Cluster effective field theory and nuclear reactions

Shung-Ichi Ando

School of Mechanical and ICT convergence engineering, Sunmoon University, Asan, Chungnam 31460, Republic of Korea

Abstract. An effective field theory (EFT) for a nuclear reaction at low energies is studied. The astrophysical S-factor of radiative α capture on 12C at the Gamow-peak energy, $T_G = 0.3$ MeV, is a fundamental quantity in nuclear-astrophysics, and we construct an EFT for the reaction. To fix parameters appearing in the effective Lagrangian, the EFT is applied to the study for three reactions: elastic α-12C scattering at low energies, $E_1$ transition of radiative α capture on 12C, and β delayed α emission from 16N. We report an estimate of the $S_{E1}$-factor of the reaction through the $E_1$ transition at $T_G$ by employing the EFT. We also discuss applications of EFTs to nuclear reactions at low energies.

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1 Introduction

In the late 1970s, a phenomenological Lagrangian method is suggested by Weinberg [1] as an alternative of current algebra to calculate a hadronic matrix element at low energies. Skillful techniques are required for calculations of the current algebra to derive a hadronic reaction amplitude, e.g., the electron pion production on a nucleon, considering CVC and PCAC and satisfying the results from the low energy theorem and the relations among vertex functions required by Ward-Takahashi identities [2,3]. Those results, including the radiative and non-radiative vector and axial-vector matrix elements of nucleon, nonetheless, are straightforwardly calculated from the chiral Lagrangian, which embodies the chiral SU(2) × SU(2) symmetry and the explicit symmetry breaking patterns, in heavy-baryon chiral perturbation theory [4,5,6].

The idea of the method has been developed in various ways, and the framework of those methods are now known as effective field theories (EFTs). (For a review of EFTs, see, e.g., Refs. [7,8,9,10,11].) In the present work, we discuss a construction of an EFT for nuclear reactions at low energies based on our recent works [12,13,14,15,16].

When one studies a reaction by means of EFT, one may expect the characteristics for an EFT listed below: (1) It is a model independent approach. (2) One needs to introduce a momentum scale to separate relevant degrees of freedom at low energies from irrelevant degrees of freedom at high energies. (3) The theory provides us a perturbative expansion scheme around a specified theoretical limit; counting rules in powers of $Q/A_H$, where $Q$ denotes a typical momentum scale of a reaction and $A_H$ does a high momentum scale, will be available. By using the counting rules, one can expand the amplitude order by order, and, up to a given order, one has a finite number of diagrams to calculate. (4) Coefficients appearing in an effective Lagrangian may not be constrained by its mother theory but can be fixed by using experimental data.

One might regard the construction of an EFT for a nuclear reaction as a formidable task because a nucleus is a many-body nucleon system and its structure is not still easily described from first principles. At low energies, e.g., a long wavelength limit for an external probe, however, an amplitude constructed from an effective Lagrangian, which is represented in terms of relevant low energy degrees of freedom and embodies symmetry requirements, exhibits a specific expression in the low energy limit and may describe well the reaction through the external probe. This might be a consequence of the low energy theorem [17] for nuclear reactions. As discussed above, to construct an EFT for a nuclear reaction is possible; the important two conditions are (2) a clear separation scale and (4) an availability of experimental data for a reaction. For application in nuclear-astrophysics, nonetheless, one encounters an additional task. Because experimental data are available at relatively high energies due to the Coulomb barrier, it is necessary to work out an extrapolation of a reaction rate to a low energy.

In the following sections, we discuss and evaluate the $S_{E1}$ factor of radiative α capture on 12C at stellar energy through the $E_1$ transition by constructing an EFT [16]. Our approach is rather new compared to popular methods for nuclear-astrophysics studies, such as R-matrix or K-matrix approach [18] and potential models. Many studies of the $E1$ transition of radiative α capture on 12C have actually been carried out by using the conventional methods, in which a crucial observation has been made: The importance of indirect measurement of a reaction, β delayed α emission from 16N, was pointed out [19,20]. An
interference between a subthreshold state and a resonant state (also including so called background levels) for \( l = 1 \) channel is sensitive to a secondary peak at lower energy side of the \( \alpha \) energy spectrum from \( \beta \)-delayed \( \alpha \) emission from \( {}^{16}\text{N} \). That is an essential input to deduce the \( S_{E1} \) factor when one employs the \( R \)-matrix approach. Meanwhile, this feature appears quite differently in our approach: the subthreshold and resonant states for the \( l = 1 \) channel are represented within a single dressed \( ^{16}\text{O} \) propagator while the secondary peak of the \( \alpha \) energy spectrum from \( \beta \)-delayed \( \alpha \) emission from \( ^{16}\text{N} \) is described by an interference between the amplitudes from a pole diagram and a non-pole one. Because one might argue about the applicability of the present approach to studies of nuclear reactions (e.g., because of no interference between the subthreshold and resonant states for the \( l = 1 \) channel), the related reactions to estimate the \( S_{E1} \)-factor should be studied simultaneously using the same formalism. In addition, because the \( S_{E1} \)-factor has recently been estimated by using mainly a single method, the \( R \)-matrix approach, the model dependence of the method is worth questioning. An examination employed by another method would be called for.

In the present work, we review a series of the calculations for the \( E1 \) transition of radiative \( \alpha \) capture on \( ^{12}\text{C} \) by constructing an EFT, in which three reactions, elastic \( ^{12}\text{C} \) scattering \([12,13,14,15]\), \( S_{E1} \) factor of the radiative \( \alpha \) capture on \( ^{12}\text{C} \) \([16]\), and \( \beta \) delayed \( \alpha \) emission from \( ^{16}\text{N} \), are discussed. The construction of the reaction amplitudes for those three sectors, 1) nuclear elastic scattering, 2) an electromagnetic probe, and 3) an electroweak probe for nuclear reactions, is systematically carried out by introducing external vector (or minimally coupled photon) and axial-vector fields, which preserve symmetry requirement for constructing nuclear reaction amplitudes. In addition, a part of the reaction amplitudes, a dressed \( ^{16}\text{O} \) propagator, is shared by those three reactions, which can easily be identified by drawing Feynman diagrams. Once parameters for the dressed \( ^{16}\text{O} \) propagator are fitted to the elastic scattering data, it can be used for making nuclear reaction amplitudes for the radiative \( \alpha \) capture on \( ^{12}\text{C} \) and the \( \beta \) delayed \( \alpha \) emission from \( ^{16}\text{N} \). Additional coupling constants appearing in the reaction amplitudes are fixed by using experimental data for the corresponding reactions, and the \( S_{E1} \) factor is extrapolated to stellar energies. Reviewing those calculations, as an example of application of EFTs to nuclear reactions, one may see the reliability of the present approach.

This paper is organized as follows. In Sec. 2, a short overview of EFTs for the present work is presented. In Sec. 3 a construction of an EFT for the radiative \( \alpha \) capture on \( ^{12}\text{C} \) at stellar energy is discussed; counting rules of the reaction are mentioned, and an effective Lagrangian is displayed. In Sec. 4 an application of the EFT to the study of the elastic \( ^{12}\text{C} \) scattering at low energies is discussed, and in Sec. 5 the EFT is applied to the study of the \( E1 \) transition of radiative \( \alpha \) capture on \( ^{12}\text{C} \). In Sec. 6 an application of the EFT to the \( \beta \) delayed \( \alpha \) emission from \( ^{16}\text{N} \) is explored. In Sec. 7 results and discussion of the present work are summarized. In Appendix, the propagators and the vertex functions for the elastic \( ^{12}\text{C} \) scattering are displayed.

## 2 EFTs for nuclear reactions at low energies

EFTs are now a popular method for the studies of hadron and nuclear physics. In this section, we review the progress of EFTs as regards to their application to nuclear reactions at low energies.

The most representative example of an EFT is chiral perturbation theory (\( \chi \)PT), a low energy effective field theory of QCD. \([12,21,22,23]\) Hadrons are composite particles consisting of quarks and gluons, which are described by \( SU(3) \) color gauge theory. The quarks and gluons, however, areconfined in the hadrons and never appear as free particles. Meanwhile, the QCD Lagrangian approximately has a global \( SU(3)_R \times SU(3)_L \) flavor symmetry because of the light \( uds \) quark masses. The symmetry is spontaneously broken down to \( SU(3)_V \) involving eight massless Goldstone bosons in the ground state of QCD. Because of the nature of the Goldstone bosons, its interactions vanish in the zero momentum and zero light quark mass limits, and one can expand a reaction amplitude perturbatively in powers of the number of derivative and/or light meson mass factor around the vanishing interaction limit. One may notice that the picture of a hadron is completely altered between QCD and \( \chi \)PT; a hadronic system, for example, a proton is described by QCD as a many body system consisting of strongly interacting quarks and gluons and by \( \chi \)PT as a heavy core surrounded by a cloud of the massless Goldstone bosons in the chiral limit.

Weinberg suggested the first application of \( \chi \)PT to nuclear physics. \([24,25]\). Because of an enhancement effect from two nucleon propagation, which can alter the counting rules, one may construct a nuclear potential according to the chiral order counting rules from two-nucleon irreducible diagrams in the time ordered perturbation theory. To obtain a reaction amplitude, one solves the Lippmann-Schwinger equation with the chiral potential. This approach matches to the traditional view of the nuclear potential; the long range part of the nuclear potential consists of the one-pion exchange, the intermediate range does of the two-pion exchange, and the short range part is parameterized by models and has a hard repulsion core. This picture can be reproduced order by order in the Weinberg’s counting scheme; at leading order (LO) the nuclear potential consists of the two nucleon contact interactions and the one-pion exchange contribution, at next-to leading order (NLO) one includes the two pion exchange contributions and contact interactions with two derivatives or two meson mass factors, and so on. However, because of a singular interaction due to the tensor force, modification of the Weinberg’s counting rules is necessary in order to make the scattering amplitudes cutoff-independent.

Instead of a perturbative expansion of the nuclear potential, Kaplan, Savage and Wise(KSW) suggested to expand the two-nucleon scattering amplitudes. \([27,28]\). An elusive point to construct a perturbation theory for the
two-nucleon systems is how to treat unnaturally small scales in the s-wave states, the scattering length for $^1S_0$ channel, $a_{np} \approx -23.7$ fm, and the deuteron binding energy for $^3S_1$ channel, $B_0 \approx 2.22$ MeV, compared to the typical scale of $\chi$PT, the pion mass, $m_\pi \approx 140$ MeV. To take account of those small scales in the s-wave two-nucleon scattering amplitudes, the LO two-nucleon contact interactions are resummed up to infinite order while the one-pion exchange is perturbatively included. This scheme is known as the KSW counting scheme. For studies of low energy reactions, the theory is easily simplified as a pionless EFT in which the pions are regarded as irrelevant degrees of freedom at high energy and integrated out of the theory. The pionless EFT is subsequently applied to the study of three nucleon systems. Bedaque, Hammer and van Kolck studied the triton system by using the pionless theory. They found a cyclic singularity, the so-called limit cycle, in the triton channel when scaling a momentum cutoff introduced in the coupled integral equations. Along with the limit cycle, a universal feature emerges in the three-body system, known as the Efimov effect: an infinite number of three-body bound states, whose binding energies appear as a geometrical series, are accumulated at the threshold. The Efimov states in the triton system appear in a theoretical limit, the so-called unitary limit, where the s-wave scattering lengths become infinite and the two-body binding energy vanish in the two-body propagators. Thus, one can make a perturbative expansion around the unitary limit, which appears in the inverse of the two-body propagators. It actually coincides with the well-known effective range expansion. It is conjectured that this vanishing point appears in QCD at a slightly large ulphonpion mass, $m_\pi \approx 200$ MeV, (see, e.g., Figs. 11 and 12 in Ref. 32). Such an infrared point as a function of the pion mass for the triton system is studied by Braaten and Hammer. To renormalize the cyclic singularity, one needs to promote the three-body contact interaction at LO. For the study of the triton system, one can fix the coupling constant of the three-body contact interaction by using the triton binding energy. In addition, one can choose a triton wavefunction as a LO constituent of an amplitude and make a perturbative expansion around it.

Along with various theoretical developments of EFTs (a part of which we have briefly discussed above), EFTs are also employed for phenomenological studies at low energies, for which error estimates are important. An error estimate for a reaction can be controlled by using the perturbative expansion scheme of EFTs. EFTs have been employed for the phenomenological studies of, for example, neutron $\beta$-decay, solar neutrino reactions on the deuterons, radiative neutron capture on a proton at BBN energies, proton-proton fusion in the Sun. In the following sections, we discuss an application of an EFT to nuclear reactions for heavier nuclei at low energies.

3 EFT for radiative $\alpha$ capture on $^{12}$C at $T_G$

In this section, we discuss a construction of an EFT for the study of radiative $\alpha$ capture on $^{12}$C at $T_G = 0.3$ MeV. We first briefly review previous studies and then construct an EFT for this reaction. In the following subsections, we analyze the typical and high energy-momentum scales for the radiative $\alpha$ capture reaction at $T_G$, we also discuss which are the relevant and irrelevant physical degrees of freedom, and from those we derive the power counting rules for our EFT description. We also write down the effective Lagrangian for three reactions: elastic $\alpha$-$^{12}$C scattering, $E1$ transition of the radiative $\alpha$ capture on $^{12}$C, and $\beta$ delayed $\alpha$ emission from $^{16}$N.

3.1 Introduction to the radiative $\alpha$ capture on $^{12}$C

The radiative $\alpha$ capture on $^{12}$C, $^{12}$C$(\alpha, \gamma)^{16}$O, is one of the fundamental reactions in nuclear astrophysics, which determines the ratio $^{12}$C/$^{16}$O produced in helium burning. The reaction rate, equivalently the astrophysical S-factor, of the process at the Gamow peak energy, $T_G = 0.3$ MeV however, cannot experimentally be determined due to the Coulomb barrier. A theoretical model is necessary to be employed in order to extrapolate the cross section down to $T_G$ by fitting model parameters to available experimental data measured at a few MeV or large. During the last half century, a number of experimental and theoretical studies for the process have been carried out. For reviews, see, e.g., Refs. 19, 20, 61, and references therein.

In constructing a model for the radiative capture process, one needs to take into account the excited states of $^{16}$O, particularly, two excited bound states for $l_{\pi \rightarrow \gamma} = 1^+$ and $2^+$ just below the $\alpha$-$^{12}$C breakup threshold at $T \approx -0.045$ and $-0.24$ MeV respectively, as well as $1^+$ and $2^+$ resonant (second excited) states at $T = 2.42$ and 2.68 MeV, respectively. Thus, the capture reaction to the ground state of $^{16}$O at $T_G$ is expected to be $E1$ and $E2$ transitions dominant due to the threshold $1^+$ and $2^+$. While the resonant $1^-$ and $2^+$ states play a dominant role in the available experimental data at low

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1 There are other approaches, e.g., which treat the one-pion-exchange piece non-perturbatively while the two-pion-exchange is treated perturbatively. See, e.g., Ref. 29.

2 The Efimov system in the pionless EFT, including the Coulomb interaction, is studied in Ref. 32.

3 A dibaryon field, which has baryon number 2, is introduced by Kaplan 36, and it is useful to implement the effective range expansion in a theory without pions 37,38 and with perturbative pions 39,40.

4 Recently, a study for nuclear few-body systems around the unitary limit was reported by König et al. 11.

5 See the footnote 6.

6 The kinetic energy $T$ denotes that of the $\alpha$-$^{12}$C system in the center of mass frame.
energies, typically $1 \leq T \leq 3$ MeV. A minor contribution to the $S$-factor appears from so-called cascade transitions in which the initial $\alpha$ and $^{12}$C form an excited state of $^{16}$O emitting a photon, and it subsequently decays to the ground state of $^{16}$O. Experimental data pertaining to processes for nuclear astrophysics are compiled, known as NACRE-II compilation [65], in which the $S$-factor of the $^{12}$C($\alpha,\gamma$)$^{16}$O reaction is estimated employing a potential model, and reported uncertainty of the process is less than 20%.

Theoretical frameworks employed for the study are categorized mainly into two: the cluster models using generalized coordinate method [66] or potential model [67] and the phenomenological models using the parameterization of Breit-Wigner, $R$-matrix [15], or $K$-matrix [68]. A recent trend of the study is to rely on intensive numerical analysis, in which a larger amount of the experimental data relevant to the study are accumulated, such as those from $^{12}$C($\alpha,\gamma$)$^{16}$O, radiative proton capture on $^{15}$N to the ground state of $^{16}$O, $^{15}$N($p,\gamma$)$^{16}$O, $\beta$ delayed emission from $^{16}$N, $^{16}$N($\beta\alpha$)$^{12}$C, elastic $\alpha$-$^{12}$C scattering, $^{12}$C($\alpha,\alpha$)$^{12}$C, and $\alpha$ transfer reactions, $^{12}$C($^6$Li,$d$)$^{16}$O, $^{12}$C($^7$Li,$t$)$^{16}$O, $^6$Li($^{12}$C,$d$)$^{16}$O, and $^7$Li($^{12}$C,$t$)$^{16}$O, up to relatively high energies, $T \approx 7$ MeV, and a significant number of parameters of the models are fitted to the data by using computing power [20,65,69]. In the following, we discuss an alternative approach to estimate the $S$-factor at $T_G$; we discuss counting rules of the EFT at $T_G$ and display an effective Lagrangian for the reactions.

### 3.2 An EFT for the radiative $\alpha$ capture on $^{12}$C

In the study of the radiative $\alpha$ capture on $^{12}$C at $T_G = 0.3$ MeV employing an EFT, at such a low energy, one may regard the ground states of $\alpha$ and $^{12}$C as point-like particles whereas the first excited states of $\alpha$ and $^{12}$C are chosen as irrelevant degrees of freedom, by which a large momentum scale of the theory is determined. An effective Lagrangian for the process is constructed by using two spinless scalar fields for $\alpha$ and $^{12}$C, and terms of the Lagrangian are expanded in terms of the number of derivatives. The expansion parameter of the theory is $Q/A_H \sim 1/\beta$ where $Q$ denotes a typical momentum scale $Q \approx k_G$; $k_G$ is the Gamow peak momentum, $k_G = \sqrt{2\mu L}$, $\approx 41$ MeV, where $\mu$ is the reduced mass of $\alpha$ and $^{12}$C. $A_H$ denotes a large momentum scale $A_H \approx 2\mu_4(4)$ or $2\mu_{12}(12) \approx 150$ MeV where $\mu_4$ is the reduced mass of one and three-nucleon system and $\mu_{12}$ is that of four and eight-nucleon system. $T(4)$ and $T(12)$ are the first excited energies of $\alpha$ and $^{12}$C, respectively; $T(4) = 20.21$ MeV for $0^+_1$ state of $^4$He and $T(12) = 4.44$ MeV for $2^+_1$ state of $^{12}$C. By including the terms up to next-to-next-to-leading order (NNLO), for example, one may obtain about 10% theoretical uncertainty for the process.

Coefficients appearing in an effective Lagrangian are fixed by using the experimental data which are measured at significantly higher energies than $T_G$. In the following sections, we employ the data from three processes, elastic $\alpha$-$^{12}$C scattering measured at $T \approx 2 - 5$ MeV, $S_{E1}$ factor for $^{12}$C($\alpha,\gamma$)$^{16}$O at $T \approx 1 - 3$ MeV, and $\beta$ delayed emission from $^{16}$N at $T \approx 0.8 - 3.2$ MeV. Thus, the perturbative scheme of the theory may not be reliable at the energies where those experimental data are measured. We first fix some parameters of the dressed $^{16}$O propagators, parameterized in terms of effective range expansion, by using the binding energies of the excited $^{16}$O states and fit the other parameters of the propagators to the phase shift data of the elastic scattering measured at $T \approx 2 - 5$ MeV. Because of non-perturbative nature of a propagator we treat it as a non-perturbative quantity; the dressed $^{16}$O propagator commonly appears in the three processes, and we keep it as a non-perturbative one and expand reaction amplitudes perturbatively around it. For the radiative capture process, two additional parameters for the $E1$ transition amplitude of the radiative $\alpha$ capture process are fitted to the $S_{E1}$ factor data measured at $T \approx 1 - 3$ MeV, and a value of the $S_{E1}$ factor is estimated at $T_G = 0.3$ MeV.

For the $\beta$ delayed emission from $^{16}$N, parameters of the decay amplitudes are fitted to the experimental data measured at $T \approx 0.8 - 3.2$ MeV. This study would explore a validity of the present approach. One reason is that the $\beta$ delayed emission data are covered with the small energy region compared to that for the elastic scattering data. The other reason is, as mentioned before, to test a different parameterization for the dressed $^{16}$O propagator compared to that in the $R$-matrix or $K$-matrix analysis. In the conventional $R$-matrix analysis, the subthreshold $1^-_1$ state and the broad resonant $1^-_2$ state of $^{16}$O are represented by the Breit-Wigner formula and are linearly combined in the reaction matrix along with a background contribution; a secondary maximum of the $\beta$ delayed emission data is known to be important to constrain an interference pattern among those levels [70,71,72]. In the present approach, the interference between the $1^-$ subthreshold and resonant states doesn’t exist because the subthreshold $1^-_1$ state and the broad resonant $1^-_2$ state of $^{16}$O is represented by a single dressed $^{16}$O propagator (in terms of effective range expansion); instead, we will see that the secondary peak can be reproduced by an interference between the amplitudes from a non-pole diagram and a pole diagram.

An effective Lagrangian for the present study is written as [12,10]

\[
\mathcal{L} = \mathcal{L}_{ES} + \mathcal{L}_{RC(+)} + \mathcal{L}_{BD(+)} ,
\]

where $\mathcal{L}_{ES}$ is the Lagrangian for the elastic scattering process [12], $\mathcal{L}_{RC(+)}$ is for additional terms for the radiative
where α and β are related to effective range parameters along with later, because of a modification of the counting rules for n and α effective field theories. The Lagrangian L_{ES} may be written by using the composite $^{16}$O fields consisting of α and 12C as \[ L_{ES} = \phi_{\alpha} \left( iD_{\alpha} + \frac{D^2}{m_{\alpha}} + \cdots \right) \phi_{\alpha} \]
+ $\phi_{C} \left( iD_{C} + \frac{D^2}{m_{C}} + \cdots \right) \phi_{C} \]
+ \sum_{l,n} C_{n}^{(l)} \left( iD_{l} + \frac{D^2}{2(m_{\alpha} + m_{C})} \right)^{n} d_{i(l)} \]
- \sum_{l=0}^{3} y_{l}(1) \left( (\phi_{D} \phi_{C})^{\dagger} d_{l(1)} + d_{l(1)}^{\dagger} (\phi_{D} \phi_{C}) \right) \]
+ \cdots, \tag{2}

with
\[ O_{l}^{(1)} = \frac{iD_{l}}{m_{O}}, \tag{5} \]

where $m_{O}$ is the mass of $^{16}$O in the ground state. We note that because the $^{16}$O ground state appears only in the final state, $\phi_{D} \phi_{C}^{\dagger}$ is introduced as a source field for the $^{16}$O ground state in the final (initial) state. In the first term in Eq. (5), for example, $\phi_{D} \phi_{C}^{\dagger}$ destroys (creates) the $^{16}$O ground state, and $\phi_{D} \phi_{C}$ fields create (destroy) a s-wave α-12C state. The transition rate between the $^{16}$O ground state and the s-wave α-12C state is parameterized by the coupling constant $y_{l}^{(0)}$, which is fixed by using experimental data. A contact interaction, the $h^{(1)}$ term, is introduced to renormalize divergence from loop diagrams for the radiative α capture reaction.

A weak decay amplitude is described in terms of the $V - A$ type current-current interaction for semi-leptonic decay process, and its interaction Hamiltonian density is given as
\[ \mathcal{H}(x) = \frac{G_{F}}{\sqrt{2}} j_{\mu}^{(n)}(x) \cdot j_{\mu}^{(\bar{\nu})(x)}, \tag{6} \]

where $G_{F}$ is the Fermi constant, $j_{\mu}^{(n)}(x)$ is a nuclear current to be constructed by using the external vector and axial-vector fields in the effective Lagrangian while $j_{\mu}^{(\bar{\nu})(x)}$ is a lepton current. In the energy-momentum space it is given as
\[ j_{\mu}^{(\bar{\nu})}(q) = \bar{u}_{e}(p') \gamma_{\mu}(1 - \gamma_{5}) \nu_{\nu}(p), \tag{7} \]

where q is the momentum transfer, $q^{\mu} = p'^{\mu} - p^{\mu}$.

The Lagrangian $L_{BD(+)\dagger}$ is for additional terms for the study of the β delayed α emission from $^{16}$N. We construct $L_{BD(+)\dagger}$ for interactions where the initial $2_{1}^{-}$ ground state of $^{16}$N decaying to the $1_{1}^{-}$ and $3_{1}^{-}$ states of $^{16}$O and the α-12C breakup state for $l = 1$ and 3 channels through the Gamow-Teller transition, which may be written as
\[ L_{BD(+)\dagger} = C_{a}^{(l=1)} y_{l(1)} \left[ a_{i} (\phi_{\alpha} O_{1;j}\phi_{C})^{\dagger} \phi_{N,ij} \right] \]
+ $C_{b}^{(l=3)} a_{i}(d_{l(1)})^{\dagger} \phi_{N,ij}$
+ $C_{c}^{(l=3)} (d_{l(3)})^{\dagger} \phi_{N,ij}$
+ $D_{b}^{(l=3)} a_{i}(O_{1;1}^{(l)} d_{l(1)})^{\dagger} \phi_{N,ij}$
+ $D_{a}^{(l=3)} a_{i}(O_{1;1}^{(l)} d_{l(1)})^{\dagger} \phi_{N,ij}$
+ \cdots, \tag{8}

where $a_{i}$ is the external axial-vector field, which generates an axial nuclear current for the Gamow-Teller transition, and $\phi_{N,ij}$ is the source field for the initial $2_{1}^{-}$ ground state of $^{16}$N. $C_{a}^{(l=1)}$, $C_{b}^{(l=3)}$, $C_{c}^{(l=3)}$, and $C_{d}^{(l=3)}$ are coefficients for $l = 1$ and $l = 3$ at LO. and $D_{b}^{(l=1)}$ and $D_{a}^{(l=1)}$ are those for $l = 1$ at NNLO; the indices of the squared operators, $O_{1}^{2}$ and $O_{1}^{(1)}$, for the terms at NNLO are suppressed in the above equation.
4 Elastic $\alpha$-$^{12}$C scattering in the cluster EFT

In this section, we construct dressed composite $^{16}$O propagators. We first review the formalism of the effective range expansion for elastic $\alpha$-$^{12}$C scattering. We then discuss a modification of the counting rules for a low momentum expansion around the unitary limit, based on the observation in a comparison between experimental data and terms generated from the Coulomb self-energy. After including broad resonant states of $^{16}$O, we fit the effective range parameters to phase shift data. We then calculate asymptotic normalization coefficients (ANCs) for the $1^-$ and $3^-$ states of $^{16}$O and compare them to the previous results.

4.1 Differential cross section of the elastic scattering

The differential cross section of the elastic $\alpha$-$^{12}$C scattering (for two spin-0 charged particles) in terms of the phase shifts is given by (see, e.g., Ref. [18])

$$
\sigma(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2
$$

$$
= \frac{1}{k^2} - \frac{\eta}{2\sin^2 \frac{\theta}{2}} \exp \left( -2i\eta \ln \sin \frac{1}{2}\theta \right)
$$

$$
+ \frac{1}{2} \sum_{l=0}^{\infty} (2l+1) \left[ \exp (2i\omega_l) - U_l \right] P_l(\cos \theta) \right|^2
$$

(9)

where $f(\theta)$ is the scattering amplitude including both pure Coulomb part and Coulomb modified strong interaction part, $\theta$ is the scattering angle, $k$ is the absolute relative momentum, and $\eta = \kappa/k$, $\kappa$ is the inverse of the Bohr radius, $Z_\alpha Z_C \mu_C E_{\text{kin}}$ where $Z_\alpha$ and $Z_C$ are the number of protons in $\alpha$ and $^{12}$C, respectively, and $\mu_C$ is the fine structure constant. In addition, $\omega_l$ is the Coulomb scattering phase, $\omega_l = 2\pi l + \sigma_l = \sum \arctan(\eta/s)$ for $s = 1$ to $l$; $\sigma_l$ are the Coulomb phase shifts, $\sigma_l = \arg \Gamma(1 + l + i\eta)$ with $l = 0, 1, 2, \cdots$, and

$$
U_l = \exp \left[ 2i(\delta_l + \omega_l) \right].
$$

(10)

$\delta_l$ are real scattering phase shifts. $P_l(x)$ are the Legendre polynomials. The scattering amplitudes are represented in terms of $\delta_l$ as [18]

$$
A_l = \frac{2\pi}{\mu} \frac{(2l+1)P_l(\cos \theta)e^{2i\sigma_l}}{k \cot \delta_l - ik}.
$$

(11)

In the EFT, the amplitudes of the elastic scattering are calculated from diagrams depicted in Figs. 1 and 2.

\* One can see the relation between $U_l$ and $A_l$ through the relation

$$
\frac{1}{2} \left[ \exp (2i\omega_l) - U_l \right] = \exp (2i\omega_l) \frac{1}{\cot \delta_l - i}.
$$

Fig. 1. Diagrams for dressed $^{16}$O propagator. A thick (thin) dashed line represents a propagator of $^{12}$C ($\alpha$), and a thick and thin double dashed line with and without a filled circle represent a dressed and bare $^{16}$O propagator, respectively. A shaded blob represents a set of diagrams consisting of all possible one-potential-photon-exchange diagrams up to infinite order and no potential-photon-exchange one.

Fig. 2. Diagram of the scattering amplitude. See the caption of Fig. 1 as well.

We obtain the scattering amplitudes for $l$-th partial wave states as [12,74,75]

$$
A_l = \frac{2\pi}{\mu} \frac{(2l+1)P_l(\cos \theta)e^{2i\sigma_l}W_l(\eta)C_2^2}{K_l(k) - 2\kappa H_l(k)},
$$

(12)

with

$$
C_2^2 = \frac{2\pi \eta}{e^{2\pi \eta} - 1}, \quad W_l(\eta) = \kappa^{2l} \frac{1}{(l!)^2} \prod_{n=0}^{l} \left( 1 + \frac{\eta^2}{n^2} \right),
$$

$$
H_l(k) = W_l(\eta)H(\eta),
$$

(13)

and

$$
H(\eta) = \psi(\eta) + \frac{1}{2\eta} - \ln(\eta),
$$

(14)

where $\psi(z)$ is the digamma function. Note that the function, $-2\kappa H_l(k)$, the Coulomb self-energy term, in the denominator of the amplitude is obtained from the Coulomb bubble diagram for the dressed propagator of $^{16}$O in Fig. 1 and the factor, $e^{2i\sigma_l}W_l(\eta)C_2^2$, in the numerator is from the initial and final state Coulomb interactions between $\alpha$ and $^{12}$C in Fig. 2. In Appendix A, we display the renormalized dressed three-point vertices for the initial and final Coulomb interactions and the renormalized dressed composite $^{16}$O propagators for $l = 0, 1, 2, 3$.

The function $K_l(k)$ represents the interaction due to the short range nuclear force (compared with the long range Coulomb force), which is obtained in terms of the effective range parameters as

$$
K_l(k) = -\frac{1}{a_l} + \frac{1}{2} \frac{1}{k^2 - \frac{1}{4} P_l k^4 + Q_l k^6 - R_l k^8 + \cdots}.
$$

(15)

One can find that the expression obtained in Eq. (12) reproduces well the previous results reported in Refs. [76, 77, 78, 79].

By comparing the two expressions of the amplitudes $A_l$ in Eqs. (11) and (12), one has a relation between the
phase shift and the effective range parameters in the denominator of the scattering amplitudes, $D_l(k)$, as

$$W_l(\eta)C^2_\eta k^\delta \gamma = \text{Re} D_l(k),$$  \hspace{1cm} (16)

where

$$D_l(k) = K_l(k) - 2\kappa H_l(k).$$  \hspace{1cm} (17)

To estimate the ANC, $|C_b|$, for the $0^+_2$, $1^-_1$, $2^+_1$, $3^-_1$ states of $^{16}\text{O}$, we employ the definition of $|C_b|$ [80]

$$|C_b| = \gamma^l_\eta \frac{\Gamma(l + 1 + |\eta_\delta|)}{l!} \left( - \frac{dD_l(k)}{dk^2} \right)_{k^2 = -\gamma^2}^{\frac{1}{2}},$$ \hspace{1cm} (18)

where $\eta_\delta = \kappa/k_0$; $k_0 = i\gamma_l$, and $\gamma_l$ are binding momenta which will be given in the next subsection.

### 4.2 Effective range expansion and modification of the counting rules

We now discuss a modification of the counting rules [13], based on an observation in Eq. (16). The term on the left-hand-side of Eq. (16) is significantly suppressed by the factor $C^2_\eta$ at low energies. Meanwhile, the Coulomb self-energy term, $-2\kappa H_l(k)$, on the right-hand-side of Eq. (16) turns out to be two orders of magnitude larger than the term on the left-hand-side of the equation. Thus, we introduce terms in $K_l(k)$ as counter terms at LO, by which those unnaturally large terms from the Coulomb self-energy term are subtracted. We will discuss it in detail below.

The effective range parameters in $K_l(k)$ are expanded in powers of $k^2$ whereas the real part of the function $H_l(k)$ can be expanded in powers of $k^2$ as well. For the function $H_l(\eta)$ in $H_l(k)$, one has

$$\text{Re} H_l(\eta) = \frac{1}{12\kappa^2}k^2 + \frac{1}{120\kappa^4}k^4 + \frac{1}{252\kappa^6}k^6 + \frac{1}{240\kappa^8}k^8 + \cdots ,$$ \hspace{1cm} (19)

where $\kappa$ is the inverse of the Bohr radius, $\kappa \approx 245$ MeV, and regarded as another large scale of the theory. Thus, the right-hand-side of equation, $\text{Re} D_l(k)$, in Eq. (16) can be expanded as a power series of $k^2$ for both $K_l(k)$ and $2\kappa \text{Re} H_l(k)$. Meanwhile, the left-hand-side of Eq. (16) is suppressed by the factor $C^2_\eta$, due to the Gamow factor $P = \exp(-2\pi\eta)$.

In the case of the $s$-wave, for example, the reported phase shift at the smallest energy, $T_0 = 2.6$ MeV, where $T_0$ is the kinetic energy of $\eta$ in the laboratory frame [3] is $\delta_0 = -1.893^\circ$ [31]. The factor $C^2_\eta$ becomes $C^2_\eta \approx 6 \times 10^{-6}$ at $k = 104$ MeV which corresponds to $T_0 = 2.6$ MeV, and the left-hand-side of Eq. (16) numerically becomes $C^2_\eta k^2 \cot \delta_0 = -0.019$ MeV. The function $2\kappa \text{Re} H_0(k)$ is expanded as

$$2\kappa \text{Re} H_0(k) = \frac{1}{6\kappa}k^2 + \frac{1}{60\kappa^3}k^4 + \frac{1}{126\kappa^5}k^6 + \frac{1}{120\kappa^7}k^8 + \cdots$$

$$= 7.441 + 0.136 + 0.012 + 0.002 + \cdots \text{MeV},$$ \hspace{1cm} (20)

at $k = 104$ MeV. The numerical values in the third line of Eq. (20) correspond to the terms appearing in the second line of the equation in order. One can see that the power series converges well, but the first and second terms are two and one order of magnitude larger than a value estimated by using the experimental data in the left-hand-side of Eq. (16), $-0.019$ MeV. Those terms are unnaturally large, and thus it is necessary to introduce a new renormalization method, in which the counter terms remove the unnaturally large terms and make the terms in a natural size. In other words, we assume that fitting polynomial functions are represented as a natural power series at the low energy region, and to maintain such polynomial functions, large cancellations for the first and second terms with the $r_l$ and $P_l$ effective terms, respectively, are expected. So we include the three effective range parameters, $r_l$, $P_l$, and $Q_l$, for the $l = 0$ channel as the counter terms. The same tendency can be seen in the $l = 1, 2$ channels whereas one needs to include four effective range parameters for the $l = 3$ channel. Thus, we employ the three effective range parameters, $r_l$, $P_l$, $Q_l$ for the $l = 0, 1, 2$ channels and the four effective range parameters, $r_l$, $P_l$, $Q_l$, $R_l$ for the $l = 3$ channel when fitting the parameters to the phase shift data.

At the binding energies of the excited $0^+_2$, $1^-_1$, $2^+_1$, $3^-_1$ states of $^{16}\text{O}$, the amplitudes should have a pole at $k_0 = i\gamma_l$ where $\gamma_l$ are the binding momenta, $\gamma_l = \sqrt{2\mu B_l}$. $B_l$ denote the binding energies of $l^+_l$ excited states of $^{16}\text{O}$. Thus the denominator of the scattering amplitude, $D_l(k)$, should vanish at $k_0$. Using this condition, a first effective range parameter, $a_l$, is related to other effective range parameters as

$$-\frac{1}{a_l} = \frac{1}{2}\gamma_l^2 + \frac{1}{4}P_l\gamma_l^4 + Q_l\gamma_l^6 + R_l\gamma_l^8 + \cdots$$

$$+ 2\kappa H_l(k_0),$$  \hspace{1cm} (21)

and the remaining effective range parameters are fixed by using the phase shift data of the elastic scattering [33].

### 4.3 Numerical results for the elastic scattering

The effective range parameters, $r_l$, $P_l$, $Q_l$ for $l = 0, 1, 2$ and $r_l$, $P_l$, $Q_l$, $R_l$ for $l = 3$ are fitted, employing the standard $\chi^2$ fit [22] to the experimental phase shift data of the elastic $\alpha-^{12}\text{C}$ scattering reported by Tischhauer et al. [10] In Ref. [23], we have also studied an inclusion of the ground $0^+_1$ state of $^{16}\text{O}$ in the parameter fitting for the elastic $\alpha-^{12}\text{C}$ scattering at low energies.

[10] We employ a python package, emcee [54], for the fitting.
we have discussed that a large energy scale of the theory is that for the first excited $2^+_1$ state of $^{12}$C, $T_{12} = 4.44$ MeV ($T_{12} = 5.92$ MeV) in the previous section, other large scales can emerge from resonate energies of $^{16}$O. As suggested by Teichmann [18], below the resonance energies, the Breit-Wigner-type parameterization for the resonances can be expanded in powers of the energy, and one can obtain an expression of the amplitude in terms of the effective range expansion. Therefore, in the previous works [12][13], we choose the energies of the resonant states, $T_{o,0}(0^+_1) = 6.52$ MeV, $T_{o,0}(1^+_2) = 3.23$ MeV, $T_{o,0}(2^+_2) = 3.58$ MeV, $T_{o,0}(3^+_2) = 5.92$ MeV, as a large scale of the theory for each partial wave state. After fitting the parameters to the data below the resonant energies, we confirm large cancellations between the fitted parameters and those generated from the Coulomb self-energy terms, e.g., in Eq. (20) while the perturbative series of the effective range expansions at $T_{C}$ converges well as expected by the theory. See Table V in Ref. [13] and Table 2 in Ref. [14]. The perturbative series, however, does not converge at the large, experimental energies. We will come back to the issue in the next section.

In our recent works [15][16], we include broad resonant states in the fitting because one may expect that because a relatively deep pole for a broad resonance state in the complex energy plane is located at a distance from the real axis, it can possibly be represented by a polynomial function in terms of the effective range expansion. We find that the broad resonant $1^+_2$ and $3^+_2$ states with the decay widths $\Gamma(1^+_2) = 420 \pm 20$ keV and $\Gamma(3^+_2) = 800 \pm 100$ keV can be well incorporated into the fitting of the effective range parameters whereas the narrow resonant $0^+_1$ and $2^+_1$ states with the decay widths $\Gamma(0^+_1) = 1.5 \pm 0.5$ keV and $\Gamma(2^+_1) = 0.625 \pm 0.100$ keV cannot. In addition, a tail-like structure from a higher resonant $1^+_3$ state at $T_{o,0}(1^+_3) = 7.96$ MeV with the width $\Gamma(1^+_3) = 110 \pm 30$ keV cannot be incorporated into the fitting for the high energy region of the data at $T_o \geq 6$ MeV for $l = 1$ either. Thus we use the maximum energies, $T_{o,max} = 5.5$, 6.0, 3.2, and 6.62 MeV for $l = 0$, 1, 2, and 3, respectively, of the data sets for the fitting where we assume that the resonant $2^+_1$ state of $^{12}$C, which could appear at $T_{o,0}(12) = 5.92$ MeV, would be negligible for $l = 1$ and $l = 3$.

12 Input data for the parameter fitting are the phase shifts of the elastic $\alpha$-$^{12}$C scattering for $l = 0, 1, 2, 3$ [31][33], which have been generated from the $R$-matrix analysis of the elastic scattering data [32]. In the input data files [33], there are four column data: the first column is the alpha energy, the second one the phase shift as derived from the globalized Monte Carlo simulations, the third one is the same phase shift randomized by the error from the Monte Carlo simulations, and the fourth one is the error of the phase shifts from the Monte Carlo simulations. We have used the second column of the phase shift data in our previous works [12][13] and the third column of the phase shift data in our other works [14][15][16] for fitting.

4.3.1 $l = 0$ channel

As discussed above, we use the phase shift data for $l = 0$ up to the maximum energy $T_{o,max} = 5.5$ MeV due to the narrow resonant $0^+_1$ state at $T_{o,0}(0^+_1) = 6.52$ MeV with $\Gamma(0^+_1) = 1.5 \pm 0.5$ keV. To study the dependence on the choice of the data sets, we employ four data sets up to different maximum energies, $T_{o,max} = 4.0, 4.5, 5.0, \text{ and } 5.5$ MeV.

In Table I1 fitted values and errors of the effective range parameters, $r_o, P_0$, and $Q_0$, are displayed, and calculated values of $a_0$ by using Eq. (21) and those of the ANC, $|C_b|$, for the $0^+_2$ state of $^{16}$O by using Eq. (18) are also displayed. $\chi^2/N$ and numbers of the data ($N$) for the fitting are $\chi^2/N(N) = 0.354(80), 0.393(149), 0.450(167), 0.552(230)$ for $T_{o,max} = 4.0, 4.5, 5.0, 5.5$ MeV, respectively. As seen in the table, the errors of those quantities become smaller as the value of $T_{o,max}$ increases (mainly because the number of the data increases) while the center values of them are changing. The change of the center values of the coefficients may indicate an effect of truncation error in the fitting and/or that of the narrow resonance. If we include the data at high energies larger than $T_{o,max} = 5.5$ MeV into the fitting, we have large $\chi^2/N$ values: $\chi^2/N = 1.36$ with those up to $T_{o,max} = 6.0$ MeV, and $\chi^2/N = 11.3$ with those up to $T_{o,max} = 6.6$ MeV. [41]

4.3.2 $l = 2$ channel

For $l = 2$, as discussed above, we cannot incorporate the narrow resonant $2^+_2$ state at $T_{o,0}(2^+_2) = 3.58$ MeV with $\Gamma(2^+_2) = 0.625 \pm 0.100$ keV in the fitting. Thus we choose the maximum energy of the data set for the fitting as $T_{o,max} = 3.2$ MeV. Using the data of the data set we fit effective range parameters as

$$r_2 = 0.15 \pm 0.14 \text{ fm}^{-3}, \quad P_2 = -1.2 \pm 2.2 \text{ fm}^{-1},$$
$$Q_2 = 0.10 \pm 0.93 \text{ fm}. \quad (22)$$

We calculate a value of $a_2$ and that of the ANC, $|C_b|$, for the subthreshold $2^+_1$ state of $^{16}$O as $a_2 = 4.6 \times 10^3 \text{ fm}^2$ and $|C_b| = (3.1 \pm 24.5) \times 10^4 \text{ fm}^{-1/2}$, respectively, where the number of the data is $N = 23$ and $\chi^2/N = 0.22$. One can see above that because the value of $\chi^2/N$ is quite small while the error bars of those fitted quantities are large, we cannot deduce a meaningful result for $l = 2$ from the fitting.

13 In Ref. [36], we recently studied an inclusion of the sharp resonant $0^+_2$ state of $^{16}$O and the first excited $2^+_2$ state of $^{12}$C in the study of elastic $\alpha$-$^{12}$C scattering for $l = 0$ up to $T_{o,max} = 6.62$ MeV. We found that the $2^+_2$ state of $^{12}$C is redundant for fitting the phase shift data. To investigate its precise role, the inelastic open channel, $\alpha$-$^{12}$C$(2^+_2)$, would be necessary to be included in the study of the elastic scattering above the excited energy of $^{12}$C.
Table 1. Center values and errors of the effective range parameters, $r_0$, $P_0$, and $Q_0$, fitted to the data of the data sets with the energy ranges, 2.6 MeV $\leq T_o \leq T_o,max$ where $T_o,max = 4.0$, 4.5, 5.0, and 5.5 MeV. Values of $a_0$ and those of $|C_b|$ for the $0^+_2$ state of $^{16}$O are calculated by using the fitted $r_0$, $P_0$, and $Q_0$ values. For details, see the text.

| $T_o,max$ (MeV) | $r_0$ (fm) | $P_0$ (fm$^2$) | $Q_0$ (fm$^2$) | $a_0$ (fm) | $|C_b|$ (fm$^{-1/2}$) |
|----------------|-----------|----------------|----------------|-----------|-----------------|
| 4.0            | 0.26851(5) | $-0.0354(33)$  | 0.0014(15)     | 3.89 $\times 10^4$ | 4.5(89) $\times 10^4$ |
| 4.5            | 0.26851(3) | $-0.0352(13)$  | 0.0015(5)      | 4.84 $\times 10^4$ | 4.9(20) $\times 10^2$ |
| 5.0            | 0.26849(3) | $-0.0358(8)$   | 0.0013(3)      | 4.11 $\times 10^4$ | 4.1(8) $\times 10^2$  |
| 5.5            | 0.26845(2) | $-0.0375(5)$   | 0.0006(2)      | 3.10 $\times 10^4$ | 3.1(2) $\times 10^2$  |

4.3.3 $l = 1$ and $l = 3$ channels

For $l = 1$, as discussed above, we can incorporate the broad resonant $1^+_2$ state at $T_o(1^+_2) = 3.23$ MeV with $\Gamma(1^+_2) = 420 \pm 20$ keV in the fitting for the effective range parameters but cannot do a tail from the next broad resonant $1^+_1$ state at $T_o(1^+_1) = 7.96$ MeV with $\Gamma(1^+_1) = 110 \pm 30$ keV. Thus the highest energy of the data for the fitting is $T_o,max = 6.0$ MeV for $l = 1$. To study the dependence on the choice of the data sets, we employ four data sets up to the different maximum energies, $T_o,max = 3.0$, 4.0, 5.0, and 6.0 MeV.

In Table 2, fitted values and errors of the effective range parameters, $r_1$, $P_1$, and $Q_1$, are displayed, and calculated values of $a_1$ and those of the ANC, $|C_b|$, for the $1^+_1$ state of $^{16}$O are displayed as well. $\chi^2/N$ and numbers of the data ($N$) for the fitting are $\chi^2/N(N) = 0.872(13)$, 0.450(80), 0.509(167), 0.738(273) for $T_o,max = 3.0$, 4.0, 5.0, 6.0, respectively. As seen in the table, the errors of those quantities decrease as $T_o,max$ increases. It is worth pointing out that the center values of them are almost not altered even though $T_o,max$ is changed. Because the $\chi^2/N$ value for the data up to $T_o,max = 6.0$ MeV is still smaller than 1, an effect from the resonant $2^+_1$ state of $^{12}$C is not significant either. Thus we take available all energy range of those quantities decrease as $T_o,max$ increases. It is interesting to point out that, as the same as that we have seen for $l = 1$, the center values of them are almost not altered even though $T_o,max$ is changed except for the deviations of $a_3$ and $|C_b|$ and a large error of $|C_b|$ for $T_o,max = 5.0$ MeV compared to the corresponding values for the other $T_o,max$. That may stem from negative correlations between the parameters for $T_o,max = 4.6$ MeV and almost no correlations between those for $T_o,max = 5.0$ MeV. (One can find the correlations between the parameters in Table VI in Ref. [15].) The effect from the resonant $2^+_1$ state of $^{12}$C is not significant either because of the small $\chi^2/N$ value for the data up to $T_o,max = 6.62$ MeV.

4.3.4 Comparison of the ANCs

In Table 2, we summarize our results for the ANCs for the $1^+_1$ and $3^+_1$ states of $^{16}$O where we show the results obtained from the largest data sets for the fitting. Because of the change of the center values and the large uncertainty for the ANCs for the $0^+_2$ and $2^+_1$ states of $^{16}$O, respectively, we do not include those results. We suppress the comparison about the ANCs for the $0^+_2$ and $2^+_1$ states below.

The ANCs for the $1^+_1$ and $2^+_1$ states of $^{16}$O have intensively been studied because the major contribution of the $S$-factor of the radiative capture on $^{12}$C at $T_C$ come from the $E1$ and $E2$ transitions due to the threshold $1^+_1$ and $2^+_1$ states of $^{16}$O. In our previous work [23], we obtained $(1.6 - 1.9) \times 10^{14}$ fm$^{-1/2}$ for the $1^+_1$ state, which agrees well with the result presented above. Our result underestimates, by about 10%, compared to the other theoretical estimates: $(2.22 - 2.24) \times 10^{14}$ fm$^{-1/2}$ obtained from a potential model calculation by KatSUMA [27], and $2.14(6) \times 10^{14}$ fm$^{-1/2}$ and $2.073 \times 10^{14}$ fm$^{-1/2}$ from a new parameterization method by Ramirez Suarez and Sparenberg [88] and by Orkov et al. [89], respectively. Our result, on the other hand, also underestimates or agrees well with the experimental results within the reported errors: $(2.10 \pm 0.14) \times 10^{14}$ fm$^{-1/2}$ obtained from the $^6$Li($^{12}$C,$d$)$^{16}$O reaction by Avila et al. [83], $(2.00 \pm 0.35) \times 10^{14}$ fm$^{-1/2}$ from the $^{12}$C($^7$Li,t)$^{16}$O reaction by Oulebsir et al. [71], and $(2.08 \pm 0.20) \times 10^{14}$ fm$^{-1/2}$ from the $^{12}$C($^6$Li,d)$^{16}$O and $^{12}$C($^7$Li,t)$^{16}$O reactions by Brune et al. [72].

Only some studies for the ANCs for the $0^+_2$ and $3^+_1$ states of $^{16}$O have been reported so far, though those

---

14 Though the maximum energy of the data for fit is larger than the energy of the first excited $2^+_1$ state of $^{12}$C, $T_C(2^+_1) = 4.44$ MeV, there is no indication of a need to include the $2^+_1$ state of $^{12}$C. See the footnote 13 as well.

15 See the footnote 14.
\[
T_{\alpha,\text{max}} \text{ (MeV)} \quad r_1 \text{ (fm)} \quad P_1 \text{ (fm)} \quad Q_1 \text{ (fm)} \quad a_1 \text{ (fm)} \quad |C_b| \text{ (fm}^{-1/2})
\]

| 3.0 | 0.4157(9) | −0.568(11) | 0.022(4) | −1.316 × 10^3 | 1.63(3) × 10^{14} |
| 4.0 | 0.415266(49) | −0.57481(56) | 0.02015(19) | −1.665 × 10^3 | 1.835(25) × 10^{14} |
| 5.0 | 0.415272(20) | −0.57474(20) | 0.02018(22) | −1.658 × 10^3 | 1.832(10) × 10^{14} |
| 6.0 | 0.415273(9) | −0.57473(9) | 0.02018(3) | −1.658 × 10^3 | 1.832(5) × 10^{14} |

Table 2. Center values and errors of the effective range parameters, \( r_1 \), \( P_1 \) and \( Q_1 \), fitted to the data of the data sets with the energy ranges, 2.6 MeV ≤ \( T_\alpha \) ≤ \( T_{\alpha,\text{max}} \) where \( T_{\alpha,\text{max}} = 3.0, 4.0, 5.0, \) and 6.0 MeV. Values of \( a_1 \) and those of \( |C_b| \) for the subthreshold \( 1^-_1 \) state of \( ^{16}\text{O} \) are calculated by using the fitted \( r_1 \), \( P_1 \), and \( Q_1 \) values. For details, see the text.

\[
T_{\alpha,\text{max}} \text{ (MeV)} \quad r_3 \text{ (fm)} \quad P_3 \text{ (fm)} \quad Q_3 \text{ (fm)} \quad R_3 \text{ (fm)} \quad a_3 \text{ (fm)} \quad |C_b| \text{ (fm}^{-1/2})
\]

| 4.6 | 0.032(1) | −0.50(12) | 0.28(9) | −0.17(9) | −2.8 × 10^3 | 2.3(82) × 10^2 |
| 5.0 | 0.0321(4) | −0.507(57) | 0.276(42) | −0.175(35) | −3.9 × 10^3 | 4.3(265) × 10^2 |
| 6.0 | 0.0320(3) | −0.494(11) | 0.285(6) | −0.167(4) | −2.8 × 10^3 | 2.2(7) × 10^2 |
| 6.62 | 0.0320(2) | −0.495(6) | 0.285(3) | −0.168(2) | −2.8 × 10^3 | 2.3(4) × 10^2 |

Table 3. Center values and errors of effective range parameters, \( r_3 \), \( P_3 \), \( Q_3 \), and \( R_3 \), fitted to the data of the data sets with the energy ranges, 2.6 MeV ≤ \( T_\alpha \) ≤ \( T_{\alpha,\text{max}} \) where \( T_{\alpha,\text{max}} = 4.6, 5.0, \) and 6.62 MeV. Values of \( a_3 \) and the ANC, \( |C_b| \), for the \( 3^+_1 \) state of \( ^{16}\text{O} \) are calculated by using the fitted \( r_3 \), \( P_3 \), \( Q_3 \), and \( R_3 \) values. For details, see the text.

\[
|C_b| \text{ (fm}^{-1/2}) \quad 1^-_1 \quad 3^+_1
\]

| 1.832(5) × 10^{14} | 2.4(4) × 10^2 |

Table 4. Our result of the ANCs for the \( 1^-_1 \) and \( 3^+_1 \) states of \( ^{16}\text{O} \).

\[
\sigma_{E1}(T) = \frac{4}{3} \frac{\alpha E_\gamma E_\gamma'}{(1 + E_\gamma'/m_\gamma)^2} X^{(l=1)}(T),
\]

where \( T \) is the kinetic energy of the initial \( \alpha \)-12C state in the center of mass frame, \( T = p^2/(2m) \); \( p \) is the magnitude of relative momentum between \( \alpha \) and \( ^{12}\text{C} \). \( E_\gamma' \) is the photon energy.

\[
E_\gamma' \simeq B_0 + T - \frac{1}{2m_\gamma} (B_0 + T)^2,
\]

where \( B_0 \) is the \( \alpha \)-12C breakup energy from the ground state of \( ^{16}\text{O} \); \( B_0 = m_\gamma - m_\alpha - m_\gamma \) = 7.162 MeV. One may notice that \( B_0 \) is larger than the large energy scale of the theory, the first excited energy of \( ^{12}\text{C} \), \( T_{(12)} = 4.44 \) MeV. Because the released large energy is carried away by the outgoing photon, the final nuclear state remains in a state with a typical energy scale. We will discuss how one can avoid invoking a resonant state originated from the first excited \( 2^+_1 \) state of \( ^{12}\text{C} \) below. \( X^{(l=1)} \) is a transition amplitude which will also be shown in the following.

In Fig. 3 diagrams of the radiative \( \alpha \) capture process from the initial \( \alpha \)-12C state for \( l = 1 \) to the \( ^{16}\text{O} \) ground state are depicted, in which the Coulomb interaction between \( \alpha \) and \( ^{12}\text{C} \) is taken into account, and in Fig. 1 those for dressed composite propagators of \( ^{16}\text{O} \) consisting of \( \alpha \) and \( ^{12}\text{C} \) for \( l = 1 \) are depicted. The propagator is obtained in the previous section. (See Appendix as well.)

The radiative \( \alpha \) capture amplitude for the initial \( l = 1 \) state is presented as

\[
A^{(l=1)} = \epsilon_\gamma^{(\alpha \gamma)} \cdot \tilde{p} X^{(l=1)},
\]

where \( \epsilon_\gamma^{(\alpha \gamma)} \) is the polarization vector of outgoing photon and \( \tilde{p} = \hat{p}/|p| \); \( p \) is the relative momentum of the initial \( \alpha \) and \( ^{12}\text{C} \). The amplitude \( X^{(l=1)} \) is decomposed as

\[
X^{(l=1)} = X^{(l=1)}_{(\alpha + b)} + X^{(l=1)}_{(c)} + X^{(l=1)}_{(d + e + f)} + X^{(l=1)}_{(f)}.
\]
where those amplitudes correspond to the diagrams depicted in Fig. 3.

We follow the calculation method suggested by Rynberg et al. [93], in which Coulomb Green’s functions are represented in the coordinate space satisfying appropriate boundary conditions. Thus we obtain the expression of those amplitudes in the center of mass frame as

\[
X_{(a+b)}^{(l=1)} = 2g^{(0)}e^{i\sigma} \Gamma(1 + \kappa/\gamma_0) \int_0^\infty dr |W_{-\kappa/\gamma_0 + \frac{1}{2}}(2\gamma_0 r)|
\]

\[
\times \left[ \frac{Z_{\alpha\mu}}{m_\alpha} j_0 \left( \frac{m_\alpha}{m_C} k'r \right) - \frac{Z_C \mu}{m_C} j_0 \left( \frac{\mu}{m_C} k'r \right) \right]
\times \left\{ \frac{\partial}{\partial r} \left[ F_1(\eta, pr) \right] + \frac{2}{pr} \right\},
\]

(28)

where \( k' \) is the magnitude of outgoing photon momentum, \( Z_O \) is the number of protons in \(^{16}\)O, and \( \gamma_0 \) is the binding momentum of the ground state of \(^{16}\)O. \( \gamma_0 = \sqrt{2\mu E_0} \) and \( j_l(x) \) are gamma function and spherical Bessel function, respectively, while \( F_1(\eta, r) \) and \( W_{\eta}(\mu, \rho) \) are regular Coulomb function and Whittaker function, respectively. The functions, \( H_1(p) \) and \( K_1(p) \) (where \( \eta = \kappa/p \)), and \( K_1(p) \), have been introduced in Eqs. (13), (14), and (15), respectively.

Regarding the divergence from the loop integrals, the loops of the diagrams (a) and (b) in Fig. 3 are finite while those of the diagrams (d) and (e) lead to a log divergence in \( X_{(d+e)}^{(l=1)} \) in the limit, \( r \to 0 \). We introduce a short range cutoff \( r_C \) in the \( r \) integral in Eq. (30), and the divergence is renormalized by the counter term, \( h^{(1)} \). The loop of the diagram (f) diverges and is renormalized by the \( h^{(1)} \) term as well. Thus we have

\[
h^{(1)}_R = h^{(1)} - \mu m_O \left( \frac{Z_O}{m_\alpha} - \frac{Z_C}{m_C} \right) \left[ J^{(d+e)}_0 + J^{(div)}_0 \right]
\]

(32)

where \( J^{(div)}_0 \) is the divergence term from the diagrams (d) and (e) and \( J^{(div)}_0 \) is that from the diagram (f); we have

\[
J^{(div)}_0 = \kappa \mu \left\{ \frac{1}{e} - 3C_E + 2 + \ln \left( \frac{\mu B}{4r^2} \right) \right\},
\]

(33)

where, when we derive the expression of \( J^{(div)}_0 \), the dimensional regularization in \( 4 - 2\epsilon \) space-time dimensions is used; \( C_E = 0.577 \cdots \) and \( \mu B_r \) is a scale factor from the dimensional regularization. \( h^{(1)}R \) is a renormalized coupling constant which is fixed by experiment.

We have used the two regularization methods when calculating the loop diagrams (d) and (e) and the loop diagram (f). Some different regularization methods result in different expressions for divergent terms and constant terms but the same expression for functional terms (such as mass or momentum dependence terms) [93]. Thus the different regularization methods may be adapted by adjusting a value of the coefficient \( h^{(1)} \). When we send a value of \( r_C \) sufficiently small. Recently, Higa, Rupak, and Vagnelli reported that the divergent terms are exactly canceled with each other among those diagrams when they calculate one-photon-exchange diagrams for the \( \alpha_E \)-order terms using the dimensional regularization [50].

We now discuss how one can avoid invoking a resonant state due to the released energy from the radiative \( \alpha \) capture reaction. As mentioned above, the both initial and final nuclear states remain in the states at the typical energies because the photon carries away almost all of the released energy. A large energy gap, then, appears in the intermediate state, i.e., in the loop diagrams; the initial \( p \)-wave \(^{12}\)C state is in a typical energy state while, after the photon is emitted, the excited energies for the resonant \(^{2}\)He and \(^{12}\)C are located at farther above than that for the \(^{12}\)C breakup threshold, thus an effect from the resonant
2\(^+\) state of \(^{12}\)C will hardly be seen. Meanwhile the loop integrals may pick up the deep momentum scale. From the loop diagram (f), for example, when the Coulomb interaction is ignored, the large momentum scale \(\gamma_0 \simeq 200\) MeV is picked up in the numerator of the amplitude. It causes the emergence of a term which does not obey the counting rules.\(^{16}\) In the present case, the large momentum scale \(\gamma_0\) from the ground state energy of \(^{16}\)O appears as a ratio \(\kappa/\gamma_0\), due to the non-perturbative Coulomb interaction, where \(\kappa\) is another large momentum scale, \(\kappa \simeq 245\) MeV. The finite term \(-2\kappa H(\eta_0)\) in \(X^{(f)}\) from the loop of the diagram (f) with \(\eta_0 = \kappa/(i\gamma_0)\) is reduced to a typical momentum scale, \(-2\kappa H(\eta_0) = 25.8\) MeV.

5.2 Suppression of the E1 transition and mixture of isospin \(I = 1\) state

Before fitting the parameters to available experimental data, we discuss three issues: non-perturbative treatment of the dressed \(^{16}\)O propagator for \(l = 1\), suppression of the E1 transition amplitude, and a mixture of isospin \(I = 1\) state.

For the radiative \(\alpha\) capture amplitudes, whose expressions are displayed in Eqs. (25), (26), (30), (31), we have two limits for perturbative expansion: The one appears in the denominator of the transition amplitudes; the dressed \(^{16}\)O propagator is expanded around the unitary limit. The other appears in the numerator of the transition amplitudes as loop and vertex corrections. As discussed above, the perturbative expansion in the denominator in terms of the effective range expansion is valid at \(T_C\) and does not converge at the energies where the experimental data are available for fitting. Meanwhile, because the phase shift data for \(l = 1\) are reproduced very well (as we will see in Fig. 4) by means of the effective range expansion, we treat the dressed \(^{16}\)O propagator as a non-perturbative quantity and perturbatively expand the transition amplitudes around it.

An order of an amplitude from each of the diagrams is found by counting the number of momenta of vertices and propagators in a Feynman diagram; one has a LO amplitude from the diagram (c) because the contact \(\gamma\)-\(d\)-\(\phi\)O vertex of the \(h^{(1)R}\) term does not have a momentum dependence, and NLO amplitudes from the other diagrams in Fig. 3. One may notice a large suppression factor, \(Z_a/m_a-Z_C/m_C\), appearing in \(X^{(l=1)}\). \((m_\eta/Z_\eta)(Z_a/m_a-Z_C/m_C)\) \(\simeq -6.5 \times 10^{-4}\). Similar suppression effect can be found in \(X^{(l=1)}\) and \(X^{(l=1)}\) as well; we denote those amplitudes as \(X^-\), and when changing the minus sign to the plus one in the front of the spherical Bessel function \(j_0(z)\) in Eqs. (25) and (30), we do them as \(X^+\). We thus have

\[
|X^{(l=1)}_a/X^{(l=1)}_a| \simeq 8.7 \times 10^{-4} \quad \text{and} \quad |X^{(l=1)}_d/X^{(l=1)}_d| \simeq 3.6 \times 10^{-4}
\]

at the energy range, \(T = 0.9-3\) MeV, at which we fit the parameters to the experimental \(S_{E1}\) data in the next subsection. The suppression effect is common among those amplitudes from the diagrams (a), (b), (d), (e), (f) at NLO. Thus, the radiative \(\alpha\) capture rate will be well controlled by the coefficient \(h^{(1)R}\) from the diagram (c) at LO.

The strong suppression effect mentioned above is well known; the E1 transition is strongly suppressed between isospin-zero \((N = Z)\) nuclei. This mechanism is recently reviewed and studied for \(\alpha(d,\gamma)^{\text{6}}\text{Li}\) reaction by Baye and Tursunov.\(^{97}\) In the standard microscopic calculations with the long-wavelength approximation, the term proportional to \(Z_1/m_1-Z_2/m_2\) vanishes because of the standard choice of mass of nuclei as \(m_\eta = A_m N\) where \(A_i\) is the mass number of \(i\)-th nucleus and \(m_N\) is the nucleon mass. We have strongly suppressed but non-zero contribution above because of the use of the physical masses for \(\alpha\) and \(^{12}\)C. The small but non-vanishing E1 transition for the \(N = Z\) cases has intensively been studied in the microscopic calculations and can be accounted by two effects: The one is the second order term of the E1 multipole operator in the long-wavelength approximation,\(^{98}\) and the other is due to the mixture of the small \(I = 1\) configuration in the actual nuclei.\(^{99}\) In the present approach, the first one may be difficult to incorporate in the point-like particles while the second one could be introduced from a contribution at high energy: At \(T \simeq 5\) MeV and 8.5 MeV above the \(\alpha-^{12}\)C breakup threshold, \(p-\)\(^{15}\)N and \(n-\)\(^{15}\)O breakup channels, respectively, are open, and \(I = 1\) resonant states of \(^{16}\)O start emerging (along with the \(I = 1\) isobars, \(^{16}\)N, \(^{16}\)O, and \(\text{^{16}F}\)). We might have introduced the \(p-\)\(^{15}\)N and \(n-\)\(^{15}\)O fields as relevant degrees of freedom in the theory. The \(p-\)\(^{15}\)N and \(n-\)\(^{15}\)O fields, then, appear in the intermediate states, as \(p-\)\(^{15}\)N or \(n-\)\(^{15}\)O propagation, in the loop diagrams (d), (e), (f) in Fig. 3 instead of the \(\alpha-\)\(^{12}\)C propagation. One may introduce a mixture of the isospin \(I = 0\) and \(I = 1\) states in the \(p-\)\(^{15}\)N or \(n-\)\(^{15}\)O propagation, and the strong E1 suppression is circumvented in the loops. (The contribution from the \(p-\)\(^{15}\)N and \(n-\)\(^{15}\)O channels for the \(\text{^{12}C} (\gamma,\alpha)^{\text{16}}\text{O}\) reaction has already been studied in the microscopic approach\(^{100}\).) In our work, however, the \(p-\)\(^{14}\)N and \(n-\)\(^{15}\)O fields are regarded as irrelevant degrees of freedom at high energy and integrated out of the effective Lagrangian. Its effect, thus, is embedded in the coefficient of the contact interaction, the \(h^{(1)R}\) term, in the diagram (c) while the \(h^{(1)R}\) term is fitted to the experimental \(S_{E1}\) data in the next subsection.

5.3 Numerical results for the radiative \(\alpha\) capture reaction

We have five parameters in the radiative \(\alpha\) capture amplitudes to fit to the data; three parameters, \(r_1, P_1, Q_1\), are fitted to the phase shift data of the elastic scattering, and the other two parameters, \(h^{(1)R}\) and \(\eta^{(0)}\), are to the experimental \(S_{E1}\) data. The standard \(\chi^2\)-fit is performed by

\[\chi^2 = \sum_{\text{data}}\frac{(\text{data } - \text{fit})^2}{\text{uncertainty}}\]
employing a Markov chain Monte Carlo method for the parameter fitting. The phase shift data for \( l = 1 \) are taken from Tischhauser et al’s paper [51], and the experimental \( S_{11} \) data are from the literature summarized in Tables V and VII in Ref. [20]: Dyer and Barnes (1974) [101], Redder et al. (1987) [102], Ouellet et al. (1996) [103], Roters et al. (1999) [104], Gialanella et al. (2001) [105], Kunz et al. (2001) [106], Fey (2004) [107], Makii et al. (2009) [108], and Plag et al. (2012) [109].

As discussed in Sec. 3, we fitted the effective range parameters to the phase shift data for \( l = 1 \) at \( T_\alpha = 2.6 - 6.0 \) MeV and displayed the fitted values of \( r_1, P_1, \) and \( Q_1 \) in Table 2 where the number of the data is \( N = 273 \) and \( \chi^2/N = 0.74 \). The uncertainties of the fitted values stem from those of the experimental data. In Fig. 4, we plot a curve of the phase shift \( \delta_1 \) calculated by using the fitted effective range parameters as a function of \( T_\alpha \). We display the experimental data in the figure as well. One can see that the theory curve reproduces well the experimental data. In our work, we choose the results of \( S_{11} \) with \( \chi^2/N \approx 1.7 \) at \( r_C \leq 0.1 \) fm for our estimate of \( S_{11} \) at the Gamow-peak energy, \( T_G = 0.3 \) MeV, thus, we have

\[
S_{11} = 59 \pm 3 \text{ keV b,}
\]

where the small, about 5%, uncertainty stems from those of \( h^{(1)R} \) and \( y^{(0)} \) as well as that of the \( r_C \) dependence of \( S_{11} \) within \( \chi^2/N \leq 1.7 \). The previous estimates of the \( S_{11} \) factor at \( T_G \) are well summarized in Table IV in Ref. [20].

In Fig. 5, we plot a curve of \( S_{11} \) calculated by using the fitted parameters, \( h^{(1)R} = -0.0695(11) \times 10^4 \) MeV \(^4\) and \( y^{(0)} = 0.495(18) \) MeV \(^{1/2}\) with \( r_C = 0.1 \) fm (where \( \chi^2/N = 1.715 \)). We display the experimental data in the figure as well. One can see that the theory curve reproduces well the experimental data.

We fit the parameters, \( h^{(1)R} \) and \( y^{(0)} \), to the experimental data of \( S_{11} \) at the energy range, \( T = 0.9 - 3.0 \) MeV using some values of the cutoff \( r_C \) in the range, \( r_C = 0.01 - 0.35 \) fm, in the \( r \) integral in \( \chi^{(\text{fit})} \) in Eq. (30). The number of the data is \( N = 151 \). We find a significant cut-off dependence of the couplings, \( h^{(1)R} \) and \( y^{(0)} \), as well as the \( S_{11} \) factor at \( T_G \) when varying the short range cutoff, \( r_C = 0.01 - 0.35 \) fm; as the values of \( r_C \) become larger, \( \chi^2/N \) become larger while the \( S_{11} \) values at \( T_G \) become smaller. (See Table 1 in Ref. [16].)

In Fig. 5, we plot a curve of \( S_{11} \) calculated by using the fitted parameters, \( h^{(1)R} = -0.0695(11) \times 10^4 \) MeV \(^4\) and \( y^{(0)} = 0.495(18) \) MeV \(^{1/2}\) with \( r_C = 0.1 \) fm (where \( \chi^2/N = 1.715 \)). We display the experimental data in the figure as well. One can see that the theory curve reproduces well the experimental data.

In this section, we study the \( \beta \) delayed \( \alpha \) emission from \(^{16}\text{N}\) by employing an EFT. For the \( R \)-matrix analysis this is an important input to estimate the \( S_{11} \)-factor at \( T_G \) while for the EFT approach this is not the case; we will discuss that though the experimental data of the \( \beta \) delayed \( \alpha \) emission are well described in the EFT approach, it is notably different from those in the \( R \)-matrix analysis. In the following subsections, the formalism of \( \beta \)-decay and \( \beta \) delayed \( \alpha \) emission from \(^{16}\text{N}\) is first discussed, and the decay amplitudes up to NNLO are derived from the effective Lagrangian. After fitting parameters to each of existing two data sets for the \( \alpha \) energy distributions, we discuss numerical results we obtained.
6.1 β-decay and β delayed α emission from 16N

14N and 15N are stable nuclei while 16N is radioactive whose half-lifetime is 7.13 ± 0.02 sec decaying through the Gamow-Teller transition for β−decay [113,114,115]. It shows a first-forbidden character of the β−decay, and the ground state of 16N is identified as Jπ = 2+. Branching ratios of the decaying channels to the 01+, 02+, 21+, 22+ states of 16O and the α12C breakup channel are experimentally known as b = 0.28, 10−4, 0.66, 0.01, 0.05, 10−5, respectively. p- and f-waves are dominant for the α12C breakup channel, and its Q value is Qm = 3.257 MeV. Recently, the branching ratios are updated by experiment as b_{b,11} = (5.02 ± 0.10) × 10−2 for the bound 1− state and b_{b,α} = (1.59 ± 0.06) × 10−5 for the β delayed α emission [116].

Yields at the energy bins for the α kinetic energy, Tα, for the β delayed α emission from 16N may be obtained as

\[ n(T_\alpha) = C C_\alpha^2 p I(T_\alpha) \left[ W_1(\eta) \left| \tilde{A}_1 \right|^2 + \frac{28}{75} W_3(\eta) \left| \tilde{A}_3 \right|^2 \right], \]

(35)

where we have defined n(T_\alpha) as a dimensionless quantity, and C is an overall constant, C (MeV−6), which is fitted to the experimental data later in addition. A phase space integral I(T_\alpha) is given as

\[ I(T_\alpha) = \int_{p_{c,\max}} dp dE F(Z, E_\alpha), \]

(36)

where F(Z, E_\alpha) is Fermi function, and

\[ E_\alpha = Qm - (Ee - mc) - T, \]

(37)

\[ p_{c,\max} = \sqrt{(Qm - mc - T)^2 - m^2_e}, \]

(38)

with T = \frac{4}{3}T_\alpha = p^2/(2p), and E_\alpha (mc) is the electron energy (mass). The decay amplitudes, \tilde{A}_1 and \tilde{A}_3, to the final α12C state for the l = 1 and l = 3 channels, respectively, are calculated from the diagrams depicted in Fig.6 and we have

\[ \tilde{A}_1 = C_a^{(l=1)} + D_a^{(l=1)} \frac{p^2}{\mu^2} + \frac{C_b^{(l=1)}}{C_a^{(l=1)}} K_1(p) - 2\kappa H_1(p), \]

(39)

\[ \tilde{A}_3 = \left( C_a^{(l=3)} + \frac{C_b^{(l=3)}}{C_a^{(l=3)}} \right) \frac{1}{K_3(p) - 2\kappa H_3(p)}, \]

(40)

where we have introduced six parameters in those amplitudes; four of them, C_a^{(l=1)}, C_b^{(l=1)}, C_a^{(l=3)}, and C_b^{(l=3)} are coefficients of the contact vertices for (a) non-pole and (b) pole diagrams for the l = 1 and l = 3 channels at LO. We also introduce two coefficients, D_a^{(l=1)} and D_b^{(l=1)}, for the vertex corrections for the diagrams (a) and (b) for the l = 1 channel at NNLO. Thus, we have seven additional parameters, including the overall constant C, appearing in n(T_\alpha) while the effective range parameters appearing in the functions K_1(p) in Eq. (39) and K_3(p) in Eq. (40) have already been obtained in Tables 2 and 3. We use those values for the effective range parameters in the following.

In Fig. 7 a diagram for the β-decay from 16N is depicted. Here we assume the perturbation expansion for the vertex correction and include the leading ones, C_b^{(l=1)} and C_b^{(l=3)} only. Thus, the decay rates to the final 1− and 3− states of 16O are obtained as

\[ \Gamma(1−) = \frac{8}{3} \frac{G_F^2}{(2\pi)^3} Z_1 I_1 C_b^{(l=1)2}, \]

(41)

\[ \Gamma(3−) = \frac{28}{3} \frac{G_F^2}{(2\pi)^3} Z_1 I_1 C_b^{(l=3)2}, \]

(42)

where I_1 is the same integral as that in Eq. (30) while

\[ E_\alpha = m_N - m_o^* - E_\pi, \]

(43)

\[ p_{c,\max} = \sqrt{(m_N - m_o^*)^2 - m^2_e}, \]

(44)

where m_N is the mass of 16N in the ground state, and m_o^* are the masses of the excited 1− and 3− states of 16O. In addition, Z_1 and Z_3 are wave-function normalization factors for the 1− and 3− states of 16O, and we have

\[ Z_3^{-1} = \mu \left( r_3 + P_3 \gamma_3 + 6Q_1 \gamma_1 \right) - 4\mu \kappa \left( H(\eta_3) + \frac{\kappa}{2\gamma_3} \left( \frac{\kappa - \gamma_3}{2\kappa - \gamma_3} \right) \right), \]

(45)

\[ Z_3^{-1} = \mu \left( r_3 + P_3 \gamma_3 + 6Q_3 \gamma_3 + 8R_3 \gamma_3 \right) - 4\mu \kappa \left( H(\eta_3) + \frac{\kappa}{2\gamma_3} \left( \frac{\kappa - \gamma_3}{2\kappa - \gamma_3} \right) \right), \]

(46)

where H(\eta) \gamma_3 \kappa \rho \gamma_3 are the poly-gamma function. In the following, we fix the two parameters, C_b^{(l=1)} and C_b^{(l=3)}, by using

Fig. 6. Diagrams for β delayed α emission from 16N. A thick dashed line denotes the 16N field in the initial state, and a filled box does a weak contact vertex at which the nuclear current and the lepton current interact. See the captions of Figs. 1 and 2 as well.

Fig. 7. Diagram for β-decay from 16N. See the caption of Fig. 6 as well.
the branching ratios of the $\beta$-decay, and fit the remaining five parameters to the $\beta$ delayed $\alpha$ emission data.

### 6.2 Numerical results for the $\beta$ delayed $\alpha$ emission from $^{16}$N

Using the experimental data for the branching ratios of the $\beta$-decay from $^{16}$N, we obtain

$$ C_b^{(l=1)} = 11.4 \text{ MeV}, \quad C_b^{(l=3)} = 7.13 \times 10^5 \text{ MeV}^3. \quad (47) \quad (48) $$

Thus, five unfixed parameters, $C$, $C_a^{(l=1)}$, $D_a^{(l=1)}$, $D_b^{(l=1)}$, and $C_b^{(l=3)}$ remain in $\alpha(T_\alpha)$, and we fit them to the experimental data. Two sets of the experimental data for the $\beta$ delayed $\alpha$ emission from $^{16}$N are available; one is from a paper by Azuma et al. [117], and the other is from that by Tang et al. [110].

In Table 5, fitted values and errors of the parameters are displayed. We include the numbers of the data ($N$) and values of $\chi^2/N$ in the table as well. In Figs. 8 and 9, curves of the $\beta$ delayed $\alpha$ emission from $^{16}$N are plotted as a function of the $\alpha$ energy. A blue curve is our fitted result, and the experimental data of Azuma et al. are included in the figure as well.

![Fig. 8. $\beta$ delayed $\alpha$ emission from $^{16}$N as a function of the $\alpha$ energy. A blue curve is our fitted result, and the experimental data of Azuma et al. are included in the figure as well.](image)

![Fig. 9. $\beta$ delayed $\alpha$ emission from $^{16}$N as a function of the $\alpha$ energy. A blue curve is our fitted result, and the experimental data of Tang et al. are included in the figure as well.](image)

7 Results and discussion

In the present work, we discuss the application of EFTs to nuclear reactions at low energies. We study the elastic scattering, the radiative $\alpha$ capture reaction, and the $\beta$ delayed $\alpha$ emission from $^{16}$N for the $\alpha$-$^{12}$C systems by constructing an EFT. This is a typical reaction for an application of EFTs because one can have a separation scale
and many experimental data are available. EFTs, thus, may provide us a new theoretical method, as an alternative of $R$-matrix or potential model analysis, for the study of nuclear-astrophysics where one needs to extrapolate a reaction rate down to a low energy by fitting some parameters of theory to experimental data measured at higher energies.

In the study of the elastic scattering, we discuss a modification of the counting rules for an expansion around the unitary limit due to the large suppression factor from the Coulomb interaction at low energies; we include up to $Q^6$ order terms for $l = 0, 1, 2$ and up to $Q^8$ order terms for $l = 3$ in the effective range expansion. We find that, after fixing the first effective range term $a_1$ by using the binding energies of $^{16}\text{O}$ and fitting the remaining effective range parameters to the phase shift data, the broad resonances, $1^+_2$ and $3^+_2$ states of $^{16}\text{O}$ can be described by the effective range parameters, but the narrow resonances, $0^+_1$ and $2^+_2$ states of $^{16}\text{O}$ cannot. As has been studied in the previous works \cite{13}, the expansion series converges well at $T_G$ while it doesn’t at the energies where the experimental data are available (though the data are well reproduced by means of the effective range expansion). In the studies for the radiative $\alpha$ capture on $^{12}\text{C}$ and the $\beta$ delayed $\alpha$ emission from $^{16}\text{N}$, thus, we assume that the dressed $^{16}\text{O}$ propagators for $l = 1$ and $l = 3$ are non-perturbative quantities, and the capture and decay amplitudes are expanded around them.

In the study of the radiative $\alpha$ capture reaction, we calculate the radiative $\alpha$ capture amplitudes up to NLO for the initial $p$-wave $\alpha$-$^{12}\text{C}$ system and confirm the suppression of the $E1$ transition between the iso-singlet ($I = 0$) nuclei. We discuss that a mixture of the $I = 1$ contribution could be introduced in the present approach by including the $p-^{15}\text{N}$ and $n-^{15}\text{O}$ channels, which are open at $T \approx 5$ MeV and $8.5$ MeV above the $\alpha$-$^{12}\text{C}$ breakup threshold. In the present work, those states are regarded as irrelevant degrees of freedom at high energy and integrated out of the Lagrangian, and its effect is embedded in the contact term, the $h^{(1)}R$ term. After fitting the two parameters, $h^{(1)}R$ and $y^{(0)}$, to the $S_{E1}$ data, we make an estimate of the $S_{E1}$-factor at $T_G$ by employing an EFT.

In the study of the $\beta$ delayed $\alpha$ emission from $^{16}\text{N}$, we calculate the $\alpha$ decay amplitudes up to NNLO. We confirm that the primary peak of the data is accounted by the broad resonant $1^+_2$ state of $^{16}\text{O}$ while we find that the secondary peak is obtained from an interference between a non-pole amplitude and a pole amplitude in the present approach. Though the study of the $\beta$ delayed $\alpha$ emission from $^{16}\text{N}$ is crucial in the $R$-matrix analysis, now one may see that, except for sharing the dressed $^{16}\text{O}$ propagators in the two reaction amplitudes, the coefficients of vertex functions for the radiative capture and the $\beta$ delayed $\alpha$ emission are independently fixed by using the corresponding experimental data because the nuclear current for the radiative capture reaction is coupled to a vector current (or a minimally coupled photon) while that for the $\beta$ delayed $\alpha$ emission reaction is to an axial-vector current. Thus, the study of the $\beta$ delayed $\alpha$ emission cannot be a constraint on an estimate of the $S_{E1}$-factor in our approach. A remarkable difference between our approach and the $R$-matrix approach can be seen in the concept for the non-pole contribution for $\beta$ delayed $\alpha$ emission from $^{16}\text{N}$ (whose diagram is displayed in the diagram (a) in Fig. 9). In the present approach, the non-pole contribution is systematically derived; the contact vertex functions are obtained from the effective Lagrangian in Eq. (1), and the reaction amplitudes in Eqs. (39) and (40) are calculated straightforwardly. In the $R$-matrix or $K$-matrix approach, there is no non-pole contribution while one necessarily introduces the so-called “background levels” \cite{71,72,119}. Though the coefficients of the background levels are fitted to the data as free parameters, they play the same role in reproducing the interference pattern for the secondary peak. Thus, our result may indicate that the non-pole contributions account for an origin of the background levels in the $R$-matrix or $K$-matrix calculations. We also find that the values of the coefficients in the $\beta$ delayed $\alpha$ emission amplitudes are quite different when fitting the coefficients to each of the two experimental data sets. Thus, to see a convergence of the experimental data would be important by performing new experiments in the future.

Though we have reported a first result of the radiative $\alpha$ capture on $^{12}\text{C}$ employing an EFT, some issues remain to be explained: In the study of the elastic scattering, we have introduced a modification of the counting rules from the observation of anomaly of the expansion series compared to the experimental data while we have not investigated how the anomalous terms come out. It could appear because of our assumption of the point-like particles; a real nucleus has a finite size, and the short range contributions due to the assumption may need to be subtracted by introducing counter terms. In the study of the radiative $\alpha$ capture reaction, we reproduce the suppression of the $E1$ transition while the non-vanishing contribution, the $h^{(1)}R$ term, is merely fitted to the experimental data. Thus it would be interesting to study a mixture of the $I = 1$ state for the $E1$ transition by including the $p-^{15}\text{N}$ and $n-^{15}\text{O}$ channels in the framework of EFT. It is also important to include higher order terms at NNLO in order to estimate a theoretical uncertainty of the $S_{E1}$-factor at $T_G$.

As we have discussed above, an application of EFTs for nuclear reactions would be possible, provided that one can choose a clear separation scale for an observable of a reaction and the experimental data are available to fix coefficients appearing in an effective Lagrangian. Thus, those nuclear reactions at low energies, which are important in nuclear-astrophysics, are possible candidates for the application of EFTs, especially when the accuracy and the error estimate of a reaction are important. It would also be interesting to study the $E2$ transition and the cascade transitions of the radiative $\alpha$ capture on $^{12}\text{C}$ at $T_G$ by employing an EFT. A study toward this direction is now underway.
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Appendix

The elastic scattering amplitudes are calculated from the renormalized dressed three-point vertices and the renormalized dressed composite $^{16}$O propagators as

$$iA_l = -i\Gamma^{(l)}(k')D^{(l)}(T)\Gamma^{(l)}(k),$$

(49)

where $k' = k$ and $T = k^2/(2\mu)$, and we have suppressed the indices from the Cartesian tensors.

We have the renormalized dressed three-point vertices for the initial and final Coulomb interaction for $l = 0, 1, 2, 3$ as

$$\Gamma^{(l=0)}(k) = y(0)e^{ix_0}C_\eta,$$

(50)

$$\Gamma^{(l=1)}(k) = \frac{y(1)}{\mu}k_\eta e^{ix_1}\sqrt{1 - \eta^2}C_\eta,$$

(51)

$$\Gamma^{(l=2)}(k) = \frac{y(2)}{\mu^2}15\epsilon^{i\sigma_3}C_2 \left(k_i k_j - \frac{1}{3}\delta_{ij}k^2\right),$$

(52)

$$\Gamma^{(l=3)}(k) = \frac{y(3)}{\mu^3}105\epsilon^{i\sigma_3}C_3$$

$$\times \left[k_i k_j k_k - \frac{1}{5}k^2(\delta_{ij}k_k + \delta_{jk}k_j + \delta_{kj}k_i)\right],$$

(53)

where

$$C_2 = \frac{1}{30}C_\eta \sqrt{(1 + \eta^2)(4 + \eta^2)},$$

(54)

$$C_3 = \frac{1}{630}C_\eta \sqrt{(1 + \eta^2)(4 + \eta^2)(9 + \eta^2)}.$$  

(55)

We have the renormalized dressed composite $^{16}$O propagators for $l = 0, 1, 2, 3$ as

$$D_{(l=0)}^{(l=0)}(T) = \frac{2\pi}{\mu y(0)} - \frac{1}{K_0(k) + 2\kappa H_0(k)},$$

(56)

$$D_{(l=1)}^{(l=1)}(T) = \frac{y(1)}{\mu} - \frac{1}{K_1(k) + 2\kappa H_1(k)},$$

(57)

$$D_{(l=2)}^{(l=2)}(T) = \frac{3}{2}P_{(l=2)}^{(l=2)} - \frac{1}{y(2)} - \frac{1}{K_2(k) + 2\kappa H_2(k)},$$

(58)

$$D_{(l=3)}^{(l=3)}(T) = \frac{5}{2}P_{(l=3)}^{(l=3)} - \frac{1}{y(3)} - \frac{1}{K_3(k) + 2\kappa H_3(k)},$$

(59)

where $P_{(l=1)}^{(l=1)}$, $P_{(l=2)}^{(l=2)}$, and $P_{(l=3)}^{(l=3)}$ are the projection operators which satisfy the relation, $P = PP$, and we have

$$D_{(l=1)}^{(l=1)}(T) = \delta_{l_1},$$

(60)

$$D_{(l=2)}^{(l=2)}(T) = \frac{1}{2} \left[\delta_{l_2}\delta_{l_3} + \delta_{l_3}\delta_{l_2} - \frac{2}{3}\delta_{l_2}\delta_{l_3}\right],$$

(61)

$$D_{(l=3)}^{(l=3)}(T) = \frac{1}{6} \left[\delta_{l_2}\delta_{l_3}\delta_{l_4} + 5\text{ terms} - \frac{2}{5}(\delta_{l_2}\delta_{l_4}\delta_{l_5} + 5\text{ terms})\right].$$

(62)

In addition, the couplings, $y(l)$, are redundant when one fixes them by using the effective range parameters, conventionally one may choose them as

$$y(0) = \sqrt{\frac{2\pi}{\mu}}, \quad y(1) = \sqrt{\frac{6\pi\mu}{\mu}},$$

(63)

$$y(2) = \sqrt{\frac{10\pi\mu^2}{\mu}}.$$  

References

1. S. Weinberg, Physica A 96, (1979) 327.
2. K. Ohta, Phys. Rev. C 46, (1992) 2519.
3. K. Ohta, Phys. Rev. C 47, (1993) 2344.
4. V. Bernard, N. Kaiser, T. S. H. Lee, and U.-G. Meißner, Phys. Rep. 246, (1994) 315.
5. S. Ando and D.-P. Min, Phys. Lett. B 417, (1998) 177.
6. S. Ando, F. Myhrer, and K. Kubodera, Phys. Rev. C 63, (2001) 015203.
7. P.F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52, (2002) 339.
8. E. Braaten and H.-W. Hammer, Phys. Rept. 383, (2004) 259.
9. U.-G. Meißner, Phys. Scripta 91, (2016) 033005.
10. H.-W. Hammer, C. Ji, D.R. Phillips, J. Phys. G 44, (2017) 103002.
11. H.-W. Hammer, S. Konig, and U. van Kolck, Rev. Mod. Phys. 92, (2020) 025004.
12. S.-I. Ando, Eur. Phys. J. A 52, (2016) 130.
13. S.-I. Ando, Phys. Rev. C 97, (2018) 014604.
14. S.-I. Ando, J. Korean Phys. Soc. 73, (2018) 1452.
15. H.-E. Yoon and S.-I. Ando, J. Korean Phys. Soc. 75, (2019) 202.
16. S.-I. Ando, Phys. Rev. C 100, (2019) 015807.
17. C. Ecker and U.-G. Meißner, Comments Nucl. Part. Phys. 21, (1995) 347.
18. A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, (1958) 257.
19. L. R. Buchmann and C. A. Barnes, Nucl. Phys. A 777, (2006) 254.
20. R. J. deBoer et al., Rev. Mod. Phys. 89, (2017) 035007.
21. J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) 158, (1984) 142.
22. J. Gasser and H. Leutwyler, Nucl. Phys. B 250, (1985) 465.
23. J. F. Donoghue, E. Golowich, and B. R. Holstein, Dynamics of the Standard Model (Cambridge University Press, 1992).
24. S. Weinberg, Phys. Lett. B 251, (1990) 288.
107. M. Fey, Ph.D. thesis (Universitat Stuttgart) (2004).
108. H. Makii et al., Phys. Rev. C 80, (2009) 065802.
109. R. Plag et al., Phys. Rev. C 86, (2012) 015805.
110. X. D. Tang et al., Phys. Rev. C 81, (2010) 045809.
111. D. Schurmann et al., Phys. Lett. B 711, (2012) 35.
112. N. Oulebsir et al., Phys. Rev. C 85, (2012) 035804.
113. D. E. Alburger, Phys. Rev. 111, (1958) 1586.
114. J. K. Bienlein and E. Kalsch, Nucl. Phys. 50, (1964) 202.
115. D. R. Tilley, H. R. Weller, and C. M. Cheves, Nucl. Phys. A 564, (1993) 1.
116. O. S. Kirsebom et al., Phys. Rev. Lett. 121, (2018) 142701.
117. R. E. Azuma et al., Phys. Rev. C 50, (1994) 1194.
118. S. Sanfilippo et al., AIP Conf. Proc. 1645, (2015) 387.
119. L. Buchmann et al., Phys. Rev. Lett. 70, (1993) 726.
