Abstract

We offer a formula for the probability distribution of the number of misseated airplane passengers resulting from the presence of multiple absent-minded passengers, given the number of seats available and the number of absent-minded passengers. This extends the work of Henze and Last on the absent-minded passenger problem.

1 Introduction

A recent article by Henze and Last, Absent-Minded Passengers [2], considers the problem of \( k \) absent-minded passengers on an airplane with \( n \) passengers assigned to \( n \) seats. The absent-minded passengers are assigned seats \( \{1, 2, \ldots, k\} \), with the other passengers assigned seats \( \{k+1, \ldots, n\} \). The passengers are seated in order of passenger number. When it is time for one of the absent-minded passengers to choose a seat, that passenger chooses an unoccupied seat at random, with an equal likelihood for each of the unoccupied seats. When it is time for a non-absent-minded passenger to choose a seat, that passenger sits where assigned, if the assigned seat is available, otherwise choosing an unoccupied seat at random. The authors of [2] determine the probability distribution in the case where \( k \), the number of misseated passengers, is one, as well as the expected value and variance for all \( k \geq 1 \). In this paper, we find the probability distribution for all positive integers \( k \).

We claim that, with \( n \) passengers, the first \( k \) of whom are absent-minded, the probability that exactly \( m \) of them will be misseated is given by the following result.

**Theorem 1** (Main Result). The probability of \( m \) misseated passengers is

\[
P_{n,k}(m) = \frac{(-1)^m(n-k)!}{n!} \binom{k}{m} + \frac{1}{n!} \sum_{s=1}^{k} \left[ \begin{array}{c} n-k+1 \\ m-s+1 \end{array} \right] \frac{k}{s!} \sum_{\ell=1}^{s} \frac{(-1)^{s-\ell}(m-s)}{(s-\ell)!}.
\]

Here, \( \left[ \begin{array}{c} i \\ j \end{array} \right] \) is the unsigned Stirling number of the first kind, which is the number of permutations of \( i \) elements with \( j \) disjoint cycles, with the convention that \( \left[ \begin{array}{c} p \\ 0 \end{array} \right] = 0 \) and \( \left[ \begin{array}{c} p \\ q \end{array} \right] = 0 \) for positive \( p \) and \( q \) [1] page 259]. The formula includes the assertion that the probability of exactly one misseated passenger is 0.

For \( k = 1, 2, \) and 3 misseated passengers this gives, respectively,

\[
P_{n,1}(m) = \frac{1}{n!} \left[ \begin{array}{c} n \\ m \end{array} \right], \text{ for } m \geq 2
\]

\[
P_{n,2}(m) = \frac{(-1)^m}{n(n-1)} \left( \frac{2}{m} \right) + \frac{1}{n!} \left( 2 \left[ \begin{array}{c} n-1 \\ m \end{array} \right] + (2^{m-1} - 2) \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right] \right)
\]

\[
P_{n,3}(m) = \frac{(-1)^m}{n(n-1)(n-2)} \left( \frac{3}{m} \right) + \frac{1}{n!} \left( 3 \left[ \begin{array}{c} n-2 \\ m \end{array} \right] + 3 (2^{m-1} - 2) \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right] + (2 \cdot 3^{m-2} - 3 \cdot 2^{m-2} + 3) \left[ \begin{array}{c} n-2 \\ m-2 \end{array} \right] \right).
\]

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2 How the passengers can be misseated

In preparation for the proof of the theorem, we prove the following lemma.

Lemma 1.

\[
\sum_{k<i_1<i_2<\cdots<i_{m-s}\leq n} \left( \prod_{j=1}^{m-s} \frac{1}{n-(i_j-1)} \right) = \frac{1}{(n-k)!} \left[ \frac{n-k+1}{m-s+1} \right].
\]

Proof. To prove this, set \( \ell_j = n - (i_j - 1) \). Then the original sum becomes

\[
\frac{1}{(n-k)!} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{m-s}\leq n-k} (n-k)! \ell_1 \ell_2 \cdots \ell_{m-s}.
\]

For a fixed positive integer \( N \), let \( g_N(x) \) be the generating function of the Stirling numbers of the first kind \[ page 263 \]; that is,

\[
g_N(x) = x(x+1) \cdots (x+N-1) = \sum_{i=0}^{N} \left[ N \atop i \right] x^i.
\]

By equating coefficients of \( x^t \) in this equation, we find that

\[
\left[ N \atop i \right] = \sum_{0\leq a_1<a_2<\cdots<a_{N-i}<N} a_1 a_2 \cdots a_{N-i},
\]

and therefore

\[
\frac{1}{(n-k)!} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{m-s}\leq n-k} (n-k)! \ell_1 \ell_2 \cdots \ell_{m-s} = \frac{1}{(n-k)!} \left[ \frac{n-k+1}{m-s+1} \right].
\]

Before proving the main theorem, we first prove the formula below. We later simplify this result to give Theorem \[ \]

Theorem 2.

\[
P_{n,k}(m) = \frac{1}{n!} \sum_{s=0}^{k} \binom{k}{s} \sum_{t=0}^{s} (t!)^2 \binom{m-s}{t} \left[ \frac{n-k+1}{m-s+1} \right] \sum_{r=t}^{s} \binom{s}{r} L(r,t) d_{s-r}.
\]

Here, \( \left[ \begin{array}{c} i \\ j \end{array} \right] \) is the Stirling number of the second kind, which counts the number of ways to partition a set of \( i \) labeled objects into \( j \) nonempty unlabeled subsets \[ page 258 \]; \( L(i,j) \) is the Lah number, which counts the number of ways a set of \( i \) elements can be partitioned into \( j \) nonempty linearly-ordered subsets \[ 3, 4 \]; and \( d_i \) is the number of derangements of a set of \( i \) elements, that is, the number of permutations with no fixed points \[ page 194 \]. Following \[ pages 262 \], we adopt the following conventions for positive integers \( p \) and \( q \):

\[
\left\{ \begin{array}{c} -p \\ q \end{array} \right\} = 0, \left\{ \begin{array}{c} -p \\ 0 \end{array} \right\} = 0, \left\{ \begin{array}{c} 0 \\ q \end{array} \right\} = 0, \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = 1, \left[ \begin{array}{c} p \\ 0 \end{array} \right] = 0 \text{ and } \left[ \begin{array}{c} p \\ -q \end{array} \right] = 0.
\]

Proof. Since the absent-minded passengers are those with the lowest numbers, we associate them with the first-class cabin and the non-absent-minded passengers with the main cabin. The probability that exactly \( m \) passengers are misseated is the sum over \( s \) of the probabilities that a total of exactly \( m \) passengers, including \( s \) from first class and \( m-s \) from the main cabin, are misseated.

The probability of a specific arrangement of the \( k \) first-class passengers is

\[
\frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{n-(k-1)} = \frac{(n-k)!}{n!},
\]
Figure 1: Airplane passengers are misseated in threads. Here, $n = 30$, $k = 5$ and $m = 13$. Furthermore, $s = 4$ and $r = 2$. The threads terminate when passenger 19 sits in either seat 1 or 2. Passenger 30 must then sit in whichever of these two seats remain.

and the probability of a specific sequence $i_1 < i_2 < \cdots < i_{m-s}$ of misseated main cabin passengers is given by

$$\prod_{j=1}^{m-s} \frac{1}{n-(i_j-1)},$$

since when it is time for passenger $i_j$ to be seated, there are $n-(i_j-1)$ seats available.

The total probability of the outcome is thus

$$\frac{(n-k)!}{n!} \cdot \prod_{j=1}^{m-s} \frac{1}{n-(i_j-1)}.$$

We now count the number of outcomes with exactly $m$ misseated passengers including exactly $s$ first-class passengers and the particular passengers $i_1 < i_2 < \cdots < i_{m-s}$ from the main cabin. There are $\binom{k}{s}$ ways of choosing which first-class passengers are misseated.

The misseating of main cabin passengers $i_1, i_2, \ldots, i_{m-s}$ occurs in threads, with a thread consisting of a non-empty sequence of first-class passengers followed by a non-empty sequence of main cabin passengers. The number of threads is at least zero (in the case that no main-cabin passengers are misseated) and at most $s$. For a given number $t$ of threads, at least $t$ and at most $s$ of the misseated first-class passengers are elements of these threads. Let the number of these absent-minded passengers be $r$. There are then $s-r$ misseated first-class passengers who are not part of a thread.

There are $\binom{s}{r}$ choices for the $r$ first-class passengers who are in threads. These $r$ passengers can be placed into $t$ threads in $L(r,t)$ ways. The $i_1, \ldots, i_{m-s}$ passengers can be placed into these $t$ threads in $(t!)\binom{m-s}{t}$ ways.

Each thread ends with a main cabin passenger sitting in the seat of a first-class passenger who is seated first in a thread. This can happen in $t!$ ways. The remaining $s-r$ misseated passengers permute their seats, with none fixed. This can happen in $d_{s-r}$ ways. A visualization of this can be seen in Figure 1. Thus,
\[
\mathbb{P}(m \text{ misseated, including the main-cabin passengers } i_1, i_2, \ldots, i_{m-s})
\]

\[
= \left( \sum_{l=0}^{s} \sum_{r=t}^{s} \binom{k}{s} \binom{s}{r} (r, t) (t!)^2 \binom{m-s}{t} d_{s-r} \right) \frac{(n-k)!}{n!} \frac{m-s}{\prod_{j=1}^{m-s} \frac{1}{n-(i_j-1)}}.
\]

and

\[
\mathbb{P}(m \text{ misseated, including } s \text{ first-class passengers})
\]

\[
= \left( \sum_{l=0}^{s} \sum_{r=t}^{s} \binom{k}{s} \binom{s}{r} (r, t) (t!)^2 \binom{m-s}{t} d_{s-r} \right) \frac{(n-k)!}{n!} \sum_{k<i_1<i_2<\ldots<i_{m-s}} \left( \prod_{j=1}^{m-s} \frac{1}{n-(i_j-1)} \right)
\]

\[
= \frac{1}{n!} \left( \sum_{l=0}^{s} \sum_{r=t}^{s} \binom{k}{s} \binom{s}{r} (r, t) (t!)^2 \binom{m-s}{t} d_{s-r} \right) \left[ n-k+1 \right] \left[ m-s+1 \right].
\]

Summing over \( s \) gives

\[
P_{n,k}(m) = \frac{1}{n!} \sum_{s=0}^{k} \binom{k}{s} \left[ n-k+1 \right] \left[ m-s+1 \right] \sum_{l=0}^{s} (t!)^2 \binom{m-s}{t} \sum_{r=t}^{s} \binom{s}{r} L(r, t) d_{s-r},
\]

as claimed. \( \square \)

### 3 Proof of main result

We proceed to obtain Theorem 1 from Theorem 2. To do so, we begin with the sum over \( r \) using formulas for the Lah numbers [4] and the derangements [1, page 195]. For \( t \geq 1 \), we have

\[
\sum_{r=t}^{s} \binom{s}{r} (r, t) L(r, t) d_{s-r} = \sum_{r=t}^{s} \binom{s}{r} (r-1) t! (s-r)! \sum_{j=0}^{s-r} (-1)^j \frac{1}{j!}
\]

\[
= \frac{s!}{t!} \sum_{j=0}^{s-t} \frac{(-1)^j}{j!} \sum_{r=t}^{s-j} \binom{r-1}{t-1}
\]

\[
= \frac{s!}{t!} \sum_{j=0}^{s-t} \frac{(-1)^j}{j!} \binom{s-j}{t}.
\]

We note that if \( t = 0 \), then \( \sum_{r=t}^{s} \binom{s}{r} (r, t) L(r, t) d_{s-r} \) and \( \frac{s!}{t!} \sum_{j=0}^{s-t} \frac{(-1)^j}{j!} \binom{s-j}{t} \) both equal \( d_s \), so we can use the result of the above calculation in that case as well.

The following result is simple but useful. We record it as a lemma.

**Lemma 2.** For positive integers \( J, K, L \), with \( L \leq K \),

\[
\sum_{J=L}^{K} (-1)^J \binom{K-L}{J-L} = (-1)^L \delta_{L,K},
\]

where \( \delta_{L,K} \) is 1 if \( L = K \) and 0 otherwise.

**Proof.** Make the change of variables \( I = J - L \) to get

\[
(-1)^L \sum_{I=0}^{K-L} (-1)^I \binom{K-L}{I};
\]

the sum is the expansion of \( (1 - 1)^{K-L} \), which is 0 unless \( L = K \). \( \square \)
We now consider the sum over \( t \) in the equation of Theorem \([2]\) substituting the result obtained in equation \([1]\) above. For \( s < m \), a formula for the Stirling numbers of the second kind \([1\text{ page 265}]\) gives
\[
\sum_{t=0}^{s} \binom{m-s}{t} t! L(r,t) = (s!) \sum_{t=0}^{s} \sum_{j=0}^{s-t} (-1)^{s-t-j} \frac{m!}{j!} \binom{s-j}{t} (s-j)^{t} (t-j)^{s-j}.
\]

Using trinomial revision \([1\text{ page 174}]\) gives
\[
\binom{t}{\ell} \binom{s-j}{t} = \binom{s-j}{t} \binom{s-j}{t-\ell},
\]
so that the above becomes
\[
= (s!) \sum_{\ell=0}^{s} (-1)^{s} \frac{m!}{j!} \sum_{\ell=0}^{s-t} (-1)^{s-t-j} \frac{m!}{j!} \binom{s-j}{t} (s-j)^{t} (t-j)^{s-j}.
\]

We now address the case \( s = m \). We have
\[
\sum_{t=0}^{s} \binom{m-s}{t} t! \sum_{j=0}^{s-t} (-1)^{j} \frac{(m-j)!}{j!} \binom{n-k+1}{s} \binom{s}{t} (s-j)^{t} (t-j)^{s-j} = \sum_{t=0}^{m} \binom{n-k+1}{s} \binom{s}{t} (m-j)^{t} (t-j)^{m-j}.
\]

The above is
\[
\sum_{t=0}^{m} \binom{n-k+1}{s} \binom{s}{t} (m-j)^{t} (t-j)^{m-j} = m! \sum_{t=0}^{m} \frac{(-1)^{j}}{j!} \frac{(m-j)!}{j!} = m! \sum_{\ell=0}^{n-k} \frac{(-1)^{\ell}}{(m-\ell)!}.
\]

Substituting in the original equation now gives
\[
P_{n,k}(m) = \frac{1}{n!} \sum_{s=0}^{k} \binom{n-k+1}{m-s+1} \binom{k}{s} (s!) (-1)^{s} \frac{\delta_{s,m}}{m!} + \sum_{\ell=1}^{n-k} \frac{(-1)^{\ell} e^{m-s}}{(s-\ell)!}.
\]

Interpreting \( \binom{k}{m} \) as 0 when \( k < m \), and noting that \( n-k+1 = (n-k)! \), we can rewrite this last result as
\[
P_{n,k}(m) = \frac{(-1)^{m}(n-k)!}{n!} \binom{k}{m} + \frac{1}{n!} \sum_{s=0}^{k} \binom{n-k+1}{m-s+1} \binom{k}{s} s! \sum_{\ell=1}^{n-k} \frac{(-1)^{s-\ell} e^{m-s}}{(s-\ell)!},
\]
as required. This proves Theorem \([1]\).

For a visual interpretation of this function for several \( k \) when the number of passengers, \( n \), is 100, we direct the reader to Figure \([2]\)
Figure 2: A graph of the probability as a function of $m$ given by our formula $P_{n,k}(m)$, for an $n = 100$ passenger plane, with $k = 1, 2$ and $3$ absent-minded passengers.

References

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[2] Norbert Henze and Gnter Last. Absent-minded passengers. *The American Mathematical Monthly*, 126(10):867–875, 2019.

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