Conservation Laws on Riemann-Cartan, Lorentzian and Teleparallel Spacetimes

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Abstract

Using a Clifford bundle formalism, we examine: (a) the strong conditions for existence of conservation laws involving only the energy-momentum and angular momentum of the matter fields on a general Riemann-Cartan spacetime and the particular cases of Lorentzian and teleparallel spacetimes and (b) the conditions for the existence of conservation laws of energy-momentum and angular momentum for the matter and gravitational fields when this later concept can be rigorously defined. We examine in more details some statements concerning the issues of the conservation laws in General Relativity and Riemann-Cartan (including the particular case of the teleparallel ones) theories.

Contents

1 Introduction

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1 Introduction

Using the Clifford bundle formalism of differential forms (see Appendix A) we reexamine the origin and meaning of conservation laws of energy-momentum and angular momentum and the conditions for their existence on a general Riemann-Cartan spacetime (RCST) \((M, g, \nabla, \tau_g, \uparrow)\) and also in the particular

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1In Appendix A we give a very short introduction to the main tools of the the Clifford bundle formalism needed for this paper. A detailed and up to date presentation to the Clifford bundle formalism is given, e.g., in [55].

2See details in Appendix A.
cases of Lorentzian spacetimes $\mathfrak{M} = (M, g, D, \tau_g, \uparrow)$ which as it is well known model gravitational fields in the General Relativity Theory (GRT) \cite{57}. A RCST is supposed to model a generalized gravitational field in the so called Riemann-Cartan theories \cite{25}. The case of the so called teleparallel\footnote{A teleparallel spacetime is a particular Riemann-Cartan spacetime with null curvature and non null torsion tensor \cite{1,2,3}.} equivalent of GRT \cite{30} is also investigated and the recent claim \cite{12} that there is a genuine energy-momentum conservation law in that theory is investigated in more details.

In what follows, we suppose that a set of dynamic fields live and interact in $(M, g, \nabla, \tau_g, \uparrow)$ (or $\mathfrak{M}$). Of course, we want that the RCST admits spinor fields, which implies according to Geroch’s theorem that the orthonormal frame bundle must be trivial \cite{21,38,55}. This permits a great simplification in our calculations, in particular if use is made of the calculation procedures of the Clifford bundle formalism. Moreover, we will suppose, for simplicity that the dynamic fields of the theory $\phi^A, A = 1, 2, ..., n$, are $r$-forms \footnote{This is not a serious restriction in the formalism since as it is shown in details in \cite{38,55}, one can represent spinor fields by sums of even multiform fields once a spinorial frame is given. The functional derivative of non-homogeneous multiform fields is developed in details in, e.g., \cite{55}.} i.e., each $\phi^A \in \text{sec} \bigwedge^r T^*M \hookrightarrow \text{sec} \mathcal{Cl}(M, g)$, for some $r = 0, 1, ..., 4$.

We recall the very important fact that there are in such theories a set of ‘covariant conservation laws’ which are identities which result from the fact that Lagrangian densities of relativistic field theories are supposed to be invariant under \textit{diffeomorphisms} and \textit{active local Lorentz rotation}.\footnote{Satisfying such a condition implies in general in the use of generalized gauge connections, implying a sort of equivalence between spacetimes equipped with connections having different curvature and/or torsion tensors \cite{51,44}.} These covariant conservation laws do \textit{not} express in general any genuine conservation law of energy-momentum or angular momentum. We prove moreover, as first shown by \cite{6} that genuine conservation laws of energy-momentum and angular momentum for only the \textit{matter fields} exist for a field theory in a RCST only if there exists a set of $m$ appropriate vector fields $\xi^{(a)}, a = 1, 2, ..., m$ such that $\mathcal{L}_{\xi^{(a)}} g = 0$ and $\mathcal{L}_{\xi^{(a)}} \Theta = 0$, where $\Theta$ is the torsion tensor.

Thus, we show in Section 6 that in the teleparallel version of GRT, the existence of Killing vector fields does \textit{not} warrant (contrary to the case of GRT) the existence of conservation laws involving only the energy-momentum tensors of the \textit{matter} fields. We show moreover, still in Section 6, that in the teleparallel version of GRT (with null or non null cosmological constant) there is a genuine conservation law involving the energy-momentum tensor of matter and the energy-momentum tensor of the gravitational field, which in that theory is a well defined object.

Although this is a well known result, we think that our formalism puts it in a new perspective. Indeed, in our approach, the teleparallel equivalent of General Relativity as formulated, e.g., by \cite{30} or \cite{12}, is easily seem as consisting in the introduction of: (a) a bilinear form (a deformed metric tensor \cite{52,55})
$g = \eta_{ab} \theta^a \otimes \theta^b$, (b) a teleparallel connection (necessary to make the theory invariant under active local Lorentz transformations\footnote{On the issue on active local Lorentz invariance, see also \cite{14,51}.} in the manifold $M \simeq \mathbb{R}^4$ of Minkowski spacetime structure, and (c) a Lagrangian density differing from the Einstein-Hilbert Lagrangian density by an exact differential.

The paper is organized as follows:

Section 2 and Appendix A are aimed to give to the reader some background information needed to better understand our developments. In Section 2 we recall some mathematical preliminaries as the definition of vertical and horizontal variations, the concept of functional derivatives of functionals on a 1-jet bundle, the Euler-Lagrange equations (ELE) and the fact that the action of any theory formulated in terms of differential forms is invariant under diffeomorphisms, whereas in Appendix A we briefly describe the Clifford bundle formalism used throughout the paper. Appendix A also provides a derivation of the energy-momentum 3-forms for the electromagnetic field which in the Clifford bundle formalism (and our conventions) are expressed very elegantly by

$$-\star T^a = \star \tilde{T}^a = -\frac{1}{2} \star (F \theta^a \tilde{F}).$$

In Section 3 we recall the proof of a set of identities called ‘covariant conservation laws’ valid in a RCST \cite{6}, which as already mentioned above do not encode, in general, any genuine energy-momentum and/or angular momentum conservation laws.

In Section 4 we assume that the Lagrangian density is invariant under active local Lorentz transformations and diffeomorphisms and then recall the conditions for the existence of genuine conservation laws in a RCST which involve only the energy-momentum and angular momentum tensor of the matter fields \cite{6}.

Next, in Section 5, we recall (for completeness) with our formalism the theory of pseudo-potentials and pseudo energy-momentum tensors in GRT, and show that there are in general no conservation laws of energy-momentum and angular momentum in this theory \cite{54}. We also discuss some misleading and even wrong statements concerning this issue that appear in the literature.

Finally, in Section 6 we discuss the conservation laws in the teleparallel equivalent of General Relativity, as already mentioned above.

Our conclusions can be found in Section 7. To better illustrate the meaning of our results, we also present, in Appendix B, various examples showing that not all Killing vector fields of a teleparallel spacetime (Schwarzschild, de Sitter, Friedmann) satisfy Eq. (39) meaning that in a model of the teleparallel ‘equivalent’ of GRT there are, in general, fewer conservation laws involving only the matter fields than in the corresponding model of GRT.
2 Some Preliminaries

2.1 Variations

2.1.1 Vertical Variation

Let $X \in \sec \mathcal{C} \ell(M, g)$, be a Clifford (multiform) field. An active local Lorentz transformation sends $X \mapsto X' \in \sec \mathcal{C} \ell(M, g)$, with

$$X' = UXU.$$  \hfill (1)

Each $U \in \sec \text{Spin}_{1,3}(M)$ can be written (see, e.g., [55]) as $\pm$ the exponential of a 2-form field $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C} \ell(M, g)$. For infinitesimal transformations we must choose the $+$ sign and write $F = \alpha f$, $\alpha \ll 1$, $F^2 \neq 0$.

**Definition 1** Let $X$ be a Clifford field. The vertical variation of $X$ is the field $\delta_v X$ (of the same nature of $X$) such that

$$\delta_v X = X' - X.$$ \hfill (2)

**Remark 2** The case where $F$ is independent of $x \in M$ is said to be a gauge transformation of the first kind, and the general case is said to be a gauge transformation of the second kind.

2.1.2 Horizontal Variation

Let $\sigma_t$ be a one-parameter group of diffeomorphisms of $M$ and let $\xi \in \sec T M$ be the vector field that generates $\sigma_t$, i.e.,

$$\xi^\mu(x) = \left. \frac{d\sigma_t^\mu(x)}{dt} \right|_{t=0}.$$ \hfill (3)

**Definition 3** We call the horizontal variation of $X$ induced by a one-parameter group of diffeomorphisms of $M$ to be the quantity

$$\delta_h X = \lim_{t \to 0} \frac{\sigma_t^* X - X}{t} = -\mathcal{L}_\xi X.$$ \hfill (4)

**Definition 4** We call total variation of a multiform field $X$ to the quantity

$$\delta X = \delta_v X + \delta_h X = \delta_v X - \mathcal{L}_\xi X.$$ \hfill (5)

It is crucial to distinguish between the two variations defined above.

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8If $X = \psi \in \sec \mathcal{C} \ell^{(0)}(M, g)$ (where $\mathcal{C} \ell^{(0)}(M, g)$ is the even subbundle of $\mathcal{C} \ell(M, g)$) is a representative of a Dirac-Hestenes spinor field in a given spin frame, then an active local transformation sends $\psi \mapsto \psi'$, with $\psi' = L\psi$ [55].
2.2 Functional Derivatives

Let \( J^1(\bigwedge T^* M) \) be the 1-jet bundle over \( \bigwedge T^* M \hookrightarrow \mathcal{C}(M, g) \), i.e., the vector bundle defined by

\[
J^1(\bigwedge T^* M) = \{(x, \phi(x), d\phi(x)); x \in M, \phi \in \sec \bigwedge T^* M \hookrightarrow \sec \mathcal{C}(M, g)\}.
\]

Then, with each local section \( \phi \in \sec \bigwedge T^* M \hookrightarrow \sec \mathcal{C}(M, g) \), we may associate a local section \( j_1(\phi) \in \sec J^1(\bigwedge T^* M) \).

Let \( \{\theta^a\}, \theta^a \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, g) \), \( a = 0, 1, 2, 3 \), be an orthonormal basis of \( T^* M \) dual to the basis \( \{e_a\} \) of \( TM \) and let \( \omega^a_b \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, g) \) be the connection 1-forms of the connection \( \nabla \) in a given gauge. We introduce also the 1-jet bundle \( J^1[(\bigwedge T^* M)^{n+2}] \) over the configuration space \( (\bigwedge T^* M)^{n+2} \hookrightarrow (\mathcal{C}(M, g))^{n+2} \) of a field theory describing \( n \) different fields \( \phi^A \in \sec \bigwedge T^p M \hookrightarrow \sec \mathcal{C}(M, g) \) on a RCST, where for each different value of \( A \) we have in general a different value of \( p \).

\[
J^1[(\bigwedge T^* M)^{n+2}] := J^1(\bigwedge T^* M \times \bigwedge T^* M \times \cdots \times \bigwedge T^* M)
\]

\[
= \{(x, \theta^a(x), d\theta^a(x), \omega^a_b(x), d\omega^a_b(x), \phi^A(x), d\phi^A(x), A = 1, \ldots, n)\}
\]

Sections of \( J^1[(\bigwedge T^* M)^{n+2}] \) will be denoted by \( j_1(\theta^a, \omega^a_b, \phi) \) or simply by \( j_1(\phi) \) when no confusion arises.

A functional for a field \( \phi \in \sec \bigwedge T^* M \hookrightarrow \sec \mathcal{C}(M, g) \) in \( J^1(\bigwedge T^* M) \) is a mapping \( \mathcal{F} : \sec J^1(\bigwedge T^* M) \to \sec \bigwedge T^* M \), \( j_1(\phi) \mapsto \mathcal{F}(j_1(\phi)) \).

A Lagrangian density mapping for a field theory described by fields \( \phi^A \in \sec \bigwedge T^* M, A = 1, 2, \ldots, n \) over a Riemann-Cartan spacetime is a mapping

\[
\mathcal{L}_m : \sec J^1[(\bigwedge T^* M)^{n+2}] \to \sec \bigwedge^{4} T^* M
\]

\[
j_1(\theta^a, \omega^a_b, \phi) \mapsto \mathcal{L}_m(j_1(\theta^a, \omega^a_b, \phi)).
\]

**Remark 5** When convenient the image of \( \mathcal{L}_m \), i.e., \( \mathcal{L}_m(j_1(\theta^a, \omega^a_b, \phi)) \) (called Lagrangian density) will be represented by the sloppy notation \( \mathcal{L}_m(x, \theta^a, \omega^a_b, \phi) \) or, when the Lagrangian density does not depend explicitly on \( x \), \( \mathcal{L}_m(\theta^a, \omega^a_b, \phi) \) or simply \( \mathcal{L}_m(\phi) \) and even just \( \mathcal{L}_m \). The same observation holds for any other functional.

To simplify the notation even further consider in the next few definitions of a field theory with only one field \( \phi \in \sec \bigwedge T^* M \hookrightarrow \sec \mathcal{C}(M, g) \), in which case \( \mathcal{L}_m \) is a functional on \( J^1[(\bigwedge T^* M)^3] \).

Given a Lagrangian density \( \mathcal{L}_m(j_1(\theta^a, \omega^a_b, \phi)) \) for a given homogeneous matter field \( \phi \in \sec \bigwedge T^* M \hookrightarrow \sec \mathcal{C}(M, g) \) over a general Riemann-Cartan spacetime, we shall need (in order to apply the variational action principle) to calculate some algebraic derivatives of \( \mathcal{L}_m \). These are terms such as \( \frac{\partial \mathcal{L}_m}{\partial \phi}, \frac{\partial \mathcal{L}_m}{\partial d\phi} \).
which appears in the variation of $L_m$, i.e.,

$$\delta L_m(\phi) = \delta \phi \wedge \frac{\partial L_m(\phi)}{\partial \phi} + \delta(d\phi) \wedge \frac{\partial L_m(\phi)}{\partial d\phi}$$

$$= \delta \phi \wedge \frac{\partial L_m(\phi)}{\partial \phi} + d(\delta \phi) \wedge \frac{\partial L_m(\phi)}{\partial d\phi}$$

$$= \delta \phi \wedge \left( \frac{\partial L_m(\phi)}{\partial \phi} - (-1)^r d \left( \frac{\partial L_m(\phi)}{\partial d\phi} \right) \right) + d \left( \delta \phi \wedge \frac{\partial L_m(\phi)}{\partial d\phi} \right)$$

$$= \delta \phi \wedge \delta \phi \wedge \frac{\partial L_m(\phi)}{\partial \phi} + d \left( \delta \phi \wedge \frac{\partial L_m(\phi)}{\partial d\phi} \right).$$

(10a)

**Definition 6** The terms $\frac{\partial L_m}{\partial \phi}$ and $\frac{\partial L_m}{\partial d\phi}$ are called in what follows algebraic derivatives of $L_m$ and

$$\ast \Sigma(\phi) = \frac{\partial L_m(\phi)}{\partial \phi} - (-1)^r d \left( \frac{\partial L_m(\phi)}{\partial d\phi} \right)$$

(11)

is called the Euler-Lagrange functional of the field $\phi$. Some authors call it the functional derivative of $L_m$ and in this case write

$$\ast \Sigma(\phi) = \frac{\delta L_m(\phi)}{\delta \phi}$$

(12)

In working with these objects it is necessary to keep in mind that for $\phi \in \sec \wedge^r T^* M$, $F(\phi) \equiv F(j_1(\phi)) \in \sec \wedge^p T^* M$ and $K(\phi) \equiv K(j_1(\phi)) \in \sec \wedge^q T^* M$,

$$\frac{\partial}{\partial \phi} [F(\phi) \wedge K(\phi)] = \frac{\partial}{\partial \phi} F(\phi) \wedge K(\phi) + (-1)^{pr} F(\phi) \wedge \frac{\partial}{\partial \phi} K(\phi).$$

(13)

We recall also that if $G(j_1(\phi))$ is an arbitrary functional and $\sigma : M \rightarrow M$ is a diffeomorphism, then $G(j_1(\phi))$ is said to be invariant under $\sigma$ if and only if $\sigma^* G(j_1(\phi)) = G(j_1(\phi))$. Also, it is a well known result that $G(j_1(\phi))$ is invariant under the action of a one parameter group of diffeomorphisms $\sigma_t$ if and only if

$$\mathcal{L}_\xi G(j_1(\phi)) = 0,$$

(14)

where $\xi \in \sec TM$ is the infinitesimal generator of the group $\sigma_t$ and $\mathcal{L}_\xi$ denotes the Lie derivative.

### 2.3 Euler-Lagrange Equations from Lagrangian Densities

Recall now that the principle of stationary action is the statement that the variation of the action integral written in terms of a Lagrangian density $L_m(j_1(\theta^a, \omega^b, \phi))$

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9 This terminology was originally introduced in [55]. The exterior product $\delta \phi \wedge \frac{\partial}{\partial \phi}$ is a particular instance of the $A \wedge \frac{\partial}{\partial \phi}$ directional derivatives introduced in the multiform calculus developed in [55] with $\delta \phi = A$. 

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is null for arbitrary variations of \( \phi \) which vanish in the boundary \( \partial U \) of the open set \( U \subset M \) (i.e., \( \delta \phi \big|_{\partial U} = 0 \))

\[
\delta A(\phi) = \delta \int_U L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) = \int_U \delta L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) = 0. \tag{15}
\]

A trivial calculation gives

\[
\delta A(\phi) = \int_U \delta \phi \wedge *\Sigma(\phi). \tag{16}
\]

Since \( \delta \phi \) is arbitrary, the stationary action principle implies that

\[
*\Sigma(\phi) = \partial \frac{L_m(\phi)}{\partial \phi} - (-1)^{r} \frac{d}{d\phi} \left( \partial \frac{L_m(\phi)}{\partial d\phi} \right) = 0. \tag{17}
\]

The equation \( *\Sigma(\phi) = 0 \) is the corresponding ELE for the field \( \phi \in \sec \bigwedge^r T^* M \hookrightarrow \sec C\ell(M, g) \).

### 2.4 Invariance of the Action Integral under the Action of a Diffeomorphism

**Proposition 7** The action \( A(\phi) \) for any field theory formulated in terms of fields that are differential forms is invariant under the action of one parameters groups of diffeomorphisms if \( L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) \big|_{\partial U} = 0 \) on the boundary \( \partial U \) of a domain \( U \subset M \).

**Proof.** Let \( L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) \) be the Lagrangian density of the theory. The variation of the action which we are interested in is the horizontal variation, i.e.:

\[
\delta_h A(\phi) = \int_U L_\xi L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) \tag{18}
\]

Let

\[
\xi^* = g(\xi, \cdot) \in \sec \bigwedge^1 T^* M \hookrightarrow \sec C\ell(M, g). \tag{19}
\]

Then we have from a well known property of the Lie derivative (Cartan’s magical formula) that

\[
L_\xi L_m = d(\xi^* \lrcorner L_m) + \xi^* \lrcorner (dL_m). \tag{20}
\]

But, since \( L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) \in \sec \bigwedge^1 T^* M \hookrightarrow \sec C\ell(M, g) \) we have \( dL_m = 0 \) and then \( L_\xi L_m = d(\xi^* \lrcorner L_m) \). It follows, using Stokes theorem that

\[
\int_U L_\xi L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) = \int_U d[\xi^* \lrcorner L_m(j_1(\theta^a, \omega^b_\alpha, \phi))] \]

\[
= \int_{\partial U} \xi^* \lrcorner L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) = 0, \tag{21}
\]

since \( L_m(j_1(\theta^a, \omega^b_\alpha, \phi)) \big|_{\partial U} = 0 \).

**Remark 8** It is important to emphasize that the action integral is always invariant under the action of a one parameter group of diffeomorphisms even if the corresponding Lagrangian density is not invariant (in the sense of Eq.\((14)\)) under the action of that same group.
3 Covariant ‘Conservation’ Laws

Let \((M, g, \nabla, \tau, \dagger)\) denote a general Riemann-Cartan spacetime. As stated above we suppose that the dynamic fields \(\phi^A, A = 1, 2, ..., n\), are \(r\)-forms, i.e., each \(\phi^A \in \sec \wedge^r T^* M \hookrightarrow \sec \mathcal{C}(M, g)\), for some \(r = 0, 1, ..., 4\).

Let \(\{e_a\}\) be an arbitrary global orthonormal basis for \(TM\), and let \(\{\theta^a\}\) be its dual basis. We suppose that \(\theta^a \in \sec \wedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, g)\). Let moreover \(\{\theta_a\}\) be the reciprocal basis of \(\{\theta^a\}\). As it is well known (see, e.g., \[64, 52, 53, 55\]) it is possible to represent the gravitational field using a Clifford bundle \[6\] and it is also possible to write differential equations equivalent to Einstein equations for such objects \[\dagger\].

Here, we make the hypothesis that a Riemann-Cartan spacetime models a generalized gravitational field which must be described by \(\{\theta^a, \omega^a_b\}\), where \(\omega^a_b\) are the connection 1-forms (in a given gauge). Thus, we suppose that a dynamic theory for the fields \(\phi^A \in \sec \wedge^r T^* M\) (called in what follows matter fields) is obtained through the introduction of a Lagrangian density, which is a functional on \(J^1[\wedge^r T^* M]^{2+n}\) as previously discussed.

Active Local Lorentz transformations are represented by even sections of the Clifford bundle \(U \in \sec \text{Spin}_{1,3}(M) \hookrightarrow \sec \mathcal{C}(M, g)\), such that \(UU = UU = 1\), i.e., \(U(x) \in \text{Spin}_{1,3} \simeq \text{SL}(2, \mathbb{C})\). Under a local Lorentz transformation the fields transform as

\[
\begin{align*}
\theta^a &\rightarrow \theta'^a = U \theta^a U^{-1} = \Lambda^a_b \theta^b, \\
\omega^a_b &\rightarrow \omega'^a_b = \Lambda^a_c \omega^c_d (\Lambda^{-1})_d^b + \Lambda^a_c (d\Lambda^{-1})_d^b, \\
\phi^A &\rightarrow \phi'^A = U \phi^A U^{-1},
\end{align*}
\]

where \(\Lambda^a_b(x) \in \text{SO}_{1,3}\). In our formalism it is a triviality to see that \(\mathcal{L}_m(\theta^a, \omega^a_b, \phi) \in \sec \wedge^4 T^* M \hookrightarrow \sec \mathcal{C}(M, g)\) is invariant under local Lorentz transformations. Indeed, since \(\tau_x = \theta^a \theta^b \theta^c \theta^d \in \sec \wedge^4 T^* M \hookrightarrow \sec \mathcal{C}(M, g)\) commutes with even multiform fields, we have that a local Lorentz transformation produces no changes in \(\mathcal{L}_m\), i.e.,

\[
\mathcal{L}_m(\theta^a, \omega^a_b, \phi) \rightarrow U \mathcal{L}_m(\theta^a, \omega^a_b, \phi) U^{-1} = \mathcal{L}_m(\theta^a, \omega^a_b, \phi).
\]

However, this does not implies necessarily that the variation of the Lagrangian density \(\mathcal{L}_m(\theta^a, \omega^a_b, \phi)\) obtained by variation of the fields \(\theta^a, \omega^a_b, \phi\) is null, since

\[
\delta_v \mathcal{L}_m = \mathcal{L}_m(\theta^a + \delta_v \theta^a, \omega^a_b + \delta_v \omega^a_b, \phi + \delta_v \phi) - \mathcal{L}_m(\theta^a, \omega^a_b, \phi) \neq 0,
\]

unless it happens that for an infinitesimal Lorentz transformation,

\[
\mathcal{L}_m(U \theta^a U^{-1}, U \omega^a_b U^{-1}, U \phi U^{-1}) = U \mathcal{L}_m U^{-1} = \mathcal{L}_m.
\]

\[\dagger\] The Lagrangian density for the \(\{\theta^a\}\) for the case of General Relativity is recalled in Section 5.
In what follows we suppose that the Lagrangian of the matter field is invariant under local Lorentz transformations\(^{11}\), i.e., \(\delta_{\xi} L_m = 0\). Also \(L_m\) depends on \(\theta^a\) and \(\omega^a_b\), but not on \(d\theta^a\) and \(d\omega^a_b\) (minimal coupling).\(^{12}\) Then, \(\frac{\delta L_m}{\delta \theta^a} = \frac{\partial L_m}{\partial \theta^a}\) and we can write
\[
\int \delta L_m = \int \left[ \delta \theta^a \wedge \frac{\partial L_m}{\partial \theta^a} + \delta \omega^a_b \wedge \frac{\partial L_m}{\partial \omega^a_b} + \delta \phi^A \wedge \star \Sigma_A \right],
\]
(26)
where \(\Sigma_A\) are the Euler-Lagrange functionals of the fields \(\phi^A\).

As we just showed above the action of any Lagrangian density is invariant under diffeomorphisms. Let us now calculate the total variation of the Lagrangian density \(L_m\), arising from a one-parameter group of diffeomorphisms generated by a vector field \(\xi \in \text{sec} T^*M\) and by a local Lorentz transformation, when we vary \(\theta^a, \omega^a_b, \phi^A, d\phi^A\) independently. We have
\[
\delta L_m = \delta_{\xi} L_m - \mathcal{L}_{\xi} L_m.
\]
(27)

Under the (nontrivial) hypothesis\(^{51, 14}\) that \(\delta_{\xi} L_m = 0\),
\[
\delta L_m = -\mathcal{L}_{\xi} L_m = -\star T_a \wedge \mathcal{L}_{\xi} \theta^a - \star J^b_a \wedge \mathcal{L}_{\xi} \omega^a_b - \star \Sigma_A \wedge \mathcal{L}_{\xi} \phi^A,
\]
(28)
where we have:

**Definition 9** The coefficients of \(\delta \theta^a = -\mathcal{L}_{\xi} \theta^a\), i.e.
\[
\star T_a = \frac{\partial L_m}{\partial \theta^a} \in \text{sec} \wedge^3 T^*M
\]
(29)
are called the energy-momentum densities of the matter fields, and the \(T_a \in \text{sec} \wedge^1 T^*M\) are called the energy-momentum density 1-forms of the matter fields. The coefficients of \(\delta \omega^a_b\), i.e.,
\[
\star J^b_a = \frac{\partial L_m}{\partial \omega^a_b} \in \text{sec} \wedge^3 T^*M,
\]
(30)
are called the angular momentum densities of the matter fields.

Taking into account that each one of the fields \(\phi^A\) obey a Euler-Lagrange equation, \(\star \Sigma_A = 0\), we can write
\[
\int \mathcal{L}_{\xi} L_m = \int \star T_a \wedge \mathcal{L}_{\xi} \theta^a + \star J^b_a \wedge \mathcal{L}_{\xi} \omega^a_b
\]
(31)

Now, since all geometrical objects in the above formulas are sections of the Clifford bundle, we can write
\[
\mathcal{L}_{\xi} \theta^a = \xi^* d\theta^a + d(\xi^* \, . \theta^a).
\]
(32)

\(^{11}\)We discuss further the issue of local Lorentz invariance and its hidden consequence in\(^{51, 14}\).

\(^{12}\)See example of the electromagnetic field in Appendix B.
Moreover, recalling also Cartan’s first structure equation,

\[ d\theta^a + \omega^a_b \wedge \theta^b = \Theta^a, \]  

we get

\[ \mathcal{L}_\xi \theta^a = \xi^* \cdot \Theta^a - \xi^* \cdot (\omega^a_b \wedge \theta^b) + d(\xi^* \cdot \theta^a) \]
\[ = \xi^* \cdot \Theta^a - (\xi^* \cdot \omega^a_b) \theta^b + (\xi^* \cdot \theta^b) \omega^a_b + d(\xi^* \cdot \theta^a) \]
\[ = D(\xi^* \cdot \theta^a) + \xi^* \cdot \Theta^a - (\xi^* \cdot \omega^a_b) \theta^b, \]  

where \( D \) is the covariant exterior derivative of indexed \( p \)-form fields (for details, see, e.g., [7, 55]). To continue we need the following

**Proposition 10** Let \( \omega \) be the \( 4 \times 4 \) matrix whose entries are the connection 1-forms. For any \( x \in M \), the matrix with entries \( \xi^* \cdot \omega^a_b \in \text{spin}_{1,3} \simeq \text{sl}(2, \mathbb{C}) = \text{so}_{1,3} \) belongs to the Lie algebra of \( \text{Spin}_{1,3} \) (or of \( \text{SO}_{1,3} \)).

**Proof.** Recall that at any \( x \in M \) any infinitesimal local Lorentz transformation \( \Lambda^a_b(x) \in \text{SO}_{1,3} \) can be written as

\[ \Lambda^a_b = \delta^a_b + \chi^a_b, \quad |\chi^a_b| \ll 1, \]
\[ \chi_{ab} = -\chi_{ba}. \]  

Now, writing \( \omega^a_b = L^a_{cb} \theta^c \) we have

\[ \xi^* \cdot \omega^a_b = \xi^* \cdot (L^a_{cb} \theta^c) = (\xi^* \cdot \theta^d) \cdot (L^a_{cb} \theta^c) \]
\[ = \xi^c L^a_{cb} \]  

and the \( \xi^* \cdot \omega_{ab} \) satisfy

\[ \xi^* \cdot \omega_{ab} + \xi^* \cdot \omega_{ba} = \xi^c (L_{acb} + L_{bca}) = 0, \]  

since in an orthonormal basis the connection coefficients satisfy \( L_{acb} = -L_{bca} \). We see then that we can identify if \( |\xi^c| \ll 1 \)

\[ \chi^a_b = \xi^* \cdot \omega^a_b \]  

as the generator of an infinitesimal Lorentz transformation, and the proposition is proved. \( \blacksquare \)

Now, the term \( (\xi^* \cdot \omega^a_b) \theta^b \) has the form of a local vertical variation of the \( \theta^a \) and thus we write

\[ \delta_v \theta^a := (\xi^* \cdot \omega^a_b) \theta^b \]  

Using Eq. (39) we can rewrite Eq. (34) as

\[ \mathcal{L}_\xi \theta^a = D(\xi^* \cdot \theta^a) + \xi^* \cdot \Theta^a - \delta_v \theta^a. \]  

We see that \( \mathcal{L}_\xi \theta^a = -\delta_v \theta^a \) only if we have the following constraint

\[ D(\xi^* \cdot \theta^a) + \xi^* \cdot \Theta^a = 0. \]  

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A necessary and sufficient condition for the validity of Eq. (41) is given by Lemma 12 below. Now, let us calculate \( \mathcal{L}_\xi \omega^a_b \). By definition,
\[
\mathcal{L}_\xi \omega^a_b = \xi^* \omega + d(\xi^* \cdot \omega^a_b) \\
= \xi^* (\mathcal{R}^a_b) - (\xi^* \cdot \omega^c_e) \omega^a_b + (\xi^* \cdot \omega^a_b) \omega^a_c + d(\xi^* \cdot \omega^a_b),
\]
where in writing the second line in Eq. (42) we used Cartan’s second structure equation,
\[
d\omega^a_b + \omega^a_c \wedge \omega^c_b = \mathcal{R}^a_b.
\]
Under an infinitesimal Lorentz transformation \( \Lambda = 1 + \chi \), recalling Eq. (22), we can write
\[
\delta_v \omega = -d\chi + \chi \omega - \omega \chi,
\]
which using Eq. (38) gives for Eq. (42)
\[
\mathcal{L}_\xi \omega^a_b = \xi^* \omega - \delta_v \omega^a_b.
\]
Now, for a vertical variation,
\[
\int_U \delta_v \mathcal{L}_m := \int_U \delta_v \theta^a \wedge \frac{\partial \mathcal{L}_m}{\partial \theta^a} + \delta_v \omega^a_b \wedge \frac{\partial \mathcal{L}_m}{\partial \omega^a_b} + \delta_v \phi^A \wedge \frac{\delta \mathcal{L}_m}{\delta \phi^A}.
\]
Then, if we recall that we assumed that \( \int \delta_v \mathcal{L}_m = 0 \) and if we suppose that the field equations are satisfied, i.e., \( \Sigma A = \frac{\delta \mathcal{L}_m}{\delta \phi^A} = 0 \), Eq. (31) becomes,
\[
\int \mathcal{L}_\xi \mathcal{L}_m
\]
\[
= \int [-D(\xi^* \cdot \theta^a) - (\xi^* \cdot \omega^a_b) + \delta_v \theta^a] \wedge *T_a
\]
\[
+ \int [-\xi^* \omega - \delta_v \omega^a_b] \wedge *J^b_a
\]
\[
= \int *T_a \wedge (\xi^* \cdot \theta^a) + *J^b_a \wedge (\xi^* \cdot \mathcal{R}^a_b) - D[\ast T_a(\xi^* \cdot \theta^a)] + (D \ast T_a)(\xi^* \cdot \theta^a))
\]
\[
= \int *T_a \wedge (\xi^* \cdot \theta^a) + *J^b_a \wedge (\xi^* \cdot \mathcal{R}^a_b) + (D \ast T_a)(\xi^* \cdot \theta^a)),
\]
where we used also the fact that \( D[(\xi^* \cdot \theta^a) \ast T_a] = d[(\xi^* \cdot \theta^a) \ast T_a] \), that \( \ast T_a \big|_{\partial U} = 0 \) and
\[
\int_U d[(\xi^* \cdot \theta^a) \ast T_a] = \int_{\partial U} (\xi^* \cdot \theta^a) \ast T_a = 0
\]
Now, writing \( \xi^* = \xi^a \theta_a = \xi_a \theta^a \), and recalling that the action is invariant under diffeomorphisms (if as usual we suppose that \( \mathcal{L}_m \big|_{\partial U} = 0 \)), we have,
\[
\int -\delta \mathcal{L}_m = \int \mathcal{L}_\xi \mathcal{L}_m = [\ast T_a \wedge (\theta_c \cdot \theta^a) + *J^b_a \wedge (\theta_c \cdot \mathcal{R}^a_b) + D \ast T_c] \xi^c = 0,
\]
and since the $\xi^c$ are arbitrary, we end with
$$D \star T_c + \star T_a \wedge (\theta_c \lrcorner \Theta^a) + \star J^a_b \wedge (\theta_c \lrcorner R^a_b) = 0. \quad (51)$$

Also, using the explicit expressions for $\delta_v \theta^a$ and $\delta_v \omega^a_b$ (Eq.(39) and Eq.(45)) in Eq.(46) we get,
$$\int \star T^a \wedge \chi^a_b \theta^b + \star J^b_a \wedge (\chi^a_c \omega^c_b - \omega^a_c \chi^c_b - d\chi^a_b)$$
$$= \int \left[ \frac{1}{2} (\star T^a \wedge \theta^b - \star T^b \wedge \theta^a) - d \star J^b_a - \omega^a_c \wedge \star J^a_c - \star J^c_a \wedge \omega^c_b \right] \chi^a_b = 0, \quad (52)$$

and since the coefficients $\chi^a_b$ are arbitrary we end with
$$D \star J^b_a + \frac{1}{2} (\star T^b \wedge \theta_a - \star T^a \wedge \theta^b) = 0. \quad (53)$$

Eq.(51) and Eq.(53) are known as covariant conservation laws and first appeared in this form in [6]. They are simply identities that follows from the hypothesis utilized, namely that the Lagrangian density of the theory is invariant under diffeomorphisms and also invariant under the local action of the group Spin$_{1,3}$. Eq.(51) and Eq.(53) do not encode genuine conservation laws and a memorable number of nonsense affirmations have been generated along the years by authors that use those equations in a naive way. Some examples of these affirmations are recalled in the specific case of Einstein’s theory in Section 5 [53].

4 When Genuine Conservation Laws Do Exist?

We recall now the crucial result that when the Riemann-Cartan spacetime $\textit{M}, g, \nabla, \tau_g, \uparrow$ admits symmetries, then Eq.(51) and Eq.(53) can be used, as first shown by Trautman [66, 67, 68, 69], for the construction of closed 3-forms, which then provides genuine conservation laws involving only the energymomentum and angular momentum tensors of the matter fields. In the remaining of the section we recall these results following [6].

Proposition 11 For each Killing vector field $\xi \in \sec T \textit{M}$, such that $\mathcal{L}_\xi g = 0$ and $\mathcal{L}_\xi \Theta = 0$, where $\Theta = e_a \otimes \Theta^a$ is the torsion tensor of $\nabla$, and $\Theta^a$ the torsion 2-forms, we have
$$d \left[ (\xi^a \cdot \theta^a) \star T_a + (\theta_b \cdot \mathbf{L}_\xi \theta^a) \star J^b_a \right] = 0, \quad (54)$$
where $\mathbf{L}_\xi = \xi^a \lrcorner D + Di_\xi$ is the so called Lie covariant derivative.

In order to prove the Proposition 11 some preliminary results are needed.

Lemma 12 $\mathcal{L}_\xi \theta^a = -\delta_v \theta^a$ and $\mathcal{L}_\xi \omega^a_b = \delta_v \omega^a_b$ if and only if $\mathcal{L}_\xi g = 0$ and $\mathcal{L}_\xi \Theta = 0$.  

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**Proof.** Let us show first that if \( L_\xi \theta^a = -\delta_c \theta^a \) then \( L_\xi g = 0 \). We have

\[
L_\xi g = \eta_{ab} (L_\xi \theta^a) \otimes \theta^b + \eta_{ab} \theta^a \otimes (L_\xi \theta^b).
\]

(55)

On the other hand, since \( g \) is invariant under local Lorentz transformations, we have

\[
\delta_v g = \eta_{ab} (\delta_v \theta^a) \otimes \theta^b + \eta_{ab} \theta^a \otimes (\delta_v \theta^b) = 0.
\]

(56)

Then, it follows from Eqs. (55) and (56) that if \( L_\xi \theta^a = -\delta_v \theta^a \) then \( L_\xi g = 0 \).

Taking into account the definition of Lie derivative we can write

\[
L_\xi e_a = -\xi^b e_b, \quad L_\xi \theta^a = \chi^b_\theta \theta^b,
\]

\[
\chi^b_\theta = -[e_a(\xi^b) + \xi^m e_{ab}]
\]

(57)

Now, if \( L_\xi g = 0 \) we have from Eq. (55) that \( (\eta_{cb} \chi^c_\theta + \eta_{ac} \chi^c_\theta) \theta^a \otimes \theta^b = 0 \), i.e.,

\[
\eta_{ab} + \chi_{ba} = 0,
\]

(58)

and then it follows that for any \( x \in M \), \( \chi_{ab} \in \text{spin}^{\xi}_{1,3} \). Using Proposition 10 we can write \( \chi^b_\theta = -\chi^b_\xi = -\xi^c \cdot \omega^b_c \) and then the vertical variation can be written as \( \delta_v \theta^a = -L_\xi \theta^a \).

The proof that if \( L_\xi \omega^a_b = \delta_v \omega^a_b \) then \( L_\xi \Theta = 0 \) is trivial. In the following we prove the reciprocal, i.e., if \( L_\xi \Theta = 0 \) then \( L_\xi \omega^a_b = \delta_v \omega^a_b \). We have,

\[
L_\xi \Theta = L_\xi e_a \otimes \Theta^a + e_a \otimes L_\xi \Theta^a
\]

(59)

Then, if \( L_\xi \Theta = 0 \) we conclude that

\[
L_\xi \Theta^a = \chi^a_\theta \Theta^b,
\]

(60)

which is an infinitesimal Lorentz transformation of the torsion 2-forms. On the other hand, taking into account Cartan’s first structure equation, Eq. (57), and the fact that \( L_\xi d \theta^a = d(L_\xi \theta^a) \), we can write

\[
L_\xi \Theta^a = L_\xi d \theta^a + L_\xi \omega^a_b \wedge \theta^b + \omega^a_b \wedge L_\xi \theta^b
\]

\[
= d(\chi^a_b \theta^b) + L_\xi \omega^a_b \wedge \theta^b + \omega^a_b \wedge \chi^b_c \theta^c
\]

\[
= d(\chi^a_b) \wedge \theta^b + \chi^a_b d \theta^b + L_\xi \omega^a_b \wedge \theta^b + \chi^c_b \omega^a_b \wedge \theta^b.
\]

(61)

Also, using Eq. (60) we have

\[
L_\xi \Theta^a = \chi^a_b \theta^b + \chi^b_c \omega^c_b \wedge \theta^b.
\]

(62)

From Eqs. (61) and (62) it follows that \( L_\xi \omega^a_b \wedge \theta^b = \chi^a_c \omega^c_b \wedge \theta^b - \chi^b_c \omega^a_b \wedge \theta^b - d(\chi^a_b) \wedge \theta^b \), or

\[
L_\xi \omega^a_b = \chi^a_c \omega^c_b - \chi^b_c \omega^a_b - d\chi^a_b
\]

(63)

Thus, recalling Eq. (64) we finally have that \( L_\xi \omega^a_b = \delta_v \omega^a_b \). □

**Corollary 13** For any \( x \in M \), \( \theta_b \cdot L_\xi \theta^a \) is an element of \( \text{spin}^{\xi}_{1,3} \), if and only if, \( L_\xi g = 0 \).

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**Proof.** The Lie covariant derivative of $\theta^a$ is given by

$$L_\xi \theta^a = \xi^* \cdot D \theta^a + D (\xi^* \cdot \theta^a)$$

$$= \xi^* \cdot (d\theta^a + \omega_b^a \wedge \theta^b) + d(\xi^* \cdot \theta^a) + \omega_b^a (\xi^* \cdot \theta^b)$$

$$= L_\xi \theta^a + (\xi^* \cdot \omega_b^a) \theta^b - (\xi^* \cdot \theta^b) \omega_b^a + \omega_b^a (\xi^* \cdot \theta^b)$$

$$= L_\xi \theta^a + (\xi^* \cdot \omega_b^a) \theta^b$$

where we put $L_\xi \theta^a = \omega^a_b \theta^b$. Then,

$$\theta_b \cdot L_\xi \theta^a = \omega^a_b + \xi^* \cdot \omega^a_b.$$  \hspace{1cm} (65)

Now, we have already shown above that for any $x \in M$, the matrix of the $\xi^* \cdot \omega^a_b$ is an element of spin$_{1,3}$ and then, $\theta_b \cdot L_\xi \theta^a$ will be an element of spin$_{1,3}$ if and only if the matrix of the $\omega^a_b$ is an element of spin$_{1,3}$. The corollary is proved. \hfill \blacksquare

**Lemma 14** If $L_\xi g = 0$ and $L_\xi \Theta = 0$ then we have the identity

$$D (\theta_b \cdot L_\xi \theta^a) = \xi^* \cdot R^a_b = 0.$$ \hspace{1cm} (66)

**Proof.** Using the definitions of the exterior covariant derivative and the Lie covariant derivative we have

$$D (\theta_b \cdot L_\xi \theta^a) = d (\theta_b \cdot L_\xi \theta^a) + \omega^c_d (\theta_c \cdot L_\xi \theta^a) - \omega^c_d (\theta_b \cdot L_\xi \theta^c)$$

$$= d \{\theta_b \cdot [L_\xi \theta^a + (\xi^* \cdot \omega^a_b) \theta^c]\}$$

$$+ \{\theta_d \cdot [L_\xi \theta^a + (\xi^* \cdot \omega^a_b) \theta^c]\} \omega^d_b$$

$$- \{\theta_b \cdot [L_\xi \theta^d + (\xi^* \cdot \omega^d_c) \theta^c]\} \omega^c_d.$$  \hspace{1cm} (67)

i.e.,

$$D (\theta_b \cdot L_\xi \theta^a) = L_\xi \omega_b^a - \xi^* \cdot (d\omega^a_b + \omega^a_b \wedge \omega^a_b)$$

$$+ d(\theta_b \cdot L_\xi \theta^a) + \omega^c_d (\theta_c \cdot L_\xi \theta^a) - (\theta_b \cdot L_\xi \theta^c) \omega^c_d.$$  \hspace{1cm} (67)

If, $L_\xi g = 0$, then for any $x \in M$, $\theta_b \cdot L_\xi \theta^a \in$ spin$_{1,3}$ and the second line of Eq. (67) is an infinitesimal Lorentz transformation of the $\omega^a_b$. If besides that, also $L_\xi \Theta = 0$ then $L_\xi \omega^a_b = \delta^a_b, \omega^a_b$ and then the first term on the second member of Eq. (67) cancels the term in the second line. Then, taking into account Cartan’s second structure equation the proposition is proved. \hfill \blacksquare

**Proof.** (Proposition 11) We are now in conditions of presenting a proof of the Proposition 11. In order to do that we combine the results of Lemmas 12 and 13 with the identities given by Eqs. (66) and (67). We get,

$$d[(\xi^* \cdot \theta^a) \ast T_a] = D [(\xi^* \cdot \theta^a) \ast T_a]$$

$$= D (\xi^* \cdot \theta^a) \wedge \ast T_a + (\xi^* \cdot \theta^a) D \ast T_a$$

$$= L_\xi \theta^a \wedge \ast T_a - (\xi^* \cdot \theta^a) \wedge \ast T_a - (\xi^* \cdot \theta^a) D \ast T_a,$$
i.e.,
\[ d[(\xi^* \cdot \theta^a) \star T_a] = L_\xi \theta^a \wedge \star T_a - \star J^b_a \wedge (\xi^* \cdot J^b_a). \] (68)

Observe now that if \( A \in \sec \Lambda^1 T^*M \rightarrow \sec C\ell(M, g) \) then, \( \theta^a \wedge (\theta_a \cdot A) = A \). This permits us to write Eq. (68) as
\[ d[(\xi^* \cdot \theta^a) \star T_a] = - (\theta_b \cdot L_\xi \theta^a) \wedge \star T_a \wedge \theta^b - \star J^b_a \wedge (\xi^* \cdot J^b_a). \] (69)

If \( L_\xi g = 0 \), we have by the Corollary of Proposition 12 that for any \( x \in M \), \( \theta_a \cdot L_\xi \theta^a \in \text{spin}^1.3 \). In that case, we can write Eq. (69) as
\[ d[(\xi^* \cdot \theta^a) \star T_a] = - \frac{1}{2} (\theta_b \cdot L_\xi \theta^a) \wedge [\star T_a \wedge \theta^b - \star T^b \wedge \theta_a] - \star J^b_a \wedge (\xi^* \cdot J^b_a) \]
\[ = - (\theta_b \cdot L_\xi \theta^a) \wedge D \star J^b_a - \star J^b_a \wedge (\xi^* \cdot J^b_a). \] (70)

On the other hand, if \( L_\xi \Theta = 0 \), in view of Proposition 14 we can write
\[ d[(\xi^* \cdot \theta^a) \star T_a] = - D(\theta_b \cdot L_\xi \theta^a) \wedge \star J^b_a - (\theta_b \cdot L_\xi \theta^a) \wedge D \star J^b_a \]
\[ = - D((\theta_b \cdot L_\xi \theta^a) \wedge \star J^b_a) = - d[(\theta_b \cdot L_\xi \theta^a) \wedge \star J^b_a]. \] (71)

Finally, if \( L_\xi g = 0 \) and \( L_\xi \Theta = 0 \) we have
\[ d[(\xi^* \cdot \theta^a) \star T_a + (\theta_b \cdot L_\xi \theta^a) \wedge \star J^b_a] = 0, \]
which is the result we wanted to prove. ■

### 5 Pseudo Potentials in General Relativity

As we already said, in Einstein’s gravitational theory (General Relativity) each gravitational field is modelled by a Lorentzian spacetime \( \mathcal{M} = (M, g, D, r_g, \uparrow) \). The ‘gravitational field’ \( g \) is determined through Einstein’s equations by the energy-momentum of the matter fields \( \phi^A, \ A = 1, 2, ..., m \), living in \( \mathcal{M} \). As shown in details in, e.g., 53, 55 Einstein’s equations can be written using the Clifford bundle formalism in terms of the fields \( \theta^a \in \sec \Lambda^1 T^*M \rightarrow \sec C\ell(M, g) \), where \( \{ \theta^a \} \) is an orthonormal basis of \( T^*M \) as
\[ - (\partial \cdot \partial) \theta^a + \partial \wedge (\partial \cdot \theta^a) + \partial \cdot (\partial \wedge \theta^a) + \frac{1}{2} T \theta^a = T^a, \] (72)
where \( \partial = \theta^a D_{e_a} \) is the Dirac operator acting on sections of the Clifford bundle. An explicit Lagrangian giving that equation 13, which differs from the original Einstein-Hilbert Lagrangian 14 by an exact differential is
\[ \mathcal{L}_g = - \frac{1}{2} d\theta^a \wedge \star d\theta^a + \frac{1}{2} \delta \theta^a \wedge \star \delta \theta^a + \frac{1}{4} (d\theta^a \wedge \theta_a) \wedge \star (d\theta^b \wedge \theta_b). \] (73)

\textsuperscript{13} Eq. (73) is equivalent to a Lagrangian density first introduced by 37.

\textsuperscript{14} Recall that the Einstein-Hilbert Lagrangian density is \( \mathcal{L}_{EH} = \frac{1}{2} R_g = \frac{1}{2} R_{e\bar{a}d} \wedge \star (\theta^e \wedge \theta^d) \) and \( \mathcal{L}_{EH} = \mathcal{L}_g - d(\theta^a \wedge \star d\theta^a) \).
The total Lagrangian density of the gravitational field and the matter fields can then be written as

\[ \mathcal{L} = \mathcal{L}_g + \mathcal{L}_m, \]  

(74)

where \( \mathcal{L}_m(\theta^a, d\theta^a, \phi^A, d\phi^A) \) is the matter Lagrangian.

Now, variation of \( \mathcal{L} \) with respect to the the fields \( \theta^a \) yields after a very long calculation (see, e.g., [53]) the following Euler-Lagrange equations

\[ - * G^a = \frac{\partial \mathcal{L}_g}{\partial \theta^a} + d \left( \frac{\partial \mathcal{L}_g}{\partial d\theta^a} \right) = * t^a + d * S^a = - * T^a, \]  

(75)

where \( G^a = (\mathcal{R}^a - \frac{1}{2} R \theta^a) \in \sec \wedge^1 T^* M \hookrightarrow \mathcal{C} \ell (T^* M, g) \) are the Einstein 1-forms, \( R^a = R^a_b \theta^b \in \sec \wedge^1 T^* M \hookrightarrow \mathcal{C} \ell (T^* M, g) \) are the Ricci 1-forms, \( R \) is the scalar curvature, \( * T^a = \frac{\partial \mathcal{L}_m}{\partial \theta^a} \in \sec \wedge^1 T^* M \hookrightarrow \mathcal{C} \ell (T^* M, g) \) are the energy-momentum 1-forms of the matter fields,\(^{15}\) and where

\[ * S^c = \frac{\partial \mathcal{L}_m}{\partial d\theta^a} = \frac{1}{2} \omega_{ab} \wedge * (\theta^a \wedge \theta^b \wedge \theta^c) \in \sec \wedge^2 T^* M \hookrightarrow \mathcal{C} \ell (T^* M, g), \]  

(76)

The proof that the second and third members of Eq. (75) are equal follows at once from the fact that the connection 1-forms of the Levi-Civita connection of \( g \) can be written as it is trivial to verify as

\[ \omega^{cd} = \frac{1}{2} \left[ \theta^d \wedge d\theta^c - \theta^c \wedge d\theta^d + \theta^{[c} \wedge (\theta^{d]} \wedge d\theta^a) \theta^a \right], \]  

(77)

and that

\[ * G^d = - \frac{1}{2} R_{ab} \wedge * (\theta^a \wedge \theta^b \wedge \theta^d). \]  

(78)

Indeed, we can write

\[ \frac{1}{2} R_{ab} \wedge * (\theta^a \wedge \theta^b \wedge \theta^d) = - \frac{1}{2} * [R_{ab} \wedge (\theta^a \wedge \theta^b \wedge \theta^d)] \]

\[ = - \frac{1}{2} R_{ab} \wedge * [(\theta^c \wedge \theta^d) \wedge (\theta^a \wedge \theta^b \wedge \theta^d)] \]

\[ = - * (\mathcal{G}^d - \frac{1}{2} R \theta^d). \]  

(79)

\(^{15}\)Recall that due to our conventions in the writing of Einstein equations the true physical energy-momentum densities are \( * T^a = - * T^a \). The objects \( * a \) and \( d * S^a \) are more easily found by variation of \( \mathcal{L}_{EH} \) instead of the variation of \( \mathcal{L}_g \), which of course, give the same equations of motion.
On the other hand we have,

\[-2 \star G^d = d\omega_{ab} \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d) + \omega_{ac} \wedge \omega_B^a \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)\]

\[= d[\omega_{ab} \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)] + \omega_{ac} \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)\]

\[+ \omega_{ac} \wedge \omega_B^a \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)\]

\[= d[\omega_{ab} \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)] - \omega_{ab} \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)\]

\[= d[\omega_{ab} \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)] - \omega_{ab} \wedge [\omega_D^a \wedge \star(\theta^a \wedge \theta^b \wedge \theta^d)]\]

\[= 2(d \star S^d + \star t^d). \quad (80)\]

Now, we can write Einstein’s equation in a very interesting, but dangerous form\(^{16}\), i.e.:

\[-d \star S^a = \star T^a + \star t^a. \quad (81)\]

In writing Einstein’s equations in that way, we have associated to the gravitational field a set of 2-form fields \(\star S^a\) called superpotentials that have as sources the currents \((\star T^a + \star t^a)\). However, superpotentials are not uniquely defined since, e.g., superpotentials \((\star S^a + \star \alpha^a)\), with \(\star \alpha^a\) closed, i.e., \(d \star \alpha^a = 0\) give the same second member for Eq.\((81)\).

\[\text{5.1 Is There Any Energy-Momentum Conservation Law in GRT?}\]

Why did we say that Eq.\((81)\) is a dangerous one?

The reason is that if we are ignorant of the discussion of the previous section we may be led to think that we have discovered a conservation law for the energy momentum of matter plus gravitational field, since from Eq.\((81)\) it follows that

\[d(\star T^a + \star t^a) = 0. \quad (82)\]

This thought however is only an example of wishful thinking, because the \(\star t^a\) depends on the connection (see Eq.\((76)\)) and thus are gauge dependent. They do not have the same tensor transformation law as the \(\star T^a\). So, Stokes theorem cannot be used to derive from Eq.\((82)\) conserved quantities that are independent of the gauge, which is clear. However—and this is less known—Stokes theorem also cannot be used to derive conclusions that are independent of the local coordinate chart used to perform calculations \[8\]. In fact, the currents \(\star t^a\) are nothing more than the old pseudo energy-momentum tensor of Einstein in a new dress. Non recognition of this fact can lead to many misunderstandings.

\[\text{Note}\]

\[\text{Eq.\((81)\) is known in recent literature of GR as Sparling equations \[??\] because it appears (in an equivalent form) in a preprint \[??\] of 1982 by that author. However, it already appeared early, e.g., in a 1978 paper by Thirring and Wallner \[64\].}\]
We present some of them in what follows, in order to call our readers’ attention of potential errors of inference that can be done when we use sophisticated mathematical formalisms without a perfect domain of their contents.

(i) First, it is easy to see that from Eq.(75) it follows that
\[ D \star G = D \star T = 0, \]
where \( \star G = e_a \otimes \star G^a \in \text{sec} T M \otimes \text{sec} \Lambda^3 T^* M \) and \( \star T = e_a \otimes \star T^a \in \text{sec} T M \otimes \text{sec} \Lambda^3 T^* M \) and where
\[ D \star \phi := e_a \otimes D \star G^a, \quad D \star \Omega = e_a \otimes D \star T^a \] (84)
and \( D \) is the exterior covariant derivative of index valued forms (7, 55). Now, in [34] it is written (without proof) a ‘Stokes theorem’
\[ \int_{4\text{-cube}} D \star \Omega = \int_{3\text{-boundary of this 4-cube}} \star \Omega. \] (85)

Not a single proof (which we can consider as valid) of Eq.(85) which appears also in many other texts and scientific papers as, e.g., in [11, 73] has been given in any paper we know. The reason is the following. The first member of Eq.(85) is no more than
\[ \int_{4\text{-cube}} e_a \otimes (d \star T^a + \omega^a_b \wedge \star T^b). \] (86)
Thus it is necessary to explain what is the meaning (if any) of the integral. Since the integrand is a sum of tensor fields, this integral says that we are adding tensors belonging to the tensor spaces of different spacetime points. As it is well known, this cannot be done in general, unless there is a way of identifying the tensor spaces at different spacetime points. This requires, of course, the introduction of additional structure on the spacetime representing a given gravitational field, and such extra structure is lacking in Einstein theory. We must conclude that Eq.(85) do not express any conservation law, for it lacks as yet, a precise mathematical meaning.

In Einstein theory possible superpotentials are, of course, the \( \star S^a \) that we identified above (Eq.(76)), with
\[ \star S_c = \frac{1}{2} \omega_{ab,c}(\theta^a \wedge \theta^b \wedge \theta_c)] \theta^c. \] (87)

Then, if we integrate Eq.(81) over a ‘certain finite 3-dimensional volume’, say a ball \( B \), and use Stokes theorem we have\(^\text{17}\)
\[ P^a := -\frac{1}{8\pi} \int_B \star (T^a + t^a) = \frac{1}{8\pi} \int_B \star (T^a - t^a) = \frac{1}{8\pi} \int_{\partial B} \star S^a. \] (88)
\(^\text{17}\)The reason for the factor 8\(\pi\) in Eq.(88) is that we choose units where the numerical value gravitational constant 8\(\pi G/c^4\) is 1, where \( G \) is Newton gravitational constant.
In particular the energy or *(inertial mass)* of the gravitational field plus matter generating the field is defined by

\[ P^0 = E = m_I = \frac{1}{8\pi} \lim_{R \to \infty} \int_{\partial B} * S^0. \quad (89) \]

(ii) Now, a frequent misunderstanding is the following. Suppose that in a given hypothetical gravitational theory there exists an energy-momentum conservation law for matter plus the gravitational field expressed in the form of Eq. (82), where \( T^a \) are the energy-momentum 1-forms of matter and \( t^a \) are true energy-momentum 1-forms of the gravitational field. This means that the 3-forms \((*T^a + *t^a)\) are closed, i.e., they satisfy Eq. (82). Is this enough to warrant that the energy of a closed universe is zero? Well, that would be the case if starting from Eq. (82) we could jump to an equation like Eq. (81) and then to Eq. (89) (as done, e.g., in [64]). But that sequence of inferences in general cannot be done, for indeed, as it is well known, it is not the case that closed three forms are always exact. Take, for example, a closed universe with topology \( \mathbb{R} \times S^3 \). In this case \( B = S^3 \) and we have \( \partial B = \partial S^3 = \emptyset \). Now, as it is well known (see, e.g., [41]), the third de Rham cohomology group of \( \mathbb{R} \times S^3 \) is \( H^3(\mathbb{R} \times S^3) = H^3(S^3) = \mathbb{R} \). Since this group is non trivial it follows that in such manifold closed forms are not exact. Then from Eq. (82) it did not follow the validity of an equation analogous to Eq. (81). So, in that case an equation like Eq. (88) cannot even be written.

Despite that commentary, keep in mind that in Einstein’s theory the ‘energy’ of a closed universe \(^{19}\) supposed to be given by Eq. (89) is indeed zero, since in that theory the 3-forms \((*T^a + *t^a)\) are indeed exact (see Eq. (81)). This means that accepting \( t^a \) as the energy-momentum 1-form fields of the gravitational field, it follows that gravitational energy must be negative in a closed universe.

(iii) But, is the above formalism a consistent one? Given a coordinate chart with "Cartesian" like coordinates \( \{x^\mu\} \) of the atlas of \( M \), with some algebra (left as exercise to the reader) one can show that for a gravitational model represented by a diagonal asymptotic flat metric \(^{20}\), the inertial mass \( E = m_I \) is given by

\[ m_I = -\frac{1}{16\pi} \lim_{r \to \infty} \int_{\partial B} \frac{x_i}{r} \frac{\partial}{\partial x^j} (g_{11}g_{22}g_{33}g^{ij}) r^2 d\Omega, \quad (90) \]

where \( \partial B = S^2(r) \) is a 2-sphere of radius \( r \), \( g_{ij}x^j = x_i \) and \( d\Omega \) is the element of solid angle. If we apply Eq. (89) to calculate, e.g., the energy of the Schwarzschild space time \(^{21}\) generate by a gravitational mass \( m \), we expect to have one unique

\(^{18}\)This means that the \( t^a \) in the hypothetical theory are not pseudo 1-forms, as is the case in Einstein’s theory.

\(^{19}\)Note that if we suppose that the universe contains spinor fields, as we indeed did, then it must be a spin manifold, i.e., it is parallelizable according to Geroch’s theorem [21].

\(^{20}\)A metric is said to be asymptotically flat in given coordinates, if \( g_{\mu\nu} = n_{\mu\nu}(1 + O(r^{-k})) \), with \( k = 2 \) or \( k = 1 \) depending on the author. See, e.g., [55, 59, 74].

\(^{21}\)For a Schwarzschild spacetime we have \( g = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi). \)
and unambiguous result, namely $m_1 = m$.

However, as shown in details, e.g., in [8] the calculation of $E$ depends on the spatial coordinate system naturally adapted to the reference frame $Z = \frac{1}{\sqrt{(1 - \frac{2m}{r})}} \frac{\partial}{\partial t}$, even if these coordinates produce asymptotically flat metrics. Then, even if in one given chart we may obtain $m_1 = m$ there are others where $m_1 \neq m$.

Moreover, note also that, as shown above, for a closed universe Einstein’s theory implies on general grounds (once we accept that the $t^a$ describes the energy-momentum distribution of the gravitational field) that $m_1 = 0$. This result — it is important to quote — does not contradict the so called “positive mass theorems” of, e.g., references [58, 59, 77], because those theorems refer to the total energy of an isolated system. A system of that kind is supposed to be modelled by a Lorentzian spacetime having a spacelike, asymptotically Euclidean hypersurface. However, we emphasize, although the energy results positive, its value is not unique, since depends on the asymptotically flat coordinates chosen to perform the calculations, as it is clear from the elementary example of the Schwarzschild field commented above and detailed in [8].

In a book written in 1970, Davis [13] said:

“Today, some 50 years after the development of Einstein’s generally covariant field theory it appears that no general agreement regarding the proper formulation of the conservation laws has been reached.”

Well, we hope that the reader has been convinced that the fact is: there are in general no conservation laws of energy-momentum in General Relativity. Moreover, all discourses (based on Einstein’s equivalence principle) concerning the use of pseudo-energy momentum tensors as reasonable descriptions of energy and momentum of gravitational fields in Einstein’s theory are not convincing.

And, at this point it is better to quote page 98 of Sachs&Wu [57]:

“As mentioned in section 3.8, conservation laws have a great predictive power. It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity. Many of the attempts to resurrect it are quite interesting; many are simply garbage.”

In GRT—we already said—every gravitational field is modelled (module diffeomorphisms and according to present wisdom) by a Lorentzian spacetime. In that particular case, when this spacetime structure admits a timelike Killing vector, we may formulate a law of energy conservation for the matter fields. Also, if the Lorentzian spacetime admits three linearly independent spacelike

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22 This observation is true even if we use the so called ADM formalism [5]. To be more precise, let us recall that we have a well defined ADM energy only if the fall off rate of the metric is in the interval $1/2 < k < 1$. For details, see [40].

23 The proof also uses as hypothesis the so called energy dominance condition [20].

24 Like, e.g., in [4, 49, 34] and many other textbooks. It is worth to quote here that, at least, Anderson [4] explicitly said: "In an interaction that involves the gravitational field a system can lose energy without this energy being transmitted to the gravitational field.”
Killing vectors, we have a law of conservation of momentum for the matter fields.

This follows at once from the theory developed in the previous section. Indeed, in the particular case of General Relativity, the Lagrangian density of the matter field is not supposed to be explicitly dependent on the $\omega^a_{b\,c}$. Then, $\frac{\partial L}{\partial \omega^a_{b\,c}} = 0$ in Eq. (54) and writing $\mathcal{T}(\xi) = \xi^n T^n$, it becomes $d \ast \mathcal{T}(\xi) = 0$, or

$$\delta \mathcal{T}(\xi) = 0.$$ (91)

The crucial fact to have in mind here is that a general Lorentzian spacetime, does not admit such Killing vectors in general, as it is the case, e.g., of the popular Friedmann-Robertson-Walker expanding universes models.

At present, the authors know only one possibility of resurrecting a trustworthy conservation law of energy-momentum valid in all circumstances in a theory of the gravitational field that resembles General Relativity (in the sense of keeping Einstein’s equation). It consists in reinterpreting that theory as a field theory in flat Minkowski spacetime. Theories of this kind have been proposed in the past by, e.g., Feynman [18], Schwinger [60], Thirring [62] and Weinberg [76] among others and have been extensively studied by Logunov and collaborators in a series of papers summarized in the monographs [28, 29] and also in [52, 54].

6 Is there any Angular Momentum Conservation law in the GRT

If the $\{\theta^a\}$ and the $\{\omega^a_{b\,c}\}$ are varied independently in the Einstein-Hilbert Lagrangian then, as it is easy to verify we get the additional field equation

$$D_\ast \theta^{ab} = J^{ab}$$ (92)

From this equation we get immediately

$$d \ast \theta^a_{b\,c} = J^a_{b\,c} - \omega^a_{c\,d} \wedge \ast \theta^d_{b\,c} + \ast \theta^d_{b\,c} \wedge \omega^a_{c\,d}$$ (93)

and one is tempted to define $S^a_{b\,c} = (\omega^a_{c\,d} \wedge \ast \theta^d_{b\,c} + \ast \theta^d_{b\,c} \wedge \omega^a_{c\,d})$ as the density of spin angular momentum of the gravitational field and the (total) angular momentum of the system as

$$L^a_{b\,c} := \int_{S^2} \ast \theta^a_{b\,c}.$$ (94)

This definition, of course, has the same problems as the definition of energy in the GRT because $S^a_{b\,c}$ is gauge dependent.

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$^{25} \theta^{ab} = \theta^a \wedge \theta^b$ is known [61] as Bramson [9] superpotential
7 Conservation Laws in the Teleparallel Equivalent of General Relativity

We observe that recently it was claimed \[12\] a valid way of formulating a genuine energy-momentum conservation law in a theory equivalent to General Relativity. In that theory, the so-called teleparallel equivalent of General Relativity theory \[30\], spacetime is teleparallel (or Weitzenböck), i.e., has a metric compatible connection with non zero torsion and with null curvature.\[26\] However, the claim of \[12\] is investigated in more detail below. Indeed, we have two important comments (a) and (b) concerning this issue.

(a) First, it must be clear that the mathematical structure of the teleparallel equivalent of General Relativity consists in the introduction of: (i) a bilinear form (a deformed metric tensor) \( g = \eta_{ab} \theta^a \otimes \theta^b \) and (ii) a teleparallel connection in a manifold \( M \simeq \mathbb{R}^4 \) (the same which appears in the Minkowski spacetime structure). Indeed, taking advantage of the the discussion of the previous sections, we can present that theory with a cosmological constant term as follows. Start with the Lagrangian density \( L' = L_g + L_m \), where \[27\]

\[
L'_g = -\frac{1}{2} \mathcal{d}\theta^a \wedge \ast \mathcal{d}\theta_a + \frac{1}{2} \delta\theta^a \wedge \ast \delta\theta_a + \frac{1}{4} (d\theta^a \wedge \theta_a) \wedge \ast (d\theta^b \wedge \theta_b) + \frac{1}{2} m^2 \theta_a \wedge \ast \theta^a
\]

and write it (after some algebraic manipulations) as

\[
L'_g = -\frac{1}{2} \mathcal{d}\theta^a \wedge \ast \left[ \mathcal{d}\theta_a - \theta_a \wedge (\theta_b \ast d\theta_b) + \frac{1}{2} \star \theta_a \wedge \ast (d\theta^b \wedge \theta_b) \right] + \frac{1}{2} m^2 \theta_a \wedge \ast \theta^a,
\]

where

\[
\mathcal{d}\theta^a = (1) d\theta^a + (2) \bar{d}\theta^a + (3) \hat{d}\theta^a,
\]

\[
(1) d\theta^a = d\theta^a - (2) d\theta^a - (3) d\theta^a,
\]

\[
(2) d\theta^a = \frac{1}{3} \theta^a \wedge (\theta_b \ast d\theta_b),
\]

\[
(3) d\theta^a = -\frac{1}{3} \ast (\theta^a \wedge \ast (d\theta^b \wedge \theta_b)).
\]

Next introduce a teleparallel connection by declaring that the cobasis \( \{\theta^a\} \) fixes the parallelism, i.e., we define the torsion 2-forms by

\[
\Theta^a := d\theta^a,
\]

and \( L'_g \) becomes

\[
L'_g = -\frac{1}{2} \Theta^a \wedge \ast \left[ (1) \Theta^a - (2) \bar{\Theta}^a - (3) \hat{\Theta}^a \right] + \frac{1}{2} m^2 \theta_a \wedge \ast \theta^a,
\]

\[26\] In fact, formulation of teleparallel equivalence of General Relativity is a subject with an old history. See, e.g., \[24, 26, 39, 32, 72\].

\[27\] Field equations in Maxwell like form for \( F^a = d\theta^a \) are presented in \[43\].
where \( (1) \Theta^a = (1) d\theta^a \), \( (2) \Theta^a = (2) d\theta^a \) and \( (3) \Theta^a = (3) d\theta^a \), called tractor (four components), axitor (four components) and tentor (sixteen components) are the irreducible components of the tensor torsion under the action of \( SO^e_{1,3} \).

(b) Recalling the results of the previous sections we now show that even if the metric of a given teleparallel spacetime has some Killing vector fields there are genuine conservation laws involving only the energy-momentum and angular momentum tensors of matter only if some additional condition is satisfied. Indeed, in the teleparallel basis where \( \nabla_\xi, e_b = 0 \) and \([e_m,e_n] = e^a_{mn}e_a\) we have that the torsion 2-forms satisfies
\[
\Theta^a = d\theta^a = -\frac{1}{2} e^a_{mn} \theta^m \wedge \theta^n = \frac{1}{2} T^a_{mn} \theta^m \wedge \theta^n. \tag{99}
\]
Then, recalling once again that \( \mathcal{L}_\xi (d\theta^a) = d(\mathcal{L}_\xi \theta^a) = d(\xi^b \rho^b) \) and \( \mathcal{L}_\xi \Theta = 0 \) we can use \( \mathcal{L}_\xi \) (which express the condition \( \mathcal{L}_\xi \Theta = 0 \)) to write
\[
d(\xi^b \rho^b) = \xi^b d\rho^b, \tag{100}
\]
which implies
\[
d\xi^a \wedge \theta^b = 0. \tag{101}
\]
Then, Eq.\( \tag{101} \) is satisfied only if the torsion tensor of the teleparallel spacetime satisfy the following differential equation:
\[
T^m_{bd} e_m(\xi^a) + e_d(\xi^m T^a_{bm}) - e_b(\xi^m T^a_{dm}) = 0. \tag{102}
\]
Of course, Eq.\( \tag{102} \) is in general not satisfied for a vector field \( \xi \) that is simply a Killing vector of \( g \). This means that in the teleparallel equivalent of General Relativity even if there are Killing vector fields, this in general do not warrant that there are conservation laws as in Eq.\( \tag{54} \) involving only the energy and angular momentum tensors of matter.

Next, we remark that from \( \mathcal{L}_\xi \) we get as field equations (in an arbitrary basis, not necessarily the teleparallel one) satisfied by the gravitational field the Eq.\( \tag{81} \), i.e.,
\[
-d * \mathcal{S}^a = * T^a + * t^a, \tag{103}
\]
with
\[
* t^a = * t^a + m^2 * \theta^a
\]
and \( \mathcal{S}^a \) and \( t^a \) given in Eq.\( \tag{76} \) where it must also be taken into account that in the teleparallel equivalent of General Relativity and using the teleparallel basis the Levi-Civita connection 1-forms \( \omega^a_b \) there must be substituted by \(-\kappa^a_{bd} \), with
\[
\kappa^c_{ed} = \frac{1}{2} \left[ \theta^d \rho^c - \theta^c d\rho^d + (\theta^d \rho^c d\rho^d) \theta^a \right]
= \frac{1}{2} \left[ \theta^d \Theta^c - \theta^c \theta^d + (\theta^d \rho^c (\Theta^d \theta^a)) \theta^a \right], \tag{104}
\]

24
where $\kappa^a = K^a_{bc} \theta^c$, with $K^a_{bc}$ the components of the so called contorsion tensor. We have,

$$\star t^c = \frac{1}{2} \kappa_{ab} \wedge \ast(\kappa_c^d \wedge \ast((\theta^a \wedge \theta^b \wedge \theta^c) + \kappa^b_d \wedge \ast((\theta^a \wedge \theta^b \wedge \theta^c))).$$

Under a change of gauge, $\theta^a \mapsto U \theta^a = \Lambda^a_b \theta^b \ (U \in \text{sec Spin}^e_{1,3}(M) \hookrightarrow \mathcal{E}(M, g), \ \Lambda^a_b(x) \in \text{SO}^e_{1,3}, \ \forall \ x \in M)$, we have that $\Theta^a \mapsto \Theta' a = \Lambda^a_b \Theta^b$. It follows that the $t^a$, which are the components of the energy-momentum 1-forms $t^a = t^a_b \theta^b$ defines a tensor field.

We then conclude that for each gravitational field modelled by a particular teleparallel spacetime, if the cosmological term is null or not there is a conservation law of energy-momentum for the coupled system of the matter field and the gravitational field which is represented by that particular teleparallel spacetime. Although the existence of such a conservation law in the teleparallel spacetime is a satisfactory fact with respect of the usual formulation of the gravitational theory where gravitational fields are modelled by Lorentzian spacetimes and where genuine conservation laws (in general) does not exist because in that theory the components of $t^a$ defines only a pseudo-tensor, we cannot forget observation (a): the teleparallel equivalent of General Relativity consists in the introduction of: (i) a bilinear form (a deformed metric tensor) $g = \eta_{ab} \theta^a \otimes \theta^b$ and (ii) a teleparallel connection in the manifold $M \simeq \mathbb{R}^4$ of Minkowski spacetime structure. The crucial ingredient is still the Einstein-Hilbert Lagrangian density.

Finally we must remark that if we insist in working with a teleparallel spacetime we lose in general the other six genuine angular momentum conservation laws which always hold in Minkowski spacetime. Indeed, we do not obtain in general even the chart dependent angular momentum ‘conservation’ law of GRT. The reason is that if we write the equivalent of Eq. (81) in a chart $(U, \varphi)$ with coordinates $\{x^\mu\}$ for $U \subset M$ we did not get in general that $dx^\mu \wedge \star t^\nu = dx^\nu \wedge \star t^\mu$, which as well known is necessary in order to have a chart dependent angular momentum conservation law [63].

8 Conclusions

We recall that the problem of the conservation laws of energy-momentum and angular momentum in GRT occupied the mind of many people since Einstein [15] introduced the so called energy-momentum pseudo-tensor in 1916. Besides those papers that already have been quoted above it is worth to cite also [5, 10, 19, 20, 21, 23, 24, 27, 28, 29, 30, 32, 33, 35, 36, 42, 65], which—summed with the quote of [57] presented in Section 5—have been the inspiration for the present work, where we recalled (a) under which conditions there exists genuine conservation laws of energy-momentum and angular momentum involving only the matter fields on a general RCST and (b) under which conditions there exists genuine conservation laws involving both the energy-momentum and angular momentum tensors of the matter and the gravitational field, when this latter concept can be rigorously defined.
Using a Clifford bundle formalism it was shown that in case (a) contrary to the case of GRT the simply existence of Killing vector fields is not enough, since a new additional condition must hold. Some examples are presented in Appendix B.

Concerning case (b) our conclusion is that genuine laws involving both the energy-momentum and angular momentum tensors of the matter and the gravitational field exist only in a field theory of the gravitational field formulated in Minkowski spacetime. We analyzed also a particular case of a RCST theory, namely the so called teleparallel equivalent of GRT [30] [31] [12]. In that theory a genuine conservation law of energy-momentum is obtained through the introduction of a teleparallel connection, needed to restore active Local Lorentz invariance. However, in the teleparallel equivalent of GRT, it is not possible (in general) to formulate even a chart dependent conservation law for the angular momentum of matter or for both the matter and gravitational fields. Due to this fact, in our opinion it cannot be considered more general than a formulation of a theory of the gravitational field which uses a deformation tensor in Minkowski spacetime structure [52] [43], where the introduction of general connections are not needed.

A Clifford and Spin-Clifford Bundles

Let \( \mathcal{M} = (M, g, \nabla, \tau, \uparrow) \) be an arbitrary Riemann-Cartan spacetime. The quadruple \((M, g, \tau, \uparrow)\) denotes a four-dimensional time-oriented and space-oriented Lorentzian manifold. This means that \( g \in \sec T^3_0M \) is a Lorentzian metric of signature \((1,3)\), \( \tau \in \sec \Lambda^1(T^*M) \) and \( \uparrow \) is a time-orientation (see details, e.g., in [57]). Here, \( T^*M \) is the cotangent [tangent] bundle, \( T^*M = \cup_{x \in M} T^*_x M, \)

\[ T^*M = \cup_{x \in M} T^*_x M, \quad T_x M \simeq T^*_x M \simeq \mathbb{R}^{1,3}, \] where \( \mathbb{R}^{1,3} \) is the Minkowski vector space. \( \nabla \) is an arbitrary metric compatible connection, i.e., \( \nabla g = 0 \), but in general, \( R^\nabla \neq 0, \Theta^\nabla \neq 0, \) \( R^\nabla \) and \( \Theta^\nabla \) being respectively the curvature and torsion tensors of the connection \( \nabla \). When \( R^\nabla \neq 0, \Theta^\nabla \neq 0, \) \( \mathcal{M} \) is called a Riemann-Cartan spacetime. When \( R^\nabla = 0, \Theta^\nabla = 0, \) \( \mathcal{M} \) is called a teleparallel (or Weintzböck) spacetime. For a Lorentzian spacetime the connection is the Levi-Civita connection \( D \) of \( g \) for which \( R^D = 0, \Theta^D = 0, \) \( M \simeq \mathbb{R}^4 \). Let \( g \in \sec T^3_0M \) be the metric of the cotangent bundle. The Clifford bundle of differential forms \( \mathcal{O}(M, g) \) is the bundle of algebras, i.e., \( \mathcal{O}(M, g) = \cup_{x \in M} \mathcal{O}(T^*_x M, g) \), where \( \forall x \in M, \mathcal{O}(T^*_x M, g) = \mathbb{R}_{1,3} \), the so called spacetime algebra [54]. Recall also that \( \mathcal{O}(M, g) \) is a vector bundle associated to the orthonormal frame bundle, i.e., \( \mathcal{O}(M, g) = F_{SO^+(1,3)}(M) \times_{Ad} \mathcal{Cl}_{1,3} [27] [38] \). For any \( x \in M, \mathcal{O}(T^*_x M, g|_x) \) as a linear space over the real field \( \mathbb{R} \) is isomorphic to the Cartan algebra \( \bigwedge T^*_x M \)

\[ \bigwedge T^*_x M \]

\[ \bigwedge T^*_x M \]

\[ \bigwedge T^*_x M \]

\[ \bigwedge T^*_x M \]
of the cotangent space. \( \bigwedge T^*_x M = \oplus_{k=0}^4 \bigwedge^k T^*_x M \), where \( \bigwedge^k T^*_x M \) is the \( \binom{n}{k} \)-dimensional space of \( k \)-forms. Then, sections of \( \mathcal{O}(M, g) \) can be represented as a sum of non homogeneous differential forms, that will be called Clifford (multiform) fields. Let \( \{e_a\} \in \sec P_{SG(1,3)}(M) \) (the frame bundle) be an orthonormal basis for \( TU \subset TM \), i.e., \( g(e_a, e_a) = \eta_{ab} = \text{diag}(1, -1, -1, -1) \). Let \( \theta^a \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{O}(M, g) \) \((a = 0, 1, 2, 3)\) be such that the set \( \{\theta^a\} \) is the dual basis of \( \{e_a\} \).

**A.1 Clifford Product**

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by \( \theta^a \theta^b + \theta^b \theta^a = 2\eta^{ab} \) and if \( C \in \sec \mathcal{O}(M, g) \) we have

\[
C = s + v_a \theta^a + \frac{1}{2!} f_{ab} \theta^a \theta^b + \frac{1}{3!} t_{abc} \theta^a \theta^b \theta^c + p \theta^5 ,
\]

where \( \tau_g = \theta^5 = \theta^0 \theta^1 \theta^2 \theta^3 \) is the volume element and \( s, v_a, f_{ab}, t_{abc}, p \in \sec \bigwedge^0 T^* M \hookrightarrow \sec \mathcal{O}(M, g) \).

For \( A_r \in \sec \bigwedge^r T^* M \hookrightarrow \sec \mathcal{O}(M, g), B_s \in \sec \bigwedge^s T^* M \hookrightarrow \sec \mathcal{O}(M, g) \) we define the *exterior product* in \( \mathcal{O}(M, g) \) \((\forall r, s = 0, 1, 2, 3)\) by

\[
A_r \wedge B_s = \langle A_r B_s \rangle_{r+s},
\]

where \( \langle \quad \rangle_k \) is the component in \( \bigwedge^k T^* M \) of the Clifford field. Of course, \( A_r \wedge B_s = (-1)^r B_s \wedge A_r \), and the exterior product is extended by linearity to all sections of \( \mathcal{O}(M, g) \).

Let \( A_r \in \sec \bigwedge^r T^* M \hookrightarrow \sec \mathcal{O}(M, g), B_s \in \sec \bigwedge^s T^* M \hookrightarrow \sec \mathcal{O}(M, g) \). We define a *scalar product* in \( \mathcal{O}(M, g) \) (denoted by \( \cdot \)) as follows:

(i) For \( a, b \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{O}(M, g) \),

\[
a \cdot b = \frac{1}{2}(ab + ba) = g(a, b).
\]

(ii) For \( A_r = a_1 \wedge ... \wedge a_r, B_r = b_1 \wedge ... \wedge b_r, a_i, b_j \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{O}(M, g) \), \( i, j = 1, ..., r \),

\[
A_r \cdot B_r = (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r) = \begin{vmatrix} a_1 \cdot b_1 & \ldots & a_1 \cdot b_r \\ \ldots & \ldots & \ldots \\ a_r \cdot b_1 & \ldots & a_r \cdot b_r \end{vmatrix}.
\]

We agree that if \( r = s = 0 \), the scalar product is simply the ordinary product in the real field.

Also, if \( r \neq s \), then \( A_r \cdot B_s = 0 \). Finally, the scalar product is extended by linearity for all sections of \( \mathcal{O}(M, g) \).
For \( r \leq s \), \( A_r = a_1 \land \ldots \land a_r \), \( B_s = b_1 \land \ldots \land b_s \), we define the left contraction \( \cup : (A_r, B_s) \mapsto A_r \cup B_s \) by

\[
A_r \cup B_s = \sum_{i_1 < \ldots < i_r} e^{1_{1\ldots i_r}} (a_1 \land \ldots \land a_r) \cdot (b_1 \land \ldots \land b_r)^{\sim} b_{i_r+1} \land \ldots \land b_s \tag{110}
\]

where \( \sim \) is the reverse mapping (reversion) defined by

\[
\sim : \text{sec} \mathcal{C}(M, g) \rightarrow \text{sec} \mathcal{C}(M, g),
\]

\[
\tilde{A} = \sum_{p=0}^4 \tilde{A}_p = \sum_{p=0}^4 (-1)^{\frac{1}{2}k(k-1)} A_p,
\]

\[
A_p \in \text{sec} \bigwedge^p T^* M \mapsto \text{sec} \mathcal{C}(M, g). \tag{111}
\]

We agree that for \( \alpha, \beta \in \text{sec} \bigwedge^0 T^* M \) the contraction is the ordinary (pointwise) product in the real field and that if \( \alpha \in \text{sec} \bigwedge^0 T^* M \), \( A_r \in \text{sec} \bigwedge^r T^* M \), \( B_s \in \text{sec} \bigwedge^s T^* M \) \( \Rightarrow \) then \( (\alpha A_r) \cup B_s = A_r \cup (\alpha B_s) \). Left contraction is extended by linearity to all pairs of sections of \( \mathcal{C}(M, g) \), i.e., for \( A, B \in \text{sec} \mathcal{C}(M, g) \)

\[
A \cup B = \sum_{r,s} \langle A \rangle_r \cup \langle B \rangle_s, \quad r \leq s \tag{112}
\]

It is also necessary to introduce the operator of right contraction denoted by \( \triangleleft \). The definition is obtained from the one presenting the left contraction with the imposition that \( r \geq s \) and taking into account that now if \( A_r \in \text{sec} \bigwedge^r T^* M \), \( B_s \in \text{sec} \bigwedge^s T^* M \) then \( A_r \cup (\alpha B_s) = (\alpha A_r) \cup B_s \). See also the third formula in Eq.\( \text{[113]} \).

The main formulas used in this paper can be obtained from the following ones

\[
a B_s = a \cup B_s + a \land B_s, \quad B_s a = B_s \cup a + B_s \land a,
\]

\[
a \cup B_s = \frac{1}{2} (aB_s - (-)^s B_s a),
\]

\[
A_r \cup B_s = (-)^{(r-1)} B_s \cup A_r,
\]

\[
a \land B_s = \frac{1}{2} (aB_s + (-)^s B_s a),
\]

\[
A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r \cup B_s \rangle_{|r-s-2|} + \ldots + \langle A_r B_s \rangle_{|r+s|} = \sum_{k=0}^m \langle A_r B_s \rangle_{|r-s|+2k}, \tag{113}
\]

\[
A_r \cdot B_r = B_r \cdot A_r = \tilde{A}_r \cup B_r = A_r \cup \tilde{B}_r = \langle A_r \cup B_r \rangle_0 = \langle A_r \cup B_r \rangle_0,
\]

where \( a \in \text{sec} \bigwedge^1 T^* M \mapsto \text{sec} \mathcal{C}(M, g) \).
A.1.1 Hodge Star Operator

Let $\star$ be the Hodge star operator, i.e., the mapping

$$\star : \bigwedge^k T^* M \to \bigwedge^{4-k} T^* M, \quad A_k \mapsto \star A_k$$

where for $A_k \in \text{sec} \bigwedge^k T^* M \hookrightarrow \text{sec} \mathcal{A}(M, g)$

$$[B_k \cdot A_k]_{\tau_g} = B_k \wedge \star A_k, \forall B_k \in \text{sec} \bigwedge^k T^* M \hookrightarrow \text{sec} \mathcal{A}(M, g).$$

(114)

$$\tau_g = \theta^5 \in \text{sec} \bigwedge^4 T^* M \hookrightarrow \text{sec} \mathcal{A}(M, g)$$

is a standard volume element. Then we can verify that

$$\star A_k = \tilde{A}_k \theta^5.$$  

(115)

A.1.2 Dirac Operator

Let $d$ and $\delta$ be respectively the differential and Hodge codifferential operators acting on sections of $\mathcal{A}(M, g)$. If $A_p \in \text{sec} \bigwedge^p T^* M \hookrightarrow \text{sec} \mathcal{A}(M, g)$, then

$$\delta A_p = (-1)^p \star^{-1} d \star A_p,$$

with $\star^{-1}$ identity.

Remark 15 When there is necessity of specifying the metric field $g$ used in the definition of the Hodge star operator and the Hodge codifferential operator we use the notations $\star_g$ and $\delta_g$.

The Dirac operator acting on sections of $\mathcal{A}(M, g)$ associated to a general metric compatible connection $\nabla$ is the invariant first order differential operator

$$\mathbf{\nabla}^c = \partial^c \nabla_{e_a},$$

(116)

where $\{e_a\}$ is an arbitrary orthonormal basis for $TU \subset TM$ and $\{\theta^b\}$ is a basis for $T^* U \subset T^* M$ dual to the basis $\{e_a\}$, i.e., $\theta^b(e_a) = \delta^a_b$. $a, b = 0, 1, 2, 3$. The reciprocal basis of $\{\theta^b\}$ is denoted $\{\theta_a\}$ and we have $\theta_a \cdot \theta_b = \eta_{ab}$. Also,

$$\nabla_{e_a} \theta^b = -\omega_{abc} \theta^c$$

(117)

Defining

$$\omega_{e_a} = \frac{1}{2} \omega_{abc} \theta^c,$$

(118)

we have that for any $A_p \in \text{sec} \bigwedge^p T^* M$, $p = 0, 1, 2, 3, 4$

$$\nabla_{e_a} A_p = \partial_{e_a} A_p + \frac{1}{2} [\omega_{e_a}, A_p],$$

(119)

where $\partial_{e_a}$ is the Pfaff derivative, i.e., if $A_p = \frac{1}{p!} A_{i_1 \ldots i_p} \theta^{i_1 \ldots i_p}$,

$$\partial_{e_a} A_p := \frac{1}{p!} e_a (A_{i_1 \ldots i_p}) \theta^{i_1 \ldots i_p}.$$

(120)

Eq. (119) is an important formula which is also valid for a nonhomogeneous $A \in \text{sec} \mathcal{A}(M, g)$. It is proved, e.g., in [38, 55].
A.2 Dirac Operator Associated to a Levi-Civita Connection

Using Eq. (119) we can show the very important result which is valid for the Dirac operator associated to a Levi-Civita connection denoted $\partial$:

$$
\partial A_p = \partial \wedge A_p + \partial \llcorner A_p = dA_p - \delta A_p,
\partial \wedge A_p = dA_p,
\partial \llcorner A_p = -\delta A_p.
$$

(121)

B Maxwell Theory in the Clifford Bundle

With these results, Maxwell equations for $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}(M, g)$, $J \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, g)$ reads

$$
dF = 0, \quad \delta F = -J,
$$

(122)
or Maxwell equation\footnote{No misprint here.} reads (in a Lorentzian spacetime)

$$
\partial F = J.
$$

(123)

B.1 Energy-Momentum Densities $\star T_a$ for the Electromagnetic Field

In this Appendix, we present a suggestive formula for the energy-momentum densities $\star T_a = -\star T_a$ of the Maxwell field, namely:

$$
\star T_a = -\frac{1}{2} \star (F \theta^a \tilde{F}).
$$

(124)

We also show that $T_a \cdot \theta_b = T_b \cdot \theta_a$. The derivation of those formulas illustrates the power of the Clifford bundle formalism. In particular\footnote{No misprint here.}, simply cannot be written in the usual formalism of differential forms.

The Maxwell Lagrangian, here considered as the matter field coupled to the background gravitational field must be taken (due to our convention for the Ricci tensor and the definition of $\star T_a$ ) as

$$
\mathcal{L}_m = -\frac{1}{2} F \wedge \star F,
$$

(125)

where $F = \frac{1}{2} F_{ab} \theta^a \wedge \theta^b = \frac{1}{2} F_{ab} \theta^{ab} \in \sec \bigwedge^2 TM \hookrightarrow \sec \mathcal{C}(M, g)$ is the electromagnetic field. We recall (as it is easy to verify) that

$$
\delta \star \theta^{ab} = \delta \theta^c \wedge [\theta_c \llcorner \star \theta^{ab}].
$$
Also, for any \( A_p \in \text{sec} \bigwedge^0 TM \hookrightarrow \text{sec} \mathcal{C}(M, g) \) we have
\[
[\delta, \ast] A_p = \delta \ast A_p - \ast \delta A_p = \delta \theta^a \wedge (\theta_a \ast \ast A_p) - \ast \{ \delta \theta^a \wedge (\theta_a \ast \ast A_p) \}
\]
Multiplying both members of the last equation with \( A_p = F \) on the right by \( F \wedge \) we get
\[
F \wedge \delta \ast F = F \wedge \ast \delta F + F \wedge \{ \delta \theta^a \wedge (\theta_a \ast \ast F) - \ast \{ \delta \theta^a \wedge (\theta_a \ast \ast F) \} \}
\]
Next we sum \( \delta F \wedge \ast F \) to both members of the above equation obtaining
\[
\delta (F \wedge \ast F) = 2 \delta F \wedge \ast F + \delta \theta^a \wedge \{ F \wedge (\theta_a \ast \ast F) - (\theta_a \ast \ast F) \wedge \ast F \}.
\]
or,
\[
\delta \left( -\frac{1}{2} F \wedge \ast F \right) = -\delta F \wedge \ast F - \frac{1}{2} \delta \theta^a \wedge \{ F \wedge (\theta_a \ast \ast F) - (\theta_a \ast \ast F) \wedge \ast F \}.
\]
It follows that if \( \delta \theta^a = -\xi \theta^a \) for some diffeomorphism generated by the vector field \( \xi \), then
\[
\ast T_a = \frac{\partial L_{\xi}}{\partial \theta^a} = -\frac{1}{2} \{ F \wedge (\theta_a \ast \ast F) - (\theta_a \ast \ast F) \wedge \ast F \}.
\]
Now,
\[
(\theta_a \ast \ast F) \wedge \ast F = - \ast \{ (\theta_a \ast \ast F) \ast F \} = - \ast \{ (\theta_a \ast \ast F) \ast F \} \tau_g
\]
and we have
\[
(\theta_a \ast \ast F) \wedge \ast F = \theta_a (F \cdot F) \tau_g - F \wedge (\theta_a \ast \ast F).
\]
Using these results, we can write
\[
\frac{1}{2} \{ F \wedge (\theta_a \ast \ast F) - (\theta_a \ast \ast F) \wedge \ast F \} = \frac{1}{2} \{ \theta_a (F \cdot F) \tau_g - (\theta_a \ast \ast F) \wedge \ast F - (\theta_a \ast \ast F) \wedge \ast F \}
\]
\[
= \frac{1}{2} \{ \theta_a (F \cdot F) \tau_g - 2(\theta_a \ast \ast F) \wedge \ast F \}
\]
\[
= \frac{1}{2} \{ \theta_a (F \cdot F) \tau_g + 2(\theta_a \ast \ast F) \ast F \}
\]
\[
= \ast \left( \frac{1}{2} \theta_a (F \cdot F) + (\theta_a \ast \ast F) \ast F \right) = \frac{1}{2} \ast (\theta_a \theta^a) F,
\]
where in writing the last line we used the identity
\[
\frac{1}{2} F \theta^a \tilde{F} = (\theta_{\ast \ast F}) \ast F + \frac{1}{2} \ast \theta (F \cdot F), \tag{126}
\]
whose proof is as follows:

\[(n \circ F) \circ F + \frac{1}{2} n(F \cdot F) = \frac{1}{2} [(n \circ F)F - F(n \circ F)] + \frac{1}{2} n(F \cdot F)\]

\[= \frac{1}{4} [nFF - FnF - FnF + FFn] + \frac{1}{2} n(F \cdot F)\]

\[= -\frac{1}{2} FnF + \frac{1}{4} [-2n(F \cdot F) + n(F \wedge F) + (F \wedge F)n] + \frac{1}{2} n(F \cdot F)\]

\[= -\frac{1}{2} FnF + \frac{1}{2} n(F \cdot F) + \frac{1}{2} n(F \wedge F) + \frac{1}{2} n(F \cdot F)\]

\[= -\frac{1}{2} FnF + \frac{1}{2} F nF = 1\]

valid for any \( n \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} Cl(M, g) \) and \( F \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} Cl(M, g) \).

(b) To prove that \( \mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a \) we write:

\[\mathcal{T}_a \cdot \theta_b = \frac{1}{2} (F_a \theta_b) + \frac{1}{2} \theta_a (F \cdot F) \theta_b_0 \]

\[= \langle (F \cdot \theta_a)F \theta_b_0 + \frac{1}{2} \theta_a F \theta_b_0 \rangle_0\]

\[= \langle ((F \cdot \theta_a)F \theta_b) + (F \cdot \theta_a)(F \wedge \theta_b) \rangle_0 - \frac{1}{2} \theta_a (F \cdot F) \theta_b_0 + \frac{1}{2} \theta_a (F \wedge F) \theta_b_0\]

\[= \langle ((F \cdot \theta_a)(F \cdot \theta_b) + (F \cdot \theta_a)(F \wedge \theta_b)) \rangle_0 - \frac{1}{2} \theta_a (F \cdot F) \theta_b_0\]

\[= (F \cdot \theta_b) \cdot (F \cdot \theta_a) - \frac{1}{2} (F \cdot F) (\theta_b \cdot \theta_a) = \mathcal{T}_b \cdot \theta_a.\]

Note moreover that

\[- \mathcal{T}_{ab} = \mathcal{T}_{ab} = \mathcal{T}_a \cdot \theta_b = \eta^{cd} F_{ac} F_{bd} - \frac{1}{4} F_{cd} \eta^{ab}, \quad (127)\]

a well known result.

C Examples of Killing Vector Fields That Do Not Satisfy Eq.(102)

C.1 Teleparallel Schwarzschild spacetime

The metric of teleparallel Schwarzschild spacetime in spherical coordinates is

\[g = \zeta^2 dt \otimes dt - \zeta^{-2} dr \otimes dr - r^2 d\theta \otimes d\theta - r^2 \sin \theta d\phi \otimes d\phi, \quad (128)\]

with

\[\zeta := \left(1 - \frac{k}{r}\right)^{1/2}, \quad (129)\]

where \( k \) is a constant.

The Killing vector fields of this metric are
Introducing the orthonormal basis \( \{ e_\alpha \} \in \text{sec} P_{SO_1,3}(M) \), where
\[
e_0 = \zeta^{-1} \partial_t, \quad e_1 = \zeta \partial_r, \quad e_2 = \frac{1}{r} \partial_\theta, \quad e_3 = \frac{1}{r \sin \theta} \partial_\phi,
\]
we get the for the structure coefficients of the basis (which are equal the negative of the components of the torsion tensor in this basis),
\[
c^0_{10} = -k \zeta^{-1}/r^2, \quad c^2_{12} = \zeta/r = c^3_{13}, \quad c^3_{23} = \cot r.
\]

We then can verify that only the fourth Killing vector field in Table 1 satisfy Eq.(102).

C.2 Teleparallel de Sitter spacetime

The metric of de Sitter teleparallel spacetime in spherical coordinate is for \( \alpha < \sqrt{R} \):
\[
g = \omega^2 dt \otimes dt - \omega^2 dr \otimes dr - r^2 \sin \theta d\phi \otimes d\phi,
\]
where
\[
\omega := (1 - \alpha r^2)^{\frac{1}{2}}, \quad \alpha = 3/R^2,
\]
with \( \alpha \) the cosmological constant and \( R \) the curvature radius. The ten Killing vector fields of the de Sitter metric are \((c = \cosh(\sqrt{\alpha}t)) \text{ and } s = \sinh(\sqrt{\alpha}t))\).

| \( p \) | \( \xi^0 \) | \( \xi^1 \) | \( \xi^2 \) | \( \xi^3 \) |
|---|---|---|---|---|
| (1) | 1 | 0 | 0 | 0 |
| (2) | 0 | 0 | - \sin \phi | - \cot \theta \cos \phi |
| (3) | 0 | 0 | \cos \phi | - \cot \theta \sin \phi |
| (4) | 0 | 0 | 0 | 1 |

Table 1: Killing vectors associated with Schwarzschild metric.

| \( p \) | \( \xi^0 \) | \( \xi^1 \) | \( \xi^2 \) | \( \xi^3 \) |
|---|---|---|---|---|
| (1) | \( r \omega^{-1} \sin \theta \cos \phi \ c \) | \( \sqrt{\alpha} \omega \sin \theta \cos \phi \ s \) | \( \sqrt{\alpha} \omega \cos \theta \cos \phi \ s \) | \( - \sqrt{\alpha} \omega \cos \phi \ \sin \theta \ s \) |
| (2) | \( r \omega^{-1} \sin \theta \sin \phi \ c \) | \( \sqrt{\alpha} \omega \sin \theta \sin \phi \ s \) | \( \sqrt{\alpha} \omega \cos \theta \sin \phi \ s \) | \( - \sqrt{\alpha} \omega \cos \phi \ \sin \theta \ \sin \phi \ s \) |
| (3) | \( r \omega^{-1} \cos \theta \ c \) | \( - \sqrt{\alpha} \omega \cos \theta \ s \) | \( - \sqrt{\alpha} \omega \sin \theta \ s \) | \( 0 \) |
| (4) | \( -r \omega^{-1} \sin \theta \cos \phi \ s \) | \( - \sqrt{\alpha} \omega \sin \theta \cos \phi \ c \) | \( - \sqrt{\alpha} \omega \cos \theta \cos \phi \ c \) | \( \sqrt{\alpha} \omega \sin \phi \ \sin \theta \ \cos \phi \ c \) |
| (5) | \( -r \omega^{-1} \sin \theta \sin \phi \ s \) | \( - \sqrt{\alpha} \omega \sin \theta \sin \phi \ c \) | \( - \sqrt{\alpha} \omega \cos \theta \sin \phi \ c \) | \( - \sqrt{\alpha} \omega \cos \phi \ \sin \theta \ \sin \phi \ c \) |
| (6) | \( -r \omega^{-1} \cos \theta \ s \) | \( - \sqrt{\alpha} \omega \cos \theta \ c \) | \( \sqrt{\alpha} \omega \sin \theta \ c \) | \( 0 \) |
| (7) | \( \sqrt{\alpha} \) | 0 | 0 | 0 |
| (8) | 0 | 0 | - \cos \phi | \cot \theta \sin \phi |
| (9) | 0 | 0 | - \sin \phi | - \cot \theta \cos \phi |
| (10) | 0 | 0 | 0 | -1 |

Table 2. Killing vectors associated with de Sitter teleparallel spacetime for \( r < \sqrt{\alpha} \).
Introducing the orthonormal basis \( \{ e_a \} \in \text{sec} \, P\text{SO}_{1,3}^e(M) \), where

\[
e_0 = \omega^{-1} \partial_t, \quad e_1 = \omega \partial_r, \quad e_2 = \frac{1}{r} \partial_\theta, \quad e_3 = \frac{1}{r \sin \theta} \partial_\phi,
\]

we get that the non null structure coefficients of the basis (which are the negative of the components of the torsion tensor in this basis)

\[
c_{10} = \alpha r \omega^{-1}, \quad c_{12} = \omega / r = c_{313}, \quad c_{23} = \cot \theta / r.
\]

It can then be verified that only the seventh Killing vector field in Table 2 satisfy Eq. (102).

When \( r > \sqrt{\alpha} \) the metric of de Sitter teleparallel spacetime is

\[
g = \Omega^2 dt \otimes dt - \Omega^2 dr \otimes dr - r^2 \sin \theta d\phi \otimes d\phi, \quad (135)
\]

where

\[
\Omega := (\alpha r^2 - 1)^{1/2}, \quad \alpha = 3/R^2, \quad r > \sqrt{\alpha},
\]

As in the previous case, we have also ten Killing vector fields \( (c = \cosh(\sqrt{\alpha} t) \) and \( s = \sinh(\sqrt{\alpha} t) \).

| \( p \) | \( \xi^0 \) | \( \xi^1 \) | \( \xi^2 \) | \( \xi^3 \) |
|---|---|---|---|---|
| 1 | \( r \Omega^{-1} \sin \theta \cos \phi \) | \( \sqrt{\alpha} \Omega \sin \theta \cos \phi \) | \( \frac{\sqrt{\alpha}}{\Omega} \cos \theta \cos \phi \) | \( -\frac{\sqrt{\alpha}}{r} \Omega \sin \phi \) |
| 2 | \( r \Omega^{-1} \sin \theta \sin \phi \) | \( \sqrt{\alpha} \Omega \sin \theta \sin \phi \) | \( \frac{\sqrt{\alpha}}{\Omega} \cos \theta \sin \phi \) | \( -\frac{\sqrt{\alpha}}{r} \Omega \sin \phi \) |
| 3 | \( r \Omega^{-1} \cos \theta \) | \( -\sqrt{\alpha} \Omega \cos \theta \) | \( -\frac{\sqrt{\alpha}}{\Omega} \sin \phi \) | 0 |
| 4 | \( -r \Omega^{-1} \sin \theta \cos \phi \) | \( -\sqrt{\alpha} \Omega \sin \theta \cos \phi \) | \( -\frac{\sqrt{\alpha}}{\Omega} \cos \theta \cos \phi \) | \( \frac{\sqrt{\alpha}}{r} \Omega \sin \phi \) |
| 5 | \( -r \Omega^{-1} \sin \theta \sin \phi \) | \( -\sqrt{\alpha} \Omega \sin \theta \sin \phi \) | \( -\frac{\sqrt{\alpha}}{\Omega} \cos \theta \sin \phi \) | \( -\frac{\sqrt{\alpha}}{r} \Omega \sin \phi \) |
| 6 | \( -r \Omega^{-1} \cos \theta \) | \( -\sqrt{\alpha} \Omega \cos \theta \) | \( -\frac{\sqrt{\alpha}}{\Omega} \sin \phi \) | 0 |
| 7 | \( \sqrt{\alpha} \) | 0 | 0 | 0 |
| 8 | 0 | 0 | -\cos \phi | \cot \theta \sin \phi |
| 9 | 0 | 0 | -\sin \phi | -\cot \theta \cos \phi |
| 10 | 0 | 0 | 0 | -1 |

Table 3: Killing vectors associated with de Sitter teleparallel spacetime for \( r > \sqrt{\alpha} \).

Introducing the orthonormal basis \( \{ e_a \} \in \text{sec} \, P\text{SO}_{1,3}^e(M) \), where

\[
e_0 = \Omega^{-1} \partial_t, \quad e_1 = \Omega \partial_r, \quad e_2 = \frac{1}{r} \partial_\theta, \quad e_3 = \frac{1}{r \sin \theta} \partial_\phi,
\]

we get once again the non null structure coefficients of the basis (which are now the negative of the components of the torsion tensor in this basis)

\[
c_{10} = \alpha r \Omega^{-1}, \quad c_{12} = \Omega / r = c_{313}, \quad c_{23} = \cot \theta / r.
\]

It can then be verified that only the seventh Killing vector field in Table 3 satisfy Eq. (102).
C.3 Teleparallel Friedmann Spacetime

Consider the metric of the following particular Friedmann spacetime in comoving coordinates

\[ g = dt \otimes dt - R^2(t)(dx \otimes dx + dy \otimes dy + dz \otimes dz) \]

We see that there is no timelike Killing vector field. Introducing the orthonormal basis \( \{ e_a \} \in \sec \mathbb{P}_{SO(1,3)}(M) \), where

\[ e_0 = \partial_t, \quad e_1 = R^{-1} \partial_x, \quad e_2 = R^{-1} \partial_y, \quad e_3 = R^{-1} \partial_z. \]  

The non null structure coefficients of this basis (which are the negative of the components of the torsion tensor in this basis) are

\[ c_{10}^0 = c_{20}^2 = c_{30}^3 = R^{-1} \dot{R}. \]

and it can be verified that all Killing vector fields in Table 4 satisfy Eq. (102).

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