Regular irreducible representations of classical reductive groups over finite quotient rings

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Abstract

A parametrization of irreducible representations associated with a regular adjoint orbit of a reductive group over finite quotient rings of a non-dyadic non-archimedean local field is presented. The parametrization is given by means of (a subset of) the character group of the centralizer of a representative of the regular adjoint orbit. Our method is based upon Clifford’s theory and Weil representations over finite fields.

1 Introduction

Let $F$ be a non-dyadic non-archimedean local field. The integer ring of $F$ is denoted by $O$ with the maximal ideal $p$ generated by $\varpi$. The residue class field $\mathbb{F} = O/p$ is a finite field of odd characteristic with $q$ elements. For an integer $r > 0$ put $O_r = O/p^r$ so that $\mathbb{F} = O_1$.

Let $G$ be a connected reductive group scheme over $O$. The problem on which we will consider in this paper is to determine the set $\text{Irr}(G(O_r))$ of the equivalence classes of the irreducible complex representations of the finite group $G(O_r)$.

This problem in the case $r = 1$, that is the representation theory of the finite reductive group $G(\mathbb{F})$, has been studied extensively, starting from Green [7] concerned with $GL_n(\mathbb{F})$ to the decisive paper of Deligne-Lusztig [3].

On the other hand, the study of the representation theory of the finite group $G(O_r)$ with $r > 1$ is less complete. The systematic studies are done mainly in the case of $G = GL_n$ [8, 9, 11, 12, 17, 18]. The purpose of this paper is to show that the method used in [18] works for greater range of reductive group schemes over $O$.

In this paper, we will establish a parametrization of the irreducible representations of $G(O_r)$ ($r > 1$) associated with the regular (more precisely smoothly regular, see subsection 2.4 for the definition) adjoint orbits. Taking a representative $\beta$ of the adjoint orbit, the parametrization is given by means of a subset of the character group of $G(\mathbb{F})$ where $G_{\beta}$ is the centralizer of $\beta$ in $G$ which is assumed to be smooth commutative group scheme over $O$. Our theory is based on Clifford theory and Weil representations over finite fields.

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The main results of this paper are Theorem 2.3.1 and Theorem 2.5.1. The latter is a paraphrase of the former with more emphasis posed on the regularity of Lie elements but being restricted to the groups of type $A, B$ and $C$.

The situation is quite simple when $r$ is even, and almost all of this paper is devoted to treat the case of $r = 2l - 1$ being odd. In this case Clifford theory requires us to construct an irreducible representation of $G_\beta(O_r) \cdot K_{l-1}(O_r)$ where $K_{l-1}(O_r)$ is the kernel of the canonical surjection $G(O_r) \to G(O_{l-1})$. To construct an irreducible representation of $K_{l-1}(O_r)$, we will use Schrödinger representation of the Heisenberg group associated with a symplectic space over finite field which is associated with $\beta$ (Proposition 4.4.1). Then we will use Weil representation to extend the irreducible representation of $K_{l-1}(O_r)$ to an irreducible representation of $G_\beta(O_r) \cdot K_{l-1}(O_r)$. At this point appears a Schur multiplier as an obstruction to the extension (see subsection 4.5). The definition and a fundamental property of the Schur multiplier will be discussed in section 3.

In the case of $G = GL_n$, the extendability of the irreducible representation of $K_{l-1}(O_r)$ to that of $G_\beta(O_r) \cdot K_{l-1}(O_r)$ is proved by [17]. Based upon this result, we will prove the triviality of the Schur multiplier for general $G \subset GL_n$ under the condition that the reduction modulo $p$ of the characteristic polynomial of $\beta \in g(O) \subset gl_n(O)$ is the minimal polynomial of $\beta (\mod p) \in M_n(F)$ (Proposition 4.6.1).

We will give in section 5 some examples of classical groups where the reduction modulo $p$ of the characteristic polynomial of $\beta$ is the minimal polynomial of $\beta (\mod p) \in M_n(F)$. In these cases the parametrization is given by a subset of the character group of the unit group of a tamely ramified extension of the base field $F$. See Propositions 5.2.1 for a special linear group, Proposition 5.3.1 for a symplectic group, Propositions 5.4.1 and 5.4.2 for a special orthogonal group with respect to a quadratic form of even and odd variables respectively.

The character group of a finite abelian group $G$ is denoted by $\hat{G}$. The multiplicative group of the complex numbers of absolute value one is denoted by $\mathbb{C}^1$.

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2 Main results

2.1 Fix a continuous unitary character $\tau$ of the additive group $F$ such that

$$\{ x \in F \mid \tau(xO) = 1 \} = O,$$

and define an additive character $\hat{\tau}$ of $F$ by $\hat{\tau}(\overline{x}) = \tau(\overline{x}^{-1} x)$.

Let $G \subset GL_n$ be a closed smooth $O$-group subscheme, and $g$ the Lie algebra of $G$ which is a closed affine $O$-subscheme of $gl_n$ the Lie algebra of $GL_n$. We may assume that the fibers $G \otimes O K$ ($K = F$ or $K = \mathbb{F}$) are non-commutative algebraic $K$-group (that is smooth $K$-group scheme).

For any $O$-algebra $K$ (in this paper, an $O$-algebra means a commutative unital $O$-algebra) the set of the $K$-valued points $gln(K)$ is identified with the $K$-Lie algebra of square matrices $M_n(K)$ of size $n$ with Lie bracket $[X, Y] =
The condition I) implies that $B(l) > \text{truncations of the exponential mapping}$.

The conditions II) and III) from Lie algebras to groups can be regarded as with the smallest integer $l$ such that $0 < l' \leq l$. The smoothness of $G$ implies that we have a canonical isomorphism $g(O)/\omega^r g(O) \cong g(O_r) = g(O) \otimes O_r$ (\cite[Chap.II, §4, Prop.4.8]{H}) and that the canonical group homomorphism $G(O) \to G(O_r)$ is surjective due to Hensel’s lemma. Then for any $0 < l < r$ the canonical group homomorphism $G(O_r) \to G(O_l)$ is surjective whose kernel is denoted by $K_l(O_r)$.

For any $g \in G(O)$ (resp. $X \in g(O)$), the image under the canonical surjection onto $G(O_l)$ (resp. onto $g(O_l)$) with $l > 0$ is denoted by $g_l = g \mod p^l \in G(O_l)$ (resp. $X_l = X \mod p^l \in g(O_l)$).

Since the rational points $G(K)$ (resp. $g(K)$) of the fiber $G\otimes_K K$ (resp. $g\otimes_K K$) with $K = F$ or $K = \mathbb{F}$ plays some special roles in our theory, let us denote by $\omega \in G(K)$ (resp. $X \in g(K)$) the image of $g \in G(O)$ (resp. $X \in g(O)$) under the canonical morphism $G(O) \to G(K)$ (resp. $g(O) \to g(K)$).

We will pose the following three conditions:

I) $B : g(F) \times g(F) \to F$ is non-degenerate,

II) for any integers $r = l + l'$ with $0 < l' \leq l$, we have a group isomorphism $g(O_l) \cong K_l(O_r)$

defined by $X \mod p^l' \mapsto 1 + \omega^l X \mod p^r$,

III) if $r = 2l - 1 \geq 3$ is odd, then we have a mapping $g(O) \to K_{l-1}(O_r)$

defined by $X \mapsto (1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2) \mod p^r$.

The condition I) implies that $B : g(O_l) \times g(O_l) \to O_l$ is non-degenerate for all $l > 0$, and so $B : g(O) \times g(O) \to O$ is also non-degenerate. The mappings of the conditions II) and III) from Lie algebras to groups can be regarded as truncations of the exponential mapping.

From now on we will fix an integer $r \geq 2$ and put $r = l + l'$ with the smallest integer $l$ such that $0 < l' \leq l$. In other word

$$l' = \begin{cases} l & : r = 2l, \\ l - 1 & : r = 2l - 1. \end{cases}$$
Take a $\beta \in \mathfrak{g}(O)$ and define a character $\psi_\beta$ of the finite abelian group $K_1(O_r)$ by

$$\psi_\beta((1 + x^t X) (\text{mod } p^\ell)) = \tau(x^{-t}B(X, \beta)) \quad (X \in \mathfrak{g}(O)).$$

Then $\beta(\text{mod } p^{\ell'}) \mapsto \psi_\beta$ gives an isomorphism of the additive group $\mathfrak{g}(O_r)$ onto the character group $K_1(O_r)^\vee$. For any $g_r = g(\text{mod } p^{\ell'}) \in G(O_r)$, we have

$$\psi_\beta(g_r^{-1} h g_r) = \psi_{\mathrm{Ad}(g) \beta}(h) \quad (h \in K_1(O_r)).$$

So the stabilizer of $\psi_\beta$ in $G(O_r)$ is

$$G(O_r, \beta) = \{ g_r \in G(O_r) \mid \text{Ad}(g) \beta \equiv \beta \pmod{p^{\ell'}} \}$$

which is a subgroup of $G(O_r)$ containing $K_1(O_r)$.

Now let us denote by $\text{Irr}(G(O_r) \mid \psi_\beta)$ (resp. $\text{Irr}(G(O_r, \beta) \mid \psi_\beta)$) the set of the isomorphism classes of the irreducible complex representation $\pi$ of $G(O_r)$ (resp. $\sigma$ of $G(O_r, \beta)$) such that

$$\langle \psi_\beta, \pi \rangle_{K_1(O_r)} = \dim_{\mathbb{C}} \text{Hom}_{K_1(O_r)}(\psi_\beta, \pi) > 0$$

(resp. $\langle \psi_\beta, \sigma \rangle_{K_1(O_r)} > 0$). Then Clifford’s theory says that

1) $\text{Irr}(G(O_r)) = \bigsqcup_{\beta(\text{mod } p^{\ell'})} \text{Irr}(G(O_r) \mid \psi_\beta)$ where $\bigsqcup_{\beta(\text{mod } p^{\ell'})}$ is the disjoint union over the representatives $\beta(\text{mod } p^{\ell'})$ of the $\text{Ad}(G(O_r))$-orbits in $\mathfrak{g}(O_r)$,

2) a bijection of $\text{Irr}(G(O_r, \beta) \mid \psi_\beta)$ onto $\text{Irr}(G(O_r) \mid \psi_\beta)$ is given by

$$\sigma \mapsto \text{Ind}_{G(O_r, \beta)}^{G(O_r)}(\psi_\beta).$$

So our problem is to give a good parametrization of the set $\text{Irr}(G(O_r, \beta) \mid \psi_\beta)$.

2.3 For any $\beta \in \mathfrak{g}(O)$, let us denote by $G_\beta = Z_G(\beta)$ the centralizer of $\beta$ in $G$ which is a closed $O$-group subscheme of $G$. The Lie algebra $\mathfrak{g}_\beta = Z_G(\beta)$ of $G_\beta$ is a closed $O$-subscheme of $\mathfrak{g}$ such that

$$\mathfrak{g}_\beta(K) = \{ X \in \mathfrak{g}(K) \mid [X, \overline{\beta}] = 0 \}$$

for any $O$-algebra $K$ where $\overline{\beta} \in \mathfrak{g}(K)$ is the image of $\beta \in \mathfrak{g}(O)$ under the canonical morphism $\mathfrak{g}(O) \to \mathfrak{g}(K)$.

Now our main result is

**Theorem 2.3.1** Take a $\beta \in \mathfrak{g}(O)$ such that

1) $G_\beta$ is commutative smooth $O$-group scheme, and

2) the characteristic polynomial $\chi_{\overline{\beta}}(t) = \det(t \cdot 1_n - \overline{\beta})$ of $\overline{\beta} \in \mathfrak{g}(\mathbb{F}) \subset \mathfrak{g}_n(\mathbb{F})$ is the minimal polynomial of $\overline{\beta} \in M_n(\mathbb{F})$.

Then we have a bijection $\theta \mapsto \sigma_{\beta, \theta}$ of the set

$$\{ \theta \in G_\beta(O_r) \cap K_1(O_r) \mid \text{s.t. } \theta = \psi_\beta, \text{ on } G_\beta(O_r) \cap K_1(O_r) \}$$

onto $\text{Irr}(G(O_r, \beta) \mid \psi_\beta)$. 4
The explicit description of the representation $\sigma_{\beta, \theta}$ is given by (1) if $r$ is even, and by (5) if $r$ is odd.

The proof of this theorem in the case of even $r$ is quite simple, and it will be given in subsection 2.6. The remaining part of this paper is devoted to the proof in the case of odd $r$.

These proves show that the second condition in the theorem is required only in the case of $r$ being odd. The second condition is related with the smooth regularity of $\beta \in \mathfrak{g}(O)$ as presented in the next subsection.

2.4 We will present a sufficient condition on $\beta \in \mathfrak{g}(O)$ under which $G_{\beta}$ is commutative and smooth over $O$.

Let us assume that the connected $O$-group scheme $G$ is reductive, that is, the fibers $G \otimes O (K = F, F)$ are reductive $K$-algebraic groups. In this case the dimension of the maximal torus in $G \otimes O K$ is independent of $K$ which is denoted by $\text{rank}(G)$. For any $\beta \in \mathfrak{g}(O)$ we have

$$\dim_K \mathfrak{g}_{\beta}(K) = \dim \mathfrak{g}_{\beta} \otimes O K \geq \dim G_{\beta} \otimes O K \geq \text{rank}(G).$$

We say $\beta$ to be smoothly regular with respect to $G$ over $K$ (or $\beta \in \mathfrak{g}(K)$ is smoothly regular with respect to $G \otimes O K$) if $\dim_K \mathfrak{g}_{\beta}(K) = \text{rank}(G)$ (see [16, 1.4]). In this case $G_{\beta} \otimes O K$ is smooth over $K$. If $\beta$ is smoothly regular with respect to $G$ over $F$ and over $F$, then $\beta$ is said to be smoothly regular with respect to $G$.

We say $\beta$ to be connected with respect to $G$ if the fibers $G_{\beta} \otimes O K (K = F, F)$ are connected. See Remark 2.4.3 for a sufficient condition for the connectedness of $G_{\beta} \otimes O K$.

**Proposition 2.4.1** If $\beta \in \mathfrak{g}(O)$ is smoothly regular and connected with respect to $G$, then $G_{\beta}$ is commutative and smooth over $O$.

**[Proof]** Let $G_{\beta}^0$ be the neutral component of $O$-group scheme $G_{\beta}$ which is a group functor of the category of $O$-scheme (see §3 of Exposé VI in [5]). The following statements are equivalent:

1) $G_{\beta}^0$ is representable as a smooth open $O$-group subscheme of $G_{\beta}$,
2) $G_{\beta}$ is smooth at the points of unit section,
3) each fibers $G_{\beta} \otimes O K (K = F, F)$ are smooth over $K$ and their dimensions are constant

(see Th. 3.10 and Cor. 4.4 of [5]). So if $\beta$ is smoothly regular with respect to $G$, then $G_{\beta}^0$ is smooth open $O$-group subscheme of $G_{\beta}$. If further $\beta$ is connected with respect to $G$, then $G_{\beta}^0 = G_{\beta}$ is smooth over $O$. Let $\beta = \beta_s + \beta_n$ be the Jordan decomposition of $\beta \in \mathfrak{g}_{\beta}(F)$. Then the identity component $G$ of the centralizer $Z_{G \otimes O F}(\beta_s)$ is a reductive $F$-algebraic group and

$$G_{\beta} \otimes O F = Z_F(\beta_n)$$

because $G_{\beta} \otimes O F$ is connected. Then [15] shows that $G_{\beta}(F)$ is commutative ($F$ is the algebraic closure of $F$), and hence $G_{\beta}$ is a commutative $O$-group scheme. ■
Let us present more detail description of the smooth regularity of Lie element. Assume that the characteristic of $K = F, \mathbb{F}$ is not bad with respect to $G \otimes O K$. The list of the bad primes is

| type of $G \otimes O K$ | $A_r$ | $B_r, D_r$ | $C_r$ | $E_6, E_7, F_4$ | $E_8$ | $G_2$ |
|-------------------------|-------|------------|-------|-----------------|-------|-------|
| bad prime               | $\emptyset$ | $2$       | $2$   | $2, 3$          | $2, 3, 5$ | $2, 3$ |

(see [2] p.178, I-4.3).

Take a $\beta \in g(O)$ and let $\overline{\beta} = \beta_s + \beta_n$ be the Jordan decomposition of $\beta \in g(K)$ into the semi-simple part $\beta_s \in g(K)$ and the nilpotent part $\beta_n \in g(K)$ ($\overline{\beta} \in g(K)$ is the image of $\beta \in g(O)$ under the canonical mapping $g(O) \to g(K)$). The identity component $L = Z_{G \otimes O K}(\beta_s)$ of the centralizer of $\beta_s$ in $G \otimes O K$ is a reductive group over $K$ and there exists a maximal torus $T$ of $G \otimes O K$ such that

$$\beta_s \in \text{Lie}(T)(K)$$

(see [1] Prop.13.19 and its proof). Then $T \subset L$ and $\text{rank}(L) = \text{rank}(G)$. Put $l = \text{Lie}(L)$, then $l(K) = Z_{g(K)}(\beta_s)$. So $\overline{\beta} \in g(K)$ is smoothly regular with respect to $G \otimes O K$ if and only if $\beta_n \in l(K)$ is smoothly regular with respect to $L$.

Now fix a system of positive roots $\Phi^+$ in the root system $\Phi(T, L)$ of $L$ with respect to $T$ such that

$$\beta_n = \sum_{\alpha \in \Phi^+} c_\alpha \cdot X_\alpha$$

where $X_\alpha$ is a root vector of the root $\alpha$. Then the result of [2] p.228,III-3.5] implies

**Proposition 2.4.2** $\overline{\beta} \in g(K)$ is smoothly regular with respect to $G$ over $K$ if and only if $c_\alpha \neq 0$ for all simple $\alpha \in \Phi^+$.

**Remark 2.4.3** Assume that $\overline{\beta} \in g(K)$ is smoothly regular with respect to $G \otimes O K$. Then $G \otimes O K$ is connected if $Z_{G \otimes O K}(\beta_s)$ and its center are connected (see Theorem 5.9 b) of [15]).

**Remark 2.4.4** If $G \otimes O K$ is of type $A_r, B_r$ or $C_r$, then, putting $G \subset GL_n$ with suitable $n$ (that is $n = r + 1, 2r + 1, 2r$ for type $A_r, B_r, C_r$ respectively), an element $\beta \in g(O) \subset gl_n(O)$ is smoothly regular with respect to $G$ over $K$ if and only if $\beta$ is smoothly regular with respect to $GL_n$ over $K$.

Let us consider the case of $GL_n$ ($n \geq 2$) which is a connected smooth reductive $O$-group scheme. For $\beta \in gl_n(O)$, the following statements are equivalent:

1) $\beta \in gl_n(O)$ is smoothly regular with respect to $GL_n$ over $\mathbb{F}$,

2) $\overline{\beta} \in M_n(\mathbb{F})$ is $GL_n(\mathbb{F})$-conjugate to

$$J_{n_1}(\alpha_1) \oplus \cdots \oplus J_{n_r}(\alpha_r) = \begin{bmatrix} J_{n_1}(\alpha_1) & & \\ & \ddots & \\ & & J_{n_r}(\alpha_r) \end{bmatrix},$$
where $\alpha_1, \ldots, \alpha_r$ are distinct elements of the algebraic closure $\overline{F}$ of $F$ and

$$J_m(\alpha) = \begin{bmatrix} \alpha & 1 \\ \alpha & \ddots \\ \ddots & \ddots & \ddots \\ & & \alpha \\ & & & 1 \end{bmatrix}$$

is a Jordan block of size $m$.

3) the characteristic polynomial $\chi(\beta) = \det(tI_n - \beta) \in \mathbb{F}[t]$ is the minimal polynomial of $\beta \in M_n(F)$,

4) $\{X \in M_n(\mathbb{F}) \mid X \beta = \beta X\} = \mathbb{F}[\beta]$,

5) $\{X \in M_n(\mathbb{O}) \mid X \beta \equiv \beta X \pmod{p^l}\} = \mathbb{O}[\beta]$ for all $l > 0$,

6) $\mathbb{O}^n$ is a cyclic $\mathbb{O}[\beta]$-module, that is, there exists a vector $v \in \mathbb{O}^n$ such that $\mathbb{O}^n = \mathbb{O}[\beta]v$.

In this case we have

1) $\{X \in M_n(\mathbb{O}) \mid X \beta = \beta X\} = \mathbb{O}[\beta]$,

2) $\beta \in \mathfrak{gl}_n(\mathbb{O})$ is smoothly regular with respect to $GL_n$ over $F$ and

3) the centralizer $GL_{n,\beta}$ is commutative and smooth over $O$.

2.5 In order to give a presentation of the results directly connected with the regularity, let us put $G = GL_n, SL_n$ (with $n$ prime to the characteristic of $\mathbb{F}$), $Sp_n$ (with even $n$) or $SO(S)$ with a symmetric matrix $S \in \mathrm{Sym}_n(\mathbb{O})$ of odd size and

$$SO(S)(K) = \{g \in SL_n(K) \mid g^t S g = g\}$$

for any $O$-algebra $K$. Then $G$ is a smooth $O$-group scheme which fulfills three conditions I), II) and III) of subsection 2.1.

Take a $\beta \in \mathfrak{g}(O)$ which is smoothly regular and connected with respect to $G$. Then the results of the preceding subsection show that $G_{\beta}$ is a commutative smooth $O$-group scheme, and that the characteristic polynomial of $\beta \in \mathfrak{g}(\mathbb{F}) \subset \mathfrak{gl}_n(\mathbb{F})$ is the minimal polynomial of $\beta \in M_n(\mathbb{F})$. Then Theorem 2.3.1 gives the following

**Theorem 2.5.1** Take a $\beta \in \mathfrak{g}(O)$ which is smoothly regular and connected with respect $G$. Then we have a bijection $\theta \mapsto \sigma_{\beta, \theta}$ of the set

$$\{\theta \in G_{\beta}(O_r) \mid \text{s.t. } \theta = \psi_{\beta} \text{ on } G_{\beta}(O_r) \cap K_l(O_r)\}$$

onto $\mathrm{Irr}(G(O_r, \beta) \mid \psi_{\beta})$. 

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2.6 Assume that \( r = 2l \) is even so that \( l' = l \). Since \( G_\beta \) is a smooth O-group scheme, the canonical map \( G_\beta(O_r) \to G_\beta(O_l) \) is surjective. Then we have

\[
G(O_r, \beta) = G_\beta(O_r) \cdot K_l(O_r)
\]

where \( G_\beta(O_r) \) is a commutative finite group. Let \( \theta \) be a character of \( G_\beta(O_r) \) such that \( \theta = \psi_\beta \) on \( G_\beta(O_r) \cap K_l(O_r) \). Then we have an one-dimensional representation \( \sigma_{\beta, \theta} \) of \( G(O_r, \beta) \) defined by

\[
\sigma_{\beta, \theta}(gh) = \theta(g) \cdot \psi_\beta(h) \quad (g \in G_\beta(O_r), h \in K_l(O_r)). \tag{1}
\]

Then \( \theta \mapsto \sigma_{\beta, \theta} \) is an injection of the set

\[
\{ \theta \in G_\beta(O_r) \mid \text{s.t. } \theta = \psi_\beta \text{ on } G_\beta(O_r) \cap K_l(O_r) \}
\]

into \( \text{Irr}(G(O_r, \beta) \mid \psi_\beta) \).

Take any \( \sigma \in \text{Irr}(G(O_r, \beta) \mid \psi_\beta) \) with representation space \( V_\sigma \). Then

\[
V_\sigma(\psi_\beta) = \{ v \in V_\sigma \mid \sigma(g)v = \psi_\beta(g)v \text{ for } \forall g \in K_l(O_r) \}
\]

is a non-trivial \( G(O_r, \beta) \)-subspace of \( V_\sigma \) so that \( V_\sigma = V_\sigma(\psi_\beta) \). Then, for any one-dimensional representation \( \chi \) of \( G(O_r, \beta) \) such that \( \chi = \psi_\beta \) on \( K_l(O_r) \), we have \( K_l(O_r) \subset \text{Ker}(\chi^{-1} \otimes \sigma) \). On the other hand \( G(O_r, \beta) / K_l(O_r) \) is commutative, we have \( \dim(\chi^{-1} \otimes \sigma) = 1 \) and then \( \dim \sigma = 1 \). Then \( \theta = \sigma|_{G_\beta(O_r)} \) is a character of \( G_\beta(O_r) \) such that \( \theta = \psi_\beta \) on \( G_\beta(O_r) \cap K_l(O_r) \), and we have \( \sigma = \sigma_{\beta, \theta} \).

3 Schur multiplier

Let \( G \subset GL_n \) be a closed \( \mathbb{F} \)-algebraic subgroup and \( \mathfrak{g} \) the Lie algebra of \( G \) which is a closed affine \( \mathbb{F} \)-subscheme of the Lie algebra \( \mathfrak{gl}_n \) of \( GL_n \). Let us assume that the trace form

\[
B : \mathfrak{g}(\mathbb{F}) \times \mathfrak{g}(\mathbb{F}) \to \mathbb{F} \quad ((X, Y) \mapsto \text{tr}(XY))
\]

is non-degenerate. Fix a \( \beta \in \mathfrak{g}(\mathbb{F}) \) such that \( \mathfrak{g}_\beta(\mathbb{F}) \leq \mathfrak{g}(\mathbb{F}) \).

3.1 The non-zero \( \mathbb{F} \)-vector space \( \mathbb{V}_\beta = \mathfrak{g}(\mathbb{F}) / \mathfrak{g}_\beta(\mathbb{F}) \) has a symplectic form

\[
\langle \dot{X}, \dot{Y} \rangle_\beta = B([X, Y], \overline{\beta})
\]

where \( \dot{X} = X \pmod{\mathfrak{g}_\beta(\mathbb{F})} \in \mathbb{V}_\beta \) with \( X \in \mathfrak{g}_\beta(\mathbb{F}) \). Then \( g \in G_\beta(\mathbb{F}) \) gives an element \( \sigma_g \) of the symplectic group \( Sp(\mathbb{V}_\beta) \) defined by

\[
X \pmod{\mathfrak{g}_\beta(\mathbb{F})} \mapsto \text{Ad}(g)^{-1}X \pmod{\mathfrak{g}_\beta(\mathbb{F})}.
\]

Note that the group \( Sp(\mathbb{V}_\beta) \) acts on \( \mathbb{V}_\beta \) from right. Let \( v \mapsto [v] \) be a \( \mathbb{F} \)-linear section on \( \mathbb{V}_\beta \) of the exact sequence

\[
0 \to \mathfrak{g}_\beta(\mathbb{F}) \to \mathfrak{g}(\mathbb{F}) \to \mathbb{V}_\beta \to 0. \tag{2}
\]

For any \( v \in \mathbb{V}_\beta \) and \( g \in G_\beta(\mathbb{F}) \), put

\[
\gamma(v, g) = \gamma_g(v, g) = \text{Ad}(g)^{-1}[v] - [v \sigma_g] \in \mathfrak{g}_\beta(\mathbb{F}).
\]
Take a character \( \rho \in \mathfrak{g}_\beta(\mathbb{F}) \). Then there exists uniquely a \( v_\rho \in \mathbb{V}_\beta \) such that
\[
\rho(\gamma(v, g)) = \hat{\tau}(v, v_\rho)\beta
\]
for all \( v \in \mathbb{V}_\beta \). Note that \( v_\rho \in \mathbb{V}_\beta \) depends on \( \rho \) as well as the section \( v \mapsto [v] \).
Let
\[
G_\beta(\mathbb{F})^{(c)} = \{ g \in G(\mathbb{F}) \mid \text{Ad}(g)Y = Y \text{ for } \forall Y \in \mathfrak{g}_\beta(\mathbb{F}) \}
\]
be the centralizer of \( \mathfrak{g}_\beta(\mathbb{F}) \) in \( G(\mathbb{F}) \), which is a subgroup of \( G_\beta(\mathbb{F}) \). Then for any \( g, h \in G_\beta(\mathbb{F})^{(c)} \), we have
\[
v_{gh} = v_h\sigma_g^{-1} + v_g
\]
because \( \gamma(v, gh) = \gamma(v, g) + \gamma(v\sigma_g^{-1}, h) \) for all \( v \in \mathbb{V}_\beta \). Put
\[
c_{\beta, \rho}(g, h) = \hat{\tau}(2^{-1}(v_g, v_{gh})\beta)
\]
for \( g, h \in G_\beta(\mathbb{F})^{(c)} \). Then the relation (3) shows that \( c_{\beta, \rho} \in Z^2(G_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times) \) is a 2-cocycle with trivial action of \( G_\beta(\mathbb{F})^{(c)} \) on \( \mathbb{C}^\times \). Moreover we have

**Proposition 3.1.1** The cohomology class \([c_{\beta, \rho}] \in H^2(G_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times)\) is independent of the choice of the \( \mathbb{F} \)-linear section \( v \mapsto [v] \).

**Proof** Take another \( \mathbb{F} \)-linear section \( v \mapsto [v]' \) with respect to which we will define \( \gamma'(v, g) \in \mathfrak{g}_\beta \) and \( v'_\rho \in \mathbb{V}_\beta \) as above. Then there exists a \( \delta \in \mathbb{V}_\beta \) such that \( \rho([v] - [v]) = \hat{\tau}((v, \delta)\beta) \) for all \( v \in \mathbb{V}_\beta \). We have \( v'_\rho = v_\rho + \delta - \delta\sigma_g \) for all \( g \in G_\beta(\mathbb{F})^{(c)} \). So if we put \( \alpha(g) = \hat{\tau}(2^{-1}(v'_\rho - v_\rho, \delta)\beta) \) for \( g \in G_\beta(\mathbb{F})^{(c)} \), then we have
\[
\hat{\tau}(2^{-1}(v'_\rho, v_{gh})\beta) = \hat{\tau}(2^{-1}(v_\rho, v_{gh})\beta) \cdot \alpha(h)\alpha(g)^{-1}\alpha(g)
\]
for all \( g, h \in G_\beta(\mathbb{F})^{(c)} \). \( \blacksquare \)

### 3.2
Let us assume that there exists a closed smooth \( O \)-group subscheme \( H \subset GL_n \) of which our \( G \) is a closed \( O \)-group subscheme and that the trace form
\[
B : \mathfrak{h}(\mathbb{F}) \times \mathfrak{h}(\mathbb{F}) \rightarrow \mathbb{F}
\]
is non-degenerate where \( \mathfrak{h} \) is the Lie algebra of \( H \). Then we have
\[
\mathfrak{h}(\mathbb{F}) = \mathfrak{g}(\mathbb{F}) \oplus \mathfrak{g}(\mathbb{F})^\perp
\]
where \( \mathfrak{g}(\mathbb{F})^\perp = \{ X \in \mathfrak{g}(\mathbb{F}) \mid B(X, \mathfrak{g}(\mathbb{F})) = 0 \} \) is the orthogonal complement of \( \mathfrak{g}(\mathbb{F}) \) in \( \mathfrak{h}(\mathbb{F}) \).

Take a \( \beta \in \mathfrak{g}(O) \) such that \( \mathfrak{g}_\beta(\mathbb{F}) \subseteq \mathfrak{g}(\mathbb{F}) \). Then \( \beta \in \mathfrak{h}(O) \) and \( \mathfrak{h}_\beta(\mathbb{F}) \subseteq \mathfrak{h}(\mathbb{F}) \) where \( \mathfrak{h}_\beta = Z_\mathfrak{h}(\beta) \) is the centralizer. We have decompositions
\[
\mathfrak{h}_\beta(\mathbb{F}) = \mathfrak{g}_\beta(\mathbb{F}) \oplus (\mathfrak{g}(\mathbb{F})^\perp)_\beta
\]
where \( (\mathfrak{g}(\mathbb{F})^\perp)_\beta = \mathfrak{h}_\beta(\mathbb{F}) \cap \mathfrak{g}(\mathbb{F})^\perp \), and
\[
\tilde{\mathbb{V}}_\beta = \mathfrak{h}(\mathbb{F})/\mathfrak{h}_\beta(\mathbb{F}) = \mathbb{V}_\beta \oplus (\mathfrak{g}(\mathbb{F})^\perp / (\mathfrak{g}(\mathbb{F})^\perp)_\beta)
\]
is an orthogonal decomposition of symplectic spaces.

Let \( v \mapsto [v] \) be a \( \mathbb{F} \)-linear section of the exact sequence

\[
0 \to \mathfrak{h}_\beta(\mathbb{F}) \to \mathfrak{h}(\mathbb{F}) \to \mathcal{V}_\beta \to 0
\]

of \( \mathbb{F} \)-vector space such that \([\mathcal{V}_\beta] \subset \mathfrak{g}(\mathbb{F})\) and \([\mathfrak{g}(\mathbb{F})^\perp / (\mathfrak{g}(\mathbb{F})^\perp)_\beta] \subset \mathfrak{g}(\mathbb{F})^\perp\).

Take \( \rho \in \mathfrak{g}_\beta(\mathbb{F}) \) and put

\[
\tilde{\rho} : \mathfrak{h}_\beta(\mathbb{F}) = \mathfrak{g}_\beta(\mathbb{F}) \oplus (\mathfrak{g}(\mathbb{F})^\perp)_\beta \xrightarrow{\text{projection}} \mathfrak{g}_\beta(\mathbb{F}) \cong \mathbb{C}^\times.
\]

For any \( g \in G_\beta(\mathbb{F}) \subset H_\beta(\mathbb{F}) \), there exists uniquely a \( v_g \in \mathcal{V}_\beta \) such that

\[
\rho(\gamma_g(v, g)) = \tilde{\tau}(\langle v, v_g \rangle_\beta)
\]

for all \( v \in \mathcal{V}_\beta \). Then we have

\[
\tilde{\rho}(\gamma_h(v, v_g)) = \tilde{\tau}(\langle v, v_g \rangle_\beta)
\]

for all \( v \in \mathcal{V}_\beta \). In fact if we put \( v = v' + v'' \) with \( v' \in \mathcal{V}_\beta \) and \( v'' \in (\mathfrak{g}(\mathbb{F})^\perp / (\mathfrak{g}(\mathbb{F})^\perp)^\perp)_\beta \), then we have \( \gamma_h(v, g) = \gamma_h(v', g) + \gamma_h(v'', g) \) with \( \gamma_h(v'', g) \in (\mathfrak{g}(\mathbb{F})^\perp)_{\beta'} \), since

\[
\text{Ad}(g)\mathfrak{g}(\mathbb{F})^\perp = \mathfrak{g}(\mathbb{F})^\perp, \quad \text{Ad}(g) (\mathfrak{g}(\mathbb{F})^\perp)^\perp = (\mathfrak{g}(\mathbb{F})^\perp)^\perp.
\]

Then we have

\[
\tilde{\rho}(\gamma_h(v, g)) = \rho(\gamma_h(v', g)) = \tilde{\tau}(\langle v', v_g \rangle_\beta)
\]

\[
= \tilde{\tau}(\langle v, v_g \rangle_\beta)
\]

because \( \langle v'', v_g \rangle_\beta = 0 \). Hence we have

**Proposition 3.2.1** If \( G_\beta(\mathbb{F})^{(c)} \subset H_\beta(\mathbb{F})^{(c)} \) then the Schur multiplier \([c_{\beta, \rho}] \in H^2(G_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times)\) is the image under the restriction mapping

\[
\text{Res} : H^2(H_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times) \to H^2(G_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times)
\]

of the Schur multiplier \([c_{\beta, \rho}] \in H^2(G_\beta(\mathbb{F})^{(c)}, \mathbb{C}^\times)\).

## 4 Weil representation

Assume that \( r = 2l - 1 \geq 3 \) is odd so that \( l' = l - 1 \geq 1 \).

### 4.1 We have a chain of canonical surjections

\[
\varnothing : K_{l-1}(O_r) \to K_{l-1}(O_{r-1}) \to \mathfrak{o}(O_{l-1}) \to \mathfrak{g}(\mathbb{F}) \tag{4}
\]

defined by

\[
1 + \alpha^{l-1}X \pmod{p^r} \mapsto 1 + \alpha^{l-1}X \pmod{p^{r-1}}
\]

\[
\mapsto X \pmod{p^{r-1}} \mapsto X \pmod{p}.
\]
Let us denote by $Z(O_r, \beta)$ the inverse image under the surjection $\triangleright$ of $g_\beta(F)$. Then $Z(O_r, \beta)$ is a normal subgroup of $K_{l-1}(O_r)$ containing $K_l(O_r)$ as the kernel of $\triangleright$.

Let us denote by $Y_\beta$ the set of the group homomorphisms $\psi$ of $Z(O_r, \beta)$ to $\mathbb{C}^\times$ such that $\psi = \psi_\beta$ on $K_l(O_r)$. Then a bijection of $g_\beta(F)^\sim$ onto $Y_\beta$ is given by

$$\rho \mapsto \psi_{\beta, \rho} = \tilde{\psi}_\beta \circ (\rho \circ \triangleright),$$

where a group homomorphism $\tilde{\psi}_\beta : Z(O_r, \beta) \to \mathbb{C}^\times$ is defined by

$$1 + \varpi^{l-1}X \pmod{p^r} \mapsto \tau (\varpi^{-1}B(X, \beta) - (2\varpi)^{-1}B(X^2, \beta))$$

with $\bar{X} = X \pmod{p} \in g_\beta(F)$.

Take a $\psi \in Y_\beta$. For two elements

$$x = 1 + \varpi^{l-1}X \pmod{p^r}, \quad y = 1 + \varpi^{l-1}Y \pmod{p^r}$$

of $K_{l-1}(O_r)$, we have $x^{-1} = 1 - \varpi^{l-1}X + 2^{-1}\varpi^{2l-2}X^2 \pmod{p^r}$ so that we have

$$xyz^{-1}y^{-1} = 1 + \varpi^{l-1}[X, Y] \pmod{p^r} \in K_{r-1}(O_r) \subset K_l(O_r)$$

and so $\psi_{\beta}(xyz^{-1}y^{-1}) = \tau (\varpi^{-1}B(X, \text{ad}(Y) \beta))$. Hence we have

$$\psi(xyz^{-1}y^{-1}) = \psi_{\beta}(xyz^{-1}y^{-1}) = 1$$

for all $x \in K_{l-1}(O_r)$ and $y \in Z(O_r, \beta)$ so that we can define

$$D_\psi : K_{l-1}(O_r)/Z(O_r, \beta) \times K_{l-1}(O_r)/Z(O_r, \beta) \to \mathbb{C}^\times$$

by

$$D_\psi(\hat{g}, \hat{h}) = \psi(ghg^{-1}h^{-1}) = \psi_{\beta}(ghg^{-1}h^{-1}) = \tau (\varpi^{-1}B([X, Y], \beta))$$

for $g = (1 + \varpi^{l-1}X)(\pmod{p^r}), h = (1 + \varpi^{l-1}Y)(\pmod{p^r}) \in K_{l-1}(O_r)$. Note that $D_\psi$ is non-degenerate. Then Proposition 3.1.1 of [15] gives

**Proposition 4.1.1** For any $\psi = \psi_{\beta, \rho} \in Y_\beta$ with $\rho \in g_\beta(F)^\sim$, there exists unique irreducible representation $\pi_{\psi}$ of $K_{l-1}(O_r)$ such that $(\psi, \pi_{\psi})_{Z(O_r, \beta)} > 0$. Furthermore

$$\text{Ind}^{K_{l-1}(O_r)}_{Z(O_r, \beta)} \psi = \bigoplus_{\dim \pi_{\psi}} \pi_{\psi}$$

and $\pi_{\psi}(x)$ is the homothety $\psi(x)$ for all $x \in Z(O_r, \beta)$.

Fix a $\psi = \psi_{\beta, \rho} \in Y_\beta$ with $\rho \in g_\beta(F)^\sim$. Our problem is to extend the representation $\pi_{\psi}$ of $K_{l-1}(O_r)$ to a representation of $G(O_r, \beta) = G_\beta(O_r) \cdot K_{l-1}(O_r)$. Now for any $g_r = g(\pmod{p^r}) \in G_\beta(O_r)$ and $x = (1 + \varpi^{l-1}X)(\pmod{p^r}) \in Z(O_r, \beta)$, we have

$$g_r^{-1}xg_rx^{-1} = (1 + \varpi^{l-1}g^{-1}Xg)(1 - \varpi^{l-1}X + 2^{-1}\varpi^{2l-2}X^2)(\pmod{p^r})$$

$$= 1 + \varpi^{l-1} (\text{Ad}(g)^{-1}X - X)(\pmod{p^r}) \in K_l(O_r),$$

and

$$\psi(g_r^{-1}xg_rx^{-1}) = \psi_{\beta}(g_r^{-1}xg_rx^{-1}) = \tau (\varpi^{-1}B(X, \text{Ad}(g)\beta - \beta)) = 1,$$
that is \( \psi(g^{-1}xg) = \psi(x) \) for all \( x \in Z(O_r, \beta) \). This means that, for any \( g \in G_\beta(O_r) \), the \( g \)-conjugate of \( \pi_\psi \) is isomorphic to \( \pi_\psi \), that is, there exists a group homomorphism \( U(g) \in GL_C(V_\psi) \) (\( V_\psi \) is the representation space of \( \pi_\psi \)) such that

\[
\pi_\psi(g^{-1}xg) = U(g)^{-1} \circ \pi_\psi(x) \circ U(g)
\]

for all \( x \in K_{l-1}(O_r) \), and moreover, for any \( g, h \in G_\beta(O_r) \), there exists a \( c_U(g, h) \in C^\times \) such that

\[
U(g) \circ U(h) = c_U(g, h) \cdot U(gh).
\]

Then \( c_U \in Z^2(G_\beta(O_r), C^\times) \) is a \( C^\times \)-valued 2-cocycle on \( G_\beta(O_r) \) with trivial action on \( C^\times \), and the cohomology class \([c_U] \in H^2(G_\beta(O_r), C^\times)\) is independent of the choice of each \( U(g) \).

In the following subsections, we will construct \( \pi_\psi \) by means of Schrödinger representations over the finite field \( F \) (see Proposition 4.6.1), and will show that we can construct \( U(g) \) by means of Weil representation so that we have

\[
c_U(g, h) = c_{\beta, \rho}(\overline{\nu}, \overline{\nu})
\]

for all \( g, h \in G_\beta(O_r) \), where \( \overline{\nu} \in G_\beta(F) \) is the image of \( g \in G_\beta(O_r) \) under the canonical surjection \( G(O_r) \to G(F) \) (see subsection 4.3). Furthermore, Proposition 4.6.1 tells us that the Schur multiplier \([c_U] \in H^2(G_\beta(F), C^\times)\) is trivial.

So let us assume that the Schur multiplier \([c_U] \in H^2(G_\beta(F), C^\times)\) is trivial. Then we have

**Proposition 4.1.2** There exists a group homomorphism \( U_\psi : G_\beta(O_r) \to GL_C(V_\psi) \) such that

1) \( \pi_\psi(g^{-1}xg) = U_\psi(g)^{-1} \circ \pi_\psi(x) \circ U_\psi(g) \) for all \( g \in G_\beta(O_r) \) and \( x \in K_{l-1}(O_r) \)

and

2) \( U_\psi(h) = 1 \) for all \( h \in G_\beta(O_r) \cap K_{l-1}(O_r) \).

**Proof** Since the Schur multiplier \([c_U] \in H^2(G_\beta(O_r), C^\times)\) is trivial, there exists a group homomorphism \( U : G_\beta(O_r) \to GL_C(V_\psi) \) such that \( \pi_\psi(g^{-1}xg) = U(g)^{-1} \circ \pi_\psi(x) \circ U(g) \) for all \( g \in G_\beta(O_r) \) and \( x \in K_{l-1}(O_r) \). Then for any \( h \in G_\beta(O_r) \cap K_{l-1}(O_r) \) there exists a \( c(h) \in C^\times \) such that \( U(h) = c(h) \cdot \pi_\psi(h) \). On the other hand we have

\[
G_\beta(O_r) \cap K_{l-1}(O_r) \subset Z(O_r, \beta)
\]

since \((1 + \omega^{l-1}X)_r \in G_\beta(O_r) \cap K_{l-1}(O_r) \) means that

\[
\beta \equiv (1 + \omega^{l-1}X)\beta(1 + \omega^{l-1}X)^{-1} \pmod{p^l}
\]

\[
\equiv (\beta + \omega^{l-1}X\beta)(1 - \omega^{l-1}X) \pmod{p^l}
\]

\[
\equiv \beta + \omega^{l-1}[X, \beta] \pmod{p^l}
\]

and then \([X, \beta] \equiv 0 \pmod{p} \), that is \( X \pmod{p} \in g_\beta(F) \). Then \( \pi_\psi(h) \) is the homothety \( \psi(h) \) for all \( h \in G_\beta(O_r) \cap K_{l-1}(O_r) \). Extend the group homomorphism \( h \mapsto c(h)\psi(h) \) of \( G_\beta(O_r) \cap K_{l-1}(O_r) \) to a group homomorphism
\[ \theta : G_\beta(O_r) \to \mathbb{C}^\times. \] Then \( g \mapsto U_\psi(g) = \theta(g)^{-1}U(g) \) is the required group homomorphism. ■

Let us denote by \( G_\beta(O_r) \times_{K_{l-1}(O_r)} g_\beta(\mathbb{F}) \) the set of \((\theta, \rho) \in G_\beta(O_r) \times g_\beta(\mathbb{F}) \) such that \( \theta = \psi_{\beta, \rho} \) on \( G_\beta(O_r) \cap K_{l-1}(O_r) \). Then \((\theta, \rho) \in G_\beta(O_r) \times_{K_{l-1}(O_r)} g_\beta(\mathbb{F}) \) defines an irreducible representation \( \sigma_{\theta, \rho} \) of \( G(O_r, \beta) = G_\beta(O_r) \cdot K_{l-1}(O_r) \) by

\[
\sigma_{\theta, \rho}(gh) = \theta(g) \cdot U_\psi(g) \circ \pi_\psi(h)
\]

for \( g \in G_\beta(O_r) \) and \( h \in K_{l-1}(O_r) \) with \( \psi = \psi_{\beta, \rho} \). Then we have

**Proposition 4.1.3** A bijection of \( C \times_{K_{l-1}(O_r)} g_\beta(\mathbb{F}) \) onto \( \text{Irr}(G(O_r, \beta) \mid \psi_\beta) \) is given by \((\theta, \rho) \mapsto \sigma_{\theta, \rho} \).

**Proof:** Clearly \( \sigma_{\theta, \rho} \in \text{Irr}(G(O_r, \beta) \mid \psi_\beta) \) for all \((\theta, \rho) \in C \times_{K_{l-1}(O_r)} g_\beta(\mathbb{F}) \). Take a \( \sigma \in \text{Irr}(G(O_r, \beta) \mid \psi_\beta) \). Then

\[
\sigma \leftrightarrow \text{Ind}_{K_{l-1}(O_r)}^{G(O_r, \beta)} \psi_\beta = \text{Ind}_{Z(O_r, \beta)}^{G(O_r, \beta)} \left( \text{Ind}_{K_{l-1}(O_r)}^{Z(O_r, \beta)} \psi_\beta \right) = \bigoplus_{\psi \in \text{Irr}(Z(O_r, \beta))} \text{Ind}_{Z(O_r, \beta)}^{G(O_r, \beta)} \psi
\]

so that there exists a \( \psi = \psi_{\beta, \rho} \in Y_\beta \) with \( \rho \in g_\beta(\mathbb{F}) \) such that

\[
\sigma \leftrightarrow \text{Ind}_{Z(O_r, \beta)}^{G(O_r, \beta)} \psi = \text{Ind}_{K_{l-1}(O_r)}^{G(O_r, \beta)} \left( \text{Ind}_{Z(O_r, \beta)}^{K_{l-1}(O_r)} \psi \right) = \bigoplus_{\psi \in \text{Irr}(Z(O_r, \beta))} \text{dim} \pi_{\psi} \bigoplus_{\theta} \text{Ind}_{K_{l-1}(O_r)}^{G(O_r, \beta)} \pi_{\psi} = \bigoplus_{\theta} \sigma_{\theta, \psi},
\]

where \( \bigoplus \) is the direct sum over \( \theta \in G_\beta(O_r) \) such that \( \theta = \psi_{\beta, \rho} \) on \( G_\beta(O_r) \cap K_{l-1}(O_r) \). Then we have \( \sigma = \sigma_{\theta, \rho} \) for some \((\theta, \rho) \in G_\beta(O_r) \times_{K_{l-1}(O_r)} g_\beta(\mathbb{F}) \).

This proposition combined with the following proposition gives the bijection presented in our main Theorem 23.1 in the case of \( r \) being odd.

**Proposition 4.1.4** \((\theta, \rho) \mapsto \theta \) gives a bijection of \( G_\beta(\mathbb{F}) \times_{K_{l-1}(O_r)} g_\beta(\mathbb{F}) \) onto the set

\[
\{ \theta \in G_\beta(O_r) \mid \text{s.t.} \theta = \psi_\beta \text{ on } G_\beta(O_r) \cap K_{l-1}(O_r) \}.
\]

**Proof:** Take a \((\theta, \rho) \in G_\beta(\mathbb{F}) \times_{K_{l-1}(O_r)} g_\beta(\mathbb{F}) \). The smoothness of \( G_\beta \) over \( O \) implies that the canonical mapping \( g_\beta(O) \to g_\beta(\mathbb{F}) \) is surjective. So take a \( X \in g_\beta(\mathbb{F}) \) with \( X \in g_\beta(O) \). Then we have

\[
g = 1 + \omega^{l-1}X + 2^{-l-1}\omega^{2l-2}X^2 \pmod{p^r} \in K_{l-1}(O_r) \cap G_\beta(O_r)
\]

so that

\[
\theta(g) = \psi_{\beta, \rho}(g) = \tau(\omega^{l-1}B(X + 2^{-l-1}\omega^{l-1}X^2, \beta) - 2^{-l-1}\omega^{-1}B(X, \beta)) \cdot \rho(X)
\]

\[
= \tau(\omega^{l-1}B(X, \beta)) \cdot \rho(X).
\]
Hence we have
\[ \rho(X) = \tau \left( -\omega^{-1}B(X,\beta) \right) \cdot \theta \left( 1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r} \right). \]

This means that the mapping \((\theta, \rho) \mapsto \theta\) is injective. Take \(X, X' \in \mathfrak{g}_\beta(O)\) such that \(X \equiv X' \pmod{p}\). Then we have \(X' = X + \omega T\) with \(T \in \mathfrak{g}_\beta(O)\) and
\[
\begin{align*}
1 + \omega^{l-1}X' + 2^{-1}\omega^{2l-2}X'^2 & \quad \pmod{p^r} \\
= 1 + \omega^{-1}X + 2^{-1}\omega^{2l-2}X^2 + \omega^lT & \quad \pmod{p^r} \\
= (1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2)(1 + \omega^lT) & \quad \pmod{p^r},
\end{align*}
\]
where \(1 + \omega^lT \pmod{p^r} \in K_l(O_r)\) and hence
\[ \theta(1 + \omega^lT \pmod{p^r}) = \psi_\beta(1 + \omega^lT \pmod{p^r}) = \tau \left( \omega^{-l-1}B(T, \beta) \right). \]

This and the commutativity of \(G_\beta\) show that
\[ \rho(X) = \tau \left( -\omega^{-1}B(X,\beta) \right) \cdot \theta \left( 1 + \omega^{l-1}X + 2^{-1}\omega^{2l-2}X^2 \pmod{p^r} \right) \]
with \(X \in \mathfrak{g}_\beta(F)\) with \(X \in \mathfrak{g}_\beta(O)\) gives an well-defined group homomorphism of \(\mathfrak{g}_\beta(F)\) to \(C^\times\). Then \((\theta, \rho) \in G_\beta(O_r)^{\times} \times K_{l-1}(O_r)\mathfrak{g}_\beta(F)^{\times}\) and our mapping in question is surjective.

4.2 A group extension

\[ 0 \to \mathfrak{g}(O_{l-1}) \xrightarrow{\phi} K_{l-1}(O_r) \xrightarrow{\psi} \mathfrak{g}(F) \to 0 \] (6)

is given by the canonical surjection \([\phi]\), whose kernel is \(K_l(O_r)\), with the group isomorphism
\[ \phi : \mathfrak{g}(O_{l-1}) \xrightarrow{\sim} K_l(O_r) \]
defined by \(S(\mod{p^{l-1}}) \mapsto (1 + \omega^lS)(\mod{p^r})\).

In order to determine the 2-cocycle of the group extension \([\phi]\), choose any mapping \(\lambda : \mathfrak{g}(F) \to \mathfrak{g}(O)\) such that \(X = \lambda(X) (\mod{p})\) for all \(X \in \mathfrak{g}(F)\) and \(\lambda(0) = 0\), and define a section
\[ l : \mathfrak{g}(F) \to K_{l-1}(O_r) \]
of \([\phi]\) by \(X \mapsto 1 + \omega^{l-1}\lambda(X) + 2^{-1}\omega^{2l-2}\lambda(X)^2 (\mod{p^r})\). Then we have
\[ l(X)^{-1} = 1 - \omega^{l-1}\lambda(X) + 2^{-1}\omega^{2l-2}\lambda(X)^2 (\mod{p^r}) \]
for all \(X \in \mathfrak{g}(F)\) and
\[ l(X)(1 + \omega^lS)(X)^{-1} \equiv 1 + \omega^lS (\mod{p^r}) \]
for all \(S_{l-1} \in \mathfrak{g}(O_{l-1})\). Furthermore we have
\[ l(X)(Y)l((X + Y)^{-1} = 1 + \omega^l \left\{ \mu(X,Y) + 2^{-1}\omega^{l-2}\lambda(X),\lambda(Y) \right\} (\mod{p^r}) \]
for all \(X, Y \in \mathfrak{g}(F)\) where \(\mu : \mathfrak{g}(F) \times \mathfrak{g}(F) \to \mathfrak{g}(O)\) is defined by
\[ \lambda(X) + \lambda(Y) - \lambda(X + Y) = \omega \cdot \mu(X, Y) \]

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for all \(X, Y \in g(F)\). Now we have two elements (2-cocycle)
\[
\mu = [(X, Y) \mapsto \mu(X, Y)_{l-1}], \quad c = [(X, Y) \mapsto 2^{-l-2}[X, Y]_{l-1}]
\]
of \(Z^2(g(F), g(O_{l-1}))\) with trivial action of \(g(F)\) on \(g(O_{l-1})\).

Let us consider two groups \(M\) and \(G\) corresponding to the two 2-cocycles \(\mu\) and \(c\) respectively. That is the group operation on \(M = g(F) \times g(O_{l-1})\) is defined by
\[
(X, S_{l-1}) \cdot (Y, T_{l-1}) = (X + Y, (S + T + \mu(X, Y))_{l-1})
\]
and the group operation on \(G = g(F) \times g(O_{l-1})\) is defined by
\[
(X, S_{l-1}) \cdot (Y, T_{l-1}) = (X + Y, (S + T + 2^{-l-2}[X, Y])_{l-1}).
\]
Let \(G \times g(F) M\) be the fiber product of \(G\) and \(M\) with respect to the canonical projections onto \(g(F)\). In other word
\[
G \times g(F) M = \{(X; S, T) = ((X, S), (X, T)) \in G \times M\}
\]
is a subgroup of the direct product \(G \times M\). We have a surjective group homomorphism
\[
(*) : G \times g(F) M \to K_{l-1}(O_r)
\]  
(7)
defined by
\[
(X; S_{l-1}, T_{l-1}) \mapsto m(X) \cdot (1 + \omega^l(S + T) \pmod{p^r})
\]
\[
= 1 + \omega^{l-1}\lambda(X) + 2^{-1}\omega^{2l-2}\lambda(X)^2 + \omega^l(S + T) \pmod{p^r}.
\]

4.3 The group homomorphism \(B_{\beta} : g(O_{l-1}) \to O_{l-1} (X \mapsto B(X, \beta_{l-1}))\) induces a group homomorphism
\[
B^{\beta}_2 : H^2(g(F), g(O_{l-1})) \to H^2(g(F), O_{l-1}).
\]
Let us denote by \(H_{\beta}\) the group associated with the 2-cocycle
\[
c_{\beta} = B_{\beta} \circ c = [(X, Y) \mapsto 2^{-1}\omega^{l-2}B([X, Y], \beta)_{l-1}] \in Z^2(g(F), O_{l-1}).
\]
That is \(H_{\beta} = g(F) \times O_{l-1}\) with a group operation
\[
([X, s], [Y, t]) = ([X + Y, s + t + 2^{-l-2}B([X, Y], \beta)_{l-1}]).
\]
Then the center of \(H_{\beta}\) is \(Z(H_{\beta}) = g_{\beta}(\mathbb{F}) \times O_{l-1}\), the direct product of two additive groups \(g_{\beta}(\mathbb{F})\) and \(O_{l-1}\).

The inverse image of \(Z(H_{\beta})\) with respect to the surjective group homomorphism
\[
\blacktriangle : G \times g(F) M \to H_{\beta} \quad ((X; S_{l-1}, T_{l-1}) \mapsto (X, B(S, \beta)_{l-1}))
\]  
(8)
is \( (G \times g(F) M)_{\beta} = \{(X; S, T) \in G \times g(F) M \mid X \in g(F)_{\mathbb{F}}\}\) which is mapped onto \(Z(O_{l-1}, \beta) \subset K_{l-1}(O_r)\) by the surjection \(\mathbb{F}\).

Take a \(\rho \in g_{\beta}(\mathbb{F})\) which defines group homomorphisms
\[
\chi_{\rho} = \rho \otimes [x_{l-1} \mapsto \tau(\omega^{-(l-1)}x)] : Z(H_{\beta}) = g_{\beta}(\mathbb{F}) \times O_{l-1} \to \mathbb{C}\)
By the polarization $\pi$ defined in subsection 4.1. and $\beta(L)$ is isomorphic to the Heisenberg group of the symplectic $\mathbb{F}$-space $\mathbb{F}(\beta,\rho)$.

Then we have a group homomorphism

$$\xi: H_{\beta} \to \mathbb{C} \times (0,\chi_{\beta}).$$

Let us denote $H_{\beta} = \mathbb{C} \times Z(\mathbb{H}_{\beta})$.

Then $\mathbb{H}_{\beta}$ is isomorphic to $H_{\beta}$ by $(v, (Y, s)) \mapsto ([v] + Y, s)$.

Let $H_{\beta}$ be the Heisenberg group of the symplectic $\mathbb{F}$-space $\mathbb{V}_{\beta}$, that is $H_{\beta} = \mathbb{V}_{\beta} \times \mathbb{C}$. with a group operation

$$(u, s) \cdot (v, t) = (u + v, s \cdot t \cdot \hat{\tau}(2^{-1}(u, v)_{\beta})).$$

Then we have a group homomorphism

$$\mathbb{H}_{\beta} = \mathbb{V}_{\beta} \times Z(\mathbb{H}_{\beta}) \to H_{\beta} \quad ((v, z) \mapsto (v, \chi_{\beta}(z))).$$

Fix a polarization $\mathbb{V}_{\beta} = \mathbb{W}' \oplus \mathbb{W}$ of the symplectic $\mathbb{F}$-space $\mathbb{V}_{\beta}$. Let us denote $L^{2}(\mathbb{W}')$ the complex vector space of the complex-valued functions $f$ on $\mathbb{W}'$ with inner product $\langle f, f' \rangle = \sum_{w \in \mathbb{W}'} f(w) \overline{f'(w)}$. The Schrödinger representation $(\pi^{\beta}, L^{2}(\mathbb{W}'))$ of $H_{\beta}$ associated with the polarization is defined for $(v, s) \in H_{\beta}$ and $f \in L^{2}(\mathbb{W}')$ by

$$\langle \pi^{\beta}(v, s)f \rangle (w) = s \cdot \hat{\tau}(2^{-1}(v, w)_{\beta}) \cdot f(w + v_{-}).$$

4.4 Fix a $\rho \in \mathfrak{g}_{\beta}(\mathbb{F})$. Let us determine the 2-cocycle of the group extension

$$0 \to Z(\mathbb{H}_{\beta}) \to \mathbb{H}_{\beta} \to \mathbb{V}_{\beta} \to 0$$

(9)

where $\mathbb{H}_{\beta} : \mathbb{H}_{\beta} \to \mathbb{V}_{\beta}$ is defined by $(\mathbb{X}, \mathbb{Y}) \mapsto \mathbb{X}(\mathbb{X}, \mathbb{Y})$ of $\mathbb{F}$-vector spaces and define a section $l : \mathbb{V}_{\beta} \to \mathbb{H}_{\beta}$ of the group extension (9) by $l(v) = ([v], 0)$. Then we have

$$l(u)l(v)l(u + v)^{-1} = (0, 2^{-1} \omega l^{-2}B([X, Y], \beta) \mod p^{-1})$$

for $u = \mathbb{X}, v = \mathbb{Y} \in \mathbb{V}_{\beta}$ so that the 2-cocycle of the group extension (9) is

$$[(\mathbb{X}, \mathbb{Y}) \mapsto 2^{-1} \omega l^{-2}B([X, Y], \beta) \mod p^{-1}] \in Z^{2}(\mathbb{V}_{\beta}, \mathbb{O}_{1-1}).$$

Define a group operation on $H_{\beta} = \mathbb{V}_{\beta} \times Z(\mathbb{H}_{\beta})$

$$(\mathbb{X}, z) \cdot (\mathbb{Y}, w) = (\mathbb{X} + \mathbb{Y}, z + w + 2^{-1} \omega l^{-2}B([X, Y], \beta) \mod p^{-1}).$$

Then $\mathbb{H}_{\beta}$ is isomorphic to $H_{\beta}$ by $(v, (Y, s)) \mapsto ([v] + Y, s)$.

If we denote $\pi^{\beta} = \pi^{\beta}(\mathbb{W}')$ then $L^{2}(\mathbb{W}')$ is the vector space of the complex-valued functions $f$ on $\mathbb{W}'$ with inner product $\langle f, f' \rangle = \sum_{w \in \mathbb{W}' f(w) \overline{f'(w)}$. The Schrödinger representation $(\pi^{\beta}, L^{2}(\mathbb{W}'))$ of $H_{\beta}$ associated with the polarization is defined for $(v, s) \in H_{\beta}$ and $f \in L^{2}(\mathbb{W}')$ by

$$\langle \pi^{\beta}(v, s)f \rangle (w) = s \cdot \hat{\tau}(2^{-1}(v, w)_{\beta}) \cdot f(w + v_{-}).$$
where \( v = v_− + v_+ \in V_\beta \) with \( v_- \in W', v_+ \in W' \).

Now an irreducible representation \((\pi^{β,ρ}, L^2(W'))\) of \( H_\beta \) is defined by \( \pi^{β,ρ}(v, z) = \pi^β(v, χ_\rho(z)) \), and an irreducible representation \((\tilde{\pi}^{β,ρ}, L^2(W'))\) of \( G \times g(\mathbb{F}) \bar{M} \) is defined by

\[ \tilde{\pi}^{β,ρ} : G \times g(\mathbb{F}) \bar{M} \to H_\beta \to H_\beta \to GLC(L^2(W')). \]

Then \( \tilde{ψ}_0 \cdot \tilde{π}_β, ρ \) is trivial on the kernel of \( (\ast) : G \times g(\mathbb{F}) \bar{M} \to K_{l-1}(O_r) \) so that it induces an irreducible representation \( π_{β, ρ} \) of \( K_{l-1}(O_r) \) on \( L^2(W') \).

**Proposition 4.4.1** Take a \( g = 1 + \varpi^{l-1}T (\text{mod } p^r) \in K_{l-1}(O_r) \) with \( T \in gl_n(O) \). Then we have \( T \equiv \varpi^{-1}B(T, β) - 2^{-1}\varpi^{-1}B(T^2, β) \cdot ρ(Y) \cdot π^β(v, 1) \)

where \( T = [v] + Y \in g(\mathbb{F}) \) with \( v \in V_\beta \) and \( Y \in g_δ(\mathbb{F}) \). In particular \( π_{β, ρ}(h) \) is the homothety \( ψ_{β, ρ}(h) \) for all \( h \in Z(O_r, β) \).

**[Proof]** By the definition we have

\[ π_{β, ψ}(l((X))(1 + \varpi^lS \ (\text{mod } p^r))) = \psi_0(X, 0) \cdot \tilde{π}^{β,ρ}([v] + Y; S_{l-1}, 0) \]

\[ = \psi(l((Y))(1 + \varpi^lS \ (\text{mod } p^r))) \cdot \tau(\varpi^{-l}B(λ(X) - λ(Y), β) \cdot π^β(v, 1)) \]

\[ = \varpi^{-l}B(S, β) + \varpi^{-l}B(λ(X), β) \cdot ρ(Y) \cdot π^β(v, 1) \]

where \( X = [v] + Y \in g(\mathbb{F}) \) with \( v \in V_β \) and \( Y \in g_δ(\mathbb{F}) \) and put \( 1 + \varpi^{l-1}T = l((X))(1 + \varpi^lS) \ (\text{mod } p^r) \) with \( X \in g(\mathbb{F}) \) and \( S \in g(O) \). Then we have

\[ 1 + \varpi^{l-1}T \equiv 1 + \varpi^{l-1}λ(X) \ (\text{mod } p^r) \]

so that we have \( T \ (\text{mod } p) = X \in g(\mathbb{F}) \) and

\[ \varpi S \equiv T - λ(T) - 2^{-1}\varpi^{-l}λ(T^2) \ (\text{mod } p^r). \]

Then we have

\[ π_{β, ρ}(g) = π^{β,ρ}(l((T))(1 + \varpi^lS)), \]

\[ = \varpi^{-l}B(S, β) + \varpi^{-l}B(λ(T), β) \cdot ρ(Y) \cdot π^β(v, 1) \]

\[ = \varpi^{-l}B(T, β) - 2^{-1}\varpi^{-l}B(T^2, β) \cdot ρ(Y) \cdot π^β(v, 1). \]

\[ \blacksquare \]

This proposition shows that the irreducible representation \((π_{β, ρ}, L^2(W'))\) \( K_{l-1}(O_r) \) is exactly the irreducible representation \( π_ψ \) with \( ψ = ψ_{β, ρ} \in Y_β \)

defined in Proposition 4.4.1.

4.5 Fix a \( ρ \in g_δ(\mathbb{F})^{-1} \). In this subsection we will study the conjugate action of \( g_ρ = g \ (\text{mod } p^r) \in G(O_r, β) \) on \( K_{l-1}(O_r) \) and on \( π_{β, ρ} \). For any \( X \in g(\mathbb{F}) \), we have

\[ g^{-1}_ρ l((X))g_ρ = l\left(\text{Ad}(ρ)^{-1}X + \varpi^lν(X, g) \ (\text{mod } p^r)\right) \]
Then we have
\[ g_t^{-1}l(X)(1 + \omega^I(S + T))g_r \]
\[ = l(Ad(\overline{\pi}^{-1})^{-1} X) (1 + \omega^I(Ad(g)^{-1} S + Ad(g)^{-1} T + \nu(X,g)) \mod p^r) \]
and an action of \( g_r \in G(O_r, \beta) \) on \( (X; S, T, S_l, T_l) \in G \times g(\overline{F}, \mathbb{M}) \) is defined by
\[ (X; S_l, T_l)^{g_r} = (Ad(\overline{\pi})^{-1} X; (Ad(g)^{-1}) S_l, (Ad(g)^{-1} T + \nu(X,g))T_l) . \tag{10} \]
The action (10) is compatible with the action
\[ (X, s)^{\eta} = (Ad(\overline{\pi})^{-1} X, s) \]
of \( g_r \in G(O_r, \beta) \) on \( (X, s) \in H_\beta \) via the surjection \[3\]. If we put \( X = [v] + Y \in g(\overline{F}) \) with \( v \in \mathcal{V}_\beta \) and \( Y \in g(\overline{F}) \), then we have
\[ Ad(\overline{\pi})^{-1} X = [v \sigma_{\overline{\pi}}] + \gamma(v, \overline{\pi}) + Ad(\overline{\pi})^{-1} Y \]
in the notations of subsection \[3\]. So \( g_r \in G(O_r, \beta) \) acts on \( (v, (Y, s)) \in H_\beta \) by
\[ (v, (Y, s))^{g_r} = (v \sigma_{\overline{\pi}}, (Ad(\overline{\pi})^{-1} Y + \gamma(v, \overline{\pi}), s) . \]
In particular \( g_r \in G_\beta(O_r) \) acts on \( (v, z) \in H_\beta \) by
\[ (v, z)^{g_r} = (v \sigma_{\overline{\pi}}, (\gamma(v, \overline{\pi}), 0) \cdot z) . \]
There exists a group homomorphism \( T : Sp(\mathcal{V}_\beta) \to GL_C(L^2(\mathcal{W}')) \) such that
\[ \pi^\beta(v, s, T) = T(\sigma)^{-1} \circ \pi^\beta(v, T(\sigma) \circ T(\sigma) \]
for all \( \sigma \in Sp(\mathcal{V}_\beta) \) and \( (v, s) \in H_\beta \) (see \[6\] Th.2.4). Then we have
\[ \pi^\beta, (v, z)^{g_r} = \pi^\beta(v \sigma_{\overline{\pi}}, (\gamma(v, \overline{\pi}), 0) \cdot z) \]
\[ = \pi^\beta(v \sigma_{\overline{\pi}}, \left(\gamma(v, \overline{\pi}), \chi_{\rho}(z)\right) \]
\[ = \pi^\beta(v \sigma_{\overline{\pi}}, \left(\left(v, v_{\overline{\pi}}\right)_{\beta}, \chi_{\rho}(z)\right) \circ T(\sigma_{\overline{\pi}}) \]
\[ = T(\sigma_{\overline{\pi}})^{-1} \circ \pi^\beta(\left(v, \left(v, v_{\overline{\pi}}\right)\right) \circ T(\sigma_{\overline{\pi}}) \]
\[ = T(\sigma_{\overline{\pi}})^{-1} \circ \pi^\beta(\left(v, v_{\overline{\pi}}\right), (v, \chi_{\rho}(z))) \circ T(\sigma_{\overline{\pi}}) \]
\[ = T(\sigma_{\overline{\pi}})^{-1} \circ \pi^\beta(\left(v, v_{\overline{\pi}}\right), 1) \circ \pi^\beta(v, z) \circ T(\sigma_{\overline{\pi}}). \]
If we put
\[ U(g_r) = \pi^\beta(v, 1) \circ T(\sigma_{\overline{\pi}}) \in GL_C(L^2(\mathcal{W}')) \]
for \( g_r \in G_\beta(O_r) \) then we have
\[ U(g_r) \circ U(h_r) = c_{\beta, \rho}(g, h) \cdot U((gh)r) \]
for all \(g_r, h_r \in G_\beta(O_r)\), in fact
\[
U(g_r) \circ U(h_r) = \pi^\beta(v_\beta, 1) \circ T(\sigma_\beta) \circ \pi^\beta(v_\beta, 1) \circ T(\sigma_\beta)
\]
\[
= \pi^\beta(v_\beta, 1) \circ \pi^\beta(v_\beta^{-1}, 1) \circ T(\sigma_\beta) \circ T(\sigma_\beta)
\]
\[
= \pi^\beta(\psi_\beta^{-1}, 1) \circ \beta(2^{-1}(v_\beta, v_\beta^{-1})_\beta) \circ T(\sigma_\beta)
\]
\[
= c_{\beta, \rho}(\beta, \beta) \cdot \pi^\beta(v_\beta^{-1}, 1) \circ T(\sigma_\beta).
\]

On the other hand
\[
\bar{\psi}_0 \left((X; S_{l-1}, T_{l-1})^{g_r^{-1}}\right) = \tau \left(\omega^{-1}B(\lambda(X) + \omega T, \text{Ad}(g))\right)
\]
\[
= \tau \left(\omega^{-1}B(\lambda(X) + \omega T, \beta)\right)
\]
for all \(g_r \in G_\beta(O_r)\). That is \(\bar{\psi}_0\) is invariant under the conjugate action of \(G_1(O_r, \beta)^{(1)}\). Hence we have
\[
\bar{\psi}_0(g_r^{-1}h g_r) = \bar{\psi}_0(g_r) \circ \bar{\psi}_0(h) \circ \bar{\psi}_0(g_r)
\]
for all \(g_r \in G_\beta(O_r)\) and \(h \in K_{l-1}(O_r)\).

### 4.6

The following proposition is the key stone of this paper.

**Proposition 4.6.1** If the characteristic polynomial of \(\beta \in g\mathfrak{g}(F) \subset gl_n(F)\) is the minimal polynomial of \(\beta \in M_n(F)\), then the Schur multiplier \([c_{\beta, \rho}] \in H^2(G_\beta(F), \mathbb{C}^\times)\) is trivial for all \(\rho \in g\mathfrak{g}(F)^\times\).

**Proof** We will divide the proof into two parts.

1) The case of \(G = G_n\). In this case, Corollary 5.1 of [17] shows that the Schur multiplier \([c_\nu] \in H^2(G_\beta(O_r), \mathbb{C}^\times)\) is trivial. On the other hand we have the inflation-restriction exact sequence
\[
1 \rightarrow H^1(G_\beta(F), \mathbb{C}^\times) \xrightarrow{\text{inf}} H^1(G_\beta(O_r), \mathbb{C}^\times) \xrightarrow{\text{res}} H^1(K_1(O_r), \mathbb{C}^\times)^{G_\beta(F)}
\]
\[
\rightarrow H^2(G_\beta(F), \mathbb{C}^\times) \xrightarrow{\text{inf}} H^2(G_\beta(O_r), \mathbb{C}^\times)
\]
induced by the exact sequence
\[
1 \rightarrow K_1(O_r) \rightarrow G_\beta(O_r) \rightarrow G_\beta(F) \rightarrow 1.
\]
Since we have
\[
H^1(G_\beta(O_r), \mathbb{C}^\times) = \text{Hom}(G_\beta(O_r), \mathbb{C}^\times),
\]
\[
H^1(K_1(O_r), \mathbb{C}^\times)^{G_\beta(F)} = \text{Hom}(K_1(O_r), \mathbb{C}^\times)
\]
and \(K_1(O_r) \subset G_\beta(O_r)\) are finite commutative groups, the restriction mapping
\[
\text{res} : H^1(G_\beta(O_r), \mathbb{C}^\times) \rightarrow H^1(K_1(O_r), \mathbb{C}^\times)^{G_\beta(F)}
\]
is surjective. Hence the inflation mapping
\[
\text{inf} : H^2(G_\beta(F), \mathbb{C}^\times) \rightarrow H^2(G_\beta(O_r), \mathbb{C}^\times)
\]
is injective. Since the results of the preceding subsections show that the Schur multiplier \([c_{β,ρ}] \in H^2(G_β(O_n),\mathbb{C}_0^\times)\) is the image of \([c_{β,ρ}] \in H^2(G_β(F),\mathbb{C}_0^\times)\) under the inflation mapping, the statement of the proposition is established for the group \(G = GL_n\).

2) The general case of \(G \subset GL_n\). We have \(G_β(F) \subset GL_{n,β}(F)\). Then Proposition [2.2.4] says that the Schur multiplier \([c_{β,ρ}] \in H^2(G_β(F),\mathbb{C}_0^\times)\) is the image of the Schur multiplier \([c_{β,ρ}] \in H^2(GL_{n,β}(F),\mathbb{C}_0^\times)\) under the restriction mapping

\[
\text{res} : H^2(GL_{n,β}(F),\mathbb{C}_0^\times) \to H^2(G_β(F),\mathbb{C}_0^\times).
\]

Since we have shown in the part one of the proof that \([c_{β,ρ}] \in H^2(GL_{n,β}(F),\mathbb{C}_0^\times)\) is trivial, so is \([c_{β,ρ}] \in H^2(G_β(F),\mathbb{C}_0^\times)\). ■

It may be quite interesting if we can find a counter example to the following statement:

Let \(G\) be a connected reductive algebraic group defined over \(F\) and \(g\) the Lie algebra of \(G\). Take a \(β \in g(F)\) which is regular with respect to \(G\) and \(G_β\) is commutative. Then the Schur multiplier \([c_{β,ρ}] \in H^2(G_β(F),\mathbb{C}_0^\times)\) is trivial for all \(ρ \in g(F)^\ast\).

5 Examples

5.1 Let \(K/F\) be a tamely ramified field extension of degree \(n\) and \(O_K \subset K\) the integer ring with the maximal ideal \(p_K = \varpi_K O_K\). The residue class field \(F = O/p\) is identified with a subfield of \(K = O_K/p_K\). A prime element \(\varpi_K\) is chosen so that we have \(\varpi_K^e \in O_{K_{\varpi}}\) where \(K_0\) is the maximal unramified subextension of \(K/F\) and \(e = (K : K_0)\) is the ramification index of \(K/F\). Then we have \(O_K = O_{K_0}[\varpi_K]\).

For a \(β = \sum_{i=0}^{e-1} b_i \varpi_K^i \in O_K\) with \(b_i \in O_{K_0}\), [14, p.545, Lemma 4-7] shows that the following two statements are equivalent:

1) \(O_K = O[β]\),

2) \(b_0^e \neq b_0^e\) for all \(1 \neq σ \in \text{Gal}(K/F)\), and \(b_1 \in O_K^\times\) if \(e > 1\).

By means of the regular representation with respect to an \(O\)-basis of \(O_K\), we will identify \(K\) with a \(F\)-subalgebra of the matrix algebra \(M_n(F)\) where \(O_K = K \cap M_n(O)\). Take a \(β \in O_K\) such that \(O_K = O[β]\). Then [13, p.545, Cor.1] shows that the characteristic polynomial \(χ_β(t) = \det(t \cdot 1_n - β)\) of \(β \in M_n(O)\) has the following properties:

1) \(χ_β(t) (mod \ p) \in F[t]\) is the minimal polynomial of \(β \in M_n(F)\),

2) \(χ_β(t) (mod \ p) = p(t)^e\) with an irreducible polynomial \(p(t) \in F[t]\),

3) \(χ_β(t) (mod \ p^2) \in O_2[t]\) is irreducible.

By the abuse of the notation, the residue class of \(α \in O_K\) modulo \(p_K^n\) is denoted by \(\overline{α} \in O_K/p_K^n\).

[1]This argument is presented by the referee.
5.2 \( G = SL_n \) (\( n \geq 2 \)) is a smooth \( O \)-group scheme. If \( n \) is prime to the characteristic of \( \mathbb{F} \), then \( G \) fulfills three conditions I), II) and III) of subsection 2.1.

Let \( K/F \) be a field extension of degree \( n \) so that it is a tamely ramified extension. Take a \( \beta \in O_K \) such that \( O_K = O[\beta] \) and \( T_K/F(\beta) = 0 \). Under the identification of subsection 5.1, we have \( \beta \in \mathfrak{g}(O) \) such that \( G_\beta \) is commutative smooth \( O \)-group scheme. In this case, we have

\[
G_\beta(O_r) = \left\{ \varpi \in (O_K/p_K^n)^\times \mid \varepsilon \in U_{K/F} \right\}
\]

where \( e \) is the ramification index of \( K/F \) and

\[
U_{K/F} = \{ \varepsilon \in O_K \mid N_{K/F}(\varepsilon) = 1 \}.
\]

We have also

\[
G_\beta(O_r) \cap K_1(O_r) = \left\{ 1 + \varpi^t x \in (O_K/p_K^n)^\times \mid x \in O_K, T_{K/F}(x) \equiv 0 \pmod{p'} \right\}
\]

and \( \psi_\beta \left( 1 + \varpi^t x \right) = \tau \left( \varpi^{-t} T_{K/F}(x) \beta \right) \) for \( x \in O_K \) such that \( T_{K/F}(x) \equiv 0 \pmod{p'} \). Then Theorem 2.3.1 gives

**Proposition 5.2.1** There exists a bijection \( \theta : \text{Ind}_{G(O_\beta)}^{G(O_r)} \sigma_{\beta,\theta} \) of the set

\[
\left\{ \theta : U_{K/F} \rightarrow (O_K/p_K^n)^\times, \tau(\gamma) = \tau(\varpi^{-t} T_{K/F}(\beta x)) \text{ s.t. } \forall \gamma \in U_{K/F} \text{ s.t. } \gamma \equiv 1 + \varpi^t x \pmod{p_K^n}, x \in O_K \right\}
\]

onto \( \text{Irr}(G(O_r) \mid \psi_\beta) \).

5.3 \( G = Sp_{2n} \) be the \( O \)-group scheme such that

\[
Sp_{2n}(L) = \{ g \in GL_{2n}(L) \mid g J_n g = J_n \}
\]

\( (J_n = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}) \) for all \( O \)-algebra \( L \). Then \( G \) is a connected smooth reductive \( O \)-group scheme. The Lie algebra \( \mathfrak{g} = \mathfrak{sp}_{2n} \) of \( G \) is an affine \( O \)-subscheme of \( \mathfrak{gl}_{2n} \) such that

\[
\mathfrak{sp}_{2n}(L) = \{ X \in \mathfrak{gl}_{2n}(L) \mid X J_n + J_n X = 0 \}
\]

for all \( O \)-algebra \( L \). The \( O \)-group scheme \( G \) satisfies the conditions I), II) and III) of the subsection 2.1.

Let \( K_+/F \) be a tamely ramified extension of degree \( n \) and \( K/K_+ \) a quadratic extension. Take a \( \omega \in O_K \) such that

\[
O_K = O_{K_+} \oplus \omega O_{K_+}, \quad \omega^\rho = -\omega
\]

where \( \rho \in \text{Gal}(K/K_+) \) is the non-trivial element. Then

\[
D(x, y) = \frac{1}{2} T_{K/F} \left( \omega^{-1} \omega_{K_+}^{-1} x \omega y \right) \quad (x, y \in K)
\]
with the ramification index $e_+$ of $K_+/F$ is a symplectic form on the $F$-vector space $K$. Fix an $O$-basis $\{u_1, \ldots, u_n\}$ of $O_K$. Since $K_+/F$ is a tamely ramified extension, there exists $u^*_j \in p_{K_+}^{1-e_+}$ ($1 \leq j \leq n$) such that $T_{K_+/F}(u_j u^*_j) = \delta_{ij}$. If we put $v_j = \omega \cdot \varpi_{K_+}^{e_+ - 1} \cdot u^*_j \in O_K$, then we have

$$D(u_i, u_j) = D(v_i, v_j) = 0, \quad D(u_i, v_j) = \delta_{ij} \quad (1 \leq i, j \neq n).$$

Identify the $F$-algebra $K$ with a $F$-subalgebra of $M_{2n}(F)$ by means of the $O$-basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ of $O_K$.

Take a $\beta \in O_K$ such that $O_K = O[\beta]$ and $\beta^p + \beta = 0$. Then $\beta \in \mathfrak{g}(O)$ such that $G_\beta$ is a commutative smooth $O$-group subscheme of $G$. We have

$$G_\beta(O_\tau) = \{ \tau \in (O_K/p_K^{e+})^\times \mid \tau \in U_{K/K_+} \}$$

where $e$ is the ramification index of $K/F$ and

$$U_{K/K_+} = \{ \varepsilon \in O_K^\times \mid N_{K/K_+}(\varepsilon) = 1 \}.$$

We have also

$$G_\beta(O_\tau) \cap K_i(O_\tau) = \left\{ \frac{1 + \varpi^x}{1 + \varpi^y} \in (O_K/p_K^{e+})^\times \mid x \in O_K, T_{K/K_+}(x) \equiv 0 \pmod{p_K^{e+}} \right\}$$

and $\psi_\beta \left( \frac{1 + \varpi^x}{1 + \varpi^y} \right) = \tau \left( \varpi^{-y} T_{K/F}(\beta x) \right)$ for $x \in O_K$ such that $T_{K/K_+}(x) \equiv 0 \pmod{p_K^{e+}}$. Then Theorem 2.3.1 gives

**Proposition 5.3.1** There exists a bijection $\theta \mapsto \text{Ind}^{G(O_\tau)}_{G(O_\beta)} \sigma_{\theta, \theta}$ of the set

$$\left\{ \theta : U_{K/K_+} \to (O_K/p_K^{e+})^\times \to \mathbb{C}^\times : \text{group homomorphism} \right\}$$

s.t. $\theta(\gamma) = \tau(\varpi^{-y} T_{K/F}(\beta x))$

for $\forall \gamma \in U_{K/K_+}, \gamma \equiv 1 + \varpi^y x \pmod{p_K^{e+}}, x \in O_K$

onto $\text{Irr}(G(O_\tau)) \mid \psi_\beta$.

**5.4** Take a $S \in M_n(O)$ such that $^t S = S$ and det $S \in O^\times$. Let $G = SO(S)$ be the $O$-group scheme such that

$$G(L) = \{ g \in SL_n(L) \mid g S g = S \}$$

for all $O$-algebra $L$. Then $G$ is a connected smooth reductive $O$-group scheme. The Lie algebra $\mathfrak{g} = \mathfrak{so}(S)$ of $G$ is an affine $O$-subscheme of $\mathfrak{gl}_n$ such that

$$\mathfrak{g}(L) = \{ X \in \mathfrak{gl}_n(L) \mid X S + S^t X = 0 \}$$

for all $O$-algebra $L$. The $O$-group scheme $G$ satisfies the conditions I), II) and III) of the subsection 2.3.

Take a $\beta \in \mathfrak{g}(O)$ and assume that $n$ is odd or that det $\beta \neq 0 \pmod{p}$. Let $\beta_s \in \mathfrak{g}(L)$ be the semisimple part of $\beta \in \mathfrak{g}(L)$ ($L = F$ or $L = \overline{F}$). Then the centralizer $Z_{G_{O_\beta}}(\beta_s)$ is connected and its center is also connected. Hence if $\beta \in \mathfrak{g}(O)$ is smoothly regular with respect to $G$, then $G_{\beta}$ is a smooth commutative $O$-group scheme.
Assume that \( n = 2m \) is even. Let \( K/F \) be a tamely ramified Galois extension of degree \( 2m \). Fix an intermediate field \( F \subset K_- \subset K \) such that \((K : K_-) = 2\), and assume that \( K/K_- \) is unramified. Take an \( \varepsilon \in O_{K_-}^* \) and put
\[
S_\varepsilon(x, y) = T_{K/F} \left( \varepsilon \cdot \varpi_{K_-}^{1-\varepsilon} \cdot xy^\rho \right) \quad (x, y \in K)
\]
where \( \rho \in \text{Gal}(K/K_-) \) is the non-trivial element and \( \varepsilon \) is the ramification index of \( K/F \). Then \( S_\varepsilon \) is a regular \( F \)-quadratic form on \( K \). Take an \( O \)-basis \( \{u_1, \cdots, u_n\} \) of \( K \) and put \( B = (u_i^{Tj})_{1 \leq i,j \leq n} \) with \( \text{Gal}(K/F) = \{\sigma_1, \cdots, \sigma_n\} \). Then we have
\[
(S_\varepsilon(u_i, u_j))_{1 \leq i,j \leq n} = B \begin{bmatrix} (\varepsilon \varpi_{K_-}^{1-\varepsilon})^{\sigma_1} & \cdots & (\varepsilon \varpi_{K_-}^{1-\varepsilon})^{\sigma_n} \end{bmatrix} \tau B^\rho
\]
so that the discriminant of the quadratic form \( S_\varepsilon \) is
\[
\det (S_\varepsilon(u_i, u_j))_{1 \leq i,j \leq n} = \pm (\det B)^2 \left( \varepsilon \varpi_{K_-}^{1-\varepsilon} \right)^{n}.
\]
Note that \((\det B)^\sigma = \pm \det B\) for any \( \sigma \in \text{Gal}(K/F) \). Since \( K/F \) is tamely ramified, its discriminant is
\[
D(K/F) = (\det B^2) = p^f(e-1)
\]
where \( n = ef \). Hence \( \det (S_\varepsilon(u_i, u_j))_{1 \leq i,j \leq n} \in O^* \). So the \( O \)-group scheme \( G = SO(S_\varepsilon) \) and its Lie algebra \( g = so(S_\varepsilon) \) is defined by
\[
G(L) = \left\{ g \in SL_L(O_K \otimes L) \mid S_\varepsilon(xg, yg) = S_\varepsilon(x, y) \quad \text{for all } x, y \in O_K \otimes L \right\}
\]
and by
\[
g(L) = \left\{ X \in \text{End}_L(O_K \otimes L) \mid S_\varepsilon(xX, y) + S_\varepsilon(x, yX) = 0 \quad \text{for all } x, y \in O_K \otimes L \right\}
\]
for all \( O \)-algebra \( L \). Note that \( \text{End}_F(K) \) acts on \( K \) from the right side.

Take a \( \beta \in O_K^* \) such that \( O_K = O[\beta] \) and \( \beta^o + \beta = 0 \). Identify \( \beta \in K \) with the element \( x \mapsto x\beta \) of \( g(O) \subset \text{End}_O(O_K) \). Then we have
\[
G_\beta(O_r) = \left\{ \tau \in (O_K/O_K^o)^{\times} \mid \varepsilon \in U_{K/K_+} \right\}
\]
where \( e \) is the ramification index of \( K/F \) and
\[
U_{K/K_+} = \{ \varepsilon \in O_K \mid N_{K/K_+}(\varepsilon) = 1 \}.
\]
We have also
\[
G_\beta(O_r) \cap K_i(O_r) = \left\{ \frac{1}{1 + \varpi^i} x \in (O_K/O_K^o)^{\times} \mid x \in O_K, T_{K/K_+}(x) \equiv 0 \pmod{p_{K_+}^{e+1}} \right\}
\]
and \( \psi_\beta \left( \frac{1}{1 + \varpi^i} x \right) = \tau (\varpi_{K/F}^{i\varepsilon} / (\beta x)) \) for \( x \in O_K \) such that \( T_{K/K_+}(x) \equiv 0 \pmod{p_{K_+}^{e+1}} \). Then Theorem 2.3.1 gives
Proposition 5.4.1 \( \text{There exists a bijection} \theta \mapsto \text{Ind}_{G(O_r)}^{G(O_{r,3})} \sigma_{\beta, \theta} \text{ of the set} \)

\[
\left\{ \begin{array}{l}
\theta : U_{K/K_+} \to (O_K/\mathfrak{p}_K^{\sigma})^\times \to C^\times : \text{group homomorphism} \\
\text{s.t.} \ \theta(\gamma) = \tau(\varpi^{-t} T_{K/F}(\beta x)) \\
\text{for } \forall \gamma \in U_{K/K_+}, \gamma \equiv 1 + \varpi^t x \pmod{\mathfrak{p}_K^{\tau}}, \ x \in O_K
\end{array} \right\}
\]

onto \( \text{Irr}(G(O_r) \mid \psi_{\beta}) \).

Let us consider the case of \( n = 2m + 1 \) being odd. Take a \( \eta \in O^\times \) and define a \( F \)-quadratic form \( S_{\epsilon, \eta} \) on the \( F \)-vector space \( K \times F \) by

\[ S_{\epsilon, \eta}((x, s), (y, t)) = S_\epsilon(x, y) + \eta \cdot s. \]

Then the \( O \)-group scheme \( G = SO(S_{\epsilon, \eta}) \) and its Lie algebra \( \mathfrak{g} = \mathfrak{so}(S_{\epsilon, \eta}) \) is defined by

\[ G(L) = \left\{ g \in SL_2((O_K \times O) \otimes L) \mid S_{\epsilon, \eta}(ug, vg) = S_{\epsilon, \eta}(u, v) \text{ for } \forall u, v \in (O_K \times O) \otimes O \right\} \]

and

\[ \mathfrak{g}(L) = \left\{ X \in \text{End}_L((O_K \times O) \otimes L) \mid S_{\epsilon, \eta}(uX, v) + S_{\epsilon, \eta}(u, vX) = 0 \text{ for } \forall u, v \in (O_K \times O) \otimes O \right\} \]

for all \( O \)-algebra \( L \). An element \( X \in \text{End}_L((O_K \times O) \otimes O) \) is denoted by

\[ X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ with } \begin{cases} A \in \text{End}_L(O_K \otimes O), & B \in \text{Hom}_L(O_K \otimes O, L), \\ C \in \text{Hom}_L(L, O_K \otimes O), & D \in \text{End}_L(L) = L. \end{cases} \]

Put

\[ \tilde{\beta} = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{g}(O) \subset \text{End}_O(O_K \times O) \]

which corresponds to the element \( (x, s) \mapsto (x; \tilde{\beta} 0) \) of \( \mathfrak{g}(O) \). The characteristic polynomial of \( \tilde{\beta} \in \text{End}_O(O_K \times O) \) is \( \chi_{\tilde{\beta}}(t) = t \cdot \chi_{\tilde{\beta}}(t) \). Since \( \tilde{\beta} \in O^\times_{K_+} \), the reduction modulo \( p \) of \( \chi_{\tilde{\beta}}(t) \in O[t] \) is the minimal polynomial of \( \tilde{\beta} (\text{mod } p) \in \text{End}_F(K \times F) \). We have

\[ G_{\tilde{\beta}}(O_r) = \left\{ \gamma \pmod{\mathfrak{p}_K^{\tau}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid \gamma \in U_{K/K_+} \right\} \]

and \( \psi_{\beta}(h) = \tau(\varpi^{-t} T_{K/F}(\beta_0 x)) \) for all

\[ h = \begin{bmatrix} 1 + \varpi^t x & (\text{mod } \mathfrak{p}_K^{\tau}) \\ 0 & 1 \end{bmatrix} \in K_I(O_r) \cap G_{\tilde{\beta}}(O_r). \]

Then Theorem 2.3.1 gives

Proposition 5.4.2 \( \text{There exists a bijection} \theta \mapsto \text{Ind}_{G(O_r)}^{G(O_{r,3})} \sigma_{\beta, \theta} \text{ of the set} \)

\[
\left\{ \begin{array}{l}
\theta : U_{K/K_+} \to (O_K/\mathfrak{p}_K^{\tau})^\times \to C^\times : \text{group homomorphism} \\
\text{s.t.} \ \theta(\gamma) = \tau(\varpi^{-t} T_{K/F}(\beta x)) \\
\text{for } \forall \gamma \in U_{K/K_+}, \gamma \equiv 1 + \varpi^t x \pmod{\mathfrak{p}_K^{\tau}}, \ x \in O_K
\end{array} \right\}
\]

onto \( \text{Irr}(G(O_r) \mid \psi_{\beta}) \).

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