QUANTITATIVE QUASISYMMETRIC UNIFORMIZATION OF COMPACT SURFACES

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Abstract. Bonk and Kleiner showed that any metric sphere which is Ahlfors 2-regular and linearly locally contractible is quasisymmetrically equivalent to the standard sphere, in a quantitative way. We extend this result to arbitrary metric compact orientable surfaces.

1. Introduction and statement of results

Through isothermal coordinates, every Riemannian metric on a compact orientable surface $S$ determines a Riemann surface structure on $S$. By the classical uniformization theorem, $S$ carries a conformally equivalent Riemannian metric of constant curvature 1 (in the case of a sphere), 0 (for a torus), or $-1$ (for higher genus surfaces). Hence, the original Riemannian metric on $S$ can be conformally deformed to a metric of constant curvature.

The purpose of this note is to extend the above discussion to certain classes of possibly non-smooth distances on compact orientable surfaces. In this setting we have a metric, but no smooth structure, and the appropriate category replacing the class of conformal mappings is the class of quasisymmetric mappings.

In a metric space $X$ we will denote the distance between $a$ and $b$ by $|a - b|_X$, or $|a - b|$ if the metric space $X$ is clear from the context. Let $X$ and $Y$ be metric spaces. An embedding $\phi: X \to Y$, i.e., a homeomorphism from $X$ onto its image $\phi(X)$, is a quasisymmetric embedding if there is a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that for all triples of distinct points $a$, $b$, and $c$ in $X$,

$$\frac{|\phi(a) - \phi(b)|_Y}{|\phi(a) - \phi(c)|_Y} \leq \eta \left( \frac{|a - b|_X}{|a - c|_X} \right).$$

The homeomorphism $\eta$ is called a distortion function for the mapping $\phi$. If a quasisymmetric embedding $\phi$ has a distortion function $\eta$, it will be called an $\eta$-quasisymmetric embedding. The inverse of an $\eta$-quasisymmetric map is $\eta'$-quasisymmetric with $\eta'(t) = 1/\eta^{-1}(1/t)$. A proof of this fact can be found in the book of Heinonen [Hei01, Proposition 10.6], which also serves as an excellent introduction to the theory of quasisymmetric mappings on metric spaces. A quasisymmetric mapping distorts relative distances by a controlled amount. While it may distort distances, it must do so more-or-less isotropically.

A motivating example arises from Cannon’s Conjecture in geometric group theory. The boundary at infinity of a Gromov hyperbolic group carries a natural family of visual metrics, any two of which are quasisymmetrically equivalent. For this reason, one might say that the metric on the boundary of a Gromov hyperbolic group is defined only up to quasisymmetry. Cannon’s Conjecture can be phrased as follows: If the boundary $\partial_\infty G$ of a Gromov hyperbolic group $G$ is homeomorphic to

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the sphere $\mathbb{S}^2$, then each visual metric on $\partial_\infty G$ is quasisymmetry equivalent to the standard metric on $\mathbb{S}^2$. If Cannon’s conjecture is true, then a Gromov hyperbolic group whose boundary is homeomorphic to $\mathbb{S}^2$ acts discretely and co-compactly by isometries on 3-dimensional hyperbolic space. See [Bon06] and the references therein for a more detailed discussion of this problem.

The following quasisymmetric uniformization theorem of Bonk and Kleiner [BK02] led to significant progress on Cannon’s Conjecture:

**Theorem 1** (Bonk-Kleiner). Let $(X, d)$ be an Ahlfors 2-regular metric space that is homeomorphic to $\mathbb{S}^2$. Then $(X, d)$ is quasisymmetrically equivalent to $\mathbb{S}^2$ equipped with the smooth metric of constant curvature 1 if and only if $(X, d)$ is linearly locally contractible.

A metric space $X$ is **linearly locally contractible**, if there is a constant $\Lambda \geq 1$ such that for each point $x \in X$ and radius $0 < r < (\text{diam } X)/\Lambda$, the ball $B(x, r)$ is contractible inside the dilated ball $B(x, \Lambda r)$. The metric space $X$ is **Ahlfors 2-regular** if there is a constant $K \geq 1$ such that for each $x \in X$ and radius $0 < r < 2 \text{diam } X$, the two-dimensional Hausdorff measure $\mathcal{H}^2$ satisfies

$$\frac{r^2}{K} \leq \mathcal{H}^2(B(x, r)) \leq Kr^2.$$

Theorem 1 is quantitative in the sense that the quasisymmetric map can be chosen to have a distortion depending only on the data of $X$, i.e., the constants $\Lambda$ and $K$ appearing in the linear local contractibility and Ahlfors 2-regularity conditions.

The linear local contractibility condition is a quantitative quasisymmetric invariant, i.e., if $f : X \to Y$ is $\eta$-quasisymmetric and $X$ is linearly locally contractible with constant $\Lambda$, then $Y$ is linearly locally contractible with a constant that depends only on $\Lambda$ and $\eta$. In contrast, Ahlfors 2-regularity is not a quasisymmetric invariant.

Given a Gromov-hyperbolic group $G$ with $\partial_\infty G$ homeomorphic to $\mathbb{S}^2$, each visual metric on $\partial_\infty G$ is linearly locally contractible. However, it is not known whether there always is an Ahlfors 2-regular visual metric on $\partial_\infty G$. Simple examples show that Theorem 1 fails without the assumption of Ahlfors 2-regularity.

Problems outside of geometric group theory, such as the search for an intrinsic characterization of $\mathbb{R}^2$ up to bi-Lipschitz equivalence (note that Ahlfors 2-regularity is a bi-Lipschitz invariant), led to the development of versions of Theorem 1 for non-compact simply connected surfaces [Wil08] and for a large class of planar domains [MW13], as well as a local version [Wil10]. In this work, we provide a version of Theorem 1 that applies to all orientable compact surfaces, including those of higher genus. This is part of an ongoing program to complete the classification of Ahlfors 2-regular and linearly locally contractible metric surfaces up to quasisymmetry.

**Theorem 2.** Let $(X, d)$ be a metric compact orientable surface. Assume that $(X, d)$ is Ahlfors 2-regular and linearly locally contractible. Then there exists a Riemannian metric $\hat{d}$ of constant curvature 1, 0, or $-1$ on $X$ such that the identity map $\text{id} : (X, d) \to (X, \hat{d})$ is quasisymmetric with a distortion function depending only on the data of $X$.

Here the data of $X$ consists of the topological genus of $X$, and the constants in the Ahlfors 2-regularity and linear local contractibility conditions.
Except for the statement of dependence of the distortion function only on the data of \( X \), Theorem 2 follows immediately from the local uniformization theorem of [Wil10] and an elementary local-to-global result for quasisymmetric mappings due to Tukia and Väisälä [TV80, Theorem 2.23]. While a quantitative proof is significantly more involved, it provides much more information about the “space” of quasisymmetric structures on surfaces. For example, let us suppose that \( (X, d) \) satisfies the assumptions of Theorem 2 and is homeomorphic to a torus. All smooth Riemannian metrics of constant curvature 0 on the torus are quasisymmetrically equivalent, but not with a uniform distortion function. In essence, the quantitative Theorem 2 allows us to assign to \( (X, d) \) a point in the Riemann moduli space of the torus that is “roughly optimal”. Finding a smooth metric on the torus that is quasisymmetrically equivalent to \( (X, d) \) with “minimal” quasisymmetric distortion seems to be both a difficult and ill-defined question, but one of obvious interest.

Our proof of Theorem 2 proceeds as follows. Let \( (X, d) \) be a metric surface as given in the theorem. By the local uniformization result of [Wil10], \( (X, d) \) has an atlas of uniformly quasisymmetric mappings. Our first step is to create a compatible conformal atlas, giving \( (X, d) \) the structure of a Riemann surface. This step can be thought of as creating “quasi-isothermal” coordinates; while the chart transitions are conformal, the chart mappings themselves are only quasisymmetries. The classical uniformization theorem then provides a globally defined conformal homeomorphism \( F: X \to Y \) to a Riemann surface that is the quotient of the sphere, plane, or disk by an appropriate group of Möbius transformations. The Riemann surface \( Y \) inherits a Riemannian metric \( d_Y \) of constant curvature 1, 0, or \(-1\), respectively. Although conformal maps are locally quasisymmetric, this does not immediately give us global information about the metric properties of the uniformizing map \( F: (X, d) \to (Y, d_Y) \). We show that the uniformizing mapping \( F \) is in fact globally quasisymmetric with a distortion function that depends only on the data of \( X \). The key idea in this step is a type of Harnack inequality. The claim in the theorem then follows by taking \( \hat{d} \) to be the pull-back of the metric \( d_Y \) under \( F \).

2. Background and preliminary results

2.1. Notation. In a metric space \( X \), we will denote the distance between points \( x \) and \( y \) of \( X \) by \( |x - y|_X \) or \( |x - y| \) when the space \( X \) is understood. We denote by

\[
B_X(x, r) = B(x, r) = \{y : |x - y| < r\}
\]

the open ball of radius \( r \) centered at \( x \) and by \( \overline{B}(x, r) \) the corresponding closed ball. For an open or closed ball \( B \) of radius \( r \) and a number \( \lambda > 0 \), the notation \( \lambda B \) denotes the same type of ball with the same center and radius \( \lambda r \).

We denote the complex plane with the standard Euclidean metric by \( \mathbb{C} \), and the disk model of hyperbolic space equipped with the standard hyperbolic metric of curvature \(-1\) by \( \mathbb{D} \). The open ball of radius \( \alpha > 0 \) centered at 0 is denoted by \( \mathbb{C}_\alpha \) in \( \mathbb{C} \) and by \( \mathbb{D}_\alpha \) in \( \mathbb{D} \), i.e.,

\[
\mathbb{C}_\alpha = B_\mathbb{C}(0, \alpha), \quad \mathbb{D}_\alpha = B_\mathbb{D}(0, \alpha).
\]

Note that \( \alpha \) is the Euclidean radius for \( \mathbb{C}_\alpha \), whereas it is the hyperbolic radius for \( \mathbb{D}_\alpha \).
2.2. Quasisymmetric mappings. Let \( \phi: X \to Y \) be an embedding of metric spaces. For \( x \in X \) and \( 0 < r < \frac{1}{2} \text{diam} \ X \), define

\[
L_\phi(x, r) := \sup\{|\phi(x) - \phi(y)|_X : |x - y|_X \leq r\}, \quad \text{and} \quad l_\phi(x, r) := \inf\{|\phi(x) - \phi(y)|_Y : |x - y|_X \geq r\}.
\]

If \( \phi \) is an \( \eta \)-quasisymmetric embedding, then for \( x \in X \) and \( 0 < r_1, r_2 < \frac{1}{2} \text{diam} \ X \),

\[
\frac{L_\phi(x, r_1)}{l_\phi(x, r_2)} \leq \eta \left( \frac{r_1}{r_2} \right).
\]

We will need a statement about the equicontinuity of quasisymmetric maps, which is a slightly more general version of [Hei01 Corollary 10.27].

**Lemma 3.** Let \( \phi: U \to V \) be an \( \eta \)-quasisymmetric embedding of metric spaces and let \( D_U \) and \( D_V \) be positive real numbers. If \( \text{diam} \ U \geq D_U \) and \( \text{diam} \ V \leq D_V \), then

\[
\omega(t) = \max \left\{ \frac{3D_V t}{D_U}, D_V \eta \left( \frac{3t}{D_U} \right) \right\}
\]

is a modulus of continuity for \( \phi \).

**Proof.** Given \( x_1, x_2 \in U \) with \( 0 < |x_1 - x_2| < D_U/3 \), there exists \( x_3 \in U \) such that \( |x_1 - x_3| \geq D_U/3 \). Then

\[
|\phi(x_1) - \phi(x_2)| \leq |\phi(x_1) - \phi(x_3)| \eta \left( \frac{|x_1 - x_2|}{|x_1 - x_3|} \right) \leq D_V \eta \left( \frac{3|x_1 - x_2|}{D_U} \right).
\]

If \( |x_1 - x_2| \geq D_U/3 \), the trivial estimate \( |\phi(x_1) - \phi(x_2)| \leq D_V \) yields the desired estimate.

Note that the modulus of continuity provided by Lemma 3 is not scale-invariant. If the metrics on \( U \) and \( V \) are scaled by the same quantity, the resulting modulus of continuity may still change; see subsection 2.4.

The following result, which is a slight variation of [TV80 Theorem 2.23], gives a local-to-global result for quasisymmetric homeomorphisms between compact spaces in terms of Lebesgue numbers. We include a proof for the reader’s convenience. Recall that \( L > 0 \) is a Lebesgue number for a covering \( \{X_j\} \) if for every set \( E \) with \( \text{diam} \ E \leq L \) there exists \( j \) such that \( E \subseteq X_j \).

**Theorem 4** (Tukia-Väisälä). Let \( F: X \to Y \) be a homeomorphism between compact connected metric spaces \( X \) and \( Y \). Suppose that

- \( \{X_j\}_{j=1}^n \) is a finite open covering of \( X \) with Lebesgue number \( L_X < \text{diam} \ X \).
- \( \delta > 0 \) satisfies the implication
  \[
  |x - x'| = L_X/2 \implies |F(x) - F(x')| \geq \delta,
  \]
- there is a quasisymmetric distortion function \( \eta \) such that for each \( j = 1, \ldots, n \), the restricted mapping \( F|_{X_j} \) is \( \eta \)-quasisymmetric.

Then \( F \) is quasisymmetric with distortion function depending only on \( \eta \) and the ratios \( \text{diam}(X)/L_X \) and \( \text{diam}(Y)/\delta \).
Proof. Let \( a, b, \) and \( c \) be distinct points of \( X \). Set
\[
\rho = \frac{|a-b|}{|a-c|}, \quad \text{and} \quad \rho' = \frac{|F(a) - F(b)|}{|F(a) - F(c)|}.
\]
We consider four cases.

1. Suppose that \( \max\{|a-b|, |a-c|\} \leq L_X/2 \). Then \( \{a,b,c\} \) is contained in some \( X_j \), and so \( \rho' \leq \eta(\rho) \).

2. Suppose that \( |a-b| \leq L_X/2 \) but \( |a-c| > L_X/2 \). Since \( X \) is connected with \( \text{diam} X > L_X \), there exists \( x \in X \) with \( |a-x| = L_X/2 \). Then there exists \( j \) such that \( \{a,b,x\} \subseteq X_j \) and
\[
|F(a) - F(x)| \geq \delta \quad \text{and} \quad \frac{|a-b|}{|a-x|} \leq \frac{2 \text{diam} X}{L_X}.
\]
This implies that
\[
\rho' = \frac{|F(a) - F(b)|}{|F(a) - F(x)|} \leq \frac{|F(a) - F(x)|}{|F(a) - F(c)|} \leq \eta \left( \frac{2 \text{diam} X}{L_X} \right) \frac{\text{diam} Y}{\delta}.
\]

3. Suppose that \( |a-b| > L_X/2 \) but \( |a-c| \leq L_X/2 \). By the same argument as in the previous case, there exists \( x \in X \) with \( |a-x| = L_X/2 \), as well as an index \( j \) such that \( \{a,c,x\} \subseteq X_j \). This gives
\[
|F(a) - F(x)| \geq \delta \quad \text{and} \quad \frac{|a-x|}{|a-c|} \leq \frac{2 \text{diam} X}{L_X}.
\]
which implies that
\[
\rho' = \frac{|F(a) - F(b)|}{|F(a) - F(x)|} \leq \frac{\text{diam} Y}{\delta} \eta \left( \frac{2 \text{diam} X}{L_X} \right).
\]

4. Suppose that \( \min\{|a-b|, |a-c|\} > L_X/2 \). Then \( \rho \geq \frac{L_X}{2 \text{diam} X} \) and \( \rho' \leq \frac{\text{diam} Y}{\delta} \), so
\[
\rho' \leq 2 \left( \frac{\text{diam} X}{L_X} \right) \left( \frac{\text{diam} Y}{\delta} \right) \rho.
\]
Combining these estimates on \( \rho' \) in terms of \( \rho \) yields the desired result. \( \square \)

2.3. Conformality and quasisymmetry. A conformal mapping \( f: U \to V \) between domains in \( \mathbb{C} \) is quasisymmetric when restricted to a relatively compact subdomain \( U' \subset U \), with quasisymmetric distortion depending only on \( U \) and \( U' \). This fact, which should be compared to Koebe's distortion theorem, will play a key role in the proof of Theorem 12. A more general statement is even true: one may consider quasiconformal mappings in higher dimensional Euclidean spaces, although the quasisymmetric distortion then also depends on the maximal quasiconformal dilatation. See [Väi81, Theorem 2.4] for a proof of this fact, which we will use in the following form:

**Proposition 5.** Let \( G: \mathbb{C}_1 \to \mathbb{C} \) be a conformal embedding. For each \( \beta \in (0,1) \), the restriction \( G: \mathbb{C}_\beta \to \mathbb{C} \) is a quasisymmetric embedding with distortion function that depends only on \( \beta \).

An easy consequence of Proposition 5 is the following.

**Corollary 6.** There is a universal constant \( 0 < \beta_1 < 1/2 \) such that for any conformal embedding \( G: \mathbb{C}_1 \to \mathbb{C} \),
\[
G(\mathbb{C}_{\beta_1}) \subseteq B_{\mathbb{C}} \left( G(0), \frac{l_G(0,1/2)}{6} \right).
\]
Proof. By Proposition 5 there is a universal quasisymmetric distortion function $\eta$ for $G: \mathbb{C}_{1/2} \to \mathbb{C}$. Hence, if $\beta \in (0, 1/2)$, then
\[ L_G(0, \beta) \leq \eta(2\beta) l_G(0, 1/2) \]
Thus, choosing $0 < \beta_1 < 1/2$ so small that $\eta(2\beta_1) < 1/6$ fulfills the requirements of the statement.

We will also need a hyperbolic version of this result. Recall that the hyperbolic disk $\mathbb{D}$ is equipped not just with a conformal structure, but also with the standard hyperbolic metric. Our proof employs Koebe’s distortion theorem for specificity, but could also be carried out using Proposition 5 alone.

**Proposition 7.** Let $G: \mathbb{C}_1 \to \mathbb{D}$ be a conformal embedding of the Euclidean unit disk into the hyperbolic plane. Then the restriction $G: \mathbb{C}_{1/10} \to \mathbb{D}$ is a quasisymmetric embedding with a universal distortion function.

**Proof.** The map $f(z) = \frac{G(z) - G(0)}{G'(0)} = z + O(z^2)$ is conformal in $\mathbb{C}_1$, so by Koebe’s distortion theorem [Dur83], the image $f(\mathbb{C}_1)$ contains the Euclidean disk $\mathbb{C}_{1/4}$, and the image of the Euclidean disk $\mathbb{C}_{1/10}$ is contained in the Euclidean disk $\mathbb{C}_{10/81} \subseteq \mathbb{C}_{1/8}$. This implies
\[ B_{\mathbb{C}} \left( G(0), \frac{|G'(0)|}{4} \right) \subseteq G(\mathbb{C}_1) \subseteq \mathbb{D}, \]
and
\[ G \left( \mathbb{C}_{1/10} \right) \subseteq B_{\mathbb{C}} \left( G(0), \frac{|G'(0)|}{8} \right) \subseteq B_{\mathbb{C}} \left( G(0), \frac{1 - |G(0)|}{2} \right). \]
It follows from the definition of the hyperbolic metric that the identity mapping
\[ \text{id}: B_{\mathbb{C}} \left( G(0), \frac{1 - |G(0)|}{2} \right) \to \mathbb{D} \]
is, up to scaling, a bi-Lipschitz embedding with universal constants, so it is quasisymmetric with universal distortion. Since the composition of quasisymmetric maps is quasisymmetric, quantitatively, the result follows from Proposition 5.

The following statement is an easy corollary of Proposition 7.

**Corollary 8.** There is a universal constant $0 < \beta_2 < 1/10$ such that for any conformal embedding $G: \mathbb{C}_1 \to \mathbb{D}$,
\[ G(\mathbb{C}_{\beta_2}) \subseteq B_{\mathbb{D}} \left( G(0), \frac{l_G(0, 1/2)}{6} \right). \]

**Proof.** By Proposition 7 there is a universal quasisymmetric distortion function $\eta$ for $G: \mathbb{C}(1/10) \to \mathbb{D}$. Hence, if $\beta \in (0, 1/10)$, then
\[ L_G(0, \beta) \leq \eta(10\beta) l_G(0, 1/10) \leq \eta(10\beta) l_G(0, 1/2). \]
Choosing $\beta_2$ so small that $\eta(10\beta_2) < 1/6$, we get the claim of the corollary.

For the remainder of the paper, for convenience we define $\beta = \min\{\beta_1, \beta_2\}$. 
2.4. The data of $X$, scalings, and normalization. Let $X$ be a metric space that is Ahlfors 2-regular with constant $K$, linearly locally contractible with constant $\Lambda$, and homeomorphic to a compact orientable surface. We will refer to $\Lambda$ and $K$ as the data of $X$.

For $\lambda > 0$, we may form a new metric space $X_\lambda$ by multiplying the original metric on $X$ by $\lambda$. Then $X_\lambda$ is again Ahlfors 2-regular and linearly locally contractible, and $X_\lambda$ has the same data as $X$. Moreover, the identity mapping from $X$ to $X_\lambda$ is $\eta$-quasisymmetric with $\eta(t) = t$. Hence, in the proof of Theorem 2, we may scale the domain as we see fit.

For the remainder of this article, let $X$ be a metric space that is Ahlfors 2-regular with constant $K$, linearly locally contractible with constant $\Lambda$, homeomorphic to a compact orientable surface, and has unit diameter. When we state that a quantity depends only the data of $X$, we are implicitly assuming this normalization.

The main reason for this convention (aside from notational convenience) is that Lemma 3 is not scale-invariant. Without this normalization, we would not be able to say that certain moduli of continuity depend only on the data of $X$.

3. Finding a conformal structure

By [Wil10] Theorem 4.1], $X$ possesses a quasisymmetric atlas:

**Theorem 9.** There is a quantity $A_0 \geq 1$ and a quasisymmetric distortion function $\eta$, each depending only on the data of $X$, such that for each $0 < R \leq 1/A_0$ there is a neighborhood $U$ of $x$ such that

1. $B(x_0, R/A_0) \subseteq U \subseteq B(x_0, A_0 R)$,
2. there exists an $\eta$-quasisymmetric map $\phi : U \to \mathbb{C}_1$ with $\phi(x_0) = 0$,

The original theorem in [Wil10] does not include the normalization $\phi(x_0) = 0$. However, since $\phi : U \to \mathbb{C}_1$ is $\eta$-quasisymmetric, the basic distortion estimates [Hei01 Proposition 10.8] imply that $\phi(x_0) \in \mathbb{C}_\alpha$ where $\alpha < 1$ depends only on $\eta$. As the Möbius transformation $T(z) = \frac{z - \phi(x_0)}{1 - \phi(x_0) \bar{z}}$ has quasisymmetric distortion depending only $|\phi(x_0)|$, we may assume that $\phi(x_0) = 0$. Accordingly, given a pair $(U, \phi)$ where $\phi : U \to \mathbb{C}_1$ is a homeomorphism, we will call $\phi^{-1}(0)$ the center of $(U, \phi)$.

We use the atlas provided by Theorem 9 to produce a conformal atlas on $X$ that is adapted to its metric. We have separated the construction into two lemmas. The first is purely metric; the second modifies the output of the first.

**Lemma 10.** Let $\rho \in (0, 1)$ be given. Then there exists a quasisymmetric distortion function $\eta$, radii $\alpha$, $r_0 > 0$, and a positive integer $n \in \mathbb{N}$, such that the following statements hold:

1. There exists an atlas $A_\rho = \{(U_j, \mathbb{C}_1, \phi_j) \mid j = 1, \ldots, n\}$ of $X$, where each mapping $\phi_j : U_j \to \mathbb{C}_1$ is an $\eta$-quasisymmetric homeomorphism with center denoted by $x_j$.
2. The collection $\{B(x_j, r_0)\}_{j=1}^n$ is pairwise disjoint.
3. The collection $\{B(x_j, 2r_0)\}_{j=1}^n$ covers $X$.
4. For each $j = 1, \ldots, n$, it holds that $B(x_j, 10r_0) \subseteq U_j$, and $\mathbb{C}_\alpha \subseteq \phi_j(B(x_j, r_0)) \subseteq \phi_j(B(x_j, 10r_0)) \subseteq \mathbb{C}_\rho$. 


Moreover, \( \eta \) depends only on the data of \( X \), while \( \alpha \), \( r_0 \), and \( n \) depend only on the data of \( X \) and \( \rho \).

Note that the collection \( \{ B(x_j, 10r_0) \}_{j=1}^n \) forms an open cover of \( X \) for which \( 8r_0 \) is a Lebesgue number; cf. Theorem 6. Moreover, Lemma 8 implies that for each \( j = 1, \ldots, n \), the restriction \( \phi_j|_{B(x_j, 10r_0)} \) and its inverse have moduli of continuity that depend only on \( \rho \) and the data of \( X \).

**Proof.** For each \( x \in X \), let \( \phi_x : U_x \rightarrow \mathbb{C}_1 \) be the \( \eta \)-quasisymmetric mapping provided by Theorem 9 with \( R = 1/A_0 \), so that

\[
B(x, 1/A_0^2) \subseteq U_x \subseteq B(x, 1/A_0).
\]

Applying Lemma 8 to \( \phi_x \) and its inverse, we see that there is a common modulus of continuity \( \omega \) for all of the mappings \( \{ \phi_x, \phi_x^{-1} \} \)\( x \in X \), depending only on the data of \( X \).

Hence, there is a radius \( 0 < r_0 < 1/10A_0^2 \) and a number \( 0 < \alpha < \rho \), each depending only on \( \rho \) and the data of \( X \), such that for each \( x \in X \),

\[
\phi_x(B(x, 10r_0)) \subseteq \mathbb{C}_\rho, \quad \text{and} \quad \phi_x(B(x, r_0)) \supseteq \mathbb{C}_\alpha.
\]

Let \( \{x_1, \ldots, x_n\} \) be a maximal \( 2r_0 \)-separated set in \( X \). Then the open balls \( \{ B(x_j, r_0) \}_{j=1}^n \) are pairwise disjoint, while the open balls \( \{ B(x_j, 2r_0) \}_{j=1}^n \) cover \( X \). Since \( X \) is Ahlfors 2-regular and we have assumed that \( \text{diam } X = 1 \), this implies that \( n \) is comparable to \( r_0^{-2} \) and therefore depends only on \( \rho \) and the data. Moreover, as \( 10r_0 < 1/(A_0)^2 \), for each \( j = 1, \ldots, n \), it holds that \( B(x_j, 10r_0) \subseteq U_{x_j} \). In particular \( \{ U_{x_j} \}_{j=1}^n \) is also a cover of \( X \). This shows that \( \{(U_{x_j}, \mathbb{C}_1, \phi_{x_j})\}_{j=1}^n \) is the desired atlas.

We now adapt the atlas given in Lemma 10 so that the transition mappings are conformal. This step is similar to the proof that a quasiconformal structure on a surface has a compatible conformal structure; see [Knu67] and [Can69].

We recall that \( \beta \in (0, 1) \) is a universal constant given by Corollaries 6 and 8.

**Lemma 11.** There exists a quasisymmetric distortion function \( \eta \), radii \( \alpha \), \( r_0 > 0 \), and a positive integer \( n \in \mathbb{N} \), all depending only on the data of \( X \), such that the following statements hold:

1. There exists an atlas \( \mathcal{B} = \{(U_j, \mathbb{C}_1, \psi_j)\}_{j=1}^n \) of \( X \), where each mapping \( \psi_j : U_j \rightarrow \mathbb{C}_1 \) is an \( \eta \)-quasisymmetric homeomorphism with center denoted by \( x_j \).
2. The collection \( \{ B(x_j, r_0) \}_{j=1}^n \) is pairwise disjoint.
3. The collection \( \{ B(x_j, 2r_0) \}_{j=1}^n \) covers \( X \).
4. For each \( j = 1, \ldots, n \), it holds that \( B(x_j, 10r_0) \subseteq U_j \), and

\[
\mathbb{C}_\alpha \subseteq \psi_j(B(x_j, r_0)) \subseteq \psi_j(B(x_j, 10r_0)) \subseteq \mathbb{C}_\beta.
\]
5. The transition maps \( \psi_j \circ \psi_k^{-1} \) are conformal wherever defined.

**Proof.** We begin by letting \( 0 < \rho < 1 \) be a number which will be determined below and will depend only on the data of \( X \). Consider the atlas \( \mathcal{A}_\rho \) provided by Lemma 10. Let us say that this atlas is given by \( \eta' \)-quasisymmetric charts \( \{ \phi_j : U_j \rightarrow \mathbb{C}_1 \}_{j=1}^n \) where \( \eta' \) depends only on the data of \( X \). The associated radii \( \alpha', r_0' \) and the number of charts \( n \) depend only on the data of \( X \) and \( \rho \).

Since \( X \) is connected, the charts can be relabeled to satisfy

\[
U_{j+1} \cap (U_1 \cup \ldots \cup U_j) \neq \emptyset
\]
for \( j = 1, \ldots, n - 1 \).

Define \( \psi_1 : U_1 \to \mathbb{C}_1 \) by setting \( \psi_1 = \phi_1 \). We will iteratively construct \( \psi_j : U_j \to \mathbb{C}_1 \) for \( j = 2, \ldots, n \) as follows. For \( k < j \), we assume that the already constructed map \( \psi_k \) is quasisymmetric on \( U_k \), has center \( x_k \), and that if \( k, k' < j \), then the transition functions \( \psi_k \circ \psi_k^{-1} \) are conformal where defined.

In the following, we will write \( D_{a,b} := \phi_a(U_a \cap U_b) \) for indices \( a > b \) in \( \{1, \ldots, n\} \).

For \( k < j \), define \( T_{j,k} = \psi_k \circ \phi_j^{-1} \). Then \( T_{j,k} \) is quasisymmetric and hence quasiconformal on \( D_{j,k} \). Therefore, the complex dilatation \( \mu_{j,k} \) of \( T_{j,k} \) is well-defined (up to a.e. equivalence) on \( D_{j,k} \). Given another index \( k' < j \), it holds that \( T_{j,k} = (\psi_k \circ \phi_k^{-1}) \circ T_{j,k'} \) on \( D_{j,k} \cap D_{j,k'} \). It therefore follows from the conformality of \( \psi_k \circ \phi_k^{-1} \) that \( \mu_{j,k} = \mu_{j,k'} \) a.e. on the intersection \( D_{j,k} \cap D_{j,k'} \).

This shows that there exists a measurable Beltrami coefficient \( \mu_j : \mathbb{C}_1 \to \mathbb{C}_1 \) with \( \mu_j = \mu_{j,k} \) a.e. on \( D_{j,k} \) for each index \( 1 \leq k < j \), and \( \mu_j = 0 \) on \( \mathbb{C}_1 \setminus \bigcup_{k=1}^{j-1} D_{j,k} \). We extend \( \mu_j \) to the whole plane by \( \mu_j(z) = (z/\overline{z})^2 \mu_j(1/\overline{z}) \) for \( |z| > 1 \). By the Measurable Riemann Mapping Theorem there exists a unique quasiconformal map \( h_j : \mathbb{C} \to \mathbb{C} \), normalized by \( h_j(0) = 0, h_j(1) = 1 \), with complex dilatation \( \mu_{h_j} = \mu_j \) a.e. By the symmetry of \( \mu_j \) and the chosen normalization, \( h_j(1/\overline{z}) = 1/h_j(\overline{z}) \), and so \( h_j(\mathbb{C}_1) = \mathbb{C}_1 \). We define \( \psi_j = h_j \circ \phi_j \). The transformation formula for Beltrami coefficients (see, e.g., [LV73 IV.5.1]) shows that if \( k < j \), then \( \psi_k \circ \phi_j^{-1} = T_{j,k} \circ h_j^{-1} \) is conformal. Moreover, \( \psi_j(x_j) = 0 \).

We claim that for each \( j = 1, \ldots, n \), the mapping \( \psi_j \) has a quasisymmetric distortion function that depends only on the data of \( X \); as this is true of \( \phi_j \), it suffices to prove the same of \( h_j \), and we may also assume that \( j > 1 \). As a normalized quasiconformal self-map of the unit disk, \( h_j \) is quasisymmetric with distortion controlled by the maximal dilatation \( \|\mu_j\|_\infty \), see e.g. [Väi81 Theorem 2.4]. By an inductive argument, it is easy to show that this dilatation is bounded by a constant depending only on \( \eta' \) (which depends only on the data of \( X \)) and the number of charts \( n \) (which depends on the data and \( \rho \)). However, the bound is actually independent of \( \rho \), by the following argument.

Fix \( j > 1 \). By the uniform quasisymmetry of \( \{\phi_k\}_{k=1}^n \), there is a quantity \( 0 \leq \kappa < 1 \), depending only on the data of \( X \), such that for all indices \( k < j \) with \( D_{j,k} \neq \emptyset \), the complex dilatation \( \mu_{\phi_k \circ \phi_j^{-1}} \) of \( \phi_k \circ \phi_j^{-1} : D_{j,k} \to \phi_k(D_{j,k}) \) satisfies \( \|\mu_{\phi_k \circ \phi_j^{-1}}\|_\infty \leq \kappa \). For \( k \in \{1, \ldots, j-1\} \), define

\[
F_{j,k} = D_{j,k} \setminus \bigcup_{l=1}^{k-1} D_{j,l}, \quad F_k = \mathbb{C}_1 \setminus \bigcup_{l=1}^{k-1} D_{k,l}, \text{ and } F_j = \mathbb{C}_1 \setminus \bigcup_{k=1}^{j-1} D_{j,k}.
\]

If \( z \in F_j \), then \( \mu_j(z) = 0 \). If \( z \in F_{j,k} \), then \( \phi_k \circ \phi_j^{-1}(z) \in F_k \), and hence

\[
\mu_k|_{\phi_k \circ \phi_j^{-1}(F_{j,k})} = 0.
\]

The transformation formula for Beltrami coefficients now shows that for almost-every point \( z \in F_{j,k} \),

\[
\mu_j(z) = \mu_{j,k}(z) = \mu_{\psi_k \circ \phi_j^{-1}}(z) = \mu_{h_k \circ \phi_k \circ \phi_j^{-1}}(z) = \mu_{\phi_k \circ \phi_j^{-1}}(z).
\]

Since \( \mathbb{C}_1 = \bigcup_{k=1}^{j-1} F_{j,k} \cup F_j \), we see that \( \|\mu_j\|_\infty \leq \kappa \). As discussed above, we may now conclude that each of the mappings \( \{\psi_j\}_{j=1}^n \) has a quasisymmetric distortion function that depends only on the data of \( X \).
We have now seen the atlas \( \mathcal{B} := \{(U_j, \mathbb{C}, \psi_j)|j = 1, \ldots, n\} \) of \( X \) satisfies conditions (1) and (5) of the statement. Moreover, setting \( r_0 := r_0' \), the conditions (2) and (3) follow directly from the corresponding statements for the atlas \( \mathcal{A}_{X, \rho} \). Note that only condition (4) involves the constant \( \beta \). We now show how to choose \( \rho \) so that condition (4) is satisfied.

Let us make the convention that \( h_1: \mathbb{C}_1 \to \mathbb{C}_1 \) is the identity. As discussed above, the mappings \( \{h_j\}^n_{j=1} \) are uniformly quasisymmetric with a distortion function depending only on the data of \( X \). Hence, by Lemma 3 there is a common modulus of continuity for all of the mappings \( \{h_j, h_j^{-1}\}^n_{j=1} \) that depends only on the data of \( X \). Since \( \beta \) is a universal constant, we may choose \( \rho > 0 \) depending only on the data of \( X \) such that for each \( j = 1, \ldots, n \), it holds that \( h_j(\mathbb{C}_\rho) \subseteq \mathbb{C}_\beta \). Having so chosen \( \rho \), the radius \( \alpha' \) depends only on the data of \( X \), and so we may also choose \( \alpha > 0 \) depending only on the data of \( X \) such that for for each \( j = 1, \ldots, n \), \( h_j(\mathbb{C}_{\alpha'}) \supseteq \mathbb{C}_\alpha \). This establishes condition (4). \( \Box \)

3.1. Uniformizing to a standard metric. The atlas \( \mathcal{B} \) given by Lemma 11 determines a conformal structure on the compact orientable surface \( X \), i.e., the pair \((X, \mathcal{B})\) determines a Riemann surface. By the classical uniformization theorem, \((X, \mathcal{B})\) is conformally equivalent to a standard Riemann surface \( Y = U/\Gamma \), where \( U \) denotes the standard Riemann surface structure on the sphere, the plane, or the unit disk, and \( \Gamma \) is a discrete group of Möbius transformations acting freely and properly discontinuously on \( U \). The standard spherical, plane, or hyperbolic Riemannian metric on \( U \) then descends to a Riemannian metric of constant curvature +1, 0, or −1 on \( Y \), compatible with the conformal structure. We fix a uniformizing conformal homeomorphism \( F: (X, \mathcal{B}) \to Y \), and equip \( Y \) with the distance function \( d_Y \) arising from the Riemannian metric of constant curvature. Denote by \( \pi \) the quotient mapping from \( U \) to \( Y \). Recall that \( d_Y \) may be expressed as \( d_Y(p, q) = \inf_{|\gamma|} \text{length}(\gamma) \), where the infimum is taken over all smooth paths \( \gamma \) in \( U \) such that the projected path \( \pi \circ \gamma \) connects \( p \) and \( q \). A priori, it is not clear how the properties of the distance \( d_Y \) or the map \( F \) depend on the original metric space \((X, d)\). The following statement is the main result of this paper, and completes the proof of Theorem 12.

**Theorem 12.** The uniformizing map \( F: (X, d) \to (Y, d_Y) \) is \( \eta \)-quasisymmetric, with distortion \( \eta \) depending only on the data of \( X \).

**Proof of Theorem 12.** In the case that \( U = \mathbb{C} \), then \( X \) itself must be homeomorphic to \( \mathbb{C} \), and so Theorem 11 implies Theorem 12. We consider the remaining planar and hyperbolic cases together.

Define \( g_j: \mathbb{C}_1 \to F(U_j) \) by \( g_j = F \circ \psi_j^{-1} \). Then \( g_j \) is a conformal homeomorphism that lifts to a conformal embedding \( G_j: \mathbb{C}_1 \to U \).

We use a sequence of claims to complete the proof.

**Claim 1.** Let \( j \in \{1, \ldots, n\} \). The projection \( \pi: U \to Y \) maps \( G_j(\mathbb{C}_\beta) \) isometrically onto \( g_j(\mathbb{C}_\beta) \).

In other words, for each \( z, z' \in \mathbb{C}_\beta \), \( |g_i(z) - g_i(z')|_Y = |G_i(z) - G_i(z')|_U \).

**Proof of Claim 1.** For \( u \in U \), define
\[
r(u) = \min\{|u - \gamma(u)|_U : \gamma \in \Gamma \setminus \{\text{id}\}\}.
\]

If \( U = \mathbb{C} \), then \( r(u) \) is independent of \( u \in U \) and depends only on the group \( \Gamma \). If \( U = \mathbb{D} \), then \( r(u) \) is bounded below by the minimal translation distance of a non-identity element of \( \Gamma \setminus \{\text{id}\} \), and is a 2-Lipschitz function of \( z \), see e.g. [Bea83] Section 7.35]. These facts imply that for any \( u \in U \) and \( r > 0 \),
• if \( r \leq r(u)/6 \), then \( \pi|_{B_U(u, r)} \) is an isometry,
• if \( \pi|_{B_U(u, r)} \) is injective, then \( r \leq r(u) \).

Since \( \pi: G_j(C_1) \rightarrow g_j(C_1) \) is injective, it follows that \( l_{G_j}(0, 1) \leq r(G_j(0)) \). Corollaries 5 and 8 now complete the proof of this claim.

**Claim 2.** For each \( j = 1, \ldots, n \), the mapping \( F|_{B(x_j, 10r_0)} \) is quasisymmetric with distortion that depends only on the data of \( X \).

**Proof of Claim 2** By Propositions 5 and 7, the mapping \( G_j \) restricted to \( C_\beta \) is quasisymmetric with a universal distortion function. By Claim 1 this is also true of \( g_j \). Since we may write \( F = g_j \circ \psi_j \), and \( \psi_j \) is \( \eta \)-quasisymmetric where \( \eta \) depends only on the data, it follows that for each \( j = 1, \ldots, n \), the mapping \( F|_{\psi_j^{-1}(C_\beta)} \) is quasisymmetric with distortion that depends only on the data of \( X \). According to Lemma 11, \( \psi_j^{-1}(C_\beta) \supseteq B(x_j, 10r_0) \), implying the claim.

**Claim 3.** There is a constant \( C \geq 1 \) depending only on the data of \( X \) such that for each \( j = 1, \ldots, n \),

\[
(2) \quad l_{g_j}(0, \alpha) \geq \frac{\text{diam } Y}{C}.
\]

**Proof of Claim 3** As \( \{B_X(x_j, 2r_0)\}_{j=1}^n \) covers \( X \) and \( \psi_j(B_X(x_j, 10r_0)) \subseteq C_\beta \), it holds that \( \{\psi_j^{-1}(C_\beta)\}_{j=1}^n \) also covers \( X \). Since \( F \) is a homeomorphism, this implies that \( Y = \bigcup_{j=1}^n g_j(C_\beta) \). As \( Y \) is connected, it follows that

\[
(3) \quad \max\{\text{diam } g_j(C_\beta) : j = 1, \ldots, n\} \geq \frac{\text{diam } Y}{n}.
\]

Recall that the number of charts \( n \) depends only on the data of \( X \).

Consider indices \( j \) and \( k \) in \( \{1, \ldots, n\} \), and suppose that \( |x_j - x_k| < 4r_0 \). Then \( \psi_j(x_k) \in C_\beta \), and so \( d_Y(F(x_j), F(x_k)) \leq l_{g_j}(0, \beta) \). On the other hand, \( |x_j - x_k| \geq r_0 \), and so \( F(x_k) \notin g_j(C_\alpha) \), implying \( d_Y(F(x_j), F(x_k)) \geq l_{g_j}(0, \alpha) \). Moreover, since \( g_j : C_\beta \rightarrow Y \) is a quasisymmetric embedding with universal distortion function, there is a quantity \( C_0 \geq 1 \) depending only on the ratio of \( \beta \) to \( \alpha \), and hence only on the data of \( X \), such that

\[
l_{g_j}(0, \alpha) \geq \frac{L_{g_j}(0, \beta)}{C_0} \geq \frac{\text{diam } g_j(C_\beta)}{2C_0}.
\]

Since in addition,

\[
\text{diam } g_j(C_\beta) \geq L_{g_j}(0, \beta) \geq l_{g_j}(0, \alpha),
\]

we have now shown that the quantities

\[
l_{g_j}(0, \alpha), \ L_{g_j}(0, \beta), \ \text{diam } g_j(C_\beta), \ \text{and } d_Y(F(x_j), F(x_k))
\]

are all comparable with constants that depend only on the data of \( X \). The same is true with the roles of \( j \) and \( k \) reversed.

Since the open sets \( \{B(x_j, 2r_0)\}_{j=1}^n \) cover the connected space \( X \), for any pair of indices \( j, j' \in \{1, \ldots, n\} \), there is a sequence of distinct indices \( j = j_1, \ldots, j_k = j' \) of length at most \( n \) so that \( |x_{j_i} - x_{j_{i+1}}| < 4r_0 \) for each \( i = 1, \ldots, k - 1 \). Since \( n \) depends only on the data of \( X \), (3) proves the claim.
We complete the proof of Theorem \[12\] by employing Theorem \[4\]. We consider the covering of \(X\) given by \(\{B(x_j,10r_0) : j = 1,\ldots,n\}\). By Claim \[2\] the mapping \(F\) is quasisymmetric on each element of this cover with a distortion function \(\eta\) that depends only on the data of \(X\). Moreover, Lemma \[11\] implies that this cover has Lebesgue number \(8r_0\). Suppose that \(x, x' \in X\) satisfy \(|x - x'| = 4r_0\).

We may find indices \(j\) and \(k\) in \(\{1,\ldots,n\}\) such that \(|x - x_j| < 2r_0\) and \(|x' - x_k| < 2r_0\). Then 
\[
2r_0 \leq |x' - x_j| \leq 6r_0, \text{ so } x, x', x_j \in B(x_j,10r_0) \text{ and }
\]
\[
|F(x') - F(x)| \geq \frac{|F(x') - F(x_j)|}{\eta\left(\frac{|x' - x_j|}{|x - x_j|}\right)} \geq \frac{|F(x') - F(x_j)|}{\eta\left(\frac{2}{3}\right)}.
\]

Since \(x' \notin B(x_j, r_0)\), we see that \(F(x') \notin g_j(C_\alpha)\), so by Claim \[3\]
\[
|F(x') - F(x_j)| = |F(x') - g_j(0)| \geq I_{g_j}(0,\alpha) \geq \frac{\text{diam } Y}{C}.
\]

Now Theorem \[4\] implies that \(F : X \to Y\) is quasisymmetric with distortion function depending only on the data of \(X\). \(\square\)

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