One Dimensional Schrödinger Equation With Two Moving Boundaries

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Abstract

In this letter, we consider the Schrödinger equation for a well potential with varying width. We solve one dimensional Schrödinger equation subject to time-dependent boundary conditions for a spinless particle inside infinite potential well, both wall of which move opposite direction with different velocities $v_1$ and $v_2$, respectively.

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I. INTRODUCTION

A well-known feature of the Schrödinger equation is its property of Galilean covariance. In section II, we briefly review Galilean covariance and give the transformation properties [1]. Recently, Makowski et al. [2–4] have reported exact solutions for the schrödinger equation submitted to time dependent boundary conditions. The case they solved describe a particle bouncing between two infinitely walls, one wall of which is fixed and the other is allowed to move according to a function L(t). To our knowledge, most of attempts have been focused on solution of the Schrödinger equation for a particle between two infinitely walls, but one of which is fixed [5,6].

There are two motivations for this study. The first one, we will see the group of transformations admitted by the problem. In quantum mechanics, one of the main task is to solve the Schrödinger equation. The second motivation is physical; the solutions of the Schrödinger equation for one-dimensional system with two moving boundaries as seen in Fig.1 are not cited in literature and this subject is worthy to be studied [7]. We solve the Schrödinger equation for a particle between two infinitely walls, but both of which move in opposite directions. If the velocities of the moving walls are constant, there exits a set of exact solutions for any values of the velocities of the moving walls [8].

II. GALILEAN COVARIANCE IN THE CASE OF A SCALAR POTENTIAL

We assume there exists some inertial frame S, relative to which the Schrödinger equation is valid for a spinless particle in the presence of a scalar potential V. The Schrödinger equation

\[ \frac{i\hbar}{\partial t} \psi(x, t) = \left[ \frac{1}{2m} (-i\hbar \nabla)^2 + V(x, t) \right] \psi(x, t). \]  

(1)

is to be solved under Dirichlet boundary conditions [2]. Relative to a new inertial frame S' moving at velocity \( v_1 \) with respect to S, the value of the wave function at an arbitrary
space-time location is related to that of $\psi$ at the same location by a phase factor to ensure invariance of the probability density at that position:

$$\psi(x, t) = e^{-i\phi} \psi'(x', t').$$  \hspace{1cm} (2)

In general, it is difficult to work out the exact solutions for a moving boundary system due to the moving-boundary conditions

$$\psi(-v_1 t, t) = 0 \quad \text{and} \quad \psi(a + v_2 t, t) = 0$$  \hspace{1cm} (3)

It is well-known that Eq. (1) is covariant under the following transformations:

$$x' = x - v_1 t, \quad t = t', \quad \partial_x = \partial_{x'}, \quad \partial_t = \partial_{t'} - v_1 \partial_{x'}$$  \hspace{1cm} (4)

$$V(x, t) = V'(x', t')$$  \hspace{1cm} (5)

We consider the case the potential $V=0$. As we know, under the Galilean transformations the time derivative of $\psi$ and $\frac{\partial}{\partial x}$ will be changed as follows:

$$\dot{\psi}(x, t) = -i\dot{\phi} e^{-i\phi} \psi'(x', t') + e^{-i\phi} \psi'(x', t')$$  \hspace{1cm} (7)

Where $\dot{\phi} = \frac{\partial \phi}{\partial t}$ and $\dot{\psi} = \frac{\partial \psi}{\partial t}$ and

$$\frac{\partial^2}{\partial x^2} \psi(x, t) = e^{-i\phi} \left[ -i \frac{\partial^2 \phi}{\partial x'^2} + \left( \frac{\partial \phi}{\partial x'} \right)^2 \right] \psi'(x', t') - 2ie^{-i\phi} \frac{\partial \phi}{\partial x'} \frac{\partial \psi'(x', t')}{\partial x'} + e^{-i\phi} \frac{\partial^2 \psi'(x', t')}{\partial x'^2}$$  \hspace{1cm} (8)

We get the Schrödinger equation under the Galilean transformation;

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi'(x', t')}{\partial x'^2} + \left( \frac{i\hbar^2}{m} \frac{\partial \phi}{\partial x'} + i\hbar v_1 \right) \frac{\partial \psi'(x', t')}{\partial x'} + \left[ \frac{\hbar^2}{2m} \left( \frac{\partial \phi}{\partial x'} \right)^2 + \frac{i\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x'^2} - \hbar \frac{\partial \phi}{\partial t'} + \hbar v_1 \frac{\partial \phi}{\partial x'} \right] \psi'(x', t') = i\hbar \frac{\partial \psi'(x', t')}{\partial t'}$$  \hspace{1cm} (9)

The problem posed in this way is unsolvable, so we define a new coordinate as given in section III.
III. FUNDAMENTAL TRANSFORMATION

The main purpose is to change the unsolvable moving boundary problem into a solvable one side fixed-boundary problem as seen in Fig.2 by using the following transformation.

Let us define a new rescaled space coordinate and from now on we use $t'$ in stead of $t$.

$$\bar{x} = \frac{x'}{L(t')}$$

(10)

where $L(t') = \left[1 + \frac{(v_1 + v_2)t'}{a}\right]$. Then the derivative can be written as follows:

$$\frac{\partial}{\partial x'} = \frac{\partial \bar{x}}{\partial x'} \frac{\partial}{\partial \bar{x}} = \frac{1}{L(t')} \frac{\partial}{\partial \bar{x}}$$

(11)

and

$$\frac{\partial^2}{\partial x'^2} = \frac{1}{L^2(t')} \frac{\partial^2}{\partial \bar{x}^2}$$

(12)

We have got time derivative

$$\frac{\partial}{\partial t'} \rightarrow \frac{\partial}{\partial t'} - \bar{x} \frac{\partial L(t')}{\partial \bar{x}}$$

(13)

The new form of Schrödinger equation by using these transformations and substituting $x' = \bar{x}L(t')$ is

$$-\frac{\hbar^2}{2m} \frac{1}{L^2} \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} + \left[\frac{i\hbar^2}{mL^2} \frac{\partial \bar{\phi}}{\partial \bar{x}} + \frac{i\hbar v_1}{L} + \frac{i\hbar \bar{x}L}{L} \frac{\partial \bar{\psi}}{\partial \bar{x}}\right]$$

$$+ \left[\frac{\hbar^2}{2mL^2} \left(\frac{\partial \bar{\phi}}{\partial \bar{x}}\right)^2 + \frac{i\hbar^2}{mL^2} \frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} - \hbar \frac{\partial \bar{\phi}}{\partial t'} + \frac{\bar{x} L}{L} \frac{\partial \bar{\phi}}{\partial \bar{x}} + \frac{\hbar v_1}{L} \frac{\partial \bar{\phi}}{\partial \bar{x}}\right] \bar{\psi} = i\hbar \frac{\partial \bar{\psi}}{\partial t'}$$

(14)

the second term and third term of Eq. (14) equal zero, separately. So from the second term it is easy to find $\bar{\phi}$ as follows;

$$\left[\frac{i\hbar^2}{mL^2} \frac{\partial \bar{\phi}}{\partial \bar{x}} + \frac{i\hbar v_1}{L} + \frac{i\hbar \bar{x}L}{L}\right] = 0$$

(15)

$$\bar{\phi} = -\frac{m}{\hbar} \frac{\bar{x}^2}{2} L + v_1 \bar{x}]L + f(t')$$

(16)

One can easily calculate $\bar{\phi}'' = \frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2}$ and $\bar{\phi}' = \frac{\partial \bar{\phi}}{\partial t'}$ as follows;
\[ \ddot{\phi} = -\frac{m}{\hbar} \ddot{L}L \] (17)

and

\[ \dot{\phi} = -\frac{m}{\hbar} \left( \ddot{\bar{x}}^2 \right) \dot{L} - \frac{m}{\hbar} LL \ddot{\bar{x}}^2 + \dot{f}(t') \] (18)

The second term of Eq. (18) will be zero because of \( \ddot{\bar{L}} = 0 \). The third term of Eq. (14) can be rewritten as follows;

\[ \frac{1}{L^2} \left\{ \frac{i\hbar^2}{2m} \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} + \frac{\hbar^2}{2m} \left( \frac{\partial \bar{\psi}}{\partial \bar{x}} \right)^2 + \hbar (v_1 + \bar{x} \bar{L}) L \frac{\partial \bar{\psi}}{\partial \bar{x}} - \hbar L^2 \frac{\partial \bar{\psi}}{\partial t} \right\} \bar{\psi} = 0 \] (19)

When we substitute Eq. (16,17,18) into Eq. (19), we get the following equation;

\[ \frac{1}{2} m L^2 v_1^2 - \frac{i\hbar}{2} \ddot{\bar{L}} \bar{L} - \hbar L^2 \dot{f}(t') = 0 \] (20)

after manipulation we get,

\[ \dot{f}(t') = \frac{m}{2\hbar} v_1^2 t - \frac{i\dot{\bar{L}}}{2L} \] (21)

and it is easy to calculate \( f(t') \) by using \( L = 1 + \frac{v_1 + v_2}{a} t' \) and

\[ f(t') = \frac{m}{2\hbar} v_1^2 t' - \frac{i\dot{\bar{L}} a}{2} \int \frac{dt'}{a + (v_1 + v_2)t'} \] (22)

The remain terms in Eq. (14) which are different zero are following;

\[ -\frac{\hbar^2}{2m} \frac{1}{L^2} \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} = i\hbar \frac{\partial \bar{\psi}}{\partial t'} \] (23)

Eq. (23) must be equal to a constant in order to be solved,

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} = i\hbar L^2 \frac{\partial \bar{\psi}}{\partial t} = E \bar{\psi} \] (24)

where \( E \) is energy. By using separation of variables we get

\[ \bar{\psi}(\bar{x}, t') = X(\bar{x}) T(t') \] (25)

space part of equation Eq. (24) can be rewritten as follows,
\[-d^2X(\bar{x}) \over \partial \bar{x}^2 = k^2X(\bar{x})\] (26)

where \(k^2 = \frac{2mE}{\hbar^2}\) and the solution

\[X(\bar{x}) = A \sin k\bar{x} + B \cos k\bar{x}\] (27)

and the time-dependent part of Eq. (24) can be rewritten as follows,

\[i\hbar L^2 \dot{T}(t') = ET(t')\] (28)

and solution of time part is:

\[T(t') = \exp \left( -\frac{iE}{\hbar} \int \frac{dt'}{L^2(t')} \right)\] (29)

one can easily found the solution:

\[\bar{\psi}(\bar{x}, t') = \exp \left[ -\frac{iE}{\hbar} \int \frac{dt'}{L^2(t')} \right] [A \sin k\bar{x} + B \cos k\bar{x}]\] (30)

The function (30) is a correct solution of Eq. (14) for both cases if it vanishes at \(\bar{x} = 0\) and \(\bar{x} = a\). For \(\bar{x} = 0\) the function \(\bar{\psi}(\bar{x}, t') = 0\) is obviously. To fulfil the second boundary condition the constant \(k\) have to be chosen in such a way that \(\bar{\psi}(\bar{x}, t') = 0\) when \(\bar{x} = a\).

Under Eq. (4 and 10) transformation, boundary conditions in Eq. (3) transform as follows

\[\bar{\psi}(0, t') = 0 \quad \text{and} \quad \bar{\psi}(a, t') = 0\] (31)

\(k\) can be find by using boundary conditions,

\[at \quad \bar{x} = 0, \quad \bar{\psi}(\bar{x}, t') = \psi' \left( \frac{x'}{L(t')}, t' \right) = 0 \implies B = 0\] (32)

and

\[at \quad \bar{x} = a, \quad \bar{\psi}(\bar{x}, t') = 0 \implies k = \frac{n\pi}{a}\] (33)

finally, the exact solution of our problem is
\[ \psi(x, t) = A \exp \left[ -\frac{iE}{\hbar} \int \frac{dt}{L^2} \right] \sin \left[ \frac{n\pi}{a} \frac{(x - v_1 t)}{1 + \frac{(v_1 + v_2) t}{a}} \right] e^{-i\phi} \]  

where A is a normalization constant and, \( \phi \) and \( f(t) \) are follows, respectively.

\[ \phi = -\frac{m}{\hbar} \left[ \frac{1}{2} \frac{(x - v_1 t)^2}{L} + v_1(x - v_1 t) \right] + f(t) \]  

\[ f(t) = \frac{m}{2\hbar} v_1^2 t - \frac{iLa}{2} \int \frac{dt}{a + (v_1 + v_2) t} \]  

**IV. CONCLUSION**

As a conclusion of this study, we notice that the general problem becomes much simpler by using fundamental transformation. However, the moving-boundary condition renders the equation unsolvable by the usual means. The reader can easily be convinced of the difficulty of this problem by trying a method of separation of variables on Eq. (9) without using the fundamental transformation. It can be easily seen that this fundamental transformation leads to an ordinary differential equation, the solution of which can be achieved by using separation of variables.

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FIGURES

FIG. 1. Potential well with two moving boundaries.

FIG. 2. Potential well with one fixed-wall and one moving wall.