Stochastically generated turbulence with improved kinematic properties

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Abstract

We present a stochastic turbulence generator based on a vorticity formulation where the generated turbulent field implicitly fulfills the kinematic constraints of an incompressible flow. The generator allows direct access to the turbulent velocity and vorticity field. Enforcing additional constraints such as a divergence-free vorticity field and a specified differentiability of the flow field can also be implemented directly within this formulation. The resulting turbulent field contain improved kinematic properties and may be imported into numerical simulations without an excessive loss of energy.

Keywords: Turbulence generator, Stochastic simulations, Synthesized turbulence, Vorticity formulation

1. Introduction

A stochastic turbulent field is a randomly generated field that possesses the same statistical correlation properties as a corresponding real turbulent field but is generally not a solution to the governing equations of fluid flow i.e. the Navier-Stokes equation. Stochastic turbulence generators are used in fluid dynamics simulations to emulate the effect of turbulence in the oncoming flow. A common use is simulating a fluid-structure interaction with a turbulent oncoming flow in order to investigate the effect of high-frequency fluctuations on the aero-elastic coupling \cite{1,2,3,4}. Stochastic turbulence generators are also being used in large eddy simulations for generating non-deterministic boundary conditions for the resolved (i.e. simulated) turbulent scales \cite{5}.

Current stochastic turbulence generators \cite{6,2} are based on the method of Shinozuka \cite{7} for simulating a statistically homogeneous process of the velocity field. However creating the velocity field by a stochastic process does not generally comply with the kinematic constraints of a flow field. Hence the generated turbulent energy is distributed into the components of an unconstrained velocity field. Consequently a significant part of the kinetic energy may be lost in a numerical simulation when applying kinematic constraints on the unconstrained velocity field e.g. enforcing the divergence-free velocity field of an incompressible flow.

We show that by formulating the stochastic method in a vorticity formulation some of these constraints are implicitly fulfilled and additional constraints may be enforced by a re-projection method. This approach ensures that the kinetic energy is contained in components which comply to the kinematics of the flow. Furthermore, we discuss how to improve the numerical and physical properties of the generated field to ensure that it can be introduced into a numerical simulation without having an excessive loss of turbulent energy.

2. Methodology

A vector field $\mathbf{v}$ that has at least two continuous derivatives (i.e. $\mathbf{v} \in C^2$) can be viewed as consisting of two intrinsic components according to the Helmholtz decomposition:

$$\mathbf{v} = \nabla \times \psi - \nabla \phi \quad \text{with} \quad \nabla \cdot \psi = 0 \quad (1)$$
Here $\phi$ is a scalar potential and $\psi$ is a vector potential which is divergence-free in order to ensure the uniqueness of the decomposition. The two terms represents a divergence-free and a irrotational component of the velocity field, respectively. For a constant density fluid the conservation of mass imposes the constraint of a divergence-free velocity field $\nabla \cdot \mathbf{v} = 0$. Hence the irrotational component of the velocity field is nullified $\nabla \phi = 0$ and the velocity field can be formulated by the vorticity field alone. Using the definition of vorticity $\omega \equiv \nabla \times \mathbf{v}$ we thus obtain the relation:

$$
\omega = \nabla(\nabla \cdot \psi) - \nabla^2 \psi = -\nabla^2 \psi \quad \text{by which} \quad \nabla \times \omega = -\nabla^2 \mathbf{v}
$$

Hence the velocity field can be obtained from the vorticity field by solving a Poisson equation. We see that by generating a turbulent vorticity field, from which the velocity field is then calculated, the constraint imposed by mass conservation is implicitly fulfilled, and the turbulent energy is thus fully contained in a divergence-free velocity field. This is not the case when generating the velocity field directly and supplementary numerical schemes must then be used to enforce this constraint.

A correlated random vector field can be generated by convolving a random white noise vector field $\phi$ of unit variance with the desired covariance tensor function $H$:

$$
\omega(x) = \int H(x - y) \cdot \phi(y) \, dy \quad \text{and thus} \quad \hat{\omega}(k) = \hat{H}(k) \cdot \hat{\phi}(k)
$$

Here $\hat{\cdot}$ denotes the Fourier transform and $k$ is the angular wave-number. In this work the multi-variate formulation simply represents the components of the vorticity vector. However the multi-variate formulation may also be used to introduce a statistically inhomogeneous direction in the turbulence field c.f. \[8\]. Furthermore, for a turbulent field which is advected with a constant velocity $U$ (e.g. in the $x$-direction) we may use Taylor’s frozen turbulence assumption to transform a spatial dimension into a temporal one e.g. by $x = (U t, y, z)$.

In order to find an expression for the covariance tensor $H$ we start with the expression for the spectral power density tensor of the turbulent vorticity field $\hat{\Omega}$ combined with the spectral correlation properties of the components of the random vector field $\phi$:

$$
\hat{\Omega} = \hat{\omega} \hat{\omega}^* = \frac{\langle \hat{H} \cdot \hat{\phi} \rangle}{V} \frac{(\hat{H} \cdot \hat{\phi})^*}{V} \quad \text{with} \quad \hat{\phi} \hat{\phi}^* = I
$$

As $\Omega(k)$ is a symmetric matrix consisting of real, positive and even functions of $k$ we get:

$$
V \hat{\Omega} = \hat{H} \hat{H}^\dagger
$$

Here the right-hand-side is a matrix multiplication and the $\dagger$ denotes the conjugate transpose by which we see that the covariance tensor function $\hat{H}$ may be found by a Cholesky decomposition of $\Omega$. The turbulent vorticity field can thus be generated by Eq. \[3\] given a well-defined $\hat{\Omega}$.

We may relate the vorticity power density function to the kinetic energy of the velocity field by specifying the velocity field as an even function i.e. $\hat{\mathbf{v}}^\dagger(k) = \hat{\mathbf{v}}(k)$, by which the vorticity field effectively becomes an odd function i.e. $\hat{\omega}^\dagger(k) = -\hat{\omega}(k)$. This specification formally changes the periodic boundary condition of the velocity field to a symmetrical one. The spectral vorticity power density tensor may thus be obtained:

$$
\hat{\Omega}(k) = -\hat{\omega} \hat{\omega}^* = \frac{|\hat{\omega}|^2|k|^2 I - k k} V
$$

Here we have used the spectral definition of vorticity $\hat{\omega} = i k \times \hat{\mathbf{v}}$ and the incompressibility constraint $i k \cdot \hat{\mathbf{v}} = 0$. By further assuming an isotropic turbulence, the spectral vorticity power density tensor may be stated as:

$$
\hat{\Omega}(k) = \frac{E(|k|)}{4\pi \rho |k|^2} |k|^2 I - k k
$$

where the $E(|k|)$ is the kinetic energy spectrum (Eq. \[7\] is also given by \[8\]).
The resulting field that is generated by Eq. (3), is at this point a pure Monte Carlo method and thus powered by a random number generator. Consequently, the generated field does not possess any properties which allows the field to comply to a numerical simulation of differential equations, e.g. smoothness. Evidently this non-compliance will result in a large numerical dissipation and thus loss of kinetic energy when introducing the turbulent field in a numerical simulation. To amend this, we propose to perform a high order energy conserving smoothing of the turbulence field obtained by the approximate de-convolution method [10, 11, 12, 13]. Using a Gaussian filter we obtain and m-th order filter from:

$$\hat{\zeta}(k) = D_m \exp\left(-\frac{\sigma^2 k^2}{2}\right) \quad \text{with} \quad D_m = \sum_{n=0}^{m/2-1} \frac{(\sigma^2 k^2/2)^n}{n!}$$

where \(D_m\) is the m-th order Taylor approximation of the inverse Gaussian, \(k^2 = k \cdot k\) and \(\sigma\) is the smoothing radius which should not be smaller than the discretization length \(h\). The smoothed turbulent vorticity field may now be obtained by:

$$\hat{\omega}(k) = \hat{\zeta}(k) \hat{\phi}(k) \cdot \hat{H}(k)$$

Once smoothed, we apply an additional correction to the generated vorticity field in order to enforce the constraint of a divergence-free vorticity field. From the kinematic relations of Eq. (2) we may deduce that a vorticity field \(\omega^*\) where \(\nabla \cdot \omega^* \neq 0\) should be corrected accordingly to:

$$\omega = \omega^* + \nabla(\nabla \cdot \psi^*) \quad \text{and} \quad \nabla \cdot \omega^* = -\nabla^2(\nabla \cdot \psi^*)$$

The divergence of the uncorrected vector potential \(\psi^*\) is found by solving the latter equation which conveniently constitutes a Poisson equation and can thus be obtained using spectral differentiation. As seen in Eq. (3) we also obtain the velocity field by solving the Poisson equation in a similar way.

Not only does the vorticity formulation of the turbulence generator provides a field with improved kinematic properties it also allows for an a priori determination of the Kolmogorov length which is the smallest length scale allowed by the viscous dissipation. By dimensional analysis we reformulate the conventional estimation of the Kolmogorov length as:

$$\eta = \left(\frac{\nu^2 \epsilon}{\sigma^4}\right)^{1/4}$$

where \(\nu\) is the kinematic viscosity of the fluid, \(\epsilon = \omega \cdot \omega\) is the enstrophy density and \(\sigma\) its mean value which can be determined directly from the generated field. The reformulated Kolmogorov length should be larger than the discretization length i.e. \(\eta > h\) for a direct numerical simulation in order to avoid an excessive numerical dissipation. The Kolmogorov length should thus be used as the smoothing length \(\sigma\) in order to ensure that the generated kinetic energy is contained in length scales which is allowed by the viscous dissipation of the fluid. If scales below the Kolmogorov length are introduced into a numerical simulation the viscous terms of the governing equations will dissipate the energy of these scales within a few time steps.

3. Results

The kinetic energy spectrum and the enstrophy spectrum of a generated isotropic turbulence field are shown in Fig. 1. The turbulent vorticity field is generated through Eq. (7) using the von Kármán spectrum [14] for the turbulent kinetic energy \(E(k)\). The von Kármán spectrum is here determined by defining an integral length \(L\), and a scaling parameter \(K = \rho \text{trace}(\Sigma)/2\) which is the total turbulent kinetic energy:

$$E(k) = C \frac{L^4 k^4}{(1 + L^2 k^2)^{17/6}} \quad \text{with} \quad C = K \left(\int_0^{\infty} \frac{L^4 k^4}{(1 + L^2 k^2)^{17/6}} \, dk\right)^{-1}$$

Hence we have the two input parameters \(L\) and \(\Sigma\) representing the length scale of the largest coherent flow structures and the covariance tensor of the generated velocity field, respectively. For the presented example the turbulent field of a cubed domain of \(256 \times 256 \times 256\) cells is generated using \(L = 0.1\) and \(\Sigma = I\). The
re-sampled spectra of the turbulence energy and enstrophy are shown in Fig. 1. The generated turbulence spectrum is found to converge to the design spectrum when averaging multiple realizations of the turbulence field. Furthermore, the calculated divergence in the velocity and the vorticity field was observed to be at the order of machine precision.

4. Conclusion

A vorticity formulated stochastic turbulence generator was presented which improved the kinetic properties of the generated turbulent field compared to present methods. Additional measures, such as explicit high order smoothing of the flow field, was introduced to ensure that the generated field can be introduced into numerical simulations without an excessive loss of turbulence energy caused by numerical dissipation. The turbulence generator allows for a direct access to both the velocity and vorticity field without implementing supplementary schemes.

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