ON THE EXISTENCE OF STABLE BUNDLES WITH PRESCRIBED
CHERN CLASSES ON CALABI-YAU THREEFOLDS

BJÖRN ANDREAS AND GOTTFRIED CURIO

Abstract. We prove a case of the conjecture of Douglas, Reinbacher and Yau about
the existence of stable vector bundles with prescribed Chern classes on a Calabi-Yau
threefold. For this purpose we prove the existence of certain stable vector bundle
extensions over elliptically fibered Calabi-Yau threefolds.

1. Introduction

The present note is concerned with the question of existence of stable bundles $V$ with
prescribed Chern class $c_2(V)$ on a given Calabi-Yau threefold $X$. The ‘DRY’-conjecture
of Douglas, Reinbacher and Yau in [1] gives a sufficient condition for cohomology classes
on $X$ to be equal to the Chern classes of a stable sheaf $V$. In this note we consider
the case that $V$ is actually a vector bundle and $c_1(V) = 0$. In [4] we showed that
infinitely many classes on an $X$ which is elliptically fibered over a base surface $B$ exist
for which the conjecture is true. A weak form of the conjecture (considered already in
[7]) asserts the existence of $V$ with a suitable prescribed second Chern class $c$. In [8]
we showed that rank 4 polystable vector bundles exist for such suitable cohomology classes
$c$ (‘DRY-classes’, cf. the definition below). In this note we prove the weak form of the
conjecture by providing corresponding stable bundles $V$, for all ranks $N \geq 4$, for $B$ a
Hirzebruch surface $F_g$ or a del Pezzo surface $dP_k$, in all but finitely many cases.

We recall the following definition (for use in this note) and the weak DRY-conjecture

Definition 1.1. Let $X$ be a Calabi-Yau threefold of $\pi_1(X) = 0$ and $c \in H^4(X, \mathbb{Z})$,

(i) $c$ is called a Chern class if a stable $SU(N)$ vector bundle $V$ on $X$ exists with
$c = c_2(V)$,

(ii) $c$ is called a DRY class if an ample class $H \in H^2(X, \mathbb{R})$ exists (and an integer $N$) with

$c = N \left( H^2 + \frac{c_2(X)}{24} \right). \quad (1.1)$

Conjecture 1.2. On a Calabi-Yau threefold $X$ with $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbb{Z})$ is a Chern class.

B. A. is supported by DFG SFB 647: Space-Time-Matter. Analytic and Geometric Structures.
G. C. is supported by DFG grant CU 191/1-1; ASC Report-Nr. LMU-ASC 09/11.
Here it is understood that the integer $N$ occurring in the two definitions is the same.

In this paper we will construct a class of stable bundle extensions on an elliptically fibered Calabi-Yau threefold $\pi: X \to B$ with section $\sigma$ (we also denote by $\sigma$ the image divisor in $X$ and its cohomology class). We will consider $B$ to be a surface with ample $K_B^{-1}$ such as the Hirzebruch surface $F_g$ with $g = 0, 1$ or the del Pezzo surface $dP_k$ with $k = 0, \ldots , 8$ (note that $dP_1 \cong F_1$).

The main result of this note is (using the decomposition $H^4(X, \mathbb{Z}) \cong H^2(B, \mathbb{Z})\sigma \oplus H^4(B, \mathbb{Z})$; we usually identify $H^4(B, \mathbb{Z})$ with $\mathbb{Z}$).

**Theorem 1.3.** Let the class $c = \phi \sigma + \omega$ be a DRY class. Then $c$ is a Chern class

(i) up to finitely many exceptions in $c$
   (a) for $N \geq 4$ and $B = F_0$,
   (b) for $N \geq 6$ and $B = dP_k$ for $k = 1, \ldots , 8$

(ii) without any exceptions in $c$
   (a) for $N \equiv 2 \pmod{4}$ with $N \neq 2$ and either $B = F_g$ or $B = dP_k$ with $k = 1, \ldots , 6$,
   (b) for $N \equiv 0 \pmod{4}$ and $B = F_0$.

This note has two parts. In section 2 we prove the existence of a stable bundle extensions under certain conditions (equations (2.5)-(2.7)) on the input data.

In section 3 we apply this to the weak DRY conjecture by showing that, for a class of cases described precisely below, a DRY class fulfills the assumptions of section 2.

### 2. Stable Bundle Extensions

In this section we will construct a class of stable bundle extensions which will later, in section 3, serve as the bundles which realize a given DRY class as Chern class.

We consider the following extension

$$0 \to \pi^* E \otimes \mathcal{O}_X(-nD) \to V \to W \otimes \mathcal{O}_X(rD) \to 0$$

(2.1)

where $E$ is a stable rank $r$ vector bundle on $B$ with Chern classes $c_1(E) = 0$; these bundles exist on rational surfaces if $c_2(E) \geq r + 2$ [3]; $D = \pi^* \alpha$ is a divisor in $X$ with $\alpha$ a divisor in $B$ and $W$ a rank $n$ spectral cover bundle with $c_1(W) = 0$. Let $C$ be an irreducible surface in the linear system $|n \sigma + \pi^* \eta|$ (where we denote by $\eta$ a divisor class in $B$ and likewise its cohomology class) and $i: C \to X$ the immersion of $C$ into $X$ and let $L$ be a rank one sheaf on $C$. We say $W$ is a spectral cover bundle [2] of rank $n$ if $W = \pi_{1*}(\pi_2^*(i_* L) \otimes \mathcal{P})$ where $\mathcal{P}$ is the Poincaré sheaf on the fiber product $X \times_B X$ and $\pi_{1,2}$ are the respective projections on the first and second factor. The condition $c_1(W) = 0$ leads to (here and in the sequel $c_1$ denotes $c_1(B)$) (cf. [2])

$$c_1(L) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^* \eta + \left(\frac{1}{2} + n\lambda\right)\pi^* c_1.$$  

(2.2)
Since $c_1(L)$ must be an integer class it follows that: if $n$ is odd, then $\lambda$ is strictly half-integral and for $n$ even an integral $\lambda$ requires $\eta \equiv c_1 \pmod{2}$ while a strictly half-integral $\lambda$ requires $c_1$ even. Moreover note that, to assure that the linear system \([n\sigma + \pi^*\eta]\) contains an irreducible surface $C$ it is sufficient to demand that the linear system \(|\eta|\) is base-point free in $B$ and that the divisor corresponding to the cohomology class $\eta - nc_1$ is effective $[2]$.

Now let $H_0$ and $H_B$ be fixed ample divisors in $X$ and $B$, respectively. We have that $\pi^*E$ and $W$ are stable with respect to $H = \epsilon H_0 + \pi^*H_B$ for $\epsilon > 0$ chosen sufficiently small. For $W$ this is due to Theorem 7.2. in $[3]$ and for $\pi^*E$ Theorem 3.1 in $[6]$.

Two necessary conditions for $V$ to be stable are:

1. $DH^2 > 0$,
2. $\text{Ext}^1(W \otimes \mathcal{O}_X(rD), \pi^*E \otimes \mathcal{O}_X(-nD)) \neq 0$.

Here the first condition assures that $\pi^*E \otimes \mathcal{O}_X(-nD)$ is not a destabilizing subbundle of $V$ and the second condition (which is equivalent to $H^1(X, \pi^*E \otimes W^* \otimes \mathcal{O}_X(-mD)) \neq 0$ with $m = r + n = N$) assures that a non-split extension exists and shows that $W \otimes \mathcal{O}(X(rD))$ is not a destabilizing subbundle $V$.

**Theorem 2.1.** Assume (i) and (ii) are satisfied and $\alpha \cdot H_B = 0$. Then $V$ as defined in (2.1) is stable with respect to $H = \epsilon H_0 + \pi^*H_B$ for sufficiently small $\epsilon > 0$.

**Proof.** To prove stability of $V$ consider the diagram of exact sequences

$$
\begin{array}{cccc}
0 & \rightarrow & \pi^*E/F_s \otimes \mathcal{O}_X(-nD) & \rightarrow \ V/V'_{s+t} & \rightarrow & W/G_t \otimes \mathcal{O}_X(rD) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & \pi^*E \otimes \mathcal{O}_X(-nD) & \rightarrow & V & \rightarrow & W \otimes \mathcal{O}_X(rD) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & F_s \otimes \mathcal{O}_X(-nD) & \rightarrow & V'_{s+t} & \rightarrow & G_t \otimes \mathcal{O}_X(rD) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & & 0 & & 
\end{array}
$$

with $F_s \otimes \mathcal{O}_X(-nD) = i^{-1}V'_{s+t}$ and $G_t \otimes \mathcal{O}_X(nD) = j(V'_{s+t})$ of ranks $0 \leq s \leq r$ and $0 \leq t \leq n$ for a subsheaf $V'_{s+t}$ of $V$.

Note that $s = 0$ or $t = 0$ implies $F_s = 0$ or $G_t = 0$, respectively. Moreover, note for $0 < s < r$ we have $c_1(F_s) = -A_1 \sigma + \pi^*\lambda$ with $A_1 \geq 0$ and $\lambda \cdot H_B < 0$. To see this consider a $F_s$, a subsheaf of $\pi^*E$ where we can assume that $\pi^*E/F_s$ is torsion free. We have $0 \rightarrow F_s \rightarrow E$ and $c_1(F_s\mid_E)H < 0$. Similarly we get for restriction to the fiber $F'$ that $0 \rightarrow F_s \mid_F \rightarrow \mathcal{O}_F'$ thus $\deg(F_s \mid_F) \leq 0$ as $\mathcal{O}_F'$ is semistable. Thus $A_1 \geq 0$ and $\lambda H_B < 0$.

For spectral cover bundles and $0 < t < n$ we have $c_1(G_t) = -A_2 \sigma + \pi^*\beta$ with $A_2 > 0$ (Theorem 7.2. $[3]$).
We need to show for all subheaves $V'_{s+t}$ of $V$ with $0 \leq s \leq r$ and $0 \leq t \leq n$ and $0 < s + t < n + r$ that $\mu(V'_{s+t}) < 0$. We can assume that the quotient $V/V'_{s+t}$ is torsion free thus all subheaves $V'_{s+t}$ with $s = r$ need not be considered since $\pi^*E/F_r$ is a torsion sheaf thus $0 \leq s < r$.

i) For subheaves $V'_{s+t}$ with $0 \leq s < r$ and $0 < t < n$ the slope is given by
\[(s + t)\mu(V'_{s+t}) \leq -H^2_B \sigma + O(\epsilon) < 0\]
where the latter inequality holds for $\epsilon$ sufficiently small. If $s = 0$ and $0 < t < n$ we find similarly $t\mu(V'_{0+t}) = -H^2_B \sigma + O(\epsilon)$ and $\mu(V'_{0+t}) < 0$ for $\epsilon$ sufficiently small.

ii) For subheaves $V'_{s+t}$ with $0 < s < r$ and $t = 0$ we get
\[\mu(V'_{s}) = -nDH^2 + \mu(F_s)\]
but since $DH^2 > 0$ by assumption and $\mu(F_s) < 0$ by stability of $\pi^*E$ we get $\mu(V'_{s}) < 0$.

So we are left with the cases where $t = n$. In this case $W/G_n$ is a torsion sheaf. Let us write $c_1(W/G_n) = D$; since $W/G_n$ is a torsion sheaf one has $D \cdot H^2 \geq 0$ and $D \cdot H^2 = 0$ if and only if $W/G_n$ is supported in codimension $\geq 2$. Moreover, $D$ is an effective divisor in $X$. From $G_n \to W$, we get the map $(\Lambda^nG_n)^* \to \Lambda^nW^*$ and that moreover $rk(\Lambda^nG_n) = rk(\Lambda^nW) = 1$ and $(\Lambda^nG_n)^*$ is a reflexive torsion free sheaf of rank one with $c_1(G_n) = c_1((\Lambda^nG_n)^*)$. Furthermore, we have $(\Lambda^nW)^* = L$ with $c_1(L) = 0$. Thus $c_1(G_n) = -D$ with $D$ effective.

iii) For subheaves $V'_{s+n}$ with $0 < s < r$ and $t = n$ we get for $\epsilon$ sufficiently small
\[(s + n)\mu(V'_{s+n}) \leq \epsilon \left(2H_0\pi^*(\lambda H_B) + \epsilon H^2_B(\pi^*\lambda + n(r - s)\pi^*\alpha)\right) < 0\]
since $\lambda H_B < 0$ and $H^2_B \lambda$ is bounded.

iv) For subheaves $V'_{s+n}$ with $s = 0$ and $t = n$ we find
\[\mu(V'_{0+n}) = rDH^2 - \bar{D}H^2\]
For $\bar{D}H^2 > 0$ we get (with $D = y\sigma + \pi^*\beta$)
\[\mu(V'_{0+n}) \leq \epsilon \left(-2H_0\pi^*(\beta H_B) + \epsilon rH^2_B\pi^*\alpha\right)\]
and for $\epsilon$ sufficiently small that $\mu(V'_{0+n}) < 0$.

It remains to prove that the case $\bar{D}H^2 = 0$ cannot possibly happen. Note that $V$ does not admit a destabilizing subsheaf $V'_{0+n} = G_n \otimes \mathcal{O}_X(rD)$ if the map
\[f : Ext^1(W \otimes \mathcal{O}_X(rD), \pi^*E \otimes \mathcal{O}_X(-nD)) \to Ext^1(G_n \otimes \mathcal{O}_X(rD), \pi^*E \otimes \mathcal{O}_X(-nD))\]
is injective [4, Lemma 2.3]. To see this in our case, we apply $Hom(\ , \pi^*E \otimes \mathcal{O}_X(-nD))$ to the exact sequence
\[0 \to G_n \otimes \mathcal{O}_X(rD) \to W \otimes \mathcal{O}_X(rD) \to W/G_n \otimes \mathcal{O}_X(rD) \to 0\]
to obtain (set $E = \pi^*E \otimes \mathcal{O}_X(-nD)$ and $T = W/G_n$)
\[Ext^1(T \otimes \mathcal{O}_X(rD), E) \to Ext^1(W \otimes \mathcal{O}_X(rD), E) \to Ext^1(G_n \otimes \mathcal{O}_X(rD), E)\]
Proof. (i): For a given spectral cover bundle \( F \) is the zero element in the group law on the fibre \( b \in B \), we find that 
\[
\pi^* F = 0.
\]
Finally, let us analyze when the above extension can be chosen non-split. To simplify notation set \( F = E \otimes \mathcal{O}_B(-m\alpha) \). The Hirzebruch-Riemann-Roch theorem on \( X \) gives
\[
I_X = \sum_{i=0}^3 (-1)^i \dim H^i(X, \pi^* F \otimes W^*) = \int_X ch(\pi^* F \otimes W^*) Td(X) = r(-\lambda \eta + m\alpha)(\eta - nc_1).
\]
The following lemma will be helpful for proving the next proposition.

**Lemma 2.2.** Let \( W \) be a spectral cover bundle over \( X \) then

1. \( \pi_* W = \pi_* W^* = 0 \).
2. The sheaves \( R^1 \pi_* W \) and \( R^1 \pi_* W^* \) are supported on \( A = C \cap \sigma \).

**Proof.** (i): For a given spectral cover bundle \( W \) one has \( \pi_* W = 0 \). At a generic point \( b \in B \), one has the stalk \( (\pi_* W)_b = H^0(F, W_b) = \bigoplus_{i=1}^n H^0(F, \mathcal{O}_F(q_i - p)) \) where \( p = \sigma F \) is the zero element in the group law on the fibre \( F \) over \( b \in B \) and \( q_i \) are the points at which the spectral cover of \( V_n \) intersects \( F \). Now \( \mathcal{O}(q_i - p) \) is generically a non-trivial bundle of degree zero which over an elliptic curve admits no global sections. Thus \( H^0(F, \mathcal{O}_F(q_i - p)) = 0 \) for all \( i \) and so \( (\pi_* W)_b = 0 \). However, since \( V_n \) is torsion free, \( \pi_* W \) is also torsion free. Thus \( (\pi_* W)_b = 0 \) for generic \( b \in B \) gives \( \pi_* W = 0 \) everywhere. As \( W^* \) is again a spectral cover bundle one has also \( \pi_* W^* = 0 \).

(ii): For generic points \( b \in B \) where \( p \) is distinct from the \( n \) points \( p_i \) we have \( (R^1 \pi_* W)_b = 0 \). However, at points in \( B \) at which \( p \) is equal to one of the points \( p_i \), we find that \( (R^1 \pi_* W)_b \neq 0 \). The locus of such points form a curve \( C \cap \sigma \) in \( B \).

**Proposition 2.3.** \( H^3(X, \pi^* F \otimes W^*) = 0 \).

**Proof.** The Leray spectral sequence applied to \( \pi: X \to B \) leads to (using the fact that \( \pi_* W^* = 0 \))
\[
H^3(X, \pi^* F \otimes W^*) \cong H^2(B, R^1 \pi_* W^* \otimes F)
\]
As \( R^1 \pi_* W^* \) is a torsion sheaf on \( B \) supported on the curve of class \( A = C \cap \sigma \) we get \( H^2(B, R^1 \pi_* W^* \otimes F) = 0 \) and conclude.

Thus imposing \( I_X < 0 \) one gets a non-split extension \((2.1)\) as \( H^1(X, \pi^* F \otimes W^*) \neq 0 \).

In summary, we get the following list of conditions (where \( \alpha \neq 0 \) because of \((2.7)\)):
\[
\left[ \lambda \eta - m\alpha \right] (\eta - nc_1) > 0 \quad \text{(2.5)}
\]
\[
\alpha H_B = 0 \quad \text{($\Rightarrow \alpha \neq \pm\text{effective}$) (2.6)}
\]
\[
\pi^* \alpha H^2_B > 0 \quad \text{(2.7)}
\]
To solve (2.7) we note

**Lemma 2.4.** Let \( X \) be an elliptically fibered Calabi-Yau threefold with section \( \sigma \), then \( H_0 = x \sigma + \pi^* \rho \) is ample if and if \( x > 0 \) and \( \rho - xc_1 \) is ample in \( B \).

The proof of this Lemma is given in appendix A in [7]. Thus condition (2.7) becomes

\[
(2 \rho - xc_1) \alpha > 0.
\]

(2.8)

In the following, (2.8) will not be considered as a condition on \( \alpha \) but rather as a condition on \( x \) and \( \rho \) (for each respective \( \alpha \)). To show that it is possible to solve this for \( x \) and \( \rho \) note that it is enough to show that an ample class \( h \) exists with \( h \alpha > 0 \): on the one hand the expression in brackets in (2.8) \( h := \rho + (\rho - xc_1) \) is such a class, and on the other hand any \( h \) can be written in such a way.\(^1\) Now the existence of a nonzero \( \alpha \) which is neither effective nor anti-effective, cf. equ. (2.6), presupposes that \( h^{1,1}(B) \geq 2 \). Let \( \alpha = \beta - \gamma \) be a decomposition with \( \beta \) and \( \gamma \) effective, then it is possible to choose in the open ample cone an element \( h \) with \( h \beta = 2 \) and \( h \gamma = 1 \).

Finally let us give the expressions for the second Chern classes of \( V \) and \( W \)

\[
c_2(V) = c_2(W) + c_2(\pi^*E) - \frac{rn(r+n)}{2} \pi^* \alpha^2,
\]

(2.9)

\[
c_2(W) = \pi^* \eta \cdot \sigma - \frac{n^3 - n}{24} \pi^* c_1^2 + (\lambda^2 - \frac{1}{4} n \pi^* \eta \cdot (\pi^* \eta - n \pi^* c_1)).
\]

(2.10)

We also note that for the special case \( n = r \) a simplified version of the construction is possible where the twisting is as follows

\[
0 \to \pi^* E \otimes \mathcal{O}_X(-D) \to V \to W \otimes \mathcal{O}_X(D) \to 0
\]

(2.11)

and the stability proof of \( V \) is complete analogous. In this case we get

\[
c_2(V) = c_2(W) + c_2(\pi^*E) - \pi^* \alpha^2.
\]

(2.12)

### 3. Proof of main result

The aim of this section is to prove Theorem 1.3. For this let us first recall a result characterizing DRY classes on elliptically fibered Calabi-Yau threefolds obtained in [7]

**Theorem 3.1.** A class \( c = \phi \sigma + \omega \in H^4(X, \mathbb{Z}) \) is a DRY class if and only if the following condition is fulfilled (where \( b \) is some \( b \in \mathbb{R}^{>0} \) and \( \omega \in H^4(B, \mathbb{Z}) \cong \mathbb{Z} \):

\( \phi - \mathcal{N}(\frac{1}{2} + b)c_1 \) is ample and \( \frac{1}{\mathcal{N}} \omega > \omega_0(\phi; b) := R + \frac{c_1^2}{4}(b + \frac{q}{2}) \).

Here we use the following abbreviations (cf. [7]): \( R := \frac{1}{2N} \phi c_1 + \frac{1}{8} c_1^2 + \frac{1}{2}, \ q := \frac{(\phi - \frac{N c_1}{2})^2}{\mathcal{N} c_1^2} \).

Note that under the hypothesis that \( \phi - \frac{N c_1}{2}c_1 = A + b\mathcal{N} c_1 \) with an ample class \( A \) on \( B \) (and with \( c_1 \neq 0 \) effective) one has \( (\phi - \frac{N c_1}{2} c_1)^2 > b^2 \mathcal{N} c_1^2 \), thus one has \( b < \sqrt{q} \).

\(^1\)one has \( h = (A + xc_1) + A = 2[A + \frac{x}{2} c_1] \) with the ample class \( A := \rho - xc_1 \); conversely one can, if an ample \( h \) is given, solve this for an ample class \( A \) as \( \frac{1}{2} h \) will be also ample and thus also \( A = \frac{1}{2} h - \frac{1}{2} c_1 \) for \( x > 0 \) sufficiently small (the ample cone is an open set)
On the other hand $\omega_0$ reaches its minimum as a function of $b$ for $b = \sqrt{q}$, giving that
\[
\omega_0(\phi) > R + \frac{c_1^2}{2}\sqrt{q} = R + \frac{1}{2\sqrt{\pi}}\sqrt{c_1^2} \sqrt{(\phi - \frac{N^2}{2}c_1)^2}.
\]

Important for us will be especially the following (recall that we assume that $c_1$ is ample)

**Corollary 3.2.** For a DRY class $c = \phi\sigma + \omega$ one has that

(i) $\phi - \frac{N}{2}c_1$ is ample,

(ii) $\omega > N\left(R + \frac{1}{2\sqrt{\pi}}\sqrt{c_1^2} \sqrt{(\phi - \frac{N^2}{2}c_1)^2}\right)$.

Our goal is to show that, for a certain class of cases the conditions for a DRY class $c$ imply indeed the conditions under which the stable extension bundle $V$ of the previous section exist (the latter will have the prescribed class $c$ as $c_2(V)$).

Before we come to the general discussion of all cases with $N \geq 4$ we have to mention two special cases. This is caused by a restriction in the relation between the ranks $n$ and $r$ of the spectral bundle and the pullback bundle, respectively. From the Corollary we know given a DRY class $c$, that $\eta - nc_1$ is effective (we will put $\eta := \phi$); on the other hand for the spectral construction we need that $\eta - nc_1$ is effective. Thus one is lead to the condition

\[ r \geq n \quad (3.1) \]

### 3.3. The cases $N = 4, 5$ and $B \neq F_0$.

In this case one would have to assume that $n$ is even (actually $n = 2$) and so that $\eta \equiv c_1 (\mod 2)$; thus one would not be sufficiently general in the choice of $\eta$ to match a given $\phi$ in the DRY class $c = \phi\sigma + \omega$. Thus, if one is forced to have $n$ even, we can only cover the case $B = F_0$. On the other hand this will be no problem, of course, if $N = n + r \geq 6$ as one can then choose $n = 3$. For $N = 4, 5$, however, we can only cover the case $B = F_0$.

### 3.4. The cases $N \geq 6$ or $N \geq 4$ and $B = F_0$.

Let us therefore assume from now on that we are in the case that one has either $N \geq 6$, and thus $n$ can be choosen odd (actually $n = 3$), or that $N = 4$, and thus $n$ is even (actually $n = 2$), but $B = F_0$. In both cases we can then use a strictly half-integral $\lambda$; we will have to choose $\lambda = \pm 1/2$ to kill the corresponding contribution in $c_2(V)$ (which would lift the socle of realisable $\omega$). We will discuss the nonsplit-condition (2.5) (where $H_B$ has to be chosen orthogonal to $\alpha$). As $\eta(\eta - nc_1) > 0$ we will just choose (for each $\eta$) a suitable $\alpha$ such that $\alpha \cdot (\eta - nc_1) \leq 0$.

**The case of $B$ being a Hirzebruch surface $F_g$**

Because of our assumption that $c_1$ is ample we have actually $g = 0$ or 1. Let us take first $\alpha = (-1, 1)$ such that $-\alpha^2 = g + 2$ and $H_B := (e, f) = (e, 1, g + 1)$ (which is ample as $f > ge$). The evaluation $\alpha \cdot (\eta - nc_1) = (g + 1)a - b - gn$ shows that one chooses this $\alpha$ if the latter expression is $\leq 0$; if it is $> 0$ one chooses just $-\alpha$ instead of $\alpha$. 
Let us now also investigate the situation for the del Pezzo surfaces $dP_k$ with $k = 0, \ldots, 8$ and $c_1 = 3l - \sum_k E_k$ (where $l$ is of course the class of the pull-back of the line). On $P^2$ the condition (2.6) amounts, with $\alpha = a\ell$, to $a = 0$, giving a contradiction. The case of $dP_1 \cong F_1$ is already settled. Now in the case of $2 \leq k \leq 8$ let $\alpha = k\ell - 3\sum_i E_i$ such that $-\alpha^2 = k(9 - k)$ and $H_B = c_1$. Then one finds with $\eta = a\ell + \sum b_i E_i$ that $\alpha(\eta - nc_1) = ka + 3\sum b_i$: if this expression is $\leq 0$ the choice of $\alpha$ was already successful; otherwise one just takes $-\alpha$.

3.5. The realizability of DRY classes by Chern classes of stable bundles. Let us now look at the conditions under which we can realize a given DRY class $c = \sigma + \omega$ as $c_2(V)$ for the extension bundle $V$ we have constructed. In the $\sigma$ term one just takes $\eta := \phi$ where under our assumed condition $r \geq n$ the spectral requirement that $\eta - nc_1$ is effective follows from the DRY property that $\phi - \frac{n+r}{2}c_1$ is effective. Now let us compare the $\omega$ terms, i.e. the lower bound for $\omega_{DRY}$ which follows from the DRY condition with the lower bound of the $\omega$ term in the expression $c_2(V)$ which comes from a stable bundle. Here one finds (with $n = 3, \lambda = 1/2$; note that we had a different choice for $\alpha$ for $F_1$ and $dP_1$, respectively)

$$
\omega_{DRY} > \left\{ \begin{array}{ll}
\frac{1}{2} \eta c_1 + \frac{m}{6} c_1^2 + \frac{m}{2} + \frac{1}{2m} \sqrt{c_1^2 \sqrt{(\eta - \frac{m}{2} c_1)^2}} & \\
> \frac{1}{2} \eta c_1 + \frac{m}{6} c_1^2 + \frac{m}{2} & \text{for } B = F_g \\
\text{for } B = dP_k
\end{array} \right. \quad (3.2)
$$

$$
\omega_V \geq -c_1^2 + r + 2 - \frac{nr(n+r)}{2} \al^2
$$

$$
= \left\{ \begin{array}{ll}
-8 + m - 1 + \frac{3}{2}(m-3)m(g+2) & \text{for } B = F_g \\
-(9-k) + m - 1 + \frac{3}{2}(m-3)mk(9-k) & \text{for } B = dP_k
\end{array} \right. \quad (3.3)
$$

Thus one finds that, as the bound for $\omega_{DRY}$ depends on $\eta$ while the bound for $\omega_V$ does not, only for finitely many choices of $\eta$ the corresponding bound for $\omega_{DRY}$ does not lie above the bound for $\omega_V$ (keeping $m$ fixed). This finishes the proof of part (i) of Theorem 1.3.

For part (a) of (ii) note that for $\mathcal{N} = m$ being $\equiv 2 \pmod{4}$ one can take $n = r$ (being odd) and use (still with $\lambda = 1/2$) the modified construction (2.11). In this case one gets

$$
\omega_V \geq -c_1^2 + \frac{m}{2} + 2 - \alpha^2 = \left\{ \begin{array}{ll}
\frac{m}{2} + g - 4 & \text{for } B = F_g \\
\frac{m}{2} + k + 1 & \text{for } B = dP_k
\end{array} \right. \quad (3.4)
$$

where this time for $dP_k$ we choose $\alpha = l - 3E_1$ (which is also orthogonal to $c_1$) with $-\alpha^2 = 8$. To finish the proof for $B = F_g$ one just has to note that $\frac{11}{6} m > \frac{m}{2} - 4 + g$. For $B = dP_k$ note that, using $\eta c_1 > \frac{m}{2} (9-k)$ because $\eta - nc_1$ is ample, one has $\omega_{DRY} > \left[ \frac{11}{12} (9-k) + \frac{1}{2} \right] m$. For $k = 1, \ldots, 6$ the lower bound for $\omega_{DRY}$ is greater than the lower bound for $\omega_V$ because $\frac{5}{12} (9-k) m > k + 1$. For $k = 7, 8$ only a small number of the possible choices for $m$ is excluded.
For part (b) of (ii) where one takes \( n = r \) with \( n \) even and \( \lambda = 1/2 \) thus \( B = F_0 \) one notes that \( \frac{1}{6}m > \frac{1}{2}m - 4 \).

Acknowledgements. B. A. thanks the SFB 647 for support. G. C. thanks the DFG for support in the grant CU 191/1-1 and the FU Berlin for hospitality.

REFERENCES

[1] M.R. Douglas, R. Reinbacher and S.-T. Yau, Branes, Bundles and Attractors: Bogomolov and Beyond, math.AG/0604597.
[2] R. Friedman, J. Morgan and E. Witten, Vector Bundles and F-Theory, hep-th/9701162, Comm. Math. Phys. 187 (1997) 679.
[3] R. Friedman, J. Morgan and E. Witten, Vector Bundles over Elliptic Fibrations, J. Algebraic Geom. 8 (1999), pp. 279401, arXiv:alg-geom/9709029.
[4] B. Andreas and G. Curio, Stable Bundle Extensions on elliptic Calabi-Yau threefold, J. Geom. Phys. 57, 2249-2262, 2007, math.AG/0611762.
[5] B. Andreas and M. Garcia-Fernandez, Solution of the Strominger System via Stable Bundles on Calabi-Yau threefolds, arXiv:1008.1018 [math.DG].
[6] B. Andreas and M. Garcia-Fernandez, Heterotic Non-Kähler Geometries via Polystable Bundles on Calabi-Yau Threefolds, arXiv:1011.6246 [hep-th].
[7] B. Andreas and G. Curio, On possible Chern Classes of stable Bundles on Calabi-Yau threefolds, Jour. of Geom. and Phys. Vol 61, Issue 8, (2011) 1378-1384, arXiv:1010.1644 [hep-th].
[8] B. Andreas and G. Curio, Spectral Bundles and the DRY-Conjecture, arXiv:1012.3858 [hep-th].
[9] I.V. Artamkin, Deforming Torsion-free Sheaves on an Algebraic Surface, Math.USSR.Izv. 36 (1991) 449.
[10] R. Friedman, Algebraic Surfaces and Holomorphic Vector Bundles, Springer (1998).

Institut für Mathematik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin.

E-mail address: andreasb@mi.fu-berlin.de

Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität München, Theresienstr. 37, 80333 München.

E-mail address: Gottfried.Curio@physik.uni-muenchen.de