GEODESIC INTERSECTIONS AND ISOXIAL FUCHSIAN GROUPS.

GREG MCSHANE

Abstract. The set of axes of hyperbolic elements in a Fuchsian group depends on the commensurability class of the group. In fact, it has been conjectured that it determines the commensurability class and this has been verified in for groups of the second kind by G. Mess and for arithmetic groups by by D. Long and A. Reid. Here we show that the conjecture holds for almost all Fuchsian groups and explain why our method fails for arithmetic groups.

1. Introduction

Let \( \Sigma \) be a closed orientable hyperbolic surface. The free homotopy classes of closed geodesics on \( \Sigma \) conjugacy classes of hyperbolic elements in \( \Gamma \). If \( \gamma \in \Gamma \) is a hyperbolic element, then associated to \( \gamma \) is an axis \( \text{ax}(\gamma) \subset \mathbb{H} \). The projection of \( \text{ax}(\gamma) \) to \( \Sigma \) determines a closed geodesic whose length is \( \ell_\gamma \). We shall denote the set of axes of all the hyperbolic elements in \( \Gamma \) by \( \text{ax}(\Gamma) \). It’s easy to check that if \( g \in \text{PSL}(2, \mathbb{R}) \) then we have the relation

\[
\text{ax}(g) = g \text{ax}(\Gamma).
\]

1.1. Isoaxial groups. Following Reid [10] we say that a pair of Fuchsian groups \( \Gamma_1 \) and \( \Gamma_2 \) are isoaxial iff \( \text{ax}(\Gamma_1) = \text{ax}(\Gamma_2) \). One obtains a a trivial example of an isoaxial pair by taking \( \Gamma_1 \) any Fuchsian group and \( \Gamma_2 < \Gamma_1 \) any finite index subgroup. This example can be extended to a more general setting as follows. Recall that a pair of subgroups \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable if \( \Gamma_1 \cap \Gamma_2 \) is finite index in both \( \Gamma_1 \) and \( \Gamma_2 \). Thus if \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable then they are isoaxial because:

\[
\text{ax}(\Gamma) = \text{ax}(\Gamma_1 \cap \Gamma_2) = \text{ax}(\Gamma_2),
\]

It is natural to ask whether the converse is true:

If \( \Gamma_1 \) and \( \Gamma_2 \) are isoaxial then are they commensurable?

In what follows we shall say simply that the group \( \Gamma_1 \) is determined (up to commensurability) by its axes. We shall show that this conjecture holds for almost all Fuchsian groups:

**Theorem 1.1.1.** For almost every point \( \rho \) in Teichmüller space of a hyperbolic surface \( \Sigma \) the corresponding Fuchsian representation the fundamental group \( \Gamma \) is determined by its axes.

2010 Mathematics Subject Classification. Primary 57M27, Secondary 37E30, 57M55 .

Key words and phrases. Fuchsian groups, commensurability.
1.2. Spectra. We define the length spectrum of $\Sigma$ to be the collection of lengths $\ell_\alpha$ of closed geodesics $\alpha \subset \Sigma$ counted with multiplicity. In fact, since $\Sigma$ is compact, the multiplicity of any value in the spectrum is finite and moreover the set of lengths is discrete. Let $\alpha, \beta$ be primitive closed geodesics which meet at a point $z \in \Sigma$, we denote by $\alpha \angle_z \beta$, the angle measured in the counter-clockwise direction from $\alpha$ to $\beta$. Let $\alpha, \beta$ be primitive closed geodesics which meet at a point $z \in \Sigma$, we denote by $\alpha \angle_z \beta$, the angle measured in the counter-clockwise direction from $\alpha$ to $\beta$. Following Mondal [8],[9] we define an angle spectrum to be the collection of all such angles (counted with multiplicity).

The length spectrum has proved useful in studying many problems concerning the geometry of hyperbolic surfaces. The angle spectrum is very different from the length spectrum: the set of angles is obviously not discrete and, as we shall see, the there are surfaces for which every value has infinite multiplicity. However, when considering the question of whether groups are isoaxial, the angle spectrum has a distinct advantage for it is easy to see that:

- There are isoaxial groups which do not have the same set of lengths, that is, the same angle spectrum without multiplicities.
- If two groups are isoaxial then they have the same set of angles, that is, the same angle spectrum without multiplicities.

Using properties of angles we will deduce Theorem 1.1.1 from the following lemma inspired by a result of G. Mess (see paragraph 2.1).

**Lemma 1.2.1.** Define the group of automorphisms of $ax(\Gamma)$ to be the group of hyperbolic isometries which preserve $ax(\Gamma)$. If $\Sigma$ has a value in its angle spectrum with finite multiplicity then $\Gamma$ is finite index in the group of automorphisms of $ax(\Gamma)$.

It remains to prove that there are such points of $T(\Sigma)$, we show in fact that they are generic:

**Theorem 1.2.2.** For almost every point $\rho \in T(\Sigma)$ there is a value in the angle spectrum which has multiplicity exactly one.

Our method applies provided there is some value in the angle spectrum that has finite multiplicity. Unfortunately, for arithmetic surfaces, the multiplicity of every value is infinity (Lemma 2.2.4).

1.3. Sketch of proof. The method of proof of Theorem 1.1.1 follows the proof of the first part of Theorem 1.1 in [6]. This says that the set of surfaces in Teichmüller space where every value in the simple length spectrum has multiplicity exactly one is dense and its complement is measure zero (for the natural measure on Teichmüller space.)

1.3.1. Two properties of (simple) length functions. Recall that the simple length spectrum is defined to be the collection of lengths of simple closed geodesics counted with multiplicity.

- The analyticity of the geodesic length $\ell_\alpha$ as a function over Teichmüller space;
- The fact that if $\alpha, \beta$ are a pair of distinct simple closed geodesics then the difference $\ell_\alpha - \ell_\beta$ defines a non constant (analytic) function on the Teichmüller space $T(\Sigma)$.
It is clear that the set of surfaces where every value in the simple length spectrum has multiplicity exactly one is the complement of
\[ Z := \cup_{(\alpha, \beta)} \{ \ell_\alpha - \ell_\beta = 0 \}, \]
where the union is over all pairs \( \alpha, \beta \) of distinct closed simple geodesics. Each of the sets on the left is nowhere dense and its intersection with any open set is measure zero. Since \( Z \) is countable union of such sets, its complement is dense and meets every open set in a set of full measure.

We note in passing that the second of these properties is not true without the hypothesis "simple". Indeed, there are pairs of distinct closed unoriented geodesics \( \alpha \neq \beta \) such that \( \ell_\alpha = \ell_\beta \) identically on \( \mathcal{T}(\Sigma) \) (see [2] for an account of their construction).

1.3.2. Analogues for angles. We will deduce Theorem 1.1.1 using the same approach but instead of geodesic length functions we use angle functions. The most delicate point is to show that if \( \alpha_1, \alpha_2 \) are a pair of simple closed geodesics that meet in a single point \( z \) and \( \beta_1, \beta_2 \) are a pair of closed geodesics that meet in a point \( z' \) then the difference \( \alpha_1 \angle z \alpha_2 - \beta_1 \angle z' \beta_2 \) defines a non constant function on Teichmueller space.

We do this by establishing the analogue of the following property of geodesic length functions:

**Fact 1.3.1.** A closed geodesic \( \alpha \subset \Sigma \) is simple if and only if the image of the geodesic length function \( \ell_\alpha \) is \([0, \infty]\).

Our main technical result (Theorem 6.1.3) is an analogue of this property. We consider pairs of simple closed geodesics \( \alpha_1, \alpha_2 \) which meet in a point \( z \) - this configuration will be the analogue of a simple closed geodesic. Now, for any such pair we find a subset \( X \subset \mathcal{T}(\Sigma) \) such that, for any other pair of closed geodesics \( \beta_1, \beta_2 \) which meet in \( z' \neq z \):
- the image of \( X \) under \( \beta_1 \angle z' \beta_2 \) is a proper subinterval of \([0, \pi]\)
- whilst its image under \( \alpha_1 \angle z \alpha_2 \) is the whole of \([0, \pi]\).

1.4. Further remarks. Since one objective of this work is to compare systematically the properties of geodesic length and angle functions we include an exposition of geodesic length functions and give an account of the characterisation of simple geodesics mentioned above our Proposition 3.1.2.

Mondal [8] has obtained a rigidity result by using a richer collection of data than we use here. He defines a length angle spectrum and proves that this determines a surface up to isometry. However, the set of axes does not determine the lengths of closed geodesics and so commensurability is the best one can hope for in the context we consider here.

In paragraph 2.2.1 we answer a question of Mondal in [9] concerning multiplicities by observing that arithmetic surfaces are very special: the multiplicity of any angle in the angle spectrum is infinite.

2. Automorphisms and commensurators

To study this question we define, following Reid, two auxiliary groups. The first is the group of automorphisms of \( \text{ax}(\Gamma) \):

\[ \text{Aut}(\text{ax}(\Gamma)) := \{ \gamma \in \text{PSL}(2, \mathbb{R}), \gamma(\text{ax}(\Gamma)) = \text{ax}(\Gamma) \}. \]
The second is the *commensurator of* $\Gamma$ defined as:

$$\text{Comm}(\Gamma) := \{ \gamma \in \text{PSL}(2, \mathbb{R}) : \gamma \Gamma \gamma^{-1} \text{ is directly commensurable with } \Gamma \}.$$ 

We leave it to the reader to check that $\text{Aut}(\text{ax}(\Gamma))$ and $\text{Comm}(\Gamma)$ are indeed groups and that they contain $\Gamma$ as a subgroup. In fact any element $\gamma \in \text{Comm}(\Gamma)$ is an automorphism of $\text{ax}(\Gamma)$. To see this, if $\gamma \in \text{Comm}(\Gamma)$, then $\Gamma$ and $\gamma \Gamma \gamma^{-1}$ are commensurable so are isoaxial. Now by (1) one has

$$\text{ax}(\Gamma) = \text{ax}(\gamma \Gamma \gamma^{-1}) = \gamma \text{ax}(\Gamma)$$

so $\gamma \in \text{Aut}(\text{ax}(\Gamma))$. In summary one has a chain of inclusions of subgroups:

$$\Gamma < \text{Comm}(\Gamma) < \text{Aut}(\text{ax}(\Gamma)) < \text{PSL}(2, \mathbb{R}).$$

We shall be concerned with two cases:

1. $\Gamma$ is finite index in $\text{Aut}(\text{ax}(\Gamma))$.
2. $\text{Aut}(\text{ax}(\Gamma))$ is dense in $\text{PSL}(2, \mathbb{R})$ so that $\Gamma$ is necessarily an infinite index subgroup.

The first case arises for the class of Fuchsian groups of the second kind studied by G. Mess and the second for arithmetic groups.

### 2.1. Fuchsian groups of the second kind.

G. Mess in an IHES preprint studied a variety of questions relating to $\text{ax}(\Gamma)$ notably proving the following result

**Theorem 2.1.1** (Mess). If $\Gamma_1$ and $\Gamma_2$ are isoaxial Fuchsian groups of the second kind then they are commensurable.

The proof of this result is a consequence of the fact that, under the hypotheses, $\text{Aut}(\text{ax}(\Gamma))$ is a discrete, convex cocompact Fuchsian group. It is easy to deduce from this that $\Gamma$ is finite index in $\text{ax}(\Gamma)$.

To show that $\text{Aut}(\text{ax}(\Gamma))$ is discrete it suffices to find a discrete subset of $\mathbb{H}$, containing at least two points, on which it acts. Recall that the *convex hull of the limit set* of $\Gamma$, is a convex subset $C(\Lambda) \subset \mathbb{H}$. If $\Gamma$ is a Fuchsian groups of the second kind then its limit set $\Lambda$ is a proper subset of $\partial \mathbb{H}$ and $C(\Lambda)$ is a proper subset of $\mathbb{H}$ whose frontier $\partial C(\Lambda)$ consists of countably many complete geodesics which we call *sides*. By definition $\text{ax}(\Gamma)$ is $\text{Aut}(\text{ax}(\Gamma))$-invariant and so $C(\Lambda)$ is too since, in fact, it is the minimal convex set containing $\text{ax}(\Gamma)$. Now choose a minimal length perpendicular $\lambda$ between edges of $C(\Lambda)$; such a minimising perpendicular exists because the double of $C(\Lambda)/\Gamma$ is a compact surface without boundary, every perpendicular between edges of $C(\Lambda)$ gives rise to a closed geodesic on the double and the length spectrum of the double is discrete. Let $L$ be the $\text{Aut}(\text{ax}(\Gamma))$-orbit of $\lambda$ and observe that $L \cap \partial C(\Lambda)$ is a discrete set which contains at least two points.

### 2.2. Arithmetic groups.

In the case of Fuchsian groups of the first kind Long and Reid [4] proved the conjecture for arithmetic groups.

**Theorem 2.2.1** (Long-Reid). If a Fuchsian group is arithmetic then its commensurator is exactly the group of automorphisms of the group.

Also notice that if $\Gamma_1$ and $\Gamma_2$ are isoaxial Fuchsian groups, then for any $\gamma \in \Gamma_2$

$$\text{ax}(\Gamma_1) = \text{ax}(\gamma \Gamma_1 \gamma^{-1}),$$

and therefore $\gamma \in \text{Aut}(\text{ax}(\Gamma))$. Hence $\Gamma_2 < \text{Aut}(\text{ax}(\Gamma))$.

So by the above discussion $\Gamma_2 < \text{Comm}(\Gamma_1)$, and if $\Gamma_2$ is also arithmetic, then $\Gamma_1$ and $\Gamma_2$ are commensurable. Thus they obtain as a corollary:
Corollary 2.2.2. Any pair of isoaxial arithmetic Fuchsian groups is commensurable.

2.2.1. Multiplicities for arithmetic groups. Let $\Gamma$ be an arithmetic Fuchsian group since its commensurator is dense in $SL(2,\mathbb{R})$ set of geodesic intersections is “locally homogenous” in the following sense:

Lemma 2.2.3. Let $\theta = \alpha \angle z \beta$ be an intersection of closed geodesics then for any open subset $U \subset \Sigma$ there is a pair of closed geodesics $\alpha_u, \beta_u$ such that:

$$\alpha_u \angle z_u \beta_u, \ z_u \in U.$$

Proof. Choose hyperbolic elements $a, b \in \Gamma$ such that the axis of $a$ (resp. $b$) is a lift of $\alpha$ (resp $\beta$) to $\mathbb{H}$ and so that the axes meet in a lift $\hat{z} \in \mathbb{H}$ of $z$. Since Comm($\Gamma$) is dense in $SL(2,\mathbb{R})$, there is some element $g \in$ Comm($\Gamma$) so that $g(\hat{z}) \in U$ for some lift of $U$ to $\mathbb{H}$. By the commensurability of the groups $\Gamma$ and $g\Gamma g^{-1}$ there is a positive integer $m$ such that $(gag^{-1})^m, (gbg^{-1})^m \in \Gamma$ so that the axes of these elements project to closed geodesics $\alpha_u, \beta_u$ on $\Sigma$ meeting in a point $z_u$ as required.

An immediate corollary is:

Corollary 2.2.4. The multiplicity of any angle $\theta$ in the spectrum of an arithmetic surface $\Sigma/\Gamma$ is infinite.

3. Functions on Teichmüller space

Recall that the Teichmüller of a surface $\Sigma$, $T(\Sigma)$, is the set of marked complex structures and that, by Riemann’s Uniformization Theorem, this is identified with a component of the character variety of PSL(2, $\mathbb{R}$)-representations of $\pi_1(\Sigma)$. Thus we think of a point $\rho \in T(\Sigma)$ as an equivalence class of PSL(2, $\mathbb{R}$)-representations of $\pi_1(\Sigma)$. We remark that PSL(2, $\mathbb{R}$) := $SL(2,\mathbb{R})/\langle -I_2 \rangle$ so that although the trace $tr\rho(a)$ is not well defined for $a \in \pi_1(\Sigma)$, the square of the trace $tr^2\rho(a)$ is and so is $|tr\rho(a)|$. In fact, there is a natural topology $T(\Sigma)$ such that for each $a \in \pi_1(\Sigma)$, $\rho \mapsto tr^2\rho(a)$ is a real analytic function.

3.1. Geodesic length. If $a \in \pi_1(\Sigma)$ is non trivial then there is a unique oriented closed simple geodesic $\alpha$ in the conjugacy class $[a]$ determined by $a$. The length of $\alpha$, measured in the Riemannian metric on $\Sigma = \mathbb{H}/\rho(\pi_1(\Sigma)))$, can be computed from $tr\rho(a)$ using the well-known formula

$$(2) \quad |tr\rho(a)| = 2 \cosh(\ell_\alpha/2).$$

There is a natural function,

$$\ell : T(\Sigma) \times \{ \text{homotopy classes of loops} \} \rightarrow ]0, +\infty[,$$

which takes the pair $\rho, [a]$ to the length $\ell_\alpha$ of the geodesic in the homotopy class $[a]$. It is an abuse, though common in the literature, to refer merely to the length of the geodesic $\alpha$ (rather than, more properly, the length of the geodesic in the appropriate homotopy class).

We define the length spectrum of $\Sigma$ to be the collection of lengths $\ell_\alpha$ of closed geodesics $\alpha \subset \Sigma$ counted with multiplicity. In fact, since $\Sigma$ is compact, the multiplicity of any value in the spectrum is finite and moreover the set of lengths is discrete.
3.1.1. Analyticity. A careful study of properties of length functions was made in [6] where one of the key ingredients is the analyticity of this class of functions:

**Fact 3.1.1.** For each closed geodesic $\alpha$, the function 
\[ T(\Sigma) \to ]0, +\infty[, \rho \mapsto \ell_\alpha \]
is a non constant, real analytic function.

See [1] for a proof of this. Note that, to prove that such a function is non constant, it is natural to consider two cases according to whether the geodesic $\alpha$ is simple or not:

1. If $\alpha$ is simple then by including it as a curve in a pants decomposition one can view $\ell_\alpha$ as one of the Fenchel-Nielsen coordinates so it is obviously non constant and, moreover, takes on any value in $]0, +\infty[$.

2. If $\alpha$ is not simple then it suffices to find a closed simple geodesic $\beta$ such that $\alpha$ and $\beta$ meet and use the inequality (see Buser [?])
\[ \sinh(\ell_\alpha/2) \sinh(\ell_\beta/2) \geq 1 \]
to see that if $\ell_\beta \to 0$ then $\ell_\alpha \to \infty$ and so is non constant.

3.1.2. Characterization of simple geodesics. There is always a simple closed geodesic shorter than any given closed geodesic. More precisely, if $\beta \subset \Sigma$ is a closed geodesic which is not simple then by doing surgery at the double points one can construct a simple closed geodesic $\beta' \subset \Sigma$ with $\ell_{\beta'} < \ell_\beta$.

For $\epsilon > 0$ define the $\epsilon$-thin part of the Teichmüller space $T(\Sigma)$ to be the set 
\[ T_{<\epsilon}(\Sigma) := \{ \ell_\beta < \epsilon, \forall \beta \text{ closed simple} \} \subset T(\Sigma). \]

By definition, on the complement of the thin part $\ell_\beta \geq \epsilon$ for all simple closed geodesics and since, by the preceding remark, there is always a simple closed geodesic shorter than any given closed geodesic, $\ell_\beta \geq \epsilon$ for all closed geodesics.

**Proposition 3.1.2.** Let $\Sigma$ be a finite volume hyperbolic surface. Then a closed geodesic $\alpha \subset \Sigma$ is simple if and only if the infimum over $T(\Sigma)$ of the geodesic length function $\ell_\alpha$ is zero.

**Proof.** In one direction, if $\alpha$ is simple then $\ell_\alpha$ is one of the Fenchel-Nielsen coordinates for some pants decomposition of $\Sigma$ so there is some (non convergent) sequence $\rho_n \in T(\Sigma)$ such that $\ell_\alpha \to 0$.

Now suppose that $\alpha$ is not simple and we seek a lower bound for its length. There are two cases depending on whether there exists a closed simple geodesic $\beta$ disjoint from $\alpha$ or not. If there is no such geodesic then $\alpha$ meets every simple closed geodesic $\beta \subset \Sigma$ and it is cusomary to call such a curve a filling curve. Choose $\epsilon > 0$ and consider the decomposition of the Teichmüller space into the $\epsilon$-thin part and its complement. On the thick part $\ell_\alpha \geq \epsilon$ whilst on the thin part, by the inequality [3], it is bounded below by $\arcsinh(1/\sinh(\epsilon/2))$.

If there is an essential simple closed geodesic disjoint from $\alpha$ then we cut along this curve to obtain a possibly disconnected surface with geodesic boundary. We repeat this process to construct a compact surface $C(\alpha)$ such that $\alpha$ is a filling curve in $C(\alpha)$. By construction $C(\alpha)$ embeds isometrically as a subsurface of $\Sigma$ and since $\alpha$ is not simple $C(\alpha)$ is not an annulus. On the other hand, by taking the Nielsen extension of $C(\alpha)$ then capping off with a punctured disc we obtain a conformal embedding $C(\alpha) \hookrightarrow C(\alpha)^*$ where $C(\alpha)^*$ is a punctured surface with a
natural Poincaré metric. By the Ahlfors-Pick-Schwarz Lemma there is a contraction between the metrics induced on $C(\alpha)$ from the metric on $\Sigma$ and from the Poincaré metric on $C(\alpha)^*$. A consequence of this is that the geodesic in the homotopy class determined by $\alpha$ on $C(\alpha)$ is longer than the one in $C(\alpha)^*$. So, to bound $\ell_\alpha$ it suffices to bound the length of every filling curve on a punctured surface. There are two cases.

- If $C(\alpha)$ has an essential simple closed curve then we have already treated this case above.
- If $C(\alpha)$ has no essential simple closed curves then it is a 3 punctured sphere and the bound is trivial since the Teichmüller space consists of a point.

\[\square\]

4. **Fenchel-Nielsen twist deformation**

Whilst make no claim as to the originality of the material in this section it is included to set up notation give an exposition of two results which we use in Section 6.1.

4.1. **The Fenchel-Nielsen twist.** We choose a simple closed curve $\alpha \subset \Sigma$. Following $[3]$, cut along this curve, and take the completion of the resulting surface with respect to the path metric to obtain a possibly disconnected surface with geodesic boundary $\Sigma'$.

Obviously, one can recover the original surface from $\Sigma'$ by identifying pairs of points of one from each of the boundary components. More generally, if $t \in \mathbb{R}$ then a \textit{(left) Fenchel-Nielsen twist along $\alpha$} allows one to construct a new surface $\Sigma_t$, homeomorphic to $\Sigma$ by identifying the two boundary components with a left twist of distance $t$, i.e. the pair of points which are identified to obtain $\Sigma$ are now separated by distance $t$ along the image of $\alpha$ in $\Sigma_t$. Thus this construction gives rise to a map, which we will call the \textit{time $t$ twist along $\alpha$},

$$\tau^t_\alpha : \Sigma \to \Sigma_t,$$

discontinuous for $t \neq 0$ and mapping $\Sigma \setminus \alpha$ isometrically onto $\Sigma_t \setminus \alpha$. Note that $\tau^t_\alpha$ is not unique but this will not be important for our analysis, what is important, and easy to see from the construction, is that the geometry of $\Sigma_t \setminus \alpha$ does not vary with $t$ as we will exploit this to obtain our main result.

4.2. **The lift of the twist to $\mathbb{H}$.** Let $\Gamma$ be Fuchsian group such that $\Sigma := \mathbb{H}/\Gamma$ is a closed surface, $\alpha \subset \Sigma$ a non separating simple closed geodesic and $x \notin \alpha$ a basepoint for $\Sigma$. Now let $A \subset \mathbb{H}$ denote the set of all lifts of $\alpha$ and $\hat{x} \in \mathbb{H}$ a lift of $x$.

Then the complement of $A$ consists of an infinite collection of pairwise congruent, convex sets. Moreover, if $P$ denotes the connected component of the complement of $A$ containing $\hat{x}$, then $P$ can be identified with the universal cover of the surface $\Sigma \setminus \alpha$ and the subgroup $\Gamma^P \subset \Gamma$ that preserves $P$ is isomorphic to the fundamental group of this subsurface. Since the geometry of $\Sigma_t$ does not change with $t \in \mathbb{R}$ the geometry of $P$ does not change either. This observation is the key to establishing uniform bounds in the proof of Theorem 6.1.3

Each of the other connected components of $\mathbb{H} \setminus P$ can be viewed as a translate of $g_i(P)$ for some element $g_i$ of $\Gamma$ and so $\mathbb{H}$ is \textit{tiled} by copies of $P$. Let us consider how this tiling evolves under the time $t$ twist $\tau^t_\alpha$ along $\alpha$. There is a unique lift
$\hat{\tau}_n^t : \mathbb{H} \to \mathbb{H}$ which fixes $\hat{x}$ and hence $P$. We can calculate the image of a translate of $P$ under the lift of $\hat{\tau}_n^t$ by a recursive procedure. Suppose that for some $g_1, \ldots, g_n \in \Gamma$;

- $\cup g_i(\hat{P})$ is connected,
- we have determined the images of $g_1(\hat{P}), \ldots, g_n(\hat{P})$.

Let $g_{n+1}(P)$ be a translate of $P$ such that $g_{n+1}(\hat{P}) \cap g_n(\hat{P}) = \hat{\alpha}$. and we consider two cases:

1. If $g_n(P) = P$ then the image of $g_{n+1}(P)$ is $\phi^t(g_{n+1}(P))$ where $\phi^t$ is a hyperbolic translation of length $t$ with axis $\hat{\alpha}$.
2. If $g_n(P) \neq P$ and its image under $\hat{\tau}_n^t$ is $h(P)$ then the image of $g_{n+1}(P)$ is $h \circ \phi^t \circ g_n^{-1}(g_{n+1}(P))$ where $\phi^t$ is a hyperbolic translation of length $t$ with axis $g_n^{-1}(\hat{\alpha}) \subset A$.

This procedure allows us to prove the following:

**Lemma 4.2.1.** Let $\Lambda^P \subset \partial \mathbb{H}$ denote the limit set of $\Gamma^P$. Then $\hat{\tau}_n^t$ admits a canonical extension $\hat{\tau}_n^t : \mathbb{H} \cup \partial \mathbb{H} \to \mathbb{H} \cup \partial \mathbb{H}$ which is continuous on $\partial \mathbb{H}$. Further:

1. For any $w \in \Gamma^P$ one has $\hat{\tau}_n^t(w) = w$;
2. For any $w \in \partial \mathbb{H}$ one has $\lim_{t \to \pm \infty} \hat{\tau}_n^t(w) \in \Lambda^P$ and further this is an endpoint of an edge of $\partial \mathbb{P}$.

**Proof.** It is standard from the theory of negatively curved groups that the lift admits a unique extension to $\mathbb{H} \cup \partial \mathbb{H}$, continuous on the boundary $\partial \mathbb{H}$, since $\mathbb{H}/\Gamma$ is compact and so the restriction of the lift to the set of lifts of a base point $x \in \Sigma$, $\Gamma \cdot \{x\}$ is Lipschitz.

Since the extension is continuous, to prove (1) it suffices to note that the lift of the Fenchel-Nielsen deformation fixes the endpoints of the edges of $\partial \mathbb{P}$ and these are dense in $\Lambda^P$.

For (2) let $w \in \partial \mathbb{H}$ and suppose that it is not a point of $\partial \mathbb{P}$. Then there is an edge $\hat{\alpha}$ of $\partial \mathbb{P}$ such that $w$ is a point of the interval determined by the endpoints of this geodesic. It is easy to check using our recursive description of the action of $\hat{\tau}_n^t$ on $\mathbb{H}$ that $w$ converges to the appropriate endpoint of $\hat{\alpha}$. □

We note that (2) can also be proved as follows. For $t = n\ell_\alpha, n \in \mathbb{Z}$ the Fenchel-Nielsen twist coincides with a Dehn twist. If $\beta$ is a loop, disjoint from $\alpha$ then (up to homotopy) it is fixed by the Dehn twist. If $\beta$ is a loop which crosses $\alpha$ then under iterated Dehn twists $\text{tw}_\alpha^n$ it limits to a curve on $\Sigma$ that spirals to $\alpha$. That is, lifting to $\mathbb{H}$ and considering the extension of the lift of the Dehn twist $\text{tw}_\alpha^n : \mathbb{H} \cup \partial \mathbb{H} \to \mathbb{H} \cup \partial \mathbb{H}$, an endpoint of $\text{tw}_\alpha^n(\beta)$ converges to an endpoint of some lift of $\alpha$. It is not difficult to pass to general $t$ using the fact that the $\hat{\tau}_n^t$ extends to a homeomorphism on $\mathbb{H} \cup \partial \mathbb{H}$.

### 4.3. Separated geodesics

We say that a pair of geodesics $\hat{\gamma}_1, \hat{\gamma}_2 \subset \mathbb{H}$ are separated by a a geodesic $\hat{\gamma}$ with end points $\hat{\gamma}^\pm \in \partial \mathbb{H}$ if the ideal points of $\hat{\gamma}_1, \hat{\gamma}_2$ are in different connected components of $\partial \mathbb{H} \setminus \{\hat{\gamma}^\pm\}$. Note that $\hat{\gamma}_1, \hat{\gamma}_2$ are necessarily disjoint.

If $\gamma_1, \gamma_2 \subset \mathbb{H}$ are a pair of simple closed geodesics, such that $\alpha, \gamma_1, \gamma_2$ are disjoint and we choose an arc $\beta$ between $\gamma_1$ and $\gamma_2$ that meets $\alpha$ transversely in a single point then this configuration lifts to $\mathbb{H}$ as $\hat{\gamma}_1, \hat{\gamma}_2$ separated by a lift $\hat{\alpha}$ of $\alpha$. It is easy to convince oneself that, as we deform by the Dehn twist $\text{tw}_\alpha^n$, the length of $\beta$ goes to infinity. Essentially, our next lemma says that this is true for any pair of geodesics $\gamma_1, \gamma_2$ in $\Sigma$ admitting an arc that meets $\alpha$ in an essential way.
Lemma 4.3.1. Let \( \hat{\gamma}_1, \hat{\gamma}_2 \subset \mathbb{H} \) be a pair of geodesics which are separated by some lift of \( \alpha \) then the distance between \( \hat{\tau}_1^*(\hat{\gamma}_1) \) and \( \hat{\tau}_2^*(\hat{\gamma}_2) \) tends to infinity as \( t \to \pm \infty \).

Proof. Let \( \hat{\alpha} \) be a lift of \( \alpha \) which separates \( \hat{\gamma}_1, \hat{\gamma}_2 \subset \mathbb{H} \). Let \( P_1 \) and \( P_2 \) be the pair of complementary regions which have \( \hat{\alpha} \) as a common edge and we label these so that \( \hat{\gamma}_1 \) is on the same side of \( \hat{\alpha} \) as \( P_1 \) for \( i = 1, 2 \). We choose the lift of the base point to be in \( P_1 \) and lift the Fenchel-Nielsen deformation.

First consider the orbit \( \hat{\tau}_i^*(\hat{\gamma}_i) \) of an ideal endpoint \( y \) of \( \hat{\gamma}_2 \) as \( t \to \infty \). Since \( y \in P_1 \), the region \( P_2 \) gets translated and so, for any side \( \beta \) of \( P_2 \), the sequence \( \hat{\tau}_i^*(\beta) \) converges to the endpoint \( \hat{\alpha}^+ \). Now there is a pair of edges \( \beta_1, \beta_2 \) such that the endpoints of \( \hat{\gamma}_2 \) are contained in the closed interval containing the endpoints of \( \beta_1, \beta_2 \). Since each of the \( \beta_i \) converge to \( \hat{\alpha}^+ \) under the deformation it is easy to see that \( \hat{\tau}_1^*(\hat{\gamma}_2) \) must converge to \( \hat{\alpha}^+ \).

Now consider the orbit of an endpoint \( y \) of \( \hat{\gamma}_1 \) under the deformation. It suffices to show that, under this deformation, \( y \) does not converge to \( \hat{\alpha}^+ \). There are two cases according to whether or not \( y \) belongs to the limit set \( \Lambda^{P_1} \) of the subgroup of \( \Gamma \) which stabilises \( P_1 \).

1. If \( y \in \Lambda^{P_1} \) then it is invariant under the Fenchel-Nielsen deformation.
2. If \( y \notin \Lambda^{P_1} \) then it limits to a point in \( y_\infty \in \Lambda^{P_2} \) which is an endpoint of one of the edges of \( P_1 \). By hypothesis \( \hat{\gamma}_1 \) does not meet \( \hat{\alpha} \) and so \( y_\infty \) is not \( \hat{\alpha}^+ \).

\( \square \)

5. Geodesic angle functions

We present two methods for computing (functions of) the angle \( \alpha_1 \angle_z \alpha_2 \) between \( \alpha_1, \alpha_2 \) at \( z \). The first method, just like the formula \( (2) \) for geodesic length, is a closed formula in terms of traces (equation \( (4) \)) whilst the second is in terms of end points of lifts of \( \alpha_1, \alpha_2 \) to the Poincaré disk (equation \( (5) \)). This second formula will prove useful for obtaining estimates for the variation of angles along a Fenchel-Nielsen deformation. In either case, we start as before by identifying \( \Sigma \) with the quotient \( \mathbb{H}/\Gamma \) where \( \Gamma = \rho(\pi_1(\Sigma)), \rho \in \mathcal{T}(\Sigma) \). We choose \( z \) as a basepoint for \( \Sigma \) and associate elements \( \alpha_1, \alpha_2 \in \pi_1(\Sigma, z) \) such that \( \alpha_i \) is the unique oriented closed geodesic in the conjugacy class \( [\alpha_i] \) in the obvious way.

5.1. Traces and analyticity. As explained in the introduction we shall need an analogue of Fact 3.1.1 so we give a brief account of the analyticity of the angle functions:

Proposition 5.1.1. If \( \rho \in \mathcal{T}(\Sigma) \) is a point in Teichmüller space then

\[
\mathcal{T}(\Sigma) \to [0, 2\pi[, \rho \mapsto \alpha_1 \angle_z \alpha_2,
\]

is a real analytic function.

Proof. With the notation above we have the following expression for the angle:

\[
(4) \quad \sin^2(\alpha_1 \angle_z \alpha_2) = \frac{4(2 - \text{tr}[\rho(a_1), \rho(a_2)])}{(\text{tr}^2\rho(a_1) - 4)(\text{tr}^2\rho(a_2) - 4)}.
\]

This equation is actually implicit in \([7]\) but it is not claimed to be new there and seems to have been well known. The left hand side of \( (4) \) is clearly an analytic function on \( \mathcal{T}(\Sigma) \) and it follows from elementary real analysis the the angle varies real analytically too. \( \square \)
Note that, though we will not need this, (4) shows that the square of the sine is in fact a rational function of traces (see Mondal [9] for applications of this).

5.1.1. Cross ratio formula. It will useful to have another formula for the angle in terms of a cross ratio. This formula is well-known, see for example, The Geometry of Discrete Groups, by A.F. Beardon but we since we will use it extensively to obtain bounds we give a short exposition. If $\theta$ is the angle between two hyperbolic geodesics $\hat{\alpha}, \hat{\beta} \subset \mathbb{H}$ then \( \tan^2 \left( \frac{\theta}{2} \right) \) can be expressed as a cross ratio. One can prove this directly by taking $\hat{\alpha}$ to have endpoints $\alpha^\pm = \pm 1$ and $\hat{\beta}$ endpoints $\beta^\pm = \pm e^{i\theta}$ in the Poincaré disc model. Then

\[
\left( \frac{\alpha^+ - \beta^+}{\alpha^- - \beta^-} \right) \left( \frac{\alpha^-- \beta^-}{\alpha^+ - \beta^+} \right) = \left( \frac{1 - e^{i\theta}}{1 + e^{i\theta}} \right) \left( \frac{-1 + e^{i\theta}}{-1 - e^{i\theta}} \right) = \left( \frac{1 - e^{i\theta}}{1 + e^{i\theta}} \right)^2 = \tan^2 \left( \frac{\theta}{2} \right).
\]

6. Angles defined by closed geodesics

6.1. Variation of angles. In this paragraph we give an improved version of the following well known fact:

**Fact 6.1.1.** Let $\alpha, \beta \subset \Sigma$ be a pair of closed simple geodesics that meet in a point $z \in \Sigma$. If $\alpha$ is simple then for any $\theta \in [0, \pi]$ there exists $\rho \in \mathcal{T}(\Sigma)$ such that $\alpha \angle_x \beta = \theta$.

Under the hypothesis, there is a convex subsurface $\Sigma' \subset \Sigma$ homeomorphic to a holed torus which contains $\alpha \cup \beta$. The fact follows by presenting $\Sigma'$ as the quotient of $\mathbb{H}$ by a Schottky group.

Using the preceding discussion of the Fenchel-Nielsen deformation we can relax the hypothesis on $\beta$ even whilst taking the restriction of the angle function to a one dimensional submanifold of $\mathcal{T}(\Sigma)$. The proof will should also serve to familiarise the reader with the notation and provide intuition as to why this case is different to that of an intersection of a generic pair of closed geodesics treated in Theorem 6.1.3.

**Lemma 6.1.2.** Let $\alpha, \beta \subset \Sigma$ be a pair of closed geodesics that meet in a point $z \in \Sigma$. If $\alpha$ is simple then for any $\theta \in [0, \pi]$ and any $\rho_0 \in \mathcal{T}(\Sigma)$ there exists $\rho_t \in \mathcal{T}(\Sigma)$ obtained from $\rho_0$ by a time $t$ Fenchel-Nielsen twist along $\alpha$ such that $\alpha \angle_x \beta = \theta$.

Moreover,

\[
\lim_{t \to \pm\infty} \alpha \angle_x \beta \in \{0, \pi\}.
\]

**Proof.** With the notation of subsection 4.2 there is a convex region $P$ in $\mathbb{H}$ bounded by lifts of $\alpha$ as before. Let $\hat{\alpha}$ be an edge of $\partial P$, and choose a corresponding lift $\hat{\beta}$ which intersects $\hat{\alpha}$. There is an element of the covering group $g \in \Gamma$ such that $\hat{\alpha} = \partial P \cap g(\partial P)$.

We lift the Fenchel-Nielsen deformation and consider, as before, its extension

\[
\hat{\tau}^t_\alpha : \mathbb{H} \sqcup \partial \mathbb{H} \to \mathbb{H} \sqcup \partial \mathbb{H}.
\]

Now, arguing as in Lemma 1.2.4, we see that;

- the endpoints of $\hat{\alpha}$ are fixed by $\hat{\tau}^t_\alpha$,
• the endpoint of \( \hat{\beta} \) on the same side of \( \hat{\alpha} \) as \( P \) converges to a point \( z \neq \alpha^+ \) as \( t \to -\infty \),
• the other endpoint of \( \hat{\beta} \) converges to \( \alpha^+ \) as \( t \to -\infty \).

It follows that, after possibly changing the orientation of \( \beta \), that the angle between \( \hat{\alpha} \) and \( \hat{\beta} \), and hence \( \alpha \angle \beta \), tends to 0. Likewise, as \( t \to +\infty \) the angle between \( \hat{\alpha} \) and \( \hat{\beta} \), and hence \( \alpha \angle \beta \), tends to \( \pi \).

Thus, by continuity, the range of the angle function is \([0, \pi]\).

\[ \square \]

**Theorem 6.1.3.** Let \( \beta_1, \beta_2 \) a pair of closed geodesics and \( y \in \beta_1 \cap \beta_2 \). Then for any simple closed geodesic \( \alpha \), different from both \( \beta_1 \) and \( \beta_2 \), the angle function \( \beta_1 \angle y \beta_2 \) is bounded away from \( \pi \) along the Fenchel-Nielsen orbit of \( \rho \in T(\Sigma) \).

**Proof.** If \( \alpha \) and \( \beta_1 \cup \beta_2 \) are disjoint then \( \beta_1 \angle y \beta_2 \) is constant along the \( \hat{\tau}^t \alpha \)-orbit so the result is trivial.

Suppose now that \( \alpha \) and \( \beta_1 \cup \beta_2 \) are not disjoint and choose \( x \) as a basepoint of \( \Sigma \). Then, with the notation of paragraph ??, there is a convex region \( P \) in \( \mathbb{H} \) bounded by lifts of \( \alpha \). We now consider three cases according to the number of edges of \( \partial P \) that \( \hat{\beta}_1 \cup \hat{\beta}_2 \) meets.

![Figure 1. Case of 4 intersections.](image)

We first deal with the simplest case. Suppose that \( \hat{\beta}_1 \cup \hat{\beta}_2 \) meets \( \partial P \) in four distinct edges denoted \( C_1, C_2, C_3, C_4 \subset \mathbb{H} \), and, after possibly relabelling these, \( \hat{\beta}_1 \) meets \( C_1, C_2 \) whilst \( \hat{\beta}_2 \) meets \( C_3, C_4 \) as in Figure [6.1]. Now we deform \( \rho_0 \) by a Fenchel-Nielsen twist along \( \alpha \) to obtain a 1-parameter family of \( \rho_t \in T(\Sigma), t \in \mathbb{R} \). As we have seen above, under such a deformation the length of \( \alpha \) does not change nor does the geometry of \( \partial P \) in particular the positions of the \( C_i \) remain unchanged. From our discussion of the \( \hat{\tau}^t \alpha \) and its extension to \( \mathbb{H} \cup \partial \mathbb{H} \) it is clear that, \( \forall t \in \mathbb{R} \), \( \hat{\tau}^t \alpha(\beta_0) \) meets \( C_1, C_2 \) whilst \( \hat{\tau}^t \alpha(\beta_2) \) meets \( C_3, C_4 \). Thus, if the diameters of the circles were small, the angle at \( \hat{z} \) cannot not vary much from its value at \( \rho_0 \) since the radii of the circles are small. More generally, we can bound the size of the angle
using the cross ratio formula. Labeling the endpoints as in Figure 6.1 one has:

\[
\tan^2\left(\frac{\theta}{2}\right) = \frac{|\beta_1^+ - \beta_2^+|}{|\beta_1^- - \beta_2^-|} \cdot \frac{|\beta_1^- - \beta_2^-|}{|\beta_1^+ - \beta_2^+|}
\]

Note first that each of the four points lies on the unit circle and so that its diameter, that is 2, is a trivial upper bound for each of the four distances appearing on the left hand side of this equation. Now under the deformation each of the endpoints \( \hat{\tau}_t(\beta_i^\pm) \) stays in one of four disjoint euclidean discs defined by one of the \( C_j \). In particular, there exists \( \delta_4 > 0 \) such that for all \( t \in \mathbb{R} \)

\[
\delta_4 \leq |\hat{\tau}_t(\beta_1^+ \pm \beta_2^+) - \hat{\tau}_t(\beta_1^- \pm \beta_2^-)| \leq 2
\]

and this is sufficient to obtain bounds on the cross ratio:

(6) \[ \frac{1}{2\delta_4} \leq \tan\left(\frac{\theta}{2}\right) \leq 2/\delta_4 \]

If \( \beta_1 \cup \beta_2 \) meets \( \partial P \) in just two edges, \( C_1, C_2 \subset \mathbb{H} \) say. Although we no longer have a uniform lower bound for \( |\hat{\tau}_t(\beta_1^+ \pm \beta_2^+) - \hat{\tau}_t(\beta_1^- \pm \beta_2^-)| \) in this case, there still exists \( \delta_2 > 0 \) such that for all \( t \in \mathbb{R} \),

\[
\delta_2 \leq |\hat{\tau}_t(\beta_1^+ \pm \beta_2^+) - \hat{\tau}_t(\beta_1^- \pm \beta_2^-)|.
\]

Thus, for all \( t \in \mathbb{R} \),

(7) \[ 0 \leq \tan\left(\frac{\theta}{2}\right) \leq 2/\delta_2. \]

Finally, if \( \beta_1 \cup \beta_2 \) meets \( \partial P \) in exactly three edges then it is easy to see that, using the same reasoning as for the two edge case, there is \( \delta_3 \) such that

\[
\delta_3 \leq |\hat{\tau}_t(\beta_1^+ \pm \beta_2^+) - \hat{\tau}_t(\beta_1^- \pm \beta_2^-)|.
\]

\[ \square \]

**Corollary 6.1.4.** Let \( \alpha_1, \alpha_2 \) pairs of simple closed geodesics which meet in a single point \( z \) and \( \beta_1, \beta_2 \) primitive closed geodesics which meet in \( z' \). If the difference \( \alpha_1 \angle z \alpha_2 - \beta_1 \angle z' \beta_2 \) is constant then the angles are equal and, after possibly relabelling the geodesics, \( \alpha_i = \beta_i \) and \( z = z' \).
Note that we cannot suppose that $z, z'$ are distinct because of the following phenomenon. Jorgenson studied intersections of closed geodesics proving in particular that if $z \in \Sigma$ was the intersection of a pair of distinct closed geodesics then it is the intersection of infinitely many pairs of distinct closed geodesics. Such intersections are stable in that, if $(\alpha_i)_i$ is a family of geodesics obtained from Jorgenson’s procedure that meet in a point $z$ on some hyperbolic surface $\Sigma$ then, for any $\rho \in \mathcal{T}(\Sigma)$, there is $z_\rho \in \mathbb{H}/\rho(\pi_1(\Sigma))$ such that the family $(\alpha_i)_i$ meet in $z_\rho$.

Proof. We first consider the case where four geodesics are distinct then, under the Fenchel Nielsen twist along $\alpha_1$, the image of $\alpha_1 \angle z \alpha_2$ is $[0, \pi]$ whilst, by Theorem 6.1.3, the image $\beta_1 \angle z \beta_2$ is a strict subinterval. It is easy to see $\alpha_1 \angle z \alpha_2 - \beta_1 \angle z \beta_2$ cannot be a constant.

Now suppose, $\alpha_1 = \beta_1$, if $\alpha_2 = \beta_2$ then, since $\alpha_1$ and $\alpha_2$ meet in a single point, we must have $z = z'$ and the angles must be the same.

On the other hand, if $\alpha_2 \neq \beta_2$ then $z, z'$ may or may not be distinct

- If $z = z'$ then, by Lemma 6.1.2 both $\alpha_1 \angle z \alpha_2$ and $\beta_1 \angle z \beta_2$ tend to $0$ or $\pi$ as the Fenchel-Nielsen parameter $t \to \pm \infty$. Therefore, if the difference is constant it must be $0$ or $\pi$ and so, up to switching orientation, $\beta_1 = \beta_2$.
- If $z = z'$ then, by Lemma 6.1.2 and Theorem 6.1.3 $\beta_1 \angle z \beta_2$ is a proper subinterval of the range of $\alpha_1 \angle z \alpha_2$ so the difference cannot be constant.

Proof. of Lemma 1.2.1 Suppose that $\Sigma$ has a value in its angle spectrum, $\theta$ say, with finite multiplicity. Let $x_1, x_2 \ldots x_n \in \Sigma$ be a complete list of points such that there are pair of closed geodesics meeting at $x_i$ at angle $\theta$. Then the set of preimages of the $x_i$ under the covering map $\mathbb{H} \to \Sigma$ is a discrete set which is invariant under $\text{Aut}(\text{ax}(\Gamma))$. Thus $\text{Aut}(\text{ax}(\Gamma))$ is discrete and has $\Gamma$ as a finite index subgroup.

References
[1] Alan F. Beardon The Geometry of Discrete Groups, Springer
[2] P. Buser, Geometry and spectra of compact Riemann surfaces, Birkhauser
[3] S. Kerckhoff, The Nielsen Realization problem, Ann. of Math. (2) 117:2 (1983), 235265. MR 85e:32029 Zbl 0528.57008
[4] D. Long, A. Reid, On Fuchsian groups with the same set of axes, Bull. L.M.S, 30 (1998), 535538.
[5] G. Mess, IHES preprint circa 1991.
[6] Greg McShane, Hugo Parlier, Multiplicities of simple closed geodesics and hypersurfaces in Teichmuller space, Geometry and Topology
[7] Greg McShane. Length series in teichmuller space. Pacific Journal of Math, 231(2):461–479, 2007.
[8] S. Mondal, Rigidity of length-angle spectrum for closed hyperbolic surfaces https://arxiv.org/abs/1701.08829
[9] S. Mondal, An arithmetic property of the set of angles between closed geodesics on hyperbolic surfaces of finite type. https://arxiv.org/abs/1703.02478
[10] Alan W. Reid, Annales de la faculté des sciences de Toulouse Mathématiques (2014), Volume: 23, Issue: 5, page 1103-1118

UFR de Mathématiques, Institut Fourier 100 rue des maths, BP 74, 38402 St Martin d’Hères cedex, FRANCE
E-mail address: Greg.McShane@ujf-grenoble.fr