The cohomology of the full directed graph complex

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Abstract

In his seminal paper “Formality conjecture”, M. Kontsevich introduced a graph complex $G_{1ve}$ closely connected with the problem of constructing a formality quasi-isomorphism for Hochschild cochains. In this paper, we express the cohomology of the full directed graph complex $dfGC$ explicitly in terms of the cohomology of $G_{1ve}$. Applications of our results include a recent work by the first author which completely characterizes homotopy classes of formality quasi-isomorphisms for Hochschild cochains in the stable setting.

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1 Introduction

Graph complexes provide us with a large supply of intriguing questions and conjectures\cite{1}, \cite{2}, \cite{3}, \cite{4}, \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{14}. One source of the motivation for working with graph complexes comes from the study of embedding spaces\cite{2}, \cite{3}, \cite{4}, \cite{5}, \cite{6}, \cite{7}, \cite{8}. Another source\cite{10}, \cite{11}, \cite{12} comes from the study of moduli spaces of smooth complex algebraic curves.

This paper is devoted to the full directed graph complex $\text{dfGC}$ which generalizes Kontsevich’s graph complex $\text{GC}$ from \cite{19}, Section 5.2 and its study is motivated by the fact that it “acts on” homotopy classes of stable formality quasi-isomorphisms \cite{7}. More precisely, using the full directed graph complex $\text{dfGC}$, one can describe all homotopy classes of formality quasi-isomorphisms for Hochschild cochains in the “stable setting”.

The graded vector space of $\text{dfGC}$ is “assembled from” directed graphs (possibly with loops) with some additional data. $\text{dfGC}$ is naturally a graded Lie algebra and the graph $\Gamma_{\cdot - \cdot}$ with a single edge connecting two vertices is a Maurer-Cartan element in $\text{dfGC}$. So the differential on $\text{dfGC}$ is defined as the adjoint action of $\Gamma_{\cdot - \cdot}$.

It is easy to see that the graph $\Gamma_{\cdot \triangleright}$ which consists of the single loop and the polygon $\Gamma_{\ast \ast}^{4m+1}$ with $4m+1$ edges (for $m \geq 1$) shown in figure 1.1 are non-trivial cocycles in $\text{dfGC}$.

![Fig. 1.1: The graph $\Gamma_{\cdot \triangleright}^{4m+1}$](image)

There is an obvious embedding

$$\text{GC}_{1\text{ve}} \hookrightarrow \text{dfGC}$$

which upgrades to the following map of cochain complexes:

$$\Psi : s^{-2}\hat{S}\left(s^{2}\text{GC}_{1\text{ve}} \oplus \bigoplus_{m \geq 0} s^{2m}v_{4m-1}\right) \to \text{dfGC},$$

where $s$ (resp. $s^{-1}$) denotes the operator which shifts the degree up by 1 (resp. down by 1), $\hat{S}$ is the completed (and truncated) symmetric algebra

$$\hat{S}(V) := \prod_{n \geq 1} S^n(V),$$

$v_{4m-1}$ is a vector of degree $4m-1$ which gets mapped to the cocycle $\Gamma_{4m+1}^{\circ}$ in $\text{dfGC}$ if $m \geq 1$ and $v_{-1}$ is a vector of degree $-1$ which gets mapped to $\Gamma_{\circ \cdot}$. It turns out that the map $\Psi$ is a quasi-isomorphism of cochain complexes and the goal of this paper is to give a careful proof of this statement\footnote{In this paper, we assume that $\mathbb{K}$ is a field of characteristic zero.} (see Theorem 3.1). Using this statement, we deduce the following corollary

**Corollary 1.1** For the full directed graph complex $\text{dfGC}$, we have

$$H^\ast(\text{dfGC}) \cong s^{-2}\hat{S}\left( s^{2}H^\ast(\text{GC}_{1\text{ve}}) \oplus \bigoplus_{m \geq 0} s^{4m+1}\mathbb{K}\right).$$

\footnote{This list of references is far from complete.}

\footnote{To get a vector in $\text{dfGC}$ from the undirected graph $\Gamma_{4m+1}^{\circ}$, we have to choose a total order on the set of edges and take the sum of directed graphs which are obtained from $\Gamma_{4m+1}^{\circ}$ by choosing all possible directions on edges.}

\footnote{To get a vector in $\text{dfGC}$ from the undirected graph $\Gamma_{4m+1}^{\circ}$, we have to choose a total order on the set of edges and take the sum of directed graphs which are obtained from $\Gamma_{4m+1}^{\circ}$ by choosing all possible directions on edges.}
Combining Corollary 1.1 with [28, Theorem 1.1], we conclude that

**Corollary 1.2**

\[ H^0(\text{dfGC}) \cong \text{grt}_1, \quad H^{-1}(\text{dfGC}) \cong \mathbb{K}[\Gamma_\diamond], \quad \text{and} \quad H^{-2}(\text{dfGC}) = 0, \]

where \( \text{grt}_1 \) is the Grothendieck-Teichmüller Lie algebra introduced by V. Drinfeld in [11, Section 6] and \([\Gamma_\diamond]\) is the cohomology class of the graph \( \Gamma_\diamond \) which consists of a single loop.

### 1.1 Organization of the paper

The material of the paper is presented in a way which requires a minimal knowledge of prerequisites. For example, a reader, who is unfamiliar with algebraic operads [22], can easily understand most of this paper.

In Section 2, we introduce the full directed graph complex \( \text{dfGC} \), its “uncompleted” version \( \text{dfGC}^\oplus \), its undirected version \( \text{fGC} \) and the important subcomplexes \( \text{GC}_{\text{ev}} \subset \text{GC} \subset \text{fGC} \). At the end of this section, we recall the interpretation of \( \text{dfGC} \) in terms of the convolution Lie algebra [9, Section 4].

In Section 3, we formulate the main result of this paper (see Theorem 3.1) and its variant for the “uncompleted” version \( \text{dfGC}^\oplus \) of \( \text{dfGC} \) (see Theorem 3.2). We deduce Theorem 3.1 from Theorem 3.2 and prove the version of Theorem 3.1 for the subcomplex \( \text{dfGC}^\oplus \subset \text{dfGC} \) which is “assembled from” directed graphs without loops (see Proposition 3.3). It is Proposition 3.3 which is used in paper [7] to express the coformality quasi-isomorphisms for Hochschild cochains in the “stable setting”.

In Section 4, we also recall the version \( \text{dfGC}_d \) of the full directed graph complex for an arbitrary even integer \( d \) (\( \text{dfGC} := \text{dfGC}_d \)) and generalize Theorem 3.1 to the case of arbitrary even \( d \).

The proof of Theorem 3.2 is broken into several parts (see Subsection 3.2 for the outline of the proof) and the two major parts are presented in Sections 3 and 5 respectively.

Section 4 is devoted to the subcomplex \( \text{dfGC}_{\text{conn}, \geq 3}^\oplus \) which is spanned by connected graphs with at least one vertex having valency \( \geq 3 \). In this section, we prove that the natural embedding \( \text{GC}_{\text{ev}}^\oplus \hookrightarrow \text{dfGC}_{\text{conn}, \geq 3}^\oplus \) is a quasi-isomorphism of cochain complexes.

In Section 5, we prove that the natural embedding \( \text{GC}_{\text{ev}} \hookrightarrow \text{GC}^\oplus \) is also a quasi-isomorphism of cochain complexes. This statement plays an important role in the proof of Willwacher’s theorem [28, Theorem 1.1] which links the zeroth cohomology of \( \text{GC} \) to the Grothendieck-Teichmüller Lie algebra \( \text{grt}_1 \) [11, Section 4.2], [11, Section 6]. For this reason, we decided to write a careful proof of this statement.

The easier parts of the proof of Theorem 3.2 are presented in Appendix B. Finally, Appendix A is devoted to auxiliary statements which are used in Section 4 and in Appendix B.

### 1.2 Notational conventions

In this paper, the ground field \( \mathbb{K} \) has characteristic zero and \( \text{grVect}_\mathbb{K} \) denotes the category of \( \mathbb{Z} \)-graded \( \mathbb{K} \)-vector spaces. The notation \( s \) (resp. \( s^{-1} \)) is reserved for the operator which shifts the degree up by 1 (resp. down by 1), i.e. \( (sV)^\bullet := V^{\bullet -1} \) and \( (s^{-1}V)^\bullet := V^{\bullet +1} \). For a set \( X \), we denote by \( \text{span}_\mathbb{K}(X) \) the \( \mathbb{K} \)-vector space of finite \( \mathbb{K} \)-linear combinations of elements in \( X \).

The notation \( S_n \) is reserved for the symmetric group on \( n \) letters and \( S_{p_1, \ldots, p_k} \) denotes the subset of \( (p_1, \ldots, p_k) \)-shuffles in \( S_{p_1+p_2+\cdots+p_k} \), i.e. \( S_{p_1, \ldots, p_k} \) consists of elements \( \sigma \in S_{p_1+p_2+\cdots+p_k} \), such that \( \sigma(1) < \sigma(2) < \cdots < \sigma(p_1), \sigma(p_1 + 1) < \sigma(p_1 + 2) < \cdots < \sigma(p_1 + p_2), \ldots, \sigma(n - p_k + 1) < \sigma(n - p_k + 2) < \cdots < \sigma(n) \). For a group \( G \) acting on a vector space \( V \), we denote by \( V^G \) (resp. \( V_G \)) the space of invariants (resp. the space of coinvariants).

For a graded vector space (or a cochain complex) \( V \), the notation \( S(V) \) (resp. \( S(V) \), resp. \( \hat{S}(V) \)) is reserved for the space of the symmetric algebra (resp. the truncated symmetric algebra, resp. the completed and truncated symmetric algebra). Namely,

\[
S(V) := \mathbb{K} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \ldots, \quad \hat{S}(V) := V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,
\]

and

\[
\hat{S}(V) := \prod_{n \geq 1} S^n(V), \quad S^n(V) := (V^\otimes n)_{S_n}.
\]
coCom denotes the cooperad (in the category of $\mathbb{K}$-vector spaces) which governs cocommutative (and coassociative) coalgebras without counit. In other words, $\text{coCom}(n)$ is the trivial representation $\mathbb{K}$ of $S_n$ for every $n \geq 1$ and $\text{coCom}(0) = 0$. The notation $\Lambda$ is reserved for the underlying collection of the endomorphism operad $\text{End}_K$. It is known that $\Lambda$ is naturally a cooperad (in the category $\text{grVect}_K$). So, for a cooperad $C$ (in the category $\text{grVect}_K$) and an integer $m$, we denote by $\Lambda^m C$ the cooperad $\Lambda \otimes_m C$. For example, $\Lambda^2 \text{coCom}(n)$ is the trivial representation of $S_n$ placed in degree $2 - 2n$, if $n \geq 1$ and $\Lambda^2 \text{coCom}(0) = 0$.

By a graph $\Gamma$ we typically mean an undirected graph with a finite set of vertices $V(\Gamma)$ and a finite set of edges $E(\Gamma)$. Multiple edges as well as loops (i.e. cycles of length 1) are allowed. A graph $\Gamma$ is directed, if each edge of $\Gamma$ is equipped with a direction. A forest is a finite (undirected) graph $F$ without cycles. A tree is a connected forest.

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2 The graph complexes $\text{dfGC}^{\oplus}$, $\text{dfGC}$ and their modifications

For a pair of integers $(n,r)$ with $n \geq 1$ and $r \geq 0$, the set $\text{dgra}_n^r$ consists of directed graphs $\Gamma$ with the set of vertices $V(\Gamma) = \{1, 2, \ldots, n\}$ and the set of edges $E(\Gamma) = \{1, 2, \ldots, r\}$. An example of an element $\Gamma$ in $\text{dgra}_5^4$ is shown in figure 2.1.

The set $\text{dgra}_n^r$ is equipped with the obvious action of the group $S_n \times S_r$. Using this action we define the graded vector space

$$\text{dfGC}^{\oplus} := \bigoplus_{n \geq 1, r \geq 0} (s^{2n-2} \text{span}_K(\text{dgra}_n^r) \otimes \text{sgn}_r)_{S_n \times S_r},$$

where $\text{sgn}_r$ denotes the sign representation of $S_r$.

Since we take coinvariants with respect to the action of $S_n$ on $\text{span}_K(\text{dgra}_n^r)$, the labels on vertices do not play any essential role. On the other hand, the presence of the sign representation of $S_r$ tells us that the change in the labels on edges results in a sign factor. For example, in $\text{dfGC}^{\oplus}$, we have

$$\begin{align*}
\begin{array}{c}
\begin{array}{c}
3 \quad 2 \quad 2 \\
1 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
= 
\begin{align*}
\begin{array}{c}
\begin{array}{c}
3 \quad 2 \quad 2 \\
1 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
= - 
\begin{align*}
\begin{array}{c}
\begin{array}{c}
3 \quad 2 \quad 2 \\
1 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
= 
\begin{align*}
\begin{array}{c}
\begin{array}{c}
3 \quad 2 \quad 2 \\
1 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\begin{array}{c}
3 \quad 2 \quad 2 \\
1 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\begin{array}{c}
3 \quad 2 \quad 2 \\
1 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\begin{array}{c}
3 \quad 2 \quad 2 \\
1 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
(2.2)

It is clear that a graph $\Gamma \in \text{dgra}_n^r$ gives us the zero vector in $\text{dfGC}^{\oplus}$ if and only if it has an automorphism

\[\text{In figures, the labels for vertices are shown in gray color and the labels for edges are underlined numbers.}\]
which induces an odd permutation in $S_r$. For example,

$$\Gamma = \begin{array}{ccc}
3 & 2 & 1 \\
4 & 1 & 2
\end{array} = - \begin{array}{ccc}
3 & 2 & 1 \\
1 & 2 & 4
\end{array} = - \begin{array}{ccc}
3 & 2 & 1 \\
1 & 2 & 4
\end{array} = - \Gamma. \quad (2.3)
$$

So $\Gamma = 0$ in $\text{dfGC}^\oplus$.

**Definition 2.1** We will say that a graph $\Gamma \in \text{dgra}_n^r$ is odd if it has an automorphism which induces an odd permutation in $S_r$. Otherwise, we say that $\Gamma$ is even.

For example, the pentagon shown in (2.2) is even and the square shown in (2.3) is odd.

It is clear that $\text{dfGC}^\oplus$ is the span of isomorphism classes of all even graphs with the rule that an even graph $\Gamma \in \text{dgra}_n^r$ carries the degree $2n - 2 - r$.

It is easy to see that every graph which has multiple edges with the same direction is odd. Thus, the graph shown in figure 2.1 is odd. Hence it corresponds to the zero vector in $\text{dfGC}^\oplus$. On the other hand, there are even graphs with double edges of opposite directions. An example of such a graph is shown in figure 2.2.

**Definition 2.2** Let $\Gamma, \tilde{\Gamma}$ be two even elements of $\text{dgra}_n^r$ for which there exists an isomorphism of directed graphs $\varphi : \Gamma \to \tilde{\Gamma}$. We say that $\Gamma$ and $\tilde{\Gamma}$ are concordant (resp. opposite) if $\tilde{\Gamma} = \Gamma$ (resp. $\tilde{\Gamma} = -\Gamma$) in $\text{dfGC}^\oplus$.

**Remark 2.3** Since edges of $\Gamma$ and $\tilde{\Gamma}$ are labeled, any isomorphism (of directed graphs) $\varphi : \Gamma \to \tilde{\Gamma}$ gives us a bijection $\{1, \ldots, r\} \to \{1, \ldots, r\}$ and hence an element $\sigma_\varphi \in S_r$. Moreover, since $\Gamma$ (and hence $\tilde{\Gamma}$) is even, the sign of the permutation $\sigma_\varphi$ does not depend on the choice of the isomorphism $\varphi$. So $\Gamma$ and $\tilde{\Gamma}$ are concordant (resp. opposite) if and only if the permutation $\sigma_\varphi$ is even (resp. odd).

For $\Gamma \in \text{dgra}_n^r, \tilde{\Gamma} \in \text{dgra}_m^q$, and $1 \leq i \leq n$, we denote by $\Gamma \circ_i \tilde{\Gamma}$ the vector in $\text{span}_K(\text{dgra}_m^q \oplus \text{dgra}_m^q + n - 1)$ which is obtained by following these steps:

- we shift the labels $i + 1, i + 2, \ldots, n$ on vertices of $\Gamma$ up by $m - 1$;
- we shift all the labels on vertices of $\tilde{\Gamma}$ up by $i - 1$;
- we shift all the labels on edges of $\tilde{\Gamma}$ up by $r$;
- finally, we replace vertex $i$ of $\Gamma$ by the graph $\tilde{\Gamma}$ (with the new labels) and sum over all possible ways of attaching the resulting free edges to vertices of $\tilde{\Gamma}$ (with the new labels).

Although $\Gamma \circ_i \tilde{\Gamma}$ is defined as a vector in $\text{span}_K(\text{dgra}_m^q \oplus \text{dgra}_m^q + n - 1)$, we will often use the same notation $\Gamma \circ_i \tilde{\Gamma}$ for its image in the space $\text{dfGC}^\oplus$ of coinvariants. For example, a computation of $\Gamma \circ_2 \tilde{\Gamma} \in \text{span}_K(\text{dgra}_4^4)$ for two concrete graphs $\Gamma$ and $\tilde{\Gamma}$ is shown in figure 2.4 and the image of $\Gamma \circ_2 \tilde{\Gamma}$ in $\text{dfGC}^\oplus$ is shown in figure 2.4. This image has only three terms because the third graph in the second line of figure 2.3 is odd.

It is not hard to see that the map

$$\Gamma \bullet \tilde{\Gamma} := \sum_{i=1}^n \Gamma \circ_i \tilde{\Gamma} : \text{span}_K(\text{dgra}_n^r) \otimes \text{span}_K(\text{dgra}_m^q) \to \text{span}_K(\text{dgra}_m^q \oplus \text{dgra}_m^q + n - 1) \quad (2.4)$$

Note that an even graph $\Gamma$ without labels on edges determines a vector in $\text{dfGC}^\oplus$ only up to a sign factor. To specify the sign factor, one has to fix a total order on $E(\Gamma)$ up to any even permutation in $S_{E(\Gamma)}$. 

5
Fig. 2.3: An example of computing $\circ_2$

Fig. 2.4: The image of $\Gamma \circ_2 \tilde{\Gamma}$ in $\text{dfGC}^\oplus$

descends to coinvariants and we get a degree 0 bilinear operation on $\text{dfGC}^\oplus$:

$$\bullet : \text{dfGC}^\oplus \otimes \text{dfGC}^\oplus \to \text{dfGC}^\oplus.$$ 

We recall that

**Proposition 2.4** The bracket

$$[\Gamma, \tilde{\Gamma}] = \Gamma \bullet \tilde{\Gamma} - (-1)^{|\Gamma||\tilde{\Gamma}|} \tilde{\Gamma} \bullet \Gamma \quad (2.5)$$

equips $\text{dfGC}^\oplus$ with the structure of a graded Lie algebra.

Proof. This statement follows directly from [10, Proposition C.2]. More details about the relationship between $\text{dfGC}^\oplus$ and a certain convolution Lie algebra [9, Section 4], [23] are given in Section 2.2 below. □

It is clear that

$$\Gamma_{\bullet \bullet} := 1 \overset{2}{\longrightarrow} \quad (2.6)$$

is a non-zero vector in $\text{dfGC}^\oplus$ of degree 1. Moreover, a direct computation shows that $\Gamma_{\bullet \bullet}$ satisfies the MC equation

$$[\Gamma_{\bullet \bullet}, \Gamma_{\bullet \bullet}] = 0. \quad (2.7)$$

So we use $\Gamma_{\bullet \bullet}$ to define the following differential on $\text{dfGC}^\oplus$:

$$\partial = [\Gamma_{\bullet \bullet}, \cdot]. \quad (2.8)$$

To define the (full) directed graph complex $\text{dfGC}$, we denote by $\text{dfGC}(n)$ the subspace of $\text{dfGC}^\oplus$ which is spanned by isomorphism classes of even graphs with exactly $n$ vertices. It is obvious that $\partial(\text{dfGC}(n)) \subset \text{dfGC}(n+1)$ and it allows us to give the following definition:

**Definition 2.5** The full directed graph complex $\text{dfGC}$ is the following completion of $\text{dfGC}^\oplus$

$$\text{dfGC} := \prod_{n \geq 1} \text{dfGC}(n). \quad (2.9)$$
Example 2.6 Let $\Gamma_\bullet$ be the graph in $\text{dgra}_0^1$ which consists of a single vertex and $\Gamma_\bigcirc$ be the graph in $\text{dgra}_1^1$ which consists of a single loop. For these graphs we have

$$\partial \Gamma_\bullet = \Gamma_\bullet, \quad \partial \Gamma_\bigcirc = 1 \begin{array}{c} \circ \end{array} 2 = 0.$$

Thus $\Gamma_\bigcirc$ represents a degree $-1$ (non-trivial) cocycle in $\text{dfGC}$.

Another example of the computation of the differential in $\text{dfGC}$ is given in figure 2.5. In this example, only the uni-bivalent graphs coming from the insertion of $\Gamma_\bullet$ into vertex 2 survive. All the other graphs cancel each other.

$$\partial \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4 = \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4 + \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4 + \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4 + \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4 + \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4 = \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4 + \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array} 4$$

Fig. 2.5: An example of computing $\partial(\Gamma)$

Let us observe that

Proposition 2.7 If $\Gamma$ is a connected (even) graph in $\text{dgra}_r^n$ with $r \geq 1$, then

$$\partial \Gamma = -(-1)^{|\Gamma|} \sum_{i=1}^{n} \gamma_i,$$  \hspace{1cm} (2.10)

where $\gamma_i$ is obtained from the linear combination $\Gamma \circ_i \Gamma_\bullet$ by discarding all graphs in which either vertex $i$ or vertex $i+1$ has valency 1.

Proof. Since each vertex of $\Gamma$ is adjacent to at least one edge, in $\text{dfGC}^\oplus$, we have

$$\Gamma_\bullet \circ_1 \Gamma + \Gamma_\bullet \circ_2 \Gamma = (-1)^{|\Gamma|} \sum_{i=1}^{n} \gamma'_i$$

where $\gamma'_i$ is obtained from the linear combination $\Gamma \circ_i \Gamma_\bullet$ by keeping only the graphs in which either vertex $i$ or vertex $i+1$ has valency 1.

Thus (2.10) follows. \hfill \Box
2.1 fGC, GC, and other variants of dfGC. Examples of cocycles

For a pair of integers \( n \geq 1, \ r \geq 0 \), we introduce the auxiliary set \( \text{gra}_n^r \). An element \( \Gamma \) of \( \text{gra}_n^r \) is an undirected graph with the set of vertices \( V(\Gamma) = \{1, 2, \ldots, n\} \) and the set of edges \( E(\Gamma) = \{1, 2, \ldots, r\} \).

The set \( \text{gra}_n^r \) is equipped with the obvious action of the group \( S_n \times S_r \). So, by analogy with \( (2.1) \), we set

\[
\text{fGC}^\oplus := \bigoplus_{n \geq 1, \ r \geq 0} \left( s^{2n-2-r} \text{span}_K(\text{gra}_n^r) \otimes \text{sgn}_r \right)_{S_n \times S_r}.
\]

(2.11)

Just as for \( \text{dfGC}^\oplus \), a graph \( \Gamma \in \text{gra}_n^r \) gives us the zero vector in \( \text{fGC}^\oplus \) if and only if \( \Gamma \) has an automorphism which induces an odd permutation in \( S_r \). So, by analogy with directed graphs, we say that a graph \( \Gamma \in \text{gra}_n^r \) is odd if it has an automorphism which induces an odd permutation in \( S_r \). Otherwise, we say that \( \Gamma \) is even. For example, the tetrahedron shown in figure 2.6 is even and the triangle shown in figure 2.7 is odd.

It is easy to see that \( \text{fGC}^\oplus \) is the span of isomorphism classes of all even (undirected) graphs with the rule that an even graph \( \Gamma \in \text{gra}_n^r \) carries the degree \( 2n - 2 - r \).

![Fig. 2.6: This graph is even](image1)

![Fig. 2.7: This graph is odd](image2)

We should mention that in figures 2.6 and 2.7 as well as in some further figures, we often omit labels for edges. The reader should keep in mind that an even graph without labels on edges define a vector in \( \text{fGC}^\oplus \) only up to a sign factor.

Using the analogous map \( \alpha_i : \text{span}_K(\text{gra}_n^r) \otimes \text{span}_K(\text{gra}_m^t) \rightarrow \text{fGC}^\oplus \), we define the degree 0 binary operation \( \bullet : \text{fGC}^\oplus \otimes \text{fGC}^\oplus \rightarrow \text{fGC}^\oplus \) by setting

\[
\Gamma \bullet \tilde{\Gamma} := \sum_{i=1}^{n} \Gamma \alpha_i \tilde{\Gamma}.
\]

(2.12)

Moreover, we claim that the same formula \( (2.5) \) defines a Lie bracket on the graded vector space \( \text{fGC}^\oplus \).

A direct computation shows that the (non-zero) degree 1 vector

\[
\Gamma^{un} := \frac{1}{2} 1 \quad 2
\]

satisfies the MC equation

\[
[ \Gamma^{un}, \Gamma^{un} ] = 0.
\]

So we define the differential on \( \text{fGC}^\oplus \) by the formula:

\[
\partial = [ \Gamma^{un}, \Gamma^{un} ].
\]

(2.13)

Just as for \( \text{dfGC}^\oplus \), we denote by \( \text{fGC}(n) \) the subspace of \( \text{fGC}^\oplus \) which is spanned by isomorphism classes of even graphs with exactly \( n \) vertices. We also observe that \( \partial(\text{fGC}(n)) \subset \text{fGC}(n+1) \). So we define the full graph complex \( \text{fGC} \) as the following completion of \( \text{fGC}^\oplus \)

\[
\text{fGC} := \prod_{n \geq 1} \text{fGC}(n).
\]

(2.14)

2.1.1 GC and Kontsevich’s graph complex GC_{1ve}

Recall that a vertex \( v \) of a graph \( \Gamma \) is called a cut vertex if \( \Gamma \) becomes disconnected upon deleting \( v \). A graph \( \Gamma \) without cut vertices is called 1-vertex irreducible. For example, the tetrahedron shown in figure 2.6 is 1-vertex irreducible while the graph shown in figure 2.8 is not.

\(^6\)It is convenient to have the factor \( 1/2 \) in the definition of \( \Gamma^{un} \).
Let us denote by \( GC(n) \) the subspace of \( fGC(n) \) which is spanned by isomorphism classes of even graphs \( \Gamma \) satisfying these technical conditions:

- \( \Gamma \) is connected;
- each vertex of \( \Gamma \) has valency \( \geq 3 \).

We denote by \( GC_{1ve}(n) \) the subspace of \( GC(n) \) which is spanned by isomorphism classes of even 1-vertex irreducible graphs \( \Gamma \). Finally, we set

\[
\begin{align*}
GC^\oplus & := \bigoplus_{n \geq 1} GC(n), & GC & := \prod_{n \geq 1} GC(n), \\
GC^\oplus_{1ve} & := \bigoplus_{n \geq 1} GC_{1ve}(n), & GC_{1ve} & := \prod_{n \geq 1} GC_{1ve}(n).
\end{align*}
\]

(2.15) (2.16)

We claim that

**Proposition 2.8**

- If \( \Gamma \) is a connected (even) graph in \( gra^r_n \) with \( r \geq 1 \), then

\[
\partial \Gamma = -(-1)^{|\Gamma|} \sum_{i=1}^{n} \gamma_i,
\]

(2.17)

where \( \gamma_i \) is obtained from the linear combination \( \Gamma \circ_i \Gamma^{un} \) by discarding all graphs in which either vertex \( i \) or vertex \( i + 1 \) has valency 1.

- If \( \Gamma \) is a connected (even) graph with all its vertices having valencies \( \geq 3 \), then

\[
\partial \Gamma = -(-1)^{|\Gamma|} \sum_{i=1}^{n} \gamma_i^{\geq 3},
\]

(2.18)

where \( \gamma_i^{\geq 3} \) is obtained from the linear combination \( \Gamma \circ_i \Gamma^{\bullet \bullet} \) by discarding all graphs which have a univalent or a bivalent vertex.

- The subspaces \( GC^\oplus \) and \( GC^\oplus_{1ve} \) (resp. \( GC \) and \( GC_{1ve} \)) are dg Lie subalgebras of \( fGC^\oplus \) (resp. \( fGC \)). In particular, \( GC^\oplus_{1ve} \) (resp. \( GC_{1ve} \)) is a subcomplex of \( fGC^\oplus \) (resp. \( fGC \)).

Proof. The proof of (2.17) is almost identical to the proof of (2.10). So we skip it.

It is easy to see that the subspaces

\[
GC^\oplus \subset fGC^\oplus, \quad GC \subset fGC
\]

are closed with respect to the binary operation (2.12). Hence \( GC^\oplus \) (resp. \( GC \)) are Lie subalgebras of \( fGC^\oplus \) (resp. \( fGC \)).

Let \( \Gamma \) be a connected (even) graph in \( gra^r_n \) whose all vertices have valencies \( \geq 3 \) (hence \( r > 1 \)). It is clear that every graph in \( \partial \Gamma \) is connected.

Let \( 1 \leq q \leq r \) and the edge labeled by \( q \) in \( \Gamma \) connects two distinct vertices \( i \) and \( j \) with \( i < j \). We denote by \( \Gamma^+_q \) and \( \Gamma^-_q \) the elements of \( gra^r_{n+1} \) which are obtained from \( \Gamma \) via replacing the edge \( q \) by two edges which connect the additional vertex \( n + 1 \) to \( i \) and \( j \) respectively. For \( \Gamma^+_q \), the edge connecting \( n + 1 \) to \( j \) is labeled by...
\[ \Gamma_q^+ = \quad \begin{array}{c} i \quad n + 1 \quad j \\ \bullet \quad \bullet \quad \bullet \end{array} \quad \begin{array}{c} r + 1 \quad q \\ \bullet \quad \bullet \quad \bullet \end{array} \]

\[ \Gamma_q^- = \quad \begin{array}{c} i \quad n + 1 \quad j \\ \bullet \quad \bullet \quad \bullet \end{array} \quad \begin{array}{c} q \quad r + 1 \\ \bullet \quad \bullet \quad \bullet \end{array} \]

Fig. 2.9: Producing \( \Gamma_q^+ \) and \( \Gamma_q^- \) from \( \Gamma \)

According to (2.17),

\[ \partial(\Gamma) + (-1)^{|\Gamma|} \sum_{i=1}^{n} q_i \geq 3 = - (-1)^{|\Gamma|} \sum_{q} (\Gamma_q^+ + \Gamma_q^-), \]

where the summation goes over all edges with distinct end point.

On the other hand, \( \Gamma_q^- = -\Gamma_q^+ \) in the space of coinvariants. Thus (2.18) follows. In particular, the subspaces \( GC^B \) and \( GC \) are closed with respect to the differential \( \partial \).

Let \( \Gamma \) and \( \tilde{\Gamma} \) be (connected) 1-vertex irreducible graphs. It is clear that graphs with a cut vertex in \( \Gamma \circ \tilde{\Gamma} \)

are obtained only when we connect all edges, which were adjacent to vertex \( i \), to the same vertex of \( \tilde{\Gamma} \). It is not hard to see that all such graphs cancel each other in the sum

\[ \Gamma \circ \tilde{\Gamma} - (-1)^{|\Gamma||\tilde{\Gamma}|} \tilde{\Gamma} \circ \Gamma. \]

Thus, \( GC_{1ve}^B \) (resp. \( GC_{1ve} \)) is closed with respect to the bracket in \( f_{GC}^B \) (resp. \( f_{GC} \)).

Since \( \Gamma_{1un} \) is 1-vertex irreducible, \( GC_{1ve}^B \) and \( GC_{1ve} \) are also closed with respect to the differential. □

Equation (2.18) implies that

\[ \text{Corollary 2.9} \quad \text{Any linear combination of connected trivalent graphs is a cocycle in } GC. \]

\[ \text{Remark 2.10} \quad \text{It is the graph complex } (GC_{1ve}, \partial) \text{ which was introduced in [19, Section 5.2]. In this paper, we refer to } (GC_{1ve}, \partial) \text{ as Kontsevich’s graph complex.} \]

2.1.2 \( fGC \) as a subcomplex of \( dGC \)

Let \( \Gamma \) be an element in \( d\text{gra}_n \). We denote by \( \rho_j(\Gamma) \) the graph which is obtained from \( \Gamma \) by changing the direction of the edge with label \( j \).

It is convenient to draw the linear combination \( \Gamma + \rho_j(\Gamma) \) as a graph which is obtained from \( \Gamma \) by forgetting the direction of the edge with label \( j \). For example,

\[ \begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \end{array} : = \quad \begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \end{array} + \quad \begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \end{array} \quad (2.19) \]

Similarly, if the graph \( \Gamma' \) is obtained from \( \Gamma \) by forgetting the directions on the edges with labels \( j_1, \ldots, j_p \in \{1, \ldots, n\} \), then \( \Gamma' \) denotes the sum

\[ \Gamma' = \sum_{(k_1, \ldots, k_p) \in \{0,1\}^n} (\rho_{j_1})^{k_1} (\rho_{j_2})^{k_2} \cdots (\rho_{j_p})^{k_p}(\Gamma). \]

\[ \text{Remark} \quad \text{The unwanted terms in } \gamma_i \text{ should be analyzed separately in the case when } \Gamma \text{ has a loop based at } i. \text{ This analysis is straightforward and we leave it to the reader.} \]
For example,
\[
    \begin{array}{c}
        1 \quad 2 \\
        \end{array}
    =
    \begin{array}{c}
        1 \quad 2 \\
        \end{array}
    +
    \begin{array}{c}
        1 \quad 2 \\
        \end{array}
    .
\]
\hspace{1cm} (2.20)

This way, we may view undirected graphs as well as graphs with both directed and undirected edges as vectors in \(\text{d} \Gamma\text{GC}\) (and \(\text{d} \Gamma\text{GC}\)).

This identification is compatible with the binary operations \(2.23\) and \(2.24\). So we will view \(\Gamma\text{GC} \subset \text{d} \Gamma\text{GC}\) (resp. \(\Gamma\text{GC} \subset \text{d} \Gamma\text{GC}\) as Lie subalgebras of \(\text{d} \Gamma\text{GC}\) (resp. \(\text{d} \Gamma\text{GC}\)). In addition equation \(2.19\) implies that \(\Gamma_{\text{un}} = \Gamma_{\text{un}}\). So \(\Gamma\text{GC} \subset \text{d} \Gamma\text{GC}\) (resp. \(\Gamma\text{GC} \subset \text{d} \Gamma\text{GC}\)) are, in fact, \(\text{d} \text{G} \text{a}\) Lie subalgebras of \(\text{d} \Gamma\text{GC}\) (resp. \(\text{d} \Gamma\text{GC}\)).

### 2.2 \text{d} \Gamma\text{GC} as the convolution Lie algebra

According to \([7]\) Section 3], the graded vector spaces \((n \geq 1)\)
\[
    \text{d} \Gamma\text{GC}(n) := \bigoplus_{r \geq 0} \mathbb{S}^{n-r} (\text{span}_{\mathbb{R}}(\text{gra}_{n}^{r}) \otimes \text{sgn}_{r})_{\mathbb{S}_{n}}.
\]
\hspace{1cm} (2.21)

assemble to form an operad \(\text{d} \Gamma\text{GC}\) in the category \(\text{grVect}_{\mathbb{R}}\).

We denote by \(\text{Conv}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC})\) the convolution Lie algebra \([9]\) Section 4] corresponding to the cooperad \(\Lambda^{2}\text{coCom}\). Moreover, we denote by \(\text{Conv}^{\Box}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC})\) the following subspace of \(\text{Conv}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC})\):
\[
    \text{Conv}^{\Box}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC}) := \bigoplus_{n=1}^{\infty} \text{Hom}_{\mathbb{S}_{n}}(\Lambda^{2}\text{coCom}(n), \text{d} \Gamma\text{GC}(n)).
\]
\hspace{1cm} (2.22)

It is clear that
\[
    \text{Conv}^{\Box}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC}) = \bigoplus_{n=1}^{\infty} s^{2n-2}(\text{d} \Gamma\text{GC}(n))_{\mathbb{S}_{n}}
\]
\hspace{1cm} (2.23)

and
\[
    \text{Conv}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC}) = \bigoplus_{n=1}^{\infty} s^{2n-2}(\text{d} \Gamma\text{GC}(n))_{\mathbb{S}_{n}}.
\]
\hspace{1cm} (2.24)

Since the space of invariants \((\text{d} \Gamma\text{GC}(n))_{\mathbb{S}_{n}}\) can be identified with the quotient space of coinvariants \((\text{d} \Gamma\text{GC}(n))_{\mathbb{S}_{n}}\) via the isomorphism
\[
    \text{Av}(v) := \sum_{\sigma \in \mathbb{S}_{n}} \sigma(v) : (\text{d} \Gamma\text{GC}(n))_{\mathbb{S}_{n}} \to (\text{d} \Gamma\text{GC}(n))_{\mathbb{S}_{n}},
\]
we have the obvious isomorphisms of graded vector spaces
\[
    \text{d} \Gamma\text{GC} \xrightarrow{\cong} \text{Conv}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC}), \quad \text{d} \Gamma\text{GC}^{\Box} \xrightarrow{\cong} \text{Conv}^{\Box}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC}).
\]
\hspace{1cm} (2.25)

Due to \([10]\) Proposition C.2, the isomorphisms in \(2.25\) send the bracket \(2.19\) to the Lie bracket on \(\text{Conv}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC})\) and \(\text{Conv}^{\Box}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC})\), respectively. Thus equation \(2.19\) indeed defines a Lie bracket and \(\text{d} \Gamma\text{GC}\) (resp. \(\text{d} \Gamma\text{GC}^{\Box}\)) can be identified with \(\text{Conv}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC})\) (resp. \(\text{Conv}^{\Box}(\Lambda^{2}\text{coCom}, \text{d} \Gamma\text{GC})\)).

### 3 The main theorem and the outline of the proof

Let us denote by \(\text{d} \Gamma\text{GC}_{\text{conn}}(n)\) the subspace of \(\text{d} \Gamma\text{GC}(n)\) spanned by (even) connected graphs with exactly \(n\) vertices. It is clear that for any pair of connected graphs \(\Gamma\) and \(\tilde{\Gamma}\), every term in the linear combination \([\Gamma, \tilde{\Gamma}]\) is a connected graph. So setting
\[
    \text{d} \Gamma\text{GC}_{\text{conn}} := \bigoplus_{n \geq 1} \text{d} \Gamma\text{GC}_{\text{conn}}(n) \quad \text{and} \quad \text{d} \Gamma\text{GC}^{\Box}_{\text{conn}} := \bigoplus_{n \geq 1} \text{d} \Gamma\text{GC}^{\Box}_{\text{conn}}(n)
\]
\hspace{1cm} (3.1)

\[\text{From now on, we will drop the superscript un in } \Gamma^{\text{un}}.\]
\[\text{In } [7], \text{ the author considers directed graphs without loops. It is easy to see that, after adding loops, } \text{d} \Gamma\text{GC} \text{ is still an operad.}\]
we get the dg Lie subalgebra $dfG_{\text{conn}}$ of $dfG$ and the dg Lie subalgebra $dfG_{\text{conn}}^\oplus$ of $dfG^\oplus$, respectively.

Since every disconnected graph is a union of (finitely many) connected graphs, it is clear that

$$dfG \cong s^{-2} \widehat{S}(s^2dfG_{\text{conn}}) \quad \text{and} \quad dfG^\oplus \cong s^{-2} \widehat{S}(s^2dfG_{\text{conn}}^\oplus).$$

(3.2)

Let us denote by $GC^\circ_1$ the cochain complex

$$GC^\circ_1 := GC_1 \oplus \bigoplus_{m \geq 0} \mathbb{K}v_{4m-1},$$

(3.3)

where $v_{4m-1}$ is a vector of degree $4m - 1$ and $\mathbb{K}v_{4m-1}$ is considered as the cochain complex with the zero differential.

Next we upgrade the embedding $GC_1 \hookrightarrow dfG$ to the map of cochain complexes

$$\Psi : GC^\circ_1 \rightarrow dfG_{\text{conn}}$$

by setting

$$\Psi(v_1) := \Gamma^{\circ} = \quad \text{and} \quad \Psi(v_{4m-1}) := \Gamma^{\circ}_{4m+1} \quad \text{for} \quad m \geq 1,$$

where $\Gamma^{\circ}_{4m+1}$ is the graph shown in figure 3.1.

![Fig. 3.1: The graph $\Gamma^{\circ}_{4m+1}$](image)

Due to the first isomorphism in (3.2), the map $\Psi$ upgrades further to the map of cochain complexes

$$\Psi : s^{-2} \widehat{S}(s^2GC^\circ_1) \rightarrow dfG.$$  

(3.4)

Moreover, the restriction of $\Psi$ to

$$s^{-2} \widehat{S}(s^2GC_{1\vee}^\oplus \bigoplus_{m \geq 0} s^2\mathbb{K}v_{4m-1})$$

(3.5)

gives us the map of cochain complexes

$$\Psi^\oplus : s^{-2} \widehat{S}(s^2GC_{1\vee}^\oplus \bigoplus_{m \geq 0} s^2\mathbb{K}v_{4m-1}) \rightarrow dfG^\oplus.$$  

(3.6)

In this paper, we give a careful proof of the following statements about the full directed graph complex $dfG$ and its subcomplex $dfG^\oplus$:

**Theorem 3.1** The map (3.4) is a quasi-isomorphism of cochain complexes.

**Theorem 3.2** The map (3.6) is a quasi-isomorphism of cochain complexes.
3.1 Theorem 3.2 implies Theorem 3.1

In this section, we tacitly identify $v_1$ with the (isomorphism class of the) graph $\Gamma_\odot$ and $v_{4m-1}$ (for $m \geq 1$) with the (isomorphism class of the) graph $\Gamma_{4m+1}^\circ$ shown in figure 3.1. We also denote by $\mathcal{GC}^\circ_{1ve}(n)$ the following subspace of $\mathfrak{fGC}(n)$:

$$\mathcal{GC}^\circ_{1ve}(n) := \mathfrak{fGC}(n) \cap \mathcal{GC}^\circ_{1ve}.$$  \hspace{1cm} (3.7)

In other words,

$$\mathcal{GC}^\circ_{1ve}(n) := \begin{cases} \mathcal{GC}_{1ve}(n) \oplus \mathbb{K}[\Gamma_{4m+1}^\circ] & \text{if } n = 4m + 1 \text{ for some } m \geq 1, \\ \mathbb{K}[\Gamma_\odot] & \text{if } n = 1, \\ \mathcal{GC}_{1ve}(n) & \text{if } n \not= 1 \mod 4, \end{cases}$$

where $[\Gamma_{4m+1}^\circ]$ and $[\Gamma_\odot]$ denotes the isomorphism class of $\Gamma_{4m+1}^\circ$ and $\Gamma_\odot$, respectively.

3.1.1 The map (3.4) induces a surjective map on the level of cohomology.

Every vector $\gamma \in \partial\mathfrak{fGC}$ of a fixed degree $d$ can be written as an infinite sum

$$\gamma = \sum_{n \geq 1} \gamma_n,$$

where $\gamma_n \in \mathfrak{fGC}(n)^d$. In other words, $\gamma_n$ is a necessarily finite linear combination of graphs with $n$ vertices and $2n - 2 - d$ edges. Let us assume that $\gamma$ is a cocycle in $\mathfrak{fGC}$.

Since $\partial(\mathfrak{fGC}(n))^d \subset \mathfrak{fGC}(n+1)^{d+1}$ for every $n \geq 1$, the condition $\partial \gamma = 0$ is equivalent to

$$\partial(\gamma_n) = 0, \quad \forall \ n \geq 1.$$

Since each $\gamma_n$ belongs to $\mathfrak{fGC}^\circ_\gamma$, Theorem 3.2 implies that there exists a degree $d$ cocycles $\kappa_n$ in (3.3) and degree $d - 1$ vectors $\theta_{n-1} \in \mathfrak{fGC}(n-1)$ such that

$$\gamma_n = \begin{cases} \Psi(\kappa_n) + \partial(\theta_{n-1}) & \text{if } n \geq 2, \\ \Psi(\kappa_1) & \text{if } n = 1. \end{cases}$$

For every $n \geq 1$, $\kappa_n$ is a finite linear combination of monomials of the form

$$s^{-2}(s^2w_1s^2w_2 \cdots s^2w_q), \quad w_j \in \mathcal{GC}^\circ_{1ve}(n_j),$$

where $n_1 + n_2 + \cdots + n_q = n$.

Therefore, since

$$s^{-2}S(s^2\mathcal{GC}^\circ_{1ve}) = \prod_{q \geq 1} \prod_{n_1, \ldots, n_q \geq 1} s^{-2}(s^2\mathcal{GC}^\circ_{1ve}(n_1) \otimes \cdots \otimes s^2\mathcal{GC}^\circ_{1ve}(n_q)) \otimes_{\mathbb{Z}_q}$$

the vector $\kappa := \sum_{n=1}^{\infty} \kappa_n$ belongs to (3.9) (i.e. the source of (3.4)). The vector $\kappa$ is a cocycle in (3.9) and we have

$$\gamma = \Psi(\kappa) + \partial \theta, \ \text{where} \ \theta := \sum_{n=1}^{\infty} \theta_n.$$

3.1.2 The map (3.4) induces an injective map on the level of cohomology.

Since every vector $\kappa$ in (3.4) can written as the sum $\kappa := \sum_{n=1}^{\infty} \kappa_n$, where $\kappa_n$ is a finite linear combination of monomials of the form (3.8), we have

$$\Psi(\kappa_n) \in \mathfrak{fGC}(n) \quad \forall \ n \geq 1.$$
Let us now assume that $\kappa$ is a degree $d$ cocycle in (3.9) such that

$$\Psi(\kappa) = \partial \theta,$$  

(3.10)

where $\theta := \sum_{n=1}^{\infty} \theta_n$ and $\theta_n \in \text{dfGC}(n)$.

Since $\partial(\text{dfGC}(n)) \subset \text{dfGC}(n+1)$ for every $n \geq 1$, equation (3.10) is equivalent to

$$\Psi(\kappa_n) = \partial \theta_n - 1 \quad \forall \ n \geq 2 \quad (3.11)$$

and

$$\Psi(\kappa_1) = 0.$$ 

In particular $\kappa_1 = 0$.

Since $\kappa_{n+1}$ belongs to (3.5) and $\theta_n \in \text{dfGC}^\oplus$, equation (3.11) implies that $\kappa_n$ is exact in (3.5) for every $n \geq 2$. The exactness of $\kappa$ in (3.9) follows.

### 3.2 The cohomology of $\text{dfGC}^\oplus$ and $\text{dfGC}$ and the loopless version $\text{dfGC}^\triangleright$ of $\text{dfGC}$

Recall that, in characteristic zero, the functor $H^\bullet$ commutes with taking coinvariants (and invariants) with respect to an action of a finite group. Therefore, combining Theorem 3.2 (resp. Theorem 3.1) with the Künneth theorem, we get the following corollaries:

**Corollary 3.3** For the graph complex $\text{dfGC}^\oplus$, we have

$$H^\bullet(\text{dfGC}^\oplus) \cong s^{-2} \hat{S}(s^2 H^\bullet(\text{GC}^\oplus_{1\text{ve}}) \oplus \bigoplus_{m \geq 0} s^{4m+1} K).$$

□

**Corollary 3.4** For the full directed graph complex $\text{dfGC}$, we have

$$H^\bullet(\text{dfGC}) \cong s^{-2} \hat{S}(s^2 H^\bullet(\text{GC}_{1\text{ve}}) \oplus \bigoplus_{m \geq 0} s^{4m+1} K).$$

□

It is clear that, if $\Gamma \in \text{dgra}_r$ does not have loops, then the linear combination $\partial \Gamma$ cannot involve graphs with loops. So we denote by $\text{dfGC}^\triangleright$ the “loopless versions” of $\text{dfGC}$, i.e.

$$\text{dfGC}^\triangleright := \prod_{n \geq 1} \text{dfGC}^\triangleright(n),$$  

(3.12)

where $\text{dfGC}^\triangleright(n)$ is the subspace of $\text{dfGC}(n)$ which is spanned by graphs (with $n$ vertices) without loops.

Let us also denote by $\text{dfGC}^\triangleright_-$ the following graded vector space

$$\text{dfGC}^\triangleright_- := \prod_{n \geq 1} \text{dfGC}^\triangleright_- (n),$$  

(3.13)

where $\text{dfGC}^\triangleright_- (n)$ is the subspace of $\text{dfGC}(n)$ which is spanned by graphs with $n$ vertices and with at least one loop.

Let $\Gamma$ be an element in $\text{dgra}_r$ with exactly one loop based at vertex 1 which has valency $\geq 3$. Figure 3.2 shows that terms without loops in the linear combination $\partial(\Gamma)$ form the zero vector in $\text{dfGC}$.

Combining this observation with the fact that $\Gamma_\triangleright$ is a cocycle in $\text{dfGC}$, we conclude that the cochain complex $(\text{dfGC}, \partial)$ splits into the direct sum of its subcomplexes:

$$\text{dfGC} = \text{dfGC}^\triangleright \oplus \text{dfGC}^\triangleright_-.$$  

(3.14)
Let us also observe that the cochain complex (3.9) splits into the direct sum of subcomplexes:

\[ s^{-2} \hat{S}(s^2 \text{GC}_{1ve} \oplus \bigoplus_{m \geq 0} s^2 \mathbb{K}v_{4m-1}) = s^{-2} \hat{S}(s^2 \text{GC}_{1ve}^\circ) \oplus \left( \mathbb{K}v_{-1} \oplus \hat{S}(s^2 \text{GC}_{1ve}^\circ) \otimes \mathbb{K}v_{-1} \right), \] (3.15)

where

\[ \text{GC}_{1ve}^\circ := \text{GC}_{1ve} \oplus \mathbb{K}v_3 \oplus \mathbb{K}v_7 \oplus \mathbb{K}v_{11} \oplus \ldots \]

It is clear\(^\text{10}\) that the map \( \Psi \) is compatible with the splittings (3.14) and (3.15). Hence we proved the following statement:

**Proposition 3.5** The restriction of the map \( \Psi \) to the subspace

\[ s^{-2} \hat{S}(s^2 \text{GC}_{1ve} \oplus \bigoplus_{m \geq 1} s^2 \mathbb{K}v_{4m-1}) \] gives us a quasi-isomorphism

\[ \Psi : s^{-2} \hat{S}(s^2 \text{GC}_{1ve} \oplus \bigoplus_{m \geq 1} s^2 \mathbb{K}v_{4m-1}) \rightarrow \text{dfGC}^\circ. \] (3.17)

In particular,

\[ H^\ast(\text{dfGC}^\circ) \cong s^{-2} \hat{S}(s^2 H^\ast(\text{GC}_{1ve}) \oplus \bigoplus_{m \geq 1} s^{4m+1} \mathbb{K}). \] (3.18)

Combining Proposition 3.5 with [28, Theorem 1.1], we deduce that

**Corollary 3.6** For the loopless version \( \text{dfGC}^\circ \) of the full directed graph complex \( \text{dfGC} \), we have

\[ H^0(\text{dfGC}^\circ) \cong \text{grt}_1, \quad \text{and} \quad H^{\leq -1}(\text{dfGC}^\circ) = 0. \]

\( \square \)

### 3.3 The version \( \text{dfGC}_d \) for an arbitrary even dimension \( d \)

The cochain complex \( \text{dfGC} \) (resp. \( \text{dfGC}^\oplus \)) is a member of the family of graph complexes \( \{\text{dfGC}_d\}_{d \in \mathbb{Z}} \) (resp. \( \{\text{dfGC}^\oplus_d\}_{d \in \mathbb{Z}} \)) indexed by an even integer \( d \).

As the graded vector space,

\[ \text{dfGC}_d^\oplus := \bigoplus_{n \geq 1, r \geq 0} (s^{dn-d+r(1-d)} \text{span}_{\mathbb{K}}(\text{dgra}_n^r) \otimes \text{sgn}_r)_{S_n \times S_r}. \] (3.19)

The Lie bracket on \( \text{dfGC}^\oplus_d \) is defined by the same formula (2.5). It is easy to see that the vector \( \Gamma_{\ast \ast} \) (see (2.6)) has degree 1 in (3.19) and satisfies the MC equation (2.7). So we use the same formula (see (2.8)) for the differential \( \partial \) on (3.19).

Just as for \( \text{dfGC}^\oplus_d \), we denote by \( \text{dfGC}_d(n) \) the subspace of \( \text{dfGC}_d^\oplus \) spanned by isomorphism classes of even graphs with exactly \( n \) vertices and observe that \( \partial(\text{dfGC}_d(n)) \subset \text{dfGC}_d(n+1) \). So we define \( \text{dfGC}_d \) as the following completion of \( \text{dfGC}_d^\oplus \):

\[ \text{dfGC}_d := \prod_{n \geq 1} \text{dfGC}_d(n). \] (3.20)

\(^{10}\)Note that every 1-vertex irreducible graph \( \Gamma \) with all vertices having valencies \( \geq 3 \) cannot have a loop.
It is clear that \( dfGC = dfGC_2 \) and \( dfGC \oplus = dfGC_\oplus \).

Similarly, by using undirected graphs, we define the subcomplexes \( fGC_d, GC_d \) and \( GC_{1\text{ve},d} \) of \( dfGC_d \) and the their “uncompleted” version \( fGC_d \oplus, GC_d \oplus \) and \( GC_{1\text{ve},d} \oplus \), respectively.

To describe a link between \( dfGC_d \) (resp. \( dfGC_d \oplus \)) and \( dfGC_\oplus \) (resp. \( dfGC_\oplus \)), we remark that, for every even graph \( \Gamma \) of Euler characteristic \( \chi \), \( \partial(\Gamma) \) is the linear combination of (even) graphs of the same Euler characteristic. Thus \( dfGC_\oplus \) splits into the direct sum of cochain complexes

\[
dfGC_\oplus = \bigoplus_{\chi \in \mathbb{Z}} dfGC_{d,\chi}
\]

and \( dfGC_d \) is isomorphic to the direct product

\[
dfGC_d = \prod_{\chi \in \mathbb{Z}} dfGC_{d,\chi},
\]

where \( dfGC_{d,\chi} \) is the subcomplex of \( dfGC_\oplus \) spanned by isomorphism classes of even graphs whose Euler characteristic is \( \chi \).

Note that the degree of an even graph \( \Gamma \) can be expressed in terms of the number of vertices \( n \) and its Euler characteristic \( \chi \) by the formula

\[
deg(\Gamma) = n + (d - 1)\chi - d.
\]

Using this isomorphism, we easily obtain the following generalizations of statements of Theorems 3.1 and 3.2:

**Theorem 3.7** Let \( d \) be any even integer and \( v_{4m+1-d} \) be a symbol of degree \( 4m+1-d \). The natural embeddings

\[
s^{-d}S(s^dGC_{1\text{ve},d} \oplus \bigoplus_{m \geq 0} s^d K_{v_{4m+1-d}}) \hookrightarrow dfGC_{d,\chi}
\]

\[
s^{-d}S(s^dGC_{1\text{ve},d} \oplus \bigoplus_{m \geq 0} s^d K_{v_{4m+1-d}}) \hookrightarrow dfGC_d
\]

are quasi-isomorphisms of cochain complexes.

\[\square\]

**Remark 3.8** One can define families of graph complexes \( \{fGC_d\}_{d \in \mathbb{Z}}, \{GC_d\}_{d \in \mathbb{Z}}, \) and \( \{GC_{1\text{ve},d}\}_{d \in \mathbb{Z}} \) indexed by an arbitrary (not necessarily even) integer \( d \) and we refer the reader to \[2], \[6], \[8], \[16], \[17], \[27], \[28], \[29\] for more details about these families of graph complexes and their generalizations. For odd \( d \), the directions on edges play a special role. So Theorem 3.7 does not have a generalization to the case when \( d \) odd.

### 3.4 The proof of Theorem 3.2

In this section, we deduce Theorem 3.2 from several auxiliary statements. The remainder of this paper is devoted to the proofs of these auxiliary statements.

Recall that \( dfGC_{\text{conn}}(n) \) is the subspace of \( dfGC(n) \) spanned by (even) connected graphs. Moreover, \( dfGC_{\text{conn}} \) is the dg Lie subalgebra of \( dfGC \) introduced in 3.1.

Let us split the cochain complex \((dfGC_{\text{conn}},\partial)\) into the following direct sum of subcomplexes:

\[
dfGC_{\text{conn}} = dfGC_{\text{conn},\geq 3} \oplus dfGC_{\text{conn},0} \oplus dfGC_{\text{conn},-1}.
\]
• $dfGC_{\text{conn}, \geq 3}^{\oplus}$ is the subcomplex spanned by (even) connected graphs with at least one vertex having valency $\geq 3$,

• $dfGC_{\text{conn}, o}^{\oplus}$ is the subcomplex spanned by (even) connected graphs with all vertices having valency 2 (i.e. $\Gamma_o$ and various polygons), and

• $dfGC_{\text{conn}, -}^{\oplus}$ is the subcomplex spanned by $\Gamma \in dgra_0^0$ and uni-bivalent (even) connected graphs, i.e. a path graphs (an example of a path graph is shown in figure 3.3).

Fig. 3.3: An example of an even uni-bivalent graph

It is clear that the restriction of $\Psi^{\oplus}$ (see (3.6)) to $\bigoplus_{m \geq 0} \mathbb{K}v_{4m-1}$ gives us a chain map

$$\Psi_o : \bigoplus_{m \geq 0} \mathbb{K}v_{4m-1} \to dfGC_{\text{conn}, o}^{\oplus}.$$  (3.28)

Thus Theorem 3.2 follows from the following propositions:

**Proposition 3.9** The natural embeddings

$$GC^{\oplus} \hookrightarrow dfGC_{\text{conn}, \geq 3}^{\oplus}$$ (3.29)

and

$$GC_{\text{1ve}}^{\oplus} \hookrightarrow GC^{\oplus}$$ (3.30)

are quasi-isomorphisms of cochain complexes.

**Proposition 3.10** The chain map (3.28) is a quasi-isomorphism of cochain complexes.

**Proposition 3.11** The subcomplex $dfGC_{\text{conn}, -}^{\oplus}$ is acyclic.

The first part of Proposition 3.9 is proved in Section 4 and the second part is proved in Section 5. Propositions 3.10 and 3.11 are proved in Appendix B.

**Remark 3.12** We should remark that the proof of the statement which is very similar to the first part of Proposition 3.9 is sketched in Appendix K of [28]. The sketch of the proof given in loc. cit. is admittedly very brief. So we decided to give a careful proof of this statement.

## 4 Analyzing the subcomplex $dfGC_{\text{conn}, \geq 3}^{\oplus}$

In this section, we prove that the embedding (3.29) is a quasi-isomorphism of cochain complexes.

Let $\Gamma$ be an even connected graph with at least one vertex of valency $\geq 3$. Let us denote by $\nu_2(\Gamma)$ the number of bivalent vertices of $\Gamma$.

It is clear that the linear combination

$$\partial \Gamma$$

may involve only graphs $\Gamma'$ with $\nu_2(\Gamma') = \nu_2(\Gamma)$ or $\nu_2(\Gamma') = \nu_2(\Gamma) + 1$.

Thus we may introduce on the complex $dfGC_{\text{conn}, \geq 3}^{\oplus}$ an ascending filtration

$$\cdots \subset \mathcal{F}^{m-1}dfGC_{\text{conn}, \geq 3}^{\oplus} \subset \mathcal{F}^m dfGC_{\text{conn}, \geq 3}^{\oplus} \subset \mathcal{F}^{m+1} dfGC_{\text{conn}, \geq 3}^{\oplus} \subset \cdots$$  (4.1)

where $\mathcal{F}^m dfGC_{\text{conn}, \geq 3}^{\oplus}$ consists of vectors $\gamma \in dfGC_{\text{conn}, \geq 3}^{\oplus}$ which only involve graphs $\Gamma$ satisfying the inequality $\nu_2(\Gamma) - |\gamma| \leq m$.

It is clear that

$$\mathcal{F}^m dfGC_{\text{conn}, \geq 3}^{\oplus}$$
does not have non-zero vectors in degree $< -m$. Therefore, the filtration (4.1) is locally bounded from the left. Furthermore,

$$
\text{df}G\oplus_{\geq 3} = \bigcup_m \mathcal{F}^m \text{df}G\oplus_{\geq 3}.
$$

In other words, the filtration (4.1) is cocomplete.

It is also clear that the differential $\partial$ on the associated graded complex

$$
\text{Gr}(\text{df}G\oplus_{\geq 3}) = \bigoplus_m \mathcal{F}^m \text{df}G\oplus_{\geq 3} / \mathcal{F}^{m-1} \text{df}G\oplus_{\geq 3}. 
$$

is obtained from $\partial$ by keeping only the terms which raise the number of the bivalent vertices.

Thus, since $G\oplus$ is a subcomplex of $\text{df}G\oplus_{\geq 3}$, we conclude that

$$
(G\oplus)^k \subset F^{-k}(\text{df}G\oplus_{\geq 3})^k \cap \ker \partial_{\text{Gr}},
$$

where $(G\oplus)^k$ (resp. $F^{-k}(\text{df}G\oplus_{\geq 3})^k$) denotes the subspace of degree $k$ vectors in $G\oplus$ (resp. in $F^{-k}\text{df}G\oplus_{\geq 3}$).

We need the following technical lemma which is proved in Section 4.1 below.

**Lemma 4.1** For the filtration (4.1) on $\text{df}G\oplus_{\geq 3}$ we have

$$
H^k(\mathcal{F}^m \text{df}G\oplus_{\geq 3} / \mathcal{F}^{m-1} \text{df}G\oplus_{\geq 3}) = 0
$$

for all $m > -k$. Moreover,

$$
(G\oplus)^k = F^{-k}(\text{df}G\oplus_{\geq 3})^k \cap \ker \partial_{\text{Gr}}.
$$

It is easy to see that the restriction of (4.1) to the subcomplex $G\oplus$ gives us the “silly” filtration:

$$
\mathcal{F}^m(G\oplus)^k = \begin{cases} (G\oplus)^k & \text{if } m \geq -k, \\ 0 & \text{otherwise} \end{cases}
$$

and the associated graded complex $\text{Gr}(G\oplus)$ for this filtration has the zero differential.

Since

$$
\mathcal{F}^m(G\oplus)^k = 0 \quad \forall \ m < -k,
$$

we have

$$
F^{-k}(\text{df}G\oplus_{\geq 3})^k \cap \ker \partial_{\text{Gr}} = H^k(\mathcal{F}^{-k} \text{df}G\oplus_{\geq 3} / \mathcal{F}^{-k-1} \text{df}G\oplus_{\geq 3}).
$$

Thus, Lemma 4.1 implies that, the embedding (3.29) induces a quasi-isomorphism of cochain complexes $\text{Gr}(G\oplus) \simeq \text{Gr}(\text{df}G\oplus_{\geq 3})$. On the other hand, both filtrations (4.1) and (4.5) are locally bounded from the left and cocomplete. Therefore the embedding (3.29) satisfies all the conditions of Lemma A.3 from [9, Appendix A] and hence it is a quasi-isomorphism.

### 4.1 Proof of Lemma 4.1

#### 4.1.1 Frames

To every positive integer $n$, we assign an auxiliary groupoid $\text{Frame}_n$. An object of this groupoid is a connected directed graph $\mathcal{J} \in \text{dg}_{r,n}$ for some $r \geq 2$ satisfying the following properties:

- $\mathcal{J}$ does not have bivalent vertices and at least one vertex of $\mathcal{J}$ has valency $\geq 3$; for example the graphs shown in figure 4.1 are not frames while the graphs shown in figures 4.2 and 4.3 are frames;
- each edge adjacent to a univalent vertex (if any) of $\mathcal{J}$ originates at this univalent vertex;
- the edges of $\mathcal{J}$ are labeled in such a way that edges incident to univalent vertices (if any) precede all the remaining edges;
- finally, loops of $\mathcal{J}$ (if any) go after all the remaining edges.
A morphism from a frame $\mathcal{F}$ to a frame $\mathcal{F}'$ is an isomorphism of the underlying graphs which respects neither the total order on the set of edges, nor the directions of edges. For example, there are exactly two isomorphism classes of frames with 2 edges and the corresponding representatives are shown in figures 4.2 and 4.3.

**Example 4.2** Note that frames may have multiple edges with the same direction. For example, the frame shown in figure 4.4 has a double edge. It is clear that the frame $\mathcal{F}$ has only one non-trivial automorphism, i.e. the one which interchanges edges 5 and 6.

The total number of edges $e$ of any frame $\mathcal{F}$ splits into the sum $e = e_\star + e_- + e_\circ$, where $e_\star$ is the number of edges of $\mathcal{F}$ adjacent to univalent vertices (if any), $e_\circ$ is the number of loops of $\mathcal{F}$ (if any) and $e_-$ is the number of the remaining edges.

To describe the associated graded complex

$$\text{Gr}(\text{dfGC}_{\text{conn}, \geq 2}^\oplus),$$

we denote by $V_2$ the two dimensional vector space spanned by symbols $a$ and $b$ placed in degree 1. We consider the truncated tensor algebra of $V_2$

$$\mathcal{T}(V_2) := V_2 \oplus (V_2)^{\otimes 2} \oplus (V_2)^{\otimes 3} \oplus \ldots,$$

introduce on (the underlying vector space of) $\mathcal{T}(V_2)$ the following two differentials

$$b_1(v_1v_2\ldots v_n) := \sum_{i=1}^{n} (-1)^{i+1} v_1 \ldots v_i(a+b)v_{i+1} \ldots v_n,$$

$$b_2(v_1v_2\ldots v_n) := -(a+b)v_1 \ldots v_n + b_1(v_1v_2\ldots v_n), \quad v_i \in V_2,$$

and denote by $U$ and $R$ the corresponding cochain complexes:

$$U = (\mathcal{T}(V_2), b_1), \quad R = (\mathcal{T}(V_2), b_2).$$

For our purposes, it is convenient to use the simplified notation $v_1v_2\ldots v_n$ for the tensor monomial $v_1\otimes v_2\otimes \ldots \otimes v_n$ in $(V_2)^{\otimes n}$. Of course, in general, $v_1\ldots v_i v_{i+1} \ldots v_n \neq -v_1\ldots v_{i+1} v_i \ldots v_n$ in $(V_2)^{\otimes n}$.

We observe that the cochain complex $R$ carries the following action of the group $S_2$

$$\sigma(v_1v_2\ldots v_n) := (-1)^{\frac{n(n-1)}{2}} v_n^{-1} v_{n-1} \ldots v_1,$$

11. We call objects of the groupoid Frame$_n$ frames.
12. $U$ stands for “univalent” and $R$ stands for “remaining”.

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where \( \sigma = (1, 2) \in S_2 \), each \( v_j \) is either \( a \) or \( b \), and
\[
\overline{v_j} = \begin{cases} 
  b & \text{if } v_j = a, \\
  a & \text{if } v_j = b.
\end{cases}
\]

Given a frame \( \mathcal{I} \in \text{Frame}_n \) with \( e = e_* + e_- + e_o \) edges and a tensor product of monomials
\[
P = p_1 \otimes \cdots \otimes p_{e_*} \otimes p_{e_* + 1} \otimes \cdots \otimes p_e \in U^{\otimes e_*} \otimes R^{\otimes (e_- + e_o)}
\]
we can form a graph \( \Gamma \in \text{dgra}_m \), where
\[
m = n + \sum_{i=1}^{e} (\deg(p_i) - 1) \quad \text{and} \quad r = \deg(p_1) + \deg(p_2) + \cdots + \deg(p_e).
\]

To better visualize this construction, it is convenient to think of the tensor product of monomials \( P \) as the sequence of disjoint arrows: the symbol \( a \) corresponds to the arrows pointing to the right and the symbol \( b \) corresponds to the arrow pointing to the left. Such a sequence of arrows has exactly \( 2e \) vertices. We label these vertices in the natural order from left to right. For example, the tensor product
\[
b \otimes ab \otimes a \otimes b \otimes ba \otimes a \otimes aba \otimes b
\]
corresponds to the disjoint graph:

To form the graph \( \Gamma \) corresponding to the monomial (4.11) and the frame \( \mathcal{I} \), we follow these steps:

- For every \( 1 \leq i \leq e \), we introduce vertices \( s_i \) and \( t_i \) and convert the symbols of the monomial \( p_i \) into \( k_i \) directed edges which connect \( s_i \) with \( t_i \) by these rule: if a symbol \( w \) in \( p_i \) stands to the left of the symbol \( w' \) then the edge corresponding to \( w \) is closer to \( s_i \). Then the edge corresponding to \( w' \), if the \( j \)-th symbol in \( p_i \) is \( a \) then we direct corresponding edge toward \( t_i \); otherwise, we direct the corresponding edge towards \( s_i \). For example, if \( p_i = aaba \) then we assign to \( p_i \) this string of edges

\[
\begin{array}{c}
s_i \\
\cdots \\
t_i
\end{array}
\]

- If the \( i \)-th edge of \( \mathcal{I} \) is not a loop, then we replace it with the above string of directed edges corresponding to \( p_i \) via identifying \( s_i \) (resp. \( t_i \)) with what was the source (resp. the target) of the \( i \)-th edge of \( \mathcal{I} \).
- If the \( i \)-th edge of \( \mathcal{I} \) is a loop based, say, at vertex \( v \) then we remove it and identify both \( s_i \) and \( t_i \) with \( v \).
- Since the resulting graph \( \Gamma \) is obtained by gluing edges corresponding to symbols in the tensor product \( P \), we get the obvious map
\[
\psi : \{1, 2, \ldots, 2e\} \to V(\Gamma),
\]
where \( V(\Gamma) \) is the set of vertices of \( \Gamma \). We order these vertices according to this rule: \( v_1 < v_2 \iff \min(\psi^{-1}(v_1)) < \min(\psi^{-1}(v_2)) \).

- Since the set of edges of \( \Gamma \) is in bijection with symbols in (4.11), it is already equipped with a total order.

In what follows, we will say that \( \Gamma \) is the graph reconstructed from the tensor product of monomials in (4.11) using the frame \( \mathcal{I} \).

It is easy to see that the assignment \( F_2(P) := \Gamma \) defines a degree zero map
\[
F_2 : s^{2n-2-2e} U^{\otimes e_*} \otimes R^{\otimes (e_- + e_o)} \to \text{Gr}(\text{dfGC}_{\text{conn} \geq 3}).
\]

**Example 4.3** Let \( \mathcal{I} \) be the frame shown in figure 4.4 and \( P \) be the tensor product of monomials
\[
P = b \otimes (ab \otimes a \otimes b \otimes ba \otimes a \otimes aba \otimes b) \in U \otimes R^{\otimes 7}.
\]

Then \( F_2(P) = \Gamma \), where \( \Gamma \) is the labeled graph shown in figure 4.5.
Remark 4.4 It may happen that the graph $\Gamma$ reconstructed from a tensor product of monomials $P$ in $U \otimes e \otimes R \otimes (e^- + e_\circ)$ is odd and hence $F_3(P) = 0$. For example if $\triangledown$ is the frame shown in figure 4.4 then the graph reconstructed from the monomial $P' = b \otimes (ab \otimes a \otimes b \otimes b \otimes aba \otimes b) \in U \otimes R \otimes 7$

has a pair of edges with the same direction connecting the same vertices. Hence $F_3(P') = 0$.

Here is another example. If $Q = aba \otimes aba$

then the graph reconstructed from $Q$ using the frame $\triangledown \bigodot \bigodot$ in figure 4.2 is shown in figure 4.6. This graph has the obvious automorphism which “switched the triangles” and this automorphism gives us the odd permutation $(1, 4)(2, 5)(3, 6)$ in $S_6$. Hence $F_3(Q) = 0$.

Let us denote by $\partial^{Gr}$ the differential on the associated graded complex $Gr(dfG_{\text{conn,} \geq 3})$. It is clear from the definition of the filtration (4.1) on $dfG_{\text{conn,} \geq 3}$ that $\partial^{Gr}$ is obtained from $\partial$ by keeping only the terms which raise the number of the bivalent vertices. Hence the image of the map $F_3$ is closed with respect to the action the differential $\partial^{Gr}$.

Using equation (2.10) it is easy to show that for every frame $\triangledown$,

$$\partial^{Gr} \circ F_3 = F_3 \circ \delta,$$

where $\delta$ is the differential on the vector space

$$s^{2n-2-2e} U \otimes e \otimes R \otimes (e^- + e_\circ)$$

given by the formula $\delta = b_1 \otimes 1 + 1 \otimes b_2$.

4.1.2 The kernel of $F_3$

To describe the kernel of the map $F_3$, we introduce the semi-direct product $S_e \ltimes (S_2)^e$ of the groups $S_e$ and $(S_2)^e$ with the multiplication rule:

$$(\tau; \sigma_1, \ldots, \sigma_e) \cdot (\lambda; \sigma_1', \ldots, \sigma_e') = (\tau \lambda; \sigma_{\lambda(1)} \sigma_1', \ldots, \sigma_{\lambda(e)} \sigma_e'),$$

where, as above, $e$ is the number of edges of the frame $\triangledown$.

The subgroup

$$G := (S_e \times S_{e^-} \times S_{e_\circ}) \ltimes (\text{id})^e \times S_2^{(e^- + e_\circ)}$$

of $S_e \ltimes (S_2)^e$ acts on the graded vector space (4.15) in the following way:
• If $\sigma = (1, 2) \in S_2$ and $e_* < i \leq e$ then
  \[(1, \ldots, 1, \sigma_{i-1}, 1, \ldots, 1)(p_1 \otimes \cdots \otimes p_{e_*} \otimes p_{e_*+1} \otimes \cdots \otimes p_e) := p_1 \otimes \cdots \otimes p_{e_*} \otimes p_{e_*+1} \otimes \cdots \otimes \sigma(p_i) \otimes \cdots \otimes p_e,
\]
  where $\sigma(p_i)$ is defined in (4.10).

• For every $\tau \in S_{e_*} \times S_{e_-} \times S_{e_0}$ we set
  \[\tau(p_1 \otimes p_2 \otimes \cdots \otimes p_e) := (-1)^{\epsilon(\tau)} p_{\tau^{-1}(1)} \otimes p_{\tau^{-1}(2)} \otimes \cdots \otimes p_{\tau^{-1}(e)},
\]
  where the sign factor $(-1)^{\epsilon(\tau)}$ is determined by the usual Koszul rule.

We will now prove the following claim:

**Claim 4.5** Let $\mathcal{J}$ be a frame with $e = e_* + e_- + e_0$ edges and $\Gamma, \Gamma'$ be the graphs reconstructed from the tensor products of monomials

\[X = p_1 \otimes p_2 \otimes \cdots \otimes p_e, \quad X' = p'_1 \otimes p'_2 \otimes \cdots \otimes p'_e
\]

in (4.10), respectively, using $\mathcal{J}$. Let $k$ be the number of edges of $\Gamma$ and $\varphi : \Gamma \to \Gamma'$ be an isomorphism of graphs which induces a permutation $\sigma_{\varphi} \in S_k$. If $\sigma_{\varphi}$ is even, then there exists an element $g \in G_2$ such that $g(X) = X'$. If $\sigma_{\varphi}$ is odd, then there exists an element $g \in G_2$ such that $g(X) = -X'$.

**Proof.** Let $\hat{\varphi}$ be the automorphism of $\mathcal{J}$ corresponding to the isomorphism $\varphi : \Gamma \to \Gamma'$ and $\tau$ be the permutation in $S_{e_*} \times S_{e_-} \times S_{e_0}$ coming from $\hat{\varphi}$.

By construction, the edges of $\Gamma$ (resp. $\Gamma'$) are in bijection with symbols of the monomial in $X$ (resp. $X'$). Hence the isomorphism $\varphi$ gives us a bijection from the set of symbols of $X$ to the set of symbols of $X'$. Furthermore, for every $1 \leq i \leq e$ the isomorphism $\varphi$ gives us a bijection from the set of symbols of $p_i$ to the set of symbols of $p'_{\tau(i)}$.

Thus we set

\[g = (\tau; \sigma_1, \ldots, \sigma_e) \in G_2,
\]

where the elements $\sigma_i \in S_2$ are specified by considering these three cases:

**Case 1:** $1 \leq i \leq e_*$, i.e. the $i$-th edge of $\mathcal{J}$ originates at a univalent vertex. Since the isomorphism $\varphi$ is compatible with the directions of edges (in $\Gamma$ and $\Gamma'$), in this case we have $p'_{\tau(i)} = p_i$.

**Case 2:** $e_* + 1 \leq i \leq e_* + e_-$, i.e. the $i$-th edge of $\mathcal{J}$ connects two distinct vertices of valencies $\geq 3$. In this case we have two possibilities: if the automorphism $\hat{\varphi}$ of $\mathcal{J}$ respects the directions of edges $i$ and $\tau(i)$, then $p'_{\tau(i)} = p_i$; if the automorphism $\hat{\varphi}$ of $\mathcal{J}$ does not respect the directions of edges $i$ and $\tau(i)$, then $p'_{\tau(i)}$ coincides with $\sigma(p_i)$ up to the appropriate sign factor, where $\sigma = (12) \in S_2$ and $\sigma(p_i)$ is defined in (4.10).

If the first possibility realizes, we set $\sigma_i = \text{id} \in S_2$. Otherwise, we set $\sigma_i = (12) \in S_2$.

**Case 3:** $e_* + e_- + 1 \leq i \leq e$, i.e. the $i$-th edge of $\mathcal{J}$ is a loop. Here we have four possibilities. First, if $p'_{\tau(i)} = p_i = a$ or $p'_{\tau(i)} = p_i = b$ then we set $\sigma_i = \text{id} \in S_2$. Second, if $p'_{\tau(i)} = a$, $p_i = b$ or $p'_{\tau(i)} = b$, $p_i = a$, then we set $\sigma_i = (12) \in S_2$. Third, if $p_i$ has more than one symbol and $\varphi$ respects the orders on the edges coming from symbols of $p_i$ and $p'_{\tau(i)}$, then $p'_{\tau(i)} = p_i$ and we set $\sigma_i = \text{id} \in S_2$. Fourth, if $p_i$ has more than one symbol and $\varphi$ does not respect the orders on the edges coming from symbols of $p_i$ and $p'_{\tau(i)}$, then $p'_{\tau(i)}$ coincides with $\sigma(p_i)$ up to the appropriate sign factor and we set $\sigma_i = (12) \in S_2$.

This analysis shows that, for the element $g \in G_2$ constructed in this way, $g(X) = X'$ or $g(X) = -X'$ depending on whether the permutation $\sigma_{\varphi} \in S_k$ is even or odd, respectively.

Setting $X = X'$ in Claim 4.4 we get the following obvious corollary:

**Corollary 4.6** Let $\mathcal{J}$ be a frame with $e = e_* + e_- + e_0$ edges and $\Gamma$ be the graph reconstructed from the tensor product of monomials

\[X = p_1 \otimes p_2 \otimes \cdots \otimes p_e
\]

in (4.10) using $\mathcal{J}$. If $\Gamma$ has an automorphism which induces an odd permutation on the set of edges (i.e. $\Gamma$ is odd) then there exists an element $g \in G_2$ such that $g(X) = -X$. □
**Example 4.7** Let $\mathcal{J}$ be the frame shown in figure 4.4 and $P, P'$ be the following tensor products of monomials in $U \otimes R^\otimes 7$

$$P = b \otimes (ab \otimes a \otimes b \otimes ba \otimes a \otimes aba \otimes b), \quad P' = b \otimes (ab \otimes a \otimes b \otimes a \otimes ba \otimes bab \otimes a),$$

The graph $\Gamma$ (resp. $\Gamma'$) reconstructed from $P$ (resp. $P'$) using the frame $\mathcal{J}$ is shown in figure 4.5 (resp. in figure 4.7). We have the obvious isomorphism $\varphi: \Gamma \to \Gamma'$. This isomorphism acts by identity on the set of vertices $\{1, 2, \ldots, 9\}$ and the action on the ordinals of edges is defined by this permutation in $S_{12}$

$$\sigma_\varphi = (6, 7, 8)(9, 11).$$

The automorphism $\tilde{\varphi} \in \text{Aut}(\mathcal{J})$ corresponding to the isomorphism $\varphi$ is the only non-trivial automorphism, i.e. the one which flips the double edges 5 and 6 of $\mathcal{J}$. So the permutation $\tau \in S_8$ is the transposition $(5, 6)$.

Finally, we see that $P' = - (\tau; 1, 1, 1, 1, 1, 1, 1, \sigma, \sigma)(P)$, where $\tau = (5, 6) \in S_8$ and $\sigma = (1, 2) \in S_2$. It agrees with the fact that the permutation $\sigma_\varphi = (6, 7, 8)(9, 11)$ is odd. Thus, $F_2(P) = - F_2(P')$.

**Example 4.8** Let $\mathcal{J} \otimes \otimes$ be the frame shown in figure 4.2 and $Q = aba \otimes aba$. The graph $\Gamma_Q$ reconstructed from $Q$ using $\mathcal{J} \otimes \otimes$ is shown in figure 4.6. As we saw above, $\Gamma_Q$ has the obvious automorphism $\varphi$ which “switched the triangles” and induces the permutation $\sigma_\varphi = (1, 4)(2, 5)(3, 6) \in S_6$. It is clear that $Q = - (\tau; 1, 1)(Q)$, where $\tau = (1, 2)$.

Let now use Claim 4.5 and Corollary 4.6 to describe the kernel of the map $F_2(4.12)$:

**Claim 4.9** Let $\mathcal{J} \in \text{Frame}_n$ be a frame with $e$ edges

$$e = e_+ + e_- + e_\circ,$$

where $e_+$ is the number of edges of $\mathcal{J}$ adjacent to univalent vertices, $e_\circ$ is the number of loops and $e_- = e - e_+ - e_\circ$.

Then the kernel of $F_2$ is spanned by vectors of the form

$$X - g(X),$$

where $X$ is a vector in (4.15) and $g \in \mathcal{G}_2$.

Proof. Let us show that, for every tensor product of monomials $X$ in (4.15) and for every $g \in \mathcal{G}_2$,

$$X - g(X) \in \ker F_2.$$  \hspace{1cm} (4.19)

If the graph $\Gamma$ reconstructed from $X$ is odd then $F_2(X) = 0$ and, similarly, $F_2(g(X)) = 0$. So let us assume that $\Gamma$ is even.

If the element $g$ induces an even permutation on the ordinal of edges of $\Gamma$ then

i) $g(X)$ is also a tensor product of monomials,

ii) $g$ gives us an isomorphism from $\Gamma$ to the graph $\Gamma'$ reconstructed from $g(X)$, and
iii) this isomorphism induces an even permutation in $S_k$, where $k$ is the number of edges of $\Gamma$ (i.e. the graphs $\Gamma$ and $\Gamma'$ are concordant).

Thus $\Gamma = \Gamma'$ in $\text{dfGC}^\oplus$ and, in this case, inclusion (4.19) holds.

If the element $g$ induces an odd permutation on the ordinal of edges of $\Gamma$ then

i) $-g(X)$ is a tensor product of monomials,

ii) $g$ gives us an isomorphism from $\Gamma$ to the graph $\Gamma'$ reconstructed from $-g(X)$, and

iii) this isomorphism induces an odd permutation in $S_k$, where $k$ is the number of edges of $\Gamma$. (i.e. the graphs $\Gamma$ and $\Gamma'$ are opposite).

Thus $\Gamma + \Gamma' = 0$ in $\text{dfGC}^\oplus$ and, in this case, inclusion (4.19) also holds.

Let us now consider a tensor product of monomials $Y$ in (4.15) satisfying the property

$$F_2(Y) = 0. \quad (4.20)$$

The latter means that the graph $\Gamma$ reconstructed from $Y$ is odd, i.e. there exists an automorphism $\varphi$ of $\Gamma$ which induces an odd permutation on the set of edges of $\Gamma$. Corollary 4.6 implies that there exists $g \in G_2$ such that $Y = -g(Y)$. Hence, $Y = \frac{1}{2}(Y - g(Y))$. Thus every tensor product of monomials $Y$ in (4.15) satisfying equation (4.20) belongs to the span of vectors of the form (4.18).

Let us now consider a linear combination

$$c_1Y_1 + c_2Y_2 + \cdots + c_mY_m, \quad c_i \in \mathbb{K} \quad (4.21)$$

of tensor products of monomials $Y_1, \ldots, Y_m$ in (4.15) such that

$$\sum_i c_iF_2(Y_i) = 0. \quad (4.22)$$

Due to the above observation about monomials satisfying (4.20) we may assume, without loss of generality, that

$$F_2(Y_i) \neq 0 \quad \forall \quad 1 \leq i \leq m.$$ 

In other words, the graph $\Gamma_i$ reconstructed from $Y_i$ is even for every $1 \leq i \leq m$.

We may also assume, without loss of generality, that the graphs $\{\Gamma_i\}_{1 \leq i \leq m}$ have the same number of vertices $n + n'$.

Thus, for every $1 \leq i \leq m$, the graph $\Gamma_i \in \text{dgra}_{n+n'}$ is even.

By definition of $\text{dfGC}^\oplus$, the number $m$ is even and the set of graphs $\{\Gamma_i\}_{1 \leq i \leq m}$ splits into pairs

$$(\Gamma_i, \Gamma'_i), \quad t \in \{1, \ldots, m/2\}$$

such that for every $t$ the graphs $\Gamma_i$ and $\Gamma'_t$ are either opposite or concordant. For every pair $(\Gamma_i, \Gamma'_t)$ of opposite graphs we have

$$c_i = c'_t. \quad (4.23)$$

For every pair $(\Gamma_i, \Gamma'_t)$ of concordant graphs we have

$$c_i = -c'_t. \quad (4.24)$$

Let $e_t$ denote the number of edges of $\Gamma_i$ (or $\Gamma'_t$) and let $\varphi_t$ be the isomorphism from $\Gamma_i$ to $\Gamma'_t$ which induces an odd or even permutation in $S_{e_t}$ depending on whether $\Gamma_i$ and $\Gamma'_t$ are opposite or concordant. Let $g_t \in G_2$ be the element induced by the isomorphism $\varphi_t$ as in Claim 4.5. Equations (4.22) and (4.24) and Claim 4.5 imply that

$$\sum_{i=1}^m c_i Y_i = \sum_{i=1}^{m/2} c_i (Y_{it} - g_t(Y_{it})).$$

In other words, the linear combination (4.21) belongs to the span of vectors of the form (4.18) and the claim follows.

\footnote{See Definition 2.2 and Remark 2.3.}
4.1.3 Description of the associated graded complex $\text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3})$

Now we are ready to give a convenient description of $\text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3})$.

**Claim 4.10** Let us choose a representative $\mathcal{J}_z$ for every isomorphism class $z \in \pi_0(\text{Frame}_n)$. Let

$$e^z = e^z_\circ + e^z_\circ - e^z_0$$

be the number of edges of $\mathcal{J}_z$, where $e^z_\circ$ is the number of edges adjacent to univalent vertices, $e^z_0$ is the number of loops and $e^z_\circ$ is the number of the remaining edges. Then the cochain complex $\text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3})$ splits into the direct sum

$$\text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3}) \cong \bigoplus_{n \geq 1} \bigoplus_{z \in \pi_0(\text{Frame}_n)} s^{2n-2-2e^z} (U^\otimes e^z_\circ \otimes R^\otimes (e^z_\circ + e^z_0))_{\mathcal{G}_{2z}},$$

(4.25)

where $U$ and $R$ are the cochain complexes introduced in (4.10).

Proof. Let us recall that the map $F_{\mathcal{J}_z}$ (4.12) is a morphism from the cochain complex

$$s^{2n-2-2e^z} U^\otimes e^z_\circ \otimes R^\otimes (e^z_\circ + e^z_0)$$

to $\text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3})$.

Thus, Claim 4.9 implies that $F_{\mathcal{J}_z}$ induces an isomorphism from the cochain complex of coinvariants

$$s^{2n-2-2e^z} (U^\otimes e^z_\circ \otimes R^\otimes (e^z_\circ + e^z_0))_{\mathcal{G}_{2z}}$$

to the subcomplex

$$\text{Im}(F_{\mathcal{J}_z}) \subset \text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3}).$$

On the other hand, the cochain complex $\text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3})$ is obviously the direct sum

$$\text{Gr}(\text{dfGC}^{\oplus}_{\text{conn}, \geq 3}) = \bigoplus_{n \geq 1} \bigoplus_{z \in \pi_0(\text{Frame}_n)} \text{Im}(F_{\mathcal{J}_z}).$$

(4.26)

Thus, the desired statement follows. □

4.1.4 The end of the proof of Lemma 4.1

We now have all we need to prove Lemma 4.1. Let $\mathcal{J}_z$ be a representative of an isomorphism class $z \in \pi_0(\text{Frame}_n)$ and let

$$e^z = e^z_\circ + e^z_\circ + e^z_0$$

be the number of edges of $\mathcal{J}_z$, where $e^z_\circ$ is the number of edges adjacent to univalent vertices, $e^z_0$ is the number of loops and $e^z_\circ$ is the number of the remaining edges.

Let us consider the cochain complex

$$s^{2n-2-2e^z} U^\otimes e^z_\circ \otimes R^\otimes (e^z_\circ + e^z_0).$$

(4.27)

Due to Claim A.1 from Appendix A, the cochain complex $U$ is acyclic,

$$H^k(U) = \begin{cases} \mathbb{K} & \text{if } k = 1, \\ 0 & \text{otherwise}, \end{cases}$$

and the $H^1(R)$ is spanned by the class represented by the cocycle $(a+b)$.

Thus Künneth’s theorem implies that

$$H^k \left( s^{2n-2-2e^z} U^\otimes e^z_\circ \otimes R^\otimes (e^z_\circ + e^z_0) \right) = \begin{cases} \mathbb{K} & \text{if } e^z_\circ = 0 \text{ and } k = 2n - 2 - e^z, \\ 0 & \text{otherwise}. \end{cases}$$

Combining this observation with Claim 4.10, we conclude that (4.3) and (4.4) indeed hold. Thus Lemma 4.1 is proved. □
5 On the subcomplex of 1-vertex irreducible graphs $GC_{1ve}$

The goal of this section is to prove the second part of Proposition 3.9, i.e. that the chain map

\[ GC_{1ve}^0 \to GC^0 \] (5.1)

is a quasi-isomorphism of cochain complexes.

The corresponding statement for the (commutative) graph complex introduced in [20] was proved in [5]. However, since the (commutative) graph complex from [20] is quite different from $GC^0$, we decided to write a detailed proof of the fact that (5.1) is a quasi-isomorphism. Needless to say, this proof is similar to the one given in [5] for the (commutative) graph complex from [20].

5.1 The filtration by the number of separating edges

Let $\Gamma$ be a connected graph whose all vertices have valencies $\geq 3$. Recall that $i \in V(\Gamma)$ is a cut vertex if $\Gamma$ becomes disconnected upon deleting $i$. Furthermore, we call $e \in E(\Gamma)$ a separating edge if $\Gamma$ becomes disconnected upon deleting $e$. For a graph $\Gamma$ with at least one cut vertex, we denote by $\Gamma^{\text{red}}$ the graph which is obtained from $\Gamma$ by contracting all separating edges. If $\Gamma$ does not have separating edges, we simply set $\Gamma^{\text{red}} := \Gamma$.

To prove that (5.1) is a quasi-isomorphism, we denote by $QC^0$ the quotient complex

\[ QC^0 := GC^0 / GC_{1ve}^0 \] (5.2)

and observe that $QC^0$ is the span of isomorphism classes of even connected graphs with at least one cut vertex and with all vertices having valency $\geq 3$.

Next, we equip the complex $QC^0$ with the following ascending filtration

\[ \cdots \subset F_{m-1}QC^0 \subset F_mQC^0 \subset F_{m+1}QC^0 \subset \cdots \] (5.3)

where $F_mQC^0$ consists of vectors $\gamma \in QC^0$ which only involve graphs $\Gamma$ satisfying the inequality

\[ \# \text{ of separating edges of } \Gamma - |\gamma| \leq m. \]

To describe the associated graded cochain complex, we consider graphs $\mathfrak{D}$ which fulfill the condition:

**Condition 5.1** $\mathfrak{D}$ is a connected graph with all vertices having valency $\geq 3$. $\mathfrak{D}$ has at least 1 cut vertex but it does not have separating edges.

An example of a graph satisfying the above condition is shown in figure 5.1.

We denote by $\text{Home}$ the set of isomorphism classes of such graphs $\mathfrak{D}$. For every $z \in \text{Home}$, we denote by $\mathfrak{D}_z$ once and for all chosen representative of $z$. Finally, we denote by

\[ QC^0_{\mathfrak{D}} \] (5.4)

the subspace of $QC^0$ spanned by graphs $\Gamma$ for which $\Gamma^{\text{red}}$ is isomorphic to $\mathfrak{D}$.

Since the differential $\partial_{\text{Gr}}$ of the associated graded complex can only raise the number of separating edges, the subspace $QC^0_{\mathfrak{D}}$ is closed with respect to $\partial_{\text{Gr}}$. Thus the associated graded complex $Gr F QC^0$ of $QC^0$ splits into the direct sum

\[ Gr F QC^0 \cong \bigoplus_{z \in \text{Home}} QC^0_{\mathfrak{D}_z}. \] (5.5)

We claim that

**Proposition 5.2** For every $z \in \text{Home}$ the cochain complex

\[ (QC^0_{\mathfrak{D}_z}, \partial_{\text{Gr}}) \]

is acyclic.
This proposition is proved in Section 5.3 below. Let us now use it to prove that the quotient complex $\mathbb{Q}C^\oplus$ is acyclic.

Indeed, it is easy to see that

$$\mathbb{Q}C^\oplus = \bigcup_m \mathcal{F}_m \mathbb{Q}C^\oplus.$$ 

Moreover, $\mathcal{F}_m(\mathbb{Q}C^\oplus)^d = 0$ for every $m < -d$.

Thus, applying [9, Claim A.2] and Proposition 5.2 we deduce that $\mathbb{Q}C^\oplus$ is acyclic.

This concludes a proof of the second part of Proposition 3.9 modulo Proposition 5.2.

5.2 Assembling a connected graph from a “separating” forest and islands

Let us consider a connected graph $\Gamma$ with at least one cut vertex (and with all vertices having valency $\geq 3$). It is clear that all separating edges and cut vertices of $\Gamma$ form a forest $F$ and we call $F$ the separating forest of $\Gamma$. We will now explain how any connected graph $\Gamma$ with at least one cut vertex can be reconstructed from its separating forest and 1-vertex irreducible graphs which we call islands. For this purpose, we need a more precise terminology.

An island is a 1-vertex irreducible graph $\Upsilon$ with a non-zero number of marked vertices. Each marked vertex of $\Upsilon$ must have valency $\geq 2$ and all the remaining vertices of $\Upsilon$ must have valency $\geq 3$.

Let $S = \{c_1, c_2, \ldots, c_m\}$ be an auxiliary set with $\geq 2$ elements. An $S$-decorated forest $F$ is a forest with a possibly empty set of internal vertices of valencies $\geq 3$, a non-empty set of external vertices $V_{ext}(F)$ and a surjective map

$$p : S \rightarrow V_{ext}(F)$$

satisfying these properties:

- each connected component of $F$ has at least one external vertex;
- if a connected component of $F$ has exactly one external vertex $v$ then $p^{-1}(v)$ has at least 2 elements.

Note that, since each internal vertex has valency $\geq 3$, all univalent and bivalent vertices of any $S$-decorated forest $F$ are external. So if a connected component of $F$ has exactly one external vertex $v$ then this connected component consists of only one vertex and it does not have any edges.

Figure 5.2 shows an example of a $\{c_1, \ldots, c_7\}$-decorated forest $F$ and figure 5.3 shows a collection of islands with $\{c_1, \ldots, c_7\}$ being the set of its marked vertices. In pictures, internal vertices of forests are depicted by
gray bullets, edges of forests are depicted by dashed lines, and external vertices are depicted by white vertices with inscribed pre-images of $p$. Using this forest $F$ and the collection of islands from figure 5.3 we assemble the graph shown in figure 5.4 via merging every external vertex $v \in V_{\text{ext}}(F)$ with the vertices of the islands marked by elements in $p^{-1}(v)$. For example, the bivalent vertex of $F$ is merged with the marked vertex $c_1$ of the tetrahedron and the marked vertex $c_4$ of the “square”.

![Fig. 5.2: A $\{c_1, \ldots, c_7\}$-decorated forest $F$.](image1)

The only internal vertex is shown as the gray bullet

![Fig. 5.3: A collection of islands](image2)

![Fig. 5.4: The graph assembled from the forest in figure 5.2 and the islands in figure 5.3. The cut vertices are shown as gray bullets and separating edges are shown as dashed lines](image3)

It is clear that every connected graph $\Gamma$ with at least one cut vertex (and all vertices having valency $\geq 3$) can be disassembled into a collection of islands (with at least 2 elements) and its separating forest decorated by the set of marked vertices of the islands.

An example of this process is shown in figures 5.5, 5.6 and 5.7. More precisely, we start with a graph $\Gamma$ shown in figure 5.5. The cut vertices of $\Gamma$ are shown as gray bullets and separating edges of $\Gamma$ are shown as dashed lines.

By detaching the separating forest from $\Gamma$ we get a disconnected graph with some edges having “free” ends shown in figure 5.6. To get the islands, we attach the “free” ends to appropriate marked vertices $c_1, c_2, \ldots, c_7$ and decorate the separating forest by the set $\{c_1, c_2, \ldots, c_7\}$. The resulting collection of islands and the $\{c_1, c_2, \ldots, c_7\}$-decorated forest are shown in figure 5.7.

### 5.3 The proof of Proposition 5.2

Let $\square$ be a graph satisfying Condition 5.1 with $n_0$ non-cut vertices, $n_1$ cut vertices and $r_0$ edges.

Let us fix integers $r \geq 0$, $n \geq n_1$ and denote by $\text{gra}^{r_0 + r}_{n_0 + n}$ the set of connected graphs $\Gamma \in \text{gra}^{r_0 + r}_{n_0 + n}$ which satisfy the following conditions:

- every vertex of $\Gamma$ has valency $\geq 3$,
- $\Gamma$ has at least one cut vertex and $\Gamma^{\text{red}}$ is isomorphic to $\square$,
- $\Gamma$ has exactly $r$ separating edges and they are labeled by $\{r_0 + 1, r_0 + 2, \ldots, r_0 + r\}$,
- $\Gamma$ has exactly $n$ cut vertices and they are labeled by $\{n_0 + 1, n_0 + 2, \ldots, n_0 + n\}$.
The last two conditions imply that \( n_0 \) non-cut vertices (resp. \( r_0 \) non-separating edges) of \( \Gamma \) are labeled by \( \{1, 2, \ldots, n_0\} \) (resp. \( \{1, 2, \ldots, r_0\} \)).

For example, the graph \( \Gamma \) shown in figure 5.8 belongs to \( \text{gra}(\mathcal{S})^{2+1}_{11+2} \), where \( \mathcal{S} \) is depicted in figure 5.1.

The group \( S_n \times S_r \) acts on \( \text{gra}(\mathcal{S})^{r_0+r}_{n_0+n} \) in the obvious way by rearranging the labels

\[
\begin{align*}
n_0 + 1, n_0 + 2, \ldots, n_0 + n & \quad \text{and} \quad r_0 + 1, r_0 + 2, \ldots, r_0 + r.
\end{align*}
\]

Using this action, we define the graded vector space

\[
C_\mathcal{S} := \bigoplus_{n \geq n_1, \ r \geq 0} \left( s^2(n_0+n)-2-(r_0+r) \, \text{span}_K (\text{gra}(\mathcal{S})^{r_0+r}_{n_0+n}) \otimes \text{sgn}_r \right)_{S_n \times S_r}.
\]

(5.7)

Let us denote by \( \delta \) the linear map

\[
\delta : \text{span}_K (\text{gra}(\mathcal{S})^{r_0+r}_{n_0+n}) \rightarrow \text{span}_K (\text{gra}(\mathcal{S})^{r_0+r+1}_{n_0+n+1})
\]

defined by the formula

\[
\delta(\Gamma) = -(-1)^{||\Gamma||} \sum_{i=n_0+1}^{n_0+n} \delta_i(\Gamma),
\]

(5.8)

where \( \Gamma \in \text{gra}(\mathcal{S})^{r_0+r}_{n_0+n} \) and \( \delta_i(\Gamma) \) is obtained from \( \Gamma \circ_i \Gamma \) by retaining only graphs with vertices of valencies \( \geq 3 \) for which the additional edge is separating.

It is easy to see that \( \delta \) descends to a linear map of degree 1 on \( C_\mathcal{S} \). For example, if \( \Gamma \) is the graph shown in figure 5.8 then

\[
\delta(\Gamma) = \delta_{12}(\Gamma)
\]

because \( \Gamma \circ_{13} \Gamma \) involves only “unwanted” graphs. Moreover, \( \delta_{12}(\Gamma) \) is the sum of the graphs depicted in figures 5.9 and 5.10.

Let us prove that

**Claim 5.3** The map \( \delta : C_\mathcal{S} \rightarrow C_\mathcal{S} \) satisfies the identity

\[
\delta^2 = 0.
\]
Fig. 5.8: An example of a graph $\Gamma \in \text{gra}(\mathbb{Z}_{11+2}^{22+1})$. The (only separating) edge connecting 12 to 13 is labeled by 23. The labels 1, 2, \ldots, 22 for non-separating edges are not shown.

Fig. 5.9: The edge (12, 14) is labeled by 23 and the edge (12, 13) is labeled by 24. The labels 1, 2, \ldots, 22 for non-separating edges are not shown.
Fig. 5.10: The edge (13, 14) is labeled by 23 and the edge (12, 13) is labeled by 24. The labels 1, 2, ..., 22 for non-separating edges are not shown

Proof. Let $\Gamma \in \text{gra}(\mathcal{D})_{n_0+r_0}^{r_0}$. It is easy to see that the terms coming from linear combinations $(\Gamma \circ_i \Gamma \star \star \circ_j \Gamma \star \star)$ for $j \neq i$ and $j \neq i + 1$ appear twice with opposite signs. So they cancel each other and it remains to consider contributions coming from $(\Gamma \circ_i \Gamma \star \star \circ_i \Gamma \star \star)$ and $(\Gamma \circ_i \Gamma \star \star \circ_{i+1} \Gamma \star \star)$. Let us denote by $X = \{x_1, \ldots, x_m\}$ the union of $p^{-1}(i)$ and $\{k_1, k_2, \ldots\}$, where $k_1, k_2, \ldots$ are all the separating edges (if any) incident to $i$ in $\Gamma$. It is clear that, if $m \leq 2$, then we do not have contributions coming from $(\Gamma \circ_i \Gamma \star \star \circ_i \Gamma \star \star)$ and $(\Gamma \circ_i \Gamma \star \star \circ_{i+1} \Gamma \star \star)$. If $m \geq 3$, then for every partition of $X$ into three (non-empty) subsets $X_1, X_2, X_3$, we get 6 terms coming from $(\Gamma \circ_i \Gamma \star \star \circ_i \Gamma \star \star)$ and $(\Gamma \circ_i \Gamma \star \star \circ_{i+1} \Gamma \star \star)$. These terms are all shown schematically in figure 5.11. It is easy to see that the terms in the two columns cancel each other when we pass to coinvariants. Thus the claim is proved.

Let $\mathcal{D}$ be the graph satisfying Condition 5.1 and let $n_0$ (resp. $r_0$) be the number of non-cut vertices (resp. edges) of $\mathcal{D}$. The group $S_{n_0} \times S_{r_0}$ acts on the graded vector space $C_{\mathcal{D}}$ by rearranging labels on non-cut vertices and non-separating edges. Moreover, the differential $\delta$ commutes with this action.

Let us now observe that both the space of coinvariants

$$(C_{\mathcal{D}})_{S_{n_0} \times S_{r_0}}$$

and the space $QC_{\mathcal{D}}^\oplus$ from (5.4) is the span of isomorphism classes of even connected graphs $\Gamma$ satisfying these conditions

- every vertex of $\Gamma$ has valency $\geq 3$,
- $\Gamma$ has at least one cut vertex, and
- $\Gamma_{\text{red}}$ is isomorphic to $\mathcal{D}$.

Moreover, such $\Gamma$ has the same degree in (5.9) and in $QC_{\mathcal{D}}^\oplus$.

Therefore the space of coinvariants (5.9) is isomorphic to $QC_{\mathcal{D}}^\oplus$ as the graded vector space.

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Fig. 5.11: The decoration of a cut vertex $v$ by $X_t$ means that $X_t$ is the union of $p^{-1}(v)$ and the set of non-separating edges incident to $v$.

It is not hard to see that the differential induced on (5.9) by $\delta$ corresponds to the differential $\partial$ on $\mathbb{C} \oplus \mathbb{Q}$. Thus, since the base field has characteristic zero and the group $S_{n_0} \times S_{r_0}$ is finite, Proposition 5.2 follows directly from Proposition 5.4.

**Proposition 5.4** For every graph $\mathcal{G}$ satisfying Condition 5.1, the cochain complex $(\mathbb{C} \mathcal{G}, \delta)$ is acyclic.

Proof. Let, as above, $n_0$ (resp. $r_0$) be the number of non-cut vertices (resp. non-separating edges) of $\mathcal{G}$ and $n_1$ be the number of cut vertices of $\mathcal{G}$. Clearly, the separating forest of every graph $\Gamma \in \mathcal{G}(\mathbb{C})^{|r_0|+n}$ (for some $r \geq 0$ and $n \geq n_1$) has exactly $n_1$ connected components. Every connected component of the separating forest of $\Gamma$ is incident to $\geq 2$ islands. These islands may have common (necessarily cut) vertices but they do not share edges.

Let us order the islands of $\mathcal{G}$ according to this rule: $\Upsilon_1 < \Upsilon_2 \Leftrightarrow$ the smallest edge of $\Upsilon_1$ precedes the smallest edge of $\Upsilon_2$.

Let $\Upsilon_0$ be the smallest island of $\mathcal{G}$. If the island $\Upsilon_0$ has exactly one marked vertex then we call this marked vertex special and denote it by $c_0$.

Let us now consider the case when $\Upsilon_0$ has several marked vertices. For a marked vertex $c$ of $\Upsilon_0$, we denote by $X(c)$ the set of islands of $\mathcal{G}$ which are different from $\Upsilon_0$ and attached to $c$ in $\mathcal{G}$. Let us also denote by $\Upsilon(c)$ the smallest island from the set $X(c)$. Clearly, for two distinct marked vertices $c_1$ and $c_2$ of $\Upsilon_0$ the sets $X(c_1)$ and $X(c_2)$ are disjoint (otherwise, the vertex of $\mathcal{G}$ corresponding to $c_1$ is not a cut vertex). So $\Upsilon(c_1) \neq \Upsilon(c_2)$ if $c_1 \neq c_2$.

We call the marked vertex $c_0$ of $\Upsilon_0$ special if $\Upsilon(c_0)$ is the smallest island of the set

$$\{ \Upsilon(c) \mid c \text{ is a marked vertex of } \Upsilon_0 \}.$$

Thus we conclude that, for every graph $\mathcal{G}$ satisfying Condition 5.1, the (unique) smallest island $\Upsilon_0$ of $\mathcal{G}$ has exactly one special marked vertex. For example, if $\mathcal{G}$ is the graph shown in figure 5.1 (with the lexicographic order on the set of edges), then the pentagon is the smallest island and the unique marked vertex of this island is special.
Let $c_0$ be the special marked vertex of the smallest island of $\mathcal{I}$, $\Gamma \in \text{gra}(\mathcal{I})^{r_0+r}$, and $F$ be the separating forest of $\Gamma$ decorated by marked vertices of the islands of $\mathcal{I}$. If
\[ p^{-1}(p(c_0)) = \{c_0\} \tag{5.10} \]
and the vertex $p(c_0)$ has valency 1 in $F$ then we denote by $\tilde{\Gamma}$ the graph in $\text{gra}(\mathcal{I})^{r_0+r-1}$ which is obtained by contracting the (only) separating edge incident to $p(c_0)$. If this separating edge is labeled by $j$ and it connects the (cut) vertices with labels $i_1 < i_2$ then we label vertices and edges of $\tilde{\Gamma}$ in the following way:

- we shift the labels $j+1, \ldots, r_0+r$ down by 1,
- the vertex obtained by contracting edge $j$ is labeled by $i_1$,
- we shift the labels $i_2+1, \ldots, n_0+n$ down by 1.

We set
\[ h(\Gamma) := (-1)^j \tilde{\Gamma}, \tag{5.11} \]
if condition (5.10) is satisfied and $p(c_0)$ has valency 1 in $F$. Otherwise, we set
\[ h(\Gamma) := 0. \tag{5.12} \]
It is easy to see that equations (5.11) and (5.12) define a degree $-1$ linear map $h : C_{\mathcal{I}} \to C_{\mathcal{I}}$.\[ h : C_{\mathcal{I}} \to C_{\mathcal{I}}. \tag{5.13} \]
We will conclude the proof of the proposition by showing that
\[ \delta \circ h + h \circ \delta = \text{id}_{C_{\mathcal{I}}}. \tag{5.14} \]
The set $\text{gra}(\mathcal{I})^{r_0+r}$ splits into the disjoint union of two subsets:

- The first subset consists of graphs for which condition (5.10) is satisfied and the vertex $p(c_0)$ is incident to exactly one separating edge.
- The second subset consists of graphs for which condition (5.10) is not satisfied or it is satisfied but the vertex $p(c_0)$ in the separating forest of $\Gamma$ is not univalent.

For example the graphs shown in figures 5.1, 5.8 and 5.9 belongs to the second subset while the graph shown in figure 5.10 belongs to the first subset.

Let us assume that $\Gamma$ belongs to the first subset of $\text{gra}(\mathcal{I})^{r_0+r}$. It is easy to see that $\delta(h(\Gamma))$ has exactly one term whose the underlying graph belongs to the first subset of $\text{gra}(\mathcal{I})^{r_0+r}$. Moreover, this term coincides with $\Gamma$.

Let $i$ be the label of the cut vertex $p(c_0)$ in $\Gamma$. Since the linear combination $\Gamma \circ_1 \Gamma \circ_\bullet \circ_\bullet \circ_\bullet$ does not involve graphs in which the additional edge is separating, $\delta_i(\Gamma)$ in (5.8) is zero. Hence,
\[ -h(\delta(\Gamma)) \]
coincides with the sum of terms in $\delta(h(\Gamma))$ whose underlying graphs belong to the second subset of $\text{gra}(\mathcal{I})^{r_0+r}$. Thus $\delta \circ h(\Gamma) + h \circ \delta(\Gamma) = \Gamma$.

Let us now assume that $\Gamma$ belongs to the second subset of $\text{gra}(\mathcal{I})^{r_0+r}$. For such $\Gamma$, we have $h(\Gamma) = 0$. It is easy to see that $\delta(\Gamma)$ has exactly one term $v$ whose underlying graph belongs to the first subset of $\text{gra}(\mathcal{I})^{r_0+r+1}$. Moreover, $h(v) = \Gamma$. Since all the remaining terms of $\delta(\Gamma)$ are annihilated by $h$,
\[ \delta \circ h(\Gamma) + h \circ \delta(\Gamma) = \Gamma. \]

Proposition 5.4 is proved. \[ \square \]
As we explained above, Proposition 5.4 implies Proposition 5.2 Thus the second part of Proposition 3.9 is proved.
A Cohomology of auxiliary complexes

Let us denote by \( \mathcal{V}_2 \) the two dimensional vector space spanned by the degree 1 symbols \( a \) and \( b \) and denote by
\[
U := (T(\mathcal{V}_2), b_1), \quad R := (T(\mathcal{V}_2), b_2) \quad \text{and} \quad P := (T(\mathcal{V}_2), b_3)
\]
the cochain complexes with the following differentials:
\[
b_1(v_1v_2\ldots v_n) := \sum_{i=1}^{n} (-1)^{i+1} v_1 \ldots v_i (a + b) v_{i+1} \ldots v_n, \quad (A.1)
\]
\[
b_2(v_1v_2\ldots v_n) := - (a + b)v_1 \ldots v_n + b_1(v_1v_2\ldots v_n), \quad (A.2)
\]
\[
b_3(1) := (a + b)/2, \quad b_3(a) = b_3(b) := 0, \quad b_3(v_1v_2\ldots v_n) := \sum_{i=1}^{n-1} (-1)^{i+1} v_1 \ldots v_i (a + b) v_{i+1} \ldots v_n. \quad (A.3)
\]

To analyze these complexes, we consider the truncated tensor algebra \( T(\mathcal{V}_2) \), where \( \mathcal{V}_2 \) is the vector space spanned by two symbols \( a \) and \( x \) of degree 1. We identify \( T(\mathcal{V}_2) \) (resp. \( T(\mathcal{V}_2) \)) with \( T(\mathcal{V}_2) \) (resp. \( T(\mathcal{V}_2) \)) via the obvious isomorphism which sends \( a \) to \( a \) and \( x \) to \( a + b \). The differentials on (the underlying vector spaces of) \( T(\mathcal{V}_2) \) and \( T(\mathcal{V}_2) \) corresponding to \( (A.1), (A.2), (A.3) \) take the form\(^{14}\)
\[
b_1(v_1v_2\ldots v_n) := \sum_{i=1}^{n} (-1)^{i+1} v_1 \ldots v_i x v_{i+1} \ldots v_n, \\
b_2(v_1v_2\ldots v_n) := - x v_1 \ldots v_n + b_1(v_1v_2\ldots v_n), \\
b_3(1) := x/2, \quad b_3(a) = b_3(x) := 0, \quad b_3(v_1v_2\ldots v_n) := \sum_{i=1}^{n-1} (-1)^{i+1} v_1 \ldots v_i x v_{i+1} \ldots v_n.
\]

Let us prove the following statements about the cochain complexes \( U \) and \( R \):

**Claim A.1** The cochain complex \( U \) is acyclic. As for \( R \), we have
\[
H^n(R) \cong \begin{cases} 
\mathbb{K} & \text{if } n = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover, \( H^1(R) \) is spanned by the cohomology class of the cocycle \( a + b \).

Proof. The cochain complex \( (T(\mathcal{V}_2), b_1) \) splits into the direct sum of subcomplexes
\[
\bigoplus_{m \geq 0} K_m, \quad (A.4)
\]
where \( K_m \) is the subspace of \( T(\mathcal{V}_2) \) spanned by monomials of degree \( m \) in the symbol \( a \).

Since \( b_1(x^{2^{m-1}}) = x^{2^m} \) and \( b_1(x^{2^m}) = 0 \), the cochain complex \( (K_0, b_1) \) is acyclic.

To analyze \( (K_m, b_1) \) for \( m \geq 1 \), we denote by \( K^\circ \) the following cochain complex
\[
\ldots \to 0 \to 0 \to \mathbb{K} \xrightarrow{-x} \mathbb{K}x^{2} \xrightarrow{0} \mathbb{K}x^{3} \xrightarrow{x} \mathbb{K}x^{4} \xrightarrow{0} \mathbb{K}x^{5} \xrightarrow{-x} \ldots
\]

It is easy to see that (for every \( m \geq 1 \)), \( K_m \) splits into the direct sum
\[
K_{m,x} \oplus K_{m,a},
\]

\(^{14}\)By abuse of notation, we use the same symbols for the corresponding differentials on \( T(\mathcal{V}_2) \) and \( T(\mathcal{V}_2) \), respectively.
where \( K_{m,x} \) (resp. \( K_{m,a} \)) is the subspace of \( K_m \) spanned by monomials of the form \( x \ldots (\text{resp. } a \ldots) \). Moreover, the cochain complex \( K_{m,x} \) is isomorphic to \( K_0 \otimes (sK^\vee)^\otimes m \) and the cochain complex \( K_{m,a} \) is isomorphic to \( (sK^\vee)^\otimes m \).

Since the cochain complexes \( K_0 \) and \( K^\vee \) are acyclic, \( U \) is also acyclic.

Let us now consider the cochain complex \( T(W_2), b_2 \). Just as \( U \), the cochain complex \( T(W_2), b_2 \) splits into the direct sum of subcomplexes \( (K_m, b_2) \) for \( m \geq 0 \).

Since \( b_2(x^{2n-1}) = 0 \) and \( b_2(x^{2n}) = x^{2n+1} \) for every positive integer \( n \), \( H^m(K_0, b_2) = 0 \) for every \( m \neq 1 \), \( H^1(K_0, b_2) \cong \mathbb{K} \) and it is spanned by the cohomology class of \( x \).

It is easy to see that, for every \( m \geq 1 \), the cochain complex \( K_m \) is isomorphic to the tensor product
\[
K^\vee \otimes (sK^\vee)^\otimes m,
\]
where \( K^\vee \) is the cochain complex defined in \( (A.5) \).

Since the cochain complex \( K^\vee \) is acyclic, so is \( K_m \) for every \( m \geq 1 \).

The claim is proved.\( \square \)

The following claim takes care of the cochain complex \( (T(V_2), b_3) \):

**Claim A.2** The cochain complex \( P \cong (T(W_2), b_3) \) has the following cohomology
\[
H^n(P) \cong \begin{cases} \mathbb{K} & \text{if } n = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

The cocycles \( a \) and \( -b \) are cohomologous and \( H^1(P) \) is spanned by the cohomology class of \( a \).

Proof. Just as for \( U \) and \( R \) the cochain complex \( P \) splits into the direct sum of subcomplexes
\[
\bigoplus_{m \geq 0} K_m^-,
\]
where \( K_m^- \) is the subspace of \( T(W_2) \) spanned by monomials of degree \( m \) in the symbol \( a \).

Since \( b_3(1) = x/2, b_3(x^{2t-1}) = 0 \) and \( b_3(x^{2t}) = x^{2t+1} \) for every \( t \geq 1 \), the complex \( (K_0^-, b_3) \) is acyclic.

As for \( m \geq 2 \), it is easy to see that, up to the shift of degree, \( (K_m^-, b_3) \) is isomorphic to the cochain complex
\[
(sK^\vee)^\otimes m-1
\]
where \( K^\vee \) is the cochain complex defined in \( (A.5) \).

Finally, \( K_1^- \) is the following cochain complex with the zero differential
\[
\cdots \rightarrow 0 \rightarrow \mathbb{K}a \rightarrow 0 \rightarrow 0 \rightarrow \cdots
\]

The desired statements follow.\( \square \)

**B** The subcomplexes of polygons and path graphs

Let us recall that \( \text{dfGC}_{\text{conn,o}} \) is the subcomplex of \( \text{dfGC}_{\text{conn}} \) spanned by (even) connected graphs with all vertices having valency 2 (i.e. \( \Gamma_0 \) and various polygons) and \( \text{dfGC}_{\text{conn,-}} \) is the subcomplex of \( \text{dfGC}_{\text{conn}} \) spanned by path graphs and the graphs \( \Gamma_* \in \text{gra}_o^0 \).

In this appendix, we prove Propositions \( 8.11 \) and \( 8.11 \) which take care of the complexes \( \text{dfGC}_{\text{conn,o}} \) and \( \text{dfGC}_{\text{conn,-}} \).

First, we observe that the degree \( n \) term \( U^n \) of the cochain complex \( U \cong (T(W_2), b_1) \) is equipped with the obvious action of the cyclic group \( G_n \) of order \( n \). The generator \( g_n \) of this group acts as
\[
g_n(v_1, v_2, \ldots, v_n) = (-1)^{n-1}(v_2, v_3, \ldots, v_n, v_1).
\]

Moreover, for every \( X \in U^n \), the vector \( b_1(X - g_n(X)) \) belongs to the subspace spanned by vectors of the form
\[
Y - g_{n+1}(Y), \quad Y \in U^{n+1}.
\]
Hence $b_1$ induces a differential on the cochain complex

$$0 \to U^1_G \to U^2_G \to \cdots \to U^n_G \to \cdots$$  \hspace{1cm}  \text{(B.2)}

By abuse of notation, we will use the same symbol $b_1$ for the differential on the cochain complex (B.2). We denote by the resulting cochain complex by $K_G$.  

Hence where $\psi$ we get a surjective map of cochain complexes

$$K \to \cdots \to K \to U \to \cdots$$

Thus, since our base field has characteristic zero and $K^{\otimes}$ is acyclic, the Künneth theorem implies that $K_{o,r}$ is acyclic for every $r \geq 1$.

It is easy to see that, for every $r \geq 1$, $K_{o,r}$ is obtained from the cochain complex $(s K^{\otimes})^{\otimes r}$ by taking coinvariants with respect to the action of the cyclic group $G_r$. Here, $K^{\otimes}$ is the cochain complex defined in (A.5).

Therefore $K_{o,0}$ is the following cochain complex with the zero differential

$$\cdots \to 0 \to 0 \to K \to 0 \to K \to 0 \to 0 \to \cdots$$

Thus the cochain complex (B.2) has the following cohomology:

$$H^m(K) \cong \begin{cases} \mathbb{K} & \text{if } m \text{ is positive and odd,} \\ 0 & \text{otherwise.} \end{cases}$$  \hspace{1cm}  \text{(B.3)}

Moreover, $H^{2n+1}(K_o)$ (for $n \geq 0$) is spanned by the cohomology class represented by $(a+b)^{2n+1}$.

### B.1 The end of the proof of Proposition 3.10

Let us recall (see (1.10)) that the cochain complex $(U, b_1)$ is equipped with an action of $S_2$. It is easy to see that this action descends to the cochain complex $K_G$ (see (B.2)).

To every monomial $X = v_1 v_2 \cdots v_n$ in $U$ we assign the cycle graph $\Gamma_X \in \text{dgra}_n$ which is obtained as follows: if $v_i = a$ and $i \leq n - 1$ then the $i$-th edge originates from vertex $i$ and terminates at vertex $i + 1$; if $v_i = b$ and $i \leq n - 1$ then the $i$-th edge originates from vertex $i + 1$ and terminates at vertex $i$; finally, the $n$-th edges connects vertex $n$ to vertex $1$; it originates from vertex $n$ (resp. $1$) if $v_n = a$ (resp. $v_n = b$).

It is clear that the assignment $X \mapsto \Gamma_X$ extends to the surjective map of cochain complexes

$$K \to \text{dfGC}_{\text{conn, o}}$$  \hspace{1cm}  \text{(B.4)}

Moreover a vector $Y \in K_o$ belongs to the kernel of this map if and only if $Y$ belongs to the span of vectors of the form $X - \sigma(X)$, where $X \in K_o$ and $\sigma$ is the only non-trivial element of $S_2$.

Thus (B.4) induces an isomorphism of cochain complexes $K_o \cong \text{dfGC}_{\text{conn, o}}$ and Proposition 3.10 follows from (B.3) and the fact that the image of $(a+b)^{2n+1}$ in $(K_o)_{S_2}$ is non-zero if and only if $n$ is divisible by $2$.  \hspace{1cm}  \Box

### B.2 The subcomplex $\text{dfGC}_{\text{conn, -}}$ is indeed acyclic

To every monomial $X = v_1 v_2 \cdots v_n \in V^n_2$, we assign the path graph $\Gamma_X \in \text{dgra}_{n+1}$ by declaring that, if $v_i = a$ (resp. $v_i = b$), then edge $i$ originates at vertex $i$ (resp. $i + 1$) and terminates at vertex $i + 1$ (resp. $i$). Setting,

$$\psi(X) := \Gamma_X, \hspace{1cm} \psi(1) := \Gamma_1$$

we get a surjective map of cochain complexes $\psi : P \to \text{dfGC}_{\text{conn, -}}$.

It is easy to see that a vector $Y \in P$ belongs to the kernel of $\psi$ if and only if $Y$ belongs to the span of vectors of the form $X - \sigma(X)$, where $X$ is a monomial in $P$, $\sigma = (1,2) \in S_2$ and the action of $S_2$ on $P$ is defined by (1.10).

Due to Claim A.2 $H^*(P)$ is spanned by the cohomology class of $a$. Since, in the space of coinvariants $P_{S_2}$, we have $a = (a+b)/2$, the cochain complex $P_{S_2}$ is acyclic. Thus the cochain complex $\text{dfGC}_{\text{conn, -}}$ is also acyclic.  \hspace{1cm}  \Box
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