Stability of the inverse problem for Dini continuous conductivities in the plane

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ABSTRACT
We show that the inverse problem of Calderon for conductivities in a two-dimensional Lipschitz domain is stable in a class of conductivities that are Dini continuous. This extends previous stability results when the conductivities are known to be Hölder continuous.

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1. Introduction

We study the inverse problem of Calderon in two-dimensions, namely the determination of the conductivity function $\gamma$ in a bounded domain $U \subset \mathbb{R}^2$ from the Dirichlet to Neumann (or 'voltage to current') map

$$\Lambda_\gamma : H^{1/2}(\partial U) \to H^{-1/2}(\partial U),$$

which associates to the Dirichlet data $f \in H^{1/2}(\partial U)$ the function $\Lambda_\gamma (f) = \gamma \frac{\partial u}{\partial n}$ where $n$ is the exterior unit normal on $\partial U$ and $u$ is the unique solution of the Dirichlet problem

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } U, \quad u = f \quad \text{on } \partial U.$$  \hspace{1cm} (2)

The uniqueness of the inverse of $\gamma \mapsto \Lambda_\gamma$ for conductivities $\gamma \in L^\infty(U)$ was proved in [1]: this means that $\Lambda_\gamma_1 = \Lambda_\gamma_2$ implies $\gamma_1 = \gamma_2$. Stability is the stronger statement that

$$\| \gamma_1 - \gamma_2 \|_{L^\infty(\partial U)} \leq V(\| \Lambda_\gamma_1 - \Lambda_\gamma_2 \|_{\partial U})$$ \hspace{1cm} (3)

where $\| \cdot \|_{\partial U}$ denotes the operator norm $H^{1/2}(\partial U) \to H^{-1/2}(\partial U)$ and $V(\rho) > 0$ is a non-decreasing 'stability function' that satisfies $V(\rho) \to 0$ as $\rho \to 0$. It is known that stability (3) does not hold for all $\gamma \in L^\infty(U)$ (see the example of Alessandrini [2]), so it becomes interesting to inquire what is the most general class of functions $\gamma$ for which stability holds?
In [3] it was shown that if $\partial U$ is Lipschitz and the $\gamma_i$ are Hölder continuous functions on $\overline{U}$ satisfying $\varepsilon < \gamma_i < 1/\varepsilon$, then (3) holds with $V(\rho) = C(\log \rho)^{-a}$ for some $a > 0$. In this paper we want to prove stability for Dini continuous conductivities. Because we need fairly sharp estimates, we will assume that our conductivities have modulus of continuity $\varpi$ of the form

$$\varpi(r) = |\log r|^{-a} \quad \text{for } 0 < r \leq 1/2.$$  

Note that $\varpi$ satisfies the Dini condition $\int_{1/2}^0 r^{-1} \varpi(r) \, dr < \infty$ if $\alpha > 1$; however, for technical reasons, we shall require $\alpha > 3/2$.

**Main Theorem:** Suppose $U$ is a bounded Lipschitz domain and $\gamma_1, \gamma_2$ are functions on $\overline{U}$ which have modulus of continuity $\varpi$ as in (4) with $\alpha > 3/2$ and satisfy $\varepsilon < \gamma_i(z) < 1/\varepsilon$ for some $\varepsilon > 0$ and all $z \in U$. Then there exists a stability function $V(\rho)$ satisfying $V(\rho) \to 0$ as $\rho \to 0$ such that (3) holds.

2. Some function spaces

Recall that a *modulus of continuity* $\omega(r)$ is a function $\omega : [0, \infty) \to [0, \infty)$ which is strictly increasing for $r$ near 0 and satisfies $\omega(0) = 0$. We shall also assume

$$\omega(r) \geq c r^\varepsilon \quad \text{for some } \varepsilon \in (0, 1) \text{ and all } 0 < r \leq 1/2,$$

and, for convenience, we assume $\omega$ is constant for $r \geq 1/2$. For any bounded domain $U$ in $\mathbb{R}^2$, we define the following function spaces:

**Definition 2.1:** Let $C^{\omega}(\overline{U})$ denote the Banach space of functions $f \in C(\overline{U})$ for which $|f(x) - f(y)| \leq C \omega(|x - y|)$ for all $x, y \in \overline{U}$ with the norm

$$\|f\|_{C^{\omega}(\overline{U})} := \sup_{x \in \overline{U}} |f(x)| + \sup_{x, y \in \overline{U}, x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)}.$$  

Let $C^{1,\omega}(\overline{U})$ denote the Banach space of functions $f \in C^1(\overline{U})$ whose first order derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ are in $C^{\omega}(\overline{U})$ with the norm

$$\|f\|_{C^{1,\omega}(\overline{U})} := \|\partial f\|_{C^{\omega}(\overline{U})} + \|\overline{\partial f}\|_{C^{\omega}(\overline{U})} + \|f\|_{C^{\omega}(\overline{U})}.$$  

**Definition 2.2:** For any domain $\Omega$, let $C^{\omega}(\Omega)$ and $C^{1,\omega}(\Omega)$ denote respectively the union of all $C^{\omega}(\overline{U})$ and $C^{1,\omega}(\overline{U})$ where $U$ is compactly contained in $\Omega$. We also let $C^{\omega}_0(\Omega)$ and $C^{1,\omega}_0(\Omega)$ denote those functions with compact support in $\Omega$. Frequently we take $\Omega = \mathbb{D}$, the unit disk, or $\mathbb{D}_R := \{x : |x| < R\}$. 

When $\omega(r) = r^\gamma$ for $\gamma \in (0, 1)$, then $C^\omega(\Omega)$ is traditionally written as $C^\gamma(\Omega)$, the functions which are Hölder continuous of order $\gamma$. For another example, we can extend the function $\varpi(r)$ as in (4) to be constant on $[1/2, \infty)$ and, since $\varpi(r) \to 0$ as $r \to 0$, we can define $\varpi(0) = 0$ to make $\varpi(r)$ a modulus of continuity. Thus we may consider the function spaces $C^\varpi(\Omega)$, etc. Notice that $C^\varpi(\Omega)$ is larger than any Hölder space $C^\gamma(\Omega)$ for $\gamma \in (0, 1)$.

For $1 \leq p < \infty$, we let $H^{1,p}(\Omega)$ denote the 1st-order $L^p$-Sobolev space for $\Omega$ and $H^{1,loc}_p(\Omega)$ functions that are in $H^{1,p}(U)$ for any compact subset $U \subset \Omega$. However, we will also be interested in less regular functions. Suppose $\vartheta : [0, \infty) \to [0, \infty)$ is increasing with $\vartheta(r) \to \infty$ as $r \to \infty$. Then we use the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{2\pi i \xi \cdot f(x)} dx$ to define the following Banach (and Hilbert) space.

**Definition 2.3:** For $\vartheta : [0, \infty) \to [0, \infty)$ nondecreasing with $\vartheta(r) \to \infty$ as $r \to \infty$,

$$W^{\vartheta,2}(\mathbb{R}^2) := \{f \in L^2(\mathbb{R}^2) : \|f\|^2_{W^{\vartheta,2}} := \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 (1 + \vartheta(|\xi|)) d\xi < \infty\}. \quad (8)$$

We are interested in a $\vartheta$ that is associated with a modulus of continuity $\omega$ as follows:

**Definition 2.4:** For a given modulus of continuity $\omega(r)$, let us define

$$\vartheta(r) = \begin{cases} \int_1^r \frac{ds}{s \omega^2(s/r)} & \text{for } r > 1, \\ 0 & \text{for } 0 \leq r \leq 1. \end{cases} \quad (9)$$

For example, in the Hölder case $\omega(r) = r^\gamma$ for $\gamma \in (0, 1)$, we have $\vartheta(r) \approx c r^{2\gamma}$ as $r \to \infty$ and $W^{\vartheta,2}(\mathbb{R}^2)$ coincides with the fractional-order Sobolev space $H^{\gamma,2}(\mathbb{R}^2)$ defined as Bessel potentials. On the other hand, for $\varpi(r)$ as in (4), we get

$$\vartheta(r) = \int_1^r \frac{ds}{s \log(r/s))^{-2\alpha}} \approx \frac{\log r |2\alpha+1}{2\alpha+1} \quad \text{as } r \to \infty. \quad (10)$$

Note that $H^{\gamma,2}(\mathbb{R}^2) \subset W^{\vartheta,2}(\mathbb{R}^2)$ for any $\gamma \in (0, 1)$.

Now let us explain why we are interested in $W^{\vartheta,2}(\mathbb{R}^2)$. In the Hölder case $\omega(r) = r^\gamma$, functions in $H^{\gamma,2}(\mathbb{R}^2)$ coincide with functions in $L^2(\mathbb{R}^2)$ for which the $L^2$-modulus of continuity

$$M_2(f, y) = \|f(\cdot + y) - f(\cdot)\|_{L^2} \quad (11)$$

is small enough as $|y| \to 0$ that $M_2(f, y)|y|^{1-\gamma} \in L^2(\mathbb{R}^2)$; cf. [4]. Functions in $W^{\vartheta,2}$ can be similarly characterized; for this purpose we need to introduce

$$\tilde{\omega}(r) = \begin{cases} \omega(r) & \text{for } 0 < r < 1, \\ \frac{1}{\omega(1/r)} & \text{for } r > 1. \end{cases} \quad (12)$$

Note that $\tilde{\omega}(r) \to \infty$ as $r \to \infty$. In fact, for any $\alpha > 0$ we have

$$\omega(r) = |\log r|^{-\alpha} \quad \text{for } 0 < r \leq 1/2 \quad \Rightarrow \quad \tilde{\omega}(r) = (\log r)^\alpha \quad \text{for } r \geq 2.$$
However, in order to characterize functions in $W^{\theta,2}$ in terms of the modulus of continuity $\omega$, we need to assume that $\omega$ satisfies the 'square-Dini condition'

$$\int_0^\varepsilon \frac{(\omega(r))^2}{r} \, dr < \infty \quad \text{for some } \varepsilon > 0. \quad (13)$$

**Lemma 2.1:** If $\omega$ satisfies (13), then $f \in W^{\theta,2}$ if and only if $f \in L^2$ and

$$\int_{\mathbb{R}^2} \frac{(M_2(f, y))^2}{|y|^2 \omega^2(|y|)} \, dy < \infty. \quad (14)$$

Here $\nabla$ is defined in terms of $\omega$ by (9).

**Proof:** Proceeding as in [4], we first use the Plancherel Theorem to obtain

$$M_2^2(f, y) = (M_2(f, y))^2 = \int_{\mathbb{R}^2} \hat{f}(\xi)^2 \, e^{-2\pi i \xi \cdot y} - 1 \, d\xi.$$ 

Now let us consider the integral

$$\int_{\mathbb{R}^2} \frac{M_2^2(f, y)}{|y|^2 \omega^2(|y|)} \, dy = \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 I(\xi) \, d\xi,$$

where

$$I(\xi) = \int_{\mathbb{R}^2} \frac{|e^{-2\pi i \xi \cdot y} - 1|^2}{|y|^2 \omega^2(|y|)} \, dy. \quad (15)$$

We note that the integral defining $I(\xi)$ converges: write

$$\int_{\mathbb{R}^2} \frac{|e^{-2\pi i \xi \cdot y} - 1|^2}{|y|^2 \omega^2(|y|)} \, dy = \int_{|y|<1} \frac{|e^{-2\pi i \xi \cdot y} - 1|^2}{|y|^2 \omega^2(|y|)} \, dy + \int_{|y|>1} \frac{|e^{-2\pi i \xi \cdot y} - 1|^2 \omega^2(|y|^{-1})}{|y|^2} \, dy.$$

Use $|e^{-2\pi i \xi \cdot y} - 1| \leq c|y|$ for $|y| < 1$ and $\omega(r) \geq cr^\theta$ for $0 < r < 1$ to conclude that the first integral converges, and for the second integral use $|e^{-2\pi i \xi \cdot y} - 1| \leq 2$ with

$$\int_{|y|>1} \frac{\omega^2(|y|^{-1})}{|y|^2} \, dy = c \int_1^\infty \frac{\omega^2(r^{-1})}{r} \, dr = c \int_0^1 \frac{\omega^2(s)}{s} \, ds < \infty.$$

Moreover, since $I(\xi)$ is rotation-invariant we can define the radial function $I_0(r)$:

$$I_0(|\xi|) := I(\xi). \quad (16)$$

The proof is complete if we can show that $\theta(r) \approx I_0(r)$ as $r \to \infty$, i.e. there exists a constant $c > 0$ such that

$$c I_0(r) \leq \theta(r) \leq (1/c) I_0(r) \quad \text{for } r \text{ sufficiently large.} \quad (17)$$

To estimate $I_0(r)$ as $r \to \infty$, let us choose $\xi = r(1, 0)$ and let $\tilde{y} = ry$. Then $I_0(r) = I_1(r) + I_2(r)$ where

$$I_1(r) = \int_{|y|<1} \frac{|e^{-2\pi i r y_1} - 1|^2}{|y|^2 \omega^2(|y|)} \, dy = \int_{|\tilde{y}|<r} \frac{|e^{-2\pi i \tilde{y}_1} - 1|^2}{|\tilde{y}|^2 \omega^2(|\tilde{y}|/r)} \, d\tilde{y}.$$
and

\[ I_2(r) = \int_{|\gamma| > 1} \frac{|e^{-2\pi i y_1} - 1|^2}{|y|^2} \frac{1}{|y_1|} \, dy = \int_{|\gamma| > r} \frac{|e^{-2\pi i \tilde{y}_1} - 1|^2}{|\tilde{y}|^2} \, d\tilde{y}. \]

Recall that \( \vartheta(r) \to \infty \) whereas \( I_2(r) \) is decreasing as \( r \to \infty \), so we need only concern ourselves with \( I_1(r) \). Moreover, it is clear that \( I_1(r) - I_1(1) \leq c \vartheta(r) \) since

\[ \int_{1 < |\gamma| < r} \frac{|e^{-2\pi i \tilde{y}_1} - 1|^2}{|\tilde{y}|^2} \, d\tilde{y} \leq 4 \int_{1}^{r} \frac{d\rho}{\rho \omega^2(\rho/r)} = 4 \vartheta(r), \]

so we need only show \( I_1(r) \geq c \vartheta(r) \) for some \( c > 0 \).

To show \( I_1(r) \geq c \vartheta(r) \), let us write \( \tilde{y}_1 = s \cos \theta \) and compute \( |e^{-2\pi i \tilde{y}_1} - 1|^2 = 2(1 - \cos[2\pi s \cos \theta]) \). Consequently,

\[ I_1(r) = \int_{0}^{r} \int_{-\pi}^{\pi} \frac{2(1 - \cos[2\pi s \cos \theta])}{s \omega^2(s/r)} \, d\theta \, ds \]

\[ \geq 2 \int_{0}^{r} \int_{-\pi}^{\pi} \frac{1}{s \omega^2(s/r)} \left( \int_{-\pi}^{\pi} (1 - \cos[2\pi s \cos \theta]) \, d\theta \right) \, ds. \]

Thus it suffices to have \( \int_{-\pi}^{\pi} (1 - \cos[2\pi s \cos \theta]) \, d\theta \geq c \) for all \( s \geq 1 \). This is proved in Lemma A.1 in the Appendix.

Since \( \vartheta(r) \) is increasing in \( r > 1 \), it is elementary to verify the following:

**Lemma 2.2:** If \( f \in W^{\theta,2}(\mathbb{R}^2) \), \( R_0 > 1 \), and \( 0 \leq \nu \leq 1 \), then \( \tilde{f} \) satisfies

\[ \int_{|\xi| \geq R_0} |\hat{f}(\xi)|^2 \vartheta(|\xi|)^\nu \, d\xi \leq \frac{\|f\|^2_{W^{\theta,2}}}{\vartheta(R_0)^{1-\nu}}. \]

Now, for the given \( \alpha > 1 \) in \( \vartheta \), we want to consider a modulus of continuity \( \omega \) satisfying

\[ \omega(r) = |\log r|^{-\beta} \quad \text{for} \quad 0 < r \leq 1/2, \quad \text{where} \quad 0 < \beta < \alpha. \]

(18)

Note that \( \omega \) is weaker than \( \vartheta \): \( \vartheta(r) \leq \omega(r) \) for \( 0 < r \leq 1/2 \) (and hence for all \( r > 0 \)). We not only want \( \omega \) to satisfy the Dini condition, but we want \( \vartheta/\omega \) to satisfy the square-Dini condition (13) at \( r = 0 \). In fact, if we assume that

\[ 1 < \beta < \alpha - 1/2, \]

(19)

which is possible since \( \alpha > 3/2 \), then we have

\[ C_{\alpha,\beta} := \int_{0}^{\infty} \frac{\vartheta^2(r)}{r \omega^2(r)} \, dr < \infty. \]

(20)

We will also use the notation \( C_{\alpha,\beta} \) for \( KC_{\alpha,\beta} \) where \( K \) is a constant that might depend on other parameters. As in (10) we find that for any constant \( c > 0 \)

\[ \vartheta(r) \approx \vartheta(c r) \approx \frac{|\log r|^{2\beta+1}}{2\beta+1} \quad \text{as} \quad r \to \infty. \]

(21)
In particular, we have

$$\vartheta (r) \geq C|\log r|^{3+\delta} \quad \text{for } r \geq 2,$$

for $$\delta = 2(\beta - 1) > 0$$ and some $$C > 0$$.

**Lemma 2.3:** Suppose $$\mu \in C_0^\infty (\mathbb{D})$$.

(a) For $$1 \leq p < \infty$$, the $$L^p$$-modulus of $$\mu$$ is uniformly bounded by $$\varpi$$:

$$M_p(\mu, y) := \|\mu(\cdot + y) - \mu(\cdot)\|_{L^p} \leq C \varpi(|y|)\|\mu\|_{C^\infty} \quad \text{for any } y \in \mathbb{R}^2.$$

(b) For $$\varpi$$ as in (18), (19) with associated $$\vartheta$$, we have $$\mu \in W^{\theta, 2}(\mathbb{R}^2)$$ and

$$\|\mu\|_{W^{\theta, 2}} \leq C_{\alpha, \beta}\|\mu\|_{C^\infty}.$$

(c) If $$\varpi$$ and $$\vartheta$$ are as in (b) and $$f \in W^{\theta, 2}(\mathbb{R}^2)$$, then $$\mu f \in W^{\theta, 2}(\mathbb{R}^2)$$ and

$$\|\mu f\|_{W^{\theta, 2}} \leq C_{\alpha, \beta}\|\mu\|_{C^\infty}\|f\|_{W^{\theta, 2}}.$$

**Proof:** To prove (a), we begin with $$|\mu(x + y) - \mu(x)| \leq \varpi(|y|)\|\mu\|_{C^\infty}$$. For $$|y| \leq 1$$,

$$M_p^p(\mu, y) = \int_{\mathbb{D}_2} |\mu(x + y) - \mu(x)|^p \, dx \leq \varpi^p(|y|)\|\mu\|_{C^\infty}^p \int_{\mathbb{D}_2} \, dx.$$

For $$|y| \geq 1$$ we can use $$\varpi(|y|) = \varpi(1/2)$$ to conclude

$$M_p(\mu, y) \leq 2\|\mu\|_{L^p} \leq 2\|\mu\|_{C^\infty} \left(\int_{\mathbb{D}_1} \, dx\right)^{1/p} \leq C_p \varpi(|y|)\|\mu\|_{C^\infty},$$

where $$\|\cdot\|_{C^\infty}$$ denotes the sup-norm.

To prove (b), we use (a) and some of the formulas in the proof of Lemma 2.1. We first use (17) and then (a) to estimate

$$\int_{|\xi| > 1} |\hat{\mu}(\xi)|^2 \vartheta(|\xi|) \, d\xi \leq C \int_{\mathbb{R}^2} |\hat{\mu}(\xi)|^2 I(\xi) \, d\xi = C \int_{\mathbb{R}^2} \frac{M_2^2(\mu, y)}{|y|^2 \omega^2(|y|)} \, dy \leq C \int_{\mathbb{R}^2} \frac{\varpi^2(|y|)}{|y|^2 \omega^2(|y|)} \, dy \|\mu\|_{C^\infty}^2 = C_{\alpha, \beta}\|\mu\|_{C^\infty}^2.$$

To prove (c), note that

$$\|\mu f\|_{W^{\theta, 2}}^2 \approx \int_{\mathbb{R}^2} \frac{M_2^2(\mu f, y)}{|y|^2 \omega^2(|y|)} \, dx + \|\mu f\|_{L^2}^2$$

But we have

$$M_2^2(\mu f, y) = \int_{\mathbb{R}^2} |(\mu(z + y) - \mu(z))f(z + y) + \mu(z)(f(z + y) - f(z))|^2 \, dz \leq C \left(\varpi^2(|y|)\|\mu\|_{C^\infty}^2 \|f\|_{L^2}^2 + \|\mu\|_{C^\infty}^2 M_2^2(f, y)\right).$$
Hence (23) gives,
\[
\| \mu f \|_{W^{0,2}}^2 \leq C \int_{\mathbb{R}^2} \frac{\sigma^2(|y|) \| \mu \|_{C^0}^2 \| f \|_{L^2}^2 + \| \mu \|_{L^\infty}^2 M_2^2(f, y)}{|y|^2 \omega^2(|y|)} dy + \| \mu \|_{L^\infty}^2 \| f \|_{L^2}^2
\]
\[
\leq C \| \mu \|_{C^0}^2 \int_{\mathbb{R}^2} \frac{\sigma^2(|y|) \| f \|_{L^2}^2 + M_2^2(f, y)}{|y|^2 \omega^2(|y|)} dy + \| \mu \|_{C^0}^2 \| f \|_{L^2}^2
\]
\[
\leq C_{\alpha, \beta} \| \mu \|_{C^0}^2 \| f \|_{W^{0,2}}^2.
\]
This completes the proof. ■

3. Estimating the complex geometric optics solutions as \( |k| \to \infty \)

The proof of the Main Theorem uses the so-called ‘complex geometric optics solutions’ which were initiated by Calderon [5]. As in [1], these can be constructed by solving the associated \( \mathbb{R} \)-linear Beltrami equation
\[
\overline{\partial} f = \mu \partial f, \quad \text{where } \mu = (1 - \gamma)/(1 + \gamma). \tag{24}
\]
Here \( \overline{\partial} = \frac{i}{2}(\partial_x + i \partial_y) \) and \( \partial = \frac{i}{2}(\partial_x - i \partial_y) \). In fact, a complex solution \( u_\gamma \) of (2) can be recovered from a solution \( f_\mu \) of (24) simply by letting
\[
u_\gamma = \text{Re}(f_\mu) + i \text{Im}(f_\mu). \tag{25}
\]
As in [1,3] (see also Theorem 6.1 below), the problem can be reduced to the case that \( U = \mathbb{D} \) and \( \gamma = 1 \) in a neighborhood of \( \partial \mathbb{D} \), so \( \mu \) has compact support in \( \mathbb{D} \) and we may consider (24) on \( \mathbb{C} \). Henceforth we use the notation \( \| \cdot \|_{\infty} \) for the sup-norm on \( \mathbb{C} \) (or \( \mathbb{R}^2 \)).

Now, for each \( k \in \mathbb{C} \), [1] shows that there is a unique solution \( f_\mu(z, k) \) of (24) in the form
\[
f_\mu(z, k) = e^{ik\phi(z, k)}, \tag{26}
\]
where, for each fixed \( k \in \mathbb{C} \), \( \phi(z, k) \) is a quasiconformal homeomorphism in \( z \) and satisfies the nonlinear Beltrami equation
\[
\partial \bar{z} \phi = -\frac{k}{k} \mu(z) e^{-k}(\phi(z)) \partial \bar{z} \phi, \tag{27}
\]
where \( e_k(z) = \exp(i(kz + \bar{k}z)) \), and the boundary condition
\[
\phi(z) = z + O(z^{-1}) \quad \text{as } |z| \to \infty. \tag{28}
\]
In (27) and (28) we consider \( k \) fixed, but the dependence of \( \phi \) on \( k \) is important. For \( \mu \in L^\infty \), it was shown in [1] that \( \phi(z, k) \to z \) as \( |k| \to \infty \) uniformly for \( z \in \mathbb{C} \):
\[
|\phi(z, k) - z| \leq C(k), \tag{29}
\]
where \( C(k) \to 0 \) as \( |k| \to \infty \). For more regular conductivities, \( C(k) \) can be described more precisely. For Hölder continuous conductivities \( \mu \in C^\gamma \), it was shown in [3] that (29) holds.
with $C(k) = c|k|^{-a}$, where $c, a > 0$ depend on $\gamma$ (and other parameters). For the Dini continuous conductivities that we consider, $\mu \in C^\sigma$, $C(k)$ will depend on $\omega$ and $\vartheta$ as in the previous section.

As in [1,3], the study of the solutions of (27)–(29) is reduced to the study of solutions $\psi(z, k)$ of a $C$-linear Beltrami equation with boundary condition:

$$\partial_z \psi = -\frac{k}{k} \mu(z) e^{-k(z)} \partial_z \psi,$$

$$\psi(z, k) = z + O(z^{-1}) \quad \text{as} \quad |z| \to \infty.$$  \hfill (30)

By writing $\psi = z + \eta$, this is reduced to solving a nonhomogenous Beltrami equation with boundary condition

$$\eta = O(z^{-1}) \quad \text{as} \quad |z| \to \infty.$$  \hfill (31)

The function $\eta$ is uniquely determined in $H_{1, p}^2$ for all $p > 2$, so $\psi \in H_{1, 2}^{1, \ell}$ is uniquely determined.

A tool used in the study of (30) is the Beurling transform

$$T(g)(z) = -\frac{1}{\pi} \int \frac{g(w)}{(w-z)^2} \, dw,$$  \hfill (32)

which has the well-known properties (cf. [6]) that

$$\lim_{p \to 2} \|T\|_{L^p \to L^p} = \|T\|_{L^2 \to L^2} = 1,$$  \hfill (33)

and $T(\partial_z \eta) = \partial_z \eta$ for all $\eta \in H^{1,p}(\mathbb{C})$. As in [1], there is a convergent series representation

$$\partial_z \psi = \sum_{n=0}^\infty (aT)^n a, \quad \text{where} \quad a(z, k) = -\frac{k}{k} \mu(z) e^{-k(z)},$$  \hfill (34)

provided $\|aT\|_{L^p \to L^p} < 1$. Since $|a(z, k)| \leq \|\mu\|_\infty \leq \kappa < 1$, this condition will be met provided $p$ is sufficiently close to 2. Now choose $p = p(\kappa) > 2$ so that

$$\kappa_1 := \kappa \|T\|_{L^p \to L^p} < 1 \quad \text{for all} \quad p \in [2, p(\kappa)].$$  \hfill (35)

Then (33) converges in $L^p$ for all $p \in [2, p(\kappa)]$. For a chosen integer $n_0 > 0$, write

$$\partial_z \psi = g_k(z) + h_k(z) := \sum_{n=0}^{n_0-1} (a(z, k)T)^n a(z, k) + \sum_{n=n_0}^\infty (a(z, k)T)^n a(z, k).$$  \hfill (36)

For $2 \leq p \leq p(\kappa)$, a geometric series can be used to estimate

$$\sup_{k \in \mathbb{C}} \|h_k\|_{L^p} \leq \pi^{-1/p} \kappa \frac{\kappa_1^{n_0}}{1 - \kappa_1}.$$  \hfill (37)

By choosing $n_0$ sufficiently large, $\sup_{k} \|h(-, k)\|_{L^p}$ can be made arbitrarily small. Similarly, we can estimate

$$\sup_{k \in \mathbb{C}} \|g_k\|_{L^p} \leq \pi^{-1/p} \kappa \frac{1 - \kappa_1^{n_0}}{1 - \kappa_1} \leq C_1(\kappa).$$  \hfill (38)

We cannot make this small so we will require estimates on the Fourier transform of $g$. The following result is analogous to Lemma 3.6(c) in [3]:

$$\sup_{k \in \mathbb{C}} \|\hat{g}_k\|_{L^p} \leq \pi^{-1/p} \kappa \frac{1 - \kappa_1^{n_0}}{1 - \kappa_1} \leq C_2(\kappa).$$  \hfill (39)
Lemma 3.1: Suppose \( \mu \in C^\infty_0 (\mathbb{D}) \) with \( \| \mu \|_\infty \leq \kappa \) and \( \| \mu \|_{C^\omega} \leq \Gamma \). Let \( \omega \) satisfy (18), (19) with associated \( \vartheta \) as in (9). Let \( \psi \in H_{loc}^{1,2} (\mathbb{C}) \) be the unique solution of (30) and for any fixed \( n_0 \in \mathbb{N} \) consider the decomposition (35). Then, for any \( R_0 > 1 \) and \( |k| \geq 2R_0 \), we can estimate

\[
\int_{|\xi| < R_0} |\widehat{g}_k(\xi)|^2 \, d\xi \leq \frac{n_0 (C_{\alpha,\beta} \Gamma)^{n_0}}{\vartheta (|k|/2)}. \tag{38}
\]

**Proof:** Write \( g_k(z) \) as

\[
g_k(z) = \sum_{n=0}^{n_0-1} (-\bar{k}/k)^{n+1} e^{-(n+1)k} f_n(z), \tag{39a}
\]

where \( f_0 = \mu \) and

\[
f_n(z) = \mu T_n \mu T_{n-1} \mu \ldots \ldots \mu T_1 (\mu) \quad \text{for} \ n > 0. \tag{39b}
\]

Here \( T_j = e^{jk} T e^{-jk} \) is the Fourier multiplier with symbol \((\xi - jk)/(\xi - jk)\), which is unimodular, so \( \| T_j \|_{W^{\theta,2} \rightarrow W^{\theta,2}} = 1 \) and \( \| f_n \|_{W^{\theta,2}} \leq \| \mu \|_{W^{\theta,2}} \leq C_{\alpha,\beta}^{n+1} \| \mu \|_{C^\omega}^{n+1} \) where we have also used Lemma 2.3(c). But this means

\[
\| g_k \|_{W^{\theta,2}} \leq \sum_{n=0}^{n_0-1} \| f_n \|_{W^{\theta,2}} \leq n_0 C_{\alpha,\beta}^{n_0} \| \mu \|_{C^\omega}^{n_0}. \tag{40}
\]

Now let us turn to (38):

\[
\int_{|\xi| < R_0} |\widehat{g}_k(\xi)|^2 \, d\xi \leq \sum_{n=0}^{n_0-1} \int_{|\xi| < R_0} |\widehat{f}_n (\xi - (n+1)k)|^2 \, d\xi
\]

and by Lemma 2.2

\[
\int_{|\xi| < R_0} |\widehat{f}_n (\xi - (n+1)k)|^2 \, d\xi \leq \int_{|\xi| > (n+1)|k| - R_0} |\widehat{f}_n (\xi)|^2 \, d\xi \leq \frac{||f_n||_{W^{\theta,2}}^2}{\vartheta ((n+1)|k| - R_0)}. \tag{41}
\]

For \( |k| \geq 2R_0 \), since \( \vartheta (|x|) \) is an increasing function, \( \vartheta ((n+1)|k| - R_0) \geq \vartheta (|k|/2) \), so we obtain

\[
\int_{|\xi| < R_0} |\widehat{g}_k(\xi)|^2 \, d\xi \leq \sum_{n=0}^{n_0-1} \frac{(C_{\alpha,\beta} \Gamma)^{n+1}}{\vartheta (|k|/2)} \leq \frac{n_0 (C_{\alpha,\beta} \Gamma)^{n_0}}{\vartheta (|k|/2)}. \tag{42}
\]
Proposition 3.1: Suppose \( \mu \in C^\infty_0(\mathbb{D}) \) with \( \|\mu\|_\infty \leq \kappa \) and \( \|\mu\|_{C^\infty} \geq \Gamma \). Let \( \omega \) and \( \theta \) be as in Lemma 3.1 and let \( \psi \in H^{1,2}_{\text{loc}}(\mathbb{C}) \) be the unique solution of (30). Then there exist constants \( C = C(\alpha, \beta, \kappa, \Gamma) \) and \( a = a(\kappa, \Gamma) > 0 \) such that for all \( z \in \mathbb{C} \) we have
\[
|\psi(z, k) - z| \leq \frac{C}{\theta(|k|)^a} \text{ as } |k| \to \infty. \tag{43}
\]

Proof: We observe that \( \partial_\nu \psi \) has compact support in \( \mathbb{D} \), so we may use (35) to write for fixed \( k \in \mathbb{C} \):
\[
\psi(z, k) - z = P[\partial_\nu \psi](z, k) = P[g_k](z) + P[h_k](z).
\]
Let us choose \( p = p(\kappa) \) as in (34) and let \( q = p' \) be the conjugate index; in particular, we have \( p > 2 \) so \( 1 < q < 2 \). Let us take
\[
\varepsilon = \frac{C}{\theta(R_0)^a} \approx \frac{C}{(\log R_0)^{(2\beta+1)a}}, \tag{44}
\]
where \( R_0, a > 0 \) are to be determined so that \( \|P[h_k]\|_\infty < \varepsilon / 3 \) and \( \|P[g_k]\|_\infty < 2\varepsilon / 3 \).

Estimate \( P[h] \). Using (36), we have
\[
\|P[h_k]\|_\infty \leq C(\kappa, q)(\kappa_1)^{n_0},
\]
so we will have \( \|P[h_k]\|_\infty < \varepsilon / 3 \) provided
\[
n_0 \geq \frac{\log(C_0 \varepsilon)}{\log(\kappa_1)} \text{ where } C_0 = C_0(\kappa, p). \tag{45}
\]

Estimate \( P[g] \). Since \( g_k(z) \) has compact support in \( \mathbb{D} \), we can write
\[
P[g_k](z) = \int_{\mathbb{C}} K_z(y) g_k(y) \, dy \text{ where } K_z(y) = \frac{1}{\pi} \frac{\chi(|y|)}{z - y};
\]
here \( \chi(r) \) is a smooth cut-off function satisfying \( \chi(r) = 1 \) for \( 0 \leq r \leq 1 \) and \( \chi(r) = 0 \) for \( r \geq 3/2 \). Notice that \( K_z \in L^q(\mathbb{C}) \) for all \( 1 \leq q < 2 \). We want to use the Fourier transform to estimate \( P[g_k] \). The Fourier transform of \( g_k \) is well-behaved since \( g_k \in L^p \) and has compact support, so \( g_k \in L^2 \) and hence \( \hat{g}_k \in L^2 \). We also know that \( \hat{K}_z \) is in \( L^p(\mathbb{C}) \) for all \( p > 2 \) (by the Hausdorff-Young inequality) so \( \hat{K}_z \in L^2(\mathbb{D}) \); but we do not know that \( \hat{K}_z \in L^2(\mathbb{C}) \), so we cannot just use Plancherel's theorem to conclude
\[
P[g_k](z) = \int_{\mathbb{C}} \hat{K}_z(\xi) \hat{g}_k(\xi) \, d\xi. \tag{46}
\]
However, the integral in (46) converges by the Hölder inequality (since \( \hat{K}_z \in L^p(\mathbb{C}) \) and \( \hat{g}_k \in L^q(\mathbb{C}) \)); then we can use an approximation argument to show its equality with \( P[g_k](z) \). Now, for \( R_0 > 0 \) let us write
\[
|P[g_k](z)| \leq \int_{|\xi| > R_0} |\hat{K}_z(\xi)\hat{g}_k(\xi)| \, d\xi + \int_{|\xi| < R_0} |\hat{K}_z(\xi)\hat{g}_k(\xi)| \, d\xi =: I_1 + I_2.
\]
Want \( R_0 \) large so that \( I_1 < \varepsilon / 3 \) and \( I_2 < \varepsilon / 3 \) for all \( |k| \geq R_0 \) (uniformly in \( z \)).
To estimate $I_1$, we use the following estimate that is proved in the Appendix:

$$|\hat{K}_z(\xi)| \leq C \frac{\log |\xi|}{|\xi|} \quad \text{for } |\xi| > 2 \text{ (uniformly in } z \in \mathbb{C}). \quad (47)$$

Using (47) and (22), we conclude that $\hat{K}_z(\xi)/\theta^{1/2}(|\xi|) \in L^2(|\xi| > 2)$ since

$$\int_{|\xi|>2} \frac{|\hat{K}_z(\xi)|^2}{\theta(|\xi|)} d\xi \leq \int_2^{\infty} \frac{(\log r)^2}{r \theta(r)} dr \leq \int_2^{\infty} \frac{1}{r (\log r)^{1+\delta}} dr < \infty.$$

Now by Cauchy-Schwarz, (22), and (40):

$$(I_1)^2 \leq \|g_k\|^2_{W^{0,2}} \int_{|\xi|>R_0} \frac{|\hat{K}_z(\xi)|^2}{\theta(|\xi|)} d\xi \leq n_0 (C_{\alpha,\beta \Gamma})^{n_0} (\log R_0)^{-\delta},$$

for any $\delta < 2(\beta - 1)$. So we can choose $R_0$ large enough that $I_1 < \varepsilon/3$ for all $|k| \geq R_0$.

To estimate $I_2$, we use Cauchy-Schwarz, (47), (21), and Lemma 3.1:

$$(I_2)^2 \leq \int_{|\xi|<R_0} |\hat{K}_z(\xi)|^2 d\xi \int_{|\xi|<R_0} \left| \hat{g}_k(\xi) \right|^2 d\xi$$

$$\leq C \left( \int_{|\xi|<1} |\hat{K}_z(\xi)|^2 d\xi + \int_{1}^{R_0} \frac{(\log r)^2}{r} dr \right) \left( \frac{n_0 (C_{\alpha,\beta \Gamma})^{n_0}}{\theta(|k|)} \right)$$

$$\leq C \left( 1 + (\log R_0)^3 \right) \frac{(C_{\alpha,\beta \Gamma})^{n_0+1}}{\theta(|k|)}.$$

So, by (22) we can choose $R_0$ sufficiently large that $I_2 < \varepsilon/3$ for all $|k| \geq R_0$.

Finally, we need to confirm that the choices of $n_0$ and $R_0$ are compatible. First, let $A := 2C_{\alpha,\beta \Gamma}$ so that $n_0 (C_{\alpha,\beta \Gamma})^{n_0} \leq A^{n_0}$ and then take equality in (45). We find

$$n_0 (C_{\alpha,\beta \Gamma})^{n_0} \leq B(\log R_0)^{\tau (2\beta + 1)a}, \quad \text{where } \tau = \frac{\log A}{\log \kappa^{-1}}.$$

(Here $B$ is also independent of $n_0, R_0, \varepsilon$.) If we use this in our estimate for $I_1$ we find $I_1^2 \leq C(\log R_0)^{\tau (2\beta + 1)a - \delta}$, where $\delta \leq 2(\beta - 1)$. So we can achieve $I_1 < \varepsilon/3 = C(\log R_0)^{-(2\beta + 1)a}$ provided

$$a \leq \frac{\beta - 1}{(2\beta + 1)(1 + \tau/2)} \quad (48)$$

Similarly, we have $I_2^2 \leq C(\log R_0)^{3 + (2\beta + 1)(\tau a - 1)}$. We find that (48) is exactly the condition we need to make $I_2 < \varepsilon/3$. This completes the proof. \[ \blacksquare \]

We now want to obtain an estimate like (43) for solutions $\phi(z,k)$ of the nonlinear Beltrami equation (27)-(28). This analysis uses the fact that for each $k \in \mathbb{C}$, $\phi(z,k)$ is a quasiconformal homeomorphism, and hence has an inverse function $\psi : \mathbb{C} \to \mathbb{C}$ defined
\[ \psi \circ \phi(z) = z \]  
which is also quasiconformal. Differentiating (49) with respect to \( z \) and \( \bar{z} \) shows that \( \psi \) satisfies
\[
\partial_{\bar{z}} \psi = -\frac{k}{k} \mu(\psi(z,k)) e^{-k(z)} \partial_z \psi,
\]
\[
\psi(z,k) = z + O(z^{-1}) \quad \text{as } |z| \to \infty.
\]

Of course, this is of the form (30). So, provided we can show that the coefficient \( \mu(\psi(z,k)) \) satisfies the conditions of Proposition 3.1, we may use it to conclude the desired estimate for \( \psi \); these estimates then apply to \( \phi = \psi^{-1} \). We use this line of reasoning to prove the following:

**Theorem 3.1:** Suppose \( \mu \in C^\infty(\mathbb{D}) \) and \( \| \mu \|_{C^\infty} \leq \Gamma \). Let \( \omega \) and \( \vartheta \) be as in Lemma 3.1 and let \( \phi \in H^{1,2}_{\text{loc}}(\mathbb{C}) \) be the unique solution of (27)–(28). Then there exist positive constants \( C_* = C_*(\kappa, \Gamma) \) and \( a = a(\kappa, \Gamma) \) such that
\[
|\phi(z,k) - z| \leq C_* \vartheta(|k|)^a \quad \text{as } |k| \to \infty.
\]

**Proof:** Let \( \tilde{\mu}(z,k) = \mu(\psi(z,k)) \). As in [1], since \( \psi \) is Hölder continuous and \( \mu \) has support in \( \mathbb{D} \), \( \tilde{\mu} \) has support in \( \mathbb{D}_4 \) (by the 1/4-Koebe theorem) for each fixed \( k \in \mathbb{C} \). So we can apply Proposition 3.1 in \( \mathbb{D}_4 \) provided we can show \( \tilde{\mu} \in C^\infty(\mathbb{D}_4) \). Let \( \gamma \) be the Hölder coefficient for \( \psi \). (Recall that \( \gamma = K^{-1} \) where \( K = (1 - \kappa)/(1 + \kappa) \).) Then
\[
\frac{|\tilde{\mu}(z) - \tilde{\mu}(y)|}{\sigma(|z - y|)} \leq \| \mu \|_{C^\infty} \sup_{z,y} \frac{\sigma(|\psi(z) - \psi(y)|)}{\sigma(|z - y|)}.
\]

But we know \( |\psi(z) - \psi(y)| \leq C_1 |z - y|^{\gamma} \) for all \( z, y \in \mathbb{D}_4 \), where we may assume \( C_1 > 1 \). Since \( \sigma(r) \) is nondecreasing, we have \( \sigma(|\psi(z) - \psi(y)|) \leq \sigma(C_1 |z - y|^{\gamma}) \), so it suffices to show that
\[
\sup_{0 < r < \infty} \frac{\sigma(C_1 r^{\gamma})}{\sigma(r)} < \infty. \tag{50}
\]

But since we know explicitly that \( \sigma(r) = |\log r|^{-\alpha} \) near \( r = 0 \) and is constant for large \( r \), condition (50) is easily verified. \( \blacksquare \)

### 4. Regularity of the complex geometric optics solutions

Now let us turn to the regularity of the complex geometric optics solutions \( f_\mu \) and \( u_\gamma \). Similar to the result obtained in [3], we show that the assumption \( \| \mu \|_{C^\infty} \leq \Gamma \) gives an upper bound for \( f_\mu \) and lower bound for the Jacobian, \( J_{f_\mu} \). We obtain interior estimates on compact subsets which can then be used to prove the main theorem when \( \mu_j \) has compact support in \( \mathbb{D} \). To obtain these estimates, we first obtain estimates for a more general Beltrami equation which is also \( \mathbb{R} \)-linear. To analyse such an equation, we use a modulus of continuity \( \sigma \) as defined in [7].
Definition of $\sigma(r)$: Define a continuous function $\sigma$ on $[0, \infty)$ by

$$
\sigma(r) := \int_0^r \frac{\sigma(s)}{s} \, ds = \frac{|\log r|^{1-\alpha}}{\alpha - 1} \quad \text{for } 0 < r \leq 1/2,
$$

with $\sigma(0) = 0$ and $\sigma(r)$ is a constant for $r \geq 1/2$. (Note that $\sigma$ needs not satisfy the Dini condition at $r = 0$.)

Using $\sigma$ as the modulus of continuity, $C^\sigma(U)$ and $C^{1,\sigma}(U)$ are Banach spaces defined as in Section 2. It is easy to see that $\psi(r) \leq \sigma(r)$ and hence for any domain $\Omega$,

$$
C^\sigma(\Omega) \subset C^\sigma(U).
$$

**Proposition 4.1:** For a bounded domain $\Omega$, let $\mu, \nu \in C^\sigma(\Omega)$ satisfying $|\mu(\xi)| + |\psi(\xi)| \leq \kappa < 1$ for all $\xi \in \Omega$. Let $v \in H^{1,2}_{\text{loc}}(\Omega)$ be a solution to the equation

$$
\overline{\partial}v - \mu \partial v - \nu \overline{\partial}v = 0.
$$

Consider domains $D, U$ such that $\overline{D} \subset U$ and $\overline{U} \subset \Omega$. If $\|\mu\|_{C^\sigma(U)} + \|\nu\|_{C^\sigma(U)} < \Gamma$, then we have

(a) $v \in C^{1,\sigma}(\Omega)$. In particular, we have $v \in C^{1,\sigma}(\overline{D})$ and there exists $K_1 = K_1(\kappa, \Gamma, D, U)$ such that

$$
\|v\|_{C^{1,\sigma}(\overline{D})} \leq K_1 \|v\|_{C^0(U)}.
$$

(b) If $v$ is a quasiconformal homeomorphism in $\mathbb{C}$, let $M = M(U)$ satisfy

$$
M = \max_{x \in U} |v(x)|.
$$

Then there exists a constant $K_2 = K_2(\kappa, \Gamma, D, U, M) > 0$ such that

$$
\inf_{z \in D} J_v(z) = \inf_{z \in D} (|\partial v(z)|^2 - |\overline{\partial}v(z)|^2) \geq K_2.
$$

**Proof:** For the proof of (a), refer to [8], where a similar result is proved for the non-homogeneous equation corresponding to (53).

To prove (b), we proceed as in [3]. For $z \in D$, we use (53) to obtain

$$
J_v(z) = |\partial v(z)|^2 - |\overline{\partial}v(z)|^2 \geq (1 - \kappa^2)|\partial v(z)|^2.
$$

Consider the inverse function $v^{-1}$ of $\xi = v(z)$, which satisfies the Beltrami equation

$$
\partial_\xi (v^{-1}) - (\mu \circ v^{-1})\overline{\partial_\xi (v^{-1})} - (\nu \circ v^{-1})\partial_\xi (v^{-1}) = 0.
$$

Since $v^{-1}$ is quasiconformal, it is Hölder continuous, so the coefficients $\mu \circ v^{-1}$ and $\nu \circ v^{-1}$ in this Beltrami equation are in $C^\sigma(v(D))$, by a similar argument as in the proof
of Theorem 3.1. Therefore $v^{-1}$ satisfies the conditions in (a) to obtain the corresponding estimate (54). In particular, we have

$$|\partial_\xi v^{-1} \circ v(z)| \leq K_3 \quad \text{for } z \in D,$$

(58)

where $K_3 = K_3(\kappa, \Gamma, D, U, M)$. On the other hand, differentiating $z = v^{-1} \circ v(z)$ by the chain rule, we obtain $1 = |(\partial_\xi v^{-1} \circ v)(\partial_2 v)| = |(\partial_\xi v^{-1} \circ v)| |(\partial_2 v)|$. So, by (58), we have

$$|\partial_2 v(z)| = \frac{1}{|\partial_\xi v^{-1} \circ v|} \geq \frac{1}{K_3} \quad \text{for } z \in D,$$

(59)

Since $J_\nu(z) \geq |\partial_2 v(z)|^2$, we obtain (56).

We can now obtain the following result:

**Theorem 4.1:** Suppose $\mu \in C^\infty_0(\mathbb{D})$ with $\|\mu\|_{C^\infty} \leq \Gamma$. There exist positive constants $C_1(\kappa, \Gamma, |k|)$ and $C_2(\kappa, \Gamma, |k|)$ so that the complex geometric optics solution (26) satisfies

$$\|f_\mu(\cdot, k)\|_{C^{1,\sigma}(\mathbb{D})} \leq C_1 \quad \text{and} \quad \inf_{z \in \mathbb{D}} |J_{f_\mu}(z, k)| \geq C_2.$$  

(60)

**Proof:** Recall from (26) that $f_\mu(z, k) = e^{ik\varphi(z, k)}$ where $\phi = z + \varepsilon(z, k)$, with $\varepsilon(z, k)$ uniformly bounded for fixed $k$, is Hölder continuous. So the coefficient $\frac{\Gamma}{\mu} e_{-k}(\phi)$ in (27) is in $C^\infty(\mathbb{D})$. Also, for $z \in \mathbb{D}$, $\max |\phi(z, k)| = C$ for $C = C(k, \kappa, \mathbb{D})$ gives us bounds for $f_\mu$ as $1/C \leq |f_\mu(z, k)| \leq C$. Hence we can apply Proposition 4.1 to $\varphi(z, k)$ to obtain $\|\varphi\|_{C^{1,\sigma}(\mathbb{D})} \leq K_1\|\varphi\|_{C^0(\mathbb{D})}$ for some constant $K_1 = K_1(\kappa, \Gamma)$. This in turn shows that there exists a constant $C_1(\kappa, \Gamma, |k|)$ such that $\|f_\mu(\cdot, k)\|_{C^{1,\sigma}(\mathbb{D})} \leq C_1$.

Proposition 4.1 can also be used to obtain the lower estimate

$$\inf_{z \in \mathbb{D}} |\partial_\phi \varphi(z)| \geq K_2$$  

(61)

for some constant $K_2 = K_2(\kappa, \Gamma)$. But $\partial_2 f_\mu = ikf_\mu \partial_2 \phi$. Hence, using (61) and the lower bound for $f_\mu$ for $z \in \mathbb{D}$ as mentioned above, we get $\inf_{z \in \mathbb{D}} |J_{f_\mu}(z, k)| \geq C_2$ for some constant $C_2 = C_2(\kappa, \Gamma, |k|)$.

### 5. Stability of the complex geometric optics solutions

In this section we consider two conductivities $\gamma_1, \gamma_2 \in C^\infty(\mathbb{D})$ that are 1 near $\partial \mathbb{D}$ so that $\mu_j = (1 - \gamma_j)/(1 + \gamma_j)$ has compact support in $\mathbb{D}$ and we can apply the results of the previous two sections; this restriction on $\gamma_j$ will be removed in the next section. For fixed $k \in \mathbb{C}$ we want to study the stability of the geometric optics solutions (26) but, as in [1,3], we will work with the associated solutions $u_1, u_2$ of $\nabla \cdot \gamma \nabla u = 0$ defined by (25). Let $\omega$ and $\vartheta$ be as in Lemma 3.1 and let $\rho = \rho_{12} = \|\Lambda \gamma_1 - \Lambda \gamma_2\|_{\partial \mathbb{D}}$ where $\|\cdot\|_{\partial \mathbb{D}}$ denotes the operator norm $H^{1/2}(\partial \mathbb{D}) \rightarrow H^{-1/2}(\partial \mathbb{D})$. The stability function that we seek will be of the form

$$V_k(\rho) := C_1(k) [\vartheta (|\log \rho|/C_2)]^{-a}$$  

(62)

for positive constants $C_1(k), C_2, a$. Recalling (21), we see that $V_k(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ like a negative power of $\log |\log \rho|$; since we are only interested in $\rho \rightarrow 0$, we henceforth assume $0 < \rho < 1/2$ so that $|\log \rho| > 0$. We want to prove the following.
**Theorem 5.1:** Suppose \( \gamma_1, \gamma_2 \in C^\sigma (\mathbb{D}) \) such that \( \mu_j = (1 - \gamma_j) / (1 + \gamma_j) \) has compact support in \( \mathbb{D} \) and satisfies \( \| \mu_j \|_\infty \leq \kappa < 1 \) and \( \| \mu_j \|_{C^\sigma} \leq \Gamma \). Then, for every \( k \in \mathbb{C} \) there exists \( V_k(\rho) \) of the form (62) with constants \( C_1(k), C_2, \) and \( a \) (depending on \( \kappa \) and \( \Gamma \)) such that

\[
\| u_1(\cdot, k) - u_2(\cdot, k) \|_{C^\sigma(\mathbb{D})} \leq V_k(\rho) \quad \text{as } \rho \to 0. \tag{63}
\]

We can write

\[
u_j(z, k) = e^{ik(z + \varepsilon_j(z, k))} \tag{64a}\]

where (by Theorem 3.1) we have \( C_* \), \( a > 0 \) such that

\[|\varepsilon_j(z, k)| \leq C_* [\vartheta (|k|)]^{-a} \quad \text{for all } z \in \mathbb{C} \text{ and all } |k| > 2. \tag{64b}\]

As in [1], let us introduce

\[g(z, w, k) := i(z - w) + k \varepsilon_{z,w}(k), \tag{65a}\]

where

\[\varepsilon_{z,w}(k) := i(\varepsilon_1(z, k) - \varepsilon_2(w, k)). \tag{65b}\]

We claim the following is true:

**Proposition 5.1:** For \( C_* \) and \( a \) as in (64b), there is a constant \( C_1 > 0 \) so that \( g(z, w, k) = 0 \) for some \( k \neq 0 \) implies \( |z - w| \leq C_1 [\vartheta (|\log \rho|/4C_*)]^{-a} \).

This proposition was obtained as Prop. 5.3 in [3] for the Hölder case \( \omega (r) = r^r, \vartheta (r) \approx cr^{2\gamma}, 0 < \gamma < 1 \). The proof of Proposition 5.1 follows the same outline; but, for completeness, we explain this in the Appendix, including some details that were missing in [3]. Now let us use Proposition 5.1 to prove our theorem.

**Proof of Theorem 5.1:** For \( k = 0 \), \( u_j(z, 0) = 1 \), so the left hand side of (63) is zero. Hence let us fix \( k \neq 0 \) and pick \( z \in \mathbb{D} \). Using the fact that \( \delta_1(\cdot, k) \) is onto \( \mathbb{C} \) (cf. Prop. 5.2 in [1]), there is a \( w \in \mathbb{C} \) such that \( \delta_1(w, k) = \delta_2(z, k) \) and hence \( g(z, w, k) = 0 \). Let \( U \) be a bounded, open set containing both \( z \) and \( w \). Then by Theorem 4.1 in the previous section, we know that \( u_1(z, k) \) is \( C^1 \) on \( \overline{U} \), so

\[|u_1(z, k) - u_2(z, k)| = |u_1(z, k) - u_1(w, k)| \leq C(k)|w - z|. \]

Proposition 5.1 shows that \( |w - z| \leq C_1 [\vartheta (|\log \rho|/4C_*)]^{-a} \), so we have (63).

---

**6. Proof of the main theorem**

Now we return to a bounded Lipschitz domain \( U \) which we may assume satisfies \( \overline{U} \subset \mathbb{D} \). For \( \gamma_1, \gamma_2 \in C^\sigma (\overline{U}) \), we want to be able to assume that \( \gamma_j \in C^\sigma (\mathbb{D}) \) with \( \gamma_j = 1 \) near \( \partial \mathbb{D} \) so we can apply our results from Sections 3–5. This can be achieved using the Whitney extension (cf. [4]). As in the Introduction, let \( \| \cdot \|_{\partial U} \) denote the norm of an operator \( H^{1/2}(\partial U) \to H^{-1/2}(\partial U) \), but now also let \( \| \cdot \|_{\partial \mathbb{D}} \) denote the norm of an operator \( H^{1/2}(\partial \mathbb{D}) \to H^{-1/2}(\partial \mathbb{D}) \). The following is the analogue of Theorem 6.2 in [3].
Theorem 6.1: Let $U$ be a Lipschitz domain satisfying $\overline{U} \subset \mathbb{D}$ and $\gamma_1, \gamma_2 \in C^\infty(\overline{U})$ satisfying $\|\gamma_1\|_{C^\infty(\overline{D})} \leq \Gamma$ and $\|\gamma_2\|_{\infty} \leq \kappa < 1$. There exists a constant $C = C(\kappa, U)$ and extensions $\bar{\gamma}_1, \bar{\gamma}_2$ to $\mathbb{D}$ such that $\text{supp}(\bar{\gamma}_j - 1) \subset \mathbb{D}$, $\|\bar{\gamma}_j\|_{C^\infty(\overline{D})} \leq C\Gamma$, and

$$\|\Lambda_{\bar{\gamma}_1} - \Lambda_{\bar{\gamma}_2}\|_{a\mathbb{D}} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{aU}. \quad (66)$$

The proof of this theorem follows the same steps as in [3] so we will not discuss it.

Proof of Main Theorem.: Given $\gamma_1, \gamma_2 \in C^\infty(\overline{U})$, we may use Theorem 6.1 to consider $\gamma_1, \gamma_2 \in C^\infty(\mathbb{D})$ such that $\mu_j = (1 - \gamma_j)/(1 + \gamma_j)$ has compact support in $\mathbb{D}$. For $k \in \mathbb{C}$, let $f_j(z, k) = f_{\mu_j}(z, k)$ be the complex geometric optics solution of the associated Beltrami equation $\tilde{\partial}f = \mu \tilde{\partial}f$ that was discussed in Section 3. Let

$$F(k) = F(\cdot, k) = f_1(\cdot, k) - f_2(\cdot, k). \quad (67)$$

By (60) we know that

$$\|F(k)\|_{C^{1, \sigma}(\mathbb{D})} \leq C(|k|). \quad (68a)$$

On the other hand, by Theorem 5.1 there exists $V_k(\rho)$ of the form (62) so that

$$\|F(k)\|_{C^0(\mathbb{D})} \leq V_k(\rho), \quad (68b)$$

where $\rho = \rho_\mathbb{D} = \|\Lambda_{\bar{\gamma}_1} - \Lambda_{\bar{\gamma}_2}\|_{a\mathbb{D}}$. We need to interpolate between (68a) and (68b) to show that $\|F(k)\|_{C^{1, \sigma}(\mathbb{D})} \to 0$ as $\rho \to 0$. In fact, we only need this for one nonzero value of $k$, so let us fix $k = 1$ and indicate the $\rho$-dependence by $F_{\rho}$. Then (68a) and (68b) become

$$\|F_{\rho}\|_{C^{1, \sigma}(\mathbb{D})} \leq C(1), \quad (69a)$$

$$\|F_{\rho}\|_{C^0(\mathbb{D})} \leq V_1(\rho). \quad (69b)$$

We want to interpolate to show the spatial derivatives $D F_{\rho}$ satisfy

$$\|D F_{\rho}\|_{C^0(\mathbb{D})} \leq V^*(\rho), \quad (70)$$

where $V^*(\rho)$ is a nondecreasing positive function satisfying $V^*(\rho) \to 0$ as $\rho \to 0$.

Proving (70) is somewhat technical and uses a proposition that we have proved in the Appendix. Multiplying $F_{\rho}$ by a smooth cutoff function which is 1 on $\mathbb{D}$, we can assume $F_{\rho} \in C_0^{1, \sigma}(\mathbb{C})$. Applying Proposition A.1 in the Appendix, we obtain

$$\|D F_{\rho}\|_{C^0} \leq 2\sigma \left( \xi^{-1} \left( \|F_{\rho}\|_{C^0}/[D F_{\rho}]_\sigma \right) \right) [D F_{\rho}]_\sigma, \quad (71)$$

where $\xi(r) := r\sigma(r)$ is strictly increasing $[0, \infty) \to [0, \infty)$ and surjective so its inverse $\xi^{-1}$ is well-defined and also strictly increasing; the notation $[f]_\sigma$ is defined in (A16). We need to show the right hand side tends to zero as $\rho \to 0$ in order to conclude (70). We know by (69b) that $\|F_{\rho}\|_{C^0} \to 0$ as $\rho \to 0$, so $\sigma(\xi^{-1}(\|F_{\rho}\|_{C^0}/[D F_{\rho}]_\sigma)) \to 0$ as $\rho \to 0$, provided $[D F_{\rho}]_\sigma \geq \varepsilon > 0$ as $\rho \to 0$. However, if $[D F_{\rho}]_\sigma \to 0$ as $\rho \to 0$, we still know the right hand side of (71) tends to zero because $\sigma$ is bounded on $[0, \infty)$. So (70) holds.
Now we can use $\mu = \partial_z f_\mu / \partial_z f_\mu$ and the lower bound $\inf_D |\partial_z f| \geq m$ provided by (61) to estimate
\[
\|\gamma_1 - \gamma_2\|_{C^0(U)} \leq \|\gamma_1 - \gamma_2\|_{C^0(D)} \\
\leq \frac{4}{1 - \kappa^2} \|\mu_1 - \mu_2\|_{C^0(D)} \\
\leq \frac{4}{(1 - \kappa^2)m} \|Df_1 - Df_2\|_{C^0(D)}.
\]
But finally we use (70), (66), and the fact that $V^*$ is nondecreasing to conclude
\[
\|\gamma_1 - \gamma_2\|_{C^0(U)} \leq V^*(\rho_D) \leq V^*(C \rho_U)
\]
where $\rho_U = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\partial U}$ and similarly for $\rho_D$. This completes the proof. ■

**Note**

1. A homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ is $K$-quasiconformal if it is orientation-preserving, $\phi \in H^{1,2}_{\text{loc}}(\mathbb{C})$, and the directional derivatives $\partial_{\nu}\phi$ satisfy the $\max_{\nu}|\partial_{\nu}\phi(z)| \leq K \min_{\nu}|\partial_{\nu}\phi(z)|$ for almost every $z \in \mathbb{C}$. If $\phi$ is $K$-quasiconformal, then it is locally $K^{-1}$-Hölder continuous. See [6].

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Appendix. Additional lemmas and proofs

Lemma A.1:

\[ \int_{-\pi}^{\pi} (1 - \cos(2\pi s \cos \theta)) \, d\theta \geq c \quad \text{for all } s \geq 1. \]

**Proof:** Since \( \cos \theta \) is an even function, it suffices to prove

\[ F(s) := \int_0^\pi \cos(2\pi s \cos \theta) \, d\theta \leq \pi - \varepsilon \quad \text{for all } s \geq 1. \]  

(A1)

To do this, let us introduce a change of variables \( x = s \cos \theta \), so \( x \) ranges from \( s \) to \(-s\) as \( \theta \) ranges from 0 to \( \pi \), and \( d\theta = -\frac{dx}{\sqrt{s^2 - x^2}} \). This means that we can write

\[ F(s) = \int_{-s}^{s} \cos(2\pi x) \frac{dx}{\sqrt{s^2 - x^2}} = 2 \int_{0}^{s} \cos(2\pi x) \frac{dx}{\sqrt{s^2 - x^2}}. \]  

(A2)

If we estimate \( F(s) \) using \( |\cos(2\pi x)| \leq 1 \), then we obtain \( F(s) = \pi \), which is not good enough. So we need to make use of values of \( x \) for which \( \cos(2\pi x) \) is negative.

For fixed \( s \geq 1 \), we note that \( f_s(x) = \frac{1}{\sqrt{s^2 - x^2}} \) is increasing in \( x \) for \( 0 < x < s \). Consequently, although \( \cos(2\pi x) \) is positive for \( 0 < x < 1/4 \), we may conclude that

\[ \int_0^{1/4} \cos(2\pi x) \frac{dx}{\sqrt{s^2 - x^2}} + \int_{1/4}^{3/4} \cos(2\pi x) \frac{dx}{\sqrt{s^2 - x^2}} < 0. \]

Trivially, we then conclude that \( \int_0^{3/4} \cos(2\pi x) \frac{dx}{\sqrt{s^2 - x^2}} < 0 \). For the same reason, if \( s > 7/4 \), we have \( \int_{3/4}^{7/4} \cos(2\pi x) \frac{dx}{\sqrt{s^2 - x^2}} < 0 \) and we may add the integrals together to conclude the negativity of the integral over \((0, 7/4)\). Generalizing this, we conclude that

\[ \int_{0}^{[s]^{-1/4}} \cos(2\pi x) \frac{dx}{\sqrt{s^2 - x^2}} < 0, \]

where \( s \geq 1 \) and \([s]\) denotes the greatest integer less than or equal to \( s \). Thus

\[ F(s) < 2 \int_{[s]^{-1/4}}^{s} \frac{dx}{\sqrt{s^2 - x^2}} = \pi - 2 \sin^{-1} \frac{[s]}{s}. \]  

(A3)

But it is easy to see that \( \frac{[s]}{s} \geq \eta > 0 \) for all \( s \geq 1 \), and \( \sin^{-1}(t) \) is a positive and increasing function for \( 0 < t < 1 \), so from (A3) we conclude that \( F(s) \leq \pi - \varepsilon \) as desired. \( \blacksquare \)

As in the proof of Proposition 3.1, let \( K_z(y) = \frac{1}{\pi} \chi(|y|)(z - y)^{-1} \), where \( \chi(r) \) is a smooth cut-off function satisfying \( \chi(r) = 1 \) for \( 0 \leq r \leq 1 \) and \( \chi(r) = 0 \) for \( r \geq 3/2 \). Since \( K_z \in L^1(\mathbb{R}^2) \) we know that \( \hat{K}_z(\xi) \) is bounded for \( \xi \in \mathbb{R}^2 \) and even \( \hat{K}_z(\xi) \to 0 \) as \( |\xi| \to \infty \) by Riemann-Lebesgue. But we need more precise decay as \( |\xi| \to \infty \).

Lemma A.2: The Fourier transform \( \hat{K}_z(\xi) \) satisfies the estimate (47).

**Proof:** For \( |z| > 3/2 \), \( K_z(y) \) is a smooth function of \( y \in \mathbb{R}^2 \), so \( \hat{K}_z(\xi) \) decays rapidly as \( |\xi| \to \infty \), uniformly for \( |z| \geq 2 \). Se we restrict our attention to \( |z| \leq 2 \). It suffices to consider \( \xi = (s, 0) \) for \( s > 2 \) and use the equivalent definition of the Fourier transform, \( \hat{f}(\xi) = \int e^{i\xi \cdot y} f(y) \, dy \). In the following,
we use polar coordinates \( re^{i\theta} \) for \( u = y-z \) and observe that \( |y| < 2, |z| \geq 2 \) imply \( |u| < 3 \): 

\[
-\pi \tilde{K}_z(\xi) = \int_{|y|<2} \frac{e^{iy\xi} \chi(|y|)}{y-z} \, dy = e^{iz\xi} \int_0^{2\pi} e^{-i\theta} \int_0^4 e^{isr \cos \theta} \chi(re^{i\theta} + z) \, dr \, d\theta.
\]

Now let us integrate by parts:

\[
\int_0^4 e^{isr \cos \theta} \chi(re^{i\theta} + z) \, dr = -\frac{1}{is \cos \theta} \int_0^4 \left( e^{isr \cos \theta} - 1 \right) \frac{d}{dr} \chi(re^{i\theta} + z) \, dr,
\]

where we observe that \( \chi(re^{i\theta} + z) \) is constant for \( 0 < r < \varepsilon \). For \( \theta \in (0, \pi/2) \) let us make the substitution \( t = \cos \theta \), \( dt = -\sin \theta \, d\theta \), to find

\[
\left| \int_0^{\pi/2} \int_0^4 e^{isr \cos \theta} \chi(re^{i\theta} + z) \, dr \, d\theta \right| \leq \frac{1}{s} \int_0^1 \int_0^4 \frac{|e^{isrt} - 1|}{t \sqrt{1-t^2}} \left| f(r) \right| \, dr \, dt \leq \frac{C}{s} \int_0^4 \int_0^1 \frac{|\cos(srt) - 1| + |\sin(srt)|}{t \sqrt{1-t^2}} \, dt \, dr,
\]

where \( f(r) = f_{t,z}(r) = \frac{d}{dr} \chi(re^{i\theta} + z) \) and the constant \( C \) depends on the maximum of \( f \). Let us focus on the integral involving \( \sin(srt) \). If \( sr < 1 \) then

\[
\int_0^1 \frac{|\sin(srt)|}{t \sqrt{1-t^2}} \, dt \leq \int_0^1 \frac{dt}{\sqrt{1-t}} = 1.
\]

If \( sr > 1 \) then

\[
\int_0^1 \frac{|\sin(srt)|}{t \sqrt{1-t^2}} \, dt \leq \int_0^{1/sr} \frac{sr \, dt}{\sqrt{1-t^2}} + \int_{1/sr}^1 \frac{dt}{t \sqrt{1-t}}.
\]

We can evaluate the first of these integrals and then estimate as \( sr \to \infty \):

\[
\int_{1/sr}^1 \frac{sr \, dt}{\sqrt{1-t^2}} = sr \sin^{-1} \frac{1}{sr} \approx 1.
\]

For the second integral, we can use an integral table and \( \sqrt{1-a} \approx 1 - \frac{a}{2} \) as \( a \to 0 \):

\[
\int_{1/sr}^1 \frac{dt}{t \sqrt{1-t}} = \log \left| \frac{\sqrt{1-1/sr + 1}}{\sqrt{1-1/sr - 1}} \right| = \log sr + O(1) \leq \log s + \log |r| + C \quad \text{for } s > 2, 0 < r < 4.
\]

We conclude that

\[
\int_0^1 \frac{|\sin(srt)|}{t \sqrt{1-t^2}} \, dt \leq C(\log s + \log r) \quad \text{for } s > 2, 0 < r < 4.
\]

We can similarly show

\[
\int_0^1 \frac{|\cos(st) - 1|}{|t| \sqrt{1-t^2}} \, dt \leq C(\log s + \log r) \quad \text{for } s > 2, 0 < r < 4,
\]

so we conclude

\[
\left| \int_0^{\pi/2} \int_0^4 e^{isr \cos \theta} \chi(re^{i\theta} + z) \, dr \, d\theta \right| \leq C \frac{\log s}{s} \quad \text{for } s > 2.
\]

The substitution \( t = \cos s \) can be used again for \( \theta \in (\pi/2, \pi), (\pi, 3\pi/2) \) and \( (3\pi/2, 2\pi) \), so we can put these all together to obtain the desired estimate:

\[
\pi |\tilde{K}_z(\xi)| \leq C \frac{\log |\xi|}{|\xi|} \quad \text{for } |\xi| > 2.
\]
Now we begin the preparations to prove Proposition 5.1 on the complex geometric optics solutions $u_t(z, k) = \exp[i k (z + \varepsilon_j(z, k))]$ which satisfy $|\varepsilon_j(z, k)| \leq C_{\theta} (|k|)^{-\alpha}$ for all $z \in \mathbb{C}$ and $|k| \geq 2$.

To study $g(z, w, k) = i(z - w) + k \varepsilon_{z,w}(k)$ where $\varepsilon_{z,w}(k) = i(\varepsilon_1(z, k) - \varepsilon_2(z, k))$, we need to treat its behavior for large $|k|$ differently from small $|k|$; but what is 'large' and what is 'small' depends on $\lambda := z - w$.

As in [3], we want to define a function $R : \mathbb{C} \to \mathbb{R}$ so that for $|k| \geq R(\lambda)$ we have $|\varepsilon_{z,w}(k)| \leq |\lambda|/2$ and hence $g(z, w, k) \neq 0$. In fact, since $\varepsilon : [1, \infty) \to [0, \infty)$ is strictly increasing, let us denote its inverse by $\varepsilon^{-1} : [0, \infty) \to [1, \infty)$. If we use the above constants $C_\alpha$ and $a$ to define

$$R(\lambda) := \varepsilon^{-1} \left( \left| \frac{\lambda}{4C_\alpha} \right|^{-1/a} \right), \quad (A4)$$

then $|k| \geq R(\lambda)$ indeed implies

$$|\varepsilon_{z,w}(k)| \leq 2C_\alpha \left[ \theta (|k|) \right]^{-a} \leq 2C_\alpha \left[ \theta (R(\lambda)) \right]^{-a} = \frac{|\lambda|}{2}.$$  

The proof of Proposition 5.1 is then reduced to finding a constant $C_1$ so that if

$$|\lambda| > C_1 \left[ \theta (|\log \rho|/4C_\alpha) \right]^{-a}, \quad (A5)$$

then the only zero of $g(z, w, k)$ in the set $|k| \leq R(\lambda)$ is at $k = 0$. The following is a simple relationship between $\lambda$ and $\rho$ that is useful in subsequent proofs.

**Lemma A.3:** There is a constant $C_1$ such that if $\lambda$ satisfies (A5), then

$$\rho < |\lambda| e^{-C_1R(\lambda)}, \quad (A6)$$

**Proof:** Note that (A5) implies $|\lambda/C_1|^{-1/a} < \theta (|\log \rho|/4C_\alpha)$, and the strict monotonicity of $\varepsilon^{-1}$ implies $\varepsilon^{-1}(|\lambda/C_1|^{-1/a}) < |\log \rho|/4C_\alpha$. Now, provided $C_1 \geq 4C_\alpha$, we have $|\lambda/4C_\alpha|^{-1/a} \leq |\lambda/C_1|^{-1/a}$, so by the monotonicity of $\varepsilon^{-1}$ we have

$$R(\lambda) = \varepsilon^{-1} \left( \left| \frac{\lambda}{4C_\alpha} \right|^{-1/a} \right) \leq \varepsilon^{-1} \left( \left| \frac{\lambda}{C_1} \right|^{-1/a} \right) < |\log \rho|/4C_\alpha.$$  

Consequently, we have $e^{-C_1R(\lambda)} > \rho^{1/4}$. Thus to obtain (A6) it suffices to show $\rho^{3/4} < |\lambda|$. Using (A5) again, we see that it suffices that $C_1$ is an upper bound for

$$f(\rho) = \rho^{3/4} \left[ \theta (|\log \rho|/4C_\alpha) \right]^{a} \quad \text{for } 0 < \rho < e^{-4C_\alpha}. \quad (A7)$$

But we know from (21) that $\theta (|\log \rho|/4C_\alpha)$ grows only like a power of $\log |\log \rho|$ as $\rho \to 0$, so $f(\rho) \to 0$ as $\rho \to 0$. Thus such a $C_1$ may be found. \[\blacksquare\]

For fixed $z, w$, the function $g$ satisfies a $\check{\partial}$-equation in the variable $k$. Since this does not involve the regularity of $\mu$, we may import results from the Hölder case. The following appears as Lemma 5.4 in [3]:

**Lemma A.4:** For fixed $z, w$, the function $g$ as in (65a) satisfies

$$\partial_k^\alpha g = \sigma g + E, \quad (A8)$$

where $\sigma = \sigma_{z,w}$ and $E = E_{z,w}$ satisfy

$$|\sigma (k)| \leq 2, \quad |E(k)| \leq \rho e^{c_1(|k|)}, \quad |DE(k)| \leq e^{c_1(|k|)}, \quad (A9)$$

for some constant $c_1 = c_1(k) > 0$.

We want to obtain conclusions about the behavior of $g$ from the fact that it satisfies (A8). This requires inverting the operator $\partial_k^\alpha$, but we do not have sufficient decay at infinity to directly apply
the Cauchy transform, so we need to multiply by a cut-off function. For the moment, let us ignore \( \lambda \). We fix \( R \geq 2 \) and consider a cut-off function \( \chi_R \in C_0^\infty(\overline{\mathbb{D}_2}) \) with \( \chi_R(k) = 1 \) for \( |k| \leq R \). Then, for the functions \( \sigma \) and \( E \) in Lemma A.4, let us introduce
\[
\eta_R(k) := P(\sigma \chi_R) \quad \text{and} \quad S_R(k) := P(e^{-\eta_R} E \chi_R).
\] (A10)

Here, \( P \) denotes the Cauchy transform (in the variable \( k \)), so
\[
\partial_k \eta_R = \sigma \chi_R \quad \text{and} \quad \partial_k S_R = e^{-\eta_R} E \chi_R.
\]
The functions \( \eta_R \) and \( S_R \) have the following global estimates: cf. Lemma 5.5 in [3].

**Lemma A.5:** For fixed \( z, w \), there is a constant \( c_2 = c_2(\kappa) \) such that
\[
\|\eta_R\|_{L^\infty(\mathbb{C})} \leq c_2 R \quad \text{and} \quad \|S_R\|_{L^\infty(\mathbb{C})} \leq \rho \, e^{c_2 R}.
\] (A11)

In fact, for any \( 0 < \theta < 1 \), \( c_2 \) may be chosen so that
\[
\|\nabla S_R\|_{L^\infty(\mathbb{C})} \leq \rho^\theta \, e^{c_2 R}.
\] (A12)

Now let \( \tilde{S}_R(k) = S_R(k) - S_R(0) \) so that \( \tilde{S}_R(0) = 0 \). Then \( \tilde{S}_R \) satisfies (A12) and \( \|	ilde{S}\|_{L^\infty} \leq 2\rho \, e^{c_2 R} \).

Let us define
\[
F(k) \equiv F(z, w, k) := e^{-\eta_R(k)} g(z, w, k) - \tilde{S}_R(k).
\] (A13)

A straightforward calculation shows \( \partial_k F(z, w, k) = 0 \) for \( |k| \leq R \), so for fixed \( z, w \) the function \( F \) is analytic for \( k \in \mathbb{D}_R \). By construction, \( F(0) = 0 \) and the following result shows that this is the only zero in \( \mathbb{D}_R \) when \( \lambda \) is sufficiently large.

**Lemma A.6:** There is a constant \( C_1 \) such that for \( \lambda \) satisfying (A5) the function \( F(z, w, k) \) has a unique zero at \( k = 0 \) in the set \( |k| \leq R(\lambda) \).

This appears as Proposition 5.6 in [3], which is proved by showing that \( F(k) \) is homotopic to \( k \) through nonvanishing functions on \( |k| = R(\lambda) \) and uses the estimate (A6); thus it may be repeated in our case.

We shall need some additional properties of \( F \).

**Lemma A.7:** For \( \lambda \) satisfying (A5) with \( C_1 \) given in Lemma A.6, we can write
\[
F(z, w, k) = \lambda k \, e^{v(k)} \quad \text{for} \ |k| \leq R(\lambda),
\]
where \( v(k) \) is analytic and satisfies \( |v(k)| \leq c_2 R(\lambda) \) with \( c_2 \) from Lemma A.5.

This appears as Lemma 5.7 in [3], which is proved using the analyticity of \( F(k)/k \) in \( |k| < R(\lambda) \); the estimate (A6) is used with the maximum principle and may be repeated in our case. The next two results follow from Lemma A.7 exactly as in [3].

**Corollary A.1:** For \( \lambda \) satisfying (A5) with \( C_1 \) given in Lemma A.6 and any \( \delta > 0 \), we have
\[
F^{-1}(\mathbb{D}_\delta) \subset \mathbb{D}_{\delta_1} \quad \text{where} \ \delta_1 = \delta \, e^{c_2 R(\lambda)}/|\lambda|,
\]
with \( c_2 \) from Lemma A.5.

**Corollary A.2:** For \( \lambda \) satisfying (A5) with \( C_1 \) given in Lemma A.6, there exists \( d > 0 \) so that
\[
\inf_{|k| < d} |F'(k)| > \frac{1}{2} |\lambda| e^{-c_2 R(\lambda)}.
\]
Now in order to reach our desired conclusion that, for \( \lambda = z - w \) satisfying (A5), \( g(z, w, k) \) only vanishes at \( k = 0 \), it suffices to show that the function
\[
H_{z, w}(k) = e^{-\eta(k)} g(z, w, k) = F_{z, w}(k) + \tilde{S}_{z, w}(k)
\]  
(A14)
has a unique zero at \( k = 0 \) in the set \( |k| \leq R(\lambda) \). Note that \( H \) is not analytic in \( |k| \leq R(\lambda) \), so we cannot use the principle of the argument as we did in the proof of Lemma A.6; instead we shall apply degree theory to \( H \). For this we need to know more about the zeros of \( H \); for given values of \( z, w \) let
\[
Z(H_{z, w}) = \{ k \in \mathbb{C} : H(z, w, k) = 0 \}.
\]
The following two Lemmas and Proof of Proposition 5.1 follow the ideas in [3].

**Lemma A.8:** There exists \( C_1 \) such that for \( \lambda \) satisfying (A5) we have
\[
Z(H) \subset \mathbb{D}_d,
\]
where \( d \) is given in Corollary A.2.

**Proof:** For \( k \in Z(H) \), \( F(k) = -S(k) \). But by Lemma A.5, we have \( \|S\|_{\infty} \leq \rho e^{c_2 R(\lambda)} \). So by Corollary A.1, \( |k| < \rho e^{c_2 R(\lambda)}/|\lambda| \). Thus if we have chosen \( \lambda \) so that \( \rho e^{c_2 R(\lambda)}/|\lambda| < d \), then we will have \( Z(H) \subset \mathbb{D}_d \). But, recalling that \( R(\lambda) \rightarrow 1 \) as \( |\lambda| \rightarrow \infty \), this can be arranged by requiring \( |\lambda| > C_1 \theta (|\log \rho|/4C_+)^{-\delta} \) for \( C_1 \) sufficiently large.

We also need to know about the Jacobian determinant of \( H \), which can be expressed (cf. [6]) as
\[
\det DH = |H_k|^2 - |H_k|^2.
\]  
(A15)

**Lemma A.9:** There is a constant \( C_1 \) such that for \( \lambda \) satisfying (A5) we have \( \det DH(k) > 0 \) for all \( |k| < d \), where \( d \) is as in Corollary A.2.

**Proof:** Since \( H = F + S \) with \( F \) analytic for \( |k| \leq R(\lambda) \), we have \( H_k = F_k + S_k = F' + S_k \) and \( H_k = S_k \). Using \(-2 \text{Re}(F'S_k) \leq 2|F'| |S_k| \leq \frac{1}{2}|F'|^2 + 2|S_k|^2 \), we can easily show
\[
\det DH \geq \frac{1}{2} |F'|^2 - |DS|^2.
\]
Now assuming \( |k| < d \) so that we can use Corollary A.2 and using (A12) we have
\[
\frac{1}{2} |F'|^2 - |DS|^2 > \frac{1}{2} |\lambda|e^{-c_2 R(\lambda)} - \rho^{2\delta} e^{2c_2 R(\lambda)}.
\]
Thus we can prevent \( \det DH \) from vanishing by choosing \( |\lambda| \) large enough that
\[
|\lambda|e^{-3c_2 R(\lambda)} > 2 \rho^{2\delta}.
\]
Recalling that \( R(\lambda) \rightarrow 1 \) as \( |\lambda| \rightarrow \infty \), we see that this can be achieved by taking \( |\lambda| > C_1 [\theta (|\log \rho|/4C_+)]^{-\delta} \) with \( C_1 \) sufficiently large.

Now we are finally ready to give the proof of Proposition 5.1.

**Proof of Proposition 5.1:** To begin with, let \( \mathbb{S}_R = \partial \mathbb{D}_R \) denote the circle of radius \( R \). For \( \lambda \) satisfying (A5) we know that \( g = i\lambda k + \varepsilon(k)k \) is homotopic to \( i\lambda k \), and hence to \( k \), through nonvanishing functions on \( \mathbb{S}_R \). Since \( H = e^{-\eta} g \) is homotopic to \( g \) through nonvanishing functions on \( \mathbb{S}_R \), we have
\[
\deg(H, \mathbb{D}_{R(\lambda)}, 0) = \deg(g, \mathbb{D}_{R(\lambda)}, 0) = 1.
\]
On the other hand, by the degree formula we have
\[
\deg(H, \mathbb{D}_{R(\lambda)}, 0) = \sum_{k \in Z(H)} \text{sign} \det DH(k_i).
\]
However, we know by Lemmas A.8 and A.9 that \( \text{sign} \det DH(k_i) = +1 \) for all \( k_i \in Z(H) \). So, in order to have \( \deg(H, \mathbb{D}_{R(\lambda)}, 0) = 1 \), we must have only one zero, namely at \( k = 0 \).
The following lemma and proposition are used in the proof of the main theorem. Let $\sigma(r)$ be a modulus of continuity; for example, using (51) as in Section 4. (Note that we not require $\sigma$ to satisfy the Dini condition at $r = 0$.) For $f \in C_0^\sigma(\mathbb{R}^2)$, let us introduce

$$[f]_\sigma = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\sigma(|x - y|)}.$$  \hspace{1cm} (A16)

The following result and its proof are taken from [9] (cf. (10.3) in [9]).

**Lemma A.10**: Suppose $f \in C_0^{1,\sigma}(\mathbb{R}^2)$. Then for any $r > 0$ and any $i = 1, \ldots, n$,

$$\|f_{x_i}\|_{C^0} \leq \sigma(r)[f_{x_i}]_\sigma + \frac{1}{r}\|f\|_{C^0}. \hspace{1cm} (A17)$$

**Proof**: For any $y \in \mathbb{R}^2$ and $r > 0$ we can find $y_1, y_2 \in \partial B(y, r)$ and $\bar{y} \in B(y, r)$ such that

$$|f_{x_i}(\bar{y})| = \frac{1}{2r}||f(y_1) - f(y_2)|| \leq \frac{1}{r}\|f\|_{C^0}.$$

Thus

$$|f_{x_i}(y)| \leq |f_{x_i}(y) - f_{x_i}(\bar{y})| + |f_{x_i}(\bar{y})|$$

$$\leq \sigma(r)[f]_\sigma + \frac{1}{r}\|f\|_{C^0}.$$  

Taking supremum over $y \in \mathbb{R}^2$ yields (A17). \hfill \blacksquare

Note that $\zeta(r) = r\sigma(r)$ is strictly increasing, so its inverse function $\zeta^{-1}(r)$ is defined and also strictly increasing. The following is a sort of interpolation inequality.

**Proposition A.1**: If $f \in C_0^{1,\sigma}(\mathbb{R}^2)$, then

$$\|f_{x_i}\|_{C^0} \leq 2\sigma(\zeta^{-1}\left(\|f\|_{C^0}/[f_{x_i}]_\sigma\right))[f_{x_i}]_\sigma. \hspace{1cm} (A18)$$

**Proof**: Since $f$ cannot be identically constant unless it is identically zero, we can assume $[f_{x_i}]_\sigma \neq 0$. Starting from $r = 0$, increase $r$ until $\sigma(r)[f_{x_i}]_\sigma = \frac{1}{2r}\|f\|_{C^0}$. For this value of $r$, (A17) becomes

$$\|f_{x_i}\|_{C^0} \leq 2\sigma(r)[f_{x_i}]_\sigma. \hspace{1cm} (A19)$$

We need to eliminate $r$ from this estimate. But we know

$$\zeta(r) = r\sigma(r) = \frac{\|f\|_{C^0}}{[f_{x_i}]_\sigma}.$$

So

$$r = \zeta^{-1}\left(\frac{\|f\|_{C^0}}{[f_{x_i}]_\sigma}\right).$$

Plugging this into (A19) yields (A18). \hfill \blacksquare

For example, if $\sigma = r^\alpha$ for $\alpha \in (0, 1)$, then $\zeta(r) = r\sigma(r) = r^{1+\alpha}$ and $\zeta^{-1}(r) = r^{1/(1+\alpha)}$. Consequently, (A18) implies the more familiar interpolation inequality

$$\|f_{x_i}\|_0 \leq 2\|f\|_0\|f\|_1^{1-\theta} \|f\|_1^{\theta} \quad \text{where } \theta = \alpha/(1 + \alpha).$$