Color confinement and color singlet structure of quantum states in Yang-Mills theory

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We consider two fundamental long-standing problems in quantum chromodynamics (QCD): the origin of color confinement and structure of a true vacuum and color singlet quantum states. There is a common belief that resolution to these problems needs a knowledge of a strict non-perturbative quantum Yang-Mills theory and new ideas. Our principal idea in resolving these problems is that structure of color confinement and color singlet quantum states must be determined by a Weyl symmetry which is an intrinsic symmetry of the Yang-Mills gauge theory, and by properties of a selected class of solutions satisfying special requirements. Following this idea we construct for the first time a space of color singlet one particle quantum states for primary colorless gluons and quarks and reveal the structure of color confinement in quantum Yang-Mills theory. As an application we demonstrate formation of physical observables in a pure QCD, pure glueballs.

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I. INTRODUCTION

Color confinement represents the most amazing phenomenon in quantum chromodynamics. Despite on tremendous progress in QCD since its invention the origin and mechanism of color confinement remains unclear. The first deep insight on the nature of the color confinement was revealed long time ago in 1, 2 where it was stressed that the color confinement phenomenon is intimately related to the gauge invariance of the vacuum. Based on this ’t Hooft conjectured that QCD vacuum structure in the confinement phase must be described by Abelian fields which supposed to be color neutral. First scenarios for the color confinement mechanism based on monopole condensation were proposed in 3, 4 and developed in subsequent studies. Existence and microscopic structure of a stable QCD vacuum is another long-standing problem since finding in 1977 by Savvidy 5 that a non-trivial QCD vacuum can be generated by radiative corrections. Quantum instability of Savvidy vacuum established in the seminal paper by Nielsen and Olesen 6 triggers active extensive searches of stable vacuum field configurations in the subsequent several decades. A stable QCD vacuum has been remained unknown despite on significant progress in approximate description of the vacuum in various approaches starting from seminal works by Nielsen and Olesen 6, 7 and in other works 9, 10.

In the present paper we propose a novel approach to resolution of the problems of color confinement and vacuum stability elaborating an idea that Weyl symmetry of the color group SU(3) is a principal symmetry which determines all color attributes of vacuum and quantum states, and provides microscopic description of a true stable color invariant vacuum. In Section II we formulate main requirements that solutions must meet for proper definition of one particle quantum states. Based on this we construct an ansatz for Weyl symmetric stationary solutions of magnetic and dual electric type and construct solutions to equations of motion of SU(3) Yang-Mills theory. We demonstrate that obtained Weyl symmetric solutions represent fixed points in the configuration field space under Weyl transformation and possess a vanishing total color charge. We prove that each solution space with a fixed set of quantum numbers is one-dimensional and provides color singlet one particle quantum states. The Weyl symmetric solutions manifest an Abelian dominance effect, which allows to classify non-Abelian solutions and construct a full Hilbert space of color singlet quantum states, at least in principle. In Section III we apply the Weyl symmetric ansatz to Dirac equation in a QCD with one flavor quark. Obtained results reveal unexpected feature: there are three independent Weyl symmetric quark solutions which are colorless, contrary to commonly accepted view that one has three color quarks. Section IV is devoted to a persistent problem in all known QCD vacuum models, the quantum vacuum stability against quantum fluctuations. We prove that vacuum Abelian Weyl symmetric solutions are stable under quantum gluon fluctuations. Thus, the Weyl symmetric solutions provide microscopic description of stable vacuum gluon and quark condensates. In Section V we consider equations of motion corresponding to quantum one-loop effective action of a pure QCD. We show that equations with quantum corrections admit a localized stationary solution in a finite space region. It has been demonstrated that the lightest pure glueball is formed due to interaction of the primary gluon with corresponding generated vacuum gluon condensate with a proper zero angular momentum. A qualitative spectrum of lightest scalar glueballs is calculated in agreement with the Regge theory. In Appendix we consider a
reduced system of equations for propagating off-diagonal and Abelian modes $K_{2,4}$ in 1 + 1 dimensional space-time. We demonstrate how the energy conservation law in a finite space domain leads to correlation of solution modes $K_{2,4}$. As a result the Weyl symmetric solution forms a one dimensional space and provides a color singlet quantum state. Implications of our results and discussions are enclosed in the last section.

II. WEYL SYMMETRIC SOLUTIONS

1. A basic idea and requirements to classical solutions

Yang-Mills theory is formulated on a basis of a strict mathematical scheme of fiber bundle supplied with a structural group. We consider Yang-Mills theory with a gauge group $SU(3)$ which represents a classical theory for a standard pure gluodynamics, a pure QCD. So the structure of the gauge group is the only mathematical structure which determines all symmetry properties of the theory defined by a standard Lagrangian. An important role of the gauge principle is that gauge symmetry defines dynamics of the physical system by means of Euler equations of motion, and governs all properties of corresponding solutions which describe the classical system. From a formal point of view the gauge symmetry looks redundant and usually during quantization procedure one has to fix the gauge symmetry to select one field representative in each gauge equivalence class of the fields. This can be done by numerous ways, and a basis set of dynamical solutions can be chosen by numerous ways. Consistent quantization procedure provides physical quantities to be independent on a choice of gauge fixing condition for quantum fields. This is true for quantum virtual fields since the functional integration is performed over space of all possible quantum fluctuations in a gauge invariant manner, and it is clear, that choice of basis fields is not important. However, in general, the physical properties of a system depends on which class of classical solutions is selected contrary a commonly accepted opinion that all gauges are equivalent. Especially this is important in non-Abelian theory which does not admit linear superposition principle, and typically it is not possible to construct a complete basis in the space of non-linear solutions. So that an improper choice of basis classical solutions for construction of Hilbert space of quantum states will lead to inconsistent concepts of particles, and physical observables.

We adopt the following requirements to the classical solutions in the Yang-Mills theory which are used for further description and construction of vacuum and quantum states:

(i) after fixing the local gauge symmetry only symmetries corresponding to global or finite subgroups of the original structural group survive. We follow an idea that a Weyl group of $SU(3)$ is the only proper color symmetry group which survives after gauge fixing, and determine all color attributes of solutions. We require that solutions describing vacuum and quantum states must be defined by an ansatz invariant under Weyl transformation. Global color symmetries are not acceptable since they imply spontaneous color symmetry breaking which prevents appearance of color confinement phase.

(ii) Consistence with quantum mechanical principles implies that at microscopic space-time level the solutions must depend on time and admit stationary classical states with a conserved energy. This was observed in early papers [7–8], so the static solutions must be excluded unless the gauge symmetry is broken.

(iii) solutions must admit localization of particle states in a finite space regions, since all hadrons are localized objects and there is no massless hadrons like free photons which are not localized in space.

(iv) solutions must be exact solutions to exact equations of motion, otherwise some important non-perturbative features will be lost.

(v) vacuum solutions describing the microscopic structure of vacuum gluon and quark condensate must be stable against quantum fluctuations and possess classical stability, solutions like saddle-points are not acceptable.

(vi) solutions must be regular, possess finite energy density and a conserved total energy inside hadrons.

Based on these requirements we construct Weyl symmetric solutions which provide microscopic description of the vacuum and color singlet structure of the Hilbert space of quantum states.

2. Weyl symmetric $SU(3)$ ansatz for stationary magnetic type solutions

Let us consider first the $SU(2)$ Yang-Mills theory with a standard Lagrangian ($\mu, \nu = 0, 1, 2, 3; a = 1, 2, 3$)

$$\mathcal{L}_0 = -\frac{1}{4} F^{\alpha \beta} F_{\alpha \beta}.$$  \hspace{1cm} (1)

A generalized Dashen-Hasslacher-Neveu (DHN) ansatz [14] for time dependent axially symmetric solutions of magnetic type is defined by means of the following non-vanishing components of the gauge potential $A_{\mu}^a$ [15]

$$A_1^a = K_0 (r, \theta, t), \quad A_2^a = K_1 (r, \theta, t), \quad A_3^a = K_2 (r, \theta, t),$$

$$A_4^a = K_3 (r, \theta, t).$$  \hspace{1cm} (2)

The ansatz is invariant under residual $U(1)$ transformations with a gauge parameter $\lambda (r, \theta, t)$ [16–18]

$$K_0' = K_0 + \partial_t \lambda, \quad K_1' = K_1 + \partial_\theta \lambda, \quad K_2' = K_2 + \partial_\theta \lambda,$$

$$K_3' = K_3 \cos \lambda + K_4 \sin \lambda.$$  \hspace{1cm} (3)

One can fix the local $U(1)$ symmetry by adding a gauge fixing term $\mathcal{L}_{gf}$ to the original Yang-Mills Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{2} (\partial_t K_0 - \partial_\theta K_1 - \frac{1}{r^2} \partial_\theta K_2)^2.$$  \hspace{1cm} (4)
After fixing a gauge the Yang-Mills Lagrangian is still invariant under global color $SO(2)$ transformations with a constant parameter $\lambda$ in $\lambda$. One can fix the global symmetry and define a minimal ansatz by imposing a constraint $K_3 = c_3K_4$ (we set $c_3 = \sqrt{2}/2$ without loss of generality). With this, five equations of motion for the fields $K_i$ ($i = 0, 1, 2, 4$) reduce to four equations

$$r^2 \partial_t^2 K_1 - r^2 \partial_r^2 K_1 - \partial_\theta^2 K_1 + 2r \cot(\partial_\theta K_0 - \partial_r K_1) + \frac{9}{2} \csc^2 \theta K_1^4 K_1 = 0,$$

(5)

$$r^2 \partial_t^2 K_2 - r^2 \partial_r^2 K_2 - \partial_\theta^2 K_2 + 2r \cot(\partial_\theta K_0 - \partial_r K_1) - \cot \theta \partial_\theta K_2 + \frac{9}{2} \csc^2 \theta K_2^4 K_2 = 0,$$

(6)

$$r^2 \partial_t^2 K_4 - r^2 \partial_r^2 K_4 - \partial_\theta^2 K_4 + \cot \theta \partial_\theta K_4 + 3r^2(K_4^2 - K_0^2)K_4 + 4K_0^2K_4 = 0,$$

(7)

$$r^2 \partial_t^2 K_0 - r^2 \partial_r^2 K_0 - \partial_\theta^2 K_0 + 2r \cot(\partial_\theta K_0 - \partial_r K_0) + \frac{9}{2} \csc^2 \theta K_0^4 K_0 = 0,$$

(8)

and one quadratic constraint

$$2r^2(K_0\partial_0K_4 - K_1\partial_1K_4) + K_2(\cot K_4 - 2\partial_0K_4) + K_4(-\partial_0K_2 + r^2(\partial_1K_0 - \partial_r K_1)) = 0.$$  

(9)

A total Yang-Mills Lagrangian with gauge fixing terms is simplified as follows

$$\mathcal{L}_{tot} = \mathcal{L}_0(K) + \mathcal{L}_{gf}$$

$$= \frac{1}{2r^2} \left( r^2(\partial_0K_1 - \partial_rK_0)^2 - (\partial_\theta K_1)^2 + (\partial_\theta K_0)^2 \right)$$

$$+ \frac{1}{2r^2} \left( \partial_0K_2(\partial_0K_2 - \partial_\theta K_0) - \partial_1K_2(\partial_0K_2 - \partial_\theta K_0) \right) + \frac{3}{4r^2 \sin^2 \theta} \left( r^2((\partial_0K_4)^2 - (\partial_rK_4)^2) - (\partial_\theta K_4)^2 \right)$$

$$- \frac{3}{4r^2 \sin^2 \theta} \left( K_0^2(K_4^2 + 2r^2(K_0^2 - K_0^2)) \right).$$

(10)

Using the $SU(2)$ ansatz $[3]$ we construct a Weyl symmetric ansatz for $SU(3)$ Yang-Mills theory by setting non-vanishing elements of the gauge potential $A_\mu$ corresponding to $I, U, V$ type subgroups $SU(2)$ as follows $[3]$

$$I: \quad A_1^I = K_0, \quad A_2^I = K_1, \quad A_3^I = K_2, \quad A_4^I = K_4,$$

$$U: \quad A_1^U = -Q_0, \quad A_2^U = -Q_1, \quad A_3^U = -Q_2, \quad A_4^U = Q_4,$$

$$V: \quad A_1^V = S_0, \quad A_2^V = S_1, \quad A_3^V = S_2, \quad A_4^V = S_4,$$

$$A_\mu^V = A_\mu^V r_\mu^P, \quad A_3^V = K_3, \quad A_4^V = K_8, \quad (11)$$

where $r_\mu^P$ ($p = I, U, V$, $\alpha = 3, 8$) are root vectors $r_1^I = (1, 0), \quad r_2^I = (-1/2, \sqrt{3}/2), \quad r_3^I = (-1/2, -\sqrt{3}/2)$. The Weyl group acts on color components $A_\mu^V$ as a symmetric permutation group $S_3$ and realizes eight dimensional reducible representation. One can define a minimal ansatz by imposing additional constraints $(i = 0, 1, 2)$

$$Q_i = S_i = K_i,$$

$$Q_4 = \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} \right) K_4, \quad S_4 = \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} \right) K_4,$$

(12)

$$K_3 = -\frac{\sqrt{3}}{2} K_4, \quad K_3 = K_8,$$

(13)

which extract three non-trivial one-dimensional irreducible representations $\{\Gamma_1\}_i$ of $S_3$ acting on field components $K_i, Q_i, S_i$ in color space spanned by $\{T^2, 5, 7\}$ and one two-dimensional standard irreducible representation $\Gamma_2$ acting on fields $\{K_4, Q_4, S_4\}$ in the space formed by $\{T^{1, 4, 6}\}$, the fields $\{K_4, Q_4, S_4\}$ satisfy an equation

$$K_4 + Q_4 + S_4 = 0,$$

(13)

which defines a two-dimensional plane and implies a vanishing total color charge in a similar way as for $I, U, V$-vectors $A_\mu^V$ in $[11]$. The last two constraints in $[12]$ provide consistency with equations of motion. With this, the original $SU(3)$ Yang-Mills Lagrangian with gauge fixing terms, $\mathcal{L}_0 + \sum_{I,U,V} \mathcal{L}_{gf}^{I,U,V}$ can be written in an explicit Weyl symmetric form

$$\mathcal{L}_{Weyl}^{I,U,V} = \mathcal{L}_0 + \sum_{I,U,V} \mathcal{L}_{gf}^{I,U,V}$$

$$= \sum_p \left\{ -\frac{1}{3}(\partial_\mu A_\mu^p)^2 - |D_\mu W_\mu^p|^2 - \frac{3}{2} \left( (W^{*\rho\mu} W_\mu^p)^2 - (W^{*\rho\mu} W_\mu^p)(W^{\rho\nu} W_\nu^p) \right) \right\},$$

(14)

with

$$W_\mu^I = \frac{1}{\sqrt{2}}(A_\mu^I + i A_\mu^V), \quad W_\mu^U = \frac{1}{\sqrt{2}}(A_\mu^I + i A_\mu^V),$$

$$W_\mu^V = \frac{1}{\sqrt{2}}(A_\mu^V - i A_\mu^I), \quad D_\mu^p = \partial_\mu + i A_\mu^V r_\mu^p,$$

which coincides with $SU(2)$ reduced Lagrangian $\mathcal{L}_{tot}$, after rescaling $K_i \rightarrow 1/\sqrt{3}K_i$. This implies that Weyl symmetric $SU(3)$ Lagrangian produces the same Euler equations and solutions as in the case of $SU(2)$ Yang-Mills theory. However, there is a principal difference: $SU(2)$ solutions are degenerated due to the presence of the global color symmetry $SO(2)$ in $[3]$, which causes spontaneous color symmetry breaking. Contrary to this, solutions defined by the ansatz $[11][12]$ are non-degenerate due to the rigid relationship for the fields $K_3, K_4, K_8$ which prevents appearance of a global symmetry.

Let us consider eigenvalues of a Lie algebra valued Abelian vector field, $A_\mu = (A_\mu^I T^I, A_\mu^V T^V)$, acting in an adjoint representation in the Cartan basis. One can find

$$[A_\mu^V T^p] = K_3 r_\mu^p T^p_+,$$

(15)
where eigenvalues $K_3 r^p$ define color charges of irreducible representations $\{\Gamma_i\}_i$. The eigenvalues $K_3 r^p$ match the root system with a common field factor $K_2$ and imply zero color charges of representations $\{\Gamma_i\}_i$. With this a Weyl invariant total color charge for the solution $A_3^\mu$ vanishes, as a consequence, all cubic interaction terms are mutually canceled in the Lagrangian $\mathcal{L}_{\text{tot}}$. All field functions $K_i$ represent fixed points under the Weyl transformations, so every Weyl symmetric solution $A_3^\mu$ defined by the ansatz (11,12) represents a fixed point in the configuration space of fields. Note that, despite on a fact that irreducible multiplet $(K_i, Q_i, S_i)$ is composed from one independent field $K_i$, which is a fixed point under Weyl transformation, we do not name it as a singlet following the accepted terminology in the group theory for $\Gamma_2$ defined as a two-dimensional representation since it is defined on two-dimensional plane $[13]$. In general, a Weyl symmetric Lagrangian admits solutions which are not Weyl symmetric. Such a case is realized, for example, in one-loop effective Lagrangian with a constant magnetic field background $[19]$ where solutions form a Weyl sextet. In our case each solution transforms into itself in a non-trivial way while $I, U, V$-components of the solution permute with each other. We constrain our consideration mostly with magnetic type solutions. Similar results are valid for electric solutions defined by a dual ansatz in subsection 5.

3. Abelian solutions and Abelian dominance

A Hilbert space of Abelian Weyl symmetric solutions is defined by a complete basis of transverse vector spherical harmonics which are eigenfunctions of a total angular momentum operator with quantum numbers $J = l$, $J_z = m$ $[20]$, 

\[ \begin{align*}
\hat{A}_m^r &= \frac{1}{\sqrt{l(l+1)}} \hat{L}_{jl}(kr) Y_{lm}(\theta, \varphi) e^{i\omega t}, \\
\hat{A}_m^i &= \frac{-i}{\sqrt{l(l+1)}} \nabla \times (\hat{L}_{jl}(kr) Y_{lm}(\theta, \varphi)) e^{i\omega t},
\end{align*} \]

where $j_l(r)$ is a spherical Bessel function, $Y_{lm}(\theta, \varphi)$ is a spherical harmonic, $\omega = k \equiv M$ (in units $c = 1$) due to conformal invariance, $M$ is a conformal mass scale parameter and superscripts $m, i$ denote magnetic and electric type, respectively. Non-Abelian solutions can be obtained only numerically by solving four partial differential equations and one constraint $[5]-[9]$. To solve these equations we apply a method which transforms hyperbolic equations $[5]-[9]$ defined on a three-dimensional space-time to a system of $4 \times N$ elliptic equations in the two-dimensional space. First, we decompose fields $K_i(r, \theta, t)$ in a Fourier series

\[ \begin{align*}
K_{1,2,4}(r, \theta, t) &= \sum_{n=1}^{N} \tilde{K}^{(n)}_{1,2,4}(Mr, \theta) \cos(nMt), \\
K_0(r, \theta, t) &= \sum_{n=1}^{N} \tilde{K}^{(n)}_0(Mr, \theta) \sin(nMt).
\end{align*} \]  

(17)

After averaging the Lagrangian $\mathcal{L}_{\text{tot}}$ in $[10]$ over the time period $T = 2\pi/M$, one obtains a system of $4 \times N$ two-dimensional equations for Fourier modes $\tilde{K}^{(n)}_i(r, \theta)$. A further simplification is achieved by setting all even Fourier modes to zero. This resolves the quadratic constraint $[9]$ and selects a subclass of solutions with a definite parity.

We solve equations in a spherical space domain $\{0 \leq r \leq L, 0 \leq \theta \leq \pi\}$, constrained by radius values $L = \{m_n, \nu_n\}$ where $m_n, \nu_n$ are nodes and antinodes of the spherical Bessel function $r j_l(r)$. We impose the following boundary conditions

\[ \begin{align*}
\tilde{K}_i(r, \theta)^{(n)}|_{r=0} &= 0, \\
\tilde{K}_i(r, \theta)^{(n)}|_{\theta=0, \pi} &= 0,
\end{align*} \]

(18)

and on a spherical boundary one has

\[ \begin{align*}
\tilde{K}_{1,2,4}(r, \theta)^{(n)}|_{r=L} &= 0, & \text{if } L = m_n, \\
\partial_r \tilde{K}_{1,2,4}(r, \theta)^{(n)}|_{r=L} &= 0, & \text{if } L = \nu_n, \\
\tilde{K}_{0,1}(r, \theta)^{(n)}|_{r=L} &\approx 0.
\end{align*} \]

(19)

In addition, periodic and antiperiodic boundary conditions are used for even and odd field modes respectively. One has to solve a non-linear stationary boundary value problem (BVP). As it is known, a regular solution to non-linear BVP exists not for arbitrary boundary conditions, and not for arbitrary size of the space domain. To solve the non-linear BVP we apply iterative numerical methods which generate a convergent solution starting from approximate initial profile functions for fields $K_i$. Initial profile functions can be found in analytic form in term of series expansion in local vicinity near $r = 0$ and in the asymptotic region $r \rightarrow \infty$. An advantage of iterative methods is that a final convergent solution is not much sensitive to a choice of initial profile functions and values of integration constants in boundary conditions. The initial profile functions for the Abelian field $K_4$ is provided by vector spherical harmonics with quantum numbers $(l, m)$. Polar angle modes of the propagating off-diagonal field $K_2(r, \theta)$ are characterized by number $k = 0, 1, 2, \cdots$ of zeros in the interval $[0 \leq \theta \leq \pi]$. So one can construct an initial profile for these modes by modifying the Legendre polynomial $P_k(\cos \theta)$. A numeric solution of magnetic type with the lowest non-trivial polar angle modes is presented in FIG. 1 in the leading order of Fourier series decomposition which provides sufficiently high accuracy due to structure of the equations.
defining generalized Jacobi type functions and additional reflection symmetry of solutions at the origin \( r = 0 \) \[15\]

\[
K_{1,2,4}(r, \theta, t) = \tilde{K}_{1,2,4}(Mr, \theta) \cos(Mt),
\]

\[
K_0(r, \theta, t) = \tilde{K}_0(Mr, \theta) \sin(Mt).
\] \[20\]

The Lagrangian \( \mathcal{L}_{\text{tot}}, \) \[10\], does not contain interaction terms composed from only off-diagonal fields \( K_i \). Due to this, non-Abelian solutions exist only in the presence of Abelian field \( K_3 \). This implies an Abelian dominance effect for low energy solutions. Indeed, the Abelian numeric profile function \( \tilde{K}_4 \) of the numeric solution in the leading order: (a) \( \tilde{K}_1 \); (b) \( \tilde{K}_2 \); (c) \( \tilde{K}_4 \); (d) \( \tilde{K}_0 \); (e) time averaged radial magnetic field \( \langle F_{\theta \rho}^3 \rangle_t = -\langle F_{\rho \theta}^3 \rangle_t = -\frac{3}{4} \tilde{K}_2 \tilde{K}_4 \); (f) time averaged energy density \( \bar{\rho}(\rho, z) \) in cylindrical coordinates \( (g = 1, M = 1) \). FIG. 1: Numeric solution in the leading order: (a) \( \tilde{K}_1 \); (b) \( \tilde{K}_2 \); (c) \( \tilde{K}_4 \); (d) \( \tilde{K}_0 \); (e) time averaged radial magnetic field \( \langle F_{\theta \rho}^3 \rangle_t = -\langle F_{\rho \theta}^3 \rangle_t = -\frac{3}{4} \tilde{K}_2 \tilde{K}_4 \); (f) time averaged energy density \( \bar{\rho}(\rho, z) \) in cylindrical coordinates \( (g = 1, M = 1) \). FIG. 1: Numeric solution in the leading order: (a) \( \tilde{K}_1 \); (b) \( \tilde{K}_2 \); (c) \( \tilde{K}_4 \); (d) \( \tilde{K}_0 \); (e) time averaged radial magnetic field \( \langle F_{\theta \rho}^3 \rangle_t = -\langle F_{\rho \theta}^3 \rangle_t = -\frac{3}{4} \tilde{K}_2 \tilde{K}_4 \); (f) time averaged energy density \( \bar{\rho}(\rho, z) \) in cylindrical coordinates \( (g = 1, M = 1) \).

The presence of the Abelian field component in the non-Abelian solution is important since it allows to classify all regular finite energy non-Abelian Weyl symmetric solutions, at least in principle, what is usually not possible in non-linear non-integrable theories. Each Abelian solution defined by a spherical harmonic with quantum numbers \( M, J = l \) and \( J_z = 0 \) determines an infinite countable set of non-Abelian solutions numerated by number “\( k \)” of zeros of the polar angle mode of the field \( \tilde{K}_2 \). Setting approximate initial profile functions for \( K_i \) and appropriate boundary conditions providing finite energy density a numeric solution is defined uniquely and can be obtained by using iterative numerical methods. Solution with the lowest non-trivial polar angle mode of the field \( \tilde{K}_2 \) is shown in FIG. 1. The next solution with \( l = 1, k = 1 \) is depicted in FIG. 3. The solution has the same energy density function and an opposite parity of the off-diagonal field \( \tilde{K}_2 \) compare to the solution with quantum numbers \( l = 1, k = 0 \). To show clearly the location of zeros of the polar angle mode of the field \( \tilde{K}_2 \) we present density plots for higher mode solutions with \( l = 1, k = 2, 3 \) in FIGs. 4, 5. One can observe, that lowest mode solution in FIG. 1 contains a non-vanishing radial component of the magnetic field \( \langle F_{\theta \rho}^3 \rangle_t = -\frac{3}{4} \tilde{K}_2 \tilde{K}_4 \) of Coulomb type, FIG. 1(e), which can be treated as a magnetic field of the magnetic stationary monopole. The solution with quantum numbers \( (l = 1, k = 1) \) has non-vanishing radial magnetic field of higher mode, FIG. 3(e), which can be treated as a field of a monopole-antimonopole pair.
We have solved equations in space domain with radial size in the interval $\nu_1 \leq L \leq 200$, all solutions show fast convergence with number of iterations less than 10.

The most intriguing question is: "what is the dimension of the solution space defined by a given set of quantum numbers $(M, l, m, k)$?" Due to the presence of local $U(1)$ gauge symmetry $(3)$, one would expect that there are two dynamic degrees of freedom corresponding to two long-distance propagating modes $K_2, K_4$. On the other hand, due to the Abelian dominance effect it is clear that off-diagonal components cannot represent independent dynamic degrees of freedom due to non-linear interaction between fields $K_i$. Moreover, in the limit $K_4 \to 0$ the equations for off-diagonal fields $K_i$ turn precisely into Maxwell equations for electric type vector harmonics. However, the magnetic ansatz does not describe non-Abelian electric dual solutions, since the radial non-Abelian electric field strength vanishes within the magnetic ansatz. A dual electric ansatz is presented in the next subsection. Therefore, off-diagonal field $\tilde{K}_2$ must be correlated with the Abelian field $\tilde{K}_4$. Indeed, it is surprising, a careful numeric analysis of solutions with quantum numbers $l = 1, k = 0, 1, 2, 3$ shows that amplitudes of the fields $K_i$ are correlated with the amplitude of the Abelian field: once fixed the value of the asymptotic amplitude of the Abelian field, the amplitudes of other modes $K_{0,1,2}$ are uniquely determined by the numeric procedure irrespective of initial amplitude values of $K_{0,1,2}$. A source of such a non-trivial feature is related to the fact that spherical harmonics can describe localized states in specific space domains constrained by a sphere with selected radius values, $R = \{\nu, \mu\}$, since only in these cases the total energy inside space domain is conserved and classically stable stationary solutions exist. In addition, for small amplitudes of the Abelian
field $K_3$ the field profiles $K_{1,2}$ are given approximately by the electric vector harmonic with the same spherical Bessel function $j_l(r)$. So that the radial parts of modes $K_{2,4}$ in the non-Abelian solution must have the same at least, one node or antinode to provide energy conservation inside finite space domain. This causes correlation between amplitudes of fields $K_{2,4}$ since for non-Abelian solutions the localization of nodes/antinodes depend on field amplitudes. This is demonstrated in a simplified 1+1 dimensional model obtained by averaging equations for $K_{2,4}$ over time period and polar angle in Appendix FIG. 10(a),(b).

We conclude Weyl symmetric solutions defined by ansatz (11,12) have a total color charge zero, and a space of solutions for a given set of quantum numbers $(M,l,m,k)$ is one-dimensional and defined by one normalization constant, an asymptotic amplitude of the Abelian field. This lead to color singlet one-particle quantum states after quantization.

5. Duality, Weyl symmetric ansatz for electric solutions

Non-Abelian field strength components $F_{\mu\nu}^a$ are not gauge invariants like the magnetic and electric fields in the Maxwell theory. So that, the duality symmetry between non-Abelian magnetic and electric solutions cannot be realized as a symmetry with respect to mutual exchange of color electric and magnetic fields $\vec{B}_a \rightarrow \pm \vec{E}_a$. We define dual non-Abelian electric and magnetic fields by imposing minimal requirements that: (i) dual fields have the same energy density function; (ii) the gauge invariants $(F_{\mu\nu}^a)^2$ corresponding to vacuum gluon condensates must be equal by module and opposite by sign; (iii) magnetic non-Abelian solution contains a non-vanishing time averaged radial magnetic field $\langle F_{\theta\phi}^a \rangle_t$ of Coulomb type, and a dual electric field contains time averaged non-vanishing radial electric field $\langle F_{\theta\phi}^a \rangle_t$ of Coulomb type.

With this one can construct $SU(3)$ Weyl symmetric ansatz for dual electric fields. A minimal axially symmetric ansatz contains the following non-vanishing components of the gauge potential

\[
A_{r,\theta,\varphi}^{3,8} = -\frac{\sqrt{3}}{2} K_{1,2,3}, \\
A_{r,\theta,\varphi}^1 = K_{1,2,3}, \\
A_{r,\theta,\varphi}^4 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) K_{1,2,3}, \\
A_{r,\theta,\varphi}^6 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) K_{1,2,3}, \\
A_t^2 = K_0, \quad A_t^5 = -K_0, \quad A_t^7 = K_0.
\]  

(21)

To solve equations of motion it is suitable to remove redundant gauge symmetry and pure gauge degrees of freedom. So one should impose gauge conditions in such a way that the Weyl symmetry survives and equations for independent fields $K_i$ keep consistence with equations for the full gauge potential and with Maxwell equations in the Abelian limit. In a case of the electric ansatz (21) all requirements are satisfied if we introduce gauge fixing terms corresponding to the covariant Lorenz gauge

\[
L_{gf} = -\frac{1}{2} \sum_a \left(-\partial_t A_t^a + (\text{div} \vec{A})^a\right)^2,
\]

(22)

which differs from the gauge fixing term for magnetic solutions (4). A Weyl symmetric electric type solution with lowest non-trivial polar modes is presented in FIG. 6 in the leading order of Fourier series decomposition. The solution has the same energy density profile func-

FIG. 6: Solution profile functions of electric solution: (a) $\tilde{K}_1$; (b) $\tilde{K}_2$; (c) $\tilde{K}_0$; (d) Time averaged energy density ($g = 1, \omega = 1$).
III. WEYL SYMMETRIC DIRAC FERMIONS

Now we consider an effect of the Weyl symmetry on matter fields described by Lagrangian \( \mathcal{L}_f \) for fermions

\[
\mathcal{L}_f = \bar{\Psi} \left[ i \gamma^\mu (\partial_\mu - \frac{ig}{2} A^\mu \lambda^a) - m \right] \Psi.
\]

The Euler equations to the total Yang-Mills Lagrangian \( \mathcal{L}_{tot} = \mathcal{L}_0 + \mathcal{L}_f \) read

\[
(D^\mu F_{\mu\nu})^\alpha = -\frac{g}{2} J^\alpha \equiv -\frac{g}{2} \bar{\Psi} \gamma^\alpha \Psi,
\]

\[
\left[ i \gamma^\mu (\partial_\mu - \frac{ig}{2} A^\mu \lambda^a) - m \right] \Psi = 0.
\]

A usual simple Abelian projection with two independent Abelian fields \( A^{3,8}_\mu \) corresponding to the Cartan generators leads to Dirac equations for three independent color quarks

\[
\left[ i \gamma^\mu \partial_\mu - \frac{ig}{2} \sum_p A^\mu_p w^p_\alpha - m \right] \Psi_p = 0,
\]

where \( w^p_\alpha \) are the weight vectors \( w^p = \{(1,1/\sqrt{3}), (-1,1/\sqrt{3}), (0, -2/\sqrt{3})\} \). The equations have three independent solutions for quarks forming a color triplet. Note that the simple Abelian projection is not consistent with Weyl symmetric structure of the full Yang-Mills Lagrangian including off-diagonal gluon fields. An important feature of the Weyl symmetric ansatz \( 11, 12 \) is that it implies a non-trivial Abelian projection with a true stable vacuum, and then we prove explicitly the quantum stability of Abelian vector harmonics by solving a system of eigenvalue equations for unstable (tachyonic) modes. The quantum instability of constant color magnetic QCD vacuum was established in \( 9 \), and later it was found for constant color electric background as well \( 23 \). The source of quantum instability is the presence of anomalous magnetic moment interaction terms in the classical Yang-Mills Lagrangian which implies that

\[
(i \gamma^\mu \partial_\mu - m) \Psi^0 = 0.
\]

The solution implies a vanished color current \( j^a \) and full decoupling of equations for gluons and quarks. So that a complete basis of solutions describing the free gluon and quark solutions is given by vector and spinor spherical harmonics with the same quantum numbers \( J, m, \mu_n, n_l \) which provide the total energy conservation in a finite space domain. The zero mode solutions describe non-interacting vacuum gluon and quark condensates and primary gluons and quarks corresponding to vacuum excitations. Solutions \( \Psi^\pm = \psi^\pm(x) u^k_\alpha \) corresponding to eigenvalues \( \lambda^\pm \) belong to a standard two-dimensional representation \( \Gamma_2 \) and are colorless due to equation \( \Psi_1 + \Psi_2 + \Psi_3 = 0 \) defining \( \Gamma_2 \). The coordinate part \( \psi^\pm \) satisfies a Dirac equation with a Weyl invariant color charge \( \hat{g} \) which represents an effective coupling constant \( \hat{g} = \sqrt{6} \).

\[
(i \gamma^\mu \partial_\mu - m) \pm g \hat{g} \gamma^\mu A^3_\mu) \psi^\pm = 0,
\]

Solutions \( \Psi^0 \) form three one-dimensional invariant Weyl symmetric subspaces and represent an orthogonal basis in the vector space of the representation \( \Gamma_2 \oplus \Gamma_1 \). It is clear that after quantization such solutions lead to primary color singlet one-particle quantum states for quarks. It is remarkable, the Weyl symmetric solutions describe only color singlet quantum states for gluons and quarks, contrary to commonly accepted wisdom on color nature of gluons and quarks. Color gluons and quarks are not observable objects because their concepts are artifacts of perturbation theory and applied a simple Abelian projection which is inconsistent with non-Abelian Weyl symmetric structure \( 11, 12 \) of the full Yang-Mills Lagrangian. Certainly, some special solutions, like ordinary Abelian plane wave or non-linear plane waves, exist, however, such solutions are not physical due to another important condition for physical vacuum and states - quantum stability against vacuum fluctuations.

IV. QUANTUM STABLE VACUUM

The vacuum stability has been a persistent problem in all vacuum models in QCD since 1977 \( 3, 4 \). We outline in short the main difficulties on the way of constructing a true stable vacuum, and then we prove explicitly the quantum stability of Abelian vector harmonics by solving a system of eigenvalue equations for unstable (tachyonic) modes. The quantum instability of constant color magnetic QCD vacuum was established in \( 9 \), and later it was found for constant color electric background as well \( 23 \). The source of quantum instability is the presence of anomalous magnetic moment interaction terms in the classical Yang-Mills Lagrangian which implies that
one-loop effective action gains an imaginary part, i.e., vacuum is unstable. Numerous attempts to construct a stable vacuum from static field configurations fail in the class of regular solutions in Minkowski space-time. It was noticed that at small space-time scale the elementary field configurations should be vibrating due to quantum mechanical principle [7, 8]. A simple analysis of plane wave solutions shows the presence of quantum instability [24], since the eigenvalue problem for unstable modes represents a quantum mechanical bound state problem in one-dimensional periodic potential which admits bound states for any shallow potential well. It is clear, that stationary solution like a spherical wave in three-dimensional space remove such obstacle and can provide a stable vacuum configuration. Indeed, a spherically symmetric stationary wave solution possesses quantum stability [25, 26]. Unfortunately, the spherical wave solution has classical instability and can not describe a physical state. So, an axially-symmetric non-linear stationary solution of Yang-Mills theory has been proposed as a stable vacuum field in [15]. It is very difficult to prove rigorously classical stability of such solution, however, later it was realized that in the Abelian limit the solutions turn into the known vector spherical harmonics which represent free photons, which are obviously classically stable since they represent a free gas. An important role in providing the deepest vacuum is played by Weyl symmetry which selects Abelian color singlet solution due to the constraint on Abelian fields $A_{\mu}^a = A_{\mu}^b$.

Weyl symmetric solutions have a vanished total color charge which implies mutual cancellation of all cubic interaction terms in the Lagrangian $\mathcal{L}_{\text{red}}$. As it is known, the cubic interaction terms correspond to anomalous magnetic moment interaction which causes the vacuum instability. The absence of such terms plays a principal role in providing quantum stability of Weyl symmetric solutions. Quantum stability of the lowest energy non-Abelian Weyl symmetric solution has been proved numerically in [15]. We demonstrate quantum stability of vacuum Abelian solutions localized in a finite space region. We solve a “Schrödinger” type eigenvalue equation for quantum gluon fluctuations $\Psi_{\mu}$

$$\hat{\mathcal{K}}^{ab}_{\mu\nu} \Psi_{\nu}^{a} = \Lambda \Psi_{\mu}^{a},$$

$$\hat{\mathcal{K}}^{ab}_{\mu\nu} = -\delta^{ab} \delta_{\mu\nu} \partial_{t}^{2} - \delta_{\mu\nu} (\mathcal{D}_{\rho} \mathcal{D}^{\rho})^{ab} - 2 f_{abc} \mathcal{F}^{c}_{\mu\nu},$$

where the operator $\hat{\mathcal{K}}^{ab}_{\mu\nu}$ corresponds to the gluon contribution to one-loop effective action [15], and $\mathcal{D}_{\mu}, \mathcal{F}_{\mu\nu}$ are defined by means of a background vector harmonic [16]. The existence of a negative eigenvalue $\Lambda$ would imply the vacuum instability. An analysis of vacuum quantum stability is similar to one performed in [15]. Due to the presence of Weyl symmetry the vacuum stability problem in SU(3) QCD reduces to a corresponding problem in SU(2) theory. An explicit vector harmonic for axially-symmetric Abelian solution of magnetic type reads

$$B_{\mu}^{a} = \delta_{\mu3} \delta^{a3} c_{0} r j_{1}(Mr) \sin^{2} \theta \cos(Mt)$$

$$\equiv \delta_{\mu3} \delta^{a3} B_{r} (r, \theta, t).$$

Substitution of the solution $B_{\mu}^{a}$ into the eigenvalue equations [32] leads to factorization of the initial twelve equations to three systems of equations for three sets of fluctuating fields: (I) $\{\Psi_{1}, \Psi_{1}, \Psi_{3}, \Psi_{3}, \Psi_{5}, \Psi_{5}\}$, (II) $\{\Psi_{1}, \Psi_{2}, \Psi_{3}, \Psi_{3}, \Psi_{5}, \Psi_{5}\}$, (III) $\{\Psi_{j}\}$. The equations from group (III) represent free equations and does not admit solutions with negative eigenvalues. The second system of equations is equivalent to the first one after changing variables $\Psi_{1}^{a} \rightarrow \Psi_{1}^{a}, \Psi_{2}^{a} \rightarrow \Psi_{2}^{a}, \Psi_{3}^{a} \rightarrow \Psi_{3}^{a}, \Psi_{3}^{a} \rightarrow \Psi_{3}^{a}$ and reflection of the background field, $B_{r} \rightarrow -B_{r}$. So one has to solve only one system of four eigenvalue equations

$$\Delta \Psi_{1}^{1} + \frac{1}{r^{2}} (2 + \csc^{2} \theta B_{r}^{2}) \Psi_{1}^{1} + 2(cot \theta + \partial_{t}) \Psi_{2}^{1}$$

$$- 2 \csc \theta (B_{r} - r \partial_{r} B_{r}) \Psi_{2}^{3} = \lambda \Psi_{1}^{1},$$

$$\Delta \Psi_{1}^{2} + \frac{1}{r^{2}} (\csc^{2} \theta (1 + B_{r}^{2}) \Psi_{2}^{1} - 2 \partial_{t} \Psi_{1}^{1})$$

$$- 2 \csc \theta (cot \theta B_{r} - \partial_{r} B_{r}) \Psi_{2}^{3} = \lambda \Psi_{1}^{2},$$

$$\Delta \Psi_{3}^{2} + \frac{1}{r^{2}} ((1 + cot^{2} \theta + \csc^{2} \theta B_{r}^{2}) \Psi_{3}^{2}$$

$$- 2 \csc \theta (B_{r} - r \partial_{r} B_{r}) \Psi_{3}^{1}$$

$$- 2 \csc \theta (cot \theta B_{r} - \partial_{r} B_{r}) \Psi_{3}^{1} = \lambda \Psi_{1}^{2},$$

$$\Delta \Psi_{0}^{1} + \frac{1}{r^{2} \sin^{2} \theta} B_{r} \Psi_{0}^{1} + \frac{2}{r \sin \theta} \partial_{r} B_{r} \Psi_{3}^{2} = \lambda \Psi_{0}^{1},$$

with

$$\Delta \equiv - \left( \partial_{t}^{2} + \partial_{r}^{2} + \frac{2}{r} \partial_{r} + \frac{1}{r^{2} \sin^{2} \theta} \partial_{\theta} + \cot \theta \frac{\partial_{\theta}^{2}}{r^{2}} \right),$$

where $\Delta$ is a common part of the vector Laplace operator acting on the gluon fluctuation function. The obtained numeric solutions with the lowest eigenvalues confirm the absence of negative modes in the case of parameter values ($l \leq 4; c_{0} \leq 6; M \leq 10$) for size of the space domain $\mu_{11} \leq L \leq 100$. Quantum stability of Abelian solutions for finite size space domains constrained by lowest nodes and antinodes has been demonstrated in [27, FIG. 2].

V. LOCALIZATION OF QUANTUM STATES AND FORMATION OF LIGHTEST G אליוボールS

We quantize the Abelian solutions [16] in a finite space region constrained by a sphere of radius $a_{0}$ corresponding to an effective glueball size. It is suitable to introduce dimensionless units $M = M a_{0}, x = r/a_{0}, \tau = t/a_{0}$. To find
proper boundary conditions we require that the Pointing vector $\mathbf{s} = \mathbf{E} \times \mathbf{B}$ vanishes on the sphere. This implies two possible types of boundary conditions

(I) \[ \hat{A}_{lm}^{\mu \nu}(\hat{M}x)|_{x=1} = 0, \quad \hat{M}_{nl} = \mu_{nl}, \]

(II) \[ \partial_{r}(r \hat{A}_{lm}^{\mu \nu}(\hat{M}x))|_{x=1} = 0, \quad \hat{M}_{nl} = \nu_{nl}, \]

where $\hat{M}_{nl}$ stands for nodes $\mu_{nl}$ and antinodes $\nu_{nl}$ of the Bessel function $\psi_{jl}(r)$. We choose the following normalization condition for the vector harmonics $\hat{A}_{lm}^{\mu \nu}(\hat{M}x)$

\[ \frac{1}{4\pi} \int_0^1 \int d\theta d\varphi x^2 \sin \theta (\hat{A}_{lm}^{\mu \nu}(\hat{M}x))^2 = \frac{1}{M_{nl}}. \]  

The standard canonical quantization results in the following Hamiltonian expressed in terms of the creation and annihilation operators $c_{nlm}^\pm$

\[ H = \frac{1}{2} \sum_{n,l,m} \hat{M}_{nl}(c_{nlm}^+ c_{nlm}^- + c_{nlm}^- c_{nlm}^+). \]  

One particle states $\{c_{nlm}^+ |0\}$ describe free primary gluons which are not observable quantities since we have not taken into account their interaction to vacuum gluon condensate. We follow an idea that vacuum gluon and quark condensates represent inevitable attributes of hadrons.

We apply a simple model based on one-loop effective Lagrangian of QCD which describes appearance of localized solutions corresponding to the lightest glueballs. Certainly, one loop effective potential is not a much appropriate tool for quantitative description of glueballs, nevertheless, it contains a non-perturbative part originated from summation of contributions from infinite number of one-loop quantum corrections. This provides qualitative description of formation of glueballs as a result of interaction of primary gluon with corresponding generated vacuum gluon condensates.

We apply one-loop effective Lagrangain for $SU(3)$ pure QCD with Abelian background field corresponding to Abelian projection defined by ([27], (6)-(9)). In approximation of slowly varied fields one can calculate a one-loop effective Lagrangian

\[ \mathcal{L}^{1-1} = -\frac{1}{4} \tilde{F}^2 - k_0 g^2 F^2 \left( \log \left( \frac{g^2 F^2}{\Lambda_{\text{QCD}}^2} \right) - c_0 \right), \]  

where the number factor $k_0$ in front of logarithmic term is three times larger to compare with the effective Lagrangian for $SU(3)$ Yang-Mills theory with gluons in adjoint representation and quarks in fundamental representation. The factor three appears due to contribution of Weyl symmetric $L, U, V$ type Abelian gluon fields and contribution of quark solutions is also three times larger due to the effective color charge $\tilde{g} = \sqrt{6}$. So that the coefficient in front of logarithmic term is still proportional to the standard beta function of $SU(3)$ QCD with the same critical number of flavor quarks. The field $\tilde{F}_{\mu \nu}$ is an external Weyl symmetric Abelian color magnetic field, corresponding to Abelian projection defined by Weyl symmetric ansatz ([27], (6)-(9)). We will treat the parameters $k_0, c_0$ as free model parameters. The most important property of the effective Lagrangian is the presence of the non-perturbative logarithmic term which generates a non-trivial minimum of the effective potential $V^{1-1} = -\mathcal{L}^{1-1}$ at non-zero value of the vacuum condensate $\bar{g}^2$

\[ g^2 B_{\mu \nu}^2 = \Lambda_{\text{QCD}} \exp \left( c_0 - 1 - \frac{1}{2k_0 g^2} \right). \]  

We split the field $\tilde{F}_{\mu \nu}$ into two parts

\[ \tilde{F}_{\mu \nu} = B_{\mu \nu} + F_{\mu \nu}, \]  

where the background field $B_{\mu \nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}$ describes the magnetic vacuum gluon condensate, and $F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ contains an Abelian potential which describes the primary gluon interacting with the vacuum gluon condensate, we will treat the potential $A_{\mu}$ as a wave function of a pure glueball formed as a system of interacting primary gluon and vacuum gluon condensate. We decompose the Lagrangian $\mathcal{L}^{1-1}$ around the vacuum condensate field $B_{\mu \nu}$ and obtain an effective Lagrangian for physical Abelian glueballs in the lowest quadratic approximation

\[ \mathcal{L}_{\text{eff}}^{(2)}(A) = -2k_0 g^2 \frac{(B_{\mu \nu} F_{\mu \nu})^2}{B^2} \equiv -\kappa (B_{\mu \nu} F_{\mu \nu})^2, \]  

where we neglect a term corresponding to an absolute value of the vacuum energy, and $\kappa$ is a free parameter.
The effective Lagrangian $S^{(2)}_{\text{eff}}[A]$ is strikingly different from the effective Lagrangians obtained in quantum electrodynamics. Namely, the expression $\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ does not contain the classical kinetic term $-1/4 F_{\mu\nu}F^{\mu\nu}$ which is disappeared due to the non-perturbative expression for the vacuum gluon condensate $\langle A_{\mu}\rangle$ realizing the minimum of the effective potential.

Consider a case of the lightest magnetic glueball which is formed from the primary gluon in the presence of vacuum gluon condensate described by magnetic vector harmonic $A_{a1,m=0}$, $\omega = \nu_{11}$, which contains one non-zero magnetic potential $B_x$

$$B_x(x,t) = N_{11}xj_1(\nu_{11}x) \sin^2 \theta \sin(\nu_{11}x),$$

where $N_{11}$ is a renormalization constant, and we introduce dimensionless variables $x = r/\nu_{11}$, $t/\nu_{11}$, $\nu_{11} = 2.74\cdots$ is the first antinode of the radial function $rj_1(r)$. The lightest magnetic glueball is described by the gauge potential $A_x(x,t)$ which assumed to be time-coherent to the vacuum condensate field

$$A_x(x,t) = a(x,t) \sin(\nu_{11}x + \phi_0),$$

with a constant phase shift $\phi_0$. A time-averaged effective Lagrangian $S^{(2)}_{\text{eff}}[A]$ leads to the following Euler equation for the coordinate function $a(x,t)$

$$-\dot{\bar{m}} r^2 \sin \theta \left( \bar{m} r^2 b_{03}^2 + r^2 b_{13}^2 \xi b_{03}^2 
+ \xi b_{03}(r^2 b_{13}^2 - 2r b_{13}^2 - 2b_{23}^2) \right) a_x$$

$$+ \cot \theta \csc \theta \left( 2b_{23}^2 \cos(2\theta) - 3 + 2r^2 b_{23}^2 b_{13}^2 \sin^2 \theta 
- 4r b_{23}^2 b_{13} \sin^2 \theta (rb_{23}^2 - 4b_{23}^2) \right) \partial_0 a_x$$

$$+ 4b_{23}^2 \cos \theta \cot \theta b_{23}^2 a_x$$

$$+ 2r^2 b_{13}^2 \sin \theta (r^2 b_{13}^2 - rb_{13} b_{23}) \partial_0 a_x$$

$$+ 4r^2 b_{13}^2 b_{23} \cos \theta \partial_0 b_{23}^2 a_x + r^2 b_{23}^2 \sin^2 \theta \partial_0^2 b_{23}^2 a_x = 0,$$  

where $\xi \equiv \cos(2\phi_0)/(2 + \cos(2\phi_0))$, and $b_{\mu\nu}(r)$ are coordinate parts of the field strength $F_{\mu\nu}$. The equation looks quite complicate. Surprisingly, for the ground state the equation is separable and admits a spherically symmetric solution $a_x(r,\theta) = f(r)$ which describes a glueball state with zero total angular momentum. With this one results

$$x^2 b_{13}^2 b_{23}^2 - 2b' (b + xb' - x^2 b'' f')$$

$$- \nu_{11}^2 \left( \nu_{11}^2 x^2 b_{23}^2 - \xi \nu_{11}^2 x^2 j_2(\nu_{11}x)b_{23}^2 \right) f = 0,$$  

where $b(x) = \nu_{11}xj_1(\nu_{11}x)$, $\xi \equiv \cos(2\phi_0)/(2 + \cos(2\phi_0))$. The coefficient functions in front of the first and second derivative terms in (48) vanish at $x = 0$ (or $r = \nu_{11}$). This implies that a regular solution is localized inside a finite interval $0 < x < 1$, FIG. 8(a). Since the point $r_0 = \nu_{11}$ represents a singularity it is suitable to apply the “shooting” numeric method which allows to verify the type of singularity $r_0 = \nu_{11}$. The regular structure of the solution has been checked in the small vicinity of the singularity $r < \nu_{11} - 1.0 \cdot 10^{-6} = 2.743705$, and implies that singularity belongs to removable type. This provides a smooth structure of the energy density which has vanished first and second radial derivatives at $r_0$, FIG. 8(c), (d). The solution has a removable singularity at $x_0 = 1$. To verify that solution is physical we check the properties of the energy density averaged over the time and polar angle

$$\bar{\mathcal{E}} = \frac{\pi \kappa}{4x\nu^2} \left( 2 + \cos(2\phi_0) \right) b_{23}^2 f' f''$$

$$+ (2 + \cos(2\phi_0)) b_{23}^2 f'^2 \right).$$  

The solution has a minimal energy at the phase shift

$$\phi_0 = \pi/2,$$

and only at this value the averaged over time effective Lagrangian vanishes as well, like in a case of free Lagrangian for photon plane waves. So that the obtained solution describes a stable ground state for the scalar glueball. Quantum numbers of the glueball interacting to vacuum magnetic gluon condensate are defined in the same way as for two free photons system and lead to two lightest magnetic glueballs $0^+\oplus, 0^+\ominus$.

Note that result (44) is model independent, and it can be obtained from a class of Lagrangian functions (like in Ginsburg-Landau model) which admit series expansion around a non-trivial vacuum. This allows to obtain a qualitative estimate of lightest glueball spectrum in [27]. Qualitative estimates of the lightest scalar glueball spectrum can be performed in a model independent way assuming that vacuum gluon condensate is a universal order parameter for glueballs with different quantum numbers. The knowledge of explicit solutions for the vector potential (16) allows to find analytical expressions for the radial density of the vacuum gluon condensate performing averaging over the time and polar angle. Averaged over the time and polar angle vacuum gluon condensates
\(\alpha_s((F_{\mu\nu})^2)\) corresponding to magnetic modes \(\tilde{A}_{\nu_1 l}^m\) and \(\tilde{A}_{\mu_1 l}^m\) are depicted in FIG. 8 (\(\alpha_s = 0.5\)). The oscillating behavior of the vacuum gluon condensate density was obtained before within the instanton approach to QCD \[31\].

Integrating the radial density over the interval \(0 \leq x \leq 1\) one can fit a value of the obtained vacuum gluon condensate parameter to the known value \(\alpha_s((F_{\mu\nu})^2) = (540[\text{MeV}])^4\), and obtain an explicit dependence of the glueball size on quantum number \(\tilde{M}_{nl}\)

\[
a_{nl}[\text{fm}] = \frac{197}{\nu_0} f_c^{1/4}(\tilde{M}) \approx \frac{107\alpha_s^{1/4}}{\nu_0} \sqrt{\tilde{M}_{nl}},
\]

\[
f_c(\tilde{M}) = \frac{N_{nl}^2}{12\pi M^2}\left((3 - 4\tilde{M}^2 + 2\tilde{M}^4)\cos(2\tilde{M}) - 3\right.
\]
\[
- 2\tilde{M}^2 + 2\tilde{M}(3 - \tilde{M}^2)\sin(2\tilde{M}) + 4\tilde{M}^3\sin(2\tilde{M})\right),
\]

where \(\nu_0 = 540[1/\text{fm}]\), \(N_{nl}\) is the normalization factor of the vector harmonic, and \(\sin(2\tilde{M})\) is the sine integral function. With this one can find the energy spectrum of light scalar glueballs \(J^{PC} = 0^{++}\)

\[
E_{nl}[\text{MeV}] = \tilde{k}\nu_0 f_c^{1/4} \tilde{M}_{nl} \approx \left(\frac{80}{r\alpha_s}\right)^{1/4} \tilde{k}\nu_0 \sqrt{\tilde{M}_{nl}},
\]

where \(\tilde{k}\) is a free model parameter which can be fixed by fitting the energy value of the lightest glueball. For \(\tilde{k} = 1.01\) the lightest glueball \(J^{PC} = 0^{++}\) has energy \(E_{nl} = 1440[\text{MeV}]\). The energy spectrum \[51\] agrees with the Regge theory of hadrons.

VI. DISCUSSION

We have demonstrated that Weyl symmetric solutions provide color singlet primary quantum states for gluons and quarks. Instead of eight free color gluons defined in the framework of a perturbative QCD one has an infinite number of colorless primary gluons of magnetic or electric type and colorless quarks with quantum numbers \((l, m, k)\) which are localized in finite space domains constrained by nodes/antinodes \(\tilde{M}_{nl}\). Physical observables, hadrons, are formed as systems of interacting primary gluons and quarks with corresponding generated vacuum gluon and quark condensates. Certainly, this implies that one has to construct an improved quark model of hadrons which might be successful in resolving a problem of proton mass and spin structure. This problem and other related issues will be considered elsewhere.

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VII. APPENDIX: SINGLET STRUCTURE OF WEYL SYMMETRIC SOLUTION SPACE

We demonstrate the origin of this phenomenon by solving a reduced system of equations for two long distance propagating modes \(K_2, K_4\) \[6\]. \[7\]. In the asymptotic region fields \(K_{0,1}\) vanish and one has in the leading order the following factorized structure of solutions for \(K_{2,4}\)

\[
\begin{align*}
K_2(r, \theta) &= f_2(Mr) T_2(\theta) \cos(Mt), \\
K_4(r, \theta) &= f_4(Mr) T_4(\theta) \cos(Mt).
\end{align*}
\]  

We set for simplicity \(M = 1\), and consider the lowest energy solution with quantum number \(J = l = 1\) and lowest polar modes \(T_2(\theta) = 1, T_4(\theta) = \sin^2 \theta\). With this, the equations for \(K_{2,4}\) are simplified as follows

\[
\begin{align*}
f''_2 + f_2 - \frac{9}{4\nu_0^2} f_2 f_2^2 \sin^2 \theta &= 0, \\
f''_4 + f_4 - \frac{2}{\nu_0^2} f_4 - \frac{3}{4\nu_0^2} f_4 f_4^2 \cos^2 \theta &= 0.
\end{align*}
\]  

To obtain qualitative estimate we perform averaging over polar angle which leads to a simple system of ordinary differential equations

\[
\begin{align*}
\tilde{f}''_2 + \tilde{f}_2 &= \frac{1}{\nu_0^2} \tilde{f}_2 \tilde{f}_2^2 = 0, \\
\tilde{f}''_4 + \tilde{f}_4 &= \frac{2}{\nu_0^2} \tilde{f}_4 = 0.
\end{align*}
\]  

Where \(\tilde{f}_2 = \sqrt{3}/2 f_2, \tilde{f}_4 = 3/2 \sqrt{2} f_4\). All possible solutions to that system of equations can be easily obtained by applying a “shooting” numeric method by setting initial values and first derivatives of functions \(\tilde{f}_{2,4}\) at the origin \(r = 0\). Local solution near the origin \(r = 0\) implies that \(\tilde{f}_{2,4}(0) = 0\), so a general solution is defined by two integration constants corresponding to normal
derivatives \( \tilde{f}_{2,4}(0) = c_{2,4} \), and the space of solutions with given quantum numbers is two-dimensional in agreement with a simple counting degrees of freedom in the presence of local \( U(1) \) symmetry. An example of a general solution is presented in FIG. 4(a). Nodes and extremums of the fields \( \tilde{f}_2, \tilde{f}_4 \) do not coincide, so that such solutions do not possess a conserved energy. Moreover, the solutions are not classically stable under small fluctuations and represent saddle point solutions. Stable stationary solutions with a conserved localized energy are extracted by a constraint that fields \( f_{2,4} \) must have the same, at least, one node or antinode., FIG. 4(b). This constraint implies correlation of amplitudes of \( f_{2,4} \) since location of nodes/antinodes of interacting fields \( f_{2,4} \) depends on their amplitudes. Therefore, a space of solutions with a given quantum numbers \( M, l, m, k \) and conserved energy in a finite space region is one-dimensional and the norm of the solution is determined by one normalization constant which can be assign to the amplitude of the Abelian field \( K_4 \). Due to classical electric-magnetic duality one has similar results for electric type solutions.

![Figure 10: Solution for \( \tilde{f}_2 \) (in red), \( \tilde{f}_4 \) (in blue): (a) \( \tilde{f}_2, \tilde{f}_4 \) do not have common zeros; (b) \( \tilde{f}_{2,4} \) have the same zeros and extremums.](image)

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