On some probabilistic properties of periodic \textit{GARCH} processes

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\textbf{Abstract}

This paper examines some probabilistic properties of the class of periodic \textit{GARCH} processes (\textit{PGARCH}) which feature periodicity in conditional heteroskedasticity. In these models, the parameters are allowed to switch between different regimes, so that their structure shares many properties with periodic \textit{ARMA} process (\textit{PARMA}). We examine the strict and second order periodic stationarities, the existence of higher-order moments, the covariance structure, the geometric ergodicity and \(\beta\)-mixing of the \textit{PGARCH}(\(p,q\)) process under general and tractable assumptions. Some examples are proposed to illustrate the various concepts.

\textbf{Keywords.} Periodic \textit{GARCH} Processes; Periodic Stationarity; Geometric Ergodicity; Higher-Order Moments.

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\section{Introduction}

Consider a periodic \textit{GARCH}(\(p_1,...,p_s,q_1,...,q_s\)) process \((x_t)_{t\in\mathbb{Z}}\) with period \(s > 0\) and orders \(p = (p_1,...,p_s)\) and \(q = (q_1,...,q_s)\), defined on some probability space \((\Omega,\mathcal{A},P)\) with the non-linear periodic difference equation:

\begin{equation}
\begin{aligned}
\forall t \in \mathbb{Z}: & \quad x_{st+v} = \varepsilon_{st+v} \sqrt{h_{st+v}} \\
& \quad h_{st+v} = \alpha_0(v) + \sum_{i=1}^{p_v} \alpha_i(v)x_{st+v-i}^2 + \sum_{j=1}^{q_v} \beta_j(v)h_{st+v-j}
\end{aligned}
\end{equation}

where \((\varepsilon_t)_{t\in\mathbb{Z}}\) is a sequence of independent and identically distributed (i.i.d.) random variables defined on the same probability space \((\Omega,\mathcal{A},P)\) such that \(E\{\varepsilon_t\} = E\{\varepsilon_t^3\} = 0\), and \(E\{\varepsilon_t^2\} = 1\) (these conditions are obviously satisfied if \((\varepsilon_t)_{t\in\mathbb{Z}}\) is Gaussian).

In the difference equation (1.1) \(x_{st+v}\) refers to \(x_t\) during the \(v - th\) ‘season’, \(1 \leq v \leq s\) of period \(t\), \(\alpha_0(v), \alpha_1(v), ..., \alpha_{p_v}(v)\) and \(\beta_1(v), ..., \beta_{q_v}(v)\) are the model coefficients at season \(v\) such that for all
$v = 1, \ldots, s$, $\alpha_0(v) > 0$, $\alpha_i(v) \geq 0$, $i = 1, \ldots, p_v$, and $\beta_j(v) \geq 0$, $j = 1, \ldots, q_v$. Moreover, we assume that $\varepsilon_k$ is independent of $x_t$ for $k > t$. We use the periodic notations $(x_{st+v})$, $(\varepsilon_{st+v})$, $(\alpha_i(v), 0 \leq i \leq p_v)$, and $(\beta_j(v), 1 \leq i \leq q_v)$ to emphasize the periodicity in the model. There is no loss of generality in taking $p_v$ and $q_v$ to be constant in $v$. If $p_v$ or $q_v$ change with $v$, one can set $p = \max_{1 \leq v \leq s} p_v$, $q = \max_{1 \leq v \leq s} q_v$ and take $\alpha_k(v) = 0$ for $p_v < k \leq p$ and $\beta_k(v) = 0$ for $q_v < k \leq q$, so in the sequel, we shall consider the periodic GARCH with constant orders $p$ and $q$. Since Bollerslev and Ghysels (1996), this type of non-linear models has become an appealing tool for investigating both volatility and distinct seasonal patterns, and has been applied in various disciplines such as finance and monetary economics (see e.g. Bollerslev and Ghysels, 1996 and Franses and Paap, 2000).

When we consider a periodic model as a data generating process, it is important to find conditions ensuring the (periodic) stationarity, ergodicity and the existence of higher moments for further statistical analysis. Various probabilistic properties of standard GARCH models have been studied extensively by many authors (see e.g., Chen and An, 1998; Bougerol and Picard, 1992a, 1992b and Carrasco and Chen, 2002 and the references therein). In the present paper, we focus on studying the fundamental probabilistic properties of the PGARCH process $(x_t)_{t \in \mathbb{Z}}$ generated by (1.1) so, in Section 2, we present a vectorial representation from which we derive some sufficient conditions for the strict stationarity. In Sections 3 and 4, necessary and sufficient conditions for the second order stationarity and the existence of higher order moments are given. Section 5 is devoted to covariance structure. In Section 6 we provide conditions under which strictly stationary solutions are exponential $\beta$-mixing with finite higher order moments. We conclude in Section 7.

Some notations are used throughout the paper: $I_{(k)}$ denotes the identity matrix of order $k$ and $O_{(k \times l)}$ denotes the matrix of order $k \times l$ whose elements are zeroes, for simplicity we set $O_{(k)} := O_{(k \times k)}$, $\rho(A)$ refers to the spectral radius of a square matrix $A$, i.e., the maximum eigenvalue of a matrix $A$ in absolute value, $Vec(A)$ is the usual column-stacking vector of the matrix $A$, $\|\|_p$ refers to the standard (Euclidean) norm in $\mathbb{R}^n$ or the uniform induced norm in the space $\mathcal{M}(n)$ of $n \times n$ matrices, $\otimes$ denotes the Kronecker product of matrices, and $A \otimes^m = A \otimes A \otimes \ldots \otimes A$ ($m$-times), for any integer $m \geq 1$. For any $p \geq 1$, $L^p = L^p(\Omega, \mathcal{A}, P)$ denotes the Hilbert space of random variables $X$ defined on the probability space $(\Omega, \mathcal{A}, P)$ such that $\|X\|_p = (E|X|^p)^{1/p} < \infty$. We also use the following property of matrix operation, $Vec(ABC') = (C \otimes A)Vec(B)$, where $'$ is the matrix transpose.
2 The Markovian representation and strict stationarity

Let \((x_t)_{t \in \mathbb{Z}}\) be a process conforming to the model \((1.1)\). Setting \(y_{st+v} = x_{st+v}^2\) and \(\eta_{st+v} = \varepsilon_{st+v}^2\), we obtain from \((1.1)\) the following representation

\[
y_{st+v} = \sum_{i=1}^{p} \alpha_i(v) \eta_{st+v} y_{st+v-i} + \sum_{j=1}^{q} \beta_j(v) \eta_{st+v-h_{st+v-j}} + \alpha_0(v) \eta_{st+v}. \tag{2.1}
\]

Equation \((2.1)\) is intractable when we want to examine the probabilistic structure of this representation. Instead, we will work with the corresponding state-space representation. Let \(d = p + q\) and define \(\eta_t = (\eta_{st+1}, \ldots, \eta_{st+s})'\), \(y_{st+v} = (y_{st+v}, \ldots, y_{st+v-p+1}, h_{st+v}, \ldots, h_{st+v-q+1})'\) and \(B_v(\eta_t) = (\alpha_0(v) \eta_{st+v}, 0, \ldots, 0, \alpha_0(v), 0, \ldots, 0)'\) as vectors in \(\mathbb{R}^s\), \(\mathbb{R}^d\) and \(\mathbb{R}^d\) respectively, and set

\[
\phi_v(\eta_t) = \begin{pmatrix} A_v(\eta_t) & B_v(\eta_t) \end{pmatrix}_{d \times d}
\]

where

\[
A_v(\eta_t) = \begin{pmatrix} \alpha_1(v) \eta_{st+v} & \alpha_2(v) \eta_{st+v} & \ldots & \alpha_p(v) \eta_{st+v} \\ 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}_{p \times p}
\]

and

\[
B_v(\eta_t) = \begin{pmatrix} \beta_1(v) \eta_{st+v} & \ldots & \beta_q(v) \eta_{st+v} \\ 0 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{p \times q}
\]

are \(p \times p\) and \(p \times q\) matrix valued polynomial functions of \(\eta_t\) and where

\[
A_v = \begin{pmatrix} \alpha_1(v) & \ldots & \alpha_p(v) \\ 0 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{q \times p}
\]

and

\[
B_v = \begin{pmatrix} \beta_1(v) & \beta_2(v) & \ldots & \beta_q(v) \\ 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}_{q \times q}
\]

this means that \(A_v(\varepsilon_t), B_v(\varepsilon_t)\) and \(B_v(\varepsilon_t)\) have entries and coordinates, respectively, which are polynomial functions of the coordinates of \(\varepsilon_t\). Using the notations above, Equation \((2.1)\) can now be written as

\[
y_{st+v} = \phi_v(\eta_t)y_{st+v-1} + B_v(\eta_t). \tag{2.2}
\]

with \(y_{st+v} = H' y_{st+v}\) where \(H' = (1, 0, \ldots, 0)_{1 \times d}\). Equation \((2.2)\), is the same as the defining equation for multivariate generalized periodic autoregressive process introduced recently by Franses and Paap (2000). However, since Gladychev (1961), with periodic time-varying coefficients, it is possible to embed seasons into a multivariate stationary process. More precisely \((\mathcal{G}_t)_{t \in \mathbb{Z}}\) where \(\mathcal{G}_t = (y_{st+1}', y_{st+s}')'\) is a generalized autoregressive process, i.e.,

\[
\mathcal{G}_t = A(\eta_t) \mathcal{G}_{t-1} + B(\eta_t). \tag{2.3}
\]
where $A(\eta)$ and $B(\eta)$ are defined by blocks as

\[
A(\eta) = \begin{pmatrix}
O(d) & \cdots & O(d) \\
O(d) & \cdots & O(d) \\
\vdots & \ddots & \vdots \\
O(d) & \cdots & O(d)
\end{pmatrix} \cdot \phi_1(\eta) \cdot \phi_2(\eta) \cdot \phi_3(\eta) \\
\vdots & \ddots & \vdots \\
O(d) & \cdots & O(d)
\end{pmatrix}_{ds \times ds}
\]
\[
B(\eta) = \begin{pmatrix}
B_1(\eta) \\
\phi_2(\eta) B_1(\eta) + B_2(\eta) \\
\vdots
\end{pmatrix}_{ds \times 1}
\]

where, as usual, empty products are set equal to 1.

In this section, we are interested in strictly stationary and causal solutions for \( \mathbf{1} \). The results of this section are based on theorems proved by Bougerol and Picard (1992a) for generalized autoregressive representation. Since \( \{\eta_t\}_{t \in \mathbb{Z}} \) is an i.i.d. process, \( \{A(\eta), B(\eta)\}_{t \in \mathbb{Z}} \) is a strictly stationary and ergodic process and since \( E \{\log^+ \|A(\eta_0)\|\} \leq E \|A(\eta_0)\| \) and \( E \{\log^+ \|B(\eta_0)\|\} \leq E \|B(\eta_0)\| \), then both \( E \{\log^+ \|A(\eta_t)\|\} \) and \( E \{\log^+ \|B(\eta_t)\|\} \) are finite where for any \( x > 0 \), \( \log^+ x = \max(\log x, 0) \).

**Theorem 1** [Strict stationary solution] Equation (2.3) has a unique strictly stationary and ergodic solution if and only if the top Lyapunov exponent \( \gamma_L(A) \) associated with the sequence matrices \( \{A(\eta)\}_{t \in \mathbb{Z}} \)

\[
\gamma_L(A) := \inf_{t>0} \left\{ \frac{1}{t} \log \left\| \prod_{j=0}^{t-1} A(\eta_{j-1}) \right\| \right\} \overset{a.s.}{=} \lim_{t \to \infty} \left\{ \frac{1}{t} \log \left\| \prod_{j=0}^{t-1} A(\eta_{j-1}) \right\| \right\}
\]

is strictly negative. The unique stationary solution is causal, ergodic and given by

\[
\mathbf{Y}_t = \sum_{k=1}^{\infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{j-1}) \right\} B(\eta_{t-k}) + B(\eta_t)
\]

where the above series converges almost surely (a.s).

**Proof.** The theorem is a multidimensional extension of Theorem 1.3 by Bougerol and Picard (1992a).

**Remark 1** Since \( \gamma_L(A) \) is independent of the norm, thus we can work with some norms that make it rather straightforward to show \( \gamma_L(A) < 0 \). However, sufficient conditions which ensure \( \gamma_L(A) < 0 \) are

1. \( E \{\log \|A(\eta_0)\|\} < 0 \).

2. \( E \left\{ \left\| \prod_{j=0}^{t-1} A(\eta_{t-j}) \right\|^r \right\} < 1 \) for some \( r > 0 \) and \( t \geq 1 \), in which case \( E \{\|\mathbf{Y}_t\|^r\} < +\infty \).
Now, a simple computation shows that

\[
\prod_{j=0}^{t} A(\eta_{t-j}) = A(\eta_t) = \left( \begin{array}{ccc}
O(d) & \cdots & O(d) \\
O(d) & \cdots & O(d) \\
\vdots & \ddots & \vdots \\
O(d) & \cdots & O(d)
\end{array} \right)
\prod_{i=1}^{t-1} \left\{ \prod_{v=0}^{s-1} \phi_{s-v}(\eta_{t-i}) \right\}.
\]

Therefore, because the top-Lyapunov exponent is independent of the norm, by choosing a multiplicative norm it is straightforward to show that

\[
\gamma_L(A) \leq \gamma_L(\Phi) := \inf_{t>0} \left\{ E \frac{1}{t} \log \left\| \prod_{i=1}^{t} \Phi \left( \eta_{t-i} \right) \right\| \right\}.
\]

where \( \Phi \left( \eta_t \right) := \left\{ \prod_{v=0}^{s-1} \phi_{s-v}(\eta_t) \right\} \). We have thus shown our first result which gives a sufficient condition for strict stationarity.

**Theorem 2** Suppose that \( \gamma_L(\Phi) < 0 \). Then for all \( t \in \mathbb{Z} \) the series

\[
\sum_{k=0}^{\infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{t-j}) \right\} B(\eta_{t-k})
\]

converges a.s. and the process \((Y_t)_{t \in \mathbb{Z}}\) defined by (2.5) is the unique strict stationary, causal and ergodic solution of (2.5).

**Example 1** In PGARCH(1,1), writing \( \phi_v \left( \eta_t \right) = (\eta_{st+v}\alpha_1(v) + \beta_1(v)) \), we obtain

\[
\prod_{v=0}^{s-1} \phi_{s-v}(\eta_t) = \left\{ \prod_{v=1}^{s-1} \left( \eta_{st+v}\alpha_1(v+1) + \beta(v+1) \right) \right\} (\eta_{st+s},1)^{\prime} (\alpha_1(1), \beta_1(1)).
\]

Hence

\[
\log \left\| \prod_{i=1}^{t} \Phi \left( \eta_{t-i} \right) \right\| = \sum_{v=1}^{s-1} \sum_{i=1}^{t} \log \left( \eta_{s(t-i)+v}\alpha_1(v+1) + \beta_1(v+1) \right) + \log \left\| \prod_{i=1}^{t} \left( \eta_{s(t-i)+s},1 \right)^{\prime} (\alpha_1(1), \beta_1(1)) \right\|.
\]

By the law of large numbers, a sufficient condition which ensure \( \gamma_L(A) < 0 \) is

\[
\sum_{v=1}^{s} E \left\{ \log \left( \eta_{st+v}\alpha_1(v) + \beta_1(v) \right) \right\} < 0
\]

which reduces to the classical condition when \( s = 1 \). It is worth noting that the existence of explosive regimes (i.e., \( E \left\{ \log \left( \eta_{st+v}\alpha_1(v) + \beta_1(v) \right) \right\} > 0 \)) does not preclude (periodic) strict stationarity.
The top-Lyapunov exponent $\gamma_L(.)$ criterion seems difficult to obtain explicitly, however a potential method to verify whether or not $\gamma_L(.) < 0$ is via a Monte-Carlo simulation using Equation (2.3). This fact heavily limits the interests of the criterion in statistical applications. Indeed, the solution need to have some moments to make an estimation theory possible and Condition (2.4) does not guarantee the existence of such moments. Therefore, we have to search for conditions ensuring the existence of moments for the stationary solution, for which, the top-Lyapunov exponent $\gamma_L(.)$ will be automatically negative (see Remark (2)).

3 Necessary and sufficient second-order stationarity conditions

In this section we examine the necessary and sufficient conditions ensuring the existence of a unique causal periodically correlated (PC) solution to (1.1). This is equivalent to examining conditions of the existence of solutions in $L^1$ of the process $(Y_t)_{t \in \mathbb{Z}}$ defined in equation (2.3). Let $A = E\{A(\eta_t)\}$ and $B = E\{B(\eta_t)\}$ be the expectations of the random matrix $A(\eta_t)$ and the random vector $B(\eta_t)$, then $A$ and $B$ are defined element-wise as

$$A = \begin{pmatrix} O_{(d)} & \ldots & O_{(d)} & \phi_1 \\ O_{(d)} & \ldots & O_{(d)} & \phi_2 \phi_1 \\ \vdots & \vdots & \vdots & \vdots \\ O_{(d)} & \ldots & O_{(d)} & \prod_{v=0}^{s-1} \phi_{s-v} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{B_1}{\phi_2 B_1 + B_2} \\ \vdots \\ \sum_{k=1}^{s} \left\{ \prod_{v=0}^{s-k-1} \phi_{s-v} \right\} B_k \end{pmatrix}$$

where

$$\phi_v := \begin{pmatrix} A_v(1) & B_v(1) \\ A_v & B_v \end{pmatrix}_{d \times d}, \quad B_v := \begin{pmatrix} \alpha_0(v) \\ O_{(p-1) \times 1} \\ \alpha_0(v) \\ O_{(q-1) \times 1} \end{pmatrix}_{d \times 1}$$

with $1 = (1, 1, \ldots, 1)' \in \mathbb{R}^s$. To verify that the process $(Y_t)_{t \in \mathbb{Z}}$ defined by (2.5) is well-defined in $L^1$, it is sufficient to show that the coefficients $\sum_{k=0}^{\infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{t-j}) \right\} B(\eta_{t-k})$ converge to zero in $L^1$ (endowed with any matrix norm) at an exponential rate, as $k \to \infty$.

**Lemma 1** Assume that

$$\rho \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) < 1 \quad (3.1)$$

then the series $\sum_{k=0}^{\infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{t-j}) \right\} B(\eta_{t-k})$ converges a.s. Furthermore, the process $(Y_t)_{t \in \mathbb{Z}}$ defined by (2.5) is stationary in $L^1$ and satisfying (2.3).
Proof. First, we notice that \( A(\eta_i) \) and \( B(\eta_i) \) are sequences of independent, non-negative random matrices and vectors respectively, and \( A(\eta_{i-j}) \) is independent of \( B(\eta_{i-j}) \) for all \( i \neq j \). Therefore, we have

\[
E \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k}) = A^k B. \]

It can be shown that the characteristic polynomial of \( A \) is \( \det(I(s) - \lambda A) = \det(I(s) - \lambda s \prod_{v=0}^{s-1} \phi_{s-v}) \), hence \( \rho(A) = \rho \left( s \prod_{v=0}^{s-1} \phi_{s-v} \right) \). If \( \rho \left( s \prod_{v=0}^{s-1} \phi_{s-v} \right) < 1 \), then \( \sum_{k=1}^{\infty} A^k < \infty \), and

\[
\sum_{k=0}^{\infty} E \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k}) \right\} < \infty \]

and thus \( \sum_{k=0}^{\infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k}) \right\} \) converges a.s. This further implies that \( \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k}) \) converges a.s. to the zero matrix as \( k \to \infty \). It is obvious that the process \( (Y_t)_{t \in \mathbb{Z}} \) defined in (2.5) is stationary in \( \mathbb{L}^1 \). Furthermore, we obtain

\[
Y_t = B(\eta_i) + A(\eta_i) \left\{ B(\eta_{i-1}) + \sum_{k=2}^{\infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k}) \right\}
\]

\[
= B(\eta_i) + A(\eta_i) \left\{ B(\eta_{i-1}) + \sum_{k=1}^{\infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k-1}) \right\}
\]

\[
= B(\eta_i) + A(\eta_i)Y_{i-1}.
\]

Lemma 2 If (2.3) admits a stationary solution in \( \mathbb{L}^1 \), then \( \rho \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) < 1 \). Moreover, the stationary solution of (2.3) is unique, causal and ergodic.

Proof. From (2.3), we obtain, by recursion, for any \( n \geq 1 \)

\[
Y_t = B(\eta_i) + \sum_{k=1}^{n} \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k}) + \left\{ \prod_{j=0}^{n} A(\eta_{i-j}) \right\} Y_{i-n-1}. \tag{3.2}
\]

Since, all \( A(\eta_i) \), \( B(\eta_i) \) and \( Y_t \) are non-negative, then

\[
E \{ Y_t \} \geq \sum_{k=0}^{n} E \left\{ \left\{ \prod_{j=0}^{k-1} A(\eta_{i-j}) \right\} B(\eta_{i-k}) \right\} = \sum_{k=0}^{n} A^k B.
\]
This implies that \( \sum_{k=0}^{\infty} A^k B < +\infty \). Hence, \( \lim_{k \to \infty} A^k B = 0 \). Since

\[
A^k = \begin{pmatrix}
O(d) & \cdots & O(d) & \phi_1 \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^{k-1} \\
O(d) & \cdots & O(d) & \phi_2 \phi_1 \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^{k-1} \\
\vdots & \vdots & \vdots & \vdots \\
O(d) & \cdots & O(d) & \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k
\end{pmatrix},
\]

then to show that \( \lim_{k \to \infty} A^k = 0 \), it is sufficient to show that \( \lim_{k \to \infty} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k = 0 \) which implies that

\[
\rho \left( \prod_{i=0}^{s-1} \phi_{s-i} \right) = \rho(A) < 1.
\]

To do this we shall prove that

\[
\lim_{k \to \infty} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k e_i = 0, \quad i = 1, \ldots, d
\]  

(3.3)

where \((e_v)_{1 \leq v \leq d}\) is the canonical basis of \(\mathbb{R}^d\), i.e., \(e_v = (\delta_{1,v}, \delta_{2,v}, \ldots, \delta_{d,v})^T\) where \(\delta_{i,j} = 1\) if \(i = j\) and 0 otherwise. Because the \(d \times d\) \(j\)-th block of vector \(A^k B\) is \(A^k B_j = \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) \sum_{i=1}^{s} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) B_i\), \(j = 1, \ldots, s\) we deduce that for \(j = s\),

\[
\lim_{k \to \infty} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) B_{s-l} = 0, \quad l = 1, \ldots, s.
\]  

(3.4)

Since \(B_{s-l} = \alpha_0(v)e_1 + \alpha_0(v)e_{p+1}\) for \(v = 1, \ldots, s\) and \(\alpha_0(v) > 0\), we obtain from (3.4) for \(l = 1, \ldots, s\)

\[
\lim_{k \to \infty} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) e_1 = 0,
\]

\[
\lim_{k \to \infty} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) e_{p+1} = 0
\]

and we use the relationships \(\phi_v e_i = \alpha_i(v) (e_1 + e_{p+1}) + e_{i+1}\) for \(i = 1, \ldots, p\) and \(\phi_v e_{p+i} = \beta_i(v) (e_1 + e_{p+1}) + e_{p+i+1}\) for \(i = 1, \ldots, q - 1\) we obtain

\[
0 = \lim_{k \to \infty} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k \left( \prod_{v=0}^{s-l-1} \phi_{s-v} \right) e_{s+1-l}, \quad l = 1, \ldots, s
\]

\[
0 = \lim_{k \to \infty} \left( \prod_{i=0}^{s-1} \phi_{s-i} \right)^k \left( \prod_{v=0}^{s-l-1} \phi_{s-v} \right) e_{p+s+1-l}, \quad l = 1, \ldots, s
\]
and hence (3.3) holds for \( i = 1, \ldots, s \) and \( i = p + 1, \ldots, p + s \). On the other hand, from (3.4) we have

\[
0 = \lim_{k \to \infty} \left\{ s_{i+} - 1 \prod_{i=0}^{s-1} \phi_{s-i} e_{i+} \right\}^k = \lim_{k \to \infty} \left\{ s_{i+} - 1 \prod_{i=0}^{s-1} \phi_{s-i} e_{i+} \right\}^{k-1} e_{p+i+1}, \quad i = 1, \ldots, s
\]

This concludes the proof of the lemma. To prove the uniqueness, let \((Z_t)_{t \in \mathbb{Z}}\) be another stationary process conforming to (2.3). Then \((Z_t)_{t \in \mathbb{Z}}\) satisfies an equation similar to (3.2). By setting \(W_t = Z_t - Y_t\), we obtain, \( \forall m \geq 1 : W_t = \sum_{j=0}^{m} A(\eta_{t-j})W_{t-m-1} \). Defining \( W = E \{ W_t \} \), we have \( W = A^{m+1}W \), since \( \lambda := \rho(A) < 1 \) we conclude that \( W = 0 \) (with probability 1). Hence the uniqueness follows.

**Theorem 3** A necessary and sufficient condition for the existence of unique stationary solution in \( \mathbb{L}^1 \) of equation (2.3) is that (3.1) holds. Moreover, this stationary solution is causal and ergodic.

**Proof.** The proof follows from Lemma 1 and 2. The fact that stationary solution is ergodic is obtained by the same argument as in Bougerol and Picard (1992a).

**Remark 2** Since \( \gamma_L(\Phi) < \log \rho \left( \prod_{v=0}^{s-1} \phi_{s-v} \right) \) (see Kesten and Spitzer, 1984), the condition (3.1) is necessary and sufficient for the existence of a strictly stationary solution in \( \mathbb{L}^1 \) of (2.3).

4 Existence of the higher-order moments

In this section, we present a necessary and sufficient conditions for the existence of finite higher order moments for a PGARCH process.

**Theorem 4** Let \((Y_t)_{t \geq 0}\) be the stationary solution of model (2.3). Assume \( \kappa_2 < +\infty \) where \( \kappa_k = E \{ \varepsilon_t^{2k} \} \).

1. If

\[
\rho \left( \prod_{v=0}^{s-1} E \left\{ \phi_{s-v}^{\otimes 2}(\eta_t) \right\} \right) < 1
\]

then \( Y_t \in \mathbb{L}^2 \).

2. Conversely, if \( \rho \left( \prod_{v=0}^{s-1} E \left\{ \phi_{s-v}^{\otimes 2}(\eta_t) \right\} \right) \geq 1 \), then there is no strictly stationary solution \((Y_t)_{t \in \mathbb{Z}}\) to model (2.3) such that \( E \{ Y_t^{\otimes 2} \} < +\infty \).

**Proof.**
1. We first define the following \( \mathbb{R}^{sd} \)-valued stochastic processes

\[
\mathcal{S}_n(t) = \begin{cases} 
0 & \text{if } n < 0 \\
B(\eta_t) + A(\eta_t)\mathcal{S}_{n-1}(t-1) & \text{if } n \geq 0,
\end{cases}
\]

and for all \( n \in \mathbb{Z} \): \( \Delta_n(t) = \mathcal{S}_n(t) - \mathcal{S}_{n-1}(t) \). It can be easily shown that, for all \( n \geq 0 \), \( \mathcal{S}_n(t) \) and \( \Delta_n(t) \) are measurable functions of \( \eta_t, \eta_{t-1}, \ldots, \eta_{t-n} \). Hence the processes \( (\mathcal{S}_n(t))_{t \in \mathbb{Z}} \) and \( (\Delta_n(t))_{t \in \mathbb{Z}} \) are stationary. From the definition of \( \Delta_n(t) \) and \( \Delta_n(t) \), we can verify that

\[
\Delta_n(t) = \begin{cases} 
0 & \text{if } n < 0 \\
B(\eta_t) & \text{if } n = 0 \\
A(\eta_t)\Delta_{n-1}(t-1) & \text{if } n > 0
\end{cases}
\]

For all \( n \in \mathbb{Z} \), define \( \Gamma_n^{(2)}(t) = \Delta_n^{\otimes 2}(t) = \text{Vec} \{ \Delta_n(t)\Delta_n(t) \} \). Using the properties of Kronecker product, we obtain \( \Gamma_n^{(2)}(t) = A^{\otimes 2}(\eta_t)\Gamma_n^{(2)}(n-1)(t-1) \) for \( n > 0 \) and \( E \{ \Gamma_n^{(2)}(t) \} = (A^{(2)})^n E \{ \Gamma_0^{(2)}(t-n) \} = (A^{(2)})^n E \{ B^{(2)}(\eta_{n-n}) \} \), where \( A^{(2)} := E \{ A^{\otimes 2}(\eta_t) \} \). Since \( \rho(A^{(2)}) = \rho \left( \prod_{r=0}^{n-1} E \left\{ \phi_{s-r}(\eta_t) \right\} \right) < 1 \), we conclude that \( \mathcal{S}_n(t) \) converges in \( L^2 \) and almost surely to some limit \( Y_t \in L^2 \) which is the solution of equation (2.3). This completes the proof.

2. From (3.2) we obtain

\[
E \{ Y^{\otimes 2} \} \geq \sum_{k=0}^{\infty} E \left\{ \left( \prod_{j=0}^{k-1} A(\eta_{t-j}) \right) B(\eta_{t-k}) \right\} = \sum_{k=0}^{\infty} (A^{(2)})^k B^{(2)}
\]

where \( B^{(2)} := E \{ B^{\otimes 2}(\eta_t) \} \) and the conclusion follows.

The result of the above theorem can be further extended to the higher-order moments.

**Theorem 5** Let \( (Y_t)_{t \geq 0} \) be the stationary solution of model (2.3). Assume that \( \kappa_{2(r-1)} < +\infty \) where \( r > 2 \).

1. If \( \rho \left( \prod_{r=0}^{n-1} E \left\{ \phi_{s-r}(\eta_t) \right\} \right) < 1 \) then \( Y_t \in L^r \).

2. Conversely, if \( \rho \left( \prod_{r=0}^{n-1} E \left\{ \phi_{s-r}(\eta_t) \right\} \right) \geq 1 \), then there is no strictly stationary solution \( (Y_t)_{t \in \mathbb{Z}} \) to model (2.3) such that \( E \{ Y^{\otimes r} \} < +\infty \).

**Proof**: Define \( \mathcal{S}_n(t) \) and \( \Delta_n(t) \) as in Theorem 4 and let \( \Gamma_n^{(r)}(t) = \Delta_n^{\otimes r}(t) \). Since \( \Gamma_n^{(r)}(t) = A^{\otimes r}(\eta_t)\Gamma_n^{(r)}(n-1)(t-1) \) for \( n > 0 \), the proof is similar to that for the Theorem 4 and thus we omit the details.
5 Covariance structure

To get the covariance structure of the squared PGARCH process, we first assume that the Condition \(4.1\) holds, this implies that \(2.2\) has a unique (in \(L^2\) sense) PC solution. Taking expectation on both sides of \(2.2\) and using the notation \(\mu_k(v) = E \{y_{st+v}^{\otimes k}\}, k = 1, 2\) gives

\[
\mu_1(v) = \phi_v \mu_1(v-1) + B_v, \quad v = 1, \ldots, s.
\]

Iterating \(5.1\) \(s\) times and requiring \(\mu_k(0) = \mu_k(s)\), we obtain

\[
\mu_i(s) = \left( I(d) - \prod_{v=0}^{s-1} \phi_{s-v} \right)^{-1} \sum_{j=0}^{s-1} \prod_{i=0}^{j-1} \phi_{s-v} \{ B_{s-j} \}.
\]

Further manipulations in \(5.1\) provide the desired seasonal mean

\[
\mu_1(v) = \sum_{j=0}^{v-1} \left\{ \prod_{i=0}^{j-1} \phi_{v-i} \right\} B_{s-j} + \left\{ \prod_{i=0}^{v-1} \phi_{v-i} \right\} \mu_1(s), \quad v = 1, \ldots, s.
\]

The seasonal variance can be obtained as follow

\[
\mu_2(v) = E \left\{ \left( \phi_v(y_{n}) y_{st+v-1} + B_v(y_{n}) \right) \otimes \left( \phi_v(y_{n}) y_{st+v-1} + B_v(y_{n}) \right) \right\}
\]

\[
= \phi_b^{(2)} \mu_1(v-1) + \varphi_v
\]

where \(\varphi_v = B_v^{(2)} + \left( E \{ \phi_v(y_{n}) \otimes B_v(y_{n}) + B_v(y_{n}) \otimes \phi_v(y_{n}) \} \right) \mu_1(v-1)\) with \(\phi_b := E \{ \phi_b^{(2)}(y_{n}) \}\) and \(B_v^{(2)} = E \{ B_v^{(2)}(y_{n}) \}\). Thus

\[
\mu_2(s) = \left( I(d^2) - \prod_{v=0}^{s-1} \phi_{s-v}^{(2)} \right)^{-1} \sum_{j=0}^{s-1} \prod_{i=0}^{j-1} \phi_{s-v}^{(2)} \varphi_{s-j},
\]

\[
\mu_2(v) = \sum_{j=0}^{v-1} \left\{ \prod_{i=0}^{j-1} \phi_{v-i}^{(2)} \right\} \varphi_{v-j} + \left\{ \prod_{i=0}^{v-1} \phi_{v-i}^{(2)} \right\} \mu_2(s).
\]

Now, noting that for any \(h > 0\), we have

\[
\gamma_v(h) = E \left\{ y_{st+v} \otimes y_{st+v-h} \right\}
\]

\[
= E \left\{ \phi_v(y_{n}) \otimes I(d) \right\} E \left\{ y_{st+v-1} \otimes y_{st+v-h} \right\} + B_v \otimes \mu_1(v-h)
\]

\[
= (\phi_v \otimes I(d)) \gamma_v-1(h-1) + B_v \otimes \mu_1(v-h)
\]

\[
= \sum_{k=0}^{h-1} \left\{ \prod_{i=0}^{k-1} (\phi_{v-i} \otimes I(d)) \right\} B_v-1 \otimes \mu_1(v-h-k) + \left\{ \prod_{i=0}^{h-1} (\phi_{v-i} \otimes I(d)) \right\} \mu_2(v-h).
\]

In the above equations, \(\phi_v, \mu_1(v)\) and \(\mu_2(v)\) are interpreted periodically in \(v\) with period \(s\).
The covariance structure of $\text{PGARCH}$ process is the $\text{PGARCH}(1,1)$ process. However, it is worth noting that the existence of explosive regimes (i.e. $\theta_1(v) > 1$) does not preclude the periodic stationarity. The seasonal moments are

$$
\mu_1(s) = \left(1 - \prod_{v=1}^{s} \theta_1(v)\right)^{-1} \sum_{j=0}^{s-1} \prod_{v=0}^{j-1} \theta_1(s-v) \alpha_0(s-j)
$$

$$
\mu_1(v) = \sum_{j=0}^{v-1} \left\{ \prod_{i=0}^{j-1} \theta_1(v-i) \right\} \alpha_0(v-j) + \left\{ \prod_{i=0}^{v-1} \theta_1(v-i) \right\} \mu_1(s)
$$

$$
\mu_2(s) = \kappa_2 \left(1 - \prod_{v=1}^{s} \theta_2(v)\right)^{-1} \sum_{j=0}^{s-1} \prod_{v=0}^{j-1} \theta_2(s-v) \left\{ \alpha_0^2(s-j) + 2\alpha_0(s-j)\theta_1(s-v)\mu_1(s-j-1) \right\}
$$

$$
\mu_2(v) = \sum_{j=0}^{v-1} \left\{ \prod_{i=0}^{j-1} \theta_2(v-i) \right\} \left\{ \alpha_0^2(v-j) + 2\alpha_0(v-j)\theta_1(v-j)\mu_1(v-j-1) \right\} + \left\{ \prod_{i=0}^{v-1} \theta_2(v-i) \right\} \mu_2(s).
$$

The covariance structure of $\text{PGARCH}(1,1)$ process can be obtained as

$$
\gamma_v(h) = \left\{ \begin{array}{ll}
\alpha_0(v)\mu_1(v-1) + \theta_1(v)\mu_2(v-1), & h = 1 \\
\alpha_0(v)\mu_1(v-1) + \theta_1(v)\gamma_{v-1}(h-1), & h \geq 2.
\end{array} \right.
$$

When compared to the covariance function of second order $\text{GARCH}(1,1)$ process, the above formulas are quite complex. One can verify that these expressions reduce to the classical $\text{GARCH}(1,1)$ forms when the $\text{PGARCH}(1,1)$ parameters are constant in $v$. In general, calculation in $\text{PGARCH}$ process are very tedious.

6 Geometric ergodicity and strong mixing

The basic tools presented here are drawn from the monograph by Doukhan (1994). In this section we analyze the statistical properties of $\text{PGARCH}$ process, such as the geometric ergodicity and the strong mixing. These concepts are fundamental in central limit theorem and law of large numbers which can be employed to derive asymptotic normality, consistency of maximum likelihood style estimators and inference with the model.

**Definition 1** Let $(X_t)_{t \geq 0}$ be a discrete Markov chain taking values in $\mathbb{R}^k$ $k \geq 1$ with time homogeneous $t$-step transition probabilities i.e., $P^t(x, A) = P(X_t \in A | X_0 = x)$ where $x \in \mathbb{R}^k$, $A \in \mathcal{B}_{\mathbb{R}^k}$ and where $\mathcal{B}_{\mathbb{R}^k}$ is a Borel $\sigma$-field on $\mathbb{R}^k$ with $P^1(x, A) = P(x, A)$. Let $\pi$ be the invariant probability measure on $(\mathbb{R}^k, \mathcal{B}_{\mathbb{R}^k})$, i.e., $\forall A \in \mathcal{B}_{\mathbb{R}^k}$: $\pi(A) = \int P(x, A) \pi(dx)$.
a. The chain \((X_t)_{t \geq 0}\) is said to be geometrically ergodic, if there exists some \(0 < r < 1\) such that
\[
\forall x \in \mathbb{R}^k, \quad \| P^t (x, \cdot) - \pi(\cdot) \|_V = o(r^t)
\]
where \(\| \cdot \|_V\) is the total variation norm. Furthermore, if the chain \((X_t)_{t \in \mathbb{Z}}\) is started with an initial distribution \(\pi\), the process is strictly stationary.

b. The \(\beta\)-mixing coefficients are defined by
\[
\beta_X(k) = E \left\{ \sup_{B \in \sigma(X_t, t \geq k)} |P(B| \sigma(X_t, t \leq 0)) - P(B)| \right\} = \frac{1}{2} \sup (\sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)|)
\]
where the supremum of the last equality is taken over all pairs of partitions \(\{A_1, ..., A_I\}\) and \(\{B_1, ..., B_J\}\) of \(\Omega\) such that \(A_i \in \sigma(X_t, t \leq 0)\) for all \(i\) and \(B_j \in \sigma(X_t, t \geq k)\) for all \(j\). The process \((X_t)_{t \in \mathbb{Z}}\) is called \(\beta\)-mixing if \(\lim_{k \to \infty} \beta_X(k) = 0\).

This powerful result helps us to develop asymptotic results, since convergence in distribution is ensured for all measurable functions of the chain. Perhaps one of the most well-known criterion used in establishing the geometric ergodicity of a Markov chain is the drift condition developed in Tweedie (1975) and employed for the analysis of stochastic stability.

**Definition 2** A drift function \(g : \mathbb{R}^k \to [1, \infty]\) satisfies a 1-step geometrical drift criterion (relative to a Markov chain) if there exists a compact \(K \subset \mathbb{R}^k\) and a positive constants \(b\) and \(0 < \lambda < 1\),
\[
E \left\{ g(X_t)|X_{t-1} = x \right\} \leq \begin{cases} 
\lambda g(x) & \text{if } x \notin K \\
\ b & \text{if } x \in K 
\end{cases}
\]
where \(g\) is interpreted as a generalized energy function and the compact set \(K\) as the centre of attraction.

**Remark 3** One consequence of the geometric ergodicity is that the stationary Markov chain \((X_t)_{t \in \mathbb{Z}}\) is \(\beta\)-mixing, and hence strongly mixing, with geometric rate. Indeed, Davidov (1973) showed that for an ergodic Markov chain \((X_t)_{t \in \mathbb{Z}}\) with invariant probability measure \(\pi\), \(\beta_X(k) = \int \| P^k (x, \cdot) - \pi(\cdot) \|_V \pi(dx)\). Thus \(\beta_X(k) = o(r^k)\) if \((6.1)\) holds.

In what follows, we shall assume, without loss of generality, that \(p = q\), otherwise zeros can be filled. To derive the geometric ergodicity and \(\beta\)-mixing results, we must assume the following conditions for the sequence \((\eta_t)_{t \in \mathbb{Z}}\).

**[B.0]** The i.i.d. sequence \((\eta_t)_{t \in \mathbb{Z}}\) has a probability distribution function absolutely continuous with respect to the Lebesgue measure and such that the density function take positive values almost surely on its support \(E \subset \mathbb{R}^s\).

We are now ready to state the main result of this section which establishes statistical properties of \(PGARCH\) processes, in particular, we focus on the \(\beta\)-mixing property with exponential decay.
Theorem 6 Under Conditions (B.0) and (3.1) and assume that

1. \( \rho \left( \prod_{v=1}^{s} B_v \right) < 1 \)

2. There are \( 0 < r \leq 1 \) and \( \rho < 1 \) such that \( E \left\{ \left\| A(\eta_n) \right\|^r \right\} \leq \rho \) and \( \eta_n \in L^r \).

then the process \((Y_t)_{t \in \mathbb{Z}}\) defined by (2.3) is geometrically ergodic. Moreover, if initialized from its invariant measure, \((Y_t)_{t \in \mathbb{Z}}\) is strictly stationary and \(\beta\)-mixing with exponential decay.

Remark 4 The condition \( E \left\{ \left\| A(\eta_n) \right\|^r \right\} < 1 \) for some \( 0 < r \leq 1 \) in some neighborhood of zero is satisfied if \( E \left\{ \log \left\| A(\eta_n) \right\| \right\} < 0 \) and \( E \left\{ \left\| A(\eta_n) \right\|^\epsilon \right\} < \infty \) for some \( \epsilon > 0 \). Indeed, the function \( \vartheta(v) = E \left\{ \left\| A(\eta_n) \right\|^v \right\} \) has derivative \( \vartheta'(0) = E \left\{ \log \left\| A(\eta_n) \right\| \right\} < 0 \), hence \( \vartheta(v) \) decreases in small neighborhood of zero, and since \( \vartheta(0) = 1 \) it follows that \( \vartheta(r) < 1 \) for \( 0 < r \leq 1 \). On the other hand \( E \left\{ \left\| A(\eta_0) \right\|^r \right\} < 1 \) implies that \( E \left\{ \log \left\| A(\eta_0) \right\| \right\} < 0 \) by Jensen’s inequality.

Proof. The proof is based on Carrasco and Chen (2002, Theorem 1). Assumption B.0 ensures that Carrasco and Chen’s (2002 p. 20) condition on the innovation term holds. We must then prove that \( A \left( \eta_n \right) \) and \( B \left( \eta_n \right) \) satisfy their assumptions \((A_0), (A_1), (A_2^*)\) and \((A_3^*)\)

[A.0] Both \( A \left( \eta_n \right) \) and \( B \left( \eta_n \right) \) are polynomial functions of \( \eta_n \), therefore the measurability condition is trivially satisfied.

[A.1] We must show that \( \rho \left( A(0) \right) < 1 \). It turns out however that the nonzero eigenvalues of \( A(0) \) are the nonzero eigenvalues of \( \Phi(0) \). It is not hard to see that then \( \rho \left( \Phi(0) \right) = \rho \left( \prod_{v=1}^{s} B_v \right) \) (see Horn and Johnson, 1999, p. 68).

[A*.2] That \( \sum_{k \geq 0} \left( \prod_{j=0}^{k-1} A \left( \eta_{n-j} \right) \right) \left( B \left( \eta_{n-k} \right) \right) \) converges almost surely to some constant immediately follows from Lemma. It remains to show that \( \prod_{j=0}^{k-1} A \left( \eta_{n-j} \right) \) converges almost surely to zero for all \( X \in \mathbb{R}^{dn} \) as \( k \to \infty \). As \( X \) is nonrandom, it suffices to show that \( \prod_{j=0}^{k-1} A \left( \eta_{n-j} \right) \) converges almost surely to zero as \( k \to \infty \). Since \( E \left\{ \prod_{j=0}^{k-1} A \left( \eta_{n-j} \right) \right\} = A^k \), then the condition (3.1) implies that \( \sum A^k < +\infty \) and hence \( \lim_{k \to \infty} A^k = 0 \). Combining this result with the fact that \( \lim_{k \to \infty} \prod_{j=0}^{k-1} A \left( \eta_{n-j} \right) \geq 0 \) yields that almost surely
\[
\lim_{k \to \infty} \left\{ \prod_{j=0}^{k-1} A(\eta_{t-j}) \right\} = 0.
\]

\[\text{[A*.3]}\] Let \( \tilde{z} \in \mathbb{R}^{sd} \) and the Lyapunov function \( g : \mathbb{R}^{sd} \to [1, \infty[ \) by \( g(\tilde{z}) = \|\tilde{z}\|^r + 1 \), we have

\[
E\left\{ g(Y_t) \mid Y_{t-1} = y \right\} = E\left\{ \|A(\eta_t)Y_{t-1} + B(\eta_t)\|^r \mid Y_{t-1} = y \right\} + 1
\]

\[
\leq E\left\{ \|A(\eta_t)\|^r \|y\|^r + E\left\{ B(\eta_t) \right\|^r \right\} + 1
\]

\[
\leq \rho \|y\|^r + \kappa
\]

where \( \kappa \) is a positive constant such that \( E\left\{ \|B(\eta_t)\| \right\}^r \leq \kappa - 1 \). Choose \( \lambda > 0 \) with \( 1 - \lambda > \rho \) and set \( M = \frac{\kappa}{1 - \lambda - \rho} \) and consider the compact \( K = \left\{ y \in \mathbb{R}^{sd} : \|y\| \leq M \right\} \). For all \( y \notin K \), we have \( (1 - \lambda - \rho) \|y\|^r \geq \kappa \) and therefore

\[
E\left\{ g(Y_t) \mid Y_{t-1} = y \right\} \leq \rho \|y\|^r + (1 - \lambda - \rho) \|y\|^r \leq (1 - \lambda) g(y). \tag{6.2}
\]

When \( y \in K \), we have

\[
E\left\{ g(Y_t) \mid Y_{t-1} = y \right\} \leq \rho \|y\|^r + \kappa \leq \frac{\rho \kappa}{1 - \lambda - \rho} + \kappa = b \tag{6.3}
\]

Combining (6.2) and (6.3) we obtain

\[
E\left\{ g(Y_t) \mid Y_{t-1} = y \right\} \leq (1 - \lambda) g(y) + b I_K(y)
\]

where \( I_K \) denotes the indicator function of the \( K \).

\section{Conclusion}

The paper partially extends \( \mathbb{L}^2 \) structures of the usual GARCH model to periodic ones which allow the volatility of time series to have different dynamics according to the model parameters which switches between \( s \)-regimes. Our study is based (in a multivariate framework) on a generalized autoregressive representation which we are preferred for a periodic ARMA (PARMA) representation. The main advantage of the approach is that, besides its simplicity, it preserve the mathematically tractable GARCH structure when \( s = 1 \). A thorough examination of the \( \mathbb{L}^2 \) structures of the series and its powers revealed that, under appropriate moment conditions, these structures were those of periodic AR (PAR) processes. Beside the conditions ensuring the existence and uniqueness of strictly stationary solution of PGARCH, we have also gave sufficient conditions for the PGARCH processes to belong to \( \mathbb{L}^p \), \( p \geq 1 \). Some fundamental probabilistic properties such as the \( \beta \)-mixing and the geometric ergodicity with exponential decay have been studied.
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