Abstract

We define a cup product on the cochain complex of a multisimplicial set, that is compatible with the classical cup product on the cochain complex of the diagonal simplicial set via the Eilenberg-Zilber map. This helps to speed up cochain level computations for multisimplicial complexes.

1 Introduction

The cochain complex $C^*(K)$ of a simplicial set $K$ is equipped with the classical Alexander-Whitney cup product that makes into a differential graded algebra. Our primary goal is to extend this product to the case of multisimplicial sets. Let us consider a $k$-fold simplicial set $X$, that is a contravariant functor from $(\Delta)^k$ to the category of sets. The restriction to the diagonal $\Delta \subset (\Delta)^k$ defines a simplicial set $X^D$. There is a notion of geometric realization $|X|$ of a $k$-fold simplicial set $X$, that is a CW complex with a cell $e_x$ for each non-degenerate multisimplex $x$, with a characteristic map from a product of simplexes

$$\Delta_{i_1} \times \cdots \times \Delta_{i_k} \to e_x$$

This extends the classical case where the characteristic map has a single simplex as domain. Quillen proved in [4] that there is a natural homeomorphism of realizations $|X| \cong |X^D|$. Under this homeomorphism the cells of $|X^D|$ arise from those of $|X|$ by subdividing $k$-fold products of simplexes into simplexes. This procedure is described combinatorially by the Eilenberg-Zilber quasi-isomorphism

$$EZ : C_*(X) \to C_*(X^D)$$

that induces a quasi-isomorphism on normalized chains $N_*(X) \to N_*(X^D)$ after quotienting out degenerate chains. As in the classical simplicial case, the projection $C_*(X) \to N_*(X)$ onto normalized chains is a quasi-isomorphism.

We prove in Theorem 4.8 that the cochain complex $C^*(X)$ is equipped with a differential graded algebra structure. The product is the natural extension to the multisimplicial case of the cup product defined by the Alexander-Whitney formula, by evaluating cochains on front and rear faces in all multisimplicial directions. We prove in section 6 that the dual Eilenberg-Zilber map

$$EZ^* : C^*(X^D) \to C^*(X)$$

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is a quasi-isomorphism of differential graded algebras, where the source is equipped with the classical cup product. Furthermore the Eilenberg-Zilber map restricts to a quasi-isomorphism

\[ N^\ast(X^D) \to N^\ast(X) \]

of sub-algebras of normalized cochains. Our result is very useful for computations, since multisimplicial models of spaces have a significantly smaller number of non-degenerate cells than their simplicial models. So \( N^\ast(X) \) is much smaller than \( N^\ast(X^D) \), but it contains the same information up to homotopy, allowing for example to calculate Massey products, and to detect its formality. As an example we consider a family of multisimplicial sets \( Sur(k) \) defined by McClure and Smith, see [5], modelling euclidean configuration spaces. The proof by the second author of the non-formality of the cochain algebra of planar configuration spaces in [6] used the Barratt-Eccles simplicial model and the classical cup product. Our new product on the multisimplicial McClure-Smith models makes the computation much simpler and faster, paving the way for an extension to higher dimensions.

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2 Normalized complexes of multisimplicial modules

We always consider modules over a commutative ring \( R \) that will be usually be dropped from the notation.

**Definition 2.1.** Let \( X \) be a simplicial module, i.e. a contravariant functor from \( \Delta \) to the category of \( R \)-modules. The chain complex associated to \( X \) is \( C^\ast(X) = X^\ast \) with differential \( \partial_n : C_n(X) \to C_{n-1}(X) \) given by the formula

\[ \partial_n = \sum_{i=0}^{n} (-1)^i d_i \]

where each \( d_i : X_n \to X_{n-1} \) is a face map, the image of \( \delta_i : [n-1] \to [n] \) via \( X \).

**Definition 2.2.** A multisimplicial (or \( k \)-fold simplicial) module (resp. set) \( X \) is a functor from \((\Delta)^k\) to the category of \( R \)-modules (resp. sets), for some positive integer \( k \). An element \( x \in X_{i_1, \ldots, i_k} \) is called a \((i_1, \ldots, i_k)\)-multisimplex. The face map

\[ d_j^i : X_{i_1, \ldots, i_{j-1}, i_j, \ldots, i_k} \to X_{i_1, \ldots, i_{j-1}, i_j-1, \ldots, i_k} \]

in direction \( j \) for \( 1 \leq j \leq k \) and \( 0 \leq i_j \leq j \), is the image via \( X \) of \((id, \ldots, id, \delta_{i_j}, id, \ldots, id)\). The degeneracy map

\[ s_l^i : X_{i_1, \ldots, i_{j-1}, i_j, \ldots, i_k} \to X_{i_1, \ldots, i_{j-1}, i_j+1, \ldots, i_k} \]

in direction \( l \) is the image via \( X \) of \((id, \ldots, id, \sigma_l, id, \ldots, id)\).

Clearly faces and/or degeneracies in different directions commute with each other.
**Definition 2.3.** The associated chain complex $C_*(X)$ of a multisimplicial module is defined by

$$C_n(X) = \bigoplus_{i_1, \ldots, i_k} X_{i_1, \ldots, i_k}$$

If we define

$$\partial_{i_j} = \sum_{t=0}^{i_j} (-1)^{t+i_1+\cdots+i_{j-1}} d_{i_j}^t : X_{i_1, \ldots, i_j, \ldots, i_k} \to X_{i_1, \ldots, i_{j-1}, i_j, \ldots, i_k}$$

then the multisimplicial differential on $C_*(X)$ is defined on the summand $X_{i_1, \ldots, i_j, \ldots, i_k}$ by the formula

$$\partial = \partial_{i_1} + \cdots + \partial_{i_k}$$

As usual the cochain complex $C^*(X)$ of a (multi)simplicial module $X$ is the linear dual of $C_*(X)$. A standard example of multisimplicial $R$-module is $X = RK$, with $K$ multisimplicial set, and $X_{i_1, \ldots, i_k}$ the free $R$-module on $K_{i_1, \ldots, i_k}$.

The following definition extends the standard definition of the simplicial case.

**Definition 2.4.** A multisimplex of a multisimplicial module (or multisimplicial set) is said to be degenerate if it is in the image of a degeneracy.

We define next the normalized chain complex of a multisimplicial module.

**Definition 2.5.** Let $X$ a multisimplicial module. We define $D_*(X)$ to be the subcomplex of $C_*(X)$ generated by all degenerate elements, in each dimension $n$, i.e. $D_0(X) = 0$ and:

$$D_n(X) = \sum_{0 \leq i_1 \leq \cdots \leq i_k \leq n - 1} s^l_{i_1} X_{i_1, \ldots, i_k}$$

By the multisimplicial identities $D_*(X)$ is closed under the differential $\partial$, so it is actually a subcomplex of $C_*(X)$. The quotient $C_*(X)/D_*(X) = N_*(X)$ is called the normalized chain complex of $X$.

A classical result is the following normalization theorem.

**Theorem 2.6.** *(Normalization Theorem 6.1, Chap. 8 in [3])* For each simplicial module $X$ the canonical projection $\pi : C_*(X) \to N_*(X)$ is a chain equivalence.

The result extends to multisimplicial modules.

**Theorem 2.7.** *(Generalized Normalization Theorem)* For each multisimplicial module $X$ the canonical projection $\pi : C_*(X) \to N_*(X)$ is a chain equivalence.

**Proof.** The proof is a slight variation of the proof in [2]. Consider the degree one map $t^l_j : C_*(X) \to C_{*+1}(X)$ given by $t^l_j(x) = (-1)^{j+i_1+\cdots+i_{l-1}} s^l_{i_1} x$ for a multisimplex $x \in X_{i_1, \ldots, i_k}$, $1 \leq l \leq k$ and $0 \leq j \leq i_l$. Then

$$h^l_j = 1 - \partial t^l_j - t^l_j \partial : C_*(X) \to C_*(X)$$

is a chain map, and $t^l_j$ a homotopy between $h^l_j$ and the identity. We set $h^l = h^l_0 \circ h^l_1 \circ \ldots$ for $l = 1, \ldots, k$. Then we set $h = h^1 \circ \cdots \circ h^k$. Notice that the chain maps $h^1, \ldots, h^k$ commute with each other. We conclude following the proof by Mac Lane. 🅽
3 Eilenberg-Zilber maps

We define first the classical Eilenberg-Zilber map, and then the multisimplicial version. Let us recall the definition of shuffle.

Definition 3.1. Let \( a_1, \ldots, a_k \in \mathbb{N} \). A permutation \( \sigma \in S_{a_1 + \cdots + a_k} \) such that

\[
\sigma(1) < \cdots < \sigma(a_1), \quad \sigma(a_1 + 1) < \cdots < \sigma(a_1 + a_2) \cdots \sigma(a_1 + \cdots + a_{k-1} + 1) < \cdots < \sigma(a_1 + \cdots + a_k)
\]

is called an \((a_1, \ldots, a_k)\)-shuffle.

Equivalently an \((a_1, \ldots, a_k)\)-shuffle corresponds to a collection of monotone maps

\[
\pi_i : [a_1 + \cdots + a_k] \to [a_i]
\]

for \( i = 1, \ldots, k \) such that for each \( j = 0, \ldots, a_1 + \cdots + a_k - 1 \) there is exactly one index \( l \in \{1 \ldots k\} \) satisfying \( \pi_i(j + 1) = \pi_l(j + 1) \) and \( \pi_i(j + 1) = \pi_l(j) \) for all \( i \neq l \).

Geometrically this collection \((\pi_1, \ldots, \pi_k)\) represents a sequence of \((a_1 + \cdots + a_k)\) moves in a lattice of integral points with \((k+1)\)-coordinates, starting at the origin, and moving in a single direction at each stage, until the point \((a_1, \ldots, a_k)\) is reached.

The associated permutation sends \( j + 1 \) to \( a_1 + \cdots + a_{i-1} + \pi_i(j + 1) \). We denote the set of \((a_1, \ldots, a_k)\)-shuffles by \( sh(a_1, \ldots, a_k) \).

Example 3.2. The permutation \( \pi = (1, 2, 5, 3, 4, 6) \) is a \((4, 2)\)-shuffle. Indeed we have that \( \pi(1) = 1 < \pi(2) = 2 < \pi(3) = 4 < \pi(4) = 5 < \pi(5) = 3 < \pi(6) = 6 \). The corresponding monotone maps are

\[
\pi_1 : [0, 6] \to [0, 4] \quad \text{such that} \quad \pi_1(0) = 0, \pi_1(1) = 1, \pi_1(2) = 2 = \pi_1(3), \pi_1(4) = 3, \pi_1(5) = 4 = \pi_1(6) \quad \text{and} \quad \pi_2 : [0, 6] \to [0, 2] \quad \text{such that} \quad \pi_2(0) = 0 = \pi_2(1) = \pi_2(2), \pi_2(3) = 1 = \pi_2(4) = \pi_2(5), \pi_2(6) = 2.
\]

The corresponding path is indicated in red in the figure.

![Figure 1: (4,2)-shuffle of example 3.2](image)

Definition 3.3. For simplicial modules \( X \) and \( Y \) the classical Eilenberg-Zilber map

\[
EZ : C_\ast(X) \otimes C_\ast(Y) \to C_\ast(X \otimes Y)
\]

is defined by

\[
EZ(x \otimes y) = \sum_{\pi \in sh(p,q)} sgn(\pi)X(\pi_1)(x) \otimes X(\pi_2)(y)
\]

for \( x \in X_p \) and \( y \in Y_q \).
Theorem 3.4. (Classic Eilenberg-Zilber Theorem) For simplicial modules $X$ and $Y$

$$EZ : C_\ast(X) \otimes C_\ast(Y) \to C_\ast(X \otimes Y)$$

is a natural chain equivalence, inducing on the quotient a chain equivalence

$$N_\ast(X) \otimes N_\ast(Y) \to N_\ast(X \otimes Y)$$

One can choose as natural inverse equivalence the Alexander-Whitney map that we will introduce later.

Definition 3.5. For simplicial modules $X_1, \ldots, X_k$ the multivariable Eilenberg-Zilber chain map $EZ : C_\ast(X_1) \otimes \cdots \otimes C_\ast(X_k) \to C_\ast(X_1 \otimes \cdots \otimes X_k)$ is defined by

$$EZ(x_1 \otimes \cdots \otimes x_k) = \sum_{\pi \in sh(a_1, \ldots, a_k)} sgn(\pi)X_1(\pi_1)(x_1) \otimes \cdots \otimes X_k(\pi_k)(x_k)$$

It is easy to verify that this chain map can also be obtained by iterating the classical map. In particular $EZ$ induces a map on normalized cochains. By applying repeatedly theorem 3.4 and using the normalization theorem 2.6 we obtain the following corollary.

Corollary 3.6. (Extension of Eilenberg-Zilber Theorem) For simplicial modules $X_1, \ldots, X_k$

$$EZ : C_\ast(X_1) \otimes \cdots \otimes C_\ast(X_k) \to C_\ast(X_1 \otimes \cdots \otimes X_k)$$

is a natural chain equivalence, and so is the induced chain map

$$EZ : N_\ast(X_1) \otimes \cdots \otimes N_\ast(X_k) \to N_\ast(X_1 \otimes \cdots \otimes X_k)$$

Similarly as before, a natural inverse equivalence can be chosen to be an iterated Alexander-Whitney map.

For the multisimplicial version we need the following definition.

Definition 3.7. Given a multisimplicial module (or set) $X : \Delta^k \to \text{Set}$ its diagonal $X^D : \Delta \to \text{Set}$ is the simplicial module (or set) that is the restriction of $X$ to the diagonal copy $\Delta \subset \Delta^k$.

Observe now that if $X$ is a multisimplicial module, $\pi$ an $(a_1, \ldots, a_k)$-shuffle, and $x \in X_{a_1, \ldots, a_k}$ a multisimplex, then

$$X(\pi_1, \ldots, \pi_k)(x) \in X_{a_1 + \cdots + a_k}^D$$

is in the diagonal. We can now define the multisimplicial Eilenberg-Zilber map.

Definition 3.8. Let $X$ be a multisimplicial module. The multisimplicial Eilenberg-Zilber map, $EZ : C_\ast(X) \to C_\ast(X^D)$, is defined as follows: For a given $(a_1, \ldots, a_k)$-multisimplex $x$

$$EZ(x) = \sum_{\pi \in sh(a_1, \ldots, a_k)} sgn(\pi)X(\pi_1, \ldots, \pi_k)(x)$$

We can extend Theorem 2.5, chap. 4 in [2], that is about bisimplicial modules, to the general multisimplicial case as follows:
Theorem 3.9. Let $X$ be a multisimplicial module. Then $EZ : C_*(X) \rightarrow C_*(X^D)$ is a natural chain equivalence.

Proof. The proof is similar to that of Theorem 2.5, chap.4 in [2] adapted to multisimplicial modules, using corollary 3.6. Consider the standard multisimplicial set

$$\Delta_{i_1,\ldots,i_k} := \text{Hom}(\_,[i_1],\ldots,[i_k])$$

Corollary 3.6 proves the theorem for the standard multisimplicial module $X = R\Delta_{i_1,\ldots,i_k}$. Namely the simplicial set $\Delta_{i_1} \times \cdots \times \Delta_{i_k}$ is isomorphic to the diagonal $R\Delta_{i_1,\ldots,i_k}$, and $R\Delta_{i_1} \otimes \cdots \otimes R\Delta_{i_k}$ is isomorphic to the diagonal $R\Delta_{i_1,\ldots,i_k}$, so it is sufficient to choose $X_l = R\Delta_{i_l}$ for $l = 1,\ldots,k$. The remainder of the proof follows [2], writing a generic multisimplicial module as colimit of standard multisimplicial modules.

Even in this case a natural inverse equivalence can be constructed in terms of Alexander-Whitney maps, by naturality of the chain homotopies in the proof [2] and theorem 2.7.

Corollary 3.10. The multisimplicial Eilenberg-Zilber map induces a natural chain map of normalized chain complexes $EZ^N : N_*(X) \rightarrow N_*(X^D)$.

We state next an important result by Quillen about realizations of multisimplicial sets, that is a companion of theorem 3.9. We need first to define the realization of a multisimplicial set $X$, that is the natural generalization of the realization of a simplicial set. Let us denote the standard topological $i$-simplex by $\Delta_i$, and the face and degeneracy maps between topological simplexes respectively by $\delta_i$ and $\sigma_i$.

Definition 3.11. The realization $|X|$ of a $k$-fold simplicial set $X$ is the CW complex

$$\prod X_{i_1,\ldots,i_k} \times \Delta_{i_1} \times \cdots \times \Delta_{i_k} / \sim$$

where

$$(d^i_l(x), y_1, \ldots, y_l, \ldots, y_k) \sim (x, y_1, \ldots, \delta_i(y_l), \ldots, y_k),$$

$$(s^i_l(x), y_1, \ldots, y_l, \ldots, y_k) \sim (x, y_1, \ldots, \sigma_i(y_l), \ldots, y_k)$$

Theorem 3.12. (Quillen) [7] For a multisimplicial set $X$ there is a natural homeomorphism $|X| \cong |X^D|$.

The theorem is proved similarly as Theorem 3.9 first when $X = \Delta_{i_1,\ldots,i_k}$. In this case the multisimplicial realization is $\Delta_{i_1} \times \cdots \times \Delta_{i_k}$, and the realization of the diagonal $X^D$ provides a subdivision of this product of simplexes into various simplexes. The general case is then obtained via colimits. Notice that the (non-degenerate) simplices of $|X^D|$ correspond to the summands appearing in the formula of the normalized Eilenberg-Zilber map applied to the (non-degenerate) multisimplices of $X$.

4 The multisimplicial Alexander-Whitney map

The task of this section is to define the Alexander-Whitney map for multisimplicial modules, that will allow us to define a cup product on the multisimplicial cochain
level, yielding a differential graded algebra structure that generalizes the classical cup product.

We recall that for a simplicial module $X$ and a simplex $x \in X_n$, its front $i$-face is $x \lfloor_i := X(F_i)(x) \in X_i$, with $F_i : [i] \to [n]$, $F_i(l) = l$, and the back $j$-face of $x$ is $j \lfloor x := X(B_j)(x) \in X_j$, with $B_j : [j] \to [n]$, $B_j(l) = l + n - j$.

The classical Alexander-Whitney map is defined as follows.

**Definition 4.1.** (Simplicial Alexander-Whitney map) Let $X$ be a simplicial module. The Alexander-Whitney map is a chain map:

$$ \text{AW} : C_\ast(X) \to C_\ast(X) \otimes C_\ast(X) $$

which acts on $n$-simplexes $x$ according to the formula

$$ \text{AW}_n(x) = \sum_{i=0}^{n} x \lfloor_i \otimes n-i \lfloor x $$

**Lemma 4.2.** (8.6 Chap. 8 in [3]) The Alexander-Whitney map induces a chain transformation on the normalized chain complexes

$$ N_\ast(X) \to N_\ast(X) \otimes N_\ast(X) $$

The key point for the definition of the multisimplicial Alexander-Whitney map is to extend what 'front' and 'back' faces mean, because a multisimplicial set has more coordinates (or directions) to manage. We can do this by picking front and back faces independently in each coordinate as follows.

**Definition 4.3.** Let $X$ be a multisimplicial module and $x \in X_{a_1, \ldots, a_n}$. Given indices $i_l \in \{0, \ldots, a_l\}$ for $l = 1, \ldots, k$ the front $(i_1, \ldots, i_k)$-face of $x$ is the multisimplex

$$ x \lfloor_{(i_1, \ldots, i_k)} := X(F_{i_1}, \ldots, F_{i_k})(x) \in X_{i_1, \ldots, i_k} $$

and the back $(i_1, \ldots, i_k)$-face of $x$ is the multisimplex

$$ (i_1, \ldots, i_k) \lfloor x := X(B_{i_1}, \ldots, B_{i_k})(x) \in X_{i_1, \ldots, i_k} $$

Proceeding from here, emulating the process used to obtain the simplicial Alexander-Whitney map, we have the following definition.

**Definition 4.4.** The Alexander-Whitney of a multisimplicial module $X$ is the chain map

$$ \text{AW}_{msimp} : C_\ast(X) \to C_\ast(X) \otimes C_\ast(X) $$

which assigns to any $(a_1, \ldots, a_k)$-simplex $x$

$$ \text{AW}_{msimp}(x) = \sum_{i_j=0, \ldots, a_j}^{a_j} \sum_{j=1, \ldots, k} \left( \sum_{l<h} (-1)^{\sum_{i_h=a_h-i_h}^{a_h}} x \lfloor_{(i_1, \ldots, i_h)} \otimes (a_1-i_1, \ldots, a_k-i_k) \right) $$

The following properties can be verified similarly as in the classical simplicial case.

**Lemma 4.5.** The homomorphism $\text{AW}_{msimp} : C_\ast(X) \to C_\ast(X) \otimes C_\ast(X)$ is a chain map
Lemma 4.6. The homomorphism $AW = AW_{msimp}$ satisfies coassociativity, in the sense that

$$(AW \otimes id)AW = (id \otimes AW)AW : C_*(X) \to C_*(X) \otimes C_*(X) \otimes C_*(X).$$

Lemma 4.7. The multisimplicial Alexander-Whitney map induces a chain transformation on the associated normalized chain complexes

$$N_*(X) \to N_*(X) \otimes N_*(X).$$

The dual homomorphism of $AW_{msimp}$ gives a pairing $C_*(X) \otimes C_*(X) \to C_*(X)$ that we call multisimplicial cup product.

Theorem 4.8. For a multisimplicial module $X$, the cup product defines a graded differential algebra structure on $C_*(X)$, inducing a graded algebra structure on $H_*(X)$. Furthermore the quasi-isomorphic subcomplex of normalized cochains $N_*(X) \subset C_*(X)$ is a subalgebra.

Proof. By lemma 4.6 Alexander-Whitney map is coassociative, and so its dual is associative. The compatibility with the differential follows from lemma 4.5. The second statement follows from lemma 4.7. \qed

5 Surjection and Barratt-Eccles complexes

We study an important example that will help us to understand how to prove our main theorem.

Definition 5.1. We recall the definition of the surjection multisimplicial set $\text{Sur}(k)$ by McClure and Smith [5]. $\text{Sur}(k)_{i_1,\ldots,i_k}$ is the set of surjective maps $f : \{1, \ldots, i_1 + \cdots + i_k + k\} \to \{1, \ldots, k\}$ such that the cardinality of $f^{-1}(l)$ is $i_l$, for $l = 1, \ldots, k$. We represent such maps by the sequence $f(1) \ldots f(i_1 + \cdots + i_k + k)$

The multisimplicial structure is defined as follows: $d_j$ removes the $(j + 1)$-th occurrence of $l$ in a sequence, and $s_j$ doubles the $(j + 1)$-th occurrence of $l$ in a sequence. So for example

$$d_0(12321) = 1321, \quad d_2(12321) = 1231, \quad s_0(121) = 1121$$

Degenerate multisimplices are exactly the sequences containing two equal adjacent terms.

The front (resp. back) $(i_1, \ldots, i_k)$-face of a sequence is the subsequence containing only the first (resp. last) $(i_l + 1)$-values of $l$, for each $l = 1, \ldots, k$.

There is an interesting connection between $\text{Sur}$ and the Barratt-Eccles simplicial sets $W\Sigma_k$. Here $\Sigma_k$ is the symmetric group of permutations of $\{1, \ldots, k\}$, $(W\Sigma_k)_i = (\Sigma_k)^i$, a face $d_j$ removes the $(j + 1)$-st permutation, and a degeneracy $s_j$ doubles the $(j + 1)$-st permutation. The normalized chain complex of $W\Sigma_k$ is the Barratt-Eccles chain complex

$$BE(k) := N_*(W\Sigma_k).$$
Similarly the normalized chain complex of $\text{Sur}$ is the surjection chain complex

$\chi(k) := N_*(\text{Sur}(k))$

The collections of these complexes over all $k$ are operads bearing the same name. Berger and Fresse construct in [1] chain maps $T_C : \chi(k) \rightarrow BE(k)$ (respecting the operad structures). We claim that $T_C$ is induced by a map of simplicial sets

$tc : \text{Sur}(k)^D \rightarrow W\Sigma_k$

that we define.

**Definition 5.2.** Let $s$ be an $i$-simplex of the diagonal $\text{Sur}(k)^D$, i.e. a sequence containing any value in $\{1, \ldots, k\}$ exactly $i + 1$ times. Then $tc(s) := (\sigma_0, \ldots, \sigma_i)$ is a sequence of permutations where each $\sigma_j$ is the subsequence of $s$ containing the $(j + 1)$-st occurrence of each value in $\{1, \ldots, k\}$.

For example

$tc(122333112) = (123, 231, 312)$

**Proposition 5.3.** The homomorphism $T_C$ by Berger-Fresse satisfies

$T_C = N_*(tc) \circ EZ$

where

$EZ : \chi(k) = N_*(\text{Sur}(k)) \rightarrow N_*(\text{Sur}(k)^D)$

and

$N_*(tc) : N_*(\text{Sur}(k)^D) \rightarrow N_*(W\Sigma_k) = BE(k)$

Both $W$ and $\text{Sur}$ are filtered [1], i.e. there is a family of nested simplicial sets

$W_1\Sigma_k \subset W_2\Sigma_k \subset \ldots$

such that $W\Sigma_k = \text{colim} d W_d\Sigma_k$, and a family of nested $k$-fold simplicial sets

$\text{Sur}_1(k) \subset \text{Sur}_2(k) \subset \ldots$

such that $\text{Sur}(k) = \text{colim}_d \text{Sur}_d(k)$. The normalized chain functor defines

$BE_d(k) := N_*(W_d\Sigma_k)$

so that $BE_d$ is a suboperad of $BE$ for each $d$, and similarly

$\chi_d(k) := N_*(\text{Sur}_d(k))$

so that $\chi_d$ is a suboperad of $\chi$ for each $c$. We observe that the simplicial map $tc$ respects the filtration, sending $W_d\Sigma_k$ to $\text{Sur}_d(k)^D$. The geometric realizations satisfy

$|\text{Sur}_d(k)| \simeq |W_d\Sigma_k| \simeq F_k(R^d)$

where $F_k(R^d)$ is the ordered configuration space of $k$-tuples of points in $\mathbb{R}^d$. We stress that the number of generators in $\chi_d(k)$, corresponding to non-degenerate surjections,
is much smaller than the corresponding number of generators in $BE_d(k)$. Let us consider the generating polynomial functions counting the generators

$$PB_d^k(x) = \sum_i \text{rank}(BE_d(k)_i) x^i$$

$$P\chi_d^k(x) = \sum_i \text{rank}(\chi_d(k)_i) x^i$$

Then for example

$$PB_d^4(x) = 24(1 + 23x + 104x^2 + 196x^3 + 184x^4 + 86x^5 + 16x^6)$$

$$PB_d^4(x) = 24(1 + 6x + 10x^2 + 5x^3)$$

$$PB_d^3(x) = 6(1 + 5x + 25x^2 + 60x^3 + 70x^4 + 38x^5 + 8x^6)$$

$$PB_d^3(x) = 6(1 + 3x + 7x^2 + 9x^3 + 6x^4 + x^5)$$

Therefore the multisimplicial approach using $\chi$ is much more efficient than the simplicial approach using $BE$, when performing computations as in [6].

6 Compatibility of the multisimplicial cup product

The aim of this section is to compare the multisimplicial cup product on $C^*(X)$ for a multisimplicial module $X$ with the classical cup product on $C^*(X^D)$, where $X^D$ is the diagonal simplicial module, by showing that the (dual) Eilenberg-Zilber map $EZ^*: C^*(X^D) \rightarrow C^*(X)$ is a homomorphism of differential graded algebras. This happens if and only if the following diagram of chain maps is commutative.

$$\begin{array}{ccc}
C_*(X) & \xrightarrow{EZ} & C_*(X^D) \\
\downarrow AW_{msimp} \quad & & \quad \downarrow AW_{simp} \\
C_*(X) \otimes C_*(X) & \xrightarrow{EZ \otimes EZ} & C_*(X^D) \otimes C_*(X^D)
\end{array}$$

**Example 6.1.** Let us consider the multisimplicial module $X = \mathbb{Z}_2 \text{Sur}(3)$. The $(1,1,0)$-multisimplex $12321$ of $\text{Sur}(3)$ represents a generator of $C_2(X)$. 

\[ \begin{array}{c}
1 \\
\Delta^1 \\
\downarrow \\
2 \\
\times \\
\Delta^1 \\
\downarrow \\
3 \\
\times \\
\Delta^0 \\
\downarrow \\
2 \\
\times \\
1
\end{array} \]
Let us calculate \( AW_{simp}(EZ(12321)) \) and \((EZ \otimes EZ)(AW_{msimp}(12321))\).

\[
EZ(12321) = 11233221 + 12233211 \in C_3(X^D).
\]

\[
AW_{simp}(EZ(12321)) = AW_{simp}(11233221) + AW_{simp}(12233211) =
\]
\[
= 123 \otimes EZ(12321) + EZ(12321) \otimes 321 +
+ 112332 \otimes 133221 + 122331 \otimes 233211.
\]

\[
AW_{msimp}(12321) = 123 \otimes 12321 + 1231 \otimes 2321 +
+ 1232 \otimes 1321 + 12321 \otimes 321.
\]

\[
(EZ \otimes EZ)(AW_{msimp}(12321)) = 123 \otimes EZ(12321) + 122331 \otimes 233211 +
+ 112332 \otimes 133221 + EZ(12321) \otimes 321.
\]

So the arrows of diagram \( \Delta \) commute when evaluated on 12321.

**Example 6.2.** Consider the \((2,1)\)-multisimplex 12121 of \( X = \mathbb{Z}_2 Sur(2) \), that is a generator of \( C_3(X) \).

Let us compute \( AW_{simp}(EZ(12121)) \) and \((EZ \otimes EZ)(AW_{msimp}(12121))\). We draw the grids used to calculate \( EZ \) and write down the indices in \( AW \). We have

\[
EZ(12121) = 12221211 \underbrace{+ 12211221}_{A} + 11212221 \underbrace{+ 11212221}_{C}
\]

where \( A, B \) and \( C \) are obtained respectively through shuffles associated to the following paths in the associated grids.

![Figure 2: Grids representing shuffles respectively of \( A, B \) and \( C \).](image)

We compute the summands of \( AW_{simp}(EZ(12121)) \) by indicating the index \( i \) of the front face for each of the 3-simplexes \( A, B, C \).
\begin{align*}
AW_{simp}(A) & \quad i = 0 \quad 12 \otimes 1221211 \\
& \quad i = 1 \quad 1221 \otimes 22111 \\
& \quad i = 2 \quad 122211 \otimes 22111 \quad \text{degenerate by } s_0 \\
& \quad i = 3 \quad 122211 \otimes 21 \\
AW_{simp}(B) & \quad i = 0 \quad 12 \otimes 12211221 \\
& \quad i = 1 \quad 1221 \otimes 211221 \\
& \quad i = 2 \quad 122112 \otimes 1221 \\
& \quad i = 3 \quad 12211221 \otimes 21 \\
AW_{simp}(C) & \quad i = 0 \quad 12 \otimes 11212221 \\
& \quad i = 1 \quad 1122 \otimes 112221 \quad \text{degenerate by } s_0 \\
& \quad i = 2 \quad 112122 \otimes 1221 \\
& \quad i = 3 \quad 11212221 \otimes 21 
\end{align*}

So we have twelve summands, and two of these have a degenerate factor.

We calculate now \((EZ \otimes EZ)(AW_{simp}(1212121))\). Starting from a \((2, 1)\)-multisimplex, to calculate \(AW_{simp}\) we need two indices, \(k = 0, 1, 2 \) e \( j = 0, 1 \). We outline couples of indices associated to each summand of \(AW_{simp}(1212121)\).

\begin{itemize}
\item \((k = 0, j = 0)\) \quad 12 \otimes 12121 \quad \text{risp. (0, 0) and (2, 1) multisimplex}
\item \((k = 0, j = 1)\) \quad 122 \otimes 12121 \quad \text{risp. (0, 1) and (2, 0) multisimplex}
\item \((k = 1, j = 0)\) \quad 121 \otimes 2121 \quad \text{risp. (1, 0) and (1, 1) multisimplex}
\item \((k = 1, j = 1)\) \quad 1212 \otimes 121 \quad \text{risp. (1, 1) and (1, 0) multisimplex}
\item \((k = 2, j = 0)\) \quad 1211 \otimes 221 \quad \text{risp. (2, 0) and (0, 1) multisimplex}
\item \((k = 2, j = 1)\) \quad 12121 \otimes 21 \quad \text{risp. (2, 1) and (0, 0) multisimplex}
\end{itemize}

Now, for each couple of indices, we write down all the shuffles we need for \(EZ \otimes EZ\): for \((k = 0, j = 0)\) the shuffles determining \((EZ \otimes EZ)(12 \otimes 12121)\) are given by:

\begin{itemize}
\item \(id \otimes \) \quad \Rightarrow \quad 12 \otimes 1221211
\item \(id \otimes \) \quad \Rightarrow \quad 12 \otimes 12211221
\end{itemize}
for \((k = 0, j = 1)\) the shuffles determining \((EZ \otimes EZ)(122 \otimes 1121)\) are given by:

\[
\begin{array}{c}
1 \\
\otimes \\
0 \\
\end{array} \Rightarrow 
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]

\[
\begin{array}{c}
12 \\
\otimes \\
112122 \\
\end{array}
\]

for \((k = 1, j = 0)\) the shuffles determining \((EZ \otimes EZ)(121 \otimes 2121)\) are given by:

\[
\begin{array}{c}
0 \\
1 \\
\otimes \\
0 \\
1 \\
\end{array} \Rightarrow 
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]

\[
\begin{array}{c}
1221 \\
\otimes \\
212121 \\
\end{array}
\]

for \((k = 1, j = 1)\) the shuffles determining \((EZ \otimes EZ)(1212 \otimes 121)\) are given by:

\[
\begin{array}{c}
1 \\
\otimes \\
0 \\
1 \\
\end{array} \Rightarrow 
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]

\[
\begin{array}{c}
122112 \\
\otimes \\
1221 \\
\end{array}
\]

for \((k = 2, j = 0)\) the shuffles determining \((EZ \otimes EZ)(1211 \otimes 221)\) are given by:

\[
\begin{array}{c}
0 \\
1 \\
2 \\
\otimes \\
0 \\
1 \\
\end{array} \Rightarrow 
\begin{array}{c}
1 \\
0 \\
\end{array}
\]

\[
\begin{array}{c}
122111 \\
\otimes \\
221 \\
\end{array}
\]
finally for \((k = 2, j = 1)\) the shuffles determining \((EZ \otimes EZ)(12121 \otimes 21)\) are given by:

\[
\begin{array}{ccc}
1 & \otimes & id \\
0 & 0 & 1 \\
& & 2 \\
\end{array} \Rightarrow \begin{array}{c}
12221211 \otimes 21 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & \otimes & id \\
0 & 0 & 1 \\
& & 2 \\
\end{array} \Rightarrow \begin{array}{c}
12211221 \otimes 21 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & \otimes & id \\
0 & 0 & 1 \\
& & 2 \\
\end{array} \Rightarrow \begin{array}{c}
11212221 \otimes 21 \\
\end{array}
\]

So \((EZ \otimes EZ) \circ AW_{\text{msimp}}(12121)\) has the same summands as \(AW_{\text{simp}} \circ EZ(12121)\) (and two of them have a degenerate factor). Therefore the arrows of diagram \((\Pi)\) commute when evaluated on 12121.

From this examples we can see that each summand of \(AW_{\text{simp}} \circ EZ\) in a diagram like \((\Pi)\) is indexed by a path on a grid

\[
\begin{array}{cccccccc}
3 & | & | & | & | & | & 6 \\
\hline
2 & | & | & | & | & | & 5 \\
\hline
1 & | & | & | & | & | & 4 \\
\hline
0 & | & | & | & | & | & 3 \\
\hline
0 & | & | & | & | & | & 2 \\
\hline
0 & | & | & | & | & | & 1 \\
\hline
0 & | & | & | & | & | & 0 \\
\hline
\end{array}
\]

together with a splitting of this path as concatenation of two paths. The corresponding summand of \((EZ \otimes EZ) \circ AW_{\text{msimp}}\) is indexed by the two maximal sub-grids containing respectively these two paths
We formalize this fact in order to prove the commutativity of diagram (1).

**Definition 6.3.** For $0 \leq i_l \leq a_l, l = 1, \ldots, k$ there is a concatenation product of shuffles

$$sh(i_1, \ldots, i_k) \times sh(a_1 - i_1, \ldots, a_k - i_k) \to sh(a_1, \ldots, a_k)$$

sending a pair $(\pi], [\pi)$ to $\pi := \pi] * [\pi$ such that

$$(\pi)]_l(j) = \pi_l(j)$$

for $j \in \{0, \ldots, i_1 + \cdots + i_k\}$, and

$$([\pi)_l(j) = \pi_l(j + i_1 + \cdots + i_k) - i_l$$

for $j \in \{0, \ldots, a_1 - i_1 + \cdots + a_k - i_k\}$

It is easy to see the following.

**Lemma 6.4.** There is a bijection

$$\prod_{0 \leq i_l \leq a_l, \atop 1 \leq l \leq k} (sh(i_1, \ldots, i_k) \times sh(a_1 - i_1, \ldots, a_k - i_k)) \cong sh(a_1, \ldots, a_k)$$

induced by the concatenation product.

We are now ready to prove the commutativity of diagram (1) in full generality.

**Proposition 6.5.** Let $X$ be a multisimplicial module. Then the following diagram commutes

$$\begin{array}{ccc}
C_*(X) & \xrightarrow{EZ} & C_*(XD) \\
\downarrow AW_{msimp} & & \downarrow AW_{simp} \\
C_*(X) \otimes C_*(X) & \xrightarrow{EZ \otimes EZ} & C_*(XD) \otimes C_*(XD)
\end{array}$$

**Proof.** Suppose that $X$ is a $k$-fold simplicial module, and pick $x \in X_{a_1, \ldots, a_k}$.

For $x \in X_{a_1, \ldots, a_k}, EZ(x)$ is a sum over the $(a_1, \ldots, a_k)$-shuffles $\pi$ of $X(\pi_1, \ldots, \pi_k)(x)$. The summands of $(EZ \otimes EZ)(AW_{msimp}(x))$ are indexed over pairs $(\pi], [\pi)$ with $\pi] \in sh(i_1, \ldots, i_k)$ and $[\pi \in sh(a_1 - i_1, \ldots, a_k - i_k)$. The summand corresponding to a pair, up to sign, is

$$X(\pi], [\pi)(x)_{i_1, \ldots, i_k} \otimes X(\pi_{[1, \ldots, [\pi)(x)_{a_1 - i_1, \ldots, a_k - i_k}) =$$
but this is a summand, up to sign, of $AW_{simp}(EZ(x))$, and it is easy to see that there is a bijective correspondence between such summands by lemma \[6.3\]. By careful tracking signs of corresponding summands it turns out that they agree. This concludes the proof.

**Corollary 6.6.** The following diagram commutes

\[
\begin{array}{c}
N_*(X) \xrightarrow{EZ} N_*(X^D) \\
\downarrow AW_{simp} \quad \quad \quad \quad \quad \quad \quad \quad \downarrow AW_{simp} \\
N_*(X) \otimes N_*(X) \xrightarrow{EZ \otimes EZ} N_*(X^D) \otimes N_*(X^D)
\end{array}
\]

**Proof.** It is a direct consequence of theorem \[2.6\], corollary \[3.10\] and lemmas \[4.2\] and \[4.7\].

We can finally prove the compatibility theorem.

**Theorem 6.7.** The dual Eilenberg-Zilber map $EZ^*: C^*(X^D) \rightarrow C^*(X)$ and its restriction $EZ^*: N^*(X^D) \rightarrow N^*(X)$ are homomorphisms of differential graded algebras inducing in cohomology an isomorphism of graded algebras

\[ H^*(X^D) \cong H^*(X) \]

**Proof.** The fact that $EZ^*$ induces homomorphisms of differential graded algebras follows by dualizing the diagrams of proposition \[6.5\] and corollary \[6.6\]. The fact that they induce isomorphism in cohomology follows from theorem \[3.9\].
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