Free Fourier Multipliers associated with the first Segment

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Abstract

We study Fourier multipliers on free group \( F_\infty \) associated with the first segment of the reduced words, and prove that they are completely bounded on the noncommutative \( L^p \) spaces \( L^p(\hat{F}_\infty) \) iff their restriction on \( L^p(\hat{F}_1) = L^p(T) \) are completely bounded. As a consequence, we get an analogue of the classical Mikhlin multiplier theorem for this class of Fourier multipliers on free groups.

Notation

\( F_n \) : free group of rank \( n \in \mathbb{N} \cup \{ \infty \} \) with given generators \( g_k \)’s.
\( \lambda_g \) : the left translation operator on \( \ell_2(F_n) \) sending \( \delta_h \) to \( \delta_{gh} \).
\( \mathcal{L}(F_n) \) : the group von Neumann algebra is the weak *-closure of the space of linear combinations of \( \lambda_g \) in \( B(\ell_2(F_n)) \).
\( \tau \) : the canonical trace on \( \mathcal{L}(F_n) \) is the linear functional such that \( \tau(\lambda_e) = 1 \) and \( \tau(\lambda_h) = 0 \) if \( h \neq e \).

Set \( L^\infty(\hat{F}_n) = \mathcal{L}(F_n) \) by convention.
\( L^p(\hat{F}_n), 1 \leq p < \infty \) : the noncommutative \( L^p \) space is the completion of \( \mathcal{L}(F_n) \) with respect to the norm \( \| x \|_{L^p(\hat{F}_n)} = (\tau|x|^p)^{\frac{1}{p}}. \)

For a reduced word \( h = g_{i_1}^{k_1}g_{i_2}^{k_2} \cdots g_{i_m}^{k_m} \), we denote by \( \| h \| \) the block length \( m \). Let \( L_0 = \{ e \} \). Denote by \( L_{k\pm} \) the set of all reduced words \( h \) that starts with a power of \( g_k \), i.e. \( h = g_{i_1}^{k_1}g_{i_2}^{k_2} \cdots g_{i_m}^{k_m} \) with \( i_1 = k \).

Let \( L_{k\pm} \) be the projection on to \( \lambda(L_k) \) in \( L_2(\hat{F}_2) \). Let \( e_{kj} \) be the canonical basis of \( B(\ell_2) \). Let \( L^p(\ell_2^\infty), 2 \leq p \leq \infty \) be the space of operator valued sequences \( x = (x_k)_k \) such that

\[ \| x \|_{L^p(\ell_2^\infty)} = \max\{ \| (\sum_k |x_k|^2)^{\frac{1}{2}} \|_p, \| (\sum_k |x_k|^2)^{\frac{3}{2}} \|_p \} < \infty. \]
1 Introduction

The Fourier multiplier operators have been a central object in analysis. Their boundedness on $L^p$-spaces of $\mathbb{R}^n$ has been extensively studied. The so-called Mikhlin multiplier theorem says that, if $m$ is a smooth function on $\mathbb{R}^n$ such that

$$\sup_{0 \leq |j| \leq \frac{n}{2} + 1} |\xi^j \nabla^j m(\xi)| < C$$

for all $\xi \neq 0$, then the multiplier operator

$$T_m : e^{i\xi x} \mapsto m(\xi)e^{i\xi x}$$

extends to a bounded operator on $L^p(\mathbb{R}^n)$ (resp. $L^p(T^n)$) for all $1 < p < \infty$. This result was originally proved by S. Mikhlin ([M56], [M65]) and is now a fundamental theorem in the Calderón-Zygmund-Stein Singular integral theory.

Murray and von Neumann's work ([MvN36]) demonstrates von Neumann algebras as a natural framework to do noncommutative analysis. The elements in a von Neumann algebra $\mathcal{M}$ can be "integrated over" the equipped trace $\tau$ and "measured" by the associated $L^p$-norms. For a (nonabelian) discrete group $\Gamma$, the von Neumann algebra is the closure of the linear span of left regular representation $\lambda_g$'s w.r.t. a weak operator topology. The trace $\tau$ is simply defined as

$$\tau x = c_x,$$

for $x = \sum_g c_g \lambda_g$. The associated $L^p$ norm is defined as

$$\|x\|_p = (\tau|x|^p)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. When $p = \infty$, the $L^p$ space is set to be the von Neumann algebra itself. When $\Gamma = \mathbb{Z}$, the obtained $L^p$ space is the $L^p$ space on the unit circle $\mathbb{T} = \hat{\mathbb{Z}}$.

The theory of noncommutative $L^p$-spaces was laid out in the work of Segal ([Seg53]) and Dixmier ([Dix53]). Fourier multipliers on noncommutative $L^p$-spaces have been used as fundamental tools in operator algebras theory, noncommutative geometry, and mathematical physics, and have grown up to a new studying-object in functional analysis with its own interest. Fourier multipliers on nonabelian groups $\Gamma$ are linear maps $M$ on the left regular representation of $\Gamma$ such that

$$M : \lambda_g \mapsto m(g)\lambda_g$$

with $m$ a scalar-valued bounded function on $\Gamma$. The boundedness of $M$ is tested on the associated noncommutative $L^p$ spaces.

The study of Fourier multipliers on free groups has a long history. Please refer to the works of Bözejko, Figá-Talamanca/Picardello, Haagerup, Pytlik-Szwarc, Junge/Le Merdy/Xu etc. These works usually rely on the theory of positive definite functions and restrict the study on the radial multipliers. In
[MR17], Mei and Ricard studied an important non-radial multiplier, the so-called free Hilbert transform, and answered a question of P. Biane and G. Pisier about its \(L^p\) boundedness on free groups.

The key of Mei-Ricard’s argument is a free analogue of the classical Cotlar’s formula. This type of formula fits Hilbert transform type multipliers but does not work for general Fourier multipliers. This raises the question whether Mei-Ricard’s result is a lucky case or there is a more general rule on the \(L^p\)-boundedness of Fourier multipliers. In this note, we study the class of Fourier multipliers on the free group associated with the first segment of reduced words, and give a general rule for their complete-\(L^p\)-boundedness.

2 A Translation Group

Given \(z = (z_1, \ldots, z_n)\), a sequence of complex numbers with modular 1, we use \(T_z, T_z^0\) to denote the linear maps on \(L^2(\hat{\mathbb{F}}_n)\) such that

\[
T_z(\lambda e) = \lambda e = T_{z^0}(\lambda e)
\]

and

\[
T_z(\lambda h) = z^{k_1} \lambda h, \quad T_z^0(\lambda h) = z^{k_m} \lambda h
\]

for

\[
h = g^{k_1}_{i_1} g^{k_2}_{i_2} \cdots g^{k_m}_{i_m}.
\]

Note we have that

\[
[T_z(\lambda h)]^* = T_z^0 \lambda_h^{-1}.
\]

We will prove that \(T_z\) is a uniformly bounded group of operators on \(L^p(\hat{\mathbb{F}}_n)\) for all \(1 < p < \infty\). Therefore, an analogue of the classical Mikhlin’s multiplier theorem follows by Coifman/Weiss/Zygmund’s transference principle.

Denote by \(\pi_z\) the \(*\)-homeomorphism on \(L(\mathbb{F}_n)\) that sends \(\lambda g_i\) to \(z_i \lambda g_i\). Let \(P_1\) be the projection onto the subspace of \(L^2(\hat{\mathbb{F}}_n)\) spanned by reduced words with block length \(\leq 1\). We see that

\[
P_1 \pi_z = P_1 T_z = P_1 T_z^0 = P_1 T_z T_z^0,
\]

so

\[
\|P_1 T_z\| = \|P_1 T_z^0 T_z\| = \|P_1\| \leq 3.
\]

(1)

The following Lemma is from [MR17], Corollary 3.10. One can check the proofs there and find the upper bound \(p^2\) for \(p > 2\).

**Lemma 1.** For \(x \in L^p(\hat{\mathbb{F}}_n)\), we have

\[
(\sqrt{2}p^2)^{-1} \|x\|_p \leq \max\{\sum_k e_{k1} \otimes L_k x\|_p, \sum_k e_{lk1} \otimes L_k x\|_p\} \leq p^2 \|x\|_p,
\]

for all \(2 < p < \infty\).
Lemma 2. For $g, h \in \mathbb{F}_\infty, g \in \mathcal{L}_{kz}, h^{-1} \in \mathcal{L}_{jz}, k, j \geq 0$ we have that

(i) if the block length $\|gh\| \leq 1$ and $k = j$

$$T_z(\lambda_g)T_z^\circ(\lambda_h) = T_z(\lambda_g\lambda_h) = T_z^\circ(\lambda_g\lambda_h);$$

(ii) otherwise,

$$T_z(\lambda_g)T_z^\circ(\lambda_h) = T_z(\lambda_g^T_z\lambda_h^T_z) + T_z^\circ(T_z(\lambda_g)\lambda_h) - T_z^\circ(T_z(\lambda_gh)).$$

Proof. In case (i), suppose $T_z(\lambda_g) = z_k^i\lambda_g$ and $T_z^\circ(\lambda_h) = z_k^i\lambda_h$, we have $T_z(\lambda_g\lambda_h) = T_z^\circ(\lambda_gh) = z_k^{i+j}\lambda_gh = T_z(\lambda_g)T_z^\circ(\lambda_h)$ since $gh$ has block length 1. We get (3).

In case (ii), we have either the identity

$$T_z(\lambda_g\lambda_h) = T_z(\lambda_g)\lambda_h$$

or

$$T_z^\circ(\lambda_g\lambda_h) = \lambda_gT_z^\circ(\lambda_h).$$

Assuming (5), we must have

$$T_z(\lambda_gT_z^\circ(\lambda_h)) = T_z(\lambda_g)T_z^\circ(\lambda_h),$$

because $T_z^\circ(\lambda_h)$ is merely a multiplication of $\lambda_h$ by a constant. We then get (4). Assuming (6), we have

$$T_z^\circ(T_z(\lambda_g)\lambda_h) = T_z(\lambda_g)T_z^\circ(\lambda_h),$$

because $T_z(\lambda_g)$ is merely a multiplication of $\lambda_g$ by a constant. We get (4) again.

Theorem 3. For $1 < p < \infty$, we have

$$\|T_zx\|_{L^p} \simeq^p \|x\|_{L^p},$$

for any $x \in L^p(\hat{\mathbb{F}}_\infty)$.

Proof. Assume $\|T_z\|_{L^p(\hat{\mathbb{F}}_\infty) \rightarrow L^p(\hat{\mathbb{F}}_\infty)} \leq c_p$ for some $p = 2j, j \geq 1$. For $x = \sum_g c_g\lambda_g$, denote by $x_k = L_{kz}x$. Let $P_k$ be the projection onto the linear space corresponding to reduced words with block length smaller or equals to 1 in $L^2(\hat{\mathbb{F}}_n)$. Lemma 2 implies that

$$P_k^\perp[T_z(x)T_z^\circ(x^*)] = P_k^\perp[T_z(xT_z^\circ(x^*)) + T_z^\circ(T_z(x)x^*) - T_z^\circ T_z(xx^*)]$$

for $k \neq j$, and

$$P_k^\perp[T_z(x_k)T_z^\circ(x_k^*)] = P_k^\perp[T_z(x_kx_k^*)] = P_k^\perp[T_z^\circ(x_kx_k^*)]$$

(8)
Therefore,
\[
T_z(x)T_z^o(x^*) = [T_z(xT_z^o(x^*)) + T_z^o(T_z(x)x^*) - T_z^oT_z(x)T_z(x^*)] - P_1 \sum_k [T_z(x_kT_z^o(x_k^*)) + T_z^o(T_z(x_k)x_k^*) - T_z^oT_z(x_k)x_k^*) - T_z(x_k)x_k^*)].
\]

Denote by \( y = \sum_k e_k \otimes x_k^* \). Note \( P_1 T_z = P_1 T_z^o \) and
\[
P_1 T_z(x_k x_k^*) = P_1 |T_z^o(x_k^*)|^2
\]
because of (8), we have
\[
P_1 T_z(|y - T_z^o y|^2) = P_1 \sum_k [T_z^oT_z(x_k x_k^*) + T_z(x_k x_k^*) - T_z(x_k T_z^o(x_k^*)) - T_z^o(T_z(x_k) x_k^*)].
\]

Therefore,
\[
T_z(x)T_z^o(x^*) = \left[ T_z(xT_z^o(x^*)) + T_z^o(T_z(x)x^*) - T_z^oT_z(x)T_z(x^*) \right] + P_1 T_z(|y - T_z^o y|^2).
\]

By Lemma 1, we have
\[
||y||_{L^{2p}} = ||T_z y||_{L^{2p}} \leq (2p)^2 ||x||_{L^{2p}}
\]
for \( p > 1 \). Taking \( L^p \) norms on both sides of (9) and applying (11) and Hölder’s inequality, we get that
\[
||T_z x||_{L^{2p}}^2 \leq 2c_p||x||_{L^{2p}}||T_z(x)||_{L^{2p}} + (c_p^2 + 192p^4)||x||_{L^{2p}}^2.
\]
Therefore,
\[
||T_z(x)||_{L^{2p}} \leq (c_p + \sqrt{2c_p^2 + 192p^4})||x||_{L^{2p}}.
\]
By induction, we have
\[
||T_z x||_{L^p} \leq 8p^2||x||_{L^p}
\]
for \( p = 2^n \). Applying the fact that \( T_z T_z = id, ||T_z x||_{L^2} \leq ||x||_{L^2} \), by interpolation and passing to the dual, we then get the desired result for all \( 1 < p < \infty \). \( \square \)

**Remark.** Theorem 3 fails for \( p = 1, \infty \), see the remark after Theorem 4.

### 3 Transference via the Translation Group

Given a bounded map \( m \) from \( \mathbb{Z} \) to \( \mathbb{C} \). Let \( M_m \) be the linear multiplier on \( L^2(\mathbb{R}_n) \) such that
\[
M_m(\lambda_h) = m(k_1)\lambda_h
\]
for \( h = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_n}^{k_n} \).
Theorem 4. For any $1 < p < \infty$, $M_m$ extends to a completely bounded linear operator on $L^p(\mathbb{F}_n)$ iff the restriction of $M_m$ on $\mathbb{F}_1$, denoted by $\hat{M}_m$, is completely bounded Fourier multiplier on $L^p(\mathbb{T})$. Moreover,

$$\|M_m\| \leq c_p \|\hat{M}_m\|$$

with $c_p$ the equivalence constant in Theorem 3.

Proof. By Theorem 3 we have

$$T_1: \lambda_h \mapsto e^{ikt_1}$$

for $h = \hat{G}_1 \hat{G}_2 \cdots \hat{G}_r$ is a uniformly bounded $c_0$-group of operators on $L^p$. The desired result follows by Coifman/Weiss/Zygmund’s transference principle. Assume $\hat{M}_m$ extends to a completely bounded Fourier multiplier on $L^p(\mathbb{T})$. By approximation, we may assume that $m$ has a finite support $[-N, N]$. Define the scalar valued function $\phi_N$ on the unit circle as $\phi_N = \sum_k m(k)e^{ik\theta}$. Then we have, for any $L^p(\mathbb{F}_n)$-valued function $F$,

$$\|F \ast \phi_N\|_{L^p(\mathbb{T}, L^p(\mathbb{F}_n))} = \|M_m(F)\|_{L^p(\mathbb{T}, L^p(\mathbb{F}_n))} \leq \|\hat{M}_m\| \|F\|_{L^p(\mathbb{T}, L^p(\mathbb{F}_n))}.$$ 

For any $x \in L^p(\mathbb{F}_n)$ with norm 1, we have

$$M_m(x) = \frac{1}{2\pi} \int_0^{2\pi} T_{-t} x \phi_N(t) dt = \frac{1}{2\pi} \int_0^{2\pi} T_s T_{-t} x \phi_N(t) dt.$$ 

Let $F(t) = T_t x$. Then $\|F\|_{L^p([0, 2\pi], L^p(\mathbb{F}_n))} \leq c_p$ and

$$\|M_m(x)\|_{L^p(\mathbb{F}_n)} \leq c_p \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{2\pi} \int_0^{2\pi} T_s T_{-t} x \phi_N(t) dt \right| ds$$

$$= c_p \|F \ast \phi_N\|_{L^p(\mathbb{F}_n)}^p \leq c_p \|\hat{M}_m\| \|F\|_{L^p(\mathbb{T}, L^p(\mathbb{F}_n))} \leq c_p \|\hat{M}_m\|.$$ 

Remark. Theorem 4 fails for $p = 1, \infty$. Take $m(k) = \chi_{[-2, 2]}(k)$, which is the symbol of a c.b multiplier on $L^\infty(\mathbb{F}_1)$. Then the multiplier $M_m$ is the projection onto the set $\{s; |\lambda_k(s)| \leq 2\}$ of $L^\infty(\mathbb{F}_\infty)$, which is certainly unbounded. To see this, let

$$x = \sum_{-N < k < N} c_k (\lambda g_1 g_2)^k g_1$$

and note

$$M_m(x) = \sum_{0 \leq k < N} c_k (\lambda g_1 g_2)^k g_1.$$ 

Theorem 3 fails for $p = 1, \infty$ too, since Theorem 4 would follow from it.

Let $A_0 = \{0\}$, $A_k = [2^{k-1}, 2^k)$ for $k \in \mathbb{N}$ and $A_k = -A_{-k}$ for $k \in -\mathbb{N}$.
Corollary 5. Let $\chi_k$ be the characteristic function on $A_k$ for $k \in \mathbb{Z}$. Then
\[
\|x\|_{L^p} \simeq \|(M_{\chi_k}(x))_k\|_{L^p(\ell^2_{cr})}.
\] (10)

Proof. Let $m = \sum_{k \in \mathbb{Z}} \varepsilon_k \chi_k$ with $\varepsilon_k = \pm$. Then $M_m$ is a completely bounded multiplier on $L^p(\mathbb{T})$. The Khitchine inequality and Theorem above imply that
\[
\|(M_{\psi_k}(x))_k\|_{L^p(\ell^2_{cr})} \lesssim \|x\|_{L^p}.
\] (11)
The other direction of the inequality follows from the duality between $L^p$ spaces and the identity
\[
\langle x, y \rangle = \sum_k \langle M_{\chi_k}(x), M_{\chi_k}(y) \rangle.
\] (12)

Corollary 6. Suppose $M$ is a Mikhlin multiplier in the sense that
\[
\sup_{k \in \mathbb{Z}} \{|m(k)|, |m(k) - m(k-1)|\} < C.
\]
Then $M$ extends to a completely bounded linear operator on $L^p(\mathbb{T}^n)$ for all $1 < p < \infty$.

Proof. This follows from Theorem 4 and the classical Mikhlin multiplier theorem.

Corollary 7. Let $\psi$ be a $C^2$ function supported on $[\frac{1}{2}, \frac{3}{2}]$ and $\psi(t) = 1$ for $t \in [\frac{2}{3}, \frac{4}{3}]$. Let $\psi_k(t) = \psi(\frac{t}{2^k})$. Then
\[
\|x\|_{L^p} \simeq \|(M_{\psi_k}(x))_k\|_{L^p(\ell^2_{cr})}.
\] (13)

Proof. Let $m_1 = \sum_{k \text{ odd}} \varepsilon_k \psi_k$ and $m_2 = \sum_{k \text{ even}} \varepsilon_k \psi_k$ with $\varepsilon_k = \pm$. The Khitchine inequality and Corollary 6 imply that
\[
\|(M_{\psi_k}(x))_k\|_{L^p(\ell^2_{cr})} \lesssim \|x\|_{L^p}.
\] (14)
The other direction of the inequality follows from the duality between $L^p$ spaces and the identity
\[
\langle x, y \rangle = \sum_k \langle M_{\psi_k}(x), M_{\psi_k}(y) \rangle.
\] (15)

Corollary 8. Let $1 < p < \infty$. Then the unbounded linear operator $L : \lambda_h \mapsto k_1(h)\lambda_h$ has a bounded $H^\infty$-functional calculus on $L^p(\mathbb{G})$ of any positive angle $\mu$. Moreover, we have that
\[
\|\Phi(L)\| \lesssim (\sin \mu)^{-1}\|\Phi\|_{H^\infty},
\] (16)
for all $\Phi \in H^\infty(\Sigma_\mu)$. 

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Proof. Applying Cauchy’s integral formula, it is easy to check that
\[
\sup_{t \in \mathbb{R}} \{|\Phi(t)|, |t|\Phi'(t)|\} < C(\sin \mu)^{-1} \|\Phi\|_{\infty}
\]
for \( \Phi \in H^\infty(\Sigma_\mu) \). The desired result follows from Corollary \( \Box \)

Recall that we say a subset \( \Lambda \) of \( \mathbb{Z} \) is a c. b. \( \Lambda_p \) set with constant \( C_\Lambda \) if, for any operator valued sequence \( x_k, k \in \Lambda \), we have the equivalence
\[
\| \sum_{k \in \Lambda} x_k e^{ik\theta} \| \simeq C_\Lambda \| (x_k)_{k \in \Lambda} \|_{L^p(\mathbb{Z}_n)}.
\]
This is equivalent to that, for any subset \( A \subset \Lambda \), the Fourier multiplier
\[
\tilde{M}_{\chi A} : e^{ik\theta} \mapsto \chi_A(k) e^{ik\theta}
\]
extends to a completely bounded map on \( L^p(\mathbb{Z}) = L^p(\mathbb{T}) \) with a bound \( \leq C_\Lambda \).

Corollary 9. Suppose \( \Lambda \subset \mathbb{Z} \) is a c. b. \( \Lambda_p \) set. Then, for any \( A \subset \Lambda \), \( M_{\chi A} \) extends to a completely bounded map on \( L^p(\mathbb{F}_n) \) with a bound \( \leq c_p^2 C_\Lambda \).

Proof. This follows from Theorem \( \Box \)

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