Anomalies and nonperturbative results.

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Abstract

We investigate nonperturbative effects in N=1 and N=2 supersymmetric theories using a relation between perturbative and exact anomalies as a starting point. For N=2 supersymmetric SU(n) Yang-Mills theory we derive the general structure of the Picard-Fuchs equations; for N=1 supersymmetric Yang-Mills theories we find holomorphic part of the superpotential (with gluino condensate) exactly.

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1 Introduction

The presence of quantum anomalies in the field theory is known for a long time and plays an important role in the high energy physics \[^1\]. However, they were usually studied in the frames of perturbation theory. Only recently the exact expression for R-anomaly was found in N=2 supersymmetric Yang-Mills theory \[^2, 3\]. Due to the instanton contributions it differs from the perturbative result. Such possibility was pointed rather long ago \[^4\], but a series of instanton corrections with unknown coefficients produced considerable difficulties, in particular, in the research of anomalies cancellation. So, obtaining of exact results becomes very important. Their derivation in a number of papers \[^2, 3, 5, 6\] is based on the exact results of Seiberg and Witten \[^7\], but the result appeared to have a very simple interpretation: exact anomaly is a vacuum expectation value of the perturbative one. Nevertheless, for checking this relation one should essentially use the exact expression for the prepotential, found in \[^7\] by completely different methods. Thus, we come to the question, whether it is possible to solve the inverse problem, i.e. to derive exact results from the form of anomalies. In the present paper we try to do it for supersymmetric Yang-Mills theories. For N=2 SU\( (N_c) \) the presented approach allows to derive the general structure of Picard-Fuchs equations. The investigation of N=1 theories turns out to be very similar. In particular, the holomorphic part of the N=1 superpotential is found to satisfy Picard-Fuchs equation, that can be solved exactly.

Our paper is organized as follows: Section 2 is devoted to the brief review of necessary information concerning N=1 and N=2 supersymmetric Yang-Mills theories (for details see \[^8, 9\]). In the Section 3 we show, that the collective coordinate measure is not invariant under \( U(1)_R \)-transformations. Using its transformation law we are able to define the general structure of nonperturbative corrections, that agrees with instanton calculations. In the Section 4 we derive the relation between perturbative and exact anomalies. First, in the Subsection 4.1 we reobtain the exact expression for the R-anomaly in the N=2 supersymmetric SU\( (2) \) Yang-Mills theory \[^2, 3\] as an indication to the result. Then in the Subsection 4.2 suggested relation is formulated and proven. The Section 5 is devoted to consequences. In the Subsection 5.1 from the relation between perturbative and exact anomalies we derive the general structure of Picard-Fuchs equations and restrictions on their form. Then the presented approach is applied to N=1 supersymmetric Yang-Mills theory. In the Subsection 5.2 we investigate the holomorphic part of the superpotential and find its structure. The exact result is obtained in the Subsection 5.3. Conclusion is devoted to the brief review and discussion of the results. Some auxiliary facts are given in the Appendix.

2 Supersymmetric Yang-Mills theories

2.1 N=1 supersymmetry
The massless N=1 supersymmetric Yang-Mills theory with $SU(N_c)$ gauge group and $N_f$ matter multiplets is described by the action

$$S = \frac{1}{16\pi} \text{tr} \, \text{Im} \left( \tau \int d^4xd^2\theta \, W^2 \right) + \frac{1}{4} \int d^4xd^4\theta \sum_{A=1}^{N_f} \left( \phi_A^+ e^{-2V} \phi_A^A + \tilde{\phi}_A^+ e^{2V} \tilde{\phi}_A^A \right)$$

where the matter superfields $\phi$ and $\tilde{\phi}$ belong to fundamental and antifundamental representations of the gauge group $SU(N_c)$.

Here we use the following notations

$$A_\mu = eA^a_\mu T^a \quad \text{and so on,} \quad \text{tr}T^a T^b = \delta^{ab},$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}; \quad (2)$$

$$V(x, \theta) = -\frac{i}{2} \theta \gamma^\mu \gamma_5 \theta A_\mu(x) + i\sqrt{2}(\tilde{\theta}\theta)(\tilde{\theta}\gamma_5 \lambda(x)) + \frac{i}{4}(\tilde{\theta}\theta)^2 D;$$

$$W(y, \theta) = \frac{1}{2}(1 + \gamma_5) \left(i\sqrt{2}\lambda(y) + i\theta D(y) + \frac{1}{2} \Sigma_{\mu\nu}\theta F_{\mu\nu}(y) + \frac{1}{\sqrt{2}} \tilde{\theta}(1 + \gamma_5)\theta \gamma^\mu D_\mu \lambda(y) \right);$$

$$\phi(y, \theta) = \varphi(y) + \sqrt{2}\lambda(y) \psi(y) + \frac{1}{2} \tilde{\theta}(1 + \gamma_5) \theta f(y);$$

$$y^\mu = x^\mu + \frac{i}{2} \theta \gamma^\mu \gamma_5 \theta. \quad (3)$$

Eliminating auxiliary fields we find that in components the action (1) is written as

$$S = \frac{1}{e^2} \text{Re} \, \text{tr} \left( d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda(1 + \gamma_5) \gamma^\mu D_\mu \lambda + \frac{\theta e^2}{32\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu} \right) \right.$$

$$+ \sum_A \int d^4x \left( D_\mu \varphi_A^+ D^\mu \varphi^A + D_\mu \tilde{\varphi}_A^+ D^\mu \tilde{\varphi}_A^A + i \bar{\Psi} \gamma^\mu D_\mu \Psi \right.$$ $$- i \bar{\Psi}_A(1 - \gamma_5) \lambda \varphi^A + i \varphi_A^+ \tilde{\lambda}(1 - \gamma_5) \Psi^A - i \tilde{\varphi}_A^+ \Psi_A(1 + \gamma_5) \lambda + i \tilde{\lambda}(1 - \gamma_5) \Psi^A \tilde{\varphi}_A$$

$$+ \frac{1}{2} \left( \varphi_A^+ T^n \varphi^A - \tilde{\varphi}_A^+ T^n \tilde{\varphi}_A^A \right)^2 \right) \left. \right)$$

where we introduced the Dirac spinor

$$\Psi \equiv \frac{1}{2} \left[ (1 + \gamma_5) \psi + (1 - \gamma_5) \tilde{\psi} \right]$$

In the massless case the action is invariant under the transformations
$U(1)_1: \quad W(\theta) \to e^{i\alpha} W(e^{-i\alpha \gamma_5} \theta), \quad \phi(\theta) \to \phi(e^{-i\alpha \gamma_5} \theta), \quad \tilde{\phi}(\theta) \to \tilde{\phi}(e^{-i\alpha \gamma_5} \theta);$

$U(1)_2: \quad W(\theta) \to W(\theta), \quad \phi(\theta) \to e^{i\beta} \phi(\theta), \quad \tilde{\phi}(\theta) \to e^{i\beta} \tilde{\phi}(\theta).$ (6)

that in components are written as

$U(1)_1: \quad A_{\mu} \to A_{\mu}; \quad \varphi \to \varphi; \quad \tilde{\varphi} \to \tilde{\varphi};$

$\lambda \to e^{i\alpha \gamma_5} \lambda; \quad \Psi \to e^{-i\alpha \gamma_5} \Psi.$

$U(1)_2: \quad A_{\mu} \to A_{\mu}; \quad \varphi \to e^{i\beta} \varphi; \quad \tilde{\varphi} \to e^{i\beta} \tilde{\varphi};$

$\lambda \to \lambda; \quad \Psi \to e^{i\beta \gamma_5} \Psi.$ (7)

The conservation of corresponding currents

\[ J^\mu_1 = \bar{\lambda}^a (1 + \gamma_5) \gamma^\mu \lambda^a + \sum_A \bar{\Psi}_A \gamma^\mu \gamma_5 \Psi_A; \]

\[ J^\mu_2 = - \sum_A \bar{\Psi}_A \gamma^\mu \gamma_5 \Psi_A - i \sum_A \left( \bar{\varphi}^*_A D^\mu \varphi_A - D^\mu \bar{\varphi}^*_A \varphi_A + \bar{\tilde{\varphi}}^*_A D^\mu \tilde{\varphi}_A - D^\mu \tilde{\varphi}^*_A \tilde{\varphi}_A \right). \] (8)

is destroyed at the quantum level by anomalies. In the perturbation theory

\[ \partial_\mu J^\mu_1 = (-N_f + N_c) \frac{1}{16\pi^2} \varepsilon^{\mu\alpha\beta} \text{tr} F_{\mu\nu} F_{\alpha\beta} = (N_f - N_c) \frac{1}{16\pi^2} \text{Im} \text{tr} \int d^2\theta W^2; \]

\[ \partial_\mu J^\mu_2 = N_f \frac{1}{16\pi^2} \varepsilon^{\mu\alpha\beta} \text{tr} F_{\mu\nu} F_{\alpha\beta} = -N_f \frac{1}{16\pi^2} \text{Im} \text{tr} \int d^2\theta W^2. \] (9)

Nevertheless, it is possible to construct an anomaly free symmetry. Really, from (7) we conclude, that

\[ J^\mu_R \equiv J^\mu_1 + \frac{N_f - N_c}{N_f} J^\mu_2 = \bar{\lambda}^a (1 + \gamma_5) \gamma^\mu \lambda^a + \frac{N_c}{N_f} \sum_A \bar{\Psi}_A \gamma^\mu \gamma_5 \Psi_A \]

\[ - i \sum_A \left( 1 - \frac{N_c}{N_f} \right) \left( \bar{\varphi}^*_A D^\mu \varphi_A - D^\mu \bar{\varphi}^*_A \varphi_A + \bar{\tilde{\varphi}}^*_A D^\mu \tilde{\varphi}_A - D^\mu \tilde{\varphi}^*_A \tilde{\varphi}_A \right) \] (10)

is conserved even at the quantum level.

This current is produced by the transformations

$U(1)_R: \quad W(\theta) \to e^{i\alpha_R} W(e^{-i\alpha_R \gamma_5} \theta);$

$\phi(\theta) \to \exp \left( i\alpha_R \frac{N_f - N_c}{N_f} \right) \phi(e^{-i\alpha_R \gamma_5} \theta);$

$\tilde{\phi}(\theta) \to \exp \left( i\alpha_R \frac{N_f - N_c}{N_f} \right) \tilde{\phi}(e^{-i\alpha_R \gamma_5} \theta).$ (11)
Below we will also use the combination of $U(1)_1$ and $U(1)_2$ with $\beta = x\alpha$ in (7), i.e.

\[
\begin{align*}
U(1)_x : & \quad W(\theta) \to e^{i\alpha}W(e^{-i\alpha\gamma_5}\theta); \\
& \quad \phi(\theta) \to e^{i\alpha}\phi(e^{-i\alpha\gamma_5}\theta); \\
& \quad \tilde{\phi}(\theta) \to e^{i\alpha}\tilde{\phi}(e^{-i\alpha\gamma_5}\theta). 
\end{align*}
\]

(12)

where $x$ is an arbitrary constant.

In particular, for $x = (N_f - N_c)/N_f$ we obtain $U(1)_R$ transformations; for $x = 0$ - $U(1)_1$ and for $x \to \infty$ (after redefinition $\alpha \to \alpha/x$) $U(1)_2$.

The corresponding current is

\[
J^\mu_x \equiv J^\mu_1 + xJ^\mu_2 = \bar{\lambda}(1 + \gamma_5)\gamma^\mu\lambda + (1 - x)\sum_A \bar{\Psi}_A\gamma^\mu\gamma_5\Psi_A
\]

\[-i\sum_A x(\varphi_A^* D^\mu\varphi_A - D^\mu\varphi_A^*\varphi_A + \tilde{\varphi}_A^* D^\mu\tilde{\varphi}_A - D^\mu\tilde{\varphi}_A^*\tilde{\varphi}_A).
\]

(13)

In the perturbation theory

\[
\partial_\mu J^\mu_x = \left(-N_f + N_c + xN_f\right)\frac{1}{16\pi^2}e^{\mu\nu\alpha\beta}\text{tr}F_{\mu\nu}F_{\alpha\beta}
\]

\[-i\sum_A x\left(\varphi_A^* D^\mu\varphi_A - D^\mu\varphi_A^*\varphi_A + \tilde{\varphi}_A^* D^\mu\tilde{\varphi}_A - D^\mu\tilde{\varphi}_A^*\tilde{\varphi}_A\right)
\]

\[
= \left(N_f - N_c - xN_f\right)\frac{1}{16\pi^2}\text{Im tr} \int d^2\theta W^2. \quad (14)
\]

2.2 $N=2$ supersymmetry

In the superspace $N=2$ supersymmetric Yang-Mills theory is described by the action

\[
S = \frac{1}{32\pi}\text{tr \text{Im} } (\tau \int d^4d^2\theta_1d^2\theta_2\frac{1}{2}\Phi^2) \quad (15)
\]

where

\[
\Phi(y, \theta_1, \theta_2) = \phi(y, \theta_1) - i\bar{\theta}_2(1 + \gamma_5)W(y, \theta_1) + \frac{1}{2}\bar{\theta}_2(1 + \gamma_5)\theta_2G(y, \theta_1);
\]

\[
y^\mu = x^\mu + \frac{i}{2}\bar{\theta}_1\gamma^\mu\gamma_5\theta_1;
\]

\[
G(y, \theta_1) = \frac{1}{2}\int d^2\bar{\theta}_1e^{2V}\bar{\phi}^+\phi - 2V.
\]

(16)

where $\phi$ and $W$ were defined in (3).

Denoting $\varphi \equiv P + iS$, $\psi_1 \equiv \lambda$ and $\psi_2 \equiv \psi$, we find that in components
\[ S = \frac{1}{e^2} \text{tr} \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^2 - i\bar{\psi}_i \gamma^\mu D_\mu \psi_i + \frac{1}{2} (D_\mu P)^2 + \frac{1}{2} (D_\mu S)^2 - \frac{1}{2} [P, S]^2 \right. \\
- i\epsilon_{ij} \{\bar{\psi}_i, \gamma_5 \psi_j\} P - \epsilon_{ij} \{\bar{\psi}_i, \psi_j\} S + \frac{\theta e^2}{32\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu} \right) \] (17)

In this paper we will consider only the case of SU(n) gauge group. The action (15) is invariant under the transformations

\[ U(1)_R : \Phi(\theta) \rightarrow e^{2i\alpha} \Phi(e^{-i\alpha\gamma} \theta) \] (18)

In components they are written as

\[ \varphi \rightarrow e^{2i\alpha} \varphi; \quad \psi_i \rightarrow e^{i\alpha\gamma} \psi_i; \quad A_\mu \rightarrow A_\mu \] (19)

So, R-symmetry leads to the chiral transformations for fermions. Using the expression for the axial anomaly we find that in the perturbation theory for SU(n) gauge group

\[ \langle \partial_\mu j_R^\mu \rangle_{\text{pert}} = \frac{n}{16\pi^2} e^{\mu\nu} \text{tr} F_{\mu\nu} F_{\alpha\beta} = -\frac{n}{32\pi^2} \text{Im} \text{tr} \int d^2\theta_1 d^2\theta_2 \Phi^2, \] (20)

where

\[ j_R^\mu = \bar{\psi}_i^a \gamma^\mu \gamma_5 \psi_i^a - 4D^\mu P^a S^a + 4D^\mu S^a P^a. \] (21)

Below we will see, that (20) is no longer valid beyond the frames of the perturbation theory. Although the existence of instanton corrections was predicted rather long ago [4], it is much better to have an exact result. Its derivation requires information about the vacuum structure and low energy limit of the theory. Here we would like to remind some key points.

The classical potential for the scalar superfield component \( \varphi \) is given by

\[ -\frac{1}{4} [\varphi, \varphi^+]^2_a = \frac{1}{2} [P, S]^2_a. \] (22)

It leads to a continuous family of unequivalent ground states, which constitutes the classical moduli space \( M_0 \). In order to characterize \( M_0 \) we note, that one can always rotate \( \varphi \) into Cartan sub-algebra

\[ \varphi = \sum_{k=1}^r a_k H_k. \] (23)

Here \( r \) denotes the rank of gauge group G. Below we will consider only \( G = \text{SU}(n) \), so that \( r = n-1 \). In the generic point of \( M_0 \) it is spontaneously broken down to \( U(1)^{n-1} \).

The Cartan sub-algebra variables \( a_i \) are not gauge invariant, and, therefore, one should introduce other variables for parametrizing the classical moduli space. It can be done in the following way:

Let us consider
\[ W_{A_{n-1}} \equiv \langle \det(\mathbf{x} - \varphi) \rangle \] (24)

whose coefficients are gauge invariant. If (in the case of SU(n))

\[ \varphi = \text{diag}(a_1, a_2, \ldots, a_n), \quad \sum_i a_i = 0, \] (25)

we find that classically

\[ W_{A_{n-1}} = x^n + x^{n-2} \sum_{i<j} a_i a_j - x^{n-3} \sum_{i<j<k} a_i a_j a_k + \ldots + (-1)^n \prod_i a_i. \] (26)

\[ \text{From the other hand} \]

\[ W_{A_{n-1}} = \langle x^n \det(1 - \frac{\varphi}{x}) \rangle = x^n \langle \exp \left[ \text{tr} \ln \left( 1 - \frac{\varphi}{x} \right) \right] \rangle = \\
= x^n + x^{n-2} \frac{1}{2} \langle \text{tr} \varphi^2 \rangle - x^{n-3} \frac{1}{3} \langle \text{tr} \varphi^3 \rangle + O(x^{n-4}) \equiv x^n + \sum_{k=1}^{n-1} x^{n-1-k} u_k(a). \] (27)

Therefore, the gauge invariant description of the theory can be made in terms of

\[ u_k(a) = \prod_{n_1 < n_2 < \ldots < n_k} a_{n_1} a_{n_2} \ldots a_{n_k}. \] (28)

In particular,

\[ u_1 = \frac{1}{2} \langle \text{tr} \varphi^2 \rangle; \quad u_2 = -\frac{1}{3} \langle \text{tr} \varphi^3 \rangle. \] (29)

In the cases of SU(2) and SU(3) it is easy to see, that

\[ SU(2) : \quad W_{A_1} = x^2 - u; \]

\[ u_1 \equiv u = \frac{1}{2} \langle \text{tr} \varphi^2 \rangle = \frac{1}{2} a^2; \]

\[ SU(3) : \quad W_{A_2} = x^3 - xu - v; \]

\[ u_1 \equiv u = \frac{1}{2} \langle \text{tr} \varphi^2 \rangle = -\sum_{i<j} a_i a_j = a_1^2 + a_2^2 + a_1 a_2; \]

\[ u_2 \equiv v = -\frac{1}{3} \langle \text{tr} \varphi^3 \rangle = -a_1 a_2 a_3 = a_1 a_2 (a_1 + a_2). \] (30)

As we mentioned above, in the low energy limit the theory is described by \( r = n - 1 \) N=2 abelian superfields \( \Phi_i \). N=2 supersymmetry constrains the form of effective action to be

\[ \Gamma = \frac{1}{32\pi} \text{Im} \int d^4 x d^2 \theta_1 d^2 \theta_2 F(\Phi_i) \] (31)
where $F$, called prepotential, depends only on $\Phi$ and not on $\Phi^+$. The low energy effective action was shown \cite{7} to be invariant under duality transformations

$$
F(\Phi) \rightarrow F_D(\Phi_D) = F(\Phi) - \Phi_i \Phi^i_D \big|_{\Phi = \Phi(\Phi_D)};
$$

$$
\Phi \rightarrow \Phi_D^i = \frac{\partial F(\Phi)}{\partial \Phi_i}.
$$

(32)

Vacuum expectation values of dual superfields we denote as $a_i^D$.

The explicit form of $a_i$ and $a_i^D$ is usually found by Seiberg-Witten elliptic curve method \cite{4, 10}. However, there is another approach. Let us note, that

$$
\vec{a} = \left( \begin{array}{c} a_i \\ a_i^D \end{array} \right)
$$

(33)

satisfies a system of second order differential equations (so called Picard-Fuchs equations). Its explicit form was found to be

$$
\left(4(u^2 - \Lambda^4)\partial_u^2 + 1\right)\vec{a} = 0
$$

(34)

for the case of SU(2) \cite{11} and

$$
\left((27\Lambda^6 - 4u^2 - 27v^2)\partial_u^2 - 12u^2v\partial_u \partial_v - 3uv\partial_v - u\right)\vec{a} = 0
$$

(35)

$$
\left((27\Lambda^6 - 4u^2 - 27v^2)\partial_v^2 - 36uv\partial_u \partial_v - 9v\partial_v - 3\right)\vec{a} = 0
$$

for the SU(3) \cite{12} gauge group. (Here $\Lambda$ is the instanton generated scale.)

Below we will see, that Picard-Fuchs equations play a crucial role in the consideration of anomalies. Actually, they assure, that the relation between perturbative and exact anomalies is satisfied.

In the SU(2) case one can easily solve (34) and finds, that taking into account perturbative asymptotics

$$
a(u) = \frac{\sqrt{2}}{\pi} \int_{-\Lambda^2}^{\Lambda^2} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - \Lambda^4}};
$$

$$
a_D(u) = \frac{\sqrt{2}}{\pi} \int_{\Lambda^2}^{u} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - \Lambda^4}}.
$$

(36)

The prepotential $F$ can be then found by

$$
\frac{dF}{du} = a_D \frac{da}{du}
$$

(37)

and its perturbative asymptotics.
3 Instanton contributions to anomalies and the structure of the effective potential

3.1 N=2 SUSY SU(2) Yang-Mills theory

First we would like to discuss how instantons contribute to anomalies. On the one hand anomalies can be defined from the effective action and, therefore, instanton corrections to the effective action lead to the instanton corrections to anomalies. However, in this section we will try to investigate the mechanism of their appearance and show, that nonperturbative contributions arise due to the noninvariance of collective coordinate measure. The developed approach extends the method, presented in [4]. However, comparing the results with the form of the effective action allows to predict the structure of nonperturbative superpotential, which will be used below.

Nonperturbative contributions to the effective action are obtained by the expansion of the generating functional near the instanton solution [13]

\[ \Delta \Gamma = \frac{1}{Z_0} \int d\mu D\phi \exp\left(-S[\phi_0 + \phi]\right). \] (38)

Here \( \phi \) denotes the whole set of fields, \( \phi_0 \) is a classical instanton solution and \( d\mu \) is a collective coordinate measure.

At the one-instanton level there are 8 bosonic zero modes, due to the invariances under 4 translations (the corresponding collective coordinates are \( a^\mu \)), rescaling (\( \rho \)) and 3 gauge transformations (\( \omega \)). Moreover, N=2 supersymmetry adds 4 supersymmetries (\( \epsilon_i \)) and 4 superconformal transformations (\( \beta_i \)) [14]. If the scalar field has nonzero vacuum expectation value, the superconformal modes are lifted due to the conformal symmetry breaking. Nevertheless, we still keep the integration over the corresponding parameters in the instanton measure following [13, 16, 17].

The final expression for the one-instanton measure [17] is

\[ d\mu = \text{const} \int d^4a \frac{d\rho d^3\omega}{\rho^5 2\pi^2} M^8 \rho^8 \times \frac{1}{M^4 \rho^4} \int d^2\epsilon_1 d^2\epsilon_2 d^2\beta_1 d^2\beta_2 \] (39)

A next step to obtain nonperturbative corrections is the calculation of the exponent in the constant field limit [15]. However, it can be omitted, because we are going to investigate only the general structure of instanton corrections.

Really, let us perform \( U(1)_R \)-transformations in the generating functional [38]. Because

\[ \epsilon_i \rightarrow e^{i\alpha \gamma_5} \epsilon_i; \quad \rho \rightarrow \rho; \]
\[ \beta_i \rightarrow e^{i\alpha \gamma_5} \beta_i; \quad a^\mu \rightarrow a^\mu; \]
\[ \delta \Phi \rightarrow e^{2i\alpha \Phi} \]  

\[ ^1 \text{Of course, this structure can be obtain from dimensional arguments, but the presented approach seems more relevant in the paper, devoted to anomalies.} \]
the collective coordinate measure is not invariant. It is easy to see, that the exponent in (38), being a function of collective coordinates, is not invariant too. For example, the scalar field contribution \(4\pi^2 \rho^2 \Phi^2\) has evidently nontrivial transformation law.

However, the invariance of the exponent can be easily restored by additional variable substitution

\[
\begin{align*}
\theta &\rightarrow e^{-i\alpha \gamma_5 \theta}; \\
\rho &\rightarrow e^{-2i\alpha \rho}; \\
a^\mu &\rightarrow e^{-2i\alpha a^\mu},
\end{align*}
\]

because the overall transformations

\[
\begin{align*}
x^\mu &\rightarrow e^{-2i\alpha x^\mu}; \\
a^\mu &\rightarrow e^{-2i\alpha a^\mu}; \\
\rho &\rightarrow e^{-2i\alpha \rho};
\end{align*}
\]

\[
\begin{align*}
(1 + \gamma_5) \epsilon &\rightarrow e^{i\alpha (1 + \gamma_5) \epsilon}; \\
(1 - \gamma_5) \beta &\rightarrow e^{-i\alpha (1 - \gamma_5) \beta}
\end{align*}
\]
do not effect any dimensionless function of collective coordinates.

Thus, only the instanton measure is transformed nontrivially under (42)

\[
d\mu \rightarrow \left(e^{-2i\alpha}\right)^4 \left(e^{-i\alpha}\right)^4 \left(e^{i\alpha}\right)^4 d\mu = e^{-8i\alpha} d\mu.
\]

For n-instanton contribution we should consider the limit, where a multiinstanton solution is presented as a sum of n instantons, distant from each other. Then we immediately conclude, that

\[
d\mu^{(n)} \rightarrow e^{-8i\alpha} d\mu^{(n)}.
\]

Taking into account that

\[
\int d^4 x d^2 \theta_1 d^2 \theta_2
\]

(which is present in the prepotential definition) remains invariant under (12) we find

\[
\langle \partial_\mu j_R^\mu \rangle = -\frac{\partial \Gamma}{\partial \alpha} \bigg|_{\alpha=0} = \sum_{n=0}^\infty \frac{1}{32\pi} \text{Im}\left(8i n \int d^2 \theta_1 d^2 \theta_2 \Delta F^{(n)}\right),
\]

where \(\Delta F^{(n)}\) is n-instanton contribution to the prepotential.

On the other hand, transforming (31) we obtain

\[
\langle \partial_\mu j_R^\mu \rangle = -\frac{1}{32\pi} \text{Im} \int d^2 \theta_1 d^2 \theta_2 \left(2i\Phi \frac{\partial}{\partial \Phi} - 4i\right) \sum_{n=0}^\infty \Delta F^{(n)}.
\]

Solving the equation

\[
\Phi \frac{\partial}{\partial \Phi} \Delta F^{(n)} = \left(-4n + 2\right) \Delta F^{(n)}
\]

Note, that the transformations of \(\theta\) and \(x^\mu\) are not independent

\(^3\)Of course, our method is in a deep connection with dimensional arguments, although it does not completely repeat them. For example, \(\theta\)-transformation law does not correspond to its dimension.
we obtain the final structure of the nonperturbative anomaly to be

$$
\langle \partial_\mu J^R_\mu \rangle = -\frac{1}{16\pi^2} \text{Im} \int d^4 x d^2 \theta_1 d^2 \theta_2 \sum_{n=0}^{\infty} c_n \Phi^{2(-2n+1)},
$$

where $c_n \epsilon \text{Re}$ and $c_0 = 1$

### 3.2 N=1 SUSY Yang-Mills with matter

Now let us consider N=1 supersymmetric $SU(N_c)$ Yang-Mills theory with $N_f$ matter supermultiplets and find the general possible structure of instanton corrections to superpotential and anomalies.

The effective Lagrangian can be split into the following parts

$$
L_{\text{eff}} = L_k + L_a + L_m,
$$

where

$$
L_k = \int d^4 \theta K(S, S^*, \Phi, \Phi^*);
$$

$$
L_a = \text{Re} \int d^2 \theta w(S, \Phi)
$$

and $S \equiv \text{tr}W^2$

Here $L_k$ denotes kinetic terms, that do not contribute to the anomaly, $L_a$ is a holomorphic part of the superpotential and $L_m$ is a mass term. Below we will consider only massless case ($L_m = 0$). Therefore, the only nontrivial contribution to anomalies comes from $L_a$ and it is the only part, that we are able to investigate. (Our method can not give any information about a possible kinetic term.)

As above we will calculate anomalies by 2 different ways and compare the results. The action is invariant under $U(1)_1 \times U(1)_2$ group. However, it is more convenient to investigate the anomaly of $U(1)_x$ symmetry, constructed in Section 2.1. Performing $U(1)_x$ transformation in the effective action we obtain

$$
\langle \partial_\mu J^\mu_x \rangle = -\frac{\partial \Gamma}{\partial \alpha} = -\text{Im} \int d^2 \theta \left( 2w - 2 \frac{\partial w}{\partial S} S - x \frac{\partial w}{\partial v} v \right),
$$

where we substituted $\phi$ and $\tilde{\phi}$ by their vacuum expectation values $v$. (For simplicity we assume, that all $v_i$ are equal; a brief review of the moduli space structure is given in the Appendix 4.)

On the other hand, the anomaly can be found from the transformation law of the collective coordinate measure.

At the one-instanton level in this case there are 8 bose zero modes (exactly as above), $2N_c$ gluino zero modes (corresponding to supersymmetric ($\epsilon_a$) and superconformal ($\beta_a$) transformations) and $2N_f$ zero modes for matter multiplets (supersymmetry $\epsilon_A$). Each
zero mode should be removed by integration over the corresponding collective coordinate. The measure is written as \([14]\)

\[
dµ = \text{const} \int d^4a \frac{dρ}{ρ^3} (Mρ)^{4Nc} d(\text{gauge}) \frac{1}{M^{Nc+Nf} ρ^{Nc}} \prod_{a=1}^{Nc} dε_a dβ_a \prod_{A=1}^{Nf} \frac{dε_A d\tilde{ε}_A}{ρ^2 v^2} \exp \left( -\frac{8π^2}{e^2} \right) \]

\[
= \text{const} A^{3Nc-Nf} \int d^4a dρ ρ^{3Nc-2Nf-5} \frac{1}{v^{2Nf}} d(\text{gauge}) \prod_{a=1}^{Nc} dε_a dβ_a \prod_{A=1}^{Nf} dε_A d\tilde{ε}_A, \quad (53)
\]

where we take into account normalization of all zero modes. The gauge part and constant factors are written only schematically, because they are not important in our discussion. As above we need not know the explicit form of the action in the constant field limit. We should only emphasize, that it is a dimensionless function of collective coordinates, \(φ\) and, in principle, \(W\). Of course, it is not invariant under \(U(1)_x\)-transformations

\[
W \rightarrow e^{iαγ_5} W; \quad θ \rightarrow e^{-iαγ_5} θ
\]

\[
φ \rightarrow e^{iα} φ; \quad \tilde{φ} \rightarrow e^{iα} \tilde{φ};
\]

\[
ε_a \rightarrow e^{iαγ_5} ε_a; \quad β_a \rightarrow e^{iαγ_5} β_a;
\]

\[
ε_A \rightarrow e^{i(x-1)αγ_5} ε_A; \quad \tilde{ε}_A \rightarrow e^{i(x-1)αγ_5} \tilde{ε}_A;
\]

\[
ρ \rightarrow e^{-2iα ρ}; \quad a^μ \rightarrow e^{-2iα a^μ},
\]

(54)

as above.

Similarly to \(N=2\) supersymmetric Yang-Mills theory we perform an additional substitution

\[
θ \rightarrow e^{-iαγ_5} θ; \quad x^μ \rightarrow e^{-2iα} x^μ;
\]

\[
ε_A \rightarrow e^{-iαγ_5} ε_A; \quad \tilde{ε}_A \rightarrow e^{-iαγ_5} \tilde{ε}_A;
\]

\[
ρ \rightarrow e^{-2iα ρ}; \quad a^μ \rightarrow e^{-2iα a^μ};
\]

(55)

so that the final transformations

\[
(1 + γ_5)ε_a \rightarrow e^{iα} (1 + γ_5)ε_a; \quad (1 - γ_5)β_a \rightarrow e^{-iα} (1 - γ_5)β_a;
\]

\[
(1 + γ_5)ε_A \rightarrow e^{-iα} (1 + γ_5)ε_A; \quad (1 + γ_5)\tilde{ε}_A \rightarrow e^{-iα} (1 + γ_5)\tilde{ε}_A;
\]

\[
ρ \rightarrow e^{-2iα ρ}; \quad a^μ \rightarrow e^{-2iα a^μ}; \quad θ \rightarrow e^{-2iαγ_5} θ
\]

(56)

(−except for \(θ\)) correspond to dimension of the fields. The dimensionless action would have been invariant, if we had made additional rotation

13
\[ v \to e^{i(2-x)\alpha} v; \quad W \to e^{2i\alpha\gamma_5} W. \] (57)

However, we cannot make it because \( v \) and \( W \) are not collective coordinates (and, therefore, integration variables). It means, that under (56)

\[
S(v, W) \to S(e^{i(x-2)\alpha} v, e^{-2i\alpha\gamma_5} W);
\]

\[
d\mu(v) \to \exp \left[ i\alpha \left( -2(3N_c - 2N_f) - 2N_f(x + 2N_f(x - 2) - 2N_f) \right) \right] \quad d\mu(e^{i(x-2)\alpha} v)
\]

\[
= \exp \left[ i\alpha \left( -2(3N_c - N_f) \right) \right] \quad d\mu(e^{i(x-2)\alpha} v).
\] (58)

It is quite evident, that the \( n \)-instanton collective coordinate measure is transformed as

\[
d\mu(v) \to \exp \left[ i\alpha \left( -2n(3N_c - N_f) \right) \right] \quad d\mu(e^{i(x-2)\alpha} v).
\] (59)

Moreover, we should also perform the inverse substitution in the remaining integral (see the definition of the superpotential)

\[
\int d^4x d^2\theta \to e^{4i\alpha} \int d^4x d^2\theta,
\] (60)

so that finally from (57), (59) and (60) we conclude, that

\[
w(v, W) \to \exp \left[ i\alpha \left( -2n(3N_c - N_f) \right) + 4 \right] \quad w(e^{i(x-2)\alpha} v, e^{-2i\alpha\gamma_5} W).
\] (61)

Taking into account that the action contains only \((1 + \gamma_5)W\), we find the anomaly to be

\[
\langle \partial_{\mu} J_{R}\rangle = - \left. \frac{\partial \Gamma}{\partial \alpha} \right|_{\alpha = 0} =
\]

\[
= \text{Im} \int d^2\theta \left[ -2n(3N_c - N_f) + (-2 + x)v \frac{\partial}{\partial v} - 2W \frac{\partial}{\partial W} + 4 \right] w.
\] (62)

Comparing (52) and (52), we obtain the following equation for \( n \)-instanton contribution to the superpotential:

\[
\left( 2v \frac{\partial}{\partial v} + 3W \frac{\partial}{\partial W} - 6 \right) w^{(n)} = -2n(3N_c - N_f)w^{(n)}.
\] (63)

It is easily verified, that the solution is

\[
w^{(n)} = W^2 g_n \left( \frac{v^3}{W^2} \right)^{n(3N_c - N_f)} \left( \frac{\Lambda}{v} \right) = S g_n \left( \frac{v^3}{S} \right)^{\frac{n(3N_c - N_f)}{S}}
\] (64)

where \( g_n \) is an arbitrary function. Its explicit form will be found below from the relation between perturbative and exact anomalies.

Of course, the result (64) is in a complete agreement with dimensional arguments and does not depend on the particular choice of symmetry (i.e. \( x \)).
4 The relation between perturbative and exact anomalies

4.1 Exact anomaly in the N=2 SUSY SU(2) case

Let us briefly remind the calculation of exact R-anomaly following [2, 20]. The anomaly can be found from the effective action

\[ \langle \partial_\mu j^R_\mu \rangle = - \left. \frac{\partial \Gamma}{\partial \alpha} \right|_{\alpha=0} = - \frac{1}{32\pi} \text{Im} \frac{\partial}{\partial \alpha} \int d^4x d^2\theta_1 d^2\theta_2 F\left( e^{2i\alpha} \Phi(e^{-4i\alpha}\theta) \right) \bigg|_{\alpha=0}. \] (65)

Taking into account, that \( \int d\theta \) is actually a differentiation over the anticommuting variables and it is possible to perform the additional rotation \( \theta \to e^{i\alpha\gamma_5} \theta \), we obtain

\[ \langle \partial_\mu j^R_\mu \rangle = - \frac{1}{32\pi} \text{Im} \frac{\partial}{\partial \alpha} \int d^4x d^2\theta_1 d^2\theta_2 e^{-4i\alpha} F\left( e^{2i\alpha} \Phi(\theta) \right) \bigg|_{\alpha=0} = \]
\[ = \frac{1}{16\pi} \text{Re} \int d^2\theta_1 d^2\theta_2 \left( 2F(\Phi) - \frac{\partial F}{\partial \Phi} \right) = \frac{1}{16\pi} \text{Re} \int d^2\theta_1 d^2\theta_2 (F + F_D). \] (66)

This expression can be found explicitly, really

\[ \frac{d}{du}(F + F_D) = \frac{dF}{du} + \frac{dF_D}{du} = a_D \frac{da}{du} - a \frac{da_D}{du}; \]
\[ \frac{d^2}{du^2}(F + F_D) = a_D \frac{d^2a}{du^2} - a \frac{d^2a_D}{du^2} = 0, \] (67)

where we used (34).

Therefore, taking into account perturbative asymptotics finally we have

\[ F + F_D = \text{const } u = \frac{2i}{\pi} u, \] (68)

so that the anomaly can be written as

\[ \langle \partial_\mu j^R_\mu \rangle = - \frac{1}{8\pi^2} \text{Im} \int d^2\theta_1 d^2\theta_2 u, \] (69)

while in the perturbation theory

\[ A \equiv \langle \partial_\mu j^R_\mu \rangle_{\text{pert}} = - \frac{1}{16\pi^2} \text{Im tr} \int d^2\theta_1 d^2\theta_2 \Phi^2. \] (70)
4.2 Derivation

Of course, the expressions (69) and (70) are quite different. The former is a series over \( \Lambda^4 \) produced by instanton contributions. In particular, taking into account one instanton correction we have \[21, 17\]

\[
\langle \partial_{\mu} j_{R}^{\mu} \rangle = -\frac{1}{16\pi^2} \text{Im} \text{tr} \int d^2 \theta_1 d^2 \theta_2 \left[ \Phi^2 + \frac{\Lambda^4}{2\Phi^2} + O(\Lambda^8) \right],
\]
(71)
that in components can be written as

\[
\langle \partial_{\mu} j_{R}^{\mu} \rangle = \frac{e^2}{4\pi^2} (1 + \frac{3\Lambda^4}{2e^4\varphi^4}) F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{3\Lambda^4}{e^2\pi^2\varphi^5} F_{\mu\nu} \bar{\Psi}_{D} \Sigma_{\mu\nu} \gamma_5 \Psi_{D} + \frac{60\Lambda^4}{e^2\pi^2\varphi^6} (\bar{\Psi}_{D} \Sigma \Psi_{D} \gamma_5 \bar{\Psi}_{D} + O(\Lambda^8)),
\]
(72)
where

\[
\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}
\]
(73)
and we introduced a Dirac spinor

\[
\Psi_{D} = \frac{1}{2} (1 + \gamma_5) \psi_1 + \frac{1}{2} (1 - \gamma_5) \psi_2.
\]
(74)

And nevertheless, nonperturbative result is only a vacuum expectation value of the perturbative one, that in particular produces a natural solution of anomalies cancellation problem in the realistic models.

This result is not unexpected. Really, performing, for example, chiral transformation in the generating functional we have

\[
0 = \frac{1}{Z} \frac{\delta Z}{\delta \alpha} \bigg|_{\alpha=0} = \frac{1}{Z} \frac{\delta}{\delta \alpha} \int DAD \bar{\psi}' D\psi' \exp (iS - \partial_{\mu} j_{5}^{\mu}) \bigg|_{\alpha=0}
\]
\[
= \frac{1}{Z} \frac{\delta}{\delta \alpha} \int DAD \bar{\psi} D\psi \exp (iS - \partial_{\mu} \alpha j_{5}^{\mu} - \alpha A) \bigg|_{\alpha=0} = \langle \partial_{\mu} j_{5}^{\mu} - A \rangle,
\]
(75)
where \( A \) denotes the perturbative anomaly, produced by the measure noninvariance \[22\]. Finally

\[
\langle \partial_{\mu} j_{5}^{\mu} \rangle = \langle A \rangle.
\]
(76)
(R-transformation are considered similarly).

It is just the relation, mentioned above. Of course, it is valid for a wide range of models and is really a point to start with. Let us note, that the derivation presented in Section 4.1 essentially used the form of exact results. So, we are tempted to reverse the arguments. In the next section we will try to develop this approach.
5 Consequences and results

5.1 N=2 SUSY SU(n): Picard - Fuchs equations versus anomalies

We will start the investigation of the SU(n) case with the relation (76). It means that

\[ \langle \partial_\mu j^\mu_R \rangle = -\frac{n}{16 \pi^2} \text{Im} \int d^2\theta_1 d^2\theta_2 u_1. \]  

From the other hand

\[ \langle \partial_\mu j^\mu_R \rangle = \frac{1}{16 \pi} \text{Re} \int d^2\theta_1 d^2\theta_2 (F + F_D) \]  

as above. Comparing (77) and (78) we find that

\[ F + F_D = \frac{in}{\pi} u_1 \]  

and, therefore,

\[ \frac{\partial}{\partial u_k} (F + F_D) = \frac{\partial F}{\partial a_i} \frac{\partial a_i}{\partial u_k} + \frac{\partial F_D}{\partial a_i} \frac{\partial a_i}{\partial u_k} = a^i_D \frac{\partial a_i}{\partial u_k} - a_i \frac{\partial a^i_D}{\partial u_k} = \frac{in}{\pi} \delta_{k1}. \]  

In the case of SU(2) gauge group

\[ W \equiv a_D \frac{da}{du} - a \frac{da_D}{du} \]  

can be identified with the Wronsky determinant for a linear second order differential equation, \( a \) and \( a_D \) being 2 its linear independent solutions. The condition \( W = \text{const} \) constrains a form of the equation to be

\[ \left( \partial^2_u + L(u) \right) \begin{pmatrix} a \\ a_D \end{pmatrix} = 0 \]  

due to the Liouville formula

\[ W(u) = W(u_0) \exp \left( -\int_{u_0}^u du' K(u') \right), \]  

where \( K(u) \) is a coefficient of a first derivative term in the equation.

The SU(n) case can be considered similarly. Let us differentiate (80) once more and contract the result with a symmetric matrix \( M_{km}(u) \)

\[ M_{km} \frac{\partial}{\partial u_m} \left( a^i_D \frac{\partial a_i}{\partial u_k} - a_i \frac{\partial a^i_D}{\partial u_k} \right) = a^i_D M_{km} \frac{\partial^2 a_i}{\partial u_k \partial u_m} - a_i M_{km} \frac{\partial^2 a^i_D}{\partial u_k \partial u_m} = 0. \]  

It means, that \( a \) and \( a_D \) should satisfy a system of the form
\[
\left( \delta_{ij} M_{km}(u) \frac{\partial^2}{\partial u_k \partial u_m} + \delta_{ij} K_m(u) \frac{\partial}{\partial u_m} + L(u)_{ij} \right) \left( \begin{array}{c} a_j \\ a_D^i \end{array} \right) = 0. \quad (85)
\]

Substituting it to (84) leads to
\[
K_m \left( a_i \frac{\partial a_D^i}{\partial u_m} - a_D^i \frac{\partial a_i}{\partial u_m} \right) = 0,
\]
so that using (80) we obtain the constrain \( K_1(u) = 0 \). Thus the Picard-Fuchs system should contain no first derivatives with respect to \( u_1 \).

Note, that here \( M_{km} \) is an arbitrary symmetric matrix. By a special choice of \( M_{km} \) we are able to diagonalize \( L_{ij} \). Really, there are \( r(r+1)/2 \) linear independent symmetric matrixes. Adding the corresponding equations (85) with unknown coefficients we should set to zero \( r(r-1)/2 \) nondiagonal elements of \( L_{ij} \). Thus we obtain a system of \( r(r+1)/2 \) linear algebraic equations with \( r(r+1)/2 \) variables. It has \( r = n - 1 \) linear independent solutions.

Therefore, (85) can be rewritten as
\[
\left( \sum_{k,m} M_{km}^{(p)}(u) \frac{\partial^2}{\partial u_k \partial u_m} + \sum_{m \neq 1} K_m^{(p)}(u) \frac{\partial}{\partial u_m} + L^{(p)}(u) \right) \left( \begin{array}{c} a_i \\ a_D^i \end{array} \right) = 0,
\]
\( p=1, \ldots, n-1. \)

Let us check, that this system gives true and unique solution for
\[
W_k(u) \equiv a_D^i \frac{\partial a_i}{\partial u_k} - a_i \frac{\partial a_D^i}{\partial u_k}.
\]

It is easily verified, that for \( W \) (87) gives
\[
\sum_{k,m} M_{km} \frac{\partial W_k}{\partial u_m} + \sum_{m \neq 1} K_m W_m = 0.
\]

Initial conditions are defined by the perturbative result, that is valid if \( u_1 \rightarrow \infty \) [7, 11]. Its form, of course, coincides with (80):
\[
W_m(u_1 \rightarrow \infty) = \frac{i \pi}{\pi} \delta_{m1}.
\]

However, the relation between perturbative and nonperturbative anomalies assumes, that (80) is satisfied for arbitrary \( u_i \). (87) (and, therefore, in (89)) should automatically produce it.

Really, (90) is a solution of (89): there are \( n - 1 \) variables \( W_m \), which are uniquely determined from \( n - 1 \) equations.

To conclude, the relation between perturbative and nonperturbative anomalies in the N=2 supersymmetric SU(n) gauge theory leads to the system
\[
\left( \sum_{k,m} M_{km}^{(p)}(u) \frac{\partial^2}{\partial u_k \partial u_m} + \sum_{m \neq 1} K_m^{(p)}(u) \frac{\partial}{\partial u_m} + L^{(p)}(u) \right) \left( \begin{array}{c} a_i \\ a_D^i \end{array} \right) = 0,
\]
\( p=1, \ldots, n-1. \)
These equations are in a complete agreement with the results of explicit calculations (34) and (35) for SU(2) and SU(3) gauge groups. (In particular, we explained the absence of first derivatives over $u_1 = u$, that seems accidental at the first sight.)

5.2 Effective Lagrangian for N=1 supersymmetric theories. General structure

Let us apply (76) to N=1 supersymmetric SU($N_c$) Yang-Mills theory with $N_f$ matter supermultiplets. The vacuum expectation value of the perturbative anomaly (14) is given by

$$\langle \partial_\mu J_\mu^x \rangle = (N_f - N_c - xN_f) \frac{1}{16\pi^2} \text{Im} \int d^2 \theta u.$$  (92)

where $u \equiv \langle \text{tr} W^2 \rangle$. Therefore, the effective Lagrangian should depend in particular on $S = \text{tr} W^2$. This result is not new. At the perturbative level the similar investigation was made in [19]. However, in this paper we do not intend to restrict ourselves by the frames of perturbation theory. Therefore, we can not assume, that $u = \text{tr} W^2$ (Here we would like to remind (71)).

Comparing (92) with (52) and taking into account, that the equality should be satisfied for all $x$, we obtain

$$2w - 2 \frac{\partial w}{\partial S} S = - \frac{1}{16\pi^2}(N_f - N_c)w;$$  
$$\frac{\partial w}{\partial v} v = - \frac{1}{16\pi^2} N_f u.$$  (93)

This equation corresponds to the exact conservation of R-symmetry at nonperturbative level. The similar condition was used in [15, 23], although the dependence $w = w(S)$ was ignored. Of course, it is quite clear, that integrating out $S$ yields ADS superpotential [24] and corresponds to imposing the condition

$$\frac{\partial w}{\partial S} = 0.$$  (95)

We do not intend to discuss here the legitimacy of this operation and send the reader to [24], although we dare to suggest, that the situation is more complicated. For the complete analysis we need to know the kinetic part of the effective action, while now we
can say nothing about it, except for the general assumptions [19]. It is worth to mention, that in the case $N_c > N_f$ the gauge group is not completely broken and, therefore, the low energy theory contains massless degrees of freedom at the perturbative level. We believe, that integrating them out should be substantiated more thoroughly.

However this discussion, although being very interesting, is far beyond the frames of the present paper. We would like only to stress, that the presence of $S$-field is a strict consequence of (93).

The solution of (94) should agree with instanton calculations. It is easily verified, that the only solution of (94), agreeing with (64)

$$w = \frac{1}{32\pi^2} S f(z); \quad f(z) = f_{\text{pert}}(z) + \sum_{n=1}^{\infty} c_n z^n,$$

where

$$z = \frac{\Lambda^{3N_c-N_f}}{v^{2N_f} S^{N_c-N_f}},$$

is a dimensionless parameter.

In the final result $z$ should be written in terms of gauge invariant variables. Of course, the result will depend on the structure of moduli space, that is briefly reviewed in the Appendix A. The derivation, made in the Appendix B, gives

$$z = \frac{\Lambda^{3N_c-N_f}}{v^{2N_f} S^{N_c-N_f}}, \quad N_f < N_c;$$

$$z = \frac{\Lambda^{3N_c-N_f} S^{N_f-N_c}}{\det M - (B^{A_1 A_2 \ldots A_{N_f-N_c} M_{A_1} B_1 M_{A_2} B_2 \ldots M_{A_{N_f-N_c}} B_{N_f-N_c} B_{B_1 B_2 \ldots B_{N_f-N_c}})}, \quad N_f \geq N_c. \quad (98)$$

In order to define $f_{\text{pert}}$ we note, that at the perturbative level $u = S$. Therefore, in this case (93) gives

$$\frac{\partial f_{\text{pert}}}{\partial z} z = 1,$$

so that

$$w = \frac{1}{32\pi^2} S \left( \ln z + \sum_{n=1}^{\infty} c_n z^n \right). \quad (100)$$

Substituting it to (93), we obtain

$$u = S \left( 1 + \sum_{n=1}^{\infty} n c_n z^n \right), \quad (101)$$

\footnote{It corresponds to $g_n(x) = \frac{1}{32\pi^2} c_n x^{n(N_c-N_f)}, \quad n \geq 1$ in (94).}
that defines all anomalies in the theory according to (92). At the perturbative level both (100) and (101) are certainly in agreement with [19].

In the end of this section we should mention, that (100) is not in a complete agreement with instanton calculations. Really, although \( \Lambda^{(3N_c-N_f)n}/v^{2N_f n} \) is already present in the instanton measure, the result of integration over collective coordinates will differ from the required form due to the factor \( \exp(-4\pi \rho^2 v^2) \) in the exponent. However the agreement can be achieved by changing the form of the instanton vertex, but this problem is far beyond the frames of present paper and we do not intend to discuss it here.

5.3 Effective Lagrangian for N=1 supersymmetric theories.

Exact result

Let us try to define \( f \) exactly. The general structure of the holomorphic superpotential, found in section 5.2, is similar to the structure of the nonperturbative prepotential in the N=2 supersymmetric Yang-Mills theory [25]. In the latter case the relation between perturbative and nonperturbative anomalies leads to Picard-Fuchs equations, that can be used for derivation of exact results. Is it possible to extend this approach to the case of N=1 supersymmetry?

First we substitute (96) into (93), that gives

\[
S \frac{df}{dz} = u
\]

(and therefore \( u/S \) depends only on \( z \)).

The way to solve this equation is indicated by the analogy with N=2 supersymmetric SU(2) Yang-Mills theory. In terms of N=1 superfields the action (31) is written as

\[
\frac{1}{16\pi} \text{Im} \int d^4 x d^2 \theta \left( \frac{d^2 F}{d \phi^2} W^2 + \frac{1}{2} \int d^2 \theta \frac{dF}{d \phi} \phi^+ \right). \tag{103}
\]

Let us compare it with

\[
S_a = \frac{1}{32\pi^2} \text{Re} \int d^4 x d^2 \theta \text{ } S f(z) \tag{104}
\]

and introduce \( a \equiv z^{-1/4} \) (this choice of the power will be explained below). The first term in (103) will coincide with (104) if

\[
-2\pi i \frac{d^2 F}{d a^2} \equiv f; \quad \frac{d^2 U}{d a^2} \equiv \frac{u}{S}. \tag{105}
\]

Then (102) takes the form

\[
F + F_D = \frac{2i}{\pi} U, \tag{106}
\]

where
\[ F_D = F - a a_D; \quad a_D = \frac{dF}{da}. \] (107)

As above this leads to the Picard-Fuchs equation

\[
\left( \frac{d^2}{da^2} + L(U) \right) \left( \begin{array}{c} a \\ a_D \end{array} \right) = 0,
\] (108)

where \( L(U) \) is an undefined function.

At the perturbative level (see (99))

\[
\begin{align*}
    f_{\text{pert}} &= -4 \ln a; \\
    u_{\text{pert}} &= W^2 = S,
\end{align*}
\] (109)

so that

\[
\begin{align*}
    F &= \frac{i}{\pi} a^2 \left( \frac{3}{2} - \ln a \right); \\
    U &= a^2 / 2
\end{align*}
\]

and, therefore

\[
\begin{align*}
    a &= \sqrt{2U}; \\
    a_D &= -\frac{2i}{\pi} (a \ln a - a) = -\frac{i}{\pi} \sqrt{2U} \left( \ln(2U) - 2 \right)
\end{align*}
\] (110)

are 2 independent solutions of the Picard-Fuchs equation

\[
\left( \frac{d^2}{dU^2} + \frac{1}{4U^2} \right) \left( \begin{array}{c} a \\ a_D \end{array} \right) = 0.
\] (111)

However, the perturbative solution does not satisfy the requirement \[ \text{Im } \tau > 0 \]

that is derived exactly as in the N=2 case. Therefore, two singularities (at \( U = 0 \) and \( U = \infty \)) are impossible.

To find the structure of singularities let us note, that the solution (100) should contain all positive powers of \( z \) and, therefore, is invariant under \( Z_4 \) transformations \( a \to e^{i\pi k/2} a \). Taking into account (103) and (104) we conclude, that the corresponding transformations in the \( U \)-plane are \( U \to e^{i\pi k} U \). Thus, singularities of \( L(U) \) in the Picard-Fuchs equation (108) should come in pairs: for each singularity at \( U = U_0 \) there is another one at \( U = -U_0 \).

Therefore, the considered model is completely equivalent to N=2 supersymmetric SU(2) Yang-Mills theory without matter and the only possible form of Picard-Fuchs equation (up to the redefinition of \( \Lambda \)) is

\[
\left( \frac{d^2}{dU^2} + \frac{1}{4(U^2 - 1)} \right) \left( \begin{array}{c} a \\ a_D \end{array} \right) = 0
\] (113)

with the solution \[ \text{Im } \tau > 0 \]
\[
\begin{align*}
  a(U) &= \frac{\sqrt{2}}{\pi} \int_{-1}^{1} dx \frac{\sqrt{x - U}}{\sqrt{x^2 - 1}} , \\
  a_D(U) &= \frac{\sqrt{2}}{\pi} \int_{1}^{U} dx \frac{\sqrt{x - U}}{\sqrt{x^2 - 1}} .
\end{align*}
\]

(114)

Its uniqueness and, therefore, the uniqueness of the choice (113) was proven in [26].

The function \( F \) can be found by

\[
\frac{dF}{dU} = a_D \frac{da}{dU} .
\]

(115)

Its general structure is well known to be

\[
F = -\frac{i}{\pi} a^2 \left( \ln a + \sum_{n=0}^{\infty} F_n a^{-4n} \right) ,
\]

(116)

so that

\[
f = -2\pi i \frac{d^2 F}{da^2} = -4 \ln a + \sum_{n=0}^{\infty} f_n a^{-4n} = \ln z + \sum_{n=0}^{\infty} f_n z^n .
\]

(117)

And now it is quite clear, that the choice \( a = z^{-1/4} \) was made to obtain the true structure of instanton corrections (96).

6 Conclusion.

In the present paper we tried to investigate the structure of quantum anomalies beyond the frames of perturbation theory. Although nontrivial corrections exist due to the instanton effects, it is not difficult to treat nonperturbative expressions. The matter is that the exact anomalies turned out to be the vacuum expectation values of the perturbative ones. However, the explicit check of this statement should essentially use the structure of the result. Thus we are able to research nonperturbative effects starting with this relation between perturbative and exact anomalies. The approach was illustrated by deriving the Picard-Fuchs equations for the exact solution of the SU(n) N=2 supersymmetric Yang-Mills theory. However, the most interesting results were found when the method was applied to N=1 theories. We managed to predict the general structure of holomorphic superpotential and even to obtain the exact solution. Unfortunately, the kinetic part can not be found by the presented approach. It complicates the research significantly, because we are not able to solve some important problems. In our opinion, the key question is when we can describe the theory by the gauge invariant superfield \( S \) and when it is necessary to use original fields. It is really important, because the problem is tightly bound with the quark confinement. We believe, that the solution can
be found only by the investigation of nonperturbative kinetic term, although there are implications, that $S$ can be considered as a quantum field if $N_f \geq N_c$ (see the brief discussion in the Section 5.2). Of course, it would be wonderful to solve this problem and we do not lose the hope.

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Note added

After sending this paper to hep-th we were informed by professor M.Alishahiha and professor H.Schnitzer, that the general form of the Picard-Fuchs equations for $N=2$ supersymmetric Yang-Mills theories was already obtained in [28] by other methods. Our results agree with it, so as with the similar results of [29].

Appendix

A The classical moduli spaces of $N=1$ supersymmetric theories

To describe the vacuum states it is convenient to introduce two $N_f \times N_c$ matrixes of the form

$$\phi \equiv (\phi^1, \phi^2, \ldots, \phi^N_f) ; \quad \tilde{\phi} \equiv (\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_N_f).$$

(Their rows correspond to different values of color index.) The energy is minimal if $\phi = \tilde{\phi} \equiv v$. Performing rotations in the color and flavor spaces we can always reduce the matrix $v$ to the form
if $N_f < N_c$ and

$$v = \begin{pmatrix} v_1 & 0 & \ldots & 0 \\ 0 & v_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & v_{N_f} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \end{pmatrix}$$

(119)

if $N_f > N_c$.

1. $N_f < N_c$.

In the generic point the gauge group $SU(N_c)$ is broken down to $SU(N_f - N_c)$. Therefore,

$$\left(N_c^2 - 1\right) - \left((N_c - N_f)^2 - 1\right) = 2N_cN_f - N_f^2$$

(121)

chiral superfields are eaten up by super-Higgs mechanism. Taking into account that originally there are $2N_cN_f$ chiral matter superfields, we conclude that only

$$2N_cN_f - \left(2N_cN_f - N_f^2\right) = N_f^2$$

(122)

ones remain massless.

The flat direction can be described in the gauge invariant way by $N_f^2$ composite chiral superfields

$$M_{AB} = \tilde{\phi}_{Aa}\phi^{Ba}.$$  

(123)

(Here $a$ denotes a color index.)

2. $N_f \geq N_c$.

If the number of flavors is equal to or larger than the number of colors, the original gauge group is completely broken in the generic point. Therefore, the number of remaining massless chiral superfields is

$$2N_cN_f - \left(N_c^2 - 1\right) = 2N_cN_f - N_c^2 + 1.$$  

(124)

In this case the gauge invariant description is provided by "mesons"

$$M_{AB} = \tilde{\phi}_{Aa}\phi^{Ba}.$$  

(125)

and "barions"
\[ B_{A_{Nc+1}A_{Nc+2}...A_{Nf}} = \frac{1}{N_c!} \varepsilon_{A_{1}A_{2}...A_{Nf}} \varepsilon^{a_{1}a_{2}...a_{Nc}} \phi^{A_{1}A_{2}a_{2}} \cdots \phi^{A_{Nc}a_{Nc}}; \]
\[ \tilde{B}_{A_{Nc+1}A_{Nc+2}...A_{Nf}} = \frac{1}{N_c!} \varepsilon_{A_{1}A_{2}...A_{Nf}} \varepsilon^{a_{1}a_{2}...a_{Nc}} \phi_{A_{1}A_{2}a_{2}} \cdots \phi_{A_{Nc}a_{Nc}}. \] (126)

However, their overall number is greater than \(2N_cN_f - N_c^2 + 1\). The matter is that at the classical level these fields are not independent and satisfy some constraints. For example, if \(N_f = N_c\) the number of massless superfields is \(N_f^2 + 1\) while \(N_M + N_B = N_f^2 + 2\). The constraint eliminating the redundant chiral variable is

\[ \det M = \tilde{B}B. \] (127)

Similarly, for \(N_f = N_c + 1\)

\[ B_A M_A^B = M_B A_B = 0; \]
\[ \det M (M^{-1})_A^B = B_A \tilde{B}^B. \] (128)

However, at the quantum level these constraints are violated by instanton corrections and are no longer valid \[23\].

### B On the gauge invariant form of parameter z

#### B.1 \( N_f < N_c \)

In this case the only gauge invariant parameter of \(v^{2N_f}\) order is \(\det M\), so that

\[ z = \frac{\Lambda^{3N_c - N_f}}{\det M S^{N_c - N_f}}. \] (129)

#### B.2 \( N_f \geq N_c \)

For \(N_f \geq N_c\) the moduli space is parametrized by mesons \(M_A^B\) and barions \(B_{B_{1}...B_{Nf-N_c}}\), \(\tilde{B}^{A_{1}...A_{Nf-N_c}}\), satisfying some classical constrains. At the quantum level these constrains are broken by instanton contributions. In the effective action approach the modifications should be produced automatically. It can be achieved by integrating out the \(S\)-superfield. \[\] (In particular, for \(N_f = N_c\) \(S\) is a natural Lagrange multiplier). The result should have the following form \[27\]:

---

5This implies, that \(S\) becomes a massive quantum field and the theory is described by the gauge invariant variables or, by other words, confines.
\[ \det M - \hat{B}B = \text{const } \Lambda^{2N_f}, \quad N_f = N_c; \]

\[ w_{\text{eff}} = \text{const } \Lambda^{\frac{3N_c - N_f}{N_f - N_c}} \left( \det M - \left( \hat{B}^{A_1A_2\ldots A_{N_f - N_c}} M_{A_1} B_1 B_2 \ldots M_{A_{N_f - N_c}} B_{N_f - N_c} \right) \right)^{-1} \frac{1}{N_f - N_c} + \text{h.c.}, \quad N_f > N_c. \]  

(130)

It can be achieved if and only if \( v^{2N_f} \) is substituted by

\[ \det M - \left( \hat{B}^{A_1A_2\ldots A_{N_f - N_c}} M_{A_1} B_1 B_2 \ldots M_{A_{N_f - N_c}} B_{N_f - N_c} B_{B_1B_2\ldots B_{N_f - N_c}} \right), \]  

(131)

so that finally

\[ z = \frac{\Lambda^{3N_c - N_f} S^{N_f - N_c}}{\det M - \left( \hat{B}^{A_1A_2\ldots A_{N_f - N_c}} M_{A_1} B_1 B_2 \ldots M_{A_{N_f - N_c}} B_{N_f - N_c} B_{B_1B_2\ldots B_{N_f - N_c}} \right)}. \]  

(132)

We would like to mention, that in the presented approach (130) certainly contains multiinstanton corrections, that contribute to the overall constant factor in the RHS.

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