Adaptive Refinement for $hp$–Version Trefftz Discontinuous
Galerkin Methods for the Homogeneous Helmholtz Problem

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Abstract

In this article we develop an $hp$–adaptive refinement procedure for Trefftz discontinuous
Galerkin methods applied to the homogeneous Helmholtz problem. Our approach combines
not only mesh subdivision ($h$–refinement) and local basis enrichment ($p$–refinement), but also
incorporates local directional adaptivity, whereby the elementwise plane wave basis is aligned
with the dominant scattering direction. Numerical experiments based on employing an em-
pirical a posteriori error indicator clearly highlight the efficiency of the proposed approach for
various examples.

Keywords Homogeneous Helmholtz problem, Discontinuous Galerkin methods, Trefftz meth-
ods, Adaptivity, $hp$–finite element methods

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1 Introduction

Trefftz discontinuous Galerkin (TDG) methods are finite element schemes which employ discon-
tinuous test and trial functions whose restriction to each mesh element belongs to the kernel of
the differential operator to be discretized. For time-harmonic wave problems, Trefftz discretiza-
tion spaces are made of oscillating functions with the same frequency as that of the underlying
analytical solution. This results in improved approximation properties, as compared to standard
piecewise polynomial spaces. Moreover, based on Trefftz spaces, one can construct discontinu-
ous Galerkin methods which feature unconditional unique solvability, as well as coercivity of the
discrete bilinear forms in suitable (mesh-dependent) norms. We focus here on the case of the
Helmholtz problem and refer, e.g., to the survey [17] for a review of the construction, properties,
and relevant literature of Trefftz methods for its numerical approximation.

The purpose of this article is to develop an efficient $hp$–adaptive refinement algorithm for TDG
methods applied to the homogeneous Helmholtz problem; we will specifically consider the ultra-
weak variational formulation with plane wave basis functions [8]. Within the adaptive procedure,
elements will be marked for refinement based on employing an empirical a posteriori error indicator,
stimulated by the upper bounds derived in [21] for the $h$–version of the TDG method. For the
$h$–version of the plane wave discontinuous Galerkin method, incorporating Lagrange multipliers,
a similar error indicator has been presented in [2]. Once an element has been marked for refinement,
a decision must then be made regarding the type of refinement to be undertaken, i.e., whether
the element should be subdivided ($h$–refinement), or whether the local basis should be enriched
($p$–refinement). The choice of whether to $h$– or $p$–refine an element is typically based on the
observation that when the underlying solution is smooth, then $p$–refinement will be more efficient

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in terms of reducing the error, for a given increase in the number of degrees of freedom, than if the element is subdivided. On the other hand, if the solution is not smooth, then \( h \)-refinement should be employed. In general, \textit{a posteriori} error estimators only provide an estimate of the local elementwise error, but do not indicate which type of refinement should be employed. Within the existing literature a number of algorithms have been devised for determining the type of refinement (\( h \)- or \( p \)-) to be undertaken. For a comprehensive review of this subject, we refer to [24]–[26], and the references cited therein. In the present context, given the oscillatory nature of solutions to high-frequency scattering problems, the exploitation of \( hp \)-strategies based on local regularity estimation techniques is not generally applicable. Thereby, we consider an alternative approach based on estimating the predicted decay rate of the \textit{a posteriori} estimator, given the refinement history of each element; see, for example, [23]. For \textit{a posteriori} error estimation of conforming finite element approximations of the Helmholtz problem, we refer, e.g., to [3]–[4] and [9]; analogous bounds have been established for polynomial-based discontinuous Galerkin finite element methods in [27]–[30].

In addition to standard \( h \)- and \( hp \)-adaptivity, we also consider the issue of directional refinement of the underlying plane wave basis employed within our TDG scheme. In particular, we rotate the underlying elementwise plane wave basis in order that the first basis function is aligned with the local dominant propagation direction; strategies for determining the local dominant propagation direction have been proposed in [1]–[5], cf., also, [13], for example. Stimulated by the work undertaken on anisotropic mesh adaptation in [10]–[11], we propose an alternative approach based on studying the properties of the Hessian of the computed TDG solution. More precisely, the principal eigenvector of the Hessian of the solution indicates the dominant direction of wave propagation. However, since eigenvectors are only unique up to scalar multiples, the precise wave direction must be fixed, based on exploiting an impedance condition. In this way, we can locally orientate the elementwise plane wave basis to reduce the error in the underlying computed TDG solution in a simple and computationally cheap manner. When combined with \( hp \)-refinement, the resulting adaptive procedure is capable of generating highly optimized \( hp \)-refined Trefftz spaces. Indeed, the efficiency of the proposed strategy is illustrated for a number of test problems, where we compare the performance between an \( h \)- and \( hp \)-refinement algorithm, both with and without directional adaptivity.

The outline of this article is as follows: in Section 2 we introduce the model problem to be studied within this article, together with its TDG discretization. Then in Section 3 we develop an \( hp \)-refinement algorithm, based on employing both local mesh subdivision and local basis enrichment, together with directional adaptivity for the underlying Trefftz space. The performance of this procedure is studied in Section 4 through a series of two- and three-dimensional examples. Finally, in Section 5 we summarize the work undertaken within this article and highlight potential future directions of research.

## 2 Model problem and TDG discretization

In this section we state the model problem to be studied in this article, together with its TDG discretization; for further details, we refer to [17], for example.

### 2.1 Model problem

We study the homogeneous Helmholtz equation; to this end, we let \( \Omega \subset \mathbb{R}^d, d = 2, 3 \), be an open bounded, Lipschitz domain with boundary \( \partial \Omega \). Thereby, we seek \( u : \Omega \mapsto \mathbb{C} \) such that

\[
-\Delta u - k^2 u = 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} + ik d u = g_R \quad \text{on } \Gamma_R, \\
u = g_D \quad \text{on } \Gamma_D,
\]

(1)
where \( n \) denotes the unit outward normal vector on the boundary \( \partial \Omega \), and \( \Gamma_R \) and \( \Gamma_D \) are non-overlapping open subsets of \( \partial \Omega \), such that \( \partial \Omega = \Gamma_R \cup \Gamma_D \). Furthermore, \( i \) is the imaginary unit, \( \vartheta = \pm 1 \), \( g_R \in L^2(\Gamma_R) \), and we assume, for the moment, that the (real-valued) wavenumber \( k \) is constant in \( \Omega \).

### 2.2 Meshes and spaces

We partition \( \Omega \) into computational meshes \( \{T_h\}_{h>0} \) consisting of non-overlapping (curvilinear) polygons/polyhedra \( K \), which potentially include hanging nodes, such that \( \Omega = \bigcup_{K \in T_h} K \). Moreover, we assume that the family of subdivisions \( \{T_h\}_{h>0} \) is shape regular [7, pp. 61, 114, and 118]. For each element \( K \in T_h \), we write \( h_K \) to denote its diameter and \( n_K \) signifies the unit outward normal vector to \( K \) on \( \partial K \); we set \( h := \max_{K \in T_h} h_K \). Furthermore, we introduce the mesh skeleton \( F_h \), defined by \( F_h = \bigcup_{K \in T_h} \partial K \); we write \( F_h^I \) and \( F_h^B \) to denote the interior and boundary skeletons, respectively, defined by \( F_h^I = F_h \setminus \partial \Omega \) and \( F_h^B = \partial \Omega \). Implicitly, we assume that the finite element mesh \( T_h \) respects the decomposition of the boundary, in the sense that, given an element face \( f \subset \partial K, K \in T_h \), which lies on the boundary \( \partial \Omega \), i.e., \( f \subset \partial \Omega \), then \( f \) is entirely contained within either \( \Gamma_R \) or \( \Gamma_D \).

Let \( K \) and \( K' \) be two adjacent elements of \( T_h \), and \( x \) an arbitrary point on the interior face \( f \subset F_h^I \) given by \( f = (\partial K \cap \partial K')^\circ \). Furthermore, let \( v \) and \( w \) be scalar- and vector-valued functions, respectively, that are sufficiently smooth inside each element \( K, K' \). Then, the averages of \( v \) and \( w \) at \( x \in f \) are given by

\[
\mathbb{\{v\}} = \frac{1}{2}(v|_K + v|_{K'}) , \quad \mathbb{\{w\}} = \frac{1}{2}(w|_K + w|_{K'}) ,
\]

respectively. Similarly, the jumps of \( v \) and \( w \) at \( x \in f \) are given by

\[
[v] = v|_K n_K + v|_{K'} n_{K'} , \quad [w] = w|_K \cdot n_K + w|_{K'} \cdot n_{K'} ,
\]

respectively.

Given \( K \in T_h \) the local Trefftz space is defined by

\[
T(K) := \{ v \in H^1(K) : -\Delta v - k^2 v = 0 \} ;
\]

with this notation, we write

\[
T(T_h) := \{ v \in L^2(\Omega) : v|_K \in T(K), K \in T_h \} .
\]

Thereby, given a local space \( V_{pK}(K) \subset T(K) \), of finite dimension \( p_K \geq 1 \), the corresponding TDG finite element space is defined by

\[
V_p(T_h) := \{ v \in T(T_h) : v|_K \in V_{pK}(K), K \in T_h \} ,
\]

where \( p = \{p_K : K \in T_h \} \).

### 2.3 TDG discretization

Equipped with the TDG finite element space \( V_p(T_h) \) defined on the mesh partition \( T_h \) of \( \Omega \), the TDG approximation of (1) is given by: find \( u_{hp} \in V_p(T_h) \) such that

\[
\mathcal{A}_h(u_{hp}, v_{hp}) = \ell_h(v_{hp}) \quad (2)
\]
for all \( v_{hp} \in V_p(T_h) \), where
\[
A_h(u, v) = \int_{\Gamma_d} \left( \|u\| \|\nabla_h v\| - \beta (ik)^{-1} \left[ \nabla_h u \|\nabla_h v\| - \|\nabla_h u\| \cdot |v| \right] + \alpha ik u \cdot |v| \right) ds
\]
\[
+ \int_{\Gamma_R} \left( (1 - \delta)(u \nabla_h \bar{v} \cdot n + i k \theta u \bar{v}) - \delta ((ik\theta)^{-1}(\nabla_h u \cdot n)(\nabla_h \bar{v} \cdot n) + \nabla_h u \cdot n \bar{v}) \right) ds
\]
\[
+ \int_{\Gamma_D} (-\nabla_h u \cdot n \bar{v} + \alpha i ku \bar{v}) ds,
\]
\[
\ell_h(v) = \int_{\Gamma_R} g_R((1 - \delta)\bar{v} - \delta (ik\theta)^{-1}\nabla_h \bar{v} \cdot n) ds + \int_{\Gamma_D} g_D(\alpha iku - \nabla_h v \cdot n) ds,
\]
and \( \nabla_h \) denotes the broken gradient operator, defined elementwise. Here, \( \alpha > 0 \), \( \beta > 0 \) and \( 0 < \delta \leq \frac{1}{2} \) are given penalty parameters. We note that the selection of these penalty parameters has been studied in a number of different contexts within the literature; in particular, here we mention the ultra-weak variational formulation (UWVF), cf. [8], the DG-type scheme studied in [13], and [16] which considered their selection on locally refined meshes; cf. [17, Table 1]. For the purposes of this article we consider the UWVF, corresponding to the choice \( \alpha = \beta = \delta = \frac{1}{2} \).

### 2.4 Plane wave basis functions

Finally, in this section we outline the choice of the underlying discrete space \( V_{pK}(K) \), \( K \in T_h \). To this end, we select \( V_{pK}(K) \) to be a local space consisting of plane waves in \( pK \) different directions, all with the same wavenumber \( k \). We note that, under suitable assumptions on \( K \) and the choice of plane wave directions, \( V_{pK}(K) \) approximates smooth Trefftz functions with the same order of convergence as polynomials of degree \( q_K \), where
\[
p_K = \begin{cases} 2q_K + 1, & d = 2, \\ q_K + 1, & d = 3; \end{cases}
\]
(3)

see [26]. Thereby, \( q_K \) is referred to as the effective polynomial degree of the discrete Trefftz space; we set \( q = \{q_K : K \in T_h\} \). More precisely, we write
\[
V_{pK}(K) := \left\{ v \in T(K) : v(x) = \sum_{\ell=0}^{pK-1} \alpha_{\ell} e^{ikd_{K,\ell} \cdot (x-x_K)}, \alpha_{\ell} \in \mathbb{C} \right\},
\]
(4)

where \( x_K \) is the center of mass of element \( K \) and \( d_{K,\ell}, \ell = 0, \ldots, pK - 1 \), are \( pK \) evenly distributed unit direction vectors (with respect to the unit ball). For \( d = 2 \) we can simply define
\[
d_{K,\ell} = (\cos(2\pi\ell/p_K), \sin(2\pi\ell/p_K))^T, \quad \ell = 0, \ldots, pK - 1;
\]
(5)
for \( d = 3 \) we employ the directions determined by the extremal (maximum determinant) points on \( S^2 \), cf. [28] [29].

### 3 Adaptive mesh refinement

In this section we develop an automatic adaptive refinement algorithm which is capable of not only marking elements for refinement, but also determining the type of refinement to be undertaken. In particular, here we consider both \( h \)- and \( p \)-refinement, whereby the local element is subdivided, or the number of elementwise plane wave directions is enriched, respectively, as well as directional refinement which seeks to rotate the local plane wave basis in order to align it with the principal scattering direction.
3.1 A posteriori error indicator

In the absence of rigorous a posteriori error bounds for the numerical approximation of $[1]$ by the TDG scheme $[2]$, which are sharp with respect to both the local mesh size $h_K$ and the number of local plane waves $p_K$ employed on each element $K \in \mathcal{T}_h$, we employ an empirical error estimator stimulated by the work undertaken in $[21]$ in the $h$-version setting. To this end, we first introduce the dual problem: find $z \in H^1(\Omega)$, such that

$$
-\Delta z - k^2 z = u - uh_p \quad \text{in} \quad \Omega ,
$$

$$
\frac{\partial z}{\partial n} + ikz = 0 \quad \text{on} \quad \Gamma_R,
$$

$$
z = 0 \quad \text{on} \quad \Gamma_D.
$$

Noting that $z \in H^{3/\beta + \epsilon}(\Omega), 0 < s \leq 1/2, \text{cf. } [16]$, we recall the following (second) a posteriori error bound from $[21]$.

**Theorem 3.1.** Assume that the mesh $\mathcal{T}_h$ is shape-regular, locally quasi-uniform, in the sense that, for two elements $K$ and $K'$ which share a face $f \subset \mathcal{F}_h$, there is a constant $\tau$, independent of $h$, such that

$$
\tau^{-1} \leq h_K/h_{K'} \leq \tau
$$

for all choices of $K$ and $K'$, and that $\mathcal{T}_h$ is quasi-uniform in the vicinity of $\Gamma_R$, i.e., for all $K \in \mathcal{T}_h$ which lie on the boundary $\Gamma_R$, i.e., so that $\partial K \cap \Gamma_R \neq \emptyset$, there exists $\tau_R$ such that

$$
h/\tau_R \leq \tau_R.
$$

Then, for $g_D \equiv 0$ and fixed $p_K, K \in \mathcal{T}_h$, the following a posteriori bound holds:

$$
\|u - uh_p\|_{L^2(\Omega)} \leq \mathcal{E}(uh_p, h) \equiv C \left( \sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2},
$$

where

$$
\eta_K^2 = \left\| \alpha^{1/2} h_K^s [uh_p] \right\|_{L^2(\partial K \setminus \partial \Omega)}^2 + k^{-2} \left\| \beta^{1/2} h_K^s \nabla uh_p \right\|_{L^2(\partial K \setminus \partial \Omega)}^2 + k^{-2} \left\| \delta^{1/2} h_K (g_R - \nabla uh_p \cdot n_K + ikuh_p) \right\|_{L^2(\partial K \cap \Gamma_R)}^2 + \left\| \alpha^{1/2} h_K^s uh_p \right\|_{L^2(\partial K \cap \Gamma_D)}^2,
$$

(6)

where $C$ is a positive constant, which is independent of $h$.

We stress that the a posteriori error bound stated in Theorem $3.1$ depends on the regularity index $s$; thereby, a priori knowledge of $s$ is required in order to yield a fully computable bound. Moreover, the dependence of $C$ on $p$, or equivalently $q$, is unclear; indeed, to the best of our knowledge, an $h^p$-version generalization of Theorem $3.1$ is not currently available within the literature. Thereby, we propose the following empirical error estimator, where for simplicity of notation we also denote it by $\mathcal{E}$, for the $h^p$-version TDG method:

$$
\mathcal{E}(uh, h, p) = \left( \sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2},
$$

(7)

where

$$
\eta_K^2 = \left\| \alpha^{1/2} h_K^s [uh_p] \right\|_{L^2(\partial K \setminus \partial \Omega)}^2 + \left\| \beta^{1/2} h_K^s \nabla uh_p \right\|_{L^2(\partial K \setminus \partial \Omega)}^2 + \left\| \delta^{1/2} h_K (g_R - \nabla uh_p \cdot n_K + ikuh_p) \right\|_{L^2(\partial K \cap \Gamma_R)}^2 + \left\| \alpha^{1/2} h_K^s uh_p \right\|_{L^2(\partial K \cap \Gamma_D)}^2,
$$

(8)

5
In this section, we discuss the design of a practical algorithm for determining the direction vectors \( d_{K,\ell}, \ell = 0, \ldots, p_K - 1 \), used to define the plane wave basis within each element \( K \) in the computational mesh \( T_h \). The key observation is that, many wave propagation problems typically exhibit a dominant direction of propagation of the underlying wave within each element in \( T_h \). Thereby, by aligning the plane wave basis in an appropriate fashion, we expect to attain a significant reduction of the error in the computed TDG solution. Indeed, in the simple case when the analytical solution is a plane wave, then if the direction for one of the plane wave basis functions is selected such that it is aligned with this plane wave direction, then the TDG method will exactly recover the analytical solution, subject to rounding errors.

The essential idea here is to simply rotate the element basis according to the predicted elementwise dominant direction. For simplicity of presentation, let us consider the two-dimensional case, i.e., \( d = 2 \); we note that \( d = 3 \) follows in an analogous manner, cf. Remarks 3.2 and 3.3 below. In two-dimensions, the standard plane wave directions are generally selected to be evenly spaced, with the first direction \( d_{K,0} = (1,0) \top \) always pointing along the \( x \)-axis, cf. [5] (in the three-dimensional setting, the first direction vector typically points along the \( z \)-axis). Alternatively, assuming that a dominant elementwise direction, denoted by \( \tilde{d}_K \), can be determined within each \( K \in T_h \), then the direction vectors for the plane wave basis functions in \( K \) are chosen such that the first plane wave direction is aligned with \( \tilde{d}_K \), i.e., (5) is replaced by

\[
\begin{align*}
    d_{K,\ell} &= (\cos(2\pi\ell/p_K + \theta_K), \sin(2\pi\ell/p_K + \theta_K)) \top, \quad \ell = 0, \ldots, p_K - 1,
\end{align*}
\]

where \( \theta_K \) is the angle between \( \tilde{d}_K \) and the \( x \)-axis.

Clearly, in general, the dominant elementwise direction \( \tilde{d}_K, K \in T_h \), cannot be determined a priori, but instead must be numerically estimated as part of the solution process. In this regard, a number of algorithms have been proposed within the literature; here we mention the ray-tracing approach developed in [5, 6], though this includes terms involving integrals over the elements within the underlying TDG formulation. In [1], the optimal angle of rotation was numerically estimated based on adding an extra unknown into the problem; however, this leads to a system of nonlinear equations to be computed. Finally, [13] uses an approximation of

\[
    \frac{\nabla e(x_0)}{i k e(x_0)},
\]

at a given point \( x_0 \in K, K \in T_h \), where \( e \) denotes the error.

Stimulated by the work undertaken in [10, 11], cf. also [12, 13], on the design of anisotropically refined computational meshes, in this section we compute an estimate of \( \tilde{d}_K, K \in T_h \), based on the properties of the Hessian of the TDG solution \( u_{hp} \). Indeed, we note that the principal eigenvector, i.e., the eigenvector corresponding to the largest eigenvalue in absolute value, of the Hessian of a given function indicates the direction of most rapid variation, and thereby, in our context, the dominant direction of wave propagation. With this in mind, writing \( H(\varphi, x_0) \) to denote the Hessian matrix of a given function \( \varphi \), evaluated at the point \( x_0 \in \mathbb{R}^d \), in Algorithm 1 we outline the steps involved in computing a potential dominant plane wave direction \( \tilde{d}_K \) for a given element \( K \in T_h \). Table I summarizes how this potential first plane wave direction \( \tilde{d}_K \) is selected; for the numerical experiments presented in Section 4, we set \( \Lambda = 2 \). We note that in the case when no primary propagation direction is determined, then we leave the first plane wave direction unchanged.
Algorithm 1 Computation of the potential first plane wave direction $\hat{d}_K$ for element $K$.

1: Input: the TDG solution $\mathbf{u}_{hp}$ of the discrete problem (2) and the parameter $\Lambda > 1$.
2: Writing $x_K$ to denote the centroid of $K$, $K \in T_h$, evaluate the eigenpairs $(\lambda_1, v_1), (\lambda_2, v_2)$ of $\mathbf{H} \left( \text{Re}(\mathbf{u}_{hp}|_K), x_K \right)$, and $(\mu_1, w_1), (\mu_2, w_2)$ of $\mathbf{H} \left( \text{Im}(\mathbf{u}_{hp}|_K), x_K \right)$, such that $|\lambda_1| \geq |\lambda_2|$ and $|\mu_1| \geq |\mu_2|$.
3: if $|\lambda_1| \geq \Lambda |\lambda_2|$ then
   4: if $|\mu_1| \geq \Lambda |\mu_2|$ then
      5: if $|\lambda_1| \geq \Lambda |\mu_1|$ then
         6: $\hat{d}_K \leftarrow v_1$
      7: else if $|\mu_1| \geq \Lambda |\lambda_1|$ then
         8: $\hat{d}_K \leftarrow w_1$
      9: else
         10: $\hat{d}_K \leftarrow \frac{v_1 + w_1}{\|v_1 + w_1\|}$
   11: end if
   12: else
      13: if $|\lambda_1| \geq \Lambda |\mu_1|$ then
         14: $\hat{d}_K \leftarrow v_1$
      15: else
         16: No primary propagation direction
      17: end if
   18: end if
   19: else
      20: if $|\mu_1| \geq \Lambda |\mu_2|$ then
         21: if $|\mu_1| \geq \Lambda |\lambda_1|$ then
            22: $\hat{d}_K \leftarrow w_1$
         23: else
            24: No primary propagation direction
         25: end if
      26: else
         27: No primary propagation direction
      28: end if
   29: end if

| $|\lambda_1| \geq C|\lambda_2|$ | $|\mu_1| \geq C|\mu_2|$ | $|\lambda_1| \geq C|\mu_1|$ | $|\mu_1| \geq C|\lambda_1|$ | First Plane Wave $\hat{d}_K$ |
|------------------------|------------------------|------------------------|------------------------|------------------------|
| $\checkmark$ | $\checkmark$ | $\checkmark$ | $\times$ | $v_1$ |
| $\checkmark$ | $\checkmark$ | $\times$ | $\times$ | $w_1$ |
| $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\frac{v_1 + w_1}{\|v_1 + w_1\|}$ |
| $\checkmark$ | $\times$ | $\times$ | $\times$ | $v_1$ |
| $\times$ | $\checkmark$ | $\times$ | $\checkmark$ | $w_1$ |
| $\times$ | $\checkmark$ | $\times$ | $\times$ | $\times$ |
| $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |

Table 1: Summary of selection of first plane wave direction $\hat{d}_K$ using Algorithm 1.

Remark 3.2. We note that in the case when $d = 3$, $\mathbf{H} \left( \text{Re}(\mathbf{u}_{hp}|_K), x_K \right)$ and $\mathbf{H} \left( \text{Im}(\mathbf{u}_{hp}|_K), x_K \right)$ each have a third eigenpair, $(\lambda_3, v_3)$ and $(\mu_3, w_3)$, respectively. However, if the eigenpairs are sorted such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$ and $|\mu_1| \geq |\mu_2| \geq |\mu_3|$, the third eigenpairs never represent a dominant direction, and thereby Algorithm 1 can be used to identify $\hat{d}_K$, $K \in T_h$, without modification.

Noting that eigenvectors are only unique up to scalar multiples, the vector $\hat{d}_K$, $K \in T_h$, evaluated according to Algorithm 1 may be pointing in precisely the opposite direction to the
Algorithm 2: Evaluation of the first plane wave direction $\mathbf{d}_K$ for element $K$.

1: Input: the TDG solution $u_{hp}$ of the discrete problem 2, the parameter $0 \leq \delta \to 0$, and $\mathbf{d}_K$ computed by Algorithm 1.

2: The first plane wave direction $\mathbf{d}_K$ on element $K$, $K \in \mathcal{T}_h$, is given by

$$
\mathbf{d}_K = \begin{cases} 
-\mathbf{d}_K, & \text{if } \text{Re} \left( \nabla u_{hp}(x_K + \mathbf{d}_K) \cdot \mathbf{d}_K + ik u_{hp}(x_K + \mathbf{d}_K) \right) < e^{ik\delta}, \\
\mathbf{d}_K, & \text{if } \text{Re} \left( \nabla u_{hp}(x_K + \mathbf{d}_K) \cdot \mathbf{d}_K + ik u_{hp}(x_K + \mathbf{d}_K) \right) \geq e^{ik\delta}.
\end{cases}
$$

primary wave propagation direction. Thereby, to ensure that $\mathbf{d}_K$, $K \in \mathcal{T}_h$, is correctly oriented, we study the impedance trace on the boundary of a ball $B_\delta(x_K)$ of radius $\delta$, centered at $x_K$, of both the numerical solution and a plane wave with (the desired) propagation direction $\mathbf{d}_K$. As we let $\delta \to 0$, we expect that the numerical solution should be closely approximated by the plane wave in the primary propagation direction.

More precisely, given $K \in \mathcal{T}_h$, the impedance trace of the plane wave

$$
\tilde{u}_K(x) = e^{ik\mathbf{d}_K \cdot (x - x_K)}
$$

on $\partial B_\delta(x_K)$ is given by

$$
(\nabla \tilde{u}_K(x) \cdot n_{B_\delta})|_{\partial B_\delta(x_K)} = (ik (\mathbf{d}_K \cdot n_{B_\delta} + 1) e^{ik\mathbf{d}_K \cdot (x - x_K)})|_{\partial B_\delta(x_K)}, \quad (10)
$$

where $n_{B_\delta}$ denotes the unit outward normal vector on $\partial B_\delta(x_K)$. Setting $x = x_K + \mathbf{d}_K$ in (10) and noting that, at this point of evaluation, $n_{B_\delta} = \hat{\mathbf{d}}_K$, we deduce that

$$
\nabla \tilde{u}_K(x_K + \mathbf{d}_K) \cdot n_{B_\delta} + ik \tilde{u}_K(x_K + \mathbf{d}_K) = \begin{cases} 
2e^{ik\delta}, & \text{if } \hat{\mathbf{d}}_K = \mathbf{d}_K, \\
0, & \text{if } \hat{\mathbf{d}}_K = -\mathbf{d}_K.
\end{cases}
$$

Thereby, the (potential) dominate direction of propagation $\hat{\mathbf{d}}_K$, $K \in \mathcal{T}_h$, predicted according to Algorithm 1, this direction will then be selected as the first plane wave direction on element $K$, $K \in \mathcal{T}_h$. For simplicity, throughout this article we set $\delta = 0$.

Remark 3.3. In the three-dimensional setting, once the selection of the primary wave propagation direction $\mathbf{d}_K$ has been computed on the basis of Algorithms 1 & 2, we then select the remaining wave directions, $d_{K,\ell}$, $\ell = 1, \ldots, p_K - 1$, by applying a transformation matrix $T \in \mathbb{R}^{3 \times 3}$ to the original ‘reference’ directions $\tilde{d}_{K,\ell}$, $\ell = 1, \ldots, p_K - 1$, respectively, where $\tilde{d}_{K,0}$ points along the $z$-axis, cf. above. Thereby,

$$
d_{K,\ell} = T \tilde{d}_{K,\ell},
$$

$\ell = 1, \ldots, p_K - 1$, where $T$ is selected such that

$$
\mathbf{d}_K = d_{K,0} = T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv T \tilde{d}_{K,0}.
$$

We note that the selection of $T$ is not unique; writing $\mathbf{d}_K = (d_x, d_y, d_z)^T$, we define $T$ to be the identity matrix if $d_x = d_y = 0$; otherwise, we set

$$
T = \begin{pmatrix}
\frac{d_x}{\sqrt{d_x^2 + d_z^2}} & \frac{d_y}{\sqrt{d_y^2 + d_z^2}} & d_z \\
\frac{d_x}{\sqrt{d_x^2 + d_y^2}} & \frac{d_y}{\sqrt{d_y^2 + d_z^2}} & d_x \\
-\frac{d_x}{\sqrt{d_x^2 + d_y^2}} & -\frac{d_y}{\sqrt{d_y^2 + d_z^2}} & 0
\end{pmatrix}.
$$
Here set to zero; thereby, this ensures that $p$ is achieved, then of convergence based on the last type of refinement employed. If the expected rate of convergence is selected based on checking if the local error estimate has decayed according to the expected rate the refinement history of the current element, cf. [23]. More precisely, following [23] refinements are $a$ posteriori the to be unreliable due to the oscillatory nature of the computed numerical solution. 

In this section we discuss the design of an automatic algorithm for generating sequences of $hp$–refinement strategies is to employ local mesh subdivision ($h$–refinement) in regions where the solution is not smooth, while local enrichment of the finite element space ($p$–refinement) is undertaken elsewhere. Given that such regularity information is generally unknown $a$ priori, several strategies have been developed to $a$ posteriori estimate the local smoothness of the analytical solution, based on its numerical approximation; cf. [19], for example. However, in the context of TDG schemes for the numerical approximation of high-frequency time-harmonic wave problems, the extraction of such regularity information is expected to be unreliable due to the oscillatory nature of the computed numerical solution. 

Thereby, as an alternative to directly estimating local smoothness of the solution, we employ the $a$ posteriori error indicators $\eta_{K,i}$ to select the type of refinement to be undertaken on the basis of the refinement history of the current element, cf. [23]. More precisely, following [23] refinements are selected based on checking if the local error estimate has decayed according to the expected rate of convergence based on the last type of refinement employed. If the expected rate of convergence is achieved, then $p$–refinement is performed; otherwise, $h$–refinement is undertaken. The variant of [23] Algorithm 4.4 we employ here is summarized in Algorithm 3. Here, we note that $\gamma_h$, $\gamma_p$, and $\gamma_n$ are control parameters; for the purposes of this article, we select $\gamma_h = 4$, $\gamma_p = 0.4$, and $\gamma_n = 1$. Furthermore, the number of child elements, $N$, cf. step 10. in Algorithm 3 is dependent on the type of subdivision, i.e., isotropic/anisotropic, undertaken, as well as the element shape; for isotropic refinement of tensor-product elements, we have that $N = 2^d$.

Remark 3.4. We note that in [23] the initial values of the predicted error indicator $\eta_{K,0}^{pred}$, $K \in \mathcal{T}_{h,0}$, are set to zero; thereby, this ensures that $h$–refinement is undertaken the first time an element

### Algorithm 3 $hp$–Adaptive refinement algorithm.

1: Input the control parameters $\gamma_h$, $\gamma_p$, and $\gamma_n$.
2: Choose a coarse initial mesh $\mathcal{T}_{h,0}$ of $\Omega$ and a corresponding low-order starting (effective) polynomial degree vector $q_0$, together with the total dimension vector $p_0$ defined as in (3).
3: Set the initial predicted error indicator $\eta_{K,0}^{pred} = \infty$ for all $K \in \mathcal{T}_{h,0}$.
4: for $i = 0, 1, \ldots$, until sufficiently many iterations have been performed. do
5: Solve (2) for $u_{hp} \in V_{p_0}(T_{h,i})$.
6: Compute the $a$ posteriori error indicators $\eta_{K,i} \equiv \eta_K$, $K \in \mathcal{T}_{h,i}$, and mark elements for refinement based on their relative magnitude.
7: for $K \in \mathcal{T}_{h,i}$ do
8: if $K$ is marked for refinement then
9: if $\eta_{K,i} > \eta_{K,i}^{pred}$ then
10: Perform $h$–refinement: Subdivide $K$ into $N$ children $K_s$, $s = 1, \ldots, N$, and set
11: $(\eta_{K_{s,i+1}}^{pred})^2 \leftarrow \gamma_h \left( \frac{1}{2} \right) q_K \eta_{K,i}^{pred}$, $1 \leq s \leq N$.
12: else
13: Perform $p$–refinement: $q_K \leftarrow q_K + 1$
14: $(\eta_{K,i+1}^{pred})^2 \leftarrow \eta_K \eta_{K,i}^{pred}$
15: end if
16: else
17: $(\eta_{K,i+1}^{pred})^2 \leftarrow \eta_n (\eta_{K,i}^{pred})^2$
18: end if
19: end for
20: Construct the new mesh $\mathcal{T}_{h,i+1}$ and corresponding Trefftz space $V_{p_{i+1}}(T_{h,i+1})$.
21: end for

### 3.3 $hp$–Adaptive mesh refinement

In this section we discuss the design of an automatic algorithm for generating sequences of $hp$–adaptively refined TDG finite element spaces in an efficient manner. This topic has been extensively studied within the finite element element literature in the case when the local element spaces consist of polynomial functions; for a comprehensive review, we refer to [24, 25]. In general, the key underlying principle of most $hp$–refinement strategies is to employ local mesh subdivision ($h$–refinement) in regions where the solution is not smooth, while local enrichment of the finite element space ($p$–refinement) is undertaken elsewhere. Given that such regularity information is generally unknown $a$ priori, several strategies have been developed to $a$ posteriori estimate the local smoothness of the analytical solution, based on its numerical approximation; cf. [19], for example. However, in the context of TDG schemes for the numerical approximation of high-frequency time-harmonic wave problems, the extraction of such regularity information is expected to be unreliable due to the oscillatory nature of the computed numerical solution.

Thereby, as an alternative to directly estimating local smoothness of the solution, we employ the $a$ posteriori error indicators $\eta_{K,i}$ to select the type of refinement to be undertaken on the basis of the refinement history of the current element, cf. [23]. More precisely, following [23] refinements are selected based on checking if the local error estimate has decayed according to the expected rate of convergence based on the last type of refinement employed. If the expected rate of convergence is achieved, then $p$–refinement is performed; otherwise, $h$–refinement is undertaken. The variant of [23] Algorithm 4.4 we employ here is summarized in Algorithm 3. Here, we note that $\gamma_h$, $\gamma_p$, and $\gamma_n$ are control parameters; for the purposes of this article, we select $\gamma_h = 4$, $\gamma_p = 0.4$, and $\gamma_n = 1$. Furthermore, the number of child elements, $N$, cf. step 10. in Algorithm 3 is dependent on the type of subdivision, i.e., isotropic/anisotropic, undertaken, as well as the element shape; for isotropic refinement of tensor-product elements, we have that $N = 2^d$. 

Remark 3.4. We note that in [23] the initial values of the predicted error indicator $\eta_{K,0}^{pred}$, $K \in \mathcal{T}_{h,0}$, are set to zero; thereby, this ensures that $h$–refinement is undertaken the first time an element
Table 2: Plane Wave Refinement: Comparison of the relative $L^2$-error for uniform $p$–refinement (without direction adaptivity), and $p$–refinement with direction adaptivity (Algorithm 2).

is refined. In contrast, in Algorithm 3 we set $\eta_{K,0}^\text{pred} = \infty$ for all $K \in T_{h,0}$ which instead leads to $p$–enrichment being undertaken as the first refinement of a given element, since the TDG method for the numerical approximation of the Helmholtz equation is intrinsically a high-order method.

Remark 3.5. Plane wave directional adaptivity can be performed at different stages within Algorithm 3; for example, the following options are available:

- undertake directional adaptivity only on elements marked for $p$–refinement,
- undertake directional adaptivity on all elements marked for refinement, with $h$–refinement performed after plane wave direction adaptivity, or
- undertake directional adaptivity on every element $K \in T_h$, even if the element $K$ has not been marked for refinement.

In Section 4 we shall numerically investigate each of these approaches in order to assess their relative computational performance in terms of error reduction.

Remark 3.6. As a final remark, we note that within Algorithm 3 we employ the fixed fraction refinement strategy to select elements for refinement, cf. step 6; throughout this article we set the refinement fraction equal to 25%.

4 Numerical experiments

In this section, we present a series of numerical experiments to highlight the practical performance of the $hp$–refinement algorithm, with directional adaptivity, proposed in Algorithm 3. Throughout this section we shall compare the performance of the proposed $hp$–adaptive refinement strategy with the corresponding algorithm based on exploiting only local mesh subdivision, i.e., $h$–refinement. The numerical experiments presented within this section have been undertaken using the AptoFEM software package [18].

4.1 Plane wave direction adaptivity

In this first example, we study the effect of adjusting the plane wave directions while employing a fixed computational mesh with uniform $p$–refinement. To this end, we consider problem (1) with $\Omega = (0,1)^2$, $\Gamma_R = \partial \Omega$, and $\Gamma_D \equiv \emptyset$; furthermore, the Robin boundary condition $g_R$ is selected such that the analytical solution $u$ of (1) is given by

$$u(x,y) = H_0^{(1)} \left( k \sqrt{(x + \frac{1}{4})^2 + y^2} \right),$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind of order 0. Throughout this section, we set $k = 20$; note that for this problem the analytical solution $u$ is smooth in $\Omega$. 

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Here, the underlying computational mesh consists of 16 uniform square elements; on each element we initially select the effective polynomial degree $q = 2$, i.e., $p = 5$. In Table 2 we compare the computed relative $L^2$-error based on employing uniform $p$-refinement of the underlying TDG space $V_p(T_h)$ in the two cases when the standard TDG scheme is employed, i.e., when the local plane wave directions are kept fixed, and when plane wave directional adaptivity is utilised, based on exploiting Algorithm 2 (direction adaptivity). We note that, since uniform $p$-refinement is employed in both cases, then at each step of the refinement, both schemes possess the same number of degrees of freedom. At each step of the refinement algorithm, we observe that the exploitation of directional adaptivity leads to roughly 50% reduction in the relative $L^2$-error when compared to the corresponding quantity computed for the standard TDG method (without direction adaptivity). We note, however, that in the case when $q = 3$, the relative $L^2$-error is only reduced by a small amount when directional adaptivity is employed; this is due to the fact that the local plane wave directions are computed based on the numerical solution evaluated with $q = 2$, which is numerically too inaccurate to reliably predict the correct local direction of wave propagation. Furthermore, we also note that, as the number of plane waves increases, the improvement in the relative $L^2$-error decreases; this is caused by the fact that, as the number of plane waves increases for the standard TDG scheme, one of the directions will get closer to the actual dominant direction.

In Figure 1 we plot, for each element, the initial plane wave directions and the plane wave directions computed after 1, 2, and 3 uniform $p$-refinements employing directional adaptivity. We emphasize the first plane wave direction with a larger arrow, i.e., the dominant wave direction as determined by Algorithm 2. Moreover, we overlay the directions on top of a contour plot showing the real part of the analytical solution \[11\]. From Figure 1 we can clearly observe that the directional adaptivity algorithm is able to accurately determine the dominant wave direction after a few refinements.

Finally, in this section we consider performing more than one directional adaptivity step after
each uniform $p$-refinement. To this end, in Table 3 we compare the relative $L^2$-error for the initial directions, as well as after one and two steps of directional adaptivity have been performed, for the case when $q = 3, \ldots, 8$. Here, we observe that additional application of the direction adaptivity algorithm does not lead to a significant reduction in the relative $L^2$-error; indeed, most of the reduction, when compared to the standard TDG scheme, without directional adaptivity, is attained after one step of Algorithm 2. Moreover, we emphasise that this first step may be undertaken in a very computationally cheap manner.

4.2 Efficiency of the a posteriori error indicator

The selection of the exponents of $h_K$ and $q_K$ in the weights present in \([8]\), together with the independence on the wavenumber $k$, have been determined by numerical experimentation. To this end, we considered the example presented in the previous section, cf. \([11]\), whereby the numerical approximation is computed on a series of uniform computational meshes, with uniform effective polynomial degrees $q$, for a range of wave numbers $k$. In each case, we computed the effectivity index of each constituent term arising in $E(u_h,h,p)$, whereby the dependency on $h_K$, $q_K$, and $k$ was eliminated; note that with the removal of $h_K$, $q_K$, and $k$, the effectivity index of each term is computed by dividing by $\|u - u_{hp}\|_{L^2(\Omega)}$. More precisely, effectivity indices were computed for $q = 3, \ldots, 8$ and $k = 20, 30, 40, 50$, based on starting from a uniform $4 \times 4$ mesh consisting of square elements. On the basis of these results, the dependence of each term on $h_K$, $q_K$, and $k$ was established. The final effectivity indices for the correctly scaled empirical a posteriori error indicator $E(u_h,h,p)$, i.e., $E(u_h,h,p)/\|u - u_{hp}\|_{L^2(\Omega)}$ are presented in Figure 2. Here, we observe that the effectivity indices have roughly the same values for all the selected values of $h$, $q$, and $k$; however, at higher wave numbers, pre-asymptotic behaviour leads to some increase in the effectivity indices as the mesh is refined, due to the fact that the mesh size is too large for the wavelength. We note that this behaviour is more noticeable in the case when $k = 3$.

Finally, we compute the effectivity index for each individual term arising in the definition of the error indicator $E(u_h,h,p)$, cf. \([8]\). More precisely, we define

$$E_{[u_{hp}]} := \left( \frac{\sum_{K \in \mathcal{T}_h} \| \alpha^{1/2} h_K^{1/2} q_F^{1/2} \|_{L^2(\partial K \setminus \partial \Omega)} \| u - u_{hp}\|_{L^2(\Omega)}^2 }{ \| u - u_{hp}\|_{L^2(\Omega)}^2 } \right)^{1/2}$$

$$E_{[\nabla u_{hp}]} := \left( \frac{\sum_{K \in \mathcal{T}_h} \| \beta^{1/2} h_K^{1/2} q_K^{1/2} \|_{L^2(\partial K \setminus \partial \Omega)} \| u - u_{hp}\|_{L^2(\Omega)}^2 }{ \| u - u_{hp}\|_{L^2(\Omega)}^2 } \right)^{1/2}$$

$$E_{R} := \left( \frac{\sum_{K \in \mathcal{T}_h} \| \delta^{1/2} h_K^{1/2} q_K^{1/2} (g_{K} - \nabla u_{hp} \cdot \mathbf{n}_F + ik u_{hp}) \|_{L^2(\partial K \cap \Gamma_n)}^2 }{ \| u - u_{hp}\|_{L^2(\Omega)}^2 } \right)^{1/2};$$

the results for the case when $k = 20, 30, 40, 50$ are depicted in Figure 3. Here, we observe that each
Figure 2: Effectivities for $h$–refinement with fixed effective polynomial degree of the smooth analytical Hankel solution with different wavenumbers.

individual effectivity index is roughly constant for all the selected values of $h$, $q$, and $k$, except within the pre-asymptotic region. For this smooth problem, we clearly observe that the dominant part of the error indicator involves the jump in the gradient of the numerical solution.

Remark 4.1. We note there that we have only computed the weightings for the interior and Robin faces. In the case of Dirichlet boundary conditions we assume that the weighting scales the same as the term involving $[u_{hp}]$.

4.3 $hp$–Adaptive refinement

In this section we consider computational performance of the proposed $hp$–adaptive refinement algorithm, with directional adaptivity, for a range of test problems in both two- and three-dimensions. To this end, employ the fixed fraction refinement strategy to mark elements for refinement; throughout this section, we set the refinement fraction equal to 25% of the elements with the largest contribution to the error bound. Furthermore, we allow the meshes $T_h$ to be ‘1-irregular’, i.e., each face of any element $K \in T_h$ contains at most one hanging node (which, for simplicity, we assume to be at the barycenter of the corresponding face) and each edge of each face contains at most one hanging node (yet again assumed to be at the barycenter of the edge). We also only allow the effective polynomial degree $q_K$ to vary by one between neighbouring elements.

For each test problem, we compare the performance of employing $hp$–adaptive refinement with $h$–adaptivity. In the latter case, we consider a standard $h$–adaptive algorithm, i.e., adaptive mesh refinement without directional adaptivity, as well as an $h$–adaptive strategy which incorporates directional adaptivity; here, we shall consider the two cases when directional adaptivity is either undertaken only on the elements marked for refinement, as well as the case when it is performed on all elements in the computational mesh. In the $hp$-setting, similar comparisons will be made, in addition to studying the case when directional adaptivity is only performed on elements marked
Figure 3: Effectivities of individual components of the error indicators for $h$–refinement with fixed effective polynomial degree of the smooth analytical Hankel solution with different wavenumbers.
for \( p \)-refinement.

We note that when \( hp \)-refinement is exploited we often reach a point where the \( L^2 \)-norm of the error and \( a \) posteriori error bound stagnates, in the sense that both quantities no longer tend to zero, and indeed may start to oscillate, as further refinement is undertaken. This is caused by the fact that as the relative magnitude of \( q_K \), with respect to \( h_K k \), becomes large, the local plane wave bases are very ill-conditioned. In this situation, we simply stop the numerical experiments and discard further results; however, possible improvements based on ensuring \( q_K \) is well behaved with respect to \( h_K k \) could be implemented; cf. [8, 20, 22] for details.

4.3.1 Example 1 — Smooth solution (Hankel function)

In this section, we again consider the problem outlined in Section 4.1. Furthermore, we select the initial mesh to consist of \( 8 \times 8 \) uniform square elements and set \( q_K = 3 \) on each \( K \in \mathcal{T}_h \). Firstly, in Figures 4a and 4e we compare the relative error in the \( L^2 \)-norm to the number of degrees of freedom in the TDG space \( V_p(\mathcal{T}_h) \), when \( h \)-refinement is employed, with the wavenumbers \( k = 20 \) and \( k = 50 \), respectively. In each case, we consider the performance of the underlying adaptive algorithm when both the standard TDG scheme (without direction adaptivity) is employed, as well as the corresponding method with directional adaptivity: in this latter setting, we consider the cases when either directional adaptivity is undertaken on only the elements marked for refinement, as well as when it is exploited on every element in the computational mesh \( \mathcal{T}_h \). Analogous results are presented in Figures 4c and 4g in the \( hp \)-setting, respectively; here, we compare standard \( hp \)-refinement, with \( hp \)-adaptivity incorporating directional adaptivity. In the latter case, different directional adaptivity strategies are considered: firstly, directional adaptivity is performed only on elements marked for \( p \)-refinement; secondly, directional adaptivity is undertaken on all elements marked for refinement; finally, directional adaptivity is applied to every element in \( \mathcal{T}_h \). In the \( hp \)-setting we observe exponential convergence of the error as the finite element space is adaptively enriched: indeed, on a linear-log scale, the convergence lines are roughly straight. Thereby, it is clear that the exploitation of the proposed \( hp \)-refinement algorithm, with directional adaptivity, leads to a significant reduction in the \( L^2 \)-norm of error, for a given number of degrees of freedom, when compared to the same quantity computed with \( h \)-refinement alone; cf. Figure 5.

In both the \( h \)- and \( hp \)-refinement cases, we generally observe that the error is decreased when directional refinement is employed. Moreover, it is evident in the \( hp \)-setting that selecting more elements for directional refinement generally leads to a smaller error, for a given number of degrees of freedom; this is particularly noticeable in the case when \( k = 50 \). In Figures 4b, 4d, 4f, and 4h we plot the effectivity indices for each of the above refinement strategies for the case when \( k = 20, 50 \); here, we observe that they remain roughly constant during adaptive \( h \)-/\( hp \)-mesh refinement, and are roughly the same for the two different wavenumbers, with the notable exception of the pre-asymptotic region for \( k = 50 \).

Finally, in Figures 4i and 4j, we show the meshes after 8 \( h \)- and \( hp \)-refinements, with directional adaptivity employed on all elements, for both \( k = 20 \) and \( k = 50 \); here, the \( hp \)-meshes show the effective polynomial degree \( q_K \) for each element. Given the smoothness of the analytical solution on \( \Omega \), we observe that the resulting computational meshes are almost uniform; indeed, in the \( hp \)-setting almost uniform \( p \)-refinement has been undertaken.

4.3.2 Example 2 — Singular solution

In this second example, we consider problem (1) posed on the L-shaped domain \( \Omega = (-1, 1)^2 \setminus (0, 1) \times (-1, 1), \Gamma_R = \partial \Omega, \) and \( \Gamma_D = \emptyset \), with Robin boundary condition \( g_R \) selected so that the analytical solution is given, in polar coordinates \( (r, \varphi) \), by

\[
 u(r, \theta) = J_{2/3}(kr) \sin(2\theta/a);
\]

we note that the gradient of \( u \) has a singularity at the origin.

As in the previous example, we again compare the performance of the \( h \)- and \( hp \)-adaptive refinement algorithms, both in the standard setting, as well as when directional adaptivity is
Figure 4: Example 1: (a) $L^2$-error and (b) Effectivity index for $h$-refinement with wavenumber $k = 20$; (c) $L^2$-error and (d) Effectivity index for $hp$-refinement with $k = 20$; (e) $L^2$-error and (f) Effectivity index for $h$-refinement with $k = 50$; (g) $L^2$-error and (h) Effectivity index for $hp$-refinement with $k = 50$. 
Figure 5: Example 1: Comparison of relative $L^2$-error for $h$- and $hp$-refinement, with direction adaptivity on all elements, for wavenumbers (a) $k = 20$ and (b) $k = 50$.

Figure 6: Example 1: Meshes after 8 (a) $h$- and (b) $hp$-refinements for wavenumber $k = 20$; meshes after 8 (c) $h$- and (d) $hp$-refinements for wavenumber $k = 50$. 
Figure 7: Example 2: (a) $L^2$-error and (b) Effectivity index for $h$–refinement with wavenumber $k = 20$; (c) $L^2$-error and (d) Effectivity index for $hp$–refinement with $k = 20$; (e) $L^2$-error and (f) Effectivity index for $h$–refinement with $k = 50$; (g) $L^2$-error and (h) Effectivity index for $hp$–refinement with $k = 50$. 

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employed; here, we again consider the analogous directional refinement strategies employed in Section 4.3.1. To this end, in Figures 7a and 7e we compare the relative error in the $L^2$-norm with the number of degrees of freedom in the TDG space $V_p(T_h)$ when $h$–refinement is employed for $k = 20$ and $k = 50$, respectively; the respective convergence plots in the $hp$–setting are given in Figures 7a and 7e. Here, we observe that although exploiting $hp$–refinement leads to exponential convergence of the relative $L^2$-norm of the error as $V_p(T_h)$ is enriched, in both the $h$– and $hp$–settings, we observe that the magnitude of the error, computed both with and without directional refinement, is roughly identical; i.e., directional refinement does not lead to any reduction in the computed TDG solution when either $h$–/ $hp$–refinement is employed. We note that, for this particular problem, this behaviour is expected, since the error in the computed TDG solution is dominated by the influence of the singularity at the origin, rather than local wave propagation.

In Figures 7b, 7d, 7f, and 7h we plot the effectivity indices when both $h$– and $hp$–refinement is employed for the case when $k = 20, 50$. In all cases, we observe that the effectivity indices are roughly constant for this singular problem, though when $h$–refinement is employed, on highly refined meshes, we see a slight drop in the computed effectivity indices. Finally, in Figures 8a and 8d we show the meshes after $8$ $h$– and $hp$–refinements, with direction adaptivity employed on all elements, for both $k = 20$ and $k = 50$. As we would expect, in both the $h$– and $hp$–settings, mesh subdivision is concentrated in the vicinity of the singularity located at the origin; away from this region, the $h$–refinement algorithm employs almost uniform mesh subdivision, while the $hp$–refinement strategy employs the necessary combination of local mesh refinement and local polynomial enrichment, as required, to reduce the error in the computed TDG solution.

4.3.3 Example 3 — Transmission/internal reflection

We now consider the case of transmission and internal reflection of a plane wave across a fluid-fluid interface in the domain $\Omega = (-1, 1)^2$, $\Gamma_R \equiv \emptyset$, and $\Gamma_D = \partial \Omega$, with two different refractive indices,
Figure 9: Example 3: Analytical solutions (real part) when (a) $\theta_i = 29^\circ$ resulting in internal reflection, and (b) $\theta_i = 69^\circ$ resulting in refraction.

cf. [21, Section 6.3]. The interface between the two regions is located at $y = 0$; in this setting the wavenumber $k$ is given by the piecewise constant function

$$k(x, y) = \begin{cases} k_1 := \omega n_1 & \text{if } y \leq 0, \\ k_2 := \omega n_2 & \text{if } y > 0, \end{cases}$$

where, we select $\omega = 11$, $n_1 = 2$, and $n_2 = 1$. Throughout this section we impose an appropriate inhomogeneous Dirichlet boundary condition, so that the analytical solution $u$ to (1) is given, for a constant $0 \leq \theta_i \leq \pi/2$, by

$$u(x, y) = \begin{cases} T e^{i(K_1 x + K_2 y)} & \text{if } y > 0, \\ e^{ik_1(x \cos(\theta_i) + y \sin(\theta_i))} + Re^{ik_1(x \cos(\theta_i) - y \sin(\theta_i))} & \text{if } y < 0, \end{cases}$$

where $K_1 = k_1 \cos(\theta_i)$, $K_2 = \sqrt{k_2^2 - k_1^2 \sin^2(\theta_i)}$, 

$$R = -\frac{K_2 - k_1 \sin(\theta_i)}{K_2 + k_1 \sin(\theta_i)},$$

and $T = 1 + R$. We note that there exists a critical angle $\theta_{\text{crit}}$, such that when $\theta_i > \theta_{\text{crit}}$ the wave is refracted, while $\theta_i < \theta_{\text{crit}}$ results in internal reflection, cf. [21, Section 6.3]. As in [21] we perform numerical experiments for both internal reflection ($\theta_i = 29^\circ$) and refraction ($\theta_i = 69^\circ$). To highlight the reflection and refraction behaviour, in Figures 9a and 9b we show the analytical solution when $\theta_i = 29^\circ$ and $\theta_i = 69^\circ$, respectively.

To account for the jump in the wavenumber $k$, the value of $k$ present in the integrals along the interface $y = 0$ in the TDG scheme [2] is replaced by $\omega$. We select the initial mesh to consist of $8 \times 8$ uniform square elements, so that the interface between the two materials is captured by the mesh; thereby, the wavenumber is constant in every element, and hence the TDG space [4] and error indicators [8] can be easily modified to treat this example by setting the wavenumber for each element equal to the wavenumber of the material within which the element is contained. Firstly, we consider the case when there is an internal reflection, i.e., when $\theta_i = 29^\circ$; to this end, in Figures 10a and 10c we plot the relative error in the $L^2$-norm against the number of degrees of freedom in $V_p(T_h)$ using both $h$- and $hp$-refinement, respectively. As for the previous numerical experiments, here we again observe exponential convergence of the error when $hp$-refinement is employed. Furthermore, in both the $h$- and $hp$-version settings, we observe that employing directional adaptivity does not improve the magnitude of the error; indeed, in the $hp$-version setting, initially the standard refinement approach is superior, though as $V_p(T_h)$ is enriched, we again observe the benefits of employing directional adaptivity. This behaviour is perhaps expected, since for the internal reflection case, no waves are present above the $y = 0$ line and moreover it
Figure 10: Example 3: (a) $L^2$-error and (b) Effectivity index for $h$–refinement with reflection ($\theta_i = 29^\circ$); (c) $L^2$-error and (d) Effectivity index for $hp$–refinement with reflection ($\theta_i = 29^\circ$); (e) $L^2$-error and (f) Effectivity index for $h$–refinement with refraction ($\theta_i = 69^\circ$); (g) $L^2$-error and (h) Effectivity index for $hp$–refinement with refraction ($\theta_i = 69^\circ$).
Figure 11: Example 3: Meshes after 7 (a) $h$– and (b) $hp$–refinements for reflection ($\theta_i = 29^\circ$); meshes after 7 (c) $h$– and (d) $hp$–refinements for refraction ($\theta_i = 69^\circ$).

does not possess a dominant wave propagation direction below the $y = 0$ line due to the reflected waves, cf. Figure 9a. In Figures 10b and 10d, we plot the effectivity indices for both refinement strategies, respectively; here we observe that, apart from an initial pre-asymptotic region, the effectivity indices are roughly constant.

The corresponding convergence plots for the refraction case, i.e., when $\theta_i = 69^\circ$, are presented in Figures 10e and 10g when both $h$– and $hp$–refinement are employed, respectively; in the latter setting, we again observe exponential convergence of the computed relative $L^2$-norm of the error. Moreover, in contrast to the case when there is an internal reflection, here we observe the computational benefits of employing directional adaptivity, in the sense that this typically leads to a reduction in the error, for a given fixed number of degrees of freedom, when compared to the standard refinement strategy; this is particularly evidenced in the $hp$–setting. Indeed, in this case there is a dominant propagation direction throughout the domain, cf. Figure 9b. Figures 10f and 10h show the effectivity indices computed using both $h$– and $hp$–refinement, respectively; analogous behaviour is observed as for the internal reflection case, i.e., the effectivity indices become roughly constant, after an initial pre-asymptotic region.

Finally, in Figures 11a & 11b we show the meshes after 7 $h$– and $hp$–adaptive mesh refinements have been performed, respectively, in the case of an internal reflection, i.e., $\theta_i = 29^\circ$. Here, the $h$–refinement strategy concentrates most of the elements in the $y < 0$ region; although, there is some refinement above $y = 0$ to resolve the exponentially decaying solutions present there. Additional mesh smoothing has also been undertaken here to ensure that there is only one hanging node per face, cf. [21]. The $hp$–refinement algorithm also performs some $h$–refinement below the $y = 0$ line, though this region is largely $p$–refined; however, most of the refinement occurs around the $y = 0$ line to resolve the exponentially decaying solutions. Some $p$–refinement occurs in the rest of the $y > 0$ region, which is caused by enforcing the condition that the effective polynomial degree may only vary by one between neighboring elements. In the refraction case, i.e., $\theta_i = 69^\circ$, cf.
Figures 11c & 11d, we note a sharp boundary at the $y = 0$, with more refinement undertaken in the $y < 0$ region than the region $y > 0$.

4.3.4 Example 4 — 3D smooth solution (plane wave)

In this final example, we consider problem (1) posed on the domain $\Omega = (0,1)^3$, $\Gamma_R = \partial \Omega$, and $\Gamma_D \equiv \emptyset$, with Robin boundary condition $g_R$ selected so that the analytical solution $u$ to (1) is given by

$$u(x) = e^{ik \cdot d \cdot x},$$

where $d_j = 1/\sqrt{3}$ for $j = 1, 2, 3$.

In Figures 12a and 12e we present the performance of the proposed directional adaptivity algorithm employing $h$-refinement with wavenumbers $k = 20$ and $k = 50$, respectively; the analogous results for $hp$-refinement are given in Figures 12c and 12g, respectively. As in the two-dimensional setting, we observe that selecting more elements for directional adaptivity at each step of the proposed refinement strategy, leads to a greater reduction in the relative $L^2$-norm of the error, for a fixed number of degrees of freedom, when compared to the standard case when directional adaptivity is not employed. Of course, given the simple nature of the analytical solution for this problem, we clearly expect directional adaptivity to be advantageous. In the case when the wavenumber $k = 50$ we note that both $h$- and $hp$-refinement strategies are essentially in the pre-asymptotic region; however, performing directional adaptivity ensures that the method leaves this pre-asymptotic region after only a few mesh refinements. Finally, in Figures 12b, 12d, 12f, and 12h we plot the effectivity indices of both the $h$- and $hp$-refinement algorithms for the case when $k = 20, 50$. We note, especially in the $hp$-refinement case, that the effectivity indices are roughly constant but do slightly rise after the pre-asymptotic region.

5 Concluding remarks

In this article we have developed an automatic $hp$-adaptive refinement algorithm for the TDG approximation of the homogeneous Helmholtz equation. In addition to employing both local mesh subdivision and local basis enrichment, we also locally rotate the underlying plane wave basis in such a manner so that the first basis function is aligned with the dominant wave direction. The choice to $h$- or $p$-refine an element is based on a prediction of how much reduction we expect to observe in the elementwise error indicator, when a particular refinement is performed. The alignment of the local basis with the dominant wave direction is undertaken on the basis of an eigenvalue analysis of the Hessian of the numerical solution, together with a correction computed from an impedance condition. The computational efficiency of the proposed adaptive strategy has been studied through a series of numerical examples; indeed, the application of $hp$-refinement, with directional adaptivity, leads to a significant reduction in the computed error compared to standard refinement strategies. We also note that performing directional adaptivity on all elements generally leads to a greater reduction in the error than the corresponding case when only elements marked for refinement are directionally adapted; clearly, this error reduction is attained while keeping the number of degrees of freedom in the underlying TDG space fixed. Future work will be devoted to the derivation of robust $hp$-version a posteriori error bounds, as well as the application to problems of engineering interest.

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Figure 12: Example 4: (a) $L^2$-error and (b) Effectivity index for $h$–refinement with wavenumber $k = 20$; (c) $L^2$-error and (d) Effectivity index for $hp$–refinement with $k = 20$; (e) $L^2$-error and (f) Effectivity index for $h$–refinement with $k = 50$; (g) $L^2$-error and (h) Effectivity index for $hp$–refinement with $k = 50$. 
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