BIMODULE MONOMORPHISM CATEGORIES 
AND RSS EQUIVALENCES VIA COTILTING MODULES 

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ABSTRACT. The monomorphism category \( S(A, M, B) \) induced by a bimodule \( A \mathcal{M} B \) is the subcategory of \( \Lambda \text{-mod} \) consisting of \( \{ X \} \phi \) such that \( \phi : M \otimes_B Y \to X \) is a monic \( A \)-map, where \( \Lambda = \left[ \begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix} \right] \). In general, it is not the monomorphism categories induced by quivers. It could describe the Gorenstein-projective \( \Lambda \)-modules. This monomorphism category is a resolving subcategory of \( \Lambda \text{-mod} \) if and only if \( M_B \) is projective. In this case, it has enough injective objects and Auslander-Reiten sequences, and can be also described as the left perpendicular category of a unique basic cotilting \( \Lambda \)-module. If \( M \) satisfies the condition (IP), then the stable category of \( \mathcal{S}(A, M, B) \) admits a recollement of additive categories, which is in fact a recollement of singularity categories if \( \mathcal{S}(A, M, B) \) is a Frobenius category. Ringel-Schmidmeier-Simson equivalence between \( \mathcal{S}(A, M, B) \) and its dual is introduced. If \( M \) is an exchangeable bimodule, then an RSS equivalence is given by a \( \Lambda \text{-} \Lambda \) bimodule which is a two-sided cotilting \( \Lambda \)-module with a special property; and the Nakayama functor \( N_\Lambda \) gives an RSS equivalence if and only if both \( A \) and \( B \) are Frobenius algebras.

Keywords: monomorphism category induced by bimodule, Auslander-Reiten sequence, cotilting module, recollement of additive categories, exchangeable bimodule, RSS equivalence, Frobenius algebra, Nakayama functor

1. Introduction and preliminaries

1.1. Throughout, algebras mean Artin algebras, modules are finitely generated, and a subcategory is a full subcategory closed under isomorphisms. For an algebra \( A \), let \( \text{mod} A \) (resp. \( \text{mod} A \)) be the category of left (resp. right) \( A \)-modules. So there is a duality \( D : \text{mod} A \to \text{mod} A \).

This paper is to draw attention to the monomorphism category \( \mathcal{S}(A, M, B) \) induced by an \( A \text{-}B \)-bimodule \( M \). It is defined to be the subcategory of \( \Lambda \text{-mod} \) consisting of left \( \Lambda \)-modules \( \{ X \} \phi \) such that \( \phi : M \otimes_B Y \to X \) is a monic \( A \)-map, where \( \Lambda \) is the triangular matrix algebra \( \left[ \begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix} \right] \). When \( A \mathcal{M} B = AA \), it is the classical submodule category \( \mathcal{S}(A) \) in [RS1-RS3]. This submodule category is initiated in [Bir]. C. Ringel and M. Schmidmeier [RS2] establish its Auslander-Reiten theory; and D. Simson ([S1]-[S3]) studies its representation type. By D. Kussin, H. Lenzing and H. Meltzer ([KLM1, KLM2]; see also [C]), it is related to the singularity theory. It has been generalized via quivers to the filtered chain category and the separated monomorphism category ([S1-S3], [Z1], [LZ], [ZX]). However, all these generalizations cannot include monomorphism categories induced by bimodules (this will be clarified in Example 5.8). Another motivation is that \( \mathcal{S}(A, M, B) \) can describe the Gorenstein-projective \( \Lambda \)-modules ([Z2, Thms. 1.4., 2.2]).

1.2. To study \( \mathcal{S}(A, M, B) \), first, we need it to be a resolving subcategory of \( \Lambda \text{-mod} \). So we work under the condition that \( M_B \) is projective: this is a necessary and sufficient condition such that \( \mathcal{S}(A, M, B) \) is a resolving subcategory. Then \( \mathcal{S}(A, M, B) \) has enough projective objects and enough injective objects, and it

Supported by the NSF of China (11431010, 11301019, 11271251).

2010 Mathematics Subject Classification. Primary 16G70; Secondary 16D40; 16E30

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is a Frobenius category if and only if $A$ and $B$ are selfinjective and $\mathcal{A}M$ and $\mathcal{M}B$ are projective (Corollary 2.4). This monomorphism category $\mathcal{J}(A, M, B)$ enjoys the functorially finiteness and Auslander-Reiten sequences, and it closely relates to the tilting theory. Here we use the classical cotilting modules of injective dimension at most 1 ([HR], [R, p.167], [AR], [ASS, p.242]). For a left $\Lambda$-module $Z$, let $Z^\perp$ denote the subcategory \{\$L \in \Lambda\text{-mod} \mid \text{Ext}^m_{\Lambda}(L, Z) = 0, \forall m \geq 1\}. 

**Theorem 1.1.** Let $M$ be an $A$-$B$-bimodule. Then

(1) The following are equivalent:

(i) $M_B$ is projective;

(ii) $\mathcal{J}(A, M, B)$ is a resolving subcategory of $\Lambda\text{-mod}$;

(iii) $\Lambda T := \left[\frac{\mathcal{D}(A\Lambda)}{\mathcal{D}(B\Lambda)}\right]_{\text{c}}$ is a unique cotilting left $\Lambda$-module, up to multiplicities of indecomposable direct summands, such that $\mathcal{J}(A, M, B) = \perp T$, where $E_{\mathcal{D}(B\Lambda)}$ is an injective envelope of the left $\Lambda$-module $\mathcal{A}M \otimes_B \mathcal{D}(B\Lambda)$ with inclusion $e : M \otimes_B \mathcal{D}(B\Lambda) \hookrightarrow E_{\mathcal{D}(B\Lambda)}$.

(2) $\mathcal{J}(A, M, B)$ is a contravariantly finite subcategory of $\Lambda\text{-mod}$. Moreover, if $M_B$ is projective, then $\mathcal{J}(A, M, B)$ is a functorially finite subcategory of $\Lambda\text{-mod}$, and has Auslander-Reiten sequences.

**Corollary 1.2.** If $A\mathcal{M}B$ satisfies the condition (IP), then $\Lambda T = \left[\frac{\mathcal{D}(A\Lambda)}{\mathcal{D}(B\Lambda)}\right]_{\text{c}}$ is a unique cotilting left $\Lambda$-module, up to multiplicities of indecomposable direct summands, such that $\mathcal{J}(A, M, B) = \perp T$.

1.3. A recollement is first introduced for triangulated categories ([BBD]), and then for abelian categories ([MV], [PS], [Ku]). It becomes a powerful tool in triangulated categories and in abelian categories (see e.g. [Kö], [H2], [IKM], [FP], [PV], [FZ]). One can also consider recollements of additive categories in the similar way. For a subcategory $\mathcal{X}$ of an additive category $\mathcal{A}$, recall that the objects of the stable category $\mathcal{A}/\mathcal{X}$ are the objects of $\mathcal{A}$, and $\text{Hom}_{\mathcal{A}/\mathcal{X}}(M, N) := \text{Hom}_{\mathcal{A}}(M, N)/(\mathcal{X}, N)$, where $(\mathcal{X}, N)$ is the subgroup consisting of those morphisms factoring through objects of $\mathcal{X}$. For an algebra $A$, denote $\Lambda\text{-mod}/\text{inj}(A)$ by $\Lambda\text{-mod}$, where $\text{inj}(A)$ is the subcategory of the injective $A$-modules. Similarly, $\mathcal{J}(A, M, B)$ is the stable category of $\mathcal{J}(A, M, B)$ respect to the subcategory of the injective objects of $\mathcal{J}(A, M, B)$.

**Theorem 1.3.** An $A$-$B$-bimodule satisfying the condition (IP) induces a recollement of additive categories

$$
\begin{array}{cccc}
A\text{-mod} & \xrightarrow{\perp} & \mathcal{J}(A, M, B) & \xrightarrow{\perp} \mathcal{J}(A, M, B) & \xrightarrow{\perp} B\text{-mod}.
\end{array}
$$

If in addition $A$ and $B$ are selfinjective algebras, then it is in fact a recollement of singularity categories.

Here the singularity category $\mathcal{D}_{eq}^b(\Lambda)$ of an algebra $\Lambda$ is defined to be the Verdier quotient $\mathcal{D}_{eq}^b(\Lambda) := \mathcal{D}^b(\Lambda\text{-mod})/K^b(\text{proj}(\Lambda))$, where $\mathcal{D}^b(\Lambda\text{-mod})$ is the bounded derived category, and $K^b(\text{proj}(\Lambda))$ is the bounded homotopy category. See R. Buchweitz [Buch] and D. Orlov [O].

1.4. The dual of $\mathcal{J}(A, M, B)$ is the epimorphism category $\mathcal{F}(A, M, B)$. The right module version of $\mathcal{J}(A, M, B)$ is $\mathcal{J}(A, M, B)_{\perp}$, and $\mathcal{J}(A, M, B)$ is a resolving subcategory if and only if $\mathcal{A}M$ is projective; in this case, there is a unique basic cotilting right $\Lambda$-module $U$ such that $\mathcal{J}(A, M, B)_{\perp} = \perp(U\Lambda)$. Then $\mathcal{F}(A, M, B)$ can also be described as $\mathcal{D}\mathcal{J}(A, M, B)_{\perp}$. Ringel-Schmidmeier-Simson equivalence $\mathcal{J}(A, M, B) \cong$
\( \mathcal{F}(A,M,B) \) is studied. Such an equivalence implies a strong symmetry, and was first observed by C. Ringel and M. Schidmeier [RS2] for the case of \( _AM_B = _AA_A \), and by D. Simson [S1] for a chain without relations, and then developed to acyclic quivers with monomial relations in [ZX].

We introduce exchangeable bimodules. If \( _AM_B \) is exchangeable, then the unique left cotilting \( \Lambda \)-module \( T \) with \( \mathcal{F}(A,M,B) = \dagger T \) (cf. Corollary 1.2) can be endowed with a \( \Lambda \)-\( \Lambda \)-bimodule structure via the exchangeable bimodule isomorphism, such that the right module \( T_{\Lambda} \) coincides with the unique right cotilting \( \Lambda \)-module \( U \) with \( \mathcal{F}(A,M,B)_r = \dagger (U_{\Lambda}) \). This two-sided cotilting \( \Lambda \)-module \( _AT_{\Lambda} \) enjoys a good property in the sense that \( \text{End}_\Lambda(_AT)^{op} \cong \Lambda \) as algebras, and under this isomorphism of algebras, \( T_{\text{End}_\Lambda(_AT)^{op}} \) coincides with \( T_{\Lambda} \). These good properties of \( T \) induce an RSS equivalence:

**Theorem 1.4.** Let \( _AM_B \) be an exchangeable bimodule. Then \( T = \begin{bmatrix} D(A) & 0 \\ M \otimes_B D(B) \\ \text{Id} \end{bmatrix} \) can be endowed with a \( \Lambda \)-\( \Lambda \)-bimodule such that \( D \text{Hom}_\Lambda(-,T) : \mathcal{F}(A,M,B) \cong \mathcal{F}(A,M,B) \) is an RSS equivalence.

The Nakayama functor \( N_\Lambda \) gives an RSS equivalence if and only if both \( A \) and \( B \) are Frobenius algebras (Proposition 5.5). Examples show that if \( _AM_B \) is not exchangeable, then an RSS equivalence cannot be guaranteed. Examples also show that the monomorphism category \( \mathcal{F}(A,M,B) \) is not the separated monomorphism category of the corresponding quiver in the sense of [ZX], in general (see Example 5.8). However, we do not know a sufficient and necessary condition and the uniqueness of an RSS equivalence. See Subsection 5.5.

1.5. Let \( M \) be an \( A \)-\( B \)-bimodule. The multiplication of the associated matrix algebra \( \Lambda = [A \, M] \) is given by

\[
\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix} = \begin{bmatrix} aa' + mb' \\ b'm + mb \end{bmatrix}.
\]

Each left \( \Lambda \)-module is identified with a triple \( [X \, Y \, \phi] \), where \( X \in A\text{-mod}, Y \in B\text{-mod}, \) and \( \phi : M \otimes_B Y \to X \) is an \( A \)-map; and a \( \Lambda \)-map is identified with a pair \( f_1, f_2 : [X_1 \, Y_1 \, \phi_1] \to [X_2 \, Y_2 \, \phi_2] \), where \( f_1 : X_1 \to X_2 \) is an \( A \)-map, and \( f_2 : Y_1 \to Y_2 \) a \( B \)-map, such that the diagram

\[
\begin{array}{ccc}
M \otimes_B Y_1 & \overset{1 \otimes f_2}{\rightarrow} & M \otimes_B Y_2 \\
\phi_1 \downarrow & & \downarrow \phi_2 \\
X_1 & \underset{f_1}{\rightarrow} & X_2
\end{array}
\]

commutes. Under this identification, the indecomposable projective \( \Lambda \)-modules are exactly \([P \, Q] \) and \([M \otimes_B Q] \), where \( P \) and \( Q \) run over the indecomposable projective \( A \)-modules and \( B \)-modules, respectively. The indecomposable injective \( \Lambda \)-modules are \([\text{Hom}_A I, J]\) and \([\text{Id}] \), where \( I \) and \( J \) run over the indecomposable injective \( A \)-modules and \( B \)-modules, respectively ([ARS, p.73]). Throughout, for any left \( A \)-module \( X \), we denote by \( \phi = \phi_X \) the left \( A \)-map \( M \otimes_B \text{Hom}_A(M,X) \to X \) given by \( \phi(m \otimes f) = f(m) \), i.e., the adjunction isomorphism \( \text{Hom}_A(M \otimes_B \text{Hom}_A(M,X), X) \cong \text{Hom}_B(\text{Hom}_A(M,X), \text{Hom}_A(M,X)) \) sends \( \phi \) to \( \text{Id}_{\text{Hom}_A(M,X)} \). We will call \( \phi \) the involution map.

1.6. **Conditions on a bimodule.** A bimodule \( _AM_B \) satisfies the condition (IP), if \( M \otimes_B D(B_B) \) is an injective left \( A \)-module and \( M_B \) is projective.

A bimodule \( _AM_B \) is \textit{exchangeable}, if both \( _AM \) and \( M_B \) are projective and there is an \( A \)-\( B \)-bimodule isomorphism \( D(AA_A) \otimes_A M \cong M \otimes_B D(B_B) \), which is called an \textit{exchangeable bimodule isomorphism}. 

By adjunction isomorphisms we have $A$-$B$-bimodule isomorphisms

$$D(DA \otimes_A M) \cong \text{Hom}_A(M, A), \quad D(M \otimes_B D B) \cong \text{Hom}_B(M, B).$$

Thus $\Lambda M_B$ is exchangeable if and only if there is an $A$-$B$-bimodule isomorphism $\text{Hom}_A(M, A) \cong \text{Hom}_B(M, B)$; and if and only if there is an $A$-$B$-bimodule isomorphism $N_A(A M) \cong N_B(M_B)$, where $N_A$ denotes the Nakayama functor $D \text{Hom}_A(-, A)$.

For $A N$, let $\text{add}(N)$ be the subcategory of $A$-mod of direct summands of finite direct sums of $N$.

**Example 1.5.** (1) An exchangeable bimodule $\Lambda M_B$ satisfies the condition (IP), and $D(A A) \otimes_A M$ is an injective right $B$-module.

In fact, since $\Lambda M$ is projective, $D(AA) \otimes_A M \in \text{add}(D(AA) \otimes_A A) = \text{add}(D(AA))$, so $D(AA) \otimes_A M$ is an injective left $A$-module. Thus $M \otimes_B D(BB) \cong D(AA) \otimes_A M$ is an injective left $A$-module. Similarly, one can prove that $D(AA) \otimes_A M$ is an injective right $B$-module.

(2) If $B = A$ and $M = A \oplus \cdots \oplus A$, then $\Lambda M_A$ is an exchangeable bimodule.

(3) For an algebra $A$, let $B := A \oplus \cdots \oplus A$, and $\Lambda M_B := AB$. Then $\Lambda M_B$ is an exchangeable bimodule.

(4) For an algebra $B$, let $A := B \oplus \cdots \oplus B$, and $\Lambda M_B := AB$. Then $\Lambda M_B$ is an exchangeable bimodule.

(5) An algebra $A$ over field $k$ is symmetric, if $D(AA) \cong AA$ as $A$-$A$-bimodules. If both $A$ and $B$ are symmetric algebras and $M = AP \otimes_k QB$, where $AP$ and $QB$ are projective, then $\Lambda M_B$ is an exchangeable bimodule.

(6) An algebra $B$ is Frobenius, if $D(BB) \cong BB$ as left $B$-modules. If $B$ is a Frobenius algebra and $\Lambda M_B$ is a bimodule with $\Lambda M$ injective and $MB$ projective, then $\Lambda M_B$ satisfies the condition (IP).

(7) Let $B$ be a selfinjective algebra, and $\Lambda M_B$ a bimodule with $MB$ projective. Then $\Lambda M_B$ satisfies the condition (IP) if and only if $\Lambda M$ is injective. In particular, if both $A$ and $B$ are selfinjective $k$-algebras and $M = AP \otimes_k QB$, where $AP$ and $QB$ are projective, then $\Lambda M_B$ satisfies the condition (IP).

In fact, since $B$ is a selfinjective algebra, $D(BB) \in \text{add}(BB)$, and hence $\Lambda M \otimes_B D(BB) \in \text{add}(\Lambda M)$. So $M \otimes_B D(BB)$ is an injective $A$-module if and only if $\Lambda M$ is injective.

2. Monomorphism categories induced by bimodules

2.1. Recall that the monomorphism category $\mathcal{I}(A, M, B)$ induced by bimodule $\Lambda M_B$ is the subcategory of $A$-mod consisting of $[\frac{X}{Y}]_\phi$ such that $\phi : M \otimes_B Y \longrightarrow X$ is a monic $A$-map. So it contains all the projective $A$-modules and is closed under direct sums and direct summands.

**Lemma 2.1.** Let $\Lambda M_B$ be a bimodule. Then $\mathcal{I}(A, M, B)$ is closed under extensions. Thus $\mathcal{I}(A, M, B)$ is an exact category with the canonical exact structure, and hence a Krull-Schmidt category.
Proof. For an exact sequence $0 \rightarrow [\frac{X_1}{Y_1}]_{\phi_1} \xrightarrow{\frac{f_1}{g_1}} [\frac{X_2}{Y_2}]_{\phi_2} \rightarrow 0$ in $\Lambda$-mod with $[\frac{X_1}{Y_1}]_{\phi_1} \in \mathcal{S}(A, M, B)$ and $[\frac{X_2}{Y_2}]_{\phi_2} \in \mathcal{S}(A, M, B)$, we get a commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
M \otimes_B Y_1 & \xrightarrow{1 \otimes g_1} & M \otimes_B Y & \xrightarrow{1 \otimes g_2} & M \otimes_B Y_2 & \rightarrow 0 \\
\phi_1 & \downarrow & \phi & \downarrow & \phi_2 & \\
0 & \rightarrow & X_1 & \rightarrow & X & \rightarrow & 0
\end{array}
\end{array}
\]

with exact rows. It follows from the Snake Lemma that $\phi$ is monic, i.e., $[\frac{X}{Y}]_\phi \in \mathcal{S}(A, M, B)$. \qed

**Proposition 2.2.** Let $\mathcal{S}(A, M, B)$ be a bimodule such that $M_B$ is projective. Then

1. $\mathcal{S}(A, M, B)$ has enough projective objects; and the indecomposable projective objects of $\mathcal{S}(A, M, B)$ are exactly the indecomposable projective $\Lambda$-modules.

2. $\mathcal{S}(A, M, B)$ has enough injective objects; and the indecomposable injective objects are exactly $[\frac{I}{J}]_e$, where $I$ (resp. $J$) runs over indecomposable injective left $A$-modules (resp. $B$-modules), and $E_J$ is an injective envelope of the left $A$-module $M \otimes_B J$ with inclusion $e : M \otimes_B J \rightarrow E_J$.

In particular, if $M$ satisfies the condition (IP), then the indecomposable injective objects of $\mathcal{S}(A, M, B)$ are exactly $[\frac{I}{J}]_e$ and $[\frac{M \otimes_B J}{I}]_e$.

Proof. (1) Projective $\Lambda$-modules are clearly projective objects of $\mathcal{S}(A, M, B)$. For any object $[\frac{X}{Y}]_\phi \in \mathcal{S}(A, M, B)$, taking projective covers $\pi_Y : BQ \rightarrow Y$ and $\pi_C : AP \rightarrow A \text{Coker}(\phi)$, we get exact sequences $0 \rightarrow \text{Ker}(\pi_Y) \xrightarrow{\iota_Y} Q \xrightarrow{\pi_Y} Y \rightarrow 0$ and $0 \rightarrow \text{Ker}(\pi_C) \xrightarrow{\iota_C} P \xrightarrow{\pi_C} A \text{Coker}(\phi) \rightarrow 0$. Consider the exact sequence $0 \rightarrow M \otimes_B Y \xrightarrow{\phi} X \xrightarrow{\pi} A \text{Coker}(\phi) \rightarrow 0$. We get an $A$-map $\theta : P \rightarrow X$ such that $\pi_C = \pi \theta$, and hence the commutative diagram in $\Lambda$-mod with exact rows

\[
\begin{array}{c}
\begin{array}{ccc}
0 & \rightarrow & M \otimes_B Q & \xrightarrow{[\frac{1}{0}]} & (M \otimes_B Q) \oplus P & \xrightarrow{[0,1]} & P & \rightarrow & 0 \\
1 \otimes \pi_Y & \downarrow & [\phi(1 \otimes \pi_Y), \theta] & \downarrow & \pi_C & & & & \\
0 & \rightarrow & M \otimes_B Y & \xrightarrow{\phi} & X & \xrightarrow{\pi} & A \text{Coker}(\phi) & \rightarrow & 0.
\end{array}
\end{array}
\]

Since $M_B$ is projective, $0 \rightarrow M \otimes_B \text{Ker}(\pi_Y) \xrightarrow{1 \otimes \iota_Y} M \otimes_B Q \xrightarrow{1 \otimes \pi_Y} M \otimes_B Y \rightarrow 0$ is an exact sequence of $A$-module. By the Snake Lemma we get the commutative diagram in $A$-mod with exact rows and columns

\[
\begin{array}{c}
\begin{array}{ccc}
0 & \rightarrow & M \otimes_B \text{Ker}(\pi_Y) & \xrightarrow{- \rightarrow} & N & \xrightarrow{- \rightarrow} & \text{Ker}(\pi_C) & \rightarrow & 0 \\
1 \otimes \iota_Y & \downarrow & \psi & \downarrow & \iota_C & & & & \\
0 & \rightarrow & M \otimes_B Q & \xrightarrow{[\frac{1}{0}]} & (M \otimes_B Q) \oplus P & \xrightarrow{[0,1]} & P & \rightarrow & 0 \\
1 \otimes \pi_Y & \downarrow & [\phi(1 \otimes \pi_Y), \theta] & \downarrow & \pi_C & & & & \\
0 & \rightarrow & M \otimes_B Y & \xrightarrow{\phi} & X & \xrightarrow{\pi} & A \text{Coker}(\phi) & \rightarrow & 0.
\end{array}
\end{array}
\]
So the left and middle columns give the exact sequence
\[
0 \to [\text{Ker}(\pi_Y)]_{\phi} \to \left[\frac{P}{0} \oplus \left[\frac{M \otimes_B Q}{Q}\right]_{\text{Id}}\right] \to [X]_{\phi} \to 0
\]
(\ast)
in \mathcal{S}(A, M, B). This shows that \mathcal{S}(A, M, B) has enough projective objects.

Let \([\frac{X}{Y}]_{\phi}\) be an indecomposable projective object of \mathcal{S}(A, M, B). Then the exact sequence (\ast) splits and \([\frac{Y}{X}]_{\phi}\) is a direct summand of \([\frac{P}{0}] \oplus \left[\frac{M \otimes_B Q}{Q}\right]_{\text{Id}}\). By Lemma 2.2, \mathcal{S}(A, M, B) is a Krull-Schmidt category, so \([\frac{Y}{X}]_{\phi}\) is isomorphic to \([\frac{P'}{0}]\) or \([\frac{M \otimes_B Q'}{Q'}]_{\text{Id}}\), where \(P'\) (resp. \(Q'\)) is an indecomposable projective \(A\)-module (resp. \(B\)-module). Thus \([\frac{Y}{X}]_{\phi}\) is a projective \(A\)-module.

(2) It is clear that \([\frac{I}{0}]\) and \([\frac{E_J}{J}e]\) are indecomposable objects of \(\mathcal{S}(A, M, B)\). We show that they are injective objects of \(\mathcal{S}(A, M, B)\). Put \([\frac{W}{V}]_{\phi}\) to be \([\frac{I}{0}]\) or \([\frac{E_J}{J}e]\). For an exact sequence in \(\mathcal{S}(A, M, B)\)

\[
0 \to [\frac{X_1}{Y_1}]_{\phi_1} \to [\frac{X_2}{Y_2}]_{\phi_2} \to [\frac{X_3}{Y_3}]_{\phi_3} \to 0
\]
with \([\frac{Y}{X}]_{\phi}\) \(\in \text{Hom}_A([\frac{X_1}{Y_1}]_{\phi_1}, [\frac{X_2}{Y_2}]_{\phi_2})\), we need looking for \([\frac{Y}{X}]_{\phi}\) such that \([\frac{X}{Y}]_{\phi} = [\frac{Y}{X}]_{\phi}\) \([\frac{Y}{X}] = [\frac{Y}{X}]\).

Since \(B V\) is an injective module and \(g_1 : Y_1 \to Y_2\) is monic, there is a \(B\)-map \(\delta : Y_2 \to V\) such that \(\beta = \delta g_1\). Consider the \(A\)-map \(\phi(1 \otimes \delta) : M \otimes_B Y_2 \to W\). Since \(A W\) is an injective module and \(\phi_2 : M \otimes_B Y_2 \to X_2\) is monic, there is an \(A\)-map \(\gamma' : X_2 \to W\) such that \(\phi(1 \otimes \delta) = \gamma' \phi_2\). Since \(\alpha g_1 = \phi(1 \otimes \beta)\) and \(g_1 \phi_1 = \phi_2(1 \otimes g_1)\), we have

\[
\alpha g_1 = \phi(1 \otimes \beta) = (1 \otimes \delta)(1 \otimes g_1) = \gamma' \phi_2(1 \otimes g_1) = \gamma' f_1 \phi_1.
\]

So \(\alpha - \gamma' f_1 = \eta \pi_1\) for some \(\eta : \text{Coker}(\phi_1) \to W\). Since \(h_1 : \text{Coker}(\phi_1) \to \text{Coker}(\phi_2)\) is monic and \(W\) is injective, there is an \(A\)-map \(\eta' : \text{Coker}(\phi_2) \to W\) such that \(\eta = \eta' h_1\). We present the process as the diagram with exact rows and columns:

```
0 \to M \otimes_B Y_1 \to M \otimes_B Y_2 \to M \otimes_B Y_3 \to 0
\phi_1 \downarrow \downarrow \downarrow \downarrow \downarrow \phi_2 \phi_3
0 \to X_1 \to X_2 \to X_3 \to 0
\pi_1 \downarrow \downarrow \downarrow \downarrow \downarrow \pi_2 \pi_3
0 \to \text{Coker}(\phi_1) \to \text{Coker}(\phi_2) \to \text{Coker}(\phi_3) \to 0
\eta \downarrow \downarrow \downarrow \downarrow \downarrow \eta'
\gamma \downarrow \downarrow \downarrow \downarrow \downarrow \gamma'
0 \to Coker(\phi_1) \to Coker(\phi_2) \to Coker(\phi_3) \to 0
```

Put \(\gamma := \gamma' + \eta' \pi_2 \in \text{Hom}_A(X_2, W)\). Then

\[
\gamma \phi_2 = (\gamma' + \eta' \pi_2) \phi_2 = \gamma' \phi_2 = \phi(1 \otimes \delta)
\]

and

\[
\gamma f_1 = (\gamma' + \eta' \pi_2) f_1 = \gamma' f_1 + \eta' \pi_2 f_1 = \gamma' f_1 + \eta' h_1 \pi_1 = \gamma' f_1 + \eta \pi_1 = \alpha.
\]

This shows \([\frac{Y}{X}]_{\phi}\) \(\in \text{Hom}_A([\frac{X_2}{Y_2}]_{\phi_2}, [\frac{W}{V}]_{\phi_2})\) and \([\frac{Y}{X}]_{\phi} = [\frac{Y}{X}]_{\phi}\).

\(\mathcal{S}(A, M, B)\) has enough projective objects.
Next, we show that $\mathcal{S}(A, M, B)$ has enough injective objects. For $[\frac{X}{Y}]_\phi \in \mathcal{S}(A, M, B)$, taking injective envelopes $\iota_Y : Y \to J$ and $\iota_C : \text{Coker}(\phi) \to I$, we get exact sequences $0 \to Y \overset{\iota_Y}{\to} J \overset{p_Y}{\to} \text{Coker}(\iota_Y) \to 0$ and $0 \to \text{Coker}(\phi) \overset{\iota_C}{\to} I \overset{p_C}{\to} \text{Coker}(\iota_C) \to 0$. We take an injective envelope of the left $A$-module $M \otimes_B J$ with inclusion $e : M \otimes_B J \to E_J$. Since $\phi : M \otimes_B Y \to X$ is monic and $E_J$ is an injective module, there is an $A$-map $\alpha : X \to E_J$ satisfying $\alpha \phi = e(1 \otimes \iota_Y)$, and hence we get a commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \to & M \otimes_B Y & \phi & \to & X & \overset{\pi}{\to} & \text{Coker}(\phi) & \to & 0 \\
\downarrow{1 \otimes \iota_Y} & & \downarrow{\iota_C} & & \downarrow{\alpha} & & \downarrow{\beta} & & \\
0 & \to & M \otimes_B J & e & \to & E_J & \overset{f}{\to} & \text{Coker}(e) & \to & 0
\end{array}
$$

Since $M_B$ is projective, we have a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccc}
\begin{array}{cccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M \otimes_B Y & \phi & \to & X & \overset{\pi}{\to} & \text{Coker}(\phi) & \to & 0 \\
\downarrow{1 \otimes \iota_Y} & & \downarrow{[\iota_C]} & & \downarrow{[\iota_\phi]} & & \downarrow{[\iota_\psi]} & & \\
0 & \to & M \otimes_B J & e & \to & E_J \oplus I & \overset{f \oplus I}{\to} & \text{Coker}(e) \oplus I & \to & 0 \\
\downarrow{1 \otimes \iota_Y} & & \downarrow{[\iota_C]} & & \downarrow{[\iota_\phi]} & & \downarrow{[\iota_\psi]} & & \\
0 & \to & M \otimes_B \text{Coker}(\iota_Y) & \psi & \to & L & \overset{C'}{\to} & 0 & & \\
\end{array}
\end{array}
$$

(since $\iota_C$ is monic, applying the Snake Lemma to the upper two rows we see that $\psi$ is monic). The left and middle columns give a short exact sequence in $\mathcal{S}(A, M, B)$

$$
0 \to [\frac{X}{Y}]_\phi \to [\frac{J}{E_J}]_e \to [\frac{\text{Coker}(\iota_Y)}{C'}]_\psi \to 0.
$$

This shows that $\mathcal{S}(A, M, B)$ has enough injective objects.

Finally, if $[\frac{X}{Y}]_\phi$ is an indecomposable injective object of $\mathcal{S}(A, M, B)$, then (**) splits. By Lemma 2.1, $\mathcal{S}(A, M, B)$ is a Krull-Schmidt category. So $[\frac{X}{Y}]_\phi$ is either of the form $[\frac{J}{0}]_e$ or of the form $[\frac{E_J}{I}]_e$. \hfill \Box

**Corollary 2.3.** Let $AM_B$ be a bimodule with $M_B$ projective. Then $\mathcal{S}(A, M, B)$ is a Frobenius category (with the canonical exact structure) if and only if both $A$ and $B$ are selfinjective algebras and $AM$ is projective.

**Proof.** By Proposition 2.2, $\mathcal{S}(A, M, B)$ is a Frobenius category if and only if

$$
\text{add}([A] \oplus [M_B]_{\text{id}}) = \text{add}([D(A_A) \oplus E_{D(B_B)}]_e),
$$

where $E_{D(B_B)}$ is an injective envelope of the $A$-module $M \otimes_B D(B_B)$ with embedding $e : M \otimes_B D(B_B) \to E_{D(B_B)}$. Thus, if $\mathcal{S}(A, M, B)$ is Frobenius, then $A_A$ is injective (thus $\text{add}(D(A_A)) = \text{add}(A_A)$), $AM$ is injective (thus $AM \in \text{add}(D(A_A)) = \text{add}(A_A)$, hence $AM$ is projective), and $B_B$ is injective. Conversely, if $A$ and $B$ are selfinjective and $AM$ is projective, then $\text{add}(B_B) \cong \text{add}(D(B_B))$, hence $M \otimes_B D(B_B) \in \text{add}(M \otimes_B B) = \text{add}(AM)$, so $M \otimes_B D(B_B)$ is a projective left $A$-module, and hence an injective left $A$-module. Thus $M \otimes_B D(B_B) = E_{D(B_B)}$. So $\text{add}([E_{D(B_B)}]_e) = \text{add}([M \otimes_B D(B_B)]_e) = \text{add}([M_B]_{\text{id}})$, thus $\text{add}([A] \oplus [M_B]_{\text{id}}) = \text{add}([D(A_A)] \oplus [E_{D(B_B)}]_e)$). Hence $\mathcal{S}(A, M, B)$ is a Frobenius category. \hfill \Box
2.2. To prove Theorem [1], we need some preparations. A subcategory is a resolving subcategory of \( \Lambda\)-mod, if it contains all the projective \( \Lambda\)-modules and is closed under extensions, kernels of epimorphisms and direct summands ([AR]).

**Lemma 2.4.** Let \( \mathcal{A}M_B \) be a bimodule. Then \( \mathcal{J}(A, M, B) \) is a resolving subcategory of \( \Lambda\)-mod if and only if \( M_B \) is projective.

**Proof.** By Lemma 2.1 it suffices to prove that \( \mathcal{J}(A, M, B) \) is closed under kernels of epimorphisms if and only if \( M_B \) is projective. Suppose that \( M_B \) is projective. Let \( f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : [X_1]_{\phi_1} \to [X_2]_{\phi_2} \) be an epimorphism in \( \Lambda\)-mod with both \([X_1]_{\phi_1}\) and \([X_2]_{\phi_2}\) in \( \mathcal{J}(A, M, B) \). So \( X_1 \xrightarrow{f_1} X_2 \) and \( Y_1 \xrightarrow{f_2} Y_2 \) are epic with \( \text{Ker}(f_1) \hookrightarrow X_1 \) and \( \text{Ker}(f_2) \hookrightarrow Y_1 \). Since \( M_B \) is projective, we get the commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & M \otimes_B \text{Ker}(f_2) & \to & M \otimes_B Y_1 & \to & M \otimes_B Y_2 & \to & 0 \\
0 & \to & \text{Ker}(f_1) & \xrightarrow{i} & X_1 & \xrightarrow{f_1} & X_2 & \to & 0.
\end{array}
\]

Thus there is a unique \( A\)-map \( \phi : M \otimes_B \text{Ker}(f_2) \to \text{Ker}(f_1) \) such that \( i \phi = \phi_1 (1 \otimes f) \). It is clearly that \( \phi \) is monic, and hence \( \text{Ker}(f) = \begin{bmatrix} \text{Ker}(f_1) \\ \text{Ker}(f_2) \end{bmatrix} \) in \( \mathcal{J}(A, M, B) \).

Conversely, suppose that \( \mathcal{J}(A, M, B) \) is closed under kernels of epimorphisms. We claim that \( M \otimes_B - \) is an exact functor. In fact, let \( 0 \to K \xrightarrow{j} Y_1 \xrightarrow{i} Y_2 \to 0 \) be an arbitrary exact sequence of \( B\)-modules. Then \( \begin{bmatrix} 1 \otimes f \\ f \end{bmatrix} : \begin{bmatrix} M \otimes_B Y_1 \\ Y_2 \end{bmatrix} \to \begin{bmatrix} M \otimes_B Y_1 \\ Y_2 \end{bmatrix} \) is an epimorphism in \( \Lambda\)-mod with \( \begin{bmatrix} M \otimes_B Y_1 \\ Y_2 \end{bmatrix} \) in \( \mathcal{J}(A, M, B) \) and \( \text{Ker} \begin{bmatrix} 1 \otimes f \\ f \end{bmatrix} = \begin{bmatrix} \text{Ker}(1 \otimes f) \\ \text{Ker}(f) \end{bmatrix} \), where \( M \otimes_B K \xrightarrow{\phi} \text{Ker}(1 \otimes f) \) is the unique \( A\)-map such that \( \sigma \phi = 1 \otimes j \), and \( \sigma : \text{Ker}(1 \otimes f) \hookrightarrow M \otimes_B Y_1 \) is the embedding. Since \( \mathcal{J}(A, M, B) \) is closed under kernels of epimorphisms, \( \begin{bmatrix} \text{Ker}(1 \otimes f) \\ \text{Ker}(f) \end{bmatrix} \) in \( \mathcal{J}(A, M, B) \), i.e., \( \phi \) is monic, and thus \( 1 \otimes j \) is monic. This proves the claim and hence \( M_B \) is flat. Since \( B \) is an Artin algebra and \( M_B \) is finitely generated, it follows that \( M_B \) is projective. \( \square \)

**Lemma 2.5.** ([XZ], Lemma 1.2) For \( X \in A\)-mod and \( Y \in B\)-mod, we have

1. \( \text{Ext}^1_A([0,0],[X]) \cong \text{Hom}_A(M \otimes_B Y, X) \).

2. If \( A I \) is an injective \( A\)-module, then \( \text{Ext}^{i+1}_A([0,0],[I]) \cong \text{Ext}^i_B(Y, \text{Hom}_A(M, I)) \) for \( i \geq 0 \).

**Proof.** For the convenience we include a short justification.

1. Let \( 0 \to K \xrightarrow{j} Q \xrightarrow{p} Y \to 0 \) be an exact sequence with \( Q \) a projective left \( B\)-module. Then \( 0 \to \begin{bmatrix} M \otimes Q \\ K \end{bmatrix} \xrightarrow{1 \otimes i} \begin{bmatrix} M \otimes Q \\ Q \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 \\ \text{Id} \end{bmatrix}} \begin{bmatrix} Q \\ 0 \end{bmatrix} \to 0 \) is an exact sequence with \( \begin{bmatrix} M \otimes Q \\ Q \end{bmatrix} \) a projective left \( \Lambda\)-module. Applying \( \text{Hom}_A(-,[0,0]) \), since \( \text{Hom}_A\left(\begin{bmatrix} M \otimes Q \\ Q \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 \\ \text{Id} \end{bmatrix}},[X]_0\right) = 0 \), we see

\[
\text{Ext}^1_A([0,0],[X]) \cong \text{Hom}_A\left(\begin{bmatrix} M \otimes Q \\ K \end{bmatrix} \xrightarrow{1 \otimes i},[X]_0\right) = \{ f \in \text{Hom}_A(M \otimes_B Q, X) \mid f(1 \otimes i) = 0 \} \\
\cong \text{Hom}_A(M \otimes_B Y, X).
\]

2. If \( i = 0 \) then the assertion follows from (1) and the Tensor-Hom adjunction isomorphism. Let \( i \geq 1 \). Using the abbreviation \( (M, I) = \text{Hom}_A(M, I) \), by the exact sequence \( 0 \to [I]_0 \to \left(\begin{bmatrix} f \\ \text{Id} \end{bmatrix}\right) \to [I]_0 \to 0 \), \( \text{Ext}^{i+1}_A([0,0],[I]) \cong \text{Ext}^i_B(Y, \text{Hom}_A(M, I)) \).
we get the exact sequence
\[ \operatorname{Ext}^i_A([1 \varphi], [\cdot, (M, I)]) \to \operatorname{Ext}^1_A([0 \varphi], [\cdot, (M, I)]) \to \operatorname{Ext}^i_A([1 \varphi], [\cdot, (M, I)]) \to \operatorname{Ext}^i_A([1 \varphi], [\cdot, (M, I)]) \times \operatorname{Ext}^1_A([1 \varphi], [\cdot, (M, I)]) \times \cdots \]
Since \([\cdot, (M, I)]) is an injective left \(A\)-module, we have \(\operatorname{Ext}^i_A([1 \varphi], [\cdot, (M, I)]) \cong \operatorname{Ext}^1_A([1 \varphi], [\cdot, (M, I)]) \times \cdots \), and then the assertion follows from \(\operatorname{Ext}^i_A([1 \varphi], [\cdot, (M, I)]) \cong \operatorname{Ext}^1_B(Y, \operatorname{Hom}_A(M, I))\), by using the injective resolution of \(\operatorname{Hom}_A(M, I)\).

**Lemma 2.6.** If \(M_B\) is projective, then \(\mathcal{J}(A, M, B) = \perp \frac{\operatorname{D}(A)}{0} \)

**Proof.** We need to prove that \([\cdot, \varphi] \in \perp \frac{\operatorname{D}(A)}{0} \) if and only if \([\cdot, \varphi] \in \mathcal{J}(A, M, B)\). Since \(M_B\) is projective, \(\operatorname{Hom}_A(M, \operatorname{D}(A_A)) \cong \operatorname{D}(M)\) is an injective left \(B\)-module, and hence \(\operatorname{Ext}^i_B(Y, \operatorname{Hom}_A(M, \operatorname{D}(A_A))) = 0\) for all \(i \geq 1\). Applying \(\operatorname{Hom}_A(\cdot, \varphi)\) on the exact sequence \(0 \to [1 \varphi] \to [1 \varphi] \to [\cdot, \varphi] \to 0\), by Lemma 2.4 we get the commutative diagram with the upper row being exact
\[
\begin{array}{ccc}
\operatorname{Hom}_A([1 \varphi], [\cdot, (M, I)]) & \to & \operatorname{Ext}^1_A([1 \varphi], [\cdot, (M, I)]) \\
\downarrow \cong & & \downarrow \cong \\
\operatorname{Hom}_A(X, \operatorname{D}(A)) & \to & \operatorname{Hom}_A(M \otimes_B Y, \operatorname{D}(A))
\end{array}
\]
and the following exact sequence for \(i \geq 1\)
\[
\begin{array}{ccc}
\operatorname{Ext}^i_A([1 \varphi], [\cdot, (M, I)]) & \to & \operatorname{Ext}^i_A([1 \varphi], [\cdot, (M, I)]) \\
\downarrow \cong & & \downarrow \cong \\
0 & \to & \operatorname{Ext}^i_B(\phi, D A) \\
0 & \to & \operatorname{Ext}^i_B(Y, \operatorname{Hom}_A(M, D A)) = 0.
\end{array}
\]
So \([\cdot, \varphi] \in \perp \frac{\operatorname{D}(A)}{0} \) if and only if \(\phi^* : \operatorname{Hom}_A(X, \operatorname{D}(A_A)) \to \operatorname{Hom}_A(M \otimes_B Y, \operatorname{D}(A_A))\) is an epimorphism, and if and only if \(\phi : M \otimes_B Y \to X\) is monic. \(\square\)

2.3. **Proof of Theorem 1.1**

1. By Lemma 2.4 we have the implications (i) \(\iff\) (ii).
2. \(( \text{iii) } \implies ( \text{ii) } ):\) Since \(\mathcal{J}(A, M, B) = \perp T\), it is clear that \(\mathcal{J}(A, M, B)\) is a resolving subcategory of \(A\)-mod.
3. \(( \text{i) } \implies ( \text{iii) } ):\) Since \(M_B\) is projective, \(\operatorname{Hom}_A(M, \operatorname{D}(A_A)) \cong \operatorname{D}(M)\) is an injective left \(B\)-module, and hence \([0 \varphi] \in \operatorname{D}(A_A) \) is an injective left \(A\)-module. By the exact sequence in \(A\)-mod
\[
0 \to [\cdot, (M, I)] \to \operatorname{D}(A_A) \to \operatorname{Hom}_A(M, \operatorname{D}(A_A)) \to 0
\]
we see that \(\operatorname{inj. dim}_A \frac{\operatorname{D}(A)}{0} \leq 1\).

Let \(\alpha : \operatorname{D}(B_B) \to \operatorname{Hom}_A(M, E_{D(D_B)})\) be the image of \(e \in \operatorname{Hom}_A(M \otimes_B \operatorname{D}(B_B), E_{D(D_B)})\) under the adjunction isomorphism
\[
\operatorname{Hom}_A(M \otimes_B \operatorname{D}(B_B), E_{D(D_B)}) \cong \operatorname{Hom}(B_B, \operatorname{Hom}_A(M, E_{D(D_B)})).
\]
By the naturality of the adjunction isomorphisms we have the commutative diagram
\[
\begin{array}{ccc}
\operatorname{Hom}_A(M \otimes_B \operatorname{D}(B_B), E_{D(D_B)}) & \cong & \operatorname{Hom}(B_B, \operatorname{Hom}_A(M, E_{D(D_B)}) \cong \operatorname{Hom}(B_B, \operatorname{Hom}_A(M, E_{D(D_B)}) \\
\downarrow \cong & & \downarrow \cong \\
\operatorname{Hom}_A(M \otimes_B \operatorname{D}(B_B), E_{D(D_B)}) & \cong & \operatorname{Hom}(B_B, \operatorname{Hom}_A(M, E_{D(D_B)})).
\end{array}
\]
Let $\varphi : \text{Hom}_A(M \otimes_B \text{Hom}_A(M, E_{D(B)})_e), D(B))_e$ be the involution map. By the above commutative diagram we can get $\varphi(1 \otimes_B \alpha) = e$. So we get a $\Lambda$-map $[\alpha] : \left[ \frac{E_{D(B)}}{D(B)} \right]_e \rightarrow \left[ \frac{E_{D(B)}}{\text{Hom}_A(M, E_{D(B)})}_e \right]_\varphi$, and we have the exact sequence in $\Lambda$-mod

\[
0 \rightarrow \left[ \frac{E_{D(B)}}{D(B)} \right]_e \rightarrow \left[ \frac{E_{D(B)}}{\text{Hom}_A(M, E_{D(B)})}_e \right]_\varphi \oplus \left[ \frac{0}{D(B)} \right]_e \rightarrow \left[ \frac{0}{\text{Hom}_A(M, E_{D(B)})}_e \right]_\varphi \rightarrow 0.
\]

Since $E_{D(B)}$ is an injective left $\Lambda$-module, it follows that $\text{Hom}_A(M, E_{D(B)})_e \in \text{add}(\text{Hom}_A(M, D(A)_A))$, so $\text{Hom}_A(M, E_{D(B)})$ is an injective left $\Lambda$-module, and hence $\left[ \frac{\text{Hom}_A(M, E_{D(B)})}{0} \right]_e$ is an injective $\Lambda$-module. This shows $\text{inj.dim}_\Lambda \left[ \frac{E_{D(B)}}{D(B)}_e \right] \leq 1$. Thus $\text{inj.dim}_\Lambda \left[ \frac{D(A)_A}{0} \right] \oplus \left[ \frac{E_{D(B)}}{D(B)}_e \right] \leq 1$.

By Proposition 2.2(2), $T$ is an injective object of $\mathcal{J}(A, M, B)$, so $\text{Ext}^1_\Lambda(T, T) = 0$. It is clear that the number of the pairwise non-isomorphic indecomposable direct summands of $T$ is the number of the simple $\Lambda$-modules. So $\Lambda T$ is a cotilting $\Lambda$-module.

Since $T$ is an injective object of $\mathcal{J}(A, M, B)$ and $\text{inj.dim}_\Lambda T \leq 1$, we have $\mathcal{J}(A, M, B) \subseteq \perp T$. By Lemma 2.6 $\mathcal{J}(A, M, B) = \perp \left[ \frac{D(A)_A}{0} \right] \supset \perp T$. Thus $\mathcal{J}(A, M, B) = \perp T$.

If there is another cotilting $\Lambda$-module $L$ such that $\mathcal{J}(A, M, B) = \perp L$. Then $T \oplus L$ is also a cotilting $\Lambda$-module. By comparing the number of pairwise non-isomorphic indecomposable direct summands of $T \oplus L$ and $T$, we see the uniqueness of $T$, up to the multiplicities of indecomposable direct summands.

(2) The following construction is from C. Ringel and M. Schmidmeier [RS2]. Let $[\chi]_\phi \in \Lambda$-mod. Define $\text{Mimo}(\phi)$ to be the $\Lambda$-module $\left[ \frac{X \oplus \text{Ker}(\phi)}{Y}_e \right]$, where $e : M \otimes_B Y \rightarrow \text{Ker}(\phi)$ is an extension of the injective envelope $\text{Ker}(\phi) \hookrightarrow \text{I Ker}(\phi)$. Then it is clear that $\text{Mimo}(\phi)$ is well-defined (i.e., independent of the choice of $e$) and it is in $\mathcal{J}(A, M, B)$. For any $[\chi]_\phi \in \Lambda$-mod, by the similar argument as in [RS2, Prop. 2.4], one can see that $\left[ \frac{\chi}{\phi} \right] : \text{Mimo}(\phi) = \left[ \frac{X \oplus \text{Ker}(\phi)}{Y}_e \right] \rightarrow [X]_\phi$ is a minimal right $\mathcal{J}(A, M, B)$-approximation of $[\chi]_\phi$. Thus $\mathcal{J}(A, M, B)$ is a contravariantly finite subcategory of $\Lambda$-mod.

By [KS, Corol. 0.3], a resolving contravariantly finite subcategory of $\Lambda$-mod is functorially finite, and by [AS, Thm. 2.4], an extension-closed functorially finite subcategory of $\Lambda$-mod has Auslander-Reiten sequences. Thus, if $M_B$ is projective, then by (1), $\mathcal{J}(A, M, B)$ is functorially finite in $\Lambda$-mod, and hence $\mathcal{J}(A, M, B)$ has Auslander-Reiten sequences. □

2.4. Recall that each right $\Lambda$-module is identified with a triple $(X, Y)_\phi$, where $X \in \text{mod}A$, $Y \in \text{mod}B$, and $\phi : X \otimes_A M \rightarrow Y$ is a right $B$-map; and a right $\Lambda$-module is identified with a pair $(f_1, f_2) : (X_1, Y_1)_\phi \rightarrow (X_2, Y_2)_{\phi_2}$, where $f_1 : X_1 \rightarrow X_2$ is an $A$-map and $f_2 : Y_1 \rightarrow Y_2$ a $B$-map, such that $f_2 \phi_1 = \phi_2(f_1 \otimes 1)$. The injective right $\Lambda$-modules are exactly $(I, 0)$ and $(\text{Hom}_B(M, J), J)_\varphi$, where $I$ (resp. $J$) runs over the injective right $\Lambda$-modules (resp. $B$-modules), and $\varphi : \text{Hom}_B(M, J) \otimes_A M \rightarrow J$ is the involution map given by $\varphi(f \otimes m) = f(m)$. See [ARS, p.73].

All the results obtained so far have the right module versions. We only write down what is needed later. The right module version of $\mathcal{J}(A, M, B)$ is $\mathcal{J}(A, M, B)_r$, which is the subcategory of $\text{mod}A$ consisting of the triple $(U, V)_\phi$, where $X \in \text{mod}A$, $Y \in \text{mod}B$, and $\phi : X \otimes_A M \rightarrow Y$ is a monic right $B$-map. Then $\mathcal{J}(A, M, B)_r$ is a resolving subcategory of $\Lambda$-mod if and only if $A_M$ projective, and if and only if
$\mathscr{I}(A,M,B)_r = \perp(D(\Lambda A),0)$. The following result is only a part of the right module version of Theorem 1.1 which is what we will need later.

A right module version of Theorem 1.1. Let $AM_B$ be a bimodule with $AM$ projective. Then $U_A := (D(\Lambda A), E_{D(\Lambda A)}e) \oplus (0, D(B))$ is a unique cotilting right $\Lambda$-module, up to multiplicities of indecomposable direct summands, such that $\mathscr{I}(A,M,B)_r = \perp(U_A)$, where $E_{D(\Lambda A)}$ is an injective envelope of $D(\Lambda A) \otimes_A M$ with embedding $e : D(\Lambda A) \otimes_A M \hookrightarrow E_{D(\Lambda A)}$.

In particular, if $D(\Lambda A) \otimes_A M$ is an injective right $B$-module, then $U_A = (D(\Lambda A), D(\Lambda A) \otimes_A M)_{\text{id}} \oplus (0, D(B))$ is a unique cotilting right $\Lambda$-module, up to multiplicities of indecomposable direct summands, such that $\mathscr{I}(A,M,B)_r = \perp(U_A)$.

3. Proof of Theorem 1.3

3.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be additive categories. The diagram of functors

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\mathbf{1}} & \mathcal{A} \\
\downarrow{i} & & \downarrow{j} \\
\mathcal{C} & & \mathcal{A}
\end{array}
$$

is a recollement of $\mathcal{A}$ relative to $\mathcal{B}$ and $\mathcal{C}$, if the conditions (R1), (R2), (R3) are satisfied:

1. (R1) $(i^*, i_*)$, $(i_*, i'^*)$, $(j_*, j'^*)$ and $(j^*, j_*)$ are adjoint pairs;
2. (R2) $i_*$, $j_*$ and $j_*$ are fully faithful;
3. (R3) $\text{Im} i_* = \text{Ker} j^*$.

Since the functors in an adjoint pair between additive categories are additive functors, all the six functors in a recollement of additive categories are additive.

3.2. The following fact is easy.

Lemma 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories with subcategories $\mathcal{X}$ and $\mathcal{Y}$, respectively, $(F,G)$ an adjoint pair with $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$. If $F \mathcal{X} \subseteq \mathcal{Y}$ and $G \mathcal{Y} \subseteq \mathcal{X}$, then there is an induced adjoint pair $(\overline{F}, \overline{G})$ with $\overline{F} : \mathcal{A} / \mathcal{X} \to \mathcal{B} / \mathcal{Y}$ and $\overline{G} : \mathcal{B} / \mathcal{Y} \to \mathcal{A} / \mathcal{X}$.

Let $\Lambda = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be an Artin algebra. Define five functors as follows.

$$
\begin{align*}
i^* : \mathscr{I}(A,M,B) & \to A\text{-mod}, \quad [\begin{array}{c} \phi \\ \phi \end{array}] \mapsto \text{Coker}(\phi); \\
i_* : A\text{-mod} & \to \mathscr{I}(A,M,B), \quad X \mapsto [\begin{array}{c} \phi \\ 0 \end{array}]; \\
i_! : \mathscr{I}(A,M,B) & \to A\text{-mod}, \quad [\begin{array}{c} \phi \\ \phi \end{array}] \mapsto X; \\
j^* : \mathscr{I}(A,M,B) & \to B\text{-mod}, \quad [\begin{array}{c} \phi \\ 0 \end{array}] \mapsto Y; \\
j_* : B\text{-mod} & \to \mathscr{I}(A,M,B), \quad X \mapsto [\begin{array}{c} \phi \\ 0 \end{array}]; \\
j_! : \mathscr{I}(A,M,B) & \to B\text{-mod}, \quad [\begin{array}{c} \phi \\ 0 \end{array}] \mapsto Y;
\end{align*}
$$

Lemma 3.2. Let $\Lambda = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be an Artin algebra. Then

1. $(i^*, i_*)$, $(i_*, i'^*)$, and $(j_!, j^*)$ are adjoint pairs;
2. $i_*$ and $j_!$ are fully faithful;
3. $\text{Im} i_* = \text{Ker} j^*$.
Lemma 1.2). Since the functor \( \text{Hom}_A(\phi, \text{Coker}(\phi), W) = \text{Hom}_A(\text{Coker}(\phi), W) \cong \text{Hom}_A([\phi], [W]) = \text{Hom}_A([\phi], i(W)); \)

\( \text{Hom}_A(i_*(W), [\phi]) = \text{Hom}_A([W], [\phi]) \cong \text{Hom}_A(W, X) = \text{Hom}_A(W, i([\phi])); \)

\( \text{Hom}_A(j!(V), [\phi]) = \text{Hom}_A([\phi], j(V)) \cong \text{Hom}_B(V, Y) = \text{Hom}_B(V, j^*(i([\phi])). \)

These show that \((i^*, i_*)\), \((i_*, i^*)\), \((j!*, j!)\) are adjoint pairs.

(2) For any \( X_1, X_2 \in A\text{-mod} \) and \( Y_1, Y_2 \in B\text{-mod} \), we have the isomorphisms:

\[ \text{Hom}_A(i_*(X_1), i_*(X_2)) = \text{Hom}_A([X_1], [X_2]) \cong \text{Hom}_A(X_1, X_2); \]

\[ \text{Hom}_A(j!(Y_1), j!(Y_2)) = \text{Hom}_A([M_{\otimes B}Y_1], [M_{\otimes B}Y_2]) \cong \text{Hom}_B(Y_1, Y_2). \]

These show that \( i_* \) and \( j! \) are fully faithful.

(3) This is clear. \[ \square \]

Suppose that \( A_{M_B} \) satisfies the condition (IP). By Proposition 2.2, the injective objects of \( \mathcal{S}(A, M, B) \) are exactly \([0]\) and \([M_{\otimes B}I]\), where \( I \) runs over injective \( A\)-modules, and \( J \) runs over injective \( B\)-modules. Thus, by the constructions all the functors \( i^*, i_*, i^!, j^! \) preserve injective objects. It follows from Lemmas 3.3 and 3.2 that there are the induced functors:

\[ \overline{\iota} : \mathcal{S}(A, M, B) \to A\text{-mod}, \quad [\phi] \mapsto \text{Coker}(\phi); \]

\[ \overline{j} : A\text{-mod} \to \mathcal{S}(A, M, B), \quad X \mapsto [\phi]; \]

\[ \overline{i} : A\text{-mod} \to \mathcal{S}(A, M, B), \quad [\phi] \mapsto X; \]

\[ \overline{\iota} : \mathcal{S}(A, M, B) \to B\text{-mod}, \quad [\phi] \mapsto Y; \]

\[ \overline{j} : \mathcal{S}(A, M, B) \to B\text{-mod}, \quad [\phi] \mapsto Y; \]

such that the following fact holds:

**Lemma 3.3.** If \( A_{M_B} \) satisfies the condition (IP), then \((\overline{\iota}, \overline{i_*}, \overline{i^*}, \overline{i^!})\) and \((\overline{j}, \overline{j^*})\) are adjoint pairs.

Moreover, if in addition both \( A \) and \( B \) are self-injective algebras, then all the functors \( \overline{\iota}, \overline{i_*}, \overline{i^*}, \overline{j}, \overline{j^*} \) are triangle functors between triangulated categories.

**Proof.** We only need to justify the last assertion. In this case, both \( A\text{-mod} \) and \( B\text{-mod} \) are triangulated categories. By Example 1.3, \( A\text{-mod} \) is projective; and then by Corollary 2.3, \( \mathcal{S}(A, M, B) \) is a Frobenius category, hence \( \mathcal{S}(A, M, B) \) is also a triangulated category. See [H1, p.16]. Recall the distinguished triangles in the stable category of a Frobenius category. Each exact sequence \( 0 \to X_1 \xrightarrow{\nu} X_2 \xrightarrow{\nu} X_3 \to 0 \) in \( A\text{-mod} \) gives rise to a distinguished triangle \( X_1 \xrightarrow{\nu} X_2 \xrightarrow{\nu} X_3 \to X[1] \) of \( A\text{-mod} \), and conversely, each distinguished triangle of \( A\text{-mod} \) is of this form up to an isomorphism of triangles (see [H1], Chap. 1, Sect. 2; also [CZ], Lemma 1.2). Since the functor \( i_* : A\text{-mod} \to \mathcal{S}(A, M, B) \) given by \( X \mapsto [\phi] \) preserves exact sequences, \( i_* : A\text{-mod} \to \mathcal{S}(A, M, B) \) preserves the distinguished triangles, i.e., \( i_* \) is a triangle functor. Note that in an adjoint pair \((F, G)\) between triangulated categories, \( F \) is a triangle functor if and only if so is \( G \) ([Ke, 6.7], [N, p.179]). Thus, \( \overline{\iota} \) and \( \overline{i_*} \) are triangle functors.

Similarly, \( \overline{j} : \mathcal{S}(A, M, B) \to B\text{-mod} \) is a triangle functor, and then so is \( \overline{j^*} \). \[ \square \]

The following lemma is crucial in the proof of Theorem 1.3.

**Lemma 3.4.** Assume that \( A_{M_B} \) satisfies the condition (IP). Then there exists a fully faithful functor \( \overline{j^*} : B\text{-mod} \to \mathcal{S}(A, M, B) \) such that \((\overline{j^*}, \overline{j_*})\) is an adjoint pair.
Proof. The following construction is similar as [Z2, Thm. 3.5]. Define a functor \( j_* : \text{B-mod} \to \mathcal{J}(A, M, B) \) as follows. For \( Y \in \text{B-mod} \), define \( j_*(Y) := \left[ E_Y \right]_{\psi} \), where \( E_Y \) is an injective envelope of the \( A \)-module \( M \otimes_B Y \) with embedding \( \psi : M \otimes_B Y \to E_Y \). (This is clearly well-defined. One can see this also from the argument below, by taking \( h = \text{Id}_Y \).) For \( h : Y \to Y' \), there is a commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_B Y & \xrightarrow{\psi} & E_Y & \xrightarrow{\pi} & \text{Coker}(\psi) & \longrightarrow & 0 \\
0 & \longrightarrow & M \otimes_B Y' & \xrightarrow{\psi'} & E_{Y'} & \xrightarrow{\pi'} & \text{Coker}(\psi') & \longrightarrow & 0
\end{array}
\]

and we define \( j_*(h) := \left[ \frac{E_Y}{\psi} \right] \cdot \left[ \frac{E_{Y'}}{\psi'} \right] \). We claim that \( j_*(h) \) is well-defined, and hence the functor \( j_* : \text{B-mod} \to \mathcal{J}(A, M, B) \) is well-defined.

In fact, if there is another \( A \)-map \( f' : E_Y \to E_{Y'} \), such that \( f' \psi = \psi'(1 \otimes h) \), then \( f' \psi = f \psi \), i.e., \( (f - f') \psi = 0 \) and hence \( f - f' : E_Y \to E_{Y'} \), factors through \( \text{Coker}(\psi) \). Furthermore, \( f - f' \) factors through an injective envelope \( I \) of \( \text{Coker}(\psi) \). Then it is clear that \( \left[ \frac{f'}{h} \right] - \left[ \frac{f'}{h} \right] = \left[ \frac{f - f'}{0} \right] \cdot \left[ \frac{E_Y}{\psi} \right] \cdot \left[ \frac{E_{Y'}}{\psi'} \right] \). We consider the injective object \( \left[ \frac{I}{h} \right] \) of \( \mathcal{J}(A, M, B) \). So \( \left[ \frac{I}{h} \right] = \left[ \frac{f'}{h} \right] \) in \( \mathcal{J}(A, M, B) \).

Next, we claim that the functor \( j_* : \text{B-mod} \to \mathcal{J}(A, M, B) \) induces a functor \( j_* : \text{B-mod} \to \mathcal{J}(A, M, B) \). For this, assume that \( h : Y \to Y' \) factors through an injective \( B \)-module \( J \) via \( h = h_2 h_1 \) with some \( h_1 \in \text{Hom}_B(Y, J) \) and some \( h_2 \in \text{Hom}_B(J, Y') \). Taking an injective envelope \( E_J \) of the \( A \)-module \( M \otimes_B J \) with embedding \( \eta : M \otimes_B J \to E_J \), then there are \( \sigma \in \text{Hom}_A(E_Y, E_J) \) and \( \delta \in \text{Hom}_A(E_J, E_{Y'}) \), such that \( \eta(1 \otimes h_1) = \sigma \psi \) and \( \psi'(1 \otimes h_2) = \delta \psi \). Then \( f \psi = \psi'(1 \otimes h) = \psi'(1 \otimes h_2)(1 \otimes h_1) = \delta \eta(1 \otimes h_1) = \delta \sigma \psi \), i.e., \( (f - \delta \sigma) \psi = 0 \). Hence \( f - \delta \sigma \) factors through \( \text{Coker}(\psi) \) via \( f - \delta \sigma = \gamma \pi \) with some \( \gamma \in \text{Hom}_A(\text{Coker}(\psi), E_{Y'}) \). Consider an injective envelope \( I \) of \( \text{Coker}(\psi) \) with embedding \( \alpha : \text{Coker}(\psi) \to I \). Then there is a \( \beta \in \text{Hom}_A(I, E_{Y'}) \) such that \( \gamma = \beta \alpha \), and then we have \( f = \delta \sigma + \beta \alpha \pi \). We present this process as the diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_B Y & \xrightarrow{\psi} & E_Y & \xrightarrow{\pi} & \text{Coker}(\psi) & \longrightarrow & 0 \\
0 & \longrightarrow & M \otimes_B Y' & \xrightarrow{\psi'} & E_{Y'} & \xrightarrow{\pi'} & \text{Coker}(\psi') & \longrightarrow & 0
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_B J & \xrightarrow{\eta} & E_J & \xrightarrow{\delta} & \text{Coker}(\psi) & \longrightarrow & 0 \\
0 & \longrightarrow & M \otimes_B J & \xrightarrow{\eta'} & E_{Y'} & \xrightarrow{\delta} & \text{Coker}(\psi') & \longrightarrow & 0
\end{array}
\]

Now, we get a \( \Lambda \)-map \( \left( \frac{\sigma}{h_1 \alpha} \right) : \left[ E_Y \right]_{\psi} \to \left[ E_J \right]_{\eta} \oplus \left[ I \right]_{\delta} \) and a \( \Lambda \)-map \( \left( \frac{\delta}{h_2} \right) : \left[ E_J \right]_{\eta} \oplus \left[ I \right]_{\delta} \to \left[ E_{Y'} \right]_{\psi'} \) with composition

\[
\left( \frac{\delta}{h_2} \right) \cdot \left( \frac{\sigma}{h_1 \alpha} \right) = \left( \frac{\delta \sigma}{h_1 h_2} \right) + \left( \frac{\beta \alpha \pi}{h_2 h_1} \right) = \left( \frac{\delta \sigma + \beta \alpha \pi}{h_1 h_2} \right) = \left( \frac{h}{h} \right).
\]

This shows that \( \left[ \frac{f}{h} \right] : \left[ E_Y \right]_{\psi} \to \left[ E_{Y'} \right]_{\psi'} \) factors through the injective object \( \left[ E_J \right]_{\eta} \oplus \left[ I \right]_{\delta} \) of \( \mathcal{J}(A, M, B) \), and hence \( j_*(h) = \left[ \frac{I}{h} \right] = 0 \) in \( \mathcal{J}(A, M, B) \). Thus \( j_* \) induces a functor

\[
j_* : \text{B-mod} \to \mathcal{J}(A, M, B)
\]

given by \( j_*(Y) := \left[ E_Y \right]_{\psi} \) and \( j_*(h) := \left[ \frac{h}{h} \right] \).
By construction it is clear that \( \overline{\mathcal{J}} \) is full. For \( h : Y \to Y' \), assume that \( \overline{\mathcal{J}}(h) = [\overline{j}] = 0 \). Then \( \overline{[j]} : [E_Y] \to [E_{Y'}] \) factors through an injective object \( [E_j] \otimes [I] \) of \( \mathcal{S}(A, M, B) \), and hence \( h : Y \to Y' \) factors through the injective \( B \)-module \( J \). So \( \overline{\mathcal{J}} \) is also faithful.

It remains to prove that \( (\overline{\mathcal{J}}, \overline{\mathcal{J}}) \) is an adjoint pair. Thus, for \( \overline{[Y]}_\phi \in \mathcal{S}(A, M, B) \) and \( Y' \in B\text{-mod} \), we need to show that there is a bi-functorial isomorphism \( \overline{\text{Hom}}_B(j_\overline{\mathcal{J}}(\overline{[Y]}_\phi), Y') \cong \overline{\text{Hom}}_A(\overline{[Y]}_\phi, \overline{\mathcal{J}}(Y')) \), i.e.

\[
\overline{\text{Hom}}_B(Y, Y') \cong \overline{\text{Hom}}_A(\overline{[Y]}_\phi, [E_{Y'}] \psi) \]

where \( \psi : M \otimes_B Y' \to E_{Y'} \) is an injective envelope of \( M \otimes_B Y' \). We claim that the map \( \overline{h} \mapsto \overline{[h]} \) gives such an isomorphism, where \( f : X \to E_{Y'} \) is an \( A \)-map such that \( f \phi = \psi (1 \otimes h) \). In fact, a \( \Lambda \)-map \( \overline{[f]} : \overline{[Y]}_\phi \to \overline{[E_{Y'}]} \psi \) factors through an injective object \( [E_j] \otimes [E_{Y'}] \psi \) of \( \mathcal{S}(A, M, B) \) if and only if \( h : Y \to Y' \) factors through the injective \( B \)-module \( J \). This shows that the given map above is well-defined and injective; and by the construction it is clearly surjective. This completes the proof. □

3.3. Proof of Theorem 1.3. To see the diagram (1.1) forms a recollement of additive categories, by Lemmas 3.3 and 3.4 it remains to prove that \( \overline{\mathcal{J}} \) and \( \overline{\mathcal{J}} \) are fully faithful, and that \( \text{Im} \overline{\mathcal{J}} = \text{Ker} \overline{\mathcal{J}} \).

Since \( A \text{-mod}_B \) satisfies the condition (IP), by Proposition 2.3(2), \([I]\) and \([M \otimes_B J]_{\text{Id}}\) are injective objects of \( \mathcal{S}(A, M, B) \), where \( I \) is an injective \( A \)-module and \( J \) is an injective \( B \)-module. Recall that \( \overline{\mathcal{J}} : A\text{-mod} \to \mathcal{S}(A, M, B) \) is given by \( X \mapsto \overline{[Y]}_\phi \) and \( \overline{\mathcal{J}} \) is an injective object of \( \mathcal{S}(A, M, B) \) if and only if \( h : Y \to Y' \) factors through the injective \( B \)-module \( J \). This shows that the given map above is well-defined and injective; and by the construction it is clearly surjective. This completes the proof.

4. The dual version: the epimorphism category induced by a bimodule

4.1. We briefly state the dual version for the next section. The epimorphism category \( \mathcal{F}(A, M, B) \) induced by a bimodule \( A \text{-mod}_B \) is the subcategory of \( \Lambda \text{-mod} \) consisting of \( [\overline{Y}]_\phi \) such that \( \eta_{Y, X}(\phi) : Y \to \text{Hom}_A(M, X) \) is an epic left \( B \)-map, where

\[
\eta_{Y, X} : \text{Hom}_A(M \otimes_B Y, X) \cong \text{Hom}_B(Y, \text{Hom}_A(M, X))
\]

is the adjunction isomorphism. Then \( \mathcal{F}(A, M, B) \) contains all the injective \( \Lambda \)-modules and is closed under direct sums, direct summands and extensions. Thus \( \mathcal{F}(A, M, B) \) is a Krull-Schmidt exact category with the canonical exact structure.

Proposition 2.3 Let \( A \text{-mod}_B \) be a bimodule with \( A \text{-mod} \) projective. Then
Corollary 1.2 Let \( \mathcal{F}(A, M, B) \) have enough injective objects, which are exactly projective left \( A \)-modules.

(2) \( \mathcal{F}(A, M, B) \) has enough projective objects, and the projective objects of \( \mathcal{F}(A, M, B) \) are exactly \([\frac{0}{P}]_{\mathcal{F}}\) and \([\frac{0}{Q}]_\theta\), where \( P \) (resp. \( Q \)) runs over projective \( A \)-modules (resp. \( B \)-modules), and \( \theta : C \to \text{Hom}_A(M, P) \) is a projective cover of the left \( B \)-module \( \text{Hom}_A(M, P) \). In particular, if in addition \( \text{Hom}_A(M, A) \) is a projective left \( B \)-module, then the projective objects of \( \mathcal{F}(A, M, B) \) are exactly \([\frac{0}{P}]_{\text{Hom}_A(M, P)}\) and \([\frac{0}{Q}]_\theta\).

Corollary 2.4 Let \( \Lambda M_B \) be a bimodule with \( \Lambda M \) projective. Then \( \mathcal{F}(A, M, B) \) is a Frobenius category (with the canonical exact structure) if and only if \( A \) and \( B \) are selfinjective algebras and \( M_B \) is projective.

Theorem 1.1\' Let \( \text{mod} \Lambda \) be an \( A-B \)-bimodule. Then

1. The following are equivalent:
   (i) \( \Lambda M \) is projective;
   (ii) \( \mathcal{F}(A, M, B) \) is a coresolving subcategory of \( \Lambda \)-mod;
   (iii) \( L := [\frac{A}{C}]_{\mathcal{F}}^{\perp} \oplus [\frac{0}{B}] \) is the unique tilting left \( \Lambda \)-module, up to multiplicities of indecomposable direct summands, such that \( \mathcal{F}(A, M, B) = L^\perp \), where \( C \) is a projective cover of the \( B \)-module \( \text{Hom}_A(M, A) \) with projection \( \theta : C \to \text{Hom}_A(M, A) \).

2. \( \mathcal{F}(A, M, B) \) is a covariantly finite subcategory of \( \Lambda \)-mod. Moreover, if \( \Lambda M \) is projective, then \( \mathcal{F}(A, M, B) \) is a functorially finite subcategory of \( \Lambda \)-mod, and has Auslander-Reiten sequences.

Corollary 1.2\' If \( \Lambda M \) is projective and \( \text{Hom}_A(M, A) \) is a projective left \( B \)-module, then \( L = [\frac{A}{\text{Hom}_A(M, A)}]_\phi \oplus [\frac{0}{\text{id}}] \) is a cotilting left \( \Lambda \)-module such that \( \mathcal{F}(A, M, B) = L^\perp \).

Theorem 1.3. Let \( \Lambda M_B \) be a bimodule such that \( \Lambda M \) is a projective \( A \)-module and \( \text{Hom}_A(M, A) \) is a projective left \( B \)-module. Then there is a recollement of additive categories

\[
\text{B-mod} \quad \longleftrightarrow \quad \mathcal{F}(A, M, B) \quad \longleftrightarrow \quad \text{A-mod}.
\]

If in addition \( A \) and \( B \) are selfinjective algebras, then it is in fact a recollement of singularity categories.

Note that if \( \Lambda M_B \) is an exchangeable bimodule then \( \text{Hom}_A(M, A) \) is a projective left \( B \)-module.

4.2. Another description of the epimorphism category induced by a bimodule. Recall that the right module version of \( \mathcal{F}(A, M, B) \) is \( \mathcal{F}(A, M, B)_r \), which is the subcategory of \( \text{mod} \Lambda \) consisting of the triple \( (U, V)_\psi \), where \( U \in \text{mod} A \), \( V \in \text{mod} B \), and \( \psi : U \otimes_A M \to V \) is a monic \( B \)-map.

Proposition 4.1. The restriction of \( D : \text{mod} \Lambda \to \Lambda \)-mod gives a duality \( D : \mathcal{F}(A, M, B)_r \to \mathcal{F}(A, M, B)_l \).

Proof. For a right \( A \)-module \( U \), denote by \( \alpha_U \) the adjunction isomorphism

\[
D(U \otimes_A M) = \text{Hom}_R(U \otimes_A M, J) \cong \text{Hom}_A(M, \text{Hom}_R(U, J)) = \text{Hom}_A(M, D(U))
\]

where \( D = \text{Hom}_R(-, J) \) is the duality. For a left \( A \)-module \( X \) and a left \( B \)-module \( Y \), denote by \( \eta_{V \otimes X} \) the adjunction isomorphism \( \text{Hom}_A(M \otimes_B Y, X) \cong \text{Hom}_B(Y, \text{Hom}_A(M, X)) \).

For a right \( A \)-module \( (U, V)_\psi \) with a right \( B \)-map \( \psi : U \otimes_A M \to V \), we have \( D(V) \xrightarrow{D(\psi)} D(U \otimes_A M) \xrightarrow{\alpha_U} \text{Hom}_A(M, D(U)) \), and \( \eta^{-1}_{D(\psi)} : \text{Hom}_B(D(V), \text{Hom}_A(M, D(U))) \to \text{Hom}_A(M \otimes_B D(V), D(U)) \). Then

\[
D : \text{mod} \Lambda \to \Lambda \text{-mod}, \quad (U, V)_\psi \mapsto [\frac{D(U)}{D(V)}]_{\eta_{D(\psi)}^{-1}(\alpha_U D(\psi))}
\]
with $\eta_{D,X,Y}^{-1}(\alpha, D(\psi)) : M \otimes_B D V \rightarrow D U$. For a left $\Lambda$-module $[\bar{X}]_\alpha$ with a left $A$-map $\phi : M \otimes_B Y \rightarrow X$, we have $Y \eta_{Y,X}^{-1}(\phi) \text{Hom}_A(M, X) \otimes_{\Lambda} D(X \otimes_A M)$. Then a quasi-inverse of $D : \text{mod}\Lambda \rightarrow \text{mod}\Lambda$ is

\[ D : \text{mod}\Lambda \rightarrow \text{mod}\Lambda, \quad [X]_\phi \mapsto (D(X, D Y)D(\alpha^{-1} \eta_{Y,X}(\phi))) \]

with $D(\alpha^{-1} \eta_{Y,X}(\phi)) : D X \otimes_A M \rightarrow D Y$. In fact,

\[ D D(U,V) = D\left[D \left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi))\right] \cong (U,V) \in \text{mod}\Lambda \]

and $D D\left[D\left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi))\right] \cong \left[D\left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi))\right] \cong \left[D\left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi))\right] \cong \left[D\left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi))\right]$. 

If a right $\Lambda$-module $(U,V) \in \mathcal{S}(A,M), B)_r$, i.e., $\psi : U \otimes_A M \rightarrow V$ is a monic right $B$-map, then

\[ D(U,V) = \left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi)) \in \mathcal{S}(A,M,B), \]

for $\eta_{D,V,D,U}(\alpha, D(\psi)) = \alpha D(\psi) : D V \rightarrow \text{Hom}_A(M, D U)$ is an epic left $B$-map.

If a left $\Lambda$-module $[\bar{X}]_\phi \in \mathcal{S}(A,M)_r$, i.e., $\eta_{Y,X}(\phi) : Y \rightarrow \text{Hom}_A(M,X)$ is an epic left $B$-map, then $D\left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi)) \in \mathcal{S}(A,M,B)_r$, since $D\left[D(U,V)\right]^{-1}\eta_{D,V,D,U}(\alpha, D(\psi)) : D X \otimes_A M \rightarrow D Y$ is monic. □

5. Ringel-Schmidmeier-Simson equivalences via cotilting modules

**Definition 5.1.** A Ringel-Schmidmeier-Simson equivalence, in short, an RSS equivalence, induced by a bimodule $A M_B$ is an equivalence $F : \mathcal{S}(A,M,B) \equiv \mathcal{S}(A,M,B)$ of categories, such that

\[ F [\bar{X}]_\phi \equiv \left[\text{Hom}_A(M,X)\right]_\phi, \forall X \in \text{mod}\Lambda, \quad \text{and} \quad F [M \otimes_B Y]_{Id} \equiv \left[0 \otimes Y\right]_{Id}, \forall Y \in \text{mod}\Lambda, \]

where $\phi : M \otimes_B \text{Hom}_A(M,X) \rightarrow X$ is the involution map.

An RSS equivalence implies a strong symmetry. It was first observed by C. Ringel and M. Schmidmeier [RS2] for $A M_B = A A$. For an RSS equivalence induced by a chain without relations we refer to D. Simson [S1-S3], and for RSS equivalences induced by acyclic quivers with monomial relations we refer to [ZX].

5.1. Special cotilting modules induced by exchangeable bimodules. Let $M$ be an $A$-$B$-bimodule. If $M \otimes_B D B$ is an injective left $A$-module, then by Corollary 1.2, $A T = \left[D(A_A) + \left[M \otimes_B D(B)\right]_{Id}\right]$ is a cotilting left $\Lambda$-module with $\mathcal{S}(A,M,B) = \perp T$. If $D(A) \otimes_A M$ is an injective right $B$-module, then by a right module version of Theorem 1.1 (cf. Subsection 2.4), $U_A = (D(A_A), D(A_A) \otimes_A M)_{Id} \oplus (0, D(B_B))$ is a cotilting right $\Lambda$-module with $\mathcal{S}(A,M,B)_r = \perp (U_A)$. If $A M_B$ is exchangeable, then both the conditions are satisfied; and by $D(A A_A) \otimes_A M \cong M \otimes_B D(B_B)$, we can regard that $T$ and $U$ have the same underlying abelian group $D(A_A \oplus (M \otimes_B D B) \oplus D B)$. The following lemma claims that in this case $T$ (and $U$) can be endowed with a $\Lambda$-$\Lambda$-bimodule structure such that $\mathcal{S}(A,M,B) = \perp (A T)$ and $\mathcal{S}(A,M,B)_r = \perp (T_A)$ (thus $T_A \cong U_A$).

**Lemma 5.2.** Let $A M_B$ be an exchangeable bimodule with an $A$-$B$-bimodule isomorphism $g : D(A A_A) \otimes_A M \cong M \otimes_B D(B_B)$, and $\perp A T = \left[D(A_A) + \left[M \otimes_B D(B)\right]_{Id}\right]$. Then $T$ has a $\Lambda$-$\Lambda$-bimodule structure such that both $\Lambda T$ and $T_A$ are cotilting modules, $T_A \equiv (D(A_A), D A_A M)_{Id} \oplus (0, D(B_B))$ as right $\Lambda$-modules, $\mathcal{S}(A,M,B) = \perp (A T)$ and $\mathcal{S}(A,M,B)_r = \perp (T_A)$.

Write $T = \left[D(A_A) M \otimes_B D(B_B)\right]_{Id}$. Then the right $\Lambda$-action is given by

\[
\begin{pmatrix}
\alpha m' \otimes_B b' \\
0 & B \end{pmatrix} = \begin{pmatrix}
\alpha m' \otimes_B b' \\
\alpha m \otimes_B b \end{pmatrix}.
\]
Proof. By Example 1.5(1), $A\mathcal{M}_B$ satisfies the condition (IP). It follows from Corollary 1.2 that $\Lambda T$ is a cotilting left $\Lambda$-module with $\mathcal{S}(A,M,B) = ^\perp(A\mathcal{T})$.

Since $g$ is a right $B$-isomorphism, we get a right $\Lambda$-isomorphism

$$(\text{Id}_{D(A\mathcal{A})},g^{-1}) : (D(A\mathcal{A}), M \otimes_B D B)_g \cong (D(A\mathcal{A}), D A \otimes_A M)_{\text{Id}}.$$ We endow the abelian group $T = DA \oplus (M \otimes_B D B) \oplus D B$ with a right $\Lambda$-module structure via $g$, i.e., $T_\Lambda = (D(A\mathcal{A}), M \otimes_B D B)_g \oplus (0, D(BB))$. Then we get a right $\Lambda$-isomorphism:

$$T_\Lambda = (D(A\mathcal{A}), M \otimes_B D B)_g \oplus (0, D(BB)) \cong (D(A\mathcal{A}), D A \otimes_A M)_{\text{Id}} \oplus (0, D(BB)).$$ since $g$ is also a left $\Lambda$-map, one can easily verify that $T$ is a $\Lambda$-$A$-bimodule. We omit the details.

By Example 1.5(1), $D(A\mathcal{A}) \otimes A M$ is an injective right $B$-module. It follows from the right module version of Theorem 1.1 (cf. Subsection 2.4) that $\text{End}(T)$ is also a left $\Lambda$-module, and there is an algebra isomorphism $\text{End}(T) \cong \Lambda$, such that under this algebra isomorphism, the left module $\text{End}(T) \Lambda$ coincides with the right module $T_\Lambda$.

The following fact will play a crucial role in proving the existence of an RSS equivalence.

**Lemma 5.3.** Let $A\mathcal{M}_B$ be an exchangeable bimodule, and $\Lambda T_\Lambda$ the $\Lambda$-$A$-bimodule $T$ given in Lemma 5.2. Then there is an algebra isomorphism $\rho : \text{End}(\Lambda T) \cong \Lambda$, such that under $\rho$, the right module $T_{\text{End}(\Lambda T)\Lambda}$ coincides with the right module $T_\Lambda$; and there is an algebra isomorphism $\text{End}(T_\Lambda) \cong \Lambda$, such that under this algebra isomorphism, the left module $\text{End}(T_\Lambda) \Lambda$ coincides with the left module $\Lambda T$.

**Proof.** Since $T_\Lambda$ is a right $\Lambda$-module, we get the canonical algebra homomorphism $\rho : \Lambda \to \text{End}(\Lambda T)^\Lambda$, $\lambda \mapsto "t \mapsto t\lambda"$. By Lemma 5.2, $T_\Lambda$ is cotilting, so $T_\Lambda$ is faithful (Suppose $T_\Lambda = 0$ for $\lambda \in \Lambda$. By a surjective $\Lambda$-map $T_0 \to DA$ with $T_0 \in \text{add}T$, we see $(D\Lambda)\lambda = 0$, i.e., $(D\lambda A) = 0$. So $\lambda = 0$). Thus $\rho$ is an injective map. On the other hand, we have algebra isomorphisms

$$\text{End}(\Lambda T)^\Lambda \cong \left[\begin{array}{c|c}
\text{End}\Lambda\left[D A\right] & \text{Hom}_\Lambda\left[D A\right] \\
\text{Hom}_\Lambda\left[M \otimes_B D B\right]_{\text{Id}} & \text{End}\Lambda\left[M \otimes_B D B\right]_{\text{Id}}
\end{array}\right]_{\text{Id}}^\Lambda \cong \left[\begin{array}{c|c}
\Lambda^\perp & 0 \\
0 & M_B^\perp
\end{array}\right]_{\text{Id}} \cong \left[\begin{array}{c|c}
\Lambda & 0 \\
0 & B
\end{array}\right] = \Lambda.$$

Denote this algebra isomorphism $\text{End}(\Lambda T)^\Lambda \cong \Lambda$ by $\rho$. Since $\Lambda$ is an Artin $R$-algebra, where $R$ is a commutative artinian ring, $\rho : \Lambda \to \Lambda$ is an $R$-endomorphism of artinian $R$-module $\Lambda$. Since $\rho$ is an injective map, it follows that $\rho$ is surjective, and hence $\rho$ is surjective (since $h$ is an $R$-module isomorphism). Thus $\rho$ is an algebra isomorphism. By the construction of $\rho$, $T_{\text{End}(\Lambda T)^\Lambda}$ is exactly $T_\Lambda$.

Since $\rho : \text{End}(\Lambda T)^\Lambda \cong \Lambda$ as algebras, and under $\rho$, $T_{\text{End}(\Lambda T)^\Lambda} = T_\Lambda$. By the tilting theory, the homomorphism $\Lambda \to \text{End}(T_\Lambda)$ given by $\lambda \mapsto "t \mapsto \lambda t\"$ is an algebra isomorphism ([HR, p.409]), and hence $\text{End}(T_\Lambda) \Lambda$ is exactly $\Lambda T$ (one can also prove this by the same argument as above). □

5.2. **Existence of RSS equivalences.** For any left $\Gamma$-module $L$, following [AR], let $\mathcal{X}_{\ell L}$ be the subcategory of $\Gamma$-mod consisting of $\Gamma$-modules $\Gamma X$ such that there is an exact sequence

$$0 \to X \to L_0 \xrightarrow{f_0} L_1 \to \cdots \to L_j \xrightarrow{f_j} L_{j+1} \to \cdots$$

with $L_j \in \text{add}(\Gamma L)$ and $\text{Im}f_j \in \perp(\Gamma L)$ for $j \geq 0$. The following fact is in T. Wakamatsu [W, Prop. 1].
Lemma 5.4. ([W]) For any $\Gamma$-module $\Gamma L$ with $C := \mathrm{End}_\Gamma(L)^{op}$, we have a contravariant functor
\[
\mathrm{Hom}_\Gamma(-, \Gamma L) : \mathcal{X}_{\Gamma L} \to \mathcal{X}^{op}(
\Gamma C)
\]
such that if $\Gamma X \in \mathcal{X}_{\Gamma L}$, then the canonical $\Gamma$-map $\Gamma X \to \mathrm{Hom}_C(\mathrm{Hom}_\Gamma(\Gamma X, \Gamma L), \n\Gamma C)$ is an isomorphism.

Proof of Theorem 1.4. Step 1. We first prove that $D \mathrm{Hom}_A(-, A T_A) : \mathcal{S}(A, M, B) 
\to \mathcal{S}(A, M, B)$ is an equivalence of categories. By Proposition 4.1, $D : \mathcal{S}(A, M, B) \to \mathcal{S}(A, M, B)$ is a duality. So, it suffices to prove that $\mathrm{Hom}_A(-, A T_A) : \mathcal{S}(A, M, B) \to \mathcal{S}(A, M, B)$ is a duality.

By Lemma 5.3 $T$ is a $\Lambda$-$\Lambda$-bimodule such that $\mathcal{S}(A, M, B) = \mathcal{S}(\Lambda) = \mathcal{S}(\Lambda T_A)$. Since $\Lambda T$ is cotilting, it follows from M. Auslander and I. Reiten [AR, Thm. 5.4(b)] that $\mathcal{S}_\Lambda = \mathcal{S}(\Lambda T)$. Thus $\mathcal{S}(A, M, B) = \mathcal{S}(\Lambda T) = \mathcal{S}(\Lambda T_A)$.

By Lemma 5.3 there is an algebra isomorphism $\rho : \mathrm{End}_\Lambda(\Lambda T)^{op} \cong \Lambda$, such that $T_{\mathrm{End}_\Lambda(\Lambda T)^{op}} = T_A$ under $\rho$. So we can apply Lemma 5.4 to $\Lambda T$ to get a contravariant functor:
\[
\mathrm{Hom}_A(-, \Lambda T) : \mathcal{X}_\Lambda T = \mathcal{S}(A, M, B) \to \mathcal{S}(A, M, B)_r = \mathcal{S}(\Lambda T)
\]
such that for $X \in \mathcal{S}(A, M, B) = \mathcal{X}_\Lambda T$, the canonical left $\Lambda$-map $\gamma X \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\Lambda X, \Lambda T), T_A)$ is an isomorphism.

Similarly, $T_A$ is a cotilting module and $\mathcal{S}(A, M, B)_r = \mathcal{S}(\Lambda T) = \mathcal{S}(\Lambda T_A)$. By Lemma 5.3 there is an algebra isomorphism $\mathrm{End}_\Lambda(T_A) \cong \Lambda$, such that $\mathrm{End}_\Lambda(T_A)^{op} = T_A$ under this isomorphism. So we can apply the right module version of Lemma 5.4 to $T_A$ to get a contravariant functor
\[
\mathrm{Hom}_A(-, T_A) : \mathcal{X}_T = \mathcal{S}(A, M, B)_r \to \mathcal{S}(A, M, B) = \mathcal{S}(\Lambda T)
\]
such that for each $Y \in \mathcal{S}(A, M, B)_r = \mathcal{X}_T$, the canonical left $\Lambda$-map $\gamma Y \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\Lambda X, T), T_A)$ is an isomorphism. Thus $\mathrm{Hom}_A(-, T_A) : \mathcal{S}(A, M, B) \to \mathcal{S}(A, M, B)_r$ is a duality with a quasi-inverse $\mathrm{Hom}_A(-, T_A) : \mathcal{S}(A, M, B)_r \to \mathcal{S}(A, M, B)$.

Step 2. Put $F := D \mathrm{Hom}_A(-, A T_A) : \mathcal{S}(A, M, B) \cong \mathcal{S}(A, M, B)$ (cf. Proposition 4.1). It remains to prove $F([X]) \cong [\mathrm{Hom}_A(M,X)]_\varphi$ and $F([M \otimes_B Y]) \cong [Y]_\varphi$ for $X \in \mathcal{A}$-mod and $Y \in \mathcal{B}$-mod. This is true by direct computations, which are included as an Appendix at the end of this paper. □

5.3. RSS equivalences and the Nakayama functors. It is natural to ask when the Nakayama functor $N_A = D \mathrm{Hom}_A(-, A A_A) : \mathcal{A}$-mod $\to \mathcal{A}$-mod induces an RSS equivalence $\mathcal{S}(A, M, B) \cong \mathcal{S}(A, M, B)$ if and only if both $A$ and $B$ are Frobenius algebras. If this is the case, we have $\mathcal{S}(A, M, B) = \mathcal{G}(\Lambda)$ and $\mathcal{S}(A, M, B) = \mathcal{G}(\Lambda)$.

Proposition 5.5. Let $\Lambda M_B$ be an exchangeable $\Lambda B$-bimodule. Then the restriction of the Nakayama functor $N_A$ gives an RSS equivalence $\mathcal{S}(A, M, B) \cong \mathcal{S}(A, M, B)$ if and only if both $A$ and $B$ are Frobenius algebras. If this is the case, we have $\mathcal{S}(A, M, B) = \mathcal{G}(\Lambda)$ and $\mathcal{S}(A, M, B) = \mathcal{G}(\Lambda)$.

Proof. If the restriction of $N_A$ gives an RSS equivalence, then we have $N_A([A]) = [\mathrm{Hom}_A(M,M,A)]_\mathrm{Id}$ and $N_A([M]) = [M]_\mathrm{Id}$. However
\[
N_A([A]) = D \mathrm{Hom}_A([A], A) = D \mathrm{Hom}_A([A], [A] \oplus [M]_\mathrm{Id}) = D(A, M)_\mathrm{Id} = [D(A,A)]_\mathrm{Id},
\]

\[
N_A([M]) = D \mathrm{Hom}_A([M], A) = D \mathrm{Hom}_A([M], [A] \oplus [M]_\mathrm{Id}) = D(A, M)_\mathrm{Id} = [D(A,M)]_\mathrm{Id},
\]
and

\[ \mathcal{N}_\Lambda [\frac{M}{B}]_{\text{Id}} = \text{DHom}_\Lambda ( [\frac{M}{B}]_{\text{Id}}, \Lambda) = \text{DHom}_\Lambda ( [\frac{M}{B}]_{\text{Id}}, [0] \oplus [\frac{M}{B}]_{\text{Id}}) = \text{D}(0, B) = [\frac{0}{D(B_B)}]. \]

So \( \text{D}(A_A) \cong A_A \) and \( \text{D}(B_B) \cong B_B \), i.e., \( A \) and \( B \) are Frobenius algebras.

Conversely, if both \( A \) and \( B \) are Frobenius algebras, then replacing \( A T_A \) by \( \Lambda A \Lambda \) and using the same arguments as in the proofs of Lemmas 5.2 and 5.3 and Theorem 1.4, we get an RSS equivalence

\[ \mathcal{N}_\Lambda = \text{DHom}_\Lambda (-, \Lambda \Lambda) : \mathcal{F}(A, M, B) \rightarrow \mathcal{F}(A, M, B). \]

If both \( A \) and \( B \) are Frobenius algebras and both \( A M \) and \( M_B \) are projective, then \( \Lambda \) is a Gorenstein algebra (cf. [Z2, Lemma 2.1]), and then by [Z2, Thm. 2.2] and its dual, \( \mathcal{F}(A, M, B) = \mathcal{G}P(\Lambda) \) and \( \mathcal{F}(A, M, B) = \mathcal{G}I(\Lambda) \). This completes the proof. \( \square \)

**Remark.** For any algebra \( \Gamma \), the Nakayama functor always gives an equivalence \( \mathcal{N}_\Gamma : \mathcal{G}P(\Gamma) \cong \mathcal{G}I(\Gamma) \) (see [Bel, Prop. 3.4]). Also, if \( A \) and \( B \) are Frobenius, then there is a left \( A \)-isomorphism \( f_1 : \text{D}(A_A) \cong A_A \) and a left \( B \)-isomorphism \( g_1 : \text{D}(B_B) \cong B_B \), so \( A T = [\frac{\text{D}(A_A)}{0}] \oplus [\frac{M_B \oplus \text{D}(B_B)}{\text{D}(A_A)}]_{\text{Id}} \cong [\frac{A A}{0}] \oplus [\frac{M_B}{B}]_{\text{Id}} = \Lambda A \). By the symmetry a Frobenius algebra, there is a right \( A \)-isomorphism \( f_2 : \text{D}(A_A) \cong A_A \) and a right \( B \)-isomorphism \( g_2 : \text{D}(B_B) \cong B_B \). So \( T_A \cong (\text{D}(A_A), D(A_A) \otimes_A M_B)_{\text{Id}} \oplus (0, D(B_B)) \cong (A_A, M_B)_{\text{Id}} \oplus (0, B_B) = \Lambda A \). But \( T \neq \Lambda \) as \( A \)-\( \Lambda \)-modules in general, since a Frobenius algebra is not necessarily a symmetric algebra. Thus, the “if part” of Proposition 5.5 is not a corollary of Theorem 1.4.

### 5.4. We illustrate Theorem 1.4 and Proposition 5.5. The conjunction of paths of a quiver is from the right to the left.

**Example 5.6.** Let \( B \) be the path algebra \( k(b \rightarrow a) \). We write the indecomposable \( B \)-modules as \( \frac{0}{0} = S(a) = P(a) \), \( \frac{1}{0} = P(b) = I(a) \), \( \frac{1}{0} = S(b) = I(b) \). Let \( A := B \oplus B \) and \( \Lambda B := A A_B \). So \( \Lambda B \) is an exchangeable bimodule, and \( \Lambda := \left[ \begin{array}{cc} A & M \\ 0 & B \end{array} \right] \left[ \begin{array}{cc} 0 & B \\ B & 0 \end{array} \right] = B \otimes_k Q \), where \( Q \) is the quiver \( 1 \overset{\alpha}{\leftarrow} 3 \overset{\beta'}{\rightarrow} 2 \). Thus \( \Lambda \) is given by the quiver

![Quiver Diagram]

with relations \( \gamma_1 \alpha - \alpha \gamma_3 \), \( \beta \gamma_3 - \gamma_2 \beta' \). We will write a \( \Lambda \)-module as a representation of \( Q \) over algebra \( B \) (see e.g. [ZX], [LZ]). Thus a \( \Lambda \)-module is written as \( X_1 \overset{X_3}{\leftarrow} X_3 \overset{X_5}{\rightarrow} X_2 \), where \( X_1, X_2, X_3 \in B \text{-mod} \), and \( X_\alpha \) and \( X_\beta \) are \( B \)-maps. For example, the indecomposable projective \( \Lambda \)-module \( P(6) = \frac{1}{1} \frac{1}{1} \) at vertex 6 is

\[ (\frac{1}{1} \overset{0}{\leftarrow} \frac{1}{1} \overset{0}{\rightarrow} \frac{1}{1}) = (P(b) \overset{0}{\leftarrow} P(b) \overset{0}{\rightarrow} P(b)) \]
With this notation, the Auslander-Reiten quiver of $\Lambda$ is

Since $\Lambda$ is of the form $[\frac{A M}{B}]$, a $\Lambda$-module $X_1 \rightleftharpoons X_3 \rightarrow X_2$ is also written as a triple $[\frac{X_1 \oplus X_3}{X_2}]_\phi$, where $X_1 \oplus X_2 \in A$-mod and $\phi: M \otimes_B X_3 \rightarrow X_1 \oplus X_2$ is exactly the $A$-map $\begin{pmatrix} X_\alpha & 0 \\ 0 & X_\beta \end{pmatrix}: X_3 \oplus X_3 \rightarrow X_1 \oplus X_2$. Thus it is in $\mathcal{F}(A, M, B)$ if and only if $X_\alpha$ and $X_\beta$ are monic. So the Auslander-Reiten quiver of $\mathcal{F}(A, M, B)$ is:

Note that $(X_2 \xrightarrow{X_\alpha} X_1 \xrightarrow{X_\beta} X_4) = [\frac{X_1 \oplus X_3}{X_2}]_\phi \in \mathcal{F}(A, M, B)$ if and only if $(X_\alpha) : X_3 \rightarrow X_1 \oplus X_2$ is an epic $B$-map (in particular, $X_\alpha$ and $X_\beta$ are epic; but this is not sufficient). So the Auslander-Reiten quiver of $\mathcal{F}(A, M, B)$ is:

There is a unique RSS equivalence, sending an indecomposable object in $\mathcal{F}(A, M, B)$ to the one in $\mathcal{F}(A, M, B)$, in the same positions of the Auslander-Reiten quivers. Note that this RSS equivalence is not given by the Nakayama functor $N_{\Lambda}$, since it does not send projective $\Lambda$-modules to injective $\Lambda$-modules.

**Example 5.7.** Let $\Lambda$ be the algebra given by the quiver $2 \xrightarrow{\beta} 1 \xrightarrow{\alpha}$ with relation $\alpha^2$. Then $\Lambda := [\frac{A M}{B}]$, where $A := k[[\alpha]]/((\alpha^2))$, $B := e_2 A e_2 \cong k$, and $A M_k = e_1 A e_2 = k\beta \oplus k\alpha \beta \cong A_k$. Then $A M_k$ is an exchangeable
The Auslander-Reiten quiver of $\Lambda$ is

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1 \\
\end{array}
\]

where the two 1's represents the same module, and the two 2's also represents the same module. To compute $\mathcal{S}(A,M,B)$ and $\mathcal{F}(A,M,B)$, we need to write a $\Lambda$-module $N$ as the form $\left[ \frac{e_1 N}{e_2 N} \right]_\phi$, where $\phi : M \otimes_k e_2 N \rightarrow e_1 N$ is the $\Lambda$-map given by the $\Lambda$-actions:

$$
\beta \otimes_k e_2 n \mapsto \beta e_2 n, \quad \alpha \beta \otimes_k e_2 n \mapsto \alpha \beta e_2 n, \quad \forall \ n \in N.
$$

For example, the $\Lambda$-module $\begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array}$ has a basis $e_1, \alpha, u, v$ with the $\Lambda$-actions

\[
\begin{array}{cccc}
e_1 & e_2 & \alpha & u \\
e_1 & 0 & 0 & v \\
\alpha & 0 & 0 & 0 \\
\beta & 0 & 0 & \alpha e_1 \\
\alpha \beta & 0 & 0 & \alpha \\
\end{array}
\]

it follows that $\begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array} = \left[ \frac{e_1 \oplus \alpha e_1}{k u \oplus k v} \right]_\phi$ with the $\Lambda$-map $\phi$ given by the $\Lambda$-actions:

$$
\beta \otimes_k u \mapsto \beta u = \alpha, \quad \beta \otimes_k v \mapsto \beta v = e_1, \quad \alpha \beta \otimes_k u \mapsto \alpha \beta u = 0, \quad \alpha \beta \otimes_k v \mapsto \alpha \beta v = \alpha.
$$

So $\phi$ is not monic and hence $\begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array} \notin \mathcal{S}(A,M,B)$. Also, since the $\Lambda$-module $\begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array}$ has a basis $e_1, \alpha, u$ with the $\Lambda$-actions given by the table above, it follows that $\begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array} = \left[ \frac{e_1 \oplus \alpha e_1}{k u \oplus k v} \right]_\phi$ with the $\Lambda$-map $\phi$ given as above. So $\phi$ is not monic and hence $\begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array} \notin \mathcal{S}(A,M,B)$. In this way we see that $\mathcal{S}(A,M,B)$ has 3 indecomposable objects: $1, \begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array}$.

Similarly a $\Lambda$-module $N = \left[ \frac{e_1 N}{e_2 N} \right]_\phi \in \mathcal{F}(A,M,B)$ if and only if $\varphi : e_2 N \rightarrow \text{Hom}_A(M,e_1 N) = e_1 N$ is epic, where $\varphi$ is the image of $\phi : M \otimes_k e_2 N \rightarrow e_1 N$ under the adjunction isomorphism. Note that $\varphi : e_2 N \rightarrow e_1 N$ is exactly given by the actions of $\beta$. For example, since the $\Lambda$-module $\begin{array}{c}
2 \\
\end{array}$ has a basis $\alpha, u$ with the $\Lambda$-actions given by the table above, it follows that $\begin{array}{c}
2 \\
\end{array} = \left[ \frac{k \alpha}{k u} \right]_\phi$ with $\varphi : k u \rightarrow k \alpha$ given by $u \mapsto \beta u = \alpha$. So $\begin{array}{c}
2 \\
\end{array} \in \mathcal{F}(A,M,B)$. In this way we see that $\mathcal{F}(A,M,B)$ has 3 indecomposable objects: $\begin{array}{c}
2 \\
\end{array}, \begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array}, \begin{array}{c}
2 \\
\end{array}$.

There is a unique RSS equivalence given by $1 \mapsto \begin{array}{c}
2 \\
\end{array}, \begin{array}{c}
1 \\
\end{array} \mapsto \begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array}, \begin{array}{c}
2 \\
\end{array} \mapsto \begin{array}{c}
2 \\
\end{array}$. Note that $\mathcal{S}(A,M,B) = GP(\Lambda)$ and $\mathcal{F}(A,M,B) = GL(\Lambda)$, and this RSS equivalence is given by the Nakayama functor $N_\Lambda$. 


5.5. The following examples show that if $A M_B$ is not exchangeable, then the existence of an RSS equivalence can not be guaranteed. They also show that $\mathcal{S}(A, M, B)$ is not the separated monomorphism category of the corresponding quiver in the sense of [ZX], in general.

**Example 5.8.**

1. Let $A$ be the path algebra $k(2 \rightarrow 1)$, $B := k$, and $A M_k := A(e_1) = 01$. Then

   $$\Lambda := \begin{bmatrix} A M_k \end{bmatrix} = \begin{bmatrix} k & k \\ k & 0 \\ 0 & k \\ 0 & 0 \\ k & 0 \end{bmatrix}$$

   is just the path algebra $kQ$, where $Q$ is the quiver $2 \rightarrow 1 \leftarrow 3$. The Auslander-Reiten quiver of $\Lambda$-mod is

   ![Diagram](image)

   where we denote a $\Lambda$-module $V_2 \rightarrow V_1 \leftarrow V_3$ by $\dim V_2 \dim V_1$. Since

   $$D(AA_A \otimes_A M) = D(AA_A) \otimes_A A \sim D(e_1 A) \not\cong A \sim e_1 A = M \otimes_B D(B B_B),$$

   $A M_k$ is not an exchangeable bimodule. In fact, $A M_k$ also does not satisfy the condition (IP). The monomorphism category $\mathcal{S}(A, M, k)$ induced by $A M_k$ is

   $$\mathcal{S}(A, M, k) = \left\{ \begin{bmatrix} Y \end{bmatrix}_{\phi} \in \Lambda \text{-mod} \mid X \in \Lambda \text{-mod}, Y \in k \text{-mod}, M \otimes_k Y \xrightarrow{\phi} X \text{ is a monic } A \text{-map} \right\}.$$ 

   Thus $\mathcal{S}(A, M, k)$ has 5 indecomposable objects $01 \to 11 \to 01 \to 11 \to 00$. While the epimorphism category $\mathcal{F}(A, M, k)$ induced by $A M_k$ is

   $$\mathcal{F}(A, M, k) = \left\{ \begin{bmatrix} Y \end{bmatrix}_{\phi} \in \Lambda \text{-mod} \mid X \in \Lambda \text{-mod}, Y \in k \text{-mod}, Y \xrightarrow{\phi} e_1 X \text{ is an epic } k \text{-map} \right\}$$

   where $\phi := \eta_{X, Y}(\phi)$ and $\eta_{X, Y} : \text{Hom}_A(M \otimes_k Y, X) \cong \text{Hom}_k(Y, \text{Hom}_A(M, X))$ is the adjunction isomorphism.

   So $\mathcal{F}(A, M, k)$ has only 4 indecomposable objects $01 \to 11 \to 00 \to 10$. Thus $\mathcal{S}(A, M, k) \not\cong \mathcal{F}(A, M, k)$.

   Note that the indecomposable objects of the separated monomorphism category $\text{smor}(Q, 0, k)$ are exactly the indecomposable projective $\Lambda$-modules. See [ZX, Exam. 2.3]. So $\mathcal{S}(A, M, k) \not\cong \text{smor}(Q, 0, k)$.

2. Let $A$ and $B := k$ be as in (1), and $A M_k := S(2) = 10$. Since $A M$ is not projective, $A M_k$ is not exchangeable (note that $D(AA_A) \otimes_A M = D(AA_A) \otimes_A S(2) = D(AA_A) \otimes_A e_2 S(2) \cong D(e_2 A) = S(2) \cong M \otimes_B D(B B_B)$; and $A M_k$ satisfies the condition (IP)). Then

   $$\Lambda := \begin{bmatrix} A M_k \end{bmatrix} = \begin{bmatrix} k & k \\ k & 0 \\ 0 & k \\ 0 & 0 \\ k & 0 \end{bmatrix} \cong kQ/I$$

   with $Q = 3 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 1$ and $I = \langle \alpha \beta \rangle$. The Auslander-Reiten quiver of $\Lambda$-mod is

   ![Diagram](image)

   where a $\Lambda$-module $V_3 \rightarrow V_2 \rightarrow V_1$ is denoted by $\dim V_3 \dim V_2 \dim V_1$. The monomorphism category $\mathcal{S}(A, M, k) = \left\{ \begin{bmatrix} Y \end{bmatrix}_{\phi} \in \Lambda \text{-mod} \mid M \otimes_k Y \xrightarrow{\phi} X \text{ is a monic } A \text{-map} \right\}$ has 4 indecomposable objects $01 \to 11 \to 10 \to 00$. The epimorphism category $\mathcal{F}(A, M, k) = \left\{ \begin{bmatrix} Y \end{bmatrix}_{\phi} \in \Lambda \text{-mod} \mid Y \xrightarrow{\phi} \text{Hom}_A(S(2), X) \text{ is an epic } k \text{-map} \right\}$ has also 4
indecomposable objects \([01, 11, 10, 00]\). We claim that there are no RSS equivalences \(F : \mathcal{S}(A, M, k) \cong \mathcal{S}(A, M, k)\). Otherwise, by the definition of an RSS equivalence we get a contradiction

\[0 \neq \text{Hom}_A \left( \begin{bmatrix} 11 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \end{bmatrix} \right) = \text{Hom}_A \left( F \left( \begin{bmatrix} 11 \\ 0 \end{bmatrix} \right), F \left( \begin{bmatrix} 10 \\ 1 \end{bmatrix} \right) \right) = \text{Hom}_A \left( \begin{bmatrix} 11 \\ 0 \end{bmatrix}, \begin{bmatrix} 00 \\ 1 \end{bmatrix} \right) = 0.\]

By [ZX, Exam. 2.3], the indecomposable objects of the separated monomorphism category \(\text{sm}(Q, I, k)\) are exactly the indecomposable projective \(\Lambda\)-modules. So \(\mathcal{S}(A, M, k) \not\cong \text{sm}(Q, 0, k)\).

We propose the following problems.

1. What is a sufficient and necessary condition for the existence of an RSS equivalence?
2. Whether or not an RSS equivalence is unique?

**Appendix: computations of Step 2 in the proof of Theorem 1.3**

Let \(X \in A\)-mod and \(Y \in B\)-mod. To prove \(F(\begin{bmatrix} X \\ 0 \end{bmatrix}) \cong \left[ \text{Hom}_A(\begin{bmatrix} X \\ 0 \end{bmatrix}, \begin{bmatrix} M, X \end{bmatrix}) \right]_\varphi \) and \(F(\begin{bmatrix} M \otimes B Y \\ Y \end{bmatrix}) \cong \left[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right]_\varphi \), where \(\varphi : M \otimes_B \text{Hom}_A(M, X) \rightarrow X\) is the involution map, by the proof of Proposition 4.1 it suffices to prove

\[
\text{Hom}_A \left( \begin{bmatrix} X \\ 0 \end{bmatrix}, \begin{bmatrix} T \end{bmatrix} \right) \cong (DX, \text{DHom}_A(M, X))_{D(\alpha^{-1}_D X)} (6.1)
\]

where \(\alpha : D(DX \otimes M) \cong \text{Hom}_A(M, X)\) is the adjunction isomorphism, \(D(\alpha^{-1}_D X) : DX \otimes A M \rightarrow D\text{Hom}_A(M, X)\) is a right \(B\)-map, and

\[
\text{Hom}_A \left( \begin{bmatrix} M \otimes B Y \\ Y \end{bmatrix}_{\text{Id}}, \begin{bmatrix} T \end{bmatrix} \right) \cong (0, DY). (6.2)
\]

Put \(e_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}\). Then as left \(\Lambda\)-modules we have \(Te_1 = \begin{bmatrix} D(A) \end{bmatrix}_{\text{Id}}\) and \(Te_2 = \begin{bmatrix} M \otimes_B D(B) \end{bmatrix}_{\text{Id}} \cong \begin{bmatrix} D(A) \otimes_A B \end{bmatrix}_{\text{Id}} \).

For any left \(\Lambda\)-module \(L\), \(\text{Hom}_A(L, T)\) is a right \(\Lambda\)-module. Thus \(\text{Hom}_A(L, T)e_1 \cong \text{Hom}_A(L, T)e_2\) is a right \(A\)-module and \(\text{Hom}_A(L, T)e_2 \cong \text{Hom}_A(L, T)e_2\) is a right \(B\)-module, in the obvious way, and we have a right \(\Lambda\)-isomorphism

\[
\text{Hom}_A(L, T) \cong (\text{Hom}_A(L, T)e_1, \text{Hom}_A(L, T)e_2)_{\phi} (6.3)
\]

where \(\phi\) is explicit given by:

\[
\phi : \text{Hom}_A(L, T)e_1 \otimes A M \rightarrow \text{Hom}_A(L, T)e_2, \quad f \otimes A m \mapsto “l \mapsto f(l) \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) e_2“. (6.4)
\]

Applying (6.3) to \(L = \begin{bmatrix} M \otimes B Y \\ Y \end{bmatrix}_{\text{Id}}\) we get right \(\Lambda\)-isomorphisms:

\[
\text{Hom}_A \left( \begin{bmatrix} M \otimes B Y \\ Y \end{bmatrix}_{\text{Id}}, \begin{bmatrix} T \end{bmatrix} \right) = (\text{Hom}_A(\begin{bmatrix} M \otimes B Y \\ Y \end{bmatrix}_{\text{Id}}, \begin{bmatrix} T e_1 \end{bmatrix}), \text{Hom}_A(\begin{bmatrix} M \otimes B Y \\ Y \end{bmatrix}_{\text{Id}}, \begin{bmatrix} T e_2 \end{bmatrix})) = (0, \text{Hom}_B(Y, D(B))). \cong (0, DY).
\]

This proves (6.2).
Applying (6.3) to $L = [\frac{X}{0}]$ we get right $\Lambda$-isomorphisms:

$$\text{Hom}_{\Lambda}([\frac{X}{0}], T) = (\text{Hom}_{\Lambda}([\frac{X}{0}], T_{e1}), \text{Hom}_{\Lambda}([\frac{X}{0}], T_{e2}))$$

$$= (\text{Hom}_{\Lambda}([\frac{X}{0}], \left[\frac{D(Ae)}{0}\right]), \text{Hom}_{\Lambda}([\frac{X}{0}], \left[\frac{M\otimes B\otimes D(Be)}{D(Be)}\right]_{\alpha^{-1}}))$$

$$\cong (D X, \text{Hom}_{\Lambda}([\frac{X}{0}], \left[\frac{D(A)\otimes M}{D(Be)}\right]_{\alpha^{-1}}))$$

$$\cong (D X, \text{Hom}_{\Lambda}(X, D(A) \otimes_{A} M))_{\phi}$$

and by (6.4), $\phi$ is explicitly given by

$$\phi : D X \otimes_{A} M \longrightarrow \text{Hom}_{\Lambda}(X, D(A) \otimes_{A} M), \quad \alpha \otimes_{A} m \mapsto "x \mapsto \alpha_{x} \otimes_{A} m" \quad (6.5)$$

where $\alpha_{x} \in D A$ sends $a$ to $\alpha(ax)$.

To prove (6.1), it is clear that we need a right $B$-isomorphism $\psi : D \text{Hom}_{A}(M, X) \cong \text{Hom}_{A}(X, D(A) \otimes_{A} M)$. For this, we need to use the assumption that $A M$ is projective. Without loss of the generality, one can take $A M = Ae$ for some idempotent element $e \in A$. First, we have group isomorphisms

$$D \text{Hom}_{A}(M, X) = D \text{Hom}_{A}(Ae, X) \cong D(X)e \cong \text{Hom}_{A}(X, D(eA)) \cong \text{Hom}_{A}(X, D(A) \otimes_{A} M).$$

This isomorphism $\psi : D \text{Hom}_{A}(M, X) \cong \text{Hom}_{A}(X, D(A) \otimes_{A} M)$ of abelian groups is explicitly given by

$$\gamma \mapsto "x \mapsto \gamma_{x} \otimes_{A} e" \quad (6.6)$$

where $\gamma_{x} \in D A$ sends $a \in A$ to $\gamma(f_{a,x})$, and $f_{a,x} \in \text{Hom}_{A}(M, X)$ sends $m = ce \in M = Ae$ to $m$. We claim that $\psi$ is a right $B$-map, and hence $\psi$ is a right $B$-module isomorphism.

In fact, for each $b \in B$, suppose $eb = v_{b}e \in M = Ae$ for some $v_{b} \in A$. Then for each $x \in X$ we have

$$\psi(\gamma b)(x) = (\gamma b)_{x} \otimes e, \quad \text{with} \quad (\gamma b)_{x} \in D A$$

and

$$(\psi(\gamma b))(x) = \psi(\gamma)(x)b = \gamma_{x} \otimes_{A} eb = \gamma_{x} \otimes_{A} v_{b}e = \gamma_{x} v_{b}e \otimes_{A} e, \quad \text{with} \quad \gamma_{x} v_{b} \in D A.$$ 

Thus, it suffices to show $(\gamma b)_{x}(a) = (\gamma_{a} v_{b}e)(a) = \gamma_{x} v_{b}e(a)$ for each $a \in A$, i.e., $(\gamma b)(f_{a,x}) = \gamma(f_{v_{b}ea,x})$. That is $\gamma(bf_{a,x}) = \gamma(f_{v_{b}ea,x})$. This is really true, since both $bf_{a,x}$ and $f_{v_{b}ea,x}$ sends $m$ to

$$(bf_{a,x})(m) = f_{a,x}(mb) = mbax = ce(eb)ax = ce(v_{b}e)ax = mv_{b}ea = f_{v_{b}ea}(m).$$

This proves that $\psi : D \text{Hom}_{A}(M, X) \cong \text{Hom}_{A}(X, D(A) \otimes_{A} M)$ is a right $B$-module isomorphism.

So, we get the right $A$-isomorphism

$$(\text{Id}_{D X}, \psi^{-1}) : (D X, \text{Hom}_{A}(X, D(A) \otimes_{A} M))_{\psi, D(\alpha_{D X}^{-1})} \cong (D X, \text{D} \text{Hom}_{A}(M, X))_{D(\alpha_{D X}^{-1})},$$

where $\alpha_{D X} : D(D X \otimes M) \cong \text{Hom}_{A}(M, X)$ sends $\beta \in D(D X \otimes M)$ to $f \in \text{Hom}_{A}(M, X)$ such that

$$\beta(\alpha \otimes_{A} m) = \alpha f(m), \quad \forall \alpha \in D X, m \in M,$$

$\alpha_{D X}^{-1} : \text{Hom}_{A}(M, X) \longrightarrow D(D X \otimes M)$ is given by $f \mapsto "\beta : \alpha \otimes_{A} m \mapsto \alpha f(m)"$, and $D(\alpha_{D X}^{-1}) : D X \otimes_{A} M \longrightarrow D \text{Hom}_{A}(M, X)$ is given by

$$\alpha \otimes_{A} m \mapsto "\delta : f \mapsto \alpha f(m)".$$
ψ : D\text{Hom}_A(M, X) \cong \text{Hom}_A(X, D(A) \otimes_A M) is given by (6.6), and ψD(α_{D,X}^{-1}) : D(X \otimes_A M) \to \text{Hom}_A(X, D(A) \otimes_A M). To prove (6.1), it suffices to prove φ = ψD(α_{D,X}^{-1}). Thus by (6.6) we have

\[
(\psi D(α_{D,X}^{-1})(α \otimes_A m))(x) = ψ(δ)(x) = δ_x \otimes_A e
\]

where δ_x ∈ DA sends a ∈ A to δ(f_{a,x}) = αf_{a,x}(m) = α(max). Comparing with (6.5) we see δ_x = α_xce = α_xm, since α_xm(a) = α_x(α_xea) = α(max) = δ_x(a). It follows that

\[
(\psi D(α_{D,X}^{-1})(α \otimes_A m))(x) = δ_x \otimes_A e = α_xce \otimes_A e = α_x \otimes_A m = φ(α \otimes_A m)(x).
\]

This proves ψD(α_{D,X}^{-1}) = φ, and hence completes the proof.

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