THERE MAY BE NO HAUSDORFF ULTRAFILTERS

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Abstract. An ultrafilter $U$ is Hausdorff if for any two functions $f, g \in \omega^\omega$, $f(U) = g(U)$ iff $f|X = g|X$ for some $X \in U$. We will show that it is consistent that there are no Hausdorff ultrafilters.

1. Introduction

For $f \in \omega^\omega$ and an ultrafilter $U$ on $\omega$ define $f(U) = \{ X \subseteq \omega : f^{-1}(X) \in U \}$. Let $\text{FtO}$ be the collection of all finite-to-one functions $f \in \omega^\omega$.

Definition 1. Let $U$ be an ultrafilter on $\omega$. We say that

(1) $U$ is Hausdorff if for any two functions $f, g \in \omega^\omega$, if $f(U) = g(U)$ then $f|X = g|X$ for some $X \in U$.

(2) $U$ is weakly Hausdorff if for any two functions $f, g \in \text{FtO}$, if $f(U) = g(U)$ then $f|X = g|X$ for some $X \in U$.

It is easy to see that

(1) Ramsey ultrafilters are Hausdorff.

(2) $q$-points are weakly Hausdorff.

(3) Weakly Hausdorff $p$-points are Hausdorff.

It is worth mentioning that the following appears as an exercise in [9].

Lemma 2. If $f(U) = U$ then there exists $X \in U$ such that $f(n) = n$ for $n \in X$.

Therefore, if $U$ is not Hausdorff, then this is witnessed by two functions, both not one-to-one mod $U$.

The notion of a Hausdorff ultrafilters was reintroduced and studied by Mauro Di Nasso, Marco Forti and others in a sequence of papers ([8], [7], [10] and [6]) in context of topological extensions. They used the name Hausdorff because Hausdorff ultrafilters are precisely those ultrafilters whose ultrapowers equipped with the standard topology are Hausdorff topological spaces. They asked whether the existence of a Hausdorff ultrafilter can be proved in ZFC. We will show that, at least for ultrafilters on $\omega$, the answer is negative. However such ultrafilters (with various extra properties) may be constructed under from additional set theoretical assumptions (see [7]).
2. Construction of the model

In this section we will show how to build a model where there are no Hausdorff ultrafilters modulo the proofs of theorems 4 and 6 below.

**Definition 3.** An ultrafilter \( U \) is strongly non-Hausdorff if for every \( f \in \text{FtO} \), \( f(U) \) is not weakly Hausdorff.

**Theorem 4.** Assume CH. There exists a strongly non-Hausdorff \( p \)-point.

**Definition 5.** [2], [3], [5]. Let NCF stand for the following statement:
for any ultrafilters \( U, V \) on \( \omega \) there exists \( h \in \text{FtO} \) such that \( h(U) = h(V) \).

**Theorem 6.** There exists a proper forcing notion \( P \) such that

1. If \( V \models \text{GCH} \) then \( V^P \models 2^\omega = \aleph_2 \).
2. If \( U \) is a \( p \)-point then \( V^P \models U \) generates a \( p \)-point.
3. If \( U \) is strongly non-Hausdorff filter then \( V^P \models U \) generates a strongly non-Hausdorff filter.
4. \( V^P \models \text{NCF} \).

**Theorem 7.** Suppose that \( V \models \text{GCH} \). Then in \( V^P \) there are no weakly Hausdorff ultrafilters. In particular, there are no Hausdorff ultrafilters in this model.

**Proof.** Let \( U_0 \) be a strongly non-Hausdorff \( p \)-point in \( V \) given by theorem 4. By theorem 6, \( U_0 \) generates a strongly non-Hausdorff \( p \)-point in \( V^P \), and \( V^P \) satisfies NCF. So suppose that \( U \) is an ultrafilter in \( V^P \). By NCF there exists \( h \in \text{FtO} \) such that \( h(U) = h(U_0) \). Since \( U_0 \) is strongly non-Hausdorff in \( V^P \) it follows that \( h(U_0) \) is not Hausdorff. On the other hand if \( U \) was Hausdorff then the following lemma would imply that \( h(U) \) is Hausdorff as well, a contradiction.

**Lemma 8.** If \( U \) is Hausdorff then \( h(U) \) is also Hausdorff.

**Proof.** Let \( f, g \in \text{FtO} \) be such that \( f(h(U)) = g(h(U)) \). It follows that there is \( X \in U \) such that \( f \circ h|X = g \circ h|X \). Thus \( f[h[X] = g[h[X] \) and \( h[X] \in h(U) \).

□

3. A strongly non-Hausdorff ultrafilter

Let \( I \subset \omega \) be a finite set and let \( \Delta = \{ (n, n) : n \in \omega \} \). Denote by \( [I]^2 = (I \times I) \setminus \Delta \).

For a set \( X \subseteq [I]^2 \) define
\[
\|X\|_I = \min \left\{ k : \exists \{A_i, B_i : i \leq k\} \forall i \leq k A_i \cap B_i = \emptyset \text{ and } X \subseteq \bigcup_{i \leq k} A_i \times B_i \right\}.
\]

We will drop the subscript \( I \) if it is clear from the context what it is.

**Lemma 9.**

1. \( \|I\|^2 \rightarrow \infty \text{ as } |I| \rightarrow \infty \).
2. \( |X \cup Y|_I \leq |X|_I + |Y|_I \).
3. If \( Z \subseteq I \) and \( X \subseteq [I]^2 \), \( |X|_I > 2 \), then either \( \|Z\|^2 \cap X\|_I \geq |X|_I / 2 - 1 \)
or \( \|I \setminus Z\|^2 \cap X\|_I \geq |X|_I / 2 - 1 \).

**Proof.** If (1) fails then there is \( k \in \omega \) and sets \( \{A^n_i, B^n_j : n, j \leq k\} \) such that \( A^n_j \cap B^n_j = \emptyset \) for \( j \leq k \) and \( [n]^2 = \bigcup_{j \leq k} A^n_j \times B^n_j \). By compactness we get sets
Lemma 12. If $f, g \in \text{FtO}$ and $U$ is an ultrafilter then the following conditions are equivalent:

(1) $f(U) \neq g(U)$,
(2) $f[X] \cap g[X] = \emptyset$ for some $X \in U$. $\square$

We will build an ultrafilter $V_0$ on the set $\bigcup_k [I_k]^2$ which we identified with $\omega$. Let 
$\{Z_{\alpha} : \alpha < \omega_1\}$ be enumeration of $[\omega]^\omega$.
We will build by induction a sequence $\{X_\alpha : \alpha < \omega_1\}$ so that

(1) $\forall \beta < \alpha \ X_\alpha \subseteq^* X_\beta$,
(2) \( X_{\alpha+1} \cap Z_\alpha = \emptyset \) or \( X_{\alpha+1} \subseteq Z_\alpha \) for all \( \alpha \).

(3) for every \( \alpha < \omega_1 \), \( f^0[X_\alpha] \cap g^0[X_\alpha] \neq \emptyset \).

(4) for every \( \alpha < \omega_1 \), \( \limsup_k |X_\alpha \cap J_k|_{I_k} = \infty \).

Let \( V_0 = \{ X : \exists \alpha X_\alpha \subseteq^* X \} \). Note that the conditions (1) and (2) guarantee that \( V_0 \) is a p-point, and lemma 12 and (3) implies that \( f^0(V_0) = g^0(V_0) \). Finally, (4) is the requirement that (by lemma 10) implies (3).

**Successor step.** Suppose that \( X_\alpha \) is given. Find a strictly increasing sequence \( \{ \ell_k : k \in \omega \} \) such that the set \( A = \{ k : |X_\alpha \cap J_k|_{I_k} = \ell_k \} \) is infinite. Let \( A_0 = \{ k : |X_\alpha \cap Z_\alpha \cap J_k|_{I_k} \geq \ell_k/2 - 1 \} \) and \( A_1 = \{ k : |(X_\alpha \setminus Z_\alpha) \cap J_k|_{I_k} \geq \ell_k/2 - 1 \} \). By lemma 9(3), one of these sets, say \( A_0 \), is infinite. Let \( X_{\alpha+1} = \bigcup_{k \in A_0} X_\alpha \cap Z_\alpha \cap J_k \). The other case is the same.

**Limit step.** Given \( \{ X_\beta : \beta < \alpha < \omega_1 \} \) let \( \{ \beta_k : k \in \omega \} \) be an increasing sequence cofinal in \( \alpha \). By finite modifications we can assume that \( X_{\beta_k+1} \subseteq X_{\beta_k} \) for all \( k \). Build by recursion a strictly increasing sequence \( \{ u_k : k \in \omega \} \) such that

\[
\forall k \forall j \leq k \exists i \in [u_k, u_{k+1}) \quad |X_{\beta_j} \cap J_i|_{I_i} \geq k,
\]

and let

\[
X_\alpha = \bigcup_k \left( X_{\beta_k} \cap \bigcup_{i \in [u_k, u_{k+1})} J_i \right).
\]

It is clear that \( X_\alpha \) satisfies (1) and (4). \( \square \)

Observe that CH was only needed in the limit step. If we do not require that that \( U \) is a p-point then we have the following:

**Theorem 13.** There exists an ultrafilter that is not weakly Hausdorff.

**Proof.** As in lemma 11, we will build an ultrafilter on the set \( \bigcup_k |I_k|^2 \). Let

\[
\mathcal{I} = \left\{ X \subseteq \bigcup_k |I_k|^2 : \limsup_k |X \cap J_k|_{I_k} < \infty \right\}.
\]

Note that \( \mathcal{I} \) is an ideal, and let \( U \) be any ultrafilter orthogonal to \( \mathcal{I} \). Functions \( f^0, g^0 \) witness that \( U \) is not Hausdorff. \( \square \)

**Proof of Theorem 4.**

Now we are ready to construct a p-point ultrafilter \( U_0 \) whose all finite-to-one images are not weakly Hausdorff.

Let \( \{ h_\alpha, Z_\alpha : \alpha < \omega_1 \} \) be enumeration of \( \text{FtO} \) and \( [\omega]^{\omega_1} \) respectively. We will build by induction sequences \( \{ f^\alpha, g^\alpha : \alpha < \omega_1 \}, \{ X_\alpha : \alpha < \omega_1 \} \) so that

1. \( \forall \beta < \alpha X_\alpha \subseteq^* X_\beta \)
2. \( X_{\alpha+1} \cap Z_\alpha = \emptyset \) or \( X_{\alpha+1} \subseteq Z_\alpha \) for all \( \alpha \).
3. for every \( \alpha \), \( f^{\alpha+1}, g^{\alpha+1} \) witness that \( h_\alpha(U_0) \) is not Hausdorff, where \( U_0 = \{ X \subseteq \omega : \exists \alpha X_\alpha \subseteq^* X \} \).
4. \( \forall \beta \forall \alpha \geq \beta f^{\beta+1} \circ h_\beta[X_\alpha] \cap g^{\beta+1} \circ h_\beta[X_\alpha] \neq \emptyset \).

As before, (1) and (2) guarantee that \( U_0 \) is a p-point, and (3) implies that \( U_0 \) is strongly non-Hausdorff, and (4) is a specific form of (3). Note that at the limit stages we only have to preserve the induction hypothesis. At the successor step we will first define an auxiliary function \( e_{\alpha+1} \), and put \( f^{\alpha+1} = f^0 \circ e_{\alpha+1} : h_\alpha[X_\alpha] \rightarrow \omega \) and \( g^{\alpha+1} = g^0 \circ e_{\alpha+1} : h_\alpha[X_\alpha] \rightarrow \omega \). In other words, \( f^{\alpha+1}, g^{\alpha+1} \) are copies of \( f^0, g^0 \) on the image of \( X_\alpha \) via \( e_{\alpha+1} \circ h_\alpha \).
Therefore we need to clarify condition (3) by imposing conditions on \( e_\alpha \) and specifying the induction hypothesis.

**Definition 14.** Let us say that a finite set \( Y \subseteq X_\alpha \) is a \((n, \beta, \alpha)\)-witness if there exists \( k \in \omega \) such that \( |e_{\beta+1} \circ h_\beta[Y] \cap J_k|_{l_k} \geq n \).

To satisfy (3), we demand that for \( \beta < \alpha < \omega_1 \),

5. \( \limsup_k |e_{\beta+1} \circ h_\beta[X_\alpha] \cap J_k|_{l_k} = \infty \), or equivalently

6. \( \forall n \ \exists Y \in [X_\alpha]^{\omega} \text{ is a } (n, \beta, \alpha)\)-witness.

**LIMIT step.**

Suppose that \( \{X_\beta : \beta < \alpha\} \) are defined and \( \alpha \) is a limit ordinal. Let \( \{\beta_k : k \in \omega\} \) be an increasing sequence cofinal in \( \alpha \), and let \( \{\gamma_k : k \in \omega\} \) be an enumeration of \( \alpha \) such that \( \gamma_j \leq \beta_k \) for \( j \leq k \). Without loss of generality we can assume that \( X_{\beta_n} \subseteq X_{\beta_m} \) for \( n \geq m \).

Build by recursion a strictly increasing sequence \( \{u_k : k \in \omega\} \) such that

\[
\forall k \ \forall l, j \leq k \ \exists i \in [u_k, u_{k+1}) \ | e_{\gamma_{i+1}} \circ h_{\gamma_i}[X_{\beta_j}] \cap J_i|_{l_i} \geq k,
\]

and let

\[
X_\alpha = \bigcup_k \left( X_{\beta_k} \cap \bigcup_{i \in [u_k, u_{k+1})} J_i \right).
\]

It is clear that \( X_\alpha \) satisfies (1) and (4).

**SUCCESSOR step.**

Suppose that \( X_\alpha \) satisfying (4) is already defined and we want to define \( X_{\alpha+1} \) and \( e_{\alpha+1} \) satisfying (2) and (5). Recall that by the induction hypothesis, for \( \beta < \alpha \),

\[
\forall n \ \exists Y \in [X_\alpha]^{\omega} \text{ is a } (n, \beta, \alpha)\)-witness.
\]

Let \( \{\beta_k : k \in \omega\} \) be an enumeration of \( \alpha \). Find a sequence \( \{E_k : k \in \omega\} \) of consecutive intervals such that

1. \( \forall k \ \forall j \leq k \ h_{\alpha}^{-1}(E_k) \) contains a \((k, \beta_j, \alpha)\)-witness.

2. \( \forall k \ E_k \cap h_{\alpha}[X_\alpha] \neq \emptyset \).

Let \( e_{\alpha+1}(j) = k \iff j \in E_k \) for \( j \in \omega \). Condition (2) implies that \( e_{\alpha+1} \circ h_\alpha[X_\alpha] = \omega \). In particular, either \( \limsup_k |e_{\alpha+1} \circ h_\alpha[Z_\alpha \cap X_\alpha] \cap J_k|_{l_k} = \infty \), or \( \limsup_k |e_{\alpha+1} \circ h_\alpha[X_\alpha \setminus Z_\alpha] \cap J_k|_{l_k} = \infty \). Let \( X_{\alpha+1} \) be the appropriate set. It remains to check that for \( \beta \leq \alpha \), \( \limsup_k |e_{\gamma_{i+1}} \circ h_{\gamma_i}[X_{\alpha+1}] \cap J_k|_{l_k} = \infty \). If \( \beta = \alpha \) it follows immediately from the definition of \( X_{\alpha+1} \), so suppose that \( \beta = \beta_j < \alpha \).

Let \( m \in e_{\alpha+1} \circ h_\alpha[X_{\alpha+1}] \setminus j \) and note that \( (e_{\alpha+1} \circ h_\alpha)^{-1}(m) \) contains a \((m, \beta_j, \alpha)\)-witness. It follows that \( \limsup_k |e_{\beta+1} \circ h_\beta[X_{\alpha+1}] \cap J_k|_{l_k} = \infty \), which finishes the construction.

\[ \textbf{4. Forcing} \]

Since known models for \( \text{NCF} \) are obtained by countable support iteration we will look for a proper forcing notion \( P \) such that the iteration of \( P \) has the required properties.

Suppose that \( P \) is a proper forcing notion. \( P \) preserves non-meager sets if for every countable elementary submodel \( N \prec H(\chi) \) containing \( P \), a condition \( p \in P \in N \) and a Cohen real \( c \) over \( N \) there exists \( q \geq p \) such that \( q \) is \((N, P)\) generic and \( q \upharpoonright p \ c \) is Cohen over \( N[\check{c}] \).

By [1], 6.3.16 this is equivalent to the property \( \subseteq \text{Cohen} \) defined in [1].
Let $$\mathbb{P}$$ be a proper forcing notion such that:

1. $$\mathbb{P}$$ preserves $$p$$-points,
2. $$\mathbb{P}$$ preserves non-meager sets, is it preserves $$\subseteq^\text{Cohen}$$.

Let $$\mathcal{P} = \mathbb{P}_{\omega_2}$$ be the countable support iteration of $$\mathbb{P}$$ of length $$\omega_2$$. We have the following:

1. $$\mathcal{P}$$ preserves $$p$$-points (see [4] or [1] 6.2.6),
2. $$\mathcal{P}$$ preserves non-meager sets ([1], 6.3.20).

Recall that if $$\mathbb{P}$$ is either Blass-Shelah forcing from [4] or Miller superperfect forcing, then $$\mathcal{P}$$ has the above properties (7.3.46 and 7.3.48 of [1]) and $$\mathbf{V}^\mathbb{P} \models \text{NCF}, \quad [5]$$ or [4]. Therefore the following theorem concludes the proof of theorem 6.

**Theorem 15.** Suppose that $$\mathcal{P}$$ is a proper forcing that preserves non-meager sets and $$U_0 \in \mathbf{V}$$ is a strongly non-Hausdorff ultrafilter defined earlier. Then

$$\mathbf{V}^\mathbb{P} \models U_0 \text{ generates a strongly non-Hausdorff filter.}$$

**Proof.** Clearly, $$U_0$$ may not generate an ultrafilter in the extension, for example when $$\mathcal{P}$$ is Cohen forcing.

Let $$\mathcal{C}$$ be the Cohen forcing interpreted as adding a function $$c \in \mathbf{FtO}$$. Specifically, the conditions are finite sequences of consecutive intervals $$\{I_k : k < n\}$$ and $$c(i) = k \iff i \in I_k$$.

**Lemma 16.** Let $$c$$ be Cohen reals over $$\mathbf{V}$$. Then for every $$h \in \mathbf{V} \cap \mathbf{FtO}$$, $$h(U_0)$$ is not Hausdorff as witnessed by $$f^h = f^0 \circ c$$ and $$g^h = g^0 \circ c$$.

**Proof.** This is quite easy. Given $$s = \{L_k : k < n\} \in \mathcal{C}$$, $$X = X_\alpha \in U_0$$ we extend $$s$$ by adding an interval $$L_n$$ so large that $$L_n \supseteq (e_{\alpha+n})^{-1}(k)$$ for some $$k > n$$. That means that $$f^0 \circ c$$ and $$g^0 \circ c$$ agree with $$f^0 \circ e_{\alpha+1}$$ and $$g^0 \circ e_{\alpha+1}$$ on long enough segments to witness that $$f^0 \circ c[X] \cap g^0 \circ c[X] \neq \emptyset$$. \hfill \Box

Let $$\dot{h}$$ be a $$\mathcal{P}$$-name for an element of $$\mathbf{FtO}$$. Let $$N \prec H(\chi)$$ be a countable submodel containing $$U_0, h, p, \mathcal{P}$$ and let $$c \in \mathbf{V} \cap \mathbf{FtO}$$ be a Cohen real over $$N$$. Since $$\mathcal{P}$$ preserves non-meager sets there is $$q \geq p$$ which is $$\langle N, \mathcal{P} \rangle$$ generic and $$q \models \mathcal{P} c$$ is Cohen over $$N[\dot{G}]$$. In particular, by lemma 16,

$$q \models \mathcal{P} h(U_0)$$ is not Hausdorff as witnessed by $$f^0 \circ c$$ and $$g^0 \circ c$$.

By elementarity, it means that $$\mathbf{V}^\mathbb{P} \models h(U_0)$$ is not Hausdorff. \hfill \Box

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