On the tensor product construction for 
$q$-differential algebras

Andrzej Sitarz*

Instut für Physik, Johannes-Gutenbery Universität
55099 Mainz, Germany

March 31, 2022

Abstract: We show that for $q \neq -1$ the $q$-graded tensor product fails to
preserve the $q$-differential structure of the product algebra and therefore there
is no natural tensor product construction for $q$-differential algebras.

1 Introduction

Recently there has been much interest in noncommutative generalizations
of the standard differential geometry. An interesting point is the attempt
to study the $Z_N$-graded differential structure and the $N$-nilpotent ($d^N \equiv 0$)
linear maps satisfying a deformed Leibniz rule [3, 2]. This could be extended
to a $q$-generalization of complexes and cohomology [1].
Special examples of $Z_3$-graded structures have been discussed earlier (see [4]
and references therein) with many examples and a view to possible physical
realizations [5, 6].
Although the generalization of differential algebras opens new possibilities,
one should not forget that it also brings difficulties. Not every construction
that is possible in the classical geometry can be automatically extended to

*e-mail:sitarz@higgs.physik.uni-mainz.de
a more general setup. This Letter discusses one of the significant constructions, the tensor product of \( q \)-differential algebras, proving that within the commonly assumed rules there is no room for it.

2 \( q \)-differential algebras

A \( \mathbb{Z} \) graded \( q \)-differential algebra \( \Omega_{q} \) is an unital algebra with the following properties:

- \( \Omega = \bigoplus_{i=0}^{\infty} \Omega^{i} \) and \( \Omega^{i} \cdot \Omega^{j} \subset \Omega^{i+j} \) (\( \mathbb{Z} \)-grading),
- there exists a linear operator \( d : \Omega^{i} \rightarrow \Omega^{i+1} \) such that \( d^{N} \equiv 0 \),
- \( d \) obeys a \( q \)-graded Leibniz rule, for \( q \neq 1 \) and \( q^{N} = 1 \):

\[
d(\omega^{i}\omega^{j}) = (d\omega^{i})\omega^{j} + q^{i}\omega^{i}(d\omega^{j})
\]
where \( \omega^{i,j} \in \Omega^{i,j} \).

Note that \( q \) can be any primitive \( N \)-th root of unity, i.e., \( N \) is the smallest integer such that \( q^{N} = 1 \). This follows from the compatibility of the \( q \)-Leibniz rule with the nilpotency of \( d \). Iterating the Leibniz rule we obtain:

\[
d^{N}(ab) = \sum_{n=0}^{N} \left[ \begin{array}{c} N \\ n \end{array} \right]_{q} d^{n}a d^{N-n}b,
\]
where

\[
\left[ \begin{array}{c} N \\ n \end{array} \right]_{q} = \frac{N!_{q}}{n!_{q}(N-n)!_{q}},
\]
and

\[
n!_{q} = \frac{1 - q^{n}}{1 - q} \cdots \frac{1 - q^{2}}{1 - q}.
\]

It is easy to see that for \( q \) a primitive root of unity the \( q \)-deformed binomial vanishes for \( 0 < n < N \) and therefore (\( \square \)) agrees with \( d^{N} = 0 \). Of course, for \( N = 2 \) (\( q = -1 \) being the only possibility), we recover the standard differential algebras. However, for \( N > 2 \) we have a whole family
of differential algebras, which differ by the value of $q$ in the Leibniz rule, as for different values of $q$ (for the same $N$) we have different objects. We can make this statement more precise using the notion of homomorphisms between $q$-differential algebra, which are algebra homomorphisms preserving the grading and commuting with $d$:

**Observation 1** Let $\Omega_q$ and $\Omega_p$ be two $q$- and $p$-differential algebras for $q^N = p^N = 1$. If there exists a differential homomorphism $\psi : \Omega_q \to \Omega_p$ then $p = q$.

The proof of this observation is simple, one should apply $\phi$ to the Leibniz rule in $\Omega_q$ and compare the result with the Leibniz rule in $\Omega_p$.

If the subalgebra of 0-forms has a star operation then it extends naturally (provided that $\Omega^n$ is generated as a bimodule by $\Omega^{n-1}$) on the whole $\Omega$ by the rule:

$$(d\omega^i)^* = q^{-i}d(\omega^*) .$$

## 3 Tensor products of $q$-differential algebras

So far, we have seen that the definition and properties of the standard differential algebras extend without problems to the $q$-differential case. Our task will now be to verify whether the tensor product construction extends to the $q$-differential case.

The construction of the differential algebra over the tensor product of algebras (which corresponds to the construction of differential geometry over Cartesian products of manifolds) uses the graded tensor product. Let us remind that for the ordinary differential algebras $\Omega(A)$ and $\Omega(B)$ we construct $\Omega(A \otimes B)$ in the following way. We take $\Omega(A) \otimes \Omega(B)$ as a vector space, with the following product:

$$(\omega_1 \otimes \eta_1)(\omega_2 \otimes \eta_2) = (-1)^{|\eta_1||\omega_2|} (\omega_1 \omega_2 \otimes \eta_1 \eta_2)$$

for $\omega_1, \omega_2 \in \Omega(A)$ and $\eta_1, \eta_2 \in \Omega(B)$.

Then one finds that the following linear operator $d$:

$$d(\omega \otimes \eta) = (d\omega \otimes \eta) + (-1)^{|\omega|}(\omega \otimes d\eta),$$

(2)
satisfies both the graded Leibniz rule and $d^2 = 0$, and therefore our graded tensor product of differential algebras is a differential algebra itself.

In this section we shall attempt to generalize the above scheme for $q$-differential algebras.

### 3.1 $q$-graded tensor product

First, we have to introduce the notion of a braided tensor product, with braiding set by a root of unity $q$, $q^N = 1$.

Let us assume that $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{Z}$-graded algebras. Then we introduce the tensor product $\mathcal{A} \otimes \mathcal{B}$ with the following algebra structure:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = q^{|b_1||a_2|}(a_1a_2 \otimes b_1b_2),$$

where $q$ is any $N$-th root of unity, and $|a|$ denotes the grades of the algebra elements. We shall denote this tensor product at $\otimes_q$. For the proof that the construction is well-defined and the above defined braided tensor multiplication $\otimes_q$ is associative we refer to [?] (In the case of the above braiding one uses often the name of anyonic vector spaces and anyonic tensor product).

Before we proceed with the extension of differential structures on the tensor product, let us make a simple observation:

**Lemma 1** $\mathcal{A} \otimes_q \mathcal{B}$ is isomorphic to $\mathcal{B} \otimes_q \mathcal{A}$.

**Proof:** Let us define $\phi(a \otimes_q b) = q^{-|a||b|}b \otimes_q a$.

Then

$$\phi((a_0 \otimes_q b_0)(a_1 \otimes_q b_1)) =$$

$$\phi\left(q^{||a_1||a_0a_1}b_0b_1\right) =$$

$$q^{-(|a_0||a_1|)(|b_0|+|b_1|)}q^{||a_1||a_1}(b_0b_1 \otimes_q a_0a_1) =$$

$$q^{-|a_0||b_0|+|a_1||b_1|}(b_0 \otimes_q a_0)(b_1 \otimes_q a_1) =$$

$$\phi(a_0 \otimes_q b_0)\phi(a_1 \otimes_q b_1).$$

Therefore the anyonic tensor product (for $q \neq -1$) is strictly noncommutative (for $q = -1$ it is $\mathbb{Z}_2$-graded commutative).

Now we shall prove the main lemma.
Lemma 2 Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{Z}$-graded $q$-differential algebras, with the deformed Leibniz rule determined by $q_\mathcal{A}$ and $q_\mathcal{B}$ respectively, both primitive $N$-th roots of unity. Then, no anyonic tensor product admits a $q$-differential structure.

To prove this no-go lemma we shall verify the restrictions set upon the differential structure on $\mathcal{A} \otimes_q \mathcal{B}$, where $q$ is again a $N$-th root of unity.

Let us rewrite the Leibniz rules for $\mathcal{A}$ and $\mathcal{B}$:

$$
\begin{align*}
\quad d(a_1a_2) & = da_1a_2 + q_\mathcal{A}^{\lvert a_1 \rvert}a_1da_2, \quad a_1, a_2 \in \Omega(\mathcal{A}), \\
\quad d(b_1b_2) & = db_1b_2 + q_\mathcal{B}^{\lvert b_1 \rvert}b_1db_2, \quad b_1, b_2 \in \Omega(\mathcal{B}).
\end{align*}
$$

(3) (4)

Note, that a priori $q_\mathcal{A}$ might be different from $q_\mathcal{B}$.

Now, we pose the following question: can one find a $q$, $p$, $s$ (all $N$-th roots of unity) such that the $q$-tensor product of $\mathcal{A}$ and $\mathcal{B}$, admits the differential structure defined as:

$$
\quad d(a \otimes_q b) = da \otimes_q b + s^{\lvert a \rvert}a \otimes_q db
$$

and this $d$ obeys the $p$-deformed Leibniz rule.

We leave as much parameters free as possible, as a priori we know nothing about the restrictions (if any) that may appear between them.

Of course, from the natural embedding:

$$
\begin{align*}
\mathcal{A} & \sim \mathcal{A} \otimes_q 1 \hookrightarrow \mathcal{A} \otimes_q \mathcal{B} \\
\mathcal{B} & \sim 1 \otimes_q \mathcal{B} \hookrightarrow \mathcal{A} \otimes_q \mathcal{B}
\end{align*}
$$

(5) (6)

we learn immediately that $q_\mathcal{A} = p = q_\mathcal{B}$. It remains now to look for possible values of $q$ and $s$. The test of consistency is, as usually, the Leibniz rule.

We take the product $(a_1 \otimes_q b_1)(a_2 \otimes_q b_2)$ and apply $d$ (defined as above) to it. This could be done in two ways, first, we multiply the components and then apply $d$, using the Leibniz rule for each of the algebra. The other way is that we first apply $d$, using the above definition and then multiply the result.

The table below shows the coefficients obtained in each way:

|       | $da_1a_2 \otimes_q b_1b_2$ | $a_1da_2 \otimes_q b_1b_2$ | $a_1a_2 \otimes_q db_1b_2$ | $a_1a_2 \otimes_q b_1db_2$ |
|-------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| I     | $q^{\lvert a_2 \rvert \lvert b_1 \rvert}$ | $q^{\lvert a_2 \rvert \lvert b_1 \rvert}p^{\lvert a_1 \rvert}$ | $q^{\lvert a_2 \rvert \lvert b_1 \rvert}s^{\lvert a_1 \rvert+\lvert a_2 \rvert}$ | $q^{\lvert a_2 \rvert \lvert b_1 \rvert}s^{\lvert a_1 \rvert+\lvert a_2 \rvert}p^{\lvert b_1 \rvert}$ |
| II    | $q^{\lvert a_2 \rvert \lvert b_1 \rvert}$ | $p^{\lvert a_1 \rvert+\lvert b_1 \rvert}q^{\lvert b_1 \rvert\lvert a_1 \rvert+1}$ | $q^{\lvert b_1 \rvert+1\lvert a_2 \rvert}s^{\lvert a_1 \rvert}$ | $p^{\lvert a_1 \rvert+\lvert b_1 \rvert}s^{\lvert a_2 \rvert}q^{\lvert b_1 \rvert\lvert a_2 \rvert}$ |

5
Of course, the consistency requires that the values in the first row must coincide with these in the second row. Let us compare: from the second column we get that \( p = q^{-1} \), the third determines \( s = q \) and from the last one we obtain \( p = s \). However, this is possible only in the case \( p = q = s = -1 \)!. Therefore, the procedure of tensoring the differential algebras is natural only in the standard \( d^2 = 0 \) situation.

### 4 An example - but not a counterexample.

Let us see a simple example illustrating the problem. We take \( N = 3 \), then we have two primitive roots of unity \( j \) and \( j^2 = \bar{j} \). Let us take the simplest (but nontrivial) \( d^3 \equiv 0 j \)-differential algebra on the real line. Our starting point is the standard definition of \( d \) on the algebra of \( C^\infty(\mathbb{R}) \). For \( f \in C^\infty(\mathbb{R}) \) we define:

\[
\frac{df}{dx} = df',
\]

where \( f' \) is the usual derivative and \( dx \) is the one-form that generates the bimodule of one-forms. Applying \( d \) to both sides we get:

\[
d^2f = d^2xf' + jdxdf'.
\]

where we have assumed the \( j \)-Leibniz rule. The forms \( d^2x \) and \( dxdx \) are two independent generators of \( \Omega^2_j(\mathbb{R}) \). Consider now the general \( j \)-Leibniz rule. First, for the product of two functions \( f \) and \( g \) we have:

\[
d(fg) = dx(fg)' = dx(f'g + fg'),
\]

however, on the other hand, from the \( j \)-Leibniz rule:

\[
d(fg) = (df)g + f(dg) = dxf'g + fdxg'.
\]

This gives us the first rule for the left and right multiplication of differential forms:

\[
fdx = dxf,
\]

which is the same as for the standard calculus. However, if we apply \( d \) to the both sides of (9) we get:

\[
dxdf' + fdx = d^2xf + jdxdf',
\]

6
where we have already used (9) when appropriate. Finally, we obtain the second relation:

$$fd^2 x = d^2 xf + (j - 1)dxdxf'.$$

(11)

To obtain some further relations between the differential forms, we apply $d$ for the third time, to both sides of (11), then taking into account that $d^3 \equiv 0$ we see:

$$dxf'd^2 x = j^2 d^2 xdx f' + (j - 1)d^2 xdx f' + j(j - 1)dxd^2 x + j^2 (j - 1)dxdxdxf''.$$

(12)

Again, using both (9) and (11) we get:

$$-3dxdx dx'' - 2j^2 (dx^2 x - jdx^2 x) f' = 0,$$

(13)

from which we immediately get:

$$dxdx dx = 0,$$

(14)

$$dxd^2 x = jd^2 xdx.$$  

(15)

These are the only rules that are necessary for the construction of a consistent $j$-differential algebra on $\mathbb{R}$. The obtained differential complex is infinite-dimensional, as there are no restrictions on products of $d^2 x$. However, without breaking the consistency, one may impose more rules, for instance, one may postulate that $d^2 x d^2 x = 0$ or $d^2 x d^2 y d^2 x = 0$.

The one-form $dx$ is hermitian, $dx^* = dx$ whereas $d^2 x$ is antihermitian: $(d^2 x)^* = j^2 d^2 x$.

Of course, the whole procedure could be carried out with $\bar{j}$ instead of $j$, giving us a different $d^3 = 0$ differential structure on $\mathbb{R}$.

So far, we have been able to construct the $j$- (or $\bar{j}$-) differential structures on the real line. The natural question is then, how can one extend this construction to the plane or, more generally, $\mathbb{R}^n$. As we have seen in the previous section, there is no canonical way to do it. This does not mean, however, that such structure does not exist.

Below, we shall provide an example of the $j$-differential structure on the plane, which is built out of $j$ and $\bar{j}$ differential structures on the line. This will not be, however, a counterexample to our no-go lemma.

Consider the algebra of functions on the plane, and the differential calculus defined similarly as in the one-dimensional case:

$$df = dx f_x + dy f_y,$$

$$d^2 f = d^2 x f_x + d^2 y f_y + j(dx)^2 f_{xx} + + j(dy)^2 f_{yy} + j(dx dy + dy dx)f_{xy}$$

(17)
We notice here that the $j$-differential algebras on the real line are embedded in a natural way in $\Omega_j(\mathbb{R}^2)$ as constructed above. However, we encounter problems when we come to the commutation relations between $x,y$ and their derivatives. From Eq.(16) we have

$$x \, dy = dy \, x, \quad y \, dx = dx \, y.$$  \hspace{1cm} (18)

If we differentiate it (using the $j$-Leibniz rule) we obtain first set of the restrictions:

$$dx \, dy - jdy \, dx = d^2 y x - x d^2 y, \hspace{1cm} (19)$$

$$dy \, dx - jdx \, dy = d^2 x y - y d^2 x. \hspace{1cm} (20)$$

We immediately see that at least in one row the left-hand side does not vanish and therefore either $x$ does not commute with $d^2 y$ or $y$ with $d^2 x$. This proves that such structure does not come from the anyonic tensor product, even though it is well-defined and contains $j$-differential algebras on the real line.

5 Conclusions

What we have shown in this Letter has some profound consequences for the theory of $q$-differential algebras. First, we have no natural procedure of tensoring such structures, while keeping all the natural rules. Of course, one could attempt to relax one or another of them, however, there is still ambiguity, which should be kept and which could be allowed to change.

Having no canonical construction for the $q$-differential structures on the products of spaces is a major drawback of the theory.

Of course, let us notice that the problem arises only if we take into account the algebra structure, as for the simple $q$-generalizations of cochains and cohomology the construction of the tensor product might be possible.

The world of $q$-deformations, and $q$-differential structures seems, at first sight, similar to the well-known commutative and anticommutative objects. What we see at least in the case of the $q$-differential algebras is that $q = -1$ case is very special.

Acknowledgements: It is a pleasure to thank R.Kerner for discussions and M.Dubois-Violette for remarks on this topic.
References

[1] M.Dubois-Violette, R.Kerner *Universal q-differential calculus and q-analog of homological algebra*, LPTHE 96/48

[2] M.Dubois-Violette, R.Kerner, *Universal $\mathbb{Z}_N$-graded differential calculus*, LPTHE 96/58

[3] M.Dubois-Violette *Generalized differential spaces with $d^N = 0$ and the q-differential calculus*, Czech. J. Phys. 46 (1996) 1227-1233

[4] R.Kerner, *$\mathbb{Z}_3$-graded algebras and the cubic root of the supersymmetry translations*, J.Math.Phys. 33(1), (1992), 403,

[5] R.Kerner, Lett.Math.Phys. 36, (1996) 441-454,

[6] V.Abramov, R.Kerner, B. Le Roy, *Hypersymmetry: a $\mathbb{Z}_3$-graded generalization of supersymmetry*, J.Math.Phys., to appear,

[7] Shahn Majid, *Foundations of Quantum Group Theory*, Cammbridge University Press, 1995

[8] M.M.Kapranov, *On the q-analog of homological algebra*, q-alg/9611003