SIMULATION OF HIGH TEMPERATURE SUPERCONDUCTORS AND EXPERIMENTAL VALIDATION

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ABSTRACT. In this work, we present a numerical framework to simulate high temperature superconductor devices. We select the so-called $H$-formulation, which uses the magnetic field as a state variable, in order to develop an augmented formulation that guarantees the divergence-free condition. However, nodal elements fail in the $H$-formulation modelling of electromagnetic fields along interfaces between regions with high contrast medium properties, thus Nédélec elements (of arbitrary order) are preferred. The composition of a customized, robust, nonlinear solver completes the exposition of the developed tools to tackle the problem. A set of detailed numerical experiments in 2D/3D shows the availability of the finite element approximation to model the phenomena, including a validation against experimental data.

Keywords: High Temperature Superconductors, Maxwell equations, Nédélec Finite Elements

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1. INTRODUCTION

High Temperature Superconductor (HTS) devices possess a number of unique properties that make them attractive for use in a wide range of engineering applications. In order to design and optimize devices using superconducting tapes, computational tools are a powerful technique to simulate its electromagnetic behaviour by solving the system of partial differential equations (PDEs) that governs the problem. Finite Element (FE) methods are commonly used in this context because they can handle complicated geometries whilst providing a rigorous mathematical framework. However, HTS modelling is an extremely challenging simulation process that stresses many aspects of a numerical code such as multiphysics modelling, multiscale modelling, highly nonlinear behaviour, and a large number of time steps involved. Hence an appropriate definition of the formulation, the FE method, and the solver will play a crucial role in order to obtain meaningful results.

Many formulations exist in subsequent approaches to the problem, which can be mainly classified in three kinds of formulations, named after the variables used in the system of PDEs that one aims to solve: the A-V formulation [1], which is based on the magnetic vector potential, the T-\Phi formulation [2], which is based on the current vector potential, and the H-formulation [3], which is based directly on the magnetic field. In this work, the authors have selected the generalized Coulomb gauged H-formulation. The H-formulation provides the direct solution to the magnetic vector field, and has the advantage of dealing with boundary conditions in the model in a simple fashion. External magnetic fields can be applied directly by setting boundary values of the magnetic field, while currents in the superconductor device can be injected through Ampère’s law. The use of a gauged formulation allows one to enforce the divergence-free constraint and make the problem well-posed in the limit of infinite resistivity.

The FE discretization of the magnetic field relies on the curl-conforming edge (or Nédélec) element [4] (of arbitrary order). Edge elements are preferred over grad-conforming Lagrangian ones for the magnetic field, since it facilitates the modelling of the field near singularities, as they allow normal fields components to jump across interfaces between two different media with highly contrasting properties [5]. In general, Lagrangian elements with a weak imposition of the divergence constraint can converge to singular solutions for homogeneous problems (see, e.g., [6]), but these methods are not robust for heterogeneous problems like the ones in HTS modelling.

It is well known that the curl operator has null space (the gradient of any scalar field), thus the uniqueness of the solution is not guaranteed. To remove the kernel associated to the curl operator, one needs: a) to rely on the mass matrix term in the transient formulation, or b) to add the Coulomb gauge condition. Although the first option is usually found in Maxwell transient equations and available for a wide range of cases, it is not suitable for HTS modelling conditions. The mass matrix term is scaled by the inverse of the resistivity in the equations, which is practically zero in non-conducting regions [7]. Further, neither nodal elements nor edge elements make the solution to satisfy the divergence-free constraint [5]. In other words, none of these elements guarantees that the solution obtained will be free of spurious modes. To make the problem well-posed, we must consider the (discrete) Helmholtz decomposition to enforce the solution to be
discrete divergence-free \[8\]. To do so, we will make use of a scalar magnetic pressure in the system, acting as a Lagrange multiplier.

Current works on the simulation of the HTS problem focus on the application case, lack of theory, do not provide mechanisms to enforce the divergence constraint, consider small test cases (mainly 2D problems), and are mainly developed with commercial software and linear (at most quadratic) edge elements, see e.g. \[9, 3, 10\]. The current work is one step beyond, since we consider advanced electromagnetic discretizations, i.e., high-order edge elements, provide a detailed formulation with magnetic pressure, and explore advanced linear and nonlinear parallel solvers for this particular problem. Particular emphasis is put in aspects that the authors have not found in the literature, such as a straight way to construct arbitrary order curl-conforming spaces. In this sense, contributions of this work are twofold: first, it aims to shed some light on well-posed high-order FE formulations in superconductivity and the design of efficient nonlinear solvers, and second, it describes a new numerical framework for HTS modelling of realistic applications. The specific FE formulation and the customized robust nonlinear solver presented in this work have been implemented in FEMPAR \[11, 12\], an open source object oriented Fortran 2008 scientific software library developed by the Large Scale Scientific Computing (LSSC) group at CIMNE-UPC. Therefore, this work also aims to introduce FEMPAR to the superconductivity modelling community as a new and powerful alternative for their simulations. In order to show its applicability to realistic cases, a validation with the Hall probe mapping experiment \[13\] is performed, obtaining a good agreement between the simulation results and the experimental data.

The outline of the article is as follows. The problem is defined in Sect. \[2\] and some notation is introduced. The FE approximation of the problem is developed in Sect. \[3\]. In Sect. \[4\] we present a customized nonlinear parallel solver suitable for the problem at hand. We present a detailed set of numerical experiments in Sect. \[5\] which include a validation phase against experimental data. Finally, some conclusions are drawn in Sect. \[6\].

2. The system of equations

2.1. Notation. In this section, we introduce the problem to be solved and its particularities. Let \(\Omega \subset \mathbb{R}^d\) be a bounded domain with \(d = 2, 3\) the space dimension. Let us denote by \(L^2(\Omega)\) the space of square integrable functions, and its associated norm \(\|u\|_2 = (\int_{\Omega} |u|^2)^{1/2}\). Further, we will make use of the spaces

\[
(1) \quad \mathcal{H}^1(\Omega) := \{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega)^d \}
\]

\[
(2) \quad \mathcal{H}(\text{curl}, \Omega) := \{ v \in L^2(\Omega)^d \mid \nabla \times v \in L^2(\Omega)^d \}
\]

and the subspaces

\[
(3) \quad \mathcal{H}^1(\Omega) := \{ v \in \mathcal{H}^1(\Omega) \mid v = 0 \text{ in } \partial\Omega \}
\]

\[
(4) \quad \mathcal{H}(\text{curl}, \Omega) := \{ v \in \mathcal{H}(\text{curl}, \Omega) \mid n \times v = 0 \text{ in } \partial\Omega \}
\]

where \(n\) denotes the outward unit normal to the boundary of the domain \(\Omega\). In the sequel, bold characters will be used to describe vector functions or tensors, while regular ones will determine scalar functions or values. No difference is intended by using upper-case or lower-case letters for functions. Calligraphic letters are used to describe functional spaces and bold calligraphic letters will denote bilinear operators.

2.2. Maxwell equations in electromagnetics. Let us first state the Maxwell equations, which physically describe magnetostatics. Let us consider \(\Omega \subset \mathbb{R}^d\) to be a simply
connected nonconvex polyhedral domain with a connected Lipschitz continuous boundary \( \partial \Omega \). Maxwell equations in differential form read

\[
\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \quad \text{Maxwell-Faraday equation}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \text{Ampère’s circuital law}
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad \text{Gauss’s law for magnetism}
\]

\[
\nabla \cdot \mathbf{E} = 4\pi \sigma, \quad \text{Gauss’ law}
\]

in \( \Omega \times (0, T] \), where \( \mathbf{E} \) is the electric field, \( \mathbf{B} \) is the magnetic field, \( \mathbf{J} \) is the electric current density, \( \mu_0 > 0 \) is the magnetic permeability of the vacuum, and \( \sigma \) is the electric charge density. Further, we add the constitutive law that specifies the relationship between the electric field and the current density in a material by

\[
\mathbf{E} = \rho \mathbf{J}, \quad \text{Ohm’s law}
\]

being \( \rho > 0 \) the material resistivity (inverse of conductivity). We also consider the constitutive law between the magnetic fields \( \mathbf{B} = \mu_0 \mathbf{H} \). After some trivial manipulation of Eqs. (6, 7) and the integration of the constitutive law Eq. (10) one can obtain the so-called \( H \)-formulation for solving the magnetic field \( \mathbf{H} \) directly as

\[
\frac{\partial (\mu_0 \mathbf{H})}{\partial t} + \nabla \times \rho \nabla \times \mathbf{H} = 0 \quad \text{in } \Omega \times (0, T],
\]

together with the constraint \( \nabla \cdot (\mu_0 \mathbf{H}) = 0 \) given by Eq. (8) and appropriate boundary conditions for the magnetic field \( \mathbf{H} \) over the boundary \( \partial \Omega \).

2.3. The Coulomb gauged \( H \)-formulation. Let us consider the Maxwell problem recast as a saddle-point problem by enforcing the divergence constraint with a Lagrange multiplier that we will call magnetic pressure \( p \). The purpose of the addition of the magnetic pressure in this problem is twofold. On one hand, it allows to explicitly impose the divergence-free constraint to the double curl-problem by adding an extra unknown. On the other hand, although the double curl kernel (gradient of any scalar field) is theoretically removed by the addition of transient terms, in the case where high resistivity parameters are used (as usually occurs in dielectric regions)

\[
\lim_{\rho \to \infty} (\mu_0 \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \rho \nabla \times \mathbf{H}) = \nabla \times \rho \nabla \times \mathbf{H},
\]

and therefore the addition of the divergence-free constraint may be of paramount importance. An extensive analysis for the steady version of this formulation can be found in [6]. Then, the proposed Coulomb gauged formulation reads as follows: seek a magnetic field \( \mathbf{H} \) and a magnetic pressure \( p \) solution of

\[
\frac{\partial \mu_0 \mathbf{H}}{\partial t} + \nabla \times \rho \nabla \times \mathbf{H} - \nabla p = f \quad \text{in } \Omega \times (0, T],
\]

\[
-\nabla \cdot (\mu_0 \mathbf{H}) = 0 \quad \text{in } \Omega \times (0, T],
\]

assuming that \( f \) is a divergence-free datum. Eqs. (13, 14) need to be supplied with appropriate boundary and initial conditions. The boundary of the domain \( \partial \Omega \) is divided into its Dirichlet boundary part, i.e., \( \partial \Omega_D \), and its Neumann boundary part, i.e., \( \partial \Omega_N \), such that \( \partial \Omega_D \cup \partial \Omega_N = \partial \Omega \) and \( \partial \Omega_D \cap \Omega_N = \emptyset \). Then, boundary and initial conditions for the
problem at hand read
\begin{align}
(15) & \quad H \times n = g & \text{on } \partial \Omega_D \times (0, T], \\
(16) & \quad p = 0 & \text{on } \partial \Omega_D \times (0, T], \\
(17) & \quad n \times (\rho \nabla \times H) = 0 & \text{on } \partial \Omega_N \times (0, T], \\
(18) & \quad H \cdot n = 0 & \text{on } \partial \Omega_N \times (0, T], \\
(19) & \quad H(x, t = 0) = 0 & \text{in } \Omega.
\end{align}

Note that Dirichlet boundary conditions prescribe the tangent component of the magnetic field in the boundary of the domain, while Neumann boundary conditions prescribe the tangent component of the electric field \(E\), (see Eq. \((10)\)) and the normal component of the magnetic field \(H\) (zero-flux condition). The variational functional setting corresponding to the established curl formulation reads as follows: find \(H \in \mathcal{H}(\text{curl}, \Omega), p \in \mathcal{H}_1^1(\Omega)\) such that
\begin{align}
(20) & \quad (\frac{\partial \mu_0 H}{\partial t}, v) + (\rho \nabla \times H, \nabla \times v) - (\nabla p, v) = (f, v) & \forall v \in \mathcal{H}_0(\text{curl}, \Omega) \\
(21) & \quad (\mu_0 H, \nabla q) = 0 & \forall q \in \mathcal{H}_1^0(\Omega)
\end{align}

Proposition 2.1. Function \(p\) vanishes in the appropriate functional setting and thus Eq. \((20)\) leads to the same solution for \(H\) as the double-curl equation \((11)\) in weak form (for \(\mu_0 \neq 0\)).

Proof. Let us take \(v = \nabla p\) in Eq. \((20)\). Then, \(v \in \mathcal{H}_0(\text{curl}, \Omega)\) and we have
\begin{align}
(22) & \quad (\frac{\partial \mu_0 H}{\partial t}, \nabla p) + (\rho \nabla \times H, \nabla \times \nabla p) - (\nabla p, \nabla p) = (f, \nabla p).
\end{align}
After integrating by parts, we get:
\begin{align}
(23) & \quad (\frac{\partial}{\partial t}(\nabla \cdot \mu_0 H), p) + (\rho \nabla \times H, \nabla \times \nabla p) - (\nabla p, \nabla p) = (\nabla \cdot f, p).
\end{align}

Now, taking the time derivative of Eq. \((26)\) (assuming that the initial condition is divergence-free), and the fact that \(\nabla \times \nabla p = 0\) and \(\nabla \cdot f = 0\) in \(\Omega\), we obtain \(\int_\Omega |\nabla p|^2 = 0\). Since \(p\) vanishes on \(\partial \Omega_D\), it implies that \(p \equiv 0\) at all times in \(\Omega\) by virtue of Poincaré’s inequality. \(\Box\)

2.4. Augmented Coulomb gauged \(H\)-formulation. Let us consider an augmented formulation of the Maxwell problem based on \([5]\). The idea is to introduce to the second equation of our formulation a scaled term \(\gamma \Delta p\), with \(\gamma > 0\), so that, without affecting the solution, we gain control over the diagonal dominance of the resulting operator, which can be desired for the design of robust and efficient physics-based linear solvers. Let us state the augmented formulation in strong form as
\begin{align}
(24) & \quad \frac{\partial \mu_0 H}{\partial t} + \nabla \rho \nabla \times H - \nabla p = f & \text{in } \Omega \times (0, T], \\
(25) & \quad -\nabla \cdot (\mu_0 H) - \gamma \Delta p = 0 & \text{in } \Omega \times (0, T],
\end{align}
satisfying \(H \times n = 0\) and \(p = 0\) over \(\partial \Omega\). The weak form for this formulation reads: find \(H \in \mathcal{H}(\text{curl}, \Omega), p \in \mathcal{H}_1^1(\Omega)\) such that
\begin{align}
(26) & \quad (\frac{\partial \mu_0 H}{\partial t}, v) + (\rho \nabla \times H, \nabla \times v) - (\nabla p, v) = (f, v) & \forall v \in \mathcal{H}_0(\text{curl}, \Omega) \\
(27) & \quad (\mu_0 H, \nabla q) + \gamma (\nabla p, \nabla q) = 0 & \forall q \in \mathcal{H}_1^0(\Omega).
\end{align}

Proposition 2.2. In the augmented formulation, the function \(p\) also vanishes in the appropriate functional setting and thus Eqs. \((26)-(27)\) lead to the same solution for \(H\) as the double-curl equation \((11)\) in weak form (for \(\mu_0 \neq 0\)).
Proof. Let us take \( \mathbf{v} = \nabla p \) in Eq. (26) and \( q = p \) in Eq. (27). Then, \( \mathbf{v} \in \mathcal{H}_0(\text{curl}, \Omega) \), \( q \in \mathcal{H}_0^1(\Omega) \), and we have

\[
\frac{\partial}{\partial t} (\nabla \cdot \mu_0 \mathbf{H}, p) + (\rho \nabla \times \mathbf{H}, \nabla \times \nabla p) - (\nabla p, \nabla p) = (\nabla \cdot \mathbf{f}, p),
\]

\[
(\nabla \cdot \mu_0 \mathbf{H}, p) + \gamma (\nabla p, \nabla p) = 0.
\]

Using the fact that \( \nabla \times \nabla p = 0 \), \( \nabla \cdot \mathbf{f} = 0 \), and combining both equations we obtain

\[
\gamma (\frac{\partial}{\partial t} \nabla p, \nabla p) + (\nabla p, \nabla p) = 0.
\]

Let us now write the expression in the \( L^2 \) norm as

\[
\gamma \int_0^{T_f} \frac{\partial}{\partial t} \| \nabla p \|^2 + \int_0^{T_f} \| \nabla p \|^2 = \gamma \| \nabla p \|^2_{t=t_f} - \gamma \| \nabla p \|^2_{t=t_0} + \int_0^{T_f} \| \nabla p \|^2 = 0.
\]

where we have omitted the subindex in the norms for simplicity. Since \( p(t = 0) = 0 \), the term \( \| \nabla p \|^2_{t=0} \) vanishes and we end up with a norm. Then, as \( p \) vanishes on \( \partial \Omega_D \), it implies that \( p \equiv 0 \) in \( \Omega \) at all times. \( \square \)

2.5. Transmission conditions. Consider now two different non-overlapping regions on the domain \( \Omega \), namely \( \Omega_1 \) and \( \Omega_2 \) such that \( \Omega = \Omega_1 \cup \Omega_2 \) and \( \Gamma = \Gamma_1 \cap \Gamma_2 \). Natural boundary conditions appear on the formulation after integrating by parts Eq. (20):

\[
\int_\Omega (\nabla \times \rho \nabla \mathbf{n}) \cdot \mathbf{v} - \int_{\partial \Omega_N} (\rho \nabla \times \mathbf{H}) \cdot (\mathbf{n} \times \mathbf{v}),
\]

where we can identify the condition \( \mathbf{n} \times (\rho \mathbf{n} \nabla \mathbf{H}) \) to be introduced in the Neumann boundary \( \partial \Omega_N \). The same procedure for the divergence-free constraint leads to the normal component condition for the magnetic field

\[
\int_\Omega \nabla \cdot (\mu_0 \mathbf{H}) q = - \int_\Omega \mu_0 \mathbf{H} \cdot \nabla q + \int_{\partial \Omega_N} \mu_0 (\mathbf{H} \cdot \mathbf{n}) q.
\]

where clearly the flux \( \mathbf{H} \cdot \mathbf{n} \) has to be prescribed. Consider now the domain with the two regions described above, with different media. Let us denote by \( \{ \mathbf{n}_{\Gamma_1}, \mathbf{n}_{\Gamma_2} \} \) the unit pointing outwards normal of \( \{ \Omega_1, \Omega_2 \} \). Clearly, \( \mathbf{n}_{\Gamma_1} = -\mathbf{n}_{\Gamma_2} \) and we state the natural interface conditions (transmission conditions) for the Eqs. (13)-(14) as

\[
\mathbf{n} \times (\rho_1 \nabla \mathbf{H}_1 - \rho_2 \nabla \mathbf{H}_2) = 0 \quad \text{on} \ \Gamma
\]

\[
(\mu_0 \mathbf{H}_2 - \mu_0 \mathbf{H}_1) \cdot \mathbf{n} = 0 \quad \text{on} \ \Gamma
\]

where \( \mathbf{n} \) holds in this case for \( \mathbf{n} = \mathbf{n}_{\Gamma_1} \). Note that, with no other sources, Eq. (34) enforces the continuity of the tangent component of the electric field \( \mathbf{E} \) over the interface, i.e., \( \mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \). Eq. (35) enforces the normal continuity of the magnetic field \( \mathbf{B} = \mu_0 \mathbf{H} \). For the problem at hand, the domain will be composed of HTS device \( \Omega_{\text{hts}} \) and a surrounding dielectric region \( \Omega_{\text{air}} \).

On the other hand, currents in the superconductor device are injected through Ampère’s circuital law, Eq. (7), in a closed surface as (by the Stokes theorem)

\[
\int_S (\nabla \times \mathbf{H}) \cdot \mathbf{n} dS = \oint_{\partial S} \mathbf{H} \cdot \mathbf{\tau} dl = I_{\text{app}},
\]

where now \( \mathbf{n} \) denotes here the unit normal pointing outwards to the surface \( S \) defined by a section of \( \Omega_{\text{hts}} \) (the domain itself in a 2D case). On the other hand, \( \mathbf{\tau} \) is the unit tangent to the surface boundary. The scalar value \( I_{\text{app}} \) is the net current enforced in the superconductor in the perpendicular direction to the surface.

The final HTS problem will read as follows: find \( \mathbf{H} \in \mathcal{H}(\text{curl}, \Omega) \) and \( p \in \mathcal{H}^1(\Omega) \) such that Eqs. (20)-(21) (or Eqs. (26-27) in the augmented formulation) hold in \( \Omega_{\text{hts}} \) and \( \Omega_{\text{air}} \), together with the transmission conditions (34)-(35) and the constraint (36).
2.6. Critical state modelling. In previous sections, a suitable $H$–formulation (applicable to general electromagnetics) has been presented. In this section, the general formulation is extended to superconductivity by means of the constitutive law definition.

For HTS materials, when the applied field $H_{\text{app}}$ is above the lower critical field $H_c$, the current penetrates from the surface inwards, and follows the critical-state model. Some analytical models can be found in the literature to calculate magnetization curves under external magnetic fields loads. The Bean model, first presented in [14], states the existence of a limiting macroscopic superconducting current density $J_c$ that a hard superconductor can carry. It assumes that the critical current density is independent of the magnetic field, i.e., $J_c$ is a fixed value and satisfies the condition $\|J\| \leq J_c$. Besides, we find the Kim model, presented in [15], in which the critical current density is no longer a constant value but depends on the magnetic field, i.e., $J_c = J_c(B)$. A good comprehensive work in the magnetization problems in superconductivity can be found in [16]. However, in both models, $E - J$ relations are non-smooth and can be multi-valued. Therefore, they should be approximated by an analytical and smoother function, e.g., a power law, for computational purposes.

The superconductivity problem at hand consists of a superconductor region $\Omega_{hts}$ surrounded by a non-conducting region $\Omega_{\text{air}}$, large enough for neglecting boundary effects associated to the magnetization of the superconductor. In order to model a non-conducting region, we consider a conductivity (inverse of resistivity) value that ideally tends to 0. However, the dramatic jump of resistivity on the interface introduces a boundary layer on the interface that would require huge computational resources to be captured, whereas we are mainly interested in the superconductor behaviour. Thus, it is common practice to consider a regularized problem, in which the resistivity in the air domain $\Omega_{\text{air}}$ is fixed to a value that, whilst maintaining a large magnitude ratio with regard to the superconducting material, allows the computation to take place with a desired level of precision [13]. On the other hand, a critical state model is used in the superconducting device to describe the penetration of the magnetic flux and induced currents. We will use the following electric field-current density relation power law

$$E = \frac{E_c}{J_c} \left( \frac{\|J\|}{J_c} \right)^n J,$$

(37)

where $E_c$ is the critical electric field, $J_c$ is the critical current density, and $n$ is the exponent of the power law. This kind of expression tends to the analytical Bean’s model when $n$ tends to infinity. One can identify Ohm’s law (Eq. (10)) in this $E - J$ relation, with the following expression for the resistivity parameter in the superconducting region:

$$\rho_{hts}(H) = \frac{E_c}{J_c} \left( \frac{\|\nabla \times H\|}{J_c} \right)^n, \quad J_c(B) = \frac{J_{c0}B_0}{B_0 + \mu_0\|H\|},$$

(38)

where $J_{c0}$ is a parameter determined by the physical properties of the superconducting material. In turn, $J_c$ may be considered a fixed current density value or dependent on the magnetic strength $B = \mu_0H$ (as in Kim’s model). In this second case, the expression for $J_c(B)$ for magnetization of type-II superconductors depends on the magnetic field $B$ and $J_{c0}$.

3. Numerical approximation

3.1. FE approximation. In this section, we discuss conforming Sobolev FE spaces associated with the spaces $H^1(\Omega)$ and $\mathcal{H}(\text{curl}, \Omega)$. For this purpose, we will start by presenting different types of FE spaces involved in our approximation. Once defined, we will be able to define the global FE spaces.
Let $\mathcal{T}_h$ be a partition of $\Omega$ into a set of hexahedra or tetrahedra (quadrilaterals and triangles in 2D) geometrical cells $K$. The presentation of FEs follows the classical approach of Ciarlet. In this approach, a FE is represented by the triplet \{K, \mathcal{V}, \Sigma\}, where $\mathcal{V}$ is the space of functions on $K$, and $\Sigma$ is a set of functionals on $\mathcal{V}$, that forms a basis. This functionals are called degrees of freedom (DoFs) of the FE. The definition of the space of functions and DoFs relies on a reference FE \{K, \mathcal{V}, \Sigma\}. Then, in the physical space, the FE triplet on a cell $K$ relies on its reference FE, a geometrical mapping $\Phi_K$ such that $K = \Phi(K)$ and a linear bijective function mapping $\Psi_K : \mathcal{V} \to \mathcal{V}$. Let us denote by $J_K$ the Jacobian of the geometrical mapping, i.e. $J_K = \frac{\partial \Phi_K}{\partial x}$. The functional space is defined as $\mathcal{V} = \{\Psi_K(\hat{\psi}) \circ \Phi_K^{-1} : \hat{\psi} \in \mathcal{V}\}$; we will also use the mapping $\Psi_K : \hat{\mathcal{V}} \to \mathcal{V}$ defined as $\Psi_K(\hat{\psi}) = \Psi_K(\hat{\psi}) \circ \Phi_K^{-1}$. Finally, the set of DoFs on the physical FE is defined as $\Sigma = \{\hat{\sigma} \circ \Psi_K^{-1} : \hat{\sigma} \in \hat{\Sigma}\}$ from the set of the reference FE linear functionals.

3.1.1. **Lagrangian (nodal) FEs.** Let us first define polynomial spaces that will be needed in forthcoming definitions. The space of polynomials of degree less than or equal to $L$ is denoted by $\mathcal{P}_L$. Let us also define the corresponding truncated polynomial space $\mathcal{P}_L(K)$ as the span of the monomials with total degree less or equal to $L$. The reference FE space for Lagrangian FEs is $\mathcal{P}_0(K)$, which relies on a reference FE, a geometrical mapping $\Phi_K : \hat{K} \to K$, and DoFs (basis functions) of the FE. The definition of the space of functions and DoFs relies on a reference FE $\{\hat{K}, \mathcal{V}, \hat{\Sigma}\}$. This functionals are called degrees of freedom (DoFs) of the FE. The definition of the space of functions and DoFs relies on a reference FE $\{\hat{K}, \mathcal{V}, \hat{\Sigma}\}$. Let us note that the proposed basis contains all the functions in the first set of functions are indeed linearly independent; the two sets have different non-zero components. Finally, the last subset of functions contributes with components that contain the $x_1^0 = 1$ term, i.e., they do not depend on $x_1$, whereas all functions in the previous subset do depend on $x_1$.

3.1.2. **Edge tetrahedral FEs.** Let us define the homogeneous polynomial space $\mathcal{P}_k^d := \mathcal{P}_k \setminus \mathcal{P}_k^{-1}$. In [4], Nédélec introduces the function space of order $k$ on a tetrahedron

$$ (39) \quad \mathcal{V}_k = \mathcal{P}_k^d \oplus S_k, $$

where $S_k$ is the auxiliary space of polynomials

$$ (40) \quad S_k := \{p(x) \in \mathcal{P}_k^d \text{ such that } p(x) \cdot x = 0\}, $$

of dimension $(k+2)k$ in 3D and $k$ in 2D [17]. Among all the possible forms of representing the space $S_k$ in 3D, we propose to consider the spanning set

$$ (41) \quad S = \left( \bigcup_{\beta=1}^{k+1} \bigcup_{\alpha=1}^{k} \left\{ \begin{bmatrix} -x_1^{\alpha-1} x_2^{\beta-1} x_3^{\gamma-1} \\ x_1 x_2 x_3 \\ -x_1^{\alpha+1} x_2^{\beta+1} x_3^{\gamma+1} \\ x_1 x_2 x_3 \\ 0 \\ x_1^{\alpha-1} x_2^{\beta+1} x_3^{\gamma+1} \\ x_1 x_2 x_3 \\
\end{bmatrix} \right\} \right), $$

$$ (42) \quad S = \left( \bigcup_{\alpha=1}^{k} \left\{ \begin{bmatrix} 0 \\ -x_1^{\alpha} x_2^{\beta-1} x_3^{\gamma} \\ x_1^{\alpha-1} x_2 x_3^{\gamma+1} \\ x_1 x_2 x_3^{\gamma} \\
\end{bmatrix} \right\} \right). $$

**Proposition 3.1.** The set of vector functions defined in Eq. [44] forms a basis of the space $S_k$

**Proof.** First, we note that the proposed basis contains $\{p_i(x)\}_{i=1}^{(k+2)k}$ vector functions. Clearly, the total degree of the monomials found on each component for all functions is $k$, thus $S \subset \mathcal{P}_k^d$. Further, it is easy to check that $p_i(x) \cdot x = 0$ for any $p_i(x) \in S$ is readily shown. The proof is completed by showing that all the functions are linearly independent. It is trivial to see that all the functions in the first set of functions are indeed linearly independent; the two sets have different non-zero components. Finally, the last subset of functions contributes with components that contain the $x_1^0 = 1$ term, i.e., they do not depend on $x_1$, whereas all functions in the previous subset do depend on $x_1$. $\square$
In two dimensions, the analytical expression of the spanning set simply reduces to

\[
S = \bigcup_{\alpha=1}^{k} \left[ -x_1^{\alpha-1} x_2^{k-\alpha+1} \right] .
\]

The set of functionals \( \hat{\Sigma} \) on \( \hat{V}_k \), associated to the reference element \( \hat{K} \) and defined over all edges \( \hat{E} \in \hat{K} \), faces \( \hat{F} \in \hat{K} \), or \( \hat{K} \) itself ([8], ch.6), are in 2D

\[
\frac{1}{\|\hat{E}\|} \int_{\hat{E}} (u_h \cdot \tau) q \quad \forall q \in \mathcal{P}_{k-1}(\hat{E}),
\]

while in three dimensions the set is defined as

\[
\frac{1}{\|\hat{F}\|} \int_{\hat{F}} u_h \cdot q \quad \forall q \in \mathcal{P}^{3}_{k-2}(\hat{F}) \quad \text{such that} \quad q \cdot n = 0,
\]

\[
\frac{1}{\|\hat{K}\|} \int_{\hat{K}} u_h \cdot q \quad \forall q \in \mathcal{P}^{3}_{k-3}(\hat{K}).
\]

In previous definitions, \( \tau \) is the unit vector along the edges and \( n \) the unit outwards normal to the faces of the reference FE \( \hat{K} \). In the case of the lowest order elements, i.e., \( k = 1 \), only DoFs associated to edges occur. For second order elements, i.e., \( k = 2 \), we additionally find face DoFs (inner DoFs in 2D). For orders \( k > 2 \), all types of DoFs are found in both dimensions.

### 3.1.3. Edge hexahedral FEs

Let us now denote by \( \mathcal{V}_k \) the vector space defined with the scalar spaces of polynomials with variable degree in directions \( \{ x_i \}_{i=1}^{d} \)

\[
\mathcal{V}_k := \{ Q_{k-1,k} \times Q_{k,k-1} \}, \quad \mathcal{V}_k := \{ Q_{k-1,k,k} \times Q_{k,k-1,k} \times Q_{k,k,k-1} \},
\]

in the 2D and 3D case respectively. The set of functionals that form the basis in two dimensions reads (see [8], ch.6)

\[
\frac{1}{\|\hat{E}\|} \int_{\hat{E}} (u_h \cdot \tau) q \quad \forall q \in \mathcal{P}_{k-1}(\hat{E}),
\]

while in three dimensions the set is defined as

\[
\frac{1}{\|\hat{F}\|} \int_{\hat{F}} (u_h \times n) \cdot q \quad \forall q \in \mathcal{Q}_{k-2,k-1}(\hat{F}) \times \mathcal{Q}_{k-1,k-2}(\hat{F}),
\]

\[
\frac{1}{\|\hat{K}\|} \int_{\hat{K}} u_h \cdot q \quad \forall q \in \mathcal{Q}_{k-1,k-2,k-2}(\hat{K}) \times \mathcal{Q}_{k-2,k-1,k-2}(\hat{K}) \times \mathcal{Q}_{k-2,k-2,k-1}(\hat{K})
\]

where \( \tau \) is the unit vector along the edge and \( n \) the unit normal to the face. Note that in the case of the lowest order elements, i.e., \( k = 1 \), only DoFs associated to edges occur. For higher order elements, i.e., \( k \geq 2 \), we find all kinds of DoFs in both dimensions.

Note that local spaces \( \mathcal{V}_k \) lie between the full polynomial spaces of order \( k-1 \) and \( k \) in both tetrahedral and hexahedral cases, i.e., \( \mathcal{P}^{d}_{k-1} \subseteq \mathcal{V}_k \subseteq \mathcal{P}^{d}_{k} \) (resp. \( \mathcal{Q}^{d}_{k-1} \subseteq \mathcal{V}_k \subseteq \mathcal{Q}^{d}_{k} \)). This kind of elements are found in literature as Nédélec of the first kind [4]. Another edge FE based on full polynomial spaces, the so-called second kind, was introduced also
by Nédélec in [18].

3.1.4. **Global FE spaces and conformity.** The space of continuous piecewise polynomials is defined as

\[ \mathcal{R}_k(\Omega) = \{ v_h \in C^0(\Omega) \text{ such that } v_h|_K \in \mathcal{V}_k(K) \forall K \in \mathcal{T}_h \}, \]

for \( \mathcal{V}_k = \{ P_k \} \) (resp., \( Q_k \)) on tetrahedral (resp., hexahedral) Lagrangian FEs, where \( C^0(\Omega) \) is the space of continuous functions in \( \Omega \), and thus, grad-conforming. In this work, this type of FE space is considered for the scalar field. Therefore, any function \( p_h \in \mathcal{R}_k \) can be uniquely determined by its values on a set of points (nodes) in \( \Omega \), so this is a nodal FE approximation. This space \( \mathcal{R}_k \) is the discrete counterpart of the \( H^1(\Omega) \) space.

On the other hand, a space is conforming in \( H(\text{curl}) \) if the tangential components are continuous at the interface between elements, i.e., they do not have to satisfy normal continuity over element faces. Any function \( H_h \) can be uniquely determined by the set of DoFs defined for the edge FEs. Therefore, this spaces are the discrete counterpart of \( H(\text{curl}, \Omega) \), and the discrete FE-space where the magnetic field solution \( H_h \) lies is defined as

\[ \mathcal{N}_k(\Omega) = \{ v_h \in H(\text{curl}, \Omega) \text{ such that } v_h|_K \in \mathcal{V}_k(K) \forall K \in \mathcal{T}_h \}. \]

where \( \mathcal{V}_k \) has been defined for the tetrahedral and hexahedral edge FE in Subsects. 3.1.2 and 3.1.3. Local FE space information is transferred from the reference element to physical elements through the so-called covariant Piola mapping \( \hat{\Psi}_K(\hat{v}) = \hat{\Psi}_K(\hat{v}) \circ \Phi^{-1}_K \), where

\[ \hat{\Psi}_K(v) = J^{-T}_K v. \]

The Piola mapping, which preserves tangential traces of vector fields [19], is the key to achieve a curl-conforming global space. It can be checked that, by enforcing the continuity of DoF values on edges/faces for all the elements that contain them, one already enforces continuity of the tangent component across elements.

Let us shortly comment on the special care to be taken in order to guarantee global DoF continuity when assembling edge FEs. One must assure consistency in DoFs definition (tangent and normal directions) for all different elements that may share the same geometric entity. There exist mainly two kinds of approaches dealing with this issue. The first one is to define a unique global orientation of the geometric entities that all local entities must follow [8], i.e., to use properly oriented meshes. It is our preferred approach on tetrahedral meshes for its simplicity. As presented, e.g., in [19], an unique orientation can be defined for tetrahedral meshes. In a nutshell, if an edge \( E \) adjoins two vertices \( v_i \) and \( v_j \), the direction is set from the lowest global numbering index to the other, i.e., from \( v_i \) to \( v_j \) if \( i < j \). Then, sorting the vertices of every element based on their global indices, common geometric entities directions will always agree on the orientation of the shared edge or face. Unfortunately, it is not possible to define properly oriented hexahedral meshes in general. The second approach, being used in hexahedral meshes, is to solve the so-called sign conflict [20, 21], i.e., to do a sign flip \( \pm 1 \) over basis functions that are not in accordance to a chosen criteria (e.g., a master cell sign). This second strategy is easy to implement for DoFs associated to edges, but the situation becomes more involved when dealing with face related DoFs.

3.2. **Time discretization.** Let us consider a partition of the time interval \( [0, T] \) into \( N \) time slabs. We denote the \( n \)-th time slab by \( \Delta t^n = (t^{n-1}, t^n) \), for \( n = 1, \cdots, N \). We also denote each time slab size by \( |\Delta t^n| \). Time integration is performed with a \( \theta \)-method integrator, even though the use of other time integrators is straightforward. For the sake
of clarity, we use the Backward-Euler time integration in the presentation of the method. The already discretized in time problem reads: Given \( \mathbf{H}(t^0) = 0 \), find at every time step \( n = 1, \ldots, N \) the pair \( [\mathbf{H}^n_h, p^n_h] \in [\mathcal{R}_k, \mathcal{N}\mathcal{D}_k] \) such that

\[
\left(\mathbf{f}^n_h, \mathbf{v}_h\right) + \frac{\mu_0}{\Delta t^n} \left(\mathbf{H}^n_h, \mathbf{v}_h\right) + \int_{\Omega} \left(\rho \left(\mathbf{H}^n_h\right) \nabla \times \mathbf{H}^n_h, \nabla \times \mathbf{v}_h\right) - \left(\nabla p^n_h, \mathbf{v}_h\right) = 0, \forall \mathbf{v}_h \in \mathcal{N}\mathcal{D}_k
\]

where \( \mathbf{f}^n_h \) is the discrete version of the source term evaluated at time \( t^n \). Note that non-homogeneous initial conditions are readily imposed by considering \( \mathbf{H}_0^h = \mathbf{H}_0^h(t^0) \), where \( \mathbf{H}_0^h(t^0) \) is the projection of the initial value onto the FE space \( \mathcal{N}\mathcal{D}_k \).

3.3. Dirichlet Boundary Conditions. The \( H \)-formulation is preferred over other formulations, among other reasons, due to its straightforward manner of integrating magnetic fields and currents. External magnetic fields can be applied directly by setting boundary values of the magnetic field on Dirichlet boundaries, while currents in the superconductor device can be injected through Ampere’s law (see Eq. (7)) through constraints. Dirichlet boundary conditions will be strongly imposed in the resulting system (usual implementation in FE codes), hence we need the DoF values over the boundary to be fixed. The magnetic pressure field is \( H^1 \)-conforming and thus nodal interpolation is possible (\( p = 0 \)). However, the magnetic field case is more elaborated. \( H_h \) DoFs are obtained by means of moments defined over edges, faces, and elements (see Eqs. (44), (47), (50), and (53)). As a result, one has to use the corresponding edge FE projector, which consists in evaluating the moments (e.g., Eqs. (44), (47), (50), or (53)) for the continuous boundary function in the reference FE element, using the mapping (56) from the physical to the reference space (see [8, p. 134]).

4. Nonlinear transient solver

HTS modelling requires not only a reliable constitutive model but also a robust and efficient nonlinear solver. Superconductor phenomena may occur in a very short time period and thus abrupt changes of behaviour are found in a very small time scale. Further, the nonlinearity associated to the constitutive law \( E - J \) presents extreme parameters (\( n \sim 30, 50 \)). Spatial scales, time scales, and nonlinearity make superconductivity modelling a very challenging task from a computational point of view. The expression for the nonlinearity is very stiff, since the exponent in (37) is usually \( n = 30 \), which makes the Lipschitz continuity constant of the nonlinear PDE operator at hand large. For this purpose, a specific nonlinear transient solver is proposed in this section. The time integration is performed with the Backward Euler technique (Sect. 3.2), thus at every time step one aims to solve a nonlinear problem. A suitable formulation for solving the electromagnetic problem has been presented in Sect. 2.3 whereas in this section we present the nonlinear solver employed, based on a Newton-Raphson (NR) nonlinear solver. It involves a linearization of the problem at every time step and nonlinear iterations, until convergence is attained and one can proceed to the following time step.

4.1. Algebraic form. For the sake of clarity in forthcoming sections, let us write the problem in algebraic form. The magnetic vector field \( \mathbf{H}_h \) is expanded by means of the vector shape functions \( \{\phi^1_i\}_{i=1}^{N_H} \) related to curl-conforming edge element, whereas the magnetic pressure is expanded with scalar shape functions \( \{\varphi^1_i\}_{i=1}^{N_p} \), corresponding to the grad-conforming Lagrangian elements. Let us define the following element-wise matrices in the
element $K$:

\begin{equation}
\mathcal{M}_K := \sum_{i=1}^{N_H} \sum_{j=1}^{N_H} \int_K \phi_i \cdot \phi_j \quad \mathcal{B}_K := \sum_{i=1}^{N_H} \sum_{j=1}^{N_H} \int_K \nabla \phi_i \cdot \phi_j \tag{59}
\end{equation}

\begin{equation}
\mathcal{K}_K := \sum_{i=1}^{N_H} \sum_{j=1}^{N_H} \int_K \rho(H_h)(\nabla \times \phi_i) \cdot (\nabla \times \phi_j) \quad \mathcal{S}_K := \sum_{i=1}^{N_H} \sum_{j=1}^{N_H} \int_K \nabla \phi_i \cdot \nabla \phi_j. \tag{60}
\end{equation}

Consider also the right-hand side discrete vector $F^i_K := \int_K f_K \phi_i$. Then, the usual assembly (see Sect. [3.1.4] for more details) is performed to obtain global matrices and arrays.

Once the operators have been introduced in algebraic form, the problem for a single time step $t^n$ in algebraic form reads:

\begin{equation}
\begin{bmatrix}
\frac{\mu_0}{\Delta t^n} \mathcal{M} + \mathcal{K}(H_h^n) & -\mathcal{B} & \mathcal{C}^t \\
\mathcal{B} & \frac{1}{\mu_0} \mathcal{S} & 0 \\
\mathcal{C} & 0 & \lambda^n \\
\end{bmatrix}
\begin{bmatrix}
H_h^n \\
\rho_h^n \\
\lambda^n \\
\end{bmatrix}
= \begin{bmatrix}
F_h(t^n) + \frac{\mu_0}{\Delta t^n} \mathcal{M} H_h^{n-1} \\
0 \\
I_{app} \\
\end{bmatrix}, \tag{61}
\end{equation}

where $\mathcal{C}$ is the matrix corresponding to the Lagrange multiplier values used for enforcing a current value $I_{app}$ over the $\Omega_{hts}$ domain through Eq. (36). $H_h$ and $\rho_h$ are the discrete functions containing the DoF values for the magnetic field and the magnetic pressure, respectively. Note that $\mathcal{C}$ imposes a relation among only some of the DoFs on the interface $\Gamma = \partial \Omega_{hts} \cap \partial \Omega_{air}$.

4.2. **Adaptive time stepping.** Time scales may be very small in this problem due to the applied fields frequency. However, the process of magnetization of the superconductor allows to identify different needs in different periods of the process. Although restrictions in the time step size are severe in concrete periods (when $\|J\|$ becomes larger than $J_c$ in some region), the time step can be relaxed in monotone magnetization curves. The same effect occurs for the validation model that will be considered in Sect. [5.4] where an injected current is kept constant for a period of time before proceeding to the following current load increment. A simple adaptive time stepping will be used to accelerate convergence. The time step size is updated with the nonlinear solver convergence history for the last converged time step as

\begin{equation}
|\Delta t^n| = \frac{\#\text{iters}}{\kappa} |\Delta t^{n-1}|, \tag{62}
\end{equation}

where $\kappa$ stands for a selected growing ratio. Usually, one may select $\kappa$ as the "ideal" number of iterations to convergence sought in the nonlinear algorithm. Finally, the trial time step size is restricted to upper and lower bound values

\begin{equation}
|\Delta t^n| = \begin{cases} 
\Delta t_{min} & \text{if } |\Delta t^n| \leq \Delta t_{min}, \\
|\Delta t^n| & \text{if } \Delta t_{min} \leq |\Delta t^n| \leq \Delta t_{max}, \\
\Delta t_{min} & \text{if } |\Delta t^n| \leq |\Delta t_{max}|.
\end{cases} \tag{63}
\end{equation}

4.3. **Linearization.** The problem at hand contains an extreme nonlinearity, with an exponent of the $E - J$ constitutive law $n \approx 30$. The problem and its residual are stated in an algebraic form as

\begin{equation}
\mathcal{A}(x) x = b, \quad R = \mathcal{A}(x) x - b = 0, \tag{64}
\end{equation}

where the full vector of unknowns $x$ and the right-hand side $b$ have been presented in (61). It is therefore essential to build a robust nonlinear solver together with an effective adaptive time stepping technique. Note that combined with a highly nonlinear problem in the superconductor region $\Omega_{hts}$, the resistivity takes a constant value $\rho_{hts}$ in the air region $\Omega_{air}$ and the problem is linear. Therefore, our strategies focus on the linearization of the problem associated to the extreme nonlinearity given by the resistivity $\rho_{hts}(H)$. For that purpose, we will make use of a composition of nonlinear solvers (see [22]). Our nonlinear
sampler is the composition of a Newton-Raphson (NR) method with an exact derivation of the Jacobian and a cubic backtracking (BT) line search algorithm (see [22]).

We obtain by means of the NR method (for the time step \( t^n \)) the direction of the solution update at the iterate \( k \), i.e., \( \delta x^{n,k} \), solving the linearized problem

\[
\frac{\partial R^{n,k}}{\partial (H_{h}^{n,k}, p_{h}^{n,k}, \lambda^{n,k})} (\delta H_{h}^{n,k}, \delta p_{h}^{n,k}, \delta \lambda^{n,k}) = J \delta x^{n,k} = -R^{n,k}.
\]

Later, the BT technique tries to minimize the residual of the iterate \( x^{n,k+1} = x^{n,k} + \beta \delta x^{n,k} \) by means of the step length \( \beta \), i.e.,

\[
\beta = \arg\min_{0 < \beta \leq 1} \| R(x^{n,k} + \beta \delta x^{n,k}) \|^2.
\]

It is not our intention to define nor the BT technique neither the basic NR algorithm, which can be found in [22], but we will introduce the forms of the Jacobian matrix \( J \) that are specific to our gauged double curl formulation. First of all, let us define the discrete residual of the resulting algebraic system associated to each variable \( (H_h, p_h, \lambda) \), which we represent with \( R = [R^{H_h}, R^{p_h}, R^{\lambda}] \), where we have omitted the indices \( n, k \) for simplicity. The component-wise definition of the residual \( R \) follows

\[
R^{H_h} = \sum_{j=1}^{N_H} \int_{\Omega_{hts}} \frac{\mu_0}{\Delta I} \phi^j \cdot \phi^i H_{h}^i + \sum_{j=1}^{N_H} \int_{\Omega_{hts}} \rho_{hts}(H_{h})(\nabla \times \phi^j) \cdot (\nabla \times \phi^i) H_{h}^i - \sum_{j=1}^{N_p} \int_{\Omega_{hts}} f^i \cdot \phi^j
\]

\[
R^{p_h} = \sum_{j=1}^{N_H} \int_{\Omega_{hts}} \phi^i \cdot \nabla \phi^j H_{h}^j + \sum_{j=1}^{N_H} \int_{\Omega_{hts}} \frac{1}{\mu_0} \nabla \phi^j \cdot \nabla \phi^i p_{h}^j
\]

\[
R^{\lambda} = -I_{app} + \sum_{j=1}^{N_H} \int_{\Omega_{hts}} (\nabla \times \phi^j) \cdot n H_{h}^j
\]

for the magnetic field DoFs \( \{H_{h}^i\}_{i=1}^{N_H} \), the magnetic pressure related DoFs \( \{p_{h}^j\}_{j=1}^{N_p} \), and, finally, the Lagrange multiplier \( \lambda \) used to enforce the applied current \( I_{app} \). In this case, \( n \) denotes the unit normal to the surface \( \Omega_{hts} \). Once the full residual has been defined, the Jacobian \( J \) can be derived as

\[
J = \frac{\partial R}{\partial (H_h, p_h, \lambda)} = A + \frac{\partial K(H_h)}{\partial H_h} H_h.
\]

It becomes clear in this expression the fact that the tangent terms are given by the discrete magnetic field \( H_h \), i.e., the nonlinearity is found in the magnetic field, while the magnetic pressure and the Lagrange multiplier enforcing the current are linear relations. Let us give the exact expressions used for the Jacobian computation at each nonlinear iterate. Following the notation proposed in Sect. [1], the terms of \( J \) that depend on the current iterate \( H_{h}^{n,k} \) for the magnetic field are

\[
J_{ij} = \frac{\partial R^{n,k}_{H_{h}^{n,k}}}{\partial H_{h}^{n,k}_{ij}} = A_{ij}(H_{h}^{n,k}) + \int_{\Omega_{hts}} \frac{\partial \rho_{hts}(H_{h}^{n,k})}{\partial H_{h}^{n,k}_{ij}} (\nabla \times H_{h}^{n,k}) \cdot (\nabla \times \phi^i),
\]

where \( \{i, j\} = 1, \cdots, N_H \), i.e., the magnetic field \( H \) diagonal block. All the other blocks of \( J \) are the same as \( A \) (see Eq. (61)). If we go one step further, for the constitutive law presented in Eq. [10], and considering \( J_c \) independent of the magnetic field, we obtain
the expression for the tangent resistivity with respect to the magnetic field
\begin{equation}
\frac{\partial \rho_{\text{hts}}(H^{n,k}_h)}{\partial H^{n,k}_h} = \frac{E_c}{J_c} \left( \frac{\| \nabla \times H^{n,k}_h \|}{J_c} \right)^{n-2} \left( \frac{\nabla \times H^{n,k}_h}{J_c} \right) \cdot \left( \nabla \times \phi_j \right).
\end{equation}

4.4. **Parallel linear solver.** The presented nonlinear solver (Sect. 4.3) can rely on sparse parallel direct solvers (e.g., PARDISO) or iterative solvers (e.g., Krylov methods with domain decomposition preconditioning) to solve the linearized problems that arise at every nonlinear iteration. E.g., one can consider block-preconditioning techniques to solve the linear system (65) or a direct application of the original Balancing Domain Decomposition by Constraints (BDDC) preconditioner [23] to the multiphysics problem. Grad-conforming problems, e.g., the magnetic pressure part, can easily be handled with these algorithms, leading to robust and scalable linear solvers (see, e.g., [24]). However, the treatment of curl-conforming problems is much more involved. An extension to the curl-conforming case has been presented in [24] (for a very related solver); see also the deluxe scaling BDDC method in [26] for problems with jumps of coefficients. **FEMPAR** has a highly scalable implementation of BDDC preconditioner, as previously shown in [24, 27]. An extension of the BDDC preconditioner to the curl-conforming space has recently been introduced in the code. Our BDDC preconditioner implementation can deal with curl-conforming spaces of arbitrary order, tetrahedral/hexahedral meshes and structured/unstructured partitions. However, the development of highly scalable and robust solvers for HTS applications involve non-constant physical coefficients, block-preconditioning techniques, and indefinite problems, and high contrast problems. As a result, we postpone the presentation of these linear solvers and a detailed both practical and theoretical weak scalability analysis for a subsequent work.\(^1\)

In any case, we show some preliminary analyses obtained in MareNostrum IV supercomputer for a modest number of processors, with structured hexahedral meshes and manufactured solutions. With the aim of showing the potential of the parallel framework, we include some results obtained for the transient double-curl formulation; the treatment of the gauged formulation is more straightforward, since the magnetic pressure is discretized with a grad-conforming space. The problem is solved in a unit cubic domain, with all parameters set to unity. A structured partition of the domain is performed in each direction into \(P\) parts, to end up with a \(P^3\) number of subdomains (=cores). The number of iterations until convergence is presented in Fig. 1 for different element orders and local problem sizes \(H/h\) (subdomain/element size ratio). As predicted by theory, the number of iterations to solve the system does not increase for an increasing number of subdomains, whilst maintaining the local problem size \(H/h\).

5. **Numerical experiments**

In this section we test the Coulomb gauged \(H−\) formulation (see Sects. 2.3 and 2.5) and the nonlinear solver presented before. In general, our simulation domains consist of a superconducting bulk completely surrounded by a dielectric box (see Fig. 6 for an illustrative example). Dirichlet boundary conditions are applied on the boundary of the outer domain. Thus, we make sure that this boundary is far enough from the superconducting device to avoid interior magnetic fields generated by it to reach the boundary. The stopping criteria for the nonlinear solver is the reduction of the discrete \(L_2\)–norm of the nonlinear residual relative to the right-hand side of the system below a value of \(10^{-8}\). Time integration, as stated in Sect. 4.2, will be performed with a variable time step. We will present different magnetic fields and current profiles for two selected cases, where the

\[^1\text{Weak scalability is the ability to exploit further computational resources (i.e., solving larger problems) without an increasing metrics of the preconditioned system (i.e., number of iterations, time to solution), keeping constant the load per processor.}\]
Figure 1. BDDC-CG Weak scalability results for the transient Maxwell problem.

effect of two external variables, such as an imposed magnetic field or an injected current, is presented. We present results for both 2D and 3D models. In the last subsection, we also include a validation of the model against experimental data, based on the Hall probe mapping experiment in [13].

5.1. Experimental framework. **FEMPAR**, developed by the Large Scale Scientific Computing (LSSC) group at CIMNE-UPC, is an open source object oriented Fortran 2008 scientific software library for the high-performance scalable simulation of complex multiphysics problems governed by partial differential equations at large scales. **FEMPAR** is released under the GNU GPL v3 license. All the proposed algorithms for the solution of the HTS problem have been implemented in **FEMPAR** [11, 12], providing the results presented in this section. Results reported in this section were obtained on two High Performance Computing platforms: the MareNostrum IV, located in the Barcelona Supercomputing Center (BSC) and the Finis Terrae II, located in the Centro Tecnológico de Supercomputación de Galicia (CESGA). All the results here use the multi-threaded parallel sparse direct solver provided by Intel MKL PARDISO for the solution of the global system (in serial runs) or the local subdomain problems (in parallel runs with domain decomposition preconditioning).

5.2. External applied magnetic field $H_{app}$. In this first case, an homogeneous external time-dependent magnetic field is applied. Consider a squared domain $\Omega = [0, 48] \text{ mm}^2$ in the 2D case such that the superconductor fills the volume $\Omega_{hts} = [18, 30] \times [23.73, 24.27] \text{ mm}^2$. The air domain is defined as $\Omega_{air} = \Omega \setminus \Omega_{hts}$. There is no source term $f$ and Dirichlet-type
boundary conditions are imposed over the entire boundary, as a time-dependent vertical magnetic field \( \mathbf{H}_{\text{app}} = H_0 \sin(2\pi \omega t) \hat{e}_y \), with an amplitude of \( H_0 = 10^6 \) Am\(^{-1}\) and a frequency of \( \omega = 50 \) Hz. Initial conditions are simply \( \mathbf{H}(t = 0) = 0 \). On the other hand, no net current flow condition \( (I_{\text{app}} = 0 \) A) is imposed in the superconductor. Remaining parameter values that complete the problem definition are defined in Table 1. We show results for a full cycle with the specified frequency. An extra first quarter cycle is considered for the first magnetization of the superconductor. Computations are performed with a mesh consisting of 180 \( \times \) 99 elements mesh (90 \( \times \) 9 in the HTS tape region). Second order spaces are considered for both the magnetic field and pressure.

From the plots, it can be seen how the superconducting tape tries to oppose to the penetration of the magnetic flux as the external field increases. At the end of the cycle, the superconducting tape has trapped some of the magnetic field and remains magnetized. Even the external magnetic field at this point is null. The magnetization of the HTS tape in the vertical (\( y^- \)) direction is computed by means of

\[
M = \frac{1}{|\Omega_{\text{hts}}|} \int_{\Omega_{\text{hts}}} \mu_0 (H_y - H_{\text{app}}).
\]

The magnetization loop for a full cycle is plotted in Fig. 11a. At a first stage, the tape is in the diamagnetic state. At some point, some currents start to penetrate into the superconductor and the magnetization increases. Even though the applied field increases further, the value of the magnetization tends to be stable.

It is also important to note where the nonlinear solver convergence is becoming critical. In the magnetization plot in Fig. 11a, one can clearly observe that time step decreases drastically when the magnetization cycle is reversing its direction. However, during flat transitions in the magnetization cycle, the time step raises to a bound maximum value. It is therefore of paramount importance the consideration of an adaptive time stepping in order to reduce it in the parts where the nonlinear solver convergence may be put at risk and increase it in other parts to reduce computational cost.

Fig. 2 present the distribution of the current density over the entire HTS domain. Additionally, the magnetic field is presented by means of vector fields on top of the current density color map. This allows one to appreciate the magnetic field distribution and the currents induced into the superconductor in a single plot. Further, current density and magnetic field profiles are presented for a section parallel to the \( x^- \)-axis that passes through the center of the HTS tape in Figs. 3-4.

Fig. 5 depicts one full hysteresis cycle for the magnetization of the superconductor device computed for different edge elements orders. We consider a mesh of 40 \( \times \) 33 elements and first to fifth order. Clearly, exponential growth is expected in both problem sizes and computing times with the element order. It is clearly shown how the proposed regular refinement in order makes the hysteresis loop converge to a solution.

5.2.1. 3D results. The presented case extends the range of applicability to any 3D geometry. For this purpose, the computational domain is discretized with a tetrahedral unstructured mesh. First order in both magnetic field and magnetic pressure is considered. We have selected the magnetization of a superconducting bulk cylinder without injection of net current. Although an axis-symmetric model could be used to reduce the computational domain, no symmetry is exploited with the aim of presenting a full 3D model with a non-trivial geometry. The cylinder dimensions are \( R = 12.5 \) mm and \( h = 10 \) mm, and it is centered in a surrounding air box of \( L = 50 \) mm. \( I_c \) value is kept from the previous case. The external applied magnetic field increases from \( \mathbf{H}_{\text{ext}} = 0 \) A/m to \( \mathbf{H}_{\text{ext}} = 10^6 \hat{e}_z \) A/m in 5 s, and keeps the constant peak value from this point on. In Fig. 6 the meshed geometry is shown.
In Fig. 7 captures at four different times are presented. Only half of the bulk cylinder is shown so the magnetization process can be observed beneath the bulk cylinder surface. In this example, streamlines show the magnetic field $\mathbf{H}$ surrounding the bulk cylinder. On the other hand, the electric current density $\mathbf{J}$ is shown through a colormap of its $x$-direction component. As expected, magnetic field cannot penetrate into the bulk when magnetization has not occurred yet.
\( \omega t = \frac{1}{4} \).

\( \omega t = \frac{1}{2} \).

\( \omega t = \frac{3}{4} \).

\( \omega t = 1.0 \).

Figure 3. Magnetic strength profiles in a section parallel to \( x \)-axis and 300\( \mu \)m above the HTS tape surface. 2D problem with inputs \( H_{\text{app}} = 10^6 \sin(2\pi \omega t) e_y \) A/m and \( I_{\text{app}} = 0 \) A.

\( \omega t = \frac{1}{4} \).

\( \omega t = \frac{1}{2} \).

\( \omega t = \frac{3}{4} \).

\( \omega t = 1.0 \).

Figure 4. Current profiles in a section that passes through the center of the tape and parallel to \( x \)-axis. 2D problem with inputs \( H_{\text{app}} = 10^6 \sin(2\pi \omega t) e_y \) A/m and \( I_{\text{app}} = 0 \) A.

5.3. **Injected current** \( I_{\text{app}} \). The 2D HTS tape is now magnetized by a current flowing through it. For this purpose, we consider a time-dependent electrical AC current perpendicular to the plane where the cross section of the device is located. Now homogeneous Dirichlet boundary conditions are applied over the entire boundary, i.e., \( H_{\text{app}} = 0 \) A.
Figure 5. Hysteresis cycle for different edge element orders. 2D problem with inputs $H_{app} = 10^6 \sin(2\pi\omega t)$ A/m and $I_{app} = 0$ A.

net current of $I_{app} = I_0 \sin(2\pi\omega t) e_z$ is fed into the superconducting tape. In this case, $I_0 = 2700$ A and $\omega = 50$ Hz. Computations are performed in a mesh consisting of $180 \times 99$ elements ($90 \times 9$ in the HTS tape region). In this case, second order is considered for both fields.

Fig. 8 shows the current distribution and the magnetic flux vector field for one complete cycle. It can be noted that the HTS tape carrying the AC current contains a region with positive current, a region with reverse current, and a current-free region. We can evaluate the AC loss per cycle by means of

$$Q_{AC} = \int_{\Omega_{hts}} E \cdot J = \int_{\Omega_{hts}} \rho_{hts} \| J \|^2.$$  (74)

Fig. 11b shows the AC loss cycle for the problem at hand. A detailed definition of the parameters used can be found in Table 1.

Results obtained are qualitatively in good agreement with the theoretical exposition of losses in hard superconductors carrying AC currents by Norris [28]. According to that
5.4. Model validation. In this section, the computational model is validated against some experimental data, obtained in a Hall scanning magnetometer experiment. The experimental data is taken from the tests by Granados et al in [13], where a detailed exposition of the experiment can be found. Our goal is to compare the experimental data on a 2D tape sample (see table I for properties) with our implementation results in FEMPAR. In the experiment, the current is applied in the HTS tape by a sequence of step functions, with an interval of duration of 100 s for each one before proceeding to the following increment. Applied current is gradually incremented up to $I_{\text{app}} = 460$ A, which corresponds to $1.03 \cdot I_c$. See Fig. [12a] for a clear exposition of the injected current. The constant current during this limited period of time allows the flux creep effects to pass, and therefore, the current distribution is stabilized along the superconductor specimen. The experiment is carried out at a temperature of 77 K.

We will compare the magnetic field vertical component $B_y = \mu_0 H_y$ profiles from both the experimental and the computational data at the distance of 400 $\mu$m above the HTS tape surface, where the active part of the sensor is located in the physical experiment. For this modelling case, a critical current dependence with the field is introduced. For the sake of simplicity, we will not take into account the dependence of the critical current with the incidence angle of the magnetic field. Therefore, only $B$-dependence will be introduced.
in our model. It is done by means of tabulated values (see Fig. 12b), where (normalized with the critical current for $B = 0$ T) effective $J_c$ values for parallel and perpendicular directions to the HTS tape surface are obtained. This fact certainly introduces some deviation in the behaviour of the sample, but the effect of the observed anisotropy in the experimental data is not severe for magnetic flux densities below 100 mT, which is our case.

Fig. 13 presents the comparison between the vertical magnetic field $\mu_0 H_y$ profile obtained from the experiment and the computed magnetic field profile. Computation is performed with a mesh of 500x430 first order elements, which includes a 100x30 elements mesh of the surroundings of the HTS tape and a 50x3 elements mesh of the superconductor tape. The rest of the elements discretize the air domain. The plots show the profile at a distance of $d = 400$ μm above the HTS tape surface (where the experimental data is obtained), and the x-axis section contains the full HTS tape and 6 additional mm to each side. Therefore, with the reference domain $\Omega = 100^2$ mm, presented profiles correspond to the line $y = 50.455, 38 < x < 62$ mm. Out of these plots, some conclusions can be drawn. First and most important, the experimental data is in good agreement with the performed
Figure 9. Magnetic strength profiles in a section parallel to $x$-axis at a distance of 300 µm above the HTS tape surface. Problem inputs: $H_{\text{app}} = 0$ A/m and $I_{\text{app}} = 2700 \sin(2\pi\omega t)$ A perpendicular to the plane.

(a) $\omega t = \frac{1}{4}$.

(b) $\omega t = \frac{1}{2}$.

(c) $\omega t = \frac{3}{4}$.

(d) $\omega t = 1.0$.

Figure 10. Current profiles for different time steps at a section parallel to $x$-axis that passes through the center of the HTS tape. Problem inputs: $H_{\text{app}} = 0$ A/m and $I_{\text{app}} = 2700 \sin(2\pi\omega t)$ A perpendicular to the plane.

(a) $\omega t = \frac{1}{4}$.

(b) $\omega t = \frac{1}{2}$.

(c) $\omega t = \frac{3}{4}$.

(d) $\omega t = 1.0$.

Simulations and therefore the proposed formulation is able to reproduce the phenomena. Second, the magnetic field peak values are, in all cases, in an excellent agreement in both experimental and computed data. It is important to note that the experimental data does not possess a perfect symmetry from the central point, while the computed data respects
this symmetry. This fact can be attributed to some small imperfections in the sample of the experiment and introduces uncertainty. Out of the validation test we can identify three phases: a first one where the computed data reproduces with a high accuracy the experimental data (see Figs. 13a, 13b, and 13c). A second stage, in which, even though the peak values are respected, the experimental data presents small variations in the superconductor area, (see Figs. 13d-13g), and finally a third stage, coinciding with the unloading of the sample, where slightly higher discrepancy is observed. However, peak values are correctly captured and the model predicts the physical phenomenon throughout the entire simulation.

Figure 11. Cycles corresponding to two first variable problems.

(A) Hysteresis cycle for the applied magnetic field problem. (B) AC loss cycle for the injected current problem.

Figure 12. Validation problem inputs definition.

6. Conclusions

In this article, we present mixed FE formulations suitable for the solution of nonlinear electromagnetic problems modelling HTS devices. With the Maxwell equations as a starting point, we developed the transient Augmented Coulomb Gauged $H$-formulation, which does not affect the magnetic field solution. We model the HTS characteristic relation between electric fields and current densities by means of an extremely stiff power law, which certainly makes the problem highly nonlinear. For the FE approximation, we choose edge (or Nédélec) elements of arbitrary order to represent the magnetic field. These elements are favoured in electromagnetics simulations due to their sound mathematical structure. However, high order (e.g., larger than two) implementations of these FEs are relative scarce, due to their high complexity. We provide an accurate definition of edge FEs, presenting a straightforward manner of constructing its associated functional spaces. In order to end
up with a well-posed linear system and eliminate spurious modes, we explicitly impose the divergence-free constraint by means of a magnetic pressure acting as a Lagrange multiplier. For this scalar field, common Lagrangian elements were considered. A tailored nonlinear parallel solver, which includes a linearization with a Newton-Raphson method (with exact Jacobian derivation) and advanced domain decomposition preconditioners for the tangent problem, was designed to solve the nonlinear problem. Time integration was performed

| Parameter                        | Problems 1,2 | Validation model |
|----------------------------------|--------------|-----------------|
| air domain size                  | $48^2$       | $100^2$         |
| HTS tape width                   | 12           | 12              |
| HTS tape thickness               | 540          | 110             |
| $n$ power law exponent           | 29           | 32              |
| $\mu_0$                          | $4\pi \cdot 10^{-7}$ | $4\pi \cdot 10^{-7}$ |
| $\mu_r$                          | 1            | 1               |
| $E_0$                            | $10^{-4}$    | $10^{-4}$       |
| $J_c$                            | $4.08 \cdot 10^8$ | $3.38 \cdot 10^8$ |
| $\rho_{air}$                     | 1.0          | $1.0 \cdot 10^{-2}$ |

**Table 1.** Geometric and electrical parameters of the HTS tape used in different problems.

**Figure 13.** Magnetic field $B_y$ profiles for experimental (Hall probe mapping) and computed data in a full load-unload cycle for the validation test.
with the Backward Euler integrator and a variable time step, taking advantage of the convergence history of the nonlinear solver and thus reducing time to solution. The work here presented has been implemented in FEMPAR, which becomes a very powerful tool for the HTS modelling community. We have also provided a detailed validation of the framework for a set of 2D/3D numerical experiments for the magnetization of superconducting bulks and a model validation that showed an excellent agreement between computational and experimental data. The presentation and analysis of effective preconditioners for large scale 3D HTS problems will be addressed in a subsequent work.

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