INTEGRABLE CHAIN MODELS WITH STAGGERED R-MATRICES

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Abstract
The technique of construction on Manhattan lattice (ML) the fermionic action for Integrable models is presented. The Sign-Factor of 3D Ising model (SF of 3DIM) and Chalker-Coddington-s phenomenological model (CCM) for the edge excitations in Hall effect are formulated in this way. The second one demonstrates the necessity to consider the inhomogeneous models with staggered R-matrices. The disorder over the $U(1)$ phases is taken into account and staggered Hubbard type of model is obtained. The technique is developed to construct the integrable models with staggered disposition of R-matrices.

Keywords: Hall effect, 3D Ising Model, Integrable models, Bethe Ansatz, ladder models

1. Introduction

The goal of the talk is twofold. First is the demonstration how the integrable models, which can be solved via the Bethe Ansatz (BA) technique and by definition are in Hamiltonian formalism, can be formulated in the action (Lagrangian) formalism on so called Manhattan lattice (ML) exactly. More precisely it will be shown how the partition function of the model, which is a trace of $N$-degree of the Transfer matrix, can be represented exactly as some functional integral over classical Grassmann fields $\psi_n$ with two dimensional action $S(\bar{\psi}_n; \psi_n)$ defined on
We start by demonstrating that two interesting problems of modern physics, namely the so called Sign-factor of three dimensional Ising model (SF of 3DIM) \cite{1, 2, 3} and the edge excitations in Hall effect (more precisely the Chalker-Coddington (CCM) phenomenological model before taking into account the disorder over random phases \cite{4}) can be described by the same type of 2D model on \( ML \) \cite{5}, but have a different amount of degrees of freedom (correspondingly different gauge groups of symmetries) and are in different points of the space of hopping parameters.

The formulation of CCM as a field theory of scalar fermions on \( ML \) in the \( U(1) \) gauge field background exhibits chess like structure and demonstrates the necessity to consider and investigate an inhomogeneous integrable models with staggered disposition of the R-matrices along a chain and time directions.

It is turning out, that this formalism on \( ML \) is very appropriate for taking into account the disorder over \( U(1) \) phases in the CCM in order to analyze its Lyapunov index (which defines the correlation length index for the edge excitations). In a result the Hubbard type model with staggered disposition of R-matrices is appearing.

In the action formalism also becomes evident how the models can be formulated on the random \( ML \), which will allow to develop the string model corresponding to them.

In a second part of the talk the integrable models will be analyzed, where the Monodromy matrix is defined as a two row product of staggered R-matrices. The corresponding Yang-Baxter equations (\( YBE \)), which ensures the commutativity of Transfer matrices of different values of the spectral parameter will be presented \cite{9}. It appeared, that the modified \( YBE \)’s have a solution for \( U_q(sl(n)) \) groups giving rise of the models with staggered signs of the anisotropy parameter \( \Delta \). Since in this construction the \( R(u) \)-matrices in the product has also staggered shift of the spectral parameter \( u \) by new model parameter \( \theta \), as the calculations of the Hamiltonian shows, they can be regarded as a models on the zig-zag ladder chains. In the XXZ \cite{10}, anisotropic t-J \cite{12} and Hubbard cases \cite{13} the Hamiltonian is found explicitly. The quantum group structure, which is behind of this construction in the \( sl(n) \) case was analyzed in the article \cite{9}.
2. The SF of 3DIM and Field Theory Formulation of CCM on 2D ML

In the article [3] the model for SF of 3DIM [1, 2, 3] was formulated on the random ML, which is induced by the random closed surface in 3D regular lattice. But for simplicity we will consider now the flat ML and outlined the essential characteristics of the model.

The Manhattan lattice (ML) is the lattice, where there are continuous arrows on the links with the opposite directions on the neighbor parallel lines (Fig.1). The arrows form a set of vectors $\vec{\mu}_{ij} \in S$. ML originally was defined by Kasteleyn [14] in connection with the problem of single Hamiltonian walk.

Figure 1. Manhattan lattice.

The plaquettes of ML are divided into four groups, $A_a$ and $B_a$ (a=1,2), destined in the chess like order. The A-plaquettes differ from B-plaquettes by the fact, that arrows are circulating around them, while
there is no circulation for B-plaquettes. $A_1(A_2)$ has clockwise (counterclockwise) circulation, while $B_1$ differs from the $B_2$ by rotation on $\pi/4$.

Consider the field of Grassmann variables $\Psi_{\vec{n}} = \left( \psi_{\vec{n},L} \psi_{\vec{n},L} \right)$ at the sites $\vec{n}$ of $ML$, which is spinor irrep of $SO(3)$ (or fundamental irrep of $SU(2)$), but forbid the double occupancy of all sites by fermions. This can be achieved, for example, by putting the projectors $\Delta_{\vec{n}} = \psi_{\vec{n},L} \psi_{\vec{n},L} + \psi_{\vec{n},R} \psi_{\vec{n},R}$ to the sites. Following the article [3] let us write the action of this fields as for fermions, hoping only along arrows of $ML$ and being in the external $SU(2)$ gauge field, which is induced by the immersion of the 2D surface into the 3D Euclidean space (see details in [3]). Then this action defines the model for $SF$ of $3DIM$.

In 1988 J. Chalker and P.D. Coddington [4] have defined a phenomenological model in the Transfer matrix formalism in order to describe the edge excitations in Hall effect, responsible for plateau–plateau transitions. Remarkably, the numerical simulations give the desired experimental value for the correlation length index, approximately (may be exactly) equal to $7/3$.

We will see now, that if one will consider on $ML$ an action of scalar Grassmann fields $\psi_{\vec{n}}$, which are hopping in the $U(1)$ gauge field along arrows with appropriate hopping parameters and by use of coherent states [15, 16] pass to Transfer matrix (Hamiltonian) of discrete time evolution (as it is done in [5]), then in one particle sector the Transfer matrix of $CCM$ before averaging over random phases will be reproduced. The action of the model is

$$\mathcal{-S}(\Psi_{\vec{n}}; \Psi_{\vec{n}}) = \sum_{\vec{n}, \beta = 1, 2} t_{\vec{n}, \vec{n} + \vec{\mu}_\beta(\vec{n})} \psi_{\vec{n}} U_{\vec{n}} \psi_{\vec{n} + \vec{\mu}_\beta(\vec{n})} + \sum_{\vec{n}} \bar{\psi}_{\vec{n}} \psi_{\vec{n}} + \bar{\psi}_{\vec{n}} \psi_{\vec{n}}. \quad (2)$$

In this expression $\mu_\beta(\vec{n})$, $\beta = 1, 2$ are the fields of unit vectors on $ML$ defined at each site $\vec{n}$ and directed along two exiting arrows and $t_{\vec{n}, \vec{n} + \vec{\mu}_\beta(\vec{n})}$ are the hopping parameters between the points $\vec{n}$ and $\vec{n} + \vec{\mu}_\beta(\vec{n})$. Because the structure of $ML$ is translational invariant on two lattice spacing in both(time and space) directions, which we would like to maintain, in most general case one can consider only eight different hopping parameters. Below, in correspondence with notations on Fig.1, we will mark the hopping parameters from $j$ to $i$ ($i, j = 1, 2, 3, 4$) as $t_{ij}$.

The field of phase factors $U_{\vec{n}} = e^{i\alpha_{\vec{n}}}$ is independent of $\beta = 1, 2$ (the phase factors on the exiting from the site $\vec{n}$ two links are the same). This distribution of phases on $ML$ is in exact correspondence with $CCM$ and defines the $U(1)$- curvature equal to zero for the all B-plaquettes, while random curvatures are located in the A-plaquetts. It is also in clear
correspondence with the random ML picture for the SF of 3DIM [3], where the curvatures, induced by immersions of 2d surfaces in 3D regular lattice, are located in the A-plaquettes.

Let us introduce now the fermionic coherent states according to articles [15, 16] and pass to fermionic Transfer matrix as it is done in [5].

\[ |\psi_{2j}\rangle = e^{\psi_{2j}c_{2j}^+}|0\rangle, \quad \langle\bar{\psi}_{2j}| = \langle 0|e^{c_{2j}\bar{\psi}_{2j}} \]

for the even sites of the chain and

\[ |\bar{\psi}_{2j+1}\rangle = (c_{2j+1}^+ - \bar{\psi}_{2j+1})|0\rangle, \quad \langle\psi_{2j+1}| = \langle 0|(c_{2j+1} + \psi_{2j+1}) \]

for the odd sites.

This states are designed as an eigenstates of creation-annihilation operators of fermions \( c_j^+, \ c_j \) with eigenvalues \( \psi_j \) and \( \bar{\psi}_j \)

\[
\begin{align*}
    c_{2j} | \psi_{2j}\rangle &= -\psi_{2j} | \psi_{2j}\rangle, \quad \langle \bar{\psi}_{2j} | c_{2j}^+ \rangle = -\langle \bar{\psi}_{2j} | \bar{\psi}_{2j} \rangle, \\
    c_{2j+1}^+ | \bar{\psi}_{2j+1}\rangle &= \bar{\psi}_{2j+1} | \bar{\psi}_{2j+1}\rangle, \quad \langle \psi_{2j+1} | c_{2j+1} \rangle = -\langle \psi_{2j+1} | \psi_{2j+1} \rangle.
\end{align*}
\]

It is easy to calculate the scalar product of this states

\[
\langle \bar{\psi}_{2j} | \psi_{2j}\rangle = e^{\bar{\psi}_{2j}\psi_{2j}}, \\
\langle \psi_{2j+1} | \bar{\psi}_{2j+1}\rangle = e^{\bar{\psi}_{2j+1}\psi_{2j+1}}
\]

and find the completeness relations

\[
\int d\bar{\psi}_{2j} d\psi_{2j} | \psi_{2j}\rangle \langle \bar{\psi}_{2j} | e^{\bar{\psi}_{2j}\psi_{2j}} = 1, \\
\int d\bar{\psi}_{2j+1} d\psi_{2j+1} | \psi_{2j+1}\rangle \langle \bar{\psi}_{2j+1} | e^{\bar{\psi}_{2j+1}\psi_{2j+1}} = 1.
\]

Let us attach the Fock spaces \( V_j \) of scalar fermions \( c_j^+, \ c_j \) to each site of the chain and consider two type of \( R \)-matrices in the braid formalism in the operator form

\[
\begin{align*}
    R_{2j, 2j+1} &= a_{\pm 1}n_{2j}n_{2j+1} + a_{\pm 2}(1 - n_{2j})(1 - n_{2j+1}) + n_{2j}(1 - n_{2j+1}) \\
    &= (a_{\pm 1}a_{\pm 2} + b_{\pm 1}b_{\pm 2})n_{2j+1}(1 - n_{2j}) + b_{\pm 1}c_{2j}^+c_{2j+1} + b_{\pm 2}c_{2j+1}^+c_{2j} \\
    &= e^{[b_{\pm 1}c_{2j+1}^+c_{2j} + b_{\pm 2}c_{2j}^+c_{2j+1} + (a_{\pm 1} - 1)c_{2j}^+c_{2j} + (1 - a_{\pm 2})c_{2j+1}^+c_{2j}]};
\end{align*}
\]

corresponding to two type of \( B \)-plaquettes on ML, with

\[
\begin{align*}
    a_{+1} &= e^{i\alpha_3 t_{43}}, \quad a_{+2} = e^{i\alpha_1 t_{21}}, \quad b_{-1} = -e^{i\alpha_3 t_{23}}, \quad b_{+2} = e^{i\alpha_1 t_{41}}, \quad \text{for } B_1, \\
    a_{-1} &= e^{i\alpha_4 t_{34}}, \quad a_{-2} = e^{i\alpha_2 t_{12}}, \quad b_{-1} = -e^{i\alpha_4 t_{14}}, \quad b_{-2} = e^{i\alpha_2 t_{32}}, \quad \text{for } B_2.
\end{align*}
\]
and where the symbol \( :: \) means normal ordering of fermionic operators in the even sites and anti-normal (hole) ordering for the odd sites. This operators are acting on the direct product of two neighbor Fock spaces \( V_{2j} \otimes V_{2j+1} \) and are nothing, but the fermionized versions of \( R \)-matrices of the ordinary \( XX \) models

\[
\hat{R}_\pm = \begin{pmatrix}
a_{\pm 1} & 0 & 0 & 0 \\
0 & 1 & b_{\pm 1} & 0 \\
0 & b_{\pm 2} & (a_{\pm 1}a_{\pm 2} + b_{\pm 1}b_{\pm 2}) & 0 \\
0 & 0 & 0 & a_{\pm 2}
\end{pmatrix},
\]

which can be found by Jordan-Wigner transformation [19] or by the alternative technique, developed in [20]. Considering now two Monodromy matrices \( M_1 \) and \( M_2 \) as a product of \( R \)-matrices (corresponding to \( B \)-plaquettes) along the neighbor rows

\[
M_1 = \prod_j \hat{R}_{2j, 2j+1}, \quad M_2 = \prod_j \hat{R}_{2j, 2j-1},
\]

one can show that the Transfer matrix \( T = Tr M_1 M_2 \) defines the partition function \( Z \) according to formulas (1) and (2). Indeed, let us in the space of states \( \prod_j V_j \) of the chain pass to the coherent basis and calculate the matrix elements of the \( \hat{R}_{2j, 2j\pm 1} \)-operators between the initial \( | \psi_{2j} \rangle, \langle \bar{\psi}_{2j\pm 1} | \) and final \( | \psi'_{2j} \rangle, \langle \bar{\psi}'_{2j\pm 1} | \) states. By use of properties of coherent states it is easy to find from the formula (8), that

\[
R_{\bar{\psi}_{2j}, \psi_{2j}\pm 1}^{\bar{\psi}'_{2j}, \psi'_{2j}\pm 1} = \langle \psi'_{2j\pm 1}, \bar{\psi}'_{2j} | \hat{R}_{2j, 2j\pm 1} | \psi_{2j}, \bar{\psi}_{2j\pm 1} \rangle = e^{[a_{\pm 1} \bar{\psi}_{2j} \psi_{2j} + a_{\pm 2} \bar{\psi}_{2j\pm 1} \psi_{2j\pm 1} - b_{\pm 1} \bar{\psi}_{2j\pm 1} \psi_{2j} + b_{\pm 2} \bar{\psi}_{2j} \psi'_{2j\pm 1}],}
\]

which, together with multiplication rules due to completeness relations (7), demonstrates the correctness of the formula (1) (see [5] for details).

Let us consider now the one-particle sector of Fock space of the chain

\[
| i \rangle = c_i^+ | 0 \rangle, \quad i = 1, \ldots, 2N
\]

and calculate the matrix elements of the operators \( M_0 \) and \( M_1 \) (11) in this basis. Then after parameterizing the hopping parameters as

\[
-t_{23} = t_{12} = t_{34} = t_{41} = 1 / \cosh \theta,
-t_{14} = t_{43} = t_{32} = t_{21} = \tanh \theta
\]

and unessential rescaling of \( M_0 M_1 \) by factor \( (t_{12}t_{21})^N \), one easily can recover the \( 2N \times 2N \) Transfer matrix by Chalker and Coddington before
Integrable chain models

Figure 2. Scattering of particles in CCM corresponding a) to $B_2$ - b) to $B_1$ plaquettes.

the averaging over phases, introduced in the article [4]. In order to make the correspondence totally obvious one should change in the Fig.1 the $B$-plaquettes bye the act of scattering, as it is drown by dashed lines in the Fig.2.

It appears that the action formalism by use of fermionic fields on $ML$ is quite appropriate for taking now into account the disorder over the $U(1)$ phases in the model and investigate, for example, the Lyapunov index, which defines the correlation length index. For Lyapunov index one should investigate the average over $\phi = -i \log U$ phases of the square of the partition function $\langle Z \otimes Z^+ \rangle$ and we will consider the Gaussian distribution $P(\{\phi_i\}) = \prod \frac{1}{\sqrt{\pi}} \exp \left(-\frac{\phi^2_i}{\kappa} \right)$ for them. It is clear, that since phases are defined locally and there is no correlations in averaging between different points, we will have

$$\langle Z \otimes Z^+ \rangle = Tr \left( \prod_j \langle \tilde{R}_{2j,2j+1} \otimes \tilde{R}_{2j,2j+1} \rangle \right)^N \prod_i \langle \tilde{R}_{2i-1,2i} \otimes \tilde{R}_{2i-1,2i} \rangle$$

(15)

The average $\langle \tilde{R}_{2j,2j+1} \otimes \tilde{R}_{2j,2j+1} \rangle$ is defining the $R$-operator of the new model and it is easy to calculate it in the $\psi$-basis of coherent states. Simple Gaussian integration by use of expressions (12) gives us the $R$-matrix of the averaged model

$$R_{\psi_{2j,\sigma_{-1,\pm}} \psi_{2j,\sigma_{+1,\pm}}} = exp \left\{ \sum_{\sigma=\uparrow,\downarrow} \left[ a_{\sigma} (\psi'_{2j,\sigma} \psi_{2j,\sigma} + \psi_{2j,\sigma} \psi_{2j,\sigma}^r) + (-)^{\sigma} b_{\sigma} (\psi_{2j,\sigma} \psi_{2j,\sigma} - \psi_{2j,\sigma} \psi_{2j,\sigma}^r) \right] \right\}$$

(16)

$$+ 2 \sinh \kappa \left[ \psi_{2j,\sigma_{\downarrow}} (\bar{a}_{\sigma} \psi_{2j,\sigma_{\downarrow}} - \bar{b}_{\sigma} \psi_{2j,\sigma_{\downarrow}}) (\bar{a}_{\sigma} \psi_{2j,\sigma_{\downarrow}} - \bar{b}_{\sigma} \psi_{2j,\sigma_{\downarrow}}) \psi_{2j,\sigma_{\downarrow}} \right]$$
\[ + \psi_{2j, \uparrow}^\dagger (\bar{a}_{1} \pm \psi_{2j, \uparrow} + \bar{b}_{1} \pm \psi_{2j, \downarrow}) (\bar{a}_{2} \pm \psi_{2j, \downarrow} + \bar{b}_{2} \pm \psi_{2j, \uparrow}) \}\{ \psi_{2j, \downarrow} \}. \]

For simplicity we have written here the expression only for the case \( \bar{a}_{\pm} = e^{-\kappa/2t_{43}}, \bar{a}_{-} = e^{-\kappa/2t_{34}}, \bar{b}_{\pm} = e^{-\kappa/2t_{41}}, \bar{b}_{\pm} = e^{-\kappa/2t_{32}} \) are the average values of hopping parameters.

The fermionic fields \( \psi_{\uparrow} \) and \( \psi_{\downarrow} \) in the expression (16) appeared because the operators \( R \) and \( R^+ \) in the direct products in (15) are acting on independent spaces and we should introduce different coherent fields for them.

What is left now to say, that the (16) is the expression for the \( R \)-operator of the generalization of the Hubbard model

\[ \bar{R}_{12} = e^{-h_{L}(u)(2n_{1, \uparrow} - 1)(2n_{1, \downarrow} - 1)} \bar{R}_{XX} \bar{R}_{12, \uparrow} e^{-h_{R}(u)(2n_{2, \uparrow} - 1)(2n_{2, \downarrow} - 1)} \] (17)

with the condition \( h_{L}(u) = h_{R}(u) = \kappa(u)/4 \), written in the basis of coherent states, as it was described above.

It is necessary now to mention two things:

a) The \( R \)-matrix of ordinary Hubbard model contained the exponent in (17), which is responsible for the interaction, only in the right (or left) hand sides of the product of two \( XX \) models \( R \)-matrices with \( \uparrow \) and \( \downarrow \) spins [17, 18, 19];

b) Averaging the \( CCM \) we have obtained Hubbard type model with staggered disposition of \( R \)-matrices. A similar type of integrable model is developed in [13].

3. The action on \( ML \) for any model described by \( R \)-matrix

It is not hard to realize now, that the formulated above technique is quite general and allows to pass from Hamiltonian to the Action (Transfer matrix) formalism for any 2D model, which has a description via \( R \)-matrix. Let \( R_{a, \alpha, \gamma}^{\gamma, \beta} \), \( \alpha, \gamma = 1, \ldots, l \) is the \( R \)-matrix of some model, which has \( l \)-degrees of freedom at the sites of the chain.

In a beginning we should fermionize the model (see [20] for details) by considering Fock space of \( r \)-scalar fermions (with \( l \leq 2r \)) \( c_{i,s}^{\dagger} \), \( c_{i,s} \), \( s = 1, \ldots, r \) at each site \( i \) of the chain with basis

\[ | n_{i,1}, \ldots, n_{i,r} \rangle = c_{i,1}^{\dagger} \ldots c_{i,r}^{\dagger} | 0 \rangle, \] (18)

and restrict the appearance of the \( (2^r - l) \) basic states by applying with appropriate projectors on them as

\[ | \alpha \rangle = \Delta_{1} \ldots \Delta_{2^r - l} | n_{i,1}, \ldots, n_{i,r} \rangle. \] (19)
As an example one can mention the 3-state $t-J$ model of two fermions (spin $\uparrow$ and spin $\downarrow$) with restriction on double occupancy ($\Delta_1 = (1 - n_{\uparrow}n_{\downarrow})$).

Let us define now the fermionic $R$-operator
\[
\hat{R}_{ij} = \hat{R}_{ij,\alpha\gamma} X_{i,\alpha}^{\alpha'} X_{j,\gamma}^{\gamma'} (-1)^{p(\alpha)p(\gamma)},
\]
where $X_{i,\alpha}^{\alpha'} = |\alpha\rangle \langle \alpha'|$ is the Hubbard operator and $p(\alpha)$ is the fermionic parity of the state $|\alpha\rangle$.

One can now consider the coherent states for the all $r$-copies of fermions, extend the definition (12) for the $R$-operator and express it as an exponent of some action term, written for the $B$-plaquettes
\[
\hat{R}_{\{\bar{\psi}_{j,r}\},\{\psi'_{j+1,r}\}} = \langle\{\bar{\psi}_{j,r}\},\{\psi'_{j+1,r}\} | \hat{R} | \{\psi_{j,r}\},\{\bar{\psi}'_{j+1,r}\}\rangle = \exp \left\{ -S_{j,r}\{\psi_{j,r}\},\{\bar{\psi}_{j+1,r}\}\right\}.
\]
(21)

Then the full action of the model will be
\[
S = \prod_{B-\text{plaquettes}} S_{j,r}\{\psi_{j,r}\},\{\bar{\psi}_{j+1,r}\},\{\psi'_{j,r}\},\{\bar{\psi}'_{j+1,r}\}\rangle + \sum_{j,s} \bar{\psi}_{j,s} \psi_{j,s}.
\]
(22)

4. Integrable $\mathcal{U}_q(gl(n))$ models with staggered disposition of $R$-matrices

In this section we will present the main results [9, 10, 11, 12] of construction of integrable models with staggered disposition of $R$-matrices along chain and time directions.\(^1\)

Let us consider now $\mathbb{Z}_2$ graded quantum $V_{j,\rho}(v)$ (with $j = 1, ..., N$ as a chain index) and auxiliary $V_{a,\sigma}(u)$ spaces, where $\rho, \sigma = 0, 1$ are the grading indices. Consider $R$-matrices, which act on the direct product of spaces $V_{a,\sigma}(u)$ and $V_{j,\rho}(v)$, ($\sigma, \rho = 0, 1$), mapping them on the intertwined direct product of $V_{\bar{a},\bar{\sigma}}(u)$ and $V_{j,\bar{\rho}(v)}$ with the complementary $\bar{\sigma} = (1 - \sigma)$, $\bar{\rho} = (1 - \rho)$ indices

\[
R_{aj,\sigma\rho}(u, v) : V_{a,\sigma}(u) \otimes V_{j,\rho}(v) \rightarrow V_{j,\rho}(v) \otimes V_{a,\sigma}(u).
\]
(23)

It is convenient to introduce two transmutation operations $\iota_1$ and $\iota_2$ with the property $\iota_1^2 = \iota_2^2 = id$ for the quantum and auxiliary spaces correspondingly, and to mark the operators $R_{aj,\sigma\rho}$ as follows

\[
R_{aj,00} \equiv R_{aj}, \quad R_{aj,01} \equiv R_{aj}^{\iota_1}, \quad R_{aj,10} \equiv R_{aj}^{\iota_2}, \quad R_{aj,11} \equiv R_{aj}^{\iota_1\iota_2}.
\]
(24)
The introduction of the $\mathbb{Z}_2$ grading of quantum spaces in time direction means, that we have now two Monodromy operators $T_{\rho}, \rho = 0, 1$, which act on the space $V_{\rho}(u) = \prod_{j=1}^{N} V_{j,\rho}(u)$ by mapping it on $\bar{V}_{\rho}(u) = \prod_{j=1}^{N} V_{j,\bar{\rho}}(u)$

$$T_{\rho}(v, u) : V_{\rho}(u) \to \bar{V}_{\rho}(u), \quad \rho = 0, 1.$$  \hspace{1cm} (25)

It is clear now, that the Monodromy operator of the model, which is defined by translational invariance in two steps in the time direction and determines the partition function, is the product of two Monodromy operators

$$T(v, u) = T_0(v, u)T_1(v, u).$$  \hspace{1cm} (26)

The $\mathbb{Z}_2$ grading of auxiliary spaces along the chain direction means that the $T_0(u, v)$ and $T_1(u, v)$ Monodromy matrices are defined as a staggered product of the $R_{\alpha \beta}(v, u)$ and $\bar{R}_{\alpha \beta}(v, u)$ matrices:

$$T_1(v, u) = \prod_{j=1}^{N} R_{a,2j-1}(v, u)\bar{R}_{a,2j}(v, u)$$

$$T_0(v, u) = \prod_{j=1}^{N} \bar{R}_{a,2j-1}(v, u)R_{a,2j}(v, u),$$  \hspace{1cm} (27)

where the notation $\bar{R}$ denotes a different parameterization of the $R(v, u)$-matrix via spectral parameters $v$ and $u$ and can be considered as an operation over $R$ with property $\bar{R} = R$. For the integrable models where the intertwiner matrix $R(v - u)$ simply depends on the difference of the spectral parameters $v$ and $u$ this operation means the shift of its argument $u$ as follows

$$\bar{R}(u) = R(\bar{u}), \quad \bar{u} = \zeta - u,$$  \hspace{1cm} (28)

where $\zeta$ is an additional model parameter.

This definitions of the Monodromy matrices can be obtained from the disposition of $B$-plaquettes on the $ML$, when we are considering a chains under the angle $\pi/4$ with respect to those of (11) and staggering corresponds to $CCM$.

As it is well known in Bethe Ansatz Technique [6, 7, 8], the sufficient condition for the commutativity of transfer matrices $\tau(u) = TrT(u)$ with different spectral parameters is the YBE. For our case we have a two sets of equations [10]

$$R_{12}(u, v)\bar{R}_{13}(u)R_{23}(v) = R_{23}(v)\bar{R}_{13}(u)\bar{R}_{12}(u, v)$$  \hspace{1cm} (29)
Integrable chain models

\[ \tilde{R}_{12}(u,v) R_{13}^{i+1}(u) \tilde{R}_{23}^{i-1}(v) = \tilde{R}_{23}^{i+1}(v) R_{13}^{i-1}(u) R_{12}(u,v), \]  

(30)

with \( \tilde{R}(u) \equiv R(\bar{u}) \) and \( R^\pm(u) = R^\pm(-u) \).

From \( R(u) \) above, we follow a procedure which is the inverse of the Baxterisation (debaxterisation) [21]. Let

\[ R_{12}(u) = \frac{1}{2i}(z R_{12} - z^{-1} R_{21}^{-1}), \]  

(31)

with \( z = e^{iu} \) and the constant \( R_{12} \) and \( R_{21}^{-1} \) matrices are spectral parameter independent. Then the Yang–Baxter equations (29)–(30) for the spectral parameter dependent \( R \)-matrix \( R(u) \) and \( R^\pm(u) \) are equivalent to the following equations for the constant \( R \)-matrices

\[ R_{12} R_{13}^i R_{23} = R_{23}^i R_{13} R_{12} \]  

(32)

\[ R_{12} R_{13}^i R_{23} = R_{23}^i R_{13} R_{12} \]  

(33)

\[ R_{12} (R_{31}^i)^{-1} R_{23} = (R_{21}^i)^{-1} R_{13} (R_{32}^i)^{-1} = \]  

(34)

\[ R_{12} (R_{31}^i)^{-1} R_{23} = (R_{21}^i)^{-1} R_{13} (R_{32}^i)^{-1} = \]  

(35)

assuming \( \tilde{R} = R^i \).

If this modified YBE’s have a solution, then one can formulate a new integrable model on the basis of existing ones. It appeared that in the \( gl(N) \) case the two constant \( R \)-matrices \( R \) and \( R^i \) given by

\[ R = \sum_{i=1}^{N} q e_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^{N} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1 \atop i > j}^{N} e_{ij} \otimes e_{ji} \]  

(36)

\[ R^i = \sum_{i=1}^{N} q e_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^{N} b_{ij} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1 \atop i > j}^{N} e_{ij} \otimes e_{ji} \]  

(37)

satisfy the four equations (32)–(35) provided that \( b_{ij} = b_{ik} b_{kj} \) and \( b_{ij}^2 = 1 \).

By construction, all this models are of the ladder type.

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