Functoriality of Rieffel’s Generalised Fixed-Point Algebras for Proper Actions

Astrid an Huef, Iain Raeburn, and Dana P. Williams

Abstract. We consider two categories of \( C^* \)-algebras; in the first, the isomorphisms are ordinary isomorphisms, and in the second, the isomorphisms are Morita equivalences. We show how these two categories, and categories of dynamical systems based on them, crop up in a variety of \( C^* \)-algebraic contexts. We show that Rieffel’s construction of a fixed-point algebra for a proper action can be made into functors defined on these categories, and that his Morita equivalence then gives a natural isomorphism between these functors and crossed-product functors. These results have interesting applications to non-abelian duality for crossed products.

Introduction

Let \( \alpha \) be an action of a locally compact group \( G \) on a \( C^* \)-algebra \( A \). In [39], Rieffel studied a class of proper actions for which there is a Morita equivalence between the reduced crossed product \( A \rtimes_{\alpha, r} G \) and a generalised fixed-point algebra \( A^\alpha \) sitting inside the multiplier algebra \( M(A) \). Rieffel subsequently proved that \( \alpha \) is proper whenever there is a free and proper \( G \)-space \( T \) and an equivariant embedding \( \varphi : C_0(T) \to M(A) \) [40, Theorem 5.7]. In [15], inspired by previous work of Kaliszewski and Quigg [13], it was observed that Rieffel’s hypothesis says precisely that \( ((A, \alpha), \varphi) \) is an object in a comma category of dynamical systems. It therefore becomes possible to ask questions about the functoriality of Rieffel’s construction, and about the naturality of his Morita equivalence.

These questions have been tackled in several recent papers [15, 7, 8], which we believe contain some very interesting results. In particular, they have substantial applications to non-abelian duality for \( C^* \)-algebraic dynamical systems. However, these papers also contain a confusing array of categories and functors. So our goal here is to discuss the main categories and explain why people are interested in them. We will then review some of the main results of the papers [13, 15, 7, 8], and try to explain why we find them interesting.

In all the categories of interest to us, the objects are either \( C^* \)-algebras or dynamical systems involving actions or coactions of a fixed group on \( C^* \)-algebras. But when we decide what morphisms to use, we have to make a choice, and what we choose depends on what sort of theorems we are interested in. Loosely speaking,
we have to decide whether we want the isomorphisms in our category to be the usual isomorphisms of $C^*$-algebras, or to be Morita equivalences. We think that, once we have made that decision, there is a “correct” way to go forward.

We begin in §1 with a discussion of commutative $C^*$-algebras; since Morita equivalence does not preserve commutativity, it is clear that in this case we want isomorphisms to be the usual isomorphisms. However, even then we have to do something a little odd: we want the morphisms from $A$ to $B$ to be homomorphisms $\varphi : A \to M(B)$. Once we have the right category, we can see that operator algebraists have been implicitly working in this category for years. The motivating example for Kaliszewski and Quigg was a duality theory for dynamical systems due to Landstad [19], and our main motivation is, as we said above, to understand Rieffel’s proper actions. We discuss Landstad duality in §2. In §3, we discuss its analogue for crossed products by coactions, which is due to Quigg [30], and how this makes contact with Rieffel’s theory of proper actions.

We begin §4 by showing how the search for naturality results leads us to a different category $C^*$ of $C^*$-algebras in which the morphisms are based on right-Hilbert bimodules. Categories of this kind have been round much longer, and [2, 3], for example, contain a detailed discussion of how imprimitivity theorems provide natural isomorphisms between functors with values in $C^*$. In §5, we discuss a theorem from [7] which says that Rieffel’s Morita equivalences give a natural isomorphism between a crossed-product functor and a fixed-point-algebra functor. This powerful result implies, for example, that the version in [9] of Mansfield imprimitivity for arbitrary subgroups is natural. We finish with a brief survey of one of the main results of [8] which uses an approach based on Rieffel’s theory to establish induction-in-stages for crossed products by coactions.

1. The category $C^*$ and commutative $C^*$-algebras

In our first course in $C^*$-algebras, we learned that commutative unital $C^*$-algebras are basically the same things as compact topological spaces. To make this formal, we note that the assignment $X \mapsto C(X)$ is the object map in a contravariant functor $C$ from the category $Cpct$ of compact Hausdorff spaces and continuous functions to the category $CommC^*$ of unital commutative $C^*$-algebras and unital homomorphisms (which for us are always $*$-preserving); the morphism $C(f)$ associated to a continuous map $f : X \to Y$ sends $a \in C(Y)$ to $a \circ f \in C(X)$. Then the Gelfand-Naimark theorem implies that the functor $C$ is an equivalence of categories. (This result goes back to [25], and we will go into the details of what it means in Theorem 2 below.)

The Gelfand-Naimark theorem for non-unital algebras says that commutative $C^*$-algebras are basically the same things as locally compact topological spaces. However, it is not so easy to put this version in a categorical context, and in doing so we run into some important issues which are very relevant to problems involving crossed products and non-abelian duality. So we will discuss these issues now as motivation for our later choices.

There is no doubt what the analogue of the functor $C$ does to objects: it takes a locally compact Hausdorff space $X$ to the $C^*$-algebra $C_0(X)$ of continuous functions $a : X \to \mathbb{C}$ which vanish at infinity. However, there is a problem with morphisms: composing with a continuous function $f : X \to Y$ does not necessarily map $C_0(Y)$ into $C_0(X)$. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined
by \( f(x) = (1 + x^2)^{-1} \); any function \( a \in C_0(\mathbb{R}) \) which is identically 1 on \([0,1]\) satisfies \( a \circ f = 1 \), and hence \( a \circ f \) does not vanish at infinity. One way out is to restrict attention to the category in which the morphisms from \( X \) to \( Y \) are the proper functions \( f : X \to Y \) for which inverse images of compact sets are compact, and then on the \( C^* \)-algebra side one has to restrict attention to the homomorphisms \( \varphi : A \to B \) such that the products \( \varphi(a)b \) span a dense subspace of \( B \). In [28], Pedersen does exactly this, and calls these proper homomorphisms. It turns out, though, that there is a very satisfactory way to handle arbitrary continuous functions between locally compact spaces, in which we allow morphisms which take values in \( C_b(X) \).

A homomorphism \( \varphi \) of one \( C^* \)-algebra \( A \) into the multiplier algebra \( M(B) \) of another \( C^* \)-algebra \( B \) is called nondegenerate if \( \varphi(A)B \coloneqq \overline{\varphi(a)b} \) is all of \( B \). (This notation is suggestive: the Cohen factorisation theorem says that everything in the closed span factors as \( \varphi(a)b \).) We want to think of the nondegenerate homomorphisms \( \varphi : A \to M(B) \) as morphisms from \( A \) to \( B \). Every nondegenerate homomorphism \( \varphi \) extends to a unital homomorphism \( \bar{\varphi} : M(A) \to M(B) \) (see [36, Corollary 2.51], for example); the extension has to satisfy \( \bar{\varphi}(m)(\varphi(a)b) = \varphi(\bar{\varphi}(m)a)b \), and hence the non-degeneracy implies that there is exactly one such extension, and that it is strictly continuous.

The following fundamental proposition is implicit in [13, §1].

**Proposition 1.** There is a category \( C*_{nd} \) in which the objects are \( C^* \)-algebras, the morphisms from \( A \) to \( B \) are the nondegenerate homomorphisms from \( A \) to \( M(B) \), and the composition of \( \varphi : A \to M(B) \) and \( \psi : B \to M(C) \) is \( \psi \circ \varphi := \bar{\psi} \circ \varphi \). The isomorphisms in this category are the usual isomorphisms of \( C^* \)-algebras.

**Proof.** It is easy to check that the composition \( \bar{\psi} \circ \varphi : A \to M(C) \) is non-degenerate, and hence defines a morphism in \( C*_{nd} \). Since \( \bar{\psi} \circ \varphi \) is a homomorphism from \( M(A) \) to \( M(C) \) which extends \( \bar{\psi} \circ \varphi \), it must be the unique extension \( \bar{\psi} \circ \varphi \). Thus if \( \theta : C \to M(D) \) is another nondegenerate homomorphism, we have

\[
\theta \circ (\psi \circ \varphi) = \bar{\theta} \circ (\psi \circ \varphi) = \bar{\theta} \circ (\bar{\psi} \circ \varphi) = (\bar{\theta} \circ \bar{\psi}) \circ \varphi = (\theta \circ \psi) \circ \varphi = (\theta \circ \psi) \circ \varphi = \varphi \circ \psi \circ \varphi,
\]

and composition in \( C*_{nd} \) is associative. The identity maps \( \text{id}_A : A \to A \), viewed as homomorphisms into \( M(A) \), satisfy \( \text{id}_A = \text{id}_{M(A)} \), and hence have the properties one requires of the identity morphisms in \( C*_{nd} \). Thus \( C*_{nd} \) is a category, as claimed.

For the last comment, notice first that every isomorphism is trivially nondegenerate, and hence defines a morphism in \( C*_{nd} \), which is an isomorphism because it has an inverse. Conversely, suppose that \( \varphi : A \to M(B) \) and \( \psi : B \to M(A) \) are inverses of each other in \( C*_{nd} \), so that \( \psi \circ \varphi = \text{id}_A \) and \( \varphi \circ \psi = \text{id}_B \). Using first the non-degeneracy of \( \psi \) and then the non-degeneracy of \( \varphi \), we obtain

\[
\varphi(A) = \varphi(\psi(B))A = \bar{\varphi}(\psi(B))\varphi(A) = B\varphi(A) = B.
\]

Thus \( \varphi \) has range \( B \), and since \( \bar{\psi}|_B = \psi \), we have \( \psi \circ \varphi = \text{id}_A \). The same arguments show that \( \varphi \circ \psi = \text{id}_B \), so \( \varphi \) is an isomorphism in the usual sense. \( \square \)

If \( f : X \to Y \) is a continuous map between locally compact spaces and \( a \in C_0(Y) \), then \( a \circ f \) is a continuous bounded function which defines a multiplier of \( C_0(X) \). For every \( b \) in the dense subalgebra \( C_c(X) \), we can choose \( a \in C_c(Y) \) such that \( a = 1 \) on \( f(\text{supp} \, b) \), and then \( b = (a \circ f)b \), so \( C_0(f) : a \mapsto a \circ f \) is a
nondegenerate homomorphism from $C_0(Y)$ to $M(C_0(X))$: the extension $\overline{C_0(f)}$ to $C_0(X) = M(C_0(X))$ is again given by composition with $f$. We now have a functor $C_0$ from the category $\text{LCpct}$ of locally compact spaces and continuous maps to the full subcategory $\text{Comm}^*_{\text{nd}}$ of $\text{Comm}^*_{\text{nd}}$ whose objects are commutative $C^*$-algebras. This functor has the properties we expect:

**Theorem 2.** The functor $C_0 : \text{LCpct} \to \text{Comm}^*_{\text{nd}}$ is an equivalence of categories.

**Proof.** To say that $C_0$ is an equivalence means that there is a functor $G : \text{Comm}^*_{\text{nd}} \to \text{LCpct}$ such that $C_0 \circ G$ and $G \circ C_0$ are naturally isomorphic to the identity functors. To verify that it is an equivalence, though, it suffices to show that every object in $\text{Comm}^*_{\text{nd}}$ is isomorphic to one of the form $C_0(X)$, which is exactly what the Gelfand-Naimark theorem says, and that $C_0$ is a bijection on each set $\text{Mor}(X,Y)$ of morphisms (see [21, page 91]). Injectivity is easy: since $C_0(Y)$ separates points of $Y$, $a \circ f = a \circ g$ for all $a \in C_0(Y)$ implies that $f(x) = g(x)$ for all $x \in X$. For surjectivity, we suppose that $\varphi : C_0(Y) \to C_0(X)$ is a nondegenerate homomorphism. Then for each $x \in X$, the composition $\epsilon_x \circ \varphi$ with the evaluation map is a homomorphism from $C_0(Y)$ to $C$, and the non-degeneracy of $\varphi$ implies that $\epsilon_x \circ \varphi$ is non-zero. Since $y \mapsto \epsilon_y$ is a homeomorphism of $Y$ onto the maximal ideal space of $C_0(Y)$, there is a unique $f(x) \in Y$ such that $\epsilon_x \circ \varphi = \epsilon_{f(x)}$, and $f = \epsilon^{-1} \circ \varphi \circ \epsilon$ is continuous. The equation $\epsilon_x \circ \varphi = \epsilon_{f(x)}$ then says precisely that $\varphi = C_0(f)$. \[\square\]

The result in [21, page 91] which we have just used is a little unnerving to analysts. (Well, to us, anyway.) Its proof, for example, makes carefree use of the axiom of choice. So it is perhaps reassuring that in the situation of Theorem 2, there is a relatively concrete inverse functor $\Delta$ which takes a commutative $C^*$-algebra $A$ to its maximal ideal space $\Delta(A)$. (We say “relatively concrete” here because the axiom of choice is also used in the proof that the Gelfand transform is an isomorphism.) The argument on page 92 of [21] shows that, once we have chosen isomorphisms $\eta_A : A \to C(\Delta(A))$ for every commutative $C^*$-algebra $A$, there is exactly one way to extend $\Delta$ to a functor in such a way that $\eta := \{\eta_A : A \in \text{Obj}(\text{Comm}^*_{\text{nd}})\}$ is a natural isomorphism. If we choose $\eta_A : A \to C_0(\Delta(A))$ to be the Gelfand transform, then the functor $\Delta$ takes a morphism $\varphi : A \to M(B)$ to the map $\Delta(\varphi) : \omega \mapsto \omega \circ \varphi$. So we have the following naturality result:

**Corollary 3.** The Gelfand transforms $\{\eta_A : A \in \text{Obj}(\text{Comm}^*_{\text{nd}})\}$ form a natural isomorphism between the identity functor on $\text{Comm}^*_{\text{nd}}$ and the composition $C_0 \circ \Delta$.

Of course, modulo the existence of the isomorphisms $\eta_A$, which is the content of the (highly non-trivial) Gelfand-Naimark theorem, this result can be easily proved directly: we just need to check that for every morphism $\varphi : A \to M(B)$ the following diagram commutes in $\text{Comm}^*_{\text{nd}}$:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & C_0(\Delta(A)) \\
\varphi \downarrow & & \downarrow C_0(\Delta(\varphi)) \\
B & \xrightarrow{\eta_B} & C_0(\Delta(B)).
\end{array}
$$
2. Crossed products and Landstad duality

Although the category \( C_G^* \) has only been studied in recent years, possibly for the first time in [13], nondegenerate homomorphisms have been around for years. For example, the unitary representations \( U \) of a locally compact group \( G \) on a Hilbert space \( H \) are in one-to-one correspondence with the nondegenerate representations \( \pi_U \) of the group algebras \( L^1(G) \) or \( C^*(G) \) on \( H \). In this context, “nondegenerate” usually means that the elements \( \pi_U(a)h \) span a dense subspace of \( H \), but this is equivalent to the nondegeneracy of \( \pi_U \) as a homomorphism into \( B(H) = M(K(H)) \).

More generally, if \( u : G \to UM(B) \) is a strictly continuous homomorphism into the unitary group of a multiplier algebra, then there is a unique nondegenerate homomorphism \( \pi_u : C^*(G) \to M(B) \), called the integrated form of \( u \), from which we can recover \( u \) by composing with a canonical unitary representation \( k_G : G \to UM(C^*(G)) \). The composition here is taken in the spirit of the category \( C_G^* \): it is the composition in the usual sense of the extension of \( \pi_u \) to \( M(C^*(G)) \) with \( k_G \).

We say that \( k_G \) is universal for unitary representations of \( G \).

One application of this circle of ideas which will be particularly relevant here is the existence of the comultiplication \( \delta_G \) on \( C^*(G) \), which is the integrated form of the unitary representation \( k_G \otimes k_G : G \to UM(C^*(G) \otimes C^*(G)) \). Thus \( \delta_G \) is by definition a nondegenerate homomorphism of \( C^*(G) \) into \( M(C^*(G) \otimes C^*(G)) \). Its other crucial property is coassociativity: \( (\delta_G \otimes \text{id}) \circ \delta_G = (\text{id} \otimes \delta_G) \circ \delta_G \), where the compositions are interpreted as being those in the category \( C_G^* \).

Now suppose that \( \alpha : G \to \text{Aut} A \) is an action of a locally compact group \( G \) on a \( C^* \)-algebra. Nondegeneracy is then built into the notion of covariant representation of the system: a covariant representation \((\pi, u)\) of a dynamical system \((A, G, \alpha)\) in a multiplier algebra \( M(B) \) consists of a nondegenerate homomorphism \( \pi : A \to M(B) \) and a strictly continuous homomorphism \( u : G \to UM(B) \) such that \( \pi(\alpha_t(a)) = u_t \pi(a) u_t^* \). The crossed product is then, either by definition [33] or by theorem [42, 2.34–36], a \( C^* \)-algebra \( A \rtimes G \) which is generated (in a sense made precise in those references) by a universal covariant representation \((i_A, i_G)\) of \((A, G, \alpha)\) in \( M(A \rtimes G) \). Each covariant representation \((\pi, u)\) in \( M(B) \) has an integrated form \( \pi \times u \) which is a nondegenerate homomorphism of \( A \rtimes G \) into \( M(B) \) such that \( \pi = (\pi \times u) \circ i_A \) and \( u = (\pi \times u) \circ i_G \).

The crossed product \( A \rtimes \alpha G \) carries a dual coaction \( \hat{\alpha} \), which is the integrated form of \( i_G \otimes k_G : G \to UM((A \rtimes \alpha G) \otimes C^*(G)) \). This is another nondegenerate homomorphism, and the crucial coaction identity \((\hat{\alpha} \otimes \text{id}) \circ \hat{\alpha} = (\text{id} \otimes \delta_G) \circ \hat{\alpha} \) again has to be interpreted in the category \( C_G^* \). (Makes you wonder how we ever managed without \( C_G^* \).)

There is another version of the crossed-product construction which can be more suitable for spatial arguments, and which is particularly important for the issues we discuss in this paper. For any representation \( \pi : A \to B(H_\pi) \), there is a regular representation \((\hat{\pi}, U)\) of \((A, G, \alpha)\) on \( L^2(G, H_\pi) \) such that \( (\hat{\pi}(a)h) (r) = \pi(\alpha_r^{-1}(a)) (h(r)) \) and \( \lambda_r h(r) = h(s^{-1}r) \) for \( h \in L^2(G, H_\pi) \). The reduced crossed product \( A \rtimes_{\alpha, r} G \) is the quotient of \( A \rtimes \alpha G \) which has the property that every \( \hat{\pi} \times \lambda \) factors through a representation of \( A \rtimes_{\alpha, r} G \), and then \( \hat{\pi} \times \lambda \) is faithful whenever \( \pi \) is [42, §7.2]. The reduced crossed product is also generated by a canonical covariant representation \((i_A^*, i_G^*)\), and the dual coaction \( \hat{\alpha} \) factors through a coaction
\[ \hat{\alpha}^*: \hat{A} \times_{\alpha,r} G \to M((\hat{A} \times_{\alpha,r} G) \otimes C^*(G)) \] characterised by
\[ (1) \quad \overline{\hat{\alpha}^*} \circ \hat{i}_A^*(a) = \hat{i}_A^*(a) \otimes 1 \quad \text{and} \quad \overline{\hat{\alpha}^*} \circ \hat{i}_G^*(s) = \hat{i}_G^*(s) \otimes k_G(s). \]
This coaction is called the normalisation of \( \hat{\alpha} \), and is in particular normal in the sense that the canonical map \( j_{A \times G} \) of \( A \times_{\alpha,r} G \) into \( M((\hat{A} \times_{\alpha,r} G) \times_{\hat{\alpha}} G) \) is injective (see Proposition A.61 of \([3]\)).

Kaliszewski and Quigg’s motivation for working in the category \( C^*_\text{nd} \) came from the following characterisation of the \( C^* \)-algebras which arise as reduced crossed products.

**Theorem 4 (Landstad, Kaliszewski-Quigg).** Suppose that \( B \) is a \( C^* \)-algebra and \( G \) is a locally compact group. Then there is a dynamical system \( (A, \alpha, G) \) such that \( B \) is isomorphic to \( A \times_{\alpha,r} G \) if and only if there is a morphism \( \pi : C^*(G) \to UM(B) \) in \( C^*_\text{nd} \) and a nondegenerate (see Remark 6 below) normal coaction \( \delta : B \to M(B \otimes C^*_\alpha(G)) \) such that
\[ (\pi \otimes \text{id}) \circ \delta_G = \delta \circ \pi. \]

In \([13]\) the authors say that this result follows from a theorem of Landstad \([19]\), and it is certainly true that most of the hard work is done by Landstad’s result. But we think it is worth looking at the proof; those who are not interested in the subtleties of coactions should probably skip to the end of the proof below. We begin by stating Landstad’s theorem in modern terminology.

**Theorem 5 (Landstad, 1979).** Suppose that \( B \) is a \( C^* \)-algebra and \( G \) is a locally compact group. Then there is a dynamical system \( (A, G, \alpha) \) such that \( B \) is isomorphic to \( A \times_{\alpha,r} G \) if and only if there are a strictly continuous homomorphism \( u : G \to UM(B) \) and a reduced coaction \( \delta : B \to M(B \otimes C^*_\alpha(G)) \) such that
\begin{enumerate}[\( (a) \)]  
  \item \( \delta(u_s) = u_s \otimes \lambda_s \) for \( s \in G \), and
  \item \( \delta(A)(1 \otimes C^*_\alpha(G)) = A \otimes C^*_\alpha(G). \)
\end{enumerate}

The “reduced coaction” appearing in Landstad’s theorem is required to have slightly different properties from the full coactions which we use elsewhere in this paper, and which are used in \([3]\) and \([13]\), for example. A reduced coaction on \( B \) is an injective nondegenerate homomorphism of \( B \) into \( M(B \otimes C^*_\alpha(G)) \) rather than \( M(B \otimes C^*(G)) \), and it is required to be coassociative with respect to the comultiplication \( \delta_G^* \) on \( C^*_\alpha(G) \).

**Remark 6.** Nowadays, the second condition \( (b) \) in Theorem 5 is usually absorbed into the assertion that \( \delta \) is a coaction. Everyone agrees that for \( \delta \) to be a coaction \( \delta(A)(1 \otimes C^*_\alpha(G)) \) must be contained in \( A \otimes C^*_\alpha(G) \), and Landstad described the requirement of equality as “nondegeneracy”, which in view of our emphasis on \( C^*_\text{nd} \) has turned out to be unfortunate terminology. Coactions of amenable or discrete groups are automatically nondegenerate in Landstad’s sense, and dual coactions are always nondegenerate. We therefore follow modern usage and assume that all coactions satisfy \( (b) \), or its analogue in the case of full coactions. (So \( (b) \) can now be deleted from Theorem 5 and the word “nondegenerate” from Theorem 4.)

**Proof of Theorem 4.** For \( B = A \times_{\alpha,r} G \), we take \( \delta = \hat{\alpha}^* \) and \( \pi = \pi_G^* \). The second equation in (1) implies that
\[ (\pi \otimes \text{id}) \circ \delta_G(k_G(s)) = (\pi \otimes \text{id})(k_G(s) \otimes k_G(s)) = \hat{i}_G^*(s) \otimes k_G(s) \]
\[ = \overline{\hat{\alpha}^*} \circ \pi(k_G(s)) \]
Now we compute:

\[ \delta^r(u_s) = (\text{id} \otimes \pi_{\lambda}) \circ \delta(\pi(k_G(s))) = \text{id} \otimes \pi_{\lambda} \circ \bar{\delta}(\bar{k}_G(s)) \]

\[ = \text{id} \otimes \pi_{\lambda} \circ \pi \otimes \text{id} \circ \bar{\delta}(\bar{k}_G(s)) = \bar{\pi} \otimes \pi(k_G(s) \otimes k_G(s)) \]

\[ = \bar{\pi} \circ k_G(s) \otimes \lambda_s = u_s \otimes \lambda_s. \]

Thus \( u \) and \( \delta^r \) satisfy the hypotheses of Landstad’s theorem (Theorem 5), and we can deduce from it that \( B \) is isomorphic to a reduced crossed product.

Kaliszewski and Quigg then made two further crucial observations. First, they recognised that there is a category of coactions associated to \( \mathbb{C}^* \): the objects in \( \mathbb{C}^*_{\text{coact}_{\text{nd}}}(G) \) consist of a full coaction \( \delta \) on a \( \mathbb{C}^* \)-algebra \( B \), and the morphisms from \( (B, \delta) \) to \( (C, \epsilon) \) are nondegenerate homomorphisms \( \varphi: B \to M(C) \) such that \( (\varphi \otimes \text{id}) \circ \delta = \epsilon \circ \varphi \). Then (2) says that the homomorphism \( \pi \) in Corollary 4 is a morphism in \( \mathbb{C}^*_{\text{coact}_{\text{nd}}}(G) \) from \( (\mathbb{C}^*(G), \delta_G) \) to \( (B, \delta) \). Second, they knew that for every object \( a \) and every subcategory \( D \) in a category \( C \) there is a comma category \( a 
\downarrow \ D \) in which objects are morphisms \( f: a \to x \) in \( C \) from \( a \) to objects in \( D \), and the morphisms from \( (f, x) \) to \( (y, g) \) are morphisms \( h: x \to y \) in \( D \) such that \( h \circ f = g \). Thus Landstad’s theorem identifies the reduced crossed products as the \( \mathbb{C}^* \)-algebras which can be augmented with a coaction \( \delta \) and a homomorphism \( \pi \) to form an object in the comma category \( (\mathbb{C}^*(G), \delta_G) \downarrow \mathbb{C}^*_{\text{coact}_{\text{nd}}}(G) \).

The main results in [13] concern crossed-product functors defined on the category \( \mathbb{C}^*_{\text{coact}_{\text{nd}}}(G) \) whose objects are dynamical systems \( (A, G, \alpha) \) and whose morphisms \( \varphi: (A, \alpha) \to (B, \beta) \) are nondegenerate homomorphisms \( \varphi: A \to M(B) \) such that \( \varphi \circ \alpha_s = \beta_s \circ \varphi \) for \( s \in G \) (where yet again the compositions are taken in \( \mathbb{C}^*_{\text{nd}} \)). The following theorem is Theorem 4.1 of [13].

**Theorem 7** (Kaliszewski-Quigg, 2009). There is a functor \( \mathbb{C}P^r \) from \( \mathbb{C}^*_{\text{coact}_{\text{nd}}}(G) \) to the comma category \( (\mathbb{C}^*(G), \delta_G) \downarrow \mathbb{C}^*_{\text{coact}_{\text{nd}}}(G) \) which takes the object \( (A, \alpha) \) to \( (A \rtimes_{\alpha, r} G, \bar{\alpha}, \bar{\delta}_G) \), and this functor is an equivalence of categories.

Landstad’s theorem, in the form of Theorem 4, says that \( \mathbb{C}P^r \) is essentially surjective: every object in the comma category is isomorphic to one of the form \( \mathbb{C}P^r(A, \alpha) = A \rtimes_{\alpha, r} G \). Thus Theorem 7 can be viewed as an extension of Landstad’s theorem, and Kaliszewski and Quigg call it “categorical Landstad duality for actions”. They also obtain an analogous result for full crossed products.

### 3. Proper actions and Landstad duality for coactions

Quigg’s version of Landstad duality for crossed products by coactions [30] is also easy to formulate in categories based on \( \mathbb{C}^*_{\text{nd}} \). Suppose that \( \delta \) is a coaction of \( G \) on \( C \), and let \( w_G \) denote the function \( s \mapsto k_G(s) \), viewed as a multiplier of \( C_0(G, \mathbb{C}^*(G)) \). A covariant representation of \( (C, \delta) \) in a multiplier algebra \( M(B) \) consists of nondegenerate homomorphisms \( \pi: C \to M(B) \) and \( \mu: C_0(G) \to M(B) \) such that

\[ (\pi \otimes \text{id}) \circ \delta(c) = \mu \otimes \text{id}(w_G)(\pi(c) \otimes 1)\mu \otimes \text{id}(w_G)^* \quad \text{for} \quad c \in C, \]

where, as should seem usual by now, the composition is interpreted in \( \mathbb{C}^*_{\text{nd}} \). The crossed product \( C \times_{\delta} G \) is generated by a universal covariant representation \((j_C, j_G)\)
in $M(C \rtimes_\delta G)$, in the sense that products $j_C(c)j_G(f)$ span a dense subspace of $C \rtimes_\delta G$. The crossed product carries a dual action $\delta$ such that $\delta_s(j_C(c)j_G(f)) = j_C(c)j_G(rt_s(f))$, where $rt$ is defined by $rt_s(f)(t) = f(ts)$. Quigg’s theorem identifies the $C^*$-algebras which are isomorphic to crossed products by coactions.

**Theorem 8** (Quigg, 1992). Suppose that $G$ is a locally compact group and $A$ is a $C^*$-algebra. There is a system $(C, \delta)$ such that $A$ is isomorphic to $C \rtimes_\delta G$ if and only if there are a nondegenerate homomorphism $\varphi : C_0(G) \to A$ and an action $\alpha$ of $G$ on $A$ such that $(A, \alpha, \varphi)$ is an object in the comma category $(C_0(G), rt) \downarrow C^*\text{act}_{\alpha}(G)$.

When $A = C \rtimes_\delta G$, we can take $\varphi := j_G$ and $\alpha := \hat{\delta}$, and the hard bit is to prove the converse. This is done in [30, Theorem 3.3]. It is then natural to look for a “categorical Landstad duality for coactions” which parallels the results of [13]. However, triples $(A, \alpha, \varphi)$ of the sort appearing in Theorem 8 had earlier (that is, before [13]) appeared in important work of Rieffel on proper actions, and it has proved very worthwhile to follow up this circle of ideas in Rieffel’s context. To explain this, we need to digress a little.

If $\alpha : G \to \text{Aut}A$ is an action of a compact abelian group, then information about the crossed product can be recovered from the fixed point algebra $A^\alpha$, and, more generally, from the spectral subspaces

$$A^\alpha(\omega) := \{ a \in A : a_s(a) = \overline{\omega(a)a} \} \quad \text{for } \omega \in \hat{G}.$$  

A fundamental result of Kishimoto and Takai [16, Theorem 2] says that if the spectral subspaces are large in the sense that $A^\alpha(\omega)^*A^\alpha(\omega)$ is dense in $A^\alpha$ for every $\omega \in \hat{G}$, then $A \rtimes_\alpha G$ is Morita equivalent to $A^\alpha$. There is as yet no completely satisfactory notion of a free action of a group on a $C^*$-algebra (see [29], for example), but having large spectral subspaces is one example of such a notion.

When $G$ is locally compact, the fixed-point algebra is often trivial. For example, if $rt$ is the action of $G = \mathbb{Z}$ on $\mathbb{R}$ by right translation, then $f \in C_0(\mathbb{R})^w$ if and only if $f$ is periodic with period 1, which since $f$ vanishes at $\infty$ forces $f$ to be identically zero. However, if the orbit space for an action is nice enough, then the algebra of continuous functions on the orbit space can be used as a substitute for the fixed-point algebra. A right action of a locally compact group $G$ on a locally compact space $T$ is called proper if the map $(x, s) \mapsto (x, x \cdot s) : T \times G \to T \times T$ is proper. The orbit space $T/G$ for a proper action is always Hausdorff [42, Corollary 3.43], and a classical result of Green [5] says that if the action of $G$ on $T$ is free and proper, then $C_0(T) \rtimes_{rt} G$ is Morita equivalent to $C_0(T/G)$ (for this formulation of Green’s result see [42, Remark 4.12]). We want to think of $C_0(T/G)$ as a subalgebra of the multiplier algebra $M(C_0(G)) = C_b(T)$ which is invariant under the extension $rt$.

In the past twenty-five years, many researchers have investigated analogues of free and proper actions for noncommutative $C^*$-algebras [35, 39, 4, 24, 10, 40, 11]. Here we are interested in the notion of proper action $\alpha : G \to \text{Aut}A$ introduced by Rieffel [39]. He assumes that there is an $\alpha$-invariant subalgebra $A_0$ of $A$ with properties like those of the subalgebra $C_c(T)$ of $C_0(T)$, and that there is an $M(A)^\alpha$-valued inner product on $A_0$. The completion $Z(A, \alpha)$ of $A_0$ in this inner product is a full Hilbert module over a subalgebra $A^\alpha$ of $M(A)^\alpha$, which Rieffel calls the generalized fixed point algebra for $\alpha$. The algebra $K(Z(A, \alpha))$ of generalized compact operators on $Z(A, G, \alpha)$ sits naturally as an ideal $E(\alpha)$ in the reduced crossed product $A \rtimes_{\alpha, r} G$ [39, Theorem 1.5]. The action $\alpha$ is saturated when
$E(\alpha)$ is all of the reduced crossed product. Thus when $\alpha$ is proper and saturated, $A \rtimes_{\alpha, r} G$ is Morita equivalent to $A^\alpha$. Saturation is a freeness condition: if $G$ acts properly on $T$, then $rt : G \to \text{Aut}(C_0(T))$ is proper with respect to $C_0(G)$, and the action is saturated if and only if $G$ acts freely [23, §3]. On the face of it, though, Rieffel’s bimodule $Z(A, \alpha)$ and the fixed-point algebra $A^\alpha$ depend on the choice of subalgebra $A_0$, and it seems unlikely that Rieffel’s process is functorial.

The connection with our categories lies in a more recent theorem of Rieffel which identifies a large family of proper actions for which there is a canonical choice of the dense subalgebra $A_0$ [40, Theorem 5.7].

**THEOREM 9** (Rieffel, 2004). Suppose that a locally compact group $G$ acts freely and properly on the right of a locally compact space $T$, and $(A, G, \alpha)$ is a dynamical system such that there is a nondegenerate homomorphism $\varphi : C_0(T) \to M(A)$ satisfying $\varphi \circ rt = \alpha \circ \varphi$ (with composition in the sense of $C^*_\text{nd}$). Then $\alpha$ is proper and saturated with respect to the subalgebra $A_0 = \varphi(C_c(T))A\varphi(C_c(T))$.

**EXAMPLE 10.** A closed subgroup $H$ of a locally compact group $G$ acts freely and properly on $G$, and hence we can apply Theorem 9 to the pair $(T, G) = (G, H)$ and to the canonical map $j_G : C_0(G) \to M(C \rtimes_G G)$. In this case, highly nontrivial results of Mansfield [22] can be used to identify the fixed-point algebra $(C \rtimes_G G)^\delta$ with the crossed product $C \rtimes_{\delta,r} (G/H)$ by the homogeneous space [9, Remark 3.4]. (These crossed products were introduced in [1]; the relationship with the crossed product $C \rtimes_G (G/H)$ by the restricted coaction, which makes sense when $H$ is normal, is discussed in [1, Remark 2.2].) Then Theorem 3.1 of [9] shows that Rieffel’s Morita equivalence between $(C \rtimes_{\delta} G) \rtimes_{\delta,r} H$ and $(C \rtimes_G G)^\delta$ extends Mansfield’s imprimitivity theorem for coactions to arbitrary closed subgroups (as opposed to the amenable normal subgroups in Mansfield’s original theorem [22, Theorem 27] and the normal ones in [12]).

From our categorical point of view, the hypotheses on $\varphi$ in Theorem 9 say precisely that $(A, \alpha, \varphi) := ((A, \alpha), \varphi)$ is an object in the comma category $(C_0(T), rt) \downarrow \mathcal{C}^*\text{act}_{\text{nd}}(G)$. Then Rieffel’s theorem implies that $(A, \alpha, \varphi) \mapsto A^\alpha$ is a construction which takes objects in the comma category to objects in the category $\mathcal{C}^*\text{nd}$. One naturally asks: is this construction functorial? More precisely, is there an analogous construction on morphisms which which makes $(A, \alpha, \varphi) \mapsto A^\alpha$ into a functor from $(C_0(T), rt) \downarrow \mathcal{C}^*\text{act}_{\text{nd}}(G)$ to $\mathcal{C}^*\text{nd}$?

This question was answered in [15, §2] using a new construction of Rieffel’s generalized fixed-point algebra. The crucial ingredient is an averaging process $E$ of Olesen and Pedersen [26, 27], which was subsequently developed by Quigg in [31, 32] and used extensively in his proof of Theorem 8. This averaging process $E$ makes sense on the dense subalgebra $A_0 = \varphi(C_c(T))A\varphi(C_c(T))$, and satisfies

$$\varphi(f)E(\varphi(g)\alpha_s(\varphi(h))) = \int_G \varphi(f)\alpha_s(\varphi(g)\alpha_r(\varphi(h))) ds \quad \text{for } f, g, h \in C_c(T);$$

the integral on the right has an unambiguous meaning because properness implies that $s \mapsto \int rt_s(g)$ has compact support. It is shown in [15, Proposition 2.4] that the closure of $E(A_0)$ is a $C^*$-subalgebra of $M(A)$, which we denote by $\text{Fix}(A, \alpha, \varphi)$ to emphasise all the data involved in the construction. It is shown in [15, Proposition 3.1] that $\text{Fix}(A, \alpha, \varphi)$ and Rieffel’s $A^\alpha$ are exactly the same subalgebra of $M(A)$. If $\sigma : (A, \alpha, \varphi) \to (B, \beta, \psi)$ is a morphism in the comma category, so that
in particular $\sigma$ is a nondegenerate homomorphism from $A$ to $M(B)$, then the
extension $\tilde{\sigma}$ maps $\text{Fix}(A, \alpha, \varphi)$ into $M(\text{Fix}(B, \beta, \psi))$, and is nondegenerate. (This is
Proposition 2.6 of [15]; a gap in the proof of nondegeneracy is filled in Corollary 2.3
of [8].)

**Theorem 11** (Kaliszewski-Quigg-Raeburn, 2008). Suppose that a locally compact
group $G$ acts properly on the right of a locally compact space $T$. Then the
assignments $(A, \alpha, \varphi) \mapsto \text{Fix}(A, \alpha, \varphi)$ and $\sigma \mapsto \tilde{\sigma}|_{\text{Fix}(A, \alpha, \varphi)}$ form a functor from
$(C_0(T), \text{rt}) \downarrow \text{C*act}_{\text{nd}}(G)$ to $\text{C*nd}^\ast$.

To return to the setting of Quigg-Landstad duality, we take $(T, G) = (G, G)$
in this theorem. This gives us a functor $\text{Fix}$ from $(C_0(G), \text{rt}) \downarrow \text{C*act}_{\text{nd}}(G)$ to $\text{C*nd}^\ast$.
Because the fixed-point algebra $\text{Fix}(A, \alpha, \varphi)$ is defined using the same averaging
process $E$ as Quigg used in [30, §3], $\text{Fix}(A, \alpha, \varphi)$ is the same as the algebra $C$
constructed by Quigg (unfortunately for us, he called it $B$). So Quigg proves in
[30] that
$$\delta_A(e) = \varphi \otimes \pi_\lambda(w_G)(e \otimes 1)\varphi \otimes \pi_\lambda(w_G)^*$$
defines a reduced coaction of $G$ on $C = \text{Fix}(A, \alpha, \varphi)$, and that $A$ is isomorphic to
the crossed product $C \rtimes_A G$. An examination of the proof of [31, Theorem 4.7]
shows that the similar formula
$$\delta_A^\ast(e) = \varphi \otimes \text{id}(w_G)(e \otimes 1)\varphi \otimes \text{id}(w_G)^*$$
defines the unique full coaction with reduction $\delta_A$. The argument on page 2960
of [15] shows that this construction respects morphisms, so that $\text{Fix}$ extends to a
functor $\text{Fix}_G$ from $(C_0(G), \text{rt}) \downarrow \text{C*act}_{\text{nd}}(G)$ to $\text{C*coact}_{\text{nd}}^\ast(G)$. The following very
satisfactory “categorical Landstad duality for coactions” is Corollary 4.3 of [15].

**Theorem 12** (Kaliszewski-Quigg-Raeburn, 2008). Let $G$ be a locally compact
group. Then $(C, \delta) \mapsto (C \rtimes_G G, \delta, j_G)$ and $\pi \mapsto \pi \rtimes \text{id}$ form a functor from
$\text{C*coact}_{\text{nd}}^\ast(G)$ to $(C_0(G), \text{rt}) \downarrow \text{C*act}_{\text{nd}}^\ast(G)$. This functor is an equivalence of cate-
gories with quasi-inverse $\text{Fix}_G$.

In fact, this is a much more satisfying theorem than its analogue for actions
because we have a specific construction of a quasi-inverse. We would be interested
to see an analogous process for Fixing over coactions.

4. Naturality

Now that we have a functorial version $\text{Fix}$ of Rieffel’s generalised fixed-point
algebra, we remember that the main point of Rieffel’s paper [39] was to construct
a Morita equivalence between $A^\alpha = \text{Fix}(A, \alpha, \varphi)$ and the reduced crossed product
$A \rtimes_{\alpha, r} G = \text{RCP}(A, \alpha, \varphi)$. This equivalence is implemented by an $(A \rtimes_{\alpha, r} G) -
\text{Fix}(A, \alpha, \varphi)$ imprimitivity bimodule $Z(A, \alpha, \varphi)$. There is another category $\text{C*}$ of
$C^*$-algebras in which the isomorphisms are given by imprimitivity bimodules, so
it makes sense to ask whether these isomorphisms are natural. Of course, before
discussing this problem, we need to be clear about what the category $\text{C*}$ is.

If $A$ and $B$ are $C^*$-algebras, then a right-Hilbert $A - B$ bimodule is a right
Hilbert $B$-module $X$ which is also a left $A$-module via a nondegenerate homomor-
phism of $A$ into the algebra $\mathcal{L}(X)$ of bounded adjointable operators on $X$. (These
are sometimes called $A - B$ correspondences.) The objects in $\text{C*}$ are $C^*$-algebras,
and the morphisms from $A$ to $B$ are the isomorphism classes $[X]$ of full right-
Hilbert $A - B$ bimodules. Every nondegenerate homomorphism $\varphi : A \rightarrow M(B)$
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gives a right-Hilbert bimodule: view $B$ as a right Hilbert $B$-module over itself with \langle b_1, b_2 \rangle_B := b_1^* b_2$, and define the action of $A$ by $a \cdot b := \varphi(a) b$. We denote the isomorphism class of this bimodule by $[\varphi]$. In [2], it is shown that $[\varphi] = [\psi]$ if and only if there exists $u \in UM(B)$ such that $\psi = (\text{Ad} u) \circ \varphi$, so we are not just adding more morphisms to $C^*_\text{nd}$, we are also slightly changing the morphisms we already have.

If $A \times_B X$ and $B \times_C Y$ are right Hilbert bimodules, then we define the composition using the internal tensor product: $[Y] \times [X] := [X \otimes_B Y]$. The identity morphism $1_A$ on $A$ is $[A A] = [\text{id}_A]$. Now we can see why we have had to take isomorphism classes of bimodules as our morphisms: the bimodule $A \otimes_X X$ representing $1_A [X] = [X][\text{id}_A]$ is only isomorphic to $X$. A similar subtlety arises when checking that composition of morphisms is associative. The details are in [2, Proposition 2.4].

In [2, Proposition 2.6], it is shown that the isomorphisms from $A$ to $B$ in $C^*$ are the classes $[X]$ in which $X$ is an imprimitivity bimodule, so that $X$ also carries a left inner product $\langle x, y \rangle$ such that $\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle$. Similar results were obtained independently by Landsman [17, 18] and by Schweizer [41], and a slightly more general category in which the bimodules are not required to be full as right Hilbert modules was considered in [3].

Theorem 3.2 of [15] says that, for every nondegenerate homomorphism $\sigma : A \to M(B)$, the diagram

\[
\begin{array}{ccc}
A \rtimes_{\alpha, r} G & \xrightarrow{[Z(A, \alpha, \varphi)]} & \text{Fix}(A, \alpha, \varphi) \\
| & | \\
B \rtimes_{\beta, r} G & \xrightarrow{[Z(B, \beta, \psi)]} & \text{Fix}(B, \beta, \psi)
\end{array}
\]

commutes in $C^*$, which means that

$$Z(A, \alpha, \varphi) \otimes_{\text{Fix}(A, \alpha, \varphi)} \text{Fix}(B, \beta, \psi) \text{ and } (B \rtimes_{\beta, r} G) \otimes_{B \rtimes_{\beta, r} G} Z(B, \beta, \psi)$$

are isomorphic as right-Hilbert $(A \rtimes_{\alpha, r} G) - \text{Fix}(B, \beta, \psi)$ bimodules. Thus Rieffel’s bimodules (or rather, the morphisms in $C^*$ which they determine) implement a natural isomorphism between the functors $RCP$ and $\text{Fix}$ from $(C_0(T), \text{rt}) \downarrow C^*_\text{act,nd}(G)$ to $C^*$.

This naturality theorem certainly has interesting applications to nonabelian duality, where it gives naturality for the extension in [9] of Mansfield’s imprimitivity theorem to closed subgroups (see [15, Theorem 6.2]). However, it is slightly unsatisfactory in that the functors involved go from a category built from $C^*_\text{nd}$ to $C^*$: we were forced to go into $C^*$ because the bimodules $Z$ do not define morphisms in $C^*_\text{nd}$, but in the diagram (3) we have not fully committed to the move. Our goal in [7] was to find versions of the same functors defined on a category built from $C^*$ — that is, ones in which the morphisms are implemented by bimodules — to establish that Rieffel’s Morita equivalence gives a natural isomorphism between these functors, and to apply the results to nonabelian duality. We will describe our progress in the next section.
5. Upgrading to \( \mathbb{C}^* \)

Proposition 3.3 of [2] says that for every locally compact group \( G \), there is a category \( \mathbb{C}^* \text{act}(G) \) whose objects are dynamical systems \((A, \alpha) = (A, G, \alpha)\) and whose morphisms are obtained by adding actions to the morphisms of \( \mathbb{C}^* \). Formally, if \((A, \alpha)\) and \((B, \beta)\) are objects in \( \mathbb{C}^* \text{act}(G) \) and \( A X_B \) is a right-Hilbert bimodule, then an action of \( G \) on a right-Hilbert bimodule \( X \) is a strongly continuous homomorphism of \( G \) into the linear isomorphisms of \( X \) such that
\[
    u_s(a \cdot x \cdot b) = \alpha_s(a) \cdot u_s(x) \cdot \beta_s(b) \quad \text{and} \quad \langle u_s(x), u_s(y) \rangle_b = \beta_s(\langle x, y \rangle_a),
\]
and the morphisms in \( \mathbb{C}^*(G) \) are isomorphism classes of pairs \((X, u)\).

Next we consider a free and proper action of \( G \) on a locally compact space \( T \) and look for an analogue of the comma category for the system \((A, \alpha) \to (B, \beta)\) in \( \mathbb{C}^* \text{act}(G) \). In [7, Remark 2.4] we have discussed our reasons for adding the maps \( \varphi \) to our objects and then ignoring them in our morphisms, and the discussion below of how we define morphisms should help convince sceptics that this is appropriate.

We know how to \( \text{Fix} \) objects in the semi-comma category \( \mathbb{C}^*(G, (C_0(T), rt)) \), and we need to describe how to define \( \text{Fix} \) a morphism \([(X, u)] \) from \((A, \alpha, \varphi) \) to \((B, \beta, \psi)\). We begin by factoring the morphism \([X]\) in \( \mathbb{C}^* \) as the composition \( [\kappa(X) X_B][\kappa_A] \) of the isomorphism associated to the imprimitivity bimodule \( \kappa(X) X_B \) with the morphism coming from the nondegenerate homomorphism \( \kappa_A : A \to M(K(X)) = \mathcal{L}(X) \) describing the left action of \( A \) on \( X \). (see Proposition 2.27 of [3]). The action \( u \) of \( G \) on \( X \) gives an action \( \mu \) of \( G \) on \( \kappa(X) \) such that \( \mu_s(\Theta_{x,y}) = \Theta_{u_s(x), u_s(y)} \). and then \( \kappa_A \) satisfies \( \kappa_A \circ \alpha_s = \mu_s \circ \kappa_A \). The morphism \([[(X, u)](X, u)(B, \beta)]\) in \( \mathbb{C}^*(G) \) factors as \([[(\kappa(X), \mu)(X, u)(B, \beta)]\][\kappa_A] \). Now \( \kappa_A \) is a morphism in \( \mathbb{C}^* \text{act}(G) \) from \((A, \alpha, \varphi) \) to \((\kappa(X), \mu, \kappa_A \circ \varphi)\), and hence by Theorem 11 restricts to a morphism \( \kappa_A \) from \( \text{Fix}(A, \alpha, \varphi) \) to \( \text{Fix}(\kappa(X), \mu, \kappa_A \circ \varphi) \). We want to define \( \text{Fix} \) so that it is a functor, so our definition must satisfy
\[
(4) \quad \text{Fix}([(X, u)]) = \text{Fix}([(\kappa(X), \mu)(X, u)(B, \beta)]) \text{Fix}([\kappa_A]).
\]
Since we don’t want to change the meaning of \( \text{Fix} \) on morphisms in \( \mathbb{C}^* \text{act} \), our strategy is to define \( \text{Fix}([\kappa_A]) := [\kappa_A] \), figure out how to define \( \text{Fix} \) imprimitivity bimodules, and then use (4) to define \( \text{Fix}([(X, u)]) \).

So we suppose that \((A, \alpha, \varphi) \) to \((B, \beta, \psi)\) are objects in the semi-comma category \( \mathbb{C}^*(G, (C_0(T), rt)) \), and that \([(X, u)]\) is an equivariant \((A, \alpha) \to (B, \beta)\) imprimitivity bimodule. We emphasise that, because of our choice of morphisms in \( \mathbb{C}^*(G, (C_0(T), rt)) \), we do not make any assumption relating the actions of \( \varphi \) and \( \psi \) on \( X \). We let \( \tilde{X} := \{ \varphi(x) : x \in X \} \) be the dual bimodule, and form the linking algebra
\[
L(X) := \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix},
\]
as in the discussion following [36, Theorem 3.19]. Then
\[
L(u) := \begin{pmatrix} \alpha & u \\ \beta(u) & \beta \end{pmatrix} \quad \text{and} \quad \varphi_L := \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}.
define an action \( L(u) \) of \( G \) on \( L(X) \) and a nondegenerate homomorphism \( \varphi_L \) of \( C_0(T) \) into \( M(L(X)) \) which intertwines \( rt \) and \( L(u) \). Then \( (L(X), L(u), \varphi_L) \) is an object in \( \mathcal{C}^*(G, (C_0(T), rt)) \), and (reverting to Rieffel’s notation to simplify the formulas) we can form \( L(X)^{L(u)} := \text{Fix}(L(X), L(u), \varphi_L) \). It follows quite easily from the construction of \( \text{Fix} \) in [15, §2] that the diagonal corners in \( L(X)^{L(u)} \) are \( A^\alpha \) and \( B^\beta \), and we define \( X^u \) to be the upper right-hand corner, so that

\[
L(X)^{L(u)} = \begin{pmatrix} A^\alpha & X^u \\ * & B^\beta \end{pmatrix}
\]

with the actions and inner products coming from the operations in \( L(X)^{L(u)} \), \( X^u \) becomes an \( A^\alpha - B^\beta \)-imprimitivity bimodule (see [36, Proposition 3.1]). We now define \( \text{Fix}([X, u]) := X^u \), and use (4) to define \( \text{Fix} \) in general, as described above.

With this definition, Theorem 3.3 of [7] says:

**Theorem 13.** Suppose that \( T = \{t \in G \} \) is a free and proper right \( G \)-space. Then the assignments

\[
(A, \alpha, \varphi) \mapsto \text{Fix}(A, \alpha, \varphi) \quad \text{and} \quad ([X, u]) \mapsto \text{Fix}([X, u])
\]

form a functor \( \text{Fix} \) from the semi-comma category \( \mathcal{C}^*(G, (C_0(T), rt)) \) to \( \mathcal{C}^* \).

Proving that \( \text{Fix} \) preserves the composition of morphisms is surprisingly complicated, and involves several non-trivial steps. For example, we needed to show that if \( (A, \alpha)(X, u)_{(B, \beta)} \) and \( (B, \beta)Y_{(C, \gamma)} \) are imprimitivity bimodules implementing isomorphisms in \( \mathcal{C}^*(G, (C_0(T), rt)) \), then \( (X \otimes_B Y)^{u \otimes v} \) is isomorphic to \( X^u \otimes_{B^\beta} Y^v \) as \( A^\alpha - C^\gamma \) imprimitivity bimodules.

It follows from [3, Theorem 3.7] that \( \text{RCP} \) is a functor from \( \mathcal{C}^*(G, (C_0(T), rt)) \) to \( \mathcal{C}^* \) which takes a morphism \([X, u]\) to the class of the Combes bimodule \([X \rtimes_{u, r} G]\). We can now state the main naturality result, which is Theorem 3.5 of [7].

**Theorem 14.** Suppose that a locally compact group \( G \) acts freely and properly on a local compact space \( T \). Then the Morita equivalences \( Z(A, \alpha, \varphi) \) form a natural isomorphism between the functors \( \text{RCP} \) and \( \text{Fix} \) from \( \mathcal{C}^*(G, (C_0(T), rt)) \) to \( \mathcal{C}^* \).

The proof of Theorem 14 relies on factoring morphisms: then Theorem 3.2 of [15] gives the result for the nondegenerate homomorphism, and standard linking algebra techniques give the other half.

We saw in Example 10 that Rieffel’s Morita equivalence can be used to generalise Mansfield’s imprimitivity theorem to crossed products by homogeneous spaces, and we want to deduce from Theorem 14 that this imprimitivity theorem gives a natural isomorphism. To get the imprimitivity theorem in Example 10, we applied Rieffel’s Theorem 9 to a crossed product \( C \rtimes_{\delta} G \). So the naturality result we seek relates the compositions of \( \text{RCP} \) and \( \text{Fix} \) with a crossed-product functor.

Suppose as in Example 10 that \( H \) is a closed subgroup of a locally compact group \( G \). We know from Theorem 2.15 of [3] that there is a category \( \mathcal{C}^*\text{coact}^2(G) \) whose objects are normal coactions \( (B, \delta) \), and whose morphisms are isomorphism classes of suitably equivariant right-Hilbert bimodules. We also know from Theorem 3.13 of [3] that there is a functor \( \text{CP} : \mathcal{C}^*\text{coact}^2(G) \to \mathcal{C}^*\text{act}(H) \), and adding the canonical map \( j_\delta \) makes \( \text{CP} \) into a functor with values in the comma category \((C_0(G), rt) \downarrow \mathcal{C}^*\text{act}(H)) \). We show in [7, Proposition 5.5] that there is a functor \( \text{RCP}_{G/H} \) which sends \( (B, \delta) \) to the crossed product \( B \rtimes_{\delta,r} (G/H) \) by the homogeneous space \( G/H \), and that this functor coincides with \( \text{Fix} \circ \text{CP} \). We saw in Example 10
that Rieffel’s bimodules \( Z(B \times_\delta G, \delta|H, j_G) \) implement an Morita equivalence between \( (B \times_\delta G) \times_\delta H \) and \( B \times_\delta G/H \). Write \( \text{RCP}_\delta \) for the functor from \( \text{C}^*\text{act}(G) \) to \( \text{C}^* \) sending \((C, \gamma) \mapsto C \times_\gamma H \). Then the general naturality result above gives the following theorem, which is Theorem 5.6 of \([7]\).

**Corollary 15.** Let \( H \) be a closed subgroup of \( G \). Then Rieffel’s Morita equivalences \( Z(G \times_\delta G, \delta|H, j_G) \) implement a natural isomorphism between the functors \( \text{RCP}_\delta \circ \text{CP} \) and \( \text{RCP}_\delta |_{\text{coact}^\delta}(G) \) to \( \text{C}^* \).

Corollary 15 extends Theorem 4.3 of \([3]\) to non-normal subgroups, and extends Theorem 6.2 of \([15]\) to categories based on \( \text{C}^* \) rather than ones based on \( \text{C}^*_\text{red} \).

**6. Induction-in-stages and fixing-in-stages**

Rieffel’s theory of proper actions seems to be a powerful tool for studying systems in the comma or semi-comma category associated to a pair \((T, G)\). Corollary 15 is, we think, an impressive first example. As another example, we discuss an approach to induction-in-stages which works through the same general machinery, and which we carried out in \([8]\).

The original purpose of an imprimitivity theorem was to provide a way of recognising induced representations (as in, for example, \([20]\)), and Rieffel’s theory of Morita equivalence for \( \text{C}^*\)-algebras was developed to put imprimitivity theorems in a \( \text{C}^*\)-algebraic context \([37, 38]\). One can reverse the process: a Morita equivalence \( X \) between a crossed product \( C \rtimes_\alpha G \) and another \( \text{C}^*\)-algebra \( B \) gives an induction process \( X-\text{Ind} \) which takes a representation of \( B \) on \( \mathcal{H} \) to a representation of \( C \) on \( X \rtimes_\alpha \mathcal{H} \), and for which there is a ready-made imprimitivity theorem (see, for example, \([6, \text{Proposition 2.1}]\)). The situation is slightly less satisfactory when one has a reduced crossed product, but one can still construct induced representations and prove an imprimitivity theorem.

Mansfield’s imprimitivity theorem, as extended to homogeneous spaces in \([9]\), gives an induction process \( \text{Ind}^{G/H}_{G/K} \) from \( B \rtimes_{\delta, r} (G/H) \) to \( B \rtimes_\delta G \) which comes with an imprimitivity theorem. One then asks whether this induction process has the other properties which one would expect. For example, we ask whether we can induce-in-stages: if we have subgroups \( H, K \) and \( L \) with \( H \subset K \subset L \), is \( \text{Ind}^{G/H}_{G/K} \left( \text{Ind}^{G/K}_{G/L} \pi \right) \) unitarily equivalent to \( \text{Ind}^{G/H}_{G/L} \pi \)? If the subgroups are normal and amenable, then the induction processes are those defined by Mansfield \([22]\), and induction-in-stages was established in \([14, \text{Theorem 3.1}]\). For non-normal subgroups, not much seems to be known. There are clearly issues: for example, the subgroups \( H \) and \( K \) have to be normal in \( L \) for the three induction processes to be defined.

We tackled this problem in \([8]\) using our semi-comma category. Suppose that \((T, G)\) is as usual, \( N \) is a closed normal subgroup of \( G \), and \((A, \alpha, \varphi)\) is an object in \( \text{C}^*\text{act}(G, (C_0(T), \text{rt})) \). Then \( N \) also acts freely and properly on \( T \), so we can form the fixed-point algebra \( \text{Fix}_N(A, \alpha|_N, \varphi) \). The quotient \( G/N \) has a natural action \( \alpha^{G/N} \) on \( A^{\alpha|N} := \text{Fix}(A, \alpha|_N, \varphi) \), and the map \( \varphi \) induces a homomorphism \( \varphi_N : C_0(T/N) \to M(A^{\alpha|N}) \) such that \((A^{\alpha|N}, \alpha^{G/N}, \varphi_N)\) is an object in the semi-comma category \( \text{C}^*\text{act}(G/N, (C_0(T/N), \text{rt})) \). We prove in \([8]\) that \( \text{Fix}_N \) extends to a functor

\[
\text{Fix}_N^{G/N} : \text{C}^*\text{act}(G, (C_0(T), \text{rt})) \to \text{C}^*\text{act}(G/N, (C_0(T/N), \text{rt})),
\]
and that the functors $\text{Fix}_{G/N} \circ \text{Fix}^G_{N}$ and $\text{Fix}_G$ are naturally isomorphic (see [8, Theorem 4.5]). The first difficulty in the proof is showing that the functor $\text{Fix}_N$ has an equivariant version: because the functor $\text{Fix}$ is defined using the factorisation of morphisms, we have to track carefully through the constructions in [7] to make sure that they all respect the actions of $G/N$.

Applying this result on “fixing-in-stages” with $(T, G) = (L/H, K/H)$, gives the following version of induction-in-stages, which is Theorem 7.3 of [8].

**Theorem 16.** Suppose that $\delta$ is a normal coaction of $G$ on $B$, and that $H$, $K$ and $L$ are closed subgroups of $G$ such that $H \subset K \subset L$ with both $H$ and $K$ normal in $L$. Then for every representation $\pi$ of $B \rtimes_{\delta,r} (G/L)$, the representation $\text{Ind}^G_{G/K}(\text{Ind}^G_{G/L} \pi)$ is unitarily equivalent to $\text{Ind}^G_{G/L} \pi$.

Obviously this is not the last word on the subject, and the normality hypotheses on subgroups are irritating. However, Mansfield’s induction process is notoriously hard to work with, and it seems remarkable that one can prove very much at all about an induction process which is substantially more general than his. We think that Rieffel’s theory of proper actions is proving to be a remarkably malleable and powerful tool.

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School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia
E-mail address: astrid@unsw.edu.au

School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia
E-mail address: raeburn@uow.edu.au

Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA
E-mail address: dana.williams@dartmouth.edu