MW-MOTIVE OF GRASSMANNIAN BUNDLES AND COMPLETE FLAG BUNDLES

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Abstract. We study the notion of split Milnor-Witt motives, which corresponds to varieties with only 2-torsions in cohomology. Moreover, we compute the Milnor-Witt motive of Grassmannian bundles and complete flag bundles, which turn out to fit the split pattern we desired.

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1. Introduction

The category of Milnor-Witt motives (abbr. MW-motives), defined by B. Calmès, F. Déglise and J. Fasel, is the Chow-Witt refinement of the classical category of motives defined by Voevodsky. In [Yan21b], we have shown the Milnor-Witt motive of \( \mathbb{P}^n \) splits, i.e., is a direct sum of \( \mathbb{Z}(i)[2i] \) and \( \mathbb{Z}/\eta(i)[2i] \), where \( \eta \) is the Hopf element (see [Yan21b, Definition 4.1]), deducing the projective bundle theorem from this computation. The aim of this paper is to push this split pattern further and show that the Grassmannians and complete flags behave in the same way, as well as corresponding fiber bundles.

More precisely, suppose that \( X \) is a smooth scheme and that \( L \in \text{Pic}(X) \). Define

\[
E^n(X, L) = \frac{\text{Ker}(\text{Sq}^2 L)}{\text{Im}(\text{Sq}^2 L)}
\]

where \( \text{Sq}^2 L = \text{Sq}^2 + c_1(L) \cup \) is the twisted Steenrod operation. These groups have a good functorial behaviour. They admit products, pullbacks, push-forwards and satisfy a projective bundle theorem (see Proposition 3.3). We have a diagram

\[
\begin{array}{ccc}
\text{Ch}^{n-1}(X) & \xrightarrow{\beta} & H^n(X, \mathcal{P}(L)) \\
\downarrow & & \downarrow \\
H^n(X, \mathcal{P}(L)) & \xrightarrow{\eta} & \text{Ker}(\text{Sq}^2 L)_n \\
\downarrow & & \downarrow \\
H^n(X, \mathcal{W}(L)) & \xrightarrow{0} & E^n(X, L)
\end{array}
\]

where the first column is exact. We then obtain a natural homomorphism

\[
\delta : H^n(X, \mathcal{W}(L)) \otimes \mathcal{W}(k) \mathbb{Z}/2\mathbb{Z} \rightarrow E^n(X, L).
\]

The central theorem of split MW-motives is the following (see Theorem 2.15):

**Theorem 1.1.** Suppose that \( X \in \text{Sm}/k \), that \( L \in \text{Pic}(X) \) and that \( \text{Th}(L) \) splits as an MW-motive.

1. The group \( H^\ast(X, \mathcal{W}(L)) \) is a free \( \mathcal{W}(k) \)-module and \( \delta \) is an isomorphism of vector spaces.

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(2) The natural map \( \widetilde{CH}^*(X, \mathcal{L}) \to CH^*(X) \) yields a decomposition

\[
\widetilde{CH}^*(X, \mathcal{L}) \cong I(k) \cdot H^*(X, W(\mathcal{L})) \oplus \text{Ker}(Sq^1 \circ \pi)_* \]

where \( \pi : CH^*(X) \to Ch^*(X) \) is the reduction modulo 2 map.

The first statement is the algebraic geometric explanation of the fact (see [Mc01, Theorem 10.3]) that for any topological space \( X \) with finitely generated cohomologies and only 2-torsions, we have

\[
\text{Ker}(Sq^1)_* \cong H^*(X, \mathbb{Z})_{\text{free}} \otimes \mathbb{Z}/2\mathbb{Z}.
\]

The second statement says that for split MW-motives, computing Chow-Witt groups is reduced to computing Chow groups and the Steenrod square operation.

Define \( \overline{DM}_n = DM[\eta^{-1}] \) (So \( Z(1)[1] = \mathbb{Z} \)), whose (tensor) unit represents the cohomology of Witt sheaves. So, in order to decompose an MW-motive, it suffices to write down its decompositions in both \( DM \) and \( \overline{DM}_n \).

A Young tableau \( T \) (a series of rows of boxes with decreasing length) is called untwisted (resp. twisted) if it is filled by the checkboard pattern such that the first box in the first row is black (resp. white). We thus define \( \overline{DM} \) (see Theorem 5.6). In the statement, the symbol \( \Lambda \) is completely even.

The stage being set, we can now state the following theorem, which describes the decomposition of the Grassmannians \( Gr(k, n) \) in the category of MW-motives (see Theorem 4.119).

**Theorem 1.2.**

(a) We have

\[
Z(Gr(k, n)) \cong \bigoplus_{\Lambda \text{ even}} Z((|\Lambda|)) \oplus \bigoplus_{\Lambda \text{ irred. not full}, i_1 > 1} Z/\eta(|\Lambda_{i_1, \ldots, i_l}|).
\]

(b) Suppose that all tableaux are twisted. We have

\[
\text{Th}(O_{Gr(k, n)}(1)) \cong \bigoplus_{\Lambda \text{ even}} Z((|\Lambda| + 1)) \oplus \bigoplus_{\Lambda \text{ irred. not full}, i_1 > 1} Z/\eta(|\Lambda_{i_1, \ldots, i_l}| + 1)).
\]

We derive from this result the Grassmannian bundle theorem, by the method developed in [Yan21b] (see Theorem 6.10). In the statement, the symbol \( Gr(k, \mathcal{E}) \) denotes the variety of quotient bundles of \( \mathcal{E} \) of rank \( k \).

**Theorem 1.3.** Let \( S \in Sm/k \) and let \( X \in Sm/S \) be quasi-projective, \( \mathcal{L} \in \text{Pic}(X) \) and \( \mathcal{E} \) be a vector bundle of rank \( n \) over \( X \). Denote by \( p : Gr(k, \mathcal{E}) \to X \) is the structure map.

(1) We have

\[
Z(Gr(k, \mathcal{E}))/\eta \cong \bigoplus_{\Lambda \text{ (k, n)-truncated}} Z(X)/\eta(|\Lambda|)
\]

in \( \overline{DM}(S, \mathbb{Z}) \).

(2) If \( k(n - k) \) is even, we have

\[
\text{Th}(p^* \mathcal{L}) \cong \bigoplus_{\Lambda \text{ even}} \text{Th}(\mathcal{L})(|\Lambda|) \oplus \bigoplus_{\Lambda \text{ irred. not full}, i_1 > 1} Z(X)/\eta(|\Lambda_{i_1, \ldots, i_l}| + 1))
\]

in \( \overline{DM}^{\text{eff}}(S, \mathbb{Z}) \).

(3) If both \( k \) and \( n \) are even, we have

\[
\text{Th}(p^* \mathcal{L} \otimes O(1)) \cong \bigoplus_{\Lambda = \sigma_{k, T}} \text{Th}(\det(\mathcal{E})^\vee \otimes \mathcal{L})(|\Lambda|) \oplus \bigoplus_{\Lambda = \sigma_{k, T}} \text{Th}(\mathcal{L})(|\Lambda|) \oplus \bigoplus_{i_1 > 1} Z(X)/\eta(|\Lambda_{i_1, \ldots, i_l}| + 1))
\]

in \( \overline{DM}(S, \mathbb{Z}) \), where \( T \) is completely even.

(4) If \( n - k \) is odd, we have

\[
\text{Th}(p^* \mathcal{L} \otimes O(1)) \cong \bigoplus_{\Lambda \text{ even}} \text{Th}(\mathcal{L})(|\Lambda|) \oplus \bigoplus_{\Lambda \text{ irred. not full}, i_1 > 1} Z(X)/\eta(|\Lambda_{i_1, \ldots, i_l}| + 1))
\]

in \( \overline{DM}(S, \mathbb{Z}) \).
(5) If $k$ and $n$ are odd, we have
\[ Th(p^*L \otimes O(1)) \cong \bigoplus_{\Lambda \text{ even}} Th(L \otimes det(\xi)^\vee)(([\Lambda]) \oplus \bigoplus_{\Lambda \text{ irred. not full}, i > 1} \mathbb{Z}(X)/\eta([\Lambda_{i_1}, \ldots, i_n] + 1)) \]
\[ \text{in } \overline{DM}(S, \mathbb{Z}). \]

(6) If $k$ is odd, $n$ is even and $e(E) = 0 \in C\bar{H}^n(X, det(E)^\vee)$, there is an isomorphism
\[ Th(p^*L) \cong \bigoplus_{\Lambda \in \mathcal{R}T} Th(L \otimes det(\xi)^\vee)(([\Lambda]) \oplus \bigoplus_{\Lambda = T} Th(L)(([\Lambda])) \oplus \bigoplus_{\Lambda \text{ irred. not full}, i > 1} \mathbb{Z}(X)/\eta([\Lambda_{i_1}, \ldots, i_n] + 1)) \]
\[ \text{in } \overline{DM}(S, \mathbb{Z}), \text{ where } T \text{ is the longest hook.} \]

Here $\overline{DM}^{eff}(S, \mathbb{R})$ (resp. $\overline{DM}(S, \mathbb{R})$) is the category of effective (resp. stable) MW-motives over $S$ with coefficients in $\mathbb{R}$ (see Section 2 of [Yan21] or [BDFO20, §3]) and $(ii) := (i)[2i]$.

Last but not the least, we show that the complete flag also splits as an MW-motive. Moreover, we have a complete flag bundle theorem if all Pontryagin classes and the Euler class vanish in Witt cohomology (see Theorem 5.8). For any vector bundle $\xi$, $Fl(\xi)$ parametrizes complete flags
\[ 0 \leq \xi_1 \leq \cdots \leq \xi_n = \xi \]
where $rk(\xi_i) = i$ and successive quotients are line bundles.

**Theorem 1.4.** For any $\mathcal{L} \in Pic(Gr(1, \ldots, n))$, $Th(\mathcal{L})$ splits as an MW-motive. Moreover, we have

1. If $n$ is odd, define $deg(a) = 4a - 1$. We have
\[ Th(\mathcal{L}) = \begin{cases} \bigoplus_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} \bigoplus_{a_1 < \cdots < a_i \leq \lfloor \frac{n}{2} \rfloor} \mathbb{Z}[1 + \sum a_k(a)] & \mathcal{L} = 0 \\ 0 & \text{else} \end{cases} \]
\[ \text{in } \overline{DM}_\eta. \]

2. If $n$ is even, define $deg(a) = \begin{cases} 4a - 1 & 1 \leq a \leq \frac{n}{2} - 1 \\ n - 1 & a = \frac{n}{2} \end{cases} \cdot$

We have
\[ Th(\mathcal{L}) = \begin{cases} \bigoplus_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} \bigoplus_{a_1 < \cdots < a_i \leq \lfloor \frac{n}{2} \rfloor} \mathbb{Z}[1 + \sum a_k(a)] & \mathcal{L} = 0 \\ 0 & \text{else} \end{cases} \]
\[ \text{in } \overline{DM}_\eta. \]

So $Th(\mathcal{L})$ are mutually isomorphic in $\overline{DM}^{eff}(pt, \mathbb{Z})$ if $\mathcal{L} \neq 0$. Denote by $F$ this common object.

Suppose that $X \in Sm/S$ is quasi-projective, $\mathcal{M} \in Pic(Fl(\xi))$ and that $\xi$ is a vector bundle of rank $n$ on $X$. Denote by $p : Fl(\xi) \rightarrow X$ the structure map. We have
\[ Th(\mathcal{M}) = \begin{cases} Th(\mathcal{L}) \otimes \mathbb{Z}(Gr(1, \ldots, n)) & p_!(\xi), e(\xi) = 0 \in H^*(X, W(-)), \forall i, \mathcal{M} = p^*\mathcal{L} \\ R(X) \otimes F & \mathcal{M} \notin Pic(X) \end{cases} \]
\[ \text{in } \overline{DM}(S, \mathbb{Z}). \]

We can write down the generators of $H^*(Gr(1, \ldots, n), W)$ in the aid of Theorem 1.4 (see Remark 5.8).

**Proposition 1.5.** Suppose that $1 \leq a \leq \lfloor \frac{n}{2} \rfloor$. If $n$ is odd or $n$ is even and $a < \frac{n}{2}$, define
\[ T^{\frac{a}{2}} = h_{2a}(x_1, \ldots, x_{n-2a})h_{2a-1}(x_1, \ldots, x_{n-2a+1}) + u(x_1, \ldots, x_{n-2a}) \]
where $u$ satisfies $Sq^2(u) = h_{2a}(x_1, \ldots, x_{n-2a})^2$. If $n$ is even, define
\[ T^{\frac{n}{2}} = x_1^{n-1}. \]

Then we have
\[ E^*(Fl(k^{[n]})) = \wedge [T_n]. \]

Here the $\{x_i\}$ are the first Chern classes of quotient line bundles of a complete flag and $h_i(x_1, \ldots, x_j)$ is the complete homogenous polynomial of degree $i$. Our computation is compatible with the Cartan model (see [Ter11, Theorem 1]) and [BF12].

There has been many works related to the Witt or Chow-Witt groups of Grassmannians and complete flags, for example [BC08, BF12, Wen18a and Wen18b]. The article [BC08] was devoted to Witt groups and defined the notion of even Young tableaux. In [Wen18a], the Chow-Witt ring of Grassmannians was computed in terms of generators and relations as an algebra. In [BF12], they pointed out some cases where the twisted Witt group of the complete flag of some linear algebraic group vanishes, but they did
not compute the Witt group without twist. An explanation of these results in terms of MW-motives is completely new, needless to say any characterization of their fiber bundles.

For convenience, we give a list of frequently used notations in this paper:

| Symbol | Notation |
|--------|----------|
| $p$    | $Spec(k)$ |
| $Sm/S$ | Smooth and separated schemes over $S$ |
| $R(X)$ | Motive of $X$ with coefficients in $R$ |
| $\text{Th}(\mathcal{E})$ | Thom space of $\mathcal{E}$ |
| $\pi$  | The reduction modulo 2 map $CH \to Ch$ |
| $h$    | The hyperbolic quadratic form |
| $\eta$ | The Hopf element |
| $A/\eta$ | $A \otimes \text{cone}(\eta)$ |
| $|D|$ | Cardinality of the set $D$ |
| $|\Lambda|$ | Number of boxes in the Young tableau $\Lambda$ |
| $R$    | The orientation class of $G_r(k, n)$ |
| $\lfloor \cdot, \cdot \rfloor$ | $\text{Hom}_{SM(k)}(\cdot, \cdot)$ |
| $\lfloor \cdot, \cdot | K$ | $\text{Hom}_{DM}_{k,l}(\cdot, \cdot | K)$ |
| $h_i(x_1, \ldots, x_n)$ | The complete homogeneous polynomial of degree $i$ |
| $c_i(x_1, \ldots, x_n)$ | The elementary homogeneous polynomial of degree $i$ |

Throughout, our base field $k$ is infinite perfect with $\text{char}(k) \not= 2$.

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## 2. Split Milnor-Witt motives

In this section, we discuss the splitting pattern in MW-motives. Let us briefly recall the language of four motivic theories established in [Yan21b §2] and [BCDFØ20 §3]. Let $Sm/S$ be the category of smooth and separated schemes over $S$ and $K_*$ be one of the homotopy modules $K_*^{MW}, K_*^M, K_*^W, K_*^M/2$. For every $X \in Sm/k$, if $T \subset X$ is a closed set and $n \in \mathbb{N}$, define

$$C^n_{RS,T}(X; K_m; \mathcal{L}) = \bigoplus_{y \in X^{(n)} \cap T} K_{m-n}(k(y), \Lambda^*_y \otimes_{k(y)} \mathcal{L}_y),$$

where $X^{(n)}$ means the points of codimension $n$ in $X$. Then $C^*_{RS,T}(X; K_m; \mathcal{L})$ form a complex (see [Mor12]), which is called the Rost-Schmid complex with support on $T$. Define (see [BCDFØ20] and [Yan21b])

$$KCH^n(X, \mathcal{L}) = H^n(C^*_{RS,T}(X; K_m; \mathcal{L})).$$

Thus we see that $KCH = \overline{CH}, CH, Ch$ when $K_* = K_*^{MW}, K_*^M, K_*^W, K_*^M/2$ respectively.

Suppose that $\mathcal{E}$ is a vector bundle of rank $n$ on $X$. It admits the Euler class $e(E) \in C^nH(X, \text{det}(\mathcal{E})^\vee)$ (see [Fas08 Définition 13.2.1]) and Pontryagin classes (or Borel classes) $p_i(\mathcal{E}) \in C^nH(X)$ (see [DF20 Definition 2.2.6]). So do the other three cohomology theories.

**Proposition 2.1.** Let $X \in Sm/k$ and $\mathcal{E}$ be a vector bundle over $X$. Then for every odd $i$, we have

$$p_i(\mathcal{E}) = 0 \in H^{2i}(X, W), \quad p_i(\mathcal{E}) = 0 \in E^{2i}(X).$$

**Proof.** Denote by $p : X \times \mathbb{G}_m \to X$ the projection. By [DF20 Proposition 2.2.8], we have

$$p^*(p_i(\mathcal{E})) \cdot < t > = p^*(p_i(\mathcal{E}))$$

where $t$ is the parameter in $\mathbb{G}_m$. But

$$H^{2i}(X \times \mathbb{G}_m, W) = H^{2i}(X, W) \oplus H^{2i}(X, W) \cdot (< t > = < 1 >)$$

by [BCDFØ20 Lemma 3.1.8]. Then we have

$$(p_i(\mathcal{E}), 0) = p^*(p_i(\mathcal{E})) = p^*(p_i(\mathcal{E})) \cdot < t >= (p_i(\mathcal{E}), 0) \cdot < t > = (p_i(\mathcal{E}), p_i(\mathcal{E})).$$

So $p_i(\mathcal{E}) = 0$. The second statement follows from [HW19 Remark 7.9].

By Jouanolou’s trick, there is a Whitney sum formula (see [DF20 2.2.4]) for even Pontryagin classes for vector bundles on quasi-projective smooth schemes. Namely, for any exact sequence of bundles

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

we have

$$\sum_i p_{2i}(\mathcal{E}_1)t^i \sum_j p_{2j}(\mathcal{E}_3)t^j = \sum_k p_{2k}(\mathcal{E}_2)t^k$$

where $t$ is an indeterminant.
For any $S \in Sm/k$ and $X, Y \in Sm/S$, define $\mathcal{S}(X, Y)$ to be the poset of closed subsets in $X \times_S Y$ such that each of its component is finite over $X$ and of dimension $\dim X$. Suppose that $R$ is a commutative ring. Let

$$KC_{rS}(X, Y, R) := \lim_{T} KCH^{\dim Y - \dim S}(X \times_S Y, \omega_{X \times \hat{Y}/X}) \otimes_{Z} R,$$

be the finite correspondences between $X$ and $Y$ over $S$ with coefficients in $R$, where $T \in \mathcal{S}(X, Y)$. Hence $KCor$ just means $\mathcal{C}or$, $Cor$ and $WCor$ (see \cite{BCDFO20} §3) when $K_* = K_M, K^n_M, K^n_W$, respectively. This produces an additive category $KCor$ whose objects are the same as $Sm/S$ and whose morphisms are defined above. There is a functor $Sm/S \longrightarrow KCor$ sending a morphism to its graph.

We say that an abelian Nisnevich sheaf over $Sm/S$ is a sheaf with $K$-transfers if it admits extra functoriality from $KCor$ to $Ab$. For any smooth scheme $X$, let $R(X)$ be the representable sheaf with $K$-transfers of $X$, which is called the motive of $X$ in $R$-coefficients. The Tate twist is denoted by $R(1)$.

Denote by $C_{K}(S, R)$ (resp. $D_{K}(S, R)$) the (resp. derived) category of cochain complexes of sheaves with $K$-transfers. Define the category of effective $K$-motives over $S$ with coefficients in $R$

$$DM^{eff}_{K}(S, R) = D_{K}(S, R)[\{R(X \times \hat{Y}) \rightarrow R(Y )\}^{-1}].$$

Furthermore, the category $DM_{K}(S, R)$ of stable $K$-motives is the homotopy category of $\mathcal{G}_m$-spectra of $C_{K}(S, R)$, with respect to stable $A^1$-equivalences. When $K_* = K^M$ (resp $K_* = K^n_M$), we will write $DM_{K}$ as $D\tilde{M}$ (resp. $DM$) conventionally. We have full faithful embedding of categories $DM^{eff}_{K}(pt, Z) \subseteq DM_{K}(pt, Z)$ and $DM^{eff}_{K}(pt, \hat{Z}) \subseteq D\tilde{M}_{K}(pt, \hat{Z})$ by cancellation (see \cite{Yan21b} Proposition 2.5).

There are functorialities of $DM^{eff}_{K}(S, R)$ (resp. $DM_{K}(S, R)$) with respect to $K, S$ and $R$ (see \cite{Yan21b} Proposition 2.1.2.2.3).

The stable $A^1$-homotopy category $SH(k)$ is defined to be the homotopy category of $\mathbb{P}^1$-spectra of Nisnevich simplicial sheaves, localized with respect to morphisms $X \times A^1 \longrightarrow X$. We set $[\_ , \_ ] = Hom_{SH(k)}(\_ , \_ )$ and $[\_ , \_ ]_{K} = Hom_{DM^{eff}_{K}}(\_ , \_ )$ for convenience (see \cite{Yan21b} §2). Suppose that $\varepsilon$ is a vector bundle on $X$, define $Th(\varepsilon) = R(\varepsilon)/R(\varepsilon^\times)$.

**Proposition 2.2.** Suppose that $X, Y \in Sm/k, Y$ is a closed subset of $X$ with $\text{codim}_XY = n$ and $\varepsilon$ is a vector bundle over $X$. We have a distinguished triangle

$$Th(det(\varepsilon)|_{X \backslash Y}) \longrightarrow Th(det(\varepsilon)) \longrightarrow Th(det(\varepsilon)|Y \otimes det(N_{Y/X})[n][2n] \longrightarrow [1]$$

in $D\tilde{M}(S, R)$.

**Proof.** We have an exact sequence

$$0 \longrightarrow Th(\varepsilon)|_{X \backslash Y} \longrightarrow Th(\varepsilon) \longrightarrow Th(N_{Y/X}) \longrightarrow 0$$

by the inclusions $Y \subseteq X \subseteq \varepsilon$. Then we apply [Yan21a] Proposition 2.6. \qed

Recall that the Hopf map $\mathbb{A}^2 \setminus 0 \longrightarrow \mathbb{P}^1$ $\{x,y\} \rightarrow [x:y]$ gives the Hopf element (see \cite{Yan21b} Definition 4.1) $\eta \in [\mathbb{Z}(1)[1], \mathbb{Z}]_{MW}$ up to a $\mathbb{P}^1$-suspension. We write $A/\eta = A \otimes \text{cone}(\eta)$ for any MW-motive $A$.

**Proposition 2.3.** For any $X \in Sm/k, \mathcal{L} \in Pic(X)$ and $n, m \in \mathbb{Z}$, we have

$$[Th(\mathcal{L}), H_{\mathcal{W}}Z(n)[m]] = H^{m-n-1}(X, K_{\mathcal{W}}^{+1}(\mathcal{L}))$$

$$[Th(\mathcal{L}), H_{\mathcal{W}}Z(n)[m]] = H^{m-n-1}(X, K_{\mathcal{W}}^{+1}(\mathcal{L}))$$

if $m \geq 2n - 1$.

**Proof.** The statement is an analogue of [BCDFO20] Remark 4.2.7. The proof of both equations are the same so we do the first one. The right hand side is $H^{m-n}(\mathcal{L}, K_{\mathcal{W}}^{+1})$ by the push-forward along the zero section of $\mathcal{L}$. We have the Postnikov spectral sequence (see [Mor03] Remark 4.3.9)

$$H^{p}_{X}(\mathcal{L}, \pi_{-q}(H_{W}Z)_{n}) \Rightarrow [Th(\mathcal{L}), H_{W}Z(n)[n + p + q]].$$

Then use that

$$H^{p}_{X}(\mathcal{L}, \pi_{-q}(H_{W}Z)_{n}) = 0$$

if $q > 0$ or $p \geq n$ and $q \neq 0$ (see [Yan21b] Proposition 4.5). \qed

**Proposition 2.4.** Suppose that $X \in Sm/k, n \in \mathbb{N}, 2CH^{n}(X) = 0$ and that $\mathcal{L} \in Pic(X)$. Define

$$\eta_{MW}^{n}(X, \mathcal{L}) = [Th(\mathcal{L}), Z/\eta(n+1)[2n+2]_{MW}].$$
We have a natural Cartesian square
\[
\begin{array}{ccc}
\eta_{MW}^n(X,\mathcal{L}) & \longrightarrow & CH^{n+1}(X) \\
\downarrow & & \downarrow \\
CH^n(X) & \xrightarrow{Sq^2 \circ p} & CH^{n+1}(X)
\end{array}
\]
where $Sq^2 = Sq^2 + c_1(\mathcal{L})U$ is defined by the composition (see \[AF15\] Theorem 3.4.1)
\[
CH^n(X) \longrightarrow H^{n+1}(X, \mathbb{P}^{n+1}(\mathcal{L})) \longrightarrow CH^{n+1}(X).
\]

Proof. The proof is essentially the same as in \[Yan21b\] Theorem 4.13, by using Proposition 2.3. □

If $2CH^*(X) = 0$ and $x \in CH^n(X,\mathcal{L})$, we will write $x = (a, b)$ if $(a, b)$ is the image of $x$ in $CH^n(X) \oplus CH^{n+1}(X)$.

Proposition 2.5. Suppose that $X \in Sm/k$ and $\mathcal{L},\mathcal{M} \in Pic(X)$. The ring structure of $\mathbb{Z}/\eta$ gives a product
\[
\eta_{MW}^n(X,\mathcal{L}) \times \eta_{MW}^n(X,\mathcal{M}) \longrightarrow \eta_{MW}^{n+m}(X,\mathcal{L} \otimes \mathcal{M})
\]
with the identity being $(1, 0) \in \eta_{MW}^n(pt)$. If $u = (a, b), v = (c, d)$ and $2CH^*(X) = 0$, we have
\[
(a, b) \cdot (c, d) = (ac, bc + ad).
\]

Proof. By \[Yan21b\] Proposition 5.4, the map $(\mathbb{Z}/\eta)^2 \rightarrow \mathbb{Z}/\eta$ gives the ring structure of $\mathbb{Z}/\eta$. Moreover, the map $DM(pt, Z) \rightarrow DM(pt, Z)$ preserves ring structures. So it suffices to look at the case in $DM(pt, Z)$, which is clear. □

Definition 2.6. For any $A, B \in DM^{eff}(pt, Z)$, we define
\[
[A, B]_{\eta} = \lim_{n \rightarrow +\infty} [A(n)[n], B]_{MW}
\]
where the limit is taken with respect to the Hopf map $\eta$. Define $\overline{DM}_{\eta}$ to be the category with same objects as $DM^{eff}(pt, Z)$ and morphisms as defined above. There is a canonical functor
\[
L : \overline{DM}^{eff}(pt, Z) \longrightarrow \overline{DM}_{\eta}.
\]

We will write $L(Z)$ as $Z$.

Proposition 2.7. The category $\overline{DM}_{\eta}$ is the Verdier localization of the category $\overline{DM}^{eff}(pt, Z)$ with respect to the morphisms $Z(X) \otimes \eta$ for every $X \in Sm/k$. Thus it is a triangulated category and the functor $L$ is monoidal and exact.

Proof. Suppose $\Sigma$ is the smallest thick triangulated subcategory (with arbitrary coproducts) of $\overline{DM}^{eff}(pt, Z)$ containing $Z(X) \otimes \eta$ for every $X \in Sm/k$. If
\[
X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]
\]
is a distinguished triangle in $\overline{DM}^{eff}(pt, Z)$ with $Z \in \Sigma$, we claim that $L(f)$ is an isomorphism. It suffices to show that $[T, f]_{\eta}$ is an isomorphism for any $T$. The functor $[-, -]_{\eta}$ also induces a long exact sequence with respect to the triangle. So it suffices to show that $T, Z]_{\eta} = 0$, which is reduced to show that $[T, Z(X)]_{\eta} = 0$. It is easy to check that the morphism $Z(X) \otimes \eta$ has the inverse $cId_{Z(X)[1]}$ (see \[BCDF020\] Lemma 2.0.7, §4 for the sign $c$) in $\overline{DM}_{\eta}$, so $L(Z(X) \otimes \eta)$ is an isomorphism. Hence we have proved the claim.

Then it is easy to show that the pair $(L, \overline{DM}_{\eta})$ is the universal one such that $L(f)$ is an isomorphism if $C(f) \in \Sigma$. The $\overline{DM}_{\eta}$ has a natural symmetric monoidal structure such that $L$ is symmetric monoidal. Hence we have done. (see \[Kra10\] Proposition 4.6.2) □

Hence we see that $L(Z(1)[1]) = L(Z)$ so there is no need to consider the Tate twist in $\overline{DM}_{\eta}$.

Proposition 2.8. For any $X \in Sm/k$, $\mathcal{L} \in Pic(X)$ and $m, n \in \mathbb{N}$, we have
\[
[Th(\mathcal{L}), Z(m)[m + n + 1]]_{\eta} = H^n(X, W(\mathcal{L})).
\]

Proof. We have the distinguished triangle
\[
H^n Z \longrightarrow H^n \mathcal{L} \longrightarrow H^n W Z \longrightarrow H^n Z[1]
\]
by \[Bac17\] Lemma 19. If $i > m$, we have
\[
[Th(\mathcal{L})[i][m + n + 1]]_{MW} = [\Sigma^i Th(\mathcal{L}) \otimes \mathbb{G}_m^i, H_W Z[n + 1]] = H^n(X, W(\mathcal{L}))
\]
by Proposition \[BCDF020\] and \[BCDF020\] Lemma 3.1.8, §2. □
Definition 2.9. We say that an element \( A \in \overline{DM}'(pt, \mathbb{Z}) \) splits if it is a finite direct sum of elements like \( \mathbb{Z}(i)[2i] \) or \( \mathbb{Z}/\eta(i)[2i] \) \( (i \in \mathbb{N}) \).

Definition 2.10. If the \( A \) splits, define its Witt weights \( \WW(A) \) to be the set
\[
\{i \in \mathbb{N} | \mathbb{Z}(i)[2i] \text{ is a direct summand of } A\}.
\]
We have
\[
\WW(A) = \{i \in \mathbb{N} | [\mathbb{Z}(i+1)[2i+2], A[1]]_{MW} \neq 0\}
\]
by \[Yan21b\] Proposition 5.4 hence it is well defined.

Hence we have for example
\[
\WW(\mathbb{Z}(\mathbb{P}^n)) = \begin{cases} 
\{0, n\} & \text{if } n \text{ is odd} \\
\{0\} & \text{if } n \text{ is even} 
\end{cases}
\]
by \[Yan21b\] Theorem 5.11.

Recall that in \[Bac17\], T. Bachmann defined the motivic cohomology spectra \( H\mathbb{Z}, H_\mu\mathbb{Z}, H_W\mathbb{Z}, H_\nu\mathbb{Z}/2 \) as the effective cover of the homotopy modules \( K_*^{MW}, K_*^M, K_*^W, K_*^{M/2} \), respectively. The readers may also refer to \[Yan21b\] §4.

Lemma 2.11. Suppose that \( A \in \overline{DM}'(pt, \mathbb{Z}) \) splits as
\[
A = \oplus_{i \in \mathbb{N}} \mathbb{Z}(i)[2i]^{\oplus w_i(A)} \oplus \oplus_{j \in \mathbb{N}} \mathbb{Z}/\eta(j)[2j]^{\oplus t_j(A)}.
\]
Then
\[
L(A) = \oplus_{i \in \mathbb{N}} \mathbb{Z}(i)^{\oplus w_i(A)}
\]
in \( \overline{DM}_\eta \). Suppose that \( \gamma^*: \overline{DM}' \to DM' \) is the functor defined in \[BCDFO20\] 3.2.4, §3 and that
\[
\gamma^*(A) = \oplus_{k \in \mathbb{N}} \mathbb{Z}(i)[2i]^{\oplus s_k(A)}.
\]
We have
\[
t_j(A) = \sum_{i \leq j} (-1)^i (s_{j-i}(A) - w_{j-i}(A)).
\]
Proof. The first statement follows from \( L(\mathbb{Z}/\eta) = 0 \) and \( L(\mathbb{Z}(1)[1]) = \mathbb{Z} \). The second statement follows from \( r^*\mathbb{Z}/\eta = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \).

Remark 2.12. Suppose that \( k = \mathbb{R} \) and that \( X \in Sm/k \) is cellular, we have
\[
[SX^X, H_W\mathbb{Z}(i)[2i]] = H^i(X, I^i) = H^i(X(\mathbb{R}), \mathbb{Z})
\]
by \[Yan21b\] Proposition 4.5 and \[HWXZ19\] Theorem 5.6. On the other hand, we have
\[
[Z/\eta(i)[2i], H_W\mathbb{Z}(j)[2j]] = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & \text{if } j - i = 1 \\
0 & \text{else}
\end{cases}
\]
\[
[\mathbb{I}(i)[2i], H_W\mathbb{Z}(j)[2j]] = \begin{cases} 
\mathbb{Z} & \text{if } j = i \\
0 & \text{else}
\end{cases}
\]
by \[Yan21b\] Proposition 5.3, 5.6 and the distinguished triangle (see \[Bac17\] Lemma 19)
\[
H_\mu\mathbb{Z} \to \overline{H}\mathbb{Z} \to H_W\mathbb{Z} \to \ldots [1].
\]
Hence we see that if \( Z(X) \) also splits as an MW-motive, then we have
\[
H^i(X(\mathbb{R}), \mathbb{Z})_{free} = Z^{\oplus w_i(Z(X))}
\]
\[
H^i(X(\mathbb{R}), \mathbb{Z})_{tor} = \mathbb{Z}/2\mathbb{Z}^{\oplus t_{i-1}(Z(X))}.
\]
Whereas
\[
CH^i(X) = H^{2i}(X(\mathbb{C}), \mathbb{Z}) = Z^{\oplus w_i(Z(X))} + \mathbb{Z}^{\oplus t_i(Z(X))} + \mathbb{Z}^{\oplus t_{i-1}(Z(X))}
\]
by \[HWXZ19\] Proposition 5.2.

Now we want to give a general method to detect splitting in MW-motives.

Proposition 2.13. Suppose \( f : X \to Y \) is a morphism in \( \overline{DM}'(pt, \mathbb{Z}) \). If both \( X \) and \( C(f) \) split as MW-motives, then \( f \) is injective in \( \overline{DM}'(pt, \mathbb{Z}) \) if and only if \( L(f) \) is injective.

Proof. This is because by \[Yan21b\] Proposition 5.4, we have an isomorphism
\[
L : [C(f), X[1]]_{MW} \to [C(f), X[1]]_{\eta}
\]
hence the injectivity of \( f \) is equivalent to that of \( L(f) \) under our setting.
Definition 2.14. Suppose that \( X \in \text{Sm}/k \), \( \mathcal{L} \in \text{Pic}(X) \) and that \( n \in \mathbb{Z} \). Define
\[
E^n(X, \mathcal{L}) = \frac{\text{Ker}(Sq^n_{\mathcal{L}})}{\text{Im}(Sq^n_{\mathcal{L}})}
\]
where the right hand side is an \( E_3 \)-term in the Pardon spectral sequence (see \cite[Theorem 2.2.1]{AF15}).

The ring structure of Chow groups makes \( \oplus_{n, \mathcal{L}} E^n(X, \mathcal{L}) \) an \( \mathbb{N} \times \text{Pic}(X)/2 \)-graded ring. Given a morphism \( f : X \to Y \), there is a pullback
\[
f^* : E^n(Y, \mathcal{L}) \to E^n(X, f^* \mathcal{L})
\]
by the naturality of Steenrod operations. Moreover, if \( f \) is projective with relative dimension \( c \), the pushforward of Chow groups gives a pushforward
\[
f_* : E^n(X, \omega_X \otimes f^* \mathcal{L}) \to E^{n-c}(Y, \omega_Y \otimes \mathcal{L}),
\]
which is well-defined by \cite[Theorem 4.5]{Para11}. For every vector bundle \( \mathcal{E} \), we have an Euler class \( e(\mathcal{E}) \in E^n(X, det(\mathcal{E})^\vee) \) and Pontryagin classes \( p_n(\mathcal{E}) \in E^{2n}(X) \) inherited by those of Chow groups.

Suppose \( X \in \text{Sm}/k \) and \( \mathcal{L} \in \text{Pic}(X) \). We have an exact sequence
\[
Ch^{n-1}(X) \xrightarrow{\beta} H^n(X, I^n(\mathcal{L})) \xrightarrow{\partial} H^n(X, W(\mathcal{L})) \to 0.
\]
The composite
\[
Ch^{n-1}(X) \to H^n(X, I^n(\mathcal{L})) \to \text{Ker}(Sq^n_{\mathcal{L}}) \to E^n(X, \mathcal{L})
\]
vanishes hence it induces a map
\[
\delta : H^n(X, W(\mathcal{L})) \otimes_{W(k)} \mathbb{Z}/2\mathbb{Z} \to E^n(X, \mathcal{L}).
\]
So we obtain a map
\[
\delta : H^n(X, W(\mathcal{L})) \otimes_{W(k)} \mathbb{Z}/2\mathbb{Z} \to E^n(X, \mathcal{L}).
\]

Theorem 2.15. Suppose that \( X \in \text{Sm}/k \), \( \mathcal{L} \in \text{Pic}(X) \) and that \( Th(\mathcal{L}) \) splits in \( \overline{\text{DM}}^{eff}(pt, \mathbb{Z}) \).

1. The \( H^*(X, W(\mathcal{L})) \) is a free \( W(k) \)-module. There is a natural isomorphism of \( \mathbb{Z}/2\mathbb{Z} \)-vector spaces
\[
\delta : H^*(X, W(\mathcal{L})) \otimes_{W(k)} \mathbb{Z}/2\mathbb{Z} \cong E^*(X, \mathcal{L}).
\]

2. The natural map \( \overline{CH}^*(X, \mathcal{L}) \to CH^*(X) \) yields a decomposition
\[
\overline{CH}^*(X, \mathcal{L}) \cong I(k) \cdot H^*(X, W(\mathcal{L})) \oplus \text{Ker}(Sq^2_{\mathcal{L}} \circ \pi)_*.
\]

Proof. (1) Suppose we have
\[
Th(\mathcal{L}) = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}((i)[2i]^{\text{even}}) \oplus \bigoplus_{j \in \mathbb{N}} \mathbb{Z}/\eta(j)[2j]^{\text{odd}}.
\]
The \( H^*(X, W(\mathcal{L})) \) is clear free by applying the functor \( L \) on \( Th(\mathcal{L}) \).

For the second statement, it suffices to prove that the map
\[
\rho : H^n(X, I^n(\mathcal{L})) \to \text{Ker}(Sq^n_{\mathcal{L}})
\]
is surjective and count the dimension on both sides of \( \delta \). The surjectivity of \( \rho \) is equivalent to \( \text{Ker}(\beta) = \text{Ker}(Sq^2_{\mathcal{L}}) \). We have a commutative diagram with exact rows
\[
\begin{array}{c}
\begin{array}{c}
CH^{n-1}(X) \oplus CH^n(X) \to \eta_{MW}(X, \mathcal{L}) \to Ch^{n-1}(X) \to 0 \\
CH^n(X) \to \overline{CH}^n(X, \mathcal{L}) \to H^n(X, I(\mathcal{L})) \to 0 \\
CH^n(X) \to CH^n(X) \to CH^n(X) \to 0
\end{array}
\end{array}
\]
induced by the commutative diagram of distinguished triangles
\[
\begin{array}{c}
\begin{array}{c}
H_\mu \mathbb{Z}/\eta \xrightarrow{h} H\hat{\mathbb{Z}}/\eta \xrightarrow{} H_W \mathbb{Z}/\eta \\
H_\mu \mathbb{Z} \wedge \mathbb{P}^1 \xrightarrow{h} H\hat{\mathbb{Z}} \wedge \mathbb{P}^1 \xrightarrow{} H_W \mathbb{Z} \wedge \mathbb{P}^1 \\
H_\mu \mathbb{Z} \wedge \mathbb{P}^1 \xrightarrow{2} H_\mu \mathbb{Z} \wedge \mathbb{P}^1 \xrightarrow{} H_\mu \mathbb{Z}/2 \wedge \mathbb{P}^1 \\
\end{array}
\end{array}
\]
We have \( \beta' = \beta \) by the proof of [Yan21b, Theorem 4.13]. Since \( Th(\mathcal{L}) \) splits and \( \mathbb{P}^2 = \mathbb{Z}/\eta(1)[2] \), we see that for the surjectivity of \( \rho \), it suffices to prove the cases when \( X = pt \) and \( X = \mathbb{P}^2 \) \( (\mathcal{L} = \mathcal{O}_X) \). If \( X = pt \), both \( \beta \) and \( Sq^2 \) vanish, so there is nothing to prove. If \( X = \mathbb{P}^2 \), only the case
\[
\beta : CH^1(\mathbb{P}^2) \to H^2(\mathbb{P}^2, \mathcal{I}^2)
\]
is interesting. But \( Ker(Sq^2) = 0 \) in this case, which implies that \( Ker(\beta) = 0 \) also. So we have proven the surjectivity of \( \rho \).

Now we count the dimension of both sides of \( \delta \). Define \( \varphi_n \) to the composite
\[
\eta_{\text{Min}}^{-1}(X, \mathcal{L}) \to \overline{CH}^n(X, \mathcal{L}) \to CH^n(X)
\]
for any \( n \in \mathbb{N} \). We have
\[
Im(\pi \circ \varphi_n) = Im(Sq^2)_{\mathcal{L}} \quad \text{Ker}(\varphi_{n+1}) = Ker(Sq^2 \circ \pi)_{\mathcal{L}}
\]
by Proposition 2.4. Since \( Th(\mathcal{L}) \) splits, it suffices to consider the composite
\[
\psi_n : [A, H\tilde{\mathcal{L}}/\eta((n))] \to [A, H\tilde{\mathcal{L}}]/((n+1))] \to [A, H\tilde{\mathcal{L}}]/((n+1))
\]
where \( A = \mathbb{Z}/((i)), \mathbb{Z}/\eta((i)) \) and \( i \in \mathbb{N} \).

Now we apply the computation in [Yan21b Proposition 5.6]. Set \( [A, H\tilde{\mathcal{L}}/\eta((n))] = F \) and \( [A, H\tilde{\mathcal{L}}]/((n+1))] = G \).

Suppose \( A = \mathbb{Z}/((i)) \). If \( i - n > 1 \) or \( i < n \), \( F = 0 \) so \( \psi_n = 0 \) and \( Ker(\psi_n) = 0 \). If \( i = n \), we have \( G = 0 \) and \( F = \mathbb{Z} / \mathbb{Z} \) so \( \psi_n = 0 \) and \( Ker(\psi_n) = 0 \). If \( i = n + 1 \), we have \( F = 2\mathbb{Z} \) so \( \pi \circ \psi_n = 0 \) and \( Ker(\psi_n) = 0 \).

Suppose \( A = \mathbb{Z}/\eta((i)) \). If \( |i - n| > 1 \), we have \( F = 0 \) so \( \psi_n = 0 \) and \( Ker(\psi_n) = 0 \). If \( i = n + 1 \), we have \( F = 2\mathbb{Z} \) and \( G = \mathbb{Z} \), so \( Im(\pi \circ \psi_n) = \mathbb{Z}/2\mathbb{Z} \) and \( Ker(\psi_n) = 2\mathbb{Z} \). If \( i = n - 1 \), we have \( F = \mathbb{Z} \) and \( G = 0 \) so \( \psi_n = 0 \) and \( Ker(\psi_n) = 0 \).

Summarize, we have
\[
Im(Sq^2)_{\mathcal{L}} = \mathbb{Z}/2\mathbb{Z} \quad Ker(Sq^2)_{\mathcal{L}} = \mathbb{Z}/2\mathbb{Z}^{\oplus w_n+f_n}.
\]
Hence the result follows.

(2) By the Cartesian square in [Wen18a 2.3], the \( \overline{CH}^n(X, \mathcal{L}) \to Ker(Sq^2 \circ \pi)_{\mathcal{L}} \) is split surjective and its kernel is isomorphic to the kernel of \( \rho : H^n(X, \mathcal{L}) \to CH^n(X) \). We have an exact sequence
\[
Ch^{n-1}(X) \to H^n(X, \mathcal{I}^n(\mathcal{L})) \xrightarrow{\varphi} H^n(X, \mathcal{W}(\mathcal{L})) \to 0.
\]
Since \( H^n(X, \mathcal{W}(\mathcal{L})) \) is free, \( \varphi \) has a section \( s \), which satisfies \( Im(\rho \circ s) \subseteq Ker(Sq^2)_{\mathcal{L}} \). So we obtain a map
\[
\overline{\varphi}: H^n(X, \mathcal{W}(\mathcal{L})) \to E^n(X, \mathcal{L})
\]
which is equal to \( \Delta \) given before. Now given \( x \in Ker(\rho) \), we have
\[
(\rho \circ s \circ \varphi)(x) \in Im(Sq^2)_{\mathcal{L}}.
\]
But by (1), \( \delta \) is an isomorphism. So we see that
\[
Im(\rho \circ s) \cap Im(Sq^2)_{\mathcal{L}} = 0
\]
and the sequence
\[
0 \to I(k) \cdot H^n(X, \mathcal{W}(\mathcal{L})) \to H^n(X, \mathcal{W}(\mathcal{L})) \xrightarrow{\rho \circ s} Ker(Sq^2)_{\mathcal{L}}
\]
is exact. Hence we have \( \rho \circ s \circ \varphi = 0 \) and \( \varphi(x) \in I(k) \cdot H^n(X, \mathcal{W}(\mathcal{L})) \). On the other hand,
\[
I(k \cdot H^n(X, \mathcal{W}(\mathcal{L})) \subseteq Ker(\rho)
\]
via \( s \). So we have proved that
\[
Ker(\rho) = I(k) \cdot H^n(X, \mathcal{W}(\mathcal{L})).
\]

\[\square\]

**Corollary 2.16.** In the context above, we have Cartesian squares
\[
\begin{array}{ccc}
\overline{CH}^1(X) & \longrightarrow & Ker(Sq^2 \circ \pi), \\
\downarrow & & \\
H^1(X, \mathcal{I}^n(\mathcal{L})) & \longrightarrow & Ker(Sq^2)_{\mathcal{L}}i, \\
\downarrow & & \\
H^1(X, \mathcal{W}(\mathcal{L})) & \longrightarrow & E^n(X, \mathcal{L})
\end{array}
\]
for every $i \in \mathbb{N}$.

So we see that the (twisted) Chow-Witt groups of schemes with splitting MW-motives are completely determined by its Chow groups and the vector space $E^*(-,-)$.

**Proposition 2.17.** Suppose that $X \in \text{Sm}/k$, $\mathcal{L} \in \text{Pic}(X)$ and that $\text{Th}(\mathcal{L})$ splits in $\tilde{D}_{\text{eff}}M^e(k,\mathbb{Z})$. Suppose $\lambda$ (resp. $\mu$) is a generator of $[\mathbb{Z}/\eta_{\mathbb{Z}}(1)[2]]_{\text{MW}} = \mathbb{Z}$ (resp. $[\mathbb{Z}/\eta_{\mathbb{Z}}][22]$). We have maps

$$
\partial : \eta_{\text{MW}}^n(X, \mathcal{L}) \rightarrow \widetilde{CH}^n(X, \mathcal{L})
$$

$$
h : \eta_{\text{MW}}^n(X, \mathcal{L}) \rightarrow \widetilde{CH}^{n-1}(X, \mathcal{L})
$$

induced by $\lambda$ and $\mu$, respectively. Denote the natural map $\widetilde{CH}^n(X, \mathcal{L}) \rightarrow CH^n(X)$ by $\gamma$.

Given a morphism in $\tilde{D}_{\text{eff}}M^e(k,\mathbb{Z})$

$$
f = \{s_i, t_j\} : \text{Th}(\mathcal{L}) \rightarrow \text{Th}(\mathcal{L}^\vee) \cong \oplus_i \mathbb{Z}(a_i) \oplus \oplus_j \mathbb{Z}/\eta[2b_j],
$$

the following statements are equivalent:

1. The $f$ is an isomorphism;
2. For every $n \in \mathbb{N}$, we have

$$
\widetilde{CH}^n(X, \mathcal{L}) = \bigoplus_{a_k=n+1} GW(k) \cdot s_k \oplus \bigoplus_{b_k=n+1} \mathbb{Z} \cdot h(t_k) \oplus \bigoplus_{b_k=n} \mathbb{Z} \cdot \partial(t_k);
$$

3. For every $n \in \mathbb{N}$, we have

$$
\text{Ker}(\text{Sq}^2_{\mathcal{L}} \circ \pi)_n = \bigoplus_{a_k=n+1} \mathbb{Z} \cdot \gamma(s_k) \oplus \bigoplus_{b_k=n+1} \mathbb{Z} \cdot \gamma(h(t_k)) \oplus \bigoplus_{b_k=n} \mathbb{Z} \cdot \gamma(\partial(t_k))
$$

$$
H^n(X, W(\mathcal{L})) = \bigoplus_{a_k=n+1} W(k) \cdot w(s_k)
$$

where $w$ is the composite

$$
\widetilde{CH}^n(X, \mathcal{L}) \rightarrow H^n(X, I^*(\mathcal{L})) \rightarrow H^n(X, W(\mathcal{L})).
$$

**Proof.** We have maps

$$
h_1 : \eta_{\text{MW}}^n(X, \mathcal{L}) \rightarrow \eta_{\text{MW}}^n(X, \mathcal{L})
$$

$$
h_2 : \eta_{\text{MW}}^n(X, \mathcal{L}) \rightarrow \eta_{\text{MW}}^{n-1}(X, \mathcal{L})
$$

$$
p : \eta_{\text{MW}}^n(X, \mathcal{L}) \rightarrow \eta_{\text{MW}}^n(X, \mathcal{L})
$$

$$
g_1 : \widetilde{CH}^n(X, \mathcal{L}) \rightarrow \eta_{\text{MW}}^n(X, \mathcal{L})
$$

Here the maps come from the following. The $g_1$ (resp. $h_2$) is induced by the generator in $[\mathbb{Z}, \mathbb{Z}/\eta]_{\text{MW}}$ (resp. $[\mathbb{Z}/\eta(1)[2], \mathbb{Z}/\eta]_{\text{MW}}$). Moreover,

$$
p = g_1 \circ \partial, h_1 = g_1 \circ h.
$$

(1) $\Rightarrow$ (2): If $f$ is an isomorphism, applying $[-, \mathbb{Z}(n+1)[2n+2]]_{\text{MW}}$ we obtain an isomorphism

$$
\bigoplus_k [\mathbb{Z}(a_k), \mathbb{Z}(n+1)]_{\text{MW}} \oplus \bigoplus_k [\mathbb{Z}/\eta(b_k), \mathbb{Z}(n+1)]_{\text{MW}} \cong \widetilde{CH}^n(X, \mathcal{L}).
$$

So the statement follows from [Yan21b Proposition 5.6].

(2) $\Rightarrow$ (1): Suppose $\widetilde{CH}^n(X, \mathcal{L})$ has the required decomposition for every $n$. Then $[f, \mathbb{Z}(n)[2n]]_{\text{MW}}$ is an isomorphism for every $n$. So it suffices to prove that $[f, \mathbb{Z}/\eta(n)[2n]]_{\text{MW}}$ is an isomorphism for every $n$, which induces an endomorphism of a free abelian group of finite rank by loc. cit.. Hence it suffices to prove that it is surjective.

Denote by $\pi$ the reduction modulo 2 map as before. By Theorem 2.15, we have an exact sequence

$$
0 \rightarrow \text{Ker}(\text{Sq}_2^2 \circ \pi)_{n-1} \rightarrow \eta_{\text{MW}}^n(X, \mathcal{L}) \rightarrow \pi^{-1}(\text{Im}(\text{Sq}_2^2)_n) \rightarrow 0
$$

and the elements

$$
\{\gamma(s_k)\}_{a_k=n} \cup \{\gamma(h(t_k))\}_{b_k=n} \cup \{\gamma(\partial(t_k))\}_{b_k=n-1}
$$

generate $\text{Ker}(\text{Sq}_2^2 \circ \pi)_{n-1}$. Suppose $t_k = (x_k, y_k)$. We have

$$
\gamma(h(t_k)) = 2x_k, \quad \gamma(\partial(t_k)) = y_k.
$$

On the other hand, we have

$$
\gamma_1(s_k) = (\gamma(s_k), 0), \quad \gamma_1(h(t_k)) = (2x_k, 0), \quad \gamma_1(\partial(t_k)) = (y_k, 0),
$$

which shows that $\text{Ker}(\text{Sq}_2^2 \circ \pi)_{n-1} \subseteq \text{Im}([f, \mathbb{Z}/\eta(n)[2n]]_{\text{MW}})$ for every $n$. 

Again by the proof of Theorem 2.15 the $\text{Im}(Sq^2_{\mathbb{Z}})_{n}$ is freely generated by $\pi(\{\gamma(\partial(t_k))\}_{b_k=n-1})$ for every $n$. So suppose we have $(x, y) \in \eta^{-1}_{MW}(X, \mathcal{L})$. We could find $(x_1, y_1) \in Z(t_k)_{b_k=n-1}$ such that $\gamma(x - y) = 0$, so $y - y_1 = 2y_2$. Then there is a $(u, v) \in Z(t_k)_{b_k=n}$ such that $\gamma(v) = Sq^2_{\mathbb{Z}}(y_2)$. Hence $(x, y) - (x_1, y_1) - h_2(u, v) = (x - x_1, 2y_2 - 2u)$ and $y_2 - u \in \text{Ker}(Sq^2_{\mathbb{Z}} \circ \gamma)$. Then $(x - x_1, 0), (2y_2 - 2u) = h_2(y_2 - u, 0) \in \text{Im}([f, Z/\eta(n)[2n]_{MW}])$ so we have proved that $(x, y) \in \text{Im}([f, Z/\eta(n)[2n]_{MW}])$.

(2) $\Rightarrow$ (3): The $w$ is surjective because it is the composite of surjective maps. In the proof of [Yan21b, Theorem 4.13], we showed that there is a commutative diagram

$$
\eta^{-1}_{MW}(X, \mathcal{L}) \xrightarrow{\partial} \widetilde{CH}^n(X, \mathcal{L})
$$

where the upper horizontal arrow is induced by $\partial$ and the lower horizontal arrow is the Bockstein map. So $w(\partial(t_k)) = 0$. The commutative diagram

$$
\eta^{-1}_{MW}(X, \mathcal{L}) \xrightarrow{\eta} H^n(X, \mathcal{L})
$$

shows that $w(\eta(t_k)) = 0$. So $\{w(s_k)\}_{a_k=n+1}$ freely generate $H^n(X, \mathcal{W}(\mathcal{L}))$. On the other hand, by Theorem 2.15 the elements given generate $\text{Ker}(Sq^2_{\mathbb{Z}} \circ \gamma)_n$ and the kernel of $\gamma$ is $\bigoplus_{a_k=n+1} I(k) \cdot s_k$. So they freely generate $\text{Ker}(Sq^2_{\mathbb{Z}} \circ \gamma)_n$.

(3) $\Rightarrow$ (2): Suppose $x \in \widetilde{CH}^n(X, \mathcal{L})$, we could find a $y \in Z[s_k, h(t_k), \partial(t_k)]$ such that $\gamma(x - y) = 0$. Then $x - y \in I(k)s_k$ by Theorem 2.15. Suppose $\sum e_k \cdot s_k + \sum h(t_k) + \sum m_k \cdot h(t_k) = 0$, where $e_k \in GW(k), n_k, m_k \in \mathbb{Z}$. Then applying $\gamma$, we find $n_k = m_k = 0$ and $e_k \in I(k)$ by the conditions. But $\sum e_k \cdot w(s_k) = 0$ implies that $e_k \in I(k)$ as we have done. 

**Proposition 2.18.** In the context above, the following sequence

$$
\eta^{-1}_{MW}(X, \mathcal{L}) \xrightarrow{\partial} \widetilde{CH}^n(X, \mathcal{L}) \rightarrow H^n(X, \mathcal{W}(\mathcal{L}))
$$

is exact. Moreover, we have

$$
\text{Im}(\partial) = \bigoplus_{a_k=n+1} \mathbb{Z} \cdot h(t_k) \oplus \bigoplus_{b_k=n+1} \mathbb{Z} \cdot \partial(t_k).
$$

**Proof.** We have a commutative diagram

$$
\widetilde{CH}^n(X, \mathcal{L}) \xrightarrow{\eta} H^{2n+1,n}(Th(\mathcal{L}), Z)
$$

where $\text{Ker}(\eta) = \text{Im}(\partial)$ and $w$ is an isomorphism by Proposition 2.3 and $H^{2n+1,n}(Th(\mathcal{L}), Z) = H^n(X, K_{MW}^{n-1}(\mathcal{L})) = H^n(X, \mathcal{W}(\mathcal{L}))$.

The statements then follow from $\text{Ker}(\eta') = \text{Im}(\beta)$ and Proposition 2.17. 

3. **Geometry of Grassmannians and Complete Flags**

Let us state a collection of results on the geometry of Grassmannians. Suppose $V$ is a vector space of dimension $n$ with a flag

$$
0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V
$$

where $\text{dim} V_i = i$. Denote by $\mathcal{U}_{k,n}$ (resp. $\mathcal{U}_{k,n}^c$) the (resp. complement) tautological bundle of $Gr(k, n)$. For a morphism $f : X \rightarrow Y$, we denote by $\Omega_f$ (resp. $N_f$) the relative cotangent bundle (resp. normal bundle), when it makes sense.

**Proposition 3.1.** We have closed imbeddings

$$
i : Gr(k, n-1) \rightarrow Gr(k, n) \hspace{1cm} j : Gr(k-1, n-1) \rightarrow Gr(k, n)
$$

$$
\Lambda \rightarrow \Lambda \hspace{1cm} \Lambda \rightarrow \Lambda + c_n
$$
with \( \det(N) = O_{Gr(k,n-1)}(1) = det(\mathcal{U}_{k,n-1}) \), \( det(N) = O_{Gr(k-1,n-1)}(1) \) and \( Gr(k,n-1) \cap Gr(k-1,n-1) = \emptyset \). Suppose \( p : V \to V_{n-1} \) is a projection. The morphism

\[
\begin{align*}
f : & \quad Gr(k,n) \setminus Gr(k-1,n-1) \to Gr(k,n-1) \\
& \quad \Lambda \to p(\Lambda)
\end{align*}
\]

(resp. 
\[
\begin{align*}
g : & \quad Gr(k,n) \setminus Gr(k-1,n-1) \to Gr(k-1,n-1) \\
& \quad \Lambda \to \Lambda \cap V_{n-1}
\end{align*}
\]

corresponds to \( \mathcal{U}_{k,n-1} \) (resp. \( \mathcal{U}_{k-1,n-1}^\perp \)) which is factored through by \( i \) (resp \( j \)) as the zero section. Moreover, we have

\[
i^*\mathcal{U}_{k,n} = \mathcal{U}_{k,n-1}, \quad i^*\mathcal{U}_{k,n}^\perp = \mathcal{U}_{k-1,n-1}^\perp \oplus O_{Gr(k,n-1)},
\]

\[
j^*\mathcal{U}_{k,n} = \mathcal{U}_{k-1,n-1} \oplus O_{Gr(k-1,n-1)}, \quad j^*\mathcal{U}_{k,n}^\perp = \mathcal{U}_{k-1,n-1}^\perp.
\]

**Proof.** Obvious. \( \square \)

**Proposition 3.2.** Let \( p : V_n \to V_{n-2} \) be the projection. Define

\( S = \{ \Lambda \in Gr(k,n) | \Lambda \cap ker(p) \neq 0 \} \).

We have imbeddings

\[
\begin{align*}
Gr(k,n-2) & \to Gr(k,n) & Gr(k-2,n-2) & \to S \\
\Lambda & \to \Lambda & \Lambda & \to \Lambda + ker(p)
\end{align*}
\]

The map

\[
\begin{align*}
Gr(k,n) \setminus S & \to Gr(k,n-2) \\
\Lambda & \to p(\Lambda)
\end{align*}
\]

(resp. 
\[
\begin{align*}
S \setminus Gr(k-2,n-2) & \to Gr(1,2) \times Gr(k-1,n-2) \\
\Lambda & \to (\Lambda \cap ker(p), p(\Lambda))
\end{align*}
\]

corresponds to the vector bundle \( \mathcal{U}_{k,n-2} \oplus \mathcal{U}_{k,n-2} \) (resp \( p_2 \mathcal{U}_{k-1,n-2} \)) where \( p_2 : Gr(1,2) \times Gr(k-1,n-2) \to Gr(k-1,n-2) \) is the projection.

**Proof.** Note that

\[
S \setminus Gr(k-2,n-2) = \{ V \in Gr(k,n) | \dim(V \cap ker(p)) = 1 \}.
\]

Suppose that \( \{t_1, t_2\} \) is a basis of \( ker(p) \), \( T = span(t_1) \) and that \( \Lambda \in Gr(k-1,n-2) \). We have an identification

\[
\begin{align*}
\Lambda^* & \to \{ V \in S \setminus Gr(k-2,n-2) | V \cap ker(p) = T, p(V) = \Lambda \} \\
f & \to \{ x + f(x)t_2 | x \in \Lambda \} + T
\end{align*}
\]

which is independent of the choice of \( t_2 \). So \( S \setminus Gr(k-2,n-2) = p_2 \mathcal{U}_{k-1,n-2} \). Similarly, suppose \( \Lambda \in Gr(k,n-2) \). The identification

\[
\begin{align*}
\Lambda^* \oplus \Lambda^* & \to \{ V \in Gr(k,n) \setminus S | p(V) = \Lambda \} \\
(f,g) & \to \{ z + f(z)t_1 + g(z)t_2 | z \in \Lambda \}
\end{align*}
\]

shows that \( Gr(k,n) \setminus S = \mathcal{U}_{k,n-2} \oplus \mathcal{U}_{k,n-2} \). \( \square \)

It turns out that \( E^*(-, -) \) admits a projective bundle theorem as in [Nen09].

**Proposition 3.3.** Suppose that \( X \in Sm/k, \mathcal{L} \in Pic(X) \) and that \( \mathcal{E} \) is a vector bundle of rank \( n \) on \( X \). Denote by \( p : P(\mathcal{E}) \to X \) the structure map.

1. If \( n \) is odd, we have

\[
E^i(P(\mathcal{E}), p^*\mathcal{L}) \cong E^i(X, \mathcal{L})
\]

\[
E^i(X, \mathcal{L}) \xrightarrow{e(\mathcal{E})(1)} E^{i-n+1}(P(\mathcal{E}), p^*(\mathcal{L} \otimes \det(\mathcal{E})^*)(1))
\]

for any \( i \).

2. If \( n \) is even, for any \( i \) we have an exact sequence

\[
E^i(P(\mathcal{E}), p^*\mathcal{L}) \xrightarrow{e(\mathcal{E})(1)} E^{i-n+1}(X, \mathcal{L} \otimes \det(\mathcal{E})^*)
\]

\[
e(\mathcal{E}) \neq 0 \in E^n(X, \det(\mathcal{E})^*) \),
\]

the above sequence gives a decomposition

\[
E^i(P(\mathcal{E}), p^*\mathcal{L}) \cong E^i(X, \mathcal{L}) \oplus E^{i-n+1}(X, \mathcal{L} \otimes \det(\mathcal{E})^*) \cdot R(\mathcal{E})
\]

for any \( i \). Here

\[
R(\mathcal{E}) = e(\mathcal{E}) + p^*(u) \in E^{n-1}(P(E), p^*\det(\mathcal{E})^*)
\]

is the orientation class of \( \mathcal{E} \) where \( u \) is any element such that \( S^2_{\det(\mathcal{E})}(u) = e(\mathcal{E}) \).
Proof. Any element $x \in \text{Ch}^m(\mathbb{P}(\mathcal{E}))$ can be uniquely written as $\sum_{i=0}^{n-1} a_i c^{n-1-i}$ where $c = c_1(O_E(1))$ and $a_i \in \text{Ch}^{m-n-i+1}(X)$ by the projective bundle theorem of Chow. Suppose $x \in E^m(\mathbb{P}(\mathcal{E}), p^* \mathcal{L})$ with $\{a_i\}$ determined as above.

(1) We have

$$Sq^2_{\mathcal{Z}}(x) = \sum_{i=0}^{n-1} Sq^2_{\mathcal{Z}}(a_i)c^{n-1+i} + \sum_{i=1}^{n-1} a_{2i-1} c^{n-2i+1}$$

So $Sq^2_{\mathcal{Z}}(x) = 0$ is equivalent to the equations

$$\begin{cases} Sq^2_{\mathcal{Z}}(a_i) = a_{i+1} & i < n - 1 \text{ even} \\
Sq^2_{\mathcal{Z}}(a_i) = 0 & \text{else} \end{cases}$$

If furthermore $a_{n-1} = 0$, we have

$$Sq^2_{\mathcal{Z}}(\sum_{i=0}^{n-2} a_{2i} c^{n-2-2i}) = x.$$

So it is easy to conclude that the pullback $E^i(X, \mathcal{L}) \to E^i(\mathbb{P}(\mathcal{E}), p^* \mathcal{L})$ is an isomorphism. The second statement follows similarly.

(2) The $Sq^2_{\mathcal{Z}}(x) = 0$ is equivalent to the equations

$$\begin{cases} Sq^2_{\mathcal{Z}}(a_i) + a_0 c_{i+1}(E) = 0 & i \text{ even} \\
Sq^2_{\mathcal{Z}}(a_i) + a_0 c_{i+1}(E) + a_{i+1} = 0 & i < n - 1 \text{ odd} \\
Sq^2_{\mathcal{Z}}(a_{n-1}) + a_0 e(E) = 0 \end{cases}$$

Suppose $a_0 = a_{n-1} = 0$, we have

$$Sq^2_{\mathcal{Z}}(\sum_{i=0}^{n-2} a_{2i+1} c^{n-3-2i}) = x.$$

Since $p_*(x) = a_0$, the first statement follows by direct computation. Now suppose $e(\mathcal{E}) = 0$. It is easy to check that

$$Sq^2_{\mathcal{Z}}(a_0 R(\mathcal{E})) = 0 \quad Sq^2_{\mathcal{Z}}(a_{n-1} + a_0 u) = 0.$$ 

So $x = a_0 R(\mathcal{E}) + a_{n-1} + a_0 u$ in $E^i(\mathbb{P}(\mathcal{E}), p^* \mathcal{L})$. On the other hand, if

$$a_0 R(\mathcal{E}) + a_{n-1} + a_0 u = 0 \in E^m(\mathbb{P}(\mathcal{E}), p^* \mathcal{L}),$$

we have

$$a_0 = 0 \in E^{m-n+1}(X, \mathcal{L} \otimes \text{det}(\mathcal{E})^\vee).$$

Thus $a_{n-1} = 0 \in E^m(\mathbb{P}(\mathcal{E}), p^* \mathcal{L})$. So there are $b, b_0$ such that

$$Sq^2_{\mathcal{Z}}(b) + b_0 e(E) = a_{n-1}$$

by the calculation of $Sq^2_{\mathcal{Z}}(x)$ before, where $Sq^2_{\mathcal{Z} \otimes \text{det}(\mathcal{E})^\vee}(b_0) = 0$. But again we have

$$b_0 e(E) = Sq^2_{\mathcal{Z}}(b_0 u)$$

so we conclude that $a_{n-1} = 0 \in E^m(X, \mathcal{L})$. The second statement then follows.

\[\square\]

Definition 3.4. Given any vector bundle $E$ on a scheme $X$, we take the projective bundle $p_1 : Y_1 = \mathbb{P}(E) \to X$. Then inductively define $p_i : Y_i = \mathbb{P}(\Omega_{p_{i-1}}(1)) \to Y_{i-1}$. The

$$F \ell(\mathcal{E}) = Y_{\text{rk}(\mathcal{E})-1}$$

is defined as the complete flag of $\mathcal{E}$, parametrizing filtrations

$$\mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_i \subseteq \cdots \subseteq \mathcal{E}$$

where $\text{rk}(\mathcal{E}_i) = i$ and $\mathcal{E}_i/\mathcal{E}_{i-1}$ is locally free.

Proposition 3.5. Suppose $X \in Sm/k$ and $E$ is a vector bundle on $X$. We have structure maps

$$\mathbb{P}(\Omega_p(1)) \to \mathbb{P}(E) \to X.$$

For any $M \in \text{Pic}(X)$, we have an isomorphism

$$\text{Th}(q^*(p^* M(1))) = \text{Th}((q^* p^* M)(1))$$

in $\text{DM}^{eff}(X, \mathbb{Z})$. 
Proof. The Euler sequence
\[ 0 \to \Omega_{p}(1) \to p^{*}E \to O_{E}(1) \to 0 \]
implies that \( \mathbb{P}(p^{*}E) \) has a section \( \mathbb{P}(E) \to \mathbb{P}(p^{*}E) \), whose complement \( C \) is a vector bundle over \( \mathbb{P}(\Omega_{p}(1)) \). There is an automorphism \( \sigma \) of \( \mathbb{P}(p^{*}E) \) over \( X \) which locally comes from the swapping map
\[
\begin{align*}
X \times \mathbb{P}^{rk(E)-1} & \to X \times \mathbb{P}^{rk(E)-1} \\
((x_{i}),(y_{i}),(z_{i})) & \mapsto ((x_{i}),(z_{i}),(y_{i})).
\end{align*}
\]
The \( \sigma \), which satisfies \( \sigma^{*}(\mathcal{O}_{p^{*}E}(1)) = q^{*}(\mathcal{O}_{E}(1)) \), induces an automorphism of \( C \) because locally \( C \) is defined by the equation
\[ (y_{i}) \neq (z_{i}). \]
So we have
\[ Th((q|_{C})^{*}O_{E}(1)) \otimes Th(M) = Th(\mathcal{O}_{p^{*}E}(1)|_{C}) \otimes Th(M) \]
in \( \text{DM}^{eff}(X,Z) \) via \( \sigma \), which gives the statement. \( \square \)

**Proposition 3.6.** Suppose that \( X \in Sm/k, \mathcal{L} \in \text{Pic}(X) \) and that \( \mathcal{E} \) is a vector bundle of odd rank \( n \) on \( X \). Denote by \( p : \mathbb{P}(E) \to X, q : \mathbb{P}(\Omega_{p}(1)) \to \mathbb{P}(E) \) the structure maps. If \( p_{n-1}(\mathcal{E}) = 0 \in E^{2n-2}(X) \), we have
\[ E^{i}(\mathbb{P}(\Omega_{p}(1)), q^{*}p^{*}\mathcal{L}) \cong E^{i}(X, \mathcal{L}) \oplus E^{i-2n+3}(X, \mathcal{L}) \cdot T. \]
Here
\[ T = q^{*}(e(\Omega_{p}(1))) e(\Omega_{q}(1)) + q^{*}(u) \in E^{2n-3}(\mathbb{P}(\Omega_{p}(1))) \]
where \( u \) is any element such that \( Sq^{2}(u) = e(\Omega_{p}(1))^{2} \) in \( Ch^{2n-2}(\mathcal{E}) \).

**Proof.** The Whitney sum formula for even Pontryagin classes tells us that
\[ p^{*}(p_{n-1}(\mathcal{E})) = p_{n-1}(\Omega_{p}(1)) = e(\Omega_{p}(1))^{2} \]
hence they all vanish. The composite
\[ E^{i}(X, \mathcal{L}) \xrightarrow{\alpha(\Omega_{p}(1))} E^{i+n-1}(\mathbb{P}(\mathcal{E}), \mathcal{E}(\det(\mathcal{E})(1) \otimes \mathcal{L})) \xrightarrow{\alpha(\Omega_{p}(1))^{2}} E^{i+2n-2}(\mathbb{P}(\mathcal{E}), p^{*}\mathcal{L}) \]
is zero by \( e(\Omega_{p}(1))^{2} = 0 \), so the second arrow is zero for any \( i \). Hence we have an exact sequence
\[ 0 \to E^{i}(\mathbb{P}(\mathcal{E}), p^{*}\mathcal{L}) \xrightarrow{q^{*}} E^{i}(\mathbb{P}(\Omega_{p}(1)), q^{*}p^{*}\mathcal{L}) \xrightarrow{\alpha(\Omega_{p}(1))} E^{i+n-2}(\mathbb{P}(\mathcal{E}), \mathcal{E}(\det(\mathcal{E}) \otimes \mathcal{L}))(1) \to 0 \]
for any \( i \) by Proposition 3.3. So there is an element \( T \in E^{2n-3}(\mathbb{P}(\Omega_{p}(1))) \) such that \( q_{*}(T) = e(\Omega_{p}(1)) \).
Then for any \( x \in E^{i-2n+3}(X, \mathcal{L}) \), we have
\[ q_{*}(q^{*}p^{*}(x) \cdot T) = p^{*}(x) \cdot e(\Omega_{p}(1)). \]
Since \( n \) is odd, the sequence above is thus isomorphic the sequence
\[ 0 \to E^{i}(X, \mathcal{L}) \xrightarrow{\alpha(\Omega_{p}(1))} E^{i}(\mathbb{P}(\Omega_{p}(1)), q^{*}p^{*}\mathcal{L}) \xrightarrow{\alpha(\Omega_{p}(1))^{2}} E^{i-2n+3}(X, \mathcal{L}) \to 0. \]
by loc. cit. where the third arrow has a section \( T \cdot q^{*}p^{*} \). So we obtain the first statement.

For the second statement, it suffices to check \( q_{*}(T) = e(\Omega_{p}(1)) \) and \( Sq^{2}(T) = 0 \), which are straightforward. \( \square \)

**Proposition 3.7.** Suppose that \( X \in Sm/k, \mathcal{L} \in \text{Pic}(X) \) and that \( \mathcal{E} \) is a vector bundle of rank \( n \) on \( X \). Denote by \( p : F(I\mathcal{E}) \to X \) the structure map. Suppose \( p_{0}(\mathcal{E}) = 0 \in E^{2i}(X) \) for all \( i \).

1. If \( n \) is odd, we have classes \( T_{a} \in E^{4a-1}(F(I\mathcal{E})), 1 \leq a \leq \frac{n-1}{2} \) such that
   \[ E^{i}(F(I\mathcal{E}), p^{*}\mathcal{L}) = \bigoplus_{1 \leq i \leq \frac{n-1}{2}} \bigoplus_{1 \leq a_{1} < \cdots < a_{i} \leq \frac{n-1}{2}} E^{i-\sum_{a} \deg(T_{a})}(X, \mathcal{L}) \cdot T_{a_{1}} \cdots T_{a_{i}}. \]

2. If \( n \) is even and \( e(\mathcal{E}) = 0 \), we have classes \( T_{a} \in E^{4a-1}(F(I\mathcal{E})), 1 \leq a \leq \frac{n}{2} - 1 \) and \( T_{\frac{n}{2}} \in E^{n-1}(F(I\mathcal{E})) \) such that
   \[ E^{i}(F(I\mathcal{E}), p^{*}\mathcal{L}) = \bigoplus_{1 \leq i \leq \frac{n}{2}} \bigoplus_{1 \leq a_{1} < \cdots < a_{i} \leq \frac{n}{2}} E^{i-\sum_{a} \deg(T_{a})}(X, \mathcal{L}) \cdot T_{a_{1}} \cdots T_{a_{i}}. \]

**Proof.** Follows from Proposition 3.3 and Proposition 3.6. \( \square \)

**Remark 3.8.** In case \( X = pt \), we could write down the \( \{T_{a}\} \) explicitly. In the context of Definition 3.3 denote by \( x_{i}, 1 \leq i \leq n-1 \) the first Chern class of tautological line bundle on \( Y_{i} \) and \( x_{n} \) by that of complement bundle on \( Y_{n-1} \). Then
\[ Ch^{*}(Fl(k^{\otimes n})) = Z/2Z[x_{1}, \cdots, x_{n}] / S \]
where $S$ is the ideal generated by symmetric functions, with $Sq^2(x_i) = x_i^2$ for all $i$. By inductively using Whitney sum formula, one shows that
\[ c_i(\Omega_{p_j}(1)) = h_i(x_1, \ldots, x_j) \]
for every $i,j$.

Suppose $1 \leq a \leq \lfloor \frac{n}{2} \rfloor$. If $n$ is odd or $n$ is even and $a < \frac{n}{2}$, by Proposition 1.6 we have
\[ T_a = h_{2a}(x_1, \ldots, x_{n-2a})h_{2a-1}(x_1, \ldots, x_{n-2a+1}) + u(x_1, \ldots, x_{n-2a}) \]
where $u$ satisfies $Sq^2(u) = h_{2a}(x_1, \ldots, x_{n-2a})^2$.

If $n$ is even, we have
\[ T_a^n = x_1^{n-1}. \]
It is clear that $T_a^n = 0 \in E^*(Fl(k^{\oplus n}))$ (but not true for general $X$ and $\mathcal{E}$). So we conclude
\[ E^*(Fl(k^{\oplus n})) = \wedge([T_a]). \]

4. MW-Motivic Decomposition of Grassmannians

Recall that a Young tableaux (see [Wen18b, 3.1]) is a collection of left aligned rows of boxes with (non strictly) decreasing row lengths. For example, the tableau
\[
\begin{array}{ccc}
\hline
\hline
\hline
\end{array}
\]
is denoted by $(3, 2, 1)$.

**Definition 4.1.** Suppose $k, n \in \mathbb{N}$ and $k \leq n$. We say a Young tableau is $(k, n)$-truncated if it is supposed that however adding boxes, it admits at most $k$ rows and $n-k$ columns. It’s truncated if it is $(k, n)$-truncated for some $(k, n)$, otherwise it is called untruncated.

Each $(k, n)$-truncated tableau $(a_1, \ldots, a_t)$ is associated with a Schubert cycle $\sigma(a_1, \ldots, a_t) \in CH^S_{\sum a_i}(Gr(k, n))$. All Young tableaux we will consider are filled by a checkboard pattern (see [Wen18b, Theorem 4.2]). If the first box in the first row of a tableau is black (resp. white), it is called untwisted (resp. twisted).

The following is an example (both are $(3, 6)$-truncated):

Un twisted Young tableau $T$  
Twisted Young tableau $T'$

\[
\begin{array}{ccc}
\hline
\hline
\hline
\end{array}
\]

\[
\begin{array}{ccc}
\hline
\hline
\hline
\end{array}
\]

**Definition 4.2.** Suppose $\Lambda$ is a Young tableau (twisted or not). Define
\[ A(\Lambda) = \{ \Lambda' | \Lambda' = \Lambda + \text{a white box} \} \]
\[ \bar{A}(\Lambda) = \{ \Lambda' | \Lambda' = \Lambda + \text{white boxes} \} \]
\[ D(\Lambda) = \{ \Lambda' | \Lambda' = \Lambda + \text{a white box} \} \]

Here the boxes are added in the end of rows, under the corresponding coloring rule. We say that $\Lambda$ is irredundant (resp. full) if $D(\Lambda) = \emptyset$ (resp. $A(\Lambda) = \emptyset$).

For example, the $T$ and $T'$ above satisfy
\[ A(T) = \{ (3, 3, 1), (3, 2, 2) \} \]
\[ \bar{A}(T) = \{ (3, 2, 1), (3, 3, 1), (3, 2, 2), (3, 3, 2) \} \]
\[ D(T) = \emptyset; \]
\[ A(T') = \emptyset; \quad \bar{A}(T') = \{ (3, 2, 1) \} \]
\[ D(T') = \{ (2, 2, 1), (3, 1, 1), (3, 2) \}. \]

So $T$ is irredundant and $T'$ is full.

Hence we see that the Steenrod square $Sq^2_{\mathcal{E}} : Ch^{i-1}(Gr(k, n)) \longrightarrow Ch^i(Gr(k, n))$ can be written as
\[ Sq^2_{\mathcal{E}}(T) = \sum_{\Lambda \in A(T)} \Lambda \]
by [Wen18b] Theorem 4.2. There is a truncation morphism
\[ t_{k,n} : \mathbb{Z}\{\text{untruncated Young tableaux}\} \longrightarrow \mathbb{Z}\{(k, n)-\text{truncated Young tableaux}\} \]
which sends $\Lambda$ to itself if the tableau’s size is correct otherwise it sends $\Lambda$ to zero. We have
\[ t_{k,n} (\sigma_1) t_{k,n} (\sigma_2) = t_{k,n} (\sigma_1 \sigma_2) \quad t_{k,n} \circ Sq^2_{\mathcal{E}} = Sq^2_{\mathcal{E}}. \]

**Lemma 4.3.** Suppose that all tableaux are untwisted.

(1) An untruncated Young tableau $\Lambda$ is irredundant and full if and only if it is completely even (see [Wen18b, Definition 3.8]).
(2) If \(k(n - k)\) is even, a \((k, n)\)-truncated Young tableau \(\Lambda\) is irredundant and full if and only if it is completely even.

(3) If \((n - k)\) is odd, a \((k, n)\)-truncated Young tableau \(\Lambda\) is irredundant and full if and only if it is completely even or of the form

\[
\sigma\cdots 2a_i, 2a_i, \cdots \sigma_{n-k,1}^{-1} = \sigma_{n-k,1} \cdots 2a_i + 1, 2a_i + 1, \cdots
\]

Proof. It suffices to prove the necessity for all those statements.

(1) Suppose \(\lambda_i\) is the last box in the \(i^{th}\) row of \(\Lambda\). Then either \(\lambda_i\) is black with a row of the same length above it or \(\lambda_i\) is white with a row of the same length adjacent to it. So we see that \(\Lambda\) is completely even. The converse is clear.

(2) If \(n - k\) is even, the first row of \(\Lambda\) must have even length otherwise it is not full, which implies that the second row exists and must have the same length. Then every row started with a black box must have even length, being adjacent above a row with the same length. So \(\Lambda\) is completely even.

If \(k\) is even, suppose the \(i^{th}\) row is the first row whose length is even. If no such row exists, the number of rows must be odd otherwise \(\Lambda\) is not irredundant. But we could still add a white box in the first empty row so it is not full. So such \(i\) exists. If \(i\) is even, \(\Lambda\) is not full. If \(i > 1\) is odd, \(\Lambda\) is not irredundant. Hence \(i = 1\) and \(\Lambda\) is completely even as above. So we have completed the proof.

(3) If the length of the first row is even, then \(\Lambda\) is complete even as above. Otherwise its length must be \(n - k\). So the second row must have odd length otherwise \(\Lambda\) is not full. Moreover, there is a row adjacent under it with the same length otherwise \(\Lambda\) is not irredundant. By induction we see \(\Lambda\) must be the form \(\sigma_{n-k,1} \cdots 2a_i + 1, 2a_i + 1, \cdots\).

\(\square\)

**Lemma 4.4.** Suppose that all tableaux are twisted.

1. If \(k\) and \(n\) are even, a \((k, n)\)-truncated Young tableau \(\Lambda\) is irredundant and full if and only if it is like \(\sigma_{n-k} T\) or \(\sigma^T_k\) where \(T\) is completely even.
2. If \(k\) and \(n\) are odd, a \((k, n)\)-truncated Young tableau \(\Lambda\) is irredundant and full if and only if it is like \(\sigma_{n-k} T\) where \(T\) is completely even.
3. If \(k\) is even and \(n\) is odd, a \((k, n)\)-truncated Young tableau \(\Lambda\) is irredundant and full if and only if it is like \(\sigma^T_k\) where \(T\) is completely even.
4. If \(k\) is odd and \(n\) is even, there is no \((k, n)\)-truncated Young tableau \(\Lambda\) which is both irredundant and full.

Proof. (1) It suffices to prove the necessity. If the first row has odd length, there must be a row with same length adjacent above it. Inductively we see that \(\Lambda = \sigma_{1}^T\) where \(T\) is completely even. Otherwise it has length \(n - k\). The second row must have even length otherwise \(\Lambda\) is not full, being adjacent above a row with same length. Inductively we see that \(\Lambda = \sigma_{n-k} T\) where \(T\) is completely even.

(2) The same reasoning as in (1).

(3) The same reasoning as in (1).

(4) The same reasoning as in (1).

\(\square\)

**Definition 4.5.** Slightly abusing the notation, we say that a Young tableau \(\Lambda\) (twisted or not) is even if it is irredundant and full.

Denote the free abelian group generated by the set of all those (resp. twisted) tableaux by \(N_{k,n}\) (resp. \(M_{k,n}\)), regarded as a subgroup of \(CH^*_d(Gr(k,n))\). Its degree \(i\) part is denoted by \(N^i_{k,n}\) (resp. \(M^i_{k,n}\)).

Our definition of even Young tableau is essentially the same as [Wen18b, Definition 3.8] by Lemma 4.3, but it depends on \(k, n\) and whether the tableau is twisted.

**Remark 4.6.** Here are the irredundant untwisted Young tableaux in \(Gr(2,4)\) or \(Gr(3,6)\):
Suppose as desired. Λ

So (resp. σ ’a white box added’ and 0 otherwise. Suppose |Supp| = ∑ N

\[ \text{Proposition 4.10.} \text{ Suppose } \Lambda \text{ (twisted or not) is irredundant.} \]

\[ \text{Proof.} \text{ Clear from the definition of irredundance.} \]

Here are the generators of \( M_{2,4} \) (\( M_{3,6} = 0 \)):

Here are the irredundant twisted Young tableaux in \( \text{Gr}(2, 4) \) or \( \text{Gr}(3, 6) \):

\[ \text{Lemma 4.7.} \text{ We have decompositions} \]

\[ \text{Proof.} \text{ It suffices to prove the equation in each component } C = \mathbb{Z}/2\mathbb{Z} \Lambda(\Lambda) \text{ generated by an irredundant Young tableau } \Lambda \text{ as in Lemma 4.7. Hence we may assume that } \Lambda \text{ is not full and prove that} \]

\[ \text{Proposition 4.8.} \text{ Suppose } \sigma = \sum_i \Lambda_i \in \text{Ker}(Sq^2_{2 \mathcal{X}}) \text{ where } \Lambda_i \text{ are } (k, n)\text{-truncated (twisted or not) Young tableaux. Then } \sigma \in \text{Im}(Sq^2_{2 \mathcal{X}}) \text{ if and only if none of } \Lambda_i \text{ is even. In other word, we have} \]

\[ \text{Ker}(Sq^2_{2 \mathcal{X}}) = \bigoplus_{\Lambda \text{ irredundant}} \mathbb{Z}/2\mathbb{Z} \Lambda(\Lambda) \cap \text{Ker}(Sq^2_{2 \mathcal{X}}); \]

\[ \text{Im}(Sq^2_{2 \mathcal{X}}) = \bigoplus_{\Lambda \text{ irredundant}} \mathbb{Z}/2\mathbb{Z} \Lambda(\Lambda) \cap \text{Im}(Sq^2_{2 \mathcal{X}}). \]

Suppose \(|A(\Lambda)| = n > 0\) hence each \( \Lambda_i \) can be seen as a sequence of \( \{0, 1\} \) with length \( n \) where 1 means ‘a white box added’ and 0 otherwise. Suppose \( \sigma = C0 + D1 \) where \( C \) and \( D \) are of length \( n - 1 \) and \( C0 \) (resp. \( D1 \)) means concatenating \( C \) (resp. \( D \)) by 0 (resp. 1). Then

\[ 0 = Sq^2_{2 \mathcal{X}}(C0 + D1) = Sq^2_{2 \mathcal{X}}(C)0 + C1 + Sq^2_{2 \mathcal{X}}(D)1. \]

So \( C = Sq^2_{2 \mathcal{X}}(D) \) thus

\[ Sq^2_{2 \mathcal{X}}(D0) = C0 + D1, \]

as desired.

\[ \text{Definition 4.9.} \text{ Suppose } \Lambda \text{ is an irredundant Young tableau (maybe twisted). There are } |A(\Lambda)| \text{ ‘positions’ in } \Lambda \text{ on which a white box could be added. Denote by } \Lambda_{i_1, \ldots, i_l}, 1 \leq i_1 < \cdots < i_l \leq |A(\Lambda)| \text{ (defined by } \Lambda \text{ if } l = 0) \text{ the Young tableau with a white box added on the } i_l^{th}, \ldots, i_1^{th} \text{ positions of } \Lambda. \]

For example, the (3, 6)-truncated tableau \( T = (3, 2, 1) \) has two positions available, namely we have

\[ T_1 = (3, 3, 1) \quad T_2 = (3, 2, 2) \quad T_{1,2} = (3, 3, 2). \]

We could also define

\[ Sq^2_{2 \mathcal{X}} : CH^\bullet(Gr(k, n)) \longrightarrow CH^\bullet(Gr(k, n)) \longrightarrow \sum_{T \in A(\Lambda)} T, \]

which is a lift of the Steenrod square.
(1) If $\Lambda$ is even, we set $S = \{\Lambda\}$ otherwise

$$S = S^2_{/Z}(\{\Lambda_i, \cdots, \Lambda_i| i_1 > 1\}).$$

It's a $\mathbb{Z}/2\mathbb{Z}$-basis of Ker$(S^2_{/Z}) \cap \mathbb{Z}/2\mathbb{Z}$A($\Lambda$).

(2) If $\Lambda$ is even, we set $S_\Lambda = \{\Lambda\}$ otherwise

$$S_\Lambda = S^2_{/Z}(\{\Lambda_i, \cdots, \Lambda_i| i_1 > 1\}) \cup \{2\Lambda_i, \cdots, \Lambda_i| i_1 > 1\}.$$

It's a $\mathbb{Z}$-basis of Ker$(S^2_{/Z} \circ \pi) \cap \mathbb{Z}$A($\Lambda$).

**Proof.** We may assume $\Lambda$ is not full.

(1) We have

$$\text{Ker}(S^2_{/Z}) \cap \mathbb{Z}/2\mathbb{Z}$A($\Lambda$) = Im$(S^2_{/Z}) \cap \mathbb{Z}/2\mathbb{Z}$A($\Lambda$)

by Proposition 4.12. Hence there is no generator of $\text{Ker}(S^2_{/Z}) \cap \mathbb{Z}/2\mathbb{Z}$A($\Lambda$).

(2) Denote by $C = \mathbb{Z}\{\Lambda_i, \cdots, \Lambda_i| i_1 > 1\}$ and $T = \{\Lambda_i, \cdots, \Lambda_i| i_1 > 1\}$. So a system of generators is $\{2\Lambda_i, \cdots, \Lambda_i| i_1 > 1\}$ and $T \in C$. The $\varphi$ is injective. If $\sum_j \alpha_j \varphi(t_j) = 0$ where $t_j \in T$ are distinct, we have $\sum_j \alpha_j \varphi(t_j) = 0$ hence $\alpha_j = 0$ for all $j$. So $S_\Lambda$ is linearly independent. On the other hand, suppose $i_2 > 1$. We have

$$\sum_{j \notin \{i_2, \cdots, i_1\}} \Lambda_{j, i_2, \cdots, i_1} \in T,$$

which shows that $2\Lambda_{i_2, \cdots, i_1} \in \text{Ker}(S^2_{/Z} \circ \pi)$. Hence $S_\Lambda$ is a basis.

**Corollary 4.11.** There are decompositions

$$\text{Ker}(S^2_{/Z} \circ \pi) = \bigoplus_{\Lambda \text{ irredundant}} \mathbb{Z}$A_.$$

For convenience, we define

$$\text{Ker}(S^2_{/Z} \circ \pi) = \text{Ker}(S^2_{/Z} \circ \pi)/N_{k,n}$$

$$\text{Ker}(S^2_{/Z} \circ \pi) = \text{Ker}(S^2_{/Z} \circ \pi)/M_{k,n}.$$

Recall the notations in [Wen18] that we define $c_i = c_i(\mathbb{Z}_{k,n})$, $c_I = c_i(\mathbb{Z}_{k,n}^I)$, $C_k = c(\mathbb{Z}_{k,n})$, $e_{n-k} = e(\mathbb{Z}_{k,n}^I)$ and similarly for $p_i$ and $p_I$.

**Proposition 4.12.**

(1) If $(k-n-k)$ is odd, we have isomorphisms

$$\gamma : CH^i(Gr([\frac{k}{2}, \frac{n}{2}] - 1)) \rightarrow \mathbb{Z}$A_.$$

$$\gamma' = \sigma_{n-k,1} \cdot \gamma : CH^i(Gr([\frac{k}{2}, \frac{n}{2}] - 1)) \rightarrow \mathbb{Z}$A_{n-k,1}$$

Moreover, there is a $W(k)$-algebra isomorphism

$$\lambda : CH^*(\mathbb{Z}_{k,n}) \otimes \mathbb{W}(k) \rightarrow H^*(Gr(k,n), \mathbb{W})$$

$$\lambda' : CH^*(\mathbb{Z}_{k,n}) \otimes \mathbb{W}(k) \rightarrow H^*(Gr(k,n), \mathbb{W}).$$

(2) If $(k-n-k)$ is even, we have an isomorphism

$$\gamma : CH^i(Gr([\frac{k}{2}, \frac{n}{2}] - 1)) \rightarrow \mathbb{Z}$A_.$$

Moreover, there is an $W(k)$-algebra isomorphism

$$\lambda : CH^*(\mathbb{Z}_{k,n}) \otimes \mathbb{W}(k) \rightarrow H^*(Gr(k,n), \mathbb{W})$$
(3) If \( k \) is even and \( n \neq 0 \) (mod 4), we have isomorphisms

\[
\begin{align*}
\gamma_1 &= \sigma_{1,n} \cdot \gamma : CH^i(Gr([k \over 2], [n-1 \over 2])) \longrightarrow M_{k,n}^{4i+k}
\lambda_1 : CH^i(Gr([k \over 2], [n-1 \over 2])) \otimes \mathbb{Z} W(k) \longrightarrow H^{4i+k}(Gr(k,n), W(O(1))) \quad (\lambda(\sigma) \cdot e_k)
\end{align*}
\]

(4) If \( n - k \) is even and \( n \neq 0 \) (mod 4), we have isomorphisms

\[
\begin{align*}
\gamma_2 &= \sigma_{n-k} \cdot \gamma : CH^i(Gr([k - 1 \over 2], [n-1 \over 2])) \longrightarrow M_{k,n}^{4i+n-k}
\lambda_2 : CH^i(Gr([k - 1 \over 2], [n-1 \over 2])) \otimes \mathbb{Z} W(k) \longrightarrow H^{4i+n-k}(Gr(k,n), W(O(1))) \quad (\lambda(\sigma) \cdot e_{k-n})
\end{align*}
\]

(5) If \( k(n - k) \) is even and \( n \equiv 0 \) (mod 4), we have isomorphisms

\[
\begin{align*}
\gamma_3 &= \gamma_1 \oplus \gamma_2 : CH^i(Gr([k \over 2], [n-1 \over 2])) \oplus CH^{i+2n}(Gr([k - 1 \over 2], [n-1 \over 2])) \longrightarrow M_{k,n}^{4i+k}
\lambda_3 : CH^i(Gr([k \over 2], [n-1 \over 2])) \oplus CH^{i+2n}(Gr([k - 1 \over 2], [n-1 \over 2])) \otimes \mathbb{Z} W(k) \longrightarrow H^{4i+k}(Gr(k,n), W(O(1))).
\end{align*}
\]

In all other cases, \( M_{k,n}^i \) and \( N_{k,n}^i \) vanishes.

**Proof.** Follows from [Wen18a, Theorem 6.4], [Wen18b, Proposition 3.14], Lemma 4.3 and 4.4. \(\square\)

**Theorem 4.13.**

(1) If \( k \) is odd and \( n \) is even, \( Th(O_{Gr(k_1)}(1)) \) splits in \( \tilde{MM}^{eff}(pt, \mathbb{Z}) \) and we have

\[
Th(O_{Gr(k,n)}(1)) = Th(O_{Gr(k-2,n-2)}(1))(2n - 2k)[4n - 4k] \oplus Th(O_{Gr(k-2,n)}(1)) \oplus \mathbb{Z}(Gr(k - 1, n - 2))/(\eta(n - k)[2n - 2k]).
\]

Moreover, we have

\[
WW(Th(O_{Gr(k,n)}(1))) = 0.
\]

(2) If \( n - k \) is even, \( Th(O_{Gr(k,n)}(1)) \) splits \( \tilde{MM}^{eff}(pt, \mathbb{Z}) \) and we have

\[
Th(O_{Gr(k,n)}(1)) = Th(O_{Gr(k,n-1)}(1)) \oplus \mathbb{Z}(Gr(k - 1, n - 1))(n - k + 1)[2n - 2k + 2].
\]

Moreover, if \( k \) is even, we have

\[
WW(Th(O_{Gr(k,n)}(1))) \leq 4\mathbb{Z} + n - k + 1 \cup (4\mathbb{Z} + k + 1)
\]

otherwise

\[
WW(Th(O_{Gr(k,n)}(1))) \leq 4\mathbb{Z} + n - k + 1.
\]

(3) If \( k \) is even and \( n \) is odd, \( Th(O_{Gr(k,n)}(1)) \) splits \( \tilde{MM}^{eff}(pt, \mathbb{Z}) \) and we have

\[
Th(O_{Gr(k,n)}(1)) = Th(O_{Gr(k-1,n-1)}(1)) \oplus \mathbb{Z}(Gr(k, n - 1))(k + 1)[2k + 2].
\]

Moreover, we have

\[
WW(Th(O_{Gr(k,n)}(1))) \leq 4\mathbb{Z} + k + 1.
\]

(4) If \( n - k \) is odd, \( \mathbb{Z}(Gr(k,n)) \) splits \( \tilde{MM}^{eff}(pt, \mathbb{Z}) \) and we have

\[
\mathbb{Z}(Gr(k,n)) = \mathbb{Z}(Gr(k, n - 1)) \oplus Th(O_{Gr(k-1,n-1)}(1))(n - k - 1)[2n - 2k - 2].
\]

Moreover, if \( k \) is even, we have

\[
WW(\mathbb{Z}(Gr(k,n))) \subseteq 4\mathbb{Z}
\]

otherwise

\[
WW(\mathbb{Z}(Gr(k,n))) \subseteq 4\mathbb{Z} \cup (4\mathbb{Z} + n - 1).
\]

(5) If \( n - k \) is even, \( \mathbb{Z}(Gr(k,n)) \) splits \( \tilde{MM}^{eff}(pt, \mathbb{Z}) \) and we have

\[
\mathbb{Z}(Gr(k,n)) = \mathbb{Z}(Gr(k - 1, n - 1)) \oplus Th(O_{Gr(k,n-1)}(1))(k - 1)[2k - 2].
\]

Moreover, we have

\[
WW(\mathbb{Z}(Gr(k,n))) \subseteq 4\mathbb{Z}.
\]

**Proof.** Define the partial order \( \leq \) on \((k, n)\) by \((k', n') \leq (k, n)\) iff \(k' \leq k\) and \(n' \leq n\). We suppose that all the statements are true for any \((k', n') < (k, n)\). Let us prove the statements for \((k, n)\). The cases for \(k = 0, 1\) have already known.
(1) Set $M = Th(O_{\mathcal{Gr}(k,n)}(1))_{|\mathcal{Gr}(k,n)\setminus \mathcal{Gr}(k-2,n-2)}}$ and $p_1 : Gr(1,2) \times Gr(k-1,n-2) \to Gr(1,2)$. By Proposition 3.2 we have Gysin triangles (see Proposition 2.2).

\[
\begin{align*}
Th(O_{\mathcal{Gr}(k,n-2)}(1)) & \to M \to Th(p_1)(O(1))(n-k-1)[2n-2k-2] \to \cdots [1]; \\
M & \to Th(O_{\mathcal{Gr}(k,n)}(1)) \to Th(O_{\mathcal{Gr}(k-2,n-2)}(1))(2n-2k)[4n-4k] \to \cdots [1].
\end{align*}
\]

Now we know that

\[
Th(p_1)(O(1)) = \mathbb{Z}([Gr(k-1,n-2)]) \otimes Th(O_{\mathcal{Gr}(k,n)}(1)) = \mathbb{Z}([Gr(k-1,n-2)]/[\eta](1)[2])
\]

so $WW(Th(p_1)(O(1))) = 0$. Hence the first triangle splits. Moreover, $WW(Th(O_{\mathcal{Gr}(k-2,n-2)}(1))) = \emptyset$ by induction, hence the second triangle splits as well.

(2) By the Gysin triangle related to $Gr(k-1,n-1) \to Gr(k,n)$ and Proposition 2.14.

(3) By the Gysin triangle related to $Gr(k,n-1) \to Gr(k,n)$.

(4) By the Gysin triangle related to $Gr(k-1,n-1) \to Gr(k,n)$ and Proposition 2.13.

(5) By the Gysin triangle related to $Gr(k,n-1) \to Gr(k,n)$ and Proposition 2.13.

□

Remark 4.14. Since all Thom spaces of Grassmannians split, we could apply Theorem 2.7, 2 to give an additive basis of $\widehat{CH}^*(\mathcal{Gr}(k,n), \mathbb{Z})$. For example, suppose $(k,n) = (2,4)$ or $(k,n) = (3,6)$.

| $GW(k)$ | $\mathbb{Z}$ |
|-----|-----|
| $CH^*(\mathcal{Gr}(2,4))$ | $1, \sigma_2$ |
| $CH^*(\mathcal{Gr}(3,6))$ | $1, \sigma_2, \sigma_3, 1, \sigma_3^2$ |
| $CH^*(\mathcal{Gr}(2,4), O(1))$ | $2, \sigma_1, 2, \sigma_2, \sigma_2^2$ |
| $CH^*(\mathcal{Gr}(3,6), O(1))$ | $2, \sigma_1, 2, \sigma_3, 2, \sigma_2^2, \sigma_1 \sigma_3, 2, \sigma_2^2, 2, \sigma_2^3, \sigma_1 \sigma_2^2, 2, \sigma_2^2 \sigma_2, 2, \sigma_2^3 \sigma_2^2, 2, \sigma_2^3 \sigma_3, 2, \sigma_2^3 \sigma_2^3$ |

Proposition 4.15. (1) For every $s \in \widehat{CH}^i(\mathcal{Gr}(k,n))$, it could be written as $s_1 + s_2$ where $s_1 \in GW(k)(p_j, p_j^i) \otimes \Lambda[R]$ ($R$ won’t appear if $k(n-k)$ is even) and $s_2 \in Ker(Sq^2 \circ \pi)_j$.

(2) For every $s \in \widehat{CH}^i(\mathcal{Gr}(k,n), O(1))$, it could be written as $s_1 + s_2$ where $s_1 \in GW(k)(p_j, p_j^i, e_k, e_{n-k}^i)$ and $s_2 \in Ker(Sq^2 \circ \pi)_j$.

Proof. (1) Suppose that $s$ corresponds to $(u, v) \in H^i(\mathcal{Gr}(k,n), \Gamma^i) \oplus \widehat{CH}^i(\mathcal{Gr}(k,n))$. By [Wen18a] Theorem 6.4, $u$ can be written as $u_1 + u_2$ where

\[
u_1 = \varphi(p_j, p_j^i)
\]

for some polynomial $\varphi$ with coefficients in $W(k)$ and $u_2 \in Im(\beta)$. The $\varphi$ can be lifted to

\[
\psi \in GW(k)(p_j, p_j^i) \otimes \Lambda[R] \subseteq \widehat{CH}^*(\mathcal{Gr}(k,n)).
\]

Hence $s - \psi \in 2N_{k,n}^i + Ker(Sq^2 \circ \pi)_j \subseteq \widehat{CH}^i(\mathcal{Gr}(k,n))$ (see Proposition 2.18). Then we could find

\[
\psi' \in h \cdot GW(k)(p_j, p_j^i) \otimes \Lambda[R]
\]

such that $s - \psi - \psi' \in Ker(Sq^2 \circ \pi)_j \subseteq \widehat{CH}^i(\mathcal{Gr}(k,n))$.

(2) Same proof as (1).

□

Proposition 4.16. For every $j \in \mathbb{N}$, we have a map $\varphi_j$ such that the following diagram commutes

\[
\begin{array}{ccc}
N_{k,n}^j & \xrightarrow{\varphi_j} & H^i(\mathcal{Gr}(k,n), W) \\
\downarrow \quad Ker(Sq^2)_j & & \downarrow E^i(\mathcal{Gr}(k,n)) \\
\end{array}
\]

Thus we obtain an injection

\[
u : N_{k,n}^j \otimes \mathbb{Z} \to \widehat{H}^j(\mathcal{Gr}(k,n)).
\]

When $k(n-k)$ is odd, the element $\nu(\sigma_{n-k,1}^{n-1})$ is denoted by $\mathcal{R}$, which is called the orientation class (see [Wen18a] Theorem 5.6).

Proof. Let us prove the first statement. The second follows from Corollary 2.16.
(1) Suppose \( j = 4i \) and let \( \varphi_j = \lambda \circ \gamma^{-1} \). The \( Im(Sq^2) \) is an ideal of \( Ker(Sq^2) \), which is a subalgebra of \( Ch^*(Gr(k, n)) \). Denote by \( \sim \) the equivalent relation on \( Ker(Sq^2) \) modulo by \( Im(Sq^2) \). So it suffices to prove that for every \( n - k \geq a_1 \geq \cdots \geq a_k \geq 0 \), we have

\[
d := \left| \begin{array}{ccc}
s_{2a_1} & s_{2a_1+2} & \cdots \\
s_{2a_2} & \cdots & \\
& & \\
\end{array} \right|^2 \sim \gamma(\sigma_{a_1, \ldots, a_k})
\]

by the Giambelli formula. Suppose that \( a_1 = 0 \) if \( t > l \). Let us prove by induction on \( t \). If \( l = 1 \), we have

\[
\sigma_{2j}^2 = \gamma(\sigma_j) + Sq^2(\sigma_{j-1} + \sigma_{j-3} + \cdots + \sigma_{j+1})
\]

In general case, the Cramer’s rule and induction gives

\[
d \sim \sum_{j=1}^{l} \gamma(\sigma_{a_j-1})\gamma(\sigma_{a_1+1, \ldots, a_j-1+a_j+1, \ldots, a_l})
\]

Again the Cramer’s rule says

\[
\sigma_{a_1, \ldots, a_l} = \sum_{j=1}^{l} \sigma_{a_j-1}\sigma_{a_1+1, \ldots, a_j-1+a_j+1, \ldots, a_l}
\]

so it remains to prove that

\[
\gamma(\sigma_a\sigma_{a_1, \ldots, a_l}) \sim \gamma(\sigma_a\gamma(\sigma_{a_1, \ldots, a_l})
\]

for arbitrary \( a, a_1, \ldots, a_l \). It suffices to prove the untruncated case at first, then apply the truncation map. But by Proposition \ref{prop:4.16} it suffices to compare their coefficients on even Young tableaux, which is obvious.

(2) Suppose \( j = 4i + n - 1 \) and let \( \varphi_j = \lambda' \circ \gamma'^{-1} \). The statement follows from (1) since we have

\[
Sq^2(x) \cdot \sigma_{n-k,1^{k-1}} = Sq^2(x \cdot \sigma_{n-k,1^{k-1}}).
\]

(3) Otherwise \( N^j_{k,n} \) vanishes.

\[
\square
\]

**Proposition 4.17.** For every \( j \in \mathbb{N} \), we have a map \( \psi_j \) such that the following diagram commutes

\[
\begin{array}{ccc}
M^j_{k,n} & \xrightarrow{\psi_j} & H^j(Gr(k, n), W(O(1))) \\
\downarrow & & \downarrow \\
Ker(Sq^2_{O(1)}) & \xrightarrow{\epsilon^j} & E^j(Gr(k, n), O(1))
\end{array}
\]

Thus we obtain an injection

\[
v : M^j_{k,n} \otimes_{\mathbb{Z}} GW(k) \longrightarrow CH^j(Gr(k, n), O(1)).
\]

*Proof.*

(1) If \( k \) is even, \( n \not\equiv 0 \pmod{4} \) and \( j = 4i + k \), we set \( \psi_j = \lambda_1 \circ \gamma_1^{-1} \). Then the statement follows from Proposition \ref{prop:4.16} because we have

\[
Sq^2(x) \cdot \sigma_{1^k} = Sq^2_{O(1)}(x \cdot \sigma_{1^k})
\]

(2) If \( n - k \) is even, \( n \not\equiv 0 \pmod{4} \) and \( j = 4i + n - k \), we set \( \psi_j = \lambda_2 \circ \gamma_2^{-1} \). Then the statement follows from Proposition \ref{prop:4.16} because we have

\[
Sq^2(x) \cdot \sigma_{n-k} = Sq^2_{O(1)}(x \cdot \sigma_{n-k})
\]

(3) If \( k(n - k) \) is even, \( n \equiv 0 \pmod{4} \) and \( j = 4i + k \), we set \( \psi_j = \lambda_3 \circ \gamma_3^{-1} \). Then the statement follows from (1) and (2).

(4) Otherwise \( M^j_{k,n} \) vanishes.

\[
\square
\]

The Proposition \ref{prop:4.16} and \ref{prop:4.17} were partially announced in [Wen18b Proposition 3.6]. We gave the complete version for clarity.

**Definition 4.18.** Suppose \( \Lambda = (a_1, \ldots, a_l) \) is an irredundant Young diagram, which is untwisted and \( \mathcal{L} = O_{Gr(k, n)} \) (resp. twisted and \( \mathcal{L} = O(1) \)).

If \( \Lambda \) is even, define

\[
p(\Lambda) = u(\Lambda) \in CH^{|\Lambda|}(Gr(k, n), \mathcal{L})
\]

by Proposition \ref{prop:4.16}. 

Otherwise for every $1 < i_1 < \cdots < i_t \leq |\Lambda(\Lambda)|$, define
\[ c(\Lambda_{i_1, \ldots, i_t}) \in \eta^{\lfloor \Lambda_{i_1, \ldots, i_t} \rfloor}_{\text{MW}}(Gr(k, n), \mathbb{Z}) \]
to be the unique morphism corresponding to
\[ (\Lambda_{i_1, \ldots, i_t}, Sq^2_\mathbb{Z}(\Lambda_{i_1, \ldots, i_t})) \]
by Theorem 2.17, where the $Sq^2$ here is the integral lift (see remarks before Proposition 4.10).

**Theorem 4.19.** In the context above, we have:

1. The morphism
   \[ Z(Gr(k, n)) \xrightarrow{(p(\Lambda), c(\Lambda_{i_1, \ldots, i_t}))} \bigoplus_{\Lambda \text{ even}} Z(|\Lambda|) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_t > 1} Z/\eta(|\Lambda_{i_1, \ldots, i_t}|) \]
   is an isomorphism in $\widetilde{DM}^\text{eff}(pt, \mathbb{Z})$.

2. The morphism
   \[ Th(O_{\text{Gr}(k, n)}(1)) \xrightarrow{(p(\Lambda), c(\Lambda_{i_1, \ldots, i_t}))} \bigoplus_{\Lambda \text{ even}} Z(|\Lambda| + 1) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_t > 1} Z/\eta(|\Lambda_{i_1, \ldots, i_t}| + 1) \]
   is an isomorphism in $\widetilde{DM}^\text{eff}(pt, \mathbb{Z})$.

**Proof.** We adopt the notations in Proposition 2.17. The [Wen18b, Proposition 3.14] and Proposition 4.10 describe the basis of $H^*(Gr(k, n), W(\mathbb{Z}))$ and $Ker(Sq^2_\mathbb{Z} \circ \pi)$, respectively. So the indexes are correct by Proposition 2.17.

1. Suppose $\Lambda$ is even. We have
   \[ \gamma(u(\Lambda)) = \Lambda \]
   by Proposition 4.10. Suppose $(x, y) \in \eta^\text{MW}_{\text{Gr}}(Gr(k, n))$. We showed in Proposition 2.17 that
   \[ \gamma(h(x, y)) = 2x, \quad \gamma(\partial(x, y)) = y. \]
   Then the statement follows from Proposition 2.17, 4.12 and Corollary 4.11.

2. Same proof as above.

As an application of Theorem 4.19, we could discuss the invariance of Chow-Witt cycles of trivial Grassmannians under linear automorphisms.

**Proposition 4.20.** Let $X \in Sm/k$, $\mathcal{E}$ be a vector bundle of rank $n$ on $X$ and $\varphi \in Aut_{\mathcal{O}_X}(\mathcal{E})$. The morphism $Gr(k, \varphi) : Gr(k, n) \to Gr(k, n)$ is an invertible function.

**Proof.** It suffices to prove that $Gr(k, \varphi)$ keeps Chern classes of the tautological bundle $\mathcal{W}$ invariant by the Grassmannian bundle theorem in Voevodsky’s motives (see [Sem06, Proposition 2.4] and the construction of $c_\mathcal{E}$ in Proposition 5.3, which follows from construction.

**Proposition 4.21.** Suppose that $X \in Sm/k$, $\varphi \in GL_n(G^n_X)$ and that $p : X \times Gr(k, n) \to Gr(k, n)$ is the projection. We have
\[ Gr(k, \varphi) = \text{Id} \]
in $\widetilde{DM}^\text{eff}(pt, \mathbb{Z})$ if one of the following conditions hold:

1. The $\varphi$ is an elementary matrix (see [Ana16, Definition 3]);
2. The $k(n - k)$ is even and $2CH^*(X) = 0$.
3. The $k(n - k)$ is odd, $2CH^*(X) = 0$ and $\varphi = \begin{pmatrix} g^2 & \text{Id} \end{pmatrix}$ where $g : X \to \mathbb{G}_m$ is an invertible function.

**Proof.** The Theorem 4.19 gives us the formulas
\[ Z(X \times Gr(k, n)) \cong \bigoplus_{\Lambda \text{ even}} Z(X)(|\Lambda|) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_t > 1} Z(X)/\eta(|\Lambda_{i_1, \ldots, i_t}|) \]
and
\[ CH^i(X \times Gr(k, n)) \cong \bigoplus_{\Lambda \text{ even}} CH^i(|\Lambda|)(X) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_t > 1} \eta^i_{\text{MW}}(\Lambda_{i_1, \ldots, i_t}| - 1)(X) \]
where $\eta^i_{\text{MW}} = 2CH^i$ and $\eta^i_{\text{MW}} = 0$ if $i < -1$ (see [Yan21b, Proposition 5.8]).

1. Essentially the same as in [Ana16, Lemma 1].
(2) Suppose \( s \in \overline{CH}^i(X \times Gr(k, n)) \) is equal to the pullback of \( t \in \overline{CH}^i(Gr(k, n)) \) along \( p \). Let us consider its behaviour under \( Gr(k, \varphi)^* \). By Proposition 4.15 and Proposition 2.18, \( t = t_1 + t_2 \) where \( t_1 \in GW[k]p_1, p_2^i \) and \( t_2 \in Ker(Sq^2 \circ \pi)_i \subseteq \text{Im}(\partial) \). But \( j^{-1}_{MW}(X \times Gr(k, \varphi)) = \text{Id} \) by Proposition 4.20 and Yan21b, Theorem 4.13. Hence \( Gr(k, \varphi)^*(p^*(t_2)) = p^*(t_2) \) by the commutative diagram

\[
\begin{array}{ccc}
\eta^{-1}_{MW}(X \times Gr(k, n)) & \xrightarrow{a} & \overline{CH}^i(X \times Gr(k, n)) \\
Gr(k, \varphi)^* & \downarrow & Gr(k, \varphi)^* \\
\eta^{-1}_{MW}(X \times Gr(k, n)) & \xrightarrow{a} & \overline{CH}^i(X \times Gr(k, n))
\end{array}
\]

While the corresponding result for \( t_1 \) is clear. So \( Gr(k, \varphi)^*(s) = s \). Then the statement follows from Yan21b, Theorem 4.13.

(3) The \( Gr(k, \varphi) \) is equal to the composite

\[
X \times Gr(k, n) \xrightarrow{\Gamma_g \times \text{Id}_{Gr(k, n)}} X \times G_m \times Gr(k, n) \xrightarrow{\text{Id}_X \times u} X \times Gr(k, n)
\]

where \( \Gamma_g \) is the graph of \( g \) and \( u \) is the morphism

\[
G_m \times Gr(k, n) \xrightarrow{t} \left( (x_1, \ldots, x_n) \in V \quad \mapsto \quad (t^2x_1, x_2, \ldots, x_{n-1}) \in t \cdot V \right)
\]

But the squaring morphism \( -^2 : G_m \rightarrow G_m \) is the multiplication by \( h \) (see the proof of Yan21b, Proposition 5.16) and \( [Z(Gr(k, n))(1)]_1, Z(Gr(k, n))]_{MW} \) is a direct sum of \( W(k) \) by Yan21b, Proposition 5.4. Hence \( u \) coincides with the projection map in \( \overline{DM}^{eff}(pt, \mathbb{Z}) \). Hence \( Gr(k, \varphi) = \text{Id} \) in \( \overline{DM}^{eff}(pt, \mathbb{Z}) \) by loc. cit..

\[ \square \]

5. Theorems of Fiber Bundles

Definition 5.1. For every Young tableau \( \Lambda = (a_1, \ldots, a_t) \), we define its transpose \( \Lambda^T = (b_1, \ldots, b_s) \) by

\[ b_i = \# \text{ of boxes in } i\text{-th column of } \Lambda \]

where \( s = \# \text{ of columns in } \Lambda \). If \( \Lambda \) is \((k, n)\)-truncated, \( \Lambda^T \) is understood as \((n - k, n)\)-truncated.

For every Young tableau \( \Lambda \), we write \( \Lambda_i \) the length of its \( i\)-th row. For two tableaux \( S, \Lambda \), we say \( S \leq \Lambda \) if and only if \( S_i \leq \Lambda_i \) for all \( i \). Suppose \( x^{(n)} \) is a set of \( n \) indeterminants. Define the Schur function

\[ s_{\Lambda}(x^{(n)}) = \det(h_{\Lambda_i - i + j}(x^{(n)})) = \det(e_{\Lambda^T_j - j}(x^{(n)})). \]

Let us state a result between Schur functions and the Steenrod square, which has been essentially proved in Wen18b, Theorem 4.2.

Proposition 5.2. Let \( R \) be a commutative ring with identity and \( x_1, \ldots, x_n \) are indeterminants. Define

\[ x_{a_1, \ldots, a_t} = \begin{vmatrix} x_{a_1} & x_{a_1+1} & \cdots \\ x_{a_2} & x_{a_2+1} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}_{1 \times l} \]

where \( a_i \in \mathbb{N} \) for every \( 1 \leq i \leq l \), \( x_0 := 1 \) and \( x_i := 0 \) if \( i \notin [0, n] \).

If \( R = \mathbb{Z}/2\mathbb{Z} \), define \( Sq^2 : R[x_1, \ldots, x_n] \rightarrow R[x_1, \ldots, x_n] \) to be the \( R \)-derivation satisfying

\[ Sq^2(x_i) = (i + 1)x_{i+1} + x_1x_i. \]

We have

1. The \( \{x_{a_1, \ldots, a_t} \}_{n \geq a_1 \geq \ldots \geq a_t \geq 0} \) form an \( R \)-basis of \( R[x_1, \ldots, x_n] \).
2. Suppose \( a_1 \geq \cdots \geq a_t \), we have the Pieri formula

\[ x_{t}x_{a_1, \ldots, a_t} = \sum_{b = (b_1, \ldots, b_{t+1})} x_{b} \]

where \( b_1 \geq \cdots \geq b_{t+1} \), \( \sum b_i = t + \sum a_i \) and \( a_i \leq b_i \leq a_{i+1} \) for every \( i \).

3. \( Sq^2(x_{a_1, \ldots, a_t}) = \sum_{i=1}^{l} (a_i - i + 1)x_{\cdots a_{i-1}, a_i, a_{i+1}, \cdots} + l \cdot x_{a_1, \ldots, a_t} \).

4. We have

\[ \text{Ker}(Sq^2) = \text{Im}(Sq^2) \oplus N \]

where \( N \) is the vector space generated by \( x_{a_1, \ldots, a_t} \) where \( (a_1, \ldots, a_t) \) is completely even.
Proof. Set
\[ s_i : \mathbb{N}^\times l \to \mathbb{N}^\times l \]
\[ (a_1, \ldots, a_l) \mapsto (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots) . \]
Recall we have the Cramer’s rule
\[ x_{a_1, \ldots, a_l} = \sum_{i=1}^l (-1)^{i-1} x_{a_i - i + 2} x_{(s_1 \ldots s_{i-1})}(\ldots \tilde{a}_i \ldots) . \]
It implies that
\[ \sum_{i=1}^l (-1)^{i-1} x_{a_i - i + 2} x_{(s_1 \ldots s_{i-1})}(\ldots \tilde{a}_i \ldots) = 0 . \]
The \( e_i(y_1, \ldots, y_n), i = 1, \ldots, n \) are algebraically independent so we may as well set \( x_i = e_i \). Then the \( R[x_1, \ldots, x_n] \) is identified with symmetric polynomials in \( R[y_1, \ldots, y_n] \) and \( \{ x_{a_1, \ldots, a_l} \} \) are Schur polynomials.

(1) This is because the Schur polynomials form a basis of the symmetric polynomials.

(2) The statement follows from the Pieri formula of Schur polynomials.

(3) It’s easy to verify the statement when \( l = 1 \). Let us prove by induction on \( l \) and suppose \( l \geq 2 \).

We have
\[ S^2(x_{a_1, \ldots, a_l}) \]
\[ = \sum_{i=1}^l S^2(x_{a_i - i + 2} x_{(s_1 \ldots s_{i-1})}) \]
\[ = \sum_{i=1}^l ((a_i - i + 2)x_{a_i - i + 2} + x_{a_i - i + 2}) x_{(s_1 \ldots s_{i-1})} \]
\[ + \sum_{j=i+1}^l ((a_j - j + 2)x_{s_j \ldots s_{i-1}}) \]
\[ = \sum_{i=1}^l ((a_i - i + 2)x_{a_i - i + 2} + x_{a_i - i + 2}) x_{(s_1 \ldots s_{i-1})} \]
\[ + \sum_{j=i+1}^l (a_j - j + 2)x_{s_j \ldots s_{i-1}} \]
\[ = \sum_{i=1}^l (a_i - i + 1)x_{a_i - i + 2} x_{(s_1 \ldots s_{i-1})} + (l - 1)x_{a_1, \ldots, a_{i-1}} \]
\[ + \sum_{j=i+1}^l (a_j - j + 2)x_{s_j \ldots s_{i-1}} \]
\[ = \sum_{i=1}^l (a_i - i + 1)x_{a_i - i + 2} x_{(s_1 \ldots s_{i-1})} \]
\[ + \sum_{j=i+1}^l (a_j - j + 1)x_{s_j \ldots s_{i-1}} \]
\[ = \sum_{i=1}^l (a_i - i + 1)x_{a_i - i + 2} x_{(s_1 \ldots s_{i-1})} + l \cdot x_{a_1, \ldots, a_{i-1}} \]
\[ + \sum_{j=i+1}^l (a_j - j + 1)x_{s_j \ldots s_{i-1}} \]
\[ = \sum_{i=1}^l (a_i - i + 1)x_{a_i - i + 1} x_{(s_1 \ldots s_{i-1})} + l \cdot x_{a_1, \ldots, a_{i-1}} \]
\[ + \sum_{j=i+1}^l (a_j - j + 1)x_{s_j \ldots s_{i-1}} . \]
(4) It is easy to show that \( S^2(S^2(x_i)) = 0 \) for every \( i \). Then
\[ S^2(S^2(\prod_j x_{i_j})) = \sum_k S^2(S^2(x_{i_k})) \prod_{j \neq k} x_{i_j} = 0 . \]
Hence \( S^2 \circ S^2 = 0 \). Then the proof is essentially the same as Lemma 1.7 and Proposition 1.8.

Given a vector bundle \( \mathcal{E} \) on \( X \), we have a tautological exact sequence on \( Gr(k, \mathcal{E}) \) which relates \( \mathcal{E} \), \( \mathcal{U}_\mathcal{E} \) and \( \mathcal{U}_\mathcal{E}^\perp \). It is interesting to express Schur functions of Chern roots of one of them in terms of other two.

**Proposition 5.3.** Let \( f(t) = 1 - a_1 t + \cdots + (-1)^k a_k t^k \), \( g(t) = 1 - b_1 t + \cdots + (-1)^{n-k} b_{n-k} t^{n-k} \) and \( h(t) = 1 - c_1 t + \cdots + (-1)^n c_n t^n \), where \( t \) is an indeterminate and all coefficients live in some commutative ring with identity. We could define \( a_\Lambda, b_\Lambda, c_\Lambda \) for every Young tableau \( \Lambda \) as in Proposition 5.3. Suppose that
\[ f(t)g(t) = h(t) . \]
Proposition 5.4.

Suppose that
\[ a_i = c_i = \frac{1}{i!} \sum_{n=0}^{\infty} \frac{1}{n!} x^n \]
for every \( m \in \mathbb{N} \) (\( a_0 := 1 \) and \( a_i := 0 \) if \( i \neq [0, k] \), similar convention for \( b \) and \( c \)).

(2) For every untruncated Young tableau \( \Lambda \), we have
\[
eq \sum_{\Lambda : \lambda, \mu \subseteq \Lambda} c_{\lambda, \mu} a_{\lambda} b_{\mu}
\]
where \( c_{\lambda, \mu} \) is the Littlewood-Richardson coefficient.

Proof. (1) We could regard \( a_i = c_i = e_i(x_1, \ldots, x_k) \) as the elementary symmetric function of degree \( i \) (similar for \( b_i = e_i(y_1, \ldots, y_{n-k}) \) and \( c_i = e_i(z_1, \ldots, z_n), z = (x, y) \)). Then the statements come from the fact that
\[
\frac{1}{x^i} = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n.
\]

(2) The \( c_{\lambda, \mu} \) is the coproduct of Schur function \( s_{\lambda \tau}(z^{(n)}) \). So the first equation follows from [Zel81 4.18].

The \( b_{\lambda} \) is the super Schur polynomial \( s_{\lambda \tau}(z^{(n)})/(x^{(k)}) \) (see [PT92 Definition 1.3]). By [PT92 Theorem 3.1], we have
\[
s_{\lambda \tau}(z^{(n)})/(x^{(k)}) = \sum_{\lambda, \mu \subseteq \Lambda} \alpha_{\lambda, \mu} s_{\lambda \tau}(z^{(n)})
\]
where \( \alpha_{\lambda, \mu} \in \mathbb{Z} \). Hence we have
\[
b_{\lambda} = \sum_{\lambda, \mu \subseteq \Lambda} \alpha_{\lambda, \mu} s_{\lambda \tau} c_{\mu} = \sum_{\lambda, \mu \subseteq \Lambda} \alpha_{\lambda, \mu} s_{\lambda \tau} c_{\mu}
\]
and
\[
b_{\lambda} = \sum_{\lambda, \mu \subseteq \Lambda} \alpha_{\lambda, \mu} s_{\lambda \tau} c_{\mu} = \sum_{\lambda, \mu \subseteq \Lambda} \alpha_{\lambda, \mu} s_{\lambda \tau} c_{\mu} + \sum_{\lambda, \mu \subseteq \Lambda} \alpha_{\lambda, \mu} s_{\lambda \tau} c_{\mu}.
\]
But \( a_{S} \) is a multiple of \( a_{k} \) if \( S \geq k \) (which is equivalent to \( (S^T)_k > 0 \)) and
\[
a_{\lambda} = \sum_{\lambda, \mu \subseteq \Lambda} \beta_{\lambda, \mu} b_{\lambda \tau} c_{\mu}.
\]
where \( \beta_{\lambda, \mu} \in \mathbb{Z} \) by the same arguments before. So we have proved the second equation. The third one follows symmetrically.

\[\square\]

Proposition 5.4. Suppose that \( X \in \text{Sm}/k \), \( 2CH^*(X) = 0 \) and that \( \mathcal{E} \) is a vector bundle of rank \( n \) on \( X \) trivialized by an acyclic covering \( \{ U_a \} \) (see [Yam21 Def. 5.12]). Denote by \( p : \text{Gr}(k, \mathcal{E}) \to X \) the structure map and \( w : \tilde{CH}^* \to H^*(-, \mathcal{E}) \) the canonical map.

(1) For every element \( s \in \eta_{MW}(\text{Gr}(k, n)) \) there is a canonical element \( \varphi(s) \in \eta_{MW}(\text{Gr}(k, \mathcal{E})) \) such that \( \varphi(s)|_{U_a} \) is the pullback of \( s \) along \( \text{Gr}(k, n) \times U_a \to \text{Gr}(k, n) \).

(2) Suppose both \( k \) and \( n \) are even, \( \mathcal{L} \in \text{Pic}(X) \) and \( \varphi(s) \in \eta_{MW}(\text{Gr}(k, n), \mathcal{O}(1)) \). There is a canonical element \( \varphi(s) \in \eta_{MW}(\text{Gr}(k, \mathcal{E}), p^*\mathcal{L} \otimes \mathcal{O}(1)) \) such that \( \varphi(s)|_{U_a} \) is the pullback of \( s \) along \( \text{Gr}(k, n) \times U_a \to \text{Gr}(k, n) \).

(3) Suppose \( k(n-k) \) is even. For every element \( s \in \text{CH}^i(\text{Gr}(k, n)) \) there is an element \( \psi(s) \in \text{CH}^i(\text{Gr}(k, \mathcal{E})) \) such that \( \psi(s)|_{U_a} \) is the pullback of \( s \) along \( \text{Gr}(k, n) \times U_a \to \text{Gr}(k, n) \).

(4) Suppose both \( k \) and \( n \) are even and \( s \in \text{CH}^i(\text{Gr}(k, n), \mathcal{O}(1)) \).

(a) If \( w(s) \in \mathcal{O}_{-k} : H^*(\text{Gr}(k, n), \mathcal{W}) \), there is an element \( \psi(s) \in \text{CH}^i(\text{Gr}(k, \mathcal{E}), \mathcal{O}(1)) \) such that \( \psi(s)|_{U_a} \) is the pullback of \( s \) along \( \text{Gr}(k, n) \times U_a \to \text{Gr}(k, n) \).

(b) If \( w(s) \in \mathcal{O}_{-k} : H^*(\text{Gr}(k, n), \mathcal{W}) \), there is an element \( \psi(s) \in \text{CH}^i(\text{Gr}(k, \mathcal{E}), p^*\text{det}(\mathcal{E}) \otimes \mathcal{O}(1)) \) such that \( \psi(s)|_{U_a} \) is the pullback of \( s \) along \( \text{Gr}(k, n) \times U_a \to \text{Gr}(k, n) \).
Proof. (1) By [Yan21b, Theorem 4.13] and Proposition 4.10, \( \eta_{MW}^{-1}(Gr(k, n)) \subseteq CH^1(Gr(k, n)) \oplus CH^{k+1}(Gr(k, n)) \) is freely generated by elements like 
\[
u_{a_1, \ldots, a_l} = (\sigma_{a_1, \ldots, a_l}, SQ^2(\sigma_{a_1, \ldots, a_l})) \quad \text{and} \quad \nu_{a_1, \ldots, a_l} = (0, 2\sigma_{a_1, \ldots, a_l}),
\]
We define a morphism 
\[
\varphi_{\mathcal{E}} : CH^*(Gr(k, n)) \longrightarrow CH^*(Gr(k, \mathcal{E}))
\]
\[
\sigma_{a_1, \ldots, a_l} \quad \mapsto \quad \begin{pmatrix} c_{a_1}(\mathcal{U}_\mathcal{E}^+)+c_{a_1+1}(\mathcal{U}_\mathcal{E}^+) & \cdots & c_{a_2}(\mathcal{U}_\mathcal{E}^+) & \cdots \\ \vdots & & \vdots & \ddots
\end{pmatrix}_{l \times 1}
\]
Define 
\[
\varphi_{\mathcal{E}}(\nu_{a_1, \ldots, a_l}) = (0, 2\varphi_{\mathcal{E}}(\sigma_{a_1, \ldots, a_l})).
\]
We have 
\[
SQ^2(c_i(\mathcal{U}_\mathcal{E}^+)) = (i + 1)c_{i+1}(\mathcal{U}_\mathcal{E}^+) + c_1(\mathcal{U}_\mathcal{E}^+)c_i(\mathcal{U}_\mathcal{E}^+)
\]
in \( CH^*(Gr(k, \mathcal{E})) \) by [Kio12, Proposition 5.5.2]. If \((a_1, \ldots, a_l) = T \cdot \sigma_1\) where \(T\) is completely even, \(l\) is even and \(T_{l+1} = 0\), we define 
\[
\varphi_{\mathcal{E}}(\nu_{a_1, \ldots, a_l}) = (\varphi_{\mathcal{E}}(T)c_1(\mathcal{U}_\mathcal{E}), \varphi_{\mathcal{E}}(T)(c_{l+1}(\mathcal{U}_\mathcal{E}) + c_1(\mathcal{U}_\mathcal{E})c_1(\mathcal{U}_\mathcal{E}))).
\]
Otherwise define 
\[
\varphi_{\mathcal{E}}(\nu_{a_1, \ldots, a_l}) = (\varphi_{\mathcal{E}}(\sigma_{a_1, \ldots, a_l}), \varphi_{\mathcal{E}}(\sum_{i=1}^l \mu(a_i - i + 1)\sigma_{a_{i-1}a_{i+1}a_{i+2} \ldots} + \mu(l)\sigma_{a_1, \ldots, a_l}))
\]
where 
\[
\mu : \begin{cases} \mathbb{Z} & \rightarrow \{0, 1\} \\ \text{odd numbers} & \rightarrow 1 \\ \text{even numbers} & \rightarrow 0
\end{cases}
\]
We have \( \varphi_{\mathcal{E}}(\nu_{a_1, \ldots, a_l}) \in \eta_{MW}^{-1}(Gr(k, \mathcal{E})) \) by Proposition 5.5.2. Then we expand the definition linearly to obtain a map 
\[
\varphi_{\mathcal{E}} : \eta_{MW}^{-1}(Gr(k, n)) \longrightarrow \eta_{MW}^{-1}(Gr(k, \mathcal{E})).
\]
(2) The proof is completely the same as (1). By Proposition 2.14 and 4.10, \( \eta_{MW}^{-1}(Gr(k, n), O(1)) \subseteq CH^{k+1}(Gr(k, n)) \oplus CH^1(Gr(k, n)) \) is freely generated by elements like 
\[
u_{a_1, \ldots, a_l} = (\sigma_{a_1, \ldots, a_l}, SQ^2_{O(1)}(\sigma_{a_1, \ldots, a_l})) \quad \text{and} \quad \nu_{a_1, \ldots, a_l} = (0, 2\sigma_{a_1, \ldots, a_l}).
\]
We define 
\[
\varphi_{\mathcal{E}}(\nu_{a_1, \ldots, a_l}) = (0, 2\varphi_{\mathcal{E}}(\sigma_{a_1, \ldots, a_l})).
\]
If \((a_1, \ldots, a_l) = T \cdot \sigma_1\) where \(T\) is completely even, \(l\) is even and \(T_{l+1} = 0\), we define 
\[
\varphi_{\mathcal{E}}(\nu_{a_1, \ldots, a_l}) = (\varphi_{\mathcal{E}}(T)c_1(\mathcal{U}_\mathcal{E}), \varphi_{\mathcal{E}}(T)(c_{l+1}(\mathcal{U}_\mathcal{E}) + c_1(\mathcal{U}_\mathcal{E})c_1(\mathcal{U}_\mathcal{E}))).
\]
Otherwise we define 
\[
\varphi_{\mathcal{E}}(\nu_{a_1, \ldots, a_l}) = (\varphi_{\mathcal{E}}(\sigma_{a_1, \ldots, a_l}), \varphi_{\mathcal{E}}(\sum_{i=1}^l \mu(a_i - i + 1)\sigma_{a_{i-1}a_{i+1}a_{i+2} \ldots} + \mu(l)\sigma_{a_1, \ldots, a_l} + \delta(\mathcal{E})))
\]
where \(\delta(\mathcal{E}) = c_1(p^*\mathcal{L} \otimes O(1))c_0(\sigma_{a_1, \ldots, a_l})\).
(3) By Proposition 2.15, there is a \(u'_1 = \psi(p_j, p_j^+)\) such that \(\psi\) is a polynomial with coefficients in \(\mathbf{GW}(k)\) and \(s - u'_1 \in Ker_1(Sq^2 \circ \pi_1)\). Suppose 
\[
t : Ker_1(Sq^2 \circ \pi_1) \longrightarrow \eta_{MW}^{-1}(Gr(k, n))
\]
is a section of \(\partial\) (see Proposition 2.18). Then \((\pi \circ \varphi_{\mathcal{E}} \circ t)(s - u'_1)\) is a global section of \(s - u'_1\). The global version of \(u'_1\) exists when \(k(n - k)\) is even, which is 
\[
x = \psi(p_j, p_j^+)(\mathcal{U}_\mathcal{E}).
\]
Then we set \(\psi_{\mathcal{E}}(s) = x + (\pi \circ \varphi_{\mathcal{E}} \circ t)(s - u'_1)\).
(4) (a) By Proposition 4.15, there is a \(u'_1 = \psi(p_j, p_j^+, c_k)\) such that \(\psi\) is a polynomial with coefficients in \(\mathbf{GW}(k)\) and \(s - u'_1 \in Ker_1(Sq^2_{O(1)} \circ \pi_1)\). Then proceed the same proof as (3). 
(b) By Proposition 4.15, there is a \(u'_1 = \psi(p_j, p_j^+, c_k^+)\) such that \(\psi\) is a polynomial with coefficients in \(\mathbf{GW}(k)\) and \(s - u'_1 \in Ker_1(Sq^2_{O(1)} \circ \pi_1)\). Then proceed the same proof as (3). \(\square\)

Now in order to perform more precise computation, we have to globalize the elements \(p(A)\) defined in Definition 4.18 in a canonical way, in addition to the \(\varphi_{\mathcal{E}}\) defined above. Note that the \(\psi_{\mathcal{E}}\) above is not canonical.
Proposition 5.5. Suppose that $X \in Sm/k$ and that $E$ is a vector bundle of rank $n$ over $X$. Recall the
morphisms $\gamma : CH^* \rightarrow CH$ and $w : CH^* \rightarrow H^*(-, W)$ defined in Proposition 2.14.

1. If $\mathbb{Z}(Gr(k, E))$ splits as an MW-motive, for every completely even Young tableau
   $\Lambda = (\cdots, 2a_1, 2a_2, \cdots)$,
   there is a unique element $p_\epsilon(\Lambda) \in \overline{CH}^{|\Lambda|}(Gr(k, E))$ such that
   $\gamma(p_\epsilon(\Lambda)) = c_\epsilon(\Lambda)$
   $w(p_\epsilon(\Lambda)) = \begin{vmatrix} p_{2a_1}(\mathcal{O}_E^+) & p_{2a_1+2}(\mathcal{O}_E^+) & \cdots \\ p_{2a_2}(\mathcal{O}_E^+) & p_{2a_2+2}(\mathcal{O}_E^+) & \cdots \end{vmatrix}.$

2. If $Th(O_{Gr(k, E)}(1))$ splits as an MW-motive, for every even Young tableau
   $\Lambda = (\cdots, 2a_1, 2a_2, \cdots) \cdot \sigma_1$ (when makes sense), there is a unique element $p_\epsilon(\Lambda) \in \overline{CH}^{|\Lambda|}(Gr(k, E), O(1))$ such that
   $\gamma(p_\epsilon(\Lambda)) = c_\epsilon(\sigma_1, 2a_1, \cdots) \cdot c_k(\mathcal{O}_E)$
   $w(p_\epsilon(\Lambda)) = \begin{vmatrix} p_{2a_1}(\mathcal{O}_E) & p_{2a_1+2}(\mathcal{O}_E) & \cdots \\ p_{2a_2}(\mathcal{O}_E) & p_{2a_2+2}(\mathcal{O}_E) & \cdots \end{vmatrix} \cdot e(\mathcal{O}_E)$.

3. If $Th(O_{Gr(k, E)}(1) \otimes p^* det(E))$ splits as an MW-motive, for every Young tableau
   $\Lambda = (\cdots, 2a_1, 2a_2, \cdots) \cdot \sigma_{n-k}$ (when makes sense), there is a unique element $p_\epsilon(\Lambda) \in \overline{CH}^{|\Lambda|}(Gr(k, E), p^* det(E) \otimes O(1))$ such that
   $\gamma(p_\epsilon(\Lambda)) = c_\epsilon(\sigma_1, 2a_1, \cdots) \cdot c_{n-k}(\mathcal{O}_E)$
   $w(p_\epsilon(\Lambda)) = \begin{vmatrix} p_{2a_1}(\mathcal{O}_E) & p_{2a_1+2}(\mathcal{O}_E) & \cdots \\ p_{2a_2}(\mathcal{O}_E) & p_{2a_2+2}(\mathcal{O}_E) & \cdots \end{vmatrix} \cdot e(\mathcal{O}_E)$.

where $p : Gr(k, E) \rightarrow X$ is the structure map.

Proof. (1) Suppose $|\Lambda| = n$. We have a commutative diagram with squares being Cartesian

$$\begin{array}{ccc} CH^n(Gr(k, E)) & \rightarrow & Ker(Sq^2 \circ \pi)_n \\
\downarrow & & \downarrow \\
H^n(Gr(k, E), W) & \rightarrow & E^n(Gr(k, E))
\end{array}$$

by Corollary 2.10. Then the proof is essentially the same as Proposition 4.10 by Proposition 5.2.

(2) The Cartesian squares in (1) hold if we replace $Sq^2$ by $Sq^2_{O(1)}$. Then the statement follows from
same method as in Proposition 4.17.

(3) The Cartesian squares in (1) hold if we replace $Sq^2$ by $Sq^2_{p^* det(E) \otimes O(1)}$. Then the statement follows from same method as in Proposition 4.17.

△

Suppose $f, g \in Hom_\mathcal{C}(A, B)$ are morphisms in some triangulated category $\mathcal{C}$. We say $f$ could be
simplified to $g$ if $g = h \circ f$ where $h \in Aut(B)$. If $B = C \oplus D$, any morphism $C \rightarrow D$ gives an element of
$Aut(C \oplus D)$ by an elementary matrix. Simplification of morphisms does not change their mapping cone.

Theorem 5.6. Let $S \in Sm/k$, $X \in Sm/S$ being quasi-projective, $\mathcal{L} \in Pic(X)$ and $E$ be a vector bundle
of rank $n$ over $X$. Denote by $p : Gr(k, E) \rightarrow X$ is the structure map.

1. We have
   $$R(Gr(k, E))/\eta \cong \bigoplus_{\Lambda \ (k, n)\text{-truncated}} R(X)/\eta(|\Lambda|)$$
in $\overline{DM}(S, R)$.

2. If $k(n - k)$ is even, we have
   $$Th(p_* \mathcal{L}) \cong \bigoplus_{\Lambda \ \text{even}} Th(\mathcal{L})(|\Lambda|) \oplus \bigoplus_{\Lambda \ \text{irred. not full, } i_i > 1} R(X)/\eta(|\Lambda|, i_i + 1))$$
in $\overline{DM}^{eff}(S, R)$.
may suppose trick we may assume

\( \text{Th}(p^* \mathcal{L} \otimes O(1)) \cong \bigoplus_{\Lambda = \pi_{k,n}} \text{Th}(\det(\mathcal{E})^\vee \otimes \mathcal{L})(((|\Lambda|)) \oplus \bigoplus_{\Lambda \text{ even}} \text{Th}(\mathcal{L})(((|\Lambda|)) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_1 > 1} R(X)/\eta((|\Lambda_{i_1}, \ldots, i_1| + 1)) \)

in \( \mathcal{D}\mathcal{M}(S, R) \), where \( T \) is completely even.

(4) If \( n - k \) is odd, we have

\( \text{Th}(p^* \mathcal{L} \otimes O(1)) \cong \bigoplus_{\Lambda = \pi_{k,n}} \text{Th}(\mathcal{L})(((|\Lambda|)) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_1 > 1} R(X)/\eta((|\Lambda_{i_1}, \ldots, i_1| + 1)) \)

in \( \mathcal{D}\mathcal{M}(S, R) \).

(5) If \( k \) and \( n \) are odd, we have

\( \text{Th}(p^* \mathcal{L} \otimes O(1)) \cong \bigoplus_{\Lambda = \pi_{k,n}} \text{Th}(\mathcal{L} \otimes \det(\mathcal{E})^\vee)((|\Lambda|)) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_1 > 1} R(X)/\eta((|\Lambda_{i_1}, \ldots, i_1| + 1)) \)

in \( \mathcal{D}\mathcal{M}(S, R) \).

(6) If \( k \) is odd, \( n \) is even and \( e(E) = 0 \in C \hat{H}^n(X, \det(E)^\vee) \), there is an isomorphism

\( \text{Th}(p^* \mathcal{L} \otimes O(1)) \cong \bigoplus_{\Lambda = \pi_{k,n}} \text{Th}(\mathcal{L} \otimes \det(\mathcal{E}))((|\Lambda|)) \oplus \bigoplus_{\Lambda \text{ irreducible, not full, } i_1 > 1} R(X)/\eta((|\Lambda_{i_1}, \ldots, i_1| + 1)) \)

in \( \mathcal{D}\mathcal{M}(S, R) \), where \( T \) is completely even.

Proof. It suffices to prove the case \( R = \mathbb{Z} \) and \( \mathcal{L} = \mathcal{O}_X \). Since \( X \) is quasi-projective, by Jouanolou’s trick we may assume \( X \) is affine. So there is a map \( f : X \to Gr(n, m) \) such that \( f^* \mathcal{V}_{n,m} = \mathcal{E} \). So we may suppose \( X = S = Gr(n, m) \), in particular, \( \mathbb{Z}(X) \) splits.

(1) By tensoring the formulas in Theorem 4.19 with \( \mathbb{Z}/\eta \) we obtain an isomorphism

\( \mathbb{Z}(Gr(k, n))/\eta \overset{\delta_k}{\to} \bigoplus_{\Lambda \text{ (k, n)-truncated}} \mathbb{Z}/\eta((|\Lambda|)) \)

in \( \mathcal{D}\mathcal{M}_\eta^{eff}(pt, \mathbb{Z}) \) by [Yan21b Proposition 5.4]. We have

\( [\mathbb{Z}(Gr(k, n))/\eta, \mathbb{Z}/\eta(|\Lambda|)]_{MW} = \eta_{MW}^{-1}(Gr(k, n)) \oplus \eta_{MW}^1(Gr(k, n)) \)

by strong duality of \( \mathbb{Z}/\eta \) (see [Yan21b Proposition 5.8]). So by Proposition 5.4, we could find

\( (t_\Lambda)_{|U_i} = (p|U_i)^* (t_\Lambda) \)

under the chosen trivialization of \( E|_{U_i} \), where \( p : Gr(k, \mathcal{E}) \to X \) is the structure map. Then the map

\( \mathbb{Z}(Gr(k, \mathcal{E}))/\eta \overset{(t_\Lambda)_{|U_i}}{\to} \mathbb{Z}/\eta(|\Lambda|) \)

is an isomorphism in \( \mathcal{D}\mathcal{M}(X, \mathbb{Z}) \) by [Yan21b Proposition 2.4]. Finally we apply \( q_\# \) where \( q : X \to S \) is the structure map.

(2) We have morphisms

\( \psi(T) : \mathbb{Z}(Gr(k, \mathcal{E})) \to \mathbb{Z}(X)/(|\Lambda|)) \quad \varphi(\Lambda) : \mathbb{Z}(Gr(k, \mathcal{E})) \to \mathbb{Z}(X)/\eta(|\Lambda|) \)

by Proposition 5.4, which gives the isomorphism desired using [Yan21b Proposition 2.4], where \( T \) is completely even and \( \Lambda \) is not full as in Theorem 4.19. Finally we tensor the equation with \( \text{Th}(\mathcal{L}) \) and apply \( q_\# \).

(3) We have morphisms

\( \psi_{\sigma_{k,T}}(\text{Th}(O_{Gr(k, \mathcal{E})}(1)) \to \mathbb{Z}(X)/(|\sigma_{k,T}| + 1)) \quad \varphi(\Lambda) : \mathbb{Z}(Gr(k, \mathcal{E})) \to \mathbb{Z}(X)/\eta(|\Lambda|) \)

by Proposition 5.4, where \( T \) and \( \Lambda \) are same as above. Then proceeds in the same way as (2).

(4) We have morphisms

\( \psi_{\sigma_{k,T}}(\text{Th}(O_{Gr(k, \mathcal{E})}(1)) \to \mathbb{Z}(X)/(|\sigma_{k,T}| + 1)) \quad \varphi(\Lambda) : \mathbb{Z}(Gr(k, \mathcal{E})) \to \mathbb{Z}(X)/\eta(|\Lambda|) \)

by Proposition 5.4, where \( T \) and \( \Lambda \) are same as above. Then proceeds in the same way as (2).

(5) We have morphisms

\( \psi_{\sigma_{k,T}}(\text{Th}(O_{Gr(k, \mathcal{E})}(1)) \to \mathbb{Z}(X)/(|\sigma_{k,T}| + 1)) \quad \varphi(\Lambda) : \mathbb{Z}(Gr(k, \mathcal{E})) \to \mathbb{Z}(X)/\eta(|\Lambda|) \)

by Proposition 5.4, where \( T \) and \( \Lambda \) are same as above. Then proceeds in the same way as (2).
(6) We will adopt the notation in Proposition 2.17. We have a distinguished triangle

\[ Th(O_{Gr(k-1, \mathcal{E})}(1)) \xrightarrow{i^*} Th(O_{Gr(k, \mathcal{E} \otimes O_X)}(1)) \xrightarrow{\varphi} \mathbb{Z}(Gr(k, \mathcal{E}))(|k + 1|) \rightarrow \cdots \]

where \( i : Gr(k - 1, \mathcal{E}) \rightarrow Gr(k, \mathcal{E} \otimes O_X) \) is the canonical embedding.

By the discussion above, we have already had the splitting formula for the first two terms, which split by \( X = Gr(n, m) \). Then we could apply the precise classes defined in Proposition 5.4. The idea is to compute \( i^* \) with respect to their splitting and simplify \( i^* \). We will denote by \( T \) a completely even Young tableau.

Suppose \( \Lambda = T \cdot \sigma_{n+1-k} \) is a \((k, n+1)\)-truncated twisted even Young tableau. If \( \Lambda_k = 0 \), we have

\[ i^*(p_\mathcal{E}(\Lambda)) = p_\mathcal{E}(\Lambda) \]

by Proposition 5.5. Let us suppose \( \Lambda_k > 0 \), so \( T_{k-1} > 0 \). By Proposition 2.14, we have

\[ \sum_a p_{2a}(\mathcal{H}_2)|t_{2a}|^2 \sum_b p_{2b}(\mathcal{H}_2)|t_{2b}|^2 = \sum_c p_{2c}(q^* \mathcal{E})|t_{2c}|^2 \]

in \( H^*(Gr(k-1, \mathcal{E}), \mathbb{W})[t] \), where \( q \) is the structure map of \( Gr(k-1, \mathcal{E}) \). Let \( s = -t^2 \). We obtain the following equation by Proposition 5.3:

\[ w(p_\mathcal{E}(T)) \equiv \sum_{C \leq T, C_{k-1} = 0} w(p_\mathcal{E}(C)) \cdot Z[p_{2c}(q^* \mathcal{E})] \pmod{e(\mathcal{H}_2)}. \]

Hence

\[ w(p_\mathcal{E}(\Lambda)) \equiv \sum_{C \leq T, C_{k-1} = 0} w(p_\mathcal{E}(C) \cdot \sigma_{n-k+1}) \cdot f_C(p_{2c}(q^* \mathcal{E})) \pmod{e(q^* \mathcal{E})}. \]

where \( f_C \) is a polynomial with integer coefficients. Hence we could simplify \( i^* \) to some \( \varphi \) via the morphisms

\[ Th(det(\mathcal{E}))(|[C] + n - k + 1|) \xrightarrow{\varphi} Th(det(\mathcal{E}))(|\Lambda|) \]

for every \( C \) above, so that \( w(p_\mathcal{E}(\Lambda) \circ \varphi) \equiv 0 \pmod{e(q^* \mathcal{E})} \). Thus

\[ (\pi \circ \gamma)(p_\mathcal{E}(\Lambda) \circ \varphi) \in Im(Sq^0_{q^* \mathcal{E} \otimes O(1)}) \pmod{e(q^* \mathcal{E})} \]

by Corollary 2.10. So \( p_\mathcal{E}(\Lambda) \circ \varphi \) is equal, up to a multiple of \( e(q^* \mathcal{E}) \), to a composite

\[ Th(O_{Gr(k-1, \mathcal{E})}(1)) \xrightarrow{\varphi} Th(det(\mathcal{E})/\eta(\Lambda - 1)) \xrightarrow{\varphi} Th(det(\mathcal{E}))(\Lambda), \]

where the first arrow comes from a sum composites of three kinds, according to the decomposition given by (3). Let us discuss case by case:

(a) A composite induced by \( C = T \cdot \sigma_{n-k+1} \).

In this case, the composite factors through \( i^* \) and \( C \neq \Lambda \).

(b) A composite induced by \( C = T \cdot \sigma_{1-k} \).

In this case, \( C \) is an irredundant non-even \((k, n+1)\)-truncated twisted tableau. There is a commutative diagram

\[ Th(O_{Gr(k-1, \mathcal{E})}(1)) \xrightarrow{i^*} Th(O_{Gr(k, \mathcal{E} \otimes O_X)}(1)) \]

\[ \xrightarrow{p_\mathcal{E}(C)} \xrightarrow{\varphi_\mathcal{E}(C)} \]

\[ \mathbb{Z}(X)(|[C] + 1|) \xrightarrow{\varphi_\mathcal{E}(C)} \mathbb{Z}(X)(|[C] + 1|)/\eta \]

where the lower horizontal arrow is given by the quotient map. Hence the composite factors through \( i^* \) and \( C \neq \Lambda \).

(c) A composite induced by \( C \) which is irredundant and not even.

In this case, the composite factors through \( i^* \) and \( C \neq \Lambda \).

Hence we see that \( \varphi \) could be simplified so that

\[ p_\mathcal{E}(\Lambda) \circ \varphi = 0 \pmod{e(q^* \mathcal{E})}. \]

Suppose \( \Lambda = C_{i_1, \ldots, i_1} \cdot i_1 > 1 \) where \( C \) is a \((k, n+1)\)-truncated irredundant twisted Young tableau. If \( \Lambda_k = 0 \) and \( |A(\Lambda)| > 1 \), there is an obvious naturality. If \( \Lambda_k = 0 \) and \( |A(\Lambda)| = 1 \), \( \Lambda = T \cdot \sigma_{1-k} \) is even, regarded as a \((k-1, n)\)-truncated twisted Young tableau. Then \( \varphi_\mathcal{E}(\Lambda) \circ \varphi \) is the composite

\[ Th(O_{Gr(k-1, \mathcal{E})}(1)) \xrightarrow{p_\mathcal{E}(\Lambda)} \mathbb{Z}(X)(|\Lambda| + 1) \xrightarrow{\varphi} \mathbb{Z}(X)/\eta(|\Lambda| + 1) \]

by Proposition 5.4. If \( \Lambda_k > 0 \), we could simplify \( \varphi \) according to the decomposition of \( Th(O_{Gr(k-1, \mathcal{E})}(1)) \) as above so that \( \varphi_\mathcal{E}(\Lambda) \circ \varphi = 0 \).
Now pullback everything along $f$, the $e(q^*\mathcal{E})$ all vanish. Summarizing the computation above, we see that

\[
Z(Gr(k, \mathcal{E})) = \bigoplus_{\Delta \in S_1} Z(X)((|\Delta| - k + 1)) \oplus \bigoplus_{\Delta \in S_2} Th(det(\mathcal{E}^\vee))((|\Delta| - k - 1)) \oplus \bigoplus_{\Delta \in S_3} Z(X)/\eta(|\Delta| - k)
\]

where

\[
S_1 = \{\sigma_{1-k} \cdot T\} \quad S_2 = \{\sigma_{n-k+1} \cdot T|T_{k-1} > 0\} \quad S_3 = \{C_{1, \ldots, 1} \mid i_1 > 1, C_k > 0\}
\]

with $C$ irredundant non-even and $T$ completely even. Then we obtain the statement by erasing the first column of every tableau in $S_1, S_2, S_3$, in order to get bijections between indexes. \hfill \qed

Suppose $A \in \tilde{DM}^{\text{eff}}(pt, \mathbb{Z})$ splits. In order to compute $A$, it suffices to compute its image in $DM$ and $\tilde{DM}_\mathfrak{g}$ by Lemma 2.11. For flag varieties $Gr(d_1, \ldots, d_t)$, their decomposition in $DM$ is well-known.

**Proposition 5.7.** For any odd $n$ and $\mathcal{M} \in \text{Pic}(Gr(n-2, n-1, n))$, $Th(\mathcal{M})$ splits as an MW-motive and we have

\[
Th(\mathcal{M}) = \begin{cases} Z[1] \oplus Z[2n-2] & \mathcal{M} = 0 \\ 0 & \text{else} \end{cases}
\]

in $\tilde{DM}_\mathfrak{g}$. So $Th(\mathcal{M})$ are mutually isomorphic in $\tilde{DM}^{\text{eff}}(pt, R)$ if $\mathcal{M} \neq 0$. Denote by $F$ this common object.

Suppose that $S \in Sm/k, X \in Sm/S$ is quasi-projective, $\mathcal{M} \in \text{Pic}(\mathbb{P}(\Omega_p(1)))$ and that $\mathcal{E}$ is a vector bundle of odd rank $n$ on $X$. Denote by $p: P(E) \to X, q: \mathbb{P}(\Omega_p(1)) \to \mathbb{P}(E)$ the structure maps. We have

\[
Th(\mathcal{M}) \cong \begin{cases} Th(\mathcal{L}) \otimes R(Gr(n-2, n-1, n)) = p_n-1(\mathcal{E}) = 0 \in H^{2n-2}(X, \mathcal{W}), & \mathcal{M} = q^*p^*\mathcal{L} \\ R(X) \otimes F & \mathcal{M} \notin \text{Pic}(X) \end{cases}
\]

in $\tilde{DM}(S, R)$.

**Proof.** If $\mathcal{M}$ does not come from $X$, Proposition 3.8 shows that we may suppose $\mathcal{M} = (q^*\mathcal{L})(1)$. Then the statements follow from applying Theorem 5.6 on both $\mathcal{E}$ and $\Omega_p(1)$. Otherwise we may suppose $S = X = Gr(n, m)$ and $R = \mathbb{Z}$ as before. The Whitney sum formula for even Pontryagin classes tells us that

\[
p^*(p_n-1(\mathcal{E})) = p_n-1(\Omega_p(1)) = e(\Omega_p(1))^2 \in H^{2n-2}(\mathbb{P}(\mathcal{E}), \mathcal{W})
\]

hence they all vanish. The composite

\[
\mathbb{Z}(\mathbb{P}(\Omega_p(1))) \xrightarrow{e(\Omega_p(1))} Th(p^*det(\mathcal{E}^\vee)(1))(n-3)[2n-5] \xrightarrow{e(\Omega_p(1))p^*} \mathbb{Z}(\mathbb{P}(\mathcal{E})) \xrightarrow{\mathbb{Z}(\mathbb{P}(\mathcal{E}))/\eta(2)} \mathbb{Z}(\mathbb{P}(\mathcal{E}))/\eta(2i-1))
\]

is zero by $e(\Omega_p(1))^2 = 0$ and the second arrow is an isomorphism after applying $L$ by Theorem 5.6 so the first arrow is zero. Hence we have

\[
\mathbb{Z}(\mathbb{P}(\Omega_p(1))) \cong Th(p^*det(\mathcal{E}^\vee)(1))(n-3) \oplus \mathbb{Z}(\mathbb{P}(\mathcal{E})) \oplus \bigoplus_{i=1}^{\frac{n-3}{2}} \mathbb{Z}(\mathbb{P}(\mathcal{E})) / \eta(2i-1)
\]

by [Yan21b Corollary 5.15]. Then the statement follows from Theorem 5.6. \hfill \qed

**Theorem 5.8.** Suppose $\mathcal{L} \in \text{Pic}(Gr(1, \ldots, n))$. The $Th(\mathcal{L})$ splits as an MW-motive. Moreover, we have

1. If $n$ is odd, define $\text{deg}(a) = 4a - 1$. We have

\[
Th(\mathcal{L}) = \begin{cases} \bigoplus_{1 \leq t \leq \frac{n+1}{2}} \bigoplus_{1 \leq a_1 < \ldots < a_t \leq \frac{n+1}{2}} \mathbb{Z}[1 + \sum_a \text{deg}(a)] & \mathcal{L} = 0 \\ 0 & \text{else} \end{cases}
\]

in $\tilde{DM}_\mathfrak{g}$.

2. If $n$ is even, define

\[
\text{deg}(a) = \begin{cases} 4a - 1 & 1 \leq a \leq \frac{n}{2} - 1 \\ n - 1 & a = \frac{n}{2} \end{cases}
\]

We have

\[
Th(\mathcal{L}) = \begin{cases} \bigoplus_{1 \leq t \leq \frac{n}{2}} \bigoplus_{1 \leq a_1 < \ldots < a_t \leq \frac{n}{2}} \mathbb{Z}[1 + \sum_a \text{deg}(a)] & \mathcal{L} = 0 \\ 0 & \text{else} \end{cases}
\]

in $\tilde{DM}_\mathfrak{g}$. 

\hfill \qed
So $\text{Th}(\mathcal{L})$ are mutually isomorphic in $\overline{DM}_{eff}(pt,R)$ if $\mathcal{L} \neq 0$. Denote by $F$ this common object.

Suppose $X \in Sm/S$ is quasi-projective, $\mathcal{M} \in \text{Pic}(\text{Fl}(\mathcal{E}))$ and $\mathcal{E}$ is a vector bundle of rank $n$ on $X$. Denote by $p : \text{Fl}(\mathcal{E}) \rightarrow X$ the structure map. We have

$$\text{Th}(\mathcal{M}) \cong \begin{cases} \text{Th}(\mathcal{L}) \otimes R(\text{Gr}(1, \cdots, n)) & p_!(\mathcal{E}), e(\mathcal{E}) = 0 \in H^*(X, W(-)), \forall i, \mathcal{M} = p^* \mathcal{L} \smallsetminus \text{Pic}(X) \\ R(X) \otimes F & \mathcal{M} \notin \text{Pic}(X) \end{cases}$$

in $\overline{DM}(S,R)$.

**Proof.** Follows from Proposition 5.7 and Theorem 5.6. \qed

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