A NON-AUTONOMOUS MODEL PROBLEM FOR THE OSEEN-NAVIER-STOKES FLOW WITH ROTATING EFFECTS

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Abstract. Consider the Navier-Stokes flow past a rotating obstacle with a general time-dependent angular velocity and a time-dependent outflow condition at infinity. After rewriting the problem on a fixed domain, one obtains a non-autonomous system of equations with unbounded drift terms. It is shown that the solution to a model problem in the whole space case \( \mathbb{R}^d \) is governed by a strongly continuous evolution system on \( L^p(\mathbb{R}^d) \) for \( 1 < p < \infty \).

The strategy is to derive a representation formula, similar to the one known in the case of non-autonomous Ornstein-Uhlenbeck equations. This explicit formula allows to prove \( L^p - L^q \) estimates and gradient estimates for the evolution system. These results are key ingredients to obtain (local) mild solutions to the full nonlinear problem by a version of Kato’s iteration scheme.

1. Introduction and main result

In this paper we consider a model problem in \( \mathbb{R}^d \) for the flow of an incompressible, viscous fluid past a rotating obstacle with an additional time-dependent outflow condition at infinity. The equations describing this problem are the Navier-Stokes equations in an exterior domain varying in time with an additional condition for the velocity field at infinity.

In order to motivate our model problem, let \( O \subset \mathbb{R}^d \) be a compact obstacle with smooth boundary, let \( \Omega := \mathbb{R}^d \setminus O \) be the exterior of the obstacle and let \( m \in C([0, \infty); \mathbb{R}^{d \times d}) \) be a continuous matrix-valued function. Then, the exterior of the rotated obstacle at time \( t > 0 \) is represented by \( \Omega(t) := Q(t)\Omega \) where \( Q(t) \) solves the ordinary differential equation

\[
\begin{align*}
\partial_t Q(t) &= m(t)Q(t), \quad t > 0, \\
Q(0) &= \text{Id}.
\end{align*}
\]

(1.1)

With a prescribed velocity field \( v_\infty \in C^1([0, \infty); \mathbb{R}^d) \) at infinity, the equations for the fluid on the time-dependent domain \( \Omega(t) \) with no-slip boundary condition take the form

\[
\begin{align*}
v_t - \Delta v + v \cdot \nabla v + \nabla q &= 0 & \text{in } \Omega(t) \times (0, \infty), \\
\text{div } v &= 0 & \text{in } \Omega(t) \times (0, \infty), \\
v(t, y) &= m(t)y & \text{on } \partial\Omega(t) \times (0, \infty), \\
\lim_{|y| \to \infty} v(t, y) &= v_\infty(t) & \text{for } t \in (0, \infty), \\
v(0, y) &= u_0(y) & \text{in } \Omega,
\end{align*}
\]

(1.2)

where \( v \) and \( q \) are the unknown velocity field and the pressure of the fluid, respectively.

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The disadvantage of this description is the variability of the domain \( \Omega(t) \), and the fact that the equations do not fit into the \( L^p \)-setting, due the velocity condition at infinity. Assume for the time being that \( m(t) \) is skew symmetric for \( t > 0 \); this implies that for all \( t > 0 \) the matrix \( Q(t) \) is orthogonal. Then, by setting 
\[
x = Q(t)^T y, \quad u(t, x) = Q(t)^T (v(t, y) - v_\infty(t)), \quad p(t, x) = q(t, y),
\]
the above equations can be transformed to the reference domain \( \Omega \) and the new velocity field \( u \) vanishes at infinity. Then (1.2) is equivalent to the following system of equations
\[
\begin{aligned}
u_t - \Delta u - \mathcal{M}(t)x \cdot \nabla u + \mathcal{M}(t)u + Q(t)^T v_\infty(t) \cdot \nabla u - Q(t)^T \partial_t v_\infty(t) + u \cdot \nabla u + \nabla p &= 0 & \text{in } \Omega \times (0, \infty), \\
\text{div } u &= 0 & \text{in } \Omega \times (0, \infty), \\
\lim_{|x| \to \infty} u(t, x) &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(0, x) &= u_0(x) & \text{in } \Omega,
\end{aligned}
\]
where \( \mathcal{M}(t) := Q(t)^T m(t) Q(t) \). The main difficulty in dealing with this problem arises since the term \( \mathcal{M}(t)x \cdot \nabla \) has unbounded coefficients. In particular, the lower order terms cannot be treated by classical perturbation theory for the Stokes operator.

Note that even if we assume that \( m(t) \equiv m \) is independent of time (this implies that also \( \mathcal{M}(t) \equiv \mathcal{M} \) is independent of time), equation (1.3) is still non-autonomous due to the time-dependent first order term \( Q(t)^T v_\infty \cdot \nabla \) (except in some special cases discussed below).

However, by using localization techniques similar to [GH06], this problem is finally reduced to a model problem in \( \mathbb{R}^d \) and a model problem in a bounded domain. Since \( Q(t)\partial_t v_\infty(t) \equiv F(t), t > 0 \), i.e. it is constant in space, we may put this term in the pressure \( p \). Hence, in this paper we discuss the following linearized model problem in \( \mathbb{R}^d \)
\[
u_t - \Delta u - (M(t)x + f(t)) \cdot \nabla u + M(t)u + \nabla p = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\text{div } u &= 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
u(0) &= u_0 & \text{in } \mathbb{R}^d,
\]
where we allow general coefficients \( M \in C([0, \infty); \mathbb{R}^{d \times d}) \) and \( f \in C([0, \infty); \mathbb{R}^d) \). If we set \( M(t) := Q(t)^T m(t) Q(t) \) and \( f(t) := -Q(t)^T v_\infty(t) \) then we obtain the linearization of equation (1.4) with \( \Omega = \mathbb{R}^d \). Such a model problem also arises in the analysis of a rotating body with translational velocity \(-v_\infty(t)\), see [Far05].

Existence and uniqueness of a mild solution of an autonomous variant of problem (1.2) without an outflow condition, i.e. \( v_\infty \equiv 0 \), and \( m(t) \equiv m \), was investigated in quite a few papers, see [His99a, His99b, GH06] and [HS05]. Hishida was even able to deal with a time dependent rotation in [His01], however only for angular velocities of a special form.

For the problem including an additional outflow condition at infinity, there are only a few results. Indeed, in the special case, where \( m(t)x = \omega(t) \times x \) and \( \omega : [0, \infty) \to \mathbb{R}^3 \) is the angular velocity of the obstacle and \( v_\infty : [0, \infty) \to \mathbb{R}^3 \) a time-dependent outflow velocity, Borchers [Bor92] constructed weak non-stationary solutions for the equations (1.4). Moreover, Shibata [Shi08] studied the special case where \( m(t) \equiv m \), \( v_\infty(t) = v_\infty \) and \( mv_\infty = 0 \). The
condition \( mv_\infty = 0 \), i.e. \( Q(t)^T v_\infty = kv_\infty \) for \( k \in \{-1,1\} \), ensures that (1.4) is still an autonomous equation and the solution of (1.4) is governed by a \( C_0 \)-semigroup which is not analytic. The physical meaning of the additional condition \( mv_\infty = 0 \) is that the outflow direction of the fluid is parallel to the axis of rotation of the obstacle. The stationary problem of this latter situation was analysed in [Far05].

The assumption \( mv_\infty = 0 \) was recently relaxed by the second author in [Han10]. Indeed, he was able to deal with the model problem in \( \mathbb{R}^d \) where \( m(t)v_\infty \neq 0 \) and \( v_\infty(t) \equiv v_\infty \). However he assumes that \( m(t) \) and \( m(s) \) commute for all \( t, s > 0 \) which can physically be interpreted by the fact that the axis of rotation is fixed.

The aim of this work is to remove the latter additional condition, i.e. \( m(t) \) and \( m(s) \) need not to commute and \( v_\infty \) may be time-dependent.

As usual the Helmholtz projection \( \mathbb{P} \) allows us to rewrite (1.3) as an abstract Cauchy problem in \( L^p(\mathbb{R}^d) \), where \( L^p(\mathbb{R}^d) \) denotes the space of all solenoidal vector fields in \( L^p(\mathbb{R}^d) \):

\[
\begin{align*}
    u'(t) - A(t)u(t) &= 0, \quad t > 0, \\
    u(0) &= u_0.
\end{align*}
\]

(1.6)

Here:

\[
A(t) := \mathbb{P}(\Delta u + (M(t)x + f(t)) \cdot \nabla u + M(t)u)
\]

\[
D(A(t)) := \{ u \in W^{2,p}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) : M(t)x \cdot \nabla u \in L^p(\mathbb{R}^d) \}.
\]

Note that it immediately follows from [HS05] that for fixed \( t > 0 \), the operator \( A(t) \) is the generator of a \( C_0 \)-semigroup, which is not analytic. The fact that the semigroup is not analytic prevents us from employing standard generation results for evolution systems, see [Paz83 Chapter 5] and references therein. For the same reason, \( L^p-L^q \) estimates and gradient estimates don’t follow from standard arguments.

Therefore, we first derive a representation formula for the solution of (1.5). In order to derive this representation formula we transform (1.5) to a non-autonomous heat equation which can be explicitly solved, see Section 3. It turns out that the transformation to a non-autonomous heat equation is crucial to deal with our problem in this generality since the different transformation used in [Han10] caused the additional assumption that \( M(t) \) and \( M(s) \) commute for all \( t, s > 0 \).

In the following we denote by \( \{U(t,s)\}_{t,s \geq 0} \) the evolution system on \( \mathbb{R}^d \) generated by the family of matrices \( \{-M(t)\}_{t \geq 0} \), i.e.

\[
\begin{align*}
    \partial_t U(t,s) &= -M(t)U(t,s), \\
    U(s,s) &= \text{Id}.
\end{align*}
\]

(1.7)

Note that \( \partial_s U(t,s) = U(t,s)M(s) \).

We are now ready to present our main result.

**Theorem 1.1.** Let \( 1 < p < \infty \), \( M \in C([0,\infty); \mathbb{R}^{d \times d}) \) and \( f \in C([0,\infty); \mathbb{R}^d) \). The the solution of (1.6) is governed by a strongly continuous evolution system \( \{T(t,s)\}_{t \geq s \geq 0} \subset L(L^p(\mathbb{R}^d)) \). Moreover, the evolution system \( \{T(t,s)\}_{t \geq s \geq 0} \) admits the following properties:
(a) For $T_0 > 0$ set $M_{T_0} := \sup\{\|U(t,s)\| : t, s \in [0, T_0]\}$. Then for $1 < p < \infty$ and $p \leq q \leq \infty$ there exists $C := C(M_{T_0}, d) > 0$ such that for $u \in L^p_q(\mathbb{R}^d)$

$$\|T(t,s)u\|_{L^q_p(\mathbb{R}^d)} \leq C(t - s)^{-\frac{d}{2} \left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^q_p(\mathbb{R}^d)}, \quad 0 \leq s < t < T_0,$$

(1.8)

$$\|\nabla T(t,s)u\|_{L^q_p(\mathbb{R}^d)} \leq C(t - s)^{-\frac{d}{2} \left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^q_p(\mathbb{R}^d)}, \quad 0 \leq s < t < T_0.$$  (1.9)

In particular, if the evolution system $\{U(t,s)\}_{s,t \geq 0}$ is uniformly bounded, i.e. $M_{T_0} \leq M$, for some $M > 0$ and all $T_0 > 0$, we may set $T_0 = \infty$.

(b) For $1 < p < q < \infty$, $s \geq 0$ and $u \in L^p_q(\mathbb{R}^d)$ we have

$$\lim_{t \to s, t > s} (t - s)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|T(t,s)u\|_{L^q_p(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t \to s, t > s} (t - s)^{\frac{d}{2}} \|\nabla T(t,s)u\|_{L^q_p(\mathbb{R}^d)} = 0.$$

Next we consider the nonlinear problem

$$u'(t) - A(t)u(t) + \mathbb{P}((u(t) \cdot \nabla)u(t)) = 0, \quad t > 0,$$

(1.10)

$$u(0) = u_0,$$

with initial value $u_0 \in L^p_q(\mathbb{R}^d)$.

For given $0 < T_0 \leq \infty$, we call a function $u \in C([0,T_0]; L^p_q(\mathbb{R}^d))$ a mild solution of (1.10) if $u$ satisfies the integral equation

$$u(t) = T(t,0)u_0 - \int_0^t T(t,s)\mathbb{P}((u(s) \cdot \nabla)u(s))ds, \quad t > 0,$$  (1.11)

in $L^p_q(\mathbb{R}^d)$. By adjusting Kato’s iteration scheme (see [Kat82]) to our situation the existence of a unique (local) mild solution follows, cf. [Han10] for details.

**Corollary 1.2.** Let $2 \leq d \leq p \leq q < \infty$, $M \in C([0,\infty); \mathbb{R}^{d \times d})$, $f \in C([0,\infty); \mathbb{R}^d)$ and $u_0 \in L^p_q(\mathbb{R}^d)$. Then there exists $T_0 > 0$ and a unique mild solution $u \in C([0,T_0]; L^p_q(\mathbb{R}^d))$ of (1.10), which has the properties

$$t^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{2}\right)}u(t) \in C([0,T_0]; L^p_q(\mathbb{R}^d)), \quad (1.12)$$

$$t^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{2}\right)}\nabla u(t) \in C([0,T_0]; L^p(\mathbb{R}^d)^{d \times d}).$$  (1.13)

If $p < q$, then in addition

$$t^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{2}\right)}\|u(t)\|_{L^q_p(\mathbb{R}^d)} + t^{\frac{d}{2}}\|\nabla u(t)\|_{L^q_p(\mathbb{R}^d)} \to 0 \quad \text{as} \ t \to 0. \quad (1.14)$$

Moreover, in the case $d = p$ we may set $T_0 = +\infty$ provided $\|u_0\|_{L^q_p(\mathbb{R}^d)}$ is small enough and $\{U(t,s)\}_{s,t \geq 0}$ is uniformly bounded.

**Remark 1.3.** In particular, $\{U(t,s)\}_{s,t \geq 0}$ is uniformly bounded if $M(t)$ is skew symmetric for all $t > 0$.

2. **Proof of Theorem 1.1**

Let $M$ be as in Theorem 1.1 and let $\{U(t,s)\}_{s,t \geq 0}$ be the evolution system on $\mathbb{R}^d$ that satisfies (1.7). We consider the system of parabolic equations of the form

$$\begin{cases}
\partial_t u(t,x) - A(t)u(t,x) &= 0, \quad t > s, \ x \in \mathbb{R}^d, \\
u(s,x) &= \varphi(x), \ x \in \mathbb{R}^d,
\end{cases} \quad (2.1)$$
for \( s \geq 0 \) fixed, initial value \( \varphi \in L^p(\mathbb{R}^d)^d \) and some \( p \in (1, \infty) \). Here the family of operators \( \mathcal{A}(t) \) is of the form

\[
\mathcal{A}(t)u(x) := \left( \Delta u_i(t, x) + \langle M(t)x + f(t), \nabla u_i(t, x) \rangle \right)_{i=1}^d - M(t)u(t, x), \quad t > 0, \ x \in \mathbb{R}^d.
\]

As in [DPL07] Lemma 3.2 or [Han10], we first develop an explicit representation formula. To be more precise, we show in Section 3 that for \( p \in (1, \infty) \) and \( \varphi \in L^p(\mathbb{R}^d)^d \) the solution \( u \) to (2.1) is governed by a strongly continuous evolution system \( \{\mathcal{T}(t, s)\}_{t \geq s} \subset L(L^p(\mathbb{R}^d)^d) \) which is explicitly given by

\[
u(t, x) := (\mathcal{T}(t, s)\varphi)(x) := (k(t, s, \cdot) \ast \varphi)(U(s, t)x + g(t, s)), \quad t > s, \ x \in \mathbb{R}^d, \tag{2.2}
\]

where

\[
k(t, s, x) := \frac{1}{(4\pi)^{d/2}(\det Q_{t, s})^{1/2}} U(t, s)e^{-\frac{1}{4}(Q_{t, s}^{-1})_{x, x}}dy, \quad t > s \geq 0, \ x \in \mathbb{R}^d, \tag{2.3}
\]

\[
g(t, s) := \int_s^t U(s, r) f(r)dr, \quad Q_{t, s} := \int_s^t U(s, r)U^*(s, r)dr, \quad t \geq s \geq 0.
\]

Similar to [DPL07] one can show that for \( \varphi \in C_c^\infty(\mathbb{R}^d)^d \) the solution \( u \) of (2.1) given by (2.2) is a classical solution.

A simple calculation shows that \( \bar{T}(t, s)\varphi = 0 \) for \( \varphi \in C_c^\infty(\mathbb{R}^d)^d \) and \( t \geq s \geq 0 \). Hence, the restriction \( T(t, s) := \bar{T}(t, s)|_{L^p(\mathbb{R}^d)^d} \) is an evolution system on \( L^p(\mathbb{R}^d)^d \). In particular, \( u(t) := (T(t, 0)0) \) is a solution to (1.1).

By similar arguments as in the proofs of [DPL07] Lemma 3.2 or [Han10] Lemma 2.4, for \( T_0 > 0 \) there exists \( C := C(d, M_{T_0}) > 0 \) (see Theorem 1.1 for the definition of \( M_{T_0} \)) such that

\[
\|Q_{t, s}^{-\frac{1}{2}}\| \leq C(t - s)^{-\frac{1}{2}}, \quad 0 \leq s < t < T_0, \tag{2.4}
\]

\[
(\det Q_{t, s})^\frac{1}{2} \geq C(t - s)^{-\frac{1}{2}}, \quad 0 \leq s < t < T_0.
\]

Moreover, if \( M_{T_0} \) is uniformly bounded in \( T_0 \) we may write \( T_0 = \infty \) in (2.4).

**Proof of Theorem 1.1.** We start by showing the estimate (1.8). Let \( T_0 > 0 \). By the change of variables \( \xi = U(s, t)x \) and by Young’s inequality we obtain

\[
\|T(t, s)u\|_{L^p(\mathbb{R}^d)} \leq \|\det U(s, t)\|^\frac{1}{p}\|k(t, s, \cdot)\|_{L^r(\mathbb{R}^d)}\|u\|_{L^q(\mathbb{R}^d)}, \quad t > s \geq 0,
\]

where \( 1 < r < \infty \) with \( \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q} \). Further, by the change of variable \( y = Q_{t, s}^{1/2}z \) we obtain

\[
\|k(t, s, \cdot)\|_{L^r(\mathbb{R}^d)} = \|U(t, s)\| \int_{\mathbb{R}^d} \left( \frac{1}{(4\pi)^\frac{d}{2}} e^{-\frac{|z|^2}{4}} \right)^\frac{r}{2} (\det Q_{t, s})^{\frac{1-r}{2}}dz \\
\leq C \|U(t, s)\| (\det Q_{t, s})^{\frac{1-r}{2}}, \quad t \geq s \geq 0,
\]

for some \( C > 0 \). Now (2.4) yields (1.8).

To prove the gradient estimate (1.9), we first observe that

\[
\nabla T(t, s)u(x) = \int_{\mathbb{R}^d} u(U(s, t)x + g(t, s)k(t, s, y)(U^T(s, t)Q^{-1}_{s,t}y)^T dy, \quad t > s \geq 0, \ x \in \mathbb{R}^d.
\]

Now, (1.9) follows similarly as above.
Since \([2.1]\) is uniquely solvable for \(\varphi \in C_c^\infty(\mathbb{R}^d)^d\), see Section 3 the law of evolution is valid, i.e.

\[
\hat{T}(t,s)\varphi = \hat{T}(t,r)\hat{T}(r,s)\varphi,
\]

holds for \(0 \leq s \leq r \leq t\) and every \(\varphi \in C_c^\infty(\mathbb{R}^d)^d\). The density of \(C_c^\infty(\mathbb{R}^d)^d\) in \(L^p(\mathbb{R}^d)^d\) yields that \(2.5\) even holds for all \(\varphi \in L^p(\mathbb{R}^d)^d\).

In order to prove the strong continuity of the map \((t,s) \mapsto \hat{T}(t,s)\) on \(0 \leq s \leq t\) we apply the change of the variables \(y = Q_{t,s}^{-1/2}z\), to see that

\[
\hat{T}(t,s)\varphi(x) = \frac{1}{(4\pi)^{d/2}} U(t,s) \cdot \int_{\mathbb{R}^d} \varphi(U(s,t)x + g(t,s) - Q_{t,s}^{1/2}z)e^{-|z|^2/4}dz
\]

holds. For \(t > s\) fixed, we pick two sequences \((t_n)_{n \in \mathbb{N}}\) and \((s_n)_{n \in \mathbb{N}}\) such that \(t_n \geq s_n\) holds for every \(n \in \mathbb{N}\) and \((t_n, s_n) \to (t, s)\) as \(n \to \infty\). For every \(\varphi \in C_c^\infty(\mathbb{R}^d)^d\) and every \(x \in \mathbb{R}^d\) we now obtain

\[
\varphi(U(s_n, t_n)x + g(t_n, s_n) - Q_{t_n,s_n}^{1/2}z) \to \varphi(U(s,t)x + g(t,s) - Q_{t,s}^{1/2}z)
\]
as \(n \to \infty\). Lebegue’s theorem now yields \(\hat{T}(t_n, s_n)\varphi \to \hat{T}(t,s)\varphi\) as \(n \to \infty\) for every \(\varphi \in C_c^\infty(\mathbb{R}^d)^d\). The density of \(C_c^\infty(\mathbb{R}^d)^d\) in \(L^p(\mathbb{R}^d)^d\) implies the strong continuity.

In order to prove Theorem 1.1 b) let \(u \in L^p_0(\mathbb{R}^d), t-s \leq 1\) and choose \((u_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^d) \subset L^p_0(\mathbb{R}^d),\) such that \(\lim_{n \to \infty} \|u - u_n\|_{L^p(\mathbb{R}^d)} = 0\). The triangle inequality together with the \(L^p-L^q\) estimates \(1.8\) imply that there exist constants \(C_1, C_2 > 0\) such that

\[
(t-s)^\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\|T(t,s)u\|_{L^q_0(\mathbb{R}^d)} \leq (t-s)^\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\|T(t,s)(u - u_n)\|_{L^q_0(\mathbb{R}^d)} + (t-s)^\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\|T(t,s)u_n\|_{L^q_0(\mathbb{R}^d)} \leq C_1\|u - u_n\|_{L^p_0(\mathbb{R}^d)} + C_2(t-s)^\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|_{L^p_0(\mathbb{R}^d)}, \quad 0 \leq t - s \leq 1, \quad n \in \mathbb{N}.
\]

Hence, \(\lim_{t \to s}(t-s)^\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\|T(t,s)u\|_{L^q_0(\mathbb{R}^d)} = 0\) by letting first \(t \to s\) and then \(n \to \infty\). The second assertion in Theorem 1.1 b) is proved in a similar way.

\[\square\]

3. Representation Formula

In this section the representation formula \(2.2\) is derived. The general idea is to do a coordinate transformation in order to eliminate the unbounded drift and the zero order term of the operator \(A(t)\). For this purpose we set

\[
z := U(s,t)x + g(t,s),
\]

where

\[
g(t,s) := \int_s^t U(s,r)f(r)dr,
\]

and we look for a solution \(u\) of \(2.1\) with initial value \(\varphi \in C^\infty_c(\mathbb{R}^d)^d\) in the form

\[
u(t,x) = U(t,s)w(t,U(s,t)x + g(t,s)).
\]
By recalling (1.7) we obtain from a straightforward computation that
\[ \partial_t u(t, x) = -M(t) U(t, s) w(t, z) + U(t, s) \left( \langle U(s, t) x, \nabla w_i(t, z) \rangle \right) \]
holds. Moreover, we can write equation (3.1) component-wise as
\[ u_i(t, x) = \sum_{j=1}^d U_{ij}(t, s) w_j(t, U(s, t) x + g(t, s)), \quad \text{for } i = 1, \ldots, d, \]
and thus for the spatial derivatives of \( u \) we obtain
\[ \nabla u_i(t, x) = \sum_{j=1}^d U_{ij}(t, s) U^*(s, t) \nabla w_j(t, z), \]
\[ \nabla^2 u_i(t, x) = \sum_{j=1}^d U_{ij}(t, s) U^*(s, t) \nabla^2 w_j(t, z) U(s, t). \]
In particular, the drift term can be written as
\[ \langle M(t)x + f(t), \nabla u_i(t, x) \rangle = \sum_{j=1}^d U_{ij}(t, s) \langle U(s, t) M(t)x + U(s, t) f(t), \nabla w_j(t, z) \rangle. \]
Thus, the function \( u \) solves problem (2.1) if and only if for every \( i = 1, \ldots, d \), the function \( w_i : \mathbb{R}^d \to \mathbb{R} \) is a solution to
\[ \begin{cases} \partial_t w_i(t, z) = \text{Tr}[U(s, t) U^*(s, t) \nabla^2 w_i(t, z)], & t > s, z \in \mathbb{R}^d, \\ w_i(s, z) = \varphi_i(z), & z \in \mathbb{R}^d. \end{cases} \] (3.2)
By our transformation we now obtained an uncoupled system of parabolic equations with
coefficients only depending on \( t \). More precisely, for \( i = 1, \ldots, d \), the equation (3.2) is a
non-autonomous heat equation. It is well known that such a problem can be uniquely solved
(cf. [DPL07, Proposition 2.1]) and that for every \( \varphi_i \in C^\infty_c(\mathbb{R}^d) \) its unique solution is explicitly
given by the formula
\[ w_i(t, z) = \frac{1}{(4\pi)^{d/2} (\det Q_{t,s})^{1/2}} \int_{\mathbb{R}^d} \varphi_i(z - y) e^{-\frac{1}{2}(Q_{t,s}^{-1} y, y)} dy, \] (3.3)
where
\[ Q_{t,s} = \int_{s}^{t} U(s, r) U^*(s, r) dr. \] (3.4)
Now, via (3.1), the unique solution to our original problem (2.1) is given by the representation
formula
\[ u(t, x) = (k(t, s, \cdot) * u)(U(s, t) x + g(t, s)), \] (3.5)
where the kernel \( k(t, s, x) \) is defined in (2.3).
Note that the right hand side of (3.5) is even well defined for each \( L^p(\mathbb{R}^d)^d \)-function \( \varphi \). Thus, this explicit formula can be used to define an evolution system on \( L^p(\mathbb{R}^d)^d \) in the following
way. For $\varphi \in L^p(\mathbb{R}^d)^d$ we set

$$\tilde{T}(t,s)\varphi := \begin{cases} 
\varphi & \text{for } t = s, \\
(k(t,s,x) \ast \varphi)(U(s,t)x + g(t,s)) & \text{for } t > s.
\end{cases}$$

Since problem (3.2) is uniquely solvable it follows via (3.1) that $\tilde{T}(t,s)\varphi$ is the unique solution of (2.1) for initial value $\varphi \in C^\infty_c(\mathbb{R}^d)^d$.

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