Abstract: It is shown that harmonic functions from a simply connected domain in $\mathbb{R}^3$ to $\mathbb{R}^3$ cannot always be expressed as a sum of a monogenic (hyperholomorphic) function and an antimonogenic function, in contrast to the situation for complex numbers or quaternions. Harmonic functions orthogonal in $L^2$ to all such sums are termed “contragenic” and their properties are studied. A “Bergman kernel” and is derived, whose corresponding operator vanishes precisely on the contragenic functions. A graded orthonormal basis for the contragenic function in the ball $B^3$ is given.

Keywords: monogenic function, hyperholomorphic function, spherical harmonic, quaternion, contragenic function, homogeneous polynomial, Bergman kernel, Legendre polynomial, Riesz system.

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0 Introduction

The following fact in elementary complex analysis is well known.

**Theorem A.** Every harmonic function $u: \{ |z| < 1 \} \to \mathbb{C}$ of a complex variable is expressible as the sum of a holomorphic function and an antiholomorphic function.

This principle has many uses. In particular, when a holomorphic solution to a problem is sought and a first attempt is made as a harmonic function, one may “throw away” the antiholomorphic part to obtain a holomorphic approximation. A classical example of this principle in conformal mapping theory is found in the method of Fornberg [10], in which a guess as to the boundary values of the mapping is expressed on the boundary of the unit disk as a Fourier series, whose coefficients give the sum of a power series in $z$ (positive powers of $e^{i\theta}$) and in $\overline{z}$ (negative powers) in the interior of the disk. The sum of these two is a harmonic function, whose antiholomorphic part is discarded in the algorithm.

Theorem A holds in many generalizations of the field $\mathbb{C}$ of complex numbers, for example monogenic (hyperholomorphic) functions on quaternions [28] or on Clifford algebras [4,13]. It also holds for monogenic functions from $\mathbb{R}^3$ to $\mathbb{H}$ [4] [12].

In this paper we show (Theorem 3.1) that the natural generalization of Theorem A does not hold for monogenic functions from $\mathbb{R}^3$ to $\mathbb{R}^3$. Therefore, the class of harmonic functions which have no such decomposition is new. We will use the term “contragenic” for harmonic

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functions which are orthogonal to the monogenic and the antimonogenic functions in the sense of $L_2$. We believe they are of interest because of the relevance of monogenic functions in $\mathbb{R}^3$ to physical systems [24].

Thus we initiate here the study of contragenic functions. The precise definitions for monogenic functions and related notions in $\mathbb{R}^3$ are specified in section 1 based on work of [2, 5, 18, 17]. In section 2 we summarize the necessary facts about the standard basis for homogeneous monogenic polynomials in the unit ball of $\mathbb{R}^3$ and related spaces. We also calculate an orthonormal basis for the vector parts of the monogenic functions. In section 3 we prove the existence of contragenic functions and prove some of their basic properties. In particular we derive a Bergman kernel which annihilates precisely the contragenics. Finally, in section 4 we give an explicit construction for a graded basis for the space of contragenic functions.

1 Monogenic functions in $\mathbb{R}^3$

1.1 Notation

We will use fairly standard notation for the skew field of quaternions $\mathbb{H} = \mathbb{R}^4 = \{x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, x_j \in \mathbb{R}\}$. Here $e_0 = 1$ is the unit and the multiplication is determined by $e_j^2 = -1$ and $e_{j_1}e_{j_2} = \pm e_{j_3}$ where $\{j_1, j_2, j_3\}$ is the set of indices $\{1, 2, 3\}$ and the + sign is taken precisely when the cyclic order matches that of 1, 2, 3. See texts such as [12, 15, 28] for further general information. The $\mathbb{R}$-subspace of quaternions $x \in \mathbb{H}$ such that $e_3 = 0$ will be denoted $\mathbb{R}^3 \oplus \{0\}$ or more simply just $\mathbb{R}^3$ when there is no danger of confusion.

Let us write $\partial_j = \partial/\partial x_j$ for $j = 0, 1, 2, 3$. There is a great deal of literature (see [28] and references in [4]) on the Cauchy-Riemann type differential operators operators

$$\tilde{\partial}_3 = \sum_{j=1}^{3} \partial_j e_j,$$

$$D_\mathbb{H} = \partial_0 - \tilde{\partial}_3,$$

$$\overline{D}_\mathbb{H} = \partial_0 + \tilde{\partial}_3,$$

which act both from the left and from the right on differentiable functions $f = f_0 + f_1e_1 + f_2e_2 + f_3e_3$ defined in open subsets of $\mathbb{H}$. The operator $\overline{D}_\mathbb{H}$ is a generalization of the operator $\partial/\partial \overline{\mathbf{z}}$ on which complex analysis is based: functions for which $\overline{D}_\mathbb{H} f = 0$ (resp. $f \overline{D}_\mathbb{H} = 0$) are variously called left (resp. right) Fueter-regular, monogenic, hyperholomorphic, among others; occasionally the roles of $D_\mathbb{H}$ and $\overline{D}_\mathbb{H}$ are interchanged in the terminology.

In recent years some work has been done [2, 12] on the analogous functions from $\mathbb{R}^3$ to $\mathbb{R}^4$ and from $\mathbb{R}^4$ to $\mathbb{R}^3$, with a view to expressing and studying operators relevant to physics. Relatively little has been done for functions from $\mathbb{R}^3$ to $\mathbb{R}^3$; in this regard we mention [9, 18]. This setting is particularly interesting because even though such functions are not conserved under multiplication by elements of $\mathbb{R}^3 \oplus \{0\}$, i.e., the algebraic structure of an algebra or
ring is lost, in many ways the monogenic functions behave more like standard holomorphic functions in $\mathbb{C}$ (cf. Proposition 1.1). To be precise, for $x = x_0 + x_1 e_1 + x_2 e_2 \in \mathbb{R}^3$, let us write $\text{Sc } x = x_0$, $\text{Vec } x = x_1 e_1 + x_2 e_2$, $\overline{x} = x - \overline{x}$. Let $\Omega \subset \mathbb{R}^3$ be an open set, and $\mathcal{H}_\mathbb{R}(\Omega)$ the space of real-valued harmonic functions defined in $\Omega$. We consider the (Moisil-Theodorescu type) operators $D$ and $\overline{D}$, defined by

$$
\tilde{\partial} = \partial_1 e_1 + \partial_2 e_2, \quad D = \partial_0 - \tilde{\partial}, \quad \overline{D} = \partial_0 + \tilde{\partial},
$$

and define the set of (left-)monogenic (or hyperholomorphic) functions

$$
\mathcal{M}(\Omega) = \{ f = f_0 + f_1 e_1 + f_2 e_2 \in C^1(\Omega, \mathbb{R}^3) : \overline{D}f = 0 \}. \quad (1.2)
$$

The fact that $\mathcal{M}(\Omega) \subseteq \mathcal{H}(\Omega)$, where $\mathcal{H}(\Omega) = \mathcal{H}_\mathbb{R}(\Omega) \times \mathcal{H}_\mathbb{R}(\Omega) \times \mathcal{H}_\mathbb{R}(\Omega)$ is the set of $\mathbb{R}^3$-valued harmonic functions in $\Omega$, follows immediately from the factorization $\Delta = D\overline{D} = \overline{D}D$ of the Laplacian on $\mathbb{R}^3$.

### 1.2 Antimonogenic and ambigenics

We say that $f$ is (left) antimonogenic when $Df = 0$. A monogenic constant is a function which is simultaneously monogenic and antimonogenic: $Df = \overline{D}f = 0$, or equivalently, $\partial_0 f = \tilde{\partial} f = 0$.

It is unavoidable that $Df$ and $\overline{D}f$ need not take their values in $\mathbb{R}^3$ even when $f$ does. However, due to the fact that

$$
-e_3 \overline{D}f e_3 = -e_3 (\partial_0 + \tilde{\partial}) e_3 (-e_3)(f_0 + \tilde{f}) e_3 \\
= (\partial_0 - \tilde{\partial})(f_0 - \tilde{f}) \\
= D\tilde{f}
$$

we have

$$
\overline{D}f = 0 \iff D\tilde{f} = 0 \iff f\overline{D} = 0,
$$

and consequently,

**Proposition 1.1.** A function is left monogenic if and only if it is right monogenic. The set of conjugates of monogenic functions

$$
\overline{\mathcal{M}}(\Omega) = \{ \overline{f} : f \in \mathcal{M}(\Omega) \} \quad (1.3)
$$

coincides with the set of antimonogenic functions in $\Omega$.

This is of course quite different from the situation for monogenic functions in $\mathbb{H}$, where left- and right-monogenicity are different. The proposition allows us to write $\mathcal{M}(\Omega) \cap \overline{\mathcal{M}}(\Omega)$ for the set of monogenic constants. If $f \in \mathcal{M}(\Omega) \cap \overline{\mathcal{M}}(\Omega)$, then $\partial_0 f = \tilde{\partial} f = 0$. Thus a monogenic constant $f$ does not depend on $x_0$ and can be expressed as

$$
f = c_0 + f_1 e_1 + f_2 e_2 \quad (1.4)
$$
where $c_0 \in \mathbb{R}$ is constant and the quantity $f_1 - if_2$ is an ordinary holomorphic function of the complex variable $x_1 + ix_2$. There are natural projections of $\mathcal{M}(\Omega)$ onto the subspaces

$$
\text{Sc} \mathcal{M}(\Omega) = \{ \text{Sc} f : f \in \mathcal{M}(\Omega) \} \subseteq \mathcal{H}_\mathbb{R}(\Omega),
$$

$$
\text{Vec} \mathcal{M}(\Omega) = \{ \text{Vec} f : f \in \mathcal{M}(\Omega) \} \subseteq \mathcal{H}_{\{0\} \oplus \mathbb{R}^2}(\Omega),
$$

and by Proposition 1.1 we see that $\text{Sc} \mathcal{M}(\Omega) = \text{Sc} \mathcal{M}(\Omega)$, $\text{Vec} \mathcal{M}(\Omega) = \text{Vec} \mathcal{M}(\Omega)$. An element of $\mathcal{M}(\Omega) + \mathcal{M}(\Omega)$ will be called an ambigenic function; its decomposition as a sum of a monogenic and an antimonogenic function is unique up to the addition of a monogenic constant.

Consider $L_2(\Omega, \mathbb{R}^3)$ with the real-valued inner product $\langle f, g \rangle = \text{Sc} \int_\Omega \overline{f}g \, dV = \int_\Omega (f_0g_0 + f_1g_1 + f_2g_2) \, dV$. Write

$$
\mathcal{M}_2(\Omega) = \mathcal{M}(\Omega) \cap L_2(\Omega).
$$

It is somewhat inconvenient that $\mathcal{M}_2(\Omega)$ is not orthogonal to $\mathcal{M}_2(\Omega)$. However, we have automatically that $\text{Sc} \mathcal{M}_2(\Omega) \perp \text{Vec} \mathcal{M}_2(\Omega)$ since any scalar function multiplied by $e_0$ is by definition orthogonal to any combination of $e_1$ and $e_2$. This fact gives us an orthogonal direct sum decomposition of the space of square-integrable ambigenic functions,

$$
\mathcal{M}_2(\Omega) + \mathcal{M}_2(\Omega) = \text{Sc} \mathcal{M}_2(\Omega) \oplus \text{Vec} \mathcal{M}_2(\Omega). \quad (1.5)
$$

We will always assume that $\Omega \subseteq \mathbb{R}^3$ is connected.

**Lemma 1.2.** [17] Suppose $\Omega$ is simply connected. (a) Let $f_0 \in \mathcal{H}_\mathbb{R}(\Omega)$. Then there exists $f \in \mathcal{M}(\Omega)$ such that $\text{Sc} f = f_0$. This $f$ is unique up to an additive monogenic constant.

(b) Let $f_1, f_2 \in \mathcal{H}_\mathbb{R}(\Omega)$. A necessary and sufficient condition for there to exist $f = f_0 + \tilde{f} \in \mathcal{M}(\Omega)$ such that $\text{Sc} f = f_0$, $\tilde{f} = f_1 e_1 + f_2 e_2$, is that $\partial_2 f_1 = \partial_1 f_2$. When it exists, this $f$ is unique up to an additive scalar constant.

When the operator $D$ is applied exclusively to scalar-valued harmonic functions, i.e.

$$
D : \mathcal{H}_\mathbb{R}(\Omega) \to \mathcal{M},
$$

we see from (1.5) and Lemma 1.2 that this operator splits naturally to give exact sequences

$$
0 \to \mathbb{R} \to \mathcal{H}_\mathbb{R}(\Omega) \xrightarrow{\partial} \text{Sc} \mathcal{M} \to 0, \quad (1.6)
$$

$$
0 \to \ker \tilde{\partial} \to \mathcal{H}_\mathbb{R}(\Omega) \xrightarrow{\tilde{\partial}} \text{Vec} \mathcal{M} \to 0. \quad (1.7)
$$

Here $\ker \tilde{\partial}$ is two-dimensional over $\mathbb{R}$, consisting only of polynomials $h(x) = c_0 + c_1 x_0$.

**Note 1.3.** One could equally well embed $\mathbb{R}^3$ in $\mathbb{H}$ differently, for example by considering $\{0\} \oplus \mathbb{R}^3 \subseteq \mathbb{H}$, thus writing

$$
\tilde{x} = \tilde{x}_1 e_1 + \tilde{x}_2 e_2 + \tilde{x}_3 e_3 = -e_3 x = x_2 e_1 - x_1 e_2 + x_0 e_3
$$

$$
\tilde{f} = \tilde{f}_1 e_1 + \tilde{f}_2 e_2 + \tilde{f}_3 e_3 = e_3 f = -f_2 e_1 + f_1 e_2 - f_0 e_3
$$

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and using the operators

\[
\bar{D} = e_1 \partial_1 - e_2 \partial_2 - e_3 \partial_3, \quad \overline{D} = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3.
\]

The relationship \(\overline{D} \tilde{f}|_{\tilde{x}} = Df|_{e \tilde{x}}\) implies

\[
Df = 0 \iff \overline{D} \tilde{f} = 0 \iff \text{grad } \tilde{f} = 0, \text{ curl } \tilde{f} = 0
\]

and consequently this alternative embedding, which is used for example in [13, 9], is equivalent to the form we are using in this article. Our spaces \(\mathcal{M}(\Omega), \overline{\mathcal{M}(\Omega)}\) correspond to the spaces of left- and right-monogenic functions \(\tilde{f}\) in the sense of \(\overline{D}\). In this context the equations defining monogenicity are also known as a Riesz system [7, 17, 18, 22].

## 2 Homogeneous monogenics

We will mostly work in the ball \(B^3 = \{ |x| < 1 \} \subseteq \mathbb{R}^3\). The real vector space \(\mathcal{H}_R(\mathbb{B}^3) \cap L_2(\mathbb{B}^3)\) is stratified into the subsets \(\mathcal{H}_R^{(n)}(\mathbb{B}^3)\) comprised of real harmonic functions which are homogeneous of successive degrees \(n = 0, 1, \ldots\). It is well known that the elements of \(\mathcal{H}_R^{(n)}(\mathbb{B}^3)\) are polynomials of degree \(n\) and that these real linear subspaces are orthogonal with respect to the inner product \(\langle \cdot, \cdot \rangle\). Further, every square integrable harmonic function on \(\mathbb{B}^3\) has a unique expression as a series formed of elements of these sets.

### 2.1 Basis for \(\mathcal{M}_2(\mathbb{B}^3)\)

Many schemes have been devised to construct bases for spaces of homogeneous monogenic functions of given degree, from the classical construction of Fueter to diverse applications of symmetric sums of products; see [5, 12, 10]. For \(\mathbb{R}^3\) one way is to proceed as follows. A well known orthonormal basis of \(\mathcal{H}_R^{(n)}(\mathbb{B}^3), n \geq 0,\) is the system of \(2n + 1\) solid spherical harmonics

\[
\hat{U}_0^n, \hat{U}_1^n, \ldots, \hat{U}_n^n, \hat{V}_1^n, \ldots, \hat{V}_n^n,
\]

where \(\hat{U}_m^n = r^n U_m, \hat{V}_m^n = r^n V_m\), are defined in terms of spherical coordinates \(x_0 = r \cos \theta, x_1 = r \sin \theta \cos \varphi, x_2 = r \sin \theta \sin \varphi\) via the relations

\[
\begin{align*}
U_0^n(\theta, \varphi) &= P_n(\cos \theta), \\
U_m^n(\theta, \varphi) &= P_n^m(\cos \theta) \cos(m\varphi), \\
V_m^n(\theta, \varphi) &= P_n^m(\cos \theta) \sin(m\varphi), \quad m = 1, 2, \ldots, n.
\end{align*}
\]

Here \(P_n\) is the Legendre polynomial of degree \(n\) and \(P_n^m\) is the associated Legendre function is given by

\[
P_n^m(t) = (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_n(t).
\]
We will need the following standard identities \([21]\):

\[
(1 - t^2)(P_{n+1}^m)'(t) = (n + m + 1)P_n^m(t) - (n + 1)tP_{n+1}^m(t),
\]
\[
(1 - t^2)^{1/2}(P_{n+1}^m)'(t) = P_{n+1}^m(t) - m(1 - t^2)^{-1/2}tP_{n+1}^m(t),
\]
\[
(1 - t^2)^{1/2}P_{n+1}^m(t) = \frac{1}{2n + 3}(P_{n+2}^{m+1}(t) - P_{n+1}^{m-1}(t)),
\]
\[
2mtP_{n+1}^m(t) = (1 - t^2)^{1/2}(P_{n+1}^{m+1}(t) + (n + 1)(n - m + 2)P_{n+1}^{m-1}(t)),
\]
\[
(n - m + 1)P_{n+1}^m(t) = (2n + 1)tP_n^m(t) - (n + m)P_{n-1}^m(t).
\]

The spherical harmonics \([2,3]\) are polynomials when expressed in cartesian coordinates \((x_0, x_1, x_2)\). One obtains a basis for the space \(\mathcal{M}^{(n)}(\mathbb{B}^3)\) of homogeneous monogenic functions of degree \(n\) formed by the \(2n + 3\) polynomials:

\[
X_n^m = D[\hat{U}_{n+1}^m], \quad m = 0, \ldots, n + 1,
\]
\[
Y_n^m = D[\hat{V}_{n+1}^m], \quad m = 1, \ldots, n + 1,
\]

These are monogenic by construction due to the factorization of the Laplacian. A detailed explanation of the analogous construction for \(\mathbb{R}^3 \rightarrow \mathbb{H}\) is found in \([5,12]\); the specific construction for \(\mathbb{R}^3 \rightarrow \mathbb{R}^3\) given here appears in \([17,18]\) and it is shown that \(X_n^m, Y_n^m\) form an orthogonal basis for \(\mathcal{M}^{(n)}(\mathbb{B}^3)\). For reference we recall that the derivation \([2,8]\) produces the monogenic basis elements in the following form,

\[
X_n^m = r^n \left( A_n^m \cos m\varphi + (B_n^m \cos \varphi \cos m\varphi - C_n^m \sin \varphi \sin m\varphi) e_1 
+ (B_n^m \sin \varphi \cos m\varphi + C_n^m \cos \varphi \sin m\varphi) e_2 \right),
\]
\[
Y_n^m = r^n \left( A_n^m \sin m\varphi + (B_n^m \cos \varphi \sin m\varphi + C_n^m \sin \varphi \cos m\varphi) e_1 
+ (B_n^m \sin \varphi \sin m\varphi - C_n^m \cos \varphi \sin m\varphi) e_2 \right),
\]

where

\[
A_n^m = \frac{1}{2} \left( (1 - t^2)(P_{n+1}^m)'(t) + (n + 1)tP_{n+1}^m(t) \right) |_{t = \cos \theta},
\]
\[
B_n^m = \frac{1}{2} \left( \sqrt{1 - t^2} t(P_{n+1}^m)'(t) - (n + 1)\sqrt{1 - t^2}P_{n+1}^m(t) \right) |_{t = \cos \theta},
\]
\[
C_n^m = \frac{m}{2\sqrt{1 - t^2}} P_{n+1}^m(t) |_{t = \cos \theta}.
\]

(We observe that other authors have used notation such as \(X_n^{m,\dagger}, Y_n^{m,\dagger}\) where we write \(X_n^m, Y_n^m\), in order to stress that these functions are defined in \(\mathbb{B}^3\) rather than on \(S^2\).) Finally in \(L_2(\mathbb{B}^3, \mathbb{R}^3)\) we consider the norm \(\|f\|_2 = \sqrt{\langle f, f \rangle}\), where the scalar product is over \(\mathbb{B}^3\) as in section \([1,2]\) (rather than over \(S^2\) as is the case of some other authors). The norms of the
solid spherical harmonics and orthogonal monogenic functions are given by

\[ \|\hat{U}_n\|_2 = \sqrt{\frac{4\pi}{(2n+1)(2n+3)}} \]
\[ \|\hat{V}_m\|_2 = \sqrt{\frac{2\pi}{(2n+1)(2n+3)} \frac{(n+m)!}{(n-m)!}} \],

(2.11)

\[ \|X_n\|_2 = \sqrt{\frac{\pi(n+1)}{2n+3}} \]
\[ \|Y_m\|_2 = \sqrt{\frac{\pi(n+1)(n+m+1)!}{2(2n+3)(n-m+1)!}} \],

(2.12)

when \( m \geq 1 \).

The following explicit representation of the basis elements of \( \mathcal{M}^{(n)}(\mathbb{B}^3) \) in terms of spherical harmonics is stated in [18] without proof. Since our results depend on this representation, we will prove it here in detail.

**Theorem 2.1.** Write

\[ c_m^n = \frac{(n+m)(n+m+1)}{4} \]

For each degree \( n \geq 1 \), the basis elements for the homogeneous monogenic polynomials of degree \( n \) are given by

\[ X_n = \frac{(n+1)}{2} \hat{U}_n + \frac{1}{2} \hat{U}_1 e_1 + \frac{1}{2} \hat{V}_1 n e_2, \]
\[ X_m^n = \frac{(n+m+1)}{2} \hat{U}_m - (c_m^n \hat{U}_{m-1} - \frac{1}{4} \hat{V}_{m+1}^n) e_1 + (c_m^n \hat{V}_{m-1} - \frac{1}{4} \hat{U}_{m+1}^n) e_2, \]
\[ Y_m^n = \frac{(n+m+1)}{2} \hat{V}_m - (c_m^n \hat{V}_{m-1} - \frac{1}{4} \hat{U}_{m+1}^n) e_1 - (c_m^n \hat{U}_{m-1} - \frac{1}{4} \hat{V}_{m+1}^n) e_2, \]

(2.13)

where \( 1 \leq m \leq n + 1 \).

**Proof.** Recalling (1.6), (1.7) we see that since \( \hat{U}_m^n \) and \( \hat{V}_m^n \) are scalar valued, the definition (2.8) may be expressed as

\[ X_m^n = \partial_0 \hat{U}_m^n - \partial_1 \hat{U}_m^n e_1 - \partial_2 \hat{U}_m^n e_2, \]
\[ Y_m^n = \partial_0 \hat{V}_m^n - \partial_1 \hat{V}_m^n e_1 - \partial_2 \hat{V}_m^n e_2, \]

so the components of \( X_m^n, Y_m^n \) given in (2.13) are precisely the partial derivatives of \( \hat{U}_m^n, \hat{V}_m^n \). Consider the formula proposed for \( X_0^n \) in (2.13); i.e., assume for the moment \( m = 0 \). By (2.9), it is necessary to prove the three equalities

\[ \frac{(n+1)}{2} \hat{U}_0^n = A_0^n, \]
\[ \frac{1}{2} \hat{U}_1^n = B_0^n \cos \varphi, \]
\[ \frac{1}{2} \hat{V}_1^n = B_0^n \sin \varphi. \]

(2.14)
Substitute the definitions (2.2) and (2.10) into these equations. We see that (2.14) is given immediately by (2.3). To verify equations (2.15), (2.16) we must prove that

\[(1 - t^2)^{1/2}t(P_{n+1}') - (n + 1)(1 - t^2)^{1/2}P_{n+1} - P_n = 0.\]

where for brevity we write \(P_{n+1}\) in place of \(P_{n+1}(t)\). The left hand side may be expressed as

\[(1 - t^2)^{1/2}(t(P_{n+1}') - (n + 1)P_{n+1} - (P_n)')\]

and when we substitute the values for \((P_n)', (P_{n+1})'\) given by (2.3) we the result is zero according to (2.7). Thus (2.15) and (2.16) hold.

Similarly the desired formula for \(X_m^n, 1 \leq m \leq n + 1\), is equivalent to proving the three equalities

\[
\frac{(n + m + 1)}{2}U_m^n = A_m^n \cos m\varphi, \quad (2.17)
\]

\[-c_m^n U_{m-1}^n + \frac{1}{4}U_m^n = B_m^n \cos \varphi \cos m\varphi - C_m^n \sin \varphi \sin m\varphi, \quad (2.18)
\]

\[c_m^n V_{m-1}^n + \frac{1}{4}V_m^n = B_m^n \sin \varphi \cos m\varphi + C_m^n \cos \varphi \sin m\varphi. \quad (2.19)
\]

As before, substituting the definitions (2.2) and (2.10) we obtain again that (2.17) reduces immediately to (2.3). For equation (2.18) we equate the coefficients of \(\cos \varphi \cos m\varphi\) and of \(\sin \varphi \sin m\varphi\) on both sides; for (2.19) we use the coefficients of \(\sin \varphi \cos m\varphi\) and \(\cos \varphi \sin m\varphi\). From this it is seen that proving equations (2.18) and (2.19) is equivalent to proving

\[
(n + m)(n + m + 1)P_n^{m-1} - P_n^{m+1} + 2(1 - t^2)^{1/2}t(P_{n+1}^m)' - 2(n + 1)(1 - t^2)^{1/2}P_{n+1}^m = 0, \quad (2.20)
\]

\[
(n + m)(n + m + 1)P_n^{m-1} + P_n^{m+1} - 2m(1 - t^2)^{-1/2}P_{n+1}^m = 0. \quad (2.21)
\]

Let us verify (2.20). Eliminate the derivative \((P_{n+1}^m)'\) via (2.3) to convert the left hand side into

\[
(n + m)(n + m + 1)P_n^{m-1} - P_n^{m+1} + 2(n + m + 1)t(1 - t^2)^{-1/2}P_{n+1}^m - 2(n + 1)(1 - t^2)^{1/2}P_{n+1}^m.
\]

We note that (2.6) provides a value for \((n + m)(n - m + 1)P_n^{m-1}\), so we are led to decompose \(n + m + 1 = 2(n + 1) - (n - m + 1)\) and to arrange the terms as follows,

\[2(n + m)(n + 1)P_n^{m-1} - (P_n^{m+1} + (n + m)(n - m + 1)P_n^{m-1}) + 2(1 - t^2)^{-1/2}((n + m + 1)tP_{n+1}^m - (n + 1)P_{n+1}^m), = 2(n + m)(n + 1)P_n^{m-1} - (1 - t^2)^{-1/2}((n - m + 1)tP_{n+1}^m - 2(n + 1)P_{n+1}^m).\]

Multiply this by \((1 - t^2)^{1/2}\), and then substitute in the first term the value for \((1 - t^2)^{1/2}P_{n+1}^{m-1}\) provided by (2.5) to arrive at

\[(n + m)(P_{n+1}^m - P_{n-1}^m) + (2n + 1)tP_n^m - (2n + 1)P_{n+1}^m\]
which clearly vanishes by (2.7), thus proving (2.20).

Now we verify equation (2.21). Its left hand side can be written as

\[ 2m(n+m)P_{n}^{m-1} - 2m(1-t^2)^{-1/2}P_{n+1}^{m} + P_{n}^{m+1} + (n+m)(n-m+1)P_{n}^{m-1} \]

and by applying (2.6) to the last two terms it is equal to

\[ 2m(n+m)P_{n}^{m-1} - 2m(1-t^2)^{-1/2}(P_{n+1}^{m} - tP_{n}^{m}). \]

We may divide by \(2m(1-t^2)^{1/2}\). Using (2.5) (with \(n-1, m-1\) in place of \(m, n\)) to replace the first term, the result is zero by (2.7). This proves (2.21) and thus the formula for \(X_m^n\). The formula for \(Y_m^n\) likewise reduces to (2.20) and (2.21), so (2.13) is verified for all cases. \(\square\)

Theorem 2.1 immediately yields a basis for \(\text{Vec } \mathcal{M}\), which turns out to be orthogonal:

**Proposition 2.2.** For each \(n \geq 0\), the set

\[ \{ \text{Vec } X_m^n = \tilde{\partial}U_{m+1}, \ 0 \leq m \leq n+1 \} \cup \{ \text{Vec } Y_m^n = \tilde{\partial}V_{m+1}, \ 1 \leq m \leq n+1 \} \]

is an orthogonal basis for \(\text{Vec } \mathcal{M}^{(n)}(\mathbb{B}^3)\). The union of these sets over all \(n \geq 0\) is an orthogonal basis for \(\text{Vec } \mathcal{M}(\mathbb{B}^3)\), and the norms of the basis elements are given by

\[
\| \text{Vec } X_0^n \| = \sqrt{\frac{\pi n(n+1)}{(2n+1)(2n+3)}}, \\
\| \text{Vec } X_m^n \| = \| \text{Vec } Y_m^n \|^2 = \sqrt{\frac{\pi(n^2 + m^2 + n)(n+m+1)!}{2(2n+1)(2n+3)(n-m+1)!}}, \quad m \geq 1.
\]

**Proof.** The given set is clearly a basis of \(\text{Vec } \mathcal{M}^{(n)}(\mathbb{B}^3)\) because of (1.7). We need to see that it is orthogonal. Choose elements \(f, g\) in the basis for the monogenics \(\{X_0^n, X_m^n, Y_m^n\}\), and express them as \(f = f_0 + \hat{f}\), \(g = g_0 + \hat{g}\), with scalar parts \(f_0, g_0\). Clearly

\[ \langle f, g \rangle = \langle f_0, g_0 \rangle + \langle \hat{f}, \hat{g} \rangle. \]

By Theorem 2.1 the scalar parts of the monogenics run through (scalar multiples of) the spherical harmonics, and thus are orthogonal. Suppose \(f \neq g\). Then \(\langle f, g \rangle = 0\) and \(\langle f_0, g_0 \rangle = 0\), so \(\langle \hat{f}, \hat{g} \rangle = 0\) as desired. To calculate the norms, now suppose \(f = g\). Then \(\langle \hat{f}, \hat{f} \rangle = \|f\|^2 - \|f_0\|^2\). Thus if \(f = X_0^n\), then by (2.11), (2.12), and (2.13)

\[ \langle \hat{f}, \hat{f} \rangle = \frac{\pi(n+1)}{(2n+3)} - \frac{(n+1)^2}{4} \frac{4\pi}{(2n+1)(2n+3)} = \frac{\pi n(n+1)}{(2n+1)(2n+3)}. \]

Similarly, if \(f = X_m^n\) or \(f = Y_m^n\) for \(m \geq 1\),

\[
\langle \hat{f}, \hat{f} \rangle = \frac{\pi(n+1)(n+m+1)!}{2(2n+3)(n-m+1)!} - \frac{(n+m+1)^2}{4} \frac{2\pi(n+m)!}{(2n+1)(2n+3)(n-m)!} = \frac{\pi(n^2 + m^2 + n)(n+m+1)!}{2(2n+1)(2n+3)(n-m+1)!}. \]

\(\square\)
We now give a basis for the ambigenic functions. It must be noted that the monogenic constants $X_{n+1}^n$ and $Y_{n+1}^n$ are the negatives of their conjugates, so care must be taken to count them only once. Thus the following functions thus form a (not orthogonal) basis for the ambigenic functions on $\mathbb{B}^3$:

$$X_0^n, X_1^n, \ldots, X_n^n, X_{n+1}^n, Y_1^n, \ldots, Y_n^n, \overline{X_0^n}, \overline{X_1^n}, \ldots, \overline{X_n^n}, \overline{Y_1^n}, \ldots, \overline{Y_n^n}, Y_{n+1}^n.$$  \hspace{1cm} (2.22)

**Proposition 2.3.** Let $n > 0$. The $4n + 4$ functions

$$X_{m+}^n := X_m^n, \quad m = 0, \ldots, n + 1,$n+1

$$Y_{m+}^n := Y_m^n, \quad m = 1, \ldots, n,$$n+1

$$X_{m-}^n := \overline{X_m^n} - a_m^n X_m^n, \quad m = 0, \ldots, n,$$n+1

$$Y_{m-}^n := \overline{Y_m^n} - a_m^n Y_m^n, \quad m = 1, \ldots, n + 1,$$n+1

where

$$a_m^n = \frac{n - 2m^2 + 1}{(n + 1)(2n + 1)} \quad (0 \leq m \leq n), \quad a_{n+1}^n = 0,$n+1

form an orthogonal basis for the space of square integrable ambigenic functions on $\mathbb{B}^3$ which are homogeneous of degree $n$.

**Proof.** Since these are $4n + 4$ ambigenic functions, it suffices to prove the orthogonality. First we calculate the scalar products of each $X_{m+}^n, Y_{m+}^n$ and its conjugate. By Theorem 2.1 and (2.11) we obtain

$$\langle X_0^n, X_0^n \rangle = \frac{\pi (n + 1)}{(2n + 1)(2n + 3)},$$

$$\langle X_m^n, X_m^n \rangle = \langle Y_m^n, Y_m^n \rangle = \frac{\pi (n - 2m^2 + 1)(n + m + 1)!}{2(2n + 1)(2n + 3)(n - m + 1)!}.$$n+1

for $1 \leq m \leq n + 1$. Since the set $\{X_0^n, X_m^n, Y_m^n, m = 1, \ldots, n + 1\}$ is an orthogonal basis of $\mathcal{M}^{(n)}(\mathbb{B}^3)$, it follows at once that

$$\langle X_{m+}^n, Y_{l+}^n \rangle = \langle X_{m+}^n, Y_{l-}^n \rangle = \langle Y_{m+}^n, X_{l+}^n \rangle = \langle Y_{m+}^n, X_{l-}^n \rangle = \langle X_{m-}^n, Y_{l-}^n \rangle = 0.$$

Further, by (2.12)

$$\langle X_{m+}^n, X_{l+}^n \rangle = \left\{ \begin{array}{ll}
\frac{\pi (n + 1)}{2n + 3} & \text{if } m = l = 0, \\
\frac{\pi (n + 1)(n + m + 1)!}{2(2n + 3)(n - m + 1)!} & \text{if } m = l \neq 0, \\
0 & \text{if } m \neq l.
\end{array} \right.$$n+1

Substituting (2.12) and (2.24) we find that

$$\langle X_{m+}^n, X_{l-}^n \rangle = \langle X_m^n, X_l^n \rangle - a_l^n \langle X_m^n, X_l^n \rangle = 0$$n+1
for all $l,m$, and also that
\[
\langle X^n_m, X^n_l \rangle = \langle X^n_m, X^n_l \rangle - a^m_l (X^n_m, X^n_l) + a^m_n a^n_l (X^n_m, X^n_l)
\]
\[
= \begin{cases} 
\frac{4\pi n(n+1)^2}{(2n^2+3)(2n^2+1)} & \text{if } m = l = 0, \\
\frac{2\pi(n^2+m^2+n(n+m+1)(n+m+1)!}{(n+1)(2n^2+3)(2n^2+1)^2(n-m)!} & \text{if } m = l \neq 0, \\
0 & \text{if } m \neq l.
\end{cases}
\]

The calculation of the scalar products for $\{Y^n_m\}$, as well as for the mixed cases, is similar.

3 Contragenic functions in $\mathbb{R}^3$

3.1 Existence of contragenics

From now on we will abbreviate $M^{(n)} = M^{(n)}(\mathbb{B}^3)$, $M^{(n)} = \overline{M}^{(n)}(\mathbb{B}^3)$, $H^{(n)} = \mathcal{H}^{(n)}(\mathbb{B}^3)$. Because of the correspondence of monogenic constants with holomorphic functions described in section 1, and since the real homogeneous harmonic polynomials of degree $n$ in the complex variable $x + iy$ are linear combinations of $\text{Re} (x + iy)^n$ and $\text{Im} (x + iy)^n$, the space $M^{(n)} \cap \overline{M}^{(n)}$ is 2-dimensional over $\mathbb{R}$ for $n \geq 1$. We summarize in Table 1 the dimensions over $\mathbb{R}$ of the relevant spaces of functions in $\mathbb{B}^3$. Recall that $\mathcal{H}$ denotes $\mathbb{R}^3$-valued functions.

| Space of polynomials | $n = 0$ | $n \geq 1$ |
|----------------------|---------|------------|
| $\mathcal{H}^{(n)}$  | 1       | $2n + 1$   |
| $M^{(n)}$, $\overline{M}^{(n)}$ | 3 | $2n + 3$ |
| $M^{(n)} \cap \overline{M}^{(n)}$ | 3 | 2 |
| $M^{(n)} + \overline{M}^{(n)}$ | 3 | $4n + 4$ |
| $\mathcal{H}^{(n)}$ | 3 | $6n + 3$ |

Table 1: Dimensions over $\mathbb{R}$ of spaces related to monogenic polynomials in $\mathbb{B}^3$.

From the last two rows of this table comes the following notable fact.

**Theorem 3.1.** Not all $L^2$-harmonic functions are ambigenic: $M_2 + \overline{M}_2$ is a proper subspace of $\mathcal{H}(\mathbb{B}^3) \cap L^2(\mathbb{B}^3)$.

**Definition 3.2.** In any domain $\Omega$, a harmonic function $h \in H(\Omega) \cap L^2(\Omega)$ is called contragenic when it is orthogonal to all square-integrable ambigenic functions, i.e., if it lies in
\[
\mathcal{N}(\Omega) = (M_2(\Omega) + \overline{M}_2(\Omega))^\perp
\]
where the orthogonal complement is taken in $\mathcal{H}(\Omega) \cap L_2(\Omega)$. In $\mathbb{B}^3$ we have the orthogonal complements in the spaces of homogeneous harmonic polynomials $\mathcal{H}^{(n)}$

$$\mathcal{N}^{(n)} = (\mathcal{M}_{2}^{(n)} + \overline{\mathcal{M}_{2}^{(n)}})^{\perp}.$$  

As observed in the introduction, there is no direct analogy to be found for contragenic functions in hypercomplex analysis on $\mathbb{C}$ or $\mathbb{H}$ or more general Clifford algebras since in those contexts all harmonic functions are known to be ambigenic.

Let $n \geq 1$. It follows from (1.4) that $\mathcal{M}^{(n)}(\Omega) \cap \overline{\mathcal{M}^{(n)}(\Omega)} \subseteq \text{Vec} \mathcal{M}^{(n)}(\Omega)$. By Lemma 1.2 when $\Omega$ is simply connected we have $\text{Sc} \mathcal{M}^{(n)}(\Omega) = \mathcal{H}^{(n)}(\Omega)$. Returning to $\mathbb{B}^3$, we have specifically $\dim \text{Sc} \mathcal{M}^{(n)} = \dim \mathcal{H}^{(n)} = 2n + 1$. From (1.5) it follows that $\dim \text{Vec} \mathcal{M}^{(n)} = 2n + 3$. (Since this is equal $\dim \mathcal{M}^{(n)}$, this means that given the vectorial part $\vec{f}$, the scalar part $f_0$ is uniquely determined.) Thus

$$\dim \mathcal{N}^{(n)} = 2n - 1$$

when $n \geq 1$, while $\dim \mathcal{N}^{(0)} = 0$.

The following result is a simple consequence.

**Theorem 3.3.** Let $h \in \mathcal{N} = \mathcal{N}(\mathbb{B}^3)$. Then $h$ can be uniquely expressed as a sum in $L_2$

$$h = \sum_{n=1}^{\infty} h^{(n)}$$

where $h^{(n)} \in \mathcal{N}^{(n)}$.

**Note 3.4.** It is easily checked by a dimension count that the analogue of Theorem 3.1 for “clasical” monogenic functions $\mathbb{H} \to \mathbb{H}$ does not hold; i.e., all harmonics are ambigenic over $\mathbb{H}$. For $n \geq 0$ one has that the homogeneous monogenics in $\mathbb{B}^4$ form a right vector space over $\mathbb{H}$ of dimension $\frac{1}{2}(n+1)(n+2)$ [28], and the same is true for the antimonogenics. The monogenic constants have dimension $n + 1$ over $\mathbb{H}$ (See example [2]). Since the dimension of the harmonics from $\mathbb{H}$ to $\mathbb{H}$ is $(n + 1)^2$ over $\mathbb{H}$ [28], it follows that every harmonic function can be expressed as a sum of a monogenic function and an antimonogenic function.

It may also shed light on the situation to see what fails when one attempts to express a harmonic $\mathbb{R}^3$-valued function in terms of monogenics and antimonogenics. For scalar-valued $f_0 \in \mathcal{H}_R(\Omega)$ there is in fact no problem, since

$$f_0 = \frac{1}{2}(f_0 + \bar{g}_0) + \frac{1}{2}(f_0 - \bar{g}_0)$$

where $f_0 + \bar{g}_0$ is the completion of $f_0$ to a monogenic function as given by Lemma 1.2. When we are given a general harmonic function $f_0 + f_1 e_1 + f_2 e_2 \in \mathcal{H}(\Omega)$ and complete each component separately, we obtain analogously

$$f_0 + f_1 e_1 + f_2 e_2 = \frac{1}{2}((f_0 + \bar{g}_0) + (f_1 + \bar{g}_1)e_1 + (f_2 + \bar{g}_2)e_2)$$

$$+ \frac{1}{2}((f_0 - \bar{g}_0) + (f_1 - \bar{g}_1)e_1 + (f_2 - \bar{g}_2)e_2).$$
By the orthogonal decomposition (1.5),

$$h(x_0, x_1, x_2) = h_1(x_0, x_1, x_2)e_1 + h_2(x_0, x_1, x_2)e_2.$$  \hfill (3.2)

In particular, contragenic functions are never invertible. Equation (1.5) also gives the property $h \perp \text{Vec} \mathcal{M}_2(\Omega)$, which by Lemma 1.2 can be expressed as

$$\int_{\Omega} (h_1f_1 + h_2f_2) \, dV = 0$$  \hfill (3.3)

whenever $\partial_1 f_2 = \partial_2 f_1$ with $f_1$ and $f_2$ harmonic.

3.2 Basic properties of contragenics

Although our main interest is in $\mathbb{B}^3$, we observe also some relations which hold in more general domains. Consider an arbitrary contragenic function $h = h_0e_0 + h_1e_1 + h_2e_2 \in \mathcal{N}(\Omega)$. By the orthogonal decomposition (1.5), $h \perp \text{Sc} \mathcal{M}_2(\Omega)$, which implies $\langle h_0, h_0 \rangle = 0$ so in fact $h_0 = 0$. Thus $h$ is of the form

$$h(x_0, x_1, x_2) = h_1(x_0, x_1, x_2)e_1 + h_2(x_0, x_1, x_2)e_2.$$  \hfill (3.2)

Let $\Omega$ be a connected domain which enjoys the symmetry that $(x_0, x_1, x_1) \in \Omega$ whenever $x = (x_0, x_1, x_2) \in \Omega_1$. For $f : \Omega_1 \rightarrow \mathbb{H}$ write

$$f^*(x) = f_0(x_0, x_2, x_1) + f_2(x_0, x_2, x_1)e_1 + f_1(x_0, x_2, x_1)e_2 - f_3(x_0, x_2, x_1)e_3$$

for $x \in \Omega_1$. Thus $(f^*)^* = f$, and it is easily seen that

$$(fg)^* = g^*f^*$$  \hfill (3.4)

for $f, g : \Omega_1 \rightarrow \mathbb{R}^3$, and

$$\overline{D}(f^*) = (\overline{D}f)^*$$  \hfill (3.5)

when $f$ is differentiable. From this we have

**Proposition 3.5.** The involution $f \mapsto f^*$ preserves $\mathcal{M}(\Omega_1)$ and $\overline{\mathcal{M}}(\Omega_1)$. If $\Omega_1$ is simply connected, the involution preserves $\mathcal{N}(\Omega_1)$ as well.

**Proof.** Let $f \in \mathcal{M}(\Omega_1)$. By (3.5), $\overline{D}(f^*) = 0^*_0 = 0$, so $f^* \in \mathcal{M}(\Omega_1)$. Similarly $\overline{\mathcal{M}}$ is invariant. Now let $h \in \mathcal{N}(\Omega_1)$. For $g \in \mathcal{H}(\Omega_1) \cap L_2(\Omega_1)$ take $f = \partial_1 g e_1 + \partial_2 g e_2 \in \text{Vec} \mathcal{M}_2(\Omega_1)$. We find that

$$\langle h^*, f \rangle = \int_{\Omega_1} (h_2(x_0, x_2, x_1) \partial_1 g(x) + h_1(x_0, x_2, x_1) \partial_2 g(x)) \, dV$$

$$= \int_{\Omega_1} (h_2(x_0, x_2, x_1) \partial_2 g^*(x_0, x_2, x_1) + h_1(x_0, x_2, x_1) \partial_1 g^*(x_0, x_2, x_1)) \, dV = 0$$

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by (3.3), where \( g^*(x) = g(x_0, x_2, x_1) \). Assuming \( \Omega \) is simply connected, it follows from Lemma 1.2 that \( f \) ranges over all of \( \text{Vec} \mathcal{M}_2(\Omega) \); thus \( h^* \in \mathcal{N}(\Omega) \) as claimed.

We conjecture that the simple connectedness is not necessary for the invariance of \( \mathcal{N}(\Omega) \) in Proposition 3.5.

In the unit ball we have the following characterization of contragenic functions via integration over the unit sphere.

**Proposition 3.6.** Let \( h \in \mathcal{H}^{(n)}(\mathbb{B}^3) \), \( h = h_1 e_1 + h_2 e_2 \). Then \( h \in \mathcal{N}^{(n)}(\mathbb{B}^3) \) if and only if the equality

\[
\int_{S^2} h_1 g \, dx_0 \wedge dx_2 = \int_{S^2} h_2 g \, dx_0 \wedge dx_1
\]

holds for all \( g \in \mathcal{H}^{n+1}_{\mathbb{R}}(\mathbb{B}^3) \).

**Proof.** We have the following relation of differential forms,

\[
(h_1 \partial_1 g + h_2 \partial_2 g) \, dx_0 \, dx_1 \, dx_2 =
\]

\[
d(-h_1 g \, dx_0 \wedge dx_2 + h_2 g \, dx_0 \wedge dx_1) - (\partial_1 h_1 + \partial_2 h_2) g \, dx_0 \, dx_1 \, dx_2.
\]

Since harmonic functions of differing degrees are orthogonal over \( \mathbb{B}^3 \), integration leaves

\[
\langle h, \bar{\partial} g \rangle = \int_{\mathbb{B}^3} d(-h_1 g \, dx_0 \wedge dx_2 + h_2 g \, dx_0 \wedge dx_1)
\]

which is equal to 0 whenever \( h \in \mathcal{N} \), and Stokes’ theorem gives (3.6). Conversely, if (3.6) holds, then \( \langle h, \bar{\partial} g \rangle = 0 \). Since \( \bar{\partial} g \) ranges over all of \( \text{Vec} \mathcal{M}^{(n)} \), \( h \) is orthogonal to \( \mathcal{M}^{(n)} \). Further, since \( h \) is trivially orthogonal to \( \mathcal{M}^{(m)} \) for \( m \neq n \), it is contragenic.

**Corollary 3.7.** Let \( h \in \mathcal{H}^{(n)}(\mathbb{B}^3) \), \( h = h_1 e_1 + h_2 e_2 \). Then \( h \in \mathcal{N}(\mathbb{B}^3) \) if and only if (3.6) holds for all \( g \in \mathcal{H}_{\mathbb{R}}(\mathbb{B}^3) \).

Proposition 3.6 and Corollary 3.7, together with the bases given in section 2, provide an algorithmic method for determining when a given harmonic function, expressed as a convergent series in \( \mathcal{H} \cap \mathcal{L}^2 \), is contragenic. However, it is not likely that there is a simple characterization of contragenics purely in terms of derivatives for general \( \Omega \), and indeed we know of none even for \( \mathbb{B}^3 \).

### 3.3 Bergman kernel for \( \text{Vec} \mathcal{M} \)

The natural generalization of the holomorphic Bergman kernel [14] from the context of holomorphic functions in \( \mathbb{C} \) to that of monogenic functions in \( \mathbb{H} \) is described in [25, 26]. A generalization for functions in \( \mathbb{R}^3 \) was defined and studied more recently in [9]. We restate some of the main facts in the present terminology, and then give a new “Bergman kernel” which is more appropriate to the subject at hand, as it provides another characterization of contragenic functions.

The following result establishes that evaluation at a fixed point is a continuous linear functional on \( \text{Vec} \mathcal{M}(\Omega) \); it suffices to work in \( \mathbb{B}^3 \) for the basic estimate.
Proposition 3.8. Let \( f \in \text{Vec} \mathcal{M} \). Then for all \( x \in \mathbb{B}^3 \),

\[
|f(0)| \leq C\|f\|_2
\]

where \( C = \sqrt{3/(4\pi)} \).

Proof. Since \( f \) is harmonic, using the orthogonal basis of Proposition 2.2 we see that the constant \( f(0) \) is orthogonal to \( f - f(0) \) in \( L_2(\mathbb{B}^3) \). (More simply, one may just observe that the constants are orthogonal to all harmonic functions which vanish at the origin, a statement of the Mean Value Property of harmonic functions.) Therefore

\[
|f(0)|^2 = C^2 \int_{\mathbb{B}^3} |f(0)|^2 \, dV = C^2 (\|f\|_2^2 - \|f - f(0)\|_2^2) \leq C^2 \|f\|_2^2.
\]

The underlying idea, given a closed subspace \( \mathcal{A} \subseteq L_2 \) and an orthonormal basis \( \{\varphi_k\} \) of \( \mathcal{A} \), is to form an integral kernel \( B(x, y) = \sum_k \varphi_k(x)\varphi_k(y) \) which automatically enjoys the reproducing property \( f(x) = \int B(x, y)f(y) \, dy \) for \( f \in \mathcal{A} \), and projects \( L_2 \) orthogonally onto \( \mathcal{A} \). However, given an orthonormal basis \( \{\varphi_k\} \) of \( \mathcal{M}_2(\Omega) \), there is a problem if we try to construct a Bergman kernel in this way because \( \mathcal{M}_2 \) is not closed under multiplication: the integrand of

\[
\int (B_0(x, y)e_0 + B_1(x, y)e_1 + B_2(x, y)e_2 + B_3(x, y)e_3) \cdot (f_0(y)e_0 + f_1(y)e_1 + f_2(y)e_2) \, dV_y
\]

contains the term \((B_1f_2 - B_2f_1 + B_3f_0)e_3\), whereas \( f \) should be \( \mathbb{R}^3 \)-valued. Thus one needs an additional condition

\[
\int (B_1f_2 - B_2f_1 + B_3f_0) \, dy = 0. \tag{3.7}
\]

In [9] this is dealt with by working in the subspace of \( L_2 \) corresponding to functions \( f \) for which this property holds. We will take a different approach here, constructing a Bergman kernel for \( \text{Vec} \mathcal{M}_2 \) rather than \( \mathcal{M}_2 \), and not requiring a special condition such as (3.7).

The scalar product restricted to \( L_2(\Omega, \{0\} \oplus \mathbb{R}^2 \oplus \{0\}) \), a Hilbert space which contains \( \text{Vec} \mathcal{M}_2(\Omega) \) as a closed subspace, is

\[
\langle f, g \rangle = \int (f_1 g_1 + f_2 g_2) = -Sc \int fg
\]

since \( f = -f \). Take an orthonormal basis \( \{\psi_k\}_{k=1}^{\infty} \) of \( \text{Vec} \mathcal{M}_2(\Omega) \) over \( \mathbb{R} \), and write \( \psi_k = \psi_{k,1}e_1 + \psi_{k,2}e_2 \). Define the following functions \( \Omega \times \Omega \to \{0\} \oplus \mathbb{R}^2 \oplus \{0\} \),

\[
\begin{align*}
b_{\Omega,1}(x, y) &= -\sum_{k=1}^{\infty} \psi_{k,1}(x)\psi_k(y), \\
b_{\Omega,2}(x, y) &= -\sum_{k=1}^{\infty} \psi_{k,2}(x)\psi_k(y);
\end{align*}
\]

(3.8)
it can be shown by means of Proposition 3.8 that these series converge uniformly on compact subsets of \( \Omega \times \Omega \).

**Definition 3.9.** The **Bergman operator** \( B_\Omega \) for \( \text{Vec}\mathcal{M}(\Omega) \) is defined by

\[
B_\Omega[f](x) = \text{Sc} \left( \int_\Omega b_{\Omega,1}(x,y) f(y) \, dV_y \right) e_1 + \text{Sc} \left( \int_\Omega b_{\Omega,2}(x,y) f(y) \, dV_y \right) e_2
\]

for all \( f \in L_2(\Omega, \{0\} \oplus \mathbb{R}^2 \oplus \{0\}) \) and \( x \in \Omega \).

It is shown in the traditional way that \( b_{\Omega,1}(x,y) \) and \( b_{\Omega,2}(x,y) \) are independent of the orthonormal basis chosen. Since we can express elements of \( \text{Vec}\mathcal{M}(\Omega) \) as \( f = \sum a_k \psi_k \) (\( a_k \in \mathbb{R} \)), the following reproducing property is easily checked.

**Theorem 3.10.** The linear operator \( B_\Omega \) projects \( L_2(\Omega, \{0\} \oplus \mathbb{R}^2 \oplus \{0\}) \) orthogonally onto \( \text{Vec}\mathcal{M}_2(\Omega) \). In particular, \( B_\Omega[f] = f \) for \( f \in \text{Vec}\mathcal{M}_2(\Omega) \).

Note that two separate integral kernels (3.8) are necessary in the definition of the operator \( B_\Omega \) because the scalar product on \( \text{Vec}\mathcal{M}(\Omega) \) is only bilinear over the reals.

For \( \mathbb{R}^3 \), in terms of the specific basis \( u^n_m := \bar{\partial}U^n_m, v^n_m := \bar{\partial}\bar{V}^n_m \) of \( \text{Vec}\mathcal{M}^{(n)} \) given in Proposition 2.2, one can define kernels for each degree \( n \),

\[
b_1^n(x,y) = -\sum_{m=0}^n u^n_{m,1}(x)u^n_m(y) - \sum_{m=1}^n v^n_{m,1}(x)v^n_m(y),
\]

\[
b_2^n(x,y) = -\sum_{m=0}^n u^n_{m,2}(x)u^n_m(y) - \sum_{m=1}^n v^n_{m,2}(x)v^n_m(y),
\]

(3.9)

and then can form operators \( B^{(n)} \) analogously to Definition 3.9. These operators project the harmonic functions onto \( \text{Vec}\mathcal{M}^{(n)} \), and the Bergman operator \( B \) for \( \mathbb{R}^3 \) is their sum \( B = \sum_{n=0}^\infty B^{(n)} \). It would be interesting to express (3.9) in closed form.

The following is an immediate consequence of the foregoing, and with the formulas (3.9) allows one to detect computationally when a harmonic function is contragenic, or close to contragenic in the \( L_2 \)-sense.

**Corollary 3.11.** Let \( h = h_1e_1 + h_2e_2 \in L_2(\Omega) \) be harmonic. Then \( h \in \mathcal{N}(\Omega) \) if and only if \( B_\Omega[h] = 0 \).

## 4 Construction of homogeneous contragenic polynomials

In this section we will give an explicit construction of a basis of \( \mathcal{N}^{(n)} \) for every \( n = 0, 1, \ldots \)

One possible approach would be as follows. The basis for \( \mathcal{M}^{(n)} + \overline{\mathcal{M}}^{(n)} \) given in Proposition 2.3 is orthogonal, so one may extend this to a basis of \( \mathcal{H}^{(n)} \) by choosing suitable linearly independent triples of spherical harmonics and applying the Gram-Schmidt process.
to produce the contragenic polynomials of degree $n$. However, this procedure is quite costly numerically\(^2\) and leads to little insight regarding contragenic functions.

Here we give a direct construction of the contragenic homogeneous functions. From Table \ref{table:contragenic} it is clear that $\mathcal{N}^{(0)} = \{0\}$.

**Theorem 4.1.** Let $n \geq 1$. Write $d_m^n = (n-m)(n-m+1)$. The $2n-1$ functions

$$
\begin{align*}
Z_0^n & := \hat{V}_1^n e_1 - \hat{U}_1^n e_2, \\
Z_{m,+}^n & := (d_m^n \hat{V}_{m-1}^n + \hat{V}_m^n) e_1 + (d_m^n \hat{U}_{m-1}^n - \hat{U}_m^n) e_2, \\
Z_{m,-}^n & := (d_m^n \hat{U}_{m-1}^n + \hat{U}_m^n) e_1 + (-d_m^n \hat{V}_{m-1}^n + \hat{V}_m^n) e_2,
\end{align*}
$$

\quad (4.1)

for $1 \leq m \leq n-1$, form an orthogonal basis for $\mathcal{N}^{(n)}$ over $\mathbb{R}$.

**Proof.** First we show that the functions (4.1) are contragenic: it is sufficient to show that each one is orthogonal to $M$, numerically, when attempting to calculate contragenic homogeneous polynomials of degree $n$. Let

$$
\langle \hat{U}_m^n, \hat{V}_m^n \rangle = \frac{1}{2} \left( \langle d_m^n \hat{V}_{m-1}^n, \hat{U}_1^n \rangle + \langle \hat{V}_{m+1}^n, \hat{U}_1^n \rangle + \langle d_m^n \hat{U}_{m-1}^n, \hat{V}_1^n \rangle - \langle \hat{U}_{m+1}^n, \hat{V}_1^n \rangle \right).
$$

By the orthogonality of the spherical harmonics $\hat{U}_m^n$ and $\hat{V}_m^n$, this scalar product is equal to zero. Next we observe that for $1 \leq k \leq n$,

$$
\langle Z_{m,+}^n, \text{Vec} X_k^n \rangle = \langle d_m^n \hat{V}_{m-1}^n + \hat{V}_m^n, -c_k^n \hat{V}_{k-1} + \frac{1}{4} \hat{V}_{k+1} \rangle
$$

$$
- \langle d_m^n \hat{U}_{m-1}^n - \hat{U}_m^n, c_k^n \hat{V}_{k-1} + \frac{1}{4} \hat{V}_{k+1} \rangle
$$

$$
= 0,
$$

since $\hat{U}_{m+1}$ is orthogonal to $\hat{V}_{m-1}$.

Finally, it remains to check that

$$
\langle Z_{m,+}^n, \text{Vec} Y_k^n \rangle = -d_m^n c_k^n \langle \hat{V}_{m-1}^n, \hat{V}_{k-1} \rangle + \frac{1}{4} d_m^n \langle \hat{V}_{m-1}^n, \hat{V}_{k+1} \rangle
$$

$$
- c_k^n \langle \hat{V}_{m+1}^n, \hat{V}_{k-1} \rangle + \frac{1}{4} \langle \hat{V}_{m+1}^n, \hat{V}_{k+1} \rangle
$$

$$
- d_m^n c_k^n \langle \hat{U}_{m-1}^n, \hat{U}_{k-1} \rangle - \frac{1}{4} d_m^n \langle \hat{U}_{m-1}^n, \hat{U}_{k+1} \rangle
$$

$$
+ c_k^n \langle \hat{U}_{m+1}^n, \hat{U}_{k-1} \rangle + \frac{1}{4} \langle \hat{U}_{m+1}^n, \hat{U}_{k+1} \rangle
$$

$$
= 0.
$$

\(^2\) Calculations in Mathematica on a desktop computer with 4 Gb of RAM have saturated the memory when attempting to calculate contragenic homogeneous polynomials of degree $n \geq 7$ via Gram-Schmidt as described here.
Once again it is immediate that this is true under the condition \( k \neq m, k \neq m + 2 \) and \( k \neq m - 2 \) since all of the scalar products involved vanish. Now consider \( k = m \). Substituting the equation \((2.11)\) we obtain that

\[
\langle Z^m_{m,+}, \text{Vec} Y^n_m \rangle = -d^m_m c^m_m (\hat{V}^n_{m+1}, \hat{V}^n_{m+1}) - d^m_m c^m_m (\hat{U}^n_{m-1}, \hat{U}^n_{m-1}) \\
+ \frac{1}{4} (\hat{V}^n_{m+1}, \hat{V}^n_{m+1}) + \frac{1}{4} (\hat{U}^n_{m+1}, \hat{U}^n_{m+1}) \\
= -2(n - m)(n - m + 1) \frac{(n + m)(n + m + 1) 2\pi(n + m - 1)!}{4(2n + 1)(2n + 3)(n - m + 1)!} \\
+ \frac{2 \cdot 2\pi}{4(2n + 1)(2n + 3)(n - m + 1)!} \\
= 0.
\]

When \( k = m + 2 \), we see by \((2.11)\) that

\[
\langle Z^m_{m,+}, \text{Vec} Y^n_{m+2} \rangle = -c^m_{m+2} (\hat{V}^n_{m+1}, \hat{V}^n_{m+1}) + c^m_{m+2} (\hat{U}^n_{m+1}, \hat{U}^n_{m+1}) \\
= 0
\]

and the case \( k = m - 2 \) is similar. Therefore \( \langle Z^m_{m,+}, \text{Vec} Y^n_k \rangle = 0 \) for all \( k \). This completes the proof that \( Z^m_{m,+} \) is contragenic. The proofs that \( Z^m_{m,-} \) and \( Z^m_{m,-} \) are contragenic are analogous.

Now we show that these functions form an orthogonal basis. Since \( \dim_{\mathbb{R}} \mathcal{N}(n) = 2n - 1 \), it suffices to show that they form an orthogonal collection. The only nontrivial cases are \( \langle Z^n_0, Z^n_{2,+} \rangle, \langle Z^n_{m,+}, Z^n_{m+2,+} \rangle, \langle Z^n_{m,+}, Z^n_{m-2,+} \rangle, \langle Z^n_{m,-}, Z^n_{m+2,-} \rangle \) and \( \langle Z^n_{m,-}, Z^n_{m-2,-} \rangle \), all of which are seen to be zero by repeated applications of \((2.11)\).

**Corollary 4.2.** The set

\[\{Z^n_0, Z^n_{m,\pm}, 1 \leq m \leq n - 1, \ n \geq 1\}\]

is an orthogonal basis for \( \mathcal{N} = \mathcal{N}(\mathbb{B}^3) \). The norms of the basis elements are

\[
\|Z^n_0\| = \sqrt{\frac{4\pi n(n + 1)}{(2n + 1)(2n + 3)}},
\]

\[
\|Z^n_{m,\pm}\| = \sqrt{\frac{8\pi(n^2 + m^2 + n)(n + m - 1)!}{(2n + 1)(2n + 3)(n - m - 1)!}}
\]

for \( 1 \leq m \leq n - 1 \).

**Proof.** It is only necessary to establish the values of the norms. By \((2.11)\)

\[
\|Z^n_0\|^2 = \langle Z^n_0, Z^n_0 \rangle = \langle \hat{V}^n_1, \hat{V}^n_1 \rangle + \langle \hat{U}^n_1, \hat{U}^n_1 \rangle = \frac{4\pi n(n + 1)}{(2n + 1)(2n + 3)}.
\]
and,

\[
\|Z_{m,+}^n\|^2 = \langle Z_{m,+}^n, Z_{m,+}^n \rangle = (d_m^n)^2 \langle \hat{V}_{m-1}^n, \hat{V}_{m-1}^n \rangle + (d_m^n)^2 \langle \hat{U}_{m-1}^n, \hat{U}_{m-1}^n \rangle + \langle \hat{V}_{m+1}^n, \hat{V}_{m+1}^n \rangle + \langle \hat{U}_{m+1}^n, \hat{U}_{m+1}^n \rangle \\
= \frac{4\pi(n-m)^2(n-m+1)^2(n+m-1)!}{(2n+1)(2n+3)(n-m+1)!} + \frac{4\pi(n+m+1)!}{(2n+1)(2n+3)(n-m-1)!} \\
= \frac{8\pi(n^2+m^2+n)(n+m+1)!}{(2n+1)(2n+3)(n+m)(n+m+1)(n-m-1)!} \\
= \frac{8\pi(n^2+m^2+n)(n+m-1)!}{(2n+1)(2n+3)(n-m-1)!}.
\]

The calculation for \(Z_{m,-}^n\) is similar. \(\square\)

**Note 4.3.** The involution \(f \mapsto f^*\) of Proposition 3.5 sends \(Z_{m,+}^n\) to (a multiple of) \(Z_{m,-}^n\) for some, but not all \(m\).

## 5 Conclusions

Consider a triple \(f = f_0e_0 + f_1e_1 + f_2e_2\) of harmonic functions in a domain \(\Omega\). We have shown that \(f\) has a natural decomposition \(f = g + h\) where \(g\) is ambigenic and \(h\) is orthogonal in \(L_2(\Omega)\) to all ambigenic functions. The existence of nontrivial contragenic functions raises the following question. Suppose that \(\Omega\) has smooth boundary and \(f\) is defined only on \(\partial \Omega\). When is the harmonic extension of \(f\) to the interior of \(\Omega\) monogenic, ambigenic, or contragenic? How do the boundary values of the monogenic, ambigenic, or contragenic part of the extension relate to the original \(f\)?

Further, it remains to investigate bases of contragenic functions in domains of \(\mathbb{R}^3\) other than \(\mathbb{B}\), as well as analogous notions of contragenicity with respect to other scalar products, for example in weighted inner product spaces or with respect to the Fischer product \([23, 27]\).

## References

[1] S. Bock, *Über funktionentheoretische Methoden in der räumlichen Elastizitätstheorie*, doctoral dissertation, Bauhaus-University, Weimar (2009)

[2] S. Bock, “Orthogonal Appell bases in dimensions 2, 3 and 4,” in: T. E. Simos, G. Psihoyios, Ch. Tsitouras (Eds.), *Numerical Analysis and Applied Mathematics, AIP Conference Proceedings*, 1281 American Institute of Physics, Melville, NY (2010) 1447–1450

[3] S. Bock, K. Gürlebeck, “On a generalized Appell system and monogenic power series,” *Math. Methods Appl. Sci.* 33:4 (2010) 394–411
[4] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman Advanced Publishing Program (1982)

[5] I. Caçao, *Constructive Approximation by Monogenic polynomials*, doctoral dissertation, Universidade de Aveiro (2004)

[6] C. Constales, R. S. Krausshar, “The Bergman kernels for the half-ball and for fractional wedge-shaped domains in Clifford analysis,” *Math. Meth. Appl. Sci.* 25 (2002) 1509–1526

[7] R. Delanghe, “On homogeneous polynomial solutions of the Moisil-Théodoresco system in $\mathbb{R}^3$” (English summary), *Comput. Methods Funct. Theory* 9:1 (2009) 199–212

[8] R. Delanghe, F. Brackx, “Hypercomplex function theory and Hilbert modules with reproducing kernel,” *Proc. London Math. Soc.* 3:37 (1978) 545–76

[9] J. O. González, M. E. Luna, M. Shapiro, “On the Bergman theory for solenoidal and irrotational vector fields, I: General theory,” *Oper. Theory Adv. Appl.* 210 (2010) 79–106

[10] B. Fornberg, “A Numerical Method for Conformal Mappings,” *SIAM J. Sci. Statist. Comput.* 1 (1980) 386–400

[11] R. Fueter, “Analytische Funktionen einer Quaternionenvariablen,” *Comment. Math. Helv.* 4 (1932) 9–20

[12] K. Gürlebeck, K. Habetha, W. Sprössig, *Holomorphic Functions in the Plane and n-dimensional space*, Birkhäuser Verlag, Basel-Boston-Berlin (2008)

[13] K. Gürlebeck, W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, Wiley: Chichester (1997)

[14] E. Hille, *Analytic Function Theory* vol. I, AMS Chelsea Publishing (1959)

[15] V. Kravchenko, *Applied Quaternionic Analysis*, Research and Exposition in Mathematics 28 Heldermann Verlag: Lemgo (2003)

[16] H. R. Malonek, “Power series representation for monogenic function in $\mathbb{R}^{n+1}$ based on a permutational product,” *Complex Variables, Theory and Application* 15 (1990) 181–191

[17] J. Morais, *Approximation by homogeneous polynomial solutions of the Riesz system in $\mathbb{R}^3$*, doctoral dissertation, Bauhaus-Universität, Weimar (2009)

[18] J. Morais, K. Gürlebeck, “Real-Part Estimates for Solutions of the Riesz System in $\mathbb{R}^3$,” *Complex Var. Elliptic Equ.* 57:5 (2012) 505–522

[19] J. Morais, H. T. Le, “Orthogonal Appell systems of monogenic functions in the cylinder,” *Math. Meth. Appl. Sci.* 34:12 (2011) 1472–1486

[20] C. Müller, *Spherical Harmonics*, Lectures Notes in Mathematics 17 Berlin: Springer-Verlag (1966)
[21] F. Olver, “Legendre functions with both parameters large,” *Phil. Trans. R. Soc.* **278** (1975) 175–185

[22] M. Riesz, *Clifford numbers and spinors*. With the author’s private lectures to E. Folke Bolinder, *Fundamental Theories of Physics* **54**, Kluwer Academic Publishers Group, Dordrecht (1993)

[23] M. Rösler, “Dunkl operators: theory and applications,” in *Orthogonal Polynomials and Special Functions (Leuven, 2002)*, *Lecture Notes in Math* **1817** (2002) 93–135

[24] I. Sabadini, M. V. Shapiro, D. C. Struppa, “Algebraic analysis of the Moisil-Theodorescu system,” *Complex Variables* **40** (2000) 333–357

[25] M. Shapiro, N. Vasilevski, “Quaternionic Ψ-hyperholomorphic functions, singular integral operators and boundary value problems I. Ψ-hyperholomorphic function theory,” *Complex Variables* **27** (1995) 17–46

[26] M. Shapiro, N. Vasilevski, “On the Bergman kernel function in hyperholomorphic analysis,” *Acta Appl. Math.* **46**:1 (1997) 1–27

[27] F. Sommen, “Clifford analysis on the level of abstract vector variables,” in F. Brackx et. al. (eds.) *Clifford Analysis and Its Applications*, Kluwer Academic Publishers (2001) 303–322

[28] A. Sudbery, “Quaternionic analysis,” *Math. Proc. Cambridge Phil. Soc.* **85** (1979) 199–225