THE QUANTIZATION OF THE SPACETIME GEOMETRY GENERATED BY PLANCKIAN ENERGY PARTICLES

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Abstract: We study the quantization of the curved spacetime created by ultrarelativistic particles at Planckian energies. We consider a minisuperspace model based on the classical shock wave metric generated by these particles, and for which the Wheeler - De Witt equation is solved exactly. The wave function of the geometry is a Bessel function whose argument is the classical action. This allows us to describe not only the semiclassical regime ($S \to \infty$), but also the strong quantum regime ($S \to 0$). We analyze the interaction with a scalar field $\phi$ and apply the third quantization formalism to it. The quantum gravity effects make the system to evolve from a highly curved semiclassical geometry (a gravitational wave metric) into a strongly quantum state represented by a weakly curved geometry (essentially flat spacetime).

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1. Introduction

The spacetime geometry created by ultrarelativistic sources, that is, the gravitational shock wave spacetimes, have arisen great interest recently\textsuperscript{[1,2,3,4,5,6,7,8,9,10,11,12]}. These backgrounds are relevant to describe the particle scattering at the Planck energy scale. Quantum scattering of particle fields and strings by this class of metrics have been studied in the approximation in which the geometry is treated classically, i.e., as a background field.

A further step in this direction is to quantize the geometry itself. As it is known, so far there is not a quantum theory of gravitation to fully carry out this program. Although a conventional quantum field theory of gravitation lacks to be renormalizable, information about the quantization of the spacetime geometry can be obtained by solving the Wheeler-De Witt equation\textsuperscript{[13]}:

\[ G_{ijkl} \frac{\delta}{\delta g_{ij}} \frac{\delta}{\delta g_{kl}} + \sqrt{(3)g}^{(3)} R \Psi^{(3)G} = 0. \]  
\[(1)\]

In the canonical description of General Relativity, the space-time metric has the 3 + 1 decomposition

\[ d^2 s = (N^2 - N_i N^i) dt^2 - 2 N_i dx^i dt + g_{ij} dx^i dx^j, \]  
\[(2)\]

that is

\[ g_{\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2} N^i \\ N^{-2} N^i & g^{ij} - N^{-2} N_i N_j \end{pmatrix}. \]  
\[(3)\]

In the region between two space like hypersurfaces \( t = t_i \) and \( t = t_f \), the Einstein equations determine the sequence of three-geometries \( ^{(3)}G \) on the space-like surfaces of constant \( t \), the dynamical important object being the 3-geometry \( ^{(3)}G \). The dynamics of the gravitational field is entirely described by the so called Hamiltonian constraint \( \mathcal{H} = 0 \). In the quantum theory, this becomes an equation for the state vector \( \mathcal{H} \Psi = 0 \), which takes the form of the functional differential equation (1). There are also the others constraints of the classical theory, but at the quantum level, they express merely the gauge invariance on \( \Psi \). Classically, one can know both \( ^{(3)}G(t_0) \) and \( \frac{\partial}{\partial t} ^{(3)}G(t_0) \) at some time parameter \( t_0 \) and determine the 4-geometry \( ^{(4)}G \), but quantum mechanically, one can only know \( ^{(3)}G(t_0) \)
or \( \frac{\partial}{\partial t} \mathcal{G}(t_0) \) and therefore, one has a certain probability for \( (3)\mathcal{G}(t) \). The manifold of all possible \( (3)\mathcal{G} \)- the so called superspace - in which each point is a metric \( g_{ij}(\vec{x}) \), has the metric

\[
G_{ijkl} = \sqrt{g}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) ,
\]

with signature (-, +, +, +, +, +). In order to solve Eq. (1), one considers spacetime symmetries and restricts the degrees of freedom to be quantized (minisuperspace models).

2. Pure Gravity Model

Let us quantize now the gravitational shock wave geometries. The minisuperspace metric to quantize is

\[
ds^2 = -(N^2 - N_u N^u)dv^2 + 2N_u dudv + F(\rho, u)du^2 + dx^2 + dy^2
\]

where \( u = z - t, v = z + t \) are null variables and \( \rho = \sqrt{x^2 + y^2} \). The classical shock wave metrics have the form of Eq. (5) with\ped{10}

\[
F(u, \rho) = f(\rho)\delta(u) ,
\]

and the Lagrange multipliers taken \( N_u = 1, N = 0 \). This expression of the metric is generic for any ultrarelativistic source; its particularities entering only in the form of the function \( f(\rho) \). This expression can be extended to D - dimensions and to shock waves superimposed to curved backgrounds\ped{4,7}. Eq. (5) can also represent a sourceless plane gravitational wave. In such case

\[
F(x, y, u) = (x^2 - y^2)D(u) ,
\]

\( D(u) \) being an arbitrary function of \( u \).

The metric on superspace has the components:

\[
G_{1111} = \frac{1}{2}F^{3/2} ,
\]

\[
G_{1122} = G_{1133} = -\frac{1}{2}F^{1/2} ,
\]

\[
G_{2222} = G_{3333} = -G_{2233} = \frac{1}{2}F^{-1/2} ,
\]
the others components vanish.

The three-curvature \((3)R\) is given by

\[
(3)R = -g^{11}\nabla^2 g_{11} + \frac{1}{2}(g^{11})^{-2}(\nabla_\perp g_{11})^2 = \frac{1}{F}[-\nabla^2_{\perp}F + \frac{1}{2F}(\nabla_\perp F)^2]
\]

where the subscript \(\perp\) refers to the (2, 3) spatial transverse coordinates.

Thus, the Eq. (1) in our case reads,

\[
\left[ -\frac{1}{2}F^{3/2} \frac{\delta^2}{\delta F^2} + F^{1/2} \left( F^{-1}\nabla^2_\perp F - \frac{F^{-2}}{2}(\nabla_\perp F)^2 \right) \right] \Psi(F) = 0 .
\]

Here, for the sake of convenience, we have taken a 3+1 decomposition with respect to the hypersurface \(v = \text{const}\), that will play the role of time. Notice also that the supermomentum constraint is already satisfied classically, since for the metric (5), one has \(R_{vu} = 0\). Thus, we do not need to consider the quantization of this equation\(^{[14]}\).

Let us consider now a minisuperspace model by freezing out all the transversal degrees of freedom and quantizing the longitudinal one, i.e. we write,

\[
F(u, \rho) = f(\rho)D(u) ,
\]

where \(f(\rho)\) is the profile function characterizing the classical shock wave metric\(^{[10]}\). The function \(D(u)\) will represent the degree of freedom to quantize (classically, \(D(u) \equiv \delta(u)\)).

By discretizing the variable \(u \rightarrow u_n\) and making the change of variable \(s^2(u_n) = D(u_n)\) we can transform the functional differential equation (7) into the following ordinary differential equation,

\[
\left[ \frac{d^2}{ds^2} + (2p - 1)s^{-1}\frac{d}{ds} + 4C(\rho) \right] \Psi(s_n) = 0 ,
\]

where \(p\) accounts for the arbitrariness in the operator ordering, and

\[
C(\rho) = -2\left( \nabla^2_\perp f - \frac{f^{-1}}{2}(\nabla_\perp f)^2 \right) = 2f(3)R(\rho).
\]

The whole wave function will read

\[
\Psi(D) = \prod_n \Psi(D(u_n)) .
\]
For $p = 1$ (which corresponds to the Laplacian ordering prescription), Eq. (9) can be brought into a Bessel equation of index $\nu = 0$. We choose the solution that remains finite when $D \to 0$. Thus, our solution reads,

$$
\Psi(F_n) = J_0 \left( 2 \sqrt{C(\rho)D(u_n)} \right) = J_0 \left( 2 \sqrt{2 (3) R F(\rho, u_n)} \right).
$$

(11)

We can obtain the semiclassical limit by considering large arguments of the Bessel function in equation (11). Thus,

$$
\Psi(F) \simeq \left( \pi \sqrt{2 (3) R F(\rho, u_n)} \right)^{-1/2} \cos \left[ 2 \sqrt{2 (3) R F(\rho, u_n)} - \pi/4 \right].
$$

(12)

On the other hand, the semiclassical regime can also be directly studied from the Hamilton - Jacobi equation for our system,

$$
-\frac{1}{2} F^{3/2} \left( \frac{dS}{dF} \right)^2 + F^{1/2} (3) R(\rho) = 0,
$$

where $S$ is the Hamilton - Jacobi principal function that by direct integration reads,

$$
S = \pm 2 \sqrt{2 (3) R F}.
$$

(13)

We see that the argument of the wave function (11) is the action of the classical Hamilton - Jacobi equation. Thus, for large $S$, the Bessel function has an oscillatory behavior and the wave function Eq. (11), is given by

$$
\Psi \simeq \frac{A}{S} \exp\{iS/\hbar\} + \frac{B}{S} \exp\{-iS/\hbar\}.
$$

(14)

that is, Eq. (12) gives the right semiclassical limit.

Semiclassically, the wave function $\Psi$ is picked around the classical metric geometry. It appears natural to interpret the oscillatory behavior Eq. (14) as describing $\Psi$ in the classical allowed regime. The classically forbidden regime, instead, appears with a real exponential behavior (Euclidean signature region).

For small $S$, $(S \ll \hbar)$, the behavior is non - oscillatory. In particular, the Bessel function reaches a finite value in the limit $S \to 0$, i.e. $J_0(0) = 1$. This can be interpreted
as a genuine quantum behavior, as opposite to the large \( S \) sector which describes the semiclassical regime. Notice that \( S \to 0 \), i.e. \( D(u) = 0 \), is flat spacetime and that this appears in the strong quantum regime. For \( F > 0 \) the action is real and Eq. (14) describes \( \Psi \) in the classically allowed regime. For each \( S \), i.e. each \( F \), \( \Psi \) describes a classical configuration. All configurations are classically allowed.

There, \(| \Psi |^2 \) (which to some extent can be seen as proportional to a probability), is a maximum. We will come back to this interpretation by the end of the paper.

For \( (3) R > 0 \) the action is real and Eq. (12) describes the classically allowed regime. If \( (3) R < 0 \) then \( \Psi(S \to \infty) \sim \exp\{S\}/S \). This is an exponential growing of the semiclassical wave function. It is the opposite of what happens in the case of tunnel effect, and like the situation of the falling of a particle into a well potential, indicating the presence of an instability. That is, the classically forbidden regime does not corresponds to tunnel effect, but to an unstable (exponentially growing) behavior.

### 3. Gravity plus matter

We will include a scalar field in the analysis of the wave function. The Wheeler - De Witt equation in this case reads

\[
\left\{ -\frac{\delta^2}{\delta F^2} - \frac{2}{F} (3)^2 R(\rho) + F^{-2} \frac{\delta^2}{\delta \phi^2} + (\phi_{,u})^2 + \frac{1}{F} [(\phi_{,\rho})^2 + m^2 \phi^2] \right\} \Psi(F, \phi) = 0. \tag{15}
\]

here it is understood that the discretization of the variable \( u \) is already made in an analogous way to that of the previous section.

To make an analysis of this equation we use the following decomposition in modes of the scalar field motivated by the solution of the Klein - Gordon equation, i.e. the semiclassical regime

\[
\phi_k = \phi_k^0 \exp\{ -ik_+ \int_{-\infty}^{u} F(\rho, \tilde{u}) d\tilde{u} \} ; \quad \phi_k^0 = e^{-ik_+ \nu} e^{ik_- u} e^{ik \rho},
\]

where \( -k_+ k_- + k^2 = -m^2 \).
Thus, replacing $\phi, u$ and $\phi, \rho$ into Eq. (15) we can write approximately (up to a negligible logarithmic term proportional to $\ln |\phi_k/\phi_k^0|$),

$$\left\{-\frac{\delta^2}{\delta F^2} + F^{-2} \frac{\delta^2}{\delta \phi^2} - U_k(F, \phi)\right\} \Psi(F, \phi) = 0 \ .$$

(16)

where the effective potential $U_k(F, \phi)$ is given by

$$U_k(F, \phi) = \frac{2}{F} \frac{(3) R(\rho)}{F} + \left[k_+^2 - 2k_+k_- F + k_+^2 F^2 + (k^2 - m^2)F^{-1}\right] \phi^2 \ ,$$

(17)

If we use the following transformation of variables

$$X = F \sinh(\phi) \ , \ T = F \cosh(\phi) \ ,$$

this equation can be interpreted as a Klein-Gordon equation for our wave function with a time-dependent potential $U_k(X, T)$. When $U_k < 0$ the wave function $\Psi$ will have a real exponential behavior, this corresponds to a classically forbidden regime, in which geometries have Euclidean signature. For $U_k > 0$, instead, the wave function has an exponentially oscillatory behavior corresponding to a Lorentzian signature (classically allowed) regime.

The analysis of the potential can be done directly in terms of the variables $F$ and $\phi$.

To this end we define:

$$\alpha \equiv -\frac{k_+}{\phi^2 k_-} \left[2 (3) R + (k^2 - m^2)\right] \phi^2 \ , \ \beta \equiv \frac{k_+ F}{k_-} .$$

Thus,

$$U_k = k_+^2 \phi^2 \left[-\frac{\alpha}{\beta} + (1 - \beta)^2\right] .$$

(18)

The zeros of the potential can thus be obtained from the condition

$$\alpha_0 = \beta_0 (1 - \beta_0)^2 \ ,$$

(19)

and the extrema of the potential from

$$\alpha_m = 2\beta_m^2 (1 - \beta_m) .$$

(20)
Let us consider now the potential as a function of $F$ for slices of $\phi = constant$. There are two main cases:

a) $\alpha < 0$

In this case the potential $U_k$ will always be bigger than zero having a minimum at a value $\beta_m > 1$ (solving the third order equation (20)). Thus, we have here the normal oscillatory behavior of the wave function.

b) $\alpha > 0$

Here we have three subcases depending on the value of $\alpha$:

i) $\alpha > 8/27$: The potential grows monotony from $-\infty$ at $\beta = 0$ to $+\infty$ at $\beta \to \infty$ passing through zero at $\beta_0 > 1$.

ii) $4/27 < \alpha < 8/27$: In this case the potential shows a local maximum and minimum, both in the region $U_k < 0$. For $\beta > \beta_0 > 1$, $U_k > 0$, and we have a steady growing. The maximum and minimum are located at $0 < \beta_{max} < 2/3$ and $2/3 < \beta_{min} < 1$ respectively, and they can be obtained by from Eq. (20).

iii) $0 < \alpha < 4/27$: This case presents two regions where $U_k > 0$. These two regions are $\beta_0^1 < \beta < \beta_0^2$, where $0 < \beta_0^1 < 1/3$ and $1/3 < \beta_0^2 < 1$; and $\beta > \beta_0^3 > 1$. The two local maxima and minima are of course located at $\beta_0^1 < \beta_{max} < \beta_0^2$ and $\beta_0^2 < \beta_{min} < \beta_0^3$.

The dependence of $U_k$ on $\phi$ is clearly a growing parabola (see Eq. (17)). The minimum value is reached at the origin $\phi = 0$ and it will eventually cross a zero of the potential iff $(3) R < 0$.

Thus, we have seen that we can classify the two regimes: Euclidean (exponential behavior of the wave function) and Lorentzian (oscillatory behavior) according to the sign of the effective potential $U_k$. A weak potential $U_k$ corresponds to the semiclassical regime. This is precisely what happens when $F$ is large, i.e. $S \to \infty$, namely a strong curvature regime, for instance a state picked around a singular gravitational wave. A strong potential
$U_k$ corresponds to a truly quantum regime in which $S \to 0$, i.e. $F \to 0$, that is, a smooth and weak curvature state.

If we prepare the system in a classical configuration, for instance $|\phi| = 1$ and $F \to \infty$ (that would correspond to the semiclassical problem studied in Ref. 6,10 of a Klein-Gordon field $\phi$ in a classical shock wave geometry), we observe that we fall into the case a), i.e. $U_k > 0$, since $(3)R > 0$. Due to the particle creation (that we will consider in the next section) of the field $\phi$, the system evolves towards the minimum of the potential $U_k$ making $F$ small and $|\phi|$ big, that is, the system evolves towards a quantum regime of smooth and weak curvature.

4. Third Quantization

In this section we study the possibility of third quantizing the wave function of the ultrarelativistic particles. A motivation for the third quantization is to overcome the problem of negative probabilities just as was the case for the Klein-Gordon equation (see for example Ref. 15).

Let us then consider the Eq. (15) for an ultrarelativistic particle and an homogeneous massless scalar field $\phi$, that up to the operator ordering ambiguity reads,

$$\left[ \partial_t^2 - \partial_\phi^2 + 72 \ (3)R \exp\{-6t\} \right] \Psi(t, \phi) = 0,$$

where we have made the change of variables (at each point $u_n$)

$$t_n = -\frac{1}{6} \ln(F_n),$$

which allows the variable $t_n$ to run from $-\infty$ to $+\infty$ and to give to it a time-like interpretation.

The general solution to Eq. (21) can be written as

$$\Psi_j(t, \phi) = \prod_n Z_{2ij} \left( \sqrt{8 \ (3)R \ e^{3t_n}} \right) \exp\{ij\phi\},$$

where $Z_\nu$ is a Bessel function of first or second kind.
It is interesting to note here that the case of pure gravity is recovered for \( j = 0 \). As \( j \) is related to the energy of the matter field \( \phi \), it seems that the presence of matter fields excite the gravitational modes and allow for particle production as we shall immediately see.

Following Ref. 16 we can define \( in \) states proportional to the Bessel function
\[
J_{2ij} \left( \sqrt{8^{(3)}Re^{3t_n}} \right) \quad \text{for} \quad t \to -\infty
\]
(which are natural positive frequency modes in the \( in \) region), and \( out \) states proportional to the Hankel function
\[
H_{2ij}^{(2)} \left( \sqrt{8^{(3)}Re^{3t_n}} \right) \quad \text{for} \quad t \to +\infty.
\]
Then, one can compute the Bogoliubov transformation coefficients between these two basis, and since the Hankel function is a linear combination of positive and negative subindex (frequency) Bessel functions one obtains particle production of the outgoing modes with respect to the \( in \) vacuum. It appears that the spectrum of produced particles has a Planckian distribution at temperature,
\[
T = 18 \sqrt{8\pi G/3},
\]
of the order of the Planck temperature.

Thus, the interpretation of our results can be made in terms of the creation of ultrarelativistic particles carrying with them its own geometry. The interaction with the matter fields deplete the gravitational energy bringing the ingoing semiclassical state, for instance picked around a shock wave metric, (that is, a strongly curved geometry) into a state in the quantum regime represented by a weakly curved geometry (essentially flat spacetime). The evolution of the system is from the classical into the quantum regime. The initial semiclassical configuration is a highly curved geometry, the final configuration is a weakly curved one.

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