Non-abelian Eikonals

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Abstract

A functional formulation and partial solution is given of the non-abelian eikonal problem associated with the exchange of non-interacting, charged or colored bosons between a pair of fermions, in the large $s$/small $t$ limit. A simple, functional “contiguity” prescription is devised for extracting those terms which exponentiate, and appear to generate the leading, high-energy behavior of each perturbative order of this simplest non-abelian eikonal function; the lowest non-trivial order agrees with the corresponding SU(N) perturbative amplitude, while higher-order contributions to this eikonal generate an “effective Reggeization” of the exchanged bosons, resembling previous results for the perturbative amplitude. One exact and several approximate examples are given, including an application to self-energy radiative corrections. In particular, for this class of graphs and to all orders in the coupling, we calculate the leading-log eikonal for SU(2). Based on this result, we conjecture the form of the eikonal scattering amplitude for SU(N).

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I Introduction

One of the most persistent problems in the application of field theory methods to particle scattering has been the inability to generalize, in a direct functional, non-perturbative way, abelian eikonal models to their non-abelian counterparts. Many efforts in this direction have of course been made over the last several decades, using the partial, perturbative summation of an eikonal function \[^{[1]}\), or a variety of non-perturbative approximations \[^{[2]}\). In ref. \[^{[2]}\), for example, a “mean-field” approximation was made to the relevant functional integrals corresponding to the exchange of neutral vector mesons (NVMs) between scattering nucleons, which include the restrictions of SU(2) isospin; and the result, while “approximately correct”, left a certain unease in its wake. A more modern example is the problem of how to include SU(3) color restrictions in QCD\[^{[2]}\), which must be faced if one is to attempt any functional calculation using the recent, exact and approximate Greens’ functions \[^{[3]}\) \[^{[4]}\) of that theory \[^{[3]}\); or, indeed, the new dimensional-transmutation/flux-string expansion of quark-quark scattering amplitudes \[^{[4]}\).

We give in this paper a complete, if formal, representation of the simplest non-abelian eikonal, corresponding to multiple gluon exchange between scattering quarks \[^{without}\) virtual gluon-gluon interactions; we extract that portion which can be easily isolated, and define a particular, ordered-exponential (OE) representation of the remainder, which can be expanded or approximated in various ways. In particular, we define a simple, functional procedure called “contiguity”, which, in an immediate way, isolates at least a subset of those terms that are definitely exponentiated, and can be represented to all orders by a perturbative expansion of the eikonal. These terms correspond to the leading \[^s\)-dependence in the lowest, non-trivial order, and we argue that they correspond to the extraction of the leading \[^s\)-dependence in every perturbative order of the eikonal function. For quark-quark scattering, the result duplicates the essence of well-known, leading-log perturbative estimates previously calculated for amplitudes \[^[4]\). The method will be illustrated in two contexts, and its applicability discussed for more general, non-abelian problems; in particular, for this class of graphs and to all orders in the coupling, we calculate the leading-log eikonal of SU(2).

To our knowledge this is the first time that such estimates have been obtained in a purely functional context, while the contiguity technique opens the way for an attack on other, more complicated, non-abelian eikonal problems, such as those which involve virtual gluon-gluon interactions (in particular, the so-called “towers” and their generalizations), as well as self-energy and vertex effects of non-abelian, virtual-gluon emission and absorption by a single quark. However, by treating only boson exchange, without self-interactions between the exchanged bosons, we are apparently going to violate requirements of gauge invariance, which for perturbative, Yang-Mills gluons, require the simultaneous computation of all relevant graphs of a given order, and not just the simple eikonal graphs considered here. Surely the same sort of inclusion must eventually be true for any non-perturbative attempt. We ask the reader to suspend judgment on this point until the final discussion presented in the Summary of Section 6; and to realize that, while gauge invariance must of course be insured in any com-
putation whose results are going to be compared with experiment, we are proposing a
functional attach on that part of the problem of immediate concern to the scattering
quarks. This is important because a functional treatment contains all powers of the
coupling; and it is useful because there exists an additional, computational step by
which gauge invariance can be re-established - including the relevant contributions
generated by all gluon-gluon interactions - later on. The main thrust of the present
paper is the functional extraction of leading-log, energy dependence of the simplest,
non-Abelian eikonal.

We begin at that stage of a quark-quark scattering amplitude where mass-shell
amputation (MSA) has already been carried out on the fermion Green’s functions
\langle p_{1,2}|G_c[A]|p'_{1,2} \rangle approximated in a no-recoil fashion \([5]\), and the essential structure of
the eikonal function which describes non-abelian NVM exchange between a pair of
fermions (quarks, for SU(3)) has been recognized \([6]\) as:

\[ e^{i\chi} = e^{-i \int \frac{\delta}{\delta A_I} Q \frac{\delta}{\delta A_I} (e^{-i g_1 \int_{-\infty}^{+\infty} ds p_1 \cdot A_I^a(z_1 - sp_1) \lambda^I_a})} + \]

\[ e^{-i g_2 \int_{-\infty}^{+\infty} dt p_2 \cdot A^b_{II}(z_2 - tp_2) \lambda^I_b} + |A_I = A_{II} = 0, g_1 = g_2 = g \]

where \(z_{1,2}\) and \(p_{1,2}\) are the fermions’ configuration and momentum coordinates, and
\(Q_{\mu\nu}^{ab}\) is the appropriate boson propagator. Eq. (1.1) defines “linkages” between a pair
of OEs, and the result will necessarily be a “doubly-ordered-exponential”. How this
can be transformed into a pair of single OEs; how the leading-logs of the latter may
be extracted, leaving but a single OE; and how that OE can, for SU(2), be summed
explicitly over all perturbative orders, is the main content of this paper.

More precisely, the preferred method of obtaining the eikonal in the conventional
case, where the conventional, no-recoil approximation of \(G_c[A]\) destroys coordinate
symmetry of this Green’s function, is to calculate not \(T_{eik}\) but, before MSA:

\[ \frac{\partial^2 T_{eik}}{\partial g_1 \partial g_2} = \frac{i}{g_1 g_2} \frac{\delta}{\delta \phi(0)} \frac{\delta}{\delta \psi(0)} e^{-i \int \frac{\delta}{\delta A_I} Q \frac{\delta}{\delta A_{II}}} \left( e^{-i g_1 \int_{-\infty}^{+\infty} ds \phi(s) p_1 \cdot A_I(z_1 - sp_1) \cdot \lambda^I} \right) + \]

\[ e^{-i g_2 \int_{-\infty}^{+\infty} dt \psi(t) p_2 \cdot A_{II}(z_2 - tp_2) \cdot \lambda^II} + |\phi(s) = \psi(s) = 1, A_I = A_{II} = 0 \]

and integrate over \(g_{1,2}\) (with the boundary conditions \(T_{eik}(g_1, 0) = T_{eik}(0, g_2) = 0\)
after the necessary functional linkages have been performed \([6]\); it has been assumed
that the RHS of (1.2) is a function of $z_1 - z_2$, and the subsequent $\delta^{(4)}(q_1 + q_2)$ statement of 4-momentum conservation has been suppressed. For simplicity we consider the quantity of (1.1) as representative of the correct eikonal – it is exactly the eikonal in the absence of non-abelian complications – even though it is quite possible to produce, upon integration over the couplings of (1.2), combinations which are more complicated than that of (1.1). However, eq. (1.1) is representative of the full, non-abelian structure of the problem, and we here restrict attention to this quantity. The non-commuting objects $\lambda_a$ are taken to be the Gell-Mann matrices of SU(N). We again emphasize that more complicated eikonal graphs, such as the "tower graphs" of Cheng and Wu, are not inc luded in this analysis, although they can be formally inserted by the functional methods outlined in the last chapters of references [1] and [3].

In the abelian case, where $A^a_\mu \rightarrow A_\mu$, and the $\lambda_a$ are missing, the functional operation of (1.1) may be performed immediately, yielding:

$$i\chi = ig^2(p_1 \cdot p_2) \int \int_{-\infty}^{+\infty} ds \, dt \Delta_c(z_1 - z_2 - sp_1 + tp_2)$$

(1.3)

with a propagator $Q_{\mu\nu}(x_1, x_2) = \delta_{\mu\nu}\Delta_c(x_1 - x_2)$. The proper-time integrals are easily performed when a Fourier representation of $\Delta_c$ is inserted into (1.3); and one finds:

$$i\chi = -ig^2 \frac{\gamma(s)}{2\pi} K_0(\mu b)$$

(1.4)

where $\gamma(s) = \frac{(s - 2m^2)}{\sqrt{s(s - 4m^2)}}$ is that factor depending on the spin of the exchanged boson, of mass $\mu$; the fermion mass is denoted by $m$, and in this equation, $s$ denotes the total CM (energy)$^2$ of the two quarks. In all subsequent expressions, we shall assume the high-energy limit, where $\gamma(s) \rightarrow 1$.

We give in the next Section a new, functional formulation of the eikonal of (1.1), and, in an appropriate kinematical situation, display one exact solution. More generally, a perturbative expansion of this eikonal functional may be defined, and certain obvious terms (which are the most elementary generalizations of the abelian eikonal) are summed to all orders. In Section III, we define the statement of “functional contiguity”, which isolates those terms of (1.1) that are definitely exponentiated, and which appears to generate the leading $\ln(E/m)$ dependence of every perturbative term of the non-abelian eikonal, when the necessary, doubly-ordered-exponential is defined in a moderately elegant way. In the next Section, we discuss the leading-log approximation, and show how the extraction of such terms (from “nested” momentum integrals) can reduce the complexity of the computations to operations upon a single OE; for SU(2), these operations are performed and summed to all orders, and suggest a conjecture for the corresponding eikonal scattering amplitude of SU(N). In Section V, we apply the analysis to self-energy processes, as well as to eikonal tower graphs and their generalizations, while Section VI contains a summary of our present understanding of this eikonal construction.
II Formulation

In order to perform the functional operation of (1.1), it is useful to introduce for each OE the functional representation:

\[
\left( -ig \int_{-\infty}^{+\infty} ds p_\mu A_\mu^a(z - sp) \lambda_a \right) \equiv (2.1)
\]

\[
N' \int d[\alpha] \int d[u] e^i \int_{-\infty}^{+\infty} ds \alpha_a(s) [u_a(s) - gp_\mu A_\mu^a(z - sp)] \cdot \left( e^{i \int_{-\infty}^{+\infty} ds \lambda_a u_a(s)} \right)
\]

or, more simply, rewriting (2.1) as: 

\[
I \otimes \exp \left[ -i \int_{-\infty}^{+\infty} ds p_\mu A_\mu^a(z - sp) \alpha_a(s) \right],
\]

where \( N' \) is an appropriate normalization constant. That (2.1) is trivially true can be seen by breaking up the \(-\infty < s < +\infty\) range into small intervals, and integrating over the \( \alpha_a(s_i) \) which leads to a delta functional of the \( u_a(s) \), whose integration immediately produces the LHS of (2.1). The advantage of this procedure is that the functional linkages of (1.1) are now abelian, and may be performed immediately, yielding:

\[
e^{i\eta} = I_1 \otimes \cdot I_2 \otimes \exp \left[ i \int \int_{-\infty}^{+\infty} ds dt \alpha_a(s) Q_{ab}(s, t) \beta_b(t) \right] \]  

(2.2)

with \( Q_{a,b} = g^2 p_\mu^a Q_{ab}^\mu p_\mu^b \), and where the \( I_{1,2} \) denote, from (2.1), simultaneous functional operations to be performed on the \( \alpha_a(s) \) and \( \beta_b(t) \) variables.

These final operations are what is now needed, and may be delineated by the insertion of relevant source and parameter dependence, followed by a “Schwingerian search” for an appropriate “differential characterization”. With the definition:

\[
R(s, t|\xi, \eta) = N' \int d[\alpha] \int d[u] e^{i \int_{-\infty}^{+\infty} \alpha \cdot u \left( e^{i \int_{-\infty}^{+\infty} \lambda \cdot u} \right) + e^{i \int_{-\infty}^{+\infty} u \cdot \xi}} \]

\[
\cdot N' \int d[\beta] \int d[v] e^{i \int_{-\infty}^{+\infty} \beta \cdot v \left( e^{i \int_{-\infty}^{+\infty} \lambda \cdot v} \right) + e^{i \int_{-\infty}^{+\infty} v \cdot \eta}} \]

\[
\cdot \exp \left[ i \int_{-\infty}^{+\infty} ds' dt' \alpha_a(s') Q_{ab}(s', t') \beta_b(t') \right]
\]

(2.3)

comparison with (2.2) shows that the quantity needed is \( \ln R(+\infty, +\infty|0, 0) \). One can create a variety of differential equations involving the proper-time parameters \( s, t \) and the sources \( \xi_a(s), \eta_b(t) \); but for present purposes, it seems to be sufficient to work with only \( s \) and \( \eta \), so that we consider \( R(s, +\infty|0, \eta) = R(s|\eta) \).

We next outline the steps which result in the differential equation (2.6), stated below. Calculation of \( (\partial/\partial s) R(s|\eta) \) brings down under the integrals the quantity \( i\lambda^\mu_a u_a(s) \), standing to the left of its OE, which may be represented as \( \lambda^\mu_a \partial / \partial \alpha_a \) acting upon \( \exp \left[ i \int \alpha \cdot \xi \right] \); then a functional integration-by-parts moves this \( \partial / \partial \alpha_a \) to act upon the last line of (2.3), which generates under the functional integrals the net
quantity \((-i) \int_{-\infty}^{+\infty} dt \lambda_{ab}^I Q_{ab}(s,t) \beta_b(t)\). The procedure may now be reversed, representing \((-i)\beta_b(t)\) by the operation \(-\delta/\delta v_b(t)\) acting upon \(\exp[i v \cdot \beta]\); and using another functional integration-by-parts to convert this to the operation:

\[
\frac{\delta}{\delta v_b(t)} \left[ \left( e^{i \int_{-\infty}^{+\infty} \lambda^I u \cdot v} + e^{i \int_{+\infty}^{1/2} \lambda u \cdot v} \right) \right]
= i \left[ \eta_b(t) + \left( e^{i \int_{+\infty}^{1/2} \lambda u \cdot v} + \lambda^I_b \left( e^{-i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) \right) \right] \left( e^{i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) + e^{i \int_{+\infty}^{1/2} \lambda u \cdot v} \tag{2.4}\]

written in terms of the anti-ordered quantity:

\[
\left( e^{-i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) = \left[ \left( e^{i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) \right] = \left[ \left( e^{i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) \right]^{-1}
\]

We introduce the notation:

\[
\Lambda^H_b (t|iv) = \left( e^{i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) + \lambda^H_b \left( e^{-i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) \tag{2.5}\]

and observe that (2.4) may be rewritten as:

\[
i \left[ \eta_b(t) + \Lambda^H_b (t|\delta/\delta \eta) \right] \left( e^{i \int_{-\infty}^{+\infty} \lambda u \cdot v} \right) + e^{i \int_{-\infty}^{+\infty} \lambda u \cdot v}
\]

so that, finally, one obtains the differential equation:

\[
\frac{\partial R(s|\eta)}{\partial s} = i \int_{-\infty}^{+\infty} dt \lambda^I_a Q_{ab}(s,t) \left[ \eta_b(t) + \Lambda^H_b \left( t|\delta \right) \right] \cdot R(s|\eta) \tag{2.6}\]

With the boundary condition \(R(-\infty|\eta) = 1\), easily seen as appropriate from the definition of \(R(s|\eta)\), the solution to (2.6) may be written as an OE:

\[
R(s,t) = \left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' \lambda^I_a Q_{ab} \left( s', t' \right) \left[ \eta_b(t') + \Lambda^H_b \left( t'|\delta \right) \right] \right] \right) \bigg|_{s'} \tag{2.7}\]

with the ordering indicated for the \(s'\) variables only. With \(s \to +\infty\) and \(\eta \to 0\), we then have a representation of (2.3) which apparently involves a single OE; however, it should be noted that the second ordering will be found in the definition of \(\Lambda^H_a\), eq.(2.5), so that there do exist two sets of “orderings”, although they can now be addressed separately. In fact, the “\(t\)-orderings” can be defined from the integral solution to the differential equation satisfied by \(\Lambda^H_a \left( t|\delta \right)\); the latter may immediately be obtained from its definition (2.5):

\[
\frac{\partial}{\partial t} \Lambda^H_a \left( t|\delta \right) = 2i \int f_{acd} \frac{\delta}{\delta \eta_c(t)} \Lambda^H_d \left( t|\delta \right)
\]

which, together with the boundary condition at \(t = \infty\), generates:
\[
\Lambda^H_a \left( t \big| \frac{\delta}{\delta \eta} \right) = \Lambda^H_a - 2i f_{abc} \int_t^\infty dt' \frac{\delta}{\delta \eta_c(t')} \Lambda^H_d \left( t' \big| \frac{\delta}{\delta \eta} \right) \tag{2.8}
\]
whose repeated iterations contain all the \(t\)-orderings of the problem, and where the \(f_{abc}\) are the structure constants of the SU(N) algebra.

Conventional eikonal models replace \(Q_{a,b}\) by \(\delta_{a,b} Q(s, t)\), and in the absence of any other isospin/color vector, we may expect that the result will generate the products \(\lambda^I \cdot \lambda^H\). The latter may then be replaced by eigenvalues appropriate to the scattering problem; for example, in the SU(2) isospin scattering of two nucleons, those eigenvalues are given by \(I (I+1)/2-3/4\), for singlet \((I=0)\) or triplet \((I=1)\) total isospin; for SU(3), the situation is somewhat more complicated, as one tries to extract the overall, contribution of the eikonal to the singlet scattering amplitude \([\text{II}]\).

While (2.7) is a formal solution of the problem, certain terms of its expansion can be summed without difficulty. To see this, consider the expansion of (2.7) up to quadratic \(Q\)-dependence:

\[
R \bigg|_{s \to \infty} \simeq 1 + i \int \int_{-\infty}^{+\infty} ds' dt' \lambda^I_a Q_{ab}(s', t') \left[ \eta_b(t') + \Lambda^H_b \left( t' \big| \frac{\delta}{\delta \eta} \right) \right] +
\]

\[
+ i^2 \int \int_{-\infty}^{+\infty} ds_1 dt_1 \lambda^I_a Q_{ab} C_{s_1, t_1} \int_{-\infty}^{s_1} ds_2 \int_{-\infty}^{t_1} dt_2 \lambda^H_a Q_{ab} \left( s_2, t_2 \right) \cdot \left[ \eta_{b_1}(t_1) + \Lambda^H_{b_1} \left( t_1 \big| \frac{\delta}{\delta \eta} \right) \right] \cdot \left[ \eta_{b_2}(t_2) + \Lambda^H_{b_2} \left( t_2 \big| \frac{\delta}{\delta \eta} \right) \right] n \to 0 + \cdots \tag{2.9}
\]

With the definition of \(\Lambda^H_b(t \big| \frac{\delta}{\delta \eta})\), it is clear that the only contribution of the linear \(Q\)-terms is:

\[
i \int \int_{-\infty}^{+\infty} ds \, dt \lambda^I_a Q_{ab}(s, t) \lambda^H \tag{2.10}
\]
while the \(\frac{\delta}{\delta \eta}\)-independent part of the quadratic \(Q\)-terms of (2.9) yields:

\[
i^2 \int \int_{-\infty}^{+\infty} ds_1 dt_1 \lambda^I_a Q_{ab} C_{s_1, t_1} \lambda^H \int_{-\infty}^{s_1} ds_2 \int_{-\infty}^{t_1} dt_2 \lambda^I_a Q_{ab} \lambda^H \tag{2.11}
\]

This structure, obtained from the first term, \(\lambda^H\), in the iterative expansion of \(\Lambda^H\), eq. (2.8):

\[
\Lambda^H_a \left( t \big| \frac{\delta}{\delta \eta} \right) \simeq \lambda^H_a - 2i f_{bcd} \lambda^H_d \int_t^\infty dt' \frac{\delta}{\delta \eta_c(t')} + \cdots \tag{2.12}
\]
will appear in every term of the complete expansion of \(R\), and generates the OE:

\[
\left( \exp \left[ i \int \int_{-\infty}^{+\infty} ds \, dt \lambda^I_a Q_{ab}(s, t) \lambda^H \right] \right)_{+(s)} \tag{2.13}
\]
If, as typical, \( Q_{ab} = \delta_{ab} Q(s, t) \), all the \( \lambda^I \cdot \lambda^H \) terms in the expansion of (2.13) combine to form the products \( \lambda^I \cdot \lambda^H \), at which point the OE becomes an ordinary exponential (oe):

\[
\exp \left[ i \left( \lambda^I \cdot \lambda^H \right) \int_{-\infty}^{+\infty} ds \, dt \, Q(s, t) \right]
\] (2.14)

where the combination \( \lambda^I \cdot \lambda^H \) may be replaced by its appropriate eigenvalue. The value of the integrals of (2.14) may be read off from (1.3) and (1.4).

It is instructive to continue with the example of (2.9) and calculate the first commutator-term, as in (2.12), to this quadratic \( Q \)-dependence; it is:

\[
2i f_{b_1 b_2} \int \int_{-\infty}^{+\infty} ds_1 dt_1 \lambda^I_{a_1} Q_{a_1 b_1}(s_1, t_1) \int_{-\infty}^{s_1} ds_2 \int_{t_1}^{\infty} dt_2 \lambda^I_{a_2} Q_{a_2 b_2}(s_2, t_2) \lambda^H_d .
\] (2.15)

If, again \( Q_{a,b} = \delta_{a,b} Q(s, t) \), the antisymmetry of (2.15) under \( b_1, b_2 \) exchange is converted to a like antisymmetry under \( a_1, a_2 \) exchange, so that the pair \( \lambda^I_{a_1} \lambda^I_{a_2} \) may be replaced by \( i f_{a_1 a_2 c} \lambda^I_c \). One then finds the double summation \( \sum_{a_1 a_2} f_{a_1 a_2 c} f_{a_1 a_2 d} = C_2 \delta_{cd} \), where \( C_2(N) = N \) denotes the value of the quadratic Casimir invariant of the adjoint representation; and (2.15) becomes:

\[
-2C_2 \left( \lambda^I \cdot \lambda^H \right) \int \int_{-\infty}^{+\infty} ds \, dt \, Q(s, t) \int_{-\infty}^{s} ds_1 \int_{t}^{\infty} dt_1 Q(s_1, t_1)
\] (2.16)

In a typical eikonal situation corresponding to NVM exchange, \( Q(s, t) = g^2 (p_1 \cdot p_2) \Delta_c (z_1 - z_2 - sp_1 + tp_2) \), and the integrals of (2.16) may be evaluated to yield the leading \( \ln(E/m) \) dependence:

\[
i \frac{C_2}{2\pi} \left( \frac{g^2}{\pi} \right)^2 \left( \lambda^I \cdot \lambda^H \right) \ln(E/m) K^2_0(\mu b)
\] (2.17)

where \( 4E^2 \) denotes the total CM (energy)^2 of the scattering quarks. The form of (2.17) is worth noting, for it contains the new feature of a \( \ln(E/m) \) dependence multiplying reasonable, impact-parameter dependence; as explained in great detail in reference [1], it is the first appearance of an effective Reggeization of the exchanged gluon, and it appears directly in the eikonal function.

Before discussing how such contributions may be extracted and summed in this functional context, it may be appropriate to note that there is at least one kinematical context in which (2.13) is the exact result. This is the special case where \( Q_{a,b}(s, t) = Q_{a,b}(s) \delta(s - t) \), when the functional derivatives of (2.12) can never appear (due to a mis-ordering of subsequent, proper-time variables).

Another example where differences may be expected from the usual eikonal forms results from the appearance of a \( Q_{a,b} = f_{abc} \xi_c Q(s, t) \), where \( \xi_c \) is a color vector in the flux-string model of reference [4]. Because this \( Q \) is proportional to a delta function of the square of the \( x_1 - x_2 \) variables of (1.3), it produces an OE with only \( s \) dependence, and the kinematical forms which appear are quite different from the examples noted above.
Other formulations of the solution to (2.6) are possible, such as the representation of $R(s|\eta)$ by a Fourier functional transform, and the subsequent conversion of (2.6) to a differential equation linear in parametric and functional derivatives. However, because of the non-commutation of the $\lambda_a$, this route does not seem to lead to any real simplification.

III Contiguity

A representation for the general structure of all such terms may be obtained by the following argument. Return to the differential equation (2.6) for $R(s|\eta)$ and make the ansatz:

$$R_0(s) \equiv \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' \lambda_a^I Q(s', t') \left[ \eta_a(t') + \lambda_a^H \right] \right]_{+(s')} \quad (3.1)$$

and substitution of (3.1) into (2.6) then yields:

$$\frac{\partial U_0}{\partial s} = i \int_{-\infty}^{+\infty} dt R_0^{-1}(s) \lambda_a^I Q(s, t) \Delta \Lambda_a^H \left( t \frac{\delta}{\delta \eta} \lambda_a \right) R_0(s) \cdot U_0(s), \quad \Delta \Lambda_a^H = \Lambda_a^H - \lambda_a^H$$

with solution:

$$U_0(s) = \left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' R_0^{-1}(s') \lambda_a^I Q(s', t') \Delta \lambda_a^H \left( t' \frac{\delta}{\delta \eta} \lambda_a \right) R_0(s') \right] \right)_{+(s')} \quad (3.2)$$

from which we require the limits $s \to \infty, \eta \to 0$. To quadratic order in $Q$, one finds that the expansion of $U_0$ generates (2.16), as it must; but because of the $R_0$ factors inside the OE of (3.2), higher-order terms will, at least in part, involve commutators of the $\lambda$-dependence of $R_0$ with neighboring $\lambda^I, \lambda^H$ dependence of (3.2); those terms will be different from the simple exponentiation of (2.16), but they will always be of higher perturbative order than that of (2.16), and are not the leading terms of their own perturbative order. Note that the combination $\Delta \Lambda_a^H$ of (3.2) contains all the multiple commutators, indicated in (2.12), whose functional derivatives act upon the $\eta$-dependence of $R_0$.

To find that term in the eikonal of order $g^2(n+1)$ which is the leading term of that order, let us now write:

$$\Delta \Lambda_a^H \left( t \frac{\delta}{\delta \eta} \lambda_a \right) = \sum_{n=1}^{\infty} \Delta_n \Lambda_a^H \left( t \frac{\delta}{\delta \eta} \lambda_a \right),$$

$$\Delta_n \Lambda_a^H \left( t \frac{\delta}{\delta \eta} \lambda_a \right) = (-2i)^n f_{a_1 c_1} f_{a_2 c_2} \cdots f_{a_n c_n} \lambda_a^H \lambda_{a_n} \lambda_{a_{n-1}} \cdot \int_t^\infty dt_1 \int_{t_1}^\infty dt_2 \cdots \int_{t_{n-1}}^\infty dt_n \frac{\delta}{\delta \eta_{c_1}(t_1)} \cdots \frac{\delta}{\delta \eta_{c_n}(tn)}$$

(3.3)
and set $U_0 = R_1 U_1$, where we define:

$$R_1(s) = \left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' R_0^{-1}(s') \lambda_a^l Q(s', t') \Delta_n A_a^H \left( t'\delta \eta \right) R_0(s') \right] \right)_{+(s')}$$

Then, by again solving the appropriate differential equation, we find:

$$U_1(s) = \left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' \left[ R_0(s') R_1(s') \right]^{-1} \lambda_a^l Q(s', t') \cdot \sum_{n=2}^{\infty} \Delta_n A_a^H \left( t'\delta \eta \right) \left[ R_0(s') R_1(s') \right] \right) \right)_{+(s')}$$

Performing this operation sequentially, it is clear that the general structure of the result may be written as:

$$R(s|\eta) = R_0(s) \cdot R_1(s) \cdots R_n(s) \cdot U_n(s) \equiv [s_n] U_n(s)$$

where:

$$R_n(s) = \left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' \left[ t'|\delta \eta \right] \Delta_n A_a^H \left( t'|\delta \eta \right) \left[ s'_n \right] \right] \right)_{+(s')}$$

and:

$$U_n(s) = \left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' \left[ t'|\delta \eta \right] \lambda_a^l Q(s', t') \cdot \sum_{t=n+1}^{\infty} \Delta_t A_a^H \left( t'|\delta \eta \right) \left[ s'_n \right] \right] \right)_{+(s')}$$

Because each functional derivative $\frac{\delta}{\delta \eta}$ will generate a term (when operating on $R_0(s)$) proportional to $Q \sim g^2 \Delta_c$, the log of $R_n$ contains all powers of $g^{2m}$, with $m \geq n + 1$. The lowest order term, with $m = n + 1$, will contain the largest power of $\ln^n(E/m)$, while higher-order terms constructed from the same $R_n$ will have no higher-order log; rather, the terms containing $\ln^m(E/m)$, $m > n + 1$, will come from the corresponding, lowest-order terms of $R_m$.

In order to define “contiguity”, imagine that $R_n$ is expanded in powers of $g^2$, by expanding its OE:

$$R_n \mid_{s \to \infty} \approx 1 + i \int \int_{-\infty}^{+\infty} ds dt \left[ s \right]^{-1} \lambda_a^l Q(s, t) \Delta_n A_a^H \left( t\delta \eta \right) \left[ s \right]_{n-1}$$

$$+ i^2 \int \int_{-\infty}^{+\infty} ds dt \left[ s \right]^{-1} \lambda_a^l Q(s, t) \Delta_n A_a^H \left( t\delta \eta \right) \left[ s \right]_{n-1}$$

$$+ \cdot \int_{-\infty}^{s} ds_1 \int_{-\infty}^{+\infty} dt_1 \left[ s_1 \right]^{-1} \lambda_a^l Q(s_1, t_1) \Delta_n A_a^H \left( t_1\delta \eta \right) \left[ s_1 \right]_{n-1} + \cdots$$

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where \([s]_{n-1} = R_0(s) \cdots R_{n-1}(s)\). “Contiguity” suggests that the leading dependence of \(\ln(R_n)\) will be obtained if each \(\Delta_n \Lambda^U_a(t_j \frac{\delta}{\delta \eta})\) operates directly upon the \([s]_{n-1}\) factor contiguous to it, that is, immediately to its right. This can be seen in the simplest, non-trivial terms of order \(g^4\) and \(g^6\), and, we subsequently argue, is true for all terms; however, what is clear from this definition is that terms contributing to each order of the contiguity operation can be summed and calculated directly from the OE form of \(R_n\), writing:

\[
R_n \mid_{s \to \infty} = \left( \exp \left[ i \int_{-\infty}^{+\infty} dsdt \left[ [s]_{n-1}^{-1} \lambda^I Q(s, t) \Delta_n \Lambda^U_a \left( t \frac{\delta}{\delta \eta} \right) \right] \right] \right)_{+}(s) \tag{3.7}
\]

where the factor-pairing notation is meant to express the subset of terms extracted by contiguity.

The entire \(g^{2n}\) dependence of the eikonal, that is, of \(\ln(R)\), can be obtained by considering the following sequence of ascending powers of \(g^2\), in the limit of \(s \to \infty, \eta \to 0\):

- All \(g^2\) dependence is given by \(R_0, \ln(R_0) = i \left( \lambda^I \cdot \lambda^U \right) \int_{-\infty}^{+\infty} dsdtQ(s, t)\).
- All \((g^2)^2\) dependence is given by the contiguity calculation of \(R_1\), which generates our previous result, \(\ln(R_1) = -2C_2 \left( \lambda^I \cdot \lambda^U \right) \int_{-\infty}^{+\infty} dsdtQ(s, t) \int_{-\infty}^{+\infty} ds_1 \int_{-\infty}^{+\infty} dt_1 Q(s_1, t_1)\).
- All \((g^2)^3\) dependence is given by the contiguity calculation of \(R_2\), and by the \(g^2\) expansion of the \([s]_0^{-1}\) and \([s]_0\) factors of \(R_1\).
- All \((g^2)^4\) dependence is given by the contiguity calculation of \(R_3\), by the \(g^2\) expansion of the \([s]_1^{-1}\) and \([s]_1\) factors of \(R_2\), and by the \(g^4\) expansion of the \([s]_0^{-1}\) and \([s]_0\) factors of \(R_0\); etc.

In this way, one constructs the complete \(g^{2(n+1)}\) dependence of \(\ln(R) = \ln(R_0 \cdots R_n)\). Those exponential, eikonal terms obtained directly from contiguity will contain one or more terms proportional to a single factor of \(\lambda^I \cdot \lambda^U\); while the \(g^{2p}\) expansion of the \([s]_j\) and \([s]_j^{-1}\) appear to generate more complicated group factors, similar to those found in the perturbative calculations of the amplitude \([\Pi]\). We argue in the next Section that the leading \(\ln(E/m)\) dependence of the eikonal of order \(g^{2(n+1)}\) comes only from the contiguity calculation of \(R_n\), when the functional differentiation is performed only on the \(R_0(s)\) factor of \([s]_n\). Using simple functional techniques, the sum of these leading contributions over all orders \(n\) is constructed for the eikonal of SU(2).

### IV Leading Logs

We here give a qualitative discussion of the leading \(\ln(E/m)\) dependence of this class of non-abelian eikonals (where, we again remind the reader, interacting gluons are not included). For this, consider first those terms of order \(g^{2(n+1)}\) in the expression for \(\ln(R_n)\) arising from the contiguity operation of \(\Delta_n \Lambda^U_a\) upon the factor \([s]_{n-1}\) standing to its immediate right, as in (3.7). In particular, the leading terms of that order
will come from the $\Delta_n \Lambda_a^H$ operation upon the $R_0(s)$ functional in $[s]_{n-1}$ (rather than the same-$g^2$-order contribution to the eikonal from $\ln(R_{n-1})$, with $\Delta_n \Lambda_a^H$ acting upon $R_1(s)$ in $[s]_{n-2}$, etc).

For clarity, we carry the discussion through for $n = 2$, and then generalize to arbitrary $n$; for the moment, we suppress the $f_{abc}$ factors arising in the $t$-dependent iterations of $\Lambda_a^H \left( t \frac{\delta}{\delta n} \right)$, but we explicitly write the $s$-dependent permutations that are generated by the functional differentiation of $\Delta_2 \Lambda_a^H \left( t \frac{\delta}{\delta n} \right)$ upon $R_0(s)$, which are proportional to:

$$
\left. \int_t^\infty dt_1 \int_t^\infty dt_2 \int_{\delta c_1(t_1)}^\infty \frac{\delta}{\delta c_2(t_2)} \left( \exp \left[ i \int_{-\infty}^s ds' \int_{-\infty}^{+\infty} dt' Q(s', t') \lambda^t_{a} \eta_a(t') \right] \right) \right|_{\eta \to 0}
$$

(4.1)

We have neglected in this $R_0(s)$ its exponential $i \left( \lambda^I \cdot \lambda^H \right) \int_{-\infty}^s ds' \int_{-\infty}^{+\infty} dt' Q(s', t')$ dependence because, as explained below, it can only contribute to orders $g^{2p}, p > n + 1$, and carries no additional $\ln(E/M)$ factors. Suppressing the superscript $I$ for each $\lambda_c^I$, the functional operations of (4.1) yield:

$$
i^2 \int_t^\infty dt_1 \int_t^\infty dt_2 \int_{-\infty}^s ds_1 \int_{-\infty}^s ds_2 Q(s_1, t_1) Q(s_2, t_2) [\lambda_{c_1} \lambda_{c_2} \theta (s_1 - s_2) + \lambda_{c_2} \lambda_{c_1} \theta (s_2 - s_1)]
$$

(4.2)

and suggest the obvious generalization for $n > 2$ as:

$$
i^n \int_t^\infty dt_1 \cdots \int_{t_{n-1}}^\infty \frac{\partial}{\partial c_j(t_1)} \cdots \frac{\partial}{\partial c_n(t_n)} \left[ \int_{-\infty}^s ds_1 \cdots \int_{-\infty}^{s_{n-1}} ds_n \cdot \lambda_{c_1} \cdots \lambda_{c_n} Q(s_1, t_1) \cdots Q(s_n, t_n) \right]
$$

(4.3)

in which the $n$ $c_i$ indices are permuted, with a corresponding permutation of the $s_i$, in $n!$ different ways.

For our estimates of the $\ln(E/m)$ dependence, we use the standard Fourier propagator representation, $\Delta_c(x) = (2\pi)^{-4} \int d^4k (k^2 + \mu^2 - i\epsilon)^{-1} e^{ikx}$, and (improperly) take the kinematic limits for each (mass-shell) quark: $E - p = 0$, rather than the more accurate $E - p \approx m^2/2E$. Any integral that we find containing an UV log divergence is really proportional to a corresponding factor of $\ln(E/m)$, which dependence appears when proper (but much more complicated) kinematics are used.

For $n = 2$, let us examine both permutations, and include the $i\lambda_{c}^I \int ds dt Q(s, t)$ factor of (3.7), whose $[s]_1^{-1}$ has been replaced by unity (because it can only contribute to higher orders with no corresponding increase in the number of $\ln(E/m)$ factors). Each factor of $Q$ carries with it $p_1 \cdot p_2 \sim E^2$ dependence, which is removed by the explicit $E$-factors associated with the $s$- and $t$-integrations, in standard eikonal fashion; and we suppress all such cancelling $E$-dependence. With $Q(s, t) = g^2 (p_1 \cdot p_2) \Delta_c (z - s p_1 + t p_2)$, where $z = z_1 - z_2 = (b, z_3, z_0)$ is the difference of configuration coordinates of the scattering quarks, the first of the two permutations of (4.1) will lead to:

$$
\int d^4k^{(+)} e^{ikz} \delta (k^{(+)} \delta (k^{(-)} \int d^2k_1 \int d^2k_2 \int dk_1^{(+)} \int dk_1^{(-)} \int dk_2^{(+)} \int dk_2^{(-)}
$$

11
\[
\left[\omega^2 (\vec{k} - k_1 - k_2) + (k_1^{(+)} + k_2^{(+)})(k_1^{(-)} + k_2^{(-)}) - i\epsilon\right]^{-1} \left[\omega^2 + k_1^{(+)}k_1^{(-)} - i\epsilon\right]^{-1} \\
\left[\omega^2 + k_2^{(+)}k_2^{(-)} - i\epsilon\right]^{-1} (k_2^{(-)} + i\epsilon)^{-1} (k_1^{(+)} + k_2^{(+)} - i\epsilon)^{-1}
\]

where \(\vec{k} = k + k_1 + k_2\), \(k^{(\pm)} = k_3 \pm i k_0, \omega^2 = \mu^2 + k_1^{2\perp}\), and \(\omega^2 (\vec{k} - k_1 - k_2)^2 = \mu^2 + (\vec{k} - k_1 - h_2)^2\perp\), with \(\perp\) components referring to the transverse 1,2 directions (the impact parameter vector \(b\)) in the CM of the scattering quarks; all momentum integrals run from \(-\infty\) to \(+\infty\). The \(-i\epsilon\) factors are important, and - aside from the \(-i\epsilon\) of the standard Feynman propagator denominators - arise upon calculating \(\int_{-\infty}^{+\infty} ds_1 \int_{-\infty}^{+\infty} ds_2\) and \(\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\), when one insists upon the proper definition of the integrand at the \(\pm \infty\) limits of integration. The second permutation of (4.1) leads to the same form with the interchange of \(k_1^{(-)}\) and \(k_2^{(-)}\), and, it will become clear immediately, to the same leading-log dependence.

It is best to begin by performing the \(\int d k_{1,2}^{(\pm)}\) integrations, which by simple contour evaluation require \(k_1^{(+)} > 0\) and \(k_1^{(+)} + k_2^{(+)} > 0\), and generate:

\[
(-2\pi i)^2 \int \frac{d^2 k}{\omega^2} e^{i b \cdot k_{\perp}} \int \frac{d^2 k_1}{\omega_1^2} e^{i b \cdot k_1} \int \frac{d^2 k_2}{\omega_2^2} e^{i b \cdot k_2} \\
\cdot \int_{\epsilon}^{K} \frac{d k_{2}^{(+)}}{k_{2}^{(+)}} \int_{\epsilon}^{K} \frac{d k_{1}^{(+)}}{k_{1}^{(+)}} \left\{ 1 + \frac{k_{2}^{(+)}}{k_{2}^{(+)}} \cdot \frac{\omega_1^2}{k_{1}^{(+)}} \cdot \frac{\omega_2}{k_{2}^{(+)}} \cdot \frac{\omega_1}{k_{2}^{(+)}} \right\}
\]

(4.4)

where we have inserted upper \((K)\) and lower \((\epsilon)\) cut offs for the \(k_{1,2}^{(\pm)}\) integrations, and have replaced the transverse \(k_{\perp}\) variable by \((k + k_1 + k_2)_{\perp}\). Each of the three factors \(\int d^2 k \omega^{-2} e^{i k \cdot b}\) generates a term \((2\pi) K_0(\mu b)\), and the “1” of the curly bracket of (4.4) produces a “nested” contribution for the \(k_{1,2}^{(\pm)}\) integrals of amount \((1/2) \ln^2 (K/\epsilon) \rightarrow (1/2) \ln^n (E^2/m^2)\), when the replacement \(E - p_3 \simeq m^2/2E\) is used. In contrast, the second term of the curly bracket of (4.4) generates a contribution proportional to \(\ln (E/m)\), and can be dropped as sub-leading. Quite generally, a “nesting” of the \(k_i^{(+)}\) momenta follows directly from the ordered \(t\)-limits of the iterates of \(\Delta_n \Lambda^H_n\), while the sum over all permutations of the \(\lambda_{c_1} \cdots \lambda_{c_n}\) follows from the ordered \(s\)-limits of the terms obtained upon functional differentiation of \(R_0(s)\) by \(\Delta_n \Lambda^H_n\). The leading-log result for each \(\lambda_{c_1} \cdots \lambda_{c_n}\) permutation is proportional to \((1/n!) \ln^n (E^2/m^2)\).

One can easily see that any expansion of the \(i \left( \lambda^I \cdot \lambda^H \right) \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dt Q(s, t)\) portion of the exponent of \(R_0(s)\), in conjunction with the above forms, must always produce sub-leading dependence, because at least one of the nested \(k^{(\pm)}\) denominators needed for leading-log behavior will be missing. Further, one can also see the reason for the importance of the contiguity prescription, for - when the OEs defining each \(R_n\) are each expanded in powers of \(g^2\) - all the non-contiguous \(\Delta t \Lambda^H \left( t, \frac{\Delta}{2g} \right)\) operations will display “improper”, or out-of-sequence limits for the \(s\)-integrals, which will generate a similar sort of sub-leading behavior. For this standard choice of propagator, contiguity generates a first sub-division of terms containing the desired, leading-log dependence; and the latter are then isolated by the retention of only \(R_0(s)\) in each factor of \([s]_{n-1}\), and the neglect of every \([s]_{n-1}^{-1}\), in each \(R_n(s)\).
Perturbative eikonal analyses quite similar to the above have appeared long ago, in connection with multiperipheral processes of scalar “tower” exchange. There also one expects \( k_i^{(-)} \sim 0 \) and large, nested, \( k_i^{(+)} \) momenta. What is different here (aside from trivial, complex, multiplicative factors) is that one must also include the sums over all \( \lambda_{c_1} \cdots \lambda_{c_1} \) permutations, and the general form of such a sum is not clear for SU(N).

For SU(2), however, this computation can be carried through, and we now sketch that calculation. Its essence is to replace the leading-log dependence by another method of extraction which does not arise from the nested \( k_i^{(+)} \) integrations, but yields, term-for-term and order-by-order, the same results. This method is defined by retaining the same \( f_{abc} \) factors obtained from the \( \Delta_n \Lambda_n^H \) iterations, and multiplying those that contribute to order \( n \) by the terms:

\[
\sum \frac{i^n}{n! \text{perms}} \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{s_{n-1}} ds_n \int_{t_1}^{\infty} dt_1 \cdots \int_{t}^{\infty} dt_n Q(s_1, t_1) \cdots Q(s_n, t_n) \lambda_{c_1} \cdots \lambda_{c_n}
\]

and then summing over all \( n \). It is easy to see that the leading-log contributions of (4.5) are identical to those of (4.4); the only difference is that the \( \int_k^\lambda k^{(+)} / k^{(+)} \) contributions of (4.5) are not nested, and that the \((n!)^{-1}\) which follows from (4.4) because of nesting is, in (4.5), inserted by hand. This replacement can be made for arbitrary SU(N); but the next step, summing over all permutations of the s-ordering, seems to be straightforward only for SU(2).

Because the \( t \)-integrals of (4.5) are not ordered, we introduce the symbol \( A_c(t) = \int_{t_1}^{\infty} dt' \frac{\delta}{\delta u_c(t')} \), and \( \Sigma^H[A] \) as the sum of all functional operations, which when performed on \( R_0(s) \), generate the correct sequence of \( \epsilon_{abc} \) coefficients multiplying (4.5). \( \Delta \Sigma^H[A] \) corresponds to the set of all the iterations of the SU(2) version of \( \Delta \Lambda^H \left( \frac{t}{\delta} \frac{\delta}{\delta u} \right) \), where a factor of \((n!)^{-1}\) is inserted for each \( n \)th order, and the operators \( A_{c_1} \cdots A_{c_n} \) replace the \( t \)-ordered \( \int_{t_1}^{\infty} dt_1 \int_{t_2}^{\infty} dt_2 \cdots \int_{t_n}^{\infty} dt_n \delta_{\delta_{u_{c_1}(t_1)} \cdots \delta_{u_{c_n}(t_n)}} \) of the expansion of (2.8). We work directly with the contiguity approximation to \( U_0 \) (rather than to the \( R_n \) separately), in which \( R_0^{-1} \) is replaced by unity, and the leading-log simplification of \( R_0 \) is used, as in (4.1); everywhere, the \( f_{abc} \rightarrow \epsilon_{abc} \), and \( \lambda_c \rightarrow \sigma_c \). One may now examine the first four terms of this expansion, and it then becomes clear, by inspection, that the full sum over all such \( A \)-dependence may be written as:

\[
\Delta \Sigma^H[A] = \left[A^2 \delta_{ab} - A_a A_b \right] \sigma_b \cdot \frac{1}{A^2} \{ \cosh(A) - 1 \} - i \epsilon_{acd} \sigma_d \cdot A_c \frac{\sinh(A)}{A}
\]

(4.6)

where \( A^2 = \sum \epsilon_{abc} \epsilon_{ade} \delta_{bd} - \delta_{ab} \delta_{cd} \). To obtain (4.6), one repeatedly uses the SU(2) property \( \sum \epsilon_{abc} \epsilon_{ade} = \delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd} \).

The functional expression of contiguity, of \( \Delta \Sigma^H[A] \) operating on \( R_0(s) \), can be performed by first introducing the representations:

\[
A_a \frac{\sinh(A)}{A} = \frac{1}{2\pi} \int d^3 u \delta \left( \bar{u}^2 - 1 \right) \frac{\partial}{\partial u_a} e^{\bar{u} \bar{A}}
\]

(4.7)
and
\[
\left[ A^2 \delta_{ab} - A_a A_b \right] \frac{1}{A^2} \{ \cosh(A) - 1 \} \\
= \frac{1}{2\pi} \int_{0}^{1} \frac{d\lambda}{\lambda} \int d^3 u \delta(\vec{u}^2 - 1) \left[ \delta_{ab} \left( \frac{\partial}{\partial \vec{u}} \right)^2 - \frac{\partial}{\partial u_a} \frac{\partial}{\partial u_b} \right] e^{\lambda \vec{u} \cdot \vec{A}}
\]
(4.8)

where \((\lambda, u)\) are dummy integration variables. The quantity \(e^{\lambda \vec{u} \cdot \vec{A}} R_0(s)|_{\eta \to 0}\) is then the OE:
\[
\left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{t}^{\infty} dt' Q(s', t') \lambda \left( \hat{\sigma}^I \cdot \vec{u} \right) \right] \right)_{+(s')}
\]
(4.9)
and may be replaced by the oe:
\[
\exp \left[ i \lambda \left( \hat{\sigma}^I \cdot \vec{u} \right) K(s, t) \right]
\]
(4.10)
where \(K(s, t) = \int_{-\infty}^{s} ds' \int_{t}^{\infty} dt' Q(s', t')\). In effect, the lack of \(A\)-ordering for these leading-log terms has transformed their operation upon \(R_0(s)\) into an ordinary exponential with weightings to be determined by the integrations of (4.7) and (4.8). These last steps are now easily performed, by the replacement of (4.10) by \(\cos(\lambda u K) + i \left( \hat{\sigma}^I \cdot \vec{u} \right) \frac{\sin(\lambda u K)}{u}\), and its substitution into (4.7) and (4.8), whose evaluations yield:
\[
A_a \frac{\sinh(A)}{A} \cdot R_0(s)|_{\eta \to 0} = \frac{i}{3} \sigma^I_a (K \cos K + 2 \sin K)
\]
(4.11)
and:
\[
\left[ A^2 \delta_{ab} - A_a A_b \right] \frac{1}{A^2} \{ \cosh(A) - 1 \} R_0(s)|_{\eta \to 0} = \frac{4}{3} \delta_{ab} \left[ \cos(K) - 1 - \frac{K}{2} \sin(K) \right]
\]
(4.12)

From (4.6) and (4.10), (4.11) and (4.12), one obtains:
\[
\Delta \sum_a^n [A] R_0(s)|_{\eta \to 0} = \frac{1}{3} \epsilon_{acd} \sigma^I_d \sigma^I_c [K \cos K + 2 \sin K]
\]
\[
+ \frac{4}{3} \sigma^I_a \left[ \cos K - 1 - \frac{K}{2} \sin K \right]
\]
(4.13)

Multiplying (4.13) on the left by \(\sigma^I_a\), antisymmetrizing where appropriate (together with the Casimir relation \(\sum_{ac} \epsilon_{acd} \epsilon_{ace} = 2 \delta_{de}\), and including the \(R_0\) contribution of the product \(R = R_0 U_0\), one finds the eikonal given by:
\[
\chi = \left( \sigma^I \cdot \sigma^I \right) \int_{-\infty}^{+\infty} dsdt Q(s, t) \left\{ 1 - \frac{4}{3} \left[ 1 - \cos K + \frac{K}{2} \sin K \right] \right\}
\]
\[
+ \frac{2}{3} i \left[ K \cos K + 2 \sin K \right]
\]
(4.14)
Finally, if one imagines expanding (4.14) in powers of \( K(s,t) \), and combines each \( K^n(s,t) \) with the remaining integrand of \( U_0 \), one may use the easily-verified property, correct for the leading-log dependence of each order:

\[
\int \int_{-\infty}^{+\infty} dsdt \, Q(s,t)K^n(s,t) \simeq \left[ -i \frac{g^2}{\pi^2} \ln(E/m)K_0(\mu b) \right]^n \equiv [-iL]^n
\]

so that, upon resumming these terms into the equivalent of (4.14), in effect the quantity \( K(s,t) \) may be replaced by \(-iL\) of (4.15), yielding:

\[
\chi = -\frac{g^2}{2\pi} \left( \sigma^I \cdot \sigma^H \right) K_0(\mu b) \left\{ 1 - \frac{4}{3} \left[ 1 - e^L \right] + \frac{2}{3} L e^L \right\}
\]

as the complete eikonal in leading-log approximation for the SU(2) problem (e.g., of nucleon-nucleon scattering by the exchange of neutral and charged vector mesons, with conserved isospin).

Perhaps the most obvious feature of (4.16) is its proportionality to \( \sigma^I \cdot \sigma^H \), which quantity takes on isoscalar or isovector eigenvalues depending on the nature of the initial scattering states. A second interesting property is that, by expressing the exp\([L]\) factors of (4.16) in terms of:

\[
e^L = \left( s/m^2 \right)^{g^2/2\pi^2} K_0(\mu b)
\]

one finds an “effective Reggeization” of the eikonal, where again denotes total CM (energy)\(^2\). For \( \mu \neq 0 \), there is little contribution to the scattering amplitude for small \( b \); and hence if \( K_0(\mu b) \) is approximated as \( \sim \exp[-\mu b] \), one obtains forms similar to those found in the Regge-eikonal approximation of multiperipheral scattering, except that this eikonal is real. In fact, the amplitude, constructed in the generic form (and suppressing all inessential factors):

\[
T \sim is \int_0^\infty db \, J_0(qb) \cdot \left[ 1 - e^{i\chi(s,b)} \right]
\]

exhibits a variant of a “hard disc” scattering solution, in that there are two regions of impact parameter, \( b < b_0 \), which produce different contributions to the amplitude. This can be seen by defining \( b_0 \) as that impact parameter where \( L(b_0) = 1 \), \( b_0 = \mu^{-1} \ln \left( (g^2/2\pi^2)Y \right) > \mu^{-1} \), \( Y = 2 \ln(E/m) \), and writing the contributions to the amplitude of (4.17) in terms of integrations over these two regions of \( b \). Since \( L(b) = \exp[\mu(\mu_0 - b)] \), and we assume that \( Y \) is large, when \( b < b_0 \), \( L \) is large, as is the eikonal of (4.16), and the only significant contribution to the amplitude comes from the "1" of the first term of (4.17). When \( b > b_0 \), \( L \) is small, and the only significant contribution to the eikonal comes from the "1" of the bracket of (4.16), which we shall call \( \chi_0 \); this is the contribution coming from the original \( R_0 \) term of (2.14). This argument leads to the representation of the amplitude of (4.17) as the sum of two parts:

\[
T \sim is \int_0^{b_0} db \, J_0(qb) + is \int_{b_0}^\infty db \, J_0(qb) \left[ 1 - e^{i\chi_0} \right]
\]
or as:

\[ T \sim is \int_0^{b_0} b db J_0(qb)e^{i\chi_0} + is \int_0^\infty b db J_0(qb) \left[ 1 - e^{i\chi_0} \right] \quad (4.19) \]

in which the amplitude is characterized by its simplest eikonal approximation, \( \chi_0 \), and by the range parameter \( b_0(E/m) \) which defines that impact parameter beyond which leading-log corrections force the eikonal to become extremely large and oscillatory, thereby removing its contribution from the amplitude.

Could the same mechanism be operative for the general case of SU(N)? Even though we cannot perform the closed sum over all orders of leading-log contributions for \( N > 2 \), one can anticipate that for a similar \( b_0(E/m) \) the eikonal becomes very large, contributing a rapidly oscillating and negligible contribution to the amplitude, which may be written in the form of (4.18) or (4.19), with the \( \sigma^I \cdot \sigma^{II} \) invariant of \( \chi_0 \) replaced by \( \lambda^I \cdot \lambda^{II} \). We think it a reasonable conjecture that this simple form is the actual result of the complete SU(N) calculation. Of course, this point is somewhat academic, since when energies are large enough to take leading-logs seriously, other processes which have here been neglected (e.g., multiperipheral production) are going to appear. Nevertheless, it is of some theoretical interest to examine an amplitude constructed from the eikonal of (4.16), under the assumption that \( \ln(E/m) \gg 1 \); and it will be most interesting to see if similar structures and simplifying approximations are going to appear in the study of other eikonal processes which reflect the growth of inelastic particle production with increasing energies.

V Other Processes

An important variation of the non-abelian eikonal scattering problem is found when self-energy processes (as in radiative corrections to other QCD \( n \)-point functions) are attempted. Here, one may make use of the new, exact and approximate representations for the needed Green’s functions of reference [3] in which dependence on the source fields, \( A_\mu \) and \( F_{\mu\nu} \) is that of an OE of linear form; for the simplest example, we omit the \( F_{\mu\nu} \) terms, and work in a quenched approximation, so that the sum of all radiative corrections to the fermion propagator will require evaluation of the quantity:

\[ R(s|\xi) = N' \int d[u] \int d[\alpha] e^{i \int \alpha \cdot u} \left( e^{i \int_{-\infty}^s \lambda \cdot u} \right) + e^{i \int \alpha a Q_{ab} \alpha_b \cdot \cdot \cdot e^{i \int u \xi}} \quad (5.1) \]

in the limit of \( s \to \infty \) and \( \xi_a(s') \to 0 \). Here, \( Q_{ab}(s, t) \) is considerably more complicated than the corresponding function of an eikonal scattering amplitude (although the resemblance becomes closer if an improper, no-recoil approximation is adopted), but must satisfy \( Q_{ab}(s, t) = Q_{ba}(t, s) \).

Using techniques modeled after those sketched above, it is easy to see that a representation of (5.1) is given by the formal OE:
\[ R(s|\xi) = \left( \exp \left[ i \int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' \lambda_a Q_{ab}(s', t') \left( \xi_b(t') + \theta(t' - t') \Lambda_b \left( s', t' | \frac{\delta}{\delta \xi} \right) \right) \right] \right)_{+}^{(s')} \]  

where \( \Lambda_b(s, t | iu) = \left( e^{i \int_{s}^{t} \lambda \cdot u} \right)_{+} \lambda_b \left( e^{-i \int_{s}^{t} \lambda \cdot u} \right)_{-} \). The same, formal expansion corresponding to (3.4) and (3.5) may be defined, except that \( R_0 \) is now multiplied by the exponential factor \( \exp \left[ i \frac{1}{2} \int \xi Q \xi \right] \), which has the effect of inserting polynomial \( \xi \)-dependence into all the exponents of subsequent \( R_n \), and the power-counting arguments given above must be appropriately modified.

Perhaps the most interesting generalization of the forms of Section III should appear in eikonal quark-scattering models when gluon-gluon interactions (e.g., the "tower graphs" and their generalizations) are taken into account. Before a functional treatment can be attempted, even in the relatively simple models described in the last chapters of references [1] and [5], it is necessary to have a decent representation - as a functional of an equivalent gluon source used to represent internal, "s-channel" gluon exchanges - for the Green’s function corresponding to the \( t \)-channel gluons exchanged between quarks. For the eikonal situation where different spin-one bosonic fields are used to describe distinct \( t \)- and \( s \)-channel exchanges, respectively, such a representation now exists [7], and can be written down without undue complications; for the single gluonic field of real QCD, the situation is similar but not as straightforward.

If these calculations can be carried through for the tower graphs (corresponding to two-gluon, \( t \)-channel exchange between scattering quarks) in a functional context, using contiguity as appropriate, there should then be an immediate functional generalization which includes multiple, \( t \)-channel gluon exchanges. Such estimates of the QCD eikonal would be most relevant to high-energy particle scattering experiments.

VI Summary and Acknowledgements

In this paper we have shown how the formidable, non-abelian eikonal combination (1.1) may be written as the OE \( R(s|\eta) \) in the limit as \( s \rightarrow \infty \), and \( \eta \rightarrow 0 \); and have, by contiguity, isolated a sub-set of terms which exponentiate and contribute directly to the eikonal function, and which contain appropriate \( \ln(E/m) \) dependence associated with the leading-log behavior of every perturbative order. For SU(2), these terms may be summed to all orders, generating an eikonal dependent on the total isospin of the scattering channel, which displays a form of Reggeization peculiar to this set of graphs summed.

Contiguity may also be phrased in terms of the original ansatz, \( R(s|\eta) = R_0 U_0 \), by replacing the exact \( U_0 \) of (3.2) by its contiguity approximation, as used for the SU(2) calculation. However, at least for the specifically perturbative estimates of \( \ln(U) \), it appears to be simpler to adopt contiguity in the context of the \( R_n \). As explained in Section III, contiguity together with the elimination of obviously sub-leading terms, provides a straightforward method for the estimation of the eikonal’s leading-log terms in every perturbative order. We have found an elementary method
for summing all such terms in SU(2), and conjecture the form of a simplified eikonal amplitude for all $N$.

In summary, we cannot here claim to have given the complete solution to the problem of non-abelian field-theory structure; but, rather, a new and complete functional formulation (for eikonals and related self-energy graphs), and a “contiguity” method of extracting those terms which are certainly going to be exponentiated, and which seem to correspond to the identification of leading $\ln(E/m)$ dependence appearing in the construction of specifically non-abelian eikonals. It is hoped that these new techniques will be useful for other processes, as discussed in the previous Sections.

In particular, it is now appropriate to explain to the patient reader how this procedure - which lacks manifest gauge invariance in a Yang-Mills context - can be incorporated within a larger scheme, in order to obtain strictly gauge-invariant results for physical scattering amplitudes. There are three separate issues involved. In any eikonal calculation, one is searching for the proper separation of longitudinal/timelike momenta from transverse momenta - this is the problem attempted from first principles by Verlinde and Verlinde [8] - while at the same time, one is trying to sum over the contributions of all perturbative orders for the classes of graphs considered; and, simultaneously, one must insist on the restrictions of gauge invariance.

The eikonal calculation of the present paper, with its ability to extract leading-$\ln(s)$ dependence, is intended to be used as an initial step in a complete functional expression for the scattering of a pair of quarks, which includes all gluonic self-interactions as part of a “gluonic sector” described by the methods of Halpern [9], or its slight generalization by Fried [10]. The $A_\mu$-dependence of these formulations takes the form of an exponential of linear and quadratic forms, so that the $Q(s,t)$-propagator of (1.1) is now dependent upon auxiliary fields, and is linked to subsequent functional integrals which describe the gluon self-interactions; extra functional integrations maintain gauge restrictions. The insertion of the forms of this paper then leads, as an intermediate step, to a rather complicated set of functional integrals; but in the integrands of these functional integrals, one has already extracted the leading $\ln(s)$ behavior of the simple eikonal where $s$ is essentially given by quark kinematics.

For large $s$, by a rescaling of the auxiliary functional integrands, one can now try to approximate and to extract relevant gluon self-interaction structure, in this large $s$/small $t$ limit; and in a gauge invariant way. These calculations are presently underway, and whether they will succeed is not yet known; but this is the reason why a functional evaluation of the leading-log behavior of the simple eikonal form of (1.1) can be relevant to quarks and gluons.

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