Orlicz version of the mixed width integrals

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Abstract. In the paper, our main aim is to generalize the width integrals to the Orlicz space. Under the framework of Orlicz Brunn-Minkowski theory, we introduce a new affine geometric quantity by calculating Orlicz first order variation of the width integrals, and call as Orlicz mixed width integrals. The fundamental notions and conclusions of the width integrals and Minkowski and Brunn-Minkowski inequalities for the width integrals are extended to an Orlicz setting and the related concepts and inequalities of $L_p$-mixed width integrals of convex body are also derived.

Keywords. convex body, width integrals, Orlicz mixed width integrals, first order variation, Orlicz Brunn-Minkowski theory.

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1 Introduction

One of the most important operations in geometry is vector addition. As an operation between sets $K$ and $L$, defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

it is usually called Minkowski addition and plays an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to $L_p$-Brunn-Minkowski theory. For compact convex sets $K$ and $L$ containing the origin in its interior and $1 \leq p < \infty$, the $L_p$ addition of $K$ and $L$, introduced by Firey in [3] or [4], is defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p,$$

(1.1)

for all $x \in \mathbb{R}^n$ and compact convex sets $K$ and $L$ in $\mathbb{R}^n$ containing the origin. When $p = \infty$, (1.1) is interpreted as $h(K +_\infty L, x) = \max\{h(K, x), h(L, x)\}$. Here the functions are the support functions. If

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$K$ is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then the support function $h(K, x)$ of $K$ is defined by

$$h(K, x) = \max\{x \cdot y : y \in K\},$$

(1.2)

for $x \in \mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. $L_p$ addition and inequalities are the fundamental and core content in the $L_p$ Brunn-Minkowski theory. For recent important results and more information from this theory, refer to [8], [9], [10], [11], [15], [16], [18], [19], [20], [21], [22], [25], [26], [29], [30] and [31]. In recent years, a new extension of $L_p$-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak, Yang, and Zhang [23] and [24]. Gardner, Hug and Weil [6] constructed a general framework for the Orlicz-Brunn-Minkowski theory, and made clear for the first time the relation to Orlicz spaces and norms. The Orlicz addition of convex bodies was introduced and the Orlicz Brunn-Minkowski inequality is obtained (see [32]). The Orlicz centroid inequality for convex bodies was introduced in [39] which is an extension from star to convex bodies. The Orlicz-Brunn-Minkowski theory and its dual theory have attracted people’s attention. The other articles to promote the theory can be found in literatures [7], [13], [14], [27], [33], [38], [34], [35], [36] and [37].

For $u \in S^{n-1}$, the half width of convex body $K$ in the direction $u$, defined by

$$b(K, u) = \frac{1}{2}(h(K, u) + h(K, -u)).$$

(1.3)

Convex bodies $K, L$ are said to have similar width if there exist a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$, for all $u \in S^{n-1}$. Lutwak [17] introduced the width integrals: For $0 \leq i < n$, the width integral of convex body $K$, denotes by $A_i(K)$, defined by

$$A_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u).$$

(1.4)

In the paper, our main aim is to generalize the width integrals to Orlicz space. Under the framework of Orlicz Brunn-Minkowski theory, we introduce a new affine geometric quantity-Orlicz mixed width integrals. The fundamental notions and conclusions of the width integrals and Minkoswki and Brunn-Minkowski inequalities for the width integrals are extended to an Orlicz setting. The related concepts and inequalities of $L_p$-mixed width integrals of convex body are also derived. The new Orlicz Minkowski and Brunn-Minkowski inequalities in special case yield the $L_p$-dual Minkowski, and Brunn-Minkowski inequalities for the $L_p$-mixed width integrals.

In Section 3, we introduce a notion of Orlicz width addition $K + \phi L$ of convex bodies $K$ and $L$, defined by

$$\phi \left( \frac{b(K, x)}{b(K + \phi L, x)} \cdot \frac{b(L, x)}{b(K + \phi L, x)} \right) = 1.$$

(1.5)
Here \( \phi \in \Phi_2 \), the set of convex function \( \phi : [0, \infty)^2 \to (0, \infty) \) that are decreasing in each variable and satisfy \( \phi(0,0) = \infty \) and \( \phi(\infty,1) = \phi(1,\infty) = 1 \). The particular instance is \( \phi(x_1, x_2) = \phi_1(x_1) + \varepsilon \phi_2(x_2) \) for \( \varepsilon > 0 \) and some \( \phi_1, \phi_2 \in \Phi \), where the sets of convex functions \( \phi_1, \phi_2 : [0, \infty) \to (0, \infty) \) that are decreasing and satisfy \( \phi_1(0) = \phi_2(0) = \infty \), \( \phi_1(\infty) = \phi_2(\infty) = 0 \) and \( \phi_1(1) = \phi_2(1) = 1 \).

Complying with the basic spirit of Aleksandrov [1], mixed quermassintegrals [2] and \( L_\nu \)-mixed quermassintegrals [16], we concentrate on the study of the first order Orlicz variational of the width integrals. In Section 4, we prove that the first order Orlicz variation of the width integrals can be expressed as: For massintegrals \([16]\), we concentrate on the study of the first order Orlicz variational of the width integrals.

We also prove the new affine geometric quantity has an integral representation.

In Section 5, we establish an Orlicz Minkowski inequality for the Orlicz mixed width integrals: If \( K \) and \( L \) are convex bodies, \( 0 \leq i < n \) and \( \varepsilon > 0 \),

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0^+} A_i(K + \varepsilon \cdot L) = \frac{n-i}{(\phi_1)'(1)} \cdot A_{\phi_2,i}(K,L). \tag{1.6}
\]

In this first order variational equation (1.6), we define a new geometric quantity, Orlicz mixed width integrals \( A_{\phi_2,i}(K,L) \), is defined by

\[
A_{\phi_2,i}(K,L) := \frac{(\phi_1)'(1)}{n-i} \cdot \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0^+} A_i(K + \varepsilon \cdot L). \tag{1.7}
\]

We also prove the new affine geometric quantity has an integral representation.

\[
A_{\phi,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} \phi \left( \frac{b(L,u)}{b(K,u)} \right) b(K,u)^{n-i} dS(u). \tag{1.8}
\]

In Section 5, we establish an Orlicz Minkowski inequality for the Orlicz mixed width integrals: If \( K \) and \( L \) are convex bodies, \( 0 \leq i < n \) and \( \phi \in \Phi \), then

\[
A_{\phi,i}(K,L) \geq A_i(K) \cdot \phi \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)}. \tag{1.9}
\]

If \( \phi \) is strictly convex, equality holds if and only if \( K \) and \( L \) have similar width. In Section 6, we establish an Orlicz Brunn-Minkowski inequality for the Orlicz width addition and the width integrals. If \( K \) and \( L \) are convex bodies, \( 0 \leq i < n \) and \( \phi \in \Phi_2 \), then

\[
1 \geq \phi \left( \frac{A_i(K)}{A_i(K + \varepsilon L)} \right)^{1/(n-i)} \cdot \frac{A_i(L)}{A_i(K + \varepsilon L)}^{1/(n-i)}. \tag{1.10}
\]

If \( \phi \) is strictly convex, equality holds if and only if \( K \) and \( L \) have similar width.

2 Preliminaries

The setting for this paper is \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). A body in \( \mathbb{R}^n \) is a compact set equal to the closure of its interior. A set \( K \) is called a convex body if it is compact and convex subsets with non-empty interiors. Let \( K^n \) denote the class of convex bodies containing the origin in their interiors in \( \mathbb{R}^n \). For a compact set \( K \subset \mathbb{R}^n \), we write \( V(K) \) for the \((n\text{-dimensional}) \) Lebesgue measure of \( K \) and call
this the volume of $K$. The letter $B$ denotes the unit ball centered at the origin. The support function of convex body $K$ is homogeneous of degree 1, that is (see e.g. [28]),

$$h(K, ru) = rh(K, u),$$

for all $u \in S^{n-1}$ and $r > 0$. Let $\delta$ denote the Hausdorff metric, as follows, if $K, L \in \mathcal{K}^n$, then

$$\delta(K, L) = |h(K, u) - h(L, u)|_\infty.$$

If $K \in \mathcal{K}^n$ and $A \in \text{GL}(n)$, then (see e.g. [5], p.17)

$$h(AK, x) = h(K, A^tx), \quad (2.1)$$

for all $x \in \mathbb{R}^n$.

For $K_i \in \mathcal{K}^n, i = 1, \ldots, m$, define the real numbers $R_{K_i}$ and $r_{K_i}$ by

$$R_{K_i} = \max_{u \in S^{n-1}} b(K_i, u), \quad r_{K_i} = \min_{u \in S^{n-1}} b(K_i, u). \quad (2.2)$$

Obviously, $0 < r_{K_i} < R_{K_i}$, for all $K_i \in S^n$, and writing $R = \max\{R_{K_i}\}$ and $r = \min\{r_{K_i}\}$, where $i = 1, \ldots, m$.

2.1 Mixed width integrals

If $K_1, \ldots, K_n \in \mathcal{K}^n$, the mixed width integral of $K_1, \ldots, K_n$, denoted by $A(K_1, \ldots, K_n)$, defined by (see [17])

$$A(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} b(K_1, u) \cdots b(K_n, u) dS(u).$$

If $K_1 = \cdots = K_{n-i} = K, \ K_{n-i+1} = \cdots = K_n = B$, the mixed width integral $A(K_1, \ldots, K_n)$ is written as $A_i(K)$ and call width integral of $K$. Obviously, For $K \in \mathcal{K}^n$ and $0 \leq i < n$, we have

$$A_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u). \quad (2.3)$$

If $K_1 = \cdots = K_{n-i-1} = K, \ K_{n-i} = \cdots = K_{n-1} = B$ and $K_n = L$, the mixed width integral $A(K_1, \ldots, K_i, B, \ldots, B, L)$ is written as $A_i(K, L)$ and call $i$-th mixed width integral of $K$ and $L$. For $K, L \in \mathcal{K}^n$ and $0 \leq i < n$, it is easy that

$$A_i(K, L) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i-1} b(L, u) dS(u). \quad (2.4)$$

This integral representation (2.4), together with Hölder inequality, immediately gives: The Minkowski inequality for the $i$-th mixed width integral. If $K, L \in \mathcal{K}^n$ and $0 \leq i < n$, then

$$A_i(K, L)^{n-i} \leq A_i(K)^{n-i-1} A_i(L), \quad (2.5)$$

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with equality if and only if $K$ and $L$ have similar width.

### 2.2 $L_p$-mixed width integrals

Putting $\phi(x_1, x_2) = x_1^{-p} + x_2^{-p}$ and $p \geq 1$ in (1.5), the Orlicz width addition $+_{\phi}$ becomes the $L_p$-width addition, denoted by $+_p$, and call as $L_p$-width addition of convex bodies $K$ and $L$.

$$b(K +_p L, u)^{-p} = b(K, u)^{-p} + b(L, u)^{-p},$$

(2.6)

for $u \in S^{n-1}$. The following result follows immediately form (2.6) with $p \geq 1$.

$$-\frac{np}{n - i} \lim_{\varepsilon \to 0^+} \frac{A_i(K +_p L, \varepsilon \cdot L) - A_i(L)}{\varepsilon} \geq \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i+p} b(L, u)^{-p} dS(u).$$

**Definition 2.1** Let $K, L \in \mathbb{K}^n$, $0 \leq i < n$ and $p \geq 1$, the $L_p$-width integral of convex bodies $K$ and $L$, denoted by $A_{-p, i}(K, L)$, defined by

$$A_{-p, i}(K, L) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i+p} b(L, u)^{-p} dS(u).$$

(2.7)

Obviously, when $K = L$, the $L_p$-mixed width integral $A_{-p, i}(K, K)$ becomes the width integral $A_i(K)$. This integral representation (2.7), together with Hölder inequality, immediately gives:

**Proposition 2.2** If $K, L \in \mathbb{K}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$A_{-p, i}(K, L)^{n-i} \geq A_i(K)^{n-i+p} A_i(L)^{-p},$$

(2.8)

with equality if and only if $K$ and $L$ have similar width.

**Proposition 2.3** If $K, L \in \mathbb{K}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$A_i(K +_p L)^{-p/(n-i)} \geq A_i(K)^{-p/(n-i)} + A_i(L)^{-p/(n-i)},$$

(2.9)

with equality if and only if $K$ and $L$ are dilates.

**Proof** From (2.6) and (2.7), it is easily seen that the $L_p$-width integrals is linear with respect to the $L_p$-width addition, and together with inequality (2.8) show that for $p \geq 1$

$$A_{-p, i}(Q, K +_p L) = A_{-p, i}(Q, K) + A_{-p, i}(Q, L) \geq A_i(Q)^{(n-i+p)/(n-i)} (A_i(K)^{-p/(n-i)} + A_i(L)^{-p/(n-i)}),$$

with equality if and only if $K$ and $L$ have similar width.

Take $K +_p L$ for $Q$, recall that $A_{-p, i}(Q, Q) = A_i(Q)$, inequality (2.9) follows easy. $\Box$

### 3 Orlicz width addition
Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Let $\Phi_n$, $n \in \mathbb{N}$, denote the set of convex function $\phi : [0, \infty)^n \to (0, \infty)$ that are strictly decreasing in each variable and satisfy $\phi(0) = \infty$ and $\phi(e_j) = 1$, $j = 1, \ldots, n$. When $n = 1$, we shall write $\Phi$ instead of $\Phi_1$. The left derivative and right derivative of a real-valued function $f$ are denoted by $(f)_l'$ and $(f)_r'$, respectively. We first define the Orlicz width addition.

**Definition 3.1** Let $m \geq 2, \phi \in \Phi_m$, $K_j \in \mathcal{K}^n$ and $j = 1, \ldots, m$, define the Orlicz width addition of $K_1, \ldots, K_m$, denoted by $+\phi(K_1, \ldots, K_m)$, defined by

$$b(+\phi(K_1, \ldots, K_m), u) = \sup \left\{ \lambda > 0 : \phi \left( \frac{b(K_1, u)}{\lambda}, \ldots, \frac{b(K_m, u)}{\lambda} \right) \leq 1 \right\},$$

(3.1)

for $u \in S^{n-1}$. Equivalently, the Orlicz width addition $+\phi(K_1, \ldots, K_m)$ can be defined implicitly by

$$\phi \left( \frac{b(K_1, u)}{b(+\phi(K_1, \ldots, K_m), u)}, \ldots, \frac{b(K_m, u)}{b(+\phi(K_1, \ldots, K_m), u)} \right) = 1,$$

(3.2)

for all $u \in S^{n-1}$.

An important special case is obtained when

$$\phi(x_1, \ldots, x_m) = \sum_{j=1}^{m} \phi_j(x_j),$$

for some fixed $\phi_j \in \Phi$ such that $\phi_1(1) = \cdots = \phi_m(1) = 1$. We then write $+\phi(K_1, \ldots, K_m) = K_1 + \phi \cdots + \phi K_m$. This means that $K_1 + \phi \cdots + \phi K_m$ is defined either by

$$b(K_1 + \phi \cdots + \phi K_m, u) = \sup \left\{ \lambda > 0 : \sum_{j=1}^{m} \phi_j \left( \frac{b(K_j, u)}{\lambda} \right) \leq 1 \right\},$$

(3.3)

for all $u \in S^{n-1}$, or by the corresponding special case of (3.2).

**Lemma 3.2** The Orlicz width addition $+\phi : (\mathcal{K}^n)^m \to \mathcal{K}^n$ is monotonic and has the identity property.

**Proof** Suppose $K_j \subset L_j$, $j = 1, \ldots, m$, where $K_j, L_j \in \mathcal{K}^n$. By using (3.1), and in view of $K_1 \subset L_1$ and the fact that $\phi$ is decreasing in the first variable, we obtain

$$b(+\phi(L_1, K_2 \ldots, K_m), u)$$

$$= \sup \left\{ \lambda > 0 : \phi \left( \frac{b(L_1, u)}{\lambda}, \frac{b(K_2, u)}{\lambda}, \ldots, \frac{b(K_m, u)}{\lambda} \right) \leq 1 \right\}$$

$$\geq \sup \left\{ \lambda > 0 : \phi \left( \frac{b(K_1, u)}{\lambda}, \frac{b(K_2, u)}{\lambda}, \ldots, \frac{b(K_m, u)}{\lambda} \right) \leq 1 \right\}$$

$$= b(+\phi(K_1, K_2, \ldots, K_m), u).$$

By repeating this argument for each of the other $(m - 1)$ variables, we have $b(+\phi(K_1, \ldots, K_m), u) \leq b(+\phi(L_1, \ldots, L_m), u)$. 

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The identity property is obvious from (3.2).

**Lemma 3.3** The Orlicz width addition $+_{\phi} : (K^n)^m \to K^n$ is $GL(n)$ covariant.

**Proof** From (2.1), (3.1) and let $A \in GL(n)$, we obtain

$$b(+_{\phi}(AK_1, AK_2, \ldots, AK_m), u)$$

$$= \sup \left\{ \lambda > 0 : \phi \left( \frac{b(AK_1, u)}{\lambda}, \frac{b(AK_2, u)}{\lambda}, \ldots, \frac{b(AK_m, u)}{\lambda} \right) \leq 1 \right\}$$

$$= \sup \left\{ \lambda > 0 : \phi \left( \frac{b(K_1, A^t u)}{\lambda}, \frac{b(K_2, A^t u)}{\lambda}, \ldots, \frac{b(K_m, A^t u)}{\lambda} \right) \leq 1 \right\}$$

$$= b(+_{\phi}(K_1, \ldots, K_m), A^t u)$$

$$= b(+_{\phi}(K_1, \ldots, K_m), u).$$

This shows Orlicz width addition $+_{\phi}$ is $GL(n)$ covariant.

**Lemma 3.4** Suppose $K_1, \ldots, K_m \in K^n$. If $\phi \in \Phi^n$, then

$$\phi \left( \frac{b(K_1, u)}{t} \right) + \cdots + \phi \left( \frac{b(K_m, u)}{t} \right) = 1$$

if and only if

$$b(+_{\phi}(K_1, \ldots, K_m), u) = t$$

**Proof** This follows immediately from Definition 3.1.

**Lemma 3.5** Suppose $K_1, \ldots, K_m \in K^n$. If $\phi \in \Phi^m$, then

$$\frac{r}{\phi^{-1}(\frac{1}{m})} \leq b(+_{\phi}(K_1, \ldots, K_m), u) \leq \frac{R}{\phi^{-1}(\frac{1}{m})}.$$

**Proof** Suppose $b(+_{\phi}(K_1, \ldots, K_m), u) = t$. From Lemma 3.4 and noting that $\phi$ is strictly deceasing on $(0, \infty)$, we have

$$1 = \phi \left( \frac{b(K_1, u)}{t} \right) + \cdots + \phi \left( \frac{b(K_m, u)}{t} \right)$$

$$\leq \phi \left( \frac{rK_1}{t} \right) + \cdots + \phi \left( \frac{rK_m}{t} \right)$$

$$\leq m\phi \left( \frac{r}{t} \right).$$

Noting that $\phi^{-1}$ is strictly deceasing on $(0, \infty)$, we obtain the lower bound for $b(+_{\phi}(K_1, \ldots, K_m), u)$:

$$t \geq \frac{r}{\phi^{-1}(\frac{1}{m})}.$$

To obtain the upper estimate, observe that from the lemma 3.4, together with the convexity and the fact $\phi$ is strictly deceasing on $(0, \infty)$, we have

$$1 = \phi \left( \frac{b(K_1, u)}{t} \right) + \cdots + \phi \left( \frac{b(K_m, u)}{t} \right)$$
Then we obtain the upper estimate:
\[ t \leq \frac{R}{\phi^{-1}(\frac{1}{m})}. \]

\[ \square \]

**Lemma 3.6** The Orlicz width addition \( +_\phi : (\mathcal{K}^n)^m \to \mathcal{K}^n \) is continuous.

**Proof** To see this, indeed, let \( K_{ij} \in \mathcal{K}^n, i \in \mathbb{N} \cup \{0\}, j = 1, \ldots, m \), be such that \( K_{ij} \to K_{0j} \) as \( i \to \infty \).

Let
\[ b(+_\phi(K_{i1}, \ldots, K_{im}), u) = t_i. \]

Then Lemma 3.5 shows
\[ \frac{r_{ij}}{\phi^{-1}(\frac{1}{m})} \leq t_i \leq \frac{R_{ij}}{\phi^{-1}(\frac{1}{m})}, \]
where \( r_{ij} = \min\{r_{K_{ij}}\} \) and \( R_{ij} = \max\{R_{K_{ij}}\} \). Since \( K_{ij} \to K_{0j} \), we have \( R_{K_{ij}} \to R_{K_{0j}} < \infty \) and \( r_{K_{ij}} \to r_{K_{0j}} > 0 \), and thus there exist \( a, b \) such that \( 0 < a \leq t_i \leq b < \infty \) for all \( i \). To show that the bounded sequence \( \{t_i\} \) converges to \( b(+_\phi(K_{01}, \ldots, K_{0m}), u) \), we show that every convergent subsequence of \( \{t_i\} \) converges to \( b(+_\phi(K_{01}, \ldots, K_{0m}), u) \). Denote any subsequence of \( \{t_i\} \) by \( \{t_i\} \) as well, and suppose that for this subsequence, we have
\[ t_i \to t_* \]

Obviously \( a \leq t_* \leq b \). Noting that \( \phi \) is continuous function, we obtain
\[ t_* \to \sup \left\{ t_* > 0 : \phi \left( \frac{b(K_{01}, u)}{t_*}, \ldots, \frac{b(K_{0m}, u)}{t_*} \right) \leq 1 \right\} \]
\[ = b(+_\phi(K_{01}, \ldots, K_{0m}), u). \]

Hence
\[ b(+_\phi(K_{i1}, \ldots, K_{im}), u) \to b(+_\phi(K_{01}, \ldots, K_{0m}), u) \]
as \( i \to \infty \).

This shows that the Orlicz width addition \( +_\phi : (\mathcal{K}^n)^m \to \mathcal{K}^n \) is continuous. \( \square \)

Next, we define the Orlicz width linear combination on the case \( m = 2 \).

**Definition 3.7** Orlicz width linear combination \( +_\phi(K, L, \alpha, \beta) \) for \( K, L \in \mathcal{K}^n \), and \( \alpha, \beta \geq 0 \) (not both zero), defined by
\[ \alpha \cdot \phi_1 \left( \frac{b(K, u)}{b(+_\phi(K, L, \alpha, \beta), u)} \right) + \beta \cdot \phi_2 \left( \frac{b(L, u)}{b(+_\phi(K, L, \alpha, \beta), u)} \right) = 1, \quad (3.4) \]
for all \( u \in S^{n-1} \).

We shall write \( K + \phi \varepsilon \cdot L \) instead of \( +_{\phi}(K, L, 1, \varepsilon) \), for \( \varepsilon \geq 0 \) and assume throughout that this is defined by (3.1), if \( \alpha = 1, \beta = \varepsilon \) and \( \phi \in \Phi \). We shall write \( K + \phi L \) instead of \( +_{\phi}(K, L, 1, 1) \) and call the Orlicz width addition of \( K \) and \( L \).

### 4 Orlicz mixed width integrals

In order to define Orlicz mixed width integrals, we need the following lemmas.

**Lemma 4.1** Let \( \phi \in \Phi \) and \( \varepsilon > 0 \). If \( K, L \in \mathcal{K}^n \), then \( K + \phi \varepsilon \cdot L \in \mathcal{K}^n \).

**Proof** Let \( u_0 \in S^{n-1} \), for any subsequence \( \{u_i\} \subset S^{n-1} \) such that \( u_i \to u_0 \) as \( i \to \infty \).

Let

\[
b(K + \phi L, u_i) = \lambda_i.
\]

Then Lemma 3.5 shows

\[
\frac{r}{\phi^{-1}(\frac{1}{2})} \leq \lambda_i \leq \frac{R}{\phi^{-1}(\frac{1}{2})},
\]

where \( R = \max\{R_K, R_L\} \) and \( r = \min\{r_K, r_L\} \).

Since \( K, L \in \mathcal{K}^n \), we have \( 0 < r_K \leq R_K < \infty \) and \( 0 < r_L \leq R_L < \infty \), and thus there exist \( a, b \) such that \( 0 < a \leq \lambda_i \leq b < \infty \) for all \( i \). To show that the bounded sequence \( \{\lambda_i\} \) converges to \( b(K + \phi L, u_0) \), we show that every convergent subsequence of \( \{\lambda_i\} \) converges to \( b(K + \phi L, u_0) \). Denote any subsequence of \( \{\lambda_i\} \) by \( \{\lambda_i\} \) as well, and suppose that for this subsequence, we have

\[
\lambda_i \to \lambda_0.
\]

Obviously \( a \leq \lambda_0 \leq b \). From (3.4) and note that \( \phi_1, \phi_2 \) are continuous functions, so \( \phi_1^{-1} \) is continuous, we obtain

\[
\lambda_i \to \frac{b(K, u_0)}{\phi_1^{-1}\left(1 - \varepsilon \phi_2\left(\frac{b(L, u_0)}{\lambda_0}\right)\right)}
\]

as \( i \to \infty \). Hence

\[
\phi_1\left(\frac{b(K, u_0)}{\lambda_0}\right) + \varepsilon \phi_2\left(\frac{b(L, u_0)}{\lambda_0}\right) = 1.
\]

Therefore

\[
\lambda_0 = b(K + \phi \varepsilon \cdot L, u_0).
\]

Namely

\[
b(K + \phi \varepsilon \cdot L, u_i) \to b(K + \phi \varepsilon \cdot L, u_0).
\]

as \( i \to \infty \).

This shows that \( K + \phi \varepsilon \cdot L \in \mathcal{S}^n \). \( \square \)
Lemma 4.2 If $K, L \in K^n$, $\varepsilon > 0$ and $\phi \in \Phi$, then

$$K + \phi \varepsilon \cdot L \to K$$  \hspace{1cm} (4.1)

as $\varepsilon \to 0^+$.

Proof From (3.4) and noting that $\phi_2$, $\phi_1^{-1}$ and $b$ are continuous functions, we obtain

$$\lim_{\varepsilon \to 0^+} b(K + \phi \varepsilon \cdot L, u) = \lim_{\varepsilon \to 0^+} \phi_1^{-1} \left( 1 - \varepsilon \phi_2 \frac{b(L, u)}{b(K + \phi \varepsilon \cdot L, u)} \right).$$

Since $\phi_1^{-1}$ is continuous, $\phi_2$ is bounded and in view of $\phi_1^{-1}(1) = 1$, we have

$$\lim_{\varepsilon \to 0^+} \phi_1^{-1} \left( 1 - \varepsilon \phi_2 \frac{b(L, u)}{b(K + \phi \varepsilon \cdot L, u)} \right) = 1. \hspace{1cm} (4.2)$$

This yields

$$b(K + \phi \varepsilon \cdot L, u) \to b(K, u)$$

as $\varepsilon \to 0^+$.

Lemma 4.3 If $K, L \in K^n$, $0 \leq i < n$ and $\phi_1, \phi_2 \in \Phi$, then

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0^+} b(K + \phi \varepsilon \cdot L, u)^{n-i} = \frac{n-i}{(\phi_1)'(1)} \cdot \phi_2 \left( \frac{b(L, u)}{b(K, u)} \right) \cdot b(K, u)^{n-i}. \hspace{1cm} (4.3)$$

Proof Form (3.4), (4.1), Lemma 4.2 and notice that $\phi_1^{-1}$, $\phi_2$ are continuous functions, we obtain for $0 \leq i < n$

$$\lim_{\varepsilon \to 0^+} \frac{b(K + \phi \varepsilon \cdot L, u)^{n-i} - b(K, u)^{n-i}}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\phi_1^{-1} \left( 1 - \varepsilon \phi_2 \frac{b(L, u)}{b(K + \phi \varepsilon \cdot L, u)} \right)}{\varepsilon} \cdot b(K, u)^{n-i} - b(K, u)^{n-i}$$

$$= \lim_{\varepsilon \to 0^+} \frac{(n-i)b(K, u)}{(\phi_1)'(1)} \cdot \phi_2 \left( \frac{b(L, u)}{b(K, u)} \right) \cdot \lim_{y \to 1^+} \frac{y - \phi_1^{-1}(1)}{y - 1}$$

$$= \frac{n-i}{(\phi_1)'(1)} \cdot \phi_2 \left( \frac{b(L, u)}{b(K, u)} \right) \cdot b(K, u)^{n-i},$$

where

$$y = 1 - \varepsilon \phi_2 \left( \frac{b(L, u)}{b(K + \phi \varepsilon \cdot L, u)} \right),$$

and note that $y \to 1^+$ as $\varepsilon \to 0^+$. \hfill \Box
Lemma 4.4 If $\phi \in \Phi_2$, $0 \leq i < n$ and $K, L \in \mathcal{K}^n$, then
\[
\left. \frac{d}{d\varepsilon} \left|_{\varepsilon=0^+} \right. \frac{(\phi_1)_n(1)}{n-i} \cdot A_i(K + \phi \varepsilon \cdot L) = \frac{1}{n} \int_{S^{n-1}} \phi_2 \left( \frac{b(L, u)}{b(K, u)} \right) \cdot b(K, u)^{n-i} dS(u). \right. \tag{4.4}
\]

Proof This follows immediately from (2.1) and Lemma 4.2.

Denoting by $A_{\phi,i}(K, L)$, for any $\phi \in \Phi$ and $0 \leq i < n$, the integral on the right-hand side of (4.4) with $\phi_2$ replaced by $\phi$, we see that either side of the equation (4.4) is equal to $A_{\phi,i}(K, L)$ and hence this new Orlicz mixed width integrals $A_{\phi,i}(K, L)$ has been born.

Definition 4.5 For $\phi \in \Phi$ and $0 \leq i < n$, Orlicz mixed width integrals of convex bodies $K$ and $L$, $A_{\phi,i}(K, L)$, defined by
\[
A_{\phi,i}(K, L) := \frac{1}{n} \int_{S^{n-1}} \phi \left( \frac{b(L, u)}{b(K, u)} \right) \cdot b(K, u)^{n-i} dS(u). \tag{4.5}
\]

Lemma 4.6 If $\phi_1, \phi_2 \in \Phi$, $0 \leq i < n$ and $K, L \in \mathcal{K}^n$, then
\[
A_{\phi_2,i}(K, L) = (\phi_1)_n(1) \lim_{\varepsilon \to 0^+} \frac{A_i(K + \phi \varepsilon \cdot L) - A_i(K)}{\varepsilon}. \tag{4.6}
\]

Proof This follows immediately from Lemma 4.4 and (4.5).

Lemma 4.7 If $K, L \in \mathcal{K}^n$, $\phi \in \Phi$ and any $A \in \text{SL}(n)$, then for $\varepsilon > 0$
\[
A(K + \phi \varepsilon \cdot L) = (AK) + \phi \varepsilon \cdot (AL). \tag{4.7}
\]

Proof For any $A \in \text{SL}(n)$, from (2.1), we have
\[
b(AK, u) = b(K, A^t u), \quad b(AL, u) = \rho(L, A^t u), \quad b(A(K + \phi \varepsilon \cdot L, u) = b((K + \phi \varepsilon \cdot L), A^t u).
\]

Hence
\[
b((AK + \phi \varepsilon \cdot AL), u)
\]
\[
= \sup \left\{ \lambda > 0 : \phi \left( \frac{b(AK, u)}{\lambda} \right) + \phi \left( \frac{b(AL, u)}{\lambda} \right) \leq 1 \right\}
\]
\[
= \sup \left\{ \lambda > 0 : \phi \left( \frac{b(K, A^t u)}{\lambda} \right) + \phi \left( \frac{b(L, A^t u)}{\lambda} \right) \leq 1 \right\}
\]
\[
= b(K + \phi \varepsilon \cdot L, A^t u)
\]
\[
= b(A(K + \phi \varepsilon \cdot L), u). \tag*{□}
\]

Lemma 4.8 If $\phi \in \Phi$, $0 \leq i < n$ and $K, L \in \mathcal{K}^n$, then for $A \in \text{SL}(n)$,
\[
A_{\phi,i}(AK, AL) = A_{\phi,i}(K, L). \tag{4.8}
\]

Proof From (1.7), Lemma 3.3 and Lemma 4.7, we have for $A \in \text{SL}(n)$,
\[
A_{\phi,i}(AK, AL) = (\phi_1)_n(1) \cdot \frac{d}{d\varepsilon} \left|_{\varepsilon=0^+} \right. A_i(AK + \phi \varepsilon \cdot AL)
\]
\[\begin{align*}
&= \frac{(\phi_1^f(1))}{n-i} \cdot \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0^+} A_i(A(K + \phi \cdot L)) \\
&= \frac{(\phi_1^f(1))}{n-i} \cdot \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0^+} A_i(K + \phi \cdot L) \\
&= A_{\phi,i}(K,L).
\end{align*}\]

5 Orlicz width Minkowski inequality

In this section, we need to define a Borel measure in \(S^{n-1}\), denoted by \(A_{n,i}(K,v)\), called as width measure of convex body \(K\).

**Definition 5.1** Let \(K \in K^n\) and \(0 \leq i < n\), the width measure, denoted by \(A_{n,i}(K,v)\), defined by

\[
dA_{n,i}(K,v) = \frac{b(K,v)^{n-i}}{nA_i(K)} dS(v).
\]  (5.1)

**Lemma 5.2** (Jensen’s inequality) Let \(\mu\) be a probability measure on a space \(X\) and \(g : X \to I \subset \mathbb{R}\) is a \(\mu\)-integrable function, where \(I\) is a possibly infinite interval. If \(\psi : I \to \mathbb{R}\) is a convex function, then

\[
\int_X \psi(g(x))d\mu(x) \geq \psi \left( \int_X g(x)d\mu(x) \right).
\]  (5.2)

If \(\psi\) is strictly convex, equality holds if and only if \(g(x)\) is constant for \(\mu\)-almost all \(x \in X\) (see [12, p.165]).

**Lemma 5.3** Suppose that \(\phi : [0, \infty) \to (0, \infty)\) is decreasing and convex with \(\phi(0) = \infty\). If \(K, L \in K^n\) and \(0 \leq i < n\), then

\[
\frac{1}{nA_i(K)} \int_{S^{n-1}} \phi \left( \frac{b(L,u)}{b(K,u)} \right) b(K,u)^{n-i} dS(u) \geq \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right).
\]  (5.3)

If \(\phi\) is strictly convex, equality holds if and only if \(K\) and \(L\) have similar width.

**Proof** For \(K \in K^{n-1}\), \(0 \leq i < n\) and any \(u \in S^{n-1}\), since

\[
\int_{S^{n-1}} dA_{n,i}(K,v) = 1,
\]

so the width measure \(\frac{b(K,u)^{n-i}}{nA_i(K)} dS(u)\) is a probability measure on \(S^{n-1}\).

Hence, from (2.4), (2.5), (5.1) and by using Jensen’s inequality (5.2), and in view of \(\phi\) is decreasing, we obtain

\[
\frac{1}{nA_i(K)} \int_{S^{n-1}} \phi \left( \frac{b(L,u)}{b(K,u)} \right) b(K,u)^{n-i} dS(u) = \int_{S^{n-1}} \phi \left( \frac{b(L,u)}{b(K,u)} \right) dA_{n,i}(K,u) \geq \phi \left( \frac{A_i(K,L)}{A_i(K)} \right)
\]

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\[ \geq \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right). \]

Next, we discuss the equal condition of (5.3). Suppose the equality holds in (5.3) and \( \phi \) is strictly convex, form the equality condition of (2.5), so there exist \( r > 0 \) such that

\[ b(L, u) = rb(K, u), \]

for all \( u \in S^{n-1} \). On the other hand, suppose that \( K \) and \( L \) have similar width, i.e. there exist \( \lambda > 0 \) such that

\[ b(L, u) = \lambda b(K, u) \]

for all \( u \in S^{n-1} \). Hence

\[ \frac{1}{nA_i(K)} \int_{S^{n-1}} \phi \left( \frac{b(L, u)}{b(K, u)} \right) b(K, u)^{n-i} dS(u) \]

\[ = \frac{1}{nA_i(K)} \int_{S^{n-1}} \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right) b(K, u)^{n-i} dS(u) \]

\[ = \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right). \]

This implies the equality in (5.3) holds. \( \square \)

**Theorem 5.4** (Orlicz width Minkowski inequality) If \( K, L \in K^n \), \( 0 \leq i < n \) and \( \phi \in \Phi \), then

\[ A_{\phi,i}(K, L) \geq A_i(K) \cdot \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right). \]

(5.4)

If \( \phi \) is strictly convex, equality holds if and only if \( K \) and \( L \) have similar width.

**Proof** This follows immediately from (4.5) and Lemma 5.3. \( \square \)

**Corollary 5.5** If \( K, L \in K^n \), \( 0 \leq i < n \) and \( p \geq 1 \), then

\[ A_{-p,i}(K, L)^{n-i} \geq A_i(K)^{n-i+p} A_i(L)^{-p}, \]

(5.5)

with equality if and only if \( K \) and \( L \) have similar width.

**Proof** This follows immediately from Theorem 5.4 with \( \phi_1(t) = \phi_2(t) = t^{-p} \) and \( p \geq 1 \). \( \square \)

Taking \( i = 0 \) in (5.6), this yields \( L_p \)-Minkowski inequality is following: If \( K, L \in K^n \) and \( p \geq 1 \), then

\[ A_{-p}(K, L)^n \geq A(K)^n \cdot A(L)^{-p}, \]

with equality if and only if \( K \) and \( L \) have similar width.

**Corollary 5.6** Let \( K, L \in M \subset K^n \), \( 0 \leq i < n \) and \( \phi \in \Phi \), and if either

\[ A_{\phi,i}(Q, K) = A_{\phi,i}(Q, L), \text{ for all } Q \in M \]

(5.6)

or

\[ \frac{A_{\phi,i}(K, Q)}{A_i(K)} = \frac{A_{\phi,i}(L, Q)}{A_i(L)}, \text{ for all } Q \in M, \]

(5.7)
then $K = L$.

Proof Suppose (5.6) hold. Taking $K$ for $Q$, then from (2.3), (4.5) and (5.3), we obtain

$$A_i(K) = A_{\phi,i}(K, L) \geq A_i(K) \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right)$$

with equality if and only if $K$ and $L$ have similar width. Hence

$$1 \geq \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right),$$

with equality if and only if $K$ and $L$ have similar width. Since $\phi$ is decreasing function on $(0, \infty)$, this follows that

$$A_i(K) \leq A_i(L),$$

with equality if and only if $K$ and $L$ have similar width. On the other hand, if taking $L$ for $Q$, we similar get $A_i(K) \geq A_i(L)$, with equality if and only if $K$ and $L$ have similar width. Hence $A_i(K) = A_i(L)$, and $K$ and $L$ have similar width, it follows that $K$ and $L$ must be equal.

Suppose (5.7) hold. Taking $L$ for $Q$, then from from (2.3), (4.5) and (5.3), we obtain

$$1 = A_{\phi,i}(K, L) A_i(K) \geq A_{\phi,i}(K, L) A_i(L),$$

with equality if and only if $K$ and $L$ are dilates. Hence

$$1 \geq \phi \left( \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \right),$$

with equality if and only if $K$ and $L$ have similar width. Since $\phi$ is decreasing function on $(0, \infty)$, this follows that

$$A_i(K) \leq A_i(L),$$

with equality if and only if $K$ and $L$ have similar width. On the other hand, if taking $K$ for $Q$, we similar get $A_i(K) \geq A_i(L)$, with equality if and only if $K$ and $L$ have similar width. Hence $A_i(K) = A_i(L)$, and $K$ and $L$ have similar width, it follows that $K$ and $L$ must be equal.

When $\phi_1(t) = \phi_2(t) = t^{-p}$ and $p \geq 1$, Corollary 5.6 becomes the following result.

**Corollary 5.7** Let $K, L \in \mathcal{M} \subset \mathcal{K}_n$, $0 \leq i < n$ and $p \geq 1$, and if either

$$A_{-p,i}(K, Q) = A_{-p,i}(L, Q), \text{ for all } Q \in \mathcal{M}$$

or

$$\frac{A_{-p,i}(K, Q)}{A_i(K)} = \frac{A_{-p,i}(L, Q)}{A_i(L)}, \text{ for all } Q \in \mathcal{M},$$
6 Orlicz width Brunn-Minkowski inequality

Lemma 6.1 If $K, L \in \mathbb{K}^n$, $0 \leq i < n$, and $\phi_1, \phi_2 \in \Phi$, then

$$A_i(K +_\phi L) = A_{\phi_1,i}(K +_\phi L, K) + A_{\phi_2,i}(K +_\phi L, L).$$  \hspace{1cm} (6.1)

Proof From (2.3), (3.1), (3.4) and (4.5), we have for $K +_\phi L = Q \in \mathbb{K}^n$

$$A_{\phi_1,i}(Q, K) + A_{\phi_2,i}(Q, L)
= \frac{1}{n} \int_{S^{n-1}} \left( \phi_1 \left( \frac{b(K, u)}{b(Q, u)} \right) + \phi_2 \left( \frac{b(L, u)}{b(Q, u)} \right) \right) b(Q, u)^{n-i} dS(u)
= \frac{1}{n} \int_{S^{n-1}} \phi \left( \frac{b(K, u)}{b(Q, u)} \frac{b(L, u)}{b(Q, u)} \right) b(Q, u)^{n-i} dS(u)
= A_i(Q).$$  \hspace{1cm} (6.2)

This completes the proof. □

Theorem 6.2 (Orlicz width Brunn-Minkowski inequality) If $K, L \in \mathbb{K}^n$, $0 \leq i < n$ and $\phi \in \Phi_2$, then

$$1 \geq \phi \left( \frac{A_i(K)}{A_i(K +_\phi L)} \right)^{1/(n-i)}, \left( \frac{A_i(L)}{A_i(K +_\phi L)} \right)^{1/(n-i)}.$$  \hspace{1cm} (6.3)

If $\phi$ is strictly convex, equality holds if and only if $K$ and $L$ have similar width.

Proof From (5.4) and Lemma 6.1, we have

$$A_i(K +_\phi L) = A_{\phi_1,i}(K +_\phi L, K) + A_{\phi_2,i}(K +_\phi L, L)
\geq A_i(K +_\phi L) \left( \phi_1 \left( \frac{A_i(K)}{A_i(K +_\phi L)} \right)^{1/(n-i)} \right) + A_i(K +_\phi L) \left( \phi_2 \left( \frac{A_i(L)}{A_i(K +_\phi L)} \right)^{1/(n-i)} \right)
= A_i(K +_\phi L) \phi \left( \frac{A_i(K)}{A_i(K +_\phi L)} \right)^{1/(n-i)}, \left( \frac{A_i(L)}{A_i(K +_\phi L)} \right)^{1/(n-i)}.$$  \hspace{1cm}

This is just inequality (6.3). From the equality condition of (5.4), if follows that if $\phi$ is strictly convex, equality in (6.3) holds if and only if $K$ and $L$ have similar width. □

Corollary 6.3 If $K, L \in \mathbb{K}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$A_i(K +_p L)^{-p/(n-i)} \geq A_i(K)^{-p/(n-i)} + A_i(L)^{-p/(n-i)},$$  \hspace{1cm} (6.4)

with equality if and only if $K$ and $L$ have similar width.

Proof The result follows immediately from Theorem 6.2 with $\phi(x_1, x_2) = x_1^{-p} + x_2^{-p}$ and $p \geq 1$. □
Taking $i = 0$ in (6.4), this yields the $L_p$-Brunn-Minkowski inequality for the width integrals. If $K, L \in \mathcal{K}^n$ and $p \geq 1$, then

$$A(K + p L)^{-p/n} \geq A(K)^{-p/n} + A(L)^{-p/n},$$

with equality if and only if $K$ and $L$ have similar width.

**Corollary 6.4** If $K, L \in \mathcal{K}^n$, $0 \leq i < n$ and $\phi \in \Phi$, then

$$A_{\phi, i}(K, L) \geq A_i(K) \cdot \phi \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)}.$$  

(6.5)

If $\phi$ is strictly convex, equality holds if and only if $K$ and $L$ have similar width.

**Proof**

Let $K_\varepsilon = K + \phi \varepsilon \cdot L$.

From (4.6) and in view of the Orlicz-Brunn-Minkowski inequality (6.3), we obtain

$$\frac{n - i}{(\phi_1)'(1)} \cdot A_{\phi_2, i}(K, L) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0^+} A_i(K_\varepsilon)$$

$$= \lim_{\varepsilon \to 0^+} \frac{A_i(K_\varepsilon) - A_i(K)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \frac{1 - \frac{A_i(K)}{A_i(K_\varepsilon)}}{1 - \phi_1 \left( \frac{A_i(K)}{A_i(K_\varepsilon)} \right)^{1/(n-i)}} \cdot \frac{1 - \phi_1 \left( \frac{A_i(K)}{A_i(K_\varepsilon)} \right)^{1/(n-i)}}{\varepsilon} \cdot A_i(K_\varepsilon)$$

$$= \lim_{t \to 1^-} \frac{1 - t}{\phi_1(1) - \phi_1(t^{n-i})} \cdot \lim_{\varepsilon \to 0^+} \frac{1 - \phi_1 \left( \frac{A_i(K)}{A_i(K_\varepsilon)} \right)^{1/(n-i)}}{\varepsilon} \cdot \lim_{\varepsilon \to 0^+} A_i(K_\varepsilon)$$

$$\geq \frac{n - i}{(\phi_1)'(1)} \cdot \lim_{\varepsilon \to 0^+} \phi_2 \left( \frac{A_i(L)}{A_i(K_\varepsilon)} \right)^{1/(n-i)} \cdot \lim_{\varepsilon \to 0^+} A_i(K_\varepsilon)$$

$$= \frac{n - i}{(\phi_1)'(1)} \cdot \phi_2 \left( \frac{A_i(L)}{A_i(K)} \right)^{1/(n-i)} \cdot A_i(K).$$  

(6.6)

Replacing $\phi_2$ by $\phi$, (6.6) becomes (6.5). If $\phi$ is strictly convex, from the equality condition of (6.3), it follows that the equality holds in (6.5) if and only if $K$ and $L$ have similar width.

This proof is complete.  

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