DEGREE SUBTRACTION ADJACENCY EIGENVALUES AND ENERGY OF GRAPHS OBTAINED FROM REGULAR GRAPHS

HARISHCHANDRA S. RAMANE¹, HEMARADDI N. MARADDI

ABSTRACT. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$ and let $d_G(v_i)$ be the degree of a vertex $v_i$ in $G$. The degree subtraction adjacency matrix of $G$ is a square matrix $DSA(G) = [d_{ij}]$, in which $d_{ij} = d_G(v_i) - d_G(v_j)$, if $v_i$ is adjacent to $v_j$ and $d_{ij} = 0$, otherwise. In this paper we express the eigenvalues of the degree subtraction adjacency matrix of subdivision graph, semitotal point graph, semitotal line graph and total graph of a regular graph in terms of the adjacency eigenvalues of $G$. Further we obtain the degree subtraction adjacency energy of these graphs.

Mathematics Subject Classification: Primary: 05C50; Secondary: 05C07.
Key words and phrases: Degree subtraction adjacency matrix; Eigenvalues; Energy; Regular graphs.

1. Introduction

Let $G$ be a simple, undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The degree of a vertex $v_i$ denoted by $d_G(v_i)$ is the number of edges incident to it. If all vertices have same degree equal to $r$ then $G$ is called an $r$-regular graph.

The adjacency matrix of $G$ is a square matrix of order $n$, defined as $A = A(G) = [a_{ij}]$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$, otherwise. The characteristic polynomial of $A(G)$ is denoted by $\phi(G : \lambda)$, that is, $\phi(G : \lambda) = \det(\lambda I - A(G))$, where $I$ is an identity matrix. The characteristic polynomial of the adjacency matrix of a complete graph $K_n$ is $\phi(K_n : \lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}$. The roots of the equation $\phi(G : \lambda) = 0$ are called the adjacency eigenvalues of $G$.

Received 30-04-2018. Revised 18-08-2018. Accepted 21-09-2018.

¹ Corresponding Author

© 2018 Harishchandra S. Ramane and Hemaraddi N. Maraddi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
and they are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. Two non-isomorphic graphs are said to be cospectral if they have same eigenvalues. For any graph $G$, $-\Delta \leq \lambda_i \leq \Delta$, where $\Delta$ is the maximum degree. Thus for an $r$-regular graph, $\lambda_i + r \geq 0$ for $i = 1, 2, \ldots, n$.

The vertex-edge incidence matrix of $G$ is defined as $B = B(G) = [b_{ij}]$, where $b_{ij} = 1$ if the vertex $v_i$ is incident to an edge $e_j$ and $b_{ij} = 0$, otherwise.

It is easy to observe that

$$BB^T = A + D,$$

where $D = \text{diag}[d_G(v_1), d_G(v_2), \ldots, d_G(v_n)]$ is a diagonal degree matrix of $G$ and $B^T$ is the transpose of $B$.

If $G$ is an $r$-regular graph, then

$$BB^T = A + rI. \tag{1}$$

The other matrices of a graph exists in the literature such as distance matrix [2], Laplacian matrix [3], Laplacian distance matrix [4], sum-eccentricity matrix [5, 6], degree sum matrix [7, 8], degree sum adjacency matrix [9], Zagreb matrix [10], degree subtraction matrix [11], degree product matrix [12], degree square sum matrix [13] and average-degree eccentricity matrix [14].

In [15] the degree subtraction adjacency (DSA) matrix is defined as $\text{DSA}(G) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} d_G(v_i) - d_G(v_j), & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $\text{DSA}(G)$ is called the DSA-polynomial and is denoted by $\psi(G : \xi)$. Thus $\psi(G : \xi) = \det(\xi I - \text{DSA}(G))$, where $I$ is an identity matrix of order $n$.

For any regular graph of order $n$, $\psi(G : \xi) = \xi^n$. The line graph $L(G)$ of a regular graph is regular. Hence $\psi(L(G) : \xi) = \xi^m$, where $m$ is the number of edges of $G$.

The eigenvalues of $\text{DSA}(G)$, denoted by $\xi_1, \xi_2, \ldots, \xi_n$ are called DSA-eigenvalues of $G$. Two non-isomorphic graphs are said to be DSA-cospectral if they have same DSA-eigenvalues. Since $\text{DSA}(G)$ is a skew-symmetric matrix, its eigenvalues are purely imaginary or zero.

The DSA-energy of a graph $G$ is defined as

$$\text{DSAE}(G) = \sum_{i=1}^{n} |\xi_i|. \tag{2}$$

The Eq. (2) is analogous to the ordinary graph energy defined as

$$E_A(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$. The ordinary graph energy is well studied by many researchers [17].
In [13] the DSA-polynomial and DSA-energy of a path, complete bipartite graph, wheel, windmill graph and corona graph have been obtained. In this paper we obtain the DSA-eigenvalues and DSA-energy of subdivision graph, semitotal point graph, semitotal line graph and of total graph of regular graphs.

2. DSA-eigenvalues

A subdivision graph of $G$ is a graph $S(G)$ obtained from $G$ by inserting a new vertex on each edge of $G$ [18]. Thus if $G$ has $n$ vertices and $m$ edges, then $S(G)$ has $n + m$ vertices and $2m$ edges. If $u \in V(G)$ then $d_{S(G)}(u) = d_G(u)$ and if $v$ is subdivided vertex then $d_{S(G)}(v) = 2$.

Lemma 2.1. [1] If $M$ is a non-singular matrix, then we have

$$\det(MNPQ) = |M||Q - PM^{-1}N|.$$  

Theorem 2.2. Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Then

$$\psi(S(G): \xi) = \begin{cases} 
\xi^{\frac{n}{2}}(\xi^2 + 2)^{\frac{n}{2}}, & \text{if } r = 1 \\
\xi^{2n}, & \text{if } r = 2 \\
(-1)^n(r - 2)^{2n}\xi^{m-n}\phi(G: \frac{\xi^2 - r(\xi^2 + 2)}{(r - 2)^2}), & \text{if } r \geq 3.
\end{cases}$$

Proof. (i) If $r = 1$, then $G$ is a union of $k \geq 1$ edges. Thus $G$ has $n = 2k$ vertices and $k$ edges. The vertices of $S(G)$ can be labeled in such a way that

$$\text{DSA}(S(G)) = \begin{bmatrix} O & B^T \\
-B & O \end{bmatrix},$$

where $B$ is vertex-edge incidence matrix of $G$ and $O$ is zero matrix. Therefore by Lemma [2.1] and Eq. [1]

\[
\psi(S(G): \xi) = \begin{vmatrix} \xi I_m & -B^T \\
B & \xi I_n \end{vmatrix}
\]

\[
= \xi^m \begin{vmatrix} \xi I_n + B\frac{I_m}{\xi}B^T \end{vmatrix}
\]

\[
= \xi^{m-n} \begin{vmatrix} \xi^2 I_n + BB^T \end{vmatrix}
\]

\[
= \xi^{m-n} \begin{vmatrix} \xi^2 I_n + A + rI_n \end{vmatrix}
\]

\[
= (-1)^n \xi^{m-n} \begin{vmatrix} -(\xi^2 + r)I_n - A \end{vmatrix}
\]

\[
= (-1)^n \xi^{m-n} \phi(G: -(\xi^2 + r))
\]

\[
= (-1)^{2k}\xi^{k-2k} \left((\xi^2 + 1)^2 - 1\right)^k
\]

\[
= \xi^k(\xi^2 + 2)^k
\]

\[
= \xi^n(\xi^2 + 2)^{\frac{n}{2}}.
\]
Hence

Proof.

(i) If \( r = 2 \), then each component of \( G \) is cycle. Therefore \( S(G) \) is 2-regular graph on \( 2n \) vertices. Hence

\[
\psi(S(G) : \xi) = \xi^{2n}.
\]

(ii) Let \( r \geq 3 \). The vertices of \( S(G) \) can be labeled in such a way that

\[
DSA(S(G)) = \begin{bmatrix}
O & (2 - r)B^T \\
-(2 - r)B & O
\end{bmatrix},
\]

where \( B \) is vertex-edge incidence matrix of \( G \) and \( O \) is zero matrix. Therefore by Lemma 2.1 and Eq. 1

\[
\psi(S(G) : \xi) = \xi^m |\xi_n + (2 - r)^2 B^T \frac{I_m}{\xi} B^T| = \xi^{m-n} \left| \xi^2 I_n + (r - 2)^2 (A + rI) \right|
\]

\[
= (-1)^n (r - 2)^{2n} \xi^{m-n} \left( -\frac{\xi^2 - r(r - 2)^2}{(r - 2)^2} \right) I_n - A \]

\[
= (-1)^n (r - 2)^{2n} \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r(r - 2)^2}{(r - 2)^2} \right).
\]

\( \square \)

By Theorem 2.2 we have following corollary.

Corollary 2.3. Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( m \) edges. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of \( G \) and \( i = \sqrt{-1} \).

(i) If \( r = 1 \) then DSA-eigenvalues of \( S(G) \) are \( 0 \) (\( \frac{n}{2} \) times) and \( \pm \sqrt{2} \) (\( \frac{n}{2} \) times).

(ii) If \( r = 2 \) then DSA-eigenvalues of \( S(G) \) are all zeros.

(iii) If \( r \geq 3 \) then DSA-eigenvalues of \( S(G) \) are \( 0 \) (\( m-n \) times) and \( \pm (r - 2)^{\sqrt{r + \lambda_i}} \), \( i = 1, 2, \ldots, n \).

The semitotal point graph of \( G \), denoted by \( T_1(G) \), is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( T_1(G) \) are adjacent if they are adjacent vertices in \( G \) or one is a vertex and other is an edge incident to it in \( G \).

Note that if \( u \in V(G) \) then \( d_{T_1(G)}(u) = 2d_G(u) \) and if \( e \in E(G) \) then \( d_{T_1(G)}(e) = 2 \).

Theorem 2.4. Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( m \) edges. Then

\[
\psi(T_1(G) : \xi) = \begin{cases}
\xi^{m+n}, & \text{if } r = 1 \\
(-1)^n (2r - 2)^2 \xi^{m-n} \phi \left( G : \frac{-\xi^2 + (2r - 2)^2}{(2r - 2)^2} \right), & \text{if } r \geq 2.
\end{cases}
\]

Proof. (i) If \( r = 1 \), then \( T_1(G) \) is a regular graph of degree two on \( m+n \) vertices. Hence

\[
\psi(T_1(G) : \xi) = \xi^{m+n}.
\]
(ii) Let \( r \geq 2 \). The vertices of \( T_1(G) \) can be labeled in such a way that
\[
\text{DSA}(T_1(G)) = \begin{bmatrix} O & (2-2r)B^T \\ -(2-2r)B & O \end{bmatrix},
\]
where \( B \) is vertex-edge incidence matrix of \( G \) and \( O \) is a zero matrix. 
Therefore by Lemma 2.1 and Eq. (1)
\[
\psi(T_1(G) : \xi) = \left| \begin{array}{cc}
\xi I_m & -(2-2r)B^T \\
(2-2r)B & \xi I_n
\end{array} \right| = \xi^m \left| \xi I_n + (2r-2)^2 B I_m B^T \right|
\]
\[
= \xi^m \left| \xi I_n + (2r-2)^2 B I_m B^T \right|
\]
\[
= (-1)^n (2r-2)^2 \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r(2r-2)^2}{(2r-2)^2} \right).
\]

By Theorem 2.4, we have following corollary.

**Corollary 2.5.** Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( m \) edges. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of \( G \) and \( i = \sqrt{-1} \).

(i) If \( r = 1 \) then DSA-eigenvalues of \( T_1(G) \) are all zeros.

(ii) If \( r \geq 2 \) then DSA-eigenvalues of \( T_1(G) \) are \( 0 \) (\( m - n \) times) and \( \pm i (2r - 2) \sqrt{r + \lambda_i}, i = 1, 2, \ldots, n \).

Semitotal line graph of \( G \), denoted by \( T_2(G) \), is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( T_2(G) \) are adjacent if one is a vertex and other is an edge incident to it in \( G \) or both are edges adjacent in \( G \) \([1]\). Note that if \( u \in V(G) \) then \( d_{T_2(G)}(u) = d_G(u) \) and if \( e = uv \in E(G) \) then \( d_{T_2(G)}(e) = d_G(u) + d_G(v) \).

**Theorem 2.6.** Let \( G \) be an \( r \)-regular graph \( (r \geq 1) \) on \( n \) vertices and \( m \) edges. Then
\[
\psi(T_2(G) : \xi) = (-1)^n r^2 n \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r^3}{r^2} \right).
\]

**Proof.** The vertices of \( T_2(G) \) can be labeled in such a way that
\[
\text{DSA}(T_2(G)) = \begin{bmatrix} O & rB^T \\ -rB & O \end{bmatrix},
\]
where \( B \) is vertex-edge incidence matrix of \( G \) and \( O \) is a zero matrix. 
Therefore by Lemma 2.1 and Eq. (1)
\[
\psi(T_2(G) : \xi) = \left| \begin{array}{cc}
\xi I_m & -rB^T \\
rB & \xi I_n
\end{array} \right| = \xi^m \left| \xi I_n + r^2 B I_m B^T \right|
\]
\begin{align*}
\xi^{m-n} | & \xi^2 I_n + r^2 (A + r I) | \\
= & ( -1 )^r 2 n \xi^{m-n} \left| \left( \frac{-\xi^2 - r^3}{r^2} \right) I_n - A \right| \\
= & ( -1 )^r 2 n \xi^{m-n} \phi \left( G : \frac{-\xi^2 - r^3}{r^2} \right).
\end{align*}

By Theorem 2.6, we have following corollary.

**Corollary 2.7.** Let \( G \) be an \( r \)-regular graph (\( r \geq 1 \)) on \( n \) vertices and \( m \) edges. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of \( G \) and \( i = \sqrt{-1} \). Then DSA-eigenvalues of \( T_2(G) \) are 
\( 0 \) (\( m - n \) times) and 
\( \pm i r \sqrt{r + \lambda_1}, i = 1, 2, \ldots, n \).

Total graph of \( G \), denoted by \( T(G) \), is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( T(G) \) are adjacent if and only if they are adjacent vertices of \( G \) or adjacent edges of \( G \) or one is a vertex and other is an edge incident to it in \( G \) [18]. Total graph of a regular graph is regular. Hence if \( G \) is regular, then 
\( \psi(T(G) : \xi) = \xi^{m+n} \).

Two different graphs having same eigenvalues are called cospectral. If \( G_1 \) and \( G_2 \) are adjacency cospectral graphs with same regularity, then by Corollaries 2.3, 2.5 and 2.7, the graphs \( S(G_1) \) and \( S(G_2) \); \( T_1(G_1) \) and \( T_1(G_2) \); \( T_2(G_1) \) and \( T_2(G_2) \) form a pair of DSA-cospectral graphs.

3. DSA-energy

By Corollaries 2.3, 2.5 and 2.7 and by Eq. (2) we get the following proposition.

**Proposition 3.1.** Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( m \) edges. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of \( G \). Then 
(i) 
\[
\text{DSAE}(S(G)) = \begin{cases} 
 n \sqrt{2}, & \text{if } r = 1 \\
0, & \text{if } r = 2 \\
2(r - 2) \sum_{i=1}^{n} \sqrt{r + \lambda_i}, & \text{if } r \geq 3.
\end{cases}
\]

(ii) 
\[
\text{DSAE}(T_1(G)) = \begin{cases} 
 0, & \text{if } r = 1 \\
2(2r - 2) \sum_{i=1}^{n} \sqrt{r + \lambda_i}, & \text{if } r \geq 2.
\end{cases}
\]

(iii) \( \text{DSAE}(T_2(G)) = 2r \sum_{i=1}^{n} \sqrt{r + \lambda_i} \) for \( r \geq 1 \).

(iv) \( \text{DSAE}(T(G)) = 0 \) for \( r \geq 1 \).

By Proposition 3.1 we have following result.
Proposition 3.2. If $G$ is an $r$-regular graph ($r \geq 3$) on $n$ vertices, then

(i) $(2r-2)\text{DSAE}(S(G)) = (r-2)\text{DSAE}(T_1(G))$;
(ii) $r\text{DSAE}(S(G)) = (r-2)\text{DSAE}(T_2(G))$;
(iii) $r\text{DSAE}(T_1(G)) = (2r-2)\text{DSAE}(T_2(G))$.

Proposition 3.3. Let $G$ be an $r$-regular graph ($r \geq 3$) on $n$ vertices. Then

\[ \text{DSAE}(S(G)) < \text{DSAE}(T_2(G)) < \text{DSAE}(T_1(G)). \]

Proof. For $r \geq 3$, we see that

\[ r - 2 < r < 2r - 2. \]

Hence by Proposition 3.2, this implies

\[ \text{DSAE}(S(G)) < \text{DSAE}(T_2(G)) < \text{DSAE}(T_1(G)). \]

References

1. Cvetković, D. M., Doob, M., & Sachs, H. (1980). Spectra of graphs: theory and application (Vol. 87). Academic Pr.
2. Indulal, G. (2009). Distance spectrum of graph compositions. *Ars Mathematica Contemporanea*, 2, 93-100.
3. Mohar, B., Alavi, Y., Chartrand, G., & Oellermann, O. R. (1991). The Laplacian spectrum of graphs. *Graph theory, combinatorics, and applications*, 2(871-898), 12.
4. Aouchiche, M., & Hansen, P. (2013). Two Laplacians for the distance matrix of a graph. *Linear Algebra and its Applications*, 439(1), 21-33.
5. Revankar, D. S., Patil, M. M., & Ramane, H. S. (2017). On eccentricity sum eigenvalue and eccentricity sum energy of a graph. *Ann. Pure Appl. Math*, 13, 125-130.
6. Sowaity, M. I., & Sharada, B. (2017). The sum-eccentricity energy of a graph. *Int. J. Rec. Innovat. Trends Comput. Commun*, 5, 293-304.
7. Hosamani, S. M., & Ramane, H. S. (2016). On degree sum energy of a graph. *European Journal of Pure and Applied Mathematics*, 9(3), 340-345.
8. Ramane, H. S., Revankar, D. S., & Patil, J. B. (2013). Bounds for the degree sum eigenvalues and degree sum energy of a graph. *International Journal of Pure and Applied Mathematical Sciences*, 6(2), 161-167.
9. Shinde, S. S. (2016). *Spectral Graph Theory*(Ph.D. thesis), Visvesvaraya Technological University, Belagavi.
10. Rad, N. J., Jahanbani, A., & Gutman, I. (2018). Zagreb energy and Zagreb Estrada index of graphs. *MATCH Commun. Math. Comput. Chem*, 79, 371-386.
11. Ramane, H. S., Nandeesh, K. C., Gudodagi, G. A., & Zhou, B. (2018). Degree subtraction eigenvalues and energy of graphs. *Computer Science*, 26(2), 146-162.
12. Ramane, H. S., & Gudodagi, G. A. (2017). Degree product polynomial and degree product energy of specific graphs. *Asian J. Math. Comput. Res.*, 15, 94-102.
13. Basavanagoud, B., & Chitra. (2018). Degree square sum energy of graphs, *Int. J. Math. And Appl.*, 6 ,193-205.
14. Gutman, I., Mathad, V., Khalaf, S. I., & Mahde, S. S. (2018). Average Degree-Eccentricity Energy of Graphs. *Mathematics Interdisciplinary Research*, 3(1), 45-54.
15. Ramane, H. S., Maraddi, H. N., & Jummannaver R. B. (Preprint 2018). Degree subtraction adjacency eigenvalues and energy of graphs.
16. Gutman, I. (1978). The Energy of a Graph. *Ber. Math. Stat. Sekt. Forschungsgr. Graz*, 103, 1-22.
17. Li, X., Shi, Y., & Gutman, I. (2012). *Graph Energy*. Springer, New York, NY.
18. Harary, F. (1999). *Graph Theory*. Narosa Publishing House, New Delhi.
19. Sampathkumar, E., & Chikkodimath, S. B. (1973). Semitotal graphs of a graph-I. *J. Karnatak Univ. Sci.*, 274-280.

**Harishchandra S. Ramane**
1. Department of Mathematics, Karnatak University, Dharwad - 580003, India.
   e-mail: hsramane@yahoo.com

**Hemaraddi N. Maraddi**
2. Department of Mathematics, Karnatak University, Dharwad - 580003, India.
   e-mail: hemaraddi.kud@gmail.com