4}}{4\pi R} \sum_{m=0}^{+\infty} B_m (\omega, k_{p0}) I_m(f p0 r X_1) \cos \left[ m (\varphi - \varphi X_1) \right] \\
\chi &= -\frac{e^{i\omega R/\beta \cdot \pi/4}}{4\pi R} \sum_{m=0}^{+\infty} D_m (\omega, k_{s0}) I_m(f s0 r X_1) \sin \left[ m (\varphi - \varphi X_1) \right] \\
\Gamma &= -\frac{e^{i\omega R/\beta \cdot \pi/4}}{4\pi R} \sum_{m=0}^{+\infty} F_m (\omega, k_{f0}) I_m(f f0 r X_1) \cos \left[ m (\varphi - \varphi X_1) \right]
\end{align*}
\]

where \( k_{p0} = (\omega/\alpha) \cdot \cos \theta_t \) and \( k_{s0} = (\omega/\beta) \cdot \cos \theta_t \) denote, respectively, the steepest-descent solutions of the formation \( P \)- and \( S \)-wave wavenumbers; \( \theta_t \) indicates the angle between the direction of the radiated wave and the \( z \)-axis; \( f_{p0} = \sqrt{k_{p0}^2 - (\omega/\alpha)^2} \) and \( f_{s0} = \sqrt{k_{s0}^2 - (\omega/\beta)^2} \) correspond, respectively, to the fluid radial wavenumbers calculated using \( k_{p0} \)
and \( k_0 \) \( R = \sqrt{r^2 + z^2} \), note that the terms related to \( r_{X1} \) are omitted under the condition of \( r_{X1} \ll r \). Substituting equation (7) into equation (3) and ignoring \( O(1/R^2) \) results in the radiated far-field displacements of the P-, SH- and SV-waves, as follows

\[
\begin{align*}
 u_P &= \left\{ (-i\rho\omega) \sum_{m=0}^{+\infty} B_m(\omega, k_{p0}) I_m(f_{p0}r_{X1}) \cos \left[ m(\varphi - \varphi_{X1}) \right] \right\} e^{i\omega R/\alpha - i \pi /4} \frac{e^{i\omega R/\beta - i \pi /4}}{4\pi \rho \alpha^2 R}, \\
 u_{SH} &= \left\{ (i\rho \beta \cos \theta) \sum_{m=0}^{+\infty} D_m(\omega, k_{s0}) I_m(f_{s0}r_{X1}) \sin \left[ m(\varphi - \varphi_{X1}) \right] \right\} e^{i\omega R/\beta - i \pi /4} \frac{e^{i\omega R/\beta - i \pi /4}}{4\pi \rho \beta^2 R}, \\
 u_{SV} &= \left\{ (\rho \omega^2 \sin \theta) \sum_{m=0}^{+\infty} F_m(\omega, k_{s0}) I_m(f_{s0}r_{X1}) \cos \left[ m(\varphi - \varphi_{X1}) \right] \right\} e^{i\omega R/\alpha - i \pi /4} \frac{e^{i\omega R/\beta - i \pi /4}}{4\pi \rho \beta^2 R},
\end{align*}
\]

where \( \rho \) denotes the formation density. The bracketed parts of equation (8) give the far-field radiation directivities of the P-, SH- and SV-waves, respectively, as follows

\[
\begin{align*}
 R_P(\omega; \theta, \varphi) &= (-i\rho\omega) \sum_{m=0}^{+\infty} B_m(\omega, k_{p0}) I_m(f_{p0}r_{X1}) \cos \left[ m(\varphi - \varphi_{X1}) \right], \\
 R_{SH}(\omega; \theta, \varphi) &= (i\rho \beta \cos \theta) \sum_{m=0}^{+\infty} D_m(\omega, k_{s0}) I_m(f_{s0}r_{X1}) \sin \left[ m(\varphi - \varphi_{X1}) \right], \\
 R_{SV}(\omega; \theta, \varphi) &= (\rho \omega^2 \sin \theta) \sum_{m=0}^{+\infty} F_m(\omega, k_{s0}) I_m(f_{s0}r_{X1}) \cos \left[ m(\varphi - \varphi_{X1}) \right].
\end{align*}
\]

By repeating the derivation to obtain the solution for the other three eccentric point sources in figure 2c and combining the \((X1, X2)\) and \((Y1, Y2)\) pairs to form a cross-dipole source, the far-field wave solutions of the source system are determined. The P-, SH- and SV-wave potentials are

\[
\begin{align*}
 \phi_{XY} &= -\frac{e^{i\omega R/\alpha - i \pi /4}}{4\pi R} \sum_{m=0}^{+\infty} B_m(\omega, k_{p0}) \left[ I_m(f_{p0}r_{X1}, Y1) \cos m(\varphi - \varphi_{X1}, Y1) \right. \\
 &\left. \quad - I_m(f_{p0}r_{X2}, Y2) \cos m(\varphi - \varphi_{X2}, Y2) \right], \\
 \chi_{XY} &= -\frac{e^{i\omega R/\beta - i \pi /4}}{4\pi R} \sum_{m=0}^{+\infty} D_m(\omega, k_{s0}) \left[ I_m(f_{s0}r_{X1}, Y1) \sin m(\varphi - \varphi_{X1}, Y1) \right. \\
 &\left. \quad - I_m(f_{s0}r_{X2}, Y2) \sin m(\varphi - \varphi_{X2}, Y2) \right], \\
 \Gamma_{XY} &= -\frac{e^{i\omega R/\beta - i \pi /4}}{4\pi R} \sum_{m=0}^{+\infty} F_m(\omega, k_{s0}) \left[ I_m(f_{s0}r_{X1}, Y1) \cos m(\varphi - \varphi_{X1}, Y1) \right. \\
 &\left. \quad - I_m(f_{s0}r_{X2}, Y2) \cos m(\varphi - \varphi_{X2}, Y2) \right],
\end{align*}
\]

and their respective far-field radiation directivities are

\[
\begin{align*}
 R_{P}^{XY}(\omega; \theta, \varphi) &= (-i\rho\omega) \sum_{m=0}^{+\infty} B_m(\omega, k_{p0}) \left[ I_m(f_{p0}r_{X1}, Y1) \cos m(\varphi - \varphi_{X1}, Y1) \right. \\
 &\left. \quad - I_m(f_{p0}r_{X2}, Y2) \cos m(\varphi - \varphi_{X2}, Y2) \right], \\
 R_{SH}^{XY}(\omega; \theta, \varphi) &= (i\rho \beta \cos \theta) \sum_{m=0}^{+\infty} D_m(\omega, k_{s0}) \left[ I_m(f_{s0}r_{X1}, Y1) \sin m(\varphi - \varphi_{X1}, Y1) \right. \\
 &\left. \quad - I_m(f_{s0}r_{X2}, Y2) \sin m(\varphi - \varphi_{X2}, Y2) \right], \\
 R_{SV}^{XY}(\omega; \theta, \varphi) &= (\rho \omega^2 \sin \theta) \sum_{m=0}^{+\infty} F_m(\omega, k_{s0}) \left[ I_m(f_{s0}r_{X1}, Y1) \cos m(\varphi - \varphi_{X1}, Y1) \right. \\
 &\left. \quad - I_m(f_{s0}r_{X2}, Y2) \cos m(\varphi - \varphi_{X2}, Y2) \right],
\end{align*}
\]

where the superscript \( X \) \( Y \) denotes the excitation of the \( X \) \( Y \) dipole source of the system, and correspondingly, the radius and azimuth parameters of the point source pair, as denoted by subscripts \( X1 \) and \( X2 \) \( (Y1 \) and \( Y2) \), should be used. The radiation patterns of the radiated waves can be calculated by evaluating the amplitude of the expressions in equation (11) relative to \( \theta \) in the vertical plane.
2.2. Reception response of the borehole to the incident elastic waves

After reflection in the formation, the spherical wave for the (centred or off-centred) borehole source was returned to the borehole. The incident wave can be expressed as a coupling of multiple functions as follows

\[
I_{\text{wave}} = S \cdot RD \cdot RF \cdot e^{i\omega D/v}/D, \tag{12}
\]

where \(v\) denotes the velocity of the formation P- or S-wave; \(S\) stands for the source spectrum; \(RD\) stands for the borehole radiation function, which is obtained by removing the spherical-wave propagation factor \(e^{i\omega D/v}/R\) from equation (10); \(RF\) stands for the reflection coefficient at the reflector, which can be calculated using the Zoeppritz equation; \(RD\) and \(RF\) are determined by the type of reflected wave being considered, \(P\), \(SV\) or \(SH\), and \(D\) indicates the distance the wave travels in the formation. By treating the reflected wave from the reflector as a spherical wave generated by a ‘virtual source’ whose location coincides with the mirror image of the eccentric source relative to the reflector (as shown in figure 1), we can express the spherical-wave factor \(e^{i\omega D/v}/D\) of equation (12) in a similar way to equation (2) (Li et al. 2021b)

\[
e^{i\omega D/v}/D = \frac{1}{\pi} \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \varepsilon_m I_m(k_v r)K_m(k_v r') \cos \left[m(\varphi - \varphi_r')\right] e^{ik_v(z-z')} dk_v, \tag{13}
\]

where \(k_v = \sqrt{k^2 - (c/v)^2}\) stands for the radial wavenumber of the incident P-, SV- or SH-wave and \((r', \varphi_r', z_r')\) denotes the virtual-source location in the borehole cylindrical coordinates and \(D = \sqrt{r^2 + r'^2 - 2rr' \cos(\varphi - \varphi_r) + (z - z_r)^2}\). The incident waves on the borehole cause wave responses in the borehole fluid and formation, and the potentials of these waves given in equation (4) become

\[
\phi_j = \sum_{m=0}^{+\infty} \left[A_m \cos(m\varphi) + A'_m \sin(m\varphi)\right] I_m(r)K_m(k_v r') \quad (0 < r < r_b),
\]

\[
\phi = \sum_{m=0}^{+\infty} \left[B_m \cos(m\varphi) + B'_m \sin(m\varphi)\right] K_m(pr)K_m(k_v r'),
\]

\[
\chi = \sum_{m=0}^{+\infty} \left[D_m \sin(m\varphi) - D'_m \cos(m\varphi)\right] K_m(sr)K_m(k_v r'), \quad (r > r_b),
\]

\[
\Gamma = \sum_{m=0}^{+\infty} \left[F_m \cos(m\varphi) + F'_m \sin(m\varphi)\right] K_m(sr)K_m(k_v r'),
\]

Correspondingly, equation (5), which describes the boundary conditions, needs to be changed to determine the amplitude coefficients in equation (14)

\[
u^i(r_b^+) + u^s(r_b^+) = 0
\]

\[
\sigma^i_{rr}(r_b^+) + \sigma^s_{rr}(r_b^+) = \sigma_{rr}(r_b^+)
\]

\[
\sigma^i_{rr}(r_b^+) + \sigma_{rr}(r_b^+) = 0
\]

\[
\sigma^i_{rz}(r_b^+) + \sigma_{rz}(r_b^+) = 0
\]

where superscripts \(i\) and \(s\) refer to the incident and scattered waves in the elastic formation, respectively, and \(f\) refers to the transmitted wave in the borehole fluid. Substituting equation (13) into equation (12) yields the displacement potential...
functions of the incident waves

\[
\begin{align*}
\phi^i &= S(\omega) \cdot RD_{xy}^P(\omega) \cdot RF^P(\omega) \sum_{m=0}^{+\infty} \epsilon^m I_m(pr)K_m(\psi_0) \cos \left[ m(\varphi - \varphi_0) \right], \\
\chi^i &= S(\omega) \cdot RD_{xy}^SH(\omega) \cdot RF_{SH}^i(\omega) \sum_{m=0}^{+\infty} \epsilon^m I_m(sr)K_m(\psi_0) \cos \left[ m(\varphi - \varphi_0') \right], \\
\Gamma^i &= S(\omega) \cdot RD_{xy}^SV(\omega) \cdot RF_{SV}^i(\omega) \sum_{m=0}^{+\infty} \epsilon^m I_m(sr)K_m(\psi_0) \cos \left[ m(\varphi - \varphi_0') \right],
\end{align*}
\]

where the subscripts \( P, SH \) and \( SV \) represent the type of incident wave.

The displacements and stresses of wave motions in and out of the borehole are calculated from equations (3), (14) and (16). After substitution into the boundary conditions of equation (15), this results in two matrix equations

\[
M \times \begin{bmatrix} A_m B_{mr} D_{mr} F_m \end{bmatrix}^T = c \quad \text{and} \quad M \times \begin{bmatrix} A'_m B'_{mr} D'_{mr} F'_m \end{bmatrix}^T = c',
\]

where the matrix \( M \) is the same as equation (6) (see Appendix B), and \( c \) and \( c' \) are 4×1 vectors representing the contribution of the incident wave from the virtual source (see Appendix C for a detailed description). After determining wavefields in and out of the borehole with equation (17), incident-wave-induced fluid pressure and radial and azimuthal displacements inside the borehole could be obtained based on the formulae \( p = \rho_j \omega^2 \phi_j \), \( u'_r = \frac{\partial \phi_j}{\partial r} \), and \( u'_\varphi = (1/r) \frac{\partial \phi_j}{\partial \varphi} \)

\[
\begin{align*}
p &= \frac{\rho_j \omega^2}{\pi} \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \left[ A_m \cos(m\varphi) + A'_m \sin(m\varphi) \right] I_m(fr)K_m(k_0r_0)e^{ik(z-z_0')} \, dk, \\
u'_r &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \left[ A_m \cos(m\varphi) + A'_m \sin(m\varphi) \right] \left[ f_{m+1}(fr) + \frac{m}{r} f_m(fr) \right] K_m(k_0r_0)e^{ik(z-z_0')} \, dk, \\
u'_\varphi &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \left[ -A_m \sin(m\varphi) + A'_m \cos(m\varphi) \right] \frac{m}{r} f_m(fr)K_m(k_0r_0)e^{ik(z-z_0')} \, dk,
\end{align*}
\]

where \( \rho_j \) denotes fluid density. The radial distance of the virtual source is usually greater than the wavelength, \( |k_0r_0| \gg 1 \), and the wavenumber integration can be replaced with the steep-descent method

\[
\begin{align*}
p &= \rho_j \omega^2 \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \left[ A_m \cos(m\varphi) + A'_m \sin(m\varphi) \right] f_0(r) + \frac{m}{r} f_m(r), \\
u'_r &= \frac{D}{\pi} \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \left[ A_m \cos(m\varphi) + A'_m \sin(m\varphi) \right] \left[ f_{m+1}(fr) + \frac{m}{r} f_m(fr) \right], \\
u'_\varphi &= \frac{-D}{\pi} \int_{-\infty}^{+\infty} \sum_{m=0}^{+\infty} \left[ -A_m \sin(m\varphi) + A'_m \cos(m\varphi) \right] \frac{m}{r} f_m(r),
\end{align*}
\]

Where \( f_0 = \sqrt{k_0^2 - (\omega/\alpha)^2} \), \( k_0 = (\omega/\nu) \cdot \cos \theta_i \) represents the steep-descent solution of the incident-wave wavenumber, \( \theta_i \) indicates the angle between the direction of the incident wave and the negative z-axis; \( D = \sqrt{k_0^2 + (z-z_0')^2} \). Note that the terms related to \( r \) are omitted under the condition of \( r \ll r_0 \). Equation (19) calculates the fluid response in the borehole coordinate system. For the wavefield measured by the tool, the results from equation (19) need to be converted to the tool coordinate system. Because pressure is a scalar, the results are the same in the two coordinate system cases, so only displacements need to be converted. The off-centring of the receiver is the same as that of the source (see figure 2). Figure 3 illustrates the projection method to obtain the fluid radial displacement in the tool coordinate system. Thus, the fluid pressure and radial displacement measured by the tool are calculated using

\[
\begin{align*}
\frac{\hat{p}}{\hat{u}'_r} &= \left\{ \begin{array}{ll}
p, & r = r_{Re}, \\
u'_r \cos(\varphi_{R_0} + \varphi - \varphi_{Re}) + u'_\varphi \sin(\varphi_{R_0} + \varphi - \varphi_{Re}), & \varphi = \varphi_{Re}
\end{array} \right. 
\end{align*}
\]

where \( R_0(r_{RD}, \varphi_{RD}, z_{RD}) \) and \( R_e(r_{Re}, \varphi_{Re}, z_{RD}) \) are the receiver locations in the centred and off-centred tool cases, respectively.
Figure 3. The projection method to obtain the radial displacement in the tool coordinate system.

Table 1. Borehole and formation parameters for simulation

|                     | P-velocities (m s$^{-1}$) | S-velocities (m s$^{-1}$) | Densities (kg m$^{-3}$) | Radii (m) |
|---------------------|---------------------------|---------------------------|-------------------------|-----------|
| Borehole fluid      | 1500                      | –                         | 1000                    | 0.1       |
| Fast formation      | 4000                      | 2300                      | 2500                    | –         |
| Slow formation      | 2074                      | 869                       | 2250                    | –         |
| Formation 2         | 4500                      | 3000                      | 2600                    | Infinity  |

3. Result validation

Having obtained the analytical solution to model the received reflection data from a reflector in formation, we now verify the analytical solution result of equation (20) using the 3D finite-difference numerical method (Wei & Tang 2012). The borehole model in figure 1 and the parameters in Table 1 were used in the simulation, where the interface between formations 1 (fast formation used) and 2 was considered the reflector. The dip angle and azimuth of the reflector are 45° and 0°, respectively. The radial distance ($r'_0$, as shown in figure 1) of the virtual source was assumed to be 10 m. Receivers were placed 0.035 m from the tool centre and circumferentially separated by 45° angles. The tool was offset along the x-axis by 0.05 m. The vertical distance between the planes where the source and receivers were located is 1 m. We selected the Ricker wavelet with a frequency of 3 kHz to actuate the X-dipole source, assuming the radius ($r_0$, as shown in figure 2a) of the dipole source is 0.035 m. Set the grid size and the iteration time step to 0.005 m and 0.6 μs for the finite-difference simulation, respectively. Figure 4 shows the borehole-fluid radial displacement (figure 4a) and pressure (figure 4b) calculated from the asymptotic analytical solution (solid curves) and numerical method (dashed curves), respectively. For the dipole-source orientation and reflector strike, P- and SV-waves will be simultaneously excited, which results in a finite-difference simulation containing the P- and SV-waves reflections, and the P-SV and SV-P converted wave reflections. In contrast, equation (20) is only used to model an individual reflected wave. For clarity, figure 4 only compares the SV-wave reflection results. Despite the interferences, the SV-wave results were in good agreement between the two methods for both radial displacement and pressure, demonstrating the accuracy and validity of the analytical solution. Note that the 3D finite-difference method requires several tens of hours for the simulation, whereas the analytical solution of equation (20) requires only a few seconds to generate accurate results. Thus, our approach provides an efficient algorithm for simulating borehole eccentric dipole reflection surveys.

4. Far-field radiation characteristics of an eccentric dipole source

These theoretical results allow us to calculate the far-field radiation patterns of an off-centred dipole acoustic tool and investigate its dependence on the eccentric offset and source frequency as well as on the formation properties. For simplicity, we present modelling examples for the $\varphi_r = \varphi_x$ case. Without loss of generality, the cross-dipole source shown in figure 2
Figure 4. Comparisons of the borehole-fluid (a), radial displacement and (b) pressure responses calculated using the 3D finite-difference numerical method (dashed curves) and asymptotic analytical solution (solid curves) for SV-wave incidence. The dipole acoustic tool is offset along the x-axis by 0.05 m.

Figure 5. P (a), SV (b) and SH (c) radiation patterns of a borehole cross-dipole source in a fast formation for different eccentricity offsets at 0.5 kHz frequency.

was eccentrically placed along the x-axis (i.e. $\varphi_r = \varphi_o = 0^\circ$). The X-dipole source was excited to model the P- and SV-wave radiations in the $\varphi = 0^\circ$ (or 180°) plane and the Y-dipole source was excited to model the SH-wave radiation in the same plane.

4.1. Dependence on eccentric offset

Figure 5 compares the radiation patterns of an eccentric cross-dipole source for four eccentricity offset values of 0, 0.01, 0.03 and 0.05 m at 0.5 kHz frequency in the fast formation scenario. The types of radiated waves were P- (figures 5a and 6a), SV- (figures 5b and 6b) and SH- (figures 5c and 6c) waves. The circumferential label represents the radiation angle $\theta_t$, and the radial label marks the amplitude of the radiation directivities. Figure 5 shows that, for the low-frequency value of 0.5 kHz, the patterns are hardly affected by tool eccentricity and closely resemble those from a centred dipole source, indicating that the influence of eccentricity on the radiated wavefield can be ignored at low frequencies. For a frequency value of 3 kHz (figure 6), the results show that the directivity patterns of an eccentric dipole source are asymmetric relative to the borehole axis, as compared to their centred counterparts. With increasing eccentric offsets from 0 to 0.05 m, the P- and SH-wave patterns enlarge (shrink) toward (away from) the eccentric direction. In comparison, the eccentricity had only a small effect on the SV-wave patterns. It is worth noting that compared to P- and SV-waves, the SH-wave had better radiation performance and coverage, which means that the SH-wave dominated the radiated wavefield of the eccentric dipole source and was the optimal type of wave used for dipole imaging in the fast formation of the eccentric tool condition.

The radiation analysis for the same model in the slow formation case (see Table 1) is shown in figures 7 and 8. For the low-frequency value of 0.5 kHz (figure 7), the P- and SH-wave radiation patterns were hardly affected by eccentricity, whereas the SV-wave radiation pattern changed, and there were four evident mutation regions when the dipole source was eccentric.
With increasing eccentric offsets, the SV-wave directivity factor increased sharply near 24° and 156°, whereas it decreased sharply near 204° and 336°. For the frequency value of 3 kHz, the asymmetry of the patterns is enhanced relative to the fast formation case in figure 6, especially for the P- and SV-wave, as depicted in figure 8. Figure 8 shows that the amplitude of the radiated P-wave is stronger than that of the radiated SV- and SH-waves, indicating that the P-wave can also be used for eccentric dipole reflection imaging in slow formation. More importantly, for the typical 3 kHz dipole logging frequency, the radiation performances of P-, SV- and SH-waves are better in the off-centring direction than in the opposite direction in the fast and slow formation scenarios. An important implication of eccentric-source modelling is that the asymmetry of the source radiation can help resolve the 180° azimuth uncertainty of dipole shear reflection imaging (Li et al. 2021b).
4.2. Dependence on frequency

Figures 9 and 10 compare the radiation patterns of an eccentric cross-dipole source for the five frequencies of 2, 3, 4, 5 and 6 kHz in the fast and slow formation scenarios, respectively. The eccentric offset of the dipole-source system was 0.05 m. In general, radiation patterns exhibit complex frequency-dependent characteristics. In the fast formation case (figure 9), as the frequency increased from 2 to 6 kHz, the P-, SV- and SH-wave radiation factors increased gradually to a maximum and then decreased. When the frequency was 2 or 3 kHz, radiation performance was better in the off-centring direction than the opposite direction, whereas the trend was reversed when the frequency increased to 5 or 6 kHz. For the slow formation case shown in figure 10, with increasing frequency, the P-wave radiation factor increased gradually to a maximum and then decreased, whereas the SV- and SH-wave radiation factors gradually decreased. Comparing the radiation patterns at different frequencies shows that the optimal excitation frequency for an eccentric cross-dipole source is \( \sim 4 \) kHz in the fast and 2 kHz in the slow formation case. This is almost consistent with the results from their centred counterpart, suggesting the influence of tool eccentricity on the optimal excitation frequency is small. A detailed comparison of the radiation patterns with eccentric offset and frequency showed that the formation elasticity substantially influences the radiation pattern.

5. Application to eliminate the azimuth ambiguity

As a significant application of the theoretical solution of equation (20), it can be used to calculate the reflection data generated and recorded by an off-centred dipole tool for any reflector location in the formation. Figure 11 (left panel) shows a 100 m model containing two reflectors with opposite dipping directions at 0° (relative to the borehole x-axis). The cross-dipole acoustic tool was offset along the x-axis by 0.05 m. We calculated the SH-wave reflection data of the receiver array by actuating the Y-dipole source with a 3 kHz Ricker wavelet. The dipole receivers were placed 0.035 m from the tool centre. The dipole source was at a distance of 3 m below the array receivers, and the data-sample depth and time were 0.1524 m and 36 \( \mu \)s, respectively. The formation properties in Table 1 were used, and the slow formation was used as formation 1 in figure 1. The pressure difference between the two receivers on opposite sides of the tool was calculated to simulate dipole acquisition.
Figure 11. A model of two reflectors (lines in the left panel) for SH-wave reflection modelling. Note the apparent amplitude difference between the reflection signals measured using centred (middle panel) and off-centred (right panel) dipole tools.

Figure 11 compares the simulated first-receiver data for the centred (middle panel) and off-centred (right panel) scenarios. The data contain the flexural wave along the borehole and the reflected waves from the reflectors. The modelled data are displayed as a variable density diagram, and the intensity of the colour represents the wave amplitude strength. As can be seen from the middle and right panels, the amplitude of the borehole flexural waves is larger than that of the reflected waves from the reflectors. For convenience, the two cross-well reflectors in the left panel were divided into sections I, II, III and IV. The corresponding wave data are labelled ➀, ➁, ➂ and ➃. For the centred tool (middle panel), the reflection data from both sides of the borehole are similar owing to the 180° azimuth ambiguity of the dipole tool. In contrast, for the eccentric tool, the reflection data (❼ and ⬋) from reflectors (II and IV) on the eccentric side show higher amplitudes relative to their centred tool counterparts, whereas the reverse is valid for the reflection data (➀ and ➁) from reflectors (I and IV) on the opposite side of the off-centre direction. The wave phenomena are consistent with the radiation analysis results shown in figure 8c, which have the potential to resolve the 180° uncertainty.

6. Conclusion

This paper presents a theoretical analysis of wave radiation and signal acquisition for a dipole acoustic reflection survey using an off-centred tool. We provide a fast and accurate algorithm to simulate borehole responses for the azimuthally placed off-axis receivers of an eccentric dipole tool. Compared with a centred tool, a prominent feature of an eccentric dipole source is the asymmetry of the source radiation. Thus, reflection data from the eccentric side of the borehole tended to show higher amplitude than the data from the opposite side. This provides a potential application for resolving 180° azimuth ambiguity in dipole shear-wave imaging. Our results also show that the asymmetry effect depends strongly on the degree of eccentricity, source frequency and formation elasticity. These factors should also be considered when using the effect to help resolve reflector azimuth ambiguity.
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Appendix A. Determining the coordinates of the source and receiver locations for an off-centred dipole acoustic tool

Consider an off-centre point source in the borehole fluid, as shown in figure 2. Assuming the original coordinates are $(r_0, \varphi_0, 0)$, after offsetting along the direction of $\varphi_0$ by the distance $D_\circ$ and rotating by the angle $\varphi_r$, the point source is located at $(r_\varphi, \varphi_\varphi, 0)$

\[
\begin{align*}
  r_\varphi &= \sqrt{x_r^2 + y_r^2}, \\
  \varphi_\varphi &= a \tan \left( \frac{y_r}{x_r} \right),
\end{align*}
\]

(A-1)

where $(x_r, y_r, 0)$ denotes the location of the eccentric point source in Cartesian coordinates and is given by

\[
\begin{align*}
  x_r &= r_0 \cos (\varphi_0 + \varphi_r) + D_\circ \cos \varphi_0, \\
  y_r &= r_0 \sin (\varphi_0 + \varphi_r) + D_\circ \sin \varphi_0
\end{align*}
\]

(A-2)

According to equations (A-1) and (A-2), the coordinates of the four eccentric point sources shown in figure 2c are given by

\[
\begin{align*}
  r_{X1} &= \sqrt{r_0^2 + D_\circ^2 + 2r_0D_\circ \cos(\varphi_0 - \varphi_r)}, \\
  \varphi_{X1} &= a \tan \left( \frac{r_0 \sin \varphi_r + D_\circ \sin \varphi_0}{r_0 \cos \varphi_r + D_\circ \cos \varphi_0} \right), \\
  r_{Y1} &= \sqrt{r_0^2 + D_\circ^2 + 2r_0D_\circ \sin(\varphi_0 - \varphi_r)}, \\
  \varphi_{Y1} &= a \tan \left( \frac{r_0 \cos \varphi_r + D_\circ \sin \varphi_0}{-r_0 \sin \varphi_r + D_\circ \cos \varphi_0} \right), \\
  r_{X2} &= \sqrt{r_0^2 + D_\circ^2 - 2r_0D_\circ \cos(\varphi_0 - \varphi_r)}, \\
  \varphi_{X2} &= a \tan \left( \frac{r_0 \sin \varphi_r - D_\circ \sin \varphi_0}{r_0 \cos \varphi_r - D_\circ \cos \varphi_0} \right), \\
  r_{Y2} &= \sqrt{r_0^2 + D_\circ^2 - 2r_0D_\circ \sin(\varphi_0 - \varphi_r)}, \\
  \varphi_{Y2} &= a \tan \left( \frac{-r_0 \cos \varphi_r + D_\circ \sin \varphi_0}{r_0 \sin \varphi_r + D_\circ \cos \varphi_0} \right)
\end{align*}
\]

(A-3)

Note that the eccentric angles calculated using equations (A-1) and (A-3) are periodic functions with a period of $\pi$, which means that the function values need to be selected according to the actual location of the eccentric point source. For example, $\pi/2$ should be added to $\varphi$, for the $x_r < 0$ ($x_r > 0$) case. The coordinate solution method for the receiver location is similar to that for the source location.

Appendix B. Details of matrix equation (6) for borehole radiation

In equation (6), the expressions of the matrix $M$ and the vector $b$ are given next

\[
\begin{align*}
  M_{11} &= -f_{m+1}(r_b) - ml_m(r_b) / r_b, \\
  M_{12} &= -pK_{m+1}(r_b) + mK_m(r_b) / r_b, \\
  M_{13} &= mK_m(r_b) / r_b, \\
  M_{14} &= -i k \left[ sK_{m+1}(r_b) - mK_m(r_b) / r_b \right], \\
  M_{21} &= \rho_\circ \omega^2 l_m(r_b), \\
  M_{22} &= 2 \rho \beta^2 \left[ pK_{m+1}(r_b) / r_b + m(m - 1)K_m(r_b) / r_b^2 \right] + \rho(2k^2 \beta^2 - \omega^2)K_m(r_b), \\
  M_{23} &= -2m\rho \beta^2 \left[ sK_{m+1}(r_b) / r_b - (m - 1)K_m(r_b) / r_b^2 \right],
\end{align*}
\]

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where $r_b$ is the borehole radius; $\omega$ is the circular frequency; $k$ stands for the axial wavenumber; $f = \sqrt{k^2 - (\alpha/\beta)^2}$ stands for the radial wavenumber of the borehole fluid, $\alpha$ is the fluid velocity; $p = \sqrt{k^2 - (\alpha/\beta)^2}$ and $s = \sqrt{k^2 - (\alpha/\beta)^2}$ stand for the radial wavenumbers of the formation P- and S-waves, $\alpha$ and $\beta$ are the formation P- and S-wave velocities, respectively; $I_m (K_m)$ represents the modified Bessel functions of the first (second) kind with order $m$; the symbol $\epsilon_m$ represents Neumann’s factor, which is defined as 1 for $m = 0$ and 2 for $m \neq 0$ and the symbol $r_0$ denotes the dipole-source radius. Note that when calculating the radiation pattern using equations (9) and (11), the vector $b$ needs to be divided by $(2\rho_0 \omega^2 r_0)$ to unify dimensions.

**Appendix C. Details of matrix equation (17) for borehole reception**

In matrix equation (17), the left-hand-side matrix $M$ is the same as equation (6) (see Appendix B), whereas the right-hand side vectors $c$ and $c'$ are different, which depends on the type of the incident wave, as given next.

For the P incident wave (with the common factor $[S(\omega) \cdot RD_{P}^{X,Y}(\omega) \cdot RF_{P}(\omega)]$),

\[
M_{24} = 2ik\rho \beta^2 \left\{ sK_{m+1}(sr_b) / r_b + \left[ m(m-1) / r_b^2 + s^2 \right] K_m(sr_b) \right\},
M_{31} = 0,
M_{32} = 2m\rho \beta^2 \left[ pK_{m+1}(pr_b) / r_b - (m-1)K_m(pr_b) / r_b^2 \right],
M_{33} = -\rho \beta^2 \left\{ 2sK_{m+1}(sr_b) / r_b + \left[ 2m(m-1) / r_b^2 + s^2 \right] K_m(sr_b) \right\},
M_{34} = 2ikm\rho \beta^2 \left[ sK_{m+1}(sr_b) / r_b - (m-1)K_m(sr_b) / r_b^2 \right],
M_{41} = 0,
M_{42} = -2ik\rho \beta^2 \left[ pK_{m+1}(pr_b) - mK_m(pr_b) / r_b \right],
M_{43} = ikm\rho \beta^2 K_m(sr_b) / r_b.
\]

\[
M_{44} = (s^2 + k^2)\rho \beta^2 \left[ sK_{m+1}(sr_b) - mK_m(sr_b) / r_b \right],
\]

\[
\begin{align*}
c_1 &= -\epsilon_m \left[ pI_{m+1}(pr_b) + mI_m(pr_b) / r_b \right] \cos(m\phi_0'), \\
c_2 &= \epsilon_m \beta \left\{ 2\beta^2 \left[ pI_{m+1}(pr_b) / r_b - m(m-1)I_m(pr_b) / r_b^2 \right] \right\} \cos(m\phi_0'), \\
c_3 &= 2\epsilon_m \rho \beta^2 \left[ pI_{m+1}(pr_b) / r_b + (m-1)I_m(pr_b) / r_b^2 \right] \cos(m\phi_0'), \\
c_4 &= -2\epsilon_m ik\rho \beta^2 \left[ pI_{m+1}(pr_b) + mI_m(pr_b) / r_b \right] \cos(m\phi_0'), \\
c'_1 &= -\epsilon_m \left[ pI_{m+1}(pr_b) + mI_m(pr_b) / r_b \right] \sin(m\phi_0'), \\
c'_2 &= \epsilon_m \beta \left\{ 2\beta^2 \left[ pI_{m+1}(pr_b) / r_b - m(m-1)I_m(pr_b) / r_b^2 \right] \right\} \sin(m\phi_0'), \\
c'_3 &= 2\epsilon_m \rho \beta^2 \left[ pI_{m+1}(pr_b) / r_b + (m-1)I_m(pr_b) / r_b^2 \right] \sin(m\phi_0'), \\
c'_4 &= -2\epsilon_m ik\rho \beta^2 \left[ pI_{m+1}(pr_b) + mI_m(pr_b) / r_b \right] \sin(m\phi_0').
\end{align*}
\]
For the SH incident wave (with the common factor \([S(\omega) \cdot RD_{SH}^{XY}(\omega) \cdot RF_{SH}(\omega)\])

\[
c_1 = -m\epsilon_m s_m^b (s_{rb}) \sin(m\varphi'_0) / r_{rb},
\]

\[
c_2 = -2m\epsilon_m \rho^2 \left[ s_{l_{m+1}}(s_{rb}) / r_b + (m - 1)l_m(s_{rb}) / r_b^2 \right] \sin(m\varphi'_0),
\]

\[
c_3 = -\epsilon_m \rho^2 \left[ 2s_{l_{m+1}}(s_{rb}) / r_b - \left[ 2m(m - 1) / r_b^2 + s^2 \right] l_m(s_{rb}) \right] \sin(m\varphi'_0),
\]

\[
c_4 = -ik\epsilon_m \rho^2 l_m(s_{rb}) \sin(m\varphi'_0) / r_{rb},
\]

\[
\gamma_1 = m\epsilon_m s_m^b (s_{rb}) \cos(m\varphi'_0) / r_{rb},
\]

\[
\gamma_2 = 2m\epsilon_m \rho^2 \left[ s_{l_{m+1}}(s_{rb}) / r_b + (m - 1)l_m(s_{rb}) / r_b^2 \right] \cos(m\varphi'_0),
\]

\[
\gamma_3 = \epsilon_m \rho^2 \left[ 2s_{l_{m+1}}(s_{rb}) / r_b - \left[ 2m(m - 1) / r_b^2 + s^2 \right] l_m(s_{rb}) \right] \cos(m\varphi'_0),
\]

\[
\gamma_4 = ik\epsilon_m \rho^2 l_m(s_{rb}) \cos(m\varphi'_0) / r_b.
\]

For the SV incident wave (with the common factor \([S(\omega) \cdot RD_{SV}^{XY}(\omega) \cdot RF_{SV}(\omega)\])

\[
c_1 = -ike_m \left[ s_{l_{m+1}}(s_{rb}) + ml_m(s_{rb}) / r_b \right] \cos(m\varphi'_0),
\]

\[
c_2 = 2ike_m \rho^2 \left[ s_{l_{m+1}}(s_{rb}) / r_b - \left[ m(m - 1) / r_b^2 + s^2 \right] l_m(s_{rb}) \right] \cos(m\varphi'_0),
\]

\[
c_3 = 2ik\epsilon_m \rho^2 \left[ s_{l_{m+1}}(s_{rb}) / r_b + (m - 1)l_m(s_{rb}) / r_b^2 \right] \cos(m\varphi'_0),
\]

\[
\gamma_1 = -ike_m \left[ s_{l_{m+1}}(s_{rb}) + ml_m(s_{rb}) / r_b \right] \sin(m\varphi'_0),
\]

\[
\gamma_2 = 2ike_m \rho^2 \left[ s_{l_{m+1}}(s_{rb}) / r_b - \left[ m(m - 1) / r_b^2 + s^2 \right] l_m(s_{rb}) \right] \sin(m\varphi'_0),
\]

\[
\gamma_3 = 2ik\epsilon_m \rho^2 \left[ s_{l_{m+1}}(s_{rb}) / r_b + (m - 1)l_m(s_{rb}) / r_b^2 \right] \sin(m\varphi'_0),
\]

\[
\gamma_4 = \epsilon_m (s^2 + k^2) \rho^2 \left[ s_{l_{m+1}}(s_{rb}) + ml_m(s_{rb}) / r_b \right] \sin(m\varphi'_0),
\]

where \(S\) stands for the source spectrum; \(RD\) stands for the radiation function; \(RF\) stands for the reflection coefficient; the superscript \(X\) \((Y)\) denotes the excitation of the \(X\) \((Y)\) dipole source of the system; the subscripts \(P\), \(SH\) and \(SV\) represent the types of incident waves, \(P\)-, \(SH\)- or \(SV\)-wave; \(\varphi'_0\) is the virtual-source azimuth and other symbols are defined in Appendix B.

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