Closed-form Approximations for Coverage and Rate in a Multi-tier Heterogeneous Network in Nakagami-m Fading

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Abstract—In this paper, we consider the downlink in a K-tier heterogeneous network in the presence of Nakagami-m fading and noise. For such a system, we derive closed-form approximations of coverage probability and average rate achievable. A piece-wise linear approximation is employed in obtaining the simplified expressions. The proposed results are verified numerically through simulations. A comparison with existing work shows that the proposed work is a good approximation.

Index Terms—Average rate achievable, coverage probability, Nakagami fading, path-loss, piece-wise linear approximation.

I. INTRODUCTION

The demand for higher data rates in cellular networks has lead to the deployment of small cells along with the macro base stations (BS) resulting in multi-tier heterogeneous networks (HetNets). The irregular BS deployment in such HetNets can be suitably modeled as a Poisson point process (PPP) as shown in [1]. In presence of Rayleigh fading, a stochastic geometry based approach has been used to determine coverage probability and average rate achievable for multi-tier HetNets in [1]. In the absence of simple analytical expressions of coverage and rate, ordering results for various transmission techniques in multi-antenna HetNets with Rayleigh fading for an interference limited scenario have been presented in [2]. In [3], average rate for generalized fading channels were derived using an MGF approach. An expression for the coverage probability was derived in semi-closed form for the dual branch in multi-antenna single tier network in [4]. Coverage and rate were derived using the gil-peleaz inversion formula in [5]. Analytical results in the above literature were expressed in either single or two fold integrals. A closed form expression for outage in the presence of rayleigh fading for a single tier network in terms of a toepztix matrix was obtained in [6]. This problem was extended and solved for MIMO in [7], where the decision variable involved Gamma random variables. However, both the above papers focused on interference limited systems.

Thus, simplified analytical expressions for coverage and rate in the presence of Nakagami-m fading for both noise and interference limited scenarios are required. This is the motivation of this work. The coverage expressions obtained in our work are approximate, but the approach is extremely simple and the results quite accurate. The system model considered in this paper is presented next.

II. SYSTEM MODEL

We consider a K-tier HetNet such that each tier i’s BSs are distributed according to a PPP $\Phi_a$ of density $\lambda_i$. The BSs in a tier $i$ have same transmit power $P_i$ and signal-to-interference-plus-noise ratio (SINR) threshold $\beta_i$. The path loss from a location $x_i$ to the origin is defined as $L(x_i) = ||x_i||^{-\alpha}$. The SINR for a typical user equipment (UE) at the origin from a BS located at $x_i$, in the tier $i$, is

$$\text{SINR}(x_i) = \frac{P_i h_{x_i} ||x_i||^{-\alpha}}{\sum_{j=1}^{K} \sum_{x_j \in \Phi_j \backslash x_i} P_j h_{x_j} ||x_j||^{-\alpha} + \sigma^2} \quad (1)$$

where, $h_{x_i}$ is the fading power between the UE and the BS at location $x_i$, and $\sigma^2$ is the noise power. The fading power from all the BSs is assumed to be independently distributed such that $h_{x_i} \sim \Gamma(M_i, 1)$ has a Gamma distribution, i.e., Nakagami fading. Further, for every tier $i$, $h_{x_i}$’s are independent and identically distributed (i.i.d). A coverage event for the typical user $C(\beta_i)$ is defined for the set $\{\beta_i\}$ as

$$C(\beta_i) \triangleq \bigcup_{x_i \in \Phi_a \cap \Phi_i} S\text{INR}(x_i) > \beta_i \quad (2)$$

Then, from [1], the coverage probability $P_e$ for Nakagami- $m$ fading is expressed as

$$P_e = \mathbb{P}(C(\beta_i)) = \mathbb{P} \left( \bigcup_{x_i \in \Phi_a \cap \Phi_i} \text{SINR}(x_i) > \beta_i \right) \quad (3)$$

Under the assumption $\beta_i > 1$, i.e., at most one BS in the entire network can provide SINR greater than the required threshold and using [1], [5] simplifies to [11]

$$P_e = \sum_{i=1}^{K} \lambda_i \int_{\mathbb{R}^2} \mathbb{P} \left( \frac{P h_{x_i} L(x_i)}{L_{x_i} + \sigma^2} > \beta_i \right) d x_i \quad (4)$$

Given that the user is in coverage, the average rate achievable for Nakagami-m fading is expressed in [11] as

$$R = \mathbb{E} \left[ \log \left( 1 + \max_{x_i \in \Phi_i} \text{SINR}(x_i) \right) \right] C(\beta_i) \right]$$

which simplifies to

$$R = \int_{0}^{\infty} \frac{\mathbb{P}(X > y | C(\beta_i))}{1 + y} dy \quad (5)$$

where,

$$\mathbb{P}(X > y | C(\beta_i)) = \frac{\mathbb{P}(C(max(x_i, \beta_i)))}{\mathbb{P}(C(\beta_i)))} \quad (6)$$

Next, we present the main results of this paper.


\[ I_i = \sum_{k=0}^{m-1} \frac{1}{k!} \sum_{\ell=0}^{k} \left( \binom{\ell}{k} \right) I^{\ell-k-1}(1) \sum_{t=0}^{\ell-k-1} (-1)^t B_{t+1}(D_1, D_2, \ldots, D_{t+1}) \begin{cases} \gamma \left( r + \frac{\alpha}{2} (k-l) + 1 - \gamma \left( r + \frac{\alpha}{2} (k-l) + 1, \frac{A}{(\sigma^2)^{2/\alpha}} x_2 \right) \right) \\ + \gamma \left( r + \frac{\alpha}{2} (k-l) + 1, \frac{A}{(\sigma^2)^{2/\alpha}} x_1 \right) \end{cases} \]  
\[ + \gamma \left( r + \frac{\alpha}{2} (k-l) + 2 - \gamma \left( r + \frac{\alpha}{2} (k-l) + 2, \frac{A}{(\sigma^2)^{2/\alpha}} x_2 \right) \right) \]  
\[ + \gamma \left( r + \frac{\alpha}{2} (k-l) + 2, \frac{A}{(\sigma^2)^{2/\alpha}} x_1 \right) \]  
\[ = m \left( e^{-\gamma x_1} \left( 1 + \frac{U^{2/\alpha}}{V} \right) - e^{-\gamma x_2} \left( 1 + \frac{U^{2/\alpha}}{V} \right) \right) \]  
\[ \text{where, } V = \frac{2\gamma x_0}{\alpha} \Gamma(2/\alpha) \Gamma(1 - 2/\alpha) \sum_{k=1}^{K} \lambda_k \frac{P_m}{\alpha^2} \]  
\[ \text{and } zF_1(.) \text{ is the Gauss Hypergeometric function as in } [10]. \]

**Proof:** Substituting \( M_i = 1 \) in (10) and using \( \gamma(1+z, \alpha) = z! \left( 1 - e^{-\gamma} \sum_{z=0}^{\infty} \frac{\gamma^z}{\alpha z!} \right), \forall z \in \mathbb{Z} \) results in (12).

**Theorem IV.1.** The average rate achievable of a typical UE in coverage of a K-tier HetNet in Nakagami fading, when \( \beta_i > 1 \), is

\[ R = \frac{\sum_{i=1}^{K} \lambda_i P_i^{2/\alpha} P_m^{2-2/\alpha} \mathcal{A}_i I_i}{\sum_{i=1}^{K} \lambda_i P_i^{2/\alpha} P_m^{2-2/\alpha} I_i} \]  

**Proof:** See Appendix C.

**Corollary IV.2.** In the presence of Rayleigh fading, i.e., \( M_i = 1 \), the average rate achievable by a typical UE in coverage is

\[ R = \frac{\sum_{i=1}^{K} \lambda_i P_i^{2/\alpha} P_m^{2-2/\alpha} \mathcal{A}_i}{\sum_{i=1}^{K} \lambda_i P_i^{2/\alpha} P_m^{2-2/\alpha} I_i} \]  

**Proof:** Substituting \( M_i = 1 \) in (10) results in (11) being a constant with respect to (w.r.t.) \( i \) which together with (12) results in (15).

**V. Numerical Results**

We consider the simulation setup as in (11), a two-tier HetNet consisting of macro BSs and small cells. We performed Monte Carlo simulations in MATLAB to obtain the simulation results which are averaged over 10^4 location realizations each with 10^3 channel realizations. We also numerically computed the integrals presented in (11) and the proposed approximate expressions in MATLAB. In Fig. 1 we present the variation of \( P_c \) w.r.t. \( \beta_i \).

The curve generated using (10) matches closely with the existing result in (12) and the simulation results for various values of the Nakagami parameters, \( M_1 \) and \( M_2 \). The variation of \( P_c \) w.r.t. \( \sigma^2 \) in Rayleigh fading is presented in Fig. 2. A good match is observed even for high values of \( \sigma^2 \), i.e., the noise limited regime.
In Fig. 3 the variation of $R$ w.r.t. $\beta_1$ when UE is in coverage ($K = 2, \alpha = 3, P_1 = 25P_2, \lambda_2 = 5\lambda_1, \beta_2 = 1dB$)

VI. Conclusion

We have proposed closed-form approximations for coverage probability and average rate achievable in a $K$-tier HetNet in the presence of noise and Nakagami fading. Further, through simulation results we have shown that the proposed simplified expressions match closely with existing results.

References

[1] H. Dhillon, R. Ganti, F. Baccelli, and J. G. Andrews, “Modeling and analysis of k-tier downlink heterogeneous cellular networks,” IEEE J. Sel. Areas Commun., vol. 30, no. 3, pp. 550–560, April 2012.

[2] H. S. Dhillon, M. Kountouris, and J. G. Andrews, “Downlink mimo het-nets: Modeling, ordering results and performance analysis,” IEEE Trans. Wireless Commun., vol. 12, no. 10, pp. 5208–5222, October 2013.

[3] M. D. Renzo, A. Guidotti, and G. Corazza, “Average rate of downlink heterogeneous cellular networks over generalized fading channels: A stochastic geometry approach,” IEEE Trans. Commun., vol. 61, no. 7, pp. 3050–3071, July 2013.

[4] R. Tanbourgi, H. S. Dhillon, J. G. Andrews, and F. K. Jondral, “Dual-branch mrc receivers under spatial interference correlation and nakagami fading,” IEEE Trans. Commun., vol. 62, no. 6, pp. 1830–1844, June 2014.

[5] M. D. Renzo and P. Guan, “Stochastic geometry modeling of coverage and rate of cellular networks using the gil-pelaez inversion theorem,” IEEE Commun. Lett., vol. 18, no. 9, pp. 1575–1578, Sept. 2014.

[6] C. Li, J. Zhang, and K. Letaief, “Success probability and area spectral efficiency in multiuser mimo hetnets,” IEEE Trans. Wireless Commun., vol. 13, no. 5, pp. 2505 – 2517, May 2014.

[7] C. Li, J. Zhang, J. G. Andrews, and K. Letaief, “Success probability and area spectral efficiency in multiuser mimo hetnets,” IEEE Trans. Wireless Commun., vol. 64, no. 4, pp. 1544 – 1556, April 2016.

[8] S. Gradshyten and I. Ryzhik, Table of Integrals, Series, and Products, 7th ed. Academic Press, 2007.

[9] M. E. Hazewinkel, Encyclopaedia of Mathematics. Kluwer Academic Publishers, 1994.

Appendix A

Proof of Theorem III.1

The expression in (3) can be simplified to

$$\int_{0}^{\infty} e^{-U^{\alpha / 2}} e^{-V t} dt = \frac{1}{U} \int_{0}^{\infty} e^{\frac{-U^{\alpha / 2}}{U}} e^{-V U^{\alpha / 2} \frac{t}{U}} dy. \quad (16)$$

It is difficult to obtain an exact closed form solution for (16) for arbitrary $\alpha$. From Fig. 4 it can be seen that $f(x) = e^{-x^{\alpha / 2}}$ is monotonically decreasing w.r.t. $x$, $f(x) \in (0, 1) \forall x \geq 0$, and has a single point of inflection. Hence, for $f(x) = e^{-x^{\alpha / 2}}$ a piece-wise linear approximation (PLA) is given by

$$e^{-x^{\alpha / 2}} \approx \begin{cases} 
1 & x \leq x_1 , \\
mx + c & x_1 < x < x_2 , \\
0 & x_2 \leq x , 
\end{cases} \quad (17)$$
Given $h_\xi \sim \Gamma(M_1, 1)$, the conditional probability in (4) is
\[ P\left( \frac{P_i h_\xi L(x_i)}{I_{\xi_i} + \sigma^2} > \beta_i \middle| I_{\xi_i} = I \right) = \int_{I_{\xi_i} + \sigma^2}^{\infty} y^{M_1 - 1} e^{-y} \frac{1}{\Gamma(M_1)} \, dy, \]  
which using [8] 2.321, simplifies to
\[ = \frac{1}{\Gamma(M_1)} \beta_i^{M_1 - k} k! \frac{\beta_i (I + \sigma^2)}{P_i L(x_i)} e^{-\frac{(I + \sigma^2) \beta_i}{P_i L(x_i)}} \]  
\[ = e^{-\frac{\beta_i}{P_i L(x_i)}} \sum_{k=0}^{M_1 - 1} \frac{\beta_i^k}{k!} \sum_{l=0}^{k} \frac{k!}{l!} (\sigma^2)^{l-1} I e^{-\frac{\beta_i}{P_i L(x_i)}}, \]  
where, (a) is obtained through the binomial expansion. Averaging (22) over interference results in
\[ P\left( \frac{P_i h_\xi L(x_i)}{I_{\xi_i} + \sigma^2} > \beta_i \right) = e^{-\frac{\beta_i}{P_i L(x_i)}} \sum_{k=0}^{M_1 - 1} \frac{\beta_i^k}{k!} \sum_{l=0}^{k} \frac{k!}{l!} \frac{I}{(\sigma^2)^{l-1}} \frac{d^l}{dI^l} \frac{\mathbb{E}_I [e^{-\beta_i/I}]}{\beta_i} \]  
We simplify $\mathbb{E}_I [I e^{-\beta_i/I}]$ as follows
\[ \mathbb{E}_I [I e^{-\beta_i/I}] = \int_{-\infty}^{\infty} y e^{-\frac{\beta_i y}{P_i L(x_i)}} dy = \mathcal{L}\left\{ y f_I(y) \right\}(s) \]  
\[ = (-1)^{d} \frac{d^l}{ds^l} \frac{\mathbb{E}_I [e^{-\beta_i/I}]}{\beta_i} \]  
where $\mathcal{L}[\cdot]$ is the Laplace transform. Substituting (24) in (23), and $s = \beta_i/(P_i L(x_i))$ results in
\[ e^{-\frac{\beta_i}{P_i L(x_i)}} \sum_{k=0}^{M_1 - 1} \frac{\beta_i^k}{k!} \sum_{l=0}^{k} \frac{k!}{l!} \frac{I}{(\sigma^2)^{l-1}} \frac{d^l}{dI^l} \frac{\mathbb{E}_I [e^{-\beta_i/I}]}{\beta_i} \]  
An expression of $\mathbb{E}_I [e^{-\beta_i/I}]$ has been expressed in [2] as follows
\[ \exp\left\{ -\left( s \frac{2}{\alpha} \sum_{m=1}^{M_1} \frac{M_m}{p} \frac{2}{\alpha} \sum_{p=1}^{M_1} \frac{M_m}{p} (M_m - 2p + 2 \alpha) \right) \right\} \]  
where, $\mathcal{B}(\cdot, \cdot)$ is the Beta function as given in [9] 8.380. The Faa Di Bruno formula [9] can be used to obtain,
\[ \frac{d^l}{ds^l} \frac{\mathbb{E}_I [e^{-\beta_i/I}]}{\beta_i} = \sum_{r=0}^{l} f^{(r)}(g) B_{l,r}(g', g'', \ldots, g^{r+1}) \]  
where
\[ f = e^g, g = -As^{2/\alpha}, \]  
and $A$, $D_r$, $B_{l,r}(g', g'', \ldots, g^{r+1})$ are as expressed in [9]. Bell...
polynomial in \(27\) can be further simplified as,
\[
B_{i,j}(g', \ldots, g^{i-r+1}) = \sum_{j_1, j_2, \ldots, j_{i-r+1}} \frac{i!}{j_1! j_2! \cdots j_{i-r+1}!} \prod_{l=1}^{i-r+1} (-AD, s^{j_l-1} t^l)^h \\
= \sum_{j_1, j_2, \ldots, j_{i-r+1}} s^{j_1+j_2+\cdots+j_{i-r+1}} \times (-A)^{j_1+j_2+\cdots+j_{i-r+1}} \prod_{l=1}^{i-r+1} (D, t^l)^h \\
= (-A)^{j} s^{j-i} B_{i,j}(D_1, D_2, \ldots, D_{i-r+1}). \tag{30}
\]
Substituting \(27, 29,\) and \(30\) in \(25\) results in
\[
\mathbb{P} \left( \frac{P, k, L(x)}{1 + \sigma^2} > \beta \right) = \sum_{k=0}^{M-1} \sum_{l=0}^{k} (\frac{\beta}{P})^l (\sigma^2)^{k-l} (-1)^l \times \int_{\mathbb{R}} e^{-\frac{\beta}{\sigma} x^2 - \frac{A}{\sigma} \|x\|^2} dx. \tag{31}
\]
Using \(31,\) the probability of coverage in \(4\) is expressed as
\[
P_c = \sum_{i=1}^{K} \lambda_i \int_{\mathbb{R}^2} \sum_{k=0}^{M-1} \sum_{l=0}^{k} (\frac{\beta}{P})^l (\sigma^2)^{k-l} (-1)^l \times \int_{\mathbb{R}^2} e^{-\frac{\beta}{\sigma} x^2 - \frac{A}{\sigma} \|x\|^2} dx. \tag{32}
\]
Converting \(32\) to polar form along with transformation of variable results in
\[
P_c = \sum_{i=1}^{K} \pi \lambda_i P^2/\sigma \beta_i^{-2/\alpha} \int_{\mathbb{R}^2} e^{-\frac{\beta}{\sigma} x^2 - \frac{A}{\sigma} \|x\|^2} dx. \tag{33}
\]
Substituting the result obtained in \(8\) in \(33\) results in \(10.\) This completes the proof of Theorem IV.1.

**Appendix C**

**Proof of Theorem IV.1**

Substituting \(P_c\) from \(10\) in \(6\) we have,
\[
\mathbb{P}(X > y|C(\beta_i)) \sim \sum_{i=1}^{K} \pi \lambda_i P^2/\sigma \beta_i^{-2/\alpha} I_i. \tag{34}
\]
Using \(34\) in \(5\) gives,
\[
R = \int_{0}^{\infty} \frac{1}{1 + y} \sum_{i=1}^{K} \pi \lambda_i P^2/\sigma \beta_i^{-2/\alpha} I_i \left(\rho^2/\sigma \max(y, \beta_i)^{-2/\alpha}\right) dy. \tag{35}
\]
where,
\[
\mathcal{A}_i = \int_{0}^{\infty} \frac{\max(\beta_i, y)^{-2/\alpha}}{\beta_i^{-2/\alpha}(1 + y)} dy
\]