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The symplectic area of a geodesic triangle in a Hermitian symmetric space of compact type

Mads Aunskjær Bech, Jean-Louis Clerc & Bent Ørsted

Abstract

Let $M$ be an irreducible Hermitian symmetric space of compact type, and let $\omega$ be its Kähler form. For a triplet $(p_1, p_2, p_3)$ of points in $M$ we study conditions under which a geodesic triangle $T(p_1, p_2, p_3)$ with vertices $p_1, p_2, p_3$ can be unambiguously defined. We consider the integral $A(p_1, p_2, p_3) = \int_{\Sigma} \omega$, where $\Sigma$ is a surface filling the triangle $T(p_1, p_2, p_3)$ and discuss the dependence of $A(p_1, p_2, p_3)$ on the surface $\Sigma$. Under mild conditions on the three points, we prove an explicit formula for $A(p_1, p_2, p_3)$ analogous to the known formula for the symplectic area of a geodesic triangle in a compact Hermitian symmetric space.

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Key words : compact Hermitian symmetric space, Kähler form, geodesic triangle, automorphy kernel

Introduction

On a Kählerian manifold $M$, the existence of a Riemannian (Hermitian) metric and of a Kähler form $\omega$ suggests to define the symplectic area of a geodesic triangle. Given three points $p_1, p_2, p_3 \in M$, consider the geodesic triangle $T(p_1, p_2, p_3)$ obtained by joining $p_1, p_2$ (resp. $(p_2, p_3), (p_3, p_1)$) by a geodesic segment. Choose a surface $\Sigma(p_1, p_2, p_3)$ in $M$ having the geodesic triangle as its (oriented) boundary and define

$$A(p_1, p_2, p_3) = \int_{\Sigma(p_1, p_2, p_3)} \omega$$

as the symplectic area of the geodesic triangle built on the vertices $(p_1, p_2, p_3)$. Loosely speaking, as $\omega$ is a closed form, a continuous variation of the surface does not change the value of the integral and so the formula defines a
real-valued 3-points function on $M$, which is in particular invariant under any holomorphic isometry of $M$.

However, there are two main obstacles to a rigorous definition of the symplectic area of a geodesic triangle: some coming from the Riemannian geometry of $M$, some coming from the topology of $M$.

In fact, to build the geodesic triangle in a uniquely defined way, there should exist a unique minimizing geodesic segment between any two points of $M$. Whether this is true for “small” triangles, it is in general not true globally, due to the existence of a cut locus on $M$. Next, when the manifold $M$ has a non-trivial topology, the integral may depend on the choice of the surface filling the triangle.

When $M$ is a Hermitian symmetric space of the non compact type, there exists a unique geodesic segment between two arbitrary points of $M$ (a consequence of the negative curvature) and the topology of $M$ is trivial, as $M$ can be realized as the open ball of a complex vector space $\mathbb{C}^N$ for some Banach norm (a consequence of the Harish Chandra embedding). Hence the symplectic area of a geodesic triangle is well defined. An explicit expression for this area was obtained by Domic and Toledo (see [11]) for classical domains and in general for all domains by the two last present authors (see [8]) which amounts to

$$\int_{\Sigma(z_1,z_2,z_3)} \omega = - \left( \arg k(z_1,z_2) + \arg k(z_2,z_3) + \arg k(z_3,z_1) \right),$$

where $k(z,w)$ is a normalized version of the Bergman kernel of $M$ in its Harish Chandra realization. The three points function thus obtained defines a bounded cocycle, invariant under the group of holomorphic isometries of $M$, which turned out to be quite useful for various geometric problems (see e.g. [11, 8, 22]).

The goal of this article is to clarify the definition of the symplectic area of a geodesic triangle for a Hermitian symmetric space of compact type and to find an analogue of the Domic-Toledo formula. For previous work on this question, see [2, 3, 4, 12].

The geometry forces to restrict the definition to regular triangles, avoiding the cut-locus phenomena, and the topology suggests that one should consider rather the quantity

$$e^{\frac{1}{2} A(p_1,p_2,p_3)}$$

as it can be shown to be independent of the surface chosen to fill the triangle, see Theorem 4.2 for the precise formulation. The final result is a (singular)
three-points function, which is a (multiplicative $U(1)$-valued) cocycle invariant under holomorphic isometries of the manifold. An appropriate version of the Domic-Toledo formula is then formulated and proved along lines similar to the original proof. See Theorem 6.2 for the main formula.

Some results in this paper first appeared in the thesis ([1]) defended by the first author at Aarhus University. The second author would like to thank Aarhus University for welcoming him during the time this paper was started.

Acknowledgement: Joseph Wolf has many contributions to Lie theory and geometry, in particular that of Hermitian symmetric spaces; it is a pleasure to let this paper be part of a tribute to him.

1 The geometry of Hermitian symmetric spaces of compact type

Any Hermitian symmetric space $M$ of compact type belongs to the class of symmetric $R$-spaces, which means in particular that one can enlarge the group $U$ of holomorphic isometries of the space to a larger finite dimensional Lie group of diffeomorphisms of $M$. In the present case, this larger group turns out to be a complexification $G$ of $U$. So it is convenient to introduce the corresponding complex data, also relevant for the dual (non compact) Hermitian symmetric space $M^d$. We consider only irreducible Hermitian symmetric spaces, which correspond to simple Lie groups $G$ (or equivalently simple Lie algebras) of Hermitian type. Our main references for this section are [14, 19] and [16].

We first introduce the infinitesimal data. So let $g_0$ be a simple real Lie algebra, choose a Cartan involution $\theta$ of $g_0$ and let $g_0 = \kappa_0 \oplus \e_0$ be the corresponding Cartan decomposition. The algebra is said to be of Hermitian type if the center $\z$ of $\kappa_0$ is non trivial, in which case it can be shown to be one-dimensional. Then, up to a sign $\pm$, there exists a unique element $H_0$ in $\z$ such that $(\text{ad}_{\e_0} H_0)^2 = -\text{id}_{\e_0}$. Then $J = \text{ad}_{\e_0} H_0$ defines a complex structure on $\e_0$. Let $g, \kappa, \e, p$ be the complexifications of, respectively $g_0, \kappa_0, \e_0, p_0$.

Let $u = \kappa_0 \oplus i\e_0$, and let $\tau$ be the conjugation of $g$ with respect to $u$. Then $u$ is a simple real Lie algebra of compact type, and the pair $(u, \theta)$ where, abusing somewhat notation, $\theta$ is used for the restriction to $u$ of the complexification of $\theta$ is a simple symmetric Lie algebra of compact type.

Let $G$ be the simply connected Lie group with Lie algebra $\text{Lie}(G) = g$. The involution $\tau$ can be lifted to an involution of $G$ (still denoted by $\tau$) and the fixed points set of $\tau$ is a maximal compact subgroup of $U = G^\tau$ of $G$ with
Lie algebra \( \mathfrak{u} \). The complex automorphism \( \theta \) can be lifted to an involution (still denoted by \( \theta \)) of \( \mathbf{G} \) which preserves \( \mathfrak{u} \). The tangent space \( T_o M \) to \( M \) at \( o \) is identified with \( i\mathfrak{u}_0 \). The restriction of \( \text{ad} H_0 \) to \( i\mathfrak{u}_0 \) induces a complex structure on \( T_o M \). Let \( \mathbf{K}_0 = \mathbf{K}^\theta \) be the fixed points subgroup of \( \theta \) in \( \mathbf{G} \) with \( \text{Lie}(\mathbf{K}_0) = \mathfrak{k}_0 \). Form the quotient \( M = \mathbf{U}/\mathbf{K}_0 \). We refer to the element \( o = e\mathbf{K}_0 \) as the origin of \( M \). The tangent space \( T_o M \) to \( M \) at \( o \) is identified with \( i\mathfrak{u}_0 \). The restriction of \( \text{ad} H_0 \) to \( i\mathfrak{u}_0 \) induces a complex structure on \( T_o M \). Let \( \mathbf{P}_\pm \) be the eigenspaces of \( \text{ad} p H_0 \). Then \( \mathfrak{k}, \mathfrak{p}_\pm \subset \mathfrak{p}_\pm \) turn out to be Abelian subspaces of \( \mathfrak{g} \). Let \( \mathbf{P}_\pm = \exp \mathfrak{p}_\pm \) be the corresponding Lie subgroups of \( \mathbf{G} \), and let \( \mathbf{K} = \mathbf{G}^\theta \) be the fixed points subgroup of \( \theta \) in \( \mathbf{G} \) with \( \text{Lie}(\mathbf{K}) = \mathfrak{k} \). Then \( \mathbf{K} \mathbf{P}_\pm \) are maximal parabolic subgroups of \( \mathbf{G} \) and \( \mathbf{U} \cap \mathbf{K} \mathbf{P}_- = \mathbf{U} \cap \mathbf{K}_0 \), so that

\[
\mathbf{G}/\mathbf{K} \mathbf{P}_- \simeq \mathbf{U}/\mathbf{K}_0 = M .
\]

The map \( \Xi : \mathfrak{p}_+ \longrightarrow M \) defined for \( z \in \mathfrak{p}_+ \) by

\[
\Xi(z) = \exp(z)\mathbf{K} \mathbf{P}_-
\]

is a holomorphic diffeomorphism of \( \mathfrak{p}_+ \) onto a dense open subset of \( M \), which will be used as a chart on \( M \).

These two realizations of a Hermitian symmetric space of compact type are related through the Harish Chandra construction of a Cartan subspace of \( \mathfrak{p}_0 \) from a Cartan subalgebra of \( \mathfrak{g} \). Start with a maximal Abelian Lie subalgebra \( \mathfrak{h}_0 \) contained in \( \mathfrak{h}_0 \). Notice that \( \mathfrak{h}_0 \supset \subset \mathfrak{e} \supset \mathfrak{h}_0 \). Let \( \mathfrak{h} \) be its complexification, which is a Cartan subalgebra of \( \mathfrak{g} \). Let \( \Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \) be the corresponding root system, which is viewed as a subset of \( (i\mathfrak{h}_0)^\ast \). If \( \alpha \) is a root, then \( \alpha(\mathfrak{H}_0) \in \{-i, 0, i\} \). Choose a linear order on \( (i\mathfrak{h}_0)^\ast \) such that

\[
\alpha(\mathfrak{H}_0) = i \quad \Longrightarrow \quad \alpha > 0 .
\]

There is a corresponding partition of \( \Delta \)

\[
\Delta = \Delta_c \cup \Delta^+_\text{nc} \cup \Delta^-\text{nc}
\]

where \( \Delta_c \) is the set of compact roots (those for which \( \alpha(\mathfrak{H}_0) = 0 \)) and \( \Delta^+_\text{nc} \) (resp. \( \Delta^-\text{nc} \)) the set of positive (resp. negative) non-compact roots (those
which satisfy \( \alpha(H_0) = i \), resp. \( \alpha(H_0) = -i \). For \( \gamma \) any positive non-compact root, it is possible to choose elements \( X_{\pm \gamma} \in \mathfrak{g}_{\pm \gamma} \) such that

\[
X_{\gamma} - X_{-\gamma} \in \mathfrak{u}, \quad i(X_{\gamma} + X_{-\gamma}) \in \mathfrak{u}
\]

normalized such that

\[
[X_{\gamma}, X_{-\gamma}] = \frac{2}{\gamma(H_\gamma)} H_\gamma
\]

where \( H_\gamma \) is the unique vector in \( \mathfrak{h} \) satisfying \( B(H_\gamma, H) = \gamma(H) \) for all \( H \in \mathfrak{h} \).

**Proposition 1.1.** There exists a set \( \Gamma = \{\gamma_1, \ldots, \gamma_r\} \) of strongly orthogonal non-compact positive roots such that

\[
a_0 = \sum_{j=1}^{r} \mathbb{R}(X_{\gamma_j} + X_{-\gamma_j})
\]

is a Cartan subspace of the pair \((\mathfrak{g}_0, \mathfrak{k}_0)\).

It will be useful to also introduce

\[
a_+ = \sum_{j=1}^{r} \mathbb{C}X_{\gamma_j} \subset \mathfrak{p}_+.
\]

As \( X \mapsto \frac{1}{2}(X - iJX) \) is an \( \text{Ad} \mathcal{K}_0 \)-covariant real isomorphism of \( \mathfrak{p}_0 \) onto \( \mathfrak{p}_+ \), it follows that

\[
\bigcup_{k \in \mathcal{K}_0} \text{Ad} \ k \ a_+ = \mathfrak{p}_+.
\]

2 Fine Riemannian geometry of a CHSS

2.1 Helgason spheres

Let \( N \) be an irreducible Riemannian (not necessarily Hermitian) symmetric space of compact type. Let \( \kappa > 0 \) be the maximum of the sectional curvatures of \( N \). Then a **Helgason sphere** is a totally geodesic submanifold of \( N \), of constant curvature \( \kappa \) and of maximal dimension among submanifolds of this type. The Helgason spheres are conjugate under the isometry group of \( M \). See [15] for more details.

When \( M \) is a compact irreducible Hermitian symmetric space, a Helgason sphere turns out to be a complex submanifold of \( M \), of complex dimension 1, and of constant curvature equal to the maximum of the holomorphic
sectional curvature $\kappa$ and bihomorphically isomorphic to the Riemann sphere $\mathbb{C}P_1$.

Let us recall the construction of a Helgason sphere in this special case. Let $a_0$ be a Cartan subspace of $p_0$ and let $r = \dim a_0 = \text{rank } M$. Clearly, the space $ia_0$ is a Cartan subspace for the pair $(u, ip_0)$. The system $\Sigma$ of restricted roots of the pair $(u, ia_0)$ is known to be of type $C_r$ or $BC_r$. Hence there a basis of $ia_0^*,$ say $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$, where $r$ is the rank of $M$ such that the roots are given by

$$\pm \gamma_k, 1 \leq k \leq r, \quad \text{with multiplicity 1}$$

$$\pm \frac{1}{2} \gamma_k \pm \frac{1}{2} \gamma_l, 1 \leq k < l \leq r, \quad \text{with multiplicity } a$$

and possibly $\pm \frac{1}{2} \gamma_k, 1 \leq k \leq r, \quad \text{with even multiplicity } 2b$.

Now extend the Cartan subspace $ia_0$ to a Cartan subalgebra of $u$, say $b_0 = a_0 + ia_0$ with $b_0 \subset \mathfrak{b}_0$. Let $\mathfrak{h}$ be its complexification and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the associated root system. For $1 \leq k \leq r$, $\gamma_k$ is of multiplicity 1, hence is the restriction to $ia_0$ of a unique root $\tilde{\gamma}_k \in \Delta$. The Lie algebra $\mathfrak{g}(k)$ generated by the root spaces $\mathfrak{g}_{\pm \tilde{\gamma}_k}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, stable by $\tau$ and $u(k) = \mathfrak{g}(k)^\tau$ is isomorphic to $\mathfrak{su}(2)$. Let $G^{(k)}$ be the closed analytic subgroup of $G$ with Lie algebra $\mathfrak{g}_k$. Then $U^{(k)} = (G^{(k)})^\tau$ is a maximal compact subgroup of $G^{(k)}$. The orbit of $o$ under $G^{(k)}$ coincides with its orbit under $U^{(k)}$ and is isomorphic to a Riemann sphere $\mathbb{C}P_1$. It is in fact a Helgason sphere (see [15, 20]).

There is another way to obtain a Helgason sphere, which we will use in Section 3. This time, we choose a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{b}_0$, and we already noticed that its complexification $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta$ the corresponding root system, and let $\Pi$ be the set of simple roots with respect to some choice of positive roots. Among the simple roots, one and only one is non-compact, say $\alpha_1$. Let $\mathfrak{g}_{\pm \alpha_1}$ be the corresponding root spaces. Then, as $\tau(iH) = -iH$, $\tau(\mathfrak{g}_{\alpha_1}) = \mathfrak{g}_{-\alpha_1}$. The Lie algebra generated by $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{-\alpha_1}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and stable by $\tau$. As in the previous construction, the corresponding analytic subgroup $G^{(\alpha_1)}$ of $G$ is closed and stable by $\tau$. The orbit of $o$ under $G^{(\alpha_1)}$ is again a Helgason sphere. It can be shown by using the Harish Chandra construction of a Cartan subalgebra in $p_0$ from $\mathfrak{h}$, followed by a Cayley transform (see [16] III.2). After these operations, we are back to the first construction and the conclusion follows. This last construction has a more general version, valid for any symmetric $R$-space, presented in [20].
2.2 The polysphere embedding

Going back to notations of the beginning of subsection 2.1, observe that as the roots $γ_k, 1 ≤ k ≤ r$ are mutually strongly orthogonal, the subalgebras $g^{(k)}$ mutually commute to each other. Form

$$g(Γ) = \bigoplus_{k=1}^{r} g^{(k)}, \quad u(Γ) = \bigoplus_{k=1}^{r} u^{(k)},$$

and let $G(Γ)$ and $U(Γ)$ be the corresponding analytic subgroups of $G$.

**Proposition 2.1** (polysphere embedding). The orbit of $o$ under the action of $U(Γ)$ is a complex totally geodesic submanifold $S(Γ)$ of $M$, isomorphic to $S^{(1)} \times \cdots \times S^{(k)} \times \cdots \times S^{(r)}$, where for $1 ≤ k ≤ r$, $S^{(k)}$ is a Helgason sphere, isomorphic to $\mathbb{CP}_{1}$.

See [23]. Let

$$a_{0}^{(k)} = a_{0} \cap g^{(k)}, \quad a_{+}^{(k)} = a_{+} \cap g^{(k)},$$

and observe that $a_{0} = \bigoplus_{k=1}^{r} a_{0}^{(k)}$, and

$$T(Γ) = \exp(ia_{0}) = \prod_{k=1}^{r} \exp(i a_{0}^{(k)})$$

is a maximal torus $T(Γ)$ in $M$. For further reference, let $o = (o_{1}, \ldots, o_{r})$ where, for $1 ≤ k ≤ r$, $o_{k}$ is the origin point in $S^{(k)}$.

**Corollary 2.1.** Let $γ : [0, 1] \rightarrow M$ be a geodesic curve starting from $γ(0) = p$. Then there exists an element $u ∈ U$ such that $u(p) = o$ and $u \circ γ$ is contained in $S(Γ)$.

**Proof.** First, as $U$ is transitive on $M$, there is an element of $U$ which maps $p$ to the origin $o$. Now any geodesic curve through $o$ can be mapped by an element of $K_{0}$ to a geodesic curve contained in $T(Γ) \subset S(Γ)$. The lemma follows by composing the two elements of $U$.

2.3 First conjugate locus

Let $N$ be a compact Riemannian manifold. For $p \in N$ let $T_{p}N$ be the tangent space to $N$ at $p$, and let $\text{Exp}_{p} : T_{p}N \rightarrow N$ be the exponential
map with source $p$. The tangent conjugate locus of $p$ is the space $C_p \subset T_p N$ defined by

$$X \in C_p \iff d\text{Exp}_p(X) \text{ is singular}.$$  

The tangent first conjugate locus of $p$ is the subset $C_p^{(1)}$ defined as

$$C_p^{(1)} = \{ x \in C_p, tX \notin C_p, \text{ for any } t, 0 \leq t < 1 \}.$$  

The conjugate locus $C_p$ (resp. first conjugate locus $C_p^{(1)}$) of $p$ is the image under the exponential map with source at $p$ of the tangent conjugate locus (resp. first tangent conjugate locus).

For an irreducible Riemannian symmetric space of compact type, it is enough to determine the tangent conjugate locus at the origin $o$. Its description is known (see [14] Ch. VII, Prop. 3.1). In our situation, with the notation introduced above, first

$$C_o = \cup_{k \in K} \text{Ad } k \left( C_o \cap i \mathfrak{a}_0 \right).$$

Further,

$$X \in C_o \cap i \mathfrak{a}_0 \iff \exists \alpha \in \Sigma, \alpha(X) \in i \pi (\mathbb{Z} \setminus \{0\}).$$

**Proposition 2.2.** Let $M$ be a Hermitian symmetric space of compact type. The first tangent conjugate locus at the origin $o$ is given by

$$C_o^{(1)} = \cup_{k \in K} \text{Ad } k \left( C_o^{(1)} \cap i \mathfrak{a}_0 \right), \quad C_o^{(1)} \cap i \mathfrak{a}_0 = \{ X \in i \mathfrak{a}_0, \max_{1 \leq j \leq r} |\gamma_j(X)| = \pi \}.$$

**Proof.** For any restricted root $\alpha$

$$|\alpha(X)| \leq \max_{1 \leq j \leq r} |\gamma_j(X)|,$$

so that

$$\max_{\alpha \in \Sigma} |\alpha(X)| = \max_{1 \leq j \leq r} |\gamma_j(X)|,$$

and the proposition follows. \qed

We may now combine this result with the polysphere embedding (cf. Proposition 2.1).

**Proposition 2.3.** Let $q \in M$. Then $q$ belongs to the first conjugate locus of $o$ if and only if there exists a polysphere $S(\Gamma) \simeq S^{(1)} \times \cdots \times S^{(r)}$ such that $q = (q_1, \ldots, q_r)$ and there exists some $j, 1 \leq j \leq r$ such that $q_j$ is antipodal to $o_j$.  

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**Proof.** By using the action of $K_0$, we may assume that $q \in T(\Gamma) \subset \mathbf{S}(\Gamma)$. Then $q$ is in the first conjugate locus of $o$ if and only if there exists $H \in \mathfrak{a}_0$ such that $\exp iH(o) = q$ and there exists $j, 1 \leq j \leq r$ such that $\gamma_j(H) = \pm \pi$. Let $q = (q_1, \ldots, q_r)$ with $q_k \in S^{(k)}$, and let $H = \sum_{k=1}^{r} H_k$ be the decomposition of $X$ with respect to the decomposition $\mathfrak{a}_0 = \bigoplus_{k=1}^{r} \mathfrak{a}_0^{(k)}$. Now $\gamma_j(H_j) = \gamma_j(H) = \pm \pi$, hence $q_j = \exp iH_j(o_j) = q_j$ is in the first conjugate locus of $o_j$ in the sphere $S^j \simeq \mathbb{C}P^1$, that is to say $q_j$ is antipodal to $o_j$. \hfill \Box

### 2.4 Cut locus

Let $N$ be a Riemannian manifold and let $p \in N$. A tangent vector $X$ at $p$ is said to belong to the *tangential cut locus* of $p$ is the geodesic segment starting from $o$ in the direction $X$ is arc-length minimizing up to $\text{Exp}_p X$, but not further.

The *cut locus* of $p$ is the image under $\text{Exp}_p$ of the tangent cut locus of $p$.

**Proposition 2.4.** Let $M$ be an irreducible Hermitian symmetric space of compact type. Then for any $p \in M$, the cut locus of $p$ is equal to the first conjugate locus of $p$.

**Proof.** Any Hermitian symmetric space of compact type is simply connected (see [14] chapter VIII, Theorem 4.6). Hence the statement follows from a more general theorem due to Crittenden (see [9] Theorem 5). \hfill \Box

As a consequence, Proposition 2.3 is valid when the first conjugate locus is replaced by the cut locus. Let us mention that another proof of this result can be obtained for Hermitian symmetric spaces of compact type by using the more sophisticated result obtained by Tasaki, based on the notion of *diastasis* (see [21] Corollary 8).

### 2.5 Unique minimizing geodesic

Given two points $p, q \in M$, there always exists a minimizing geodesic segment joining $p$ to $q$. The next proposition describes the situation where the segment is not unique.

**Proposition 2.5.** Let $p, q \in M$. The following assertions are equivalent :

i) there exists (at least) two distinct minimizing geodesic segments joining $p$ to $q$.

ii) there exists an infinity of distinct minimizing geodesic segments joining $p$ to $q$. 

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iii) \( q \) belongs to the cut locus of \( p \).

iv) there exists \( u \in U \) such that \( u(p) = o \), \( q' = u(q) \) belongs to \( S(\Gamma) \) and for some \( j, 1 \leq j \leq r \), \( q_j' \) is the antipodal point of \( o_j \) in \( S^{(j)} \).

Proof. The fact that i) implies iii) is a general result valid for any Riemannian manifold (see e.g. [10] Proposition 2.2, chap. 13). Now let \( q \) belong to the cut locus of \( p \). Combine Proposition 2.4 and Proposition 2.3 to conclude that iii) \( \implies \) iv). As two antipodal points on a sphere can be joined by an infinite number of minimizing distinct geodesic segment, iv) implies ii) and ii) trivially implies i). \( \square \)

2.6 The big cell

We now use the realization of \( M \) as \( G/KP_- \) and consider the mapping

\[ \Xi : p_+^+ \rightarrow G/KP_-, \quad z \mapsto \exp(z)KP_- . \]

Then the image of \( \Xi \) is a dense open subset of \( M \).

**Proposition 2.6.** The image of \( \Xi \) is equal to \( M \setminus C_0^{(1)} \).

**Proof.** First remark that both \( \text{im}(\Xi) \) and \( M \setminus C_0^{(1)} \) are invariant under the action of \( K \).

Next recall the Lie algebra \( \mathfrak{g}(\Gamma) \) introduced in (3), which is isomorphic to a product of \( r \) copies of \( \mathfrak{sl}(2, \mathbb{C}) \), and let

\[ p_+^{(k)} = p_+ \cap \mathfrak{g}^{(k)}, \quad p_+^{(\Gamma)} = \mathfrak{g}(\Gamma) \cap p_+ = \bigoplus_{j=1}^{r} p_+^{(k)} , \]

and consider the restriction \( \Xi|_{p_+^{(k)}} \) of \( \Xi \) to \( p_+^{(k)} \). By an elementary \( SL(2, \mathbb{C}) \) calculation, the image of \( \Xi|_{p_+^{(k)}} \) is equal to \( S^{(k)} \setminus \{ -o_k \} \), where \( -o_k \) is the antipodal point to \( o_k \) in \( S^{(k)} \). Hence the image of \( \Xi|_{p_+^{(\Gamma)}} \) is equal to

\[ (S^{(1)} \setminus \{ -o_1 \}) \times \cdots \times (S^{(r)} \setminus \{ -o_r \}) , \]

and is equal to \( S(\Gamma) \setminus C_0^{(1)} \). By \( K \)-invariance, \( \text{im}(\Xi) = M \setminus C_0^{(1)} \). \( \square \)

3 The homology of Hermitian symmetric spaces of compact type

To understand the ambiguity in the definition of the symplectic area of a geodesic triangle, it is necessary to study its topology and more precisely
the homology with coefficients in \( \mathbb{Z} \) in degree 2. In general, the homology \( H_*(M, \mathbb{Z}) \) is explicitly known and related to Schubert cells. Our main reference for this section is [5].

Let \( h \) be a complex Cartan subalgebra of \( k \). Notice that \( z \subset h \) and hence \( h \) is also a Cartan subalgebra of \( g \). Let \( \Delta \) be the root system of \( (g, h) \) and choose a system of positive roots \( \Delta^+ \) such that any \( \alpha \in \Delta \) satisfying \( \alpha(H_0) = i \) is a positive root. Let \( B^- \) be the Borel subgroup of \( G \) associated to \(-\Delta^+\), and let \( N^- \) be its unipotent radical. Let \( W \) be the Weyl group of \( G \). For \( w \in W \), let \( l(w) \) be the length with respect to the set \( \Pi \) of simple roots in \( \Delta^+ \). Let \( s \) be the unique element of maximal length \( l(s) = r \), and let \( N = sN^-s^{-1} \). For \( w \in W \), set

\[
N^-_w = wNw^{-1} \cap N^-.
\]

Then \( N^-_w \) is a unipotent subgroup of \( N^- \) of (complex) dimension \( l(w) \).

Among the simple roots, one and only one is non-compact, say \( \alpha_1 \). In particular, \( \alpha_1(H_0) = i \). Let \( \Theta = \Pi \setminus \{\alpha_1\} \) be the set of compact simple roots, and let \( W_\Theta \) be the subgroup of \( W \) generated by the reflexions \( s_\alpha, \alpha \in \Theta \). \( W_\Theta \) is the Weyl group of the reductive subgroup \( K \).

The Borel subgroup \( B^- \) acts on \( M \) with a finite number of orbits. They are indexed by the coset space \( W/W_\Theta \). More precisely,

\[
M = \bigcup_{w \in W/W_\Theta} B^-w_0.
\]

**Lemma 3.1.** Let \( W^\Theta \) be the set of \( w \in W \) such that \( w\Theta \subset \Delta_+ \). Then any coset in \( W/W_\Theta \) contains a unique representative in \( W^\Theta \).

For \( w \in W^\Theta \), let \( X_w = B^-w_0 \).

**Lemma 3.2.** Let \( w \in W^\Theta \). The map

\[
N^-_w \ni n \mapsto nwo \in X_w
\]

is an isomorphism of algebraic varieties.

Denote by \( \overline{X_w} \) the image in \( H_*(M, \mathbb{Z}) \) of the fundamental cycle of \( X_w \) under the mapping induced by \( \overline{X_w} \mapsto M \).

**Theorem 3.1.** The elements \( \overline{X_w}, w \in W^\Theta \) form a free basis of \( H_*(M, \mathbb{Z}) \).

See [5], Proposition 5.2, attributed to A. Borel (see [6]).

**Lemma 3.3.** The variety \( X_w \) is of complex dimension 1 if and only if \( w = s_{\alpha_1} \).
Proof. As \( \dim(X_w) \) is equal to \( l(w) \), it suffices to show that the reflexion \( s_\alpha \) for \( \alpha \) a simple root belongs to \( W^\Theta \) if and only if \( \alpha = \alpha_1 \). Now if \( \alpha \) is a simple compact root, \( s_\alpha \alpha = -\alpha \) so that \( s_\alpha(\Delta_\Theta) \nsubseteq \Delta^+ \). On the other hand \( s_{\alpha_1} \) transforms any positive compact root into a positive root, so that \( s_{\alpha_1}(\Delta_\Theta) \subseteq \Delta^+ \).

**Proposition 3.1.** The variety \( \overline{X_{s_{\alpha_1}}} \) is a Helgason sphere, isomorphic to \( \mathbb{C}P_1 \).

Proof. As \( \alpha_1 \) is a simple root, \( s_{\alpha_1} \) permutes \( \Delta_+ \setminus \{\alpha_1\} \) and maps \( \alpha_1 \) to \( -\alpha_1 \), so that

\[
N_{s_{\alpha_1}}^- = \exp(g_{-\alpha_1}) .
\]

Let \( g^{(\alpha_1)} \) be the Lie algebra (isomorphic to \( \mathfrak{sl}(2, \mathbb{C}) \)) generated by \( g_{\alpha_1} \) and \( g_{-\alpha_1} \), and let \( G^{(\alpha_1)} \) be the corresponding analytic subgroup of \( G \). Then as explained in subsection 2.1, the orbit \( G^{(\alpha_1)}o \) is a Helgason sphere. Now

\[
N_{s_{\alpha_1}}^- s_{\alpha_1} o = s_{\alpha_1} \exp(g_{\alpha_1})o ,
\]

and \( \exp(g_{\alpha_1})o \) is dense in \( G^{(\alpha_1)}o \). Hence

\[
\overline{\overline{X_{s_{\alpha_1}}}} = N_{s_{\alpha_1}}^- s_{\alpha_1} o = s_{\alpha_1} G^{(\alpha_1)}o = G^{(\alpha_1)}o
\]

and the proposition follows.

Theorem 3.1 can now be formulated in degree 2.

**Theorem 3.2.** The group \( H_2(M, \mathbb{Z}) \) is equal to \( \mathbb{Z}[\overline{X_{s_{\alpha_1}}}]. \)

### 4 The symplectic area of geodesic triangles

#### 4.1 The normalized Kähler form on \( M \)

On \( M \) there is a canonical Kähler form \( \omega \). As \( M \) is assume to be irreducible, \( \omega \) is unique up to a constant. The form \( \omega \) is related to the invariant Hermitian metric \( q \) on \( M \) by the relation

\[
\omega_m(X, Y) = q_m(JX, Y)
\]

for \( X, Y \in T_m M \). As will appear in the sequel, it is preferable to use a different normalization, chosen such that the sectional holomorphic curvature
has maximal value 1. It is sufficient to choose the normalization of \( q_0 \), and under the identification of \( T_e M \cong i \mathfrak{p}_0 \), the normalization is given by
\[
\tilde{q}_0 = \frac{2}{p} q_0 , \tag{5}
\]
where \( p = (r - 1) a + b + 2 \). As similar computations for the dual space \( M^d \) were carefully done in [8], we skip the details. The maximum of the scalar curvature is reached for the tangent space to the complex Riemann spheres \( S^{(j)} \) associated to any root \( \gamma_j \) and, as a particular case to the sphere called \( X_\alpha \) in section 3. Notice that the restriction of the renormalized Hermitian metric (and consequently of the renormalized Kähler form \( \tilde{\omega} = \frac{2}{p} \omega \)) to such a sphere coincides with the usual metric (resp. area form) on the sphere \( \mathbb{C}P_1 \cong S^2 \). The following proposition is an immediate consequence of this remark.

**Proposition 4.1.** Let \( S \cong \mathbb{C}P_1 \) be a Helgason sphere. Then
\[
\int_S \tilde{\omega} = 4\pi . \tag{6}
\]

A triplet \((p_1, p_2, p_3)\) of points in \( M \) is said to be **regular** if there exists a unique minimizing geodesic segment between any two of the three points. Given a regular triplet \((p_1, p_2, p_3)\) we can form without ambiguity the oriented triangle \( T(p_1, p_2, p_3) \), by joining \( p_1 \) to \( p_2 \) via the unique minimizing geodesic from \( p_1 \) to \( p_2 \) and similarly for \((z_1, z_2)\) and \((p_3, p_1)\). Let \( \Sigma = \Sigma(p_1, p_2, p_3) \) be any 2-simplex whose boundary is the oriented triangle \( T(p_1, p_2, p_3) \). Then define
\[
A(\Sigma) = \int_{\Sigma(p_1, p_2, p_3)} \tilde{\omega} .
\]

Now, by Stokes theorem, as \( \omega \) is a closed form, this integral does not change if the simplex \( \Sigma \) is changed in a smooth way. However, as \( \omega \) is not exact, the corresponding global statement is not true.

**Theorem 4.1.** Let two 2-simplices \( \Sigma_1 \) and \( \Sigma_2 \), both having \( T(p_1, p_2, p_3) \) as boundary. Then
\[
A(\Sigma_1) \equiv A(\Sigma_2) \mod 4\pi .
\]

**Proof.** Consider two simplices \( \Sigma_1 \) and \( \Sigma_2 \) whose boundary is the oriented triangle \( T(p_1, p_2, p_3) \). Then the boundary of \( \Sigma_1 \) and \( \Sigma_2 \) is the same, which we can translate in homological terms as
\[
\partial(\Sigma_1 - \Sigma_2) = 0 ,
\]
where \( \partial \) is the boundary operator. Hence \( \Sigma_1 - \Sigma_2 \) is a 2-cycle with integer coefficients, which by our computation of \( H_2(M, \mathbb{Z}) \) implies that the corresponding homology class is given by

\[
[\Sigma_1 - \Sigma_2] = n[S]
\]

for some \( n \in \mathbb{Z} \). Hence, as \( \omega \) is closed,

\[
\int_{\Sigma_1} \tilde{\omega} - \int_{\Sigma_2} \tilde{\omega} = n \int_S \tilde{\omega} = n(4\pi),
\]

and the statement of the theorem follows.

For \((p_1, p_2, p_3)\) a regular triplet, form the geodesic triangle \( T(p_1, p_2, p_3) \). Let \( \Sigma \) be a 2-simplex whose boundary is \( T(p_1, p_2, p_3) \). Then the quantity \( e^{i2\int_{\Sigma} \tilde{\omega}} \) is independent of the simplex \( \Sigma \) and thus defines a \( U(1) \)-valued function \( \Psi(p_1, p_2, p_3) \) depending only of \((p_1, p_2, p_3)\).

**Theorem 4.2.** The function \( \Psi \) defined on regular triplets by

\[
\Psi(p_1, p_2, p_3) = e^{i2\int_{\Sigma} \tilde{\omega}}
\]

where \( \Sigma \) is a 2-simplex whose boundary is the oriented geodesic triangle \( T(p_1, p_2, p_3) \) satisfies the following properties

i) for any permutation \( \sigma \) of \( \{1, 2, 3\} \)

\[
\Psi(z_{\sigma(0)}, z_{\sigma(1)}, z_{\sigma(2)}) = \Psi(z_0, z_1, z_2)^{\epsilon(\sigma)},
\]

where \( \epsilon(\sigma) \) is the signature of the permutation \( \sigma \).

ii) for any quadruplet \((p_0, p_1, p_2, p_3)\) such that the four triplets \((p_0, p_1, p_2), (p_1, p_2, p_3), (p_2, p_3, p_0)\) and \((p_3, p_0, p_1)\) are regular

\[
\Psi(p_0, p_1, p_2)\Psi(p_1, p_2, p_3)^{-1}\Psi(p_2, p_3, p_0)\Psi(p_3, p_0, p_1)^{-1} = 1.
\]

**Proof.** For i), observe that the triangle \( T(p_{\sigma(0)}, p_{\sigma(1)}, p_{\sigma(2)}) \) has the same orientation as \( T(p_0, p_1, p_2) \) when \( \sigma \) is an even permutation and the opposite orientation if \( \sigma \) is an odd permutation. Hence \( \Sigma \) is a 2-simplex with boundary equal to \( T(p_{\sigma(0)}, p_{\sigma(1)}, p_{\sigma(2)}) \) if \( \sigma \) is even and to its opposite if \( \sigma \) is odd. The values of \( \Psi(p_{\sigma(0)}, p_{\sigma(1)}, p_{\sigma(2)}) \) is equal to \( \Psi(p_0, p_1, p_2) \) if \( \sigma \) is even and to \( \Psi(p_0, p_1, p_2)^{-1} \) if \( \sigma \) is odd.

For ii) choose four 2-simplices \( \Sigma(p_0, p_1, p_2), \ldots, \Sigma(p_3, p_0, p_1) \) such that the boundary of \( \Sigma(p_0, p_1, p_2) \) is equal to the oriented triangle \( T(p_0, p_1, p_2) \),
the boundary of $\Sigma(p_3, p_0, p_1)$ is equal to the oriented triangle $T(p_3, p_0, p_1)$. Then observe that

$$\partial(\Sigma(p_0, p_1, p_2) - \Sigma(p_1, p_2, p_3) + \Sigma(p_2, p_3, p_0) - \Sigma(p_3, p_0, p_1)) = 0.$$ 

Hence

$$\int_{\Sigma(p_0, p_1, p_2)} \tilde{\omega} - \int_{\Sigma(p_1, p_2, p_3)} \tilde{\omega} + \int_{\Sigma(p_2, p_3, p_0)} \tilde{\omega} - \int_{\Sigma(p_3, p_0, p_1)} \tilde{\omega} \equiv 0 \mod 4\pi.$$

The identity $ii)$ follows from this last equation.

5 The canonical kernel and the Kähler potential

The symplectic form $\omega$ on $M$ is closed, but not exact. Hence, it is not possible to find a 1-form $\rho$ such that $\omega = d\rho$ and hence it is also impossible to find a global Kähler potential, i.e. a function $k(z)$ on $M$ such that $\omega = i\partial\bar{\partial} k$.

We are forced to restrict to an open set of $M$, which is topologically trivial. It seems reasonable to use the chart $(\Xi, p_+)$.

5.1 The canonical kernel for the compact space

The main ingredient is the automorphy kernel. We first recall its definition in the case of a noncompact Hermitian symmetric domain. We essentially follow [19].

Let $\sigma$ be the involution of $\mathfrak{g}$ with respect to $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ and denote by $\sigma$ its lift to $G$.

The map

$$P_+ \times K \times P_- \ni (p_+, k, p_-) \mapsto p_+ k p_-$$

is a diffeomorphism on a dense open subset of $G$, and we write the inverse map as $g = g_+ g_0 g_-$. When $g \in P_+ K P_-$. We can define a partial action of $G$ on $p_+$ by

$$g(z) = (g \exp(z))_+,$$

which is nothing else than the expression in the chart $(\Xi, p_+)$ of the action of $G$ on $M$. Where defined, the differential $J(g, z)$ of the map $z \mapsto g(z)$ is given by

$$J(g, z) = (g \exp(z))_0.$$
For $z, w \in p_+$ such that
\[
\exp(-\sigma w) \exp(z) \in P_+KP_-
\]
the \textit{canonical automorphy kernel} $K(z, w)$ is defined by
\[
K(z, w) = J(\exp(-\sigma w), z)^{-1} = (\exp(-\sigma w)\exp(z))^{-1}_0.
\]
To pass to the case of a compact Hermitian symmetric space, we replace $\sigma$ by $\tau$.

For $z, w$ in $p_+$ such that $\exp(-\tau w)\exp(z)$ belongs to $P_+KP_-$, define the \textit{compact automorphy kernel} $K_c(z, w)$ as
\[
K_c(z, w) = (\exp(-\tau w)\exp(z))^{-1}_0.
\]
An elementary but crucial observation is that on $p^+$, $\tau$ and $\sigma$ differ by a sign. So for $z, w \in p_+$ where $K_c$ is defined,
\[
K_c(z, w) = K(z, -w).
\]
The results to follow are stated without proofs, as they are mostly consequence of this remark and can be deduced from the analogous result in the non-compact case.

\textbf{Proposition 5.1.} Let $(z, w) \in p_+$ such that $K_c$ is defined at $(z, w)$ and let $g \in G$ such that $g$ is defined at $z$ and $\tau(g)$ is defined at $w$. Then $K_c$ is defined at $(g(z), \tau(g)(w))$ and
\[
K_c(g(z), \tau(g)(w)) = J(g, z)K_c(z, w)\tau(J(\tau(g), w))^{-1}.
\]
The automorphy kernel allows to express the Riemannian metric $q$ on $M$ in the chart $(\Xi, p_+)$.

\textbf{Proposition 5.2.} For $z \in p_+, \zeta, \eta \in p_+
\[
q_z(\zeta, \eta) = -\frac{1}{2}B(\Ad K_c(z, z)^{-1}\zeta, \tau\eta).
\]

\textbf{Proposition 5.3.} Let $(z, w) \in p_+$ such that $K(z, w)$ is defined. Then on $p_+$
\[
\Ad_{p_+} K_c(z, w) = \id - \ad[z, \tau w] + \frac{1}{4}(\ad z)^2(\ad \tau w)^2.
\]
Next let for $z, w \in p_+$

$$k_c(z, w) = \det \text{Ad}_{p_+} K_c(z, w) = k(z, -w)$$

Define for $(z, w) \in p_+$

$$k_c(z, w) = \det \text{Ad}_{p_+} K_c(z, w),$$

where $\det$ refers to the complex determinant, and observe that the definition makes sense for all $(z, w) \in p_+$. The covariance property of the automorphy kernel implies the following covariance property for the kernel $k_c$.

**Proposition 5.4.** Let $z, w \in p_+$ and let $g \in G$ such that $g$ is defined at $z$ and $\tau(g)$ is defined at $w$. Then

$$k_c(g(z), \tau(g)(w)) = j(g, z) k_c(z, w) \overline{j(\tau(g), w)}. \quad (7)$$

The kernel $k_c$ can be explicitly expressed on $a_+ \times a_+$.

**Proposition 5.5.** Let $z = \sum_{j=1}^r z_j X_j$, $w = \sum_{j=1}^r w_j X_j$. Then

$$k_c(z, w) = \prod_{j=1}^r (1 + z_j \overline{w_j})^p. \quad (8)$$

The last proposition shows in particular that $k_c(z, z) > 0$ for $z \in p_+$. In turn, the kernel $k_c$ allows to describe a Kähler potential for $\omega$ on $p_+$.

**Proposition 5.6.** For $z \in p_+$,

$$\omega_z = i\partial \overline{\partial} \log k_c(z, z). \quad (9)$$

The ingredients for the proof of the analog of the Domic-Toledo formula are now available. But a new difficulty occurs, namely the formula ought to involve an argument of $k_c(z, w)$ for $z, w \in p_+$. But $k_c(z, w)$ may vanish (see (8)). Let us consider first the situation on the simplest example, namely the case where $M = \mathbb{CP}_1$.

### 5.2 The case of the projective space

Let $M = \mathbb{CP}_1$ be the complex projective space, i.e. the space of complex lines in $\mathbb{C}^2$. As origin, choose the origin the complex line generated by $(1, 0)$. The map

$$\mathbb{C} \ni z \quad \mapsto \quad \text{the complex line generated by } (z, 1) \in M$$
gives a local chart on $M$ and actually coincides with $\Xi$. As usual, it is convenient to consider $M$ as $\mathbb{C} \cup \infty$ where $\infty$ is the line generated by $(0, 1)$.

The group $\text{SL}(2, \mathbb{C})$ acts naturally (projectively) on $M$, and the expression of this action in the chart is given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}), \quad z \in \mathbb{C}, \quad g(z) = \frac{az + b}{cz + d},$$

extended to $\mathbb{C} \cup \infty$ by

$$g(\infty) = \frac{a}{c}, \quad g\left(-\frac{d}{c}\right) = \infty \quad \text{if} \quad c \neq 0, \quad g(\infty) = \infty \quad \text{if} \quad c = 0.$$

Its standard maximal compact subgroup is

$$U = \text{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

The stabilizer of the origin in $U$ is the subgroup

$$K_0 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}.$$

Let $\beta \in \mathbb{C}, |\beta| = 1$. Consider the one-parameter subgroup of $\text{SU}(2)$ given by

$$g_\beta(t) = \begin{pmatrix} \cos t & \beta \sin t \\ -\beta \sin t & \cos t \end{pmatrix} = \exp t \begin{pmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{pmatrix}.$$

Then $t \mapsto g_\beta(t)0 = (\tan t)\beta$ is the expression in the chart of the geodesic curve through $0$ with tangent vector at $0$ equal to $\beta$.

**Lemma 5.1.**

i) the antipodal point of $0$ is $\infty$

ii) the antipodal point of $z \neq 0$ is equal to $-\bar{z}^{-1}$.

**Proof.** Using the notation introduced above, for $t = \pi$ we get $g_\beta(\pi) = \infty$ for any $\beta$, so that i) follows. Next let

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \left| \alpha \right|^2 + \left| \beta \right|^2 = 1$$

be an element of $\text{SU}(2)$, so that

$$g(0) = \frac{\beta}{\bar{\alpha}}, \quad g(\infty) = -\frac{\alpha}{\beta}.$$

As $g$ acts by isometrically on $M$, the antipodal point of $z = \frac{\beta}{\bar{\alpha}}$ is equal to $-\frac{\alpha}{\beta} = -\bar{z}^{-1}$. \[\square\]
The automorphy kernel is given by
\[ K(z, w) = \begin{pmatrix} 1 - zw & 0 \\ 0 & (1 - zw)^{-1} \end{pmatrix} \]
and the canonical kernel \( k(z, w) \) is then given by
\[ k(z, w) = (1 - zw)^2. \]

The compact canonical kernel is given by
\[ k_c(z, w) = (1 + zw)^2. \]

In order to be able to define a continuous argument for \( k_c \) we introduce the set
\[ S = \{(z, w) \in \mathbb{C} \times \mathbb{C}, 1 + zw \notin (-\infty, 0]\}. \]

The geometric significance of the space \( S \) is given by the following propositions.

**Proposition 5.7.** The pair \((z, w)\) belongs to \( S \) if and only if there exists a unique minimizing geodesic segment contained in \( \mathbb{C} \) joining \( z \) and \( w \).

**Proof.** First observe that \( 1 + zw \in \mathbb{R} \) is equivalent to \((z, w)\) linearly independent over \( \mathbb{R} \) and hence connected by a line through the origin. Assume now that \( 1 + zw \notin \mathbb{R} \). Then the antipodal point of \( z \) is not equal to \( w \), so that there exists a unique minimizing geodesic segment on \( M \) from \( \Xi(z) \) to \( \Xi(w) \). If the segment is not contained in \( \mathbb{C} \), then \( \infty \) belongs to this segment. But any geodesic line which passes through \( \infty \) passes through its antipodal point \( 0 \), so that \( z, w \) would be linearly dependent over \( \mathbb{R} \). Hence if \( 1 + zw \notin \mathbb{R} \), there exists a unique minimizing geodesic segment contained in \( \mathbb{C} \) joining \( z \) and \( w \).

Now assume that \( 1 + zw \in \mathbb{R} \). It follows that \( z, w \neq 0 \) and using the action of \( K_0 \), we may even assume that \( z > 0 \) and \( w \in \mathbb{R} \setminus \{0\} \). For convenience set \( z = x \) and \( w = y \). The antipodal point of \( z \) is equal to \(-\frac{1}{x}\). Now if \( y \in ]-\frac{1}{x}, \infty) \), the geodesic segment from \( x \) to \( y \) does not contain the antipodal point of \( z \) and hence is the minimizing geodesic segment from \( z \) to \( w \). On the opposite, if \( y < -\frac{1}{x} \), the geodesic segment from \( z \) to \( w \) contains the antipodal point of \( z \) and hence is not minimizing. This finishes the proof of the Proposition. \( \square \)

**Proposition 5.8.** There exists a global smooth determination of \( \arg k_c(z, w) \) on the set \( S \). It can be chosen as \( 2 \text{Arg} \left( k_c(z, w) \right) \), where \( \text{Arg} \) is the principal determination of the argument on \( \mathbb{C} \setminus (-\infty, 0] \).

Notice that the space \( S \), seen as \( \{(z, w), zw \notin (-\infty, -1]\} \) is star-shaped with respect to \((0,0)\) in \( \mathbb{C}^2 \), so that \( S \) simply connected.
5.3 Geodesics and triangles in \( \mathfrak{p}_+ \)

Inspired by the previous case, we now define a certain subset \( S \) of \( \mathfrak{p}_+ \times \mathfrak{p}_+ \).

**Definition.** Let \( S \) be the set of all \( (z, w) \in \mathfrak{p}_+ \times \mathfrak{p}_+ \) for which

i) there exists a unique minimizing geodesic segment from \( \Xi(z) \) to \( \Xi(w) \)

ii) the minimizing segment is contained in \( \text{im}(\Xi) \).

In the sequel for \( (z, w) \in S \), denote by \( \beta_{z,w} : [0,1] \rightarrow \mathfrak{p}_+ \) the unique curve such that \( \Xi \circ \beta_{z,w} \) is the unique minimizing geodesic segment joining \( \Xi(z) \) and \( \Xi(w) \).

**Lemma 5.2.** Let \( (z, w) \in S \). Then there exists \( u \in U \) such that \( u \) is defined along the curve \( \beta_{z,w} \) and \( u(z) = 0 \).

**Proof.** By definition, there exists \( \beta : [0,1] \rightarrow \mathfrak{p}_+ \) such that \( z = \beta(0), w = \beta(1) \) and \( \Xi \circ \beta \) is a unique minimizing geodesic segment joining \( \Xi(z) \) and \( \Xi(w) \). By Corollary 2.1, there exists \( u \in U \) such that \( u \) maps the geodesic segment into \( S(\Gamma) \) and \( u(z) = 0 \). The point \( u(\Xi(w)) \) is the endpoint of a unique minimizing geodesic segment starting at \( o \), hence, by Proposition 2.5 does not belong to the cut locus of \( o \) and by Proposition 2.6 does belong to \( \text{im}(\Xi) \). In other words, \( u \) is defined at \( w \). But the same argument applies for \( z_t = \beta(t) \) for any \( t \in [0,1] \), so that \( u \) is defined at any point of the minimizing segment.

**Proposition 5.9.** Let \( (z, w) \in S \). Then \( k_c(z, w) \neq 0 \).

**Proof.** Let \( (z, w) \in S \). By Lemma 5.2 there exists \( u \in U \) such that \( u(z) = 0 \) and \( u \) is defined at \( w \). Now \( k_c(u(z), u(w)) = k_c(0, u(w)) = 1 \). Hence by (7) \( k_c(z, w) \neq 0 \).

**Proposition 5.10.** The subset \( S \) is simply connected.

**Proof.** The map \( (z, w) \mapsto (w, w) \) is a deformation retract from \( S \) onto the diagonal \( \text{diag}(\mathfrak{p}_+) \) in \( \mathfrak{p}_+ \times \mathfrak{p}_+ \). In fact \( \beta_{z,w}(t) \) is a continuous family of continuous maps from \( S \) into \( S \) which satisfies \( \beta_{z,w}(1) = (w, w) \). As \( \text{diag}(\mathfrak{p}_+) \simeq \mathfrak{p}_+ \) is simply connected, \( S \) is also simply connected.

For \( (z, w) \) in \( S \) define \( \text{arg} \ k(z, w) \) as the unique continuous determination of the argument of \( k(z, w) \) which satisfies \( \text{arg}(k(z, z)) = 0 \). The existence of such an argument is guaranteed by the two last propositions.
6 The formula for the symplectic area of geodesic triangles

We are now ready for the proof of the main formula. Recall the formula (9) for expressing the Kähler form as
\[ \omega_z = i\partial\bar\partial \log k(z, z). \]

Let \( dC = -i(\partial - \bar\partial) \), and let \( \rho \) be the 1-differential form defined by
\[ \rho_z = dC \log k(z, z). \]

Then (9) is equivalent to
\[ \omega = \frac{1}{2} d\rho. \]

The next step is to compute \( \int_\gamma \rho \) where \( \gamma \) is a minimizing geodesic segment between two points \( z, w \) such that \( (z, w) \in \mathcal{S} \). We follow an argument due to A. Wienhard ([22]).

**Lemma 6.1.** Let \( \gamma: [a, b] \to p_+ \) be a smooth curve segment, and suppose that \( k_c(\gamma(a), \gamma(b)) \) is defined. Assume that \( g \in U \) is an element such that the action of \( g \) is defined on all points of \( \gamma \), that is, \( g\gamma \) is another smooth curve segment in \( p_+ \). Then we have
\[ \frac{k_c(g\gamma(a), g\gamma(b))}{k_c(g\gamma(b), g\gamma(a))} \exp i \int_{g\gamma} \rho = \frac{k_c(\gamma(a), \gamma(b))}{k_c(\gamma(b), \gamma(a))} \exp i \int_{\gamma} \rho \quad (10) \]

**Proof.** Let \( z \) be any point on \( \gamma \). It follows from (7) that
\[ k_c(g(z), g(z)) = j(g, z)k_c(z, z)\overline{j(g, z)} \]
and hence
\[ \log k_c(g(z), g(z)) = \log |j(g, z)|^2 + \log k_c(z, z). \]

Now from \( dC \log k(z, z) = \rho_z \) and the above we see that
\[ \int_{g\gamma} \rho = \int_{\gamma} dC g^* \log k_c(z, z) \]
\[ = \int_{\gamma} dC \log |j(g, z)|^2 + \int_{\gamma} dC \log k(z, z), \]

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and since the action of \( g \) is defined along \( \gamma \) we may choose a holomorphic logarithm of \( z \mapsto j(g, z) \) along \( \gamma \). This logarithm, denoted \( \log j(g, z) \), has real part \( \log |j(g, z)| \) and it follows from the Cauchy-Riemann equations that \( d_C \log |j(g, z)| = d\bar{\Im} \log j(g, z) \). Hence, \( d_C \log |j(g, z)|^2 = 2d\bar{\Im} \log j(g, z) \) and

\[
\int_{g\gamma} \rho = 2\Im \log j(g, \gamma(b)) - 2\Im \log j(g, \gamma(a)) + \int_{\gamma} \rho
\]

follows. After taking exponentials we obtain

\[
\exp i \int_{g\gamma} \rho = \frac{j(g, \gamma(b)) j(g, \gamma(a))}{j(g, \gamma(b)) j(g, \gamma(a))} \exp i \int_{\gamma} \rho.
\]  

(11)

Using (7) again, we see that

\[
\frac{k_c(g\gamma(a), g\gamma(b))}{k_c(g\gamma(b), g\gamma(a))} = \frac{j(g, \gamma(a)) k_c(\gamma(a), \gamma(b)) j(g, \gamma(b))}{j(g, \gamma(a)) k_c(\gamma(b), \gamma(a)) j(g, \gamma(b))}
\]

(12)

and upon combining (11) with (12) we obtain (10).

Lemma 6.2. Let \( \gamma : [a, b] \to \mathbb{P}_+ \) be a geodesic segment passing through 0. Then \( \rho(\gamma(t)) = 0 \) for all \( t \).

Proof. This proof is a variation of the proof of Theorem 9.1 in [8] in which a similar statement played a key role. Since \( k_c(z, w) \) is \( \operatorname{Ad}(K_0) \)-invariant, so if \( \rho \) and we may therefore assume that \( \gamma \) runs in \( a_+ \). Write \( \gamma(t) = \sum_{k=1}^r \gamma_k(t) X_k \) where \( \gamma_k : [a, b] \to \mathbb{C} \) is a geodesic in the Riemann sphere \( \mathbb{C}P_1 \). Put

\[
k_{C^r}(z, w) = \prod_{k=1}^r (1 + z_k w_k)^2
\]

for \( z = (z_1, \ldots, z_r), w = (w_1, \ldots, w_r) \) in \( \mathbb{C}^r \). Then

\[
(d_C \log k_c)(\gamma) = \frac{p}{2} (d_C \log k_{C^r})(\gamma_1, \ldots, \gamma_r)
\]

\[
= \frac{p}{2} \sum_{k=1}^r (d_C \log k_c)(\gamma_k)
\]

and the calculation reduces to the situation \( \mathbb{C}P_1 \). Here each \( \gamma_k \) is a line through the origin and \( k_{C^r}(z, z) = (1 + |z|^2)^2 \) and it is straightforward to verify that \( (d_C \log k_{C^r}(z, z))(\gamma) \) vanishes.
Theorem 6.1. Let \((z, w) \in S\) and let \(\gamma = \beta_{z,w}\). Then
\[
\exp \frac{1}{i} \int_{\gamma} \rho = \frac{k_c(z, w)}{k_c(w, z)} \tag{13}
\]
and
\[
\frac{1}{2} \int_{\gamma} \rho = - \arg k_c(z, w) \tag{14}
\]

Proof. Let \(u \in U\) be as in Lemma 5.2. As \(u(z) = o\) we combine Lemma 6.2 and 6.1 to obtain
\[
1 = \frac{k_c(z, w)}{k_c(w, z)} \exp i \int_{\gamma} \rho, \tag{15}
\]
proving (13). As \(k_c(w, z) = \overline{k_c(z, w)}\) we have, for all \((z, w) \in S\)
\[
\exp \left( -i \int_{\beta_{z,w}} \rho \right) = \exp 2i \arg k_c(z, w),
\]
Hence \(- \int_{\beta_{z,w}} \rho\) and \(2 \arg k_c(z, w)\) are two continuous functions on \(S\) which differ by a multiple of \(2\pi\), are equal to 0 on the diagonal of \(p_+ \times p_+\), hence coincide everywhere on \(S\), so that (14) holds.

Now, if we are given a triple \((z_0, z_1, z_2)\) of points in \(p_+\) such that each of the pairs \((z_i, z_j)\) belong to \(S\), then we may form an oriented geodesic triangle \(T = T(z_0, z_1, z_2)\) as follows: The triangle \(T\) is made up of the three unique shortest geodesic segments connecting the three vertices \(z_0, z_1,\) and \(z_2\) with orientation given by traversing the boundary in the order \(z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow z_0\). If \(\Sigma\) is a smooth surface in \(p^+\) with \(T = \partial \Sigma\), then \(\int_{\Sigma} \omega\) only depends on the boundary \(T\) and therefore we will not specify any particular "filling" of \(T\).

Theorem 6.2. Let \((z_0, z_1, z_2)\) be a triple of points in \(p_+\) and suppose that each pair \((z_i, z_j)\) belongs to \(S\). Construct the oriented geodesic triangle \(T = T(z_0, z_1, z_2)\) as above. Then
\[
\int_{\Sigma} \omega = - \left( \arg k_c(z_0, z_1) + \arg k_c(z_1, z_2) + \arg k_c(z_2, z_0) \right) \tag{16}
\]
holds for any smooth surface \(\Sigma \subset p_+\) with \(T\) as its boundary.
Proof. We have $\omega = \frac{1}{2} d\rho$ so the result follows after an application of Stoke’s theorem and (14) on each of the three geodesic segments of $T$. \hfill \Box

In particular, for a geodesic triangle $T(0, z_1, z_2)$ with $(z_1, z_2) \in S$ we see that the symplectic area of $T$ is given by $-\arg k_c(z_1, z_2)$. This result essentially appears in several articles by S. Berceanu, see e.g. [2] and [3]. It is proven by direct calculation for the complex Grassmannian. See also [4] where the authors use an embedding of $U/K_0$ into projective space $\mathbb{CP}_N$ and give an interpretation of the argument of $k_c$ in terms of the symplectic area of geodesic triangles in the ambient space $\mathbb{CP}_N$. Hangan and Masala [12] already proved an exponentiated version of Theorem 6.2.

At this point, in order to make connection with the global point of view adopted in Section 4, it is convenient to renormalize the metric and the Kähler form so that the metric has maximal holomorphic sectional curvature equal to $+1$. The new metric $\tilde{q}$ is equal to $\frac{2}{p} q$ (see details in [8]). Let $\tilde{\omega}$ be the corresponding normalized Kähler form.

One can show (see e.g. [16]) that there is a unique $K$-invariant $h(z, w)$ on $p_+ \times p_+$, holomorphic in $z$ and conjugate-holomorphic in $w$ such that

$$h(z, w) = \prod_{j=1}^{r} (1 + z_j \overline{w}_j),$$

when $z = \sum_{k=1}^{r} z_k X_k$ and $w = \sum_{k=1}^{r} w_k X_k$ and that $k_c(z, w) = h(z, w)^p$ for any pair of $(z, w)$. Now set

$$\tilde{k}_c(z, w) = h(z, w)^2.$$

Recalling the fact that the factor of normalization from $\omega$ to $\tilde{\omega}$ is $\frac{2}{p}$, (9) becomes

for $z \in p_+$, \[ \tilde{\omega}_z = i \partial \overline{\partial} \log \tilde{k}(z, z) . \]

and consequently, the main formula (15) remains valid with $\tilde{\omega}$ (resp. $\tilde{k}_c$) replacing $\omega$ (resp. $k_c$).

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