Application of the $\tau$-Function Theory of Painlevé Equations to Random Matrices: PIV, PII and the GUE

P.J. Forrester$^1$, N.S. Witte$^{1,2}$

1 Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia.
E-mail: P.Forrester@ms.unimelb.edu.au E-mail: N.Witte@ms.unimelb.edu.au,
2 School of Physics, University of Melbourne, Victoria 3010, Australia.

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Abstract: Tracy and Widom have evaluated the cumulative distribution of the largest eigenvalue for the finite and scaled infinite GUE in terms of a PIV and PII transcendent respectively. We generalise these results to the evaluation of $\tilde{E}_N(\lambda; a) := \left\langle \prod_{l=1}^{N} \chi_{(-\infty, \lambda]}^{(l)}(\lambda - \lambda_l)^a \right\rangle$, where $\chi_{(-\infty, \lambda]}^{(l)} = 1$ for $\lambda_l \in (-\infty, \lambda]$ and $\chi_{(-\infty, \lambda]}^{(l)} = 0$ otherwise, and the average is with respect to the joint eigenvalue distribution of the GUE, as well as to the evaluation of $F_N(\lambda; a) := \left\langle \prod_{l=1}^{N} (\lambda - \lambda_l)^a \right\rangle$. Of particular interest are $\tilde{E}_N(\lambda; 2)$ and $F_N(\lambda; 2)$, and their scaled limits, which give the distribution of the largest eigenvalue and the density respectively. Our results are obtained by applying the Okamoto $\tau$-function theory of PIV and PII, for which we give a self contained presentation based on the recent work of Noumi and Yamada. We point out that the same approach can be used to study the quantities $\tilde{E}_N(\lambda; a)$ and $F_N(\lambda; a)$ for the other classical matrix ensembles.

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1. Introduction and Summary

Hermitian random matrices $X$ with a unitary symmetry are defined so that the joint distribution of the independent elements $P(X)$ is unchanged by the similarity transformation $X \mapsto U^\dagger X U$ for $U$ unitary. For example, an ensemble of matrices with $P(X) := \exp \sum_{j=0}^\infty \alpha_j \text{Tr}(X^j) =: \prod_{j=1}^N g(\lambda_j)$ for general $g(x) \geq 0$ possesses a unitary symmetry. Such ensembles have the property that the corresponding eigenvalue probability density function $p(\lambda_1, \ldots, \lambda_N)$ is given by the explicit functional form

$$p(\lambda_1, \ldots, \lambda_N) = \frac{1}{C} \prod_{i=1}^N g(\lambda_i) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^2,$$

(1.1)

$C$ denoting the normalization. (Throughout the symbol $C$ will be used to denote some constant, i.e. a quantity independent of the primary variable(s) of the equation.) The choice $g(x) = e^{-x^2}$, which is realized by choosing each diagonal element of $X$ independently from the normal distribution $N[0, 1/\sqrt{2}]$, and each off diagonal independently with distribution $N[0, 1/2] + i N[0, 1/2]$, is referred to as the Gaussian Unitary Ensemble (GUE) and is the main focus of the present article. Specifically our interest is in the distribution of the largest eigenvalue, and the average values of powers (integer and fractional) of the characteristic polynomial $\prod_{l=1}^N (\lambda - \lambda_l)$ for such matrices.

Let $E_N(0; (s, \infty))$ denote the probability that there are no eigenvalues in the interval $(s, \infty)$ for $N \times N$ GUE matrices. The distribution of the largest eigenvalue $p_{\text{max}}(s)$ is given in terms of $E_N(0; (s, \infty))$ by

$$p_{\text{max}}(s) = \frac{d}{ds} E_N(0; (s, \infty)).$$

(1.2)

With $R_N(s)$ specified by the solution of the nonlinear equation

$$(R_N''(s))^2 + 4(R_N'(s))^2(R_N' + 2N) - 4(sR_N' - R_N)^2 = 0,$$

(1.3)

(an example of the Jimbo-Miwa-Okamoto $\sigma$-form of the Painlevé IV differential equation; see Eq. (2.18) below) subject to the boundary condition

$$R_N(s) \sim \frac{2^{N-1} s^{2N-2} e^{-s^2}}{\pi^{1/2} (N-1)!},$$

(1.4)

it has been shown by Tracy and Widom \cite{33} that

$$E_N(0; (s, \infty)) = \exp \left( - \int_s^\infty R_N(t) \, dt \right).$$

(1.5)
The derivation in [35] uses functional properties of Fredholm determinants (a subsequent derivation using the KP equations and Virasoro algebras has been given by Adler et al. [1]). In this work we will give a derivation of (1.5) based on the \( \tau \)-function theory of the Painlevé IV equation due to Okamoto [26], and refined by Noumi and Yamada [21]. Our principal observation in employing this body of theory to problems in random matrix theory is that there is a deep and fundamental relationship between \( \tau \)-functions relating to the Hamiltonian formalism of the Painlevé theory, and particular multiple integrals specifying averages with respect to the probability density function (1.1) in the case that \( g(x) \) takes the form of a classical weight function. From the random matrix perspective the classical weights are

\[
g(x) = \begin{cases} 
e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} (x > 0), & \text{Laguerre} \\ (1-x)^a (1+x)^b (-1 < x < 1), & \text{Jacobi} \\ (1+x^2)^{-\alpha}, & \text{Cauchy}. \end{cases} \tag{1.6}
\]

To summarise the correspondence, which applies to any of the cases, we set out the following schematic table:

| Painlevé Theory | Random Matrix Ensembles |
|-----------------|-------------------------|
| \( \tau \)-function \( \tau[N](s;a) \) | Gap probability \( E_N(s) \) |
| Hamiltonian \( H[N](s;a) \) | Averages \( \tilde{E}_N(s;a), F_N(s;a) \) |
| Classical solutions - Weyl chamber walls | Classical weights - Determinant structure |

\[ \text{Table 1. Correspondence between random matrix theory and Painlevé theory.} \]

The quantity \( \tilde{E}_N(s;a) \) in Table 1 is specified by

\[
\tilde{E}_N(s;a) = \langle \prod_{l=1}^{N} \chi^{(l)}_{(-\infty,\lambda]}(\lambda - \lambda_l)^a \rangle,
\]

where \( \chi^{(l)}_{(-\infty,\lambda]} = 1 \) for \( \lambda_l \in (-\infty, \lambda] \) and \( \chi^{(l)}_{(-\infty,\lambda]} = 0 \) otherwise, and the average is with respect to the eigenvalue probability density function (1.1). For general \( a \) we obtain the evaluation

\[
\tilde{E}_N(s;a) = \tilde{E}_N(s_0;a) \exp \left( \int_{s_0}^{s} U_N(t;a) \, dt \right) \tag{1.8}
\]

(Eq. (4.14) with the substitution (4.10)), where \( U_N(t;a) \) satisfies the nonlinear equation

\[
(U_N')^2 - 4(tU_N' - U_N)^2 + 4U_N' (U_N' - 2a)(U_N' + 2N) = 0, \tag{1.9}
\]

(Eq. (4.15)) subject to the boundary condition

\[
U_N(t;a) \mathop{\sim}_{t \to -\infty} -2Nt - \frac{N(a + N)}{t} + O \left( \frac{1}{t^3} \right), \tag{1.10}
\]
For \((N+1)\times(N+1)\) dimensional GUE matrices \(p_{\text{max}}(s)\) is proportional to \(e^{-s^2} \hat{E}_N(s;2)\). We therefore have
\[
p_{\text{max}}(s) \bigg|_{N\to N+1} = p_{\text{max}}(s_0) \bigg|_{N\to N+1} \exp \left( \int_{s_0}^{s} [-2t + U_N(t;2)] \, dt \right),
\]
(Eq. (1.11)).

The quantity \(F_N(s;a)\) in Table 1 is specified by
\[
F_N(\lambda;a) := \left\langle \prod_{l=1}^{N} (\lambda - \lambda_l)^a \right\rangle,
\]
(Eq. (1.12)). For general positive integers \(a\) (1.12) has been computed by Brézin and Hikami in terms of the determinant of an \(a \times a\) matrix involving Hermite polynomials. Note that for \(a\) not equal to a positive integer, (1.12) is well defined provided \(\lambda\) has a non-zero imaginary part. For general \(a\) we obtain the evaluation
\[
F_N(\lambda;a) = F_N(\lambda_0;a) \exp \left( \int_{\lambda_0}^{\lambda} V_N(t;a) \, dt \right),
\]
(Eq. (1.13)) where \(V_N(t;a)\) also satisfies the nonlinear equation (4.15), but now with the boundary conditions
\[
V_N(t;a) \sim \chi \frac{Na}{t} \left( 1 + O(1/t) \right) \quad \text{as} \quad t \to \infty
\]
(Eq. (1.14)) where \(\chi = 1\) for \(t \to \infty\) and \(|\chi| = 1\) for \(t \to -\infty\). In the case \(a = 2\) this average is proportional to the polynomial part of the eigenvalue density for \((N+1)\times(N+1)\) dimensional GUE matrices, which in terms of the Hermite polynomial \(H_N(\lambda)\) is proportional to each of the 2 \(\times\) 2 determinants termed Turánians [17]
\[
\begin{bmatrix}
H_N(\lambda) & H_{N+1}(\lambda) \\
H_N'(\lambda) & H_{N+1}'(\lambda)
\end{bmatrix}, \quad \begin{bmatrix}
H_{N+1}(\lambda) & H_{N+1}'(\lambda) \\
H_{N+1}'(\lambda) & H_{N+1}''(\lambda)
\end{bmatrix}, \quad \begin{bmatrix}
H_N(\lambda) & H_{N+1}(\lambda) \\
H_{N+1}(\lambda) & H_{N+2}(\lambda)
\end{bmatrix},
\]
(1.15)
(which are of course proportional to each other). The result (1.33) with \(a = 2\) implies
\[
\rho(\lambda) \bigg|_{N\to N+1} = \rho(\lambda_0) \bigg|_{N\to N+1} \exp \left( \int_{\lambda_0}^{\lambda} [-2t + V_N(t;2)] \, dt \right),
\]
(Eq. (1.16)).

In Sect. 2 we review the \(\tau\)-function theory of the Painlevé IV equation, revising relevant aspects of the work of Okamoto [24,26], Noumi and Yamada [21,22] and Kajiwara et al. [16]. The culmination of this theory from our perspective is the derivation of determinant formula expressions for the \(\tau\)-function corresponding to special values of the parameters in the Painlevé IV equation. On the other hand, it follows easily from the definitions that \(\hat{E}_N\) and \(F_N\) can be written as determinants. These are presented in Sect. 4. The determinant formulas in fact precisely coincide with those occurring in Sect. 2, so consequently we can characterise both \(\hat{E}_N\) and \(F_N\) in terms of solutions of the nonlinear equation (4.13). The theory presented in Sect. 2 also allows \(\hat{E}_N, F_N\) to be characterised as solutions of a certain fourth order difference equation (Eq. (2.82)), and \(U_N, V_N\) as solutions of a particular third order difference equation (4.19).

Also of interest is the scaling limit of (1.7) and (1.12) with \(\lambda \mapsto \sqrt{2N} + \lambda / \sqrt{2N}^{1/6}\). This choice of coordinate corresponds to shifting the origin to the edge of the leading order support of the eigenvalue density, then scaling
the coordinate so as to make the spacings of order unity as $N \to \infty$. We find the scaled quantities can be expressed in terms of particular solutions of the general Jimbo-Miwa-Okamoto $\sigma$ form of the Painlevé II equation

$$ (u')^2 + 4u'\left((u')^2 - su' + u\right) - a^2 = 0, \quad (1.17) $$

(Eq. (5.10)). Specifically, as already known from [36],

$$ E^\text{soft}(s) := \lim_{N \to \infty} E_N \left( 0; \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \right) = \exp \left( - \int_s^{\infty} r(t) \, dt \right), \quad (1.18) $$

where $r(s)$ satisfies (5.10) with $a = 0$. Also

$$ E^\text{soft}(s; a) := \lim_{s \to \sqrt{2N^{1/6}}} \left( Ce^{-as^2/2} \tilde{E}_N(s; a) \right) = E^\text{soft}(s_0; a) \exp \left( \int_{s_0}^s u(t; a) \, dt \right), \quad (1.19) $$

where $u(s; a)$ satisfies (5.11) subject to the boundary condition

$$ u(s; a) \sim 1/4s^2 + \frac{4a^2 - 1}{8s} + \frac{(4a^2 - 1)(4a^2 - 9)}{64s^4} + \ldots, \quad (1.20) $$

(Eq. (5.11)). In the case $a = 2$ (Eq. (5.7)) gives the formula

$$ r^\text{soft}_{\max}(s) = r^\text{soft}_{\max}(s_0) \exp \left( \int_{s_0}^s u(t; 2) \, dt \right), \quad (1.21) $$

(Eq. (5.20)) for the scaled distribution of the largest eigenvalue in the GUE.

Analogous to the formula (5.7), for the scaled limit of $F_N(\lambda; a)$ we have

$$ F^\text{soft}(\lambda; a) := \lim_{\lambda \to \sqrt{2N^{1/3}}} \left( Ce^{-a\lambda^2/2} F_N(\lambda; a) \right) = F^\text{soft}(\lambda_0; a) \exp \left( \int_{\lambda_0}^\lambda v(t; a) \, dt \right), \quad (1.22) $$

(Eq. (5.31)) where $v(s; a)$, like $u(s; a)$, satisfies (5.11). The difference between $u$ and $v$ is in the boundary condition; for the latter we require

$$ v(t; a) \sim -at^{1/2} - \frac{a^2}{4t} + \frac{a(4a^2 + 1)}{32t^{5/2}} \quad (1.23) $$

(Eq. (5.35)). The case $a = 2$ corresponds to the scaled eigenvalue density at the spectrum edge, which has the known evaluation [10]

$$ \rho^\text{soft}(s) = - \begin{vmatrix} \text{Ai}(s) & \text{Ai}'(s) \\ \text{Ai}'(s) & \text{Ai}''(s) \end{vmatrix}, \quad (1.24) $$

where $\text{Ai}(s)$ denotes the Airy function. In fact for all $a \in \mathbb{Z}_{\geq 0}$ we have the determinantal form

$$ F^\text{soft}(\lambda; a) = (-1)^{a(a-1)/2} \det \left[ \frac{d^{j+k}}{d\lambda^{j+k}} \text{Ai}(\lambda) \right]_{j,k=0,\ldots,a-1}, \quad (1.25) $$

(Eq. (5.33)).

In Sect. 3 we present the $\tau$-function theory of the Painlevé II equation in an analogous fashion to the theory presented in Sect. 2 for the Painlevé IV equation. In particular we derive the second order second degree equation satisfied by the Hamiltonian (which is known from [14] and [21]) as well as a fourth order difference equation satisfied by the $\tau$-functions. Also derived is the fact that the right-hand side of (5.33) corresponds to a $\tau$-function sequence in the PII theory, which is a result of Okamoto [21]. In Sect. 5 the results (5.3), (5.7), (1.31) and (5.33) are derived from a limiting process applied to the corresponding finite $N$ results.

A programme for further study is outlined in Sect. 6.
2. \( \tau \)-Function Theory for PIV

2.1. Affine Weyl group symmetry. It has been demonstrated in the works of Okamoto [26] (in a series of papers treating all the Painlevé equations), Noumi and Yamada [21] (see also their works [20,23,22]) and the earlier work of Adler [2] that the fourth Painlevé equation

\[
y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y},
\]

(2.1)
can be recast in a way which reveals its symmetries in a particularly manifest and transparent form.

**Proposition 1** ([21,22]). The fourth Painlevé equation is equivalent to the coupled set of autonomous differential equations (where \( \frac{d}{dt} \) is used)

\[
\begin{align*}
f_0' &= f_0(f_1 - f_2) + 2\alpha_0, \\
f_1' &= f_1(f_2 - f_0) + 2\alpha_1, \\
f_2' &= f_2(f_0 - f_1) + 2\alpha_2,
\end{align*}
\]

(2.2)

with \( y = -f_1 \) and where the parameters \( \alpha_j \in \mathbb{R} \) with \( \alpha_0 + \alpha_1 + \alpha_2 = 1 \) are related by

\[
\alpha = \alpha_0 - \alpha_2, \quad \beta = -2\alpha_1^2,
\]

(2.3)

and the constraint taken conventionally as

\[
f_0 + f_1 + f_2 = 2t.
\]

(2.4)

**Proof.** Equation (2.4) reduces the three first order equations of (2.2) down to two. Eliminating a further variable by introducing a second derivative shows that \( y = -f_1 \) satisfies the PIV equation. The form of these equations implies

\[
(f_0 + f_1 + f_2)' = 2\alpha_0 + 2\alpha_1 + 2\alpha_2 = k,
\]

(2.5)
k \( \neq 0 \) constant, thus permitting the normalization given above. \( \Box \)

**Note.** Many differing conventions are in use for such a description of the PIV system and for example we have written \( 2\alpha_j \) \( (j = 0, 1, 2) \) in place of the \( \alpha_j \) used in [21,23,22] in order to eliminate unnecessary factors of two appearing in the ensuing theory.

The hyperplane \( \alpha_0 + \alpha_1 + \alpha_2 = 1 \) in parameter space \( (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3 \) is associated with the simple roots \( \alpha_0, \alpha_1, \alpha_2 \) spanning the root system of type \( A_2^{(1)} \). From this perspective the parameters \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) define a triangular lattice in the plane (see Fig. 1).
Let the fundamental reflections $s_i$ ($i = 0, 1, 2$) represent the automorphism of the lattice specified by a reflection with respect to the line $\alpha_i = 0$. Their action on the simple roots are given by

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad (2.6)$$

where $a_{ij}$ are the elements of the Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \quad (2.7)$$

Let $\pi$ represent the lattice automorphism corresponding to a rotation by $120^\circ$ degrees around the barycentre of the fundamental alcove $C$ defined by $\alpha_i > 0$ ($i = 0, 1, 2$). Then

$$\pi(\alpha_j) = \alpha_{j+1}, \quad (2.8)$$

$j \in \mathbb{Z}/3\mathbb{Z}$. The operators $\pi$, $s_i$ obey the algebra

$$s_j^2 = 1, \quad (s_j s_{j+1})^3 = 1, \quad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}, \quad \pi^3 = 1, \quad \pi s_j = s_{j+1} \pi, \quad (2.9)$$

and generate $\widetilde{W} = \langle \pi, s_0, s_1, s_2 \rangle$ defining an extension of the affine Weyl group associated with the $A_2^{(1)}$ root system.

**Proposition 2 ([20, 23]).** The Bäcklund transformations of the PIV system are given by the actions of the extended affine Weyl group $\widetilde{W}$ on the parameters as specified by (2.4) and (2.3), and on the functions as specified by

$$s_i(f_j) = f_j + \frac{2\alpha_i}{f_i} u_{ij}, \quad \pi(f_j) = f_{j+1} \quad (i, j = 0, 1, 2), \quad (2.10)$$
where the $u_{ij}$ are the elements of the orientation matrix

$$U = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad (2.11)$$

associated with the boundary of the fundamental alcove \cite{21}.

**Proof.** Let $V$ denote one of $\pi, s_0, s_1, s_2$ and let $\beta_i := V(\alpha_i)$. Using (2.6) and (2.8) it’s a simple exercise to explicitly verify that $g_i := V(f_i)$ of the form (2.10) satisfy the structurally identical equations

$$
\begin{align*}
g'_0 &= g_0(g_1 - g_2) + 2\beta_0, \\
g'_1 &= g_1(g_2 - g_0) + 2\beta_1, \\
g'_2 &= g_2(g_0 - g_1) + 2\beta_2,
\end{align*}
$$

thus giving rise to the stated Bäcklund transformation. $\square$

Following \cite{22,18}, in Tables 2, 3 the actions (2.6), (2.8) and (2.10), are listed in tabular format.

|     | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ |
|-----|------------|------------|------------|
| $s_0$ | $-\alpha_0$ | $\alpha_0 + \alpha_1$ | $\alpha_0 + \alpha_2$ |
| $s_1$ | $\alpha_1 + \alpha_0$ | $-\alpha_1$ | $\alpha_1 + \alpha_2$ |
| $s_2$ | $\alpha_2 + \alpha_0$ | $\alpha_2 + \alpha_1$ | $-\alpha_2$ |
| $\pi$ | $\alpha_1$ | $\alpha_2$ | $\alpha_0$ |
| $T_1$ | $\alpha_0 + 1$ | $\alpha_1 - 1$ | $\alpha_2$ |
| $T_2$ | $\alpha_0$ | $\alpha_1 + 1$ | $\alpha_2 - 1$ |
| $T_3$ | $\alpha_0 - 1$ | $\alpha_1$ | $\alpha_2 + 1$ |

**Table 2.** Action of the generators of the extended affine Weyl group associated with the root system $A_2(1)$ on the simple roots.

|     | $f_0$ | $f_1$ | $f_2$ |
|-----|------|------|------|
| $s_0$ | $f_0$ | $f_1 + \frac{2\alpha_0}{f_0}$ | $f_2 - \frac{2\alpha_0}{f_0}$ |
| $s_1$ | $f_0 - \frac{2\alpha_1}{f_1}$ | $f_1$ | $f_2 + \frac{2\alpha_1}{f_1}$ |
| $s_2$ | $f_0 + \frac{2\alpha_2}{f_2}$ | $f_1 - \frac{2\alpha_2}{f_2}$ | $f_2$ |
| $\pi$ | $f_1$ | $f_2$ | $f_0$ |

**Table 3.** Bäcklund transformations for the PIV system.
From the earlier work of Okamoto it has been known that the PIV system, as for all the Painlevé transcendents, admits a Hamiltonian formulation and that from this viewpoint the Bäcklund transformations are birational canonical transformations \( \{q, p; H\} \mapsto \{\tilde{q}, \tilde{p}; \tilde{H}\} \).

**Proposition 3** \([26, 16]\). The PIV dynamical system is a Hamiltonian system \( \{q, p; H\} \) with the Hamiltonian

\[
H = (2p - q - 2t)pq - 2\alpha_1 p - \alpha_2 q ,
\]

and canonical variables \( q, p \)

\[
-f_1 = q, \quad f_2 = 2p .
\]

**Proof.** With \( H \) specified by (2.13), Hamilton’s equations of motion read

\[
q' = \frac{\partial H}{\partial p} = q(4p - q - 2t) - 2\alpha_1 , \quad p' = -\frac{\partial H}{\partial q} = p(2q - 2p + 2t) + \alpha_2 .
\]

Substituting for \( p \) and \( q \) according to (2.14) shows that these equations are identical to the final two equations in (2.2). \( \square \)

**Note.** Because \( -f_1 \) satisfies the PIV equation (2.1), it follows immediately from the first equation in (2.14) that \( q \) satisfies the PIV equation (2.1). Furthermore, use of the first equation in (2.15) shows

\[
p = \frac{1}{4q}(q' + q^2 + 2tq + 2\alpha_1) ,
\]

so \( H \) is completely specified in terms of the Painlevé IV transcendent (2.1) with parameters (2.3).

There is a degree of ambiguity in constructing a Hamiltonian in that arbitrary functions of time can be added, and in fact there is a more symmetrical form

\[
H_S = \frac{1}{2} f_0 f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2 ,
\]

which is central to the Okamoto theory (termed the auxiliary Hamiltonian). However for our purposes this complicates some later results so we prefer the unsymmetrical form. Furthermore, in the full theory of PIV \([21, 22]\) the Hamiltonian \( H_0 \equiv H \) is associated with two additional Hamiltonians \( H_1 = \pi(H_0), H_2 = \pi^2(H_0) \) but these are not required in the random matrix context.

It is also true that \( H(t) \) can be specified as the solution of a certain second order second degree equation.

**Proposition 4** \([26, 14]\). The Hamiltonian (2.13) satisfies the second order second degree differential equation of the Jimbo-Miwa-Okamoto \( \sigma \) form for PIV,

\[
(H'')^2 - 4(tH' - H)^2 + 4H'(H' + 2\alpha_1)(H' - 2\alpha_2) = 0 .
\]

**Proof.** Making use of Hamilton’s equations (2.15), we have for \( H(t) = H(t; q(t), p(t)) \),

\[
H' = f_1 f_2 ,
\]

\[
H'' = f_1 f_2 (f_2 - f_1) + 2\alpha_2 f_1 + 2\alpha_1 f_2 .
\]
Use of (2.13) and (2.19) in (2.20) shows

\begin{align*}
f_1 &= -\frac{\sqrt{2}H'' + tH' - H}{H' - 2\alpha_2}, \\
f_2 &= \frac{\sqrt{2}H'' + tH' - H}{H' + 2\alpha_1}.
\end{align*}

Substituting (2.21) in (2.13) gives the desired equation (2.18). \(\square\)

For future reference we note that use of Tables 2, 3 shows that under the action of the generators of \(\tilde{W}\), \(H\) transforms according to

\begin{align*}
s_0(H) &= H + \frac{2\alpha_0}{f_0}, \\
s_1(H) &= H + 2\alpha_1 t, \\
s_2(H) &= H - 2\alpha_2 t, \\
\pi(H) &= H + f_2 - 2\alpha_2 t.
\end{align*}

2.2. Toda lattice equation. The \(\tau\)-function \(\tau = \tau(t)\) is defined in terms of the Hamiltonian \(H(t)\) by

\[H(t) = \frac{d}{dt} \log \tau(t).\] (2.23)

It is possible to derive a Toda lattice equation for the sequences of \(\tau\)-functions \(\{\tau_k[n]\}_{n=0,1,...} (k = 1, 3)\) associated with the Hamiltonians

\[H\bigg|_{\alpha_0 \to \alpha_0 + n, \alpha_1 \to \alpha_1 - n, \alpha_2 \to \alpha_2 + n}, \quad H\bigg|_{\alpha_0 \to \alpha_0 + n, \alpha_2 \to \alpha_2 - n},\] (2.24)

respectively (the reason for the subscripts 1 and 3 on \(\tau\) will become apparent subsequently). An essential point is that there exist shift operators from the algebra \(\tilde{W}\) which after \(n\) applications on \(H\) generate the shifts required by (2.24). There are in fact three fundamental shift operators \(T_1, T_2, T_3\) corresponding to translations on the root lattice by the fundamental weights \(\tilde{\omega}_j, j = 1, 2, 3\) of the root system \(A_2^{(1)}\). As can be checked from Tables 2, 3 and (2.22) these operators have the property that

\[T_1 H = H\bigg|_{\alpha_0 \to \alpha_0 + 1, \alpha_1 \to \alpha_1 - 1, \alpha_2 \to \alpha_2 + 1}, \quad T_3^{-1} H = H\bigg|_{\alpha_0 \to \alpha_0 + 1, \alpha_2 \to \alpha_2 - 1}.
\] (2.25)

Table 2 also shows that when acting on the parameters themselves, the same shifts occurring in the transformed Hamiltonian results, and thus

\begin{align*}
T_1(\alpha_0, \alpha_1, \alpha_2) &= (\alpha_0 + 1, \alpha_1 - 1, \alpha_2), \\
T_3^{-1}(\alpha_0, \alpha_1, \alpha_2) &= (\alpha_0 + 1, \alpha_1, \alpha_2 - 1).
\end{align*}

(2.26)

After a further \(n\) iterations the equations (2.25) can be written in the form

\[T_{1}^{n+1} H - T_{1}^{n} H = f_{(1)[n]}, \quad T_{3}^{-(n+1)} H - T_{3}^{-n} H = -f_{(3)[n]},\]

(2.27)

where the subscripts (1) (3) refer to the system of Eqs. (2.2) with the parameters replaced as in the first (second) Hamiltonian (2.24) and use has been made of (2.13).
We remark that the two results of (2.25) are inter-related. Thus consider the mapping \( \omega \) defined by multiplication by \(-1\) together with the replacements

\[(\alpha_0, \alpha_1, \alpha_2) \mapsto (-\alpha_0, -\alpha_2, -\alpha_1), \quad (f_0, f_1, f_2) \mapsto (-f_0, -f_2, -f_1).\] (2.28)

We see immediately that the system (2.2) is unchanged by \( \omega \), as is the Hamiltonian (2.13), while we can check from Table 2 that

\[\omega T_1^w = T_3^{-1}.\] (2.29)

Applying \( \omega \) to the first equation of (2.25) using (2.28) and (2.29) gives the second equation. With the \( \tau \)-functions \( \tau_1[n] \) and \( \tau_3[n] \) defined by

\[T_n^1 H = \frac{d}{dt} \log \tau_1[n], \quad T_3^{-n} H = \frac{d}{dt} \log \tau_3[n],\] (2.30)

application of (2.29) shows

\[\omega \tau_1[n] = C \tau_3[n].\] (2.31)

In light of the relation (2.31), let us focus attention on the first equation of (2.25) only.

**Proposition 5** ([26, 16]). The \( \tau \)-function sequence \( \tau_1[n] \) corresponding to the parameter sequence \( (\alpha_0 + n, \alpha_1 - n, \alpha_2) \) obeys the Toda lattice equation

\[\frac{d^2}{dt^2} \log \sigma_1[n] = \frac{\sigma_1[n+1] \sigma_1[n-1]}{\sigma_1^2[n]},\] (2.32)

where

\[\sigma_1[n] := Ce^{t^2(\alpha_1-n)} \tau_1[n].\] (2.33)

**Proof.** Following [26, 16] we make use of the first equation in (2.27) and consider the difference

\[(T_{n+1}^1 H - T_n^1 H) - (T_{n}^1 H - T_{n-1}^1 H) = f_{(1)2}[n] - f_{(1)2}^{-1}[n].\] (2.34)

A crucial fact, which follows from Table 3 and (2.13), is that this difference is a total derivative

\[f_{(1)2}[n] - f_{(1)2}^{-1}[n] = \frac{d}{dt} \log \left(f_{(1)1}[n] f_{(1)2}[n] + 2(\alpha_1 - n)\right).\] (2.35)

But it follows from (2.19) and (2.13) that

\[f_{(1)1}[n] f_{(1)2}[n] + 2(\alpha_1 - n) = \frac{d}{dt} \left(T_n^1 H + 2t(\alpha_1 - n)\right) = \frac{d^2}{dt^2} \log \left(e^{t^2(\alpha_1-n)} \tau_1[n]\right),\] (2.36)

and hence the right-hand side of (2.33) is equal to

\[\frac{d}{dt} \log \left(\frac{d^2}{dt^2} \log \left(e^{t^2(\alpha_1-n)} \tau_1[n]\right)\right).\] (2.37)

On the other hand (2.34) and (2.30) shows the left-hand side of (2.33) is equal to

\[\frac{d}{dt} \log \left(\frac{\tau_1[n+1] \tau_1[n-1]}{\tau_1^2[n]}\right).\] (2.38)
Equating (2.37) and (2.38), and integrating shows that
\[
\frac{d^2}{dt^2} \log \left( e^{t^2(\alpha_1 - n)} \tau_1[n] \right) = C \frac{\tau_1[n+1] \tau_1[n-1]}{\tau_1[n]},
\]
and the stated result (2.32) follows. There is the ambiguity of a multiplicative constant C, possibly dependent on \( n \) but not on \( t \), and this can be chosen freely, for example to render the Toda lattice equation in a simple form. □

The Toda lattice equation obeyed by the \( \tau_3[n] \) with parameters \((\alpha_0 + n, \alpha_1, \alpha_2 - n)\) is obtained by applying the mapping \( \omega \) to both sides of (2.39) and making use of (2.31). This shows
\[
\frac{d^2}{dt^2} \log \left( e^{-t^2(\alpha_2 - n)} \tau_3[n] \right) = C \frac{\tau_3[n+1] \tau_3[n-1]}{\tau_3[n]},
\]
which with
\[
\sigma_3[n] := C e^{-t^2(\alpha_2 - n)} \tau_3[n],
\]
gives the Toda lattice equation
\[
\frac{d^2}{dt^2} \log \sigma_3[n] = \frac{\sigma_3[n+1] \sigma_3[n-1]}{\sigma_3[n]^2}.
\]

Another way to deduce (2.40) is via (2.39) and the differential equation (2.18). Now, the second Hamiltonian in (2.24) is obtained from the first by simply interchanging \( \alpha_1 \) and \( \alpha_2 \). On the other hand \( \alpha_1 \) and \( \alpha_2 \) are interchanged in (2.18) if we replace \( t \) by \( it \) then replace \( H(it) \) by \( -iH(t) \). This tells us that in (2.39) we can make the replacements
\[
t \mapsto it, \quad \tau_1(it) \mapsto \tau_3(t), \quad \alpha_1 \mapsto \alpha_2,
\]
which indeed gives (2.40). Furthermore, since (2.43) shows \( \tau_1 \) and \( \tau_3 \) are simply related, it suffices to consider one sequence only, \( \tau_3[n] \) say.

2.3. Classical solutions. For a special initial choice of the parameters it is possible to choose \( \tau_3[0] = 1 \), and then to determine \( \tau_3[1] \) in terms of a classical function, that is to say the solution of a second order linear differential equation. What is essential here is the condition for the decoupling of the two independent first order differential equations so that what remains is a Riccati equation.

**Proposition 6** ([26]). *For the special initial choice of the parameters such that \( \alpha_2 = 0 \), i.e. for parameters \((1 - \alpha_1, \alpha_1, 0)\) the first nontrivial member of the \( \tau \)-function sequence \( \tau_3[1] \) satisfies the Hermite-Weber equation,*
\[
\tau_3'[1] = -2t\tau_3[1] - 2\alpha_1 \tau_3[1].
\]

*Proof.* With \( n = 0 \), \((1 - \alpha_1 + n, \alpha_1, -n)\) implies \( \alpha_2 = 0 \). Now we see from (2.13) that
\[
H \Big|_{\alpha_2=0} = pq(2p - q - 2t) - 2\alpha_1 p,
\]
which allows us to take
\[
H \Big|_{\alpha_2=0} = 0,
\]
provided we set $p = 0$. Recalling (2.30) this implies

$$\tau_3[0] = 1.$$  \hfill (2.47)

We read off from the $n = 1$ case of the second equation in (2.27), together with (2.14), that

$$T_3^{-1} H \bigg|_{\alpha_2 = 0} - H \bigg|_{\alpha_2 = 0} = q,$$  \hfill (2.48)

and thus, after recalling the second equation in (2.30) and (2.46),

$$\frac{d}{dt} \log \tau_3[1] = q.$$  \hfill (2.49)

The first of Hamilton’s equations (2.15) gives, with $\alpha_2 = 0$ and $p = 0$ (after the differentiation), the Riccati equation

$$q' = -q^2 - 2tq - 2\alpha_1.$$  \hfill (2.50)

Substituting (2.49) this reduces to the linear equation (2.44) first obtained in the present context by Okamoto [26]. □

**Proposition 7.** Two linearly independent solutions to the Toda lattice equation (2.40) for sequences of $\tau$-functions with parameters $(\alpha_0 + n, \alpha_1, -n)$, $n \geq 0$, starting from the Weyl chamber wall $\alpha_2 = 0$ are given by the determinant forms

$$\tau_3[n](t; \alpha_1) = C \det \left[ \int_{-\infty}^{t} (t - x)^{-\alpha_1 + i+j} e^{-x^2} dx \right]_{i,j=0,\ldots,n-1},$$  \hfill (2.51)

and

$$\bar{\tau}_3[n](t; \alpha_1) = C \det \left[ \int_{-\infty}^{\infty} (t - x)^{-\alpha_1 + i+j} e^{-x^2} dx \right]_{i,j=0,\ldots,n-1}.$$  \hfill (2.52)

**Proof.** In the special case $\alpha_1 = 0$, we observe that a solution of (2.44) is

$$\tau_3[1] = C \int_{-\infty}^{t} e^{-x^2} dx.$$  \hfill (2.53)

In fact it is possible to solve (2.44) in a form analogous to (2.53) for general $\alpha_1$. Thus consider the integral

$$I_a(t) := \int_{-\infty}^{t} (t - x)^a e^{-x^2} dx,$$  \hfill (2.54)

and suppose temporarily that Re($a$) > −1. Simple manipulation gives

$$I_a(t) = tI_{a-1}(t) + \frac{1}{2} \int_{-\infty}^{t} (t - x)^{a-1} \frac{d}{dx} e^{-x^2} dx$$

$$= tI_{a-1}(t) + \frac{(a-1)}{2} I_{a-2}(t).$$  \hfill (2.55)

But

$$(a - 1)I_{a-2}(t) = \frac{1}{a} \frac{d^2}{dt^2} I_a(t), \quad I_{a-1}(t) = \frac{1}{a} \frac{d}{dt} I_a(t).$$  \hfill (2.56)
Thus we see that \( I_{-\alpha_1}(t) \) satisfies (2.43) and this implies
\[
\tau_3[1] = C \int_{-\infty}^{t} (t-x)^{-\alpha_1} e^{-x^2} \, dx ,
\] (2.57)
where we require \( \text{Re}(\alpha_1) < 1 \).

Starting with (2.47) and (2.57), up to a multiplicative constant the \( \tau \)-functions \( \tau_3[n] \) \( (n = 2, 3, \ldots) \) are uniquely specified by the Toda lattice equation (2.42). In fact it was known to Sylvester ([19] pp. 115-117) that the solution of (2.42) with initial condition \( \sigma_3[0] = 1 \) is the double Wronskian or Hankel determinant
\[
\sigma_3[n] = \det \left[ d^{i+j} \sigma_3[1] \right]_{i,j=0,...,n-1} ,
\] (2.58)
Recalling (2.41) and (2.57) we therefore have
\[
\tau_3[n] := \tau_3[n](t; \alpha_1) = C \det \left[ e^{-t^2} \frac{d^{i+j}}{dt^{i+j}} \left( e^{t^2} \int_{-\infty}^{t} (t-x)^{-\alpha_1} e^{-x^2} \, dx \right) \right]_{i,j=0,...,n-1} .
\] (2.59)
Making use of (2.57) we can check that
\[
\frac{d^p}{dt^p} \left( e^{t^2} \int_{-\infty}^{t} (t-x)^{-\alpha_1} e^{-x^2} \, dx \right) = 2^p e^{t^2} \int_{-\infty}^{t} (t-x)^{-\alpha_1+p} e^{-x^2} \, dx ,
\] (2.60)
so (2.59) can also be written in the final form of (2.51).

The second linearly dependent solution of (2.44) can also be written in an integral form similar to (2.54). Thus we see the integral
\[
\bar{I}_a(t) := \int_{-\infty}^{\infty} (t-x)^a e^{-x^2} \, dx ,
\] (2.61)
satisfies the formulas (2.54) and (2.56), and thus satisfies (2.44) with \( a = -\alpha_1 \). Hence in addition to (2.57) we have the solution
\[
\bar{\tau}_3[1] = C \int_{-\infty}^{\infty} (t-x)^{-\alpha_1} e^{-x^2} \, dx ,
\] (2.62)
(note that for \( \alpha_1 \) not equal to a non-positive integer, this is well defined only if \( t \notin \mathbb{R} \)). Proceeding as in the derivation of (2.51), we deduce from the Toda lattice equation (2.42), and the initial values (2.47), (2.62), the sequence of \( \tau \)-functions given by (2.63).

Let us now consider the sequence of \( \tau \)-functions \( \bar{\tau}_1[n] \).

**Proposition 8 ([26]).** The sequence of \( \tau \)-function solutions to the Toda lattice equation (2.39) \( \bar{\tau}_1[n] \), \( n \geq 0 \), corresponding to the parameter sequence \((\alpha_0 + n, -n, \alpha_2)\) starting from the line \( \alpha_1 = 0 \) has the determinantal form
\[
\bar{\tau}_1[n](t; -p) = C \det \left[ H_{p+i+j}(t) \right]_{i,j=0,...,n-1} \]
(2.63)
for \( -\alpha_2 = p \in \mathbb{Z}_{\geq 0} \).
Proof. This sequence can be obtained from $\bar{\tau}_3[n]$ by the (inverse of) the mappings (2.43). Replacing $t$ by $-it$ in (2.51) does not lead to an integral of interest in random matrix applications, but doing the same in (2.52) gives

$$\bar{\tau}_1[n](t; \alpha_2) = C \det \left[ \int_{-\infty}^{\infty} (t - ix)^{-\alpha_2+i+j} e^{-x^2} \, dx \right]_{i,j=0,\ldots,n-1},$$

(2.64)

which is of interest. We recall that for $\bar{\tau}_1^{(n)}$ the parameters $(\alpha_0, \alpha_1, \alpha_2)$ in the corresponding Hamiltonian are given by

$$(1 - \alpha_2 + n, -n, \alpha_2),$$

(2.65)

For $p \in \mathbb{Z}_{\geq 0}$ we know

$$\int_{-\infty}^{\infty} (t - ix)^p e^{-x^2} \, dx = \sqrt{\pi} 2^{-p} H_p(t),$$

(2.66)

and thus setting $\alpha_2 = -p$ equation (2.64) yields (2.63). $\square$

Note that with $p = N, n = 2$, this is precisely the final determinant in (1.15).

2.4. Bäcklund transformations and discrete Painlevé systems. It has been known that some of the Bäcklund transformations of the PIV transcendent can be identified with discrete Painlevé equations [6,11], although no systematic study has been undertaken for this class. We will find that the difference equations for $f_j[n], H[n], \tau[n]$ which are generated by the Bäcklund transformations for the two shift operations $T_3^{-1}, T_1$ are in fact manifestations of discrete Painlevé equations.

Proposition 9 ([11]). The Bäcklund transformations of the PIV system corresponding to the shift operator $T_3^{-1}$ generating the parameter sequence of $(\alpha_0 + n, \alpha_1, -n)$ with $n \in \mathbb{Z}$, $0 < \alpha_0, \alpha_1 < 1$ and $\alpha_0 + \alpha_1 = 1$ are second order difference equations of the first discrete Painlevé equation dPI, namely

$$\chi_{k+1} + \chi_k + \chi_{k-1} = 2t + \frac{k - (1/2 + \alpha_1) + (-1)^k(1/2 - \alpha_1)}{\chi_k} \chi_k \geq 1,$$

(2.67)

where

$$\chi_{2n+1} = f_{(3)2}[n], \quad \chi_{2n+2} = f_{(3)0}[n] \quad n \geq 0.$$

(2.68)
Proof. The action of the shift operators on the \( f_i \) is expressible in a terminating continued fraction, which for \( T_3 \) and its inverse takes the form

\[
T_3(f_0) = f_1 - \frac{2\alpha_2}{f_2}, \quad (2.69)
\]

\[
T_3(f_1) = f_2 + \frac{2(\alpha_1 + \alpha_2)}{f_1 - \frac{2\alpha_2}{f_2}}, \quad (2.70)
\]

\[
T_3(f_2) = f_0 + \frac{2\alpha_2}{f_2} - \frac{2(\alpha_1 + \alpha_2)}{f_1 - \frac{2\alpha_2}{f_2}}, \quad (2.71)
\]

\[
T_3^{-1}(f_0) = f_2 - \frac{2\alpha_0}{f_0} + \frac{2(\alpha_0 + \alpha_1)}{f_1 + \frac{2\alpha_0}{f_0}}, \quad (2.72)
\]

\[
T_3^{-1}(f_1) = f_0 - \frac{2(\alpha_0 + \alpha_1)}{f_1 + \frac{2\alpha_0}{f_0}}, \quad (2.73)
\]

\[
T_3^{-1}(f_2) = f_1 + \frac{2\alpha_0}{f_0}, \quad (2.74)
\]

as one can verify using the action of the affine Weyl group reflections and diagram rotations as given in Tables 2, 3. For simplicity of notation we suppress the subscript \((3)\) labelling the sequence \((\alpha_0 + n, \alpha_1, -n)\) during the discussion of our proofs as there is no risk of confusion. Taking the first and last members of this set, now at the \( n \)th rung of the \( T_3 \) ladder, and adding their unshifted \( f \)-variable we have

\[
f_0[n] + f_0[n-1] = 2t - f_2[n] + \frac{2n}{f_2[n]}, \quad (2.75)
\]

so that one has a closed system. This can be recognised as the two components of a staggered system of difference equations and employing the definitions of \( \chi_k \) above we arrive at the discrete Painlevé equation dPI. \( \square \)

In terms of the coordinate and momenta of the Hamiltonian system this difference system was found by Okamoto [26, 28] and can be expressed as

\[
q[n+1] = (2t + q[n] - 2p[n])q[n](2t + q[n] - 2p[n]) + 2\alpha_1 q[n][2t + q[n] - 2p[n]], \quad (2.76)
\]

\[
q[n-1] = -2p[n]q[n][p[n] - \alpha_1] q[n][p[n] - n], \quad (2.77)
\]

\[
p[n+1] = -\frac{1}{2}q[n] + \alpha_0 + n 2t + q[n] - 2p[n], \quad (2.78)
\]

\[
p[n-1] = t + \frac{q[n][p[n] - n] - p[n]q[n][p[n] - \alpha_1]}{2p[n] - p[n]q[n][p[n] - n]}, \quad (2.79)
\]

Consequently a third order difference equation exists for the Hamiltonian through the relation

\[
H[n+1] - H[n] = -f_1[n]. \quad (2.80)
\]
Eliminating \( p[n] \) between (2.70), (2.77) we find a second order difference equation for \( q[n] \),

\[
\begin{align*}
\quad & = 2 \left[ (n+1)q[n+1] - nq[n-1] - (2t + q[n] + q[n+1] + q[n+1]) \right] \\
\quad & \times \left[ (n+1)q[n+1] - nq[n-1] - (2t + q[n] + q[n+1] + q[n+1]) \right].
\end{align*}
\]

(2.81)

Use of \( H[n+1] - H[n] = q[n] \) leads to the third order equation in \( H[n] \).

In addition we have a higher order difference equation for the \( \tau \)-function.

**Proposition 10.** The \( \tau \)-function sequence, appropriately normalised, associated with the shift operator \( T^{-1}_3 \) with parameter values \((\alpha_0 + n, \alpha_1, -n)\), \( n \in \mathbb{Z}_{\geq 0}, \alpha_0, \alpha_1 < 1 \) and \( \alpha_0 + \alpha_1 = 1 \) satisfies the fourth order difference equation

\[
4t^2 \left( 2n\tau^2[n] - \tau[n+1]\tau[n-1] \right) \times \left( 2(n-\alpha_1)\tau^2[n] - \tau[n+1]\tau[n-1] \right)
\]

\[
= \left\{ \tau[n+2]\tau[n-2]\tau^3[n] - 16n^2(\alpha_1-n)^2\tau^5[n] \ight. \\
+ 16n(\alpha_1-n)(\alpha_1-2n)\tau^3[n]\tau[n+1]\tau[n-1] - 4(2n^2-2\alpha_1n+1)\tau[n]\tau^2[n]\tau[n+1]\tau[n-1] \\
+ \tau[n+2]\tau^2[n-1]\left( \tau[n+1]\tau[n-1] + 2(\alpha_1+1-2n)^2\tau^2[n] \right) \\
\left. + \tau[n-2]\tau^2[n+1]\left( \tau[n+1]\tau[n-1] + 2(\alpha_1-1-2n)^2\tau^2[n] \right) \right\}^2.
\]

(2.82)

**Proof.** We first seek to express all the fundamental quantities in terms of the product \( f_1[n]f_2[n] \). By multiplying the two transformations (2.69) and (2.70) we find a quadratic relation for \( f_1[n] \) (and \( f_1[n-1] \)),

\[
f_1[n](2t - f_1[n]) = f_1[n+1]f_2[n+1] + f_1[n]f_2[n] + 2\alpha_1.
\]

(2.83)

Next we multiply (2.70) by \( f_1[n] \) which yields a relation for the product

\[
f_1[n]f_1[n-1] = f_1[n]f_2[n] \frac{f_1[n]f_2[n] + 2\alpha_1}{f_1[n]f_2[n] + 2n}.
\]

(2.84)

One can verify then that a linear proportionality exists between \( f_1[n] \) and \( f_1[n-1] \) via the product \( f_1[n]f_2[n] \),

\[
f_1[n] \left\{ f_1[n]f_2[n] + f_1[n-1]f_2[n-1] + 2\alpha_1 - f_1[n]f_2[n] \frac{f_1[n]f_2[n] + 2\alpha_1}{f_1[n]f_2[n] + 2n} \right\}
\]

\[
= f_1[n-1] \left\{ f_1[n+1]f_2[n+1] + f_1[n]f_2[n] + 2\alpha_1 - f_1[n]f_2[n] \frac{f_1[n]f_2[n] + 2\alpha_1}{f_1[n]f_2[n] + 2n} \right\},
\]

(2.85)

so that \( f_1[n] \) and \( f_1[n-1] \) may now be linearly related to \( f_1[n]f_2[n] \). Multiplying these two later relations and using

\[
C \frac{\tau_3[n+1]\tau_3[n-1]}{\tau_3[n]} = 2n + f_1[n]f_2[n],
\]

(2.86)
with $C = 1$ to introduce the $\tau$-functions we arrive at (2.82). □

Note. The difference equations (2.81) and (2.82) have the advantage of being of the lowest order we have found possible, but the disadvantage of not being linear in the highest order terms ($q[n+1]$ and $\tau[n+2]$ respectively). In fact difference equations linear in the highest order terms can be given by increasing by one the order of the equations in each case [28].

Applying the operator $\omega$ (recall (2.28)) we obtain analogous results for the sequence generated by $T_1$.

**Proposition 11 ([11]).** The Bäcklund transformations generated by the shift operator $T_1$ corresponding to the parameter sequence $(\alpha_0 + n, -n, \alpha_2)$ with $n \in \mathbb{Z}$, $0 < \alpha_0, \alpha_2 < 1$, and $\alpha_0 + \alpha_2 = 1$, are second order difference equations of the first discrete Painlevé equation dPI, that is

$$
\eta_{k+1} + \eta_k + \eta_{k-1} = 2t - \frac{k - [1+(-1)^k]\alpha_2 - \frac{1}{2}[1-(-1)^k]}{\eta_k}, \quad k \geq 1 ,
$$

(2.87)

where

$$
\eta_{2n+1} = f_{(1)1}[n] , \quad \eta_{2n+2} = f_{(1)0}[n] , \quad n \geq 0 .
$$

(2.88)

**Proof.** This follows immediately upon applying $\omega$ to both sides of (2.67). □

The analogue of (2.81) for the parameter sequence generated by the shift $T_1$ can be found by applying the $\omega$ map to this relation,

$$
-2np[n]p[n-1]\left(2t - 2p[n] - p[n+1] - p[n-1]\right)^2
= \left[(n+1)p[n+1] - np[n-1] + (t - p[n] - p[n+1])(\alpha_2 + 2p[n](t - p[n]))\right]
\times \left[(n+1)p[n+1] - np[n-1] + (t - p[n] - p[n-1])(-\alpha_2 + 2p[n](t - p[n] - p[n+1] - p[n-1]))\right]
$$

(2.89)

and this implies a third order difference equation for the Hamiltonian via

$$
H[n+1] - H[n] = f_2[n] = 2p[n] .
$$

(2.90)

There is also a higher order difference equation for the $\tau$-function which can be derived using the relation

$$
\frac{\tau_1[n+1]\tau_2[n-1]}{\tau_2^2[n]} = f_1[n]f_2[n] - 2n ,
$$

(2.91)

although we do not reproduce this here.

### 3. $\tau$-Function Theory for PII

#### 3.1. Affine Weyl group symmetry.

In the general Painlevé theory the second Painlevé equation naturally appears as a coalescence limit of PIV. From the work of [36] it is known that in random matrix theory PII occurs in the edge scaling limit of the GUE. This suggests that before studying this limit we should develop a theory of PII analogous to that developed for PIV in the previous section. We take the PII equation to be defined in the standard manner

$$
y'' = 2y^3 + ty + \alpha .
$$

(3.1)
Proposition 12. The second Painlevé equation with the transcendent \( y = q(t) \) and parameter \( \alpha \) is equivalent to the system of first order differential equations
\[
\begin{align*}
f_0' &= -2q f_0 + \alpha_0 , \\
f_1' &= 2q f_1 + \alpha_1 ,
\end{align*}
\] (3.2) (3.3)
where \( f_0 + f_1 = 2q^2 + t \) and \( \alpha_0 + \alpha_1 = 1 \) with \( \alpha = \alpha_1 - 1/2 = 1/2 - \alpha_0 \).

Proof. This is established by eliminating \( p \) through the substitutions \( f_0 = 2q^2 - p + t \) and \( f_1 = p \). \( \square \)

Proposition 13. Let \( \tilde{W} = \langle s_0, s_1, \pi \rangle \) be the extended affine Weyl group of the root system of type \( A_1^{(1)} \) generated by the reflections \( s_0, s_1 \) and the diagram rotation \( \pi \), with action on the roots \( \alpha_0, \alpha_1 \) as given in Table 4.

\[
\begin{array}{ccc}
\alpha_0 & \alpha_1 \\
\hline
s_0 & -\alpha_0 & \alpha_1 + 2\alpha_0 \\
s_1 & \alpha_0 + 2\alpha_1 & -\alpha_1 \\
\pi & \alpha_1 & \alpha_0 \\
\end{array}
\]

Table 4. Action of the generators of the extended affine Weyl group associated with the root system \( A_1^{(1)} \) on the simple roots.

The coupled system (3.2), (3.3) is symmetric under the Bäcklund transformations induced by the elements of the above affine Weyl group as specified in Table 5.

\[
\begin{array}{cccc}
 & f_0 & f_1 & q \\
\hline
s_0 & f_0 & f_1 - \frac{4\alpha_0 q}{f_0} + \frac{2\alpha_0^2}{f_0^2} & q - \frac{\alpha_0}{f_0} \\
s_1 & f_0 + \frac{4\alpha_1 q}{f_1} + \frac{2\alpha_1^2}{f_1^2} & f_1 & q + \frac{\alpha_1}{f_1} \\
\pi & f_1 & f_0 & -q \\
\end{array}
\]

Table 5. Bäcklund transformations for the PII system.

Proof. This can be directly verified using the equations of motion (3.2), (3.3). \( \square \)

Underlying the dynamics of the PII system is a Hamiltonian structure.

Proposition 14. The PII dynamical system is equivalent to a Hamiltonian system \( \{ q, p; H \} \) with Hamiltonian
\[
H = -\frac{1}{2} f_0 f_1 - \alpha_1 q ,
\] (3.4)
and canonical coordinates and momenta \( q, p \) defined by
\[
p = f_1 , \quad 2q^2 = f_0 + f_1 - t .
\] (3.5)
Proof. Using the symmetrised differential equations (3.2,3.3) the Hamilton equations of motion

\[ q' = p - q^2 - \frac{1}{2} t, \quad p' = 2qp + \alpha_1, \]  

(3.6)
can be verified. □

Remark. The fundamental domain or Weyl chamber for PII can be taken as the interval \( \alpha \in (-\frac{1}{2}, 0] \) or \( \alpha \in [0, \frac{1}{2}) \), and so there exist identities relating the transcendents and related quantities at the endpoints of these intervals. In particular, denoting the transcendent \( q(t, \alpha) \) and with \( \epsilon^2 = 1, \ t = -2^{1/3}s \) we have [12]

\[ -\epsilon 2^{1/3} q^2(s, 0) = \frac{d}{dt} q(t, \frac{1}{2}\epsilon) - \epsilon q^2(t, \frac{1}{2}\epsilon) - \frac{1}{2}\epsilon t, \]

\[ q(t, \frac{1}{2}\epsilon) = \epsilon 2^{-1/3} \frac{1}{q(s, 0)} \frac{d}{ds} q(s, 0). \]  

(3.7)

The action of the affine Weyl group on the Hamiltonian is given in Table 6.

| \( s_0 \) | \( H_0 \) | \( H_1 = \pi(H_0) = H_0 + q \) |
|---|---|---|
| \( s_1 \) | \( H_0 + \frac{\alpha_0 f_0}{f_0} \) | \( H_1 \) |
| \( \pi \) | \( H_1 \) | \( H_0 \) |

Table 6. Bäcklund transformations of the Hamiltonian.

We define two shift operators corresponding to translations by the fundamental weights of the affine Weyl group \( A_1^{(1)} \),

\[ T_1 = \pi s_1, \quad T_2 = s_1 \pi, \]  

(3.8)

although \( T_1T_2 = 1 \) so only one is independent. Their action on the parameter space is given in Table 7.

| \( T_1 \) | \( \alpha_0 \) | \( \alpha_1 \) |
|---|---|---|
| \( T_2 \) | \( \alpha_0 + 1 \) | \( \alpha_1 - 1 \) |
| \( \alpha_0 - 1 \) | \( \alpha_1 + 1 \) |

Table 7. Action of the shift operators on the simple roots of the root system \( A_1^{(1)} \).
Proposition 15. The Bäcklund transformations corresponding to the shifts are given by

\[
\begin{align*}
T_1(f_0) &= f_1 - \frac{4\alpha_0 q}{f_0} + \frac{2\alpha_1^2}{f_0^2}, \\
T_1(f_1) &= f_0, \\
T_2(f_0) &= f_1, \\
T_2(f_1) &= f_0 + \frac{4\alpha_1 q}{f_1} + \frac{2\alpha_1^2}{f_1^2}, \\
T_1(H_0) &= H_0 + q, \\
T_1(H_1) &= H_1 - q + \frac{\alpha_0}{f_0}, \\
T_2(H_0) &= H_0 + q + \frac{\alpha_1}{f_1}, \\
T_2(H_1) &= H_1 - q.
\end{align*}
\]

Proposition 16. The Hamiltonian \( H(t) \) satisfies the second order second degree differential equation of Jimbo-Miwa-Okamoto \( \sigma \)-form for PII,

\[
(H'')^2 + 4(H')^3 + 2H'[tH' - H] - \frac{1}{4}\alpha_1^2 = 0.
\]

Proof. Using first two derivatives of \( H \)

\[
\begin{align*}
H' &= -\frac{1}{2} f_1, \\
H'' &= -q f_1 - \frac{1}{2}\alpha_1,
\end{align*}
\]

one can solve for \( q, f_0, f_1 \) and then substitute these back into the expression for \( H \). The result (3.17) then follows after simplification. \( \Box \)

With \( H \) given by (3.4), and \( p \) and \( q \) specified by (3.5), Hamilton’s equation for \( q' \) implies

\[
p = q' + q^2 + \frac{1}{2} t.
\]

Thus \( H \) can be expressed in terms of the Painlevé II transcendent \( q \) according to

\[
H = \frac{1}{2}(q')^2 - \frac{1}{2}(q^2 + \frac{1}{2} t)^2 - (\alpha + \frac{1}{2})q.
\]

3.2. Toda lattice equation. The \( \tau \)-functions are defined as before (2.23) and corresponding to each sequence generated by the shift operators is a Toda lattice equation.

Proposition 17. The \( \tau \)-function sequence generated by the shift operator \( T_1 \) with the parameter sequence \((\alpha_0 + n, \alpha_1 - n)\) for \( n \geq 0 \)

\[
T_1^n(H) = H[n] = \frac{d}{dt} \ln \tau[n],
\]

obeys the Toda lattice equation

\[
C \frac{\tau[n+1]\tau[n-1]}{\tau^2[n]} = \frac{d^2}{dt^2} \ln \tau[n].
\]
Proof. This parallels the argument employed for PIV case, by utilising the relations

\[ H[n+1] - H[n] = q[n] , \tag{3.23} \]

and

\[ q[n] - T^{-1}_1[q[n]] = \frac{d}{dt} \ln f_1[n] , \tag{3.24} \]

along with

\[ \frac{d}{dt} H = -\frac{1}{2} f_1 . \tag{3.25} \]

\[ \square \]

3.3. Classical solutions. When the parameter values are those on a chamber wall (a point) \( \alpha_1 = n \in \mathbb{Z} \) then the \( \tau \)-functions are known to be expressible in terms of Airy functions \[26\].

**Proposition 18.** The solution for the first non-trivial member of the \( \tau \)-function sequence \( \tau[1](t) \) generated by the shift operator \( T_1 \) with initial parameters \( (\alpha_0, \alpha_1) = (1, 0) \) that is bounded as \( t \to -\infty \) is

\[ \tau[1](t) = CAi(-2^{-1/3}t) . \tag{3.26} \]

The \( n^{th} \) member of this sequence is

\[ \tau[n](t) = C \det \left[ \frac{d^{i+j}}{dt^{i+j}} Ai(-2^{-1/3}t) \right]_{i,j=0,\ldots,n-1} . \tag{3.27} \]

**Proof.** Starting from \( \alpha_1 = 0 \) at \( n = 0 \) one can take \( p[0] = 0 \) so that \( H[0] = 0 \) and conventionally \( \tau[0] = 1 \). Using (3.13) we find that \( H[1] = q[0] \) and so the equation of motion (3.6) gives the second order linear differential equation

\[ \tau''[1] + \frac{1}{2} t \tau[1] = 0 , \tag{3.28} \]

and thus (3.26). The determinant formula (3.27) follows from (2.58). \( \square \)

Another special parameter value of the Hamiltonian system (3.4) is \( \alpha_1 = -\frac{1}{2} \) \[27\] when Hamilton’s equation (3.6) permit the solution

\[ (q, p) = (t^{-1}, \sqrt{2}t) . \tag{3.29} \]

However the corresponding value of \( H \) is not zero so in this case we do not have \( \tau[0] = 1 \) (rather \( H[0] \) and thus \( \log \tau[0] \) is a rational function of \( t \), and thus the sequence of \( \tau \)-functions generated by \( T_1 \) is not given by a determinant \[13\]. Nonetheless the Bäcklund transformations of Prop. 15 show that \( H[n] \) and thus \( \log \tau[n] \) remain rational functions of \( t \) for all \( n = 1, 2, 3, \ldots \).
3.4. Bäcklund transformations and discrete dPI. The discrete dynamical system generated by the Bäcklund transformations is also integrable and can be identified with a discrete Painlevé system.

Proposition 19. The members of the sequence \(\{q[n]\}\), \(n \geq 0\) generated by the shift operator \(T_1\) with the parameters \((\alpha_0 + n, \alpha_1 - n)\) are related by a second order difference equation which is the alternate form of the first discrete Painlevé equation, a-dPI,

\[
\frac{\alpha + \frac{1}{2} - n}{q[n] + q[n-1]} + \frac{\alpha - \frac{1}{2} - n}{q[n+1] + q[n]} = -2q^2[n] - t.
\] (3.30)

Proof. We deduce from (3.8) and Table 3 that

\[
T_1(q) = -q + \frac{\alpha - \frac{1}{2}}{p - 2q^2 - t},
\] (3.31)

\[
T_2(q) = -q - \frac{\alpha + \frac{1}{2}}{p},
\] (3.32)

so eliminating \(p\) through the combination of these two we arrive at the stated result. □

The full set of forward and backward difference equations are [26]

\[
q[n+1] = -q[n] + \frac{\alpha - \frac{1}{2} - n}{p[n] - 2q[n]^2 - t},
\] (3.33)

\[
q[n-1] = -q[n] - \frac{\alpha + \frac{1}{2} - n}{p[n]},
\] (3.34)

\[
p[n+1] = -p[n] + 2q[n]^2 + t,
\] (3.35)

\[
p[n-1] = t - p[n] + 2\left(q[n] + \frac{\alpha + \frac{1}{2} - n}{p[n]}\right)^2.
\] (3.36)

The discrete Painlevé equation (3.31) implies a third order difference equation for the Hamiltonian

\[
\frac{\alpha + \frac{1}{2} - n}{H[n+1] - H[n-1]} + \frac{\alpha - \frac{1}{2} - n}{H[n+2] - H[n]} = -2(H[n+1] - H[n])^2 - t,
\] (3.37)

because \(q[n] = H[n+1] - H[n]\). Equations (3.35) and (3.36) also imply

\[
q[n] = \frac{1}{4\alpha_n^2}p[n] \left(p[n-1] - p[n+1]\right) - \frac{\alpha_n}{p[n]}, \quad \alpha_n := \alpha + \frac{1}{2} - n.
\] (3.38)

Using this to eliminate \(q[n]\) (in say (3.35)) yields a second order difference equation for the \(p[n]\),

\[
\frac{1}{4\alpha_n^2}p^2[n] \left(p[n+1] - p[n-1]\right)^2 + \frac{\alpha_n^2}{p^2[n]} - 2p[n] - p[n+1] - p[n-1] + 2t = 0.
\] (3.39)

Furthermore, Eqs. (3.21), (3.22) and (3.25) give

\[
C \frac{\tau[n+1] - \tau[n-1]}{\tau^2[n]} = p[n].
\] (3.40)

So substituting in (3.39) (with say \(C = -2\)) implies a fourth order difference equation for \(\tau[n]\),

\[
\frac{1}{2\alpha_n^2} \left(\tau[n-2] \tau^2[n+1] - \tau[n+2] \tau^2[n-1]\right)^2 + \frac{1}{s\alpha_n^2} \tau^6[n] + \tau[n-2] \tau^3[n+1] + \tau[n+2] \tau^3[n] + \tau[n-1] + 2\tau^3[n-1] + 2 \tau^3[n-1] + t \tau^2[n] = 0.
\] (3.41)
While (3.39) and the corresponding equation for \( \tau[n] \) provide a polynomial relation between the smallest set of consecutive sequence members \( \{p[n]\} \) and \( \{\tau[n]\} \) we have found possible, they have the disadvantage of not being linear in the highest order term \( (p[n+1] \) and \( \tau[n+2] \) respectively). This disadvantage can be remedied by increasing by one the order of the equation in each case. Thus by replacing \( n \) by \( n-1 \) in (3.38), then adding the result to the original equation and using (3.34) implies a third order difference equation for the \( p[n] \) \[29, \] 

\[
0 = \frac{1}{4\alpha_n} p[n] (p[n-1] - p[n+1]) + \frac{\alpha_n}{2p[n]} + \frac{1}{4\alpha_{n-1}} p[n-1] (p[n-2] - p[n]) - \frac{\alpha_{n-1}}{2p[n]}, \tag{3.42}
\]

which is indeed linear in \( p[n+1] \). Substituting (3.41) gives a fifth order equation for \( \tau[n] \), linear in the highest order term \( \tau[n+2] \).

### 3.5. Coalescence from PIV.

Since the earliest works on the Painlevé transcendents \[31, \] it was known how to obtain the PII system from a limiting procedure or coalescence applied to the PIV, however there is in fact more than one such coalescence path. In fact our analysis of the scaled GUE requires the application of a second coalescence path rather than the one commonly employed. In the first limit the parameters \((\alpha_0, \alpha_1, \alpha_2)\) and variables \( t_{IV}, q_{IV}, p_{IV}, H_{IV} \) in the PIV system scale in a way that \( \alpha_2 \) is fixed so that

\[
\alpha_0 = \frac{1}{2} - \alpha_{II} - \frac{1}{2} \epsilon^{-6}, \tag{3.43}
\]

\[
\alpha_1 = \frac{1}{2} \epsilon^{-6}, \tag{3.44}
\]

\[
\alpha_2 = \alpha_{II} + \frac{1}{2}, \tag{3.45}
\]

\[
t_{IV} = -\epsilon^{-3} + 2^{-2/3} \epsilon t_{II}, \tag{3.46}
\]

\[
q_{IV} = \epsilon^{-3} + 2^{2/3} \epsilon^{-1} q_{II}, \tag{3.47}
\]

\[
p_{IV} = 2^{-2/3} \epsilon p_{II}, \tag{3.48}
\]

\[
H_{IV} = -\epsilon^{-3} (\alpha_{II} + \frac{1}{2}) + 2^{2/3} \epsilon^{-1} H_{II}, \tag{3.49}
\]

as \( \epsilon \to 0 \) then the function \( q_{II}(t_{II}) \) satisfies the PII differential equation with parameter \( \alpha_{II} \) \[13, \]

The second limiting procedure is obtained from the first by the mapping \( \omega \) introduced in (2.28).

**Proposition 20.** If \( \alpha_1 \) is fixed so that the variables scale like

\[
\alpha_0 = \alpha_{II} + \frac{1}{2} + \frac{1}{2} \epsilon^{-6}, \tag{3.50}
\]

\[
\alpha_1 = -\alpha_{II} - \frac{1}{2}, \tag{3.51}
\]

\[
\alpha_2 = -\frac{1}{2} \epsilon^{-6}, \tag{3.52}
\]

\[
t_{IV} = \epsilon^{-3} - 2^{-2/3} \epsilon t_{II}, \tag{3.53}
\]

\[
q_{IV} = 2^{1/3} \epsilon q_{II}, \tag{3.54}
\]

\[
p_{IV} = \epsilon^{-3} + 2^{2/3} \epsilon^{-1} q_{II}, \tag{3.55}
\]

\[
H_{IV} + \alpha_{II} t_{IV} = -2^{2/3} \epsilon^{-1} H_{II}, \tag{3.56}
\]

as \( \epsilon \to 0 \) then the function \( q_{II}(t_{II}) \) satisfies the PII differential equation with parameter \( \alpha_{II} \). Furthermore the third order difference equation for \( H_{IV} \) \[28,29, \] related to the discrete Painlevé equation dPI, corresponding to the Bäcklund transformation under the shift operator \( T_1 \) transforms into the third order difference equation for \( H_{II} \) \[37, \] related to the alternate discrete Painlevé equation a-dPI, under this scaling.
Proof. Under the mapping $H_{IV} \mapsto -H_{IV}$, $2p_{IV} \leftrightarrow q_{IV}$, $\alpha_1 \leftrightarrow -\alpha_2$ and $t_{IV} \leftrightarrow -t_{IV}$. The only equation which isn’t immediate from the mapping is (3.56). Substituting (3.53) for $t_{IV}$ and ignoring the term proportional to $\epsilon t_{II}$ shows this equation is equivalent to

$$H_{IV} = \epsilon^{-3}(\alpha_{II} + \frac{1}{2}) - 2^{2/3} \epsilon^{-1} H_{II}, \quad (3.57)$$

which is precisely what results from applying the mapping $\omega$ to (3.49). The scaling of the third order difference equation for $H_{IV}$ associated with the discrete Painlevé equation, $dPI$, (2.89) to (3.37) can be verified directly.

4. Application to Finite GUE Matrices

In this section we will show that the determinants in (2.51) and (2.52) occur in the calculation of the quantities $\tilde{E}_N$ and $F_N$, introduced in the Introduction, relating to GUE random matrices.

4.1. Calculation of $E_N(0; (s, \infty))$ and $\tilde{E}_N(s; a)$. Consider first the probability $E_N(0; (s, \infty))$.

**Proposition 21.** The gap probability $E_N(0; (s, \infty))$ is identical to the $N^{th}$ $\tau$-function of the sequence generated by $T_{a}^{-1}$ from the corner of the Weyl chamber $(\alpha_0, \alpha_1, \alpha_2) = (1, 0, 0)$,

$$E_N(0; (s, \infty)) = \tau_3[N](s; 0), \quad (4.1)$$

where the normalization of $\tau_3[N]$ must be such that

$$\lim_{s \to \infty} \tau_3[N](s; 0) = 1. \quad (4.2)$$

The resolvent kernel function $R = R_N(t)$ occurring in (4.3) is the $N^{th}$ Hamiltonian associated with this sequence,

$$R_N(t) = H(t) \bigg|_{(\alpha_0, \alpha_1, \alpha_2) = (1+\frac{N}{2}, 0, -\frac{N}{2})}. \quad (4.3)$$

**Proof.** From the meaning of $E_N$ we see from (1.1) with $g(x) = e^{-x^2}$ that

$$E_N(0; (s, \infty)) = \frac{1}{C} \int_{-\infty}^{s} dx_1 \cdots \int_{-\infty}^{s} dx_N \prod_{j=1}^{N} e^{-x_j^2} \prod_{1 \leq j \leq k \leq N} (x_k - x_j)^2. \quad (4.4)$$

Introducing the Vandermonde determinant

$$\prod_{1 \leq j < k \leq N} (x_k - x_j) = \det[x_{j+1}^{k}]_{j,k=0,\ldots,N-1}, \quad (4.5)$$

standard manipulations following Heine (see [33], p. 27) allow this to be rewritten as the $N \times N$ determinant

$$E_N(0; (s, \infty)) = \frac{1}{C} \det \left[ \int_{-\infty}^{s} x^{j+k} e^{-x^2} dx \right]_{j,k=0,\ldots,N-1},$$

$$= \frac{1}{C} \det \left[ \int_{-\infty}^{s} (s-x)^{j+k} e^{-x^2} dx \right]_{j,k=0,\ldots,N-1}. \quad (4.6)$$
where the second equality follows by repeating the procedure which led to the first equality but starting from the Vandermonde determinant formula with \( x_j \mapsto x_j - s \) \((j = 1, \ldots, N)\). Comparing with (2.51) gives (4.1). Recalling (2.30) it follows from (4.1) that

\[
\rho(\tau) \text{ where the normalization by } C \text{ is such that } C = \det \left[ \int_{-\infty}^{s} (s - x)^{\alpha_1} e^{-x^2} dx \right]_{j,k=0,\ldots,N-1}. \tag{4.10}
\]

Recalling (2.51) we thus have (4.9). It follows from this that, with \( \rho(t) \) denoting the eigenvalue density,

\[
R(t) \sim \rho(t) = \frac{2^{N}}{\pi^{1/2}(N-1)!} e^{-t^2} \left[ H_N^2(t) - H_{N+1}(t)H_{N-1}(t) \right]. \tag{4.8}
\]

Considering next \( \tilde{E}_N(s; a) \) we have a generalisation of the previous case.

**Proposition 22.** The average \( \tilde{E}_N(s; a) \) is the \( N \)th member of the \( \tau \)-function sequence \( \{ \tau_3[N](s; -a) \} \) generated by \( T_3^{-1} \) from a general position on the \( \alpha_2 = 0 \) wall of the Weyl chamber, namely \( (\alpha_0, \alpha_1, \alpha_2) = (1 + a, -a, 0) \), and so

\[
\tilde{E}_N(s; a) = \tau_3[N](s; -a). \tag{4.9}
\]

The logarithmic derivative \( U_N(t; a) \) is identical to the Hamiltonian associated with this sequence,

\[
U_N(t; a) = H(t) \bigg|_{(\alpha_0, \alpha_1, \alpha_2) = (1 + a, -a, -N)}. \tag{4.10}
\]

**Proof.** According to (4.3) \( \tilde{E}_N(s; a) \) is given in terms of a multiple integral by

\[
\tilde{E}_N(s; a) = \frac{1}{C} \int_{-\infty}^{s} dx_1 \cdots \int_{-\infty}^{s} dx_N \prod_{j=1}^{N} e^{-x_j^2} (s - x_j)^{\alpha_j} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2, \tag{4.11}
\]

where the normalization \( C \) is such that

\[
\tilde{E}_N(s; a) \sim s^{N^a} \left( 1 + O(1/s) \right) \tilde{E}_N(s; 0) \text{ as } s \to \infty. \tag{4.12}
\]

The method of derivation of (4.6) shows that

\[
\tilde{E}_N(s; a) = \frac{1}{C} \det \left[ \int_{-\infty}^{s} (s - x)^{\alpha_j + k} e^{-x^2} dx \right]_{j,k=0,\ldots,N-1}. \tag{4.13}
\]

Recalling (2.51) we thus have (4.9). It follows from this that

\[
\tilde{E}_N(s; a) = \tilde{E}_N(s_0; a) \exp \left( \int_{s_0}^{s} H(t) \bigg|_{(\alpha_0, \alpha_1, \alpha_2) = (1 + a, -a, -N)} \ dt \right), \tag{4.14}
\]

and consequently (4.10). □

According to (2.18) \( U_N(t; a) \) satisfies the nonlinear equation

\[
(U_N')^2 - 4(tU_N' - U_N)^2 + 4U_N' (U_N' - 2a)(U_N' + 2N) = 0, \tag{4.15}
\]

which considering (4.12) and (4.3) is to be solved subject to the boundary condition

\[
U_N(t; a) \sim \frac{Na}{t} \left( 1 + O(1/t) \right) + R(t). \tag{4.16}
\]
In light of (4.18) the term $R(t)$ decays as a Gaussian and so is negligible with respect to the inverse powers in (4.16) (here we assume $a \neq 0$). This means the boundary condition (4.16) cannot be distinguished from the boundary condition with $R(t)$ removed; however we will see below that this latter boundary condition relates to a solution of (4.15) distinct from the one required in (4.14). To overcome this ambiguity we specify the $t \to -\infty$ behaviour of $U(t)$ rather than the $t \to \infty$ behaviour. Now, replacing $s$ by $-s$ in (4.11) and changing variables $x \mapsto x + s$ shows

$$
\tilde{E}_N(-s; a) = \frac{1}{C} e^{-Ns^2} s^{-Na-N^2} \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \prod_{j=1}^N e^{-(x_j/s)^2} e^{-2x_j/x_j^2} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 
\sim \frac{1}{C} e^{-Ns^2} s^{-Na-N^2} \left( 1 + O(1/s^2) \right).
$$

(4.17)

In combination with (4.14) this implies

$$
U_N(t; a) \sim_{t \to -\infty} -2Nt - \frac{N(a+N)}{t} + O\left(\frac{1}{t^3}\right).
$$

(4.18)

Under the action of the Bäcklund transformations $U_N(t; a)$ will satisfy two recurrence relations corresponding to the shift operators $T^{-1}_3$ and $T_1$. For the $T^{-1}_3$ sequence we relate the Painlevé and GUE parameters by $n = N$ and $\alpha_1 = -a$, and with (2.81) and (2.80) we have a third order recurrence relation in $N$ with $a$ fixed (suppressing the additional variable dependence in $U$)

$$
\begin{align*}
-N(U_{N+1} - U_N)(U_N - U_{N-1}) &\left( 4t + U_{N+2} + U_{N+1} - U_N - U_{N-1} \right)^2 \\
+2 \left[ (N+1)(U_{N+2} - U_{N+1}) - (N(U_N - U_{N-1}) \\
- \frac{1}{2}(2t + U_{N+2} - U_N)(-2a + (U_{N+1} - U_N)(2t + U_{N+1} - U_N)) \right] \times \left[ (N+1)(U_{N+2} - U_{N+1}) - (N(U_N - U_{N-1}) \\
- \frac{1}{2}(2t + U_{N+1} - U_{N-1})(2a + (U_{N+1} - U_N)(2t + U_{N+2} - U_{N-1})) \right] &= 0.
\end{align*}
$$

(4.19)

In contrast the $T_1$ sequence has the parameter correspondence $n = a$ and $\alpha_2 = -N$ which is an interchange of $\alpha_1$ and $\alpha_2$ with respect to the $T_3$ sequence. A third order recurrence relation in $a$, with fixed $N$, can be derived from (4.16) using the mapping (2.43) in which $N \leftrightarrow a$, $t \leftrightarrow it$ and then $U_N(it; a) \leftrightarrow -iU_N(t; a)$. Alternatively this difference equation can be found directly from (2.89) and (2.90), with the identification (4.10).

As noted in the Introduction, for $(N+1) \times (N+1)$ dimensional GUE matrices, the distribution of the largest eigenvalue $p_{\text{max}}(s)$ is proportional to $e^{-s^2} \tilde{E}_N(s; 2)$. Hence, using (4.14),

$$
p_{\text{max}}(s) \big|_{N \to N+1} = p_{\text{max}}(s_0) \big|_{N \to N+1} \exp\left( \int_{s_0}^s [-2t + U_N(t; 2)] \, dt \right).
$$

(4.20)

Comparison with (1.2), after putting $N \mapsto N + 1$ therein and substituting (1.3), gives an identity between the Hamiltonians $R(t) \big|_{N \to N+1}$ and $U(t; 2)$. Using the theory developed here this relation can be independently verified.

**Proposition 23.** The logarithmic derivative of the average $\tilde{E}_N(t; 2)$ is related to the resolvent kernel $R_N(t)$ by the identity

$$
U_N(t; 2) = 2t + R_{N+1}(t) + \frac{R'_{N+1}(t)}{R_{N+1}(t)}.
$$

(4.21)
Proof. From the identifications made above we have

\[ U_N(t; 2) = H(t) \bigg|_{(N+3, -2, -N)} , \]

\[ R_{N+1}(t) = H(t) \bigg|_{(N+2, 0, -N-1)} , \]

and we seek to express the first in terms of the second. We note that

\[ H(t) \bigg|_{(N+3, -2, -N)} = T_1 T_2^{-1} H(t) \bigg|_{(N+2, 0, -N-1)} , \]

\[ = 2t + H + \frac{2(N+2)}{f_0} - \frac{2(N+1)}{f_0} \]

\[ = 2t + \frac{2(N+2)}{f_0} - \frac{2(N+1)}{f_0} , \]

where the right-hand side is evaluated at \((N + 2, 0, -N - 1)\). We recognise in this expression the factors of

\[ H \bigg|_{(N+2, 0, -N-1)} = f_1 \frac{f_2}{f_0} , \]

\[ H' \bigg|_{(N+2, 0, -N-1)} = f_1 f_2 , \]

so that it can be simplified to

\[ H(t) \bigg|_{(N+3, -2, -N)} = 2t + H + \frac{H'}{H} \bigg|_{(N+2, 0, -N-1)} . \]

The result then follows upon making the appropriate identifications. \( \square \)

One can verify that this transformation \( T_1 T_2^{-1} \) (and its inverse) is the only nontrivial one which can map the Hamiltonian \( H \) into a rational function of \( H \) and \( H' \). We have

\[ H \bigg|_{(\alpha_0+1, \alpha_1-2, \alpha_2+1)} = T_1 T_2^{-1} H \bigg|_{(\alpha_0, \alpha_1, \alpha_2)} = 2t + H + \frac{(1-\alpha_1)H'}{H + \alpha_1 f_2} , \]

and this is rational if \( \alpha_1 = 1 \) (trivial case) or if \( \alpha_1 = 0 \) (this case). All other transformations are algebraic functions of \( H \) and \( H' \).

4.2. Calculation of \( F_N(s; a) \). Turning our attention to \( F_N \), we can make the following identifications.

**Proposition 24.** The average of the power of the characteristic polynomial is given by the \( N^{th} \) member of the \( \tau \)-function sequence generated by the shift operator \( T_3^{-1} \) from the initial parameters \((1 + a, -a, 0)\),

\[ F_N(\lambda; a) = \overline{\tau}_3[N](\lambda; -a) , \]

with the normalization of \( \overline{\tau}_3^{(N)} \) chosen so that

\[ F_N(\lambda; a) \sim \lambda^{Na} \quad \text{as} \; \lambda \to \infty , \]

(Eq. \([4.13])\). The logarithmic derivative of the average is related to the Hamiltonian by

\[ V_N(t; a) = H \bigg|_{(1+a+N, -a, -N)} , \]

(Eq. \([4.10])\).
Proof. We see from (1.12) that

$$F_N(\lambda; a) = \frac{1}{C} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \prod_{j=1}^{N} e^{-x_j^2} (\lambda - x_j)^a \prod_{1 \leq j < k \leq N} (x_k - x_j)^2,$$  \hspace{1cm} (4.31)

where the normalization is such that (1.29) is satisfied. Proceeding as in the derivation of (4.6) and (4.13) we see that this can be written in terms of determinants according to

$$F_N(\lambda; a) = \frac{1}{C} \det \left[ \int_{-\infty}^{\infty} (\lambda - x)^{a+j+k} e^{-x^2} \, dx \right]_{j,k=0,\ldots,N-1}. \hspace{1cm} (4.32)$$

This is precisely the determinant occurring in (2.52) so we have (1.28). Recalling (2.30), we thus have

$$F_N(\lambda; a) = F_N(\lambda_0; a) \exp \left( \int_{\lambda_0}^{\lambda} V_N(t; a) \, dt \right), \hspace{1cm} (4.33)$$

where $V_N(t; a)$ is given in terms of $H$ with $(\alpha_0, \alpha_1, \alpha_2)$ equal to $(1 + N + a, -a, -N)$ as in the formula (4.10) relating $U_N(t; a)$ to $H$. \hfill $\square$

Thus $V_N(t; a)$, like $U_N(t; a)$ (recall (4.15)), satisfies the nonlinear equation

$$(V_N'')^2 - 4(tV_N' - V_N)^2 + 4V_N'(V_N - 2a)(V_N + 2N) = 0. \hspace{1cm} (4.34)$$

The asymptotic behaviour (4.29) together with (4.33) implies (4.34) is to be solved subject to the boundary condition

$$V_N(t; a) \sim \frac{Na}{t} \left(1 + O(1/t)\right) \quad \text{as} \quad t \to \infty. \hspace{1cm} (4.35)$$

Apart from the quantity $R(t)$, which we know decays as a Gaussian, this boundary condition is the same as (1.14). Thus $U_N(t; a)$ and $V_N(t; a)$ satisfy the same differential equation, and up to a term which decays as a Gaussian, the same boundary condition as $t \to \infty$. However the $t \to -\infty$ behaviours are very different: for $U_N(t; a)$ it is given by (4.18), while for $V_N(t; a)$ it is (up to a possible phase) again given by (4.35). In addition $V_N(t; a)$ satisfies the $N$-difference equation (1.19) and its $a$-difference analogue but with the appropriate boundary conditions.

It was noted in the Introduction that for $(N+1) \times (N+1)$ dimensional GUE matrices the density, $\rho(\lambda) \big|_{N \to N+1}$ say, is proportional to $e^{-\lambda^2} F_N(\lambda; 2)$. Hence, analogous to (4.28), we see from (4.33) that

$$\rho(\lambda) \big|_{N \to N+1} = \rho(\lambda_0) \big|_{N \to N+1} \exp \left( \int_{\lambda_0}^{\lambda} [-2t + V_N(t; 2)] \, dt \right). \hspace{1cm} (4.36)$$

On the other hand, we know $\rho(\lambda) \big|_{N \to N+1}$ is proportional to the $2 \times 2$ determinants of (1.17). In fact $F_N(\lambda; a)$, for general $a \in \mathbb{Z}_{>0}$ can be written as an $a \times a$ determinant, a fact which can be understood in the present setting by considering the $\tau$-function sequence (2.64).

**Proposition 25.** The average of the powers of the characteristic polynomial $F_N(\lambda; a)$ obey the duality relation

$$\frac{F_N(\lambda; a)}{F_N(\lambda_0; a)} = \frac{F_a(i\lambda; N)}{F_a(i\lambda_0; N)}, \hspace{1cm} (4.37)$$

for all $a, N \in \mathbb{Z}$. 
Proof. First, note that by reversing the steps which led to (4.6) and recalling (4.31) the determinant of (2.64) which specifies \( \bar{\tau}_1 [n] \) can be written as a multiple integral to give
\[
\bar{\tau}_1 [n] (\lambda; \alpha_2) = CF_n (i\lambda; -\alpha_2),
\]
(4.38)
where \(|C| = 1\). On the other hand, we see from (2.13) that the Hamiltonian corresponding to \( \bar{\tau}_1 [n] \),
\[
H (\alpha_0, \alpha_1, \alpha_2) = (1 - \alpha_2 + n, -n, \alpha_2),
\]
(4.39)
satisfies the differential equation (4.15) with
\[
N = -\alpha_2, \quad a = n.
\]
(4.40)
(Note that for the latter identification to be possible, we require \( a \in \mathbb{Z}_{\geq 0} \).) Furthermore, from (4.38) and (4.29), we have that
\[
\frac{d}{dt} \log \bar{\tau}_1 [n] (t; \alpha_2) \sim t \to \infty - n\alpha_2 t,
\]
(4.41)
which with the substitutions (4.40) is identical to the boundary condition of (4.35). It follows from these facts that
\[
F_n (i\lambda; N) = F_n (i\lambda_0; N) \exp \left( \int_{\lambda_0}^{\lambda} V_N (t; a) \, dt \right).
\]
(4.42)
Comparison with (4.33) then yields (4.37). \( \square \)

From the definition (4.31) this identity implies the integral identity
\[
\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \prod_{j=1}^{N} e^{-x_j^2} (\lambda - x_j)^a \prod_{1 \leq j < k \leq N} (x_k - x_j)^2
\]
\[
= C \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_a \prod_{j=1}^{a} e^{-x_j^2} (\lambda - ix_j)^N \prod_{1 \leq j < k \leq a} (x_k - x_j)^2,
\]
(4.43)
and from (4.32) gives the determinant identity
\[
\det \left[ \int_{-\infty}^{\infty} (\lambda - x)^a x^{j+k} e^{-x^2} \, dx \right]_{j,k=0,\ldots,N-1} = C \det \left[ \int_{-\infty}^{\infty} (\lambda - ix)^N x^{j+k} e^{-x^2} \, dx \right]_{j,k=0,\ldots,a-1}.
\]
(4.44)
The integral identity (4.43) has been derived earlier in the context of a theory of generalized Hermite polynomials based on symmetric Jack polynomials \( [4] \), and in fact can be generalized so that on the left-hand side the exponent 2 in the product of differences is replaced by \( 2c \) and \( x_j^2 \to cx_j^2 \) in the Gaussian, while on the right-hand side this same exponent is replaced by \( 2/c \). Regarding the determinant identity, noting that the right-hand side is proportional to
\[
\det \left[ \int_{-\infty}^{\infty} (\lambda - ix)^{N+j+k} e^{-x^2} \, dx \right]_{j,k=0,\ldots,a-1} = C \det \left[ H_{N+j+k} (\lambda) \right]_{j,k=0,\ldots,a-1},
\]
(4.45)
this gives a determinant formula for \( F_N (\lambda; a) \), equivalent to that given by Brézin and Hikami \( [4] \). Determinants with a Hankel structure constructed with orthogonal polynomial elements are termed Turánians and their positivity and other properties such as relations with novel Wronskians have been investigated extensively, and reviewed in Karlin and Szegő \( [17] \). Explicit evaluations of Turánians of the Hermite polynomials in terms of the Barnes \( G \)-function where the initial degree of the polynomial is zero \( (N = 0) \) have been given by Radoux \( [32] \).
4.3. $U_N(t; a)$ and $V_N(t; a)$ as Painlevé transcendents. The formula (4.10) relates $U_N(t; a)$ to the Hamiltonian (2.13). Substituting the appropriate values of $\alpha_1$ and $\alpha_2$ in the first equality of (2.13) and recalling (2.16) shows

$$U_N(t; a) = \frac{1}{8q} (q')^2 - \frac{1}{8} (q + 2t)(q^2 + 2tq - 4a) - \frac{a^2}{2q} + Nq,$$  (4.46)

where $q$ satisfies the PIV equation (2.1) with

$$\alpha = 2N + 1 + a, \quad \beta = -2a^2.$$  (4.47)

In the case $a = 0$ the functional expression (4.46) agrees with that presented earlier [35, 38], although the transcendent $q$ in the earlier works is the PIV transcendent with $\alpha = 2N - 1, \beta = 0$ rather than $\alpha = 2N + 1, \beta = 0$ as given by (4.47). In fact it follows from the work [8] that in general Eq. (2.18) has more than one expression in terms of Painlevé transcendents. In the case $\alpha_1 = N, \alpha_2 = 0$ of this equation the results of [8] give the functional expression implied (4.46) with $q$ the PIV transcendent specified with the parameters $\beta = 0$ and either $\alpha = 2N + 1$ or $\alpha = 2N - 1$ thus reconciling (4.46) in the case $a = 0$ with the results of [35, 38]. We remark that the theory of [8] gives distinct functional forms for the derivative,

$$U'_N(t; 0) = -\frac{1}{2} q' - \frac{1}{2} q^2 - tq,$$  (4.48)

in the two case $\alpha = 2N + \epsilon$ ($\epsilon = \pm 1$).

Comparison of the formulas (4.30) and (4.10) shows $V_N(t; a)$ is given by the same Hamiltonian as $U_N(t; a)$. Thus (4.46) remains true with the function $U_N(t; a)$ replaced by $V_N(t; a)$ on the left-hand side.

5. Edge Scaling in the GUE

5.1. Calculation of $E_{\text{soft}}(s)$ and $\tilde{E}_{\text{soft}}(s; a)$. To leading order the support of the eigenvalue density for $N \times N$ GUE matrices is the interval ($-\sqrt{2N}, \sqrt{2N}$). To study distributions in the neighbourhood of the largest eigenvalue one shifts the origin to the edge at $\sqrt{2N}$ and then scales the coordinate so as to make the spacings of order unity in the $N \to \infty$ limit. This is achieved by the mapping [10]

$$\lambda \mapsto \sqrt{2N} + \frac{\lambda}{\sqrt{2N^{1/6}}}.$$  (5.1)

Suppose we make this replacement (in the $s$-variable) in the probability $E_N(0; (s, \infty))$ as specified by (4.7). Then with

$$E_{\text{soft}}(s) := \lim_{N \to \infty} E_N \left( 0; \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \right),$$  (5.2)

(because the eigenvalue density is not strictly zero outside the leading order of its support the edge is referred to as a soft edge) we see that

$$E_{\text{soft}}(s) = \exp \left( - \int_s^\infty r(t) \, dt \right),$$  (5.3)

where

$$r(t) = \lim_{N \to \infty} \frac{1}{\sqrt{2N^{1/6}}} R \left( \sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}} \right).$$  (5.4)
Furthermore, it follows from changing variables $s \mapsto \sqrt{2N} + s/\sqrt{2N^{1/6}}$ in \[4.13\], replacing $R(\sqrt{2N} + s/\sqrt{2N^{1/6}})$ by $\sqrt{2N^{1/6}} r(s)$ and taking the limit $N \to \infty$ that $r(s)$ satisfies the differential equation
\[
(r'')^2 + 4r' (r')^2 - sr' + r = 0,
\]
a result first obtained by Tracy and Widom \[36\]. Equation \[5.3\] is a particular case of the Jimbo-Miwa-Okamoto $\sigma$ form of the Painlevé II equation. We will find that the edge scaling is essentially the second coalescence limit of the PIV system to the PII as discussed in Subsect. 3.5.

**Proposition 26.** Define the scaling limit of the quantity $\tilde{E}_N(s; a)$ by
\[
\tilde{E}^{\text{soft}}(s; a) := \lim_{N \to \infty} \left( Ce^{-as^2/2} \tilde{E}_N(s; a) \right) \Big|_{s \to \sqrt{2N} + s/\sqrt{2N^{1/6}}},
\]
Then
\[
\tilde{E}^{\text{soft}}(s; a) = \tilde{E}^{\text{soft}}(s_0; a) \exp \left( \int_{s_0}^s u(t; a) \, dt \right),
\]
where
\[
u(t; a) = \lim_{N \to \infty} \frac{1}{\sqrt{2N^{1/6}}} \left( -at + U_N(t; a) \right) \Big|_{t \to \sqrt{2N^{1/6}} + t/\sqrt{2N^{1/6}}},
\]
\[
-u(t; a) = -2^{1/3} H(-2^{1/3} t) \big|_{t=(1-a, a)},
\]
with $H(t)$ is given by \[3.4\]. The function $u(s; a)$ satisfies a second order second degree differential equation of the general Jimbo-Miwa-Okamoto $\sigma$ form of the Painlevé II equation
\[
(u'')^2 + 4u' (u')^2 - su' + u - a^2 = 0,
\]
subject to the boundary condition
\[
u(s; a) \sim 1/4s^2 + \frac{4a^2-1}{8s} + \frac{(4a^2-1)(4a^2-9)}{64s^4} + \ldots.
\]
The function $u(s; a)$ also satisfies the third order difference equation, related to the alternate discrete Painlevé $a$-dPI equation, equation
\[
\frac{a}{u(s; a+1) - u(s; a-1)} + \frac{a+1}{u(s; a+2) - u(s; a)} = s - (u(s; a+1) - u(s; a))^2.
\]

**Proof.** Unlike the probability $E_N(0; (s, \infty))$, we do not expect the soft edge scaling limit of the quantities $\tilde{E}_N(s; a)$ as specified by \[4.13\] to be well defined. For example, in the case $a = 2$, it is the combination $e^{-s^2} \tilde{E}_N(s; 2)$ which is proportional to $p_{\max}(s)$, and thus which should have a well defined scaling limit. This suggests that for general $a$ we consider the scaling limit of
\[
Ce^{-as^2/2} \tilde{E}_N(s; a).
\]
According to \[4.13\] we have
\[
e^{-as^2/2} \tilde{E}_N(s; a) = e^{-as_0^2/2} \tilde{E}_N(s_0; a) \exp \left( \int_{s_0}^s [-at + U_N(t; a)] \, dt \right).
\]
Relation (5.18) follows from the definitions of $\tilde{E}_{\text{soft}}(s; a)$ and $u(t; a)$ and (5.14). The scaling in (5.8) is identical to the coalescence of the PIV system to the PII defined in Prop. 20, with the identifications $\epsilon = (2N)^{-1/6}$, $\alpha_{(IV)} = -a$ and (4.10) for the relationship of the PIV Hamiltonian and $U_N(t; a)$, and the scale changes $t_{\Pi} = -2^{1/3}t$ and $H_{\Pi}(t_{\Pi}) = -2^{-1/3}u(t; a)$. Proceeding as in the derivation of (5.5) we find from (4.15), or from (3.17) using the change of scale (5.9), that $u$ satisfies (5.10).

To formulate the boundary condition for $u(t; a)$, we first recall [36] that the $s \to -\infty$ boundary condition of $r(s)$ in (5.5) is given by

$$r(s) \sim s \to -\infty \frac{1}{4s^2} - \frac{1}{8s} + \frac{9}{64s^4} + O\left(\frac{1}{s^7}\right),$$

(5.15)

and this corresponds to the asymptotic behaviour [36]

$$E_{\text{soft}}(s) \sim s \to -\infty \exp\left(\frac{s^3}{12} + \frac{1}{8}\log(-s)\right).$$

(5.16)

Also, we know that $\tilde{E}_{\text{soft}}(s; 2)$ is proportional to the derivative of $E_{\text{soft}}(s)$, which implies

$$\tilde{E}_{\text{soft}}(s; 2) \sim s \to -\infty \exp\left(\frac{s^3}{12} + C\log(-s)\right).$$

(5.17)

This suggests that for general $a$ we seek a solution of (5.10) with the $s \to -\infty$ boundary condition

$$u(s; a) \sim s \to -\infty \frac{1}{4} s^2 + \sum_{j=1}^{\infty} \frac{c_j}{s^j},$$

(5.18)

Substitution shows that in fact (5.10) has a unique solution of this form, and furthermore

$$c_1 = \frac{4a^2 - 1}{8}, \quad c_2 = c_3 = 0, \quad c_4 = \frac{(4a^2 - 1)(4a^2 - 9)}{64}, \quad \ldots.$$  

(5.19)

The third order difference equation is just (3.37) with the scale change (5.9).

We see from (5.11) that the asymptotic expansion of $u(s; a)$ terminates for $a$ equal to half an odd integer. Recalling (3.9) this is the case of $a_1$ half an odd integer of the PII Hamiltonian (3.4). From the text about (3.20) we know that this is precisely the parameter value for which the PII Hamiltonian can be expressed as a rational function of $t$.

The fact that $e^{-s^2} E_N(s; 2)$ is proportional to $p_{\text{max}}(s)$ implies that the corresponding quantities in the scaled limit, $\tilde{E}_{\text{soft}}(s; 2)$ and $p_{\text{soft}}(s)$ are proportional. Hence

$$p_{\text{soft}}(s) = p_{\text{soft}}(s_0) \exp\left(\int_{s_0}^{s} u(t; 2) \, dt\right).$$

(5.20)

The relation

$$p_{\text{soft}}(s) = \frac{d}{ds} E_{\text{soft}}(s)$$

(Eq. (5.21)) with $E_{\text{soft}}(s)$ specified by (5.3) then implies an identity between transcendentals analogous to that of Prop. 29.

**Proposition 27.** The quantity $u(t; 2)$ of (5.20) and the quantity $r(t)$ of (5.3) are related by

$$u(t; 2) = \frac{d}{dt} \log r(t) + r(t).$$

(5.22)
Proof. With
\[ H(t)\bigg|_{\alpha_1=n} := H[n], \] (5.23)
it follows from (5.3) and the substitution \( r(t) = u(t; 0) \) that (5.22) is equivalent to
\[ H[2] = \frac{d}{dt} \log H[0] + H[0]. \] (5.24)
But from the analogue of the first equality in (3.21), together with (3.15) and Table 5,
\[ H[2] = T_2^2 H[0] = H[0] + q[0] + \frac{\alpha_1}{f_1[0]} + T_2 \left( q[0] + \frac{\alpha_1}{f_1[0]} \right) \bigg|_{\alpha_1=0} \]
\[ = H[0] + \frac{1}{2q^2[0] + t - p[0]}. \] Furthermore the first equality in (3.18) together with (3.4) in the case \( \alpha_1 = 0 \) give
\[ \frac{1}{2q^2[0] + t - p[0]} = \frac{d}{dt} \log H[0]. \] (5.25)
\[
\qed
\]
Note. According to (5.9) we therefore have
\[ -2^{-1/3}u(-2^{-1/3}t; a) = \frac{1}{2} \left[ q'(t, a-\frac{1}{2}) \right]^2 - \frac{1}{2} \left[ q^2(t, a-\frac{1}{2}) + \frac{t}{2} \right]^2 - aq(t, a-\frac{1}{2}), \] (5.26)
where \( q = q(t, a) \) satisfies the PH equation (3.1). Also the first member of (3.18), along with (3.5) and (3.6), gives
\[ -2^{-1/3} \frac{d}{dt} u(-2^{-1/3}t; a) = -\frac{1}{2} \left[ q'(t, a-\frac{1}{2}) + q^2(t, a-\frac{1}{2}) + \frac{t}{2} \right]. \] (5.27)
In the special case \( a = 0 \) we see from (3.7) that (5.27) simplifies to read
\[ u'(t; 0) = -q^2(t, 0) \] (5.28)
while (5.26) simplifies to read
\[ u(t; 0) = [q'(t, 0)]^2 - tq^2(t, 0) - q^4(t, 0). \] (5.29)
The results (5.28) and (5.29), deduced in a different way, can be found in [36].

5.2. Calculation of \( F_{\text{soft}}(\lambda; a) \). Let us next consider the scaled quantity
\[ F_{\text{soft}}(\lambda; a) := \lim_{N \to \infty} \left( C e^{-a\lambda^2/2} F_N(\lambda; a) \right) \bigg|_{\lambda \to \sqrt{2N} + \lambda + \sqrt{2N}^{-1/6}}, \] (5.30)
where \( F_N(\lambda; a) \) is specified by (4.31). Because of the analogy with \( \tilde{E}_{\text{soft}}(s; a) \), which follows from the identical structure of (4.14), (4.17) and (4.33), (4.34), analogous to (5.10) we have
\[ F_{\text{soft}}(\lambda; a) := F_{\text{soft}}(\lambda_0; a) \exp \left( \int_{\lambda_0}^{\lambda} v(t; a) \, dt \right), \] (5.31)
where
\[ v(t; a) = \lim_{N \to \infty} \frac{1}{\sqrt{2N}^{1/6}} \left( -at + V_N(t; a) \right) \bigg|_{t \to \sqrt{2N} + t + \sqrt{2N}^{1/6}} \] (5.32)
satisfies the differential equation (5.10) and the difference equation (5.12). The only difference between the logarithmic derivatives \( v(t; a) \) and \( u(t; a) \) is the boundary condition.
Proposition 28. The scaled averages of the powers of the characteristic polynomial $F^{\text{soft}}(\lambda; a)$ for $a \in \mathbb{Z}_{\geq 0}$ have the determinantal form

$$F^{\text{soft}}(\lambda; a) = (-1)^a(a-1)/2 \det \left[ \frac{d^{j+k}}{d\lambda^{j+k}} \Lambda_i(\lambda) \right]_{j,k=0,\ldots,a-1}, \quad (5.33)$$

which was shown by Okamoto [23] to be a $\tau$-function sequence for Painlevé II (recall Prop. 18). Furthermore this scaled average has a multiple integral representation of the Kontsevich form [18],

$$F^{\text{soft}}(\lambda; a) = (-1)^a(a-1)/2 (-2\pi i)^a \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \prod_{j=1}^{a} e^{v_j/3-\lambda v_j} \prod_{1 \leq j < k \leq a} (v_k - v_j)^2. \quad (5.34)$$

The logarithmic derivative $v(t; a)$ has the asymptotic expansion

$$v(t; a) \sim -at^{1/2} - \frac{a^2(4a^2+1)}{32t^{5/2}}. \quad (5.35)$$

Proof. For positive integer $a$ we can determine the $\lambda \to \infty$ behaviour of $F^{\text{soft}}(\lambda; a)$, and thus the corresponding behaviour of $v(t; a)$, by making use of the scaled form of the right-hand side of (4.31). To determine this scaled form, we first require the explicit values of the constants in (4.31), (4.44) and (5.30). Let us denote these constants by $C_1, C_2, C_3$ respectively. Then from (4.31) and (4.29) we read off that

$$C_1 := c(N) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \prod_{j=1}^{N} e^{-x_j^2} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 = 2^{-N^2/2}(2\pi)^{N/2}G(N+2), \quad (5.36)$$

where $G(x)$ denotes the Barnes $G$-function, characterised for $x$ a positive integer by the functional property $G(x+1) = \Gamma(x)G(x)$ and the initial value $G(1) = 1$. The integral evaluation in (5.36) can be derived by making use of the Vandermonde determinant identity (4.4) written in terms of Hermite polynomials.

The proportionality constant $C$ in (4.44) is the same as that in (4.43) and thus from (5.36) given by

$$C_2 = \frac{c(N)}{c(a)}. \quad (5.37)$$

Finally, we seek the value of the constant in (5.30). We know that in the case $a = 2$, $e^{-a^2x^2/2}F_N(\lambda; a)$ is proportional to the eigenvalue density for $(N+1) \times (N+1)$ dimensional GUE matrices. Specifically

$$\rho(\lambda)_{N \to N+1} = (N+1) \frac{c(N)}{c(N+1)} e^{-\lambda^2} F_N(\lambda; a). \quad (5.38)$$

Since it is the combination $\rho(\lambda) d\lambda$ which has a scaled limit, it follows that in the case $a = 2$, $C_3 = (N+1)c(N)/(c(N+1)\sqrt{2N^{1/6}})$. This suggests that for general $a \in \mathbb{Z}_{\geq 0}$ we should choose

$$C_3 = c_a \frac{\Gamma(N+1+a/2)}{\Gamma(N+1)} \frac{c(N)}{c(N+a/2)} \left( \frac{1}{\sqrt{2N^{1/6}}} \right)^p, \quad (5.39)$$

where $p$ is a power to be determined and $c_a$ depends only on $a$ ($p = 1, c_a = 1$ for $a = 2$).

Substituting (4.31), (5.37), (5.39) in (4.31), (4.44), (5.30) respectively we see that for $a \in \mathbb{Z}_{>0}$,

$$F^{\text{soft}}(\lambda; a) = c_a \lim_{N \to \infty} \frac{\Gamma(N+1+a/2)}{\Gamma(N+1)} \frac{c(N)}{c(N+a/2)c(a)} \left( \frac{1}{\sqrt{2N^{1/6}}} \right)^p e^{-a^2/2} \times \det \left[ \int_{-\infty}^{\infty} (\lambda - ix)^{j+k} e^{-x^2} dx \right]_{\lambda \to \sqrt{2N^{1/6}}+\sqrt{2N^{1/6}}} j,k=0,\ldots,a-1. \quad (5.40)$$
But analogous to the equality in (5.41) we have
\[
\det \left[ \int_{-\infty}^{\infty} (\lambda - ix)^{N+j+k} e^{-x^2} \, dx \right]_{j,k=0,\ldots,a-1} = (-1)^{a(a-1)/2} \det \left[ \int_{-\infty}^{\infty} (\lambda - ix)^{N+j+k} e^{-x^2} \, dx \right]_{j,k=0,\ldots,a-1}.
\]
(5.41)

This can be further rewritten by noting that analogous to (2.60),
\[
\int_{-\infty}^{\infty} (\lambda - ix)^{N+j+k} e^{-x^2} \, dx = (-2)^{-(j+k)} \lambda^{2} \frac{d^{j+k}}{d\lambda^{j+k}} \left( e^{-\lambda^2} \int_{-\infty}^{\infty} (\lambda - ix)^{N} e^{-x^2} \, dx \right)
\]
\[
= (-2)^{-(j+k)} \lambda^{2} 2^{-N} \sqrt{\pi} \frac{d^{j+k}}{d\lambda^{j+k}} \left( e^{-\lambda^{2}} H_{N}(\lambda) \right).
\]
(5.42)

Making use of the asymptotic expansion for the Barnes G-function [3]
\[
\log G(x + 1) \sim \frac{x^2}{2} \log x - \frac{3}{4} x^2 + \frac{x}{2} \log 2\pi - \frac{1}{12} \log x + O(1),
\]
(5.43)
and the Plancherel-Rotach asymptotic expansion of the Hermite polynomials [33]
\[
\exp(-x^2/2) H_{N}(x) = \pi^{-3/4} 2^{N/2+1/4} (N!)^{1/2} N^{-1/12} \{ \pi \operatorname{Ai}(-t/3^{1/3}) + O(N^{-2/3}) \},
\]
(5.44)
where \( x = (2N)^{1/2} - 2^{-1/3} t^{-1/3} N^{-1/6} t \) and with \( \operatorname{Ai}(x) \) denoting the Airy function, we see from Eqs. (5.44), (5.41) and (5.42) that with \( p = a/2 \) in (5.39) and appropriate \( c_{a} \), the determinantal representation (5.33) holds. Furthermore, in the case \( a = 2 \) we read off the functional form
\[
\left( \operatorname{Ai}'(x) \right)^{2} - \operatorname{Ai}(x) \operatorname{Ai}''(x),
\]
(5.45)
which is the known expression [10] for the scaled soft edge density in the GUE. Another point of interest, which follows from the integral formula
\[
\operatorname{Ai}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp \left( \frac{1}{3} v^3 - xv \right) \, dv,
\]
(5.46)
is that (5.33) can be written
\[
F_{\text{soft}}(\lambda; a) = \frac{(-1)^{a(a-1)/2}}{(-2\pi i)^{a}} \det \left[ \int_{-\infty}^{\infty} \exp \left( \frac{1}{3} v^3 - \lambda v \right) v^{j+k} \, dv \right]_{j,k=0,\ldots,a-1}.
\]
(5.47)
Thus, reversing the reasoning leading from (1.4) to (1.6) we have the multiple integral representation (5.34) for \( F_{\text{soft}}(\lambda; a) \), which is an example of the class of integrals studied by Kontsevich [18].

Consider now the asymptotic form of (5.33). In the case \( a = 1 \) this is just the Airy function, which has the known \( x \to \infty \) asymptotic form (see e.g. [30], p. 116)
\[
\operatorname{Ai}(x) \sim \frac{e^{-\xi}}{2\pi^{1/2}x^{1/4}} \sum_{k=0}^{\infty} (-1)^{k} u_{k} \frac{\xi^{k}}{\xi^{k}},
\]
(5.48)
where \( \xi := \frac{2}{3} x^{3/2} \), \( u_{0} = 1 \) and
\[
u_{k} = \frac{2k+1)(2k+3) \cdots (6k-1)}{(216)^{k} k!}, \quad k \geq 1.
\]
(5.49)
Application of the $\tau$-Function Theory of PIV, PII to the GUE

It follows from this and (5.33) that for general $a \in \mathbb{Z}_{>0}$,

$$\log F_{\text{soft}}(\lambda; a) \sim -\frac{2a}{3} \lambda^{3/2} + C \log \lambda + c_0 + \sum_{j=1}^{\infty} \frac{\tilde{c}_j}{\lambda^{3j/2}}.$$  \hspace{1cm} (5.50)

which in combination with (5.31) implies that we must seek a solution of (5.10) (with $u$ replaced by $v$) subject to the boundary condition

$$v(t; a) \sim -at^{1/2} + \frac{C}{t} + \sum_{j=1}^{\infty} \frac{c_j}{t^{3j/2+1}}.$$  \hspace{1cm} (5.51)

Substitution of (5.51) in (5.10) shows there is a unique solution of this form, with

$$C = -\frac{a^2}{4}, \quad c_1 = \frac{a(1+4a^2)}{32}, \quad \ldots$$ \hspace{1cm} (5.52)

given by (5.35). \qed

6. Conclusions - A Programme

We have applied the Okamoto $\tau$-function theory of PIV and PII to the computation of $\tilde{E}_N(s; a)$ and $F_N(s; a)$ for the GUE and its scaled soft edge limit. As noted in the Introduction, the Okamoto $\tau$-function theory applies equally as well to the computation of $\tilde{E}_N(s; a)$ and $F_N(s; a)$ for all matrix ensembles with a unitary symmetry and classical weight functions (1.6). Thus we expect to be able to compute $\tilde{E}_N(s; a)$ and $F_N(s; a)$ in the cases of the Laguerre, Jacobi and Cauchy ensembles (special cases of $F_N(s; a)$ have been evaluated in terms of Painlevé transcendents for the Laguerre ensemble [34], and for the Jacobi ensemble [3]). In future studies we will undertake this task by following the programme used here for the GUE, the main steps of which can be itemised as follows:

- From the definitions of the gap probability $E_N(0; I)$, $I$ a single interval including the boundary of the eigenvalue support, and $\tilde{E}_N(s; a), F_N(s; a)$ as $N$-dimensional multidimensional integrals they can be converted into $N \times N$ determinants analogous to (4.6), (4.13) and (4.32) respectively.
- Using an identity analogous to (2.60), the determinants can be put into the double Wronskian form (2.58), with $d/dt$ replaced by

$$t \frac{d}{dt}, \quad t(1-t) \frac{d}{dt}$$  \hspace{1cm} (6.1)

in the Laguerre and Jacobi ensembles respectively.
- The Okamoto $\tau$-function theory of PV and PVI [23, 27] gives these same determinants as $\tau$-function sequences, in which the initial members are $\tau[0] = 1$, and $\tau[1]$ the solution of the particular classical equation associated with relevant Painlevé transcendent when the parameter sequences begin on a wall of the Weyl chamber in the affine space of parameters. The classical solutions, and their polynomial specialisations, are noted for each of the Painlevé transcendents in Table 8.
Table 8. Classical solutions of the Painlevé transcendents.

| PJ | Classical Solution | Classical Orthogonal Polynomial |
|----|-------------------|-------------------------------|
| PI | -                 | -                            |
| PII | Airy             | -                            |
| PIII | Bessel          | -                            |
| PIV | Hermite-Weber  | Hermite                      |
| PV  | Confluent Hypergeometric | Laguerre                  |
| PVI | Gauß Hypergeometric | Jacobi                      |

– The logarithmic derivatives (with \(d/dt\) replaced by (6.1) as appropriate) \(R_N(s), U_N(s), V_N(s)\) coincide with the Hamiltonians in the Painlevé theory and as such satisfy certain second order second degree ODEs of the Painlevé type.

– The \(\tau\)-function sequence \(\{\tau_0[N](t;a)\}_{N \geq 0}\), say corresponding to \(F_N(s;a)\), is simply related to another \(\tau\)-function sequence \(\{\tau_1[a](t;N)\}_{a \geq 0}\). Both \(\tau\)-functions relate to the same Hamiltonian but result from the action of different shift operators. Because the shifts are commutative one has

\[
\frac{\tau_0[N](t;a)}{\tau_0[N](t_0;a)} = \frac{\tau_1[a](t;N)}{\tau_1[a](t_0;N)},
\]

(6.2)

Identities of this type for the Laguerre and Jacobi ensembles, written as multiple integrals, are already known from [4].

– For all the independent shift operators and sequences of \(q[n], p[n], H[n], \tau[n]\) there exist difference equations generated by the Bäcklund transformations of these shifts. It has been conjectured that all the difference equations arising in this way are discrete Painlevé equations satisfying integrability criteria such as singularity confinement analogous to the Painlevé criteria.

– In the appropriate edge scaling limit, the analogues of \(r(s), u(s), v(s)\) are Hamiltonian functions for PII or PIII, and satisfy the corresponding second order second degree equation.

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