Abstract: In this paper we first construct an analytic realization of the $C_\lambda$-extended oscillator algebra with the help of difference-differential operators. Secondly, we study families of $d$-orthogonal polynomials which are extensions of the Hermite and Laguerre polynomials. The underlying algebraic framework allowed us a systematic derivation of their main properties such as recurrence relations, difference-differential equations, lowering and rising operators and generating functions. Finally, we use these polynomials to construct a realization of the $C_\lambda$-extended oscillator by block matrices.

Keywords: vector orthogonal polynomials, deformed oscillator algebra.

1. INTRODUCTION

In [20], Quesne defined the $C_\lambda$-extended oscillator algebra ($\lambda$ being an integer bigger than 1) as the associative algebra generated by four elements $a_-, a_+, N, s$ that obey the commutation relations

$$[a_-, a_+] = 1 + \sum_{i=1}^{\lambda-1} \beta_i s^i, \quad s^\lambda = 1,$$

$$[N, a_-] = -a_-, \quad [N, a_+] = a_+, \quad [N, s] = 0,$$

$$a_- s = \varepsilon_\lambda s a_- \quad a_+ s = \varepsilon_\lambda^{-1} s a_+, \quad \varepsilon_\lambda = e^{\frac{2\pi}{\lambda^2}},$$

$$N^* = N, \quad a^*_+ = a_-, \quad s^* = s^{-1},$$

where the constants structure $\beta_1, \ldots, \beta_{\lambda-1}$ are complex numbers restricted by the conditions

$$\overline{\beta_i} = \beta_{\lambda-i}, \quad i = 1, \ldots, \lambda.$$

The connection between orthogonal polynomials and harmonic oscillator as well as the quantum algebra is well known [28] [21]. The oscillator algebra of creation, annihilation, and number operators plays a central role in the investigation of many class of orthogonal polynomials, and provides a useful tool to derive their operational properties such as recurrence relations, difference equation,
lowering and rising operators and generating functions. It particular, when \(\lambda = 2\), the \(C_\lambda\)-extended Heisenberg algebra is reduced to the Calogero–Vasiliev algebra \([16]\), which is given by the generators \(a_-, a_+, R\), 1 that satisfy the commutation relations
\[
[a_-, a_+] = 1 + 2\nu R, \quad R^2 = 1, \\
\{a_\pm, R\} = 0, \quad [1, a_\pm] = [1, R] = 0.
\]
A realization of the Calogero–Vasiliev algebra working in the Schrödinger representation, \(\Psi = \Psi(x)\), is given by the operators
\[
a_- = \frac{1}{\sqrt{2}}(Y_\nu + x), \quad a_+ = \frac{1}{\sqrt{2}}(-Y_\nu + x),
\]
where \(Y_\nu\) is the Dunkl operator (corresponding to the root system \(A_1\), see \([13\text{, Definition 4.4.2}]\)), which is a differential-difference operator, depending on a parameter \(\nu \in \mathbb{R}\) and acting on \(C^\infty(\mathbb{R})\) as follows:
\[
Y_\nu := \frac{d}{dx} + \frac{\nu}{x}(1 - R),
\]
where \(R\) is the Klein operator acting on function \(\psi\) of the real variable \(x\) as
\[
(R\psi)(x) = \psi(-x)
\]
(see \([16]\) for more details). Note that the operator \(Y_\nu\) is also related by a simple similarity transformation to the Yang-Dunkl operator used in \([16]\), were the authors show that the Calogero–Vasiliev algebra is intimately related to parabosons and parafermions, and to the \(osp(1|2)\) and \(osp(2|2)\) superalgebras.

The Hamiltonian takes the form
\[
H_\nu = -\frac{1}{2}Y_\nu^2 + \frac{1}{2}x^2 = -\frac{1}{2} \frac{d^2}{dx^2} - \frac{\nu}{x} \frac{d}{dx} + \frac{\nu^2}{2x^2}(1 - R) + \frac{1}{2}x^2
\]
and the wave functions corresponding to the well-known eigenvalues
\[
\lambda_n = n + \nu + \frac{1}{2}, \quad n = 0, 1, 2, \ldots
\]
are given by
\[
\psi^{(\nu)}_n(x) = \gamma_n \, e^{-x^2/2} H^{(\nu)}_n(x),
\]
where
\[
\gamma_n = \left(2^n \Gamma\left(\frac{n}{2} + 1\right) \Gamma\left(\frac{n + \frac{1}{2}}{2}\right) \right)^{-\frac{1}{2}}
\]
and $H_{2n}^{(\nu)}(x)$ is the generalized Hermite polynomial introduced by Szegő [26] and obtained in [26] from Laguerre polynomials $L_n^{\nu}(x)$ by the means:

$$
\begin{cases}
H_{2n}^{(\nu)}(x) = (-1)^n 2^{2n} n! L_n^{\nu - \frac{1}{2}}(x^2), \\
H_{2n+1}^{(\nu)}(x) = (-1)^n 2^{2n+1} n! x L_n^{\nu + \frac{1}{2}}(x^2).
\end{cases}
$$

(12)

It is well known that for $\nu > -\frac{1}{2}$, these polynomials satisfy the orthogonality relations:

$$
\int_{\mathbb{R}} H_n^{(\nu)}(x) H_m^{(\nu)}(x) |x|^{2\nu} e^{-x^2} dx = \frac{1}{\gamma_n^2} \delta_{nm},
$$

(13)

where $\gamma_n$ is given in (11). Many of the known generalized Hermite polynomials are also the eigenfunctions of the energy operator for a deformed oscillator (see [15, 3]).

In this paper, we discuss the connection of some class of $d$-orthogonal polynomials with the $C_\lambda$-extended oscillator algebra (for $\lambda = d + 1$). The $d$-orthogonal polynomials can be obtained from general multiple orthogonal polynomials under some restrictions upon their parameters (see [27]). Applications of $d$-orthogonal polynomials include the simultaneous Padé approximation problem where the multiple orthogonal polynomials appear (see [6]). Also they play important role in random matrix theory (see [4]). The $d$-orthogonal polynomials have been intensively studied in the last 30 years due to their intriguing properties and applications (see [5, 7] and further references in the literature).

One problem that deserves attention is to relate $d$-orthogonal polynomials to some oscillator algebras. In [29, 8, 9], the authors have found some $d$-orthogonal polynomials related to the deformed harmonic oscillator and share a number of properties with the classical orthogonal polynomials. In this paper, we discuss the connection of some class of $d$-orthogonal polynomials with the $C_\lambda$-extended oscillator algebra ($\lambda = d + 1$). We use a Klein type operator $S$ of finite order to construct a realization of the $C_\lambda$-extended oscillator algebra in terms of difference-differential operator and a system of vector orthogonal polynomials obtained from $d$-orthogonal polynomials, in order to provide realizations of the $C_\lambda$-extended oscillator by block matrices. Note that the obtained $d$-orthogonal polynomials are eigenfunctions of the operator $S$. 

The paper is organized as follows. In Section 2, we review the definition of the $C_\lambda$-extended oscillator algebra and introduce a Bergmann realization associated to this oscillator. In section 3, we deal with the formalism of the exponential of the Dunkl type operators, which are the most commonly used operators in the context of evolution problems. We establish the relevant rules to the action of an exponential operator on a given function and those for the disentanglement of exponential operators. We also establish there the most important properties of $d$-orthogonal polynomials. In section 4, we construct a family of vector orthogonal polynomial and analyse the effect of the generators of the $C_\lambda$-extended algebra on it.

2. BERGMANN REALIZATION OF THE $C_\lambda$-EXTENDED OSCILLATOR ALGEBRA

2.1. The $C_\lambda$-extended oscillator algebra. The element $s$ introduced in (1), is a generator of the cyclic group

$$C_\lambda = \{1, s, s^2, \ldots, s^{\lambda-1}\}$$

(14)
of order $\lambda$. The primitive idempotents related to this group are denoted by $\Pi_0, \Pi_1, \ldots, \Pi_{\lambda-1}$ and are given by

$$\Pi_i = \frac{1}{\lambda} \sum_{j=0}^{\lambda-1} \varepsilon_{\lambda}^{-ij} s^j, \quad i = 0, \ldots, \lambda - 1, \quad \varepsilon_{\lambda} = e^{2i\pi/\lambda}$$

(15)

and the bosonic $C_\lambda$-extended oscillator Hamiltonian is defined by

$$H = \frac{1}{2}\{a_-, a_+\}. $$

(16)

According to (18), it may therefore be rewritten in terms of $a_-, a_+, N, \Pi_0, \ldots, \Pi_{\lambda-1}$ as follows:

$$[a_-, a_+] = 1 + \sum_{j=0}^{\lambda-1} \beta_j \Pi_j,$$

(17)

$$[N, a_-] = -a_- \quad [N, a_+] = a_+, \quad a_+ \Pi_i = \Pi_{i+1} a_+,\quad i = 0, \ldots, \lambda - 1,$$

(18)

$$\sum_{i=1}^{\lambda} \Pi_i = 1, \quad \Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \Pi_i^\dagger = \Pi_i,$$

(19)

where, according to condition (5), the discrete Fourier transforms $\hat{\beta}_0, \ldots, \hat{\beta}_{\lambda-1}$, are real numbers given by

$$\hat{\beta}_j = \sum_{i=0}^{\lambda-1} \varepsilon_{\lambda}^{ij} \beta_i, \quad j = 1, \ldots, \lambda - 1,$$
and are restricted by the condition $\sum_{i=0}^{\lambda-1} \hat{\beta}_i = 0$.

Next proposition will be needed in the sequel and gives a relation in the $C_\lambda$-extended oscillator algebra.

**Proposition 1.** Let $n \in \mathbb{N}$, we have

1. $[a_-^n, a_+] = (n + \sum_{i=0}^{\lambda-1} \beta_i \frac{\varepsilon_{ni} - 1}{\varepsilon_\lambda - 1} s^i) a_-^{n-1}$,

2. $[a_-, a_+^n] = a_+^{n-1} (n + \sum_{i=0}^{\lambda-1} \beta_i \frac{\varepsilon_{ni} - 1}{\varepsilon_\lambda - 1} s^i)$,

3. $[N, a_-^n] = na_-^{n-1}$, $[N, a_+^n] = na_+^{n-1}$.

In particular, it holds that $[a_-^{n\lambda}, a_+] = n\lambda a_-^{n\lambda-1}$ and $[a_-, a_+^{n\lambda}] = n\lambda a_+^{n\lambda-1}$.

**Proof.** The proof is easy and follows from the identity $[a^n, b] = \sum_{i=0}^{n-1} a^{n-1-i}[a, b]a^i$.

According to [18], the $C_\lambda$-extended oscillator algebra possesses a canonical irreducible representation defined on the orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots$, endowed with the following actions:

$$N|k\lambda + i\rangle = (k\lambda + i)|k\lambda + i\rangle, \quad \Pi_j|k\lambda + i\rangle = \delta_{ij}|k\lambda + i\rangle,$$

$$a_+|k\lambda + i\rangle = \sqrt{k\lambda + i + 1 + \hat{\beta}_{i+1}}|k\lambda + i + 1\rangle,$$

$$a_-|k\lambda + i\rangle = \sqrt{k\lambda + i + \hat{\beta}_i}|k\lambda + i - 1\rangle.$$

**2.2. Dunkl type operator.** Let $\nu = (\nu_1, \ldots, \nu_{\lambda-1})$, where $\nu_1, \ldots, \nu_{\lambda-1}$ are complex numbers satisfying the condition

$$\sum_{i=0}^{\lambda-1} \nu_i = 0.$$ (23)

Consider the Klein-type reflection $S$ acting on functions $f(z)$ of complex variable $z$ as follows

$$(Sf)(z) := f(\varepsilon_\lambda z).$$ (24)
and its associated differential-difference operator

\[ Y_\nu := \frac{d}{dz} + \frac{1}{z} \sum_{j=1}^{\lambda-1} \nu_j S^j. \]  

(25)

Recall that the complex reflection group \( G(r, 1, N) \) consists of the \( N \times N \) permutation matrices with the nonzero entries being powers of \( \varepsilon = e^{\frac{2i\pi}{r}} \). The group \( G(r, 1, N) \) is also generated by the transpositions \( (i, i+1), \ i = 1, \ldots, N-1 \), and by the complex reflections \( S_i \) defined by

\[ S_i z = (z_1, \ldots, \varepsilon z_i, \ldots, z_N). \]

When \( N = 1 \), it happens that:

- the group \( G(r, 1, N) \) is isomorphic to the cyclic group \( C_\lambda \) defined in (14);
- the operator defined in (25) is a particular case of the complex Dunkl operator \( Y_i \) associated to \( G(r, 1, N) \) given by \[ 13\] :

\[ Y_i = \frac{\partial}{\partial z_i} + \kappa_0 \sum_{j=1}^{r-1} \frac{1 - S_j^{-1}(i, j) S_j^j}{z_i - \varepsilon^j z_j} + \sum_{j=1}^{r-1} \kappa_j \sum_{l=0}^{r-1} \frac{\varepsilon^{-rl} S_j^l}{z_i}, \]  

(26)

where \( \kappa_0, \kappa_1, \ldots, \kappa_r \) are real numbers.

2.3. Some results on deformed factorial numbers. Let \( \nu_i, i = 0, 1, \ldots, \lambda-1 \) be a sequence of complex numbers and \( \hat{\nu}_s \) their discrete Fourier transforms defined as

\[ \hat{\nu}_s = \sum_{l=0}^{\lambda-1} \nu_l \varepsilon_{\lambda}^{sl}. \]  

(27)

One can notice that

\[ \hat{\nu}_n = \hat{\nu}_s, \ \text{if} \ n = s \ (\text{mod} \ \lambda). \]  

(28)

Then, for nonnegative integers \( n \), we introduce the deformed numbers \([n]_\nu \) and the deformed factorial numbers by

\[ [n]_\nu = n + \hat{\nu}_n, \]  

(29)

\[ [0]_\nu! = 1, \quad [n+1]_\nu! = [n+1]_\nu [n]_\nu!. \]  

(30)

For

\[ \alpha_k = \frac{k + \hat{\nu}_k}{\lambda}, \quad k = 1, 2, \ldots, \lambda - 1. \]  

(31)

and for \( 0 \leq s \leq \lambda - 1 \), we need to introduce the multi-index numbers

\[ \Delta(\nu, s) = \begin{cases} (1, \alpha_1, \ldots, \alpha_{\lambda-1}) & \text{if} \ s = 0, \\ (1, \alpha_1 + 1, \ldots, \alpha_s + 1, \alpha_{s+1}, \ldots, \alpha_{\lambda-1}) & \text{otherwise}. \end{cases} \]  

(32)
We obtain the following result:

**Proposition 2.** The initial problem

\[
\begin{aligned}
Y_\nu f(z) &= \varrho f(z), \\
f(0) &= 1 \\
\end{aligned}
\]  

(33)

has a unique analytic solution given by

\[
E_\lambda(\varrho z, \nu) = \sum_{n=0}^{\infty} \frac{(\varrho z)^n}{[n]_\nu!}. \\
\]  

(34)

Furthermore, the generalized exponential function \(E_\lambda(\varrho . , \nu)\) has the following hypergeometric representation

\[
E_\lambda(z, \nu) = \frac{1}{\lambda - 1} \sum_{s=0}^{\lambda - 1} \frac{(z/\lambda)^s}{[s]_\nu!} \left( \Delta(\nu, s) \left| \left( \frac{z}{\lambda} \right)^\lambda \right. \right). \\
\]  

(35)

**Proof.** It is easily seen from the action of the operator \(Y_\nu\) on monomials \(z^n\):

\[
Y_\nu z^n = [n]_\nu z^{n-1}, \\
\]  

(36)

that the function \(E_\lambda(\varrho . , \nu)\) is the unique solution of the system (34). To prove (35), it suffice to write \([n\lambda + s]_\nu!\) in terms of the Pochhammer Symbol \((a)_n\) defined for \(a \in \mathbb{C}\) and \(n = 1, \ldots\) by

\[
(a)_0 := 1, \quad (a)_n := a(a + 1) \ldots (a - n + 1). \\
\]

For \(s = 1, 2, \ldots \lambda - 1\), and with \(\alpha_k\) defined in (31), we get

\[
[n\lambda + s]_\nu! = \prod_{l=1}^{\lambda-1} \prod_{k=0}^{s} \prod_{l=1}^{n} (l\lambda + k + \tilde{\nu}_k) \prod_{k=s+1}^{\lambda-1} \prod_{l=0}^{n-1} (l\lambda + k + \tilde{\nu}_k), \\
\]

\[
= \lambda^{n\lambda + s} n! \prod_{k=1}^{s} \frac{(k + \tilde{\nu}_k)_{n+1}}{\lambda} \prod_{k=s+1}^{\lambda-1} \frac{(k + \tilde{\nu}_k)_{n}}{\lambda}, \\
\]

\[
= \lambda^{n\lambda + s} n! \prod_{k=1}^{s} \frac{\Gamma(\alpha_k + n + 1)}{\Gamma(\alpha_k)} \prod_{k=s+1}^{\lambda-1} \frac{\Gamma(\alpha_k + n)}{\Gamma(\alpha_k)}. \\
\]

Similarly,

\[
[n\lambda]_\nu! = \lambda^{n\lambda} n! \prod_{k=1}^{\lambda-1} \frac{\Gamma(\alpha_k + n + 1)}{\Gamma(\alpha_k)} = \lambda^{n\lambda} n! \prod_{k=1}^{\lambda-1} \frac{\Gamma(\alpha_k + n)}{\Gamma(\alpha_k)}. \\
\]
If we decompose the sum (34) in the form
\[ E_\lambda(z, \nu) = \sum_{s=0}^{\lambda-1} \sum_{n=0}^{\infty} \frac{(zt)^{n\lambda+s}}{n\lambda+s}_\nu! \] (37)
we obtain
\[ E_\lambda(z, \nu) = \sum_{s=0}^{\lambda-1} \frac{(zt/\lambda)^s}{[s]_\nu!} 0 F_{\lambda-1} \left( \begin{array}{c} - \\ \alpha_1 + 1, \ldots, \alpha_s + 1, \alpha_{s+1}, \ldots, \alpha_{\lambda-1} \end{array} \right| \left( \frac{z}{\lambda} \right)^4 \right), \] (38)
and this allows to conclude. \[ \square \]

2.4. Bergmann realization. Let \( G_{m,n}^{p,q} \) be the Meijer’s \( G \)-function [22], \( m_s \) be the measure defined by
\[ dm_s(z) = \frac{\lambda^{s+1}}{\prod_{k=1}^{\lambda-1} \Gamma(\alpha_k)} G_{0,0}^{\lambda,0} \left( \begin{array}{c} r_{\lambda}^{2\lambda} \\ \lambda \end{array} \Delta(\nu, s) \right) dr d\theta, \quad z = re^{i\theta}, \] (39)
and \( L^2_{\nu,\lambda}(\mathbb{C}) \) be the space of measurable functions \( f \) on \( \mathbb{C} \) that satisfy
\[ \|f\|_{\nu,\lambda}^2 = \sum_{s=0}^{\lambda-1} \int_{\mathbb{C}} |\pi_s(f)(z)|^2 dm_s(z) < \infty, \] (40)
The generalized Bergmann space \( \mathcal{B}_{\nu,\lambda}(\mathbb{C}) \) is the pre-Hilbert space of analytic functions in \( L^2_{\nu,\lambda}(\mathbb{C}) \), equipped with the inner product
\[ \langle f, g \rangle_\nu = \sum_{s=0}^{\lambda-1} \int_{\mathbb{C}} \pi_s(f)(z) \overline{\pi_s(g)(z)} dm_s(z), \] (41)
where
\[ \pi_s = \frac{1}{\lambda} \sum_{l=0}^{\lambda-1} e^{-sl} S^l. \] (42)

Theorem 1. If \( f, g \in \mathcal{B}_{\nu,\lambda}(\mathbb{C}) \) expanded in the form
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \]
then, their inner product is given by
\[ \langle f, g \rangle_\nu = \sum_{n=0}^{\infty} a_n \overline{b_n}[n]_\nu!. \] (43)
Proof. Let \( 0 \leq s, s' \leq \lambda - 1 \) and \( n, m = 0, \ldots \). From (43), it is clear that for \( s \neq s' \), we have
\[
\pi_l(z^{n+\lambda+s}) \pi_l(z^{n+\lambda+s'}) = 0 \quad \text{for} \quad l = 0, \ldots \lambda - 1,
\]
hence,
\[
\langle z^{n+\lambda+s}, z^{m+\lambda+s'} \rangle_\nu = 0.
\]
For \( s = s' \), we have
\[
\langle z^{n+\lambda+s}, z^{m+\lambda+s} \rangle_\nu = \frac{1}{\pi} \frac{\lambda^{s+1}}{\prod_{k=1}^{\lambda-1} \Gamma(\alpha_k)} \int_0^\infty \frac{r^{(n+m)\lambda+2s}}{\lambda^\lambda} \left| \frac{r^{2\lambda}}{\Delta(\nu, s)} \right| \, dr \times \int_0^{2\pi} e^{i\lambda \theta (n-m)} \, d\theta.
\]
Now using the following well known formula [22]
\[
\int_0^\infty x^{n-1} G_{p,q}^{r,s} \left( \frac{x}{a}, \beta_1, \ldots, \beta_p ; \frac{\gamma_1, \ldots, \gamma_q}{\gamma_1, \ldots, \gamma_q} \right) \, dx = a^n \prod_{i=r+1}^{s-1} \Gamma(1-\gamma_i+n) \prod_{i=s+1}^{p} \Gamma(\gamma_i+n) \prod_{i=1}^{p} \Gamma(1-\beta_i-n),
\]
we obtain
\[
\langle z^{n+\lambda+s}, z^{m+\lambda+s} \rangle_\nu = \delta_{nm} \lambda^{n+\lambda+s} n! \prod_{k=1}^{s} (\alpha_k)_n + 1 \prod_{k=s+1}^{\lambda-1} (\alpha_k)_n = \delta_{nm} [n \lambda + s]_\nu !.
\]

It is oblivious that the space \( \mathfrak{B}_{\nu,\lambda}(\mathbb{C}) \) is a Hilbert space equipped with the inner product (41) and the monomials
\[
\{ e_n(z) = z^n / \sqrt{[n]_\nu !}, \quad n = 0, 1, 2, \ldots \}
\]
constitute an orthonormal basis for this space. The space \( \mathfrak{B}_{\nu,\lambda}(\mathbb{C}) \) has the kernel function
\[
K_\nu(w, z) = \sum_{n=0}^\infty e_n(z) e_n(w) = \mathcal{E}_\lambda(z \overline{w}, \nu).
\]

**Proposition 3.** (i) Let \( Z \) be the multiplication operator \((Zf)(z) := zf(z)\). The operators \( Y_\nu \) and \( Z \) are closed densely defined operators on the space \( \mathfrak{B}_{\nu,\lambda}(\mathbb{C}) \) on the common domain
\[
D = \{ f(z) = \sum_{n=0}^\infty a_n z^n : \sum_{n=0}^\infty |a_n|^2 [n + 1]_\nu < \infty \}.
\]
Furthermore, for $f, g \in D$ one has
\[ \langle Y_\nu f, g \rangle_\nu = \langle f, Z g \rangle_\nu. \]

**Proposition 4.** The following holds:

(i) 
\[ S^* = S^{-1}, \quad Y_\nu S = \varepsilon_\lambda S Y_\nu, \quad SZ = \varepsilon_\lambda ZS. \]

(ii) 
\[
[Y_\nu, z] = (n + \sum_{i=1}^{\lambda-1} \nu_i (\varepsilon_\lambda^{i n} - 1) S^i) Y_\nu^{n-1} \quad (46)
\]
\[
[Y_\nu, z^n] = z^{n-1} \left( n + \sum_{i=1}^{\lambda-1} \nu_i (\varepsilon_\lambda^{i n} - 1) S^i \right). \quad (47)
\]

Now, assume that the complex numbers $\nu_i$ are restricted by the conditions
\[ \nu_{\lambda-i} = -\varepsilon_\lambda^i \nu_i, \quad i = 1, \ldots, \lambda - 1. \quad (48)\]

The case where the structure constants $\beta_i$ in (1) take the values
\[ \beta_i = \nu_i (\varepsilon_\lambda^i - 1), \quad i = 1, \ldots, \lambda - 1, \]
we get a convenient one variable model of the representation of $C_\lambda$-extended oscillator algebra given by the Hilbert space $B_{\nu,\lambda}(\mathbb{C})$ and the action of the $C_\lambda$-extended oscillator algebra is given by
\[
(s f)(z) := f(\varepsilon z), \quad (a f)(z) := (Y_\nu f)(z), \quad (a_+ f)(z) := Z f(z) \quad (49)
\]
and the related bosonic extended Hamiltonian takes the form
\[
H = z \frac{d}{dz} + \frac{1}{2} + \frac{1}{2} \sum_{j=1}^{\lambda-1} \nu_j (\varepsilon_\lambda^j + 1) S^j. \quad (50)
\]

### 3. Generalized Hermite Polynomials

For fixed integer $m$, Gould and Hopper [14] defined the generalized Hermite polynomials $H_n(x, m)$ by the operational identity
\[
H_n(x, m) = e^{(\frac{x^2}{2m})^m} x^n, \quad n = 0, 1, \ldots.
\]

Similarly, a new family of generalized Hermite polynomials $\{H_n^{(\lambda, \nu)}(x)\}_{n=0}^{\infty}$ can be determined by means of the following operational formula
\[
H_n^{(\lambda, \nu)}(x) = e^{-Y_\nu^{\lambda}/\lambda} x^n. \quad (51)
\]

Due to the fact that $Y_\nu$ reduces the degrees of the polynomials, see equation (36) then the latter series (51) contains only a finite number of terms. The operational definition (51) greatly simplifies the study
of the generalized Hermite polynomials. From Proposition 4, we deduce that
\[ [Y^\nu, x] = n\lambda x^{n\lambda - 1}, \quad [Y^\nu, x^{n\lambda}] = n\lambda Y^\nu x^{n\lambda - 1} \] (52)
and that
\[ [e^{\mu Y^\nu}, x] = \mu \lambda Y^{\lambda - 1} e^{\mu Y^\nu}, \quad [Y^\nu, e^{\mu x^\lambda}] = \mu \lambda x^{\lambda - 1} e^{\mu x^\lambda}. \] (53)

**Proposition 1.** The following holds true
\[ Y^\nu H^{(\lambda, \nu)}_n(x) = [n] \nu H^{(\lambda, \nu)}_{n-1}(x), \]
\[ (x - Y^\nu x^{\lambda - 1})H^{(\lambda, \nu)}_n(x) = H^{(\lambda, \nu)}_{n+1}(x). \] (54)

**Proof.** Performing the Dunkl operator \( Y^\nu \) of both sides of (51) with respect to \( x \), we obtain
\[ Y^\nu H^{(\lambda, \nu)}_n(x) = [n] \nu H^{(\lambda, \nu)}_{n-1}(x). \] (54)

The equation (53) can be exploited to derive the operational formula
\[ e^{-Y^\nu / \lambda} x^n = e^{-Y^\nu / \lambda} x e^{-Y^\nu / \lambda} x^{\nu - 1} = (x - Y^\nu) e^{-Y^\nu / \lambda} x^{\nu - 1}. \]

Hence
\[ H^{(\lambda, \nu)}_n(x) = (x - Y^\nu) H^{(\lambda, \nu)}_{n-1}(x). \] (55)

We also obtain

**Theorem 2.** The polynomials \( H^{(\lambda, \nu)}_n(x) \) satisfy the following higher order differential-difference equations
\[ Y^\nu (x - Y^\nu x^{\lambda - 1})H^{(\lambda, \nu)}_n(x) = [n + 1] \nu H^{(\lambda, \nu)}_n(x), \]
\[ (x - Y^\nu x^{\lambda - 1})Y^\nu H^{(\lambda, \nu)}_n(x) = [n] \nu H^{(\lambda, \nu)}_{n+1}(x). \] (56)

**Proof.** Applying the operator \( Y^\nu \) to the both side of (55), we get the following differential-difference equation:
\[ Y^\nu (x - Y^\nu x^{\lambda - 1})H^{(\lambda, \nu)}_n(x) = -[n + 1] \nu H^{(\lambda, \nu)}_n(x). \] (56)

**Proposition 2.** The polynomials \( H^{(\lambda, \nu)}_n \) are generated by the series
\[ e^{-t^\lambda / \lambda} \mathcal{E}(xt, \nu) = \sum_{n=0}^{\infty} H^{(\lambda, \nu)}_n(x) \frac{t^n}{[n] \nu !}. \]

Furthermore,
\[ H^{(\lambda, \nu)}_n(x) = \sum_{k=0}^{[\frac{n}{\nu}]} \frac{(-1)^k [n] \nu !}{\lambda^k \lambda [n - k \lambda] \nu !} x^{n-k\lambda}. \] (57)
Proof. According to (55) and (33), we can write
\[
\sum_{n=0}^{\infty} H_n^{(\lambda,\nu)}(x) \frac{t^n}{[n]_\nu!} = \sum_{n=0}^{\infty} e^{-Y_\lambda/\lambda} t^n \frac{n!}{n!} = e^{-Y_\lambda/\lambda} E_\lambda(xt, \nu) = e^{-t^\lambda/\lambda} E_\lambda(xt, \nu)
\]

Hence, the polynomials \( \{H_n^{(\lambda,\nu)}(x)\} \) are generated by
\[
e^{-t^\lambda/\lambda} E_\lambda(xt, \nu) = \sum_{n=0}^{\infty} H_n^{(\lambda,\nu)}(x) \frac{t^n}{[n]_\nu!}.
\]

An explicit formula for the generalized Hermite polynomials \(H_n^{(\lambda,\nu)}(x)\) is obtained by expanding in power series the generation function given in (58):
\[
e^{-t^\lambda/\lambda} E_\lambda(xt, \nu) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda^k k!} \sum_{m=0}^{\infty} \frac{(xt)^m}{[m]_\nu!}.
\]

Substituting \( n = k\lambda + m \), then \( 0 \leq k \leq \lfloor n/\lambda \rfloor \) and we
\[
e^{-t^\lambda/\lambda} E_\lambda(xt, \nu) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/\lambda \rfloor} \frac{(-1)^k}{\lambda^k k!} \frac{x^{n-k\lambda}}{[n-k\lambda]_\nu!} \frac{t^n}{n!}.
\]

Finally,
\[
H_n^{(\lambda,\nu)}(x) = \sum_{k=0}^{\lfloor n/\lambda \rfloor} \frac{(-1)^k [n]_\nu!}{\lambda^k k! [n-k\lambda]_\nu!} x^{n-k\lambda}.
\]

\[\square\]

Proposition 3. The polynomials \(H_n^{(\lambda,\nu)}\) satisfy the following three terms recurrence relations
\[
x H_n^{(\lambda,\nu)}(x) = H_{n+1}^{(\lambda,\nu)}(x) + [n]_\nu \ldots [n-\lambda]_\nu H_{n-k}^{(\lambda,\nu)}(x).
\]

Proof. By taking into account (54) and (55) we get the three terms recurrence relations
\[
x H_n^{(\lambda,\nu)}(x) = H_{n+1}^{(\lambda,\nu)}(x) + [n]_\nu \ldots [n-\lambda]_\nu H_{n-k}^{(\lambda,\nu)}(x).
\]

\[\square\]
4. Matrix realization of the $C_\lambda$-extended oscillator

In sequel we assume that $d = \lambda - 1$. We propose here a realization of the $C_\lambda$-extended oscillator by matrices.

Let us first clarify the relationship between the notion of $d$-orthogonality of a family of polynomials and orthogonality of vector polynomials. Recall that $d$-orthogonal polynomials are system $\{P_n(x)\}$ of monic polynomials (with $\deg P_n = n$) such that there exists a vector linear functional $U = [u_0 \ldots u_{d-1}]^T$ satisfying the conditions:

\begin{align}
&i) \quad \langle u_j, x^k P_n \rangle = 0, \quad 0 \leq k \leq \left\lfloor \frac{n-j}{d} \right\rfloor, \\
&ii) \quad \langle u_j, x^n P_{n+d} \rangle \neq 0, \quad n = 0, \ldots.
\end{align}

where $\langle u, P \rangle$ is the effect of a linear functional $u$ on a polynomial $P$ and $\lfloor x \rfloor$ denotes the greatest integer function. Note that the case $d = 1$ corresponds to the ordinary notion of orthogonal polynomials.

According to \cite{24,25} the vector orthogonality relations is equivalent to the existence of a linear $(d+2)$-term recurrence relation

$$x P_n(x) = P_{n+1} + \sum_{j=0}^{d} a_j(n) P_{n-j}(x)$$

(63)

with constants $a_j(n), a_j(d) \neq 0$.

- A system of polynomials $\{P_n\}$ is said to be $d$-symmetric when it verifies

$$P_n(\varepsilon_{d+1} x) = \varepsilon_{d+1}^n P_n(x), \quad n \geq 0, \quad \varepsilon_{d+1} = e^{\frac{2\pi i}{d+1}}.$$

(64)

- We say that the vector of linear functionals $U = [u_1, \ldots, u_d]^T$ is said to be $d$-symmetric when the moments of its entries satisfy, for every $n \geq 0$,

$$\langle u_j, x^{(d+1)n+k-1} \rangle = 0$$

(65)

$$1 \leq j \leq d \quad 1 \leq k \leq d+1, \quad j \neq k.$$

(66)

According to \cite{11}, for every sequence of monic polynomials $\{P_n\},$ $d$-orthogonal with respect to the vector of linear functionals $U = [u_0 \ldots u_{d-1}]^T$, the following statements are equivalent:

(i) the vector of linear functionals $U$ is $d$-symmetric;
(ii) the sequence $\{P_n\}$ is $d$-symmetric;
(iii) the sequence $\{P_n\}$ satisfies

$$x P_{n+d}(x) = P_{n+d+1}(x) + \gamma_{n+1} P_n(x),$$

(67)

with $P_n(x) = x^n$ for $0 \leq n \leq d$. 

\[\]
Within this context, we obtain the following result on the generalized Hermite polynomials:

**Theorem 3.** For \( d = \lambda - 1 \), the family \( \{ H_n^{(\lambda,\nu)}(x) \} \) are \( d \)-orthogonal polynomials with respect to the functionals \( u_0, \ldots, u_{d-1} \), which are determined by their moments:

\[
\langle u_k, x^{n\lambda+s} \rangle = \delta_{ks} \int_0^\infty u^{n\lambda+s} v_s(u) \, du = \frac{[n\lambda + s]_\nu!}{n! \lambda^n},
\]

where

\[
v_s(u) = \frac{\lambda u^{-s-1}}{\prod_{i=1}^{\lambda-1} \Gamma(\alpha_i)} \left( \frac{u^\lambda}{\lambda^\lambda} \right)^{-1} \Delta(\nu, s).
\]

**Proof.** From (61), the polynomials \( \{ H_n^{(\lambda,\nu)}(x) \} \) satisfy a \( (\lambda+1) \)-term recursion relation of the form (67) and then, are \( d \)-orthogonal \( d \)-symmetric polynomials. Furthermore there exist \( \lambda - 1 \) symmetric functionals \( u_k, k = 0, \ldots, d-1 \), on the space of all polynomials \( \mathcal{P} \) such that

\[
\begin{cases}
  u_k(P_m P_n) = 0, m < \left\lfloor \frac{n - k}{d} \right\rfloor, \\
  u_k(P_n P_{n(d+1)+k}) \neq 0, n \geq 0.
\end{cases}
\]

To determine the moments of the functional \( u_k \), we will need the following inversion formula

\[
x^m = \sum_{n=0}^{\left\lfloor m/\lambda \right\rfloor} \frac{[m]_\nu!}{\lambda^n n! [m - n\lambda]_\nu!} H_{m-n\lambda}(x).
\]

To prove (71), we can use the generating function (58) to first obtain

\[
\mathcal{E}_\lambda(x t, \nu) = \sum_{n,k=0}^{\infty} \frac{H_k^{(\lambda,\nu)}(x)}{\lambda^n n! [k]_\nu!} t^{n\lambda+k},
\]

then, substituting \( m = n\lambda + k \), then \( 0 \leq n \leq \left\lfloor m/\lambda \right\rfloor \), we get

\[
\mathcal{E}_\lambda(x t, \nu) = \sum_{m=0}^{\infty} \sum_{n=0}^{\left\lfloor m/\lambda \right\rfloor} \frac{H_{m-n\lambda}(x)}{\lambda^n n! [m - n\lambda]_\nu!} t^m.
\]

and it remains to compare the coefficient of \( t^m \) in the two expansion (34) and (73) of \( \mathcal{E}_\lambda(x t, \nu) \).

Following Maroni [24], the moments of the functionals \( u_k, k = 0, \ldots, d-1 \) related to \( \{ H_n^{(\lambda,\nu)}(x) \} \) can be obtained from the inversion formula (71)

\[
\langle u_k, x^{n\lambda+s} \rangle = \langle u_k, x^{n\lambda+s} \rangle = \delta_{ks} \frac{[n\lambda + s]_\nu!}{n! \lambda^n}.
\]
It remains to prove the integral representation of the functionals \( u_k \), so it suffices to substitute

\[
p = s = 0, \quad q = r = \lambda - 1,
\]

\[
\gamma_i = \alpha_i + 1, i = 1, \ldots, s, \quad \gamma_i = \alpha_i, \quad i = s + 1, \ldots, \lambda - 1,
\]

in the integral (44), in order to get

\[
\int_0^\infty x^{n-1} G_{0,\lambda-1}^\lambda \left( \frac{x}{\lambda-1} \right) \Delta(\nu, s) \, dx = \lambda^{-s} \prod_{i=1}^{\lambda-1} \Gamma(\alpha_i) \frac{[n\lambda + s]!}{n!\lambda^n}.
\]

The latter yields

\[
\int_0^\infty u^{n\lambda+s} v_s(u) \, du = \frac{[n\lambda + s]!}{n!\lambda^n},
\]

where

\[
v_s(u) = \frac{\lambda u^{-s-1}}{\prod_{i=1}^{\lambda-1} \Gamma(\alpha_i)} G_{0,\lambda-1}^\lambda \left( \frac{u^{\lambda-1}}{\lambda^{\lambda-1}} \right) \Delta(\nu, s),
\]

so that

\[
\langle u_k, x^{n\lambda+s} \rangle = \delta_{ks} \int_0^\infty u^{n\lambda+s} v_s(u) \, du = \frac{[n\lambda + s]!}{n!\lambda^n}.
\]

Let \( \{P_n\} \) be a \( d \)-symmetric family of \( d \)-orthogonal polynomials with respect to the vector of functionals \( U = [u_0, \ldots, u_{d-1}]^T \). We define the family of vector polynomials \( \{P_n\} \) related to \( \{P_n\} \) as follows:

\[
P_n = [P_{nd}, \ldots, P_{(n+1)d-1}]^T
\]

and we extend the action of the vector of functionals \( U \) in vector polynomials \( \{P_n\} \) as follows:

\[
U(P) = \begin{bmatrix}
u_0(P_1) & \cdots & \nu_{d-1}(P_1) \\
\vdots & \ddots & \vdots \\
\nu_0(P_d) & \cdots & \nu_{d-1}(P_d)
\end{bmatrix}
\]

Within this context, \( \{P_n\} \) is said to be a vector orthogonal polynomial sequence with respect to the vector of functionals \( U \), if

\[
\begin{align*}
\text{i)} \quad (x^k U)(P_n) &= 0_{d \times d}, \quad k = 0, 1, \ldots, n - 1, \\
\text{ii)} \quad (x^0 U)(P_n) &= \Delta_n,
\end{align*}
\]

where \( \Delta_n \) is a regular upper triangular \( d \times d \) matrix.

We can see easily that the \( d \)-orthogonality of a family of polynomials \( \{P_n\} \) defined in (62) is equivalent to the vector orthgonality of
the related family \( \{ \mathbb{P}_n \} \) defined as above. For a deeper account of the theory (in a more general framework, considering quasi–diagonal multi–indices) we refer the reader to \cite{1}.

Let consider the family of vector polynomials \( \{ \mathbb{H}_n^{(\lambda, \nu)}(x) \} \) given by
\[
\mathbb{H}_n^{(\lambda, \nu)}(x) = \left[ \tilde{H}_{nd}^{(\lambda, \nu)}(x), \ldots, \tilde{H}_{(n+1)d-1}^{(\lambda, \nu)}(x) \right]^T, \quad n \in \mathbb{N}. \tag{80}
\]
where
\[
\tilde{H}_{n}^{(\lambda, \nu)}(x) = \left( [n]_\nu \right)^{-1/2} H_n^{(\lambda, \nu)}(x). \tag{81}
\]
From (61), we easily check the three terms recurrence relations
\[
x \tilde{H}_{n}^{(\lambda, \nu)}(x) = \sqrt{n+1} \tilde{H}_{n+1}^{(\lambda, \nu)}(x) + \alpha_n \tilde{H}_{n-\lambda+1}^{(\lambda, \nu)}(x). \tag{82}
\]
where
\[
\alpha_n = \sqrt{\frac{[n]_\nu}{[n-\lambda+1]_\nu}}. \tag{83}
\]
We also extend the action of the operator \( Y_\nu \) on the vector polynomial \( \mathbb{H}_n^{(\lambda, \nu)}(x) \) through
\[
Y_\nu \mathbb{H}_n^{(\lambda, \nu)}(x) = \left[ Y_\nu \tilde{H}_{nd}^{(\lambda, \nu)}(x), \ldots, Y_\nu \tilde{H}_{(n+1)d-1}^{(\lambda, \nu)}(x) \right]^T. \tag{85}
\]

**Theorem 4.** Under the above notations, we have
\[
x \mathbb{H}_n^{(\lambda, \nu)} = A_n \mathbb{H}_{n+1}^{(\lambda, \nu)} + B_n \mathbb{H}_n^{(\lambda, \nu)} + C_n \mathbb{H}_{n-1}^{(\lambda, \nu)}, \quad n = 0, 1, \ldots, \tag{84}
\]
\[
Y_\nu \mathbb{H}_n^{(\lambda, \nu)} = A_{n-1}^T \mathbb{H}_n^{(\lambda, \nu)} + B_n^T \mathbb{H}_n^{(\lambda, \nu)}, \quad n = 0, 1, \ldots \tag{85}
\]
\[
S \mathbb{H}_n^{(\lambda, \nu)} = R_n \mathbb{H}_n^{(\lambda, \nu)}. \tag{86}
\]
with \( \mathbb{H}_{-1}^{(\lambda, \nu)} = [0, \ldots, 0]^T \), \( \mathbb{H}_0^{(\lambda, \nu)} = \left[ \tilde{H}_0^{(\lambda, \nu)}(x), \ldots, \tilde{H}_{d-1}^{(\lambda, \nu)}(x) \right]^T \),
and matrix coefficients \( A_n, B_n, C_n, R_n \in \mathcal{M}_{d \times d} \) given by
\[
A_n = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & \sqrt{[n+1]d}_\nu & 0 & \cdots & 0 \\
\end{bmatrix}
\]
\[
B_n = \begin{bmatrix}
0 & \sqrt{nd+1}_\nu \\
\vdots & \ddots \\
0 & \sqrt{nd+d-1}_\nu \\
\end{bmatrix}
\]
\[
C_n = \text{diag} \left[ \gamma_{nd}, \gamma_{nd+1}, \ldots, \gamma_{(n+1)d-1} \right],
\]
\[
R = \text{diag} \left[ 1, \varepsilon_\lambda, \ldots, \varepsilon_{\lambda, d-1} \right].
\]
The position operator $x$ can be identified with the following Block matrix

\[
X = \begin{bmatrix}
B_0 & C_1 & & \\
A_0 & B_1 & C_2 & \\
A_1 & B_2 & C_3 & \\
& & \ddots & \ddots & \ddots
\end{bmatrix}
\]

and the momentum operator $Y_\nu$ with

\[
Y = \begin{bmatrix}
B_0^T & A_0^T & & \\
0 & B_1^T & A_1^T & \\
0 & B_2^T & A_2^T & \\
& & \ddots & \ddots & \ddots
\end{bmatrix}
\]

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$C_\lambda$: EXTENDED OSCILLATOR ALGEBRA AND $d$-ORTHOGONAL POLYNOMIALS

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