The $v$-number and Castelnuovo–Mumford regularity of graphs

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Abstract

We prove that for every integer $k \geq 1$, there exists a connected graph $H_k$ such that $v(H_k) = \text{reg}(H_k) + k$, where $v(G)$ and $\text{reg}(G)$ denote the $v$-number and the (Castelnuovo–Mumford) regularity of a graph $G$ respectively.

Keywords $v$-number · Regularity · Collapsibility

1 Introduction

A clutter $C$ on a vertex set $V = \{x_1, \ldots, x_n\}$ is a family of subsets (edges or circuits) of $V$ which are pairwise incomparable with respect to the inclusion. We identify the set of edges with the clutter $C$ itself. When $R = \mathbb{k}[x_1, \ldots, x_n] = \bigoplus_{i=0}^{\infty} R_i$ is a polynomial ring over a field $\mathbb{k}$ with the standard grading, the edge ideal $I(C)$ of the clutter $C$ is defined to be the ideal of $R$ generated by all squarefree monomials $x_e := \prod_{x_i \in e} x_i$ such that $e \in C$.

Consider the minimal free graded resolution of $R/I(C)$ as an $R$-module:

$$0 \to \bigoplus_j R(-j)^{\beta_{j,i}} \to \cdots \to \bigoplus_j R(-j)^{\beta_1,j} \to R \to R/I(C) \to 0.$$ 

The Castelnuovo–Mumford regularity or simply the regularity of $R/I(C)$ is defined as

$$\text{reg}_k(C) := \text{reg}_k(R/I(C)) = \max\{j - i : \beta_{i,j} \neq 0\}.$$
Most of the recent work in the area has been focused on either finding applicable bounds on the (Castelnuovo–Mumford) regularity $\text{reg}(\mathcal{C})$ in terms of other parameters or performing exact computation of the regularity in specific cases [2]. Recently, a new invariant associated to the edge ideal $I(\mathcal{C})$ is introduced in [4] to study the asymptotic behavior of the minimum distance of projective Reed–Muller-type codes that we recall next. If $\mathcal{C}$ is a clutter, a subset $A \subseteq V$ is an independent set if $e \cap A$ for every $e \in \mathcal{C}$. Furthermore, a subset $W \subseteq V$ is said to be a vertex cover provided that $V \setminus W$ is an independent set.

When $A$ is an independent set, a vertex $w \in V \setminus A$ is a neighbor of $A$ if $A \cup \{w\}$ contains an edge of $\mathcal{C}$. We denote by $N_{\mathcal{C}}(A)$, the set of all neighbors of $A$ in $\mathcal{C}$, and by $A(\mathcal{C})$, the family of all independent sets $A$ such that $N_{\mathcal{C}}(A)$ is a minimal vertex cover.

**Definition 1** [4, 8] The $v$-number of a clutter $\mathcal{C}$ is defined by

$$v(\mathcal{C}) := \min\{|A| : A \in A(\mathcal{C})\}.$$

The subject of three recent papers [5, 8, 12] is the comparison of the $v$-number and the regularity of graphs. Under suitable restrictions, it is proved that the $v$-number of a graph $G$ provides a lower bound to $\text{reg}(G)$. On the other hand, Jaramillo and Villarreal [8] show that there exists a graph $G$ satisfying $v(G) = \text{reg}(G) + 1$ (over the field of rationals), and ask whether or not the inequality $v(H) \leq \text{reg}(H) + 1$ holds for every graph $H$.

We prove that there exist even connected graphs such that their $v$-numbers are far larger than their regularities.

**Theorem 2** For every integer $k \geq 1$, there exists a connected graph $H_k$ such that $v(H_k) = \text{reg}(H_k) + k$.

### 2 Preliminaries

A simplicial complex $X$ on a vertex set $V$ is simply a family of subsets of $V$, closed under inclusion such that $\{x\} \in X$ for every $x \in V$. A set $A \in X$ is said to be a face (or a simplex) of $X$, and the dimension of a face $A$ is $\dim(A) = |A| − 1$. The dimension $\dim(X)$ of a complex $X$ is the maximum dimension of a face in $X$. A face $F$ of $X$ is said to be a facet of $X$ if it is a maximal face with respect to the inclusion.

For a given face $A \in X$, the deletion and link subcomplexes of $X$ at the face $A$ is defined by $\text{del}(X; A) := \{S \in X : A \cap S = \emptyset\}$ and $\text{lk}(X; A) := \{T \in X : A \cap T = \emptyset\}$ and $T \cup A \in X\}$. When $X$ is a simplicial complex, a subset $S \subseteq V$ is said to be a circuit (minimal non-face) of $X$ if $S$ is not a face of $X$ while any proper subset of $S$ is. We denote by $\mathcal{C}(X)$, the family of all circuits of $X$. On the other hand, the family of all independent sets of a clutter $\mathcal{C}$ forms a simplicial complex, the independence complex $\text{Ind}(\mathcal{C})$ of

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1 Unless otherwise stated, our results are independent of the characteristic of the coefficient field. So, wherever it is appropriate, we drop $k$ from our notation.
Finally, we denote by \( c(X) \) of its circuits is a clutter, and \( X = \text{Ind}(c(X)) \). On the other side, we have that \( c(\text{Ind}(C)) = C \) for a clutter \( C \) on \( V \).

When \( G = (V, E) \) is a (finite and simple) graph, we denote by \( N_G(x) := \{ y \in V : x y \in E \} \), the (open) neighborhood of \( x \) in \( G \), whereas \( N_G[x] := N_G(x) \cup \{ x \} \) is its closed neighborhood. In particular, we set \( N_G(S) := \bigcup_{s \in S} N_G(s) \) for \( S \subseteq V \). The size of the set \( N_G(v) \) is called the degree of \( x \) in \( G \) and denoted by \( \deg_G(v) \). Furthermore, \( \overline{G} \) denotes the complement of the graph \( G \). For a given subset \( S \subseteq V \), the subgraph \( G[S] \) of \( G \) induced by the set \( S \) is the graph on \( S \) with \( E(G[S]) = E \cap (S \times S) \). Finally, we denote by \( K_n, P_n \) and \( C_k \), the complete, path and cycle graphs on \( n \geq 1 \) and \( k \geq 3 \) vertices respectively.

We say that \( G \) is \( H \)-free if no induced subgraph of \( G \) is isomorphic to \( H \). A graph \( G \) is called chordal if it is \( C_r \)-free for every \( r > 3 \). Moreover, a graph \( G \) is said to be co-chordal if its complement \( \overline{G} \) is a chordal graph.

The following provides an inductive bound on the regularity of graphs.

**Lemma 3** [6] Let \( G \) be a graph and let \( v \in V \) be given. Then

\[
\text{reg}(G) \leq \max\{\text{reg}(G - v), \text{reg}(G - N_G[v]) + 1\}.
\]

Moreover, \( \text{reg}(G) \) always equals to one of \( \text{reg}(G - v) \) or \( \text{reg}(G - N_G[v]) + 1 \).

A matching in a graph is a subset of edges no two of which share a vertex. An induced matching is a matching \( M \) if no two vertices belonging to different edges of \( M \) are adjacent. The maximum size of an induced matching of \( G \) is known as the induced matching number \( \text{im}(G) \) of \( G \). The induced matching number provides a lower bound to regularity, that is, \( \text{im}(G) \leq \text{reg}(G) \) holds for every graph \( G \) [9]. Finally, if we denote by \( \text{co-chord}(G) \), the least number of co-chordal subgraphs \( G_1, \ldots, G_k \) of \( G \) satisfying \( E(G) = \bigcup_{i=1}^k E(G_i) \), then the inequality \( \text{reg}(G) \leq \text{co-chord}(G) \) holds for every graph \( G \) [14].

### 3 The \( v \)-number of simplicial complexes and graphs

Using the above stated correspondence between simplicial complexes and clutters, we set \( v(X) := v(\text{c}(X)) \) for every simplicial complex \( X \). We prove that the \( v \)-number of simplicial complexes is closely related to a known parameter appearing in the collapsibility theory of Wegner [13]. In particular, we verify that the \( v \)-number of the independence complex of a graph corresponds to a domination parameter on the underlying graph.

We next rephrase the \( v \)-number in the language of simplicial complexes as follows. Let \( A \in X \) be a face, and define \( U_X(A) := \{ v \in V - A : A \in \text{lk}(X; v) \} \cup A \). Observe that if \( F \in X \) is a facet, then \( U_X(F) = F \).

**Proposition 4** \( v(X) = \min\{|A| : A \in X \text{ and } U_X(A) \text{ is a facet of } X\} \) for every simplicial complex \( X \).
The construction of the graph $G$ is denoted by $\gamma_0(X)$ in [11, Section 4].

The number $\beta(X)$ for a simplicial complex $X$ is firstly considered in [7, Theorem 5.4].

The construction of the graph $G$ is due to R. Woodrooffe, and it was devised over a discussion with the author.

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vertex-wise dominates $e$ (see [10] for details). Furthermore, a ve-dominating set $S$ of $G$ is called minimal, if no proper subset of $S$ is ve-dominating for $G$. When $S$ is a minimal ve-dominating set for $G$, every vertex in $S$ has a private neighbor in $E$. In other words, $e$ is a vertex-wise private neighbor of $s \in S$ if $s$ ve-dominates $e$ while no vertex in $S \setminus \{s\}$ ve-dominates the edge $e$ in $G$.

The independent vertex-wise domination number and the upper independent vertex-wise domination number of $G$ are defined by

$$i_{ve}(G) := \min\{|S| : S \text{ is an independent vertex-wise dominating set of } G\},$$

$$\beta_{ve}(G) := \max\{|T| : T \text{ is a minimal independent vertex-wise dominating set of } G\}$$

respectively. Notice that the inequality $i_{ve}(G) \leq \beta_{ve}(G)$ holds for every graph $G$.

**Theorem 9** $v(G) = i_{ve}(G)$ for every graph $G$.

**Proof** Suppose that $i_{ve}(G) = |S|$, where $S$ is an independent ve-dominating set. Observe that $N_G(S)$ is a vertex cover. Indeed, assume otherwise that there exists an edge $e = xy$ in $G - N_G(S)$. Since $S$ is a ve-dominating set, there exists $s \in S$ such that either $sx \in E$ or $sy \in E$. However, this implies that either $x \in N_G(S)$ or else $y \in N_G(S)$, a contradiction. The fact that $N_G(S)$ is a minimal vertex cover follows, since $S$ is an independent ve-dominating set in $G$. This shows that $v(G) \leq i_{ve}(G)$.

Next, assume that $v(G) = |A|$ for some independent set $A$ in $G$. If $f = uv \in E$, we must have that either $u \in N_G(A)$ or else $v \in N_G(A)$, since $N_G(A)$ is a vertex cover. However, this means that $A$ is an independent ve-dominating set in $G$. Therefore, we conclude that $i_{ve}(G) \leq |A| = v(G)$. \hfill $\square$

**Corollary 10** $\beta_{ve}(G) = \beta(\text{Ind}(G))$ for every graph $G$. 

\[Fig. 1\] The graph $G$ in Remark 8
4 Proof of Theorem 2

We consider a 17-vertex flag triangulation of the dunce hat \([3]\) illustrated as in Fig. 2, where vertices with the same label are identified. Denote by \(D\), the graph whose independence complex is isomorphic to given triangulation.

Observe first that \(\text{reg}(D) = 2\) and \(v(D) = 3\). The latter follows from the fact that neither any vertex nor any edge is a free face of \(\text{Ind}(D)\). For the former, we note that \(C_{12}\) on the vertex subset \(\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}\) is an induced subgraph of \(D\) so that \(2 = \text{reg}(C_{12}) \leq \text{reg}(D)\). On the other hand, we have \(\text{reg}(D) \leq \text{co-chord}(D) = 2\). Indeed, we set \(V_1 := \{5, 7, 9, 11, 13, 15\}\) and \(V_2 := \{6, 8, 10, 12, 14, 16\}\), and define two graphs \(R_i\) on \(V_1 \cup V_2\) by \(E(R_i) := \{pq \in E(D) : p \in V_i\} \cap E(D)\). We then let \(Q_1\) and \(Q_2\) be the subgraphs of \(D\) on \(V(D)\) with

\[
E(Q_1) := E(R_1) \cup \{uv \in E(D) : u \in \{2, 3\}\},
\]

\[
E(Q_2) := E(R_2) \cup \{xy \in E(D) : x \in \{1, 4\}\}.
\]

It is rather easy to check that both subgraphs \(Q_1\) and \(Q_2\) are co-chordal and satisfy that \(E(D) = E(Q_1) \cup E(Q_2)\).

Now, we construct a graph \(G_n\) for each \(n \geq 2\) as follows. The vertex set of \(G_n\) is given by \(V(G_n) = A_n \cup \bigcup_{i=1}^n B_i\), where \(A_n = \{a_1, \ldots, a_n\}\) with \(G_n[A_n] \cong K_n\),
and \( B_i = \{ y_{i1}^i, \ldots, y_{i17}^i \} \) such that \( D_i := G_n[B_i] \cong D \) via the mapping \( y_{ij}^i \mapsto j \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq 17 \). Furthermore, the edge set of \( G_n \) is given by

\[
E(G_n) := E(K_n) \cup \bigcup_{i=1}^n E(D_i) \cup \{ a_i y_1^i : 1 \leq i \leq n \}.
\]

We remark that as the graph \( D_i \) is connected for each \( 1 \leq i \leq n \), so is the graph \( G_n \).

**Proof of Theorem 2** We initially verify that \( v(G_n) = 3n \) and \( \text{reg}(G_n) = 2n + 1 \) for every \( n \geq 2 \).

**Claim 1** 1: \( v(G_n) = 3n \) for every \( n \geq 2 \).

**Proof of the Claim** 1: Firstly, the set \( S_n := \bigcup_{i=1}^n \{ y_{i1}^i, y_{i2}^i, y_{i3}^i \} \) is an independent ve-dominating set in \( G_n \). Since the set \( \{ y_{i1}^i, y_{i2}^i, y_{i3}^i \} \) forms a triangle in \( \text{Ind}(D_i) \) which is a free face of it, the set \( S_n \) is a free face of \( \text{Ind}(G_n) \) from which we conclude that \( v(G_n) \leq 3n \).

Suppose next that \( v(G_n) = i_{ve}(G_n) = |S| \) for some subset \( S \subseteq V(G_n) \). Since \( S \) is an independent set and \( A_n \) induces a complete subgraph, we have that \( |S \cap A_n| \leq 1 \). If \( S \cap A_n = \emptyset \), it then follows that \( |S| \geq 3n \) as \( v(D_i) = i_{ve}(D_i) = 3 \) for each \( i \in [n] \). We may therefore assume that \( S \cap A_n = \{ a_1 \} \) without loss of generality. Since the vertex \( a_1 \) cannot ve-dominate any edge in the induced subgraph \( D_i \), we conclude that \( |S \cap B_i| = 3 \) for each \( 2 \leq i \leq n \). On the other hand, if we consider the graph \( L_1 := G_n[B_1 \cup \{ a_1 \}] \), we conclude that \( v(L_1) = i_{ve}(L_1) = 3 \). This readily follows from the fact that neither any vertex nor any edge in \( \text{Ind}(L_1) \) is a free face of it. However, this shows that \( |S \cap V(L_1)| = 3 \); hence, \( |S| = 3n \).

**Claim 2** 2: \( \text{reg}(G_n) = 2n + 1 \) for every \( n \geq 2 \).

**Proof of the Claim** 2: We first show that \( \text{reg}(G_n) \leq 2n + 1 \) by applying to Lemma 3 together with an induction on \( n \).

For \( n = 2 \), if we set \( L_2 := G_2[B_2 \cup \{ a_2 \}] \), we have that \( \text{reg}(G_2 - a_1) = \text{reg}(D_1) + \text{reg}(L_2) \). However, since \( \text{deg}_{L_2}(a_2) = 1 \), it follows from [1, Lemma 6.2] that we have either \( \text{reg}(L_2) = \text{reg}(L_2 - a_2) = \text{reg}(B_2) = 2 \) or else \( \text{reg}(L_2) = \text{reg}(L_2 - N_{L_2}[y_1^2]) + 1 \). As a result, we conclude that \( \text{reg}(L_2) \leq 3 \), which in turn implies the upper bound \( \text{reg}(G_2 - a_1) \leq 2 + 3 = 5 \). On the other hand, there is the isomorphism \( G_2 - N_{G_2}[a_1] \cong (D_1 - y_1^1) \cup D_2 \cup \cdots \cup D_n \) so that \( \text{reg}(G_2 - N_{G_2}[a_1]) \leq 2 + 2 = 4 \). Altogether, these imply that \( \text{reg}(G_2) \leq 5 \), which completes the case \( n = 2 \).

For every \( n \geq 3 \), notice the following isomorphisms

\[
G_n - a_1 \cong G_{n-1} \cup D_1,
\]
\[
G_n - N_{G_n}[a_1] \cong (D_1 - y_1^1) \cup D_2 \cup \cdots \cup D_n.
\]
Fig. 3 A vertex decomposable graph $R$ with $v(R) = 1 < 2 = \text{col}(\text{Ind}(R))$

Now, it follows from (11) together with the induction that

\[
\begin{align*}
\text{reg}(G_n - a_1) &= \text{reg}(G_{n-1}) + 2 \leq 2(n - 1) + 1 + 2 = 2n + 1, \\
\text{reg}(G_n - NG_n[a_1]) &= 2n.
\end{align*}
\]

Thus, we conclude that $\text{reg}(G_n) \leq 2n + 1$ for each $n \geq 2$ by Lemma 3. On the other hand, the set

\[M_n := \{a_1a_2\} \cup \{y_i^jy_{17}^j, y_i^jy_9^j : 1 \leq i \leq n\}\]

forms an induced matching in $G_n$ of size $2n + 1$. Therefore, it follows that $2n + 1 \leq \text{im}(G_n) \leq \text{reg}(G_n) \leq 2n + 1$; hence, $\text{reg}(G_n) = 2n + 1$ as claimed.

Finally, in order to complete the proof, we set $H_k := G_{k+1}$ for each $k \geq 1$. It then follows that $v(H_k) = 3k + 3$ and $\text{reg}(H_k) = 2k + 3$; thus, $v(H_k) = \text{reg}(H_k) + k$. □

5 Further comments

We recall that a simplicial complex $X$ is $k$-collapsible if it can be reduced to the void complex by repeatedly removing a free face of size at most $k$. The collapsibility number $\text{col}(X)$ of $X$ is the smallest integer $k$ such that it is $k$-collapsible. The family of $k$-collapsible simplicial complexes were introduced by Wegner [13], where he also proved that $\text{reg}(X) \leq \text{col}(X)$ holds for every simplicial complex $X$.

As a result of Corollary 5, the inequality $v(X) \leq \text{col}(X)$ holds for every simplicial complex $X$. However, it could be strict even for vertex decomposable simplicial complexes. Recall that a simplicial complex $X$ is said to be vertex-decomposable if it is either a simplex or else there exists a vertex $z$ such that $\text{del}(X; z)$ and $\text{lk}(X; z)$ are vertex decomposable, and every facet of $\text{del}(X; z)$ is a facet of $X$. In the latter, the vertex $z$ is called a shedding vertex of $X$. Now, for the graph $R$ depicted in Fig. 3, its independence complex is vertex decomposable, while $v(R) = 1 < 2 = \text{col}(\text{Ind}(R))$. 

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Finally, we point out that the graph $H_k$ constructed in the proof of Theorem 2 also provides the first example of a connected graph satisfying that $\text{col}(\text{Ind}(H_k)) \geq \text{reg}(H_k) + k$ for every $k \geq 1$ (compare to [11, Theorem 1.1(b)]).

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